Chapter 3

Polynomial Approximation and Interpolation in Besov Spaces

Approximation of functions in fractional order Sobolev spaces is treated as well as the usual integer order spaces and several nonstandard Sobolev-like spaces. We have the results of the determination of the interpolation spaces between a pair of Besov spaces; an atomic decomposition for functions in Besov space; the characterization of the class of functions which have certain prescribed degree of approximation by dyadic splines . We study Besov spaces$B_q^\alpha(L_p(\Omega))$, $0 < p, q, \alpha < \infty$, on domains Ω in R^d . This is then used to derive various properties of the Besov spaces such as interpolation Theorems for a pair of

$B_q^\alpha(L_p(\Omega))$, atomic decompositions for the elements of $B_q^\alpha(L_p(\Omega))$, and a description of the Besov spaces by means of spline approximation.

Sec (3.1): Functions in Sobolev Spaces:

Approximation properties of finite element spaces are often derived using variations of the so-called Bramble-Hilbert Lemma [91], [92]. This lemma is based on an inequality of the form

$$\inf_{P\in\mathcal{P}}\|f-P\| \le C\sum_{\alpha\in A}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} f\right|, \qquad (1)$$

where$\mathcal{P}$is a class of polynomials, A is an associated class of multi-indices, and $\|\cdot\|$ and $|\cdot|$ denote certain Sobolev norms. An inequality of the form (1) can be found in Morrey [103] (and implicitly in Sobolev [105]) for the case of $\mathcal{P}$ being all polynomials of degree at most r, A being all multi-indices of length $r+1, \|\cdot\|$ being the norm on W_p^m and $|\cdot|$ being the norm on L_p. In the second Bramble-Hilbert [92], (1) is derived for certain classes P that range from the polynomials of degree at most r to the polynomials that are of degree at most r in each variable separately. Motivated by particular applications, we extend (1) by allowing more general collections P and A and, further, by deriving inequalities of the form

$$\inf_{P\in\mathcal{P}}\|f-P\| \le C\sum_{j}\left|P_j\left(\frac{\partial}{\partial x}\right) \; f\right|, \qquad (2)$$

Where $\{P_j\}$ is a collection of homogeneous polynomials of degree l_j and $\mathcal{P}$ is the intersection of the kernels of the operators $P_j(\partial/\partial x)$.

The proofs of Bramble and Hubert used the results of Morrey and generalizations thereof. The proofs of these results are non constructive and cannot be used to estimate the size of C in (1) or to determine how C would vary as a function of the domain. Sobolev's approach to imbedding theorems is based on an explicit representation. of a function as a polynomial plus a remainder term.

The results presented here use a related representation that is derived as an averaged Taylor's series . This representation can be manipulated in various ways to get bounds of the form (1) and (2). Although we do not explicitly calculate the associated constants here, it is easy to see what parameters they depend on, and, in particular cases, the proofs could be used to bound them. (The results of are somewhat of an exception to this; see Remark (3.1.8).) We have calculated these constants in one special case [96]. Further, the form of proof used here allows the dependence of the constant on the underlying domain to be clarified. The basic results of this section are derived initially for domains that are star-shaped with respect to (each point of) a ball and in these cases the constants are seen to depend on the domain only through its diameter and the diameter of the associated ball. Having this type of dependence makes it easier (or possible) to treat the perturbations of the domain that are frequently needed to handle curved boundaries by finite element methods . Our results are also extended to regions that may be viewed as a finite union of domains that are star-shaped with respect to balls. Polynomial approximation results for such regions have been derived by Jamet [101] using an entirely different approach. These regions include ones satisfying the cone condition used by Bramble and Hubert [91], [92], but are slightly more general. When functions are approximated by piecewise polynomials on a mesh of size $h > 0$, the bound for the error typically involves Atoa positive power. In most cases, the power decreases by one for each additional order of differentiation applied to the error. One purpose of our results on tensor-product polynomial approximation is to show under what conditions one should expect not to lose a power of h when differentiating the error. An application is given to illustrate this point .

There are situations in which it is necessary to approximate a function satisfying a homogeneous, constant coefficient differential equation by polynomials which also satisfy that equation. The approximation results following from (2) can be used to treat such cases. An application is given in which harmonic functions are approximated by harmonic polynomials.

Our proofs of (1) and (2) are based on a basic representation formula of a function as a polynomial projection plus a remainder derived. An important property of the projection operator is that it commutes with differentiation, that is, a derivative of the polynomial projection of a function is the same as an associated (lower order) polynomial projection of that derivative of the function. This commutativity property is used in a crucial way to derive the results described in the previous two paragraphs.

Frequently, one is interested in the best possible approximation of a function subject to the constraint that a function and its approximation agree at certain points [90], [95], [104]. Restricting to integer index Sobolev spaces excludes certain interesting cases from study. Most of the results in this section are proved for the integer case, estimates of the form (1) involving fractional order Sobolev norms are proved, and an illustration of their application is given.

Several of the questions we discuss here have been treated from different points of view by many authors. Our interest in these questions comes from studying the approximation results that are needed to analyze finite element methods. In this area, the work of Bramble and Hubert [91], [92] is fundamental. The work of Ciarlet and Wagschal on multipoint Taylor formulas [94] is another approach to giving constructive proofs of approximation results needed for finite element analysis and their results played an important role in the evolution of this section. The basic representation given, which we call a Sobolev representation, is quite similar to one used by Sobolev [105] in proving imbedding theorems .However, it appears to be different from the one used in [105] for which, in particular, it is not clear that the commutativity property mentioned above is valid. A more recent treatment of related representations, as well as some discussion of their applications in other work, can be found in an article by Burenkov [93]. In [102], Meinguet gave a constructive polynomial approximation process that is closely related to the Sobolev representation in [89], Arcangeli and Gout applied Meinguet's ideas to Lagrange interpolation in R^n.

Notation: Let $x, y, \ldots$ denote points in R^n, and let $dx, dy, \ldots$ denote Lebesgue measure. If D is a measurable set, $p \in [1, \infty]$, and f is a (real or complex valued) measurable function, we say $f \in L_p(D)$ if

$$\|f\|_{L_p(D)} \equiv \left(\int_D |f(x)|^p dx\right)^{1/p} < \infty \tag{3}$$

with the usual modification when $p = \infty$. When $p = \infty$, $1/p$ is defined to be zero.

Let N denote the set of nonnegative integers. A multi-index α is an n-tuple of nonnegative integers: $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in N, i = 1, \ldots, n$. We have the following definitions:

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \tag{4}$$

$$\alpha \le \beta \quad \text{iff} \alpha_i \le \beta_i, i = 1, \ldots, n, \tag{5}$$

$$(\alpha + \beta)_i = \alpha_i + \beta_i, i = 1, \ldots, n, \tag{6}$$

$$(\alpha - \beta)_i = \max\{\alpha_i - \beta_i, 0\}, i = 1, \ldots, n, \tag{7}$$

$$\alpha! = (\alpha_1!)(\alpha_2!) \ldots (\alpha_n!), \tag{8}$$

$$x^\alpha = \left(x_1^{\alpha_1}\right)\left(x_2^{\alpha_2}\right) \ldots \left(x_n^{\alpha_n}\right), \qquad \text{and} \tag{9}$$

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}. \tag{10}$$

We let $\delta^i, i = 1, \ldots, n$, denote the multi-index whose ith component is 1 and the rest are zero:

$$\delta_j^i = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases} \tag{11}$$

When D is an open set, denote by $C^\infty(D)$ the space of infinitely differentiable

functions in D. For $f \in C^\infty(D)$, we use the notation

$$f^{(\alpha)}(x) = \left(\frac{\partial}{\partial x}\right)^\alpha f(x)(x \in D). \tag{12}$$

Let $C_0^\infty(D)$ denote the subset of $C^\infty(D)$ functions that have compact support in D.

Let $\mathcal{D}(D)$ denote $C_0^\infty(D)$ topologized with the usual inductive limit topology [100]. The dual $\mathcal{D}'(D)$ of $\mathcal{D}(D)$ is called the set of distributions on D. If $\phi \in \mathcal{D}'(D)$ and if α is a multi-index, $\phi^{(\alpha)}$ is called a distributional or weak derivative of ϕ, where $\phi^{(\alpha)}$ is defined by

$$\phi^{(\alpha)}(f) = (-1)^{|\alpha|}\phi\left(f^{(\alpha)}\right), \; f \in \mathcal{D}(D).$$

A distribution $\phi \in \mathcal{D}'(D)$ is identified with a function ψ defined on D if for each $f \in \mathcal{D}(D), \psi f \in L_1(D)$and $\phi(f) = \int_D \psi f \, dx$. In this case we shall let ϕ denote the identified function, ψ, as well.

If $m \in N$ and if for each $\alpha \in N^n$ with $|\alpha| \leq m, \phi^{(\alpha)}$ is given by a function such that

$$\|\phi\|_{W_p^m(D)} = \sum_{|\alpha| \leq m} \left\|\phi^{(\alpha)}\right\|_{L_p(D)} < \infty, \tag{13}$$

then $\phi \in W_p^m(D)$. Note that $C^\infty(D) \cap W_p^m(D)$ is dense in $W_p^m(D)$provided $p < \infty$. (See [98] for a proof.) If D has finite measure, then $W_p^m(D) \subset W_q^m(D)$ if $1 \leq q \leq p \leq \infty$ (by Holder's inequality). For $\phi \in W_p^m(D)$ let

$$|\phi|_{W_p^m(D)} = \sum_{|\alpha| = m} \left\|\phi^{(\alpha)}\right\|_{L_p(D)}. \tag{14}$$

Let r be a positive integer, and denote by $\mathcal{P}_r$ the space of polynomials in n

variables of degree less than r. Let $\mathcal{P}_\infty = \bigcup_{r=1}^\infty \mathcal{P}_r$.

Let D be a bounded set in R^n with diameter d. Suppose D is star-shaped with respect to every point in an open ball B. Let $\phi \in C_0^\infty(B)$ have integral one. Throughout , D, d, B, and ϕ will remain the same.

Sobolevs Representation.If $f \in C^\infty(D)$, I is a positive integer, and $x \in D$, then

$$f(x) = Q^l f(x) + R^l f(x), \tag{15}$$

where

$$Q^l f(x) = \sum_{|\alpha|<l} \int_B \phi(y) f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!} dy \tag{16}$$

is a polynomial of degree less than l and

$$\mathcal{R}^l f(x) = \sum_{|\alpha|=l} \int_D k_\alpha(x,y) f^{(\alpha)}(y)\, dy. \tag{17}$$

The kernels k_α are given by

$$k_\alpha = (l/\alpha!)(x-y)^\alpha k(x,y), \tag{18}$$

where

$$k(x,y) = \int_0^1 s^{-n-1} \phi\big(x + s^{-1}(y-x)\big)\, ds. \tag{19}$$

Remark (3.1.1) [87]:

As a function of $y, k(x,\cdot)$, and, therefore, each $k_\alpha(x,\cdot)$, is supported in the convex hull of $\{x\} \cup \operatorname{supp} \phi$; in particular, the region of integration in (17) is contained in a compact subset of D.

Remark (3.1.2) [87]:

Integration by parts shows that Q^l is defined for all $f \in \mathcal{D}'(B)$ and that, in particular, for f in $L_1(B)$

$$\|Q^l f\|_{W_\infty^{l-1}(D)} \le C(n,l,d,\phi) \|f\|_{L_1(B)}. \tag{20}$$

Proof of the Representation:

Let $x \in D, y \in B$, and use Taylor's theorem:

$$f(x) = \sum_{|\alpha|<l} \frac{(x-y)^\alpha}{\alpha!} f^{(\alpha)}(y) + l \sum_{|\alpha|<l} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 s^{l-1} f^{(\alpha)}\big(x + s(y-x)\big)\, ds.$$

Multiply by $\phi(y)$ and integrate with respect to y:

$$f(x) = Q^l f(x) + l \sum_{|\alpha|=l} \frac{1}{\alpha!} \int \phi(y)(x-y)^\alpha \int_0^1 s^{l-1} f^{(\alpha)}\big(x + s(y-x)\big)\, ds\, dy.$$

Using Fubini's theorem and the change of variables $z = x + s(y - x)$, one finds

$$\int \phi(y)(x-y)^\alpha \int_0^1 s^{l-1} f^{(\alpha)}\big(x + s(y-x)\big)\, ds\, dy$$

$$= \int_0^1 \int \phi(y)(x-y)^\alpha\, s^{l-1} f^{(\alpha)}\big(x + s(y-x)\big)\, dy\, ds$$

$$= \int_0^1 \int \phi\big(x + s^{-1}(z-x)\big)\,(x-z)^\alpha s^{-1} f^{(\alpha)}(z) s^{-n} dz\, ds$$

$$= \int (x-z)^\alpha f^{(\alpha)}(z) \left(\int_0^1 \phi\big(x + s^{-1}(z-x)\big) s^{-n-1} ds \right) dz$$

$$= \frac{\alpha!}{l} \int k_\alpha (x-z) f^{(\alpha)}(z)\, dz.$$

The use of Fubini's theorem is justified because

$$|k(x,y)| \;=\; \int_0^1 \phi\big(x + s^{-1}(z-x)\big) s^{-n-1} ds$$

$$\le c_1 |x-z|^{-n} = \left| \int_{|z-x|/d}^1 \phi\big(x + s^{-1}(z-x)\big) s^{-n-1} ds \right| \qquad (21)$$

where$c_1 = \|\phi\|_{L_\infty(B)}\, d^n / n$

Remark (3.1.3) [87]:

In view of (18), it follows as in (21) that

$$\left| \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial y}\right)^\gamma \right| \le C(n.l.d.\phi, |\beta|, |\gamma|)|x-y|^{|\alpha|-n-|\beta|-|\gamma|}. \qquad (22)$$

In view of the Sobolev representation and (22), estimates of the approximation error $u - Q^l u$ may be reduced to consideration of the Riesz potentials

$$I^l f(x) \equiv \int_D |x-y|^{l-n} f(y)\, dy. \qquad (23)$$

The following proposition collects several known results in the form we find useful.

Proposition (3.1.4) [87]:

Let l be a positive integer and let p and q be in $[1, \infty]$. Suppose that $1/q - 1/p + l/n \ge 0$ and that a is a positive lower bound for

$$\max\left\{\left[\frac{l}{n}\right],\frac{1}{q}-\frac{1}{p}+\frac{l}{n},\min\left\{1-\frac{1}{q},\frac{1}{p}\right\}\right\},$$

where$[x]$is the largest integer not greater than x. Then L^l maps $L_p(D)$ to $L_q(D)$ and for all $f \in L_p(D)$

$$\|I^l f\|_{L_q(D)} \leq C(n,l,d,\phi)\|f\|_{L_q(D)}.$$

Proof:

Let

$$R_l(x) = \begin{cases} |x|^{l-n} & \text{for } x \leq d, \\ 0 & \text{for } x > d. \end{cases}$$

Then, for $x \in D$,

$$I^l f(x) = R_l * f(x),$$

where$f$ is extended to R^n by zero outside D.

For $l \geq n, R_l$ is bounded and, hence,

$$\|I^l f\|_{L_\infty(D)} \leq C(n,l,d)\|f\|_{L_1(D)}.$$

The result then follows from Holder's inequality.

Now suppose that $l < n$ and that $1/q - 1/p + l/n \geq \sigma > 0$. For this case we apply Young's inequality [108] to obtain

$$\|I^l f\|_{L_q(D)} = \|R_l * f\|_{L_q(D)} \leq \|R_l\|_{L_r(R^n)}\|f\|_{L_p(D)},$$

where $1/r = 1 - 1/p + 1/q \geq 1\ - l/n + \sigma$. But

$$\|R_l\|_{L_r(R^n)} = C(n)\left[\frac{d^{(l-n)r+n}}{(l-n)r+n}\right]^{1/r} < \infty,$$

since $(f -\ n)r\ + n \geq\ \sigma nr$. Hence, the proposition holds in this case as well.

Now suppose that / < n and 1/q - 1/p + l/n = 0. Then it is a standard result

(cf. Stein [107]) that

$$\|I^l f\|_{L_q(R^n)} \leq C(n,l,p,q)\|f\|_{L_q(R^n)},$$

provided that $p > 1$ and $p <\ n/l$, i.e., $q < \infty$. It is clear from the proof in Stein referenced above that the constant $C(n,l,p,q)$ can be chosen to be continuous in $p$and q, and hence bounded by

$$C(n,l,\sigma) = \sup\left\{C(u,l,p,q) \colon \max\left(1-\frac{1}{p},\frac{1}{q}\right) \geq \sigma\right\} < \infty.$$

This yields the proposition in the case $1/q - 1/p + l/n = 0$and $\max(1 - 1/p, 1/q) \geq \sigma > 0$.

Now suppose $1/q - 1/p + l/n$ is positive, but arbitrarily small, and $\max(1 - 1/p, 1/q) \geq \sigma > 0$. If $1 - 1/p \geq 1/q$, choose$\tilde{q}$ such that $1/\tilde{q} - 1/p + l/n = 0$. Since $\max(1 - 1/p, \sqrt[1]{q}) \geq \sigma$, the previous case implies that

$$\|I^l f\|_{L_{\tilde{q}}(D)} \leq C(n,l,d,\sigma)\|f\|_{L_p(D)},$$

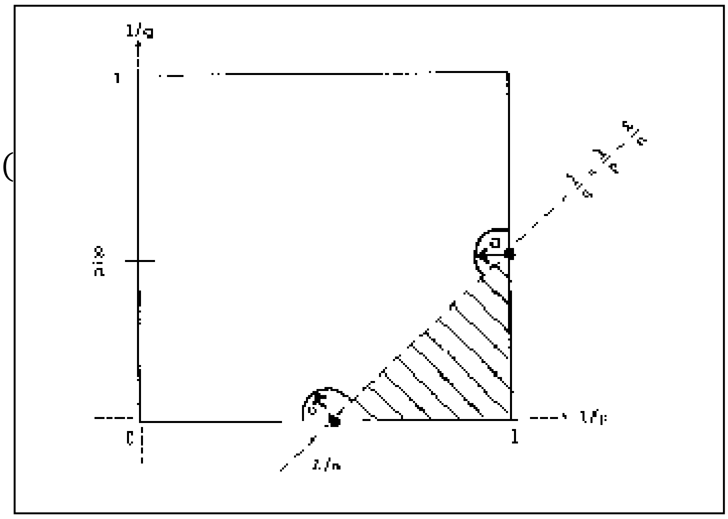
for all $f \in L_p(D)$. Since $1/\tilde{q} < 1/q$, Holder's inequality yields the desired result. If on the other hand $1 - 1/p \leq 1/q$, choose p such that $1/q - 1/\tilde{p} + l/n = 0$. The previous case again implies that

$$\|I^l f\|_{L_q(D)} \leq C(n,l,d,\sigma)\|f\|_{L_{\tilde{p}}(D)},$$

and Holder's inequality applied to the right-hand side yields the desired conclusion. This completes the proof of the proposition.

Remark (3.1.5) [87]:

The above proposition may be interpreted via the following $1/q$ v.s. $1/p$ diagram:



(24)

The proposition holds for all pairs $(1/p, 1/q)$ in the closed unit square excluding the shaded region lying below the line $1/q = 1/p - l/n$ and excluding the two points

$(l/n, 0)$and $(1, l/n)$. Furthermore, the norm of $I^l: L_p(D) \to L_q(D)$ can be bounded uniformly in the closed subset of the unit square excluding the shaded region and excluding discs of radius σ around the points $(l/n, 0)$ and $(1, l/n)$. However, as σ is allowed to tend to zero, the norm of f tends to infinity. (If $l \geq n$, then I is bounded uniformly for all p and q.) Notice that for all l, n, p, and q for which the proposition is applicable, it is also applicable for I, n, p', and q for some $p' < \infty$. This observation will be used later to restrict attention to finite p in order to allow the use of a density argument.

The restriction that $p \neq 1$ and $q \neq \infty$ when $1/q = 1/p - l/n$ is necessary since the Riesz potential of order / does not map $L_1(R^n)$ (respectively, $L_{\frac{n}{l}}(R^n)$) into $L_{n/n-1}(R^n)$ (respectively, $L_\infty(R^n)$); see Stein [106]. However that the case $p = 1$ may be treatable by another argument is indicated by the fact that the Sobolev embedding holds in this case; see Stein [106].

Viewing the Sobolev representation of f as giving a polynomial approximation $(Q^l f)$, there are now two natural polynomial approximations to the derivatives off, namely, $(\partial/\partial x)^\alpha Q^l f$ and $Q^{l-|\alpha|}(\partial/\partial x)^\alpha f$. Both are polynomials of degree less than $l - |\alpha||$. Schematically,

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\left(\frac{\partial}{\partial x}\right)^\alpha} & \mathcal{D}' \\ \downarrow & Q^l \quad \downarrow & Q^{l-|\alpha|} \\ \mathcal{P}_l & \xrightarrow{\left(\frac{\partial}{\partial x}\right)^\alpha} & \mathcal{P}_{l-|\alpha|} \end{array} \tag{25}$$

Theorem (3.1.6) [87]:

The diagram (25) commutes, i.e. for $f \in \mathcal{D}'(B)$, $(\partial/\partial x)^\alpha Q^l f = Q^{l-|\alpha|}(\partial/\partial x)^\alpha f$.

Proof:

Let $f \in C^\infty(B)$ and let $x, y \in B$. We write the Taylor polynomial of f as

$$T_y^l f(x) = \sum_{|\alpha|<l} f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!}.$$

Then

$$\left(\frac{\partial}{\partial x}\right)^\beta (T_y^l f)(x) = T_y^{l-|\beta|}\left(\left(\frac{\partial}{\partial x}\right)^\beta f\right)(x),$$

as is easily proved by induction. But since

$$Q^l f(x) = \int \phi(y) T_y^l f(x)\, dy,$$

the result follows by differentiation under the integral.

The result then follows for $f \in \mathcal{D}'(B)$ since $(\partial/\partial x)^\alpha$is continuous on $\mathcal{D}'(B) Q^k$ is continuous from $\mathcal{D}'(B)$ into $\mathcal{P}_k$, and $C^\infty(B)$ is dense in $\mathcal{D}'(B)$ [100].

Combining Proposition (3.1.4) with Theorem (3.1.6) gives the following:

Theorem (3.1.7) [87]:

Let m and l be integers such that $0 \leq m < l$, and let p and q be in $[1, \infty]$. Suppose that, with $\tilde{l} = l - m$, $1/q - 1/p + \tilde{l}/n \geq 0$ and that

$$0 < \sigma \leq \max\left\{\lceil \tilde{l}/n \rceil, \frac{1}{q} - \frac{1}{p} + \frac{\tilde{l}}{n}, \min\left\{1 - \frac{1}{p}, \frac{1}{q}\right\}\right\}.$$

Then for $f \in W_p^l(D)$

$$\|f - Q^l f\|_{W_q^m(D)} \leq C(n, l, d, \phi) \|f\|_{W_p^l(D)}.$$

Proof:

In view of Remark (3.1.5), it suffices to assume that $p < \infty$, for then the general case follows from Holder's inequality. Since $C^\infty(D) \cap W_p^l(D)$ is dense in $W_p^l(D)$, it suffices to prove the estimate for $f \in C^\infty(D) \cap W_p^l(D)$. Take $\alpha \in N^n$ such that $|\alpha| \leq m$. Then by Theorem 1,

$$\left(\frac{\partial}{\partial x}\right)^\alpha (f - Q^l f) = f^{(\alpha)} - Q^{l-|\alpha|} f^{(\alpha)} = \mathcal{R}^{l-|\alpha|} f^{(\alpha)}.$$

Thus the result follows from (22) and Proposition (3.1.4)

Remark (3.1.8) [87]:

Sobolev's proof [105] of the imbedding theorems was essentially via Theorem (3.1.72), Remark (3.1.2), and the triangle inequality:

$$\|f\|_{W_q^m(D)} \leq \|f - Q^l f\|_{W_q^m(D)} + \|Q^l f\|_{W_q^m(D)}$$

$$\leq C_1 |f|_{W_p^l(D)} + C_2 \|f\|_{L_1(D)}.$$

It is not clear that Theorem (3.1.6) holds for the representation used by Sobolev, but as noted below, Theorem (3.1.7) does not really rely on the commutativity.

Remark (3.1.9) [87]:

The estimate in Theorem (3.1.7) could likewise be derived without using Theorem (3.1.6) simply by differentiating under the integral, in view of (22). However, the use of the commutativity becomes crucial in the next two sections.

Remark (3.1.10) [87]:

Note that if $m = l$, then the conclusion of Theorem (3.1.7) remains valid for $q \leq p$. This follows because

$$|f - Q^l f|_{W_q^l(D)} = |f|_{W_q^l(D)}.$$

Let A be a set of multi-indices, and let the polar of A, A^0, be the set of multi-indices given by

$$A^0 = \left\{\beta \in N^n : \left(\frac{\partial}{\partial x}\right)^\alpha x^\beta \equiv 0 \text{ for } all\ \alpha \in A\right\}. \tag{26}$$

If A and s are two sets of multi-indices such that $A \supset B$, then $A^0 \subset B^0$.

Two sets of multi-indices that play important roles are the following:

$$A = \{\alpha \in N^n : |\alpha| = l\},\ \ B = \{l\delta^1, \dots, l\delta^n\}.$$

In these cases

$$A^0 = \{\beta : |\beta| < l\}, \qquad B^0 = \{\beta : \beta_i < l \ \text{ for } i = 1, \dots, n\}$$

The set A^0 is naturally associated with complete polynomials of degree less than l while B^0 is naturally associated with polynomials that are of degree less than l in each variable separately.

For any set of multi-indices A define the base of A, A_-, as the collection of all $\alpha \in A$ such that $\beta \in A$ and $\beta \leq \alpha$ implies that $\beta = \alpha$. Note that $A^0 = (A_-)^0$ since $\alpha \in A$ implies that there is $\gamma \in A_-$ such that $\gamma \leq \alpha$, and hence $(\partial/\partial x)^\gamma x^\beta \equiv 0$ implies $(\partial/\partial x)^\alpha x^\beta \equiv 0$

Lemma (3.1.11) [87]:

$A^0$is a finite set if and only if there are nonnegative integers $r_i, i = 1, \dots, n$ such that

$$\{r_i\delta^i : i = 1, \dots, n\} \subset A.$$

Proof:

The "if' is obvious, since

$$A^0 \subset \{r_i\delta^i : i = 1, \dots, n\}^0 = \{\alpha : \alpha_i < r_i, i = 1, \dots, n\}.$$

To prove "only if, suppose that, for some i, $\{n\delta^i : n \in N\} \cap A = \emptyset$. Then for $\alpha \in A$, there is some $j \neq i$ such that $\alpha_j \neq 0$, and so if f is any function that is constant as a function of x_j, then $f^{(\alpha)} \equiv 0$. In particular, if f depends on x_i alone, then $f^{(\alpha)} \equiv 0$ for all $\alpha \in A$. Thus, $\{n\delta^i : n \in N\} \subset A^0$, and hence, A^0 is not finite.

Remark (3.1.12) [87]:

If A^0 is a finite set, then it follows from Lemma (3.1.11) that

$$\max_{\beta \in A^0} |\beta| < n \left(\max_{\alpha \in A_-} |\alpha| - 1 \right).$$

Note that it also follows from Lemma (3.1.11) that if A^0 is finite then the class of polynomials spanned by x^β for $\beta \in A^0$ is a subset of the tensor product space of polynomials which are of degree less than r_i in $x_i$for $i = 1, \dots, n$.

Extended Tensor Product Representation. Given a finite set A of multi-indices such that A^0 is finite and given $f \in C^\infty(D)$,

$$f(x) = Q^A f(x) + \mathcal{R}^A f(x), \quad x \in D, \tag{27}$$

where

$$Q^A f(x) = \sum_{\alpha \in A^0} \int_B \phi(y) f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!} dy \tag{28}$$

and

$$\mathcal{R}^A f(x) = \sum_{\alpha \in A_-} \int_D \bar{k}_\alpha(x,y) f^{(\alpha)}(y) dy \tag{29}$$

If $A^0 = \emptyset$, then the sum over A^0 is identically zero. The kernels $\bar{k}_\alpha$ satisfy

$$\left| \left(\frac{\partial}{\partial x}\right)^\beta \left(\frac{\partial}{\partial y}\right)^\gamma \bar{k}_\alpha(x,y) \right| \leq C(l, n, \phi, d, |\beta|, |\gamma|) |x-y|^{|\alpha|-n-|\beta|-|\gamma|}, \tag{30}$$

where $l = 1 + \max_{\alpha \in A^0} |\alpha|$.

Proof:

Consider Sobolev's representation of order l:

$$f(x) = Q^l f(x) + \mathcal{R}^l f(x) = \sum_{|\alpha|<l} \phi(y) f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!} dy + \sum_{|\alpha|=l} \int k_\alpha(x,y) f^{(\alpha)}(y)\, dy$$

The set of all multi-indices N^n decomposes into two disjoint sets, namely A^0 and $\bar{A} \equiv \{\beta: \exists\alpha \in A \ni \alpha \leq \beta\} = A + N^n = A_- + N^n$. Since $l > \left(\max_{\beta\in A^0}|\beta|, |\alpha| = l\right)$ implies that $\alpha \in \bar{A}$. Thus, we have

$$f(x) = Q^A f(x) + \sum_{\substack{|\alpha|<l \\ \alpha\notin A^0}} \int \phi(y) f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!} dy + \sum_{|\alpha|<l} \int k_\alpha(x,y) f^{(\alpha)}(y) dy$$

and the sums in the remainder terms are over $\alpha \in \bar{A}$. It remains to convert these terms to the form (29). But, for each $\alpha \in \bar{A}$, there is some $\beta \in A_-$ such that $\beta \leq \alpha$, so we may write $f^{(\alpha)} = (\partial/\partial y)^{\alpha-\beta} f^{(\beta)}$ integrate by parts $|\alpha - \beta|$ times, and obtain

$$\int \phi(y) f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!} dy = (-1)^{|\alpha-\beta|} \int \left\{ \left(\frac{\partial}{\partial y}\right)^{\alpha-\beta} \left[\phi(y) \frac{(x-y)^\alpha}{\alpha!}\right] \right\} f^{(\beta)}(y) dy$$

or

$$\int k_\alpha(x,y) f^{(\alpha)}(y) = (-1)^{|\alpha-\beta|} \int \left\{ \left(\frac{\partial}{\partial y}\right)^{\alpha-\beta} k_\alpha(x,y) \right\} f^{(\beta)}(y) dy.$$

Summing over all α, we obtain (29); it is not clear whether the $\bar{k}_\beta$ are uniquely determined by the above process. Estimate (30) follows from (22).

We now consider the commutativity of Q^A with differentiation. For two multiindices $\alpha$and β, (7) defines a new multi-index $\alpha - \beta$. Note that $\alpha - \beta$ is defined even if $\alpha \ngeq \beta$ and that $(\alpha - \beta) + \beta \geq \alpha$, with equality if and only if $\alpha \geq \beta$. Given a set A of multi-indices and a multi-index β, define a new set $A - \beta \subset N^n$ by

$$A - \beta = \{\alpha - \beta: \alpha \in A\}.$$

Since $(A-\beta)^0 \subset A^0$, we see that if A^0 is finite then so is $(A-\beta)^0$.

Theorem (3.1.13) [87]:

Let A be a finite set of multi-indices such that A^0 is finite and let β be a multi-index. Then

$$\left(\frac{\partial}{\partial x}\right)^\beta Q^A f = Q^{A-\beta} f^{(\beta)} \text{for any } f \in \mathcal{D}'(B). \tag{31}$$

Proof:

It is easy to see that $(\partial/\partial x)^{\gamma-\beta} x^\delta \equiv 0$ if and only if $(\partial/\partial x)^\gamma x^{\delta+\beta} \equiv 0$. Thus, $\delta \in (A-\beta)^0$ if and only if $\delta + \beta \in A^0$. Hence,

$$(A-\beta)^0 = \{\delta - \beta: \delta \in A^0, \delta \geq \beta\}.$$

Thus, for $f \in C^\infty(B)$,

$$\left(\frac{\partial}{\partial x}\right)^{\beta} Q^{A} f(x)=\sum_{\substack{\gamma\in A^{0}\\ \gamma\geq\beta}}\int \phi(y) f^{(\gamma)}(y)\frac{(x-y)^{\gamma-\beta}}{(\gamma-\beta)!}$$

$$=\sum_{\delta\in(A-\beta)^{0}}\int \phi(y) f^{(\delta+\beta)}(y)\frac{(x-y)^{\delta}}{\delta!}dy=Q^{A-\beta} f^{(\beta)}(x).$$

The result follows for $f\in\mathcal{D}'(B)$ by density.

Remark (3.1.14) [87]:

Note that both sides in (31) are zero unless $\beta\in A^{0}$. The remaining results in this section and the results ,hold for functions that are in a function space described in Remark(3.1.20). In each case it suffices to prove these results for functions in $C^{\infty}(D)$ such that all relevant norms are finite. The definition of the function space is delayed so that it need be given only once and because the intervening results make the appropriateness of the norm used much more apparent.

Theorem (3.1.15) [87]:

Let A be a finite set of multi-indices such that A^{0} is a finite set; let s be any multi-index; let $l=\min_{\alpha\in A-\beta\text{let}}|\alpha|$; let m be a nonnegative integer less than l; and let $\{q\}\cup\{p_{\alpha}:\alpha\in(A-\beta)_{-}\}\subset[1,\infty]$. Suppose that

$$\min\left\{\frac{1}{q}-\frac{1}{p_{\alpha}}+(|\alpha|-m)/n:\alpha\in(A-\beta)_{-}\right\}\geq 0,$$

and that $0<\sigma<\min\{\mu_{\alpha}:\alpha\in(A-\beta)_{-}\}$, where, with $\tilde{\alpha}=|\alpha|-m$,

$$\mu_{\alpha}=\max\left\{\lceil\tilde{\alpha}/n\rceil,\frac{1}{q}-\frac{1}{p_{\alpha}}+\frac{\tilde{\alpha}}{n},\min\left\{1-\frac{1}{p_{\alpha}},\frac{1}{q}\right\}\right\}.$$

Then

$$\left\|\left(\frac{\partial}{\partial x}\right)^{\beta}(f-Q^{A} f)\right\|_{W_{q}^{m}(D)}\leq C(n,A,\beta,m,d,\phi,\sigma)\sum_{\alpha\in(A-\beta)_{-}}\left\|f^{(\alpha+\beta)}\right\|_{L_{p_{\alpha}}(D)}.$$

Proof:

Use Theorem (3.1.13) to see that

$$\left(\frac{\partial}{\partial x}\right)^{\beta}(f-Q^{A} f)=\mathcal{R}^{A-\beta} f^{\{\beta\}}.$$

Thus, it suffices to prove the result for $\beta=0$. Differentiating under the integral in (29) and using (30) and Proposition (3.1.4) completes the proof.

Theorem (3.1.16) [87]:

Let $P_1, \dots, P_k$ be nontrivial homogeneous polynomials (in n variables) of degrees $l_1, \dots, l_k$, respectively, having no common (nonzero) complex zero ithis forces $k \geq n$. Define

$$K = \left\{ f \in \mathcal{D}'(R^n) : P_j\left(\frac{\partial}{\partial x}\right) f \equiv 0 \quad \text{for } j = 1, \dots, k \right\}.$$

Then $K \subset \mathcal{P}_r$ for some integer r. Let $l = \min_{1 \leq j \leq k} l_j$; let m be a nonnegative integer less than l; and let $\{p_j : j = 1, \dots, k\} \subset [1, \infty]$. Suppose that

$$\frac{1}{q} - \frac{1}{p} + (l_j - m)/n \geq 0, \quad j = 1, \dots, k,$$

and that $0 < \sigma < \min\{\mu_j : j = 1, \dots, k\}$ where, with $\tilde{l}_j = l_j - m$,

$$\mu_j = \max\left\{ [\tilde{l}_j/n], \frac{1}{q} - \frac{1}{p_j} + \tilde{l}_j/n, \min\left\{1 - \frac{1}{p_j}, \frac{1}{q}\right\} \right\}.$$

Then

$$\inf_{Q \in K} \|f - Q\|_{W_q^m(D)} \leq C(n, m, \{P_j\}, d, \phi, \sigma) \sum_{j=1}^{k} \left\| P_j\left(\frac{\partial}{\partial x}\right) f \right\|_{L_{p_j}(D)}.$$

Proof:

As is Agmon [88], it follows from Hubert's Nullstellensatz that there is an integer r such that for all $|a| = r$,

$$\xi^\alpha = \sum_{j=1}^{k} R_j^\alpha(\xi) P_j(\xi) \tag{32}$$

for some polynomials R_j^α that are homogeneous of degree $r - l_j$. Thus each $f \in K$ satisfies $f^{(\alpha)} \equiv 0$ for all $|a| = r$, i.e., $K \subset p_r$.

Since $K \subset p_r$, it follows that for any $P \in p_r$

$$\inf_{Q \in K} \|P - Q\|_{W_q^m(D^*)} \leq C \sum_{j=1}^{k} \left\| P_j\left(\frac{\partial}{\partial x}\right) P \right\|_{L_{P_j}(B)},$$

whereD^* is the ball of diameter $2d$ concentric with B, because of the equivalence of norms on the finite-dimensional space p_r/K. Therefore,

$$\inf_{Q\in K}\|P-Q\|_{W_q^m(D)} \le C\sum_{j=1}^{k}\left\|P_j\left(\frac{\partial}{\partial x}\right)P\right\|_{L_{P_j}(D)}, \tag{33}$$

With C depending only on the diameter d of D, the diameter of B, and $\{P_j\}$. (The independence from q and $\{p_j\}$ can be achieved using Hölder's inequality.) Using Sobolev's representation of order r and the triangle inequality, we get

$$\inf_{Q\in K}\|f-Q\|_{W_q^m(D)} \le \|f-\mathcal{Q}^r f\|_{W_q^m(D)} + \inf_{Q\in K}\|\mathcal{Q}^r f-Q\|_{W_q^m(D)}$$

$$\le \|R^r f\|_{W_q^m(D)} + c\sum_{j=1}^{k}\left\|P_j\left(\frac{\partial}{\partial x}\right)\mathcal{Q}^r f\right\|_{L_{P_j}(D)}.$$

Because of Theorem (3.1.6) and the linearity of $\mathcal{Q}^r$,

$$P_j\left(\frac{\partial}{\partial x}\right)\mathcal{Q}^r f = \left[\sum_{|\alpha|=\iota_t} c_{\alpha,j}\left(\frac{\partial}{\partial x}\right)^{\alpha}\right]\mathcal{Q}^r f = \sum_{|\alpha|=\iota_t} c_{\alpha,j}\mathcal{Q}^{r-\iota}f^{(\alpha)} = \mathcal{Q}^{r-\iota_j}P_j\left(\frac{\partial}{\partial x}\right)f.$$

Thus, Remark (3.1.2) shows that

$$\left\|P_j\left(\frac{\partial}{\partial x}\right)\mathcal{Q}^r f\right\|_{W_q^m(D)} \le C\left\|P_j\left(\frac{\partial}{\partial x}\right)f\right\|_{L_1(D)} \le C\left\|P_j\left(\frac{\partial}{\partial x}\right)f\right\|_{L_{P_j}(D)}.$$

It now remains to estimate $R^r f$. Using (32), we have

$$R^r f(x) = \sum_{|\alpha|=r}\int k_\alpha(x,y)f^{(\alpha)}(y)\,dy$$

$$= \sum_{|\alpha|=r}\sum_{j=1}^{k}\int k_\alpha(x,y)\left[R_j^\alpha\left(\frac{\partial}{\partial y}\right)P_j\left(\frac{\partial}{\partial y}\right)f\right](y)\,dy$$

$$= \sum_{j=1}^{k}\int\left[\sum_{|\alpha|=r}(-1)^{\deg R_j^\alpha}R_j^\alpha\left(\frac{\partial}{\partial y}\right)k_\alpha(x,y)\right]P_j\left(\frac{\partial}{\partial y}\right)f(y)dy.$$

Since $\deg R_j^\alpha = r-l_j$, (22) and Proposition (3.1.4) imply that

$$\|R^r f\|_{W_q^m(D)} \le C\sum_{j=1}^{k}\left\|P_j\left(\frac{\partial}{\partial x}\right)f\right\|_{L_{P_j}(D)},$$

and this completes the proof of the theorem.

Remark (3.1.17) [87]:

If $A = \{\alpha^1, \ldots, \alpha^k\}$ is a finite set of multi-indices and $\{P_j\}$ is defined by $P_j(x) = x^{\alpha^j}$ for $j = 1, \ldots, k$, then A^0 is a finite set if and only if $P_1, \ldots, P_k$ have no common (nonzero) complex zero. This is because $K \cap P_\infty$ is the space spanned by $\{x^\beta : \beta \in A^0\}$; hence, $\dim K \cap P_\infty = \text{card}A^0$. The proof of Theorem (3.1.16) thus contains the "if" part of our assertion. To prove the "only if part, suppose there is some$\xi \neq 0$ such that $P_j(\xi) = \xi\alpha^j = 0$ for $j = 1, \ldots, k$. Then some component, say ξ_i of ξ must be nonzero, and so none of the α^j's can be of the form $r\delta^i, r \in N$; hence, A^0 is not finite (Lemma (3.1.11)).

Remark (3.1.18) [87]:

The proof of Theorem (3.1.16) is constructive to the extent that the constant C in (33) can in principle be computed (it is a finite-dimensional problem). The integer$r$ guaranteed by Hubert's Nullstellensatz depends only on $l_1, \ldots, l_k$; cf. van der Waerden [107].

Remark (3.1.19) [87]:

If $0 \neq \xi \in C^n$ is such that $P_j(\xi) = 0$for $j = 1, \ldots, k$, then $P_j(\partial/\partial x)e^{\xi x} = 0$, so that $K \not\subset p_\infty$. Thus, this condition is necessary for polynomial approximation theory. (Note that even if the P_j's have real coefficients, there is a real-valued non polynomial function in K, namely,

$$P_j(\partial/\partial x)\left(Re\ e^{\xi x}\right) = Re\left(P_j(\partial/\partial x)e^{\xi x}\right) = 0,$$

so it is necessary to consider all complex zeros of the P_j's.)

Remark (3.1.20) [87]:

The estimates of Sections (4) and (5) are valid for functions f in the space H defined as follows: Let $\{P_j\}$ and $\{p_j\}$ be finite sets of polynomials and extended real numbers as in Theorem (3.1.16). Take H to be the subset of $L_1(D)$ consisting of functions f such that the distributional derivatives $P_j(\partial/\partial x)f$are elements of $L_{p_j}(D)$.

This is a Banach space with the norm

$$\|f\|_{L_1(D)} + \sum_j \left\|P_j\left(\frac{\partial}{\partial x}\right)f\right\|_{L_{P_j}(D)}.$$

Further, when all the p_j's are finite, the set $C^\infty(D) \cap H$ (see below), and this allows to be carried through in view of Remark (3.1.15).

The claimed density of $C^\infty(D) \cap H$ is not easily seen by the standard partition of unity argument (cf. [98]), but can be demonstrated as follows: Assume that $0 \in B$, and for $r \geq 1$ let $f_r(x) = f(x/r)$ and $D_r = \{x\colon\ x/r \in D\}$. Since D is star-shaped with respect to $B, D \subset\subset D_r$ for $r > 1$. Given $f \in H$, it is easily seen that $f_r \in H$ and that $f_r \to r$ in H as $r \downarrow 1$. Thus, it suffices to approximate f_r, by a function in C^∞. If $\psi \in C_0^\infty(\mathbb{R}^n), \int \psi(x)dx = 1$, and $\psi_\epsilon(x) = \epsilon^{-n}\psi(x/\epsilon)$, then as $\epsilon \downarrow 0$ $\psi_\epsilon * f_r \to f_r$ in L_p of any compact subset of D_r provided $P < \infty$. Hence, $\psi_\epsilon * f_r \to f_r$in $L_1(D)$.

Finally, since $P_j(\partial/\partial x)(\psi_\epsilon * f_r) = \psi_\epsilon * P_j(\partial/\partial x)f_r$, we see that $\psi_\epsilon * f_r \to f_r$ in H as $\epsilon \downarrow 0$.

For $m > 0$let $m = \bar{m} + \theta$, where in is the integer part of m; i.e., $\bar{m} \in N$ and $0 \leq \theta < 1$. For positive nonintegral m and $1 < p < \infty$ define

$$|f|_{W_p^m(D)}^p = \sum_{|\alpha|=\bar{m}} \iint_{D\times D} \frac{\left|f^{(\alpha)}(x) - f^{(\alpha)}(y)\right|^p}{|x-y|^{n+\theta P}}\, dx\, dy. \tag{34}$$

For $p = \infty$ define the seminorm by

$$|f|_{W_\infty^m(D)} = \sum_{|\alpha|=\bar{m}} \operatorname*{ess\,sup}_{D\times D} \frac{\left|f^{(\alpha)}(x) - f^{(\alpha)}(y)\right|}{|x-y|^\theta}. \tag{35}$$

The space $W_p^m(D)$ is the set of all $W_p^{\bar{m}}(D)$ functions such that $|f|_{W_p^m(D)} < \infty$, and its norm is defined by

$$\|f\|_{W_p^m(D)} = \|f\|_{W_p^{\bar{m}}(D)} + |f|_{W_p^m(D)}.$$

Proposition (3.1.21) [87]:

Suppose that $1 \leq p \leq \infty$, $m = \bar{m} + \theta$ where $\bar{m} \in N$ and $0 < \theta < 1$, and $l = \bar{m} + 1$. Then there is a constant $C = C(n, \phi, d, m)$ such that for $f \in W_p^m(D)$

$$\|f - \mathcal{Q}^l f\|_{L_p(D)} \leq C|f|_{W_p^m(D)}, \tag{36}$$

where$\mathcal{Q}^l$ is defined in (16).

Proof:

First take $1 \leq p < \infty$. Then, we can assume without loss of generality that $f \in C^\infty(\mathbb{R}^n)$. (See [99].)

Suppose that a is a multi-index such that $|a| = l$, and take j to be such that $\alpha = \beta + \delta^j$ where β is a multi-index. Let

$$R_\alpha(x) = \int_D f^{(\alpha)}(y) k_\alpha(x,y)\, dy, \tag{37}$$

wherek_α is defined in (18). This can be written as follows:

$$R_\alpha(x) = \int_D \frac{\partial}{\partial y_j}\left[f^{(\beta)}(y) - f^{(\beta)}(x)\right] k_\alpha(x,y)\,dy$$

$$= \lim_{e \searrow 0} \left\{ \begin{array}{l} -\int_{\{y \in D : |x-y| > \epsilon\}} \left[f^{(\beta)}(y) - f^{(\beta)}(x)\right] \frac{\partial}{\partial y_j} k_\alpha(x,y)\,dy \\ + \int_{|x-y|=\epsilon} \left[f^{(\beta)}(y) - f^{(\beta)}(x)\right] k_\alpha(x,y)\left(x_j - y_j\right)\epsilon^{-1}\,ds \end{array} \right\} \tag{38}$$

whereds is surface measure.

The surface integral in (38) tends to zero as $e \searrow 0$. To see this note that, for $|x-y| = \epsilon$, (22) implies

$$|k_\alpha(x,y)| \le C\epsilon^{1-n},$$

that, for $|x-y| = \epsilon, f \in C^\infty(\mathbb{R}^n)$ implies

$$\left|f^{(\beta)}(y) - f^{(\beta)}(x)\right| \le C\epsilon,$$

and that

$$\int_{|x-y|=\epsilon} 1\,ds = C\epsilon^{n-1}.$$

Using (22) again, we see that

$$|R_\alpha(x)| \le C_0 \int_D \frac{\left|f^{(\beta)}(y) - f^{(\beta)}(x)\right|}{|x-y|^n}. \tag{39}$$

Note that the integrand is in L^1 since it is bounded by $C|x-y|^{-n+1}$. Hölder's inequality and (39) imply that

$$|R_\alpha(x)|^p \le C \int_D \frac{\left|f^{(\beta)}(y) - f^{(\beta)}(x)\right|^p}{|x-y|^{n+\theta p}}\,dy. \tag{40}$$

Integrating this with respect to x and summing on $|a| = l$ gives (36) for the case $p < \infty$.

Note that the C in (40) is just

$$C_0^p \left(\int_D |x-y|^{-n+\theta P'} dy\right)^{p/p'} \le \left\{C_0 \left(\frac{\omega_n}{\theta}\left(1 - \frac{1}{p}\right)\right)^{1-1/p} d^\theta\right\}^p,$$

whereω_n is the measure of the unit $(n-l)$-sphere. Thus the constant C in (36) can be taken to be independent of $p \in (1,\infty)$, and it is bounded for θ in the interval $(\epsilon, 1)$ where ϵ is positive.

The estimate for $p = \infty$ is complicated by the facts that, for nonintegral m, $W_\infty^m(D) \not\subset W_p^m(D)forp < \infty$ and $C^\infty(D)$ is not dense in $W_\infty^m(D)$. In this case, note that $C^{\bar{m}} \supset W_\infty^m(D)$. For $\bar{m} \geq 1$ and for $f \in C^{\bar{m}}(D)$,

$$f(x) = \sum_{|\alpha|=\bar{m}} \frac{(x-y)^\alpha}{\alpha!} f^{(\alpha)}(y) + \bar{m}$$

$$+ \sum_{|\alpha|=\bar{m}} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 s^{\bar{m}-1} \left[f^{(\alpha)}\big(x + s(y-x)\big) - f^{(\alpha)}(y)\right] ds. \quad (41)$$

This representation is just the first line of the proof of the Sobolev representation given in ,except l was decreased to $\bar{m}$ and zero was added in a convenient form. Each term in the second sum can be bounded by

$$\frac{\bar{m}}{\alpha!} |(x-y)^\alpha| \int_0^1 s^{\bar{m}-1} [(1-s)|x-y|]^\theta \, ds \, |f|_{W_\infty^m(D)}.$$

Multiplying (41) by $\phi(y)$, integrating with respect to y, and applying the above bound gives the conclusion for $p = \infty$ and $\bar{m} \geq 1$.

For $\bar{m} = 0$ replace (41) by the trivial relation

$$f(x) = f(y) + [f(x) - f(y)].$$

Then proceed as above.

Proposition (3.1.21), when combined with Theorem (3.1.15), gives

Theorem (3.1.22) [87]:

Suppose that $m = \bar{m} + 9$, where $0 < \theta < 1$ and $\bar{m}$ is a nonnegative integer. Let $l = \bar{m} + 1$, and let Q^l be defined by (16). Then there exists a constant $C = C(n, \phi, d, m)$ such that, for $1 \leq p \leq \infty$and $f \in W_p^m(D)$,

$$\|f - Q^l f\|_{W_p^m(D)} \leq C |f|_{W_p^m(D)}. \quad (42)$$

The results, were derived under the assumption that the domain was star-shaped with respect to each point in a ball. In this section we show how this constraint can be weakened. In particular, the previous results can be extended to bounded domains which satisfy the restricted cone condition (see below) that was used in [91], [92]. In addition certain domains which fail to satisfy the restricted cone condition can be treated (for example a slit disk in $\mathbb{R}^2$). The principal result of this section states roughly that a domain has good approximation properties if it is a finite union of domains with good approximation properties. In [101] Jamet uses a different method to relax geometric constraints associated with polynomial approximation.

First, we remark on the relation between domains which satisfy the restricted cone condition and those which are star-shaped with respect to a ball. A bounded open set Ω is said to satisfy the restricted cone condition if there exists a finite open cover $\{\mathcal{O}_j\}_{j=1}^{J}$ of $\overline{\Omega}$ and a corresponding collection $\{C_j\}_{j=1}^{J}$ of truncated right circular cones with vertices at the origin such that if $x \in \Omega \cap \mathcal{O}_j$ then $x + C_j \subset \Omega$. The following remark is easily verified.

Remark (3.1.23) [87]:

If a bounded open set Ω satisfies the restricted cone condition then it is the finite union of open sets D_j each of which is star-shaped with respect to a B_j.

That the converse of this result is not valid is easily seen by considering

$$\Omega = \{re^{i\theta} : 0 < r < 1, 0 < \theta < 2\pi\},$$

where we identify C with $\mathbb{R}^2$. This domain fails to satisfy the restricted cone condition, $\Omega = D_1 \cup D_2$where

$$D_1 = \{re^{i\theta} : 0 < r < 1, 0 < \theta < 3\pi/2\},$$

$$D_2 = \{re^{i\theta} : 0 < r < 1, \pi/2 < \theta < \pi\}.$$

The domains D_j are star-shaped with respect to balls $B_j = \{z: |z - z_j| < 1/4\}$ where $z_1 = {}^1/_2\, e^{i3\pi/4}$and $z_2 = {}^1/_2\, e^{i5\pi/4}$.

For each bounded nonvoid open set D let $H(D)$ denote a linear space of functions, and let $H(D)$ be equipped with two seminorms $\|.\|_D$ and $|||.|||_D$. Suppose that these spaces and seminorms have the following properties:

a) The restriction of each element of $H(D_1 \cup D_2)$ to D_1 is in $H(D_1)$.
b) For each $f \in H(D_1 \cup D_2)$,

$$\|f\|_{D_1 \cup D_2} \le \|f\|_{D_1} + \|f\|_{D_2} \le 2\|f\|_{D_1 \cup D_2}$$

$$\text{and} |||f|||_{D_1} + |||f|||_{D_2} \le 2|||f|||_{D_1 \cup D_2}.$$

c) $p_\infty \subset H(D)$.
d) If $P \in p_\infty$and $\|P\|_{D_1} = 0$, then $\|P\|_{D_1 \cup D_2} = 0$.

In the use of the results of this section, $\|f\|_D$ will be a finite sum of terms of the form$\|f^\alpha\|_{L_p(D)}$ and $|||f|||$ will include in addition terms of the form $\|P(\partial/\partial x)f\|_{L_p(D)}$ and $|f^{(\gamma)}|_{W_r^\theta(D)}$.

Theorem (3.1.24) [87]:

Suppose that $\Omega = \bigcup_{j=1}^{N} D_j$ is connected and that each D_j is a bounded nonvoid open set. Let P be a finite-dimensional subspace of P_∞, and suppose that there exist Q_j and C_j for $j = 1, \dots, N$ such that, for $f \in H(D_j), Q_j f \in P$ and

$$\|f - Q_j f\|_{D_j} \leq C_j |\|f\|\|_{D_j}. \tag{43}$$

Then there exists C_0 such that, for $j = 1, \dots, N$ and $f \in H(\Omega)$,

$$\|f - Q_j f\|_{\Omega} \leq C_0 |\|f\|\|_{\Omega}. \tag{44}$$

Proof:

It suffices to consider the case $N = 2$, since the general case follows easily by induction. Let $B = D_1 \cap D_2$; $B \neq \phi$ since Ω is connected. By properties (b) and (d) above, the seminorms $\|P\|_{D_1} + \|P\|_{D_2}$ and $|\|P\|_B$ on $P \in p$ have the same kernel. Using the equivalence of norms on the corresponding quotient space yields

$$\|P\|_{D_1} + \|P\|_{D_2} \leq C\|P\|_B \text{for all } P \in p \tag{45}$$

for some constant $C = C(D_1, D_2, p)$.

Suppose that $f \in H(\Omega)$ and that $P_j = Q_j f$ for $j = 1,2$. Note that

$$\|f - P_1\|_\Omega \leq \|f - P_1\|_{D_1} + \|f - P_2\|_{D_2} + \|P_2 - P_1\|_{D_2},$$

using property (b) and the triangle inequality. By (45),

$$\|P_2 - P_1\|_{D_2} \leq C\|P_2 - P_1\|_B \leq C[\|P_2 - f\|_B + \|f - P_1\|_B].$$

with the previous inequality, applying (43), and using property (b) yields

$$\|f - P_1\|_\Omega \leq (1 + C)\big(C_1 |\|f\|\|_{D_1} + C_2 |\|f\|\|_{D_2}\big)$$

$$\leq (1 + C) \max\{C_1, C_2\} |\|f\|\|_\Omega.$$

Remark (3.1.25) [87]:

In those cases in which the norm $\|.\|_D$ is translation invariant, as is the case for all the Sobolev-type seminorms used so far in this section, the constant C in (45) can be taken to depend only on $\underline{d}$ and $\overline{d}$ instead of D_1 and D_2, where $\underline{d} > 0$ is such that some ball of radius $\underline{d}$ is contained in $D_1 \cap D_2$ and $\overline{d} = \operatorname{diam}(D_1 \cup D_2)$.

Remark (3.1.26) [87]:

It follows from Theorem (3.1.24) that Theorems (3.1.7), (3.1.14), (3.1.16), and (3.1.22), hold if D is any connected open set that is the union of a finite collection of domains that are star-shaped with respect to balls. In Theorems (3.1.7), (3.1.15), and (3.1.22), one chooses Q to be defined with respect to any ball contained in D. Note that Theorem (3.1.16) still holds because Theorem (3.1.24) does not require the mapping Q . to be linear or even continuous; hence, we may define Q_j by taking anything reasonably close to the infimum in Theorem (3.1.16).

Examples (3.1.27) [87]:

This section contains four simple examples that are based on the results . The purpose here is to show how the refinements in those sections yield results that would not be easily derived by results based on complete polynomial approximation or on more restrictive tensor product results. First the results are used to show an error bound for approximation by polynomials that are constant in one variable and linear in another. Next, the results are used to show how well harmonic polynomials can approximate harmonic functions. In the third example, the results are used to bound the interpolation error in a case in which the function being interpolated does not have enough derivatives to be able to apply the Theorems . The fourth example shows how triangles with curved edges can be treated using our results.

Example (3.1.28) [87]:

In this example we consider approximation in two variables by polynomials that are constant in one variable and linear in the other. One interesting question in this context is whether differentiation of the approximating polynomial in the direction in which it is linear gives a good constant approximation to the derivative of the function being approximated. The commutativity of the operator Q^A with the differentiation operator allows an affirmative conclusion.

Let h be a positive parameter, and let $D_h = (0,h) \times (0,h)$. Take $\beta = (1,0)$ and consider, for $f \in C^\infty(\mathbb{R}^2)$,

$$\eta(f;h) = \inf\left\{\|f-P\|_{L_2(D_h)} + \left\|\left(\frac{\partial}{\partial x}\right)^\beta (f-P)\right\|_{L_2(D_h)} : P \in p\right\}, \tag{46}$$

where $p = \{a + bx_1 : a, b \in \mathbb{R}\}$.

For any function g defined on D_h let $\tilde{g}(x_1,x_2) = g(hx_1,hx_2)$ be defined on

D_1. Note that

$$\left\|\left(\frac{\partial}{\partial x}\right)^\alpha (f-P)\right\|_{L_2(D_h)} = h^{1-|\alpha|}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha (\tilde{f}-\tilde{P})\right\|_{L_2(D_1)} \tag{47}$$

Thus, since $P \in p$ if and only if $\tilde{P} \in p$,

$$\eta(f;h)=\inf\left\{h\|\tilde{f}-P\|_{L_2(D_1)}+\left\|\left(\frac{\partial}{\partial x}\right)^{\beta}(\tilde{f}-P)\right\|_{L_2(D_1)}:P\in p\right\}.$$

Fix $\phi\in C_0^{\infty}$ such that $\int\phi=1$, where B is a ball contained in D_1. Take $A=\{(0,1),(2,0)\}$ and let $Q=Q^A$ be defined .Then, by Theorem (3.1.15),

$$\begin{aligned} h\|\tilde{f}-Q\tilde{f}\|_{L_2(D_1)} &\le hC\left[\|\tilde{f}^{(0,1)}\|_{L_2(D_1)}+\|\tilde{f}^{(2,0)}\|_{L_2(D_1)}\right],\\ \left\|\left(\frac{\partial}{\partial x}\right)^{\beta}(\tilde{f}-Q\tilde{f})\right\|_{L_2(D_1)} &\le C\left[\|\tilde{f}^{(1,1)}\|_{L_2(D_1)}+\|\tilde{f}^{(2,0)}\|_{L_2(D_1)}\right]. \end{aligned} \tag{48}$$

Adding the two inequalities in (48) and applying (47) with $P\equiv 0$, we see that for $0<h\le 1$

$$\eta(f;h)\le Ch\left[\|f^{(0,1)}\|_{L_2(D_h)}+\|f^{(1,1)}\|_{L_2(D_h)}+\|f^{(2,0)}\|_{L_2(D_h)}\right]. \tag{49}$$

It is interesting to note that if we had restricted ourselves to the direct application of the results .We would have not been able to show that $\eta(f;h)=O(h)$since the largest class of complete polynomials contained in p is p_1 and $\eta(f;h)\ge\|\partial/\partial x\|_{L_2(D)}$if p is replaced by p_1, in the inf. Our results are related to those used by Ewing [97] in deriving a similar cross derivative approximation bound.

To further illustrate the possible uses of Theorem (3.1.15), suppose that $q,p_1,p_2\in[1,\infty], f^{(0,1)}\in L_{p_1}(D_h)$, and $f^{(2,0)}\in L_{p_2}(D_h)$ where $1/q-1/p_1+1/2>0$. Then

$$\inf\left\{\|f-P\|_{L_q(D_h)}:P\in p\right\}\le Ch^{1-2/P_1+2/q}\|f^{(0,1)}\|_{L_{P_1}(D_h)}+h^{2-2/q_2}\|f^{(2,0)}\|_{L_{P_2}(D_h)}.$$

In general, direct application of results ,would not yield such a bound.

Example (3.2.29) [87]:

To construct an example of the use of Theorem (3.1.16), let

$P_{j+1}(x)=x^{(r+1-j,j)}$, for $j=0,\dots,r+1$, and let $P_{r+3}(x_1,x_2)=x_1^2+x_2^2$. The set K consists of all harmonic polynomials of degree less than or equal to r. If we proceed exactly as in the previous example, we see that for all $f\in C^{\infty}(\mathbb{R}^2)$

$$\inf_{P\in K}\|f-P\|_{L_2(D_h)}\le C\left[h^{r+1}\sum_{|\alpha|=r+}\|f^{(\alpha)}\|_{L_2(D_h)}+h^2\|\Delta f\|_{L_2(D_h)}\right]. \tag{50}$$

We can conclude that if f is harmonic on D_h then it can be approximated by elements of K with an error Ch^{r+1}.

Example (3.1.30) [87]:

To give an application of Theorem (3.1.22) we consider the question of bounding the error in an interpolation process. Suppose that Ω is a bounded domain in $\mathbb{R}^2$ with a polygonal boundary and that F is a family of triangulations of Ω. For $\mathcal{T} \in F$, let $h = h(\mathcal{T}) = \max_{T\in\mathcal{T}} \operatorname{diam}(T)$. Denote by $M_{\mathcal{T}} = M_0(1,\mathcal{T})$ the space of functions that are continuous on $\bar{\Omega}$ and linear (affine) on each $T \in \mathcal{T}$. Assume that there is a $\rho < \infty$ such that for all $\mathcal{T} \in F, T \in \mathcal{T} \implies (\operatorname{diam}(T))^2/\operatorname{area}(T) \le \rho$; this says that the triangles do not degenerate.

For any function $f \in (\bar{\Omega})$ let $If = I_{\mathcal{T}}f$ be the element of $M_{\mathcal{T}}$ which agrees with f at each vertex of $\mathcal{T}$; i.e., If is the piecewise linear interpolant of f. It is well known (and follows easily from Theorem (3.1.7)) that if $f \in W_2^2(\Omega)$, then

$$\|f - If\|_{L_2(\Omega)} \le Ch^2\|f\|_{W_2^2(\Omega)}.$$

However, if $0 < \epsilon < 1$ and if $f \in W_2^{1+\epsilon}(\Omega)$, the results of give no error bound. A natural approach would be to try to use the theory of interpolation of Banach spaces and use results for W_2^2 and W_2^1; however, this fails because If is not defined on W_2^1 since the elements of this space are not in general continuous.

Define $T_R = \{(x_1,x_2): x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}$. For T a triangle in $\mathcal{T} \in$ Flet A be an affine map taking $T_R$onto T. Assume, without loss of generality, that A is linear, and note that $\|A\| \le Ch$. For a function $\tilde{g} \in C(\bar{T}_R)$ define $\tilde{I}\tilde{g}$ to be the affine function of x that agrees with $\tilde{g}$ at the vertices of T_R; i.e.,

$$\tilde{I}\tilde{g}(x_1,x_2) = \tilde{g}(0,0)(1 - x_1 - x_2) + \tilde{g}(1,0)x_1 + \tilde{g}(0,1)x_2.$$

Note that $(\tilde{f} - \tilde{I}\tilde{f})(x) = (f - I)(Ax)$, where $\tilde{f}(x) \equiv f(Ax)$. Thus

$$\|f - If\|_{L_2(T)}^2 = |\det A| \|\tilde{f} - \tilde{I}\tilde{f}\|_{L_2(T_R)}^2. \tag{51}$$

Next note that for any $P \in \mathcal{P}_2$

$$\|\tilde{f} - \tilde{I}\tilde{f}\|_{L_2(T_R)} = \|\tilde{f} - P - \tilde{I}(\tilde{f} - P)\|_{L_2(T_R)},$$

since$\tilde{I}$ is a linear map which reproduces polynomials in $\mathcal{P}_2$. Letting $\|I\|$ denote the norm of $\tilde{I}$ as a map of $W_2^{1+\varepsilon}(T_R)$ into $L_2(T_R)$ (which is finite by Sobolev's inequality) we see that

$$\|\tilde{f} - \tilde{I}\tilde{f}\|_{L_2(T_R)} \le (1 + \|\tilde{I}\|) \inf_{P\in\mathcal{P}_2} \|\tilde{f} - P\|_{W^{\frac{1}{2}+\varepsilon}(T_R)}.$$

From Theorem (3.1.22) and the fact that $|P|_{W^{\frac{1}{2}+\varepsilon}} = 0$ for $P \in \mathcal{P}_2$, it follows that

$$\|\tilde{f} - \tilde{I}\tilde{f}\|_{L_2(T_R)} \le C|\tilde{f}|_{W^{\frac{1}{2}+\varepsilon}(T_R)}. \tag{52}$$

To estimate the right-hand side of (52) represent A as a 2×2 matrix that acts on column vectors $(x_1, x_2)^T$ and then note that

$$\nabla \tilde{f} = \begin{pmatrix} \tilde{f}^{(1,0)} \\ \tilde{f}^{(0,1)} \end{pmatrix} = A^* \begin{pmatrix} f^{(1,0)} \\ f^{(0,1)} \end{pmatrix} = A^* \nabla f,$$

Where $\tilde{f}^{(\alpha)}$ and $f^{(\alpha)}$ are evaluated at x and Ax, respectively. Thus

$$\iint\limits_{T_R \times T_R} \frac{\left|\nabla \tilde{f}(x) - \nabla \tilde{f}(y)\right|^2}{|x-y|^{2+2\epsilon}} dx\, dy$$

$$= |\det A|^{-2} \iint\limits_{T \times T} \frac{\left|A^*\left(\left(\nabla f(x)\right) - \nabla f(y)\right)\right|^2}{|x-y|^{2+2\epsilon}} \left(\frac{|x-y|}{|A^{-1}(x-y)|}\right)^{2+2\epsilon} dx\, dy$$

$$\leq \frac{\|A\|^{4+2\epsilon}}{|\det A|^2} \iint\limits_{T \times T} \frac{|\nabla f(x) - \nabla f(y)|^2}{|x-y|^{2+2\epsilon}} dx\, dy.$$

From this result, (51) and (52) we see that

$$\|f - If\|_{L_2(T)}^2 \leq C \left[\frac{\|A\|^2}{|\det A|}\right] h^{2+2\epsilon} \iint\limits_{T \times T} \frac{|\nabla f(x) - \nabla f(y)|^2}{|x-y|^{2+2\epsilon}} dx\, dy.$$

Sum this result over triangles and use the nondegeneracy of $\mathcal{F}$ to bound the term in brackets to obtain

$$\|f - If\|_{L_2(\Omega)} \leq C h^{1+\epsilon} \left(\sum_{T \in \mathcal{T}} \iint\limits_{T \times T} \frac{|\nabla f(x) - \nabla f(y)|^2}{|x-y|^{2+2\epsilon}} dx\, dy\right)^{1/2} \leq C h^{1+\epsilon} |f|_{W^{\frac{1}{2}+\epsilon}(\Omega)} \quad (53)$$

If we had not needed to estimate the interpolation error but merely the error in the best possible approximation in $\mathcal{M}_{T\mathcal{T}}$ a bound like (53) could be obtained by interpolating between $L_2(\Omega)$ and $W_2^2(\Omega)$. However, it is frequently the case that one needs to know how well a function that vanishes on the boundary can be approximated by function spaces that vanish on the boundary. In such cases bounds like (53) extend the error estimates to their natural lower limits. One such example can be found in Douglas, Dupont, Percell and Scott [95].

Example (3.1.31) [87]:

We now show how the above results can be applied to certain families of curved domains. Suppose Ω is a bounded domain in R^2 with smooth boundary $\partial\Omega$. Let $\mathcal{F}$ be a family of triangulations of Ω having straight interior edges and (possibly) curved edges lying on $\partial\Omega$, and suppose that $\mathcal{F}$ satisfies the nondegeneracy assumption of Example (3.1.30)

$$\sup_{\mathcal{T} \in \mathcal{F}} \sup_{T \in \mathcal{T}} (\operatorname{diam}(T))^2 / \operatorname{area}(T) \leq p < \infty.$$

Such families of triangulations were considered in [104], where approximation properties for the boundary triangles having a curved side were derived in a very complicated way. The main difficulty is that now there is no fixed reference traingle, but rather a family of reference domains. For each $T \in \mathcal{T} \in \mathcal{F}$, define an affine mapping by sending the vertices of T onto the set $\{(0,0),(1,0),(0,1)\}$, and let the image of T be denoted T_R. Again define $h(\mathcal{T}) = \max_{T\in\mathcal{T}} \operatorname{diam}(T)$ for all $\mathcal{T} \in \mathcal{F}$ G F. For h_0 sufficiently small (depending only on $\Omega$and ρ), if $h(T) \le h_0$ and $T \in \mathcal{T}$, then T_R is contained in the disc $\{|x| \le 2\}$ and is star-shaped with respect to the disc $\{|x - x_0| \le 1/8\}$, where $x_0 = (1/4,1/4)$. Thus, the above approximation results apply to $T_R$and, via the affine mapping, to each T, with the constant C in the estimates depending only on Ω and ρ (as well as the degree and type of polynomial approximation).

Sec (3.2): Besov Spaces

The Besov space $B_q^\alpha(L_p)$ is a set of functions f from L_p which have smoothness α. The parameter q gives a finer gradation of smoothness (see (57) for a precise definition). These spaces occur naturally in many fields of analysis. Many of their applications require knowledge of their interpolation properties, i.e. a description of the spaces which arise when the real method of interpolation is applied to a pair of these spaces. There are two definitions of Besov spaces which are currently in use. One uses Fourier transforms in its definition and the second uses the modulus of smoothness of the function f. These two definitions are equivalent only with certain restrictions on the parameters; for example they are different when $p < 1$ and α is small. The first and simplest interpolation theorems for Besov spaces were for interpolation between a pair $B_q^\alpha(L_p)$ and $B_\gamma^\beta(L_p)$ with $p \ge 1$ fixed. In this case, the real method of interpolation for the parameters (θ, s) applied to these spaces gives the Besov space $B_s^\gamma(L_p)$ with, $\gamma = \theta\alpha + (1-\theta)\beta$. Hence, when p is held fixed the Besov spaces are invariant under interpolation.

More interesting and somewhat ‘more difficult to describe are the interpolation spaces when p is not fixed. Such a program has been carried out in the book of Petrel [121] using the Fourier transform definition of the Besov spaces. The main tool in describing these interpolation spaces is to correspond to each f in the Besov space a sequence of trigonometric polynomials obtained from the Fourier series of. In this way, the Besov space $B_q^\alpha(L_p)$ is identified with a weighted sequence space$l_q^\alpha(L_p)$. Interpolation properties of the Besov spaces are then derived from the interpolation between sequence spaces (when these are known). The success of this approach when $p < 1$ rests on the fact that the corresponding Besov spaces are defined using H_p norms so that each f in the Besov space is a distribution and therefore has a Fourier series.

The Besov spaces defined by the modulus of smoothness occur more naturally in many areas of analysis including approximation theory. Especially important is the case when $p < 1$ since these spaces are needed in the description of approximation classes for the classical methods of nonlinear approximation such as rational approximation and approximation by splines with free knots (see [121]).

The purpose of the present section is to describe the interpolation of the Besov spaces defined by the modulus of smoothness. This is established by developing the connections between Besov spaces and approximation by dyadic splines. We shall show that a function is in $B_q^\alpha(L_p)$ if and only if it has a certain rate of approximation by dyadic splines . In this way, we can identify $B_q^\alpha(L_p)$ with certain sequence spaces in a manner similar to that described above for the Fourier transform definition. While the basic ideas behind such identification is rather simple, some of the details become technical when dealing with the case $p < 1$. One of the main difficulties encountered is that in contrast to the Fourier transform case, the mapping which we use to associate to each $f \in L_p$ a dyadic spline is nonlinear when $p < 1$.

In the process of proving our 'interpolation theorem, we shall develop several interesting results about dyadic spline approximation and about the representation of a function $f \in B_q^\alpha(L_p)$ as a series of dyadic splines (see the atomic decomposition in Corollary (3.2.14)).

Let Ω be the unit cube inR^d. If$f \in L_p(\Omega)$, $0 < p \le \infty$, we let $w_r(f,t)_p$, $t > 0$, denote the modulus of smoothness of order r of $f \in L_p(\Omega)$:

$$w_r(f,t)_p := \sup_{|h|\le t} \Delta_h^r(f,.)_p\big(\Omega(rh)\big) \tag{54}$$

Where $|h|$ is the Euclidean length of the vector h; Δ_h^r is the rth order difference with step $h \in R^d$; and the norm in (54) is the L_p "norm" on the set $\Omega(rh) := \{x: x + rh \in \Omega\}$. Of course, when $p < 1$, this is not really a norm, it is only a quasi-norm, i.e. in place of the triangle inequality, we have

$$\|f + g\|_p \le 2^{1/p}\big[\|f\|_p + \|f\|_q\big] \tag{55}$$

Also useful is the fact that for any $\mu \le \min(1,p)$ and any sequence (f_i)

$$\left\|\sum f_i\right\|_p \le \left(\sum \|f_i\|_p^{\mu}\right)^{1/\mu} \tag{56}$$

If $\alpha, p, q > 0$, we say f is in the Besov space $B_q^\alpha(L_p)$ whenever

$$\left\|\sum f_i\right\|_p \le \left(\int_0^\infty \big(t^{-\alpha}\omega_r(f,t)_p\big)^{q\frac{dt}{t}}\right)^{1/q} \tag{57}$$

Is finite. Here, r is any integer larger than a. When $q = \infty$, the usual change from integral to sup is made in (56).

We also define the following "norm" for $B_q^\alpha(L_p)$:

$$\|f\|_{B_q^\alpha(L_p)} = \|f\|_p + |f|_{B_q^\alpha(L_p)} \tag{58}$$

Different values of $r > \alpha$ result in norms (58) which are equivalent. This is proved by establishing inequalities between the moduli of smoothness ω_r and $\omega_{r'}$ when $r' \leq r$, A simple inequality is $\omega_r \leq c\omega_{r'}$ which follows readily from (55). In the other direction, we have the Marchaud type inequality:

$$\omega_{r'}(f,t)_p \leq \left[\|f\|_p + \left(\int_t^\infty \left(s^{-r'}\omega_r(f,t)\right)^{\mu\frac{ds}{s}}\right)^{1/\mu}\right] \tag{59}$$

Which holds for every $\mu \leq \min(1,p)$. This inequality can be proved by using standard identities for differences , we give a different proof of (59) using dyadic spline approximation. Using these two inequalities for moduli together with the Hardy inequality [113]J, one shows that any two norms (58) are equivalent provided that $r > \alpha$.

There are many other norms which are equivalent to (58). We shall have occasion to use several of these which we describe in later sections. A simple observation is

$$\|f\|_{B_q^\alpha(L_p)} \simeq \|f\|_p + \left(\sum_{k=1}^{\infty}\left[2^{k\alpha}\omega_r(f,2^{-k})_p\right]^q\right)^{1/q} \tag{60}$$

In fact, since ω_r is bounded, the integral in (57) is equivalent to the integral of the same integrand taken over [O,IJ. Now, the monotonicity properties of ω_r allow us to discretize the integral and obtain that (60) is equivalent to (57).

We want to show that $\omega_r(f,2^{-k})_p$ in (60) can be replaced by the error of dyadic spline approximation. This requires inequalities between the modulus of smoothness and the degree of spline approximation. These will be given .To-estimate the degree of spline approximation by the modulus of smoothness, we first need estimates for local polynomial approximation. We define the local error of approximation by polynomials by

$$E_r(f,I)_p = \inf_{deg(Q) < r}\|f-Q\|_p(I) \tag{61}$$

With deg (Q) the coordinate degree of Q. We use the notation $\| \quad \|_p(I)$ to denote the L_p norm over I; when I is omitted the norm is understood to be taken over Ω.

$$E_r(f,I)_p \leq c\omega_r(f,l_I)_p \tag{62}$$

With l_I the side length of I. Here and in what follows, c is a constant which depends only on r,d (and p, if p appears) unless otherwise stated, the value of c may vary at each appearance.

Whitney's theorem is best known for univariate functions and $p \geq 1$. It has also been proved by Yu. Brudnyi [112] for multivariate functions and $p \geq 1$. A proof of (62) for all p and all dimensions d can be found in the section of Storozhenko and Oswald [123]. We would also like to mention that the ideas used in the univariate proof for $p \geq 1$ carry over to the general case. For example, in the forthcoming book of Popov and Petrushev [124], the reader will find a proof of this type for $p < 1$ for univariate functions.The modulus of smoothness is not suitable when we want to add up estimates over several intervals. We therefore introduce the following modified modulus:

$$\omega_r(f,t)_p = \omega_r(f,t,I)_p = \left[\|f\|_p + \left(t^{-d}\int_{\Omega_t}\int_{I(rs)} |\Delta_s^r(f,x)|^p dxds\right)^{1/p}\right] \tag{63}$$

Where $\Omega_t = [-t,t]^d$. Using identities for differences, it can be shown that ω_r and ω_r are equivalent, i.e., $c_1\omega_r(f,t)_p \leq \omega_r(f,t)_p \leq c_2\omega_r(f,t)_p$ with constants $c_1, c_2 > 0$ which depend only on r, p and d (see [124] for a proof of this in the univariate case; the same proof applies to the multivariate case as well). From this, we have the following result

Lemma (3.2.1) [109]:

If $f \in L_p(I)$, with $0 < P \leq \infty$ and if I is a cube with side length l_I, then

$$E_r(f,I)_p \leq c\omega_r(f,l_I,I)_p \tag{64}$$

This result in a slightly different form can also be found in [123].

There always exist polynomials Q of best $L_p(I)$ approximation of coordinate degree $< r$: $\|f - Q\|_p(I) = E_r(f,I)_p$. In the present section we shall also find it very useful to use the concept of "near best" approximation. We say Q is a near best $L_p(I)$ approximation to f from polynomials of coordinate degree $< r$ with constant A if

$$\|f - Q\|_p(I) \leq AE_r(f,I)_p \tag{65}$$

It follows that if P is any polynomial of coordinate degree$< r$, then

$$\|f - Q\|_p(I) \leq A\|f - P\|_p(I) \tag{66}$$

One method for coustructing near best approximants of f is as follows. We let $\rho \leq p$ and we let Q_ρ be any polynomial of near best $L_\rho(I)$ approximation to f of coordinate degree $< r$, i.e, $\|f - Q_\rho\|_\rho(I) \leq AE_r(f,I)_\rho$.

Lemma (3.2.2) [109]:

if $\rho \leq p$, and Q_ρ is as above, we have

$$\|f - Q\|_p(I) \leq cAE_r(f,I)_p \tag{67}$$

with the constant c depending only on r, d and ρ.

Proof.

Let Q be a best $L_p(I)$ approximation to f of coordinate degree $< r$. Then, from elementary properties of polynomials (see[119], we have with $\theta = 1/p - 1/\rho$,

$$\|f - Q_\rho\|_p \le c\left(E_r(f,I)_p + \|Q - Q_\rho\|_p\right)$$

$$\le c\left(E_r(f,I)_p + |I|^\theta \|Q - Q_\rho\|_\rho\right)$$

$$\le c\left(E_r(f,I)_p + |I|^\theta \left[\|f - Q\|_\rho(I) + \|f - Q_\rho\|_\rho(I)\right]\right)$$

$$\le c\big(E_r(f,I)_p + |I|^\theta (A+1)\|f - Q\|_\rho(I)\big)$$

$$\le c\big(E_r(f,I)_p + (A+1)\|f - Q\|_\rho(I)\big) \le cAE_r(f,I)_p$$

Here, the first inequality uses the quasi-norm property (55); the second inequality is a comparison of polynomial norms; the third again uses (55); the fourth uses (66); and the fifth inequality is HOlder's inequality.

We introduce the following notation. If I is any cube, we let P_I denote any near best $L_\rho(I)$ approximation to f from polynomials of coordinate degree $< r$ with constant A. The following lemma shows that P_I is also a near best approximation on larger cubes.

Lemma (3.2.3) [109]:

For any $p \ge \rho$ and any cube $J \supset I$with $|J| \le a|I|$, we have

$$\|f - P_I\|_p(J) \le cE_r(f,J)_p \tag{68}$$

withc depending at most on r, d, a and A.

Proof.

If P is the best L_p approximation to f on J from polynomials of coordinate degree$< r$, then from (55) and Lemma (3.2.2),

$$\|P_I - P\|_p(I) \le c\big[\|f - P_I\|_p(I) + \|f - P\|_p(I)\big]$$

$$\le c\big[E_r(f,I)_p + E_r(f,J)_p\big]$$

$$\le cE_r(f,J)_p$$

This estimate can be enlarged to J (see [115]):

$$\|P_I - P\|_p(J) \le cE_r(f,J)_p$$

Hence,

$$\|f - P_I\|_p(J) \le c\left[\|f - P\|_p(J) + \|P - P_I\|_p(J)\right] \le cE_r(f,J)$$

Dyadic spline approximation. We want in this section to describe the connection between Besov spaces and dyadic spline approximation. Our main goal is to show that ω_r in (59) can be replaced by an error in dyadic spline approximation with a resulting equivalent seminorm. This means that the Besov spaces $B_q^\alpha(L_p)$ are the approximation spaces for tile approximation by dyadic splines inL_p . Such characterizations are known when $p \ge 1$ (see [114]; also [118]) and when $p < 1$ and $d = 1$ (see [115]).

We let D_k denote the collection of dyadic cubes of R^d of side length 2^{-k} and we let $D_k(\Omega)$denote the set of those cubes $I \in D_k$with $I \in \Omega$. We introduce two spline spaces for this partition. The first of these is $\prod_k = \prod_k(r)$, the space of all piecewise polynomials of coordinate degree $< r$ on the partition D_k . That is, $S \in \prod_k$ means that in the interior of each cube $I \in \prod_k$, S is a polynomial of coordinate degree $< r$. We denote by $\prod_k(\Omega)$ the restrictions of splines S in $\prod_k$to Ω.

A best (or near best) approximation s_k to I in $L_p(\Omega)$ from $\prod_k(\Omega)$ is gotten by simply taking $S = P_I, x \in I$, .where $P_I, I \in D_k(\Omega)$, is a best (or near best) approximation to f in $L_p(I)$ by polynomials of coordinate degree $< r$ on each cube I from $D_k(\Omega)$. However, we shall also need to construct good approximations from $\prod_k(\Omega)$ which have smoothness. For this, we shall use the tensor of product B-splines and the quasi-interpolants of de Boor-Fix.

Let N be the univariate B-spline of degree $r - 1$ which has knots at the points $0,1,\dots,r$, i.e., $N(x) = r[0,1,\dots,r](x - .)_+^{r-1}$ with the usual divided difference notation. For higher dimensions, we define N by

$$N(x) \coloneqq N(x_1) \dots N(x_d) \tag{69}$$

These are the tensor product of B-splines. They are piecewise polynomials of coordinate degree $< r$ which have continuous derivatives $D^v N, 0 \le v \le r - 2$, and derivatives $D^v N$ in L_∞ for $0 \le v < r - 1$. We use the notation $k = (k,k,\dots,k)$. The splines N are nonnegative and are supported on the cube $[0,r]^d$.

To get splines in the space $\prod_k$, we let

$$N_k(x) \coloneqq N(2^k x), \;\; k = 0,1,\dots \tag{70}$$

and

$$N_{j,k}(x) \coloneqq N_k(x - x_j), \;\; j \in Z^d \tag{71}$$

Where the $X_j = 2^{-k_j}$ are the vertices of the cubes in D_k. The B-splines $N_{j,k}$ are a partition of unity, i.e

$$\sum_{j\in Z^d} N_{j,k} \equiv 1\,,\quad onR^d \tag{72}$$

Each spline S in the span of the $N_{j,k}$ can be written in a B-spline series:

$$S = \sum_{j\in Z^d} \alpha_j(S) N_{j,k} \tag{73}$$

with the $\alpha_j = \alpha_{j,k}$ the dual functionals of the $N_{j,k}$. The functionals $\alpha_j$can be expressed in terms of the univariate functionals:

$$\alpha_j(S) = \alpha_{ji}((\dots \alpha_{jd}(S)) \tag{74}$$

where the univariate functional α_{jv}. is applied to a multivariate function g by considering g as a function of x_v with the other variables held fixed. There are many representations for the functional α_j .We mention in particular, the de Boor-Fix formula [111]. This representation gives that for any point $\xi_j$in the supp (N_j), we can write

$$S\alpha_j = \sum_{0\leq r\leq r-1} a_v D^v(S)(\xi_j)\, j \in A \tag{75}$$

for certain coefficients a_v depending on ξ_j and r.

For approximation on Ω, we need only the B-splines $N_{j,k}$ which do not vanish identically on Ω. We let $A = A(k)$ denote the set of j for which this is the case and we let $\sum_k = \sum_k(\Omega)$ denote the linear span of the B-splines $N_{j,k}, j \in A$. Then any $S \in \sum_k can$ be written

$$S = \sum_{j\in A} \alpha_j(S)\, N_{j,k} \tag{76}$$

For the representation of α_j , $j \in A$, we shall choose the points ξ_j as the center of a cube $J_j = J_{j,k} \ \in D_k$ such that

$$\xi_j \in J_j \subset supp(N_j) \cap \Omega j \in A \tag{77}$$

With this choice, we can define $\alpha_j(f)$ for any f which is suitably differentiable at ξ_j. In particular, in this way, we have that α_j is defined for any $S \in \prod_k$ From (75), it is easy to estimate the coefficients of a spline $S \in \prod_k$

Lemma (3.2.4) [109]:

We have for any $0 < P \leq \infty$ and any $S \in \prod_k$,

$$|\alpha_j(S)| \leq c2^{kd/p} \|S\|_p(J_j) \tag{78}$$

Proof.

This is well known for one variable and $p \geq 1$. A similar proof applies in the general case. For example, from Markov's inequality applied to S on J_j, and estimates for the coefficients a_v (see [115]), it follows that

$$|\alpha_j(S)| \leq c\|S\|_\infty(J_j) \tag{79}$$

Since$|J_j| = 2^{-k}$, (78) follows from (79) and the well-known inequality between L_p and L_∞ norms for polynomials (see [119]). Closely related to (78) is the following.

Lemma (3.2.5) [109]:

If $S = \sum_{j\in \mathrm{A}} \alpha_j\, N_{j,k}$ is in $\sum_k$ *then* for any $0 < p \leq \infty$ we have

$$c_1\|S\|_p \leq \left(\sum_{j\in A} |\alpha_j(S)|^p 2^{-kd}\right) \leq c_2\|S\|_p \tag{80}$$

With c_1, c_2 depending at most on d and r.

Proof.

Again this is well known (see [110]) when $P \geq 1$ and the general case is proved in the same manner. For example, since $\sum_k \subset \prod_k$, the right side of (80) follows from (78) and the fact that a point x falls in at most r^d of the cubes J_j.For the left inequality, we use the fact that at most r^d terms in the representation of S are nonzero at a given point x. Hence

$$|S(x)|^p \leq c \sum_{j\in A} |\alpha_j|^p N_{j,k}\,(x)^p$$

Integrating with respect to x and using the fact that the integral of $N_{j,k}^p$ is less than $c2^{-kd}$ (because $N_{j,k} \leq 1$) gives the desired result.

Now, let f be any function which is $r - 1$ times continuously differentiable at each of the points ξ_j. Then $\alpha_j(f)$.is defined for all j and we define

$$Q_k(f) := \sum_{j\in A} \alpha_j(f)\, N_{j,k} \tag{81}$$

The Q_k are called quasi-interpolant operators. In particular Q_k is defined for all $S \in \prod_k$, and it follows that Q_k is a projector from $\prod_k$ onto $\sum_k$: $Q_k(S) = S$ whenever $S \in \sum_k$: .

We want to examine the approximation properties of the Q_k. For this, we introduce the following notation. If $\in D_k$, we let $\hat{I}$ be the smallest cube which contains each of the J_j, for which supp $N_{j,k} \cap I \neq \emptyset$. Then, $\hat{I} \subset \Omega$ and $|\hat{I}| \leq c|I|$with c depending only on d and r,

Lemma (3.2.6) [109]:

If $S \in \prod_k$ and $0 < p \leq \infty$, then for each $I \in D_k(\Omega)$, we have

$$\|Q_k(S)\|_p \leq c\|S\|_p(\hat{I}) \quad (82)$$

And

$$\|S - Q_k(S)\|_p(I) \leq cE_r(S, \hat{I})_p \quad (83)$$

Proof.

We let A_I be the set of j such that $N_{j,k}$ does not vanish identically on I, We use the representation (81) and the estimate (78) for the functionals α_j, to find

$$\|Q_k(S)\|_p(I) \leq \max_{j \in A_I} |\alpha_j(S)| \left\| \sum_{j \in A_I} N_{j,k} \right\|_p (I) \quad (84)$$

$$\leq c|I|^{1/p} \max_{j \in A_I} 2^{kd/p} \|S\|_p(J_j) \leq c\|S\|_p(\hat{I})$$

Because of (72). This is (82).

To prove (83), we let $I \in D_k$ and let P be a polynomial of best $L_p(\breve{I})$ approximation to S of coordinate degree$\leq r$. Since $Q_k(P) = P$, we have by (55) and (82)

$$\|S - Q_k(S)\|_p(I) \leq \left[c\|S - p\|_p(I) + \|Q_k(S - p\|_p(I)\right] \quad (85)$$

$$\leq c\|S - p\|_p(\breve{I}) = cE_r(S, \hat{I})_p$$

Corollary (3.2.7) [109]:

If $0 < p \leq \infty$, then $\|Q_k(S)\|_p \leq c\|p\|_p$. for all $S \in \sum_k$:

Proof.

This follows immediately from (82) when $p = \infty$. When $0 < p < \infty$, we raise both sides of (82) to the power p and then we sum over $I \in D_k(\Omega)$. Since each point $x \in \Omega$ appears in at most c of the cubes $\breve{I}$, with e depending only on r and d, the corollary follows.

We want to describe a class of methods for approximating each f in $L_p(\Omega)$ by smooth dyadic splines from $\sum_k$. *For* each $I \in D_k$and $f \in L_p(\Omega)$, we let $P_I = P_I(f)$ be a near best $L_p(I)$approximation to f from polynomials of coordinate degree $< r$ with an absolute constant A. We then define $S_k = S_k(f) \in \prod_k, k = 0, 1, \ldots$, to be the piecewise polynomial

$$S_k = P_1(x), \quad x \in interior(I), forallI \in D_k \tag{86}$$

From (68), we have

$$\|f - P_I\|_p(\check{I}) \in cE_r(S, \hat{I})_p I \in D_k \tag{87}$$

with e depending only on r, d and A.

Going further, for each $f \in L_p(\Omega)$, we define

$$T_k = T_k(f) = Q_k(S_k(f)), \quad k = 0,1, \ldots \tag{88}$$

Then T_k is in $\sum_k$. and we have

$$\|T_k(f)\|_p \le c\|f\|_p \tag{89}$$

With c depending only on r, d and A. Indeed, since P_I is a near best approximation to f, we have $\|P_I\|_p(I) \le c\|f\|_p(I), I \in D_k(\Omega)$. Hence, $\|S_k(f)\|_p \le c\|f\|_p$. And therefore (89) follows from Corollary (3.2.7).

Theorem (3.2.8) [109]:

For any of the operators T_k in (88) and lor each $f \in L_p(\Omega)$, , we have

$$\|f - T_k(f)\|_p \le c\omega_r(f, 2^{-k})_p, \quad k = 0,1, \ldots \tag{90}$$

With c depending only on r, d, p and A.

Proof.

From (83), we have for each $I \in D_k(\Omega)$,

$$\begin{aligned} \|f - T_k\|_p(I) &\le [c\|f - S_k\|_p(I) + \|S_k - Q_k(S_k)\|_p(I)] \qquad (91) \\ &\le c[\|f - p_I\|_p(I) + E_r(S_k, \hat{I})_p] \\ &\le c\left[E_r(f, \hat{I})_p + E_r(S_k, \hat{I})_p\right] \end{aligned}$$

Now, for any cube $J \subseteq \hat{I}$with $\in D_k$, we have from (87)

$$\begin{aligned} \|S_k - P_I\|_p(J) = \|P_J - P_I\|_p(J) &\le c\left[\|f - P_J\|_p(J) + \|f - P_I\|_p(J)\right] \qquad (92) \\ &\le c\left[E_r(f, J)_p + E_r(f, \hat{I})_p\right] \le cE_r(f, \hat{I})_p \end{aligned}$$

Since the number of cubes $J \in D_k$ contained in Î depends only on dand r, (92) gives $E_r(S_k, \hat{I})_p \leq cE_r(f, \hat{I})_p$ If we use this in (91), we obtain

$$\|f - T_k\|_p(I) \leq cE_r(f, \hat{I})_p \tag{93}$$

Now, each point x ∈ Ω appears in only a constant depending only on r and d number of cubes Î. Hence, if we raise both sides of (93) to the power p and sum over all Iin $D_k(\Omega)$, and use (64); we obtain

$$\|f - T_k\|_p^p(\Omega) \leq c \sum_{I \in D_k(\Omega)} \omega_r(f, l_{\tilde{I}}, \tilde{I})_p^p \quad \leq ct^{-d} \int_{\Omega_t} \int_{\Omega(rs)} |\Delta_s^r (f, x)|^p dxds \tag{94}$$

With $t = \max l_{\tilde{I}} \leq c2^{-k}$. Here, we have used the fact that $\omega_r(f, t') \leq c\omega_r(f, t)_p$provided $t' \leq t \leq ct'$, Finally, (90) follows from (94)because each of the interior integrals on the right side of (94) does not exceed $\omega_r(f, t)_p^p$ which from the usual properties of modulus is $\leq c\omega_r(f, 2^{-k})_p^p$.

Theorem (3.2.9) [109]:

Shows that the error of dyadic approximation can be majorized by the modulus of smoothness. Namely, if we let

$$S_k(f)_p = \inf_{s \in \Sigma_k} \|f - s\|_p \tag{95}$$

Then we have

Corollary (3.2.10) [109]:

For each $f \in L_p(\Omega)$and for each $r = 1,2,\ldots$, we have

$$S_k(f)_p \leq c\omega_r(f, 2^{-k})_p k = 0,1,\ldots. \tag{96}$$

It is also important to note that $T_k(f)$is a near best approximation from Σ_k

Corollary (3.2.11) [109]:

If $f \in L_p(\Omega)$, then

$$\|f - T_k(f)\|_p \leq cS_k(f)_p$$

With $c$depending only on r, d, p and A.

Proof.

Let S be a best $L_p(\Omega)$ approximation to f from Σ_k. Then since

$Q_k(S) = S$, we have$f - T_k(f) = f - S + Q_k(S - S_k(f))$.If we use the fact that $Q_k(S)$is bounded (Corollary 4.4), we obtain

$$\|f - T_k(f)\|_p \le c\big[\|f - S)\|_p + \|S - S_k(f)\|_p\big]$$

$$\le c\big[\|f - S)\|_p + \|f - S_k(f)\|_p\big] \le cS_k(f)_p \ .$$

Here, the last inequality uses the fact that $S_k(f)$is a near best approximation from $\Pi_k$with constant A and the error of approximating f from Π_k is smaller than the error in approximating f from Σ_k (because $\Sigma_k \subset \Pi_k$).

We also need inverse estimates to (96). We let $S_{-1}(f)_p = \|f\|_p$,

Theorem (3.2.12) [109]:

For each $k > 0$, and each $r = 1,2,\ldots$, we have for $\lambda := \min(r, r-1+1/p)$ and for each $f \in L_p$,

$$\omega_r(f, 2^{-k})_p \le cU_k \left(\sum_{j=1}^{k} \left[2^{j\lambda} S_j(f)_p\right]^{\mu} \right)^{\frac{1}{\mu}} \qquad (97)$$

Provided $\mu \le \min(1, p)$.

Proof:

We let U_k be a best approximation to f from Σ_k and let $\mathcal{U}_k = U_k - U_{k-1}, k = 0,1,\ldots$,with $U_{-1} := 0$. If $|h| \le r^{-1}2^{-k}$ and $x \in \Omega(rh)$, we write

$$\Delta_h^r(f,x) = \Delta_h^r(f - U_k, x) + \sum_{j=0}^{k} \Delta_h^r(\mathcal{U}_j, x) \qquad (98)$$

Then, from (63),

$$\|\Delta_h^r(f)\|_p\big(\Omega(rh)\big) \le c \left(S_k(f)^{\mu} + \sum_{j=0}^{k} \left\|\Delta_h^r(\mathcal{U}_j)\right\|_p \big(\Omega(rh)\big) \right)^{1/\mu} \qquad (99)$$

To estimate $\left\|\Delta_h^r(\mathcal{U}_j)\right\|_p\big(\Omega(rh)\big)$, we write $\mathcal{U}_j$in its B −splineseries:

$$\mathcal{U}_j = \sum_{v \in \Lambda(j)} \alpha_{v,j}(\mathcal{U}_j) N_{v,j} \qquad (100)$$

For any x,at mostcB −splines (4.32) are nonzero at $x$with $c$depending only on r and d.Hence,

$$\left|\Delta_h^r\left(\mathcal{U}_j,x\right)\right|^p \leq \sum_{v\in\Lambda(j)} \left|\alpha_{v,j}(\mathcal{U}_j)\right|^p \left|\Delta_h^r\left(N_{v,j},x\ \right)\right|^p \tag{101}$$

Now, we shall give two estimates for $\Delta_h^r\left(N_{v,j},x\ \right)$. The first of these is for the set Γwhich consists of all $x$such that $x$and $x+rh$ are in the same cube $I\in D_j$ and $N_{v,j}$ does not vanish identically on$I$. Since $N_{v,j}$is a polynomial on$I$ whose rth order derivatives do not exceed $c2^{jr}$, we have

$$\left|\Delta_h^r\left(N_{v,j},x\ \right)\right|^p \leq c(2^j|h|)^r x\in\Gamma \tag{102}$$

Our second estimate is for the set Γ' which consists of all x such that x and $x+rh$ are in different cubes from D_j and$N_{v,j}$does not vanish identically on both of these cubes. Since$N_{v,j}\in W_\infty^{r-1}$ (Sobolev space) has $(r-1)\ th$ derivatives whose $L_P(\Omega)$ norms do not exceed $c2^{j(r-1)}$, we have

$$\left|\Delta_h^r\left(N_{v,j},x\ \right)\right| \leq c(2^j|h|)^{r-1} x\in\Gamma' \tag{103}$$

Now, the set Γ has measure $\leq c2^{-jd}$ because the support of $N_{v,j}$ has measure $\leq c2^{-jd}$.Also ,Γ' has measure$\leq c|h|2^{-j(d-1)}$. Indeed, for x to be in Γ' , we must have $dist(x,bound(I)\ \leq |rh|$ for the cube I which contains$x$. The measure of all such$x\in I$ is less than $c|h|2^{-j(d-1)}$. Since $N_{v,j}$vanishes on all but c cubes with depending only on $r$and d, we have $\Gamma'\leq c|h|2^{-j(d-1)}$ as claimed.

Using these two estimates for the measure of Γ and Γ' together with (102) and (103), we obtain

$$\int_{\Omega(rh)} \left|\Delta_h^r\left(N_{v,j},x\ \right)\right| \leq c[|h|^{rp}2^{jrp}2^{-d} + |h|^{(r-1)p}2^{j(r-1)p}|h|2^{-j(d-1)} \tag{104}$$

$$\leq c[|h|^{\lambda p}2^{j\lambda p}2^{-jd}$$

Because$|h|2^{-k}\leq r^{-1}\leq 1$.

Now, we integrate (101) and use (104) to find

$$\left\|\Delta_h^r\left(\mathcal{U}_j\right)\right\|_p \leq c|h|^{\lambda p}2^{j\lambda p}\left(\sum\left|\alpha_{v,j}(\mathcal{U}_j)\right|^p 2^{-jd}\right)^{1/p}$$

$$\leq c|h|^{\lambda p}2^{j\lambda p}\left\|\mathcal{U}_j\right\|_p \leq c|h|^{\lambda p}2^{j\lambda p}[\mathrm{S}_j(\mathrm{f})_p + \mathrm{S}_{j-1}(\mathrm{f})_p] \tag{105}$$

where the next to last inequality is (80) and the last inequality is the triangle inequality applied to $\mathcal{U}_j = f - U_{j-1} -\ (f - U_j)$.

If we use (105) in (99), we obtain

$$\left\|\Delta_h^r\left(\mathcal{U}_j\right)\right\|_p(\Omega(rh) \leq c\left(\mathrm{S}_k(\mathrm{f})^\mu+|h|^{\lambda\mu} \sum_{j=-1}^k\left[2^{j\lambda} \mathrm{S}_j(f)\right]^\mu\right)^{\frac{1}{\mu}} \quad (106)$$

If we now take a sup over all $|h| \leq r^{-1}2^{-k}$ (106) gives

$$\omega_r(f, 2^{-k})_p \leq c\omega_r(f, r^{-1}, 2^{-k})_p \leq c2^{-k\lambda}\left(2^{k\lambda\mu}\mathrm{S}_k(\mathrm{f})^\mu+\sum_{j=-1}^k\left[2^{j\lambda} \mathrm{S}_j(f)\right]^\mu\right)^{1/\mu}$$

Since the term $2^{k\lambda\mu}\mathrm{S}_j(f)^\mu$ can be incorporated into the sum, we obtain (97).

It is also possible to estimate $\omega_{r'}$ for each $r' < r$:

$$\omega_{r'}(f, 2^{-k})_p \leq c2^{-kr'}\left(\sum_{j=-1}^k (2^{jr'}S_j(f)_p)^\mu\right)^{\frac{1}{\mu}} \quad (107)$$

Indeed, this is proved in exactly the same way as we have derived (4.29), except that, in place of (102) and (103), we use the inequality

$$\left|\Delta_h^{r'}\left(N_{v,j}, x\right)\right| \leq c|h|^{r'}2^{ir'} \quad (108)$$

Which follows from the fact that $N_{v,j}$has all derivatives of orderr'inL_∞. With (97), we can easily prove the Marchaud type inequality (59).

Corollary (3.2.13) [109]:

There is a constant c depending only on p, r, and d such that for each $f \in L_p$ we have the inequality (59).

Proof.

We have by (96):$S_j(f)_p \leq c\omega_r(f, 2^{-j})_p j = 0,1$`, ... Also $S_{-1}(f)_p := \|f\|_p$Using this in (107) gives for$2^{-k-1} \leq t \leq 2^{-k}$,

$$\omega_{r'}(f, t)_p \leq c\omega_{r'}(f, 2^{-k})_p \leq c2^{-kr'}\left(\|f\|_p^\mu + \sum\nolimits_{j=0}^k\left[\omega_{r'}(f, 2^{2-i})_p\right]^\mu\right)$$

and (59)then follows from the monotonicity of ω_r .

The estimates of the last section allow us to introduce several norms which are equivalent to$\|f\|_{B_q^\alpha(L_p)} If a := (a_k)$ is a sequence whose component functions are in the quasi-normed space X ,we use the $l_q^\alpha(X)$ norms

$$\|a\|_{l_p^\alpha}(X) := \left(\sum_{k=0}^{\infty}[2^{k\alpha}\|a_k\|_X]^q\right)^{1/q} \tag{109}$$

With the usual change to a supremum norm when $q = \infty$.When (a_k) is a sequence of real numbers, we replace $\|a_k\|_X$by $|a_k|$ in (109) and denote the resulting norm $\|a_k\|_{l_q^\alpha}$.

Useful for us will be the discrete Hardy inequalities

$$\|b_k\|_{l_q^\alpha} \le c\|(a_k)\|_{l_q^\alpha} \tag{110}$$

Which hold if either

(i)$|b_k| \le c2^{-k\lambda}\left(\sum_{j=k}^{\infty}\left[2^{j\lambda}|a_j|\right]^{\mu}\right)^{1/\mu}$ or

(ii) $|b_k| \le c(\sum_{j=k}^{\infty}|a_j|\mu$

(111)

With$\mu \le q$ and (in (i)) $\alpha < \lambda$. Here, the constant c in (110) depends only on r, d and $1/(\lambda - \alpha)$ in case of (i) and $1/\alpha$ in the case of(ii).

in the following theorem, we let $T_k = T_k(f)$ be defined as in (4.20) for a given $r = 1,2, \ldots$ and given near best approximations P_I with constant A. We let $t_k = t_k(f) = T_k - T_{k-1}$ with $T_{-1} = 0$ and let $\lambda = \min(r - 1 + 1/p, r)$, as before.

Theorem (3.2.14) [109]:

Let $0 < \alpha and$ $0 < q, p < \infty$. If $\alpha < \lambda$, then the lollowing norms are equivalent to $N(f) = \|f\|_{B_q^\alpha(L_p)}$ with constants of equivalency depending only on d, r and A and the constant of (110):

(i)$N_1(f) := \|s_k(f)\|_{l_q^\alpha} + \|f\|_p$

$(ii)N_2(f) := \|f - T_k(f)\|_{l_q^\alpha(L_p)} + \|f\|_p$ (112)

$(iii)N_3(f) := \|t_k(f)\|_{l_q^\alpha(L_p)}$.

Proof.

From Theorem (3.2.9), $s_k(f)_p \le \|f - T_k(f)\|_p \le c\omega_r(f, 2^{-k})_p$ .Hence, $N_1(f) \le N_2(f) \le cN(f)$. On the other hand, from Theorem (3.2.12) and the Hardy ineqnality (110) above, we have $N(f) \le cN_1(f)$ .This shows that $N(f), N_2(f)$ and $N_2(f)$ are all equivalent .Since $\|t_k\|_p \le c\left[\|f - T_k(f)\|_p + \|f - T_k(f)\|_p\right]$ we have $N_3(f) \le cN_2(f)$ .In the other direction $f - T_k = \sum_{k+1}^{\infty} t_j$ and therefore from (55), we obtain for $k = -1,0,1, \ldots,$

$$\|f - T_k\|_p \leq \left(\sum_{k+1}^{\infty} \|t_j\|_p^{\mu}\right)^{1/\mu}$$

Note, when $k = -1$, this is an estimate for $\|f\|_p$ Now, from the Hardy inequality (110), we have $N_2(f) \leq cN_3(f)$ and therefore $N_2(f)$ and $N_3(f)$ are equivalent.

The norm N_1 of Theorem (3.2.12) shows that a function $f$is in$B_q^{\alpha}(L_p)$. if and only if $(S_k(f))$ is in l_q^{α}. In the terminology of [116],we have that the approximation class A_q^{α} for dyadic spline approximation in L_p.is the sameasthe Besov space $B_q^{\alpha}(L_p)$. Related to this is the following Bernstein type inequality for dyadic splines.

Corollary (3.2.15) [109]:

If $r = 1,2,\ldots$ and$\alpha < \lambda$,then for each $S \in \Sigma_n$

$$\|S\|_{B_q^{\alpha}(L_p)} \leq c2^{\alpha n}\|S\|_p \tag{113}$$

With$c$ in dependent of $S$and n.

Proof.

Since $S \in \Sigma_n, S_k(S) = 0, k \geq n$,and for $k < n$,we have$S_k(S)_p \leq \|S\|_p$.

Hence , for $q < \infty$,

$$N_1(S)^q \leq c \sum_{k=-1}^{n} \left[2^{\alpha k} S_k(S)_p\right]^q \leq c2^{\alpha n}\|S\|_p^q$$

and (5.5)follows from Theorem(3.2.12),Similarly for $q = \infty$.

Another interesting application of Theorem (3.2.12) is the following atomic decomposition for functions $f$in $B_q^{\alpha}(L_p)$ AccordingtoTheorem(3.2.12),we can writ $f = \sum \mathrm{t}_k$ with the notation of that theorem .Since $\mathrm{t}_k \in \Sigma_k$, we have

$$\mathrm{t}_k = \sum_{v \in \Lambda(\mathrm{k})} \alpha_{v,k}\, N_{v,k} \tag{114}$$

With $N_{v,k}$The B-splines for D_k .Hence,

$$f = \sum_{k=0}^{\infty} \sum_{v \in \Lambda(\mathrm{k})} \alpha_{v,k}\, N_{v,k} \tag{115}$$

With convergence in the sense of L_p.

Corollary (3.2.16) [109]:

Let $0 < q, p \leq \infty$ and $r = 1,2,\ldots$ .If $0 < \alpha < \lambda$, then a function$f \in L_p$.is in $B_q^\alpha(L_p)$ if and only if f can be represented as in (5.7)with

$$N_4(f) = \left(\sum_{k=0}^{\infty} 2^{\alpha k q}\left(\sum_{v\in\Lambda(\mathrm{k})} \left|\alpha_{v,k}\right|^p 2^{-kd}\right)^{q/p}\right)^{q/q} < \infty \tag{116}$$

(and the usual modification if either $p$or $q = \infty$) and $N_4(f)$is equivalent to$\|f\|_{B_q^\alpha(L_p)}$.

Proof .

From Lemma (3.2.5),

$$\mathrm{t_k} \simeq \left(\sum_{v\in\Lambda_\mathrm{k}} \left|\alpha_{v,k}\right|^p 2^{-kd}\right)^{1/\mathrm{p}}$$

Hence from Theorem (3.2.12),$N_4(f)$is equivalent to $N_3(f)$which is in turn equivalent to $N(f)$.

A different atomic decomposition was given by M. Frazier and B.J aewerth [120] for Besov spaces defined by the Fourier transform. In the case$d = 1$, there is also an atomic decomposition using spline functions by Ciesielski[114].

We are now interested in proving interpolation theorems forBesov spaces. If $\alpha_0, \alpha_1, > 0$, and $0 < p_0, p_1, q_0, q_1 \leq \infty$, we introduce the abbreviated notation $B_i := B_{q_i}^{\alpha_i}(L_{p_i})$ and $l_i := l_{q_i}^{\alpha_i}(L_{p_i}), i = 0, 1$.

We recall that if X_0, X_1, is a pair of quasi-nonmed spaces which are continuously embedded in a Hausdorff space$X$, then the K-functional

$$K(f, t, X_0, X_1) = \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\} \tag{117}$$

is defined for all $f \in X_0 + X_1$,This K-functional determines new function spaces. If $0 < \theta < 1$ and $0 < q \leq \infty$, we define the space $X_{\theta,q} := (X_0, X_1)_{\theta,q}$. as the set of all f such that

$$\|f\|_{X_{\theta,q}} = \|f\|_{X_0+X_1} + \left(\int_0^\infty \left[t^{-\theta}K(f,t)\right]^q \frac{dt}{t}\right)^{1/q} \tag{118}$$

is finite.

We wish to establish a connection between the K-functional for $B_0, B_1$and the K-functional forl_0, l_1. For this, we fix a number $0 < \rho \leq p_0, p_1$ and an integer r such that $\alpha_0, \alpha_1 \leq r - 1$. We let$P_I(f)$be the best $L_p(I)$ approximation to$f$ from polynomials of coordinate degree$< r$. A ccording to Lemma (3.2.2), if $S_k(f)$ is defined by (86),and$T_k(f)$ is defined by (88) then $N_2(f)$ and $N_3(f)$of Theorem (3.2.12) are equivalent to the norm of $B_q^{\alpha}(L_p)$.

If$f \in L_p$, we let $Tf = (t_k(f))$. In this way ,we associate to each $f \in L_p$ a

Sequence of dyadic splines and $f \in L_p$if and only if $B_q^{\alpha}(L_p)$and from Theorem (3.2.12)

$$\|f\|_{B_q^{\alpha}(L_p)} \cong \|Tf\|_{l_q^{\alpha}(L_p)} \tag{119}$$

for all$\alpha, q > 0$,provided $p \geq \rho$.

Theorem (3.2.17) [109]:

There are constants $c_0, c_1 > 0$ which depend only on ρ, r, d, α_0, and α_0 such that

$$c_1 K(f, t, B_0, B_1) \leq K(Tf, t, l_0, l_1) \leq c_2 K(f, t, B_0, B_1) \quad , t > 0 \tag{120}$$

Whenever $\in B_0 + B_1$.

Proof of lower inequality . We suppose that $a = (a_k) \in l_1$ is such that $Tf - a$ is in l_0, We define $g_k = T_k(a_k) = Q_k(S_k(a_k))$ as in (88). Then by (89), $\|g_k\|_{p_1} \leq c\|a_k\|_{p_1}$.Now, we let $g \coloneqq \sum_0^{\infty} g_k$with convergence in$L_{p_1}$. Since $\sum_0^{\infty} g_j$ is in $\Sigma_k$we have from (55)

$$S_k(g)_{p_1} \leq \left\|\sum_{k+1}^{\infty} g_j\right\|_{p_1} \leq c\left(\sum_{k+1}^{\infty} \|g_j\|_{p_1}^{\mu}\right)^{\frac{1}{\mu}} \quad , \; k = -1, 0, \ldots,$$

Provided $\mu \leq p_1$. Here, when $= -1 S_1(f)_p = \|f\|_p$, as usual . If we take also $\mu \leq p_1$we have from the Hardy inequality (110) and the equivalence of the norms N and N_1, in Theorem (3.2.12) that

$$\|g\|_{B_1} \leq c\|a\|_{l_1} \tag{121}$$

We can prove a similar estimate for $f - g$.Namely,

$$S_k(f - g)_{p_0} \leq \left\|\sum_{k+1}^{\infty} (t_j - g_j)\right\|_{p_0} \leq \left(\sum_{k+1}^{\infty} \|t_j - g_j\|_{p_0}^{\mu}\right)^{\frac{1}{\mu}} \tag{122}$$

Now $t_j = Q_j(t_j)$ because$Q_j$is a projector. Also since$t_j \in \Pi_j$, we have $S_j(a_j - t_j) = S_j(a_j) - t_j$. Hence,

$$\|t_j - g_j\|_{p_0} = \|Q_j(t_j - S_j(a_j))\|_{p_0} = \|Q_j(S_j(t_j - a_j))\|_{p_0}$$
$$\leq c\|t_j - a_j\|_{p_0}$$

because of (89).If we use our last inequality in (6.6) and then argue as in the proof of (6.5), we obtain

$$\|f-g\ \ \|_{B_0} \le c\|Tf-a\ \ \|_{l_0} \tag{123}$$

Since $a \in l_1$ is arbitrary,(121), (123) and the definition of the K-functional give the lower inequality in (120).

For the proof of the upper inequality in (120), we shall need a result about Approximation in a quasi-normed space X .We suppose that Z is a linear subspace of X such that each element $x \in X$ has a best approximation from Z. We let

$$E(x) = \inf_{z \in Z}\|x-z\|_X \tag{124}$$

We say that z is a near best approximation to x with constant A if

$$\|x-z\| \ \ \le \ AE(x) \tag{125}$$

Lemma (3.2.18) [109]:

Let X and Z be as above. If $x \in X$ and $z \in Z$ is a near best approximation to x with constant A ,then for each $y \in X$,there is a $z' \in Z$ such that z' is a near best approximation to $y$ond $z - z'$ is a near best approximation to $x-y$ with constants c depending only on X and A.

Proof.

Let γ be such that $\|u+v\| \le \gamma(\|u\|+\|v\|)$ for all $u, v \in X$ (all norms in this proof are for X).

Case:$E(x-y) \le E(y)$.We let $z' = z" + z$ with $z"$ a best approximation to $y-z$. Then, by definition $z-z'$ is near best for $x- \ y$ with constant1. On the other hand,

$$\|y-z'\| = \|y-z-z"\| \ \le \gamma(\|y-x-z"\| + \|x- \ z\| \le \gamma(E(x-y)+AE(x))$$

$$\le \gamma(E(x- \ y)+\gamma AE(y)+\gamma AE(x-y)) \le \gamma + 2r^2A)E(y)$$

Case: $E(y) \le E(x-y)$. The same as the previous case with $x- \ y$ and y interchanged.

Proof of the upper inequality in (120) . We suppose that g is any function in B_1 for which $f-g$is in B_0. We let P_I be the polynomials which make up $S_k := S_k(f)$. Then P_I is a best $L_\rho(I)$ approximation of f from polynomials of coordinate degree$< r$. Therefore, we can apply Lemma 6.2 to obtain a near best $L_\rho(I)$ approximation Q_I to g from polynomials of coordinate degree $< r$ such that $P_I - Q_I$ is also a near best $L_\rho(I)$ approximation to$f-g$.

We let U_k, R_k be obtained from Q_I and $P_I - Q_I$,$I \in D_k$, by using quasi-interpolants in the same way that T_k was defined from the P_I. Since Q_k is linear, we have $R_k = T_k -$

U_kThen, by Corollary4.7, U_k and R_k are respectively near best $L_{p_1}$and L_{p_0} approximations to$g$ and $f-g$from Σ_p, $k=0,1,\ldots$.

We let $t_k = T_k - T_{k-1}, u_k = U_k - U_{k-1}, r_k = R_k - R_{k-1}, k = 0,1,\ldots$, with

Our usual convention $R_{k-1} = 0, R_{k-1} = 0$.We then have for $k = 0, 1, \ldots$,

$\|u_k\|_{p_1} \le c\left[S_k(g)_{p_1} + S_{k-1}(g)_{p_1}\right]$,

$\|r_k\|_{p_0} \le c\left[S_k(f-g)_{p_0} + S_{k-1}(f-g)_{p_0}\right]$,

With $u \coloneqq (u_k)$, it follows from Theorem (3.2.12) that

$$\|Tf - u\|_{l_0} + t\|u\|_{l_1} \le c\left[\|f-g\|_{B_0} + t\|g\|_{B_1}\right]$$

The upper estimate in (120) then follows from the definition of the K-functional.

ForB_0, B_1, l_0, l_1andTfas above, we have for any $q > 0$ and $0 < \theta < 1$,

$f \in (B_0, B_1)_{\theta,q}$if and only if $Tf \in (l_0, l_1)_{\theta,q}$.

$$\|f\|_{(B_0,B_1)_{\theta,q}} \cong \|Tf\|_{(l_0,l_1)_{\theta,q}} \tag{126}$$

Indeed, this follows immediately from the definition of the spaces $X_{\theta,q}$.

Now (126) allows us to deduce information about the interpolation spaces between B_0 and $B_1$from known theorems (see [121]) about the interpolation between $l_0$andl_1. The simplest case to describe is when $p_0 = p_1 = p$. We then have

$$(l_{q_0}^{\alpha_0}(L_p), l_{q_1}^{\alpha_1}(L_p))_{\theta,q} = l_q^{\alpha}(L_p) \tag{127}$$

Where $\alpha = \theta\alpha_0 + (1-\theta)\alpha_1$.

From this, (6.10), andTheorem (3.2.12),we obtain

Corollary (3.2.19) [109]:

If $0 < \alpha_0, \alpha_1$, and $0 < p, q_0, q_1, \le \infty$, we have for each $0 < \theta < 1$ and $0 < q \le \infty$,

$$(B_{q_0}^{\alpha_0}(L_p), B_{q_1}^{\alpha_1}(L_p))_{\theta,q} = B_q^{\alpha}(L_p), \ \text{with} \alpha = \theta\alpha_0 + (1-\theta)\alpha_1 \tag{128}$$

When $p_0 \ne L_p$ the interpolation spaces between L_{p_o} and L_{p_1}can be described in terms of the Lorentz spaces $L_{p,q}$ (see [113]J for their definition and properties).We have for $0 < q_0, q_1, \le \infty$ (see [121]),

$$(l_0, l_1)_{\theta,q} = l_q^{\alpha}(L_p) \tag{129}$$

With $\alpha = \theta\alpha_0 + (1-\theta)\alpha_1$, $1/q = \theta/q_0 + (1-\theta)/q_1$, and $1/p = \theta/p_0 + (1-\theta)/p_1$.

In the special case when $q = p$, we have $L_{p,q} = L_p$ and therefore, we obtain

Corollary (3.2.20) [109]:

If $0 < \alpha_0, \alpha_1$, and $0 < p_0, p_1, q_0, q_1, \leq \infty$, then for each $0 < \theta < 1$ and for $1/q = \theta/q_0 + (1-\theta)/q_1; 1/p = \theta/p_0 + (1-\theta)/p_1$ we have

$$(B_{q_0}^{\alpha_0}(L_{p_0}), B_{q_1}^{\alpha_1}(L_{p_1}))_{\theta,q} = B_q^{\alpha}(L_p), \text{ with } \alpha = \theta\alpha_0 + (1-\theta)\alpha_1 \tag{130}$$

Provided $p = q$.

An embedding theorem for Beaov spaces. As an application of the results of the previous sections, we shall prove Sobolev type embedding theorems for Besov spaces. These have important applications in nonlinear approximation (see [117]). We fix a value of p with$0 < p < \infty$. Given$\alpha > 0$, we determine σ from the equation

$$1/\sigma = 1/d + 1/p \tag{131}$$

We shall prove that $B_q^{\alpha}(L_\sigma)$is continuously embedded inL_p.For this, we shall

use the following simple inequality for splines $S \in \Pi_k(r)$:

$$\|S\|_p \leq c2^{k\alpha p}\|S\|_\sigma \tag{132}$$

Indeed, on each cube $I \in D_k$, $S = P$ with$P$a polynomial of coordinate degree$< r$. Hence (see[119]), $\|S\|_p(I) \leq c|I|^{1/p-1/\sigma}\|S\|_\sigma(I) = 2^{k\alpha}\|S\|_\sigma(I)$ .Therefore,

$$\|S\|_p^p \leq c2^{k\alpha p} \sum_{I \in D_{k(\Omega)}} \|S\|_\sigma(I)^p \leq c2^{k\alpha p} \left(\sum_{I \in D_{k(\Omega)}} \|S\|_\sigma(I)^\sigma \right)^{p/\sigma}$$

where the last inequality uses the fact that the $l_{\sigma/p}$ norm is larger than the l_1 norm because $\sigma/p < 1$.

Theorem (3.2.21) [109]:

If α, σ, p are related as in (131), then $B_q^{\alpha}(L_\sigma)$ is continuously embedded in L_p,that is,

$$\|f\|_p \leq c\|f\|_{B_q^{\alpha}(L_\sigma)} \tag{133}$$

holds for $f \in B_q^{\alpha}(L_\sigma)$.

Proof.

We choose $r > \alpha + 1$ and let $t_j \in \Sigma_j(r)$be as in Theorem (3.2.12). Then $f = \sum_{j=0}^{\infty} t_j$ in the sense of convergence in L_σ.From (56), it follows that for

$\mu = \min(1, p)$,

$$\|f\|_p \leq \left(\sum_{j=0}^{\infty} \|t_j\|_p^\mu \right)^{\frac{1}{\mu}} \leq c \left(\sum_{j=0}^{\infty} \left(2^{\alpha j} \|t_j\|_\sigma \right)^\mu \right)^{\frac{1}{\mu}} \leq c\|f\|_{B_q^\alpha(L_\sigma)} \tag{134}$$

where the second inequality follows from (132) and the last from Theorem (3.2.12).

Inequality (134) shows that $B_q^\alpha(L_\sigma)$ is continuously embedded in L_p which is the desired result when $p \leq 1$. When $p > 1$, we choose $1 \leq p_0 < p < p_1 < \infty$ and for $i = 0,1$, we let α_i be determined by formula (131) for p_i and ourσ. Then by (134)

$$\|f\|_{p_i} \leq c\|f\|_{B_1^{\alpha_i}(L_\sigma)}, \qquad i = 0,1 \tag{135}$$

If we now apply Corollary (3.2.19) with θ chosen so that $1/p = \theta/p_0 + (1-\theta)/p_1$and $q = p$, we obtain by interpolation

$$\|f\|_p \leq c\|f\|_{B_1^{\alpha'}(L_\sigma)}$$

With $\alpha' = \theta\alpha_0 + (1-\theta)\alpha_1$ Here, we have used the fact that $L_{p,p} = L_p$.Now using (131) for the pairs (α, p), (α_0, p_0) and (α_1, p_1)shows that $\alpha' = \alpha$, as desired.

Sect(3.3): Besov Spaces On $\boldsymbol{R^d}$:

Besov spaces $B_q^\alpha(L_p(\Omega))$ are being applied to a variety of problems in analysis and applied mathematics. Applications frequently require knowledge of the interpolation and approximation properties of these spaces. These properties are well understood when $p \geq 1$ or when the underlying domainΩ, is R^d . The purpose of the present section is to show that these properties can be extended to general nonsmooth domains Ω of R^d and for all $0 < p \leq \infty$. Besov spaces with $p < 1$ are becoming increasingly more important in the study of nonlinear problems.

To a large extent the present section is a sequel to [127 and 129] which established various properties of the spaces $B_q^\alpha(L_p(\Omega))$, Ω a cube. Among these are atomic decompositions for the functions in $B_q^\alpha(L_p(\Omega))$, a characterization of $B_q^\alpha(L_p(\Omega))$ through spline approximation, and a description of interpolation spaces for a pair of Besov spaces. We establish similar results for more general domains.

Our approach is to first define an extension operatorξ, which extends functions in $\mathrm{B}_\mathrm{q}^\alpha(\mathrm{L}_\mathrm{p}(\Omega))$ to all of R^d . Similar extension operators for $\mathrm{p} \geq 1$ have been introduced by

Calderón and Stein (see [132]). Our main departure from these earlier approaches is that by necessity our extension operators are nonlinear. Moreover, whereas in the case p ≥ 1, it is possible to take ξ so that $\omega_r(\xi f,t)_p \leq C\,\omega_r(f,t)_p$ with ω_r the rth order modulus of smoothness (at least when Ω is minimally smooth [130]), in the case$0 < p < 1$, we only obtain a weak comparison between $\omega_r(\xi f,t)_p$and $\omega_r(f,t)_p$ .We shall establish our results for two important classes of no smooth domains: the Lipschitz graph domains, and the (ε,δ) domains introduced by Jones [131]. We begin with the case of Lipschitz graph domains since the geometric arguments in this case are the most obvious. We later generalize these arguments to (ε,δ) domains . Although the results of contain those of, we feel that this two tier presentation makes the essential arguments much clearer.

Let Ω be an open subset of R^{d} . We can measure the smoothness of a function $f \in L_p(\Omega)$, $0 < p < \infty$, by its modulus of smoothness. For any $h \epsilon \mathrm{R}^{\mathrm{d}}$, let I denotes the identity operator, $\tau(h)$ the translation operator $(\tau(h)(f,x) = f(x+h))$ and $\Delta_h^r = (\tau(h) - I)^r, r = 1,2,\ldots$, be the difference operators. We shall also use the notation

$$\Delta_h^r(f,x,\Omega) = \begin{cases} \Delta_h^r(f,x), & x, x+h, \ldots, x+rh \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

The modulus of smoothness of order r of a function $f \in L_p(\Omega)$is then

$$\omega_r(f,t)_p = \omega_r(f,t,\Omega)_{p=\sup_{|h\leq t|}\|\Delta_h^r(f,.,\Omega)\|_{L_P(\Omega)}} \tag{136}$$

For any $h \in \mathrm{R}^{\mathrm{d}}$, we define

$$\Omega(\mathrm{h}) := \{x : [x, x+\mathrm{h}] \subset \Omega\}.$$

A Besov space is a collection of functions / with common smoothness. If $0 < \alpha \leq r$ And $0 < q ,p \leq \infty$, the Besov space $\mathrm{B}_{\mathrm{q}}^{\alpha}(\mathrm{L}_{\mathrm{p}}(\Omega))$consists of all Functions f such that

$$|f|_{\mathrm{B}_{\mathrm{q}}^{\alpha}(\mathrm{L}_{\mathrm{p}}(\Omega))} := \left(\int_0^1 [t^{-\alpha}\omega_r(f,t,\Omega]_p dt\right)^{1/q} < \infty \tag{137}$$

With the usual change to sup when $q = \infty$. It follows that (137) is a semi (quasi)-norm for $\mathrm{B}_{\mathrm{q}}^{\alpha}(\mathrm{L}_{\mathrm{p}}(\Omega))$. (Frequently, the integral in (137) is taken over $(0,\infty)$; While this results in a different semi norm, the norms given below are equivalent.) If we add $\|f\|_{l_p}(\Omega)$ to (137), we obtain the (quasi)norm for $\mathrm{B}_{\mathrm{q}}^{\alpha}(\mathrm{L}_{\mathrm{p}}(\Omega))$. It is well Known in the case $p \geq 1$ that different values of $r > \alpha$ give equivalent norms. This remains true for $p < 1$ as well and can be derived from the 'Marchaud Inequalities', which compare moduli of smoothness of different orders. These Inequalities have been proved for all $p > 0$ and Ω, either a cube or all of $\mathrm{R^d}$ in [133] (See also [127]), and for more general domains Ω and $p \geq 1$ by Johnen and Scherer [130] (among others). We address this topic later for the remaining Case $0 < p \leq 1$ and more general Ω.

There are fundamental connections between smoothness and approximation (See [127] and the references therein, especially [133]). We now describe these without Proofs (which can be found in [127] or [133]). If $f \in L_p(Q), 0 < p \leq \infty$, Qa Cube in $\mathrm{R^d}$, we let

$$E_r(f,Q)_p = \inf_{p \in P_r} \|f - P\|_{p(Q)} \tag{138}$$

Be the error of approximation by the elements from the space $P_r$of polynomials of total degree less than r where $\|.\|_{P(Q)}$ denotes the $L_P(Q)$ (quasi)norm.

We then have Whitney's inequality

$$E_r(f,Q)_p \leq C\, \omega_r\big(f, l(Q)\big)_p \tag{139}$$

Where $I(Q)$ is the side length of Q and C is a constant which depends only on r and d (also p if p is close to 0).

Sometimes (139) is not sufficient because it is not possible to add these estimates For different cubes Q. For this purpose, the following averaged moduli of smoothness is more convenient. For any domain Ω and $t > 0$, we define

$$W_r(f,t,\Omega)_p := \left(t^{-d} \int_{|s| \leq t} \int_{\Omega} |\Delta_s^r(f,x,\Omega)|^p \, dxds\right)^{1/p} \tag{140}$$

Where $p < \infty$. Then, returning once again to cubes Q, ω_r and W_r are equivalent:

$$C_1 \omega_r(f,t,Q) \leq W_r(f,t,\Omega)_p \leq C_2 \omega_r(f,t)_p \tag{141}$$

Where $C_1$and C_2 depend only on d , r and p if p is small. Therefore, the estimate (139) can be improved by replacing ω_r by W_r :

$$E_r(f,Q)_p \leq C\, W_r(f, l(Q), Q)_p \tag{142}$$

We shall use the generic notation $P_Q = P_Q(f)$ to denote a polynomial in P_r which satisfies

$$\|f - P_Q\|_p(Q) \leq \lambda\, E_r(f,Q)_p \tag{143}$$

Where $\lambda \geq 1$ is a constant which we fix. The polynomial P_Q is then called a near best approximation to f with constant λ. When $\lambda = 1$, P_Q is a best approximation. It follows from (142) and (143) that

$$\|f - P_Q\|_p(Q) \leq C\, W_r(f, l(Q), Q)_p \tag{144}$$

We shall use the following observation (see [127]) about near best approximation in the sequel. Let $r > 0$. If $P_Q \in \mathrm{P}_r$ is a near best approximation to f with constant λ on Q in the L_γ norm, then it is also a near best approximation to f for all $P \geq \gamma$:

$$\left\|f - P_Q\right\|_p(Q) \leq C\lambda\, E_r(f,Q)_p \tag{145}$$

where the constant C depends only on γ, r and d .

The estimate (145) leads to a characterization of Besov spaces in terms of spline approximation. For $n \in Z$, let D_n , be the collection of dyadic cubes Q of side length 2^{-n} and let $D := \cup_{n\in z} D_n$ be the collection of all dyadic Cubes. For $n \in Z$, let $\Pi_n := \Pi_{n,r}$ be the space of piecewise polynomials S on D_n which have degrees less than r. The error of approximation to a function $f \in L_p(\Omega)$ by elements of Π_n is

$$s_n(f)_p := \inf_{s\in\Pi_n}\|f - S\|_p(\Omega) \tag{146}$$

It follows from [127] that a function $f \in L_p(\Omega)$ is in $B_q^\alpha \in L_p(\Omega)$, Ω a cube, if and only if

$$\|f\|_{\mathcal{A}_q^\alpha(L_p)} := \left(\sum_{n\in z}\left(2^{n\alpha}s_n(f)_p\right)^q\right)^{1/q} < \infty \tag{147}$$

Moreover, (147) is an equivalent semi norm for$\mathrm{B}_\mathrm{q}^\alpha(\mathrm{L}_\mathrm{p}(\Omega))$. Let us emphasize for later use that this same result holds in the case $\Omega = \mathrm{R}^\mathrm{d}$ with the same proof.

It will be useful to mention briefly some well-known properties of polynomials which we shall use frequently in what follows. If Q is a cube, we let, for $0 < P \leq \infty$,

$$\|f\|_p^*(Q) := |Q|^{-1/p}\|f\|_p(Q) \tag{148}$$

be the normalized L_p norms. We also introduce the notation ρQ to denote the cube with the same center as Q and side length $\rho l(Q)$ where $l(Q)$ is the side length of Q.

If r is a nonnegative integer, $\rho > 1$ and P is a polynomial of $degree \leq r$, then (see for example [129]) for a constant C depending only on d, r (this constant and other constants in this section also depend on the distance of p to 0), we have for any $q \geq p$:

$$\|P\|_q^*(\rho Q) \leq C\|P\|_q^*(Q) \leq C\|P\|_p^*(\rho Q) \tag{149}$$

We often apply this inequality in the following way. Let Q_1, Q_2 be two cubes with $l(Q_1) \leq l(Q_2)$ and $Q_1 \subset \rho Q_2$ for some $\rho \geq 1$. Then for a constant c depending only on d, ρ, p, r we have, for all $q \geq p$,

$$\|P\|_q^*(Q_1) \leq C\|P\|_p^*(Q_2) \tag{150}$$

Indeed, it is enough to compare the left side of (150) with$\|P\|_p^*(Q_1)$, compare this with $\|P\|_p^*(\rho Q_2)$, and then finally make a comparison with $\|P\|_p^*(Q_2)$.

We shall define an extension operator ξ (similar to that introduced in [4]) which extends each function $f \in L_p(\Omega)$to all of Rd and has the property that if $f \in \mathrm{B}_{\mathrm{q}}^{\alpha}(\mathrm{L_p}(\Omega))$, then $\xi f \in \mathrm{B}_{\mathrm{q}}^{\alpha}(\mathrm{L_p}(\mathrm{R^d}))$, (with suitable restrictions on α, p, q, and Ω. We assume at the outset that Ω is a Lipschitz graph domain and treat more general domains in the next section. This means that $\Omega = \{(\mathrm{u,v}) : \mathrm{u} \in \mathrm{R^{d-1}}, \mathrm{v} \in \mathrm{R}$ and $\mathrm{v} > \phi(u)\}$ where ϕ i s a fixed Lip 1 function.

That is , ϕ atisfies $|\phi(\mathrm{u_1}) - \phi(\mathrm{u_2})| \leq M|u_1 - u_2|$for all$\mathrm{u_1}, \mathrm{u_2} \in \mathrm{R^{d-1}}$, where M is a fixed constant (which we can assume is greater than one).

We let F denote the Whitney decomposition of Q into dyadic cubes (see [132]). Similarly we denote by Fc the Whitney decomposition of $\Omega^c \setminus \partial\Omega$. Then,

(i) $diam\ (Q) \leq dist\ (Q, \partial\Omega) \leq 4\ diam\ (Q), Q \in F \cup F_c$,

(ii) $if\ Q\ , Q_0 \in F \cup F_c\ touch\ , then\ l(Q_0) \leq 4l(Q))$,

(151)

(iii) $\sup_{(u,v)\in Q} |v - \phi(\mathrm{u})| \leq \ Cl(Q)$

Where C depends only on the Lipschitz constant M and the dimension d. Here, $diam(Q) = \sqrt{d}\ l(Q)$ with $l(Q)$the side length of Q. For each cube Q in $F \cup F_c$ let$Q^* = \frac{9}{8}Q$. If $Q \in\ F$, then$Q^* \subset 3Q \subset \Omega$.

According to [132] there is a partition of unity $\{\phi_Q\}_{Q\in F_c}$for the open set $\Omega^c \setminus \partial\Omega$ with the properties:

(i) $0\ \leq \phi_Q \leq 1$,

(ii) $\sum_{Q\in F_c} \phi_Q \equiv 1\ , on\ \Omega$,

(iii) $\phi_Q\ is\ supported\ in\ int(Q^*)$,

(152)

(iv) $\|D^v\phi_Q\|_\infty \leq c\ [l(Q)]^{-|v|}, |v| \leq m$,

(v) $if\ Q_1\ , Q_2 \in F \cup F_c\ with\ \ Q_1^* \cap Q_2^*\ \neq\ \phi\ , then\ Q_1\ and\ Q_2\ touch$

(vi) at most $N_0 \coloneqq\ 12^d$cubes from either F or F_c may touch a given cube from either family.

Properties (i)-(iv) and (vi) are proved in [132], while a proof of (v) can be found in [129]. Here m is an arbitrary integer and c depends only on d, Ω, and m . We are using standard multivariate notation for the derivatives $D^v \coloneqq D_{X_1}^{v_1} \dots D_{X_d}^{v_d}$.

If $Q \in F_c$ has center (u, v), we let Q^s denote the cube in F which contains the point $(u, ,2\phi(\mathrm{u}) - \mathrm{v})$. We speak of Q^s as being the cube symmetric to Q a cross $\partial\Omega$. The symmetric cubes Q^s were introduced in [129] and we recall now some of their properties proved in [129]. While Q and Q^s need not have the same size, they are comparable; i.e. there is a constant $C > 0$ for which there holds (see [129]).

(i) $C^{-1}l(Q) \le l(Q^s) \le Cl(Q$ (153)

(ii) $dist(Q, Q^s) \le Cl(Q$

(iii) each cube in F can be the symmetric cube Q^s of at most C cubes $Q \in F_c$. To define our extension operators ξ, we fix a value $\gamma > 0$ (which in application is chosen smaller than all p under consideration), and a value r (which in application is larger than all the α under consideration) and we let $\gamma \ge 0$.

If $f \in L_\gamma(loc)$ and Q is a cube, we let $P_Q(f)$ be a polynomial which satisfies (143). we then define ξ by

$$\xi f(x) := \begin{cases} f(x), & x \in \Omega \\ \sum_{Q \in F_c} \mathrm{P}_{Q^s}\mathrm{f(x)}\phi_Q(x), & x \in \Omega^c \backslash \partial\Omega \end{cases} \tag{154}$$

Actually, (154) defines a family of extension operators, since each choice of near Best approximants $\mathrm{P}_{Q^s}f$ give an extension ξ . The results that follow apply to any such extension operator ξ with the restriction that the constant $\lambda \ge 1$ of (143) is fixed.

We have shown in [129] that ξ is a bounded mapping from $\mathrm{L_p(\Omega)}$ into $\mathrm{L_p(R^d)}$, $\gamma \le p \le \infty$, and also from $\mathrm{B_q^\alpha(L_p(\Omega))}$. Into $\mathrm{B_q^\alpha(L_p(R^d))}$. Whenever $1 < p \le \infty$,. We shall prove now the same result when $0 < p \le 1$. To study the smoothness of ξf, we shall need estimates of how well ξf can be approximated by polynomials on cubes R in the L_p norm for $p \ge \gamma$. We fix $0 < p \le \infty$ and r and use the abbreviated notation $E(Q) := \mathrm{E_r(f, Q)_p}$

Lemma (3.3.1) [125]:

There exists a constant $C > 0$ so that if Q_1 , Q_2 belong to F and touch, then

$$\|P_{Q_1} - P_{Q_2}\|_\infty(Q_1) \le C|Q_1|^{-1/p}[E(Q_1^*) + E(Q_2^*)] \tag{155}$$

Proof.

By property (151) (ii) of the Whitney decomposition, Q_1 and Q_2 have comparable side lengths and so we may select a cube $\tilde{Q} \subset Q_1^* \subset Q_2^*$ whose side length is comparable to that of either cube:

$$l(\tilde{Q}) = \frac{1}{16}\min\{l(Q_1), l(Q_2)\}$$

Applying the triangle inequality in $L_\infty(Q_j)$ and using the elementary estimates for polynomials (150), we have for $j = 1,2$

$$\left\|P_{Q_j} - P_{\tilde{Q}}\right\|_\infty (Q_j) \le C\left\{\left\|P_{Q_j} - P_{Q_j^*}\right\|_p^* (Q_j) + \left\|P_{Q_j^*} - P_{\tilde{Q}}\right\|_p^* (\tilde{Q})\right\}$$

Using this inequality and two applications of Lemma 3.3 of [127] (applied once to Q_j and Q_j^* and again to $\tilde{Q}$and Q_j^*) gives

$$\left\|P_{Q_j} - P_{\tilde{Q}}\right\|_\infty (Q_j) \le C(Q_j)^{-1/P} E(Q_j^*) \tag{156}$$

Again using (150), we obtain

$$\|P_{Q_2} - P_{\tilde{Q}}\|_\infty(Q_1) \le C\|P_{Q_2} - P_{\tilde{Q}}\|_\infty(Q_2)$$

and so together with (156) (applied with $j = 2$) and the modified triangle inequality we obtain the desired result (155).

To estimate the smoothness of ξf, we shall approximate ξf on cubes Q from $\mathrm{R^d}$. We consider first the approximation of ξf f on cubes close to $\partial\Omega$.

Lemma (3.3.2) [125]:

There exists a constant $c > 0$ so that if ξ is any of the extension operators (154) and R is a cube with $dist(R, \partial\Omega) \le diam(R)$, then for$f \in \mathrm{L_p}(\Omega), \gamma \le p \le 1$, we hav

$$\mathrm{E_r}(\xi f, R)_\mathrm{p} \le \left(\sum_{\substack{\mathrm{s \in F} \\ \mathrm{s \subset cR}}} \mathrm{E(S^*)^p}\right)^{1/\mathrm{p}} \tag{157}$$

Where c , C depend only on d, r, γ, λ, and Ω.

Proof.

For such an R, if (u_0, v_0) denotes its center, then we let R_0 be the member of F containing a point of the form (u_0, v_0) such that $l(R_0) \ge 16l(R)$ and v is smallest. It is clear (see property (151) (i)) that R and Ro have comparable side lengths and so we may choose a constant $c > 0$ so that $cR \supset R_0$. Let $Q \in F$ intersect R. We shall estimate$\left\|f-P_{R_0}\right\|_p(Q)$. Since $dist(Q,\partial\Omega)\le diam(R)+dist(R,\partial\Omega)\le 2diam(R)$, from (151)(i) it follows that $l(Q)\le 2l(R)$.

Our next step is to construct a 'chain' of cubes $\{R_j\}_0^m$ from F which connect R_0 To $Q=R_m$ with each R_j touching R_{j+1}. We accomplish this as follows.

Let $x_1=(u_1,v_1)$ be the center of R_0 and $x_3=(u_3,v_3)$ be a point from $Q\cap R$ We consider the path consisting of a 'horizontal' followed by a 'vertical' Linear segment which connects first x_1 to the point $x_2=(u_3,v_1)$ and then x_2 to x_3. The point x_2 is in $\frac{9}{8}R_0=R_0^*$ and is therefore in a cube $\tilde{R}\in F$ which touches R_0 . If $\tilde{R}\neq R_0$, we define $R_1:=\widetilde{R}$, otherwise R_1 is not yet defined.

The remaining cubes R_j are obtained from the vertical segment which connects x_2 to x_3, namely the cubes we encounter (in order) as v changes from v_1 to v_3. Since all these cubes are in F, they have disjoint interiors. From property (151)(iii), we obtain $\sum_{j=1}^m l(R_j)$ is comparable to $l(R_0)$; moreover,

$$l(R_k)\le\sum_{j=1}^m l(R_j)\le c\,l(R_k)\ ,\qquad 0\le k\le m\ , \tag{158}$$

In particular, we have $Q\subset cR_j$ and $R_j\subset cR$, where c has been increased as necessary but remains independent of f.

Since $Q\subset cR_j$, the inequalities (150) for polynomials, give that for any polynomial P, $\|P\|_\infty(Q)\le C\|P\|_p^*(R_j)$, $j=\ 0,\ldots,m$, for a constant C depending only on p, d, Ω and the degree of P but not on j. We now write $P_Q-P_{R_0}=\left(P_{R_m}-P_{R_{m-1}}\right)+\cdots+(P_{R_1}-P_{R_0})$ and find from Lemma 4.1 that

$$\begin{aligned}\left\|P_Q-P_{R_0}\right\|_\infty(Q)&\le C\sum_{j=0}^{m-1}\left\|P_{R_{j+1}}-P_{R_j}\right\|_p(R_j)\\&\le C\sum_{j=0}^{m-1}\left|R_j\right|^{-\frac{1}{p}}\left[E\left(R_j^*\right)+E\left(R_{j+1}^*\right)\right]\\&\le C\sum_{j=0}^{m}\left|R_j\right|^{-\frac{1}{p}}E\left(R_j^*\right)\end{aligned} \tag{159}$$

Hence, $|Q|^{-\frac{1}{p}}\left\|P_Q-P_{R_0}\right\|_p(Q)$ also does not exceed the right side of (159). If we write $f-P_{R_0}=(f-P_Q)+(P_Q-P_{R_0})$, we obtain

$$\left\|f-P_{R_0}\right\|_p(Q)\le C|Q|^{\frac{1}{p}}\sum_{j=0}^{m}\left|R_j\right|^{-1}E\left(R_j^*\right) \tag{160}$$

Since an l_1 norm does not exceed an l_p norm for $0<\ p\ \le 1$, we have

$$\|f - P_{R_0}\|_p^p(Q) \le C|Q| \sum_{j=0}^{m} |R_j|^{-1} E(R_j^*)^p \tag{161}$$

We denote the 'chain' from Q to R_0 by $T_Q := (R_j)_{j=0}^m$ Summing (161) over all Q belonging to F such that$Q \cap R \neq \Phi$, we then obtain

$$\sum_{\substack{Q\in F \\ Q\cap R\neq\Phi}} \|f - P_{R_0}\|_p^p(Q) \le C \sum_{\substack{Q\in F \\ Q\cap R\neq\Phi}} \sum_{S\in T_Q} |Q||S|^{-1} E(S^*)^p \tag{162}$$

Next we interchange the order of summation in (162) and note that while an S that appears in the sum of (162) may occur in more than one T_Q , each such Q is contained in cS and therefore $\sum_{\{Q:S\in T_Q\}} |Q| \le C|S|$. Since $\xi f = f$ on such$Q$, we obtain

$$\sum_{\substack{Q\in F \\ Q\cap R\neq\Phi}} \|\xi f - P_{R_0}\|_p^p(Q) \le C \sum_{\substack{S\in F \\ S\subset cR}} E(S^*)^p \tag{163}$$

We can prove a similar estimate to (163) for cubes $\tilde{Q} \in F_c$for which $\tilde{Q} \cap R \neq \Phi$:

$$\sum_{\substack{\tilde{Q}\in F_c \\ \tilde{Q}\cap R\neq\Phi}} \|\xi f - P_{R_0}\|_p^p(\tilde{Q}) \le C \sum_{\substack{S\in F \\ S\subset cR}} E(S^*)^p \tag{164}$$

Indeed, for a cube $\tilde{Q}$ which appears in the left sum of (164), we have from the definition of ξ in (154):

$$\|\xi f - P_{R_0}\|_p^p(\tilde{Q}) \le \sum_{\substack{\tilde{Q}\cap Q^*\neq\Phi \\ Q\in F_c}} \|P_{Q^s} - P_{R_0}\|_p^p(\tilde{Q})$$

$$\le \sum_{\substack{Q^*\cap\tilde{Q}\neq\Phi \\ Q\in F_c}} \|P_{Q^s} - P_{R_0}\|_p^p(Q^s) \tag{165}$$

where we have used the fact that the Φ_Q are positive and sum to one and we have used (150) (for $q = p$) to replace $\|P_{Q^s} - P_{R_0}\|_p^p(\tilde{Q})$ by $\|P_{Q^s} - P_{R_0}\|_p^p(Q^s)$ (recall that Q, $\tilde{Q}$, and Q^s all have comparable size and the distance between any two of these cubes does not exceed $C\, diam(Q)$). Now, by $(152)(v)$, $Q^* \cap \tilde{Q} \neq \Phi$ only if Q and $\tilde{Q}$ touch. Therefore by $(4.2)(iv)$ there are at most N terms in the sum (165) and N depends only on d and Ω. Also a given Q^s appears for at most C cubes $\tilde{Q}$ (see the remark following (156)). Furthermore Q^s is contained in cR and therefore the estimate (159) holds (with the Q there replaced by Q^s). Finally, if we use (149) to replace the $L_\infty(Q^s))$ norm by an $L_q(Q^s))$ norm on the left side of (159) and then use this in the terms of the right sum of (165), we arrive at (164) in the same way that we have derived (4.13).

To complete the proof, it is enough to add the estimates (164) and (165).

We are now in a position to give an estimate for $\omega_r(\xi f, t)_p$ for each of the extension operators ξ .

Theorem (3.3.3)[125]:

If $\gamma \leq p \leq 1$ and $t > 0$ then

$$\omega_r(\xi f, t)_p^p \leq C^p \left[\sum_{2^j \leq c_1 t} W_r(f, 2^j)_p^p + t^{rp} \sum_{2^j \leq t} 2^{-jrp} W_r(f, 2^j)_p^p \right] \qquad (166)$$

where W_r is the averaged modulus of smoothness (140) and the constants c_1 and C depend only on d, r, γ, λ, and Ω.

Proof.

We write $R^d \setminus \partial\Omega = \Omega_0 \cup \Omega_- \cup \Omega_+$ where $\Omega_0 := \cup \{Q \in F \cup F_c : l(Q) \leq 16rt\}$, $\Omega_+ := \Omega \setminus (\Omega_0 \cup \partial\Omega)$, $\Omega_- := \Omega^c \setminus (\Omega_0 \cup \partial\Omega)$. It follows that for each $x \in \Omega_0$ and for the appropriate cube $Q \in F \cup F_c$ which contains x , we have

$$dist(x, \partial\Omega) \leq diam(Q) + dist(Q, \partial\Omega) \leq 5\ diam(Q) \leq 80\sqrt{d}rt \qquad (167)$$

We shall consider three cases. Let $|h| \leq t$.

Case 1 ($x \in \Omega_+$). In this instance, there is a cube $Q \in F$ containing x and $l(Q) > I6rt$. Therefore the expanded cube $Q^* := \frac{9}{8} Q \subset \Omega$ contains the line Segment $[x,\ x + rh]$, which shows for ($x \in \Omega_+$, that $\Delta_h^r(\xi f, x) = \Delta_h^r(f, x)$.

Hence, by (141),

$$\int_Q |\Delta_h^r(\xi f, x, \Omega)|^p\, dx \leq \int_{Q^*} |\Delta_h^r(\xi f, x, Q^*)|^p\, dx \leq \omega_r(f, t, Q^*)_p^p$$
$$\leq W_r(f, t, Q^*)_p^p$$

We now sum over all Q which intersect Ω_+ and use the fact that a point $x \in R^d$ can appear in at most N_0 of the cubes Q^* (see (4.2)(vi)) to find

$$\int_{\Omega_+} |\Delta_h^r(\xi f, x)|^p\, dx \leq W_r(f, t)_p^p \qquad (168)$$

Case 2 $x \in \Omega_0$. In this case we are near the boundary and employ Lemma (3.3.2). We take a tiling Λ_0 of R^d into pairwise disjoint cubes R of side length 80 rt . Next we obtain additional staggered tilings by translating Λ_0 in coordinate directions. Namely, if v is a vector in R^d with coordinates 0 or1, then $\Lambda_v := \{40rtv + R\}_{R \in \Lambda_0}$ is also a tiling. We let Λ denote the collection of those R such that $R \cap \Omega_0 \neq \Phi$ and $R \in \Lambda_v$ for one of these v . We note that there are 2^d such v and for each point $x \in \Omega_0$ there is a cube $R \in \Lambda$ such that $[x, x + rh] \subset R$. Hence,

$$\int_{\Omega_0} |\Delta_h^r(\xi f, x)|^p\, dx \leq \sum_{R \in \Lambda} \int_{R(rh)} |\Delta_h^r(\xi f, x)|^p\, dx$$

$$\leq 2^{\,d} \sum_{R\in\Lambda} E(\xi f,R)_p^p \tag{169}$$

where the last inequality follows since the $r\,th$ difference annihilates polynomials of degree less than r. The multiple 80 was chosen so that the cubes R in Λ satisfy $dist(R,\partial\Omega) \leq diam(R)$ as follows from (167) because $\Omega_0 \cap R \neq \Phi$. We may therefore estimate $E(\xi f,R)_p$ by Lemma (3.3.2) to give

$$\int_{\Omega_0} |\Delta_h^r\,(\xi f,x)|^p\,dx \leq C \sum_{R\in\Lambda} \sum_{\substack{S\in F\\ S\subset cR}} E(S^*)^p \tag{170}$$

Next, we observe that F is the disjoint union of the $F_j := F \cap \mathbb{D}_j$ and so (170) becomes

$$\int_{\Omega_0} |\Delta_h^r\,(\xi f,x)|^p\,dx \leq C \sum_{j=-\infty}^{\infty} \left(\sum_{R\in\Lambda} \sum_{\substack{S\in F_j\\ S\subset cR}} E(S^*)^p \right) =: C \sum_{j=-\infty}^{\infty} I_j \tag{171}$$

Let $S_j = \cup \{S^*: S \in F_j$. By properties $(152)(v)$ and (vi) of Whitney decompositions, it follows that for each j

$$\sum_{R\in\Lambda} \sum_{\substack{S\subset cR\\ S\in F_j}} \mathcal{X}_{S^*} \leq CN_0 \mathcal{X}_{S_j} \tag{172}$$

where N_0 is the constant of $(152)(vi)$, and C is a constant which depends only on d and c counting the number of times a cube $S \in F$ can appear in distinct cubes cR, $R \in \Lambda$. Therefore, from (142), we obtain for each$j \in Z$,

$$I_j \leq CN 2^{jd} \int_{|h|\leq \frac{9}{8}2^{-j}} \int_{S_j} \left|\Delta_h^r\,(f,x,S_j)\right|^p dxdh \leq CW_r\,(f\,,2^{-j+1})_p^p \tag{173}$$

Furthermore, if $S \in F_j$ satisfies $S \subset R$ for some$R \in \Lambda$, then $l(S) \leq cl(R) = 80crt$. Hence, if $c_1 \geq 160cr$ we have from $(151)(i)$ that$2^{-j+1} \leq c_1 t$. Using this together with inequalities (171) and (173), we obtain

$$\int_{\Omega_0} |\Delta_h^r\,(\xi f,x)|^p\,dx \leq C \sum_{2^{-j}\leq 80crt} I_j \leq C \sum_{2^{-j}\leq c_1 t} W_r\,(f\,,2^{-j})_p^p \tag{174}$$

Case $(x \in \Omega_-)$. Let $R \in F_c$ with$R \cap \Omega_- \neq \Phi$, then $l(R) > 16rt$ and so $[x,\ x + rh] \subset R^*$ whenever $x \in R$. We consider any other cube $Q \in F_c$ such that Q^* intersects $[x, x + rh]$ for some $x \in R$ and $|h| \leq t$. By $(152)(v)$, we have that Q and R touch. Next we let $\Lambda_R := \{Q \in F_c\colon\ Q$ touches $R\}$ denote the collection consisting of R and its neighbors from Fc, then all cubes $Q \in \Lambda_R$ have side length comparable to $l(R)$. The number of cubes in Λ_R does not exceed the constant N_0 of $(152)(vi)$. We can use $(152)(iv)$ to majorize derivatives of the Φ_Q. Hence, from the definition of ξ and Leibniz' formula, we have for $|\mu| = r$:

$$\|D^{\mu}\xi f\|_{\infty}(R^*) = \|D^{\mu}[\xi f - P_{R^s}]\|_{\infty}(R^*)$$
$$\leq C \max_{0\leq k\leq r} \sum_{Q\in\Lambda_R} l(R)^{-k} \max_{|v|=r-k} \left\|D^{v}[P_{Q^s} - P_{R^s}]\right\|_{\infty}(Q^*)$$
$$\leq C\, l(R)^{-r} \sum_{Q\in\Lambda_R} \left\|P_{Q^s} - P_{R^s}\right\|_{\infty}(R^s) \qquad (175)$$

where the last inequality uses Markov's inequality and (150). We next choose a constant $c > 0$ so large that it exceeds the constant in (153) and also cR^s contains each of the cubes Q^s, for $Q \in \Lambda_R$. We shall possibly increase the size of the constant c in the remainder of the proof but it will end up to be a fixed constant depending at most on d, Ω, and previous constants.

For each Q^s, such that $Q \in \Lambda_R$, there is a 'chain' T_Q connecting R^s with Q^s which can be obtained from the proof of Lemma (3.3.2). Namely, if the constant $C > 0$ is large enough then $\bar{R} = CR$ will contain R^s and all of the Q^s, . We choose $\mathrm{R}_0 \in F$ as in Lemma (3.3.2) for the cube $\bar{R}$. The chain T_Q then consists of the cubes in F which connect Q^s to R_0 and then R_0 to R^s. Each cube in the chain T_Q will have side length larger than $c^{-1}l(R)$ where c may have to be increased appropriately. Of course each cube in the chain also has side length $< Cl(\mathrm{R}_0) < Cl(R)$. Because of the size condition on the cubes in T_Q , the fact that they have disjoint interiors, and $dist(Q^s, R^s) < Cl(R^s)$, the number of cubes in T_Q is no larger than a fixed constant depending only on d and Ω.

Therefore, we can estimate $P_{Q^s} - P_{R^s}$ as in (159) of Lemma (3.3.2) and obtain

$$\left\|P_{Q^s} - P_{R^s}\right\|_{\infty}(R^s) \leq C\left\|P_{Q^s} - P_{R^s}\right\|_{\infty}(Q^s)$$
$$\leq C|R|^{-\frac{1}{p}} \left(\sum_{S\in T_Q} E(S^*)^p\right)^{\frac{1}{p}} \qquad (176)$$

Now, from (175) and (176), we obtain for $x \in R$,

$$|\Delta_h^r(\xi f, x)| \leq \max_{|\mu|=r} \|D^{\mu}\xi f\|_{\infty}(R^*)|h|^r \leq Ct^r l(R)^{-r}|h|^{-\frac{1}{p}} \left(\sum_{S\in T_Q} E(S^*)^p\right)^{\frac{1}{p}} \qquad (177)$$

Now let $\widetilde{\Lambda}_R$ denote the collection of all cubes S from F which are contained in cR^s and have side length $l(S) \geq c^{-1}l(R$. Then, by again enlarging c if necessary, we can guarantee that any cube S appearing on the right side of (177) is contained in $\widetilde{\Lambda}_R$. Therefore, if we take $p\,th$ powers of (177) and integrate over R and then sum over all R, we obtain

$$\int_{\Omega-} |\Delta_h^r(\xi f, x)|^p\, dx \leq Ct^{rp} \sum_{R\cap\Omega-\neq\Phi} l(R)^{-rp} \sum_{S\in\widetilde{\Lambda}_R} E(S^*)^p \qquad (178)$$

where we have used the fact that the number of cubes in Λ_R is bounded index-pendent of R.

We now proceed in a similar fashion to the way we derived (174). Since (as we have derived earlier) $cl(R) \leq l(S) \leq Cl(R)$, every cube S appearing in the sum of (178) satisfies $ct \leq l(S) \leq c_1 t$ provided c_1 is sufficiently large. We majorize $E(S^*)$ by (140) and (142). This gives (compare with the derivation of (171) through (174)):

$$\sum_{R\cap\Omega-\neq\Phi} l(R)^{-rp} \sum_{S\in\widetilde{\Lambda}_R} E(S^*)^p = \sum_j \sum_{\substack{R\cap\Omega-\neq\Phi \\ R\in\mathbb{D}_j}} 2^{jrp} \sum_{S\in\widetilde{\Lambda}_R} E(S^*)^p$$

$$\leq C \sum_{2^{-j}\geq c_1 t} 2^{jrp} W_r(f, 2^{-j})_p^p \tag{179}$$

We use (179) in (178) to obtain

$$\int_{\Omega-} |\Delta_h^r(\xi f, x)|^p\, dx \leq C t^{rp} \sum_{2^{-j}\geq c_1 t} W_r(f, 2^{-j})_p^p \tag{180}$$

The proof of the theorem is completed by adding the estimates (168), (174), and (180) and making the observation that $W_r(f, s, \Omega) \leq a^{d/p} W_r(f, as, \Omega)_p$ any $a \geq 1$ to put the resulting sum in the form (166). The techniques of apply to more general domains. We shall indicate in this section the adjustments required in to execute the extension theorem for (ε, δ) domains as introduced by $P.\ Jones$ [6]. Such domains include as special cases the minimally smooth domains in the sense of [189]. The latter are equivalent to domains with the uniform cone property $[Sh]$. We say an open set Ω is called an (ε, δ)domain if:

for any $x, y \in \Omega$satisfying $|x - y| \leq \delta$, there exists a rectifiable path Γ,of length $\leq C_0|x - y|$, connecting x and y, such that for each $z \in \Gamma$,

$$dist(z, \partial\Omega \geq \varepsilon\, min(|z - x|, |z - y|) \tag{181}$$

We shall also assume that the diameter of Ω is larger than δ which, of course, will be true, if we take δ small enough.

Let F be a Whitney decomposition of Ω and F_c be a Whitney decomposition of $\Omega^c/\partial\Omega$; that is (151)(i) and (ii) hold for the cubes $Q \in F \cap F_c$. We shall often make use of the following two properties which hold for a constant C depending only on d :

(i) If $Q, Q' \in F$ do not touch, then $Cdist(Q, Q') \geq diam(Q)$,
(ii) if $Q \in F$, then
$C\, dist(Q, \partial\Omega) \geq sup_{z\in Q}\ d(z, \partial\Omega)$. (182)

The first of these properties follow from the fact that the neighbors of Q all have size comparable to that of Q (property (151)(ii)), while the second is a consequence of (151)(i). For a cube $Q \in F_c$, we let Q^s be any cube from F of maximal diameter such that $dist(Q^s, Q) < 2dist(Q, \partial\Omega)$. The cube Q^s will be called the reflection of Q and plays the same role as the reflected cubes for the Lipschitz graph domains . We note for further use that from (151)(i) and the definition of reflected cubes, it follows that if $Q_1, Q_2 \in F_c$, then

$$dist(Q_1^s, Q_2^s) < C\big(dist(Q_1, Q_2) + \max\big(diam(Q_1), diam(Q_2)\big)\big) \tag{183}$$

With C depending only on d .

Since there are not necessarily arbitrarily large cubes in Ω, for large cubes $Q \in F_c$, the reflected cube Q^s may have small diameter compared to that of Q . On the other hand, if $\mathcal{F}_c$ denotes the collection of cubes $Q \in F_c$ whose diameters are no larger than δ, then for each Q in $\mathcal{F}_c$ its reflection will satisfy properties (153) for a fixed constant C depending only on ε , δ, and d. To see this, we take a point $x_0 \in \partial\Omega$ which is closest to Q from the boundary and let $x \in \Omega$ be a point close to x_0 (to be described in more detail shortly). Since $diam(\Omega) \geq \delta \geq diam(Q)$, there is a $y \in \Omega$ such that $\delta \geq |x - y| \geq \delta/2 \geq dist(Q, \partial\Omega)/8$. Let Γ be a path connecting x to y satisfying the (ε, δ) property. Then, we can find a point $z \in \Gamma$ such that $|x - z| = dist(Q, \partial\Omega)/16$ and $|y - z| \geq dist(Q, \partial\Omega)/16$. Therefore, by (181), $dist(z, \partial\Omega) \geq Cdist(Q, \partial\Omega)$. Now let $Q' \in F$ be the cube which contains z. Then by $(151)(ii)$ and $(182)(ii) diam(Q') \geq Cdist(Q, \partial\Omega) \geq Cdiam(Q)$.

If x is close enough to x_0 (e.g., $|x - x_0| < \frac{1}{2} dist(Q, \partial\Omega)$ will be fine), then $dist(Q', Q) \leq 2dist(Q, \partial\Omega)$. Hence Q' is one of the candidates for Q^s which means that $diam(Q^s) \geq diam(Q') \geq Cdiam(Q)$ from which the properties in (153) easily follow.

The key to generalizing the extension theorem from Lipschitz graph domains to (ε, δ) domains is to find chains which connect cubes of F. For this we shall use the following.

Lemma (3.3.4) [125]:

Let R_0 and Q be two cubes from F with $diam(Q) \leq diam(R_0)$ and $dist(Q, R_0) \leq min(\delta, C_1 diam(R_0))$ with C_1 a fixed constant. Then, there is a sequence of cubes $Q =: R_m, R_{m-1}, \ldots, R_0$, from F, such that each R_j touches R_{j-1}, $j = 0, 1, \ldots, m-1$, and for each $j = 1, \ldots, m, R_j \subset cR_0$ and for each $j = 0, \ldots, m-1$, $Q \subset cR_j$ with c depending only on C_1 and Ω .

Proof.

Let $z \in Q$ and $z_0 \in R_0$ satisfy $|z - z_0| \leq \delta$ and let $\Gamma(t)$, $0 \leq t \leq 1$, is a path connecting z to z_0 guarantee by the definition of (ε, δ) domains. We claim that any cube $S \leq F$ which intersects Γ has $diameter \geq Cdiam(Q)$. Indeed, if S touches Q orR_0, this is clear. If S does not touch Q or $R_0$and$\omega \in \Gamma \cap S$, then, by $(4.1)(ii)$, $|\omega - z_0| \geq l(R_0)/4$ and $|\omega - z| \geq l(Q)/4$. Hence, by (181), $dist(\omega, \partial\Omega) \geq \varepsilon l(Q)/4$ and therefore our claim follows from $(182)(ii)$ and $(151)(i)$.

We let $S_0, S_1, S_2, \ldots$ be the cubes from F met by the path Γ as t increases; by the above remarks this sequence is finite. If two cubes are identical,$S_i = S_j$ we delete

S_{i+1}, , S_j from this sequence. It is clear that R_j touches R_{j-1} for each $j = 1,2, \dots , m$. We take points $z_j \in \Gamma \cap R_j j = 0,\dots,m$. Since the path Γ has length $\leq C|z_0 - z| \leq Cdiam(R_0)$, all points z_j satisfy dist $(z_j, \partial\Omega) \leq C$ diam(R_0). Therefore, properties (151) (i) and (182) (ii) give that diam $(R_i) \leq C$ diam(R_0). Hence $R_j \subset c\, R_0$ for some constant depending only on C_1 and Ω. We also claim that $Q \subset cR_j$. This is clear if R_j touches Q or R_0 (see (151)(ii)). On the other hand, if R_j does not touch Q or R_0, then by (181) and (151) (ii), we have dist$(z_j, \partial\Omega) \geq \varepsilon \min(|z - z_j|, |z_j - z_0|) \geq C\, l(Q)$. Hence, by (181) (ii) and (151) (i), $diam\ (R_j) \geq C$diam(Q) and our claim follows in this case as well.

We shall now define our extension operator for the (ε, δ) domain Ω. Let ϕ_Q, $Q \in F_c$, be a partition of unity for Ω^c which satisfies (152). Recall that $\mathcal{F}_c$ is collection of all cubes $Q \in F_c$ for which diam$(Q) \leq \delta$. If $\gamma > 0$ and r is a positive integer, we define

$$\xi f := f\chi_\Omega + \sum_{Q \in \mathcal{F}_c} P_{Q^s}\phi_Q \tag{184}$$

Where as before P_{Q^s} denotes a near best approximation to f in the metric $L_\gamma(Q^s)$. we let $\Omega_1 = \{x \in \mathcal{R}^d :$ dist$(x, \Omega) \leq \delta/4\}$ and $\Omega_2 = \{x \in \mathcal{R}^d :$ dist$(x, \Omega) \leq 6\delta\}$. Then, $\xi f(x) = 0$ for $x \in \Omega_2^c$, while on Ω_1, we have $\sum_{Q \in \mathcal{F}_c} \phi_Q(x) = 1$. For example, to prove the first of these statements, let $Q \in \mathcal{F}_c$. Then supp$(\phi_Q) \subset Q^*$. Since any point $x \in Q^*$ satisfies dist$(z_j, \partial\Omega) \leq \frac{9}{8}$ diam$(Q) +$ dist$(Q, \Omega) \leq \frac{41}{8}$ diam(Q) our claim follows. A similar argument proves the second statement.

The proof of the smoothness preserving property of the extension operator $\mathcal{E}$ is now very similar. We first consider the analogue of Lemma (3.3.2).

Lemma (3.3.5) [125]:

Let Ω be an (ε, δ) domain, $\gamma > 0$, r be a positive integer and $\mathcal{E}$ be an extension operator defined by (184). Let R be a cube with dist$(R, \partial\delta) \leq$ diam$R) \leq a\delta$ where a is a fixed sufficiently small constant depending only on ε, δ, and d. Then for $f \in L_P(\Omega), \gamma \leq p \leq 1$, we have

$$E_r(\mathcal{E}\, f, R)_p^p \leq C \sum_{\substack{S \in F \\ S \subset cR}} E(S^*)^p \tag{185}$$

Where c, C depend only on $d, r, \gamma, \lambda, \varepsilon$ and δ.

Proof.

Let $\mathcal{Q} = \{Q : Q \in F \text{ and } Q \cap R \neq \emptyset\} \cup \{Q^s : \mathrm{Q} \in \mathcal{F}_\mathrm{c} \text{ and } \mathrm{Q} \cap \mathrm{R} \neq \emptyset\}$.

If a is small enough then the properties (151) and (183) give that dist$(x_0, x_1) \leq \sqrt{a}\,\delta$ for the centers x_0, x_1 of Q_0, Q_1 respectively with these cubes chosen arbitrarily from $\mathcal{Q}$. We want to find the cube R_0 to be used in conjuction with lemma (3.3.4). Let Q_0, we can take $R_0 = \mathrm{Q}_0$. Otherwise, we pick a cube $Q_1 \in \mathcal{Q}$ such that the centers x_0, x_1 of Q_0, Q_1

respectively have the largest distance, say $|x_0 - x_1| = \eta$. If $\Gamma$is a path that connects the centers x_0, x_1 of these cubes and satisfies the (ε, δ) condition, then there is a point $z \in \Gamma$ such that $|z - x_i| \geq \eta/2,\ \ i = 0, 1$. If S is the cube in F which contains z, then we can take R_0 as the largest of the cubes $S,\ Q_0$.

We next check that R_0 satisfies the conditions of Lemma (3.3.4) relative to any $Q \in Q$. It is clear that $diam(Q) < diam(Q_0) < diam(R_0)$ for all $Q \in Q$. Since $\eta{:}\, |x_0 - x_1| \leq \sqrt{a}\delta$ and the length of Γ is $< C_\eta$, we have

$$dist(Q, R_0) < dist(Q, Q_0) + diam(Q_0) + dist(Q_0, R_0) < \eta + 2C_\eta \leq \delta \qquad (186)$$

Provided a is sufficiently small. Also, by (151)(i) and (181)

$$diam(R_0) \geq diam(R) \geq dist(R, \partial\Omega)/4 \ > \ \varepsilon\eta/8.$$

Hence, as in (186)$dist(Q, R_0) \leq (C + 1)\eta \leq C_1 diam(R_0)$ with C_1a fixed constant. We have verified the hypothesis of Lemma 5.1. Therefore, there is a chain of cubes $R_j,\ j = 0, \dots, m$, connecting R_0 to Q . By our assumptions, $Q \subset \Omega_1$ whenever $Q \in \mathcal{F}_c$ and $Q \cap R \neq \Phi$ (provided a is sufficiently small). Hence $\sum_{Q \in \mathcal{F}_c} \Phi_Q \equiv 1$ OnR. We can therefore apply exactly the same proof as for Lemma (3.3.2) (namely from (152) on) to derive (185).

Theorem (3.3.6) [125]:

Let Ω be an$(\varepsilon,\ \delta)$ domain and let $\gamma > 0$ and r be a positive integer. If $\mathcal{E}$ is any extension operator defined by (184), then for each $1 \geq p \geq \gamma$ and $f \in L_p(\Omega)$, we have for $0 < t \leq 1$,

$$\omega_r(\xi f, t)_p^p \leq C^p \left[\sum_{2^j \leq C_1 t} W_r(f, 2^j)_p^p + t^{rp} \left(\|f\|_p^p + \sum_{1 \geq 2^j \geq t} 2^{-jrp} W_r(f, 2^j)_p^p \right) \right] \qquad (187)$$

With the constants C and c_1 depending only on $d,\ r,\ \lambda,\ \gamma,\ \varepsilon$, and δ .

Proof.

The proof of (187) is very similar to that of (166) and we shall only Highlight the differences. We first observe that (187) automatically holds if $t \geq a\delta$ and a is a fixed constant because $\|f\|_p \leq C\|\xi f\|_p$. Therefore, we need only consider $t \leq a\delta$ with a a sufficiently small but fixed constant to be prescribed in more detail as we proceed. As in the proof of Theorem (3.3.3), we write $R^d \backslash \partial\Omega = \Omega_0 \cup \Omega_- \cup \Omega_+$, where $\Omega_0 \coloneqq \cup \{Q \in F \cup F_c{:}\, l(Q) \leq 16rt\}$, $\Omega_+ \coloneqq \Omega \backslash (\Omega_0 \cup \partial\Omega)$, $\Omega_- \coloneqq \Omega^c \backslash (\Omega_0 \cup \partial\Omega)$. We estimate $\int_S |\Delta_h^r (\xi f)|^p\, dx$ for the three sets $= \Omega_\pm$, Ω_0 and for $|h| \leq t$.

We proceed as in the proof of Theorem (3.3.3) and consider three cases. Case 1 which estimates the integral over Ω_+ is identical to the proof in Theorem (3.3.3) and yields the estimate (168). Case 2 is also the same since if a is small enough the cube R which contains $[x,\ x + rh]$ will be one of the cubes to which we can apply Lemma 5.2. We obtain in this way the estimate (174) for the integral over Ω_0.

In Case 3, that is $x \in \Omega_-$, we let $R \in F_c$ have nontrivial intersection with Ω_- . If $x \in R$, then $[x, x + rh] \subset R^*$. we have two possibilities for R . If $dist(R, \partial\Omega) < a\delta$ and a is small enough, then $\sum_{Q\in\mathcal{F}} \Phi_Q \equiv 1$ on R^* .We consider $\mathcal{Q} = \{Q^s: Q \in \mathcal{F}_c, Q \ touches \ R\}$. we can take R_0 as the largest cube in$\mathcal{Q}$. Then R_0 and any other cube Q^s in $\mathcal{Q}$ will satisfy the hypothesis of Lemma (3.3.4). We take a chain (R_j) connecting Q^s and R_0 and proceed as in Theorem (3.3.3) to obtain

$$\sum_R \|\Delta_h^r (\xi f)\|_p^p \leq Ct^{rp} \sum_{1\geq 2^j \geq t} 2^{-jrp} W_r(f, 2^j)_p^p \tag{188}$$

where the sum is taken over all cubes R of this type.

The second possibility is that $dist(R, \partial\Omega) \geq a\delta$. Whenever $Q \in \mathcal{F}_c$ is such that $\Phi_Q$does not identically vanish on R, then $6\delta \geq l(Q)$, $Cl(Q) \geq \delta$ and therefore from (4.2), $\|D^v\Phi_Q\|_\infty(Q^s) \leq C$, $|v| \leq r$, with C a constant depending only on δ and$r$. Also $\|P_{Q^s}\|_p(Q^s) \leq C\|f\|_p(Q^s)$ by the definition of P_{Q^s} as a near best approximation. From this and by Markov's inequality for polynomials, we obtain $\|D^vP_{Q^s}\|_\infty(Q^s) \leq C\|f\|_p(Q^s)$, $|v| \leq r$. Therefore, Leibniz' rule for differentiation gives that

$$\|D^v(\mathcal{E}f)\|_\infty(R) \leq C\|f\|_p R')$$

where R' is the union of all the cubes Q^s such that Φ_Q does not vanish on R .

Here we are using the fact that the number of cubes which appear nontrivially in $\xi f(x)$ does not exceed a constant which depends only on d. This gives

$$\|\Delta_h^r (\xi f)\|_p(R) \leq |h|^r \max_{|v|=r} \|D^v(\xi f)\|_\infty(R) \leq C\,|h|^r \|f\|_p(R') \tag{189}$$

Since a point $x \in \Omega$ can appear in at most C of the sets R' with C depending only on d, we can raise the inequality (189) to the power p and sum over all R of this type and obtain

$$\sum_R \|\Delta_h^r (\xi f)\|_p^p \leq C|h|^{rp}\|f\|_p^p(\Omega) \leq Ct^{rp}\|f\|_p^p(\Omega) \tag{190}$$

We add (188) and (190) to obtain that $\int_{\Omega_-} |\Delta_h^r (\xi f)|^p \, dx$ does not exceed the sum of the right sides of (188) and (190). The proof is then completed by adding the estimates in the three cases.

In this section, we establish the roundedness of the extension operator $\mathcal{E}$ on Besov spaces and apply this to obtain other characterizations of these spaces. Given $0 < \alpha < \infty$ and $0 < q < \infty$ and a sequence $\{a_k\}_{k\in N}$of real numbers, we define

$$\|(a_k)\|_{l_q^\alpha} := \left(\sum_{k\in\boldsymbol{N}}[2^{k\alpha}|a_k|]^q\right)^{\frac{1}{q}} \tag{191}$$

With the usual adjustment when $q=\infty$. We shall need the following discrete Hardy inequalities (for a proof see [127]). If for sequences $(a_k)_{k\in N}$ and $(b_k)_{k\in N}$ of real numbers, we have either

$(i)|b_k| \le c2^{-kr}\left(\sum_{j=0}^{\infty}\left[2^{jr}|a_j|\right]^{\mu}\right)^{\frac{1}{\mu}}$Or

$$(ii)|b_k| \le \left(\sum_{j=0}^{\infty}|a_j|^{\mu}\right)^{\frac{1}{\mu}}, \tag{192}$$

Then for all $q\ge\mu$ and $0<\alpha<r$, in case (i), and all $q\ge\mu$ and $0<\alpha\le\infty$, in case (ii), we have

$$\|(b_k)\|_{l_q^\alpha} \le C\|(a_k)\|_{l_q^\alpha} \tag{193}$$

Therefore, (193) holds for $q\ge\mu$ and $0<\alpha<r$, if $|b_k|$does not exceed the sum of the right sides of (192).

Theorem (3.3.7) [125]:

If Ω is an (ε,δ)domain, $\gamma>0$ and r is positive integer, then the extension operator $\mathcal{E}$of (184) is a bounded mapping from $B_q^\alpha(L_p(\Omega)$into $B_q^\alpha(L_p(\mathrm{R}^d)$For all $\gamma\le p\le 1, 0<q\le\infty$ and $\alpha<r$:

$$\|\xi f\|_{B_q^\alpha(L_p(\mathrm{R}^d)} \le C\|f\|_{B_q^\alpha(L_p(\Omega)} \tag{194}$$

With the constant C depending only on $d,\ r,\ \lambda,\ \gamma,\ \varepsilon,$ and δ.

Proof.

Let $\mu<min(q,\ p)$. Since an $l_p$norm is less than an $l_\mu$norm and since $W_r\le\omega_r$, from (187) for $t=2^{-k}$, we have

$$\omega_r(\xi f,2^{-k},R^d)_p \le C\left[\sum_{j=ck}^{\infty}\omega_r(f,2^{-j},\Omega)_p^{\mu}\right]^{\frac{1}{\mu}} + C2^{-kr}\left[\|f\|_p^{\mu}(\Omega)+\sum_{j=0}^{k}\left[2^{jr}\omega_r(f,2^{-j},\Omega)_p\right]^{\mu}\right]^{1/\mu} \tag{195}$$

We can therefore apply (193) and obtain

$$\left\|(\omega_r(\xi f,2^{-k},R^d)_p)\right\|_{l_q^\alpha} \le C\left[\|f\|_p(\Omega)+\left\|(\omega_r\left(f,2^{-k},R^d\right)_p)\right\|_{l_q^\alpha}\right] \tag{196}$$

The monotonicity of ω_r shows that the left side of (196) is equivalent to $|f|_{B_q^\alpha(L_p(\mathrm{R}^d)}$While the right side is equivalent $\|f\|_{B_q^\alpha(L_p(\Omega)}$ .Since $\mathcal{E}$ is a bounded map from $L_p(\Omega)$ into$L_p(\mathrm{R}^d)$, (196) establishes the theorem.

If follows from Theorem (3.3.7) that for each $0 < p \leq 1$,$0 < q \leq \infty$,$\alpha > 0$ and any (ε, δ) domain Ω , we have

$$\|f\|_{B_q^\alpha(L_p(\Omega)} \leq \|\xi f\|_{B_q^\alpha(L_p(\mathrm{R}^d)} \leq C\|f\|_{B_q^\alpha(L_p(\Omega)} \tag{197}$$

With constant C depending only on $d,\ r,\ \lambda,\ \gamma,\ \varepsilon,$ andΩ

We next show that functions in$B_q^\alpha(L_p(\Omega)$have atomic or wavelet decompositions. Let $\boldsymbol{N} = \boldsymbol{N}_r$ be the tensor product B spline in R^d obtained from the univariate B spline of degree $r-1$ which has knots at $0,\ 1,\ \dots,\ r$.

Let $\mathbb{D}_k$ denote the collection of all dyadic cubes for R^d which have side length $2^{-k}$and $\mathbb{D}_+ = \cup_{k\geq 0}\ \mathbb{D}_k$. With $\boldsymbol{N}$ /V, we can associate to any dyadic cube $I = \left[j^{2^{-k}}, (j+1)2^{-k}\right] \in\ \mathbb{D}_k$, $j \in \mathbb{Z}^d$, $k \in \boldsymbol{N}$, the dilated functions $\boldsymbol{N}_{\boldsymbol{I}}(x) \coloneqq \boldsymbol{N}(2^k x - j)$. This function has support on an expansion of the cube .

Theorem (3.3.8) [125]:

Let Ωbe a (ε, δ) domain and$0 < p \leq 1$,$0 < q \leq \infty$,$\alpha > 0$. Then each function $f \in B_q^\alpha(L_p(\Omega)$ has decomposition

$$f(x) = \sum_{\boldsymbol{I}\in\mathbb{D}_+} a_{\boldsymbol{I}}(f)\boldsymbol{N}_{\boldsymbol{I}}(x) \qquad\qquad x \in \Omega \tag{198}$$

Where the coefficients $a_{\boldsymbol{I}}(f)$ satisfy

$$\|f\|_{B_q^\alpha(L_p(\Omega)} \approx \left(\left(\sum_{k=0}^{\infty} 2^{k\alpha q} \sum_{\boldsymbol{I}\in\mathbb{D}_+} |a_{\boldsymbol{I}}(f)|^p |\boldsymbol{I}|\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \tag{199}$$

with constants of equivalency independent of f and the usual change on the right side of (199) when $q = \infty$.

Proof.

By (6.7), $f \in B_q^\alpha(\Omega)$ if and only $\xi f \in B_q^\alpha(R^d)$ with equivalent norms. It was shown in [129] that ξf has a decomposition (6.8) on R^d with coefficients $a_{\boldsymbol{I}}(\xi f)$ satisfying (199). Since $\xi f = f$ on Ω, the theorem follows. We next discuss the interpolation of Besov spaces using the real method of Peetre. If X_0 and X_1 are a pair of quasi-normed spaces which are continuously embedded in a linear Hausdorff space ξ, their K −functional is defined for any $f \in X_0 + X_1$ by

$$K(f,t) := K(f,t;X_0,X_1) := \inf_{f=f_0+f_1} \|f_0\|_{X_0} + t\|f_1\|_{X_1} \tag{200}$$

For each $0 < \theta < 1$, $0 < q \le \infty$, the space $X_{\theta,q} := (X_0, X_1)_{\theta,q}$ is the collection of all functions $f \in X_0 + X_1$ for which

$$\|f\|_{\theta,q} := \left(\int_0^\infty \left(t^{-\theta} K(f,t) \frac{dt}{t} \right)^q \right)^{\frac{1}{q}} \tag{201}$$

is finite (with again the usual adjustment on the right side of (201) when $q = \infty$.This is an interpolation space since it follows easily from the definition of the K −functional that each linear operator which is bounded on X_0 and X_1 is also bounded on $X_{\theta,q}$.

We are interested in interpolation for a pair of Besov spaces. Suppose that $0 < p_0$, $p_1 \le \infty$. And $0 < q_0$, $q_1 \le \infty$ and $\alpha_0, \alpha_1 \ge 0$. We let $X_i(\Omega) := B_{qi}^{\alpha i}(L_{pi}(\Omega))$, $i = 0{,}1$, with the understanding that this space is $L_{pi}(\Omega)$ when $\alpha_i = 0$. If we choose $r > max(\alpha_0, \alpha_1)$ and $\gamma < min(p_0, p_1)$ then the extension operators ξ of (184) are defined and (197) holds for each of these extensions. In fact, we observe that

$$K(f,t;X_0(\Omega)) \le \mathrm{K}(\xi f,t;X_0(R^d),X_1(R^d)) \le \mathrm{CK}\big(f,t;X_0(\Omega),X_1(\Omega)\big) \tag{202}$$

The left inequality in (202) is clear. The usual proof of the right inequality relies on the linearity of the operator, which as we have previously mentioned may fail for ξ since near best approximations $P_Q(f)$ are used in its definition (184). However, given any decomposition $f = f_0 + f_1$, we may decompose ξf as. $F_0 + F_1$ Where F_i is a norm bounded extension (in X_i) of f_i, $(i = 0,1)$. To see this, we recall Lemma 6.2 of [2] which established that if $f = f_0 + f_1$ and $P_Q(f)$ is any near best approximation to f, then there exist near best approximations R_Q^i to f_i $(i = 0,1)$ so that $P_Q(f) = R_Q^0 + R_Q^1$ We then use $R_{Q^s}^i$ in place of P_{Q^s} in (184) to define F_i from which we may conclude that (202) holds. From (202) it follows, therefore, that the interpolation spaces $\big(X_0(\Omega), X_1(\Omega)\big)_{\theta,q}$ and $(X_0(R^d), X_1(R^d))_{\theta,q}$ are identical with equivalent norms. From known results for the latter spaces (see [129]) we obtain the following.

Theorem (3.3.9) [125]:

Let Ω be a (ε,δ) domain. If $0 < p \le 1$ and α , $q_0 > 0$, then for any $0 < \theta < 1$,$0 < q \le \infty$ we have

$$(L_p(\Omega), B_{q_0}^{\alpha}(L_p))_{\theta,q} = B_q^{\theta\alpha}\big(L_p\big) \tag{203}$$

With equivalent norms .If $0 < p \le 1$, we let $\tau(\beta) = (\beta/d + 1/p)^{-1}, \beta > 0$, then for any $\alpha > 0$ and $0 < \theta < 1, 0 < q \le \infty$ we have

$$(L_p(\Omega), B_{\tau(\alpha)}^{\alpha}(L_{\tau(\alpha)}(\Omega)))_{\theta,\tau(\theta\alpha)} = B_{\tau(\theta\alpha)}^{\theta\alpha}\left(L_{\tau(\theta\alpha)}(\Omega)\right) \tag{204}$$

With equivalent norms.

Remark (3.3.10) [125]:

The proof in [127] of interpolation of Besov spaces relies on estab-lishing the equivalence of the K −functional of f with that of its retract. We take this opportunity to correct the proof of the lower inequality of that equivalence. The sentences in lines 3 through 7 on [127] should be replaced by: " We may estimate each term of the last sum as

$$\|t_j - g_j\|_{p_0} \le \left(\|t_j - a_j\|_{p_0} + \|a_j - T_j(a_j)\|_{p_0}\right) ,$$

And apply Corollary 4.7 to obtain

$$\|a_j - T_j(a_j)\|_{p_0} \le cS_j(a_j)_{p_0} \le c\|t_j - a_j\|_{p_0} .$$

Hence,

$$\|t_j - g_j\|_{p_0} \le c\|t_j - a_j\|_{p_0} \quad ,''$$

While preparing the present section, Ridgway Scott posed to us a question concerning interpolation of Besov spaces for$0 < p \le \infty$. It is rather easy to settle this question given the machinery developed of the present section. We shall from here on assume that Ω is a minimally smooth domain in the sense of Stein (it may be that Theorem (3.3.3) that follows also holds for (ε, δ)domains, however our proof does not seem to apply in this generality). Minimally smooth domain in R^d is an open set for which there is a number $\eta > 0$ and open sets U_i , $i = 1, 2, \ldots$, such that: (i) for each $x \in \delta\Omega$, the ball $B(x, \eta)$ is contained in one of the U_i ; (ii) a point $x \in R^d$ is in at most N of the sets U_i where N is an absolute constant; and (iii) for each $i, U_i \cap \Omega = U_i \cap \Omega_i$ for some domain Ω_i , which is the rotation of a Lipchitz graph domain with Lipchitz constant M independent of i .

We recall the fractional order Sable spaces. Let$0 < p \le \infty$. And$\alpha > 0$.

If is not an integer, we write $\alpha = \beta + r$ where $0 < \beta < 1$ and r is a nonnegative integer. Let W_p^α be the collection of all functions f in the Sobolev space $W_p^r(\Omega)$, for which

$$|f|^p_{W_p^\alpha(\Omega)} := \sum_{|v|=r} \int_{\Omega\times\Omega} \frac{|D^v f(x) - D^v f(y)|^p}{|x-y|^{\beta p + d}} dxdy \qquad (205)$$

Is finite.

If $\Omega = R^d$and a is not an integer, then it is well known that (205) is equivalent to $|f|^p_{B_p^\alpha(\Omega)}$ We want to show this remains true for minimally smooth domains Ω. For this purpose, we define for$f \in W_p^r(\Omega)$,

$$\bar{\omega}_{r+1}(f,t)^p := t^{rp} \sum_{|v|=1} W_1(D^v f, t)_p^p \qquad (206)$$

with W_1 , as before, the averaged modulus of smoothness (140).

Lemma (3.3.11) [125]:

Let Ω be any open set. For $0 < p \leq \infty$ and $\alpha > 0$not an integer, we have

$$|f|^p_{W^\alpha_p(\Omega)} = (\beta p + d)^{-1} \int_0^\infty \left[t^{-\alpha} \overline{\omega}_{r+1}(f,t)_p\right]^p \frac{dt}{t} \qquad (207)$$

Where as above.

Proof.

For any $g \in L_p(\Omega)$, we have for $0 < \beta < 1$, by a change of variables and Fubini's theorem,

$$\int_0^\infty \left[t^{-\beta p} W_1(g,t)_p\right]^p \frac{dt}{t}$$

$$= \int_0^\infty \int_{|s|\leq 1} \int_\Omega |\Delta_s\,(g,x,\Omega)|^p t^{-\beta p-d-1}\, dxdsdt$$

$$= \int_\Omega \int_\Omega \int_{|x-y|}^\infty t^{-\beta p-d-1} dt |g(x)-g(y))|^p\, dxdy$$

$$= (\beta p + d)^{-1} \int_\Omega \int_\Omega |x-y|^{-\beta p-d} dt |g(x)-g(y))|^p dxdy \qquad (208)$$

We take $g = D^v f, |v| = r$, and add the identities (208) to obtain (207).

We shall next show that an analogue of inequality (187) holds for $p \geq 1$. It is well known that if $f \in W^{r-1}_p$ then for the error $E(S)_p$ for approximating f in the norm $L_p(S)$ on a cube S by polynomials of $degree < r$, we have

$$E(S)^p \leq C\, l(S)^{p(r-1)} \sum_{|v|=r-1} \omega_1(D^v f\,, l(S), S)^p_p$$

$$\leq C\, l(S)^{p(r-1)} \sum_{|v|=r-1} W_1(D^v f\,, l(S), S)^p_p$$

$$= \overline{\omega}_r(f\,, l(S), S)^p_p \qquad (209)$$

Where as before W is the averaged modulus of smoothness given by (140) and ω_r is defined by (206).

Theorem (3.3.12) [125]:

Let Ω be a minimally smooth domain, let r be a positive integer and let $1 \leq p < \infty$. Then for any $f \in W^{r-1}_p$ and$0 < t < 1$, we have

$$\omega_r(\xi f,t)_p^p \le C^p\left[\sum_{2^j\le t}\overline{\omega}_r(f\,,2^j)_p^p\,+t^{rp}\left(\|f\|_p^p(\Omega)\right)\sum_{2^j\le t}2^{-jrp}\overline{\omega}_r\big(f\,,2^j\big)_p^p\big)\right] \qquad (210)$$

With C a constant depending only on d, r, λ andΩ.

Proof.

We first recall that a minimally smooth domain is an $(\varepsilon,\ \delta)$ domain. Since Ω will be an $(\varepsilon,\ \delta)$ domain for any e and ô sufficiently small, we can assume that η in the definition of minimally smooth domains is $\ge C_0\delta$ with C_0 Arbitrary but fixed. We shall prescribe η in more detail as we continue through the proof.

We proceed as in Theorems (3.3.3) and (3.3.6). The first case, namely the estimate of $\int_{\Omega_+}|\Delta_h^r\,(\xi f\ ,x)|^p\,dx$ is as before, but we use standard estimates of $r\,th$ differences in terms of a first order difference of $(r-1)\ th$ derivatives . This gives that the integral does not exceed $\overline{\omega}_r\big(f\ ,t\ ,\Omega)_p^p\big)$.

For the estimate in the second case, that is over Ω_0, we need first to derive an analogue of Lemma (3.3.5) for $\overline{\omega}_r$. With the same constructions and notation as in Lemma (3.3.5) and the same argument, we arrive at the estimate (160), where now $1\le p<\infty$. We need to observe that for each k, at most C of the cubes R_j appearing in (160) belong to $\mathbb{D}_k$. To see this, we recall that these cubes meet the path Γ which connects a point $z\in Q$ to a point$z_0\in Q_0$ From (151)(i), letting S be such an R_j, any point $\omega\in S\cap\Gamma$ satisfies

$$dist(\omega,\partial\Omega)<diam(S)+dist(S,\partial\Omega))\le 5\ \ diam(S)=5\sqrt{d}2^{-k}\quad .$$

Therefore, by the definition of $(\varepsilon,\ \delta)$domain (property (181)), we have

$$\min(|\omega-z|\,,|\omega-z_0|)\le 5\sqrt{d}\varepsilon^{-1}2^{-k}\quad .$$

That is, each of these cubes S meets one of the balls of radius $5\sqrt{d}\varepsilon^{-1}2^{-k}$ about z andz_0. Since the cubes S are disjoint there are at most C of them with C depending only on ε and d .

We now write $|R_j|^{-1/p}=|R_j|^{-a/p}|R_j|^{-b/p}$ where $a\,+\,b=1$ and $ad>d-1$.

We then apply Holder's inequality to (160) and use the observation above for$l\big(R_j\big)=2^k l(Q)$ to conclude that

$$\|f-P_{R_0}\|_p^p(Q)\le|Q|\left(\sum_{j=0}^m|R_j|^{-\frac{bp'}{p}}\right)^{\frac{p}{p'}}\left(\sum_{j=0}^m|R_j|^{-a}\,E(R_j^*)^p\right)$$

$$\le C|Q|^{1-b}\left(\sum_{j=0}^m|R_j|^{-a}\,E(R_j^*)^p\right)=C|Q|^a\left(\sum_{j=0}^m|R_j|^{-a}\,E(R_j^*)^p\right) \qquad (211)$$

We now sum over all $Q \in F$ such that $Q \cap R \neq \Phi$, reverse the order of summation to obtain that (185) is valid for this range of p provided that we can show that for fixed $S = R_j$, we have

$$\sum_{\substack{Q \in F \\ Q \subset cS}} |Q|^a \leq C\,|S|^a \tag{212}$$

With $C \geq 1$ a fixed constant and C depending only on d, ε, δ and η.

We postpone for a moment the proof of (212) and conclude the proof of the theorem. Now that we have established (185) of Lemma (3.3.5) for $1 \leq p < \infty$, the estimate of $\int_{\Omega_0} |\Delta_h^r (f, x)|^p\, dx$ can be made exactly as in the proof of Theorem (3.3.3) with (6.19) used in place of (142) and $\bar{\omega}_r$ used in place of wr. Finally, the proof in Case 3, that is the estimate of $\int_{\Omega_-} |\Delta_h^r (f, x)|^p\, dx$, can be made exactly as in the proof of Theorem (3.3.3) because the number of cubes in the sums appearing in (176), (177), and (178) is bounded by a constant C depending only on d, ε, and δ. This then completes the proof of the theorem subject to the verification of (212).

To prove (212), we count the number N_k of cubes $Q \in F$ with $Q \subset cS$ and $l(Q) = 2^k l(S))$. There are only a finite number of values of $k \leq 0$ and for each of these $N_k \leq C$ with C depending only on d (because the cubes Q are pairwise disjoint). Therefore, this portion of the sum appearing in (212) does not exceed the right side of (212).

To estimate N_k for $k \geq 1$, we recall that the cubes S have side length $\leq l(R_0) \leq Cl(R) \leq C\delta$. Therefore, by choosing δ sufficiently small, we can assume that $2c\,diam(S) \leq \eta$ with c the constant in the summation index of (212) and η of course the constant in the definition of minimally smooth domains. Therefore, by property (ii) of minimally smooth domains, we may assume that $(4cdS) \cap \Omega = (4cdS) \cap \Omega_j$ for one of the domains Ω_j. Since $c \geq 1$ and $dist(Q, \delta\Omega) \leq 4diam(Q) \leq 2diam(S)$, we have $dist(Q, \delta\Omega) = dist(Q, \delta\Omega_j)$. From property $(151)(i)$ of Whiney cubes, we have $Q \subset A_k := \{x : dist(x, \delta\Omega_j) \leq 52^{-k}\, diam(S)\}$. Now from the fact that Ω_j is a Lipschitz graph domain, we have that $|A_k| \leq C2^{-k}\,|S|$ with the constant C depending only on d and the Lipschitz constant M. Henc A_k can contain at most $C2^{k(d-1)}$ cubes Q of side length $2^{-k}\,l(S)$. This shows that $N_k \leq C2^{k(d-1)}$. Using this estimate for N_k, we find that the portion of the sum on the left side of (212) that remains to be estimated does not exceed

$$\sum_{k=1}^{\infty} N_k (\, 2^{-k}\, l(S))^{da} \leq C \sum_{k=1}^{\infty} 2^{k(d-1)}\; 2^{-da}\, |S|^a \leq C|S|^a$$

because $ad > d - 1$.

Using Theorem (3.3.13) we are able to easily establish the equivalent of the fractional Sobolev spaces$W_p^\alpha(\Omega)$ with the special family of Besov spaces $B_p^\alpha(L_p(\Omega))$.

Theorem (3.3.13) [125]:

Let Ω be a minimally smooth domain inR^d, and$1 \le p < \infty$, $0 < \alpha$, then $W_p^\alpha(\Omega) = B_p^\alpha(L_p(\Omega))$ and there exist positive constants c_1 , c_2 independent of f so that

$$c_1\|f\|_{W_p^\alpha(\Omega)} = \|f\|_{B_p^\alpha(L_p(\Omega))} \le c_2\|f\|_{W_p^\alpha(\Omega)} \tag{213}$$

Proof.

The upper inequality in (213) is obtained by applying thel_p^α norm to both sides of inequality (210) and using Hardy's inequality (193) together with Lemma (3.3.11). The lower inequality is confirmed by recalling that the corresponding result holds on R^d, and then following with an application of Theorem (3.3.7):

$$\|f\|_{W_p^\alpha(\Omega)} \le \|\xi f\|_{W_p^\alpha(R^d)} \le c\|\xi f\|_{B_p^\alpha(L_p(R^d))} \le c\|f\|_{W_p^\alpha(\Omega)} .$$

As we previously mentioned, when $1 \le p$ the extension operators may be taken to be linear. It then follows that $\|\xi f\|_{B_p^\alpha(L_p(R^d))}$ is equivalent (within constants independent of f) to $\|f\|_{B_p^\alpha(L_p(R^d))}$. Applying the interpolation theorem Corollary in [127] to $B_p^\alpha(L_p(R^d)$, we obtain the following interpolation resultfor the fractional order Sobolev spaces $W_p^\alpha(\Omega)$:

Corollary (3.3.14) [125]:

Let Ω be a minimally smooth domain inR^d, and$1 \le p_0$,$p_1 \le \infty$,$0 < \alpha_0$, α_1, then for p satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, we have

$$\left(W_{p_0}^{\alpha_0}(\Omega) , W_{p_1}^{\alpha_1}(\Omega)\right)_{\theta,p} = W_p^\alpha(\Omega) \tag{214}$$

with equivalent norms.