Chapter 4

Extension of Sampling Inequalities in Sobolev Spaces

The extensions established in this Chapter allow us to bound fractional order semi-norms and to incorporate, if available, values of partial derivatives on the discrete set. Both the cases of a bounded domain and the Euclidean space are considered .Sampling inequalities in the Sobolev space $W^{r,p}(\Omega)$, where Ω is a domain of R^n, are defined as relations like

$$|u|_{l,q,\Omega} \leq C\left(d^{r-l-n\left(\frac{1}{p}-\frac{1}{q}\right)}|u|_{l,q,\Omega} + d^{\frac{n}{q}-l}\left(\sum_{a\in A}|u(a)|^p\right)^{\frac{1}{p}}\right), l \leq \ell,$$

for suitable values of r , p, q and ℓ. In this statement, u denotes a function in $W^{r,p}(\Omega)$, A is a discrete set in $\bar{\Omega}$ and $d = \sup_{x\in\Omega} \inf_{a\in A}|x-a|$.

Sec(4.1):Sobolev Semi-norms of Fractional order and Derivative data:

Sampling in equalities in Sobolev spaces have received increasing gattention in fields like variational splines, surface reconstruction, machine learning or numerical solution of partial differential equations. Given a domain Ω and suitable values of $p, q \in [1,\infty]$,these inequalities s typically yield bounds of the$|\cdot|_{l,q,\Omega}$Sobolev semi-norm of a function uon Ω interms of the$|\cdot|_{r,q,\Omega}$semi-normof$u$, with$r \geq l$and$l \in \mathbb{N}$,the Hausd(or fill)distance d between $\bar{\Omega}$ and a discrete set $A \subset \bar{\Omega}$, and the values of u on A, thatis, abound of the form

$$|u|_{l,q,\Omega} \leq \mathfrak{C}^{(dr-l-n(1/p-1/q))_+}|u|_{r,q,\Omega} + d^{\frac{n}{\gamma}-l}\|u|_A\|_x \quad (1)$$

(see later the relation (19) for a full explanation). Most section on this subject assume that Ω is bounded. Let us quote, for instance, Narcowich et al. [149], Wendland and Rieger [157], Arcangéli et al. [136] and the references therein. A few articles deal with the case Ωunbounded (Madych and Potter [146], Madych [145] and Arcangéli et al. [137]). For a survey on sampling inequalities and their applications, we referred to Rieger et al. [151].

The purpose of this section is to extend the results of [136] and [137], for both cases Ω bounded and $= \mathbb{R}^n$, in two directions: to allow fractional order semi-norms on the left-hand side of the sampling inequality (1), and to admit derivative pointwise values on the second term of the right-hand side of (1), so increasing the order of the approximation parameter d .

First, let us consider the problem of the extension to fractional order semi-norms. The proof of (1) for $l \in \mathbb{N}$ commonly follows a strategy which dates back to Duchon [143]: first, a special case of the sampling inequality is proven for a class of sets with simple geometry (like balls); then, for any set A with fill distance d small enough, one considers a finite family $\{B_i\}_{i\in I}$, formed by sets belonging to such a class, that covers the set Ω ; next, for each $u \in W^{r,p}(\Omega)$, one considers an extension $\tilde{u}$ of u to $\mathbb{R}^n$and uses the relation

$$|\tilde{u}|_{l,q,\Omega} \leq |\tilde{u}|_{l,q,\bigcup_{i\in I} B_i} \leq \left(\sum_{i\in I} |\tilde{u}|_{l,q,B_i}^q\right)^{1/q} \quad (2)$$

finally, one applies the sampling inequality on each $B_i$and collects terms to derive (1). If l is non-integer, this workflow breaks precisely when using (2), because Sobolev semi-norms of non-integer order are subject to a super-additivity rule. This means that, if Ω is the union of two disjoint sets Ω_1 and Ω_2 , then

$$v|_{l,q,\Omega}^q \geq |v|_{l,q,\Omega_1}^q + |v|_{l,q,\Omega_2}^q.$$

So, it is clear that(2) cannot hold .This is there as on why we have a doped a different strat egy, now based on the interpolation theory between Banach spaces. We have chosen the K-method which see m stousra ther well adapted to explicit calculus. Thus, we obtain as extensions of previous results in [136] and[137]. It that interpolation is applied for the first time to obtain interpolation sampling inequalities in a section by Schaback (cf.[152]).

If r is large enough, the space $W^{r,p}(\Omega)$ is imbedded into a space of functions having continuous partial derivatives up to a given order. Hence, it is natural to try to use values of partial derivatives as pointwise data and incorporate them into sampling inequalities. This idea was introduced by Corrigan et al. [142]. We exploit it, with corrections and improvements, as we later express [140],[147],[148].

For any $x \in \mathbb{R}$, we shall write $\lfloor x \rfloor$ and $\lceil x \rceil$ for the floor (or integer part) and ceiling of x , that is, the unique integers satisfying $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lceil x \rceil - 1 < x \leq \lceil x \rceil$. Likewise, we shall write $(x)_+ = \max\{x, 0\}$.

The letter n will always stand for an integer belonging to $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ (by convention, $0 \in \mathbb{N}$). The Euclidean norm in $\mathbb{R}^n$will be denoted by $|\cdot|$. For any $t \in \mathbb{R}^n$and for any $\delta > 0$, we shall denote by $B(t, \delta)$ the open ball with centre t and radius δ. In addition, S_{n-1} will stand for the unit sphere in $\mathbb{R}^n$ (i.e., $S_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$) and $|S_{n-1}|$ will denote its area.

For any $k \in \mathbb{N}$, we shall denote by P_k the space of polynomial functions defined on $S_{n-1}$of total degree less than or equal to k.

For any $= (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})$,$x_1, \dots, x_n$ being the generic independent variables in $\mathbb{R}^n$. From now on, the term domain means a non-empty, connected open set in likewise, we shall use the expression Lipschitz-continuous boundary in the sense of Nečas [150]. It can be seen (cf., for example, Adams and Fournier [135]) that any bounded domain $\Omega \subset \mathbb{R}^n$with a Lipschitz-continuous boundary satisfies, for some $\theta \in (0, \pi/2]$and $\rho > 0$, the cone property with radius ρ and angle θ , that is, for every $x \in \Omega$, there exists a unit vector $\xi(x) \in \mathbb{R}^n$ such that the cone

$$C\ (x, \xi\ (x), \theta, \rho) = \{x + h\eta : \eta \in \mathbb{R}^n, |\eta| = 1, \eta \cdot \xi(x) \geq cos\ \theta, 0 \leq h \leq \rho\}$$

is contained in Ω (above, the dot symbol $\cdot$ is the Euclidean scalar product in $\mathbb{R}^n$). We recall that the radius of the greatest ball contained in $C(x, \xi(x), \theta, \rho)$ is just ρ/τ , with $\tau = 1 + 1/\sin\theta$.

For any domain $\Omega \subset \mathbb{R}^n$and for any discrete subset $A \subset \bar{\Omega}$ (i.e., made up of isolated points), we define the fill distance

$$\delta(\bar{\Omega}, A) = \sup_{x \in \Omega} dist(x\ , A), \tag{3}$$

where$\mathrm{dist}(x, A) = \inf_{a \in A}|x - a|$ stands for the Euclidean distance between x and A. Clearly, $\delta(\bar{\Omega}, A)$ may be infinite when Ω is unbounded, and is just the Hausdorff distance between A and $\bar{\Omega}$ when Ω is bounded. Moreover, for any function v defined on A and for any $x \in [1, \infty]$, we shall use the notation

$$\|v|_A\|_x \begin{cases} \left(\sum_{a \in A} |v(a)|^x\right)^{\frac{1}{x}} & , \text{ if } x < \infty, \\ \max_{a \in A}|v(a)|\ , & \text{if } x = \infty. \end{cases} \tag{4}$$

Hereafter, the expression $\mathcal{L}[\rho, \theta\ ]$-domain will be a shorthand for bounded domain with a Lipschitz-continuous boundary satisfying the cone property with radius ρ and angleθ .

For the sake of completeness, we include here a brief account of the K-method for real interpolation of Banach spaces. The reader is referred to Adams and Fournier [135] for details. In particular, we recall below an adapted version of Theorems in [135].

Let $\{X_0, X_1\}$ be an interpolation pair of Banach spaces (i.e., X_0 and X_1 have non-trivial intersection and are continuously imbedded in the same Hausdorff topological vector space). The spaces $X_0 \cap X_1$ and $X_0 + X_1$ are themselves Banach spaces with respect to the norms

$$\|u\|_{X_0 \cap X_1} = \max(\|u\|_{X_0}, \|u\|_{X_1})$$

and

$$\|u\|_{X_0 + X_1} = \inf\{\|u_0\|_{X_0} + \|u_1\|_{X_1}: u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\},$$

where, for $i = 0,1$, $\|\cdot\|_{X_i}$ is the norm in X_i . For each $t > 0$, the functional $(t\ ;\cdot) : X_0 + X_1 \to \mathbb{R}$ is defined by

$K(t; u) = \inf\{\|u_0\|_{X_0} + t\|u_1\|_{X_1}: u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\}$.

Then,for any$\sigma \in (0,1)$and$q \in [1, \infty]$(oreven$\sigma \in [0,1]$if$q = \infty$),onewrites

$$[X_0, X_1]_{\sigma,q} = \left\{u \in X_0 + X_1 :\ \|u\|_{[X_0,X_1]_{\sigma,q}} < \infty\right\},$$

where

$$\|u\|_{[X_0,X_1]_{\sigma,q}} = \begin{cases} \left(\left(\int_0^{\infty} (t^{-\sigma}K(t;u))^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ \underset{0<t<\infty}{ess\ sup}\{t^{-\sigma}K(t;u)\}, & \text{if } q = \infty. \end{cases}$$

Theorem (4.1.1)[134]:

The space $[X_0, X_1]_{\sigma,q}$ is a nontrivial Banach space with norm $\|\cdot\|_{[X_0,X_1]_{\sigma,q}}$. Furthermore,

$X_0 + X_1 \le (q\sigma(1-\sigma))^{\frac{1}{q}}\|\cdot\|_{[X_0,X_1]_{\sigma,q}} \le \|\cdot\|_{X_0\cap X_1}$,

if $q < \infty$, or, for $q = \infty$,

$$X_0 + X_1 \le [X_0, X_1]_{\sigma,q} \le X_0 \cap X_1$$

Thus, the following embeddings hold:

$$X_0 \cap X_1 \to [X_0, X_1]_{\sigma,q} \to X_0 + X_1$$

(i.e., $[X_0, X_1]_{\sigma,q}$ is an intermediate space between X_0 and X_1).

Theorem(4.1.2) [134]:

Let$\{X_0, X_1\}$and$\{Y_0, X_1\}$be two interpolation pairs of Banach spaces, and let$T$be abounded linear operator from$X_0 + X_1$into$Y_0 + Y_1$,having the property that , for $i = 0,1, T$ is bounded from X_i to Y_i with norm at most M_i, thatis,

$$\forall u_i \in X_i\ ,\ \ T\|u_i\|_{Y_i} \le M_i\|u_i\|_{X_i}\,.$$

Then, for any $q \in [1, \infty]$ and $\sigma \in (0,1)$,T maps $[X_0, X_1]_{\sigma,q}$ into $[Y_0, Y_1]_{\sigma,q}$and

$$\forall u \in [X_0, X_1]_{\sigma,q}\ ,\ \ \|Tu\|_{[Y_0,Y_1]_{\sigma,q}} \le M_0^{1-\sigma} M_0^{\sigma}\|u\|_{[X_0,X_1]_{\sigma,q}}\,.$$

Let Ω be a domain of $\mathbb{R}^n$, $q \in [1, \infty]$ and $s \in \mathbb{N}$ (resp. $s \in (0, \infty)\setminus\mathbb{N}$). We shall denote by $W^{s,q}$ (Ω) the usual Sobolev space of integer order (resp. of non-integer order) s defined by

$$W^{s,q}\ (\Omega) = \{v \in L^q(\Omega) \colon \forall\alpha \in \mathbb{N}^n, |\alpha| \le s, \partial^\alpha v \in L^q(\Omega)\}$$

(resp. by $W^{s,q}\ (\Omega) = \{v\ \in\ W^{\lfloor s\rfloor,q}(\Omega) \colon |v|_{s,q\ ,\Omega}\ <\ \infty\}$)

In the latter relation, $|\cdot|_{s,q\ ,\Omega}$ is the semi-norm defined by

$$|v|^q_{s,q\ ,\Omega} \ = \sum_{|\alpha|=\lfloor s\rfloor} \int_{\Omega\times\Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^q}{|x-y|^{n+q(s-\lfloor s\rfloor)}} dxdy \tag{5}$$

if $q < \infty$, and

$$|v|_{s,\infty,\Omega} = \max_{|\alpha|=\lfloor s\rfloor} \operatorname*{ess\,sup}_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^q}{|x-y|^{s-\lfloor s\rfloor}} \tag{6}$$

We recall that the derivatives $\partial^\alpha v$ are taken in the distributional sense.

The integer order Sobolev space $W^{s,q}$ (Ω) is endowed with the semi-norms $|\cdot|_{j,q\ ,\Omega}$, with $j \in \{0, \dots, s\}$, and the norm $\|\cdot\|_{s,q\ ,\Omega}$ given, if $q < \infty$, by

$$|v|_{j,q,\Omega} = \left(\sum_{|\alpha|=j} \int_{\Omega} |\partial^{\alpha} v(x)|^{q} \, dx \right)^{1/q}$$

and

$$\|v\|_{s,q,\Omega} = \left(\sum_{j=0}^{s} |v|_{j,q,\Omega}^{q} \right)^{1/q},$$

or, if $q = \infty$, by

$$|v|_{j,\infty,\Omega} = \max_{|\alpha|=j} \operatorname*{ess\,sup}_{x\in\Omega} |\partial^{\alpha} v(x)|$$

and

$$\|v\|_{s,\infty,\Omega} = \max_{0\le j\le s} |v|_{j,\infty,\Omega}.$$

The non-integer order space $W^{s,q}(\Omega)$ is endowed with the semi-norms $|\cdot|_{j,q,\Omega}$, with $j \in \{0, \dots, s\}$, the semi-norm $|\cdot|_{s,q,\Omega}$ defined by (5) or (6), and the norm

$$\|v\|_{s,q,\Omega} = \begin{cases} \left(\|v\|_{\lfloor s\rfloor,q,\Omega}^{q} + |v|_{s,q,\Omega}^{q} \right)^{1/q}, & \text{if } 1 \le q < \infty, \\ \max\left\{ \|v\|_{\lfloor s\rfloor,q,\Omega}, |v|_{s,q,\Omega} \right\}, & \text{if } q = \infty. \end{cases}$$

It is well-known that non-integer order Sobolev spaces can be equivalently defined, at least on suitable domains, by real interpolation of Sobolev spaces of integer consecutive orders. In particular, let Ω be $\mathbb{R}^n$ or a bounded domain with a Lipschitz-continuous boundary, and let $s \in (0,\infty)\backslash\mathbb{N}$ and $q \in [1,\infty)$. Then,

$$W^{s,q}(\Omega) = [W^{l,q}(\Omega), W^{l+1,q}(\Omega)]_{\sigma,q},$$

where $l = \lfloor s \rfloor$ and $\sigma = s - l$, and there exist constants γ_s and β_s such that, for all $u \in W^{s,q}(\Omega)$,

$$\gamma_s \|u\|_{[W^{l,q}(\Omega),W^{l+1,q}(\Omega)]_{\sigma,q}} \le \|v\|_{s,q,\Omega} \le \beta_s \|u\|_{[W^{l,q}(\Omega),W^{l+1,q}(\Omega)]_{\sigma,q}} \quad (7)$$

(cf., for example, Bergh and Löfström [139], Brenner and Scott [141] or Tartar [154]).

To deduce the results in the next sections, we need to know how the constant β_s depends on the order s of the corresponding Sobolev space. We have not found in the literature a clear account of this question, so we study it here, providing some theorems. Proofs can be skipped in a first reading. For this reason and for the sake of brevity in this preliminary section, we defer the proofs to Appendix A.

Theorem (4.1.3) [134]:

Let $q \in [1,\infty)$, $s \in (0,\infty)\backslash\mathbb{N}$, $l = \lfloor s \rfloor$ and $\sigma = s - l$. Then,

$$\forall u \in W^{s,q}(\mathbb{R}^n), \quad \|u\|_{s,q,\mathbb{R}^n} \le b_s \|u\|_{[W^{l,q}(\mathbb{R}^n),W^{l+1,q}(\mathbb{R}^n)]_{\sigma,q}}, \quad (8)$$

where

$$b_s = (q\sigma(1-\sigma) + (2+\nu_l n^{(1/2-1/q)_+})^{\frac{1}{q}}. \quad (9)$$

In the above expression, $|S_{n-1}|$, we recall, stands for the area of the unit sphere in $\mathbb{R}^n$ and $\nu_l = \min\{n, l+1\}$.

Theorem (4.1.4) [134]:

Let Ω be a bounded domain of $\mathbb{R}^n$ with a Lipschitz continuous boundary. Let $q \in [1,\infty)$ and $\ell \in \mathbb{N}$. Then, there exists a family $\{\beta_s\}_{s\in(0,\ell)\setminus\mathbb{N}}$ of positive real numbers such that, for each $s \in (0,\ell)\setminus\mathbb{N}$ and for all $u \in W^{s,q}(\Omega)$, the right bound in (7) holds, with $l = \lfloor s \rfloor$ and $\sigma = s - l$. Moreover, there exists a constant $\beta^* \in [1,\infty)$, dependent on $\Omega, n, \ell,$ and q, such that, for all $s \in (0,\ell)\setminus\mathbb{N}, \beta_s \leq \beta^*$.

We conclude this section with two lemmas about the asymptotic behaviour of fractional order semi-norms, extensions of previous results in Bourgain et al. [140]), Maz'ya and Shaposhnikova [147] and Milman [148]. The interested reader may find proofs and comments in Arcangéli and Torrens [138].

Lemma (4.1.5) [134]:

Let Ω be a bounded domain of $\mathbb{R}^n$ with a Lipschitz continuous boundary. Let $q \in [1,\infty)$ and $l \in \mathbb{N}$.

(i) Let $\sigma_0 \in (0,1)$. For any $v \in W^{l+\sigma_0}, q(\Omega)$, the semi-norm

$$|v|_{l,\mathrm{dini}(q),\Omega} = \left(\sum_{|\alpha|=l} \int_{\Omega\times\Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^q}{|x-y|^n} dxdy\right)^{1/q} \quad (10)$$

is finite and

$$\lim_{\sigma\to 0^+} |v|_{l+\sigma,q,\Omega} = |v|_{l,\mathrm{dini}(q),\Omega}. \quad (11)$$

(ii) For any $v \in W^{l+1,q}(\Omega)$,

$$\lim_{\sigma\to 0^-}(1-\sigma)|\nabla v|^q_{l+\sigma,q,\Omega} = q^{-1}K_{q,n}\,|\nabla v|^q_{l,q,\Omega}, \quad (12)$$

where

$$K_{q,n} = \int_{S_{n-1}} |\omega\cdot\nu|^q d\omega, \quad (13)$$

ν being any unit vector in $\mathbb{R}^n$, and $|\nabla v|_{l,q,\Omega}$ is the semi-norm, equivalent to $|v|_{l+1,q,\Omega}$, given by

$$|\nabla v|_{l,q,\Omega} = \left(\sum_{|\alpha|=l} \int_{\Omega} |\nabla(\partial^\alpha v)(x)|^q\, dx\right)^{\frac{1}{q}} \quad (14)$$

Lemma (4.1.6) [134]:

Let $q \in [1,\infty)$ and $l \in \mathbb{N}$.

(i) Let $\sigma_0 \in (0,1)$. For any $v \in W^{l+\sigma_0,q}(\mathbb{R}^n)$

$$\lim_{\sigma\to 0^+}\sigma \left|v\right|^q_{l+\sigma,q,\mathbb{R}^n} = 2q^{-1}|S_{n-1}|\left| v\right|_{l,q,\mathbb{R}^n}. \quad (15)$$

(ii) For any $v \in W^{l+1,q}(\mathbb{R}^n)$,

$$\lim_{\sigma\to 1^-}(1-\sigma)\left|v\right|^q_{l+\sigma,q,\mathbb{R}^n} = q^{-1}K_{q,n}\left|\nabla v\right|^q_{l,q,\mathbb{R}^n}. \quad (16)$$

where $K_{q,n}$ is given by (13) and $|\nabla v|_{l,q,\mathbb{R}^n}$ is defined by (14) with $\mathbb{R}^n$instead of Ω .

From now on, we shall make constant use of the parameters denoted by $,q$, and r . The following hypotheses state their admissible values:

$$\text{p}, x \in [1,\infty], \quad (17a)$$

$$q \in [1,\infty), \quad (17b)$$

$$r \in \begin{cases} [n,\infty), & if\ p = 1, \\ \left(\frac{n}{p},\infty\right), & \text{if } 1 < p < \infty, \\ \mathbb{N}^*, & \text{if } p = \infty. \end{cases} \quad (17c)$$

We shall also need the integer number

$$\ell = \begin{cases} l_0\ , & \text{if } r \in \mathbb{N}^* \text{andeither } p < q < \infty \text{ and} l_0 \in \mathbb{N}, \text{ or} \\ & (p,q) = (1,\infty), \text{or } p = q, \\ \lceil l_0 \rceil - 1 & \text{otherwise} \end{cases} , \quad (18)$$

if $r \in \mathbb{N}^*$ and either $p < q < \infty$ and

where $l_0 = r - n(1/p - 1/q)_+$.

Remark (4.1.7) [134]:

Given a bounded domain $\Omega \subset \mathbb{R}^n$with a Lipschitz-continuous boundary, the relations (17) and (18) serve to assure the continuous imbedding of $W^{r,p}(\Omega)$ into the following spaces: $C^0(\bar{\Omega})$ (or $C^0_B(\Omega)$ if $p = 1$ and $r = n$) and $W^{l,q}(\Omega)$, if $p \le q$ and $l \in \{0,\ldots,\}$ (cf. [136, Propositions 2.1 and 2.2]). Above, $C^0_B(\Omega)$ denotes the space of continuous bounded functions on Ω.

We first recall the sampling inequality established in [136].

Theorem (4.1.8) [134]:

Let Ω be a bounded $\mathcal{L}[\rho,\theta]$-domain of $\mathbb{R}^n$, for some $\rho > 0$ and $\theta \in (0,\pi/2]$. Suppose that p,q,x and r satisfy (17), with (17b) replaced by $q \in [1,\infty]$, let $\gamma = \max\{p,q,x\}$ and let ℓ be the integer given by (18). Then, there exist two positive constants $\mathfrak{d}$ (dependent on θ,ρ,n and r) and $\mathfrak{C}$(dependent on Ω,n,r,p,q and x) such that the following property holds: for any finite set $A \subset \bar{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $\delta(\bar{\Omega},A) \le \mathfrak{d}$, for any $u \in W^{r,p}(\Omega)$ and for any $l = 0,\ldots,\ell$,

$$|u|_{l,q,\Omega} \le \mathfrak{C}\left(d^{(r-l-n(1/p-1/q)_+}\left|u\right|_{r,p,\Omega} + d^{\frac{n}{\gamma}-l}\ u|_A\right) \quad (19)$$

where$d = \delta(\bar{\Omega},A)$ is defined by (3) and $\|u|_A\|_x$ is given by (4).

Remark (4.1.9) [134]:

The constant $\mathfrak{d}$ provided by Theorem (4.1.8) is explicitly given by $\mathfrak{d} = \rho/(2R\tau)$, where $\tau = 1 + 1/sin\theta$ and R is a constant greater than 1 which depends on n and r through the following condition: the ball $B(0, R)$ contains $\mathfrak{K}$ balls $B_1, \ldots, B_{\mathfrak{K}}$ of radius 1 such that $\prod_{i=1}^{\mathfrak{K}} \bar{B}_i$ is a compact subset of $(\mathbb{R}^n)^{\mathfrak{K}}$ formed by P_k –unisolvent tuples, with $k = \lceil r \rceil - 1$ and $\mathfrak{K} = \dim P_k$. For example, for $r = n = 2$, we have $R > 1 + 2/\sqrt{3}$. Let us point out two useful consequences:

If needed, it can always be assumed that the constant d belongs to $(0,1]$, since, for fixed values of ρ and θ , the constant R can be chosen as large as required to get $\mathfrak{d} \le 1$.

In the hypotheses of Theorem (4.1.8), any finite set $A \subset \bar{\Omega}$ such that $\delta(\bar{\Omega}, A) \le \mathfrak{d}$ contains a P_k-unisolvent subset. To prove this, let us first observe that Ω contains a ball $\tilde{B}$ with radius ρ/τ , because it is a $\mathcal{L}[\rho, \theta]$-domain. Now let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a bijective affine mapping such that $F(B(0, R)) = \tilde{B}$. Thus, F maps the balls $B_1, \ldots, B_{\mathfrak{K}}$ of radius 1 contained in $B(0, R)$ onto the balls $F(B_1), \ldots, F(B_{\mathfrak{K}})$ of radius $\rho/(\tau R) = 2\mathfrak{d}$ contained in $\tilde{B}$. Consequently, $\prod_{i=1}^{\mathfrak{K}} \bar{F}(B_i)$ is a compact subset of $\Omega^{\mathfrak{K}}$ K formed by P_k-unisolvent tuples. Since $\delta(\bar{\Omega}, A) \le \mathfrak{d}$, there is at least one point of A in each ball $F(B_i)$. Collecting these points, we get a P_k–unisolvent subset of A.

Remark (4.1.10) [134]:

Let T_2 denote the second term on the right-hand side of (19), that is, $T_2 = d^{n/\gamma - l} \|u|_A\|_{x}$, with $d = \delta(\bar{\Omega}, A)$. It can be easily shown that, for any $u \in W^{r,p}(\Omega)$, there exists a constant $C = C(u)$, such that, for any finite set A such that d is sufficiently small,

$$T_2 \ge C d^{\,n(1/\gamma - 1/x) - l} \|u\|_{C_B^0(\Omega)},$$

where $C_B^0(\Omega)$ denotes the space of continuous bounded functions on Ω . Therefore, in order for T_2 to be bounded for d near 0, it is necessary that $l = 0$ and $\gamma = x$. Thus, relation (19) makes sense in this case. It also makes sense, when u is the approximation error function by spline interpolation or smoothing (cf. [136]). We can now establish the main result of this subsection.

Theorem (4.1.11) [134]:

Let Ω be a bounded $\mathcal{L}[\rho, \theta]$-domain of $\mathbb{R}^n$, for some $\rho > 0$ and $\theta \in (0, \pi/2]$. Suppose that p, q, x , and r satisfy (17), let $\gamma = \max\{p, q, x\}$ and let ℓ be the integer given by (18). Then, there exist two constants $\mathfrak{d}^* \in (0,1]$ (dependent on θ, ρ, n and r) and $\mathfrak{C}^* > 0$ (dependent on Ω, n, r, p, q and x) such that the following property holds: for any finite set $A \subset \bar{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $\delta(\bar{\Omega}, A) \le \mathfrak{d}^*$, for any $u \in W^{r,p}(\Omega)$ and for any

$s \in [0, \ell\,]$, we have

$$|u|_{s,q,\Omega} \leq \mathfrak{C}^* \Lambda_{\sigma,q} (d^{r-s-n(1/p-1/q)_+} |u|_{r,p,\Omega} + d^{\frac{n}{\gamma}-s} \| u|_A \|_x \quad) \tag{20}$$

Where $d = \delta(\bar{\Omega}, A)$ is defined by (3),$\| u|_A \|_x$ is given by (4), $\sigma = s - \lfloor s \rfloor$ and

$$\Lambda_{\sigma,q} = \begin{cases} (q\sigma(1-\sigma))^{1/q}, & \text{if } \sigma \in (0,1), \\ 1, & \text{if } \sigma = 0. \end{cases} \tag{21}$$

Proof .

We start by applying Theorem (4.1.8)to the given set Ω and parameters p, q, x and r. This theorem provides two constants: $\mathfrak{d}$, which , by Remark (4.1.9), can be supposed to belong to (0,1], and $\mathfrak{C}$. Then, we take $\mathfrak{d}^* = \mathfrak{d}$ and we define $\mathfrak{C}^*$as follows:

$$\mathfrak{C}^* = \mathfrak{C}\beta^*(\ell + 1)^{\frac{1}{q}}, \tag{22}$$

whereβ^* is the constant given by Theorem (4.1.4) .It is easily seen that$\mathfrak{C}^*$, like $\mathfrak{C}$, only depends on $\Omega, n, r, p, q,$ and , and that $\mathfrak{C} \leq \mathfrak{C}^*$.

Now , let $s \in [0, \ell]$. Let us write $l = \lfloor s \rfloor$and $\sigma = s - l$. If $s = l$(so $\sigma = 0$), by Theorem (4.1.13), the result obviously holds, since (19) and the relation $\mathfrak{C} \leq \mathfrak{C}^*$ immediately imply (20). We assume in the sequel that $s \neq l$. Hence, $= l + \sigma$, with $\sigma \in (0,1)$ and $l \in \{0, \ldots, \ell - 1\}$.

For any finite set $A \subset \bar{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $d = \delta(\bar{\Omega}, A) \leq \mathfrak{d}^*$, let $[\![\cdot]\!]_A : W^{r,p}(\Omega) \to \mathbb{R}$ be the mapping defined by

$$[\![u]\!]_A = d^{r-n(1/p-1/q)_+} |u|_{r,p,\Omega} + d^{\frac{n}{\gamma}} \| u|_A \|_x \,. \tag{23}$$

It is readily checked that $[\![\cdot]\!]_A$ is a semi-norm on $W^{r,p}(\Omega)$. In fact, it is a norm and $W^{r,p}(\Omega)$, endowed with $[\![\cdot]\!]_A$, is a Banach space. To prove this, it suffices to show that there exist constants C_1 and C_2 such that

$$\forall u \in W^{r,p}(\Omega), \;\; C_1 \|u\|_{r,p,\Omega} \;\leq\; [\![u]\!]_A \leq\; C_2 \|u\|_{r,p,\Omega} \tag{24}$$

Since A is finite, the upper bound is a consequence of the continuous imbedding of $W^{r,p}(\Omega)$ into$C^0(\bar{\Omega})$. The lower bound can be deduced, for example, from Proposition 3.3 in [136], taking Remark (4.1.9) into account. Thus (24) holds.

We now proceed by real interpolation of Banach spaces, using the K-method. Let $X_0 = X_1 = W^{r,p}(\Omega)$, equipped with the norm $[\![\cdot]\!]_A$. Likewise, let $Y_0 = W^{l,q}(\Omega)$ and $Y_1 = W^{l+1,q}(\Omega)$, endowed with their usual Sobolev norms $\|\cdot\|_{l,q,\Omega}$ and $\|\cdot\|_{l+1,q,\Omega}$. On the one hand, by Theorem (4.1.1), it is clear that $[X_0, X_1]_{\sigma,q} = W^{r,p}(\Omega)$ and

$$\|\cdot\|_{[X_0,X_1]_{\sigma,q}} = \Lambda_{\sigma,q} [\![\cdot]\!]_A \,,$$

with $\Lambda_{\sigma,q}$ given by (21). On the other hand, as already pointed out in Sect. 2.3$[X_0, X_1]_{\sigma,q} = W^{s,q}(\Omega)$and there exists a constant β_s such that

$$\|\cdot\|_{s,q,\Omega} \le \beta_s \|\cdot\|_{[Y_0,Y_1]_{\sigma,q}} \tag{25}$$

Let T be the injection operator from $X_0 = X_1 = W^{r,p}(\Omega)$ into $Y_0 + Y_1 = W^{r,p}(\Omega)$. For $i = 0,1$ and for any $u \in W^{r,p}(\Omega)$, it follows from Theorem (4.1.13) That

$$\|Tu\|_{Y_i} = \|u\|_{l+1,q,\Omega} = \left(\sum_{j=0}^{l+i} |u|_{j,q,\Omega}^q\right)^{1/q}$$

$$\le \left(\sum_{j=0}^{l+i} (\mathfrak{C} d^{-j} [\![u]\!]_A)^q\right)^{1/q} \le \mathfrak{C}(l+1+i)^{\frac{1}{q}} d^{-(l+i)} [\![u]\!]_A.$$

Thus, by (25) and Theorem (4.1.2), for any $u \in W^{r,p}(\Omega)$, we get

$$\|u\|_{s,q,\Omega} = \|\,Tu\|_{s,q,\Omega} \le \beta_s \|Tu\|_{[Y_0,Y_1]_{\sigma,q}}$$

$$\le \beta_s (\mathfrak{C}(l+1)^{\frac{1}{q}} d^{-l})^{1-\sigma} (\mathfrak{C}(l+2)^{\frac{1}{q}} d^{-(l+1)})^{\sigma} \|u\|_{[X_0,X_1]_{\sigma,q}}$$

$$= \beta_s \mathfrak{C}((l+1)^{1-\sigma} (l+2)^{\sigma})^{\frac{1}{q}} d^{-(l+\sigma)} \Lambda_{\sigma,q} [\![u]\!]_A$$

$$\le \beta_s \mathfrak{C}((l+2)^{1/q} \Lambda_{\sigma,q} d^{-s} [\![u]\!]_A .$$

Therefore, from Theorem (4.1.4) and relations (22), (23) and $l \le \ell - 1$, we finally derive (20). The proof is complete.

Remark (4.1.12) [134]:

It follows from the preceding proof that (20) holds with $\|u\|_{s,q,\Omega}$ instead of $|u|_{s,q,\Omega}$ on the left-hand side of the bound. It is worth noting that, if $u|_A = 0$, we get

$$\|u\|_{s,q,\Omega} \le \mathfrak{C}^* \Lambda_{\sigma,q} d^{r-s-n(1/p-1/q)_+} |u|_{r,p,\Omega} .$$

Cf. a similar result in Wendland [156]. See also Krebs et al. [144] for a particular case of Theorem (4.1.11).

The presence of the number $\Lambda_{\sigma,q}$ on the right-hand side of (20) may be quite disturbing from a numerical standpoint since, by (21), $\Lambda_{\sigma,q} \to \infty$ as $\sigma \to 0^+$ or $\sigma \to 1^-$. Hence, it is natural to wonder whether the appearance of such a number is an intrinsic feature of the sampling inequality or an undesirable by-product of the interpolation technique used to derive it. To provide an answer to this query, let us temporarily assume that (20) can be written with a constant C, independent of s, instead of $\mathfrak{C}^* \Lambda_{\sigma,q}$. With the help of (23) and writing $s = l + \sigma$, with $l = \lfloor s \rfloor$ and $\sigma = s - l$, under such an assumption, the sampling inequality reads as

$$|u|_{l+\sigma,q,\Omega} \le C d^{-(l+\sigma)} [\![u]\!]_A. \tag{26}$$

Let us fix a suitable set A and a function $u \in W^{r,p}(\Omega)$. By Lemma (4.1.5)(ii), the product $(1-\sigma)^{1/q} |u|_{l+\sigma,q,\Omega}$ has a finite limit as $\sigma \to 1^-$. Hence, unless u is a polynomial of degree≤ l, it is clear that

$$\lim_{\sigma\to 1^-} |u|_{l+\sigma,q,\Omega} = \infty.$$

Likewise

$$\lim_{\sigma\to 1^-} Cd^{-(l+\sigma)}[\![u]\!]_A = Cd^{-(l+1)}[\![u]\!]_A.$$

Therefore, by taking limits in (26) as $\sigma \to 1^-$, one is led to an obvious contradiction. This means that C must depend on s. In fact, since $|u|_{l+\sigma,q,\Omega}$ grows to ∞ as $\sigma \to 1^-$ at the same rate as $(1-\sigma)^{1/q}$, this should also be the case of C.

The preceding reasoning, however, does not apply as $\sigma \to 0^+$. In this case, by Lemma (4.1.5) (i), the semi-norm $|\cdot|_{l+\sigma,q,\Omega}$ has a finite limit and so the product $\sigma^{1/q}|u|_{l+\sigma,q,\Omega}$ simply tends to 0. If we had taken limits in (26) as $\sigma \to 0^+$, we would not have been led to contradiction.

Summarizing, the sampling inequality (20), for a non-integer s, contains the number $\Lambda_{\sigma,q} = q^{-1/q}\sigma^{-1/q}(1-\sigma)^{-1/q}$ on its right-hand side. The presence of the decay factor $(1-\sigma)^{-1/q}$ is absolutely necessary, while that of $\sigma^{-1/q}$ lacks an analogous justification. Whether or not the factor $\sigma^{-1/q}$ can be avoided is an open question.

The above discussion suggests that the sampling inequality (20) could be advantageously rewritten as follows:

$$[u]_{s,q,\Omega} \le \mathfrak{C}^* \left(d^{r-s-n(1/p-1/q)_+} \Big|u\Big|_{r,p,\Omega} + d^{\frac{n}{\gamma}-s} \|u|_A\|_x \right), \tag{27}$$

where $[\cdot]_{s,q,\Omega} = \Lambda^{-1}_{\sigma,q,\Omega}|\cdot|_{s,q,\Omega}$, with $\sigma = s - \lfloor s \rfloor$. The use of a “normalized” semi- norm like $[\cdot]_{s,q,\Omega}$ is common to deal with defective semi-norms, as is the case of the intrinsic Sobolev semi-norms of fractional order. See, for example, Milman [148]. Let us observe that $[\cdot]_{\sigma+1,q,\Omega}$behaves “continuously” as $\sigma \to 1^-$, since, by Lemma (4.1.5) (ii), this semi-norm, up to a constant, tends to a semi-norm equivalent to $|\cdot|_{l+1,q,\Omega} = [\cdot]_{l+1,q,\Omega}$. however , the semi-norm $[\cdot]_{l+\sigma,q,\Omega}$ is not fully satisfactory, since, by Lemma (4.1.5) (i),

$$\lim_{\sigma\to 0^+} [u]_{l+\sigma,q,\Omega} = 0.$$

we shall see later that matters are much clearer for $\Omega = \mathbb{R}^n$

The contents of this subsection follow the account of Corrigan et al. (cf. [142]), that we have changed and improved.

We have previously obtained sampling inequalities which involve Lagrange data. Now we are interested in expressing those bounds in terms of derivative data, that is, values of derivatives at selected points. Of course, the nature of the data plays a crucial role in the matter. On the one hand, it is clear that, given a function in $W^{r,p}(\Omega)$, the derivative data cannot include derivatives of arbitrary order. In fact, assuming that r and p satisfy (17), only derivative data of order, at most, μ^* should be allowed, where

$$\mu^* = \begin{cases} \lfloor r-n \rfloor, & \text{if } p=1, \\ \lceil r-n/p \rceil - 1, & \text{if } 1<p<\infty, \\ r-1, & \text{if } p=\infty. \end{cases} \quad (28)$$

On the other hand, to keep things at a reasonable degree of complexity, we assume that, for a function u, the sets of derivative data of order $\mu \le \mu^*$ consist of the values $\partial^\alpha u(a)$ of all the derivatives of order $|\alpha| = \mu$ at the points of "dense" subsets A of $\bar{\Omega}$.

Theorem (4.1.13) [134]:

Let Ω be a bounded $\mathcal{L}[\rho,\theta]$-domain of $\mathbb{R}^n$, for some $\rho > 0$ and $\theta \in (0, \pi/2]$. Suppose that $, q, x$, and r satisfy (17), let $\gamma = \max\{p, q, x\}$ and let and μ^* be the integers given by (18) and (28), respectively. Then, there exist two constants $\mathfrak{d}^{**} \in (0,1]$ (dependent on θ, ρ, n, r and p) and $\mathfrak{C}^{**} > 0$ (dependent on Ω, n, r, p, q and x) such that the following property holds: for any $\mu = 0, \dots, \mu^*$, for any finite set $A \subset \bar{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $\delta(\bar{\Omega}, A) \le \mathfrak{d}^{**}$, for any $u \in W^{r,p}(\Omega)$ and for any $s \in [0, \ell]$, we have

$$|u|_{s,q,\Omega} \le \mathfrak{C}^{**} \Lambda_{\sigma,q} (d^{r-s-n(1/p-1/q)_+} |u|_{r,p,\Omega} + d^{\frac{n}{\gamma}+\lambda-s} \|D^\lambda u|_A\|_x), \quad (29)$$

where $d = \delta(\bar{\Omega}, A)$, $\lambda = \min\{\mu, \lfloor s \rfloor\}$, $\sigma = s - \lfloor s \rfloor$, $\Lambda_{\sigma,q}$ is given by (21) and

$$\|D^j u|_A\|_x = \begin{cases} \left(\sum_{|\alpha|=j} \sum_{a\in A} |\partial^\alpha u(a)|^x \right)^{1/x}, & \text{if } x < \infty, \\ \max_{|\alpha|=j} \max_{a\in A} |\partial^\alpha u(a)|, & \text{if } x = \infty. \end{cases} \quad (30)$$

Proof :

Let j be an integer in $\{0, \dots, \mu^*\}$. It is clear that (17c) holds with $r - j$ instead of . Hence, we can apply Theorem (4.1.11) with $r - j$ instead of . Let $\mathfrak{d}_j^*$ and $\mathfrak{C}_j^*$ be the two constants that Theorem (4.1.11) provides. Thus, for any finite set $A \subset \bar{\Omega}$ (or $A \subset \Omega$ if $p = 1$and $r - j = n$) for which $d = \delta(\bar{\Omega}, A) \le \mathfrak{d}^*$, for any $v \in W^{r-j,p}(\Omega)$ and for any $\hat{s} \in [0, \ell - j]$, we have

$$|v|_{\hat{s},q,\Omega} \le \mathfrak{C}_j^* \Lambda_{\hat{\sigma},q} (d^{r-\hat{s}-n(1/p-1/q)_+} |v|_{r-j,p,\Omega} + d^{\frac{n}{\gamma}-\hat{s}} \|v|_A\|_x), \quad (31)$$

with $\hat{\sigma} = \hat{s} - \lfloor \hat{s} \rfloor$. Then, we take

$$\mathfrak{d}^{**} = \min_{0\le j\le\mu^*} \mathfrak{d}_j^*,$$

which obviously is a constant that depends only on θ, ρ, n, r and p. Likewise, we choose

$$\mathfrak{C}^{**} = \max\left\{ v_j^{1/q} \mathfrak{C}_j^* : j = 0, \dots, \mu^* \right\}, \quad (32)$$

where $\nu_j = \binom{j+n-1}{j}$ is just the cardinal number of $\{\alpha \in \mathbb{N}^n : |\alpha| = j\}$. The constant $\mathfrak{C}^{**}$ depends on Ω, n, r, p, q and x.

Now, let $\mu \in \{0, \ldots, \mu^*\}$ and $s \in [0, \ell]$. We first assume that $\mu \leq s$, so $\lambda = \min\{\mu, \lfloor s \rfloor\} = \mu$. Let us see that (29) holds.

Let $A \subset \bar{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $d = \delta(\bar{\Omega}, A) \leq \mathfrak{d}^{**}$.

Given $u \in W^{r,p}(\Omega)$, for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| = \mu$, the function $\partial^\alpha u$ belongs to $W^{r-\mu,p}(\Omega)$. Hence, by (31) with $\hat{s} = s - \mu$ and $j = \mu$, since $\sigma = s - \lfloor s \rfloor = \hat{s} - \lfloor \hat{s} \rfloor = \hat{\sigma}$, we get

$$|\partial^\alpha u|_{s-\mu,p,\Omega} \leq \mathfrak{C}_\mu^* \Lambda_{\sigma,q} (d^{r-s-n(1/p-1/q)_+} |\partial^\alpha u|_{r-j,p,\Omega} + d^{n/\gamma - s + \mu} \|\partial^\alpha u|_A\|_x)$$

Consequently, since $|\partial^\alpha u|_{r-\mu,p,\Omega} \leq |u|_{r,p,\Omega}$, we deduce that

$$|\partial^\alpha u|_{s-\mu,p,\Omega} \leq \mathfrak{C}_\mu^* \Lambda_{\sigma,q} (d^{r-s-n(1/p-1/q)_+} |u|_{r,p,\Omega} + d^{n/\gamma - s + \mu} \|D^\mu u|_A\|_x).$$

This relation implies (29), taking into account (32) and that

$$|u|_{s,q,\Omega}^q \leq \sum_{|\alpha|=\mu} |\partial^\alpha u|_{s-\mu,p,\Omega}^q.$$

Let us finally assume that $\mu > s$. The above reasoning cannot be done, since it explicitly requires the condition $s - \mu \geq 0$. However, since $\lfloor s \rfloor \leq s < \mu \leq \mu^*$, such a reasoning still holds if μ is replaced by $\lfloor s \rfloor$, which yields (29) with $\lambda = \min\{\mu, \lfloor s \rfloor\} = \lfloor s \rfloor$.

Following Madych and Potter [146] and Madych [145], we considered in [137] the case of sampling inequalities for functions defined on $\mathbb{R}^n$ or on quite general unbounded domains. In the present section, for the sake of simplicity, we limit ourselves to the $\mathbb{R}^n$-case.

There are three main changes with respect to the framework described. Firstly, instead of finite sets A, sampling inequalities will now involve discrete sets (i.e., sets without accumulation points) whose fill distance to $\mathbb{R}^n$ is finite. Such sets, in particular, should be countably infinite. Secondly, the parameters p, q, x and r should satisfy a set of conditions more restrictive than (17), namely:

$$p, q, x \in [1, \infty), \quad \text{with} \max\{p, x\} \leq q, \tag{33a}$$

$$r \in [n, \infty), \quad \text{if } p = 1, \text{or } r \in \left(\frac{n}{p}, \infty\right), \quad \text{if } p > 1. \tag{33b}$$

Please note that (33a) implies that $(1/p - 1/q)_+ = 1/p - 1/q$. We still define by (18), which, accordingly with (33), now reads as with

$$\ell = \begin{cases} l_0, & if\ r \in \mathbb{N}^* \ and\ either\ p < q\ and\ l_0 \in \mathbb{N}, or\ p = q, \\ \lceil l_0 \rceil - 1, & otherwise, \end{cases} \tag{34}$$

with $l_0 = r - n(1/p - 1/q)$.

Finally, sampling inequalities on unbounded domains, in particular, $\mathbb{R}^n$, are established for functional spaces more general than the Sobolev ones. Specifically, as in [137], we shall use the Beppo–Levi space $\dot{W}^{r,p}(\mathbb{R}^n)$, formed by all the distributions v on $\mathbb{R}^n$whose $\lfloor r\rfloor$th order derivatives are functions such that $|v|_{r,p,\mathbb{R}^n} < \infty$. We recall that any $v \in \dot{W}^{r,p}(\mathbb{R}^n)$ is locally in $W^{r,p}(\mathbb{R}^n)$, that is, v belongs to $W^{r,p}(\omega)$ for any bounded open set $\omega \subset \mathbb{R}^n$(cf. [137]).

We shall need some additional notations and results. For any $\nu \in \mathbb{N}^*$, we write $\omega_\nu = B(0,\nu)$. For suitable values of $\rho > 0$ and $\theta \in (0,\pi/2)$ (e.g., $\rho = 1/2$ and $\theta = \pi/3$; cf. Wendland [154]), any ball ω_ν is a $\mathcal{L}[\rho,\theta]$-domain. Likewise,

$$\omega_\nu \subset \omega_{\nu+1}, \qquad \text{for all } \nu \in \mathbb{N}^*, \quad \text{and} \bigcup_{\nu\in\mathbb{N}^*} \omega_\nu = \mathbb{R}^n. \tag{35}$$

Thus, in the terminology introduced in [137], $(\omega_n)_{n\in\mathbb{N}^*}$ is a filling sequence of $\mathbb{R}^n$. We shall exploit this fact.

Lemma (4.1.14) [134]:

Let $\rho > 0$ and $\theta \in (0,\pi/2)$ such that ω_1 is a $\mathcal{L}[\rho,\theta]$-domain. Let A be a discrete subset of $\mathbb{R}^n$such that $(\mathbb{R}^n, A) \le \rho/\tau$, with $\tau = 1 + 1/\sin\theta$, and, for any $\nu \in \mathbb{N}^*$, let $A_\nu = A \cap \omega_\nu$. Then, there exists an integer ν_0 , depending on A, such that

$\forall \nu \in \mathbb{N}^*, \nu \ge \nu_0, \tau^{-1}\delta(\mathbb{R}^n, A) \le \delta(\overline{\omega}_\nu, A_\nu) \le \tau\,\delta(\mathbb{R}^n, A)$, .

Proof:

The hypothesis on ρ and θ implies that any other ball $\omega_\nu$is also a $\mathcal{L}[\rho,\theta]$- domain. Now, the existence of $\nu_0$and the lower bound come from [137], whereas the upper bound follows from [137].

Lemma (4.1.15) [134]: Suppose that p, q, x and r satisfy (34) and let ℓ be the integer defined by (35). For any $\nu \in \mathbb{N}^*$, let us apply Theorems (4.1.11) and (4.1.13) with $\Omega = \omega_\nu$. Then, the constants $\mathfrak{C}^*$and $\mathfrak{C}^{**}$ that these theorems respectively provide can be chosen independently of ν.

Proof : Given the way in which the constant $\mathfrak{C}^{**}$is chosen in Theorem (4.1.13), it suffices to prove the result for the constant $\mathfrak{C}^*$of Theorem (4.1.11).

(1) We first apply Theorem (4.1.11) to $\Omega = \omega_1$, which is a $\mathcal{L}[\rho,\theta]$-domain for, say, $\rho = 1/2$ and $\theta = \pi/3$. We get a constant $\mathfrak{d}^*$, only dependent on n and r (and the fixed values of ρ and θ), and a constant $\widehat{\mathfrak{C}}$, dependent on n, r, p, q and x (the dependence on ω_1, unit ball of $\mathbb{R}^n$, is subsumed in that on n), such that

$$|\hat{u}|_{s,q,\omega_1} \le \widehat{\mathfrak{C}}\Lambda_{\sigma,q}\,(\hat{d}^{\,r-s-n\left(\frac{1}{p}-\frac{1}{q}\right)}|\hat{u}|_{r,p,\omega_1} + \hat{d}^{\frac{n}{q}-s}\,\|\hat{u}|_{\hat{A}}\|_x\,), \tag{36}$$

where$\hat{A}$ is any finite subset of ω_1 (or ω_1 if $p = 1$ and $r = n$) such that $\hat{d} = \delta(\overline{\omega}_1, \hat{A}) \leq \mathfrak{d}^*$, $\hat{u}$ is any element of $W^{r,p}(\omega_1)$ and $s \in [0, \ell]$.

(2) We shall now use the technique of change of scale. Let $\nu \in \mathbb{N}^*$ be fixed and let $L_\nu: \mathbb{R}^n \to \mathbb{R}^n$be the linear continuous mapping given by $\hat{x} \to \nu\hat{x}$. Clearly

$\omega_\nu = L_\nu\ (\omega_1)$ and $\det L_\nu = \nu_n$.

Likewise, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq r$ and for any $u \in W^{r,p}(\omega_\nu)$, we have

$\partial^\alpha u(x\ ) = \nu^{-|\alpha|}\partial^\alpha \hat{u}(\hat{x})$ a.e. on ω_ν,

where$\hat{u} = u \circ L_\nu$and $\hat{x} = L_\nu^{-1}(x) = \nu^{-1}x$. Thus, it is readily seen that, for any$u \in W^{r,p}(\omega_\nu)$ and for any $s \in [0, \ell]$, either integer or non-integer,

$$|u|_{s,q,\omega_\nu} = \nu^{-s+n/q}|\hat{u}|_{s,q,\omega_1} \text{and} |u|_{r,p,\omega_\nu} = \nu^{-r+n/p}|\hat{u}|_{r,p,\omega_1}. \tag{37}$$

In addition, given $A \subset \omega_\nu$, we have

$$u|_A = \hat{u}|_{\hat{A}}\ and\ d = \nu\hat{d} \tag{38}$$

with$d = \delta(\overline{\omega}_\nu, A)$, $\hat{A} = L_\nu^{-1}(A)$ and $\hat{d} = \delta(\overline{\omega}_1, \hat{A})$.

(3) Let $u \in W^{r,p}(\omega_\nu)$ and $A \subset \overline{\omega}_\nu$ (or $A \subset \omega_\nu$ if $p = 1$ and $r = n$) such that $\mathfrak{d} \leq \mathfrak{d}^*$. Then, $\hat{u} \in W^{r,p}(\omega_1)$and $\hat{A}$ is such that $\hat{d} \leq \mathfrak{d}^*/\nu \leq \mathfrak{d}^*$. It follows from (36), (37) and (38) that, for any $s \in [0, \ell]$,

$$|u|_{s,q\ ,\omega_\nu} \leq \widehat{\mathfrak{C}}\Lambda_{\sigma,q}\ (d^{r\ -s-n(1/\ p-1/q\ )}\ |u|_{r,p,\omega_\nu} + d^{\ n/q\ -s}\ \|u|_A\|_x\ ),$$

Thus, we obtain the bound of Theorem (4.1.11) for $\Omega = \omega_\nu$ with $\mathfrak{C}^* = \widehat{\mathfrak{C}}$ which is independent of ν

Theorem (4.1.16) [134]:

Suppose that , q , x , and r satisfy (33) and let ℓ be the integer given by (34). Then, there exist two constants $\mathfrak{d}' \in (0,1]$ (dependent on n and) and $\mathfrak{C}' > 0$ (dependent on n, r, p, q and x) such that the following property holds: for any discrete set $A \subset \mathbb{R}^n$such that $\delta(\mathbb{R}^n, A) \leq \mathfrak{d}'$, for any $u \in \dot{W}^{r,p}(\mathbb{R}^n\ )$ and for any $s \in [0, \ell]$, we have

$$|u|_{s,q\ ,\mathbb{R}^n} \leq\ \mathfrak{C}'\ \Lambda_{\sigma,q}\ (d^{r\ -s-n\left(\frac{1}{p}-\frac{1}{q}\right)}|u|_{r,p,\mathbb{R}^n} + d^{\frac{n}{q}\ -s}\ \|u|_A\|_x), \tag{39}$$

where $d = \delta(\mathbb{R}^n, A)$ is defined by (3), $\|u|_A\|_x$ is given by (4), $\sigma = s - \lfloor s \rfloor$ and $\Lambda_{\sigma,q}$ is given by (21).

Proof :

We proceed as in the proof of [137]. Likewise, as in the proof of Lemma (4.1.15), we fix $\rho = 1/2$ and $\theta = \pi/3$, and we let $\tau = 1 + 1/\sin\theta$.

We set $' = \mathfrak{d}^*/\tau$, where $\mathfrak{d}^*$is the constant given by Theorem (4.1.11) when applied to any $\mathcal{L}[\rho, \theta]$-domain for n, r and the fixed values of ρ and θ . Hence, $\mathfrak{d}'$ depends on n and , and, since $\mathfrak{d}^* \leq 1 < \tau$, we have $\mathfrak{d}' \in (0,1]$. Likewise, remembering the definition of $\mathfrak{d}^*$given in the proof of Theorem (4.1.11) and taking Remark (4.1.12) into account, we see that $\mathfrak{d}'<\mathfrak{d}^* \leq \rho/\tau$.

Let A be a discrete subset of $\mathbb{R}^n$ such that $(\mathbb{R}^n, A) \leq \mathfrak{d}'$. It follows from Lemma(4.1.15) that, for any $\nu \in \mathbb{N}^*$ big enough, $(\overline{\omega}_\nu, A_\nu) \leq \tau\mathfrak{d}' = \mathfrak{d}^*$, with $A_\nu = A \cap \omega_\nu$. Then, by Theorem (4.1.11), for any $u \in \dot{W}^{r,p}(\mathbb{R}^n)$, which belongs to $W^{r,p}(\omega_\nu)$, and for any $s \in [0, \ell]$, we have

$$|u|_{s,q,\omega_\nu} \leq \mathfrak{C}^* \Lambda_{\sigma,q} \left(d_\nu^{r-s-n\left(\frac{1}{p}-\frac{1}{q}\right)} |u|_{r,p,\omega_\nu} + d_\nu^{\frac{n}{p}-s} \|u|_{A_\nu}\|_{x}\right), \tag{40}$$

with $d_\nu = \delta(\overline{\omega}_\nu, A_\nu)$, where, by Lemma (4.1.5), the constant $\mathfrak{C}^*$ only depends on n, r, p, q and x (it does not depend on ν). Since $A_\nu \subset A$, it is clear that $\|u|_{A_\nu}\|_{x} \leq \|u|_A\|_x$. Likewise, $|u|_{r,p,\omega_\nu} \leq |u|_{r,p,\mathbb{R}^n}$. Hence

$$|u|_{s,q,\omega_\nu} \leq \mathfrak{C}^* \Lambda_{\sigma,q} \left(d_\nu^{r-s-n(1/p-1/q)} |u|_{r,p,\mathbb{R}^n} + d_\nu^{n/p-s} \|u|_{A_\nu}\|_{x}\right),$$

Now, using again Lemma (4.1.15), we obtain

$$|u|_{s,q,\omega_\nu} \leq \mathfrak{C}^* \Lambda_{\sigma,q} \max\{\tau^{r-s-n(1/p-1/q)}, \tau^{|n/q-s|}\}$$
$$\times \left(\delta(\mathbb{R}^n, A)^{r-s-n(1/p-1/q)} |u|_{r,p,\mathbb{R}^n} + \delta(\mathbb{R}^n, A)^{n/q-s} \|u|_A\|_x\right).$$

Since $\tau > 1, r \geq n/p$ and $0 \leq s \leq r - n/p + n/q$, it then suffices to set

$$\mathfrak{E}' = \mathfrak{C}^* \tau^{r-n/p+n/q}$$

and to take limits as $\nu \to \infty$, since the sequence $(|u|_{s,q,\omega_\nu})_{\nu \in \mathbb{N}^*}$ is increasing and $|u|_{s,q,\omega_\nu} \to |u|_{s,q,\mathbb{R}^n}$ as $\nu \to \infty$ (this is a consequence of the Monotone Convergence Theorem).

Remark (4.1.17) [134]:

In view of (39), we may take up again the discussion held in Sect. 3.2 about the role of the number $\Lambda_{\sigma,q}$ in the sampling inequality. The reasoning is now based on Lemma (4.1.6), which shows that the semi-norm $|\cdot|_{l+\sigma,q,\mathbb{R}^n}$, with $l \in \mathbb{N}$ and $\sigma \in (0,1)$, blows up to infinity as $\sigma \to 0^+$ and $\sigma \to 0^-$ (except for polynomials of degree$\leq l$). Consequently, the presence in (36) of both factors $\sigma^{-1/q}$ and $(1-\sigma)^{-1/q}$, by means of $\Lambda_{\sigma,q}$, is absolutely required by the intrinsic nature of the semi-norm $|\cdot|_{s,q,\mathbb{R}^n}$.

Of course, the sampling inequality (39) can be expressed as

$$[u]_{s,q,\mathbb{R}^n} \leq C\left(d^{r-s-n(1/p-1/q)+} |u|_{r,p,\mathbb{R}^n} + d^{\frac{n}{\gamma}-s} \|u|_A\|_x\right), \tag{41}$$

where $[\cdot]_{s,q,\mathbb{R}^n} = \Lambda_{\sigma,q}^{-1} |\cdot|_{s,q,\mathbb{R}^n}$, with $\sigma = s - \lfloor s \rfloor$. In this way, the constant on the right-hand side of the sampling inequality no longer blows up. We remark that, on the left-hand side, one could also use any semi-norm equivalent to $[\cdot]_{s,q,\mathbb{R}^n}$ with equivalence constants not depending onσ . For example, for$q = 2$, an admissible choice could be the semi-norm $|\cdot|_{0,s}$ given

$$|v|_{0,s} = \left(\int_{\mathbb{R}^n} |\xi|^2 |\hat{v}\xi|^2 d\xi\right)^{1/2},$$

$\hat{v}$being the Fourier transform of v (cf. [138]).

Theorem (4.1.18) [134]:

Suppose that p, q, x and r satisfy (33) and let ℓ and μ^* be the integers given by (34) and (28), respectively. Then, there exist two constants $\mathfrak{d}'' \in (0,1]$ (dependent on n, r and p) and $\mathfrak{C}'' > 0$ (dependent on n, r, p, q and x) such that the following property holds: for any $\mu = 0, \ldots, \mu^*$, for any discrete set $A \subset \mathbb{R}^n$ such that $\delta(\mathbb{R}^n, A) \leq \mathfrak{d}''$, for any $u \in \dot{W}^{r,p}(\mathbb{R}^n)$ and for any $s \in [0, \ell]$, we have

$$|u|_{s,q,\mathbb{R}^n} \leq \mathfrak{C}'' \Lambda_{\sigma,q}(d^{r-s-n(1/p-1/q)} |u|_{r,p,\mathbb{R}^n} + d^{n/q+\lambda-s} \|D^\lambda u|_A\|_x),$$

where $d = \delta(\mathbb{R}^n, A)$ is defined by (3), $\lambda = \min\{\mu, \lfloor s \rfloor\}$, $D^j u|_A$ is given by (30), $\sigma = s - \lfloor s \rfloor$ and $\Lambda_{\sigma,q}$ is given by (21).

Proof:

Identical to that of Theorem (4.1.16), but applying Theorem (4.1.13) to the filling sequence $(\omega_v)_{v \in \mathbb{N}^*}$ instead of Theorem (4.1.11).

Sec(4.2):Sobolev Spaces and Sampling Inequalities:

The concept of sampling inequality was introduced by Rieger in his Ph.D. Thesis (cf. [169]; see also Rieger et al. [170]) in order to systematize a new kind of bounds developed, in their origins, to cope with error estimates, using Sobolev norms, in scattered data interpolation. Due to their increasing number of applications in areas like approximation theory, machine learning or mesh- less methods for PDE, the research on sampling inequalities has received an increasing attention. In our opinion, some milestones in this field previous to Rieger's dissertation are the pioneering work of Duchon [164] and the by section Wendland and Rieger [175], Narcowich et al. [18], Madych[168] and Arcangéli et al. [19]. Actual lines of research on sampling inequalities and their applications include, for example, the extension of sampling inequalities to functions defined on Riemannian manifolds (cf. Hangelbroek et al. [166]) or to spaces of infinitely smooth functions (cf. Rieger and Zwicknagl [171,172]), the derivation of Sobolev-type error estimates for several interpolation and approximation methods (cf. Fuselier and Wright [165], Lee et al. [168]), or the development of meshless methods for the numerical solution of PDE (cf. Schräder and Wend- land [173]).

In this section, we shall restrict our study to the classic frame of Sobolev spaces. We have two objectives in mind: (a) to provide new insights on the structure of sampling inequalities and, in particular, on the discrete term, and (b) to show that sampling inequalities are strongly related to and allow the recovering of existing bounds for intermediate semi-norms. We shall later precise these goals. Before that, we introduce the notations used throughout this section and formalize, in this context, the notion of sampling inequality.

For any x ∈ $\mathbb{R}$, we shall write $\lfloor x \rfloor$ and $\lceil x \rceil$ for the floor (or integer part) and ceiling of x , that is, the unique integers satisfying $\lfloor x\rfloor \leq x < \lfloor x\rfloor + 1$ and $\lceil x\rceil - 1 < x \leq \lceil x \rceil$. The letter n will always stand for an integer belonging to N $\mathbb{N}^* = N\backslash\{0\}$ (by convention, $0 \in \mathbb{N}$). Likewise, we write for the set of positive real numbers. The Euclidean norm in $\mathbb{R}^n$ will be denoted by $|\cdot|$.

For any multi-index $= (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})$, $x_1, \dots, x_n$ being the generic independent variables in $\mathbb{R}^n$.

Let Ω be a nonempty open set in $\mathbb{R}^n$. For any $r \in \mathbb{N}$ and for any $p \in [1, \infty)$, we shall denote by $W^{r,p}(\Omega)$ the usual Sobolev space defined by

$$W^{r,p}(\Omega) = \{ v \in L^p(\Omega) \colon \forall \alpha \in \mathbb{N}^n, |\alpha| \leq r, \partial^\alpha v \in L^p(\Omega)\}.$$

We recall that the derivatives $\partial^\alpha v$ are taken in the distributional sense. The space $W^{r,p}(\Omega)$ is equipped with the semi-norms $|\cdot|_{j,p,\Omega}$, with $j \in \{0, \dots, r\}$, and the norm $\|\cdot\|_{r,p,\Omega}$ given by

$$|v|_{j,p,\Omega} = \left(\sum_{|\alpha|=j} \int_\Omega |\partial^\alpha v(x)|^p dx \right)^{1/p} \quad and \quad \|v\|_{r,p,\Omega} = \left(\sum_{j=0}^{r} |v|_{j,p,\Omega}^p \right)^{1/p}.$$

We remark that $|\cdot|_{0,p,\Omega}$ is, in fact, the usual L^P-norm $\|\cdot\|_{L^P(\Omega)}$. For any $r \in \mathbb{R}_+^*\backslash\mathbb{N}$ and for any $p \in [1, \infty)$, we shall denote by $W^{r,p}(\Omega)$ the Sobolev space of noninteger order r, formed by the (equivalence classes of) functions v ∈ $W^{\lfloor r \rfloor,p}(\Omega)$ such that

$$|v|_{j,p,\Omega} = \left(\sum_{|\alpha|=\lfloor r \rfloor} \int_{\Omega\times\Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^p}{|x-y|^{n+p(r-\lfloor r \rfloor)}} dxdy \right)^{1/p} < \infty \tag{42}$$

The space $W^{r,p}(\Omega)$ is endowed with the semi-norms $|\cdot|_{j,p,\Omega}$, with $j \in \{0, \dots, \lfloor r \rfloor\}$, and $|\cdot|_{r,p,\Omega}$ and the norm

$$\|v\|_{r,p,\Omega} = \left(\|v\|_{\lfloor r \rfloor,p,\Omega}^p + |v|_{r,p,\Omega}^p \right)^{1/p}.$$

It is known that, for any $r \in \mathbb{R}_+^*\backslash\mathbb{N}$ and $p \in [1, +\infty)$ such that either $r > n/p$, if $p > 1$ or $r \geq n$, if $p = 1$, the space $W^{r,p}(\Omega)$is continuously embedded in $C_B^0(\Omega)$ (space of bounded continuous functions on Ω), provided that, for example, $\Omega = \mathbb{R}^n$ or Ω is a bounded open set with a Lipschitz-continuous boundary. For $r > n/p$, the space $C_B^0(\Omega)$can be replaced by the space $C^0(\overline{\Omega})$ of bounded and uniformly continuous functions on Ω .

Let $\Omega = \mathbb{R}^n$be a domain (i.e. a non-empty connected open set) and let $A \subset \overline{\Omega}$ be a discrete set (i.e. a closed set without accumulation points). We set

$$\delta(\overline{\Omega}, A) = \sup_{x \in \Omega} \inf_{a \in A} |x - a|.$$

It is clear that $\delta(\overline{\Omega}, A)$ may be infinite when Ω is unbounded, and is just the Hausdorff distance between A and $\overline{\Omega}$ when Ω is bounded. Moreover, for any function v defined on A and $p \in [1, \infty)$, we shall write

$$\|v|_A\|_p = \left(\sum_{a \in A} |v(a)|^p \right)^{1/p}.$$

If Ω is bounded, the discrete set A is finite and so it is $\|v|_A\|_p$. However, if Ω is unbounded, the set A is necessarily countably infinite; hence, $\|v|_A\|_p^p$ is a series which may diverge.

Let Ω be $\mathbb{R}^n$or a domain satisfying a suitable geometric condition. Likewise, let $r \in \mathbb{R}_+^*$ and $p, q \in [1, \infty)$, with $p \leq q$ and $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$. The condition $p \leq q$ is just a simplifying hypothesis that only modifies the form of forthcoming results.

We set $\nu = n(1/p - 1/q)$ and $\ell = \lceil r - \nu \rceil - 1$ (or $\ell = r - \nu$ if $r \in \mathbb{R}_+^*$ and $\nu \in \mathbb{N}$). Then, we shall write sampling inequality in the space $W^{r,p}(\Omega)$ for designating a relation such as

$$|u|_{l,q,\Omega} \leq C\left(d^{r-l-n(1/p-1/q)}\right)|u|_{r,q,\Omega} + d^{n/q-l}\|u|_A\|_p \ , \quad l \leq \ell, \tag{43}$$

where$u \in W^{r,p}(\Omega)$, $C = C(\Omega, n, r, p, q)$ and $d = \delta(\overline{\Omega}, A)$ is small enough, i.e. $d \leq \mathfrak{d}$ for some $\mathfrak{d} = \mathfrak{d}(\Omega, n, r)$. This relation corresponds to various situations like, for instance:

- the one of [134], where Ω is supposed to be a bounded domain with a Lipschitz-continuous boundary;
- the one of [161], where the domain Ω is supposed to verify a particular property, that is, the so-called property $\mathcal{C}[\rho, \theta]$.
- the one of [134], where $\Omega = \mathbb{R}^n$ (a particular case of the previous one)

Relation (43) is susceptible to several extensions. One may more generally replace $\|u|_A\|_p$ by a discrete norm defined from values of continuous linear forms on $W^{r,p}(\Omega)$. It may be a matter; for instance, of point values of derivatives (cf. [134]). The first question which must be asked about relation (43) is the following one: are these bounds meaningful, i.e. are they finite for all $d \in (0, \mathfrak{d}]$? The answer is obviously positive when Ω is bounded. However, when Ω is unbounded, we have already pointed out that $d = \delta(\overline{\Omega}, A)$may be infinite and that $\|u|_A\|_p^p$ may be a divergent series. Consequently, we are led to analyse the behaviour of the term

$$T_2 = d^{n/q - l} \|u|_A\|_p ,$$

i.e. the second term inside the parenthesis on the right-hand member of the sampling inequality (43). This will be the subject of. We shall first prove a one-sided Marcinkiewicz–Zygmund type estimate, which may have some interest on its own, and then we shall obtain sufficient conditions to assure that $T_2$is finite.

we shall briefly consider the problem of bounding the interpolation error by D^m -splines, which historically is the origin of the introduction of sampling inequalities, as already noted above.

Finally, we study the analogy and relationship between sampling inequalities and classical inequalities for intermediate semi-norms in Sobolev spaces $W^{r,p}(\Omega)$. For example, Adams and Fournier prove in [159] that, given a domain Ω satisfying the cone condition, $r \in \mathbb{N}, p \geq 1$ and $t_0 > 0$, there exists a constant C such that, for any $u \in W^{r,p}(\Omega), t \in (0, t_0)$ and $= 0, \dots, r$,

$$|u|_{l,p,\Omega} \leq C\left(t^{r-l} |u|_{r,p,\Omega} + t - l\, u|_{0,p,\Omega}\right) \tag{44}$$

We show that, in fact, (44) can be derived from (43), or even extended, under suitable conditions (cf. Theorems 4.1.16–4.1.18). Probably the inequalities so obtained may be not new in the literature or could be proved independently. But, to our knowledge, this method to recover or establish bounds for intermediate semi-norms starting from sampling inequalities is original.

We assume that Ω is a domain in $\mathbb{R}^n$, bounded or unbounded. Given a discrete subset A of $\mathbb{R}^n$, with card $A \geq 2$, we recall that the separation radius q of A is defined as

$$q = q(A) = \frac{1}{2}\inf\{|a - b| : a, b \in A, a \neq b\}$$

Theorem (4.2.1)[158]:

Let $r \in \mathbb{R}_+^*$ and $p \in [1, \infty)$ such that $r > n/p$, if $p > 1$ or $r \geq \Omega$ be a domain in $\mathbb{R}^n$having a Lipschitz boundary (cf. [162, Definition 1.4.4]). Then, there exist positive constants K and $\mathfrak{C}$such that, for any $u \in W^{r,p}(\Omega)$ and for any discrete subset $A \subset \overline{\Omega}$such that $q > 0$

$$\|u|_A\|_p^p \leq K 2^{p-1} q^{-n}\left(v_n^{-1} |u|_{0,p,\Omega}^p + \mathfrak{C} \max\{1, q^{rp}\} \|u\|_{r,p,,\Omega}^p\right),$$

where $v_n =$ meas $B(0,1)$.

Proof.

(1) Let us introduce the Stein total extension operator (cf. [174]), which exists since Ω is a domain with a Lipschitz boundary. Let $\mathfrak{S}$be such an operator. We recall that, for any

$p \in [1,\infty)$ and for all non-negative integers l , the operator $\mathfrak{S}$enjoys the following properties:

• for all $u \in W^{l,p}(\Omega)$, $(\mathfrak{S}u)|_{\Omega} = u$ (i.e., S is an extension operator);

• $\mathfrak{S}$ is a linear continuous operator from $W^{l,p}(\Omega)$ to $W^{l,p}(\mathbb{R}^n)$, and so there exists $M_l > 0$ such that

$$\forall u \in W^{l,p}(\Omega), \quad \| \mathfrak{S}u \|_{l,q,\mathbb{R}^n} \leq M_l \| u \|_{l,q,\Omega}. \tag{45}$$

In fact, for any $s \geq 0$, the operator $\mathfrak{S}$maps $W^{s,p}(\Omega)$ continuously into $W^{s,p}(\mathbb{R}^n)$, that is, there exists a constant $K_{s,p} > 0$ such that

$$\forall u \in W^{s,p}(\Omega), \quad \| \mathfrak{S}u \|_{s,p,\mathbb{R}^n} \leq K_{s,p} \| u \|_{s,p,\Omega}. \tag{46}$$

For proving this assertion, we only have to show it for any non integer $s > 0$. To this end, given $s > 0, s \notin \mathbb{N}$, we write $l = \lfloor s \rfloor$ and $\sigma = s - l$, and consider the spaces

$$[W^{l,p}(\mathbb{R}^n), W^{l+1,p}(\mathbb{R}^n)]_{\sigma,p} \text{and } [W^{l,p}(\Omega), W^{l+1,p}(\Omega)]_{\sigma,p} ,$$

obtained by real interpolation between the corresponding Sobolev spaces. It follows from (45) ,Adams and Fournier [159] that

$$\forall u \in W^{s,p}(\Omega),$$

$$\| \mathfrak{S}u \|_{[W^{l,p}(\mathbb{R}^n), W^{l+1,p}(\mathbb{R}^n)]_{\sigma,p}} \leq M_l^{1-\sigma} M_{l+1}^{\sigma} \| u \|_{[W^{l,p}(\Omega), W^{l+1,p}(\Omega)]_{\sigma,p}}. \tag{47}$$

Now, we know that $W^{s,p}(\mathbb{R}^n) = [W^{l,p}(\mathbb{R}^n), W^{l+1,p}(\mathbb{R}^n)]_{\sigma,p}$, with norm equivalence (cf [162]). Hence, there exists $C_1 > 0$ such that

$$\forall v \in W^{s,p}(\mathbb{R}^n), \|v\|_{s,p,\mathbb{R}^n} \leq C_1 \| v \|_{[W^{l,p}(\mathbb{R}^n), W^{l+1,p}(\mathbb{R}^n)]_{\sigma,p}}. \tag{48}$$

(for an estimation of C_1, see [134]). Likewise, the proof in Brenner and Scott [163] shows that there exists $C_2 > 0$ such that

$$\forall u \in W^{s,p}(\Omega), \| u \|_{[W^{l,p}(\Omega), W^{l+1,p}(\Omega)]_{\sigma,p}} \leq C_2 \|u\|_{s,p,\Omega}. \tag{49}$$

The relation (46) is then a consequence of (47)–(49).

(2) Let us fix a function $u \in W^{r,p}(\Omega)$. To simplify notations, we shall write U instead of $\mathfrak{S}u$. By the Mean Value Theorem, for any $a \in A$, we can pick a point $a' \in B(a,q)$ such that

$$\int_{B(a,q)} |U(x)|^p dx = q^n v_n |U(a')|^p, \tag{50}$$

where $v_n =$ meas $B(0,1)$. Let A' denote the set of points a' so selected. We remark that, for any $a, b \in A$, with $a \neq b$, the balls $B(a,q)$ and $B(b,q)$ are disjoint, so the corresponding points a' and b' are distinct. Moreover

$$\int_{\bigcup_{a\in A} B(a,q)} |U(x)|^p dx = \sum_{a\in A} \int_{B(a,q)} |U(x)|^p dx = q^n v_n \sum_{a\in A} |U(a')|^p \quad = q^n v_n \|U|_{A'}\|_p^p$$

which implies that

$$\|U|_{A'}\|_p^p \leq q^{-n} v_n^{-1} |U|_{o,p,\mathbb{R}^n}^p \tag{51}$$

On the other hand, we have

$$\|U|_{A'}\|_p^p \leq 2^{p-1}\big(\|U|_{A'}\|_p^p + \|U|_A - U|_{A'}\|_p^p\big). \tag{52}$$

Thus, to get the result, we are led to bound the term $\|U|_A - U|_{A'}\|_p^p$.

(3) Given $a \in A$, let $F: \mathbb{R}^n \to \mathbb{R}^n$ be the bijective affine mapping defined by $F(\hat{x}) = a + q\hat{x}$. It is clear that F transforms the ball $B(0,1)$ onto the ball $B(a,q)$ and that the Jacobian determinant of F is q^n . As usual, for any function $v: \mathbb{R}^n \to \mathbb{R}$, we write $\hat{v} = v \circ F$. Let us continue reasoning with the function $u \in W^{r,p}(\Omega)$ fixed in point (2). We have

$$U(a) - U(a') = \widehat{U}(0) - \widehat{U}(q^{-1}(a' - a),$$

and, taking into account that $|q^{-1}(a' - a)| < 1$, we obtain

$$|U(a) - U(a')| \leq 2 \sup_{\hat{x} \in B(0,1)} \big|\widehat{U}(\hat{x})\big|.$$

Now, since $W^{r,p}(B(0,1))$ is continuously embedded into $C_B^0(B(0,1))$, we have

$$|U(a) - U(a')| \leq \hat{C} \parallel \widehat{U} \parallel_{r,p,B(0,1)}, \tag{53}$$

where $\hat{C}$ is a constant that only depends on r , p and n.

Now, it easily results from the change of variables $x = F(\hat{x})$ that, for any $\alpha \in \mathbb{R}^n$ with $|\alpha| \leq r$,

$\partial^\alpha \widehat{U}(\hat{x}) = q^{|\alpha|} \partial^\alpha U(x)$ a.e. on $B(0,1)$.

Consequently, for any $s \in [0,r]$, either integer or non-integer, we readily have

$$|\widehat{U}|_{s,p,B(0,1)} = q^{s-n/p} |U|_{s,p,B(a,q)} .$$

Noting that, for all $s \in [0,r]$, $q^s \leq \max\{1, q^r\}$, we obtain

$$\parallel \widehat{U} \parallel_{r,p,B(0,1)} \leq q^{-n/q} \leq \max\{1, q^r\} \parallel U \parallel_{r,p,B(a,q)} . \tag{54}$$

Then, setting $\mathfrak{C} = \hat{C}^p$, we derive from (53) and (54) that

$$\begin{aligned}\|U|_A - U|_{A'}\|_p^p = \sum_{a \in A} |U(a) - U(a')|^p &\leq \mathfrak{C} \max\{1, q^r\} q^{-n} \sum_{a \in A} \|U\|_{r,p,B(a,q)}^p \\ &= \mathfrak{C} \max\{1, q^{rp}\} q^{-n} \|U\|_{r,p,\bigcup_{a \in A} B(a,q)}^p \\ &\leq \mathfrak{C} \max\{1, q^{rp}\} q^{-n} \|U\|_{r,p,\mathbb{R}^n}.\end{aligned} \tag{55}$$

Since $|_A = U|_A$, the theorem follows from (46), (52), (53) and (55).

Corollary (4.2.2) [158]:

Let $r \in \mathbb{R}_+^*$ and $p \in [1,\infty)$ such that $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$. Let Ω be a domain in $\mathbb{R}^n$having a Lipschitz boundary. Then, for any $u \in W^{r,p}(\Omega)$ and for any discrete subset $A \subset \overline{\Omega}$ such that $q > 0$, $u|_A$ belongs to the space $\ell^p(\mathbb{R})$ of real sequences $a = (a_i)_{i \in \mathbb{N}}$ such that $\sum_{i \in \mathbb{N}} |a_i|^p$ is finite.

Corollary (4.2.3) [158]:

Let A be a family of discrete sets in a domain $\Omega \subset \mathbb{R}^n$ satisfying the quasi-uniformity condition

$$\exists C > 0, \forall A \in \mathcal{A}, \ \ d \leq Cq,,$$

where$q$ is the separation radius of A and $d = \delta(\overline{\Omega}, A)$. Then, in the hypotheses of Theorem (4.2.1), the quantity $d^{n/p} \parallel u|_A \parallel_p$ is bounded independently of A.

As a consequence, under the conditions of Corollary (4.2.3), the relation (43) is always meaningful.

Let $f^A \in W^{r,p}(\Omega)$ be a function interpolating f on A. When setting$u = f^A - f$, the sampling inequality (43) takes the form of a relation between the semi-norms of orders l and r of the interpolation error $f^A - f$:

$$\left|f^A - f\right|_{l,q,\Omega} \leq Cd^{r-l-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left|f^A - f\right|_{r,p,\Omega}, \qquad l \leq \ell, \tag{56}$$

with $\ell = \lceil r - \nu \rceil - 1$ (or $\ell = r - \nu$ if $r \in \mathbb{N}^*$ and $\nu \in \mathbb{N}$), where $\nu = n(1/p - 1/q)$.

Above, it is a matter of Lagrange interpolation, but we could be thinking of a more general situation (as, for instance, when the discrete term in (43) involves derivatives). We could also remove the condition $\leq q$.

When Ω is a bounded domain with a Lipschitz-continuous boundary, $m \in \mathbb{N}^*$,$m > n/2$ and $p = 2$, an emblematic interpolation example is the one where f^A is the interpolating D^m – spline over Ω of f relative to A (cf. [160]), i.e. the element of $W^{m,2}(\Omega)$ such that

$$\forall v \in W^{m,2}(\Omega), \ \ v|_A = f|_A : \ |f^A|_{m,2,\Omega} \leq |v|_{m,2,\Omega} .$$

It is shown in [160] that limd →0 ∥ f A − f ∥m,2,Ω = 0. Then, for q > 2, we deduce from the relation (56) the estimate

$$|f^A - f|_{l,q,\Omega} = o(d^{m-l-n(1/2-1/q)}), \quad d \to 0, l \leq \ell.$$

When $\Omega = \mathbb{R}^n$and $p = 2$, we obtain similar results replacing the space $W^{m,2}(\Omega)$ by the space $D^{-m}L^2$ or, more generally, by the space $X^{m,s}$ (see in [160], for example, the definition of these spaces).

Let us observe that relation (56) follows results of J. Duchon, namely [164, Propositions 2 and 3]. So, we can consider that the notion of sampling inequality, as well as the main results about spline-interpolation derive from ideas of J. Duchon.

Our aim is to obtain a result analogous to the one of Adams and Fournier (cf. [159]) that we recalled in the introduction (see (44)), but which is a bound for the semi- norm $|\cdot|_{l,q,\Omega}$ of functions in $W^{r,p}(\Omega)$, with $q \geq p$.

Let us denote by $\mathcal{L}(\rho,\theta)$-domain a bounded domain of Rn with a Lipschitz-continuous boundary verifying the cone property with radius ρ and angle . Assuming that $q \geq p$and $x = p$, (see [134]) reads as

Theorem (4.2.4) [158]:

Let Ω be a $\mathcal{L}(\rho,\theta)$-domain of $\mathbb{R}^n$, with $\rho > 0$ and $\theta \in (0,\pi/2)$. We suppose that $p,q \in [1,\infty)$, with $p \leq q$, $r \in \mathbb{R}_+^*$ and $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$. We set $v = n(1/p - 1/q)$ and $\ell = \lceil r - v \rceil - 1$(or $\ell = r - v$ if $r \in \mathbb{N}^*$ and $v \in \mathbb{N}$) then, there exist two positive constants d (depending on θ, ρ, n and r) and C (depending on Ω, n, r, p and q) such that the following property holds:

(i) for any finite set $A \subset \bar{\Omega}$ such that $\delta(\bar{\Omega}, A) \leq \mathfrak{d}$,

(ii) for any $u \in W^{r,p}(\Omega)$ and for any $l = 0, \ldots, \ell$,

we have

$$|u|_{l,q,\Omega} \leq C d^{-v}\left(d^{r-l}\Big|u|_{r,p,\Omega} + d^{\frac{n}{p}-l} \parallel u|_A \parallel_p\right), \tag{57}$$

where $d = \delta(\bar{\Omega}, A)$.

The idea of the method is the following one: to consider Theorem (4.2.4) with a permutation of points (i) and (ii), as it is allowed to do, and then to choose the set A from the function u so that the norm $L^p(\Omega)$ of u appears in some upper bound of the term$d^{\frac{n}{q}-l}$ $\parallel u|_A \parallel_p$.

For any $h > 0$, let $\mathcal{R}_h$ be a set of open n-cubes $Q = Q_i$ of $\mathbb{R}^n$with sides of length h, such that any face of an n-cube $Q \in \mathcal{R}_h$ is a face of another n-cube belonging to $\mathcal{R}_h$and $\bigcup_{Q \in \mathcal{R}_h} \bar{Q} = \mathbb{R}^n$.

We set $I = \{i | \bar{Q}_i \subset \bar{\Omega}\}$. Let $\eta > 0$ such that $I \neq \phi$for $h \leq \eta$.

In (ii), we fix $u \in W^{r,p}(\Omega)$. Therefore (cf. [160]), u is continuous over $\bar{\Omega}$and it results from the Mean Value Theorem that

$$\forall i \in I, \exists c_i \in Q_i, \int_{Q_i} |u(x)|^p \, dx = h^n |u(c_i)|^p \ . \tag{58}$$

We set $A_h = \{c_i \mid i \in I\}$and $d_h = d(\bar{\Omega}, A_h)$. We want to use Theorem (4.2.4) with $A = A_h$. For that, we have to verify that, for h small enough, $d_h \leq d$. Moreover, in order to bound the right-hand member of (57), it is necessary to bound the term $d_h$below and above.

Lemma (4.2.5)[158]:

Let Ω be a $\mathcal{L}(\rho,\theta)$-domain of $\mathbb{R}^n$. There exist constants $\alpha > 1$ and $\beta \in (0,1)$ such that, for any $h \leq \eta, \beta_h \leq d_h \leq \alpha_h$.

Proof.

(1) We recall that $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$. For any $\in I$, let c_i^* be the centre of $Q_i$and let A^* be the set $\{c_i^* | \ i \in I\}$. Then, for any $h \leq \eta$ and for any $\in I$,

$$|x - c_i| \leq |x - c_i^*| + |c_i^* - c_i| \leq |x - c_i^*| + \frac{h}{2}\sqrt{n},$$

hence

$$\delta(\overline{\Omega}, A_h) \leq \sup_{x \in \Omega} \inf_{i \in I} |x - c_i^*| + \frac{h}{2}\sqrt{n} \leq \delta(\overline{\Omega}, A_h^*) + \frac{h}{2}\sqrt{n}.$$

(2) So, to get an upper bound of d_h/h, it is enough to get an upper bound of $\delta(\overline{\Omega}, A_h^*)$. First, let us point out that a $\mathcal{L}(\rho, \theta)$-domain is a fortio ri a $\mathcal{L}(\rho', \theta)$-domain for any $\rho' \leq \rho$. Now, let h be any positive number $\leq \eta$ and set $\rho' = h\tau\sqrt{n}$, with $= 1 + 1/\sin\theta$. Adjusting η if necessary, we have $\rho' \leq \rho$.

Let x be any point of Ω . Since Ω is a $\mathcal{L}(\rho', \theta)$-domain of $\mathbb{R}^n$, x is the vertex of a cone C , with radius ρ' and angle θ , included in Ω . On the other hand, we know that C contains a ball B of radius $'/\tau$, hence a cube of side $2\rho'/(\tau\sqrt{n})$. Therefore, C contains a cube Q_2 of side $2h$. We can always suppose that Q_2has its faces parallel to the ones of the generic element of $\mathcal{R}_h$, from which it is easy to verify that Q_2contains a cube $Q_1 \in \mathcal{R}_h$. Let us denote by $c^*$the centre of Q_1 . Then

$$\delta(\overline{\Omega}, A_h^*) \leq |x - c^*| \leq \sup_{y \in C} |x - y| \leq \rho',$$

and we finally obtain $\delta(\overline{\Omega}, A_h^*) \leq \tau\sqrt{n}h$. The bound follows with $\alpha = \sqrt{n}(\tau + 1/2)$

(3) We still have to get a lower bound for d_h/h. Let us fix a cube Q_j . This cube can be divided into 2^n small cubes with sides of length $h/2$. The cube Q_j contains a sole point of A_h , that is c_j . There obviously exists a small cube not containing c_j . Let T be its interior and t^* its centre. It is clear that T is a cube with sides of length $h/2$, which is contained in Q_j and does not contain c_j , hence no other point of A_h. In addition, the point of $A_h$close to t^* is, at least, on a face of , therefore from a distance $\geq h/4$ to t^*. Consequently, the distance from t^* to $A_h$is, at least, $h/4$. In short, we have $\beta_h \leq d_h$, with $\beta = 1/4$.

Remark (4.2.6) [158]:

The reasoning in point (1) of the above proof shows that one could choose $\eta = \rho/(\tau\sqrt{n})$ in the statement of Lemma (4.2.5).

Proposition (4.2.7) [158]:

For $h \leq \eta$, we have

$$d_h^{-v+n/p-l} \left\| u|_{A_h} \right\|_p \leq \alpha^{n/q} \beta^{-l} h^{-v-l} |u|_{o,p,\Omega}, \tag{59}$$

with $v = n(1/p - 1/q)$.

Proof.

From relation (58), we have

$$\sum_{i\in I}|u(c_i)|^p = h^{-n}\sum_{i\in I}\int_{Q_i}|u(c_i)|^p\,dx \le \int_\Omega |u(x)|^p\,dx.$$

Then, the proposition derives from Lemma (4.2.5).

The envisaged result will follow from Theorem (4.2.4), with $A = A_h$, Lemma (4.2.5) and Proposition (4.2.7).

Theorem (4.2.8) [158]:

Let Ω be a $\mathcal{L}(\rho,\theta)$ -domain of $\mathbb{R}^n$, with $\rho > 0$ and $\theta \in (0,\pi/2)$. We suppose that $p, q \in [1,\infty)$, with $p \le q$, r and $r > n/p$, if $p > 1$, or $r \ge n$, if $p = 1$. We set $\nu = n(1/p - 1/q)$ and $\ell = \lceil r-\nu\rceil - 1$ (or $\ell = r - \nu$ if $r \in \mathbb{N}^*$ and $\nu \in \mathbb{N}$). Then, for any $t_0 > 0$, there exists a positive constant C depending on Ω , n, r, p, q and t_0 such that, if $0 < t \le t_0$, $u \in W^{r,p}(\Omega)$ and $l = 0,\ldots,\ell$, we have the relation

$$|u|_{l,q,\Omega} \le Ct^{-\nu}\left(t^{r-l}\,|u|_{r,p,\Omega} + t^{-l}|u|_{0,p,\Omega}\right). \tag{60}$$

Proof.

Let us suppose that $h \le \eta$.

(1) By the right-hand inequality in Lemma (4.2.5) (i.e., $d_h \le \alpha_h$), we have the relation $(h \le d/\alpha) \Rightarrow (d_h \le d)$. Therefore, for h ≤ d/α, equation (58) holds with d_h and Ah instead of d and A. Remarking that $r - \nu - l \ge 0$, we also have the bound

$$d_h^{r-\nu-l} \le \alpha^{r-\nu-l}h^{r-\nu-l}. \tag{61}$$

Thus, from (57), (59) and (61), we obtain for h ≤ d/α the equation

$$|u|_{l,q,\Omega} \le C\max\left\{\alpha^{r-\nu-l}, \alpha^{\frac{n}{q}\beta-l}\right\}h^{-\nu}\left(h^{r-l}|u|_{r,p,\Omega} + h^{-l}|u|_{0,p,\Omega}\right), \tag{62}$$

where C is the constant of Theorem (4.2.4).

(2) Now, we reason as in [159]. Let t_0 be any positive number. Setting $t_1 = \min\{\eta, d/\alpha\}$, it follows that (62), written with t instead of h is verified for $t < t_1$. So, (60) (with a suitably adjusted constant C) is verified for $tt_1/t_0 < t_1$, i.e. for $t < t_0$.

Let us notice that C may be chosen as

$$C = C_0\max\{\alpha^{r-\nu}, \alpha^{n/q\,\beta-\ell}\},$$

where C_0 depends on Ω , n, r, p and q , (see [134]), and also on t_0

Remark (4.2.9) [158]:

For a bounded domain with a Lipschitz continuous boundary and $p < q$, Theorem (4.1.16) extends [159]. This is true even if $p = q$ and r is not integer. However, for $p = q$ and r integer, our result is less general than the one of Adams and Fournier: Theorem (4.2.8) requires a stronger condition on Ω and, moreover, needs the assumption $r > n/p$, unnecessary for this theorem. A similar remark can be made to Theorems 4.3 and 4.4 below.

The starting point is no longer (see [134]). In the inequality corresponding to (57), the number s replaces the integer l. Now, we have: $T_2 = d^{n/q-s} \parallel u|_A \parallel_p$, with $s \in [0, \ell]$, and, instead of (59), the inequality

$$d_h^{-\nu+n/p-s} \left\| u|_{A_h} \right\|_p \leq \alpha^{n/q} \beta^{-s} h^{-\nu-s} |u|_{0,p,\Omega}$$

Then, the following theorem, an equivalent of Theorem (4.2.8), can be derived.

Theorem (4.2.10) [158]:

Let Ω be a $\mathcal{L}(\rho, \theta)$-domain of $\mathbb{R}^n$, with $\rho > 0$ and $\theta \in (0, \pi/2)$. We suppose that $p, q \in [1, \infty)$, with $p \leq q, r$ and $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$ set $\nu = n(1/p - 1/q)$ and $\ell = \lceil r - \nu \rceil - 1$ (or $\ell = r - \nu$ if $r \in \mathbb{N}^*$ and $\nu \in \mathbb{N}$). Then, for any $t_0 > 0$, there exists a constant C depending on Ω, n, r, p, q and t_0 such that, for any $0 < t \leq t_0, u \in W^{r,p}(\Omega)$ and $s \in [0, \ell]$, we have the relation

$$|u|_{s,q,\Omega} \leq \mathfrak{C} \Lambda_{\sigma,q} t^{-\nu} \left(t^{r-s} |u|_{r,p,\Omega} + t^{-s} |u|_{0,p,\Omega} \right),$$

where $\sigma = s - \lfloor s \rfloor$ and

$$\Lambda_{\sigma,q} = \begin{cases} (q\sigma(1-\sigma))^{-1/q} & , \text{ if } \sigma \in (0,1) \\ 1, & \text{if } \sigma = 0 \end{cases} \tag{63}$$

The matter of the validity of Theorem (4.2.8) for any open set Ω having the cone property is not straightforward. The difficulty comes from the definition of the sampling inequality (57) for an unbounded domain Ω.

This problem has been studied in the article [161], where we show that the extension of (57) is possible in the following case: (1) the function u belongs to the space $\dot{W}^{r,p}(\Omega)$ of distributions v on Ω whose derivatives of order $\lfloor r \rfloor$ are functions in $L^p(\Omega)$; (2) the open set Ω verifies the so-called $\mathcal{C}[\rho, \theta]$-property, that is satisfied in different examples, the simplest one being $\Omega = \mathbb{R}^n$.

Thus, we have the following result.

Proposition (4.2.11) [158]:

We suppose that $p, q \in [1, \infty)$, with $p \leq q$, $r \in \mathbb{R}_+^*$ and $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$. We set $\nu = n(1/p - 1/q)$ and $\ell = \lceil r - \nu \rceil - 1$ (or $\ell = r - \nu$ if $r \in \mathbb{N}^*$ and $\nu \in \mathbb{N}$). Then, there exist two positive constants $\mathfrak{d}$ (depending on n and) and C (depending on n, r, p and q) such that the next property is verified: for any discrete set $A \subset \mathbb{R}^n$ such that $\delta(\mathbb{R}^n, A) \leq \mathfrak{d}$, for any $u \in \dot{W}^{r,p}(\mathbb{R}^n)$ and any $l = 0, \ldots, \ell$, we have

$$|u|_{l,p,\mathbb{R}^n} \leq C d^{-\nu} \left(d^{r-l} |u|_{r,p,\mathbb{R}^n} + d^{n/p-l} \parallel u|_A \parallel_p \right),$$

where $d = \delta(\mathbb{R}^n, A)$.

Proof.

The proposition is a particular case in [161] and [134], another analogue follows.

Theorem (4.2.12) [158]:

We suppose that $p, q \in [1, \infty)$, with $p \leq q$, $r \in \mathbb{R}_+^*$ and $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$. We set $\nu = n(1/p - 1/q)$ and $\ell = \lceil r - \nu \rceil - 1$ (or $\ell = r - \nu$ if $r \in \mathbb{N}^*$ and $\nu \in \mathbb{N}$). Then, for any $t_0 > 0$, there exist a constant $\mathfrak{C}$ depending on n, r, p, q and t_0 such that, for any $0 < t \leq t_0$, $u \in W^{r,p}(\mathbb{R}^n)$ and $s \in [0, \ell]$, we have the relation

$$|u|_{s,q,\mathbb{R}^n} \leq C\Lambda_{\sigma,q} t^{-\nu} \left(t^{r-s} |u|_{r,p,\mathbb{R}^n} + t^{-s} |u|_{0,p,\mathbb{R}^n}\right),$$

where $\sigma = s - \lfloor s \rfloor$ and $\Lambda_{\sigma,q}$ is defined by (63).