# Dedication

To My Family …

## Acknowledge meant

Thanks to Alla who gave me the patience to conduct this study. My deepest thanks goes to my teacher and supervisor professor Dr. Shawgy Hussein AbdAlla, at Sudan university of science and technology, for all his guidance inspiration, encouragement constant support and advice. I would like to thank all my friends, lately, I indebted to my family for all their help and support. Finally my deepest thanks to shendi university to this scholarship .

# Abstract

We show  Stečkin's theorem, transference, spectral decompositions, operators and examples with bounded and unbounded imaginary powers. We discuss the powers, spectrum of class operators, and the operator equations with applications .We establish the maps preserving the harmonic mean or the parallel sum of positive operators preserving Lebesgue decompositions. We study the strong convergence theorems and viscosity approximation methods for nonexpansive mappings and monotone mappings, with a general iterative method with strongly positive operators for general variational inequalities .We characterize the Krein's differential system and generalization with the effective construction of a class of positive operators, in Hilbert space which do not admit triangular factorization.

# الخلاصة

أوضحنا مبرهنة ستيكن والانتقال والتفكيك الطيفي والمؤثرات والامثلة مع المحدودية والقوي التخيلية غير المحدودة. درسنا القوي وطيف مؤثرات العائلة ومعادلات المؤثر مع التطبيقات. أ سسنا الرواسم الحافظة للوسط التوافقي او الجمع المتوازي للمؤثرات الموجبة الحافظة لتفكيكات لبيق. تم دراسة مبرهنات التقارب القوي وطرق التقارب اللزج للرواسم غير الممددة والرواسم الرتيبة مع طريقة التكرار العامة مع المؤثرات الموجبة القوية لمتباينات التغير العامة .تم تشخيص النظام التفاضلي لكرين والتعميم مع البناء الفعال لعائلة المؤثرات الموجبة في فضاء هلبرت والذي لا يعترف بالتحلل للعوامل المثلثية.

# Introduction

Let $Y$ be a closed subspace of $L^p(\mu)$, where $\mu$ is an arbitrary measure and $1 < p < \infty$. It is shown that every invertible operator $V$ on $Y$ such that $\sup\{\|V^n\|: n = 0, \pm 1, \pm 2, \dots\} < \infty$ can be expressed in the form $V = e^{iA}$, where $A$ is well bounded of type $(B)$ .This result, which fails if $Y$ is replaced by an arbitrary reflexive space, is obtained by a blend of the transference method of Coif-man and Weiss with Stečkin's Theorem and a recent result in abstract operator theory. We introduce the class of operators under consideration and discuss several examples to show its importance. Then the functional calculus for this class is presented and exploited, also we show the uniform estimates on the imaginary powers of $\varepsilon + A$. A study of examples of Unbounded Imaginary Powers of Operators are follows. We show some spectral properties of class $w\,F(p, r, q)$ operators for $p > 0, r > 0, p + r \leq 1$, and $q \geq 1$. It is shown that if $T$ is a class $wF(p, r, q)$ operator, then the Riesz idempotent $E_\lambda$ of $T$ with respect to each nonzero isolated point spectrum $\lambda$ is selfadjoint and $E_\lambda\, \mathcal{H} = ker\,(T - \lambda) = ker\,(T - \lambda)^*$. Let $H$ and $K$ be bounded positive operators on a Hilbert space, and assume that $H$ is nonsingular. Based on Pedersen and Takesaki's research on the operator equation $K = THT$, Furuta and Bach gave deep discussion on the equation $K = T^{\frac{1}{2}} \left(T^{\frac{1}{2}} T^{\frac{1}{n}} T^{\frac{1}{2}}\right)^n T^{\frac{1}{2}}$ where $n$ is a natural number. We show that any transformation is implemented by an invertible bounded linear or conjugate-linear operator on $H$. Similar results concerning the parallel sum and the arithmetic mean in the place of the harmonic mean are also presented. It is showed that every such transformation $\phi$ is of the form $\phi(A) = SAS^*$ $(A \in B(H)^+)$ for some invertible bounded linear or conjugate-linear operator $S$ on $H$. We introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly monotone mapping in a Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. Viscosity approximation methods for nonexpansive mappings are studied. Consider the iteration process $\{x_n\}$,where $x_0 \in C$ is arbitrary and $x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), f$ is a contraction on $C, S$ is a nonexpansive self-mapping of a closed convex subset $C$ of a Hilbert space $H$. We introduce and study a general iterative method with strongly positive operators for finding solutions of a general variational inequality problem with inverse strongly monotone mapping in a real Hilbert space. We investigate Krein's differential systems as well as correct some assertions both in M.G. Krein's article and in our works dedicated to the Krein's systems and their generalization.

We investigate the problem of the triangular factorization of positive operators in a Hilbert space. A class of non-factorable positive operators is constructed.

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