Chapter 1

Scattered Zeroes Extensions of Bounds for Functions in Sobolev Spaces

We apply results to obtain estimates for continuous and discrete least squares surface fits via radial basis functions (RBFs). These estimates include situations in which the target function does not belong to the native space of the RBF.We then apply the Sobolev bound to derive error estimates for interpolating and smoothing (m, s) −splines. In the case of smoothing, noisy data as well as exact data are considered.

Sec (1.1):Applications to Radial Basis Functions Surface Fitting:

The problem of effectively representing an underlying function based on its values sampled at finitely many distinct scattered sites $X = \{x_1, \dots, x_N\}$ lying in a compact region $\Omega \subset \mathbb{R}^n$ is important and arises in many applications—neural net- works, computer aided geometric design, and gridless methods for solving partial differential equations, to name a few.

There are two main ways of dealing with this problem: interpolation of the data or least squares approximation of the data. In both cases one assumes the data is generated by a function f belonging to a classical Sobolev space, $W_p^k(\Omega)$. One next needs to select an interpolating or approximating subspace of functions. One choice is to use multivariate splines or finite elements. In this approach, one needs to decompose Ω into a number of subregions and interpolate or approximate by multivariate polynomials on each subregion. One then sews together the pieces in a smooth way to construct the representing surface. This is, in $\mathbb{R}^n$ with $n \geq 3$, a nontrivial task.

Another approach, which will be the focus of this section, is to use radial basis functions (RBFs). An RBF is a radial function $\Phi(x) = \varphi(|x|)$ that is either positive definite or conditionally positive definite on $\mathbb{R}^n$. Interpolants for multivariate functions sampled at scattered sites are constructed from translates of RBFs with the possible addition of a polynomial term.

It was Duchon [5] who introduced a type of RBF, the thin-plate spline, which he constructed via a variational technique similar to those used to obtain ordinary splines. The error analysis he provided for thin-plate splines involved reproducing kernel Hilbert space (RKHS) methods and applied to both interpolation and least squares approximation. Subsequently, the theory of RBF interpolation evolved with seminal contributions from Micchelli [9], who introduced a wide class of functions for which interpolation of scattered data was always possible, and Madych and Nelson [6, 7], who obtained L_∞ error estimates for RBF interpolation. Least squares approximation by RBFs was treated by de Boor, DeVore and Ron [3, 12] in the case where the underlying domain was $\mathbb{R}^n$ and the approximating subspace had "centers" at the scaled lattice points. In particular, the theory of least squares approximation on a compact set Ω for scattered data has not gone beyond the initial work of Duchon.

In this section we seek to extend the work of Duchon in several directions. The original work of Duchon dealt with the globally supported thin-plate splines. The natural spaces to deal with in that setting were the integer-order Sobolev spaces (or the Beppo-Levi spaces which are Sobolev semi-normed spaces). We obtain similar results for the locally supported Wendland functions [15] in their natural setting of fractional order Sobolev spaces. Another aim of this section is to extend the least squares setting estimates to functions which lie outside the RKHS as has been recently done for the case of interpolation [10].

Recall that the original Duchon estimates applied to the continuous least squares setting only. That is, one approximated functions that were defined on all of Ω. We will obtain discrete least squares estimates where it is assumed the function belongs to an appropriate Sobolev space $W_p^k(\Omega)$but is only known on a discrete subset . These results are the first of their kind.

Finally we wish to provide an "intrinsic proof" of all these results which relies on basic principles.

Central to our approach will be a Theorem which gives very precise Sobolev norm estimates for functions having many zeros in a domain Ω. Note that the interpolation error function is an example of a function having many zeros. This same concept will be important in establishing the least squares error estimates as well. In general, we believe this Theorem has applications outside the realm of RBFs. In particular, a variant of the theorem below can be used to extend to more general domains some of the interpolation error estimates found in [1]. More precisely, the following will be established in the $\mathbb{R}^n$setting.

Theorem (1.1.1)[18]:

Let k be a positive integer, $0 < s \leq 1, 1 \leq p < \infty, 1 \leq q \leq \infty$ and let α be a multi-index satisfying $k > |\alpha| + n/p$ or, for $p = 1, k \geq |\alpha| + n$. Let $X \subset \Omega$ be a discrete set with mesh norm $h = h_{X,\Omega} = \sup_{x\in\Omega}\min_{x_j\in X} |x - x_j|$ where Ω is a compact set with Lipschitz boundary which satisfies an interior cone condition. If $u \in W_p^{k+s}(\Omega)$ satisfies $u|_X = 0$, then

$$|u|_{W_q^{|\alpha|}(\Omega)} \leq ch^{k+s-|\alpha|-n(1/q-1/q)}|u|_{W_p^{k+s}(\Omega)},$$

where c is a constant independent of u and h, and $(x)_+ = x$ if $x \geq 0$ and is 0 otherwise.

Here $|u|_{W_p^m(\Omega)}$ refers to the (fractional) Sobolev semi-norm (see definitions later). A precursor of this theorem may be found in [5] by Duchon, who restricted Ω to balls of certain radii and considered only the cases $p = 2$ and $s = 0$. In another direction, Madych and Potter [8] obtained a restricted version of this theorem for the case $p = q$ and for functions which vanished on the boundary of Ω.

A typical application of Theorem (1.1.1) can be described as follows. Suppose we have an interpolation process $P_X : W_p^{k+s}(\Omega) \to V_X$ that maps Sobolev functions to a finite dimensional subspace of $W_p^{k+s}(\Omega)$with the additional property Then $|P_X f|_{W_p^{k+s}(\Omega)} \leq |f|_{W_p^{k+s}(\Omega)}$. Theorem (1.1.1) immediately gives error estimates of the form

$$|f - P_X f|_{W_p^{|\alpha|}(\Omega)} \leq ch^{k+s-|\alpha|-n(1/q-1/q)}|f|_{W_p^{k+s}(\Omega)}.$$

We illustrate the above in two different cases. Probably the most prominent situation is illustrated by classical univariate splines. For example, natural cubic spline interpolants are known to minimize $|\cdot|_{W_2^2[a,b]}$amongst all interpolants from $|\cdot|_{W_2^2[a,b]}$.

The second example deals with multivariate radial basis function interpolation. In our framework the error estimates fall into two parts. Theorem (1.1.1) gives estimates on the interpolation error. Moreover, it is well known that radial basis function interpolants are also best approximants in certain associated reproducing kernel Hilbert spaces. Hence, if such a space coincides with an appropriate Sobolev space, the (semi-)norm of the interpolant can be bounded by the (semi-)norm of the target function.

Our new approach offers a new paradigm for radial basis function nterpolation error estimates, where estimates on functions with a large zero set replaces the power function approach.

We will need to work with a variety of Sobolev spaces. The def- initions used here follow those used by Brenner and Scott [1]. Let $\Omega \subset \mathbb{R}^n$be a domain. For $k \geq 0, k \in \mathbb{Z}$, and $1 \leq p < \infty$, we define the Sobolev spaces $W_p^k(\Omega)$ to be all with distributional derivatives $D^\alpha u \in L_p(\Omega), |\alpha| \leq k$. Associated with these spaces are the (semi-)norms

$$|u|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} \quad and \quad \|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

The case $p = \infty$ is defined in the obvious way

$|u|_{W_p^k(\Omega)} = \sup_{|\alpha|=k} \|D^\alpha u\|_{L_\infty(\Omega)}$ and $\|u\|_{W_\infty^k(\Omega)} = \sup_{|\alpha|\leq k} \|D^\alpha u\|_{L_\infty(\Omega)}$.

For fractional order Sobolev spaces, we use the norms below. Let $1 \leq p < \infty$, $k \geq 0, k \in \mathbb{Z}$, and let $0 < s < 1$. We define the fractional order Sobolev spaces $W_p^{k+s}(\Omega)$ to be all u for which the norms below are finite

$$|u|_{W_p^{k+s}(\Omega)} = \left(\sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{1/p},$$

$$|u|_{W_p^{k+s}(\Omega)} = \left(\|u\|_{W_p^k(\Omega)}^p + |u|_{W_p^{k+s}(\Omega)}^p \right)^{1/p}.$$

Let $X = \{x_1, \ldots, x_N\}$ be a finite, discrete subset of Ω, which we now assume to be bounded. There are three quantities that we associate with : the separation radius, the mesh norm or fill distance, and the mesh ratio. Respectively, these are given by

$$q_X = \frac{1}{2} \min_{j\neq k} |x_j - x_k|, \qquad h_{X,\Omega} = \sup_{x\in\Omega} \operatorname{dist}(x, X), \quad and \quad \rho_{X,\Omega} = \frac{h_{X,\Omega}}{q_X}.$$

Here, $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}^n$. The first is half the smallest distance between points in , the second measures the maximum distance a point in Ω can be from any point in X , and the final quantity, the mesh ratio, measures to what extent points in X uniformly cover Ω. Frequently, when the set Ω or X is understood, we will drop subscripts and write h_X or h. Other notation will be introduced along the way.

In this section we obtain Sobolev bounds on functions with scattered zeros in a bounded Lipschitz domain Ω that satisfies a uniform interior cone condition. This is done in two main steps. We first obtain results for a special class of domains that are star-shaped with respect to balls. We then use a decomposition of Ω into such domains to obtain the general results.

We will first obtain our bounds for a special class of domains. Following Brenner and Scott [1], we will say that a domain $\mathcal{D}$ is star-shaped with respect to a ball $B(x_c, r) = \{x \in \mathbb{R}^n : |x - xc| < r\}$ if for every $x \in \mathcal{D}$, the closed convex hull of $\{x\} \cup B$ is contained in $\mathcal{D}$.

We will deal only with bounded domains. Thus, there will be a ball $B(x_c, R)$that contains $\mathcal{D}$. Of course, the diameter $d_{\mathcal{D}}$ of $\mathcal{D}$ satisfies $r < d_{\mathcal{D}} < 2R$. Also, Brenner and Scott [1, Definition 4.2.16] define the chunkiness parameter γ to be the ratio of $d_{\mathcal{D}}$to the radius of the largest ball relative to which $\mathcal{D}$ is star-shaped. This parameter comes up in various estimates and it is useful to note that it can be bounded above; namely, we have

$$\gamma \le \frac{2R}{r}. \tag{1}$$

Finally, such domains satisfy a simple, interior cone condition, which we now describe.

Proposition (1.1.2)[18]:

If $\mathcal{D}$ is bounded, star-shaped with respect to $B(x_c, r)$and contained in $B(x_c, R)$, then every $x \in \mathcal{D}$ is the vertex of a cone $C \subset \mathcal{D}$ having radius r and angle $\theta =$ $2\arcsin\left(\frac{2R}{r}\right)$

Proof:

It is easy to check that when $x \in B(x_c, r)$, the condition is satisfied if the central axis of the cone is directed along a diameter of the ball $x \in B(x_c, r)$. If x is outside of that ball, then consider the convex hull of x and the intersection of the sphere $S(x, |x - x_c|) = \{y \in \mathbb{R}^n: |y - x| = |x_c - x|\}$ with $B(x_c, r)$. This is a cone, and, because $\mathcal{D}$ is star-shaped with respect to $B(x_c, r)$, it is contained in $\mathcal{D}$. Its radius is the distance from x to x_c. To find its angle θ, consider a triangle formed by x, x_c, and any point on y in the intersection of $S(x, |x - x_c|)$ and the sphere $S(x_c, r)$ This is any isosceles triangle, since $|x_c - x| = |y - x|$. The angle $\angle x_c x y = \theta$; the side opposite this angle has length r. A little trigonometry then gives us that $|x_c - x|\sin(\frac{1}{2}\theta) = \frac{1}{2}r$. Consequently, we have $\theta = 2\arcsin\left(\frac{r}{2|x_c - x|}\right)$. Moreover, since$\mathcal{D} \subset B(x_c, R)$, we also have $|x_c - x| \le R$. Thus, $\ge 2\arcsin\left(\frac{r}{2R}\right)$. Finally, $r \le |x - x_c|$ implies that the cone with vertex x, axis along $x_c - x$, and angle $\theta = 2\arcsin\left(\frac{r}{2R}\right)$ is contained in $\mathcal{D}$.

Throughout the remainder of this section, $\mathcal{D}, r, R, \gamma, \theta$, and x are related in the way described above.

What we want to do next is to prove a Bernstein inequality for polynomials restricted to $\mathcal{D}$. Let $p \in \pi_\ell(\mathbb{R}^n)$ and assume that ∇p is not identically zero. The maximum of $|\nabla p(x)|$ over $\overline{\mathcal{D}}$ occurs at some point $x_M \in \overline{\mathcal{D}}$. Obviously, the maximum is positive. Let $\eta = \frac{\nabla p(x_M)}{|\nabla p(x_M)|}$. Because $x_M \in \overline{\mathcal{D}}$, Proposition (1.2.1), which holds for $\overline{\mathcal{D}}$ as well as $\mathcal{D}$, implies that $x_M$is the vertex of a cone $C \subset \overline{\mathcal{D}}$ having radius r, axis along a direction ξ, and angle $\theta = 2\arcsin\left(\frac{r}{2R}\right)$. We may adjust the sign of p so that $\eta \cdot \xi \geq 0$. By looking at the intersection of the cone C with a plane containing ξ and η, we see that there is a unit vector ζ pointing into the cone and satisfying $\eta \cdot \zeta \geq \cos(\pi/2 - \theta) = \sin(\theta)$. It follows that

$$|\nabla p(x_M)| = \frac{\partial p}{\partial \eta}(x_M) \leq \csc(\theta) \frac{\partial p}{\partial \zeta}(x_M).$$

On the other hand, for $t \in \mathbb{R}, \tilde{p}(t) = p(x_M + t\zeta)$ is in $\pi_\ell(\mathbb{R})$. In particular, it obeys the usual Bernstein inequality on $0 \leq t \leq r$

$$|\tilde{p}'(t)| \leq (2\ell^2/r) \max_{t\in[0,r]} |\tilde{p}'(t)| \leq (2\ell^2/r)\|p\|_{L_\infty(\mathcal{D})}.$$

Since $\tilde{p}'(0) = \frac{\partial p}{\partial \zeta}(xM)$, we have for all $x \in \overline{\mathcal{D}}$,

$$|\nabla p(x)| \leq |\nabla p(x_M)| \leq \csc(\theta)\frac{\partial p}{\partial \zeta}(x_M) \leq \frac{2\ell^2}{r\sin(\theta)}\|p\|_{L_\infty(\mathcal{D})} \tag{2}$$

Noting that $|\frac{\partial p}{\partial x_j}| \leq |\nabla p(x)|$ and keeping track of polynomial degrees as we differentiate, we arrive at the following result.

Proposition (1.1.3) [18]:

With the notation and assumptions of Proposition 2.1, if $p \in \pi_\ell(\mathbb{R}^n)$ and if α is a multi-index for which $|\alpha| \leq \ell$, then

$$\|D^\alpha p\|_{L_\infty(\mathcal{D})} \leq \frac{2^{|\alpha|}(\ell!)^2}{r^{|\alpha|}\sin^{|\alpha|}(\theta)((\ell-|\alpha|)!)^2}\|p\|_{L_\infty(\mathcal{D})} \leq \left(\frac{2\ell^2}{r\sin(\theta)}\right)^{|\alpha|}\|p\|_{L_\infty(\mathcal{D})}.$$

Proposition (1.1.4) [18]:

Let $p \in \pi_\ell(\mathbb{R}^2)$ and let $\mathcal{D}$ be a bounded domain that is star- shaped with respect to a ball $B(x_c, r)$ and also contained in a ball $B(x_c, R)$. If the mesh norm h for $X = \{x_1, \dots, x_N\}$ in $\mathcal{D}$ satisfies

$$h \leq \frac{r\sin(\theta)}{4(1+\sin(\theta))\ell^2} \tag{3}$$

then there exist complex numbers $a_j(x)$ such that for any multi-index α with $|\alpha| \leq \ell$

$$D^\alpha p(x) = \sum_{j=1}^{N} a_j^\alpha(x)p(x_j)$$

where

$$\sum_j |a_j^\alpha (x)| \le \frac{2^{|\alpha|}(\ell!)^2}{r^{|\alpha|}\sin^{|\alpha|}(\theta)((\ell-|\alpha|)!)^2} \le \left(\frac{2\ell^2}{r\sin(\theta)}\right)^{|\alpha|}$$

Proof::

See [17, Proposition 3.6] and [11, Lemma 6.2].

Remark (1.1.5) [18]:

The result derived in [17] is stated with h taken to be the mesh norm of X relative to $\mathcal{D}$. In fact, in the proof of the result, h is only required to satisfy the condition that every ball $B(x,h) \subset \mathcal{D}$ contains at least one point in , rather than being the mesh norm. This will be useful later.

In [1], Brenner and Scott discuss approximating a function $u \in W_p^k(\mathcal{D})$ by averaged Taylor polynomials $Q^k u \in \pi_{k-1}(\mathbb{R}^n)$In this section, we briefly summarize their discussion and extend some of their results.

The averaged Taylor polynomials are defined as follows. Let B_ρ be a ball relative to which $\mathcal{D}$ is star-shaped and having radius $\rho \ge \frac{1}{2}\rho_{\max}$, the largest radius of a ball relative to which $\mathcal{D}$ is star-shaped. In particular, we have $d_{\mathcal{D}}/\rho \le 2\gamma$, where γ is the chunkiness parameter. The averaged Taylor polynomials are then given by

$$Q^k u(x) = \sum_{|\alpha|<k} \frac{1}{\alpha!}\int_{B_\rho} D^\alpha u(y)(x-y)^\alpha \varphi(y)dy\,.$$

Here $\varphi(y) \ge 0$ is a C^∞ "bump" function supported on $B_\rho$and satisfying both $\int_{B_\rho} \varphi(y)dy = 1$ and $\max \varphi \le C\rho^{-n}$, where $C = C_n$. Finally, the remainder $R^k u$ is defined by

$$R^k u = u - Q^k u.$$

The following result provides a bound on Rk u.

Proposition (1.1.6) [18]:

For $u \in W_p^k(\mathcal{D})$, with $1 < p < \infty$ and $k > n/p$ or with $p = 1$ and $k \ge n$,

$$\|R^k u\|_{L_\infty(\mathcal{D})} \le C_{k,n,p}(1+\gamma)^n d_{\mathcal{D}}^{k-n/p}|u|_{W_p^k(\mathcal{D})}.$$

Where $C_{k,n,p} = C_{n,p}\frac{n^{k-1}}{(k-1)!}(k-\frac{n}{p})^{1/p-1}$ if $p > 1$ and $C_{k,n,1} = C_{n,1}\frac{n^{k-1}}{(k-1)!}$ if $p = 1$.

Proof:

See Brenner and [1]. We remark that we have tracked down and made explicit the dependence on γ and k of the constant $C_{k,n,\gamma,p}$ used in [1]. In the process, we employed the identity $\sum_{|\alpha|=k}\frac{k!}{\alpha!}=n^k$.

To deal with fractional Sobolev spaces, we need a version of the previous result that applies when u belongs to $W^{k+s}(\mathcal{D})$, where $0<s<1$. We begin with this lemma.

Lemma (1.1.7) [18]:

For $1<p<\infty$ and $k>n/p$ or $p=1$ and $k\geq n$, if $u\in W^{k+s}(\mathcal{D})$, and $P\in\pi_k(\mathbb{R}^n)$, then

$$\|R^{k+1}u\|_{L_\infty(\mathcal{D})}\leq C_{k,n,p}(1+\gamma)^n d_{\mathcal{D}}^{k-n/p}|u-P|_{W_p^k(\mathcal{D})}. \tag{4}$$

Proof:

We begin by noting that if P is in $\pi_k(\mathbb{R}^n)$, then $Q^{k+1}P=P$; that is, Q^{k+1} reproduces polynomials of degree k. Thus, $R^{k+1}u=R^{k+1}(u-P)$. The obvious identity $R^{k+1}u=R^ku+Q^ku-Q^{k+1}u$ then implies that

$$R^{k+1}u=R^{k+1}(u-P)=R^k(u-P)+(Q^k-Q^{k+1})(u-P).$$

By the triangle inequality and Proposition (1.1.6), we obtain

$$\begin{aligned}\|R^{k+1}u\|_{L_\infty(\mathcal{D})}\leq{}& C_{k,n,p}(1+\gamma)^n d_{\mathcal{D}}^{k-n/p}|u-P|_{W_p^k(\mathcal{D})}\\ &+\|(Q^k-Q^{k+1})(u-P)\|_{L_\infty(\mathcal{D})}\end{aligned} \tag{5}$$

The second of the two terms can be estimated as follows. First, from the definition of Q^k, the fact that $\max\varphi\leq C\rho^{-n}$, and the identity $\sum_{|\alpha|=k}\frac{1}{\alpha!}=\frac{n^k}{k!}$ we get

$$\begin{aligned}\|(Q^k-Q^{k+1})(u-P)\|_{L_\infty(\mathcal{D})}&\leq\sup_{x\in\mathcal{D}}\sum_{|\alpha|=k}\int_{B_\rho}\frac{\varphi(y)|x-y|^k|D^\alpha(u-P)(y)|}{\alpha!}dy\\ &\leq d_{\mathcal{D}}^k\cdot C\rho^{-n}\frac{n^k}{k!}\max_{|\alpha|=k}\int_{B_\rho}|D^\alpha(u-P)(y)|dy.\end{aligned}$$

Applying Holder's inequality to the integral above, we see that

$$\begin{aligned}\int_{B_\rho}|D^\alpha(u-P)(y)|dy&\leq\mathrm{vol}(B_\rho)^{1-1/p}\|D^\alpha(u-P)\|_{L^p(B_\rho)}\\ &\leq\mathrm{vol}(B_1)^{1-1/p}\rho^{n-n/p}\|D^\alpha(u-P)\|_{L^p(\mathcal{D})}\\ &\leq\mathrm{vol}(B_1)^{1-1/p}\rho^{n-n/p}|u-P|_{W_p^k(\mathcal{D})}.\end{aligned}$$

Combining these inequalities and using $d_{\mathcal{D}}/\rho\leq 2\gamma$, we arrive at the estimate

$$\|(Q^k-Q^{k+1})(u-P)\|_{L_\infty(\mathcal{D})}\leq C\mathrm{vol}(B_1)^{1-1/p}\frac{2^{n/p}n^k}{k!}d_{\mathcal{D}}^{k-n/p}\gamma^{n/p}|u-P|_{W_p^k(\mathcal{D})}.$$

Obviously, $\gamma^{n/p} \leq (1+\gamma)^n$. Consequently, putting the inequality above together with (5) yields (4).

Proposition (1.1.8) [18]:

Let $0 < s \leq 1$. For $1 < p < \infty$ and $k > n/p$ or $p = 1$ and $k \geq n$, if $u \in W_p^{k+s}(\mathcal{D})$, then

$$\|R^{k+1}u\|_{L_\infty(\mathcal{D})} \leq C_{k,n,p}(1+\gamma)^{n(1+1/p)} d_{\mathcal{D}}^{k+s-n/p} |u|_{W_p^{k+s}(\mathcal{D})}. \tag{6}$$

Proof:

The case $s = 1$ is a consequence of Proposition (1.1.6), so we may assume that $s < 1$. Let $P = Q^{k+1}u$ and note that $P \in \pi_k(\mathbb{R}^n)$. The identity,

$$D^\beta Q^m u = Q^{-m-|\beta|} D^\beta u, \tag{7}$$

which is found in [1], holds for $|\beta| \leq m-1$. In particular, if we take $\beta = \alpha, |\alpha| = k$ and $m = k+1$, then we have

$$D^\alpha Q^{k+1} u = Q^1 D^\alpha u \int_{B_\rho} \varphi(y) D^\alpha u(y)\, dy,$$

which is of course a constant. Since $\int_{B_\rho} \varphi(y)\, dy = 1$ we note that

$$D^\alpha u - D^\alpha Q^{k+1} u = \int_{B_\rho} \varphi(y)\, (D^\alpha u(x) - D^\alpha u(y)) dy$$

From this, a simple manipulation, bounds on φ and $|x-y| \leq d_{\mathcal{D}}$, and Hölder's inequality, it follows that

$$\begin{aligned} |D^\alpha u - D^\alpha Q^{k+1} u| &\leq \int_{B_\rho} \varphi(y)\, |x-y|^{s+n/p} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{s+n/p}} dy \\ &\leq C \rho^{-n} d_{\mathcal{D}}^{s+n/p} \int_{B_\rho} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{s+n/p}} dy \\ &\leq C_{n,p} \rho^{-n} d_{\mathcal{D}}^{s+n/p} \left\| \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{s+n/p}} \right\|_{L^p(\mathcal{D},dy)} \end{aligned}$$

Raise both sides to the power p. Integrate in x over $\mathcal{D}$ and sum over all $|\alpha| = k$. The result is

$$|u-P|^p_{W_p^{k+s}(\mathcal{D})} \leq C^p_{n,p} d_{\mathcal{D}}^{sp+n} \rho^{-n} \sum_{|\alpha|=k} \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+sp}} dxdy$$

The double integral on the right is just $|u|^p_{W_p^{k+s}(\mathcal{D})}$Again using $d_{\mathcal{D}}/\rho \leq 2\gamma$ and taking the p^{th} root of both sides, we obtain

$$|u-P|_{W_p^k(\mathcal{D}),} \leq 2^{n/p} C_{n,p} d_{\mathcal{D}}^s \gamma^{n/p} |u|_{W_p^{k+s}(\mathcal{D})}$$

Applying Lemma (1.1.7) yields the result

Corollary (1.1.9) [18]:

Let $0 < s \le 1$. For $u \in W_p^{k+s}(\mathcal{D})$,

$$\|D^\alpha u - D^\alpha Q^{k+1} u\|_{L_\infty(\mathcal{D})} \le C_{k,n,p}(1+\gamma)^{n(1+1/p)} d_{\mathcal{D}}^{k+s-|\alpha|-n/p} |u|_{W_p^{k+s}(\mathcal{D})}$$

provided that $1 < p < \infty$ and $k > |\alpha| + n/p$, or $p = 1$ and $k \ge |\alpha| + n$.

Proof:

This follows directly from Proposition (1.1.8), the identity (7), and the inequality $|D^\alpha u|_{W_p^{k+s-|\alpha|}(\mathcal{D})} \le |u|_{W_p^{k+s}(\mathcal{D})}$.One can use function-space interpolation theory to prove Proposition (1.1.8) and Corollary (1.1.9). Indeed, the proofs are somewhat simpler. There is a difficulty in doing this, however. The fractional Sobolev norms then also must come from interpolation of integer Sobolev spaces. While these are known to be equivalent to the intrinsic fractional norms we employ here, determining the dependence of the equivalence constants on the parameters of $\mathcal{D}$is problematic.

We are now ready to establish Sobolev bounds for functions with scattered zeros in $\mathcal{D}$. Suppose that $X \subset \mathcal{D}$ is finite and has a mesh norm h satisfying the conditions in Proposition (1.1.4). In addition, with $0 < s \le 1$, suppose that $u \in W^{k+s}(\mathcal{D})$ satisfies $u|_X = 0$, where $k > n/p$ or, if $p = 1, k \ge n$. Let $v = u - Q^{k+1}u$. Note that if $x_j \in X$, $v(x_j) = u(x_j) - (Q^{k+1}\, u)(x_j) = -(Q^{k+1}\, u)(x_j)$. By Proposition (1.1.4), with $\ell = k$, we thus have for each $x \in \mathcal{D}$,

$$D^\alpha(Q^{k+1}\, u)(x) = -\sum_{j=1}^{N} a_j^\alpha(x) v(x_j)$$

and hence that

$$|D^\alpha(Q^{k+1}\, u)(x)| \le \left(\sum_{j=1}^{N} |a_j^\alpha(x)|\right) \max_{x_j \in X} |v(x_j)|$$

$$\le 2\left(\frac{2k^2}{r\sin(\theta)}\right)^{|\alpha|} \|u - Q^{k+1}u\|_{L_\infty(\mathcal{D})}$$

$$\le 2\left(\frac{2k^2}{r\sin(\theta)}\right)^{|\alpha|} C_{k,n,p}(1+\gamma)^{n(1+1/p)} d_{\mathcal{D}}^{k+s-n/p} |u|_{W_p^{k+s}(\mathcal{D})}.$$

where the last step follows from Proposition (1.1.8).

Next, let α be a multi-index satisfying $k > |\alpha| + n/p$, or $p = 1$ and $k \ge |\alpha| + n$.

From Corollary (1.1.9), the previous inequality, and the triangle inequality, we have

$$\|D^\alpha u\|_{L_\infty(\mathcal{D})} \leq \left\{1 + 2\left(\frac{2k^2 d_{\mathcal{D}}}{r\sin(\theta)}\right)^{|\alpha|}\right\} C_{k,n,p}(1+\gamma)^{n(1+1/p)} d_{\mathcal{D}}^{k+s-|\alpha|-n/p} |u|_{W_p^{k+s}(\mathcal{D})}$$

Now, $1 \leq \gamma \leq \frac{d_{\mathcal{D}}}{r} \leq \frac{2R}{r} = \csc(\theta/2)$, $\sin(\theta/2) \leq \sin(\theta)$, and so we have that

$$\|D^\alpha u\|_{L_\infty(\mathcal{D})} \leq 3C_{k,n,p} 2^{|\alpha|+n+n/p} k^{2|\alpha|} \csc^{2|\alpha|+n+n/p}(\theta/2)\, d_{\mathcal{D}}^{k+s-|\alpha|-n/p} |u|_{W_p^{k+s}(\mathcal{D})}$$

Collecting coefficients in this expression and simplifying, we obtain the following result.

Proposition (1.1.10) [18]:

Let k be a positive integer, $1 \leq p < \infty$, $0 < s \leq 1$, and let α be a multi-index satisfying $k > |\alpha| + n/p$, or, for $p = 1$, $k \geq |\alpha| + n$. Also, let $X \subset \mathcal{D}$ be a discrete set with mesh norm h satisfying (3). If $u \in W_p^{k+s}(\mathcal{D})$ satisfies $u|_X = 0$,

$$\|D^\alpha u\|_{L_\infty(\mathcal{D})} \leq 3C_{k,n,p,|\alpha|} \csc^{2|\alpha|+n+n/p}(\theta/2)\, d_{\mathcal{D}}^{k+s-|\alpha|-n/p} |u|_{W_p^{k+s}(\mathcal{D})},$$

where $C_{k,n,p,|\alpha|} = 3C_{k,n,p} 2^{|\alpha|+n(1+1/p)} k^{2|\alpha|}$

Corollary (1.1.11) [18]:

Let $1 \leq q < \infty$. With the notation and assumptions of Proposition (1.1.10), we have

$$|u|_{W_q^{|\alpha|}(\mathcal{D})} \leq C_{k,n,p,q} \csc^{2|\alpha|+n(1+\frac{1}{p})}(\theta/2)\, d_{\mathcal{D}}^{k+s-|\alpha|-n(1/q-1/p)} |u|_{W_p^{k+s}(\mathcal{D})}$$

Proof:

Since $\operatorname{card}\{\beta \in \mathbb{N}_0^n : |\beta| = |\alpha|\} = \binom{|\alpha|+n-1}{n-1} = O(|\alpha|^{n-1})$ and $\operatorname{vol}(\mathcal{D}) < C d_{\mathcal{D}}^n$ we find that

$$\begin{aligned}
|u|_{W_q^{|\alpha|}(\mathcal{D})} &\leq \binom{|\alpha|+n-1}{n-1}^{1/q} \operatorname{vol}(\mathcal{D})^{1/q} \max_{|\beta|=|\alpha|} \|D^\beta u\|_{L_\infty(\mathcal{D})} \\
&\leq C_{n,q,|\alpha|} d_{\mathcal{D}}^{n/p} \max_{|\beta|=|\alpha|} \|D^\beta u\|_{L_\infty(\mathcal{D})} \\
&\leq C_{k,n,p,q} \csc^{2|\alpha|+n(1+\frac{1}{p})}(\theta/2)\, d_{\mathcal{D}}^{k+s-|\alpha|-n(1/q-1/p)} |u|_{W_p^{k+s}(\mathcal{D})}.
\end{aligned}$$

We will now treat a domain $\Omega \subset \mathbb{R}^n$ that is bounded, has a Lipschitz boundary, and satisfies an interior cone condition, where the cone has a maximum radius R_0 and angle φ. Of course, the cone condition will be obeyed if we use any radius $0 < R \leq R_0$.

To begin, we need to cover Ω with domains that are star-shaped with respect to a ball. We will employ a construction due to Duchon [5]. Let

$$r = \frac{R\sin(\varphi)}{2(1+\sin(\varphi))} \text{ and } T_r = \left\{t \in \frac{2r}{\sqrt{n}}\mathbb{Z}^n : B(t,r) \subset \Omega\right\}, \tag{8}$$

where $\leq R_0$. Fix $x \in \Omega$. Duchon (see the proof of [5, Proposition 1]) shows that the cone $C_x \subset \Omega$ associated with x contains one of the balls $B(t,r)$, where $\frac{2r}{\sqrt{n}}\mathbb{Z}^n$. This of course implies that the set $T_r = \varphi$and, since $|t-x| < R$, that $C_x \subset B(t,R) \cap \Omega$. Moreover, the closed convex hull of $\{x\} \cup B(t,r)$ is contained in C_x , because C_x is itself convex.

Instead of fixing x, we now fix $t \in T_r$. Let $\mathcal{D}_t$ be the set of all $x \in \Omega$ such that the closed convex hull of $\{x\} \cup B(t,r)$ is contained in $\Omega \cap B(t,R)$. By construction, each $\mathcal{D}_t$is star-shaped with respect to $B(t,r)$. What we have shown above is that every $x \in \Omega$ is in some $\mathcal{D}_t$, so $\Omega \subset \bigcup_{t\in T_r} \mathcal{D}_t$. Of course, it is also true that $\mathcal{D}_t \subset \Omega$, so in fact we have that $\Omega = \bigcup_{t\in T_r} \mathcal{D}_t$

This implies several useful geometric facts. We have that the diameter of $\mathcal{D}_t$ satisfies $d_{\mathcal{D}_t} < 2R$ and that the angle of the cone θ in Proposition 2.1 is related to φ via $\theta = 2\arcsin(\frac{r}{2R}) = 2\arcsin(\frac{\sin(\varphi)}{4(1+\sin(\varphi)))})$. Also we have that $\#T_r$, the cardinality of T_r,, satisfies $\#T_r < \mathrm{vol}(\Omega)/\mathrm{vol}(B(t,r)) \leq C_{\Omega,n,\varphi} R^{-n}$.

There is one more thing that we need. Let χ_S denote the characteristic function of a set S. Because $\mathcal{D}_t \subset B(t,R)$, $\chi_{\mathcal{D}_t}(x)\ \chi_{B(t,R)}(x)$ for all $x \in \mathbb{R}^n$. By [5], there is a constant M_1 , which may be taken as $M_1 = M_1(\varphi,n)$, such that $\sum_{t\in T_t} \chi_{B(t,R)}(x) \leq M_1$ all $x \in \mathbb{R}^n$. Consequently, $\sum_{t\in T_t} \chi_{\mathcal{D}_t}(x) \leq M_1$. We summarize these remarks below.

Lemma (1.1.12) [18]:

With the notation introduced above, we have the following:

(1) Each $\mathcal{D}_t$is star-shaped with respect to the ball $B(t,r)$ and satisfies $B(t,r) \subseteq \mathcal{D}_t \subseteq \Omega \cap B(t,R)$, $d_{\mathcal{D}_t} < 2R$, and $\theta = 2\arcsin\left(\frac{\sin(\varphi)}{4(1+\sin(\varphi)))}\right)$.

(2) $\Omega = \bigcup_{t\in T_r} \mathcal{D}_t$ and $\#T_r < C_{\Omega,n,\varphi} R^{-n}$.

(3) There exists a constant $M_1 = M_1(\varphi,n)$ such that $\sum_{t\in T_t} \chi_{\mathcal{D}_t}(x) \leq M_1$ for all $x \in \mathbb{R}^n$.

We are now ready to obtain Sobolev bounds for functions having zeros at a finite subset $X \subset \Omega$, where we let $h = h_{X,\Omega}$ be the mesh norm of X in Ω. We will assume that h satisfies the following condition:

$$h \leq k^{-2} Q(\varphi) R_0 \quad \text{where} \quad Q(\varphi) = \frac{\sin(\varphi)\sin(\theta)}{8(1+\sin(\theta))(1+\sin(\varphi))}. \tag{9}$$

We note that $\theta = 2\arcsin\left(\frac{\sin(\varphi)}{4(1+\sin(\varphi))}\right)$, so that Q only depends on φ. If this as sumption holds, then we can take $= \frac{k^2 h}{Q(\varphi)}$, for then $R \leq R_0$. Moreover, from the definition of r in terms of $\varphi$and R given in (8), we see that $h = \frac{\sin(\theta)}{4k^2(1+\sin(\theta))}$. Hence, h satisfies (3) for $\ell = k$.

We point out that every ball $B(x, h) \subset \Omega$ contains at least one point in .
In particular, if we have $(x, h) \subset \mathcal{D}_t$, this is still the case. By Remark (1.1.5), if $h = h_{X,\Omega}$ satisfies (3), then the results proved earlier all hold with this h. That said, we now have the following estimate.

Theorem (1.1.13) [18]:

Let k be a positive integer, $0 < s \leq 1, 1 \leq p < \infty, 1 \leq q \leq \infty$, and let α be a multi-index satisfying $k > |\alpha| + n/p$, or $p = 1$ and $k \geq |\alpha| + n$. Also, let $X \subset \Omega$ be a discrete set with mesh norm h satisfying (9). If $u \in W_p^{k+s}(\Omega)$satisfies $u|_X = 0$

$$|u|_{W_q^{|\alpha|}(\Omega)} \leq C_{k,n,p,q,|\alpha|} h^{k+s-|\alpha|-n(1/q-1/p)} |u|_{W_p^{k+s}(\Omega)}, \tag{10}$$

where $(x)_+ = x$ if $x \geq 0$ and is 0 otherwise.

Proof:.

The case $q = \infty$ follows from Proposition (1.1.10) and the decomposition given in Lemma (1.1.12), $\Omega = \cup_{t\in T_r} \mathcal{D}_t$. Thus, we will assume $1 \leq q < \infty$. For such q, the decomposition $\Omega = \cup_{t\in T_r} \mathcal{D}_t$implies that we have

$$|u|_{W_q^{|\alpha|}(\Omega)} = \left(\sum_{|\beta|=|\alpha|} \int_\Omega |D^\beta u(x)|^q \, dx \right)^{1/q}$$

$$\leq \left(\sum_{t\in T_r} \sum_{|\beta|=|\alpha|} \int_{\mathcal{D}_t} |D^\beta u(x)|^q \, dx \right)^{1/q} = \left(\sum_{t\in T_r} |u|_{W_q^{|\alpha|}(\mathcal{D}_t)}^q \right)^{1/q}$$

$$\leq (\#T_r)^{(\frac{1}{q}-\frac{1}{p})_+} \left(\sum_{t\in T_r} |u|_{W_q^{|\alpha|}(\mathcal{D}_t)}^q \right)^{\frac{1}{q}},$$

where $\#T_r$ is the cardinality of $T_r$and where the last bound follows from standard inequalities relating p and q norms on finite dimensional spaces. Next, by this inequality and Corollary (1.1.11), where we use $d_{\mathcal{D}_t} < 2R = 2k^2h/Q(\varphi)$, we obtain

$$|u|_{W_q^{|\alpha|}(\Omega)} \leq C'_{k,n,p,q,|\alpha|,\varphi} (\#T_r)^{(\frac{1}{q}-\frac{1}{p})_+} h^{k+s-|\alpha|-n(1/q-1/p)} \left(\sum_{t\in T_r} |u|_{W_q^{|\alpha|}(\mathcal{D}_t)}^q \right)^{\frac{1}{q}},$$

for each $t \in T_r$. Now, since $\mathcal{D}_t \subset \Omega$, we have by Lemma 1.1.10,

$$\sum_{t\in T_r} |u|^p_{W_p^{k+s}(\mathcal{D}_t)} = \sum_{|\beta|=k} \int_\Omega \left(\sum_{t\in T_r} \chi_{\mathcal{D}_t}(x) \right) \int_{\mathcal{D}_t} \frac{\left|D^\beta u(x) - D^\beta u(y)\right|^p}{|x-y|^{n+sp}} dydx$$

$$\le M_1 \sum_{|\beta|=k} \int_\Omega \int_\Omega \frac{\left|D^\beta u(x) - D^\beta u(y)\right|^p}{|x-y|^{n+sp}} dydx$$

$$\le M_1 |u|^p_{W_p^{k+s}(\mathcal{D}_t)}$$

Putting these two inequalities together yields

$$|u|_{W_q^{|\alpha|}(\Omega)} \le M_1^{1/p} C'_{k,n,p,q,|\alpha|,\varphi} (\#T_r)^{(\frac{1}{q}-\frac{1}{p})_+} h^{k+s-|\alpha|-n\left(\frac{1}{q}-\frac{1}{p}\right)} |u|_{W_p^{k+s}(\Omega)}.$$

Now, by part (2) of Lemma 1.1.10 and $R = 2k^2h/Q(\varphi)$, we see that $\#T_r < Ch^{-n}$. Inserting this in the inequality above gives us

$$|u|_{W_q^{|\alpha|}(\Omega)} \le C_{k,n,p,q,|\alpha|,\varphi} h^{k+s-|\alpha|-n\left(\frac{1}{q}-\frac{1}{p}\right)-(\frac{1}{q}-\frac{1}{p})_+} |u|_{W_p^{k+s}(\Omega)},$$

Using $n\left(\frac{1}{q}-\frac{1}{p}\right) - n(\frac{1}{q}-\frac{1}{p})_+ = n(1/p-1/q)_+$in the previous inequality yields (10).

In practical situations, bounds on continuous norms, such as those we have investigated above, are less important than bounds on discrete norms. Our aim here is to obtain estimates similar to those in Theorem (1.1.13), again for $u|_X = 0$, but with continuous norms replaced by the discrete ones that we now define.

Let $Y = \{y_1, \ldots, y_M\}$ be a finite subset of Ω, and denote its separation radius by q_Y , its mesh norm by h_Y, and its mesh ratio by $\rho_Y = h_Y/q_Y$. Let $1 \le q \le \infty$. (Note that q is not the same quantity as q_Y.) For a continuous function u defined on Ω, define the norm $\ell_q(Y)$ by

$$\|u\|_{\ell_q(Y)} = \begin{cases} \left(\frac{1}{M} \sum_{j=1}^{M} \left|u(y_j)\right|^q \right)^{\frac{1}{q}} \text{for } 1 \le q < \infty, \\ \max\limits_{1\le j\le M} \left|u(y_j)\right| \text{for } q = \infty. \end{cases} \quad (11)$$

As before, we also define αq (Y)-derivative norms when u is in C k (Ω) and 1 ≤ q < ∞:

$$|u|_{w_q^k(Y)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{\ell_q(Y)} \right)^{1/q} \text{and} \|u\|_{w_q^k(Y)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|^q_{\ell_q(Y)} \right)^{1/q} \quad (12)$$

The $q = \infty$ norms are defined in the obvious way. We now state the analog of Theorem (1.1.13) for the discrete norms.

Theorem (1.1.14) [18]:

Let k be a positive integer, $0 < s \le 1, 1 \le p < \infty, 1 \le q \le \infty$, and let α be a multi-index satisfying $k > |\alpha| + \frac{n}{p}$, or $p = 1$ and $k \ge |\alpha| + n$. Also, let $X \subset \Omega$ be a discrete set with mesh norm $h = h_X$ satisfying (9). Let $Y = \{y_1, \dots, y_M\} \subset \Omega$ be a second discrete set, with $h_X \le h$. If $u \in W_p^{k+s}(\Omega)$satisfies $u|_X = 0$ then

$$|u|_{W_q^{|\alpha|}(Y)} \le C_{k,n,p,q,|\alpha|,\varphi,\Omega}\rho_Y^{n/q} h^{k+s-|\alpha|-n(\frac{1}{p}-\frac{1}{q})_+} |u|_{W_p^{k+s}(\Omega)}, \qquad (13)$$

where the discrete norm on the left above is defined in (12). In particular, if $|\alpha| = 0$, then

$$\|u\|_{\ell_q(Y)} \le C_{k,n,p,q,\varphi,\Omega}\rho_Y^{n/q} h^{k+s-|\alpha|-n(\frac{1}{p}-\frac{1}{q})_+} |u|_{W_p^{k+s}(\Omega)}.$$

Proof:

The $q = \infty$ case is a direct consequence of Theorem (1.1.13) and $\rho_Y \ge 1$. We therefore assume that $q < \infty$. Let $\mathcal{D}_t$be one of the star-shaped domains from the decomposition of Ω given in Lemma (1.1.12). From the L_∞ bound in Proposition (1.1.10), the conditions on $\mathcal{D}_t$in Lemma (1.1.12), and the fact that $d_{\mathcal{D}_t} \le 2R = 2k^2\, h/Q(\varphi)$, we have that

$$\sum_{y_j \in \mathcal{D}_t} \left|D^\alpha u(y_j)\right|^q \le C h^{(k+s-|\alpha|)q-nq/p} \operatorname{card}(\mathcal{D}_t \cap Y) |u|^q_{W_p^{k+s}(\mathcal{D}_t)}$$

To estimate $\operatorname{card}(\mathcal{D}_t \cap Y)$, we note that every point $y_j$in $\mathcal{D}_t \cap Y$ is the center of the ball $B(y_j, q_Y)$. Now, by construction, $\mathcal{D}_t \subset B(t, R)$ and $q_Y \le h_Y \le h \le R$, so every $B(y_j, q_Y) \subset B(t, 2R)$. Hence, the number of points in $\mathcal{D}_t \cap Y$ satisfies the bound

$$\operatorname{card}(\mathcal{D}_t \cap Y) \le \frac{\operatorname{vol}(B(t,2R))}{\operatorname{vol}\left(B(y_j, q_Y)\right)} = \left(\frac{2R}{q_Y}\right)^n.$$

Recall from the previous section that we chose $= \frac{k^2 h}{Q(\varphi)}$, and so we have

$$\sum_{y_j \in \mathcal{D}_t} \left|D^\alpha u(y_j)\right|^q \le q_Y^{-n} h^{(k+s-|\alpha|)q+n-nq/p} C' |u|^q_{W_p^{k+s}(\mathcal{D}_t)}$$

where C' depends on $n, p, q, \phi, |\alpha|$. Sum over $t \in T_r$ on both sides. Since every $y_j \in Y$ is in at least one $\mathcal{D}_t$, we have

$$\sum_{j=1}^{M} \left|D^\alpha u(y_j)\right|^q \le \sum_{t \in T_r} \sum_{y_j \in \mathcal{D}_t} \left|D^\alpha u(y_j)\right|^q$$

$$\le q_Y^{-n} h^{(k+s-|\alpha|)q+n-\frac{nq}{p}} C'{\sum_{t\in T_r} |u|^q_{W_p^{k+s}(\mathcal{D}_t)}}$$

The sum on the left above is $M\|D^\alpha u\|^q_{\ell_q(Y)}$. To deal with the sum on the right, note that standard inequalities relating p and q norms on a finite dimensional space give

$$\sum_{t\in T_r} |u|^p_{W_p^{k+s}(\mathcal{D}_t)} \le (\#T_r)^{q(\frac{1}{p}-\frac{1}{q})_+}\left(\sum_{t\in T_r} |u|^p_{W_p^{k+s}(\mathcal{D}_t)}\right)^{q/p}$$

The sum $\sum_{t\in T_r}|u|^p_{W_p^{k+s}(\mathcal{D}_t)}$was dealt with in proving Theorem (1.1.13), where we showed that it is bounded by $M_1|u|^p_{W_p^{k+s}(\Omega)}$. Also, recall that $\#T_r < Ch^{-n}$. Using these bounds in our earlier inequality and dividing by , we obtain

$$\|D^\alpha u\|^q_{\ell_q(Y)} \le M^{-1} q_Y^{-n} h^{(k+s-|\alpha|)q+n-nq(\frac{1}{p}-\frac{1}{q})_+} C' C^q M_1^{q/p} |u|^q_{W_p^{k+s}(\Omega)}.$$

Summing over all multi-indices α of fixed length, simplifying the exponent of h, and suppressing constants, we arrive at

$$|u|^q_{w_q^{|\alpha|}(Y)} \le M^{-1} q_Y^{-n} h^{(k+s-|\alpha|)q-nq(\frac{1}{p}-\frac{1}{q})_+} C'' |u|^q_{W_p^{k+s}(\Omega)}. \tag{14}$$

Our last task is to estimate , the number of points in Y , from below. Since the mesh norm of Y relative to Ω is h_Y, every $x\in\Omega$ is in one of the closed balls $\overline{B(y_j, h_Y)}$, and so their union covers Ω. It follows that the number of such balls, , satisfies $M \ge \mathrm{vol}(\Omega)/\mathrm{vol}(B(y_j, h_Y))$ or, equivalently,

$$M^{-1} \le \frac{\mathrm{vol}(B(y_j, h_Y))}{\mathrm{vol}(\Omega)} \le C_{\Omega,n} h_Y^n.$$

Insert this in (14), simplify, and collect constants. Taking the q^{th} root of both sides then completes the proof.

In this section, we will apply the estimates that we obtained in the previous section to obtain error estimates for both continuous and discrete least squares RBF surface fitting in a domain Ω in $\mathbb{R}^n$. We make the same assumptions on Ω as we did above; namely, Ω is bounded, has a Lipschitz boundary, and satisfies an interior cone condition, where again the cone is assumed to have a maximum radius R0 and angle φ.

We will concentrate on radial basis functions $\Phi: \mathbb{R}^n \to \mathbb{R}$ that have a positive, algebraically decaying Fourier transform. To be more precise, we assume that

$$c_1(1+\|\omega\|_2^2)^{-\tau} \le \widehat{\Phi}(\omega) \le c_2(1+\|\omega\|_2^2)^{-\tau}, \quad \omega\in\mathbb{R}^n, \tag{15}$$

where $c_1, c_2 > 0$ are some constants and $\tau > n/2$. In this case it is well known that the native space $\mathcal{N}_{\Phi} = \mathcal{N}_{\Phi}(\mathbb{R}^n)$ associated to Φ is the Sobolev space

$$W_2^{\tau}(\mathbb{R}^n) := \{f \in L_2(\mathbb{R}^n): \hat{f}(\cdot)(1 + \|\cdot\|_2^2)^{\tau/2} \in L_2(\mathbb{R}^n)\} \qquad (16)$$

and the native space norm

$$\|f\|_{\mathcal{N}_{\Phi}}^2 := \int_{\mathbb{R}^n} \frac{|\hat{f}(\omega)|^2}{\widehat{\Phi}(\omega)} d\omega$$

is obviously equivalent to the Sobolev norm

$$\|f\|_{W_2^{\tau}(\mathbb{R}^n)}^2 := \left\|\hat{f}(\cdot)(1 + \|\cdot\|_2^2)^{\tau/2}\right\|_{L_2(\mathbb{R}^n)}. \qquad (17)$$

Later on, we will also deal with the case of thin-plate splines. The details of treating them differ somewhat from the more usual RBF case above. So, even though their treatment is in fact easier, they will be handled separately. Until then, we assume that the RBF Φ has a Fourier transform $\widehat{\Phi}$satisfying (15).

As is well known, the great utility in RBFs is that for any finite subset $X = \{x_1, \dots, x_N\} \subset \mathbb{R}^n$and arbitrary complex numbers $\{d_1, \dots, d_N\}$, one can find a unique function v from the span of $V_{X,\Phi} = \text{span}\{\Phi(x - x_j)\}_{j=1}^N$ such that $v(x_j) = d_j, j = 1, \dots, N$. In addition, interpolants satisfy a minimum principle. If f is in the native space $\mathcal{N}_{\Phi}$and if we let the interpolant to f on X from $V_{X,\Phi}$ be $I_X f$, then

$$\min_{v \in V_{X,\Phi}} \|f - v\|_{\mathcal{N}_{\Phi}} = \|f - I_X f\|_{\mathcal{N}_{\Phi}}.$$

In particular, since we can take $v = 0$ on the left above, we also have

$$\|f - I_X f\|_{\mathcal{N}_{\Phi}} \le \|f\|_{\mathcal{N}_{\Phi}}.$$

These observations lead to the following lemma, which we will need in the sequel.

Lemma (1.1.15) [18]:

Let $\tau > n/2$, $f \in W_2^{\tau}(\Omega)$, $X = \{x_1, x_2, \dots, x_N\} \subset \Omega$, and let $I_X f \in V_{X,\Phi}$ be the unique function that interpolates f on X . If $\widehat{\Phi}$ satisfies (15), then there exists a constant $C_{\Omega,\Phi}$, depending on Ω and Φ, such that

$$\|f - I_X f\|_{W_2^{\tau}(\Omega)} \le C_{\Omega,\Phi} \|f\|_{W_2^{\tau}(\Omega)}.$$

Proof:

We will require extension theorems for $W_2^{\tau}(\Omega)$, where Ω is a bounded Lipschitz domain. For the case in which τ is a nonnegative integer we may use the extension operator $\mathfrak{E}$ constructed by Stein [14] to extend any f in $W_2^{\tau}(\Omega)$ to a function defined for $1 \le p \le \infty$. Brenner and Scott [1] give on a brief discussion concerning extensions for fractional Sobolev spaces (i.e., $\tau \notin \mathbb{Z}$). They point out that combining results of DeVore and Sharpley [4] immediately yields the existence of $\mathfrak{E}$ in the fractional case as well, provided only that $1 \le p < \infty$. In particular, $\mathfrak{E}$ exists for $p = 2$, the value of p we are concerned with here.

Since $\mathfrak{E} f = f$ on Ω and since the values of $f|_X$ uniquely determine the interpolant from$V_{X,\Phi}$, we have that$I_X \mathfrak{E} f = I_X f$. Consequently, we obtain this chain of inequalities:

$$\begin{aligned}\|f - I_X f\|_{W_2^\tau(\Omega)} &= \|\mathfrak{E} f - I_X \mathfrak{E} f\|_{W_2^\tau(\Omega)} \\ &\le \|\mathfrak{E} f - I_X \mathfrak{E} f\|_{W_2^\tau(\mathbb{R}^n)} \\ &\le c_2^{-1/2} \|\mathfrak{E} f - I_X \mathfrak{E} f\|_{\mathcal{N}_\Phi} \\ &\le c_2^{-1/2} \|\mathfrak{E} f\|_{\mathcal{N}_\Phi} \\ &\le (c_1 c_2)^{-1/2} \|\mathfrak{E} f\|_{W_2^\tau(\mathbb{R}^n)} \\ &\le (c_1 c_2)^{-1/2} \|\mathfrak{E}\| \|f\|_{W_2^\tau(\Omega)}\end{aligned}$$

Setting $C_{\Omega,\Phi} = (c_1 c_2)^{-1/2} \|\mathfrak{E}\|$ completes the proof.

We now employ this lemma and the results obtained in the previous section to derive bounds on $f - I_X f$, in both continuous and discrete norms, for the case $p = 2$.

Proposition (1.1.16) [18]:

Suppose $\tau = k + s$, where k is a positive integer and $0 < s \le 1$. Let α be a multi-index satisfying $k > |\alpha| + n/2$, and let $X \subset \Omega$ be a discrete set with mesh norm h satisfying (9). If $f \in W_2^\tau(\Omega)$ and if $1 \le q \le \infty$, then

$$|f - I_X f|_{W_q^{|\alpha|}(\Omega)} \le C_{k,n,q,|\alpha|,\Omega,\Phi} h^{\tau - |\alpha| - n(1/2 - 1/q)_+} \|f\|_{W_2^\tau(\Omega)}. \qquad (18)$$

In addition, the continuous least squares error ($q = 2$) satisfies the bound,

$$\min_{v \in V_{X,\Phi}} \|f - v\|_{L_2(\Omega)} \le C_{k,n,q,|\alpha|,\Omega,\Phi} h^\tau \|f\|_{W_2^\tau(\Omega)}. \qquad (19)$$

Proof:

Apply Theorem (1.1.13) to $= f - I_X f$, with $p = 2$. Using Lemma (1.1.15) then gives us (18). Since$I_X f \in V_{X,\Phi}$, we also have that

$$\min_{v \in V_{X,\Phi}} \|f - v\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)}. \qquad (20)$$

The estimate (19) then follows from (18) with $q = 2$and $|\alpha| = 0$.

The case where the discrete norm is to be bounded, rather than the continuous one, can be dealt with in a similar way

Proposition (1.1.17) [18]:

Suppose $\tau = k + s$, where k is a positive integer and $0 < s \leq 1$. Let α be a multi-index satisfying $k > |\alpha| + n/2$. Also, $let\ X \subset \Omega$ be a discrete set with mesh norm $h = h_X$ satisfying (9). Let $Y = \{y_1, \dots, y_M\} \subset \Omega$ be a second discrete set, with$h_Y \leq h$. If $f \in W_2^\tau(\Omega)$, and if $1 \leq q \leq \infty$, then

$$|f - I_X f|_{w_q^{|\alpha|}(Y)} \leq C_{k,n,q,|\alpha|,\Omega,\Phi} \rho_Y^{n/q} h^{\tau - |\alpha| - n(1/2 - 1/q)_+} \|f\|_{W_2^\tau(\Omega)}. \quad (21)$$

where $|\cdot|_{w_q^{|\alpha|}(Y)}$ is defined in (12). Also, the discrete least squares error satisfies the bound,

$$\min_{v \in V_{X,\Phi}} \|f - v\|_{\ell_2(\mathrm{Y})} \leq C_{k,n,\Omega,\Phi} \rho_Y^{n/2} h^\tau \|f\|_{W_2^\tau(\Omega)}. \quad (22)$$

Proof:

Apply Theorem (1.1.14) to $u = f - I_X f$. Using Lemma (1.1.15), with $p = 2$, then completes the proof. Again, because$I_X f \in V_{X,\Phi}$, we have that

$$\min_{v \in V_{X,\Phi}} \|f - v\|_{\ell_2(\mathrm{Y})} \leq \|f - I_X f\|_{\ell_2(\mathrm{Y})}. \quad (23)$$

The estimate (22) then follows from the interpolation estimate (21) with $|\alpha| = 0$ and $q = 2$

We remark that in both cases the interpolant is a good approximation to the least squares fit.

The RBFs we just discussed are all positive definite functions. The thin-plate splines, however, are RBFs that are conditionally positive definite functions. If $k > n/2$ is an integer, then we define the thin-plate spline corresponding to n and k as follows. For $\|x\|_2 \neq 0$, we let

$$\Phi_{n,k}(x) := c_{n,k} \begin{cases} \|x\|_2^{2k-n} \text{for } n \text{ odd}, \\ \|x\|_2^{2k-n} \log\|x\|_2 \text{for } n \text{ even}, \end{cases}$$

where $c_{n,k}$ is a constant chosen so that $\Phi_{n,k}$ is a fundamental solution of the iterated Laplacian. In terms of the distributional Fourier transform, this is equivalent to requiring that $\widehat{\Phi}_{n,k}(\omega) = \|\omega\|_2^{-2k}$, if $\omega = 0$.

The native space associated with $\Phi_{n,k}$is the Beppo-Levi space,

$$BL_k(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : D^\alpha f \in L_2(\mathbb{R}^n) \text{ for all } |\alpha| = k\},$$

which is equipped with the semi-inner product

$$(\mathrm{f}, \mathrm{g})_{\mathrm{BL_k}(\mathbb{R}^\mathrm{n})} = \sum_{|\alpha|=\mathrm{k}} \frac{\mathrm{k}!}{\alpha!} (\mathrm{D}^\alpha \mathrm{f}, \mathrm{D}^\alpha \mathrm{g})_{\mathrm{L_2}(\mathbb{R}^\mathrm{n})}$$

and induced semi-norm $|\cdot|_{\mathrm{BL_k}(\mathbb{R}^\mathrm{n})}$. For Beppo-Levi spaces on Ω, similar definitions apply. Both the semi-norm $|\cdot|_{\mathrm{BL_k}(\mathbb{R}^n)}$ and $|\cdot|_{\mathrm{BL_k}(\Omega)}$ are equivalent to the corresponding Sobolev semi-norms of order k.

An interpolant$I_X f$, which is associated with $\Phi_{n,k}$ and $f|_X$ from a continuous function, includes a polynomial piece $p \in \pi_{k-1}(\mathbb{R}^n)$ as well as a linear combination of $\mathrm{span}\{\Phi_{n,k}(x - x_j)\}_{j=1}^N$. That is, $I_X f$ is in

$$V_{X,n,k} = \pi_{k-1}(\mathbb{R}^n) \oplus \mathrm{span}\{\Phi_{n,k}(x - x_j)\}_{j=1}^N.$$

To insure that the interpolant exists, one must make the additional assumption that the finite set $X \subset \Omega$ is unisolvent for π_{k-1}). Under this assumption, the method reproduces polynomials in π_{k-1} . In addition, if f is in the native space $\mathrm{BL_k}(\mathbb{R}^n)$, then $I_X f$ minimizes the semi-norm $|f - v|_{\mathrm{BL_k}(\mathbb{R}^n)}$ among all $\in V_{X,n,k}$. As in the previous section, this implies that

$$|f - I_X f|_{\mathrm{BL_k}(\mathbb{R}^n)} \le |f|_{\mathrm{BL_k}(\mathbb{R}^n)}, \qquad f \in \mathrm{BL_k}(\mathbb{R}^n).$$

Now, an extension theorem of Duchon [5] shows the existence of a linear map $\mathfrak{E}_k : W_2^k(\Omega) \to \mathrm{BL_k}(\mathbb{R}^n)$such that for all $f \in W_2^k(\Omega)$we have $\mathfrak{E}_k f|_\Omega = f$ and $|\mathfrak{E}_k f|_{\mathrm{BL_k}(\mathbb{R}^n)} \le \|\mathfrak{E}_k\| |f|_{\mathrm{BL_k}(\Omega)}$. Essentially repeating the proof of our own Lemma (1.1.15) then yields the following:

Lemma (1.1.18) [18]:

Let $k > n/2$ be an integer, $X = \{x_1, x_2, \ldots, x_N\} \subset \Omega$ be unisolvent for $\pi_{k-1}(\mathbb{R}^n)$ and let $f \in W_2^k(\Omega)$. If $I_X f \in V_{X,n,k}$ is the unique function that interpolates f on X , then there exists a constant $C_{\Omega,n,k}$ such that

$$\left|f - I_X f\right|_{W_2^k(\Omega)} \le C_{\Omega,n,k} \left| f \right|_{W_2^k(\Omega)} \le C_{\Omega,n,k} \|f\|_{W_2^k(\Omega)}.$$

Recall that a finite, discrete set $X \subset \mathbb{R}^n$is unisolvent for the vanishing of p on X —i.e., $p|_X = 0$—implies that $p \equiv 0$. Suppose that we again have $X \subset \Omega$, with mesh norm h satisfying (9). We want to show that under these conditions we have the slightly stronger result that X is unisolvent with respect to $\pi_k(\mathbb{R}^n)$.

Proposition (1.1.19) [18]:

Let $k \ge 1$ be an integer. If X a finite, discrete subset of X , with mesh norm h satisfying (9), then X is unisolvent for$\pi_k(\mathbb{R}^n)$.

Proof:

In Theorem (1.1.13), take$s = 1$, $|\alpha| = 0$, $q = \infty$, $p = 2n$. If u is a polynomial in$\pi_k(\mathbb{R}^n)$, with$u|_X = 0$, then, we have that $\|u\|_{L_\infty(\Omega)} \le Ch^{k+1/2}|u|_{W_{2n}^{k+1}(\Omega)}$ Since

$D^{\alpha}u \equiv 0$ for $|\alpha| \geq k+1$, the norm $|u|_{W_{2n}^{k+1}(\Omega)}$. It then follows from Theorem (1.1.13) that $u|_{\Omega} = 0$ and, since Ω contains open sets, that $u \equiv 0$. Thus, X is unisolvent for $\pi_k(\mathbb{R}^n)$.

By our remarks above, the set X being unisolvent implies that for any $f \in C(\Omega)$, there is a unique interpolant$I_X f \in V_{X,n,k}$ for f . This plus the lemma above is precisely what we require to get the same type of estimates that we obtained in the last section. In fact, repeating the proofs of Propositions (1.1.16) and (1.1.7) yields the same estimates. We formally state these observations below.

Corollary (1.1.20) [18]:

Under the assumptions on X ,Ω, and h made in Propositions (1.1.16) and (1.1.17), the interpolant $I_X f \in V_{X,n,k}$ exists and is unique. Moreover, the estimates in both propositions also hold for$I_X f \in V_{X,n,k}$.

It is now our goal to establish discrete and continuous Sobolev-type error estimates for functions that are outside the native space, but still in a Sobolev space or a C^k -space. More precisely, if we let τ determine the decay of $\widehat{\Phi}$, we will assume either that $f \in W_2^t(\Omega)$, where $\tau \geq t > n/2$, or that $f \in C^k(\Omega)$, $\tau \geq k > n/2$.

For approximation rather than interpolation, such error estimates have been derived for integer τ in [16], using a technique introduced in [13]. We will extend this result to positive, real τ . The proof we give here is simpler than that given in [16]; it is based upon recent results from [10].

Lemma (1.1.21) [18]:

Let $t \geq r \geq 0$. If $f \in W_2^t(\mathbb{R}^n)$, then there exists a constant $c_{t,r}$ such that for every $\sigma > 0$ we can choose a band limited function $g_\sigma \in B_\sigma = \{f \in L_2(\mathbb{R}^n) : \operatorname{supp}(\hat{f}) \subseteq B(0,\sigma)\}$ with

$$\|f - g_\sigma\|_{W_2^t(\mathbb{R}^n)} \leq c_{t,r}\sigma^{r-t}\|f\|_{W_2^t(\mathbb{R}^n)}. \tag{24}$$

Obviously, this result is important mainly in the case of $\sigma > 1$, and in such a situation we will use it now.

Theorem (1.1.22) [18]:

Suppose Φ is a positive definite function satisfying (15), with $\tau \geq t > n/2$, and that $X = \{x_1, \dots, x_N\} \subset \Omega$ has mesh norm h satisfying (9). If $f \in W_2^t(\Omega)$, then there exists a function $v \in V_{X,\Phi} = \operatorname{span}\{\Phi(\cdot - x_j) : x_j \in X\}$ such that for every real $0 \leq r \leq t$,

$$\|f - v\|_{W_2^r(\Omega)} \leq Ch^{t-r}\|f\|_{W_2^t(\Omega)}.$$

Here, C is a constant independent of f and h.

Proof.

Let $\mathfrak{E}$ be the extension operator discussed in the proof of Lemma (1.1.15). We first extend the function $f \in W_2^t(\Omega)$ to a function $\mathfrak{E}f \in W_2^t(\mathbb{R}^n)$. Next we pick a band limited function g_σ that approximates $\mathfrak{E}f$ according to (24), with $\sigma = 1/h$. Finally, we let $v = I_X g_\sigma$. Then, we have

$$\|f - v\|_{W_2^r(\Omega)} \le \|\mathfrak{E}f - g_\sigma\|_{W_2^r(\mathbb{R}^n)} + \|g_\sigma - I_X g_\sigma\|_{W_2^r(\Omega}$$
$$\le c_1 h^{t-r}\|\mathfrak{E}f\|_{W_2^t(\mathbb{R}^n)} + c_2 h^{\tau-r}\|g_\sigma\|_{W_2^\tau(\Omega)}$$
$$\le c_3 h^{t-r}\|f\|_{W_2^t(\mathbb{R}^n)} + c_2 h^{\tau-r}\|g_\sigma\|_{W_2^\tau(\Omega)},$$

where we have used (24), Proposition (1.1.16), and the continuity of the extension operator $\mathfrak{E}$. To estimate the second term on the right, we observe that $\|g_\sigma\|_{W_2^\tau(\Omega)} \le \|g_\sigma\|_{W_2^\tau(\mathbb{R}^n)}$. Now, $g_\sigma$band-limited, and so $\|g_\sigma\|_{W_2^\tau(\mathbb{R}^n)} \le c\sigma^{\tau-t}\|g_\sigma\|_{W_2^t(\mathbb{R}^n)} = ch^{\tau-t}\|g_\sigma\|_{W_2^t(\mathbb{R}^n)}$. (This is trivial to show $p = 2$. It is, of course a special case of Bernstein's Theorem for functions of exponential type.) Another application of (24) and the continuity of $\mathfrak{E}$ establishes

$$\|g_\sigma\|_{W_2^t(\mathbb{R}^n)} \le \|\mathfrak{E}f\|_{W_2^t(\mathbb{R}^n)} + \|\mathfrak{E}f - g_\sigma\|_{W_2^t(\mathbb{R}^n)} \le c_4\|\mathfrak{E}f\|_{W_2^t(\mathbb{R}^n)} \le c_5\|f\|_{W_2^t(\Omega)}$$

Combining these bounds results in $\|g_\sigma\|_{W_2^\tau(\Omega)} \le c_5 h^{t-\tau}\|f\|_{W_2^t(\Omega)}$ Overall, this gives us the estimate

$$\|f - v\|_{W_2^r(\Omega)} \le (c_3 h^{t-r} + c_2 c_5 h^{\tau-r} h^{t-\tau})\|f\|_{W_2^t(\Omega)} \le Ch^{t-r}\|f\|_{W_2^t(\Omega)},$$

which is what we wished to show

We now turn to error estimates for interpolation of a function f in $W_2^k(\Omega)$ by the smoother functions in $V_{X,\varphi} \subset \mathcal{N}_\varphi$. In the special case of interpolation by means of an integer order thin-plate spline, Brown-lee and Light [2] have obtained L_p error estimates in terms of $|f|_{W_2^k(\Omega)}$ We will treat the general RBF case here, but we will need to work in the space $C^k(\overline{\Omega})$, rather than $W_2^k(\Omega)$.

We begin with a few remarks about the extension operator $\mathfrak{E}$ constructed by Stein[14]. Stein explicitly states that this operator maps $W_p^k(\Omega)$ boundedly To $W_p^k(\mathbb{R}^n)$ for $1 \le p \le \infty$ and for any integer$k \ge 0$. In fact, it does a little more than that. If $f \in C^k(\overline{\Omega})$, then Stein's construction yields $\mathfrak{E}f \in C^k(\mathbb{R}^n) \cap W_\infty^k(\mathbb{R}^n)$. Moreover, if $k > n/2$, then the fact that $\mathfrak{E}f \in W_2^k(\mathbb{R}^n)$ also implies that $\widehat{\mathfrak{E}f} \in L^1(\mathbb{R}^n)$ which in turn yields $\lim_{|x|\to\infty} \mathfrak{E}f(x) = 0$ course, we also have the norm bounds

$$\|\mathfrak{E}f\|_{W_2^k(\mathbb{R}^n)} \leq C_1\|f\|_{W_2^k(\Omega)} \text{ and } \|\mathfrak{E}f\|_{W_\infty^k(\mathbb{R}^n)} \leq C_1\|f\|_{C^k(\bar{\Omega})}$$

since$\|f\|_{W_2^k(\Omega)} \leq C_3\|f\|_{C^k(\bar{\Omega})}$,we have that

$$\max\left\{\|\mathfrak{E}f\|_{W_2^k(\mathbb{R}^n)}, \|\mathfrak{E}f\|_{W_\infty^k(\mathbb{R}^n)}\right\} \leq C\|f\|_{C^k(\bar{\Omega})}. \quad (25)$$

We will now make use of the extension $\mathfrak{E}f$ to obtain a band-limited interpolant to f on $X \subset \Omega$. For normalization purposes we will require $\operatorname{diam}(X) \leq 1$.

Lemma (1.1.23) [18]:

Let $f \in C^k(\bar{\Omega})$ and suppose that $X = \{x_1, x_2, \ldots, x_N\} \subset \Omega$ satisfies $\operatorname{diam}(X) \leq 1$. Let q_X be the separation radius of . Then, there is a constant c_n, depending only on the dimension n, such that, for any $\sigma \geq \frac{c_n}{q_X}$, there exists a band-limited function $f_\sigma \in \mathcal{B}_\sigma$ for which

$$f|_X = f_\sigma|_X \text{and} \|f_\sigma\|_{W_2^k(\mathbb{R}^n)} \leq C\|f\|_{C^k(\bar{\Omega})}. \quad (26)$$

Proof:

The extension $\mathfrak{E}f$ is in $C_0(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, so [10] gives us the existence of $f_\sigma$for which $\mathfrak{E}f|_X = f_\sigma|_X$. Since$\mathfrak{E}f|_\Omega = f$, we see that $f|_X = f_\sigma|_X$ .In addition, since$\mathfrak{E}f \in W_2^k(\mathbb{R}^n) \cap W_\infty^k(\mathbb{R}^n) \cap C^k(\mathbb{R}^n)$, [10, Proposition 3.12] provides the estimate

$$\|f_\sigma\|_{W_2^k(\mathbb{R}^n)} \leq \max\left\{\|\mathfrak{E}f\|_{W_2^k(\mathbb{R}^n)}, \|\mathfrak{E}f\|_{W_\infty^k(\mathbb{R}^n)}\right\}.$$

Applying (25) to bound the right side above then yields (26), which completes the proof

Theorem (1.1.24) [18]:

Let k and j be integers, with $0 \leq j < k \leq \tau$ and $k > n/2$, and let $f \in C^k(\bar{\Omega})$. Also suppose that $X = \{x_1, x_2, \ldots, x_N\} \subset \Omega$ satisfies $\operatorname{diam}(X) \leq 1$, with mesh norm h satisfying (9). Then,

$$|f - I_X f|_{W_q^j(\Omega)} \leq C\rho_X^{\tau-k} h^{k-j-n(1/2-1/q)_+}\|f\|_{C^k(\bar{\Omega})}, \quad (27)$$

where $\rho_X = \frac{h}{q_X}$ is the mesh ratio for X in Ω.

proof:

By Theorem (1.1.13), we have

$$|f - I_X f|_{W_q^j(\Omega)} \leq Ch^{k-j-n(1/2-1/q)_+}|f - I_X f|_{W_q^j(\Omega)} \quad (28)$$

Choosing $\sigma = \frac{c_n}{q_X}$ in Lemma (1.1.23), we have the existence of$f_\sigma \in \mathcal{B}_\sigma$ that interpolates f on X. Recall that the interpolation operator I_X depends only on $\mathrm{f}|_X = f_\sigma|_X$, so $I_X f = I_X f_\sigma$. Consequently, we have this chain of inequalities

$$|f - I_X f|_{W_2^k(\Omega)} = |f - I_X f_\sigma|_{W_2^k(\Omega)}$$
$$\leq |f - f_\sigma|_{W_2^k(\Omega)} + |f_\sigma - I_X f_\sigma|_{W_2^k(\Omega)}$$
$$\leq |f|_{W_2^k(\Omega)} + |f_\sigma|_{W_2^k(\Omega)} + |f_\sigma - I_X f_\sigma|_{W_2^k(\Omega)}.$$

By Proposition (1.1.16) (or, for the thin-plate splines, Corollary (1.1.20)), with f replaced by f_σ, q by 2, and so on, we have

$$|f_\sigma - I_X f_\sigma|_{W_2^k(\Omega)} \leq C h^{\tau-k} |f_\sigma|_{W_2^k(\Omega)}.$$

Obviously, $|f_\sigma|_{W_2^\tau(\Omega)} \leq \|f_\sigma\|_{W_2^\tau(\mathbb{R}^n)}$ By Bernstein's inequality for functions of Exponential type, $\|f_\sigma\|_{W_2^\tau(\mathbb{R}^n)} \leq c\sigma^{\tau-k} \|f_\sigma\|_{W_2^\tau(\mathbb{R}^n)}$. Hence , we have

$$|f - I_X f|_{W_2^k(\Omega)} \leq |f_\sigma|_{W_2^k(\Omega)} + (1 + C h^{\tau-k} \sigma^{\tau-k}) \|f_\sigma\|_{W_2^k(\mathbb{R}^n)}$$

However, $\sigma = \frac{c_n}{q_X}$, so

$$|f - I_X f|_{W_2^k(\Omega)} \leq |f_\sigma|_{W_2^k(\Omega)} + \left(1 + C c_n^{\tau-k} \rho_X^{\tau-k} h^{\tau-k}\right) \|f_\sigma\|_{W_2^k(\mathbb{R}^n)},$$

where we recall that $\rho_X = \frac{h}{q_X} \geq 1$ is the mesh ratio for X in Ω. By (26 In $\|f_\sigma\|_{W_2^k(\Omega)} \leq \|f\|_{C^k(\bar{\Omega})}$addition, we have the standard estimate$|f|_{W_2^k(\Omega)} \leq C' \|f\|_{C^k(\bar{\Omega})}$. Combining all of these and simplifying, we obtain

$$|f - I_X f|_{W_2^k(\Omega)} \leq \left(1 + C' + C c_n^{\tau-k} \rho_X^{\tau-k} h^{\tau-k}\right) \|f_\sigma\|_{C^k(\bar{\Omega})},$$
$$\leq C'' \rho_X^{\tau-k} \|f_\sigma\|_{C^k(\bar{\Omega})}.$$

Using this bound in (28) then gives us (27), which completes the proof.

Our final result is a corollary that deals with the discrete case, rather than the continuous one.

Corollary (1.1.25) [18]:

Let k and j be integers, with $0 \leq j < k \leq \tau$ and $k > n/2$, and let $f \in C^k(\bar{\Omega})$. Also suppose that $X = \{x_1, x_2, \ldots, x_N\} \subset \Omega$ satisfies $\operatorname{diam}(X) \leq 1$, with mesh norm h satisfying (9). In addition, let Y be a second discrete set, with $h_Y \leq h$. Then,

$$|f - I_X f_\sigma|_{w_q^j(Y)} \leq C \rho_X^{n/q} \rho_X^{\tau-k} h^{k-j-n(1/2-1/q)_+} \|f_\sigma\|_{C^k(\bar{\Omega})}, \tag{29}$$

where the discrete norm on the left above is defined in (11).

Proof:

The proof is nearly identical to the theorem above. The difference is that at the start one needs to use Theorem (1.1.14), which is the discrete version of Theorem (1.1.13).

Sec (1.2):Applications to (m, s) − Spline Interpolation and Smoothing:

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Let $r > 0$, $p \in [1, \infty)$ and $q \in [1, \infty]$, and let $W^{r,p}(\Omega)$ stand for the usual Sobolev space of order r contained in $L^p(\Omega)$. Likewise, for any finite set $\subset \overline{\Omega}$, let $\delta(A, \overline{\Omega})$ be the Hausdorff distance between A and $\overline{\Omega}$ (see (4.1)). Through the section [21] and [31],with minor corrections in [22] and [15], H. Wendland and co-workers have proved the following remarkable result (see Sect. 2 for the precise definition of the Sobolev semi-norms $| \cdot |_{l,q,\Omega}$ and $| \cdot |_{r,q,\Omega}$):

Theorem (1.2.1) [19]:

Assume that $k \geq n$, if $p = 1$, or $k > n/p$, if $p > 1$, where k stands for the integer part of r . Then, there exist two positive constants $\tilde{\delta}_r$ and C satisfying the following property: for any finite set $A \subset \overline{\Omega}$ such that $d = \delta(A, \Omega) \leq \tilde{\delta}_r$, for any $u \in W^{r,p}(\Omega)$, and for any non-negative integer l such that $l \leq k - n$, if $p = 1$, or $l < k - n/p$, if $p > 1$, we have

$$|u|_{l,q,\Omega} \leq C\left(d^{r-l-n(1/p-1/q)} + | \cdot |_{l,p,\Omega} + d^{-l}\|u|_A\|_\infty\right) \qquad (30)$$

where $\|u|_A\|_\infty = \max_{a\in A}|u(a)|$ and $(x)_+ = \max\{x, 0\}$.

As an immediate consequence, if u is null on the set A, one gets

$$|u|_{l,q,\Omega} \leq C\, d^{r-l-n(1/p-1/q)} + | \cdot |_{l,p,\Omega} \qquad (31)$$

which is really the main result in Narcowich et al. [40]. This latter bound has known several precursors in the literature. In a multivariate setting, we first quote the work of Duchon [30], where r is a positive integer and $p = 2$. For q=2, these results were extended to non-integer values of r by López de Silanes and Arcangéli [36] (see also [22]). We remark that Duchon's results and their extensions are actually particular cases of Proposition (1.2.8) and Corollary (1.2.9) in this section. We finally mention that Bezhaev and Vasilenko obtained (31) for $r \in \mathbb{N}$, $p = 2$ and $q \geq 2$
(cf. [24]).

Madych has also obtained bounds which, formally, are almost identical to (30) and (31) with l = 0 (cf. [38, 39]; the second section is co-authored by Potter). However, Madych's bounds are established in a different frame, Ω being a (possibly) unbounded open set satisfying a specific geometric condition, A a discrete set and u a function belonging to a suitable Beppo-Levi space.

In this section, our first goal is to extend Theorem (1.2.1) in two directions. On the one hand, we enlarge the set of admissible values of r , p and. This has significant consequences on the range of functions to which the result applies. On the other hand, we replace the term $d^{-l}\|u|_A\|_\infty$ by a more general one, better suited to the applications. To this end, we adapt the original approach of Duchon [30] and we develop some ideas that were already implicit in [36] (see also [22]). As a matter of fact, in [36] may be considered as a first statement of (30), in the particular case $p = q = 2$, but involving a term better than $d^{-l}\|u|_A\|_\infty$

Despite their intrinsic interest, Sobolev bounds like (30) and (31) find their main motivation in the obtaining of error estimates for approximation processes from Lagrange data, as can be easily verified from the reading of the above cited references. The present section will not be an exception. We shall derive error bounds for interpolating and smoothing (m, s)-splines, which include, as particular cases, the popular thin plate splines. Although our results are not completely new in the literature, to our knowledge, they have not been previously established in such great generality for the kind of splines considered here. (cf. [23, 37, and 48]).For any $\mathrm{x} \in \mathbb{R}$, we shall write $\lfloor \mathrm{x} \rfloor$ and $\lceil \mathrm{x} \rceil$ for the floor (or integer part) and ceiling of x , that is, the unique integers satisfying $\mathrm{x} \leq \mathrm{x} < x + 1$ and $x - 1 < x \leq x$. Likewise, as indicated in Theorem (1.2.1), we shall write$(\mathrm{x}\,)_+ = \max\{\mathrm{x}\,, 0\}$.

The letter n will always stand for an integer belonging to $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ (by convention, $0 \in \mathbb{N}$). The Euclidean norm in $\mathbb{R}^n$ will be denoted by $|\cdot|$. For any set $\mathcal{O}$ in $\mathbb{R}^n$, we shall write $\bar{\mathcal{O}}$ and $\chi_{\mathcal{O}}$, respectively, for the closure and the characteristic function of $\mathcal{O}$. The restriction to $\mathcal{O}$ of a function f defined over $\mathbb{R}^n$will be simply denoted by f instead of $|_{\mathcal{O}}$, unless this latter notation be strictly necessary. Finally, for any $t \in \mathbb{R}^n$ and for any $\delta > 0$, we shall denote by $B(t\,, \delta)$ and $\bar{B}(t\,, \delta)$, respectively, the open and closed balls with centre t and radius δ.

Given $N \in \mathbb{N}^*, x \in [1, \infty], b = (b_1\,, \ldots, b_N\,) \in (\mathbb{R}^n)^N$ and a real-valued function v defined on every b_j , we shall write

$$\|v|_b\|_x = \begin{cases} \left(\sum_{j=1}^{N} \left|v(b_j)\right|^x\right)^{1/x}, & \text{if } x < \infty, \\ \max\limits_{1\leq j\leq N} \left|v(b_j)\right| \quad, & \text{if } x = \infty. \end{cases}$$

Likewise, if the function v is defined on a finite subset B of $\mathbb{R}^n$, we let $\|v|_B\|_x = \|v|_b\|_x$, b being any card B-tuple obtained by ordering the elements of B .

Given $\mathcal{O} \subset \mathbb{R}^n$and $k \in \mathbb{N}$, we shall write $P_k(\mathcal{O})$ for the space of polynomial functions defined on $\mathcal{O}$of total degree less than or equal to k. If $= \mathbb{R}^n$, we shall simply write P_k.

For any open subset Ω of $\mathbb{R}^n$and for any $l \in \mathbb{N}$, we shall denote by $C^l(\overline{\Omega})$ the space of those functions which, together with all their partial derivatives of orders $\leq l$ are uniformly continuous and bounded in Ω . $C^l(\overline{\Omega})$ is a Banach space for the norm

$$\|v\|_{C^l(\overline{\Omega})} = \max_{|\alpha|\leq l} \sup_{x\in\Omega} |\partial^\alpha v(x)|,$$

where, for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\partial^\alpha = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}), x_1, \ldots, x_n$ being the generic independent variables in $\mathbb{R}^n$. If $v \in C^l(\overline{\Omega})$, we regard $\partial^\alpha v$, with $|\alpha| \leq l$, as defined on $\overline{\Omega}$, that is, we identify $\partial^\alpha v$ with its unique continuous extension to the closure of Ω . Finally, for any $\lambda \in (0,1]$, $C^{0,\lambda}(\overline{\Omega})$ stands for the subspace of $C^0(\overline{\Omega})$ consisting of functions satisfying in Ω a Hölder condition of exponent λ. This space is a Banach one endowed with the norm

$$\|v\|_{C^{0,\lambda}(\overline{\Omega})} = \|v\|_{C^0(\overline{\Omega})} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|v(x) - v(y)|}{|x - y|^\lambda}.$$

Let Ω be a non-empty open set in $\mathbb{R}^n$. For any $r \in \mathbb{N}$ and for any $p \in [1, \infty]$, we shall denote by $W^{r,p}(\Omega)$ the usual Sobolev space defined by

$$W^{r,p}(\Omega) = \{ v \in L^p(\Omega) | \forall\alpha \in \mathbb{N}^n, |\alpha| \leq r, \partial^\alpha v \in L^p(\Omega) \}.$$

We recall that the derivatives $\partial^\alpha v$ are taken in the distributional sense. The space $W^{r,p}(\Omega)$ is equipped with the semi-norms $|\cdot|_{j,p,\Omega}$, with $j \in \{0, \ldots, r\}$, and the norm $\|\cdot\|_{r,p,\Omega}$ given, if $p < \infty$, by

$$|v|_{j,p,\Omega} = \left(\sum_{|\alpha|=j} \int_\Omega |\partial^\alpha v(x)|^p dx \right)^{\frac{1}{p}} \; and \; \|v\|_{r,p,\Omega} = \left(\sum_{j=0}^{r} |v|_{j,p,\Omega}^p \right)^{\frac{1}{p}},$$

or, if $p = \infty$, by

$$|v|_{j,p,\Omega} = \max_{|\alpha|=j} \operatorname*{ess\,sup}_{x\in\Omega} |\partial^\alpha v(x)| \quad and \quad \|v\|_{r,\infty,\Omega} = \max_{0\leq j\leq r} |v|_{j,\infty,\Omega} .$$

If Ω is bounded, it follows from Theorem in Adams [20] that, for any $p_1, p_2 \in [1, \infty]$ such that$p_1 \leq p_2$, $W^{r,p_2}(\Omega) \subset W^{r,p_1}(\Omega)$ and, for any$v \in W^{r,p_2}(\Omega)$,

$$\forall j = 0, \cdots, r, \; |v|_{j,p_1,\Omega} \leq (\mathrm{meas}\Omega)^{1/p_1 - 1/p_2} |v|_{j,p_2,\Omega}. \tag{32}$$

For any $r \in (0,\infty)\backslash\mathbb{N}$ and for any $p \in [1,\infty]$, we shall denote by $W^{r,p}(\Omega)$ the Sobolev space of non-integer order , formed by the (equivalence classes of) functions $v \in W^{\lfloor r\rfloor,p}(\Omega)$ such that

$$|v|_{j,p,\Omega}^p = \sum_{|\alpha|=\lfloor r\rfloor} \int_{\Omega\times\Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^p}{|x-y|^{n+p(r-\lfloor r\rfloor)}} dx\, dy < \infty,$$

if $p < \infty$, and

$$|v|_{r,\infty,\Omega} = \max_{|\alpha|=\lfloor r\rfloor} \operatorname*{ess\,sup}_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|}{|x-y|^{r-\lfloor r\rfloor}} < \infty,$$

if $p = \infty$. Besides the semi-norms $|\cdot|_{j,p,\Omega}$, with $j \in \{0,\ldots,\lfloor r\rfloor\}$, and $|\cdot|_{r,p,\Omega}$, the space $W^{r,p}(\Omega)$ is endowed with the norm

$$\|v\|_{r,p,\Omega} = \begin{cases} \left(\|v\|_{\lfloor r\rfloor,p,\Omega}^p + |v|_{r,p,\Omega}^p\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max\left\{\|v\|_{\lfloor r\rfloor,\infty,\Omega}^p, |v|_{r,\infty,\Omega}\right\}, & \text{if } p = \infty. \end{cases}$$

For any $r \in [0,\infty)$, for any $p \in [1,\infty]$ and for any open subset Ω of $\mathbb{R}^n$, the following imbedding is a trivial consequence of the preceding definitions:

$$\forall l = 0,\ldots,\lfloor r\rfloor \;, W^{r,p}(\Omega) \hookrightarrow W^{l,p}(\Omega), \tag{33}$$

where the symbol $\hookrightarrow$stands, as usual, for the continuous injection.

Sobolev spaces have been intensively studied by numerous authors. For related matters in our section, we refer to Adams [20] (see also Adams and Fournier [21]) and Grisvard [31], where the main results of the theory are stated. In what follows, however, we establish certain results that we have not found in the literature formulated in the exact way that we need. As usual, we adhere to the convention that takes $1/p = 0$ if $p = \infty$.

From now on, the term domain means a non-empty, connected open set in $\mathbb{R}^n$. Likewise, we shall use the expression Lipschitz-continuous boundary in the sense of Nečas [42]. It can be seen (cf., for example, Adams [20]) that any bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz continuous boundary satisfies, for some $\theta \in (0,\pi/2]$ and $\rho > 0$, the cone property with radius ρ and angle θ , that is, for every $x \in \Omega$, there exists a unit vector $\xi(x) \in \mathbb{R}^n$ such that the cone

$$\{\, x + h\eta \mid \eta \in \mathbb{R}^n, |\eta| = 1, \eta \cdot \xi(x) \geq \cos\theta, 0 \leq h \leq \rho \,\}$$

is contained in Ω (above, the dot $\cdot$ is the Euclidean scalar product in $\mathbb{R}^n$).

Let us recall one of the most important properties of Sobolev spaces, the existence of an extension operator. Let Ω be a bounded domain in $\mathbb{R}^n$with a Lipschitz-continuous boundary. Then, for any $p \in [1, \infty)$ and $r \geq 0$, there exists a linear continuous operator P from $W^{r,p}(\Omega)$ in $W^{r,p}(\mathbb{R}^n)$ such that, for any $v \in W^{r,p}(\Omega)$, $Pv|_{\Omega} = v$. Moreover, such an operator P also exists if $p = \infty$ and $r \in \mathbb{N}$(34)

(for the proof, cf. Grisvard [31], if $p > 1$, and Sanchez [44], if $p = 1$. If $r \in \mathbb{N}$ and $p \in [1, \infty]$, see [46]).

Proposition (1.2.2) [19]:

Let Ω be a bounded domain in $\mathbb{R}^n$with a Lipschitz-continuous boundary. Let $p \in [1, \infty]$ and let r be a real number such that $r > n/p$. Then,

we have

$$\exists\lambda \in (0,1],\ W^{r,p}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega}). \tag{35}$$

In addition, we have

$$W^{n,1}(\Omega) \hookrightarrow C_B^0(\Omega), \tag{36}$$

where $C_B^0(\Omega)$ is the Banach space of bounded, continuous functions on Ω , endowed with the norm $\|v\|_{C_B^0(\Omega)} = \sup_{x\in\Omega}|v(x)|$.

Proof:

The imbedding (36) is just a particular case of (7) of Adams [20]. In what follows we shall prove (35).

We first suppose that $1 \leq p < \infty$ and r $\geq n/p + 1$. Let $l = n/p + 1$. Thus, we have $n/p < l < n/p + 1, if\ n/\ p\ \notin \mathbb{N}$, and $l = n/p + 1$, otherwise. In the first case, we choose $\lambda \in (0, l - n/p]$ and, in the second, we take $\lambda \in (0,1)$ (if $p = 1$, the value $\lambda = 1$ is also admissible). It is then clear that, by Cases C' and C'' in Adams [20], we get the imbedding $W^{l,p}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega})$. Likewise, since $\leq n/p + 1 \leq r$, it follows from (33) that $W^{r,p}(\Omega) \hookrightarrow W^{l,p}(\Omega)$. The last two imbeddings imply (35).

(b) Let us now assume that $1 < p < \infty$ and $r \in (n/p, n/p + 1)$. In this case, taking $\lambda = r - n/p$, the result directly follows from the relation (1.4.4.6) in Grisvard [31] and property (34) (cf. [31, Sect. 1.4.4]).

(c) We next assume that $p = 1$ and $r \in (n, n + 1)$. Obviously, there exists $q \in (1, \infty)$ such that $l_0 = r - n + n/q$ is a non-integer number. By Adams [20], we have $W^{r,1}(\mathbb{R}^n) \hookrightarrow W^{l_0,p}(\mathbb{R}^n)$, which implies, taking property (34) into account, that $W^{r,1}(\Omega) \hookrightarrow W^{l_0,q}(\Omega)$ (cf. [20, Remark 5 .5 (4)]). Since $l_0 \in (n/q, n/q + 1)$, it follows from point (b) that $W^{l_0,q}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega})$ with $\lambda = l_0 - n/q$. The last two imbeddings yield the result.

(d) We finally suppose that $p = \infty$. If $r \geq 1$, then (35) follows from the chain of imbeddings

$$W^{r,\infty}(\Omega) \hookrightarrow W^{1,,\infty}(\Omega) \hookrightarrow W^{1,2n}(\Omega) \hookrightarrow C^{0,1/2}(\bar{\Omega}),$$

which can be derived from (32), (33) and point (b), respectively. If $r \in (0,1)$, let us first see that

$$W^{r,\infty}(\Omega) \hookrightarrow W^{r',p'}(\Omega), \tag{37}$$

with $p' \in (2n/r, \infty)$ and $r' = r - n/p$. We observe that $1 < p' < \infty$ and that $n/p' < r' < 1 < n/p' + 1$. Let $v \in W^{r,\infty}(\Omega)$. On the one hand, it follows from relation (2.1) that $v \in L^{p'}(\Omega)$ and that

$$|\mathrm{v}|_{0,\mathrm{p}',\Omega} \leq (\mathrm{meas}\,\Omega)^{1/\mathrm{p}'} |\mathrm{v}|_{0,\infty,\Omega}.$$

On the other hand, since the set $\{(x,y) \in \Omega \times \Omega : y = x\}$ has a null measure, we have

$$|v|^{p'}_{r',p',\Omega} = \int_{\Omega\times\Omega} \frac{|v(x) - v(y)|^{p'}}{|x-y|^{n+p'r'}} dxdy$$

$$= \int_{\Omega\times\Omega} \frac{|v(x) - v(y)|^{p'}}{|x-y|^{p'r'}} dxdy \leq \mathrm{meas}(\Omega \times \Omega) |v|^{p'}_{r,\infty,\Omega}$$

We conclude that (37) holds. From this imbedding and point (b), we deduce that (35) also holds.

Proposition (1.2.3) [19]:

Let Ω be a bounded domain in $\mathbb{R}^n$with a Lipschitz-continuous boundary. Let $p, q \in [1, \infty], r > 0$ and $l_0 = r - n/p + n/q$. If $p \leq q$ and $l_0 > 0$, then, for any $l = 0, \dots, \lceil l_0 \rceil - 1$, we have

$$\mathrm{W}^{\mathrm{r,p}}(\Omega) \hookrightarrow \mathrm{W}^{\mathrm{l,q}}(\Omega). \tag{38}$$

If r $\in \mathrm{N}*$, this imbedding also holds with $l = l_0$when: (i) $1 \leq p < q < \infty$ and $l_0 \in \mathbb{N}$, or (ii) $(p, q) = (1, \infty)$ and $r \geq n$, or (iii) $1 \leq p = q < \infty$.

Proof The imbedding (36) with $l = l_0$ in the cases just mentioned is a trivial consequence of (33) (case (iii)) or follows immediately from Cases A and B in Adams [20]. Hereafter, we assume that $p \leq q$ and $l_0 > 0$. We shall consider two cases.

Case I: $1 \leq p \leq q < \infty$.

(a) Let us assume that $r \in \mathbb{N}^*$. Let $l \in \{0, \dots, \lceil l_0 \rceil - 1\}$. If $r - n/p < l$ or $-n/p = l$, Cases A or B in Adams [1] directly yield (38). If $-n/p > l$, by Proposition (1.2.2), we have $W^{r-l,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$. Thus, we get

$$\mathrm{W}^{r,p}(\Omega) \hookrightarrow \mathrm{C}^{l}(\overline{\Omega}) \hookrightarrow \mathrm{W}^{l,q}(\Omega).$$

(b) Let us now assume that $r \notin \mathbb{N}^*$ and that $l_0 \notin \mathbb{N}^*$. By Adams [1], we have $W^{r,p}(\mathbb{R}^n) \hookrightarrow W^{l_0,q}(\mathbb{R}^n)$, and so, by property (34), $W^{r,p}(\Omega) \hookrightarrow W^{l_0,q}(\Omega)$ (cf. [20, Remark 5.5 (4)] and specially, if $p > 1$, [31]). The relation (38) is then a consequence of (33) (applied with $= l_0$), taking into account that $\lfloor l_0 \rfloor = \lceil l_0 \rceil - 1$.

(c) To complete this case, we finally suppose that $r \notin \mathbb{N}^*$ and that $l_0 \in \mathbb{N}^*$. Let $q' \in (q, \infty)$, if $q \geq n$, and $q' \in (q, nq/(n-q))$, otherwise, and let $l_1 = r - n/p + n/q'$. It is readily seen that $l_0 - 1 < l_1 < l_0$. Since $l_1 \notin \mathbb{N}^*$ and $\lceil l_1 \rceil = l_0 = \lceil l_0 \rceil$, the reasoning in point (b) shows that

$$\forall \mathrm{l} = 0, \ldots, \lceil \mathrm{l}_0 \rceil - 1, \mathrm{W}^{r,p}(\Omega) \hookrightarrow \mathrm{W}^{l,q'}(\Omega).$$

But Ω is a bounded set and $' > q$. Thus, the result follows from the preceding relation and relation (32).

Case II: $1 \leq p \leq q = \infty$.

Let $l \in \{0, \ldots, \lceil l_0 \rceil - 1\}$, where $l_0 = r - n/p$. Thus, since $r - l > n/p$, by Proposition 1.2.2, we have

$$W^{r-l,p}(\Omega) \hookrightarrow C^0(\overline{\Omega}) \hookrightarrow L^\infty(\Omega) = W^{0,\infty}(\Omega).$$

This imbedding implies (38).

Remark (1.2.4) [19]:

The statement of Proposition (1.2.3) is limited to the cases needed in this section. There is, however, one exception. In the proof of Theorem (1.2.8) we shall require the relation (1,4,4,5) in Grisvard [31], which establishes that (38) holds under the following conditions: $1 < p \leq q < \infty$, and r and l are non-negative real numbers such that $r - n/p = l - n/q$.

Let us begin with a result slightly different from one by Duchon (cf. [30]).

Proposition (1.2.5) [19]:

Let Ω be a bounded open subset of $\mathbb{R}^n$satisfying the cone property with radius ρ and angle . Then, there exist constants $M > 1$ (depending on θ), $M_1 > 1$ (depending on n and θ), $\lambda_0 > 0$ (depending on ρ and θ), and $M_2 > 1$ (depending on n and diam Ω) such that, for any $\lambda \in (0, \lambda_0]$, there exists $T_\lambda \subset \Omega$ satisfying

(i) $\forall t \in T_\lambda, \bar{B}(t, \lambda) \subset \Omega$,

(ii) $\Omega \subset \bigcup_{t \in T_\lambda} \bar{B}(t, M\lambda)$,

(iii) $\sum_{t \in T_\lambda} \chi_{\bar{B}(t, M\lambda)} \leq M_1$,

(iv) card $T_\lambda \leq M_2\lambda^{-n}$.

Proof:

It has been shown by Duchon that (i), (ii) and (iii) hold if one takes $M = 2(1 + sin\theta)/sin\theta$, $M_1 = (M\sqrt{n} + 1)^n$, $\lambda_0 = \rho/M$, and, for any $\lambda \in (0, \lambda_0]$, $T_\lambda = \{t \in \nu\lambda\mathbb{Z}^n : \bar{B}(t, \lambda) \subset \Omega\}$, where $\nu = 2/\sqrt{n}$. Let us see that (iv) also holds. Since Ω is a bounded set, it is contained in an open hypercube C with sides of length $L = \text{diam}\,\Omega$. Let $M_2 = L^n\nu^{-n}$. Then, it is clear that

$$\text{card}\,T_\lambda \leq \text{card}(C \cap \nu\lambda\mathbb{Z}^n) \leq L/(\nu\lambda)^n = M_2\lambda^{-n},$$

which completes the proof.

Let us now recall a more or less classical result.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Let $p \in [1, \infty]$, $r > 0$ and $k = \lceil r \rceil - 1$. Then, there exists a positive constant C such that

$$\forall v \in W^{r,p}(\Omega), \qquad \min_{\psi \in P_k(\Omega)} \|v - \psi\|_{r,p,\Omega} \leq C|v|_{r,p,\Omega} \tag{39}$$

(for the proof, cf. Ciarlet [26], if $r \in \mathbb{N}^*$, and Sanchez and Arcangéli [45], otherwise).

Proposition (1.2.6) [19]:

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Let $p \in [1, \infty)$, $r > (n/p - n/2)_+$ and $k = \lceil r \rceil - 1$. Then, there exists a linear operator $\tilde{P}: W^{r,p}(\Omega) \to W^{r,p}(\mathbb{R}^n) + P_k$ and a positive constant C such that, for any $v \in W^{r,p}(\Omega)$, $\tilde{P}v|_\Omega = v$ and $|\tilde{P}v|_{r,p,\mathbb{R}^n} \leq C|v|_{r,p,\Omega}$. Moreover, such an operator $\tilde{P}$ also exists if $p = \infty$ and $r \in \mathbb{N}^*$.

Proof:

Let I and P be, respectively, the identity operator in $W^{r,p}(\Omega)$ and the extension operator from $W^{r,p}(\Omega)$ into $W^{r,p}(\mathbb{R}^n)$ introduced in relation (34). Likewise, let us denote by $\tilde{\Pi}_k$ and , respectively, the orthogonal projection operator from $L^2(\Omega)$ onto $P_k(\Omega)$ and the operator that assigns to any polynomial function over Ω the same polynomial function over $\mathbb{R}^n$. Finally, let T be the linear operator of the continuous imbedding of $W^{r,p}(\Omega)$ into $L^2(\Omega)$. We note that, if $2 \geq p$, the existence of T is a consequence of Proposition (1.2.3) (applied with $q = 2$ and $l = 0$), whereas, if $2 < p$, taking into account that Ω is bounded and relation (32), we obviously have $W^{r,p}(\Omega) \subset L^p(\Omega) \subset L^2(\Omega)$.

Following Geymonat (cf. Strang [28]), let us consider the operator $\tilde{P} = P(I - \Pi_k)_+ + E\Pi_k$, where $\Pi_k = \tilde{\Pi}_k T$. It is clear that $\tilde{P}$ is a linear operator from $W^{r,p}(\Omega)$ into

$W^{r,p}(\mathbb{R}^n) + P_k$. Since$\Pi_k v \in P_k(\Omega)$, for any $\in W^{r,p}(\Omega)$, we deduce that $\tilde{P}v|_\Omega = v$.

Let us now see that there exists a constant $C > 0$such that, for any $v \in W^{r,p}(\Omega)$, $|\tilde{P}v|_{r,p,\mathbb{R}^n} \leq C|v|_{r,p,\Omega}$. To this end, we first remark that

$$\forall v \in W^{r,p}(\Omega), |\tilde{P}v|_{r,p,\mathbb{R}^n} = |P(I - \Pi_k)v|_{r,p,\mathbb{R}^n} \leq \|P(I - \Pi_k)v\|_{r,p,\mathbb{R}^n} ,$$

since the semi-norm $|\cdot|_{r,p,\mathbb{R}^n}$ is null over P_k . Then, by definition of P and Π_k, there exists $C > 0$ such that

$$\forall v \in W^{r,p}(\Omega), \forall \psi \in P_k(\Omega), |\tilde{P}v|_{r,p,\mathbb{R}^n} \leq C\|(I - \Pi_k)(v - \psi)\|_{r,p,\Omega} .$$

The result then follows from relation (39) and the continuity of the operator $I - \Pi_k : W^{r,p}(\Omega) \to W^{r,p}(\Omega)$.

The following proposition is strongly based on a result due to Duchon (cf. [30 Lemma]), with additional ideas drawn from [36] (see also [22]).

Proposition (1.2.7) [19]:

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Let $p \in [1, \infty]$ and let r be a real number such that $r > n/p$, if $p > 1$, or r ≥ n, if p = 1. Likewise, let $k = \lceil r \rceil - 1$ and $R = \dim P_k$. Finally, let $B \subset \Omega^{\mathfrak{K}}$ be a compact set of P_k-unisolvent $\mathfrak{K}$ -tuples. Then, for any $x \in [1, \infty]$, there exists a constant $C > 0$ (dependent on Ω , B , r , p and x) such that

$$\forall b \in B, \forall v \in W^{r,p}(\Omega), \qquad \|v\|_{r,p,\Omega} \leq C\left(|v|_{r,p,\Omega} + \|v|_b\|_x\right) .$$

Proof:

(a) For any $= (b_1, \dots, b_{\mathfrak{K}}) \in B$, let Π^b be the Lagrange P_k-interpolating operator, defined, for any $v \in W^{r,p}(\Omega)$, by

$$\Pi^b v \in P_k(\Omega) \; and, for \; j = 1, \dots, K, \Pi^b v(b_j) = v(b_j).$$

By Proposition (1.2.4), $I - \Pi^b$ is a linear continuous operator from $W^{r,p}(\Omega)$ into $W^{r,p}(\Omega)$. Hence, for any $\in B$, there exists a constant C (b) such that

$$\forall v \in W^{r,p}(\Omega), \qquad \|v - \Pi^b v\|_{r,p,\Omega} \leq C(b)\|v\|_{r,p,\Omega} .$$

(b) Let us prove that

$$sup_{b \in B} C(b) < \infty \qquad (40)$$

To do this, it is sufficient to show that, for any $v \in W^{r,p}(\Omega)$, the set $\{ \Pi^b v : b \in B \}$is bounded in $W^{r,p}(\Omega)$. The relation (40) then follows by applying the Banach- Steinhaus Theorem to the family of operators $(I - \Pi^b)_{b \in B}$.

Let $p_1, \dots, p_{\mathfrak{K}}$ be a basis of $P_k(\Omega)$ and, for any $\in B$, let us consider the matrix

$M(b) = \left(p_j(b_i)\right)_{1\le i,j\le \mathfrak{K}}$ We remark that $M(b)$ is regular, since b is P_k–unisolvent. Denoting by $m'_{i,j}(b)$ the generic element of the inverse matrix $(b)^{-1}$, for any $v \in W^{r,p}(\Omega)$, we have

$$\Pi^b v = \sum_{i,j=1}^{\mathfrak{K}} v(b_j) m'_{ij}(b) p_i. \tag{41}$$

Now, on the one hand, $v \in W^{r,p}(\Omega)$ is continuous on $\overline{\Omega}$ (in fact, on $\overline{\Omega}$ if $r > n/p$) and, on the other hand, since matrix inversion is a continuous operation, each function m'_{ij} is bounded on the compact set B . We deduce that, for any $v \in W^{r,p}(\Omega)$, $\|\Pi^b v\|_{\mathrm{r,p,\Omega}}$ remains bounded when b varies in B and so (40) holds.

(c) Since

$$\forall b \in B, \forall \psi \in P_k(\Omega),\ \Pi^b \psi = \psi, \tag{42}$$

by points (a) and (b), there exists a constant C such that

$$\forall b \in B, \forall v \in W^{r,p}(\Omega), \forall \psi \in P_k(\Omega), \qquad \|v - \Pi^b v\|_{r,p,\Omega} \le C\|v - \psi\|_{r,p,\Omega}.$$

This inequality and relation (3.1) imply that

$$\forall b \in B, \forall v \in W^{r,p}(\Omega),\ \|v - \Pi^b v\|_{r,p,\Omega} \le C|v|_{r,p,\Omega}, \tag{43}$$

with C depending on Ω , B , r and p.

(d) Let us now see that, given $x \in [1,\infty]$,

$$\exists C > 0, \forall b \in B, \forall \psi \in P_k(\Omega), \|\psi\|_{r,p,\Omega} \le C\ \|v|_b\|_x, \tag{44}$$

with C depending on Ω , B , r , p and x .

Let $b^* = (b_1^*, \dots, b_{\mathfrak{K}}^*)$ be a fixed $\mathfrak{K}$ -tuple of . It follows from (41) and (42) that, for any $b \in B$ and for any $\psi \in P_k(\Omega)$,

$$(\psi(b_1^*), \dots, \psi(b_{\mathfrak{K}}^*))^T = M(b^*)\, M(b)^{-1}\, (\psi(b_1), \dots, \psi(b_{\mathfrak{K}}))^T,$$

where the super-index T means transposition. The compactness of B and the continuity of the operator $b \in B \longmapsto M(b^*)M(b)^{-1}$ imply the existence of a constant $C > 0$ such that

$$\forall b \in B, \forall \psi \in P_k(\Omega),\ \|\psi|_{b^*}\|_x \le C\|\psi|_b\|_x. \tag{45}$$

Likewise, the mappings $\psi \longmapsto \|\psi\|_{r,p,\Omega}$ and $\psi \longmapsto \|\psi|_{b^*}\|_x$ are both norms on the finite dimensional space $P_k(\Omega)$. Thus, there exists $C > 0$ such that

$$\forall \psi \in P_k(\Omega),\ \|\psi\|_{r,p,\Omega} \le C\ \|\psi|_{b^*}\|_x. \tag{46}$$

The relation (44) follows from (45) and (46).

(e) For any $b \in B$ and for any $v \in W^{r,p}(\Omega)$, it is clear that

$$\|v\|_{r,p,\Omega} \leq \|v - \Pi^b v\|_{r,p,\Omega} + \|\Pi^b v\|_{r,p,\Omega} .$$

The result is then a consequence of this relation, (43) and (44).

We conclude this section with a proposition that extends a well-known result by Duchon (cf. [30]; see also [22]).

Proposition (1.2.8) [19]:

Let $p, q, x \in [1, \infty]$ such that $\leq q$. Let r be a real number such that $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$. Finally, let $k = \lceil r \rceil - 1$, $\mathfrak{K} = dim\, P_k$ and $l_0 = r - n/p + n/q$. Then, there exists $R > 1$ (dependent on n and r) and, for any $M' \geq 1$, a constant C (dependent on M', n, r, p, q and x) satisfying the following property: for any $d > 0$ and any $t \in \mathbb{R}^n$, the open ball $B(t, Rd)$ contains $\mathfrak{K}$ closed balls $\mathcal{B}_1, \dots, \mathcal{B}_{\mathfrak{K}}$of radius d such that, for any $v \in W^{r,p}(\bar{B}\,(t, M'\,Rd\,))$, for any $b \in \prod_{i=1}^{\mathfrak{K}} \mathcal{B}_i$ and for $l = 0, \dots, \lceil l_0 \rceil - 1$,

$$|v|_{l,q,\bar{B}(t,M'Rd)} \leq C\big(d^{r-l-n/p-n/q}|v|_{r,p,\bar{B}(t,M'Rd)} + d^{n/q-l}\|v|_b\|_x\big). \quad (47)$$

If $r \in \mathbb{N}^*$, this bound also holds with $l = l_0$when either $p < q < \infty$ and $l_0 \in \mathbb{N}$, or $(p, q) = (1, \infty)$, or $p = q$.

Proof:

(a) Let $b^0 = (b_1^0, \dots, b_{\mathfrak{K}}^0) \in (\mathbb{R}^n)^{\mathfrak{K}}$ be a P_k-unisolvent $\mathfrak{K}$ -tuple. Since the subset of $(\mathbb{R}^n)^{\mathfrak{K}}$ formed by all the P_k-unisolvent $\mathfrak{K}$ -tuples is an open subset of $(\mathbb{R}^n)^{\mathfrak{K}}$ (cf. [30 proof of Proposition 2]), there exists $r_0 > 0$ such that any $\mathfrak{K}$ -tuple $b \in \prod_{j=1}^{\mathfrak{K}} \bar{B}(b_j^0, r_0)$ is P_k-unisolvent.

By a homothecy of reason $1/r_0$, writing $\hat{\alpha}_j = (1/r_0\,)b_j^0$, we obtain $\mathfrak{K}$ balls $\bar{B}(\hat{\alpha}_j\,, 1)$ such that the product $\hat{B} \in \prod_{j=1}^{\mathfrak{K}} \bar{B}(\hat{\alpha}_j\,, 1)$ is a compact subset of $(\mathbb{R}^n)^{\mathfrak{K}}$ formed by P_k -unisolvent $\mathfrak{K}$-tuples. The set $\bigcup_{j=1}^{\mathfrak{K}} \bar{B}(\hat{\alpha}_j\,, 1)$ is bounded and so contained in an open ball $B(\hat{a}, R)$ whose radius $R > 1$ depends on n and k, and hence, on n and r .

(b) Let $M \geq 1$ and let $l_{\max} = \lceil l_0 \rceil - 1$, except in the cases cited after (47), for which we take $l_{\max} = l_0$. Applying, in order, Proposition (1.2.5) and Proposition (1.2.7) (with $\Omega = B(\hat{a}, M\ R)$ in both cases and $B = \hat{B}$ in the second), we deduce that, for any $\hat{b} \in \hat{B}$, for any $\hat{v} \in W^{r,p}(\bar{B}(\hat{a}, M'\,R))$ and for $l = 0, \dots, l_{\max}$

$$|\hat{v}|_{l,q,\bar{B}(\hat{a},M'R)} \leq C\big(|\hat{v}|_{r,q,\bar{B}(\hat{a},M'R)} + \|\hat{v}|_{\hat{b}}\|_x\big), \quad (48)$$

with C depending on $M\ , n, r\ , p, q$ and x.

(c) For any $d > 0$ and any $\in \mathbb{R}^n$, let F_t^d d be the invertible affine mapping $x \to t +$

$d(x-\hat{a})$. This mapping transforms the ball $\bar{B}(\hat{a}, M'd)$ into the ball $\bar{B}(t, M'Rd)$ the ball $B(\hat{a}, R)$ into the ball $B(t, Rd)$ and, for any $j = 1, \dots, \mathfrak{K}$, the ball $B(\hat{a}_j, 1)$, into a closed ball $\mathcal{B}_j$ of radius d contained in the ball $B(t, Rd)$.

For any $v \in W^{r,p}(\bar{B}(t, M'Rd))$ and any $b = (b_1, \dots, b_{\mathfrak{K}}) \in \prod_{i=1}^{\mathfrak{K}} \mathcal{B}_i$, we write $\hat{v} = v \circ F_t^d$ and $\hat{b} = (\hat{b}_1, \dots, \hat{b}_{\mathfrak{K}})$, where, for $j = 0, \dots, \mathfrak{K}$, $\hat{b}_j = (F_t^d)^{-1}(b_j)$. It is clear that $v(b_j) = \hat{v}(\hat{b}_j)$. Then, it follows from the rules of change of variables in semi-norms (cf. Ciarlet [26] for the integer case, and Sanchez and Arcangéli [45], for the non-integer one) that there exists a constant C (depending on n, r, p and q) such that, for any integer $l = 0, \dots, l_{\max}$ max and for any $v \in W^{r,p}(\bar{B}(t, M'Rd))$

$$|v|_{l,q,\bar{B}(t,M'Rd)} \le C d^{n/q-l} |\hat{v}|_{l,q,\bar{B}(\hat{a},M'd)}$$

and

$$|\hat{v}|_{l,q,\bar{B}(\hat{a},M'd)} \le C d^{n-n/q} |v|_{l,q,\bar{B}(t,M'Rd)}$$

Taking into account that $\|\hat{v}|_{\hat{b}}\|_x = \|v|_b\|_x \ \Pi$, the result then follows from (47) and the last two relations.

Throughout this section we denote by Ω a bounded domain in $\mathbb{R}^n$with a Lipschitz- continuous boundary. As mentioned in Sect. 2, for some $\theta \in (0, \pi/2]$ and $\rho > 0$, the domain Ω satisfies the cone property with radius ρ and angle θ .
Likewise, for any finite subset A of $\overline{\Omega}$, we write $\delta(A, \overline{\Omega})$ for the Hausdorff distance between A and $\overline{\Omega}$, also known as fill distance, which is given by

$$\delta(A, \overline{\Omega}) = \sup_{x\in\Omega} \min_{a\in A} |x - a|. \tag{49}$$

Theorem (1.2.9) [19]:

Let $p, q, x \in [1, \infty]$. Let r be a real number such that $r \ge n$, if $p = 1, r > n/p$, if $1 < p < \infty$, or $r \in \mathbb{N}^*$, if $p = \infty$. Likewise, let $l_0 = r - n(1/p - 1/q)_+$ and $\gamma = \max\{p, q, x\}$. Then, there exist two positive constants $\mathfrak{d}_r$(dependent on θ, ρ, n and r) and C (dependent on Ω, n, r, p, q and x) satisfying the following property: for any finite set $A \subset \overline{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $d = \delta(A, \overline{\Omega}) \le \mathfrak{d}_r$, for any $u \in W^{r,p}(\Omega)$ and for any $l = 0, \dots, \lceil l_0 \rceil - 1$, we have

$$|u|_{l,q,\Omega} \le C\, d^{r-l-n(1/p-1/q)_+} |u|_{r,p,\Omega} + d^{n/\gamma - l} \|u|_A\|_x. \tag{50}$$

If $r \in \mathbb{N}^*$, this bound also holds with $l = l_0$ when either $p < q < \infty$ and $l_0 \in \mathbb{N}$, or $(p, q) = (1, \infty)$, or $p \ge q$.

Proof:

We shall consider three different cases.

Case I: $p \leq q < \infty$.

Let $M > 1, M_1 > 1, \lambda_0 > 0$ and $M_2 > 0$ be the constants given by Proposition (1.2.5), and, for any $\lambda \in (0, \lambda_0]$, let T_λ be the subset of Ω whose existence is also assured by that proposition. Likewise, let $R > 1$ be the first of the constants provided by Proposition (1.2.8). We let $\mathfrak{d}_r = \lambda_0 / R$, which is obviously a constant that depends only on θ , ρ, n and r . Finally, as in Proposition (1.2.8), we let $k = \lceil r \rceil - 1$ and $= \dim P_k$.

Let us consider any finite set $A \subset \overline{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $d = \delta(A, \overline{\Omega}) \leq \mathfrak{d}_r$. We note that $\leq \lambda_0$. Given $\in T_{Rd}$, by (49), there exists $\mathfrak{K}$-tuple $a_t^d \in \prod_{j=1}^{\mathfrak{K}} (\mathcal{B}_j, \cap A)$, where $\mathcal{B}_1, \dots, \mathcal{B}_{\mathfrak{K}}$ are the closed ball associated with d and t by Proposition (1.2.8). We remark that a_t^dbelongs to $\Omega^{\mathfrak{K}}$ K, since, by Proposition (1.2.8) and point (i) of Proposition (1.2.5), for $j = 1, \dots, \mathfrak{K}, \mathcal{B}_j \subset \bar{B}(t.Rd) \subset \Omega$.

For any $u \in W^{r,p}(\Omega)$, we write $\tilde{u} = P\tilde{u}$, where $\tilde{P}$ stands for the operator defined in Proposition (1.2.6). Since $\tilde{u} \in W^{r,p}(\mathbb{R}^n) + P_k$, it is clear that $\tilde{u}$ belongs to $W^{r,p}(\mathcal{O})$ for any bounded open subset $\mathcal{O}$ of $\mathbb{R}^n$. Let l_{max} be defined as in the proof of Proposition (1.2.8). By point (ii) of Proposition 1.2.5, for $= 0, \dots, l_{max}$, we get

$$|u|_{l,q,\Omega} \leq |\tilde{u}|_{l,q,\cup_{t \in T_{Rd}} \bar{B}(t,M\,Rd)} \leq \left(\sum_{t \in T_{Rd}} |\tilde{u}|^q_{l,q,\bar{B}(t,M\,Rd)} \right),$$

from which, applying Proposition 1.2.8 with $M' = M$, we obtain

$$|u|_{l,q,\Omega} \leq C d^{n/q-l} \leq \left(\sum_{t \in T_{Rd}} \left(d^{r-n/p} |\tilde{u}|_{l,q,\cup_{t \in T_{Rd}} \bar{B}(t,M\,Rd)} + \left\| \tilde{u}|_{a_t^d} \right\|_x \right)^q \right)^{1/q} ,$$

where C is a constant that depends on θ (through M), n, r , p, q and x. Applying Minkowski's inequality for the discrete space ℓ^q , we derive the relation

$$|u|_{l,q,\Omega} \leq C d^{n/q-l} (d^{n/q-l} \mathfrak{S}_1 + \mathfrak{S}_2) \ , \tag{51}$$

with

$$\mathfrak{S}_1 = \left(\sum_{t \in T_{Rd}} |\tilde{u}|^q_{r,p,\bar{B}(t,M\,Rd)} \right)^{1/q} \quad and \quad \mathfrak{S}_2 = \left(\sum_{t \in T_{Rd}} \left\| \tilde{u}|_{a_t^d} \right\|_x^q \right)^{1/q} .$$

Next, we shall bound above $\mathfrak{S}_1$and $\mathfrak{S}_2$.

On the one hand, by Jensen's inequality, we have

$$\mathfrak{S}_1 \leq \left(\sum_{t \in T_{Rd}} |\tilde{u}|^p_{r,p,\bar{B}(t,M\,Rd)} \right)^{1/q} \tag{52}$$

If $r \in \mathbb{N}$, we deduce from point (iii) of Proposition (1.2.5) that

$$\sum_{t \in T_{Rd}} |\tilde{u}|^p_{r,p,\bar{B}(t,M\,Rd)}$$

$$= \sum_{t \in T_{Rd}} \int_{\mathbb{R}^n} \chi_{\bar{B}(t,M\,Rd)} \left(\sum_{|\alpha|=r} |\partial^\alpha \tilde{u}(x)|^p \right) dx$$

$$= \int_{\mathbb{R}^n} \left(\sum_{t \in T_{Rd}} \chi_{\bar{B}(t,M\,Rd)} \right) \left(\sum_{|\alpha|=r} |\partial^\alpha \tilde{u}(x)|^p \right) dx$$

$$\leq M_1 \int_{\mathbb{R}^n} \left(\sum_{|\alpha|=r} |\partial^\alpha \tilde{u}(x)|^p \right) dx = M_1 |\tilde{u}|^p_{r,p,\mathbb{R}^n},$$

which obviously implies, together with (52), that

$$\mathfrak{S}_1 \leq M_1^{1/p} |u|_{r,p,\mathbb{R}^n} \tag{53}$$

If $r \notin \mathbb{N}$, a similar reasoning shows that (53) also holds, taking into account the definition of the Sobolev semi-norms of non-integer order r and that, again by Proposition (1.2.5)

$$\sum_{t \in T_{Rd}} \chi Q(t) \leq M_1,$$

Where$Q(t) = \bar{B}(t, MRd) \times B(t, MRd)$

On the other hand, using Hölder's inequality, if q ≤ x < ∞, or Jensen's inequality, if $> x$, as well as points (iii) and (iv) of Proposition (1.2.5), we get

$$\mathfrak{S}_2 \leq (\mathrm{card} T_{Rd})^{(1/q-1/x)_+} \left(\sum_{t \in T_{Rd}} \left\| \tilde{u}|_{a_t^d} \right\|_x^x \right)^{1/x}$$

$$\leq (M_2 R^{-n} d^{-n})^{(1/q-1/x)_+} \left(\sum_{t \in T_{Rd}} \left\| \tilde{u}|_{A \cap \Omega \cap \bar{B}(t,M\,Rd)} \right\|_x^x \right)^{1/x}$$

$$\leq M_1^{1/x} (M_2 R^{-n} d^{-n})^{(1/q-1/x)_+} \| \tilde{u}|_{A \cap \Omega} \|_x.$$

Of course, if $x = \infty$, we simply have

$$\mathfrak{S}_2 \leq (\mathrm{card} T_{Rd})^{1/q} \max_{t \in T_{Rd}} \left\| \tilde{u}|_{a_t^d} \right\|_\infty \leq (M_2 R^{-n} d^{-n})^{1/q} \| \tilde{u}|_{A \cap \Omega} \|_{\infty.}$$

Therefore, from (51), (53) and the two preceding relations, we deduce that

$$|u|_{l,q,\Omega} \leq C(d^{r-l-n/p+n/q})|\tilde{u}|_{r,p,\mathbb{R}^n} + M_2^{(1/q-1/x)_+} d^{n/\gamma-l}\|\tilde{u}|_A\|_x), \quad (54)$$

where $\gamma = \max\{q , x\}$ and C is a constant that depends on θ , n, r , p, q and x. The result then follows from Proposition (1.2.6).

Case II: $p \leq q = \infty$.

We keep the notations in Case I. A similar reasoning now yields

$$|u|_{l,q,\Omega} \leq |\tilde{u}|_{l,\infty,\cup_{t\in T_{Rd}} \bar{B}(t,M\,Rd)} = \max_{t\in T_{Rd}} |\tilde{u}|_{l,\infty,\bar{B}(t,M\,Rd)} \leq Cd^{-1}(d^{r-n/p}\mathfrak{S}_1 + \mathfrak{S}_2),$$

Where C is a constant that depends on $\theta , n, r , p, q = \infty$ and x,

$\mathfrak{S}_1 = \max_{t\in T_{Rd}} |\tilde{u}|_{r,p,\bar{B}(t,M\,Rd)})$ and $\mathfrak{S}_2 = \max_{t\in T_{Rd}} \left\|\tilde{u}|_{a_t^d}\right\|_x$

If $p < \infty$, it is clear that (53) still holds, which leads again to (53). This relation directly holds if $p = \infty$. Likewise, we can immediately check that $\mathfrak{S}_2 \leq \|\tilde{u}|_A\|_x$. We derive the relation

$$|u|_{l,\infty,\Omega} \leq C(d^{r-l-n/p}|\tilde{u}|_{r,p,\mathbb{R}^n} + d^{-l}\|\tilde{u}|_A\|_x),$$

from where we obtain (50) by Proposition (1.2.6).

Case III: $p > q$.

On the one hand, by the two preceding cases, there exist two positive constants $\eth_r$(depending on θ , ρ, n and) and C (depending on Ω , n, r , p and x) satisfying the following property: for any finite set $A \subset \bar{\Omega}$ such that $d = \delta(A, \bar{\Omega}) \leq \eth_r$and for any $u \in W^{r,p}(\Omega)$, we have

$$\forall l = 0, \dots , \lfloor r \rfloor, |u|_{l,p,\Omega} \leq C(d^{r-l}|\tilde{u}|_{r,p,\Omega} + d^{n/\gamma-l}\|u|_A\|_x),$$

where $\gamma = \max\{ p, x \}$. On the other hand, it is clear that, by (32),

$$\forall l = 0, \dots , \lfloor r \rfloor, |u|_{l,q,\Omega} \leq (\text{meas } \Omega)^{1/q - 1/p} |u|_{l,p,\Omega} .$$

Thus, in this case, the theorem results from the last two relations.

Remark (1.2.10) [19]:

The choice of x is completely open. When $x = \infty$, we find again, under weaker hypotheses, the result obtained by Wendland and Rieger [50]. As we shall see later, the optimal value for getting estimates for smoothing (m, s)-splines is $x = 2$.

Corollary (1.2.11) [19]:

Suppose that p, q , r and l_0 are defined as in Theorem (1.2.9). Then, there exist two positive constants $\eth_r$ (dependent on θ , ρ, n and r) and C (dep- endent on Ω , n, r , p and q) satisfying the following property: for any finite set $A \subset \bar{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and

$r = n$) such that $d = \delta(A, \overline{\Omega}) \le \mathfrak{d}_r$, for any $u \in W^{r,p}(\Omega)$ such that $u|_A = 0$ and for any $l = 0, \dots, [l_0] - 1$, we have

$$|u|_{l,q,\Omega} \le C d^{r-l-n(1/p-1/q)_+} |u|_{l,p,\Omega} .$$

If $r \in \mathbb{N}^*$, this bound also holds with $l = l_0$ when either $p < q < \infty$ and $l_0 \in \mathbb{N}$, or $(p, q) = (1, \infty)$, or $p \ge q$.

Proof :

It suffices to apply Theorem (1.2.9).Let m and s be, respectively, a positive integer and a real number such that

$$-m + \frac{n}{2} < s < \frac{n}{2} \qquad (55)$$

We write $\widetilde{H}^s$ for the space

$$\widetilde{H}^s = \left\{ v \in S' : \hat{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \ \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{v}(\xi)|^2 < \infty \right\},$$

where S' is the space of tempered distributions in $\mathbb{R}^n$ and $\hat{v}$ stands for the Fourier transform of v. Then, we denote by $X^{m,s}$ the Beppo-Levi space

$$X^{m,s} = v \in \mathcal{D}' \ \forall \alpha \in \mathbb{N}^n, |\alpha| = m, \partial^\alpha v \in \widetilde{H}^s ,$$

$\mathcal{D}'$ being the space of distributions in $\mathbb{R}^n$. Endowed with the semi-norm

$$|v|_{m,s} = \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \left| \widehat{\partial^\alpha v}(\xi) \right|^2 \right)^{1/2},$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $X^{m,s}$ is a semi-Hilbert space (cf. Duchon[10], where $X^{m,s}$ is denoted by $D^{-1}\widetilde{H}^s(\mathbb{R}^n)$). In fact, $X^{m,s}$ can be handled as a Hilbert space thanks to the following result: for any bounded domain $\Omega^* \subset \mathbb{R}^n$, the mapping

$$\|\cdot\|^{\Omega^*}_{m,s} : v \in X^{m,s} \mapsto \left(\|v\|^2_{0,2,\Omega^*} + |v|^2_{m,s} \right)^{1/2} \qquad (56)$$

is a Hilbertian norm on $X^{m,s}$ whose topology is independent of Ω^* (cf. [22]). From now on, we shall assume that $X^{m,s}$ is endowed with a norm $\|\cdot\|_{m,s}$, without making any particular reference to a which we shall simply write $\|\cdot\|_{m,s}$ particular open set Ω^* (except in Proposition 1.2.12).

Once equipped with the norm $\|\cdot\|_{m,s}$, the space $X^{m,s}$ enjoys the following property (cf.[22]).For any bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz-continuous boundary, the operator R_Ω of restriction to Ω is linear and continuous from $X^{m,s}$ onto $W^{m+s,2}(\Omega)$.

As shown in [3], the following imbedding also holds:

$$X^{m,s} \hookrightarrow C^0(\mathbb{R}^n) \tag{57}$$

We conclude this subsection with a generalization of an extension theorem by Duchon (cf. [30, Lemma 3.1]).

Proposition (1.2.12):

Suppose that (55) holds. Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz-continuous boundary. Then, there exists a linear continuous operator $\mathfrak{P}: W^{m+s,2}(\Omega) \to X^{m,s}$ such that

$$\forall f \in W^{m+s,2}(\Omega), \qquad \mathfrak{P}f|_\Omega = f.$$

Moreover, if $s \le 0$, then there exists a constant $C > 0$ such that

$$\forall f \in W^{m+s,2}(\Omega),\ |\mathfrak{P}f|_{m,s} \le C|f|_{m+s,2,\Omega} \tag{58}$$

Proof:

Since we can freely choose the open set Ω^* to define by (56) the norm $\|\cdot\|_{m,s} = \|\cdot\|_{m,s}^{\Omega^*}$, we take $\Omega^* = \Omega$. We denote by $((\cdot,\cdot))_{m,s}$ and $(\cdot,\cdot)_{m,s}$, respectively, the scalar product and the scalar semi-product associated with the norm $\|\cdot\|_{m,s}$ and the semi-norm $|\cdot|_{m,s}$.

For any $f \in W^{m+s,2}(\Omega)$, let $\widetilde{\mathcal{K}}_f\, f = \{v \in X^{m,s} \mid v|_\Omega = f\}$. Since $\|\cdot\|_{m,s}^2$ and $|\cdot|_{m,s}^2$ differ only in a constant on $\widetilde{\mathcal{K}}_f$ and this set is non-empty, convex and closed in $X^{m,s}$, by the Orthogonal Projection Theorem, there exists a unique element $\mathfrak{P}f \in \widetilde{\mathcal{K}}_f$ of minimal semi-norm $|\cdot|_{m,s}$ in $\widetilde{\mathcal{K}}_f$, which can be equivalently characterized by the relation

$$\mathfrak{P}f \in \widetilde{\mathcal{K}}_f \text{ and, all } w \in \widetilde{\mathcal{K}}_0,\ (\mathfrak{P}f, w)_{m,s} = 0, \tag{59}$$

where $\widetilde{\mathcal{K}}_0 = \{v \in X^{m,s} \mid v|_\Omega = 0\}$. In this way, we have defined an operator $: W^{m+s,2}(\Omega) \to X^{m,s}$, whose linearity follows from (59)

Now, let us see that P is continuous. Let $(f_j)_{j\in\mathbb{N}} \subset W^{m+s,2}(\Omega)$ be any sequence such that

$$\exists f \in W^{m+s,2}(\Omega), f_j \to f \text{ in} W^{m+s,2}(\Omega), \tag{60}$$

$$\exists u \in X^{m,s}, \mathfrak{P}f_j \to u \text{ in} X^{m,s}. \tag{61}$$

For any $\in \widetilde{\mathcal{K}}_0$, it follows from (59) that, for any $j \in \mathbb{N}$, $((\mathfrak{P}f_j, w))_{m,s} = 0$, which implies, together with (61), that $(u,w)_{m,s} = 0$. Likewise, by (56) and (59), we have $f_j = \mathfrak{P}f_j|_\Omega \to u|_\Omega\ in\ L^2(\Omega)$. From this fact and (60), we deduce that $= u|_\Omega$, that is, u belongs to $\mathcal{K}_f$. By (59), we conclude that $= \mathfrak{P}f$. Consequently, the graph of $\mathfrak{P}$is closed in $W^{m+s,2}(\Omega) \times X^{m,s}$. By the Closed Graph Theorem, the operator $\mathfrak{P}$ is continuous.

Let us finally suppose that $s \leq 0$ and prove (58). Let $k = \lceil m+s \rceil - 1$. Since $k \leq m-1$, for any $\psi \in P_k(\Omega)$, we have $\mathfrak{P}\psi = E\,\psi$, where E is the operator that assigns to any polynomial function over Ω the same polynomial function over $\mathbb{R}^n$. Thus, since $\mathfrak{P}$ is linear and continuous, for any $f \in W^{m+s,2}(\Omega)$, we have

$$\forall \psi \in P_k(\Omega), |\mathfrak{P}f|_{m,s} = |\mathfrak{P}f - E\,\psi\,|_{m,s} = |\mathfrak{P}(f-\psi)|_{m,s}$$

$$\leq \|\mathfrak{P}(f-\psi)\|_{m,s} \leq C\|f-\psi\|_{\,m+s,2,\Omega}\,,$$

with C independent of f and ψ. The result then follows from (39).

Assume that (55) holds. Given an ordered, finite subset A of $\mathbb{R}^n$and a vector $= (\beta_a)_{a\in A} \in \mathbb{R}^{\mathrm{card}\,A}$, we call interpolating (m,s)-spline relative to A and β any solution, if any exists, of the problem: find f^AA such that

$$f^A \in \mathcal{K}_{A,\beta} \quad \text{and} \quad |f^A|_{m,s} = \inf\nolimits_{v\in\mathcal{K}_{A,\beta}}|v|_{m,s}, \tag{62}$$

Where $\mathcal{K}_{A,\beta} = \{v \in X^{m,s} | v|_A = \beta\}$. Likewise, given, in addition, a positive real number ε, we call smoothing (m,s)-spline relative to A, β and ε any solution, if any exists, of the problem : find f_ε^A such that

$$f_\varepsilon^A \in X^{m,s} \text{ and } \mathcal{J}_{A,\beta,\varepsilon}(f_\varepsilon^A) = \inf\nolimits_{v\in X^{m,s}} \mathcal{J}_{A,\beta,\varepsilon}(v) \tag{63}$$

where the functional $\mathcal{J}_{A,\beta,\varepsilon} : X^{m,s} \to \mathbb{R}$ is given by

$$\mathcal{J}_{A,\beta,\varepsilon}(v) = \sum\nolimits_{a\in A} |v(a) - \beta_a|^2 + \varepsilon|v|_{m,s}^2.$$

We observe that $\mathcal{K}_{A,\beta}$ and $\mathcal{J}_{A,\beta,\varepsilon}$ are well defined thanks to relation (59).

We recall some relevant facts about both kinds of (m,s)-splines (cf., for example, Duchon [29] or [22]). If A contains a P_{m-1}unisolvent subset, problems (62) and (63) have unique solutions f^A and f_ε^A , respectively. Both min- imization problems admit equivalent variational formulations. Therefore, problem (62) is equivalent to

$$f^A \in \mathcal{K}_{A,\beta} \text{and, for all } w \in \mathcal{K}_{A,0}\,, (\,f^A\,, w)_{m,s} = 0, \tag{64}$$

with $\mathcal{K}_{A,0} = \{\, v \in X^{m,s} \mid v|_A = 0\,\}$, while problem (63) is equivalent to

$$f_\varepsilon^A \in X^{m,s} \text{ and , for all} v \in X^{m,s}\,, \sum\nolimits_{a\in A} f_\varepsilon^A(a)v(a) + \varepsilon(f_\varepsilon^A, v)_{m,s} = \sum\nolimits_{a\in A}\beta_a v(a). \tag{65}$$

It follows from (65) and (66) that f^A and f_ε^A belong to the space $S_A = \{v \in X^{m,s} \mid \forall w \in \mathcal{K}_{A,0}\,, (v,w)_{m,s} = 0\,\}$, called space of (m,s)-spline functions relative to A. It is well known that every element v of $S_A$can be written in a unique way in the form

$$v(x) = \sum_{a \in A} \lambda_a \, \varphi(|x-a|) + \psi(x),$$

where φ is the conditionally positive definite basis function

$$\varphi(\rho) = \begin{cases} (-1)^{\lceil v \rceil} \rho^{2v} \,, & if \ v \neq \mathbb{N} \\ (-1)^{v+1} \rho^{2v} \log \rho \,, & if \ v \in \mathbb{N} \end{cases}$$

with $v = m + s - n/2$, the function ψ belongs to P_{m-1} , and the coefficients λ_a satisfy the vanishing moment condition

$$\forall v \in P_{m-1} \,, \ \sum_{a \in A} \lambda_a v(a) \ = \ 0.$$

We recall that $S_A \subset C^{\lceil 2v \rceil - 1}(\mathbb{R}^n)$. We finally remark that the most popular (m, s)- splines are those corresponding to the case $s = 0$, also known in the literature as polyharmonic, surface or thin plate splines, as well as $\mathbb{D}^m$ -splines over $\mathbb{R}^n$. These splines belong to $C^{2m-1-n}(\mathbb{R}^n)$. For $s = (n-1)/2$, one gets the pseudo-polynomial splines, which belong to $C^{2m-2}(\mathbb{R}^n)$.

Throughout this section, we denote by Ω a bounded domain in $\mathbb{R}^n$with a Lipschitz- continuous boundary. We start with an auxiliary result and then we establish our main result on error estimates for interpolating (m, s)-splines.

Lemma (1.2.13) [19]:

Suppose that (55) holds and let $= \dim P_{m-1}$. Then, there exists a constant $\eta > 0$ verifying the following property: for any finite set $A \subset \bar{\Omega}$ such that $\delta(A, \bar{\Omega}) \leq \eta$, there exists a P_{m-1}-unisolvent M-tuple $a \in A^{\mathfrak{M}}$ satisfying

$$\forall v \in X^{m,s} \,, C_1 \|v\|_{m,s} \leq [[v]]_{a,m,s} \leq C_2 \|v\|_{m,s},$$

where $\|\cdot\|_{m,s}$ is the norm given in (56), $[[\cdot]]_{a,m,s}$ is the norm defined by

$$\forall v \in X^{m,s} \,, [v]]_{a,m,s} = \left(\|v|_a\|_2^2 + |v|_{m,s}^2 \right)^{1/2} \quad ,$$

And C_1 and C_2 are positive constants independent of A and v.

Proof:

Cf. [22].

Theorem (1.1.14) [19]:

Suppose that (55) holds and let

$$\mathfrak{d}^* = \min\{\eta, \mathfrak{d}_{m+s}\} \tag{66}$$

where η and $\mathfrak{d}_{m+s}$ are the constants provided by Lemma (1.2.13) and Theorem (1.2.9) (applied with $r = m + s$), respectively. Then, we have:

For any finite set $A \subset \overline{\Omega}$ such that $\delta(A, \overline{\Omega}) \leq \mathfrak{d}^*$ and for any $\in W^{m+s,2}(\Omega)$, there exists a unique interpolating (m, s)-spline f^A relative to A and $(f(a))_{a\in A}$.

Let $q \in [1, \infty]$ and $l_0 = m + s - n(1/2 - 1/q)_+$. Then, there exist two positive constants C_1 and C_2 satisfying, for any finite set $A \subset \overline{\Omega}$ such that $d = \delta(A, \overline{\Omega}) \leq \mathfrak{d}^*$, for any $f \in W^{m+s,2}(\Omega)$ and for all $l = 0,\ldots, l_0 - 1$,

$$|f - f^A|_{l,q,\Omega} \leq C_1 d^{m+s-l-n(1/2-1/q)_+} |f - f^A|_{m+s,2,\Omega}$$

$$\leq C_2 d^{m+s-l-n(1/2-1/q)_+} \|f\|_{m+s,2,\Omega,} \quad (67)$$

where $\|f\|_{m+s,2,\Omega,}$ can be replaced by $|f|_{m+s,2,\Omega}$ if $s \leq 0$. These bounds also hold with $l = l_0$ if $m + s \in \mathbb{N}^*, l_0 \in \mathbb{N}$ and $q < \infty$.

Proof :

Let A be any finite subset of Ω such that $d = \delta(A, \overline{\Omega}) \leq \mathfrak{d}^*$ and let $f \in W^{m+s,2}(\Omega)$.

(i) It follows from Lemma 6.1 that $A^{\mathfrak{M}}$contains a P_{m-1} -unisolvent M-tuple. In other words, A contains a P_{m-1} -unisolvent subset and, consequently, as mentioned, f A exists and is unique.

(ii) We first observe that, f^Abelongs to $W^{m+s,2}(\Omega)$. Applying Theorem (1.2.9) (or, even simpler, Corollary (1.2.11)) with $r = m + s$, $p = 2$ and $u = f - f^A$, we get the first inequality in (67). Let us prove the second one. By Proposition (1.2.12) and Lemma (1.2.13), we have

$$|f - f^A|_{l,q,\Omega} \leq C_1 \|\mathfrak{P}f - f^A\|_{m,s} \leq C_2 [\![\mathfrak{P}f - f^A]\!]_{a,m,s},$$

with $C_1$and C_2independent of f and A. Since $f^A|_A = f|_A = \mathfrak{P}f|_A$, we get $[\![\mathfrak{P}f - f^A]\!]_{a,m,s} = |f - f^A|_{m,s}$. Likewise, from (64), we derive that $|\mathfrak{P}f - f^A|_{m,s} \leq |\mathfrak{P}f|_{m,s}$. Thus, we get $|f - f^A|_{m+,s,2,\Omega} \leq c_2 |\mathfrak{P}f|_{m,s}$. Inserting this bound in (67) and applying Proposition (1.2.12), we obtain the result.

We finish with a generalization of the error estimates given in [46] and [22, Corollary II-7.1]. To state it, we may consider, without loss of generality, that any set A in Theorem (1.2.14) is, in fact, the generic element A^d of a family of subsets of $\overline{\Omega}$. More precisely, we suppose that $\mathbb{D}$ is a subset of $(0, \mathfrak{d}^*]$, with $\mathfrak{d}^*$given ,such that $0 \in \overline{\mathbb{D}}$, and $(A^d)_{d\in\mathbb{D}}$ is any family of finite subsets of $\overline{\Omega}$ such that, for any $d \in \mathbb{D}$, $d = \delta(A^d, \overline{\Omega})$.

Corollary (1.2.15) [19]:

Suppose that (55), (66) and (68) hold. Let $q \in [1,\infty]$ and $l_0 = m + s - n(1/2 - 1/q)_+$. Then, for any $f \in W^{m+s,2}(\Omega)$ and for all $l = 0, \ldots, \lceil l_0 \rceil - 1$, we have

$$|f - f^A|_{l,q,\Omega} = o\left(d^{m+s-l-n(1/2-1/q)_+}\right), \qquad d \to 0$$

Where, for any $d \in \mathbb{D}$, f^A denotes the interpolating (m,s)-spline relative to A^d and $(f(a))_{a \in A^d}$. This error estimate also holds with $l = l_0$ if $m + s \in \mathbb{N}^*$, $l_0 \in \mathbb{N}$ and $q < \infty$. In addition, we have

$$|f - f^A|_{m+s,2,\Omega} = o(1), \qquad d \to 0.$$

Proof:

For any $f \in W^{m+s,2}(\Omega)$, it directly follows from (57) and [22] (see also [29]) that $\lim_{d \to 0} \left\| f - f^{A^d} \right\|_{m+s,2,\Omega} = 0$. This relation and Theorem (1.2.14) yield the result.

Remark (1.2.16) [19]:

Results like Theorem (1.2.14) and Corollary (1.2.15) are not new in the literature. Similar results, for particular cases of the parameters q and s, can be found, for example, in Duchon [30] (case $s = 0$ and $q \geq 2$), López de Silanes and Arcangéli [46] (case q = 2), Wu and Schaback [51] (case q = ∞), Light and Wayne [35] (case s = 0), Johnson [32] (case $s = 0$), and Narcowich et al. [40] (case $s = 0$).

Remark (1.2.17) [19]:

The L^q -approximation order provided by Corollary (1.2.15) is not optimal in all cases. For $(m,0)$-splines, Johnson (cf. [33] and references therein) has proved that

$$|f - f^A|_{0,q,\Omega} = O\left(d^{m+1/q}\right), \qquad d \to 0$$

under the following conditions: Ω has the uniform C^{2m} -regularity and f belongs to the Besov space $B_{2,q}^{m+1/q}$, with $1 \leq q \leq 2$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz-continuous boundary. We first establish, for smoothing (m,s)-splines, a result analogous to Theorem (1.2.14).

Theorem (1.2.18) [19]:

Suppose that (55) holds and let $\mathfrak{d}^*$ be the constant given by (67). Then, we have:

(i) For any finite set $A \subset \overline{\Omega}$ such that $\delta(A, \overline{\Omega}) \leq \mathfrak{d}^*$, for any $f \in W^{m+s,2}(\Omega)$ and for any $\varepsilon > 0$, there exists a unique smoothing (m,s)-spline f_ε^A relative to A, $(f(a))_{a \in A}$ and ε.

(ii) Let $q \in [1,\infty]$, $x \in [2,\infty]$, $l_0 = m + s - n(1/2 - 1/q)_+$ and $\gamma = max\{q, x\}$. Then, there exist two positive constants C_1 and C_2 satisfying, for any finite set $A \subset \overline{\Omega}$ such that

$\delta(A,\overline{\Omega}) \leq \mathfrak{d}^*$, for any $f \in W^{m+s,2}(\Omega)$, for any $\varepsilon > 0$, and for all $l = 0,\dots, \lceil l_0 \rceil - 1$,

$$|f - f_\varepsilon^A|_{l,q,\Omega} \leq C_1\big(d^{m+s-l-n(1/2-1/q)_+}|f - f_\varepsilon^A|_{m+s,2,\Omega} + d^{n/\gamma - l}\varepsilon^{1/2}\|f\|_{m+s,2,\Omega,}\big) \tag{68}$$

and, whenever $d \leq 1$,

$$|f - f_\varepsilon^A|_{l,q,\Omega} \leq C_2\big(d^{m+s-l-n(1/2-1/q)_+}|f - f_\varepsilon^A|_{m+s,2,\Omega} + d^{n/\gamma - l}\varepsilon^{1/2}\big)\|f\|_{m+s,2,\Omega,} \tag{69}$$

where $\|f\|_{m+s,2,\Omega}$ can be replaced by $|f|_{m+s,2,\Omega}$ if $s \leq 0$. Both bounds also hold with $l = l_0$ if $m + s \in \mathbb{N}^*, l_0 \in \mathbb{N}$ and $q < \infty$.

Proof :

Let A be any finite set in $\overline{\Omega}$ such that $d = \delta(A,\overline{\Omega}) \leq \mathfrak{d}^*$, let $f \in W^{\mathrm{m+s,2}}(\Omega)$ and let $\varepsilon > 0$.

(i) By Lemma (1.2.13), the set A contains a P_{m-1} -unisolvent subset and so f_ε^A A exists and is unique.

(ii) We first remark that, by (64), f_ε^A belongs to $f \in W^{\mathrm{m+s,2}}(\Omega)$. Now, it follows from Proposition (1.2.12) and (64), taking $\beta(f(a))_{a\in A}$ and $= \mathfrak{P}f$, that

$$\|(f - f_\varepsilon^A)|_A\|_2^2 \varepsilon |f_\varepsilon^A|_{m,s}^2 \leq |\mathfrak{P}f|_{m,s}^2,$$

from which we deduce that

$$|f_\varepsilon^A|_{m,s}^2 \leq |\mathfrak{P}f|_{m,s}^2 \tag{70}$$

and, together with Jensen's inequality since $x \geq 2$, that

$$\|(f - f_\varepsilon^A)|_A\|_x \leq \|(f - f_\varepsilon^A)|_A\|_2 \leq \varepsilon^{1/2}|\mathfrak{P}f|_{m,s}^2, \tag{71}$$

This last bound, Proposition (1.2.11) and Theorem (1.2.9) (applied with $r = m + s, p = 2$ and $u = f - f_\varepsilon^A$) yield (68). To complete the proof, we observe that, by Proposition (1.2.12) and Lemma (1.2.13), we have

$$|f - f_\varepsilon^A|_{m+s,2,\Omega} \leq c_1\|\mathfrak{P}f - f_\varepsilon^A\|_{m,s} \leq c_2[\![\mathfrak{P}f - f_\varepsilon^A]\!]_{a,m,s} \tag{72}$$

with c_1 and $c_2$independent of f , A and ε. As $f|_A = \mathfrak{P}f|_A$, if follows from (70), (71) and (72) that

$$|f - f_\varepsilon^A|_{m+s,2,\Omega} \leq 2c_2(1 + \varepsilon^{1/2})|\mathfrak{P}f|_{m,s}. \tag{73}$$

Since $+s - n(1/2 - 1/q)_+ > n/\gamma$, whenever $d \leq 1$, we finally derive (69) from (68), (73) and Proposition (1.2.12).

Remark (1.2.19) [19]:

Theorem (1.2.18) can be considered as a generalization of previous results by López de Silanes and Arcangéli [46] (case s arbitrary, $q = x = 2$) and Wendland and

Rieger [50] (case $s = 0, q$ arbitrary, $x = \infty$).

As in Sect. 6, we now turn our attention to error estimates stated in terms of a family $(A^d)_{d\in\mathbb{D}}$ of subsets of $\overline{\Omega}$ satisfying (68). We need, in addition, two main hypotheses The first one concerns the smoothing parameter ε. We assume that $\varepsilon = \varepsilon(d)$ is a strictly positive function of d verifying the relation

$$\varepsilon = o(d^{-n}), \quad d \to 0. \tag{74}$$

Note that assumption (74), quite reasonable from a practical point of view, is not really a restricting condition, since ε has not to go to 0 as d does and may even be unbounded. This last point is very important. On the one hand, the convergence theorem (1.2.18) (see below) holds for unbounded ε's. On the other hand , in the case of noisy data, the convergence requires that ε (suitably) increases to ∞.

Later, we shall also make use of the condition

$$N = O(d^{-n}), \quad d \to 0, \tag{75}$$

where N = card A^d (to simplify, we write N instead of $N(d)$).

Remark (1.2.20) [19]:

Let us explain in some detail the meaning of hypothesis (75). From (68), it is easy to check that

$$\exists C > 0, \forall d \in \mathbb{D}, Nd^n \geq C. \tag{76}$$

Hence, hypothesis (76) implies that, as $d \to 0, N$ tends to ∞ at the same rate as d^{-n} . So, hypothesis (76) means that, asymptotically, the points of A^d should be regularly distributed in $\overline{\Omega}$.

Consider an example. Let Ω be the rectangle $(0,2) \times (0,1)$ in $\mathbb{R}^2$. First, for any$\nu \in \mathbb{N}^*$, subdivide Ω into $2\nu^2$ equal squares and define A^das the set made up of $2\nu^2$points, such that each square contains just one of these points. One easily verifies that $1/(\nu\sqrt{2}) \leq d \leq \sqrt{2}/\nu$. So $Nd^2 \leq 4$, and (75) is satisfied. Next, subdivide Ω as follows: the square $(0,1) \times (0,1)$ is subdivided into ν^2equal subsquares, and the square $(1,2) \times (0,1)$ into ν^4equal subsquares. Define A^das previously, each sub square containing just one point of the set A^d. Here, we have $N = \nu^2 + \nu^4$whereas d is unchanged as soon as $\nu > 1$, so $Nd^2 \geq (1/2)(1 + \nu^2)$ and (75) is not satisfied.Before establishing new error estimates, we recall a convergence result.

Theorem (1.2.21) [19]:

Suppose that (55), (69) and (76) hold. Then, we have

$$\forall f \in W^{m+s,2}(\Omega), \lim_{d\to 0} \left\| f_\varepsilon^{A^d} - \mathfrak{P} f \right\|_{m,s} = 0$$

where, for any $d \in \mathbb{D}$, $f_\varepsilon^{A^d}$ denotes the smoothing (m, s)-spline relative to $A^d, (f(a))_{a\in A^d}$ and ε.

Proof:

cf. [22] (see also [46] and [47]). The following theorem provides error estimates for smoothing (m, s)-splines. It has been previously proved by Utreras [48] when $s = 0$ and $q = 2$.

Theorem (1.2.22) [19]:

Suppose that (55), (68), (74) and (75) hold, and assume that

$$\exists C > 0, \varepsilon \geq C\, d^{2m+2s-n}\,,\ d \to 0. \tag{77}$$

Let $q \in [1, \infty)$ and let$l_0 = m + s - n(1/2 - 1/q)_+$. Then, for any $f \in W^{\,m+s,2}(\Omega)$ and for any $l = 0, \dots, \lceil l_0 \rceil - 1$, we have

$$\left| f - f_\varepsilon^{A^d} \right|_{l,q,\Omega} = O\,(t^{m+s-l\,-n(1/2-1/q\,)_+}),\quad d \to 0, \tag{78}$$

where $= (\varepsilon/N)^{1/(2m+2s)}$. In addition, we have

$$\left| f - f_\varepsilon^{A^d} \right|_{m+s,2,\Omega} = o(1), \qquad d \to 0. \tag{79}$$

Remark (1.2.23) [19]:

From (74) and (76), it is readily seen that

$$\varepsilon/N = o(1),\quad d \to 0. \tag{80}$$

Thus, the error estimates (78) make sense. Let us point out that the convergence result of Theorem (1.2.22) does not need $\varepsilon \to 0$.

Proof of Theorem (1.2.22):

Let $f \in W^{\mathrm{m+s,2}}(\Omega)$. It follows

$$\lim_{d\to 0} \left\| f_\varepsilon^{A^d} - f \right\|_{m+s,2,\Omega} = 0 \tag{81}$$

This implies (79). Now, let $l \in \{0, \dots, \lceil l_0 \rceil - 1\}$ and let us see (78).

Let $\lambda = l + n(1/2 - 1/q)_+$, which is a real number belonging to $[0, m + s)$. We remark that

$$W^{\lambda,2}(\Omega) \hookrightarrow W^{l,q}(\Omega) \tag{82}$$

For $q \leq 2$, this imbedding is just a consequence of (32). For $q > 2$, we can apply the relation (1,4,4,5) in Grisvard [31]. By (82), there exists a positive constant , independent of f and d , such that

$$\left|f - f_\varepsilon^{A^d}\right|_{l,q,\Omega} \leq \left\|f - f_\varepsilon^{A^d}\right\|_{l,q,\Omega} \leq C \leq \left\|f - f_\varepsilon^{A^d}\right\|_{\lambda,2,\Omega}.$$

Consequently, in order to establish (78), it suffices to prove that

$$\left\|f - f_\varepsilon^{A^d}\right\|_{\lambda,2,\Omega} = O\,(t^{m+s-\lambda}), \quad d \to 0 \tag{83}$$

with $= (\varepsilon/N)^{1/(2m+2s)}$. The case $\lambda = 0$, which happens for $l = 0$ and $q \leq 2$, is easy to handle. By (69), choosing $x = 2$, and (77), it is immediately checked that

$$\left|f - f_\varepsilon^{A^d}\right|_{\lambda,2,\Omega} = O\,(d^{n/2}\,\varepsilon^{1/2}), \quad d \to 0.$$

Thus, taking (75) into account, we get

$$\left\|f - f_\varepsilon^{A^d}\right\|_{0,2,\Omega} = O\,((\varepsilon/N)^{1/2}) = O\,(t^{m+s}), \quad d \to 0 \tag{84}$$

which is just (83) for $\lambda = 0$.

Let us finally discuss the case $\lambda > 0$. From Grisvard [31], there exists a positive constant K such that, for any $\alpha > 0$ small enough and for any $v \in W^{m+s,2}(\Omega)$

$$\|v\|_{\lambda,2,\Omega} \leq \alpha\|v\|_{m+s,2,\Omega} + K\alpha^{-\lambda/(m+s-\lambda)}\|v\|_{0,2,\Omega.} \tag{85}$$

It follows from (72) and (74) that $t \to 0$ as $d \to 0$. Thus, for any $d \in \mathbb{D}$ small enough, we can replace α by $t^{m+s-\lambda}$in (85). By taking, in addition, $v = f - f_\varepsilon^{A^d}$, for any $d \in \mathbb{D}$ small enough, we obtain

$$\left\|f - f_\varepsilon^{A^d}\right\|_{\lambda,2,\Omega} \leq t^{m+s-\lambda}\left\|f - f_\varepsilon^{A^d}\right\|_{m+s,2,\Omega} + Kt^{-\lambda}\left\|f - f_\varepsilon^{A^d}\right\|_{0,2,\Omega.}$$

This relation, together with (81) and (84), implies (83) for $\lambda > 0$. The proof is complete.

Remark (1.2.24) [19]:

With the notations and under the hypotheses of Theorem (1.2.18), if one chooses $x = 2$ (or even $x \leq q$ if $q > 2$) and assumes (67) and that

$$\exists C > 0, \varepsilon \leq C\,d^{2m+2s-n}, \; d \to 0 \tag{86}$$

it is then obvious that, for $l = 0, \ldots, \lceil l_0 \rceil - 1$ (and eventually $l = l_0$), the relation (69) yields the error estimate

$$|\,f - f_\varepsilon^{A^d}|_{l,q,\Omega} = O\,(d^{m+s-l-n(1/2-1/q)_+}), \quad d \to 0 \tag{87}$$

In [50], under the more restrictive hypothesis $\varepsilon \leq d^{2m-2n(1/2-1/q)_+}$, Wendland and Rieger obtained (87) for $s = 0$ and $l < m - n/2$.

We point out that, except for $q = \infty$, the estimate (87) is just a particular case of Theorem (1.2.22):by selecting $\varepsilon = Cd^{2m+2s-n}$ for some $C > 0$ (so (77) holds), the relation (78), together with (76), yields (87).

Now, let us notice an important point. Clearly, (86) implies that $\varepsilon \to 0$ as $d \to 0$. When the data are exact, condition (86) is sufficient (but unnecessary, according to Remark (1.2.23)) to get the convergence in $X^{m,s}$. When the data are noisy, the situation is radically different. It can be checked that the deterministic convergence over Ω cannot generally be ensured in this case. In fact, this problem needs a stochastic approach (cf. [23, 27, 28, 37, 43, 48, 49]). We shall see in Sect. 8.2 that, under a usual random noise hypothesis, the convergence of smoothing D^m -splines in quadratic mean over $\mathbb{R}^n$requires that ε must grow to ∞ as $d \to 0$ .Let Ω be a bounded domain in $\mathbb{R}^n$with a Lipschitz-continuous boundary. Assume that (55) and (67) are satisfied, and, for any $d \in \mathbb{D}$, let $v^d = (v_a^d)_{a \in A^d} = \mathbb{R}^N$ be any error vector (we recall that $N = \text{card} A^d$). Likewise, let be a given function in $W^{m+s,2}(\Omega)$. Then, for any $d \in \mathbb{D}$ and for any $\varepsilon > 0$, let $\tilde{f}_\varepsilon^{A^d}$denote the smoothing (m, s)-spline relative to A^d, $(f(a) + v_a^d)_{a \in A^d}$ and ε and let $f_\varepsilon^{A^d}$be the smoothing (m, s)-spline relative to A^d, $(f(a))_{a \in A^d}$ and ε. Finally, let $e_\varepsilon^d = \tilde{f}_\varepsilon^{A^d} - f_\varepsilon^{A^d}$. It is clear that $e_\varepsilon^d$is the smoothing (m, s)-spline relative to A^d, v^d and ε, due to the linearity of the operator that assigns to any $\beta \in \mathbb{R}^N$ the smoothing (m, s)-spline relative to A^d, β and ε.

We assume in what follows that ε is a function of d .

Let us point out a necessary condition for the deterministic convergence over Ω , i.e. such that $\lim_{d \to 0} \left\| \tilde{f}_\varepsilon^{A^d} - f \right\|_{m+s,2,\Omega} = 0$.

Proposition (1.2.25) [19]:

Suppose that (55), (67) and (74) hold. Then, a necessary condition for the deterministic convergence over Ω is that

$$\forall v \in X^{m,s}, \lim_{d \to 0} \frac{1}{N} \sum_{a \in A^d} v_a^d v(a) = 0. \qquad (88)$$

Proof For any $d \in \mathbb{D}$, it is clear that (64) holds with A^d, $\tilde{f}_\varepsilon^{A^d}$ and $f(a) + v_a^d$ instead of A, f A_ε^A and β_a . Thus, for any ∈ $X^{m,s}$, we have

$$\frac{1}{N} \sum_{a \in A^d} \left(\tilde{f}_\varepsilon^{A^d}(a) - f(a) \right) v(a) + \frac{\varepsilon}{N} \left(\tilde{f}_\varepsilon^{A^d}, v \right)_{m,s} = \frac{1}{N} \sum_{a \in A^d} v_a^d v(a) = 0. \qquad (89)$$

The convergence over Ω , together the imbedding $W^{m+s,2}(\Omega) \hookrightarrow C^0(\bar{\Omega})$, implies that the first term on the left-hand side of (89) tends to 0 when $d \to 0$. Likewise, the convergence over Ω also implies that

$$\exists C > 0, \left\| \tilde{f}_{\varepsilon}^{A^d} \right\|_{m+s,2,\Omega} \leq C \ , \quad d \to 0. \tag{90}$$

Now, it is not difficult to see that, for any $d \in \mathbb{D}$, we have $\tilde{f}_{\varepsilon}^{A^d} = \mathfrak{P}\varphi$, Where$\varphi = \tilde{f}_{\varepsilon}^{A^d}|_{\Omega}$and $\mathfrak{P}$ is the operator introduced in Proposition (1.2.12). From the continuity of $\mathfrak{P}$ and (90), we deduce that

$$\exists C > 0, \left| \tilde{f}_{\varepsilon}^{A^d} \right|_{m+s} \leq C \ , \quad d \to 0.$$

Hence, taking account of (74) and (76), the second term on the left-hand side of (91) also tends to 0 as $d \to 0$. The result follows.

Of course, condition (89) is verified under the assumption

$$\lim_{d \to 0} \sup_{a \in A^d} |v_a^d| = 0. \tag{91}$$

The deterministic convergence over Ω has even been proved (cf. [22]) under an unrealistic assumption implying, like (91), that the errors on the data decrease to 0 as $d \to 0$. But (88) is not verified in general. Consider, for example, the case where the errors are such that

$$\exists \alpha > 0, \forall d \in \mathbb{D}, \forall a \in A^d , v_a^d > \alpha,$$

and take $v = 1$. Then, the left-hand side of (88) is greater than 0. So, we cannot prove in general the deterministic convergence over Ω for noisy data.

On the contrary, (88), considered as a stochastic relation, makes sense under additional hypotheses. Assume the "random noise hypothesis" for any $d \in \mathbb{D}$, v^d is a vector of independent, identically distributed random variables, with null mean and the same positive variance η^2 , i.e., such that

$$\forall a, b \in A^d, E(v_a^d) = 0 \text{and} E\left(v_a^d v_b^d\right) = \begin{cases} \eta^2, & \text{if } a = b, \\ 0, & \text{if } a \neq b, \end{cases} \tag{92}$$

where E denotes the mathematical expectation. Moreover, suppose that the family of data sets is an increasing sequence of ordered sets A^j , with $j \in \mathbb{N}$, made up of $N = N(j)$ points. Then, it follows straightforwardly from the strong Law of Large Numbers (cf. Bouleau [25]) that

$$\forall v \in X^{m,s}, \lim_{j \to 0} \frac{1}{N} \sum_{a \in A^j} v_a^j v(a) = 0,$$

Where $v^j = (v_a^j)$ denotes the error vector on A^j , holds almost surely. Thus, we can hope for a positive conclusion in the stochastic case.

From now on, we restrict our study to the case of smoothing D^m -splines, i.e., we suppose that $s = 0$ (so condition (55) is just the inequality $> \frac{n}{2}$). Moreover, we formulate two additional hypotheses.

The first one is the quasi-uniformity condition

$$\exists C > 0, \forall d \in \mathbb{D}, d \le C\, q(A^d), \tag{93}$$

where $q(A^d) = \frac{1}{2}\min\{\, |a - b| |\, a, b \in A^d, a = b\,\}$ is the separation radius of A^d. It is not difficult to see that (93) implies (75). The second hypothesis stands as follows:

$\varepsilon = N^{n/(n+2m)}\omega(N)$, with $\lim_{N\to\infty} \omega(N)$ and

$$\omega(N) = O\left(N^{2m/(n+2m)}\right)\ ,\ N \to \infty. \tag{94}$$

It is readily seen that (75) and (94) imply (74). So (84) and (94) imply (74). The following theorem, where $\|\cdot\|_{m,0}$ denotes the norm defined by (56) and $\mathfrak{P}$ the operator introduced in generalizes results by Utreras [48] (see also Ragozin [43]).

Theorem (1.2.26) [19]:

Suppose that (55) and (67) are verified for s = 0, and that (92), (93), and (94) hold. Then, $\tilde{\mathrm{f}}_{\varepsilon}^{\mathrm{A^d}}$ converges to $\mathfrak{P}$ in quadratic mean over $\mathbb{R}^n$, i.e.,

$$E\left[\left\|\tilde{\mathrm{f}}_{\varepsilon}^{\mathrm{A^d}} - \mathfrak{P}f\right\|_{m,0}^{2}\right] = (1), \quad d \to 0. \tag{95}$$

Moreover, as $d \to 0$,

$$E\left[\left|\tilde{\mathrm{f}}_{\varepsilon}^{\mathrm{A^d}} - f\right|_{m,2,\Omega}^{2}\right] = O(1),$$

And, for any $l = 0, \dots, m-1$,

$$E\left[\left|\tilde{\mathrm{f}}_{\varepsilon}^{\mathrm{A^d}} - f\right|_{l,2,\Omega}^{2}\right] = O\left(N^{-2(m-l)/(n+2m)}(\omega(N))^{(m-l)/m}\right).$$

Proof:

For the last two relations, see [37]. Now, let us prove (95). From Theorem (1.2.21) one has $\lim_{d\to 0}\left\|f_{\varepsilon}^{\mathrm{A^d}} - \mathfrak{P}f\right\|_{m,0} = o(1)$. Since $\tilde{\mathrm{f}}_{\varepsilon}^{\mathrm{A^d}} = f_{\varepsilon}^{\mathrm{A^d}} + \mathrm{e}_{\varepsilon}^{\mathrm{A^d}}$ it is enough to prove that

$$E\left[\left\|\mathrm{e}_{\varepsilon}^{\mathrm{A^d}}\right\|_{m,0}^{2}\right] = O(1), \quad d \to 0.$$

For any $d \in \mathbb{D}$, the equation (64) for the smoothing spline $\mathrm{e}_{\varepsilon}^{\mathrm{A^d}}$, with $v = \mathrm{e}_{\varepsilon}^{\mathrm{A^d}}$, can be written as

$$\sum_{a\in A^d}\left(e_{\varepsilon}^{A^d}(a)\right)^2 + \left|e_{\varepsilon}^{A^d}\right|_{m,0}^{2} = \sum_{a\in A^d} v_a^d\, e_{\varepsilon}^{A^d}(a).$$

From (94), it follows that $\varepsilon = o(N)$ as $d \to 0$. Then, from this relation and the relation (50) of [23], we derive that there exists a constant $C > 0$ such that

$$\frac{1}{CN}\sum_{a\in A^d}\left(e_\varepsilon^{A^d}(a)\right)^2 + \left|e_\varepsilon^{A^d}\right|_{m,0}^2 \leq \sum_{a\in A^d} v_a^d e_\varepsilon^{A^d}(a) \qquad d \to 0. \tag{96}$$

where $\mathcal{R}^d$ denotes a matrix, depending on the basis interpolating D^m -splines relative to A^d, introduced by Utreras [48], and $(v^d)^t$ the transposed vector of v^d (for more details see [23]). Under assumption (93), F. Utreras showed that

$$\mathrm{Tr}(\mathcal{R}^d) = O(N/\varepsilon)^{n/2m}\ , \qquad d \to 0,$$

Where $\mathrm{Tr}(\mathcal{R}^d)$ denotes the trace of the matrix $\mathcal{R}^d$. Now, the left-hand side of (96) involves a norm on $X^{m,0}$, which is uniformly equivalent in d to the norm $\|\cdot\|_{m,0}$ (this result can be deduced from [48]). So, there exists a constant $C > 0$ such that, as $d \to 0$

$$E\left[\left\|e_\varepsilon^{A^d}\right\|_{m,0}^2\right] \leq C'E\left[\frac{1}{CN}\sum_{a\in A^d}\left(e_\varepsilon^{A^d}(a)\right)^2 + \left|e_\varepsilon^{A^d}\right|_{m,0}^2\right] \leq \frac{C'}{\varepsilon}E[(v^d)^t\mathcal{R}^d(v^d)].$$

Thus, a sufficient condition for the convergence in quadratic mean in $X^{m,0}$ of e $e_\varepsilon^{A^d}$ is that the last term tends to 0. From (92), it follows, for any $d \in \mathbb{D}$, that

$$E[(v^d)^t\mathcal{R}^d(v^d)] = \eta^2\,\mathrm{Tr}(\mathcal{R}^d).$$

Finally, using (94), we have

$$E\left[\left\|e_\varepsilon^{A^d}\right\|_{m,0}^2\right] = O\left(\eta^2\omega(N)^{-(n+2m)/2m}\right) = O(1), \qquad d \to 0,$$

and the result follows.

Remark (1.2.27) [19]:

Subject to additional stronger assumptions, we can obtain a result analogous to that of Theorem (1.2.26), but valid almost surely. We do not detail this point, for which we refer to Arcangéli and Ycart [23].

Remark (1.2.28) [19]:

It is readily seen that, under the hypotheses of Theorem (1.2.26), for any $l = 0, \ldots, m-1$, we have, as $d \to 0$,

$$E\left[\left|\tilde{f}_\varepsilon^{A^d} - f\right|_{l,2,\Omega}^2\right] = O\left((\varepsilon/N)^{(m-l)/m}\right).$$

So, as could have been conjectured, we get in Theorem (1.2.26), in the sense of mathematical expectation, the same estimations as in Theorem (1.2.22) when $s = 0$ and $q = 2$. Now, we prove our final result.

Proposition (1.2.29) [19]:

Suppose that (55) and (67) are verified for $s = 0$, that (92) and (93) hold and that $\varepsilon \leq N^{n/(n+2m)}$. Then, $\tilde{f}_\varepsilon^{A^d}$ does not converge to $\mathfrak{P}f$ in quadratic mean over $\mathbb{R}^n$, as $d \to 0$.

Proof:

We refer to the proof in [23]. The proof shows, taking $\varepsilon = N^{n/(n+2m)}$, that

$$\lim_{d\to 0} \inf E\left[\left|e_\varepsilon^{A^d}\right|_{m,0}^2\right] > 0.$$

Consider the relation (4.11) in this section. Of course, it may be written

$$E\left[\left|e_\varepsilon^{A^d}\right|_{m,0}^2\right] \geq \eta^2 \sum_{i=1}^{N} \frac{(1/N)\mu_i^d}{(1 + (\varepsilon/N)\mu_i^d)^2},$$

where the μ_i^d are positive numbers. Clearly, any term of the sum is a decreasing function of ε. So, we infer that, if $\varepsilon \leq N^{n/(n+2m)}$, i.e., if

$$\forall d \in \mathbb{D}, \varepsilon(d) \leq (N(d))^{n/(n+2m)},$$

the result follows. Notice that the proposition still holds if the preceding relation is only valid for d small enough.

Remark (1.2.30) [19]:

The results of Theorem (1.2.26) and Proposition (1.2.29) can be improved in the following form. Suppose that (55) and (67) are verified for $s = 0$, and that (74), (92) and (93) hold. Then, the condition $\lim_{d\to 0} \varepsilon N^{-\frac{n}{n+2m}} = \infty$ is necessary and sufficient for the convergence in quadratic mean of $\tilde{f}_\varepsilon^{A^d}$ to $\mathfrak{P}f$, when $d \to 0$.

As a conclusion from the preceding results, we would like to emphasize that, in the stochastic case of data perturbed by a noise verifying the random noise hypothesis (92), no assumption implying that ε goes to 0 or even remains bounded, as $d \to 0$, is acceptable.