Chapter 6

Fractional Poincare Inequalities and Fractional –Order Sobolev Semi-norms

We quantify the tightness at infinity provided by the control on the fractional derivative in terms of a weight growing at infinity. The proof goes through the introduction of the generator of the Ornstein–Uhlenbecks emigroup and some careful estimates of its powers.We the proof of fractional Poincaré inequality for measures more general than Lévy measures. Main results are mutual estimates of the three semi-norms of Sobolev-Slobodeckij, interpolation and quotient space types. In particular, we show that the former two are uniformly equivalent under affine mappings that ensure shape regularity of the domains under consideration.

Sec (6.1):General Measures:

The aim of this section is to prove a Poincaré inequality on $\mathbb{R}^n$, endowed with a measure $M(x)dx$, involving non-local quantities in the right-hand side in the spirit of Gagliardo semi-norms for Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ with fractional order $s \in (0,1)$ (see e.g. [225]).

Fractional diffusions naturally appear in many models, ranging from plasma turbulence [226] or geostrophic flows [227] in fluid dynamics, grazing collisions in kinetic theory (cf. the Boltzmann equation for long-range interactions [228-231]), all the way to stockmarket modeling based on Lévy processes [232]. They also appear naturally in mathematics: in probability, they appear in the important class of infinitely divisible Markov processes given (cf. the Lévy-Khinchin representation [233]); in analysis they naturally appear in the study of singular integral operators (e.g. for the Boltzmann equation, cf. references above) as well as in the so-called "Dirichlet-to-Neuman" boundary value problem and in the Signorini (obstacle) problem [234] (see for instance among other references [235] and [236]). The search for a Poincaré inequality for the non-local fractional energy associated with such fractional diffusion is therefore a natural and interesting question since this inequality governs the spectral gap of the underlying operator and the speed of (fractional) diffusion towards an equilibrium.

Throughout this section, we denote by M a positive weight in $L^1(\mathbb{R}^n)$. In the sequel, by $L^2(\mathbb{R}^n, M)$, we mean the space of measurable functions on $\mathbb{R}^n$ which are square integrable with respect to the measure $M(x)dx$, by $L_0^2(\mathbb{R}^n, M)$ the subspace of functions of $L^2(\mathbb{R}^n, M)$ such that $\int_{\mathbb{R}^n} f(x)M(x)\,dx = 0$, and by $H^1(\mathbb{R}^n, M)$, the Sobolev space of functions in $L^2(\mathbb{R}^n, M)$, the weak derivative of which belongs to $L^2(\mathbb{R}^2, M)$. Finally for any measurable subset $A \subset \mathbb{R}^n$ by $L^2(A, M)$ we mean the obvious restriction of the definition above to the set A.

We assume that M is a C^2 function and that this measure M satisfies the usual Poincaré inequality: there exists a constant $\lambda(M) > 0$ such that $\forall f \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} |\nabla f(y)|^2 M(y)\, dy \geq \lambda(M) \int_{\mathbb{R}^n} \left| f(y) - \int_{\mathbb{R}^n} f(x) M(x) dx \right|^2 M(y) dy. \quad (1)$$

If the measure M can be written $M = e^{-V}$, this inequality is known to hold (see [237], or also [238], Theorem (6.1.2), see also [239], proof of Theorem (6.1.2) for related criteria) whenever there exist $a \in (0,1)$, $c > 0$ and $R > 0$ such that

$$\forall |x| \geq R, \; a|\nabla V(x)|^2 - \Delta V \geq c. \quad (2)$$

In particular, the inequality (1) holds, for instance, when $M = (2\pi)^{-n/2} \exp(-|x|^2/2)$is the Gaussian measure, but also when $M(x) = e^{-|x|}$, and more generally when $M(x) = e^{-|x|^\alpha}$with $\alpha \geq 1$. Note that, when $V$is convex, and

$$\mathrm{Hess(V)} \geq \mathrm{cstId}$$

on the set where $|V| < +\infty$, the measure M(x)dx satisfies the log-Sobolev inequality, which in turn implies (1) (see [240]).

As it shall be proved to be useful later on, remark that, under a slightly stronger assumption than (1), the Poincaré inequality (2),the Poincaré inequality (1) admits the following self-improvement:

Proposition (6.1.1) [224]:

Assume that there exists $\varepsilon > 0$ such that

$$\frac{(1-\varepsilon)|\nabla V|^2}{2} - \nabla V \overrightarrow{x \to \infty} + \infty, \;\; M = e^{-V}. \quad (3)$$

Then there exists $\lambda'(M) > 0$ such that, for all functions $f \in L_0^2(\mathbb{R}^n, M) \cap H^1(\mathbb{R}^n, M)$:

$$\iint_{\mathbb{R}^n} |\nabla f(x)|^2 M(x)\, dx \geq \lambda'(M) \int_{\mathbb{R}^n} |f(x)|^2 (1 + |\nabla \mathrm{In}\, M(x)|^2)\, M(x) dx. \quad (4)$$

The proof of Proposition (6.1.1) is classical and will be given for the sake of completeness.

We want to generalize the inequality (1) in the strengthened form of Proposition (6.1.1), replacing, in the left-hand side, the H^1 semi-norm by a non-local expression in the flavour of the Gagliardo semi-norms.

We establish the following theorem:

Theorem (6.1.2) [224]:

Assume that $M = e^{-V}$ is a C^2 positive L^1 function which satisfies (3). Let $\alpha \in (0,2)$. Then there exist $\lambda_\alpha(M) > 0$ and $\delta(M)$ (constructive from our proof and the usual Poincaré constant $\lambda(M)$such that, for any function f belonging to a dense subspace of $L_0^2(\mathbb{R}^n, M)$, we have:

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)e^{-\delta(M)|x-y|}\,dx\,dy$$
$$\geq \lambda_\alpha(M)\int_{\mathbb{R}^n} |f(x)|^2(1 + |\nabla \ln M(x)|^2)\,M(x)dx \qquad (5)$$

Remark (6.1.3) [224]:

Inequality (5) could as usual be extended to any function f with zero average such that both sides of the inequality make sense. In particular it is satisfied for any function $f$with zero average belonging to the domain of the operator $L = -\Delta - \nabla V\cdot\nabla$ that we shall introduce later on. Functions of this domain with zero integral with respect to $M(x)dx$are dense in $L_0^2(\mathbb{R}^n, M)$.

Observe that the right-hand side of (5) involves a fractional moment of order αrelated to the homogeneity of the semi-norm appearing in the left-hand side. One could expect in the left-hand side of (5) the Gagliardo semi-norm for

the fractional Sobolev space $H^{\alpha/2}(\mathbb{R}^n, M)$, namely
$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)M(y)\,dx\,dy.$$
Notice that, instead of this semi-norm, we obtain a "non-symmetric" expression. However, our norm is more natural: one should think of the measure over y as the Lévy measure, and the measure over x as the a mbient measure. We emphasize on the fact that our measure is rather general and in particular, as a corollary of Theorem (6.1.2), we obtain an automatic improvement of the Poincaré inequality (1) by:

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)M(y)\,dx\,dy \geq \lambda_\alpha(M)\int_{\mathbb{R}^n} |f(x)|^2 M(x)\,dx.$$

The question of obtaining Poincaré-type inequalities (or more generally entropy inequalities) for Lévy operators was studied in the probability community in the last decades. For instance it was proved by Wu [241] and Chafaï [242] that

$$\mathrm{Ent}_\mu^\Phi(f) \leq \int \Phi^n(f)\nabla f\cdot\sigma\cdot\nabla f\,d\mu + \iint D_\Phi\big(f(x), f(x+z)\big)\,dv_\mu(z)\,dv_\mu(x)$$
(see also the use of this inequality in [243]) with
$$\mathrm{Ent}_\mu^\Phi(f) = \int \Phi(f)d\mu - \Phi\left(\int f d\mu\right),$$
and$D_\Phi$is the so-called Bregman distance associated to Φ:
$$D_\Phi(a,b) = \Phi(a) - \Phi(b) - \Phi'(b)(a-b),$$
Where Φ is some well-suited functional with convexity properties, $\sigma$the matrix of diffusion of the process, μa rather general measure, and $v_\mu$the (singular) Lévy measure associated to μ. Choosing $\Phi(x) = x^2$ and $\sigma = 0$ yields a Poincaré inequality for this choice of measure μ, v_μ. The improvement of our approach is that we do not impose any link between our measure Mon $x$and the singular measure $|z|^{-n-\alpha}$on $z = x - y$. This is to our knowledge the first result that gets rid of this strong constraint.

Remark (6.1.4)[224]:

Note that the exponentially decaying factor $e^{-\delta(M)|x-y|}$ in (5) also improvesthe inequality as compared to what is expected from Poincaré inequality for Lévy measures. This decay on the diagonal could most probably be further improved, as shall be studied in future works. Other extensions in progress are to allow more general singularities than the Martin Riesz kernel $\frac{1}{|x-y|^{n+\alpha}}$ (see the book [244]) and to develop an L^P theory of the previous inequalities.

Our proof heavily relies on fractional powers of a (suitable generalization of the) Ornstein-Uhlenbeck operator, which is defined by:

$$Lf = -M^{-1}\operatorname{div}(M\nabla f) = -\nabla f - \nabla \ln M \cdot \nabla f,$$

for all $f \in D(L) = \{g \in H^1(\mathbb{R}^n, M);\ (1/\sqrt{M})\operatorname{div}(M\nabla g) \in L^2(\mathbb{R}^n)\}$. One therefore has, for all $f \in \mathcal{D}(L)$and $g \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} Lf(x)g(x)M(x)\,dx = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla g(x)M(x)\,dx.$$

It is obvious that $L$is symmetric and nonnegative on $L^2(\mathbb{R}^n, M)$, which allows to define the usual power $L^\beta$for any $\beta \in (0,1)$by means of spectral theory. Note that $L^{\alpha/2}$ is notthe symmetric operator associated to the Dirichlet form $\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)\,dx\,dy$

We now describe the strategy of our proofs. The proof of Theorem (6.1.2) goes in three steps. We first establish L^2off-diagonal estimates of Gaffney type on theresolvent of L on $L^2(\mathbb{R}^n, M)$. These estimates are needed in our context since we do not have Gaussian pointwise estimates on the kernel of the operator L.

Then, we bound the quantity,

$$\int_{\mathbb{R}^n} |f(x)|^2(1 + |\nabla \ln M(x)|^\alpha)M(x)\,dx,$$

This will be obtained by an abstract argument of functional calculus based on rewriting in a suitable way the conclusion of Proposition (6.1.1). Finally, using the L^2 off-diagonal estimates for the kernel of L, we establish that

$$\left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n,M)} \le C \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} M(x)\,dx\,dy,$$

which concludes the proof.

As can be seen from the rough sketch previously described, we borrow methods from harmonic analysis. This seems not so common in the field of Poincaré and log-Sobolev inequalities (to the knowledge of the authors), where standard techniques rely on global functional inequalities, see for instance the powerful so-called Γ_2-calculus of Bakry and Émery [245]. We hope this section will stimulate further exchanges between these two fields.

We recall that for every $f \in \mathcal{D}(L)$, we define

$$Lf = -M^{-1}\text{div}\,(M\nabla f) = -\Delta f - \nabla \ln M \cdot \nabla f. \tag{6}$$

From the fact that L is self-adjoint and nonnegative on $L^2(\mathbb{R}^n, M)$we have:

$$\|(L-\mu)^{-1}\|_{L^2(\mathbb{R}^n,M)} \le \frac{1}{\text{dist}\,(\mu, \sum(L))}$$

where$\Sigma(L)$ denotes the spectrum of L, and $\mu \notin \Sigma(L)$. Then we deduce that $(I+tL)^{-1}$ is bounded with norm less than 1 for all $t > 0$. Since $tL(I + tL)^{-1} = I - (I + tL)^{-1}$, the same is true for $tL(I + tL)^{-1} = I - (I + tL)^{-1}$ with a norm less than 2. Moreover, $(I + tL)^{-1} f \in H^1(\mathbb{R}^n, M)$.

Actually, when $f \in L^2(\mathbb{R}^n, M)$is supported in a closed set $E \subset \mathbb{R}^n$and $F \subset \mathbb{R}^n$is a closed subset disjoint from E, a much more precise estimate on the L^2 norm of $(I + tL)^{-1} f$ and $tL(I + tL)^{-1} f$ on Fcan be given. Here are these L^2 off-diagonal estimates for the resolvent of L:

Lemma (6.1.5) [224]:
There exists $C_1 = C_1(M) > 0$ (constructive from our proof) with the following property: for all compact disjoint subsets $E, F \subset \mathbb{R}^n$, F bounded, with $\text{dist}(E, F) =: d > 0$, all functions $f \in L^2(\mathbb{R}^n, M)$ supported in E and all $t > 0$,

$$\|(I + tL)^{-1} f\|_{L^2(F,M)} + \|tL(I + tL)^{-1} f\|_{L^2(F,M)} \le 8e^{-C_1\frac{d}{\sqrt{t}}}\|f\|_{L^2(E,M)}.$$

Note that, in different contexts, this kind of estimate, originating in [246], turns out to be a powerful tool, especially when no pointwise upper estimate on the kernel of the semigroup generated by $L$is available (see for instance [247-249]). Since we found no reference for these off-diagonal estimates for the resolvent of L, we give here a proof.

Proof of Lemma (6.1.5):

We argue as in [248]. Since $(I + tL)^{-1}$ is bounded with norm less than 1 for all $t > 0$ it is clearly enough to restrict to $0 < t < d$.

Define $u_t = (I + tL)^{-1} f$, so that, for all functions $v \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} u_t(x)v(x)M(x)\,dx + t\int_{\mathbb{R}^n} \nabla u_t(x)\cdot\nabla v(x)M(x) = \int_{\mathbb{R}^n} f(x)v(x)M(x)\,dx. \tag{7}$$

Fix now a nonnegative function $\eta \in \mathcal{D}(\mathbb{R}^n)$vanishing on E. Since $f$is supported in$E$, applying (7) with $v = \eta^2 u_t$(remember that $u_t \in H^1(\mathbb{R}^n, M)$) yields,

$$\int_{\mathbb{R}^n} \eta^2(x)|u_t(x)|^2 M(x)\,dx + t\int_{\mathbb{R}^n} \nabla u_t(x)\cdot\nabla(\eta^2 u_t)M(x)\,dx = 0,$$

which implies:

$$\int_{\mathbb{R}^n} \eta^2(x)|u_t(x)|^2 M(x)\,dx + t\int_{\mathbb{R}^n} \eta^2(x)|\nabla u_t(x)|^2 M(x)\,dx$$
$$= -2t\int_{\mathbb{R}^n} \eta(x)u_t(x)\nabla\eta(x)\cdot\nabla u_t(x)M(x)\,dx$$
$$\le t\int_{\mathbb{R}^n} |u_t(x)|^2|\nabla\eta(x)|^2 M(x)\,dx + t\int_{\mathbb{R}^n} \eta^2(x)|\nabla u_t(x)|^2 M(x)\,dx,$$

hence

$$\int_{\mathbb{R}^n} \eta^2(x)|\nabla u_t(x)|^2 M(x)\,dx \le t\int_{\mathbb{R}^n} |u_t(x)|^2|\nabla\eta(x)|^2 M(x)\,dx. \tag{8}$$

Let ξbe such that $\xi = 0$ on E and ξ nonnegative so that $\eta = e^{\alpha\xi} - 1 \ge 0$ and ηvanishes on $E$for some $\alpha > 0$ to be chosen. Choosing this particular ηin (8) with$\alpha > 0$ gives:

$$\int_{\mathbb{R}^n} \left|e^{\alpha\xi} - 1\right|^2 |u_t(x)|^2 M(x)\,dx \le \alpha^2 t\int_{\mathbb{R}^n} |u_t(x)|^2|\nabla\xi(x)|^2 e^{2\alpha\xi(x)} M(x)\,dx.$$

Taking $\alpha = 1/(2\sqrt{t}\|\nabla\xi\|_\infty)$, one obtains:

$$\int_{\mathbb{R}^n} \left|e^{\alpha\xi(x)} - 1\right|^2 |u_t(x)|^2 M(x)\,dx \le \frac{1}{4}\int_{\mathbb{R}^n} |u_t(x)|^2 e^{2\alpha\xi(x)} M(x)\,dx.$$

Using the fact that the norm of $(I + tL)^{-1}$ is bounded by 1 uniformly in $t > 0$, this gives:

$$\left\|e^{\alpha\xi}u_t\right\|_{L^2(\mathbb{R}^2,M)} \le \left\|\left(e^{\alpha\xi} - 1\right)u_t\right\|_{L^2(\mathbb{R}^n,M)} + \|u_t\|_{L^2(\mathbb{R}^n,M)}$$
$$\le \frac{1}{2}\left\|e^{\alpha\xi}u_t\right\|_{L^2(\mathbb{R}^n,M)} + \|f\|_{L^2(\mathbb{R}^n,M)},$$

therefore

$$\int_{\mathbb{R}^n} \left|e^{\alpha\xi(x)}\right|^2 |u_t(x)|^2 M(x)\,dx \le 4\int_{\mathbb{R}^n} |f(x)|^2 M(x)\,dx.$$

We choose now ξsuch that $\xi = 0$ on E as before and additionally that $\xi = 1$ on F($\eta$is then compactly supported from the fact that F is bounded). It can trivially be chosen with $\|\nabla\xi\|_\infty \le C/d$, which yields the desired conclusion for the L^2 norm of $(I + tL)^{-1}f$with a factor 4 in the right-hand side. Since $tL(I + tL)^{-1}f = f - (I + tL)^{-1}f$, the desired inequality with a factor 8 readily follows.

Remark (6.1.6) [224]:

Arguing similarly, we could also obtain analogous gradient estimates for $\left\|\sqrt{t}\nabla(\mathrm{I} + tL)^{-1}f\right\|_{L^2(F,M)}$.

This section is devoted to the control of the L^2 norm of fractional powers of L. This is the cornerstone of the proof of Theorem (6.1.2). In the functional calculus theory of sectorial operators L, fractional powers (for the particular powers we are interested in) are defined as follows (see for instance [250]):

$$\forall\beta \in (0,1), \quad L^\beta f = \frac{1}{\Gamma(1-\beta)}\int_0^\infty t^{-\beta} L e^{-Lt} f\,dt. \tag{9}$$

They can also be defined in terms of the resolvent by the Balakrishnan formulation (see for instance [250]):

$$\forall \beta \in (0,1), \quad L^{\beta} f = \frac{\sin(\pi(1-\beta))}{\pi} \int_{0}^{\infty} \lambda^{\beta-1} L(L+\lambda \mathrm{I})^{-1} f \, d\lambda. \tag{10}$$

We shall in fact not need any of the representations (9) or (10); instead we shall rely on the powerful tool of the so-called "quadratic estimates" obtained in the functional calculus. This is the object of the next lemma.

Lemma (6.1.7) [224]:

Let $\alpha \in (0,2)$. There exists $C_3 = C_3(M) > 0$ such that, for all $f \in \mathcal{D}(L)$,

$$\left\|L^{\alpha/4} f\right\|_{L^2(\mathbb{R}^n,M)}^2 \leq C_3 \int_{0}^{+\infty} t^{-1-\alpha/2} \|tL(\mathrm{I}+tL)^{-1} f\|_{L^2(\mathbb{R}^n,M)}^2 \, dt. \tag{11}$$

Proof:

Let $\mu \in \left(0, \frac{\pi}{2}\right)$, and

$$\textstyle\sum_{\mu^+} = \{z \in \mathbb{C}^*; |\arg z| < \mu\}.$$

Let ψ be a holomorphic function in $H^\infty(\sum_{\mu^+})$ such that for some $C, \sigma, \tau > 0$,

$$|\psi(z)| \leq C \inf\{|z|^\sigma, |z|^{-\tau}\},$$

for any $z \in \Sigma_{\mu^+}$. Since L is positive self-adjoint operator on $L^2(\mathbb{R}^n, M)$ and L is one-to-one on $L_0^2(\mathbb{R}^n, M)$ by (6.80), one has by the spectral theorem,

$$\|F\|_{L^2(\mathbb{R}^n,M)}^2 \leq C \int_{0}^{+\infty} \|\psi(tL)F\|_{L^2(\mathbb{R}^n,M)}^2 \frac{d}{dt}$$

whenever $F \in L_0^2(\mathbb{R}^n, M)$. Choosing $\psi(z) = z^{1-\alpha/4}/(1+z)$ yields,

$$\|F\|_{L^2(\mathbb{R}^n,M)}^2 \leq C \int_{0}^{+\infty} \left\|(tL)^{1-\alpha/4}(\mathrm{I}+tL)^{-1} F\right\|_{L^2(\mathbb{R}^n,M)}^2 \frac{d}{dt}, \tag{12}$$

whenever $F \in L_0^2(\mathbb{R}^n, M)$.

Let $F \in L^2(\mathbb{R}^n, M)$. Since

$$\int_{\mathbb{R}^n} Lf(x) M(x) \, dx = 0,$$

it follows from (9) that the same is true with $L^{\alpha/4} f$. Applying now (12) with $F = L^{\alpha/4} f$ gives the conclusion of Lemma (6.1.7).

Let us draw a simple corollary of Lemma (6.1.7):

Corollary (6.1.8) [224]:

For any $\alpha \in (0,2)$ and $\varepsilon > 0$, there is $A = A(M, \varepsilon)$ such that

$$\left\|L^{\alpha/4}f\right\|^2_{L^2(\mathbb{R}^n,M)} \leq C_3 \int_0^A t^{-1-\alpha/2}\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}. \tag{13}$$

Proof:

The proof is straightforward since

$$\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)} \leq C\|F\|^2_{L^2(\mathbb{R}^n,M)}$$

and

$$\int_A^{+\infty} t^{-1-\alpha/2}\,dt \xrightarrow[A\to+\infty]{} 0.$$

We now come to the desired estimate.

Lemma (6.1.9) [224]:

Let $\alpha \in (0,2)$ and ε and A given by Corollary (6.1.8). There exist $C_4 = C_4(M,A) > 0$ and $c' = c'(A,M) > 0$ such that, for all $f \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_0^A t^{-1-\alpha/2}\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}\,dt \leq C_4 \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)e^{-c'|x-y|}\,dx\,dy.$$

Proof:

Throughout this proof, for all $x \in \mathbb{R}^n$and all $s > 0$, denote by $Q(x,s)$the closed cube centered at $x$with side length s. For fixed $t \in (0,A)$, following Lemma (6.1.7), we shall look for an upper bound for $\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}$ involving first order differences for f. Pick up a countable family of points $x_j^t \in \mathbb{R}^n, j \in \mathbb{N}$, such that the cubes $Q\left(x_j^t,\sqrt{t}\right)$ have pairwise disjoint interiors, and

$$\mathbb{R}^n = \bigcup_{j\in\mathbb{N}} Q\left(x_j^t,\sqrt{t}\right). \tag{14}$$

By Lemma (6.1.12), there exists a constant $\tilde{C} > 0$ such that for all $\theta > 1$ and all $x \in \mathbb{R}^n$, there are at most $\tilde{C}\theta^n$indexes $j$such that $|x - x_j^t \leq \theta\sqrt{t}$.

For fixed j, one has

$$tL(\mathrm{I}+tL)^{-1}f = tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t},$$

where, for all $x \in \mathbb{R}^n$,

$$\mathrm{g}^{j,t}(x) = f(x) - m^{j,t}$$

and$m^{j,t}$ is defined by:

$$m^{j,t} = \frac{1}{\left|Q\left(x_j^t,2\sqrt{t}\right)\right|}\int_{Q\left(x_j^t,2\sqrt{t}\right)} f(y)\,dy.$$

Note that, here, the mean value of f is computed with respect to the Lebesgue measure on $\mathbb{R}^n$. Since (14) holds and the cubes $Q\left(x_j^t,2\sqrt{t}\right)$ have pairwise disjoint interiors, one clearly has:

$$\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)} = \sum_{j\in\mathbb{N}} \|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)}$$
$$= \sum_{j\in\mathbb{N}} \left\|tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)},$$

and we are left with the task of estimating,

$$\left\|tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)}.$$

To that purpose, set

$$C_0^{j,t} = \mathcal{Q}\left(x_j^t, 2\sqrt{t}\right) \text{and} C_k^{j,t} = \mathcal{Q}\left(x_j^t, 2^{k+1}\sqrt{t}\right)\backslash\mathcal{Q}\left(x_j^t, 2^k\sqrt{t}\right), \quad \forall k \geq 1,$$

and $\mathrm{g}_k^{j,t} \coloneqq \mathrm{g}^{j,t}\mathbf{1}_{C_k^{j,t}}, k \geq 0$, ,where, for any subset $A \subset \mathbb{R}^n$, $\mathbf{1}_A$ is the usual characteristic function of A. Since $\mathrm{g}^{j,t} = \sum_{k\geq 0} \mathrm{g}_k^{j,t}$ one has:

$$\left\|tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)} \leq \sum_{k\geq 0} \left\|tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)} \quad (15)$$

and, using Lemma (6.1.5), one obtains (for some constants $C, c > 0$):

$$\left\|tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)} \leq C\left(\left\|\mathrm{g}_0^{j,t}\right\|_{L^2\left(C_0^{j,t},M\right)} + \sum_{k\geq 0} e^{-c2^k}\left\|\mathrm{g}_k^{j,t}\right\|_{L^2\left(C_k^{j,t},M\right)}\right) \quad (16)$$

By Cauchy-Schwarz's inequality, we deduce (for another constant $C' > 0$):

$$\left\|tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)} \leq C'\left(\left\|\mathrm{g}_0^{j,t}\right\|_{L^2\left(C_0^{j,t},M\right)} + \sum_{k\geq 0} e^{-c2^k}\left\|\mathrm{g}_k^{j,t}\right\|_{L^2\left(C_k^{j,t},M\right)}\right) \quad (17)$$

As a consequence, we have:

$$\int_0^A t^{-1-\alpha/2}\left\|tL(\mathrm{I}+tL)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)} dt \leq C'\int_0^A t^{-1-\alpha/2}\sum_{j\geq 0}\left\|\mathrm{g}_0^{j,t}\right\|_{L^2\left(C_0^{j,t},M\right)} dt)$$
$$+C'\int_0^A t^{-1-\alpha/2}\sum_{k\geq 1} e^{-c2^k}\sum_{j\geq 0}\left\|\mathrm{g}_k^{j,t}\right\|_{L^2\left(C_k^{j,t},M\right)} dt. \quad (18$$

We claim that

Lemma (6.1.10) [224]:

There exists $\bar{\mathrm{C}} > 0$ such that, for all $\mathrm{t} > 0$ and all $\mathrm{j} \in \mathbb{N}$:

A. For the first term:

$$\left\|\mathrm{g}_0^{j,t}\right\|^2_{L^2\left(C_0^{j,t},M\right)} \leq \frac{\bar{C}}{t^{n/2}} \int_{\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)}\int_{\mathcal{Q}\left(x_j^t,2\sqrt{t}\right)} |f(x)-f(y)|^2 M(x)\,dx\,dy.$$

B. For all $k \geq 1$,

$$\left\|\mathrm{g}_k^{j,t}\right\|^2_{L^2\left(C_k^{j,t},M\right)} \leq \frac{\bar{C}}{\left(\sqrt{t}\right)^2} \int_{x\in\mathcal{Q}\left(x_j^t,2^{k+1}\sqrt{t}\right)}\int_{y\in\mathcal{Q}\left(x_j^t,2^{k+1}\sqrt{t}\right)} |f(x)-f(y)|^2 M(x)\,dx\,dy.$$

We postpone the proof to the end of the section and finish the proof of Lemma (6.1.9). Using Assertion A in Lemma (6.1.10), summing up on $j \geq 0$ and integrating over $(0, A)$, we get:

$$\int_0^A t^{-1-\alpha/2} \sum_{j\geq 0} \left\| g_0^{j,t} \right\|^2_{L^2\left(C_0^{j,t}, M\right)} dt = \sum_{j\geq 0} \int_0^A t^{-1-\alpha/2} \left\| g_0^{j,t} \right\|^2_{L^2\left(C_0^{j,t}, M\right)} dt$$

$$\leq \bar{C} \sum_{j\geq 0} \int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}} \left(\int_{Q\left(x_j^t, 2\sqrt{t}\right)} \int_{Q\left(x_j^t, 2\sqrt{t}\right)} |f(x) - f(y)|^2 M(x)\, dx\, dy \right) dt$$

$$\leq \bar{C} \sum_{j\geq 0} \iint_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} |f(x) - f(y)|^2 M(x) \left(\int_{t\geq \max\left\{\frac{\left|x-x_j^t\right|^2}{n}; \frac{\left|y-x_j^t\right|^2}{n}\right\}}^{A} t^{-1-\frac{\alpha}{2}-\frac{n}{2}} dt \right) dx\, dy.$$

The Fubini theorem now shows:

$$\sum_{j\geq 0} \int_{t\geq \max\left\{\frac{\left|x-x_j^t\right|^2}{n}; \frac{\left|y-x_j^t\right|^2}{n}\right\}}^{A} t^{-1-\frac{\alpha}{2}-\frac{n}{2}} dt = \int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}} \sum_{j\geq 0} \mathbf{1}_{\left(\max\left\{\frac{\left|x-x_j^t\right|^2}{n}; \frac{\left|y-x_j^t\right|^2}{n}\right\}, +\infty\right)}(t)\, dt.$$

Observe that, by Lemma (6.1.12), there is a constant $N \in \mathbb{N}$ such that, for all $t > 0$, there are at most Nindexes $j$such that $\left|x - x_j^t\right|^2 < nt$and $\left|y - x_j^t\right|^2 < nt$. If such an index $j$exists, one has $|x - y| < 2\sqrt{nt}$. It therefore follows that

$$\sum_{j\geq 0} \mathbf{1}_{\left(\max\left\{\frac{\left|x-x_j^t\right|^2}{n}; \frac{\left|y-x_j^t\right|^2}{n}\right\}, +\infty\right)}(t) \leq N \mathbf{1}_{\left(|x-y|^2/4n, +\infty\right)}(t),$$

so that

$$\int_0^A t^{-1-\alpha/2} \sum_{j} \left\| g_0^{j,t} \right\|^2_{L^2\left(C_0^{j,t}, M\right)} dt$$

$$\leq \bar{C} N \iint_{\mathbb{R}^n\times\mathbb{R}^n} |f(x) - f(y)|^2 M(x) \left(\int_{|x-y|^2/4n}^{A} t^{-1-\frac{\alpha}{2}-\frac{n}{2}} dt \right) dx\, dy$$

$$\leq \bar{C} N \iint_{|x-y|\leq 2\sqrt{nA}} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} M(x)\, dx\, dy. \tag{19}$$

Using now Assertion B in Lemma (6.2.10), we obtain, for all $j \geq 0$ and all $k \geq 1$,

$$\int_0^A t^{-1-\alpha/2} \sum_{j\geq 0} \left\| g_k^{j,t} \right\|_2^2 dt$$

$$\leq \bar{C}\sum_{j\geq 0}\int_0^A t^{-1-\alpha/2}\left(\iint_{Q\left(x_j^t,2^{k+1}\sqrt{t}\right)\times Q\left(x_j^t,2^{k+1}\sqrt{t}\right)} |f(x)-f(y)|^2 M(x)\,dx\,dy\right)dt$$

$$\leq \bar{C}\sum_{j\geq 0}\iint_{x,y\in\mathbb{R}^n}|f(x)-f(y)|^2 M(x)\left(\int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\mathbf{1}_{\max\left\{\frac{|x-x_j^t|^2}{4^k n};\frac{|y-x_j^t|^2}{4^k n}\right\},+\infty}(t)dt\right)dx\,dy.$$

But, given $t>0$, $x,y\in\mathbb{R}^n$, by Lemma (6.1.12) again, there exist at most $\tilde{C}2^{kn}$ indexes j such that

$$\left|x-x_j^t\right|\leq 2^k\sqrt{nt}\text{ and }\left|y-x_j^t\right|\leq 2^k\sqrt{nt},$$

and for these indexes j, $|x-y|\leq 2^{k+1}\sqrt{nt}$. As a consequence we have:

$$\int_0^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\sum_{j\geq 0}\mathbf{1}_{\max\left\{\frac{|x-x_j^t|^2}{4^k n};\frac{|y-x_j^t|^2}{4^k n}\right\},+\infty}(t)\,dt\leq \tilde{C}2^{kn}\int_{t\geq\frac{|x-y|^2}{4^{k+1}n}}^A t^{-1-\frac{\alpha}{2}-\frac{n}{2}}\,dt$$

$$\leq \tilde{C}'2^{k(\alpha+n)}|x-y|^{-n-\alpha}\mathbf{1}_{|x-y|\leq 2^{k+1}\sqrt{nA}},\tag{20}$$

for some other constant $\tilde{C}'>0$, and therefore

$$\int_0^A t^{-1-\alpha/2}\sum_j\left\|g_k^{j,t}\right\|^2_{L^2\left(C_0^{j,t},M\right)}dt\leq \bar{C}\tilde{C}'2^{k(\alpha+n)}\iint_{|x-y|\leq 2^{k+1}\sqrt{nA}}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)\,dx\,dy.$$

We can now conclude the proof of Lemma (6.1.9), using Lemma (6.1.7), (16), (19) and (20). We have proved, by reconsidering (18):

$$\int_0^A t^{-1-\alpha/2}\,\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}dt\ \leq C'\tilde{C}N\iint_{|x-y|\leq 2^{k+1}\sqrt{nA}}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)\,dx\,dy$$

$$+\sum_{k\geq 1}C'\bar{C}\tilde{C}'2^{k\alpha}e^{-c2^k}\iint_{|x-y|\leq 2^{k+1}\sqrt{nA}}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)\,dx\,dy\tag{21}$$

and we deduce that

$$\int_0^A t^{-1-\alpha/2}\|tL(\mathrm{I}+tL)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}\,dt\leq C_4\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}M(x)e^{-c'|x-y|}\,dx\,dy$$

for some constants C_4 and $c'>0$ as claimed in the statement.

Proof of Lemma (6.1.10):

Observe first that, for all $x\in\mathbb{R}^n$,

$$g_0^{j,t}(x)=f(x)-\frac{1}{\left|Q\left(x_j^t,2\sqrt{t}\right)\right|}\int_{Q\left(x_j^t,2\sqrt{t}\right)}f(y)\,dy.$$

$$= \frac{1}{\left|\mathcal{Q}\left(x_j^t, 2\sqrt{t}\right)\right|} \int\limits_{\mathcal{Q}\left(x_j^t, 2\sqrt{t}\right)} \left(f(x) - f(y)\right) dy.$$

By Cauchy-Schwarz inequality, it follows that

$$\left|g_0^{j,t}(x)\right|^2 \leq \frac{C}{t^{n/2}} \int\limits_{\mathcal{Q}\left(x_j^t, 2\sqrt{t}\right)} |f(x) - f(y)|^2 \, dy.$$

Therefore,

$$\left\|g_0^{j,t}\right\|_{L^2\left(C_0^{j,t}, M\right)}^2 \leq \frac{C}{t^{n/2}} \int\limits_{\mathcal{Q}\left(x_j^t, 2\sqrt{t}\right)} \int\limits_{\mathcal{Q}\left(x_j^t, 2\sqrt{t}\right)} |f(x) - f(y)|^2 M(x) \, dx \, dy,$$

which shows Assertion A. We argue similarly for Assertion B and obtain:

$$\left\|g_k^{j,t}\right\|_{L^2\left(C_k^{j,t}, M\right)}^2 \leq \frac{C}{t^{n/2}} \int\limits_{x \in \mathcal{Q}\left(x_j^t, 2\sqrt{t}\right)} \int\limits_{y \in \mathcal{Q}\left(x_j^t, 2\sqrt{t}\right)} |f(x) - f(y)|^2 M(x) \, dx \, dy,$$

which ends the proof of Lemma (6.1.10)

We end up this section with a few comments on Lemma (6.1.10). It is a well-known fact [251] that, when $0 < \alpha < 2$, for all $P \in (1, +\infty)$,

$$\left\|(-\Delta)^{\alpha/4} f\right\|_{L^P(\mathbb{R}^n)} \leq C_{\alpha,P} \left\|S_{\alpha,P} f\right\|_{L^P(\mathbb{R}^n)}, \tag{22}$$

where

$$S_{\alpha,P} f(x) = \left(\int_0^{+\infty} \left(\int_B |f(x + ry) - f(x)| \, dy \right)^2 \frac{dr}{r^{1+\alpha}} \right)^{\frac{1}{2}},$$

and also [252]

$$\left\|(-\Delta)^{\alpha/4} f\right\|_{L^P(\mathbb{R}^n)} \leq C_{\alpha,P} \|D_\alpha f\|_{L^P(\mathbb{R}^n)} \tag{23}$$

where

$$D_\alpha f(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x+y) - f(x)|^2}{|y|^{n+\alpha}} dy \right)^{\frac{1}{2}}.$$

In [253], these inequalities were extended to the setting of a unimodular Lie group endowed with a sub-laplacian Δ, relying on semigroups techniques and Littlewood-Paley-Stein functionals. In particular, in [253], we use pointwise estimates of the kernel of the semigroup generated by Δ. The conclusion of Lemma (6.1.10) means that the norm of $L^{\alpha/4} f$ in $L^2(\mathbb{R}^n, M)$ is bounded from above by the $L^2(\mathbb{R}^n, M)$norm of an appropriate version of D_α. Note that this does not require pointwise estimates for the kernel of the semigroup generated by L, and that the L^2off-diagonal estimates given by Lemma (6.1.5), which hold for a general measure M, are enough for our argument to hold. However, we do not know if anL^P version of Lemma (6.1.10) still holds. Note also that we do not compare the $L^2(\mathbb{R}^n, M)$norm of $L^{\alpha/4} f$ with the $L^2(\mathbb{R}^n, M)$norm of a version of $S_{\alpha,P} f$. Finally, the converse inequalities to (22) and (23) hold in $\mathbb{R}^n$ and also on a unimodular Lie group [252], and we did not consider the corresponding inequalities in the present section.

Observe first that, by the definition of L, we have

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 M(x)\, dx = \int_{\mathbb{R}^n} Lf(x) f(x) M(x)\, dx,$$

for all $f \in \mathcal{D}(L)$. The inequality (4) can therefore be rewritten, in terms of operators, as

$$L \geq \lambda' \mu, \tag{24}$$

where μ is the multiplication operator by $x \longmapsto 1 + |\nabla \ln M(x)|^2$. Since μ is a nonnegative operator on $L^2(\mathbb{R}^n, M)$, using a functional calculus argument (see [254]), one deduces from (24) that, for any $\alpha \in (0,2)$,

$$L^{\alpha/2} \geq (\lambda')^{\alpha/2} \mu^{\alpha/2},$$

which implies, thanks to the fact $L^{\alpha/2} = \left(L^{\alpha/4}\right)^2$ and the symmetry of $L^{\alpha/4}$ on $L^2(\mathbb{R}^n, M)$, that

$$(\lambda')^{\alpha/2} \int_{\mathbb{R}^n} |f(x)|^2 (1 + |\nabla \ln M(x)|^2)^{\alpha/2} M(x)\, dx \leq \int_{\mathbb{R}^n} \left|L^{\alpha/4} f(x)\right|^2 M(x)\, dx$$
$$= \left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n, M)}.$$

The conclusion of Theorem (6.1.2) readily follows by using the previous inequality in conjunction with Corollary (6.1.6) and Lemma (6.1.7), and picking ε small enough.

The first author would like to thank the Award No. KUK-I1-007-43, funded by the King Abdullah University of Science and Technology (KAUST) for the funding provided in Cambridge University. In this section, we prove Proposition (6.1.1), namely:

Proposition (6.1.11) [224]:

Assume that $\mathrm{M} = \mathrm{e}^{-\mathrm{V}}$ satisfies (3). Then there exists $\lambda'(\mathrm{M}) > 0$ such that, for all functions $\mathrm{f} \in \mathrm{L}^2_0(\mathbb{R}^n, \mathrm{M}) \cap \mathrm{H}^1(\mathbb{R}^n, \mathrm{M})$:

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 M(x)\, dx \geq \lambda'(x) \int_{\mathbb{R}^n} |\nabla f(x)|^2 (1 + |\nabla \ln M(x)|^2) M(x)\, dx. \tag{25}$$

Note that of course in general the constants $\lambda(M)$ and $\lambda(M)$ in (1) and (4) are different.

Proof of Proposition (6.1.1):

Let f be as in the statement of Proposition (6.1.1) and let $\mathrm{g} \coloneqq f M^{\frac{1}{2}}$. Since

$$\nabla f = M^{-\frac{1}{2}} \nabla \mathrm{g} - \frac{1}{2} \mathrm{g} M^{-\frac{3}{2}} \nabla M,$$

assumption (3) yields two positive constants β, γ such that

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 M(x)\,dx = \int_{\mathbb{R}^n} \left(|\nabla \mathrm{g}(x)|^2 + \frac{1}{4}\mathrm{g}^2(x)|\nabla \ln M(x)|^2 - \mathrm{g}(x)\nabla \mathrm{g}(x)\cdot \nabla \ln M(x)\right) dx$$
$$= \int_{\mathbb{R}^n} \left(|\nabla \mathrm{g}(x)|^2 + \frac{1}{4}\mathrm{g}^2(x)|\nabla \ln M(x)|^2 - \frac{1}{2}\nabla \mathrm{g}^2(x)\cdot \nabla \ln M(x)\right) dx$$
$$\geq \int_{\mathbb{R}^n} \mathrm{g}^2(x)\left(\frac{1}{4}|\nabla \ln M(x)|^2 + \frac{1}{2}\Delta \ln M(x)\right) dx$$
$$\geq \int_{\mathbb{R}^n} f^2(x)(\beta|\nabla \ln M(x)|^2 - \gamma)M(x)\,dx. \qquad (26)$$

The conjunction of (1) (which holds because (2) is satisfied), and (26) yields the desired conclusion.

We prove the following lemma.

Lemma (6.1.12) [224]:

There exists a constant $\tilde{C} > 0$ with the following property: for all $\theta > 1$and all$x \in \mathbb{R}^n$, there are at most $\tilde{C}\theta^n$ indexes j such that $\left|x - x_j^t\right| \leq \theta\sqrt{t}$.

Proof:

The argument is very simple (see [255]) and we give it for the sake of completeness. Let $x \in \mathbb{R}^n$and $I(x) = \{j \in \mathbb{N};\ \left|x - x_j^t\right| \leq \theta\sqrt{t}\}$. Since, for all $j \in I(x)$,

$$Q\left(x_j^t, \sqrt{t}\right) \subset B\left(x, \left(\theta + \frac{1}{2}\right)\sqrt{nt}\right),$$

one has

$$C\left(\left(\theta + \frac{1}{2}\right)\sqrt{nt}\right)^n \geq \sum_{j\in I(x)} \left|Q\left(x_j^t, \sqrt{t}\right)\right| = |I(x)|\sqrt{t}^n,$$

we get the desired conclusion.

Lemma (6.1.13)[272]:

There exists $C_1 = C_1(M) > 0$ with the following property: for all compact disjoint subsets $E, F \subset \mathbb{R}^n$, F bounded, with $\operatorname{dist}(E,F) =: t + \epsilon, \epsilon > 0$, all functions $f \in L^2(\mathbb{R}^n, M)$ supported in E and all $t^2 > 1$

$$\|(\mathrm{I} + (t^2-1)\boldsymbol{L}^*)^{-1} f\|_{L^2(F,M)} + \|(t^2-1)\boldsymbol{L}^*(\mathrm{I} + (t^2-1)\boldsymbol{L}^*)^{-1} f\|_{L^2(F,M)}$$
$$\leq 8e^{-C_1\frac{t+\epsilon}{\sqrt{t}}}\|f\|_{L^2(E,M)}.$$

Note that, in different contexts, this kind of estimate, originating in [246], turns out to be a powerful tool, especially when no pointwise upper estimate on the kernel of the semigroup generated by $\boldsymbol{L}^*$ is available (see [247-249]). Since we found no reference for these off-diagonal estimates for the resolvent of $\boldsymbol{L}^*$, we give here a general proof [224].

Proof of Lemma (6.1.13): We argue as in [24]. Since $(\mathrm{I} + (t^2-1)\boldsymbol{L}^*)^{-1}$ is bounded with norm less than 1 for all $t^2 > 1$ it is clearly enough to restrict to $\epsilon > 0$.

Define $u(t^2-1) = (\mathrm{I} + (t^2-1)\boldsymbol{L}^*)^{-1}\mathrm{f}$, so that, for all functions $v \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} u_{t^2-1}(x_n)v(x_n)M(x_n)\,dx_n + (t^2-1)\int_{\mathbb{R}^n} \nabla u_{t^2-1}(x_n)\cdot\nabla v(x_n)M(x_n)$$

$$= \int_{\mathbb{R}^n} f(x_n)v(x_n)M(x_n)\,dx_n. \qquad (27)$$

Fix now a nonnegative function $\eta \in \mathcal{D}(\mathbb{R}^n)$ vanishing on E. Since f is supported in E, applying (27) with $v = \eta^2 u_{t^2-1}$ (remember that $u_{t^2-1} \in H^1(\mathbb{R}^n, M)$) yields,

$$\int_{\mathbb{R}^n} \eta^2(x_n)|u_{t^2-1}(x_n)|^2 M(x_n)\,dx_n + (t^2-1)\int_{\mathbb{R}^n} \nabla u_{t^2-1}(x_n)\cdot\nabla(\eta^2 u_{t^2-1})M(x_n)\,dx_n = 0,$$

which implies:

$$\int_{\mathbb{R}^n} \eta^2(x_n)|u_{t^2-1}(x_n)|^2 M(x_n)\,dx_n + (t^2-1)\int_{\mathbb{R}^n} \eta^2(x_n)|\nabla u_{t^2-1}(x_n)|^2 M(x_n)\,dx_n$$

$$= -2(t^2-1)\int_{\mathbb{R}^n} \eta(x_n)u_{t^2-1}(x_n)\nabla\eta(x_n)\cdot\nabla u_{t^2-1}(x_n)M(x_n)\,dx_n$$

$$\leq (t^2-1)\int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2|\nabla\eta(x_n)|^2 M(x_n)\,dx_n$$

$$+ (t^2-1)\int_{\mathbb{R}^n} \eta^2(x_n)|\nabla u_t(x_n)|^2 M(x_n)\,dx_n,$$

hence

$$\int_{\mathbb{R}^n} \eta^2(x_n)|\nabla u_{t^2-1}(x_n)|^2 M(x_n)\,dx_n \leq (t^2-1)\int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2|\nabla\eta(x_n)|^2 M(x_n)\,dx_n. \quad (28)$$

Let ξ be such that $\xi = 0$ on E and ξ nonnegative so that $\eta := e^{(1+\epsilon)\xi} - 1 \geq 0$ and η vanishes on E for some $\epsilon > 0$ to be chosen. Choosing this particular η in (28) with $\epsilon > 0$ gives:

$$\int_{\mathbb{R}^n} \left|e^{(2-\epsilon)\xi} - 1\right|^2 |u_{t^2-1}(x_n)|^2 M(x_n)\, dx_n$$

$$\leq (2-\epsilon)^2 (t^2-1) \int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2 |\nabla\xi(x_n)|^2 e^{2(2-\epsilon)\xi(x_n)} M(x_n)\, dx_n.$$

Taking $\epsilon = 2 - 1/(2\sqrt{t^2-1}\|\nabla\xi\|_\infty)$, one obtains:

$$\int_{\mathbb{R}^n} \left|e^{(2-\epsilon)\xi(x_n)} - 1\right|^2 |u_{t^2-1}(x_n)|^2 M(x_n)\, dx_n \leq \frac{1}{4} \int_{\mathbb{R}^n} |u_{t^2-1}(x_n)|^2 e^{2(2-\epsilon)\xi(x_n)} M(x_n)\, dx_n.$$

Using the fact that the norm of $(I + (t^2-1)\boldsymbol{L}^*)^{-1}$ is bounded by 1 uniformly in $t^2 > 1$, this gives:

$$\left\|e^{(2-\epsilon)\xi} u_{t^2-1}\right\|_{L^2(\mathbb{R}^2,M)} \leq \left\|\left(e^{(2-\epsilon)\xi} - 1\right) u_{t^2-1}\right\|_{L^2(\mathbb{R}^n,M)} + \|u_{t^2-1}\|_{L^2(\mathbb{R}^n,M)}$$

$$\leq \frac{1}{2} \left\|e^{(2-\epsilon)\xi} u_{t^2-1}\right\|_{L^2(\mathbb{R}^n,M)} + \|f\|_{L^2(\mathbb{R}^n,M)},$$

therefore

$$\int_{\mathbb{R}^n} \left|e^{(2-\epsilon)\xi(x_n)}\right|^2 |u_{t^2-1}(x_n)|^2 M(x_n)\, dx_n \leq 4 \int_{\mathbb{R}^n} |f(x_n)|^2 M(x_n)\, dx_n.$$

We choose now ξ such that $\xi = 0$ on E as before and additionally that $\xi = 1$ on F(η is then compactly supported from the fact that F is bounded). It can trivially be chosen with $\|\nabla\xi\|_\infty \leq C/\left((t^2-1)+\epsilon\right)$, which yields the desired conclusion for the L^2 norm of $(I + (t^2-1)\boldsymbol{L}^*)^{-1} f$ with a factor 4 in the right-hand side. Since $(t^2-1)\boldsymbol{L}^*(I + (t^2-1)\boldsymbol{L}^*)^{-1} f = f - (I + (t^2-1)\boldsymbol{L}^*)^{-1} f$, the desired inequality with a factor 8 readily follows.

Lemma (6.1.14)[272]:

Let $\epsilon > 0$. There exists $\tilde{C}_3 = \tilde{C}_3(M) > 0$ such that, for all $f \in \mathcal{D}(L^*)$,

$$\left\|L^{*(2-\epsilon)/4} f\right\|^2_{L^2(\mathbb{R}^n,M)}$$

$$\leq C_3 \int_0^{+\infty} (t^2-1)^{-\frac{\epsilon-4}{2}} \left\|(t^2-1)\boldsymbol{L}^*\left(\mathrm{I} + (t^2-1)\boldsymbol{L}^*\right)^{-1} f\right\|^2_{L^2(\mathbb{R}^n,M)} d\left(t^2-1\right). \quad (29)$$

Proof: Let $\mu \in \left(0, \frac{\pi}{2}\right)$, and

$$\Sigma_{\mu^+} = \{z_n \in \mathbb{C}^*; |\arg z_n| < \mu\}.$$

Let ψ be a holomorphic function in $H^\infty(\Sigma_{\mu^+})$ such that for some $C, \sigma, \tau > 0$,

$$|\psi(z_n)| \leq C \inf\{|z_n|^{\sigma}, |z_n|^{-\tau}\},$$

for any $z_n \in \Sigma_{\mu^+}$.Since $\boldsymbol{L}^*$ is positive self-adjoint operator on $L^2(\mathbb{R}^n, M)$and $\boldsymbol{L}^*$ is one-to-one on $L_0^2(\mathbb{R}^n, M)$ by (1), one has by the spectral theorem,

$$\|F\|^2_{L^2(\mathbb{R}^n,M)} \leq C \int_0^{+\infty} \left\|\psi\left((t^2-1)L^*\right)F\right\|^2_{L^2(\mathbb{R}^n,M)} \frac{d}{d(t^2-1)}$$

Whenever $F \in L_0^2(\mathbb{R}^n, M)$. Choosing $\psi(z_n) = z_n^{\frac{3-\epsilon}{4}}/(1+z_n)$yields,

$$\|F\|^2_{L^2(\mathbb{R}^n,M)} \leq C \int_0^{+\infty} \left\|\left((t^2-1)L^*\right)^{\frac{2+\epsilon}{4}} (\mathrm{I} + (t^2-1)L^*)^{-1} F\right\|^2_{L^2(\mathbb{R}^n,M)} \frac{d}{d(t^2-1)}, \tag{30}$$

Whenever $F \in L_0^2(\mathbb{R}^n, M)$.

Let $F \in L^2(\mathbb{R}^n, M)$. Since

$$\int_{\mathbb{R}^n} L^* f(x_n) M(x_n)\, dx_n = 0,$$

it follows from (9) that the same is true with $L^{*(2-\epsilon)/4} f$. Applying now (30) with $F = L^{*(2-\epsilon)/4} f$ gives the conclusion of Lemma (6.1.14).

Let us draw a simple corollary of Lemma (6.1.14) (see [224]).

Corollary (6.1.15)[272]:

For any $\in, \varepsilon > 0$, there is $A = A(M, \varepsilon)$ such that

$$\left\|L^{*(2-\epsilon)/4} f\right\|^2_{L^2(\mathbb{R}^n,M)} \leq \tilde{C}_3 \int_0^A (t^2-1)^{-\frac{\in-4}{2}} \|(t^2-1)L^*(\mathrm{I} + (t^2-1)L^*)^{-1} f\|^2_{L^2(\mathbb{R}^n,M)}. \tag{31}$$

Proof. The proof is straightforward since

$$\|(t^2-1)L^*(\mathrm{I} + (t^2-1)L^*)^{-1} f\|^2_{L^2(\mathbb{R}^n,M)} \leq C\|F\|^2_{L^2(\mathbb{R}^n,M)}$$

And

$$\int_A^{+\infty} (t^2-1)^{\frac{-3-\in}{2}} dt \xrightarrow{A \to +\infty} 0.$$

We now come to the desired estimate.

Lemma (6.1.16)[272]:

Let $\in, \varepsilon > 0$ and A given by Corollary (6.1.15). There exist $\tilde{C}_4 = \tilde{C}_4(M, A) > 0$ and $c' = c'(A, M) > 0$ such that, for all $f \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_0^A (t^2-1)^{\frac{\in-4}{2}} \|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}\, dt$$

$$\leq \tilde{C}_4 \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x_n)-f(x_{n+1})|^2}{|x_n-x_{n+1}|^{n+2-\epsilon}} M(x_n) e^{-c'|x_n-x_{n+1}|}\, dx_n\, dx_{n+1}.$$

Proof: Throughout this proof, for all $x_n \in \mathbb{R}^n$ and all $s > 0$, denote by $Q(x_n, s)$ the closed cube centered at x_n with side length s . For fixed $(t^2-1) \in (0, A)$, following Lemma (6.1.14), we shall look for an upper bound for $\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}$ involving first order differences for f. Pick up a countable family of points $(x_n)_j^{t^2-1} \in \mathbb{R}^n, j \in \mathbb{N}$, such that the cubes $Q\left(\left(x_n^{t^2-1}\right)_j, \sqrt{t^2-1}\right)$ have pairwise disjoint interiors, and

$$\mathbb{R}^n = \bigcup_{j\in\mathbb{N}} Q\left(\left(x_n^{t^2-1}\right)_j, \sqrt{t^2-1}\right). \qquad (32)$$

By Lemma (B.1) in [224], there exists a constant $\tilde{C} > 0$ such that for all $\in > 0$ and all $x_n \in \mathbb{R}^n$, there are at most $\tilde{C}(1+\in)^n$ indexes j such that $\left|x_n - \left(x_n^{t^2-1}\right)_j\right| \leq (1+\in)\sqrt{t^2-1}$.

For fixed j, one has

$$(t^2-1)L^*\left(\mathrm{I}+(t^2-1)L^*\right)^{-1}f = (t^2-1)L^*\left(\mathrm{I}+(t^2-1)L^*\right)^{-1}g^{j,t^2-1},$$

Where, for all $x_n \in \mathbb{R}^n$,

$$g^{j,(t^2-1)}(x_n) := f(x_n) - m^{j,(t^2-1)}$$

And m^{j,t^2-1} is defined by:

$$m^{j,t^2-1} := \frac{1}{\left|Q\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)\right|} \int_{Q\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)} f(x_{n+1})\, dx_{n+1}.$$

Note that, here, the mean value of f is computed with respect to the Lebesgue measure on $\mathbb{R}^n$. Since (32) holds and the cubes $Q\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)$ have pairwise disjoint interiors, one clearly has:

$$\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}$$
$$=\sum_{j\in\mathbb{N}}\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}f\|^2_{L^2Q\left((x_n)_j^{t^2-1},2\sqrt{t^2-1}\right)}$$
$$=\sum_{j\in\mathbb{N}}\left\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}\mathrm{g}^{j,t}\right\|^2_{L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)},$$

And we are left with the task of estimating,

$$\left\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}\mathrm{g}^{j,t^2-1}\right\|^2_{L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)}.$$

To that purpose, set

$$C_0^{j,t^2-1}=L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)\text{and}C_k^{j,t^2-1}$$
$$=L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)\backslash L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right),\quad\forall k\geq 1,$$

And $\mathrm{g}_k^{j,t^2-1}\coloneqq \mathrm{g}^{j,t^2-1}\mathbf{1}_{C_k^{j,t^2-1}}, k\geq 0$, where, for any subset $A\subset\mathbb{R}^n,\mathbf{1}_A$ is the usual characteristic function of A. Since $\mathrm{g}^{j,t^2-1}=\sum_{k\geq 0}\mathrm{g}_k^{j,t^2-1}$ one has:

$$\left\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}\mathrm{g}^{j,t^2-1}\right\|^2_{L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)}$$
$$\leq\sum_{k\geq 0}\left\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}\mathrm{g}^{j,t^2-1}\right\|^2_{L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)}\tag{33}$$

and, using Lemma (6.1.13), one obtains (for some constants $\tilde{C},\tilde{c}>0$):

$$\left\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}\mathrm{g}^{j,t^2-1}\right\|^2_{L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)}$$
$$\leq\tilde{C}\left(\left\|\mathrm{g}_0^{j,t^2-1}\right\|_{L^2\left(C_0^{j,t^2-1},M\right)}+\sum_{k\geq 0}e^{-\tilde{c}2^k}\left\|\mathrm{g}_k^{j,t^2-1}\right\|_{L^2\left(C_k^{j,t^2-1},M\right)}\right)\tag{34}$$

By Cauchy-Schwarz's inequality, we deduce (for another constant $C'_1>0$):

$$\left\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}\mathrm{g}^{j,t^2-1}\right\|^2_{L^2Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)}$$
$$\leq C'_1\left(\left\|\mathrm{g}_0^{j,t^2-1}\right\|_{L^2\left(C_0^{j,t^2-1},M\right)}+\sum_{k\geq 0}e^{-\tilde{c}2^k}\left\|\mathrm{g}_k^{j,t^2-1}\right\|_{L^2\left(C_k^{j,t^2-1},M\right)}\right)\tag{35}$$

As a consequence, we have:

$$\int_0^A (t^2-1)^{\frac{\epsilon-4}{2}} \left\|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}\mathrm{g}^{j,t^2-1}\right\|^2_{L^2\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)} d(t^2-1)$$

$$\leq C'_1 \int_0^A (t^2-1)^{\frac{-(3-\epsilon)}{2}} \sum_{j\geq 0} \left\|\mathrm{g}_0^{j,t^2-1}\right\|_{L^2\left(C_0^{j,t^2-1},M\right)} d(t^2-1))$$

$$+C'_1 \int_0^A (t^2-1)^{\frac{-(3-\epsilon)}{2}} \sum_{k\geq 1} e^{-\tilde{c}2^k} \sum_{j\geq 0} \left\|\mathrm{g}_k^{j,t^2-1}\right\|_{L^2\left(C_k^{j,t^2-1},M\right)} d(t^2-1). \quad (36)$$

We claim that

Lemma (6.1.17)[272]:

There exists $\bar{C}_1 > 0$ such that, for all $t^2 > 1$ and all $j \in \mathbb{N}$:

C. For the first term:

$$\left\|\mathrm{g}_0^{j,t^2-1}\right\|^2_{L^2\left(C_0^{j,t^2-1},M\right)} \leq \frac{\bar{C}_1}{(t^2-1)^{n/2}} \int_{\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)} \int_{\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)} |f(x_n)-f(x_{n+1})|^2 M(x_n)dx_n\, dx_{n+1}.$$

D. For all $k \geq 1$,

$$\left\|\mathrm{g}_k^{j,t^2-1}\right\|^2_{L^2\left(C_k^{j,t^2-1},M\right)} \leq \frac{\bar{C}_1}{\left(\sqrt{t^2-1}\right)^2} \int_{x_n\in\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j,2^{k+1}\sqrt{t^2-1}\right)} \int_{y_{n+1}\in\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j,2^{k+1}\sqrt{t^2-1}\right)} |f(x_n) - f(x_{n+1})|^2 M(x_n)\, dx_n\, dx_{n+1}.$$

We postpone the proof to the end of the section and finish the proof of Lemma (6.1.16). Using Assertion A in Lemma (6.1.17), summing up on $j \geq 0$and integrating over $(0, A)$, we get:

$$\int_0^A (t^2-1)^{\frac{-(3-\epsilon)}{2}} \sum_{j\geq 0} \left\|\mathrm{g}_0^{j,t^2-1}\right\|^2_{L^2\left(C_0^{j,t^2-1},M\right)} d(t^2-1)$$

$$= \sum_{j\geq 0} \int_0^A (t^2-1)^{\frac{-(3-\epsilon)}{2}} \left\|\mathrm{g}_0^{j,t^2-1}\right\|^2_{L^2\left(C_0^{j,t^2-1},M\right)} d(t^2-1)$$

$$\leq \bar{C}_1 \sum_{j\geq 0} \int_0^A (t^2-1)^{\frac{-(3+\epsilon+n)}{2}} \left(\int_{\mathcal{Q}\left((x_n)_j^{t^2-1}, 2\sqrt{t^2-1}\right)} \int_{\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)} |f(x_n) - f(x_{n+1})|^2 M(x_n)\, dx_n\, dx_{n+1} \right) d(t^2-1)$$

$$\leq \bar{C}_1 \sum_{j\geq 0} \iint_{(x_n,x_{n+1})\in\mathbb{R}^n\times\mathbb{R}^n} |f(x_n) - f(x_{n+1})|^2 M(x_n) \left(\int_{t\geq \max\left\{\frac{\left|x_n-\left(x_n^{t^2-1}\right)_j\right|^2}{n}; \frac{\left|x_{n+1}-\left(x_n^{t^2-1}\right)_j\right|^2}{n}\right\}}^{A} (t^2-1)^{\frac{-(3+\epsilon+n)}{2}} d(t^2-1) \right) dx_n\, dx_{n+1}.$$

The Fubini theorem now shows:

$$\sum_{j\geq 0} \int_{(t^2-1)\geq \max\left\{\frac{\left|x_n-\left(x_n^{t^2-1}\right)_j\right|^2}{n}; \frac{\left|x_{n+1}-\left(x_n^{t^2-1}\right)_j\right|^2}{n}\right\}}^{A} (t^2-1)^{\frac{-(3+\epsilon+n)}{2}} d(t^2-1)$$

$$= \int_0^A (t^2-1)^{\frac{-(3+\epsilon+n)}{2}} \sum_{j\geq 0} \mathbf{1}_{\left(\max\left\{\frac{\left|x_n-\left(x_n^{t^2-1}\right)_j\right|^2}{n}; \frac{\left|x_{n+1}-\left(x_n^{t^2-1}\right)_j\right|^2}{n}\right\}, +\infty\right)} (t^2-1)\, d(t^2-1).$$

Observe that, by Lemma (B.1) in [224], there is a constant $N \in \mathbb{N}$ such that, for all $t^2 > 1$, there are at most N indexes $j$such that $\left|x_n - \left(x_n^{t^2-1}\right)_j\right|^2 < n(t^2-1)$ and $\left|x_{n+1} - \left(x_n^{t^2-1}\right)_j\right|^2 < n(t^2-1)$. If such an index j exists, one has $|x_n - x_{n+1}| < 2\sqrt{n(t^2-1)}$. It therefore follows that

$$\sum_{j\geq 0}\mathbf{1}_{\left(\max\left\{\frac{\left|x_n-\left(x_n^{t^2-1}\right)_j\right|^2}{n};\frac{\left|x_{n+1}-\left(x_n^{t^2-1}\right)_j\right|^2}{n}\right\},+\infty\right)}(t^2-1)\leq N\mathbf{1}_{(|x_n-x_{n+1}|^2/4n,+\infty)}(t^2-1),$$

So that

$$\int_0^A (t^2-1)^{\frac{\epsilon-4}{2}}\sum_j \left\|g_0^{j,t^2-1}\right\|^2_{L^2\left(C_0^{j,t^2-1},M\right)}dt$$

$$\leq \bar{C}N\iint_{\mathbb{R}^n\times\mathbb{R}^n}|f(x_n)-f(x_{n+1})|^2 M(x_n)\left(\int_{|x_n-x_{n+1}|^2/4n}^{A}(t^2-1)^{\frac{-(4+\epsilon+n)}{2}}d(t^2-1)\right)dx_n\,dx_{n+1}$$

$$\leq \bar{C}_1 N\iint_{|x_n-x_{n+1}|\leq 2\sqrt{nA}}\frac{|f(x_n)-f(x_{n+1})|^2}{|x_n-x_{n+1}|^{n+2-\epsilon}}M(x_n)\,dx_n\,dx_{n+1}. \qquad (37)$$

Using now Assertion B in Lemma (6.1.17), we obtain, for all $j\geq 0$ and all $k\geq 1$,

$$\int_0^A (t^2-1)^{\frac{\epsilon-4}{2}}\sum_{j\geq 0}\left\|g_k^{j,t^2-1}\right\|_2^2 d(t^2-1)$$

$$\leq \bar{C}_1\sum_{j\geq 0}\int_0^A (t^2-1)^{-1-(2-\epsilon)/2}\left(\iint_{\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j,2^{k+1}\sqrt{t^2-1}\right)\times\mathcal{Q}\left(\left(x_n^{t^2-1}\right)_j,2^{k+1}\sqrt{t^2-1}\right)}|f(x_n)-f(x_{n+1})|^2 M(x_n)\,dx_n\,dx_{n+1}\right)d(t^2-1)$$

$$\leq \bar{C}_1 \sum_{j\geq 0} \iint_{x_n, x_{n+1}\in\mathbb{R}^n} |f(x_n)$$

$$- f(x_{n+1})|^2 M(x_n) \left(\int_0^A (t^2-1)^{\frac{-(4+\in+n)}{2}} 1_{\max\left\{\frac{\left|x_n-\left(x_n^{t^2-1}\right)_j\right|^2}{n};\frac{\left|x_{n+1}-\left(x_n^{t^2-1}\right)_j\right|^2}{n}\right\},+\infty} (t^2$$

$$- 1) d(t^2-1) \right) dx_n\, dx_{n+1}.$$

But, given $t^2 > 1$, $x_n, x_{n+1} \in \mathbb{R}^n$, by Lemma (B.1) in [224] again, there exist at most $\tilde{C}_1 2^{kn}$ indexes j such that

$$\left|x_n - \left(x_n^{t^2-1}\right)_j\right| \leq 2^k\sqrt{n(t^2-1)} \text{and} \left|x_{n+1} - \left(x_n^{t^2-1}\right)_j\right| \leq 2^k\sqrt{n(t^2-1)},$$

and for these indexes j, $|x_n - x_{n+1}| \leq 2^{k+1}\sqrt{n(t^2-1)}$. As a consequence we have:

$$\int_0^A (t^2-1)^{\frac{-(4+\in+n)}{2}} \sum_{j\geq 0} \mathbf{1}_{\max\left\{\frac{\left|x_n-\left(x_n^{t^2-1}\right)_j\right|^2}{4^k n};\frac{\left|x_{n+1}-\left(x_n^{t^2-1}\right)_j\right|^2}{4^k n}\right\},+\infty} (t^2-1)\, d(t^2-1)$$

$$\leq \tilde{C}_1 2^{kn} \int_{t^2 \geq \frac{|x_n-x_{n+1}|^2}{4^{k+1}n}+1}^{A} (t^2-1)^{-\frac{-(4+\epsilon+n)}{2}} dt$$

$$\leq \tilde{C}_1{}'2^{k(2-\epsilon+n)}|x_n-x_{n+1}|^{-n-(2-\epsilon)}\mathbf{1}_{|x_n-x_{n+1}|\leq 2^{k+1}\sqrt{nA}}, \quad (38)$$

for some other constant $\tilde{C}'_1 > 0$, and there for

$$\int_0^A (t^2-1)^{-1-\frac{2-\epsilon}{2}} \sum_j \left\| g_k^{j,t^2-1} \right\|^2_{L^2\left(C_0^{j,t^2-1},M\right)} dt$$

$$\leq \bar{C}_1 \tilde{C}'_1 2^{k(2-\epsilon+n)} \iint_{|x_n-x_{n+1}|\leq 2^{k+1}\sqrt{nA}} \frac{|f(x_n)-f(x_{n+1})|^2}{|x_n-x_{n+1}|^{n+2-\epsilon}} M(x_n)\, dx_n\, dx_{n+1}.$$

We can now conclude the proof of Lemma (6.1.16), using Lemma (6.1.14), (35), (37) and (38). We have proved, by reconsidering (36):

$$\int_0^A (t^2-1)^{\frac{(\epsilon-4)}{2}} \|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)} d(t^2-1)$$

$$\le C'_1\tilde{C}_1 N \iint_{|x_n-x_{n+1}|\le 2^{k+1}\sqrt{nA}} \frac{|f(x_n)-f(x_{n+1})|^2}{|x_n-x_{n+1}|^{n+2-\epsilon}} M(x_n)\,dx_n\,dx_{n+1}$$

$$+\sum_{k\ge 1} C'_1\bar{C}_1\tilde{C}'_1 2^{k(2-\epsilon)} e^{-\tilde{c}2^k} \iint_{|x_n-x_{n+1}|\le 2^{k+1}\sqrt{nA}} \frac{|f(x_n)-f(x_{n+1})|^2}{|x_n-x_{n+1}|^{n+2-\epsilon}} M(x_n)\,dx_n\,dx_{n+1} \qquad (39)$$

and we deduce that

$$\int_0^A (t^2-1)^{\frac{(\epsilon-4)}{2}} \|(t^2-1)L^*(\mathrm{I}+(t^2-1)L^*)^{-1}f\|^2_{L^2(\mathbb{R}^n,M)}\, d(t^2-1)$$

$$\le \tilde{C}_4 \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x_n)-f(x_{n+1})|^2}{|x_n-x_{n+1}|^{n+2-\epsilon}} M(x_n) e^{-c'|x_n-x_{n+1}|}\,dx_n\,dx_{n+1}$$

for some constants $\tilde{C}_4$ and $c'>0$ as claimed in the statement.

Proof of Lemma (6.1.17): Observe first that, for all $\mathrm{x_n} \in \mathbb{R}^n$,

$$\mathrm{g}_0^{j,t^2-1}(x_n) = f(x_n) - \frac{1}{\left|Q\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)\right|} \int_{Q\left(x_j^{t^2-1}, 2\sqrt{t^2-1}\right)} f(x_{n+1})\,dx_{n+1}.$$

$$= \frac{1}{\left|Q\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)\right|} \int_{Q\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)} \left(f(x_n)-f(x_{n+1})\right)dx_{n+1}.$$

By Cauchy-Schwarz inequality, it follows that

$$\left|\mathrm{g}_0^{j,t^2-1}(x_n)\right|^2 \le \frac{\tilde{C}}{(t^2-1)^{n/2}} \int_{Q\left(\left(x_n^{t^2-1}\right)_j, 2\sqrt{t^2-1}\right)} |f(x_n)-f(x_{n+1})|^2\,dx_{n+1}.$$

Therefore,

$$\left\| g_k^{j,t^2-1} \right\|_{L^2\left(C_k^{j,t},M\right)}^2$$
$$\leq \frac{\tilde{C}}{(t^2-1)^{n/2}} \int_{Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)} \int_{Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)} |f(x_n)-f(x_{n+1})|^2 M(x_n)\, dx_n\, dx_{n+1}$$

which shows Assertion A. We argue similarly for Assertion B and obtain:

$$\left\| g_k^{j,t^2-1} \right\|_{L^2\left(C_k^{j,t^2-1},M\right)}^2$$
$$\leq \frac{\tilde{C}}{t^{n/2}} \int_{x\in Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)} \int_{y\in Q\left(\left(x_n^{t^2-1}\right)_j,2\sqrt{t^2-1}\right)} |f(x_n)-f(x_{n+1})|^2 M(x_n)\, dx\,_n dx_{n+1},$$

which ends the proof of Lemma (6.1.17)

We end up this section with a few comments on Lemma (6.1.17). It is a well-known fact [250] that, when $\epsilon \geq 0$.

$$\left\|(-\Delta)^{(2-\epsilon)/4} f\right\|_{L^{1+\epsilon,}(\mathbb{R}^n)} \leq \tilde{C}_{2-\epsilon,1+\epsilon,}\left\|S_{2-\epsilon,1+\epsilon,}f\right\|_{L^{1+\epsilon,}(\mathbb{R}^n)}, \tag{40}$$

where

$$S_{2-\epsilon,1+\epsilon}f(x_n) = \left(\int_0^{+\infty} \left(\int_B |f(x_n + r x_{n+1}) - f(x_n)|\, dx_{n+1}\right)^2 \frac{dr}{r^{3-\epsilon}}\right)^{\frac{1}{2}},$$

And also [251]

$$\left\|(-\Delta)^{\frac{2-\epsilon}{4}} f\right\|_{L^{1+\epsilon}(\mathbb{R}^n)} \leq \tilde{C}_{2-\epsilon,1+\epsilon}\|D_{2-\epsilon}f\|_{L^{1+\epsilon}(\mathbb{R}^n)} \tag{41}$$

Where

$$D_{2-\epsilon}f(x_n) = \left(\int_{\mathbb{R}^n} \frac{|f(x_n + x_{n+1}) - f(x_n)|^2}{|x_{n+1}|^{n+2-\epsilon}}\, dx_{n+1}\right)^{\frac{1}{2}}.$$

In [251], these inequalities were extended to the setting of a unimodular Lie group endowed with a sub-laplacian Δ, relying on semigroups techniques and Littlewood-Paley-Stein functionals. In particular, in [251] and [224] use pointwise estimates of the kernel of the semigroup generated by Δ. The conclusion of Lemma (6.1.71) means that the norm of $L^{*(2-\epsilon)/4}f$ in $L^2(\mathbb{R}^n, M)$ is bounded from above by the $L^2(\mathbb{R}^n, M)$norm of an appropriate version of $D_{2-\epsilon}$. Note that this does not require pointwise estimates for the kernel of the semigroup generated by L^*, and that the L^2off-diagonal estimates given by

Lemma (6.1.13), which hold for a general sequence measure M, are enough for the argument to hold (see [224]) However, we do not know if an$L^{1+\epsilon}$ version of Lemma (6.1.17) still holds. Note also that we do not compare the $L^2(\mathbb{R}^n, M)$norm of $L^{*(2-\epsilon)/4}f$ with the $L^2(\mathbb{R}^n, M)$norm of a version of $S_{2-\epsilon,1+\epsilon}f$. Finally, the converse inequalities to (40) and (41) hold in $\mathbb{R}^n$ and also on a unimodular Lie group [252] and [224] did not consider the corresponding inequalities .

Corollary (6.1.18)[272] :

If L^* is self-adjoint and normal then

(i) $\|\lambda'\|_{L^2} \geq \frac{dist\ (\mu, \sum L^*)}{\|\mu\|_{L^2}} - \epsilon$.

(ii) $\|\mu\|_{L^2} \leq \frac{1}{|\lambda'|}$.

(iii) $I > 2 - t^2$.

(iv) $\|L^*\|_{L^2} < 1 + \frac{\epsilon}{t^2 - 1}$.

Proof:

(i) Since $L^* \geq \lambda'\mu$ then $\|(L^* - \mu)^{-1}\|_{L^2} \leq \|\mu^{-1}(\lambda' - 1)^{-1}\|_{L^2}$

We get $\|\mu\,(\lambda' - 1)\|_{L^2} = dist\ (\mu, \sum L^*) - \epsilon$

Thus , $\|\lambda'\|_{L^2} \geq \frac{dist\ (\mu, \sum L^*)}{\|\mu\|_{L^2}} - \epsilon$.

(ii) Let L^* be a contraction from (24) we have $\|\mu\|_{L^2} \leq \frac{1}{|\lambda'|}$.

(iii) Given $\|(I + (t^2 - 1)\,\lambda'\mu)^{-1}\| \leq 1$, using (ii) we can get $I > 2 - t^2$.

(iv) For $\|(I + (t^2 - 1)\,L^*)^{-1}\| \leq 1$, and $I + (t^2 - 1)\,L^* = 1 + \epsilon$, using (iii) we can get $\|L^*\|_{L^2} < 1 + \frac{\epsilon}{t^2 - 1}$.

Sec (6.2):Equivalence of Fractional Order:

Sobolev norms and semi-norms play a central role in the numerical analysis of discretization methods for partial differential equations. For instance, standard finite element error analysis is essentially a combination of the Bramble-Hilbert lemma and transformation properties of Sobolev (semi-) norms. These properties are also central to the area of preconditioners for (and based on) variational methods. More precisely, arguments based on finite dimensions of local spaces are inherently connected with scaling arguments to keep dimensions bounded. Norms are usually not scalable. That is, the corresponding equivalence numbers behave differently with respect to a scaling parameter like the diameter $D\mathcal{O}$ of the domain $\mathcal{O}$when the domain under consideration is transformed by an affine map that maintains shape regularity (i.e., the ratio of $D\mathcal{O}$ and the "inner diameter" of $\mathcal{O}$is bounded). This can be usually fixed only when essential boundary conditions are present. An example is using the H^1-semi-norm as norm in H_0^1. More generally, semi-norms have better scaling properties: usually they can be defined so that equivalence numbers are of the same order with respect to $D\mathcal{O}$ under shape-regular affine transformations of the domain.

Whereas properties of Sobolev (semi-) norms under smooth transformations or simple scaling are straightforward as long as their orders are integer, things are getting more complicated for fractional-order Sobolev norms. Such norms appear, e.g., in a natural way when considering boundary integral equations of the first kind [268, 266] or when studying the regularity of elliptic problems in non-convex polygonal domains [264]. There are different ways to define fractionalorderSobolev norms and they all have advantages and disadvantages (standard references are [267, 257]). Different norm variants are known to be equivalent. But dependence of the equivalence constants on the order and the domain are more involved.

There are several ways to define Sobolev norms. We use the ones defined by a double integral (Sobolev-Slobodeckij) and by interpolation. For the latter we use the so-called real K-method, cf. [258]. For $0 < s < 1$, the interpolation norm in the fractional-order Sobolev space $H^s(\mathcal{O})$ is defined by

$$\|v\|_{[L^2(\mathcal{O}),H^1(\mathcal{O})]_s} = \|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O})_s} = \left(\int_0^\infty t^{-2s} \inf_{v=v_0+v_1}\left(\|v_0\|_{0,\mathcal{O}}^2 + t^2\|v_0\|_{1,\mathcal{O}}^2\right)\frac{dt}{t}\right)^{1/2}.$$

Here and in the following, the notation $\inf_{v=v_0+v_1}\left(\|v_0\|_{0,\mathcal{O}}^2 + t^2\|v_0\|_{1,\mathcal{O}}^2\right)$ implies that the infimum is taken over $v_0 \in L^2(\mathcal{O})$and $v_1 \in H^2(\mathcal{O})$, or corresponding spaces as indicated by the respective norms.

We also define the interpolation space

$$\widetilde{H}^s(\mathcal{O}) = [L^2(\mathcal{O}), H_0^1(\mathcal{O})]_s$$

with corresponding notation for the norm. The notation $\widetilde{H}^s$ is used by Grisvard and is common in the boundary element literature, whereas the notation $H_{00}^s = \widetilde{H}^s$ is used by Lions and Magenes and is common in the finite element literature.

The Sobolev-Slobodeckij variant of these norms is defined (for $0 < s < 1$) by

$$\|v\|_{H^s(\mathcal{O})} = \|v\|_{s,\mathcal{O}} = \left(\|v\|_{L^2(\mathcal{O})}^2 + \int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\right)^{1/2} \quad (42)$$

$$\|v\|_{\widetilde{H}^s(\mathcal{O})} = \|v\|_{\sim,s,\mathcal{O}} = \left(\|v\|_{H^s(\mathcal{O})}^2 + \left\|\frac{v(x)}{\mathrm{dist}(x,\partial\mathcal{O})^s}\right\|_{L^2(\mathcal{O})}^2\right)^{1/2}$$

The corresponding semi-norms are

$$|v|_{[L^2(\mathcal{O}),H^1(\mathcal{O})]_s} = |v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} = \left(\int_0^\infty t^{-2s} \inf_{v=v_0+v_1}\left(\|v_0\|_{0,\mathcal{O}}^2 + t^2|v_1|_{1,\mathcal{O}}^2\right)\frac{dt}{t}\right)^{1/2}$$

and

$$|v|_{H^s(\mathcal{O})} = |v|_{s,\mathcal{O}} = \left(\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{1/2}.$$

Additionally, it is useful to define the semi-norm of quotient space type

$$|v|_{s,\mathcal{O},\inf} = \|v\|_{H^s(\mathcal{O})/\mathbb{R}} = \inf_{c\in\mathbb{R}}\|v+c\|_{s,\mathcal{O}}.$$

The aim of this section is to study equivalences of the semi-norms previously defined, on a fixed domain. Together with mapping properties , these estimates are needed to prove our main results . Proofs are based on a standard norm equivalence and specific Poincar´e-Friedrichs' inequalities, which are also recalled here.

It is well known that for Lipschitz domains different definitions of Sobolev norms are equivalent. However, equivalence constants depend usually on the order and the domain under consideration. In particular, for a bounded Lipschitzdomain $\mathcal{O}$, the norms $\|\cdot\|_{s,\mathcal{O}}$ and $\|\cdot\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ are equivalent for $0 < s < 1$, cf. [267, 264, 268]. Such equivalences are shown by corresponding equivalences on $\mathbb{R}^n$ and the use of appropriate extension operators, cf. [260], see also [125] for non-Lipschitz domains. In particular, the norms previously defined are uniformly equivalent for s in a closed subset of $(0, 1)$, see [267].

Here, for the norms, we don't elaborate on the dependence of the equivalence constants on $s$and $\mathcal{O}$. We rather give them specific names to be used in estimates to follow.

Proposition (6.2.1) [256]: (equivalence of norms)

For a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^n$ and for given $s \in (0,1)$ there exist constants $k(s,\mathcal{O}), K(s,\mathcal{O}) > 0$ such that

$$k(s,\mathcal{O})\|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \le \|v\|_{s,\mathcal{O}} \le K(s,\mathcal{O})\|v\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \quad \forall v \in H^s(\mathcal{O}).$$

For a proof (see [268]).

It is well known that, on bounded Lipschitz domains, lower-order norms can be bounded by higher-order semi-norms plus finite rank terms. Such estimates are referred to as Poincar´e-Friedrichs' inequalities. For integer-order norms there are direct proofs with explicit constants (depending on orders and domains) [269] and attention has received finding best constants and deriving improved weighted estimates, (see [270, 271] and [261]), respectively. We need such a Poincar´e-Friedrichs' inequality for fractional-order norms on bounded domains (for unbounded domains, see [224]), and refer to [262, Lemma 3.4] for a proof. This proof is given for two dimensions but immediately extends to the general case.

Proposition (6.2.2) [256]: (Poincar´e-Friedrichs inequality, Sobolev-Slobodeckij semi-norms)

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain, and $s \in (0,1)$. Then there holds

$$\|v\|_{0,\mathcal{O}} \leq C_{PF,SS}(s,\mathcal{O})\left(|v|_{s,\mathcal{O}} + \left|\int_{\mathcal{O}} v\right|\right) \quad \forall v \in H^s(\mathcal{O})$$

with

$$C_{PF,SS}(s,\mathcal{O}) = |\mathcal{O}|^{-1/2} \max\{1, 2^{-1/2} D_{\mathcal{O}}^{n/2+s}\}.$$

Here, $|\mathcal{O}|$ denotes the area of $\mathcal{O}$ and, as mentioned in the introduction, $D_{\mathcal{O}}$ is its diameter.

Lemma (6.2.3)[256]:

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain. Then there holds

$$|v|_{s,\mathcal{O}}^2 \leq |v|_{s,\mathcal{O},\inf}^2 = |v|_{s,\mathcal{O}}^2 + \inf_{c\in\mathbb{R}} \|v + c\|_{0,\mathcal{O}}^2 \leq \left(1 + C_{PF,SS}^2\right)|v|_{s,\mathcal{O}}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $C_{PF,SS} = C_{PF,SS}(s,\mathcal{O})$ is the number from Proposition (6.2.2).

Proof:

By definition of $|\cdot|_{s,\mathcal{O}}$ there holds for any $c \in \mathbb{R}$and any $v \in H^s(\mathcal{O})$ (we now drop $\mathcal{O}$from the notation)

$$|v|_s = |v + c|_s.$$

Therefore

$$|v|_s \leq \inf_{c\in\mathbb{R}} \|v + c\|_s = |v|_{s,\inf}$$

which is the first assertion. By the initial argument and the definition of the Sobolev-Slobodeckij norm one also finds that

$$|v|_{s,\inf}^2 = \inf_{c\in\mathbb{R}} \|v + c\|_s^2 = \inf_{c\in\mathbb{R}} \|v + c\|_0^2 + |v|_s^2.$$

This is the second assertion.

The last relation and the Poincar´e-Friedrichs' inequality (Proposition (6.2.2) lead to

$$|v|_{s,\inf}^2 \leq C_{PF,SS}^2 \inf_{c\in\mathbb{R}} \left(|v|_s + \left|\int_{\mathcal{O}} (v + c)\right|\right)^2 + |v|_s^2 = \left(1 + C_{PF,SS}^2\right)|v|_s^2.$$

This finishes the proof.

Lemma (6.2.4) [256]:

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There holds

$$k^2 |v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 \leq |v|_{s,\mathcal{O},\inf}^2 \leq 3K^2 |v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 + \frac{K^2}{s(1-s)} \inf_{c\in\mathbb{R}} \|v + c\|_{0,\mathcal{O}}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $k = k(s,\mathcal{O})$ and $K = K(s,\mathcal{O})$ are the numbers from Proposition (6.2.1).

Proof:

Let $v \in H(\mathcal{O})$, and let c_0, c_1 denote generic constants. For any $t > 0$there holds

$$\inf_{v=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2) = \inf_{v=v_0+c_0+v_1+c_1}(\|v_0 + c_0\|_0^2 + t^2|v_1|_1^2)$$
$$= \inf_{c_1, v-c_1=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2),$$

that is

$$\inf_{v=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2) = \inf_{c\in\mathbb{R}}\inf_{v+c=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2)$$
$$\leq \inf_{c\in\mathbb{R}}\inf_{v+c=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2).$$

(Recall that our convention for the notation $\inf_{v+c=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2)$ implies that the infimum is taken with respect to $v_0 \in L^2(\mathcal{O})$and $v_1 \in H^1(\mathcal{O})$.) We conclude that

$$|v|_{L^2,H^1,s}^2 = \int_0^\infty t^{-2s}\inf_{v=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2)\,dt$$
$$\leq \inf_{c\in\mathbb{R}}\int_0^\infty t^{-2s}\inf_{v+c=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2)\frac{dt}{t} = \inf_{c\in\mathbb{R}}\|v+c\|_{L^2,H^1,s}^2.$$

By Proposition (6.2.1)

$$\inf_{c\in\mathbb{R}}\|v+c\|_{L^2,H^1,s}^2 \leq k^{-2}\inf_{c\in\mathbb{R}}\|v+c\|_s^2 = k^{-2}|v|_{s,\inf}^2,$$

so that the first assertion follows.

By definition and using Proposition (6.2.1) there holds

$$|v|_{s,\inf}^2 = \inf_{c\in\mathbb{R}}\|v+c\|_s^2 \leq K^2\inf_{c\in\mathbb{R}}\|v+c\|_{L^2,H^1,s}^2$$
$$= K^2\inf_{c\in\mathbb{R}}\int_0^\infty t^{-2s}\inf_{v+c=v_0+v_1}(\|v_0\|_0^2 + t^2|v_1|_1^2)\frac{dt}{t}. \tag{43}$$

We bound the integrand separately for $t < 1$and $t \geq 1$.

For $t < 1$ we use the representation $v + c = v_0 + v_1$ to bound

$$\|v_0\|_0^2 + t^2\|v_1\|_0^2 + t^2|v_1|_1^2 \leq \|v_0\|_0^2 + 2t^2(\|v+c\|_0^2 + \|v_0\|_0^2) + t^2|v_1|_1^2$$
$$\leq 3\|v_0\|_0^2 + 2t^2\|v+c\|_0^2 + t^2|v_1|_1^2.$$

If $t \geq 1$ then we select $v_0 = v + c$ to conclude that

$$\inf_{v+c=v_0+v_1}(\|v_0\|_0^2 + t^2\|v_1\|_0^2 + t^2|v_1|_1^2) \leq \|v+c\|_0^2.$$

Together this yields

$$\int_0^\infty t^{-2s}\inf_{v+c=v_0+v_1}(\|v_0\|_0^2 + t^2\|v_1\|_0^2 + t^2|v_1|_1^2)\frac{dt}{t}$$
$$\leq \int_0^\infty t^{-2s}\inf_{v+c=v_0+v_1}(3\|v_0\|_0^2 + 2t^2\|v+c\|_0^2 + t^2|v_1|_1^2)\frac{dt}{t} + \int_1^\infty t^{-2s}\|v+c\|_0^2\frac{dt}{t}$$
$$= \int_0^\infty t^{-2s}\inf_{v+c=v_0+v_1}(3\|v_0\|_0^2 + t^2|v_1|_1^2)\frac{dt}{t} + \|v+c\|_0^2\left(\int_0^1 2t^{1-2s}\,dt + \int_1^\infty t^{-1-2s}dt\right)$$
$$\leq 3|v|_{L^2,H^1,s}^2 + \frac{1}{s(1-s)}\|v+c\|_0^2. \tag{44}$$

Therefore, recalling (44), we obtain

$$|v|^2_{s,\inf} \leq 3K^2|v|^2_{L^2,H^1,s} + \frac{K^2}{s(1-s)} \inf_{c\in\mathbb{R}} \|v+c\|^2_0,$$
which is the second assertion.

From the proof of the previous lemma one can conclude that the semi-norm $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ is indeed the principal part of a norm in $H^s(\mathcal{O})$. This will be useful to deduce a Poincar´e-Friedrichs inequality with this semi-norm. First let us specify what we mean by the semi-norm being principal part of a norm.

Corollary (6.2.5) [256]:

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There holds

$$\|v\|^2_{s,\mathcal{O}} \leq \frac{K^2}{s(1-s)} \|v\|^2_{0,\mathcal{O}} + 3K^2|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$$
for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $K = K(s,\mathcal{O})$ is the number from Proposition (6.2.1).
Proof:

This is a combination of the second bound from Proposition (6.2.1) and (44) with $c = 0$.

We are now ready to prove a second Poincar´e-Friedrichs inequality.

Proposition (6.2.6) [256]:(Poincar´e-Friedrichs inequality, interpolation semi-norms)
Let $\mathcal{O} \in \mathbb{R}^n$ be a bounded connected Lipschitz domain, and $s \in (0,1)$. Then there exists a constant $C_{PF,I} > 0$, depending on $\mathcal{O}$ and s, such that

$$\|v\|_{0,\mathcal{O}} \leq C_{PF,\mathrm{I}}(s,\mathcal{O}) \left(|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} + \left| \int_{\mathcal{O}} v \right| \right) \quad \forall v \in H^s(\mathcal{O}).$$
Proof:
Assume that the inequality is not true. Then there is a sequence $(v_j) \subset H^s(\mathcal{O})$ such that

$$\|v_j\|_{0,\mathcal{O}} = 1, \quad |v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} + \left| \int_{\mathcal{O}} v_j \right| \to 0 \ \ (j \to \infty).$$
Therefore, by Corollary (6.2.5), (v_j) is bounded in $H^s(\mathcal{O})$ with respect to the Sobolev-Slobodeckij norm. Then, by Rellich's theorem (see [268]) there is a convergent subsequence (again denoted by (v_j)) in $L^2(\mathcal{O})$. Since $|v_j|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \to 0$ this sequence is Cauchy and with limit $v$in $H^s(\mathcal{O})$. It holds $|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} = 0$ so that v is constant. Furthermore, since $\int_{\mathcal{O}} v = 0$ and $\mathcal{O}$ is connected we conclude that $v = 0$, a contradiction to $\|v_j\|_{0,\mathcal{O}} = 1$.

With the help of Proposition (6.2.6) we can now turn the estimate by Lemma (6.2.4) into a seminorm equivalence.

Lemma (6.2.7) [256]:

Let $\mathcal{O} \subset \mathbb{R}^n$ be a connected bounded Lipschitz domain. There holds

$$k^2|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \leq |v|^2_{s,\mathcal{O},\inf} \leq K^2\left(3+\frac{C^2_{PF,I}}{s(1-s)}\right)|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $k = k(s,\mathcal{O}), K = K(s,\mathcal{O})$ are the numbers from Proposition (6.2.1), and $C_{PF,I}(s,\mathcal{O})$ is the number from Proposition (6.2.6).

Proof:

The lower bound is the one from Lemma (6.2.4). The upper bound is a combination of the upper bound from the same lemma and the Poincar´eFriedrichs' inequality from Proposition (6.2.6). To this end note that the infimum $\inf_{c\in\mathbb{R}}\|v+c\|_{0,\mathcal{O}}$ is achieved by the same constant c that eliminates the integral in the bound of the Poincar´e-Friedrichs' inequality for $v + c$.

Meanwhile we have accumulated quite some parameters in the semi-norm estimates that depend on the order s and the domain $\mathcal{O}$under consideration. Our goal is to show equivalence of semi-norms which is uniform for a family of affinely transformed domains. We therefore study transformation properties of semi-norms in the following section. In this way, parameters from this section enter final results only via their values on a reference domain.

Obviously, both norms in $H^s(\mathcal{O})$ defined previously, $\|\cdot\|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$and $\|\cdot\|_{s,\mathcal{O}}$, are not scalable.This could be achieved by weighting the $L^2(\mathcal{O})$-contributions according to the diameter of $\mathcal{O}$, for instance, cf. [261]. Of course, in this way one does not obtain uniformly equivalent norms (of un-weighted and weighted variants) under transformation of the domain.

This is different for the norm in $\widetilde{H}^s(\mathcal{O})$. It can be easily fixed (to be scalable) by using that the semi-norm $|\cdot|_{1,\mathcal{O}}$ is a norm in $H^1_0(\mathcal{O})$, and re-defining

$$\|v\|_{[L^2(\mathcal{O}),H^1_0(\mathcal{O})]_s} = \|v\|_{L^2(\mathcal{O}),H^1_0(\mathcal{O}),s} = \left(\int_0^\infty t^{-2s}\inf_{v=v_0+v_1,v_1\in H^1_0(\mathcal{O})}\left(\|v_0\|^2_{0,\mathcal{O}}+t^2|v_1|^2_{1,\mathcal{O}}\right)\frac{dt}{t}\right)^{1/2}$$

in the case of interpolation. In the case of the Sobolev-Slobodeckij norm one can ensure scalability by re-defining

$$\|v\|_{\widetilde{H}^s(\mathcal{O})} = \|v\|_{\sim,s,\mathcal{O}} = \left(|v_1|^2_{H^s(\mathcal{O})}+\left\|\frac{v(x)}{\operatorname{dist}(x,\partial\mathcal{O})}\right\|^2_{L^2(\mathcal{O})}\right)^{1/2}$$

since the last term guarantees positivity. In the following we will make use of these re-defined norms.

For a domain $\hat{\mathcal{O}} \in \mathbb{R}^n$ we denote by $\mathcal{O} = F(\hat{\mathcal{O}})$ the affinely transformed domain

$$\mathcal{O} = \{F\hat{x};\ \hat{x} \in \hat{\mathcal{O}}\}\text{with } F\hat{x} = x_0 + B\hat{x},\ x_0 \in \mathbb{R}^n,\ B \in \mathbb{R}^{n\times n} \tag{45}$$

Here, B is assumed to be invertible. Correspondingly, for a given real function v defined on $\mathcal{O}$,

$$\hat{v}:\begin{cases} \hat{\mathcal{O}} \to \mathbb{R} \\ \hat{x} \to v(F\hat{x}) \end{cases}$$

is the function transformed onto $\hat{\mathcal{O}}$.

Lemma (6.2.8) [256]: (transformation properties of norms)

Let $\hat{\mathcal{O}} \subset \mathbb{R}^n$ be a bounded Lipschitz domainand let $\mathcal{O}$ be the affinely transformed domain defined by (6.2.4). Then there hold the transformation properties

$$|\det B| \|B\|^{-2s} \|\hat{v}\|^2_{L^2(\mathcal{O}),H_0^1(\mathcal{O}),s} \le \|v\|^2_{L^2(\mathcal{O}),H_0^1(\mathcal{O}),s} \le |\det B| \|B^{-1}\|^{2s} \|\hat{v}\|^2_{L^2(\mathcal{O}),H_0^1(\mathcal{O}),s}, \tag{46}$$

$$|\det B| \|B\|^{-2s} \min\{|\det B| \|B\|^{-n}, 1\} \|\hat{v}\|^2_{\sim,s,\hat{\mathcal{O}}} \le \|v\|^2_{\sim,s,\mathcal{O}} \le |\det B| \|B^{-1}\|^{2s} \max\{|\det B| \|B^{-1}\|^n, 1\} \|\hat{v}\|^2_{\sim,s,\hat{\mathcal{O}}} \tag{47}$$

for any $\hat{v} \in \tilde{H}^s(\hat{\mathcal{O}})$ and $s \in (0,1)$.

Proof:

For the interpolation norm and $\hat{\mathcal{O}}, \mathcal{O}$ being a cubes, this property (with an unspecified equivalence constant) has been shown in [264]. It is simply the scaling properties of the $L^2$and H_0^1-norms together with the exactness of the K-method of interpolation (employed here). The proof generalizes to affine mappings in a straightforward way as follows. In Euclidean norm one has $\|\nabla v(x)\| \le \|B^{-1}\| \|\nabla \hat{v}(\hat{x})\|$ so that the following relations are immediate,

$$\|v\|^2_{L^2(\mathcal{O})} = |\det B| \|\hat{v}\|^2_{L^2(\mathcal{O})}, \quad |v|^2_{H^1(\mathcal{O})} \le |\det B| \|B^{-1}\|^2 |\hat{v}|^2_{H^1(\hat{\mathcal{O}})}.$$

Then, with transformation $r = \|B^{-1}\| t$, we deduce that

$$\begin{aligned}
\|v\|^2_{L^2(\mathcal{O}),H_0^1(\mathcal{O}),s} &= \int_0^\infty t^{-2s} \inf_{v=v_0+v_1, v_1 \in H_0^1(\mathcal{O})} \left(\|v_0\|^2_{0,\mathcal{O}} + t^2 |v_1|^2_{1,\mathcal{O}}\right) \frac{dt}{t} \\
&\le |\det B| \int_0^\infty t^{-2s} \inf_{\hat{v}=\hat{v}_0+\hat{v}_1, \hat{v}_1 \in H_0^1(\mathcal{O})} \left(\|\hat{v}_0\|^2_{0,\mathcal{O}} + t^2 \|B^{-1}\|^2 |\hat{v}_1|^2_{1,\hat{\mathcal{O}}}\right) \frac{dt}{t} \\
&= |\det B| \int_0^\infty (\|B^{-1}\|^{-1} r)^{-2s} \inf_{\hat{v}=\hat{v}_0+\hat{v}_1, \hat{v}_1 \in H_0^1(\mathcal{O})} \left(\|\hat{v}_0\|^2_{0,\hat{\mathcal{O}}} + r^2 |\hat{v}_1|^2_{1,\mathcal{O}}\right) \frac{dr}{r} \\
&= |\det B| \|B^{-1}\|^{2s} \|\hat{v}\|^2_{L^2(\mathcal{O}),H_0^1(\mathcal{O}),s}.
\end{aligned}$$

This proves the upper bound in (6.2.5). The lower bound is verified by using the inverse transformation F^{-1} with matrix B^{-1}.The transformation property of the second norm is obtained similarly, see also [87] for the term of the double integral.

$$\begin{aligned}
\|v\|^2_{\sim,s,\mathcal{O}} &= \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dx\, dy + \int_{\mathcal{O}} \left(\frac{v(x)}{dis\,(x, \partial\mathcal{O})^s}\right)^2 dx \\
&\le |\det B|^2 \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|\hat{v}(\hat{x}) - \hat{v}(\hat{y})|^2}{\|B^{-1}\|^{-n-2s} |\hat{x} - \hat{y}|^{n+2s}} d\hat{x}\, d\hat{y} \\
&\quad + |\det B| \int_{\mathcal{O}} \left(\frac{\hat{v}(\hat{x})}{\|B^{-1}\|^{-s} \mathrm{dist}(\hat{x}, \partial\hat{\mathcal{O}})^s}\right)^2 d\hat{x}
\end{aligned}$$

$\leq |\det B| \|B^{-1}\|^{2s} + \max\{|\det B| \|B^{-1}\|^{n}, 1\} \|\hat{v}\|^2_{\sim,s,\hat{\mathcal{O}}}$

This is the upper bound in (6.3.6). Analogously one finds that

$$\|\hat{v}\|^2_{\sim,s,\hat{\mathcal{O}}} \leq |\det B^{-1}| \|B\|^{2s} \max\{|\det B^{-1}| \|B\|^{n}, 1\} \|v\|^2_{\sim,s,\mathcal{O}}.$$

This proves the lower bound in (6.2.6).

Lemma (6.2.9) [256]: (transformation properties of semi-norms)

Let $\hat{\mathcal{O}} \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\mathcal{O}$ be the affinely transformed domain defined by (30). Then there hold the transformation properties

$$|\det B^{-1}| \|B\|^{2s} |\hat{v}|^2_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s} \leq |v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \leq |\det B^{-1}| \|B\|^{2s} |\hat{v}|^2_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}, \quad (48)$$

$$|\det B|^2 \|B\|^{-n-2s} |\hat{v}|^2_{s,\hat{\mathcal{O}}} \leq |v|^2_{s,\hat{\mathcal{O}}} \leq |\det B|^2 \|B^{-1}\|^{n+2s} |\hat{v}|^2_{s,\hat{\mathcal{O}}} \quad (49)$$

for any $\hat{v} \in H^s(\hat{\mathcal{O}})$ and $s \in (0,1)$.

Proof:

The proof is basically identical to the one of Lemma (6.2.8).

The third semi-norm, $|\cdot|_{s,\mathcal{O},\inf}$, behaves under affine transformations as follows.

Lemma (6.2.10) [256]:

Let $\hat{\mathcal{O}} \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $\mathcal{O}$ be the affinely transformed domain defined by (30). Then there hold the transformation properties

$$|\det B|^2 \|B\|^{-n-2s} |\hat{v}|^2_{s,\hat{\mathcal{O}}} + |\det B| \inf \|\hat{v} + c\|^2_{o,\hat{\mathcal{O}}} \leq |v|^2_{s,\mathcal{O},\inf} \leq |\det B|^2 \|B^{-1}\|^{n+2s} |\hat{v}|^2_{s,\hat{\mathcal{O}}} + |\det B| \inf_{c\in\mathbb{R}} \|\hat{v} + c\|^2_{o,\hat{\mathcal{O}}}$$

for any $\hat{v} \in H^s(\hat{\mathcal{O}})$ and $s \in (0,1)$.

Proof:

This result is immediate from the representation of the semi-norm given in Lemma (6.2.3) and the transformation properties of the $|\cdot|_s$-semi-norm by Lemma (6.2.9) and of the L^2-norm.

We are now ready to state and prove our main results on certain equivalences of fractional-order

Sobolev semi-norms. We use the notation (45) from Section (2.3) for affine transformations. In particular, we assume that the domain $\mathcal{O}$under consideration is the affine image of a bounded Lipschitzdomain $\hat{\mathcal{O}}$. The following results specify how equivalence constants depend on the affine map. At the end of this section we conclude the equivalence of some semi-norms which is uniform for a family of so called shape regular domains Theorem (6.2.14) and some scaling properties Corollary (6.2.15).

These results are of importance for the approximation theory of piecewise polynomial spaces in fractional-order Sobolev spaces.

The first theorem shows the equivalence of the semi-norms $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$, and $|\cdot|_{s,\mathcal{O}}$.

Theorem (6.2.11) [256]:
Let $\hat{\mathcal{O}} \subset \mathbb{R}^2$ be a bounded, connected Lipschitz domain and let $\mathcal{O}$be the affinely transformed domain defined by (30). Then there hold the following relations.

(i)
$$|v|_{s,\mathcal{O}}^2 \le |\det B| \|B^{-1}\|^{n+2s} \|B\|^{2s} K\big(s,\hat{\mathcal{O}}\big)^2 \left(3 + \frac{\frac{C_{PF,I}\big(s,\hat{\mathcal{O}}\big)^2}{s}}{(1-s)}\right) |v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2$$
for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$ with $K(s,\hat{\mathcal{O}})$ from Proposition (6.2.1) and $C_{PF,I}\big(s,\hat{\mathcal{O}}\big)$ from Proposition (6.2.6).

(ii)
$$|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 \le |\det B|^{-1} \|B\|^{n+2s} \|B^{-1}\|^{2s} \kappa\big(s,\hat{\mathcal{O}}\big)^{-2} \left(1 + C_{PF,SS}\big(s,\hat{\mathcal{O}}\big)^2\right) |v|_{s,\mathcal{O}}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$ with $\kappa\big(s,\hat{\mathcal{O}}\big)$ from Proposition (6.2.1) and from Proposition (6.2.2).

Proof:
On a fixed domain $\hat{\mathcal{O}}$ we obtain, by combining Lemmas (6.2.3) and (6.2.7), the equivalence of semi-norms:
$$\big|\hat{\mathcal{O}}\big|_{s,\hat{\mathcal{O}}}^2 \le |\hat{v}|_{s,\hat{\mathcal{O}},\inf}^2 \quad \le K\big(s,\hat{\mathcal{O}}\big)^2 \left(3 + \frac{C_{PF,I}\big(s,\hat{\mathcal{O}}\big)^2}{s(1-s)}\right) |\hat{v}|_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}^2 \tag{50}$$
and
$$|\hat{v}|_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}^2 \le \kappa\big(s,\hat{\mathcal{O}}\big)^{-2} |\hat{v}|_{s,\hat{\mathcal{O}},\inf}^2$$
$$\le \kappa\big(s,\hat{\mathcal{O}}\big)^{-2} \Big(1 + C_{PF,SS}\big(s,\hat{\mathcal{O}}\big)^2\Big) |\hat{v}|_{s,\hat{\mathcal{O}}}^2 \tag{51}$$
The first assertion of the theorem then follows by combining (40) with the transformation properties of the semi-norms by Lemma (6.2.9):

$$|\hat{v}|_{s,\mathcal{O}}^2 \le |\det B|^2 \|B^{-1}\|^{n+2s} |\hat{v}|_{,s,\mathcal{O}}^2$$
$$\le |\det B|^2 \|B^{-1}\|^{n+2s} K\big(s,\hat{\mathcal{O}}\big)^2 \left(3 + \frac{C_{PF,I}\big(s,\hat{\mathcal{O}}\big)^2}{s(1-s)}\right) |\hat{v}|_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}^2$$
$$\le |\det B| \|B^{-1}\|^{n+2s} \|B\|^{2s} K\big(s,\hat{\mathcal{O}}\big)^2 \left(3 + \frac{C_{PF,I}\big(s,\hat{\mathcal{O}}\big)^2}{s(1-s)}\right) |\hat{v}|_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}^2$$
The second assertion of the theorem is proved by a combination of (41) with the transformation properties by Lemma (6.2.9).

The next two theorems study the other pairs of semi-norms for equivalence in combination with affine maps, $\left(|\cdot|_{s,\mathcal{O}} |\cdot|_{s,\mathcal{O},\inf}\right)$ and $\left(|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} |\cdot|_{s,\mathcal{O},\inf}\right)$

Theorem (6.2.12) [256]:

Let $\hat{\mathcal{O}} \subset \mathbb{R}^2$ be a bounded, connected Lipschitz domain and let $\mathcal{O}$ be the affinely transformed domain defined by (35). Then there hold the following relations.

(i).

$$|v|_{s,\mathcal{O}} \leq |v|_{s,\mathcal{O},\inf} \quad \forall v \in H^s(\mathcal{O}), \forall s \in (0,1),$$

(ii).

$$|v|^2_{s,\mathcal{O},\inf} \leq \left(1 + |\det B|^{-1} \|B\|^{n+2s} C_{PF,SS}\left(s,\hat{\mathcal{O}}\right)^2\right) |\hat{v}|^2_{s,\mathcal{O}} \; \forall v \in H^s(\mathcal{O}), \forall s \in (0,1)$$

with$C_{PF,SS}\left(s,\hat{\mathcal{O}}\right)$ being the number from Proposition (6.2.2).

Proof:

Assertion (i) is a repetition of the first estimate in Lemma (6.2.3).

To show the second assertion we use Proposition (6.2.2) and Lemma (6.2.9) to deduce that

$$\inf_{c\in\mathbb{R}} \|v + c\|^2_{o,\mathcal{O}} = |\det B| \inf_{c\in\mathbb{R}} \|\hat{v} + c\|^2_{o,\mathcal{O}} \leq |\det B| C_{PF,SS}\left(s,\hat{\mathcal{O}}\right)^2 |\hat{v}|^2_{s,\mathcal{O}}$$
$$\leq 1 + |\det B|^{-1} \|B\|^{n+2s} C_{PF,SS}\left(s,\hat{\mathcal{O}}\right)^2.$$

The assertion then follows by the definition of the semi-norm $|\cdot|_{s,\mathcal{O},\inf}$.

Theorem (6.2.13) [256]:

Let $\hat{\mathcal{O}} \subset \mathbb{R}^2$ be a bounded, connected Lipschitz domain and let $\mathcal{O}$ be the affinely transformed domain defined by (30). Then there hold the following relations.

(i).

$$|\hat{v}|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \leq \|B^{-1}\|^{2s} \max\{|\det B|^{-1} \|B\|^{n+2s}, 1\} \kappa\left(s,\hat{\mathcal{O}}\right)^{-2} |\hat{v}|^2_{s,\mathcal{O},\inf}$$

for any $v \in H^s(O)$ and $s \in (0,1)$ with $k(s,\mathcal{O})$ from Proposition (6.2.1),

(ii).

$$|\hat{v}|^2_{s,\mathcal{O},\inf} \leq \max\{|\det B| \|B^{-1}\|^{n+2s}, 1\} K\left(s,\hat{\mathcal{O}}\right)^2 \left(3 + \frac{C_{PF,I}\left(s,\hat{\mathcal{O}}\right)^2}{s(1-s)}\right) |\hat{v}|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$ with $K(s,\hat{\mathcal{O}})$ from Proposition (6.2.1) and $C_{PF,SS}\left(s,\hat{\mathcal{O}}\right)^2$ from Proposition (6.2.6).

Proof:

By Lemmas (6.2.9),(6.2.7) and (6.2.10) we obtain

$$|\hat{v}|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \leq |\det B|^{-1} \|B\|^{n+2s} |\hat{v}|^2_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s} \leq |\det B| \|B^{-1}\|^{2s} \kappa\left(s,\hat{\mathcal{O}}\right)^{-2} |\hat{v}|^2_{s,\mathcal{O},\inf}$$

$$\leq |\det B| \|B\|^{2s} \kappa(s,\hat{\mathcal{O}})^{-2} \left(|\det B|^{-2} \|B\|^{n+2s} \kappa(s,\hat{\mathcal{O}})^{-2} |v|_{s,\mathcal{O}}^2 + |\det B^{-1}| \inf_{c\in\mathbb{R}} \|v+c\|_{o,\mathcal{O}}^2 \right)$$
$$\leq \|B^{-1}\|^{2s} \max\{|\det B| \|B^{-1}\|^{n+2s}, 1\} \kappa(s,\hat{\mathcal{O}})^{-2} |\hat{v}|_{s,\mathcal{O},\inf}^2$$

This is the first assertion. The second one follows analogously by the same lemmas:

$$\begin{aligned} |v|_{s,\mathcal{O},\inf}^2 &\leq |\det B|^2 \|B^{-1}\|^{n+2s} \kappa(s,\hat{\mathcal{O}})^{-2} |\hat{v}|_{s,\hat{\mathcal{O}}}^2 + |\det B| \inf_{c\in\mathbb{R}} \|\hat{v}+c\|_{o,\hat{\mathcal{O}}}^2 \\ &\leq |\det B| \max\{|\det B| \|B^{-1}\|^{n+2s}, 1\} |\hat{v}|_{s,\hat{\mathcal{O}},\inf}^2 \\ &\leq |\det B| \max\{|\det B| \|B^{-1}\|^{n+2s}, 1\} K(s,\hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,I}(s,\hat{\mathcal{O}})^2}{s(1-s)}\right) |\hat{v}|_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}^2 \\ &\leq \max\{|\det B| \|B^{-1}\|^{n+2s}, 1\} \|B\|^{2s} K(s,\hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,I}(s,\hat{\mathcal{O}})^2}{s(1-s)}\right) |\hat{v}|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 \end{aligned}$$

We end this section with establishing uniform equivalence of the semi-norms $|\cdot|_{s,\mathcal{O}}$ and $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ for shape-regular domains. Three of the four remaining bounds for other combinations of semi-norms are uniform under further restrictions on the diameter of the domain.

Let us introduce some notation. We consider a bounded, connected Lipschitz domain $\hat{\mathcal{O}} \subset \mathbb{R}^n$and maps of $\hat{\mathcal{O}}$onto domains $\mathcal{O}$where the ratio $\rho\mathcal{O} := D_{\mathcal{O}}/d_{\mathcal{O}}$ is controlled. Here, $D_{\mathcal{O}}$denotes the diameter of $\mathcal{O}$and $d_{\mathcal{O}}$ is the supremum of the diameters of all balls contained in$\mathcal{O}$. In the case of finite elements (or convex polygons) boundedness of ρ is referred to as shaperegularity of $\mathcal{O}$. Also, when defining $d_{\mathcal{O}}$ with balls with respect to which $\mathcal{O}$is star-shaped, then $\rho\mathcal{O}$is referred to as chunkiness parameter.

Using the notation (30) there holds

$$\|B\| \leq \frac{D_{\mathcal{O}}}{d_{\mathcal{O}}} = \frac{D_{\mathcal{O}}}{D_{\hat{\mathcal{O}}}} \rho_{\hat{\mathcal{O}}}, \quad \|B^{-1}\| \leq \frac{D_{\hat{\mathcal{O}}}}{d_{\mathcal{O}}} = \frac{D_{\hat{\mathcal{O}}}}{D_{\mathcal{O}}} \rho\mathcal{O}, \quad \|B\| \|B^{-1}\| \leq \rho\mathcal{O}\, \rho\hat{\mathcal{O}}, \tag{42}$$

cf., e.g., [258]. Furthermore, we conclude that

$$|\det B| = \frac{|\mathcal{O}|}{|\hat{\mathcal{O}}|} \leq \frac{D_{\mathcal{O}}^n}{d_{\mathcal{O}}^n}, |\det B|^{-1} \leq \frac{D_{\hat{\mathcal{O}}}^n}{d_{\mathcal{O}}^n} = \rho_{\mathcal{O}}^n \frac{D_{\hat{\mathcal{O}}}^n}{D_{\mathcal{O}}^n}. \tag{43}$$

With this notation, the results of Theorems (6.2.11-6.2.13) imply the following.

Theorem (6.2.14) [256]:

Let $\mathcal{O}$ be the affine map of a bounded connected Lipschitz domain $\hat{\mathcal{O}} \subset \mathbb{R}^{\mathrm{n}}$, cf (2.4).

(i). The semi-norms $|\cdot|_{s,\mathcal{O}}$ and $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ are uniformly equivalent for a family of shape-regular domains $\mathcal{O}$:

$$|v|_{s,\mathcal{O}}^2 \leq \rho_{\hat{\mathcal{O}}}^n K(s,\hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,I}(s,\hat{\mathcal{O}})^2}{s(1-s)}\right) |v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2,$$
$$|v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 \leq \rho_{\mathcal{O}}^{n+2s} \rho_{\hat{\mathcal{O}}}^{n+2s} \kappa(s,\hat{\mathcal{O}})^2 \left(1 + C_{PF,SS}(s,\hat{\mathcal{O}})^2\right) |v|_{s,\mathcal{O}}^2$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $\kappa(s,\hat{\mathcal{O}}), K(s,\hat{\mathcal{O}})$ are the numbers from Proposition (6.2.1) and $C_{PF,SS}(s,\hat{\mathcal{O}}), C_{PF,I}(s,\hat{\mathcal{O}})$ are as in Propositions (6.2.2), (6.2.6), respectively.

(ii). The semi-norms $|\cdot|_{s,\mathcal{O}}$ and $|\cdot|_{s,\mathcal{O},\inf}$ are uniformly equivalent for a family of uniformly bounded, shape-regular domains $\mathcal{O}$:

$$|v|_{s,\mathcal{O}} \leq |\cdot|_{s,\mathcal{O},\inf} \quad ,$$

$$|v|^2_{s,\mathcal{O},\inf} \leq \left(1 + \frac{D^{2s}_{\mathcal{O}}}{D^{2s}_{\hat{\mathcal{O}}}} \rho^n_{\mathcal{O}} \rho^{n+2s}_{\hat{\mathcal{O}}} C_{PF,SS}(s,\hat{\mathcal{O}})^2\right) |v|^2_{s,\mathcal{O}}$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$. Here, $C_{PF,SS}(s,\hat{\mathcal{O}})$ is the number from Proposition (6.2.2).

(iii). a) For a family of shape-regular domains $\mathcal{O}$whose diameters are bounded from below by a positive constant, the semi-norm $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$is uniformly bounded by$|\cdot|_{s,\mathcal{O},\inf}$:

$$|v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s} \leq \max\{\rho^n_{\mathcal{O}} \rho^{n+2s}_{\hat{\mathcal{O}}}, D^{-2s}_{\mathcal{O}} D^{2s}_{\hat{\mathcal{O}}}\} \rho^{2s}_{\mathcal{O}} \kappa(s,\hat{\mathcal{O}})^2 |v|^2_{s,\mathcal{O},\inf}$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$.

b) For a family of uniformly bounded, shape-regular domains $\mathcal{O}$, the semi-norm $|\cdot|_{s,\mathcal{O},\inf}$ is uniformly bounded by $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$

$$|\hat{v}|^2_{s,\mathcal{O},\inf} \leq \max\{\rho^{n+2s}_{\mathcal{O}} \rho^n_{\hat{\mathcal{O}}}, D^{2s}_{\mathcal{O}} D^{-2s}_{\hat{\mathcal{O}}}\} \rho^{2s}_{\hat{\mathcal{O}}} K(s,\hat{\mathcal{O}})^2 \left(3 + \frac{C_{PF,I}(s,\hat{\mathcal{O}})^2}{s(1-s)}\right) |v|^2_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$$

for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$.

Here, $\kappa(s,\hat{\mathcal{O}}), K(s,\hat{\mathcal{O}})$ are the parameters from Proposition (6.2.1), and $C_{PF,I}(s,\hat{\mathcal{O}})$is the number from Proposition (6.2.6).

Proof:

The assertions (i)–(iii) are a combination of Theorems (6.2.11-6.2.13), respectively, with the bounds provided by (42), (43).

The uniform equivalence of the semi-norms $|\cdot|_{s,\mathcal{O}}$ and$|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ for shape-regular domains is based on what one calls their scaling property. It means that both semi-norms for functions on a domain $\mathcal{O}$are uniformly equivalent to the respective semi-norm of the affinely transformed functions onto a fixed domain $\hat{\mathcal{O}}$, when one of the semi-norms is multiplied by an appropriate number (it is a power of the diameter of $\mathcal{O}$). This property applies also to the norms $|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$and $|\cdot|_{\sim s,\mathcal{O}}$, cf. Lemma (6.2.8). Scaling properties are relevant for the error analysis of piecewise polynomial approximations. We formulate the result as a corollary to Lemmas (6.2.8) and (6.2.9).

Corollary (6.2.15) [256]:

The norms $|\cdot|_{L^2(\mathcal{O}),H_0^1(\mathcal{O}),s}$, $|\cdot|_{\sim s,\mathcal{O}}$ and semi-norms $|\cdot|_{s,\mathcal{O}}|\cdot|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}$ are scalable of order $D_{\mathcal{O}}^{n-2s}$:

$$D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{-n}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{-2s}\|\hat{v}\|_{L^2(\hat{\mathcal{O}}),H_0^1(\hat{\mathcal{O}}),s} \le \|v\|_{L^2(\mathcal{O}),H_0^1(\mathcal{O}),s}^2$$
$$\le D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{2s}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{n}\|\hat{v}\|_{L^2(\hat{\mathcal{O}}),H_0^1(\hat{\mathcal{O}}),s}^2$$
$$D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{-n}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{-2s}\min\{\rho_{\mathcal{O}}^{-n}\rho_{\hat{\mathcal{O}}}^{-n},1\}\|\hat{v}\|_{\sim,s,\hat{\mathcal{O}}}^2 \quad \le \|v\|_{\sim,s,\mathcal{O}}^2$$
$$\le D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{2s}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{n}\max\{\rho_{\mathcal{O}}^{n}\rho_{\hat{\mathcal{O}}}^{n},1\}\|\hat{v}\|_{\sim,s,\hat{\mathcal{O}}}^2$$

for any $v \in \widetilde{H}^s(\mathcal{O})$ and $s \in (0,1)$, and
$$D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{-n}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{-2s}|\hat{v}|_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}^2 \le |v|_{L^2(\mathcal{O}),H^1(\mathcal{O}),s}^2 \le D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{2s}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{n}|\hat{v}|_{L^2(\hat{\mathcal{O}}),H^1(\hat{\mathcal{O}}),s}^2,$$
$$D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{-2n}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{-n-2s}|\hat{v}|_{s,\hat{\mathcal{O}}}^2 \le |v|_{,s,\mathcal{O}}^2 \le D_{\mathcal{O}}^{n-2s}\rho_{\mathcal{O}}^{n+2s}D_{\hat{\mathcal{O}}}^{2s-n}\rho_{\hat{\mathcal{O}}}^{n}|\hat{v}|_{s,\hat{\mathcal{O}}}^2$$
for any $v \in H^s(\mathcal{O})$ and $s \in (0,1)$.

Proof:

The bounds are a combination of Lemmas (6.2.8) and (6.2.9) with (42), (43).

Remark (6.2.16) [256]:

The estimate by Theorem (6.2.14) (iii) a) breaks down when $D_{\mathcal{O}} \to 0$. In fact, for a family of scaled domains $\mathcal{O}_h$ with $D_{\mathcal{O}_h} = h$ and a non-constant function v scaled to a family $\{v_h\}$ of functions on $\{\mathcal{O}_h\}$, $|v_h|_{L^2(\mathcal{O}_h),H^1(\mathcal{O}_h),s}^2 \simeq h^{n-2s}$ by Corollary (6.2.15) whereas $|v_h|_{s,\mathcal{O}_h,\inf}^2 \ \ge \inf_{c\in\mathbb{R}}\|v_h - c\|_{o,\mathcal{O}_h}^2 \simeq h^n$. Therefore, the dependence on $D_{\mathcal{O}}$ like $D_{\mathcal{O}}^{-2s}$of the upper bound in Theorem (6.2.14) (iii) a) is optimal.

Proposition (6.2.17)[272]: (equivalence of norms) For a bounded Lipschitz series domain $\sum_{i=1}^{n}\mathcal{O}_i \subset \mathbb{R}^n$ and for given $\epsilon > 0$ there exist constants $k(1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i), K(1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i) > 0$ such that

$$k\left(1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i\right)\|v_i\|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon} \le \|v_i\|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$
$$\le K\left(1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i\right)\|v_i\|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$$
$$\forall\, v_i \in H^{1-\epsilon}\left(\sum_{i=1}^{n}\mathcal{O}_i\right).$$

It is well known that, on bounded Lipschitz series domains, lower-order norms can be bounded by higher-order semi-norms plus finite rank terms. Such estimates are referred to as Poincar´e-Friedrichs' inequalities. For integer-order norms there are direct proofs with explicit constants (depending on small orders and series domains) (see [269], [256]) and attention has received finding best constants and deriving improved weighted estimates, (see [270,271] and [261]), respectively. We need such a Poincar´e-Friedrichs' inequality for fractional-order norms on bounded series domain (for unbounded domains, see [224] and [263]) , gives for two dimensions but immediately extends to the general case.

Proposition(6.2.18)[272]: (Poincar´e-Friedrichs inequality, Sobolev-Slobodeckij semi-norm) .Let $\sum_{i=1}^{n}\mathcal{O}_i \subset \mathbb{R}^n$ be a bounded series domain, and $\epsilon > 0$. Then there holds

$$\|v_i\|_{0,\sum_{i=1}^{n}\mathcal{O}_i} \leq C_{PF,SS}\left(1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i\right)\left(|v_i|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} + \left|\int_{\sum_{i=1}^{n}\mathcal{O}_i} v_i\right|\right) \quad \forall\, v_i \in H^{1-\epsilon}\left(\sum_{i=1}^{n}\mathcal{O}_i\right)$$

With

$$C_{PF,SS}\left((1-\epsilon),\sum_{i=1}^{n}\mathcal{O}_i\right) = \left|\sum_{i=1}^{n}\mathcal{O}_i\right|^{-1/2} \max\left\{1, 2^{-1/2} D_{\sum_{i=1}^{n}\mathcal{O}_i}^{n/2+(1-\epsilon)}\right\}.$$

Here, $|\sum_{i=1}^{n}\mathcal{O}_i|$ denotes the area of $\sum_{i=1}^{n}\mathcal{O}_i$ and, as mentioned in the introduction, $D_{\sum_{i=1}^{n}\mathcal{O}_i}$ is its diameter.

Lemma (6.2.19)[272]: Let $\sum_{i=1}^{n}\mathcal{O}_i \subset \mathbb{R}^n$ be a bounded, connected Lipschitz series domain. Then there holds

$$|v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} \leq |v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf} = |v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} + \inf_{c\in\mathbb{R}}\|v_i + c_i\|^2_{0,\sum_{i=1}^{n}\mathcal{O}_i}$$
$$\leq \left(1 + C^2_{PF,SS}\right)|v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$

For any $v_i \in H^{1-\epsilon}(\sum_{i=1}^{n}\mathcal{O}_i)$and $\epsilon > 0$. Here, $C_{PF,SS} = C_{PF,SS}(1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i)$ is the number from Proposition (6.2.8).

Proof: By definition of $|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$ there holds for any $c_i \in \mathbb{R}$ and any $v_i \in H^{1-\epsilon}(\sum_{i=1}^{n}\mathcal{O}_i)$ (we now drop $\sum_{i=1}^{n}\mathcal{O}_i$ from the notation)

$$|v_i|_{1-\epsilon} = |v_i + c_i|_{1-\epsilon}.$$

Therefore

$$|v_i|_{1-\epsilon} \leq \inf_{c_i\in\mathbb{R}}\|v_i + c_i\|_{1-\epsilon} = |v_i|_{1-\epsilon,\inf}$$

Which is the first assertion by the initial argument and the definition of the Sobolev-Slobodeckij norm one also finds that

$$|v_i|^2_{1-\epsilon,\inf} = \inf_{c_i\in\mathbb{R}} \|v_i + c_i\|^2_{1-\epsilon} = \inf_{c\in\mathbb{R}} \|v_i + c_i\|^2_0 + |v_i|^2_{1-\epsilon}.$$

This is the second assertion.

The last relation and the Poincar´e-Friedrichs' inequality (Proposition (6.2.18) lead to

$$|v_i|^2_{1-\epsilon,\inf} \le C^2_{PF,SS} \inf_{c\in\mathbb{R}} \left(|v_i|_{1-\epsilon} + \left| \int_{\sum_{i=1}^n \mathcal{O}_i} (v_i + c_i) \right| \right)^2 + |v_i|^2_{1-\epsilon} = \left(1 + C^2_{PF,SS}\right) |v_i|^2_{1-\epsilon}.$$

This finishes the proof.

Lemma (6.2.20)[272]: Let $\sum_{i=1}^n \mathcal{O}_i \subset \mathbb{R}^n$ be a bounded Lipschitz series domain. There holds

$$k^2 |v_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i), H^1(\sum_{i=1}^n \mathcal{O}_i), 1-\epsilon} \le |v_i|^2_{1-\epsilon, \sum_{i=1}^n \mathcal{O}_i, \inf}$$
$$\le 3K^2 |v_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i), H^1(\sum_{i=1}^n \mathcal{O}_i), 1-\epsilon} + \frac{K^2}{\epsilon\,(1-\epsilon)} \inf_{c_i\in\mathbb{R}} \|v_i + c_i\|^2_{0, \sum_{i=1}^n \mathcal{O}_i}$$

for any $v_i \in H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$ and $\epsilon > 0$. Here, $k = k(1-\epsilon, \sum_{i=1}^n \mathcal{O}_i)$ and $K = K(1-\epsilon, \sum_{i=1}^n \mathcal{O}_i)$ are the numbers from Proposition (6.2.17).

Proof: Let $v_i \in H(\sum_{i=1}^n \mathcal{O}_i)$, and let c_n, c_{n+1} denote generic constants. For any $t > 0$ there holds

$$\inf_{v_i = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1) = \inf_{v_i = v_n + c_n + v_{n+1} + c_{n+1}} (\|v_n + c_n\|^2_0 + t^2 |v_{n+1}|^2_1)$$
$$= \inf_{c_{n+1}, v_i - c_{n+1} = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1),$$

That is

$$\inf_{v_i = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1) = \inf_{c_i\in\mathbb{R}} \inf_{v_i + c_i = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1)$$
$$\le \inf_{c_i\in\mathbb{R}} \inf_{v_i + c_i = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1).$$

(Recall that our convention for the notation $\inf_{v_i = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1)$ implies that the infimum is taken with respect to $v_n \in L^2(\sum_{i=1}^n \mathcal{O}_i)$and $v_{n+1} \in H^1(\sum_{i=1}^n \mathcal{O}_i)$.) We conclude that

$$|v_i|^2_{L^2, H^1, 1-\epsilon} = \int_0^\infty t^{-2(1-\epsilon)} \inf_{v_i = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1)\, dt$$
$$\le \inf_{c_i\in\mathbb{R}} \int_0^\infty t^{-2(1-\epsilon)} \inf_{v_i + c_i = v_n + v_{n+1}} (\|v_n\|^2_0 + t^2 |v_{n+1}|^2_1) \frac{dt}{t} = \inf_{c_i\in\mathbb{R}} \|v_i + c_i\|^2_{L^2, H^1, 1-\epsilon}.$$

By Proposition (6.2.17)

$$\inf_{c_i\in\mathbb{R}}\|v_i + c_i\|^2_{L^2,H^1,1-\epsilon} \leq k^{-2}\inf_{c_i\in\mathbb{R}}\|v_i + c_i\|^2_{1-\epsilon} = k^{-2}|v_i|^2_{1-\epsilon,\inf},$$

So that the first assertion follows

By definition and using Proposition (6.2.17) there holds

$$|v_i|^2_{1-\epsilon,\inf} = \inf_{c_i\in\mathbb{R}}\|v_i + c_i\|^2_{1-\epsilon} \leq K^2\inf_{c_i\in\mathbb{R}}\|v_i + c_i\|^2_{L^2,H^1,1-\epsilon}$$

$$= K^2\inf_{c_i\in\mathbb{R}}\int_0^\infty t^{-2(1-\epsilon)}\inf_{v_i+c_i=v_n+v_{n+1}}(\|v_n\|_0^2 + t^2|v_{n+1}|_1^2)\frac{dt}{t}. \tag{44}$$

We bound the integrand separately for $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

For $\epsilon_1 > 0$ we use the representation $v_i + c_i = v_n + v_{n+1}$ to bound

$$\begin{aligned}\|v_n\|_0^2 + (1-\epsilon_1)^2\|v_{n+1}\|_0^2 + (1-\epsilon_1)^2|v_{n+1}|_1^2 \\ &\leq \|v_n\|_0^2 + 2(1-\epsilon_1)^2(\|v_i + c_i\|_0^2 + \|v_n\|_0^2) + (1-\epsilon_1)^2|v_{n+1}|_1^2 \\ &\leq 3\|v_n\|_0^2 + 2(1-\epsilon_1)^2\|v_i + c_i\|_0^2 + (1-\epsilon_1)^2|v_{n+1}|_1^2.\end{aligned}$$

If $\epsilon_1 > 0$ then we select $v_n = v_i + c_i$ to conclude that

$$\inf_{v_i+c_i=v_n+v_{n+1}}(\|v_n\|_0^2 + (1+\epsilon_2)^2\|v_{n+1}\|_0^2 + (1+\epsilon_2)^2|v_{n+1}|_1^2) \leq \|v_i + c_i\|_0^2.$$

Together this yields

$$\begin{aligned}&\int_0^\infty (1+\epsilon_2)^{-2(1-\epsilon)}\inf_{v_i+c_i=v_n+v_{n+1}}(\|v_n\|_0^2 + (1+\epsilon_2)^2\|v_{n+1}\|_0^2 + (1+\epsilon_2)^2|v_{n+1}|_1^2)\frac{dt}{t} \\ &\leq \int_0^\infty (1+\epsilon_2)^{-2(1-\epsilon)}\inf_{v_i+c_i=v_n+v_{n+1}}(3\|v_0\|_0^2 + 2(1+\epsilon_2)^2\|v_i + c_i\|_0^2 + (1+\epsilon_2)^2|v_{n+1}|_1^2)\frac{dt}{t} \\ &\qquad + \int_1^\infty (1+\epsilon_2)^{-2(1-\epsilon)}\|v_i + c_i\|_0^2\frac{dt}{t} \\ &= \int_0^\infty (1+\epsilon_2)^{-2(1-\epsilon)}\inf_{v_i+c_i=v_n+v_{n+1}}(3\|v_n\|_0^2 + (1+\epsilon_2)^2|v_{n+1}|_1^2)\frac{dt}{t} \\ &\qquad + \|v_i + c_i\|_0^2\left(\int_0^1 2(1+\epsilon_2)^{-1+2\epsilon}\,dt + \int_1^\infty (1+\epsilon_2)^{-3+2\epsilon}dt\right) \\ &\qquad \leq 3|v_i|^2_{L^2,H^1,1-\epsilon} + \frac{1}{\epsilon\,(1-\epsilon)}\|v_i + c_i\|_0^2.\end{aligned} \tag{2.3}$$

Therefore, recalling (44), we obtain

$$|v_i|^2_{1-\epsilon,\inf} \leq 3K^2 |v_i|^2_{L^2,H^1,1-\epsilon} + \frac{K^2}{\epsilon\,(1-\epsilon)} \inf_{c_i\in\mathbb{R}} \|v_i + c_i\|_0^2,$$

Which is the second assertion

From the proof of the previous lemma one can conclude that the semi-norm $|\cdot|_{L^2\left(\sum_{i=1}^n \mathcal{O}_{i_i}\right),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon}$ is indeed the principal part of a norm in $H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$. This will be useful to deduce a Poincar´e-Friedrichs inequality with this semi-norm. First let us specify what we mean by the semi-norm being principal part of a norm.

Corollary (6.2.21)[272]: Let $\sum_{i=1}^n \mathcal{O}_i \subset \mathbb{R}^n$ be a bounded Lipschitz series domain. There holds

$$\|v_i\|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i} \leq \frac{K^2}{\epsilon\,(1-\epsilon)} \|v_i\|^2_{0,\sum_{i=1}^n \mathcal{O}_i} + 3K^2 |v_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon}$$

For any $v_i \in H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$ and $\epsilon > 0$. Here, $K = K(1-\epsilon, \sum_{i=1}^n \mathcal{O}_i)$ is the number from Proposition (6.2.17).

Proof:

This is a combination of the second bound from Proposition (6.2.17) and (6.2.19) with $c_i = 0$. We are now ready to show (see [256]) a second Poincar´e-Friedrichs inequality.

Proposition (6.2.22)[272]: (Poincar´e-Friedrichs inequality, interpolation semi-norm)
Let $\sum_{i=1}^n \mathcal{O}_i \in \mathbb{R}^n$ be a bounded connected Lipschitz series domain , and $\epsilon > 0$. Then there exists a constant $C_{PF,I} > 0$, depending on $\sum_{i=1}^n \mathcal{O}_i$ and $1-\epsilon$, such that

$$\|v_i\|_{0,\sum_{i=1}^n \mathcal{O}_i} \leq C_{PF,I}\left(1-\epsilon, \sum_{i=1}^n \mathcal{O}_i\right)\left(|v_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon} + \left|\int_{\sum_{i=1}^n \mathcal{O}_i} v_i\right|\right)$$
$$\forall\, v_i \in H^{1-\epsilon}\left(\sum_{i=1}^n \mathcal{O}_i\right).$$

Proof:

Assume that the inequality is not true. Then there is a sequence $\left((v_i)_j\right) \subset H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$ such that

$$\left\|(\mathrm{v}_i)_\mathrm{j}\right\|_{0,\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}}=1,\quad |\mathrm{v}_i|^2_{\mathrm{L}^2(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),\mathrm{H}^1(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),1-\epsilon}+\left|\int_{\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}}(\mathrm{v}_i)_\mathrm{j}\right|\to 0\quad (\mathrm{i},\mathrm{j}\to\infty).$$

Therefore, by Corollary (6.2.21), $\left((\mathrm{v}_i)_\mathrm{j}\right)$ is bounded in $\mathrm{H}^{1-\epsilon}(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i})$ with respect to the Sobolev-Slobodeckij norm. Then, by Rellich's theorem (see [268]) there is a convergent subsequence (again denoted by $\left((\mathrm{v}_i)_\mathrm{j}\right)$) in $\mathrm{L}^2(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i})$. Since $\left|(\mathrm{v}_i)_\mathrm{j}\right|_{\mathrm{L}^2(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),\mathrm{H}^1(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),1-\epsilon}\to 0$ this sequence is Cauchy and with limit v_i in $\mathrm{H}^{1-\epsilon}(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i})$. It holds $|\mathrm{v}_i|_{\mathrm{L}^2(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),\mathrm{H}^1(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),1-\epsilon}=0$ so that v_i is constant. Furthermore, since $\int_{\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}}\mathrm{v}_i=0$ and $\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}$ is connected we conclude that $\mathrm{v}_i=0$, a contradiction to $\left\|(\mathrm{v}_i)_\mathrm{j}\right\|_{0,\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}}=1.$

With the help of Proposition (6.2.22) we can now turn the estimate by Lemma (6.2.20) into a semi norm equivalence.

Lemma (6.2.23)[272]: Let $\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}\subset\mathbb{R}^\mathrm{n}$be a connected bounded Lipschitz series domain. There holds

$$\mathrm{k}^2|\mathrm{v}_i|^2_{\mathrm{L}^2(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),\mathrm{H}^1(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),1-\epsilon}\le|\mathrm{v}_i|^2_{1-\epsilon,\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i},\mathrm{inf}}$$

$$\le\mathrm{K}^2\left(3+\frac{\mathrm{C}^2_{\mathrm{PF,I}}}{\epsilon(1-\epsilon)}\right)|\mathrm{v}_i|^2_{\mathrm{L}^2(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),\mathrm{H}^1(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),1-\epsilon}$$

For any $\mathrm{v}_i\in\mathrm{H}^{1-\epsilon}(\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i})$ and $\epsilon>0$. Here, $\mathrm{k}=\mathrm{k}(1-\varepsilon,\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}),\mathrm{K}=\mathrm{K}(1-\epsilon,\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i})$ are the numbers from Proposition (6.2.17), and $\mathrm{C}_{\mathrm{PF,I}}(1-\epsilon,\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i})$ is the number from Proposition (6.2.22).

Proof: The lower bound is the one from Lemma (6.2.22). The upper bound is a combination of the upper bound from the same lemma and the Poincar´eFriedrichs' inequality from Proposition (6.2.22). To this end note that the infimum $\inf_{\mathrm{c}_i\in\mathbb{R}}\|\mathrm{v}_i+\mathrm{c}_i\|_{0,\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}}$ is achieved by the same constant c_i that eliminates the integral in the bound of the Poincar´e-Friedrichs' inequality for $\mathrm{v}_i+\mathrm{c}_i$. Meanwhile we have accumulated quite some parameters in the semi-norm estimates that depend on the order $\epsilon>0$ and the series domain $\sum_{\mathrm{i}=1}^{n}\mathcal{O}_\mathrm{i}$under consideration. Our goal is to show equivalence of semi-norms which is uniform for a family of affinely transformed series domains. We therefore study transformation properties of semi-norms in the following section. In this way, parameters from this section enter final results only via their values on a reference series domain.

Lemma (6.2.24)[272]: (transformation properties of norms) Let $\widehat{\sum_{i=1}^{n}\mathcal{O}_i} \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\sum_{i=1}^{n}\mathcal{O}_i$ be the affinely transformed series domain defined by (30). Then there hold the transformation properties

$$|\det B|\|B\|^{-2(1-\epsilon)}\|\hat{v}_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H_0^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon} \leq \|v_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H_0^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$$
$$\leq |\det B|\|B^{-1}\|^{2(1-\epsilon)}\|\hat{v}_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H_0^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon} \quad (46)$$

$$|\det B|\|B\|^{-2(1-\epsilon)}\min\{|\det B|\|B\|^{-n},1\}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}} \leq \|v_i\|^2_{\sim,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$
$$\leq |\det B|\|B^{-1}\|^{2(1-\epsilon)}\max\{|\det B|\|B^{-1}\|^{n},1\}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}} \quad (47)$$

For any $\hat{v}_i \in \widetilde{H}^{(1-\epsilon)}\left(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}\right)$ and $\epsilon > 0$.

Proof: For the interpolation norm and $\widehat{\sum_{i=1}^{n}\mathcal{O}_i}, \sum_{i=1}^{n}\mathcal{O}_i$ being a cubes, this property (with an unspecified equivalence constant) has been shown in [264]. It is simply the scaling properties of the L^2 and H_0^1-norms together with the exactness of the K-method of interpolation (employed here). The proof generalizes to affine mappings in a straightforward way as follows. In Euclidean norm one has $\|\nabla v_i(x_n)\| \leq \|B^{-1}\|\|\nabla\hat{v}_i(\hat{x}_i)\|$ so that the following relations are immediate,

$$\|v_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i)} = |\det B|\|\hat{v}_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i)}, \quad |v_i|^2_{H^1(\sum_{i=1}^{n}\mathcal{O}_i)}$$
$$\leq |\det B|\|B^{-1}\|^2|\hat{v}_i|^2_{H^1(\sum_{i=1}^{n}\mathcal{O}_i)}.$$

Then, with transformation $r = \|B^{-1}\|t$, we deduce that

$$\|v_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H_0^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$$
$$= \int_0^\infty t^{-2(1-\epsilon)} \inf_{v_i=v_n+v_{n+1},v_{n+1}\in H_0^1(\sum_{i=1}^{n}\mathcal{O}_i)}\left(\|v_n\|^2_{0,\sum_{i=1}^{n}\mathcal{O}_i} + t^2|v_{n+1}|^2_{1,\sum_{i=1}^{n}\mathcal{O}_i}\right)\frac{dt}{t}$$
$$\leq |\det B|\int_0^\infty t^{-2(1-\epsilon)} \inf_{\hat{v}_i=\hat{v}_n+\hat{v}_{n+1},\hat{v}_{n+1}\in H_0^1(\widehat{\sum_{i=1}^{n}\mathcal{O}_i})}\left(\|\hat{v}_n\|^2_{0,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}} + t^2|\hat{v}_{n+1}|^2_{1,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\right)\frac{dt}{t}$$
$$= |\det B|\int_0^\infty (\|B^{-1}\|^{-1}r)^{-2(1-\epsilon)} \inf_{\hat{v}_i=\hat{v}_n+\hat{v}_{n+1},\hat{v}_{n+1}\in H_0^1(\widehat{\sum_{i=1}^{n}\mathcal{O}_i})}\left(\|\hat{v}_n\|^2_{0,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\right.$$
$$\left. + r^2|\hat{v}_{n+1}|^2_{1,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\right)\frac{dr}{r}$$
$$= |\det B|\|B^{-1}\|^{2(1-\epsilon)}\|\hat{v}_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H_0^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}.$$

This proves the upper bound in (2.5). The lower bound is verified by using the inverse transformation F^{-1} with matrix B^{-1}.

The transformation property of the second norm is obtained similarly, (see also [87]) for the term of the double integral.

$$
\begin{aligned}
&\|v_i\|^2_{\sim,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} \\
&\quad = \int_{\sum_{i=1}^{n}\mathcal{O}_i}\int_{\sum_{i=1}^{n}\mathcal{O}_i} \frac{|v_i(x_n)-v_i(x_{n+1})|^2}{|x_n-x_{n+1}|^{n+2(1-\epsilon)}}\,dx_n\,dx_{n+1} \\
&\quad + \int_{\sum_{i=1}^{n}\mathcal{O}_i}\left(\frac{v_i(x_n)}{\operatorname{dis}(x_n,\partial\sum_{i=1}^{n}\mathcal{O}_i)^{1-\epsilon}}\right)^2 dx_n \\
&\quad \le |\det B|^2\int_{\sum_{i=1}^{n}\mathcal{O}_i}\int_{\sum_{i=1}^{n}\mathcal{O}_i}\frac{|\hat{v}_i(\hat{x}_i)-\hat{v}_i(\hat{y})|^2}{\|B^{-1}\|^{-n-2(1-\epsilon)}|\hat{x}_i-\hat{y}|^{n+2(1-\epsilon)}}\,d\hat{x}_i\,d\hat{y} \\
&\qquad + |\det B|\int_{\sum_{i=1}^{n}\mathcal{O}_i}\left(\frac{\hat{v}_i(\hat{x}_i)}{\|B^{-1}\|^{-(1-\epsilon)}\operatorname{dist}\left(\hat{x}_i,\partial\widehat{\sum_{1=1}^{n}\mathcal{O}_1}\right)^{1-\epsilon}}\right)^2 d\hat{x} \\
&\quad \le |\det B|\|B^{-1}\|^{2(1-\epsilon)} + \max\{|\det B|\|B^{-1}\|^{n},1\}\,\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{1=1}^{n}\mathcal{O}_1}}
\end{aligned}
$$

This is the upper bound in (6.2.22). Analogously one finds that

$$\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{1=1}^{n}\mathcal{O}_1}} \le |\det B^{-1}|\|B\|^{2(1-\epsilon)}\max\{|\det B^{-1}|\|B\|^{n},1\}\,\|v_i\|^2_{\sim,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}.$$

This proves the lower bound in (6.2.22).

Lemma (6.2.25)[272]: (transformation properties of semi-norms) Let $\widehat{\sum_{1=1}^{n}\mathcal{O}_1}\subset\mathbb{R}^n$ be a bounded Lipschitz series domain and let $\sum_{i=1}^{n}\mathcal{O}_i$ be the affinely transformed series domain defined by (30). Then there hold the transformation properties

$$
\begin{aligned}
|\det B^{-1}|\|B\|^{2(1-\epsilon)}|\hat{v}_i|^2_{L^2(\widehat{\sum_{1=1}^{n}\mathcal{O}_1}),H^1(\widehat{\sum_{1=1}^{n}\mathcal{O}_1}),1-\epsilon} &\le |v_i|^2_{L^2(\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon} \\
&\le |\det B^{-1}|\|B\|^{2(1-\epsilon)}|\hat{v}_i|^2_{L^2(\widehat{\sum_{1=1}^{n}\mathcal{O}_1}),H^1(\widehat{\sum_{1=1}^{n}\mathcal{O}_1}),(1-\epsilon)}, \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
|\det B|^2\|B\|^{-n-2(1-\epsilon)}|\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{1=1}^{n}\mathcal{O}_1}} &\le |v_i|^2_{1-\epsilon,\widehat{\sum_{1=1}^{n}\mathcal{O}_1}} \\
&\le |\det B|^2\|B^{-1}\|^{n+2(1-\epsilon)}|\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{1=1}^{n}\mathcal{O}_1}} \qquad (49)
\end{aligned}
$$

for any $\hat{v}_i\in H^{(1-\epsilon)}\left(\widehat{\sum_{1=1}^{n}\mathcal{O}_1}\right)$ and $\epsilon>0$.

Proof: The proof is basically identical to the one of Lemma (6.2.24).

The third semi-norm, $|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf}$, behaves under affine transformations as follows.

Lemma (6.2.26)[272]: Let $\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath} \subset \mathbb{R}^2$ be a bounded Lipschitz series domain and let $\sum_{i=1}^{n}\mathcal{O}_i$ be the affinely transformed series domain defined by (30). Then there hold the transformation properties

$$|\det B|^2\|B\|^{-n-2(1-\epsilon)}|\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}} + |\det B|\inf\|\hat{v}_i + c_i\|^2_{o,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}} \leq |v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf}$$

$$\leq |\det B|^2\|B^{-1}\|^{n+2(1-\epsilon)}|\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}} + |\det B| \inf_{c_i\in\mathbb{R}}\|\hat{v}_i + c_i\|^2_{o,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}$$

For any $\hat{v}_i \in H^{1-\epsilon}\left(\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)$ and $\epsilon > 0$.

Proof: This result is immediate from the representation of the semi-norm given in Lemma (6.2.19) and the transformation properties of the $|\cdot|_{1-\epsilon}$-semi-norm by Lemma (6.2.25) and of the L^2-norm.

Theorem (6.2.27)[272]: Let $\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath} \subset \mathbb{R}^2$be a bounded, connected Lipschitz series domain and let $(\sum_{i=1}^{n}\mathcal{O}_i)$,be the affinely transformed series domain defined by (30). Then there holds the following relations.

(i) $|v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$

$$\leq |\det B|\|B^{-1}\|^{n+2(1-\epsilon)}\|B\|^{2(1-\epsilon)}K\left(1-,\left(\sum_{i=1}^{n}\mathcal{O}_i\right),\right)^2\left(3 + \frac{C_{PF,I}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2}{\epsilon\,(1-\epsilon)}\right)|v_i|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$$

for any $v_i \in H^{1-\epsilon}(\sum_{i=1}^{n}\mathcal{O}_i)$ and $\epsilon > 0$ with $K(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath})$ from Proposition (6.2.17) and $C_{PF,I}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)$ from Proposition (6.2.22).

(ii) $|v_i|^2_{L^2(\mathcal{O}_i),H^1(\mathcal{O}_i),1-\epsilon}$

$$\leq |\det B|^{-1}\|B\|^{n+2(1-\epsilon)}\|B^{-1}\|^{2(1-\epsilon)}\kappa\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^{-2}\left(1 + C_{PF,SS}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2\right)|v_i|^2_{1-\epsilon,(\sum_{i=1}^{n}\mathcal{O}_i)}$$

For any $v_i \in H^{1-\epsilon}((\sum_{i=1}^{n} \mathcal{O}_i))$ and $\epsilon > 0$ with $\kappa\big(1-\epsilon, (\sum_{i=1}^{n} \mathcal{O}_i)\big)$ from Proposition (6.2.17) and from Proposition (6.2.18)

Proof: On a fixed series domain $\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}$ we obtain, by combining Lemmas (6.2.19) and (6.2.23), the equivalence of semi-norms:

$$\left|\widehat{\sum_{\imath=1}^{n}} \mathcal{O}_\imath\right|^2_{1-\epsilon,\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}} \leq |\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{\imath=1}^{n} \mathcal{O}_{\imath}}_i,\inf}$$

$$\leq K\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n}} \mathcal{O}_\imath\right)^2 \left(3 + \frac{C_{PF,I}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2}{\epsilon(1-\epsilon)}\right) |\hat{v}_i|^2_{L^2(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),H^1(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),1-\epsilon} \qquad (50)$$

And

$$|\hat{v}_i|^2_{L^2(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),H^1(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),1-\epsilon} \leq \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n}} \mathcal{O}_\imath\right)^{-2} |\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath},\inf}$$

$$\leq \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n}} \mathcal{O}_\imath\right)^{-2} \left(1 + C_{PF,SS}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n}} \mathcal{O}_\imath\right)^2\right) |\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}} \qquad (51)$$

The first assertion of the theorem then follows by combining (50) with the transformation properties of the semi-norms by Lemma (6.2.25):

$$|\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^{n} \mathcal{O}_i} \leq |\det B|^2 \|B^{-1}\|^{n+2(1-\epsilon)} |\hat{v}_i|^2_{,1-\epsilon,\sum_{i=1}^{n} \mathcal{O}_i}$$

$$\leq |\det B|^2 \|B^{-1}\|^{n+2(1-\epsilon)} K\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n}} \mathcal{O}_\imath\right)^2 \left(3 + \frac{C_{PF,I}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2}{\epsilon\,(1-\epsilon)}\right) |\hat{v}_i|^2_{L^2(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),H^1(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),1-\epsilon}$$

$$\leq |\det B| \|B^{-1}\|^{n+2(1-\epsilon)} \|B\|^{2(1-\epsilon)} K\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n}} \mathcal{O}_\imath\right)^2 \left(3 + \frac{C_{PF,I}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2}{\epsilon\,(1-\epsilon)}\right) |\hat{v}_i|^2_{L^2(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),H^1(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}),1-\epsilon}$$

The second assertion of the theorem is proved by a combination of (51) with the transformation properties by Lemma (6.2.25).

The next two theorems (see [256]) study the other pairs of semi-norms for equivalence in combination with affine maps $\left(|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf}\right)$ and $\left(|\cdot|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf}\right)$

Theorem (6.2.28)[272]: Let $\sum_{i=1}^{n}\mathcal{O}_i \subset \mathbb{R}^2$be a bounded, connected Lipschitz series domain and let $\sum_{i=1}^{n}\mathcal{O}_i$ be the affinely transformed series domain defined by (30). Then there holds the following relations.

(i) $|v_i|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} \leq |v_i|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_{i_i},\inf} \quad \forall\, v_i \in H^{1-\epsilon}(\sum_{i=1}^{n}\mathcal{O}_i), \forall\, \epsilon > 0,$

(ii) $|v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf}$

$$\leq \left(1 + |\det B|^{-1}\|B\|^{n+2(1-\epsilon)}C_{PF,SS}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2\right)|\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$

$$\forall\, v_i \in H^{1-\epsilon}\left(\sum_{i=1}^{n}\mathcal{O}_i\right), \forall\epsilon > 0$$

With $C_{PF,SS}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)$ being the number from Proposition (6.2.18)

Proof: Assertion (i) is a repetition of the first estimate in Lemma (6.2.19).

To show the second assertion we use Proposition (6.2.18) and Lemma (6.2.25) to deduce that

$$\inf_{c_i\in\mathbb{R}}\|v_i + c_i\|^2_{0,\sum_{i=1}^{n}\mathcal{O}_i} = |\det B| \inf_{c_i\in\mathbb{R}}\|\hat{v}_i + c_i\|^2_{0,\sum_{i=1}^{n}\mathcal{O}_i} \leq |\det B| C_{PF,SS}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2 |\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$

$$\leq 1 + |\det B|^{-1}\|B\|^{n+2(1-\epsilon)}C_{PF,SS}(1-\epsilon,\textstyle\sum_{i=1}^{n}\mathcal{O}_i)^2.$$

The assertion then follows by the definition of the semi-norm $|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf}$.

Theorem (6.2.29)[272]: Let $\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath} \subset \mathbb{R}^2$ be a bounded, connected Lipschitz series domain and let $\sum_{i=1}^{n}\mathcal{O}_i$ be the affinely transformed series domain defined by (30). Then there holds the following relations.

(ii). $|\hat{v}_i|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$

$$\leq \|B^{-1}\|^{2(1-\epsilon)} \max\{|\det B|^{-1}\|B\|^{n+2(1-\epsilon)},1\}\, \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^{-2} |\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$$

for any $v_i \in H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$ and $\epsilon > 0$ with $k(1-\epsilon, \sum_{i=1}^n \mathcal{O}_i)$ from Proposition (6.2.17),

(iii). $|\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$

$$\leq \max\{|\det B|\|B^{-1}\|^{n+2(1-\epsilon)},1\}\, K\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^{2}.$$

$$\left(3 + \frac{C_{PF,I}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2}{\epsilon\,(1-\epsilon)}\right) |\hat{v}_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon}$$

For any $v_i \in H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$ and $\epsilon > 0$ with $K(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath})$ from Proposition (6.2.17) and $C_{PF,SS}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2$ from Proposition (6.2.22).

Proof: By Lemmas (6.2.25),(6.2.23) and (6.2.27) we obtain

$$|\hat{v}_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon} \leq |\det B|^{-1}\|B\|^{n+2(1-\epsilon)} |\hat{v}_i|^2_{L^2(\widehat{\sum_{i=1}^n \mathcal{O}_i}),H^1(\widehat{\sum_{i=1}^n \mathcal{O}_i}),1-\epsilon}$$

$$\leq |\det B|\|B^{-1}\|^{2(1-\epsilon)} \kappa\left(1-\epsilon, \widehat{\sum_{i=1}^{n} \mathcal{O}_i}\right)^{-2} |\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$$

$$\leq |\det B|\|B\|^{2(1-\epsilon)} \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^{-2} \Bigg(|\det B|^{-2}\|B\|^{n+2(1-\epsilon)} \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^{-2} |v_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i}$$

$$+ |\det B^{-1}| \inf_{c_i \in \mathbb{R}} \|v_i + c_i\|^2_{o,\sum_{i=1}^n \mathcal{O}_i}\Bigg)$$

$$\leq \|B^{-1}\|^{2(1-\epsilon)} \max\{|\det B|\|B^{-1}\|^{n+2(1-\epsilon)},1\} \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^{-2} |\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$$

This is the first assertion. The second one follows analogously by the same lemmas:

$|v_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$

$$\leq |\det B|^2\|B^{-1}\|^{n+2(1-\epsilon)} \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^{-2} |\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}}$$

$$+ |\det B| \inf_{c_i \in \mathbb{R}} \|\hat{v}_i + c_i\|^2_{o,\widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}}$$

$$\le |\det B| \max\{|\det B| \|B^{-1}\|^{n+2(1-\epsilon)}, 1\} |\hat{v}_i|^2_{1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath} \inf}$$

$$\le |\det B| \max\{|\det B| \|B^{-1}\|^{n+2(1-\epsilon)}, 1\} K\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2 \left(3 + \frac{C_{PF,I}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2}{\epsilon(1-\epsilon)}\right) |\hat{v}_i|^2_{L^2(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}), H^1(\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}), 1-\epsilon}$$

$$\le \max\{|\det B| \|B^{-1}\|^{n+2(1-\epsilon)}, 1\} \|B\|^{2(1-\epsilon)} K\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2 \left(3 + \frac{C_{PF,I}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2}{\epsilon(1-\epsilon)}\right) |\hat{v}_i|^2_{L^2(\sum_{i=1}^{n} \mathcal{O}_i), H^1(\sum_{i=1}^{n} \mathcal{O}_i), 1-\epsilon}$$

We end this section with establishing uniform equivalence of the semi-norms $|\cdot|_{1-\epsilon, \sum_{i=1}^{n} \mathcal{O}_i}$ and $|\cdot|_{L^2(\sum_{i=1}^{n} \mathcal{O}_i), H^1(\sum_{i=1}^{n} \mathcal{O}_i), 1-\epsilon}$ for shape-regular series domains. Three of the four remaining bounds for other combinations of semi-norms are uniform under further restrictions on the diameter of the series domain [256].

Now , we consider a bounded, connected Lipschitz series domain $\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath} \subset \mathbb{R}^n$and maps of $\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}$ onto the series domain $\sum_{i=1}^{n} \mathcal{O}_i$where the ratio $\rho_{\sum_{i=1}^{n} \mathcal{O}_i} = D_{\sum_{i=1}^{n} \mathcal{O}_i} / d_{\sum_{i=1}^{n} \mathcal{O}_i}$ is controlled. Here, $D_{\sum_{i=1}^{n} \mathcal{O}_i}$denotes the diameter of $\sum_{i=1}^{n} \mathcal{O}_i$and $d_{\sum_{i=1}^{n} \mathcal{O}_i}$ is the supremum of the diameters of all balls contained in$\sum_{i=1}^{n} \mathcal{O}_i$. In the case of finite elements (or convex polygons) boundedness of ρ is referred to as shape regularity of $\sum_{i=1}^{n} \mathcal{O}_i$. Also, when defining $d_{\sum_{i=1}^{n} \mathcal{O}_i}$ with balls with respect to which $\sum_{i=1}^{n} \mathcal{O}_i$is star-shaped, then $\rho_{\sum_{i=1}^{n} \mathcal{O}_i}$is referred to as chunkiness parameter.

Using the notation (2.4) there holds

$$\|B\| \le \frac{D_{\sum_{i=1}^{n} \mathcal{O}_i}}{d_{\sum_{i=1}^{n} \mathcal{O}_i}} = \frac{D_{\sum_{i=1}^{n} \mathcal{O}_i}}{D_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}} \rho_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}, \quad \|B^{-1}\| \le \frac{D_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}}{d_{\sum_{i=1}^{n} \mathcal{O}_i}} = \frac{D_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}}{D_{\sum_{i=1}^{n} \mathcal{O}_i}} \rho_{\sum_{i=1}^{n} \mathcal{O}_i}, \quad \|B\| \|B^{-1}\| \le \rho_{\sum_{i=1}^{n} \mathcal{O}_i} \rho_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}, \tag{52}$$

(See [259]) Furthermore, we conclude that

$$|\det B| = \frac{|\sum_{i=1}^{n} \mathcal{O}_i|}{|\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}|} \le \frac{D^n_{\sum_{i=1}^{n} \mathcal{O}_i}}{d^n_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}}, |\det B|^{-1} \le \frac{D^n_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}}{d^n_{\sum_{i=1}^{n} \mathcal{O}_i}} = \rho^n_{\sum_{i=1}^{n} \mathcal{O}_i} \frac{D^n_{\widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}}}{D^n_{\sum_{i=1}^{n} \mathcal{O}_i}}. \tag{53}$$

With this notation, the results of Theorems (6.2.27-6.2.29) imply the following.

Corollary (6.2.30)[272]: Show that

(i) $|\det B||\det B^{-1}| \leq \|B\|\|B^{-1}\|\rho^{n-1}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{1-n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}$.

(ii) $$\frac{\|B\|\|B\|^{-1+2\epsilon}\min\{|\det B|\|B\|^{-n},1\}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}}{|\det B^{-1}|} \leq \rho^{1-n}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{1-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\|v_i\|^2_{\sim,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$
$$\leq \frac{\|B\|\|B^{-1}\|^{3-2\epsilon}\max\left\{|\det B|\|B^{-1}\|^{n},1\right\}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_1}}}{|\det B^{-1}|}.$$

Moreover if B is normal then ,

(iii) $\rho^{n-1}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{1-n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}} \geq 1$.

(iv) $\rho^{1-n}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{1-n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}\|v_i\|^2_{\sim,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} = |\det B|^2\|B\|^{-((1-\epsilon)+n)}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}$.

Proof. (i) We can easily get that

$$\frac{|\det B||\det B^{-1}|}{\|B\|\|B^{-1}\|} \leq \rho^{n}_{\sum_{i=1}^{n}\mathcal{O}_i}\frac{D^{n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}}{d^{n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}}\cdot\frac{1}{\rho_{\sum_{i=1}^{n}\mathcal{O}_i}\,\rho_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}}$$

Therefore

$$|\det B||\det B^{-1}| \leq \|B\|\|B^{-1}\|\rho^{n-1}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{1-n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}} \tag{54}$$

(ii) Appling (54) in Lemma (6.2.24) .

Here suppose B is normal , applying (50) in Lemma (6.2.24) we can find that.

(iii) $\rho^{n-1}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{1-n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}} \geq 1$.

(iv) $\rho^{1-n}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{1-n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}\|v_i\|^2_{\sim,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} = |\det B|^2\|B\|^{-((1-\epsilon)+n)}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}$.

Theorem (6.2.31)[272]: Let $\sum_{i=1}^{n}\mathcal{O}_i$ be the affine map of a bounded connected Lipschitz series domain $\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath} \subset \mathbb{R}^n$, cf (30).

(iii). The semi-norms $|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$ and $|\cdot|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$ are uniformly equivalent for a family of shape-regular series domain $\sum_{i=1}^{n}\mathcal{O}_i$:

$$|v_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i} \leq \rho^{n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}K\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2\left(3+\frac{C_{PF,I}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2}{\epsilon\,(1-\epsilon)}\right)|v_i|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon},$$

$$|v_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon}$$

$$\leq \rho^{n+2(1-\epsilon)}_{\sum_{i=1}^n \mathcal{O}_i} \rho^{n+2(1-\epsilon)}_{\widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}} \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2 \left(1 + C_{PF,SS}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2\right) |v_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i}$$

for any $v_i \in H^{1-\epsilon}(\mathcal{O}_i)$ and $\epsilon > 0$. Here, $\kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}\right), K\left(1-\epsilon, \widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}\right)$ are the numbers from Proposition (6.2.17) and $C_{PF,SS}\left(1-\epsilon, \widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}\right), C_{PF,I}\left(1-\epsilon, \widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}\right)$ are as in Propositions (6.2.18), (6.2.22), respectively.

(iv). The semi-norms $|\cdot|_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i}$ and $|\cdot|_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$ are uniformly equivalent for a family of uniformly bounded, shape-regular series domain $\sum_{i=1}^n \mathcal{O}_i$:

$$|v|_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i} \leq |\cdot|_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf} \quad ,$$

$$|v_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf} \leq \left(1 + \frac{D^{2(1-\epsilon)}_{\sum_{i=1}^n \mathcal{O}_i}}{D^{2(1-\epsilon)}_{\widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}}} \rho^{n}_{\sum_{i=1}^n \mathcal{O}_i} \rho^{n+2(1-\epsilon)}_{\widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}} C_{PF,SS}\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2\right) |v|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i}$$

for any $v_i \in H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$ and $\epsilon > 0$. Here, $C_{PF,SS}\left(1-\epsilon, \widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}\right)$ is the number from Proposition (6.2.18).

(iv). a) For a family of shape-regular series domain $\sum_{i=1}^n \mathcal{O}_i$whose diameters are bounded from below by a positive constant, the semi-norm $|\cdot|_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon}$is uniformly bounded by$|\cdot|_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$:

$$|v_i|^2_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon}$$

$$\leq \max\left\{\rho^{n}_{\sum_{i=1}^n \mathcal{O}_i} \rho^{n+2(1-\epsilon)}_{\widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}}, D^{-2(1-\epsilon)}_{\sum_{i=1}^n \mathcal{O}_i} D^{2(1-\epsilon)}_{\widehat{\sum_{\imath=1}^n \mathcal{O}_\imath}}\right\} \rho^{2(1-\epsilon)}_{\sum_{i=1}^n \mathcal{O}_i} \kappa\left(1-\epsilon, \widehat{\sum_{\imath=1}^{n} \mathcal{O}_\imath}\right)^2 |v_i|^2_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$$

for any $v_i \in H^{1-\epsilon}(\sum_{i=1}^n \mathcal{O}_i)$ and $\epsilon > 0$.

b) For a family of uniformly bounded, shape-regular series domain $\sum_{i=1}^n \mathcal{O}_i$, the semi-norm $|\cdot|_{1-\epsilon,\sum_{i=1}^n \mathcal{O}_i,\inf}$ is uniformly bounded by $|\cdot|_{L^2(\sum_{i=1}^n \mathcal{O}_i),H^1(\sum_{i=1}^n \mathcal{O}_i),1-\epsilon}$

$$|\hat{v}_i|^2_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i,\inf}$$

$$\leq \max\left\{\rho^{n+2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{n}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}, D^{2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{-2(1-\epsilon)}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}}\right\}\rho^{2(1-\epsilon)}_{\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}} K\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2\left(3 + \frac{C_{PF,I}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)^2}{\epsilon\,(1-\epsilon)}\right)|v_i|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$$

for any $v_i \in H^{1-\epsilon}(\sum_{i=1}^{n}\mathcal{O}_i)$ and $\epsilon > 0$.

Here, $\kappa\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right), K\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)$ are the parameters from Proposition (6.2.17), and $C_{PF,I}\left(1-\epsilon,\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}\right)$is the number from Proposition (6.2.22).

Proof: The assertions (i)–(iii) are a combination of Theorems (6.2.27-6.2.29), respectively, with the bounds provided by (52), (53).
The uniform equivalence of the semi-norms $|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$ and$|\cdot|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$ for shape-regular series domains is based on what one calls their scaling property (see [256]). It means that both semi-norms for functions on a series domain $\sum_{i=1}^{n}\mathcal{O}_i$are uniformly equivalent to the respective semi-norm of the affinely transformed functions onto a fixed series domain $\widehat{\sum_{\imath=1}^{n}\mathcal{O}_\imath}$, when one of the semi-norms is multiplied by an appropriate number (it is a power of the diameter of $\sum_{i=1}^{n}\mathcal{O}_i$). This property applies also to the norms $|\cdot|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$ and $|\cdot|_{\sim 1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$, cf. Lemma (6.2.24). Scaling properties are relevant for the error analysis of piecewise polynomial approximations. We formulate the result as (see [256]) a corollary to Lemmas (6.2.24) and (6.2.25).

Corollary(6.2.32)[272]: the norms $|\cdot|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1_0(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$, $|\cdot|_{\sim 1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$ and semi-norms $|\cdot|_{1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}|\cdot|_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$ are scalable of order $D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}$:

$$D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{-n}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{-2(1-\epsilon)}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\|\hat{v}_i\|_{L^2(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),H^1_0(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),1-\epsilon}$$

$$\leq \|v_i\|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1_0(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$$

$$\leq D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\|\hat{v}_i\|^2_{L^2(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),H^1_0(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),1-\epsilon}$$

$$D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{-n}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{-2(1-\epsilon)}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\min\left\{\rho^{-n}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}},1\right\}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}$$

$$\leq \|v_i\|^2_{\sim,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$

$$\leq D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\max\left\{\rho^{n}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}},1\right\}\|\hat{v}_i\|^2_{\sim,1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}$$

For any $v_i \in \widetilde{H}^{1-\epsilon}(\sum_{i=1}^{n}\mathcal{O}_i)$ and $\epsilon > 0$, and

$$D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{-n}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{-2(1-\epsilon)}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}|\hat{v}_i|^2_{L^2(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),H^1(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),1-\epsilon}$$

$$\leq |v_i|^2_{L^2(\sum_{i=1}^{n}\mathcal{O}_i),H^1(\sum_{i=1}^{n}\mathcal{O}_i),1-\epsilon}$$

$$\leq D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}|\hat{v}_i|^2_{L^2(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),H^1(\widehat{\sum_{i=1}^{n}\mathcal{O}_i}),1-\epsilon},$$

$$D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{-2n}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{-n-2(1-\epsilon)}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}|\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}} \leq |v_i|^2_{,1-\epsilon,\sum_{i=1}^{n}\mathcal{O}_i}$$

$$\leq D^{n-2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}\rho^{n+2(1-\epsilon)}_{\sum_{i=1}^{n}\mathcal{O}_i}D^{2(1-\epsilon)-n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}\rho^{n}_{\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}|\hat{v}_i|^2_{1-\epsilon,\widehat{\sum_{i=1}^{n}\mathcal{O}_i}}$$

for any $v_i \in H^{1-\epsilon}(\sum_{i=1}^{n}\mathcal{O}_i)$ and $\epsilon > 0$.

Proof: The bounds are a combination of Lemmas (6.2.24) and (6.2.25) with (52), (53).

Remark (6.2.33)[272]: The estimate by Theorem (6.2.31) (iii) a) breaks down when $D_{\sum_{i=1}^{n}\mathcal{O}_i} \to 0$. In fact, for a family of scaled series domains $\sum_{i=1}^{n}(\mathcal{O}_i)_h$ with $D_{\sum_{i=1}^{n}(\mathcal{O}_i)_h} = h$ and a non-constant function v_i scaled to a family $\{(v_i)_h\}$ of functions on $\{\sum_{i=1}^{n}(\mathcal{O}_i)_h\}$, $|(v_i)_h|^2_{L^2(\sum_{i=1}^{n}(\mathcal{O}_i)_h),H^1(\sum_{i=1}^{n}(\mathcal{O}_i)_h),1-\epsilon} \simeq h^{n-2(1-\epsilon)}$ by Corollary (6.2.39) whereas $|(v_i)_h|^2_{1-\epsilon,\sum_{i=1}^{n}(\mathcal{O}_i)_h,\inf} \geq \inf_{c_i\in\mathbb{R}}\|(v_i)_h - c_i\|^2_{0,\sum_{i=1}^{n}(\mathcal{O}_i)_h} \simeq h^n$. Therefore, the dependence on $D_{\sum_{i=1}^{n}\mathcal{O}_i}$ like $D^{2(\epsilon-1)}_{\sum_{i=1}^{n}\mathcal{O}_i}$ of the upper bound in Theorem (6.2.31) (iii) a) is optimal.