Chapter 3

Powers and Spectrum of Class $wF(p,r,q)$ Operators with an Operators Equation

In this chapter we discuss powers of class $wF\ (p,r,q)$ operators for $1 \geq p > 0$, $1 \geq r > 0$ and $q \geq 1$; and an example is given on powers of class $wF\ (p,r,q)$ operators. We show that every class $wF(p,r,q)$ operator has SVEP and property (β), and Weyl's theorem holds for $f(T)$ when $f \in H(\sigma(T))$. As a continuation, we consider the equation $K^p = H^{\frac{\delta}{2}} T^{\frac{1}{2}}(T^{\frac{1}{2}} H^{\delta+r} T^{\frac{1}{2}})^{\frac{p-\delta}{\delta+r}} T^{\frac{1}{2}} H^{\frac{\delta}{2}}$,where $p > 0, r > 0$ and $p \geq \delta > -r$. As applications, we show that the inclusion relations among class $wA(p,r)$ operators are strict and show a generalization of Aluthge's result.

Sec (3.1): Powers of Class $wF(p,r,q)$ Operators

Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators in H, and a capital letter (such as T) denote an element of $B(H)$. An operator T is said to be k-hyponormal for $k > 0$ if $(T^*T)^k \geq (TT^*)k$, where T^* is the adjoint operator of T. A k-hyponormal operator T is called hyponormal if $k = 1$; semi-hyponormal if $k = 1/2$. Hyponormal and semi-hyponormal operators have been studied by many authors, such as [119,171,159,174,135]. It is clear that every k-hyponormal operator is q-hyponormal for $0 < q \leq k$ by the Löwner-Heinz theorem ($A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $1 \geq \alpha \geq 0$). An invertible operator T is said to be log-hyponormal if $\log T^*\mathrm{T} \geq \log \mathrm{T}T^*$, see [142,158]. Every invertible k-hyponormal operator for $k > 0$ is log-hyponormal since $\log\ t$ is an operator monotone function. log-hyponormality is sometimes regarded as 0-hyponormal since $(\mathrm{X}^{\mathrm{k}} - 1)/\mathrm{k} \to \log \mathrm{X}$ as $k \to 0$ for $X > 0$.

As generalizations of k-hyponormal and log-hyponormal operators, many authors introduced many classes of operators, see the following.

Definition (3.1.1)[141,146,148]:

(1) For $p > 0$ and $r > 0$, an operator T belongs to class $A(p,r)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}.$$

(2) For $p > 0, r \geq 0$ and $q \geq 1$, an operator T belongs to class $F(p,r,q)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}.$$

For each $p > 0$ and $r > 0$, class $A(p,r)$ contains all p-hyponormal and log-hyponormal operators. An operator T is a class $A(k)$ operator ([147]) if and only if T is a class $A(k,1)$ operator, T is a class $A(1)$ operator if and only if T is a class A operator ([147]), and T is a class $A(p,r)$ operator if and only if T is a class $F\left(p,r,\frac{p+r}{r}\right)$ operator.

Aluthge-Wang [143] introduced w-hyponormal operators defined by $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$ where the polar decomposition of T is $T = U|T|$ and $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is called the Aluthge transformation of T. As a generalization of w-hyponormality, Ito [128] and Yang-Yuan [139,138] introduced the classes $wA(p,r)$ and $wF(p,r,q)$ respectively.

Definition (3.1.2)[141]:

(1) For $p > 0, r > 0$, an operator T belongs to class $wA(p,r)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r} \text{ and } |T|^{2p} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{p}{p+r}}.$$

(2) For $p > 0, r \geq 0$, and $q \geq 1$, an operator T belongs to class $wF(p,r,q)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}} \text{ and } |T|^{2(p+r)\left(1-\frac{1}{q}\right)} \geq (|T|^p|T^*|^{2r}|T|^p)^{\left(1-\frac{1}{q}\right)},$$

denoting $(1-q^{-1})^{-1}$ by q^* $(when\ q > 1)$ because q and $(1-q^{-1})^{-1}$ are a couple of conjugate exponents.

An operator T is a w-hyponormal operator if and only if T is a class $wA(\frac{1}{2},\frac{1}{2})$ operator, T is a class $wA(p,r)$ operator if and only if T is a class $wF(p,r,\frac{p+r}{r})$ operator.

Ito [129] showed that the class $A(p,r)$ coincides with the class $wA(p,r)$ for each $p > 0$ and $r > 0$, class A coincides with class $wA(1,1)$. For each $p > 0, r \geq 0$ and $q \geq 1$ such that $rq \leq p + r$, [139] showed that class $wF(p,r,q)$ coincides with class $F(p,r,q)$.

Halmos ([171, Problem 209]) gave an example of a hyponormal operator T whose square T^2 is not hyponormal. This problem has been studied by many authors, see [169,170,173,175,176]. Aluthge-Wang [169] showed that the operator T^n is (k/n)-hyponormal for any positive integer n if T *is* k-hyponormal. In this section, we firstly discuss powers of class $wF(p,r,q)$ operators for $1 \geq p > 0, 1 \geq r > 0$ and $q \geq 1$. Secondly, we shall give an example on powers of class $wF(p,r,q)$ operators.

Theorem (3.1.3)[129,141]: Let $1 \geq p > 0$, $1 \geq r > 0$. *Then* T^n *is* a class $wA(\frac{p}{n},\frac{r}{n})$ operator.

Theorem(3.1.4)[172,141]: Let $1 \geq p > 0$, $1 \geq r \geq 0, q \geq 1$ and $rq \leq p + r$. If T is an invertible class $F(p,r,q)$ operator, then T^n *is a* $F(\frac{p}{n},\frac{r}{n},q)$ operator.

Theorem (3.1.5)[139,141]: Let $1 \geq p > 0, 1 \geq r \geq 0$; $q \geq 1$ when $r = 0$ and $\frac{p+r}{r} \geq q \geq 1$ when $r > 0$. If T is a class $wF(p,r,q)$ operator, then $T^{\,n}$ is a class $wF(\frac{p}{n},\frac{r}{n},q)$ operator.

Here we generalize them in theorem (3.1.6) .

Lemma (3.1.6)[127,141]: Let $\alpha \in \mathbb{R}$ and X be invertible. Then $(X^*X)^\alpha = X^*(XX^*)^{\alpha-1}X$ holds, especially in the case $\alpha \geq 1$, Lemma (3.1.6)holds without invertibility of X.

Theorem (3.1.7)[129,141]: Let $A,B \geq 0$. Then for each $p,r \geq 0$, the following assertions hold:

(1) $\left(B^{\frac{r}{2}}A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r \Longrightarrow \left(A^{\frac{p}{2}}B^r A^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \leq A^P.$

(2) $\left(A^{\frac{p}{2}}B^r A^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \leq A^P \quad and\ N(A) \subset N(B) \Longrightarrow \left(B^{\frac{r}{2}}A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r.$

Theorem (3.1.8)[137,141]: Let T be a class wA operator. Then $|T^{\,n}|^{\frac{2}{n}} \geq \cdots \geq |T^{\,2}| \geq |T|^2$ *and* $|T^*|^2 \geq |(T^{\,2})^*| \geq \cdots \geq |(T^{\,n})^*|^{\frac{2}{n}}$ hold.

Theorem (3.1.9)[139,141]: Let T be a class $wF(p_0,r_0,q_0)$ operator for $p_0 > 0, r_0 \geq 0$ and $q_0 \geq 1$. Then the following assertions hold.

(1) If $q \geq q_0$ and $r_0 q \leq p_0 + r_0$, then T is a class $wF(p_0,r_0,q)$ operator.

(2) If $q^* \geq {q_0}^*$, $p_0 q^* \leq p_0 + r_0$ and $N(T) \subset N(T^*)$, then T is a class $wF(p_0\,,r_0,q)$ operator.

(3) If $rq \leq p + r$, then class $wF(p,r,q)$ coincides with class $F(p,r,q)$.

Theorem (3.1.10)[139,141]: Let T be a class $wF\left(p_0\,,r_0,\frac{p_0+r_0}{\delta_0+r_0}\right)$ operator for $p_0 > 0, r_0 \geq 0$ and $-r_0 < \delta_0 \leq p_0$. Then T is a class $wF\left(p,r,\frac{p+r}{\delta_0+r}\right)$ operator for $p \geq p_0$ and $r \geq r_0$.

Proposition(3.1.11)[139,141]: Let $A,B \geq 0$; $1 \geq p > 0$, $1 \geq r\ 0$; $\frac{p+r}{r} \geq q \geq 1$. Then the following assertions hold.

(1) If $\left(B^{\frac{r}{2}}A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$ and $B \geq C$, then $\left(C^{\frac{r}{2}}A^p C^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq C^{\frac{p+r}{q}}$.

(2) If $B^{\frac{p+r}{q}} \geq \left(B^{\frac{r}{2}}C^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$, $A \geq B$ and the condition

$$(*) If \lim_{n\to\infty} B^{\frac{1}{2}}x_n = 0 \text{ and } \lim_{n\to\infty} A^{\frac{1}{2}}x_n \text{ exists, then } \lim_{n\to\infty} A^{\frac{1}{2}}x_n = 0$$

holds for any sequence of vectors $\{x_n\}$, then $A^{\frac{p+r}{q}} \geq \left(A^{\frac{r}{2}}C^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$.

Theorem (3.1.12)[141]: Let $1 \geq p > 0$, $1 \geq r > 0$; $q > \frac{p+r}{r}$. If T is a class $wF(p,r,q)$ operator such that $N(T) \subset N(T^*)$, then $T^{\,n}$ is a class $wF(\frac{p}{n},\frac{r}{n},q)$ operator.

In order to prove the theorem, we require the following assertions.

Proof . Put $\delta = \frac{p+r}{q} - r$, then $-r < \delta < 0$ by the hypothesis. Moreover, if $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r+\delta}{p+r}} \geq |T^*|^{2(r+\delta)}$ and $|T|^{2(p-\delta)} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{p-\delta}{p+r}}$,

then T is a class wA operator by Theorem (3.1.10) and Theorem (3.1.7), so that the following hold by taking $A_n = |T^n|^{\frac{2}{n}}$ and $B_n = |(T^n)^*|^{\frac{2}{n}}$ in Theorem (3.1.8)

$$A_n \geq \cdots \geq A_2 \geq A_1 \, and \, B_1 \geq B_2 \geq \cdots \geq B_n . \tag{1}$$

Meanwhile, A_n and A_1 satisfy the following for any sequence of vectors $\{x_m\}$, (see [137])

if $\lim_{m\to\infty} A_1^{\frac{1}{2}} x_m = 0 \; and \; \lim_{m\to\infty} A_n^{\frac{1}{2}} x_m$ exists, then $\lim_{m\to\infty} A_n^{\frac{1}{2}} x_m = 0$.

Then the following holds by Proposition (3.1.11)

$$(A_n)^{\frac{p+r}{q^*}} \geq \left((A_n)^{\frac{p}{2}}(B_1)^r (A_n)^{\frac{p}{2}}\right)^{\frac{1}{q^*}} \geq \left((A_n)^{\frac{p}{2}}(B_n)^r (A_n)^{\frac{p}{2}}\right)^{\frac{1}{q^*}},$$

and it follows that

$$|T^n|^{\frac{2(p-r)}{nq^*}} \geq (|T^n|^{\frac{p}{n}}|(T^n)^*|^{\frac{2r}{n}}|T^n|^{\frac{p}{n}})^{\frac{1}{q^*}} .$$

We assert that $N(T) \subset N(T^*)$, implies $N(T^n) \subset N((T^n)^*)$.

In fact,

$$x \in N(T^n) \Rightarrow T^{n-1} x \in N(T) \subseteq N(T^*),$$
$$\Rightarrow T^{n-2} x \in N(T^*T) = N(T) \subseteq N(T^*)$$
$$\cdots$$
$$\Rightarrow x \in N(T) \subseteq N(T^*)$$
$$\Rightarrow x \in N(T^*) \subseteq N((T^n)^*),$$

thus

$$\left(|(T^n)^*|^{\frac{r}{n}}|T^n|^{\frac{2p}{n}}|(T^n)^*|^{\frac{r}{n}}\right)^{\frac{1}{q}} \geq |(T^n)^*|^{\frac{2(p+r)}{nq}}$$

holds by Theorem (3.1.7) and the Löwner-Heinz theorem, so that T^n is a class $wF(\frac{p}{n}, \frac{r}{n}, q)$ operator. □

Theorem (3.1.13)[141]: (Furuta inequality [124], in brief FI). *If* $A \geq B \geq 0$, *then for each* $r \geq 0$,

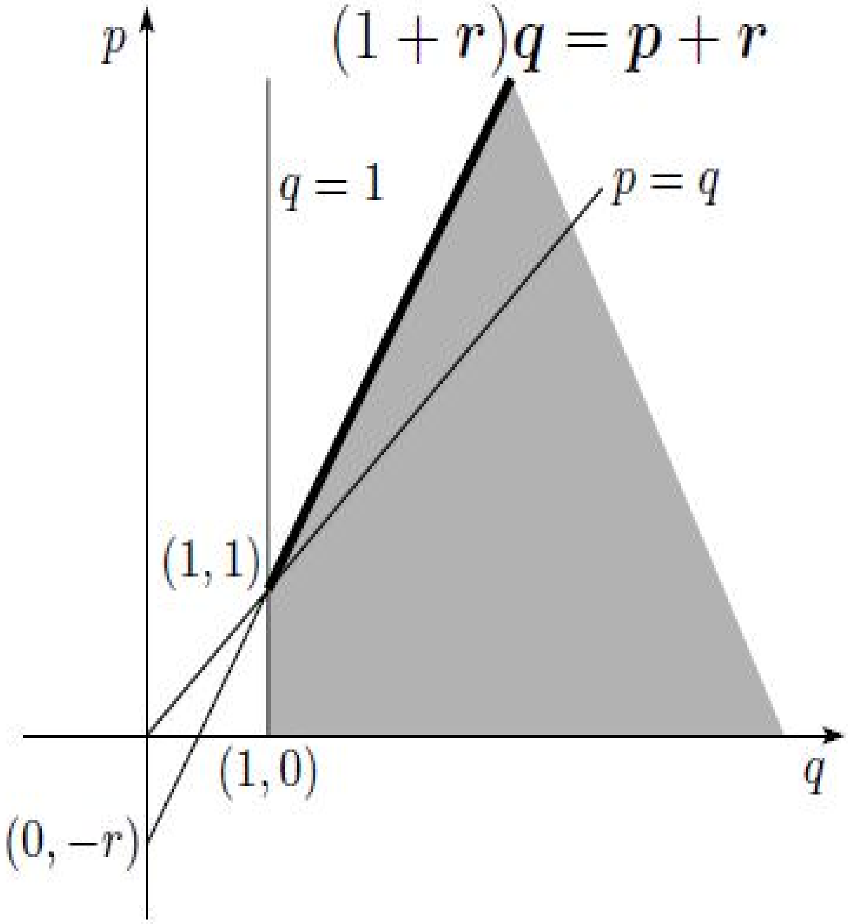
(*i*) $$\left(B^{\frac{r}{2}}A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}}B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

and

(*ii*) $$\left(A^{\frac{r}{2}}A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}}B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

Theorem(3.1.13)yields the Löwner-Heinz inequality by putting $r = 0$ in (i) or (ii) , of FI. It was shown by Tanahashi [134] that the domain drawn for p, q and r in the Figure is the best possible for Theorem (3.1.13).



Theorem (3.1.14)[141]: Let A and B be positive operators on H, U and D be operators On $\bigoplus_{k=-\infty}^{\infty} H_k$, where $H_k \cong H$, as follows

$$U = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & 1 & 0 & & & & \\ & & 1 & (0) & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

$$D = \begin{pmatrix} \ddots & & & & & & \\ & B^{\frac{1}{2}} & & & & & \\ & & B^{\frac{1}{2}} & & & & \\ & & & (A^{\frac{1}{2}}) & & & \\ & & & & A^{\frac{1}{2}} & & \\ & & & & & A^{\frac{1}{2}} & \\ & & & & & & \ddots \end{pmatrix},$$

where $(\cdot)$ shows the place of the $(0,0)$ matrix element, and $T = UD$. Then the following assertions hold.

(1)If T is a class $wF(p,r,q)$ operator for $1 \geq p > 0, 1 \geq r \geq 0, q \geq 1$ and $rq \leq p + r$, then T^n is a $wF(\frac{p}{n}, \frac{r}{n}, q)$ operator.

(2) If T is a class $wF(p,r,q)$ operator such that $N(T) \subset N(T^*)$, $1 \geq p > 0$, $1 \geq r \geq 0, q \geq 1$ and $rq > p + r$, then T^n *is a* $wF(\frac{p}{n}, \frac{r}{n}, q)$ operator.

Proof .By simple calculations ,we have

$$|T|^2 = \begin{pmatrix} \ddots & & & & & & \\ & B & & & & & \\ & & B & & & & \\ & & & (A) & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & \ddots \end{pmatrix},$$

$$|T^*|^2 = \begin{pmatrix} \ddots & & & & & & \\ & B & & & & & \\ & & B & & & & \\ & & & (B) & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & \ddots \end{pmatrix},$$

Therefore

$$\left|T^*\right|^r\left|T\right|^{2p}\left|T^*\right|^r=\begin{pmatrix}\ddots &&&&&\\ & B^{p+r} &&&&\\ && B^{p+r} &&&\\ &&& (B^{\frac{r}{2}}A^pB^{\frac{r}{2}}) &&\\ &&&& A^{p+r} &\\ &&&&& A^{p+r}\\ &&&&&& \ddots\end{pmatrix}$$

And

$$\left|T\right|^p\left|T^*\right|^{2r}\left|T\right|^p=\begin{pmatrix}\ddots &&&&&\\ & B^{p+r} &&&&\\ && B^{p+r} &&&\\ &&& (A^{\frac{p}{2}}B^rA^{\frac{p}{2}}) &&\\ &&&& A^{p+r} &\\ &&&&& A^{p+r}\\ &&&&&& \ddots\end{pmatrix},$$

Thus the following hold for $n \geq 2$

$T^{n^*}T^n$

$$=\begin{pmatrix}\ddots &&&&&&&&\\ & B^n &&&&&&&\\ && B^n &&&&&&\\ &&& B^{\frac{n-1}{2}}\mathrm{A}B^{\frac{n-1}{2}} &&&&&\\ &&&& \ddots &&&&\\ &&&&& B^{\frac{j}{2}}A^{n-j}B^{\frac{j}{2}} &&&\\ &&&&&& \ddots &&\\ &&&&&&& B^{\frac{1}{2}}A^{n-1}B^{\frac{1}{2}} &\\ &&&&&&&& (A^n) \quad A^n\\ &&&&&&&&& \ddots\end{pmatrix}$$

And

$$T^n T^{n^*}$$

$$= \begin{pmatrix} \ddots & & & & & & & & \\ & B^n & & & & & & & \\ & & (B^n) & & & & & & \\ & & & A^{\frac{1}{2}} B^{n-1} A^{\frac{1}{2}} & & & & & \\ & & & & \ddots & & & & \\ & & & & & A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}} & & & \\ & & & & & & \ddots & & \\ & & & & & & & A^{\frac{n-1}{2}} B A^{\frac{n-1}{2}} & \\ & & & & & & & & A^n \quad A^n \\ & & & & & & & & \ddots \end{pmatrix}$$

Proof of (1). T is a class $wF(p, r, q)$ operator is equivalent to the following

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \quad \text{and} \quad A^{\frac{p+r}{q^*}} \geq \left(A^{\frac{p}{2}} B^r A^{\frac{p}{2}}\right)^{\frac{1}{q^*}},$$

T^n belongs to class $wF(\frac{p}{n}, \frac{r}{n}, q)$ is equivalent to the following (2) and (3).

$$\begin{cases} \left(B^{\frac{r}{2}} \left(B^{\frac{j}{2}} A^{n-j} B^{\frac{j}{2}}\right)^{\frac{p}{n}} B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \\ \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \\ \left(\left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{r}{2n}} A^p \left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{r}{2n}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{p+r}{nq}} \\ \qquad where\ j = 1, 2, \ldots, n-1. \end{cases} \tag{2}$$

$$\begin{cases} \left(\left(B^{\frac{j}{2}} A^{n-j} B^{\frac{j}{2}}\right)^{\frac{p}{2n}} B^r \left(B^{\frac{j}{2}} A^{n-j} B^{\frac{j}{2}}\right)\right)^{\frac{1}{q^*}} \geq \left(B^{\frac{j}{2}} A^{n-j} B^{\frac{j}{2}}\right)^{\frac{p+r}{nq^*}} \\ A^{\frac{p+r}{q^*}} \geq \left(A^{\frac{p}{2}} B^r A^{\frac{p}{2}}\right)^{\frac{1}{q^*}} \\ A^{\frac{p+r}{q^*}} \geq \left(A^{\frac{p}{2}} \left(A^{\frac{j}{2}} B^{n-j} A^{\frac{j}{2}}\right)^{\frac{r}{n}} A^{\frac{p}{2}}\right)^{\frac{1}{q^*}} \\ \qquad where\ j = 1, 2, \ldots, n-1 \end{cases} \tag{3}$$

We only prove (2) because of Theorem (3.1.7).

Step 1. To show

$$\left(B^{\frac{r}{2}}\left(B^{\frac{j}{2}}A^{n-j}\,B^{\frac{j}{2}}\right)^{\frac{p}{n}}B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$$

for $j = 1,2,\dots,n-1$.

In fact, T is a class $wF(p,r,q)$ operator for $1 \geq p > 0, 1 \geq r \geq 0, q \geq 1$ and $rq \leq p + r$ implies T belongs to class $wF\left(j, n-j, \frac{n}{\delta+j}\right)$, where $\delta = \frac{p+r}{q} - r$ by Theorem (3.1.10) and Theorem (3.1.7), thus

$$\left(B^{\frac{j}{2}}A^{n-j}\,B^{\frac{j}{2}}\right)^{\frac{\delta+j}{n}} \geq B^{\delta-j} \quad and \quad A^{n-j-\delta} \geq \left(A^{\frac{n-j}{2}}B^{j}\,A^{\frac{n-j}{2}}\right)^{\frac{n-j-\delta}{n}}$$

Therefore the assertion holds by applying (i) of Theorem (3.1.13) to $\left(B^{\frac{j}{2}}A^{n-j}\,B^{\frac{j}{2}}\right)^{\frac{\delta+j}{n}}$ and $B^{\delta+j}$ for $\left(1+\frac{r}{\delta+j}\right) q \geq \frac{p}{\delta+j}+\frac{r}{\delta+j}$.

Step 2. To show

$$\left(\left(A^{\frac{j}{2}}B^{n-j}\,A^{\frac{j}{2}}\right)^{\frac{r}{2n}} A^{p}\left(A^{\frac{j}{2}}B^{n-j}\,A^{\frac{j}{2}}\right)^{\frac{r}{2n}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{j}{2}}B^{n-j}\,A^{\frac{j}{2}}\right)^{\frac{p+r}{nq}}$$

for $j = 1,2,\dots,n-1$.

In fact, similar to Step 1, the following hold

$$\left(B^{\frac{n-j}{2}}A^{j}\,B^{\frac{n-j}{2}}\right)^{\frac{\delta+n-j}{n}} \geq B^{\delta+n-j} \quad and\ A^{j-\delta} \geq \left(A^{\frac{j}{2}}B^{n-j}\,A^{\frac{j}{2}}\right)^{\frac{j-\delta}{n}} ,$$

this implies that $A^{j} \geq \left(A^{\frac{j}{2}}B^{n-j}\,A^{\frac{j}{2}}\right)^{\frac{j}{n}}$ by Theorem (3.1.7). Therefore the assertion holds by applying (i) of Theorem (3.1.13) to A^{j} and $\left(A^{\frac{j}{2}}B^{n-j}\,A^{\frac{j}{2}}\right)^{\frac{j}{n}}$ for $\left(1+\frac{r}{j}\right) q \geq \frac{p}{j}+\frac{r}{j}$.

Proof of (2). This part is similar to Proof of (1), so we omit it here. □

We are indebted to Professor K. Tanahashi for a fruitful correspondence and the referee for his valuable advice and suggestions, especially for the improvement of Theorem (3.1.12).

Corollary(3.2.15)[232]: Let $p = (1-\epsilon)$, $r = (1-\epsilon)$ and $q = (2+\epsilon)$. If T is a class $wF\big((1-\epsilon),(1-\epsilon),(2+\epsilon)\big)$ operator such that $N(T) \subset N(T^{*})$, then T^{n} is a class $wF(\frac{(1-\epsilon)}{n},\frac{(1-\epsilon)}{n},(2+\epsilon))$ operator.

In order to prove the theorem, we require the following assertions.

Proof . Put $\delta = \frac{-\epsilon(1-\epsilon)}{(2+\epsilon)}$, then $(\epsilon+1) < \delta < 0$ by the hypothesis .Moreover, if

$$\left(|T^*|^{(1-\epsilon)}|T|^{2(1-\epsilon)}|T^*|^{(1-\epsilon)}\right) \geq |T^*|^{2\frac{(1-\epsilon)^2}{(2+\epsilon)}} \, and \, |T|^{\frac{2(1-\epsilon^2)}{(2+\epsilon)}}$$
$$\geq \left(|T|^{(1-\epsilon)}|T^*|^{2(1-\epsilon)}|T|^{(1-\epsilon)}\right)^{(1+\epsilon)},$$

then T is a class wA operator by Theorem (3.1.10) and Theorem (3.1.7), so that the following hold by taking $A_n = |T^n|^{\frac{2}{n}}$ and $B_n = |(T^n)^*|^{\frac{2}{n}}$ in Theorem (3.1.8)

$$A_n \geq \cdots \geq A_2 \geq A_1 and \, B_1 \geq B_2 \geq \cdots \geq B_n .$$

Meanwhile, A_n and A_1 satisfy the following for any sequence of vectors $\{x_m\}$, (see [137])

if $\lim_{m\to\infty} A_1^{\frac{1}{2}} x_m = 0$ and $\lim_{m\to\infty} A_n^{\frac{1}{2}} x_m$ exists, $then$ $\lim_{m\to\infty} A_n^{\frac{1}{2}} x_m = 0$.

Then the following holds by Proposition (3.1.11)

$$(A_n)^{\frac{2(1-\epsilon)}{(2+\epsilon)^*}} \geq \left((A_n)^{\frac{(1-\epsilon)}{2}}(B_1)^{(1-\epsilon)}(A_n)^{\frac{(1-\epsilon)}{2}}\right)^{\frac{1}{(2+\epsilon)^*}}$$
$$\geq \left((A_n)^{\frac{(1-\epsilon)}{2}}(B_n)^{(1-\epsilon)}(A_n)^{\frac{(1-\epsilon)}{2}}\right)^{\frac{1}{(2+\epsilon)^*}},$$

and it follows that

$$|T^n|^{\frac{4(1-\epsilon)}{n(2+\epsilon)^*}} \geq \left(|T^n|^{\frac{(1-\epsilon)}{n}}|(T^n)^*|^{\frac{4(1-\epsilon)}{n}}|T^n|^{\frac{(1-\epsilon)}{n}}\right)^{\frac{1}{(2+\epsilon)^*}} .$$

We assert that $N(T) \subset N(T^*)$, implies $N(T^n) \subset N((T^n)^*)$.

In fact,

$$x \in N(T^n) \Rightarrow T^{n-1} x \in N(T) \subseteq N(T^*),$$
$$\Rightarrow T^{n-2} x \in N(T^*T) = N(T) \subseteq N(T^*)$$
$$\cdots$$
$$\Rightarrow x \in N(T) \subseteq N(T^*)$$
$$\Rightarrow x \in N(T^*) \subseteq N((T^n)^*),$$

thus

$$\left(|(T^n)^*|^{\frac{(1-\epsilon)}{n}}|T^n|^{\frac{4(1-\epsilon)}{n}}|(T^n)^*|^{\frac{(1-\epsilon)}{n}}\right)^{\frac{1}{(2+\epsilon)}} \geq |(T^n)^*|^{\frac{4(1-\epsilon)}{n(2+\epsilon)}}$$

holds by Theorem (3.1.7) and the Löwner-Heinz theorem, so that T^n is a class $wF(\frac{(1-\epsilon)}{n}, \frac{(1-\epsilon)}{n}, (2+\epsilon))$ operator. □

Sec(3.2) Spectrum of Class $wF(p, r, q)$ Operators

A capital letter (such as T) means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For $p > 0$, an operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where T^* is the adjoint operator of T. An invertible operator T is said to be log-hyponormal if $\log(T^*T) \geq \log(TT^*)$. If $p = 1, T$ is called hyponormal, and if $p = 1/2$, T is called semi-hyponormal. Log-hyponormality is sometimes regarded as 0-hyponormal since $(X^p - 1)/p \to \log X \, as \, p \to$

0 for $X > 0$. See Martin and Putinar [131] and Xia [135] for basic properties of hyponormal and semi-hyponormal operators. Log-hyponormal operators were introduced by Tanahashi [142], Aluthge and Wang [143], and Fujii et al. [144] independently. Aluthge [145] introduced p-hyponormal operators.

As generalizations of p-hyponormal and log-hyponormal operators, many authors introduced many classes of operators. Aluthge and Wang [143] introduced w-hyponormal operators defined by $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$, where the polar decomposition of T is T = U|T| and $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is called Aluthge transformation of T. For $p > 0$ and $r > 0$, Ito [128] introduced class $wA(p,r)$ defined by

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq (|T^*|^{2r}, \qquad |T|^p|T^*|^{2r}\,|T|^p)^{\frac{s}{p+r}} \leq |T|^{2p}. \tag{4}$$

Note that the two exponents $r/(p+r)$ and $p/(p+r)$ in the formula above satisfy $r/(p+r) + p/(p+r) = 1$, Yang and Yuan [138] introduced class $wF(p,r,q)$.

Definition (3.2.1) [138,139]: For $p > 0, r > 0$, and $q \geq 1$, an operator T belongs to class $wF(p,r,q)$ if

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{2(p+r)/q}\,,\ |T|^{2(p+r)\left(1-\frac{1}{q}\right)} \geq (|T|^p|T^*|^{2r}|T|^p)^{(1-1/q)} \tag{5}$$

Denote $(1-q^{-1})^{-1}$ by q^* when $q > 1$ because q *and* $(1-q^{-1})^{-1}$ are a couple of conjugate exponents. It is clear that class $wA(p,r)$ equals class $wF(p,r,(p+r)/r)$. w-hyponormality equals $wA(1/2,1/2)$ [128]. Ito and Yamazaki [129] showed that class $wA(p,r)$ coincides with class $A(p,r)$ (introduced by Fujii et al. [146]) for each $p > 0$ and $r > 0$. Consequently, class $wA(1,1)$ equals class A (i.e., $|T^2| \geq |T|^2$, introduced by Furuta et al. [147]). Reference [139] showed that class $wF(p,r,q)$ coincides with class $F(p,r,q)$ (introduced by Fujii and Nakamoto [148]) when $rq \leq p+r$.

Recently, there are great developments in the spectral theory of the classes of operators above.We cite [138, 149–157]. In this section, we will discuss several spectral properties of class

$wF(p,r,q)$ for $p > 0, r > 0, p+r \leq 1$, and $q \geq 1$.

In this Section, we prove that Riesz idempotent E_λ of T with respect to each nonzero isolated point spectrum λ is selfadjoint and $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$. also we will show that each class $wF(p,r,q)$ operator has SVEP (single-valued extension property) and Bishop's property (β). and we show that Weyl's theorem holds for class $wF(p,r,q)$. Now we show that Riesz idempotent.

Let $\sigma(T), \sigma_p(T), \sigma_{jp}(T), \sigma_a(T), \sigma_{ja}(T)$, and $\sigma_r(T)$ mean the spectrum, point spectrum, joint point spectrum, approximate point spectrum, joint approximate point spectrum, and residual spectrum of an operator T, respectively (cf. [138, 158]). $\sigma_r^{Xia}(T)$ and $\sigma_{iso}(T)$ mean the set $\sigma(T) - \sigma_a(T)$ and the set of isolated points of $\sigma(T)$, see [158, 135]. If $\lambda \in \sigma_{iso}(T)$, the Riesz idempotent E_λ of T with respect λ is defined by

$$E_\lambda = \int_{\partial\mathfrak{D}} (z - T)^{-1} dz, \tag{6}$$

where $\mathfrak{D}$ is an open disk which is far from the rest of $\sigma(T)$ and $\partial\mathfrak{D}$ means its boundary. Stampfli [159] showed that if T is hyponormal, then E_λ is selfadjoint and $E_\lambda\mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*$. The recent developments of this result are shown in [151,152,155,157], and so on.

In this section, it is shown that when $\lambda \neq 0$, this result holds for class $wF(p,r,q)$ with $p + r \leq 1$ and $q \geq 1$. It is always assumed that $\lambda \in \sigma_{iso}(T)$ when the idempotent E_λ is considered.

Theorem (3.2.2)[138,149]: Let $\lambda \neq 0$, and let $\{x_n\}$ be a sequence of vectors. Then the following assertions are equivalent.

(1) $(T - \lambda)x_n \to 0$ *and* $(T^* - \bar{\lambda})x_n \to 0$.

(2) $(|T| - |\lambda|)x_n \to 0$ *and* $(U - e^{i\theta})x_n \to 0$.

(3) $(|T|^* - |\lambda|)x_n \to 0$ *and* $(U^* - e^{-i\theta})x_n \to 0$.

Theorem (3.2.3)[138]: If T is a class $wF(p,r,q)$ operator for $p + r \leq 1$ and $q \geq 1$, then them following assertions hold.

(1) If $Tx = \lambda x,\ \lambda \neq 0, then\ T^*x = \bar{\lambda}x$.

(2) $\sigma_a(T) - \{0\} = \sigma_{ja}(T) - \{0\}$.

(3) If $Tx = \lambda x, Ty = \mu y$ and $\lambda \neq \mu$, then $(x,y) = 0$.

Theorem (3.2.4)[138,139]: If T is a class $wF(p,r,q)$ operator, then there exists $\alpha_0 > 0$, which satisfies

$$|T(p,r)|^{2\alpha_0} \geq |T|^{2\alpha_0(p+r)} \geq |T(p,r)^*|^{2\alpha_0}. \tag{7}$$

Lemma (3.2.5)[138]: If T belongs to class $wF(p,r,q)$ for $p + r \leq 1, \lambda = |\lambda|e^{i\theta} \in \mathfrak{G}$, *and* $\lambda_{p+r} = |\lambda|^{p+r}e^{i\theta}$, then $\ker(T - \lambda) = \ker(T(p,r) - \lambda_{p+r})$.

Proof. We only prove $\ker(T - \lambda) \supseteq \ker(T(p,r) - \lambda_{p+r})$ because $\ker(T - \lambda) \subseteq \ker(T(p,r) - \lambda_{p+r})$ is obvious by Theorems (3.2.2)-(3.2.3)

If $\lambda \neq 0$, let $0 \neq x \in \ker(T(p,r) - \lambda_{p+r})$. By Theorem (3.2.4), $T(p,r)$ is α_0-hyponormal and we have

$$|T(p,r)|x = |\lambda|^{p+r}x = \left|\left(T(p,r)\right)^*\right|x,$$

$$|T(p,r)|^{2\alpha_0} - |(T(p,r))^*|^{2\alpha_0} \geq |T(p,r)|^{2\alpha_0} - |T|^{2\alpha_0(p+r)} \geq 0. \tag{8}$$

Hence $(|T(p,r)|^{2\alpha_0} - |T|^{2\alpha_0(p+r)})x = 0$,

$$\left\||T|^{2\alpha_0(p+r)}x - |\lambda|^{2\alpha_0(p+r)}x\right\| \leq \left\||T|^{2\alpha_0(p+r)}x - |T(p,r)|^{2\alpha_0}x\right\| + \left\||T(p,r)|^{2\alpha_0}x - |\lambda|^{2\alpha_0(p+r)}x\right\| = 0. \tag{9}$$

On the other hand, $(T(p,r))^*x = |\lambda|^{p+r}e^{-i\theta}x$ implies that $|T|^rU^*x = |\lambda|^re^{-i\theta}x, T^* = |\lambda|e^{-i\theta}x$. Therefore,

$$\|(T-\lambda)x\|^2 = \|Tx\|^2 - \lambda(x,Tx) - \bar{\lambda}(Tx,x) + |\lambda|^2\|x\|^2$$
$$= \||T|x\|^2 - \lambda(T^*x,x) - \bar{\lambda}(x,T^*x) + |\lambda|^2\|x\|^2 = 0. \qquad (10)$$

If $\lambda = 0$, let $0 \neq x \in \ker T(p,r)$, then $x \in \ker|T| = \ker T$ by Theorem (3.2.4) so that $\ker(T-\lambda) \supseteq \ker(T(p,r) - \lambda_{p+r})$. □

Lemma (3.2.6)[138,153,160]: If A is normal, then for every operator $B, \sigma(AB) = \sigma(BA)$.

Let $\mathfrak{F}$ be the set of all strictly monotone increasing continuous nonnegative functions on $\mathfrak{R}^+ = [0,\infty)$. Let $\mathfrak{F}_0 = \{\Psi \in \mathfrak{F} : \Psi(0) = 0\}$. For $\Psi \in \mathfrak{F}_0$, the mapping $\widetilde{\Psi}$ is defined by $\widetilde{\Psi}(\rho e^{i\theta}) = e^{i\theta}\Psi(\rho)$ and $\widetilde{\Psi}(T) = U\Psi(|T|)$.

Theorem (3.2.7)[138,161]: If $\Psi \in \mathfrak{F}_0$, then for every operator T, $\sigma_{ja}(\widetilde{\Psi}(T)) = \widetilde{\Psi}(\sigma_{ja}(T))$.

Lemma (3.2.8)[138]: Let T belong to class $wF(p,r,q)$ with $p + r \leq 1, \lambda = |\lambda|e^{i\theta} \in \mathfrak{G}, T(t) = U|T|^{1-t+t(p+r)}$, and $\tau_t(\rho e^{i\theta}) = e^{i\theta}\rho^{1+t(p+r-1)}$, where $t \in [0,1]$. Then

$$\sigma_a(T(t)) = \tau_t(\sigma_a(T)),\ \sigma_r^{Xia}(T(t)) = \tau_t\left(\sigma_r^{Xia}(T)\right),$$
$$\sigma(T(t)) = \tau_t(\sigma(T)). \qquad (11)$$

Proof. We only need to show that $\sigma_a(T(t)) = \tau_t(\sigma_a(T))$ by homotopy property of the spectrum [135, page 19].

Since T belongs to class $wF(p,r,q)$ with $p + r \leq 1$, $T(t)$ belongs to class $wF(p/(1+t(p+r-1)), r/(1+t(p+r-1), q))$ with $p/(1+t(p+r-1)) + r/(1+t(p+r-1)) \leq 1$. By Theorems (3.2.3)(2) and (3.2.7),

$$\sigma_a(T(t)) - \{0\} = \sigma_{ja}(T(t)) - \{0\}$$
$$= \tau_t(\sigma_{ja}(T) - \{0\}) = \tau_t(\sigma_a(T)) - \{0\}. \qquad (12)$$

On the other hand, if $0 \in \sigma_a(T)$, then there exists a sequence $\{x_n\}$ of unit vectors such that $U|T|x_n \to 0$. Hence $|T|x_n = U^*U|T|x_n \to 0$, so that $|T|^{1/(2^m)}x_n \to 0$ for each positive integer m by induction. Take a positive integer $m(t)$ such that $1/(2^{m(t)}) \leq 1 + t(p+r-1)$, then

$$|T|^{1+t(p+r-1)}x_n = |T|^{1+t(p+r-1)-1/(2^{m(t)})}|T|^{1/(2^{m(t)})}x_n \to 0 \qquad (13)$$

and $0 \in \sigma_a(T(t))$. It is obvious that if $0 \in \sigma_a(T(t))$, then $0 \in \sigma_a(T)$ because of $p + r \leq 1$. Therefore $\sigma_a(T(t)) = \tau_t(\sigma_a(T))$. □

Theorem (3.2.9)[138,150]: If T *is* p-hyponormal or log-hyponormal, then E_λ *is* selfadjoint and $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$.

Riesz and Sz.-Nagy [162] gave the the formula $E_\lambda\mathcal{H} = \{x \in \mathcal{H} : \|(T-\lambda)^n x\|^{1/n} \to 0\}$.

Lemma(3.2.10)[138]: For any operator $T, |T|^p \ker(T-\lambda) \subseteq |T|^p E_\lambda \mathcal{H} \subseteq E_\lambda(p,r)\mathcal{H}$ for $p+r=1$.

Proof. Let $x \in E_\lambda$, by the formula above we have

$$\|(T(p,r)-\lambda)^n |T|^p x\|^{1/n} = \||T|^p (T-\lambda)^n x\|^{1/n} \to 0. \qquad (14)$$

Hence $|T|^p x \in E_\lambda(p,r)\mathcal{H}$.

Lemma(3.2.11)[138]: If T belongs to class $wF(p,r,q)$ with $p+r \leq 1$, then

$$\ker T = E_0\mathcal{H} = E_0(p,r)\mathcal{H} = \ker\big(T(p,r)\big). \qquad (15)$$

Note that $\lambda_{p+r} \in \sigma_{iso}(T(t))$ if $\lambda \in \sigma_{iso}(T)$ by Lemma (3.2.8), so the notion $E_0(p,r)$ in Lemma (3.2.10) is reasonable.

Proof. Since $T(p,r)$ is α_0-hyponormal by Theorem(3.2.4), we only need to prove that $E_0\mathcal{H} \subseteq E_0(p,r)\mathcal{H}$ for $E_0\mathcal{H} \supseteq E_0(p,r)\mathcal{H}$ holds by Lemma (3.2.5) and Theorem (3.2.9). We may also assume that $p+r=1$ by Lemma (3.2.5)
It follows from Lemma (3.2.10) that

$$|T|^p E_0(p,r)\mathcal{H} \subseteq |T|^p E_0 \subseteq E_0(p,r)\mathcal{H}, \qquad (16)$$

thus $E_0(p,r)\mathcal{H}$ is reduced by $|T|^p$.

Let $x \in E_0\mathcal{H}$ and $x = x_1 + x_2 \in E_0(p,r)\mathcal{H} \oplus (E_0(p,r)\mathcal{H})^\perp$. Then $|T|^p x \in |T|^p E_0\mathcal{H} \subseteq E_0(p,r)\mathcal{H}$, $|T|^p x_1 \in E_0(p,r)\mathcal{H}$, $|T|^p x_2 \in (E_0(p,r)\mathcal{H})^\perp$ *by* (16), and $E_0(p,r)\mathcal{H}$ is reduced by $|T|^p$.

Thus $|T|^p x_2 = |T|^p x - |T|^p x_1 \in E_0(p,r)\mathcal{H}, |T|^p x_2 \in E_0(p,r)\mathcal{H} \cap (E_0(p,r)\mathcal{H})^\perp$ so that

$$x_2 \in \ker|T|^p \subseteq \ker\big(T(p,r)\big) = E_0(p,r)\mathcal{H}, x \in E_0(p,r)\mathcal{H}.$$

Theorem (3.2.12)[138]: Let T belong to class $wF(p,r,q)$ *with* $p+r \leq 1, \lambda = |\lambda|e^{i\theta} \in \mathfrak{G}$, and $\lambda_{p+r} = |\lambda|^{p+r}e^{i\theta}$, then the following assertions hold.

(1)If $\lambda \neq 0$,then $E_\lambda = E_\lambda(p,r)$ and $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$, where $E_\lambda(p,r)$ is the Riesz idempotent of $T(p,r) = |T|^p U|T|^r$ (the generalized Aluthge transformation of T) with respect to λ_{p+r}.

(2) *If* $\lambda = 0$, then $\ker T = E_0\mathcal{H} = E_0(p,r)\mathcal{H} = \ker(T(p,r))$.

Reference [156] gave an example that the operator T is w-hyponormal, E_0 is not selfadjoint, and $kerT \neq kerT^*$.

An operator T is said to be isoloid if $\sigma_{iso}(T) \subseteq \sigma_p(T)$, is said to be reguloid if $(T-\lambda)\mathcal{H}$,is closed for each $\lambda \in \sigma_{iso}(T)$.

Proof. We only need to prove (1) for (2) holds by Lemma (3.2.11). Since $\sigma(T(p,r)) = \sigma(U|T|^{p+r}) = \{e^{i\theta}\rho^{p+r} : e^{i\theta}\rho \in \sigma(T)\}$ by Lemmas (3.2.6) and (3.2.8) , $\lambda_{p+r} \in \sigma_{iso}(T(p,r))$. Hence

$$(E_\lambda(p,r)\mathcal{H})^\perp = ker(E_\lambda(p,r)) = (I - E_\lambda(p,r))\mathcal{H} \qquad (17)$$

by Theorem (3.2.9), so $\lambda_{p+r} \notin \sigma(T(p,r)|_{(E_\lambda(p,r)\mathcal{H})^\perp})$. By Theorem (3.2.3)(1) and Lemma (3.2.5), we have $T = \lambda \oplus T_{22}$ *on* $\mathcal{H} = E_\lambda(p,r)\mathcal{H} \oplus (E_\lambda(p,r)\mathcal{H})^\perp$, where $T_{22} = T|_{(ker(T-\lambda))^\perp}$.

Since $\ker(T-\lambda)$ is reduced by T, T_{22} also belongs to class $wF(p,r,q)$ and $T_{22}(p,r) = T(p,r)|_{(E_\lambda(p,r)\mathcal{H})^\perp}$ so that $\lambda \notin \sigma(T_{22})$ because $\lambda_{p+r} \notin \sigma(T_{22}(p,r))$. Hence $T-\lambda = 0 \oplus (T_{22}-\lambda)$ and
$\ker(T-\lambda)^* = \ker(T-\lambda) \oplus \ker(T_{22}-\lambda)^* = \ker(T-\lambda)$.

Meanwhile, $E_\lambda = \int_{\partial\mathfrak{D}}(z-\lambda)^{-1} \oplus (z-T_{22})^{-1}dz = 1 \oplus 0 = E_\lambda(p,r)$. □

Theorem (3.2.13)[138]: If T belongs to class $wF(p,r,q)$ with $p+r \le 1$, then T is isoloid and reguloid.

Proof . We only need to prove that T is reguloid for T being isoloid follows by Theorem (3.2.12) easily.

If $\lambda \in \sigma_{iso}(T)$, then $\mathcal{H} = E_\lambda\mathcal{H} + (I-E_\lambda)\mathcal{H}$, where $E_\lambda\mathcal{H}$, and $(I-E_\lambda)\mathcal{H}$ are topologically complemented [163, page 94]. By $T = T|_{E_\lambda\mathcal{H}} + T|_{(I-E_\lambda)\mathcal{H}}$ on $\mathcal{H} = E_\lambda\mathcal{H} + (I-E_\lambda)\mathcal{H}$ and Theorem (3.2.12), we have

$$(T-\lambda)\mathcal{H} = \left(T|_{(I-E_\lambda)\mathcal{H}} - \lambda\right)(I-E_\lambda)\mathcal{H}. \tag{18}$$

Therefore $(T-\lambda)\mathcal{H}$ is closed because $\sigma(T|_{(I-E_\lambda)\mathcal{H}}) = \sigma(T) - \{\lambda\}$. □

Definition (3.2.14)[138]: An operator T is said to have SVEP at $\lambda \in \mathfrak{G}$ if for every open neighborhood G $of\ \lambda$, the only function $f \in H(G)$ such that $(T-\lambda)f(\mu) = 0$ on G is $0 \in H(G)$, where $H(G)$ means the space of all analytic functions on G.

When T have SVEP at each $\lambda \in \mathfrak{G}$, say that T has SVEP.

This is a good property for operators. If T has SVEP, then for each $\lambda \in \mathfrak{G}, \lambda - T$ is invertible if and only if it is surjective (cf. [164, 153]).

Definition (3.2.15)[138]: An operator T is said to have Bishop's property $(\beta)\ at\ \lambda \in \mathfrak{G}$ if for every open neighborhood G of λ, the function $f_n \in H(G)$ with $(T-\lambda)f_n(\mu) \to 0$ uniformly on every compact subset of G implies that $f_n(\mu) \to 0$ uniformly on every compact subset of G.

When T has Bishop's property (β) at each $\lambda \in \mathfrak{G}$, simply say that T has property (β). This is a generalization of SVEP and it is introduced by Bishop [165] in order to develop a general spectral theory for operators on Banach space.

Lemma (3. 2.16)[138,153]: Let G be open subset of complex plane $\mathfrak{G}$ and let $f_n \in H(G)$ be functions such that $\mu f_n(\mu) \to 0$ uniformly on every compact subset $of\ G$, then $f_n(\mu) \to 0$ uniformly on every compact subset $of\ G$.

Theorem (3.2.17)[138]: Let p and r be positive numbers. If $p+r = 1$, then T has SVEP if and only If $T(p,r)$ has SVEP, T has property (β) if and only if $T(p,r)$ has property (β). In particular, every class $wF(p,r,q)$ operator T with $p+r \le 1$ has SVEP and property (β).

This result is a generalization of [153]. Lemma (3.2.16) and the relations between T and its transformation $T(p,r)$ are important:

$$T(p,r)|T|^p = |T|^pU|T|^r|T|^p = |T|^pT,$$
$$U|T|^rT(p,r) = U|T|^r|T|^pU|T|^r = TU|T|^r. \tag{19}$$

Proof . We only prove that T has property (β) if and only if $T(p,r)$ has property (β) because the assertion that T has SVEP if and only if $T(p,r)$ has SVEP can be proved similarly.

Suppose that $T(p,r)$ has property (β). Let G be an open neighborhood of λ and let $f_n \in H(G)$ be functions such that $(\mu - T)\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G. By (19), $(T(p,r) - \mu)|T|^p\, f_n\,(\mu) = |T|^p(T - \mu)\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G. Hence $T\, f_n\,(\mu) = U|T|^r\, |T|^p\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G for $T(p,r)$ has property (β), so that $\mu\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G, and T having property (β) follows by Lemma (3. 2.16).

Suppose that T has property (β). Let G be an open neighborhood of λ and let $f_n \in H(G)$ be functions such that $(\mu - T(p,r))\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G. By (19), $(\mu - T)(U|T|^r\, f_n\,(\mu)) = U|T|^r(\mu - T(p,r))\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G. Hence $T(p,r)\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G for T has property (β) so that $\mu\, f_n\,(\mu) \to 0$ uniformly on every compact subset of G, and $T(p,r)$ having property (β) follows by Lemma (3. 2.16).

For a Fredholm operator T, ind T means its (Fredholm) index. A Fredholm operator T is said to be Weyl if ind T $= 0$.

Let $\sigma_e(T), \sigma_w(T)$, and $\pi_{00}(T)$ mean the essential spectrum,Weyl spectrum, and the set of all isolated eigenvalues of finite multiplicity of an operator T, respectively (cf. [163, 152]).

According to Coburn [166], we say that Weyl's theorem holds for an operator T if $\sigma(T) - \sigma_w(T) = \pi_{00}(T)$. Very recently, the theorem was shown to hold for several classes of operators including w-hyponormal operators and paranormal operators (cf. [152, 167, 155]).

In this section, we will prove that Weyl's theorem and Weyl spectrum mapping theorem hold for class $wF(p,r,q)$ operator T with $p + r \leq 1$. We also assume that $p + r = 1$ because of the inclusion relations among class $wF(p,r,q)$ [139].

Theorem (3.2.18)[138]: Let $p > 0, r > 0$, and $q \geq 1, s \geq p, t \geq r$. If T is a class $wF(p,r,q)$ operator and $T(s,t)$ is normal, then T is normal.

Lemma (3.2.19)[138]: If T belongs to class $wF(p,r,q)$ with $p + r = 1$ and is Fredholm, then indT ≤ 0.

This result can be regarded as a good complement of Theorem (3.2.12).

Proof. Since T is Fredholm, $|T|^p$ is also Fredholm and $\mathrm{ind}(|T|^p) = 0$. By (19),

$$\mathrm{ind}T = \mathrm{ind}(|T|^p\,\mathrm{T}) = \mathrm{ind}(\mathrm{T(p,r)}|T|^p) = \mathrm{ind}\big(\mathrm{T(p,r)}\big). \tag{20}$$

Hence, ind$T \leq 0$ for $\mathrm{ind}(T(p,r)) \leq 0$ by Theorem (3.20). □

Theorem (3.2.20)[138]: Let T belong to class $wF(p,r,q)$ with $p+r = 1$ and let $H(\sigma(T))$ be the space of all functions f analytic on some open set G containing σ(T), then the following assertions hold.

(1) $Weyl's$ theorem holds for T.

(2) $\sigma_w(f(T)) = f(\sigma_w(T))$ $when\ f \in H(\sigma(T))$.

(3) Weyl's theorem holds for $f(T)$ when $f \in H(\sigma(T))$.

This is a generalization of the related assertions of [152].

Proof . (1) Let $\lambda \in \sigma(T) - \sigma_w(T)$, then $T-\lambda$ is Fredholm, $\operatorname{ind}(T-\lambda) = 0$, and $\dim\ker(T-\lambda) > 0$.

If λ is an interior point of $\sigma(T)$, there would be an open subset $G \subseteq \sigma(T)$ including λ such that $\operatorname{ind}(T-\mu) = \operatorname{ind}(T-\lambda) = 0$ for all $\mu \in G$ [163, page 357]. So $\dim\ker(T-\mu) > 0$ for all $\mu \in G$, this is impossible for T has SVEP by Theorem (3.2.17) [164, Theorem 10]. Thus $\lambda \in \partial\sigma(T) - \sigma_w(T), \lambda \in \sigma_{iso}(T)$ by [163, Theorem 6.8, page 366], and $\lambda \in \pi_{00}(T)$ follows.

Let $\lambda \in \pi_{00}(T)$ then the Riesz idempotent E_λ has finite rank by Theorem (3.2.12), and $\lambda \in \sigma(T) - \sigma_w(T)$ follows.

(2) We only need to prove that $\sigma_w(f(T)) \supseteq f(\sigma_w(T))$ since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ is always true for any operators.

Assume that $f \in H(\sigma(T))$ is not constant. Let $\lambda \notin \sigma_w(f(T))$ and $f(z) - \lambda = (z-\lambda_1)\cdots(z-\lambda_k)g(z)$, where $\{\lambda_i\}_1^k$ are the zeros of $f(z) - \lambda$ in G (listed according to multiplicity) and $g(z) \neq 0$ for each $z \in G$. Thus

$$f(T) - \lambda = (T-\lambda_1)\ldots\ldots(T-\lambda_k)g(T). \qquad (21)$$

Obviously, $\lambda \in f(\sigma_w(T))$ if and only if $\lambda_i \in \sigma_w(T)$ for some i. Next we prove that $\lambda_i \notin \sigma_w(T)$ for every $i \in \{1,\ldots,k\}$, thus $\lambda \notin f(\sigma_w(T))$ and $\sigma_w(f(T)) \supseteq f(\sigma_w(T))$.

In fact, for each $i, T-\lambda_i$ is also Fredholm because $f(T) - \lambda$ is Fredholm. By Theorem (3.2.12) and Lemma (3.2.19), $\operatorname{ind}(T-\lambda_i) \leq 0$ for each i. Since $0 = \operatorname{ind}(f(T) - \lambda) = \operatorname{ind}(T-\lambda_1) + \cdots + \operatorname{ind}(T-\lambda_k), \operatorname{ind}(T-\lambda_i) = 0$ and $\lambda_i \notin \sigma_w(T)$ for each i.

(3) By Theorem (3.2.13), T is isoloid and it follows from [168] that

$$\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)). \qquad (22)$$

On the other hand, $f(\sigma(T) - \pi_{00}(T)) = f(\sigma_w(T)) = \sigma_w(f(T))$ by (1)-(2). The proof is complete.

Theorem (3.2.21)[138]: Let T belong to class $wF(p,r,q)$ with $p+r = 1$, then the following assertions hold.

(i) If $m_2(\sigma(T)) = 0$ where m_2 means the planar Lebesgue measure, then T is normal.

(ii) If $\sigma_w(T) = 0$, then T is compact and normal.

Theorem (3.2.21)(i) is a generalization of [161] and (ii) is a generalization of [159].

Proof . (i) By α_0-hyponormality of $T(p,r)$ and Putnam's inequality for α_0-hyponormal operators [161], $T(p,r)$ is normal. Hence, (i) follows by Theorem (3.2.18).

(ii) Since $\sigma_w(T) = 0, \sigma(T) - \{0\} = \pi_{00}(T) \subseteq \sigma_{iso}(T)$ by Theorem (3.2.20)(i). Hence $m_2(\sigma(T)) = 0$ and T is normal by (i).

Next to prove that T is compact, we may assume that $\sigma(T) - \{0\}$ is a countable infinite set for $\sigma(T) - \{0\} \subseteq \sigma_{iso}(T)$. Let $\sigma(T) - \{0\} = \{\lambda_n\}_1^\infty$ with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0$ and $\lambda_0 = \lim_{n\to\infty} |\lambda_n|$, then $\lambda_0 = 0$. Since every E_{λ_n} has finite rank by Theorems (3.2.12) and (3.2.20), for every $\varepsilon > 0, \bigoplus_{|\lambda_n|>\varepsilon} E_{\lambda_n}$ also has finite rank. Therefore T is compact [163, page 271]. □

Corollary(3.2.22)[232]: For any operator $T, |T|^{(1-r)}\ker(T - \lambda) \subseteq |T|^{(1-r)}E_{\lambda}\mathcal{H} \subseteq E_{\lambda}((1-r),r)\mathcal{H}$ for $p = 1-r$.

Proof. Let $x \in E_\lambda$, by the formula above we have

$$\left\|(T((1-r),r) - \lambda)^n |T|^{(1-r)}x\right\|^{1/n} = \left\||T|^{(1-r)}(T - \lambda)^n x\right\|^{1/n} \to 0.$$

Hence $|T|^{(1-r)}x \in E_\lambda((1-r),r)\mathcal{H}$.

Sec (3.3): The Operator Equation $K^p = H^{\frac{\delta}{2}} T^{\frac{1}{2}}(T^{\frac{1}{2}} H^{\delta+r} T^{\frac{1}{2}})^{\frac{p-\delta}{\delta+r}} T^{\frac{1}{2}} H^{\frac{\delta}{2}}$ and its Applications

A capital letter (such as T) means a bounded linear operator on a Hilbert space. $T \geq 0$ *and* $T > 0$ mean a positive operator and an invertible positive operator, respectively.

In [133], Pedersen and Takesaki developed the operator equation $K = THT$ as a useful tool for the noncommutative Radon–Nikodym theorem. By using Douglas's majorization theorem [123], Nakamoto [132] provided a simple proof.
As generalizations, Bach and Furuta [121,125] gave deep discussion on the equation $K = T(H^{\frac{1}{n}} T)^n$.

Theorem (3.3.1)[118,125]: Let H and K be bounded positive operators on a Hilbert space, and assume that H is nonsingular.

(1) The following statements are equivalent for any natural number n:

(a) $aH^{\frac{1}{n}} \geq \left(H^{\frac{1}{2n}} K H^{\frac{1}{2n}}\right)^{\frac{1}{n+1}}$ for some $a \geq 0$;

(b) there exists a unique positive operator T such that $\|T\| \leq a$,and

$$K = T^{\frac{1}{2}}\left(T^{\frac{1}{2}} T^{\frac{1}{n}} T^{\frac{1}{2}}\right)^n T^{\frac{1}{2}} \quad . \tag{23}$$

(2) If there exists a positive operator T satisfying (23) for some natural number n, then, for each natural number $m \leq n$, there exists a positive operator T_1 satisfying

$$K = T_1^{\frac{1}{2}} \left(T_1^{\frac{1}{2}} H^{\frac{1}{m}} T_1^{\frac{1}{2}} \right)^m T_1^{\frac{1}{2}} . \quad (24)$$

Lin [130] showed a generalization of Theorem (3.3.1)(1) via Furuta inequality [124] under the restriction $a = 1$.

Theorem (3.3.2)[118,121]: Given any natural number n and m with $m < n$, there exist a nonsingular positive operator H and a positive operator K such that Eq. (24) is solvable and (23) is unsolvable.

In this section , as a continuation, we consider the following equation for $p > 0, r > 0$ and $p \geq \delta > -r$

$$K^p = H^{\frac{\delta}{2}} T^{\frac{1}{2}} (T^{\frac{1}{2}} H^{\delta+r} T^{\frac{1}{2}})^{\frac{p-\delta}{\delta+r}} T^{\frac{1}{2}} H^{\frac{\delta}{2}} . \quad (25)$$

Obviously, the special case $p = 1, r = \frac{1}{n}$ and $\delta = 0$ of (25) becomes (23). Theorems (3.3.1)–(3.3.2) are extended to Theorems (3.3.4)–(3.3.5), respectively.

Some applications are obtained. We show that the inclusion relations in the following result are strict. See Theorem (3.3.3) below.

Theorem (3.3.3)[118,128,129]: Let T be a class $wA(p,r)$ operator, then T is a class $wA(p_1, r_1)$ operator for $p_1 \geq p > 0$ and $r_1 \geq r > 0$.

A kind of polar decomposition of Aluthge transformation [119] is given. See Theorems (3.3.14)–(3.3.15) below.

Theorem (3.3.4)[118,123]: The following assertions are equivalent for any operators A and B.

(1) $AA^* \leq \lambda BB^*$ for some $\lambda \geq 0$.

(2) There exists $a\ C\ with\ A = BC$ and $\|C\| \leq \lambda$.

Lemma (3.3.5)[118,126,127]: Let $\alpha \in R$ and X be invertible. Then

$$(X^*X)^\alpha = X^*(XX^*)^{\alpha-1}X,$$

especially in case $\alpha \geq 1$ the equality holds without invertibility of X.

Theorem (3.3.6)[118,137,139]: (Furuta type inequality). Let $A, B \geq 0$, $\alpha_0, \beta_0 > 0$, $-\beta_0 < \delta_0 \leq \alpha_0$, $-\beta_0 \leq \bar{\delta}_0 < \alpha_0$.

(1) If $0 \leq \delta_0 \leq \alpha_0$, then

$$\left(B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}}\right)^{\frac{\beta_0+\delta_0}{\beta_0+\alpha_0}} \geq B^{\beta_0+\delta_0} \Longrightarrow \left(B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}}\right)^{\frac{\beta+\delta_0}{\beta+\alpha}} \geq B^{\beta+\delta_0}$$

for any $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.

(2) If $-\beta_0 \leq \bar{\delta}_0 \leq 0$ and $N(A) \subset N(B)$, then

$$A^{\alpha_0+\bar{\delta}_0} \geq \left(A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}}\right)^{\frac{\alpha_0+\bar{\delta}_0}{\alpha_0+\beta_0}} \Longrightarrow A^{\alpha+\bar{\delta}_0} \geq \left(A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}}\right)^{\frac{\alpha+\bar{\delta}_0}{\alpha+\beta_0}},$$

for any $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.
Theorem (3.3.6) is important to the proof of (2) of Theorem (3.3.8).
Lemma (3.3,7)[118,134]: *Let a, b, d and θ be real numbers and satisfy $a + b > 0, ab = d^2$, and $S = \begin{pmatrix} a & de^{-i\theta} \\ de^{i\theta} & b \end{pmatrix}$. Then*

$$S^p = (a + b)^{p-1}S \quad \text{for } p > 0.$$

Theorem (3.3.8)[118]: Let H and K be bounded positive operators on a Hilbert space, and assume that H is nonsingular.
(1) The following statements are equivalent for any $p > 0, r > 0$ *and* $p \geq \delta \geq 0$:

$(a) aH^{\delta+r} \geq \left(H^{\frac{r}{2}}K^pH^{\frac{r}{2}}\right)^{\frac{\delta+r}{p+r}}$ for some $a \geq 0$;

(b) there exists a unique positive operator T satisfies $\|T\| \leq a$ and (25).
If in additional H is invertible, (1) holds for $p \geq \delta > -r$.
(2) If there exists a positive operator T satisfying (25) for fixed $p > 0, r > 0$ and $p \geq \delta \geq 0$, then, for $p_1 \geq p$ and $r_1 \geq r$, there exists a positive operator T_1 satisfying

$$K^{p_1} = H_1^{\frac{\delta}{2}} T_1^{\frac{1}{2}}(T_1^{\frac{1}{2}} H^{\delta+r_1}T_1^{\frac{1}{2}})^{\frac{p_1-\delta}{\delta+r_1}} T_1^{\frac{1}{2}} H_1^{\frac{\delta}{2}} . \quad (26)$$

Lin [130] showed case $\delta = \frac{p-nr}{n+1}$ of Theorem(3.3.8)(1) under some restrictions.
Proof . The proof is similar to [125].
$(a) \Rightarrow (b)$. By Theorem (3.3.4), there exists a S such that

$$(H^{\frac{r}{2}}K^pH^{\frac{r}{2}})^{\frac{\delta+r}{2(p+r)}} = H^{\frac{\delta+r}{2}} S = S^*H^{\frac{\delta+r}{2}} .$$

Put $T = SS^*$, then $\|T\| \leq a$ and by Lemma (3.3.7),

$$H^{\frac{r}{2}}K^pH^{\frac{r}{2}} = H^{\frac{\delta+r}{2}} T^{\frac{1}{2}}(T^{\frac{1}{2}} H^{\delta+r}T^{\frac{1}{2}})^{\frac{p-\delta}{\delta+r}} T^{\frac{1}{2}} H^{\frac{\delta+r}{2}} .$$

So (25) holds for H is singular.
$(b) \Rightarrow (a)$. For a with $\|T\| \leq a$, by Lemma (3.3.7), (25) implies

$$\left(H^{\frac{r}{2}}K^pH^{\frac{r}{2}}\right)^{\frac{\delta+r}{(p+r)}} = \left(H^{\frac{\delta+r}{2}} T^{\frac{1}{2}}\left(T^{\frac{1}{2}} H^{\delta+r}T^{\frac{1}{2}}\right)^{\frac{p-\delta}{\delta+r}} T^{\frac{1}{2}} H^{\frac{\delta+r}{2}}\right)^{\frac{\delta+r}{(p+r)}}$$

$$= H^{\frac{\delta+r}{2}}T H^{\frac{\delta+r}{2}} \leq a H^{\delta+r} . \quad (27)$$

To show the uniqueness of T. Assume that Z also satisfies (25), by (27) we have

$$H^{\frac{\delta+r}{2}}Z H^{\frac{\delta+r}{2}} = \left(H^{\frac{r}{2}}K^pH^{\frac{r}{2}}\right)^{\frac{\delta+r}{(p+r)}} = H^{\frac{\delta+r}{2}} T H^{\frac{\delta+r}{2}},$$

therefore $Z = T$.
Next to prove (2). By the assumption and (1), (a) holds for some $a > 0$, that is

$$\left(a^{\frac{p+r}{p(\delta+r)}}\ H\right)^{\delta+r} \geq \left(\left(a^{\frac{p+r}{p(\delta+r)}}\ H\right)^{\frac{r}{2}} K^{p} \left(a^{\frac{p+r}{p(\delta+r)}}\ H\right)^{\frac{r}{2}}\right)^{\frac{\delta+r}{(p+r)}} \tag{28}$$

So that the following follows from (2) of Theorem (3.3.8):

$$\left(a^{\frac{p+r}{p(\delta+r)}}\ H\right)^{\delta+r_1} \geq \left(\left(a^{\frac{p+r}{p(\delta+r)}}\ H\right)^{\frac{r_1}{2}} K^{p_1} \left(a^{\frac{p+r}{p(\delta+r)}}\ H\right)^{\frac{r_1}{2}}\right)^{\frac{\delta+r_1}{(p_1+r_1)}},$$

that is

$$a^{\frac{p+r}{p(\delta+r)}\cdot\frac{p_1(\delta+r_1)}{(p_1+r_1)}}\ H^{\delta+r_1} \geq \left(H^{\frac{r_1}{2}}K^{p_1}H^{\frac{r_1}{2}}\right)^{\frac{\delta+r_1}{(p_1+r_1)}}.$$

Therefore (26) is solvable. □

Remark (3.3.9)[118]: For each $p > 0, r > 0$ and $\min\{p,1\} \geq \delta > -r$, it is clear that the condition (a) is satisfied if H is invertible or, more generally $a^{\frac{p+r}{p(\delta+r)}} H \geq K$ for some $a \geq 0$ by (28) and Furuta inequality [124]. In the first case, the solution T to (25) is given by $T = H^{\frac{-(\delta+r)}{2}}\left(H^{\frac{r}{2}}K^{p}H^{\frac{r}{2}}\right)^{\frac{\delta+r}{(p+r)}}H^{\frac{-(\delta+r)}{2}}$ by (27).

Theorem (3.3.10)[118]: Given any positive numbers p, r, p_1 and r_1 with $r_1 > r$, there exist a nonsingular positive operator H and a positive operator K such that case $\delta = 0$ of Eq. (26) is solvable and case $\delta = 0$ of (25) is unsolvable. To give proofs, the following results are needful.

Proof . The proof is inspired by [121].

For a natural number k, let $A_k = \begin{pmatrix} 1 & 0 \\ 0 & k^{-4} \end{pmatrix}$ and $B_k = \frac{1}{1+k^2}\begin{pmatrix} 1 & k^{-1} \\ k^{-1} & k^{-2} \end{pmatrix}$. Take $H = \oplus_{k=1}^{\infty} A_k^{\frac{1}{r_1}}$ and $K = \oplus_{k=1}^{\infty} K_k^{\frac{1}{p_1}}$ where $K_k = A_k^{\frac{-1}{2}} B_k^{\frac{p_1+r_1}{r_1}} A_k^{\frac{-1}{2}}$. By Lemma (3.3.9), $K_k = \frac{1}{(1+k^2)k^{2p_1/r_1}}\begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}$, hence $\left\|K_k^{\frac{1}{p_1}}\right\| = K^{-2/r_1} \leq 1$ and K is meaningful.

Next to show that the operators H and K satisfy the conditions.

In fact, $H^{r_1} - \left(H^{\frac{r_1}{2}}K^{p_1}H^{\frac{r_1}{2}}\right)^{\frac{r_1}{(p_1+r_1)}} = \oplus_{k=1}^{\infty} (A_k - B_k) \geq 0$ and this implies case $\delta = 0$ of (26) is solvable by (1) of Theorem (3.3.8). Meanwhile, case $\delta = 0$ of (25) is unsolvable for H and K here. Otherwise, also by (1) of Theorem (3.3.8), H and K satisfy (a) for some $a > 0$. This implies that

$$aA_k^{r/r_1} \geq \left(A_k^{\frac{r}{2r_1}}K_k^{\frac{p}{p_1}}A_k^{\frac{r}{2r_1}}\right)^{\frac{r}{p+r}}.$$

By Lemma (3.3.7),

$$a \geq A_k^{\frac{-r}{2r_1}} \left\{A_k^{\frac{r}{2r_1}} \frac{1}{(1+k^2)k^{2p/r_1}} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix} A_k^{\frac{r}{2r_1}}\right\}^{\frac{r}{p+r}} A_k^{\frac{-r}{2r_1}}$$

$$= A_k^{\frac{-r}{2r_1}} \left(\frac{1}{(1+k^2)k^{2p/r_1}}\right)^{\frac{r}{p+r}} \left(\frac{1}{1+k^{2(1-2r/r_1)}}\right)^{\frac{p}{p+r}} \begin{pmatrix} 1 & k^{1-2r/r_1} \\ k^{1-2r/r_1} & k^{2(1-2r/r_1)} \end{pmatrix} A_k^{\frac{-r}{2r_1}}$$

$$= \left(\frac{1}{(1+k^2)k^{2p/r_1}}\right)^{\frac{r}{p+r}} \left(\frac{1}{1+k^{2(1-2r/r_1)}}\right)^{\frac{p}{p+r}} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix} . \tag{29}$$

Therefore,

$$a \geq \left(\frac{1+k^2}{k^{2r/r_1}\left(1+k^{2(1-2r/r_1)}\right)}\right)^{\frac{p}{p+r}} = \left(\frac{1+k^2}{\left((k^{2r/r_1}+k^{2(1-r/r_1)}\right)}\right)^{\frac{p}{p+r}} . \tag{30}$$

So that $a \geq \infty$ by letting $k \to \infty$for $\max\{2r/r_1, 2(1-r/r_1)\} < 2$. This is a contradiction. □

A fact in the proof of Theorem (3.3.10) is useful.

Theorem (3.3.11)[118]: Given any positive numbers p, r, p_1 and r_1 with $r_1 > r$, there exist invertible positive operators H and K such that

$$H^{r_1} \geq \left(H^{\frac{r_1}{2}} K^{p_1} H^{\frac{r_1}{2}}\right)^{\frac{r_1}{(p_1+r_1)}} \quad , aH^{r} \ngeq \left(H^{\frac{r}{2}} K^{p} H^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \quad ,$$

where a is an arbitrary positive number.

Proof. The operators H and K in the proof of Theorem (3.3.10) are suitable. □

We Show Some Applications . For $q > 0, T$ is called a q-hyponormal operator if $(T^*T)^q \geq (TT^*)^q$, where T^* is the adjoint operator of T . If $q = 1, T$ is called a hyponormal operator and if $q = 1/2$, T is called a semi-hyponormal operator. See Martin and Putinar [131] and Xia [135] for related topics and basic properties of hyponormal operators.

Aluthge [119] introduced Aluthge transformation $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ where the polar decomposition of T is $T = U|T|$. For each $p > 0$ and $r > 0, \tilde{T}_{p,r} = |T|^p U|T|^r$ is called generalized Aluthge transformation.

As a generalization of q-hyponormal operators, Ito [128] introduced class $wA(p,r)$ defined by

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r} \text{ and } (|T|^p|T^*|^{2r}|T|^p)^{\frac{p}{p+r}} \leq |T|^{2p} .$$

See[120,129,137,138] for related topics.

Lemma (3.3.12)[118]: For positive operators A and B on a Hilbert space $\mathcal{H}$define operators U and D on $\bigoplus_{k=-\infty}^{\infty} = \mathcal{H}_k$ where $\mathcal{H}_k \cong \mathcal{H}$ Has follows:

$$U = \begin{pmatrix} \ddots & & & & & \\ & \ddots & 0 & & & \\ & & 1 & & & \\ & & & (0) & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}, \quad D = \begin{pmatrix} \ddots & & & & \\ & B^{\frac{1}{2}} & & & \\ & & (A^{\frac{1}{2}}) & & \\ & & & A^{\frac{1}{2}} & \\ & & & & \ddots \end{pmatrix}$$

where $(\cdot)$ shows the place of the $(0,0)$ matrix element, and $T = UD$. Then the following assertions hold for each $p > 0, r > 0$ *and* $\beta > 0$:

(1) $(|T^*|^r|T|^{2p}|T^*|^r)^\beta \geq |T^*|^{2(p+r)\beta}$ if and only if $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^\beta \geq B^{(p+r)\beta}$.

(2) $|T|^{2(p+r)\beta} \geq (|T|^p|T^*|^{2r}|T|^p)^\beta$ if and only if $A^{(p+r)\beta} \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^\beta$.

This example appears in [140,141] and is a modification of [122, Theorem 2] and [136, Lemma 1].

Proof. By easy calculation,

$$|T|^2 = \begin{pmatrix} \ddots & & & \\ & B & & \\ & & (A) & \\ & & & A \\ & & & & \ddots \end{pmatrix}, \quad |T^*|^2 = \begin{pmatrix} \ddots & & & \\ & B & & \\ & & (B) & \\ & & & A \\ & & & & \ddots \end{pmatrix},$$

Therefore

$$|T^*|^r|T|^{2p}|T^*|^r = \begin{pmatrix} \ddots & & & \\ & B^{p+r} & & \\ & & (B^{\frac{r}{2}}A^pB^{\frac{r}{2}}) & \\ & & & A^{p+r} \\ & & & & \ddots \end{pmatrix}$$

and

$$|T|^{p}|T^{*}|^{2r}|T|^{p}=\begin{pmatrix} \ddots & & & \\ & B^{p+r} & & \\ & & (A^{\frac{p}{2}}B^{r}A^{\frac{p}{2}}) & \\ & & & A^{p+r} \\ & & & & \ddots \end{pmatrix},$$

By comparing the $(0,0)$ elements of the operator matrices above, the assertions hold. □

Theorem (3.3.13)[118]: Given any positive numbers p,r,p_1 and $r_1$with $r_1 > r$, there exists an operator T such that T *is a class* $wA(p_1,r_1)$ operator but not a class $wA(p,r)$ operator.Theorem (3.3.13) implies that the inclusion relations in Theorem (3.3.3) are strict.

Proof . By Theorem (3.3.11), there exist invertible positive operators H and K on a Hilbert space $\mathcal{H}$ such that

$$H^{r_1} \geq \left(H^{\frac{r_1}{2}}K^{p_1}H^{\frac{r_1}{2}}\right)^{\frac{r_1}{(p_1+r_1)}} \quad , \quad H^{r} \ngeq \left(H^{\frac{r}{2}}K^{p}H^{\frac{r}{2}}\right)^{\frac{r}{p+r}} .$$

Let $A = H$ and $B = K$, define an operator T on $\bigoplus_{k=-\infty}^{\infty} = \mathcal{H}_k$ *where* $\mathcal{H}_k = \mathcal{H}$ as Lemma (3.3.12). Then T a class $wA(p_1,r_1)$ operator but not a class $wA(p,r)$ operator by Lemma (3.3.12). □

Aluthge [119] showed a kind of polar decomposition of Aluthge transformation on invertible q-hyponormal operators via the equation

$K = THT$.

Theorem (3.3.14)[118,119]: Let T be a invertible q-hyponormal operator and the polar decomposition of $\tilde{T}$ *be* $\tilde{T} = \tilde{U}|\tilde{T}|$. *Then* $|\tilde{T}| = |T|^{1/2}S^{-1}|T|^{1/2}$ *and* $\tilde{U} = |T|^{1/2}US|T|^{-1/2}$ where S is the solution to the equation $|T| = SU^{*}|T|US$.

The following assertion say that this result holds for any invertible operator T .

Theorem (3.3.15)[118]:*Let* T *be an invertible operator and the polar decomposition of* $\tilde{T}_{p,r}$ *be* $\tilde{T}_{p,r} = \tilde{U}_{p,r}|\tilde{T}_{p,r}|$*.Then* $|\tilde{T}_{p,r}| = |T|^{r}S^{-1}|T|^{r}$ *and* $\tilde{U}_{p,r} = |T|^{p}US|T|^{-r}$ where S is the solution to the equation $|T|^{2r} = SU^{*}|T|^{2p}US$.

Proof. By Remark (3.3.9), the solution S to $|T|^{2r} = SU^{*}|T|^{2p}US$. exists and $S = H^{\frac{-1}{2}}(H^{\frac{1}{2}}KH^{\frac{1}{2}})^{\frac{1}{2}}H^{\frac{-1}{2}}$ where $H = U^{*}|T|^{2p}U$ and $= |T|^{2r}$. Hence S is invertible for T is invertible and

$$\left|\tilde{T}_{p,r}\right| = \left(|T|^{r} S^{-1} |T|^{2r} S^{-1} |T|^{r}\right)^{1/2} = |T|^{r} S^{-1} |T|^{r} .$$

Moreover, $\widetilde{U}_{p,r} = \tilde{T}_{p,r}\ |\tilde{T}_{p,r}|^{-1} = |T|^{p} US |T|^{-r}$. □

Corollary(3.3.16)[232]: Given any positive numbers $p, r_1 - \epsilon, p_1$, there exist a nonsingular positive operator H and a positive operator K such that case $\delta = 0$ *of* Eq. (26) is solvable and case $\delta = 0$ *of* (25) is unsolvable. To give proofs, the following results are needful.

Proof . The proof is inspired by [121].

For a natural number k, *let* $A_k = \begin{pmatrix} 1 & 0 \\ 0 & k^{-4} \end{pmatrix}$ and $B_k = \frac{1}{1+k^2}\begin{pmatrix} 1 & k^{-1} \\ k^{-1} & k^{-2} \end{pmatrix}$. Take $H = \oplus_{k=1}^{\infty} A_k^{\frac{1}{r_1}}$ and $K = \oplus_{k=1}^{\infty} K_k^{\frac{1}{p_1}}$ where $K_k = A_k^{\frac{-1}{2}} B_k^{\frac{p_1+r_1}{r_1}} A_k^{\frac{-1}{2}}$. By Lemma (3.3.9), $K_k = \frac{1}{(1+k^2)k^{2p_1/r_1}}\begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}$, hence $\left\|K_k^{\frac{1}{p_1}}\right\| = K^{-2/r_1} \le 1$ and K is meaningful.

Next to show that the operators H and K satisfy the conditions.

In fact, $H^{r_1} - \left(H^{\frac{r_1}{2}} K^{p_1} H^{\frac{r_1}{2}}\right)^{\frac{r_1}{(p_1+r_1)}} = \oplus_{k=1}^{\infty} (A_k - B_k) \ge 0$ and this implies case $\delta = 0$ of (26) is solvable by (1) of Theorem (3.3.8). Meanwhile, case $\delta = 0$ of (25) is unsolvable for H and K here. Otherwise, also by (1) of Theorem (3.3.8), H and K satisfy (a) for some $a > 0$. This implies that

$$aA_k^{(r_1-\epsilon)/r_1} \ge \left(A_k^{\frac{(r_1-\epsilon)}{2r_1}} K_k^{\frac{p}{p_1}} A_k^{\frac{(r_1-\epsilon)}{2r_1}}\right)^{\frac{(r_1-\epsilon)}{p+(r_1-\epsilon)}} .$$

By Lemma (3.3.7),

$$a \ge A_k^{\frac{-(r_1-\epsilon)}{2r_1}} \left\{A_k^{\frac{(r_1-\epsilon)}{2r_1}} \frac{1}{(1+k^2)k^{2p/r_1}} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix} A_k^{\frac{(r_1-\epsilon)}{2r_1}}\right\}^{\frac{(r_1-\epsilon)}{p+r}} A_k^{\frac{-(r_1-\epsilon)}{2r_1}}$$

$$= A_k^{\frac{-(r_1-\epsilon)}{2r_1}} \left(\frac{1}{(1+k^2)k^{2p/r_1}}\right)^{\frac{(r_1-\epsilon)}{p+r}} \left(\frac{1}{1+k^{2(1-2(r_1-\epsilon)/r_1)}}\right)^{\frac{p}{p+(r_1-\epsilon)}} \begin{pmatrix} 1 & k^{1-2(r_1-\epsilon)/r_1} \\ k^{1-2(r_1-\epsilon)/r_1} & k^{2(1-2(r_1-\epsilon)/r_1)} \end{pmatrix} A_k^{\frac{-(r_1-\epsilon)}{2r_1}}$$

$$= \left(\frac{1}{(1+k^2)k^{2p/r_1}}\right)^{\frac{(r_1-\epsilon)}{p+(r_1-\epsilon)}} \left(\frac{1}{1+k^{2(1-2(r_1-\epsilon)/r_1)}}\right)^{\frac{p}{p+(r_1-\epsilon)}} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix} .$$

Therefore,

$$a \ge \left(\frac{1+k^2}{k^{2(r_1-\epsilon)/r_1}\left(1+k^{2(1-2(r_1-\epsilon)/r_1)}\right)}\right)^{\frac{p}{p+(r_1-\epsilon)}} = \left(\frac{1+k^2}{\left((k^{2(r_1-\epsilon)/r_1}+k^{2(1-(r_1-\epsilon)/r_1)}\right)}\right)^{\frac{p}{p+(r_1-\epsilon)}} .$$

So that $a \ge \infty$ by letting $k \to \infty$ for $\max\{2(r_1-\epsilon)/r_1, 2(1-(r_1-\epsilon)/r_1)\} < 2$. This is a contradiction. □