Chapter 2

Embedding Derivatives and Dual Inequalities of Hardy and Littewood

A characterization is given of those measures μ on U, the upper half plane R^2_+or the unit disk, such that differentiation of order m maps H^p boundedly into $L^q(\mu)$ where $0 < p < \infty$ and $0 < q < \infty$ The cases where $0 < p = q < 2$ and $0 < q < p$ are the only two not previously known .The solution is presented in the n real variable setting R^{n+1}_+ of Fefferman and Stein [7] with an arbitrary differential monomial of order m replacing complex differentiation .

Section (2.1):Some Related Dual Inequalities:

Let $L^p(-\pi,\pi)$ denote the usual linear space of complex-valued "p- th power integrable" functions on$[-\pi,\pi]$, with the norm given by

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^p d\theta\right\}^{1/p} \quad (1 \le p \le \infty),$$

$$\|f\|_\infty = \operatorname{ess\,sup}|f|.$$

For any complex-valued function w continuous on the open unit disc U in the plane, and for $0 \le \rho < 1$, we write

$$M_p(w;\rho) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|w(\rho e^{i\theta})\right|^p d\theta\right\}^{1/p} \quad (1 < p < \infty),$$

$$M_\infty(w;\rho) = \sup_\theta\left|w\left(\rho e^{i\theta}\right)\right|.$$

It is familiar that if either w is harmonic and $1 \le p \le \infty$, or w is holomorphic and$1 < p \le \infty$, then $M_p(w;\rho)$ increases with ρ. We define

$$\mathfrak{M}_p(w) = \sup_{0\le\rho<1} M_p(w;\rho) \qquad (0 < p \le \infty),$$

the value ∞ being permitted. The class of holomorphic w for which $\mathfrak{M}_p(w) < \infty$ is the Hardy class$H^p = H^p(U)$. The class of complex-valued harmonic w for which $\mathfrak{M}_p(w) < \infty$ will be denoted by$h^p = h^p(U)$.. Clearly

$H^p \subset H^p$.

For$1 < p \le \infty$,$1 < q \le \infty$, $\gamma > 0$,wewrite

$$\mathfrak{N}_{p,q,\gamma}(w) = \begin{cases}\left\{\int_0^1 (1-\rho)^{q\gamma-1}M_p^q(w;\rho)d\rho\right\}^{1/q} & (0 < p \le \infty), \\ \sup_{0\le\rho<1}\left\{(1-\rho)^\gamma M_p(w;\rho)\right\}(q = \infty), \end{cases}$$

the value ∞ being permitted. This expression $\mathfrak{N}_{p,q,\gamma}(w)$ can be regarded as a measure of the rate of growth of $M_p(w;\rho)$ when $M_p(w;\rho)$ is unbounded. If $M_p(w;\rho)$ is increasing, then the condition $\gamma > 0$ is obviously necessary for the finiteness of $\mathfrak{N}_{p,q,\gamma}(w)$ except in the trivial case where w vanishes identically. We note in passing that if w_1, w_2 are continuous on U, then

$$\mathfrak{N}^s_{p,q,\gamma}(\ w_1 + w_2) \le \mathfrak{N}^s_{p,q,\gamma}(\ w_1) + \mathfrak{N}^s_{p,q,\gamma}(w_2) \qquad (1)$$

where $s = \min\{p, q, 1\}$. This is an easy consequence of Minkowski's inequality and the inequality

$$(a + b)^k \le a^k + b^k \qquad (a, b \ge 0\,, 0 < k < 1).$$

We use B to denote a positive constant, depending on the particular parameters $p, q, \dots, \alpha, \beta, \dots$ concerned in the particular problem in which it appears; A will denote a positive absolute constant. These constants are not necessarily the same on any two occurrences.

For any index p satisfying $1 \le p \le \infty$ we define the conjugate index p' by $p' = p/(p-1)(1 < p < \infty), p' = \infty\ (p = 1), p' = 1(p = \infty)$.

In [57] I have proved the following result.'

Theorem (2.1.1) [52]:

Let $1 < p < r \le \infty, \delta = 1/p - 1/r$, let $f \in L^p$ and let u be the Poisson integral of f on U. If either $q = \infty$, or $p \le q < \infty$ and $p > 1$, then

$$\mathfrak{N}_{p,q,\delta}(u) \le B\|f\|_p.$$

Further, $M_r(u;\rho) = o((1-\rho)^{-\delta})$as $\rho \to 1-$.

By arguments of a standard type involving subharmonic functions (cf. [57]), Theorem A gives

Theorem (2.1.2) [52]:

Let $0 < p < r \le \infty$, $p \le q < \infty$, $\delta = 1/p - 1/r$, and let $\phi \in H^p$. Then $\mathfrak{N}_{r,q,\delta}(\phi) \le B\mathfrak{M}_p(\phi)$, and $M_r(\phi;\rho) = o((1-\rho)^{-\delta})$ as $\rho \to 1-$.

Theorem B is a known result of Hardy and Littlewood([62] ,[11], [54] and [57]) , we have given a number of new applications of it. The case $p > 1$ of Theorem (2.1.2) implies the case $p > 1$ of Theorem (2.1.1), but the implication is nontrivial, since it depends on M. Riesz's theorem on conjugate functions.

In this section we show a new inequality that is the dual of Theorem (2.1.1), and consider a number of related results.

The most general form of this dual theorem involves fractional derivatives, or some multiplier transformation akin to a fractional derivative, and we use here a multiplier transformation introduced in [58] that is particularly suited to harmonic and holomorphic functions.

Consider first the case of a function ,ϕ holomorphic on U, and let $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$ $(z \in U)$. We define the multiplier transformation $\oint^{\alpha}\phi$of ϕ, where α is any real number, by

$$\oint^{\alpha}\phi(z) = \sum_{n=0}^{\infty}(n+1)^{-\alpha}\, c_n z^n (z \in U).$$

This function $\oint^{\alpha}\phi$ is clearly holomorphic on U. It may be regarded as a fractional integral (for $\alpha > 0$) or fractional derivative (for $\alpha < 0$) of $\emptyset$, and obviously

$$\oint^{\alpha}\left(\oint^{\beta}\phi\right) = \oint^{\alpha+\beta}\phi \tag{2}$$

for all real α, β. Moreover, for any positive integer m

$$\oint^{-m}\emptyset(z) = [(d/dz)z]^m\emptyset(z). \tag{3}$$

There is also an integral formula for $\oint^{\alpha}\emptyset$when $\alpha > 0$, which in its simplest form is

$$\oint^{\alpha}\phi\left(\rho e^{i\theta}\right) = \frac{1}{\Gamma\alpha}\int_0^1 (\log(\frac{1}{\sigma})^{\alpha-1}\phi(\rho\sigma e^{i\theta})\, d\sigma. \tag{4}$$

This is easily verified by term-by-term integration, using the formulae

$$\int_0^1 (\log(1/\sigma)^{\alpha-1}\sigma^n d = \int_1^{\infty} t^{\alpha-1}e^{-(n+1)t}dt = (n+1)^{-\alpha}\Gamma\alpha. \tag{5}$$

A similar definition applies to a (complex-valued) harmonic function on U.

If u is harmonic on U, then it is of the form

$$u\left(\rho e^{i\theta}\right) = \sum_{n=-\infty}^{\infty} c_n \rho^{|n|} e^{ni\theta}, \tag{6}$$

and we define $\oint^{\alpha}u$ by

$$\oint^{\alpha}\left(\rho e^{i\theta}\right) = \sum_{n=-\infty}^{\infty}(|n|+1)^{-\alpha}c_n\rho^{|n|}e^{ni\theta}.$$

It is easily verified that $\oint^{\alpha}u$ is harmonic on U, that (2) and (4) hold with u in place of ϕ, and that, if m is a positive integer, then

$$\oint^{-m}u\left(\rho e^{i\theta}\right) = [(\partial/\partial\rho)_{\rho}]^m u\left(\rho e^{i\theta}\right).$$

It is also obvious that if u is the real part of a holomorphic ϕ, then fo.u is the real part of $\oint^{\alpha}\phi$.

We can now state our dual of Theorem A, viz.,

Theorem (2.1.3) [52]:

Let $1 \le P < r < \infty,\ 1 \le q \le r,\ \gamma > 0, \delta = 1/p - 1/r$,and let u be a harmonic function on U such that $\mathfrak{N}_{p,q,\gamma}(\ \oint^{-\gamma-\delta}u) < \infty$. Then

$u \in h^r$, and

$$\mathfrak{M}_r(u) \le B\mathfrak{N}_{p,q,\gamma}(\ \oint^{-\gamma-\delta}u) \tag{7}$$

Hence also u is the Poisson integral of a function $f \in L^r$, and

$$\|f\|_r \le B\mathfrak{N}_{p,q,\gamma}(\ \oint^{-\gamma-\delta}u) \tag{8}$$

As an immediate example of the applications of Theorem 2.1.3, we may mention the well-known theorem of Hardy and Littlewood [60] on Fourier coefficients, that if $2 < r < \infty$, and $(c_n)_{-\infty<n<\infty}$ is a sequence of complex numbers such that

$$S = \left\{\sum_{n=-\infty}^{\infty} (|n|+1)^{r-2}|c_n|^r\right\}^{1/r} < \infty,$$

then the numbers $c_n$are the Fourier coefficients of a function $f \in L^r$", and

$$\|f\|_r \le BS. \tag{9}$$

To deduce this, we observe first that the condition $S < \infty$ trivially implies that the series on the right of (6) converges for $0 \le \rho < 1$. Further, if u is defined by (6), and $2 < r < \infty$, then

$$\mathfrak{N}_{2,r,\frac{1}{2}+\frac{1}{r}}(\oint^{-1}u) = \left\{\int_0^1 (1-\rho)^{r/2}\left(\sum_{n=-\infty}^{\infty}(|n|+1)^2|c_n|^2\rho^{2|n|}\right)^{r/2} d\rho\right\}^{1/r},$$

and this does not exceed BS, in virtue of the inequality

$$\int_0^1 (1-\rho)^{\alpha}\left(\sum_{n=0}^{\infty} a_n\rho^n\right)^{s} d\rho \le B\sum_{n=0}^{\infty}(n+1)^{s-\alpha-2}a_n, \tag{10}$$

which is valid for $s \ge 1, \alpha > -1, a_n \ge 0$ (Hardy and Littlewood [61, Theorem 3]; see also Mulholland [67]). Theorem 2.1.3 (with $p = 2, q = r, \gamma = 1/2 + 1/r$) now shows that if $S < \infty$, then u is the Poisson integral of a function f satisfying (9), and this gives the required result.

In the same way, by taking $r = 2, P = q$ in Theorem A, writing r in place of p, and using the reverse of (10), which holds for $0 < s \le 1, \gamma > -1$, we obtain the dual of (9), namely that if f $f \in L^r$, where $1 < r < 2$, and has Fourier series $\sum c_n e^{ni\theta}$, then

$S \leq A(r)\|f\|_r$. The same case of Theorem B gives similarly the extension of this result for a function $\emptyset \in H^r$, where$0 < r \leq 1$ (Hardy and Littlewood [60]).

Theorem (2.1.4) [52]:

Let p, q, r satisfy one of the following sets of conditions:

(i) $0 < p < r < \infty, 0 < q \leq r$;

(ii) $0 < p \leq r = \infty, 0 < q \leq 1$;

(iii) $0 < p = r < \infty, 0 < q \leq \min\{2, r\}$.

Let also $\gamma > 0, \delta = 1/p - 1/r$, and let $\emptyset$ be a holomorphic function on U whose imaginary part vanishes at the origin, and whose Teal part u satisfies $\mathfrak{N}_{p,q,\gamma}(\notin^{-\gamma-\delta}u) < \infty$. Then $\phi \in H^r$ and

$$\mathfrak{M}_r(\phi) \leq B\mathfrak{N}_{p,q,\gamma}\left(\notin^{-\gamma-\delta}u\right).$$

Theorem (2.1.4) is best possible, in the sense that the result is false for all choices of p, q, r satisfying $0 < p \leq r \leq \infty, 0 < q \leq \infty$, and not covered by one of the conditions (i)-(iii). Moreover, the result is still false in these cases if we replace u in $\mathfrak{N}_{p,q,\gamma}(\notin^{-\gamma-\delta}u)$ by ϕ.

The proof of Theorem 2.1.3 is given in 4. The various cases of Theorem (2.1.4) require widely diverse arguments, and we begin by proving an easier result (Theorem 5) in which we replace u by ϕ.Theorems 3 and 4 in 5, 6 are preliminaries to the proof of Theorem 5. The proof of Theorem 2 is completed in 9-11. We show also in 11 (Theorem 8, Corollary 2) that when $\gamma + \delta$ is a positive integer k, then there is a result similar to Theorem 2 with ,$\notin^{-\gamma-\delta}u = \notin^{-k}u$ replaced by $\rho^k(\partial^k u/\partial\rho^k)$.

Theorems 6-8 in 9-11 have applications to Lipschitz spaces of holomorphic and harmonic functions on the disc, and I hope to consider these in a further section.

We note here that the cases $p \geq 1, q \geq 1$ of some of our results have been obtained in a more general setting by Taibleson (68] and the author (58] in a discussion of Lipschitz spaces. However, the area of overlap is small, and the new cases that we have to consider, where $0 < p < 1$or $0 < q < 1$, generally require new arguments, which are often applicable also to the known cases.

In this section we give the proof of Theorem 2.1.3. For $0 \leq R < 1$ we have

$$M_r(u; R) = \sup\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(Re^{i\theta}\right)g(\theta)d\theta\right\}, \tag{11}$$

where the supremum is taken over all bounded functions g such that $\|g\|_{r'} = 1$. Let g be such a function, and let

$$g(\theta) \sim \sum_{n=-\infty}^{\infty} d_n e^{ni\theta}, \quad v(\rho e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n \rho^{|n|} e^{ni\theta}, \quad u(\rho e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n \rho^{|n|} e^{ni\theta}$$

(so that v is the Poisson integral of g). By (5),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(Re^{i\theta}) g(\theta) d\theta = \sum_{n=-\infty}^{\infty} c_n d_{-n} R^{|n|}$$

$$= \frac{2^{\gamma+\delta}}{\Gamma(\gamma+\delta)} \int_0^1 (\log 1/\rho)^{\gamma+\delta-1} \left(\sum_{n=-\infty}^{\infty} (|n|+1)^{\gamma+\delta} c_n d_{-n} R^{|n|} \rho^{2|n|} \right) \rho d\rho$$

$$= \frac{2^{\gamma+\delta}}{\Gamma(\gamma+\delta)} \int_0^1 \rho(\log 1$$

$$/\rho)^{\gamma+\delta-1} d\rho \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \oint^{-\gamma-\delta} u(\rho e^{i\theta}) v(R\rho e^{i\theta}) d\theta \right\} \rho d\rho$$

(the term-by-term integrations being justified by uniform convergence). Here$\rho(\log 1/\rho)^{\gamma+\delta-1} \le (1-\rho)^{\gamma+\delta-1}$, and, by Holder's inequality and the increasing property of $M_{p'}$,

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \oint^{-\gamma-\delta} u(\rho e^{i\theta}) v(R\rho e^{i\theta}) d\theta \right| \le M_p(\oint^{-\gamma-\delta} u; \rho) M_{p'}(v; R\rho)$$

$$\le M_p(\oint^{-\gamma-\delta} u; \rho) M_{p'}(v; \rho).$$

Hence

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} u(Re^{i\theta}) g(\theta) d\theta \right| \le B \int_0^1 (1-\rho)^{\gamma+\delta-1} M_p(\oint^{-\gamma-\delta} u; \rho) M_{p'}(v; \rho) d\rho$$

$$\le B \mathfrak{N}_{p,q,\gamma}(\oint^{-\gamma-\delta} u) \mathfrak{N}_{p',q',\delta}(v) \quad (12)$$

$$\le B \mathfrak{N}_{p,q,\gamma}(\oint^{-\gamma-\delta} u) \|g\|_{r'} = B \mathfrak{N}_{p,q,\gamma}(\oint^{-\gamma-\delta} u),$$

by Holder's inequality and Theorem 2.1.1 (note that here $1 < r' < 'p \le \infty, r' \le q' \le \infty, \delta = 1/r' - 1/p'$). On combining (12) and (11) we now obtain (7), and this in turn gives (8) by standard properties of harmonic functions.

For index $r < 1$ the key result is a theorem concerning holomorphic functions.

Theorem (2.1.5) [52]:

Let $0 < r < 1, \gamma > 0$, and let ϕ be a holomorphic function on U such that

$\mathfrak{N}_{r,q,\gamma}(\not{f}^{-\gamma}\phi)$. Then $\phi \in H^r$ and

$$\mathfrak{M}_r(\phi) \leq B\mathfrak{N}_{r,q,\gamma}(\not{f}^{-\gamma}\phi). \tag{13}$$

A similar result for a different multiplier transformation is proved in [54, Theorem 5(ii)], but the proof given there has the disadvantage that it is peculiar to the disc, and does not extend to the half-plane. An alternative proof, using yet another multiplier transform, applicable both to the disc and half-plane, is given in [55, Theorem 2]. While preparing this section, 1 have realized that a more elegant and simpler variant of the argument in [55] is implicit in a section of Hardy and Littlewood [64], and it seems worth while to give the proof of Theorem (2.1.5) explicitly using their argument.We require the following lemma (see [55]).

Lemma (2.1.6) [52]:

Let w be a nonnegative subharmonic function on U satisfying the condition that $M_1(w;p) \leq C$ for $0 \leq \rho < 1$, let $0 < \eta < 1$, and for each $z \in U$ let $w_\eta^*(z)$ be the supremum of w on the closed disc withcentre$z$ and radius $\eta(1-|z|)$. Then for $0 \leq \rho < 1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w_\eta^*(\rho e^{i\theta})d\theta \leq BC.$$

Consider now the proof of Theorem 2.1.7. Let $0 \leq R < 1$. By (5).

$$\phi(Re^{i\theta}) = \frac{2^\gamma}{\Gamma(\gamma)}\int_0^1 (\log\frac{1}{\rho})^{\gamma-1} \not{f}^{-\gamma}\phi(R\sigma^2 e^{i\theta})\sigma d\sigma, \tag{14}$$

whence also

$$|\phi(Re^{i\theta})| \leq B\int_0^1 (1-\rho)^{\gamma-1}|\not{f}^{-\gamma}\phi(R\sigma^2 e^{i\theta})|d\rho. \tag{15}$$

Let $\sigma_n = 1-2^{-n}(n = 0,1,\ldots)$. Then $1-\sigma_n = \sigma_n - \sigma_{n-1} = 2^{-n}$, and $\sigma_{n-1} \leq \sigma_n^2 \leq \sigma_n$, so that also

$$\sup_{\sigma_{n-1}\leq\sigma\leq\sigma_n} |\not{f}^{-\gamma}\phi(R\sigma^2 e^{i\theta})| \leq \sup_{\sigma_{n-2}\leq\sigma\leq\sigma_n} |\not{f}^{-\gamma}\phi(R\sigma e^{i\theta})| = \mu_n(\theta).$$

Then, by (15),

$$|\phi(Re^{i\theta})| \leq B\sum_{n=1}^{\infty}\int_{\sigma_{n-1}}^{\sigma_n}(1-\sigma)^{\gamma-1}\,|\not{f}^{-\gamma}\phi(R\sigma^2 e^{i\theta})|d\sigma\; B\sum_{n=1}^{\infty} 2^{-\gamma n}\mu_n(\theta).$$

Since $(\sum a_n)^r \leq \sum a_n^r$ whenever an $a_n \geq 0$and $0 < r < 1$, it follows that

$$\left|\phi\left(Re^{i\theta}\right)\right| \leq B\sum_{n=1}^{\infty} 2^{-r\gamma n}\mu_n^r(\theta),$$

and therefore also

$$M_r^r(\phi;R) \leq B\sum_{n=1}^{\infty} 2^{-r\gamma n}\int_{-\pi}^{\pi}\mu_n^r(\theta)\,d\theta.$$

we now apply Lemma (2.1.6) with $w\left(e^{i\theta}\right) = \left|\not{f}^{-\gamma}\phi\left(R\sigma_{n+1}\rho e^{i\theta}\right)\right|^r$. Since

$$\frac{1}{2}(\sigma_n - \sigma_{n-1})/(\sigma_{n+1} - \frac{1}{2}(\sigma_n - \sigma_{n-2})) = \frac{3}{4},$$

we see that $\mu_n^r(\theta) \leq w_\eta^*(z)$, where $z = \frac{1}{2}\sigma_{n+1}^{-1}(\sigma_n + \sigma_{n-2})e^{i\theta} = \frac{3}{4}$. Since also

$$M_1(w;\rho) \leq M_r^r(\not{f}^{-\gamma}\phi; R\sigma_{n+1}) \leq M_r^r(\not{f}^{-\gamma}\phi;\sigma_{n+1})$$

it follows that

$$\int_{-\pi}^{\pi}\mu_n^r(\theta)d\theta \leq AM_r^r(\not{f}^{-\gamma}\phi;\sigma_{n+1})$$

Hence also

$$M_r^r(\phi;R) = B\sum_{n=1}^{\infty} 2^{-r\gamma n}M_r^r(\not{f}^{-\gamma}\phi;\sigma_{n+1})$$

$$\leq B\sum_{n=1}^{\infty}\int_{\sigma_{n+1}}^{\sigma_{n+2}}(1-\sigma)^{r_{\gamma-1}}\,M_r^r(\not{f}^{-\gamma}\phi;\sigma)d\sigma$$

$$= B\int_0^1(1-\sigma)^{r_{\gamma-1}}\,M_r^r(\not{f}^{-\gamma}\phi;\sigma)d\sigma = B\mathfrak{N}_{r,r,\gamma}^r(\not{f}^{-\gamma}\phi),$$

and this gives (13).

A more detailed examination of the preceding argument shows that the constant B in (13) is bounded as $r \to 1-$ for each fixed γ. The inequality of (13) is in fact true for $r = 1$, as is shown by Theorem 4 below.

An argument exactly similar to that above gives also

Theorem (2.1.7) [52]:

Let $0 < r < 1$, and let $\emptyset$ be a holomorphic function on U such that $\mathfrak{N}_{r,r,1}(\emptyset') < \infty$. Then $\emptyset \in H^r$ and

$$\mathfrak{M}_r^r(\phi) \leq B\mathfrak{N}_{r,r,1}^r(\phi') + |\phi(0)|^r.$$

If we apply this last inequality to the function $z \rightsquigarrow \emptyset(\rho z)$where $0 \leq \rho < 1$, and use the increasing property of the mean M_r , we deduce that

$$M_r^r(\phi;\rho) \le B\int_0^1 (1-\sigma)^{r-1} M_r^r(\rho\phi';\rho\sigma)d\sigma + |\phi(0)|^r$$

$$\le BM_r^r(\rho\phi';\rho) + |\phi(0)|^r = BM_r^r(z\phi';\rho) + |\phi(0)|^r. \quad (16)$$

There is also an inequality corresponding to (5.4) for $1 \le r \le \infty$, namely,

$$M_r(\phi;\rho) \le M_r(z\phi';\rho) + |\phi(0)|. \quad (17)$$

To prove this we apply Minkowski's inequality to the relation

$$\phi(\rho e^{i\theta}) = \int_0^\rho \frac{\partial\varphi}{\partial\sigma}(\sigma e^{i\theta})\, d\sigma + \phi(0);$$

this gives

$$M_r(\phi;\rho) \le \int_0 M_r(\phi';\sigma)d\sigma + |\phi(0)|, \quad (18)$$

and (17) follows from (18) and the increasing property of M_r .

Theorem (2.1.8) [52]:

Let $1 \le r < \infty$, $\gamma > 0$, and let $\emptyset$ be a holomorphic function on U such that $\mathfrak{N}_{r,1,\gamma}(\not{f}^{-\gamma}\phi)$. Then $\phi \in H^r$ and

$$\mathfrak{M}_r(\phi) \le B\mathfrak{N}_{r,1,\gamma}(\not{f}^{-\gamma}\phi). \quad (19)$$

Moreover, if $r = \infty$, then $\emptyset$has a continuous extension to $\bar{U}$.

From Minkowski's inequality and the inequality (15), we have

$$M_r(\phi;R) \le B\int_0^1 (1-\sigma)^{\gamma-1} M_r(\not{f}^{-\gamma}\phi;R\sigma^2)d\sigma$$

and (19) follows from this and the increasing property of M_r.

To prove the second part, it is enough to prove that the integral on the right of (14) is convergent, uniformly in R,θ. Let $0 < \delta < 1$. Then, exactly as above,

$$\sup_{R,\theta}\left|\int_\delta^1 (\log 1/\sigma)^{\gamma-1} \not{f}^{-\gamma}\phi(R\sigma^2 e^{i\theta})\sigma\ d\sigma\right| \le B\int_\delta^1 (1-\sigma)^{\gamma-1} M_\infty(\not{f}^{-\gamma}\phi;\sigma)d\sigma,$$

and the required result therefore follows from the finiteness of $\mathfrak{N}_{\infty,1,\gamma}(\not{f}^{-\gamma}\phi)$.

Theorem (2.1.9) [52]:

Let p,q,r satisfy one of the following sets of conditions:

(i) $0 < p < r < \infty, 0 < q \le r$;

(ii) $0 < p \le r = \infty, 0 < q \le 1$;

(iii) $0 < p = r < \infty, 0 < q \le \min\{2,r\}$.

Let also $\gamma > 0$, $\delta = 1/p - 1/r$, and let $\emptyset$ be a holomorphic function on U such that

$\mathfrak{N}_{p,q,\gamma}(\not{f}^{-\gamma-\delta}\phi) < \infty$ Then $\phi \in H^r$ and

$$\mathfrak{M}_r(\phi) \le B\mathfrak{N}_{p,q,\gamma}(\not{f}^{-\gamma-\delta}\phi).$$

Further, in, case (ii) ϕhas a continuous extension to $\overline{U}$.

We remark here that cases (i) and (ii) and the cases $0 < p = r \le 1, 0 < q \le r$ and $1 < p = r < \infty, 0 < q \le 1$ of (iii) depend only on Theorems (2.1.3), (2.1.5) and (2.1.8), and in their proofs we have invoked none of the deeper theorems of H^p-theory, such as the Hardy-Littlewood "Complex Max" theorem or the Littlewood-Paley theorems. In contrast, the remainder of case (iii), i.e., the case $1 < p = r < \infty$, $1 < q \le \min\{2, r\}$, lies deeper, and we deduce it from Lemma (2.1.10) below. Our only application of this case is to the proof of the corresponding case of Theorem 2.1.4.

Lemma (2.1.10) [52]:

Let $1 < k \le 2, 1 < r < \infty, \gamma > 0$, and. let $\emptyset$ be a holomorphic function on U such that$g_k \in (-\pi, \pi)$, where

$$g_k(\theta) = \left\{\int_0^1 (1-\rho)^{k\gamma-1}\left|\not{f}^{-\gamma}\phi(\rho e^{i\theta})\right|^k d\rho\right\}^{1/q}.$$

Then $\emptyset \in H^r$ and

$$\mathfrak{M}_r(\phi) \le B\|g\|_r.$$

This is essentially Hirschman's extension of one of the Littlewood-Paley theorems. A proof, for a closely similar multiplier transformation, is given in [54, Theorem 4], where references can also be found. The modifications required for are minor.

To deduce Theorem (2.1.9) from these various results we require two simple lemmas.

Lemma (2.1.11) [52]:

Let $0 < p \le \infty$, $0 < q < s \le \infty, \gamma > 0$, and let $\emptyset$ be a holomorphic function on U such that $\mathfrak{N}_{p,s,\gamma}(\emptyset) < \infty$. Then

$$\mathfrak{N}_{p,s,\gamma}(\emptyset) \le B\mathfrak{N}_{p,q,s,\gamma}(\emptyset), \tag{20}$$

And $M_p(\emptyset;\rho) = 0((1-\rho)^{-\gamma})$ as $\rho \to 1-$.

Since $M_p^q(\emptyset;\sigma)$ increases with σ,

$$\int_\rho^{-1}(1-\rho)^{q\gamma-1} M_p^q(\emptyset;\sigma)d\sigma \ge M_p^q(\emptyset;\rho)\int_\rho^{-1}(1-\sigma)^{q\gamma-1}\, d\sigma$$

$$= B(1-\rho)^{q\gamma} M_p^q(\emptyset;\rho).$$

Since the integral on the left does not exceed $\mathfrak{N}_{p,q,\gamma}^q(\emptyset)$,and tends to 0 as $\rho \to 1-$, this proves both the case $s=\infty$ of (10) and the remark concerning $M_p(\emptyset;\rho)$. The deduction of the case $s<\infty$ of (10) now follows from the obvious inequality

$$\mathfrak{N}_{p,s,\gamma}^s(\emptyset) \leq \mathfrak{N}_{p,\infty,\gamma}^{s-q}(\emptyset)\mathfrak{N}_{p,q,\gamma}^q(\emptyset).$$

Lemma (2.1.12) [52]:

Let $0<p<t\leq\infty, 0<q\leq\infty, \gamma>0$, and let $\emptyset$ be a holomorphic function on Usuch that $\mathfrak{N}_{p,q,\gamma}(\emptyset)<\infty$. Then

$$\mathfrak{N}_{t,q,\gamma+1/p-1/t}(\emptyset) \leq B\mathfrak{N}_{p,q,\gamma}(\emptyset), \tag{21}$$

By the case $q=\infty$ of Theorem B applied to $z \rightsquigarrow \emptyset(\rho z)$,

$$(1-\rho)^{1/p-1/t} M_t(\emptyset;\rho^2) \leq BM_p(\emptyset;\rho),$$

and this trivially gives (21).

Proof of Theorem (2.1.9):.

Cases (i)-(iii) are covered by the following cases (not entirely mutually exclusive):

$(i)'$ $0<p<r<\infty, 1<r, 0<q\leq r$;

$(ii)'$ $0<p\leq r<\infty, 0<q<1\leq r$;

$(iii)'$ $0<p\leq r<1, 0<q\leq r$;

$(iv)'$ $1<p=r<2, 1<q\leq r$;

$(v)'$ $2<p=r<\infty, 1<q\leq 2$.

By virtue of Lemmas (2.1.11) and (2.1.12) we can reduce these cases respectively to

$(i)''$ $1\leq p<r<\infty, q=r$;

$(ii)''$ $1\leq p=r\leq\infty, q=1$;

$(iii)''$ $0<p=q=r<1$;

$(iv)''$ $1<p=q=r\leq 2$;

$(v)''$ $2<p=r<\infty, q=2$.

Here (it is contained in Theorem (2.1.3), and $(ii)'', (iii)''$ are Theorems (2.1.8) and (2.1.5). Also $(iv)''$ is the case $1<k=r\leq 2$ of Lemma (2.1.10), and to prove (v)" we have only to take $k=2$ in Lemma 2 and apply Minkowski's inequality.

Before proceeding to the proof of Theorem 2, we note that Theorem $5(i)$ and Theorem B together give a result equivalent to the Hardy- Littlewood theorem on fractional integrals [64, Theorem 336], namely, that if $0 < p < r < \infty$; $\alpha = 1/p - 1/r$, and $\emptyset \in H^p$, then $\not{f}^{\alpha}\phi \in H^r$ and

$$\mathfrak{M}_r(\not{f}^{\alpha}\phi) \leq B\mathfrak{M}_p(\phi) \qquad (22)$$

To obtain (22), let $q = \frac{1}{2}(p + r)$. By Theorem B, with the r of that theorem replaced by q.

$$BM_p(\phi) \geq \mathfrak{N}_{q,q,1/p-1/q}(\phi) = \mathfrak{N}_{q,q,\alpha-1/p+1/r}(\not{f}^{-\alpha}(\not{f}^{\alpha}\phi)),$$

and we have now only to apply Theorem (2.1.8)(i) with ϕreplaced by $\not{f}^{\alpha}\phi$.

Theorem (2.1.13) below is a further preliminary to the proof of Theorem (2.1.4). Like the theorems to be proved in 10, 11, it has also applications to the theory of Lipschitz spaces.

Theorem (2.1.13) [52] :

Let $0 < p \leq \infty, 0 < q \leq \infty, \beta > 0, \gamma > 0$, and let ϕ be holomorphic on U. Then

(i) $B\mathfrak{N}_{p,q,\gamma}(\phi) \leq \mathfrak{N}_{p,q,\beta}(\not{f}^{\gamma-\beta}(\not{f}^{\alpha}\phi) \leq B\mathfrak{N}_{p,q,\gamma}(\phi)$,

(ii)$M_p(\phi;\rho) = o((1-\rho)^{-\gamma})as\rho \to 1-$ iff $M_p\left(\not{f}^{\gamma-\beta}\phi;\rho\right) = o\left(\left(1-\rho\right)^{-\beta}\right)$.

The cases $1 \leq p, q \leq \infty$ and $0 < p = q \leq 1$ of this theorem are known (Hardy and Littlewood [64], Flett [54]), but the remaining cases appear to be new. Since the new cases require new arguments, and these arguments give also the known cases, we give the proof in full.

We observe first that the left-hand inequality in (i) follows from the right-hand inequality with $\emptyset$ replaced by $\not{f}^{\beta-\gamma}\phi$, and that the "if" in (ii) follows similarly from the "only if". Writing $\alpha = \gamma - \beta$(so that $\gamma > \alpha$), we see that it is therefore enough to prove

$(iii)\mathfrak{N}_{p,q,\gamma-\alpha}(\not{f}^{\alpha}\phi) \leq B\mathfrak{N}_{p,q,\gamma}(\phi)$,(23)

(iv)if$M_p(\emptyset;\rho) = o((1-\rho)^{-\gamma})$ as $\rho \to 1-$, then

$$M_p(\not{f}^{\alpha}\phi;\rho) = o((1-\rho)^{\alpha-\gamma}).$$

Next, to prove (iii) and (iv) it is enough to prove the cases (a) $\alpha > 0$, (b) $\alpha = -1$, for $\alpha = -\eta < 0$ and m is the integral part of $\eta + 1$, we can prove the result for 0: by m successive applications of case (b) followed by an application of case (a) with $\alpha = m - \eta$.

We prove first the case $q < \infty$ of (a), and for this we use a lemma that is a relatively simple particular case of a theorem on Riemann-Liouvill integrals (see [53] and references given there),

Lemma (2.1.14) [52]:

Let $1 \le k < \infty,\ \mu > 0, \delta > 0$, and let $f:]0,\infty][0,\infty]$ be measurable. Then

$$\int_0^\infty x^{k-k\delta-k\mu-1}\left\{\int_0^x (x-t)^{\delta-1} f(t)dt\right\}^k dx \le B\int_0^\infty x^{k-k\mu-1} f^k(x)dx. \qquad (23)$$

If in this we take $f(t) = 0$ for $0 < t \le 1, f(t) = t^{-\delta-1}h(1-1/t)$ for $t > 1$, and put $\rho = 1 - 1/x,\ \sigma = 1 - 1/t,$ we deduce that if $1 \le k < \infty,\ \mu > 0, \delta > 0,$ and $h:]0,1[\to [0,\infty]$ is measurable, then

$$\int_0^1 (1-\rho)^{k\delta-1}\left\{\int_0^\rho (\rho-\sigma)^{\delta-1}h(\sigma)d\sigma\right\}^k d\rho \le B\int_0^1 (1-\rho)^{k\mu+k\delta-1}h^k(\rho)d\rho. \qquad (24)$$

Next, we note that the substitution of $\rho^{q\alpha+1}$ for ρ, together with the increasing property of M_p, gives

$$\mathfrak{N}^q_{p,q,\gamma-\alpha}(\not f^\alpha\phi) = \int_0^1 (1-\rho)^{q(\gamma-\alpha)} M_q^p(\not f^\alpha\phi;\rho)d\rho.$$

$$= (q\alpha+1)\int_0^1 (1-\rho^{q\alpha+1})^{q(\gamma-\alpha)-1} M_q^p(\not f^\alpha\phi;\rho^{q\alpha+1})d\rho$$

$$\le B\int_0^1 (1-\rho)^{q(\gamma-\alpha)-1} M_q^p(\not f^\alpha\phi;\rho)\rho^{q\alpha}d\rho. \qquad (25)$$

We now distinguish three cases.

Case 1.1 $\le p \le \infty,\ 1 \le q < \infty$. By Theorem (2.1.8)

$$M_p(\not f^\alpha\phi;\rho) \le B\int_0^1 (1-\tau)^{\alpha-1} M_p(\phi;\rho\tau)d\tau$$

$$= B\rho^{-\alpha}\int_0^\rho (\rho-\sigma)^{\alpha-1} M_p(\phi;\sigma)d\sigma, \qquad (26)$$

and (23) follows from this, (23), and (24) with $k = q, \mu = \gamma-\alpha, \delta = \alpha, h(\sigma) = M_p(\emptyset;\ \sigma)$.

Case II. $1 \le p \le \infty,\ 0 < q < 1$ or $0 < q < p \le 1$. By Theorem (2.1.9)

$$M_p^q(\not f^\alpha\phi;\rho) \le B\int_0^1 (1-\tau)^{q\alpha-1} M_p^q(\phi;\rho\tau)d\tau$$

$$= B\rho^{-q\alpha}\int_0^\rho (\rho-\sigma)^{q\alpha-1} M_p^q(\phi;\sigma)d\sigma,$$

and (23) follows from this, (25), and (24) with $k = 1, \mu = q(\gamma - \alpha), \delta = q\alpha, h(\sigma) = M_p^q(\emptyset; \sigma)$.

CaseIII.0 $< p < 1, p \leq q < \infty$. By Theorem (2.1.15),

$$M_p^q(\not{f}^{\alpha}\phi; \rho) \leq B \int_0^1 (1-\tau)^{p\alpha-1} M_p^p(\phi; \rho\tau) d\tau$$

$$= B\rho^{-p\alpha} \int_0^{\rho} (\rho - \sigma)^{p\alpha-1} M_p^p(\phi; \sigma) d\sigma, \quad (27)$$

and we now use (25) and (24) with $k = q/p, \mu = q(\gamma - \alpha), \delta = p\alpha, h(\sigma) = M_p^p(\emptyset; \sigma)$.

To complete the proof of case (a), it remains to prove the case q = 0 of (23) and the result of (iv). These follow simply from either (26) or (27) according as $1 \leq p \leq \infty$ or $0 < p < 1$, and we omit the details .([54]).

To prove the case (b) of (iii) and (iv), where$\alpha = -1$, we use a further lemma, and we combine the proof of this lemma with that of another, which we use in II.

Lemma (2.1.15)[52]:

Let $\emptyset \in H^p$, where $0 < p < \infty$. Then for $0 \leq p < 1$

$$M_p(\phi'; \rho) \leq B(1-\rho)^{-1}\mathfrak{M}_p(\phi) \quad (28)$$

and

$$M_p(\not{f}^{-1}\phi'; \rho) \leq B(1-\rho)^{-1}\mathfrak{M}_p(\phi) \quad (29)$$

Lemma (2.1.16)[52]:

Let f $f \in L^p$, where $1 \leq p \leq \infty$, let u be the Poisson integral of f on U, and let ϕ be the holomorphic function on U with real part u and whose imaginary part 'vanishes at the origin. Then for $0 \leq \rho < 1$

$$M_p(\phi'; \rho) \leq B(1-\rho)^{-1}\|f\|_p \quad (30)$$

and

$$M_p(\not{f}^{-1}\phi'; \rho) \leq B(1-\rho)^{-1}\|f\|_p. \quad (31)$$

Let $0 \leq \rho < 1$, and let C be the circle with centre $\rho e^{i\theta}$ and radius $\frac{1}{2}(1-\rho)$. Then, by the Cauchy integral formula,

$$\left|\phi'(\rho e^{i\theta})\right| \leq 2(1-\rho)^{-1} \sup_c |\phi|,$$

and on applying Lemma 1 with $w = |\emptyset|^p, \eta = \frac{1}{2}, C = \mathfrak{M}_p^p(\phi)$, we obtain (28). Further, by (3)

$$M_p^t(\not{f}^{-1}\phi';\rho) \le M_p^t(\phi';\rho) + M_p^t(\phi;\rho), \tag{32}$$

where $t = \min\{p,1\}$, and (29) follows from (32) and (28).

In the case of Lemma 2.1.10 we have similarly

$$\left|\phi'(\rho e^{i\theta})\right| \le 2(1-\rho)^{-1}\sup_{c}|u|,$$

(this is an easy consequence of the integral formula for $\emptyset$ in terms of f, translated to C). Applying Lemma 2.1.6 with $w = |u|^p$, $\eta = \frac{1}{2}$, $C = \|f\|_p$ we thus obtain (30). Further, by (30) and (17),

$$M_p(\phi;\rho) \le B(1-\rho)^{-1}\|f\|_p + |\phi(0)|,$$

and since $|\emptyset(0)| = |u(0)| \le \|f\|_p + |\emptyset(0)|$ follows from this and (9.10).

An alternative proof of Lemma 6, using the Hardy-Littlewood maximal theorem, is given in [65, Theorem 3].

To complete the proof of Theorem 6, we apply (9.7) to $z \rightsquigarrow \emptyset(\rho z)$; we obtain

$$M_p(\not{f}^{-1}\phi';\rho^2) \le B(1-\rho)^{-1}M_p(\phi;\rho),$$

and this trivially gives the required results.

The next theorem enables us to switch from $\not{f}'s$to ordinary derivatives.

Theorem (2.1.17) [52]:

Let $0 < p \le \infty$, $0 < q \le \infty$, $\gamma > 0$, let k he a positive integer, and let $\emptyset$ be holomorphic U. Then

$$B\mathfrak{N}_{p,q,\gamma}\left(z^k\phi^{(k)}\right) \le \mathfrak{N}_{p,q,\gamma}(\not{f}^{-k}\phi) \le B\left\{\mathfrak{N}_{p,q,\gamma}\left(z^k\phi^{(k)}\right) + \max_{0\le j\le k-1}|\phi^{(j)}(0)|\right\},$$

(ii) $M_p(\not{f}^{-k}\phi;\rho) = o((1-\rho)^{-\gamma})as\rho \to 1 -$ iff $M_p\left(\phi^{(k)};\rho\right) = o((1-\rho)^{-\gamma})$.

By (3), $\not{f}^{-k}\phi$ is a linear combination$\emptyset, z\emptyset', \dots, z^k\emptyset^{(k)}$, with coefficients depending only on k. Further, by (16) and (17),

$$M_p^t(\phi;\rho) \le BM_p^t(z\phi';\rho) + |\phi(0)|^t, \tag{33}$$

Where $t = \min\{\rho, 1\}$. Replacing $\emptyset$ here by$\emptyset^{(j-1)}$, and multiplying both sides of the resulting inequality by$\rho^{(j-1)t}$, we obtain also

$$M_p^t\left(z^{j-1}\phi^{(j-1)};\rho\right) \le BM_p^t\left(z^j\phi^{(j)};\rho\right) + \left|\phi^{(j-1)}(0)\right|^t \tag{34}$$

$(j = 1,2,\dots)$. Combining the inequalities (34) for $j = 1,\dots,k$ with (33), we thus obtain

$$M_p^t(\not{f}^{-k}\phi;\rho) \le B\left\{M_p^t\left(z^k\phi^{(k)}\right) + \max_{0\le j\le k-1}|\phi^{(j)}(0)|^t\right\},$$

and this implies both the right-hand inequality in (i) and the "if" in(ii)

In the opposite direction, we note that [again by (3)] $z^k\emptyset^k$ is a linear combination of $\phi, \not{f}^{-1}\phi, \dots, \not{f}^{-k}\phi$ with coefficients depending only on k.

Further, by Theorem 6, we have

$$\mathfrak{N}_{p,q,\gamma}\left(\not{f}^{-j}\phi\right) \le \mathfrak{N}_{p,q,\gamma+h-j}(\not{f}^{-k}\phi) \le \mathfrak{N}_{p,q,\gamma}(\not{f}^{-k}\phi).$$

for $j < k$, and this easily implies the left-hand inequality in (i). The "only if" in (ii) is proved similarly.

Theorem (2.1.18) [52]:

Let $0 < p \le \infty, 0 < q \le \infty, \gamma > 0,$ and let $\emptyset$ be a holomorphic function on U whose imaginary part v vanishes at the origin and whose real part is u. Then

(i) $B\mathfrak{N}_{p,q,\gamma}(\phi) \le \mathfrak{N}_{p,q,\gamma}(u) \le \mathfrak{N}_{p,q,\gamma}(\phi)$,

(ii) $M_p(\phi;\rho) = o((1-\rho)^{-\gamma})as\rho \to 1-$ iff $M_p(u;\rho) = o((1-\rho)^{-\gamma})$.

The right-hand inequality in (i) and the "only if" in (ii) are trivial. Of the remaining results, the case $1 < p \le \infty$ is essentially known, and can be proved in several ways, For example, Lemma (2.1.16) with $f(\theta) = u(\rho,\theta)$ gives

$$(1-\rho)M_p(\not{f}^{-1}\phi;\rho^2) \le BM_p(u;\rho),$$

and the required results (for $1 \le p \le \infty$) follow from this and Theorem (2.1.13) to deal with the case $0 < p < 1$, it is enough to prove

(i) $\mathfrak{N}_{p,q,\gamma+1}(\phi') \le B\mathfrak{N}_{p,q,\gamma}(u)$,

(ii) if$M_p(\phi;\rho) = o((1-\rho)^{-\gamma})$ as $\rho \to 1-$ iff $M_p(\phi';\rho) = o((1-\rho)^{-\gamma})$.

For, suppose that (iii) and (iv) hold. Then, by (iv), Theorem (2.1.17) (ii), and Theorem (2.1.13) (ii), the "if" of (ii) holds. To prove that (i) hold, we apply Theorem (2.1.13) (i) and Theorem (2.1.17) (i) to the function$\emptyset_0 = \emptyset - \emptyset(0)$. Since$\emptyset_0' = \emptyset$, we thus get

$$\mathfrak{N}_{p,q,\gamma}(\phi_0) \le B\mathfrak{N}_{p,q,\gamma+1}(\not{f}^{-1}\phi_0) \le B\mathfrak{N}_{p,q,\gamma+1}(\phi').$$

and this, together with (iii), gives

$$\mathfrak{N}_{p,q,\gamma}(\phi_0) \le B\mathfrak{N}_{p,q,\gamma}(u).$$

Since the imaginary part of $\emptyset_0$is v, we thus have $\mathfrak{N}_{p,q,\gamma}(v) \le B\mathfrak{N}_{p,q,\gamma}(u)$,

and (i) follows from this and (1).

Next, the case $0 < q = p < 1$ of (iii) is proved in [56], and the argument given there extends to the case $0 < q \leq p < 1$ with only minor modifications.

The case $0 < p < 1, q = \infty$ of both (i) and (iii) arc also known (Hardy and Littlewood [63];1 see also Gwilliam [59]), but this case is covered by the following argument, which deals with the case $0 < p < 1, p \leq q \leq \infty$ of (iii) and the case $0 < p < 1$ of (iv]. ,reuse the analogue for the disc of Theorem (2.1.17) of [3]; this asserts that if $0 < p < 1$, then for $\frac{1}{3} \leq \rho < 1$

$$M_p^p(\phi';\rho) \leq B(1-\rho)^{-p-1}\int_{\rho-(1-\rho)}^{\rho+(1-\rho)} M_p^p(u;\sigma)d\sigma. \quad (35)$$

This inequality (35) immediately implies the case $q = \infty$ of (iii) and the result of (iv). Further, if $p \leq q < \infty$, then (35) gives

$$M_p^p(\phi';\rho) \leq B(1-\rho)^{-p}\left\{(1-\rho)^{-1}\int_{\rho-\frac{1}{2}(1-\rho)}^{\rho+\frac{1}{2}(1-\rho)} M_p^q(u;\sigma)d\sigma.\right\}^{p/q}.$$

Raising both sides to the (q/p)-th power, multiplying by $(1-\rho)^{q\gamma+q-1}$, integrating over $[\frac{1}{3},1)$, and inverting the order of integration in the resulting integral, we obtain

$$\int_{1/3}^{1}(1-\rho)^{q\gamma+q-1}M_q^p(\phi';\rho)\,d\rho \leq B\mathfrak{N}_{p,q,\gamma}^q(u).$$

Since $M_p^q(\emptyset';\rho)$ increases with ρ, we have also

$$\int_0^{1/3}(1-\rho)^{q\gamma+q-1}M_p^q(\phi';\rho)\,d\rho \leq BM_p^q\left(\phi';\frac{1}{3}\right).$$

$$\leq B\int_{1/3}^{1}(1-\rho)^{q\gamma+q-1}M_p^q(\phi';\rho)\,d\rho$$

$$\leq B\mathfrak{N}_{p,q,\gamma}^q(u),$$

and this gives (iii).

It should be remarked that the proofs of the theorems in [55) and [56] to which we have appealed are formidably long, but their use appears to be indispensable. We note two corollaries of Theorem (2.1.18).

Corollary (2.1.19) [52]:

The result of Theorem (2.1.13) continues to hold if $\emptyset$ is replaced throughout by a (complex-valued) harmonic u, this is immediate.

Corollary (2.1.20) [52]:

Let $0 < p \leq \infty, 0 < q \leq \infty, \gamma > 0$, let k be a positive integer, let u be a (complex-valued) harmonic function of the form

$$u(\rho e^{i\theta}) = \sum_{n=-x}^{x} c_n \rho^{|n|} e^{in\theta}, \tag{36}$$

and let D denote $\partial/\partial p$. Then

(i) $B\mathfrak{N}_{p,q,\gamma}(\rho^k D^k u) \leq \mathfrak{N}_{p,q,\gamma}(\oint^{-k} u) \leq B\{\mathfrak{N}_{p,q,\gamma}(\rho^k D^k u) + \max_{|j| \leq k-1} |c_j|\}$,

(ii) $M_p(\oint^{-k} u; \rho) = o((1-\rho)^{-\gamma})as\rho \to 1-$ iff $M_p(\rho^k D^k u; \rho) = o((1-\rho)^{-\gamma})$.

We note first that if u is real-valued and $\emptyset$ is the holomorphic function with real part u and imaginary part vanishing at the origin, then $\rho^k D^k u$ is the real part of $z^k \emptyset^k$. Moreover, if u is given by (36), then in this case $c_{-n} = c_{-n}$for all n, and $\emptyset(z) = c_0 + 2\sum_{n=1}^{\infty} c_n z^n$. The result for a real-valued u therefore follows from Theorem (2.1.17).

To complete the proof, we have now only to observe that if u is complex-valued and satisfies (36), v and w are the real and imaginary parts of u, and

$$v(\rho e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n \rho^{|n|} e^{in\theta}, \qquad w(\rho e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n \rho^{|n|} e^{in\theta},$$

then$\max_{|j| \leq k-1}\{|a_j|, |b_j|\} \leq \max_{|j| \leq k-1} |c_j|$. This is easily verified, for in fact

$$a_n = \bar{a}_{-n} = \frac{1}{2}(c_n - \bar{c}_{-n}), \quad b_n = \bar{b}_{-n} = -\frac{1}{2}i(c_n - \bar{c}_{-n})(n = 0,1,\dots).$$

It remains to prove the negative results mentioned in 3 and in virtue of Theorem (2.1.18) it is enough to prove that the result of Theorem (2.1.9) is false for all choices of p, q, r satisfying $0 < p \leq r \leq \infty, 0 < q \leq \infty$, and not covered by one of the conditions (i}-(iii) of that theorem. To prove this, we have to prove that the result of Theorem (2.1.9) is false when

(a) $0 < p < r < \infty, \qquad q > r,$

(b) $0 < p \leq r = \infty, \qquad q > 1,$

(c) $0 < p = r \leq 2, \qquad q > r,$

(d) $2 < p = r < \infty, \qquad q > 2.$

Further, by Lemma (2.1.12), the falsity of the theorem in the case (c) is implied by that in the case (a), and the falsity in the case (b) is implied by that in the case (b') $0 < p < r = \infty, \qquad q > 1$

Thus we have to find counter examples for cases (a), (b)', (d). For (a) we take

$$\emptyset(z) = (1-z)^{-1/r}\{(1/z)\log[1/(1-z)]\}^{-\lambda},$$

where$1/q < \lambda < 1/r$, and for(b)' we take

$$\emptyset(z) = \{(1/z)\log[1/(1-z)]\}^{1-\mu},$$

where $l/q < \mu < 1$. The arguments here are of a standard type, and we refer the reader to [66, 93-96]. Finally, for (d) we take

$$\emptyset(z) = \sum_{n=1}^{\infty} n^{-1/2} z^{2^n - 1}.$$

Since ϕ is lacunary and obviously does not belong to H^2, it does not belong to H^r for any r. Moreover, a proof similar to that of [66] shows also that $\mathfrak{N}_{r,q,\gamma}(\not{f}^{-\gamma}\phi) < \infty$ when $2 < r < \infty, q > 2, \gamma > 0$.

Section (2.2):Embedding Derivatives of Hardy and Lebesgue Spaces:

Let U denote the upper half-space $\mathbb{R}^n \times (0,\infty)$in $\mathbb{R}^{n+1}$. The reader is invited (indeed urged) to let $n = 1$ on first reading. Let H^p denote the Hardy space on . Let μ be a positive measure on U and consider the problem of determining what conditions on μ imply $|\nabla\mu| \in L^q(\mu)$ whenever$\mu \in H^p$. More generally, if β is a multi-index of order m and D^β is the corresponding differential monomial, we have the problem of determining conditions on μ so that $D^\beta u \in L^q(\mu)$ whenever $\mu \in H^p$. A standard application of the closed graph theorem leads to the following equivalent problem. characterize the μ for which there exists a constant Csatisfying

$$\left(\int |D^\beta u|^q d\mu\right)^{1/q} \le C\|u\|_{H^p} \tag{37}$$

and. estimate the size of C in terms of u.

In the case where $m = 0$ (that is, no differentiation takes place in (37) the problem is solved by the well-known theorem of Carleson [71] when $p = q$, and by a theorem of Duren [74] when $q > p$. The case where $q < p$ seems to have been a folklore theorem, at least when $n = 1$. It can be found stated in [86].

When $m > 0$ there are two solved cases: $p = q \ge 2$ due to Shirokov [83, 84] and $0 < p < q < \infty$ due to the author [69]. These two references do not consider all the possibilities implicit in (37) ([84] considers only $m = 1$, while [69] and [83] consider only $n = 1$), but the methods there easily give the complete solution. for some indications of how this could be done.

The purpose of this section is to present a solution in the remaining two cases:$0 < p = q < 2$ and $0 < q < p < \infty$.

Let us now present some basic definitions that will enable us to state the solutions. A point $z \in U$ will be written $z = (x, y)$ with $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y > 0$. We use the absolute value symbol $|\cdot|$to denote the Euclidean norm in $\mathbb{R}^n$ and in $\mathbb{R}^{n+1}$ (so $|z|^2 = |x|^2 + y^2$). When $z = (x, y)$, let $z^* = (x, -y)$. The pseudo hyperbolic metric ρ is defined on U by $\rho(z, w) = |z - w|/|z - w^*|$, Clearly ρ is invariant under rigid motions in the x-variable and under dilations in ~n+l. Let $D_\varepsilon(w) = \{z \in U: \rho(z, w) < \varepsilon\}$ when $w \in U$ and $0 < \varepsilon < 1$. Let $\Gamma_\alpha(t) = \{(x, y) \in U: |x - t| < \alpha y\}$ where $\alpha > 0$. In discussions where the actual value of ε or α is irrelevant, they may be omitted from the subscripts.

If $0 < r < \infty$ and f is a measurable function on U, define

$$A_r(f)(t) = \left(\int_{\Gamma(t)} |f|^r y^{-n-1} dx dy \right)^{1/r}$$

and

$$A_\infty(f)(t) = \sup_{z \in \Gamma(t)} |f(z)|$$

If E is an open set in $\mathbb{R}^n$, let $\hat{E} = \{(x, y) \in U: B(x, y) \subseteq E\}$, where $B(x, y)$ denotes the Euclidean ball in $\mathbb{R}^n$ centered at x with radius y. We say that E is the 'tent' over E. Define, for $0 < r < \infty$ and f measurable on U,

$$C_r(f)(t) = \sup_{t \in \mathrm{B}} \left(\frac{1}{|B|} \int_{\hat{B}} |f|^r y^{-1} dx dy \right)^{1/r},$$

where $|B|$denotes the n-dimensional volume of B and the sup is over all balls containing t.Finally, if $\beta = (\beta_1, \beta_2, \ldots, \beta_{n+1})$ is a multi index of non-negative integers with order$|\beta| = |\beta_1| + |\beta_2| + \cdots + |\beta_{n+1}|$,then D^βdenotes the differential monomial $\partial^m / \partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial y^{\beta_{n+1}}$. We are now ready to state our main theorem.

Theorem (2.2.1) [69]:

For a positive measure μ on U and a multi-index $\beta$of order m, a necessary and sufficient condition for

$$\left(\int \left|D^{\beta}u\right|^{q} d\mu\right)^{1/q} \leq C\|u\|_{H^p}$$

to hold is that the function $g(z) = y^{-qm-n}\mu(D(z))$ satisfy

(i) $C_{2/(2-q)}(g) \in L^{\infty}(\mathbb{R}^n, dt)$ if $0 < p = q < 2$,

(ii) $A_{2/(2-q)}(g) \in L^{p/(p-q)}(\mathbb{R}^n, dt)$ if $0 < q < p$ and $q < 2$,

(iii) $A_{\infty}(g) \in L^{p/(p-q)}(\mathbb{R}^n, dt)$ if $2 \leq q < p$

We will present the necessary background on H^p and tent spaces. In additional results and inequalities regarding tent spaces are presented culminating in the duality results of the key ingredient in the proof of Theorem (2.2.1) (Lemma 3) is proved, and an interpolation theorem for derivatives of H^p-functions at the points of an 'η-lattice' follows almost as a by-product. contains the proof of the sufficiency of conditions (i)-(iii) of Theorem (2.2.1). This could be read immediately after. The necessity of the conditions is shown. An interesting ingredient of the proof (not in the original proof) is the use of Khinchine's inequalities. The main result and some related results are discussed.For now we will describe 'discrete' or dyadic versions of the conditions (i), (ii) and (iii) of Theorem (2.2 .1). The equivalence between these two versions will come out in the proof of the theorem

A dyadic interval is one of the form $(m2^{-k}, (m+1)2^{-k}]$, where m and k are integers. A dyadic cube Q in $\mathbb{R}^n$ is a cube of the following form: there is an integer k such that Q is a product of dyadic intervals of length 2^{-k}. The set of all dyadic cubes of sidelength 2^{-k} will be denoted Δ_k and the set of all dyadic cubes is denoted Δ . Then the cubes of Δ_k are disjoint. For each $Q \in \Delta$ let $R(Q) = Q \times (\frac{1}{2} l(Q), l(Q)] \subseteq \mathbb{R}^{n+1}_{+}$, where $l(Q)$ is the sidelength of Q. Then clearly $\{R(Q): Q \in \Delta\}$ is a disjoint cover of $U = \mathbb{R}^{n+1}_{+}$ .The discrete version of Theorem (2.2.1) is then obtained by replacing integrals over $\Gamma(t)$ with sums over those $R(Q)$ which meet the line $\{(t, y): y > 0\}$, with a similar adjustment for integrals over $\hat{B}$. The discrete versions of (i)-(iii) are then

(i') $\sup\left\{\frac{1}{|Q|}\sum_{\substack{Q'\subseteq Q\\ Q\in\Delta}}\left[\frac{\mu\left(R(Q')\right)}{l(Q')^{qm+n}}\right]^{2/(2-q)} l(Q')^{n} : Q \in \Delta\right\}$ is finite,

(ii') the function $t \to \left(\sum_{\substack{Q\subseteq Q\\ t\in\Delta}}\left[\frac{\mu\left(R(Q')\right)}{l(Q)^{qm+n}}\right]^{2/(2-q)}\right)^{1-q/2}$ belongs to $L^{p/(p-q)}$,

(iii') the function $t \to \sup_{\substack{Q \subseteq Q \\ t \in \Delta}} \frac{\mu(R(Q))}{l(Q)^{qm+n}}$ belongs to $L^{p/(p-q)}$.

We take as our definition of $H^p(U)$ the definition by Fefferman and Stein in [76]. We are, however, more interested in certain equivalent definitions. These are summarized in Theorem (2.2.2) below. In order to state that theorem, we introduce some notation. If $j = (j_1, j_2, \ldots, j_m)$ is an m-tuple of integers with $1 \le j_k \le n+1$, let $D_j u = \partial^m u / (\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_m})$ where x_{n+1} denotes y. The m-fold gradient $\nabla^m u$ denotes the tensor $(D_j u, J \in \{1,2,\ldots,n+1\}^m)$ with $|\nabla^m u|^2 = \sum_j |D_j u|^2$. Here $\nabla^0 u$ simply means u. Thus, if β_k is the number of occurrences of k in the m-tuple J,then

$$D_j u = D^\beta u \text{ and } |\nabla^m u|^2 = \sum_{|\beta|=m} \frac{m!}{\beta!} |D^\beta u|^2,$$

Where $\beta_1! \beta_2! \ldots \beta_{n+1}!$

For $m \ge 1$,u harmonic in U and $t \in \mathbb{R}^n$, define $S^m u(t) = A_2(y^m |\nabla^m u|)(t)$. To conform with the usage of [76] we let $Su = S_1 u$ and $u^*(t) = A_\infty(u)(t)$.

Theorem (2.2.2) [69]:

For u harmonic in U the following are equivalent:

(i) $u \in H^p$;

(ii) $\lim_{y\to\infty} u(x,y) = 0$and $Su \in L^p(\mathbb{R}^n, dt)$;

(iii) for some $m \ge 1$, $\lim_{y\to\infty} |\nabla^k u(x,y)| = 0$, for all $0 \le k < m$, and $S_m u \in L^p(\mathbb{R}^n, dt)$;

(iv) same as (iii) but for every $m \ge 1$;

(v) $u^* \in L^p(\mathbb{R}^n, dt)$.

As usual L^pdenotes the Lebesque class of functions with integrable p-th power and dt denotes n-dimensional Lebesque measure on $\mathbb{R}^n$

Most of the implications in Theorem (2.2.2) can be found in [76]. Certainly one can obtain the equivalence of (i), (ii) and (v).

To get (ii)⟹(iv), we make the following observations:

(1) $u_s(x,y) = u(x, y+s)$, converges uniformly on compact sets to zero as $s \to \infty$ and hence all $\nabla^k u(x,y) \to 0$ as $y \to \infty$;

(2) $|\nabla^m u(x,y)|^2 \le C y^{-2m} \int_{D(z)} |s \nabla u(t,s)|^2 s^{-n-1} dt ds$.

Multiplying by y^{2m-n-1} in (2), integrating over $\Gamma(t)$, and applying Fubini's Theorem on the right gives $S_m u \le \tilde{S} u$; where $\tilde{S} u$ is the same as Su but defined with respect to cones $\tilde{\Gamma}(t)$with strictly larger aperture than $\Gamma(t)$. We then invoke the well-known fact that $Su \in L^p$ is independent of aperture [72,76,85].

To get (iii) $\Longrightarrow$ (ii), the approach in [85] (showing that $g_k(f)(x) \leq C_k g_{k+1}(f)(x)$ for all $k \geq 1$)is easily modified to show that $S_k u(t) \leq C_k S_{k+1} u(t)$.

Implicit in Theorem A is the fact that the quantities $\|u\|_{H^p}$ (however defined), $\|Su\|_{L^p}$, $\|S_m u\|_{L^p}$ and $\|u^*\|_{L^p}$ are equivalent. In our proof of Theorem (2.2.1), then, we will normally make use of I$\|S_m u\|_{L^p}$ with $m = \beta$ asour working norm onH^p.

Following Coifman, Meyer, and Stein [72], we define the tent spaces $T_r^s$for $0 < r,s \leq +\infty$. We will need to consider a slightly more general context than they do. Thus if v is a positive measure on U, finite on compact sets, and if $r < \infty$,let

$$A_{r,v}(f)(t) = \left(\int_{\Gamma(t)} |f|^r \, dv\right)^{1/r}$$

and

$$C_{r,v}(f)(t) = \sup_{t \in B} \left(\frac{1}{|B|} \int_{\hat{B}} |f(z)|^r y^n \, dv\right).$$

If $r = \infty$, let

$$A_{\infty,v}(f)(t) = \mathrm{v} - \operatorname*{ess\,sup}_{z \in \Gamma(t)} |f(z)|.$$

In the case where $dv = y^{-n-1} dx dy$, we follow [72] and omit the subscript v, The tent space $T_r^s(v)$ is defined to be the space of v-equivalence classes of functions f such that

(i) $A_{r,v}(f) \in L^s(\mathbb{R}^n, dt)$if $r, s < \infty$,

(ii) $C_{r,v}(f) \in L^\infty(\mathbb{R}^n, dt)ifr < s = \infty$.

We would like to use the symbol $T_\infty^s(v)$ to mean the space of functions satisfying

(i) with $r = \infty$,butthis would conflict with[72].So instead, we define$\tilde{T}_\infty^s(v)$ to be the space of functions f with

(iii) $A_{\infty,v}(f) \in L^s$.

For consistency, we let $\tilde{T}_r^s = T_r^s$when $r, s < +\infty$. (In [72], T_∞^s is defined in away that makes it the closure in $\tilde{T}_\infty^s$ of the continuous functions with compact support.) One way to view $T_r^s(v)$ when $r, s < \infty$ is as a subset of the weighted mixed norm space $L^s L^r(w dv dt)$ of functions $\varphi(x, y, t)$ on $U \times \mathbb{R}^n$ with norm

$$\|\varphi\|_{r,s} = \left[\int \left(\int |\varphi|^r \, w dv(x,y)\right)^{s/r} dt\right]^{1/s},$$

where $w(x, y, t) = \chi_{\Gamma(t)}(x, y)$. Then $T_r^s(v)$ consists of those φ independent of t. Slight (and easy) modifications of results of Benedek and Panzone [70] can then be used to obtain some duality results. Though not exactly the same, this is essentially the approach we take this is also equivalent to the approach taken independently by Harboure, Torrea, and Vivani in [77]. Unfortunately, they have chosen a different normalization: our $T_r^s(v)$ in their $T_r^s(y^n v)$. With three defendable choices of normalization, it is not surprising that we have chosen two different ones. There is also an incompatibility of notation: our $\tilde{T}_r^s$; and theirs have two different meanings.

The appropriate duality results were proven (for $v = y^{-n-t}\,dx\,dy$) in [72]. We will need the corresponding results for $T_r^s(v)$, Namely, if $1 \leq r < \infty$, $1 \leq s < \infty$, then $T_r^s(v)^* = \tilde{T}_{r'}^{s'}(v)$, where r' and s' are the usual dual exponents, that is, $r' = r/(r-1)$. We will normally be concerned only with certain discrete measures v and in that case we also need to consider the case of $r < 1$. In our application of these results to Theorem (2.2.1) we will need to consider $r = 2/q$ and $s = p/q$ so that r, s range over all of $(0, +\infty)$ and $(1, +\infty)$, respectively.

Our notations and those of [72] and [77] do not incorporate the aperture α of $\Gamma(t)$. Clearly the various $A_{r,v}(f)$ depend on , but the spaces T_r^s do not. This can be proved using a limiting argument applied to the corresponding result in [72], but we will need a stronger result which will yield it immediately .

The major result we will need from [72] is the 'atomic decomposition' of the spaces T_r^1, with $r > 1$. Define a T_r^s-atom as a function $a(x, y)$on U, supported in B for some ball $\hat{B}$ in $\mathbb{R}^n$, and satisfying $\int_{\hat{B}} |a|^r y^{-1} dxdy \leq |B|^{1-r}$. In the case where $r = \infty$, a must satisfy sup $|(x, y)| < |B|^{-1}$.

Theorem (2.2.3) [69]:

Suppose $f \in \tilde{T}_r^1$, where $1 < r \leq \infty$. Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where the $a_j$are T_r^1-atoms, the $\lambda_j$are scalars, and $\sum |\lambda_j| \leq C_r \|f\|_{\tilde{T}_r^1}$. The symbol $\|f\|_{\tilde{T}_r^1(v)}$ means the obvious: the $L^p(dt)$-norm of the appropriate functional and the omission of v in the notation means $dv = y^{-n-1} dx\,dy$. Theorem B is incorporated in Proposition 2 ($r = \infty$) and Proposition 5 ($1 < r < \infty$) of [72]. The statement there of Proposition 2 mentions only T_∞^1 (which we have not defined here) but the proof goes through unchanged for our larger space $\tilde{T}_\infty^1$.

It should be mentioned that many of the proofs and the general approach of this section are very close in spirit to work of R. Rochberg and S. Semmes. (This was even truer of the original version of this section.) In particular, there are close connections with [82]. In fact, the space $T_2^1\{z_k\}$ defined the same as QCM in [82], and a proof of, could have been based on [82].

The case where $s > 1$ of the following proposition is equivalent, via duality, with Case (i) of Proposition 4.4 of [77].

Proposition (2.2.4) [69]:

If $s > 0$ and $\lambda > \max(1,1/s)$, then there is a constant $C = C(\lambda, s, \alpha, n)$ such that, for any positive locally finite measure v on U,

$$\int_{\mathbb{R}^n}\left(\int_U\left(\frac{y}{|x-t|+y}\right)^{\lambda n} dv(x,y)\right)^s dt \le C\int_{\mathbb{R}^n} v(\Gamma_\alpha(t))^s dt. \tag{38}$$

Proof:

First suppose that $s \ge 1$ and let $0 < \psi \in L^s(\mathbb{R}^n)$ with $\|\psi\|_{L^{s'}} \le 1$. Then, defining $w^\lambda(x,y) = [y/(|x|+y)]^{n\lambda}$. and $\psi_\lambda = y^{-n}\psi * w^\lambda(\cdot, y)$, we have

$$\int_{\mathbb{R}^n}\psi(t)\int_U w^\lambda(x-t,y)\,dv(x,y)dt = C\int_U \psi_\lambda(x,y)y^2 dv(x,y)$$

$$= C_\alpha \int_{\mathbb{R}^n}\int_{\Gamma_\sigma(\mathrm{t})}\psi_\lambda(x,y)dv(x,y)dt \quad = C_\alpha\int_{\mathbb{R}^n} A_{\infty,v}(\psi_\lambda)(t)v(\Gamma_\sigma(\mathrm{t}))dt$$

$$= C_\alpha\left\|A_{\infty,v}(\psi_\lambda)\right\|_{L^s}\left(\int_{\mathbb{R}^n} v(\Gamma_\sigma(\mathrm{t}))^s dt\right)^{1/s} \tag{39}$$

The proof of Stein [83] works as well for the kernel $w^\lambda(x,y)$, $\lambda > 1$, as for the Poisson kernel .Thus $A_{\infty,v}(\psi_\lambda) \le C_{\alpha,\lambda}M\psi$ where M is the Hardy-Littlewood maximal operator. Whence [83] $\left\|A_{\infty,v}(\psi_\lambda)\right\|_{L^{s'}} \le C_{\alpha,\lambda,s}$. Taking the supremum over ψ in (39) yields (38).

Now fix $\varepsilon \in (0,1)$ and let $D(z) = D_\varepsilon(z)$. The function w^λ satisfies

$$w^\lambda(x_1 - t, y_1)/w^\lambda(x_2 - t, y_2) \le C_{\lambda,\varepsilon}$$

for any points $z_1, z_2 D \in (z)$, and $C_{\lambda,\varepsilon}$ is independent of z and t. Thus

$$y^{-n-1}\int_{D(z)} w^\lambda dxdy \sim w^\lambda(z)$$

(where $\sim$ means that the two quantities have ratios bounded independently of z).

Substituting this average into (28) in place of w^{λ} we find that

$$\int_U w^{\lambda} dv \le C_{\lambda,\varepsilon} \int_U f w^{\lambda} y^{-n-1} dxdy$$

where $f(z) = v(D(z))$. A similar argument on the right-hand side of (38) gives

$$\int_{\Gamma_\beta(t)} f y^{-n-1} dxdy \le C_\varepsilon \int_{\Gamma_\alpha} dv$$

provided β is strictly less than α and ε is chosen so that $\cup\{D(z): z \in \Gamma_\beta(t)\} \subseteq \Gamma_\beta(t)$. Thus it suffices to show (38) for $dv = y^{-n-1}dxdy$. Note that ε, β can be preassigned to depend only on α.

Now suppose $s < 1$, let $g(x,y) = f(x,y)^s$ and put $r = 1/s$. We need to show that

$$\int_{\mathbb{R}^n} \left(\int_U w^{\lambda}(x-t,y)\, g(x,y)^r y^{-n-1} dxdy \right)^{1/r} \le C\|g\|_{T_r^1}. \qquad (40)$$

with the T_r^1-norm based on $\Gamma_\beta(t)$. Because of Theorem (2.2.2), it suffices to find an upper bound for the left-hand side of (40) when $g(x,y) = a(x,y)$, a T_r^1-atom. Without loss of generality, we may suppose that the atom $a(x,y)$ is supported in $\hat{B}$ with $B = B(0,1)$ and that $\int_{\hat{B}} a^r y^{-1} dxdy \le 1$. In this case we divide the outer integral in (40) into two parts: the integral I_∞ over $|t| > 2$ and the integral I_0 over $\|t\| \le 2$. Since $y^{-n} w^{\lambda}(x-t,y) \le C_\lambda t^{-\lambda n}$ when It$|t| > 2$ and $(x,y) \in \hat{B}$, we see that

$$I_\infty \le C \int_{|t|>2} t^{-\lambda n/r} \left(\int_{\hat{B}} a^r \frac{dxdy}{y} \right)^{1/r} dt \le C$$

By Holder's Inequality followed by Fubini's Theorem,

$$I_0 \le C \left[\int_{\hat{B}} \int_{2B} \left(\frac{y}{|x-t|+y} \right)^{\lambda n} dt\, y^{-n} a(x,y)^r \frac{dxdy}{y} \right]^{1/r} |2B|^{1-1/r}$$

Clearly

$$\int_{2B} \left(\frac{y}{|x-t|+y} \right)^{\lambda n} dt \le \int_{\mathbb{R}^n} \left(\frac{y}{|t|+y} \right)^{\lambda n} dt$$

$$= y^n \int_{\mathbb{R}^n} (1-|t|)^{-\lambda n}\, dt = C y^n$$

so

$$I_0 \leq C\left[\int_{\hat{B}} a(x,y)^r\, y^{-1} dxdy\right]^{1/r} \leq C$$

By putting $|f|^r dv$ in place of v and putting s/r in place of s in (38) we obtain

$$\int\left(\int |f(z)|^r y^{\lambda n}(y+|x-t|)^{-\lambda n}\, dv\right)^{s/r} dt = C\int\left(\int_{\Gamma_\beta(\mathrm{t})} |f|^r dv\right)^{s/r} dt$$

for any aperture β. From the fact that $y^{\lambda n}/(|x-t|]+y)^{\lambda n} \geq C_\alpha \chi_{\Gamma_\alpha(\mathrm{t})}(x,y)$, we clearly get the reverse inequality as well. This shows that for $0<r,\ s<\infty$, $T_s^r(v)$ is independent of aperture.

Proposition (2.2.5) [69]:

For $1 \leq r < \infty$ and $1 \leq s < \infty$ the dual of $T_s^r(v)$ is $\tilde{T}_{s'}^{r'}(v)$.The pairing is $\langle f,g\rangle = \int_U fgy^n dv$.

Proof:

Let $s>1$. From the fact that $\int_U fgy^n dv = C\int_{\mathbb{R}^n}\int_{\Gamma(\mathrm{t})} fg\, dvdt$ plus two applications of Holder's Inequality, we see that any $g \in T_s^r(v)$ defines a continuous linear functional on $T_s^r(v)$,

Conversely, let L be a continuous linear functional on $T_s^r(v)$. Let

$L^sL^r(dvdt) = \left\{f(z,t)\colon U\times\mathbb{R}^n \to \mathbb{C}\colon \left(\int\left(\int |f(z,t)|^r\, dv(z)\right)^{s/r} dt\right)^{1/s} \equiv \|f\|_{r,s} < +\infty\right\}.$

Clearly $T_s^r(v)$ embeds in $L^sL^r(dvdt)$ by the mapping $f(z) \to f(z)\chi_{\Gamma(\mathrm{t})}(t)(z)$. By a result of Benedek and Panzone [70], $(L^sL^r)^* = L^{s'}L^{r'}$. By the Hahn-Banach Theorem there is a function $g(z,t) \in L^sL^r$ such that

$$L(f) = \int_{\mathbb{R}^n}\int_{\Gamma(t)} g(z,t)f(z)dv(z)dt$$

with $\|L\| = \|g\|_{r',s'}$. By Fubini's Theorem

$$L(f) = \int_U f(z)\left[y^{-n}\int_{|\mathrm{t}-\mathrm{x}|<y} g(z,t)dt\right] y^n dv(z)$$

It now suffices to show that the expression $P^0 g(z)$ in brackets defines a bounded operator from $L^{s'}L^{r'}(dv\, dt)$to $T_{s'}^{r'}(v)$First suppose $r' = +\infty$ and let $h(t) = v - ess\sup_{t\in U} g(z,t)$ so that $h \in L^{s'}(dt)$. Clearly $P^0 g(z) \leq CMh(t)$ whenever $z \in \Gamma(t)$. Thus $A_{\infty,v}(P^0 g(z) \leq CMh$so $\|A_{\infty,v}(P^0 g(z)\|_{L^{s'}(dt)} \leq C\|h\|_{L^{s'}}$.

Now suppose $r' = s'$. Then $T_{s'}^{r'}$is just $\{f: |f|^r y^n\, dv < +\infty\}$ and

$$\int_U |P^0 g|^{r'} y^n dv = \int_U \left| y^{-n} \int_{|t-x|<y} g(z,t)dt \right|^{r'} y^n dv(z)$$

$$\leq C \int_U \int |g(z,t)|^{r'} dt dv(z)$$

$$= C\|g\|_{r',s'}.$$

We see that $\chi_{\Gamma(t)}(z)P^0 g(z)$ defines a bounded operator from $L^{s'}L^{r'}(dv\, dt)$ to itself when $r = \infty$ and when $r = s'$. Thus, by another result of [70], it is bounded on all $L^{s'}L^{r'}$ with $1 < s' \leq r' \leq \infty$. Since $\chi_{\Gamma(t)}P^0$ is easily seen to be self-adjoint, we also see that it is bounded on $L^{s'}L^{r'}$ with $1 \leq r' \leq s' \leq \infty$

Now take $s = 1$. See [72] goes through in the weighted case to show that $T_{r'}^{\infty}(v)$is contained in the dual ofT_r^1. We again observe that a bounded linear functional corresponds to an element g of $L^\infty L^{r'}(dvdt)$. Again we need to show that $P^0 g(z) = y^{-n}\int_{|t-x|<y} g(z,t)dt$ defines a bounded operator from $L^\infty L^{r'}$ to$T_{r'}^{\infty}$. To this end let B be a ball in $\mathbb{R}^n$ and consider

$$\frac{1}{|B|}\int_{\hat{B}} |P^0 g(z)|^{r'} y^n dv(z) \leq \frac{1}{|B|}\int_{\hat{B}} \int_{|t-x|<y} |g(z,t)|^{r'} dt dv(z)$$

$$= \frac{1}{|B|}\int_{\mathbb{R}^n} \int_{\hat{B}\cap\Gamma(t)} |g(z,t)|^{r'} dv(z)dt$$

$$\leq \frac{1}{|B|}\int_B \int_U |g(z,t)|^{r'} dv(z)dt$$

$$\leq \frac{1}{|B|}\int_B \|g\|_{r',\infty}\, dt$$

Thus $\|C_{r',v}(P^0\text{ g})\| \leq \|g\|_{r',\infty}$

It is not true that the operator P^0 maps $L^\infty L^{r'}$into $\{f: A_{r',v}(f) \in L^\infty\}$. Thus the 'natural' definition of T_r^∞ is, at least for duality purposes, not the appropriate one.

We will need to identify the dual of $T_r^s(v)$ for a special class of v when $0 < r < 1$. If $\{z_k\}$ is a sequence in U, we say that it is separated if there is an $\varepsilon \in (0,1)$ such that the balls $D_\varepsilon(z_k)$ are disjoint. The separation constant will be the largest such ε. When $v = \sum \delta_{z_k}$ (where δ_z denotes a unit mass at z), we will write $T_r^s\{z_k\}$ instead of $T_r^s(v)$. Thus, for$1 < s < \infty$ and $1 \le r < \infty$, we can identify the dual to $T_r^s\{z_k\}$ with a space of sequences (for example, $T_r^s\{z_k\}^* = \left\{(c_k): \left(\int \sum_{z_k \in \Gamma(t)} |c_k|^2\right)^{s'/2} dt < +\infty\right\}$. In the case where $r \le 1$ we have the following.

Proposition (2.2.6) [69]:

If L is a continuous linear functional on $T_r^s\{z_k\}$, where $\{z_k = (x_k, y_k)\}$ is a separated sequence in U and $0 < r < 1 < s < \infty$, then $Lf = \sum f(z_k) b_k\, y_k^n$for a unique sequence $\{b_k\}$ satisfying

$$\frac{1}{C}\|L\| \le \left(\int_{\mathbb{R}^n} \left(\sup_{z_k \in \Gamma(t)} |b_k|\right)^{s'}\right)^{1/s'} \le C\|L\|$$

Conversely, any sequence in $\tilde{T}_\infty^{s'}\{z_k\}$ defines a continuous linear functional on $T_r^s\{z_k\}$·

Proof.

Let $b_k$be a sequence in $\tilde{T}_\infty^{s'}\{z_k\}$. (More precisely, $b_k = g(z_k)$ with g in that space.) Then

$$\left|\sum f(z_k) b_k\, y_k^n\right| = C\left|\int_{\mathbb{R}^n} \sum_{z_k \in \Gamma(t)} f(z_k) b_k dt\right|$$

$$\le C \int_{\mathbb{R}^n} \sup_{\Gamma(t)} |b_k| \left(\sum_{\Gamma(t)} |f(z_k)|^r\right)^{1/r}$$

$$\le C\|f\|_{T_r^s\{z_k\}} \|b_k\|_{\tilde{T}_\infty^{s'}\{z_k\}}$$

The first inequality holds because $r < 1$ so $\sum |f(z_k)| \le (\sum |f(z_k)|^r)^{1/r}$. The second comes from Holder's Inequality.

Now let L^sbe a linear functional on $T_r^s\{z_k\}$. We clearly get a candidate for the sequence

$\{b_k\}$ by putting $b_k = L(e_k)y_k^{-n}$, where $e_k$is the function with $e_k(z_j) = \delta_{jk}$. Then, for f a linear combination of the e_k we get

$$L_f = \sum f(z_k)b_k\, y_k^n = C \int_{\mathbb{R}^n} \sum_{z_k \in \Gamma(t)} f(z_k)b_k dt.$$

We need to show that $\{b_k\} \in \tilde{T}_\infty^{s'}\{z_k\}$. It suffices to obtain norm estimates when $b_k$is zero except for a finite number of indices and, by dilation and translation, it suffices to suppose b, is non-zero only if $z_k$belongs to the unit cube $Q_1^0 = (0,1]^{n+1}$in $\mathbb{R}^n$. We consider now all subcubes Q_j^1 of Q_1^0 which have as their base one of the 2^n subcubes of $(0,1]^n$ obtained by the usual bisecting of (0,1]. Similarly, Q_k^2 are the 2^{2n} subcubes obtained in the same way from all the Q_j^1. In general, Q_j^k is a cube with a dyadic cube of sidelength 2^{-k} at its base. Let $R_j^k = Q_j^k - \bigcup_m Q_m^{k+1}$ be the top half of Q_j^k. Now each R_j^k contains at most M points of $\{z_k\}$ where M is the maximum number of disjoint $D_\varepsilon(z)$ that will fit in R_1^0. We need only multiply our estimates by M if we make them for the special case where each R_j^k contains one point of $\{z_k\}$.

Now let $f \in T_r^s\{z_k\}$ and let $c_k = |f(z_k)|$. Then, by Proposition (2.2.4), the norm of f is equivalent to

$$\left(\int \left[\sum c_k^r \left(\frac{y_k}{y_k + |x_k - t|}\right)^{\lambda n}\right]^{s/r}\right)^{1/s}$$

It is clear that this is equivalent to the same expression at points $\{z_k'\}$ provided $\rho(z_k, z_k')$ is bounded away from 1. Thus, without loss of generality, we may suppose $\{z_k\}$ consists of the centres of the R_j^k We index them that way: z_{jk} is the centre of R_j^kSimilarly, index $\{c_{jk}\}$ and $\{b_{jk}\}$. Without loss of generality, we may assume that $b_{jk} \geq 0$.

Now,

$$\sum c_{jk}\, b_{jk} y_{jk}^n = C \int_{\mathbb{R}^n} \sum c_{jk}\, b_{jk} \chi_{\boldsymbol{Q}_j^k}(t)\, dt$$

Where $\boldsymbol{Q}_j^k$ is the base of Q_j^kin $\mathbb{R}^n$. Put $\varphi_k(t) = \sum_j c_{jk}\chi_{\boldsymbol{Q}_j^k}(t)$and $\psi_k = \sum_j b_{jk}\chi_{\boldsymbol{Q}_j^k}(t)$.

Then the condition on c_{jk} (that is, $f \in T_r^s\{z_k\}$) becomes $\int_{\mathbb{R}^n}(\sum_k \varphi_k^r)^{s/r}\, dt < +\infty$ and the condition on b_{jk} (the boundedness of L) becomes

$$\int \sum \varphi_k \psi_k dt \leq C \left(\int \left(\sum_k \varphi_k^r \right)^{s/r} dt \right)^{1/s} \quad for\ all \quad (\varphi_k)$$

The goal is to prove that $\sup_k \psi_k \in L^{s'}$. (We take the aperture α to be such that $z_{jk} \in \Gamma(t)$only if $t \in \boldsymbol{Q}_j^k$.)Proving this will require us to select φ_k an appropriate way. Notice that φ_k is an arbitrary function measurable with respect to the algebra $\mathcal{F}_k$ generated by $\{ \boldsymbol{Q}_j^k : j = 1,2,\ldots,2^{kn}\}$. Clearly $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$.

Lemma (2.2.7) [69]:

If φ_k is any finite sequence of non-negative $\mathcal{F}_k$-measurable functions, define a 'stopping time' as follows: $k_1(t)$ is the first k such that$\varphi_k > 0$. Once $k_1, k_2, \ldots, k_j$ are chosen let $k_{j+1}(t)$ be the first k such that$\varphi_k(t) > 2\varphi_{k_j}(t)$. Define $\tilde{\varphi}_k(t) = \varphi_k(t)$ if k is one of the k_j, otherwise let $\tilde{\varphi}_k(t) = 0$. Then $\tilde{\varphi}_k$ is $\mathcal{F}_k$-measurable and there are constants C, such that, for all t,

$$\sup_k \tilde{\varphi}_k(t) \leq \left(\sum_k \tilde{\varphi}_k(t)^r \right)^{1/r} \leq C_r \sup_k \tilde{\varphi}_k(t) \tag{41}$$

and

$$\sup_k \tilde{\varphi}_k(t) \leq \sup_k \varphi_k(t) \leq 2 \sup_k \tilde{\varphi}_k(t) \tag{42}$$

Proof.

Since for each k the choice as to whether $\tilde{\varphi}_k = 0$ or $\varphi_k$is made on the basis of $\varphi_1, \varphi_2, \ldots, \varphi_j$ which are all constant on $\boldsymbol{Q}_j^k$, then $\tilde{\varphi}_k$is also constant on $\boldsymbol{Q}_j^k$, Therefore $\tilde{\varphi}_k$is $\mathcal{F}_k$-measurable. For fixed t, it is clear that the sequence $k_j(t)$ terminates and the last element, $k_m(t)$ say, satisfies

$$\max \tilde{\varphi}_k(t) = \varphi_{k_m}(t) \geq 2 \sup_k \tilde{\varphi}_k(t)$$

for otherwise an additional $k_{m+1}(t)$ would have been chosen. This gives (42). Clearly $\varphi_{k_m}(t) > 2^{m-j} \varphi_{k_j}(t)$ by the choice of k_m. Thus

$$\left(\sum_k \tilde{\varphi}_k(t)^r \right)^{1/r} \leq \left(2^{-r(m-1)} + 2^{-r(m-2)} + \cdots + 2^{-r} + 1 \right) \varphi_{k_m}(t) \quad \leq \frac{2^r}{2^r - 1} \max_k \tilde{\varphi}_k(t)$$

This gives (41) with $C_r = 2^r/(2r-1)$.

To conclude the proof of Proposition 2.2.6 let $\varphi_k = \psi_k^{s'-1}$ and consider

$$\int \sum \tilde{\varphi}_k \, \psi_k dt \le C\left(\int \left(\sum \tilde{\varphi}_k^r\right)^{\frac{s}{r}} dt\right)^{\frac{1}{s}} \tag{43}$$

From Lemma 2.2.7 we have, for each t,

$$\sum \tilde{\varphi}_k(t)\, \psi_k(t) = \sum \tilde{\varphi}_k(t)^s \ge \left(\max_k \tilde{\varphi}_k\right)^s > 2^{-s}\left(\max_k \varphi_k\right)^s = 2^{-s} \max_k \psi_k^{s'} \tag{44}$$

and

$$\left(\sum \tilde{\varphi}_k^r\right)^{\frac{s}{r}} \le \left(C_r \max_k \tilde{\varphi}_k\right)^s \le \left(C_r \max_k \varphi_k\right)^s = C_r \max \psi_k^{s'} \tag{45}$$

Using (44) and (45) in (43) we get

$$\int 2^{-s}\left(\max_k \psi_k\right)^s dt \le C\left(\int C_r^s (\max \psi)^{s'}\, dt\right)^{1/s}$$

whence $\left(\int (\max_k \psi_k)^{s'}\, dt\right)^{1/s'} \le 2^s C C_r$. This completes the proof of Proposition (2.2.6).

It is not hard to verify that the operator R_β, defined by $(R_\beta u)_k = y_k^m D^\beta u(z_k)$, maps H^p into $T_2^p\{z_k\}$ when $\{z_k\}$ is a separated sequence ($m = |\beta|$). If $p > 1$, then the adjoint of R_β maps $T_2^p\{z_k\}$ into $H^{p'}$by duality. This gives us a bounded map S_β from $T_2^p\{z_k\}$ into H^p, for all $p > 1$, that will have useful applications. Unfortunately, this argument breaks down for $p \le 1$, so we have to construct the map we need more concretely.

First we need some facts about derivatives of the Poisson kernel. The (un normalized) Poisson kernel is given by $P(x - t, y) = y(|x - t|^2 + y^2)^{-(n+1)/2}$. If (x, y) is fixed at $(x_0, y_0) = z_0$, we write $P_{z_0}(t)$, so $P_{z_0}$is a function on $\mathbb{R}^n$. The harmonic extension of this function to $U$is $P(x_0 - x, y + y_0)$, with $(x, y)U$. Call this function $P_{z_0}(z)$. Note that $P_{z_0}(z) = P(z - z_0^*) = P(z_0 - z^*)$. Clearly $D^\beta(P_{z_0}) = (D^\beta P)(x_0 - x, y + y_0) = (D^\beta P)(z_0 - z^*)$.

Lemma (2.2.8) [69]:

(a) $D^\beta P(z) = P_\beta(z)(|x|^2 + y^2)^{-(n+1+2m)/2}$ where $P_\beta$is a polynomial of degree $m + 1$.

(b) $D^\beta P$ is homogeneous of order $-n - m$.

(c) If $\beta_1, \beta_2, \dots, n$ are even, then $D^\beta P(0,1) \ne 0$.

Proof:

Because differentiation lowers the order by 1 each time, (b) is clear. It is also clear by induction that $P_\beta$is a polynomial of degree at most $m+1$. If the degree were strictly less than $m+1$ then $D^\beta P$ would have order strictly less than $-n-m$. But $D^\beta P$ can have two different orders of homogeneity only if $D^\beta P=0$. Now

$$P(x,y)=y^{-n}(1+|x/y|^2)^{-(n+1)/2}$$

$$=\sum_{m=0}^{\infty}\binom{\frac{1}{2}(n+1)}{m}\sum_{|\alpha|=m}\frac{m!}{a!}x^{2\alpha}y^{-2m}$$

where $x^{2\alpha}=x_1^{2\alpha_1}x_2^{2\alpha_2}\dots x_n^{2\alpha_n}$. Clearly$D^\beta P\neq 0$. Moreover, it is easily seen that if $\beta=(2\alpha_1,2\alpha_2,\dots,2\alpha_n,\beta_{n+1})$ then $D^\beta P(0,1)$ is a non-zero multiple of the coefficient of $x^{2\alpha}$ and so is not zero.

To get an idea of where S_β will come from, we compute the 'adjoint' of R_β (in the case where $p>1$). Let $\{b_k\}$ belong to $T_2^{p'}\{z_k\}$, the dual of $T_2^p\{z_k\}$. Then with $m=|\beta|$,

$$\sum_k (R_\beta u)_k\, b_k y_k^n = c_n\sum_k b_k y_k^{n+m}\int_{\mathbb{R}^n} u(t)D^\beta P_{z_k}(t)dt$$

$$=c_n\int_{\mathbb{R}^n} u(t)\sum_k b_k y_k^{n+m}D^\beta P_{z_k}(t)dt$$

where$u(t)$ denotes the boundary values of u and c_n is the normalization constant in the Poisson integral formula. Since the dual of H^p is $H^{p'}$under the pairing$\langle u,v\rangle=\int_{\mathbb{R}^n} uv\,dt$,we identify $R_\beta^*((b_k))=\sum b_k y_k^{n+m}D^\beta P_{z_k}(t)$ (operating from $T_2^{p'}$ to $H^{p'}$). This therefore operates from T_2^p to H^p for all $p>1$, but not for $p\leq 1$. To remedy the situation when $p\leq 1$, define the operator S_β^λ for multi-indices β satisfying $|\beta|=m\geq 1$ and for integers $\lambda\geq 0$ by

$$S_\beta^\lambda(b_k)=\sum_k b_k y_k^{n+m+\lambda}\,\partial_y^\lambda D^\beta P_{z_k}\text{, where } \partial_y=\partial/\partial_y \tag{46}$$

Then we have

Lemma (2.2.9) [69]:

Let β be any multi-index with $|\beta|=m\geq 1$, let λ be a non-negative integer. Let $p>0$ and if $p<2$, suppose $(n+m+\lambda)p>2n$. Then S_β^λ is a bounded map from $T_2^p\{z_k\}$ into H^p whenever the sequence $\{z_k\}$ is separated.

Proof :

put$g(z)=S_{\beta}^{\lambda}(b_k)(z)=\sum_k b_k y_k^{n+m+\lambda}\partial_y^{\lambda}D^{\beta}P_{z_k}(z)$ and estimate

$|\mathrm{y}^m\nabla^m g(z)|^2$

$$\leq \sum_k |b_k|^2 y_k^{n+m+\lambda}\,\mathrm{y}^m\left|\partial_y^{\lambda}D^{\beta}\nabla^m P_{z_k}(z)\right| \sum_k y_k^{n+m+\lambda}\,\mathrm{y}^m\left|\partial_y^{\lambda}D^{\beta}\nabla^m P_{z_k}(z)\right| \quad (47)$$

Each term in the second sum can be estimated using sub harmonicity (and $P_{z_k}(z)=P_z(z'_k)$):

$$y_k^{n+m+\lambda}\mathrm{y}^m\left|\partial_y^{\lambda}D^{\beta}\nabla^m P_{z_k}(z)\right| \leq C\mathrm{y}^m \int_{D_{\varepsilon}(z_k)} \left|\partial_y^{\lambda}D^{\beta}\nabla^m P_z(w)\right| v^{m+\lambda-1}dudv \quad (48)$$

Fix an $\varepsilon\in(0,1)$ such that $D_{\varepsilon}(z_k)$ are disjoint and observe that C will depend on ε,n,m and λ only. Sum the inequality in (48) to obtain the following upper bound for the second sum in (47):

$$C\mathrm{y}^m\int_U \left|\partial_y^{\lambda}D^{\beta}\nabla^m P_z(w)\right| v^{m+\lambda-1}dudv \leq C\mathrm{y}^m\int_U \left|\partial_y^{\lambda}D^{\beta}\nabla^m P_{(0,y)}(w)\right| v^{m+\lambda-1}dudv$$

after translation by x. Now change variables via a dilation $w\to yw$ and use the homogeneity of derivatives of P to get the upper bound:

$$C\int v^{m+\lambda-1}((v+1)^2+|u|^2)^{-m-\frac{1}{2}(\lambda+n)}\,dudv$$
$$=C_{\varepsilon,n,m,\lambda} \quad (49)$$

Putting (49) into (47) for the second sum yields

$$|\mathrm{y}^m\nabla^m g(z)|^2 \leq C\sum_k |b_k|^2 y_k^{n+m+\lambda}\,\mathrm{y}^m\left|\partial_y^{\lambda}D^{\beta}\nabla^m P_{z_k}(z)\right| \quad (50)$$

To get the H^p-norm of g we integrate this over ret) with respect to $y^{-n-1}dxdy$ to obtain

$$\mathrm{A}_2(\mathrm{y}^m\nabla^m g)z(t)^2 \leq C\sum |b_k|^2 y_k^{n+m+\lambda}\int_{\Gamma(t)}\left|\partial y^{\lambda}D^{\beta}\nabla^m P_{z_k}(z)\right|\mathrm{y}^{m-n-1}\,dxdy \quad (51)$$

Taking $t=0$ and using homogeneity again we get

$$\int_{\Gamma(0)}\left|\partial_y^{\lambda}D^{\beta}\nabla^m P_{z_k}(z)\right|\mathrm{y}^{m-n-1}\,dxdy$$

$$\leq C\int_{\Gamma(0)}(|x_k-x|^2+(y_k+y)^2)^{-m-\frac{1}{2}(n+\lambda)}y^{m-n-1}\,dxdy$$

$$\leq C\int_{\Gamma(0)}(|x_k|^2+(y_k+y)^2)^{-m-\frac{1}{2}(n+\lambda)}y^{m-n-1}\,dxdy$$

$$=C\int_0^\infty(|x_k|+(y_k+y)^2)^{-m-\frac{1}{2}(n+\lambda)}y^{m-1}\,dxdy$$

$$\leq C\int_0^\infty(|x_k|+y_k+y)^{-2m-n-\lambda}y^{m-1}\,dxdy$$

$$=C(|x_k|+y_k)^{-n-m-\lambda}\int_0^\infty \xi^{m-1}(1+\xi)^{-2m-n-\lambda}\,dxdy$$

$$=C(|x_k|+y_k)^{-n-m-\lambda}$$

Translating by t and putting this in (51) gives

$$A_2(y^m\nabla^m g)z(t)^2\leq C\sum|b_k|^2\left(\frac{y_k}{|x_k-t|+y_k}\right)^{n+m+\lambda}$$

Raising this to the $\frac{1}{2}p$ power, integrating in t and applying Proposition (2.2.4)yields

$$\|g\|_{H^p}^p\leq C\|(b_k)\|_{T_2^p\{z_k\}}^p \tag{52}$$

Where C depends only on ε,n,m,λ and p. The condition required to apply Proposition (2.2.4) is that$(n+m+\lambda)/n>\max(1,2/p)$. If$p\geq 2$, this is automatically satisfied for $m+\lambda>0$.If $p<2$, we require $(n+m+\lambda)p>2n$, which is our hypothesis.

Note that the duality argument preceding the lemma allows us to take $\lambda=0$ even when $1<p<2$, while the lemma requires $m+\lambda>(2/p-1)n$.

The above lemma, combined with the results of the previous sections, will be enough to prove the main theorem. The reader interested only in the proof of that result may skip to the next section.

What follows is an interpolation theorem for the evaluation of derivatives of H^p-functions at points of an η-lattice. The first version of this section actually required this theorem in the proof of Theorem (2.2.1).

Theorem (2.2.10) [69]:

Let β be a multi-index with $|\beta| = m \geq 1$, and $\{z_k = (x_k, y_k)\}$ a separated sequence with separation constant $\gamma = \sup\{\varepsilon > 0: D_\varepsilon(z_k) \text{are disjoint}\} > 0$. Define R_β on H^p to sequences by $(R_\beta u)_k = y_k^m D^\beta u(z_k)$. Then R_β is a bounded map from H^p into $T_2^p\{z_k\}$. If γ is sufficiently close to 1, then R_β takes H^p onto $T_2^p\{z_k\}$.

Proof:

Since $|D^\beta u|^2$ is subharmonic, it follows that

$$\left|y_k^m D^\beta u(z_k)\right|^2 \leq C \int_{D_y(z_k)} \left|y^m D^\beta u\right|^2 \frac{dxdy}{y^{n+1}}$$

Summing on $z_k \in \Gamma_\alpha(t)$, we get

$$\sum_{\Gamma_\alpha(t)} \left|(R_\beta u)_k\right|^2 \leq C \int_{\tilde{\Gamma}(t)} \left|y^m D^\beta u\right|^2 \frac{dxdy}{y^{n+1}} \leq C S_m u(t)$$

Where $\tilde{\Gamma}(t) = \cup\{D_\gamma : z \in \Gamma_\alpha(t)\}$ is a cone with aperture larger than α (but depending only on γ and α). Thus R_β is bounded. To show that it is surjective, we show that for an appropriate constant A the operator norm of $I - AR_\beta S_\beta^\lambda$ tends to 0 as $\gamma \to 1-$. Thus, for large enough $\gamma, R_\beta S_\beta^\lambda$ will be invertible, whence R_β will be surjective.

To this end, fix a sequence (b_k) in $T_2^k\{z_k\}$ with norm 1 and let

$$c_j = (I - AR_\beta S_\beta^\lambda(b_k))_j = b_j - Ay_j^m D^\beta \sum_k b_k y_k^{n+m+\lambda}\, y^m \partial_y^\lambda D^\beta D_{z_k}(z_j)$$

Note that

$$\partial_y^\lambda D^{2\beta} P_{z_k}(z_j) = \partial_y^\lambda D^{2\beta} P(0,2y_k) = (2y_k)^{-2m-n-\lambda} \partial_y^\lambda D^{2\beta} P(0,1)$$

So that if $A = 2^{2m+n+\lambda} / \partial_y^\lambda D^{2\beta} P(0,1)$, then $Ay_j^m y_k^{n+m+\lambda} \partial_y^\lambda D^{2\beta} P_{z_k}(z_j) = 1$ when $k = j$. Taking this value of A, we get

$$c_j = -Ay_j^m D^\beta \sum_{k \neq j} b_k y_k^{n+m+\lambda}\, y_j^m \partial_y^\lambda D^\beta P_{z_k}(z_j)$$

Notice that this is almost $-Ay_j^m D^\beta g(z_j)$ with g as in Lemma (2.2.9). The only difference is that the sum is over $k \neq j$ instead of all k. Thus the same argument leading to (50) (where the constant is a multiple of (49)), leads to

$$\left|c_j\right|^2 \leq C_\gamma \sum_k |b_k|^2 y_k^{n+m+\lambda} \mathrm{y}_j^m \left|\partial_y^\lambda D^\beta P_{z_k}(z_j)\right|$$

where

$$C_\gamma = \int_{U_\gamma} \frac{v^{m+\lambda-1}}{((v+1)^2+|u|^2)^{m+\frac{1}{2}(\lambda+n)}} dudv$$

with$U_\gamma = \bigcup\{D_\varepsilon(z): z \notin D_\gamma(i)\}$ .The argument is based on $D_\varepsilon(z)$ being disjoint. This can be arranged by selecting any $\varepsilon \in (0,1)$, say $\varepsilon = \frac{1}{2}$ and then requiring $\gamma > \varepsilon$. It is not hard to verify that $U_\gamma \to \emptyset$as $\gamma \to 1-$, so that $C_\gamma \to 0$ as $\gamma \to 1-$. Thus

$$\left|c_j\right|^2 < C_\gamma \sum_k |b_k|^2 y_k^{n+m+\lambda} \mathrm{y}_j^m \left|\partial_y^\lambda D^\beta P_{z_k}(z_j)\right|$$

Now we sum over $z_j \in \Gamma(t)$ obtaining the discrete version of (51). Using the same argument as at the beginning of this proof, we can bound $\sum_{z_j \in \Gamma(t)} |c_j|^2$ by the right-hand side of (51) with an additional factor of C_γ Following the same argument that leads from (51) to (52), we get

$$\left\|(c_j)\right\|_{T_2^p\{z_k\}} \leq C C_\gamma^{p/2} \|(b_k)\|_{T_2^p\{z_k\}}^p$$

so that $\left\|I - AR_\beta S_\beta^\lambda\right\| \leq C C_\gamma$, and this tends to zero as $\gamma \to 1-$ (because C does not depend on γ).

Let $u \in H^p$. Then $D^\beta u$ is harmonic in U for any multi-index β and so, by Lemma (2.2.8) of [76], we have

$$\left|D^\beta u(z_0)\right|^q \leq C \frac{1}{D(z_0)} \int_{D(z_0)} \left|D^\beta u(x,y)\right|^q dxdy$$

where$C$ depends on the radius ε of $D_\varepsilon(z_0)$ and on q. From this we get

$$\int \left|D^\beta u(z_0)\right|^q d\mu \leq C \int_U \frac{1}{D(z')} \int_{D(z')} \left|D^\beta u(x,y)\right|^q dxdyd\mu(z')$$

$$\leq C \int_U \left|D^\beta u(x,y)\right|^q \mu(D(Z)) y^{-n-1} dxdy$$

$$= C \int_U \left|y^m D^\beta u(x,y)\right|^q \mu(D(Z)) y^{-mq-n} \frac{dxdy}{y}$$

where $m = |\beta|$.Since $l|y^m D^\beta u| \leq |y^m \nabla^m u|^2$, it follows that $|y^m \nabla^m u|^q$ lies in $T^{p/q}_{2/q}$, with norm at most $C\|u\|^q_{H^p}$. Thus, in order to get $\int |D^\beta u|^q \leq C\|u\|^q_{H^p}$, it suffices to have $\mu(D(z))y^{-mq-n}$ in $\left(T^{p/q}_{2/q}\right)^* = \tilde{T}^{p/(p-q)}_{2/(2-q)}$ when $q \leq 2$. This is exactly parts (i) and (ii) of Theorem (2.2.1) and the $q = 2$ case of (iii).

Suppose that now $q > 2$. Observe that $\sup_{\Gamma(t)} y^m |D^\beta u| \leq C \sup_{\tilde{\Gamma}(t)} |u| = Cu^*(t)$ provided $\tilde{\Gamma}(t)$ has larger aperture than $\Gamma(t)$. This is because

$$y^m \left|D^\beta u(z)\right| \leq C \frac{1}{D(z)} \int_{D(z)} |u| \, dxdy \quad , \quad \text{with } u \textit{ harmonic},$$

which is easily proved on a single ball, say $D(i)$, and obtained for all $D(z)$ by translation in x and dilation. Now, with $g(z) = \mu(D(z))y^{-mq-n}$,

$$\int_U \left|D^\beta u\right|^q d\mu \leq C \int_U \left|y^m D^\beta u(z)\right|^q g(z) y^{-1} dxdy$$

$$\leq C \int_{\mathbb{R}^n} \int_{\Gamma(t)} \left|y^m D^\beta u(z)\right|^q g \, y^{-n-1} dxdydt$$

$$\leq C \int_{\mathbb{R}^n} (u^*)^{q-2} \int_{\Gamma(t)} \left|y^m D^\beta u(z)\right|^q g \, y^{-n-1} dxdydt$$

$$\leq C \int_{\mathbb{R}^n} (u^*)^{q-2} S_m u(t)^2 A_\infty(g)(t) dt$$

Since u^* and $S_m u$ belong to $L^p$when $u \in H^p$, it follows that $(u^*)^{q-2}(S_m u)^2$ belongs to$L^{p/q}$, whence $A_\infty(g) \in L^{p/p-q}$is a sufficient condition. This is the rest of (iii) of Theorem (2.2.1).

One technical tool we shall need for the proof is Khinchine's Inequality (see, for example, [77]). Let r_n denote the nth Rademacher function, that is, $r_n(t) = (-1)^k$if $k2^{-n} \leq t < (k+1)2^{-n}$ where $n \geq 1, 0 \leq k < 2^n$ are integers. Khinchine's Inequality is the following: for any $p > 0$ there exist constants a_p and B_p such that, for any sequence of scalars $\{c_n\}$,

$$a_p \left(\sum |c_n|^2\right)^{p/2} \leq \int_0^1 \left|\sum c_n r_n(t)\right|^p dt \leq B_p \left(\sum |c_n|^2\right)^{p/2}$$

In order to finish the proof of Theorem (2.2.1), let μ be a positive measure on

U satisfying

$$\int \left|D^{\beta}u\right|^{q} d\mu \leq C\|u\|_{H^p}^{p} \quad , for\ u \in H^p \tag{53}$$

Let u be set equal to $S_{\beta}^{\lambda}(b_k)$ for some $(b_k) \in T_2^p\{z_k\}$ and some separated sequence $\{z_k\}$. Then by Lemma 2.2.9 and (53) we get

$$\int \left|\sum b_k y_k^{n+m+\lambda}\, \partial_y^{\lambda} D^{2\beta} P_{z_k}\right|^{q} d\mu \leq C\|(b_k)\|_{T_2^p\{z_k\}}^{q} \tag{54}$$

Now if each $b_k$is replaced by $b_k r_k(t)$ for fixed $t \in [0,1)$, the right-hand side of (54) is unchanged. We can then integrate the resulting equation in t and use the lower bound in Khinchine's Inequality to obtain

$$a_p \int \left(\sum \left|b_k y_k^{n+m+\lambda} \partial^{\lambda} D^{2\beta} P_{z_k}\right|^{2}\right)^{q/2} d\mu \leq C\|(b_k)\|_{T_2^p\{z_k\}}^{q} \tag{55}$$

Recall that $\partial_y^{\lambda} D^{2\beta} P(0,1) \neq 0$. By continuity, there are an $\varepsilon > 0$ and a $\delta > 0$ such that $\partial_y^{\lambda} D^{2\beta} P(0,1) > \delta$ on $D_{\varepsilon}(i)$. By homogeneity and translation we get a common lower bound for $|y_k^{n+2m+\lambda} \partial_y^{\lambda} D^{2\beta} P_{z_k}(z)|$ on $D_{\varepsilon}(z_k)$Call this lower bound δ_0. From (55) we get

$$a_p \delta_0^q \int \left(\sum_k |b_k y_k^{-m}|^2 \chi_{D_{\varepsilon}(z_k)}\right)^{q/2} d\mu \leq C\|(b_k)\|_{T_2^p\{z_k\}}^{q}$$

We assume that $\{z_k\}$ has separation constant at least ε so that $D_{\varepsilon}(z_k)$ are disjoint, whence

$$\sum_k |b_k|^q\, y_k^{-qm} \mu(D_{\varepsilon}(z_k)) \leq C\|(b_k)\|_{T_2^p\{z_k\}}^{q}$$

where thea_p and δ_0^q have been absorbed in the constant C. Putting $|b_k|^q = c_k$ we get

$$\sum_k c_k\, y_k^{-qm-n} \mu\big(D_{\varepsilon}(z_k)\big) y_k^{n} \leq C\|(c_k)\|_{T_{\frac{2}{q}}^{\frac{p}{q}}\{z_k\}} \tag{56}$$

for any positive $(c_k) \in T_{2/q}^{p/q}\{z_k\}$ The inequality continues to hold for non-positive (c_k), so we conclude that $\{y_k^{-qm-n} \mu(D_{\varepsilon}(z_k))\}$ belongs to the dual of $T_{2/q}^{p/q}\{z_k\}$, provided $\{z_k\}$ has separation constant at least ε. Moreover, its norm in that dual depends only on the constant in (53), p, q, λ, m, n and ε, and not on the sequence $\{z_k\}$. Note also that n, m, p, q are given, λ can be chosen to depend only on n, m and p, and ε clearly depended only on λ, β and n. The constant in (56) therefore depends only on

$p, q, n,$ and β. By adding up a finite number of inequalities like (56) for different sequences $\{z_k\}$, we can get (56) for arbitrary ε and arbitrary separated sequences with the constant now depending on $\varepsilon$and the separation constant of the new sequence.

Let us now consider the case where $q < p$ and $q < 2$. Then the dual of $T_{2/q}^{p/q}\{z_k\}$ is $T_{2/(2-q)}^{p/(p-q)}\{z_k\}$, so

$$\left[\sum_{z_k \in \Gamma(t)} \left(\frac{\mu(D_\varepsilon(z_k))}{y_k^{mq+n}}\right)^{2/(2-q)}\right]^{-\frac{1}{2}(2-q)} \in L^{\frac{p}{p-q}} d(t) \tag{57}$$

This is a discrete version of part (ii) of Theorem (2.2.1). We need only show that it implies the continuous version stated in Theorem (2.2.1) (ii).

Let $\varepsilon \in (0,1)$ be arbitrary and let $g(z) = \mu(D_\varepsilon(z))y^{-qm-n}$. Select a separated sequence $\{z_k\}$ such that the $D_\varepsilon(z_k)$ cover U. Then

$$\int_{\Gamma(t)} g(z)^{2/(2-q)} \frac{dxdy}{y^{n+1}} \leq \sum_{z_k \in \Gamma(t)} \int_{D_\varepsilon(z_k)} g(z)^{2/(2-q)} \frac{dxdy}{y^{n+1}}$$

while

$$\int_{D_\varepsilon(z_k)} g(z)^{2/(2-q)} \frac{dxdy}{y^{n+1}} \leq C \sup_{z \in D_\varepsilon(z_k)} g(z)^{2/(2-q)}$$

$$\leq C \left(\frac{\mu(D_{\varepsilon'}(z_k))}{y_k^{qm+n}}\right)^{2/(2-q)}$$

where$\varepsilon' \in (\varepsilon, 1)$ is chosen so that $\bigcup\{D_\varepsilon(w) : w \in D_\varepsilon(z)\} \subseteq D_{\varepsilon'}(z)$, namely $\varepsilon' = 2\varepsilon/(1+\varepsilon^2)$. Thus

$$\int_{\Gamma(t)} g(z)^{2/(2-q)} \frac{dxdy}{y^{n+1}} \leq C \sum_{z_k \in \Gamma(t)} \left(\frac{\mu(D_{\varepsilon'}(z_k))}{y_k^{qm+n}}\right)^{2/(2-q)}$$

Combining this with (57) gives Theorem (2.2.1) (ii). The other two cases of Theorem (2.2.1) are handled in a similar manner, part (iii) making use of Proposition (2.2.6).

If $L = \sum_{|\beta|=m} C_\beta D^\beta$ is any constant-coefficient linear partial differential operator of pure order m, it is clear that the conditions (i)-(iii) of Theorem (2.2.1) are sufficient for

$\int_U |Lu|^q \, d\mu \leq C\|u\|_{H^p}^q$, They are not necessary, as the example where L is the Laplacian $\sum_{j=1}^{n+1} \partial^2/\partial x_j^2$ shows. If we put a simple, obvious hypothesis on L then the conditions do turn out to be necessary. Suppose only that there is at least one harmonic function $u \in H^p$ with $Lu \neq 0$on U. Then $LP(x, y) \neq 0$, where P is the Poisson kernel. By homogeneity and analyticity considerations, every neighbourhood of (0,1) contains a point where $L^2 P$ is not zero. This can be used then in the proof in place of the stronger condition $D^{2\beta} P(0,1) \neq 0$. It is strong enough to complete the proof of Theorem (2.2.1).

If q and p are given, then the conclusions of Theorem (2.2.1) depend only on the function $\mu(D(z)) y^{-qm-n}$. It is clear then that there is in general no containment relationship between the families

$$C_m^{p,q} = \left\{\mu : \exists C \forall \mu \in H^p \forall |\beta| = m, \int \left|D^\beta u\right|^q \, du \leq C\|u\|_{H^p}^q\right\}$$

For different values of m. However , it μ is concentrated in a strip of the form

$\{(x, y) : 0 < y < y_0\}$ or in one of the form $\{(x, y) : 0 < y_0 < y\}$, then $C_{m+1}^{p,q} \subseteq C_m^{p,q}$ in the first case but $C_m^{p,q} \subseteq C_{m+1}^{p,q}$ in the second.

In the case where the dimension n is 1, another classical H^p is the one on the unit disk. There is no difficulty in obtaining the corresponding result in that case. In this setting we always get $C_{m+1}^{p,q} \subseteq C_m^{p,q}$.

We promised a proof of the $m = 0, q < p$ case of inequality (37). We formally state the result.

Theorem (2.2.11) [69]:

Let $q < p$. A necessary and sufficient condition on a positive measure μ on U in order that there exist a constant $C > 0$ with

$$\int |u|^q d\mu \leq \|u\|_{H^p}^q \text{for all } u \in H^p, \tag{58}$$

is that the function $t \to \int_{\Gamma(t)} y^{-n} d\mu(x, y)$ belong to $L^{p/(p-q)}$.

Theorem C can be found in [86] where it is proven only in the case where $q = 1$. The proof that the general case can be reduced to this one is omitted. To this writer the reduction is non-trivial, and the proof presented here is based on the ideas of rather than the arguments of [86].

Proof:

As in other cases, the proof of sufficiency is easy:

$$\int_U |u|^q d\mu = c\int_{\mathbb{R}^n}\int_{\Gamma(t)} |u|^q y^{-n} d\mu\, dt \le \int_{\mathbb{R}^n} u^*(t)^q \int_{\Gamma(t)} y^{-n} d\mu\, dt.$$

The sufficiency is now clear because $(u^*)^q \in L^{p/q}$ when $u \in H^p$.

The necessity seems to be much more difficult. We referred, because the case where $n = 1$ (where complex methods are available) is relatively easy. Let us present this case first, and then present the proof for $n > 1$.

In the case where $n = 1, u$ belongs to H^p if and only if there is an analytic function f on $U = \mathbb{R}^2_+$ with $\operatorname{Re} f = u$ and $\sup_{y>0}\int |f(x+iy)|^p dx < +\infty$ (the classical definition of analytic H^p). Now let $0 < \varphi \in L^{p/q}(\mathbb{R}, dt)$ and let its Poisson integral extension to U be denoted by the same letter $\varphi(x,y)$. Let g belong to $H^{p/q}$ (analytic) with $\operatorname{Re} g = \varphi$.Then $f = g^{1/q}$ belongs to H^p (where the principal branch of the root is taken). Now

$$\int \varphi d\mu \le \int |g|\, d\mu = \int |f|^q d\mu \le C\|f\|^q_{H^p} = C\|g\|_{H^{p/q}} \le C'\|\varphi\|_{L^{p/q}},$$

where the second inequality assumes (58). Now express $\varphi(x,y)$ as a convolution of $\varphi(t)$ with the Poisson Kernel and exchange integrals in the expression $\int \varphi(x,y)\, d\mu(x,y)$ to obtain

$$\int_{-\infty}^{\infty} \varphi(t)\tilde{\mu}(t)dt \le C'\|\varphi\|_{L^{p/q}}, \tag{59}$$

Where

$$\tilde{\mu}(t) = c_1 \int \frac{y}{(x-t)^2 + y^2}\, d\mu(x,y).$$

We conclude from (59) that $\tilde{\mu} \in L^{p/(p-q)}$. But clearly $\tilde{\mu}(t) \ge c\int_{\Gamma(t)} y^{-1}\, d\mu$, and so the necessity has been shown in the case where $n = 1$.

The function $\tilde{\mu}$ is called the balayage of μ and we could add the additional necessary and sufficient condition that $\tilde{\mu} \in L^{p/(p-q)}$. This follows from the above proof (when $n = 1$):, but it also follows from Proposition (2.2.4), inequality (38), with $s = p/q$ .and $\lambda = 1$, even when $n > 1$ once Theorem C is proven.

What we will actually show is that the following function belongs to $L^{p/(p-q)}$.

$$C_1(yd\mu)(t) = \sup\{\mu(\hat{B})/|B|: B \text{ a ball containing } t\}$$

It is shown [3] that C2(f) and A2(f) have comparable U-norms when $p > 2$. Since $\|A_1(f)\|_p = \|A_2(\sqrt{|f|})\|_{2p}$ and $\|C_1(f)\|_p = \|G_2(\sqrt{|f|})\|_{2p}$ it follows that $\|A_1(f)\|_p$ and $\|C_1(f)\|_p$ are comparable for $p > 1$. It is not hard to see that this continues to hold in the case where $fdxdy$ is replaced by the measure $yd\mu$. Thus, once $C_1(yd\mu)$ is shown to be in $L^{p/(p-q)}$ the proof will be finished. To simplify the notation we will write $\hat{\mu}(t)$ for $\sup\{\mu(\hat{B})/|B|: t \in B\}$instead of$C_1(yd\mu)(t)$.

As if $(\lambda + n)/n > \max(2/p, 1)$ then, for any separated sequence $z_k = (x_k, y_k)$in $U\mathbb{R}_+^{n+1}$, the function $\sum b_k y_k^{n+\lambda} \partial_y^\lambda P_{z_k}(x, y)$ has H^p-norm dominated by the $T_2^p\{z_k\}$ norm of (b_k). From the assumption (58) we get

$$\int \sum_k \left|b_k y_k^{n+\lambda} \partial_y^\lambda P_{z_k}\right|^q d\mu \leq C \left[\int \left(\sum_{z_k \in \Gamma(t)} b_k^2\right)^{p/2} dt\right]^{q/p}. \tag{60}$$

We can apply Khinchine'sInequality to (60) just as we did to obtain

$$\int \left(\sum b_k^2 F_k^2\right)^{q/2} d\mu \leq C\|(b_k)\|_{T_2^p}^q, \tag{61}$$

where $F_k = \left|y_k^{n+\lambda} \partial_y^\lambda P_{z_k}\right|$. We saw in Lemma 2 that $\partial_y^\lambda P(0,1) \neq 0$ and by continuity this persists in a neighbourhood of (0,1). Using this we can easily see that there is an $a > 0$ such that if $B_k = (x_k, \alpha y_k)$ then $F_k > C > 0$on $\hat{B}_k$. From this and (61) we obtain

$$\int \left(\sum b_k^2 \chi_{\hat{B}_k}\right)^{q/2} d\mu \leq C\|(b_k)\|_{T_2^p}^q, \tag{62}$$

We will now construct appropriate B_k and $b_k$and apply (62). Let $\mathscr{b}_k$ denote the collection of maximal dyadic cubes Q such that $(\hat{Q}) > 2^k|Q|$. Let $\mathscr{b} = \cup \mathscr{b}_k$ and let $E_k = \cup\{Q: Q \in \mathscr{b}_k\}$. We will show that

$$t \to \sup\left\{\frac{\mu(\hat{Q})}{|Q|}: Q \text{ dyadic}, t \in Q\right\}$$

belongs to $L^{p/(p-q)}$. By an observation of Fefferman and Stein [75], this will prove that $\hat{\mu}$, belongs to $L^{p/(p-q)}$. Thus, there is no harm in assuming that $\hat{\mu}$, is this dyadic supremum. Then on $E_k - E_{k+1}$ we have $2^k < \hat{\mu}(t)2^{k+1}$. If $Q \in \mathscr{b}_k$ let x_Q be its centre and let $y_Q$be

$1/(2\alpha)$ times its side length. It is not hard to verify that $\{(x_Q, y_Q): Q \in \mathscr{b}\}$ is a separated sequence in U. By assuming that μ has compact support, we may suppose this is a finite sequence. Let (b_Q) be any sequence indexed by $Q \in \mathscr{b}$. It is clear that

$$\left(\sum b_Q^2 \chi_{\hat{Q}}\right)^{q/2} \geq \sum b_Q^q \chi_{G(Q)},$$

where$G(Q) = \hat{Q} - \cup\{Q': Q' \in \mathscr{b}, Q' \subset Q\} = \hat{Q} - \cup\{Q': Q' \in \mathscr{b}_{k+1}, Q' \subseteq Q\}$ if $Q \in \mathscr{b}_k$

Let us now index the $Q \in \mathscr{b}$ according to which $\mathscr{b}_k$ they belong to, writing $\mathscr{b}_k = \{Q_j^k : j = 1, 2, \ldots\}$ and $b_{Q_j^k} = b_{k_j}$. Then, from the above and (62) we have

$$\sum_k \sum_j b_{k_j}^q \left(\mu(\hat{Q}_j^k) - \sum_{Q_j^{k+1} \subset Q_j^k} \mu(\hat{Q}_j^{k+1}) \right) \leq C \left[\int \left(\sum b_{k_j}^2 \chi_{Q_j^k} \right)^{p/2} dt \right]^{q/p}. \quad (63)$$

(Note that$\sum_{z_Q \in \Gamma(t)} b_Q^2 \leq \sum b_{k_j}^2 \chi_{Q_j^k}(t)$ because if $(x_Q, y_Q) \in \Gamma(t)$ then $t \in (Q)$.) Now we put $r = p/q$and $r' = r/(r-1) = p/(p-q)$, and set $b_{k_j}^q = 2^{k(r'-1)}$. The left-hand side of (63) becomes (if we write $\tilde{E}_k = \cup_j \hat{Q}_j^k$)

$$\sum_k 2^{k(r'-1)} \mu(\tilde{E}_k) - \sum_k 2^{k(r'-1)} \sum_j \mu(\hat{Q}_j^k) \cap \tilde{E}_{k+1})$$

$$= \sum_k 2^{k(r'-1)} \mu(\tilde{E}_k) - \sum_k 2^{k(r'-1)} \mu(\tilde{E}_{k+1})$$

$$= \sum_k 2^{k(r'-1)} - 2^{(k-1)(r'-1)} \mu(\tilde{E}_k)$$

$$= \left(1 - \frac{1}{2^{r'-1}}\right) \sum_k 2^{k(r'-1)} \sum_j \frac{\mu(\hat{Q}_j^k)}{Q_j^k} |Q_j^k|$$

$$\geq \sum_k 2^{kr'} |E_k| \geq c \int \hat{\mu}^{r'} dt \ ,$$

while the right-hand side of (63) is

$$C \left[\int \left(\sum_k 2^{k(r'-1)2/q} \chi_{E_k} \right)^{p/2} dt \right]^{1/r} = C \left[\int \left(\frac{\eta}{\eta^{-1}} \sum_k (\eta^k - \eta^{k-1}) \chi_{E_k} \right)^{p/2} dt \right]^{1/r}$$

where $\eta /= 2^{(r'-1)2/q} > 1$. Another summation by parts makes this equal to

$$const.\left[\int\left(\sum_k \eta^k(\chi_{E_k}-\chi_{E_{k+1}})\right)^{p/2} dt\right]^{1/r} = const.\left[\int\sum_k \eta^{kp/2}\,\chi_{E_k\setminus E_{k+1}} dt\right]^{1/r}$$

$$= const.\left[\int\sum 2^{kr'}\left|\chi_{E_k\setminus E_{k+1}}\right| dt\right]^{1/r}$$

$$\leq const.\left[\int \hat{\mu}^{r'} dt\right]^{1/r}$$

Combining these two estimates gives

$$\int \hat{\mu}^{r'} dt \leq const.\left[\int \hat{\mu}^{r'} dt\right]^{1/r},$$

or$\|\hat{\mu}\|_{L^{p/(p-q)}}$ const. forμ with compact support. The result for arbitrary μ follows from this by an easy limit argument. Theorem C is now proven

Many of the previously known cases were not placed in the $H^p(\mathbb{R}_+^{n+1})$ setting when the results were obtained. It makes sense to examine these cases, state the version appropriate to this setting, and verify that the results continue to hold.

Carleson's Theorem: $m = 0, p = q$. This case has been thoroughly studied in all settings and there is no problem with it here. However, for further use we will state the result and sketch a proof. The necessary and sufficient condition is that

$$\mu(\hat{B}) = C|B| \qquad for\ all\ balls \qquad B \subseteq \mathbb{R}^n \tag{64}$$

The necessity of this condition follows upon applying the inequality $\int |u|^p\, d\mu \leq C\|u\|_{H^p}^p$ to appropriate u (of the form $C\partial_y^\lambda P_{z_0}$ with $z_0 = (x_0, y_0), x$ the centre of B and y_0a multiple of the radius).

The sufficiency can be obtained by the following argument. For $\lambda > 0$, let $E_\lambda = \{t: u^*(t) > \lambda\}$. Since $E_\lambda$is open, it has a Whitney decomposition. Suffice it to say that this means $E_\lambda = \cup\{Q: Q \in \mathcal{F}\}$ such that $\hat{E}_\lambda \subseteq \cup_{Q\in\mathcal{F}}(C\cdot Q)^\wedge$, where $\mathcal{F}$is a disjoint family of cubes and $C\cdot Q$ denotes the cube with the same centre as Q but C times the size. Now, with $G_\lambda = \{(x,y): |u(x,y)| > \lambda\}$, it follows from the definition of u^* that $G_\lambda \subseteq \hat{E}_\lambda$ Thus

$$\mu(G_\lambda) \leq \sum_{Q\in\mathcal{F}_k} \mu[(C.Q)^\wedge] \leq C\sum |Q| = C|E_\lambda| \tag{65}$$

Integrating this inequality with respect to $\lambda^{p-1}d\lambda$ gives $\int |u|^p\, d\mu \leq C\|u^*\|_{L^p}^p$, Thus

(64) is sufficient.

II. Duren's Theorem: $m = 0, q > p$.The necessary and sufficient condition for $\int |u|^p \, d\mu \le C\|u\|_{H^p}^p$ is

$$\mu(\hat{B}) = C|B|^{q/p} \qquad for\ all\ balls \qquad B \subseteq \mathbb{R}^n \tag{66}$$

The necessity follows as in I by applying the inequality to appropriate u (the same one).

The sufficiency follows from something like (65): write $G_{2^k} \subseteq \bigcup_{Q \in \mathcal{F}_k} (C \cdot Q)^\wedge$ where $\mathcal{F}_k$ is the $\mathcal{F}$ for E_{2^k}. Then

$$\mu(G_{2^k}) \le C \sum_{Q \in \mathcal{F}_k} |Q|^{q/p}$$

and

$$\int u^q d\mu \le C \sum_k 2^{kq} \, \mu(G_{2^k}) \le C \sum_k 2^{kq} \sum_{\mathcal{F}_k} |Q|^{q/p}$$

On the other hand,

$$\int u^*(t)^p dt \ge C \sum_k 2^{kq} \sum_{\mathcal{F}_k} |Q|$$

Expressing $\{Q: Q \in \mathcal{F}_k\}$ as $\{Q_j^k: j = 1,2,...\}$ we need only show (with $b_{k_j} = 2^{kp}|Q_j^k|$) that $\sum_{j,k} |b_{k_j}|^{q/p}$. But this is immediate for $q/p > 1$.

Theorem (2.2.11) [69]: (Shirokov-Luecking):

$m > 0, p = q \ge 2$or $p < q$. The necessary and sufficient condition for $\int |D^\beta u|^q \, d\mu \le C\|u\|_{H^p}^q$ is that

$$\mu(\hat{B}) \le C|B|^{q/p + |\beta| q/n} \qquad for\ all\ balls \qquad B \subseteq \mathbb{R}^n \tag{67}$$

The necessity again follows on applying the inequality to u of the form $\partial_y^\lambda D^\beta P_{z_0}$. The sufficiency is obtained by the same arguments as in [78]:

$$\left| y_0^{|\beta|} D^\beta u(z_0) \right|^q \le \frac{C \int_{D(z_0)} \left| y^\lambda \nabla^\lambda u \right|^q dxdy}{y_0^{n+1}}$$

whenever$\lambda \le |\beta|$. If $q > p$, let $\lambda = 0$ and integrate both sides with respect to $y_0^{-|\beta| q} d\mu(z_0)$, using Fubini's Theorem on the right to obtain

$$\int \left| D^\beta u \right|^q d\mu \le C \int |u|^q \mu(D(z)) y^{-n-1-|\beta| q} \, dxdy$$

$$\leq C\int |u|^q y^{-nq/p-n-1}\, dxdy$$

$$\leq C\|u\|_{H^P}^q$$

where the second inequality is from (67) and the last one is Duren's Theorem (II).

If $q = p > 2$, let $\lambda = 1$ and again integrate both sides with respect to $y_0^{-|\beta|q}d\mu(z_0)$ and use Fubini's Theorem to obtain

$$\int \left|D^\beta u\right|^q d\mu \leq C\int |\nabla u|^q y^{-n-1-|\beta|q}\, dxdy$$

$$\leq C\int |u|^q y^{q-1}\, dxdy$$

$$\leq C\|u\|_{H^P}^q$$

where (67) is used in the second inequality and the Littlewood-Paley Inequality in the last.

We saw that if $\{z_k\}$is a separated sequence in U and $u \in H^p$ then $y_k^{|\beta|}D^\beta u(z_k)$ has $T_2^p\{z_k\}$-norm less than the H^p-norm of u. We also saw that if the sequence $\{z_k\}$ has large enough separation constant, then all sequences in $T_2^p\{z_k\}$ arise in this way from H^p-functions. It should come as no surprise that if $\{z_k\}$is sufficiently 'crowded' then the $T_2^p\{z_k\}$-norm of $y_k^m\partial_y^m u(z_k)$ dominates the H^p-norm of u. What this means is the following. Call a sequence $\{z_k\}$ D-dense if $\{D_\delta(z_k)\colon k = 1,2,\dots\}$ covers U. Then there is a $\delta > 0$ depending only on m, p and the aperture α such that if $\{z_k\}$is δ -dense then

$$\int \left(\sum_{z_k\in\Gamma(t)} y_k^{2m}\left|\partial_y^m u(z_k)\right|^2\right)^{p/2} dt \geq c\int \left(\int_{\Gamma(t)} \left|y_k^m\partial_y^m u(z)\right| y^{-n-1}dxdy\right)^{p/2} dt$$

$$\geq C\|u\|_{H^P}^q$$

With this inequality (and the duality between $H^p$and $H^{p'}$) there is no difficulty in showing that if $\{z_k\}$ is separated and δ-dense for sufficiently small δ, then the operator S_β (with $\beta = (0,0,\dots,0,m)$ of is surjective. That is, every $u \in H^p$ for $p > 1$ has the form $u(z) = \sum_k b_k y_k^{n+m}\partial_y^m P_{z_k}(z)$with $(b_k) \in T_2^p\{z_k\}$. Moreover, the norm of u in $H^p$is bounded above and below by the infimum of the norm in $T_2^p\{z_k\}$over all (b_k) for which this representation holds.

The proofs of these statements are essentially the same as those corresponding statements about Bergman spaces found in, for example, [78, 79, 80]. In fact, the methods in [80] easily give very general sufficient conditions on a measure v so that $\|u\|_{H^p} \leq \|y^m \partial_y^m u\|_{T_2^p(v)}$.

In [81], Shirokov actually considered the following more difficult problem. For each $x \in \mathbb{R}^n$let μ_x be a positive measure on $(0, +\infty)$. Assume $\mu_x(E)$ is measurable $E \subseteq (0, \infty)$. Characterize those μ_x for which

$$\int \left[\int_0^\infty |\nabla^m u(x,y)|^q d\mu_x(y) \right]^{p/q} dx \leq C\|u\|_{H^P}^q \tag{68}$$

(Actually in [81] Shirokov took $n = 1$and considered H^p of the unit disk, but the problems are essentially the same.)

Shirikov had four main conclusions (see [81]). The first was the necessary condition for (68):

$$\int_{B(x_0,y_0)} \mu_x \left[\frac{1}{2} y_0, y_0\right)^{p/q} dx \leq C y_0^{mp+n} \tag{69}$$

for all $(x_0, y_0) \in \mathbb{R}_+^{n+1}$. The second was the equivalence of (68) and (69) when $p = q \geq 2$. The first is a consequence of applying (68) to one of the functions $\partial_y^\lambda P z_0$ and the second is the known result discussed above. His third conclusion was the following sufficient condition in the case where $2 \leq s < q < p$: there exists $\varepsilon > 0$ such that

$$\int_{B(x_0,y_0)} \mu_x \left[\frac{1}{2} y_0, y_0\right)^{(p+\varepsilon)/q} dx \leq C y_0^{(p+\varepsilon)m+n} \tag{70}$$

for all $(x_0, y_0) \in \mathbb{R}_+^{n+1}$. Let us show how this can be obtained with our methods. In order for (68) to hold it is necessary and sufficient that for all $\psi \in L^{p/(p-q)}$ we have

$$\int_{\mathbb{R}^n} \int_0^\infty |\nabla^m u(x,y)|^q \psi(x) d\mu_x(y) dx \leq C\|\psi\|_{p/p-q} \|u\|_{H^p}^q$$

Thus $\psi(x) d\mu_x(y) dx$ must be a measure satisfying (37) and so Theorem (2.2.1) applies (part (iii) to obtain the necessary and sufficient condition

$$\sup_{(x_0,y_0)\in\Gamma(t)} y_0^{-mq-n} \int_{B(x_0,y_0)} \psi(x) \mu_x \left[\frac{1}{2} y_0, y_0\right) dx \in L^{p/p-q}(t) \tag{71}$$

with a uniform bound on the norm. Let us rewrite this as

$$\sup_{B\ni t}\frac{1}{|B|}\int_B \psi(x)g(x,y_0)dx \in L^{p/p-q}$$

where y_0 is the radius of the ball B, the sup is over all balls containing t, and $g(x,y_0)=\mu_x\left[\frac{1}{2}y_0,y_0\right)y_0^{-mq}$. The question then is to find conditions on g that give a bound on the $L^{p/(p-q)}$-norm of this function. But if we apply Holder's Inequality with exponents sand s' with $1<s<p/(p-q)$, we get

$$\frac{1}{|B|}\int_B \psi(x)g(x,y_0)\le\left(\frac{1}{|B|}\int_B\psi^s\right)^{1/s}\left(\frac{1}{|B|}\int_B g(.,y_0)^{s'}\right)^{1/s'}$$

Since $\sup_{t\in B}\left((1/|B|)\int_B\psi^s\right)^{1/s}$ has $L^{p/(p-q)}$-norm at most $C\|\psi\|_{p/(p-q)}$, the sufficient condition is that $\int_B g(\cdot,y_0)^{s'}\le C|B|$. Rewriting this in terms of μ_x, using $B=B(x_0,y_0)$, and putting $s'=(p+\varepsilon)/q$ gives (70).

(Shirokov's fourth conclusion in [81] is that a sufficient condition for (68) when $q>p$ is

$$\int_{B(x_0,y_0)}\mu_x\left[\frac{1}{2}y_0,y_0\right)dx\le Cy_0^{mq+n} \tag{72}$$

for all (x_0,y_0) in $\mathbb{R}_+^{n+1}$(or rather the $n=1$ version). Shirokov's proof of this relies on complex methods but the result remains true when $n>1$ as the following shows

$$\int_{\mathbb{R}^n}\left(\int_0^\infty|\nabla^m u|^q d\mu_x(y)\right)^{p/q}dx$$

$$\le C\int_{\mathbb{R}^n}\left(\int_0^\infty|y^m\nabla^m u|^{q-p}|\nabla^m u|^p y^{m(p-q)}d\mu_x(y)\right)^{p/q}dx$$

$$\le C\int_{\mathbb{R}^n}u^*(x)^{(q-p)p/q}\left(\int_0^\infty|\nabla^m u|^p y^{m(p-q)}d\mu_x(y)\right)^{p/q}dx$$

$$\le C\|u^*\|_p^{1-p/q}\int_{\mathbb{R}^n}\left(\int_0^\infty|\nabla^m u|^p y^{m(p-q)}d\mu_x(y)\right)^{p/q}dx$$

Now if $d\mu_x$ satisfies (72) then $d\nu_x=y^{m(p-q)}d\mu_x$ satisfies the same condition with

$q$replaced by p. This is exactly the right condition for (68) with q replaced by p. Thus, in the last expression above, the first factor is bounded by $C\|u\|_{H^p}^{1-p/q}$ and the second by $C\|u\|_{H^p}^{p/q}$, and (68) follows.)

In the case where $q < 2$with $q < p$, our methods give the following sufficient condition for (68):

$$\int_0^\infty \left(\frac{1}{y^n} \int_{B(x_0,y)} \left[\frac{\mu_x\left[\frac{1}{2}y, y\right)}{y^{mq}} \right]^{(p+\varepsilon)/q} \right)^{(q/(p+\varepsilon))(2/(2-q))} \frac{dy}{y} \leq C$$

or all $x_0 \in \mathbb{R}^n$. This is obtained just as in the $q \geq 2$ case except that Theorem (2.2.1), part (ii) is invoked instead of part (iii).

It seems clear that the approach in this section depends largely, if not entirely, on the properties of the kernel $D^\beta P$. This suggests that the whole theory could be developed for kernels defined on $X \times X \times (0, +\infty)$ where X is a space of homogeneous type. It is very likely, then, that the result corresponding to Theorem (2.2.1) for the ball in$\mathbb{C}^n$, say, would be very similar to Theorem (2.2.1).