Chapter 5

Carleson Measures and Operator Theoretic Differences

In this Chapter we show the conjecture raised by Wu is false. Indeed, we show that if $2 < p < \infty$, then there exists g analytic in $\mathbb{D}$ such that the measure $\mu_{g,p}$ on $\mathbb{D}$ defined by $d\mu_{g,p}(z) = (1-|z|^2)^{p-1}|g'(z)|^p dxdy$ is not a Carleson measure for $\mathcal{D}^p_{p-1}$ but is a classical Carleson measure. We obtain also some sufficient conditions for multipliers of the spaces $\mathcal{D}^p_{p-1}$. In particular, it is shown, on one hand, that $T_g: \mathcal{D}^p_{p-1} \to H^p$ is bounded if and only if g∈BMOA when $0 < p \leq 2$, and, on the other hand, that this equivalence is very far from being true , if $p > 2$. Those symbols g such that $T_g: \mathcal{D}^p_{p-1} \to H^q$ is bounded (or compact) when $p < q$ are also characterized. Moreover, the best known sufficient L^∞ -type condition for a positive Borel measure μ on $\mathbb{D}$ to be a p −Carleson measure for $\mathcal{D}^p_{p-1}$, $p > 2$, is significantly relaxed, and the established result is shown to be sharp in a very strong sense .

Sec(5.1): Spaces of Dirichlet Type:

We denote by $\mathbb{D}$ the unit disc $\{z \in \mathbb{C}: |z| < 1\}$ and by Hol $(\mathbb{D})$the space of all analytic functions in $\mathbb{D}$. Also,$H^p(0 < p \leq \infty)$ are the classical Hardy spaces of analytic functions in $\mathbb{D}$ (see [182] and [186]).

If E is a measurable subset of the unit circle $\mathbb{T} = \partial\mathbb{D}$, we write $|E|$ for the Lebesgue measure of E. If $I \subset \mathbb{T}$ is an interval, the Carleson square S(I) is defined as

$$S(I) = \left\{re^{it}: e^{it} \in I, 1 - \frac{|I|}{2\pi} \leq r < 1\right\}.$$

Carleson [181] (see also [182]) proved that if $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$ then $H^p \subset L^p(d\mu)$ if and only if there exists a positive constant C such that

$$\mu(S(I)) \leq C|I|, \text{for every interval } I \subset T. \quad (1)$$

The measures μ which satisfy this condition will be called *classical Carleson measures*.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in Hol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1)\int_{\mathbb{D}}(1-|z|)^\alpha\,(|f(z)|)^p dA(z) < \infty\right)^{1/p}$$

The unweighted Bergman space A^p_0 is simply denoted by A^p . Here, $dA(z) = \frac{1}{\pi}dxdy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer *to* [183] and [191] for the theory of these spaces.

The space $\mathcal{D}_\alpha^p(0 < p < \infty, \alpha > -1)$ consists of those $f \in Hol(\mathbb{D})$ such that $f' \in A_\alpha^p$. Hence, if f is analytic in $\mathbb{D}$,then $f \in \mathcal{D}_\alpha^p$if and only if

$$\|f\|_{\mathcal{D}_\alpha^p}^P \stackrel{def}{=} |f|^p + \|f'\|_{A_\alpha^p}^p < \infty .$$

If $p < \alpha + 1$ then it is well known that $\mathcal{D}_\alpha^p = A_{\alpha-p}^p$ (see [185]). It is trivial that$\mathcal{D}_1^2 = H$. The spaces $\mathcal{D}_\alpha^p$ are called Dirichlet spaces if $p \geq \alpha + 1$. In particular, the space$\mathcal{D}_0^2$is the classical Dirichlet space.

A positive Borel measure μ in $\mathbb{D}$ is said to be a Carleson measure for A_α^p (respectively, a Carleson measure for $\mathcal{D}_\alpha^p$) if $A_\alpha^p \subset L^p(d\mu)$ (respectively, $\mathcal{D}_\alpha^p \subset L^p(d\mu)$.

The Carleson measures forA_α^p are characterized in the following theorem.

Theorem (5.1.1)[176]:

Suppose that$0 < p < \infty$ and $\alpha > -1$,and let μ be a positive Borel measure on $\mathbb{D}$.Then μ is a Carleson measure for A_α^p if and only if there exists a positive constant C such that $\mu(S(I)) \leq C|I|^{\alpha+2}$, for every interval $I \subset \mathbb{T}$.

Theorem (5.1.1) was obtained by Oleinik and Pavlov [195, 196] (see also the works of Stegenga [197] and Hastings [190] where the result is proved for certain values of$p$and α). Luecking [192, 193] (see [183]) obtained another characterizazion of the Carleson measures forA_α^p which involves the p seudohyperbolic metric. $Z.Wu$ [199] and Arcozzi, Rochberg and Sawyer [177] obtained a characterization of the Carleson measures for the spaces$\mathcal{D}_\alpha^p$ for certain values of p, α .Inparticular, parts (c) and (d) of Theorem 1 of [199] (see also Theorem 2.1 of [198]), yield the following result.

Theorem (5.1.2) [176]:

Suppose that $0 < p \leq 2$ and let μ be a positive Borel measure on $\mathbb{D}$, then μ is a Carleson measure for $\mathcal{D}_{p-1}^p$ if and only if μ is a classical Carleson measure.

Wu conjectured of [199] that the conclusion of Theorem (5.1.3) is also true for $2 < p < \infty$. In this section we shall see that this conjecture is not true. Indeed, we shall prove the following result.

Theorem (5.1.3) [176]:

Suppose that $2 < p < \infty$. Then there exists a function $g \in Hol(\mathbb{D})$ such that the measure $\mu_{g,p}$ on $\mathbb{D}$ given by $d\mu_{g,p}(z) = (1 - |z|^2)^{p-1}|g'(z)|^p dA(Z)$ is not a Carleson measure for$\mathcal{D}_{p-1}^p$ but is a classical Carleson measure.

Note that ifμ is a Carleson measure for $\mathcal{D}^p_{p-1}$ then it is a classical Carleson measure (see [199]). Theorem (5.1.3) shows that μ being a classical Carleson measure is not enough to deduce that μ is a Carleson measure for $\mathcal{D}^p_{p-1}(2 < p < \infty)$. However, it is easy to prove the following result.

Proposition (5.1.4) [176]:

Suppose that $2 < p < \infty$ and let μ be a positive Borel measure on $\mathbb{D}$.Ifthere exist $C > 0$ and $\varepsilon > 0$ such that

$$\mu\big(S(I)\big) \le C|I|^{1+\varepsilon}, \tag{2}$$

for all intervals $I \subset \mathbb{T}$,then μis a Carleson measure for $\mathcal{D}^p_{p-1}$.

Theorem (5.1.3) and Proposition (5.1.4) will be proved will be devoted to obtain several results that will be needed in the proof of Theorem (5.1.3) and which may be of independent interest. In particular, Theorem (5.1.5) and Theorem (5.1.6) will be used in to obtain sufficient conditions for multipliers of the spaces $\mathcal{D}^p_{p-1}$, $0 < p < 2$.

As usual, throughout this section the letter C denotes a positive constant thatmay change from one step to the next.

We start obtaining a condition on the Taylor coefficients of a function $g \in Hol(\mathbb{D})$ which implies that the measure $\mu_{g,p}$ on $\mathbb{D}$ defined as in Theorem (5.1.3) is a classical Carleson measure.

Theorem (5.1.5) [176]:

Let g be an analytic function in , $g(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. If $0 < p < \infty$ and

$$\sum_{n=0}^{\infty}\left(\sum_{k\in I(n)} |a_k|\right)^p < \infty, \tag{3}$$

then the measure $\mu_{g,p}$ on $\mathbb{D}$ defined by $d\mu_{g,p}(z) = (1-|z|^2)^{p-1}|g'(z)|^p dA(Z)$ is a classical Carleson measure.

Here and all over the section, for $n = 0,1,\ldots,$ we let $I(n)$be the set of the integers k such that $2^n \le k < 2^{n+1}$.

Theorem(5.1.5) improves part (i) of Theorem 1 of [188] which asserts that (3) implies that $\in \mathcal{D}^p_{p-1}$.

Proof of Theorem (5.1.5). Using Lemma 3.3 in [186], we see that it suffices to prove that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)(1-|z|^2)^{p-1}}{|1-\bar{a}z|^2} |g'(z)|^p dA(z) < \infty. \tag{4}$$

Now, using Theorem 1 of [194], we deduce that there is a constant C_p which depends only on p such that, for every $a \in \mathbb{D}$,

$$\int_{\mathbb{D}} \frac{(1-|a|^2)(1-|z|^2)^{p-1}}{|1-\bar{a}z|^2} |g'(z)|^p dA(z)$$

$$\le C_p \int_0^1 (1-|a|^2)(1-r^2)^{p-1} \left(\sum_{n=1}^{\infty} n|a_k| r^{n-1}\right)^p \left(\int_0^{2\pi} \frac{1}{|1-\bar{a}re^{it}|^2} dt\right) |g'(z)|^p dr$$

$$\le C_p \int_0^1 \frac{(1-|a|^2)(1-r^2)^{p-1}}{1-|a|^2 r^2} \left(\sum_{n=1}^{\infty} n|a_k| r^{n-1}\right)^p dr$$

$$\le C_p \int_0^1 (1-r^2)^{p-1} \left(\sum_{n=1}^{\infty} n|a_k| r^{n-1}\right)^p dr$$

$$\le C_p \left(\sum_{n=0}^{\infty} 2^{-np}\right) \left(\sum_{k\in I(n)} k|a_k|\right)^p$$

$$\le C_p \sum_{n=0}^{\infty} \left(\sum_{k\in I(n)} |a_k|\right)^p = A_P < \infty .$$

Hence, we have proved (4). This finishes the proof.

Using Proposition 2.1 of [180] (see also Proposition A of [188]), we obtain that if $g \in Hol(\mathbb{D})$ is given by a power series with Hadamard gaps, $g(z) = \sum_{k=1}^{\infty} a_k z^{nk}$ $(z \in \mathbb{D})$, with $n_{k+1} \ge \lambda n_k$ for all k, for some $\lambda > 1$. Then , for every $p \in (0,\infty)$,

$$g \in \mathcal{D}_{p-1}^p \Leftrightarrow \sum_{k=1}^{\infty} |a_k|^p < \infty.$$

Our next theorem is an improvement of this result.

Theorem (5.1.6) [176]:

Suppose that $0 < p < \infty$ and let g be an analytic function in $\mathbb{D}$. Which is given by a power series with Hadamard gaps, $g(z) = \sum_{k=1}^{\infty} a_k z^{nk}$ $(z \in \mathbb{D})$ with $n_{k+1} \ge \lambda n_k$, for all k $(\lambda > 1)$, then, the following conditions are equivalent:

(a) The measure $\mu_{g,p}$ on $\mathbb{D}$ defined by $d\mu_{g,p} = (1-|z|^2)^{p-1} |g'(z)|^p dA(z)$ is a classical Carleson measure.

(b) $g \in \mathcal{D}_{p-1}^p$.

(c) $\sum_{k=1}^{\infty} |a_k|^p < \infty$..

Proof .

We already know that (b) ⇔ (c). Trivially, (a) implies that $\mu_{g,p}$ is a finite measure and, hence, $g \in \mathcal{D}^p_{p-1}$. Thus, we have seen that $(a) \Rightarrow (b)$. Consequently, it only remains to prove that $(c) \Rightarrow (a)$. So take $g \in Hol(\mathbb{D})$ which is given by a power series with Hadamard gaps

$$g(z) = \sum_{k=1}^{\infty} a_k z^{nk} \quad , \text{ with} \frac{n_{k+1}}{n_k} \geq \lambda > 1, \text{for all } k, \tag{5}$$

and suppose that $\sum_{k=1}^{\infty} |a_k|^p < \infty$. Using the gap condition, we see that there are at most $C_\lambda = log_\lambda 2 + 1$ of the n_k's in the set $I(n)$. Then there exists a constant $C_{\lambda,p} > 0$ such that

$\sum_{n=0}^{\infty} \left(\sum_{k \in I(n)} |a_k|\right)^p \leq C_{\lambda,p} \sum_{k=1}^{\infty} |a_k|^p.$

Using Theorem (5.1.5), we deduce that $\mu_{g,p}$is a classical Carleson measure. Thus, we have proved that $(c) \Rightarrow (a)$, as needed. This finishes the proof.

We need to introduce some notation to state our last result in this section.

If $f \in Hol(\mathbb{D}), 0 < p < \infty$ and $0 \leq r < 1$, we set, as usual,

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f(re^{it}\right|^p dt\right)^{\frac{1}{p}}.$$

Notice that

$g \in \mathcal{D}^p_{p-1} \Leftrightarrow \int_0^1 (1-r)^{p-1} M_p^p\,(r, g')dr < \infty.$

It is well known (see [202]) that if $f \in Hol(\mathbb{D})$ is given by a power series with Hadamard gaps and $0 < p < \infty$, then $M_2(r, f) \approx M_p(r, f)$. It follows that if $g \in Hol(\mathbb{D})$ is given by a power series with Hadamard gaps then

$g \in \mathcal{D}^p_{p-1} \Leftrightarrow \int_0^1 (1-r)^{p-1} M_2^p\,(r, g')dr < \infty.$

Our next theorem asserts that this result is sharp in a strong sense.

Theorem (5.1.7) [176]:

Suppose that $0 < p < \infty$ and let ϕ be a positive and increasing function defined in (0,1) such that

$$\int_0^1 (1-r)^{p-1} \phi^p\,(r)dr < \infty. \tag{6}$$

Then there exists a function $g \in \mathcal{D}_{p-1}^{p}$ given by a power series with Hadamard gaps such that

$$M_2^p(r, g') \geq \phi(r), \text{for all } r \in (0,1). \tag{7}$$

The proof of Theorem (5.1.7) is very similar to that of Theorem D of [186].

Proof of Theorem (5.1.7). Set $r_k = 1 - 2^{-k}, k = 1,2,\dots$. Since ϕ is increasing

$$\int_0^1 (1-r)^{p-1}\phi^p\,(r)dr \geq \sum_{k=1}^{\infty} \int_{r_k}^{r_{k+1}} (1-r)^{p-1}\phi^p\,(r)dr$$

$$\geq \sum_{k=1}^{\infty} (r_{k+1} - r_k)\,(1-r)^{p-1}\phi^p(r_k)$$

$$= 2^{-p} \sum_{k=1}^{\infty} 2^{-kp}\,\phi^p(r_k).$$

Hence,

$$\sum_{k=1}^{\infty} 2^{-kp}\,\phi^p(r_k) < \infty. \tag{8}$$

Set

$$g(z) = \phi(r_1)z + e^4 \sum_{k=1}^{\infty} 2^{-kp}\,\phi^p(r_k) z^{2^k}, z \in \mathbb{D}.$$

Then g is an analytic function in $\mathbb{D}$ which is given by a power series with Hadamard gaps. Using Theorem (5.1.6) and (8), we deduce that $g \in \mathcal{D}_{p-1}^{p}$.

We have

$$M_2^2(r, g') = \phi^2(r_1) + e^8 \sum_{k=1}^{\infty} \phi^2(r_k) r^{2^{k+1}-2} \geq \phi^2(r_1) + e^8 \sum_{k=1}^{\infty} \phi^2(r_k) r^{2^{k+1}},$$

$$0 < r < 1\,.$$

Since ϕ is increasing, we deduce that

$$M_2^2(r, g') \geq \phi^2(r_1) \geq \phi^2(r), \quad 0 < r \leq r_1 \tag{9}$$

Now, using the elementary inequality $(1 - n^{-1})^n \geq e^{-2}$ $(n \geq 2)$ and bearing in mind that ϕ is increasing, we see that, for $j \geq 1$ and $r_j \leq r \leq r_{j+1}$,

$$M_2^2(r, g') \geq e^8 \sum_{k=1}^{\infty} \phi^2(r_k) r^{2^{k+1}} \geq e^8 \phi^2\big(r_{j+1}\big) r^{2^{j+2}} \geq e^8 \phi^2(r)\big(1 - 2^j\big)^{4.2^j}$$

$\geq \phi^2(r)$.

This together with (9) implies that $M_2(r, g') \geq \phi(r)$, for all $r \in (0,1)$, and finishes the proof.

Proof of Proposition (5.1.4). Suppose p, μ, C and ε are as in Proposition (5.1.4). Take $f \in \mathcal{D}^p_{p-1}$. Then it is easy to see that

$$M_p(r, f') = O\left(\frac{1}{1-r}\right), \text{ as} r \to 1.$$

Then it follows easily that $M_p(r, f) = O\left(\log\frac{1}{1-r}\right)$, as $r \to 1$. Actually, Theorem1 of [189] implies that

$$M_p(r, f) = O\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \text{as} \quad r \to 1,$$

For all $\beta > \frac{1}{2}$. Then it is clear that $f \in A^p_\alpha$, for every $\alpha > -1$. Consequently, we have proved that $\mathcal{D}^p_{p-1} \subset A^p_\alpha$, for every $\alpha > -1$. In particular, $\mathcal{D}^p_{p-1} \subset A^p_{-1+\varepsilon}$.

Now, Theorem (5.1.1) implies that μ is a Carleson measure for $A^p_{-1+\varepsilon}$ and then it follows that μ is also a Carleson measure for $\mathcal{D}^p_{p-1}$.

Proof of Theorem (5.1.3). Suppose that $2 < p < \infty$. Take two positive numbers α and ε such that $\frac{1}{p} < \alpha < \frac{1}{2}$ and $0 < \varepsilon < \frac{1}{2} - \alpha$ and define

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k^{\frac{1}{p}+\varepsilon}} z^{2^k} \quad , z \in \mathbb{D}\,.$$

Using Theorem (5.1.6) we see that $f \in \mathcal{D}^p_{p-1}$. Also, it is easy to see that there exist $r_0 \in (0,1)$ and $C > 0$ such that

$$M_2(r, f) \geq C\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}-\varepsilon} \quad r_0 \leq r < 1. \tag{10}$$

Since$f$ is given by a power series with Hadamard gaps, , we see that there exist two absolute constants $A > 0$ and $B > 0$ such that for every $r \in (0,1)$ the set

$$E_r = \{t \in [0,2\pi]: |f(re^{it})| > BM_2(r, f)\} \tag{11}$$

has Lebesgue measure greater than or equal to A,

$$|E_r| \geq A, \qquad 0 < r < 1. \tag{12}$$

Define

$$\phi(r)=\frac{1}{(1-r)\left[log\left(\frac{e^{\alpha}}{1-r}\right)\right]^{\alpha}}\quad 0\le r<1. \tag{13}$$

Thenϕ is an increasing function defined in (0,1)and $\int_0^1(1-r)^{p-1}\phi^p(r)dr<\infty$. Using Theorem (5.1.7), we see that there exists a function $g\in\ \mathcal{D}_{p-1}^{p}$ which is given by a power series with Hadamard gaps and such that

$$M_2(r,g')\ge\phi(r), r\in(0,1). \tag{14}$$

Now, Theorem (5.1.6)implies that the measure $\mu_{g,p}$ is a classical Carleson measure. Using *Hölder's* inequality, see [202] and (12), we deduce that there exists a positive constant C_1such that

$$\int_{E_r}\left|g'(re^{it})\right|^p dt\ge|E_r|^{1-\frac{p}{2}}\left(\int_{E_r}\left|g'(re^{it})\right|^p dt\right)^{\frac{p}{2}}$$

$$\ge C_1|E_r|M_2^p(r,g')\ge C_1AM_2^p(r,g')\ ,0<r<1\,.$$

Hence, setting $C=C_1A$,we have

$$\int_{E_r}\left|g'(re^{it})\right|^p dt\ge CM_2^p(r,g')\,,\ \ 0<r<1\,. \tag{15}$$

Bearing in mind the definition of the sets E_r $(0<r<1)$ and using (15),(10), (14) and the fact that $\alpha<\frac{1}{2}-\varepsilon$, we obtain

$$\int_{\mathbb{D}}(1-|z|^2)^{p-1}\,|g'(z)|^p|f(z)|^p dA(z)$$

$$\ge C\int_{r_0}^{1}(1-r)^{p-1}\int_{E_r}\left|g'\left(re^{it}\right)\right|^p\left|f(re^{it})\right|^p dtdr$$

$$\ge C\int_{r_0}^{1}(1-r)^{p-1}\,M_2^p(r,f)\int_{E_r}\left|g'\left(re^{it}\right)\right|^p dtdr$$

$$\ge C\int_{r_0}^{1}(1-r)^{p-1}\,M_2^p(r,f)M_2^p(r,g')dr$$

$$\ge C\int_{r_0}^{1}(1-r)^{p-1}\left(\,log\frac{1}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}-\varepsilon}\ \phi^p(r)dr$$

$$\ge C\int_{r_0}^{1}\frac{dr}{(1-r)\left(\,log\frac{1}{1-r}\right)^{p\alpha-\frac{p}{2}+1-p\varepsilon}}$$

$= \infty.$

Since $\in \mathcal{D}_{p-1}^{p}$, this shows that$\mu_{g,p}$ is not a Carleson measure for $\mathcal{D}_{p-1}^{p}$ and finishes the proof.

A function$g \in Hol(\mathbb{D})$ is a multiplier for the space $\mathcal{D}_{\alpha}^{p}$ if $g\mathcal{D}_{\alpha}^{p} \subset \mathcal{D}_{\alpha}^{p}$, that is ,if $fg \in \mathcal{D}_{\alpha}^{p}$, for all $f \in \mathcal{D}_{\alpha}^{p}$. By the closed-graph theorem, g is a multiplier for $\mathcal{D}_{\alpha}^{p}$ if and only if there exists a constant $C > 0$ such that
$\|fg\|_{\mathcal{D}_{\alpha}^{p}} \leq C\|fg\|_{\mathcal{D}_{\alpha}^{p}}$, for all $f \in \mathcal{D}_{\alpha}^{p}$.

The space of all multipliers of the space $\mathcal{D}_{\alpha}^{p}$ will be denoted by $m(\mathcal{D}_{\alpha}^{p})$. Since $\mathcal{D}_{\alpha}^{p}$ contains the constant functions, we have $m(\mathcal{D}_{\alpha}^{p}) \subset \mathcal{D}_{\alpha}^{p}$. Wu obtained in Theorem 4.2 of [199] a characterization of the multipliers of the spaces $\mathcal{D}_{\alpha}^{p}(\alpha > 1, 0 < p < \infty)$. In particular, he proved the following result.

Theorem (5.1.8) [176]:
Suppose that $0 < p < \infty$ and g is an analytic function in $\mathbb{D}$.Then $g \in m(\mathcal{D}_{p-1}^{p})$ if and only if $g \in H^{\infty}$and the measure $\mu_{g,p}$ on $\mathbb{D}$ defined by $d\mu_{g,p}(z) = (1-|z|)^{p-1}|g'(z)|^{p}dA(z)$ is a Carleson measure for $\mathcal{D}_{p-1}^{p}$. Theorem (5.1.8) and Theorem (5.1.2) yield the following theorem (see [198]).

Theorem (5.1.9) [176]:
Suppose that $0 < p < 2$ and g is an analytic function in $\mathbb{D}$.Then $g \in m(\mathcal{D}_{p-1}^{p})$ if and only if $g \in H^{\infty}$ and the measure $\mu_{g,p}$ on $\mathbb{D}$ defined by $d\mu_{g,p}(z) = (1-|z|)^{p-1}|g'(z)|^{p}dA(z)$ is a classical Carleson measure.

Since $\mathcal{D}_{1}^{2} = H^{2}$,we have $m(\mathcal{D}_{1}^{2}) = m(H^{2}) = H^{\infty}$.We remark also that even though there is no relation of inclusion between $\mathcal{D}_{p-1}^{p}$ and $\mathcal{D}_{q-1}^{q}$ $(p \neq q)$, it is easy to see that

$$m(\mathcal{D}_{p-1}^{p}) \subset m(\mathcal{D}_{q-1}^{q}), \qquad \text{if } 0 < p \leq q \leq 2. \tag{16}$$

Indeed, if $0 < p \leq q \leq 2$ and $\in m(\mathcal{D}_{p-1}^{p})$, then $g \in H^{\infty}$and then it follows that $sup_{z\in\mathbb{D}}(1-|z|)|g'(z)| = A < \infty$. Then, for every interval $I \subset \mathbb{T}$, we have

$$\int_{S(I)}(1-|z|)^{q-1}|g'(z)|^{q}dA(z) \leq A^{q-p}\int_{S(I)}(1-|z|)^{p-1}|g'(z)|^{p}dA(z)$$

Since $\mu_{g,p}$ is a classical Carleson measure, it follows that $\mu_{g,p}$ is also a classical Carleson measure. This and the fact that $g \in H^{\infty}$ yield that $\in m(\mathcal{D}_{q-1}^{q})$. Using Theorem (5.1.9) and our results , we can obtain sufficient conditions for multipliers of the spaces $\mathcal{D}_{p-1}^{p}$, $0 < p < 2$.

Theorem (5.1.10)[176]:

Suppose that $0 < q < 2$ and let g be an analytic function in $\mathbb{D}$, $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, satisfying

$$\sum_{n=0}^{\infty}\left(\sum_{k\in I(n)}|a_k|\right)^{q} < \infty \tag{17}$$

(i) If $0 < q \leq 1$ and $q \leq p < 2$, then $g \in m\ (\mathcal{D}_{p-1}^{p})$.

(ii) If $0 < q \leq p < 2$ and $g \in H^{\infty}$,then $g \in m\ (\mathcal{D}_{p-1}^{p})$.

Proof.

Notice that if $0 < q \leq p$ then (18) implies that

$$\sum_{n=0}^{\infty}\left(\sum_{k\in I(n)}|a_k|\right)^{p} < \infty \quad .$$

Then, using Theorem (5.1.5) and Theorem (5.1.9) we deduce (ii).

Now, if $0 < q \leq 1$ then (18) implies $\sum_{k=1}^{\infty}|a_k| < \infty$ and, hence, $g \in H^{\infty}$. Then (i) follows from (ii).

Similarly, using Theorem (5.1.6), we obtain the following.

Theorem (5.1.11)[176]:

Suppose that $0 < q \leq 1$ and $q \leq p < 2$.Let g be an analytic function in $\mathbb{D}$ which is given by a power series with Hadamard gaps,

$g(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$ with $n_{k+1} \geq \lambda n_k$, for all k $(\lambda > 1)$,

With

$$\sum_{k=1}^{\infty}|a_k|^{q} < \infty \quad .$$

Then $g \in m\ (\mathcal{D}_{p-1}^{p})$.

We will close the section studying the connection between the multipliers of the spaces $\mathcal{D}_{q-1}^{q}$ and the spaces Q_p.

When $0 < p < \infty$, an analytic function f in $\mathbb{D}$ belongs to the space Q_p if

$$sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2\, g(z,a)^p dA(z) < \infty \ ,$$

Where g denotes the Green function for the disc given by

$$g(z,a) = log\left|\frac{1-\bar{a}z}{a-z}\right| \ , \qquad z, a \in \mathbb{D} \ , \ z \neq a \ .$$

The spaces Q_p are conformally invariant. They have their origin in [200] where it was shown that $Q_2 = \mathcal{B}$ (the Bloch space) and [178] where this result was extended by showing that $Q_2 = \mathcal{B}$ for all $p > 1$. The space Q_1 coincides with $BMOA$.When $0 < p < 1$,Qp is a proper subspace of $BMOA$ and has many Interesting properties (see, [184], [179], or the recent detailed monograph [201]).

There are various characterizations of Q_p spaces. The one that will be useful for us is expressed in terms of p −Carleson measures. Given a positive Borel measure μ on $\mathbb{D}$, we say that μ is a p −Carleson measure if there exists a positive constant C such that

$$\mu\big(S(I)\big) \leq C|I|^p, \text{for every interval } I \subset \mathbb{T} \tag{18}$$

The special case $p = 1$ yields the classical Carleson measures. The following characterization of Q_p spaces was obtained by Aulaskari, Stegenga and Xiao [179].

Theorem (5.1.12) [176]:

Let $0 < p < \infty$. Afunction f holomorphic in $\mathbb{D}$ is a member of Q_p if and only if the measure μ on $\mathbb{D}$ defined by $d\mu(z) = (1-|z|^2)^p|f'(z)|^2 dA(z)$ is a p −Carleson measure.

Vinogradov [198] proved that, for $0 < s < 2$, there are Blaschke products which do not belong to the space $\mathcal{D}^s_{s-1}$. Hence,

$$H^\infty \not\subset \mathcal{D}^s_{s-1} \text{ and } Q_1 \not\subset \mathcal{D}^s_{s-1} \ , \quad 0 < s < 2 \ .$$

However, we can prove the following result.

Theorem (5.1.13). (i) $\cup_{0<P<1} Q_p \subsetneq \cap_{0<s\leq 2} \mathcal{D}^s_{s-1}$.

(ii) If $0 < p < 1$ and $0 < s < 2$, then $H^\infty \cap Q_p \subsetneq m(\mathcal{D}^s_{s-1})$.

Proof of (i). Take p and s with $0 < p < 1$ and $0 < s < 2$, and $f \in Q_p$.We have

$$\int_{\mathbb{D}} |f'(z)|^s (1-|z|^2)^{s-1} dA(z)$$

$$= \int_{\mathbb{D}} \left[|f'(z)|^s(1-|z|^2)^{\frac{sp}{2}}\right] [1-|z|^2]^{\frac{s(2-p)-2}{2}} dA(z).$$

Applying $H\ddot{o}lder's$ inequality with the exponents $\frac{2}{s}$ and $\frac{2}{2-s}$, weobtain

$$\int_{\mathbb{D}} |f'(z)|^s (1-|z|^2)^{s-1} dA(z)$$

$$\leq \left[\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^p dA(z)\right]^{\frac{s}{2}} \left[\int_{\mathbb{D}} (1-|z|^2)^{\frac{s(2-p)-2}{2-s}} dA(z)\right]^{\frac{2-s}{2}}. \quad (19)$$

Theorem(5.1.13) [176]:

Implies that $\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^p dA(z) < \infty$. Also,$\frac{s(2-p)-2}{2-s} > -1$ and then it follows that $\int_{\mathbb{D}} (1-|z|^2)^{\frac{s(2-p)-2}{2-s}} dA(z) < \infty$. Consequently, we see that $\int_{\mathbb{D}} |f'(z)|^s (1-|z|^2)^{s-1} dA(z) < \infty$, that is ,$f \in \mathcal{D}_{s-1}^s$.Thus we have proved that $\cup_{0<P<1} Q_p \subset \cap_{0<s\leq 2} \mathcal{D}_{s-1}^s$. To see that the inclusion is strict, let S be the atomic singular inner function defined by

$$S(z) = exp\left(\frac{z+1}{z-1}\right) \quad , \quad z \in \mathbb{D}. \quad (20)$$

Then Theorem 2.7 of [198] implies that $S \in \cap_{0<s\leq 2} m(\mathcal{D}_{s-1}^s) \subset \cap_{0<s\leq 2} \mathcal{D}_{s-1}^s$ but Theorem 2.2 of [184] shows that $S \notin \cup_{0<P<1} Q_p$ the following simple lemma will be used to prove (ii).

Lemma (5.1.14) [176]:

If $\alpha > -1$ then there exists a constant $C > 0$ (which depends only on α) such that

$$\int_{S(I)} (1-|z|^2)^{\alpha} dA(z) \leq C|I|^{\alpha+2} \quad ,$$

for all intervals $I \subset \mathbb{T}$.

The proof of the lemma is elementary and will be omitted.

Proof of (ii). Suppose that $0 < p < 1$ and $0 < s < 2$ and take $f \in H^{\infty} \cap Q_p$.Let $I \subset \mathbb{T}$.be an interval. Applying $H\ddot{o}lder's$ inequality with the exponents $2/s$ and $\frac{2}{2-s}$, we obtain

$$\int_{S(I)} |f'(z)|^s (1-|z|^2)^{s-1} dA(z)$$

$$\leq \left[\int_{S(I)} |f'(z)|^2 (1-|z|^2)^p dA(z)\right]^{\frac{s}{2}} \left[\int_{S(I)} (1-|z|^2)^{\frac{s(2-p)-2}{2-s}} dA(z)\right]^{\frac{2-s}{2}}. \quad (21)$$

Since $f \in Q_p$, there is a constant C_1 such that

$$\int_{S(I)} |f'(z)|^2 (1-|z|^2)^p dA(z) \leq C_1 |I|^p . \tag{22}$$

|On the other hand, since $\frac{s(2-p)-2}{2-s} > -1$, Lemma (5.1.14)implies that

$$\int_{S(I)} (1-|z|^2)^{\frac{s(2-p)-2}{2-s}} dA(z) \leq C_1 |I|^{\frac{s(2-p)-2}{2-s}} \quad ,$$

which, together with (22) and (23), gives that there is a constant C_2 such that

$$\int_{S(I)} |f'(z)|^s (1-|z|^2)^{s-1} dA(z) \leq C_2 |I|.$$

Thus, the measure $\mu_{f,s}$ is a classical Carleson measure. Since $f \in H^\infty$,using Theorem (5.1.9) we deduce that $\in m(\mathcal{D}_{s-1}^s)$. Hence, we have proved that $Q_p \cap H^\infty \subset m(\mathcal{D}_{s-1}^s)$. As noticed above, if S is the atomic singular inner function defined by (21) then S belongs to $m(\mathcal{D}_{s-1}^s)$ but not to $Q_p \cap H^\infty$. Hence, the inclusion is strict.

Sec(5.2):Hardy and Dirichlet-type spaces:

Let $\mathcal{H}(\mathbb{D})$denote the algebra of all analytic functions in the unit disc $\mathbb{D} = \{z: |z| < 1\}$ of the complex plane C. Let $\mathbb{T}$ be the boundary of $\mathbb{D}$.The Carleson square associated with an interval $I \subset \mathbb{T}$ is the set $S(I) = \{re^{it}: e^{it} \in I , 1-|I| \leq r < 1\}$, where $|E|$denotes the normalized Lebesgue measure of the set $E \subset \mathbb{T}$. For our purposes it is also convenient to define for each $a \in \mathbb{D}\backslash\{0\}$ the interval $I_\alpha = \{e^{i\theta}: \left|\arg (ae^{-i\theta}\right| \leq \pi(1-|a|)$, and denote $S(a) = S(I_\alpha)$. For $0 < p \leq \infty$, the Hardy space H^p consists of the functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} = \lim_{r\to 1} M_p(r,f) < \infty ,$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{1/p} , 0 < p < \infty ,$$

and

$$M_\infty(r,f) = \max_{0\leq\theta\leq 2\pi} \left|f(re^{i\theta})\right| .$$

For the theory of the Hardy spaces, see [211,213].

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the Dirichlet space $\mathcal{D}_\alpha^p$ consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}_\alpha^p}^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) < \infty \quad ,$$

Where$dA(z) = \frac{dxdy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. The purpose of this study is to underline operator theoretic differences between the closely related spaces $\mathcal{D}_{p-1}^p$and H^p. Before going to that, it is appropriate to recall inclusion relations between these spaces. The classical Littlewood –Paley formula implies$\mathcal{D}_1^2 = H^2$. Moreover, it is well known [52,219] that

$$\mathcal{D}_{p-1}^p \subsetneq H^p \ , 0 < p < 2 \ , \tag{23}$$

and

$$H^p \subsetneq \mathcal{D}_{p-1}^p \ , 2 < p < \infty \ . \tag{24}$$

It is also worth mentioning that there are no inclusion relations between $\mathcal{D}_{p-1}^p$ and $\mathcal{D}_{q-1}^q$ when $p \neq q$[216]. A natural way to illustrate differences between two given spaces is to consider classical operators acting on them. For example, if $0 < p < 2$, then the behavior of the composition operator $C_\varphi(f) = f \circ \varphi$reveal that $\mathcal{D}_{p-1}^p$ is in a sense a much smaller space than H^p . Namely, it follows from Littlewood's subordination theorem that $C_\varphi: H^p \to H^p$ is bounded for each $0 < p < \infty$ and all analytic self-maps φ of $\mathbb{D}$, but in contrast to this, there are symbols φ which induce unbounded operators $C_\varphi: \mathcal{D}_{p-1}^p \to \mathcal{D}_{p-1}^p$ when $0 < p < 2$ [211]. As in the case of Hardy spaces, any composition operator $C_\varphi: \mathcal{D}_{p-1}^p \to \mathcal{D}_{p-1}^p$ is bounded when $2 \leq p < \infty$.

There are operators which do not distinguish between $\mathcal{D}_{p-1}^p$ and H^p . For agiven $g \in \mathcal{H}(\mathbb{D})$, the generalized Hilbert operator $\mathcal{H}_g$ is defined by

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t) g'(tz) dt \tag{25}$$

for any $f \in \mathcal{H}(\mathbb{D})$ such that $\int_0^1 |f(t)| dt < \infty$. If $1 < p < \infty$,then $\mathcal{H}_g: \mathcal{D}_{p-1}^p \to \mathcal{D}_{p-1}^p$ is bounded (compact) if and only if $\mathcal{H}_g: H^p \to \mathcal{D}_{p-1}^p$ is bounded (compact) by [213]. Moreover, the same condition, depending on g and p, describes the boundedness (compactness) of the operators $\mathcal{H}_g: \mathcal{D}_{p-1}^p \to H^p$ and $\mathcal{H}_g: H^p \to H^p$ when $1 < p \leq 2$. As far as we know, the problem of characterizing the symbolsgfor which $\mathcal{H}_g: \mathcal{D}_{p-1}^p \to H^p$ and $\mathcal{H}_g: H^p \to H^p$ are bounded when $2 < p < \infty$ remains unsolved.

We shall next study operator theoretic differences between $\mathcal{D}_{p-1}^{p}$andH^p by considering the integral operator

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\,\zeta\ , \qquad z \in \mathbb{D} \tag{26}$$

The bilinear operator $(f, g) \rightarrow \int fg'$ was introduced by Calderón in harmonic analysis in the 60's[208].After his research on commutators of singular integral operators, this bilinear form and its different variations, usually called "paraproducts", have been extensively studied and they have become a funda-mental tool in harmonic analysis. Pommerenke was probably one of the first complex function theorists to consider the operator T_g . He used it in late 70's to study the space $BMOA$, which consists of the functions in the Hardy space H^1that have bounded mean oscillation on the boundary $\mathbb{T}$ [221].The space $BMOA$ can be equipped with several different equivalent norms[213],here we shall use the one given by

$$\|g\|_{BMOA}^2 = \sup_{a\in\mathbb{D}} \frac{\int_{S(a)} |g'(z)|^2 (1-|z|^2) dA(z)}{1-|a|} + |g(0)|^2 .$$

Two decades later, in late 90's, the pioneering works by Aleman and Siskakis [205,206] lead to an abundant research activity on the operator T_g. In particular, the analytic symbols g such that $T_g\colon H^p \rightarrow H^q$ is bounded were characterized by Aleman, Cima and Siskakis [203,204]. Their result in the case $p = q$ says that $T_g\colon H^p \rightarrow H^p$ is bounded if and only if $g \in BMOA$. Our first result shows that whenever $0 < p \leq 2$, the domain space H^pcan be replaced by $\mathcal{D}_{p-1}^{p}$.

Theorem (5.2.1)[203]:

Let $0 < p \leq 2$ and $g \in \mathcal{H}(\mathbb{D})$. Then the following are equivalent:

$(i) T_g\colon \mathcal{D}_{p-1}^{p} \rightarrow H^p$ is bounded;

$(ii) T_g\colon H^p \rightarrow H^p$ is bounded;

$(iii) g \in BMOA$

The implication $(ii) \Rightarrow (i)$ is a direct consequence of (23), so our contribution here consists of showing $(i) \Rightarrow (iii)$. The proof of the implication $(ii) \Rightarrow (iii)$ in[204,205] relies on several powerful properties of $BMOA$ and H^p such as the conformal invariance of $BMOA$. Our proof is based on a circle of ideas developed in [220] and does not rely on these properties. Instead, the Fefferman–Stein formula [223], which states that

$$\|f\|_{H^p}^p \asymp \int_{\mathbb{T}} S_f^p(\zeta)|d\zeta| + |f(0)|^p \quad (27)$$

plays an important role in the reasoning. Here, $|d|\zeta$ denotes the arclength measure on $\mathbb{T}$, and S_f denotes the usual square function, also called the Lusin area function,

$$S_f(\mathcal{C}) = \left(\int_{\Gamma_\sigma(\mathcal{C})} |f'(z)|^2 \, dA(z) \right)^{1/2} \quad , \mathcal{C} \in \mathbb{T} \ ,$$

where$\Gamma_\sigma(\zeta)$ denotes a nontangential approach region (a Stolz angle) with vertex at $\zeta$and of aperture σ. We also show that the statement in Theorem (5.2.1)drastically fails for $p > 2$. In order to give the precise statement, we need to fix the notation. The disc algebra $\mathcal{A}$ is the space of all analytic functions in $\mathbb{D}$ that admit a continuous extension to the closed unit disc$\overline{\mathbb{D}}$. For $0 < \alpha \leq 1$, the Lipschitz space $\Lambda(\alpha)$ consists of the functions $g \in \mathcal{H}(\mathbb{D})$, having a non-tangential limit $g(e^{i\theta})$ almost everywhere on $\mathbb{T}$, such that

$$\sup_{\theta \in [0,2\pi], 0<t<1} \frac{\left|g\left(e^{i(\theta+t)}\right) - g(e^{i\theta})\right|}{t^\alpha} < \infty \ .$$

The "little oh" counterpart of this space is denoted by λ(α). The following chain of strict inclusions is known:

$$\lambda(\alpha) \subsetneq \Lambda(\alpha) \subsetneq \mathcal{A} \subsetneq H^\infty \subsetneq BMOA \subsetneq \mathcal{B}, \quad 0 < \alpha \leq 1 \ .$$

Here, as usual, Bstands for the Bloch space which consists of the functions $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) + |(f(0)| < \infty$

Theorem (5.2.2) [203]:

Let $2 < p < \infty$ and $g \in \mathcal{H}(\mathbb{D})$.

(i) If $T_g: \mathcal{D}_{p-1}^p \to H^p$ is bounded, then $g \in BMOA$.

(ii) There exists $g \in \mathcal{A}$ and $f \in \mathcal{D}_{p-1}^p$such that $T_g(f) \notin H^p$.

Part(ii) shows that $\mathcal{D}_{p-1}^p$ is in a sense a much larger space than H^p When $p > 2$, since we may choose the inducing symbol $g$to be as smooth as admitting a continuous extension to the boundary, but still a suitably chosen $f \in \mathcal{D}_{p-1}^p$ establishes $T_g(f) \notin H^p$. In contrast to this, when the inducing index of the domain space is strictly smaller than the one of the target space, that is $p < q$,then $T_g$does not distinguish between $\mathcal{D}_{p-1}^p$and H^p.

Theorem (5.2.3) [203]:

Let $0 < p < q < \infty$ and $\in \mathcal{H}(\mathbb{D})$.

(a) If $\frac{1}{p}-\frac{1}{q}\leq 1$, then the following are equivalent:

$(i)T_g:\mathcal{D}^p_{p-1}\rightarrow H^q$ is bounded;

$(ii)T_g:H^p\rightarrow H^q$ is bounded;

$(iii)g\in\Lambda\left(\frac{1}{p}-\frac{1}{q}\right)$.

(b) If $\frac{1}{p}-\frac{1}{q}>1$, then $T_g:H^p\rightarrow H^q$ is bounded if and only if g is constant .

Part (a) allows us to deduce a strengthened version of the classical result of Hardy Littlewood which states that a primitive of each function$f\in H^p$, $0<p<1$, belongs to$H^{\frac{p}{1-p}}$.

Proposition (5.2.4) [203]:

Let p, p_1 *and* p_2 be positive numbers such that $p<1<p_2$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. If $f_1\in\mathcal{D}^{p_1}_{p_1-1}f_2\in\mathcal{H}(\mathbb{D})$ such that $f=f_1 .\ f_2$ where $f_1\in\mathcal{H}(\mathbb{D})$ satisfies $|\ f_2(z)|=O\left(\frac{1}{(1-|z|)^{1/\ p_2}}\right)$ then f is the derivative of a function in $H^{\frac{p}{1-p}}$.

The statement in Proposition (5.2.4) with H^1 in place of $\mathcal{D}^{p_1}_{p_1-1}$ was proved by Aleman and Cima[203]. The strict inclusions (23) and (24)show that their result is better when $p_1<2$, which is contrary to the case $p_1>2$. An important ingredient in the proofs of both Theorems 1 and 3is the following result on a Hörmander-type maximal function

$$M(\varphi)(z)=\sup_{I:z\in S(I)}\frac{1}{|I|}\int_I|\varphi(\zeta)|\frac{|d\zeta|}{2\pi}\quad ,\ z\in\mathbb{D}\ ,$$

defined for each $2\pi-$ periodic function $\varphi(e^{i\theta})\in L^1(\mathbb{T})$.

Theorem (5.2.5) [203]:

Let $0<p\leq q<\infty$ and $0<\alpha<\infty$ such that $p\alpha>1$. Let μ be a positive Borel measure on $\mathbb{D}$. Then there exists a positive constant $C>0$ such that

$$\left\|\left[M(f)^{\frac{1}{\alpha}}\right]^{\alpha}\right\|_{L^q(\mu)}\leq C\|f\|_{L^p(\mathbb{T})}\ \text{, for all }\ f\in L^p(\mathbb{T})\,,$$

if and only if $\sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|^{\frac{q}{p}}}<\infty$.

Moreover,

$$\left\|\left[M(.)^{\frac{1}{\alpha}}\right]^{\alpha}\right\|^{q} \overset{\text{def}}{=} \sup_{\|f\|_{L^p(\mathbb{T})}=1} \left\|\left[M(f)^{\frac{1}{\alpha}}\right]^{\alpha}\right\|_{L^q(\mu)}^{q} \asymp \sup_{I\subset\mathbb{T}} \frac{\mu\big(S(I)\big)}{|I|^{\frac{q}{p}}} .$$

This result follows by the well-known works by Carleson[205,210], and hence the measures μ for which $\mu(S(I)) \le C|I|^{\frac{p}{q}}$ are known as $\frac{q}{p}$ −Carleson measures. For further references, see either [212]or the proof of [220] for a similar result. Theorem (5.2.5) has been used to characterize the so called $q-$ Carleson measures for Hardy spaces. Recall that ,for a given Banach space (or a complete metric space) $X$of analytic functions in $\mathbb{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a q −Carleson measure for$X$ if the identity operator $I_d: X \to L^q(\mu)$ is bounded. Nowadays these measures are a standard tool in the operator theory in spaces of analytic functions in $\mathbb{D}$.Let us now turn back to the two remaining cases that are not covered by Theorems (5.2.1) and (5.2.2).They are the ones in which the operator T_g acts from either H^p or $\mathcal{D}_{p-1}^p$ to $\mathcal{D}_{p-1}^p$. It is easy to see that, in terms of the language of the previous paragraph, $T_g: H^p \to \mathcal{D}_{p-1}^p$ is bounded if and only if $\mu_{g,q} = |g'(z)|^q(1-|z|^2)^{q-1}dA(z)$ is a q −Carleson measure for H^p . Therefore, in this case the symbols g that induce bounded operators get characterized by [212,Theorem 9.5] ,when $q \ge p$,and[18] if $q < p$. Analogously, it follows that $T_g: \mathcal{D}_{p-1}^p \to \mathcal{D}_{p-1}^p$ is bounded if and only if $\mu_{g,q}$ is a q −Carleson measure for $\mathcal{D}_{p-1}^p$.

Unfortunately, as far as we know, the existing literature does not offer a characterization of these measures, for the full range of parameter values, in terms of a condition depending on μ only. It is known that they coincide with q −Carleson measures for H^p and can therefore be described by the condition

$$\sup_{I\subset\mathbb{T}} \frac{\mu(S(I))}{|I|^{\frac{q}{p}}} < \infty \ , \tag{28}$$

Provided $q > p$[218, Theorem (5.2.1)(a)]. This statement remains valid also in the diagonal case $q = p$, if $p \le 2$,but fails for $p > 2$[217,222]. In more general terms, the p −Carleson measures for $\mathcal{D}_\alpha^p$ are known excepting the case $\alpha = p-1$ for $p > 2$ [207,222]. This corresponds to the diagonal case $q = p > 2$ which interests us in particular. It is known in this case that μ being a 1 −Carleson measure is a necessary but not a sufficient condition for μ to be a p −Carleson measure for $\mathcal{D}_{p-1}^p$ [217], and that the more restrictive condition

$$\sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|\left(log\frac{e}{|I|}\right)^{-p/2}}<\infty$$

is a sufficient condition for $I_d:\mathcal{D}_{p-1}^p\to L^p(\mu)$ to be bounded [215]. Our next result shows that this best known sufficient condition can be relaxed by one logarithmic factor.

Theorem (5.2.6) [203]:

Let $2<p<\infty$, and let μ be a positive Borel measure on $\mathbb{D}$.If

$$\sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|\left(log\frac{e}{|I|}\right)^{-\frac{p}{2+1}}}<\infty\ ,$$

then μ is a p −Carleson measure for $\mathcal{D}_{p-1}^p$.

We shall see in Proposition (5.2.15) that the statement in Theorem (5.2.6) is sharp in a very strong sense. The remaining part of the section is organized as follows. We state and prove some preliminary results. Theorems (5.2.1) and (5.2.3) and their expected analogues for compact operators as well as Proposition (5.2.4) are proved , we shall deal with the growth of integral means of functions $f\in\mathcal{D}_{p-1}^p$, $p>2$, and we shall prove Theorem (5.2.2).

Before proceeding further, a word about notation to be used. We shall write $\|T\|_{(X,Y)}$ for the norm of an operator $T:X\to Y$, and if no confusion arises with regards to X and Y , we shall simply write $\|T\|$. Moreover, for two real-valued functions E_1,E_2 we write$E_1\asymp E_2$ or $E_1\lesssim E_2$, if there exists a positive constantk, independent of the argument, such that $\frac{1}{k}E_1\le E_2\le kE_1$ or$E_1\le kE_2$, respectively.

We begin with a straightforward but useful estimate that will be used in proofs of Theorems 5.2.1 and 52.3.

Lemma (5.2.7) [203]:

Let $0<q,p<\infty$ and $g\in\mathcal{H}(\mathbb{D})$ .If $T_g:\mathcal{D}_{p-1}^p\to H^q$ is bounded , then

$$M_\infty(r,g')\lesssim\frac{\|T_g\|_{\mathcal{D}_{p-1}^p,H^q}}{(1-r)^{1-\frac{1}{p}+\frac{1}{q}}}\ ,\qquad 0\le r<1\ .\tag{29}$$

Proof:

The functions

$$F_{a,p,\gamma}(z)=\left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\frac{1+\gamma}{p}} \quad ,0<\gamma<\infty \qquad a\in\mathbb{D}\ ,$$

Satiafy

$$\left|F_{a,p,\gamma}(z)\right|\asymp 1 \quad , \qquad z\in S(a) \tag{30}$$

and a calculation shows that

$$\left\|F_{a,p,\gamma}(z)\right\|_{\mathcal{D}_{p-1}^p}^p\asymp 1-|a| \qquad , \ a\in\mathbb{D}$$

Since $T_g:\mathcal{D}_{p-1}^p\to H^q$ is bounded by the assumption, the well known relations$M_\infty(r,f)\lesssim M_q\left(\frac{1+r}{2},f\right)(1-r)^{-\frac{1}{q}}$ and $M_q(r,f')\lesssim M_q\left(\frac{1+r}{2},f\right)(1-r)^{-1}$, valid for all $g\in\mathcal{H}(\mathbb{D})$ (see[9]), yield

$$|g'(a)|=\left|(T_g(F_{a,p,\gamma}(z))'(a)\right|\lesssim\frac{M_q\left(\frac{1+|a|}{2},(T_g(F_{a,p,\gamma}(z))'\right)}{(1-|a|)^{\frac{1}{q}}}$$

$$\lesssim\frac{M_q\left(\frac{3+|a|}{4},(T_g(F_{a,p,\gamma}(z))\right)}{(1-|a|)^{1+\frac{1}{q}}}\lesssim\frac{\left\|(T_g(F_{a,p,\gamma})\right\|_{H^q}}{(1-|a|)^{1+\frac{1}{q}}}$$

$$\lesssim\frac{\left\|T_g\right\|_{\mathcal{D}_{p-1}^p,H^q},\left\|F_{a,p,\gamma}\right\|_{\mathcal{D}_{p-1}^p}}{(1-|a|)^{1+\frac{1}{q}}}\lesssim\frac{\left\|T_g\right\|_{\mathcal{D}_{p-1}^p,H^q}}{(1-|a|)^{1+\frac{1}{q}-\frac{1}{p}}} \quad ,a\in\mathbb{D}\ ,$$

and the assertion follows . We next recall some suitable reformulations of Lipschitz spaces $\Lambda(\alpha)$ [212].

Lemma (5.2.8) [203]:

Let $0<\alpha\le 1$ and $g\in\mathcal{H}(\mathbb{D})$. Then the following are equivalent:

$(i)g\in\Lambda(\alpha)$;

$$(ii)M_\infty(r,g')=0\left(\frac{1}{(1-r)^{1-\alpha}}\right) \quad , \qquad r\to 1^- \quad ;$$

(iii) The measure $d\mu_g(z)=|g'(z)|^2(1-|z|^2)dA(z)$ satisfies the condition

$$\sup_{I\subset\mathbb{T}}\frac{\mu(S(I))}{|I|^{2\alpha+1}}<\infty \quad .$$

We shall also need the following result[218, Theorem (5.2.1) (i)].

Theorem (5.2.9) [203]:

Let $0 < p < q < \infty$ and μ be a positive Borel measure on $\mathbb{D}$.Then μ is a q −Carleson measure for $\mathcal{D}^p_{p-1}$ if and only if μ is a $\frac{p}{q}$ −Carleson measure. Moreover,

$$\left\| I_{d(\mathcal{D}^p_{p-1}, L^q(\mu))} \right\|^q \asymp \sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|^{\frac{q}{p}}}$$

Proof of Theorem (5.2.1):

It is known that $T_g: H^p \to H^p$ is bounded if and only if $g \in BMOA$[204], and therefore (ii) and (iii) are equivalent. Moreover, since $\mathcal{D}^p_{p-1} \subset H^p$ for $0 < p \leq 2$, (ii) implies (i). To complete the proof we shall show that $g \in BMOA$, whenever $T_g: \mathcal{D}^p_{p-1} \to H^p$ is bounded. To see this, note first that $\|g\|_{\mathcal{B}} = \|T_g\|_{(\mathcal{D}^p_{p-1}, H^p)}$ by Lemma 6, and thus $g \in \mathcal{B}$.Let now $1 < \alpha, \beta < \infty$ such that $\beta/\alpha = p/2 < 1$, and let α' and β' be the conjugate indices of α and β, respectively. Assume for a moment that is g' continuous on $\overline{\mathbb{D}}$. Then (30), Fubini's theorem and Hölder's inequality yield

$$\int_{S(a)} |g'(z)|^2 (1 - |z|^2) dA(z) \asymp \int_{\mathbb{T}} \left(\int_{S(a) \cap \Gamma_\sigma(\zeta)} |g'(z)|^2 \left| F_{a,p,\gamma}(z) \right|^2 dA(z) \right)^{\frac{1}{\alpha} + \frac{1}{\alpha'}} |d\zeta|$$

$$\leq \left(\int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta)} |g'(z)|^2 \left| F_{a,p,\gamma}(z) \right|^2 dA(z) \right)^{\frac{\beta}{\alpha}} |d\zeta| \right)^{\beta}$$

$$\cdot \left(\int_{\mathbb{T}} \left(\int_{\Gamma_\sigma(\zeta) \cap S(a)} |g'(z)|^2 dA(z) \right)^{\frac{\beta'}{\alpha'}} |d\zeta| \right)^{1/\beta'}$$

$$\asymp \left\| (T_g(F_{a,p,\gamma}) \right\|_{H^p}^{\frac{p}{\beta}} \left\| S_g(\mathcal{X}S(a) \right\|_{L^{\frac{\beta'}{\alpha'}}(\mathbb{T})}^{\frac{1}{\alpha'}} \quad , a \in \mathbb{D} \qquad (31)$$

where

$$S_g(\varphi)(\zeta) = \int_{\Gamma_\sigma(\zeta)} |\varphi(z)|^2 |g'(z)|^2 dA(z) \quad , \quad \zeta \in \mathbb{T} \ ,$$

for any bounded function φ in $\mathbb{D}$.Now $\left(\frac{\beta'}{\alpha'}\right)' = \frac{\beta(\alpha-1)}{\alpha-\beta} > 1$, and hence by duality

$$\left\|S_g(\mathcal{X}S(a)\right\|_{L^{\frac{\beta'}{\alpha'}}(\mathbb{T})} = sup\left|\int_{\mathbb{T}} h(\zeta)(\mathcal{X}S(a))(\zeta)|d\zeta|\right| \tag{32}$$

where the supremum is taken on all $h$such that $\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}(\mathbb{T})} \leq 1$. To estimate the right hand side, we shall write $I(z)$for the $arc\{\zeta \in T: z \in \Gamma_\sigma(\zeta)\}$ with $|I(z)| \asymp 1-|z|$. Then Fubini's theorem, Hölder's inequality and Theorem (5.2.5) yield

$$\int_{\mathbb{T}} h(\zeta)(\mathcal{X}S(a))(\zeta)|d\zeta| \leq \int_{\mathbb{T}} h(\zeta) \int_{S(a)\cap\Gamma_\sigma(\zeta)} |g'(z)|^2 dA(z)\,|d\zeta|$$

$$\asymp \int_{S(a)} |g'(z)|^2(1-|z|^2)\left(\frac{1}{1-|z|^2}\int_{I(z)} h(\zeta)|d\zeta|\right)dA(z)$$

$$\lesssim \int_{S(a)} |g'(z)|^2(1-|z|^2)M|h|(z)\,dA(z)$$

$$\leq \left(\int_{S(a)} |g'(z)|^2(1-|z|^2)\,dA(z)\right)^{\frac{\alpha\prime}{\beta\prime}}$$

$$.\left(\int_{\mathbb{D}} M|h|^{\left(\frac{\beta\prime}{\alpha\prime}\right)'}|g'(z)|^2(1-|z|^2)\,dA(z)\right)^{1-\frac{\alpha\prime}{\beta\prime}}$$

$$\leq \left(\int_{S(a)} |g'(z)|^2(1-|z|^2)\,dA(z)\right)^{\frac{\alpha\prime}{\beta\prime}}$$

$$.\left(\sup_{b\in\mathbb{D}} \frac{\int_{S(b)}|g'(z)|^2\,(1-|z|^2)dA(z)}{1-|b|}\right)^{1-\frac{\alpha\prime}{\beta\prime}} \|h\|_{L^{\left(\frac{\beta'}{\alpha'}\right)'}(\mathbb{T})} \tag{33}$$

Since any dilated function $g_r(z) = g(rz)$, $0 < r < 1$, is analytic in $D\left(0,\frac{1}{r}\right)$, we deduce by replacing g by g_r in (30)–(33) that

$$\int_{S(a)} |g'_r(z)|^2(1-|z|^2)dA(z) \lesssim \left\|(T_{g_r}(F_{a,p,\gamma})\right\|_{H^p}^{\frac{p}{\beta}}\left(\int_{S(a)} |g'_r(z)|^2(1-|z|^2)\,dA(z)\right)^{\frac{1}{\beta\prime}}$$

$$.\left(\sup_{b\in\mathbb{D}} \frac{\int_{S(b)}|g'_r(z)|^2\,(1-|z|^2)dA(z)}{1-|b|}\right)^{\frac{1}{\alpha\prime}\left(1-\frac{\alpha\prime}{\beta\prime}\right)} \tag{34}$$

We claim that there exists $\gamma > 0$ and a constant $C = C(p,\gamma) > 0$ such that

$$\sup_{0<r<1} \left\|(T_{g_r}(F_{a,p,\gamma})\right\|_{H^p}^p \leq C\left\|T_g\right\|_{(\mathcal{D}_{p-1}^p,H^p)}(1-|a|)\,,\ \ a \in \mathbb{D} \tag{35}$$

the proof of which is postponed for a moment. Now this combined with (34)and Fatou's lemma yield

$$\sup_{a\in\mathbb{D}} \frac{\int_{S(a)}|g'(z)|^2\,(1-|z|^2)dA(z)}{1-|a|} \lesssim \left\|T_g\right\|_{(\mathcal{D}_{p-1}^p,\,H^p)}^2\ \ ,$$

and so $g \in BMOA$.

It remains to prove (35).To see this fix $\gamma > p$. Recall that

$$\left\|(T_{g_r}(F_{a,p,\gamma})\right\|_{H^p}^p \asymp \int_{\mathbb{D}} \left(\int_{\Gamma_\sigma(\zeta)} r^2|g'(rz)|^2\left|F_{a,p,\gamma}(z)\right|^2 dA(z) \right)^{\frac{p}{2}} |d\zeta|.$$

If $|a| < \frac{1}{2}$, then

$$\left\|(T_{g_r}(F_{a,p,\gamma})\right\|_{H^p}^p \lesssim (1-|a|)^{\gamma+1} \int_{\mathbb{D}} \left(\int_{\Gamma_\sigma(\zeta)} r^2|g'(rz)|^2\left|F_{a,p,\gamma}(z)\right|^2 dA(z) \right)^{\frac{p}{2}} |d\zeta|$$

$$\asymp (1-|a|)^{\gamma+1}\|g_r - g(0)\|_{H^p}^p \leq (1-|a|)\|g - g(0)\|_{H^p}^p$$

$$= (1-|a|)\left\|(T_g(1)\right\|_{H^p}^p \lesssim (1-|a|)\left\|(T_g\right\|_{(\mathcal{D}_{p-1}^p,\,H^p)}^p\ .$$

Let now $\frac{1}{2} \leq |a| < \frac{1}{2-r}$ .Then$|1-\bar{a}rz|\ \leq 2|1-\bar{a}z|$ for all $z \in \mathbb{D}$, and hence

$$\left\|(T_{g_r}(F_{a,p,\gamma})\right\|_{H^p}^p \lesssim \left\|((T_{g_r}(F_{a,p,\gamma}))_r\right\|_{H^p}^p \leq \left\|(T_g(F_{a,p,\gamma})\right\|_{H^p}^p$$

$$\leq \left\|(T_g\right\|_{(\mathcal{D}_{p-1}^p,\,H^p)}^p \left\|F_{a,p,\gamma}\right\|_{\mathcal{D}_{p-1}^p}^p \asymp \left\|(T_g\right\|_{(\mathcal{D}_{p-1}^p,\,H^p)}^p(1-|a|)$$

In the remaining case $\frac{1}{2-\mathrm{r}} \leq |\mathrm{a}| < 1$ we have $\mathrm{r} \leq 2-\frac{1}{|\mathrm{a}|} \leq |\mathrm{a}|$.Now $\gamma > p$, and hence

$$\left\|(T_{g_r}(F_a)\right\|_{H^p}^p \lesssim M_\infty^p(r,g')(1-|a|)^{\gamma+1} \int_{\mathbb{D}} \left(\int_{\Gamma_\sigma(\zeta)} \frac{dA(z)}{|1-\bar{a}z|^{\frac{2(\gamma+1)}{p}}} \right)^{\frac{p}{2}} |d\zeta|$$

$$\lesssim M_\infty^p(r,g')(1-|a|)^{\gamma+1} \left\| \frac{1}{|1-\bar{a}z|^{\frac{2(\gamma+1)}{p}}} \right\|_{H^p}^p$$

$$\asymp (M_\infty(|a|,g')(1-|a|))^p(1-|a|) \leq \|g\|_\beta^p(1-|a|)$$

$$\lesssim \|T_g\|^p_{(\mathcal{D}^p_{p-1}, H^p)}(1-|a|) \ .$$

By combining these three separate cases, we deduce (35).

Next, we shall proveTheorem3by using similar ideas to those employed in the proof of Theorem (5.2.1).

Proof of Theorem (5.2.3):

It is known that (ii) and (iii) are equivalent [204]. Further, Lemma (5.2.7)and Lemma Bgive $(i) \Rightarrow (iii)$ and (b). Moreover, if $0 < p \le 2$, then $\mathcal{D}^p_{p-1} \subset H^p$ and hence, in this case, (ii) implies (i). To complete the proof, we show that (iii) implies (i) when $2 < p < \infty$. Since $q > 2$, $L^{\frac{q}{2}}(\mathbb{T})$ canbeidentified with the dual of $L^{\frac{q}{q-2}}(\mathbb{T})$, that is ,$L^{\frac{q}{2}}(\mathbb{T}) \simeq \left(L^{\frac{q}{q-2}}(\mathbb{T})\right)^{\star}$. Therefore, $T_g: \mathcal{D}^p_{p-1} \to H^q$ is bounded if and only if

$$\left|\int_{\mathbb{T}} h(\zeta)\left(\int_{\Gamma_\sigma(\zeta)} |f(z)|^2|g'(z)|^2 dA(z)\right)|d\zeta|\right| \lesssim \|h\|_{L^{\frac{q}{q-2}}(\mathbb{T})}\|T_g\|^2_{\mathcal{D}^p_{p-1}}$$

for all $h \in L^{\frac{q}{q-2}}(\mathbb{T})$ and $f \in \mathcal{D}^p_{p-1}$. To see this, we use first Fubini's theorem to obtain

$$\left|\int_{\mathbb{T}} h(\zeta)\left(\int_{\Gamma_\sigma(\zeta)} |f(z)|^2|g'(z)|^2 dA(z)\right)|d\zeta|\right| \le \int_{\mathbb{D}} |f(z)|^2|g'(z)|^2\left(\int_{\mathrm{I}(z)} h(\zeta)|d\zeta|\right)dA(z)$$

$$\lesssim \int_{\mathbb{D}} |f(z)|^2 M(|h|)(z)|g'(z)|^2(1-|z|^2)\, dA(z)$$

Next, we estimate the last integral upwards by Hölder's inequality with exponent $x = 1 + p\left(\frac{1}{2} - \frac{1}{q}\right)$ and it's conjugate $x = 1 + \frac{2q}{p(q-2)}$,

$$\left(\int_{\mathbb{D}} |f(z)|^{2+p-\frac{2p}{q}}|g'(z)|^2(1-|z|^2)\, dA(z)\right)^{\frac{2q}{(2+p)q-2p}}$$

$$.\left(\int_{\mathbb{D}} |f(z)|^2(M(|h|)(z))^{1+\frac{2q}{p(q-2)}}|g'(z)|^2(1-|z|^2)\, dA(z)\right)^{\frac{1}{1+\frac{2q}{p(q-2)}}}$$

Since $|f(z)|^2(1-|z|^2)dA(z)$ is $2\left(\left(\frac{1}{p} - \frac{1}{q}\right) + 1\right)$ −Carleson measure by Lemma (5.2.8),

$\frac{\left(2+p-\frac{2p}{q}\right)}{p} = 2\left(\frac{1}{p}-\frac{1}{q}\right)+1$ and $\frac{\frac{1}{1+\frac{2q}{p(q-2)}}}{\frac{q}{(q-2)}} = 2\left(\frac{1}{p}-\frac{1}{q}\right)+1$ by using Theorem (5.2.9) and Theorem (5.2.5), we get

$$\left|\int_{\mathbb{T}} h(\zeta)\left(\int_{\Gamma_\sigma(\zeta)} |f(z)|^2|g'(z)|^2 dA(z)\right)|d\zeta|\right| \lesssim \|f\|^2_{\mathcal{D}^p_{p-1}} \|h\|_{L^{\frac{q}{q-2}}(\mathbb{T})},$$

and thus $T_g\colon \mathcal{D}^p_{p-1} \to H^q$ is bounded .

Proof of Proposition (5.2.4):

Let F_2 be such that $F'_2 = f_2$ then $F'_2(z) = O\left(\frac{1}{(1-|z|)^{\frac{1}{p_2}}}\right)$ by the assumption, and hence $F_2 \in \Lambda\left(1-\frac{1}{p_2}\right)$by Lemma (5.2.8). Now Theorem (5.2.3) implies that the integral operator $T_{F_2}\colon \mathcal{D}^{p_1}_{p_1-1} \to H^{\frac{p}{1-p}}$ is bounded, and since $f_1 \in \mathcal{D}^{p_1}_{p_1-1}$ by the assumption, we deduce $T_{F_2}(f_1)(z) = \int_0^z F'_2(\zeta)d\zeta = \int_0^z f(\zeta)d\zeta \in H^{\frac{p}{1-p}}$,which give the assertion

We finish this section by proving the expected versions of Theorems 5.2.1and 5.2.3for compact operators. The next auxiliary result is standard, and therefore its proof is omitted.

Lemma (5.2.10) [203]:

Let $0 < p,q < \infty$ and $g \in \mathcal{H}(\mathbb{D})$. Then the following are equivalent:

(i) $T_g\colon \mathcal{D}^p_{p-1} \to H^q$ is compact;

(ii) For any sequence of analytic functions $\{f_n\}_{n=1}^\infty$ in $\mathbb{D}$that converges uniformly to 0 on compact subsets of $\mathbb{D}$ and satisfies $\sup_{n\in N}\|f_n\|_{\mathcal{D}^p_{p-1}} < \infty$, we have

$$\lim_{n\to\infty} \left\| T_g f_n\right\|_{H^q} = 0.$$

Obviously the statement in this lemma remains valid if $\mathcal{D}^p_{p-1}$ is replaced by H^q . The space $VMOA$ consists of the functions in the Hardy space H^1that have vanishing mean oscillationon the boundary $\mathbb{T}$. It is known that this space is the closure of polynomials in $BMOA$ and is characterized by the condition

$$\sup_{a\in 1} \frac{\int_{S(a)}|g'(z)|^2 (1-|z|^2)dA(z)}{1-|a|} = 0 .$$

Theorem (5.2.11) [203]:

Let $0 < p \leq 2$ and $g \in \mathcal{H}(\mathbb{D})$. Then the following are equivalent:

$(i) T_g : \mathcal{D}^p_{p-1} \to H^p$ is compact;

$(ii) T_g : H^p \to H^p$ is compact;

$(iii) g \in VMOA$.

Proof:

It is known that (ii) and (iii) are equivalent by [204]. Moreover, by bearing in mind Lemma (5.2.10) and (23), we see that (ii) implies (i). It remains to show that $g \in VMOA$, whenever $T_g : \mathcal{D}^p_{p-1} \to H^p$ is compact. Since the proof of this implication is similar to its counterpart in the proof of Theorem (5.2.1),weonly show in detail those steps that are significantly different. First observe, that $g \in BMOA$ by Theorem (5.2.1). Let $f_{a,p,\gamma} = \frac{F_{a,p,\gamma}}{(1-|a|)^{1/p}}$, where $\gamma > 0$ and$F_{a,p,\gamma}$ are the functions defined in the proof of Lemma (5.2.7).It is clear that $\|f_{a,p,\gamma}\|_{\mathcal{D}^p_{p-1}} \asymp 1$ and $f_{a,p} \to 0$ as $|a| \to 1^-$, uniformly in compact subsets of $\mathbb{D}$. Therefore $\|(T_g(f_{a,p,\gamma})\|_{H^p} \to 0$ as $|a| \to 1^-$, by Lemma (5.2.10). Now,let $1 < \alpha, \beta < \infty$ such that $\frac{\beta}{\alpha} = \frac{p}{2} < 1$. Arguing as in (31), we deduce

$$\frac{1}{(1-|a|)^{2/p}} \int_{S(a)} |g'(z)|^2 (1-|z|^2) dA(z) \lesssim \|(T_g(f_{a,p,\gamma})\|_{H^p}^{\frac{p}{\beta}} \|S_g(\mathcal{X}S(a)\|_{L^{\frac{\beta'}{\alpha'}}(\mathbb{T})}^{\frac{1}{\alpha'}} \text{ for all } a \in \mathbb{D}.$$

Following the reasoning in the proof of Theorem (5.2.1) and bearing in mind that $g \in BMOA$, we obtain

$$\frac{\int_{S(a)} |g'(z)|^2 (1-|z|^2) dA(z)}{(1-|a|)^{2/p}} \lesssim \|(T_g(f_{a,p,\gamma})\|_{H^p}^{\frac{p}{\beta}} \frac{\left(\int_{S(a)} |g'(z)|^2 (1-|z|^2) dA(z)\right)^{\frac{\beta'}{\alpha'} \cdot \frac{1}{\alpha'}}}{(1-|a|)^{\frac{2}{p} \cdot \frac{1}{\alpha'}}}$$

which is equivalent to

$$\frac{\int_{S(a)} |g'(z)|^2 (1-|z|^2) dA(z)}{(1-|a|)} \lesssim \|(T_g(f_{a,p,\gamma})\|_{H^p}^{p} \quad .$$

Therefore $g \in VMOA$.

It is known that the "little oh" analogue of Lemma (5.2.8) is valid. This together with appropriate modifications in the proofs of Lemma (5.2.7) and Theorem (5.2.3) give the next result.

Theorem (5.2.12) [203]:

Let $0 < p < q < \infty$, and $\frac{1}{p} - \frac{1}{q} \leq 1$ $g \in \mathcal{H}(\mathbb{D})$. the following are equivalent:

$(i)T_g: \mathcal{D}^p_{p-1} \to H^q$ is compact;

$(ii)T_g: H^p \to H^q$ is compact;

$(iii)g \in \Lambda\left(\frac{1}{p}-\frac{1}{q}\right)$.

In this section we shall prove sharp estimates for the growth of $M_p(r,f)$ when $f \in \mathcal{D}^p_{p-1}$ and $2 < p < \infty$. If $f \in \mathcal{D}^p_{p-1}$ and $0 < p < 2$, then $M_p(r,f)$ is uniformly bounded due to(23).

Lemma (5.2.13) [203]:

Let $2 < p < \infty$ and $\Phi: [0,1) \to (1,\infty)$ be a differentiable increasing unbounded function such that $\frac{\Phi'(r)}{\Phi(r)}(1-r)$ is decreasing. Then the following hold:

$(i)M_p(r,f) = O\left(\left(\log\frac{e}{1-r}\right)^{\frac{1}{2}-\frac{1}{p}}\right)$, as $r \to 1^-$, for all $f \in \mathcal{D}^p_{p-1}$;

(ii) there exists $f \in \mathcal{D}^p_{p-1}$ such that

$$M_p(r,f) \gtrsim \left(\log\frac{e}{1-r}\right)^{\frac{1}{2}}\left(\frac{\Phi'(r)}{\Phi(r)}(1-r)\right)^{\frac{1}{p}} \quad ,0 < r < 1 \qquad (36)$$

for any fixed $0 < q < \infty$.

Part(i) is essentially known, but we include a proof for the sake of completeness . Part (ii), apart from showing that (i) is sharp in a very strong sense, will be used to proveTheorem (5.2.2)(ii) and the sharpness of Theorem (5.2.6). It is also worth noticing that each function

$$\Phi_{N,\alpha}(r) = \left(\log_N\frac{\exp N^2}{1-r}\right)^{\alpha}, N \in \boldsymbol{N} = \{\mathbf{1},\mathbf{2},\mathbf{3},\ldots\}, 0 < \alpha < \infty, \qquad (37)$$

Satisfies both hypotheses on the auxiliary function Φ in Lemma (5.2.13). Here, as usual, $\log_n x = \log(\log_{n-1} x)$, $\log_1 x = \log x$, $\exp_n x = \log(\log_{n-1} x)$ and $\log_1 x = e^x$. We remark that $\log_N 2$ is a normalization factor and the key point is the extremely slow growth of the iterated logarithm.

Proof of Lemma (5.2.13):

(i) First observe that[215, Theorem 1.4] yields

$$\mathcal{D}^p_{p-1} \subset A^p_{v_{\frac{p}{2}}}, \|f\|^p_{\mathcal{D}^p_{p-1}} \gtrsim \|f\|^p_{v_{\frac{p}{2}}}, f \in \mathcal{H}(\mathbb{D}) \qquad (38)$$

Where $A^p_{v_{\frac{p}{2}}}$ denotes the weighted Bergman space induced by the rapidly increasing weight $\frac{p}{2} = (1-|z|)^{-1}\left(\log\frac{e}{1-|z|}\right)^{-\frac{p}{2}}$, $z \in \mathbb{D}$, see[220]. Therefore,

$$\|f\|^p_{\mathcal{D}^p_{p-1}} \gtrsim \|f\|^p_{v_{\frac{p}{2}}} \geq \int_r^1 sM^p_p(s,f)v\frac{p}{2}(s)ds \geq M^p_p(r,f)\int_r^1 s\,v\frac{p}{2}(s)ds$$

$$\asymp M^p_p(r,f)\left(\log\frac{e}{1-r}\right)^{1-\frac{p}{2}}, 0 < r < 1\,,$$

and (i) follows. (ii) Let Φ be as in the lemma. Consider the lacunary series

$$f(z) = \sum_{k=1}^{\infty}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)^{\frac{1}{p}} z^{2^k} \quad , \quad r_k = 1-2^{-k}, k \in \boldsymbol{N} \tag{39}$$

Where $h(r) = log\Phi(r)$ is a positive function such that $h'(r)(1-r)$ is decreasing by the assumptions. By [217,Proposition 3.2],

$$\|f\|^p_{\mathcal{D}^p_{p-1}} \lesssim \sum_{k=1}^{\infty}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)$$

$$= \sum_{k=1}^{\infty}\frac{\int_{r_{k-1}}^{r_k} h'(t)\,dt}{\Phi(r_k)} \leq \int_0^1 \frac{h'(t)}{\Phi(t)}dt = \Phi(0)^{-1} < 1\,,$$

and thus $f \in \mathcal{D}^p_{p-1}$.

On the other hand,

$$M^2_2(r_N,f) = \sum_{k=1}^{\infty}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)^{\frac{2}{p}} r_N^{2^{k+1}}$$

$$\geq \sum_{k=1}^{N}\left(\frac{h(r_k)-h(r_{k-1})}{\Phi(r_k)}\right)^{\frac{2}{p}} r_N^{2^{k+1}}$$

$$\geq \frac{r_N^{2^{N+1}}}{(\Phi(r_N))^{\frac{2}{p}}}\sum_{k=1}^{N}\left(\int_{r_{k-1}}^{r_k} h'(s)(1-s)\frac{ds}{1-s}\right)^{\frac{2}{p}}$$

$$\geq \frac{r_N^{2^{N+1}}(\log 2)^{\frac{2}{p}}}{(\Phi(r_N))^{\frac{2}{p}}}\sum_{k=1}^{N}\big(h'(r_k)-h(r_{k-1})\big)^{\frac{2}{p}}$$

$$\gtrsim \frac{1}{(\Phi(r_N))^{\frac{2}{p}}}(h'(r_N)(1-r_N))^{\frac{2}{p}}N.$$

Let $r \in \left[\frac{1}{2}, 1\right)$ be given, and choose $N \in \boldsymbol{N}$ such that $r_N \leq r < r_{N+1}$. Then [222] yields

$$M_q^2(r,f) \asymp M_2^2(r,f) \geq M_2^2(r_N,f) \gtrsim \frac{1}{(\Phi(r_N))^{\frac{2}{p}}}(h'(r_N)(1-r_N))^{\frac{2}{p}}N$$

$$\gtrsim \frac{1}{(\Phi(r))^{\frac{2}{p}}}(h'(r)(1-r))^{\frac{2}{p}}\log\frac{e}{1-r}$$

$$\asymp \left(\log\frac{e}{1-r}\right)\left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{2}{p}},$$

which finishes the proof. With these preparations we are ready to prove Theorem 5.2.2.

Proof of Theorem (5.2.2) [203]:

(i) If $T_g: \mathcal{D}_{p-1}^p \to H^p$ is bounded, then $T_g: H^p \to H^p$ is bounded because $H^p \subsetneq \mathcal{D}_{p-1}^p$ for $2 < p < \infty$ by (1.2), and hence $g \in BMOA$.

(ii) In this part we use ideas from the proof of [217, Theorem 2.1]. Take a function Φ as in Lemma (5.2.13) and let $f \in \mathcal{D}_{p-1}^p$ be the lacunary series associated with Φ via (39). By using [223], we find two constants $A > 0$ and $B > 0$ such that for every $r \in (0,1)$ the set

$$E_r = \{t \in [0,2\pi]: \left|f\left(re^{it}\right)\right| > BM_2(r,f)\} \tag{40}$$

has the Lebesgue measure greater than or equal to A. Let now g be a lacunary series . By using [223] we find a constant $C_1 > 0$ such that

$$\int_{E_r} \left|g'\left(re^{it}\right)\right|^2 dt \geq C_1 A M_2^2(r,g') = C_2 M_2^2(r,g'), 0 < r < 1, \tag{41}$$

where $C_2 = C_1 A$. Bearing in mind the definition (40) of the sets E_r and using (41), we obtain

$$\left\|T_g(f)\right\|_{H^p}^2 \geq \left\|T_g(f)\right\|_2^2 \geq \int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 |1-|z|^2| dA(z)$$

$$\geq \int_0^1 r(1-r) \int_{E_r} \left|f\left(re^{it}\right)\right|^2 \left|g'\left(re^{it}\right)\right|^2 dt dr$$

$$\geq B^2 \int_0^1 r(1-r) M_2^2(r,f) \int_{E_r} \left|g'\left(re^{it}\right)\right|^2 dt dr$$

$$\geq B^2 C_2 \int_0^1 r(1-r) M_2^2(r,f) M_2^2(r,g')dr$$

$$\geq B^2 C_2 C \int_0^1 r(1-r)\left(\log\frac{e}{1-r}\right)\left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{2}{p}} M_2^2(r,g')dr \quad (42)$$

Choose now $(r) = \left(\log\frac{e}{1-r}\right)^{\varepsilon}$, where $0 < \varepsilon < \frac{p}{2} - 1$, so that

$$\left(\log\frac{e}{1-r}\right)\left(\frac{\Phi'(r)}{\Phi^2(r)}(1-r)\right)^{\frac{2}{p}} \asymp \left(\log\frac{e}{1-r}\right)^{1-\frac{2}{p}(1+\varepsilon)} .$$

Further, let

$$g(z) = \sum_{j=1}^{\infty} \frac{1}{(j+1)(\log j+1)^{\alpha}} z^{2^{2^j}} , 1 < \alpha < \infty .$$

Then, clearly, $g \in \mathcal{A}$. Moreover, since $\omega(r) = (1-r)\left(\log\frac{e}{1-r}\right)^{1-\frac{2}{p}(1+\varepsilon)}$ is a so-called regular weight [220], we deduce

$$\int_0^1 r^{2n+1}\,\omega(r)dr \asymp n^{-1}\omega(1-n^{-1}) \quad , \quad n \in \boldsymbol{N} \quad ,$$

By [220, Lemma 1.3 and (23)]. This together with (43) yield

$$\|T_g(f)\|_{H^p}^2 \gtrsim \int_0^1 r(1-r)\left(\log\frac{e}{1-r}\right)^{1-\frac{2}{p}(1+\varepsilon)} M_2^2(r,g')dr$$

$$\asymp \sum_{j=1}^{\infty} \frac{2^{2^{j+1}}}{(j+1)^2(\log j+1)^{2\alpha}}\left(\int_0^1 r^{2^{2^{j+1}}-1}(1-r)\left(\log\frac{e}{1-r}\right)^{1-\frac{2}{p}(1+\varepsilon)} dr\right)$$

$$\asymp \sum_{j=1}^{\infty} \frac{2^{(j+1)\left(1-\frac{2}{p}(1+\varepsilon)\right)}}{(j+1)^2(\log j+1)^{2\alpha}} = \infty ,$$

and finishes the proof.

The statement in Theorem (5.2.6)follows directly by (38) and [220, Theorem 2.1] with $\omega = \frac{v_p}{2}$. Next, weshow that this result is sharp in a very strong sense. For this purpose, the following lemma is needed.

Lemma (5.2.14) [203]:

Let $2 < p < \infty$, and let $\Phi: [0,1) \to (0, \infty)$ be a differentiable increasing function such that

$$\frac{\Phi(r)}{\left(\log \frac{e}{1-r}\right)^{\frac{p}{2}-1}} \to 0 \ , r \to 1^- \ , \tag{43}$$

And

$$m = \lim_{r \to 1^-} inf \frac{\Phi'(r)}{\Phi(r)} (1-r) \log \frac{e}{1-r} > 1 - \frac{p}{2} \tag{44}$$

Then

$$\int_r^1 \frac{\Phi(s) ds}{(1-s)\left(\log \frac{e}{1-r}\right)^{\frac{p}{2}}} \lesssim \frac{\Phi(r)}{\left(\log \frac{e}{1-r}\right)^{\frac{p}{2}-1}} \quad , \ r \in (0,1)$$

Proof:

By the Bernoulli–l'Hôpital theorem,

$$\lim_{r \to 1^-} \sup \frac{\int_r^1 \frac{\Phi(s) ds}{(1-s)\left(\log \frac{e}{1-r}\right)^{\frac{p}{2}}}}{\frac{\Phi(r)}{\left(\log \frac{e}{1-r}\right)^{\frac{p}{2}-1}}} \leq \left(m + \frac{p}{2} - 1\right)^{-1} \ \in (0, \infty),$$

and thusΦ_c satisfies both (43) and (44) if $c < \frac{p}{2} - 1$. Further, each function $\Phi_n(r) = log_n \frac{\exp_n(2)}{1-r}$,$n \in \boldsymbol{N}$,satisfies

$$\frac{\Phi'_n(r)}{\Phi_n(r)} (1-r) \log \frac{e}{1-r} \to 0 \ , \ r \to 1 \ ,$$

and hence satisfies all hypotheses of the next result.

Proposition (5.2.15) [203]:

Let $2 < p < \infty$, and let $\Phi: [0,1) \to (1, \infty)$be a differentiable increasing unbounded function such that$\frac{\Phi\prime(r)}{\Phi(r)}(1-r)$ is decreasing and (43) and (44) are satisfied. Then there exists a positive Borel measure μ on $\mathbb{D}$ such that

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I| \left(log \frac{e}{|I|}\right)^{-\frac{p}{2+1}} \Phi(1-|I|)} < \infty \ , \tag{45}$$

But μ is not a p −Carleson measure for $\mathcal{D}^{\mathrm{p}}_{\mathrm{p}-1}$.

Proof:

The radial measure

$$d\mu(z) = \frac{\Phi(|z|)dA(z)}{(1-|z|)\left(\log\frac{e}{1-|z|}\right)^{\frac{p}{2}}} \quad , z \in \mathbb{D} \ ,$$

satisfies (45) by Lemma (5.2.14).Toseethat μ is not a p −Carleson measure for $\mathcal{D}_{p-1}^{p}$, consider the lacunary series associated with Φ via(39).By Lemma (5.2.13), $f \in \mathcal{D}_{p-1}^{p}$

$$\|f\|_{L^p(\mu)}^p = \int_0^1 \frac{M_p^p(r,f)\Phi(r)}{(1-r)\left(\log\frac{e}{1-r}\right)^{\frac{p}{2}}} r dr$$

$$\gtrsim \int_0^1 \frac{r\Phi'(r)}{\Phi(r)} dr \gtrsim \lim_{t\to 1^-} \log \Phi(t) = \infty$$

which finishes the proof.