Chapter 1

Expressions for Hankel Determinants of the Narayana Polynomials

In this chapter we provided a Hankel transform evaluation of the linear combination of two consecutive shifted Narayana polynomials, using the same method (based on orthogonal polynomials) and previously obtained moment representation of Narayana and shifted Narayana polynomials .

Section (1.1): Hankel Transform and Series Reversion with the Shifted Narayana Polynomials

Definition (1.1.1) [1]:

Narayana numbers (In combainatorics, the Narayana numbers $N(n,k), n = 1,2,3,\dots, 1 \leq k \leq n$, form a triangular array of natural numbers, called Narayana triangule, that occur in various counting problems. They are named for T.V. Narayana (1930-1987) a mathematician from India. The Narayana numbers can be expressed in terms of binomial coefficients: $N(n,k) =$\frac {1}{n}{n\ choose k}{n\choose $k-1$}.) [5] $(N(n,k))_{n,k\in\mathbb{N}_0}$ are defined by the relations

$$N(0,\kappa) = N(\kappa,0) = \delta_{\kappa 0} \quad (\kappa \in \mathbb{N}_0), \quad N(n,\kappa) = \frac{1}{n}\binom{n}{\kappa}\binom{n}{\kappa-1} \quad (n,\kappa \in \mathbb{N}) \quad (1)$$

Here δ_{ij} denotes the Kronecker delta function.

This sequence is denoted by A001263 in the On-Line Encyclopedia of Integer Sequences. The corresponding sequence of Narayana polynomials $\big(a(n;r)\big)_{n\in\mathbb{N}_0}$ is defined by

$$a(n;r) = \sum_{\kappa=0}^{n} N(n,\kappa) r^{\kappa}. \quad (2)$$

Narayana numbers and polynomials actuate a lot of attention due to their various combinatorial interpretations.

The connection between Narayana numbers and lattice paths was studied by R.A. Sulanke. Sulanke considered lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,-1)$ to (n,n), and proved that the number of such paths of length k with step size from

$\mathbb{N} \times \mathbb{N}$, such that the whole path remains strictly above the line $y = x - 1$, is equal to $N(n, \kappa)$.

Furthermore, Brändén considered Dyck paths, i.e. paths with the step size from the set $\{(0,1), (1,0)\}$ (i.e. in each step you can move up or forward by unit distance). It is proved that the number of colored Dyck paths, such that each corner of this path is colored in one of the r colors, is $a(n; r)$.

Various other interesting properties of Narayana numbers and polynomials can be founded. Narayana polynomials are refining the famous Catalan sequence C_n, i.e. $a(n; 1) = C_n$. A lot of identities involving Narayana polynomials and Catalan numbers (In combinatorial mathematics, the Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects. They are named after the Belgian mathematician Eugéne charies Catalan (1814-1894)). [6] have been recently published by Mansour and Sun. Moreover, $\big(a(n; 2)\big)_{n \in \mathbb{N}_0}$ is the sequence of large Schröder numbers.

In order to provide an easier evaluation of the Hankel transform of Narayana polynomials, we introduce the sequences of modified Narayana numbers and polynomials.

Definition (1.1.2) [1]:

The modified Narayana numbers $\big(N_1(n, \kappa)\big)_{n,\kappa \in \mathbb{N}_0}$ are defined by the relations

$$N_1(0, \kappa) = \delta_{\kappa 0}, \quad N_1(n, \kappa) = \frac{1}{n}\binom{n}{\kappa}\binom{n}{\kappa + 1} \quad (n \in \mathbb{N}, \kappa \in \mathbb{N}_0).$$

The infinite matrix whose elements are Narayana numbers and the matrix whose members are modified Narayana numbers are given by

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 3 & 1 & 0 & \\ 0 & 1 & 6 & 6 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix}, \quad N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 3 & 1 & 0 & 0 & \\ 1 & 6 & 6 & 1 & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}.$$

The corresponding modified Narayana polynomials are

$$a_1(n;r) = \sum_{\kappa=0}^{n-1} N_1(n,\kappa) r^{\kappa}. \tag{3}$$

The connections between Narayana numbers (polynomials) and modified Narayana numbers (polynomials) are given by

$N_1(n,\kappa) = N(n,\kappa+1); \quad a(0;r) = a_1(0;r) = 1, a(n;r) = r \cdot a_1(n;r) \quad (n \in \mathbb{N})$

The sequence $\left(a_1(n;r)\right)_{n\in\mathbb{N}_0}$ reduces to the sequence of Catalan numbers for $r = 1$, and to that of little Schröder numbers for $r = 2$.

An important transform on integer sequences that has been much studied recently is the Hankel transform.

Definition (1.1.3) [1]:

The Hankel transform of a given sequence $a = (a_n)_{n\in\mathbb{N}_0}$ is the sequence of Hankel determinants $(h_n)_{n\in\mathbb{N}_0}$ where $h_n = \det\left[a_{i+j-2}\right]_{i,j=1}^{n}$, i.e.

$$a = (a_n)_{n\in\mathbb{N}_0} \xrightarrow{\mathcal{H}} h = (h_n)_{n\in\mathbb{N}_0} : h_n = \det\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{bmatrix}. \tag{4}$$

We denote the Hankel transform by H and hence we write $h = \mathcal{H}(a)$.

The Hankel transform has been widely investigated in numerous papers. For example, dealing with the Hankel transform of the sequences of the Catalan and adjusted Catalan numbers. Both the evaluations are done by using the method based on the orthogonal polynomials. Chamberland and French generalized results providing the Hankel transform of sums of two consecutive generalized Catalan numbers. They used a technique different from others. Another Hankel transform evaluation concerning Catalan numbers is given by Barry and Hennessey.

Let $(a_n)_{n\in\mathbb{N}_0}$ be the moment sequence with respect to some measure $\mathrm{d}\lambda$. In other words, let

$$a_n = \int_{\mathbb{R}} x^n \, \mathrm{d}\lambda \quad (n = 0,1,2, \dots). \tag{5}$$

Then the Hankel transform $h = \mathcal{H}(a)$ of the sequence $a = (a_n)_{n\in\mathbb{N}_0}$ can be expressed by the following relation known as the Heilermann formula (for example, see Krattenthaler):

$$h_n = a_0^{n+1}\beta_1^n\beta_2^{n-1} \dots \beta_{n-1}^2\beta_n. \tag{6}$$

The sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x), \tag{7}$$

where $(Q_n(x))_{n\in\mathbb{N}_0}$ is the monic polynomial (In algebra, monic polynomial is a univariate polynomial in which the leading coefficient (the non zero coefficient of highest degree) is equal to 1. Therefore, a monic polynomial has the form $X^{\wedge n+c} - \{n-1\}X^{\wedge}\{n-1\} +\backslash c$ dots $+ c - 2X^{\wedge 2+c} - 1X^{+c} - 0)$. [7] sequence, orthogonal with respect to the measure $\mathrm{d}\lambda$. The following theorem and corollary provide the way how to explicitly find the measure $\mathrm{d}\lambda$ with a prescribed moment sequence.

Theorem (1.1.4) [1]: (Stieltjes-Perron inversion formula)

Let $(\mu_n)_{n\in\mathbb{N}_0}$ be a sequence such that all elements of its Hankel transform are non-negative. Denote by $G(z) = \sum_{n=0}^{+\infty} \mu_n z^n$ the generating function of the sequence $(\mu_n)_{n\in\mathbb{N}_0}$ and $F(z) = z^{-1}G(z^{-1})$. Also let the function $\lambda(t)$ be defined by

$$\lambda(t) - \lambda(0) = -\frac{1}{2\pi i} \lim_{y\to 0^+} \int_0^t [F(x+iy) - F(x-iy)]\, dx. \tag{8}$$

Then holds $\mu_n = \int_{\mathbb{R}} x^n \, \mathrm{d}\lambda$, i.e. sequence $(\mu_n)_{n\in\mathbb{N}_0}$ is the moment sequence of the measure $\lambda(t)$.

Corollary (1.1.5) [1]:

Under the assumptions of the previous theorem, let in addition $F(\bar{z}) = \overline{F(z)}$ hold. Then

$$\lambda(t) - \lambda(0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_0^t \Im F(x + iy)\, dx. \tag{9}$$

We use Theorem (1.1.4) and Corollary (1.1.5) to determine the weight function (measure) such that the given sequence is the moment sequence to that measure. After that, by applying weight function transformations and using the Heilermann formula (6), we provide a closed-form expression for the Hankel transform (In mathematics the Hankel transform express any given function fcr as the weighted sum of an infinite number of Bessel functions of the first kind $J_r(kr)$. The Bessel functions in the sum are all of the same order v, but differ in a scaling factor k along the r-axis. The necessary coefficient F_v of each Bessel function in the sum, as a function of the scaling factor k constitutes the transformed function. The Hankel transform is an integral transform and was first developed by the mathematician Hermann Hankel). [8]. In order to efficiently use Theorem (1.1.4) and Corollary (1.1.5), we need the expression for the weight function of our sequence.

At last, for our further discussion we need the notion of series reversion.

Definition (1.1.6) [1]:

For a given function $v = f(u)$ with the property $f(0) = 0$, the series reversion is the sequence $\{s_\kappa\}_{\kappa \in \mathbb{N}_0}$ such that

$$u = f^{-1}(v) = s_0 + s_1 v + \cdots + s_n v^n + \cdots,$$

where $u = f^{-1}(v)$ is the inverse function of $v = f(u)$.

Define the auxiliary sequence of shifted Narayana polynomials $a_{sh}(n; r) = a_1(n + 1; r) = r^{-1} a(n; r)$. That sequence is introduced since it is the moment sequence with respect to the absolutely continuous measure.

The generating function $A_{sh}(x; r)$ of the shifted Narayana polynomials can be expressed in terms of series reversion.

Lemma (1.1.7) [1]:

Denote by $f(u)$ the function

$$v = f(u) = \frac{u}{1 + (r+1)u + ru^2}. \tag{10}$$

Series reversion of $f(u)$ satisfies

$$u = f^{-1}(v) = \sum_{n=0}^{+\infty} a_{sh}(n;r) v^{n+1}. \tag{11}$$

Denote by $A_{sh}(x;r)$ the generating function of the sequence $a_{sh}(n;r)$, i.e.

$$A_{sh}(x;r) = \sum_{n=0}^{+\infty} a_{sh}(n;r) x^n.$$

From Lemma (1.1.7), we have that $A_{sh}(x;r)$ satisfies the equation

$$\frac{xA_{sh}(x;r)}{1 + (r+1)xA_{sh}(x;r) + rx^2 A_{sh}^2(x;r)} = x.$$

This equation has two solutions for $A_{sh}(x;r)$, but only one satisfies the condition $A_{sh}(0;r) = a_{sh}(0;r) = 1$. Hence, we obtain

$$A_{sh}(x;r) = \frac{1 - (r+1)x - \sqrt{(1-(r+1)x)^2 - 4rx^2}}{2rx^2}.$$

Our next goal is to find the weight function $w_{sh}(x)$ whose moments are $a_{sh}(n;r)$, i.e.

$$a_{sh}(n;r) = \int_{\mathbb{R}} x^n w_{sh}(x)\, dx. \tag{12}$$

Lemma (1.1.8) [1]:

Sequence $\left(a_{sh}(n;r)\right)_{n\in\mathbb{N}_0}$ is the moment sequence of the weight function

$$w_{sh}(x) = \begin{cases} \dfrac{\sqrt{4r - (x-r-1)^2}}{2\pi r}, & x \in \left(\left(\sqrt{r}-1\right)^2, \left(\sqrt{r}+1\right)^2\right), \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Proof:

The proof is based on the Stieltjes-Perron inversion formula (Theorem (1.1.4)). First, let us denote by $\mathrm{F(z,r)}$ the function

$$F(z;r) = z^{-1}A_{sh}(z^{-1};r) = -\frac{(r+1) - z + \sqrt{(z-r-1)^2 - 4r}}{2r}$$

and by $\psi(x)$ the distribution function given by

$$\psi(x) - \psi(0) = -\frac{1}{\pi}\lim_{y\to 0+}\int_0^t \Im F(x+iy;r)\,dx. \tag{14}$$

Consider the function $F(z;r)$ in the half-plane $\{z \in \mathbb{C}: \Im z > 0\}$.

The function

$$\varrho(z;r) = \sqrt{(z-r-1)^2 - 4r}$$

has the branch points $(\sqrt{r}-1)^2$ and $(\sqrt{r}+1)^2$. We need to choose the regular branch of the square root such that it is positive when the expression under the square root is positive.

By explicit evaluation, we get that the integral of $F(z;r)$ is

$$\mathcal{F}(z;r) = \int F(z;r)\,dz \tag{15}$$

$$= \frac{1}{4r}\left(z^2 + (1+r-z)\varrho(z;r) - 2z(r+1)\right) + l_1(z;r)$$

where

$$l_1(z;r) = \ln\left(r+1-z+\varrho(z;r)\right).$$

Now equation (14) becomes

$$\psi(x) - \psi(0) = -\frac{1}{\pi}\lim_{y\to 0+}\left(\mathcal{F}(x+iy;r) - \mathcal{F}(iy;r)\right). \tag{16}$$

Let us now find $\lim_{y\to 0^+}\Im\mathcal{F}(x+iy;r)$. It is obvious that:

$$\lim_{y\to 0+} \Im\varrho(x+iy;r) = \begin{cases} \sqrt{4r-(x-r-1)^2}, & x \in \left(\left(\sqrt{r}-1\right)^2, \left(\sqrt{r}+1\right)^2\right), \\ 0, & \text{otherwise.} \end{cases}$$

The function $l_1(z;r)$ has one more branch point $z = r+1$. We take a branch of $l_1(z;r)$ such that the imaginary part is 0 if the value under the logarithm is real and positive. We have

$$\lim_{y\to 0+} \Im l_1(x+iy;r) = \begin{cases} \pi + \arctan\dfrac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)}, & x \in \left(\left(\sqrt{r}-1\right)^2, r+1\right), \\ \arctan\dfrac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)}, & x \in \left(r+1, \left(\sqrt{r}+1\right)^2\right), \\ 0, & \text{otherwise.} \end{cases}$$

Including the two previous results on $\mathcal{F}(x+iy;r)$ and using (16), we get the required expression for the weight function

$$w_{sh}(x) = \psi'(x) = \frac{\sqrt{4r-(x-r-1)^2}}{2\pi r}, \quad x \in \left(\left(\sqrt{r}-1\right)^2, \left(\sqrt{r}+1\right)^2\right).$$

Outside the segment $\left[\left(\sqrt{r}-1\right)^2, \left(\sqrt{r}+1\right)^2\right]$, we have $w_{sh}(x) = 0$.

Denote by $\left(Q_n^{sh}(x)\right)_{n\in\mathbb{N}_0}$ the sequence of monic polynomials, orthogonal with respect to the weight function $w_{sh}(x)$.

In order to evaluate explicitly the coefficients $(\alpha_n^{sh})_{n\in\mathbb{N}_0}$ and $(\beta_n^{sh})_{n\in\mathbb{N}_0}$ of the three-term recurrence relation

$$Q_{n+1}^{sh}(x) = (x-\alpha_n^{sh})Q_n^{sh}(x) - \beta_n^{sh}Q_{n-1}^{sh}(x), \tag{17}$$

we apply the transformation formulas. These formulas connect the coefficients α_n^{sh} and β_n^{sh} of the original and transformed weight function.

Lemma (1.1.9) [1]:

Let $w(x)$ and $\bar{w}(x)$ be weight functions and denote by $\left(\pi_n(x)\right)_{n\in\mathbb{N}_0}$ and $\left(\bar{\pi}_n(x)\right)_{n\in\mathbb{N}_0}$ the corresponding orthogonal polynomials. Also denote by

$(\alpha_n)_{n\in\mathbb{N}_0}(\beta_n)_{n\in\mathbb{N}_0}$and $(\bar{\alpha}_n)_{n\in\mathbb{N}_0}, (\bar{\beta}_n)_{n\in\mathbb{N}_0}$ the three-term relation coefficients corresponding to $w(x)$ and $\bar{w}(x)$ respectively. The following transformation formulas are valid:

(i) If $\bar{w}(x) = Cw(x)$ where $C > 0$ then $\bar{\alpha}_n = \alpha_n$ for $n \in \mathbb{N}_0$ and $\bar{\beta}_0 = C\beta_0, \bar{\beta}_n$ for $n \in \mathbb{N}$. Moreover, we have $\bar{\pi}_n(x) = \pi_n(x)$ for all $n \in \mathbb{N}_0$.

(ii) If $\bar{w}(x) = w(ax + b)$ where $a, b \in \mathbb{R}$ and $a \neq 0$ then $\bar{\alpha}_n = (\alpha_n - b)/a$ for $n \in \mathbb{N}_0$ and $\bar{\beta}_0 = \beta_0/|a|$ and $\bar{\beta}_n = \beta_n/a^2$ for $n \in \mathbb{N}$. Moreover, we have $\bar{\pi}_n(x) = a^{-n}\pi_n(ax + b)$.

Lemma (1.1.10) [1]:

The coefficients α_n^{sh} and $\beta_n^{sh}(n \in \mathbb{N}_0)$ in the three-term recurrence relation (17) are given by

$$\beta_0^{sh} = 1, \beta_n^{sh} = r \quad (n \in \mathbb{N}); \quad \alpha_n^{sh} = r + 1 \ (n \in \mathbb{N}_0).$$

Proof:

The monic Chebyshev polynomials of the second kind

$$Q_n^{(1)}(x) = S_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{2^n \cdot \sqrt{1-x^2}}$$

are orthogonal with respect to the weight $w^{(1)}(x) = \sqrt{1-x^2}$. The corresponding coefficients in the three-term relation are

$$\beta_0^{(1)} = \frac{\pi}{2}, \ \ \beta_n^{(1)} = \frac{1}{4} \ (n \geq 1), \ \ \alpha_n^{(1)} = 0 \quad (n \geq 0).$$

Let us introduce a new weight function $w^{(2)}(x) = \sqrt{4r - (x - r - 1)^2}$. It satisfies $w^{(2)}(x) = w^{(1)}(ax + b)$, where $a = 1/(2\sqrt{r})$ and $b = -(r + 1)/(2\sqrt{r})$. Hence we get (see Lemma (1.1.9)):

$$\beta_0^{(2)} = \pi\sqrt{r}, \ \ \beta_n^{(2)} = r \ (n \in \mathbb{N}), \ \ \alpha_n^{(2)} = r + 1 \quad (n \in \mathbb{N}_0).$$

Finally, since $w_{sh}(x) = w^{(2)}(x)/(\pi\sqrt{r})$, we conclude that $\beta_n^{sh} = \beta_n^{(2)} = r$ for $n \geq 1$ and $\alpha_n^{sh} = \alpha_n^{(2)} = r + 1$. The coefficient β_0^{sh} can be obtained by direct evaluation of the integral $\beta_0^{sh} = \int_{\mathbb{R}} w_{sh}(x)\, dx = 1$.

Corollary (1.1.11) [1]:

The squared norms of monic orthogonal polynomials $Q_n(x)$ have values

$$\left\|Q_n^{sh}\right\|^2 = r^n \quad (n \in \mathbb{N}_0).$$

Denote by $\left(h_{sh}(n;r)\right)_{n\in\mathbb{N}_0}$ the Hankel transform of the sequence $\left(a_{sh}(n;r)\right)_{n\in\mathbb{N}_0}$.

Now using the Heilermann formula (6), we obtain

$$h_{sh}(n;r) = a_{sh}(0;r)^{n+1} \prod_{i=1}^{n} \left(\beta_i^{sh}\right)^{n+1-i} = 1^{n+1} \prod_{i=1}^{n} r^{n+1-i} = r^{\binom{n+1}{2}}.$$

Theorem (1.1.12) [1]:

The Hankel transform of the shifted Narayana polynomials $\left(a_{sh}(n;r)\right)_{n\in\mathbb{N}_0}$ is given by

$$h_{sh}(n;r) = r^{\binom{n+1}{2}}.$$

Section (1.2): The Hankel Transform of Modified Narayana Polynomials and Two Consecutive Shifted Narayana Numbers

Denote by $\left(h(n;r)\right)_{n\in\mathbb{N}_0}$ and $\left(h_1(n;r)\right)_{n\in\mathbb{N}_0}$, respectively, the Hankel transforms of the Narayana and modified Narayana polynomials, i.e. $\left(a(n;r)\right)_{n\in\mathbb{N}_0}$ and $\left(a_1(n;r)\right)_{n\in\mathbb{N}_0}$.

From the relation

$$a_1(n;r) = a_{sh}(n-1;r) \quad (n \geq 1), \tag{18}$$

we conclude that $\left(a_1(n;r)\right)_{n\geq 1}$ can be expressed as the moment sequence in the form

$$a_1(n;r) = \int_{\mathbb{R}} x^n \widetilde{w}(x)dx \ \ (n \geq 1), \ \text{ where } \widetilde{w}(x) = \frac{w_{sh}(x)}{x}. \tag{19}$$

On the other hand, we have

$$\int_{\mathbb{R}} \frac{w_{sh}(x)}{x} dx = r^{-1} \neq a_1(0;r).$$

Hence, we introduce the sequence $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$ defined by

$$\tilde{a}(n;r) := \begin{cases} r^{-1}, & n = 0, \\ a_1(n;r), & n \geq 1, \end{cases} \tag{20}$$

which is the moment sequence for $\widetilde{w}(x)$. The next lemma shows the connection between the Hankel transforms of the sequences $\left(a_1(n;r)\right)_{n\in\mathbb{N}_0}$ and $\left(\tilde{a}(n;r)\right)_{n\in\mathbb{N}_0}$.

Lemma (1.2.1) [1]:

For the sequences $(a_k)_{n\in\mathbb{N}_0}$ and $(\tilde{a}_\kappa)_{n\in\mathbb{N}_0}$ such that $a_\kappa = \tilde{a}_\kappa$ for all $k \geq 1$, their Hankel transforms $(h_n)_{n\in\mathbb{N}_0}$ and $\left(\tilde{h}_n\right)_{n\in\mathbb{N}_0}$ are related by

$$h_n = \tilde{h}_n + (a_0 - \tilde{a}_0)\hat{h}_{n-1} \quad (n \in \mathbb{N}_0),$$

where $\{\hat{h}_n\}_{n\in\mathbb{N}_0}$ is the Hankel transform of the sequence $\{\hat{a}_\kappa\}_{\kappa\in\mathbb{N}_0}$ given by $\hat{a}_\kappa = a_{\kappa+2}$ $(\kappa \geq 0)$ and $\hat{h}_{-1} = 1$.

Proof:

By expanding the determinants over the first row, we get

$$h_n = \sum_{\kappa=0}^{n-1} a_\kappa M_{1,\kappa+1}, \quad \tilde{h}_n = \sum_{\kappa=0}^{n-1} \tilde{a}_\kappa \widetilde{M}_{1,\kappa+1}, \tag{21}$$

where $M_{1,\kappa}$ and $\widetilde{M}_{1,\kappa}$ are the minors. Note that the minors $M_{1,\kappa+1}$ and $\widetilde{M}_{1,\kappa+1}$ are equal for every $\kappa \in \mathbb{N}_0$. Hence, we have

$$h_n - \tilde{h}_n = (a_0 - \tilde{a}_0)M_{1,1}.$$

But on the other hand, $M_{1,1}$ is the $(n-1)st$ member of the Hankel transform of the sequence $\{a_2, a_3, \dots\}$, i.e. we have $M_{1,1} = \det[a_{i+j}]_{1\leq i,j\leq n-1}$ and we denote it by $\hat{h}_{n-1}$.

Applying Lemma (1.2.1) to the sequences $\big(a_1(n;r)\big)_{n\in\mathbb{N}_0}$ and $\big(\tilde{a}(n;r)\big)_{n\in\mathbb{N}_0}$, we obtain the following corollary.

Corollary (1.2.2) [1]:

If $h_1(n;r), \tilde{h}(n;r)$ and $\hat{h}(n;r)$ are the Hankel transforms of the sequences $a_1(n;r), \tilde{a}(n;r)$ and $\hat{a}(n;r) = a_1(n+2;r)$ respectively, then

$$h_1(n;r) = \tilde{h}(n;r) + (1 - r^{-1})\hat{h}(n-1;r).$$

Now we have to evaluate the Hankel transform of the sequences $\big(\hat{a}(n;r)\big)_{n\in\mathbb{N}_0}$ and $\big(\tilde{a}(n;r)\big)_{n\in\mathbb{N}_0}$.

Note that $\hat{a}(n;r) = a_{sh}(n+1;r)$, i.e. $\big(\hat{a}(n;r)\big)_{n\in\mathbb{N}_0}$ is the shifted sequence of $\big(a_{sh}(n;r)\big)_{n\in\mathbb{N}_0}$. Hence $\big(\hat{a}(n;r)\big)_{n\in\mathbb{N}_0}$ is the moment sequence of the weight function $\hat{w}(x) = xw_{sh}(x)$. The following lemma connects the Hankel transforms of the original and the shifted sequences.

Lemma (1.2.3) [1]:

Let $\omega(x)$ be the weight function and $a_n = \int_{\mathbb{R}} x^n \, \omega(x)dx$ its nth moment. Denote $\bar{\omega}(x) = x\omega(x)$ and let $\left(Q_n(x)\right)_{n\in\mathbb{N}_0}$ be the sequence of monic orthogonal polynomials corresponding to the weight $\omega(x)$. Then the Hankel transforms $(h_n)_{n\in\mathbb{N}_0}$ and $\left(\bar{h}_n\right)_{n\in\mathbb{N}_0}$ of the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(\bar{a}_n)_{n\in\mathbb{N}_0}$ are related by

$$\bar{h}_n = h_n(-1)^{n+1}\lambda_{n+1} \quad (n \in \mathbb{N}_0), \tag{22}$$

where $\lambda_n = Q_n(0)(n \in \mathbb{N}_0)$.

Proof:

The coefficients β_n and $\bar{\beta}_n$ satisfy

$$\bar{\beta}_n = \beta_n \frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2}.$$

Inserting it in the Heilermann formula (6), we obtain

$$\bar{h}_{n+1} = \bar{h}_n\bar{a}_0 \prod_{i=1}^{n+1} \bar{\beta}_i = \bar{h}_n\bar{a}_0 \prod_{i=1}^{n+1} \beta_i \frac{\lambda_{i-1}\lambda_{i+1}}{\lambda_i^2} = h_{n+1}\frac{\bar{h}_n}{h_n}\left(\frac{\lambda_0\bar{a}_0}{\lambda_1 a_0}\right)\frac{\lambda_{n+2}}{\lambda_{n+1}}. \tag{23}$$

By applying the previous equation n times, we have

$$\frac{\bar{h}_{n+1}}{h_{n+1}} = \frac{\bar{h}_0}{h_0}\left(\frac{\bar{a}_0}{a_0}\right)^{n+1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n+1}\frac{\lambda_{n+2}}{\lambda_1}. \tag{24}$$

Now, since $\bar{h}_0 = \bar{a}_0 = a_1, h_0 = a_0$ and $\lambda_0 = 1$, we have

$$\bar{h}_{n+1} = h_{n+1}\left(\frac{a_1}{\lambda_1 a_0}\right)^{n+2}\lambda_{n+2}.$$

Also note that $\lambda_1 = Q_1(0) = -a_1/a_0$, so $\bar{h}_{n+1} = (-1)^{n+2}h_{n+1}\lambda_{n+1}$. After decreasing the index by 1 we finally get by desired relation (22). Note that for $n = 0$ the relation (22) trivially holds.

By applying Lemma (1.2.3) to the sequences $\left(a_{sh}(n;r)\right)_{n\in\mathbb{N}_0}$ and $\left(\hat{a}(n;r)\right)_{n\in\mathbb{N}_0}$, we evaluate the Hankel transform of $\left(\hat{a}(n;r)\right)_{n\in\mathbb{N}_0}$.

Corollary (1.2.4) [1]:

The Hankel transform of the sequence $\left(\hat{a}(n;r)\right)_{n\in\mathbb{N}_0}$ is given by

$$\hat{h}(n;r) = r^{\binom{n+1}{2}}\frac{r^{n+2}-1}{r-1} \quad (n \in \mathbb{N}_0).$$

Proof:

Recall that from Lemma (1.1.10) we have

$$\beta_0^{sh} = 1,\ \ \beta_n^{sh} = r\ (n \in \mathbb{N});\ \ \ \alpha_n^{sh} = r+1 \quad (n \in \mathbb{N}_0)$$

Denote $\lambda_n = Q_n^{sh}(0)$. Then

$$\lambda_{n+1} + (r+1)\lambda_n + r\lambda_{n-1} = 0\ (n \in \mathbb{N}_0),\ \ \lambda_0 = 1,\ \ \lambda_1 = -(r+1)$$

The previous equation has a unique solution $\lambda_n = (-1)^n(r^{n+1}-1)/(r-1)$. Now we obtain the required formula for $\hat{h}(n;r)$ using $h_{sh}(n;\,r) = r^{\binom{n+1}{2}}$ (Theorem (1.1.12)) and Lemma (1.2.3).

Recall that $(\tilde{a}(n;\,r))_{n\in\mathbb{N}_0}$ is defined as the moment sequence of $\overline{\omega}(x) = \omega_{sh}(x)/x$. The following lemma is proved and establishes the required transformation formulas in this case.

Lemma (1.2.5) [1]:

Consider the same notation as in Lemma (1.1.9). Let the sequence $(r_n)_{n\in\mathbb{N}_0}$ be defined by

$$r_{-1} = -\int_{\mathbb{R}} \overline{\omega}(x)dx,\ \ r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0). \tag{25}$$

If $\overline{\omega}(x) = \omega(x)/(x-c)$ where $c < \inf\operatorname{supp}(\omega)$ then

$$\bar{\alpha}_0 = \alpha_0 + r_0, \qquad \bar{\alpha}_n = \alpha_n + r_n - r_{n-1} \tag{26}$$
$$\bar{\beta}_0 = -r_{-1}, \qquad \bar{\beta}_n = \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} (n \in \mathbb{N})$$

Now, we are ready to evaluate the Hankel transform of the sequence $\left(\tilde{a}(n;r)\right)_{n\in\mathbb{N}_0}$.

Lemma (1.2.6) [1]:

The coefficients $\tilde{\beta}_n$ are given by

$$\tilde{\beta}_n = \begin{cases} \beta_{n-1}, & n > 1, \\ r^{-1}, & n = 0. \end{cases}$$

The Hankel transform $\left(\tilde{h}(n;r)\right)_{n\in\mathbb{N}_0}$ of $\left(\tilde{a}(n;r)\right)_{n\in\mathbb{N}_0}$ is given by $\tilde{h}(n;r) = r^{\binom{n}{2}-1}$.

Proof:

We apply Lemma (1.2.5). The coefficients $\tilde{\beta}_n$ are given by $\tilde{\beta}_n = \beta_{n-1}^{sh} r_{n-1}/r_{n-2}$, where the sequence $(r_n)_{n\in\mathbb{N}_0}$ is determined by the recurrence relation:

$$r_n = -\alpha_n^{sh} - \frac{\beta_n^{sh}}{r_{n-1}}, \;\; r_{-1} = -a_0. \tag{27}$$

After interchanging α_n^{sh} and β_n^{sh} in Lemma (1.1.10), we have

$$r_n + r + 1 + \frac{r}{r_{n-1}} = 0 \; (n \geq 1), \qquad r_0 = -1, \;\; r_{-1} = -r^{-1}. \tag{28}$$

It can be proved by mathematical induction that $r_n = -1$ for all $n \geq 1$. Hence

$$\tilde{\beta}_n = \beta_{n-1}^{sh}\frac{r_{n-1}}{r_{n-2}} = r \;\; (n > 1), \;\; \tilde{\beta}_1 = \beta_0^{sh}\frac{r_0}{r_{-1}} = r, \;\; \tilde{\beta}_0 = \int_{\mathbb{R}} \frac{\omega_{sh}(x)}{x} dx = r^{-1}.$$

Now, the Hankel transform is

$$\tilde{h}(n;r) = \tilde{a}_0^{n+1}\tilde{\beta}_1^{n}\tilde{\beta}_2^{n-1} \dots \tilde{\beta}_{n-1}^{2}\tilde{\beta}_n = \frac{1}{r^{n+1}} r^{\binom{n+1}{2}} = r^{\binom{n}{2}-1}.$$

From the previous facts, we can formulate and prove the following

Theorem (1.2.7) [1]:

The Hankel transform $\left(h_1(n;r)\right)_{n\in\mathbb{N}_0}$ of the sequence of modified Narayana polynomials $\left(a_1(n;r)\right)_{n\in\mathbb{N}_0}$ is given by

$$h_1(n;r) = r^{\binom{n+1}{2}}. \tag{29}$$

Proof:

From Corollary (1.2.2) we have

$$\begin{aligned} h_1(n;r) &= \tilde{h}(n;r) + (1-r^{-1})\hat{h}(n-1;r) \\ &= r^{\binom{n+1}{2}-n} + (1-r^{-1})r^{\binom{n}{2}}\frac{r^{n+1}-1}{r-1} \\ &= r^{\binom{n}{2}-1} + \frac{r-1}{r}r^{\binom{n}{2}}\frac{r^{n+1}-1}{r-1} = r^{\binom{n}{2}+n} = r^{\binom{n+1}{2}}. \end{aligned}$$

The evaluation of the Hankel transform of the sequence of Narayana polynomials $\left(a(n;r)\right)_{n\in\mathbb{N}_0}$, now goes straightforward.

Theorem (1.2.8) [1]:

The Hankel transform $\left(h(n;r)\right)_{n\in\mathbb{N}_0}$ of the sequence of Narayana polynomials $\left(a(n;r)\right)_{n\in\mathbb{N}_0}$ is given by

$$h(n;r) = r^{\binom{n+1}{2}}. \tag{30}$$

Proof:

Notice that $a(n;\, r) = r\tilde{a}(n;\, r)$ for any $n \geq 0$. Therefore it can be easily concluded that the Hankel transform of Narayana polynomials $\left(a(n;r)\right)_{n\in\mathbb{N}_0}$ is given by

$$h(n;r) = r^{n+1}\tilde{h}(n;r) = r^{\binom{n}{2}+n} = r^{\binom{n+1}{2}}. \tag{31}$$

Now let us recall a few special cases mentioned in the first section.

Example (1.2.9) [1]:

In the case $r = 1$ we confirm the known result that the Hankel transform of the Catalan sequence is $C_n = a_1(n; 1) = a(n; 1)$. After inserting $r = 1$ in (30) we can easily establish the well-known result: $h_1(n; 1) = h(n; 1) = 1 (n \in \mathbb{N}_0)$.

Example (1.2.10) [1]:

Recall also that in the case $r = 2$ we have that $\left(a_1(n; 2)\right)_{n\in\mathbb{N}_0}$ is the sequence of little Schröder numbers (1,1,3,11,45,197,903,4279,...). Again, by inserting $r = 2$ in (1.30) we have the following result: $h_1(n; 2) = 2^{\binom{n+1}{2}} (n \in \mathbb{N}_0)$.

Example (1.2.11) [1]:

The numbers $\left(a(n; 2)\right)_{n\in\mathbb{N}_0}$ are the large Schröder numbers: (1,2,6,22,90, 394,...). By inserting $r = 2$ in (1.31) we also obtain a well-known result: $h(n; 2) = 2^{\binom{n+1}{2}} (n \in \mathbb{N}_0)$.

In this section we find the closed-form expression for the Hankel transform of the sequence $s(n; r, \gamma, \delta) = \gamma a_{sh}(n;\ r) + \delta a_{sh}(n + 1; r)$.

Let us start with a simplified case $s(n; r, C) = s(n; r, 1, C) = a_{sh}(n;\ r) + C a_{sh}(n\ + 1; r)$. It is clear that the elements of the sequence $s(n; r, C)$ are the moments corresponding to the weight function

$$\omega_s(x) = (x + C)\omega_{sh}(x) = \begin{cases} \dfrac{(x + C)\sqrt{4r - (x - r - 1)^2}}{2\pi r}, & x \in \left(\left(\sqrt{r} - 1\right)^2, \left(\sqrt{r} + 1\right)^2\right), \\ 0, & \text{otherwise.} \end{cases}$$

To find the Hankel transform of the sequence $\left(s(n; r)\right)_{n\in\mathbb{N}_0}$, we need to construct orthogonal polynomials $(Q_n^s(x))_{n\in\mathbb{N}_0}$ corresponding to the weight $\omega_s(x)$ and to obtain the closed form of the coefficients of the three-terms recurrence relation satisfied by these polynomials.

Lemma (1.2.12) [1]:

The coefficients α_n^s and β_n^s in the three-term relation satisfied by $(Q_n^s(x))_{n\in\mathbb{N}_0}$ are

$$\beta_n^s = r\frac{(t_2^n - t_1^n)(t_2^{n+2} - t_1^{n+2})}{(t_2^{n+1} - t_1^{n+1})^2}, \tag{32}$$

$$\alpha_n^s = -C - \frac{t_2^{n+2} - t_1^{n+2}}{t_2^{n+1} - t_1^{n+1}} - r\frac{t_2^{n+1} - t_1^{n+1}}{t_2^{n+2} - t_1^{n+2}},$$

where

$$t_{1,2} = \frac{-r - 1 - C \pm \sqrt{(r+1+C)^2 - 4r}}{2}, \quad \beta_0^s = s(0; r; C) = 1 + C(r+1).$$

Proof:

By introducing a new weight function $\omega_s(x) = (x + C)\omega_{sh}(x)$ we can derive the coefficients α_n^s and β_n^s by using the relations

$$\mu_n = Q_n^s(-C), \qquad \alpha_n^s = -C - \frac{\mu_{n+1}}{\mu_n} - \beta_{n+1}^{sh}\frac{\mu_n}{\mu_{n+1}}, \quad \beta_n^s = \beta_n^{sh}\frac{\mu_{n-1}\mu_{n+1}}{\mu_n^2}. \tag{33}$$

To obtain the values $\mu_n = Q_n^s(-C)$, let us rewrite the three-term recurrence relation for the polynomials $Q_n(x)$ and take $x = -C$:

$$\mu_{n+1} + (r + 1 + C)\mu_n + r\mu_{n-1} = 0. \tag{34}$$

Trivially we have the initial conditions $\mu_{-1} = 0$ and $\mu_0 = 1$. Relation (34), together with the stated initial conditions is a second order difference equation with constant coefficients. Hence, the solution of (34) is given by

$$\mu_n = \frac{t_2^{n+1} - t_1^{n+1}}{t_2 - t_1}. \tag{35}$$

By inserting (35) in (33), we finally obtain expressions (32).

Now we are ready to apply the Heilermann formula and evaluate the Hankel transform $h_s(n;\ r, C)$ of the sequence $s(n; r, C)$:

$$
\begin{aligned}
&h_s(n;r,C)\\
&\quad = h_s(n-1;r,C)\cdot s(0;r,C)\cdot \beta_n^s \dots \beta_1^s\\
&\quad = h_s(n-1;r,C)\cdot s(0;r,C)\cdot \beta_n^{sh} \dots \beta_1^{sh}\cdot \frac{\mu_{n+1}\mu_{n-1}}{\mu_n^2}\cdot \frac{\mu_n\mu_{n-2}}{\mu_{n-1}^2}\cdot \dots \cdot \frac{\mu_2\mu_0}{\mu_1^2}\\
&\quad = h_s(n-1;r,C)\cdot \frac{h_{sh}(n;r)}{h_{sh}(n-1;r)}\cdot \frac{\mu_{n+1}}{\mu_n}\cdot \frac{s(0;r,C)\mu_0}{a_{sh}(0;r)\mu_1}.
\end{aligned}
$$

By successive application of the previous equation we obtain

$$
h_s(n;r,C) = h_s(0;r,C)\cdot \frac{h_{sh}(n;r,C)}{h_{sh}(0;r,C)}\cdot \frac{\mu_{n+1}}{\mu_1}\cdot \left(\frac{s(0;r,C)\mu_0}{a_{sh}(0;r)\mu_1}\right)^n,
$$

where from

$$
h_s(n;r,C) = h_{sh}(n;r,C)\cdot \mu_{n+1}\cdot \left(\frac{s(0;r,C)\mu_0}{a_{sh}(0;r)\mu_1}\right)^{n+1}. \tag{36}
$$

Now replacing

$$
\begin{gathered}
h_s(0;r,C) = s(0;r,C) = a_{sh}(0;r) + Ca_{sh}(1;r) = 1 + C(r+1),\\
\mu_1 = t_1 + t_2 = -(r+1+C),\ \ \mu_0 = 1,
\end{gathered}
$$

and using formulas (30) and (35), we obtain

$$
h_s(n;r,C) = (-1)^{n+1} r^{\binom{n+1}{2}} \frac{t_2^{n+2} - t_1^{n+2}}{t_2 - t_1}.
$$

Inserting the values for $t_{1,2}$ from Lemma (1.2.12), we complete the proof of the theorem:

Theorem (1.2.13) [1]:

The Hankel transform $h_s = h_s(n;r,C)$ of the sequence $\big(s(n;r,C)\big)_{n\in\mathbb{N}_0}$ is

$$
\begin{aligned}
h_s &= \frac{r^{\binom{n+1}{2}}}{2^{n+2\zeta}}\left((r+1+C+\zeta)^{n+2} - (r+1+C-\zeta)^{n+2}\right), \qquad (37)\\
\zeta &= \sqrt{(r+1+C)^2 - 4r}.
\end{aligned}
$$

The next lemma generalizes the above approach and can be proved analogously to Theorem (1.2.13).

Lemma (1.2.14) [1]:

Let $\omega(x)$ be a weight function and $a_n = \int_{\mathbb{R}} x^n \omega(x)\, dx$ its moments. Denote by $\left(Q_n(x)\right)_{n\in\mathbb{N}_0}$ the sequence of orthogonal polynomials with respect to the weight $\omega(x)$. Also denote by $(h_n)_{n\in\mathbb{N}_0}$ and $(h_n^s)_{n\in\mathbb{N}_0}$, the Hankel transforms of the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(a_n + Ca_{n+1})_{n\in\mathbb{N}_0}$, respectively. The following relation is true:

$$h_n^s = h_n \mu_{n+1} \left(\frac{Ca_0 + a_1}{a_0 \mu_1}\right)^{n+1}, \text{ where } \mu_n = Q_n(-C). \tag{38}$$

Now return to the proof of the main result of the section, i.e. the evaluation of the Hankel transform $h_s(n; r, \gamma, \delta)$ of the sequence $s(n; r, \gamma, \delta)$. Assume that $\gamma \neq 0$. Since

$$s(n; r, \gamma, \delta) = \gamma \left(a_{sh}(n; r) + \frac{\delta}{\gamma} a_{sh}(n+1; r) \right) = \gamma s\left(n; r, \frac{\delta}{\gamma}\right),$$

we directly obtain that

$$h_s(n; r, \gamma, \delta) = \gamma^{n+1} h_s\left(n; r, \frac{\delta}{\gamma}\right).$$

Now inserting $C = \delta/\gamma$ in (37) and using the previous expression we finally derive the main result of this section, i.e. the closed form expression for $h_s = h_s(n; r, \gamma, \delta)$:

$$h_s = \frac{r^{\binom{n+1}{2}}}{2^{n+2\theta}} \left((\gamma(r+1) + \delta + \theta)^{n+2} - (\gamma(r+1) + \delta - \theta)^{n+2}\right), \tag{39}$$
$$\theta = \sqrt{(\gamma(r+1) + \delta)^2 - 4r\gamma^2}.$$

Note that the formula (1.39) also holds in the case γ = 0, since $h_s(n; r, \gamma, \delta)$ is the polynomial function of γ and δ. This follows either from (39) or from the fact that $s(n;\ r, \gamma, \delta)$ is a linear function of γ and δ.

Chapter 2

Laurent Biorthogonal and q-Narayana Polynomials

In this chapter as an application, a new proof is given to the Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp concerning domino tilings of the Aztec diamonds. The proof is based on the correspondence with nonintersecting Schröder paths developed by Johansson .

Section (2.1): Laurent Biorthogonal Polynomials and T-Fractions with Schröder Pathes

Laurent biorthogonal polynomials (LBPs) are orthogonal functions which play fundamental roles in the theory of two-point Padé approximants at zero and infinity. In Padé approximants, LBPs appear as the denominators of the convergents of a T-fraction. We utilize LBPs to calculate a determinant whose entries are given by a q-analogue of the Narayana polynomials which have a combinatorial expression in Schröder paths. As an application, we give a new proof to the Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp by means of LBPs and Schröder paths.

A Schröder path P is a lattice path (A path composed of connected horizontal and vertical lines segments, each passing between adjacent lattice points. A lattice path is there a sequence of points P-0 with such that each is a lattice point and is obtained by offsetting one unit east (or west) or one unit north (or south).) [9]

In the two-dimensional plane $\mathbb{Z}^2$ consisting of up steps $(1,1)$, down steps $(1,-1)$ and level steps $(2,0)$, and never going beneath the x-axis. See Figure 1 for example. For $k \in \mathbb{N} = \{0,1,2,\dots\}$, let S_k denote the set of Schröder paths from $(0,0)$ to $(2k,0)$. The number $\#S_k$ of such paths is counted by the k-th large Schröder number. The first few of $\#S_k$ are 1,2,6,22 and 90.

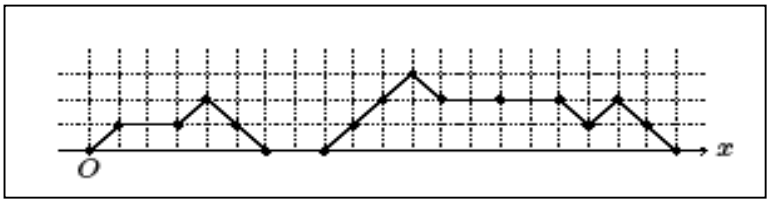

Figure 1: A Schröder path $P \in S_{10}$ such that level $(P) = 4$ and area $(P) = 20$

Enumerative or statistical properties of Schröder paths are often investigated through the Narayana polynomials

$$L_k(t) = \sum_{j=1}^{k} \frac{1}{k}\binom{k}{j}\binom{k}{j-1}(1+t)^j, \ \ k \in \mathbb{N}, \tag{1}$$

where $N_0(t) = 1$. (The coefficients $\frac{1}{k}\binom{k}{j}\binom{k}{j-1}$ are the Narayana numbers) Bonin, Shapiro and Simion interpreted the Narayana polynomials by counting the level steps in Schröder paths,

$$L_k(t) = \sum_{P \in S_k} t^{\mathrm{level}(P)}. \tag{2}$$

where $\mathrm{level}(P)$ denotes the number of level steps in a Schröder path P. For more about the Narayana polynomials and related topics, see, e.g., Sulanke's. Besides level steps, Bonin et al. also examined the area polynomials

$$A_k(q) = \sum_{P \in S_k} q^{\mathrm{area}(P)}, \ \ k \in \mathbb{N}, \tag{3}$$

with respect to the statistic $\mathrm{area}(P)$ that measures the area bordered by a path P and the x-axis. We consider the two statistics $\mathrm{level}(P)$ and $\mathrm{area}(P)$ simultaneously in the polynomials

$$N_k(t,q) = \sum_{P \in S_k} t^{\mathrm{level}(P)} q^{\mathrm{area}(P)}, \ \ k \in \mathbb{N}, \tag{4}$$

We refer to $N_k(t,q)$ by the q-Narayana polynomials. Obviously, the q-Narayana polynomials satisfy that $N_k(t,1) = L_k(t)$ and $N_k(1,q) = A_k(t)$, and reduce to the large Schröder numbers, $N_k(1,1) = \#S_k$, as well as the Catalan numbers, $N_k(0,1) = \frac{1}{k+1}\binom{2k}{k}$. We find the LBPs of which the moments are described by the q-Narayana polynomials.

The aim of this section is twofolds: (i) to calculate a determinant whose entries are described by the q-Narayana polynomials $N_k(t,q)$; (ii) to give a new proof to

the Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp by means of LBPs and Schröder paths.

Determinants whose entries are given by the large Schröder numbers, by the Narayana polynomials and by their q-analogues are calculated by many using various techniques. Ishikawa, Tagawa and Zeng found a closed-form expression of Hankel determinants of a q-analogue of the large Schröder numbers in a combinatorial way based on Gessel–Viennot's lemma (in mathematics, the lindström-Gessel–Viennot's lemma provides away to count the number of tuples of non-intersecting Lattice Paths.) [10]. Petković, Barry and Rajković calculated Hankel determinants described by the Narayana polynomials using an analytic method of solving a moment problem of orthogonal polynomials. In Section 6, we evaluate a Teoplitz determinant described by the q-Narayana polynomials $N_k(t,q)$ by means of a combinatorial interpretation of LBPs in terms of Schröder paths.

Counting domino tilings of the Aztec diamonds is a typical problem of tilings which is exactly solvable. For $n \in \mathbb{N}$, the Aztec diamond AD_n of order n is the union of all unit squares which lie inside the closed region $|x| + |y| \leq n + 1$. A domino denotes a one-by-two or two-by-one rectangle. Then, a domino tiling, or simply a tiling, of AD_n is a collection of nonoverlapping dominoes which exactly

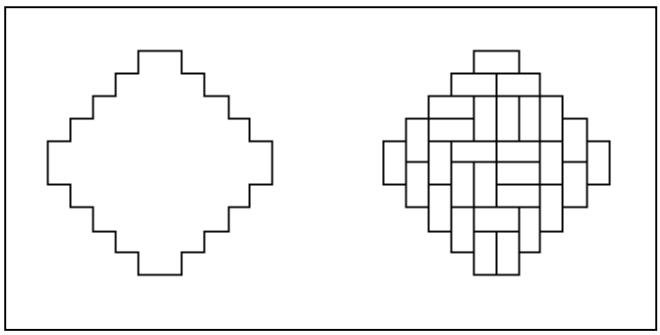

Figure 2: The Aztec diamond AD_5 (left) and a tiling of AD_5 (right).

covers AD_n. Figure 2 shows an Aztec diamond and an example of a tiling. Let T_n denote the set of all tilings of AD_n. Elkies, Kuperberg, Larsen and Propp, considered the statistics $v(T)$ and $r(T)$ of a tiling T, where $v(T)$ denotes half the number of vertical dominoes in T and $r(T)$ the rank of T. They showed that the counting polynomials

$$AD_n(t,q) = \sum_{T \in T_n} t^{v(T)} q^{r(T)}, \quad n \in \mathbb{N}, \tag{5}$$

admit the following closed-form expression.

Theorem (2.1.1) [2]: (Aztec diamond theorem)

For $n \in N$,

$$AD_n(t,q) = \prod_{k=0}^{n-1} (1 + tq^{2k+1})^{n-k}. \tag{6}$$

Especially, the number $\#T_n$ of possible tilings of AD_n equals to

$$\#T_n = AD_n(1,1) = 2^{\frac{n(n+1)}{2}}. \tag{7}$$

We give a new proof to (6) based on the one-to-one correspondence developed by Johansson. Clarifying the connection between the statistics $v(T)$ and $r(T)$ of tilings T and the statistics level(P) and area(P) of Schröder paths P, we reduce the proof to the calculation of a determinant of the q-Narayana polynomials.

Let b_{n+1} and c_n for $n \in \mathbb{N}$ be arbitrary nonzero constants. The (monic) Laurent biorthogonal polynomials (LBPs) $P_n(z), n \in \mathbb{N}$, are the polynomials determined from the recurrence

$$P_{n+1}(z) = (z - c_n)P_n(z) - b_n z P_{n-1}(z) \text{ for } n \geq 1 \tag{8}$$

with the initial values $P_0(z) = 1$ and $P_1(z) = z - c_0$. The first few of the LBPs are

$$P_0(z) = 1, \tag{9}$$

$$P_1(z) = z - c_0, \tag{10}$$

$$P_2(z) = z^2 - (b_1 + c_0 + c_1)z + c_0 c_1, \tag{11}$$

$$P_3(z) = z^3 - (b_1 + b_2 + c_0 + c_1 + c_2)z^2 + (b_1 c_2 + b_2 c_0 + c_0 c_1 + c_0 c_2 + c_1 c_2)z - c_0 c_1 c_2 \tag{12}$$

The LBP $P_n(z)$ is a monic polynomial in z exactly of degree n of which the constant term does not vanish. In fact,

$$P_n(0) = (-1)^n \prod_{j=0}^{n-1} c_j \neq 0. \tag{13}$$

The orthogonality of LBPs is described in the following theorem, that is sometimes referred to by Favard type theorem.

Theorem (2.1.2) [2]: (Favard type theorem for LBPs)

There exists a linear functional $\mathcal{F}$ defined over Laurent polynomials in z with respect to which the LBPs $P_n(z)$ satisfy the orthogonality

$$\mathcal{F}[P_n(z)z^{-k}] = h_n \delta_{n,k} \qquad \text{for } 0 \leq k \leq n \tag{14}$$

with some constants $h_n \neq 0$, where $\delta_{n,k}$ denotes the Kronecker delta. The linear functional $\mathcal{F}$ is unique up to a constant factor.

We can prove Theorem (2.1.2) in almost the same way as Favard's theorem for orthogonal polynomials.

We write the moments of the linear functional $\mathcal{F}$,

$$f_k = \mathcal{F}[z^k], \quad k \in \mathbb{Z}. \tag{15}$$

We fix the first moment $f_1 = \mathcal{F}[z]$ by

$$f_1 = \kappa \tag{16}$$

where κ is an arbitrary nonzero constant. We can show that the moment determinant of Toeplitz form

$$\Delta_n^{(s)} = \det\left(f_{s-j+k}\right) \; j,k = 0, \dots, n-1 = \begin{vmatrix} f_s & f_{s+1} & \cdots & f_{s+n-1} \\ f_{s-1} & f_s & \cdots & f_{s+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s-n+1} & f_{s-n+2} & \cdots & f_s \end{vmatrix} \tag{17}$$

does not vanish for $s \in \{0,1\}$ and $n \in \mathbb{N}$. The LBPs $P_n(z)$ have the determinant expression

$$P_n(z) = \frac{1}{\Delta_n^{(0)}} \begin{vmatrix} f_0 & f_1 & \cdots & f_{n-1} & f_n \\ f_{-1} & f_0 & \cdots & f_{n-2} & f_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{-n+1} & f_{-n+2} & \cdots & f_0 & f_1 \\ 1 & z & \cdots & z^{n-1} & z^n \end{vmatrix}. \tag{18}$$

Thus, from (1) and (14), the coefficients b_n and c_n of the recurrence (8) and the constants h_n in the orthogonality (14)

$$b_n = -\frac{\Delta_{n+1}^{(1)}\Delta_{n-1}^{(0)}}{\Delta_n^{(1)}\Delta_n^{(0)}}, \quad c_n = \frac{\Delta_{n+1}^{(1)}\Delta_n^{(0)}}{\Delta_n^{(1)}\Delta_{n+1}^{(0)}}, \quad h_n = \frac{\Delta_{n+1}^{(0)}}{\Delta_n^{(0)}}. \tag{19}$$

The inverted polynomials

$$\tilde{P}_n(z) = \frac{z^n P_n(z^{-1})}{P_n(0)}. \tag{20}$$

also make a family of LBPs which are determined by the recurrence (8) with the different coefficients

$$\tilde{b}_n = \frac{b_n}{c_{n-1}c_n}, \quad \tilde{c}_n = \frac{1}{c_n}. \tag{21}$$

We can determine a linear functional $\tilde{\mathcal{F}}$ for $\tilde{P}_n(z)$ by the moments

$$\tilde{f}_k = \tilde{\mathcal{F}}[z^k] = f_{1-k}, \quad k \in \mathbb{Z}. \tag{22}$$

Then,

$$\tilde{f}_1 = \tilde{\kappa} := f_0 = \frac{\kappa}{c_0}. \tag{23}$$

Equations (19) and (21) imply that

$$\Delta_n^{(1)} = (-1)^{\frac{n(n-1)}{2}} \kappa^n \prod_{k=1}^{n-1} \left(\frac{b_k}{c_{k-1}}\right)^{n-k}, \tag{24}$$

$$\Delta_n^{(0)} = (-1)^{\frac{n(n-1)}{2}} \kappa^n \prod_{k=1}^{n-1} \left(\frac{\tilde{b}_k}{\tilde{c}_{k-1}}\right)^{n-k}. \tag{25}$$

We make use of the formulae (24) to compute the moment determinant $\Delta_n^{(s)}$.

A T-fraction is a continued fraction

$$T(z) = \frac{\kappa \quad |}{|z - c_0} - \frac{b_1 z \quad |}{|z - c_1} - \frac{b_2 z \quad |}{|z - c_2} - \cdots. \tag{26}$$

The n-th convergent of $T(z)$

$$T_n(z) = \frac{\kappa \quad |}{|z - c_0} - \frac{b_1 z \quad |}{|z - c_1} - \cdots \frac{b_{n-1} z \quad |}{|z - c_{n-1}} \tag{27}$$

is expressed by a ratio of polynomials

$$T_n(z) = \frac{Q_n(z)}{P_n(z)} \tag{28}$$

where $P_n(z)$ is the LBP of degree n determined by the recurrence (8), and $Q_n(z)$ is the polynomial determined by the same recurrence (8) from different initial values $Q_0(z) = 0$ and $Q_1(z) = \kappa$. Thus, we can identify the LBP $P_n(z)$ with the denominator polynomial of $T_n(z)$.

In Padé approximants, the convergent $T_n(z)$ simultaneously approximates two formal power series

$$F_+(z) = \sum_{k=1}^{\infty} f_k z^{-k} \quad \text{and} \quad F_-(z) = -\sum_{k=0}^{\infty} f_{-k} k^k \tag{29}$$

in the sense that

$$T_n(z) = F_+(z) + 0(z^{-n-1}) \qquad \text{as } z \to \infty \tag{30}$$

$$= F_-(z) + 0(z^n) \qquad \text{as } z \to 0 \tag{31}$$

where $f_k = \mathcal{F}[z^k]$ are the moments of the LBPs $P_n(z)$. That is, expanded into series at $z = \infty$ and at $z = 0, T_n(z) = Q_n(z)/P_n(z)$ coincide with $F_+(z)$ and $F_-(z)$, respectively, at least in the first n terms. The approximation (29) of $F_+(z)$ and $F_-(z)$ by $T_n(z)$ is equivalent to the orthogonality (14) of LBPs.

Taking the limit $n \to \infty$ in (29), we observe that the T-fraction $T(z)$ equals to $F_+(z)$ and $F_-(z)$ as formal power series,

$$T(z) = F_+(z) \qquad \text{as } z \to \infty \tag{32}$$
$$= F_-(z) \qquad \text{as } z \to 0 \tag{33}$$

We give a combinatorial interpretation to the moments of LBPs. We review the result by providing two new simple proofs. The lattice path interpretation of LBP is quite analogous to those in the combinatorial interpretation of orthogonal polynomials by Viennot. We owe the idea of the proof by T-fractions to a combinatorial interpretation of continued fractions by Flajolet.

Let P be a Schröder path. We label each step in P by unity if the step is an up step, by b_n if a down step descending from the line $y = n$ and by c_n if a level step on the line $y = n$, where b_n and c_n are the coefficients of the recurrence (8) of the LBPs $P_n(z)$. We then define the weight $w(P)$ of P by the product of the labels of all the steps in P. For example, the path in Figure 1 weighs $w(P) = b_1^2 b_2^3 b_3 c_0 c_1 c_2^2$. In the same way, labeling each step in P using the recurrence coefficients $\tilde{b}_n$ and $\tilde{c}_n$ for $\tilde{P}_n(z)$, we define another weight $\widetilde{w}(P)$. The main statement is the following.

Theorem (2.1.3) [2]:

The moments $f_k = \mathcal{F}[z^k]$ of LBPs admit the expressions

$$f_{k+1} = \kappa \sum_{P \in S_k} w(P), \tag{34}$$

$$f_{-k} = \tilde{\kappa} \sum_{P \in S_k} \widetilde{w}(P) \quad \text{for } k \in \mathbb{N}. \tag{35}$$

For example,

$$f_{-2} = \tilde{\kappa}\left(\tilde{b}_1\tilde{b}_2 + \tilde{b}_1^2 + \tilde{b}_1\tilde{c}_1 + 2\tilde{b}_1\tilde{c}_0 + \tilde{c}_0^2\right), \tag{36}$$
$$f_{-1} = \tilde{\kappa}\left(\tilde{b}_1 + \tilde{c}_0\right), \tag{37}$$
$$f_0 = \tilde{\kappa}, \tag{38}$$
$$f_1 = \kappa, \tag{39}$$
$$f_2 = \kappa(b_1 + c_0), \tag{40}$$
$$f_3 = \kappa(b_1 b_2 + b_1^2 + b_1 c_1 + 2b_1 c_0 + c_0^2). \tag{41}$$

We show two different proofs of Theorem (2.1.3). The first proof is based on LBPs. The second proof is based on T-fractions.

Lemma (2.1.4) [2]:

For $n \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\mathcal{F}[P_n(z)z^{k+1}] = \kappa \sum_P w(P), \tag{42}$$

$$\tilde{\mathcal{F}}[\tilde{P}_n(z)z^{k+1}] = \tilde{\kappa} \sum_P \tilde{w}(P), \tag{43}$$

where both the sums range over all Schröder paths P from $(-n, -n)$ to $(2k, 0)$.

Proof:

Let us write $f_{n,k} = \mathcal{F}[P_n(z)z^{k+1}]$. From the recurrence (8) of $P_n(z)$, we obtain a recurrence of $f_{n,k}$

$$f_{n,k} = f_{n+1,k-1} + c_n f_{n,k-1} + b_n f_{n-1,k}. \tag{44}$$

for $n \in \mathbb{N}$ and $k \in \mathbb{N}$, where the boundary values $f_{-1,k} = 0$ and $f_{n,-1} = \tilde{\kappa}\delta_{n,0}$ are induced from (14) and (35). The recurrence (44) leads us to a combinatorial expression of (42),

$$f_{n,k} = \tilde{\kappa} C_0 \sum_P W(P) = \kappa \sum_p W(P) \tag{45}$$

where the sum ranges over all Schröder paths P from $(-n, -n)$ to $(2k, 0)$. In much the same way, we can derive (43) using Schröder paths labelled with $\tilde{b}_n$ and $\tilde{c}_n$.

From (15) and (22), Theorem (2.1.3) is the special case of $n = 0$ in Lemma (1.2.4). That completes the proof of Theorem (2.1.3) by LBPs.

For a Schröder path P, we define length (P) by the sum of half the number of up and down steps and the number of level steps in P. For example, the path P in Figure 1 is as long as $\text{length}(P) = 10$.

Lemma (2.1.5) [2]:

The T-fraction $T(z)$ admits the expansions into formal power series

$$T(z) = \kappa \sum_{P} w(P) z^{-\text{length}(P)-1} \quad \text{as } z \to \infty; \tag{46}$$

$$= -\tilde{\kappa} \sum_{P} \widetilde{w}(P) z^{\text{length}(P)} \qquad \text{as } z \to 0, \tag{47}$$

where the both (formal) sums range over all Schröder paths P from $(0,0)$ to some point on the x-axis.

Proof:

Let us consider partial convergents of $\mathrm{T(z)}$

$$\mathrm{T_{m,n}(z)} = \frac{1 \quad |}{|\mathrm{z} - \mathrm{c_m}} - \frac{\mathrm{b_{m+1}z} \quad |}{|\mathrm{z} - \mathrm{c_{m+1}}} - \cdots - \frac{\mathrm{b_{m+n-1}z} \quad |}{|\mathrm{z} - \mathrm{c_{m+n-1}}} \tag{48}$$

for $m \in \mathbb{N}$ and $n \in \mathbb{N}$, where $T_{m,0}(z) = 0$. We first show by induction for $n \in \mathbb{N}$ that $T_{m,n}(z) = S_{m,n}(z)$ as $n \to \infty$ where $S_{m,n}(z)$ denotes the formal power series

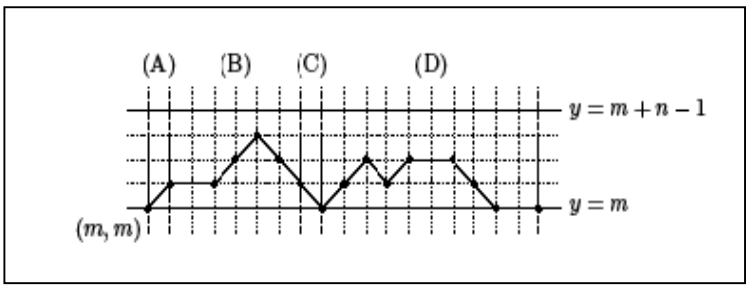

Figure 3: The decomposition of a path P_3 in the class (iii) into four parts (A), (B), (C) and (D).

$$s_{m,n}(z) = \sum_{P} w(P)\, z^{-\text{length}(P)-1} \tag{49}$$

over all Schröder paths P from (m, m) to some point on the line $y = m$ which lie in the region bounded by y=m and $y = m + n - 1$. (Hence, all the points in P have the y-coordinates $\geq m$ and $\leq m + n - 1$.) For $n = 0$, it is trivial that $S_{m,0}(z) = 0$ because the region in which P may live is empty. Hence, $T_{m,0}(z) = S_{m,0}(z) = 0$.

Suppose that $n \geq 1$. We classify Schröder paths P in the sum (49) into three classes: (i) the empty path only of one point at (m, m) (without steps) of weight 1; (ii) paths P_2 beginning by an up step; (iii) paths P_3 beginning by a level step. Thus,

$$S_{m,n}(z) = z^{-1} + \sum_{P_2} w(P_2) z^{-\mathrm{length}(P_2)-1} + \sum_{P_3} w(P_3) z^{-\mathrm{length}(P_3)-1}, \tag{50}$$

where the sums with respect to P_2 and P_3 are taken over all Schröder paths in the classes (ii) and (iii), respectively. Each path P_2 in the class (ii) consists of an initial level step on $y = m$, labelled c_m, and a subpath (maybe empty) from $(m+2, m)$ to some point on $y = m$. Hence,

$$\sum_{P_2} w(P_2) z^{-\mathrm{length}(P_2)-1} = c_m z^{-1} S_{m,n}(z). \tag{51}$$

Each path P_3 in the class (iii), as shown in Figure 3, uniquely decomposed into four parts: (A) an initial up step, labelled unity; (B) a subpath (maybe empty) from $(m+1, m+1)$ to some point on $y = m+1$ never going beneath $y = m+1$; (C) the first down step descending from $y = m+1$ to $y = m$, labelled b_{m+1}; (D) a subpath (maybe empty) both of whose initial and terminal points are on $y = m$. Hence,

$$\sum_{P_3} w(P_3) z^{-\mathrm{length}(P_3)-1} = b_{m+1} S_{m+1,n-1}(z) S_{m,n}(z). \tag{52}$$

Substituting (51) and (52) into (50), we get

$$S_{m,n}(z) = \left\{z - c_m - b_{m+1} z S_{m+1,n-1}(z)\right\}^{-1}. \tag{53}$$

From the assumption of induction, we can assume that $S_{m+1,n-1}(z) = T_{m+1,n-1}(z)$ as $z \to \infty$ and hence

$$S_{m,n}(z) = \left\{z - c_m - b_{m+1} z T_{m+1,n-1}(z)\right\}^{-1} = T_{m,n}(z) \; as \; z \to \infty. \tag{54}$$

Now, let us prove Lemma (2.1.5). In taking the limit $n \to \infty$ of the identity $T_{0,n}(z) = S_{0,n}(z)$ as $z \to \infty$, the left-hand side $T_{0,n}(z)$ tends to $T(z)$ while the

right-hand side $S_{0,n}$ to the right-hand side of (46). In order to show (47), we observe from (2.21) that $T_{m,n}(z)$ is equivalent to

$$T_{m,n}(\mathrm{z}) = -\frac{\tilde{\mathrm{c}}_m \mathrm{z}^{-1} \quad |}{|\mathrm{z}^{-1} - \tilde{c}_m} - \frac{\tilde{b}_{m+1} z^{-1} \quad |}{|z^{-1} - \tilde{c}_{m+1}} - \cdots - \frac{\tilde{b}_{m+n-1} z^{-1} \quad |}{|z^{-1} - \tilde{c}_{m+n-1}} \tag{55}$$

We can thereby show (47) as a simple corollary of (46). That completes the proof of Lemma (2.1.5).

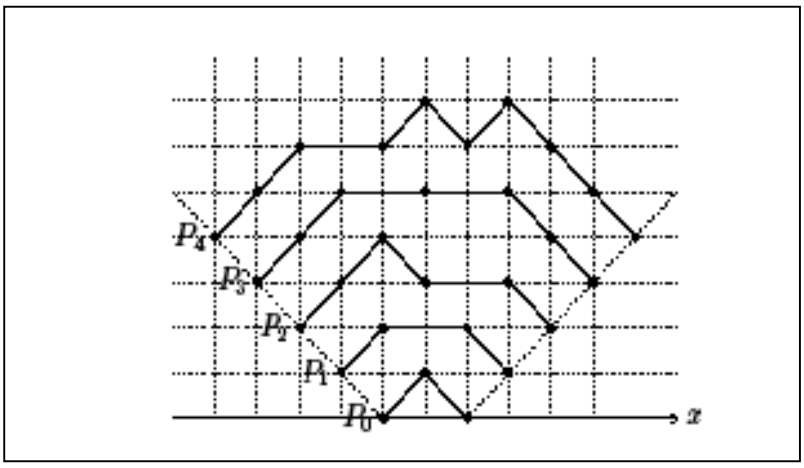

Figure 4: A quintuple $P = (P_0, \ldots, P_4) \in S_{1,5}$ of non-intersecting Schröder paths which is drawn in a plane.

The expressions (34) of moments in Theorem (2.1.3) are derived just by equating (32) and (33). Indeed, every Schröder path P from (0,0) to some point on the x-axis terminates at $(2k, 0)$ if and only if $\mathrm{length}(P) = k$. That completes the proof of Theorem (2.1.3) by T-fractions.

As a consequence of Theorem (2.1.3), we examine the moment determinant $\Delta_n^{(s)}$ from a combinatorial viewpoint. We utilize Gessel–Viennot's lemma to read the determinant in terms of non-intersecting paths.

For $m \in \mathbb{N}$ and $n \in \mathbb{N}$, let $S_{m,n}$ denote the set of n-tuples $P = (P_0, \ldots, P_{n-1})$ of Schröder paths P_k such that (i) P_k goes from $(-k, k)$ to $(2m + k, k)$ and that (ii) every two distinct paths P_j and $P_k, j \neq k$, are non-intersecting, namely $P_j \cap P_k = \emptyset$. As shown in Figure 4, each n-tuple $P \in S_{m,n}$ can be drawn in a diagram of n non intersecting Schröder paths which are pairwise disjoint. For simplicity, we write

$$w(P) = \prod_{k=0}^{n-1} w(P_k), \quad \widetilde{w}(P) = \prod_{k=0}^{n-1} \widetilde{w}(P_k). \tag{56}$$

Theorem (2.1.6) [2]:

For general b_n and c_n nonzero, the moment determinant $\Delta_n^{(s)}$ admits the expressions

$$\Delta_n^{(s)} = (-1)^{\frac{n(n-1)}{2}} \kappa^n \left(\prod_{j=1}^{n-1} b_j^{n-j} \right) \sum_{P \in S_{s-n,n}} w(P) \quad \text{if } s \geq n; \tag{57}$$

$$= (-1)^{\frac{n(n-1)}{2}} \tilde{\kappa}^n \left(\prod_{j=1}^{n-1} \tilde{b}_j^{n-j} \right) \sum_{P \in S_{|s|-n+1,n}} \widetilde{w}(P) \quad \text{if } s \leq -n+1. \tag{58}$$

Proof:

Suppose that $s \geq n \geq 0$. We rewrite $\Delta_n^{(s)}$ into Hankel form,

$$\Delta_n^{(s)} = (-1)^{\frac{n(n-1)}{2}} \det\left(f_{s-n+j+k+1}\right)_{j,k=0,\dots,n-1} \tag{59}$$

Owing to Theorem (2.1.3), the (j,k)-entry of the last Hankel determinant has the combinatorial expression

$$f_{s-n+j+k+1} = \kappa \sum_{P_{j,k}} w\left(P_{j,k}\right) \tag{60}$$

where we can assume that the sum ranges over all Schröder paths $P_{j,k}$ from $(-2j, 0)$ to $(2(s-n)+2k, 0)$. Thus, we can apply Gessel–Viennot's lemma to expand the determinant (59),

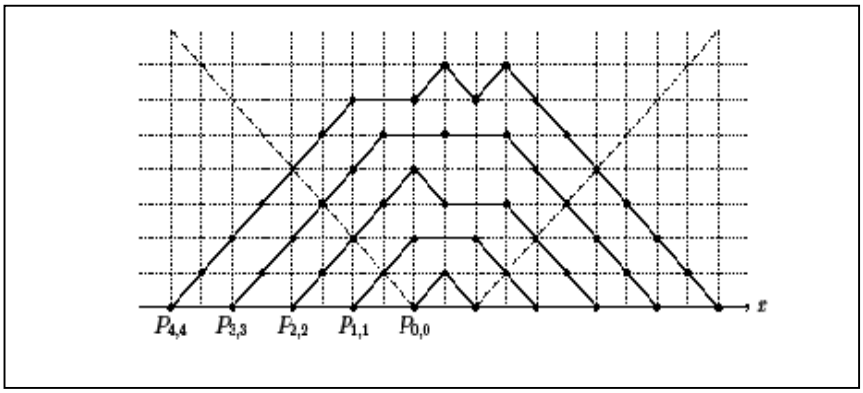

Figure 5: A quintuple $(P_{0,0}, \ldots, P_{4,4})$ of non-intersecting Schröder paths counted in the right-hand sum of (61) ($m = 1$ and $n = 5$).

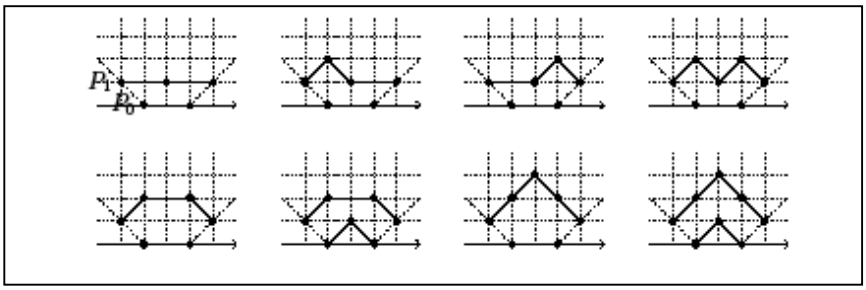

Figure 6: The eight doubles $(P_0, P_1) \in S_{1,2}$ of non-intersecting Schröder paths.

$$\det\left(f_{s-n+j+k+1}\right)_{j,k=0,\ldots,n-1} = \kappa^n \sum_{(P_{0,0},\ldots,P_{n-1,n-1})} w(P_{0,0}) \ldots w(P_{n-1,n-1}) \tag{61}$$

where the sum ranges over all n-tuples $(P_{0,0}, \ldots, P_{n-1,n-1})$ of non-intersecting Schröder paths $P_{k,k}$ such that $P_{k,k}$ goes from $(-2k, 0)$ to $(2(s-n) + 2k, 0)$ for each k. (See Figure 5 for example.) As shown in Figure 5, the first and last k steps of $P_{k,k}$ must be all up and down steps, respectively, so that the paths do not collide. Especially, $P_{k,k}$ passes the points $(-k, k)$ and $(2(s-n) + k, k)$. We thus have

$$\det\left(f_{s-n+j+k+1}\right)_{j,k=0,\ldots,n-1} = \kappa^n \left(\prod_{j=1}^{n-1} b_j^{n-1}\right) \sum_{P \in S_{s-n,n}} w(P) \tag{62}$$

and thereby (57). In the same way, we can show (58) from Theorem (2.1.3).

For example, for $m = 3$ and $n = 2$, the set $S_{1,2}$ contains exactly eight doubles (P_0, P_1) of non-intersecting Schröder paths which are shown in Figure 6. Thus, the moment determinant $\Delta_2^{(3)}$ is equal to the polynomial of eight monomials

$$\Delta_2^{(3)} = -\kappa^2 b_1 (c_0 c_1^2 + 2b_2 c_0 c_1 + b_2^2 c_0 + b_2 c_0 c_2 + b_1 b_2 c_2 + b_2 b_3 c_0 + b_1 b_2 b_3) \qquad (63)$$

of which each monomial corresponds to a diagram in Figure 6.

Section (2.2): Determinant of q-Narayana Polynomials and Aztec Diamond Theorem

We consider the special case of the LBPs whose moments are described by the q-Narayana polynomials. Let us recall from Section (2.1) the definition of the q-Narayana polynomials

$$N_k(t,q) = \sum_{P \in S_k} t^{\mathrm{level}(P)} q^{\mathrm{area}(P)} \tag{64}$$

where $\mathrm{level}(P)$ denotes the number of level steps in a Schröder path P, and $\mathrm{area}(P)$ the area bordered by P and the x-axis. For example, the first few of the q-Narayana polynomials are enumerated in

$$N_0(t,q) = 1, \tag{65}$$

$$N_1(t,q) = t + q, \tag{66}$$

$$N_2(t,q) = t^2 + 2tq + tq^3 + q^2 + q^4, \tag{67}$$

$$N_3(t,q) = t^3 + 3t^2q + 2t^2q^3 + t^2q^5 + 3tq^2 + 4tq^4 + 2tq^6 + tq^8 + q^3 + 2q^5 + q^7 + q^9. \tag{68}$$

In view of Theorem (2.1.3), it is easy to find the q-Narayana polynomials in the moments of LBPs.

Theorem (2.2.1) [2]:

Let $P_n(\mathrm{z})$ be the LBPs determined by the recurrence (8) with the coefficients

$$b_n = q^{2n-1}, \quad c_n = tq^{2n}. \tag{69}$$

Then, the linear functional $\mathcal{F}$ for $P_n(z)$ admits the moments $f_k = \mathcal{F}[z^k]$ described by the q-Narayana polynomials,

$$f_k = \kappa N_{k-1}(t,q) \qquad \text{for } k \geq 1; \tag{70}$$

$$= \kappa t^{2|k|-1} N_{|k|}(t,q^{-1}) \qquad \text{for } k \leq 0, \tag{71}$$

where κ is an arbitrary nonzero constant.

Proof:

Let $P \in S_k$. Labelled with (69), P weighs $w(P) = t^{\text{level}(P)} q^{\text{area}(P)}$. Hence, by virtue of Theorem (2.1.3), we have (70) as a special case of (234). Similarly, with

$$\tilde{b}_n = t^{-2} q^{-2n+1}, \quad \tilde{c}_n = t^{-1} q^{-2n}. \tag{72}$$

from (21), $\widetilde{w}(P) = t^{-2k+\text{level}(P)} q^{-\text{area}(P)}$. Now $\tilde{\kappa} = \kappa t^{-1}$ from (23). Hence, we obtain (71) from (35).

We remark that the q-Narayana polynomials (64) defined in a combinatorial way are already investigated by Cigler who introduced the polynomials by modifying the generating function of the q-Catalan numbers. Indeed, we can deduce from (64) a recurrence

$$N_k(t,q) = tN_{k-1}(t,q) + \sum_{j=0}^{k-1} q^{2j+1} N_j(t,q) N_{k-j-1}(t,q) \tag{73}$$

We examine a Toeplitz determinant of the q-Narayana polynomials

$$\mathcal{N}_n^{(s)}(t,q) = \det\left(N_{s+j-k-1}(t,q)\right)_{j,k=0,\dots,n-1} \tag{74}$$

where, in view of (70), we define $N_k(t,q)$ for negative k by

$$N_k(t,q) = t^{2k-1} N_{-k-1}(t,q^{-1}) \quad \text{for } k < 0 \tag{75}$$

As the special case of the moments given by the q-Narayana polynomials, Theorem (2.1.6) allows us to read the determinant (74) in the context of nonintersecting Schröder paths. The results in this section, Theorem (2.2.3) and Corollary (2.2.4), will be applied later to give a new proof of the Aztec diamond theorem (Theorem (2.1.1)).

Theorem (2.2.1) implies that

$$\Delta_n^{(s)} = \mathcal{N}_n^{(s)}(t,q) \tag{76}$$

provided that the coefficients b_n and c_n of the recurrence (8) are given by (69) where $\kappa = 1$. Hence, we can use the formulae (24) to find the exact value of $N_n^{(s)}$

for $s \in \{0,1\}$. Recall that, in using (25), we assume $\tilde{b}_n$ and $\check{c}_n$ to be given by (72) and $\tilde{\kappa} = t^{-1}$.

Lemma (2.2.2) [2]:

For $s \in \{0,1\}$ and $n \in \mathbb{N}$, the exact value of the determinant $\mathcal{N}_n^{(s)}$ is given by

$$\mathcal{N}_n^{(1)}(t,q) = (-1)^{\frac{n(n-1)}{2}} t^{-\frac{n(n-1)}{2}} q^{\frac{n(n-1)}{2}}, \tag{77}$$

$$\mathcal{N}_n^{(0)}(t,q) = (-1)^{\frac{n(n-1)}{2}} t^{-\frac{n(n-1)}{2}} q^{-\frac{n(n-1)}{2}}. \tag{78}$$

In order to find the value of $N_n^{(s)}(t,q)$ for further $s \in \mathbb{Z}$ and $n \in \mathbb{N}$, we can use Sylvester's determinant identity:

$$X \cdot X(i,j,k,\ell) - X(i;k) \cdot X(j;\ell) + X(i;\ell) \cdot X(j;k) = 0 \tag{79}$$

where X is an arbitrary determinant and $X(i,j;k,\ell)$ denotes the minor of X obtained by deleting the i-th and the j-th rows and the k-th and the ℓ-th columns; $X(i;k)$ the minor of X with respect to the i-th row and the k-th column. Applying Sylvester's determinant identity, we get

$$\mathcal{N}_{n+1}^{(s)} \cdot \mathcal{N}_{n-1}^{(s)} - \mathcal{N}_n^{(s)} \cdot \mathcal{N}_n^{(s)} + \mathcal{N}_n^{(s+1)} \cdot \mathcal{N}_n^{(s-1)} = 0 \tag{80}$$

for $s \in Z$ and $n \in N$, where $\mathcal{N}_n^{(s)} = \mathcal{N}_n^{(s)}(t,q)$ except that $\mathcal{N}_{-1}^{(s)} = 0$. Using (80) as a recurrence from appropriate initial value, we can compute the value of $N_n^{(s)}(t,q)$ for each $s \in \mathbb{Z}$ and $n \in \mathbb{N}$. Especially, we find a closed form of $\mathcal{N}_n^{(s)}(t,q)$ for $-n \le s \le n+1$ as follows.

Theorem (2.2.3) [2]:

For $-n \le s \le n+1$, the exact value of the determinant $\mathcal{N}_n^{(s)}(t,q)$ is given by

$$\mathcal{N}_n^{(s)}(t,q) = \varphi_n^{(s)}(t,q) \prod_{k=1}^{s-1} (t + q^{2k-1})^{s-k} \quad \text{for } 1 \le s \le n+1; \tag{81}$$

$$= \varphi_n^{(s)}(t,q) \prod_{k=1}^{|s|} (t + q^{-2k+1})^{|s|-k+1} \quad \text{for } -n \le s \le 0, \tag{82}$$

where

$$\varphi_n^{(s)}(t,q) = (-1)^{\frac{n(n-1)}{2}} t^{-\frac{(n-s)(n-s+1)}{2}} q^{\frac{n(n-1)(2s-1)}{2}}. \tag{83}$$

Proof:

Using Sylvester's identity (80) from the initial value (77), we can easily show (81) by induction.

Note that Cigler found a closed-form expression of the Hankel determinant $\det(N_{s+j+k}(t,q))_{j,k=0,\dots,n-1}$ of the q-Narayana polynomials for $s \in \{0,1\}$ and $n \in \mathbb{N}$ by means of orthogonal polynomials. (The Hankel determinant coincides with $\mathcal{N}_n^{(s)}(t,q)$ for $s \in \{n, n+1\}$ without sign.) Theorem (2.2.3) generalizes Cigler's result for further s and n.

As a corollary of Theorem (2.2.3), equating (81) with (57) in Theorem (2.1.6), we obtain the following result about non-intersecting Schröder paths.

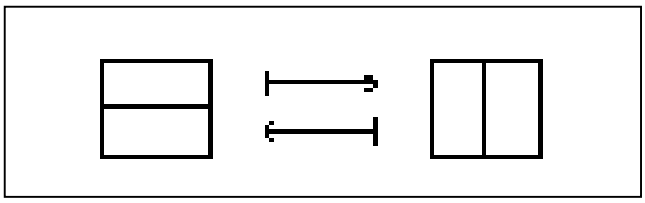

Figure 7: Rotation of a two-by-two block of two horizontal or vertical dominoes in an elementary move.

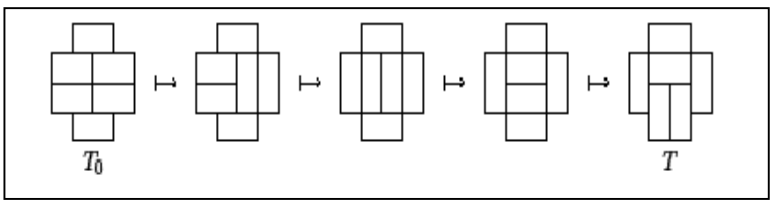

Figure 8: A sequence of elementary moves from T^0 to T of AD_2. At least four elementary moves are required to reach from T^0 only of horizontal dominoes to the rightmost , and thereby $r(T\ ) = 4$.

Corollary (2.2.4) [2]:

For $m \in \{0,1\}$ and $n \in \mathbb{N}$,

$$\sum_{P\in S_{m,n}} t^{\mathrm{level}(P)} q^{\mathrm{area}(P)} = q^{\frac{n(n-1)(3m+2n-1)}{3}} \prod_{k=1}^{m+n-1} (t+q^{2k-1})^{m+n-k} \qquad (84)$$

where $\mathrm{level}(P) = \sum_{k=0}^{n-1} \mathrm{level}(P_k)$ and $\mathrm{area}(P) = \sum_{k=0}^{n-1} \mathrm{area}(P_k)$ with $P = (P_0, \dots, P_{n-1})$.

Finally, we give a new proof of the Aztec diamond theorem (Theorem (2.1.1)) based on the discussions Elkies, Kuperberg, Larsen and Propp, the Aztec diamond theorem is proven by the technique of the domino shuffling. Whereas, the proof [2] is based on the one-to-one correspondence between tilings of the Aztec diamonds and tuples of non-intersecting Schröder paths developed by Johansson who derived the formula (3) with $q = 1$ with the help of Gessel-Viennot's lemma Eu and Fu used the one-to-one correspondence to give a simple proof to (7).

We give the definition of the rank statistic. Let $T \in T_n$ be a tilingof the Aztec diamond AD_n. If $n \geq 1, T$ certainly contains one or more two-by-two blocks of two horizontal or vertical dominoes. Thus, choosing one from such two-by-two blocks and rotating it by ninety degrees, we obtaina new tiling $T' \in T_n$. (See Figure 7.) We refer by an elementary move to this operation of transforming T into T' by rotating a two-by-two block. It can be shown that any tiling of AD_n can be reached from any other tiling of AD_n by a sequence of elementary moves. The rank $r(T)$ of T denotes the minimal number of elementary moves required to reach T from the "all horizontal" tiling T^0 consisting only of horizontal dominoes, where $r(T^0) = 0$. For example, in Figure 8, the rightmost tiling T of AD_2 has the rank $r(T) = 4$ since at least four elementary moves are required to reach from the leftmost T^0.

The one-to-one correspondence between T_n and $S_{1,n}$ by Johansson is given by the following bijection from T_n to $S_{1,n}$. We color the Aztec diamond AD_n in a black-white checker-board fashion so that all unit squares on the upper-left border of AD_n are white. We say that a horizontal domino (resp. a vertical domino) put into AD_n is even if the left half (resp. the upper half) of the domino covers a white unit square. Otherwise, the domino is odd. The bijection mapping a tiling $T \in T_n$ to an n-tuple $P = (P_0, \dots, P_{n-1}) \in S_{1,n}$ of non-intersecting Schröder paths is described by the following procedure: For each domino in T, as shown in Figure 9, draw an up step (resp. a down step, a level step) that goes through the center of the

domino if the domino is even vertical (resp. odd vertical, odd horizontal). (For even horizontal dominoes, we do nothing.) Then, we find n non-intersecting Schröder paths $P_0, \ldots, P_{n-1}$ on T of which the n-tuple $P = (P_0, \ldots, P_{n-1})$ belongs to $S_{1,n}$. For example, see Figure 10.

The bijection connects the statistics $v(T)$ and $r(T)$ for tilings and the statistics $\mathrm{level}(P)$ and $\mathrm{area}(P)$ for Schröder paths as follows. Recall from Section (2.1) that $v(T)$ denotes half the number of vertical dominoes in a tiling T.

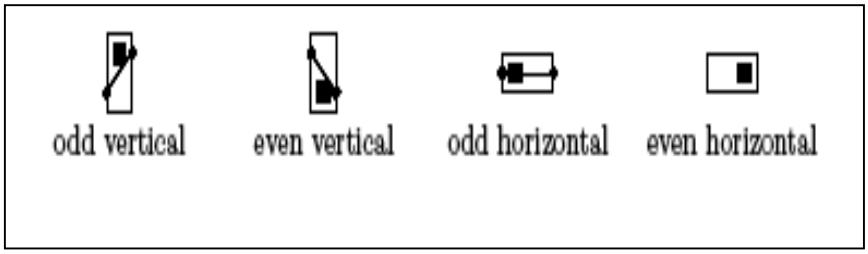

Figure 9: The rule to draw a step on a domain

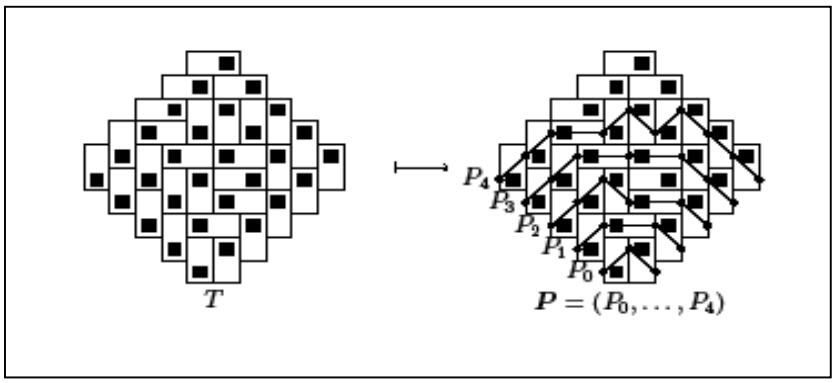

Figure 10: The bijection mapping a tiling $T \in T_5$ of AD_5 to a quintuple $P = (P_0, \ldots, P_4) \in S_{1,5}$ of non-intersecting Schröder paths. (The Aztec diamond is colored in a checkerboard fashion.)

Figure 11: A rotation of a two-by-two block in an elementary move raises the rank of the tiling T by one (left-to-right, respectively) if and only if the corresponding deformation of a tuple P of non-intersecting Schröder paths increases $\mathrm{area}(P)$ by one.

Lemma (2.2.5) [2]:

Suppose that a tiling $T \in T_n$ and an n-tuple $P = (P_0, \ldots, P_{n-1}) \in S(1, n)$ of non-intersecting Schröder paths are in the one-to-one correspondence by the bijection. Then,

$$v(T) = \frac{n(n+1)}{2} - \text{level}(P), \tag{85}$$

$$r(T) = \text{area}(P) - \frac{2n(n+1)(n-1)}{3}, \tag{86}$$

where $\text{level}(P) = \sum_{k=0}^{n-1} \text{level}(P_k)$ and $\text{area}(P) = \sum_{k=0}^{n-1} \text{area}(P_k)$.

Proof:

The bijection implies that $v(T)$ is equals to half the number of up and down steps in P. The sum of half the number of up and down steps and the number of level steps in P is a constant independent of P that equals to $n(n+1)/2$. Thus, we have (85).

As shown in Figure 11, each elementary move of a tiling T raising the rank by one gives rise to a deformation of some path in P increasing the area by one. Thus, $r(T)$ and $\text{area}(P)$ differ by a constant independent of T and P. Since $r(T^0) = 0$ then the constant equals to $\text{area}(P^0) = 2n(n+1)(n-1)/3$, where T^0 denotes the "all-horizontal" tiling of AD_n and $P_0 \in S_{1,n}$ the n-tuple of non-intersecting Schröder paths only of level steps that corresponds to T^0. Thus, we have (86).

Now, we give a proof of the Aztec diamond theorem:

As a consequence of Lemma (2.1.5), we can substitute (85) and (86) into (6) to obtain

$$AD_n(t, q) = t^{\frac{n(n+1)}{2}} q^{-\frac{2n(n-1)(n+1)}{3}} \sum_{P \in S_{1,n}} t^{-\text{level}(P)} q^{\text{area}(P)}. \tag{87}$$

From Corollary (2.2.4), the sum in the right-hand side of (87) is equated with

$$\sum_{P\in S_{1,n}} t^{-\mathrm{level}(P)} q^{\mathrm{area}(P)} = q^{\frac{2n(n-1)(n+1)}{3}} \prod_{k=1}^{n} (t^{-1} + q^{2k-1})^{n-k+1}. \tag{88}$$

Substituting (88) into the right-hand side of (87), we have

$$AD_n(t,q) = t^{\frac{n(n+1)}{2}} \prod_{k=1}^{n} (t^{-1} + q^{2k-1})^{n-k+1} = \prod_{k=1}^{n} (1 + tq^{2k-1})^{n-k+1}. \tag{89}$$

That completes the proof of Theorem (2.1.1).

We evaluated a determinant whose entries are given by the q-Narayana polynomials (Theorem (2.2.3)). In order to find the value of the determinant, we utilized Laurent biorthogonal polynomials which allow of a combinatorial interpretation in terms of Schröder paths (Theorem (2.1.3) and Theorem (2.2.1)). As an application, we exhibited a new proof of the Aztec diamond theorem (Theorem (2.1.1)) by Elkies, Kuperberg, Larsen and Propp with the help of the one-to-one correspondence developed by Johansson between tilings of the Aztec diamonds and tuples of non-intersecting Schröder paths.

We remark that, in Theorem (2.2.3), we can evaluate the determinant $\mathcal{N}_n^{(s)}(t,q)$ of the q-Narayana polynomials also for $s < -n$ and $s > n+1$ by using the formula (2.80) from Sylvester's identity. For example, if $s = n+2$,

$$\mathcal{N}_n^{(n+2)}(t,q) = (-1)^{\frac{n(n-1)}{2}} q^{\frac{n(n-1)(2n+3)}{2}} \prod_{k=1}^{n} (t + q^{2k-1})^{n-k+1} \sum_{\ell=0}^{n} t^{n-\ell} q^{\ell^2} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{q^2}, \tag{90}$$

where $\begin{bmatrix} m \\ n \end{bmatrix}_q$ denotes the q-binomial coefficient

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \prod_{k=1}^{n} \frac{1 - q^{m-k+1}}{1 - q^k}. \tag{91}$$

From Theorem (2.1.6), we can read (90) in terms of non-intersecting Schröder paths,

$$\sum_{P\in S_{2,n}} t^{-\mathrm{level}(P)} q^{\mathrm{area}(P)} = q^{\frac{n(n-1)(2n+5)}{3}} \prod_{k=1}^{n} (t+q^{2k-1})^{n-k+1} \sum_{\ell=0}^{n} t^{n-\ell} q^{\ell^2} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{q^2}. \quad (92)$$

We can readily observe that the bijection gives a one-to-one correspondence between n-tuples of non-intersecting Schröder paths in $S_{2,n}$ and tilings of the region $AD_{2,n}$, the Aztec diamond AD_{n+1} from which two unit squares at the south corner are removed. (See Figure 12). Therefore, as a variant of (6), we have

$$\sum_{T} t^{\nu(T)} q^{r(T)} = \prod_{k=0}^{n-1} (t+tq^{2k+1})^{n-k} \sum_{\ell=0}^{n} t^{\ell} q^{\ell^2} \begin{bmatrix} n+1 \\ \ell \end{bmatrix}_{q^2}, \quad (93)$$

where the sum in the left-hand side ranges over all tilings T of $AD_{2,n}$. (The $\mathrm{rank}(T)$ is defined in the same way as AD_n to be the minimal number of elementary moves required to reach from "all-horizontal" tilings of $AD_{2,n}$.) Similarly, calculating the determinant $\mathcal{N}_n^{(m+n)}(t,q)$, we can obtain in principle variant formulae of (6) for tilings of the Aztec diamond AD_{m+n} from which $m(m-1)$ unit squares at the south corner are removed. However, the value of $\mathcal{N}_n^{(m+n)}(t,q)$ seems much complicated for large m, and exact formulae have not been found yet for general m and n.

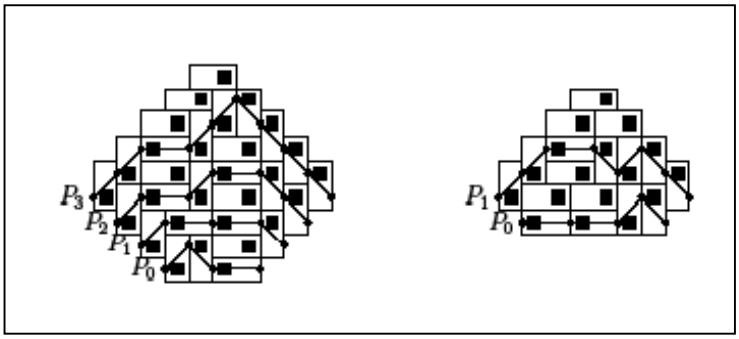

Figure 12: The one-to-one correspondence between a tiling of $AD_{m,n}$ and an n-tuple $P = (P_0, \ldots, P_{n-1}) \in S_{m,n}$ of non-intersecting Schr¨oder paths. (The left figure shows an instance for $m = 2$ and $n = 4$ while the right for $m = 3$ and $n = 2$.)

Chapter 3

Equivariant Quantum Schubert Polynomials

In this chapter we give new proofs of the presentation of the equivariant quantum cohomology ring, as well as Graham-positivity of the structure constants in equivariant quantum Schubert calculus.

Section (3.1): Background and Notations

Classical Schubert calculus is concerned with the cohomology rings of Grassmannians (In mathematics, the Grassmannians $Gr(r, V)$ is aspace which Paeameterizes all linear subspaces of a vector space V of given dimension r. For example, the Grassmannians $Gr(1, V)$ is the space of lines through the origin in V, so V it is the same as the projective space of one dimension lower than V.) [11], flag varieties, and related spaces. In recent years, equivariant and quantum versions of Schubert calculus have been developed. Two key ingredients in each setting are a presentation of the ring, and a "Giambelli formula" (In mathematics, Giambelli formula, named after Giovanni Giambelli, expresses Schubert classes in terms of special Schubert classes, or Schur functions in terms of complete symmetric functions. It states

$$\sigma_\lambda = \det\left(\sigma_{\lambda_i} + j - i\right)_{1 \le i,j \le r}$$

where σ_λ is the Schubert class of a partition λ. Giambelli formula is a consequence of pieri's formula. The porteous formula is a generalization to morphisms of vector bundles over a variety.) [12] expressing the additive basis of Schubert classes in terms of the presentation. For example, the ordinary cohomology of the Grassmannian is generated by Chern classes; the classical Giambelli formula in this context states that a Schubert class is represented by a Schur polynomial, and gives a determinantal expression for these polynomials in terms of Chern classes.

We study the equivariant quantum cohomology (In mathematics, specifically in sympletic topology and algebraic geometry, a quantum cohomology ring is an extension of the ordinary cohomology ring of a closed symplectic manifold. It comes in two versions, called small and big, in general, the latter is more complicated and contains more information than the former. In each the choice of

coefficient ring (typically a Novikov ring, described below) significantly affects its structure, as well.) [13] ring of a partial flag variety. The main results give Giambelli formulas for Schubert classes: specifically, we show that the Schubert classes are represented by equivariant quantum Schubert polynomials (In mathematics, Schubert polynomials are generalizations of Schubert polynomials that represent cohomology classes of Schubert cycles in flag varieties. They were introduced by Lascoux & Scützenberger (1982) and are named after Hermann Schubert.) [14] $\mathfrak{S}_w^q(\sigma, t)$, and in the case of the complete flag variety. These polynomials are specializations of Fulton's universal double Schubert polynomials, by a substitution analogous to the specialization of universal (single) Schubert polynomials to the quantum Schubert polynomials of Fomin, Gelfand and Postnikov. We obtain a direct relationship between universal double Schubert polynomials, which solve a degeneracy locus problem, and their specializations to equivariant quantum Schubert polynomials.

Fix an n-dimensional vector space V and a tuple of integers $\boldsymbol{n} = (0 < n_1 < \cdots < n_m < n_{m+1} = n)$, and let $Fl(\boldsymbol{n}) = Fl(n_1, \ldots, n_m; V)$ denote the partial flag variety parametrizing quotient flags $V = W_{m+1} \twoheadrightarrow W_m \twoheadrightarrow \cdots \twoheadrightarrow W_1$, with $\dim W_p = n_p$. We write $Fl(V)$ for the complete flag variety, corresponding to the case $\boldsymbol{n} = (1, \ldots, n)$. The basis of Schubert classes for $Fl(\boldsymbol{n})$ is indexed by certain permutations $S^{\boldsymbol{n}} \subseteq S_n$, namely the minimal coset representatives for the quotient by a parabolic subgroup. There are several equivalent ways to define the corresponding Schubert polynomials. For the purposes of this introduction, we use a "quantization" approach.

For any $w \in S_n$, a classical Schubert polynomial $\mathfrak{S}_\omega(x)$ can be written uniquely as a sum

$$\mathfrak{S}_\omega(x) = \sum a_{k_1 \ldots k_{n-1}} e_{k_1}(1) \ldots e_{k_{n-1}}(n-1) \tag{1}$$

over sequences $(k_1, \ldots, k_{n-1})$ with $0 \leq k_1 \leq l$ and $\sum k_l = \ell(\omega)$, where the $a_{k_1 \ldots k_{n-1}}$ are integers and $e_k(l) := e_k(x_1, \ldots, x_l)$ is the kth elementary symmetric polynomial in the variables $x_1, \ldots, x_l$.

To form the quantum polynomials, one replaces $e_k(l)$ with quantum elementary polynomials $e_k^{\boldsymbol{n},q}(n_p)$, which are defined for $n_p \in \boldsymbol{n}$, as follows. Let

$$\sigma_1^1, \dots, \sigma_{n_1}^1, \sigma_1^2, \dots, \sigma_{n_2-n_1}^2, \dots, \sigma_1^{m+1}, \dots, \sigma_{n-n_m}^{m+1}$$

be n independent variables, with σ_i^p of degree i. Form the $n \times n$ block-diagonal matrix $M^\sigma = diag(D_1, D_2, \dots, D_{m+1})$, where D_p is the square matrix of size $n_p - n_{p-1}$ with $\sigma^{p,}s$ along the last column:

$$D_p = \begin{pmatrix} 0 & \cdots & 0 & \sigma_{n_p-n_{p-1}}^p \\ 0 & \cdots & 0 & \sigma_{n_p-n_{p-1}-1}^p \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & \sigma_2^p \\ 0 & \cdots & 0 & \sigma_1^p \end{pmatrix} \tag{2}$$

Let $M^q = (m_{ab})$ be the $n \times n$ matrix with entries

$$m_{ab} = \begin{cases} (-1)^{n_{p+1}-n_p+1} q_p & \text{if } (a,b) = (n_{p-1}+1, n_{p+1}), \\ -1 & \text{if } (a,b) = (b+1, b), \\ 0 & \text{otherwise;} \end{cases} \tag{3}$$

and let $M^{\sigma,q} = M^\sigma + M^q$. Writing $M_p^{\sigma,q}$ for the upper-left $n_p \times n_p$ submatrix of $M^{\sigma,q}$, the quantum elementary polynomial $e_k^{\boldsymbol{n},q}(n_p)$ is the coefficient of T^{n_p-k} in the determinant of $M_p^{\sigma,q} + T \cdot Id$.

For $\omega \in S^{\boldsymbol{n}}$, the quantum Schubert polynomial $\mathfrak{S}_\omega^{\boldsymbol{n},q}(\sigma)$ is formed by substituting $e_k^{\boldsymbol{n},q}(n_p)$ for $e_k(l)$ on the RHS of (1) whenever $l \in [n_p, n_{p+1})$. The equivariant quantum Schubert polynomial is defined by

$$\mathfrak{S}_\omega^{n,q}(\sigma, t) = \sum_{u,v} (-1)^{\ell(v)} \mathfrak{S}_u^{n,q}(\sigma) \mathfrak{S}_v(t), \tag{4}$$

where the sum is over $u, v \in S_n$ such that $v^{-1}u = w$ and $\ell(u) + \ell(v) = \ell(w)$. When $w \in S^{\boldsymbol{n}}$, the only u that appear in the sum are also in $S^{\boldsymbol{n}}$.

Let $\Lambda = H_T^*(pt)$, and recall that choosing an isomorphism $T \cong (\mathbb{C}^*)^n$ identifies Λ with a polynomial ring $\mathbb{Z}[t_1, \dots, t_n]$. The flag variety comes with a universal sequence of quotient bundles $V_{Fl(n)} \twoheadrightarrow F_m \twoheadrightarrow \cdots \twoheadrightarrow F_1$, where F_i is the vector bundle of rank n_i whose fiber over a point $W_\bullet \in Fl(\boldsymbol{n})$ is the vector space W_i. The

equivariant Chern classes of these bundles generate $H_T^* Fl(\boldsymbol{n})$ as an algebra over Λ, and the Schubert classes σ_w form a Λ-linear basis, as w ranges over $S^{\boldsymbol{n}}$. The equivariant quantum cohomology ring is a deformation by parameters $q_1, \ldots, q_m$, with q_i having degree $n_{i+1} - n_{i-1}$. As a $\Lambda[q]$-module, it is $QH_T^* Fl(\boldsymbol{n}) = H_T^* Fl(\boldsymbol{n}) \otimes_\Lambda \Lambda[q]$, so the same Schubert classes give a $\Lambda[q]$-linear basis. The Giambelli problem is to give a formula for σw in terms of the Chern class generators. Setting $\sigma_i^p = c_i^T(\ker(F_p \to F_{p-1}))$, our main result is the solution to this problem:

Theorem (3.1.1) [3]:

For any permutation $w \in S^{\boldsymbol{n}}$, we have

$$\sigma_w = \mathfrak{S}_w^{n,q}(\sigma, t)$$

as classes in $QH_T^* Fl(\boldsymbol{n})$.

(In the case of the complete flag variety, the variables are $x_p = \sigma_1^p$; the matrix $M_n^{\sigma,q}$ has $x_1, \ldots, x_n$ along the diagonal, $q_1, \ldots, q_{n-1}$ along the superdiagonal, and -1 along the subdiagonal. For any permutation w, we write $\mathfrak{S}_w^q(x, t)$ for the corresponding Schubert polynomial, and $x_p = c_1^T(\ker(F_p \to F_{p-1}))$ in the statement of the theorem.)

Table 1 Equivariant quantum Schubert polynomials for $Fl(\mathbb{C}^3)$ and $Fl(2; \mathbb{C}^3) = \mathbb{P}^2$.

w	$n = (1,2,3)$	$n = (2,3)$
123	1	1
213	$x_1 - t_1$	
132	$x_1 + x_2 - t_1 - t_2$	$\sigma_1^1 - t_1 - t_2$
231	$x_1 x_2 + q_1 - (x_1 + x_2)t_1 + t_1^2$	$\sigma_2^1 - \sigma_1^1 t_1 + t_1^2$
312	$x_1^2 - q_1 - x_1(t_1 + t_2) + t_1 t_2$	
321	$(x_1 - t_2)(x_1 x_2 + q_1 - (x_1 + x_2)t_1 + t_1^2)$	

As an example, the equivariant quantum Schubert polynomials $\mathfrak{S}_w^{\boldsymbol{n},q}(\sigma, t)$ for $Fl(\mathbb{C}^3) = Fl(1,2; \mathbb{C}^3)$ and $Fl(2; \mathbb{C}^3)$ are listed in Table1. Note that for $Fl(2; \mathbb{C}^3) \cong \mathbb{P}^2$, the quantum parameter q_1 is of degree 3 and does not appear in $\mathfrak{S}_w^{\boldsymbol{n},q}(\sigma, t)$ for any w.

A presentation of the equivariant quantum cohomology ring as a $\Lambda[q]$-algebra was given by Kim:

Theorem (3.1.2) [3]:

Let $e_k^{\boldsymbol{n},q}(n)$ be the quantum elementary polynomial defined above, and let $e_\kappa(t)$ be the (ordinary) elementary symmetric polynomial in $t_1, \ldots, t_n$. We have

$$QH_T^*Fl(\boldsymbol{n}) \cong \Lambda[q]\left[\sigma_1^1, \ldots, \sigma_{n_1}^1, \ldots, \sigma_1^{m+1}, \ldots, \sigma_{n-n_m}^{m+1}\right]/I_T^q$$

where I_T^q is the ideal $(e_1^{\boldsymbol{n},q}(n) - e_1(t), \ldots, e_n^{\boldsymbol{n},q}(n) - e_n(t))$.

Specializing the q variables to zero, one recovers the (Borel) presentation of $H_T^*Fl(\boldsymbol{n})$; specializing t to zero, one obtains $QH^*Fl(\boldsymbol{n})$; and sending both q and t to zero, one has $H^*Fl(\boldsymbol{n})$.

We give an alternative proof of this presentation: assuming the well-known presentation of $H_T^*Fl(\boldsymbol{n})$, one deduces Theorem (3.1.2) from the Giambelli formula of Theorem (3.1.1) (see Corollary (3.3.7) for the details).

Like their non-equivariant counterparts, the equivariant quantum Schubert polynomials possess a stability property: the same polynomial represents a Schubert class σ_w of codimension $\ell(w)$, independently of which flag variety $Fl(\mathbb{C}^n)$ it is in, for sufficiently large n. We give a precise statement to this effect. A useful computational consequence is that equivariant quantum products are computed (on the nose, not up to an ideal) by products of Schubert polynomials, at least for sufficiently large n. We include some examples, using the equivariant quantum Schubert polynomials to produce multiplication tables for $QH_T^*Fl(\mathbb{C}^n)$ (for small n).

Let $T \cong (\mathbb{C}^*)^n$ be a torus. One can find a contractible space $\mathbb{E}T$ on which T acts freely, and the quotient $\mathbb{B}T = \mathbb{E}T/T$ is then unique up to homotopy. The equivariant cohomology of a space X equipped with a T-action is defined by

$$H_T^*X = H^*(\mathbb{E}T \times^T X),$$

where $Y \times^T Z$ denotes the quotient of $Y \times Z$ by the relation $(y \cdot t, z) \sim (y, t \cdot z)$. The map $\mathbb{E}T \times^T X \to \mathbb{B}T$ makes H_T^*X an algebra over

$$\Lambda = H_T^*(\mathrm{pt}).$$

When X is a nonsingular variety, a T-invariant subvariety $Y \subseteq X$ of codimension c defines a class $[Y]^T$ in $H_T^{2c}X$.

The ring Λ is naturally identified with the symmetric algebra of the character group of T. Fixing an isomorphism $T \cong (\mathbb{C}^*)^n$ gives a standard basis for the character group, $t_1, \ldots, t_n$, defined by $t_i(z) = z_i$ for $z = (zz_1, \ldots, z_n) \in T$. This also identifies Λ with the polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$.

The diagonal torus $T \cong (\mathbb{C}^*)^n \subseteq GL_n$ acts on the flag variety via its action on $V = \mathbb{C}^n$, and when discussing equivariant cohomology of flag varieties and related spaces, we always understand this action, unless otherwise indicated.

The cohomology ring $H^*Fl(\boldsymbol{n})$ has a basis of Schubert classes, indexed by permutations in the set

$$S^n \coloneqq \{w \in S_n | w(i) < w(i+1) \text{if } i \notin n\}.$$

If $S_{\boldsymbol{n}}$ denotes the parabolic (Young) subgroup of S_n generated by adjacent transpositions $s_i = (i, i+1)$ for $i \notin \boldsymbol{n}$, then $S^{\boldsymbol{n}}$ is a set of coset representatives for $S_n/S_{\mathbf{n}}$. (For the complete flag variety $Fl(V)$, so $\boldsymbol{n} = \{1,2,\ldots,n-1,n\}$, we have $S^{\mathbf{n}} = S_n$.)

For any permutation $w \in S_n$, define

$$r_w(p,q) = \# \{i \leq p | w(i) \leq q\}.$$

This is the rank of the upper-left $p \times q$ submatrix of the permutation matrix corresponding to w (which has 1's in positions $(i, w(i))$ and 0's elsewhere).

The Bruhat order on S_n is a partial order which may be defined by $v \leq w$ iff $r_v(p,q) \geq r_w(p,q)$ for all $1 \leq p, q \leq n$. The length of w is the number

$$\ell(w) = \# \{i < j | w(i) > w(j)\}.$$

The Bruhat order is ranked by length: $v \leq w$ implies $\ell(v) \leq \ell(w)$. There is a unique permutation of greatest length, denoted $w_\circ$; it is given by $w_\circ(i) = n + 1 - i$.

The subset $S^{\mathbf{n}} \subseteq S_n$ is also characterized as the set of minimal-length coset representatives for $S_n/S_{\boldsymbol{n}}$. Given any permutation $w \in S_n$, one gets an element of $S^{\mathbf{n}}$ by taking the representative of $\omega S_{\mathbf{n}}$ of smallest length; concretely, this means sorting the entries of each block $[w(n_i+1), w(n_i+2), \ldots, w(n_{i+1})]$ into increasing order.

The Bruhat order induces partial orders on the subset $S^{\boldsymbol{n}}$ and the subgroup $S_{\boldsymbol{n}}$. The unique permutation of greatest length in $S^{\boldsymbol{n}}$ is given explicitly by

$$w^\circ = [n-n_1+1, \ldots, n, n-n_2+1, \ldots, .n-n_1, \ldots, 1,2, \ldots, n-n_m].$$

Its length is $\ell(w^\circ) = \dim Fl(\boldsymbol{n}) = \sum_{i=1}^{m} n_i(n_{i+1}-n_i)$.

There is an involution on coset representatives in $S^{\boldsymbol{n}}$, using the longest element $w_\circ^{\boldsymbol{n}}$ of the subgroup $S_{\boldsymbol{n}}$. (The permutation $w_\circ^{\boldsymbol{n}}$ can be defined by $w_\circ = w^\circ \cdot w_\circ^{\boldsymbol{n}}$.) Given any $w \in S^{\boldsymbol{n}}$, define

$$w^\vee = w_\circ \cdot w \cdot w_\circ^{n}.$$

This is an element of $S^{\boldsymbol{n}}$, with $\ell(w^\vee) = \ell(w^\circ) - \ell(w)$.

Fix a basis $\{e_1, \ldots, e_n\}$ for $V = \mathbb{C}^n$. The standard flag $E_\bullet$ is defined by $E_i = \operatorname{span}\{e_1, \ldots, e_i\}$ and the opposite flag $\tilde{E}_\bullet$ is defined by $\tilde{E}_i = \operatorname{span}\{e_n, e_{n-1}, \ldots, e_{n-i+1}\}$. The Schubert varieties in $Fl(\boldsymbol{n})$ can be described in several ways. Fixing the standard flag $E_\bullet$, we define $\Omega_w = \Omega_w(E_\bullet) \subseteq Fl(\boldsymbol{n})$ by

$$\Omega_w = \{W_\bullet \in Fl(n) | rk(E_q \to W_p) \leq r_w(n_p, q) \text{ for all } 1 \leq q \leq n, n_p \in n\}.$$

This is the closure of the Schubert cell

$$\Omega_w^\circ = \{W_\bullet \in Fl(n) | rk(E_q \to W_p) = r_w(n_p, q) \text{ for all } 1 \leq q \leq n, n_p \in n\},$$

which is isomorphic to $\mathbb{A}^{\dim Fl(\boldsymbol{n}) - \ell(w)}$. Replacing $E_\bullet$ with the opposite flag $\tilde{E}_\bullet$, we obtain the opposite Schubert varieties $\tilde{\Omega}_w$ and the opposite Schubert cells $\tilde{\Omega}_w^\circ$. Here, the Bruhat order on $S^{\boldsymbol{n}}$ is identified with the order induced by inclusions of Schubert varieties: $v \leq w$ iff $\Omega_v \supseteq \Omega_w$ iff $\tilde{\Omega}_v \supseteq \tilde{\Omega}_w$.

Regarding $\mathrm{E}_\bullet$ as a flag of trivial vector bundles on Fl(n), and using the universal quotient bundles F_i, the Schubert variety may be defined equivalently as

the degeneracy locus of points $x \in Fl(\boldsymbol{n})$ where $rk_x(Eq \to Fp) \leq r_w(n_p, q)$, and similarly for the opposite Schubert varieties.

The Schubert cells are T-invariant and give an affine cell decomposition of the partial flag variety, so the classes of Schubert varieties and opposite Schubert varieties form linear bases for the equivariant cohomology ring of $Fl(\boldsymbol{n})$. We will generally write $\sigma_w = [\Omega_w]^T$ and $\tilde{\sigma}_w = [\tilde{\Omega}_w]^T$, so we have

$$H_T^* Fl(n) = \bigoplus_{w \in S^n} \Lambda \cdot \sigma_w = \bigoplus_{w \in S^n} \Lambda \cdot \tilde{\sigma}_w.$$

Since Ω_w and $\tilde{\Omega}_w$ have codimension $\ell(w)$, their classes lie in $H_T^{2\ell(w)} Fl(\boldsymbol{n})$.

The Schubert cells may also be described as orbits, and we will need this point of view. Given $w \in S^{\boldsymbol{n}}$, let $p(w) \in Fl(\boldsymbol{n})$ be the flag $V \twoheadrightarrow W_\bullet$ with $W_i = \mathrm{span}\{e_{w(1)}, e_{w(2)}, \ldots, e_{w(n_i)}\}$. The group GL_n acts transitively on $Fl(\boldsymbol{n})$ via its action on $\mathbb{C}^n$, and the parabolic subgroup P of (block) upper-triangular matrices is the stabilizer of the partial flag $p(w^\circ) \in Fl(\boldsymbol{n})$. This gives rise to an isomorphism $Fl(\boldsymbol{n}) \cong GL_n/P$, with a decomposition into Schubert cells indexed by permutations $w \in S^{\boldsymbol{n}}$. We have

$$\Omega_w^\circ = B \cdot p(w) \quad \text{and} \quad \tilde{\Omega}_w^\circ = \tilde{B} \cdot p(w^\vee).$$

where B and $\tilde{B}$ are the groups of upper- and lower-triangular matrices, respectively. The cells Ω_w° and $\tilde{\Omega}_{w^\vee}^\circ$ intersect transversally in the point $p(w)$, and it follows from this, together with dimension considerations, that the classes $[\tilde{\Omega}_{w^\vee}^\circ]$ form a Poincaré dual basis: writing $\pi: Fl(\boldsymbol{n}) \to pt$,

$$\pi_*^T(\sigma_w \cdot \tilde{\sigma}_{v^\vee}) = \delta_{w,v} \tag{5}$$

in Λ.

The Schubert polynomials of Lascoux and Schützenberger are defined inductively, starting from $\mathfrak{S}_{w_\circ}(x) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ and moving down Bruhat order using divided difference operators. For our purposes, we only remark that for any $w \in S_n$, the polynomial $\mathfrak{S}_w(x)$ has a unique expansion in terms of elementary symmetric polynomials as in (1).

Universal double Schubert polynomials were introduced as the solution to a certain degeneracy locus problem. They specialize to double Schubert polynomials as well as to quantum Schubert polynomials for complete and partial flag varieties. In the most general form considered, the universal polynomials involve two sets of $\binom{n+1}{2}$ independent variables, but we will always specialize the second set to the n variables $t_1, \ldots, t_n$.

Let $c_k(l)$ be independent variables of degree k, for $1 \leq k \leq l \leq n$; by convention, set $c_0(l) = 1$ and $c_k(l) = 0$ when $k < 0$ or $k > l$. For any $w \in S_n$, the universal single Schubert polynomial is defined by substituting the $c_k(l)$ for $e_k(l)$ in the expression (1):

$$\mathfrak{S}_w(c) = \sum a_{k_1 \ldots k_{n-1}} c_{k_1}(1) \ldots c_{k_{n-1}}(n-1). \tag{6}$$

Fixing a tuple $\boldsymbol{n}$, for any base ring R, let $R[c]_{\boldsymbol{n}}$ be the polynomial ring generated by $c_k(n_p)$ for $1 \leq p \leq m$. Define the polynomial $\mathfrak{S}_w^{\boldsymbol{n}}(c) \in \mathbb{Z}[c]_{\boldsymbol{n}}$ to be the image of $\mathfrak{S}_w(c)$ when $c_k(l)$ is sent to $c_k(n_p)$ whenever $n_p \leq l < n_{p+1}$.

The double version, in $\Lambda[c]_{\boldsymbol{n}}$, is defined by a Cauchy-type formula, using the classical Schubert polynomials in the t-variables:

$$\mathfrak{S}_w^n(c,t) = \sum_{u,v} (-1)^{\ell(v)} \mathfrak{S}_u^n(c) \mathfrak{S}_v(t), \tag{7}$$

where the sum is over $u, v \in S_n$ such that $v^{-1}u = w$ and $\ell(u) + \ell(v) = \ell(w)$. Note that when $w \in S^{\boldsymbol{n}}$, the only u that appear in the sum are also in $S^{\boldsymbol{n}}$.

The polynomials $\mathfrak{S}_w^{\boldsymbol{n}}(c,t)$ are well-suited to degeneracy locus formulas; in applications, the variables $c_k(l)$ will be specialized to certain Chern classes. For the specialization to quantum cohomology, it is more convenient to use a different set of variables. Fix $\boldsymbol{n}$, and let $g_a[n_p - a]$ be independent variables of degree $n_p - a + 1$, for $1 \leq p \leq m+1$ and $1 \leq a \leq n_p$. As before, we write $R[g]_{\boldsymbol{n}}$ for the polynomial ring generated by these variables.

The g variables are related to the c variables inductively by

$$c_k(n_p) = c_k(n_{p-1}) + \sum_{r,j} g_j[n_p - j] c_{k-n_p+j-1}(n_{r-1}) \tag{8}$$

where the sum is over $1 \le r \le p$ and $n_{r-1} < j \le n_r$. This expression can be inverted, so that we can identify the polynomial rings $R[c]_{\boldsymbol{n}} = R[g]_{\boldsymbol{n}}$; in particular, we have $\mathbb{Z}[c]_{\boldsymbol{n}} = \mathbb{Z}[g]_{\boldsymbol{n}}$ and $\Lambda[c]_{\boldsymbol{n}} = \Lambda[g]_{\boldsymbol{n}}$.

An alternative way to see these identifications of polynomial rings also provides the link with the quantum specialization. Let $M^g = (m_{ab})$ be the $n \times n$ matrix with entries

$$m_{ab} = \begin{cases} g_a[b-a] & \text{if } b = n_p \text{ for some } p \ge 1, \text{and } a \le b, \\ -1 & \text{if } (a,b) = (b+1, b), \\ 0 & \text{otherwise.} \end{cases}$$

Writing M_p^g for the upper-left $n_p \times n_p$ submatrix of M^g, equation. (8) amounts to setting $c_k(n_p)$ equal to the coefficient of $T^{n_p - k}$ in the determinant of $M_p^g + T \cdot Id$.

Now to recover the equivariant quantum Schubert polynomial $\mathfrak{S}_w^q(\sigma, t)$, as defined in (4), from the universal double Schubert polynomial $\mathfrak{S}_w^{\boldsymbol{n}}(c, t)$, use the specialization that sends M^g to the matrix $M^{\sigma,q}$ of (3). In formulas, this is

$$g_{n_p - i}[i] \mapsto \sigma_{i+1}^p \quad \text{for} \quad 0 \le i < n_p - n_{p-1} \quad \text{and} \quad 1 \le p \le m+1,$$

$$g_{n_{p-1}+1}[n_{p+1} - n_{p-1} - 1] \mapsto (-1)^{n_{p+1} - n_p + 1} q_p \quad \text{for} \quad 1 \le i \le m, \quad \text{and}$$

$$g_i[j] \mapsto 0 \quad \text{for all other } i, j. \tag{9}$$

Because M^g specializes correctly, it is clear that this definition of $\mathfrak{S}_w^q(\sigma, t)$ is equivalent to the formulation given in the introduction.

Certain of the polynomials $\mathfrak{S}_w(c, t)$ will be useful in studying partial flag varieties. For $1 \le k \le n_p$, the cyclic permutation $\alpha_{k,p} := s_{n_p - k + 1} \cdots s_{n_p} \in S^{\boldsymbol{n}}$, has universal (single) Schubert polynomial $\mathfrak{S}_{\alpha_{k,p}}(c) = \mathfrak{S}_{\alpha_{k,p}}^{\boldsymbol{n}}(c) = c_k(n_p)$, and therefore $\Lambda[c]_{\boldsymbol{n}} = \Lambda[\mathfrak{S}_{\alpha_{k,p}}(c) | 1 \le p \le m+1, 1 \le k \le n_p]$. By (7), the corresponding universal double Schubert polynomials can be written

$$\mathfrak{S}^{n}_{\alpha_{k,p}}(c,t) = \sum_{0\le\kappa'\le\kappa} (-1)^{\kappa-\kappa'} \mathfrak{S}^{n}_{\alpha_{\kappa',p}}(c)\mathfrak{S}_{s_{n_{p-k+1}}\dots s_{n_p-\kappa'}}(t)$$

$$= c_\kappa(n_p) + \sum_{0\le\kappa'<\kappa} (-1)^{\kappa-\kappa'} c_{\kappa'}(n_p)\mathfrak{S}_{s_{n,p}-\kappa+1}\dots s_{n_p-\kappa'}(t).$$

Therefore we can recursively write each $c_k(n_p)$ in terms of polynomials $\mathfrak{S}^{\boldsymbol{n}}_{\alpha_{k,p}}(c,t)$ and variables t, and we have $\Lambda[g]_{\boldsymbol{n}} = \Lambda[c]_{\boldsymbol{n}} = \Lambda[\mathfrak{S}_{\alpha_{k,p}}(c,t)]$. In particular, by degree considerations and the fact that $\ell(\alpha_{k,p}) = k$, we obtain the following result.

Lemma (3.1.3) [3]:

Let $P(g,t)$ be a polynomial of total g-degree at most s. Then it can be written as a polynomial in $\mathfrak{S}^{\boldsymbol{n}}_{w}(c,t)$ and $t_1,\dots,t_n$, where $w \in S^{\boldsymbol{n}}$ with $\ell(w) \le s$.

As in the introduction, in the case of the complete flag variety, we write $x_i = \sigma_1^i = g_i[0]$, and the other g variables are $g_i[j-i]$ for $1 \le i < j \le n$. The universal double Schubert polynomial $\mathfrak{S}_w(c,t)$ specializes to the double Schubert polynomial $\mathfrak{S}_w(x,t)$ by sending $c_k(l)$ to $e_k(x_1,\dots,x_l)$ with the single versions obtained by setting the t variables to 0. The polynomial $\mathfrak{S}_w(c,t)$ also specializes to the quantum Schubert polynomials $\mathfrak{S}^q_w(x)$ by setting $g_i[1] = q_i$, and all other $g_i[j] = 0$.

Example (3.1.4) [3]:

For $\boldsymbol{n} = (1,2,3)$, we have $\sigma_1^i = x_i$. The following calculation shows how the quantum specialization works for $w = 231$:

$$\begin{aligned}\mathfrak{S}_{231}(c,t) &= c_2(2) - c_1(2)t_1 + t_1^2\\ &= (x_1x_2 + g_1[1]) - (x_1+x_2)t_1 + t_1^2;\\ \mathfrak{S}^q_{231}(x,t) &= x_1x_2 + q_1 - (x_1+x_2)t_1 + t_1^2\end{aligned}$$

For $\boldsymbol{n} = (2,3)$, the variable $x_1 = g_1[0]$ is set to 0 (and $c_1(1) \mapsto c_1(0) = 0$, as well), so the corresponding polynomials are given as follows:

$$\begin{aligned}\mathfrak{S}_{231}^{n}(c,t) &= c_2(2) - c_1(2)t_1 + t_1^2 \\ &= g_1[1] - g_2[0]t_1 + t_1^2; \\ \mathfrak{S}_{231}^{n,q}(\sigma,t) &= \sigma_2^1 - \sigma_1^1 t_1 + t_1^2\end{aligned}$$

The equivariant quantum cohomology ring $QH_T^*Fl(\boldsymbol{n})$ is a commutative and associative graded algebra over $\Lambda[q] = \Lambda[q_1,\ldots,q_m]$, where q_i is a parameter of degree $n_{i+1} - n_{i-1}$. As a module, $QH_T^*Fl(\boldsymbol{n})$ is simply $\Lambda[q] \otimes_\Lambda H_T^*Fl(\boldsymbol{n})$, so it has a $\Lambda[q]$-basis of Schubert classes σ_w:

$$QH_T^*Fl(n) = \bigoplus_{w \in S^n} \Lambda[q] \cdot \sigma_w.$$

The equivariant quantum product is a deformation of the usual equivariant product. For permutations $u, v \in S^{\boldsymbol{n}}$, define a product by

$$\sigma_u \circ \sigma_v = \sum_{w,\mathrm{d}} \mathrm{q}^d c_{u,v}^{w,\mathrm{d}}(t)\sigma_w,$$

where $\boldsymbol{d}$ ranges over m-tuples of nonnegative integers, and the three-point equivariant Gromov–Witten invariant $c_{u,v}^{w,\boldsymbol{d}}(t)$ is defined as follows.

Let $\overline{M}_r(\boldsymbol{d}) = \overline{M}_{0,r}(Fl(\boldsymbol{n}),\boldsymbol{d})$ be the Kontsevich moduli space of r-pointed genus-zero stable maps to $Fl(\boldsymbol{n})$ of degree $\boldsymbol{d}$, parametrizing data $(f, C, \{p_1,\ldots,p_r\})$, where C is a genus-zero curve with marked points $p_1,\ldots,p_r, f: C \to Fl(\boldsymbol{n})$ is a map of degree $\boldsymbol{d}$, and a certain stability condition is imposed. The space of stable maps has dimension

$$\dim \overline{M}_r(\mathrm{d}) = \dim Fl(n) + \sum_{i=1}^{m} d_i(n_{i+1} - n_{i-1}) + r - 3,$$

and it comes with natural evaluation morphisms

$$\mathrm{ev}_i: \overline{M}_r(\mathrm{d}) \to Fl(n),$$

for $i = 1,\ldots,r$, defined by sending $(f, C, \{p_j\})$ to $f(p_i)$.

The action of T on $Fl(\boldsymbol{n})$ induces an action on $\overline{M}_r(\boldsymbol{d})$. Let $\pi_*^T: H_T^*\overline{M}_3(\boldsymbol{d}) \to H_T^*(\mathrm{pt})$ be the equivariant pushforward. Now one defines

$$c_{u,v}^{w,\mathrm{d}}(t) = \pi_*^T(ev_1^*\sigma_u \cdot ev_2^*\sigma_v \cdot ev_3^*\tilde{\sigma}_{w^\vee}). \tag{10}$$

When t is set to zero, we recover the ordinary (small) quantum cohomology ring and product. The ordinary three-point Gromov–Witten invariant may be interpreted informally as the number of maps $f: \mathbb{P}^1 \to Fl(\boldsymbol{n})$ of degree $\boldsymbol{d}$, i.e. $f_*[\mathbb{P}^1] = d_1\sigma_{s_{n_1}^\vee} + \cdots + d_m\sigma_{s_{n_m}^\vee}$, such that $f(0), f(1)$, and $f(\infty)$ lie in general translates of Ω_u, Ω_v, and $\Omega_{w^\vee}$, respectively. Note that in contrast to the non-equivariant case, it is important to use the opposite Schubert class $\tilde{\sigma}_{w^\vee}$ as the third insertion, rather than $\sigma_{w^\vee}$.

This defines an associative product, we will call the polynomials $c_{u,v}^{w,\boldsymbol{d}}(t)$ equivariant quantum Littlewood Richardson (EQLR) coefficients, and use this term also for the coefficients defined by associativity:

$$\sigma_{v_1} \circ \sigma_{v_2} \circ \ldots \circ \sigma_{v_r} = \sum_{w,\mathrm{d}} \mathrm{q}^{\mathrm{d}} c_{v_1,\ldots,v_r}^{w,d}(t)\, \sigma_w.$$

The proof of associativity shows that the EQLR coefficients may be described equivalently as

$$c_{v_1,\ldots,v_r}^{w,d}(t) = \pi_*^T\left(\mathrm{ev}_1^*\sigma_{v_1} \ldots \mathrm{ev}_{\mathcal{T}}^*\sigma_{v_r} \cdot \mathrm{ev}_{r+1}^*\tilde{\sigma}_{w^\vee} \cdot f^*[\mathrm{pt}]\right), \tag{11}$$

where $f: \overline{M}_{r+1}(\mathbf{d}) \to \overline{M}_{0,r+1}$ is the forgetful map and $[\mathrm{pt}] \in H_T^*\overline{M}_{0,r+1}$ is the class of a point, with T acting trivially on the space of stable curves.

Section (3.2): Degeneracy Locus Formula and Mixing Group

Let T act on an algebraic Cohen–Macaulay scheme X. For a sequence of integers $\boldsymbol{n} = (0 < n_1 < \cdots < n_m < n_{m+1} = n)$, consider maps of equivariant vector bundles

$$E_1 \to \cdots \to E_n = F_{m+1} \to F_m \to \cdots \to F_1 \tag{12}$$

where F_i is of rank n_i and $E_i \cong \mathbb{C}^i \otimes \mathcal{O}_X$ is trivial, but has the nontrivial equivariant structure coming from the standard action of T on $\mathbb{C}^n$. For $w \in S^{\boldsymbol{n}}$, let D_w be the degeneracy locus

$$D_w = \{x \in X | \mathrm{rank}_x(E_q \to F_p) \leq r_w(n_p, q) \text{ for all } p, q\}.$$

The universal double Schubert polynomials give the answer to a degeneracy locus problem. In fact, the formula holds equivariantly without change.

For $w \in S^{\boldsymbol{n}}$, let $\mathfrak{S}_w^{\boldsymbol{n}}(c^T(F_\bullet), t)$ denote the result of specializing $c_k(n_p)$ to $c_k^T(F_p)$ in $\mathfrak{S}_w^{\boldsymbol{n}}(c, t)$. In this context, Fulton's degeneracy locus formula is as follows:

Theorem (3.2.1) [3]:

Given maps as in (3.12), for $w \in S^{\boldsymbol{n}}$, we have $[D_w]^T = \mathfrak{S}_w^{\boldsymbol{n}}(c^T(F_\bullet), t)$ as classes in $H_T^*(X)$, whenever $\mathrm{codim}_X(D_w) = \ell(w)$.

Fix $\boldsymbol{n}$, and recall that we have an identification of polynomial rings $\Lambda[c]_{\boldsymbol{n}} = \Lambda[g]_{\boldsymbol{n}}$. Define a map of Λ-algebras

$$\beta: \Lambda[c]_n \to H_T^* X$$

by $\beta(c_k(n_p)) = c_k^T(F_p)$, and use this to define classes

$$Q_i[n_p - i] = \beta(g_i[n_p - i]) \text{ for } 1 \leq p \leq m+1 \text{ and } 1 \leq i \leq n_p \tag{13}$$

in $H_T^* X$.

In a sense, the Q classes detect the failure of surjectivity of maps among the F_p's. We have the following.

Lemma (3.2.2) [3]:

If $F_b \to F_a$ is surjective for some $a < b$, then $Q_i[j] = 0$ for all $i < n_a + 1 \leq i + j \leq n_b$.

Let $M_{r+1}(\boldsymbol{d})$ denote the space of maps $f: \mathbb{P}^1 \to Fl(\boldsymbol{n})$ of degree $\boldsymbol{d}$ with $r+1$ marked points. Recall that the EQLR coefficients are defined using the Kontsevich compactification of $M_{r+1}(\boldsymbol{d})$. Let $V = \mathbb{C}^n$. Using the fact that a degree-$\boldsymbol{d}$ map $\mathbb{P}^1 \to Fl(\boldsymbol{n})$ corresponds to successive equotient bundles of $V^*_{\mathbb{P}^1} = V^* \otimes \mathcal{O}_{\mathbb{P}^1}$ of rank n_p and degree d_p, there is another compactification of $M_{r+1}(\boldsymbol{d})$. The hyperquot scheme $\mathcal{Q}_{\boldsymbol{d}}$ parametrizes flat families of successive quotient sheaves of $V^*_{\mathbb{P}^1}$ of rank $n-n_p$ and degree d_p, generalizing Grothendieck's Quot scheme. It is a smooth, projective variety of dimension $\dim Fl(\boldsymbol{n}) + \sum d_p\,(n_{p+1} - n_{p-1})$, and comes with a universal sequence of quotient sheaves on $\mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}$

$$V^*_{\mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}} = \mathscr{B}_{m+1} \twoheadrightarrow \cdots \twoheadrightarrow \mathscr{B}_2 \twoheadrightarrow \mathscr{B}_1,$$

where $\mathscr{B}_p$ has rank $n-n_p$ and relative degree d_p. The space of maps $M(\boldsymbol{d}) = M_3(\boldsymbol{d})$ embeds in $\mathcal{Q}_{\boldsymbol{d}}$ as the largest (open) subscheme such that each restriction $\mathscr{B}_p|_{\mathbb{P}^1 \times M(\boldsymbol{d})}$ is locally free.

To avoid torsion, we prefer to work with the locally free sheaves $\mathscr{A}_p := \ker\left(V^*_{\mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}} \to \mathscr{B}_p\right)$ which are locally free of rank n_p and degree $-dp$. There is a sequence

$$\mathscr{A}_1 \to \mathscr{A}_2 \to \cdots \to \mathscr{A}_{m+1} = V^*_{\mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}}$$

The tradeoff here is that although the map $\mathscr{A}_p \to \mathscr{A}_{p+1}$ is an inclusion of sheaves, it is not necessarily an inclusion of vector bundles (i.e., the cokernel is not locally free in general).

Dualizing again, and regarding the standard flag $E_\bullet$ as a flag of trivial vector bundles on $\mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}$, we have a sequence

$$E_1 \hookrightarrow \cdots \hookrightarrow E_{n-1} \hookrightarrow E_n = V_{\mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}} = \mathscr{A}^*_{m+1} \to \mathscr{A}^*_m \to \cdots \to \mathscr{A}^*_1 \qquad (14)$$

Note that the maps $\mathscr{A}^*_{p+1} \to \mathscr{A}^*_p$ are not necessarily surjective.

Define $D_w \subseteq \mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}$ as the degeneracy locus associated to this sequence:

$$D_w = \{x|\, rk_x(E_q \to \mathscr{A}_p^*) \leq r_w(p,q) \quad \text{for all } 1 \leq q \leq n, n_p \in n \tag{15}$$

Fix a point $z \in \mathbb{P}^1$, and let $D_w(z) = D_w \cap (\{z\} \times \mathcal{Q}_{\boldsymbol{d}})$ be the corresponding closed subscheme of $\mathcal{Q}_{\boldsymbol{d}}$. Define $\widetilde{D}_w$ and $\widetilde{D}_w(z)$ similarly, using the opposite flag $\tilde{E}_{\bullet}$ in place of $E_{\bullet}$.

For a point $z \in \mathbb{P}^1$, there is an evaluation map $\mathrm{ev}_z: M(\boldsymbol{d}) \to Fl(\boldsymbol{n})$, defined by the sequence of quotient sheaves $\mathscr{A}_p^*$ restricted to $M(\boldsymbol{d}) = \{z\} \times M(\boldsymbol{d})$. We write $\Omega_w(z) = \mathrm{ev}_z^{-1}(\Omega_w)$ and $\tilde{\Omega}_w(z) = \mathrm{ev}_z^{-1}(\tilde{\Omega}_w)$. From the definitions, one sees

$$\Omega_w(z) = D_w(z) \cap M(d) \text{ and } \tilde{\Omega}_w(z) = \widetilde{D}_w(z) \cap M(d) \tag{16}$$

The group $GL_n = GL(V)$ acts on $\mathcal{Q}_{\boldsymbol{d}}$ via its action on V: a quotient sheaf of $V_{\mathbb{P}^1}$ is sent to the quotient obtained by precomposing with an automorphism of V. The above constructions are all equivariant for appropriate subgroups; in particular, we set

$$\mu_w = [D_w(z)]^T \text{ and } \tilde{\mu}_w = \left[\widetilde{D}_w(z)\right]^T$$

in $H_T^*(\mathcal{Q}_{\boldsymbol{d}})$. Since T acts trivially on the $\mathbb{P}^1$ factor of $\mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}$, these equivariant classes are independent of the choice of z.

Write A_p for the restriction of $\mathscr{A}_p$ to $\{z\} \times \mathcal{Q}_{\boldsymbol{d}}$. (The choice of point $z \in \mathbb{P}^1$ will usually be irrelevant.) The degeneracy locus formula of Theorem (3.2.1) yields the following:

Proposition (3.2.3) [3]:

Setting $c_k(n_p) = c_k^T(A_p^*)$, we have $\mu_w = \mathfrak{S}_w^{\boldsymbol{n}}(c,t)$ in $H_T^*\mathcal{Q}_{\boldsymbol{d}}$.

Proposition (3.2.4) [3]:

The equivariant quantum product can be computed on (hyper) quot schemes. That is, given permutations $v_1, \ldots, v_r$ in $S^{\boldsymbol{n}}$, we have an equality

$$c_{v_1,\ldots,v_r}^{w,d}(t) = \tilde{\pi}_*^T\left(\mu_{v_1} \ldots \mu_{v_r} \cdot \tilde{\mu}_{w^\vee}\right),$$

where the left-hand side is the EQLR coefficient and $\tilde{\pi}_*^T: H_T^*(\mathcal{Q}_{\boldsymbol{d}}) \to H_T^*(\mathrm{pt})$ is the equivariant push forward to a point.

We also write

$$\left(\mu_{v_1} \dots \mu_{v_r} \cdot \tilde{\mu}_{w^\vee}\right)_d^T = \tilde{\pi}_*^T\left(\mu_{v_1} \dots \mu_{v_r} \cdot \tilde{\mu}_{w^\vee}\right)$$

for the EQLR coefficient $c_{v_1,\dots,v_r}^{w,\boldsymbol{d}}(t)$. With this notation, we have

$$\sigma_{v_1} \circ \sigma_{v_2} \circ \dots \circ \sigma_{v_r} = \sum_{w,d} q^d \left(\mu_{v_1} \dots \mu_{v_r} \cdot \tilde{\mu}_{w^\vee}\right)_d^T \sigma_w.$$

By linearity of equivariant quantum cohomology of flag varieties and of equivariant cohomology of quot schemes, for any polynomial $F(\sigma_v, t)$ in t and variables indexed by $v \in S^{\boldsymbol{n}}$, we obtain

Corollary (3.2.5) [3]:

$$F(\sigma_v, t) = \sum_{w,d} q^d (F(\mu_v, t) \cdot \tilde{\mu}_{w^\vee})_d^T \sigma_w \quad \text{in} \quad QII_T^*\left(Fl(n)\right).$$

Unlike the Kontsevich compactification, there is no globally defined evaluation map from $\mathcal{Q}_{\boldsymbol{d}}$ to $(\boldsymbol{n})$. However, the boundary $\mathcal{Q}_{\boldsymbol{d}} \backslash M(\mathbf{d})$ can be broken into pieces which do map to (different) partial flag varieties. We summarize the relevant facts here.

Fix $\boldsymbol{n}$. Given $\mathbf{d}$, let $\boldsymbol{e} = (e_1, \dots, e_m)$ with $e_p \geq 0$ be such that

$$e_p \leq \min\left(n_p, d_p\right) \quad \text{for} \quad 1 \leq p \leq m \quad \text{and}$$

$$e_p - e_{p-1} \leq n_p - n_{p-1} \quad \text{for} \quad 2 \leq p \leq m \tag{17}$$

In addition to $\mathcal{Q}_{\boldsymbol{d}}$ and $Fl(\boldsymbol{n})$, we will also consider quot schemes $\mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}}$ (parametrizing quotients whose ranks are still $\boldsymbol{n}$, but whose degrees are $\boldsymbol{d} - \boldsymbol{e}$) and partial flag varieties $Fl(\boldsymbol{n}')$ (where $\boldsymbol{n}' = \{n_1 - e_1 \leq n_2 - e_2 \leq \dots \leq n_m - e_m\}$).

Theorem (3.2.6) [3]:

For $\boldsymbol{e} = (e_1, \ldots, e_m)$ satisfying (17) with some $e_i > 0$, there is a smooth irreducible variety $\mathscr{U}_e$ with the following properties.

(i). There is a morphism $h_{\boldsymbol{e}}$: $\mathscr{U}_e \to Q_{\boldsymbol{d}}$, which is birational onto its image. Every point of $Q_{\boldsymbol{d}} \backslash M(\boldsymbol{d})$ lies in the image of $h_{\boldsymbol{e}}$, for some $\boldsymbol{e}$.

(ii). There is a smooth morphism $\rho: \mathcal{U}_{\boldsymbol{e}} \to \mathbb{P}^1 \times Q_{\boldsymbol{d}-\boldsymbol{e}}$, whose image contains $\mathbb{P}^1 \times M(\boldsymbol{d} - \boldsymbol{e})$.

(iii). Fix a point $z \in \mathbb{P}^1$, and write $\mathscr{U}_e(z) = \rho^{-1}(\{z\} \times Q_{\boldsymbol{d}-\boldsymbol{e}})$. There is a natural morphism

$$\psi_e(z)\colon \mathscr{U}_e(z) \to Fl(n'),$$

and for each $w \in S^{\boldsymbol{n}}$, there is a $w' \in S^{\boldsymbol{n}'}$ such that

$$h_e^{-1}\big(D_w(z)\big) = \rho^{-1}\big(\mathbb{P}^1 \times D_w^e(z)\big) \cup \psi_e(z)^{-1}(\Omega_{w'}),$$

where the superscript in $D_w^{\boldsymbol{e}}$ indicates the degeneracy locus inside $Q_{\boldsymbol{d}-\boldsymbol{e}}$. The same holds with $D_w(z)$ and $\Omega_{w'}$ replaced by $\widetilde{D}_w(z)$ and $\widetilde{\Omega}_{w'}$, respectively.

Moreover, the morphisms $h_{\boldsymbol{e}}, \rho$, and $\psi_{\boldsymbol{e}}(z)$ are equivariant for natural actions of $GL_n = GL(V)$.

In the definition of equivariant cohomology, the spaces $\mathbb{E}T$ are infinite-dimensional, but one can find finite-dimensional algebraic varieties which serve as "approximations". To describe these, we will need to pay attention to the isomorphism $T \cong (\mathbb{C}^*)^n$. Let $M = \mathrm{Hom}(T, \mathbb{C}^*)$ be the character group of T, so $M \cong \mathbb{Z}^n$, and there is the standard basis $t_1, t_2, \ldots, t_n$ used above. We will also use the positive basis

$$\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$$

for M, where $\alpha_i = t_i - t_{i+1}$ for $i < n$, and $\alpha_n = t_n$. The reason for this choice will become evident below.

Take $\mathbb{E}$ to be $(\mathbb{C}^N \backslash \{0\}^n)$, for $N \gg 0$. Let T act on the ith factor via the character α_i; that is, an element $z = (z_1, \ldots, z_n) \in T = \mathbb{C}^*$ scales the ith factor by $\alpha_i(z)$. (Explicitly, $\alpha_i(z)$ is $z_i z_{i+1}^{-1}$ for $i < n$ and z_n for $i = n$.) Set

$$\mathbb{B} = \mathbb{E}/T = (\mathbb{P}^{N-1})^n.$$

These spaces approximate $\mathbb{E}T \to \mathbb{B}T$ in the sense that

$$H^k(\mathbb{E} \times^T X) = H^k(\mathbb{E}T \times^T X) = H_T^k X$$

for any given k, using a sufficiently large N. The key point for our purposes is that one can carry out any given computation in $H_T^* X$ using $H^*(\mathbb{E} \times^T X)$.

As a matter of notation, given a T-space X, we will denote the corresponding approximation space by a bold letter $\boldsymbol{X} = \mathbb{E} \times^T X$, always understanding some fixed $N \gg 0$.

We will also need to consider certain linear subspaces of $\mathbb{B}$. Specifically, for each integer j with $0 \leq j \leq N-1$, fix transverse linear subspaces $\mathbb{P}^{N-1-j}$ and $\widetilde{\mathbb{P}}^j$ inside $\mathbb{P}^{N-1}$, and for a multi-index of such integers $J = (j_1, \ldots, j_n)$, set

$$\mathbb{B}^J = \mathbb{P}^{N-1-j_1} \times \ldots \times \mathbb{P}^{N-1-j_n} \quad \text{and} \quad \mathbb{B}_J = \widetilde{\mathbb{P}}^{j_1} \times \ldots \times \widetilde{\mathbb{P}}^{j_n}.$$

Thus $\dim \mathbb{B}_J = \operatorname{codim}(\mathbb{B}^J, \mathbb{B}) = |J| = j_1 + \cdots + j_n$. These subspaces carry the Graham-positive classes in $H_T^*(\mathrm{pt})$; hence their significance:

$$[\mathbb{B}^J] = (-\alpha_1)^{j_1} \ldots (-\alpha_n)^{j_n} \quad \text{in} \quad H^* \mathbb{B} = H_T^*(pt) \tag{18}$$

Finally, let $\boldsymbol{X}_J$ and $\boldsymbol{X}^J$ denote the preimages of $\mathbb{B}_J$ and $\mathbb{B}^J$, respectively, under the projection $\boldsymbol{X} \to \mathbb{B}$.

Let $\pi^{\mathbb{B}}$ be the map $\mathbb{B} \to \mathrm{pt}$. Note that any polynomial $c(t) \in H_T^*(\mathrm{pt}) = \mathbb{Z}[t_1, \ldots, t_n]$ can be written as

$$c(t) = \sum_J c_J \, (-\alpha_1)^{j_1} \ldots (-\alpha_n)^{j_n}, \tag{19}$$

where $c_J = \pi_*^{\mathbb{B}}(c(t) \cdot [\mathbb{B}_J])$.(This is just Poincaré duality on $\mathbb{B}$.)

In particular, we have:

Lemma (3.2.7) [3]:

Suppose $c(t) = \pi_*^T(\sigma)$, for some class $\sigma \in H_T^* X$, where $\pi_*^T: H_T^* X \to H_T^*(\mathrm{pt})$ is the equivariant pushforward. Let $\pi^{\boldsymbol{X}}$ be the map $\boldsymbol{X} \to \mathrm{pt}$. Then the coefficient c_J appearing in (19) is equal to $\pi_*^{\boldsymbol{X}}(\sigma \cdot [\boldsymbol{X}_J])$.

One can realize $\mathbf{Fl}(\boldsymbol{n}) = \mathbb{E} \times^T Fl(\boldsymbol{n})$ as a flag bundle $\mathbf{Fl}(\boldsymbol{n}; E) \to \mathbb{B}$, for a vector bundle E on $\mathbb{B}$. Specifically, let $pr_i: \mathbb{B} \to \mathbb{P}^{N-1}$ be the projection on the ith factor, and let $L_i = pr_i^* \mathcal{O}(-1)$. (One has $\alpha_i = c_1(L_i)$ in $H_T^2(\mathrm{pt}) = H^2(\mathbb{B})$.) Also let $L'_i = L_i \otimes \cdots \otimes L_n$, so $c_1(L'_i) = t_i$. Then set $E_i = L'_1 \oplus L'_2 \oplus \cdots \oplus L'_i$ and $E = E_n$, so we obtain a flag of vector bundles $E_\bullet$, as well as an opposite flag $\tilde{E}_\bullet$ defined by $\tilde{E}_i = L'_n \oplus \cdots \oplus L'_{n+1-i}$. The mixing spaces of the Schubert varieties $\Omega_w = \mathbb{E} \times^T \Omega_w$ and $\tilde{\Omega}_w = \mathbb{E} \times^T \tilde{\Omega}_w$ are thus identified with corresponding degeneracy loci. The equivariant cohomology of $Fl(\boldsymbol{n})$ is computed as the ordinary cohomology of the flag bundle $\mathbf{Fl} = \mathbf{Fl}(\boldsymbol{n}; E) \to \mathbb{B}$, and the equivariant class $[\Omega_w]^T$ is identified with the class $[\Omega_w]$.

There is a large group acting on $\mathbf{Fl}$, using the transitive automorphism group of $\mathbb{B}$ together with a "fiberwise" action of lower-triangular matrices. This group was introduced and dubbed the "mixing group". We will describe its construction concretely for the present context.

Observe that for $i \geq j$, the line bundle

$$L'_i \otimes \left(L'_j\right)^{-1} = L_j^{-1} \otimes \ldots \otimes L_{i-1}^{-1}$$

is generated by global sections. It follows that the bundle

$$End(E) = \bigoplus_{i,j} L'_i \otimes \left(L'_j\right)^{-1}$$

has global sections in lower-triangular matrices.

Let $Aut(E) \subset End(E)$ be the automorphism bundle of E; this is a group scheme over $\mathbb{B}$, whose fiber at x is $GL(E(x))$. The group $\Gamma_0 = \mathrm{Hom}_{\mathbb{B}}(\mathbb{B}, \Lambda ut(E))$ of global sections is a connected algebraic group over $\mathbb{C}$. This group acts on the total space of the bundle E, and hence also on the flag bundle $\mathbf{Fl}$, preserving the

fibers of the projection to $\mathbb{B}$. The group acts similarly on $\mathcal{Q}_{\boldsymbol{d}}$, the mixing space for the quot scheme $\mathcal{Q}_{\boldsymbol{d}}$.

The group $(PGL_N)^n$ acts naturally on Γ_0 via its transitive action on B. Let $\Gamma = \Gamma_0 \rtimes (PGL_N)^n$ be the semidirect product; this is a connected linear algebraic group acting on **Fl** and $\mathcal{Q}_{\boldsymbol{d}}$, which we call the mixing group.

Lemma (3.2.8) [3]:

Let X be a scheme with an action of $\widetilde{\mathbb{B}}$, and let X_i be the $\widetilde{\mathbb{B}}$-orbits. Let $\boldsymbol{X} = \mathbb{E} \times^T X$ and $\boldsymbol{X}_i = \mathbb{E} \times^T X_i$ be the mixing spaces. Then the orbits of Γ on $\boldsymbol{X}$ are $\boldsymbol{X}_i$.

In particular, the orbits of Γ on $\boldsymbol{Fl}$ are the cells $\widetilde{\boldsymbol{\Omega}}_w^\circ \cong \mathbb{E} \times^T \tilde{\Omega}_w^\circ$.

Section (3.3): Equivariant Moving Lemmas and Further Properties

In this section we use the action of the mixing group to put certain subvarieties of the space of maps into (generically) transverse position. Establishing that the EQLR coefficients can be computed on quot schemes. Along the way, we will obtain an enumerative interpretation of these coefficients.

We will need a generalization of Kleiman's transversality theorem. As a matter of notation, for an algebraic group G acting on a scheme X, a map $\psi: Z \to X$, and an element $g \in G$, we write $g \cdot Z$ to denote Z equipped with the composite map $Z \xrightarrow{\psi} X \xrightarrow{g\cdot} X$. (When ψ is an embedding, this is simply the translation of Z by g.)

A map $\psi: Z \to X$ is dimensionally transverse to a (locally closed) subvariety $V \subseteq X$ if $\operatorname{codim}_Z(\psi^{-1}V) = \operatorname{codim}_X(V)$.

Proposition (3.3.1) [3]:

Let X be a smooth variety with an action of an algebraic group G, and let Y and Z be Cohen–Macaulay varieties with maps $\varphi: Y \to X$ and $\psi: Z \to X$. Assume that ψ is dimensionally transverse to the orbits of G on X, that is,

$$\operatorname{codim}_z(\psi^{-1}O) = \operatorname{codim}_x(O)$$

for all orbits O. Then for general g in G, the scheme $W_g = Y \times_X (g \cdot Z)$ is pure-dimensional, of dimension

$$\dim W_g = \dim Y + \dim Z - \dim X.$$

(When ψ is an embedding of Z in X, this says $\operatorname{codim}_Y(\varphi^{-1}(g \cdot Z)) = \operatorname{codim}_X(Z)$.) If the ground field has characteristic zero, and Y and Z are reduced, then for general g,W_g is also reduced.

We will apply Proposition (3.3.1) repeatedly in the situation where $X = \mathbf{Fl}$ is the flag bundle (or approximation space), or a product of such spaces. In what follows, we work with the setup:

(i) $\Omega_1, \ldots, \Omega_r \subseteq Fl$ are Schubert varieties of codimensions $\ell_1, \ldots, \ell_r$, and $\tilde{\Omega}$ is an opposite Schubert variety, of codimension ℓ.

(ii) $p_1, \ldots, p_r, p$ are fixed, distinct points of $\mathbb{P}^1$.
(iii) $J = (j_1, \ldots, j_n) \in \{0, \ldots, N-1\}^n$ is an index.
(iv) $A = \ell_1 + \cdots + \ell_r + \ell$.

Let $M_r(\boldsymbol{d})$ denote the space of maps with r marked points, let $\bar{M}_r(\boldsymbol{d})$ be the Kontse-vich space of stable maps, and let $\boldsymbol{M}_r(\boldsymbol{d})$ and $\bar{\boldsymbol{M}}_r(\boldsymbol{d})$ be the corresponding approximate mixing spaces.

Lemma (3.3.2) [3]: (Moving Lemma A).

There are elements $\gamma_1, \ldots, \gamma_r$ in the mixing group Γ such that

$$\bar{W}' = \mathrm{ev}^{-1}\left(\gamma_1 \cdot \Omega_1 \times \ldots \times \gamma_r \cdot \Omega_r \times \tilde{\Omega}\right) \cap \left(\bar{M}_{r+1}(d)\right)_J$$

is either empty, or reduced and pure dimensional of codimension A inside $(\bar{\boldsymbol{M}}_{0,r+1}(\boldsymbol{d}))_J$. Moreover,

$$W' = \mathrm{ev}^{-1}\left(\gamma_1 \cdot \Omega_1 \times \ldots \times \gamma_r \cdot \Omega_r \times \tilde{\Omega}\right) \cap \left(M_{r+1}(d)\right)_J$$

is Zariski-dense inside $\bar{W}'$.

In fact, $\gamma_1, \ldots, \gamma_r$ can be chosen so that

$$\bar{W} = \mathrm{ev}^{-1}\left(\gamma_1 \cdot \Omega_1 \times \ldots \times \gamma_r \cdot \Omega_r \times \tilde{\Omega}\right) \cap \left(\bar{M}_{r+1}(d)\right)_J \cap \tilde{f}^{-1}(x)$$

and

$$W = \mathrm{ev}^{-1}\left(\gamma_1 \cdot \Omega_1 \times \ldots \times \gamma_r \cdot \Omega_r \times \tilde{\Omega}\right) \cap \left(M_{r+1}(d)\right)_J \cap \tilde{f}^{-1}(x)$$

are either empty, or reduced and pure dimensional of codimension A inside $\tilde{f}^{-1}(x)$, where $x \in \bar{M}_{0,r+1}$ is the point corresponding to $(p_1, \ldots, p_r, p)$, and $\tilde{f}$ is the composition $\bar{\boldsymbol{M}}_{r+1}(\boldsymbol{d}) \to \bar{\boldsymbol{M}}_{0,r+1} = \mathbb{B} \times \bar{M}_{0,r+1} \to \bar{M}_{0,r+1}$.

Proof:

Apply Proposition (3.3.1), using the group $C = \Gamma^{(r+1)} = (\Gamma_0)^{r+1} \rtimes (PGL_N)^n$, with

$$X = Fl \times_{\mathbb{B}} \ldots \times_{\mathbb{B}} \mathrm{Fl} \quad (r+1 \text{ factors}),$$

$$Z = \Omega_1 \times_{\mathbb{B}} \dots \times_{\mathbb{B}} \Omega_r \times_{\mathbb{B}} \mathrm{Fl} \hookrightarrow X,$$

and

$$Y = \mathrm{ev}^{-1}\big(\mathrm{Fl} \times_{\mathbb{B}} \dots \times_{\mathbb{B}} \mathrm{Fl} \times_{\mathbb{B}} \widetilde{\Omega}\big) \cap \big(\overline{M}_{r+1}(d)\big)_J \xrightarrow{\mathrm{ev}|Y} X.$$

Observe that Y has codimension ℓ in $(\overline{\boldsymbol{M}}_{r+1}(\boldsymbol{d}))_J$, and is therefore Cohen–Macaulay, since $\overline{\boldsymbol{M}}_{r+1}(\boldsymbol{d})$ is.

The statement about W' follows by showing that

$$\overline{W}'^{\setminus}W' = \partial\overline{W}' = \mathrm{ev}^{-1}\big(\gamma_1 \cdot \Omega_1 \times \dots \gamma_r \cdot \Omega_r \times \widetilde{\Omega}\big) \cap \big(\partial\overline{M}_{r+1}(d)\big)_J$$

has strictly smaller dimension than $\overline{W}'$, where $\partial\overline{\boldsymbol{M}}_{r+1}(\boldsymbol{d}) = \overline{\boldsymbol{M}}_{r+1}(\boldsymbol{d})\backslash\boldsymbol{M}_{r+1}(\boldsymbol{d})$ is the boundary divisor. For this, apply the proposition with Y replaced by ∂Y, defined similarly. (This divisor in Y is again Cohen–Macaulay, for the same reasons.)

In Moving Lemma A, W can be identified with $\gamma_1 \cdot \boldsymbol{\Omega}_1(p_1) \cap \dots \cap \gamma_r \cdot \boldsymbol{\Omega}_r(p_r) \cap \widetilde{\boldsymbol{\Omega}}(p) \cap (\boldsymbol{M}(\boldsymbol{d}))_J$, where $M(\boldsymbol{d})$ is the space of degree $\boldsymbol{d}$ maps $\mathbb{P}^1 \to Fl$, and $\Omega_i(p_i)$ is the locus sending p_i into Ω_i. This gives an "enumerative" interpretation of the EQLR coefficients.

Corollary (3.3.3) [3]:

Write the polynomial $c^{w,\boldsymbol{d}}_{v_1,\dots,v_r}(t)$ as $\sum_J c^{w,\boldsymbol{d},\boldsymbol{J}}_{v_1,\dots,v_r}(-\alpha)^J$. Then for general $\gamma_1, \dots, \gamma_k$ chosen as in Lemma (3.3.2), we have

$$c^{w,\boldsymbol{d},\boldsymbol{J}}_{v_1,\dots,v_r} = \begin{cases} \#W & \text{if } \dim W = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the EQLR coefficient $c^{w,\boldsymbol{d}}_{v_1,\dots,v_r}(t)$ is Graham-positive; that is, when written as a polynomial in the variables $-\alpha_i$, it has nonnegative coefficients.

Proof:

Use the description of $c^{w,\boldsymbol{d}}_{v_1,\dots,v_r}(t)$ given in (11), together with the last part of Moving Lemma A. Recalling that $f\colon \overline{M}_{r+1}(\boldsymbol{d}) \to \overline{M}_{0,r+1}$ denotes the forgetful map, note that $[\tilde{f}^{-1}(x)] = f^*[\mathrm{pt}]$ as classes in $H^*\overline{\boldsymbol{M}}_{r+1}(\boldsymbol{d}) = H^*_T\overline{M}_{r+1}(\boldsymbol{d})$.

Next we must prove an analogue of Moving Lemma A for quot schemes. Continuing the notation, we write $D_i \subseteq \mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}}$ for the degeneracy locus of (15) corresponding to the Schubert variety Ω_i, and we write $\mathcal{Q}_{\boldsymbol{d}}$ and $\boldsymbol{D}_i$ for the mixing spaces.

Lemma (3.3.4) [3]: (Moving Lemma B)

There are elements $\gamma_1, \dots, \gamma_r$ in the mixing group Γ such that

$$\overline{W} = \gamma_1 \cdot D_1(p_1) \cap \dots \cap \gamma_r \cdot D_r(p_r) \cap \widetilde{D}(p) \cap (\mathcal{Q}_d)_J$$

is either empty, or reduced and pure dimensional of codimension A inside $(\mathcal{Q}_{\boldsymbol{d}})_J$. More-over,

$$W = \gamma_1 \cdot \Omega_1(p_1) \cap \dots \cap \gamma_r \cdot \Omega_r(p_r) \cap \widetilde{\Omega}(p) \cap \big(M(d)\big)_J$$

is Zariski-dense inside $\overline{W}$.

Proof:

In Moving Lemma A, we have already seen that W is reduced and pure-dimensional of codimension A (when nonempty). To prove this lemma, it will suffice to show that $\overline{W} \cap (\partial\mathcal{Q}_{\boldsymbol{d}})_J$ has codimension greater than A in $(\mathcal{Q}_{\boldsymbol{d}})_J$, where $\partial\mathcal{Q}_{\boldsymbol{d}} = \mathcal{Q}_{\boldsymbol{d}} \backslash \boldsymbol{M}(\boldsymbol{d})$ is the boundary.

Consider $\boldsymbol{e}$ as in (17), with some $e_i > 0$. By inductionon $\boldsymbol{d}$, we may assume the statement holds for $\mathcal{Q}_{\mathbf{d}-e}$. (The base case is $\mathcal{Q}_{\mathbf{0}} = \mathbf{Fl}$, where Proposition (3.3.1) applies directly.) Let $\mathcal{U}_e$ be the mixing space for the variety $\mathscr{U}_e$ of Theorem (3.2.7). Since the map $h_{\boldsymbol{e}}\colon U_{\boldsymbol{e}} \to \mathcal{Q}_{\boldsymbol{d}}$ is birational onto its image, and $\partial\mathcal{Q}_{\boldsymbol{d}}$ is covered by such images, it will suffice to show

$$\operatorname{codim}_{\mathcal{U}_e}\left(\bigcap_{i=1}^{r} h_e^{-1}\big(\gamma_i \cdot D_i(p_i)\big) \cap h_e^{-1}\left(\widetilde{D}(p)\right)\right) > A - \operatorname{codim}_{\mathcal{Q}_d}\big(h_e(\mathcal{U}_e)\big) \quad (20)$$

for general $\gamma_1, \ldots, \gamma_r$ in Γ. Using Theorem (3.2.7)(3), the intersection on the LHS can be written as

$$\bigcap_{i=1}^{r}\left(\rho^{-1}\big(\mathbb{P}^1 \times \gamma_i \cdot D_i^e(p_i)\big) \cup \psi_e(p_i)^{-1}\left(\gamma_i \cdot \Omega_{w_i'}\right)\right)$$

$$\cap \left(\rho^{-1}\left(\mathbb{P}^1 \times \widetilde{D}^e(p)\right) \cup \psi_e(p)^{-1}\big(\widetilde{\Omega}_{w'}\big)\right). \quad (21)$$

(Here $\Omega_{w_i'} \subseteq Fl(\boldsymbol{n}')$ is the Schubert variety whose existence is claimed in Theorem (3.2.7), where $\Omega_i = \Omega_{w_i}$.) Since the points $p_1, \ldots, p_r, p$ are distinct, any nonempty component of the expansion of this intersection contains at most one factor of $\psi_{\boldsymbol{e}}(p_i)^{-1}\left(\Omega_{w_i'}\right)$ or $\psi_{\boldsymbol{e}}(p)^{-1}(\tilde{\Omega}_{w'})$.

If there are no factors of $\psi_{\boldsymbol{e}}(p_i)^{-1}(\Omega_{w_i'})$ or $\psi_{\boldsymbol{e}}(p)^{-1}(\tilde{\Omega}_{w'})$, the intersection is the inverse image of a similar intersection on $\mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}}$. By induction and the fact that ρ is smooth, the inequality (20) holds for these components.

For the case where $\psi_{\boldsymbol{e}}(p)^{-1}(\Omega_{w_r'})$ occurs, let

$$X = Fl(n'),$$

$$Z = \Omega_{w_r'} \hookrightarrow X,$$

and

$$Y = \bigcap_{i=1}^{r-1} \rho^{-1}\big(\{p_r\} \times \gamma_i \cdot D_i(p_i)\big) \cap \rho^{-1}\left(\{p_r\} \times \widetilde{D}(p)\right) \xrightarrow{\psi_e(p_r)} X.$$

By the previous case, Y has codimension $\ell_1 + \cdots + \ell_{r-1} + \ell$ in $\mathscr{U}_e(p_r) = \rho^{-1}(\{pr\} \times \mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}})$. By Proposition (3.3.1), we can choose $\gamma_r \in \Gamma$ so that

$$Y \cap \psi_e(p_r)^{-1}\big(\gamma_r \cdot \Omega_{w_r'}\big)$$

has codimension $\ell(w_r')$ in Y. The inequality (20) follows from the estimates on $\ell(w_r')$.

Finally, for the case where $\psi_{\boldsymbol{e}}(p)^{-1}(\tilde{\Omega}_{w'})$ occurs, first observe that each $D_i^{\boldsymbol{e}}(p_i) \subseteq \mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}}$ is a B-invariant subscheme (where $B \subseteq GL_n$ is upper-triangular matrices), and it is dimensionally transverse to all $\tilde{B}$-orbits in $\mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}}$. It follows that $\boldsymbol{D}_i^{\boldsymbol{e}}(p_i)$ is dimensionally transverse to Γ-orbits on $\mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}}$. Now apply Proposition (3.3.1), with

$$X = \mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}} \times_{\mathbb{B}} \dots \times_{\mathbb{B}} \mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}} \quad (r \text{ factors}),$$

$$Z = D_1^e(p_1) \times_{\mathbb{B}} \dots \times_{\mathbb{B}} D_r^e(p_r) \hookrightarrow X.$$

and

$$Y = \psi_e(p)^{-1}(\tilde{\Omega}_{w'}) \xrightarrow{\tilde{\rho}\times\dots\times\tilde{\rho}} X,$$

where ρ is the restriction of $\rho: \mathcal{U}_{\boldsymbol{e}} \to \mathbb{P}^1 \times \mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}}$ to $\psi_{\boldsymbol{e}}(p)^{-1}(\tilde{\Omega}_{w'}) \subseteq \mathcal{U}_{\boldsymbol{e}}(p) = \rho^{-1}(\{p\} \times \mathcal{Q}_{\boldsymbol{d}-\boldsymbol{e}})$. We find $\gamma = (\gamma_1, \dots, \gamma_r)$ in $\Gamma^{(r)}$ such that

$$(\tilde{\rho} \times \dots \tilde{\rho})^{-1}(\gamma \cdot Z) = \bigcap_{i=1}^{r} \rho^{-1}\big(\mathbb{P}^1 \times \gamma_i \cdot D_i^e(p_i)\big) \cap \psi_{\boldsymbol{e}}(p)^{-1}(\tilde{\Omega}_{w'})$$

has codimension $\ell_1 + \dots + \ell_r$ in Y. Since Y has codimension $\ell(w')$ in $\mathcal{U}_{\boldsymbol{e}}(p)$, the same estimate as in the previous case proves the inequality (20).

We can now complete the proof of Proposition (3.2.4).

Corollary (3.3.5) [3]:

Writing $c_{v_1,\dots,v_r}^{w,\boldsymbol{d}}(t) = \sum_J c_{v_1,\dots,v_r}^{w,\boldsymbol{d},\boldsymbol{J}} (-\alpha)^J$ as above, we have

$$c_{v_1,\dots,v_r}^{w,\boldsymbol{d},\boldsymbol{J}} = \pi_*^{\mathcal{Q}_{\boldsymbol{d}}}\big([D_1(p_1)] \dots [D_r(p_r)] \cdot [\tilde{D}(p)] \cdot [(\mathcal{Q}_{\boldsymbol{d}})_J]\big),$$

where $\pi^{\mathcal{Q}_{\boldsymbol{d}}}$ is the map $\mathcal{Q}_{\boldsymbol{d}} \to \mathrm{pt}$.

Proof:

By Moving Lemma B, the right-hand side is equal to $\#W$. However, this W is the same as the one from Moving Lemma A, where x corresponds to the point $(p_1, \ldots, p_r, p) \in \overline{M}_{0,r+1}$, so the claim follows from Corollary (3.3.3).

Fix $\boldsymbol{n}$, let $P^{\boldsymbol{n}}(g,t)$ be a polynomial in variables $g_i[j]$ of degree $j+1$, with $i+j \in \boldsymbol{n}$ and $j \geq 0$, and variables t_i of degree 1 with $1 \leq i \leq n$. We denote by $P^{\boldsymbol{n},q}(\sigma,t)$ the polynomial which results after performing the specialization (9). (Recall that the polynomial $\mathfrak{S}_w^{\boldsymbol{n},q}(\sigma,t)$ is obtained from the universal polynomial $\mathfrak{S}_w^{\boldsymbol{n}}(g,t)$ this way.)

For the sequence of equivariant vector bundles $\mathscr{A}_\bullet^*$ on the quot scheme $Q_{\boldsymbol{d}}$, recall the classes $Q_i[j] = \beta(g_i[j])$ defined in (13) via the map $\beta: \Lambda[c]_{\boldsymbol{n}} = \Lambda[g]_{\boldsymbol{n}} \to H_T^* Q_d$. The key in gradient in the proof of Theorem (3.1.1) is the following.

Proposition (3.3.6) [3]:

Let $P^{\boldsymbol{n}}(g,t)$ be a polynomial in variables g, t as above. Then

$$P^{n,q}(\sigma,t) = \sum_{d,v} q^d \left(\beta\left(P^n(g,t)\right) \cdot \tilde{\mu}_{v^\vee}\right)_d^T \sigma_v \quad \text{in } QH_T^* Fl(\boldsymbol{n}).$$

Before proceeding to the proofs, we show that the presentation of $QH_T^* Fl(\boldsymbol{n})$ given in [19] can be deduced from Proposition (3.3.6).

Corollary (3.3.7) [3]:

We have

$$QH_T^* Fl(\boldsymbol{n}) \cong \Lambda[q]\left[\sigma_1^1, \ldots, \sigma_{n_1}^1, \ldots, \sigma_1^{m+1}, \ldots, \sigma_{n-n_m}^{m+1}\right]/I_T^q$$

where I_T^q is the ideal $(e_1^{\boldsymbol{n},q}(n) - e_1(t), \ldots, e_n^{\boldsymbol{n},q}(n) - e_n(t))$.

Proof:

We sketch it here. We have $\beta(c_k(n)) = c_k^T(A_n^*) = c_k^T(V) = e_k(t_1, \ldots, t_n)$. Now apply Proposition (3.3.6) to the polynomials $P^{\boldsymbol{n}}(g,t) = c_k(n) - e_k(t)$, for $k = 1, \ldots, n$: the specializations $\beta(P^{\boldsymbol{n}}(g,t))$ are zero, so it follows from the proposition that the specializations $P^{\boldsymbol{n},q}(\sigma,t)$ vanish as well. We obtain n

relations which specialize to the know n relations defining $H_T^* Fl(\boldsymbol{n})$ when $q \mapsto 0$. The claim follows.

In the proof of Theorem (3.1.1) and Proposition (3.3.6), we will use several lemmas.

Lemma (3.3.8) [3]:

For $w \in S^{\boldsymbol{n}}, \sigma_w = \sum_{\boldsymbol{d},v} q^{\boldsymbol{d}} (\mu_w \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T \sigma_v$ in $QH_T^* Fl(\boldsymbol{n})$.

Proof:

When $\boldsymbol{d} = 0$, the quot scheme $\mathcal{Q}_{\boldsymbol{d}}$ is the flag variety $Fl(\boldsymbol{n})$, and in this case duality (see (5)) gives $(\mu_w \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{0}}^T = [\Omega_w]^T \cdot [\tilde{\Omega}_{v^\vee}]^T = \delta_{wv}$.

More generally, we have

$$(\mu_w \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T = \tilde{\pi}_*^T (\mu_w \cdot \tilde{\mu}_{v^\vee}) = \tilde{\pi}_*^T \big([D_w(0)] \cdot [\widetilde{D}_{v^\vee}(\infty)]\big).$$

For degree reasons, $(\mu_w \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T$ can be nonzero only when $\ell(w) + \ell(v^\vee) \geq \dim \mathcal{Q}_{\boldsymbol{d}}$. Using Lemma (3.3.4), one sees that $D_w(0)$ and $\widetilde{D}_{v^\vee}(\infty)$ are dimensionally transverse. It follows that

$$[D_w(0)] \cdot [\widetilde{D}_{v^\vee}(\infty)] = [D_w(0) \cap \widetilde{D}_{v^\vee}(\infty)]$$

and that $D_w(0) \cap \widetilde{D}_{v^\vee}(\infty)$ is empty when $\ell(w) + \ell(v^\vee) > \dim \mathcal{Q}_{\boldsymbol{d}}$. Therefore $(\mu_w \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T$ can be nonzero only when $\ell(w) + \ell(v^\vee) = \dim \mathcal{Q}_{\boldsymbol{d}}$; that is, when $D_w(0) \cap \widetilde{D}_{v^\vee}(\infty)$ consists of finitely many points. Moreover, Lemma (3.3.4) also implies $D_w(0) \cap \widetilde{D}_{v^\vee}(\infty) \cap M(\boldsymbol{d})$ is dense in $D_w(0) \cap \widetilde{D}_{v^\vee}(\infty)$, so these points must lie in $M(\boldsymbol{d})$.

Finally, recall there is an action of $\mathbb{C}^*$ on $\mathcal{Q}_{\boldsymbol{d}}$, coming from the standard action on $\mathbb{P}^1$, and $D_w(0) \cap \widetilde{D}_{v^\vee}(\infty)$ is stable under this $\mathbb{C}^*$ action. When $\boldsymbol{d} \neq 0$, this action has no fixed points inside $M(\boldsymbol{d})$, since a non-constant map is changed by reparametrization. Therefore $D_w(0) \cap \widetilde{D}_{v^\vee}(\infty)$ is empty whenever $\ell(w) + \ell(v^\vee) = \dim \mathcal{Q}_{\boldsymbol{d}}$, and it follows that $(\mu_w \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T = 0$ in this case.

The following lemma shows that a similar sum which involves a single $Q_i[j]$ only has one non-zero term. Let $\boldsymbol{e}_l = (0,\ldots,1,\ldots,0)$ be the m-tuple whose only nonzero entry is in the lth position.

Lemma (3.3.9) [3]:

For $1 \leq l \leq m$ and $1 \leq i \leq n_l$,

$$\sum_{d,v} q^d \left(Q_i[n_{l+1} - i] \cdot \tilde{\mu}_{v^\vee}\right)_d^T \sigma_v = \left(Q_{n_{l-1}+1}[n_{l+1} - n_{l-1} - 1] \cdot \tilde{\mu}_{w^\circ}\right)_{e_l}^T ql$$

If $i = n_{l-1} + 1$, and is equal to zero otherwise.

Proof:

Note that when $d_a = 0, A_a \to A_b$ is an inclusion of vector bundles on $\mathcal{Q}_{\boldsymbol{d}}$ for all $b > a$, or equivalently, $A_b^* \to A_a^*$ is surjective.

Let $l' < l$ be such that $n_{l'} + 1 \leq i \leq n_{l'+1}$. If $d_a = 0$ for any $l' + 1 \leq a \leq l$, then by Lemma (3.2.2) applied to $b = l + 1$, we obtain $Q_i[n_{l+1} - i] = 0$, so that $Q_i[n_{l+1} - i] \neq 0$ only if $d_{l'+1}, \ldots, d_l > 0$.

By this and degree considerations, we conclude that $\left(Q_i[n_{l+1} - i] \cdot \tilde{\mu}_{v^\vee}\right)_{\boldsymbol{d}}^T$ is zero unless

$$n_{l+1} - i + 1 + \ell(v^\vee) \geq \dim \mathcal{Q}_{\boldsymbol{d}} = \dim Fl(n) + \sum_{k=1}^{m} d_k(n_{k+1} - n_{k-1})$$

$$\geq \dim Fl(n) + n_{l+1} + n_l - n_{l'} - n_{l'+1}.$$

Since $n_{l+1} - n_{l'} + \dim Fl(\boldsymbol{n}) \geq n_{l+1} - i + 1 + \dim Fl(\boldsymbol{n}) \geq n_{l+1} - i + 1 + \ell(v^\vee)$, these inequalities can only hold when $l' = l - 1$ and every inequality is an equality. Therefore $\boldsymbol{d} = \boldsymbol{e}_l, i = n_{l-1} + 1$, and $v^\vee = w^\circ$, the permutation of longest length.

The next lemma will be used in the induction step for the main proof below.

Lemma (3.3.10) [3]:

Assume that Proposition (3.3.6) holds for all polynomials $P^{\boldsymbol{n}}(g,t)$ of g-degree up to $n_{l+1}-n_{l-1}-1$. Then

$$\left(Q_{n_{l-1}+1}[n_{l+1}-n_{l-1}-1]\cdot\tilde{\mu}_{w^\circ}\right)^T_{e_l}=(-1)^{n_{l+1}-n_l+1}$$

Proof:

Recall that $(Q_{n_{l-1}+1}[n_{l+1}-n_{l-1}-1]\cdot\tilde{\mu}_{w^\circ})^T_{\boldsymbol{e_l}}$ is defined as the equivariant push-forward $\tilde{\pi}^T_*(Q_{n_{l-1}+1}[n_{l+1}-n_{l-1}-1]\cdot\tilde{\mu}_{w^\circ})$, for $\tilde{\pi}^T_*: H^*_T\mathcal{Q}_{\boldsymbol{e}_l}\to H^*_T(pt)$. Moreover, since the degree of $Q_{n_{l-1}+1}[n_{l+1}-n_{l-1}-1]\cdot\tilde{\mu}_{w^\circ}$ is equal to $n_{l+1}-n_{l-1}+\ell(w^\circ)=\dim\mathcal{Q}_{\boldsymbol{e_l}}$, the equivariant pushforward is equal to the non-equivariant one:

$$\tilde{\pi}^T_*\left(Q_{n_{l-1}+1}[n_{l+1}-n_{l-1}-1]\cdot\tilde{\mu}_{w^\circ}\right)=\tilde{\pi}_*\left(\bar{Q}_{n_{l-1}+1}[n_{l+1}-n_{l-1}-1]\cdot\bar{\tilde{\mu}}_{w^\circ}\right)$$

in $H^0_T(pt)=H^0(pt)=\mathbb{Z}$, where for a class γ in $H^*_T\mathcal{Q}_{\boldsymbol{e}_l}$, we write $\bar{\gamma}$ for its image in $H^*\mathcal{Q}_{\boldsymbol{e}_l}$. The right-hand side is $(-1)^{n_{l+1}-n_l+1}$, as desired. (To apply this result, note that our Proposition (3.3.8) for polynomials $P^{\boldsymbol{n}}(g,t)$ of g-degree up to $n_{l+1}-n_{l-1}-1$ implies the corresponding non-equivariant.

We are now ready to prove the main results.

Proof of Proposition (3.3.8) and Theorem (3.1.1):

We will use induction on the length of w, and on the degree of $P^{\boldsymbol{n}}(g,t)$, viewed as a polynomial in $g_i[j]$ with coefficients in Λ.

We proceed by showing that Theorem (3.1.1) for $\ell(w)\le s$ implies Proposition (3.3.6) for polynomials $P^{\boldsymbol{n}}(g,t)$ of g-degree at most s, and then showing that Proposition (3.3.6) for polynomials $P^{\boldsymbol{n}}(g,t)$ of g-degree at most s, together with Lemma (3.3.9), imply Theorem (3.1.1) for permutations w of length at most $s+1$. The base case $s=0$ for Theorem (3.1.1) clearly holds, because $\sigma_{id}=1$.

First suppose that Theorem (3.1.1) holds when $\ell(w)\le s$. Let $P^{\boldsymbol{n}}(g,t)$ be a polynomial of g-degree at most s. By Lemma (3.1.3), it can be written as a

polynomial in t and $\mathfrak{S}_w^{\boldsymbol{n}}(g,t)$ with $\ell(w) \leq s$. Therefore a specialization $\beta(P(g,t))$ can be rewritten as

$$\beta\big(P^n(g,t)\big) = F\big(\beta\big(\mathfrak{S}_w^n(g,t)\big),t\big) = F(\mu_w,t),$$

for some polynomial F in μ_w and t, with $w \in S^{\boldsymbol{n}}$ and $\ell(w) \leq s$. (The second displayed equality holds by Proposition (3.2.3))

By Corollary (3.2.5), we obtain the equalities

$$\sum_{d,v} q^d(\beta\big(P^n(g,t)\big) \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T \sigma_v = \sum_{d,v} q^d(F(\mu_w,t) \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T \sigma_v = F(\sigma_w,t)$$

in $QH_T^*Fl(\boldsymbol{n})$. Since $F(\sigma_w,t)$ is a polynomial in classes σ_w with $\ell(w) \leq s$, we can apply the induction hypothesis to obtain $F(\sigma_w,t) = F(\mathfrak{S}_w^{\boldsymbol{n},q}(\sigma,t),t) = P^{\boldsymbol{n},q}(\sigma,t)$, and conclude Proposition (3.3.6) holds for polynomials $P^{\boldsymbol{n}}(g,t)$ with g-degree at most s.

Now suppose that Proposition (3.3.6) holds for polynomials of g-degree at most s. Let $w \in S^{\boldsymbol{n}}$ with $(w) = s+1$. Consider the polynomial $\mathfrak{S}_w^{\boldsymbol{n}}(g,t)$. By degree considerations, since the degree of $g_i[j]$ is $j+1$, each $g_i[s]$ can only appear linearly, and we can write $\mathfrak{S}_w^{\boldsymbol{n}}(g,t) = P^{\boldsymbol{n}}(g,t) + \sum a_i g_i[s]$, where $P^{\boldsymbol{n}}(g,t)$ is a polynomial of g-degree at most s, and $a_i \in \mathbb{Z}$. Therefore

$$\begin{aligned}
\sigma_w &= \sum_{d,v} q^d\big(\beta\big(\mathfrak{S}_w^n(g,t)\big) \cdot \tilde{\mu}_{v^\vee}\big)_d\, \sigma_v \\
&= \sum_{d,v} q^d\big(\beta\big(P^n(g,t)\big) \cdot \tilde{\mu}_{v^\vee}\big)_d\, \sigma_v + \sum_i a_i \sum_{d,v} q^d(Q_i[s] \cdot \tilde{\mu}_{v^\vee})_d\, \sigma_v \\
&= P^{\mathbf{n},q}(\sigma,t) + \sum a_i (-1)^{N_{l+1}-n_l+1} q_l \\
&= \mathfrak{S}_w^{\mathbf{n},q}(\sigma,t),
\end{aligned}$$

where the final sum is over $(i,s) = (n_{l-1}+1, n_{l+1}-n_{l-1}-1)$. The last line follows from Proposition (3.3.6) for polynomials of g-degree at most s, together with Lemma (3.3.9), and Lemma (3.3.10), which holds since $s = n_{l+1} - n_{l-1} - 1$.

We have shown that Theorem (3.1.1) holds for permutations $w \in S^{\boldsymbol{n}}$ of length at most $s+1$, completing the proof.

We conclude with some brief remarks about the stability properties of the equivariant quantum Schubert polynomials. First, the universal Schubert polynomials are independent of n.

Lemma (3.3.11) [3]:

Consider $w \in S_n$. The polynomial $\mathfrak{S}_w(c,t)$ is the same when w is considered as a permutation in $S_{n'}$, for $n' > n$, using the standard embedding of symmetric groups.

As an immediate consequence, we have a weak stability property of equivariant quantum Schubert polynomials for partial flags. Let $\boldsymbol{n} = (n_1 < \cdots < n_m < n_{m+1} = n)$, and for any $n' > n$, let $\boldsymbol{n}' = (n_1 < \cdots < n_m < n_{m+1} < n_{m+2} = n')$. The standard embedding $S_n \subset S_{n'}$ leads to a canonical inclusion $S^{\boldsymbol{n}} \subset S^{\boldsymbol{n}'}$. (Concretely, given $w \in S^{\boldsymbol{n}}$, the corresponding minimal length representative in $S^{\boldsymbol{n}'}$ is given by appending $n_{m+1}+1, n_{m+1}+2, \ldots, n_{m+2} = n'$ to w.) Then $\mathfrak{S}_w^{\boldsymbol{n},q}(\sigma,t) = \mathfrak{S}_w^{\boldsymbol{n}',q}(\sigma,t)$.

For the remainder of this section, we will focus on the complete flag case, with $\boldsymbol{n} = (1,2,\ldots,n)$. In this case, we write $e_k^q(l)$ for $e_k^{\boldsymbol{n},q}(l)$. Consider the standard embeddings of symmetric groups $S_n \subset S_{n+1} \subset \cdots \subset S_\infty$, and for each n, let $I_T^q(n)$ be the ideal $(e_1^q(n) - e_1(t), \ldots, e_n^q(n) - e_n(t))$ from Corollary (3.3.7). In analogy, the polynomials for complete flags may be characterized as follows:

Proposition (3.3.12) [3]:

For $w \in S_n$, the equivariant quantum Schubert polynomial $\mathfrak{S}_w^q(x,t)$ is the unique polynomial in $\mathbb{Z}[t_1,\ldots,t_n;q_1,\ldots,q_{n-1};x_1,\ldots,x_n]$ with the property that, for all $N \geq n$, $\mathfrak{S}_w^q(x,t)$ represents the Schubert class σ_w in the ring

$$QH_T^*(Fl(\boldsymbol{N})) = \mathbb{Z}[t_1,\ldots,t_N;q_1,\ldots,q_{N-1};x_1,\ldots,x_N]/I_T^q(N).$$

Proof:

As remarked above, in Lemma (3.3.11), the polynomial $\mathfrak{S}_w^q(x,t)$ is independent of n, so long as $w \in S_n$. That $\mathfrak{S}_w^q(x,t)$ represents σ_w is the content of Theorem (3.1.1), so the only question is uniqueness. This follows from Lemma (3.3.14) below.

As polynomials in x and q, the $e_k^q(l)$ are the same as the quantum elementary symmetric polynomials considered (and denoted E_k^l there); the following "quantum straightening lemma" therefore applies without change.

Lemma (3.3.13) [3]:

For $0 \leq j, \kappa \leq l$, we have

$$e_j^q(l)e_{k+1}^q(l+1) + e_{j+1}^q(l)e_k^q(l) + q_l e_{j-1}^q(l-1)e_k^q(l)$$

$$= e_k^q(l)e_{j+1}^q(l+1) + e_{k+1}^q(l)e_j^q(l) + q_l e_{k-1}^q(l-1)e_j^q(l).$$

The straightening relation means that any monomial in the quantum elementary symmetric polynomials can be written as a $\mathbb{Z}[q]$-linear combination of "standard" monomials $e_{i_1}^q(1) \cdot e_{i_2}^q(2) \cdots e_{i_n}^q(n)$, with $0 \leq i_l \leq l$.

Lemma (3.3.14) [3]:

For each n, the following $\mathbb{Z}[q,t]$-submodules of $\mathbb{Z}[q,t,x]$ are the same:

(i) the submodule spanned by equivariant quantum Schubert polynomials $\mathfrak{S}_w^q(x,t)$, for $w \in S_n$;
(ii) the submodule spanned by ordinary (single) quantum Schubert polynomials $\mathfrak{S}_w^q(x)$, for $w \in S_n$;
(iii) the submodule spanned by the monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \leq n - i$.

Moreover, in each case the submodule is free and the indicated spanning set is a basis.

Proof:

The argument is the same as in (3) and (13). From equation (7) defining double universal Schubert polynomials, we have

$$\mathfrak{S}_w^q(x,t) = \mathfrak{S}_w^q(x) + R,$$

where R lies in the $\mathbb{Z}[q,t]$-submodule spanned by Schubert polynomials $\mathfrak{S}_u^q(x)$ with $u < w$. It follows that the equivariant quantum Schubert polynomials lie in the span of the ordinary quantum Schubert polynomials, and since the transition matrix between these two bases is uni-triangular, the submodules are the same. The equality of the last two spans follows.

Taking n to infinity, it follows from Lemma (3.3.16) that the equivariant quantum Schubert polynomials $\mathfrak{S}_w^q(x,t)$ form a $\mathbb{Z}[q,t]$-basis for $\mathbb{Z}[q,t,x]$, as w runs over S_∞, and all three sets of variables are infinite. This means that any product $\mathfrak{S}_u^q \cdot \mathfrak{S}_v^q$ can be expanded as a $\mathbb{Z}[q,t]$-linear combination of equivariant quantum Schubert polynomials. We conclude with an observation about these products.

It will be convenient to have notation for the polynomial ring: let

$$R_N = \mathbb{Z}[t_1,\ldots,t_N; q_1,\ldots,q_{N-1}; x_1,\ldots,x_N].$$

Lemma (3.3.15) [3]:

For $n < N$, consider the standard embedding $S_n \subset S_N$. Let $w \in S_N$, so $\mathfrak{S}_w^q = \mathfrak{S}_w^q(x,t)$ is a polynomial in R_N. Let J be the ideal

$$J = I_T^q(n) \cdot R_N + (t_{n+1},\ldots,t_N) \cdot R_N + (q_n,\ldots,q_{N-1}) \cdot R_N + (x_{n+1},\ldots,x_N) \cdot R_N.$$

If $w \notin Sn$, the polynomial $\mathfrak{S}_w^q$ lies in J. Equivalently, $\mathfrak{S}_w^q$ maps to zero in $QH_T^*Fl(\mathbb{C}^n) \cong R_N/J$.

Proof:

Consider a quot scheme $\mathcal{Q}_{\boldsymbol{d}}$ compactifying maps to $Fl(n)$. For any $u \in S_N$ and $v \in S_n$, the proof of Lemma (3.3.8) shows that $(\mu_u \cdot \tilde{\mu}_{v^\vee})_{\boldsymbol{d}}^T = 0$ unless $\boldsymbol{d} = \boldsymbol{0}$ and $u = v$. Since $w \notin S_n$, we have

$$(\beta(\mathfrak{S}_w(g,t)) \cdot \tilde{\mu}_{v^\vee})^T_{\boldsymbol{d}} = (\mu_w \cdot \tilde{\mu}_{v^\vee})^T_{\boldsymbol{d}} = 0$$

for all $\boldsymbol{d}$, where for $i, j > n$ and $k > n-1$, we set the extra variables $t_i = Q_j[0] = Q_k[1] = 0$ in $\beta(\mathfrak{S}_w(g,t))$. The lemma now follows from Proposition (3.3.6).

An immediate consequence is that the equivariant quantum Schubert polynomials multiply like Schubert classes in $QH^*_T Fl(\mathbb{C}^n)$.

Corollary (3.3.16):

Given permutations $u, v \in S_n$, expand the product of equivariant quantum Schubert polynomials as

$$\mathfrak{S}^q_u(x,t) \cdot \mathfrak{S}^q_v(x,t) = \sum_w a_w \mathfrak{S}^q_w(x,t),$$

with $w \in S_\infty$ and $a_w \in \mathbb{Z}[q,t]$. Then the coefficient of $\boldsymbol{q}^{\boldsymbol{d}}$ in a_w is equal to $c^{w,\boldsymbol{d}}_{u,v}$ when $w \in S_n$.

In other words, the equivariant quantum product $\sigma_u \circ \sigma_v$ (in $QH^*_T Fl(\mathbb{C}^n)$) is equal to the product of the polynomials $\mathfrak{S}^q_u$ and $\mathfrak{S}^q_v$, after discarding the terms $a_w \mathfrak{S}^q_w$ for $w \notin S_n$.

For example, as polynomials in $\mathbb{Z}[x,q,t]$, we have (see [3])

$$\begin{aligned}\mathfrak{S}^q_{231} \cdot \mathfrak{S}^q_{231} = q_1(t_2 - t_1)\mathfrak{S}^q_{21} + (t_2 - t_1)(t_3 - t_1)\mathfrak{S}^q_{231} + q_2\mathfrak{S}^q_{312} \\ + (t_2 - t_1)\mathfrak{S}^q_{2413} + \mathfrak{S}^q_{3412},\end{aligned}$$

where $\mathfrak{S}^q_w = \mathfrak{S}^q_w(x,t)$. To get the corresponding product $\sigma_{231} \circ \sigma_{231}$ in $QH^*_T Fl(\mathbb{C}^3)$, simply discard the last two terms.

This provides an easy way to compute equivariant quantum products. Using Maple, we computed the full multiplication tables for $QH^*_T Fl(\mathbb{C}^n)$ for $n \leq 5$; individual products are also quickly computable for higher n. We include the $n = 3$ in [3].

Chapter 4

Operator Space Theory and Polynomials

The aim of this chapter is to start a metric theory of homogeneous polynomials in the category of operator spaces. For this purpose we take advantage of the basic fact that the space $P^m(E)$ of all m-homogeneous polynomials on a vector space E can be identified with the algebraic dual of the m-th symmetric tensor product $\bigotimes^{m,s} E$.

Section (4.1): Homogeneous λ-cb Polynomials

A basis-free abstract theory of polynomials on vector spaces was developed from 1909 onwards by Fréchet, Gâteaux, Michal and others, and independently also by Banach –at the same time as he invented the concept of normed spaces. Banach suggested the analysis of polynomials on normed spaces and had the intention of writing a book devoted to this non-linear part of his theory. Unfortunately he died in 1945 without realizing his project.

The definition of polynomials on vector spaces, which is now commonly accepted as the most elegant one, is a definition via multilinear mappings: A mapping p between two vector spaces E and F is called m-homogeneous polynomial, $p \in P^m(E;F)$, if there exists an m-linear mapping $\varphi: E \times \cdots \times E \to F, \varphi \in L(^m E;F)$, such that $\varphi(x,\ldots,x) = p(x)$ for all $x \in E$. Then there even is a unique symmetric m-linear mapping $\check{p}$ which on all diagonal elements $(x,\ldots,x)$ equals $p(x)$. This is a consequence of the so-called polarization formula which defines $p: E \times \cdots \times E \to F$ through

$$\check{p}(x_1,\ldots,x_m) = \frac{1}{m!\,2^m} \sum_{\delta_1,\ldots,\delta_m \in \{-1,1\}} \delta_1 \ldots \delta_m p\left(\sum_{k=1}^{m} \delta_k x_k\right). \qquad (1)$$

Symmetric m-linear mappings can be linearized by means of the m-fold symmetric tensor product (In mathematics, the tensor product, denoted by $\otimes$, may be applied in different contexts to vectors, matrices, tensors vector spaces, algebras topological vector spaces, and modules, among many other structures or objects. In each case the significance of the symbol is the same: the most general bilinear

operation. In same contexts this product is also referred to as outer product. The term "tensor product" is also used in relation to monoidal categories. The \ boxtimes variant of $\otimes$ is used in control theory.) [15]. In order to understand this define the symmetric m-linear mapping

$$\vee: E \times \ldots \times E \to \oplus^m E, \quad (x_1, \ldots, x_m) \mapsto \frac{1}{m!} \sum_{\eta \in \mathcal{L}_m} x_{\eta(1)} \otimes \ldots \otimes x_{\eta(m)}$$

where the sum is taken over all possible permutations η on $\{1, \ldots, m\}$ and $\otimes^m E$ stands for the m-fold tensor product of E. The m-fold symmetric tensor product $\otimes^{m,s} E$ then by definition is the linear span of the range of $\vee$. As a consequence, for every symmetric m-linear mapping $\varphi: E \times \cdots \times E \to F$ there is a unique linear mapping $\varphi^{L,s}: \otimes^{m,s} E \to F$ such that $\varphi^{L,s} \circ \vee = \varphi$. One ends up with the following algebraic identification:

$$P^m(E, F) = L(\otimes^{m,s} E; F),$$

$$p \longmapsto \check{p}^{L,s}. \tag{2}$$

Let now E and F be two normed spaces. Then the continuous analog of the preceding equality is slightly more complicated. Let $\mathscr{P}^m(E; F)$ be the linear space of all continuous m-homogeneous polynomials which together with

$$\|p\| := \sup_{\|x\|_E \leq 1} \|p(x)\|_F$$

again forms a normed space (clearly, we have $\mathscr{P}^1(E; F) = \mathscr{L}(E; F)$, the space of all bounded linear operators). Inspired by Grothendieck's famous "Résumé de la théorie métrique des produits tensoriels topologiques" Ryan developed the concept of studying continuous m-homogeneous polynomials on normed spaces by analyzing symmetric tensor products. Define for $x \in \otimes^{m,s} E$ the projective symmetric tensor norm to be

$$\|x\|_{\pi_s} := \inf \left\{ \sum_j \|x_j\|^m | x = \sum_j \otimes^m x_j \right\}, \tag{3}$$

and denote the symmetric projective tensor product of E by $\otimes_{\pi_s}^{m,s} E$. Then it is just a short step to prove the following continuous analog of (2):

$$\mathscr{P}^m(E;F) = \mathscr{L}\left(\otimes_{\pi_s}^{m,s} E; F\right) \text{ isometrically.} \tag{4}$$

The main aim is to study this equation within the concept of operator spaces. By now operator space theory is developing quickly and has turned out to be a potent part of functional analysis. Recall that an operator space V by definition is a normed space together with an isometric linear embedding into the space $\mathscr{L}(H)$ of all (bounded) operators on a Hilbert space H. Equivalently, an operator space V is a vector space together with a so-called operator space matrix norm, a sequence of norms $(\|\cdot\|_n)_n$ on the spaces $M_n(V)$ of all $n \times n$ matrices with entries in V, satisfying some regularity conditions. In particular, the so-called completely bounded mappings turned out to be the natural morphisms in the category of operator spaces. Recall that, given two operator spaces V and W, every linear mapping $\varphi: V \to W$ induces for every k a linear mapping $\varphi_k: M_k(V) \to M_k(W)$ on the k-th matrix level, the so-called k-th amplification of φ, by entrywise application $\varphi_k([x_{i,j}]_{i,j}) := [\varphi(x_{i,j})]_{i,j}$. The mapping φ is called completely bounded if all these amplifications are uniformly bounded, i.e. if $\sup_k \|\varphi_k\| < \infty$.

As operator spaces form the "quantized" or "non-commutative" analogues of normed spaces, naturally the question arises how to create a theory of completely bounded m-homogeneous polynomials on operator spaces. The purpose of this article is to start such a theory with the main aim to obtain a completely bounded version of (4). Here we are confronted with the problems of finding proper operator space matrix norms on the space of m-homogeneous polynomials and as well on the symmetric m-fold tensor product.

All the Banach spaces E we consider are assumed to be complex. We denote their open unit balls by B_E and their duals by E'. By $\mathscr{L}(E;F)$ resp. $\mathscr{L}(E_1,\ldots,E_m;F)$ we denote the Banach space of all bounded linear operators from E to F resp. all bounded m-linear mappings from $E_1 \times\cdots\times E_m$ into F endowed with their usual sup norms. We write $\mathscr{L}(^mE;\ F)$ whenever all E_i equal E, and $\mathscr{L}(E_1,\ldots,E_m)$ resp. $\mathscr{L}$ (^mE) for forms.

Homogeneous polynomials, recall from the introduction the definition of $P^m(E;F)$ for linear spaces E,F, and the definition of the normed space $\mathscr{P}^m(E;F)$ whenever E,F are normed. $P^m(E)$ and $\mathscr{P}^m(E)$ denote the corresponding spaces of scalar valued polynomials. For the definition of $\mathscr{P}^m(E;F)$, all integral m-homogeneous polynomials.

We denote by $\bigotimes^m E_i$ the m-th full tensor product of m vector spaces $E_1,\dots,E_m$, and write $\bigotimes^m E$ whenever $E = E_1 = \cdots = E_m$. Recall from the introduction the definition of the m-th symmetric tensor product $\bigotimes^{m,s} E$. We write $\iota_E^m : \bigotimes^{m,s} E \to \bigotimes^m E$ for the canonical embedding and $\sigma_E^m : \bigotimes^m E \longrightarrow \bigotimes^{m,s} E$ for the canonical projection (defined in (4.2)). If $\bigotimes^m E_i$ is endowed with the projective norm $\alpha = \|\cdot\|_\pi$ or injective norm $\|\cdot\|_\varepsilon$, respectively, then we write $\bigotimes_\alpha^m E_i$, whereas the symmetric tensor product together with the symmetric projective norm $\|\cdot\|_{\pi_s}$ or injective symmetric norm $\|\cdot\|_{\varepsilon_s}$ is denoted by $\bigotimes_\alpha^{m,s} E$, respectively.

For all needed information on operator space theory. For a vector space E we will denote the space of m by n matrices $[v_{i,j}]_{i,j}$ with $v_{i,j} \in E$ by $M_{m,n}(E)$, and $M_n(E) := M_{n,n}(E), M_{m,n} := M_{m,n}(\mathbb{C}), M_n := M_{n,n}(\mathbb{C})$. Furthermore we will generally use the identification $M_{m,n}(M_{p,q}(E)) = M_{mp,nq}(E)$. For matrices $A,B \in M_n$ we write $A \bullet B \in M_n$ for their Schur product, $A \otimes B \in M_{n^2}$ for their tensor = Kronecker product, and $A \odot B \in M_n$ for their (ordinary) matrix product. As usual, for $x \in M_{m,n}(E)$ and $y \in M_{p,q}(E)$

$$x \oplus y := \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in M_{m+p,n+q}(E).$$

By $\varepsilon_{i,j} := \varepsilon_{i,j}^{[k,l]} \in M_{k,l}$ we denote the matrix which is 1 in the (i,j)-th coordinate and zero elsewhere. Moreover, $\varepsilon_{i,j}^{[k]} := \varepsilon_{i,j}^{[k,k]} \dots, \varepsilon_i := \varepsilon_{i,i}$, and $\varepsilon_i^{[k]} := \varepsilon_{i,i}^{[k,k]}$. For all $\kappa \in N$ define $I_k := \sum_{i=1}^k \varepsilon_i^{[k]}$ as well as $e_j := e_j^{[k]} := \varepsilon_{j,1}^{[k,1]}$. Clearly we have

$$\varepsilon_{i,j}^{[p,q]} \odot \varepsilon_{k,l}^{[q,r]} = \begin{cases} \varepsilon_{i,l}^{[p,r]} & \text{if } j = k \\ 0 & \text{else.} \end{cases} \tag{5}$$

An operator space V is a vector space V together with a matrix norm $\|\cdot\|$, i.e. a sequence of norms $\|\cdot\|_p$ on $M_p(V)$ such that for all $v \in M_m(V), w \in M_n(V)$ and $\alpha \in M_{n,m}, \beta \in M_{m,n}$ we have

$$(M1)\|v \oplus w\|_{m+n} \leq \max\{\|v\|_m, \|w\|_n\} \quad \text{and} \quad (M2)\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|.$$

Recall that each family of mappings $\|\cdot\|_p: M_p(V) \to \mathbb{R}_{\geq 0}$ which satisfies (**M1**) and (**M2**) consists of norms on $M_p(V)$ provided $\|\cdot\|_1$ is a norm on V. Let V, W be operator spaces and let $\varphi: V \to W$ be a linear mapping. For $k \in \mathbb{N}$ the k-th amplification of φ is defined by $\varphi_k: M_k(V) \to M_k(W), v = [v_{i,j}]_{i,j} \mapsto [\varphi(v_{i,j})]_{i,j}$, or in terms of tensor products $\varphi_k = \mathrm{id} \otimes \varphi$. Since φ_k is again a linear mapping, the completely bounded norm $\|\varphi\|_{cb} := \sup_k \|\varphi_k\|$ as well as the vector space $\mathscr{CB}(V; W) := \{\varphi \in L(V; W): \|\varphi\|_{cb} < \infty\}$ of completely bounded operators from V into W is well-defined. Note that by the identification $M_k(\mathscr{CB}(V; W)) = \mathscr{CB}(V; M_k(W))$ the space $\mathscr{CB}(V; W)$ is again an operator space. Finally, we mention that for any operator space V the operator space dual V^* is given through the isometric identification $M_k(V^*) = \mathscr{CB}(V; M_k)$

We start with the following simple but crucial observation: For each m-homogeneous polynomial in vector spaces, $p \in P^m(E; F)$, and each k the following mapping

$$p_{\bullet_k}: M_k(E) \to M_k(F), \quad p_{\bullet_k}\left([x_{i,j}]_{i,j}\right) = [p(x_{i,j})]_{i,j}$$

is again an m-homogeneous polynomial. Indeed, there is an m-linear mapping

$$\varphi: E^m \to F$$

which if restricted to the diagonal gives p. For this consider the m-fold Schur product

$$\bullet_\kappa: M_k \times \cdots \times M_\kappa \to M_\kappa, \quad \left(\alpha^{(1)}, \dots, \alpha^{(m)}\right) \mapsto \left[\alpha_{i,j}^{(1)} \dots \alpha_{i,j}^{(m)}\right]_{i,j}$$

of $\kappa \times \kappa$-matrices. Tensorizing $\bullet_\kappa$ with φ leads to the m-linear mapping

$$\varphi_{\bullet_\kappa} := \bullet_\kappa \otimes \varphi: M_k(E) \times \cdots \times M_k(E) \to M_k(F),$$

$$(\alpha_1 \otimes v_1, \ldots, \alpha_m \otimes v_m) \mapsto \bullet_\kappa (\alpha_1, \ldots, \alpha_m) \otimes \varphi(v_1, \ldots, v_m)$$

which if restricted to its diagonal gives as desired $p_{\bullet_\kappa}$. If we now replace the vector spaces E, F by operator spaces V, W, then we can define the normed space

$$\mathscr{p}^m_\bullet(V;W) := \left\{p \in P^m(V;W) : \|p\|_\bullet := \sup_k \|p_{\bullet_\kappa}\| < \infty\right\}.$$

the space of all so-called m-homogeneous Schur cb-polynomials from V into W; for the proof that this in fact is a linear space and $\|\cdot\|_\bullet$ in fact is a norm use that $(p+q)_{\bullet_\kappa} = p_{\bullet_\kappa} + q_{\bullet_\kappa}, (\mu p)_{\bullet_\kappa} = \mu p_{\bullet_\kappa}$ and $\|p\|_\bullet = \sup_k \|p_{\bullet_\kappa}\| \geq \|p_{\bullet_1}\| = \|p\|$. Let us define an operator space structure on $\mathscr{P}^m_\bullet(V;W)$. For each k we use the linear bijection

$$\Phi: M_k\big(P^m(V;W)\big) \to P^m\big(V; M_k(W)\big), \quad \Phi\big([p_{i,j}]\big)(v) := [p_{i,j}(v)].$$

Since Φ maps $M_k(\mathscr{P}^m_\bullet(V;W))$ onto $\mathscr{P}^m_\bullet(V;M_kW))$, the following definition

$$\|u\|_k := \|u\|_{M_k(\mathscr{p}^m_\bullet(V;W))} := \|\Phi(u)\|_{\mathscr{p}^m_\bullet(V;M_k(W))}, \quad u \in M_k\big(\mathscr{p}^m_\bullet(V;W)\big)$$

leads to a sequence of matrix norms on $\mathscr{p}^m_\bullet(V;W)$. That these norms in fact define an operator structure on $\mathscr{p}^m_\bullet(V;W)$ is a corollary of the more general Proposition (4.1.3). In this proposition we take a general point of view which we now explain. For each fixed m we will define a whole class of m-homogeneous cb-polynomials in operator spaces which includes the Schur polynomials as a special case.

Fix a sequence of matrix products $\lambda = (\lambda_n)_{n\in\mathbb{N}}$, i.e. for each k some m-linear mapping

$$\lambda_k: M_k \times \cdots \times M_k \to M_{\tau(k)},$$

where $\tau(k) \in \mathbb{N}$ is a natural number only depending on k; in order to avoid pathologies we will always assume that $\tau(1) = 1$ and $\lambda_1(1,\ldots,1) = 1$. Our theory focuses on the following three examples.

Example (4.1.1) [4]:

For $\alpha_1, \ldots, \alpha_m \in M_k$ we consider the

(i) Schur product: $\bullet_\kappa (\alpha_1, \ldots, \alpha_m) = \alpha_1 \bullet \cdots \bullet \alpha_m$ with $\tau(k) = k$

(ii) Kronecker product: $\otimes_k (\alpha_1, \dots, \alpha_m) = \alpha_1 \otimes \cdots \otimes \alpha_m$ with $\tau(k) = k^m$
(iii) Matrix product: $\odot_k(\alpha_1, \dots, \alpha_m) = \alpha_1 \odot \cdots \odot \alpha_m$ with $\tau(k) = k$

Now for each such λ we define a class of completely bounded m-homogeneous polynomials in operator spaces. Assume first that $p \in P^m(E;F), E, F$ two vector spaces. Then we know from (1) that for each choice of finitely many scalars $\alpha_1, \dots, \alpha_l$ and vectors $x_1, \dots, x_l \in E$ we have

$$p\left(\sum_{i=1}^{l} \alpha_i x_i\right) = \frac{1}{m!\,2^m} \sum_{\delta_1,\dots,\delta_m \in \{-1,1\}} \delta_1 \dots \delta_m \sum_{i_1,\dots,i_m} \alpha_{i_1} \dots \alpha_{i_m}\, p\left(\sum_{j=1}^{m} \delta_j x_{i_j}\right).$$

Since one of the basic philosophies of operator theory (at least if compared with classical Banach space theory) is to "replace scalars by matrices" we define for each k

$$p_{\lambda_k}: M_k(E) \to M_{\tau(k)}(F)$$

by

$$p_{\lambda_k}\left(\sum_{i=1}^{l} \alpha_i \otimes v_i\right) := \frac{1}{m!\,2^m} \sum_{\delta_1,\dots,\delta_m \in \{-1,1\}} \delta_1 \dots \delta_m \sum_{i_1,\dots,i_m} \lambda_k(\alpha_{i_1}, \dots, \alpha_{i_m}) \otimes p\left(\sum_{j=1}^{m} \delta_j v_{i_j}\right).$$

Exactly as in the case of Schur multiplication, $p_{\lambda_k} \in P^m(M_k(E), M_{\tau(k)}(F))$ turns out to be an m-homogeneous polynomial: Let φ be the unique symmetric m-linear mapping associated to p. Then tensorizing λ_k with φ leads to the m-linear mapping

$$\varphi_{\lambda_k} := \lambda_k \otimes \varphi: M_k(E) \times \cdots \times M_k(E) \longrightarrow M_{\tau(k)}(F),$$

$$(\alpha_1 \otimes v_1, \dots, \alpha_m \otimes v_m) \longmapsto \lambda_k(\alpha_1, \dots, \alpha_m) \otimes \varphi(v_1, \dots, v_m),$$

which if restricted to its diagonal gives again p_{λ_k}.

If V, W are operator spaces, then

$$\|p\|_\lambda := \sup_k \|p_{\lambda_k}\| \tag{6}$$

defines a norm on the vector space

$$\mathscr{P}_{\lambda}^{m}(V;W) := \{p \in P^{m}(V;W) : \|p\|_{\lambda} < \infty\} \tag{7}$$

of all λ-cb m-homogeneous polynomials from V into W. Indeed, it is easy to see that $\|\cdot\|_{\lambda}$ is homogeneous and satisfies the triangle inequality, whereas the fact that $\|\cdot\|_{\lambda}$ is positive definite is a consequence of $\|p\|_{\lambda} \geq \|p\lambda_1\| = \|p\|$. As above we use the identification

$$M_k(\mathscr{P}_{\lambda}^{m}(V;W)) = \mathscr{P}_{\lambda}^{m}(V;M_k(W)) \tag{8}$$

to implement an operator space structure on $\mathscr{P}_{\lambda}^{m}(V;W)$.

Lemma (4.1.2) [4]:

Let $p \in M_r(\mathscr{P}_{\lambda}^{m}(V;W)), q \in M_s(\mathscr{P}_{\lambda}^{m}(V;W)$ and $k \in \mathbb{N}$.

(i) There exist permutation matrices P, Q (each column and row is identically zero except in one coordinate which is 1) such that

$$(p \oplus q)_{\lambda_k} = P(p_{\lambda_k} \oplus q_{\lambda_k})Q.$$

(ii) For $\alpha \in M_{t,r}$ and $\beta \in M_{r,t}$

$$(\alpha p \beta)_{\lambda_k} = \mathrm{diag}(\alpha, \ldots, \alpha) p_{\lambda_k} \mathrm{diag}(\beta, \ldots, \beta).$$

Proof:

Take $\check{p}$ and $\check{q}$, the unique symmetric m-linear mappings associated to p and q, respectively, and fix some

$$\upsilon = \sum_{i} \alpha_i \otimes \upsilon_i \in M_k(V).$$

For the proof of (1) we will use the simple fact that for $t, r, s \in \mathbb{N}$ there are permutation matrices P, Q such that for each choice of $A \in M_t, W_1 \in M_r(W)$ and $W_2 \in M_s(W)$ we have

$$A \otimes (W_1 \oplus W_2) = P((A \otimes W_1) \oplus (A \otimes W_2))Q.$$

Accordingly, we know that there are permutation matrices P, Q for which

$$(p \oplus q)_{\lambda_k}(v) = (\check{p} \oplus \check{q})_{\lambda_k}\left(\sum_{i_1} \alpha_{i_1} \otimes v_{i_1}, \ldots, \sum_{i_m} \alpha_{i_m} \otimes v_{i_m}\right)$$

$$= \sum_{i_1} \cdots \sum_{i_m} \lambda_k(\alpha_{i_1}, \ldots, \alpha_{i_m}) \otimes (\check{p} \oplus \check{q})(v_{i_1}, \ldots, v_{i_m})$$

$$= \sum_{i_1} \cdots \sum_{i_m} P\Big(\big(\lambda_k(\alpha_{i_1}, \ldots, \alpha_{i_m}) \otimes \check{p}(v_{i_1}, \ldots, v_{i_m})\big)$$

$$\oplus \big(\lambda_k(\alpha_{i_1}, \ldots, \alpha_{i_m}) \otimes \check{q}(v_{i_1}, \ldots, v_{i_m})\big)\Big) Q$$

$$= P\left((\check{p}_{\lambda_k} \oplus \check{q}_{\lambda_k})\left(\sum_{i_1} \alpha_{i_1} \otimes v_{i_1}, \ldots, \sum_{i_m} \alpha_{i_m} \otimes v_{i_m}\right)\right) Q$$

$$= \big(P(p_{\lambda_k} \oplus q_{\lambda_k})Q\big)(v).$$

This proves (1) – for the proof of (2) look at

$$(\alpha p \beta)_{\lambda_k}(v) = (\alpha \check{p} \beta)_{\lambda_k}\left(\sum_{i_1} \alpha_{i_1} \otimes v_{i_1}, \ldots, \sum_{i_m} \alpha_{i_m} \otimes v_{i_m}\right)$$

$$= \sum_{i_1} \ldots \sum_{i_m} \lambda_k(\alpha_{i_1}, \ldots, \alpha_{i_m}) \otimes (\alpha \check{p} \beta)(v_{i_1}, \ldots, v_{i_m})$$

$$\sum_{i_1} \ldots \sum_{i_m} \operatorname{diag}(\alpha, \ldots, \alpha)\left(\lambda_k(\alpha_{i_1}, \ldots, \alpha_{i_m}) \otimes \check{p}(v_{i_1}, \ldots, v_{i_m})\right) \operatorname{diag}(\beta, \ldots, \beta)$$

$$= \operatorname{diag}(\alpha, \ldots, \alpha)\check{p}_{\lambda_k}(v, \ldots, v)\operatorname{diag}(\beta, \ldots, \beta)$$

$$= \operatorname{diag}(\alpha, \ldots, \alpha)p_{\lambda_k}(v)\operatorname{diag}(\beta, \ldots, \beta).$$

This completes the proof.

Proposition (4.1.3) [4]:

$\mathscr{P}_{\lambda}^{m}(V; W)$ is an operator space (the definitions given in (7) and (8)).

Proof:

Since we already know that $\mathscr{P}_{\lambda}^{m}(V;W)$ is a normed space, it suffices to show (M1) and (M2). For (M1) take $p \in M_r(\mathscr{P}_{\lambda}^{m}(V;W))$ and $q \in M_s(\mathscr{P}_{\lambda}^{m}(V;W))$, then by the preceding lemma we see that

$$\|p \oplus q\|_{M_{r+s}\left(\mathscr{P}_{\lambda}^{m}(V;W)\right)} = \|p \oplus q\|_{\mathscr{P}_{\lambda}^{m}(V;M_{r+s}(W))}$$

$$= \sup_{k} \sup_{\|v\|_{M_k(V)} \leq 1} \left\|(p \oplus q)_{\lambda_k}(v)\right\|_{M_{\tau(k)}(M_{r+s}(W))}$$

$$\leq \sup_{k} \sup_{\|v\|_{M_k(V)} \leq 1} \left\|\left(p_{\lambda_k} \oplus q_{\lambda_k}\right)(v)\right\|_{M_{\tau(k)(r+s)}(W)}$$

$$= \sup_{k} \sup_{\|v\|_{M_k(V)} \leq 1} \max\left\{\left\|p_{\lambda_k}(v)\right\|_{M_{\tau(k)r}(W)}, \left\|q_{\lambda_k}(v)\right\|_{M_{\tau(k)s}}(W)\right\}$$

$$= \max\left\{\|p\|_{M_r\left(\mathscr{P}_{\lambda}^{m}(V;W)\right)}, \|q\|_{M_s\left(\mathscr{P}_{\lambda}^{m}(V;W)\right)}\right\}.$$

In order to prove (M2) let $\alpha \in M_{t,r}, \beta \in M_{r,t}$ and $p \in M_r(\mathscr{P}_{\lambda}^{m}(V;W))$. Then we have

$$\|\alpha p \beta\|_{M_t\left(\mathscr{P}_{\lambda}^{m}(V;W)\right)} = \|\alpha p \beta\|_{\mathscr{P}_{\lambda}^{m}(V;M_t(W))}$$

$$\sup_{k} \sup_{\|v_i\|_{M_k(V)} \leq 1} \left\|(\alpha p \beta)_{\lambda_k}(v)\right\|_{M_{\tau(k)}(M_t(W))}$$

$$\leq \sup_{k} \sup_{\|v\|_{M_k(V)} \leq 1} \left\|\mathrm{diag}(\alpha, \dots, \alpha) p_{\lambda_k}(v) \mathrm{diag}(\beta, \dots, \beta)\right\|_{M_{\tau(k)t}(W)}$$

$$\leq \sup_{k} \sup_{\|v\|_{M_k(V)} \leq 1} \|\mathrm{diag}(\alpha, \dots, \alpha)\| \left\|p_{\lambda_k}(v)\right\|_{M_{\tau(k)t}(W)} \|\mathrm{diag}(\beta, \dots, \beta)\|$$

$$= \|\alpha\| \|p\|_{M_r\left(\mathscr{P}_{\lambda}^{m}(V;W)\right)} \|\beta\|,$$

which completes the proof.

We obtain the following particular cases.

Example (4.1.4) [4]:

The following spaces of m-homogeneous λ-cb polynomials form operator spaces:

(i) $\mathscr{P}^{m}_{\bullet}(V;W)$, the Schur polynomials
(ii) $\mathscr{P}^{m}_{\odot}(V;W)$, the Haagerup polynomials
(iii) $\mathscr{P}^{m}_{\otimes}(V;W)$, the Kronecker polynomials

Section (4.2): Symmetric Tensor Products and Representation of m-Homogeneous λ-cb Polynomials

Fix again a sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ of m-linear mappings,

$$\lambda_k : M_k \times\cdots\times M_k \longrightarrow M_\tau(k),$$

where $\tau(k) = \tau_\lambda(k)$ is only depending on k, and define for a given operator space V and $u \in M_k(\otimes^{m,s} V)$

$$\|u\|_{\lambda,k} := \inf \|\alpha\| \|v\|^m \|\beta\| \in [0,\infty],$$

where the infimum is taken over arbitrary decompositions $u = \alpha \vee \lambda_j(v,\ldots,v)\beta$ with $\alpha \in M_{k,\tau(j)}, \beta \in M_{\tau(j),k}$ and $v \in M_j(V)$.

We now isolate three technical conditions on the family $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ which will assure that the $\|\cdot\|_{\lambda,k}$ generate an operator space structure on $\otimes^{m,s} V$:

(E1) For all $k \in \mathbb{N}$ there exist $p \in \mathbb{N}$ and matrices $S \in M_{k,\tau(p)}, T \in M_{\tau(p),k}, \alpha_1,\ldots,\alpha_k \in M_p$ such that for all $j_1,\ldots,j_m \in \{1,\ldots,k\}$:

$$S\,\lambda_p(\alpha_{j_1},\ldots,\alpha_{j_m})T = \begin{cases} \varepsilon_j^{[k]} & \text{if } j_1 = \cdots = j_m =: j, \\ 0 & \text{else.} \end{cases}$$

(E2) For all $r,s \in \mathbb{N}$ there exist matrices $P \in M_{\tau(r)+\tau(s),\tau(r+s)}, Q \in M_{\tau(r+s),\tau(r)+\tau(s)}$ with $\|P\|,\|Q\| \le 1$ such that for all $(i_k,j_k) \in \{1,\ldots,r\}^2 \cup \{r+1,\ldots,r+s\}^2$ with $1 \le k \le m$:

$$P\lambda_{r+s}\left(\varepsilon_{i_1,j_1}^{[r+s]},\ldots,\varepsilon_{i_m,j_m}^{[r+s]}\right)Q = \operatorname{diag}\left(\lambda_r\left(\varepsilon_{i_1,j_1}^{[r]},\ldots,\varepsilon_{i_m,j_m}^{[r]}\right), \lambda_s\left(\varepsilon_{i_1-r,j_1-r}^{[s]},\ldots,\varepsilon_{i_m-r,j_m-r}^{[s]}\right)\right).$$

(E3) $\lambda_1(1,\ldots,1) = 1$ and $\sup_{k\in\mathbb{N}}\left\|\lambda_k : M_k \times\cdots\times M_k \to M_{\tau(k)}\right\| < \infty$

Here the property (E1) guarantees a certain abundance of λ as it prevents the images of the λ_k's from being too sparse, and (E3) anyway seems natural. With regard to property (E2) we have to distinguish three cases: If $(i_k,j_k) \in \{1,\ldots,r\}^2$ for all $1 \le k \le m$, then the property (E2) gives

$$P\lambda_{r+s}\left(\varepsilon_{i_1,j_1}^{[r+s]},\dots,\varepsilon_{i_m,j_m}^{[r+s]}\right)Q=\operatorname{diag}\left(\lambda_r\left(\varepsilon_{i_1,j_1}^{[r]},\dots,\varepsilon_{i_m,j_m}^{[r]}\right),0\right).$$

In the case that $(i_k,j_k)\in\{r+1,\dots,r+s\}^2$ for all $1\le k\le m$ we get

$$P\lambda_{r+s}\left(\varepsilon_{i_1,j_1}^{[r+s]},\dots,\varepsilon_{i_m,j_m}^{[r+s]}\right)Q=\operatorname{diag}\left(0,\lambda_s\left(\varepsilon_{i_1-r,j_1-r}^{[s]},\dots,\varepsilon_{i_m-r,j_m-r}^{[s]}\right)\right).$$

Finally, if there are p and q such that $(i_p,j_p)\in\{1,\dots,r\}^2$ and $(i_q,j_q)\in\{r+1,\dots,r+s\}^2$, then we have

$$P\lambda_{r+s}\left(\varepsilon_{i_1,j_1}^{[r+s]},\dots,\varepsilon_{i_m,j_m}^{[r+s]}\right)Q=0.$$

We remark that condition (E2) (in view of our examples) could also be modified by demanding that for each r,s and each choice of appropriate matrices $a_1,\dots,a_n,b_1,\dots,b_n$ there are matrices P,Q like above such that

$$P\lambda_{r+s}(a_1\oplus b_1,\dots,a_n\oplus b_n)Q=\lambda_r(a_1,\dots,a_n)\oplus\lambda_s(b_1,\dots,b_n).$$

Proposition (4.2.1) [4]:

If λ fulfills (E1)–(E3), then the sequence $(\|\cdot\|_{\lambda,\mathrm{n}})_{\mathrm{n}}$ defines the operator space

$$\otimes_\lambda^{m,s}V:=(\otimes^{m,s}V,\|\cdot\|_\lambda)$$

Proof:

We start proving that for any given $u=[u_{i,j}]_{i,j}\in M_k(\otimes^{m,s}V)$ there exists at least one decomposition in the postulated form. Note first that there is a representation

$$u=\gamma\left(\sum_{q=1}^{l}\varepsilon_q^{[l]}\otimes\left(w_q\vee\dots\vee w_q\right)\right)\delta \tag{9}$$

with matrices $\gamma\in M_{k,l},\delta\in M_{l,k}$ and $w_q\in V$: indeed, we use the fact that for each $v\in\otimes^{m,s}V$ there exists a representation $v=\sum_{r=1}^{s}v_r\vee\dots\vee v_r$ with $v_i\in V$. Therefore we can write

$$v=\iota_s\operatorname{diag}(v_1\vee\dots\vee v_1,\dots,v_s\vee\dots\vee v_s)\iota_s^t$$

with $\iota_s := [1 \cdots 1] \in M_{1,s}$. So there is obviously a decomposition $v = \iota_s D \iota_s^t$ where D is a diagonal matrix whose entries are symmetric elementary tensors. Take now $u = [u_{i,j}]_{i,j} \in M_k(\bigotimes^{m,s} V)$ and define $u_{(i-1)k+j} := u_{i,j}$. Then there exist matrices $\gamma_0 \in M_{k,k^2}, \delta_0 \in M_{k^2,k}$ such that

$$u = \gamma_0 \mathrm{diag}(u_1, \ldots, u_{k^2})\delta_0$$

as we can take $\gamma_0 := \mathrm{diag}(\iota_k, \ldots, \iota_k)$ and $\delta_0 := [I_k \ldots I_k]^t$. Accordingly we have

$$u = \gamma_0 \mathrm{diag}(u_1, \ldots, u_{k^2})\delta_0$$

$$= \gamma_0 \mathrm{diag}\left(\iota_{s_1} D_1 \iota_{s_1}^t, \ldots, \iota_{s_{k^2}} D_{k^2} \iota_{s_{k^2}}^t\right)\delta_0$$

$$= \gamma_0 \mathrm{diag}(\iota_{s_1}, \ldots, \iota_{s_{k^2}})\mathrm{diag}(D_1, \ldots, D_{k^2})\mathrm{diag}\left(\iota_{s_1}^t, \ldots, \iota_{s_{k^2}}^t\right)\delta_0.$$

Note that $D_0 := \mathrm{diag}(D_1, \ldots, D_{k^2})$ is again a diagonal matrix whose entries are symmetric elementary tensors so that $D_0 = \mathrm{diag}(d_1, \ldots, d_l)$ with $d_p = w_p \vee \cdots \vee w_p$ and $w_p \in V$. With $\gamma := \gamma_0 diag(\iota_{s_1}, \ldots, \iota_{s_{k^2}})$ and $\delta := \mathrm{diag}(\iota_{s_1}^t, \ldots, \iota_{s_{k^2}}^t)\delta_0$ we see that (9) holds true. Finally, from (E1) we deduce that there are $p \in \mathbb{N}$ and appropriate matrices $S, T, \alpha_1, \ldots, \alpha_l$ such that

$$u = \gamma\left(\sum_{q=1}^{l} \varepsilon_q^{[l]} \otimes (w_q \vee \ldots \vee w_q)\right)\delta$$

$$\overset{(E1)}{=} \gamma\left(\sum_{q_1=1}^{l} \ldots \sum_{q_m}^{l} S\lambda_p(\alpha_{q_1}, \ldots, \alpha_{q_m})T \otimes (w_{q_1} \vee \ldots \vee w_{q_m})\right)\delta$$

$$\gamma S\left(\vee \lambda_p\left(\sum_{q=1}^{l} \alpha_q \otimes w_q, \ldots, \sum_{q=1}^{l} \alpha_q \otimes w_q\right)\right)T\delta,$$

which gives the desired representation of u.

It remains to check that $\|\lambda\|$ satisfies (M1) and (M2), and that $\|\cdot\|_{\lambda,1}$ is a norm.

(M1): Take $u_1 \in M_r(\otimes^{m,s} V)$ and $u_2 \in M_s(\otimes^{m,s} V)$. Then for each $\varepsilon > 0$ there exist representations $u_1 = \alpha_1 \vee \lambda_k(v,\dots,v)\beta_1$ as well as $u_2 = \alpha_2 \vee \lambda_l(w,\dots,w)\beta_2$ with $v \in M_k(V), w \in M_l(V)$ and appropriate scalar matrices $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $\|\alpha_1\|\|v\|^m\|\beta_1\| \leq \|u_1\|_\lambda + \varepsilon$ as well as $\|\alpha_2\|\|w\|^m\|\beta_2\| \leq \|u_2\|_\lambda + \varepsilon$. Without loss of generality we can take $\|v\| = \|w\| = 1$ and $\|\alpha_j\| = \|\beta_j\|$ so consequently $\|\alpha_j\| \leq \|\beta_j\|(\|u_j\|_\lambda + \varepsilon)^{\frac{1}{2}}$; indeed, if $u = \mu \vee \lambda_l(w,\dots,w)\nu \neq 0$ with $\mu \in M_{k,\tau(l)}, \nu \in M_{\tau(l),k}, w \in M_l(V)$ then $u = \mu\|w\|^m \vee \lambda_l\left(\frac{w}{\|w\|},\dots,\frac{w}{\|w\|}\right)\nu$. With $\tilde{\mu} := \mu\|w\|^m, \alpha := \|\nu\|^{\frac{1}{2}}\|\tilde{\mu}\|^{-\frac{1}{2}}\tilde{\mu}, \beta := \|\tilde{\mu}\|^{\frac{1}{2}}\|\nu\|^{-\frac{1}{2}}\nu$ and $v := \frac{w}{\|w\|}$ we get

$$u = \mu \vee \lambda_l(w,\dots,w)\nu = \tilde{\mu} \vee \lambda_l\left(\frac{w}{\|w\|},\dots,\frac{w}{\|w\|}\right)\nu = \alpha \vee \lambda_l(v,\dots,v)\beta,$$

where $\|v\| = 1, \|\alpha\| = \|\tilde{\mu}\|^{\frac{1}{2}}\|\nu\|^{\frac{1}{2}} = \|\beta\|$ and $\|\alpha\|\|v\|^m\|\beta\| = \|\mu\|\|w\|^m\|\nu\|$. Let now $R := \{1,\dots,r\}^2, S := \{r+1,\dots,r+s\}^2, x := \mathrm{diag}(v,w)$ and $\alpha := \mathrm{diag}(\alpha_1,\alpha_2), \beta := \mathrm{diag}(\beta_1,\beta_2)$. As $v = \sum_{(i,j)\in\mathcal{R}} \varepsilon_{i,j} \otimes x_{i,j}$ and $w = \sum_{(i,j)\in S} \varepsilon_{i-r,j-r} \otimes x_{i,j}$ from (E2) we deduce that there are matrices P and Q such that

$$u_1 \oplus u_2 = \mathrm{diag}(\alpha_1 \vee \lambda_k(v,\dots,v)\beta_1, \alpha_2 \vee \lambda_l(w,\dots,w)\beta_2)$$

$$= \alpha\left(\mathrm{diag}(\vee\lambda_k(v,\dots,v),0) + \mathrm{diag}\big(0,\vee\lambda_l(w,\dots,w)\big)\right)\beta$$

$$\overset{(E2)}{=} \alpha\left(\sum_{(i_1,j_1)\in\mathcal{R}\cup\mathcal{S}} \dots \sum_{(i_m,j_m)\in\mathcal{R}\cup\mathcal{S}} P\lambda_{k+l}\big(\varepsilon_{i_1,j_1}^{[k+l]},\dots,\varepsilon_{i_m,j_m}^{[k+l]}\big)\, Q \otimes \big(x_{i_1,j_1} \vee \dots \vee x_{i_m,j_m}\big)\right)\beta$$

$$= \alpha P\big(\vee\lambda_{k+l}(x,\dots,x)\big)Q\beta.$$

Hence

$$\|u_1 \oplus u_2\|_{\lambda,r+s} \leq \|\alpha P\|\|x\|^m\|Q\beta\| \leq \max\{\|\alpha_1\|,\|\alpha_2\|\}\max\{\|\beta_1\|,\|\beta_2\|\}$$

$$\leq \max\left\{(\|u_1\|_{\lambda,r} + \varepsilon)^{\frac{1}{2}}, (\|u_2\|_{\lambda,s} + \varepsilon)^{\frac{1}{2}}\right\}^2$$

$$= \max\{\|u_1\|_{\lambda,r}, \|u_2\|_{\lambda,s}\} + \varepsilon$$

(M2): Take again $\varepsilon > 0, u \in M_r(\otimes^{m,s}V)$ and $\gamma \in M_{t,r}, \delta \in M_{r,t}$. By definition there is a representation $u = \alpha \vee \lambda_k(v, \dots, v)\beta$ with $v \in M_k(V)$ and appropriate scalar matrices α, β such that $\|\alpha\|\|v\|^m\|\beta\| \leq \|u\|_\lambda + \varepsilon$. This implies that $\gamma u \delta = \gamma\alpha \vee \lambda_k(v, \dots, v)\beta\delta$, so

$$\|\gamma u \delta\|_{\lambda,t} \leq \|\gamma\alpha\|\|v\|^m\|\beta\delta\| \leq \|\gamma\|\big(\|u\|_{\lambda,r} + \varepsilon\big)\|\delta\|.$$

Finally we show that $\|\cdot\|_{\lambda,1}$ is a norm: For this purpose it suffices to show that $\|\cdot\|_{\lambda,1}$ is positive definite. So let $0 \neq u \in M_1(\otimes^{m,s}V)$. Then

$$0 \neq \|u\|_{\varepsilon_s} = \sup\{(\otimes^m f)(u) | f \in V' \text{ with } \|f\| \leq 1\},$$

and hence there is $f \in V'$ with $\|f\| \leq 1$ such that $(\otimes^m f)(u) \neq 0$. Define

$$F := \otimes^m f = f \otimes \dots \otimes f : \otimes^{m,s}V \to \mathbb{C},$$

then clearly $|F(u)| > 0$. On the other hand note that for $w_1, \dots, w_m \in V$ we have

$$F(w_1 \vee \dots \vee w_m)(\otimes^m f)\left(\frac{1}{m!}\sum_{\eta \in \mathcal{P}_m} w_{\eta(1)} \otimes \dots \otimes w_{\eta(m)}\right)$$

$$= \frac{1}{m!}\sum_{\eta \in \mathcal{P}_m} f\big(w_{\eta(1)}\big) \dots f\big(w_{\eta(m)}\big)$$

$$= f(w_1) \dots f(w_m).$$

Now take a representation $u = \alpha \vee \lambda_j(v, \dots, v)\beta$ with $\alpha \in M_{1,\tau(j)}, \beta \in M_{\tau(j),1}$ and $v \in M_j(V)$. With $v = \sum_{i=1}^{l} \alpha_i \otimes v_i =$ we have

$$\vee \lambda_j(v, \dots, v) = \sum_{i_1}^{l} \dots \sum_{i_m=1}^{l} \lambda_j\left(\alpha_{i_1}, \dots, \alpha_{i_m}\right) \otimes \big(v_{i_1} \vee \dots \vee v_{i_m}\big).$$

This implies that

$$F(u) = \alpha F_{\tau(j)}\Big(\vee \lambda_j(v, \dots, v)\Big)\beta$$

$$= \alpha \sum_{i_1=1}^{l} \dots \sum_{i_m=1}^{l} \lambda_j(\alpha_{i_1}, \dots, \alpha_{i_m}) \otimes F(v_{i_1} \vee \dots \vee v_{i_m})\, \beta$$

$$= \alpha \sum_{i_1=1}^{l} \dots \sum_{i_{m=1}}^{l} \lambda_j(\alpha_{i_1}, \dots, \alpha_{i_m}) f(v_{i_1}) \dots f(v_{i_m})\, \beta$$

$$= \alpha \sum_{i_1=1}^{l} \dots \sum_{i_m=1}^{l} \lambda_j \left(\alpha_{i_1} \otimes f(v_{i_1}), \dots, \alpha_{i_m} \otimes f(v_{i_m})\right) \beta$$

$$= \alpha \sum_{i_1=1}^{l} \dots \sum_{i_m=1}^{l} \lambda_j \left(f_j(\alpha_{i_1} \otimes v_{i_1}), \dots, f_j(\alpha_{i_m} \otimes v_{i_m})\right) \beta = \alpha \lambda_j \left(f_j(v), \dots, f_j(v)\right) \beta.$$

We know that $\|f\|_{cb} = \|f\|$, so $\|f_q\| \leq 1$ for all $q \in \mathbb{N}$, and hence

$$|F(u)| \leq \|\alpha\| \|\lambda_j\| \|f_j(v)\|^m \|\beta\| \leq \|\alpha\| \|\lambda_j\| \|v\|^m \|\beta\|.$$

Taking the supremum over all j and the infimum over all possible representations of u we get

$$0 < |F(u)| \leq \sup_j \|\lambda_j\| \|u\|_{\lambda,1}.$$

But then we deduce from (E3) that as desired $\|u\|_{\lambda,1} > 0$. This completes the proof.

For our three main examples all three conditions are fulfilled.

Proposition (4.2.2) [4]:

The Schur, Haagerup and Kronecker product satisfy the conditions (E1)–(E3).

We need the following lemma.

Lemma (4.2.3) [4]:

For all $k \in \mathbb{N}$ there exist $P \in M_{k,k^m}, Q \in M_{k^m,k}$ with $\|P\|, \|Q\| \leq 1$ such that for all $\alpha^{(l)} \in M_k$ we have $P(\alpha^{(1)} \otimes \cdots \otimes \alpha^{(m)})Q = \alpha^{(1)} \bullet \cdots \bullet \alpha^{(m)}$.

Proof:

Let $Q := [\otimes^m e_1^{[k]} \dots \otimes^m e_k^{[k]}]$ and $P := Q^t$. Then for $\alpha^{(p)} \in M_k$ we have

$$P\left(\alpha^{(1)} \otimes \dots \otimes \alpha^{(m)}\right) Q = \left[(\otimes^m e_i^t) \odot \left(\alpha^{(1)} \otimes \dots \otimes \alpha^{(m)}\right) \odot \left(\otimes^m e_j\right)\right]_{i,j}$$

$$= \left[\left(e_i^t \odot \alpha^{(1)} \odot e_j\right) \otimes \dots \otimes \left(e_i^t \odot \alpha^{(m)} \odot e_j\right)\right]_{i,j}$$

$$= \left[\alpha_{i,j}^{(1)} \dots \alpha_{i,j}^{(m)}\right]_{i,j}$$

$$= \alpha^{(1)} \bullet \dots \bullet \alpha^{(m)}$$

The Schur product satisfies (E1)–(E3): To check (E1) let $k \in \mathbb{N}$ and $p := k, S := I_k, T := I_k$ and $\alpha_l := \varepsilon_l^{[k]}$ for all $1 \le l \le k$. Then we have

$$S \bullet_k \left(\alpha_{j_1}, \dots, \alpha_{j_m}\right) T = \varepsilon_{j_1}^{[k]} \bullet \dots \bullet \varepsilon_{j_m}^{[k]} = \begin{cases} \varepsilon_j^{[k]} & \text{if } j_1 = \dots = j_m =: j \\ 0 & \text{else.} \end{cases}$$

To show (E2) let $r, s \in \mathbb{N}, P := I_{r+s} =: Q$ and $(i_q, j_q) \in \{1, \dots, r\}^2 \cup \{r+1, \dots, r+s\}^2$. Then

$$P \bullet_{r+s} \left(\varepsilon_{i_1,j_1}^{[r+s]}, \dots, \varepsilon_{i_m,j_m}^{[r+s]}\right) Q = \begin{bmatrix} \varepsilon_{i_1,j_1}^{[r]} & 0 \\ 0 & \varepsilon_{i_1-r,j_1-r}^{[s]} \end{bmatrix} \bullet \dots \bullet \begin{bmatrix} \varepsilon_{i_m,j_m}^{[r]} & 0 \\ 0 & \varepsilon_{i_m-r,j_m-r}^{[s]} \end{bmatrix}$$

$$= \begin{bmatrix} \bullet_r \left(\varepsilon_{i_1,j_1}^{[r]}, \dots, \varepsilon_{i_m,j_m}^{[r]}\right) & 0 \\ 0 & \bullet_s \left(\varepsilon_{i_1-r,j_1-r}^{[s]}, \dots, \varepsilon_{i_1-r,j_1-r}^{[s]}\right) \end{bmatrix}.$$

Finally, we get (E3) from

$$\sup_k \|\bullet_k\| = \sup_k \sup_{\|\alpha_i\| \le 1} \|\bullet_k (\alpha_1 \bullet \dots \bullet \alpha_m)\|$$

$$\overset{\text{Lemma (4.2.3)}}{=} \sup_k \sup_{\|\alpha_i\| \le 1} \|P \otimes_k (\alpha_1, \cdots, \alpha_m) Q\| \le \sup_k \sup_{\|\alpha_i\| \le 1} \|\alpha_1\| \cdots \|\alpha_m\| \le 1.$$

The matrix product satisfies (E1)–(E3): In order to verify (E1) let $k \in \mathbb{N}$ and $p := k, S := I_k, T := I_k$ and $\alpha_l := \varepsilon_l^{[k]}$ for all $1 \le l \le k$. Then we have

$$S\odot_k(\alpha_{j_1},\dots,\alpha_{j_m})T=\varepsilon_{j_1}^{[k]}\odot\dots\odot\varepsilon_{j_m}^{[k]}\overset{(5)}{=}\begin{cases}\varepsilon_j^{[k]} & \text{if } j_1=\dots=j_m=:j\\ 0 & \text{else.}\end{cases}$$

For the proof of (E2) take $r,s\in\mathbb{N}, P:=I_{r+s}=:Q$ and $(i_q,j_q)\in\{1,\dots,r\}^2\cup\{r+1,\dots,r+s\}^2$. Then

$$P\odot_{r+s}\left(\varepsilon_{i_1,j_1}^{[r+s]},\dots,\varepsilon_{i_m,j_m}^{[r+s]}\right)Q=\begin{bmatrix}\varepsilon_{i_1,j_1}^{[r]} & 0\\ 0 & \varepsilon_{i_1-r,j_1-r}^{[s]}\end{bmatrix}\odot\dots\odot\begin{bmatrix}\varepsilon_{i_m,j_m}^{[r]} & 0\\ 0 & \varepsilon_{i_m-r,j_m-r}^{[s]}\end{bmatrix}$$

$$=\begin{bmatrix}\odot_r\left(\varepsilon_{i_1,j_1}^{[r]},\dots,\varepsilon_{i_m,j_m}^{[r]}\right) & 0\\ 0 & \odot_s\left(\varepsilon_{i_1-r,j_1-r}^{[s]},\dots,\varepsilon_{i_m-r,j_m-r}^{[s]}\right)\end{bmatrix}.$$

Finally, (E3) is a simple consequence of

$$\sup_k\|\odot_k\|=\sup_k\sup_{\|\alpha_i\|\le 1}\|\alpha_1\cdots\alpha_m\|\le 1.$$

The Kronecker product satisfies (E1)–(E3): First we extend our notation. For $p,q,r,s\in M_{1,m}$ define

$$\varepsilon_{r,s}^{[p,q]}:=\otimes_{t=1}^m\varepsilon_{r_t,s_t}^{[p_t,q_t]}$$

Consequently, for $p,q,i,j\in M_{1,m}$ we have

$$\varepsilon_{i,j}^{[p,q]}\odot\varepsilon_{k,l}^{[q,r]}=\otimes_{t=1}^m\left(\varepsilon_{i_t,j_t}^{[p_t,q_t]}\odot\varepsilon_{k_t,l_t}^{[q_t,r_t]}\right)=\begin{cases}\varepsilon_{i,l}^{[p,r]} & \text{if } j=k\\ 0 & \text{else.}\end{cases}\tag{10}$$

Now define $\Theta: M_1\to M_{1,m}, x\longmapsto(x,1,\dots,1)$ and $\Delta: M_1\to M_{1,m}, x\longmapsto(x,\dots,x)$. Then, in order to show that the Kronecker product fulfills (E1), let $k\in\mathbb{N}, p:=k, S:=\sum_{s=1}^k\varepsilon_{\ominus s,\Delta s}^{[\ominus k,\Delta p]}, T:=S^t$ and $\alpha_l:=\varepsilon_l^{[k]}$ for all $1\le l\le k$. There with

$$S\otimes_k(\alpha_{j_1},\dots,\alpha_{j_m})T=\sum_{s=1}^k\varepsilon_{\ominus s,\Delta s}^{[\ominus k,\Delta p]}\odot\left(\varepsilon_{j_1}^{[k]}\otimes\dots\otimes\varepsilon_{j_m}^{[k]}\right)\odot\sum_{t=1}^k\varepsilon_{\Delta t,\ominus t}^{[\Delta k,\ominus p]}$$

$$=\sum_{s=1}^k\sum_{t=1}^k\varepsilon_{\ominus s,\Delta s}^{[\ominus k,\Delta p]}\odot\varepsilon_{(j_1,\dots,j_m)(j_1,\dots,j_m)}^{[\Delta k,\Delta k]}\odot\varepsilon_{\Delta t,\ominus t}^{[\Delta p,\ominus k]}$$

$$\overset{(10)}{=} \begin{cases} \varepsilon_j^{[k]} & \text{if } j_1 = \cdots = j_m =: j \\ 0 & \text{else.} \end{cases}$$

Proof of (E2): Let

$$P_1 := \sum_{p\in\{1,\dots,r\}^m} \varepsilon_{p,p}^{[\Delta r,\Delta(r+s)]} \quad \text{and} \quad P_2 := \sum_{p\in\{r+1,\dots,r+s\}^m} \varepsilon_{p-\Delta r,p}^{[\Delta s,\Delta(r+s)]}$$

and let $Q := [P_1^t \;\; P_2^t]$ and $P := Q^t$. In each row and each column of P and Q there is at most one 1 so $\|P\| = \|Q\| = 1$ and we have

$$P \otimes_{r+s} \left(\varepsilon_{i_1,j_1}^{[r+s]}, \dots, \varepsilon_{i_m,j_m}^{[r+s]}\right) Q = P\varepsilon_{(i_1,\cdots,i_m)(j_1,\cdots,j_m)}^{[\Delta(r+s),\Delta(r+s)]} Q$$

$$= \begin{bmatrix} \varepsilon_{(i_1,\cdots,i_m),(j_1,\cdots,j_m)}^{[\Delta r,\Delta r]} & \varepsilon_{(i_1,\cdots,i_m),(j_1,\cdots,j_m)-\Delta r}^{[\Delta r,\Delta s]} \\ \varepsilon_{(i_1,\cdots,i_m)-\Delta r,(j_1,\cdots,j_m)}^{[\Delta s,\Delta r]} & \varepsilon_{(i_1,\cdots,i_m)-\Delta r,(j_1,\cdots,j_m)-\Delta r}^{[\Delta s,\Delta s]} \end{bmatrix}$$

$$= \begin{cases} \operatorname{diag}\left(\otimes_r \left(\varepsilon_{i_1,j_1}^{[r]}, \dots, \varepsilon_{i_m,j_m}^{[r]}\right), 0\right) & \text{if } (i_k,j_k) \in \{1,\dots,r\}^2 \\ \operatorname{diag}\left(0, \otimes_s \left(\varepsilon_{i_1-r,j_1-r}^{[s]}, \dots, \varepsilon_{i_m-r,j_m-r}^{[r]}\right)\right) & \text{if } (i_k,j_k) \in \{r+1,\dots,r+s\}^2 \\ 0 & \text{else} \end{cases}$$

which is (E2). It finally remains to check (E3): We have

$$\sup_{k\in\mathbb{N}} \|\otimes_k\| = \sup_{k\in\mathbb{N}} \sup_{\|\alpha_i\|\le 1} \|\alpha_1 \otimes \cdots \otimes \alpha_m\| = \sup_{k\in\mathbb{N}} \sup_{\|\alpha_i\|\le 1} \|\alpha_1\| \dots \|\alpha_m\| = 1.$$

This completes the proof of Proposition (4.2.2).

We now prove the main result – as above we fix for a given m some sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}}$ of m-linear mappings

$$\lambda_k : M_k \times \cdots \times M_k \longrightarrow M_{\tau(k)}$$

satisfying (E1)–(E3). Recall the definition of the operator spaces $\mathscr{P}_\lambda^m(V;W)$ and $\otimes_\lambda^{m,s} V$ (from (7) and (1)).

Theorem (4.2.4) [4]:

With the identification from (2) we have completely isometrically

$$\mathscr{P}^{m}_{\lambda}(V;W) = \mathscr{CB}\left(\otimes^{m,s}_{\lambda} V;W\right).$$

Proof:

We consider the bijective mapping

$$P^m(V;W) \to L\left(\otimes^{m,s} V;W\right)$$

$$p \mapsto \check{p}^{L,s}$$

and show first that

$$\|p\|_\lambda = \|\check{p}^{L,s}\|_{cb} \text{ for every } p \in P^m(V;W) \tag{11}$$

(see (2) for the notation). For $k \in \mathbb{N}$ and $v = \sum_{i=1}^{l} \alpha_i \otimes v_i \in M_k(V)$ we have

$$p\lambda_k(v) = \frac{1}{m!\,2^m} \sum_{\delta_t \in \{-1,1\}} \delta_1 \dots \delta_m \sum_{i_1} \dots \sum_{i_m} \lambda_k\left(\alpha_{i_1}, \dots, \alpha_{i_m}\right) \otimes p\left(\sum_{j=1}^{m} \delta_j v_{i_j}\right)$$

$$= \sum_{i_1} \dots \sum_{i_m} \lambda_k\left(\alpha_{i_1}, \dots, \alpha_{i_m}\right) \otimes \check{p}\left(v_{i_1}, \dots, v_{i_m}\right)$$

$$= \sum_{i_1} \dots \sum_{i_m} \lambda_k\left(\alpha_{i_1}, \dots, \alpha_{i_m}\right) \otimes \check{p}^{L,s}\left(v_{i_1} \vee \dots \vee v_{i_m}\right)$$

$$= (\check{p}^{L,s})_{\tau(k)}\left(\sum_{i_1} \dots \sum_{i_m} \vee\, \lambda_k\left(\alpha_{i_1} \otimes v_{i_1}, \dots, \alpha_{i_m} \otimes v_{i_m}\right)\right)$$

$$= (\check{p}^{L,s})_{\tau(k)}\left(\vee\, \lambda_k(v, \dots, v)\right).$$

Let now $u \in M_k(\otimes^{m,s} V)$. For every representation $u = \alpha \vee \lambda_j(w, \dots, w)\beta$ with $w \in M_j\,(V), \alpha \in M_{k,\tau(j)}$ and $\beta \in M_{\tau(j),k}$ we then have

$$(\check{p}^{L,s})_k(u) = \alpha(\check{p}^{L,s})_{\tau(j)}\left(\vee\, \lambda_j(w, \dots, w)\right)\beta = \alpha p \lambda_j(w)\beta,$$

so

$$\|(\check{p}^{L,s})_k(u)\| \leq \|\alpha\|\|p\lambda_j\|\|w\|^m\|\beta\|,$$

and therefore

$$\|(\check{p}^{L,s})_k(u)\| \leq \|p\lambda_j\|\|u\|_{\lambda,k} \leq \|p\|_\lambda\|u\|_{\lambda,k}.$$

Hence

$$\|\check{p}^{L,s}\|_{cb} = \sup_k \sup_{\|u\|_{\lambda,k}\leq 1} \|(\check{p}^{L,s})_k(u)\| \leq \sup_k \sup_{\|u\|_{\lambda,k}\leq 1} \|p\|_\lambda\|u\|_{\lambda,k} = \|p\|_\lambda$$

On the other hand, by the definition of the norm we have

$$\|\vee \lambda_k(x,\ldots,x)\|_{\lambda,\tau(k)} = \left\|I_{\tau(k)} \vee \lambda_k(x,\ldots,x)I_{\tau(k)}\right\|_{\lambda,\tau(k)} \leq \|x\|^m$$

for all $k \in \mathbb{N}$ and $x \in M_k(V)$. Accordingly

$$\|p\|_\lambda = \sup_k \sup_{\|x\|_{M_k(V)}\leq 1} \|p\lambda_k(x)\|_{M_\tau(k)(W)}$$

$$= \sup_k \sup_{\|x\|_{M_k(V)}\leq 1} \left\|(\check{p}^{L,s})_{\tau(k)}\big(\vee \lambda_k(x,\ldots,x)\big)\right\|_{M_\tau(k)(W)}$$

$$\leq \sup_k \sup_{\|x\|_{M_k(V)}\leq 1} \left\|(\check{p}^{L,s})_{\tau(k)}\right\|\left\|\big(\vee \lambda_k(x,\ldots,x)\big)\right\|_{\lambda,\tau(k)}$$

$$\leq \sup_k \sup_{\|x\|_{M_k(V)}\leq 1} \left\|(\check{p}^{L,s})_{\tau(k)}\right\|\|x\|^m \leq \|\check{p}^{L,s}\|_{cb}$$

This proves (11). As defined in (8) we have

$$M_k(\mathscr{P}_\lambda^m(V;W)) = \mathscr{P}_\lambda^m\big(V;M_k(W)\big)$$

and also

$$M_k(\mathscr{CB}\big(\otimes_\lambda^{m,s}V;W\big)) = \mathscr{CB}\Big(\otimes_\lambda^{m,s} V;M_k(W)\Big),$$

hence the proof is complete.

As an immediate consequence of Theorem (4.2.4) and the examples from Proposition (4.2.2) we get

Corollary (4.2.5) [4]:

The following operator space equalities hold:

(i) $\mathscr{P}^m_{\odot}(V;W) = \mathscr{CB}\left(\otimes^{m,s}_{\odot} V; W\right)$
(ii) $\mathscr{P}^m_{\otimes}(V;W) = \mathscr{CB}\left(\otimes^{m,s}_{\otimes} V; W\right)$
(iii) $\mathscr{P}^m_{\bullet}(V;W) = \mathscr{CB}\left(\otimes^{m,s}_{\bullet} V; W\right)$

We are not able to distinguish Kronecker from Schur polynomials – see [3]

Lemma (4.2.6) [4]:

$$\left\| id: \otimes^{m,s}_{\bullet} V \to \otimes^{m,s}_{\otimes} V \right\|_{cb} \leq 1$$

Proof:

Let $v = \alpha \vee_{\bullet_p} (x, \ldots, x)\beta \in M_k(\otimes^{m,s} V)$ with $\alpha \in M_{k,p}, \beta \in M_{p,k}$ and $x = \sum_{j=1}^{l} \alpha_j \otimes x_j \in M_p(V)$. Then due to Lemma (4.2.3) there exist matrices P, Q with $\|P\|, \|Q\| \leq 1$ such that

$$v = \alpha \left(\sum_{j_1}^{l} \ldots \sum_{j_m}^{l} \left(\alpha_{j_1} \bullet \cdots \bullet \alpha_{j_m}\right) \otimes \left(x_{j_1} \vee \ldots \vee x_{j_m}\right) \right) \beta$$

$$\overset{Lemma\,(4.2.3)}{=} \alpha \left(\sum_{j_1}^{l} \ldots \sum_{j_m}^{l} P\left(\alpha_{j_1} \otimes \cdots \otimes \alpha_{j_m}\right) Q \otimes \left(\alpha_{j_1} \vee \ldots \vee \alpha_{j_m}\right) \right) \beta$$

$$= \alpha P \vee_{\otimes_p} (x, \ldots, x) Q \beta$$

Hence

$$\|v\|_{\otimes,k} \leq \|\alpha P\| \|x\|^m \|Q\beta\| \leq \|\alpha\| \|x\|^m \|\beta\|,$$

and therefore $\|v\|_{\otimes,k} \leq \|v\|_{\bullet,k}$

Corollary (4.2.7) [4]:

(i) $\mathscr{P}^{m}_{\otimes}(V;W) \subset \mathscr{P}^{m}_{\bullet}(V;W)$ as operator spaces

(ii) $\mathscr{P}^{m}(V;W) \subset \mathscr{P}^{m}_{\otimes}(V;W) \subset \mathscr{P}^{m}_{\bullet}(V;W) \subset \mathscr{P}^{m}(V;W)$ as Banach spaces

Proof:

Clearly, the preceding lemma combined with Theorem (4.2.4) assures (1). In order to show (2) we in view of Theorem (4.2.4) only check that

$$\|\cdot\|_{\varepsilon_s} \leq \|\cdot\|_{\otimes,1} \leq \|\cdot\|_{\bullet,1} \leq \|\cdot\|_{\pi_s} \quad \text{on} \quad \otimes^{m,s}V.$$

The first inequality was implicitly shown in the proof of Proposition (4.2.1) (proof of (M2)); even for arbitrary λ's instead of $\otimes$ only. The second estimate is a special case of statement (1), whereas the third inequality easily follows from the trivial fact that

$$\|x \otimes x\|_{\bullet,1} \leq \|x\|^2, \quad x \in V.$$

This completes the proof.

List of Symbols

Symbol	Page
det : determinant	3
sh : shift	5
inf : infimum	14
supp : support	14
LBP : Laurent biorthogonal polynomial	21
AD_n : Aztec diamond of order n	23
Gr : Grassmannians	45
dim : dimension	46
$\otimes$: Tensor product	48
ker : kernel	48
$\oplus$: Direct sum	52
EQLR : Equivariant Quantum Littlewood Richardson	57
ev_z : evaluation map	60
min : minimum	61
Hom : Homomorphism	62
codim : codimension	63
End : Endomorphism	64
Aut : Automorphism	64
Sup : Supremum	82
$A \bullet B$: Schur product	84
$\odot$: matrix product	84
max : maximum	85
diag : diagonal	89

References

[1] MARKO D. PETKOVIĆ, NiŠ, PAUL BARRY, Waterford, PREDRAG RAJKOVIĆ, NiŠ: Closed-Form expression for Hankel Determinants of the Narayana polynomials: C Zechoslovak Mathematical Journal, 62 (137) (2012), 39-57.

[2] Shuhei Kamioka: Laurent biorthogonal Polynomials, q-Narayana Polynomials and domino tilings of the Aztec domains: J. of combinatorial Theory, series A 123 (2014) 14-29.

[3] Dave Anderson [a,*,1], Linda chen [b,*]: Equivariant quantum Schubert Polynomials ⋆: Advances in Mathematics 254 (2014) 300-330.

[4] Andereas Defant, Dirk Wiesner: Plynomials in Operator space theory: J, of functional Analysis 266 (2014) 5493-5525.

[5] P. A. Mahon: Combinatorial Analysis: Vols. 1 and 2, Cambridge University press (1915, 1916).

[6] Conway and Guy (1996) The Book of Numbers. New York: Copernicus, PP.96-106.

[7] Pinter Charles C. (2010) [Unabridged republication of the 1990 second edition of the work originally published in 1982 by the Mc Graw-Hill publishing company]. A book of Abstract Algebra. Dover. ISBN 978-0486474175.

[8] Louis de Branges (1968). Hilbert spaces of entire functions. London: Prentice-Hall. P.189. ISBN 978-01338900.

[9] Narayana, T. V. lattice path Combinatorics with Statistical Applications. Toronto, Ontario, Canada: University of Toronto Press, 1979.

[10] Viennot, Xavier G. (1989), Determinants, paths and plane partitions.

[11] Hatcher, Allen (2003). “vector Bundles & k-Theory” (2.0 ed). Section 1.2

[12] Sottile, Frank (2001), “Schubert Calculus”, in Hazewinkel, Michiel, Encyclopedia of Mathematics, springer, ISBN 978-1-55608-010-4.

[13] Fulton, W; Pandharipande, R(1996). “Notes on stable maps and quantum cohomology”. Arxiv: alg-geom / 960811.

[14] Lascoux, Alain; Scützenberger, Marcel-Paul (1985), “Schubert polynomials and the Little wood- Richardson rule” Letters in Mathematical physics. A journal for the Rapid Dissemination of short contributions in the Field of mathematical physics 10(2): 111-124.

[15] Halmos, Paul (1974), Finite dimensional vector spaces, springer, ISBN 0-387-90093-4.