Chapter 1

Weighted Soblove Spaces with Zeros and Critical Points

Let $u^{\#}$ be the Fefferman-Stein sharp function of u, and for $1 < r < \infty$, let $M_r u$ be an appropriate version of the Hardy-Littlewood maximal function of u.If A is a pseudodifferential operator of order 0, then there is a constant $c > 0$ such that the pointwise estimate $(Au)^{\#}(x)cM_r u(x)$ holds for all $x \in R^n$ and all Schwartz functions u. In certain cases the zeros themselves have the sameasymptotic limit distribution, while in other cases we can only ascertain that the support of a limit distribution lies within a specified set in the complex plane. One of our tools, which is of independent interest, is a new result on zero distributions of asymptotically extremal polynomials. Our results are illustrated by numerical computations for the case of two disjoint intervals.We also describe the numerical methods that were used.

Section(1.1) :Pseudo Differential Operators with Smooth Symbols:

In this section, we show boundedness results for pseudodifferential operators on weighted L^p spaces. The methods are different from those which depend upon a point wise estimate. Since this estimate does not rely on properties of weight functions, it is of independent interest and may be of further use in discovering how pseudo differential operators preserve various classes of functions and their differentiability properties.

If $1 < p < \infty$, a nonnegative function w belongs to $A_p(R^n)$ if:

(i) $w \in L^1_{\text{loc}}(R^n)$;

(ii) $\sup_Q \left(\frac{1}{|Q|}\int_Q \; wdx\right)\left(\frac{1}{|Q|}\int_Q \; w^{-1/(p-1)}dx\right)^{p-1} < \infty$,

where the supremum is taken over all cubes Q in R.

Coifman, Fefferman, Hunt, Muckenhoupt, and Wheeden have shown [1], [2], [3] that a weight function w satisfies the A_p condition if and only if the Hardy-Littlewood maximal operator or classical singular integral operators are bounded on $L^p(R^n, wdx)$. Our boundedness results for pseudodifferential operators will also apply to spaces with A_p weight functions.

We shall say that the function $a(x,\xi) \in C^\infty(R^n \times R^n)$ is a symbol of order m if it satisfies the estimates$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta a(x,\xi)\right| \le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$ for all multi-indices α and β. A symbol of order $-\infty$ is one which satisfies the above estimates for each real number m. If $a(x,\xi)$ is a symbol of order m, then it defines a pseudodifferential operator A, of order m, by the formula

$$Au(x) = \int_{R^n} a(x,\xi)\hat{u}(\xi)e^{2\pi i x\cdot\xi}d\xi\,.$$

To begin with, A is defined only on the space of Schwartz functions, where the Fourier transform w of the function u is given by $\hat{u}(\xi) = \int_{R^n} u(x)e^{-2\pi i x\cdot\xi}dx$.

That A can be extended to a larger class of functions is the main result of this section , contain this result, which we summarize as: Suppose $1 < p < \infty$. Every pseudodifferential operator of order 0 has a bounded extension to $L^p(R^n, wdx)$ if and only if $w \in A_p(R^n)$.

The necessity of the A_p condition is proved using a modification of an argument by Coifman and Fefferman.The sufficiency is proved by controlling the pseudodifferential operator with various versions of the Hardy-Littlewood maximal operator, which appeared in [4]. With this goal in mind,we make the following definitions:

(i) $Mf(x) =$ the Hardy-Littlewood maximal function of $f = \sup_Q \frac{1}{|Q|}\int_Q |f(y)|dy,$

the supremum being taken over all cubes Q containing x;

(ii)$M_r f(x) = \sup_Q \left(\frac{1}{|Q|}\int_Q |f(y)|^r dy\right)^{\frac{1}{r}},$

the supremum being taken over all cubes Q containing x;

(iii)$f^*(x) =$ the dyadic maximal function of $f = \sup_Q \frac{1}{|Q|}\int_Q |f(y)|dy$

the supremum being taken over all dyadic cubes Q, with sides parallel to the axes, containing x;

(iv)$f^{\#}(x) = \sup_Q \frac{1}{|Q|}\int_Q \left|f(y) - f_Q\right|dy,$

where the supremum is taken over all cubes Q containing x, and f_Q is the average value of f on the cube Q.

Note that f^* enjoys many of the properties of the more usual maximal function Mf; in particular, $|f(x)| \le f^*(x)$ a.e., and the operator $f \to f^*$ is bounded on $L^p(R^n, wdx)$ whenever $1 < p < \infty$ and $w \in A_p(R^n)$ [1].

In addition to all the foregoing maximal function machinery, the proof of the result requires the following pointwise estimate.

Theorem (1.1.1)[5]. Suppose $1 < r < \infty$, and let A be a pseudodifferential operator of order 0. Then there is a constant $c > 0$ such that the pointwise estimate$(Au)^{\#}(x) \le cM_r u(x)$ holds for all $x \in R^n$ and all Schwartz functions u.

Armed with these two theorems, we then define weighted Sobolev spaces in R^n and prove the usual a priori estimates of elliptic differential operators. We also formulate the A_p condition for a compact manifold without boundary and show that the condition is invariant under coordinate changes. In the setting of a manifold, further results are the construction of weighted Sobolev spaces of fractional order, a version of Sobolev's theorem, and coercive estimates for elliptic pseudo differential operators. Note that the theorem above has been proved for classical singular integral operators by Cordoba and Fefferman. Our theorem shows that the method works for "variable coefficient" operators defined by non-homogeneous kernels and that these operators can be used to give painless constructions of weighted Sobolev spaces.

Until further notice, $\|\cdot\|_p$ will denote the norm in the space $L^p(R^n, wdx)$; w will always be a weight function of class $A_p(R^n)$. We shall prove estimates of the form $\|Au\|_p \le c\|u\|_p$ for u a Schwartz function and A a pseudodifferential operator of order 0. The next lemma shows that once this is done, A can be defined as a bounded operator on $L^p(R^n, wdx)$.

Lemma (1.1.2)[5]. $\mathcal{S}$ the set of all Schwartz functions, is dense in $L^p(R^n, wdx)$, $1 < p < \infty$.

Proof. We first show that smooth functions with compact support are dense in L^p.

Given f in L^p and $\varepsilon > 0$, choose a continuous function g with compact support such that $\|f - g\|_p < \varepsilon/2$ (see[6]).

Now let ϕ be a positive-valued C^∞ function supported in the unit ball of R^n with total integral (1).

Define $\phi_t(x) = t^{-n}\phi(x/t)$for $t > 0$.

It is standard knowledge that

(i)$\phi_t * g \in C_0^\infty(R^n)$ for all $t > 0$, and

(ii)$\phi_t * g \to g$, as $t \to 0$, uniformly on compact subsets of R^n.

If K is a large ball containing the support of g in its interior, pick t small enough that $\|g - \phi_t * g\|_\infty < \frac{\varepsilon}{2}\left(\int_K wdx\right)^{-1/p}$. Then $\|f - \phi_t * g\|_p \le \|f - g\|_p + \|g - \phi_t * g\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$; this shows that

$C_0^\infty(R^n)$ is dense in $L^p(R^n, wdx)$.It remains only to show that$\mathcal{S} \subset L^p(R^n, wdx)$.

If $w \in A_p(R^n)$, then [2] implies that $\int_{R^n} w(x)/(1+|x|)^k dx < \infty$ for large enough k. But if $u \in \mathcal{S}$, then $|w(x)| \le C_k/(1+|x|)^{k/p}$, which shown the assertion in the first sentence of this paragraph. The A_p condition is a necessary one for continuity of even the best-behaved pseudo differential operators; the proof of this fact is adapted from [1]. From now on, if p is a real number between 1 and ∞, p' will denote its conjugate, the number such that $1/p + 1/p' = 1$.

Theorem (1.1.3)[5]. Suppose w is a nonnegative locally integrable function whose zero-set has Lebesgue measure 0. If every pseudo differential operator of order- ∞ is continuous on $L^p(R^n, wdx)$, then $w \in A_p(R^n), 1 < p < \infty$.

Proof. We first establish that $w^{-1/(p-1)} \in L^1(Q, dx)$ for any cube Q in R^n. Suppose Q is a cube such that $w^{-1/(p-1)} \notin L^1(Q, dx)$; then $w^{-1/p} \notin L^{p'}(Q, dx)$, and there is a function $\phi \in L^p(Q, dx)$ such that $\int_Q \phi w^{-1/p} dx = \infty$. Let $\psi = w^{-1/p}$, and let $\tau \in C_0^\infty(R^n)$ have the value 1 in the set $Q - Q = \{x - y : x, y \in Q\}$. The operator $Tu = \tau * u$ is a pseudodifferential operator of order- ∞ (its symbol, $\hat{\tau}$, is rapidly decreasing), and by hypothesis it is continuous on $L^p(R^n, wdx)$.

Now $\psi \in L^p(R^n, wdx)$, since ϕ is supported in Q, but $T\psi(x) = \infty$ for almost all $x \in Q$.

This is impossible since w has a zero-set of Lebesgue measure 0. Hence $w^{-1/(p-1)} \in L^1(Q, dx)$.

Having disposed of this preliminary step, we now show the necessity of the A_p condition. Fix a cube Q of side length d, and let Q' be an adjoining cube of the same size. If $x \in Q$ and $y \in Q'$, then $|x - y| \le 2\sqrt{n}\, d$. Suppose $g \in C_0^\infty(R^n)$, $g \ge 0$, and $g(x) = 1/d^n$ if $|x| \le 3\sqrt{n}\, d$.

Again, $f \to f * g$ is a pseudodifferential operator of order $-\infty$ since $\hat{g} \in \mathcal{S}$.

If $f \ge 0$ is supported in Q, then$f * g(x) = \int_Q f(y) g(x - y) dy \ge \left(\frac{1}{|Q|}\int_Q f(y)dy\right)\chi_{Q'}(x)$.

Hence,

$$\left(\int_{Q'} wdx\right)\left(\frac{1}{|Q|}\int_Q f(y)dy\right)^p \le \int_{Q'} |f * g|^p w\, dx \le C \int_Q f^p w dx, \tag{1}$$

the last inequality being a consequence of the assumption on w.Now let $f \equiv 1$ on Q to get $\int_{Q'} wdx \le C \int_Q wdx$; interchanging Q and Q', we get $\int_Q wdx \le C \int_{Q'} wdx$.

Since $w^{-1/(p-1)} \in L^p(Q, wdx)$ by the first part of the proof, we can now let

$$f = w^{-1/(p-1)}\chi_Q \left(\frac{1}{|Q|}\int_Q wdx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^p \le C\left(\frac{1}{|Q'|}\int_{Q'} wdx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^p$$

$$\le \frac{C}{|Q|}\int_Q w^{-1/(p-1)}dx$$

by (1). Hence, $\left(\frac{1}{|Q|}\int_Q wdx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^p \le C$.

Having established the necessity of the A_p condition for boundedness of pseudodifferential operators, we can now turn to the sufficiency of the condition.

Lemma (1.1.4)[5]. Suppose $1 < p < \infty$. Let ψ be a radial, decreasing, positive function with total integral 1. Set $\psi_t(x) = t^{-n}\psi(x/t)$. Then:

(i) (i) $\sup_{t>0}|f * \psi_t(x)| \leq Mf(x)$ for $f \in L^p(R^n, wdx)$;

(ii) If ψ has compact support, then $f * \psi_t(x) \to f(x)$, as $t \to 0$, almost everywhere for $f \in L^p(R^n, wdx)$;

(iii)if ψ has compact support, then $\|f * \psi_t - f\|_p \to 0$ as $t \to 0$, for all $f \in L^p(R^n, wdx)$.

Proof. A proof of (i) , (ii) (see [7]).

Let B be any ball in R^n, let B' be any other ball containing B in its interior, and let δ be the distance from B to the complement of B'. We shall show that if $f \in L^p(R^n, wdx)$ then $f * \psi_t(x) \to f(x)$ as $t \to 0$ for almost every $x \in B$. Then, by expanding B, we establish (ii) for almost every $x \in R^n$.

Set $f_1(x)$ equal to $f(x)$ if $x \in B'$, and equal to 0 outside B'. Let $f_2 = f - f_1$.

Now,$f_1 \in L^1(R^n, dx)$ since$\int_{B'}|f_1|dx \leq \left(\int_{B'}|f_1|^p w dx\right)^{\frac{1}{p}}\left(\int_{B'} w^{-\frac{1}{(p-1)}}dx\right)^{\frac{1}{p'}} < \infty$.

The last integral is finite since $w^{-1/(p-1)}$ is locally integrable.

Hence $f_1 * \psi_t(x) \to f_1(x) = f(x)$ as $t \to 0$ for almost every $x \in B$ (see [7]).

To deal with f_2, we note that if $x \in B$, then

$$|f_2 * \psi_t(x)| \leq \int \psi_t(x-y)|f_2(y)|dy$$

$$\leq \left(\int |f_2(y)|^p w(y)dy\right)^{\frac{1}{p}} \cdot \left(\int_{|x-y|\geq\delta} \psi_t(x-y)^{p'} w^{-\frac{1}{(p-1)}}(y)dy\right)^{\frac{1}{p'}} = 0$$

for sufficiently small t, since ψ has compact support. This completes the proof of (ii).

Part (iii) is now easy, since $|\psi_t * f - f| \leq Mf + |f|$ by part (i).

Since $Mf \in L^p(R^n, wdx)$ (see[17]), Lebesgue's dominated convergence theorem and (ii) at once yield a proof of (iii).

Now we can use the Hardy-Littlewood maximal operator to dominate any pseudodifferential operator of order $-\infty$.

Theorem (1.1.5)[5]. Suppose A is a pseudodifferential operator of order $-\infty$, and suppose $1 < r < \infty$. Then there exists a constant $c > 0$ such that for all $x^0 \in R^n$ and all $u \in \mathcal{S}$,$(Au)^{\#}(x^0) \leq cM_r u(x^0)$.

Proof. If $a(x, \xi)$ is the symbol of A, then for any real number m, and any multi-indices α and β,

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta} a(x,\xi)\right| \leq C_{\alpha\beta m}(1 + |\xi|)^m.$$

We can therefore write the operator as follows: for any Schwartz function u,

$Au(x) = \int \hat{u}(\xi)a(x,\xi)e^{2\pi i x\cdot\xi}d\xi = \int u(y)K(x, x-y)dy$, where $K(x,y) = \int a(x,\xi)e^{2\pi i y\cdot\xi}d\xi$.

Note that for fixed x, $K(x,y)$, as a function of y, lies in $\mathcal{S}$. In fact,

$$\left|y^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta} K(x,y)\right| = C_{\alpha\beta}\left|\int a(x,\xi)\xi^{\beta}\left(\frac{\partial}{\partial \xi}\right)^{\alpha} e^{2\pi i y\cdot\xi}d\xi\right|$$

$$\leq C_{\alpha\beta}\int\left|\left(\frac{\partial}{\partial \xi}\right)^{\alpha}\left[a(x,\xi)\xi^{\beta}\right]\right|d\xi \text{ (integration by parts)}$$

$$\leq C_{\alpha\beta},$$

with $C_{\alpha\beta}$ independent of x and y.

The rapid decrease in ξ of $a(x,\xi)$ justifies the differentiation under the integral sign and the integrations by parts in the calculation above.

Hence, $\sup_{x,y}\left|y^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta}K(x,y)\right| \le C_{\alpha\beta}$. Now choose an integer $k > n$.

By the previous discussion, there is a constant $C_k > 0$ such that $|K(x,y)| \le C_k/(1+|y|)^k$for all x.

Then

$$|Au(x)| \le \int |u(y)|K(x,x-y)dy \le C_k \int \frac{|u(y)|}{(1+|x-y|)^k}dy \le C_k Mu(x),$$

by an application of Lemma (1.1.4).

Suppose that x^0 is any point in R^n, that Q is a cube containing x^0 in its interior, and that Q has diameter d and center x'.

Let $\tau \in C_0^{\infty}(R^n)$ satisfy $0 \le \tau(x) \le 1$, be 1 when $|x-x'| \le 2d$, and vanish when $|x-x'| \ge 3d$.

We have

$$\frac{1}{|Q|}\int_Q \left|Au(x) - (Au)_Q\right|dx \le \frac{2}{|Q|}\int_Q |Au(x)|dx$$
$$\le \frac{2}{|Q|}\int_Q |A(\tau u)|dx + \frac{2}{|Q|}\int_Q \left|A\big((1-\tau)u\big)\right|dx.$$

Let Q' be the cube centered at x', with sides parallel to those of Q, and with diameter $4d$. Since the Hardy-Littlewood maximal operator is bounded on $L^r(R^n,dx)$ for $1 < r < \infty$, we can control the first term in the inequality above as follows:

$$\frac{2}{|Q|}\int_Q |A(\tau u)|dx \le 2\left(\frac{1}{|Q|}\int_Q |A(\tau u)|^r dx\right)^{1/r}$$
$$\le 2\left(\frac{1}{|Q|}\int_Q |M(\tau u)|^r dx\right)^{\frac{1}{r}} \le C_r\left(\frac{1}{|Q|}\int_{R^n}|\tau u|^r dx\right)^{\frac{1}{r}}$$
$$\le C_r\left(\frac{1}{|Q'|}\int_{Q'}|u|^r dx\right)^{1/r} \le C_r M_r u(x^0).$$

To dominate the other term, we first note that there is a constant $c > 0$ such that

$|x^0 - y| \le c|x-y|$for all$x \in Q$ and y such that$|x'-y| \ge 2d$.

The constant c is independent of x, y, and the cube Q. So,

$$\frac{2}{|Q|}\int_Q \left|A\big((1-\tau)u\big)\right|dx = \frac{2}{|Q|}\int_Q \left|\int\big(1-\tau(y)\big)u(y)K(x,x-y)dy\right|dx$$
$$\le \int_{|y-x'|\ge 2d}|u(y)|\frac{1}{|Q|}\int_Q \frac{C_k}{(1+|x-y|)^k}dydx$$
$$\le C_k\int_{R^n}\frac{|u(y)|}{(1+|x^0-y|)^k}dy$$
$$\le cMu(x^0) \text{(by Lemma (1.1.4))}$$
$$\le cM_r u(x^0).$$

We have showed that $\frac{1}{|Q|}\int_Q \left|Au(x) - (Au)_Q\right|dx \le cM_r u(x^0)$.

Taking the supremum of the left side over all cubes Q containing x^0, we find that

$(Au)^*(x^0) \le cM_r u(x^0)$.

Corollary (1.1.6)[5]. A pseudodifferential operator A, of order-∞, has a bounded extension to $L^p(R^n, wdx)$ whenever $w \in A_p(R^n)$ and $1 < p < \infty$.

Proof. In the course of proving the last theorem, we showed that $|Aw(x)| < cMu(x)$ for all $x \in R^n$ and $u \in \mathcal{S}$. The constant c is independent of x and u. Since $\mathcal{S}$ is dense in $L^p(R^n, wdx)$,and the maximal operator is bounded on $L^p(R^n, wdx)$ [1], the conclusion of the corollary follows immediately.

Dealing with operators of order-0 will require a much more delicate touch than in the previous theorem; however, the variants of the maximal function defined in the Introduction together with ever-reliable integration by parts will save the day.

We shall use $f^{\#}$ to control f^*; the next lemma makes this possible.

Lemma (1.1.7)[5]. There is a constant $c > 0$ such that $\|f^*\|_p \le c\|f^{\#}\|_p$ for all $f \in L^p(R^n, wdx) \cap L^1(R^n, dx)$.

Theorem(1.1.8) [5]. Suppose $1 < r < \infty$, and let A be a pseudodifferential operator of order 0. Then there is a constant $c > 0$ such that the pointwise estimate $(Au)^{\#}(x^0) < cM_r u(x^0)$ holds for all $x^0 \in R^n$ and all $u \in \mathcal{S}$.

Proof. Given $x^0 \in R^n$, we let Q be a cube containing x^0, with center x' and diameter d.
As in Theorem (1.1.5) we also let $\tau \in C_0^\infty(R^n)$ satisfy $0 \le \tau(x) \le 1$, be 1 when $|x - x'| \le 2d$, and vanish when $|x - x'| \ge 3d$. Then for $u \in \mathcal{S}$,

$$\frac{1}{|Q|}\int_Q \left|Au(x) - (Au)_Q\right|dx \le \frac{2}{|Q|}\int_Q |A(\tau u)|dx + \frac{1}{|Q|}\int_Q \left|A\big((1-\tau)u\big)(x) - \Big(A\big((1-\tau)u\big)\Big)_Q\right|dx.$$

Letting Q' be as in Theorem (1.1.5), we can dominate the first term in the inequality above by recalling that pseudodifferential operators of order 0 are bounded on $L^r(R^n, dx)$ when $1 < r < \infty$ [8]:

$$\frac{2}{|Q|}\int_Q |A(\tau u)|dx \le 2\left(\frac{1}{|Q|}\int_Q |A(\tau u)|^r dx\right)^{1/r} \le c\left(\frac{1}{|Q|}\int_{R^n}|\tau u|^r dx\right)^{1/r}$$
$$\le c\left(\frac{1}{|Q'|}\int_{Q'}|u|^r dx\right)^{1/r} \le cM_r u(x^0).$$

To deal with the second term, we simplify notation, writing u for $(1-\tau)u$, and we assume that u has support in the set $\{x: |x - x'| \ge 2d\}$.

We must estimate the quantity $(1/|Q|)\int_Q \left|Au(x) - (Au)_Q\right|dx$. For now, we shall also assume that $a(x,\xi)$, the symbol of A, has compact ξ-support. The various constants that occur in the following argument will not depend on the support of a; at the end we show how to dispense with the assumption on the support of a. We begin by decomposing the operator A into a sum of simpler operators. Standard techniques allow us to construct a nonnegative, radial, C^∞ function Φ, whose support is contained in the set $\left\{\xi: \frac{1}{2} \le \xi \le 2\right\}$, and which satisfies $\sum_{j=0}^{\infty} \phi(2^{-j}\xi) = \begin{cases} 1 & \text{if} |\xi| > 1, \\ 0 & \text{if} |\xi| < \frac{1}{2}. \end{cases}$

Now we can write

$$\begin{aligned} Au(x) &= \int \hat{u}(\xi)a(x,\xi)e^{2\pi i x\cdot\xi}d\xi \\ &= \int \hat{u}(\xi)a(x,\xi)\big(1 - \textstyle\sum_{j=0}^{\infty}\phi(2^{-j}\xi)\big)e^{2\pi i x\cdot\xi}d\xi \\ &\quad + \textstyle\sum_{j=0}^{\infty}\int u(y)\int a(x,\xi)\phi(2^{-j}\xi)e^{2\pi i(x-y)\cdot\xi}d\xi dy \\ &= Bu(x) + \textstyle\sum_{j=0}^{\infty} A_j u(x). \end{aligned}$$

B is a pseudodifferential operator whose symbol is $a(x,\xi)\big(1 - \sum_{j=0}^{\infty}\phi(2^{-j}\xi)\big)$;
theξ-support of this symbol is always contained in the set $\{\xi: |\xi| \le 1\}$.
Hence B has order $-\infty$, and $(Bu)^{\#}(x^0) \le cM_r u(x^0)$ by Theorem (1.1.5).
Since $(Au)^{\#}(x^0) \le (Bu)^{\#}(x^0) + \left(\sum_{j=0}^{\infty} A_j u\right)^{\#}(x^0) \le cM_r u(x^0) + \left(\sum_{j=0}^{\infty} A_j u\right)^{\#}(x^0)$,

the next task is to examine the operators A_j.

$A_j u(x) = \int u(y) \int a(x,\xi)\phi(2^{-j}\xi)e^{2\pi i(x-y)\cdot\xi}d\xi dy.$

The following lemma allows us to control the inner integral.

Lemma (1.1.9)[5]. Let $q(x,\xi)$ be a symbol of order m, and suppose $\phi \in C_0^\infty(R^n)$ has support in $\left\{\xi: \frac{1}{2} \le |\xi| \le 2\right\}$. If $t \ge 0$, then there is a constant $c_t > 0$ such that the inequality $\quad |y|^t \left|\int q(x,\xi)\phi(2^{-j}\xi)e^{2\pi i y\cdot\xi}d\xi\right| \le c_t 2^{j(n+m-t)}$ holds for all x and y in R^n and every integer $j \ge 0$.

Proof. Suppose first that t is a nonnegative integer.

Letting $|y|_\infty = \max\{|y_i|: 1 \le i \le n\}$, we have

$$|y|_\infty^t \left|\int q(x,\xi)\phi(2^{-j}\xi)e^{2\pi i y\cdot\xi}d\xi\right| = c_t \left|\int q(x,\xi)\phi(2^{-j}\xi)\left(\frac{\partial}{\partial\xi_i}\right)^t e^{2\pi i y\cdot\xi}d\xi\right| \text{(where} |y_i| = |y_\infty|\text{)}$$

$$\le c_t \left|\left(\frac{\partial}{\partial\xi_i}\right)^t [q(x,\xi)\phi(2^{-j}\xi)]\right| d\xi \qquad (2)$$

(integration by parts)

Now $\left|\left(\frac{\partial}{\partial\xi_i}\right)^t [q(x,\xi)\phi(2^{-j}\xi)]\right| \le \sum_{r+s=t} c_{rs} \left|\left(\frac{\partial}{\partial\xi_i}\right)^r q(x,\xi) 2^{-js}\left(\left(\frac{\partial}{\partial\xi_i}\right)^s \phi\right)(2^{-j}\xi)\right|$

$$\le \sum_{r+s=t} c_{rs}(1+2^j)^{m-r} 2^{-js}$$

(since the support of the expression above lies in $\{\xi: 2^{j-1} \le |\xi| \le 2^{j+1}\}$) $\le c_t 2^{jm-jt}$.

Substituting this estimate into (2) and integrating over the region $\{\xi: 2^{j-1} \le |\xi| \le 2^{j+1}\}$ yields the desired inequality.

If $t > 0$ is not an integer, say $i < t < i+1$, with i an integer, then we can interpolate between the inequalities for i and $i+1$. Returning to the proof of the theorem, we now estimate $\left(\sum_{j=0}^\infty A_j u\right)^\#(x^0)$. $\frac{1}{|Q|}\int_Q \left|A_j u(x) - (A_j u)_Q\right| dx = \frac{1}{|Q|}\int_Q \left|\frac{1}{|Q|}\int_Q A_j u(x) - A_j u(z)dz\right| dx$

$$= \int_Q \left|\frac{1}{|Q|}\int_Q \int_{R^n} u(y) \int_{R^n} \phi(2^{-j}\xi).\left[a(x,\xi)e^{2\pi i(x-y)\cdot\xi} - a(z,\xi)e^{2\pi i(z-y)\cdot\xi}\right]d\xi dy dz\right| dx \quad (3)$$

To estimate this last quantity, we consider two cases:

Case i. $2^j d \ge 1$. Then (3) is dominated by

$$2\sum_{k=1}^\infty \frac{1}{|Q|}\int_Q \int_{2^k d \le |y-x'| < 2^{k+1}d} |u(y)| \left|\int_{R^n} \phi(2^{-j}\xi)a(x,\xi)e^{2\pi i(x-y)\cdot\xi}d\xi\right| dy dx$$

$$\le C\sum_{k=1}^\infty \int_Q \frac{2^{nk}}{|Q_k|}\int_{2^k d \le |y-x'| < 2^{k+1}d} \frac{|u(y)|}{|x-y|^{n+1}} \cdot |x-y|^{n+1} \cdot \left|\int_{R^n} \phi(2^{-j}\xi)a(x,\xi)e^{2\pi i(x-y)\cdot\xi}d\xi\right| dy dx$$

(Q_k is the cube with center x', sides parallel to those of Q, and radius $2^{k+1}d$)

$$\le C\sum_{k=1}^\infty d^n 2^{nk}(2^k d)^{-n-1}2^{-j} \cdot \frac{1}{|Q_k|}\int_{Q_k}|u(y)|dy$$

(by Lemma (1.1.9) with $t = n+1$ and $m = 0$)

$$\le CMu(x^0)\sum_{k=1}^\infty d^{-1}2^{-k}2^{-j} \le Cd^{-1}2^{-j}Mu(x^0).$$

Case ii. $2^j d < 1$.

We write

$$a(x,\xi)e^{2\pi i(x-y)\cdot\xi} - a(z,\xi)e^{2\pi i(z-y)\cdot\xi} = \sum_{l=1}^{n}(x_l - z_l)\int_0^1 \frac{\partial a}{\partial x_l}(x(t),\xi)e^{2\pi i(x(t)-y)\cdot\xi}$$
$$+2\pi i\xi_l a(x(t),\xi)e^{2\pi i(x(t)-y)\cdot\xi}dt,$$

where $x(t) = z + t(x-z)$.

Using this last expression and the facts:

(i) $\partial a/\partial x_l$ is a symbol of order 0;

(ii) $\xi_l, a(x,\xi)$ is a symbol of order 1;

(iii)$|x_l - z_l| \le d$ since both x and z are in Q; and

(iv)if $2^k d \le |y-x'| \le 2^{k+1}d$, then $2^{k-1}d \le |x(t)-y| \le 2^{k+2}d$ since $x(t) \in Q$,

we can invoke Lemma (1.1.9) with $m = 0$ or 1, and $t = n + \frac{1}{2}$to see that (3) is dominated by

$$C\sup_{x\in Q}\sum_{k=1}^{\infty}\frac{1}{|Q|}\int_Q \int_{2^k d\le|y-x'|<2^{k+1}d}\frac{|u(y)|}{|x-y|^{n+\frac{1}{2}}}\cdot\sum_{l=1}^{n}|x_l - z_l|\int_0^1 |x(t)-y|^{n+1/2}$$
$$\cdot\left|\int_{R^n}\phi(2^{-j}\xi)\left[\frac{\partial a}{\partial x_l}(x(t),\xi)e^{2\pi i(x(t)-y)\cdot\xi} + 2\pi i\xi_l a(x(t),\xi)e^{2\pi i(x(t)-y)\cdot\xi}\right]d\xi\right|dtdydz$$
$$\le C\sum_{k=1}^{\infty}2^{nk}\frac{1}{|Q_k|}\int_{Q_k}|u(y)|dy\, d^n(2^k d)^{-n-1/2}d\left(2^{-j/2}+2^{j/2}\right)$$
$$\le CMu(x^0)d^{1/2}2^{j/2}\sum_{k=1}^{\infty}2^{-k/2} \le Cd^{1/2}2^{j/2}Mu(x^0).$$

Putting the two cases together, we have shown that if Q is any cube containing x^0, then

$$\frac{1}{|Q|}\int_Q \left|\sum_{j=0}^{\infty}A_j u(x) - \left(\sum_{j=0}^{\infty}A_j u\right)_Q\right|dx \le \sum_{j=0}^{\infty}\frac{1}{|Q|}\int_Q \left|A_j u(x) - \left(A_j u\right)_Q\right|dx$$
$$\le C\left(\sum_{2^j d\ge 1}d^{-1}2^{-j} + \sum_{2^j d<1}d^{1/2}2^{j/2}\right)Mu(x^0).$$

Since the quantity in parentheses above is finite and independent of d, we find, after taking the supremum over all cubes Q containing x^0, that $\left(\sum_{j=0}^{\infty}A_j u\right)^{\#}(x^0) \le cMu(x^0) \le cM_r u(x^0)$.

Going back to our original notation, and summarizing, we have shown that if Q is any cube containing x^0, then

$$\frac{1}{|Q|}\int_Q |Au(x) - (Au)_Q|dx \le \left(A(\tau u)\right)^{\#}(x^0) + \left(B\left((1-\tau)u\right)\right)^{\#}(x^0) + \left(\sum_{j=0}^{\infty}A_j\left((1-\tau)u\right)\right)^{\#}(x^0)$$
$$\le cM_r u(x^0) + cM_r\left((1-\tau)u\right)(x^0) \le cM_r u(x^0),$$

the constant c being independent of Q, u, x^0, and the ξ-support of $a(x,\xi)$.

We have been working under the assumption that $a(x,\xi)$, the symbol of A, has compact ξ-support. Suppose now that this is no longer so. Let $b_j(x,\xi)$ be $a(x,\xi)$ multiplied by a smooth cutoff function which is 1 when $|\xi| \le 2^j$ and 0 when $|\xi| \ge 2^{j+1}$. Let B_j be the pseudodifferential operator whose symbol is $b_j(x,\xi)$. Since$b_j(x,\xi) \to a(x,\xi)$ as $j \to \infty$, the dominated convergence theorem implies that $B_j u(x) \to Au(x)$ for all x. Another application of the dominated convergence theorem shows that for each cube Q, $\frac{1}{|Q|}\int_Q \left|B_j u(x) - \left(B_j u\right)_Q\right|dx \to \frac{1}{|Q|}\int_Q |Au(x) - (Au)_Q|dx$.

Applying our previous result to the operators B_j, and taking the limit as $j \to \infty$, we see that $\frac{1}{|Q|}\int_Q \left|Au(x) - (Au)_Q\right| dx \le cM_r u(x^0)$.When we take the supremum of the left side over all cubes containing x^0, we finally obtain the inequality $(Au)^{\#}(x^0) \le cM_r u(x^0)$.

We are now ready to prove a basic result about pseudodifferential operators.

Theorem(1. 1.10)[5]. If $1 < p < \infty$ and $w \in A_p(R^n)$, then any pseudodifferential operator of order 0 has a bounded extension to all of $L^p(R^n, wdx)$.

Proof. Let A be a pseudodifferential operator of order 0. The proof that A is bounded depends on the following train of inequalities:

if $u \in \mathcal{S}$ then

$$\|Au\|_p \le \|(Au)^*\|_p \le C\|(Au)^{\#}\|_p \le C\|M_r u\|_p \text{ if } 1 < r < \infty$$
$$\le C\|u\|_p \text{ if } 1 < r < p.$$

The first inequality is easy, since $|Au(x)| \le (Au)^*(x)$ for every x.

Since $Au \in \mathcal{S}$, $Au \in L^p(R^n, wdx) \cap L^1(R^n, dx)$; so we can apply Lemma (1.1.7) to prove the second inequality. The third inequality is Theorem (1.1.8), while the last inequality is proved like this: $\|M_r u\|_p = \left\|[M(|u|^r)]^{1/r}\right\|_p = \left(\int [M(|u|^r)]^{p/r} w dx\right)^{1/p} \le C(\int |u|^p w\, dx)^{1/p}$since $p/r > 1$ $= C\|u\|_p$.

Since $\mathcal{S}$ is dense in $L^p(R^n, wdx)$, we can now extend A to a bounded operator on $L^p(R^n, wdx)$.

We shall introduce the weighted Sobolev spaces $L_s^p(R^n, wdx)$. It will transpire that many of the properties of the traditional unweighted spaces are still true in the weighted case; in particular, we can identify the space $L_k^p(R^n, wdx)$ (k a positive integer) with the space of functions in $L^p(R^n, wdx)$ whose distributional derivatives of all orders $\le k$ lie in $L^p(R^n, wdx)$, and we can prove a version of Sobolev's theorem. If s is any real number, we write J^s for the pseudodifferential operator of order -s whose symbol is $(1 + 4\pi^2|\xi|^2)^{-s/2}$. Clearly, J^s can be defined as a map of tempered distributions to tempered distributions; we also point out that if $w \in A_p(R^n)$, then functions in $L^p(R^n, wdx)$ are tempered distributions. We define $L_s^p(R^n, wdx)$, the Sobolev space of order s, as the image of $L^p(R^n, wdx)$ under the map J^s; i.e., $L_s^p(R^n, wdx) = J^s\big(L^p(R^n, wdx)\big)$. If $f \in L_s^p(R^n, wdx)$, then $f = J^s g$ for some $g \in L^p(R^n, wdx)$. We write the L_s^p-norm of f as $\|f\|_{p,s}$, and define it the L^p-norm of its preimage g.

So $\|f\|_{p,s} = \|g\|_p$ whenever $f = J^s g$.

The following facts about the L_s^p are easy consequences of the definitions.

(i) Since J^s is an invertible elliptic pseudodifferential operator, the definition of the norm on $L_s^p(R^n, wdx)$ is unambiguous; i.e., if $J^s g_1 = J^s g_2$ then $g_1 = g_2$.

(ii) If $s \ge 0$, then $L_s^p(R^n, wdx)$ is a subspace of $L^p(R^n, wdx)$, since J^s is a pseudodifferential operator of order 0.

(iii)For all real s and $t, J^s J^t = J^{s+t}$.

(iv)For all real s, J^s is an isomorphism of $\mathcal{S}$ to $\mathcal{S}$ and of $\mathcal{S}'$ to $\mathcal{S}'$; furthermore, $\mathcal{S}$ is dense in $L_s^p(R^n, wdx)$.

(v) For all real s and t, J^t is a norm-preserving isomorphism of $L_s^p(R^n, wdx)$ to $L_{s+t}^p(R^n, wdx)$.

(vi)The spaces $L_s^p(R^n, wdx)$ are Banach spaces.

(vii) If $s \ge t$ then $L_s^p(R^n, wdx) \subseteq L_t^p(R^n, wdx)$, and $\|f\|_{p,t} \le C_{s,t}\|f\|_{p,s}$.

That pseudodifferential operators behave correctly on Sobolev spaces is the content of the next two theorems.

Theorem (1.1.11)[5]. Suppose A is a pseudodifferential operator of order m. Then A is a bounded map from $L_s^p(R^n, wdx)$ to $L_{s-m}^p(R^n, wdx)$.

Proof. We can write $A = J^{s-m}(J^{-s+m}AJ^s)J^{-s}.J^{-s}$ maps L_s^p to $L^p; J^{-s+m}AJ^s$ is a pseudo- differential operator of order 0, and therefore maps L^p to L^p; finally, J^{s+m} maps L^p to L_{s-m}^p.

Theorem (1.1.12)[5]. Let $0 \leq m \leq s$ and suppose that A is an elliptic pseudodifferential operator of order m. Then there is a constant $c_s > 0$ such that $\|f\|_{p,s} \leq c_s(\|Af\|_{p,s-m} + \|f\|_{p,0}), f \in L_s^p$.

Proof. Since A is elliptic, we can find an elliptic operator B, of order $-m$, and an operator R, of order-∞, such that I, the identity operator, can be written I = BA + R.Theorem (1.1.11) shows that

$$\|f\|_{p,s} = \|(BA + R)f\|_{p,s} \leq \|BAf\|_{p,s} + \|Rf\|_{p,s} \leq c_s(\|Af\|_{p,s-m} + \|f\|_{p,0}).$$

Theorem (1.1.13)[5]. Suppose k is a positive integer and $1 < p < \infty$.

The space $L_k^p(R^n, wdx)$ is identical to the subspace of functions in $L^p(R^n, wdx)$ whose distributional derivatives of all orders $\leq k$ lie in $L^p(R^n, wdx)$. Furthermore, the norms $\|f\|_{p,k}$ and $\sum_{|\alpha|\leq k}\|(\partial/\partial x)^\alpha f\|_p$ are equivalent.

Proof. We can use the same proof as in [9], if we keep in mind that $(\partial/\partial x)^\alpha J^k$ is a pseudodifferential operator of order 0, and hence bounded on $L^p(R^n, wdx)$ whenever $|\alpha| \leq k$.

As in the unweighted case, the weighted Sobolev spaces can be used to compare the size of the distributional derivatives of a function and its degree of smoothness. The following is a weak form of Sobolev's theorem.

Theorem (1.1.14)[5]. Suppose that $w \in A_q(R^n)$ for some q satisfying

$1 < q < p(n-1)/n$. If $k > nq/p$, then every function in $L_k^p(R^n, wdx)$ can be modified on a set of measure 0 so that the resulting function is continuous.

Proof. Fix $f \in L_k^p(R^n, wdx)$, and suppose that $f = J^k g$, with $g \in L^p(R^n, wdx)$. Let $\{g_n\}$ be a sequence in $\mathcal{S}$ such that $g_n \to g$ in $L^p(R^n, wdx)$ (Lemma (1.1.2)), and let $f_n = J^k g_n$. If $|\alpha| \leq k$, then $(\partial/\partial x)^\alpha J^k$ is a pseudodifferential operator of order 0; consequently, $\left(\frac{\partial}{\partial x}\right)^\alpha f_n = \left(\frac{\partial}{\partial x}\right)^\alpha J^k g_n \to \left(\frac{\partial}{\partial x}\right)^\alpha J^k g = \left(\frac{\partial}{\partial x}\right)^\alpha f$, the limit being taken in $L^p(R^n, wdx)$. So $f_n \to f$ in $L_k^p(R^n, wdx)$ by Theorem (1.1.13).

Since $\frac{nq}{p} < n - 1$, we can assume, by decreasing k if necessary, that k is an integer and that $nq/p < k \leq n - 1$. Let K be any compact set. If ϕ is a function in $C_0^\infty(R^n)$ which is identically 1 on K, it is clearly enough to prove the theorem for ϕf. Since the sequence $\{f_n\}$ approximates f in $L_k^p(R^n, wdx)$, the sequence $\{\phi f_n\}$ approximates ϕf.

Now choose R large enough that if K_1 is the support of ϕ, then the set $K_1 - K_1$ is contained in the ball of radius R about the origin. If $x \in K$,see [7] repeatedly to writ

$$|f_j(x) - f_l(x)| = |\phi(x)f_j(x) - f_l(x)| \leq C\sum_{|\alpha|=k}\left|\frac{\partial^\alpha[\phi(f_j - f_l)]}{\partial x^\alpha}\right| * g * g * \cdots * g(x),$$

there are k repetitions of $g(x) = |x|^{-n+1}$. By [7], $g * g * \cdots * g(x) = C|x|^{-n+k}$($k$ repetitions of g). Hence,

$$|f_j(x) - f_l(x)| \leq C\sum_{|\alpha|=k}\int_{|y|\leq R}\left|\frac{\partial^\alpha[\phi(f_j - f_l)]}{\partial x^\alpha}|x - y|\right| |y|^{-n+k}dy$$

$$\leq C\sum_{|\alpha|=k}\left\|\frac{\partial^\alpha[\phi(f_j - f_l)]}{\partial x^\alpha}\right\|_p \left(\int_{|y|\leq R} w(x-y)^{-1/(p-1)}|y|^{(-n+k)p'}\right)^{1/p'}.$$

If we can show that the second factor on the right side of the last inequality is bounded independently of $x \in K$, it will follow that the $\{f_n\}$ converge uniformly on K, and therefore that f is continuous on K. Therefore, let q be as in the statement of the theorem, and set $r = (p-1)/(q-1)$. Since $r > 1$, we can apply Hölder's inequality to the integral in question to find that

$$\int_{|y|\le R} w(x-y)^{-1/(p-1)}|y|^{(-n+k)p'}dy \le \left(\int_{|y|\le 3R} w(y)^{-1/(p-1)}dy\right)^{1/r}\left(\int_{|y|\le R}|y|^{(-n+k)p/(p-q)}\right)^{1/r'}.$$

Since $w \in A_q(R^n)$, the first integral on the right side is finite. The second integral is finite since $(-n+k)p/(p-q) > -n$.

Example (1.1.15)[5]. Suppose $q > 1$, and let $-n < t < n(q-1)$. It is well known that $|x|^t$ is in $A_q(R^n)$, and hence in $A_q(R^n)$, for all $p \ge q$. Now choose $p > nq/(n-1)$.

By Theorem(1.1.1 3), every function in $L^p_{n-1}(R^n, |x|^t dx)$ is actually continuous.

We can now transfer the results to a compact manifold without boundary. This requires that we localize the previous results to bounded open subsets of R^n. If U and V are two open subsets of R^n, the notation $U \subset\subset V$ will mean that the closure of U is compact and contained inside V. Suppose U is a bounded open subset of R^n. The function w is said to belong to $A_p(U)$ if

(i) w is nonnegative and integrable over every compact subset of U;

(ii) for every open set $U' \subset\subset U$, there is a constant C, which may depend on U', such that

$$\left(\frac{1}{|Q|}\int_Q w dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^{p-1} \le C$$

whenever Q is a cube contained in U$'$. a constant C will be called the A_p constant for U$'$.

The next lemma shows that the A_p condition is invariant under coordinate changes.

Lemma (1.1.16)[5]. Suppose U and V are bounded open sets in R^n, and $\psi: U \to V$ is a diffeomorphism. If $w \in A_p(V)$ then $w \circ \psi \in A_p(U)$.

Proof. Suppose $U' \subset\subset U$; then $V' = \psi(U') \subset\subset V$.

Using x to denote the coordinates in U, and y to denote the coordinates in V, we have $x = \psi^{-1}(y)$ and $dx = |J\psi^{-1}(y)|dy$, where $J\psi^{-1}$ is the Jacobian determinant of ψ^{-1}.

Let V'' be an open set such that $V' \subset\subset V'' \subset\subset V$.

The following list of constants will be used in the remainder of the proof:

$b = \sup\{J\psi^{-1}(y): y \in V'\}$, $\delta =$ the distance from V' to the complement of V'',

$d =$ the A_p constant for V'', $k = \sup\{D\psi(x): x \in U'\}$, where $D\psi$ is the differential of ψ.

Now suppose S is a cube in U' with center x^0 and side length $2r$, with $r < \delta/kn$. If x is any point in S, then $|\psi(x) - \psi(x^0)| \le k|x - x^0|$ for the Mean Value Theorem $\le kr\sqrt{n} \le \delta/\sqrt{n}$.

Hence $\psi(S)$ is contained inside any cube with center $\psi(x^0)$ and side length $2k\sqrt{n}\,r$, and any such cube lies entirely inside V''. Let Q be such a cube in V''.

Then

$$\left(\frac{1}{|S|}\int_S w\circ\psi(x)dx\right)\left(\frac{1}{|S|}\int_S w\circ\psi(x)^{-\frac{1}{(p-1)}}dx\right)^{p-1}$$

$$\le \left(\frac{b\left(2k\sqrt{n}\right)^n}{|Q|}\int_Q w(y)dy\right)\left(\frac{b\left(2k\sqrt{n}\right)^n}{|Q|}\int_Q w(y)^{-\frac{1}{(p-1)}}dy\right)^{p-1}$$

$$\le b^p\left(2k\sqrt{n}\right)^{np} d.$$

The A_p condition is therefore proved for small cubes in U' with side length $< 2\delta/kn$.

Now suppose that S is a cube in U' with side length $2r \geq 2\delta/kn$.
Then

$$\left(\frac{1}{|S|}\int_S \ w\circ\psi(x)dx\right)\left(\frac{1}{|S|}\int_S \ w\circ\psi(x)^{-1/(p-1)}dx\right)^{p-1}$$
$$\leq \left(\left(\frac{kn}{2\delta}\right)^n \int_{U'} w\circ\psi(x)dx\right)\left(\left(\frac{kn}{2\delta}\right)^n \int_{U'} w\circ\psi(x)^{-1/(p-1)}dx\right)^{p-1}$$
$$\leq \left(\frac{kn}{2\delta}\right)^{np} b^p\left(\int_{V'} w(y)dy\right)\left(\int_{V'} w(y)^{-1/(p-1)}dy\right)^{p-1}$$
$$\leq C \quad \text{since } w \text{ and } w^{-1/(p-1)} \text{ are integrable on} V'.$$

The result that $w\circ\psi \in A_p(U)$ now follows. We now formulate the A_p condition for a C^∞ compact manifold without boundary. A word about notation: we say that (Ω,ϕ) is a coordinate chart when Ω is a coordinate neighborhood on the manifold, and ϕ is a C^∞ coordinate map from Ω to open subset of R^n. Let M be a compact C^∞ manifold without boundary, let $\{(\Omega_i,\phi_i)\}_{i=1}^k$ be a fixed finite atlas for M. If w is a non-negative function on M, then $w \in A_p(M)$ if $w\circ\phi_i^{-1} \in A_p(\phi_i(\Omega_i))$for $i = 1,2,\dots,k$.

Theorem (1.1.17)[5]. The definition of $A_p(M)$ is independent of the particular atlas $\{(\Omega_i,\phi_i)\}_{i=1}^k$. That is, $w \in A_p(M)$ if and only if $w\circ\phi^{-1} \in A_p(\phi(\Omega))$ for any coordinate chart (Ω,ϕ).

Proof. One implication is obvious; so we shall assume that $w \in A_p(M)$ and that (Ω,ϕ) is a randomly chosen coordinate chart. Suppose $U \subset\subset \phi(\Omega)$, and let Q be a cube in U. Since $\mathrm{cl}(U)$, the closure of U, is contained in $\bigcup_i \phi(\Omega_i\cap\Omega)$, we can pick open sets W_i in R^n such that $W_i \subset\subset \phi(\Omega_i\cap\Omega)$ and $\mathrm{cl}(U) \subset \bigcup_i W_i$. When $\{W_i\}_{i=1}^k$ is regarded as a covering of $\mathrm{cl}(U)$, it has a Lebesgue number, l. If Q has diameter $< l$, then it lies entirely inside W_i for some i. Since $\phi_i\circ\phi^{-1}$ is a diffeomorphism between $\phi(\Omega\cap\Omega_i)$ and $\phi_i(\Omega\cap\Omega_i)$, and since $W_i \subset\subset \phi(\Omega\cap\Omega_i)$, the last lemma shows that the A_p condition holds for all cubes with diameter $< l$.On the other hand, if Q is a cube inside U with diameter $\geq l$ then

$$\left(\frac{1}{|Q|}\int_Q \ w\circ\phi^{-1}dx\right)\left(\frac{1}{|Q|}\int_Q \ [w\circ\phi^{-1}]^{-1/(p-1)}dx\right)^{p-1}$$
$$\leq \frac{1}{l^{np}}\left(\int_U \ w\circ\phi^{-1}dx\right)\left(\int_U \ [w\circ\phi^{-1}]^{-1/(p-1)}dx\right)^{p-1}$$
$$\leq \frac{C}{l^{np}}\left(\sum_{i=1}^k \int_{\phi_i\circ\phi^{-1}(W_i)} w\circ\phi_i^{-1}dy\right)\cdot\left(\sum_{i=1}^k \int_{\phi_i\circ\phi^{-1}(W_i)}[w\circ\phi_i^{-1}]^{-1/(p-1)}dy\right)^{p-1} \leq C.$$

Having fixed the particular atlas $\{(\Omega_i,\phi_i)\}$, we now choose a nonnegative C^∞ partition of unity, $\{\tau_i\}$, subordinate to this atlas. If $w \in A_p(M)$ is restricted to Ω_i, we can regard it as a function in $A_p(\phi_i(\Omega_i))$. Similarly, if f is a function defined on M, we will consider $\tau_i f$ as a function with compact support defined in R^n. We say that $f \in L^p(M,wdx)$ if $\tau_i f \in L^p(\phi_i(\Omega_i),wdx)$ for each i.

The norm on $L^p(M,wdx)$ is given by, $\|f\|_p = \left(\sum_{i=1}^k \left\|\tau_i^{1/p}f\right\|_p^p\right)^{1/p}$, where the norms on the right side are given by

$$\left\|\tau_i^{1/p}f\right\|_p = \left(\int\limits_{\phi_i(\Omega_i)} \tau_i|f|^p w dx\right)^{1/p}.$$

A standard argument shows that the norms given by different atlases and different partitions of unity are all equivalent and define the same topology on $L^p(M,wdx)$. We recall that a pseudodifferential operator can be defined on the manifold M by prescribing the action of the operator on functions supported in a coordinate patch. The operator A is said to be of order m if in each coordinate patch Ω we can write

$Au(x) = \int \hat{u}(\xi) a(x,\xi) e^{2mix\cdot\xi} d\xi$, with $a(x,\xi)$ a symbol of order m, whenever $x \in \Omega$ and u is a function supported in Ω. By using a partition of unity subordinate to the covering by coordinate charts, we can extend the definition of A to all functions in $C^\infty(M)$. (See [10].)The next theorem allows us to localize the estimates of theorem (1.1.10).

Theorem (1.1.18)[5]. Suppose U is a bounded open subset of R^n, and let $w \in A_p(U)$.

If $V \subset\subset U$ and if A is a pseudodifferential operator of order 0, Then $A : L^p(V, wdx) \to L^p(V, wdx)$ boundedly.

Proof. Since $V \subset\subset U$, we can cover V with a finite number of cubes $\{Q_i\}_{i=1}^k$ such that $Q_i \subset\subset U$.

By introducing a partition of unity subordinate to these cubes, we need only show that

A: $L^p(Q_i, wdx) \to L^p(Q_i, wdx)$ boundedly for $i = 1,2, \dots, k$.

So we choose one of the cubes Q and, by translating it in the directions of its edges, we decompose R^n into a mesh of cubes the same size as Q, whose interiors are disjoint, and whose sides are parallel to those of Q.

The next step is to extend the function w from Q to the rest of R^n. We do this by reflecting the values of w through the sides of Q into its adjacent cubes, continuing in this way so that the values of w in cubes sharing a common face match up along that face. The resulting function w' lies in $A_p(R^n)$, and has an A constant no more than 3^n times the $A_p(V)$ constant for w. The rest is easy, since if u is supported in Q, then $\|Au\|_{p,Q} \le \|Au\|_{p,R^n} \le C\|u\|_{p,R^n} = C\|u\|_{p,Q}$

where the first and last norms are in $L^p(Q, wdx)$ and the middle two are in $L^p(R^n, w'dx)$.

Corollary (1.1.19)[5]. If A is a pseudodifferential operator of order 0, and $w \in A_p(M)$, then

A :$L^p(M, wdx) \to L^p(M, wdx)$boundedly.

Proof. Use a partition of unity subordinate to a coordinate covering of M.

Now we can define $L_s^p(M, wdx)$, the Sobolev potential space of order s on the manifold M.

Let E_s be an invertible elliptic pseudodifferential operator of order s defined on $C^\infty(M)$.

$L_s^p(M, wdx)$ is the set of all distributions f defined on $C^\infty(M)$ such that $E_s f \in L^p(M, wdx)$.

We define a norm on this space by $\|f\|_{p,s} = \|E_s f\|_p$.

At first glance, it seems as if $L_s^p(M, wdx)$ depends on the choice of E_s, but this is not so.For suppose that E is another invertible elliptic pseudodifferential operator of order s.Then

$$\|f\|_{p,s} = \|E_s f\|_p = \|E_s E^{-1} E f\|_p \le \|Ef\|_p,$$

since $E_s E^{-1}$ is a pseudodifferential operator of order 0, and hence bounded on $L^p(M, wdx)$.

The norms defined by different E's are therefore all equivalent.

By introducing a partition of unity, covering coordinate patches in R^n with cubes, and extending w from the cubes to all of R^n, we can transfer all the results stated to $L_s^p(M, wdx)$.

Theorem (1.1.20)[5]. Let M be a compact C^∞ manifold without boundary, and let $w \in A_p(M)$.

(i) The spaces $L_s^p(M, wdx)$ are Banach spaces.

(ii) If $s \ge t$ then $L_s^p \subseteq L_t^p$ and $\|f\|_{p,t} \le C\|f\|_{p,s}$.

(iii)Suppose A is a pseudodifferential operator of order $m \le s$.

Then A: $L_s^p \to L_{s-m}^p$ boundedly.

(iv)If A is an elliptic pseudodifferential operator of order m, and $0 \le m \le s$, then there is a constant $C_s > 0$ such that $\|u\|_{p,s} \le C_s(\|Au\|_{p,s-m} + \|u\|_p), u \in L_s^p$.

(v) Suppose k is a positive integer. The space $L_k^p(M,wdx)$ coincides with the subspace of functions in $L^p(M,wdx)$ having distributional derivatives of all orders $\leq k$ in $L^p(M,wdx)$ in any coordinate system.

(vi) Suppose that $w \in A_q(M)$ for some q satisfying $1 < q < p(n-1)/n$.

If $s > nq/p$, then every function in $L_s^p(M,wdx)$ is continuous.

Corollary (1.1.21)[236]. Let $\varepsilon > 0$ and suppose that A is an elliptic pseudodifferential operator of order $s-\varepsilon$. Then there is a constant $c_s > 0$ such that $\|f\|_{1+\varepsilon,s} \leq c_s(\|Af\|_{1+\varepsilon,\varepsilon} + \|f\|_{1+\varepsilon,0}), f \in L_s^{1+\varepsilon}$.

Proof. Since A is elliptic, an elliptic operator $A+\varepsilon$, of order $-(s-\varepsilon)$, and an operator R, of order-∞, such that I, the identity operator, can be written $I = (A+\varepsilon)A + R$.

Theorem (1.1.11) shows that

$$\|f\|_{1+\varepsilon,s} = \|((A+\varepsilon)A+R)f\|_{1+\varepsilon,s} \leq \|(A+\varepsilon)Af\|_{1+\varepsilon,s} + \|Rf\|_{1+\varepsilon,s} \leq c_s(\|Af\|_{1+\varepsilon,s-m} + \|f\|_{1+\varepsilon,0}).$$

Section (1.2).Sobolev Orthogonal Polynomials:

We consider a Sobolev inner product

$$\langle f,g\rangle = \int f(t)g(t)d\mu_0(t) + \int f'(t)g'(t)d\mu_1(t), \tag{4}$$

where μ_0 and $\mu_1$are compactly supported positive measures on the real line with finite total mass.

We put

$$\Sigma_0 := \text{supp}(\mu_0), \quad \Sigma_1 := \text{supp}(\mu_1), \qquad \Sigma := \Sigma_0 \cup \Sigma_1. \tag{5}$$

If, as we assume, μ_0 has infinite support, there exists a unique sequence of monic polynomials $\pi_n, \deg \pi_n = n$, which is orthogonal with respect to the inner product (4). These Sobolev orthogonal polynomials have properties that clearly distinguish them from ordinary orthogonal polynomials, most notably by the fact that some or many of the zeros of π_n may be outside the convex hull of Σ, or even off the real line see [11],[12]. In recent many results on zeros of special classes of Sobolev orthogonal polynomials were obtained in [13].We refer to the survys in [14] , [15].Asymptotic properties of Sobolev orthogonal polynomials were obtained by López,Marcellán, and Van Assche. These authors considered a general class of inner products, including inner products (4) with discrete measure μ_1.We study the asymptotic behavior of zeros and critical points of orthogonal polynomials in a continuous Sobolev space, i.e., when both μ_0 and μ_1 are nondiscrete measures. Our results will be stated in terms of weak* convergence of measures. We associate with a polynomial P of exact degree n its normalized zero distribution,

$$\nu(P) := \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}, \tag{6}$$

where $z_1, \dots, z_n$ are the zeros of P counted according to their multiplicities. A sequence of polynomials $\{P_n\}_{n=1}^{\infty}$, $\deg P_n = n$, is said to have asymptotic zero distribution μ if μ is a probability measure on $\overline{\mathbf{C}}$ and

$$\lim_{n\to\infty} \int f d\nu(P_n) = \int f d\mu \tag{7}$$

for every continuous function f on $\overline{\mathbf{C}}$. That is, their normalized zero distributions converge in the weak* sense to μ. Asymptotic zero distributions for orthogonal polynomials with respect to an ordinary inner product

$$\langle f,g\rangle = \int f(t)g(t)d\mu(t), \quad \Sigma := \text{supp}(\mu) \subset \mathrm{R}, \tag{8}$$

have been studied by many authors. The most comprehensive account in [17].
They introduce a class Reg of regular measures. One of their results is that for $\mu \in$ Reg, the orthogonal polynomials p_n for the inner product (8) have regular asymptotic zero distribution.
This means that $\lim_{n\to\infty} \nu(p_n) = \omega_\Sigma$, where ω_Σ is the equilibrium measure of Σ , see[16].
In case $\Sigma = \operatorname{supp}(\mu)$ is regular with respect to the Dirichlet problem in $\mathbf{C}\backslash\Sigma$, the measure μ belongs to Reg if and only if

$$\lim_{n\to\infty} \left(\frac{\|P_n\|_\Sigma}{\|P_n\|_{L^2(\mu)}} \right)^{1/n} = 1 \qquad (9)$$

for every sequence of polynomials $\{P_n\}_{n=1}^{\infty}$, deg $P_n \le n$, $P_n \not\equiv 0$. Here and in the following we use $\|\cdot\|_\Sigma$ to denote the supremum norm on Σ. Regularity of a measure indicates that it is sufficiently dense on its support. For example, it is enough that μ has a density which is positive almost everywhere on Σ. See[16] for this and other criteria for regularity of μ. Motivated by these facts, we make the following assumptions on the measures μ_0 and μ_1 in (4). Recall that $\Sigma_j = \operatorname{supp}(\mu_j)\,, j = 0, 1$.

Assumption (i) For $j = 0, 1$, the set Σ_j is compact and regular for the Dirichlet problem in $\overline{\mathbf{C}}\backslash\Sigma_j$.

Assumption (ii) The measures μ_0 and μ_1 belong to the class Reg.

Our first result concerns the asymptotic zero distribution for the derivatives π_n' of the Sobolev orthogonal polynomials.

Theorem (1.2.1)[17]. Let μ_0 and μ_1 be measures on the real line satisfying Assumptions (i) and (ii).
Let $\{\pi_n\}$ be the sequence of monic orthogonal polynomials for the inner product (4).
Then $\lim_{n\to\infty} \nu(\pi_n') = \omega_\Sigma$,where $\Sigma = \operatorname{supp}(\mu_0) \cup \operatorname{supp}(\mu_1)$ and ω_Σ is the equilibrium measure of Σ.
Thus the sequence of derivatives $\{\pi_n'\}$ has regular asymptotic zero distribution.
Note, however, that this does not imply that the zeros of π_n' are all real. In fact, we do not even know if the zeros remain uniformly bounded. In our computations we found in all cases that the zeros of π_n' are real. We feel confident about the following conjecture.

Conjecture (i). Under the same conditions as in (1.2.1), let U be an arbitrary open set containing the convex hull of Σ. Then there is an n_0 such that for every $n \ge n_0$, all zeros of π_n' are in U. To discuss the zeros of the Sobolev orthogonal polynomials π_n themselves. Set $\Omega := \overline{\mathbf{C}}\backslash\Sigma$, and let $g_\Omega(z;\infty)$ be the Green function for Ω with pole at infinity see[16],[18]. For $r > 0$, we denote by V_r the union of those components of $\{z \in \mathbf{C}: g_\Omega(z;\infty) < r\}$ having empty intersection with Σ_0, and we put $V := \bigcup_{r>0} V_r$.
Finally, we put $K := \partial V \cup (\Sigma\backslash V)$.

Corollary (1.2.2). Let ν be a weak* limit of a subsequence of $\{\nu(\pi_n)\}$.
If $K = \Sigma$ (e.g., if $\Sigma_1 \subseteq \Sigma_0$), then $\nu = \omega_\Sigma$. In this case the full sequence $\{\nu(\pi_n)\}$ converges to ω_Σ.
In our numerical examples, we found that for n up to 50, part of the zeros of π_n are still pretty far outside K. But we conjecture that they do not accumulate outside of $\overline{V}$ and the convex hull of Σ.
Let U be an arbitrary open set containing $\overline{V}$ and the convex hull of Σ.
Then there is an n_0 such that for every $n \ge n_0$, all zeros of π_n are in U.

We first present numerical results on zeros and critical points for several special cases, where Σ consists of two disjoint intervals.They depend essentially on results on zero distributions of asymptotically polynomials obtained by [19],[20].
We present numerical calculations to illustrate our results.

We consider the case where Σ consists of two disjoint intervals of equal length.

We choose $\Sigma = \left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$.

With λ_+ the Lebesgue measure restricted to $\left[\frac{1}{2}, 1\right]$ and λ_- the Lebesgue measure restricted to $\left[-1, -\frac{1}{2}\right]$, we distinguish the following four cases:

Case (i): $\mu_0 = \mu_1 = \lambda_+ + \lambda_-$;

Case (ii): $\mu_0 = \lambda_+ + \lambda_-, \mu_1 = \lambda_-$;

Case (iii): $\mu_0 = \lambda_+, +\mu_1 = \lambda_+ + \lambda_-$;

Case (iv): $\mu_0 = \lambda_+, +\mu_1 = \lambda_-$.

In all four cases, we know from Theorem (1.2.1) that the asymptotic zero distribution for the derivatives is equal to ω_Σ. In Cases (i) and (ii) we have$\Sigma_1 \subseteq \Sigma_0$. Thus, it follows from Corollary (1.2.2) that in these two cases the asymptotic zero distribution for the Sobolev orthogonal polynomials is also equal to ω_Σ. This is confirmed by our calculations.

Case (i) $\mu_0 = \lambda_+ + \lambda_-, \mu_1 = \lambda_-$ (Table1).

In our calculations for $n = 1(1)25(5)50$ we found complex zeros ofπ_n only for $n = 5, 7$, and 9.

All zeros of π_n' were found to be simple, real, and in the interval $(-1, 1)$.

Case (ii) $\mu_0 = \lambda_+ + \lambda_-, \mu_1 = \lambda_-$ (Table 2) Again, most of the zeros are real.

Only for $n = 4$ and 6 did we find complex zeros of π_n.

The zeros of π_n' are all simple, real and in $(-1, 1)$.Calculations for the same n as in Case (i)

The situation is different in Cases (iii) and (iv) . In these cases the set K of Theorem (1.2.1) may be described as follows.

The Green function $g_\Omega(z;\infty)$ of $\Omega = \overline{\mathbb{C}}\backslash\Sigma$ has one level set $\{z: g_\Omega(z;\infty) = r_c\}$ consisting of a figure eight.

TABLE 1 **:** Zeros of π_n and $\pi_n', n = 5, 10$, in Case (i)

	Zeros of π_n	Zeros of π_n'
$n = 5$	$-0.93646854 - 0.20876772i$	-0.88534979
	$-0.93646854 + 0.20876772i$	-0.46499783
0.0		0.46499783
	$0.93646854 - 0.20876772i$	0.88534979
	$0.93646854 + 0.20876772i$	
$n = 10$	-1.00052723	-0.97497028
	-0.93567713	-0.87345927
	-0.80269592	-0.71474572
	-0.62612019	-0.55444777
	-0.50181795	0.0
	0.50181795	0.55444777
	0.62612019	0.71474572
	0.80269592	0.87345927
	0.93567713	0.97497028
	1.00052723	

TABLE 2 : Zeros of π_n and π_n', $n = 5, 10$, in Case (ii)

	Zeros of π_n	Zeros of π_n'
$n = 5$	-1.01982013	-0.91709404
	-0.74396812	-0.64370369
	-0.55435292	0.14139821
	0.61214903	0.78137665
	0.90846355	
$n = 10$	-1.00290062	-0.97911875
	-0.93891943	-0.89422735
	-0.84280403	-0.75923516
	-0.66396367	-0.61066220
	-0.55481204	-0.51231989
	-0.48324766	0.16014304
	0.55639877	0.62971341
	0.71942191	0.80459125
	0.87676555	0.93865007
	0.97576614	

For symmetry reasons, this is the level set containing 0. The set K consists of two parts. It is the union of $\left[\frac{1}{2}, 1\right]$ with that part of the figure eight that encircles $\left[-1, -\frac{1}{2}\right]$.

Case (iii) : $\mu_0 = \lambda_+, +\mu_1 = \lambda_+ + \lambda_-$ (Table 3). In our calculations for $n = 1(1)25(5)50$ all zeros of π_n' were found to be simple, real, and in $(-1, 1)$. All zeros of π_n are real only for $n = 1, 2, 3, 4, 6, 8$, and 10.

All complex zeros have a negative real part and they are encircling $\left[-1, -\frac{1}{2}\right]$.

For odd n, the complex zeros are outside.

TABLE 3 : Zeros of π_n and π_n', $n = 5, 10, 15$, in Case (iii)

	Zeros of π_n	Zeros of π_n'
$n = 5$	$-1.13970225 - 0.44661459i$	-0.90932823
	$-1.13970225 + 0.44661459i$	-0.62403037
	0.50779290	0.62478703
	0.76816794	0.90887919
	1.00382819	
$n = 10$	-0.98774277	-0.97498555
	-0.95967689	-0.87349586
	-0.77454092	-0.71478191
	-0.65462781	-0.55436421
	-0.48961896	0.00056691
	0.50181827	0.55445253
	0.62612626	0.71475358
	0.80270124	0.87346371
	0.93567933	0.97497123
	1.00052715	

$n = 15$	-1.20729028	-0.99008732
	$-1.11842498 - 0.23762201i$	-0.94869995
	$-1.11842498 + 0.23762201i$	-0.87812479
	$-0.86567461 - 0.41291713i$	-0.78542939
	$-0.86567461 + 0.41291713i$	-0.68199701
	$-0.48045299 - 0.45544118i$	-0.58497964
	$-0.48045299 + 0.45544118i$	-0.51762420
	0.50000295	0.51762967
	0.54387032	0.58499199
	0.63049097	0.68200314
	0.73428763	0.78542581
	0.83481287	0.87811753
	0.91801959	0.94869496
	0.97492010	0.99008612
	0.99999844	

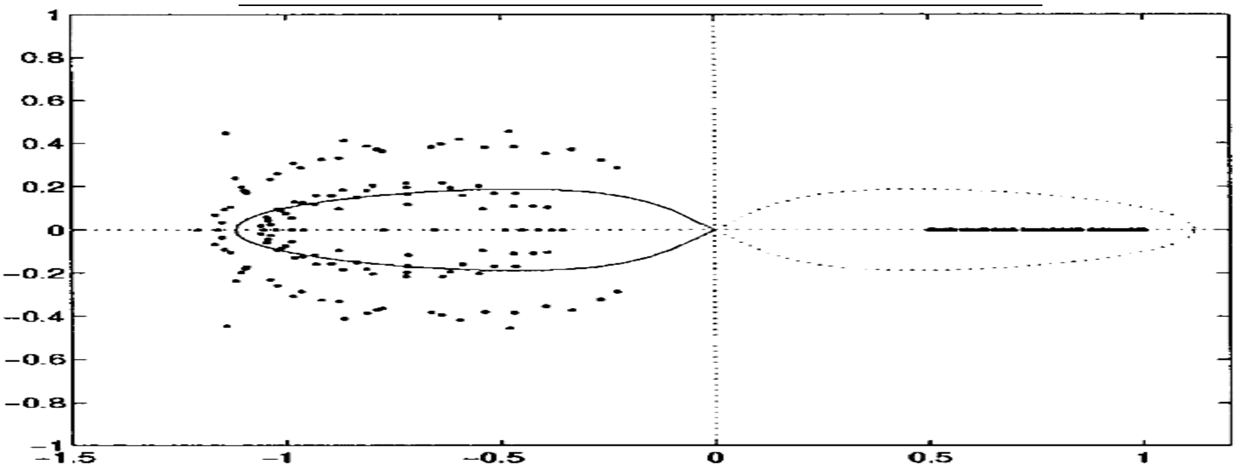

Fig. 1. Plot of the zeros of $\pi_n, n = 5(5)50$, in Case (iii).

the set K, while for even n, they are initially inside, but eventually some cross over to the outside.

It seems likely that for odd n, the zeros tend to K from the outside but the convergence is very slow. For even n, there might be a different limit distribution, although it is conceivable that also for even n, the zeros accumulate on K. It is also remarkable that the zeros

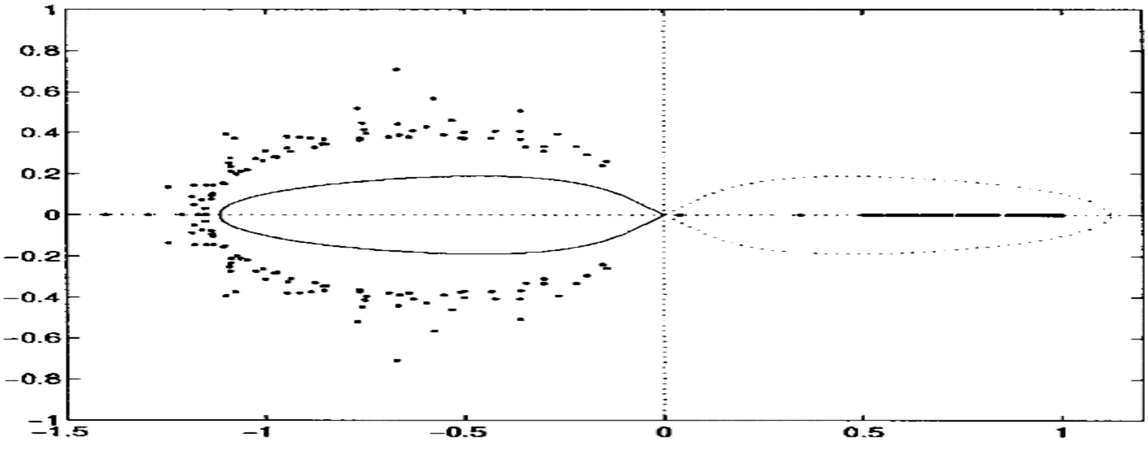

Fig. 2. Plot of the zeros of $\pi_n, n = 5(5)50$, in Case (iv).

of π'_n are very close to being symmetric around 0. We have no explanations for these phenomena.

Figure 1 depicts the zeros of $\pi_n, n = 5(5)50$, along with that part of K that encircles $\left[-1, -\frac{1}{2}\right]$.

Case (iv):

$\mu_0 = \lambda_+, \mu_1 = \lambda_-$ (Table 4) .We found complex zeros of π_n for all n, except $n = 1, 2$, and 3. Again, all the zeros ofπ'_n are simple, real, and in $(-1, 1)$. In contrast to Case (iv), we found no zeros of π_n inside the curve K (except for $n = 3$). This is illustrated in Fig. 2 with the plots of the zeros of $\pi_n, n = 5(5)50$.

Note that the zeros are pretty far from K.

TABLE 4 : Zeros of π_n and $\pi'_n, n = 5, 10, 15$, in Case (iv)

	Zeros of π_n	Zeros of π'_n
$n = 5$	-1.40237979	-0.91931357
	$-0.67193855 - 0.70835815i$	-0.64605904
	$-0.67193855 + 0.70835815i$	-0.18436141
	0.62935932	0.78712860
	0.91364079	
$n = 10$	-1.29703537	-0.98088476
	$-1.10126374 - 0.39294199i$	-0.90316848
	$-1.10126374 + 0.39294199i$	-0.77960092
	$-0.57893971 - 0.56595190i$	-0.63989830
	$-0.57893971 + 0.56595190i$	-0.53049964
	0.51468739	0.55298588
	0.60589851	0.68147141
	0.75300437	0.83024743
	0.89081502	0.94619968
	0.97842844	
$n = 15$	$-1.24663987 - 0.13488685i$	-0.99138203
	$-1.24663987 + 0.13488685i$	-0.95536746
	$-1.07914072 - 0.37346724i$	-0.89378004
	$-1.07914072 + 0.37346724i$	-0.81229432
	$-0.77108509 - 0.51962021i$	-0.71962499
	$-0.77108509 + 0.51962021i$	-0.62805945
	$-0.36124445 - 0.50773392i$	-0.55324965
	$-0.36124445 + 0.50773392i$	-0.50975247
	0.51791298	0.54446702
	0.58620377	0.63199538
	0.68402014	0.73630329
	0.78755144	0.83660513
	0.87969723	0.91913536
	0.94947423	0.97530045
	0.99024926	

Case (iiv) Another Choice for λ_+ and λ_-We also experimented with λ_+ the measure $|t|\left(t^2-\frac{1}{4}\right)^{-1/2}(1-t^2)^{-1/2}$ restricted to $\left[\frac{1}{2},1\right]$ and λ_- the same measure restricted to $\left[-1,-\frac{1}{2}\right]$.
The results, on the whole, are very similar to those for the Lebesgue measure. The differences noted were that complex zeros of π_n occur also for $n=11$ and 13 in Case (i), and for $n=8$ in Case (ii).
In Case (iii), all zeros of π_n are real only for $n=1,2,3,4,6$, and 8.

A major tool is a well-known result on zero distributions of polynomials, which we state below for the case of a set $E\subset \mathrm{R}$, and in the following, $\operatorname{cap}(E)$ denotes the logarithmic capacity of Esee[16], [18].

Lemma (1.2.3)[17]. Let $E\subset R$ be compact with $\operatorname{cap}(E)>0$ and let $\{p_n\}$ be a sequence of monic polynomials, deg $p_n=n$, such that

$$\lim_{n\to\infty}\sup\|p_n\|_E^{n/1}\le\operatorname{cap}(E). \tag{10}$$

Then

$$\lim_{n\to\infty}\nu(p_n)=\omega_E. \tag{11}$$

Proof. SeeMhaskar and Saff [19]

Monic polynomials satisfying (10) are called asymptotically minimal polynomials, since every monic polynomial p_n of degree n satisfies $\|p_n\|_E^{n/1}\ge\operatorname{cap}(E)$. Hence, if (10) holds, we have in fact equality. A weighted analogue of this theorem was obtained by Mhaskar and Saff [19] . To show the following Theorems we will need a slightly stronger result, which may be of independent interest.

To state it . Assume $E\subset\mathbf{C}$ is a closed set. A function $w\colon E\to[0,\infty)$ is an admissible weight if
(i) w is upper semicontinuous;
(ii) the set $\{z\in E\colon w(z)>0\}$ has positive capacity;
(iii) if E is unbounded, then $|z|\,w(z)\to 0$ as $|z|\to\infty, z\in E$.
Associated with an admissible weight w is a unique positive unit measure μ_w and a unique constant F_w such that $U^{\mu_w}(z)-\log w(z)=F_w$q.e.on $\operatorname{supp}(\mu_w)$,

$$U^{\mu_w}(z)-\log w(z)\ge F_w\text{q.e.on } E. \tag{12}$$

Here, U^μ denotes the logarithmic potential of the measure μ, $U^\mu(z):=\int\log\frac{1}{|z-t|}d\mu(t)$, and q.e. means quasi-everywhere, that is, except for a set of zero capacity.

In the following theorem we use S_w to denote the support of $\mu_w, Pc(S_w)$ denotes the polynomial convex hull of $S_w, D_w=\overline{\mathbf{C}}\backslash Pc(S_w)$ denotes the unbounded component of $\overline{\mathbf{C}}\backslash S_w$, and ∂D_w denotes the boundary of D_w (also known as the outer boundary of Sw).

Theorem (1.2.4)[17]. Let w be an admissible weight on the closed set $E\subset\mathbf{C}$. Let $\{p_n\}_{n=1}^\infty$ be a sequence of monic polynomials, deg $p_n=n$, such that for q.e. $z\in\partial D_w$,

$$\lim_{n\to\infty}\sup\left[w(z)|p_n(z)|^{1/n}\right]\le\exp(-F_w). \tag{13}$$

Then for every closed $A\subset D_w$,

$$\lim_{n\to\infty}\nu(p_n)(A)=0. \tag{14}$$

Furthermore, if ν is the weak* limit of a subsequence of $\{\nu(p_n)\}$, then
$\operatorname{supp}(\nu^*)\subset Pc(S_w)$ and the balayage of ν^* onto ∂D_w is equal to the balayage of μ_w onto ∂D_w.In[14] the same result was obtained from the stronger assumption $\lim\,\sup_{n\to\infty}\|w^np_n\|_{\partial D_w}^{1/n}\le\exp(-F_w)$.

Proof. In terms of potentials, the relation (14) is $F_w + \log w(z) \le \liminf_{n\to\infty} U^{\nu(p_n)}(z)$, q.e. $z \in \partial D_w$, and in view of (12) this implies

$$U^{\mu_w}(z) \le \liminf_{n\to\infty} U^{\nu(p_n)}(z), \qquad \text{q.e. } z \in \partial D_w. \tag{15}$$

Let ν_n be the balayage of $\nu(p_n)$ onto $Pc(S_w)$. Then

$$U^{\nu_n}(z) = U^{\nu(p_n)}(z) + c_n, \text{q.e. } z \in Pc(S_w), \tag{16}$$

with a constant c_n given by see [16].

$$c_n = \int g_{D_w}(z;\infty)\, d\nu(p_n)(z) \ge 0. \tag{17}$$

Let ν be the weak* limit of a subsequence of $\{\nu_n\}$, say $\nu_n \to \nu$ as $n \to \infty, n \in \Lambda$, where Λ is a subsequence of the natural numbers. Then $\mathrm{supp}(\nu) \subset Pc(S_w)$, and by the lower envelope theorem in [16] $U^{\nu}(z) = \liminf_{n\to\infty, n\in\Lambda} U^{\nu_n}(z)$, q.e. $z \in \mathbf{C}$.

Combining this with (16), (17), and (15), we find for q.e. $z \in \partial D_w$:

$$U^{\nu}(z) = \liminf_{n\to\infty,n\in\Lambda} U^{\nu_n}(z) = \liminf_{n\to\infty,n\in\Lambda}\left[U^{\nu(p_n)}(z) + c_n\right]$$
$$\ge \liminf_{n\to\infty,n\in\Lambda} U^{\nu(p_n)}(z) \ge U^{\mu_w}(z). \tag{18}$$

Since $U^{\nu} - U^{\mu_w}$ is harmonic in D_w and zero at infinity, the minimum principle and (18) give that $U^{\nu}(z) = U^{\mu_w}(z)$ for $z \in D_w$, and therefore, $U^{\nu}(z) = U^{\mu_w}(z)$, q.e. $z \in \partial D_w$.

Consequently, equality holds in every inequality in (18) for q.e. $z \in \partial D_w$. Then it follows that $\liminf_{n\in\Lambda} c_n = 0$. Since this holds for every subsequence $\Lambda \subset N$ for which $\{\nu_n\}_{n\in\Lambda}$ converges, we obtain

$$\lim_{n\to\infty} c_n = 0. \tag{19}$$

Since for a closed set $A \subset D_w$ there exists a constant $C > 0$ such that $g_{D_w}(z;\infty) \ge C$ for $z \in A$, it follows from (17) and (19) that $\lim_{n\to\infty} \nu(p_n)(A) = 0$. This proves (14).

To prove the rest of the theorem, let ν^* be the weak* limit of a subsequence of $\{\nu(p_n)\}$; say Λ is a subsequence of the natural numbers such that $\nu(p_n) \to \nu^*$ as $n \to \infty, n \in \Lambda$. Having (14), we see that ν^* is supported on $Pc(S_w)$.

Define $A := \{z \in D_w : \mathrm{dist}(z, S_w) \ge 1\}$.

Let $\zeta_{j,n}, j = 1,\dots,n$, be the zeros of p_n counted according to multiplicity, and put

$r_n(z) := \prod_{\zeta_{j,n}\in A}(z - \zeta_{j,n})$, $q_n(z) := \frac{p_n(z)}{r_n(z)} = \prod_{\zeta_{j,n}\notin A}(z - \zeta_{j,n})$.

Then, because of (14),

$$\deg q_n = n(1-\delta_n),\ \delta_n \to 0, \tag{20}$$

and the sequence $\{\nu(q_n)\}_{n\in\Lambda}$ converges to ν^* in the weak* sense. Since the measures $\nu(q_n)$ are supported on a fixed compact set, the lower envelope theorem can be applied. It gives

$$U^{\nu^*}(z) = \liminf_{n\to\infty,n\in\Lambda} U^{\nu(q_n)}(z), \qquad q.e. z \in \mathbf{C}. \tag{21}$$

Next, since $r_n(z) \ge 1$ for $z \in S_w$, we have for $z \in S_w$

$U^{\nu(p_n)}(z) = (1-\delta_n)U^{\nu(q_n)}(z) - \delta_n \log|r_n(z)| \le (1-\delta_n)U^{\nu(q_n)}(z)$; hence, by (20),(21),

$$\liminf_{n\to\infty,n\in\Lambda} U^{\nu(p_n)}(z) \le \liminf_{n\to\infty,n\in\Lambda}\left[(1-\delta_n)U^{\nu(q_n)}(z)\right] = U^{\nu^*}(z), \qquad \text{q.e. } z \in S_w.$$

Combining this with (15), we obtain $U^{\mu_w}(z) \le U^{\nu^*}(z)$, q.e. $z \in \partial D_w$.

In the same way as before, (18), this implies equality for q.e. $z \in \partial D_w$.

Now the equality of the balayages of ν^* and μ_w onto ∂D_w follows from the uniqueness of balayage. This completes the proof of Theorem (1.2.4).

Lemma (1.2.5)[17]. Let μ_0 and μ_1 be measures satisfying Assumptions (i) and (ii). Let π_n be the sequence of monic orthogonal polynomials with respect to (14). Then we have

$$\limsup_{n\to\infty} \|\pi_n\|_{\Sigma_0}^{1/n} \leq \mathrm{cap}(\Sigma) \tag{22}$$

and

$$\limsup_{n\to\infty} \|\pi'_n\|_{\Sigma_0}^{1/n} \leq \mathrm{cap}(\Sigma). \tag{23}$$

Proof. Let $\|\cdot\|_H$ denote the norm associated with the inner product (4),

$$\|f\|_H^2 = \|f\|_{L^2(\mu_0)}^2 + \|f\|_{L^2(\mu_1)}^2.$$

We first prove that

$$\limsup_{n\to\infty} \|\pi_n\|_H^{1/n} \leq \mathrm{cap}(\Sigma). \tag{24}$$

Let T_n be the monic Chebyshev polynomial of degree n for Σ. That is, $\|T_n\|_\Sigma \leq \|P_n\|_\Sigma$ for all monic polynomials P_n of degree n. It is well known that

$$\lim_{n\to\infty} \|T_n\|_\Sigma^{1/n} = cap(\Sigma). \tag{25}$$

From the regularity of Σ_1 (see Assumption (i)) it is easy to see (using the continuity of the Green function, the Bernstein-Walsh lemma and Cauchy's formula) that the Markov constants for Σ_1 have subexponential growth. This means that there exist constants M_n with $\lim_{n\to\infty} M_n^{1/n} = 1$ such that

$$\|P'_n\|_{\Sigma_1} \leq M_n \|P_n\|_{\Sigma_1}, \deg P_n \leq n. \tag{26}$$

Then, for certain constants c_1, c_2,

$$\|T_n\|_H^2 = \|T_n\|_{L_2(\mu_0)}^2 + \|T'_n\|_{L_2(\mu_1)}^2 \leq c_1 \|T_n\|_{\Sigma_0}^2 + c_2 \|T'_n\|_{\Sigma_1}^2 \leq c_1 \|T_n\|_{\Sigma_0}^2 + c_2 M_n^2 \|T'_n\|_{\Sigma_1}^2$$
$$\leq (c_1 + c_2 M_n^2) \|T_n\|_\Sigma^2 . \tag{27}$$

Using (25), (27), and $M_n^{1/n} \to 1$, we find $\limsup_{n\to\infty} \|T_n\|_H^{1/n} \leq cap(\Sigma)$.
Since π_n minimizes the Sobolev norm among all monic polynomials of degree n, we have $\|\pi_n\|_H \leq \|T_n\|_H$ for all n, and (24) follows. Now, because $\mu_0 \in \mathrm{Reg}$, we have by (9),

$$\lim_{n\to\infty} \left(\frac{\|\pi_n\|_{\Sigma_0}}{\|\pi_n\|_{L^2(\mu_0)}} \right)^{1/n} = 1. \tag{28}$$

Since $\|\pi_n\|_{L^2(\mu_0)} \leq \|\pi_n\|_H$, we get (22) from (24) and (28).

Next, using the regularity of Σ_0, we find that the Markov constants for Σ_0 grow sub exponentially. Thus, $\limsup_{n\to\infty} \left(\frac{\|\pi'_n\|_{\Sigma_0}}{\|\pi_n\|_{\Sigma_0}} \right)^{1/n} \leq 1$. Hence, from (22),

$$\limsup_{n\to\infty} \|\pi'_n\|_{\Sigma_0}^{1/n} \leq \limsup_{n\to\infty} \|\pi_n\|_{\Sigma_0}^{1/n} \leq \mathrm{cap}(\Sigma). \tag{29}$$

Further, we get from $\mu_1 \in \mathrm{Reg}$ and (9)

$$\limsup_{n\to\infty} \left(\frac{\|\pi'_n\|_{\Sigma_1}}{\|\pi'_n\|_{L^2(\mu_1)}} \right)^{1/n} \leq 1. \tag{30}$$

Since $\|\pi'_n\|_{L^2(\mu_1)} \leq \|\pi_n\|_H$, (24) and (30) give

$$\limsup_{n\to\infty} \|\pi'_n\|_{\Sigma_1}^{1/n} \leq \mathrm{cap}(\Sigma). \tag{31}$$

Combining (29) and (31), we obtain (23).
The significance of the set V is described in the following lemma.

Lemma(1.2.6)[17]. Let $z \in \mathbf{C}$. Then $z \notin V$ if and only if for every $r > g_\Omega(z;\infty)$, there is a differentiable path γ: $[0,1] \to \mathbf{C}$ such that

i. $\gamma(0) \in \Sigma_0$,

ii. $\gamma(1) = z$,

iii. $g_\Omega(\gamma(t);\infty) < r$ for all $t \in [0,1]$.

Proof. If $z \in V$, then $z \in V_r$ for some $r > g_\Omega(z;\infty)$. From the definition of V_r it follows that the connected component of $\{\zeta: g_\Omega(\zeta;\infty) < r\}$ containing z does not contain a point of Σ_0. Hence there is no path satisfying (i), (ii), and (iii).On the other hand, if $z \notin V$ and $r > g_\Omega(z;\infty)$, then $z \notin V_r$. Thus the connected component of $\{\zeta: g_\Omega(\zeta;\infty) < r\}$ does contain a point of Σ. Consequently, there is a path satisfying (i), (ii), and (iii). This allows us to estimate $|\pi_n(z)|$ for z outside V.

Lemma(1.2.7) [17]. For every $z \in \mathbf{C}\backslash V$,

$$\lim_{n\to\infty} \sup|\pi_n(z)|^{1/n} \le \operatorname{cap}(\Sigma)e^{g_\Omega(z;\infty)}. \tag{32}$$

Proof. Let $z \in \mathbf{C}\backslash V$ and $r > g_\Omega(z;\infty)$. By Lemma (1.2.6) there is a differentiable path γ: $[0,1] \to \mathbf{C}$ satisfying (i), (ii), and (iii) of Lemma (1.2.6).

By the Bernstein-Walsh lemma we have $|\pi_n'(\zeta)| \le \|\pi_n'\|_\Sigma e^{ng_\Omega(\zeta;\infty)}$, $\zeta \in \mathbf{C}$.

Using this and the properties of γ, we find

$$|\pi_n(z)| \le \left|\pi_n(\gamma(0))\right| + \left|\int_\gamma \pi_n'(\zeta)d\zeta\right| \le \|\pi_n\|_{\Sigma_0} + L(\gamma)\|\pi_n'\|_\Sigma e^{nr},$$

where $L(\gamma)$ denotes the length of γ.

Then, by (22) and (23), $\lim\ \sup_{n\to\infty}|\pi_n(z)|^{1/n} \le \operatorname{cap}(\Sigma)e^r$.

Since $r > g_\Omega(z;\infty)$ can be chosen arbitrarily close to $g_\Omega(z;\infty)$, (32) follows.

Theorem (1.2.8)[17]. Let μ_0 and μ_1 be measures on the real line satisfying Assumptions (i) and (ii). Let $\{\pi_n\}$ be the sequence of monic orthogonal polynomials for the inner product (4). Let ν be a weak* limit of a subsequence of $\{\nu(\pi_n)\}$. Then

(i) $\operatorname{supp}(\nu) \subset \bar{V} \cup \Sigma$,

(ii) the balayage of ν onto K is equal to the balayage of ω_Σ onto K see[16]. for the notion of balayage of a measure onto a compact set.

Proof: Define $w(z) := \exp(-g_\Omega(z;\infty))$, $z \in K$.

Let $\widehat{\omega}$ be the balayage of ω_Σ onto K. Since $\Sigma \subset Pc(K)$, we have $U^{\widehat{\omega}}(z) = U^{\omega_\Sigma}(z), z \in K$.

We also have $U^{\omega_\Sigma}(z) + g_\Omega(z;\infty) = -\log\operatorname{cap}(\Sigma)$, $z \in \mathbf{C}$,

so that $U^{\widehat{\omega}}(z) - \log w(z) = -\log\operatorname{cap}(\Sigma)$, $z \in K$.

Thus, by (12), $\mu_w = \widehat{\omega}$, $F_w = -\log\operatorname{cap}(\Sigma)$.

Because of (32) we can apply Theorem (1.2.1), and Theorem (1.2.2) follows.

There are two general procedures for calculating Sobolev orthogonal polynomials: the modified Chebyshev algorithm and the Stieltjes algorithm both generate the coefficients β_j^k in the recursion

$$\pi_{k+1}(t) = t\pi_k(t) - \sum_{j=0}^{k} \beta_j^k \pi_{k-j}(t), \qquad k = 0,1,2,..., \tag{33}$$

for the respective polynomials π_k. Being interested in the polynomials up to (and including) degree n, we need the coefficients $\{\beta_j^k\}_{0\le j\le k}$ for $k = 0,1,...,n-1$.

This computes the desired coefficients $\{\beta_j^k\}$ from "modified moments"

$$\nu_j^{(0)} = \int p_j(t)d\mu_0(t), \quad 0 \le j \le 2n-1,$$
$$\nu_j^{(1)} = \int p_j(t)d\mu_1(t), \quad 0 \le j \le 2n-2 \ \ (\text{if } n \ge 2), \tag{34}$$

where $\{p_j\}$ is a given set of polynomials, with p_j monic of degree j. "Ordinary moments" correspond to $p_j(t) = t^j$, but are numerically unsatisfactory. A better choice are modified moments corresponding to a set $\{p_j\}$ of orthogonal polynomials, $p_j(\cdot) = p_j(\cdot\,;\lambda)$, relative to some suitable measure λ on R. These are known to satisfy a three-term recurrence relation,

$$p_{k+1}(t) = (t-a_k)p_k(t) - b_k p_{k-1}(t), \qquad k = 0,1,2,\dots,$$
$$p_0(t) = 1, \qquad p_{-1}(t) = 0, \tag{35}$$

with coefficients $a_k = a_k(\lambda), b_k = b_k(\lambda)$ depending on λ. We need the coefficients $\{a_j\}$, $\{b_j\}$ for $0 \le j \le 2n-2$.In the context of the Sobolev orthogonal polynomials a natural choice of λ, and one that was found to work well, is $\lambda = \lambda_+ + \lambda_-$. By the orthogonality of the p_j we then have $\int_{-1}^{-\frac{1}{2}} p_j(t)d\lambda_-(t) + \int_{-\frac{1}{2}}^{1} p_j(t)d\lambda_+(t) = 0,\ j \ge 1$, so that

$$\int_{-1}^{-1/2} p_j(t)d\lambda_-(t) = -\int_{1/2}^{1} p_j(t)d\lambda_+(t). \tag{36}$$

Since, by symmetry, $p_j(-t) = (-1)^j p_j(t)$, the change of variables $t = -\tau$ in (87) yields

$$\int_{1/2}^{1} p_j(t)d\lambda_+(t) = 0 \quad \text{if } j \text{ is even} \ge 2. \tag{37}$$

Let

$$I_j = \int_{1/2}^{1} p_j(t)d\lambda_+(t), \qquad 0 \le j \le 2n-1, \tag{38}$$

so that $I_j = 0$ if $j \ge 2$ is even. We then have, in Case (i),

$$\nu_j^{(0)} = \nu_j^{(1)} = 2\delta_{j,0}I_0, \qquad j = 0,1,2,\dots, \tag{39}$$

where $\delta_{j,0}$ is the Kronecker delta. Similarly, in Case (ii),

$$\nu_j^{(0)} = 2\delta_{j,0}I_0, \quad \nu_j^{(1)} = \begin{cases} I_0, & j = 0, \\ -I_0, & j \text{ odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{40}$$

in Case (iii):

$$\nu_j^{(0)} = \begin{Bmatrix} I_j, & j = 0 \text{ or } j \text{ odd}, \\ 0, & \text{otherwise}, \end{Bmatrix}, \quad \nu_j^{(1)} = 2\delta_{j,0}I_0, \tag{41}$$

and in Case (iv):

$$\nu_j^{(0)} = \begin{Bmatrix} I_j, & j = 0 \text{ or } j \text{ odd}, \\ 0, & \text{otherwise}, \end{Bmatrix}, \quad \nu_0^{(1)} = I_0, \quad \nu_j^{(1)} = -\nu_j^{(0)}, \qquad j \ge 1. \tag{42}$$

In Case(i) - Case(iv) we have that λ_+ and λ_- are Lebesgue measure supported on $\left[\frac{1}{2}, 1\right]$and $\left[-1, -\frac{1}{2}\right]$, respectively. Here, $I_0 = \frac{1}{2}$. The coefficients $a_j(\lambda), b_j(\lambda)$ in (35) can be computed very accurately by known procedures of Stieltjes or Lanczos whereupon the integrals I_j in (38) can be computed (exactly) by (35) and n-point Gauss-Legendre quadrature.

In Case(iiv), λ_+ and λ_- are equal to the measure $|t|\left(t^2-\frac{1}{4}\right)^{-1/2}(1-t^2)^{-1/2}$ supported on $\left[\frac{1}{2},1\right]$ and $\left[-1,-\frac{1}{2}\right]$, respectively. Here, $I_0=\frac{1}{2}\pi$. The coefficients $a_j(\lambda), b_j(\lambda)$ are known explicitly

$a_j=0,\ 0\le j\le 2n-2, b_0=\pi,\ b_1=\frac{5}{8},$

$$b_j=\frac{1}{16}\begin{Bmatrix} 9\frac{1+3^{j-2}}{1+3^j}, & j \text{ even}, \\ \frac{1+3^{j+1}}{1+3^{j-1}}, & j \text{ odd}, \end{Bmatrix},\ j=2,3,\dots,2n-2. \tag{43}$$

The integrals I_j can no longer be computed exactly by numerical quadrature, but can be approximated by N-point Gauss-Chebyshev quadrature with N sufficiently large.

Indeed, if in $I_j=\int_{1/2}^{1}p_j(t)\,t\left(t^2-\frac{1}{4}\right)^{-1/2}(1-t^2)^{-1/2}dt$ one makes the change of variables $t^2=(1+3s)/4$, one gets $I_j=\frac{1}{2}\int_0^1 p_j\left(\frac{1}{2}-\sqrt{1+3s}\right)s^{-\frac{1}{2}}(1-s)^{-\frac{1}{2}}ds,$ or, transforming to the interval ,

$$I_j=\frac{1}{2}\int_{-1}^{1}p_j\left(\frac{1}{2\sqrt{2}}\sqrt{5+3x}\right)(1-x^2)^{-1/2}dx. \tag{44}$$

Gauss-Chebyshev quadrature applied to the integral in (44) converges fast.

Here the coefficients $\{\beta_j^k\}$ are computed as Fourier-Sobolev coefficients

$$\beta_j^k=\frac{(t\pi_k,\pi_{k-j})_H}{\|\pi_{k-j}\|_H^2},\qquad j=0,1,\dots,k, \tag{45}$$

where appropriate quadrature rules are used to compute the inner products in (45). The coefficients β_j^k and polynomials π_{m}intervening in (45) are computed simultaneously, the polynomials recursively by (33) using the coefficients β_j^k already obtained. The choice of quadrature rules is particularly simple in the case of Lebesgue measures. Indeed, for $k\le n-1$, the integrands in (45) are polynomials of degree $\le 2n-1$, so that n-point Gauss-Legendre rules on the respective intervals $\left[-1,-\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$ will do the job.In the other example, one has to integrate numerically as described above in connection with I_j. The zeros of π_n (including the complex ones, if any) can be conveniently computed as eigenvalues of the Hessenberg matrix

$$B_n=\begin{bmatrix} \beta_0^0 & \beta_1^1 & \beta_2^2 & \cdots & \beta_{n-2}^{n-2} & \beta_{n-1}^{n-1} \\ 1 & \beta_0^1 & \beta_1^2 & \cdots & \beta_{n-3}^{n-2} & \beta_{n-2}^{n-1} \\ 0 & 1 & \beta_0^2 & \cdots & \beta_{n-4}^{n-2} & \beta_{n-3}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_0^{n-2} & \beta_1^{n-1} \\ 0 & 0 & 0 & \cdots & 1 & \beta_0^{n-1} \end{bmatrix}. \tag{46}$$

To compute all real zeros of π_n and π_n', we scanned a suitable interval for sign changes in π_n and π_n' and used the midpoints of the smallest intervals found on which π_n (resp. π_n') changes sign as initial approximations to Newton's method.

Chapter 2
Sobolev Embeddings and Constant Functions

We provide an elementary proof of the usual concentration compactness alternative extended to the fractional Sobolev spaces H^s for any $0 < s < N/2$. We study optimizing sequences for corresponding Sobolev embedding in bounded domains, showing that they are not compact and concentrate energy at one point.

Section (2.1): Concentration-Compactness Alternative for Fractional Sobolev Spaces:

Let $N \geq 1$ and for each $s \geq 0$ let $H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \text{s.t.} |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}$ be the standard fractional Sobolev space H^s defined using the Fourier transform $\mathrm{F}(u)(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-ix\xi} u(x)dx$. As usual, the space $H^s(\mathbb{R}^N)$ can be equivalently defined as the completion of $C_0^\infty(\mathbb{R}^N \mathrm{F})$ with respect to the norm

$$\|u\|_{H^s}^2 = \left\|(Id - \Delta)^{\frac{s}{2}} u\right\|_{L^2}^2 = \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi, \tag{1}$$

where the operator $(Id - \Delta)^{\frac{s}{2}} = \mathrm{F}^{-1} \circ = M_{(1+|\xi|^2)^{s/2}} \circ \mathrm{F}$ is conjugate to the multiplication operator on $L^2(\mathbb{R}^N)$ given by the function $(1 + |\xi|^2)^{s/2}$.

It is well known that for $0 < s < N/2$ and $2^* = 2N/(N - 2s)$, the Sobolev critical exponent, the following Sobolev inequality is valid for some positive constant $S^* = S^*(N, s)$

$$\|u\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \leq S^* \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2(\mathbb{R}^N)}^{2^*} \qquad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2}$$

and the same inequality holds by density on $H^s(\mathbb{R}^N)$.

In order to discuss inequality (2), it is very natural to introduce for each $0 < s < N/2$ the homogeneous Sobolev space $H_0^s(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \text{s.t.} |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}$.
This space can be equivalently defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{H_0^s}^2 = \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2}^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \tag{3}$$

and inequality (2) holds by density on $H_0^s(\mathbb{R}^N)$.

When $0 < s < 1$, a direct calculation using Fourier transform (see,[21]) gives

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = c(N, s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy, \tag{4}$$

which provides an alternative formula for the norm on $H_0^s(\mathbb{R}^N)$. The previous equality fails for $s \geq 1$, since in that case the right hand-side in (4) is known to be finite if and only if $u$is constant [22].
When $0 < s < 1$, according to [23], [24] for the more difficult case $1 < s < N/2, s \notin \mathbb{N}$), the Sobolev inequality (2) is also equivalent to the trace Sobolev embedding $H_0^1(\mathbb{R}^N \times (0, \infty), t^{1-2s} dxdt) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Indeed, taking for simplicity $u \in C_0^\infty(\mathbb{R}^N)$ and $U \in C_0^\infty(\mathbb{R}^N [0, \infty))$ such that $U(x, 0) \equiv u(x)$ we have

$$\|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq S^{*2/2^*} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \leq C(N, s) \int_{\mathbb{R}^N} \int_0^\infty |\nabla U|^2 t^{1-2s} dxdt, \tag{5}$$

which extends to a bounded trace operator $T_r : H_0^1 \to L^{2^*}$. Moreover, the second inequality in (5) is an equality if and only if the extension U satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ U(\cdot,0) = u & \text{in } \mathbb{R}^N. \end{cases} \tag{6}$$

Actually, the solution operator to (6) allows to identify $H_0^s(\mathbb{R}^N)$ as the trace space of $H_0^1(\mathbb{R}^N \times (0,\infty), t^{1-2s}dxdt)$ and the Sobolev inequality (2) as the trace inequality in (5).

The starting point is the following theorem proved in [25] which gives the optimal constant in the Sobolev inequality (2) together with the explicit formula for those functions giving equality in the inequality.

Theorem (2.1.1)[26]. Let $0 < s < N/2$ and $2^* = 2N/(N-2s)$. Then

$$\|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \le S^* \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2(\mathbb{R}^N)}^{2^*} \qquad \forall u \in H_0^s(\mathbb{R}^N), \tag{7}$$

Where $S^* = \left(2^{-2s}\pi^{-s}\frac{\Gamma\left(\frac{N-2s}{2}\right)}{\Gamma\left(\frac{N+2s}{2}\right)}\left[\frac{\Gamma(N)}{\Gamma(N/2)}\right]^{2s/N}\right)^{\frac{2^*}{2}}$ and Γ is the Gamma function.

For $u \neq 0$, we have equality in (7) if and only if

$$u(x) = \frac{c}{(\lambda^2 + |x-x_0|^2)^{\frac{N-2s}{2}}} \qquad \forall x \in \mathbb{R}^N, \tag{8}$$

where $c \in \mathbb{R}\backslash\{0\}, \lambda > 0$ and $x_0 \in \mathbb{R}^N$ are fixed constants. The Sobolev inequality (7) as well as the previous theorem in the case $s = 1$ are proved in [27] and also in [28], where the connection with the Yamabe problem is discussed.

When $2 \le s < N/2$ is an even integer the same result was obtained some years later in [29], following the ideas in [30],[31].

Also the case $s = 1/2$ has been already studied in the equivalent form (5)-(6) in [32], in connection with the Yamabe problem on manifolds with boundary (see also [33] for the trace inequality in the case $W^{1,p}$ with a different proof using mass transportation techniques).

The proof in [25] is based on a sharp form of the Hardy-Littlewood-Sobolev inequality.
Using the moving planes method, formula (8) has been obtained independently in [34].

At least when $0 < s < 1$, a third approach through symmetrization techniques applied to the norm in the right hand-side of (4) can be found in [35].
A naive approach to the validity of (7) is to study the variational problem

$$S^* := \sup\left\{F(u) : u \in H_0^s(\mathbb{R}^N), \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \le 1\right\} \tag{9}$$

where

$$F(u) := \int_{\mathbb{R}^N} |u|^{2^*} dx. \tag{10}$$

Clearly, the validity of (7) is equivalent to show that the constant S^* defined in (9) is finite. Moreover, Theorem (2.1.1) gives an explicit formula for it as well as for the maximizers of the variational problem (9) up to normalization. Note that even the existence of a maximizer is not trivial since the embedding (2) is not compact, because of translation and dilation invariance.

Indeed, if $u \in H_0^s(\mathbb{R}^N)$ is an admissible function in (9), the same holds for $u_{x_0,\lambda}(x) = \lambda^{N-2s/2}u(x_0 + \lambda x)$ for any $x_0 \in \mathbb{R}^N$ and any $\lambda > 0$.

In addition $u_{x_0,\lambda}$ satisfies $F(u_{x_0,\lambda}) = F(u)$ and tends to zero weakly in H_0^s, as $|x_0| \to 1$ (translation invariance) or as $\lambda \to 0^+$ and $\lambda \to \infty$ (dilation invariance).Another related problem we consider is the following. Given a bounded domain $\Omega \subset \mathbb{R}^N$, one can define the Sobolev space $H_0^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H_0^s(\mathbb{R}^N)$ with the norm in (3) and the corresponding maximization problem (or Sobolev embedding), namely

$$S_\Omega^* \coloneqq \sup\left\{F_\Omega(u) : u \in H_0^s(\Omega), \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx \le 1\right\} \tag{11}$$

where

$$F_\Omega(u) \coloneqq \int_\Omega |u|^{2^*} dx. \tag{12}$$

A simple scaling argument on compactly supported smooth functions shows that $S^* = S_\Omega^*$, but in view of Theorem (2.1.1) the variational problem (11) has no maximizer. Thus, in order to study the behavior of a maximizing sequence for (9) and (11) it is very convenient to establish a concentration-compactness alternative for bounded sequences in the fractional space H_0^s, using methods and ideas introduced in the pioneering works [30] and [31] and developed extensively in literature (see [36], [37], [56]).We have the following

Theorem (2.1.2)[26]. Let $\Omega \subseteq \mathbb{R}^N$ an open subset and let $\{u_n\}$ be a sequence in $H_0^s(\Omega)$ weakly converging to u as $n \to \infty$ and such that $\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx \overset{*}{\rightharpoonup} \mu \quad and \quad |u_n|^{2^*} dx \overset{*}{\rightharpoonup} \nu \quad in \ \mathrm{M}(\mathbb{R}^N)$.

Then, either $u_n \to u$ in $L^{2^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J}$ and positive numbers $\{\nu_j\}_{j\in J}$ such that we have

$$\nu = |u|^{2^*} dx + \sum_j \nu_j \delta_{x_j}. \tag{13}$$

If, in addition, Ω is bounded, then there exist a positive measure $\tilde{\mu} \in \mathrm{M}(\mathbb{R}^N)$ with spt $\tilde{\mu} \subset \overline{\Omega}$ and positive numbers $\{u_j\}_{j\in J}$ such that

$$\mu = \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \tilde{\mu} + \sum_j \mu_j \delta_{x_j}, \qquad \nu_j \le S^*(\mu_j)^{\frac{2^*}{2}}. \tag{14}$$

Proof: Since $H_0^s(\Omega) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$ with compact embedding, passing to a subsequence if necessary, we may assume that $u_n \to u$ both in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e.. Similarly, for $v_n = u_n - u \rightharpoonup 0$ in $H_0^s(\Omega)$, up to subsequence, we may assume $\left|(-\Delta)^{\frac{s}{2}}v_n\right|^2 dx \overset{*}{\rightharpoonup} \hat{\mu}$ and $|v_n|^{2^*} dx \overset{*}{\rightharpoonup} \hat{\nu}$ in $\mathrm{M}(\mathbb{R}^N)$,for some positive measures $\hat{\mu}$ and $\hat{\nu}$ with spt $\hat{\nu} \subset \overline{\Omega}$.

In addition, when Ω is bounded, Lemma (2.1.16) easily yields spt $\hat{\mu} \subset \overline{\Omega}$. Clearly $\nu \ge |u_n|^{2^*} dx$ by Fatou's Lemma, and combining pointwise convergence and the result in [61],[62] we have

$$\int_{\mathbb{R}^N} |\varphi|^{2^*} d\nu - \int_{\mathbb{R}^N} |\varphi u|^{2^*} dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\varphi u_n|^{2^*} dx - \int_{\mathbb{R}^N} |\varphi u|^{2^*} dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\varphi v_n|^{2^*} dx$$
$$= \int_{\mathbb{R}^N} |\varphi|^{2^*} d\hat{\nu},$$

i.e. $\nu = \hat{\nu} + |u|^{2^*}dx$ because the function $\varphi \in C_0^0(\mathbb{R}^N)$ can be choosen arbitrarily.We are going to prove the structure properties in (13) and (14) assuming that Ω is bounded. Then, the structure relation (13) will be true for any Ω just by a simple localization argument.
Indeed, $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \equiv 1$ on B_1 and for $0 < \lambda < 1$ let $\psi_\lambda(x) = \psi(\lambda x)$.
For fixed $\lambda \in (0,1)$, we consider $u_n^\lambda = \psi_\lambda u_n$. Then, letting $n \to \infty$, we have $u_n^\lambda \rightharpoonup \psi_\lambda u$ in $H_0^s(\Omega)$, because ψ_λ is a multiplier on $H_0^s(\Omega)$, and $\left|u_n^\lambda\right|^{2^*} dx \overset{*}{\rightharpoonup} \nu_\lambda = |\psi_\lambda|^{2^*}\nu$ in $\mathrm{M}(\mathbb{R}^N)$.

If we assume that (13) holds for each of these limiting measures ν_λ (possibly adding further Dirac masses in $B_{\lambda^{-1}} \cap \Omega$ as λ gets smaller), then the number of atoms of ν_λ is clearly uniformly bounded and for $0 < \lambda < 1$ and, for $0 < \lambda < \lambda_0$ in the location is independent of λ in $B_{\lambda_0^{-1}}$. Thus $\nu \overset{*}{\rightharpoonup} \nu_\lambda$ as $\lambda \to 0$, hence (13) holds for νas desired. Let $\Omega \subset \mathbb{R}^N$ be bounded and let us prove (13) and (14).

Given $\varphi \in C_0^\infty(\mathbb{R}^N)$, the Sobolev inequality (2) yields

$$\left(\int_{\mathbb{R}^N} |\varphi|^{2^*}|u_n|^{2^*}dx\right)^{\frac{2}{2^*}} \leq (S^*)^{\frac{2}{2^*}} \left\|(-\Delta)^{\frac{s}{2}}(\varphi u_n)\right\|^2_{L^2(\mathbb{R}^N)} \tag{15}$$

and

$$\left(\int_{\mathbb{R}^N} |\varphi|^{2^*}|v_n|^{2^*}dx\right)^{\frac{2}{2^*}} \leq (S^*)^{\frac{2}{2^*}} \left\|(-\Delta)^{\frac{s}{2}}(\varphi v_n)\right\|^2_{L^2(\mathbb{R}^N)}. \tag{16}$$

we have $\left\|(-\Delta)^{\frac{s}{2}}(\varphi v_n)\right\|^2_{L^2(\mathbb{R}^N)} = \left\|\varphi(-\Delta)^{\frac{s}{2}} v_n\right\|^2_{L^2(\mathbb{R}^N)} + o(1) \quad as \quad n \to \infty.$
Passing to the limit in (16) we get

$$\left(\int_{\mathbb{R}^N} |\varphi|^{2^*}d\hat{\nu}\right)^{\frac{2}{2^*}} = \lim_{n\to\infty}\left(\int_{\mathbb{R}^N} |\varphi|^{2^*}|v_n|^{2^*}dx\right)^{\frac{2}{2^*}} \leq (S^*)^{\frac{2}{2^*}} \int_{\mathbb{R}^N} \varphi^2 d\hat{\mu}, \tag{17}$$

i.e. the measures $\hat{\nu}$ and $\hat{\mu}$ satisfy the reverse Holder inequality (25) with $p = 2, r = 2^*$ and $C = (S^*)^{1/2^*}$.Thus, the decomposition for $\hat{\nu}$ and in turn for $\nu = |u|^{2^*}dx + \hat{\nu}$, i.e. (13) holds.In order to prove (14), note that as $n \to \infty$ we have $v_n = u_n - u \rightharpoonup 0$ in $H_0^s(\Omega)$ (hence $(-\Delta)^{\frac{s}{2}}(u_n - u) \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$), thus Lemma 4 gives

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(\varphi u_n)\right|^2 dx &= \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(\varphi u)\right|^2 dx + \int_{\mathbb{R}^N} \left|\varphi(-\Delta)^{\frac{s}{2}}(u_n - u)\right|^2 dx + o(1) \\
&= \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(\varphi u)\right|^2 dx + \int_{\mathbb{R}^N} \left|\varphi(-\Delta)^{\frac{s}{2}}u_n\right|^2 dx - \int_{\mathbb{R}^N} \left|\varphi(-\Delta)^{\frac{s}{2}}u\right|^2 dx + o(1) \\
&= \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(\varphi u)\right|^2 dx - \int_{\mathbb{R}^N} \left|\varphi(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \int_{\mathbb{R}^N} |\varphi|^2 d\mu + o(1).
\end{aligned} \tag{18}$$

Combining (15) and (18), as $n \to \infty$ we obtain

$$\left(\int_{\mathbb{R}^N} |\varphi|^{2^*}d\hat{\nu}\right)^{\frac{2}{2^*}} \leq (S^*)^{\frac{2}{2^*}}\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(\varphi u)\right|^2 dx - \int_{\mathbb{R}^N} \left|\varphi(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \int_{\mathbb{R}^N} |\varphi|^2 d\mu\right) \tag{19}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since ν satisfies (13), choosing $\varphi_{x_j,\lambda}(x) = \varphi\left(x_j + \lambda^{-1}x\right)$ in (26) as a test function, and dominated convergence as $\lambda \to 0$ yield

$\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}\left(\varphi_{x_j,\lambda}u\right)\right|^2 dx - \int_{\mathbb{R}^N} \left|\varphi_{x_j,\lambda}(-\Delta)^{\frac{s}{2}}u\right|^2 dx = o(1),$ whence $\nu \geq \sum_j \nu_j\delta_{x_j}$ implies $\mu \geq \sum_j \mu_j\delta_{x_j}$ for some $\mu_j > 0$ such that $\nu_j \leq S^*\mu_j^{\frac{2^*}{2}}$. Note that (14) follows easily because $\sum_j \mu_j\delta_{x_j}$ and $\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx$ are mutually singular, $\mu \geq \sum_j \mu_j\delta_{x_j}$ and$\mu \geq \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx$ (the latter inequality by weak lower semicontinuity in L^2), hence (14) holds. In order to conclude, it remains to observe that spt $\bar{\mu} \subseteq \bar{\Omega}$, i.e., $\int \varphi^2 d\mu = \int \varphi^2 \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx$ for any $\varphi \in C_0^\infty(\mathbb{R}^N\backslash\bar{\Omega})$, which is a straightforward consequence of equation (18) as $n \to 1$.

The previous result extends to the case of the fractional spaces H^s a well known fact for $s = 1$ and, more generally, when s is an integer (see [30],[31]and[36]), namely that, at least locally, compactness in the Sobolev embedding fails precisely because of concentration of the L^{2^*} norm at countably many points. These results have been largely used for the variational treatment of the Yamabe problem and their higher order analogues involving the Paneitz-Branson operators and more generally for semi-linear elliptic equations with critical nonlinearities. At least when $\Omega = \mathbb{R}^N$, a proof of Theorem (2.1.2) can be deduced as a byproduct of the so-called profile-decomposition in [38].

In[38] much stronger results are obtained using Fourier analysis, Paley-Littlewood decomposition, a tricky exhaustion method and the improved Sobolev inequality in Besov spaces due to Gérard-Meyer-Oru. Here, we provide an elementary proof of Theorem (2.1.2) by following the original argument in [30],[31]; clearly, we need to operate some modifications due to the non-locality of the fractional operators $(-\Delta)^{\frac{s}{2}}$. Indeed, our approach relies on pseudodifferential calculus to control the natural error term in the localization by cut-off functions.Using a commutator estimate by Taylor [39] and a standard approximation argument, we will be able to prove the compactness of the commutator

$\left[\varphi, (-\Delta)^{\frac{s}{2}}\right] : H_0^s(\Omega) \to L^2(\mathbb{R}^N)$ when $\varphi \in C_0^\infty(\mathbb{R}^N)$, at least if Ω is bounded.

This will give us the possibility to handle the fractional differentiation giving local description of the lack of compactness in terms of atomic measures. We hope that will be of use in the variational theory of the fractional Yamabe problem firstly considered in [40,[41].

Armed with the concentration-compactness alternative given by Theorem (2.1.2), we can study maximizing sequences of the variational problems (9) and (11). We will see that concentration always occurs in problem (11) because of the classification in Theorem (2.1.1),Corollary(2.1.8)). It would be very interesting to study the existence/nonexistence of optimal functions in (9) and (11) for other equivalent norm. Indeed, even for norms equivalent to (3), (4) and (5)-(6), e.g. obtained multiplying by suitable functions $|a(\xi)|, |K(x, y)|$ and $|A(x, t)|$ bounded from above and below, we expect the existence of optimal function to depend in a nontrivial way on the choosen functions (see [42]). In addition, we expect that, as for the local case $s = 1$, optimizing sequences for the Sobolev inequality (9) look asymptotically like optimal functions. It would be interesting to prove a quantitative version of this fact in analogy with what is done in [43] for the case $s = 1$ (see [44] for Sobolev space $W^{1,p}$). Next, we consider a family of variational problems associated to suitable perturbations of the functional (12).

Let $0 < \varepsilon < 2^* - 2$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set.

We set

$$S_\varepsilon^* := \sup\left\{F_\varepsilon(u) : u \in H_0^s(\Omega), \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx \le 1\right\} \tag{20}$$

where$F_\varepsilon(u) := \int_\Omega |u|^{2^*-\varepsilon}dx$.

The previous problem is subcritical. Indeed, since Ω is a bounded open set and the embedding $H_0^s(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$ is compact, the previous problem admits a maximize $u_\varepsilon \in H_0^s(\Omega)$. Our purpose is to investigate what happens when $\varepsilon \to 0$ both to the subcritical Sobolev constant S_ε^* (i.e., the optimal constant for the embedding $H_0^s(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$ given in (20)) and to the corresponding maximizers u_ε(i.e. the corresponding optimal functions). Combining the pointwise convergence of F_ε to F_Ω together with previous Theorem (2.1.2), we are able to prove the following result.

Theorem (2.1.3)[26]. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and for each $0 < \varepsilon < 2^* - 2$ let $u_\varepsilon \in H_0^s(\Omega)$ be a maximizer for S_ε^*. Then

(i) $\lim_{\varepsilon\to 0} S_\varepsilon^* = S^*$;

(ii) As $\varepsilon = \varepsilon_n \to 0$, up to subsequences $u_n = u_{\varepsilon n}$ satisfies $u_n \rightharpoonup 0$ in $H_0^s(\Omega)$and it concentrates at some point $x_0 \in \overline{\Omega}$ both in $L^{2^*}$and in H^s, i.e.

$$|u_n|^{2^*}dx \overset{*}{\rightharpoonup} S^*\delta_{x_0} \; and \; \left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 dx \overset{*}{\rightharpoonup} \delta_{x_0} \; in \; \mathrm{M}(\mathbb{R}^N).$$

Proof: First, we claim that

$$\limsup_{\varepsilon\to 0} S_\varepsilon^* \le S^*. \tag{21}$$

Indeed, taking $u_\varepsilon \in H_0^s(\Omega)$ a maximizer for S_ε^*, by Hölder inequality we have

$$S_\varepsilon^* = F_\varepsilon(u_\varepsilon) = \int_\Omega |u_\varepsilon|^{2^*-\varepsilon}dx \le \left(\int_\Omega |u_\varepsilon|^{2^*}\right)^{\frac{2^*-\varepsilon}{2^*}} |\Omega|^{\frac{\varepsilon}{2^*}} \le (S^*)^{\frac{2^*-\varepsilon}{2^*}}|\Omega|^{\frac{\varepsilon}{2^*}}.$$

Thus, inequality (21) follows as $\varepsilon \to 0$.

The reverse inequality easily follows from the pointwise convergence of F_ε to F_Ω with a standard argument. Indeed, for every $\delta > 0$ there exists $u_\delta \in H_0^s(\Omega)$ such that $\|u_\delta\|_{H_0^s} \le 1$ and

$$F_\Omega(u_\delta) > S^* - \delta. \tag{22}$$

Clearly, for such function u_δ, we have $S_\varepsilon^* \ge F_\varepsilon(u_\delta)$. Thus, combining the previous inequality with (22) and passing to the limit as ε goes to zero, we get $\liminf_{\varepsilon\to 0} S_\varepsilon^* \ge \lim_{\varepsilon\to 0} F_\varepsilon(u_\delta) = F_\Omega(u_\delta) \ge S^* - \delta$ and claim (i) follows as $\delta \to 0$ in view of (21). The concentration result (ii) for the sequence $\{u_\varepsilon\}$ of maximizers of S_ε^* now is straightforward. Due to (i) the sequence u_ε is a maximizing sequence for F_Ω, hence, ensures that, up to subsequences, $\{u_\varepsilon\}$ concentrate at one point $x_0 \in \overline{\Omega}$, in the sense that $|u_n|^{2^*}dx \overset{*}{\rightharpoonup} S^*\delta_{x_0}$ and $\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 dx \overset{*}{\rightharpoonup} \delta_{x_0}$ in $\mathrm{M}(\mathbb{R}^N)$.

Let Ω be a bounded domain in $\mathbb{R}^N, s \in \mathbb{R}, 0 < s < N/2$. In the Introduction we have considered the following problem for $\varepsilon \in (0, 2^* - 2)$

$$S_\varepsilon^* = \sup\left\{\int_\Omega |u|^{2^*-\varepsilon}dx : u \in H_0^s(\Omega), \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx \le 1\right\}. \tag{23}$$

The goal of the present section is to refine the result given in Theorem (2.1.3) about the behavior of the maximizers u_ε of (23). Here, we describe the asymptotic behavior as $\varepsilon \to 0$ of the optimal constants S_ε^*

associated to the embeddings $H_0^s(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$ and of the corresponding maximizers, studying the family $\{F_\varepsilon\}$ of functionals

$$F_\varepsilon(u) := \int_\Omega |u|^{2^*-\varepsilon} dx, \tag{24}$$

on the set $\left\{u \in H_0^s(\Omega), \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \leq 1\right\}$. The main tool is the notion of Γ-convergence in the sense of De Giorgi (see [62]) and the crucial point is to introduce a convenient functional framework in which performing the passage to the limit. The previous concentration result is well known for $s = 1$.

The asymptotic behavior of the optimal functions has been discusses in [45] and [46], at least assuming (i) and the smoothness of the domain Ω. For the case of general possibly non-smooth domains we refer to [47],[48] for the analogous problem in $W^{1,p}$.In view of Theorem (2.1.3), it would be also interesting to understand whether the concentration point x_0 has some characterization, e.g. as critical point of some function. This is known to be the case when $s = 1$ or $s = 2$, the function being the regular part of the Green function of the Laplacian or the BiLaplacian in the domain Ω (see [45],[46],[49],[50] and [51]).Here, we also note that the maximizes $u_\varepsilon \in H_0^s(\Omega)$ discussed in Theorem (2.1.4) are in fact solutions of the semi-linear equation

$$(-\Delta)^s u_\varepsilon = \lambda |u_\varepsilon|^{2^*-2-\varepsilon} u_\varepsilon \text{in} \left(H_0^s(\Omega)\right)', \tag{25}$$

where $\lambda = (S_\varepsilon^*)^{-1}$ is a Lagrange multiplier. Indeed, (24) is the Euler-Lagrange equation for the functional F_ε among functions with H_0^s norm equal to one. Equivalently, the previous equation is the Euler-Lagrange equation for the dual variational problem, i.e. to minimize the H_0^s norm keeping F_ε constant. Our results yield a concentration phenomenon for a sequence of solutions u_ε, as ε goes to zero.In this respect, another subcritical problem that would be very natural to investigate is

$$(-\Delta)^s - \eta u = |u|^{2^*-2} \text{in} \left(H_0^s(\Omega)\right)', \tag{26}$$

where $\eta > 0$ is a parameter. Well known results for $s = 1$ (see [52]) and $s = 2m$ an even integer (see [53] and [54]) suggest that, even for fractional values of s, existence results for (26) should always depend in a delicate way on η (see [55] for a first result when $s = 1/2$, and [4] when $s \in (0,1)$, though with a slightly different definition of the fractional Laplacian; see, [56]).

Finally, we come back to the subcritical problem (20), carefully analyzing the asymptotic behavior of the energy functionals F_ε, by means of De Giorgi's Γ-convergence techniques. Here, the analysis is much in the spirit of ([50],[51],[47] and [48]), but with some relevant differences in the proofs . We have the following result.

Theorem (2.1.4)[26]. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let X be the space

$$X = X(\Omega) := \left\{(u, \mu) \in H_0^s(\Omega) \times \mathrm{M}(\mathbb{R}^N) : \mu \geq \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx, \mu(\mathbb{R}^N) \leq 1\right\},$$

endowed with the product topology τ such that

$$(u_n, \mu_n) \xrightarrow{\tau} (u, \mu) \overset{def}{\Longleftrightarrow} \mu \begin{cases} u_n \rightharpoonup u \ in \ L^{2^*}(\Omega), \\ \mu_n \overset{*}{\rightharpoonup} \mu \ \ in \ \mathrm{M}(\mathbb{R}^N). \end{cases} \tag{27}$$

Let us consider the following family of functionals

$$F_\varepsilon(u, \mu) := \int_\Omega |u|^{2^*-\varepsilon} dx \quad \forall (u, \mu) \in X \tag{28}$$

Then, as $\varepsilon \to 0$, the Γ^+-limit of the family of functionals F_ε with respect to the topology τ corresponding to (23) is the functional F defined by $F(u,\mu) = \int_\Omega |u|^{2^*} dx + S^* \sum_{j=1}^{\infty} \mu_j^{\frac{2^*}{2}}$, $\forall (u,\mu) \in X$.

Here S^* is the best Sobolev constant in $\mathbb{R}^N$ as given in Theorem (2.1.1) and the numbers μ_i are the coefficients of the atomic part of the measure μ. As a consequence of the previous Γ^+-convergence result, together with some property of the limit functional F, we can also deduce that the sequences of maximizers $\{u_\varepsilon\}$ concentrate energy at one point $x_0 \in \Omega$, as already stated in Theorem (2.1.3)-(ii). It would be interesting to prove results analogous to those of Theorem (2.1.2), (2.1.3) and (2.1.4) with respect to the equivalent norms (4) and (5)-(6), i.e. taking the measure μ as limit of energy densities in $\mathbb{R}^N \times \mathbb{R}^N$ or $\mathbb{R}^N \times (0,\infty)$, respectively, and describing the corresponding loss of compactness in terms of atomic measures.It is worth noticing that Theorem (2.1.4) could have its own relevance also to identify the location of the concentration point. Indeed, it can be read as the necessary first step in the asymptotic development by Γ-convergence (as firstly introduced in [57],[58]) of the functionals in (28). In this sense, a second order expansion of the Γ-limit could bring the desired informations on the concentration of the maximizing sequences, as in [59], where different energies involving critical growth problems have been studied (see [59] and [60]). we will prove Theorem (2.1.2) and its consequences, i.e. we establish the concentration-compactness alternative and we describe the behavior of the optimal sequences for the Sobolev inequality. We also show that in the case of bounded domain there is no energy loss in the concentration process and that the maximizing sequences for the Sobolev inequality concentrate at one point.We analyze the asymptotic behavior of the subcritical Sobolev constant S_ε^* and the corresponding optimal functions proving Theorem (2.1.3).Here we prove Theorem (2.1.4) and as a consequence we provide an alternative argument for the concentration of the corresponding maximizes u_ε.

Finally we establish two auxiliary results about H^s functions that are needed in the proof of the concentration-compactness alternative.We start with a well known lemma about pairs of positive measures in the Euclidean space. Roughly speaking, it gives control on their atomic parts whenever a reverse Hölder inequality holds.

Lemma (2.1.5)[26]. ([48]) Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let μ and ν in $\mathrm{M}(\mathbb{R}^N)$ be two nonnegative bounded measures with support in such that for some $1 \le p < r < \infty$ there exists a positive constant C such that

$$\left(\int_{\mathbb{R}^N} |\varphi|^r d\nu \right)^{\frac{1}{r}} \le C \left(\int_{\mathbb{R}^N} |\varphi|^p d\mu \right)^{\frac{1}{p}} \qquad \forall \varphi \in C_0^0(\mathbb{R}^N). \tag{29}$$

Then, there exists a number $\sigma = C^{-(p^{-1}-r^{-1})^{-1}} > 0$, a (at most countable) set of distinct points $\{x_j\}_{j\in J}$ in $\overline{\Omega}$ and positive numbers $\nu_j \ge \sigma, j \in J$, such that

$$\nu = \sum_j \nu_j \delta_{x_j} \qquad \mathit{and} \qquad \mu \ge C^{-p} \sum_j \nu_j^{\frac{p}{r}} \delta_{x_j}, \tag{30}$$

where δ_{x_j} denotes the Dirac mass at x_j.

Using the previous lemma we are able to prove the main result, i.e.

Theorem (2.1.2). Namely we show that the well-known concentration-compactness alternative holds for sequences in any Sobolev spaces $H_0^s(\Omega)$, $0 < s < N/2$. The proof follows the original arguments in [30] and [31] with some modifications to handle fractional differentiation. A simple consequence of the previous theorem is the following result, which will be useful and which shows that on bounded domains there is no energy loss in the concentration process.

Proposition(2.1.6)[26]. Let $0 < 2s < N$, let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\{u_n\} \subset H_0^s(\Omega)$ such that $u_n \rightharpoonup 0$ as $n \to \infty$. For any open set $A \subseteq \mathbb{R}^N$ such that $\bar{\Omega} \cap \bar{A} = \emptyset$ we have $\int_A \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx \to 0$ as $n \to \infty$.

Proof. In view of the energy concentration described in formula (14) of Theorem (2.1.2), the conclusion clearly holds when A is bounded, so it is enough to prove the claim when $A = \mathbb{R}^N \setminus \bar{B}$ and $B \subset \mathbb{R}^N$ is some Euclidean ball sufficiently large. Let us choose B such that $2\Omega \subset B$ and let $\varphi \in C_0^\infty(B)$ such that $\varphi \equiv 1$ on $\overline{B/2} \supset \Omega$. we have

$$\int_A \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx \le \int_{\mathbb{R}^N} (1-\varphi)^2 \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx = \int_{\mathbb{R}^N} \left|\left[1-\varphi, (-\Delta)^{\frac{s}{2}}\right] u_n\right|^2 dx$$
$$= \int_{\mathbb{R}^N} \left|\left[\varphi, (-\Delta)^{\frac{s}{2}}\right] u_n\right|^2 dx \xrightarrow{n\to\infty} 0$$

and the proof is complete.

The following result is a direct consequence of Theorem (2.1.2) and Theorem (2.1.1) and describes the behavior of optimal sequences for the variational (11) in bounded domains.

Corollary (2.1.7)[26]. Let $\Omega \subset \mathbb{R}^N$ a bounded open set and let $\{u_n\} \subset H_0^s(\Omega)$ be a maximizing sequence for the critical Sobolev inequality (11). Then, up to subsequences, $\{u_n\}$ concentrates at one point $x_0 \in \bar{\Omega}$ in the sense that $|u_n|^{2^*} dx \overset{*}{\rightharpoonup} S^* \delta_{x_0}$ and $\left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx \overset{*}{\rightharpoonup} \delta_{x_0}$ in $\mathrm{M}(\mathbb{R}^N)$.

Proof. The result easily follows from the concentration-compactness alternative in Theorem (2.1.2).

One of the key point in the proof is the well-known convexity trick by Lions.

Let $\{u_n\} \subset H_0^s(\Omega)$ be a maximizing sequence for the critical Sobolev inequality (11). Then, up to subsequences, $u_n \rightharpoonup u$ in $H_0^s(\Omega)$, $\int_\Omega |u_n|^{2^*} dx \to S^*$ and also $|u_n|^{2^*} dx \overset{*}{\rightharpoonup} \nu \in \mathrm{M}(\mathbb{R}^N)$ with $\nu(\bar{\Omega}) = S^*$.

By formula (13) in Theorem (2.1.3), we have

$$S^* = \nu(\bar{\Omega}) = \int_\Omega |u|^{2^*} dx + \sum_j \nu_j. \tag{31}$$

Combining the Sobolev inequality (2) with (13)-(14), we get

$$\int_\Omega |u|^{2^*} dx + \sum_j \nu_j \le S^* \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^{\frac{2^*}{2}} + S^* \sum_j \mu_j^{\frac{2^*}{2}}, \tag{32}$$

where μ_j are the atomic coefficients of the measure $\mu \in \mathrm{M}(\mathbb{R}^N)$, that is the limit in the sense of measures of the sequence $\left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 dx$.

Taking formula (14) and Proposition (2.1.6) into account we have

$$S^* \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^{\frac{2^*}{2}} + S^* \sum_j \mu_j^{\frac{2^*}{2}} \le S^* \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \sum_j \mu_j \right)^{\frac{2^*}{2}}$$
$$\le S^* \mu(\mathbb{R}^N) = S^*, \tag{33}$$

because $\|u_n\|_{H_0^s} = 1$ for each n and there is no loss of energy in the limit.

Therefore, combining (31),(32) and (33),we see that all the inequalities must be equalities. Since the Sobolev constant is not attained on bounded domains and the function $t \mapsto t^{\frac{2^*}{2}}$ is strictly convex, it follows that $\tilde{\mu} = 0, u$ is zero and only one of the μ_j's and ν_j's can be nonzero in (13)-(14). Hence, concentration occurs at one point $x_0 \in \overline{\Omega}$ as claimed.

We conclude this section with the asymptotic analysis of the maximizes for the variational problem in (20), proving the claims stated in Theorem (2.1.3).

Theorem (2.1.8)[26]. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let X be the space

$$X = X(\Omega) := \left\{(u,\mu) \in H_0^s(\Omega) \times \mathrm{M}(\mathbb{R}^N) : \mu \geq \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx, \mu(\mathbb{R}^N) \leq 1\right\},$$

endowed with the product topology τ such that

$$(u_n, \mu_n) \xrightarrow{\tau} (u,\mu) \overset{def}{\Longleftrightarrow} \begin{cases} u_n \rightharpoonup u \;\; in \;\; L^{2^*}(\Omega), \\ \mu_n \overset{*}{\rightharpoonup} \mu \; in \; \mathrm{M}(\mathbb{R}^N). \end{cases} \tag{34}$$

Let us consider the following family of functionals

$$F_\varepsilon(u,\mu) := \int_\Omega |u|^{2^*-\varepsilon} dx \quad \forall (u,\mu) \in X. \tag{35}$$

Then, as $\varepsilon \to 0$, the Γ^+-limit of the family of functionals F_ε with respect to the topology τ corresponding to (34) is the functional F defined by $F(u,\mu) = \int_\Omega |u|^{2^*} dx + S^* \sum_{j=1}^{\infty} \mu_j^{\frac{2^*}{2}} \quad \forall (u,\mu) \in X.$

Here S^* is the best Sobolev constant in $\mathbb{R}^N$ as given in Theorem (2.1.1) and the numbers μ_i are the coefficients of the atomic part of the measure μ. The reason for the choice of X in Theorem (2.1.8) can be described as follows. We are interested in the asymptotic behavior of the sequence $\{F_\varepsilon(u_\varepsilon)\}$ for every sequence $\{u_\varepsilon\}$ such that $\left\|(-\Delta)^{\frac{s}{2}} u_\varepsilon\right\|^2_{L^2(\mathbb{R}^N)} \leq 1$. The constraint on the "Dirichlet energy" of u_ε implies that, up to subsequences, there exists $\mu \in \mathrm{M}(\mathbb{R}^N)$ and $u \in H_0^s(\Omega)$ such that $\mu(\mathbb{R}^N) \leq 1, \left|(-\Delta)^{\frac{s}{2}} u_\varepsilon\right|^2 dx \overset{*}{\rightharpoonup} \mu$ in $\mathrm{M}(\mathbb{R}^N)$ and $u_\varepsilon \rightharpoonup u$ in H_0^s.

Clearly, by Sobolev embedding, we also have $u_\varepsilon \rightharpoonup u$ in $L^{2^*}(\Omega)$. By Fatou's Lemma, we deduce $\mu \geq \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx$ and we can always decompose μ in $\mu = \left|(-\Delta)^{\frac{s}{2}} u\right|^2 + \tilde{\mu} + \sum_{j=1}^{\infty} \mu_j \delta_{x_j}$, where $\mu_j \in [0,1]$ and $\{x_j\} \subseteq \overline{\Omega}$ are distinct points; the positive measure $\tilde{\mu}$ can be viewed as the "non-atomic part" of the measure $\left(\mu - \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx\right)$. In view of this decomposition, the definition of X given in Theorem (2.1.8) is very natural; moreover the space X is sequentially compact in the topology τ. Indeed, if $\{u_n, \mu_n\} \subseteq X$, then $\{u_n\}$ is bounded in $H_0^s(\Omega)$. Up to subsequences, $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathrm{M}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $H_0^s(\Omega)$ (and in $L^{2^*}(\Omega)$, by Sobolev embeddings) and the inequalities defining X still hold for (u,μ) by weak lower semicontinuity. Since X appears as a sort of completion of $H_0^s(\Omega)$ in the weak topology of the product $L^{2^*}(\Omega) \times \mathrm{M}(\mathbb{R}^N)$, it would be interesting to understand whether, as in the case $s = 1$ (see [51]), every pair (u,μ) in X can be actually approximated in the topology $\boldsymbol{\tau}$ by a sequence of the form $\left\{\left(u_\varepsilon, \left|(-\Delta)^{\frac{s}{2}} u_\varepsilon\right|^2 dx\right)\right\}$. We will not pursue this point here.

Note that, since the embeddings $H_0^s(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$ are compact, the functionals F_ε as extended to X by (33) are continuous and Proposition (2.1.9) below show that there are no further maximizes in the space X. As a consequence, we have that the Γ^+-convergence of functionals in this space implies the convergence of maximizers $\{u_\varepsilon\}$ of F_ε to the maxima of F; this will allow an alternative proof of the concentration for the sequences $\{u_\varepsilon\}$ already obtained in Theorem (2.1.3).

Proposition (2.1.9)[26]. For any $\varepsilon > 0$, let $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) \in X$ be such that

$\sup_{(u,\mu)\in X} F_\varepsilon(u,\mu) = F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$. Then $\bar{\mu}_\varepsilon = \left|(-\Delta)^{\frac{s}{2}}\bar{u}_\varepsilon\right|^2 dx$.

Proof. We observe that the supremum is attained at some $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$ because X is sequentially compact and F_ε is sequentially continuous (due to the compact embedding $H_0^s(\Omega) \hookrightarrow L^{2^*-\varepsilon}(\Omega)$).

Clearly, we may suppose $\bar{\mu}_\varepsilon(\mathbb{R}^N) = 1$. Indeed, if we have $\lambda_\varepsilon := \bar{\mu}_\varepsilon(\mathbb{R}^N) < 1$, then we may consider the pair $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon/\lambda_\varepsilon)$ which belongs to the space X and satisfies $(\bar{\mu}_\varepsilon/\lambda_\varepsilon)(\mathbb{R}^N) = 1$ and $F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon/\lambda_\varepsilon) = F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \max_{(u,\mu)\in X} F_\varepsilon(u,\mu)$.

Since $\bar{u}_\varepsilon \neq 0$, by the definition of X we have $0 < \left\|(-\Delta)^{\frac{s}{2}}\bar{u}_\varepsilon\right\|_{H_0^s(\Omega)} \leq 1$. Hence, if we set

$$\alpha = \alpha(\varepsilon) := \frac{1}{\|\bar{u}_\varepsilon\|_{H_0^s(\Omega)}^2} \geq 1, \tag{36}$$

we may consider a new pair $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$ given by $\bar{u}_\varepsilon := \sqrt{\alpha}\bar{u}_\varepsilon$and$\bar{\mu}_\varepsilon := \alpha\left|(-\Delta)^{\frac{s}{2}}\bar{u}_\varepsilon\right|^2 dx.$

Note that $(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$ belongs to the space X and it satisfies

$$F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \alpha^{\frac{2^*-\varepsilon}{2}} F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \alpha^{\frac{2^*-\varepsilon}{2}} \max_{(u,\mu)\in X} F_\varepsilon(u,\mu). \tag{37}$$

Clearly, (36) and (37) imply that $\alpha = 1, \left(\bar{u}_\varepsilon, \left|(-\Delta)^{\frac{s}{2}}\bar{u}_\varepsilon\right|^2 dx\right)$ is a maximize and $\|\bar{u}_\varepsilon\|_{H_0^s(\Omega)} = 1$.

Since $1 = \int \left|(-\Delta)^{\frac{s}{2}}\bar{u}_\varepsilon\right|^2 dx \leq \bar{\mu}_\varepsilon(\mathbb{R}^N) = 1$, we have $\bar{\mu}_\varepsilon = \left|(-\Delta)^{\frac{s}{2}}\bar{u}_\varepsilon\right|^2 dx$ and the proof is complete.

Definition (2.1.10)[26]. We say that the family $\{F_\varepsilon\}\Gamma^+$-converges to a functional $F: X \to [0,\infty)$; as $\varepsilon \to 0$, if for every $(u,\mu) \in X$ the following conditions hold:

(i) for every sequence $\{(u_\varepsilon, \mu_\varepsilon)\} \subset X$ such that $u_\varepsilon \rightharpoonup u$ in $L^{2^*}(\Omega)$ and $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathrm{M}(\mathbb{R}^N)$

$F(u,\mu) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \mu_\varepsilon)$;

(ii) there exists a sequence $\{(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)\} \subset X$ such that $\bar{u}_\varepsilon \rightharpoonup u$ in $L^{2^*}(\Omega)$, $\bar{\mu}_\varepsilon \overset{*}{\rightharpoonup} \mu$in $\mathrm{M}(\mathbb{R}^N)$ and

$F(u,\mu) \geq \liminf_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$.

The Γ^+-limsup inequality (i) easily follows from the concentration-compactness alternative shown see following proposition. The proof of the Γ^+-liminf inequality (ii) (i.e., the construction of a recovery sequence) is more delicate. In the case $s = 1$ it is proved in [48], following the strategy adopted in [51] and in [48], the authors prove the existence of a recovery sequence and the Γ^+-liminf inequality, working in two separate cases $(u,\mu) = (u, |\nabla u|^2 dx + \tilde{\mu})$ and $(u,\mu) = \left(0, \sum_i \mu_i \delta_{x_i}\right)$ and cover the general case by means of compactness and locality properties of the Γ^+-limit. Here, we follow a different strategy and we explicitly construct a recovery sequence using the optimal functions given by Theorem (2.1.1).

The proof of the Γ^+-limsup inequality (i) is given by the following result.

Proposition(2.1.11)[26]. For every $(u,\mu) \in X$ and for every sequence $\{(u_\varepsilon, \mu_\varepsilon)\} \subset X$ such that $(u_\varepsilon, \mu_\varepsilon) \overset{\tau}{\to} (u,\mu)$, we have $F(u,\mu) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \mu_\varepsilon)$.

Proof. Let $\{(u_\varepsilon, \mu_\varepsilon)\}$ be a sequence in X such that $(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u, \mu)$; clearly,

$\mu = \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \tilde{\mu} + \sum_{j=1}^{\infty} \mu_j \delta_{x_j}$, for some $\tilde{\mu} \in \mathrm{M}_+(\mathbb{R}^N), \{\mu_j\} \subseteq (0,1)$

and $\{\mu_j\} \subseteq \mathbb{R}^N$. Up to subsequences, there exists a measure $\nu \in \mathrm{M}(\mathbb{R}^N)$ such that $|u_\varepsilon|^{2^*}dx \overset{*}{\rightharpoonup} \nu$. and by Theorem (2.1.2) there exists a set of nonnegative numbers $\{\nu_j\}_{j\in J}$such that (up to reordering the points $\{x_j\}$ and the $\{\mu_j\}$)

$$\nu = |u|^{2^*}dx + \sum_j \nu_j \delta_{x_j} \text{ and} \nu_j \leq S^* \mu_j^{\frac{2^*}{2}}. \tag{38}$$

Using Hölder Inequality and arguing as in the proof of Proposition (2.1.11)-(i), we have

$F_\varepsilon(u_\varepsilon, \mu_\varepsilon) = \int_\Omega |u_\varepsilon|^{2^*-\varepsilon}dx \leq \left(\int_\Omega |u_\varepsilon|^{2^*}dx\right)^{\frac{2^*-\varepsilon}{2^*}} |\Omega|^{\frac{\varepsilon}{2^*}}$, hence, the definition of ν and (38) yield

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) &\leq \limsup_{\varepsilon\to 0} \left(\int_\Omega |u_\varepsilon|^{2^*}dx\right)^{\frac{2^*-\varepsilon}{2^*}} |\Omega|^{\frac{\varepsilon}{2^*}} \\ &\leq \nu(\overline{\Omega}) \leq \int_\Omega |u|^{2^*}dx + S^* \sum_{i=1}^{\infty} \mu_i^{\frac{2^*}{2}} \leq F(u,\mu).\end{aligned}$$

Now, we will prove the Γ^+-lim inf inequality (ii).

It is convenient to define a relevant subset of configurations $\tilde{X} \subset X$ as follows

$\tilde{X} := \left\{(u,\mu) \in H_0^s(\Omega) \times \mathrm{M}(\mathbb{R}^N) : \mu = \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \tilde{\mu} + \sum_{j=1}^{\infty} \mu_j \delta_{x_j}, \mu(\mathbb{R}^N) < 1\right\}$.

For any pair (u, μ) in $\tilde{X}$ we will prove the existence of a recovery sequence $\{(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)\} \subset X$ for the Γ^+-liminf inequality, as stated in the following proposition.

Proposition (2.1.12)[26]. For any $x_0 \in \overline{\Omega}$ there exists a sequence $\{v_\varepsilon\} \subset H_0^s(\Omega)$ such that

(i) $\left\{\left(v_\varepsilon, \left|(-\Delta)^{\frac{s}{2}}v_\varepsilon\right|^2 dx\right) \subseteq X\right\}$ and τ-converges to $(0, \delta_{x_0})$ as $\varepsilon \to 0$;

(ii) $\lim_{\varepsilon\to 0} \mathrm{dist}_H(\mathrm{spt} v_\varepsilon, \{x_0\}) = 0$;

(iii)$\lim_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^{2^*-\varepsilon}dx = S^*$.

Proof. We assume that x_0 is an interior point of Ω and we construct the sequence $\{v_\varepsilon\}$ modifying the extremal functions u for the Sobolev embedding S^*given by Theorem (2.1.1).

Let $u \in H_0^s(\mathbb{R}^N)$ defined as follows $u(x) = \frac{c}{(1+|x-x_0|^2)^{\frac{N-2s}{2}}}, \ \forall x \in \mathbb{R}^N$,

where the positive constant c is chosen such that $\|u\|_{H_0^s} = 1$.

If, for any positive ε, we set $w_\varepsilon(x) := \varepsilon^{-\frac{N-2s}{2}}u(x/\varepsilon)$, then we have

$$\int_{\mathbb{R}^N} |w_\varepsilon|^{2^*}dx = S^* \text{and} \|w_\varepsilon\|_{H_0^s} = 1, \tag{39}$$

by scaling invariance of L^{2^*} and H_0^s norms.

Moreover, the function w_ε satisfies $w_\varepsilon \rightharpoonup 0$ in $L^{2^*}(\mathbb{R}^N)$ and $\left|(-\Delta)^{\frac{s}{2}}w_\varepsilon\right|^2 dx \overset{*}{\rightharpoonup} \delta_{x_0}$ in $\mathrm{M}(\mathbb{R}^N)$ as $\varepsilon \to 0$, since a direct calculation for any $\rho > 0$ gives

$$w_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in} L^{2^*}\left(\mathbb{R}^N \backslash \overline{B_\rho(x_0)}\right) \text{and} \left|(-\Delta)^{\frac{s}{2}}w_\varepsilon\right|^2 \overset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in} L^2\left(\mathbb{R}^N \backslash \overline{B_\rho(x_0)}\right). \tag{40}$$

We want to localize the sequence w_ε in smaller and smaller neighborhoods of x_0.

For any fixed positive ρ, take a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in $B_\rho(x_0)$, $\varphi \equiv 0$ in $\mathbb{R}^N \backslash B_{2\rho}(x_0)$ and $0 \le \varphi \le 1$. For any $\varepsilon > 0$, we deine $\tilde{v}_\varepsilon(x) := \varphi(x) w_\varepsilon(x)$and we claim that, as $\varepsilon \to 0$,

The first convergence result in (41) is a direct consequence of (40).

$$\|\tilde{v}_\varepsilon\|_{H_0^s} = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}\big(\varphi(x) w_\varepsilon(x/\varepsilon)\big)\right|^2 dx = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}\big(\varphi(\varepsilon y) u(y)\big)\right|^2 \overset{\varepsilon \to 0}{\longrightarrow} 1.$$

The last convergence result in (41) is more delicate. We split the integral into two parts, namely $I_{1,\varepsilon}$ and $I_{2,\varepsilon}$ given by $I_{1,\varepsilon} := \int_{\Omega \cap \{w_\varepsilon < 1\}} |\varphi w_\varepsilon|^{2^*-\varepsilon} dx$ and $I_{2,\varepsilon} := \int_{\Omega \cap \{w_\varepsilon \ge 1\}} |\varphi w_\varepsilon|^{2^*-\varepsilon} dx$.

Since $|\varphi w_\varepsilon|^{2^*-\varepsilon} \le 1$ in $\Omega \cap \{w_\varepsilon < 1\}$ uniformly in ε and $|\varphi w_\varepsilon(x)|^{2^*-\varepsilon} \to 0$ a.e. as $\varepsilon \to 0$, we deduce that $I_{1,\varepsilon}$ vanishes as ε goes to 0.

For $I_{2,\varepsilon}$ first we want to prove that

$$\lim_{\varepsilon \to 0} \left\| \frac{\varphi^{2^*-\varepsilon}}{w_\varepsilon^\varepsilon} - 1 \right\|_{L^\infty(\{w_\varepsilon \ge 1\})} = 0. \tag{42}$$

Note that, for ε small enough, we have $\{w_\varepsilon \ge 1\} \subseteq B_\rho(x_0)$ and then $\frac{\varphi^{2^*-\varepsilon}}{w_\varepsilon^\varepsilon} - 1 = \frac{1}{w_\varepsilon^\varepsilon} - 1$ in $\Omega \cap \{w_\varepsilon < 1\}$. Hence, (42) follows once we show that

$$\lim_{\varepsilon \to 0} \|w_\varepsilon^\varepsilon - 1\|_{L^\infty(\{w_\varepsilon \ge 1\})} = 0. \tag{43}$$

Clearly, on $\Omega \cap \{w_\varepsilon \ge 1\}$ the function w_ε satisfies $1 \le w_\varepsilon^\varepsilon \le (\max w_\varepsilon)^\varepsilon = \left(c\varepsilon^{-\frac{N-2s}{2}}\right)^\varepsilon$ and thus we obtain (43) and in turn (42) as $\varepsilon \to 0$.

Combining (42) with (39), $I_{2,\varepsilon}$ can be estimating as follows

$$I_{2,\varepsilon} = \int_{\Omega \cap \{w_\varepsilon \ge 1\}} |\varphi w_\varepsilon|^{2^*-\varepsilon} dx = \int_{\Omega \cap \{w_\varepsilon \ge 1\}} \left| \frac{|\varphi|^{2^*-\varepsilon}}{w_\varepsilon^\varepsilon} \right| |w_\varepsilon|^{2^*} dx = \int_{\Omega \cap \{w_\varepsilon \ge 1\}} |w_\varepsilon|^{2^*} dx + o(1) \overset{\varepsilon \to 0}{\longrightarrow} S^*.$$

Thus, (41) holds for any $\rho > 0$ small enough, whence a diagonal argument as $\rho \searrow 0$ gives a sequence $\{\tilde{v}_\varepsilon\}$ such that (41) holds, since $\tilde{v}_\varepsilon \rightharpoonup 0$ in $L^{2^*}(\Omega)$ and $\lim_{\varepsilon \to 0} \operatorname{dist}_H(\operatorname{spt}\tilde{v}_\varepsilon, \{x_0\}) = 0$.

Note that, by Proposition (2.1.12), $\tilde{v}_\varepsilon$ also satisfies

$$\left|(-\Delta)^{\frac{s}{2}} \tilde{v}_\varepsilon\right|^2 dx \to 0 \ \text{ in } L^2\big(\mathbb{R}^N \backslash \overline{B_\rho(x_0)}\big) \text{as } \varepsilon \to 0. \tag{44}$$

Finally, for any $\varepsilon > 0$, we set

$$v_\varepsilon(x) := \frac{\tilde{v}_\varepsilon(x)}{\|\tilde{v}_\varepsilon\|_{H_0^s}}. \tag{45}$$

Claim (i) follows readily from (41) and (44). Claim (ii) holds by construction, since the function $\tilde{v}_\varepsilon$ has the same property. Finally, a simple calculation of the $L^{2^*-\varepsilon}$ norm of the function v_ε gives $\int_\Omega |v_\varepsilon|^{2^*-\varepsilon} dx = \|\tilde{v}_\varepsilon\|_{H_0^s}^{-(2^*-\varepsilon)} \int_\Omega |\tilde{v}_\varepsilon|^{2^*-\varepsilon} dx \to S^*$as $\varepsilon \to 0$,which proves claim (iii).

To complete the proof, we observe that the case of $x_0 \in \partial\Omega$ can be obtained by a standard diagonal argument taking an approximating sequence of points $\{x_k\} \subseteq \Omega$ converging to x_0 and the optimal sequences corresponding to each x_k.

Corollary(2.1.13)[26]. For any finite set of distinct points $\{x_1, x_2, \dots, x_n\} \subset \overline{\Omega}$ and for any set of positive numbers $\{\mu_1, \mu_2, \dots, \mu_n\} \subseteq \mathbb{R}, \sum_j \mu_j < 1$, there exists a sequence $\{u_\varepsilon^A\} \subset H_0^s(\Omega)$ such that

(i) $\left(u_\varepsilon^A, \left|(-\Delta)^{\frac{s}{2}} u_\varepsilon^A\right|^2 dx\right) \subseteq X$ and τ-converges to $\left(0, \sum_{j=1}^n \mu_j \delta_{x_j}\right)$ as $\varepsilon \to 0$;

(ii) $\lim_{\varepsilon\to 0} \int_\Omega \operatorname{dist}_H\left(\operatorname{spt} u_\varepsilon^A, \cup_j\{x_j\}\right) = 0$.

(iii) $\lim_{\varepsilon\to 0} \int_\Omega |u_\varepsilon^A|^{2^*-\varepsilon} dx = S^* \sum_{j=1}^\infty \mu_j^{\frac{2^*}{2}}$.

Proof. Let us set $A_j := B_{r_j}(x_j) \cap \Omega$ for any $j = 1,2,\dots,n$, with radii r_j and r_i such that $\operatorname{dist}(A_j, A_i) > 0$. By Proposition (2.1.12), there exists a sequence $\{(v_\varepsilon^j, \mu_\varepsilon^j)\} \subset X$, with $\mu_\varepsilon^j = \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx$ such that $(u_\varepsilon^j, \mu_\varepsilon^j) \xrightarrow{\tau} (0, \delta_{x_i})$, $\operatorname{spt} v_\varepsilon^j \subset A_j$, $\operatorname{dist}_H(\operatorname{spt} v_\varepsilon^j, \{x_j\}) \to 0$ as $\varepsilon \to 0$ and

$$\lim_{\varepsilon\to 0} \int_\Omega \left|v_\varepsilon^j\right|^{2^*-\varepsilon} dx = S^*, \qquad \text{for } j = 1,2,\dots,n. \tag{46}$$

Let us set $u_\varepsilon^A := \sum_{j=1}^\infty \sqrt{\mu_j}\, v_\varepsilon^j$. Estimating the energy of the sequence $\left\{\left|(-\Delta)^{\frac{s}{2}} u_\varepsilon^A\right|^2 dx\right\}$ gives

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u_\varepsilon^A\right|^2 dx = \sum_{j=1}^\infty \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx + 2 \sum_{i,j=1, i<j}^n \sqrt{\mu_i \mu_j}\, \langle (-\Delta)^{\frac{s}{2}} v_\varepsilon^i, (-\Delta)^{\frac{s}{2}} v_\varepsilon^j \rangle_{L^2(\mathbb{R}^N)}. \tag{47}$$

We claim that the last sum in the formula above converges to zero as ε goes to zero.
Indeed, by Cauchy-Schwarz inequality, we get

$$\left|\langle (-\Delta)^{\frac{s}{2}} v_\varepsilon^i, (-\Delta)^{\frac{s}{2}} v_\varepsilon^j \rangle_{L^2(\mathbb{R}^N)}\right| \le \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^i\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{H_i} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{H_i} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^i\right|^2 dx\right)^{\frac{1}{2}}, \tag{48}$$

where, for i and j fixed, we have divided the whole space $\mathbb{R}^N$ into two complementary half-spaces H_i and H_j such that $A_i \subset Hi$ and $A_j \subset H_j$. Note that $\int \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx$ is smaller than 1 uniformly with respect to ε because $\left\{\left(v_\varepsilon^j, \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx\right)\right\} \subseteq X$. Thus, (49) becomes

$$\left|\langle (-\Delta)^{\frac{s}{2}} v_\varepsilon^i, (-\Delta)^{\frac{s}{2}} v_\varepsilon^j \rangle_{L^2(\mathbb{R}^N)}\right| \le \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx\right)^{\frac{1}{2}} + \left(\int_{H_i} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^i\right|^2 dx\right)^{\frac{1}{2}}.$$

On the other hand, since the measure $\left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx$ converges to δ_{x_j} in $\mathrm{M}(\mathbb{R}^N)$ as $\varepsilon \to 0$ and spt $v_\varepsilon^j \subseteq A_j$ for all $j = 1,2,\dots,n$, (2.1.8) yields $\int_{H_i} \left|(-\Delta)^{\frac{s}{2}} v_\varepsilon^j\right|^2 dx \to 0$ as $\varepsilon \to 0$ for $i \ne j$, that in turn implies

$$\langle (-\Delta)^{\frac{s}{2}} v_\varepsilon^i, (-\Delta)^{\frac{s}{2}} v_\varepsilon^j \rangle_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{49}$$

Combining (47) and (49) with the fact that each v_ε^j concentrates energy at x_j, in the sense of Proposition(2.1.12),we deduce that the constructed sequence $\left\{\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon^A\right|^2 dx\right\}$ satisfies

$\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon^A\right|^2 dx \overset{*}{\rightharpoonup} \sum_{j=1}^n \mu_j \delta_{x_j}$ in $M(\mathbb{R}^N)$.

Finally, since $\sum_j \mu_j < 1$, by (47) we also deduce that $\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon^A\right|^2 dx \leq 1$, for ε small, hence $\left\{\left(u_\varepsilon^A, \left|(-\Delta)^{\frac{s}{2}}u_\varepsilon^A\right|^2 dx\right)\right\} \subset X$, for ε small and claim (i) is completely proved.

Note that (ii) follows by construction, because of Proposition (2.1.12).
Moreover, since spt v_ε^A are mutually disjoint and v_ε^A satisfies Proposition (2.1.12)
(iii), we have $\int_\Omega |u_\varepsilon^A|^{2^*-\varepsilon}dx = \sum_{j=1}^n \mu_j^{\frac{2^*-\varepsilon}{2}} \int_{A_j}|v_\varepsilon^A|^{2^*-\varepsilon}dx \overset{\varepsilon\to 0}{\longrightarrow} S^* \sum_{j=1}^n \mu_j^{\frac{2^*}{2}}$,
which concludes the proof of claim (iii). Now, we are in position to prove the Γ^+-liminf inequality for the set of configurations $\tilde{X}$ as stated in Proposition (2.1.14). The main contribution is given by the sequence $\left\{\left(u_\varepsilon^A, \left|(-\Delta)^{\frac{s}{2}}u_\varepsilon^A\right|^2 dx\right)\right\}$ built in Corollary (2.1.13), but we have to carefully modify it in order to obtain the desired recovery sequence $\{(\tilde{u}_\varepsilon, \tilde{\mu}_\varepsilon)\}$.

Proposition(2.1.14)[26]. For any $(u,\mu) \in \tilde{X}$ there exists a sequence $\{(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)\} \subset X$ such that $(u_\varepsilon, \mu_\varepsilon) \overset{\tau}{\to} (u,\mu)$ and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) \geq F(u,\mu). \tag{50}$$

Finally, we will prove the Γ^+-liminf inequality in the whole space X, by a diagonal argument using recovery sequences for the elements of $\tilde{X}$.

Proof: for any point x_j in $\bar{\Omega}$ we construct a sequence $\{v_\varepsilon^j\}$ that concentrates energy at x_j (see Proposition (2.1.12)). Then, we show that we can glue such sequences $\{v_\varepsilon^j\}$ into a sequence $\{u_\varepsilon^A\}$ such that it concentrates at any finite set of points $\{x_j\}$ in $\bar{\Omega}$ (see Corollary (2.1.13)). Thus, the sequence $\{u_\varepsilon^A\}$ will be the recovery sequence for a pair $(0,\mu) \in \tilde{X}$ when $\mu$is purely atomic. Finally, for any pair (u,μ) in $\tilde{X}$, we will able to join the function u to the sequence $\{u_\varepsilon^A\}$, adding suitably their corresponding measures, to obtain the desired recovery sequence $\{(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)\}$ satisfying (50).
Let (u,μ) be any fixed pair in $\tilde{X}$, i. e.,

$u \in H_0^s(\Omega)$ and $\mu = \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \tilde{\mu} + \sum_{j=1}^n \mu_j \delta_{x_j} \in M(\mathbb{R}^N)$, with $\mu(\mathbb{R}^N) < 1$ and let $\{u_\varepsilon^A\}$

For $\sigma > 0$, take a cut-off function φ_σ in $C_0^\infty(\mathbb{R}^N)$ such that $\varphi_\sigma \equiv 0$ in $B_{\rho_\sigma}(x_j)$,
for $j = 1,2,\dots,n, \varphi_\sigma \equiv 1$ in $\Omega \bigcup_j B_{2\rho_\sigma}(x_j)$, with $\rho_\sigma \to 0$ as $\sigma \to 0$,
$\varphi_\sigma = 1 - \sum_{j=1}^n \bar{\varphi}\left(\frac{x-x_j}{\rho_\sigma}\right)$, $\bar{\varphi} \in C_0^\infty(B_2), \bar{\varphi} \equiv 1$ on $\bar{B}_1, 0 \leq \bar{\varphi} \leq 1$.

Now, we can define the sequence $\{(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)\}$ as follows

$$\bar{u}_\varepsilon = \bar{u}_{\varepsilon,\sigma} \coloneqq u\varphi_\sigma + u_\varepsilon^A, \qquad \bar{\mu}_\varepsilon = \bar{\mu}_{\varepsilon,\sigma} \coloneqq \tilde{\mu} + \left|(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma + u_\varepsilon^A)\right|^2 dx$$

and we claim that this is a recovery sequence for $\left(u\varphi_\sigma, \left|(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma)\right|^2 dx + \tilde{\mu} + \sum_{j=1}^n \mu_j \delta_{x_j}\right)$ as $\varepsilon \to 0$.

Note that we will play with two positive parameters, namely ε (which is the parameter for the atomic part of μ) and σ (which will control the diffuse part of μ). We will take limits in these parameters in the following order: first $\varepsilon \to 0$, then $\sigma \to 0$. The recovery sequence for (u,μ) will be actually given by a further diagonal argument. First, we claim that $\{(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)\} \subset X$ for ε and σ small enough. Since we have $\bar{u}_\varepsilon \in H_0^s(\Omega)$ (because φ_σ is a multiplier in $H_0^s(\Omega)$; see [63]) and $\bar{\mu}_\varepsilon \geq \left|(-\Delta)^{\frac{s}{2}}\bar{u}_\varepsilon\right|^2 dx$, this claim reduces to proving that

$$\bar{\mu}_\varepsilon(\mathbb{R}^N) \leq 1. \tag{51}$$

In order to check (51), for any $\varepsilon, \sigma > 0$ we compute

$$\bar{\mu}_\varepsilon(\mathbb{R}^N) = \tilde{\mu}_\varepsilon(\mathbb{R}^N) + \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma + u_\varepsilon^A)\right|^2 dx = \tilde{\mu}_\varepsilon(\mathbb{R}^N) + \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma)\right|^2 dx$$

$$+ \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(u_\varepsilon^A)\right|^2 dx + \langle(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma), (-\Delta)^{\frac{s}{2}}(u_\varepsilon^A)\rangle_{L^2(\mathbb{R}^N)}. \tag{52}$$

We can treat the last three terms in the right-handside of equation (52) as follows.

For $\sigma > 0$ fixed, Corollary (2.1.13)-(i) and Proposition(2.1.6) yield

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(u_\varepsilon^A)\right|^2 dx = \sum_{j=1}^{n} \mu_i. \tag{53}$$

Again by Corollary (2.1.16) $u_\varepsilon^A \rightharpoonup 0$ in $H_0^s(\Omega)$, hence we have

$$\lim_{\varepsilon\to 0} \langle(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma), (-\Delta)^{\frac{s}{2}}(u_\varepsilon^A)\rangle_{L^2(\mathbb{R}^N)} = 0. \tag{54}$$

Finally, from the definition of $\varphi_\sigma$and we have

$$\lim_{\sigma\to 0} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma),\right|^2 dx = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}u,\right|^2 dx. \tag{55}$$

Thus, combining (53), (54), (55) with the fact that $\mu(\mathbb{R}^N)$ is strictly less than 1 (recall that $(u,\mu) \in \tilde{X}$), we can deduce the inequality in (51) for ε and σ small enough.

We prove that $\{(\bar{u}_\varepsilon \bar{\mu}_\varepsilon)\}$ τ-converges to (u,μ), i.e.,

$$\bar{u}_\varepsilon \rightharpoonup u \text{ in } L^{2^*}(\Omega) \text{ and } \bar{\mu}_\varepsilon \overset{*}{\rightharpoonup} \mu \text{ in } \mathrm{M}(\mathbb{R}^N). \tag{56}$$

Clearly, $|u\varphi_\sigma - u|^{2^*} = |1-\varphi_\sigma|^{2^*}|u|^{2^*} \leq |u|^{2^*}$, thus, $\{u\varphi_\sigma\}$ converges strongly to u in L^{2^*}, as $\sigma \to 0$, by Lebesgue's Dominated Convergence Theorem, and then the first convergence result in (56) follows from the fact that the sequence $\{u_\varepsilon^A\}$ weakly converges to 0 in $L^{2^*}(\Omega)$ as ε goes to 0.The second convergence result in (56) is a consequence of the convergence in the sense of measures of the sequence $\left\{\left|(-\Delta)^{\frac{s}{2}}u_\varepsilon^A\right|^2 dx\right\}$ to the finite sum of Dirac masses $\sum_j \mu_j \delta_{x_j}$, together with the fact that $u_\varepsilon^A \rightharpoonup 0$ as $\varepsilon \to 0$ and $u\varphi_\sigma \to u$ in $H_0^s(\Omega)$ as $\sigma \to 0$ by Corollary (2.1.13) respectively. Indeed, by arguing as in (52), (55) and (53), for any $\psi \in C_0^0(\mathbb{R}^N)$, we have

$$\lim_{\sigma\to 0}\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \psi d\bar{\mu}_\varepsilon = \lim_{\sigma\to 0}\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \psi d\bar{\mu} + \int_{\mathbb{R}^N} \psi \left|(-\Delta)^{\frac{s}{2}}(u\varphi_\sigma + u_\varepsilon^A)\right|^2 dx$$

$$= \int_{\mathbb{R}^N} \psi d\bar{\mu} + \int_{\mathbb{R}^N} \psi \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx + \int_{\mathbb{R}^N} \psi d \sum_j^n \mu_j \delta_{x_j} = \int_{\mathbb{R}^N} \psi d\mu,$$

that completely proves (56).

In order to complete the proof, it remains to show that its energy $F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$ satisfies the lim inf inequality stated in (39). Since $\operatorname{dist}\big(\operatorname{spt}(u\varphi_\sigma), \cup_j B_{\rho_\sigma}(x_j)\big) > 0$, we can split the integral in $F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon)$ as follows

$$F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \int_\Omega |u\varphi_\sigma + u_\varepsilon^A|^{2^*-\varepsilon} dx = \int_\Omega |u\varphi_\sigma|^{2^*-\varepsilon} dx + \int_\Omega |u_\varepsilon^A|^{2^*-\varepsilon} dx. \tag{57}$$

By dominated Convergence Theorem, we have

$$\lim_{\sigma\to 0} \lim_{\varepsilon\to 0} \int_\Omega |u\varphi_\sigma|^{2^*-\varepsilon} dx = \int_\Omega |u|^{2^*} dx. \tag{58}$$

On the other hand, taking Corollary (2.1.13)-(iii) into account, we have

$$\int_\Omega |u_\varepsilon^A|^{2^*-\varepsilon} \to S^* \sum_{j=1}^n \mu_j^{\frac{2^*}{2}} \quad as\ \varepsilon \to 0. \tag{59}$$

Finally, combining (57), (58) and (59), we obtain (up to the diagonal argument on ε and σmentioned)

$$\lim\ \inf_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon, \bar{\mu}_\varepsilon) = \int_\Omega |u|^{2^*} dx + S^* \sum_{j=1}^n \mu_i^{\frac{2^*}{2}} = F(u, \mu).$$

In view of Proposition (2.1.14), the Γ^+-liminf inequality in Theorem (2.1.8) holds for any $(u, \mu) \in \tilde{X}$. Thus, it is enough to check that $\tilde{X} \subseteq X$ is τ-sequentially dense by an explicit approximation and that F is continuous with respect to this approximation, in order to conclude by a standard diagonal argument. For any pair $(u, \mu) \in X$, we consider the sequence $\{(u_\varepsilon \mu_\varepsilon)\}$ defined as $u_n := c_n u$ and $\mu_n := c_n^2 \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + c_n^2 \tilde{\mu} + c_n^2 \sum_{j=1}^n \mu_j \delta_{x_j}$, where $\{c_n\} \subset (0,1)$ is any increasing sequence such that $c_n \nearrow 1$ as $n \to \infty$. Clearly, the sequence $\{(u_n, \mu_n)\}$ is in $\tilde{X}$, since, for any $n \in \mathbb{N}, u_n \in H_0^s(\Omega)$ and μ_n is a measure with a finite number of atoms such that $\mu_n(\mathbb{R}^N) \le c_n^2 \mu(\mathbb{R}^N) \le c_n^2 < 1$.

Moreover, $(u_n, \mu_n) \overset{\tau}{\to} (u, \mu)$ as $n \to \infty$, because $u_n \to u$ in $H_0^s(\Omega)$ (hence weakly in $L^{2^*}(\Omega)$) and, for any $\psi \in C_0^0(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \psi d\mu_n &= \int_{\mathbb{R}^N} \psi c_n^2 \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \int_{\mathbb{R}^N} \psi c_n^2 d\tilde{\mu} + c_n^2 \sum_{j=1}^n \mu_j \psi(x_j) \\ &= c_n^2 \int_{\mathbb{R}^N} \psi \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \int_{\mathbb{R}^N} \psi d\tilde{\mu} + \sum_{j=1}^n \mu_j \psi(x_j) \xrightarrow{n\to\infty} \int_{\mathbb{R}^N} \psi d\mu.\end{aligned}$$

Finally, evaluating the functional F, we have

$$\begin{aligned}F(u_n, \mu_n) &= \int_\Omega c_n^{2^*} |u|^{2^*} dx + S^* \sum_{j=1}^n \left(c_n^2 \mu_j\right)^{\frac{2^*}{2}} \\ &= c_n^{2^*} \left(\int_\Omega |u|^{2^*} dx + S^* \sum_{j=1}^n \mu_j^{\frac{2^*}{2}}\right) \to F(u, \mu), \quad \text{as } n \to \infty.\end{aligned}$$

Here we show that, due to the Γ^+-convergence result, the maximizers $\{u_\varepsilon\}$ for the variational problem (23) concentrate energy at one point $x_0 \in \bar{\Omega}$ when ε goes to zero. The key result is the following optimal upper bound for the limit functional F on the space X.

Lemma (2.1.15)[26] . For every $(u, \mu) \in X$, we have

$$F(u, \mu) \le S^* \tag{60}$$

and the equality holds if and only if $(u, \mu) = \left(0, \delta_{x_0}\right)$ for some $x_0 \in \bar{\Omega}$.

Proof. We adapt the argument in [51] for the case $s = 1$, using a convexity trick as in the proof of Corollary (2.1.7). For every $(u, \mu) \in X$, by Sobolev inequality (2), we have

$$F(u,\mu) \equiv \int_{\Omega} |u|^{2^*} dx + S^* \sum_{j=1}^{n} \mu_j^{\frac{2^*}{2}} \leq S^* \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^{\frac{2^*}{2}} + S^* \sum_{j=1}^{\infty} \mu_j^{\frac{2^*}{2}}.$$

Now, by the convexity of the function $t \mapsto t^{\frac{2^*}{2}}$, for every fixed $s \in (0, N/2)$, we get

$$F(u,\mu) \leq S^* \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^{\frac{2^*}{2}} + S^* \sum_{j=1}^{\infty} \mu_j^{\frac{2^*}{2}}$$

$$\leq S^* \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \sum_{j=1}^{\infty} \mu_j \right)^{\frac{2^*}{2}} \leq S^* \left(\mu(\mathbb{R}^N)\right)^{\frac{2^*}{2}} \leq S^* \quad (61)$$

which proves (60). Note that equality clearly holds if $(u, \mu) = (0, \delta_{x_0})$, for some $x_0 \in \overline{\Omega}$. Assume that equality in (60) holds for some pair $(u, \mu) \in X$. Then, each inequality in (61) is in effect an equality. In particular, we deduce $\tilde{\mu} = 0$. If $u \neq 0$ then we also deduce by convexity that $\mu_j = 0$ for every j.

In turn, this fact yields $\mu = \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx$ and $u \in H_0^s(\Omega)$ is optimal in Sobolev inequality (7), which contradicts Theorem (2.1.1). Thus, $u = 0$, equation (61) and the strict convexity implies that $\mu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$ as claimed. Now, by Theorem(2.1.8) and Γ^+-convergence properties, it follows that every sequence of maximizes of F_ε, which is in the form $\left\{\left(u_\varepsilon, \left|(-\Delta)^{\frac{s}{2}} u_\varepsilon\right|^2 dx\right)\right\}$ in view of Proposition (2.1.9), must converge (up to subsequences) to a pair $(u, \mu) \in X$ which is a maximizer for F, i.e. $\left(u_\varepsilon, \left|(-\Delta)^{\frac{s}{2}} u_\varepsilon\right|^2 dx\right) \xrightarrow{\tau} (u, \mu)$, with $F(u, \mu) = \max_{X\overline{\Omega}} F$. We have the upper bound $F(u, \mu) \leq S^*$ for every $(u, \mu) \in X$ and the equality is achieved if and only if $(u, \mu) = (0, \delta_{x_0})$ for some $x_0 \in \overline{\Omega}$. Hence, it follows that $\left(u_\varepsilon, \left|(-\Delta)^{\frac{s}{2}} u_\varepsilon\right|^2 dx\right) \xrightarrow{\tau} (0, \delta_{x_0})$, which is the desired concentration property for the energy density.

Lemma (2.1.16)[26]. Let $0 < s < N/2$ and let $u \in H_0^s(\mathbb{R}^N)$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and for each $\lambda > 0$ let $\varphi_\lambda(x) := \varphi(\lambda^{-1} x)$. Then$u\varphi_\lambda \to 0 \ in \ H_0^s(\mathbb{R}^N) \quad as \ \lambda \to 0.$
If, in addition, $\varphi \equiv 1$ in a neighborhood of the origin, then $u\varphi_\lambda \to u \ in \ H_0^s(\mathbb{R}^N) \ as \ \lambda \to \infty.$

Proof. First, note that each function φ_λ gives a bounded multiplication operator $M_{\varphi_\lambda} \in \mathrm{L}(H_0^s, H_0^s)$ with operator norm independent on λ because of the scale invariance of the H_0^s norm (see[63],[64])where instead of H_0^s the more traditional notation h_2^s is used for the Riesz potential space of order $s$and summability two).

Thus, if $C \equiv \|\varphi_\lambda\|_{\mathrm{L}(H_0^s, H_0^s)}$ we have

$$\|v\varphi_\lambda\|_{H_0^s} \leq C\|v\|_{H_0^s} \quad (62)$$

for any $v \in H_0^s$.

By density we take a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in H_0^s, so we can estimate

$$\|u\varphi_\lambda\|_{H_0^s} \leq \|(u - u_n)\varphi_\lambda\|_{H_0^s} + \|u\varphi_\lambda\|_{H_0^s} \leq C\|(u - u_n)\|_{H_0^s} + \|u_n\varphi_\lambda\|_{H_0^s}. \quad (63)$$

Since for fixed n the function u_n gives also a bounded multiplier on H_0^s, we have

$$\|u_n\varphi_\lambda\|_{H_0^s} \le C(u_n)\|\varphi_\lambda\|_{H^s} \to 0 \tag{64}$$

as $\lambda \to 0$ by a direct scaling argument. Thus, the first statement of the lemma follows from (63) and (64) letting $\lambda \to 0$ and $n \to \infty$. In order to prove the second statement, it is enough to note that whenever $u \in C_0^\infty(\mathbb{R}^N)$ (indeed for any u which is compactly supported) we have $u\varphi_\lambda \equiv u$ for λ sufficiently large (depending on u). Thus we see that $u\varphi_\lambda \to u$ as $\lambda \to \infty$ for any $u \in C_0^\infty(\mathbb{R}^N)$ and the same holds for any $u \in H_0^s(\Omega)$ by approximation.Indeed, (62) gives

$$\begin{aligned}\|u - u\varphi_\lambda\|_{H_0^s} &\le \|(u - u_n)(1-\varphi_\lambda)\|_{H_0^s} + \|u_n(1-\varphi_\lambda)\|_{H_0^s} \\ &\le (1 + C)\|(u-u_n)\|_{H_0^s} + \|u_n(1-\varphi_\lambda)\|_{H_0^s},\end{aligned} \tag{65}$$

and the conclusion follows arguing as in the previous case.

Lemma (2.1.17)[26]. Let $0 < s < N/2$, let $\Omega \subset \mathbb{R}^N$ a bounded open set and let $\varphi \in C_0^\infty(\mathbb{R}^N)$.Then the commutator $\left[\varphi, (-\Delta)^{\frac{s}{2}}\right] : H_0^s(\Omega) \to L^2(\mathbb{R}^N)$ is a compact operator, i.e. $\varphi\left((-\Delta)^{\frac{s}{2}}u_n\right) - (-\Delta)^{\frac{s}{2}}(\varphi u_n) \to 0 \quad in \quad L^2(\mathbb{R}^N)$whenever $u_n \rightharpoonup 0$ in $H_0^s(\Omega)$ as $n \to \infty$.

Proof. Let $L = (-\Delta)^{\frac{s}{2}}$ and for each $\varepsilon > 0$ set $L_\varepsilon = (\varepsilon Id - \Delta)^{\frac{s}{2}}$. Clearly, by conjugation with Fourier transform we have $Lu = \mathrm{F}^{-1} \circ M_{|\xi|^s} \circ \mathrm{F}(u)andL_\varepsilon u = \mathrm{F}^{-1} \circ M_{(|\xi|^s+\varepsilon)^{\frac{s}{2}}} \circ \mathrm{F}(u)$.

Thus, $L_\varepsilon : H^s(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a bounded operator which in turn implies the boundedness of the operator $L_\varepsilon : H_0^s(\Omega) \to L^2(\mathbb{R}^N)$ induced by the continuous embedding $H_0^s(\Omega) \hookrightarrow L^2(\mathbb{R}^N)$.

Similarly, $L : H^s(\Omega) \to L^2(\mathbb{R}^N)$ is a bounded operator and the induced operator $L : H_0^s(\Omega) \to L^2(\mathbb{R}^N)$ is also bounded. Estimating the norm in $\mathrm{L}(H^s, L^2)$ easily yields $\|L_\varepsilon - L\| \le \sup_\xi \frac{(\varepsilon+|\xi|^2)^{\frac{s}{2}} - |\xi|^s}{(1+|\xi|^2)^{\frac{s}{2}}} \xrightarrow{\varepsilon\to 0} 0$,hence the same holds in $\mathrm{L}\left(H_0^s(\Omega), L^2(\mathbb{R}^N)\right)$.Thus, it suffices to prove that$[L_\varepsilon, \varphi] : H_0^s(\Omega) \to L^2(\mathbb{R}^N)$is a compact operator for each $\varepsilon > 0$, to deduce the same property for $[L, \varphi]$.Let $L_\varepsilon = (\varepsilon Id - \Delta)^{\frac{s}{2}}$ and $l_\varepsilon(\xi) = (|\xi|^2 + \varepsilon)^{\frac{s}{2}}$ the corresponding symbol.

Clearly, L_ε is a classical pseudodifferential operator of order s, i.e. $L_\varepsilon \in OPS_{1,0}^s$ (hence $L_\varepsilon \in OP\mathrm{B}\, S_{1,1}^s$). Since $0 < s < N/2$, according to [50] we have the following commutator estimate $\|[L_\varepsilon, \varphi]u\|_{L^2(\mathbb{R}^N)} \le C\|\varphi\|_{H^\sigma(\mathbb{R}^N)}\|u\|_{H^{s-1}(\mathbb{R}^N)}$,provided $\sigma > N/2 + 1$.

Section (2.2): Constant Functions Connections in Sobolev Spaces:

Most of the ideas in this section are coming from a series of recent collaborations (see H. Brezis and P. Mironescu [65], [66], [67], [68], H. Brezis and L. Nirenberg [69]). However we will adopt here on slightly different presentation and provide some simplified proofs. The starting point is the following

Proposition (2.2.1)[70]. Let Ω be a connected open set in $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}$ be a measurable function such that

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{N+1}} dx\, dy < \infty, \tag{66}$$

then f is a constant. The original motivation for such a proposition was twofold:

(i) Uniqueness of lifting. Given a (measurable) function $u : \Omega \to \mathbb{C}$ such that $|u| = 1$ a.e., there are many liftings φ, i.e., $u = e^{i\varphi}$. If φ_1, φ_2 are 2 liftings then $k(x) = \frac{1}{2\pi}\left(\varphi_1(x) - \varphi_2(x)\right) : \Omega \to \mathbb{Z}$.

Under further assumptions one may hope to prove that k is a constant function. For example, if φ_1, φ_2 are continuous and Ω is connected, then k is constant. The message [70] wish to convey is that the continuity assumption can be replaced by a different type of condition, such as (66), which is much more natural in the framework of Sobolev spaces .

(ii) A degree theory for classes of discontinuous maps. The possibility of defining a degree for maps in Sobolev spaces (see [71],[72]) is based on the fact deg $h_t(\cdot)$ remains constant along a homotopy $h_t(\cdot)$, as t varies in $[0, 1]$ (or more generally in a connected parameter space Λ). Such a conclusion holds possibly in situations where the dependence in t need not be continuous.

Corollary (2.2.2)[70]. Assume Ω is a connected open set in $\mathbb{R}^N$, and let $f : \Omega \to \mathbb{Z}$ be a measurable function such that

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{N+1}} dxdy < 1, \tag{67}$$

for some $1 \leq p < \infty$, then f is a constant. In [94] had obtained a similar conclusion under the stronger assumption $sp > 1$.

Corollary (2.2.3)[70]. Assume is a connected open set in $\mathbb{R}^N$ and A is measurable subset such that

$$\int_A \int_{^cA} \frac{dx\, dy}{|x - y|^{N+1}} < \infty \tag{68}$$

then either meas $(A) = 0$ or meas $(\Omega \backslash A) = 0$. It suffices to apply Proposition (2.2.1) to $f = \chi_A$, the characteristic function of A.Note that in (68), $(N + 1)$ is again optimal. If A is any subset of Ω with smooth boundary, then (68) holds if $(N + 1)$ is replaced by any $q < N + 1$ (it suffices to consider the case where ∂A is flat and to make an explicit computation).

Proposition (2.2.4)[70]. Assume Ω is a connected open set in $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ is a measurable function such that

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{N+p}} dx\, dy < \infty, \tag{69}$$

for some $1 \leq p < \infty$, then f is constant. [Proposition (2.2.1) corresponds to the case $p = 1$].
Still a further generalization

Proposition {2.2.5)[70]. Assume Ω is a connected open set in $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ is a measurable function such that

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \psi(|x - y|) dx\, dy < \infty, \tag{70}$$

where $p \geq 1$ and $\psi \in L^1_{loc}(0, \infty), \psi \geq 0$ satisfies

$$\int_0^1 \psi(r) r^{N-1} dr = \infty, \tag{71}$$

then f is a constant. [Proposition (2.2.4) corresponds to the case $\psi(r) = r^{-N}$].

Here is one important generalization of Proposition (2.2.4).

Proposition (2.2.6)[70]. Assume Ω is a connected open set in $\mathbb{R}^N$ and $f:\Omega\to\mathbb{R}$ is a measurable function such that

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+p-\varepsilon}}dx\,dy = o\left(\frac{1}{\varepsilon}\right) \quad as\ \varepsilon\to 0, \tag{72}$$

i.e.,

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+p-\varepsilon}}dx\,dy = 0 \tag{73}$$

for some $p\geq 1$, then f is a constant. Assumption (72) is clearly much weaker than (69) (when Ω is bounded) which says that $\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+p-\varepsilon}}dx\,dy = 0(1) \quad as\ \varepsilon\to 0,$

On the other hand (72) is optimal since for any Lipschitz function f on Ω

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+p-\varepsilon}}dx\,dy = 0\left(\frac{1}{\varepsilon}\right) \tag{74}$$

because $\int_0^1 \frac{1}{r^{N-\varepsilon}}r^{N-1}dr = \frac{1}{\varepsilon}$.Here is a final generalization, which brings us closer to the connection with Sobolev spaces.

Theorem (2.2.7)[70]. Assume Ω is a connected open set in $\mathbb{R}^N$ and $f:\Omega\to\mathbb{R}$ is a measurable function. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a sequence of radial mollifiers, i.e.

$$\rho_\varepsilon\in L^1_{loc}(0,\infty),\qquad \rho_\varepsilon\geq 0, \tag{75}$$

$$\int_0^\infty \rho_\varepsilon(r)r^{N-1}dr = 1 \quad \forall\varepsilon>0, \tag{76}$$

$$for\ every\ \delta>0,\qquad \lim_{\varepsilon\to 0}\int_\delta^\infty \rho_\varepsilon(r)r^{N-1}dr = 0. \tag{77}$$

Assume that, for some $p\geq 1$,

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy = 0. \tag{78}$$

Then $f$is a constant. Note that Proposition (2.2.6) is a consequence of Theorem (2.2.7) when choosing$\rho_\varepsilon(r) = \begin{cases}\varepsilon r^{-N+\varepsilon}, & r<1\\ 0, & r>1.\end{cases}$

And Proposition (2.2.5) is also a consequence of Theorem (2.2.7) when choosing

$\rho_\varepsilon(r) = \begin{cases}0 & \text{if } r<\varepsilon\\ a_\varepsilon\psi(r) & \text{if } \varepsilon<r<1\\ 0 & \text{if } r>1,\end{cases}$ where

$$a_\varepsilon = \left(\int_\varepsilon^1 \psi(r)r^{N-1}dr\right)^{-1} \to 0 \qquad as\ \varepsilon\to 0. \tag{79}$$

Note that, in view of (70), $\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \leq Ca_\varepsilon \to 0$ as $\varepsilon\to 0,$ by (79).

The proof of Theorem (2.2.7) involves an excursion into Sobolev spaces which we will now describe.

For simplicity,we start with the case of all of $\mathbb{R}^N$.
Let $f \in L^p(\mathbb{R}^N), 1 < p < \infty$. It is well-know,(see[71])that if $f \in W^{1,p}(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p dx \le |h|^p \int_{\mathbb{R}^N} |\nabla f|^p dx \text{ for every } h \in \mathbb{R}^N. \quad (80)$$

And conversely, if $f \in L^p(\mathbb{R}^N)$ and if there exists a constant C such that

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p dx \le C|h|^p \text{as } h \to 0, \quad (81)$$

then $f \in W^{1,p}(\mathbb{R}^N)$.When $p = 1, W^{1,1}$ should be replaced by BV, the space of functions in L^1 who's derivatives (in the sense of distributions) are bounded Radon measures; thus $f \in BV$ if and only if

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)| dx \le C|h| \quad as\ |h| \to 0, \quad (82)$$

and then (16) holds for all $h \in \mathbb{R}^N$ with $C = \int |\nabla f| dx$.
In particular, if ρ_ε satisfies (75), (76) and $f \in W^{1,p}$, we have

$$\int_{\mathbb{R}^N} \rho_\varepsilon(|h|) dh \int_{\mathbb{R}^N} \frac{|f(x+h) - f(x)|^p}{|h|^p} dx \le C \quad as\ \varepsilon \to 0, \quad (83)$$

Since

$$\int_{\mathbb{R}^N} \rho_\varepsilon(|h|) dh = \sigma_N \int_0^\infty \rho_\varepsilon(r) r^{N-1} dr = \sigma_N$$

where $\sigma_N = |S^{N-1}|$. Changing variables in (83) yields

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy \le C \text{ as } \varepsilon \to 0. \quad (84)$$

Similarly, if $f \in BV$, we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x-y|} \rho_\varepsilon(|x-y|) dx\, dy \le C \text{ as } \varepsilon \to 0. \quad (85)$$

The heart of the matter is that (84),(85) gives a characterization of $W^{1,p}$ when $p > 1$ (resp. BV).
Theorem (2.2.8)[70]. Assume $f \in L^p(\mathbb{R}^N)$ satisfies (84) with $p > 1$.
Let (ρ_ε) be as in (75)-(76)-(77). Then $f \in W^{1,p}$ and

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy = K_{p,N} \int_{\mathbb{R}^N} |\nabla f|^p dx \quad (86)$$

where $K_{p,N}$ depends only on p and N. Similarly for $p = 1$ we have
Proof: The original proof of Theorem (2.2.8) is to be found in [72]. We present here a simpler argument suggested by E.
Assume $f \in L^p$ satisfies (84) an let (γ_δ) be any sequence of smooth mollifiers.
Set $f_\delta = \gamma_\delta \star f$.
Note that (84) still holds when f is replaced by its translates $(\tau_h f)(x) = f(x+h)$.

Also, (84) is stable under convex combinations and thus f_δ satisfies (84) with the same constant C, i.e., we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f_\delta(x)-f_\delta(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \le C \tag{87}$$

where C is independent of ε and δ.

Next, let $g \in C^2(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \le C \text{ as } \varepsilon \to 0, \tag{88}$$

where ρ_ε satisfies (75), (66), (77).

We claim that

$$\int_{\mathbb{R}^N} |\nabla g(x)|^p dx \le C/K_{p,N}, \tag{89}$$

with C taken from (88) and

$$K_{p,N} = \int_{S^{N-1}} |(\sigma\cdot e)|^p d\sigma\,, \quad e \in S^{N-1}. \tag{90}$$

Proof of (89). Let K be any compact subset of $\mathbb{R}^N$.

For $x \in K$ and $|h| \le 1$ we have

$$|g(x+h)-g(x)-h\cdot\nabla g(x)| \le C_K|h|^2. \tag{91}$$

From (88) we have

$$\int_K dx \int_{|h|\le 1} \frac{|g(x+h)-g(x)|^p}{|h|^p}\rho_\varepsilon(|h|)dh \le C. \tag{92}$$

By (91) we have

$$|h\cdot\nabla g(x)| \le |g(x+h)-g(x)| + C_K|h|^2$$

and therefore, for every $\theta > 0$

$$|h\cdot\nabla g(x)|^p \le (1+\theta)|g(x+h)-g(x)|^p + C_{\theta,K}|h|^{2p}.$$

Combining this with (92) yields

$$\int_K dx \int_{|h|\le 1} \frac{\left|\big(h\cdot\nabla g(x)\big)\right|^p}{|h|^p}\rho_\varepsilon(|h|)dh \le (1+\theta)C + C_{\theta,K}|K| \int_{|h|\le 1}|h|^p\rho_\varepsilon(|h|)dh. \tag{93}$$

But, for any vector $V \in \mathbb{R}^N$,

$$\int_{|h|\le 1} \frac{|(h\cdot V)|^p}{|h|^p}\rho_\varepsilon(|h|)dh = K_{p,N}|V|^p \int_0^1 \rho_\varepsilon(r)r^{N-1}dr.$$

On the other hand, it is clear from (76) and (77) that

$$\lim_{\varepsilon\to 0}\int_{|h|\le 1}|h|^p\rho_\varepsilon(|h|)dh = 0.$$

Passing to the limit as $\varepsilon \to 0$ in (93) we find

$$K_{p,N} \int_K |\nabla g(x)|^p dx \le (1+\theta)C. \tag{94}$$

Since (94) holds for every $\theta > 0$ and every compact set K(with C independent of θ and K) we obtain (89), that is,

$$K_{p,N} \int_K |\nabla g(x)|^p dx \le \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)dx\, dy. \tag{95}$$

On the other hand, if $g \in C_0^2(\mathbb{R}^N)$ we have, as above,
$|g(x+h)-g(x)| \le |h\cdot\nabla g(x)| + C'|h|^2 \quad \forall x \in \mathbb{R}^N, \quad \forall h \in \mathbb{R}^N$.Hence
$|g(x+h)-g(x)|^p \le (1+\theta)|h\cdot\nabla g(x)|^p + C'_\theta|h|^{2p}$.We multiply this by $\rho_\varepsilon(|h|)/|h|^p$ and integrate over the set $\{(x,h) \in \mathbb{R}^{2N} : x \text{ or } x+h \in \operatorname{supp} g\}$ to obtain
$\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{|g(x+h)-g(x)|^p}{|h|^p} \rho_\varepsilon(|h|)dh \le (1+\theta)\int_{\mathbb{R}^N} K_{p,N}|\nabla g(x)|^p dx + 2C'_\theta|\operatorname{supp} g| \int_{\mathbb{R}^N} |h|^p \rho_\varepsilon(|h|)dh.$
We first let $\varepsilon \to 0$ and then $\theta \to 0$. This yields

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{|g(x+h)-g(x)|^p}{|h|^p} \rho_\varepsilon(|h|)dh \le K_{p,N} \int_{\mathbb{R}^N} |\nabla g(x)|^p dx. \tag{96}$$

Combining (95) and (96) yields, for every $g \in C_0^2(\mathbb{R}^N)$,
$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)dx\, dy = K_{p,N} \int_{\mathbb{R}^N} |\nabla g(x)|^p dx.$
Since $C_0^2(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, it is easy to conclude (using (80)) that (86) holds for every $f \in W^{1,p}(\mathbb{R}^N)$.We may now complete the proof of Theorem (2.2.9). Assuming $f \in L^p(\mathbb{R}^N)$ satisfies (84) and applying Claim (91) to $g = f_\delta$ we see that

$$\int_{\mathbb{R}^N} |\nabla f_\delta|^p dx \le \frac{C}{K_{p,N}}, \tag{97}$$

where C comes from (84).

Finally, we pass to the limit in (97) as $\delta \to 0$ and obtain $f \in W^{1,p}$.

Theorem (2.2.9)[70]. Assume $f \in L^1(\mathbb{R}^N)$ satisfies (86). Let (ρ_ε) be as in (75)-(76)-(77).
Then $f \in BV$ and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|} \rho_\varepsilon(|x-y|)dx\, dy = K_{1,N} \int_{\mathbb{R}^N} |\nabla f| dx \tag{98}$$

where the right-hand side denote the total mass of the measure ∇f.
An interesting consequence of Theorem (2.2.9) is the following
Proof: If $f \in L^1(\mathbb{R}^N)$ and satisfies (85) and we proceed as above we are led to
$\int_{\mathbb{R}^N} |\nabla f_\delta| dx \le C/K_{1,N}$.Therefore $f \in BV$ and $\int_{\mathbb{R}^N} |\nabla f| dx \le C/K_{1,N}$.
In other words we have proved that

$$K_{1,N} \int_{\mathbb{R}^N} |\nabla f| dx \le \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|} \rho_\varepsilon(|x-y|)dx\, dy. \tag{99}$$

On the other hand it is easy to see, using (82), that for $f \in BV$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy \le \widetilde{K}_N \int_{\mathbb{R}^N} |\nabla f|dx. \tag{100}$$

Unfortunately the constant $\widetilde{K}_N$ in (100) is not the same as $K_{1,N}$. It is also clear that (98) holds when $f \in C_0^2(\mathbb{R}^N)$. However we cannot conclude easily that (98) holds for every $f \in BV$ since $C_0^2(\mathbb{R}^N)$ is not dense in BV. It remains to be shown that, for every $f \in BV(\mathbb{R}^N)$

$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy \le K_{1,N}\int_{\mathbb{R}^N}|\nabla f|dx.$

This has been established by J. It is still true (for a general Ω) that

$$K_{p,N}\int_\Omega |\nabla f|^p \le \limsup_{\varepsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy. \tag{101}$$

However, it may happen for $p>1$ that $f \in W^{1,p}(\Omega)$ (so that the left hand side in (101) is finite) while the right-hand side in (101) is infinite. Here is such an example. Let $\Omega = D\backslash\Sigma$ where D is a disc (in $\mathbb{R}^2$) and Σ is a slit. Let f be a smooth function in Ω which is discontinuous across the slit (for example two different constants on each side of the slit). Clearly $f \in W^{1,p}(\Omega)$, but the RHS in (101) is infinite. This is so because $\int_\Omega \int_\Omega \cdots = \int_D \int_D \cdots$ and if the RHS in (101) were finite we would conclude that $f \in W^{1,p}(D)$(by Theorem (2.2.8)), which is obviously wrong.

This example suggests the following open problem (i). Let $\Omega \subset \mathbb{R}^N$ be a bounded connected set (not necessarily smooth). Let $\delta(x,y)$ denote the geodesic distance in Ω. Let $f \in L^p(\Omega)$ be such that

$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{\delta(x,y)^p}\rho_\varepsilon(\delta(x,y))dx\,dy \le C$ as $\varepsilon \to 0$.

Does it follow that $f \in W^{1,p}$ and if so, does have

$\lim_{\varepsilon\to 0}\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{\delta(x,y)^p}\rho_\varepsilon(\delta(x,y))dx\,dy = K_{p,N}\int_\Omega |\nabla f|^p dx$?

Corollary (2.2.10)[70]. Let A be a bounded measurable set in $\mathbb{R}^N$.
Then A has finite perimeter
(in the sense of De Giorgi) if and only if $\int_A \int_{^cA} \frac{1}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy \le C$ as $\varepsilon \to 0$ and then

$$\lim_{\varepsilon\to 0}\int_A \int_{^cA} \frac{1}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy = K_{1,N}\,Per(A). \tag{102}$$

Theorem (2.2.11)[70]. Assume $f \in L^p(\Omega)$ satisfies

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \le C \text{ as } \varepsilon \to 0, \tag{103}$$

with ρ_ε as in (75), (76), (77). Then $f \in W^{1,p}(\Omega)$ and

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy = K_{p,N}\int_\Omega |\nabla f|^p. \tag{104}$$

Sketch of proof. First assume that (103) holds. By a standard technique of reflection across the boundary and multiplication by a cut-off one constructs a function $\tilde{f}$ on $\mathbb{R}^N$, with compact support, such that $\tilde{f} = f$ on and satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\tilde{f}(x) - \tilde{f}(y)\right|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \le C' \text{as } \varepsilon \to 0, \tag{105}$$

By Theorem (2.2.8) we conclude that $\tilde{f} \in W^{1,p}(\mathbb{R}^N)$ and thus $f \in W^{1,p}(\Omega)$.

Next one shows that if $f \in C^2(\overline{\Omega})$, then

$$\int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \le C(\Omega)\int_\Omega |\nabla f|^p dx. \tag{106}$$

Finally one proves that if $f \in C^2(\overline{\Omega})$

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy = K_{p,N}\int_\Omega |\nabla f|^p dx. \tag{107}$$

The conclusion of Theorem (2.2.11) follows from an easy density argument.

Corollary (2.2.12)[70]. Assume Ω is a smooth bounded domain in $\mathbb{R}^N$.

Let $f \in L^p(\Omega)$be such that $\varepsilon\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+p-\varepsilon}}dx\,dy \le C$ as $\varepsilon \to 0$, then $f \in W^{1,p}(\Omega)$ and

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+p-\varepsilon}}dx\,dy = K_{p,N}\int_\Omega |\nabla f|^p. \tag{108}$$

Recall that the standard fractional Sobolev space $W^{s,p}, 0 < s < 1, 1 < p < \infty$, is equipped with Gagliardo (semi) norm

$$\|f\|^p_{W^{s,p}} = \int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}}dx\,dy. \tag{109}$$

It is well-known that $\|f\|_{W^{s,p}}$ does not converge to $\|f\|_{W^{1,p}}$ as $s \uparrow 1$; in fact it converges to 1 (unless f is constant) by Proposition (2.2.4). However in view of Corollary (2.2.12)we may now assert that

$$\lim_{s\uparrow 1}(1-s)\|f\|^p_{W^{s,p}} = \frac{K_{p,N}}{p}\int_\Omega |\nabla f|^p. \tag{110}$$

This "reinstates" $W^{1,p}$ as a continuous limit of $W^{s,p}$ as $s \uparrow 1$ provided one uses the norm $(1-s)^{1/p}\|f\|_{W^{s,p}}$ on $W^{s,p}$.

Choice (ii) $\rho_\varepsilon(r) = \begin{cases} \frac{N}{\varepsilon^N} & \text{if } r < \varepsilon \\ 0 & \text{if } r > \varepsilon \end{cases}$ This choice yields

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}\int_\Omega\int_{\substack{\Omega \\ |x-y|<\varepsilon}} \frac{|f(x) - f(y)|^p}{|x-y|^p}dx\,dy = \frac{K_{p,N}}{N}\int_\Omega |\nabla f|^p. \tag{111}$$

A variant is $\rho_\varepsilon(r) = \begin{cases} \frac{(N+p)r^p}{\varepsilon^{N+p}} & r < \varepsilon \\ 0 & r > \varepsilon \end{cases}$

and then we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+p}}\int_{\Omega}\int_{\Omega}_{|x-y|<\varepsilon}|f(x)-f(y)|^p dx\,dy=\frac{K_{p,N}}{(N+p)}\int_{\Omega}|\nabla f|^p. \tag{112}$$

Still another choice yields

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+p}}\int_{\Omega}\int_{\Omega}_{\varepsilon<|x-y|<2\varepsilon}|f(x)-f(y)|^p dx\,dy=\widetilde{K}_{p,N}\int_{\Omega}|\nabla f|^p. \tag{113}$$

Choice (iii) $\rho_\varepsilon(r)=\begin{cases}0 & r<\varepsilon\\ \frac{1}{|log\varepsilon|r^N} & \varepsilon<r<1\\ 0 & r>1.\end{cases}$ This choice yields

$$\lim_{\varepsilon\to 0}\frac{1}{|log\varepsilon|}\int_{\Omega}\int_{\Omega}_{|x-y|<\varepsilon}\frac{|f(x)-f(y)|^p}{|x-y|^{N+p}}dx\,dy=K_{p,N}\int_{\Omega}|\nabla f|^p. \tag{114}$$

Choice (iv) Let $\gamma\in L^1_{loc}(0,+\infty)$, $\gamma\geq 0$, be such that $\int_0^\infty\gamma(r)r^{N+p-1}dr=1$.

Choosing $\rho_\varepsilon(r)=\frac{1}{\varepsilon^{N+p}}\gamma\left(\frac{r}{\varepsilon}\right)r^p$ yields

$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+p}}\int_{\Omega}\int_{\Omega}|f(x)-f(y)|^p\gamma\left(\frac{|x-y|}{\varepsilon}\right)dx\,dy=K_{p,N}\int_{\Omega}|\nabla f|^p$,

for every $f\in W^{1,p}$ (with $p>1$) and for every $f\in BV$ (with $p=1$). Applying this in the BV case with $f=\chi_A$ we obtain a new characterization of sets of finite perimeter. Namely a measurable set $A\subset\Omega$ has finite perimeter if and only if $\frac{1}{\varepsilon^{N+1}}\int_A\int_{^cA}\gamma\left(\frac{|x-y|}{\varepsilon}\right)dx\,dy\leq C$ as $\varepsilon\to 0$,

and then $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+1}}\int_A\int_{^cA}\gamma\left(\frac{|x-y|}{\varepsilon}\right)dx\,dy=K_{1,N}\mathrm{Per}(A)$.

All the results of (2.2) are immediate consequences of the statements of (2.2.7) applied in a ball $B\subset\Omega$. One concludes that f is constant on B and then that f is constant on Ω since Ω is connected.

Note that the assumption

$$\lim_{\varepsilon\to 0}\int_B\int_B\frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy=0, \tag{115}$$

implies first that $f\in BV$ and then that $\nabla f=0$, so that f is a constant.

By contrast, when $p>1$, and f takes its values into $\mathbb{Z}$ it suffices to assumes that

$$\int_B\int_B\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy\leq C\quad\text{as }\varepsilon\to 0. \tag{116}$$

Indeed, (116) implies that $f\in W^{1,p}$ (attention when $p=1$, (116) only implies that $f\in BV$). Then, one may use the fact that f takes its values into $\mathbb{Z}$ to conclude that f is constant. The argument is the following: write $\Omega=\bigcup_{k\in\mathbb{Z}}A_k$ where $A_k=\{x\in\Omega; f(x)=k\}$ and use a well-known result of Stampacchia asserting that $\nabla f=0$ a.e. on A_k. Hence $\nabla f=0$ a.e. on Ω.

Alternatively, one may deduce from (49) and assumption $f:\Omega\to\mathbb{Z}$, that

$\int_\Omega\int_\Omega\frac{|f(x)-f(y)|}{|x-y|}\frac{\rho_\varepsilon(|x-y|)}{|x-y|^{p-1}}dx\,dy\leq C.$

This yields easily

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy = 0$$

and thus f is a constant. There are interesting extensions of some of the above results where the ratio $\frac{|f(x)-f(y)|^p}{|x-y|^p}$ is replaced by a more general expression $\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)$. Here are two results due to R.

Theorem(2.2.13)[70]. Assume $\omega : [0,\infty) \to [0,\infty)$ is a continuous function such that $\omega(0) = 0,\ \omega(t) > 0\ \forall t > 0$ and

$$\int_1^\infty \frac{\omega(t)}{t^2}dt = \infty. \tag{117}$$

Assume $f \in L^1(\Omega)$ satisfies $\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N} < \infty$, then f is a constant.

Theorem (2.2.14)[70]. Assume $\omega : [0,\infty) \to [0,\infty)$ is a continuous function such that $\omega(0) = 0$ and $\lim_{t\to\infty}\frac{\omega(t)}{t} = \alpha > 0$. Assume $f \in L^1(\Omega)$ satisfies

$\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(|x-y|)dx\,dy \le C$ as $\varepsilon \to 0$.

Then $f \in BV$ and

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(|x-y|)dx\,dy = \int_\Omega \bar{\omega}(|\nabla f_{ac}|)dx + \alpha K_{1,N}\int_\Omega |\nabla f_s|dx,$$

where $\bar{\omega}(t) = \int_{S^{N-1}}\omega(t|\sigma\cdot e|)d\sigma$ and $\nabla f = \nabla f_{ac} + \nabla f_s$ is the Radon–Nikodym decomposition of ∇f. Here is still another open problem ,open problem (67). Let Ω be a (smooth) connected, bounded domain in $\mathbb{R}^N$. Let $f : \Omega \to \mathbb{R}$ be a continuous (or even Hölder continuous) function. Let $\omega : [0,\infty) \to [0,\infty)$ be a continuous function such that ! $\omega(0) = 0$ and $\omega(t) > 0$ for $t > 0$.

(Here (51) might fail). Assume that $\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{1}{|x-y|^N}dx\,dy < \infty$. Can one conclude that f is a constant? We first recall the definition of VMO($\Omega;\mathbb{R}$) (= vanishing mean oscillation).We say that a function $f \in VMO(\Omega;\mathbb{R})$ if $f \in L^1_{loc}(\Omega;\mathbb{R})$ satisfies

$\lim_{\varepsilon\to 0}\frac{1}{|B_\varepsilon(x)|^2}\int_{B_\varepsilon(x)}\int_{B_\varepsilon(x)}|f(y)-f(z)|dy\,dz = 0$ uniformly for $x \in \Omega$.

Let Ω be a connected (smooth) open set in $\mathbb{R}^N$ and let $f \in VMO(\Omega;\mathbb{Z})$. Then f is a constant. Indeed if we set $\bar{f}_\varepsilon(x) = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)}f(y)dy$then $\operatorname{dist}(\bar{f}_\varepsilon(x),\mathbb{Z}) \to 0$ uniformly in Ω and thus there is some constant $k_\varepsilon \in \mathbb{Z}$ such that$|\bar{f}_\varepsilon(x) - k_\varepsilon| \to 0$ uniformly in Ω (see [90]). Hence f is a constant. Functions in $W^{s,p}(\Omega)$ belong to $VMO(\Omega)$ provided $sp \ge N$ (see[69]).Therefore one cannot apply directly this argument in our setting which corresponds roughly speaking to $sp \ge 1$. Assume for simplicity that Ω is a square in $\mathbb{R}^2$.

Let $f \in W^{s,p}(\Omega)$. Then the restrictions $f(x_1,\cdot)$ and $f(\cdot,x_2)$ still belong to $W^{s,p}(I)$ for a.e. x_1 and a.e. x_2 (where I is an interval)(see[72]).This observation is very useful when combined with the following measure theoretical tool: (see[72] Assume that $f : \Omega \to \mathbb{R}$ is measurable. Suppose that for a.e. $x_1, f(x_1,\cdot)$ and for a.e. $x_2, f(\cdot,x_2)$ are constant functions. Then $f$is a constant. The considerations above yield an alternative proof of Corollary 1 when $p > 1$. Indeed, if $p > 1$, (2) says that $f \in W^{s,p}(\Omega)$ where $s = 1/p$. The restrictions of f to almost every line still belong to $W^{s,p}$ with $s = 1/p$. Therefore, if $f : \Omega \to \mathbb{Z}$ one may conclude that the restrictions of f to almost every line are constant. The above lemma allows to conclude that f is constant [75].

Section(2.3): Composition and Products in Fractional Sobolev Spaces:

The main result is the following: let $1 \le s < \infty, 1 < p < \infty$,

and let $m = \begin{cases} s, & \text{if} s \text{ is an integer} \\ [s] + 1, & \text{otherwise.} \end{cases}$

Set $R = \{f \in C^m(\mathbb{R}); f(0) = 0, f, f', \ldots, f^{(m)} \in L^\infty(\mathbb{R})\}$.

.Here, $0 < s < \infty, 1 < p < \infty$ and $\Omega$is a smooth bounded simply connected domain in $\mathbb{R}^n$. In particular, one may ask whether $X$is path-connected and whether $C^\infty(\overline{\Omega}; S^1)$is dense in X. Several results concerning the first question were obtained in [77](and subsequently in [78] for the spaces $W^{1,p}(M; N)$, where $M, N$are compact oriented Riemannian manifolds. The second equation was studied in [79] , [80] and [78] for the spaces $W^{1,p}(M; N)$ and in [81] for the spaces $W^{s,p}(M; S^k)$.

The case where $N = S^1$ is somehow special; one may attempt to answer these questions by lifting the maps $u \in X$. Here is a strategy: given $u \in W^{s,p}(\Omega; S^1)$, one may try to find some $\varphi \in W^{s,p}(\Omega; \mathbb{R})$such that $u = e^{i\varphi}$. Then, hopefully, the path $t \in [0,1] \mapsto e^{it\varphi}$ will connect continuously $u_0 \equiv 1$ to u. Moreover, if φ_j are smooth $\mathbb{R}$-valued functions on $\overline{\Omega}$such that $\varphi_j \to \varphi$ in $W^{s,p}$, then, hopefully, the smooth maps $e^{i\varphi_j}$ converge to $u$in $W^{s,p}(\Omega; S^1)$.We are thus naturally led to the study of the mapping $W^{s,p}(\Omega) \ni \psi \mapsto f(\psi)$for "reasonable" functions f(e.g. $f(x) = e^{ix} - 1$), where Ω is either a smooth bounded domain or $\Omega = \mathbb{R}^n$and $s \ge 1$. In [82] we settle the above mentioned questions about $W^{s,p}(\Omega; S^1)$when $s \ge 1$. Another motivation for analyzing composition and products in fractional Sobolev spaces comes from the study of nonlinear evolution equations (e.g. Schrödinger equation) in H^s spaces; see[83],[84] and [85]. In fact, the Appendix in [83],[86] contains a result which is a special case of the Runst-Sickel lemma about products: it coincides below when $q = 2$.We start by recalling the Littlewood-Paley decomposition of temperate distributions.

Let $\psi_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \psi_0 \le 1, \psi_0(\xi) = 1$ for $|\xi| \le 1, \psi_0(\xi) = 0$ for $|\xi| \ge 2$. Set $\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{-j+1}\xi), j \ge 1$,

and $\varphi_j = F^{-1}(\psi_j), j \ge 0$.

Thus

$$\varphi_j(x) = 2^{nj}\varphi_0(2^j x) - 2^{n(j-1)}\varphi_0(2^{j-1}x), \quad j \ge 1, \tag{118}$$

and

$$\sum_{k \le j} \varphi_k(x) = 2^{nj}\varphi_0(2^j x), \quad j \ge 0. \tag{119}$$

For $f \in S'$, set $f_j = f\varphi_j$. We have $f = \sum_{j \ge 0} f_j$in S'.

Definition (2.3.1)[87]. For $-\infty < s < \infty, 0 < p \le \infty, 0 < q \le \infty$, set

$$\tilde{F}^s_{p,q} = \left\{f \in S'; \|f\|_{\tilde{F}^s_{p,q}} = \left\| \left\|2^{sj} f_j(x)\right\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} < \infty\right\}.$$

For $0 < p < \infty$ or $p = q = \infty$, these are the standard Triebel-Lizorkin spaces $F^s_{p,q}$.

We have added the $\tilde{F}^0_{p,q}$ to avoid confusions in the exceptional cases where they do notcoincide.

When $0 < p < \infty$, different choices of ψ_0 yield equivalent quasi-norms.

The usual function spaces are special cases of these Triebel-Lizorkin spaces:

(i) $L^p = \tilde{F}^0_{p,2}, 1 < p < \infty$;

(ii) $W^{m,p} = \tilde{F}^m_{p,2}, m = 1, 2, \ldots, 1 < p < \infty$;

(iii) $W^{s,p} = \tilde{F}^s_{p,p}, 0 < s < \infty, s$non-integer, $1 \le p < \infty$;
(iv) $L^{s,p} = \tilde{F}^s_{p,2}, s \in \mathbb{R}, 1 < p < \infty$;
(v) $L^\infty \subset \tilde{F}^0_{\infty,\infty}$, i.e.,

$$\sup_{j,x} |f_j(x)| \le C\|f\|_{L^\infty}. \tag{120}$$

In this list, when $1 \le p < \infty, 0 < s < \infty, s$ non-integer, the $W^{s,p}$ are the Sobolev-Slobodeckij spaces. An equivalent norm on these spaces may be obtained as follows: let $s = k + \sigma, k$ integer, $0 < \sigma < 1$. Then

$$\|f\|^p_{W^{s,p}} \sim \|f\|^p_{L^p} + \|D^k f\|^p_{L^p} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^k f(x) - D^k f(y)|^p}{|x-y|^{n+\sigma p}} dx\, dy \tag{121}$$

In [88] these spaces also coincide with the Besov spaces $B^s_{p,p}$ (recall that s is notan integer). We warn that, for $p \neq 2$, the spaces $W^{s,p}$do not coincide with the Bessel potential spaces $L^{s,p}$ see[68].We will often use the trivial fact that, for fixed sand p, the space $\tilde{F}^s_{p,q}$ increases with q. The following result is well-known in [69]

Lemma (2.3.2)[76]. Let $0 < s < \infty, 1 < p < \infty, 1 < q < \infty$. For every $j \ge 0$, let $f^j \in S'$be such that $\operatorname{supp} F(f^j) \subset B_{2^{j+2}}$. Then

$$\left\| \sum_j f^j \right\|_{\tilde{F}^s_{p,q}} \le C \left\| \|2^{sj} f^j(x)\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)}. \tag{122}$$

In the H^s-spaces($p = q = 2$), this result is proved in [91]. Recall that, for any $f \in L^1_{loc}$, the maximal function Mf is defined by $Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy.$ For $t > 0$, set, for $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\varphi^t(x) = t^{-n}\varphi(x/t),\ x \in \mathbb{R}^n. \tag{123}$$

We recall some classical inequalities

Proof: With $f = \sum_j f^j$, we have $f_k = \left(\sum_j f^j\right)_k = \left(\sum_{j\ge k-3} f^j\right)_k = \sum_{j \ge k-3}\left(f^j\right)_k$.
Therefore

$$\|f\|_{\tilde{F}^s_{p,q}} = \left\| \left\| 2^{sk} \sum_{j\ge k-3} \left(f^j\right)_k(x) \right\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} = \left\| \left(\sum_k 2^{sqk} \left| \sum_{j \ge k-3} \left(f^j\right)_k(x) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$
$$\le C \left\| \left(\sum_k 2^{sqk} \sum_{j\ge k-3} \left| \left(f^j\right)_k(x)\right|^q (j-k+4)^{2q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

by the Hölder inequality with exponents $q$and $q' = \frac{q}{q-1}$ applied to the inner sum.

$$\|f\|_{\tilde{F}^s_{p,q}} \le C \left\| \left(\sum_j \sum_{k \le j+3} 2^{sqk} (j-k+4)^{2q} \left|Mf^j(x)\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \le C \left\| \left(\sum_j 2^{sqj} \left|Mf^j(x)\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$
$$= C \left\| \left\| 2^{sj} Mf^j(x) \right\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)}. \tag{124}$$

The desired conclusion is a consequence of (126) and (129).

Lemma (2.3.3)[76],[92]. We have:
(i) for $1 < p \le \infty$ and any function f,

$$\|Mf\|_{L^p} \sim \|f\|_{L^p}; \tag{125}$$

(ii) for $1 < p < \infty, 1 < q < \infty$, and any sequence of function (f^j),

$$\left\| \left\| Mf^j(x) \right\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left\| f^j(x) \right\|_{\ell^q} \right\|_{L^p(\mathbb{R}^n)}; \quad (126)$$

(iii)for any fixed $\varphi \in S$ and any function f,

$$|f \star \varphi^t(x)| \leq CMf(x), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^n. \quad (127)$$

By (118), (119) and (127) we obtain the following

Corollary (2.3.4)[76]. For every $f \in L^1_{loc}$ we have

$$|f_j(x)| \leq CMf(x), \quad j \geq 0, \quad x \in \mathbb{R}^n, \quad (128)$$

$$\sum_{j \leq k} f_j(x) \leq CMf(x), \quad k \geq 0, \quad x \in \mathbb{R}^n. \quad (129)$$

In the Gagliardo-Nirenberg type inequalities for the spaces $\tilde{F}^s_{p,q}$, there is a gain in the "microscopic" parameter q; this gain is also called sometimes "precised" or "improved" Sobolev inequalities. In the context of Besov spaces, a typical Gagliardo-Nirenberg inequality asserts that

$B^s_{p,r} \cap L^\infty \subset B^{s/2}_{2p,2r}$, for $0 < s < \infty,\ 0 < p < \infty, 0 < r \leq \infty$ see [93].Here, the value $2r$ of the microscopic parameter is optimal in general. By contrast, in the scale of $\tilde{F}$-spaces we have, given

$0 < s < \infty, 0 < p < \infty, 0 < r \leq \infty, \tilde{F}^s_{p,r} \cap L^\infty \subset \tilde{F}^{\frac{s}{2}}_{2p,q}$ for every $0 < q \leq \infty$ (see [93]).

Amore general version of this phenomenon, in [94], is the following.

Let $-\infty < s_1 < s_2 < \infty, 0 < q_1, q_2 \leq \infty, 0 < p_1, p_2 \leq \infty, 0 < \theta < 1$, and define $s = \theta s_1 + (1-\theta)s_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. We state some interesting consequences.

Corollary (2.3.5)[76]. We have

(i) *for* $0 \leq s_1 < s_2 < \infty,\ 1 < p_1 < \infty,\ 1 < p_2 < \infty, s = \theta s_1 + (1-\theta)s_2, \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$,

$$\|f\|_{W^{s,p}} \leq C\|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}; \quad (130)$$

(ii) ([93]) *for* $0 < s < \infty,\ 1 < p < \infty,\ 0 < q \leq \infty$,

$$\|f\|_{\tilde{F}^{\theta s}_{p/\theta,q}} \leq C\|f\|^{\theta}_{W^{s,p}} \|f\|^{1-\theta}_{L^\infty}. \quad (131)$$

In particular, we have

(iii)*for* $0 < s < \infty, 1 < p < \infty, 0 < \theta < 1$,

$$\|f\|_{W^{\theta s, p/\theta}} \leq C\|f\|^{\theta}_{W^{s,p}} \|f\|^{1-\theta}_{L^\infty}. \quad (132)$$

Lemma (2.3.6)[76]. *Let* $-\infty < s_1 < s_2 < \infty, 0 < q < \infty, 0 < \theta < 1$, and set $s = \theta s_1 + (1-\theta)s_2$. Then for every sequence (a_j) we have

$$\|2^{sj} a_j\|_{\ell^q} \leq C\|2^{s_1 j} a_j\|^{\theta}_{\ell^\infty} \|2^{s_2 j} a_j\|^{1-\theta}_{\ell^\infty}. \quad (133)$$

Proof: Let $C_1 = \sup 2^{s_1 j}|a_j|, C_2 = \sup 2^{s_2 j}|a_j|$, so that $C_1 \leq C_2$.

We may assume $C_1 > 0$. Since $s_1 < s_2$, there is some $j_0 > 0$ such that

$\min\left\{\frac{C_1}{2^{s_1 j}}, \frac{C_2}{2^{s_2 j}}\right\} = \begin{cases} \frac{C_1}{2^{s_1 j}}, & j \leq j_0 \\ \frac{C_2}{2^{s_2 j}}, & j > j_0. \end{cases}$ Since $\frac{C_1}{2^{s_1 j_0}} \leq \frac{C_2}{2^{s_2 j_0}}$ and $\frac{C_2}{2^{s_1(j_0+1)}} \leq \frac{C_1}{2^{s_1(j_0+1)}}$ we find that

$$C_2 \sim C_1 2^{(s_2-s_1)j_0}. \quad (134)$$

Therefore

$$\|2^{s_1 j} a_j\|^{\theta}_{\ell^\infty} \|2^{s_2 j} a_j\|^{1-\theta}_{\ell^\infty} \sim C_1 2^{(s_2-s_1)j_0(1-\theta)}. \quad (135)$$

On the other hand, we have $a_j \le \min\left\{\frac{C_1}{2^{s_1 j}}, \frac{C_2}{2^{s_2 j}}\right\}$, so that

$$a_j \le \frac{C_1}{2^{s_1 j}} \text{ for } 0 \le j \le j_0, \qquad a_j \le \frac{C_2}{2^{s_2 j}} \text{ for } j > j_0. \tag{136}$$

It then follows that

$$\left\|2^{sj} a_j\right\|_{\ell^q} \le \left(\sum_{j \le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j > j_0} C_2^q 2^{(s-s_2)jq}\right)^{\frac{1}{q}}$$
$$\le C\left(\sum_{j \le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j > j_0} C_1^q 2^{-\theta(s_2-s_1)jq + (s_2-s_1)j_0 q}\right)^{\frac{1}{q}}$$

so that

$$\left\|2^{sj} a_j\right\|_{\ell^q} \le C C_1 2^{(s_2-s_1)j_0(1-\theta)} \left(\sum_{j \le j_0} 2^{-(1-\theta)(s_2-s_1)(j_0-j)q} + \sum_{j > j_0} 2^{-\theta(s_2-s_1)(j-j_0)q}\right)^{1/q}.$$

Finally, we find that

$$\left\|2^{sj} a_j\right\|_{\ell^q} \le C C_1 2^{(s_2-s_1)j_0(1-\theta)}, \tag{137}$$

and (133) follows from (135) and (137).

Lemma (2.3.7)[76]. Under the above hypotheses we have, for every $0 < q \le \infty$,

$$\|f\|_{\tilde{F}^s_{p,q}} \le C \|f\|^{\theta}_{\tilde{F}^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{\tilde{F}^{s_2}_{p_2,q_2}}, \tag{138}$$

where C depends on s_i, p_i, θ and q.

Proof: Since $\left\|a_j\right\|_{\ell^\infty} \le \left\|a_j\right\|_{\ell^q}, 0 < q \le \infty$, we find that r.h.s. of (138) is $\|f\|_{\tilde{F}^s_{p,q}} \ge C\|f\|^{\theta}_{\tilde{F}^{s_1}_{p_1,\infty}} \|f\|^{1-\theta}_{\tilde{F}^{s_2}_{p_2,\infty}}$. On the other hand, $\|f\|_{\tilde{F}^s_{p,\infty}} \le \|f\|_{\tilde{F}^s_{p,q}}, 0 < q < \infty$. It therefore suffices to prove (138) in the special case $0 < q < \infty, q_1 = q_2 = \infty$.

In this case, we have

$$\|f\|_{\tilde{F}^s_{p,q}} = \left\|\left\|2^{sj} f_j(x)\right\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)} \le \quad (\text{by}(133))$$
$$\le C \left\|\left\|2^{s_1 j} f_j(x)\right\|^{\theta}_{\ell^\infty} \left\|2^{s_2 j} f_j(x)\right\|^{1-\theta}_{\ell^\infty}\right\|_{L^p(\mathbb{R}^n)}. \tag{139}$$

Using the Hölder inequality, (139) yields

$$\|f\|_{\tilde{F}^s_{p,q}} \le C \left\|\left\|2^{s_1 j} f_j(x)\right\|_{\ell^\infty}\right\|^{\theta}_{L^{p_1}(\mathbb{R}^n)} \left\|\left\|2^{s_2 j} f_j(x)\right\|_{\ell^\infty}\right\|^{1-\theta}_{L^{p_2}(\mathbb{R}^n)}$$
$$= C\|f\|^{\theta}_{\tilde{F}^{s_1}_{p_1,\infty}} \|f\|^{1-\theta}_{\tilde{F}^{s_2}_{p_2,\infty}}.$$

The proof of Lemma (2.3.7) is complete.

We split the statement into two parts; the first one contains the fundamental estimate, the other one deals with the continuity of the product.

Let $0 < s < \infty,\ 1 < q < \infty,\ 1 < p_1 \le \infty, 1 < p_2 \le \infty, 1 < r_1 \le \infty, 1 < r_2 \le \infty$ be such that

$$0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1. \tag{140}$$

Lemma (2.3.8)[76]. We have, for $f \in \tilde{F}^s_{p_1,q} \cap L^{r_1}$ and $g \in \tilde{F}^s_{p_2,q} \cap L^{r_2}$,

$$\|fg\|_{\tilde{F}^s_{p,q}} \le C\left(\left\|Mf(x)\left\|2^{sj} g_j(x)\right\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)} + \left\|Mg(x)\left\|2^{sj} f_j(x)\right\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)}\right) \tag{141}$$

and

$$\|fg\|_{\tilde{F}^s_{p,q}} \le C\left(\|f\|_{\tilde{F}^s_{p_1,q}} \|g\|_{L^{r_2}} + \|g\|_{\tilde{F}^s_{p_2,q}} \|f\|_{L^{r_1}}\right). \tag{142}$$

Proof. We start by noting that (142) follows from (141).
Indeed, using the Hölder inequality we find

$$\left\|Mf(x)\|2^{sj}g_j(x)\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)} + \left\|Mg(x)\|2^{sj}f_j(x)\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)}$$
$$\leq \left\|\|2^{sj}g_j(x)\|_{\ell^q}\right\|_{L^{p_2}(\mathbb{R}^n)}\|Mf(x)\|_{L^{r_1}(\mathbb{R}^n)} + \left\|\|2^{sj}f_j(x)\|_{\ell^q}\right\|_{L^{p_1}(\mathbb{R}^n)}\|Mg(x)\|_{L^{r_2}(\mathbb{R}^n)}$$
$$\leq C\left(\|f\|_{\tilde{F}^s_{p_1,q}}\|g\|_{L^{r_2}} + \|g\|_{\tilde{F}^s_{p_2,q}}\|f\|_{L^{r_1}}\right),$$

by (125). We turn to the proof of (141). It relies on Lemma (2.3.2) which is valid since $1 < p < \infty$and $1 < q < \infty$. We have $fg = \sum_k G_k + \sum_j F_j$, where $G_k = \left(\sum_{j\leq k} f_j\right)g_k, F_j = \left(\sum_{k<j} g_k\right)f_j$.
Since supp $F(F_j) \subset B_{2^{j+2}}$ and supp $F(G_k) \subset B_{2^{k+2}}$, Lemma (2.3.2) yields

$$\|fg\|_{\tilde{F}^s_{p_1,q}} \leq C(A+B), \tag{143}$$

With $A = \|\|2^{sk}G_k(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}, B = \|\|2^{sk}F_k(x)\|_{\ell^q}\|_{L^p(\mathbb{R}^n)}$.We estimate, e.g.

A:$A = \left\|\left\|2^{sk}\left(\sum_{j\leq k} f_j(x)\right)g_k(x)\right\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)} \leq$ by (129)

$$C\left\|Mf_j(x)\|2^{sk}g_k(x)\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)}. \tag{144}$$

We obtain (141) by combining (143), (144) and the similar estimate for B.

Corollary (2.3.9)[76].*We have that:*

(i) *for* $1 < q < \infty, 0 < s < \infty, 1 < p_1 < \infty, 1 < p_2 < \infty, 1 < r_1 < \infty, 1 < r_2 < \infty$, $0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1$, the map$\left(\tilde{F}^s_{p_1,q} \cap L^{r_1}\right) \times \left(\tilde{F}^s_{p_2,q} \cap L^{r_2}\right) \ni (f,g) \mapsto fg \in \tilde{F}^s_{p,q}$ is continuous*;*

(ii) for$1 < q < \infty, 0 < s < \infty, 1 < p < \infty$, *if* $\begin{cases} f^\ell \to f \; in \; \tilde{F}^s_{p,q}, & \|f^\ell\|_{L^\infty} \leq C \\ g^\ell \to g \; in \; \tilde{F}^s_{p,q}, & \|g^\ell\|_{L^\infty} \leq C \end{cases}$

then $f^\ell g^\ell \to fg in \; \tilde{F}^s_{p,q}$*;*

(iii) for $1 < q < \infty, 0 < s < \infty, 1 < p_1 < \infty, 1 < r < \infty, 1 < p < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{r}$, if $\begin{cases} f^\ell \to f \; in \; \tilde{F}^s_{p_1,q}, & \|f^\ell\|_{L^\infty} \leq C \\ g^\ell \to g \; in \; \tilde{F}^s_{p,q} \cap L^r, & \end{cases}$ then $f^\ell g^\ell \to fg in \; \tilde{F}^s_{p,q}$.

Proof.(i) follows directly from (142). Some care is needed when one of the $r_j's$ is ∞. We treat, e.g. case (iii). It clearly sufficesto prove the following two assertions:

(i) if $f^\ell \to 0$ in $\tilde{F}^s_{p_1,q}$ and $\|f^\ell\|_{L^\infty} \leq C$, then$f^\ell g \to 0$ for each $g \in \tilde{F}^s_{p,q} \cap L^r$.

(ii) if $g^\ell \to 0$ in $\tilde{F}^s_{p,q} \cap L^r$, $\|f^\ell\|_{\tilde{F}^s_{p_1,q}} \leq C$, $\|f^\ell\|_{L^\infty} \leq C$, then $f^\ell g^\ell \to 0$.

Assertion (ii) is clear from (142). We prove (i) using (141). We have

$$\|f^\ell g\|_{\tilde{F}^s_{p,q}} \leq C\left(\|f^\ell\|_{\tilde{F}^s_{p_1,q}}\|g\|_{L^r} + \left\|Mf^\ell(x)\|2^{sj}g_j(x)\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)}\right)$$
$$\leq o(1) + C\left\|Mf^\ell(x)\|2^{sj}g_j(x)\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)}. \tag{145}$$

Set $(x) = Mf^\ell(x)\|2^{sj}g_j(x)\|_{\ell^q}$.Then clearly

$$|F^\ell(x)| \leq C\|2^{sj}g_j(x)\|_{\ell^q} \in L^p. \tag{146}$$

On the other hand, $\tilde{F}^{s}_{p_1,q} \hookrightarrow L^{p_1}$ (see [87]). It follows from the maximal inequality (125) that $Mf^{\ell} \to 0$ in L^{p_1} and, up to a subsequence,that $Mf^{\ell} \to 0$ a.e. Then (i) follows from (145) and (146) by dominated convergence.

Theorem (2.3.10)[76]. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $f \in C^m$ be such that $f, f', \dots, f^{(m)} \in L^{\infty}$. Then the map $W^{s,p}(\Omega) \cap W^{1,sp}(\Omega) \ni \psi \mapsto f(\psi) \in W^{s,p}(\Omega)$ is well-defined and continuous. Our original motivation in proving Theorem (2.3.10) comes from the study of properties of the space $X = W^{s,p}(\Omega; S^1) = \{u \in W^{s,p}(\Omega; \mathbb{R}^2);\ |u| = 1 \text{ a.e.}\}$

Proof: The conclusion is well-known when s is an integer (this uses the standard Gagliardo-Nirenberg inequalities). Assume $s$non integer. Clearly, the map$W^{s,p} \cap W^{1,sp} \ni u \mapsto f(u) \in L^p$ is well-defined and continuous, since $f(0) = 0$, f is Lipschitz and $W^{s,p} \hookrightarrow L^p$.

Thus it suffices to prove that the map $W^{s,p} \cap W^{1,sp} \ni u \mapsto D\big(f(u)\big) = f'(u)Du \in W^{s-1,p}$ is well-defined and continuous. With $m = [s] + 1 \geq 2$, we obtain, using (14), that the inclusion

$$W^{s,p} \cap W^{1,sp} \hookrightarrow W^{m-1,\frac{sp}{m-1}} \cap W^{1,sp} \tag{147}$$

is continuous. Applying Theorem (2.3.10) to the integer $s = m - 1 \geq 1$, we find that

if $u^{\ell} \to u$in $W^{s,p} \cap W^{1,sp}$, then $f'\big(u^{\ell}\big) \to f'(u)$in $\tilde{F}^{m-1}_{\frac{sp}{m-1},2} = W^{m-1,\frac{sp}{m-1}}$ and,

$$\big\|f'\big(u^{\ell}\big)\big\|_{L^{\infty}} \leq C. \tag{148}$$

On the other hand, we clearly have that if $u^{\ell} \to u$in $W^{s,p} \cap W^{1,sp}$, then $Du^{\ell} \to Du$in

$$W^{s-1,p} \cap L^{sp} = \tilde{F}^{s-1}_{p,p} \cap L^{sp}. \tag{149}$$

Using (148) and the Gagliardo-Nirenberg type inequality (131) (with $q = p, s = m - 1, \theta = \frac{s-1}{m-1}$, $p = \frac{sp}{m-1}$), we obtain if $u^{\ell} \to u$in $W^{s,p} \cap W^{1,sp}$, then $f'\big(u^{\ell}\big) \to f'(u)$ in $\tilde{F}^{s-1}_{\frac{sp}{s-1},p}$ and

$$\big\|f'\big(u^{\ell}\big)\big\|_{L^{\infty}} \leq C. \tag{150}$$

Finally, by (149), (150), Lemma (2.3.8) and Corollary (2.3.9), we obtain that $f'(u)Du \in \tilde{F}^{s-1}_{p,p} = W^{s-1,p}$ and that if $u^{\ell} \to u$ in $W^{s,p} \cap W^{1,sp}$, then $f'\big(u^{\ell}\big)Du^{\ell} \to f'(u)Du$ in $W^{s-1,p}$.

Theorem (2.3.11)[76]. Assume $1 < s < \infty$, s non integer, $1 < p < \infty, 1 < q < \infty$. Then, for every $f \in R$, the map $\tilde{F}^{s}_{p,q} \cap W^{1,sp} \ni \psi \mapsto f(\psi) \in \tilde{F}^{s}_{p,q}$is well-defined and continuous. There is a natural strategy for proving Theorem (2.3.10): assume, e.g. that $1 < s < 2$ and try to prove that $f'(u)Du \in W^{s-1,p}$.

Set $s = 1 + \sigma$. On the one hand, we have $Du \in W^{\sigma,p} \cap L^{(1+\sigma)p}$. On the other hand, since $u \in W^{1,(1+\sigma)p}$, we find that $f'(u) \in W^{1,(1+\sigma)p} \cap L^{\infty}$. By the "standard" Gagliardo-Nirenberg inequality, we obtain $f'(u) \in W^{\sigma,\frac{1+\sigma}{\sigma}p} \cap L^{\infty}$. The conclusion of Theorem (2.3.10) would follow if we can prove that

$$\left.\begin{matrix} U \in W^{\sigma,p} \cap L^{(1+\sigma)p} \\ V \in W^{\sigma,\frac{1+\sigma}{\sigma}p} \cap L^{\infty} \end{matrix}\right\} \Longrightarrow UV \in W^{\sigma,p}. \tag{151}$$

Using the Gagliardo norm (121), we have to estimate

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x+h)V(x+h)-U(x)V(x)|^p}{|h|^{n+\sigma p}}\,dx\,dh &\leq C\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|V(x)|^p|U(x+h)-U(x)|^p}{|h|^{n+\sigma p}}\,dx\,dh \right.\\ &\qquad \left. + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x)|^p|V(x+h)-V(x)|^p}{|h|^{n+\sigma p}}\,dx\,dh\right) \\ &\leq C\left(\|V\|^p_{L^{\infty}}\|U\|^p_{W^{\sigma,p}} + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x)|^p|V(x+h)-V(x)|^p}{|h|^{n+\sigma p}}\,dx\,dh\right). \end{aligned} \tag{152}$$

It is natural to estimate the last integral in (151) using the Hölder inequality with exponents $1+\sigma$ and $\frac{1+\sigma}{\sigma}$. We find $\|UV\|^p_{W^{\sigma,p}} \le C\left(\|V\|^p_{L^\infty}\|U\|^p_{W^{\sigma,p}} + \|V\|^p_{W^{\sigma,\frac{1+\sigma}{\sigma}p}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|U(x)|^{(1+\sigma)p}}{|h|^n} dx\, dh\right)^{\frac{1}{1+\sigma}}\right)$.

Unfortunately, the last integral diverges, but we are "close" to convergence. In fact, we suspect that (151) is wrong. It is here that the microscopic improvement of the Gagliardo-Nirenberg inequality Lemma (2.3.7), combined with the Runst-Sickel Lemma (2.3.8), magically saves the proof. We make use, in an essential way, of the additional information that $V = f'(u) \in F^{\sigma}_{\frac{1+\sigma}{\sigma}p,p}$.

We conclude this section with a brief survey of earlier results dealing with composition.

(i) if $0 < s \le 1, 1 < p < \infty, f(0) = 0, f$ Lipschitz, then
$u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$(trivial for $s < 1$; see [95] and [96] for $s = 1$);

(ii) if $s = n/p, 1 < p < \infty, f \in R$, where $m = \begin{cases} s, & \text{if } s \text{ is an integer} \\ [s]+1, & \text{otherwise}\end{cases}$,then $u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$.

This result is explicitely stated in[97] ; G. Bourdaud has pointed out that it may also be derived from a result of T. see [86], combined with a result in [98] ,which asserts that, when
$s = n/p, W^{s,p} \hookrightarrow \tilde{F}^{\theta s}_{p/\theta,q}$for $0 < \theta < 1$and every $0 < q < \infty$;

(iii) if $s > n/p, 1 < p < \infty, f(0) = 0$ and $f \in C^m$, then $u \in W^{s,p} \Longrightarrow f(u) \in W^{s,p}$; see [99] for $p = 2$ and [100] for the general case;

(iv) if $1 < s < n/p$, we have to impose additional restrictions on u, if $1 + 1/p < s < n/p$, the only $C^2 f$'s that act on $W^{s,p}$are of the form $f(t) = ct$; see [126] for ***s***integer and [93], for a general s. For $1 < s < n/p$, it follows from Remark (i) in the introduction that Rdoes not act on $W^{s,p}$, since $W^{s,p} \not\subset W^{1,sp}$. A standard additional condition on $u$is $u \in L^\infty$: if $f(0) = 0$ and $f \in C^m$, then $u \in W^{s,p} \cap L^\infty \Longrightarrow f(u) \in W^{s,p}$; (see [100] , [81]).

(v) an improvement is that, for $f$as above and $0 < \sigma < 1$ we have
$u \in W^{s,p} \cap W^{\sigma,sp/\sigma} \Longrightarrow f(u) \in W^{s,p}$;see [97]. This result implies the previous one, since
$W^{s,p} \cap L^\infty \hookrightarrow W^{\sigma,sp/\sigma}$(by Corollary (2.3.4);

(vi) a finer result asserts that, for $f$as above, we have $u \in W^{s,p} \cap \tilde{F}^1_{sp,q}$(with $q \le 2$ sufficiently small depending on $s$and p) $\Longrightarrow f(u) \in W^{s,p}$;see [86].This hypothesis on $u$is weaker than the previous one, since $W^{s,p} \cap W^{\sigma,sp/\sigma} \hookrightarrow \tilde{F}^1_{sp,q}$for all $q > 0$, by Lemma (2.3.7). This result is contained in Theorem (2.3.10), since $\tilde{F}^1_{sp,q} \hookrightarrow W^{1,sp} = \tilde{F}^1_{sp,2}$ as soon as $q \le 2$ (recall that $\tilde{F}^s_{p,q}$increases with q). However, when $p \le 2$ or $1 < s < 2$, that the above smallness condition on $q$is precisely $q \le 2$. This means that Runst and Sickel had established Theorem (2.3.10) when $p \le 2$ or $1 < s < 2$.

(vii) in the framework of Bessel potential spaces
$L^{s,p} = \{f = G_s \star g;\ g \in L^p, \hat{G}_s(\xi) = (1 + |\xi|^2)^{-s/2}\} = \tilde{F}^s_{p,2}$, there are various similar results about composition, starting with [101],[102] when $s > n/p$, [103],[104] and [91] for $H^s \cap L^\infty$ when $s \ge 1$.

The ultimate result for $s \ge 1$was obtained by Adams-Frazier in [104] .
if $1 \le s < \infty, 1 < p < \infty, f \in R$, then $u \in L^{s,p} \cap L^{1,sp} \Longrightarrow f(u) \in L^{s,p}$.
This is a special case $(q = 2)$ of Theorem (2.3.11) since $L^{1,sp} = W^{1,sp}$.

(viii) Other questions concerning composition in Sobolev spaces have been investigated e.g in [105] , [106] , [86].

We state some natural results about products which may be derived from the Runst-Sickel lemma.

Let $1 < p < \infty, 0 < s < \infty, 1 < r < \infty, 0 < \theta < 1, 1 < t < \infty$, be such that $\frac{1}{r} + \frac{\theta}{t} = \frac{1}{p}$.

Corollary (2.3.12)[76]. If $1 < s < \infty, 1 < p < \infty$ and $f \in W^{s,p} \cap L^{\infty}, g \in W^{s-1}, p \cap L^{sp}$,then $fg \in W^{s-1,p}$ and

$$\|fg\|_{W^{s-1,p}} \leq C\left(\|f\|_{L^{\infty}}\|g\|_{W^{s-1,p}} + \|g\|_{L^{sp}}\|f\|_{W^{s,p}}^{1-1/s}\|f\|_{L^{\infty}}^{1/s}\right). \quad (153)$$

In particular, if $f, g \in W^{s,p} \cap L^{\infty}$, then $Dg \in W^{s-1,p} \cap L^{sp}$, so that Corollary (2.3.12) contains as a special case the following result

Corollary (2.3.13)[76]. If $1 < s < \infty, 1 < p < \infty$ and $f, g \in W^{s,p} \cap L^{\infty}$, then $f\, Dg \in W^{s-1,p}$.

Lemma (2.3.14)[76]. For $f \in W^{s,t} \cap L^{\infty}, g \in \cap L^{r}$, we have $fg \in W^{\theta s,p}$and

$$\|fg\|_{W^{\theta s,p}} \leq C\left(\|f\|_{L^{\infty}}\|g\|_{W^{\theta s,p}} + \|g\|_{L^{r}}\|f\|_{W^{s,t}}^{\theta}\|f\|_{L^{\infty}}^{1-\theta}\right). \quad (154)$$

In the special case $s > 1, \theta = \frac{s-1}{s}$, we have $r = sp$ and we obtain the following

Proof: Let $q = 2$ if θs is an integer, $q = p$ otherwise. By (131), we find that $f \in \tilde{F}_{t/\theta,q}^{\theta s}$ and

$$\|f\|_{\tilde{F}_{t/\theta,q}^{\theta s}} \leq C\|f\|_{W^{s,t}}^{\theta}\|f\|_{L^{\infty}}^{1-\theta}. \quad (155)$$

From the Runst-Sickel lemma, we deduce that $fg \in \tilde{F}_{p,q}^{\theta s}$and

$$\|fg\|_{W^{\theta s,p}} = \|fg\|_{\tilde{F}_{p}^{\theta s},q} \leq C\left(\|f\|_{L^{\infty}}\|g\|_{\tilde{F}_{p,q}^{\theta s}} + \|g\|_{L^{r}}\|f\|_{\tilde{F}_{t/\theta,q}^{\theta s}}\right)$$

$$\leq C\left(\|f\|_{L^{\infty}}\|g\|_{W^{\theta s,p}} + \|g\|_{L^{r}}\|f\|_{W^{s,t}}^{\theta}\|f\|_{L^{\infty}}^{1-\theta}\right).$$

Corollary (2.3.15)[236]. If $f \in W^{1+\varepsilon,t} \cap L^{\infty}, g \in \cap L^{r}$, we have $fg \in W^{\varepsilon,p}$ and

$$\|fg\|_{W^{\varepsilon,p}} \leq C\left(\|f\|_{L^{\infty}}\|g\|_{W^{\varepsilon,p}} + \|g\|_{L^{r}}\|f\|_{W^{1+\varepsilon,t}}^{\theta}\|f\|_{L^{\infty}}^{\frac{1}{1+\varepsilon}}\right).$$

In the special case $\varepsilon > 0, \theta = \frac{\varepsilon}{1+\varepsilon}$, we have $r = (1+\varepsilon)p$ and we obtain the following

Proof: Let $q = 2$ if θs is an integer, $q = p$ otherwise.

By (131), we find that $f \in \tilde{F}_{t/\varepsilon/1+\varepsilon,2}^{\varepsilon}$ and

$$\|f\|_{\tilde{F}_{t/\varepsilon/1+\varepsilon,2}^{\varepsilon}} \leq C\|f\|_{W^{1+\varepsilon,t}}^{\frac{\varepsilon}{1+\varepsilon}}\|f\|_{L^{\infty}}^{\frac{1}{1+\varepsilon}}.$$

From the Runst-Sickel lemma, we deduce that $fg \in \tilde{F}_{2,2}^{\varepsilon}$ and

$$\|fg\|_{W^{\varepsilon,2}} = \|fg\|_{\tilde{F}_{2,2}^{\varepsilon}}$$

$$\leq C\left(\|f\|_{L^{\infty}}\|g\|_{\tilde{F}_{2,2}^{\varepsilon}} + \|g\|_{L^{2+2\varepsilon}}\|f\|_{\tilde{F}_{t/\varepsilon/1+\varepsilon,2}^{\varepsilon}}\right)$$

$$\leq C\left(\|f\|_{L^{\infty}}\|g\|_{W^{\varepsilon,2}} + \|g\|_{L^{2+2\varepsilon}}\|f\|_{W^{1+\varepsilon,t}}^{\theta}\|f\|_{L^{\infty}}^{\frac{1}{1+\varepsilon}}\right).$$

Chapter 3
Convergence in the Mean and Necessary Conditions

For weights p(t) and q(t) with a finite number of power-law-type singularities we obtain necessary and sufficient conditions for the inequality to hold, where $s_n^{(p)}(f)$ is a partial sum of the Fourier series function f in terms of polynomials orthogonal on $[-1,1]$ with weight p(t). Some additional properties of the orthogonal polynomials are also shown.

Section(3.1): Fourier Series in Orthogonal Polynomials:

Let $\sigma_p = \{p_k(t)\}_0^\infty$ be a system of polynomials orthonormal on $[-1,1]$ with weight $p(t)$; let $s_n^{(p)}(f) = s_n^{(p)}(f,x)$ be the n-th partial sum of the Fourier series of the function f with respect to the system σ_p; let $L^\eta(R)$ be the space of functions of summable η-th powers ($\eta \geq 1$) on the set

$$R \subset (-\infty,\infty); \|f\|_{L^\eta(R)} = \left\{\int_R |f|^n dt\right\}^{1/n}; \qquad L^\eta = L^\eta(-1,1), \qquad \|f\|_n = \|f\|_{L^\eta}.$$

Throughout this section the numbers η' and η are related by $\eta^{-1} + \eta'^{-1} = t$.

We consider the following problem. Assume that the number η, the weight $p(t)$, and the set of functions $\mathfrak{M} = \{q(t)\}$ are given. We are required to find conditions on the function $q \in \mathfrak{M}$, which are necessary and sufficient for satisfaction of the inequality

$$\left\| s_n^{(p)}(f) q \right\|_n \leq C \|fq\|_\eta \tag{1}$$

for all measurable f with the finite norm $\|fq\|_\eta$ and all $n = 0, 1, \ldots$ where C is some constant. This problem was recently solved in[107] for the case in which the weight $p = (1-t)^\alpha (1+t)^\beta$ and the set $\mathfrak{M}$ consists of the functions $q = (1-t)^A (1+t)^B$, where A and B are arbitrary real numbers and $1 < \eta < \infty$. Special cases (with $q = p^{1/\eta}$ and $q = p^{1/2}$; α and $\beta \geq -1/2$) of the inequality (1) have been treated by [108],[109]. [109] generalize Muckenhoupt's result to the case of the weight

$$p(t) = (1-t)^\alpha (1+t)^\beta \prod_{v=1}^{m} |t - x_v|^{\gamma_v} H(t), \tag{2}$$

Where $-1 < x_1 < \cdots < x_m < 1, \alpha, \beta$ and $\gamma_v > -1 (v = \overline{1,m})$

$$H(t) > 0, \ \ \omega(H,\delta)\delta^{-1} \in L^2(0,2), \tag{3}$$

and the set $\mathfrak{M}$ consists of the functions

$$q(t) = (1-t)^A (1+t)^B \textstyle\prod_{v=1}^{m} |t - x_v|^{\Gamma_v}. \tag{4}$$

We show, moreover, that if the function q is of the form (4) and the weight is that defined by relations (2) and (3), then if q is a solution of the problem stated, $q \in L^\eta$. Therefore from $fq \in L^\eta$ and the inequality (1) there follows, in a known way, the inequality $\left\| \left[f - s_n^{(p)}(f) \right] q \right\|_\eta \leq (1+C) E_n(f)$,where $E_n(f)$ is the minimum of $\|[f - Q_n]q\|_\eta$ on the set of all polynomials Q_n.Thus it follows from inequality (1) that

$$\left\| \left[f - s_n^{(p)}(f) \right] q \right\|_\eta = o(1)(n \to \infty). \tag{5}$$

From the relation (5) and in[110] with $q = p^{1/\eta}$ and it follows that

$$s_n^{(p)}(f,x) - f(x) = o_x(1)(n \to \infty) \tag{6}$$

almost everywhere in $(-1,1)$ if $\omega(H,\delta)\delta^{-1} \in L^{\eta'}(0,2)$.

Lemma(3.1.1)[109]. Let $R=(-\infty,\infty)$ and assume that $K(x,y)=|(x-y)^{-1}|yx^{-1}|^{c}-1|$, where $-1/\eta'<c<1/\eta, 1<\eta<\infty$. Then

$$\left\|\int_R K(x,y)f(y)dy\right\|_{L^\eta(R)} \le C_1(\eta)\|f\|_{L^\eta(R)}. \qquad (7)$$

Proof. In [111] it is sufficient to show that for arbitrary nonnegative $f\in L^\eta(R)$ and $g\in L^{\eta'}(R)$, we have

$$J=\int_R g(x)dx\int_R K(x,y)f(y)dy \le C_1(\eta)\|f\|_{L^\eta(R)}\|g\|_{L^{\eta'}(R)}. \qquad (8)$$

Using Hiölder's inequality for double integrals and Fubini's theorem,

we obtain $J\le\left\{\int_R g^{\eta'}(x)J_1(x)dx\right\}^{1/\eta'}\left\{\int_R f^{\eta}(y)J_2(y)dy\right\}^{1/\eta}$,

where $J_1=\int_R K(x,y)|xy^{-1}|^{1/\eta}dy,\ J_2=\int_R K(x,y)|yx^{-1}|^{1/\eta'}dx$.

Making the substitutions $y=\tau|x|$ and $x=\tau|y|$, we find that

$$2^{-1}J_1(x)\le\int_0^\infty K(|x|,y)(|x|/y)^{1/\eta}dy=|E(-c+1/\eta,1/\eta)|,$$

$$2^{-1}J_2(y)\le\int_0^\infty K(x,|y|)(|y|/x)^{1/\eta'}dx=|E(c+1/\eta',1/\eta')|,$$

where $E(a,b)=\int_0^\infty(\tau^{-a}-\tau^{-b})(1-\tau)^{-1}d\tau$ is Euler's integral, which, as we know, (see [112])assumes finite Values for a and $b\in(0,1)$. Since $-1/\eta'<c<1/\eta$, then J_1 and J_2 are bounded from above by constants and, consequently, the inequality (7) is valid.Consider now the Hilbert transform$G(f)=G\,(f,x)=\int_{-\infty}^{\infty}f(t)(x-t)^{-1}dt$.It is known (see [113]) that for $1<\eta<\infty$

$$\|G(f)\|_{L^\eta(-\infty,\infty)}\le C_2(\eta)\|f\|_{L^\eta(-\infty,\infty)}. \qquad (9)$$

Let $X_R(t)$ be the characteristic function of the set $R\subset(-\infty,\infty)$. We introduce the notation: $G_R(f)=G(f\,X_R)$. Then by virtue of the inequalities (9) and (7), for arbitrary intervals R_1 and $R_2\subset(-\infty,\infty)$ and $1<\eta<\infty$, wehave

$$\left\|G_{R_1}(f)\right\|_{L^\eta(R_1)}\le C_2(\eta)\|f\|_{L^\eta(R_1)}, \qquad (10)$$

$$\left\|\int_{R_1} K(x,y)f(y)dy\right\|_{L^\eta(R_2)} \le C_1(\eta)\|f\|_{L^\eta(R_1)}. \qquad (11)$$

We consider the weights $p(t)$ and $\tau(t)=(1-t^2)p(t)$, and the systems of polynomials, orthonormalized on $[-1,1]$, corresponding to these weights, namely, $\sigma_p=\{p_k(t)\}_0^\infty$ and

$\sigma_\tau=\{\tau_k(t)\}_0^\infty$ As pollard [114]. Showed , for the kernel $K_n^{(p)}(x,t)=\sum_0^n p_k(x)p_k(t)$, we have the valid representation

$$(x-t)K_n^{(p)}(x,t)=a_n(x-t)p_{n+1}(x)p_{n+1}(t)$$
$$+b_n\{(1-t^2)\tau_n(t)p_{n+1}(x)-(1-x^2)\tau_n(x)p_{n+1}(t)\}, \qquad (12)$$

wherein a_n and $b_n=O(1)$, if $(1-t^2)^{-1/2}$ in $p(t)\in L$.

In the considerations which follow we shall also use the following estimate[110] for polynomials of the system σ_p with weight $p(t)$ satisfying the conditions (2)-(3):

$$|p_n(x)|\left(\sqrt{1-x}+n^{-1}\right)^{\alpha+\frac{1}{2}}\left(\sqrt{1+x}+n^{-1}\right)^{\beta+\frac{1}{2}}\le C_3(p)\prod_{v=1}^{m}(|x-x_v|+n^{-1})^{-\gamma_v/2},\ |x|\le 1. \qquad (13)$$

Theorem (3.1.2)[109]. Assume that $1 < \eta < \infty$, that the weight $p(t)$ is defined by the retations (2)-(3), and that $q(t)$ is a function of the form (4). If

$$|A + \eta^{-1} - (\alpha + 1)/2| < \min(1/4, (\alpha + 1)/2), \qquad (14)$$
$$|B + \eta^{-1} - (\beta + 1)/2| < \min(1/4, (\beta + 1)/2), \qquad (15)$$
$$|\Gamma_v + \eta^{-1} - (\gamma_v + 1)/2| < \min(1/4, (\gamma_v + 1)/2), v = \overline{1, m}, \qquad (16)$$

then for all f for which $fq \in L^\eta$, and for all $n = 0, 1, ...$, the inequality (1) holds, where C depends only on p, q, and η.

Proof. From the relations (14)-(16) it follows that $A\eta, (\alpha - A)\eta', B\eta, (\beta - B)\eta', \gamma_v\eta$ and $(\gamma_v - \Gamma_v)\eta' > -1 (v = \overline{1, m})$. Consequently, $q \in L^\eta, pq^{-1} \in L^{\eta'}$, and, by virtue of Hölder's inequality $\|fp\|_1 \le \|fq\|_\eta \|pq^{-1}\|_{\eta'}$ we have $fp \in L,$ if $fq \in L^\eta$. The latter is equivalent to the existence for all $n = 0, 1, ...$ of the partial sums $s_n^{(p)}(f)$.

We now show that inequality (1) holds. We fix the points y_i, satisfying the conditions $-1 < y_0 < x_1 < y_1 < \cdots < x_m < y_m < 1$.

Putting $[-1, 1], I_v = [y_{v-1}, y_v] (v = \overline{1, m}), I_0 = [-1, y_0],\ I_{m+1} = [y_m, 1]$, we have

$$\left\| s_n^{(p)}(f) q \right\|_\eta = \left\| \sum_{v=0}^{m+1} s_n^{(p)}(f) q X_{I_v} \right\|_\eta \le \sum_{v=0}^{m+1} \left\| s_n^{(p)}(f) q \right\|_{L^\eta(I_v)}. \qquad (17)$$

Putting $A_v = [y_{v-1} - \varepsilon, y_v + \varepsilon], v = \overline{1, m}, A_0 = [-1, y_0 + \varepsilon], A_{m+1} = [y_m - \varepsilon, 1]$ ($\varepsilon > 0$, we have $A_v^{(1)} = A_v, A_v^{(2)} = I \backslash A_v\ (v = \overline{0, m + 1})$,

$$g_v = \left\| s_n^{(p)}(f) q \right\|_{L^\eta(\Gamma_v)} \le g_v^{(1)} + g_v^{(2)}, \qquad (18)$$

Where $g_v^{(i)} = \left\| g(x) \int_{A_v^{(i)}} f(t) K_n^{(p)}(x, t) p(t) dt \right\|_{L^\eta(I_v)}, \qquad i = 1,2.$

Taking into account that $|x - t| \ge \varepsilon$, we obtain

$$\varepsilon g_v^{(2)} \le \|fq\|_\eta \{ \|p_{n+1} q\|_\eta \|p_n p q^{-1}\|_{\eta'} + \|p_n q\|_\eta \|p_{n+1} p q^{-1}\|_{\eta'} \}, \qquad (19)$$

if $x \in I_v$ and $t \in A_v^{(2)}$ by virtue of the Christoffel-Darboux formula and Hölder's inequality. From[115] it follows that the conditions (14)-(16) are necessary and sufficient for the simultaneous boundedness as $n \to \infty$ of the norms $\|p_n q\|_\eta$and $\|p_{n+1} p q^{-1}\|_{\eta'}$. Consequently, the right side of inequality (19) does not exceed $C\|fq\|_\eta$.

Putting $u_n = (1 - x^2) q(x) \tau_n(x), V_n = (1 - t^2) \tau_n(t) p(t)$, by virtue of (12),

$$g_{m+1}^{(1)} \le C_1 \{ \left\| q p_{n+1} \int_{A_{m+1}} f p_{n+1} p dt \right\|_{L^\eta(I_{m+1})} + \|q p_{n+1} G_{A_{m+1}} [f v_n]\|_{L^\eta(I_{m+1})}$$
$$+ \left\| u_n G_{A_{m+1}} [f p_{n+1} p] \right\|_{L^\eta(I_{m+1})} \} = C_1 \{ l_1 + l_2 + l_3 \}. \qquad (20)$$

In accord with HiSlder's inequality we have, by virtue of the relations (55)-(57),

$$l_1 \le \|q p_{n+1}\|_\eta \|fq\|_\eta \|p_{n+1} p q^{-1}\|_{\eta'} \le C\|fq\|_\eta. \qquad (21)$$

Assume that $\alpha \ge -\frac{1}{2}$.

Putting $\alpha_1 = \frac{\alpha}{2} + \frac{1}{4} - A, \alpha_2 = \alpha_1 - \frac{1}{2}$, we have, by virtue of inequality (54):

$q|p_{n+1}| \le c(1 - x)^{-\alpha_1}$ and $|u_n| \le c(1 - x)^{-\alpha_2}$, if $x \in I_{m+1}$;

$v_n = q(t) \Phi_n(t) (1 - t)^{\alpha_1}, p_{n+1} p = q(t) \Psi_n(t) (1 - x)^{\alpha_2}$,

where Φ_n and Ψ_n are uniformly bounded for $t \in A_{m+1}$ and n.

Therefore, according to inequalities (10) and (11),

$$l_2 + l_3 \le C\left\|G_{A_{m+1}}[f\Phi_n q\{1-(1-t)^{\alpha_1}(1-x)^{-\alpha_1}\}]\right\|_{L^\eta(I_{m+1})}$$

$$+\left\|G_{A_{m+1}}[f\Psi_n q\{1-(1-t)^{\alpha_2}(1-x)^{-\alpha_2}\}]\right\|_{L^\eta(I_{m+1})}$$

$$+\left\|G_{A_{m+1}}[f\Phi_n q]\right\|_{L^\eta(I_{m+1})} + \left\|G_{A_{m+1}}[f\Psi_n q]\right\|_{L^\eta(I_{m+1})} \le C_1\|fq\|_n, \alpha_1 \text{ and } \alpha_2 \in (-1/\eta', 1/\eta), \quad (22)$$

by virtue of inequality (14), α_1 and α_2 satisfy the bounds required in inequality (22).

Assume now that $-1 < \alpha < -1/2$. We decompose A_{m+1} intothe parts

$T_1 = [y_m - \varepsilon, 1 - n^{-2}]$ and $T_2 = [1 - n^{-2}, 1]$. Then, by virtue of inequality (13), for $t \in A_{m+1}$ we have $v_n = q(t)\Phi_{n_1}(t)(1-t)^{\alpha_1}, n^{\alpha+\frac{1}{2}}v_n = q(t)\Phi_{n_2}(t)(1-t)^{-A}, |\Phi_{n_1}|$ and $|\Phi_{n_2}| \le c$; for $x \in T_1$ we have $|p_{n+1}|q \le c(1-x)^{-\alpha_1}$; for $x \in T_2$ we have

$|p_{n+1}|q \le c(1-x)^A n^{\alpha+\frac{1}{2}}$. Therefore, putting $a_1 = \alpha_1, a_2 = -A$, we obtain

$$l_2 \le \sum_{i=1}^{2}\left\|qp_{n+1}G_{A_{m+1}}[fv_n]\right\|_{L^\eta(T_i)} \le C\sum_{i=1}^{2}\left\|G_{A_{m+1}}[f\Phi_{n_i}(1-t)^{a_i}(1-x)^{-a_i}]\right\|_{L^\eta(T_i)}. \quad (23)$$

Further, by virtue of inequality (13), we have $p_{n+1}p = q(t)\Psi_{n_1}(t)(1-t)^{\alpha_2}$.

$$|\Psi_{n_1}| \le c, t \in T_1; p_{n+1}p = n^{\alpha+\frac{1}{2}}q(t)\Psi_{n_2}(t)(1-t)^{\alpha-A},$$

$$|\Psi_{n_2}| \le c, t \in T_2; |u_n| \le c(1-x)^{-\alpha_2}, n^{\alpha+\frac{1}{2}}|u_n| \le c(1-t)^{A-\alpha}, \quad x \in I_{m+1}.$$

Therefore, putting $b_1 = \alpha_2, b_2 = \alpha - A$, we have

$$l_3 \le \sum_{i=1}^{2}\left\|u_n G_{T_i}[fp_{n+1}p]\right\|_{L^\eta(I_{m+1})} \le c\sum_{i=1}^{2}\left\|G_{T_i}[fq\Psi_{n_i}(1-t)^{b_i}(1-x)^{-b_i}]\right\|_{L^\eta(I_{m+1})}. \quad (24)$$

Since in accord with inequality (14) we have a_i and $b_i \in \left(-\frac{1}{\eta'}, \frac{1}{\eta}\right)$, then, by virtue of the relations (10), (23), and (24), the estimate (22) is also valid for $-1 < \alpha < -\frac{1}{2}$. From the relations (18)-(22) it follows that $g_{m+1} \le C\|fq\|_\eta$.Similarly it may be shown that $g_0 \le C\|fq\|_\eta$. We now show that even when $v = \overline{1, m}$, we have the valid inequality

$$g_v \le C\|fq\|_\eta. \quad (25)$$

By virtue of the Christoffel-Darboux formula

$$g_v^{(1)} \le \left\|qp_{n+1}G_{A_v}[fp_np]_{L^\eta(I_v)}\right\| + \left\|qp_nG_{A_v}[fp_{n+1}p]_{L^\eta(I_v)}\right\| = l_4 + l_5.$$

If $\gamma_v \ge 0$, then, putting $d_v = \frac{\gamma_v}{2} - \Gamma_v$, we have, by virtue of inequality (13)

$$q|p_{n+1}| \le c|x - x_v|^{-d_v}, x \in I_v;\ p_{n+1}p = q(t)\Phi_n(t)|t - x_v|^{d_v}, |\Phi_n(t)| \le c,\ t \in A_v.$$

From inequality (16) it follows that $d_v \in (-1/\eta', 1/\eta)$. Therefore using the relations (10) and (11), we obtain

$$l_4 \le C\left\|G_{A_v}\left[fq\Phi_n\left|\frac{t - x_v}{x - x_v}\right|^{d_v}\right]\right\|_{L^\eta(I_v)} \le C_1\|fq\|_\eta. \quad (26)$$

In a similar way we may obtain the estimate for l_5 also.Thus for $\gamma_v \ge 0$ we may consider the estimate (25) as proven. If $-1 < \gamma_v < 0$, then putting $T_4 = \{x : |x - x_v| \le n^{-1}\}, T_3 = A_v \backslash T_4$ and notingthe estimate $\left|K_n^{(p)}(x,t)\right| \le cn^{\gamma_v+1}(x$ and $t \in T_4)$, we obtain, with the help of Hölder's inequality for $i = j = 4$,

$$\left\|q(x)\int_{T_i} f(t)K_n^{(p)}(x,t)p(t)dt\right\|_{L^\eta(T_j)} \le C\|fq\|_\eta. \quad (27)$$

By virtue of the Christoffei-Darboux formula, we have

$$\left\|q(x)\int_{T_i} f(t)K_n^{(p)}(x,t)p(t)dt\right\|_{L^\eta(T_j)} \le l_4^{i,j} + l_5^{i,j}$$

$$= \left\|qp_{n+1}G_{T_i}[fp_np]\right\|_{L^\eta(T_j)} + \left\|qp_nG_{T_i}[fp_{n+1}p]\right\|_{L^\eta(T_j)}.$$

From this, by analogy with inequality (26), we obtain inequality (27) for $i = j = 3$. Using the estimate $n^{\gamma_v/2} \le |u - x_v|^{-\gamma_v/2}, u \in T_3$, we find, by virtue of the inequality (54), by analogy with inequality (24), that $l_4^{3,4} \le C\left\|G_{T_3}fq\Phi_n|t - x_v|^{-\Gamma_v}|x - x_v|^{\Gamma_v}\right\|_{L^\eta(T_4)}, l_4^{4,3} \le C\left\|G_{T_4}fq\Phi_n|t - x_v|^{\gamma_v-\Gamma_v}|x - x_v|^{\Gamma_v-\gamma_v}\right\|_{L^\eta(T_3)}$. Since by virtue of inequality (16) the numbers $-\Gamma_v$ and $\gamma_v - \Gamma_v$ belong to the interval $(-1/\eta', 1/\eta)$, it then follows from relations (10) and (11) that the right sides of the last two estimates do not exceed $C_1\|fq\|_\eta$. Upon obtaining analogous estimates for $l_5^{3,4}$ and $l_5^{4,3}$, we may convince ourselves of the validity of the estimate (27) in each of the cases: $i = 3, j = 4$ and $i = 4, j = 3$. From (18), (19), and (27) we obtain inequality (25).

Theorem(3.1.3)[109]. Assume that $1 < \eta < \infty$, that the weight $p(t)$ is defined by the conditions (2)-(3), and that $q(t)$ is a function of the form (4). If there exists a constant C such that inequality (24) is satisfied for all measurable f with finite norm $\|fq\|_\eta$ and for all $n = 0, 1, \dots$, then the inequalities (14)-(16) are valid.

Proof. Estimating, by virtue of inequality (1), the n-th term of the Fourier series of the function f sign $\{fp_n\}$ with respect to the system σ_p, we have $\|(fq)p_npq^{-1}\|_1 \le 2C\left(\|p_nq\|_\eta\right)^{-1}\|fq\|_\eta$. From [111] it follows that

$$\|qp_n\|_\eta\|p_npq^{-1}\|_{\eta'} \le 2C. \tag{28}$$

Let $\pi_n = \pi_n(t)$ denote an arbitrary polynomial of degree $\le n$. By the inequalities of Zolotarev-Korkin and Hölder, we have $2^{1-n} \le \|t^n + \pi_{n-1}\|_1 \le \|t^n + \pi_{n-1}\|_\eta\|1\|_{\eta'}$.

From this, considering the function $q(t)$ of the form (4) as the product of a polynomial by a function bounded in absolute value from below by a positive constant, we obtain

$$\|[t^n + \pi_{n-1}]\, q\|_\eta \ge c\,(q)2^{-n} > 0. \tag{29}$$

Assume that $p_n(t) = k_n^{(p)}t^n + \cdots$. If $(1 - t^2)^{-1/2}\ln p(t) \in L$,then in [116] $2^{-n}k_n^{(p)}$ tends to a finite positive limit; therefore it follows from inequalities (70) that

$$\|p_nq\|_\eta \ge c(p,q) > 0\ (\eta \ge 1). \tag{30}$$

From inequalities (28) and (30) it follows that $\|p_nq\|_\eta$ and $\|p_npq^{-1}\|_{\eta'} = O(1)$. Necessary and sufficient conditions for simultaneous boundedness (as $n \to \infty$). Of these norms are given in [115] They consist in the satisfaction of the inequalities (14)-(16). This completes the proof of the theorem.

Corollary(3.1.4)[109]. Assume that $1 < \eta < \infty$, and that p and q are the functions appearing in Theorem (3.1.3). If at least one of the relations (14)-(16) is not satisfied, then a function f can be found, with a finite norm $\|fq\|_\eta$, for which the relation (5) is invalid.

Proof. First of all we note that violation of the conditions (14)-(16) may lead to a situation in which an f exists for which $fq \in L^\eta$, but such that $fp \in L$. One cannot then form the partial sums $s_n^{(p)}(f)$ and to speak of the relation (5). If, however, we can construct, for all f for which $fq \in L^\eta$, the Fourier series with respect to the system σ_p, such that for each such function the relation (5) holds, then by virtue of the Banach-Steinhaus theorem the inequality (1) must hold (with an absolute constant C).

However, this contradicts Theorem (3.1.3). We remark that throughout the above the function $q(t)$ of the form (4) could obviously be multiplied by a fixed measurable function $H_1(t)$, satisfying on $[-1,1]$ the conditions $0 < m_1 \le H_1(t) \le M_1 < \infty$. In particular, in(3.1.2) we could consider the case in which $q = p^{1/\eta}$, where p is the weight defined by (2),(3). If in this case it be required that the condition

$$\omega(H,\delta)\delta^{-1} \in L^{\eta'}(0,2), \ \ \eta' > 1, \tag{31}$$

be satisfied in[110] the relation (6) will follow from the relation (5) almost everywhere in $(-1,1)$ for each function $f \in L_p^\eta$, where L_p^η is the class of functions f for which $fp^{1/\eta} \in L^\eta$. Assume that $H(t) \in \mathrm{Lip}\ 1$ on $[-1,1]$. Then (31) is satisfied for all $\eta' \ge 1$. Taking $A = \frac{\alpha}{\eta}, B = \frac{\beta}{\eta}$, $\Gamma_v = \gamma_v/\eta\ (v = \overline{1,m})$, we rewrite the conditions (14),(16), which are sufficient for (46), in the form

$$(\alpha+1)\left|\eta^{-1} - \tfrac{1}{2}\right| < \min\left(\tfrac{1}{4}, (\alpha+1)/2\right), \tag{32}$$

$$(\beta+1)\left|\eta^{-1} - \tfrac{1}{2}\right| < \min\left(\tfrac{1}{4}, (\beta+1)/2\right), \tag{33}$$

$$(\gamma_v+1)\left|\eta^{-1} - \tfrac{1}{2}\right| < \min\left(\tfrac{1}{2}, (\gamma_v+1)/2\right), v = \overline{1,m}. \tag{34}$$

Let $M = M(\alpha,\beta,\gamma_1,\dots,\gamma_m)$ denote the largest of the numbers $1, 4(\alpha+1)(2\alpha+3)^{-1}, 4(\beta+1)(2\beta+3)^{-1}, 2(\gamma_v+1)(\gamma_v+2)^{-1}(v = \overline{1,m})$. Then from relations (31)-(34) it follows that if $H(t) \in \mathrm{Lip}\ 1$, then relation (6) holds almost everywhere in $(-1,1)$ for all $\eta > M$ for each function $f \in L_\eta^p$. We note that if α and $\beta \le \frac{1}{2}, \gamma_v \le 0\ (v = \overline{1,m})$, then $M = 1$. In the remaining cases, $M > 1$. Assume that $M > 1$. In [110] it was shown that in the case of the system $\sigma_p = \sigma^{\alpha,\beta} = \left\{\hat{p}_k^{\alpha,\beta}(t)\right\}_0^\infty$, where $\hat{p}_k^{\alpha,\beta}(t)$ are Jacobi polynomials, orthonormal on $[-1,1]$ with the weight $p = (1-t)^\alpha(1+t)^\beta$, for $\eta < M$ for some function $f \in L_p^\eta$ does not held almost everywhere in $(-1,1)$. We consider an analogous example here, for the case of the weight $p = (1-t^2)^\alpha |t|^\gamma$, of a function having a singular point inside $(-1,1)$. Namely, we show that if $\alpha > -\frac{1}{2}, \gamma > -1$, $\eta < 4(\alpha+t)(2\alpha+3)^{-1}$ or 2) $\alpha > -1, \gamma > 0, \eta < 2\,(\gamma+1)(\gamma+2)^{-1}$, then there exists a function $f \in L_p^\eta$, for which the relation (6) does not hold everywhere in $(-1,1)$. Indeed, for $p = (t-t^2)|t|^\gamma$, we have

$$p_{2k}(x) = 2^{(2\alpha+\gamma+1)/2}\hat{p}_k^{\alpha,(\gamma-1)/2}(2x^2-1). \tag{35}$$

By virtue of (6) the Fourier series of the functions $(1-t^2)^\mu$ and $|t|^{2\mu}$, with respect to the system σ_p, at the point $x \in (-1,1)$ are carried over, respectively, into the Fourier series of the functions $f(u) = c_1(1-u)^\mu$, with respect to the system $\sigma^{\alpha,(\gamma-1)/2}$, at the point $(2x^2-1)$. However, as was shown in [110] for $\alpha > -1, \beta > -1/2$ and $\eta < 4(\beta+1)(2\beta+3)^{-1}$ the exponent μ can be chosen so that $f \in L_p^\eta$ with $g \in L_p^\eta, p = (1-t)^\alpha$ and moreover, the Fourier series of the function f with respect to the system $\sigma^{\alpha,\beta}$ is everywhere divergent in $(-1,1)$. For the case, however, in which $\alpha > -1, \beta > -1/2$ and $\eta < 4(\beta+1)(2\beta+3)^{-1}$, the number μ can be chosen so that $g \in L_p^\eta, p = (1-t)^\alpha(1+t)^\beta$, but nevertheless the Fourier series of the function g with respect to the system $\sigma^{\alpha,\beta}$ is everywhere divergent in $(-1,1)$. Taking note of the fact that $\beta = (\gamma-1)/2$ and reverting to the variable t, we obtain interesting examples of functions for which (6) is not valid almost everywhere in $(-1,1)$. Note Added in Proof. Subsequent to the submission of this section to the printers by [117] and [118] appeared in the literature, having a direct bearing on the content of the present .

In [117] stated concerning the validity of our Theorem (3.1.3) for the particular case of it for which $H(t) \equiv 1$ and $\beta \geq -12,\ \gamma_v \geq 0\ (v = \overline{1,m}), q(t)^{1/\eta} = \{p(t)\}$. Deduced the notion of the proof of this hypothesis for $p(t) = |t\gamma|$. Using Jacobi polynomials as an example, Askey discovered the importance of statements of the type of Theorem (3.1.3) for solving the problem of convergence in the mean of Lagrange interpolational polynomials which coincide with a continuous function at the zeros of the orthogonal polynomials.

In [118],[119] and [120] the convergence almost everywhere of the trigonometric Fourier series of the function $f(x) \in L^{\eta}(-\pi,\pi)(\eta > 1)$, used a theorem on uniform convergence due to Szegö and an example due to Szegö of a diverging Fourier-Jacobi in [116] series to find that the Fourier-Legendre series of the function $f(x) \in L^{\eta}$ for $\eta > 4/3$ converges almost everywhere in $(-1, 1)$, and for $1 \leq \eta < 4/3$, that it may diverge almost everywhere in $(-1, 1)$. Pollard noted that an analogous result may be obtained also in the more general case of Jacobi polynomials. We remark that this result for $\eta \geq 2$ was obtained in [121] even for the generalized Jacobi polynomials. For $\eta > 1$ this result was obtained in[110] for the Jacobi polynomials without application of Szegö's theorem but using the results of [107] in particular, it is also contained among the results of this section.

Section(3.2): Convergence of Fourier Series in Orthogonal Polynomials:

Let $d\alpha$ be a finite positive Borel measure on the real line such that $\operatorname{supp}(d\alpha)$ is an infinite set and let $p_n(d\alpha)$ denote the corresponding orthonormal polynomials. For $f \in L^1_{d\alpha}$ let $S_n(d\alpha, f)$ denote the nth partial sum of the orthogonal Fourier expansion of f in $\{p_k(d\alpha)\}$, that is,

$$S_n(d\alpha, f) = \sum_{k=0}^{n} c_k p_k(d\alpha)\,, \qquad c_k = \int_{-1}^{1} f p_k(d\alpha) d\alpha.$$

It is well known [122] that $S_n(d\alpha, f) \to f$ in $L^2_{d\alpha}$ as $n \to \infty$ for every $f \in L^2_{d\alpha}$ if and only if the moment problem for $d\alpha$ possesses a unique solution, and the latter is certainly the case whenever $\operatorname{supp}(d\alpha)$ is bounded. The problem of weighted mean convergence of $S_n(d\alpha, f)$ to S in spaces different from $L^2_{d\alpha}$ has not yet been resolved with the exception of some specific orthogonal polynomial systems. For example, if $d\alpha$ and $d\beta$ are generalized Jacobi measures, then there is a necessary and sufficient conditions for $L^p_{d\beta}$ convergence of $S_n(d\alpha, f)$ to f for every $f \in L^p_{d\beta}$. Badkov's results generalize earlier ones by [123], [124], Newman and Rudin [125], Muckenhoupt [126]. Askey [127] , and Badkov [128]. Orthogonal Hermite and Laguerre scrics were investigated in Askey and Wainger [129] . In [130] one of us found necessary conditions for $L^p_{d\beta}$ convergence of $S_n(d\alpha, f)$ when $d\alpha$ belongs to the Szegii class [131], that is, when $\operatorname{supp}(d\alpha) = [-1,1]$ and $\log \alpha'(\cos\theta) \in L^1[0,\pi]$. In the particular case when $d\alpha$ and $d\beta$ are generalized Jacobi measures, these conditions turn out to be sufficient as well [132]. We laid foundation to a theory of orthogonal polynomials that extends Szegö's theory when $\log \alpha'(\cos\theta) \in L^1[0,\pi]$ is replaced by the weaker condition that $\alpha' > 0$ a.c. in $[-1,1]$.
Our results enable us to prove the following generalization in [132].

Theorem(3.2.1)[133]. Let α be such that $\operatorname{supp}(d\alpha) = [-1,1]$ and $\alpha' > 0$ almost everywhere in $[-1,1]$. Assume that p and q satisfy $0 < p \leq \infty$ and $1 \leq q \leq \infty$. Let u and w be Borel-measurable functions such that neither of them vanishes almost everywhere in $[-1,1]$ and u is finite on a set with positive Lebesgue measure.

Write $q' = q/(q-1)$ and $v(x) = \left(\alpha'(x)\sqrt{1-x^2}\right)^{1/2}$.

Suppose that for every function $f \in L^1_{d\alpha}$ the inequality

$$\left(\int_{-1}^{1} |w/v|^p \alpha'\right)^{1/p} \leq C\left(\int_{-1}^{1} |fu|^q d\alpha\right)^{1/q} \tag{36}$$

holds for all integers $n \geq 0$ with a finite constant C indepcndmt of n and f (if $f(x) = 0$ and $u(x) = \infty$, then $f(x)\,u(x) = 0$ is to be taken in the integral on the right-hand side).
Then $w \in L^p_{d\alpha}$, $u^{-1} \in L^q_{d\alpha}$

$$\left(\int_{-1}^{1} |w/v|^p \alpha'\right)^{1/p} < \infty, \tag{37}$$

and

$$\left(\int_{-1}^{1} |uv|^q \alpha'\right)^{1/q} < \infty. \tag{38}$$

Here what follows, for $p = \infty$ the expression $(\int |g|^p d\alpha)^{1/p}$ means the $L^\infty_{d\alpha}$ norm of g.
It may be worth pointing out that if $0 < p < \infty, 1 < q < \infty$, and $p \leq q$ then in every known case (37) and (38) are also sufficient conditions for (36) to be satisfied see[128].
Proof: For $n = 0$, inequality (36) implies

$$\left(\int_{-1}^{1} |w|^p\, d\alpha\right)^{1/p} \left|\int_{-1}^{1} f\, d\alpha\right| \leq Cp_0^{-2}\left(\int_{-1}^{1} |fu|^q\, d\alpha\right)^{1/q} \tag{39}$$

for every $f \in L^1_{d\alpha}$. Since u is finite on a set of positive measure, we can find a Borel set E and a positive number N such that $d\alpha(E) > 0$ and $u(x) \leq N$ for $x \in E$. If f is the characteristic function of this set E then (39) shows that $w \in L^p_{d\alpha}$. If $1 < q \leq \infty$ then we can apply (39) with $f = (|u| + \varepsilon)^{-q'}$, where $\varepsilon > 0$ and $q' = q/(q-1)$; if we let $\varepsilon \to 0$, then $u^{-1} \in L^{q'}_{d\alpha}$ will follow by Fatou's lemma.
If $q = 1$, then we apply (39) with $f = f_n$ being the characteristic function of the set where $|u^{-1}| > 1/n$; we obtain a contradiction unless $f_n = 0$ a.e. for large enough n; thus, we can conclude that $u^{-1} \in L^\infty_{d\alpha}$. Thus we have $u^{-1} \in L^{q'}_{d\alpha}$ for $1 \leq q \leq \infty$ (q is fixed), as claimed. Therefore
$f = (fu)u^{-1} \in L^1_{d\alpha}$ also holds whenever $fu \in L^q_{d\alpha}$ $(1 \leq q \leq \infty)$.
Moreover, it follows from (39) that $\left(\int_{-1}^{1} |[S_n(f) - S_{n-1}(f)]w|^p d\alpha\right)^{1/p} \leq 2^{1+1/p} C \left(\int_{-1}^{1} |fu|^q d\alpha\right)^{1/q}$ holds for $n \geq 1$ and $f \in L^1_{d\alpha}$. Hence we have

$$\int_{-1}^{1} |p_n w|^p d\alpha \left|\int_{-1}^{1} f p_n d\alpha\right| \leq 2^{1+1/p} C\left(\int_{-1}^{1} |fu|^q d\alpha\right)^{1/q} \tag{40}$$

for $n \geq 1$ and $f \in L^1_{d\alpha}$. Fix n and choose g such that

$$gp_n \geq 0 \quad \text{and } |gu|^q = |p_n u^{-1}|^{q'}, \tag{41}$$

i.e., $g = \left(|p_n|^{q'} u^{-q-q'}\right)^{1/q} (g(x) = 0 \text{ if } u(x) = \infty)$.
Put $E = \{x \in [-1,1]: g(x) \neq 0\}$. Let $E_k \subset E$ be a Borel set and h_k its characteristic function such that $h_k(x) \to 1$ as $k \to \infty$ for $x \in E, gu \in L^q_{d\alpha}(E_k)$, i.e., $h_k gu \in L^q_{d\alpha}[-1,1]$, for every k.

Then $h_k g \in L^1_{d\alpha}[-1,1]$ according to the last sentence of the preceding paragraph, i.e., (40) holds with $f = f_k = h_k g$. Noting that we have$f_k p_n = |f_k u||p_n u^{-1}| = |f_k u|^q = |p_n u^{-1}|^{q'}$ on E_k according to (41), the equality $\int_{E_k} f_k p_n d\alpha = \left(\int_{E_k} |f_k u|^q d\alpha\right)^{1/q} \left(\int_{E_k} |p_n u^{-1}|^{q'} d\alpha\right)^{1/q'}$ holds. Thus (40) with $f = f_k$ implies $\left(\int_{-1}^{1} |p_n w|^p d\alpha\right)^{1/p} \left(\int_{E_k} |p_n u^{-1}|^{q'} d\alpha\right)^{1/q'} \le 2^{1+1/p} C$.

Making $k \to \infty$ and replacing E with $[-1,1]$ in the second integral ($u^{-1} = 0$ outside E), we obtain $\left(\int_{-1}^{1} |p_n w|^p d\alpha\right)^{1/p} \left(\int_{-1}^{1} |p_n u^{-1}|^{q'} d\alpha\right)^{1/q'} \le 2^{1+1/p} C$for all $n \ge 1$ $\left(q' = q/(q-1)\right)$.

By (42) in Theorem (3.2.3) this implies that $\sup_{n\ge1} \left(\int_{-1}^{1} |p_n w|^p d\alpha\right)^{1/p} < \infty$ and $\sup_{n\ge1} \left(\int_{-1}^{1} |p_n u^{-1}|^{q'} d\alpha\right)^{1/q'} < \infty$, and now inequalities (37) and (38) follow from Theorem (3.2.3).

For orthogonal polynomials on the unit circle, the analogue of Theorem (3.2.3) can be derived without much difficulty from [134], and therefore one can easily formulate and prove a result similar to Theorem (3.2.1) for weighted mean boundedness of Fourier expansions in orthogonal polynomials on the unit circle.We expect that Theorem (3.2.3) and the Lemma above will have further applications. In fact, we believe that these two statements will Play a significant role in the extension of Szegö's theory we initiated in [134].

Lemma(3.2.2)[133] Let $\text{supp}(d\alpha) = [-1,1]$ and $\alpha' > 0$ a.e. in $[-1,1]$. For a given real c and a nonnegative integer n define the set $B_{c,n}(d\alpha)$ by

$$B_{c,n}(d\alpha) = \left\{x: p_n^2(d\alpha, x)\alpha'(x)\sqrt{1-x^2} \ge c\right\}. \tag{42}$$

Then for every $c > 2/\pi$

$$\lim_{n\to\infty} \left|B_{c,n}(d\alpha)\right| = 0, \tag{43}$$

where $|E|$ denotes the Lebesgue measure of the set E.

Proof. Write $\Omega_n(x) = p_n^2(x) - 2xp_n(x)p_{n-1}(x) + p_{n-1}^2(x)$.

Then $\Omega_n = (xp_n - p_{n-1})^2 + (1-x^2)p_n^2$, so that $(1-x^2)p_n^2(x) \le \Omega_n(x)$.

Therefore, if $D_{c,n}(d\alpha)$ is defined by $D_{c,n}(d\alpha) = \left\{x: \Omega_n(x)\alpha'(x)(1-x^2)^{-1/2} \ge c\right\}$ then $B_{c,n} \subset D_{c,n}$. It was shown in [134] that $\lim_{n\to\infty} \int_{-1}^{1} \left|\Omega_n(x)\alpha'(x) - \frac{2}{\pi}\sqrt{1-x^2}\right| dx = 0$.

Hence, for $c > 2/\pi$ $\lim_{n\to\infty} \int_{D_{c,n}} \left(\Omega_n(x)\alpha'(x) - \frac{2}{\pi}\sqrt{1-x^2}\right) dx = 0$ holds, so that $\lim_{n\to\infty} \left(c - \frac{2}{\pi}\right) \int_{D_{c,n}} \sqrt{1-x^2} dx = 0$, from which $\lim_{n\to\infty} \left|D_{c,n}\right| = 0 \quad (c > 2/\pi)$ follows.

Thus (43) must indeed hold.

Theorem (3.2.3)[133]. Let $\text{supp}(d\alpha) = [-1,1]$, $\alpha' > 0$ almost everywhere in $[-1,1]$,and suppose $0 < p \le \infty$.

Put $v(x) = \left(\alpha'(x)\sqrt{1-x^2}\right)^{1/2}$.

If g is a Lebesgue-measurable function in $[-1,1]$ then

$$\left(\int_{-1}^{1} |g/v|^p\right)^{1/p} \le \sqrt{\pi} 2^{\max\{1/p-1/2,0\}} \liminf_{n\to\infty} \left(\int_{-1}^{1} |g p_n(d\alpha)|^p\right)^{1/p}. \tag{44}$$

In particular, if

$$\lim_{n\to\infty}\inf\left(\int_{-1}^{1}|gp_n(d\alpha)|^p\right)^{1/p} = 0 \tag{45}$$

then $g = 0$ a.e.

Proof. First assume $0 < p \leq 2$. Define r_n and h by $r_n = v^2 p_n^2(d\alpha)$ and $h = (|g|/v)^p$, respectively.

Let

$$K = \lim_{n\to\infty}\inf\left(\int_{-1}^{1}|gp_n(d\alpha)|^p\right)^{1/p}.$$

If $K = \infty$ then there is nothing to prove, so assume $K < \infty$.

Then

$$\lim_{n\to\infty}\inf\int_{-1}^{1} hr_n^{p/2} = K^p$$

holds; therefore, if h_M is defined by $h_M(x) = \min\{h(x), M\}$ for $M > 0$, then

$$\lim_{n\to\infty}\inf\int_{-1}^{1} hr_n^{p/2} \leq K^p \tag{46}$$

is satisfied as well. Fix $c > 2/\pi$. If $c > 2/\pi$. If $B_{c,n}$ is defined by (42) then (43) is holds,in [134] implies

$$\lim_{n\to\infty}\int_{B_{c,n}} h_M r_n = 0. \tag{47}$$

Applying in [134] , we obtain

$$\lim_{n\to\infty}\int_{B_{c,n}} h_M r_n = \frac{1}{\pi}\int_{-1}^{1} h_M. \tag{48}$$

Consequently,

$$\lim_{n\to\infty}\int_{[-1,1]\backslash B_{c,n}} h_M r_n = \frac{1}{\pi}\int_{-1}^{1} h_M \tag{49}$$

holds as well. On the other hand, $0 < r_n(x) < c$ is satisfied for $x \in [-1,1]\backslash B_{c,n}$, so that $0 \leq c^{p/2-1} r_n \leq r_n^{p/2}\left(x \in [-1,1]\backslash B_{c,n}\right)$ holds.

Thus by (46) we have

$$\lim_{n\to\infty}\inf\int_{[-1,1]\backslash B_{c,n}} h_M r_n \leq c^{1-p/2}K^p,$$

and combining this inequality with (48) we obtain

$$\int_{-1}^{1} h_M \leq \pi c^{1-\frac{p}{2}}K^p$$

for every $M > 0$ and $\varepsilon > 0$.

Letting $M \to \infty$ here and applying Legesgue's Monotone Convergence Theorem, and then $c \to 2/\pi$, we can conclude that $\left(\int_{-1}^{1} h\right)^{1/p} \le 2^{1/p-1/2}\sqrt{\pi}K$, and so the theorem follows for $0 < p \le 2$. When $2 < p < \infty$ we can proceed as follows. (the arguments below closely parallel those given in [130]) Keeping the previously established notation, from Holder's inequality we obtain

$$\int_{-1}^{1} h_M r_n = \int_{-1}^{1} h_M^{(p-2)/p}\left(h_M^{2/p} r_n\right) \le \left(\int_{-1}^{1} h_M\right)^{(p-2)/p} \left(\int_{-1}^{1} h_M r_n^{p/2}\right)^{2/p}.$$

Hence

$\liminf_{n\to\infty} \int_{-1}^{1} h_M r_n \le K^2 \left(\int_{-1}^{1} h_M\right)^{(p-2)/p}$. which together with (47) implies $\left(\int_{-1}^{1} h_M\right)^{1/p} \le \sqrt{\pi}K$.

Letting $M \to \infty$, Lebesgue's Monotone Convergence Theorem entails $\left(\int_{-1}^{1} h\right)^{1/p} \le \sqrt{\pi}K$, so that the theorem follows for $2 < p < \infty$ as well. Finally. assume $p = \infty$, and let $1 < q < \infty$. Clearly, we have $\left(\int_{-1}^{1} |f|^q\right)^{1/q} \le 2^{1/q}\text{css.}\sup_{[-1,1]}|f| = 2^{1/q}\left(\int_{-1}^{1}|f|^p\right)^{1/p}$, where the equation holds in view of the convention concerning the interpretation of the right-hand side for $p = \infty$.

Therefore, inequality (44) with q replacing p implies

$$\left(\int_{-1}^{1} |g/v|^q\right)^{1/q} \le \sqrt{\pi}2^{1/q} \liminf_{n\to\infty}\left(\int_{-1}^{1} |gp_n(d\alpha)|^p\right)^{1/p} \quad (p = \infty, 1 < q < \infty).$$

Making $q \to \infty$, inequality (44) follows for $p = \infty$ as well. Thus the proof of (3.2.3) is complete.

Theorem (3.2.4)[133]. If $\operatorname{supp}(d\alpha) = [-1, 1]$, $\alpha' > 0$ almost everywhere in $[-1, 1]$then

$$\sum_{k=0}^{\infty} |c_k p_k(d\alpha, x)| \tag{50}$$

either diverges or converges almost everywhere in $[-1, 1]$, and in the latter case

$$\sum_{k=0}^{\infty} |c_k| \ge \infty \tag{51}$$

holds as well,

Proof. By Theorem (3.2.3) with $p = 1$, we have

$$\liminf_{n\to\infty} \int_E p_n(d\alpha) \ge \frac{1}{\sqrt{2\pi}} \int_E v^{-1} > 0 \tag{52}$$

for every set E with positive Lebesgue measure. Now assuming that (50) converges on a set $E \subset [-1, 1], |E| > 0$, one can apply (52) and the usual arguments used to prove the Denjoy-Lusin theorem on absolute convergence of trigonometric series [135]. These give (51), from which the convergence of (49) almost everywhere in $[-1, 1]$ follows by Lebesgue's Monotone Convergence Theorem.

Chapter 4
Location and nth Root in Weierstrass' Theorem

For a wide class of Sobolev orthogonal polynomials, it is shown that their zeros are contained in a compact subset of the complex plane and the asymptotic zero distribution is obtained. With a certain information, the nth root asymptotic behavior outside the compact set containing all the zeros is given. For a big class of weights $w_0, w_1, \ldots, w_k$ (even non-bounded) weights w_j. We allow a great deal of independence among the weights w_j.

Section (4.1): Asymptotics of Sobolev Orthogonal Polynomials:

(i) Let $\{\mu_k\}_{k=0}^{m}$ be a set of $m+1$ finite positive Borel measures.

For each $k = 0, \ldots, m$ the support Δ_k of μ_k is a compact subset of the real line $\mathbb{R}$. We will assume that Δ_0 contains infinitely many points. On the space of all polynomials, we consider

$$\langle p, q \rangle_S = \sum_{k=0}^{m} \int p^{(k)}(x) q^{(k)}(x) d\mu_k(x) = \sum_{k=0}^{m} \langle p^{(k)} q^{(k)} \rangle_{L_2(\mu_k)}, \tag{1}$$

where p, q are polynomials. As usual, $f^{(k)}$ denotes the kth derivative of a function f.

Obviously, (1) defines an inner product on the linear space of all polynomials. Therefore, a unique sequence of monic orthogonal polynomials is associated to it.By Q_n, we will denote the corresponding monic orthogonal polynomial of degree n. The sequence $\{Q_n\}$ is called the sequence of Sobolev monic orthogonal polynomials relative to (1). Sobolev orthogonal polynomials have attracted much attention in the past two decades. Recently, some important results have been obtained regarding their asymptotic behavior. In this direction in [136], an important step was taken in the study of the so-called discrete Sobolev inner product; that is, when μ_0 is the only measure containing infinitely many points in its support. When $\mu_0' > 0$ a.e. on its support which consists of an interval, the authors find the relative asymptotic behavior between the Sobolev orthogonal polynomials and the orthogonal polynomials associated with μ_0 (in fact, they consider a more general class of product not necessarily positive definite). Thus, the asymptotic behavior of discrete Sobolev orthogonal polynomials is reduced to the case when the inner product solely contains the measure μ_0.In[137] With $m = 1$, the authors assume that $\mu_0, \mu_1 \in$ **Reg** in [138] and that their supports are regular sets (a compact subset of the complex plane is said to be regular if the unbounded connected component of its complement is regular with respect to the Dirichlet problem). Under these assumptions, they find the asymptotic zero distribution of the zeros of the derivatives of the Sobolev orthogonal polynomials and also of the proper sequence of Sobolev orthogonal polynomials when $\Delta_0 \supset \Delta_1$. Finally, in [139] with $m = 1$, for a wide class of Sobolev products defined on smooth curves of the complex plane, the author gives the strong asymptotics of the corresponding Sobolev orthogonal polynomials.

In contrast with the case of classical orthogonality with respect to a measure, where it is easy to prove that the zeros of the orthogonal polynomials lie on the convex hull of the support of the measure, the location of the zeros of Sobolev orthogonal polynomials in the complex plane for general Sobolev inner products seems to be a difficult problem. Thus,it is not possible to derive from the results in [137], the (uniform) nth root asymptotic behavior of the Sobolev orthogonal polynomials. The main question considered in this section is the study of the location of the zeros of Sobolev orthogonal polynomials. Under general assumptions on the measures involved in the inner product, [140] prove that the zeros of the Sobolev orthogonal polynomials are contained in a compact subset of the complex plane.

This is done and making use of methods from the theory of bounded operators. In following the ideas in[202] we extend some results of $m \geq 2$.This extension together with the results allow us to give the nth root asymptotic behavior of Sobolev orthogonal polynomials for a wide class of Sobolev orthogonal polynomials.

(ii) Before proceeding, let us fix some assumptions and additional notation.

As above, (12) defines an inner product on the space P of all polynomials. The norm of $p \in P$ is

$$\|p\|_S = \left(\sum_{k=0}^{m} \int \left(p^{(k)}\right)^2(x) d\mu_k(x)\right)^{1/2} = \left(\sum_{k=0}^{m} \|p^{(k)}\|_{L_2(\mu_k)}^2\right)^{1/2} \tag{2}$$

We will denote by $\left(H_{2,m}, \|\cdot\|_S\right)$ the Banach space obtained completing the normed space $(P, \|\cdot\|_S)$.

As usual, this is done identifying all Cauchy sequences of polynomials whose difference tends to zero in the norm $\|\cdot\|_S$. Certainty, $H_{2,m}$ heavily depends on the measures involved in the inner product, but for simplicity in the notation we will not indicate it. For $f \in H_{2,m}$, $\|f\|_S$ is defined by continuity; that is, $\|f\|_S = \lim_{n\to\infty} \|p_n\|_S$,where $\{p_n\}$ is a representative of f. On $H_{2,m}$, we consider the inner product

$$\langle f, g\rangle_S = \frac{1}{2}[\|f + g\|_S^2 - \|f\|_S^2 - \|g\|_S^2], \qquad f, g \in H_{2,m}. \tag{3}$$

Therefore, $\left(H_{2,m}, \langle\cdot,\cdot\rangle_S\right)$ is a separable Hilbert space because by construction the space of polynomials is dense in it. In particular, we have the sequence $\{q_n\}$ of Sobolev orthonormal polynomials $\left(\langle q_n, q_k\rangle_S = \delta_{n,k}\right)$ forms a complete basis in $\left(H_{2,m}, \langle\cdot,\cdot\rangle_S\right)$ and the Parseval identity takes place

$$\|f\|_S^2 = \sum_{k=0}^{\infty} \alpha_k^2, \alpha_k = \alpha_k(f) = \langle f, q_k\rangle_S, \quad f \in H_{2,m}. \tag{4}$$

In virtue of the Riesz-Fischer Theorem, the application which places $f \in H_{2,m}$ in correspondence with $\{\alpha_n(f)\} \in \ell_2$ establishes an isometric isomorphism between $H_{2,m}$ and ℓ_2.We restrict our attention to sets of measures $\{\mu_k\}, k = 0, 1, \ldots, m$, with the property that $xf \in H_{2,m}$ for each $f \in H_{2,m}$.

By $xf \in H_{2,m}$ we mean that if two Cauchy sequences of polynomials $\{p_n\}$ and $\{l_n\}$ are representatives of f (and, therefore, $\lim_{n\to\infty} \|p_n - l_n\|_S = 0$), then the sequences of polynomials $\{xp_n\}$ and $\{xl_n\}$ are also equivalent Cauchy sequences (in the sense that $\lim_{n\to\infty} \|xp_n - xl_n\|_S = 0$).

The element in $H_{2,m}$ which they represent is what we denote xf.

In this case, it is easy to verify that the application $Mf = xf$ from $H_{2,m}$ onto $H_{2,m}$ is linear.

This property is not always fulfilled. The first result below gives a class of inner products for which M is bounded. We say that the Sobolev inner product (1) is sequentially dominated if

$\Delta_k \subset \Delta_{k-1},\ k = 1, \ldots, m,$ and $d\mu_k = f_{k-1} d\mu_{k-1},\ f_{k-1} \in L_\infty(\mu_{k-1}),$ where $k = 1, \ldots, m$.

Obviously, this is the case when all the measures in the inner product are equal.

Theorem (4.1.1)[140]. Assume that the Sobolev inner product (1) is sequentially dominated, then the application $Mf = xf$ defines a bounded linear operator on $H_{2,m}$ with norm

$$\|M\| \leq (2[C_1^2 + (m+1)^2 C_2])^{1/2}, \tag{5}$$

Where $C_1 = \max_{x\in\Delta_0} |x|, C_2 = \max_{k=0,\ldots,m-1} \|f_k\|_{L_\infty(\mu_k)}$.

The boundedness of the multiplication operator has an interesting consequence on the location of the zeros of Sobolev orthogonal polynomials.

Proof: First of all, we show that there exists a constant $C > 0$ such that for any polynomial p

$$\|xp\|_S \leq C\|p\|_S. \tag{6}$$

Take C_1 and C_2 as in the statement of this theorem.
Straightforward calculations lead to the estimates

$$\|xp\|_S^2 = \sum_{k=0}^m \left\|(xp)^{(k)}\right\|_k^2 = \sum_{k=0}^m \left\|xp^{(k)} + kp^{(k-1)}\right\|_k^2$$
$$\leq 2\sum_{k=0}^m \left(\left\|xp^{(k)}\right\|_k^2 + k^2\left\|p^{(k-1)}\right\|_k^2\right) \leq 2\sum_{k=0}^m \left(C_1^2\left\|p^{(k)}\right\|_k^2 + k^2 C_2\left\|p^{(k-1)}\right\|_{k-1}^2\right)$$
$$\leq 2[C_1^2 + (m+1)^2 C_2]\sum_{k=0}^m \left\|p^{(k)}\right\|_k^2 = C^2\|p\|_S^2,$$

which imply (6) with $C = (2[C_1^2 + (m+1)^2 C_2])^{1/2}$.

Let $f \in H_{2,m}$ and assume that $\{p_n\}$ is a representative of f. Using (6), for all $n, m \in \mathbb{Z}_+$ we have

$$\|xp_n - xp_m\|_S \leq C\|p_n - p_m\|_S.$$

This shows that $\{xp_n\}$ is also a Cauchy sequence. Moreover, if $\{l_n\}$ also represents f, from (6) we also have that for all $n \in \mathbb{Z}_+$, $\|xp_n - xl_n\|_S \leq C\|p_n - l_n\|_S$,which shows that both sequences $\{xp_n\}$ and $\{xl_n\}$ represent the same element in $H_{2,m}$. If $\{p_n\}$ is a representative of $f \in H_{2,m}$ and $\{l_n\}$ is a representative of $g \in H_{2,m}$, and $\alpha, \beta \in \mathbb{R}$ it is easy to verify, that $\{\alpha x p_n + \beta x l_n\}$ represents $x(\alpha f + \beta g)$ which amounts to the linearity of M.The boundedness of the operator follows immediately because (6) and the definition of the $\|\cdot\|_S$ norm give

$$\|xf\|_S = \lim_{n\to\infty}\|xp_n\|_S \leq C \lim_{n\to\infty}\|p_n\|_S = C\|f\|_S.$$

With this we conclude the proof of Theorem (4.1.1).

Our next goal is to connect the operator M with an infinite Hessenberg matrix. We have that $H_{2,m}$ is isometrically isomorphic to ℓ_2 through the application which identifies an element $f \in H_{2,m}$ with the sequence of its Fourier coefficients (see (4)). Thus the nth Sobolev orthonormal polynomial q_n is in correspondence with the element e_n of ℓ_2 with 1 at the coordinate $n+1$ and the rest of the coordinates equal to 0. Since the sequence $\{q_n\}$ of orthonormal polynomials with respect to the inner product $\langle\cdot,\cdot\rangle_S$ forms a basis in the space of all polynomials, we have that for each $n \in \mathbb{Z}_+$

$$xq_{n-1}(x) = \sum_{k=0}^{n} c_{k,n-1} q_k(x), \tag{7}$$

Where $c_{k,n-1} = \langle xq_{n-1}, q_k\rangle_S$, $k = 0, \ldots, n$.
From (7) we obtain that the matrix representation of M, taking in ℓ_2 the canonical basis $\{e_n\}$, is given by the infinite Hessenberg matrix

$$\mathrm{M} = \begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} & \cdots & c_{0,n-2} & c_{0,n-1} & \cdots \\ c_{1,0} & c_{1,1} & c_{1,2} & \cdots & c_{1,n-2} & c_{1,n-1} & \cdots \\ 0 & c_{2,1} & c_{2,2} & \cdots & c_{2,n-2} & c_{2,n-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2,n-2} & c_{n-2,n-1} & \cdots \\ 0 & 0 & 0 & \cdots & c_{n-1,n-2} & c_{n-1,n-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{8}$$

By M_n, we denote the nth principal section of M, and $\bar{q}_n(x) = \left(q_0(x), q_1(x), \ldots, q_{n-1}(x)\right)^t$.
Here and in the following $(\cdot)^t$ denotes the transpose of the vector or matrix $(\cdot)$.
Relation (24) for consecutive values of n indicates that

$$x\bar{q}_n(x) = \mathrm{M}_n^t \bar{q}_n(x) + c_{n,n-1}\left(0, \ldots, 0, q_n(x)\right)^t. \tag{9}$$

Theorem (4.1.2)[140]. Assume that the application $Mf = xf$ defines a bounded linear operator from $H_{2,m}$ onto $H_{2,m}$. Then, all the zeros of the Sobolev orthogonal polynomials are contained in the disk $[z: |z| \leq 2\|M\|]$.We underline that in Theorem (4.1.2) the inner product does not have to be sequentially dominated. The boundedness of M is the only requirement. Therefore, it is of interest to find other (or less restrictive) sufficient conditions for the boundedness of this operator.

(iii) We mention some concepts needed to state the result on the asymptotic zero distribution of Sobolev orthogonal polynomials. For any polynomial q of exact degree n,we denote $\nu(q) := \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}$,where $z_1, \dots, z_n$are the zeros of q repeated according to their multiplicity , δ_{z_j} is the Dirac measure with mass one at the point z_j.This is the so called normalized zero counting measure associated with q.In [138] the authors introduce a class Reg of regular measures. For measures supported on a compact set of the real line, they prove that $\mu \in$ Reg if and only if the orthogonal polynomials q_n(in the usual sense) with respect to μ have regular asymptotic zero distribution. That is, that in the weak star topology of measures $\lim_{n\to\infty}(q_n) = \omega_\Delta$,where ω_Δ is the equilibrium measure of the support Δ of the measure μ. In case that Δ is regular, the measure μ belongs to Reg (see [138]) if and only if

$$\lim_{n\to\infty}\left(\frac{\|p_n\|_\Delta}{\|p_n\|_{L_2(\mu)}}\right)^{1/n} = 1 \tag{10}$$

for every sequence of polynomials $\{p_n\}, \deg p_n \leq n, p_n \not\equiv 0$. Here and in the following $\|\cdot\|_\Delta$ denotes the supremum norm on Δ. Given a compact set Δ of the complex plane, we denote by $C(\Delta)$ the logarithmic capacity of Δ and by $g_\Delta(z{:}\,\infty)$ the corresponding Green's function with singularity at infinity. In the following, $\Delta = \bigcup_{k=0}^{m}\Delta_k$, where Δ_k is the support of μ_k in (1). Assume that there exists $l \in \{0, \dots, m\}$ such that $\bigcup_{k=0}^{l}\Delta_k = \Delta$, where Δ_k is regular, and $\mu_k \in$ **Reg** for $k = 0, \dots, l$. Under these assumptions, we say that the Sobolev inner product (1) is l-regular.The next result is in [137]

Theorem (4.1.3)[140]. Let the Sobolev inner product (1) be l-regular. Then for each fixed $k = 0, \dots, l$ and for all $j \geq k$

$$\overline{\lim_{n\to\infty}} \left\|Q_n^{(j)}\right\|_{\Delta_k}^{1/n} \leq C(\Delta). \tag{11}$$

For all $j \geq l$

$$\lim_{n\to\infty} \left\|Q_n^{(j)}\right\|_{\Delta}^{1/n} \leq C(\Delta) \tag{12}$$

and

$$\lim_{n\to\infty} \nu\left(Q_n^{(j)}\right) = \omega_\Delta, \tag{13}$$

in the weak star topology of measures. If the inner product is sequentially dominated, then $\Delta_0 = \Delta$; therefore, if Δ_0 and μ_0 are regular the corresponding inner product is 0-regular.

In the sequel,

$\mathbb{Z}_+ = \{0, 1, \dots\}$.

Proof: We start out showing

$$\overline{\lim_{n\to\infty}} \|Q_n\|_S^{1/n} \leq C(\Delta). \tag{14}$$

Since each of the sets Δ_k , $k = 0, \dots, l$ is regular, so is Δ.

Let T_n denote the monic Chebyshev polynomial of degree n for the set Δ. It is well known that $\lim_{n\to\infty}\|T_n\|_\Delta^{1/n} = C(\Delta)$.

Then, for all $j \in \mathbb{Z}_{-}$

$$\lim_{n\to\infty} \left\| T_n^{(j)} \right\|_{\Delta}^{1/n} \leq C(\Delta). \tag{15}$$

Therefore, by the minimizing property of the Sobolev norm of the polynomial Q_n,
we have

$$\|Q_n\|_S^2 \leq \|T_n\|_S^2 = \sum_{k=0}^{m} \left\| T_n^{(k)} \right\|_k^2 \leq \sum_{k=0}^{m} \mu_k(\Delta_k) \left\| T_n^{(k)} \right\|_{\Delta}^2.$$

This estimate, together with (15), gives (14).

From the regularity of the measure μ_k (see (10)), we know that for each $k = 0, \ldots, l$

$$\lim_{n\to\infty} \left(\frac{\left\| Q_n^{(k)} \right\|_{\Delta_k}}{\left\| Q_n^{(k)} \right\|_k} \right)^{1/n} = 1. \tag{16}$$

Since $\left\| Q_n^{(k)} \right\|_k \leq \|Q_n\|_S$, (14) and (16) imply

$$\overline{\lim_{n\to\infty}} \left\| Q_n^{(k)} \right\|_{\Delta_k}^{1/n} \leq C(\Delta). \tag{17}$$

Taking into consideration Lemma (4.1.3), relation (11) follows from (17).

If $j \geq l$, (15) takes place for each $k = 0, \ldots, l$.

Since

$$\left\| Q_n^{(j)} \right\|_{\Delta} = \max_{k=0,\ldots,l} \left\| Q_n^{(j)} \right\|_{\Delta_k},$$

using (14), we obtain

$$\overline{\lim_{n\to\infty}} \left\| Q_n^{(j)} \right\|_{\Delta}^{1/n} \leq C(\Delta).$$

But

$$\underline{\lim}_{n\to\infty} \left\| Q_n^{(j)} \right\|_{\Delta}^{1/n} \geq C(\Delta)$$

is always true for any sequence $\{Q_n\}$ of monic polynomials. Hence (12) follows.

The compact set Δ has empty interior and connected complement. It is well known that under such conditions (12) implies (13). The so called discrete Sobolev orthogonal polynomials have attracted particular attention in the past years. They are of the form

$$\langle f, g \rangle_S = \int f g \, d\mu_0 + \sum_{i=1}^{m} \sum_{j=0}^{N_i} A_{i,j} f^{(j)}(c_i) g^{(j)}(c_i), \tag{18}$$

where $A_{i,j} \geq 0, A_{i,N_i} > 0$, and $c_i \in \mathbb{R}$. If any of the points c_i lie in the complement of the support Δ_0 of μ_0, the corresponding Sobolev inner product cannot be l-regular.

Theorem (4.1.4)[140]. Assume that the Sobolev inner product is sequentially dominated and 0-regular. Then, for all $j \in \mathbb{Z}_{+}$

$$\overline{\lim_{n\to\infty}} \left| Q_n^{(j)}(z) \right|^{1/n} = C(\Delta) e^{g_{\Delta}(z;\infty)} \tag{19}$$

for every $z \in \mathbb{C}$ except for a set of capacity zero, and

$$\lim_{n\to\infty} \left| Q_n^{(j)}(z) \right|^{1/n} = C(\Delta) e^{g_{\Delta}(z;\infty)}, \tag{20}$$

uniformly on each compact subset of $\mathbb{C}\{z: |z| \leq 2\|M\|\}$, where $\|M\|$ satisfies (5).

Proof: We have that for all $n \in \mathbb{Z}_+$, the zeros of the Sobolev orthogonal polynomials are contained in a compact subset of the complex plane. It is well known that the zeros of the derivative of a polynomial lie in the convex hull of the set of zeros of the polynomial itself. Therefore, there exists a compact subset of the complex plane containing the zeros of $Q_n^{(j)}$ for all $n, j \in \mathbb{Z}_+$. In particular, all these zeros are contained in $\{z: |z| \leq 2\,\|M\|\}$. Thus, for each fixed $j \in \mathbb{Z}_+$ the measures $\nu_{n,j} = \nu\left(Q_n^{(j)}\right), n \in \mathbb{Z}_+$, and ω_Δ have their support contained in a compact subset of $\mathbb{C}$. Using this and (20), from the lower envelope theorem, see [138] we obtain $\underline{\lim}_{n\to\infty} \int \log\frac{1}{|z-x|} d\nu_{n,j}(x) = \int \log\frac{1}{|z-x|} d\omega_\Delta(x),$
for all $z \in \mathbb{C}$ except for a set of zero capacity. This limit is equivalent to (10) because

$$g_\Delta(z:\infty) = \log\frac{1}{C(\Delta)} - \int \log\frac{1}{|z-x|} d\omega_\Delta(x)\,.$$

In order to prove (11), notice that for each fixed $j \in \mathbb{Z}_+$, the family of functions

$$\left\{\int \log\frac{1}{|z-x|} d\nu_{n,j}(x)\right\}, \qquad n \in \mathbb{Z}_+,$$

is formed by harmonic functions in the variable z which are uniformly bounded on each compact subset of $D = \mathbb{C}\backslash\{z: |z| \leq 2\|M\|\}$. From (10), we have that any subsequence which converges uniformly on compact subsets of D must tend to $\int \log|z-x|^{-1} d\omega_\Delta(x)$. Therefore, the whole sequence converges uniformly on compact subsets of D to this function. This is equivalent to (11). To conclude, we give another consequence of Theorem (4.1.3) .We fix an inner product of the form (1). For simplicity in the notation, we write $\langle\cdot,\cdot\rangle_{L_2(\mu_k)} = \langle\cdot,\cdot\rangle_k$, $\|\cdot\|_{L_2(\mu_k)} = \|\cdot\|_k$.

Lemma (4.1.5)[140]. Assume that M defines a bounded linear operator on $H_{2,m}$. Then, the infinite Hessenberg matrix M defines a bounded linear operator on ℓ_2 and $\|\mathrm{M}\| = \|M\|$. Moreover, if $\mathrm{M}_{n,\infty}$ denotes the infinite matrix which is obtained adding zeros to M_n, then for all $n \in \mathbb{Z}_+$

$$\left\|\mathrm{M}_{n,\infty}\right\| \leq 2\|M\|. \tag{21}$$

Proof. As pointed out above, $H_{2,m}$ and ℓ_2 are isometrically isomorphic, and M is the matrix representation of the operator M on the orthonormal basis of $H_{2,m}$ (see (7) and (25)). It immediately follows that $\|\mathrm{M}\| = \|M\|$. In order to prove (21), notice that Schwarz's inequality and the boundedness of M give $\left|c_{n,n-1}\right| = |\langle xq_{n-1}, q_n\rangle_S| \leq \|xq_{n-1}\|_s \leq \|M\|$.

For any $\bar{\alpha} \in \ell_2$, let $\bar{\alpha}_n$ denote its projection over the space generated by the first $n+1$ elements $e_0, \ldots, e_n$ of the canonical basis in ℓ_2. It is easy to verify that

$$\mathrm{M}_{n,\infty}\bar{\alpha}^t = \mathrm{M}_{n,\infty}\bar{\alpha}_{n-1}^t = \mathrm{M}\bar{\alpha}_{n-1}^t - c_{n,n-1}\alpha_{n-1}e_n^t.$$

Therefore $\left\|\mathrm{M}_{n,\infty}\bar{\alpha}^t\right\|_{\ell_2} \leq \|\mathrm{M}\,\bar{\alpha}_{n-1}^t\|_{\ell_2} + \left|c_{n,n-1}\alpha_{n-1}\right| \leq 2\|M\|\|\bar{\alpha}\|_{\ell_2}$, which gives (21).

Corollary (4.1.6). Assume that the Sobolev inner product (1) is sequentially dominated, then all the zeros of the Sobolev orthogonal polynomials are contained in
$\{z: |z| \leq 2\,\|M\|\}$, where $\|M\|$ satisfies (5).

Lemma (4.1.7). Let E be a compact regular subset of the complex plane and $\{P_n\}$ a sequence of polynomials such that deg $P_n \leq n$ and $P_n \not\equiv 0$. Then, for all $k \in \mathbb{Z}_+$,

$$\overline{\lim_{n\to\infty}} \left(\frac{\left\|P_n^{(k)}\right\|_E}{\|P_n\|_E}\right)^{1/n} \leq 1. \tag{22}$$

Proof. Since P_n appears in the numerator and the denominator of the expression above, we can assume without loss of generality that P_n is monic. Fix an arbitrary $\varepsilon > 0$.
Consider the curve $\gamma_\varepsilon = \{z \in \mathbb{C} : g_E(z;\infty) = \varepsilon\}$, where $g_E(z;\infty)$ denotes Green's function with respect to the unbounded connected component of the complement of E with singularity at infinity. The curve γ_ε is closed and analytic, thus it has finite length l_ε and it is at a distance $d > 0$ from E. Since E is regular, the curve γ_ε surrounds E. By Cauchy's integral formula and the Bernstein-Walsh Lemma, we have that for each $z \in E$

$$\left|P_n^{(k)}(z)\right| = \frac{k!}{2\pi i}\int_{\gamma_\varepsilon} \frac{P_n(\zeta)}{(\zeta - z)^{k+1}}d\zeta \le \frac{k!}{2\pi}\int_{\gamma_\varepsilon} \frac{|P_n(\zeta)|}{|\zeta - z|^{k+1}}|d\zeta|$$
$$\le \frac{k!\,l_\varepsilon}{2\pi d^{k+1}}\|P_n\|_{\gamma_\varepsilon} \le \frac{k!\,l_\varepsilon}{2\pi d^{k+1}}\|P_n\|_E e^{n\varepsilon}.$$

Therefore,

$$\left(\frac{\left\|P_n^{(k)}\right\|_E}{\|P_n\|_E}\right)^{1/n} \le \left(\frac{k!l_\varepsilon}{2\pi d^{k+1}}\right)^{1/n} e^{\varepsilon}, \text{ and}$$

$$\overline{\lim_{n\to\infty}}\left(\frac{\left\|P_n^{(k)}\right\|_E}{\|P_n\|_E}\right)^{1/n} \le e^{\varepsilon}.$$

Making $\varepsilon \to 0$, (22) follows immediately.

Theorem (4.1.8)[140]. Let the discrete Sobolev inner product (18) be such that Δ_0 is regular, and $\mu_0 \in$ **Reg**. Then, (12)-(13) take place, for all $j \ge 0$, with $\Delta = \Delta_0$.

Proof. Let T_n denotes the nth monic Chebyshev polynomial with respect to Δ_0.
Set $w(z) = \prod_{i=1}^{m}(z - c_i)^{N_i+1}$.Let $N = \deg w$, and take $n \ge N$.
Then,

$$\|Q_n\|_0^2 \le \|Q_n\|_S^2 \le \|wT_{n-N}\|_S^2 = \int |wT_{n-N}|^2 d\mu_0 \le \mu_0(\Delta_0)\|w\|_{\Delta_0}^2\|T_{n-N}\|_{\Delta_0}^2.$$

Since $\mu_0(\Delta_0)\|w\|_{\Delta_0}^2 > 0$ does not depend on n, we find that $\overline{\lim}_{n\to\infty}\|Q_n\|_0^{1/n} \le C(\Delta_0)$.
From the regularity of the measure μ_0, it follows that $\overline{\lim}_{n\to\infty}\|Q_n\|_{\Delta_0}^{1/n} \le C(\Delta_0)$.
Using the regularity of the compact set Δ_0 and Lemma (4.1.7) (for $E = \Delta_0$), we obtain

$$\overline{\lim}_{n\to\infty}\left\|Q_n^{(j)}\right\|_{\Delta_0}^{1/n} \le C(\Delta_0), \text{ for all } j \ge 0.$$

This inequality is necessary and sufficient in order that (12) takes place (with $\Delta = \Delta_0$), which in turn implies (13).

Theorem (4.1.9)[140]. Assume that the Sobolev inner product is sequentially dominated and 0-regular. Then, for all $j \in \mathbb{Z}_-$

$$\lim_{n\to\infty}\frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} = \int\frac{d\omega_\Delta(x)}{z - x}, \tag{23}$$

uniformly on compact subsets of $\mathbb{C}\setminus\{z: |z| \le 2\|M\|\}$, where $\|M\|$ satisfies (5).

Proof. Let $x_{n,i}^j, i = 1,\dots,n - j$, denote the $n - j$ zeros of $Q_n^{(j)}$.
As mentioned above, all these zeros are contained in $\{z: |z| \le 2\|M\|\}$.

Decomposing in simple fractions and using the definition of $\nu_{n,j}$, we obtain

$$\frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} = \frac{1}{n}\sum_{i=1}^{n-j}\frac{1}{z - x_{n,i}^j} = \frac{n-j}{n}\int\frac{d\nu_{n,j}(x)}{z - x}. \tag{24}$$

Therefore, for each fixed $j \in \mathbb{Z}_+$, the family of functions

$$\left\{\frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)}\right\}, \qquad n \in \mathbb{Z}_+, \tag{25}$$

is uniformly bounded on each compact subset of $D = \mathbb{C}\backslash\{z: |z| \le 2\|M\|\}$. On the other hand, all the measure $\nu_{n,j}, n \in \mathbb{Z}_+$, are supported in $\{z: |z| \le 2\|M\|\}$ and for $z \in D$ fixed, the function $(z - x)^{-1}$ is continuous on $\{z: |z| \le 2\|M\|\}$ with respect to x. Therefore, from (13) and (24), we find that any subsequence of (25) which converges uniformly on compact subsets of D converges pointwise to $\int (z - x)^{-1} d\omega_\Delta(x)$. Thus, the whole sequence converges uniformly on compact subsets of D to this function as stated in (23). Due to Theorem (4.1.8), results analogous to Theorems (4.1.4) and (4.1.9) may be obtained for discrete Sobolev orthogonal polynomials. For this, we must add to the restrictions of Theorem (4.1.9) that in (18) all $A_{i,j}$ be greater than zero in order that the corresponding inner product be sequentially dominated. Nevertheless, in any discrete Sobolev inner product, it is easy to see that at least $n - N - j$ zeros of $Q_n^{(j)}$ lie in the open convex hull of Δ_0.

Section (4.2): Weighted Sobolev Spaces:

If I is any compact interval, Weierstrass' theorem says that $C(I)$ is the biggest set of functions which can be approximated by polynomials in the norm $L^\infty(I)$, if we identify, as usual, functions which are equal almost everywhere.There are many generalizations of this theorem (see [141]).Here we study the same problem with the norm $L^\infty(I, w)$ defined by

$$\|f\|_{L^\infty(I,w)} := \operatorname*{ess\,sup}_{x\in I}|f(x)|w(x), \tag{26}$$

where w is a weight, i.e., a non-negative measurable function, and we use the convention $0.\infty = 0$. Observe that (26) is not the usual definition of the L^∞ norm in the context of measure theory, although it is the correct one when we work with weights (see [142]). If $w = (w_0, \dots, w_k)$ is a vectorial weight, we also study this problem with the Sobolev norm $W^{k,\infty}(\Delta, w)$ defined by

$\|f\|_{W^{k,\infty}(\Delta,w)} := \sum_{j=0}^{k}\left\|f^{(j)}\right\|_{L^\infty(\Delta,w_j)}$,where $\Delta := \cup_{j=0}^{k}\operatorname{supp} w_j$.

It is obvious that $W^{0,\infty}(\Delta, w) = L^\infty(\Delta, w)$. (see[143],[144],and [145]). In [146], [147], and [148] study some examples of Sobolev spaces for $p = 2$ with respect to general measures instead of weights, in relation with ordinary differential equations and Sobolev orthogonal polynomials.

The [149],[150], and [151] are the beginning of a theory of Sobolev spaces with respect to general measures for $1 \le p \le \infty$.This theory plays an important role in the location of the zeroes of the Sobolev orthogonal polynomials (see[152],[153], and [150]).The location of these zeroes allows us to prove results on the asymptotic behavior of Sobolev orthogonal polynomials (see [152]).

We denote by $P^{k,\infty}(\Delta, w)(k \ge 0)$ the set of functions which can be approximated by polynomials in the norm $W^{k,\infty}(\Delta, w)$, where we identify, as usual, functions which are equal almost everywhere.

We must remark that the symbol $\mathrm{P}^{\mathrm{k},\infty}(\Delta, \mathrm{w})$ has a slightly different meaning in [149], [153], [150], and [151]. First, we have results for the case $k = 0$.

Theorem (4.2.1)[154]. Let us consider a closed interval I and a weight $w \in L_{loc}^{\infty}(I)$, such that the set S of singular points of w in I has zero Lebesgue measure.

Then we have $\overline{C^{\infty}(R) \cap L^{\infty}(I,w)} = \overline{C(R) \cap L^{\infty}(I,w)} = H$, with

$$H = \{f \in C(I \backslash S) \cap L^{\infty}(I,w) \colon \forall\, a \in S, \exists l_a \in R \text{ such that } \underset{x \in I, x \to a}{ess \lim} |f(x) - l_a|\, w(x) = 0\}$$

where the closures are taken in $L^{\infty}(I,w)$. If $a \in S$ is of type 1, we can take as la any real number. If $a \in S$ is of type 2, $la = ess \lim_{w(x) \geq \varepsilon, x \to a} f(x)\ for\ \varepsilon > 0$ small enough. Furthermore, if I is compact we also have $P^{0,\infty}(I,w) = H$.

If $f \in H \cap L^1(I)$, I is compact, and S is countable, we can approximate f by polynomials with the norm $\|\cdot\|_{L^{\infty}(I,w)} + \|\cdot\|_{L^{\infty}(I)}$ .The following are two of the main results for $k \geq 1$.

Theorem (4.2.2)[154]. Let us consider a compact interval I and a vectorial weight $w = (w_0, \ldots, w_k) \in L^{\infty}(I)$ such that $w_k^{-1} \in L^1(I)$.

Then we have $P^{k,\infty}(I,w) = \{f \colon I \to R / f^{(k-1)} \in AC(I)\ and\ f^{(k)} \in P^{0,\infty}(I, wk)\}$.

Theorem (4.2.3)[154]. Let us consider a compact interval I and a vectorial weight $w = (w_0, \ldots, w_k) \in L^{\infty}(I)$ such that the set of singular points for wk in I has zero Lebesgue measure. Assume that there exist $a_0 \in I$, an integer $0 \leq r < k$, and constants $c, \delta > 0$ such that

(i) $w_{j+1}(x) \leq c|x - a_0| w_j(x)$ in $[a_0 - \delta, a_0 + \delta] \cap I, for\ r \leq j < k$,

(ii) $\int_{I \backslash [a_0 - \varepsilon, a_0 + \varepsilon]} w_k^{-1} < \infty$, for every $\varepsilon > 0$,

(iii) if $r > 0, a_0$ is $(r-1)$-regular.Then we have

$$P^{k,\infty}(I,w) = \{ f \colon I \to R / f^{(k-1)} \in AC_{loc}(I\{a_0\}), f^{(k)} \in P^{0,\infty}(I, w_k),$$
$$\exists l \in R\ with\ \underset{x \in I, x \to a_0}{ess \lim} \left|f^{(r)}(x) - l\right| w_r(x) = 0\, , \underset{x \in I, x \to a_0}{ess \lim} f^{(J)}(x) w_J(x) = 0,$$
$$\text{for } r \leq j < k\ if\ r < k-1, and\ f^{(r-1)} \in AC(I) \text{ if } r > 0\}.$$

This result gives the characterization of $P^{k,\infty}(I,w)$ for the case of Jacobi weights .The analogue of Weierstrass' theorem with the norms $W^{k,p}(\Delta, \mu)$ (with $1 \leq p < \infty$ and μ a vectorial measure) can be founded in [153] and [151]. Throughout $k \geq 0$ denotes a fixed natural number. Also, all the weights are non-negative Borel measurable functions defined on a subset of R; ifa weight is defined in a proper subset $E \subset R$, we define it in $R \backslash E$ as zero. If the weight does not appear explicitly, we mean that we are using the weight 1. Given $0 < m < k$, a vectorial weight w and a closed set E, we denote by $W^{k,\infty}(E,w)$ the space $W^{k,\infty}(\Delta \cap E, w|_E)$ and by $W^{k-m,\infty}(\Delta, w)$ the space $W^{k-m,\infty}(\Delta, (w_m, \ldots, w_k))$. We denote by supp v v the support of the measure $v(x)dx$, the intersection of every closed set $E \subseteq R$ verifying $\int_{R \backslash E} v = 0$. If A is a Borel set, $|A|, \chi_A$, $int(A)$, and $\bar{A}$ denote, respectively, the Lebesgue measure, the characteristic function, the interior and the closure of A. If I, U are subsets of R, the symbol $\partial_I U$ denotes the relative boundary of U in I. By $f(j)$ we mean the jth distributional derivative of f.

P denotes the set of polynomials. We say that an n-dimensional vector satisfies a one-dimensional property if each coordinate satisfies this property.

Definition (4.2.5)[154]. Given a measurable set A, we define the essential closure of A as the set esscl $A := [x \in R \colon |A \cap (x - \delta, x + \delta)| > 0, \forall \delta > 0]$.

Definition (4.2.6)[154]. If A is a measurable set,**f** is a function defined in A with real values and $a \in$ ess cl A, we say that $\text{ess} \lim_{x \in A, x \to a} f(x) = l \in R$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for almost every $x \in A \cap (a - \delta, a + \delta)$. In a similar way we can define $\text{ess} \lim_{x \in A, x \to a} f(x) = -\infty$ and

$\operatorname{ess}\lim_{x\in A, x\to a} f(x) = -\infty$. We define the essential limit superior and the essential limit inferior in A as follows:

$$\operatorname{ess}\limsup_{x\in A,\ x\to a} f(x) := \inf_{\delta>0} \operatorname{ess}\sup_{x\in A\cap(a-\delta,a+\delta)} f(x), \operatorname{ess}\liminf_{x\in A,\ x\to a} f(x) := \sup_{\delta>0} \operatorname{ess}\inf_{x\in A\cap(a-\delta,a+\delta)} f(x).$$

If we do not specify the set A we are assuming that $A = R$.

Definition (4.2.7)[154]. Given an interval I and a weight w in I we say that $a \in \tilde{I}$ is a singularity of w (or singular for w) in Iif $\operatorname{ess}\liminf_{x\in I, x\to a} w(x) = 0$.

We say that a singularity a of $w$is of type 1 if $\operatorname{ess}\lim_{x\in I,x\to a} w(x) = 0$. In other cases we say that a is a singularity of type 2.

Lemma (4.2.8)[154]. Let us consider an interval I, a weight w in I, and a point $a \in \bar{I}$ which is not singular for w in I. Then there exists $\delta > 0$ such that every function in the closure of $C(R)$ with the norm $L^\infty(I, w)$ belongs to $C(\bar{I} \cap [a-\delta, a+\delta])$.

Proof. We have that $\sup_{\delta>0} \operatorname{ess}\inf_{x\in I\cap(a-\delta,a+\delta)} w(x) = l > 0$.

Therefore there exists $\delta > 0$ with $\operatorname{ess}\inf_{x\in I\cap(a-\delta,a+\delta)} w(x) > \frac{l}{2} > 0$.

Hence, we have $\|g\|_{L^\infty(I\cap(a-\delta,a+\delta),w)} \geq \frac{l}{2}\|g\|_{L^\infty(I\cap(a-\delta,a+\delta))} = \frac{l}{2}\max_{x\in I\cap[a-\delta,a+\delta]}|g(x)|$,

for every $g \in C(R)$. This inequality gives the lemma, since if f is the limit of functions $\{g_n\} \subset C(R)$ with the norm in $L^\infty(I \cap (a-\delta, a+\delta), w)$, it can be modified in a set of zero Lebesgue measure in such a way that it is the uniform limit of $\{g_n\}$ in $\bar{I} \cap [a-\delta, a+\delta]$.

Lemma (4.2.9)[154]. Let us consider an interval I, a weight w in I and a singular point a of w in I of type 1. Then every function f in the closure of $C(R)$ with the norm $L^\infty(I, w)$ verifies

$$\operatorname{ess}\lim_{x\in I,x\to a} f(x)w(x) = 0. \tag{27}$$

Proof. Let us assume that (27) is not true, i.e., $\operatorname{esslim}\sup_{x\in I,x\to a}|f(x)|w(x) = l > 0$.

Therefore for every $\delta > 0$ we have $\operatorname{ess}\sup_{x\in I\cap(a-\delta,a+\delta)}|f(x)|w(x) \geq l > 0$.

Since a is of type 1 we deduce $\operatorname{ess}\lim_{x\in I\cap(a-\delta,a+\delta)}|g(x)|w(x) = 0$,

for every $g \in C(R)$. This implies that for each $g \in C(R)$ and $\varepsilon > 0$ there exists $\delta > 0$ with

$$\operatorname{ess}\sup_{x\in I\cap(a-\delta,a+\delta)} |g(x)|\, w(x) \leq \varepsilon.$$

Consequently, for this $\delta > 0$ we have

$$\|f-g\|_{L^\infty(I,w)}\|f-g\|_{L^\infty(I\cap(a-\delta,a+\delta),w)} \geq \|f\|_{L^\infty(I\cap(a-\delta,a+\delta),w)} - \|g\|_{L^\infty(I\cap(a-\delta,a+\delta),w)} \geq l-\varepsilon,$$

for every $\varepsilon > 0$ and $g \in C(R)$. Hence we have $\|f-g\|_{L^\infty(I,w)} \geq l > 0$,for every $g \in C(R)$. This implies that f cannot be approximated by functions in $C(R)$ with the norm $L^\infty(I, w)$.

Lemma (4.2.10)[154]. Let us consider an interval I and a weight $w \in L^\infty(I)$.

Denote by S the set of singular points of w in I. Assume that $a \in S$ is of type 1 and $|S| = 0$.

Then, for any fixed $\varepsilon > 0$ and $f \in C(I\backslash S) \cap L^\infty(I, w)$ with $\operatorname{ess}\lim_{x\in I,x\to a} f(x)w(x) = 0$, there exist a relative open interval U in I with $a \in U$ and $\partial_I U \subset I\backslash S$ (and $U \subset int(I)$ if $a \in int(I)$) and a function $g \in L^\infty(I, w) \cap C(\bar{U})$ such that $g = f$ in $I\backslash U$, $\|f-g\|_{L^\infty(I,w)} < \varepsilon$ (and $\|f-g\|_{L^1(I)} < \varepsilon$ if $f \in L^1(I)$).

We can choose g with the additional condition $g(a) = 0$ or even$g(a) = \lambda$ for any fixed $\lambda \in R$.

Proof. Without loss of generality we can assume that a is an interior point of I, since the case $a \in \partial I$ is simpler.

Take n such that $[a-1/n, a+1/n] \subset int(I)$. Since $|S| = 0$, there exist $y_n(a, a+1/n)\backslash S$ and $x_n \in \left(a-\frac{1}{n}, a\right)\backslash S$ verifying

$|f(y_n)| \leq 2^{-n} + \text{ess inf}_{x\in[a,a+1/n]}|f(x)|$, $|f(x_n)| \leq 2^{-n} + \text{ess inf}_{x\in[a-1/n,a]}|f(x)|$.

Let us define now the function f_n (which is continuous in an open neighbourhood of $[x_n, y_n]$, since $x_n, y_n \notin S$) as

$$f_n(x) := \begin{cases} \dfrac{x-a}{x_n-a} f(x_n) \text{if } x \in [x_n, a], \\ \dfrac{x-a}{y_n-a} f(y_n) \text{if } x \in [a, y_n], \\ f(x) \qquad\qquad \text{if } x \in I\backslash[x_n, y_n]. \end{cases}$$

Observe that $|f_n(x)| \leq 2^{-n} + |f(x)|$ for almost every $x \in [x_n, y_n]$.

Hence $\|f - f_n\|_{L^\infty(I,w)} = \|f - f_n\|_{L^\infty([x_n,y_n],w)} \leq 2\|f\|_{L^\infty([x_n,y_n],w)} + 2^{-n}\|w\|_{L^\infty(I)}$,

and this last expression goes to 0 as $n \to \infty$, since $\text{ess lim}_{x\in I, x\to a} f(x)w(x) = 0$.

If $f \ni L^1(I)$, we also have $\|f - f_n\|_{L^1(I)} = \|f - f_n\|_{L^1([x_n,y_n])} \leq 2\|f\|_{L^1([x_n,y_n])} + 2^{-n}(y_n - x_n)$, and this expression goes to 0 as $n \to 1$. Observe that $f_n(a) = 0$; it is easy to modify fn in a small neighbourhood of a in order to have $f_n(a) = \lambda$, for fixed$\lambda \in R$. This finishes the proof .

Lemma (4.2.11)[154]. If A is a measurable set, we have:

(i) ess cl A is a closed set contained in $\bar{A}$.

(ii) $|A\backslash \text{ess cl } A| = 0$.

(iii) If f is a measurable function in $A \cup \text{ess cl } A$, $a \in \text{ess cl } A$ and there exists $\text{ess lim}_{x\in \text{ess cl } A, x\to a} f(x)$, then there exists $\text{ess lim}_{x\in A, x\to a} f(x)$ and $\text{ess lim}_{x\in A, x\to a} f(x) = \text{ess lim}_{x\in \text{esscl } A, \to a} f(x)$.

(iv) If $|A| > 0$ and f is a continuous function in R we have $\|f\|_{L^\infty(A)} = \sup_{x\in \text{ess cl } A}|f(x)|$.

Proof. (i) is direct.

(ii) is a consequence of the Lebesgue differentiation theorem, since we have $\lim_{\delta\to 0}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\chi_A = 1$, for almost every $x \in A$, and this implies $|A \cap (x-\delta, x+\delta)| > 0$ for a.e. $x \in A$ and every $\delta > 0$.

Assume now that $\text{ess lim}_{x\in \text{esscl } A, x\to a} f(x) = l \in R$. Consequently, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for almost every $x \in \text{esscl } A \cap (a-\delta, a+\delta)$ we have $|f(x) - l| < \varepsilon$.

Since $|A\backslash \text{esscl } A| = 0$, we have $|f(x) - l| < \varepsilon$, for almost every $x \in A \cap (a-\delta, a+\delta)$.

This gives (iii) if $l \in R$. The case $l = \pm\infty$ is similar. The statement (ii) gives

$\|f\|_{L^\infty(A)} \leq \|f\|_{L^\infty(\text{esscl } A)} \leq \sup_{x\in \text{esscl } A}|f(x)|$.

We have $|f(x)| \leq \|f\|_{L^\infty(A)}$ for almost every $x \in A$. Then $|f(x)| \leq \|f\|_{L^\infty(A)}$ for every $x \in \text{ess cl } A$, since f is continuous. Therefore $\sup_{x\in \text{ess cl } A}|f(x)| \leq \|f\|_{L^\infty(A)}$.

These two inequalities give (iv).

Lemma (4.2.12)[154]. Let us consider an interval I, a weight w in I, and $a \in \bar{I}$.

If $\text{ess lim sup}_{x\in I, x\to a} w(x) = l > 0$, then for every function f in the closure of $C(\boldsymbol{R}) \cap L^\infty(I, w)$ with the norm$L^\infty(I, w)$ there exists the finite limit

$$\underset{w(x)\geq\varepsilon, x\to a}{\text{ess lim}} f(x), \qquad for\ every\ 0 < \varepsilon < l.$$

Proof. We have for every $\delta > 0$ $\text{ess lim}_{x\in I\cap(a-\delta,a+\delta)} w(x) \geq l > 0$, and then $|\{x \in I \cap (a-\delta, a+\delta) : w(x) \geq \varepsilon\}| > 0$, for every $\delta > 0$ and $0 <=< l$.

This implies that a belongs to ess clA_ε, where
$A_\varepsilon := \{x \in I : w(x) \geq \varepsilon\}$. If g $\in C(\mathrm{R}) \cap L^\infty(I,w)$, $0 < \varepsilon < l$, and $\delta > 0$, we have

$$\varepsilon \|g\|_{L^\infty(A_\varepsilon \cap [a-\delta,a+\delta])} \leq \|g\|_{L^\infty(A_\infty \cap [a-\delta,a+\delta],w)} .$$

Since ess cl$(A_\varepsilon \cap [a-\delta, a+\delta])$ is a compact set and g $\in C(\mathrm{R}) \cap L^\infty(I,w)$, Lemma(4.2.11) (iv) gives

$$\varepsilon \max_{x \in \text{ess cl } (A_\varepsilon \cap [a-\delta,a+\delta])} |g^{(x)}| \leq \|g\|_{L^\infty(A_\varepsilon \cap [a-\varepsilon,a+\delta],w)}.$$

Consequently, if $\{g_n\} \subset C(\mathrm{R}) \cap L^\infty(I,w)$ converges to f in $L^\infty(I,w)$, then $\{g_n\}$ converges to f uniformly iness cl $(\mathrm{A}_\varepsilon \cap [\mathrm{a}-\delta, \mathrm{a}+\delta])$ and f $\in \mathrm{C}(\text{ess cl } (\mathrm{A}_\varepsilon \cap [\mathrm{a}-\delta, \mathrm{a}+\delta]))$ for every $\delta > 0$. Therefore $f \in C(\text{ess cl } A_\varepsilon)$. This fact and (4.2.11) (iii) give that, for $0 < \varepsilon < l$, there exists

$$\operatorname*{ess\,lim}_{x \in A_\varepsilon, x \to a} f(x) = \operatorname*{ess\,lim}_{x \in \text{ess cl } A_\varepsilon, x \to a} f(x) = \lim_{x \in \text{ess cl } A_\varepsilon, x \to a} f(x) .$$

Lemma (4.2.13)[154]. Let us consider an interval I, a weight w in I, and a singular point a of w in I. Then every function f in the closure of $C(\mathrm{R}) \cap L^\infty(I,w)$ with the norm $L^\infty(I,w)$ verifies

$$\inf_{\varepsilon > 0} \left(\operatorname*{ess\,lim\,sup}_{w(x) < \varepsilon, x \to a} |f(x)| w(x) \right) = 0. \tag{28}$$

Proof: Observe first that $a \in$ ess cl$(\{x \in I : w(x) < \varepsilon\})$ for every $\varepsilon > 0$, since a is singular for w in I. Let us assume that (28) is not true, i.e.,

$$\operatorname*{ess\,lim\,sup}_{x \in A_\varepsilon^c, x \to a} |f^{(x)}| w(x) \geq l > 0,$$

for every $\varepsilon > 0$, where $A_\varepsilon := \{x \in I : w(x) \geq \varepsilon\}$ and$A_\varepsilon^c := I / A_\varepsilon$.For every $\varepsilon, \delta > 0$ we have

$$\operatorname*{ess\,sup}_{x \in A_\varepsilon^c \cap (a-\delta,a+\delta)} |f(x)| w(x) \geq l > 0 .$$

For each $g \in C(R) \cap L^\infty(I,w)$, $\varepsilon > 0$, and $\delta > 0$, we have

$$\|g\|_{L^\infty(A_\varepsilon^c \cap (a-\delta,a+\delta),w)} \leq \varepsilon \|g\|_{L^\infty(I \cap (a-\delta,a+\delta))} < \infty.$$

Consequently

$$\|f - g\|_{L^\infty(I,w)} \geq \|f - g\|_{L^\infty(A_\varepsilon^c \cap (a-\delta,a+\delta),w)}$$
$$\geq \|f\|_{L^\infty(A_\varepsilon^c \cap (a-\delta,a+\delta),w)} - \|g\|_{L^\infty(A_\varepsilon^c \cap (a-\delta,a+\delta),w)} ,$$

and therefore $\|f - g\|_{L^\infty(I,w)} \geq l - \varepsilon \|g\|_{L^\infty(I \cap (a-\delta,a+\delta))}$,
for every g $\in C(R) \cap L^\infty(I,w)$ and $\delta, \varepsilon > 0$. Hence we obtain $\|f - g\|_{L^\infty(I,w)} \geq l > 0$,for every g $\in C(R) \cap L^\infty(I,w)$. This implies that f cannot be approximated by functions in $C(R) \cap L^\infty(I,w)$.

Lemma (4.2.14)[154]. Let us consider an interval I, a weight w in I, and $a \in \bar{I}$.
If $\operatorname{ess\,lim}_{x \in I, x \to a} w(x) = 0$ and $\inf_{\varepsilon > 0}(\operatorname{ess\,lim\,sup}_{w(x) < \varepsilon, x \to a} |f(x)| w(x)) = 0$,
then we haveess $\lim_{x \in I, x \to a} f(x) w(x) = 0$.

Proof. For each $\eta > 0$ there exist $\varepsilon, \delta_1 > 0$ such that

$$\operatorname*{ess\,sup}_{w(x) < \varepsilon, x \in I \cap (a-\delta_1,a+\delta_1)} |f(x)| w(x) < \eta.$$

We also have that there exists $\delta_2 > 0$ such that $w(x) < \varepsilon$ for almost every $x \in I \cap (a - \delta_2, a + \delta_2)$.
If we take $\delta := \min(\delta_1, \delta_2)$, we obtain

$$\operatorname*{ess\,sup}_{x \in I \cap (a-\delta,a+\delta)} |f(x)| w(x) \leq \operatorname*{ess\,sup}_{w(x) < \varepsilon, x \in I \cap (a-\delta_1,a+\delta_1)} |f(x)| w(x) < \eta ,$$

and this finishes the proof.

Lemma (4.2.15)[154]. Let us consider an interval I and a weight $w \in L^\infty(I)$.
Denote by S the set of singular points of w in I. Assume that $a \in S$ and $|S| = 0$.

Then, for any fixed $\eta > 0$ and $f \in C(I\backslash S) \cap L^{\infty}(I,w)$ such that
(i) $\inf\left(\text{ess}\limsup_{w(x)<\varepsilon, x\to a}|f(x)|w(x)\right) = 0$,
(ii) there exists the finite limitess $\lim_{w(x)\geq\varepsilon, x\to a} f(x)$, for $\varepsilon > 0$ small enough, there exist a relative open interval U in I with $a \in U$ and $\partial_I U \subset I\backslash S$ (and $U \subset \text{int}(I)$ if $a \in \text{int}(I)$) and a function
$g \in L^{\infty}(I,w) \cap C(\bar{U})$ with $g = f$ in $I\backslash U$, $\|f - g\|_{L^{\infty}(I,w)} < \eta$ (and $\|f - g\|_{L^1(I)} < \eta$ if $f \in L^1(I)$). Furthermore, we can choose g with the additional condition $g(a) = \text{ess}\lim_{w(x)\geq\varepsilon, x\to a} f(x)$, for $\varepsilon > 0$ small enough.

Proof: If a is of type 1, Lemmas (4.2.14) and (4.2.13) give the result. Assume now that a is of type 2. Without loss of generality we can assume that a is an interior point of I, since the case $a \in \partial I$ is simpler. We consider first the caseess $\limsup_{x\to a^+} w(x) > 0$ and ess $\limsup_{x\to a^-} w(x) > 0$.
For each natural number n, let us choose$\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\text{ess}\limsup_{w(x)<\varepsilon_n, x\to a}|f(x)|w(x) < \frac{1}{n}.$$

Let us consider now $\delta_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\operatorname*{ess\,sup}_{x\in(a-\delta_n, a+\delta_n)\cap A^c_{\varepsilon_n}} |f(x)|w(x) < \frac{1}{n}, \tag{29}$$

where $A_\varepsilon := \{x \in I : w(x) \geq \varepsilon\}$ and A=c := I"A= . We define $l := \text{ess}\lim_{x\in A_\varepsilon, x\to a} f(x)$, for any $\varepsilon > 0$ small enough. We can take δ_n with the additional property $|f(x) - l| < 1/n$ for almost every
$x \in (a - \delta_n, a + \delta_n) \cap A = n$. Let us choose $\gamma_n \in (a, a + \delta_n)\backslash S$ and $\gamma_n' \in (a - \delta_n, a)\backslash S$ with
$|f(\gamma_n) - l| < 1/n$ and $|f(\gamma_n') - l| < 1/n$. We define the functions an (x) and $b_n(x)$in$[\gamma_n', \gamma_n]$ as follows:

$$a_n(x) := \begin{cases} l + (x-a)\dfrac{\min\{l, f(\gamma_n)\} - l}{\gamma_n - a} \text{if} & x \in [a, \gamma_n], \\ l + (x-a)\dfrac{\min\{l, f(\gamma_n')\} - l}{\gamma_n' - a} \text{if} & x \in [\gamma_n', a], \end{cases}$$

and

$$b_n(x) := \begin{cases} l + (x-a)\dfrac{\max\{l, f(\gamma_n)\} - l}{\gamma_n - a} \text{if} & x \in [a, \gamma_n], \\ l + (x-a)\dfrac{\max\{l, f(\gamma_n')\} - l}{\gamma_n' - a} \text{if} & x \in [\gamma_n', a], \end{cases}$$

Now we can define the functions $g_n \in L^{\infty}(I,w) \cap C([\gamma_n', \gamma_n])$ in the following way:

$$g_n(x) := \begin{cases} a_n(x) \text{if } x \in [\gamma_n', \gamma_n] \text{and } f(x) \leq a_n(x), \\ b_n(x) \text{if } x \in [\gamma_n', \gamma_n] \text{and } f(x) \geq b_n(x), \\ f(x) \text{in other case.} \end{cases}$$

Observe that $a_n(x) \leq g_n(x) \leq b_n(x)$, $|a_n(x) - l| < 1/n$, and $|b_n(x) - l| < 1/n$ l, for every
$x \in [\gamma_n', \gamma_n]$. Therefore $|g_n(x) - l| < 1/n$ for $x \in [\gamma_n', \gamma_n]$ and

$$\|f - g_n\|_{L^{\infty}([\gamma_n', \gamma_n]\cap A^c_{\varepsilon_n}, w)} \leq \frac{2}{n}\|w\|_{L^{\infty}(I)}. \tag{30}$$

We prove now

$$\lim_{n\to\infty}\|f - g_n\|_{L^{\infty}([\gamma_n', \gamma_n], w)} = 0.$$

The facts

$$\|g_n\|_{L^\infty([\gamma_n',\gamma_n]\cap A^c_{\varepsilon_n},w)} \le \left(|l| + \frac{1}{n}\right)\varepsilon_n ,$$

and (29) give

$$\|f - g_n\|_{L^\infty([\gamma_n',\gamma_n]\cap A^c_{\varepsilon_n},w)} < \frac{1}{n} + \left(|l| + \frac{1}{n}\right)\varepsilon_n .$$

This inequality and (41) give

$$\|f - g_n\|_{L^\infty([\gamma_n',\gamma_n],w)} < \frac{1}{n} + \left(|l| + \frac{1}{n}\right)\varepsilon_n + \frac{2}{n}\|w\|_{L^\infty(I)} .$$

If $f \in L^1(I)$, we also have

$$\begin{aligned}\|f - g_n\|_{L^1(I)} &= \|f - g_n\|_{L^1([\gamma_n',\gamma_n])}\\ &\le \|f\|_{L^1([\gamma_n',\gamma_n])} + \left(|l| + \frac{1}{n}\right)(\gamma_n',\gamma_n).\end{aligned}$$

This finishes the proof in this case.

If $\operatorname{ess\,lim\,sup}_{x\to a^+} w(x) > 0$ and $\operatorname{ess\,lim\,sup}_{x\to a^-} w(x) = 0$,we only need to consider the functions g_n for $x > a$ and the functions f_n in the proof of Lemma (4.3.11) for $x < a$ (recall that we can choose f_n with $f_n(a) = l$).

The case $\operatorname{ess\,lim\,sup}_{x\to a^+} w(x) =$ and $\operatorname{ess\,lim\,sup}_{x\to a^-} w(x) > 0$ is symmetric.

The following result is direct.

Proposition (4.2.16)[154]. Let us consider a sequence of closed intervals $\{I_n\}_{n\in\Lambda}$ such that for each $n \in \Lambda$ there exists an open neighbourhood of $I_n$which does not intersect $\bigcup_{m\neq n} I_m$.

Denote by J the union $:= \bigcup_n I_n$. Let us consider a weight w in J.

Then we have $\overline{C(J) \cap L^\infty(J,w)} = \bigcap_n \overline{C(I_n) \cap L^\infty(I_n,w)}$,where the closures are taken in L^∞ with respect to w, in the corresponding interval. We also have a similar result for contiguous intervals.

Proposition (4.2.17)[154]. Let us consider an interval I and a weight $w \in L^\infty_{loc}(I)$.

Let us consider an increasing sequence of real numbers $\{a_n\}_{n\in\Lambda}$, where Λ is either $Z^+, Z^-, Z, or\ \{1,2,\dots,N\}$ for some $N \in \mathbf{N}$ such that $I = \bigcup_n [a_n, a_{n+1}]$ and an is not singular for w in I if an is in the interior of I. Then we have

$$\begin{aligned}\overline{C^\infty(I) \cap L^\infty(I,w)} &= \overline{C(I) \cap L^\infty(I,w)}\\ &= \left\{f \in \textstyle\bigcap_{n\in\Lambda} \overline{C([a_n,a_{n+1}])} : f \text{ is continuous in each} a_n \in \operatorname{int}(I)\right\}\\ &= \left\{f \in \textstyle\bigcap_{n\in\Lambda} \overline{C^\infty([a_n,a_{n+1}])} : f \text{ is continuous in each} a_n \in \operatorname{int}(I)\right\},\end{aligned}$$

where the closures are taken in L^∞ with respect to w, in the corresponding interval.

Remark.We can ensure $\overline{C^\infty(I) \cap L^\infty(I,w)} = \overline{C^\infty(\mathrm{R}) \cap L^\infty(I,w)}$ if I is closed. The same is obviously true for $C(I)$ instead of $C^\infty(I)$.

Proof. The third equality is true since $\overline{C^\infty([a_n,a_{n+1}])} = \overline{C([a_n,a_{n+1}])}$ is a direct consequence of Weierstrass' theorem and $w \in L^\infty([a_n,a_{n+1}])$). We are going to see that the closure of $C^\infty(I) \cap L^\infty(I,w)$ and $C(I) \cap L^\infty(I,w)$ with the norm $L^\infty(I,w)$ is the same. It is enough to prove that every $f \in C(I)$ can be approximated by functions in $C^\infty(I)$ with the norm $L^\infty(I,w)$. We can assume that $\Lambda = Z$, since the argument in the other cases is simpler. Given $\varepsilon > 0$ and $f \in C(I)$, for each $n \in Z$, there exists a function $g_n \in C^\infty(R)$ with $\|f - g_n\|_{L^\infty([a_{2n-1},a_{2n+2}],w)} < \varepsilon/2$. Let us consider functions $\theta_n \in C^\infty(R)$ with $\theta_n = 0$ in$(-\infty, a_{2n-1}]$, $\theta_n = 1$ in$[a_{2n},\infty)$ and $0 \le \theta_n \le 1$.

We define a function

$g \in C^{\infty}(I)$ by $g(x) := (1-\theta_n(x))g_{n-1}(x) + \theta_n(x)g_n(x)$, if $x \in [a_{2n-1}, a_{2n+1}]$.

We have

$$\|f-g\|_{L^{\infty}([a_{2n-1},a_{2n}],w)} \le \|(1-\theta_n)(f-g_{n-1})\|_{L^{\infty}([a_{2n-1},a_{2n}],w)}$$
$$+\|(1-\theta_n)(f-g_n)\|_{L^{\infty}([a_{2n-1},a_{2n}],w)} < \varepsilon/2+\varepsilon/2=\varepsilon,$$
$$\|f-g\|_{L^{\infty}([a_{2n-1},a_{2n}],w)} = \|f-g_n\|_{L^{\infty}([a_{2n-1},a_{2n}],w)} < \varepsilon/2,$$

and this implies $\|f-g\|_{L^{\infty}(I,w)} \le \varepsilon$. In order to see the second equality, observe that the ideas above give that the result is true if it is true for the set $\Lambda = [1,2,3]$. Let us consider $f \in \overline{C([a_1,a_2])} \cap \overline{C([a_2,a_3])}$ and continuous in a_2. Given $m \in \mathbf{N}$ there exist functions $g'_m \in C([a_1,a_2])$ and $g^2_m \in C([a_2,a_3])$ with $\|f-g^1_m\|_{L^{\infty}([a_1,a_2],w)} + \|f-g^2_m\|_{L^{\infty}([a_2,a_3],w)} < 1/m$. In order to finish the proof it is enough to construct a function $g_m \in C([a_1,a_3])$ satisfying the inequality $\|f-g_m\|_{L^{\infty}([a_1,a_3],w)} < c/m$, where c is a constant independent of m. We know that there exist positive constants δ, c_1, and c_2 such that $[a_2-\delta, a_2+\delta] \sqsubseteq [a_1,a_3]$, $|f(x)-f(a_2)| < 1/m$ if $|x-a_2| \le \delta$ and $0 < c_1^{-1} \le w(x) \le c_2$ for almost every $x \in [a_2-\delta, a_2+\delta]$. (4.2.8) gives that

$f \in C([a_2-\delta, a_2+\delta])$ and then

$$|f(x)-g^1_m(x)| < c_1/m, \qquad \text{for every } x \in [a_2-\delta, a_2],$$
$$|f(x)-g^2_m(x)| < c_1/m, \qquad \text{for every } x \in [a_2, a_2+\delta],$$

and consequently

$$|g^1_m(x)-f(a_2)| < (c_1+1)/m, \qquad \text{for every } x \in [a_2-\delta, a_2],$$
$$|g^2_m(x)-f(a_2)| < (c_1+1)/m, \qquad \text{for every } x \in [a_2, a_2+\delta].$$

Let us define g^0_m as the function whose graph is the segment joining the points $(a_2-\delta, g^1_m(a_2-\delta))$ and $(a_2+\delta, g^2_m(a_2+\delta))$. Then we have

$$|g^0_m(x)-f(a_2)| < (c_1+1)/m, \qquad \text{for every } x \in [a_2-\delta, a_2+\delta],$$
$$|g^0_m(x)-f(a_2)| < (c_1+2)/m, \qquad \text{for every } x \in [a_2-\delta, a_2+\delta],$$
$$\|g^0_m - f\|_{L^{\infty}([a_2-\delta,a_2+\delta],w)} < c_2(c_1+2)/m.$$

If we define the function $g_m \in C([a_1,a_3])$ by

$$g_m(x) := \begin{cases} g^1_m(x), & \text{if } x \in [a_1, a_2-\delta], \\ g^0_m(x), & \text{if } x \in [a_2-\delta, a_2+\delta], \\ g^2_m(x), & \text{if } x \in [a_2+\delta, a_3], \end{cases}$$

we have

$\|f-g_m\|_{L^{\infty}([a_1,a_3],w)} < (c_2(c_1+2)+1)/m$.This finishes the proof of Proposition (4.2.17).

Proposition (4.2.18)[154]. Let us consider a closed interval I and a weight $w \in L^{\infty}_{loc}(I)$ such that the set S of singular points of w in I has zero Lebesgue measure.

Then we have $\overline{C^{\infty}(\mathrm{R}) \cap L^{\infty}(I,w)} = \overline{C(\mathrm{R}) \cap L^{\infty}(I,w)} = H$, with

$$\mathrm{H} := \{\mathrm{f} \in \mathrm{C}(\mathrm{I}/\mathrm{S}) \cap \mathrm{L}^{\infty}(\mathrm{I},\mathrm{w}) : \text{for each } \mathrm{a} \in \mathrm{S}, \inf_{\varepsilon>0}\left(\operatorname*{ess\,lim\,sup}_{\mathrm{w(x)}<\varepsilon, x\to a} |\mathrm{f(x)}|\mathrm{w(x)}\right)$$
$$= 0 \text{ and, if a is of type 2, there exists the finite limit } \operatorname*{ess\,lim}_{\mathrm{w(x)}\ge\varepsilon, \mathrm{x}\to\mathrm{a}} \mathrm{f(x)},$$

$for\ \varepsilon > 0\ small\ enough\}$,where the closures are taken in $L^{\infty}(I,w)$. Furthermore, if I is compact we also have $P^{0,\infty}(I,w) = H$. If $f \in H \cap L^1(I)$, I is compact, and S is countable, we can approximate f by polynomials with the norm $\|\cdot\|_{L^{\infty}(I,w)} + \|\cdot\|_{L^1(I)}$.

Proof. Lemmas (4.2.8),(4.2.9),(4.2.14),(4.2.12),and(4.2.13) give that H contains $\overline{C(R) \cap L^{\infty}(I,w)}$.

In order to see that H is contained in $\overline{C(R) \cap L^{\infty}(I,w)}$, assume first that I is compact; then $w \in L^{\infty}(I)$. Fix $\varepsilon > 0$ and $f \in H$. Proposition (4.2.16) gives that for each $a \in S$ there exist a relative open interval U_a in I with $a \in U_a$ and $\partial_I U_a \subset I/S$ (and $U_a \subset int(I)$ if $a \in int(I)$) and a function $g_a \in L^{\infty}(I,w) \cap C(\overline{U}_a)$ such that $g_a = f$ in $\frac{I}{U_a}$ and $\|f - g_a\|_{L^{\infty}(I,w)} < \varepsilon$. The set S is compact since it is a closed set contained in the compact interval I. There exist $a_1, \dots, a_m \in S$such that $S \subset U_{a_1} \cup \dots \cup U_{a_m}$.

Without loss of generality we can assume that $U_{a_1}, \dots, U_{a_m}$ is minimal in the following sense: for each $i = 1, \dots, m$ the set $\cup_{j \neq i} U_{a_j}$does not contain to U_{a_i}.Define $[\alpha_i, \beta_i] := \overline{U}_{a_i}$. Assume that we have $U_{a_i} \cap U_{a_j} \neq \emptyset$, with$\alpha_i < \alpha_j$. The minimal property gives $\overline{U}_{a_i} \cap \overline{U}_{a_j} = [\alpha_j, \beta_i]$ and $[\alpha_j, \beta_i] \cap U_{a_k} = \emptyset$ for every $k \neq i, j$. We define the functions $g_{a_j,a_i}(x) := g_{a_i,a_j}(x) := \frac{\beta_i - x}{\beta_i - \alpha_j} g_{a_i}(x) + \frac{x - \alpha_j}{\beta_i - \alpha_j} g_{a_j}(x)$.

Observe that $g_{a_i,a_j} \in C([\alpha_j, \beta_i])$ and satisfies $g_{a_i,a_j}(\alpha_j) = g_{a_i}(\alpha_j), g_{a_i,a_j}(\beta_i) = g_{a_j}(\beta_i)$, and

$$\left\|g_{a_j,a_i} - f\right\|_{L^{\infty}([\alpha_j,\beta_i],w)} \leq \left\|\frac{\beta_i - x}{\beta_i - \alpha_j}\left(g_{a_i}(x) - f(x)\right)\right\|_{L^{\infty}([\alpha_j,\beta_i],w)} + \left\|\frac{x - \alpha_j}{\beta_i - \alpha_j}\left(g_{a_i}(x) - f(x)\right)\right\|_{L^{\infty}([\alpha_j,\beta_i],w)} < 2\varepsilon.$$

If we define the function $g \in L^{\infty}(I,w) \in C(I)$ as

$$g(x) := \begin{cases} f(x) \text{if } x \in I \setminus \cup_i U_{a_i} \\ g_{a_i}(x) \text{if } x \in U_{a_i},\ x \notin \cup_{j \neq i} U_{a_j} \\ g_{a_i,a_j}(x) \text{if } x \in U_{a_i} \cap U_{a_j}, \end{cases}$$

we have $\|f - g\|_{L^{\infty}(I,w)} < 2\varepsilon$. If $f \in L^1(I)$ and S is countable, consider $[a_1, a_2, \dots] = S$.

If we take g_{a_n} with $\left\|f - g_{a_n}\right\|_{L^1(I)} < 2^{-n}\varepsilon$, it is direct that $\|f - g\|_{L^1(I)} < 2\varepsilon$.

This finishes the proof in this case. If I is not compact, we can choose an increasing sequence $\{a_n\}_{n \in \Lambda}$ of real numbers, where Λ is either Z^+, Z^-, or Z such that$I = \cup_n [a_n, a_{n+1}]$ and an is not singular for w in I if a_n is in the interior of I. We can take $\{a_n\}_{n \in \Lambda}$ with the following additional property: $\max_{n \in \Lambda} a_n = \max I$ if there exists $\max I$ and $\min_{n \in \Lambda} a_n = \min I$ if there exists $\min I$. We can reformulate this result as follows.

Theorem (4.2.19)[154]. Let us consider a closed interval I and a weight $w \in L^{\infty}_{loc}(I)$ such that the set S of singular points of w in I has zero Lebesgue measure.

Then we have $\overline{C^{\infty}(R) \cap L^{\infty}(I,w)} = \overline{C(R) - L^{\infty}(I,w)} = H$, with

$$H = \{f \in C(I \setminus S) \cap L^{\infty}(I,w) : \text{ for each } a \in S,\ \ \exists l_a \in R \text{ such that} \operatorname*{ess\,lim}_{x \in I, x \to a} |f(x) - l_a| w(x) = 0\},$$

where the closures are taken in $L^{\infty}(I,w)$. If $a \in S$ is of type 1, we can take as l_a any real number. If $a \in S$ is of type 2, $l_a = \operatorname{ess\,lim}_{w(x) \geq \varepsilon, x \to a} f(x)$ for $\varepsilon > 0$ small enough. Furthermore, if I is compact we also have $P^{0,\infty}(I,w) = H$. If $f \in H \cap L^1(I), I$ is compact and S is countable, we can approximate f by polynomials with the norm $\|\cdot\|_{L^{\infty}(I,w)} + \|\cdot\|_{L^1(I)}$.

Proof. We only need to show the equivalence of the following conditions (a) and (b):

(i) for each $a \in S$,

(ii) $\inf_{\varepsilon > 0}\left(\operatorname{ess\,lim\,sup}_{w(x) < \varepsilon, x \to a} |f(x)| w(x)\right) = 0$,

(iii) if a is of type 2, there exists the finite limit$l_a := \operatorname{ess\,lim}_{w(x) \geq \varepsilon, x \to a} f(x)$, for $\varepsilon > 0$ small enough,

(iv) for each $a \in S$, there exists $l_a \in R$ such that ess $\lim_{x\in I, x\to a}|f(x) - l_a|w(x) = 0$.
It is clear that (iv) implies (i). Hypothesis (ii) gives that for each $\eta > 0$, there exist $\varepsilon, \delta > 0$ with &$\|f\|_{L^\infty([a-\delta,a+\delta]\cap\{w(x)<\varepsilon\},w)} < \eta/3$ and $|l_a|\varepsilon < \eta/3$.
By hypothesis (iii) we can choose δ with the additional condition
$\|f - l_a\|_{L^\infty([a-\delta,a+\delta]\cap\{w(x)<\varepsilon\},w)} < \eta/3$. These inequalities imply

$$\|f - l_a\|_{L^\infty([a-\delta,a+\delta],w)} \leq \|f\|_{L^\infty([a-\delta,a+\delta]\cap\{w(x)<\varepsilon\},w)} + |l_a|\varepsilon + \|f - l_a\|_{L^\infty([a-\delta,a+\delta]\cap\{w(x)\geq\varepsilon\},w)} < \eta.$$

Corollary (4.2.20)[154]. Let us consider a closed interval I and a weight $w \in L^\infty_{loc}(I)$ such that the set S of singular points of w in I has zero Lebesgue measure.

If $f, g \in \overline{C(R) \cap L^\infty(I,w)}$ and $\varphi \in C(I) \cap L^\infty(I)$, then we also have $|f|, f_+, f_-, \max(f,g), \min(f,g), \varphi f \in \overline{C(R) \cap L^\infty(I,w)}$.

Proof. The characterization of $\overline{C(R) \cap L^\infty(I,w)}$ given in Theorem (4.2.19) implies the result for $|f|$ and φf. This fact and $\max(f,g) = \frac{f+g+|f-g|}{2}$, $\min(f,g) = \frac{f+g-|f-g|}{2}$ gives the result for $\max(f,g)$ and $\min(f,g)$. The facts $f_+ = \max(f,0)$, $f_- = \max(-f,0)$ finish the proof.Most of our resultsfor $k \geq 1$ use tools of Sobolev spaces. We include here the definitions that we need in order to understand these tools. First of all, we explain the definition of generalized Sobolev space in [149] for the particular case $p = \infty$ (the definition in [149] covers the cases $1 \leq p \leq \infty$, even if the weights are substituted by measures). One can think that the natural definition of weighted Sobolev space (the functions f with k weak derivatives satisfying $\|f^{(j)}\|_{L^\infty(w_j)} < \infty$ for $0 \leq j \leq k$) is a good one; however this is not true (see [155] or [149]).We start with some previous definitions.

Definition (4.2.21)[154]. We say that two functions u, v are comparable on the set A if there are positive constants c_1, c_2 such that $c_1 \leq u(x)/v(x) \leq c_2$ for almost every $x \in A$. We say that two norms $\|\cdot\|_1, \|\cdot\|_2$ in the vectorial space X are comparable if there are positive constants c_1, c_2 such that $c_1 \leq \|x\|_1/\|x\|_2 \leq c_2$ for every $x \in X$. We say that two vectorial weights are comparable if they are comparable on each component. (We use here the convention that $0/0 = 1$.) In what follows the symbol $a \asymp b$ means that a and b are comparable for a and b functions or norms.Obviously, the spaces $L^\infty(A,w)$ and $L^\infty(A,v)$ are the same and have comparable norms if w and v are comparable on A. Therefore, in order to study Sobolev spaces we can change a weight w by any comparable weight v.

We shall define a class of weights which plays an important role in our results.

Definition (4.2.22)[154]. We say that a weight w belongs to $B_\infty([a,b])$ if$w^{-1} \in L^1([a,b])$. Also, if J is any interval we say that $w \in B_\infty(J)$ if $w \in B_\infty(I)$ for every compact interval $I \subseteq J$. We say that a weight belongs to $B_\infty(J)$, where J is a union of disjoint intervals $\cup_{i\in A} J_i$, if it belongs to $B_\infty(J_i)$, for $i \in A$. Observe that if $v \geq w$ in J and $w \in B_\infty(J)$, then $v \in B_\infty(J)$.

Definition (4.2.23). We denote by $AC([a,b])$ the set of functions absolutely continuous in $[a,b]$, i.e. the functions $f \in C([a,b])$ such that

$$f(x) - f(a) = \int_a^x f'(t)dt$$

for all $x \in [a,b]$.

If J is any interval, $AC_{loc}(J)$ denotes the set of functions absolutely continuous in every compact subinterval of J.

Definition (4.2.24)[154]. Let us consider a vectorial weight $w = (w_0, \dots, w_k)$.
For $0 \le j \le k$ we define the open set $\Omega_j := \{x \in R : \exists$ an open neighbourhood V of x with $w_j \in B_\infty(V)\}$.

Observe that we always have $w_j \in B_\infty(\Omega_j)$ for any $0 \le j \le k$. In fact, Ω_j is the greatest open set U with $w_j \in B_\infty(U)$. Obviously, Ω_j depends on w, although w does not appear explicitly in the symbol Ω_j. It is easy to check that if $f^{(j)} \in L^\infty(\Omega_j, w_j)$ with $1 \le j \le k$, then $f^{(j)} \in L^1_{loc}(\Omega_j)$ and $f^{(j-1)} \in AC_{loc}(\Omega_j)$.
Hypothesis. From now on we assume that w_j is identically 0 in every point of the complement of Ω_j.
We need this hypothesis in order to have complete Sobolev spaces (see [155] and [149]).

The following definitions also depend on w, although w does not appear explicitly.
Let us consider $w = (w_0, \dots, w_k)$ a vectorial weight and $y \in 2$. To obtain a greater regularity of the functions in a Sobolev space we construct a modification of the weight w in a neighbourhood of y, using the following version (see[149]) of the Muckenhoupt inequality. This modified weight is equivalent in some sense to the original one .

Muckenhoupt inequality *I* (4.2.25)[154]. Let us consider w_0, w_1 weights in (a, b).
Then there exists a positive constant c such that $\left\| \int_x^b g(t)dt \right\|_{L^\infty([a,b],w_0)} \le c\|g\|_{L^\infty([a,b],w_1)}$
for any measurable function g in $[a, b]$, if and only if $\operatorname{ess\,sup}_{a<r<b} w_0(r) \int_r^b w_1^{-1} < \infty$.

Definition (4.2.26)[154]. A vectorial weight $\overline{w} = (\overline{w}_0, \dots, \overline{w}_k)$ is a right completion of a vectorial weight w with respect to y if $\overline{w}_k := w_k$ and there is an $\varepsilon > 0$ such that $\overline{w}_j := w_j$ in the complement of $[y, y+\varepsilon]$ and $\overline{w}_j := w_j + \widetilde{w}_j$, in $[y, y+\varepsilon]$ for $0 \le j < k$, where $\widetilde{w}_j$ is any weight satisfying:
(i) $\widetilde{w}_j \in L^\infty([y, y+\varepsilon])$,
(ii) $\Lambda_\infty(\widetilde{w}_j, \overline{w}_{j+1}) < \infty$, with

$$\Lambda_\infty(u, v) := \operatorname*{ess\,sup}_{y<r<y+\varepsilon} u(r) \int_r^{y+\varepsilon} v^{-1}.$$

Muckenhoupt inequality I guarantees that if $f^{(j)} \in L^\infty(w_j)$ and $f^{(j+1)} \in L^\infty(w_{j+1})$, then $f^{(j)} \in L^\infty(\overline{w}_j)$.

Example(4.2.27)[154]: It can be shown that the following construction is always a completion: we choose $\widetilde{w}_j := 0$ if $\overline{w}_{j+1} \notin B_\infty((y, y+\varepsilon])$; if $\overline{w}_{j+1} \notin B_\infty([y, y+\varepsilon])$ we set $\widetilde{w}_j(x) := 1$ in $[y, y+\varepsilon]$; and if $\overline{w}_{j+1} \notin B_\infty((y, y+\varepsilon]) \backslash B_\infty([y, y+\varepsilon])$ we take $\widetilde{w}_j(x) := 1$ for $x \in [y+\varepsilon/2, y+\varepsilon]$, and $\widetilde{w}_j(x) := \min\left\{1, \left(\int_x^{y+\varepsilon} \overline{w}_{j+1}^{-1}\right)^{-1}\right\}$, for $x \in (y, y+\varepsilon/2)$.

Definition (4.2.28)[154]. If w is a vectorial weight, we say that a point $y \in R$ is right j-regular (respectively, left j-regular), if there exist $\varepsilon > 0$, a right completion $\overline{w}$ (respectively, left completion) of w, and $j < i \le k$ such that $\overline{w}_i \in B_\infty([y, y+\varepsilon])$ (respectively, $B_\infty([y-\varepsilon, y])$). Also, we say that a point $y \in R$ is j-regular if it is right and left j-regular.

Remarks (4.2.29)[154].
(i) A point $y \in R$ is right j-regular (respectively, left j-regular), if at least one of the following properties is verified:
(ii) There exist $\varepsilon > 0$ and $j < i \le k$ such that $w_i \in B_\infty([y, y+\varepsilon])$ (respectively, $B_\infty([y-\varepsilon, y])$). Here we have chosen $\widetilde{w}_j = 0$.
(iii) There exist $\varepsilon > 0, j < i \le k, \alpha > 0$, and $\delta < i - j - 1$ such that

$w_i(x) \geq \alpha|x-y|^{\delta}$, for almost every $x \in [y, y+\varepsilon]$ (respectively, $[y-\varepsilon, y]$). See [149].
(iv) If y is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \leq i \leq j$.
(v) We can take $i = j+1$ in this definition since by the second remark to Definition (4.2.28)we can choose $\overline{w}_l = w_l + 1 \in B_\infty([\,y, y+\varepsilon])$ for $j < l < i$, if $j+1 < i$.
(vi) If y is not singular for w_j, then $y \in \Omega_j$ and y is $(j-1)$-regular.
When we use this definition we think of a point $\{b\}$ as the union of two half-points $\{b^+\}$ and $\{b^-\}$.
With this convention, each one of the following sets

$$(a,b) \cup (b,c) \cup \{b^+\} = (a,b) \cup [b^+, c) \neq (a,c),$$
$$(a,b) \cup (b,c) \cup \{b^-\} = (a,b^-] \cup (b,c) \neq (a,c),$$

has two connected components, and the set $(a,b) \cup (b,c) \cup \{b^-\} \cup \{b^+\} = (a,b) \cup (b,c) \cup \{b\} = (a,c)$ is connected. We only use this convention in order to study the sets of continuity of functions: we want that if $f \in C(A)$ and $f \in C(B)$, where A and B are union of intervals, then $f \in C(A \cup B)$. With the usual definition of continuity in an interval, if $f \in C\big([a,b)\big) \cap C([b,c])$ then we do not have $f \in C([a,c])$.
Of course, we have $f \in C([a,c])$ if and only if $f \in C([a,b^-]) \cap C([b^+,c])$, where by definition, $C([b+,c]) = C([b,c])$ and $C([a,b^-]) = C([a,b])$. This idea can be formalized with a suitable topological space. Let us introduce some notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $y \in \Omega^{(j)}$ if and only if y is j-regular, we say that $y^+ \in \Omega^{(j)}$ if and only if y is right j-regular, we say that $y^- \in \Omega^{(j)}$ if and only if y is left j-regular. Obviously, $\Omega^{(k)} < \emptyset$ and $\Omega_{j+1} \cup \ldots \cup \Omega_k \subseteq \Omega^{(j)}$.

Definition (4.2.30)[154]. We say that a function h belongs to the class $AC_{loc}\big(\Omega^{(j)}\big)$ if $h \in AC_{loc}(I)$ for every connected component I of $\Omega^{(j)}$.

Definition (4.2.31)[154] (Sobolev space). If $w = (w_0, \ldots, w_k)$ is a vectorial weight, we define the Sobolev space $W^{k,\infty}(\Delta, w)$ as the space of equivalence classes of
$V^{k,\infty}(2,w) := \{f : \Delta \to R/f^{(j)} \in AC_{loc}\big(\Omega^{(j)}\big)$ for $0 \leq j < k$ and $\|f^{(j)}\|_{L^\infty(\Delta, w_j)} < \infty$ for $0 \leq j \leq k\}$
with respect to the seminorm $\|f\|_{W^{k,\infty}(2,w)} := \sum_{j=0}^{k} \|f^{(j)}\|_{L^\infty(2,w_j)}$.

Definition (4.2.32)[154]. If w is a vectorial weight, let us define the space $\mathcal{K}(2,w)$ as $\mathrm{K}(2,w) := \Big\{g\colon \Omega^{(0)} \to R/g \in V^{k,\infty}\big(\overline{\Omega^{(0)}}, w\big), \|g\|_{W^{k,\infty}(\Omega^{(0)},w)} = 0\Big\}$.
$\mathrm{K}(2,w)$ is the equivalence class of 0 in $W^{k,\infty}\big(\overline{\Omega^{(0)}}, w\big)$. This concept and its analogue for $1 \leq p < \infty$ play an important role in the general theory of Sobolev spaces and in the study of the multiplication operator in Sobolev spaces in particular (see [149], [153], [150] and [151]).

Definition (4.2.33)[154]. If w is a vectorial weight, we say that (Δ, w) belongs to the class C_0 if there exist compact sets M_n, which are a finite union of compact intervals, such that
(i) M_n intersects at most a finite number of connected components of $\Omega_1 \cup \ldots \cup k$,
(ii) $\mathrm{K}(M_n, w) = \{0\}$,
(iii) $M_n \subseteq M_{n+1}$,
(iv) $\bigcup_n M_n = \Omega^{(0)}$.

In this section we collect the theorems we need in or to prove the results on it.

The next results, proved in [149], [153], and [150], play a central role in the theory of Sobolev spaces with respect to measures (see [149]). We present here a weak version of these theorems which are enough for our purposes.

Theorem (4.2.34)[154]. Let $w = (w_0, \dots, w_k)$ be a vectorial weight. Let K_j be a finite union of compact intervals contained in $\Omega^{(j)}$, for $0 \le j < k$, and $\overline{w}$ a right (or left) completion of w.
If $(\Delta, w) \in C_0$, then there exist positive constants $c_1 = c_1(K_0, \dots, K_{k-1})$ and $c_2 = c_2(\overline{w}, K_0, \dots, K_{k-1})$ such that $c_1 \sum_{j=0}^{k-1} \|g^{(j)}\|_{L^\infty(K_j)} \le \|g\|_{W^{k,\infty}(\Delta,w)}, c_2 \|g\|_{W^{k,\infty}(\Delta,\overline{w})} \le \|g\|_{W^{k,\infty}(\Delta,w)}, \quad \forall g \in V^{k,\infty}(\Delta, w).$

Theorem (4.2.35)[154]. Let us consider a vectorial weight $w = (w_0, \dots, w_k)$. Assume that we have either (i) $(\Delta, w) \in C_0$ or (ii) $\Omega_1 \cup \dots \cup \Omega_k$ has only a finite number of connected components. Then the Sobolev space $W^{k,\infty}(\Delta, w)$ is complete.

Proposition (4.2.36)[154]. Let $w = (w_0, \dots, w_k)$ be a vectorial weight in $[a, b]$, with $w_{k_0} \in B_\infty((a, b])$ for some $0 < k_0 \le k$. If we construct a right completion $\overline{w}$ of w with respect to the point a taking $\varepsilon = b - a$, and $\overline{w}_j = w_j$ for $k_0 \le j \le k$, then there exist positive constants c_j such that $c_j \|g^{(j)}\|_{L^\infty([a,b],\overline{w}_j)} \sum_{i=j}^{k_0} \|g^{(i)}\|_{L^\infty([a,b],w_i)} + \sum_{i=j}^{k_0-1} |g^{(i)}(b)|$, for all $0 \le j < k_0$ and $g \in V^{k,\infty}([a, b], w)$.
In particular, there is a positive constant c such that

$$c\|g\|_{W^{k,\infty}([a,b],\overline{w})} \le \|g\|_{W^{k,\infty}([a,b],w)} + \sum_{j=0}^{k_0-1} |g^{(j)}(b)|, \text{ for all } g \in V^{k,\infty}([a, b], w).$$

The following is a particular case in [149].

Corollary (4.2.37)[149]. Let us consider a vectorial weight $w = (w_0, \dots, w_k)$. Let K_j be a finite union of compact intervals contained in $\Omega^{(j)}$, for $0 \le j < k$. If$(\Delta, w) \in C_0$, then there exists a positive constant $c_1 = c_1(K_0, \dots, K_{k-1})$ such that

$$c_1 \sum_{j=0}^{k-1} \|g^{(j+1)}\|_{L^1(K_j)} \le \|g\|_{W^{k,\infty}(\Delta,w)}, \forall g \in V^{k,\infty}(\Delta, w).$$

Corollary (4.2.38)[154]. Let us consider a vectorial weight $w = (w_0, \dots, w_k)$. For some $0 < m \le k$, assume that $(2, (w_m, \dots, w_k)) \in C_0$. Let K be a finite union of compact intervals contained in $\Omega^{(m-1)}$. Then there exists a positive constant $c_1 = c_1(K)$ such that

$$c_1 \|g\|_{L^1(K)} \le \|g\|_{W^{k-m,\infty}(\Delta,w)}, \forall g \in V^{k-m,\infty}(2, w).$$

In [153] and its remark give the following result.

Theorem (4.2.39)[154]. Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ with $(\Delta 2, w) \in C_0$. Assume that K is a finite union of compact intervals $J_1, \dots, J_n$ and that for every J_m there is an integer $0 \le k_m \le k$ verifying $J_m \subseteq \Omega^{(k_m\&-1)}$, if $k_m > 0$, and $\int_{J_m} w_j = 0$ for $k_m < j \le k$, if $k_m < k$. If $w_j \in L^\infty(K)$ for $0 < j \le k$, then there exists a positive constant c_0 such that
$c_0 \|fg\|_{W^{k,\infty}(\Delta,w)} \le \|f\|_{W^{k,\infty}(\Delta,w)} (\sup_{x\in\Delta} |g^{(x)}| + \|g\|_{W^{k,\infty}(\Delta,w)})$,for every $f, g \in V^{k,\infty}(\Delta, w)$
with $g' = g'' = \dots = g^{(k)} = 0$ in $\Delta \backslash K$.
In [153] implies the following result.
Corollary (4.2.40)[154]. Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ in $(a, b)$with $w_k \in B_\infty([a, b]), w \in L^\infty([a, b])$ and $K([a, b], w) = \{0\}$.
Then there exists a positive constant c_0 such that

$$c_0 \|fg\|_{W^{k,\infty}([a,b],w)} \le \|f\|_{W^{k,\infty}([a,b],w)} \|g\|_{W^{k,\infty}([a,b],w)}$$

for every $f, g \in V^{k,\infty}([a, b], w)$.

Lemma (4.2.41)[154]. Let us consider $w=(w_0,\dots,w_k)$ a vectorial weight with $w_{j+1}(x)\le c_1|x-a_0|w_j(x)$,for $0\le j<k, a_0\in \boldsymbol{R}$ and x in an interval I. Let $\varphi\in C^k(\boldsymbol{R})$ be such that $\sup\varphi'\subseteq[\lambda,\lambda+t]$, with $\max\{|\lambda-a_0|,|\lambda+t-a_0|\}\le c_2\, t$ and $\|\varphi^{(j)}\|_{L^\infty(I)}\le c_3t^{-j}$for $0\le j\le k$.

Then, there is a positive constant c_0 which is independent of $I, a_0, \lambda, t, w, \varphi$, and g such that $\|\varphi g\|_{W^{k,\infty}(\Delta,w)}\le c_0\|g\|_{W^{k,\infty}(I,w)}$,for every $g\in V^{k,\infty}(\Delta,w)$ with $\operatorname{supp}(\varphi g)\subseteq I$

Corollary (4.2.42)[154]. Let us consider a compact interval I and a vectorial weight $w=(w_0,\dots,w_k)\in L^\infty(I)$. Assume that there exist $a_0\in I$, an integer $0\le r<k$, and constants $c,\delta>0$ such that $w_{j+1}(x)\le c|x-a_0|w_j(x)$ in $[a_0-\delta,a_0+\delta]\cap I$, for $r\le j<k$. Then a_0 is neither right nor left r-regular.We define now the following functions,

$$\log_1 x=-\log x\ ,\log_2 x=\log(\log_1 x),\dots,\log_n x=\log(\log_{n-1} x).$$

A computation involving Muckenhoupt inequality gives the following result.

Proposition (4.2.43)[154]. Let us consider a compact interval I and a vectorial weight $w=(w_0,\dots,w_k)\in L^\infty(I)$. Assume that there exist $a_0\in I$, an integer $0\le r<k, n\in\mathbf{N}, \delta, c_i>0, \varepsilon_i\le 0$, and $\alpha_i,\gamma_1^i,\dots,\gamma_n^i\in$ R for $r\le i\le k$ such that

(i) $w_i(x)\asymp e^{-c_i|x-a_0|^{-\varepsilon_i}}|x-a_0|^{\varepsilon_i}\log_1^{\gamma_1^i}|x-a_0|\cdots\log_n^{\gamma_n^i}|x-a_0|$

for $x\in[a_0-\delta,a_0+\delta]\cap I$ and $r\le i\le k$,

(ii) $\alpha_i\notin\mathbf{N}$if$\varepsilon_i=0$ and $r<i\le k$.

Then there exists a completion $\bar{w}$ of w such that the Sobolev norms $W^{k,p}(I,w)$ and $W^{k,p}(I,\bar{w})$ are comparable and there exists $r\le r_0\le k$ with

$\bar{w}_{j+1}(x)\le c|x-a_0|\bar{w}_j(x)$ in $[a_0-\delta,a_0+\delta]\cap I$, for $r_0\le j<k$ if $r_0<k$, and $w_{r_0}\in B_\infty([a_0-\delta,a_0+\delta]\cap I)$. In particular, a_0 is (r_0-1)-regular if $r_0>0$.First of all, the next results allow us to deal with weights which can be obtained by "gluing" simpler ones.

Theorem (4.2.44)[154]. Let us consider $-\infty<a<b<c<d<\infty$.

Let $w=(w_0,\dots,w_k)$ be a vectorial weight in (a,d) and assume that there exists an interval $I\subseteq[b,c]$ with $w\in L^\infty(I)$ and $(I,w)\in C_0$. Then f can be approximated by functions of C^∞(R) in $W^{k,\infty}([a,d],w)$ if and only if it can be approximated by functions of C^∞(R) in $W^{k,\infty}([a,c],w)$ and $W^{k,\infty}([b,d],w)$.

Proof. $[\alpha,\beta]\subset I$ prove the non-trivial implication. Let us consider $J=[\alpha,\beta]\subset I$ and an integer $0\le k_1\le k$, such that $J\subset(b,c)_{k_1}\subseteq(b,c)^{(k_1-1)}$ if $k_1>0$,$\int_J w_j=0$ for $k_1<j\le k$ if$k_1<k$.

Let us consider $f\in V^{k,\infty}([a,d],w)$ and $\varphi_1,\varphi_2\in C^\infty$(R) such that φ_1 approximates f in $W^{k,\infty}([a,c],w)$ and φ_2 approximates f in $W^{k,\infty}([b,d],w)$.Set $\theta\in C^\infty$(R) a fixed function with $0\le\theta\le1, \theta=0$ in $(-\infty,\alpha]$ and $\theta=1$ in$[\beta,\infty)$. It is enough to see that $\theta\varphi_2+(1-\theta)\varphi_1$ approximates f in $W^{k,\infty}([a,d],w)$ or, equivalently, in $W^{k,\infty}(I,w)$. Theorem C with $\Delta=I$ and $K=J$ gives

$$\begin{aligned}\|f-\theta\varphi_2-(1-\theta)\varphi_1\|_{W^{k,\infty}(I,w)}&\le\|\theta(f-\varphi_2)\|_{W^{k,\infty}(I,w)}+\|(1-\theta)(f-\varphi_1)\|_{W^{k,\infty}(I,w)}\\&\le c\big(\|f-\varphi_2\|_{W^{k,\infty}(I,w)}+\|f-\varphi_1\|_{W^{k,\infty}(I,w)}\big),\end{aligned}$$

and this finishes the proof of the theorem.

Theorem (4.2.45)[154]. Let us consider strictly increasing sequences of real num- bers $\{a_n\},\{b_n\}$ (n belonging to a finite set, to Z, Z^+, or Z^-) with $a_{n+1}<b_n$ for every n. Let $w=(w_0,\dots,w_k)$ be a vectorial weight in $(\alpha,\beta):=\cup_n(a_n,b_n)$ with $-\infty<\alpha<\beta<\infty$.

Assume that for each n there exists an interval $I_n\subseteq[a_{n+1},b_n]$ with $w\in L^\infty(I_n)$ and

$(I_n, w) \in C_0$.Then f can be approximated by functions of $C^\infty(\mathrm{R})$ in $W^{k,\infty}([\alpha,\beta],w)$ if and only if it can be approximated by functions of $C^\infty(\mathrm{R})$ in $W^{k,\infty}([a_n,b_n],w)$ for each n.

Proof. We prove the non-trivial implication. Let us consider $\varphi_n \in C^\infty(\mathrm{R})$ which approximates f in $W^{k,\infty}([a_n,b_n],w)$. By the proof of Theorem (4.3.44) we know that there are $\theta_n \in C^\infty(\mathrm{R})$ and positive constants c_n such that

$$\|f-\theta_n\varphi_{n+1}-(1-\theta_n)\varphi_n\|_{W^{k,\infty}(I_n,w)} \le c_n\big(\|f-\varphi_n\|_{W^{k,\infty}(I_n,w)}+\|f-\varphi_{n+1}\|_{W^{k,\infty}(I_n,w)}\big).$$

Now, given $\varepsilon>0$, it is enough to approximate f in $[a_n,b_n]$ with error less than$\varepsilon\min\{1,c_n^{-1},c_{n-1}^{-1}\}/2$.

Theorem (4.2.46)[154]. Let us consider a compact interval I and a vectorial weight $w=(w_0,\dots,w_k)\in L^\infty(I)$ such that $w_k \in B_\infty(I)$. Then we have

$$\begin{aligned} P^{k,\infty}(I,w) = H_3 &:= \{f \in V^{k,\infty}(I,w)/f^{(k)}P^{0,\infty}(I,w_k)\} \\ &= H_0 := \{f \in V^{k,\infty}(I,w)/f^{(j)} \in P^{0,\infty}(I,w_j), \text{for } 0\le j\le k\} \\ &= \{f: I \to \mathrm{R}/f^{(k-1)} \in AC(I) \text{and} f^{(k)} \in P^{0,\infty}(I,w_k)\}. \end{aligned}$$

Proof. We prove first $H_3 \subseteq P^{k,\infty}(I,w)$. If $f\in H_3$, let us consider a sequence $\{q_n\}$ of polynomials which converges to $f^{(k)}$ in $L^\infty(I,w_k)$. Let us choose $a\in I$.

Then the polynomials $\mathrm{Q}_n(x) := f(a)+f'(a)(x-a)+\cdots+f^{(k-1)}(a)\frac{(x-a)^{k-1}}{(k-1)!}+\int_a^x q_n(t)\frac{(x-t)^{k-1}}{(k-1)!}dt$ satisfy $\mathrm{Q}_n^{(j)}(x)=f^{(j)}(a)+\cdots+f^{(k-1)}(a)\frac{(x-a)^{k-j-1}}{(k-j-1)!}+\int_a^x q_n(t)\frac{(x-t)^{k-j-1}}{(k-j-1)!}dt,$ for $0\le j<k$.

Therefore, for $0\le j<k$,

$$\begin{aligned} \left|f^{(j)}(x)-\mathrm{Q}_n^{(j)}(x)\right| = \left|\int_a^x\left(f^{(k)}(t)-q_n(t)\right)\frac{(x-t)^{k-j-1}}{(k-j-1)!}dt\right| &\le c\int_I\left|f^{(k)}(t)-q_n(t)\right|w_k(t)w_k(t)-1dt \\ &\le c\left\|f^{(k)}-q_n\right\|_{L^\infty(I,w_k)}. \end{aligned}$$

Hence, we have for $0\le j<k$, $\left\|f^{(j)}-\mathrm{Q}_n^{(j)}\right\|_{L^\infty(I,w_j)} \le c\left\|f^{(k)}-q_n\right\|_{L^\infty(I,w_k)}$,

since $w_j\in L^\infty(I)$. Then we have obtained that $f\in P^{k,\infty}(I,w)$.

Since $\Omega_k=\mathrm{int}(I),\Omega_1\cup\cdots\cup\Omega_k=\mathrm{int}(I)$ is connected and Theorem (4.2.35) gives that $W^{k,\infty}(I,w)$ is complete; therefore $P^{k,\infty}(I,w)\subseteq H_0$. The content $H_0\subseteq H_3$ is direct. The last equality is also direct since the fact $w_k\in B_\infty(I)$ gives $\Omega^{(k-1)}=I$. Then $f^{(k-1)}\in AC(I)$ for every $f\in V^{k,\infty}(I,w)$.

Theorem (4.2.47)[154]. Let us consider a compact interval I and a vectorial weight $w=(w_0,\dots,w_k)\in L^\infty(I)$, the set S of singular points for w_k in I has zero Lebesgue measure.

Assume that there exist $a_0\in I$, an integer $0\le r<k$, and constants $c,\delta>0$ such that

(i) $w_{j+1}(x)\le c|x-a_0|w_j(x)\ in\ [a_0-\delta,a_0+\delta]\cap I, for\ r\le j<k,$

(ii) $w_k\in B_\infty(I\backslash\{a_0\})$,

(iii) if $r>0, a_0\ is\ (r-1)$-regular. Then we have

$$P^{k,\infty}(I,w)=H_4:=\{f\in V^{k,\infty}(I,w)/f^{(k)}\in P^{0,\infty}(I,w_k),\exists l\in \mathrm{R}\ with$$

$$\operatorname*{ess\,lim}_{x\in I,x\to a_0}\left|f^{(r)}(x)-l\right|w_r(x)=0, and\ \operatorname*{ess\,lim}_{x\in I,x\to a_0} f^{(j)}(x)w_j(x)=0, for\ r<j<k \text{ if } r<k-1\}$$

$$\begin{aligned} H_0 &:= \{f\in V^{k,\infty}(I,w)/f^{(j)}\in P^{0,\infty}(I,w_j), for\ 0\le\ j\le k\} \\ &= \{f: I\to R/f(k-1)\in AC_{loc}(I\backslash\{a_0\})\ , f^{(k)}\in P^{0,\infty}(I,w_k),\exists l\in\mathrm{R}\ with \\ &\qquad \operatorname*{ess\,lim}_{x\in I,x\to a_0}\left|f^{(r)}(x)-l\right|w_r(x)=0, \operatorname*{ess\,lim}_{x\in I,x\to a_0} f^{(j)}(x)w_j(x)=0, \\ &\qquad \text{for } r\le j<k \text{ if } r<k-1, \text{and } f^{(r-1)}-AC(I) \text{if } r>0\}. \end{aligned}$$

Proof. We prove first $H_4 \subseteq P^{k,\infty}(I,w)$. Let us take $f \in H_4$. Without loss of generality we can assume that a_0 is an interior point of I, since the argument is simpler if $a_0 \in \partial I$. Without loss of generality we can assume also $l = 0$, since in other case we can consider $f^{(x)} - lx^r/r!$ instead of $f(x)$ (recall that $w \in L^\infty(I)$). Consider now a function $\varphi \in C_c^\infty(\mathrm{R})$ with $\varphi = 1$ in $[-1,1], \varphi = 0$ in $\mathrm{R}\backslash(-2,2)$, and $0 \le \varphi \le 1$ in R. For each $n \in \mathbf{N}$, let us define $\varphi_n(x) := \varphi\big(n(x - a_0)\big)$ and $h_n := (1 - \varphi_n)f(r)$.

We have $\|f^{(r)} - h_n\|_{W^{k-r,\infty}(I,w)} = \|\varphi_n f^{(r)}\|_{W^{k-r,\infty}(I,w)} \le c_0 \|f^{(r)}\|_{W^{k-r,\infty}([a_0-2/n,a_0+2/n],w)}$, since we are in the hypotheses of Lemma(4.2.41), where $\lambda = a_0 - 2/n, t = 4/n$, and we consider the interval $[a_0 - 2/n\,, a_0 + 2/n]$: observe that $|\lambda - a_0| = |\lambda + t - a_0| = 2/n = t/2$ and

$$\left\|\varphi_n^{(j)}\right\|_{L^\infty(\mathbf{R})} = n^j \|\varphi^{(j)}\|_{L^\infty(\mathbf{R})} \le 4^k \max\left\{\|\varphi\|_{L^\infty(\mathbf{R})}, \|\varphi'\|_{L^\infty(\mathbf{R})}, \dots, \left\|\varphi^{(k)}\right\|_{L^\infty(\mathbf{R})}\right\} t^{-j}.$$

Hence, we deduce that $\|f^{(r)} - h_n\|_{W^{k-r,\infty}(I,w)} \to 0$as $n \to \infty$, sinceess $\lim_{x\in I, x\to a_0} f^{(j)}(x) w_j(x) = 0$, for each $r \le j \le k$ (Lemma (4.2.9) gives the result for $j = k$ since hypotheses $w_r \in L^\infty(I)$ and (i) give that a_0 is a singularity of type 1 for wk in I). Therefore, in order to see that $f^{(r)}$ can be approximated by polynomials in $W^{k-r,\infty}(I,w)$ it is enough to see that each h_n can be approximated by polynomials in $W^{k-r,\infty}(I,w)$. Consider weights $w_n := \big(w_0, \dots, w_{k-1}, w_{k,n}\big)$ with $w_{k,n} := w_k + \chi_{[a_0-1/n, a_0+1/n]} \ge w_k$.

It is direct that $w_n \in L^\infty(I)$ and $w_{k,n} \in B_\infty(I)$. Observe that Corollary(4.2.20) gives $h_n^{(k-r)} \in P^{0,\infty}(I, w_k)$, since $h_n^{(k-r)} = (1 - \varphi_n) f^{(k)} + F_n$, with $F_n = -\sum_{i=1}^{k-r} \binom{k-r}{i} \varphi_n^{(i)} f^{(k-i)} \in C(I)$ and $1 - \varphi_n \in C(I)$. Hence Theorem (4.2.46) implies that each h_n can be approximated by polynomials in $W^{k-r,\infty}(I, w^n)$ and consequently in $W^{k-r,\infty}(I,w)$. Therefore, f can be approximated by polynomials in $W^{k-r,\infty}(I,w)$. This finishes the proof if $r = 0$.

In other case, hypotheses (ii) and (iii) give $\Omega^{(r-1)} = I$ and consequently $f^{(r-1)} \in AC(I)$.

Without loss of generality we can assume that there exists $\varepsilon > 0$ such that $[a_0 - \varepsilon, a_0 + \varepsilon]$ is contained in the interior of I and $w_r \ge 1$ in $I\backslash[a_0 - \varepsilon, a_0 + \varepsilon]$. In the other case we can change w by w^* with$w_*^j := w_j$ if $\neq r$ and $w_*^r := w_r + \chi_{I\backslash[a_0-\varepsilon, a_0+\varepsilon]}$. It is obvious that it is more complicated to approximate f in $W^{k,\infty}(I, w^*)$ than in $W^{k,\infty}(I,w)$. Therefore, we have $\mathrm{K}\big([a,b], (w_r, \dots, w_k)\big) = \{0\}$ and $\big([a,b], (w_r, \dots, w_k)\big) \in \mathrm{C}_0$ (see Definition (4.2.33)).

Let us consider a sequence $\{q_n\}$ of polynomials converging to $f^{(r)}$ in $W^{k-r,\infty}(I,w)$.Corollary (4.2.38) gives$\|f^{(r)} - q_n\|_{L^1(I)} \le c\|f^{(r)} - q_n\|_{W^{k-r,\infty}(I,w)}$.

The polynomials defined by

$\mathrm{Q}_n(x) := f(a) + f'(a)(x-a) + \cdots + f(r-1)(a)\frac{(x-a)^{r-1}}{(r-1)!} + \int_a^x q_n(t)\frac{(x-t)^{r-1}}{(r-1)!}dt$, Satisfy

$$\|f - \mathrm{Q}_n\|_{W^{k,\infty}(I,w)} \le c\|f^{(r)} - q_n\|_{L^1(I)} + \|f^{(r)} - q_n\|_{W^{k-r,\infty}(I,w)} \le c\|f^{(r)} - q_n\|_{W^{k-r,\infty}(I,w)},$$

and we conclude that the sequence of polynomials $\{\mathrm{Q}_n\}$ converges to f in $W^{k,\infty}(I,w)$.

Since $\Omega_k = \mathrm{int}(I)\backslash\{a_0\}, \Omega_1 \cup \dots \cup \Omega_k$ has at most two connected components and Theorem (4.2.35) gives that $W^{k,\infty}(I,w)$ is complete; therefore $P^{k,\infty}(I,w) \subseteq H_0$. Observe that hypotheses $w_r \in L^\infty(I)$ and (1) give that a_0 is a singularity of type 1 for w_j in I, for each $r < j \le k$. By Theorem (4.2.19) there exists $l \in \mathrm{R}$ with ess $\lim_{x\in I, x\to a_0} |f^{(r)}(x) - l|\, w_r(x) = 0$, if a_0 is a singularity for w_r in I; in the other case, it is a direct consequence of the continuity of $f(r)$ in a_0. This fact and Lemma (4.2.9) give $H_0 \subseteq H_4$.

The last equality is direct by the definition of $V^{k,\infty}(I,w)$; it is enough to remark that Corollary (4.2.42) and (ii) give $\Omega^{(r)} = \Omega^{(r+1)} = \cdots = \Omega^{(k-1)} = I\backslash\{a_0\}$, and (ii) and (iii) give $\Omega^{(r-1)} = I$ if $r > 0$.

If we apply Theorem (4.2.44), Theorem (4.2.47), and Proposition (4.2.43), we obtain the next result for Jacobi-type weights.

Corollary (4.2.48). Consider a vectorial weight w such that $w_j(x) \asymp (x-a)^{\alpha_j}(b-x)^{\beta_j}$ with $\alpha_j, \beta_j \geq 0$ for $0 \leq j \leq k$. Assume that there exist $0 \leq r_1, r_2 < k$ such that a is
r_1-right regular if $r_1 > 0, b$ is r_2-left regular if $r_2 > 0$, and verifying either
(i) $\alpha_{j+1} \geq \alpha_j + 1$ for $r_1 \leq j < k$ and $\beta_{j+1} \geq \beta_j + 1$ for $r_2 \leq j < k$,
(ii) $\alpha_j \in [0,\infty)\backslash Z^+$ for $r_1 < j \leq k$, and $\beta_j \in [0,\infty)\backslash Z^+$ for$r_2 < j \leq k$.
Then$P^{k,\infty}([a,b],w) = \{f \in V^{k,\infty}([a,b],w)/f^{(j)} \in P^{0,\infty}([a,b],w_j),\ \text{for}\ \ 0 \leq j \leq k\}$.

Lemma (4.2.49)[154]. Let a weight $w \in B_\infty([a-2\delta, a+2\delta]\backslash\{a\}) \cap L^\infty([a-2\delta, a+2\delta])$ and a function $f \in AC_{loc}([a-2\delta, a+2\delta]\backslash\{a\})$, continuous in a and verifying
$f' \in P^{0,\infty}([a-2\delta, a+2\delta], w)$.Assume set S of singular pointsof w in $[a-2\delta, a+2\delta]$ has zero Lebesgue measure.
Then for each $\varepsilon > 0$ there exists a function $g \in AC([a-2\delta, a+2\delta])$
with $g' \in P^{0,\infty}([a-2\delta, a+2\delta], w)$, such that $g = f$ in $[a-2\delta, a-\delta] \cup [a+\delta, a+2\delta]$ and

$$\|f - g\|_{L^\infty([a-2\delta,a+2\delta])} + \|f' - g'\|_{L^\infty([a-2\delta,a+2\delta],w)} < \varepsilon .$$

Proof. Theorem (4.2.19)gives that there exists $l \in R$ with $\operatorname{ess\,lim}_{x\to a}|f'(x) - l|\, w(x) = 0$. Without loss of generality we can assume that $l = 0$, since in the other case we can consider $f(x) - lx$ instead of $f(x)$.We construct the function g in the interval $[a-2\delta, a]$. The construction in $[a, a+2\delta]$ is symmetric. If $f' \in L^1([a-2\delta, a])$, we take $g = f$ in $[a-2\delta, a]$. If $f' \in L^1([a-2\delta, a])$, the facts $f(x) = \int_{a-2\delta}^{x} f'$ for $x \in [a-2\delta, a)$ and f continuous in a give that
$(f')_+, (f')_- \in L^1_{loc}([a-2\delta, a))\backslash L^1([a-2\delta, a])$.
Assume now that $a \in \text{ess cl}\{x \in [a-2\delta, a): f(x) < f(a)\}$.
If $a \in \text{ess cl}\{x \in [a-2\delta, a): f(x) > f(a)\}$ the argument is symmetric.
If $a \notin \text{ess cl}\{x \in [a-2\delta, a): f(x) < f(a)\} \cup \text{ess cl}\{x \in [a-2\delta, a): f(x) > f(a)\}$ then
$f(x) = f(a)$ for $x \in [a-\delta_0, a]$, which contradicts $f' \notin L^1([a-2\delta, a])$.

We claim that $a \in \text{ess cl}\{x \in [a-2\delta, a): f(x) < f(a), f'(x) \geq 0\}$. If it is not true there exists $\delta_1 > 0$ with $|\{x \in (a-\delta_1, a): f(x) < f(a), f'(x) \geq 0\}| = 0$. Consider $x_0 \in (a-\delta_1, a)$ with $f(x_0) < f(a)$. Since f is continuous in x_0, there exists $\delta_2 > 0$ with $f(x) < f(a)$ for $x \in [x_0, x_0+\delta_2)$.

Then $f' < 0$ in almost every point in $[x_0, x_0+\delta_2)$, and consequently
$f(x) - f(x_0) = \int_{x_0}^{x} f' < 0$.

By this argument it is clear that the set $\{x \in [x_0, a): f(x) \leq f(x_0)\}$ is open and closed in $[x_0, a)$; therefore $f(x) \leq f(x_0) < f(a)$ for $x \in [x_0, a)$, which contradicts f continuous in a.

Since $|S| = 0$, for each $\varepsilon > 0$ there exists $\alpha \in [a-\delta, a)\backslash S$ with
$f(\alpha) < f(a), f'(\alpha) \geq 0, \|f'\|_{L^\infty([\alpha,a],w)} < \varepsilon/4$ and $|f(x) - f(a)| < \varepsilon/4$ for $x \in [\alpha, a]$.
Consider the family of functions $p_{\lambda,\mu}$ in $[\alpha, a]$ defined as follows:
for each $\lambda \geq 0$ and $0 < \mu < (a-\alpha)/2, p_{\lambda,\mu}$is the function whose graphic is the segment joining $(\alpha, f'(\alpha))$ and $(\alpha+\mu, \lambda)$ in $[\alpha, \alpha+\mu]$, the segment joining $(a-\mu, \lambda)$ and $(a, 0)$ in $[a-\mu, a]$, and is equal to λ in $[\alpha+\mu, a-\mu]$.

It is clear that there exists $\lambda \geq 0$ and $0 < \mu < (a-\alpha)/2$ such that the function

$$h_{\lambda,\mu}(x) := \begin{cases} f'(x) & \text{if } x \in [a-2\delta, \alpha] \\ \min\left((f')+(x), p_{\lambda,\mu}(x)\right) & \text{if } x \in (\alpha, a], \end{cases}$$ verifies

$f(a) - f(\alpha) = \int_\alpha^a h_{\lambda,\mu}$, since $(f')_+ \in L^1_{loc}([a-2\delta, a)) \backslash L^1([a-2\delta, a])$. Observe that $h_{\lambda,\mu} \in L^1([a-2\delta, a]) \cap P^{0,\infty}([a-2\delta, a], w)$ (see Theorem (4.2.19) and Corollary (4.2.20)).

For this particular choice of λ and μ, we define $g(x) := f(a) + \int_a^x h_{\lambda,\mu}$ in $[a-2\delta, a]$. We define g in $[a, a+2\delta]$ in a similar way. Conditions $f(a) - f(\alpha) = \int_\alpha^a h_{\lambda,\mu}$ and $h_{\lambda,\mu} = f'$ in $[a-2\delta, \alpha]$ give g = f in $[a-2\delta, \alpha]$. Since $h_{\lambda,\mu}$ does not change its sign in $[\alpha, a]$, we have

$|g(x) - g(a)| \leq |g(\alpha) - g(a)| = \left|\int_\alpha^a h_{\lambda,\mu}\right| = |f(a) - f(\alpha)| < \varepsilon/4$ for every $x \in [\alpha, a]$.

Therefore $|g(x) - f(x)| < \varepsilon/2$ for $x \in [\alpha, a]$ and $\|f - g\|_{L^\infty([a-2\delta,a])} < \varepsilon/2$. We also have $|g' - (x)| \leq |f'(x)|$ in $[\alpha, a]$ and therefore $\|f' - g\|_{L^\infty([a-2\delta,a],w)} \leq 2\|f'\|_{L^\infty([\alpha,a],w)} < \varepsilon/2$.

This finishes the proof of the lemma.

Theorem (4.2.50)[154]. Let us consider a compact interval $I := [a, b]$ and a vectorial weight $w = (w_0, \dots, w_k) \in L^\infty(I)$. Assume that there exists a finite set $R \subset I$ such that

(i) the points of R are singularities for w_k in I,

(ii) $w_k \in B_\infty(I \backslash R)$,

(iii) the points of R are not singular for $w_k - 1$ in I,

(iv) the set S of singular points for w_k in I is countable.

Then we have

$$\begin{aligned} P^{k,\infty}(I,w) = H_5 &:= \{f \in V^{k,\infty}(I,w)/f^{(k)} \in P^{0,\infty}(I,w_k) \text{ and } f(k-1) \text{ is continuous in each point of R}\} \\ = H_0 &:= \{f \in V^{k,\infty}(I,w)/f^{(j)} \in P^{0,\infty}(I,w_j), \text{for } 0 \leq j \leq k\} \\ &= \{f: I \to R/f^{(k)} \in P^{0,\infty}(I,w_k), f(k-1) \in AC_{loc}(I\backslash R), \text{and } f^{(k-1)} \text{ is continuous in each point of R}\}. \end{aligned}$$

Proof. We prove first $H_5 \subseteq P^{k,\infty}(I, w)$. Consider a function $f \in H_5$.

Condition (ii) gives $f^{(k-1)} \in AC_{loc}(I \backslash R)$. Given $n \in \mathbf{N}$, if we apply a finite number of times Lemma (4.2.49) to the function $f^{(k-1)}$, we obtain a function $g_n \in AC(I)$ with $g_n' \in P^{0,\infty}(I, w_k)$ and

$$\left\|f^{(k-1)} - g_n\right\|_{L^\infty(I)} + \left\|f^{(k)} - g_n'\right\|_{L^\infty(I,w_k)} < \frac{1}{n},$$

since $f^{(k-1)}$ is continuous in each point of R, $w_k \in L^\infty(I)$ and $|S| = 0$. If $k \geq 2$, conditions (ii) and (iii) give $\Omega^{(k-2)} = I$; hence $f^{(k-2)} \in AC([a,b])$ and the functions

$$G_n(x) := f(a) + f'(a)(x-a) + \dots + f^{(k-1)}(a)\frac{(x-a)^{k-2}}{(k-2)!} + \int_a^x g_n(t)\frac{(x-t)^{k-2}}{(k-2)!}dt,$$

verify

$$f^{(j)}(x) - G_n^{(j)}(x) = \int_a^x \left(f^{(k-1)}(t) - g_n(t)\right)\frac{(x-t)^{k-j-2}}{(k-j-2)!}dt,$$

for $0 \leq j \leq k-2$, if $k \geq 2$.

Consequently, since $w \in L^\infty(I)$, we have for any

$k \geq 1 \|f - G_n\|_{W^{k,\infty}(I,w)} \leq c\left\|f^{(k-1)} - g_n\right\|_{L^\infty(I)} + \left\|f^{(k)} - g_n'\right\|_{L^\infty(I,w_k)} \to 0$, as $n \to \infty$.

For each $n \in \mathbf{N}$, since $g_n' \in L^1(I)$, I is compact and S is countable, by Theorem (4.2.19) we can approximate g_n' by polynomials with the norm $\|\cdot\|_{L^\infty(I,w_k)} + \|\cdot\|_{L^1(I)}$.

An integration argument finishes the proof of $H_5 \subseteq P^{k,\infty}(I,w)$.

Since $\Omega_k = \text{int}(I)\backslash R, \Omega_1 \cup \ldots \cup \Omega_k$ has at most a finite number of connected components and Theorem B gives that $W^{k,\infty}(I,w)$ is complete; therefore

$P^{k,\infty}(I,w) \subseteq H_0$. Let us take $f \in H_0$. Lemma (4.2.20) and hypothesis (iii) imply that $f^{(k-1)}$ is continuous in each point of R. This gives $H_0 \subseteq H_5$. The last equality is direct by the definition of $V^{k,\infty}(I,w)$, since (ii) gives $\Omega^{(k-1)} = I\backslash R$.

Theorem (4.2.51)[154]. Let us consider $I := [a,b]$ and a vectorial weight $w = (w_0, \ldots, w_k) \in L^\infty(I)$, with $w_k \in B_\infty\big((a,b]\big)$. Assume that a is a singularity for w_k in I, the set of singularities S for w_k in I has zero Lebesgue measure and $S \cap [a, a+\varepsilon]$ is countable for some $\varepsilon > 0$. If $k \geq 2$, assume also that a is right $(k-2)$-regular. Then we have

$$\begin{aligned} P^{k,\infty}(I,w) = H_6 &:= \{f \in V^{k,\infty}(I,w)/f^{(j)} \in P^{0,\infty}(I,w_j), for\ \ j = k-1, k\} \\ = H_0 &:= \{f \in V^{k,\infty}(I,w)/f^{(j)} \in P^{0,\infty}(I,w_j), \quad for\ 0 \leq j \leq\ k\} \\ &= \{f : I \to \mathrm{R}/f^{(k-2)} \in AC(I)\ if\ k \geq 2, f^{(k-1)} \in AC_{loc}\big((a,b]\big) \\ &\quad and\ f^{(j)} \in P^{0,\infty}(I,w_j), for\ \ j = k-1, k\}. \end{aligned}$$

Proof. We prove first $H_6 \subseteq P^{k,\infty}(I,w)$. Fix a function f in H_6. Take a closed interval $J := [\alpha,\beta] \subset (a, a+\alpha)$; we have $w_k \in B_\infty(J)$ and therefore $f^{(k-1)} \in AC(J)$. Without loss of generality we can assume that $w_0 \geq 1$ in J, since in other case we can consider $w^* := (w_0^*, w_1, \ldots, w_k)$ with $w_0^* := w_0 + \chi_J$, and it is more difficult to approximate f in $W^{k,\infty}(I,w^*)$ than in $W^{k,\infty}(I,w)$.Definition (4.2.33) gives that $(J,w) \in \mathrm{C}_0$. Theorems (4.2.45) and (4.2.46) give that it is enough to prove the inclusion in the interval $[a,\beta]$. Therefore, without loss of generality we can assume that the set S of singularities for w_k in $[a,b]$ is countable.

By Theorem (4.2.19), there exist $l_j \in \mathrm{R}$ such that ess $\lim_{x\in I, x\to a}|f^{(j)}(x) - l_j|w_j(x) = 0$, for $j = k-1, k$ (if a is not singular for w_{k-1} in I, this fact is direct for $k-1$ with $l_{k-1} = f^{(k-1)}(a)$). Without loss of generality we can assume that $l_{k-1} = l_k = 0$, i.e.,

$$\operatorname*{ess\,lim}_{x\in I, x\to a}|f^{(j)}(x)|w_j(x) = 0, \tag{31}$$

for $j = k-1, k$, since in the other case we can consider $f(x) - l_{k-1}(x-a)^{k-1}/(k-1)! - l_k(x-a)^k/k!$ instead of $f(x)$.

Observe that $w_k \in B_\infty\big((a,b]\big)$ gives $f^{(k-1)} \in AC_{loc}\big((a,b]\big)$.

Let us choose $0 < t_n \leq 1/n$ such that $a + t_n \notin S$ and

$$|f^{(k-1)}(a+t_n)| \leq \inf_{x\in(a,a+1/n]}|f^{(k-1)}(x)| + \frac{1}{n}. \tag{32}$$

Choose functions g_n verifying

$\mathrm{g}_n = f(k)$ in $[a+t_n, b]$, $\mathrm{g}_n \in C([a, a+t_n])$, $|\mathrm{g}_n| \leq |f(k)|$ in $[a, a+t_n]$,

and $\int_a^{a+t_n}|\mathrm{g}_n| < 1/n$ (recall that $f(k)$ is continuous in a neighbourhood of $a+t_n$ by (4.2.20).

Since $|S| = 0$, Theorem (4.2.19) gives $\mathrm{g}_n \in P^{0,\infty}(I,w_k)$.

Observe that $\mathrm{g}_n \in L^1(I)$,since

$|\mathrm{g}_n|_{L^1(I)} < \frac{1}{n} + \int_{a+t_n}^b |f^{(k)}|w_k w_k^{-1} \leq \frac{1}{n} + \|f^{(k)}\|_{L^\infty(I,w_k)}\|w_k^{-1}\|_{L^1([a+t_n,b])} < \infty.$

Define$f_n(x) := f(b) + \cdots + f^{(k-1)}(b)\frac{(x-b)^{k-1}}{(k-1)!} + \int_b^x \mathrm{g}_n(t)\frac{(x-t)^{k-1}}{(k-1)!}dt.$

Conditions $\int_a^{a+t_n}|g_n| < 1/n$ and(32) give

$$\left|f_n^{(k-1)}(x)\right| \leq \left|f^{(k-1)}(x)\right| + \frac{2}{n}, \tag{33}$$

for $x \in (a, a+t_n]$. By (31) we have

$$\left\|f^{(k)} - f_n^{(k)}\right\|_{L^\infty(I,w_k)} = \left\|f^{(k)} - g_n\right\|_{L^\infty(I,w_k)} \leq 2\left\|f^{(k)}\right\|_{L^\infty([a,a+t_n],w_k)} \to 0\,,$$

as $n \to \infty$. By (31) and (33), we also have as $n \to \infty$

$$f^{(k-1)} - \left\|f_n^{(k-1)}\right\|_{L^\infty(I,w_{k-1})} \leq \left\|f^{(k-1)}\right\|_{L^\infty([a,a+t_n],w_{k-1})} + \left\|f_n^{(k-1)}\right\|_{L^\infty([a,a+t_n],w_{k-1})}$$

$$\leq 2\left\|f^{(k-1)}\right\|_{L^\infty([a,a+t_n],w_{k-1})} + \frac{2}{n}\|w_{k-1}\|_{L^\infty([a,b])} \to 0.$$

These facts give that $\lim_{n\to\infty}\left\|f^{(k-1)} - f_n^{(k-1)}\right\|_{W^{1,\infty}(I,w)} = 0$. Assume now $k \geq 2$. Choose a compact interval $J_0 \subset (a,b) = \Omega_k$; we have $f^{(k-1)} \in AC(J_0)$ and then f belongs to $V^{k,\infty}([a,b],\widetilde{w})$ with $\widetilde{w} = (w_0, \dots, w_{k-2}, \widetilde{w}_{k-1}, w_k)$ and $\widetilde{w}_{k-1} = w_{k-1} + \chi_{J_0}$. Observe that $\mathrm{K}\big(I, (\widetilde{w}_{k-1}, w_k)\big) = \{0\}$ and even $\big(I, (\widetilde{w}_{k-1}, w_k)\big) \in \mathrm{C}_0$, since $\Omega_k = (a,b)$ (see Definition(4.2.33)). It is obvious that it is more complicated to approximate f in $W^{k,\infty}(I,\widetilde{w})$ than in $W^{k,\infty}(I,w)$. Therefore, without loss of generality we can assume that $\big(I, (w_{k-1}, w_k)\big) \in \mathrm{C}_0$.

Since $\Omega^{(k-1)} = (a,b]$ and a is right $(k-2)$-regular, we have $\Omega^{(k-2)} = I$,and hence Corollary (4.2.48) gives $\left\|f^{(k-1)} - f_n^{(k-1)}\right\|_{L^1(I)} \leq c\left\|f^{(k-1)} - f_n^{(k-1)}\right\|_{W^{1,\infty}(I,(w_{k-1},w_k))}$.

It is clear that

$$f_n(x) = f(b) + \cdots + f^{(k-2)}(b)\frac{(x-b)^{k-2}}{(k-2)!} + \int_b^x f_n^{(k-1)}(t)\frac{(x-t)^{k-2}}{(k-2)!}dt,$$

and consequently

$$f^{(j)}(x) - f_n^{(j)}(x) = \int_b^x \left(f^{(k-1)}(t) - f_n^{(k-1)}(t)\right)\frac{(x-t)^{k-j-2}}{(k-j-2)!}dt, \text{ for } 0 \leq j \leq k-2, \text{ if } k \geq 2.$$

Hence we have that

$$\left\|f^{(j)} - f_n^{(j)}\right\|_{L^\infty(I,w_j)} \leq c\left\|f^{(k-1)} - f_n^{(k-1)}\right\|_{L^1(I)} \leq c\left\|f^{(k-1)} - f_n^{(k-1)}\right\|_{W^{1,\infty}(I,(w_{k-1},w_k))},$$

for $0 \leq j \leq k-2$ and we conclude that $\{f_n\}$ converges to f in $W^{k,\infty}(I,w)$. Therefore, for any $k \geq 1$, in order to finish the proof of this inclusion it is enough to find $\mathrm{Q}_n \in P$ with $\lim_{n\to\infty}\left\|f_n - \mathrm{Q}_n\right\|_{W^{k,\infty}(I,w)} = 0$. Since S is countable and $g_n \in P^{0,\infty}(I,w_k) \cap L^1(I)$, Theorem (4.2.19) gives that there exists $h_n \in P$ with $\|g_n - h_n\|_{L^\infty(I,w_k)} + \|g_n - h_n\|_{L^1(I)} < 1/n$. Hence the polynomials

$$\mathrm{Q}_n(x) := f(b) + \cdots + f^{(k-1)}(b)\frac{(x-b)^{k-1}}{(k-1)!} + \int_b^x h_n(t)\frac{(x-t)^{k-1}}{(k-1)!}dt$$

satisfy the inequality $c\left\|f_n - \mathrm{Q}_n\right\|_{W^{k,\infty}(I,w)} \leq \left\|g_n - h_n\right\|_{L^1(I)} + \|g_n - h_n\|_{L^\infty(I,w_k)}$, and consequently we obtain $\lim_{n\to\infty}\left\|f_n - \mathrm{Q}_n\right\|_{W^{k,\infty}(I,w)} = 0$. Therefore $H_6 \subseteq P^{k,\infty}(I,w)$.

Since $\Omega_k = \mathrm{int}(I), \Omega_1 \cup \cdots \cup \Omega_k = \mathrm{int}(I)$ is connected and Theorem (4.2.35) gives that $W^{k,\infty}(I,w)$ is complete; therefore $P^{k,\infty}(I,w) \subseteq H_0$. The content $H_0 \subseteq H_6$ is direct.

The last equality is direct by the definition of $V^{k,\infty}(I,w)$, since $\Omega^{(k-1)} = (a,b]$, and $\Omega^{(k-2)} = [a,b]$ if $k \geq 2$.

Theorem (4.2.52)[154]. Let us consider $I := [a, b]$ and a vectorial weight $w = (w_0, \cdots, w_k) \in L^\infty(I)$, with $w_k \in B_\infty((a, b])$. Assume that a is a singularity for w_k in I, the set S of singularities for w_k in I has zero Lebesgue measure and $S \cap [a, a+\varepsilon]$ is countable for some $\varepsilon > 0$.
If $k \geq 2$, assume also that $w|_{[a,a+\varepsilon]}$ is a right completion of $(0, \cdots, 0, w_{k-1}, w_k)$. Then we have

$$P^{k,\infty}(I, w) = \{f \in V^{k,\infty}(I, w) / f^{(j)} \in P^{0,\infty}(I, w_j), \quad for\ j = k-1, k\}.$$

Proof: If $k = 1$, the result is a direct consequence of Theorem (4.2.51).Assume that $k \geq 2$. The argument follows the same lines as the one in the proof of Theorem (4.2.51). By Theorems (4.2.44) and (4.2.46) we can assume that $b = a + \varepsilon$. Given a function f with $f^{(j)} \in P^{0,\infty}(I, w_j)$, for $j = k-1, k$, let us consider the sequence $\{f_n\}$ in the proof of Theorem (4.2.51). As in the proof of Theorem (4.2.51), we also have $f_n^{(k-1)} \to f^{(k-1)}$ in $W^{1,\infty}(I, w)$, as $n \to \infty$.By Proposition(4.2.36) there is a positive constant c such that $c\|g\|_{W^{k,\infty}(I,w)} \leq \|g\|_{W^{k,\infty}(I,(0,\cdots,0,w_{k-1},w_k))} + \sum_{j=0}^{k-1} |g^{(j)}(b)|$, for all $g \in V^{k,\infty}(I, w)$.
Since $(f - f_n)^{(j)}(b) = 0$ for $0 \leq j < k$, we have$\|f - f_n\|_{W^{k,\infty}(I,w)} \leq c\left\|f^{(k-1)} - f_n^{(k-1)}\right\|_{W^{1,\infty}(I,w)}$, and we conclude that $\{f_n\}$ converges to f in $W^{k,\infty}(I, w)$. The proof finishes with the arguments in the proof of Theorem (4.2.51).

Although the main interest in this section is the case of non-bounded intervals, the following result can be applied to the case of compact intervals.

Theorem (4.2.53)[154]. Let us consider a vectorial weight $w = (w_0, \cdots, w_k)$. Assume that there exist $a \in 2$ and a positive constant c such that

$$c\|g\|_{W^{k,\infty}(\Delta,w)} \leq |g(a)| + |g'(a)| + \cdots + |g^{(k-1)}(a)| + \|g^{(k)}\|_{L^\infty(\Delta,w_k)}, \tag{34}$$

for every $g \in V^{k,\infty}(2, w)$. Then, $P^{k,\infty}(\Delta, w) = \{f : \Delta \to R / f^{(k)} \in P^{0,\infty}(\Delta, w_k)\}$.

Proof. We prove the non-trivial inclusion. Let us consider a fixed function f with $f^{(k)} \in P^{0,\infty}(\Delta, w_k)$. Choose a sequence $\{q_n\}$ of polynomials which converges to $f(k)$ in $L^\infty(\Delta, w_k)$. Then the polynomials

$$Q_n(x) := f(a) + f'(a)(x-a) + \cdots + f^{(k-1)}(a)\frac{(x-a)^{k-1}}{(k-1)!} + \int_a^x q_n(t)\frac{(x-t)^{k-1}}{(k-1)!}dt$$

satisfy

$$c\|f - Q_n\|_{W^{k,\infty}(\Delta,w)} \leq \|f^{(k)} - Q_n^{(k)}\|_{L^\infty(\Delta,wk)} = \|f^{(k)} - q_n\|_{L^\infty(\Delta,w_k)},$$

since $(f - Q_n)^{(j)}(a) = 0$ for $0 \leq j < k$, and we conclude that the sequence of polynomials $\{Q_n\}$ converges to f in $W^{k,\infty}(\Delta, w)$.We show now that Theorem (4.2.53) is very useful finding a wide class of measures satisfying (34). The following inequality is similar to the Muckenhoupt inequality which can be found in [156] and [157].

Proposition(4.2.54)[154].(Muckenhoupt inequality II) Let us consider two weights w_0, w_1in $(0, \infty)$. Then there exists a positive constant c such that

$$\left\|\int_0^x g(t)dt\right\|_{L^\infty([0,\infty),w_0)} \leq c\|g\|_{L^\infty([0,\infty),w_1)} \tag{35}$$

for any measurable function g in $(0, \infty)$, if and only if

$B := \text{ess sup}_{r>0}\, w_0(r) \int_0^r w_1(t)^1 dt < \infty.$

Furthermore, the best constant c in (35) is B.

Proof: Assume that $B < \infty$. We have

$$\left|\int_0^r g(t)dt\right| w_0(r) \leq \int_0^r |g(t)|\, w_1(t)w_1(t)^{-1}dt\; w_0(r) \leq \|g\|_{L^\infty([0,r],w_1)} w_0(r) \int_0^r w_1(t)^{-1}\,dt,$$

and this implies (35) with $c = B$. If (35) holds, the choice of the functiong $:= w_1^{-1}$ gives $B \leq c < \infty$.

Lemma (4.2.55)[154]. Assume that $w_0(x) \leq c_0 x^{-\alpha_0} e^{-\lambda x^\varepsilon}$ and $w_1(x) \geq c_1 x^{-\alpha_1} e^{-\lambda x^\varepsilon}$, for $x \geq A, w_0 \in L^\infty([0,A]), w_1 \in B_\infty([0,A])$, with $\lambda, \varepsilon, c_0, c_1, A > 0$ and $\alpha_0, \alpha_1 \in \mathbf{R}$.
If $\alpha_0 \leq \alpha_1 + \varepsilon - 1$, then w_0, w_1 satisfy Muckenhoupt inequality II.

Proof. First of all observe that $\left(x^a e^{bx^\varepsilon}\right)' = x^{a-1} e^{bx^\varepsilon}(a + b\varepsilon x^\varepsilon)$.
This implies $\left(x^a e^{bx^\varepsilon}\right) \asymp x^{a+\varepsilon-1} e^{bx^\varepsilon}$, as $x \to \infty$, if $b > 0$.
Therefore $\int_A^r x^a e^{bx^\varepsilon} dx \asymp r^{a+1-\varepsilon} e^{br^\varepsilon}$,as $r \to \infty$.
Hence, we have as $r \to \infty$ $\int_0^r w_1(x)^{-1}dx \asymp \int_A^r w_1(x)^{-1}dx \leq c \int_A^r x^{-\alpha_1} e^{\lambda x^\varepsilon} dx \asymp r^{-\alpha_1+1-\varepsilon} e^{\lambda r^\varepsilon}$.
The expression $w_0(r) \int_0^r w_1^{-1}$ is bounded for r in a compact set it is bounded for big r, if $\lim_{r\to\infty} r^{\alpha_0} e^{-\lambda r^\varepsilon} r^{-\alpha_1+1-\varepsilon} e^{\lambda r^\varepsilon} < \infty$. This condition holds since $\alpha_0 \leq \alpha_1 + \varepsilon - 1$.

Lemma (4.2.56)[154]. Assume that $w_0(x) \leq k_0 x^{\beta_0}$ and $w_1(x) \geq k_1 x^{\beta_1}$, for $0 < x < b$, with $k_0, k_1 > 0, \beta_0 > 0$ and $\beta_1 \in$ R. If $\beta_0 \geq \beta_1 - 1$, then w_0, w_1 satisfy Muckenhoupt inequality I, with $a = 0$.

Proof. If $\beta_1 > 1$, we have $\int_r^b w_1(x)^{-1}dx \leq c \int_r^b x^{-\beta_1}\, dx \asymp r^{1-\beta_1}$.
If $\beta_1 > 1$, the expression $F(r) := w_0(r) \int_r^b w_1^{-1}$ is bounded for $r \in [\varepsilon, b]$ (with $\varepsilon > 0$); it is bounded for $r \in (0, \varepsilon)$, if $\lim_{r\to 0^+} r^{\beta_0} r^{1-\beta_1} < \infty$.
This condition holds since $\beta_0 \geq \beta_1 - 1$. If $\beta_1 \leq 1$, we obtain similarly that $F(r)$ is bounded since $\beta_0 > 0$ and $F(r) \leq c r^{\beta_0} \log\frac{1}{r}$, for small r.
These lemmas give the following results.

Proposition (4.2.57)[154]. Consider a vectorial weight w in $(0, \infty)$, with
(i) $w_j(x) \leq c_j x^{\beta_j}$, for $0 \leq j < k, w_k(x) \geq c_k x^{\beta_k}$, in $(0, a)$,
(ii) $w_j(x) \leq c_j x^{\alpha+}(k-j)(\varepsilon-1) e^{-\lambda x^\varepsilon}$, for $0 \leq j < k, w_k(x) \geq c_k x^\alpha e^{-\lambda x^\varepsilon}$, in$(a, \infty)$,
where $\alpha \in \mathbf{R}, a, \varepsilon, \lambda, c_j > 0$ for $0 \leq j \leq k$, and $\beta > 0$ for $0 \leq j < k$.
If $\beta_j \geq \beta_k - (k-j)$, for $0 \leq j < k$, then

$$P^{k,\infty}([0,\infty), w) = \{f\colon [0,\infty) \to \mathrm{R}/f(k) \in P^{0,\infty}([0,\infty), w_k)].$$

Proof. An induction argument with Lemma (4.2.55) in (a, ∞) instead of (a, ∞), gives for $0 \leq j < k$ and $f \in V^{k,\infty}([a,\infty), w)$,

$$\left\| f^{(j)}(x) - f^{(j)}(a) - \cdots - f^{(k-1)}(a)\frac{(x-a)^{k-j-1}}{(k-j-1)!}\right\|_{L^\infty([a,\infty),w_j)} \leq c\left\|f^{(k)}\right\|_{L^\infty([a,\infty),w_k)},$$

and therefore $c\left\|f^{(j)}\right\|_{L^\infty([a,\infty),w_j)} \leq \left\|f^{(k)}\right\|_{L^\infty([a,\infty),w_k)} + \sum_{i=j}^{k-1}\left|f^{(i)}(a)\right|$,
for $0 \leq j < k$ and $f \in V^{k,\infty}([a,\infty), w)$. Consequently, we have

$$c\|f\|_{W^{k,\infty}([a,\infty),w)} \leq \left\|f^{(k)}\right\|_{L^\infty([a,\infty),wk)} + \sum_{j=0}^{k-1}\left|f^{(j)}(a)\right|, \tag{36}$$

for all $f \in V^{k,\infty}([a,\infty), w)$.

If we use now Lemma (4.2.56) in $(0, a)$, a similar argument gives

$$c\|f\|_{W^{k,\infty}([0,a],w)} \le \left\|f^{(k)}\right\|_{L^{\infty}([0,a],wk)} + \sum_{j=0}^{k-1}\left|f^{(j)}(a)\right|, \tag{37}$$

for all $f \in V^{k,\infty}([0, a], w)$. Theorem (4.2.53), (36), and (37) give the proposition.

We can obtain similar results for weights of fast decreasing degree.

The following results are not sharp since the sharp results are hard to write and do not involve any new idea. Define inductively the functions $\exp_{\lambda_1,\dots,\lambda_n}$ as follows:

$$\exp_{\lambda}(t) := \exp(\lambda t)\,, \exp_{\lambda_1,\dots,\lambda_n}(t) := \exp\left(\lambda_1 \exp_{\lambda_2,\dots,\lambda_n}(t)\right).$$

Lemma (4.2.58)[154]. Consider a scalar weight $w(x) \asymp \exp(_{-\lambda_1,\lambda_2,\dots,\lambda_n}(x^{\varepsilon}))$ in $(0, \infty)$, where we have $n > 1$ and $\varepsilon, \lambda_1, \lambda_2, \dots, \lambda_n > 0$.

Then $\langle w, w\rangle$ satisfy Muckenhoupt inequality II.

Proof. A straightforward computation shows that the derivative of the function

$$x^{1-\varepsilon}\prod_{i=2}^{n}\exp_{\lambda_i,\lambda_{i+1},\dots,\lambda_n}(x^{\varepsilon})\,,$$

converges to zero as $x \to \infty$. Now, if $b > 0$ we have that

$$\frac{d}{dx}\left(\exp_{b,\lambda_2,\dots,\lambda_n}(x^{\varepsilon})x^{1-\varepsilon}\prod_{i=2}^{n}\exp_{\lambda_i,\lambda_{i+1},\dots,\lambda_n}(x^{\varepsilon})\right) \asymp \exp_{b,\lambda_2,\dots,\lambda_n}(x^{\varepsilon})\,,$$

in $(1, \infty)$. Hence we have that

$$\int_0^r w^{-1} \asymp \exp_{\lambda_1,\lambda_2,\dots,\lambda_n}(r^{\varepsilon})r^{1-\varepsilon}\prod_{i=2}^{n}\exp_{-\lambda_1,\lambda_2,\dots,\lambda_n}(r^{\varepsilon})\,,$$

in $(1, \infty)$. Therefore

$$w(r)\int_0^r w^{-1} \asymp r^{1-\varepsilon}\prod_{i=2}^{n}\exp_{-\lambda_1,\lambda_2,\dots,\lambda_n}(r^{\varepsilon})\,,$$

in $(1, \infty)$. This finishes the proof, since $w \in L^{\infty}\left([0, \infty)\right)$.

Corollary (4.2.59)[236]. Assume that $w_{m-1}(x_n) \le k_{n-1}x^{\beta_{n-1}}$ and $w_{m+1}(x_n) \ge k_{n+1}x^{\beta_{n+1}}$, for $0 < x_n < b$, with $k_{n-1}, k_{n+1} > 0, \beta_{n-1} > 0$ and $\beta_{n+1} \in$ R.

If $\beta_{n-1} \ge \beta_{n+1} - 1$, then $\langle w_{m-1}, w_{m+1}\rangle$ satisfy Muckenhoupt inequality I, with $a = 0$.

Proof. If $\beta_{n+1} > 1$, we get

$$\int_r^b w_{m+1}(x_n)^{-1}dx \le c\int_r^b (x_n)^{-\beta_{n+1}}\,dx \asymp r^{1-\beta_{n+1}}.$$

For $\beta_{n+1} > 1$, the expression

$$F(r) := w_{m-1}(r)\int_r^b w_{m+1}^{-1}$$

is bounded for $r \in [\varepsilon, b]$ (with $\varepsilon > 0$); it is bounded for $r \in (0, \varepsilon)$, if

$$\lim_{r\to 0^+} r^{\beta_{n-1}}r^{1-\beta_{n+1}} < \infty.$$

This condition holds since $\beta_{n-1} \ge \beta_{n+1} - 1$. If $\beta_{n+1} \le 1$, we obtain similarly that $F(r)$ is bounded since $\beta_{n-1} > 0$ and $F(r) \le cr^{\beta_{n-1}}\log\frac{1}{r}$, for small r.

Chapter 5
The Bourgain, Brezis, and Mironescu Theorem with Best constants

The relation $\lim_{s\downarrow 0} s\int_{R^n}\int_{R^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy = 2p^{-1}|S^{n-1}|\|u\|^p_{L^p(R^n)}$, in this chapter is shown .As an application, we give a new proof of a theorem of W. Beckner concerning conformally invariant higher-order differential operators on the sphere. We believe that proofs are original and we do not make use of any interpolation techniques nor pass through the theory of Besov spaces.

Section(5.1): Limiting Embeddings of Fractional Sobolev Spaces:

Let $s \in (0,1)$ and let $p \geq 1$. We introduce the space $\mathcal{W}_0^{s,p}(R^n)$ as the completion of $C_0^\infty(R^n)$ in the norm$\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy\right)^{1/p}$.We also need the space $\mathcal{W}_\perp^{s,p}(Q)$ of functions defined on the cube $Q=\{x\in R^n: |x_i|<1/2, 1\leq i\leq n\}$ which are orthogonal to 1 and have the finite norm

$$\left(\int_Q\int_Q\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy\right)^{1/p}.$$

The main result by Bourgain *et al.* [158] is the inequality

$$\|u\|^p_{L^q(Q)} \leq c(n)\frac{1-s}{(n-sp)^{p-1}}\|u\|^p_{\mathcal{W}_\perp^{s,p}(Q)}, \tag{1}$$

where $u\in\mathcal{W}_\perp^{s,p}(Q), 1/2\leq s<1, sp<n, q=pn/(n-sp)$ and $c(n)$ depends on n.

The present article is a direct outgrowth of this result. Figuring out a similar estimate for functions in $\mathcal{W}_0^{s,p}(R^n)$, valid for the whole interval $0<s<1$, one could anticipate the appearance of the factor $s(1-s)$ in the right-hand side, since the norm in $\mathcal{W}_0^{s,p}(R^n)$ blows up both as $s\uparrow 1$ and $s\downarrow 0$. The following theorem shows that this is really the case.

Theorem (5.1.1)[159]: Let $n\geq 1$, $p\geq 1, 0<s<1$, and $sp<n$. Then, for an arbitrary function $u\in\mathcal{W}_0^{s,p}(R^n)$, there holds

$$\int_{\mathbf{R}^n}|u(x)|^p\frac{dx}{|x|^{sp}} \leq c(n,p)\frac{s(1-s)}{(n-sp)^p}\|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}. \tag{2}$$

Proof. Let $\psi(h)=|S^{n-1}|^{-1}n(n+1)(1-|h|)_+$,
where $h\in R^n$ and plus stands for the nonnegative part of a real-valued function.
We introduce the standard extension of u onto

$$R_+^{n+1}=\{(x,z):x\in R^n, z>0\}U(x,z):=\int_{\mathbf{R}^n}\psi(h)u(x+zh)dh.$$

A routine majoration implies $|\nabla U(x,z)|\leq\frac{n(n+1)(n+2)}{z|S^{n-1}|}\int_{|h|<1}|u(x+zh)-u(x)|dh$.

Hence and by Hölder's inequality one has

$$\int_0^\infty\int_{\mathbf{R}^n}z^{-1+p(1-s)}|\nabla U(x,z)|^p dx\,dz \leq \frac{n}{|S^{n-1}|}(n+1)^p(n+2)^p$$

$$\times\int_0^\infty z^{-1-ps}\int_{|h|<1}\int_{\mathbf{R}^n}|u(x+zh)-u(x)|^p dx\,dh\,dz. \tag{3}$$

Setting $\eta = zh$ and changing the order of integration, one can rewrite (3) as

$$\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p dx\, dz \le \frac{n(n+1)^p (n+2)^p}{|S^{n-1}|(sp+n)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy. \qquad (4)$$

By Hardy's inequality, $\int_0^{|x|} z^{-1-sp} \left|\int_0^z \varphi(\tau)\, dt\right|^p dz \le s^{-p} \int_0^{|x|} z^{-1+p(1-s)} |\varphi(z)|^p\, dz$ one has

$$\frac{|u(x)|^p}{|x|^{sp}} = p(1-s) \int_0^{|x|} z^{-1+p(1-s)} dz \frac{|u(x)|^p}{|x|^p} \le p(1-s) \int_0^{|x|} z^{-1-sp} dz \left(\int_0^z \left(\left|\frac{\partial U}{\partial \tau}(x,\tau)\right| + \frac{|U(x,\tau)|}{|x|}\right) d\tau\right)^p$$

$$\le \frac{p(1-s)}{s^p} \int_0^{|x|} z^{-1+p(1-s)} \left(\left|\frac{\partial U}{\partial z}(x,z)\right| + \frac{U(x,z)}{|x|}\right)^p dz.$$

Now, the integration over R^n and Minkowski's inequality imply

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\, dx \le \frac{p(1-s)}{s^p} \left(\left(\int_{\mathbf{R}^n} \int_0^\infty z^{-1+p(1-s)} \left|\frac{\partial U}{\partial z}(x,z)\right|^p dz\, dx \right)^{1/p} + A \right)^p, \qquad (5)$$

Where $A := \left(\int_{\mathbf{R}^n} \int_0^{|x|} z^{-1+p(1-s)} |x|^{-p} |U(x,z)|^p\, dz\, dx\right)^{1/p}$.

Clearly, $A^p \le 2^{p/2} \int_{\mathbf{R}^n} dx \int_0^\infty z^{-1+p(1-s)} \frac{|U(x,z)|^p}{(x^2+z^2)^{p/2}} dz\, dx$, which does not exceed

$$2^{p/2} \int_{S_+^n} (\cos\theta)^{-1+p(1-s)} \int_0^\infty |U|^p \rho^{n-1-sp}\, d\rho\, d\sigma, \qquad (6)$$

where $\rho = (x^2+z^2)^{1/2}, \cos\theta = z/\rho, d\sigma$ is an element of the surface area on the unit sphere S^n, and S_+^n is the upper half of S^n.

Using Hardy's inequality

$\int_0^\infty |U|^p \rho^{n-1-sp}\, d\rho \le \left(\frac{p}{n-sp}\right)^p \int_0^\infty \left|\frac{\partial U}{\partial \rho}\right|^p \rho^{n-1+p(1-s)}\, d\rho$, one arrives at the estimate

$$A^p \le \left(\frac{2^{1/2}p}{n-sp}\right)^p \int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p\, dx\, dz.$$

Combining this with (5), one obtains

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\, dx \le \frac{p(1-s)}{s^p} \left(1 + \frac{2^{1/2}p}{n-sp}\right)^p \int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p dx\, dz$$

which, along with (5), gives

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\, dx \le \frac{(1-s)}{(n-sp)^p} \frac{p(n+2p)^{3p}}{|S^{n-1}|s^p} \|u\|^p_{W_0^{s,p}(\mathbf{R}^n)}. \qquad (7)$$

In order to justify (2) we need to improve (2) for small values of s.

Clearly, $\frac{|S^{n-1}|}{2^{sp}sp} \int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\, dx = \int_{\mathbf{R}^n} \int_{|x-y|>2|x|} \frac{dy}{|x-y|^{n+sp}} |u(x)|^p\, dx$.

Since $|x-y| > 2|x|$ implies $2|y|/3 < |x-y| < 2|y|$, we obtain

$$\left(\frac{|S^{n-1}|}{2^{sp}sp} \int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\, dx\right)^{1/p} \le \left(\int_{R^n} \int_{|x-y|>|x|} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy\right)^{1/p} + \left(|S^{n-1}| \frac{3^{sp}-1}{2^{sp}sp} \int_{\mathbf{R}^n} \frac{|u(y)|^p}{|y|^{sp}}\, dy\right)^{1/p}.$$

Hence,

$$\left(\frac{|S^{n-1}|}{2^{sp}sp}\right)^{1/p} \left(1 - (3^{sp}-1)^{1/p}\right) \left(\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}} dx\right)^{1/p} \le 2^{-1/p} \|u\|_{W_0^{s,p}(\mathbf{R}^n)}.$$

Let δ be an arbitrary number in $(0,1)$. If $s \le (4p)^{-1}\delta^p$, we conclude

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}}dx \le \frac{2^{sp-1}sp}{|S^{n-1}|(1-\delta)^p}\|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}. \tag{8}$$

Setting $\delta = 2^{-1}$ and comparing this inequality with (6), we arrive at (2) with $c(n,p) = |S^{n-1}|^{-1}(n+2p)^{3p}p^{p+2}2^{(n+1)(n+2)}$.

The proof is complete.

From Theorem (5.1.1), we shall deduce an inequality, analogous to (2), for functions defined on the cube Q. Unlike (3), this inequality contains no factor s in the right-hand side, which is not surprising, because, for smooth u, the norm $\|u\|_{\mathcal{W}_\perp^{s,p}(Q)}$ tends to a finite limit as $s \downarrow 0$.

Corollary(5.1.2)[159]: Let $n \ge 1, p \ge 1, 0 < s < 1$, and $sp < n$. Then any function $u \in \mathcal{W}_\perp^{s,p}(Q)$ satisfies

$$\int_Q |u(x)|^p \frac{dx}{|x|^{sp}} \le c(n,p)\frac{1-s}{(n-sp)^p}\|u\|^p_{\mathcal{W}_\perp^{s,p}(Q)}. \tag{9}$$

Proof. Let us preserve the notation u for the mirror extension of $u \in \mathcal{W}_\perp^{s,p}(Q)$ to the cube $3Q$, where aQ stands for the cube obtained from Q by dilation with the coefficient a.

We choose acut-off function η, equal to 1 on Q and vanishing outside $2Q$, say, $\eta(x) = \prod_{i=1}^n \min\left\{1, 2(1-x_i)_+\right\}$. By Theorem(5.1.1), it is enough to prove that

$$\|\eta u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)} \le s^{-1}c(n,p)\|u\|^p_{\mathcal{W}_\perp^{s,p}(Q)}. \tag{10}$$

Clearly, the norm in the left-hand side is majorized by

$$\left(\int_{3Q}\int_{3Q}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,\eta(y)^p\,dy\right)^{1/p} + \left(\int_{3Q}\int_{3Q}\frac{|\eta(x)-\eta(y)|^p}{|x-y|^{n+sp}}dx|u(y)|^p\,dy\right)^{1/p} + \left(2\int_{3Q}\int_{\mathbf{R}^n\setminus 3Q}\frac{dy}{|x-y|^{n+sp}}|(\eta u)(x)|^p\,dx\right)^{1/p}.$$

The first term does not exceed $6^{n/p}\|u\|_{\mathcal{W}_\perp^{s,p}(Q)}$; the second term is not greater than

$$2n^{1/2}\left(\int_{3Q}\int_{3Q}\frac{dy}{|x-y|^{n-(1-s)}}|u(y)|^p\,dy\right)^{1/p} \le n3^{2+n/p}\left(\frac{|S^{n-1}|}{p(1-s)}\right)^{1/p}\|u\|_{L^p(Q)},$$

and the third one is dominated by $\left(2\int_{2Q}\int_{|x-y|>1/2}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p\,dx\right)^{1/p} \le \left(\frac{2^{n+1+p}}{sp}\right)^{1/p}\|u\|_{L^p(Q)}$.

Summing up these estimates, one obtains

$$\|\eta u\|_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)} \le 6^{n/p}\|u\|_{\mathcal{W}_0^{s,p}(Q)} + n3^{2+n/p}p^{-1/p}\left(s^{-1/p} + (1-s)^{-1/p}\right)\|u\|_{L^p(Q)}. \tag{11}$$

Recalling that $u \perp 1$ on Q, one has for any $z \in Q$

$$\int_Q |u(x)|^p dx \le \int_Q\int_Q |u(x)-u(y)|^p dx\,dy \le 2^p\int_Q |u(x)-u(z)|^p\,dx.$$

Hence and by the obvious inequality $\int_{2Q}\frac{dz}{|x-z|^{n-p(1-s)}} > \int_{|z-x|<1/2}\frac{dz}{|x-z|^{n-p(1-s)}} = \frac{|S^{n-1}|}{p(1-s)2^{p(1-s)}}$, where $x \in Q$, it follows that $\int_Q |u(x)|^p dx \le \frac{2^{p(2-s)}p(1-s)}{|S^{n-1}|}\int_{2Q}\int_Q\frac{|u(x)-u(z)|^p}{|x-z|^{n-p(1-s)}}dx\,dz$.

Thus, $\|u\|_{L^p(Q)} \le 2^{2+n/p}n^{1/2}\left(\frac{p(1-s)}{|S^{n-1}|}\right)^{1/p}\|u\|_{\mathcal{W}_\perp^{s,p}(Q)}$.

Combining this inequality with (10), we justify (9) and hence complete the proof.

Theorem (5.1.3)[159]: Let $n \geq 1$, $p \geq 1, 0 < s < 1$, and $sp < n$. Then, for an arbitrary function $u \in \mathcal{W}_0^{s,p}(\mathbf{R}^n)$, there holds

$$\|u\|_{L^q(\mathbf{R}^n)}^p \leq c(n,p)\frac{s(1-s)}{(n-sp)^{p-1}}\|u\|_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}^p, \tag{12}$$

where $q = pn/(n-sp)$ and $c(n,p)$ is a function of n and p.

From Theorem (5.1.1), one can derive inequality (1) for all $s \in (0,1)$ with a constant c depending both on n and p.

In the case $s \geq 1/2$ considered in [158], one has $1 < p < 2n$ and therefore the dependence of the constant c on p can be eliminated.

Thus, we arrive at the Bourgain–Brezis–Mironescu result and extend it to the values $s < 1/2$.

The proof given in [158] relies upon some advanced harmonic analysis and is quite complicated. Our proof of (12) is straightforward and rather simple.

It is based upon an estimate of the best constant in a Hardy-type inequality for the norm in $\mathcal{W}_0^{s,p}(\mathrm{R}^n)$.

Proof: It is well known that the fractional Sobolev norm of order $s \in (0,1)$ is non-increasing with respect to symmetric rearrangement of functions decaying to zero at infinity (see [160], [161], [162]).

Let $v(|x|)$ denote the rearrangement of $|u(x)|$.

Then

$$\|u\|_{L^q(\mathbf{R}^n)} = \left(\frac{|S^{n-1}|}{n}\int_0^\infty v(r)^q\, d(r^n)\right)^{1/q}, \tag{13}$$

where $|S^{n-1}|$ is the area of the unit sphere S^{n-1}. Recalling that an arbitrary non-negative non-increasing function f on the semi-axis $(0,\infty)$ satisfies

$$\int_0^\infty f(t)^\lambda d(t^\lambda) \leq \int_0^\infty \left(\int_0^t f(\tau)\,d\tau\right)^{\lambda-1} f(t)dt = \left(\int_0^\infty f(t)\,dt\right)^\lambda, \qquad \lambda \geq 1$$

the right-hand side in (13) does not exceed

$$\left(\frac{|S^{n-1}|}{n}\right)^{1/q}\left(\int_0^\infty v(r)^p\, d(r^{n-sp})\right)^{1/p} = \frac{(n-sp)^{1/p}}{n^{1/q}|S^{n-1}|^{s/n}}\left(\int_{\boldsymbol{R}^n} v(|x|)^p \frac{dx}{|x|^{sp}}\right)^{1/p}.$$

We now see that (12) results from inequality (2u-n).

Corollary (5.1.4): Let $n \geq 1, p \geq 1, 0 < s < 1$, and $sp < n$.

Then any function $u \in \mathcal{W}_\perp^{s,p}(Q)$ satisfies

$$\|u\|_{L^p(Q)}^p \leq c(n,p)\frac{1-s}{(n-sp)^{p-1}}\|u\|_{\mathcal{W}_\perp^{s,p}(Q)}^p.$$

Theorem (5.1.5)[159]: For any function $u \in \bigcup_{0<s<1}\mathcal{W}_0^{s,p}(\mathrm{R}^n)$, there exists the limit

$$\lim_{s\downarrow 0} s\|u\|_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}^p = 2p^{-1}|S^{n-1}|\|u\|_{L^p(\mathbf{R}^n)}^p.$$

Proof. Since d can be chosen arbitrarily small, inequality (9) implies

$$\liminf_{s\downarrow 0} s\|u\|_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)}^p \geq 2p^{-1}|S^{n-1}|\|u\|_{L^p(\mathbf{R}^n)}^p. \tag{14}$$

Let us majorize the upper limit.

By (14), it suffices to assume that $u \in L^p(\mathrm{R}^n)$.

Clearly,

$$s\|u\|^p_{\mathcal{W}_0^{s,p}(\mathbf{R}^n)} \le 2\left\{\left(s\int_{\mathbf{R}^n}\int_{|y|\ge 2|x|}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p\,dx\right)^{\frac{1}{p}} + \left(s\int_{\mathbf{R}^n}|u(y)|^p\int_{|y|\ge 2|x|}\frac{dx\,dy}{|x-y|^{n+sp}}\right)^{\frac{1}{p}}\right\}^p$$

$$+2s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy.$$

The first term in braces does not exceed

$$\left(s\int_{\mathbf{R}^n}\int_{|y|\ge|x|}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p\,dx\right)^{1/p} = \frac{|S^{n-1}|^{1/p}}{p^{1/p}}\left(\int_{\boldsymbol{R}^n}\frac{|u(x)|^p}{|x|^{sp}}dx\right)^{1/p}$$

hence its $\limsup_{s\downarrow 0}$ is dominated by $|S^{n-1}|^{1/p}p^{-1/p}\|u\|_{L^p(\mathbf{R}^n)}$.

The second term in braces is not greater than

$$s^{1/p}\left(2^{n+sp}\int_{\mathbf{R}^n}\frac{|u(y)|^p}{|y|^{n+sp}}dy\int_{|x|<|y|/2}dx\right)^{1/p} = 2^s\left(\frac{s}{p}|S^{n-1}|\right)^{1/p}\left(\int_{\mathbf{R}^n}\frac{|u(y)|^p}{|y|^{sp}}dy\right)^{1/p},$$

so it tends to zero as $s\downarrow 0$. We claim that

$$\limsup_{s\downarrow 0}\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy = 0. \tag{15}$$

By assumption of the theorem, $u\in\mathcal{W}_0^{\tau,p}(\mathrm{R}^n)$ for a certain $\tau\in(0,1)$.

Let N be an arbitrary number greater than 1 and let $s<\tau$.

We have

$$2s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy \le 2sN^{p(\tau-s)}\int_{\mathbf{R}^n}\int_{\substack{|x|<|y|<2|x|\\|x-y|\le N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\tau p}}dx\,dy$$

$$+\,2s\int_{\mathbf{R}^n}\int_{\substack{|x|<|y|<2|x|\\|x-y|>N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy.$$

The first term in the right-hand side tends to zero as $s\downarrow 0$ and the second one does not exceed

$$2^{p+1}s\int_{|x|>N/3}\int_{|x-y|>N}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p dx \le c(n,p)\int_{|x|>N/3}|u(x)|^p\,dx,$$

which is arbitrarily small if N is sufficiently large.

The proof is complete.

Corollary (5.1.6)[236]: If any function $\mathrm{u}\in\bigcup_{1>\varepsilon>0}\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathrm{R}^n)$, there exists the limit

$$\lim_{\varepsilon\downarrow 1}\mathrm{s}\|\mathrm{u}\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)} = 2(1+\varepsilon)^{-1}|(1-\varepsilon)^{n-1}|\|\mathrm{u}\|^{1+\varepsilon}_{\mathrm{L}^{1+\varepsilon}(\mathbf{R}^n)}.$$

Proof. Since d can be chosen arbitrarily small, inequality (9) implies

$$\liminf_{\varepsilon\downarrow 1}s\|u\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)} \ge 2(1+\varepsilon)^{-1}|(1-\varepsilon)^{n-1}|\|u\|^{1+\varepsilon}_{L^{1+\varepsilon}(\mathbf{R}^n)}.$$

Let us majorize the upper limit. By (14), it suffices to assume that $u \in L^{1+\varepsilon}(\mathbf{R}^n)$. Clearly,

$$(1+\varepsilon)\|u\|^{1+\varepsilon}_{\mathcal{W}_0^{1-\varepsilon,1+\varepsilon}(\mathbf{R}^n)} \leq 2\left\{\left((1-\varepsilon)\int\limits_{\mathbf{R}^n}\int\limits_{|y|\geq 2|x|}\frac{dy}{|x-y|^{n+(1-\varepsilon^2)}}|u(x)|^{1+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}}\right.$$

$$\left. + \left((1-\varepsilon)\int\limits_{\mathbf{R}^n}|u(y)|^{1+\varepsilon}\int\limits_{|y|\geq 2|x|}\frac{dx\,dy}{|x-y|^{n+(1-\varepsilon^2)}}\right)^{\frac{1}{1+\varepsilon}}\right\}^{1+\varepsilon}$$

$$+2(1-\varepsilon)\int\limits_{\mathbf{R}^n}\int\limits_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}}dx\,dy.$$

The first term in braces does not exceed

$$\left((1-\varepsilon)\int\limits_{\mathbf{R}^n}\int\limits_{|y|\geq|x|}\frac{dy}{|x-y|^{n+(1-\varepsilon^2)}}|u(x)|^{1+\varepsilon}\,dx\right)^{1/1+\varepsilon} = \frac{|(1-\varepsilon)^{n-1}|^{1/1+\varepsilon}}{(1+\varepsilon)^{1/1+\varepsilon}}\left(\int\limits_{R^n}\frac{|u(x)|^{1+\varepsilon}}{|x|^{1-\varepsilon^2}}dx\right)^{1/1+\varepsilon}$$

hence its $\limsup_{\varepsilon\downarrow 1}$ is dominated by $|(1-\varepsilon)^{n-1}|^{1/1+\varepsilon}(1+\varepsilon)^{-1/1+\varepsilon}\|u\|_{L^{1+\varepsilon}(\mathbf{R}^n)}$.The second term in braces is not greater than

$$(1-\varepsilon)^{\frac{1}{1}+\varepsilon}\left(2^{n+1-\varepsilon^2}\int_{\mathbf{R}^n}\frac{|u(y)|^{1+\varepsilon}}{|y|^{n+(1-\varepsilon^2)}}dy\int_{|x|<\frac{|y|}{2}}dx\right)^{\frac{1}{1}+\varepsilon} = 2^{1-\varepsilon}\left(\frac{1-\varepsilon}{1+\varepsilon}|S^{n-1}|\right)^{1/1+\varepsilon}\left(\int_{\mathbf{R}^n}\frac{|u(y)|^{1+\varepsilon}}{|y|^{1-\varepsilon^2}}dy\right)^{1/1+\varepsilon},$$

so it tends to zero as $\varepsilon \downarrow 1$. We claim that

$$\limsup_{\varepsilon\downarrow 1}\int\limits_{\mathbf{R}^n}\int\limits_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}}dx\,dy = 0.$$

By assumption of the theorem, $u \in \mathcal{W}_0^{\tau,1+\varepsilon}(\mathrm{R}^n)$ for a certain $\tau \in (0,1)$. Let N be an arbitrary number greater than 1 and let $\varepsilon > \tau - 1$. We have

$$2(1-\varepsilon)\int\limits_{\mathbf{R}^n}\int\limits_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}}dx\,dy$$

$$\leq 2(1-\varepsilon)N^{1+\varepsilon(\tau-(1-\varepsilon))}\int\limits_{\mathbf{R}^n}\int\limits_{\substack{|x|<|y|<2|x|\\|x-y|\leq N}}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+\tau(1+\varepsilon)}}dx\,dy$$

$$+\,2(1-\varepsilon)\int\limits_{\mathbf{R}^n}\int\limits_{\substack{|x|<|y|<2|x|\\|x-y|>N}}\frac{|u(x)-u(y)|^{1+\varepsilon}}{|x-y|^{n+(1-\varepsilon^2)}}dx\,dy.$$

The first term in the right-hand side tends to zero as $\varepsilon \to 1$ and the second one does not exceed

$$2^{2+\varepsilon}s\int_{|x|>N/3}\int_{|x-y|>N}\frac{dy}{|x-y|^{n+(1-\varepsilon^2)}}|u(x)|^{1+\varepsilon}dx \leq c(n,(1+\varepsilon))\int_{|x|>N/3}|u(x)|^{1+\varepsilon}\,dx,$$

which is arbitrarily small if N is sufficiently large. The proof is complete.

Section(5.2): Sobolev Inequalities for Higher Order Fractional Derivatives:

Sobolev inequalities have a wide range of applications and have been extensively studied (see[163]). Sometimes, it is also important to have precise estimates for the constants appearing in these inequalities. This has been the subject on many Studies recently (see [164] . More precisely given an integer $k \in \mathbb{N}$, the Sobolev space $H^k(\mathbb{R}^n)$is defined as the space of those functions $f \in L^2(\mathbb{R}^n)$satisfying $|\nabla^\ell f| \in L^2(\mathbb{R}^n), 1 \le \ell \le k$. The Sobolev imbedding theorem asserts that $H^k(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$for

$q = 2n/(n-2k)$. For example, when $k = 1, n \ge 3$ and $q = 2n/(n-2)$, we have the inequality

$$\|f\|_{\frac{2n}{n-2}}^2 \le C_n \|\nabla f\|_2^2, \qquad f \in C_0^\infty(\mathbb{R}^n). \tag{16}$$

The best value for the constant $C_n$in the above inequality has been estimated (in [164])

$$C_n = \pi^{-1} n^{-1} (n-2)^{-1} \left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{2/n}, \tag{17}$$

where $\Gamma(t)$is the Gamma function.

Let $\mathbb{S}^n$be the n-dimensional unit sphere and let $|\mathbb{S}^n|$denote its surface area.

Then, using the formula $\frac{\Gamma(n)}{\Gamma(n/2)} = \frac{2^{n-1}}{\pi^{1/2}} \Gamma\big((n+1)/2\big)$, we have

$$C_n = \frac{4}{n(n-2)} |\mathbb{S}^n|^{-2/n} = 2^{-2/n} \pi^{-(n+1)/n} \frac{4}{n(n-2)} \left[\Gamma\left(\frac{n+1}{2}\right)\right]^{2/n}.$$

We have equality in (16) if and only if $f(x) = c(\mu^2 + (x - x_0)^2)^{-(n-2)/2},\ x \in \mathbb{R}^n$,

where $c \in \mathbb{R}, \mu > 0$ and $x_0 \in \mathbb{R}^n$are fixed constants.

Let Δ be the Laplacian in $\mathbb{R}^n$and let $\hat{f}(\xi)$denote the Fourier transform of$f$ for the precise definitions and notations).We have $\widehat{-\Delta f}(\xi) = (2\pi|\xi|)^2 \hat{f}(\xi)$. So we can define the operators $(-\Delta)^{s/2}, s \in \mathbb{R}$, by setting

$$\left((-\Delta)^{s/2} f\right)^\wedge(k) = (2\pi|k|)^s \hat{f}(k), f \in C_0^\infty(\mathbb{R}^n).$$

We can easily verify that $\|\nabla f\|_2 = \left\|(-\Delta)^{1/2} f\right\|_2$. Using this notation, we can define Sobolev spaces $H^s(\mathbb{R}^n)$, for $s > 0$ by $H^s(\mathbb{R}^n) = \left\{f \in L^2(\mathbb{R}^n): \left\|(-\Delta)^{s/2} f\right\|_2 < \infty\right\}$.

We have the following generalization of (16), which was announced in [165].

Theorem (5.2.1)[166]: *Let* $n > 2s$ *and* $q = 2n/(n-2s)$. Then

$$\|f\|_q^2 \le S(n,s) \left\|(-\Delta)^{s/2} f\right\|_2^2, \qquad f \in H^s(\mathbb{R}^n), \tag{18}$$

where

$$S(n,s) = 2^{-2s} \pi^{-s} \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)} \left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{2s/n}. \tag{19}$$

We have equality in (18) if and only if $f(x) = c(\mu^2 + (x - x_0)^2)^{-\frac{n-2s}{2}}, x \in \mathbb{R}^n$,

where $c \in \mathbb{R}, \mu > 0$ and $x_0 \in \mathbb{R}^n$ are fixed constants. Also, by simple calculations we have that

$$S(n,s) = 2^{-\frac{2s}{n}} \pi^{-\frac{s(n+1)}{n}} \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)} \left[\Gamma\left(\frac{n+1}{2}\right)\right]^{\frac{2s}{n}} = \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)} |\mathbb{S}^n|^{-\frac{2s}{n}}.$$

Note that if $s = 1$ then we are in the case (16) and then the best value for the constant $S(n,s)$has been given in [164]. For $s = 1/2$ the best value for $S(n,s)$is given in [167], for $s = 2$ in [168]. Also the case $s \in \mathbb{N}$ has been considered in [26].

Proof: Let us first observe that since $C_0^\infty(\mathbb{R}^n)$is dense in $H^s(\mathbb{R}^n)$, it is enough to prove (18) for $f \in C_0^\infty(\mathbb{R}^n)$. Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Then, we have

$$(f,g) = (\hat{f},\hat{g}) = \int |k|^s \overline{\hat{f}(k)}|k|^{-s}\hat{g}(k)dk = \int \overline{(\widehat{-\Delta)^{s/2}}(f)(k)}(\widehat{-\Delta)^{-s/2}}(g)(k)dk$$
$$= \left((-\Delta)^{s/2}(f), (-\Delta)^{-s/2}(g)\right). \quad (20)$$

Hence,

$$|(f,g)| \le \left\|(-\Delta)^{s/2}(f)\right\|_2 \left\|(-\Delta)^{-s/2}(g)\right\|_2. \quad (21)$$

Now by the Hardy–Littlewood–Sobolev inequality we have that

$$\left\|(-\Delta)^{-s/2}(g)\right\|_2 \le 2^{-s}\pi^{-s/2}\left(\frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\right)^{1/2}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{s/n}\|g\|_p, \quad (22)$$

where $1/p + 1/q = 1$, i.e., $p = 2n/(n+2s)$. Combining (21) and (22) we have that

$$|(f,g)| \le \left(S(n,s)\right)^{1/2}\left\|(-\Delta)^{s/2}(f)\right\|_2\|g\|_p. \quad (23)$$

Now let us take $g = f^{q-1}$. Then we have $|(f,g)| = |f, f^{q-1}| = \|f\|_q^q, \|g\|_p = \|f^{q-1}\|_p = \|f\|_q^{q-1}$ and hence (23) becomes $\|f\|_q^2 \le S(n,s)\left\|(-\Delta)^{s/2}f\right\|_2^2$. Finally, let us observe that in order to have equality in (18) we must have equality in (22) and as it is well known, this happens if and only if $f(x) = c(\mu^2 + (x-x_0)^2)^{-(n-2s)/2}$ for fixed constants $c \in \mathbb{R}, \mu > 0$ and $x_0 \in \mathbb{R}^n$.

Remark(5.2.2)[166]. As we mentioned in the introduction, the case $s \in \mathbb{N}$ has been considered in [174], where it was proved that the best constant C_ℓ in the inequality

$$\|f\|_q \le C_\ell\left\|\nabla^\ell f\right\|_2 \quad (24)$$

is given by $C_\ell = \pi^{-\ell/2}\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{\ell/n}\prod_{h=-\ell}^{\ell-1}(n+2h)^{-1/2}$. This constant is related to our constant $S(n,\ell) = 2^{-2\ell}\pi^{-\ell}\frac{\Gamma\left(\frac{n-2\ell}{2}\right)}{\Gamma\left(\frac{n+2\ell}{2}\right)}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{2\ell/n}$ as follows. In [174], there exist positive number $c$and C that depend only on n, ℓ such that $c\left\|(-\Delta)^{\ell/2}f\right\|_2 \le \left\|\nabla^\ell f\right\|_2 \le C\left\|(-\Delta)^{\ell/2}f\right\|_2$.

We have $C = c = 2^{-\ell}\prod_{h=-\ell}^{\ell-1}(n+2h)^{1/2}\left[\frac{\Gamma\left(\frac{n-2\ell}{2}\right)}{\Gamma\left(\frac{n+2\ell}{2}\right)}\right]^{1/2}$ because $S(n,\ell)$is the best constant for the inequality $\|f\|_q^2 \le S(n,\ell)\left\|(-\Delta)^{\ell/2}f\right\|_2^2$and $C_\ell$is the best constant of (24).

Theorem (5.2.3)[166]: Let $2 < q < \infty$ and $q - 2 < 2qs$. Then,

$$\|f\|_q^2 \le S(q,s)\left[\left\|(-\Delta)^{s/2}f\right\|_2^2 + \|f\|_2^2\right], \ f \in H^s(\mathbb{R}), \quad (25)$$

where

$$S(q,s) < (q-1)^{-1+1/q}q^{1-2/q}\left[\frac{\Gamma\left(1+\frac{1}{2s}\right)\Gamma\left(-\frac{1}{2s}+\frac{q}{q-2}\right)}{\pi\Gamma\left(\frac{q}{q-2}\right)}\right]^{(q-2)/q}. \quad (26)$$

Let us define the operators $(I-\Delta)^{s/2}, s \in R$ by setting $\widehat{(I-\Delta)^{s/2}f}(\xi) = (1+(2\pi|\xi|)^2)^{s/2}\hat{f}(\xi)$.

Then another way to define the Sobolev space $H^s(\mathbb{R}^n), s \in \mathbb{R}$ is as the space of those functionfwhich satisfy $\left\|(I-\Delta)^{s/2}f\right\|_2 < \infty$. We set $\|f\|_{H^s(\mathbb{R}^n)} = \left\|(I-\Delta)^{s/2}f\right\|_2$.

In the case of $\mathbb{R}^2$, we have the following result.

Proof: It is enough to prove (25) for $f \in C_0^\infty(\mathbb{R})$. Let $1/p + 1/q = 1$. Then $p < 2$ and

$$\frac{p}{2-p} = \frac{q/(q-1)}{2-\big(q/(q-1)\big)} = \frac{q}{q-2}. \tag{27}$$

We have that

$$\|\hat{f}\|_p^p = \int_{\mathbb{R}} \left|\hat{f}(k)\right|^p dk = \int_{\mathbb{R}} \left|\hat{f}(k)(1+(2\pi|k|)^{2s})^{1/2}\right|^p (1+(2\pi|k|)^{2s})^{-p/2} dk. \tag{28}$$

Let us set $F(k) = \left(\left|\hat{f}(k)\right|^2 (1+(2\pi|k|)^{2s})\right)^{p/2}$, $G(k) = (1+(2\pi|k|)^{2s})^{-p/2}$.Then

$$\|F\|_{2/p} = \left(\int_{\mathbb{R}} \left|\hat{f}(k)\right|^2 (1+(2\pi|k|)^{2s} dk)\right)^{p/2} = \left(\left\|(-\Delta)^{s/2} f\right\|_2^2 + \|f\|_2^2\right)^{p/2}. \tag{29}$$

Also, since $2qs/(q-2) = 2ps/(2-p) > 1$,

$$\|G\|_{2/(2-p)}^{2/(2-p)} = \int_{\mathbb{R}} (1+(2\pi|k|)^{2s})^{-p/(2-p)} dk = \frac{\Gamma\left(1+\frac{1}{2s}\right)\Gamma\left(-\frac{1}{2s}+\frac{p}{2-p}\right)}{\pi\Gamma\left(\frac{p}{2-p}\right)}. \tag{30}$$

We have that $\frac{1}{2/(2-p)} + \frac{1}{2/p} = 1$. So, by the Hölder inequality, it follows from (29), (30) and (28) that

$$\|\hat{f}\|_p \le \left(\|F\|_{\frac{2}{p}} \|G\|_{\frac{2}{(2-p)}}\right)^{\frac{1}{p}} = \left(\left\|(-\Delta)^{s/2} f\right\|_2^2 + \|f\|_2^2\right)^{1/2} \left[\frac{\Gamma\left(1+\frac{1}{2s}\right)\Gamma\left(-\frac{1}{2s}+\frac{p}{2-p}\right)}{\pi\Gamma\left(\frac{p}{2-p}\right)}\right]^{(2-p)/2p}$$

and therefore, by (27),

$$\|\hat{f}\|_p \le \left(\left\|(-\Delta)^{\frac{s}{2}} f\right\|_2^2 + \|f\|_2^2\right)^{\frac{1}{2}} \left[\frac{\Gamma\left(1+\frac{1}{2s}\right)\Gamma\left(-\frac{1}{2s}+\frac{q}{q-2}\right)}{\pi\Gamma\left(\frac{q}{q-2}\right)}\right]^{\frac{(q-2)}{2q}}. \tag{31}$$

Now, by the sharp Hausdorff–Young inequality (24) we have

$$\|f\|_q \le C_p \|\hat{f}\|_p, \quad \text{with} C_p = \left[p^{1/p}(q)^{-1/q}\right]^{1/2}. \tag{32}$$

Combining (31) and (32) we have that

$$\begin{aligned}\|f\|_q^2 &\le \left[p^{\frac{1}{p}}(q)^{-\frac{1}{q}}\right] \left[\frac{\Gamma\left(1+\frac{1}{2s}\right)\Gamma\left(-\frac{1}{2s}+\frac{q}{q-2}\right)}{\pi\Gamma\left(\frac{q}{q-2}\right)}\right]^{\frac{(q-2)}{q}} \left[\left\|(-\Delta)^{\frac{s}{2}} f\right\|_2^2 + \|f\|_2^2\right] \\ &= (q-1)^{-1+1/q} q^{1-2/q} \left[\frac{\Gamma\left(1+\frac{1}{2s}\right)\Gamma\left(-\frac{1}{2s}+\frac{q}{q-2}\right)}{\pi\Gamma\left(\frac{q}{q-2}\right)}\right]^{(q-2)/q} \times \left[\left\|(-\Delta)^{s/2} f\right\|_2^2 + \|f\|_2^2\right],\end{aligned}$$

which proves the theorem.

Theorem (5.2.4)[166]: For all $2 < q < \infty$ we have

$$\|f\|_q^2 \le V(q,s)\left\|(I-\Delta)^{s/2} f\right\|_2^2, \qquad f \in H^s(\mathbb{R}^2), \tag{33}$$

where the constant $V(q,s)$satisfies

$$V(q,s) < (q-1)^{-2+2/q} q^{2-4/q} \left[\frac{1}{4\pi} \frac{q-2}{q(s-1)+2}\right]^{1-2/q}. \tag{34}$$

Proof: is similar to the proof of Theorem (5.2.3).We observe again that it is enough to prove (33) for $f \in C_0^\infty(\mathbb{R}^2)$. Let $1/p + 1/q = 1$.
Then $p < 2$ and

$$\frac{p}{2-p} = \frac{q/(q-1)}{2-\big(q/(q-1)\big)} = \frac{q}{q-2}. \tag{35}$$

We have that

$$\|\hat{f}\|_p^p = \int_{\mathbb{R}^2} |\hat{f}(k)|^p dk = \int_{\mathbb{R}^2} \left|\hat{f}(k)(1+(2\pi|k|)^2)^{s/2}\right|^p (1+(2\pi|k|)^2)^{-sp/2} dk. \tag{36}$$

Let us set $F(k) = \left(|\hat{f}(k)|^2(1+(2\pi|k|)^2)^s\right)^{p/2}$, $G(k) = (1+(2\pi|k|)^2)^{-ps/2}$.

Then

$$\|F\|_{2/p} = \left(\int_{\mathbb{R}^2} |\hat{f}(k)|^2(1+(2\pi|k|)^2)^s\right)^{p/2} dk = \|(I-\Delta)^{s/2} f\|_2^p. \tag{37}$$

Also

$$\|G\|_{2/(2-p)}^{2/(2-p)} = \int_{\mathbb{R}^2} (1+(2\pi|k|)^2)^{-ps/(2-p)} dk = \frac{1}{4\pi}\frac{q-2}{q(s-1)+2}. \tag{38}$$

We have that $\frac{1}{2/(2-p)} + \frac{1}{2/p} = 1$. So, by the Hölder inequality, it follows from (37), (38) and (36)

$$\|\hat{f}\|_p \le \|(I-\Delta)^{s/2} f\|_2 \left[\frac{1}{4\pi}\frac{q-2}{q(s-1)+2}\right]^{(q-2)/2q}. \tag{39}$$

Now, by the sharp Hausdorff–Young inequality (53) we have

$$\|f\|_q \le C_p \|\hat{f}\|_p, \quad \text{with} C_p = \left[p^{1/p}(q)^{-1/q}\right]^{1/2}. \tag{40}$$

Combining (39) and (40) we have that $\|f\|_q^2 \le \left[p^{1/p}(q)^{-1/q}\right]^2 \left[\frac{1}{4\pi}\frac{q-2}{q(s-1)+2}\right]^{(q-2)/2q} \|(I-\Delta)^{s/2} f\|_2$
and the theorem follows.
Let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, let $g_{\mathbb{S}^n}$ be the restriction of the Euclidean metric to $\mathbb{S}^n$ and let $\Delta_{\mathbb{S}^n}$ denote the spherical Laplacian. Let also dxbe the surface measure on $\mathbb{S}^n$ andlet us denote by $|\mathbb{S}^n|$ the surface area of $\mathbb{S}^n$. We have $\mathbb{S}^n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$. We denote by $d\sigma(x)$ the normalised measure $d\sigma(x) = (1/|\mathbb{S}^n|)dx$.

Let us consider the following operators (studied in [169],[170],[171])

$$B = \sqrt{\Delta_{\mathbb{S}^n} + \left(\frac{n-1}{2}\right)^2}, \quad A_s = \frac{\Gamma(B+(1+s)/2)}{\Gamma(B+(1-s)/2)}, \ s \in \mathbb{R}.$$

A function $F : \mathbb{S}^n \to \mathbb{R}$ with $F \in L^2(\mathbb{S}^n)$ is said to be in $H^s(\mathbb{S}^n)$ if and only if $\int_{\mathbb{S}^n} F A_{2s} F d\xi < \infty$.
The above defined operators A_s are related to the operators $(-\Delta)^s$ as follows.
Let $\pi: \mathbb{R}^n \to \mathbb{S}^n - \{0, \dots, 0, -1\}$ denote stereographic projection and let J_π be the Jacobian of π. Then,we have [171]$(A_s F) \circ \pi = |J_\pi|^{-(n+s)/(2n)}(-\Delta)^{s/2}\left(|J_\pi|^{(n-s)/(2n)}(F \circ \pi)\right)$, where $s > 0$ and $F \in L^2(\mathbb{S}^n)$.
Making use of Theorem (5.2.1) and the above formula, we have a new proof of the following result due to [169].

Theorem (5.2.5)[166]: Let $n > 2s$ and $q = \frac{2n}{n-2s}$.

Then $\|F\|_q^2 \le S(n,s) \int_{\mathbb{S}^n} FA_{2s}Fd\xi$, $F \in H^s(\mathbb{S}^n)$ where $S(n,s) = 2^{-2s}\pi^{-s}\frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{2s/n}$.

Proof: The stereographic projection $\pi : \mathbb{R}^n \to \mathbb{S}^n - \{0,\dots,0,-1\}$ is defined by $\pi(x) = \left(\frac{2x_1}{1+|x|^2},\dots,\frac{2x_n}{1+|x|^2},\frac{1-|x|^2}{1+|x|^2}\right)$. Let J_π denote the Jacobian of π. The $|J_\pi| = \left(\frac{2}{1+|x|^2}\right)^n$.

If $f \in L^p(\mathbb{R}^n)$, then we can lift f to $\mathbb{S}^n$ by setting $F(\xi) = |J_{\pi^{-1}}|^{1/p} f(\pi^{-1}(\xi))$. Note that then $\|F\|_{L^p(\mathbb{S}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$. We have

$$\begin{aligned}\int_{\mathbb{S}^n} FA_{2s}Fd\xi &= \int_{\mathbb{R}^n} F\circ\pi(A_{2s}F)\circ\pi|J_\pi|dx\\ &= \int_{\mathbb{R}^n}(F\circ\pi)|J_\pi|^{-(n+2s)/(2n)}(-\Delta)^s\left(|J_\pi|^{(n-2s)/(2n)}(F\circ\pi)\right)|J_\pi|dx\\ &= \int_{\mathbb{R}^n}|J_\pi|^{(n-2s)/(2n)}(F\circ\pi)(-\Delta)^s\left(|J_\pi|^{(n-2s)/(2n)}(F\circ\pi)\right)dx.\end{aligned}$$

Applying Theorem (5.2.1) we get

$$\begin{aligned}\int_{\mathbb{S}^n} FA_{2s}Fd\xi &\ge \frac{1}{S(n,s)}\left\||J_\pi|^{(n-2s)/(2n)}(F\circ\pi)\right\|_q^2 = \frac{1}{S(n,s)}\left(\int_{\mathbb{R}^n}|J_\pi||F\circ\pi|^{2n/(n-2s)}dx\right)^{(n-2s)/n}\\ &= \frac{1}{S(n,s)}\left(\int_{\mathbb{S}^n}|F|^{2n/(n-2s)}ds\right)^{(n-2s)/n} = \frac{1}{S(n,s)}\|F\|_q^2,\end{aligned}$$

where we have set $q = \frac{2n}{n-2s}$.

This proves the theorem.

A result, which is of great importance in applications and especially in the calculus of variations, is the Rellich-Kondrashov theorem. More precisely, let Abe a measurable set of $\mathbb{R}^n$ and let us consider a sequence of functions $f_j \in L^2(A)$ such that $\|f_j\|_{L^2(A)} < c < \infty, j \in \mathbb{N}$.

Then, as we know, by the Banach-Alaoglou theorem, there exists a weakly convergent subsequence. A strongly convergent subsequence may not exist.The Rellich–Kondrashov theorem asserts that if the sequence $(f_j)$is uniformly bounded in $H^1(A)$, i.e., $\sup_{j\in\mathbb{N}}\|\nabla f_j\|_{L^2(A)} < \infty$, then any weakly convergent subsequence of (f_j) is also strongly convergent in $L^2(A)$.Let us now assume that $f_j \in H^s(\mathbb{R}^n), j \in \mathbb{N}$, and that the sequence (f_j)converges weakly to a function $f \in H^s(\mathbb{R}^n)$, i.e., that for every

$$g \in H^s(\mathbb{R}^n) \int_{\mathbb{R}^n}\left[\overline{\hat{f}_j(k)} - \overline{\hat{f}(k)}\right]\hat{g}(k)(1+(2\pi|k|)^{2s})dk \to 0 \quad (j\to\infty).$$

Theorem (5.2.6)[166]. Let (f_j) be as above and let us assume that $2s < n$ and $p < 2n/(n-2s)$. Then for every measurable set $A \subset \mathbb{R}^n$ with finite measure, $\|f_j - f\|_{L^p(A)} \to 0 \quad (j\to\infty)$.

Proof: Let us set $g_{j,t} = e^{-t(-\Delta)^s}f_j$and $g_t = e^{-t(-\Delta)^s}f, t > 0$. Then, we have

$$\|f_j - f\|_{L^2(A)} \le \|f_j - g_{j,t}\|_2 + \|g_{j,t} - g_t\|_{L^2(A)} + \|g_t - f\|_2. \tag{44}$$

Since $f_j \to f$weakly in $H^s(\mathbb{R}^n)$, by the Banach–Alaoglou theorem, there is $c > 0$ such that

$$\|(-\Delta)^{s/2}f^j\|_2 < c, \qquad j \in \mathbb{N}.$$

Therefore, by Proposition (5.2.7),

$$\|f_j - g_{j,t}\|_2 \le c\sqrt{t}, \qquad \|f - g_t\|_2 \le c\sqrt{t}. \tag{45}$$

Let $F_s(x)$ be the function with Fourier transform $\hat{F}_s(k) = e^{-t(2\pi|k|)^{2s}}$.

Then $g_{j,t} = F_s * f_j$ and $g_t = F_s * f$.

Then, by Theorem (5.2.1), for $q = 2n/(n-2s)$ there is $c > 0$ such that

$$\|f^j\|_q \leq S(n,s)^{1/2}\|(-\Delta)^{s/2}f^j\|_2 \leq c. \tag{46}$$

Let $1/q + 1/q' = 1$ and $p' = q$. Then by the Hölder inequality

$$\|g_{j,t}\|_\infty \leq \|F_s\|_{q'}\|f_j\|_{q'}, \qquad \|g_t\|_\infty \leq \|F_s\|_{q'}\|f\|_q. \tag{47}$$

Note that

$$\|F_s\|_{L^{q'}(A)} = \left\|\hat{\hat{F_s}}\right\|_{L^{q'}(A)} \leq \left\|\hat{\hat{F_s}}\right\|_{L^\infty(A)} \leq \left\|\hat{\hat{F_s}}\right\|_{L^\infty(\mathbb{R}^n)} \leq \|\hat{F_s}\|_{L^1(\mathbb{R}^n)} < \infty. \tag{48}$$

Combining (46), (47) and (48) we have that there is $c > 0$ such that

$$\|g_t\|_{L^\infty(A)} \leq c, \quad \|g_{j,t}\|_{L^\infty(A)} \leq c, \qquad j \in \mathbb{N}. \tag{49}$$

Now let us observe that since $f_j \to f$ weakly in $L^q(\mathbb{R}^n)$ and since $F_s \in L^q(\mathbb{R}^n)$ wehave that $g_{j,t}(x) \to g_t(x)$ for all $x \in \mathbb{R}^n$. From this observation and (49) and by using thedominated convergence theorem we get that

$$\|g_{j,t} - g_t\|_{L^2(A)} \to 0 \quad (j \to \infty). \tag{50}$$

Combining (44), (45) and (50) we get that $\|f_j - f\|_{L^2(A)} \to 0 \quad (j \to \infty)$.

This proves the theorem if $p \leq 2$, since then, by the Hölder inequality,

$\|f_j - f\|_{L^p(A)} \leq |A|^{\frac{1}{r}}\|f_j - f\|_{L^2(A)}$,where $1/p = 1/r + 1/2$. If $p > 2$, then again by the Hölder inequality

$$\|f_j - f\|_{L^p(A)} \leq \|f_j - f\|^{\alpha}_{L^2(A)}\|f_j - f\|^{1-\alpha}_{L^q(A)}, \tag{51}$$

for $q > p$and $\alpha = (1/p - 1/q)/(1/2 - 1/q) > 0$.

Now since $f_j \in H^s(\mathbb{R}^n)$ and $n > 2s$, there is $c > 0$ such that

$$\|f^j - f\|_q \leq S(n,s)\|(-\Delta)^{s/2}(f^j - f)\|_2 \leq S(n,s)\left(\|(-\Delta)^{s/2}f^j\|_2 + \|(-\Delta)^{s/2}f\|_2\right) \leq c. \tag{52}$$

(51) and (52) prove the theorem when $2 < p$.If $x = (x_1, \ldots, x_n), k = (k_1, \ldots, k_n) \in \mathbb{R}^n$, then we denote $(k,x) = k_1x_1 + \cdots + x_nk_n$and $|x| = (x,x)^{1/2}$.If $f, g \in L^2(\mathbb{R}^n)$, then we denote $(f,g) = \int f(x)g(x)\,dx$.
The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by $\hat{f}(k) = \int e^{-2\pi i}(k,x)f(x)dx$.
The sharp Hausdorf inequality in [172] says that if $1 \leq p \leq 2$ and $1/p + 1/q = 1$, then

$$\|\hat{f}\|_q \leq C_p\|f\|_p, \qquad f \in L^p(\mathbb{R}^n), \tag{53}$$

Where $C_p = \left[p^{1/p}(q)^{-1/q}\right]^{1/2}$. We have $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$.
Note that$\widehat{(-\Delta)f}(k) = (2\pi|k|)^2\hat{f}(k)$.

Let us recall that the operators $(-\Delta)^{s/2}$ and $(I - \Delta)^{s/2}$ have been defined respectively by [173])

$$\widehat{(-\Delta)^{s/2}f}(\xi) = (2\pi|\xi|)^s\hat{f}(\xi), \widehat{(I-\Delta)^{s/2}f}(\xi) = (1 + (2\pi|\xi|)^2)^{s/2}\hat{f}(\xi).$$

Also $H^s(\mathbb{R}^n) = \left\{f \in L^2(\mathbb{R}^n) : \|(I-\Delta)^{s/2}f\|_2 < \infty\right\}$ and $\|f\|_{H^s(\mathbb{R}^n)} = \|(I-\Delta)^{s/2}f\|_2$.

Note that $\|\nabla f\|_2 = \|(-\Delta)^{1/2}f\|_2$ and that $\|(I-\Delta)^{1/2}f\|_2^2 = \|f\|_2^2 + \|\nabla f\|_2^2$.

The operators $(-\Delta)^{-\frac{s}{2}}, 0 < s < n$, are called Riesz potential operators in [173] andwe have $(-\Delta)^{-\frac{s}{2}}(f) = I_s * f$, where I_s is the Riesz potential $I_s(x) = \frac{1}{\gamma(s)}|x|^{-n+s}$,

where $\gamma(s) = \frac{\pi^{n/2}2^s\Gamma(s/2)}{\Gamma\left(\frac{n}{2} - \frac{s}{2}\right)}$.

The Hardy–Littlewood–Sobolev fractional integration theorem asserts that the operators $(-\Delta)^{-s/2}, 0 < s < n$, are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, for $1/q = 1/p - s/n$.

The operators $(I-\Delta)^{-s/2}$, for $s > 0$, are called Bessel potential operators in [173] andthey are given by convolution with the Bessel potential
$G_s(x) = \frac{1}{\alpha(s)}\int_0^\infty e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{(-n+s)/2}\frac{d\delta}{\delta}$, where $\alpha(s) = \Gamma(s/2)(4\pi)^{s/2}$.

We consider the operator of semigroups $e^{-t(-\Delta)^s}, t > 0, e^{-t\,(I-\Delta)^s}, t > 0$, defined respectively by

$$\left(e^{-t(-\Delta)^s}f\right)^\wedge(k) = e^{-t(2\pi|k|)^{2s}}\hat{f}(k), \left(e^{-t(I-\Delta)^s}f\right)^\wedge(k) = e^{-t\left(1+(2\pi|k|)^2\right)^s}\hat{f}(k).$$

Proposition (5.2.7)[166]: For every $f \in H^s(\mathbb{R}^n)$, we have

$$\left\|f - e^{-t(-\Delta)^s}f\right\|_2 \le \sqrt{t}\left\|(-\Delta)^{s/2}f\right\|_2, \quad (54)$$

$$\left\|f - e^{-t(I-\Delta)^s}f\right\|_2 \le \sqrt{t}\left\|(I-\Delta)^{s/2}f\right\|_2. \quad (55)$$

Proof. Let $f \in H^s(\mathbb{R}^n)$. Then, we have $\left\|f - e^{-t(-\Delta)^s}f\right\|_2^2 = \int\left|\hat{f}(k)\right|^2\left(1 - e^{-t(2\pi|k|)^{2s}}\right)^2 dk$

Now let us observe that $1 - e^{-x} \le x$, for $x \ge 0$. Hence

$$\left\|f - e^{-t(-\Delta)^s}f\right\|_2^2 \le t\int(2\pi|k|)^{2s}\left|\hat{f}(k)\right|^2 dk = t\left\|(-\Delta)^{s/2}f\right\|_2^2.$$

This proves (54). The proof of (55) is similar.

Corollary (5.2.8)[236]: For all $\varepsilon > 0$ we have

$$\sum_{j=1}^m\left\|f_j\right\|_{2+\varepsilon}^2 \le V(2+\varepsilon, s)\left\|(I-\Delta)^{s/2}f_j\right\|_2^2, \ f_j \in H^s(\mathbb{R}^2),$$

where the constant $V(2+\varepsilon, s)$ satisfies

$$V(2+\varepsilon, s) < (1+\varepsilon)^{-(2+2\varepsilon)/2+\varepsilon}(2+\varepsilon)^{2\varepsilon/2+\varepsilon}\left[\frac{1}{4\pi}\frac{\varepsilon}{(s-1)+2}\right]^{\varepsilon/2+\varepsilon}.$$

Proof: is similar to the proof of Theorem (5.2.3).We observe again that it is enough to prove (33) for $f \in C_0^\infty(\mathbb{R}^2)$.

We have that

$$\sum_{j=1}^m\left\|\hat{f}_j\right\|_{2+\varepsilon/1+\varepsilon}^{2+\varepsilon/1+\varepsilon} = \int_{\mathbb{R}^2}\sum_{j=1}^m\left|\hat{f}_j(k)\right|^{2+\varepsilon/1+\varepsilon}dk$$
$$= \int_{\mathbb{R}^2}\sum_{j=1}^m\left|\hat{f}_j(k)(1+(2\pi|k|)^2)^{s/2}\right|^{2+\varepsilon/1+\varepsilon}(1+(2\pi|k|)^2)^{-s(2+\varepsilon/1+\varepsilon)/2}dk.$$

Let us set $f_j(k) = \left(\sum_{j=1}^m\left|\hat{f}_j(k)\right|^2(1+(2\pi|k|)^2)^s\right)^{(2+\varepsilon/1+\varepsilon)s}$, $G(k) = (1+(2\pi|k|)^2)^{-s(2+\varepsilon/1+\varepsilon)/2}$.

Then $\|F\|_{2+2\varepsilon/2+\varepsilon} = \left(\int_{\mathbb{R}^2}\sum_{j=1}^m\left|\hat{f}_j(k)\right|^2(1+(2\pi|k|)^2)^s\right)^{2+\varepsilon/2+2\varepsilon}dk = \left\|(I-\Delta)^{s/2}f_j\right\|_2^{2+\varepsilon/1+\varepsilon}$.

Also $\|G\|_{6+6\varepsilon/2+\varepsilon}^{6+6\varepsilon/2+\varepsilon} = \int_{\mathbb{R}^2}(1+(2\pi|k|)^2)^{-s(2+\varepsilon)/\varepsilon}dk = \frac{1}{4\pi}\frac{\varepsilon}{(4+\varepsilon)(s-1)}$. So, by the Hölder inequality,

it follows from (37), (38) and (36) $\sum_{j=1}^m\left\|\hat{f}_j\right\|_{2+\varepsilon/1+\varepsilon} \le \sum_{j=1}^m\left\|(I-\Delta)^{s/2}f_j\right\|_2\left[\frac{1}{4\pi}\frac{\varepsilon}{(4+\varepsilon)(s-1)}\right]^{\varepsilon/4+2\varepsilon}$.

Now, by the sharp Hausdorff–Young inequality (53) we have

$$\sum_{j=1}^m\left\|f_j\right\|_{2+\varepsilon} \le C_{2+\frac{\varepsilon}{1}+\varepsilon}\sum_{j=1}^m\left\|\hat{f}_j\right\|_{2+\frac{\varepsilon}{1}+\varepsilon}, \text{with } C_{2+\varepsilon/1+\varepsilon} = \left[(2+\varepsilon/1+\varepsilon)^{-1/1+\varepsilon}(2+\varepsilon)^{-1/2+\varepsilon}\right]^{1/2}.$$

Combining (39) and (40) we have that

$$\sum_{j=1}^m\left\|f_j\right\|_{2+\varepsilon}^2 \le \left[(2+\varepsilon/1+\varepsilon)^{-1/1+\varepsilon}(2+\varepsilon)^{-1/2+\varepsilon}\right]^2\left[\frac{1}{4\pi}\frac{\varepsilon}{(4+\varepsilon)(s-1)}\right]^{\varepsilon/4+2\varepsilon}\sum_{j=1}^m\left\|(I-\Delta)^{s/2}f_j\right\|_2$$

and the theorem follows.

Section (5.3): Fractional Sobolev Spaces:

This section is for Hitchhike way from 1 to $s \in (0,1)$. To wit, for anybody who, only endowed with some basic analysis course (and knowing where his towel is), would like to pick up some quick, crash and essentially information on the fractional Sobolev spaces $W^{s,p}$.The reasons for such a Hitchhiker to start this adventurous trip might be of different kind: (s) he could be driven by mathematical curiosity, or could be tempted by the many applications that fractional calculus seems to have recently experienced.

We define the fractional Sobolev spaces $W^{s,p}$ via the Gagliardo approach and we investigate some of their basic properties.

We focus on the Hilbert case $p = 2$, dealing with its relation with the fractional Laplacian, and letting the principal value integral definition interplay with the definition in the Fourier space. Then, we analyze the asymptotic behavior of the constant factor that appears in the definition of the fractional Laplacian. We have the extension problem of a function in $W^{s,p}(\Omega)$ to $W^{s,p}(\mathbb{R}^n)$: technically, this is slightly more complicated than the classical analogue for integer Sobolev spaces, since the extension interacts with the values taken by the function in Ω via the Gagliardo norm and the computations have to take care of it.Sobolev inequalities and continuous embeddings are dealt, while is devoted to compact embeddings, then we point out that functions in $W^{s,p}$ are continuous when sp is large enough. And we present some counterexamples in non-Lipschitz domains.

This section is devoted to the definition of the fractional Sobolev spaces. No prerequisite is needed. We just recall the definition of the Fourier transform of a distribution. First, consider the Schwartz space S of rapidly decaying C^∞ functions in $\mathbb{R}^n$. The topology of this space is generated by the semi norms $p_N(\varphi) = \sup_{x\in\mathbb{R}^n}(1+|x|)^N \sum_{|\alpha|\le N}|D^\alpha\varphi(x)|$, $N = 0,1,2,\ldots$, where $\varphi \in$ S $(\mathbb{R}^n)$.

Let $S'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of S $(\mathbb{R}^n)$. As usual, for any $\varphi \in$ S $(\mathbb{R}^n)$, we denote by $F_\varphi(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx$

the Fourier transform of $\varphi$and we recall that one can extend Ffrom S $(\mathbb{R}^n)$to $S'(\mathbb{R}^n)$.

Let Ωbe a general, possibly nonsmooth, open set in $\mathbb{R}^n$. For any real $s > 0$ and for any $p \in [1,\infty)$, we want to define the fractional Sobolev spaces $W^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also called Aronszajn, Gagliardo or Slobodeckij spaces, by the name of the ones who introduced them, almost simultaneously (see [176]).

We start by fixing the fractional exponent $s$in $(0,1)$. For any $p \in [1,+\infty)$, we define $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) := \left\{u \in L^p(\Omega): \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega\times\Omega)\right\}; \tag{53}$$

i.e., an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p dx + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy\right)^{\frac{1}{p}}, \tag{54}$$

where the term $[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dx\,dy\right)^{\frac{1}{p}}$ is the so-called Gagliardo (semi)norm of u.

It is worth noticing that, as in the classical case with $s$being an integer, the space $W^{s',p}$ iscontinuously embedded in $W^{s,p}$ when $s \le s'$, as next result points out.

Proposition (5.3.1)[175]. Let $p \in [1,+\infty)$ and $0 < s \le s' < 1$. Let Ω be an open set in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then $\|u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{s',p}(\Omega)}$

for some suitable positive constant $C = C(n,s,p) \ge 1$. In particular, $W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega)$.

Proof. First, $\int_\Omega \int_{\Omega\cap\{|x-y|\ge 1\}} \frac{|u(x)|^p}{|x-y|^{n+sp}} dx\, dy \le \int_\Omega \left(\int_{|z|\ge 1} \frac{1}{|z|^{n+sp}} dz\right) |u(x)|^p dx \le C(n,s,p)\|u\|^p_{L^p(\Omega)}$, where we used the fact that the kernel $1/|z|^{n+sp}$ is integrable since $n+sp>n$.

Taking into account the above estimate, it follows

$$\int_\Omega \int_{\Omega\cap\{|x-y|\ge 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy$$

$$\le 2^{p-1} \int_\Omega \int_{\Omega\cap\{|x-y|\ge 1\}} \frac{|u(x)|^p + |u(y)|^p}{|x-y|^{n+sp}} dx\, dy \le 2^p C(n,s,p)\|u\|^p_{L^p(\Omega)} \quad (55)$$

On the other hand,

$$\int_\Omega \int_{\Omega\cap\{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy \le \int_\Omega \int_{\Omega\cap\{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} dx\, dy. \quad (56)$$

Thus, combining (55) with (56), we get

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy \le 2^p C(n,s,p)\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} dx\, dy$$

and so

$$\|u\|^p_{W^{s,p}(\Omega)} \le (2^p C(n,s,p)+1)\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}} dx\, dy \le C(n,s,p)\|u\|^p_{W^{s',p}(\Omega)},$$

which gives the desired estimate, up to relabeling the constant $C(n,p,s)$.

We will show in the following Proposition that the result in Proposition (5.3.1) holds also in the limit case, namely when $s' = 1$, but for this we have to take into account the regularity of

As usual, for any $k \in \mathbb{N}$ and $\alpha \in (0,1]$, we say that Ω is of class $C^{k,\alpha}$ if there exists $M > 0$ such that for any $x \in \partial\Omega$ there exist a ball $B = B_r(x), r > 0$, and an isomorphism $T : Q \to B$ such that

$T \in C^{k,\alpha}(\bar{Q}),\ T^{-1} \in C^{k,\alpha}(\bar{B}),\ T(Q_+) = B \cap \Omega,\ T(Q_0) = B \cap \partial\Omega$ and

$\|T\|_{C^{k,\alpha}(\bar{Q})} + \|T^{-1}\|_{C^{k,\alpha}(\bar{B})} \le M,$ where $Q := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}: |x'| < 1$ and $|x_n| < 1\}$,
$Q_+ := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}: |x'| < 1$ and $0 < x_n < 1\}$ and $Q_0 := \{x \in Q: x_n = 0\}$.
We have the following result.

Proposition(5.3.2)[175]. Let $p \in [1,+\infty)$ and $s \in (0,1)$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary and $u : \Omega \to \mathbb{R}$ be a measurable function. Then

$$\|u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \quad (57)$$

for some suitable positive constant $C = C(n,s,p) \ge 1$. In particular, $W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega)$.

Proof. Let $u \in W^{1,p}(\Omega)$.Thanks to the regularity assumptions on the domain Ω, we can extend u to a function $\tilde{u}: \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ and $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$ for a suitable constant C (see [177]).

Now, using the change of variable $z = y - x$ and the Hölder inequality, we have

$$\int_{\Omega} \int_{\Omega\cap\{|x-y|<1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy \le \int_{\Omega} \int_{B_1} \frac{|u(x)-u(z+x)|^p}{|z|^{n+sp}} dz\, dx$$

$$= \int_{\Omega} \int_{B_1} \frac{|u(x)-u(z+x)|^p}{|z|^p} \frac{1}{|z|^{n+(s-1)p}} dz\, dx$$

$$\le \int_{\Omega} \int_{B_1} \left(\int_0^1 \frac{|\nabla u(x+tz)|}{|z|^{\frac{n}{p}+s-1}} dt \right)^p dz\, dx$$

$$\le \int_{\mathbb{R}^n} \int_{B_1} \int_0^1 \frac{|\nabla \tilde{u}(x+tz)|^p}{|z|^{n+p(s-1)}} dt\, dz\, dx$$

$$\le \int_{B_1} \int_0^1 \frac{\|\nabla \tilde{u}\|^p_{L^p(\mathbb{R}^n)}}{|z|^{n+p(s-1)}} dt\, dz$$

$$\le C_1(n,s,p) \|\nabla \tilde{u}\|^p_{L^p(\mathbb{R}^n)}$$

$$\le C_2(n,s,p) \|u\|^p_{W^{1,p}(\Omega)}. \tag{58}$$

Also, by (55),

$$\int_{\Omega} \int_{\Omega\cap\{|x-y|\ge 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy \le C(n,s,p) \|u\|^p_{L^p(\Omega)}. \tag{59}$$

Therefore, from (58) and (59) we get estimate (57).

We remark that the Lipschitz assumption in Proposition (5.3.2) cannot be completely dropped we discuss the extension problem in $W^{s,p}$. Let us come back to the definition of the space $W^{s,p}(\Omega)$. Before going ahead, it is worth explaining why the definition in (53) cannot be plainly extended to the case $s \ge 1$. Suppose that $\Omega$is a connected open set in $\mathbb{R}^n$, then any measurable function $u : \Omega \to \mathbb{R}$ such that $\int_{\Omega} \int_{\Omega} \frac{|u(x)-u(u)|^p}{|x-y|^{n+sp}} dx\, dy < +\infty$ is actually constant (see [178]). This fact is a matter of scaling and it is strictly related to the following result that holds for any $u$in $W^{1,p}(\Omega)$:

$$\lim_{s\to 1^-} (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy = C_1 \int_{\Omega} |\nabla u|^p\, dx \tag{60}$$

for a suitable positive constant C_1 depending only on n and p(see [179]).

In the same spirit, in [180], Maz'ya and Shaposhnikova proved that, for a function $u \in \bigcup_{0<s<1} W^{s,p}(\mathbb{R}^n)$, it yields

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy = C_2 \int_{\mathbb{R}^n} |u|^p\, dx, \tag{61}$$

for a suitable positive constant C_2 depending only on n and p.

When $s > 1$ and it is not an integer we write $s = m + \sigma$, where $m$is an integer and $\sigma \in (0, 1)$. In this case the space $W^{s,p}(\Omega)$consists of those equivalence classes of functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{\sigma,p}(\Omega)$, namely

$$W^{s,p}(\Omega) \coloneqq \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any} \alpha \text{ s.t.} |\alpha| = m\} \quad (62)$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} \coloneqq \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}}. \quad (63)$$

If $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.

Corollary (5.3.3)[175]. Let $p \in [1, +\infty)$ and $s, s' > 1$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$. Then, if $s' \geq s$, we have $W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega)$.

Proof. We write $s = k + \sigma$ and $s' = k' + \sigma'$, with k, k' integers and $\sigma, \sigma' \in (0, 1)$. In the case $k' = k$, we can use Proposition (5.3.1) in order to conclude that $W^{s',p}(\Omega)$ is continuously embeddedin $W^{s,p}(\Omega)$. On the other hand, if $k' \geq k + 1$, using Proposition (5.3.1) and Proposition(5.3.2) we havethe following chain $W^{k'+\sigma',p}(\Omega) \subseteq W^{k',p}(\Omega) \subseteq W^{k+1,p}(\Omega) \subseteq W^{k+\sigma,p}(\Omega)$.
The proof is complete.

As in the classic case with s being an integer, any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support.

Theorem (5.3.4)[175]. For any $s > 0$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support isdense in $W^{s,p}(\mathbb{R}^n)$. A proof can be found in [181]. Let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{s,p}(\Omega)}$ defined in (63). Notethat, in view of Theorem (5.3.4), we have

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n), \quad (64)$$

but in general, for $\Omega \subset \mathbb{R}^n, W^{s,p}(\Omega) \neq W_0^{s,p}(\Omega)$, i.e. $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. Furthermore,it is clear that the same inclusions stated in Proposition (5.3.1), Proposition (5.3.2) and Corollary (5.3.4) hold for the spaces $W_0^{s,p}(\Omega)$.

In this section, we focus on the case $p = 2$. This is quite an important case since the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^n)$ and $W_0^{s,2}(\mathbb{R}^n)$ turn out to be Hilbert spaces. They are usually denoted by $H^s(\mathbb{R}^n)$ and $H_0^s(\mathbb{R}^n)$, respectively. Moreover, they are strictly related to the fractional Laplacian operator $(-\Delta)^s$ (see Proposition (5.3.8)), where, for any $u \in S$and $s \in (0, 1)$, $(-\Delta)^s$ it is defined as

$$(-\Delta)^s u(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = C(n, s) \lim_{\varepsilon \to 0^+} \int_{\complement B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy. \quad (65)$$

Here $P.V.$is a commonly used abbreviation for "in the principal value sense and $C(n, s)$ is a dimensional constant that depends on n and s, precisely given by

$$C(n, s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1}. \quad (66)$$

Now, we show that one may write the singular integral in (65) as a weighted second order differential quotient.

Lemma (5.3.5)[175]. Let $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplacian operator defined by (65). Then, for any $u \in S$,

$$(-\Delta)^s u(x) = -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy, \quad \forall x \in \mathbb{R}^n. \tag{67}$$

Proof. The equivalence of the definitions in (65) and (67) immediately follows by the standard changing variable formula. Indeed, by choosing $z = y - x$, we have

$$u(x) = -C(n,s)P.V.\int_{\mathbb{R}^n}\frac{u(y)-u(x)}{|x-y|^{n+2s}}dy = -C(n,s)P.V.\int_{\mathbb{R}^n}\frac{u(x+z)-u(x)}{|z|^{n+2s}}dz. \tag{68}$$

Moreover, by substituting $\tilde{z} = -z$ in last term of the above equality, we have

$$P.V.\int_{\mathbb{R}^n}\frac{u(x+z)-u(x)}{|z|^{n+2s}}dz = P.V.\int_{\mathbb{R}^n}\frac{u(x-\tilde{z})-u(x)}{|\tilde{z}|^{n+2s}}d\tilde{z} \tag{69}$$

and so after relabeling $\tilde{z}$ as z

$$\begin{aligned} 2P.V.\int_{\mathbb{R}^n}\frac{u(x+z)-u(x)}{|z|^{n+2s}}dz &= P.V.\int_{\mathbb{R}^n}\frac{u(x+z)-u(x)}{|z|^{n+2s}}dz + P.V.\int_{\mathbb{R}^n}\frac{u(x-z)-u(x)}{|z|^{n+2s}}dz \\ &= P.V.\int_{\mathbb{R}^n}\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}dz. \end{aligned} \tag{70}$$

Therefore, if we rename $z$as $y$in (68) and (70), we can write the fractional Laplacian operator in (65) as $(-\Delta)^s u(x) = -\frac{1}{2}C(n,s)P.V.\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy.$

The above representation is useful to remove the singularity of the integral at the origin. Indeed, for any smooth function u, a second order Taylor expansion yields $\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} \le \frac{\|D^2u\|_{L^\infty}}{|y|^{n+2s-2}}$, which is integrable near 0 (for any fixed $s \in (0,1)$). Therefore, since $u \in S$, one can get rid of the *P.V.* and write (67). Now, we take into account an alternative definition of the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ via the Fourier transform. Precisely, we may define

$$\widehat{H}^s(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n): \int_{\mathbb{R}^n}(1+|\xi|^{2s})|\mathrm{F}\,u(\xi)|^2 d\xi < +\infty\right\} \tag{71}$$

and we observe that the above definition, unlike the ones via the Gagliardo norm in (54), is valid also for any real $s \ge 1$. We may also use an analogous definition for the case $s < 0$ by setting

$$\widehat{H}^s(\mathbb{R}^n) = \left\{u \in S'(\mathbb{R}^n): \int_{\mathbb{R}^n}(1+|\xi|^2)^s|\mathrm{F}\,u(\xi)|^2 d\xi < +\infty\right\},$$

although in this case the space $\widehat{H}^s(\mathbb{R}^n)$ is not a subset of $L^2(\mathbb{R}^n)$ and, in order to use the Fourier transform, one has to start from an element of $S'(\mathbb{R}^n)$ (see also Remark(i)).

The equivalence of the space $\widehat{H}^s(\mathbb{R}^n)$ defined in (71) with the one defined in the norm (53).

First, we will prove that the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$.

The proof is standard and it can be found in [182]),[184].We will follow the one in [183] ,[185] in which it is shown how singular integrals naturally arise as a continuous limit of discrete long jump random walks.

Proposition (5.3.6)[175]. Let s ∈ $(0,1)$and let $(-\Delta)^{s}: S \to L^{2}(\mathbb{R}^{n})$ be the fractional Laplacian operator defined by (65). Then, for any u ∈ S,

$$(-\Delta)^{s} u = \mathrm{F}^{-1}\big(|\xi|^{2s}(\mathrm{F}\, u)\big) \qquad \forall \xi \in \mathbb{R}^{n}. \tag{72}$$

Proof. In view of Lemma (5.3.5), we may use the definition via the weighted second order differential quotient in (67). We denote by L $u$the integral in (67), that is

$\mathrm{L}\, u(x) = -\frac{1}{2} C(n,s) \int_{\mathbb{R}^{n}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy$, with $C(n,s)$ as in (66).

L is a linear operator and we are looking for its "symbol" (or "multiplier"), that is a function $S : \mathbb{R}^{n} \to \mathbb{R}$ such that

$$\mathbb{L}u = \mathrm{F}^{-1}\big(S(\mathrm{F}u)\big). \tag{73}$$

We want to prove that

$$S(\xi) = |\xi|^{2s}, \tag{74}$$

where we denoted by $\xi$the frequency variable.

To this scope, we point out that

$$\frac{|u(x+y)+u(x-y)-2u(x)|}{|y|^{n+2s}} \le 4\big(\chi_{B_1}(y)|y|^{2-n-2s} \sup_{B_1(x)}|D^{2}u| + \chi_{\mathbb{R}^{n}\setminus B_1}(y)|y|^{-n-2s}|u(x+y)+u(x-y)-2u(x)|\big) \in L^{1}(\mathbb{R}^{2n}).$$

Consequently, by the Fubini–Tonelli theorem, we can exchange the integral in $y$with the Fourier transform in x.

Thus, we apply the Fourier transform in the variable $x$in (73) and we obtain

$$\begin{aligned} S(\xi)(\mathrm{F}u)(\xi) = \mathrm{F}(\mathrm{L}u) &= -\frac{1}{2} C(n,s) \int_{\mathbb{R}^{n}} \frac{F\big(u(x+y)+u(x-y)-2u(x)\big)}{|y|^{n+2s}} dy \\ &= -\frac{1}{2} C(n,s) \int_{\mathbb{R}^{n}} \frac{e^{i\xi\cdot y}+e^{-i\xi\cdot y}-2}{|y|^{n+2s}} dy(\mathrm{F}u)(\xi) \\ &= C(n,s) \int_{\mathbb{R}^{n}} \frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}} dy(\mathrm{F}u)(\xi). \end{aligned} \tag{75}$$

Hence, in order to obtain (74), it suffices to show that

$$\int_{\mathbb{R}^{n}} \frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}} dy = C(n,s)^{-1}|\xi|^{2s}. \tag{76}$$

To check this, first we observe that, if $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^{n}$, we have

$$\frac{1-\cos\zeta_1}{|\zeta|^{n+2s}} \le \frac{|\zeta_1|^{2}}{|\zeta|^{n+2s}} \le \frac{1}{|\zeta|^{n-2+2s}} \text{ near } \zeta = 0.$$

Thus,

$$\int_{\mathbb{R}^{n}} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}} d\zeta \text{ is finite and positive.} \tag{77}$$

Now, we consider the function $I : \mathbb{R}^n \to \mathbb{R}$ defined as follows

$$\mathbb{I}(\xi) = \int_{\mathbb{R}^n} \frac{1-\cos(\xi \cdot y)}{|y|^{n+2s}} dy.$$

We have that Iis rotationally invariant, that is

$$I(\xi) = I(|\xi| e_1), \tag{78}$$

where e_1 denotes the first direction vector in $\mathbb{R}^n$. Indeed, when $n = 1$, then we can deduce (78) by the fact that $I(-\xi) = I(\xi)$. When $n \geq 2$, we consider a rotation $R$for which $R(|\xi|e_1) = \xi$ and we denote by R^T its transpose. Then, by substituting $\tilde{y} = R^T y$, we obtain

$$I(\xi) = \int_{\mathbb{R}^n} \frac{1-\cos\left((R(|\xi|e_1))\cdot y\right)}{|y|^{n+2s}} dy = \int_{\mathbb{R}^n} 1 - \cos \frac{\left((|\xi|e_1)\cdot(R^T y)\right)}{|y|^{n+2s}} dy = \int_{\mathbb{R}^n} \frac{1-\cos((|\xi|e_1)\cdot\tilde{y})}{|\tilde{y}|^{n+2s}} d\tilde{y} = I(|\xi|e_1),$$

which proves (78).

As a consequence of (77) and (78), the substitution $\zeta = |\xi| y$ gives that

$$I(\xi) = I(|\xi|e_1) = \frac{1-\cos(|\xi|y_1)}{|y|^{n+2s}} dy = \frac{1}{|\xi|^n} \int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta/|\xi||^{n+2s}} d\zeta = C(n,s)^{-1} |\xi|^{2s},$$

where we recall that $C(n,s)^{-1}$ is equal to $\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta$by (66).

Hence, we deduce (76) andthen the proof is complete.

Proposition (5.3.7)[175]. Let $s \in (0,1)$.Then the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined in (5.3.2)coincides with $\widehat{H}^s(\mathbb{R}^n)$ defined in (71).In particular, for any $u \in H^s(\mathbb{R}^n)$

$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |Fu(\xi)|^2 d\xi$, where $C(n,s)$is defined by (66).

Proof. For every fixed $y \in \mathbb{R}^n$, by changing of variable choosing $z = x - y$, we get

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dx \right) dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(z+y)-u(y)|^2}{|z|^{n+2s}} dz dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| \frac{u(z+y)-u(y)}{|z|^{n/2+s2}} \right| dy \right) dz$$

$$= \int_{\mathbb{R}^n} \left\| \frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}} \right\|^2_{L^2(\mathbb{R}^n)} dz = \int_{\mathbb{R}^n} \left\| F\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}} \right) \right\|^2_{L^2(\mathbb{R}^n)} dz,$$

where Plancherel's formula has been used.

Now, using (76) we obtain

$$\int_{\mathbb{R}^n} \left\| F\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}} \right) \right\|^2_{L^2(\mathbb{R}^n)} dz = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left|e^{i\xi\cdot z}-1\right|^2}{|z|^{n+2s}} |Fu(\xi)|^2 d\xi dz = 2\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(1-\cos\xi\cdot z)}{|z|^{n+2s}} |Fu(\xi)|^2 dz d\xi$$

$$= 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |Fu(\xi)|^2 d\xi.$$

This completes the proof.

Finally, we are able to prove the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space H^s.

Proposition (5.3.8)[175]. Let $s \in (0,1)$ and let $u \in H^s(\mathbb{R}^n)$. Then,

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \left\| (-\Delta)^{\frac{s}{2}} u \right\|^2_{L^2(\mathbb{R}^n)}, \tag{79}$$

where $C(n,s)$ is defined by (66).

Proof. The equality in (79) plainly follows from Proposition (5.3.6) and Proposition (5.3.7). Indeed, $\left\| (-\Delta)^{\frac{s}{2}} u \right\|^2_{L^2(\mathbb{R}^n)} = \left\| F(-\Delta)^{\frac{s}{2}} u \right\|^2_{L^2(\mathbb{R}^n)} = \| |\xi|^s Fu \|^2_{L^2(\mathbb{R}^n)} = \frac{1}{2} C(n,s) [u]^2_{H^s(\mathbb{R}^n)}$.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with continuous boundary $\partial\Omega$. Denote by T the trace operator, namely the linear operator defined by the uniformly continuous extension of the operator of restriction to $\partial\Omega$for functions in $\mathcal{D}(\bar{\Omega})$, that is the space of functions $C_0^\infty(\mathbb{R}^n)$ restricted to $\bar{\Omega}$ (see[186]).

Now, for any $x = (x', x_n) \in \mathbb{R}^n$ and for any $u \in S(\mathbb{R}^n)$, we denote by $v \in S(\mathbb{R}^{n-1})$ therestriction of u on the hyper plane $x_n = 0$, that is

$$v(x') = u(x', 0) \qquad \forall x' \in \mathbb{R}^{n-1}. \tag{80}$$

Then, we have

$$\mathrm{F}v(\xi') = \int_{\mathbb{R}} \mathbb{F}u(\xi', \xi_n) d\xi_n \, \forall \xi' \in \mathbb{R}^{n-1}, \tag{81}$$

where, for the sake of simplicity, we keep the same symbol F for both the Fourier transform in $n-1$ and in $n$variables. To check (81), we write

$$\mathrm{F}v(\xi') = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} v(x') dx' = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} u(x', 0) dx'. \tag{82}$$

On the other hand, we have

$$\int_{\mathbb{R}} \mathbb{F}u(\xi', \xi_n) d\xi_n = \int_{\mathbb{R}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i(\xi', \xi_n)\cdot(x', x_n)} u(x', x_n) dx' dx_n d\xi_n$$

$$= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} \left[\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi_n \cdot x_n} u(x', x_n) dx_n d\xi_n \right] dx'$$

$$= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} [u(x', 0)] dx,$$

where the last equality follows by transforming and anti-transforming $u$in the last variable, and this coincides with (82). Now, we are in position to characterize the traces of the function in $H^s(\mathbb{R}^n)$, as stated in the following proposition.

Proposition (5.3.9)[175]. (See [182].) Let $s > 1/2$, then any function $u \in H^s(\mathbb{R}^n)$ has a trace von the hyperplane $\{x_n = 0\}$, such that $v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Also, the trace operator T is surjective from $H^s(\mathbb{R}^n)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Proof. In order to prove the first claim, it suffices to show that there exists a universal constant Csuch that, for any $u \in S(\mathbb{R}^n)$ and any v defined as in (80),

$$\|v\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C \|u\|_{H^s(\mathbb{R}^n)}. \tag{83}$$

By taking into account (81), the Cauchy–Schwarz inequality yields

$$|\mathrm{F}v(\xi')|^2 \leq \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\mathrm{F}u(\xi', \xi_n)|^2 d\xi_n \right) \left(\int_{\mathbb{R}} \frac{d\xi_n}{(1 + |\xi|^2)^s} \right). \tag{84}$$

Using the changing of variable formula by setting $\xi_n = t\sqrt{1 + |\xi'|^2}$, we have

$$\int_{\mathbb{R}} \frac{d\xi_n}{(1 + |\xi|^2)^s} = \int_{\mathbb{R}} \frac{(1 + |\xi'|^2)^{1/2}}{\left((1 + |\xi'|^2)(1 + t^2)\right)^s} dt = \int_{\mathbb{R}} \frac{(1 + |\xi'|^2)^{\frac{1}{2}-s}}{(1 + t^2)^s} dt = C(s)(1 + |\xi'|^2)^{\frac{1}{2}-s}, \tag{85}$$

where $C(s) := \int_{\mathbb{R}} \frac{dt}{(1+t^2)^s} < +\infty$since $s > 1/2$.

Combining (84) with (85) and integrating in $\xi' \in \mathbb{R}^{n-1}$, we obtain

$\int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{s-\frac{1}{2}} |\mathrm{F}v(\xi')|^2 d\xi' \leq C(s) \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (1 + |\xi|^2)^s |\mathrm{F}u(\xi', \xi_n)|^2 d\xi_n d\xi'$, that is (83).

Now, we will prove the surjectivity of the trace operator T. For this, we show that for any $v \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ the function $u$defined by

$$F\,u(\xi', \xi_n) = F\,v(\xi')\varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\frac{1}{\sqrt{1+|\xi'|^2}}, \tag{86}$$

with $\varphi \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi(t)dt = 1$, is such that $u \in H^s(\mathbb{R}^n)$ and $T\,u = v$. Indeed, we integrate(86) with respect to $\xi_n \in \mathbb{R}$, we substitute $\xi_n = t\sqrt{1+|\xi'|^2}$ and we obtain

$$\int_{\mathbb{R}} \mathbb{F}u(\xi', \xi_n)d\xi_n = \int_{\mathbb{R}} \mathbb{F}v(\xi')\varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\frac{1}{\sqrt{1+|\xi'|^2}}d\xi_n = \int_{\mathbb{R}} \mathbb{F}v(\xi')\varphi(t)dt = \mathrm{F}v(\xi') \tag{87}$$

and this implies $v = T\,u$ because of (81).

The proof of H^s-boundedness of u is straightforward. In fact, from (86), for any $\xi' \in \mathbb{R}^{n-1}$, we have

$$\int_{\mathbb{R}} (1+|\xi|^2)^s |\mathrm{F}u(\xi', \xi_n)|^2 d\xi_n = \int_{\mathbb{R}} (1+|\xi|^2)^s |\mathrm{F}v(\xi')|^2 \left|\varphi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right)\right|^2 \frac{1}{1+|\xi'|^2} d\xi_n$$

$$= C(1+|\xi|^2)^{s-\frac{1}{2}} |\mathrm{F}v(\xi')|^2, \tag{88}$$

where we used again the changing of variable formula with $\xi_n = t\sqrt{1+|\xi'|^2}$ and the constant C is given by $\int_{\mathbb{R}} (1+t^2)^s |\varphi(t)|^2 dt$.We obtain $u \in H^s(\mathbb{R}^n)$ by integrating (88) in $\xi' \in \mathbb{R}^{n-1}$.

In this section, we go into detail on the constant factor $C(n,s)$that appears in the definition of the fractional Laplacian see (65), by analyzing its asymptotic behavior as $s \to 1^-$and $s \to 0^+$. This is relevant if one wants to recover the Sobolev norms of the spaces $H^1(\mathbb{R}^n)$and $L^2(\mathbb{R}^n)$by starting from the one of $H^s(\mathbb{R}^n)$.We recall that in Proposition (5.3.6), the constant $C(n,s)$has been defined by

$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta\right)^{-1}$.

Precisely, we are interested in analyzing the asymptotic behavior as $s \to 0^+$and $s \to 1^-$of a scaling of the quantity in the right-hand side of the above formula. By changing variable $\eta' = \zeta'/|\zeta_1|$, we have

$$\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{1-\cos(\zeta_1)}{|\zeta_1|^{n+2s}} \frac{1}{(1+|\zeta'|^2/|\zeta_1|^2)^{\frac{n+2s}{2}}} d\zeta' d\zeta_1$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{1-\cos(\zeta_1)}{|\zeta_1|^{1+2s}} \frac{1}{(1+|\eta'|^2)^{\frac{n+2s}{2}}} d\eta' d\zeta_1$$

$$= \frac{A(n,s)B(s)}{s(1-s)}$$

where

$$A(n,s) = \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|\eta'|^2)^{\frac{n+2s}{2}}} d\eta' \tag{89}$$

and

$$B(s) = s(1-s)\int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}} dt. \tag{90}$$

Proposition (5.3.10)[175]. For any $n > 1$, let A and B be defined by (89) and (90) respectively. The following statements hold:

(i) $\lim_{s\to 1^-} A(n,s) = \omega_{n-2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+1}} d\rho < +\infty$;

(ii) $\lim_{s\to 0^+} A(n,s) = \omega_{n-2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}}} d\rho < +\infty$;

(iii) $\lim_{s\to 1^-} B(s) = \frac{1}{2}$;

(iv) $\lim_{s\to 0^+} B(s) = 1$,

where ω_{n-2} denotes $(n-2)$-dimensional measure of the unit sphere S^{n-2}. As a consequence,

$$\lim_{s\to 1^-} \frac{C(n,s)}{s(1-s)} = \left(\frac{\omega_{n-2}}{2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+1}} d\rho\right)^{-1} \tag{91}$$

and

$$\lim_{s\to 0^+} \frac{C(n,s)}{s(1-s)} = \left(\omega_{n-2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}}} d\rho\right)^{-1}. \tag{92}$$

Proof. First, by polar coordinates, for any $s \in (0,1)$, we get

$$\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|\eta'|^2)^{\frac{n+2s}{2}}} d\eta' = \omega_{n-2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2s}{2}}} d\rho.$$

Now, observe that for any $s \in (0,1)$ and any $\rho \geq 0$, we have $\frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n+2s}{2}}} \leq \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}}}$ and the function in the right-hand side of the above inequality belongs to $L^1\big((0,+\infty)\big)$ for any $n > 1$. Then, the Dominated Convergence Theorem yields

$\lim_{s\to 1^-} A(n,s) = \omega_{n-2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+1}} d\rho$ and $\lim_{s\to 0^+} A(n,s) = \omega_{n-2}\int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}}} d\rho$.

This proves (i) and (ii).

Now, we want to prove (iii). First, we split the integral in (90) as follows

$$\int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}} dt = \int_{|t|<1} \frac{1-\cos t}{|t|^{1+2s}} dt + \int_{|t|\geq 1} \frac{1-\cos t}{|t|^{1+2s}} dt.$$

Also, we have that $0 \leq \int_{|t|\geq 1} \frac{1-\cos t}{|t|^{1+2s}} dt \leq 4\int_1^{+\infty} \frac{1}{t^{1+2s}} dt = \frac{2}{s}$ and

$$\int_{|t|<1} \frac{1-\cos t}{|t|^{1+2s}} dt - \int_{|t|<1} \frac{t^2}{2|t|^{1+2s}} dt \leq C\int_{|t|<1} \frac{|t|^3}{|t|^{1+2s}} dt = \frac{2C}{3-2s},$$

for some suitable positive constant C.

From the above estimates it follows that $\lim_{s\to 1^-} s(1-s)\int_{|t|\geq 1} \frac{1-\cos t}{|t|^{1+2s}} dt = 0$ and

$$\lim_{s\to 1^-} s(1-s)\int_{|t|<1} \frac{1-\cos t}{|t|^{1+2s}} dt = \lim_{s\to 1^-} s(1-s)\int_{|t|<1} \frac{t^2}{2|t|^{1+2s}} dt.$$

Hence, we get $\lim_{s\to 1^-} B(s) = \lim_{s\to 1^-} s(1-s)\left(\int_0^1 t^{1-2s} dt\right) = \lim_{s\to 1^-} \frac{s(1-s)}{2(1-s)} = \frac{1}{2}$.

Similarly, we can prove (iv). For this we notice that

$$0 \le \int_{|t|<1} \frac{1-\cos t}{|t|^{1+2s}} dt \le C \int_0^1 t^{1-2s} dt$$

which yields $\lim_{s\to 0^+} s(1-s) \int_{|t|<1} \frac{1-\cos t}{|t|^{1+2s}} dt = 0$.

Now, we observe that for any $k \in \mathbb{N}, k \ge 1$, we have

$$\begin{aligned}
\left|\int_{2k\pi}^{2(k+1)\pi} \frac{\cos t}{t^{1+2s}} dt\right| &= \left|\int_{2k\pi}^{2k\pi+\pi} \frac{\cos t}{t^{1+2s}} + \int_{2k\pi}^{2k\pi+\pi} \cos \frac{(\tau+\pi)}{(\tau+\pi)^{1+2s}} d\tau\right| \\
&= \left|\int_{2k\pi}^{2k\pi+\pi} \cos t \left(\frac{1}{t^{1+2s}} - \frac{1}{(t+\pi)^{1+2s}}\right) dt\right| \\
&\le \int_{2k\pi}^{2k\pi+\pi} \left|\frac{1}{t^{1+2s}} - \frac{1}{(t+\pi)^{1+2s}}\right| dt \\
&= \int_{2k\pi}^{2k\pi+\pi} \frac{(t+\pi)^{1+2s} - t^{1+2s}}{t^{1+2s}(t+\pi)^{1+2s}} dt \\
&= \int_{2k\pi}^{2k\pi+\pi} \frac{1}{t^{1+2s}(t+\pi)^{1+2s}} \left(\int_0^{\pi} (1+2s)(t+\vartheta)^{2s} d\vartheta\right) dt \\
&\le \int_{2k\pi}^{2k\pi+\pi} \frac{3\pi(t+\pi)^{2s}}{t^{1+2s}(t+\pi)^{1+2s}} dt \\
&\le \int_{2k\pi}^{2k\pi+\pi} \frac{3\pi}{t(t+\pi)} dt \\
&\le \int_{2k\pi}^{2k\pi+\pi} \frac{3\pi}{t^2} dt \le \frac{C}{k^2}.
\end{aligned}$$

As a consequence,

$$\left|\int_1^{+\infty} \frac{\cos t}{t^{1+2s}} dt\right| \le \int_1^{2\pi} \frac{1}{t} dt + \left|\sum_{k=1}^{+\infty} \int_{2k\pi}^{2(k+1)\pi} \frac{\cos t}{t^{1+2s}} dt\right| \le \log(2\pi) + \sum_{k=1}^{+\infty} \frac{C}{k^2} \le C,$$

up to relabeling the constant $C > 0$. It follows that

$$\left|\int_{|t|\ge 1} \frac{1-\cos t}{|t|^{1+2s}} dt - \int_{|t|\ge 1} \frac{1}{|t|^{1+2s}} dt\right| = \left|\int_{|t|\ge 1} \frac{\cos t}{|t|^{1+2s}} dt\right| 2 \left|\int_1^{+\infty} \frac{\cos t}{|t|^{1+2s}} dt\right| \le C$$

and then

$$\lim_{s\to 0^+} s(1-s) \int_{|t|\ge 1} \frac{1-\cos t}{|t|^{1+2s}} dt = \lim_{s\to 0^+} s(1-s) \int_{|t|\ge 1} \frac{1}{|t|^{1+2s}} dt.$$

Hence, we can conclude that

$$\lim_{s\to 0^+} B(s) = \lim_{s\to 0^+} s(1-s) \int_{|t|\ge 1} \frac{1}{|t|^{1+2s}} dt = \lim_{s\to 0^+} 2s(1-s) \int_1^{+\infty} t^{-1-2s} dt = \lim_{s\to 0^+} \frac{2s(1-s)}{2s} = 1.$$

Finally, (91) and (92) easily follow combining the previous estimates and recalling that $C(n,s) = \frac{s(1-s)}{A(n,s)B(s)}$. The proof is complete.

Corollary (5.3.11).[175] For any $n > 1$, let $C(n,s)$ be defined by (66).

The following statements hold:

(i) $\lim_{s\to 1^-} \frac{C(n,s)}{s(1-s)} = \frac{4n}{\omega_{n-1}}$;

(ii) $\lim_{s\to 0^+} \frac{C(n,s)}{s(1-s)} = \frac{2}{\omega_{n-1}}$,

where ω_{n-1} denotes the $(n-1)$-dimensional measure of the unit sphere S_{n-1}.

Proof. For any $\theta \in \mathbb{R}$ such that $\theta > n-1$, let us define $E_n(\theta) := \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{\theta}{2}}} d\rho$.

Observe that the assumption on the parameter θensures the convergence of the integral. Furthermore, integrating by parts we get

$$E_n(\theta) = \frac{1}{n-1}\int_0^{+\infty} \frac{(\rho^{n-1})'}{(1+\rho^2)^{\frac{\theta}{2}}} d\rho = \frac{\theta}{n-1}\int_0^{+\infty} \frac{\rho^n}{(1+\rho^2)^{\frac{\theta+2}{2}}} d\rho = \frac{\theta}{n-1} E_{n+2}(\theta+2). \tag{93}$$

Then, we set $I_n^{(1)} \coloneqq E_n(n+2) = \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}+1}} d\rho$ and $I_n^{(0)} \coloneqq E_n(n) = \int_0^{+\infty} \frac{\rho^{n-2}}{(1+\rho^2)^{\frac{n}{2}}} d\rho$.

In view of (93), it follows that $I_n^{(1)}$ and $I_n^{(0)}$ can be obtained in a recursive way, since

$$I_{n+2}^{(1)} = E_{n+2}(n+4) = \frac{n-1}{n+2} E_n(n+2) = \frac{n-1}{n+2} I_n^{(1)} \tag{94}$$

and

$$I_{n+2}^{(0)} = E_{n+2}(n+2) = \frac{n-1}{n} E_n(n) = \frac{n-1}{n} I_n^{(0)}. \tag{95}$$

Now we claim that

$$I_n^{(1)} = \frac{\omega_{n-1}}{2n\omega_{n-2}} \tag{96}$$

and

$$I_n^{(0)} = \frac{\omega_{n-1}}{2\omega_{n-2}}. \tag{97}$$

We will prove the previous identities by induction. We start by noticing that the inductive bases are satisfied, since $I_2^{(0)} = \int_0^{+\infty} \frac{1}{(1+\rho^2)^2} d\rho = \frac{\pi}{4}$, $I_3^{(0)} = \int_0^{+\infty} \frac{\rho}{(1+\rho^2)^{\frac{5}{2}}} d\rho = \frac{1}{3}$

and

$$I_2^{(0)} = \int_0^{+\infty} \frac{1}{(1+\rho^2)} d\rho = \frac{\pi}{2}, \quad I_3^{(0)} = \int_0^{+\infty} \frac{\rho}{(1+\rho^2)^{\frac{3}{2}}} d\rho = 1.$$

Now, using (94) and (95), respectively, it is clear that in order to check the inductive steps, it suffices to verify that

$$\frac{\omega_{n+1}}{\omega_n} = \frac{n-1}{n}\frac{\omega_{n-1}}{\omega_{n-2}}. \tag{98}$$

We claim that the above formula plainly follows from a classical recursive formula on ω_n, that is

$$\omega_n = \frac{2\pi}{n-1}\omega_{n-2}. \tag{99}$$

To prove this, let us denote by ϖ_n the Lebesgue measure of the n-dimensional unit ball and let us fix the notation $x = (\tilde{x}, x') \in \mathbb{R}^{n-2} \times \mathbb{R}^2$. By integrating on $\mathbb{R}^{n-2}$ and then using polar coordinates in $\mathbb{R}^2$, we see that

$$\begin{aligned}
\varpi_n = \int_{|x|^2\le 1} dx &= \int_{|x'|\le 1}\left(\int_{|\tilde{x}|^2 \le 1-|x'|^2} d\tilde{x}\right) dx' \\
&= \varpi_{n-2}\int_{|x'|\le 1} (1-|x'|^2)^{\frac{(n-2)}{2}} dx' \\
&= 2\pi\varpi_{n-2}\int_0^1 \rho(1-\rho^2)^{\frac{(n-2)}{2}} d\rho = \frac{2\pi\varpi_{n-2}}{n}.
\end{aligned} \tag{100}$$

Moreover, by polar coordinates in $\mathbb{R}^n$,

$$\varpi_n = \int_{|x|\le 1} dx = \omega_{n-1}\int_0^1 \rho^{n-1}d\rho = \frac{\omega_{n-1}}{n}. \tag{101}$$

Thus, we use (101) and (100) and we obtain $\omega_{n-1} = n\varpi_n = 2\pi\varpi_{n-2} = \frac{2\pi\omega_{n-3}}{n-2}$, which is (99), up to replacing $n$with $n-1$. In turn, (99) implies (98) and so (96) and (97).

Finally, using (96), (97) and Proposition (5.3.10) we can conclude that $\lim_{s\to 1^-}\frac{C(n,s)}{s(1-s)} = \frac{2}{\omega_{n-2}I_n^{(1)}} = \frac{4n}{\omega_{n-1}}$and $\lim_{s\to 0^+}\frac{C(n,s)}{s(1-s)} = \frac{1}{\omega_{n-2}I_n^{(0)}} = \frac{2}{\omega_{n-1}}$, as desired.

Remark (5.3.12)[175]. It is worth noticing that when $p=2$ we recover the constants C_1 and C_2 in (60) and (61), respectively. In fact, in this case it is known that

$$C_1 = \frac{1}{2}\int_{S^{n-1}}|\xi_1|^2 d\sigma(\xi) = \frac{1}{2n}\sum_{i=1}^n\int_{S^{n-1}}|\xi_i|^2 d\sigma(\xi) = \frac{\omega_{n-1}}{2n}$$

and $C_2 = \omega_{n-1}$ (see [189] and [190]).

Then, by Corollary (5.3.11) it follows that

$$\begin{aligned}\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy &= \lim_{s\to 1^-} 2(1-s)C(n,s)^{-1}\||\xi|^s \mathrm{F}\, u\|^2_{L^2(\mathbb{R}^n)}\\ &= \frac{\omega_{n-1}}{2n}\|\nabla u\|^2_{L^2(\mathbb{R}^n)}\\ &= C_1\|u\|^2_{H^1(\mathbb{R}^n)}\end{aligned}$$

and

$$\begin{aligned}\lim_{s\to 0^+} s\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy &= \lim_{s\to 0^+} 2sC(n,s)^{-1}\||\xi|^s \mathrm{F}u\|^2_{L^2(\mathbb{R}^n)}\\ &= \omega_{n-1}\|u\|^2_{L^2(\mathbb{R}^n)}\\ &= C_2\|u\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

We will conclude this section with the following proposition that one could plainly deduce from Proposition (5.3.6). We prefer to provide a direct proof, based on Lemma (5.3.5), in order to show the consistency in the definition of the constant $C(n,s)$.

Proposition (5.3.13)[175]. Let $\mathrm{n} > 1$. For any $\mathrm{u} \in \mathrm{C}_0^\infty(\mathbb{R}^\mathrm{n})$ the following statements hold

(i) $\lim_{s\to 0^+}(-\Delta)^s u = u$;

(ii) $\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u$.

Proof. Fix $x\in\mathbb{R}^n, R_0 > 0$ such that $\operatorname{supp} u \subseteq B_{R_0}$ and set $R = R_0 + |x| + 1$.

First,

$$\begin{aligned}\left|\int_{B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right| &\le \|u\|_{C^2(\mathbb{R}^n)}\int_{B_R}\frac{|y|^2}{|y|^{n+2s}}dy\\ &\le \omega_{n-1}\|u\|_{C^2(\mathbb{R}^n)}\int_0^R\frac{1}{\rho^{2s-1}}d\rho\\ &= \frac{\omega_{n-1}\|u\|_{C^2(\mathbb{R}^n)}R^{2-2s}}{2(1-s)}.\end{aligned} \tag{102}$$

Furthermore, observe that $|y|\ge R$ yields $|x\pm y| \ge |y| - |x| \ge R - |x| > R_0$ and consequently $u(x\pm y) = 0$.

Therefore,

$$-\frac{1}{2}\int\limits_{\mathbb{R}^n\setminus B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy = u(x)\int\limits_{\mathbb{R}^n\setminus B_R}\frac{1}{|y|^{n+2s}}dy$$
$$= \omega_{n-1}u(x)\int\limits_R^{+\infty}\frac{1}{\rho^{2s+1}}d\rho$$
$$= \frac{\omega_{n-1}R^{-2s}}{2s}u(x). \tag{103}$$

Now, by (102) and Corollary (5.3.11), we have $\lim_{s\to 0^+} -\frac{C(n,s)}{2}\int_{B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy = 0$ and so we get, recalling Lemma (5.3.5),

$\lim_{s\to 0^+}(-\Delta)^s u = \lim_{s\to 0^+} -\frac{C(n,s)}{2}\int_{\mathbb{R}^n\setminus B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy = \lim_{s\to 0^+}\frac{C(n,s)\omega_{n-1}R^{-2s}}{2s}u(x) = u(x),$

where the last identities follow from (103) and again Corollary (5.3.11). This proves (i).

Similarly, we can prove (ii). In this case, when s goes to 1, we have no contribution outside the unit ball, as the following estimate shows

$$\left|\int_{\mathbb{R}^n\setminus B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy\right| \le 4\|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|y|^{n+2s}}dy$$
$$\le 4\omega_{n-1}\|u\|_{L^\infty(\mathbb{R}^n)}\int_1^{+\infty}\frac{1}{\rho^{2s+1}}d\rho$$
$$= \frac{2\omega_{n-1}}{s}\|u\|_{L^\infty(\mathbb{R}^n)}.$$

As a consequence (recalling Corollary (5.3.11)), we get

$$\lim_{s\to 1^-} -\frac{C(n,s)}{2}\int\limits_{\mathbb{R}^n\setminus B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy = 0. \tag{104}$$

On the other hand, we have

$$\int_{B_1}\frac{u(x+y)+u(x-y)-2u(x)-D^2u(x)y\cdot y}{|y|^{n+2s}}dy \le \|u\|_{C^3(\mathbb{R}^n)}\int_{B_1}\frac{|y|^3}{|y|^{n+2s}}dy$$
$$\le \omega_{n-1}\|u\|_{C^3(\mathbb{R}^n)}\int_0^1\frac{1}{\rho^{2s-2}}d\rho$$
$$= \frac{\omega_{n-1}\|u\|_{C^3(\mathbb{R}^n)}}{3-2s}$$

and this implies that

$$\lim_{s\to 1^-} -\frac{C(n,s)}{2}\int\limits_{B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy = \lim_{s\to 1^-} -\frac{C(n,s)}{2}\int\limits_{B_1}\frac{D^2u(x)y\cdot y}{|y|^{n+2s}}dy. \tag{105}$$

Now, notice that if $i \ne j$ then

$$\int\limits_{B_1}\partial^2_{ij}u(x)y_i\cdot y_j dy = -\int\limits_{B_1}\partial^2_{ij}u(x)\tilde{y}_i\cdot\tilde{y}_j d\tilde{y},$$

where $\tilde{y}_k = y_k$ for any $k \ne j$ and $\tilde{y}_j = -y_j$, and thus

$$\int\limits_{B_1}\partial^2_{ij}u(x)y_i\cdot y_j dy = 0. \tag{106}$$

Also up to permutations, for any fixed i, we get

$$\int_{B_1} \frac{\partial_{ii}^2 u(x) y_i^2}{|y|^{n+2s}} dy = \partial_{ii}^2 u(x) \int_{B_1} \frac{y_i^2}{|y|^{n+2s}} dy$$

$$= \partial_{ii}^2 u(x) \int_{B_1} \frac{y_1^2}{|y|^{n+2s}} dy = \frac{\partial_{ii}^2 u(x)}{n} \sum_{j=1}^{n} \int_{B_1} \frac{y_j^2}{|y|^{n+2s}} dy$$

$$= \frac{\partial_{ii}^2 u(x)}{n} \int_{B_1} \frac{|y|^2}{|y|^{n+2s}} dy = \frac{\partial_{ii}^2 u(x) \omega_{n-1}}{2n(1-s)}. \tag{107}$$

Finally, combining (104), (105), (106), (107), Lemma (5.3.5) and Corollary (5.3.11), we can conclude $\lim_{s\to 1^-} (-\Delta)^s u = \lim_{s\to 1^-} -\frac{C(n,s)}{2} \int_{B_1} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy = \lim_{s\to 1^-} -\frac{C(n,s)}{2} \int_{B_1} \frac{D^2 u(x) y\cdot y}{|y|^{n+2s}} dy$

$= \lim_{s\to 1^-} -\frac{C(n,s)}{2} \sum_{i=1}^{n} \int_{B_1} \frac{\partial_{ii}^2 u(x) y_i^2}{|y|^{n+2s}} dy = \lim_{s\to 1^-} -\frac{C(n,s)\omega_{n-1}}{4n(1-s)} \sum_{i=1}^{n} \partial_{ii}^2 u(x) = -\Delta u(x).$

As is well known when **s** is an integer, under certain regularity assumptions on the domain Ω, any function in $W^{s,p}(\Omega)$may be extended to a function in $W^{s,p}(\mathbb{R}^n)$. Extension results are quite important in applications and are necessary in order to improve some embeddings theorems. For any $s \in (0,1)$ and any $p \in [1,\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$is an extension domain for $W^{s,p}$ if there exists a positive constant $C = C(n,p,s,\Omega)$ such that: for every function $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $\|u\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)}$.

Lemma (5.3.14)[175]. Let Ω be an open set in $\mathbb{R}^n$ and ua function in $W^{s,p}(\Omega)$ with $s \in (0,1)$ and $p \in [1,+\infty)$. If there exists a compact subset $K \subset \Omega$ such that $u \equiv 0$ in $\Omega\backslash K$, then the extension function ũdefined as

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n\backslash\Omega \end{cases} \tag{108}$$

belongs to $W^{s,p}(\mathbb{R}^n)$ and $\|u\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)}$, where C is a suitable positive constant depending on $n, p, s,$ Kand Ω.

Proof. Clearly $\tilde{u} \in L^p(\mathbb{R}^n)$. Hence, it remains to verify that the Gagliardo norm of $\tilde{u}$ in $\mathbb{R}^n$ is bounded by the one of u in Ω. Using the symmetry of the integral in the Gagliardo norm with respect to x and y and the fact that $\tilde{u} \equiv 0$ in $\mathbb{R}^n\backslash\Omega$, we can split as follows

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{n+sp}} dxdy = \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy + 2\int_{\Omega} \left(\int_{\mathbb{R}^n\backslash\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} dy \right) dx, \tag{109}$$

where the first term in the right-hand side of (109) is finite since $u \in W^{s,p}(\Omega)$. Furthermore, for any $y \in \mathbb{R}^n\backslash K$, $\frac{|u(x)|^p}{|x-y|^{n+sp}} = \frac{\chi_K(x)|u(x)|^p}{|x-y|^{n+sp}} \le \chi_K(x)|u(x)|^p \sup_{x\in K} \frac{1}{|x-y|^{n+sp}}$ and so

$$\int_{\Omega} \left(\int_{\mathbb{R}^n\backslash\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} dy \right) dx \le \int_{\mathbb{R}^n\backslash\Omega} \frac{1}{\operatorname{dist}(y,\partial K)^{n+sp}} dy \|u\|_{L^p(\Omega)}^p. \tag{110}$$

Note that the integral in (110) is finite since $\operatorname{dist}(\partial\Omega,\partial K) \ge \alpha > 0$ and $n + sp > n$.

Combining (109) with (110), we get $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)}$ where $C = C(n,s,p,K)$.

Lemma (5.3.15)[175]. Let Ωbe an open set in $\mathbb{R}^n$, symmetric with respect to the coordinate x_n, and consider the sets $\Omega_+ = \{x \in \Omega: x_n > 0\}$ and $\Omega_- = \{x \in \Omega: x_n \le 0\}$. Let ube a function in $W^{s,p}(\Omega_+)$, with $s \in (0,1)$ and $p \in [1,+\infty)$. Define

$$\bar{u}(x) = \begin{cases} u(x', x_n), & x_n \ge 0, \\ u(x', -x_n), & x_n < 0. \end{cases} \tag{111}$$

Then $\bar{u}$ belongs to $W^{s,p}(\Omega)$ and $\|\bar{u}\|_{W^{s,p}(\Omega)} \le 4\|u\|_{W^{s,p}(\Omega_+)}$.

Proof. By splitting the integrals and changing variable $\hat{x} = (x', -x_n)$, we get

$$\|\bar{u}\|^p_{L^p(\Omega)} = \int_{\Omega_+} |u(x)|^p dx + \int_{\Omega_+} |u(\hat{x}', \hat{x}_n)|^p d\tilde{x} = 2\|u\|^p_{L^p(\Omega_+)}. \tag{112}$$

Also, if $x \in \mathbb{R}^n_+$and $y \in \mathcal{C}\mathbb{R}^n_+$ then $(x_n - y_n)^2 \ge (x_n + y_n)^2$ and therefore

$$\int_\Omega \int_\Omega \frac{|\bar{u}(x) - \bar{u}(y)|^p}{|x-y|^{n+sp}} dxdy = \int_{\Omega_+} \int_{\Omega_+} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy + 2\int_{\Omega_+} \int_{\mathcal{C}\Omega_+} \frac{|u(x) - u(y', -y_n)|^p}{|x-y|^{n+sp}} dxd\,y$$
$$+ \int_{\mathcal{C}\Omega_+} \int_{\mathcal{C}\Omega_+} \frac{|u(x', -x_n) - u(y', -y_n)|^p}{|x-y|^{n+sp}} dxdy$$
$$\le 4\|u\|^p_{W^{s,p}(\Omega_+)}.$$

This concludes the proof. Now, a truncation lemma near $\partial\Omega$.

Lemma (5.3.16)[175]. Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$and $p \in [1,+\infty)$. Let us consider $u \in W^{s,p}(\Omega)$and $\psi \in C^{0,1}(\Omega)$, $0 \le \psi \le 1$. Then $\psi u \in W^{s,p}(\Omega)$ and

$$\|\psi u\|_{W^{s,p}(\Omega)} \le C\|u\|_{W^{s,p}(\Omega)}, \tag{113}$$

where $C = C(n,p,s,\Omega)$.

Proof. It is clear that $\|\psi u\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)}$ since $|\psi| \le 1$. Furthermore, adding and subtracting the factor $\psi(x)u(y)$, we get

$$\int_\Omega \int_\Omega \frac{|\psi(x)u(x) - \psi(y)u(y)|^p}{|x-y|^{n+sp}} dxdy \le 2^{p-1} \left(\int_\Omega \int_\Omega \frac{|\psi(x)u(x) - \psi(x)u(y)|^p}{|x-y|^{n+sp}} dxdy \right.$$
$$\left. + \int_\Omega \int_\Omega \frac{|\psi(x)u(y) - \psi(y)u(y)|^p}{|x-y|^{n+sp}} dxdy \right)$$
$$\le 2^{p-1} \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy + \int_\Omega \int_\Omega \frac{|u(x)|^p |\psi(x) - \psi(y)|^p}{|x-y|^{n+sp}} dxdy \right). \tag{114}$$

Since ψ belongs to $C^{0,1}(\Omega)$,
we have

$$\int_\Omega \int_\Omega \frac{|u(x)|^p |\psi(x) - \psi(y)|^p}{|x-y|^{n+sp}} dxdy \le \Lambda^p \int_\Omega \int_{\Omega \cap |x-y| \le 1} \frac{|u(x)|^p |x-y|^p}{|x-y|^{n+sp}} dxdy$$
$$+ \int_\Omega \int_{\Omega \cap |x-y| \ge 1} \frac{|u(x)|^p}{|x-y|^{n+sp}} dxdy$$
$$\le \tilde{C}\|u\|^p_{L^p(\Omega)}, \tag{115}$$

where Λ denotes the Lipschitz constant of ψ and $\tilde{C}$ is a positive constant depending on n, p and s.

Note that the last inequality follows the fact that the kernel $|x-y|^{-n+(1-s)p}$ issummable with respect to y if $|x-y| \leq 1$ since $n+(s-1)p < n$ and, on the other hand, thekernel $|x-y|^{-n-sp}$ is summable when $|x-y| \geq 1$ since $n+sp > n$. Finally, combining (114) with (115), we obtain estimate (113).

We are ready to show the main theorem of this section, that states that every open Lipschitz set $\Omega$with bounded boundary is an extension domain for $W^{s,p}$.

Theorem (5.3.17)[175]. Let $p \in [1,+\infty), s \in (0,1)$ and $\Omega \subseteq \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary.Then $W^{s,p}(\Omega)$is continuously embedded in $W^{s,p}(\mathbb{R}^n)$, namely for any $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ such that $\tilde{u}|_\Omega = u$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$ where $C = C(n,p,s,\Omega)$.

Proof. Since $\partial\Omega$ is compact, we can find a finite number of balls B_j such that $\partial\Omega \subset \bigcup_{j=1}^k B_j$ and so we can write $\mathbb{R}^n = \bigcup_{j=1}^k B_j \cup (\mathbb{R}^n \backslash \partial\Omega)$.

If we consider this covering, there exists a partition of unity related to it, i.e. there exist $k+1$ smooth functions $\psi_0, \psi_1, \dots, \psi_k$ such that $\operatorname{spt}\psi_0 \subset \mathbb{R}^n \backslash \partial\Omega, \operatorname{spt}\psi_j \subset B_j$ for any $j \in \{1,\dots,k\}, 0 \leq \psi_j \leq 1$ for any $j \in \{0,\dots,k\}$ and $\sum_{j=0}^k \psi_j = 1$. Clearly, $u = \sum_{j=0}^k \psi_j u$.

By Lemma (5.3.16), we know that $\psi_0 u$ belongs to $W^{s,p}(\Omega)$. Furthermore, since $\psi_0 u \equiv 0$ in a neighborhood of $\partial\Omega$, we can extend it to the whole of $\mathbb{R}^n$, by setting $\widetilde{\psi_0 u}(x) = \begin{cases} \psi_0 u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \backslash \Omega \end{cases}$ and $\widetilde{\psi_0 u} \in W^{s,p}(\mathbb{R}^n)$. Precisely

$$\|\widetilde{\psi_0 u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|\psi_0 u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}, \tag{116}$$

Where $C = C(n,s,p,\Omega)$ possibly different step by step,see Lemmas (5.3.14) , (5.3.16).

For any $j \in \{1,\dots,k\}$, let us consider $u|_{B_j \cap \Omega}$ and set $v_j(y) := u\left(T_j(y)\right)$ for any $y \in Q_+$, where $T_j: Q \to B_j$ is the isomorphism of class $C^{0,1}$ defined in (5.3.1). Note that such a T_j exists by the regularity assumption on the domain Ω.Now, we state that $v_j \in W^{s,p}(Q_+)$. Indeed, using the standard changing variable formula by setting $x = T_j(\hat{x})$ we have

$$\int\limits_{Q_+}\int\limits_{Q_+} \frac{|v(\hat{x}) - v(\hat{y})|^p}{|\hat{x}-\hat{y}|^{n+sp}} d\hat{x}d\hat{y} = \int\limits_{Q_+}\int\limits_{Q_+} \frac{\left|u\left(T_j(\hat{x})\right) - u\left(T_j(\hat{y})\right)\right|^p}{|\hat{x}-\hat{y}|^{n+sp}} d\hat{x}d\hat{y}$$

$$= \int\limits_{B_j\cap\Omega}\int\limits_{B_j\cap\Omega} \frac{|u(x)-u(y)|^p}{\left|T_j^{-1}(x) - T_j^{-1}(y)\right|^{n+sp}} \det\left(T_j^{-1}\right) dxdy$$

$$\leq C \int\limits_{B_j\cap\Omega}\int\limits_{B_j\cap\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy, \tag{117}$$

where (117) follows from the fact that T_j is bi-Lipschitz. Moreover, using Lemma (5.3.19) we can extend v_j to all Q so that the extension $\bar{v}_j$ belongs to $W^{s,p}(Q)$ and $\|\bar{v}_j\|_{W^{s,p}(Q)} \leq 4\|v_j\|_{W^{s,p}(Q_+)}$.

We set $w_j(x) := \bar{v}_j\left(T_j^{-1}(x)\right)$ for any $x \in B_j$.

Since T_j is bi-Lipschitz, by arguing as above it follows that $w_j \in W^{s,p}(B_j)$. Note that $w_j \equiv u$(and consequently $\psi_j w_j \equiv \psi_j u$) on $B_j \cap \Omega$. By definition $\psi_j w_j$ has compact support in B_j and therefore, as done for $\psi_0 u$, we can consider the extension $\widetilde{\psi_j w_j}$ to all $\mathbb{R}^n$ in such a way that $\widetilde{\psi_j w_j} \in W^{s,p}(\mathbb{R}^n)$.

Also, using Lemmas (5.3.14), (5.3.15), (5.3.16) and estimate (117)

we get

$$\|\widetilde{\psi_j w_j}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|\psi_j w_j\|_{W^{s,p}(B_j)} \le C\|w_j\|_{W^{s,p}(B_j)}$$
$$\le C\|\bar{v}_j\|_{W^{s,p}(Q)} \le C\|v_j\|_{W^{s,p}(Q_+)} \le C\|u\|_{W^{s,p}(\Omega\cap B_j)}, \quad (118)$$

where $C = C(n,p,s,\Omega)$ and it is possibly different step by step.

Finally, let $\tilde{u} = \widetilde{\psi_0 u} + \sum_{j=1}^{k} \widetilde{\psi_j w_j}$

be the extension of $u$defined on all $\mathbb{R}^n$. By construction, it is clear that $\tilde{u}|_\Omega = u$ and, combining (116) with (118), we get $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|u\|_{W^{s,p}(\Omega)}$ with $C = C(n,p,s,\Omega)$.

Corollary (5.3.18)[175]. Let $p \in [1,+\infty), s \in (0,1)$ and Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary. Then for any $u \in W^{s,p}(\Omega)$, there exists a sequence $\{u_n\} \in C_0^\infty(\mathbb{R}^n)$ such that $u_n \to u$ as $n \to +\infty$ in $W^{s,p}(\Omega)$, i.e., $\lim_{n\to+\infty}\|u_n - u\|_{W^{s,p}(\Omega)} = 0$.

In this section, we provide an elementary proof of a Sobolev-type inequality involving the fractional norm $\|\cdot\|_{W^{s,p}}$. The original proof is contained in Appendix of [191] and it deals with the case $= 2$. We note that when $p = 2$ and $s \in [1/2, 1)$some of the statements may be strengthened (see [178]).We also note that more general embeddings for the spaces $W^{s,p}$can be obtained by interpolation techniques and by passing through Besov spaces. For a more comprehensive treatment of fractional Sobolev-type inequalities we refer to [191] .We remark that the proof here is self-contained. Moreover, we will not make use of Besov or fancy interpolation spaces. The first of them is an elementary estimate involving the measure of finite measurable sets Ein $\mathbb{R}^n$as stated in the following lemma(see [192] and also [193]).

Lemma (5.3.19)[175]. Fix $x \in \mathbb{R}^n$. Let $p \in [1,+\infty), s \in (0,1)$ and $E \subset \mathbb{R}^n$be a measurable set with finite measure. Then,$\int_{\complement E} \frac{dy}{|x-y|^{n+sp}} \ge C|E|^{-sp/n}$, for a suitable constant $C = C(n,p,s) > 0$.

Proof. We set$\rho := \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}$and then it follows

$$|(\complement E) \cap B_\rho(x)| = |B_\rho(x)| - |E \cap B_\rho(x)| = |E| - |E \cap B_\rho(x)| = |E \cap \complement B_\rho(x)|.$$

Therefore,

$$\begin{aligned}
\int_{\complement E} \frac{dy}{|x-y|^{n+sp}} &= \int_{(\complement E)\cap B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} + \int_{(\complement E)\cap \complement B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} \\
&\ge \int_{(\complement E)\cap B_\rho(x)} \frac{dy}{\rho^{n+sp}} + \int_{(\complement E)\cap \complement B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} \\
&= \frac{|(\complement E)\cap B_\rho(x)|}{\rho^{n+sp}} + \int_{(\complement E)\cap \complement B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} \\
&= \frac{|E\cap \complement B_\rho(x)|}{\rho^{n+sp}} + \int_{(\complement E)\cap \complement B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} \\
&\ge \int_{E\cap \complement B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} + \int_{(\complement E)\cap \complement B_\rho(x)} \frac{dy}{|x-y|^{n+sp}} = \int_{\complement B_\rho(x)} \frac{dy}{|x-y|^{n+sp}}.
\end{aligned}$$

The desired result easily follows by using polar coordinates centered at x.

Now, we recall a general statement about a useful summability property (see [191], for related results, see also [194]).

Lemma (5.3.20)[175]. Let $s \in (0,1)$ and $p \in [1,+\infty)$be such that $sp < n$. Fix $T > 1$;

letN $\in \mathbb{Z}$and

$$a_k \text{ be a bounded, nonnegative, decreasing sequence with } a_k = 0 \text{ for any } k \ge N. \quad (119)$$

Then, $\sum_{k\in\mathbb{Z}} a_k^{(n-sp)/n} T^k \le C \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1} a_k^{-sp/n} T^k$,

for a suitable constant $C = C(n, p, s, T) > 0$, independent of N.

Proof. By (119),both

$$\sum_{k\in\mathbb{Z}} a_k^{(n-sp)/n} T^k \text{ and } \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1} a_k^{-sp/n} T^k \text{ are convergent series.} \tag{120}$$

Moreover, since a_k is nonnegative and decreasing, we have that if $a_k = 0$, then $a_{k+1} = 0$. Accordingly,$\sum_{k\in\mathbb{Z}} a_{k+1}^{(n-sp)/n} T^k = \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1}^{(n-sp)/n} T^k$.

Therefore, we may use the Hölder inequality with exponents

$$\alpha := n/sp \text{ and } \beta := n/(n - sp)$$

by arguing as follows

$$\begin{aligned}
\frac{1}{T}\sum_{k\in\mathbb{Z}} a_k^{\frac{(n-sp)}{n}} T^k &= \sum_{k\in\mathbb{Z}} a_{k+1}^{\frac{(n-sp)}{n}} T^k \\
&= \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1}^{(n-sp)/n} T^k \\
&= \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} \left(a_k^{sp/(n\beta)} T^{k/\alpha}\right)\left(a_{k+1}^{1/\beta} a_k^{-sp/(n\beta)} T^{k/\beta}\right) \\
&\le \left(\sum_{k\in\mathbb{Z}} \left(a_k^{sp/(n\beta)} T^{k/\alpha}\right)^{\alpha}\right)^{1/\alpha} \left(\sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} \left(a_{k+1}^{1/\beta} a_k^{-sp/(n\beta)} T^{k/\beta}\right)^{\beta}\right)^{1/\beta} \\
&\le \left(\sum_{k\in\mathbb{Z}} a_k^{(n-sp)/n} T^k\right)^{sp/n} \left(\sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1} a_k^{-sp/n} T^k\right)^{(n-sp)/n}.
\end{aligned}$$

So, recalling (120), we obtain the desired result.

We use the above tools to deal with the measure theoretic properties of the level sets of the functions (see [191]).

Lemma (5.3.21)[175]. Let $s \in (0, 1)$ and $p \in [1, +\infty)$ be such that $sp < n$. Let

$$f \in L^\infty(\mathbb{R}^n) \text{ be compactly supported.} \tag{121}$$

For any $k \in \mathbb{Z}$ let

$$a_k := |\{|f| > 2^k\}|. \tag{122}$$

Then, $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy \ge C \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1} a_k^{-sp/n} 2^{pk}$,

for a suitable constant $C = C(n, p, s) > 0$.

Proof. Notice that $\big||f(x)| - |f(y)|\big| \le |f(x) - f(y)|$,

and so, by possibly replacing f with $|f|$, we may consider the case in which $f \ge 0$.

We define

$$A_k := \{|f| > 2^k\}. \tag{123}$$

We remark that $A_{k+1} \subseteq A_k$, hence

$$a_{k+1} \le a_k. \tag{124}$$

We define $D_k := A_k \backslash A_{k+1} = \{2^k < f \le 2^{k+1}\}$ and $d_k := |D_k|$.

Notice that

$$d_k \text{ and } a_k \text{ are bounded and they become zero when } k \text{ is large enough,} \tag{125}$$

thanks to (121). Also, we observe that the D_k's are disjoint, that

$$\bigcup_{\substack{\ell\in\mathbb{Z}\\ \ell\leq k}} D_\ell = \mathcal{C}A_{k+1} \tag{126}$$

and that

$$\bigcup_{\substack{\ell\in\mathbb{Z}\\ \ell\geq k}} D_\ell = A_k. \tag{127}$$

As a consequence of (127), we have that

$$a_k = \bigcup_{\substack{\ell\in\mathbb{Z}\\ \ell\geq k}} d_\ell \tag{128}$$

and so

$$d_k = a_k - \bigcup_{\substack{\ell\in\mathbb{Z}\\ \ell\geq k+1}} d_\ell. \tag{129}$$

We stress that the series in (128) is convergent, due to (125), thus so is the series in (129). Similarly, we can define the convergent series

$$S := \sum_{\substack{\ell\in\mathbb{Z}\\ a_{\ell-1}\neq 0}} 2^{p\ell} a_{\ell-1}^{-sp/n} d_\ell. \tag{130}$$

We notice that $D_k \subseteq A_k \subseteq A_{k-1}$, hence $a_{i-1}^{-sp/n} d_\ell \leq a_{i-1}^{-sp/n} a_{\ell-1}$. Therefore

$$\{(i,\ell)\in\mathbb{Z}\ \mathbb{s.t.}\ a_{i-1}\neq 0 \text{ and } a_{i-1}^{-sp/n} d_\ell \neq 0\} \subseteq \{(i,\ell)\in\mathbb{Z}\ \mathbb{s.t.}\ a_{\ell-1}\neq 0\}. \tag{131}$$

We use (131) and (124) in the following computation:

$$\begin{aligned}
\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{\substack{\ell\in\mathbb{Z}\\ \ell\geq i+1}} 2^{pi} a_{i-1}^{-sp/n} d_\ell &= \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{\substack{\ell\in\mathbb{Z}\\ \ell\geq i+1\\ a_{i-1}^{sp/n} d_\ell\neq 0}} 2^{pi} a_{i-1}^{-sp/n} d_\ell \\
&\leq \sum_{i\in\mathbb{Z}} \sum_{\substack{\ell\in\mathbb{Z}\\ \ell\geq i+1\\ a_{\ell-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} d_\ell = \sum_{\substack{\ell\in\mathbb{Z}\\ a_{\ell-1}\neq 0}} \sum_{\substack{i\in\mathbb{Z}\\ i\leq \ell-1}} 2^{pi} a_{i-1}^{-sp/n} d_\ell \\
&\leq \sum_{\substack{\ell\in\mathbb{Z}\\ a_{\ell-1}\neq 0}} \sum_{\substack{i\in\mathbb{Z}\\ i\leq \ell-1}} 2^{pi} a_{\ell-1}^{-sp/n} d_\ell = \sum_{\substack{\ell\in\mathbb{Z}\\ a_{\ell-1}\neq 0}} \sum_{k=0}^{+\infty} 2^{p(\ell-1)} 2^{-pk} a_{\ell-1}^{-sp/n} d_\ell \leq S.
\end{aligned} \tag{132}$$

Now, we fix $i\in\mathbb{Z}$ and $x\in D_i$: then, for any $j\in\mathbb{Z}$ with $j\leq i-2$ and any $y\in D_j$ we have that $|f(x)-f(y)| \geq 2^i - 2^{j+1} \geq 2^i - 2^{i-1} = 2^{i-1}$ and therefore, recalling (126),

$$\sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dy \geq 2^{p(i-1)} \sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_j} \frac{dy}{|x-y|^{n+sp}} = 2^{p(i-1)} \int_{\mathcal{C}A_{i-1}} \frac{dy}{|x-y|^{n+sp}}.$$

This and Lemma (5.3.24) imply that, for any $i\in\mathbb{Z}$ and any $x\in D_i$, we have that

$$\sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dy \geq c_o 2^{pi} a_{i-1}^{-sp/n},$$

for a suitable $c_o > 0$.

As a consequence, for any $i \in \mathbb{Z}$,

$$\sum_{\substack{j\in\mathbb{Z}\\ j\le i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy \ge c_o 2^{pi} a_{i-1}^{-\frac{sp}{n}} d_i. \tag{133}$$

Therefore, by (129), we conclude that, for any $i \in \mathbb{Z}$,

$$\sum_{\substack{j\in\mathbb{Z}\\ j\le i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy \ge c_o \left[2^{pi} a_{i-1}^{-\frac{sp}{n}} a_i - \sum_{\substack{\ell\in\mathbb{Z}\\ \ell\ge i+1}} 2^{pi} a_{i-1}^{-\frac{sp}{n}} d_\ell\right]. \tag{134}$$

By (130) and (133), we have that

$$\sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} \sum_{\substack{j\in\mathbb{Z}\\ j\le i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy \ge c_o S. \tag{135}$$

Then, using (134), (132) and (135),

$$\begin{aligned}
\sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} \sum_{\substack{j\in\mathbb{Z}\\ j\le i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy &\ge c_o \left[\sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} 2^{pi} a_{i-1}^{-sp/n} a_i - \sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} \sum_{\substack{\ell\in\mathbb{Z}\\ \ell\ge i+1}} 2^{pi} a_{i-1}^{-sp/n} d_\ell\right] \\
&\ge c_o \left[\sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} 2^{pi} a_{i-1}^{-sp/n} a_i - S\right] \\
&\ge c_o \sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} 2^{pi} a_{i-1}^{-sp/n} a_i - \sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} \sum_{\substack{j\in\mathbb{Z}\\ j\le i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy.
\end{aligned}$$

That is, by taking the last term to the left-hand side,

$$\sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} \sum_{\substack{j\in\mathbb{Z}\\ j\le i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy \ge c_o \sum_{\substack{j\in\mathbb{Z}\\ a_{i-1}\ne 0}} 2^{pi} a_{i-1}^{-\frac{sp}{n}} a_i, \tag{136}$$

up to relabeling the constant c_0.

On the other hand, by symmetry,

$$\begin{aligned}
\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy &= \sum_{i,j\in\mathbb{Z}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy \\
&\ge 2\sum_{\substack{i,j\in\mathbb{Z}\\ j<i}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy \\
&\ge 2 \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\ne 0}} \sum_{\substack{j\in\mathbb{Z}\\ j\le i-2}} \int_{D_i\times D_j} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy. \qquad (137)
\end{aligned}$$

Then, the desired result plainly follows from (136) and (137).

Lemma (5.3.22)[175]. *Let* $q \in [1,\infty)$. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. For any $N \in \mathbb{N}$, *let*

$$f_N(x) := \max\{\min\{f(x), N\}, -N\} \qquad \forall x \in \mathbb{R}^n. \tag{138}$$

Then $\lim_{N\to+\infty}\|f_N\|_{L^q(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n)}$.

Proof. We denote by $|f|_N$ the function obtained by cutting $|f|$ at level N.
We have that $|f|_N = |f_N|$ and so, by Fatou lemma, we obtain that

$\lim\ \inf_{N\to+\infty}\|f_N\|_{L^q(\mathbb{R}^n)} = \lim\ \inf_{N\to+\infty}\left(\int_{\mathbb{R}^n}|f|_N^q\right)^{\frac{1}{q}} \geq \left(\int_{R^n}|f|^q\right)^{\frac{1}{q}} = \|f\|_{L^q(\mathbb{R}^n)}$.
The reverse inequality easily follows by the fact that $|f|_N(x) \leq |f(x)|$ for any $x \in \mathbb{R}^n$.

Taking into account the previous lemmas, we are able to give an elementary proof of the Sobolev-type inequality stated in the following theorem.

Theorem (5.3.23)[175]. Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < n$. Then there exists a positive constant $C = C(n,p,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy, \tag{139}$$

where $p^\star = p^\star(n,s)$ is the so-called "fractional critical exponent" and it is equal to $np/(n-sp)$. Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p,p^\star]$.

Proof. First, we note that if the right-hand side of (139) is unbounded then the claim in the theorem plainly follows. Thus, we may suppose that f is such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy < +\infty. \tag{140}$$

Moreover, we can suppose, without loss of generality, that

$$f \in L^\infty(\mathbb{R}^n). \tag{141}$$

Indeed, if (141) holds for bounded functions, then it holds also for the function f_N, obtained by any (possibly unbounded) f by cutting at levels $-N$ and $+N$ (see (138)). Therefore, by Lemma (5.3.22) and the fact that (140) together with the Dominated Convergence Theorem imply

$\lim_{N\to+\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_N(x)-f_N(y)|^p}{|x-y|^{n+sp}}dxdy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy$,
we obtain estimate (139) for the function f.

Now, take a_k and A_k defined by (122) and (123), respectively. We have
$\|f\|_{L^{p^\star}(\mathbb{R}^n)}^{p^\star} = \sum_{k\in\mathbb{Z}}\int_{A_k\setminus A_{k+1}}|f(x)|^{p^\star}dx \leq \sum_{k\in\mathbb{Z}}\int_{A_k\setminus A_{k+1}}(2^{k+1})^{p^\star}dx \leq \sum_{k\in\mathbb{Z}}2^{(k+1)p^\star}a_k$.

That is, $\|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq 2^p\left(\sum_{k\in\mathbb{Z}}2^{kp^\star}a_k\right)^{p/p^\star}$.

Thus, since $p/p^\star = (n-sp)/n = 1 - sp/n < 1$,

$$\|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq 2^p\sum_{k\in\mathbb{Z}}2^{kp}a_k^{(n-sp)/n} \tag{142}$$

and, then, by choosing $T = 2^p$, Lemma (5.3.25) yields

$$\|f\|_{L^{p^\star}(\mathbb{R}^n)}^p \leq C\sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}}2^{kp}a_{k+1}a_k^{-\frac{sp}{n}}, \tag{143}$$

for a suitable constant C depending on n, p and s.

Finally, it suffices to apply Lemma (5.3.26) and we obtain the desired result, up to relabeling the constant Cin (143). Furthermore , the embedding for $q \in (p, p^\star)$follows from standard application of the Hölder inequality. From Lemma (5.3.19), it follows that

$$\int_E \int_{CE} \frac{dx\, dy}{|x-y|^{n+sp}} \geq c(n,s)|E|^{(n-sp)/n} \tag{144}$$

for all measurable sets $E$with finite measure.

On the other hand, we see that (139) reduces to (144) when $f = \chi_E$, so (144) (and thus Lemma (5.3.19)) may be seen as a Sobolev-type inequality for sets.

The above embedding does not generally hold for the space $W^{s,p}(\Omega)$since it not always possible to extend a function $f \in W^{s,p}(\Omega)$ to a function $f \in W^{s,p}(\mathbb{R}^n)$. In order to be allowed to do that, we should require further regularity assumptions on Ω(see (5.3.13)).

Theorem (5.3.24)[175]. Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n,p,s,\Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have

$$\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)}, \tag{145}$$

for any $q \in [p, p^\star]$; i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p^\star]$.
If, in addition, $\Omega$is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any
$q \in [1, p^\star]$.

Proof. Let $f \in W^{s,p}(\Omega)$. Since $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$, then there exists a constant $C_1 = C_1(n,p,s,\Omega) > 0$ such that

$$\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C_1\|f\|_{W^{s,p}(\Omega)}, \tag{146}$$

with $\tilde{f}$ such that $\tilde{f}(x) = f(x)$ for xa.e. in Ω.

On the other hand, by Theorem (5.3.23), the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p^\star]$; i.e., there exists a constant $C_2 = C_2(n,p,s) > 0$ such that

$$\|\tilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)}. \tag{147}$$

Combining (146) with (147), we get

$$\|f\|_{L^q(\Omega)} = \|\tilde{f}\|_{L^q(\Omega)} \leq \|\tilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C_2C_1\|f\|_{W^{s,p}(\Omega)},$$

that gives the inequality in (145), by choosing $C = C_2C_1$.

In the case of Ω being bounded, the embedding for $q \in [1, p)$plainly follows from (145), by using the Hölder inequality.

Theorem (5.3.25)[175]. Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp = n$. Then there exists a positive constant $C = C(n,p,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{W^{s,p}(\mathbb{R}^n)}, \tag{148}$$

for any $q \in [p, \infty)$; i.e., the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, \infty)$.

Theorem (5.3.26)[175]. Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp = n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n,p,s,\Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have

$$\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)}, \tag{149}$$

for any $q \in [p,\infty)$, the space $W^{s,p}(\Omega)$is continuously embedded in $L^q(\Omega)$ for any $q \in [p,\infty)$. If, in addition, $\Omega$is bounded, then the space $W^{s,p}(\Omega)$is continuously embedded in $L^q(\Omega)$for any $q \in [1,\infty)$.

The proofs can be obtained by simply combining Proposition (5.3.1) with Theorem (5.3.23) and Theorem (5.3.24), respectively. We state and prove some compactness results involving the fractional spaces $W^{s,p}(\Omega)$ in bounded domains.

The main proof is a modification of the one of the classical theorem (see [195]) and, again, it is self-contained and it does not require to use Besov or other interpolation spaces, nor the Fourier transform and semigroup flows (see [196]). We refer to [197] for the case $p = q = 2$.

Theorem (5.3.27)[175]. Let $s \in (0,1), p \in [1,+\infty), q \in [1,p]$, $\Omega \subset \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and T be a bounded subset of $L^p(\Omega)$. Suppose that $\sup_{f\in T}\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy < +\infty$.
Then T is pre-compact in $L^q(\Omega)$.

Proof. We want to show that T is totally bounded in $L^q(\Omega)$, i.e., for any $\varepsilon \in (0,1)$ there exist $\beta_1,\dots,\beta_M \in L^q(\Omega)$ such that for any $f \in T$ there exists $j \in \{1,\dots,M\}$ such that

$$\|f-\beta_j\|_{L^q(\Omega)} \le \varepsilon. \tag{150}$$

Since Ω is an extension domain, there exists a function $\tilde{f}$ in $W^{s,p}(\mathbb{R}^n)$ such that $\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\Omega)}$. Thus, for any cube Q containing Ω, we have

$$\|\tilde{f}\|_{W^{s,p}(Q)} \le \|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\Omega)}.$$

Observe that, since Q is a bounded open set, $\tilde{f}$ belongs also to $L^q(Q)$for any $q \in [1,p]$.

Now, for any $\varepsilon \in (0,1)$, we let $C_o := 1 + \sup_{f\in T}\|\tilde{f}\|_{L^q(Q)} + \sup_{f\in T}\int_Q \int_Q \frac{|\tilde{f}(x)-\tilde{f}(y)|^p}{|x-y|^{n+sp}}dxdy$, $\rho = \rho_\varepsilon := \left(\frac{\varepsilon}{2C_o^{\frac{1}{q}}n^{\frac{n+sp}{2p}}}\right)^{\frac{1}{s}}$ and $\eta = \eta_\varepsilon := \frac{\varepsilon\rho^{\frac{n}{q}}}{2}$, and we take a collection of disjoints cubes $Q_1,\dots,Q_N$ of side ρsuch that $\Omega \subseteq Q = \bigcup_{j=1}^N Q_j$. For any $x \in \Omega$, we define

$$j(x) \text{ as the unique integer in} \{1,\dots,N\} \text{for which} x \in Q_{j(x)}. \tag{151}$$

Also, for any $f \in T$, let

$$P(f)(x) := \frac{1}{|Q_{j(x)}|}\int_{Q_{j(x)}} \tilde{f}(y)dy.$$

Notice that $P(f+g) = P(f) + P(g)$for any $f,g \in T$ and that $P(f)$ is constant, say equal to $q_j(f)$, in any Q_j, for $j \in \{1,\dots,N\}$.

Therefore, we can define $R(f) := \rho^{n/q}(q_1(f),\dots,q_N(f)) \in \mathbb{R}^N$and consider the spatial q-norm in $\mathbb{R}^N$as

$$\|v\|_q := \left(\sum_{j=1}^N |v_j|^q\right)^{\frac{1}{q}}, \quad \text{for any} v \in \mathbb{R}^N.$$

We observe that $R(f+g) = R(f) + R(g)$.
Moreover,

$$\|P(f)\|_{L^q(\Omega)}^q = \sum_{j=1}^N \int_{Q_j\cap\Omega} |P(f)(x)|^q dx \le \rho^n \sum_{j=1}^N |q_j(f)|^q = \|R(f)\|_q^q \le \frac{\|R(f)\|_q^q}{\rho^n}. \tag{152}$$

Also, by the Hölder inequality,

$$\|R(f)\|_q^q = \sum_{j=1}^N \rho^n \left|q_j(f)\right|^q = \frac{1}{\rho^{n(q-1)}} \sum_{j=1}^N \left|\int_{Q_j} \tilde{f}(y)dy\right|^q$$
$$\leq \sum_{j=1}^N \int_{Q_j} \left|\tilde{f}(y)\right|^q dy = \int_Q \left|\tilde{f}(y)\right|^q dy = \left\|\tilde{f}\right\|_{L^q(Q)}^q.$$

In particular, $\sup_{f\in \mathrm{T}}\|R(f)\|_q^q \leq C_o$, that is, the set $R(\mathrm{T})$ is bounded in $\mathbb{R}^N$ (with respect to the q-norm of $\mathbb{R}^N$ as well as to any equivalent norm of $\mathbb{R}^N$) and so, since it is finite dimensional, it is totally bounded. Therefore, there exist $b_1, \dots, b_M \in \mathbb{R}^N$ such that

$$R(\mathrm{T}) \subseteq \cup_{i=1}^M B_\eta(b_i), \tag{153}$$

where the balls B_η are taken in the q-norm of $\mathbb{R}^N$. For any $i \in \{1, \dots, M\}$, we write the coordinates of b_i as $b_i = \left(b_{i,1}, \dots, b_{i,N}\right) \in \mathbb{R}^N$. For any $x \in \Omega$, we set $\beta_i(x) := \rho^{-\frac{n}{q}} b_{i,j(x)}$, where $j(x)$ is as in (151).

Notice that β_i is constant on Q_j, i.e. if $x \in Q_j$ then

$$P(\beta_i)(x) = \rho^{-\frac{n}{q}} b_{i,j} = \beta_i(x) \tag{154}$$

and so $q_j(\beta_i) = \rho^{-\frac{n}{q}} b_{i,j}$; thus

$$R(\beta_i) = b_i. \tag{155}$$

Furthermore, for any $f \in \mathrm{T}$

$$\begin{aligned}\|f - P(f)\|_{L^q(\Omega)}^q &= \sum_{j=1}^N \int_{Q_j\cap\Omega} |f(x) - P(f)(x)|^q dx \\ &= \sum_{j=1}^N \int_{Q_j\cap\Omega} \left|f(x) - \frac{1}{|Q_j|}\int_{Q_j} \tilde{f}(y)dy\right|^q dx \\ &= \sum_{j=1}^N \int_{Q_j\cap\Omega} \frac{1}{|Q_j|^q}\left|\int_{Q_j} f(x) - \tilde{f}(y)dy\right|^q dx \\ &\leq \frac{1}{\rho^{nq}} \sum_{j=1}^N \int_{Q_j\cap\Omega} \left[\int_{Q_j} \left|f(x) - \tilde{f}(y)\right|dy\right]^q dx.\end{aligned} \tag{156}$$

Now for any fixed $j \in 1, \dots, N$, by the Hölder inequality with p and $\frac{p}{(p-1)}$ we get

$$\begin{aligned}\frac{1}{\rho^{nq}}\left[\int_{Q_j} \left|f(x) - \tilde{f}(y)\right|dy\right]^q &\leq \frac{1}{\rho^{nq}} |Q_j|^{\frac{q(p-1)}{p}} \left[\int_{Q_j} \left|f(x) - \tilde{f}(y)\right|^p dy\right]^{\frac{q}{p}} \\ &= \frac{1}{\rho^{nq/p}} \left[\int_{Q_j} \left|f(x) - \tilde{f}(y)\right|^p dy\right]^{\frac{q}{p}} \\ &\leq \frac{1}{\rho^{nq/p}} n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}} \rho^{\frac{q}{p}(n+sp)} \left[\int_{Q_j} \frac{\left|f(x) - \tilde{f}(y)\right|^p}{|x-y|^{n+sp}} dy\right]^{\frac{q}{p}} \\ &\leq n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}} \rho^{sq} \left[\int_{Q_j} \frac{\left|f(x) - \tilde{f}(y)\right|^p}{|x-y|^{n+sp}} dy\right]^{\frac{q}{p}}.\end{aligned} \tag{157}$$

Hence, combining (156) with (157), we obtain that

$$\|f-P(f)\|_{L^q(\Omega)}^q \le n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}}\rho^{sq}\int_Q\left[\int_Q \frac{|\tilde f(x)-\tilde f(y)|^p}{|x-y|^{n+sp}}dy\right]^{\frac{q}{p}}dx$$

$$\le n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}}\rho^{sq}\left[\int_Q\int_Q \frac{|\tilde f(x)-\tilde f(y)|^p}{|x-y|^{n+sp}}dydx\right]^{\frac{q}{p}}$$

$$\le C_o n^{\left(\frac{n+sp}{2}\right)\frac{q}{p}}\rho^{sq}=\frac{\varepsilon^q}{2^q}, \tag{158}$$

where (158) follows from Jensen inequality since $t\mapsto |t|^{q/p}$ is a concave function for any fixed $p$and $q$such that $q/p\le 1$.Consequently, for any $j\in\{1,\dots,M\}$, recalling (152) and (154)

$$\|f-\beta_j\|_{L^q(\Omega)} \le \|f-P(f)\|_{L^q(\Omega)}+\|P(\beta_j)-\beta_j\|_{L^q(\Omega)}+\|P(f-\beta_j)\|_{L^q(\Omega)}$$

$$\le \frac{\varepsilon}{2}+\frac{\|R(f)-R(\beta_j)\|_q}{\rho^{n/q}}. \tag{159}$$

Now, given any $f\in \mathrm{T}$, we recall (153) and (155) and we take $j\in\{1,\dots,M\}$ such that $R(f)\in B_\eta(b_j)$. Then, (154) and (159) give that

$$\|f-\beta_j\|_{L^q(\Omega)} \le \frac{\varepsilon}{2}+\frac{\|R(f)-b_j\|_q}{\rho^{n/q}}\le \frac{\varepsilon}{2}+\frac{\eta}{\rho^{n/q}}=\varepsilon. \tag{160}$$

This proves (150), as desired.

Corollary (5.3.28)[175]. Let $s\in(0,1)$ and $p\in[1,+\infty)$ be such that $sp<n$. Let $q\in[1,p^\star)$, $\Omega\subseteq\mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and T be a bounded subset of $L^p(\Omega)$. Suppose that $\sup_{f\in T}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}dxdy<+\infty$. Then T is pre-compact in $L^q(\Omega)$.

Proof. First, note that for $1\le q\le p$the compactness follows from Theorem (5.3.27).

For any $q\in(p,p^\star)$, we may take $\theta=\theta(p,p^\star,q)\in(0,1)$such that $1/q=\theta/p+1-\theta/p^\star$, thus for any $f\in\mathrm{T}$ and β_j with $j\in\{1,\dots,N\}$ as in the theorem above, using the Hölder inequality with $p/(\theta q)$ and $p^\star/((1-\theta)q)$, we get

$$\|f-\beta_j\|_{L^q(\Omega)}=\left(\int_\Omega |f-\beta_j|^{q\theta}|f-\beta_j|^{q(1-\theta)}dx\right)^{1/q}\le\left(\int_\Omega |f-\beta_j|^p dx\right)^{\theta/p}\left(\int_\Omega |f-\beta_j|^{p^\star}dx\right)^{(1-\theta)/p^\star}$$

$$=\|f-\beta_j\|_{L^{p^\star}(\Omega)}^{1-\theta}\|f-\beta_j\|_{L^p(\Omega)}^{\theta}\le C\|f-\beta_j\|_{W^{s,p}(\Omega)}^{1-\theta}\|f-\beta_j\|_{L^p(\Omega)}^{\theta}\le\tilde C\varepsilon^\theta,$$

where the last inequalities come directly from (160) and the continuous embedding (see (5.3.24)).

Notice that the regularity assumption on Ωin Theorem (5.3.27) and Corollary (5.3.28) .We will show certain regularity properties for functions in $W^{s,p}(\Omega)$ when $sp>n$ and Ω is an extension domain for $W^{s,p}$ with no external cusps. For instance, one may take Ωany Lipschitz domain (recall (5.3.17)).

Lemma (5.3.29)[175]. (See [199].) Let $p\in[1,+\infty)$ and $sp\in(n,n+p]$. Let $\Omega\subset\mathbb{R}^n$ be a domain with no external cusps and fbe a function in $W^{s,p}(\Omega)$. Then, for any $x_0\in\Omega$ and R,R', with $0<R'<R<\operatorname{diam}(\Omega)$,
we have

$$\left|\langle f\rangle_{B_R(x_0)\cap\Omega}-\langle f\rangle_{B_{R'}(x_0)\cap\Omega}\right|\le c[f]_{p,sp}|B_R(x_0)\cap\Omega|^{(sp-n)/np} \tag{161}$$

Where

$$[f]_{p,sp} := \left(\sup_{x_0 \in \Omega_\rho > 0} \rho^{-sp} \int_{B_\rho(x_0)\cap\Omega} \left| f(x) - \langle f \rangle_{B_\rho(x_0)\cap\Omega} \right|^p dx \right)^{\frac{1}{p}}$$

and $\langle f \rangle_{B_\rho(x_0)\cap\Omega} := \frac{1}{|B_\rho(x_0)\cap\Omega|} \int_{B_\rho(x_0)\cap\Omega} f(x)dx$.

Theorem (5.3.30)[245]. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$ with no external cusps and let $p \in [1, +\infty), s \in (0,1)$be such that $sp > n$. Then, there exists $C > 0$, depending on n, s, p and Ω, such that

$$\|f\|_{C^{0,\alpha}(\Omega)} \le C \left(\|f\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}, \tag{162}$$

for any $f \in L^p(\Omega)$, with $\alpha := (sp - n)/p$.

Proof. In the following, we will denote by C suitable positive quantities, possibly different from line to line, and possibly depending on $p$and s.

First, we notice that if the right-hand side of Theorem (5.3.30) is not finite, then we are done. Thus, we may suppose that $\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy \le C$, for some $C > 0$.

Second, since $\Omega$is an extension domain for $W^{s,p}$, we can extend any f to a function $\tilde{f}$ such that $\|\tilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\Omega)}$.

Now, for any bounded measurable set $U \subset \mathbb{R}^n$, we consider the average value of the function $\tilde{f}$ in U, given by $\langle \tilde{f} \rangle_U := \frac{1}{|U|} \int_U \tilde{f}(x)dx$. For any $\xi \in \mathbb{R}^n$, the Hölder inequality yields

$\left|\xi - \langle \tilde{f} \rangle_U\right|^p = \frac{1}{|U|^p} \left| \int_U \xi - \tilde{f}(y)dy \right|^p \le \frac{1}{|U|} \int_U \left|\xi - \tilde{f}(y)\right|^p dy$.

Accordingly, by taking $x_o \in \Omega$ and $U := B_r(x_o), \xi := \tilde{f}(x)$ and integrating over $B_r(x_o)$, we obtain that $\int_{B_r(x_o)} \left|\tilde{f}(x) - \langle \tilde{f} \rangle_{B_r(x_o)}\right|^p dx \le \frac{1}{|B_r(x_o)|} \int_{B_r(x_o)} \int_{B_r(x_o)} \left|\tilde{f}(x) - \tilde{f}(y)\right|^p dxdy$.

Hence, since $|x - y| \le 2r$ for any $x, y \in B_r(x_o)$, we deduce that

$$\begin{aligned} \int_{B_r(x_o)} \left|\tilde{f}(x) - \langle f \rangle_{B_r(x_o)}\right|^p dx &\le \frac{(2r)^{n+sp}}{|B_r(x_o)|} \int_{B_r(x_o)} \int_{B_r(x_o)} \frac{\left|\tilde{f}(x) - \tilde{f}(y)\right|^p}{|x-y|^{n+sp}} dxdy \\ &\le \frac{2^{n+sp} r^{sp} C \|f\|^p_{W^{s,p}(\Omega)}}{|B_1|}, \end{aligned} \tag{163}$$

that implies

$$[f]^p_{p,sp} \le C\|f\|^p_{W^{s,p}(\Omega)}, \tag{164}$$

for a suitable constant C.

Now, we will show that f is a continuous function. Taking into account (161), it follows that the sequence of functions $x \to \langle f \rangle_{B_R(x)\cap\Omega}$ converges uniformly in $x \in \Omega$ when $R \to 0$. In particular the limit function $g$will be continuous and the same holds for f, since by Lebesgue theorem we have that $\lim_{R\to 0} \frac{1}{|B_R(x)\cap\Omega|} \int_{B_R(x)\cap\Omega} f(y)dy = f(x)$ for almost every $x \in \Omega$.

Now, take any $x, y \in \Omega$ and set $R = |x-y|$. We have

$$|f(x)-f(y)| \leq \left|f(x)-\langle\tilde{f}\rangle_{B_{2R}(x)}\right| + \left|\langle\tilde{f}\rangle_{B_{2R}(x)}-\langle\tilde{f}\rangle_{B_{2R}(y)}\right| + \left|\langle\tilde{f}\rangle_{B_{2R}(y)}-f(y)\right|.$$

We can estimate the first and the third term of right-hand side of the above inequality using Lemma (5.3.29). Indeed, getting the limit in (161) as $R' \to 0$ and writing $2R$ instead of R, for any $x \in \Omega$ we get

$$\left|\langle\tilde{f}\rangle_{B_{2R}(x)}-f(x)\right| \leq c[f]_{p,sp}|B_{2R}(x)|^{(sp-n)/np} \leq C[f]_{p,sp}R^{(sp-n)/p} \qquad (165)$$

where the constant C is given by $c2^{(sp-n)/p}/|B_1|$.

On the other hand, $\left|\langle\tilde{f}\rangle_{B_{2R}(x)}-\langle\tilde{f}\rangle_{B_{2R}(y)}\right| \leq \left|f(z)-\langle\tilde{f}\rangle_{B_{2R}(x)}\right| + \left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(y)}\right|$ and so, integrating on $z \in B_{2R}(x) \cap B_{2R}(y)$, we have

$$\begin{aligned}|B_{2R}(x) \cap B_{2R}(y)|\left|\langle\tilde{f}\rangle_{B_{2R}(x)}-\langle\tilde{f}\rangle_{B_{2R}(y)}\right| &\leq \int_{B_{2R}(x)\cap B_{2R}(y)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(x)}\right|dz \\ &\quad + \int_{B_{2R}(x)\cap B_{2R}(y)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(y)}\right|dz \\ &\leq \int_{B_{2R}(x)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(x)}\right|dz + \int_{B_{2R}(y)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(y)}\right|dz.\end{aligned}$$

Furthermore, since $B_R(x) \cup B_R(y) \subset \big(B_{2R}(x) \cap B_{2R}(y)\big)$, we have

$$|B_R(x)| \leq |B_{2R}(x) \cap B_{2R}(y)| \text{and} |B_R(y)| \leq |B_{2R}(x) \cap B_{2R}(y)|$$

and so

$$\left|\langle\tilde{f}\rangle_{B_{2R}(x)}-\langle\tilde{f}\rangle_{B_{2R}(y)}\right| \leq \frac{1}{|B_R(x)|}\int_{B_{2R}(x)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(x)}\right|dz + \frac{1}{B_{2R}(y)}\int_{B_{2R}(y)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(y)}\right|dz.$$

An application of the Hölder inequality gives

$$\begin{aligned}\frac{1}{|B_R(x)|}\int_{B_{2R}(x)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(x)}\right|dz &\leq \frac{|B_{2R}(x)|^{(p-1)/p}}{|B_R(x)|}\left(\int_{B_{2R}(x)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(x)}\right|^p dz\right)^{1/p} \\ &\leq \frac{|B_{2R}(x)|^{(p-1)/p}}{|B_R(x)|}(2R)^s[f]_{p,sp} \leq C[f]_{p,sp}R^{(sp-n)/p}. \qquad (166)\end{aligned}$$

Analogously, we obtain

$$\frac{1}{|B_R(y)|}\int_{B_{2R}(y)}\left|\tilde{f}(z)-\langle\tilde{f}\rangle_{B_{2R}(y)}\right|dz \leq C[f]_{p,sp}R^{(sp-n)/p}. \qquad (167)$$

Combining (165), (166) with (167) it follows

$$|f(x)-f(y)| \leq C[f]_{p,sp}|x-y|^{(sp-n)/p}, \qquad (168)$$

up to relabeling the constant C.

Therefore, by taking into account (164), we can conclude that $f \in C^{0,\alpha}(\Omega)$, with $\alpha = (sp-n)/p$.

Finally, taking $R_0 < \operatorname{diam}(\Omega)$(note that the latter can be possibly infinity), using estimate in (165) and the Hölder inequality we have, for any $x \in \Omega$,

$$|f(x)| \leq \left|\langle\tilde{f}\rangle_{B_{R_0}(x)}\right| + \left|f(x)-\langle\tilde{f}\rangle_{B_{R_0}(x)}\right| \leq \frac{C}{|B_{R_0}(x)|^{\frac{1}{p}}}\|f\|_{L^p(\Omega)} + c[f]_{p,sp}\left|B_{R_0}(x)\right|^{\alpha}. \qquad (169)$$

Hence, by (164), (168) and (169), we get

$$\|f\|_{C^{0,\alpha}(\Omega)} = \|f\|_{L^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq C\big(\|f\|_{L^p(\Omega)} + [f]_{p,sp}\big) \leq C\|f\|_{W^{s,p}(\Omega)}$$

for a suitable positive constant C.

When the domain $\Omega$is not Lipschitz, some interesting things happen, as next examples show.

Example (5.3.31)[175]. Let $s \in (0,1)$. We will construct a function $u$in $W^{1,p}(\Omega)$ that does not belong to $W^{s,p}(\Omega)$, providing a counterexample to Corollary (5.3.3) when the domain is not Lipschitz. Take any

$$p \in (1/s, +\infty). \tag{170}$$

Due to (170), we can fix

$$\kappa > \frac{p+1}{sp-1}. \tag{171}$$

We remark that $\kappa > 1$.

Let us consider the cusp in the plane $C := \{(x_1, x_2) \text{ with } x_1 \le 0 \text{ and } |x_2| \le |x_1|^\kappa\}$ and take polar coordinates on $\mathbb{R}^2 \backslash C$, say $\rho = \rho(x) \in (0, +\infty)$ and $\theta = \theta(x) \in (-\pi, \pi)$, with $x = (x_1, x_2) \in \mathbb{R}^2 \backslash C$.

We define the function $u(x) := \rho(x)\theta(x)$ and the heart-shaped domain $\Omega := (\mathbb{R}^2 \backslash C) \cap B_1$, with B_1 being the unit ball centered in the origin.

Then, $u \in W^{1,p}(\Omega) \backslash W^{s,p}(\Omega)$. To check this, we observe that

$\partial_{x_1}\rho = (2\rho)^{-1}\partial_{x_1}\rho^2 = (2\rho)^{-1}\partial_{x_1}(x_1^2 + x_2^2) = \frac{x_1}{\rho}$ and, in the same way, $\partial_{x_2}\rho = \frac{x_2}{\rho}$.

Accordingly, $1 = \partial_{x_1}x_1 = \partial_{x_1}(\rho\cos\theta) = \partial_{x_1}\rho\cos\theta - \rho\sin\theta\partial_{x_1}\theta = \frac{x_1^2}{\rho^2} - x_2\partial_{x_1} = 1 - \frac{x_2^2}{\rho^2} - x_2\partial_{x_1}\theta$.

That is $\partial_{x_1}\theta = -\frac{x_2}{\rho^2}$.

By exchanging the roles of x_1 and x_2 (with some care on the sign of the derivatives of the trigonometric functions), one also obtains $\partial_{x_2}\theta = \frac{x_1}{\rho^2}$.

Therefore, $\partial_{x_1}u = \rho^{-1}(x_1\theta - x_2)$ and $\partial_{x_2}u = \rho^{-1}(x_2\theta + x_1)$ and so $|\nabla u|^2 = \theta^2 + 1 \le \pi^2 + 1$.

This shows that $u \in W^{1,p}(\Omega)$.

On the other hand, let us fix $r \in (0,1)$, to be taken arbitrarily small at the end, and let us define $r_0 := r$ and, for any $j \in \mathbb{N}$, $r_{j+1} := r_j - r_j^\kappa$.

By induction, one sees that r_j is strictly decreasing, that $r_j > 0$ and so $r_j \in (0, r) \subset (0,1)$.

Accordingly, we can define $\ell := \lim_{j\to+\infty} r_j \in [0,1]$.

By construction $\ell = \lim_{j\to+\infty} r_{j+1} = \lim_{j\to+\infty} r_j - r_j^\kappa = \ell - \ell^\kappa$, hence $\ell = 0$.

As a consequence,

$$\sum_{j=0}^{+\infty} r_j^\kappa = \lim_{N\to+\infty}\sum_{j=0}^{N} r_j^\kappa = \lim_{N\to+\infty}\sum_{j=0}^{N} r_j - r_{j+1} = \lim_{N\to+\infty} r_0 - r_{N+1} = r. \tag{172}$$

We define $\mathcal{D}_j := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \text{ s.t. } x_1, y_1 \in (-r_j, -r_{j+1}),$

$x_2 \in (|x_1|^\kappa, 2|x_1|^\kappa) \text{ and } -y_2 \in (|y_1|^\kappa, 2|y_1|^\kappa)\}$.

We observe that

$$\Omega \times \Omega \supseteq \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \text{ s.t. } x_1, y_1 \in (-r, 0), x_2 \in (|x_1|^\kappa, 2|x_1|^\kappa) \text{ and } -y_2 \in (|y_1|^\kappa, 2|y_1|^\kappa)\}$$
$$\supseteq \bigcup_{j=0}^{+\infty} \mathcal{D}_j,$$

and the union is disjoint. Also, $r_{j+1} = r_j(1 - r_j^{\kappa-1}) \ge r_j(1 - r^{\kappa-1}) \ge \frac{r_j}{2}$, for small r.

Hence, if $(x,y) \in \mathcal{D}_j$, $|x_1| \le r_j \le 2r_{j+1} \le 2|y_1|$ and, analogously, $|y_1| \le 2|x_1|$.

Moreover, if $(x,y) \in \mathcal{D}_j$, $|x_1 - y_1| \le r_j - r_{j+1} = r_j^\kappa \le 2^\kappa r_{j+1}^\kappa \le 2^\kappa |x_1|^\kappa$

and $|x_2 - y_2| \le |x_2| + |y_2| \le 2|x_1|^\kappa + 2|y_1|^\kappa \le 2^{\kappa+2}|x_1|^\kappa$.

As a consequence, if $(x,y) \in \mathcal{D}_j$, $|x - y| \le 2^{\kappa+3}|x_1|^\kappa$.

Notice also that, when $(x,y) \in \mathcal{D}_j$, we have $\theta(x) \geq \pi/2$ and $\theta(y) \leq -\pi/2$, so

$$u(x) - u(y) \geq u(x) \geq \frac{\pi\rho(x)}{2} \geq \frac{\pi|x_1|}{2}.$$

As a consequence, for any $(x,y) \in \mathcal{D}_j$, $\frac{|u(x)-u(y)|^p}{|x-y|^{2+sp}} \geq c|x_1|^{p-\kappa(2+sp)}$, for some $c > 0$.

Therefore,

$$\begin{aligned}\iint_{\mathcal{D}_j} \frac{|u(x)-u(y)|^p}{|x-y|^{2+sp}} dxdy &\geq \iint_{\mathcal{D}_j} c|x_1|^{p-\kappa(2+sp)} dxdy \\ &= c\int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 \int_{|x_1|^\kappa}^{2|x_1|^\kappa} dx_2 \int_{-2|y_1|^\kappa}^{-|y_1|^\kappa} dy_2 |x_1|^{p-\kappa(2+sp)} \\ &= c\int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 |x_1|^{p-\kappa(2+sp)} |x_1|^\kappa |y_1|^\kappa \\ &\leq c2^{-\kappa} \int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 |x_1|^{p-\kappa sp} \\ &\leq c2^{-\kappa} \int_{-r_j}^{-r_{j+1}} dx_1 \int_{-r_j}^{-r_{j+1}} dy_1 r_j^{p-\kappa sp} \\ &= c2^{-\kappa} r_j^{p-\kappa sp+2\kappa} = c2^{-\kappa} r_j^{\kappa-\alpha}, \text{ with}\end{aligned}$$

$$\alpha := \kappa(sp-1) - p > 1, \tag{173}$$

thanks to (171). In particular, $\iint_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{2+sp}} dxdy \geq c2^{-\kappa} r^{-\alpha} r_j^\kappa$, by summing up and exploiting (172),

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{2+sp}} dxdy \geq \sum_{j=0}^{+\infty} \iint_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{2+sp}} dxdy \geq c^{2-\kappa} r^{1-\alpha}.$$

By taking $r$as small as we wish and recalling (173), we obtain that

$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{2+sp}} dxdy = +\infty$,so $u \notin W^{s,p}(\Omega)$.

Example (5.3.32). Let $s \in (0,1)$.We will construct a sequence of functions $\{f_n\}$ bounded in $W^{s,p}(\Omega)$ that does not admit any convergent subsequence in $L^q(\Omega)$, providing a counterexample to Theorem(5.3.27) when the domain is not Lipschitz.

We follow an observation by [200]. For the sake of simplicity, fix $n = p = q = 2$. We take $a_k := 1/C^k$ for a constant $C > 10$ and we consider the set $\Omega = \bigcup_{k=1}^\infty B_k$where, for any $k \in \mathbb{N}$, B_k denotes the ball of radius a_k^2 centered in a_k. Notice that

$$a_k \to 0 \text{ as } k \to \infty \quad \text{and} a_k - a_k^2 > a_{k+1} + a_{k+1}^2.$$

Thus, Ω is the union of disjoint balls, it is bounded and it is not a Lipschitz domain.

For any $n \in \mathbb{N}$, we define the function $f_n: \Omega \to \mathbb{R}$ as follows $f_n(x) = \begin{cases} \pi^{-\frac{1}{2}} a_n^{-2}, & x \in B_n, \\ 0, & x \in \Omega \backslash B_n. \end{cases}$

We observe that we cannot extract any subsequence convergent in $L^2(\Omega)$ from the sequence of functions $\{f_n\}$, because $f_n(x) \to 0$ as $n \to +\infty$, for any fixed $x \in \Omega$ but

$\|f_n\|_{L^2(\Omega)}^2 = \int_\Omega |f_n(x)|^2 dx = \int_{B_n} \pi^{-1} a_n^{-4} dx = 1.$

Now, we compute the H^snorm of f_n in Ω. We have

$$\int_\Omega \int_\Omega \frac{|f_n(x)-f_n(y)|^2}{|x-y|^{2+2s}} dxdy = 2 \int_{\Omega\backslash B_n} \int_{B_n} \frac{\pi^{-1} a_n^{-4}}{|x-y|^{2+2s}} dxdy = 2\pi^{-1} \sum_{k \neq n} \int_{B_k} \int_{B_n} \frac{a_n^{-4}}{|x-y|^{2+2s}} dxdy. \tag{174}$$

Thanks to the choice of $\{a_k\}$ we have that $|a_n^2 + a_k^2| = a_n^2 + a_k^2 \leq \frac{|a_n - a_k|}{2}$.

Thus, since $x \in B_n, y \in B_k$, it follows

$$|x-y| \geq |a_n - a_n^2 - (a_k + a_k^2)| = |a_n - a_k - (a_n^2 + a_k^2)|$$
$$\geq |a_n - a_k| - |a_n^2 + a_k^2| \geq |a_n - a_k| - \frac{|a_n - a_k|}{2} = \frac{|a_n - a_k|}{2}.$$

Therefore,

$$\int\limits_{B_k}\int\limits_{B_n} \frac{a_n^{-4}}{|x-y|^{2+2s}} dxdy \leq 2^{2+2s} \int\limits_{B_k}\int\limits_{B_n} \frac{a_n^{-4}}{|a_n - a_k|^{2+2s}} dxdy = 2^{2+2s}\pi^2 \frac{a_k^4}{|a_n - a_k|^{2+2s}}. \quad (175)$$

Also, if $m \geq j+1$ we have

$$a_j - a_m \geq a_j - a_{j+1} = \frac{1}{C^j} - \frac{1}{C^{j+1}} = \frac{1}{C^j}\left(1 - \frac{1}{C}\right) \geq \frac{a_j}{2}. \quad (176)$$

Therefore, combining (176) with (174) and (175), we get

$$\int_\Omega \int_\Omega \frac{|f_n(x) - f_n(y)|^2}{|x-y|^{2+2s}} dxdy \leq 2^{3+2s}\pi \sum_{k\neq n} \frac{a_k^4}{|a_n - a_k|^{2+2s}}$$
$$= 2^{3+2s}\pi \left(\sum_{k<n} \frac{a_k^4}{(a_k - a_n)^{2+2s}} + \sum_{k>n} \frac{a_k^4}{(a_n - a_k)^{2+2s}}\right)$$
$$\leq 2^{5+4s}\pi \left(\sum_{k<n} \frac{a_k^4}{a_k^{2+2s}} + \sum_{k>n} \frac{a_k^4}{a_n^{2+2s}}\right)$$
$$\leq 2^{6+4s}\pi \sum_{k\neq n} a_k^{2-2s} = 2^{6+4s}\pi \sum_{k\neq n} \left(\frac{1}{C^{2-2s}}\right)^k < +\infty.$$

This shows that $\{f_n\}$ is bounded in $H^s(\Omega)$.

Corollary (5.3.33)[236]. Let Ω be an open set in $\mathbb{R}^n$, $(1-\varepsilon) \in (0,1)$ and $(1+\varepsilon) \in [1,+\infty)$. Let us consider $u \in W^{1-\varepsilon,1+\varepsilon}(\Omega)$ and $(1-\varepsilon) \in C^{0,1}(\Omega)$, $0 \leq \varepsilon \leq 1$. Then $(1-\varepsilon)u \in W^{1-\varepsilon,1+\varepsilon}(\Omega)$ and $\|(1-\varepsilon)u\|_{W^{1-\varepsilon,1+\varepsilon}(\Omega)} \leq C\|u\|_{W^{1-\varepsilon,1+\varepsilon}(\Omega)}$, where $C = C(n, 1-\varepsilon, 1+\varepsilon, \Omega)$.

Proof. It is clear that $\|(1-\varepsilon)u\|_{L^{1+\varepsilon}(\Omega)} \leq \|u\|_{L^{1+\varepsilon}(\Omega)}$ since $|1-\varepsilon| \leq 1$. Furthermore, adding and subtracting the factor $(1-\varepsilon)(x)u(y)$, we get

$$\int_\Omega \int_\Omega \frac{|(1-\varepsilon)(x)u(x) - (1-\varepsilon)(y)u(y)|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy \leq 2^\varepsilon \left(\int_\Omega \int_\Omega \frac{|(1-\varepsilon)(x)u(x) - (1-\varepsilon)(x)u(y)|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy\right.$$
$$\left. + \int_\Omega \int_\Omega \frac{|(1-\varepsilon)(x)u(x) - (1-\varepsilon)(x)u(y)|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy\right)$$
$$\leq 2^\varepsilon \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy\right.$$
$$\left. + \int_\Omega \int_\Omega \frac{|u(x)|^{1+\varepsilon}|(1-\varepsilon)(x) - (1-\varepsilon)(y)|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy\right).$$

Since ψ belongs to $C^{0,1}(\Omega)$, we get

$$\int_\Omega \int_\Omega \frac{|u(x)|^{1+\varepsilon}|(1-\varepsilon)(x) - (1-\varepsilon)(y)|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy \leq \Lambda^p \int_\Omega \int_{\Omega\cap|x-y|\leq 1} \frac{|u(x)|^{1+\varepsilon}|x-y|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy$$
$$+ \int_\Omega \int_{\Omega\cap|x-y|\geq 1} \frac{|u(x)|^{1+\varepsilon}}{|x-y|^{n+1-\varepsilon^2}} dxdy \leq \tilde{C}\|u\|_{L^{1+\varepsilon}(\Omega)}^{1+\varepsilon},$$

where Λ denotes the Lipschitz constant of $(1-\varepsilon)$ and $\tilde{C} > 0$ depending on $n, (1+\varepsilon)$ and $(1-\varepsilon)$.

Corollary (5.3.34)[236]. Let $\varepsilon > 0$, Fix $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ be a measurable set with finite measure. Then, $\int_{\complement E} \frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}} \geq C|E|^{-(1-\varepsilon^2)/n}$, for a suitable constant $C = C(n, 1+\varepsilon, 1-\varepsilon) > 0$.

Proof. We set $\rho := \left(\frac{|E|}{\omega_n}\right)^{\frac{1}{n}}$ and then it follows

$$|(\complement E) \cap B_\rho(x)| = |B_\rho(x)| - |E \cap B_\rho(x)| = |E| - |E \cap B_\rho(x)| = |E \cap \complement B_\rho(x)|.$$

Therefore,

$$\begin{aligned}\int_{CE}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}} &= \int_{(CE)\cap B_\rho(x)}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}}+\int_{(CE)\cap CB_\rho(x)}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}}\\ &\ge \int_{(CE)\cap B_\rho(x)}\frac{d(x+\varepsilon)}{\rho^{n+(1-\varepsilon^2)}}+\int_{(CE)\cap CB_\rho(x)}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}}\\ &= \frac{|(CE)\cap B_\rho(x)|}{\rho^{n+(1-\varepsilon^2)}}+\int_{(CE)\cap CB_\rho(x)}\frac{dy}{|\varepsilon|^{n+(1-\varepsilon^2)}}\\ &= \frac{|E\cap CB_\rho(x)|}{\rho^{n+(1-\varepsilon^2)}}+\int_{(CE)\cap CB_\rho(x)}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}}\\ &\ge \int_{E\cap CB_\rho(x)}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}}+\int_{(CE)\cap CB_\rho(x)}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}}\\ &= \int_{CB_\rho(x)}\frac{d(x+\varepsilon)}{|\varepsilon|^{n+(1-\varepsilon^2)}}.\end{aligned}$$

The desired result easily follows by using polar coordinates centered at x.

Corollary (5.3.35)[236]. For $\varepsilon > 0$ and $p \in [1,+\infty)$ be such that $(1-\varepsilon^2) < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{1-\varepsilon,1+\varepsilon}$. Then there exists a positive constant $\tilde{C} = \tilde{C}(n, 1+\varepsilon, 1-\varepsilon, \Omega)$ such that, for every $f_j \in W^{1-\varepsilon,1+\varepsilon}(\Omega)$, we get $\sum_{j=1}^{m}\|f_j\|_{L^{1+\varepsilon_2}(\Omega)} \le \tilde{C}\sum_{j=1}^{m}\|f_j\|_{W^{1-\varepsilon,1+\varepsilon}(\Omega)}$, for any $\varepsilon_2 > 0$; i.e., the space $W^{1-\varepsilon,1+\varepsilon}(\Omega)$ is continuously embedded in $L^{1+\varepsilon_2}(\Omega)$ for any $\varepsilon_2 > 0$. If, in addition, Ω is bounded, then the space $W^{1-\varepsilon,1+\varepsilon}(\Omega)$ is continuously embedded in $L^{1+\varepsilon_2}(\Omega)$ for any $\varepsilon_2 > 0$.

Proof. Let $f_j \in W^{1-\varepsilon,1+\varepsilon}(\Omega)$. Since $\Omega \subseteq \mathbb{R}^n$is an extension domain for $W^{1-\varepsilon,1+\varepsilon}$, then there exists a constant $\tilde{C}_1 = \tilde{C}_1(n, 1+\varepsilon, 1-\varepsilon, \Omega) > 0$ such that

$$\sum_{j=1}^{m}\|\tilde{f}_j\|_{W^{1-\varepsilon,1+\varepsilon}(\mathbb{R}^n)} \le \tilde{C}_1\sum_{j=1}^{m}\|f_j\|_{W^{1-\varepsilon,1+\varepsilon}(\Omega)},$$

with $\tilde{f}_j$ such that $\tilde{f}_j(x) = f_j(x)$ for x a.e. in Ω. On the other hand, by Theorem (5.3.23), the space $W^{1-\varepsilon,1+\varepsilon}(\mathbb{R}^n)$is continuously embedded in $L^{1+\varepsilon_2}(\mathbb{R}^n)$ for any $\varepsilon_2 > 0$; i.e., there exists a constant $\tilde{C}_2 = \tilde{C}_2(n, 1+\varepsilon, 1-\varepsilon) > 0$ such that

$$\sum_{j=1}^{m}\|\tilde{f}_j\|_{L^{1+\varepsilon_2}(\mathbb{R}^n)} \le \tilde{C}\sum_{j=1}^{m}\|\tilde{f}_j\|_{W^{1-\varepsilon,1+\varepsilon}(\mathbb{R}^n)}.$$

Combining (146) with (147), we get

$$\begin{aligned}\sum_{j=1}^{m}\|f_j\|_{L^{1+\varepsilon_2}(\Omega)} &= \sum_{j=1}^{m}\|\tilde{f}_j\|_{L^{1+\varepsilon_2}(\Omega)}\\ &\le \sum_{j=1}^{m}\|\tilde{f}_j\|_{L^{1+\varepsilon_2}(\mathbb{R}^n)}\\ &\le \tilde{C}_2\sum_{j=1}^{m}\|\tilde{f}_j\|_{W^{1-\varepsilon,1+\varepsilon}(\mathbb{R}^n)}\\ &\le \tilde{C}_2\tilde{C}_1\sum_{j=1}^{m}\|f_j\|_{W^{1-\varepsilon,1+\varepsilon}(\Omega)},\end{aligned}$$

that gives the inequality in (145), by choosing $\tilde{C} = \tilde{C}_2\tilde{C}_1$.

In the case of Ω being bounded, the embedding for $\varepsilon_2 > 0$ plainly follows from (145),by using Hölder's inequality

Chapter 6
Relative asymptotics and Fourier Series with $W^{1,p}$-Convergence

We study the pointwise convergence of the Fourier series associated to the inner product provided that m is the Jacobi measure. We generalize the work done by F. Marcellan and W. Van Assche where they studied the asymptotics for only one mass point in $[-1,1]$.The same problem with a finite number of mass points off $[-1,1]$ was solved a more general setting by consider the constants $M_{k,i}$ to be complex numbers. As regards the Fourier series, we continue the results for the Jacobi measure and mass points in $R[-1,1]$.Let $\left\{Q_n^{(\alpha)}(x)\right\}_{n\geq 0}$ denote the sequence of monic polynomials orthogonal with respect to the non-discrete Sobolev inner product, $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)d_\mu(x) + \lambda\int_{-1}^{1} f'(x)g'(x)d_\mu(x)$ where $\lambda > 0$ and $d_\mu = (1-x^2)^{\alpha-1/2}dx$ with $\alpha > -1/2$. A strong asymptotic on $(-1,1)$ a Mehler-Heine type formula as well as Sobolev norms of $Q_n^{(\alpha)}$ are obtained. Let $\{q_n\}_n \geq 0$ be the sequence of polynomials orthonormal with respect to the Sobolev inner product

$\langle f, g\rangle_s := \int_{-1}^{1} f(x)g(x)w_0(x)dx + \int_{-1}^{1} f'(x)g'(x)w_1(x)dx$ where $w_0 \in L^\infty([-1,1])$ and w_1 is a weight Kufner–Opic type .

Section (6.1): Orthogonal Polynomials With a Discrete Sobolev Inner Product:

Let μ be a finite positive Borel measure supported on the interval [-1,1] with infinitely many points at the support and let $a_k, k = 1, \dots, K$ be real number such that $a_k \in [-1,1]$ .For f and g in $L^2(\mu)$ such that there exist the derivatives in a_k we can introduce the Sobolev-type inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)d\mu(x) + \sum_{k=1}^{K}\sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k)g^{(i)}(a_k) \qquad (1)$$

Where $M_{k,i} \geq 0$ for $i = 0, \dots, N_k - 1$ and $M_k N_k > 0$ when $k = 1, \dots, K$ we assume

$\mu(\{a_k\}) = 0$ otherwise the corresponding $M_{k,0}$ should be modified. Let $\left(\hat{B}_k\right)_{k=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to this inner, $\langle \hat{B}_n, \hat{B}_k\rangle = \delta_{n,k}\ \ k, n = 0,1, \dots$.[201] deduced the relative a symptotics for the orthogonal polynomials with respect to the Sobolev inner product with mass points outside $[-1,1]$ and complex constants $M_{k,i}$ [202] analyzed such a question when there is only one mass point inside $[-1,1]$. Here we deal with an extension of this last problem with a finite number of masses. We compare the polynomials $\hat{B}_n$ with the polynomials $(p_n)_{k=0}^{\infty}$ orthonormal with respect to μ.

The technique used in this section is a generalization of the one used for obtaining estimates of the Sobolev orthogonal polynomials in [203],[204]. There, studied the pointwise convergence of the Fourier series for sequences of orthogonal polynomials with respect to the inner product (1) for the Jacobi measure and with mass points outside $[-1,1]$.The main results concerning asymptotic properties we show that $\frac{\hat{B}_n(x)}{p_n(x)}$ tends to 1, and we obtain for $\hat{B}_n(x)$ the usual weak a symptotics, and, in Theorem (6.1.10), the asymptotics for the coefficients in the recurrence relation of the Sobolev orthonormal polynomials are given. We consider the pointwise convergence of the Fourier series with respect to (1) provided that μ is the Jacobi measure. We continue the work achieved in [203], [204] and prove the point wise convergence for the Fourier series of functions which satisfy some standard sufficient conditions . From now on $k(\prod_n)$

denotes the leading coefficient of any polynomial $\prod_n$ with real coefficients and n is the degree of the polynomial.

Let N_k^* be the positive integer number defined by $N_k^* = \begin{cases} N_k + 1 \text{ if } N_k \text{ is odd} \\ N_k + 2 \text{ if } N_k \text{ is even} \end{cases}$

And let $W_N(x) = \prod_{k=1}^{K}(x - a_k)^{N_k^*}$ where $\sum_{k=0}^{K} N_{k=1}^*$ let $(P_n)_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to μ.

Lemma(6.1.1)[205]. $W_N(x)\widehat{B}_n(x) = \sum_{j=0}^{2N} A_{n,j}P_{n+N-j}(x)$, $A_{n,0} \neq 0$. Moreover $A_{n,j}$ are bounded and $A_{n,2N} = \frac{k(P_{n-N})}{k(P_{n+N})}\frac{1}{A_{n,0}} \neq 0$.

Proof: Since $W_N(x)\widehat{B}_n(x) = \sum_{j=0}^{n+N} \alpha_{n,j}P_j(x)$ and $\alpha_{n,j} = \int_{-1}^{1} W_N(x)\widehat{B}_n(x)P_j(x)d\mu(x) = \langle \widehat{B}_n, W_N P_j\rangle = 0$, $j < n - N$, we have the first assertion with $A_{n,j} = \alpha_{n,j+N-j}$ Furthermore,

$$\sum_{j=0}^{2N} A_{n,j}^2 = \int_{-1}^{1} \widehat{B}_n^2(x)W_N^2(x)d\mu(x) \leq \max_{x\in[-1,1]} W_N^2$$

And thus $(A_{n,j})$ are bounded also

$$A_{n,0} = \int_{-1}^{1} W_N(x)\widehat{B}_n(x)P_{n+N}(x)d\mu(x) = \frac{k(\widehat{B}_n)}{k(P_{n+N})}$$

As well as

$$A_{n,2N} = \int_{-1}^{1} W_N(x)\widehat{B}_n(x)P_{n-N}(x)d\mu(x)\, \langle \widehat{B}_n, W_N P_{n-N}\rangle = \frac{k(P_{n-N})}{k(\widehat{B}_n)} = \frac{k(P_{n-N})}{k(P_{n+N})}\frac{1}{A_{n,0}}$$

And the lemma holds.

Let Λ be a sequence of nonnegative integers such that $\lim_{n\in\Lambda} A_{n,j} = A_j$ for j = 0,1,…,2N

When $\mu'(x) > 0$ since $A_0 < \infty$ and $\lim_{n\to\infty} \frac{k(P_{n-N})}{k(P_{n+N})} = \frac{1}{2^{2N}}$ as it is well known (see [206], [207]), A_{2N} has to be greater than zero.

Let

$$\prod{}_n = \sum_{j=0}^{2N} \frac{A_j}{A_{2N}} T_j(x)$$

where, for each j, $T_j(x)$ is the Chebyshev polynomial of the first kind and degree j.

Lemma(6.2.2)[205]. If $\mu'(x) > 0$ then the polynomial $\prod_{2N}$ satisfies $\prod_{2N}^{(i)}(a_k) = 0$ for $i = 0,1,\ldots,N_k^* - 1$ and $k = 1,\ldots,K$

Proof: For a given $k = \{1,2,\ldots,K\}$ let $\varepsilon > 0$ and $i \in \{1,2,\ldots,N_k^*\}$ consider the function

$$\Psi_{i,\varepsilon}(x) = \begin{cases} 0 & \text{if } x \in [-1, a_k + \varepsilon \\ \dfrac{1}{(x - a_k)^i} & \text{if} x\epsilon(a_k + \varepsilon, 1) \end{cases}$$

This function is bound in $[-1,1]$ and satisfies the condition $\max_{x\in[-1,1]}\left|W_N(x)\Psi_{i,\varepsilon}(x)\right| \leq C$ for some constant C independent of ε .

As it is well known (see [280]),[281] since $\mu'(x) > 0$

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)P_{n+v}P_n(x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} f(x)T_v(x)\frac{dx}{\sqrt{1 - x^2}}$$

For all Borel measurable function f bounded on [-1,1] as a consequence the expression of $W_N(x)\widehat{B}_n(x)$ in terms of $(P_j)_{j=0}^{n+N}$ of lemma (6.1.1) gives

$$\lim_{n\to\infty}\int_{-1}^{1} W_N(x)\widehat{B}_n(x)P_{n+N}(x)\Psi_{i,\varepsilon}(x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1}\sum_{v=0}^{2N} A_v T_v(x)\Psi_{i,\varepsilon}(x)\frac{dx}{\sqrt{1-x^2}}$$

From the Cauchy-Schwarz inequality

$$\left|\int_{-1}^{1}\widehat{B}_n(x)P_{n+N}(x)W_N(x)\Psi_{i,\varepsilon}(x)d\mu(x)\right| \le c\left(\int_{-1}^{1}\widehat{B}_n^2(x)d\mu(x)\right)^{\frac{1}{2}}\left(\int_{-1}^{1}P_{n+N}^2(x)d\mu(x)\right)^{\frac{1}{2}} \le c$$

And we get

$$\lim_{\varepsilon\to 0}\sup\left|\frac{1}{\pi}\int_{-1}^{1}\sum_{v=0}^{2N} A_v T_v(x)\Psi_{i,\varepsilon}(x)\frac{dx}{\sqrt{1-x^2}}\right| \le c \quad \text{for} \quad i = 1,\dots,N_k^* \tag{2}$$

When $i = 1$ we have

$$\int_{-1}^{1}\Pi_{2N}(x)\Psi_{1,\varepsilon}(x)\frac{dx}{\sqrt{1-x^2}} = \prod_{2N}(a_k)\int_{-1}^{1}\Psi_{1,\varepsilon}(x)\frac{dx}{\sqrt{1-x^2}} + \int_{-1}^{1}\left(\prod_{2N}(x)-\prod_{2N}(a_k)\right)\Psi_{1,\varepsilon}(x)\frac{dx}{\sqrt{1-x^2}}$$

Thus condition (2) holds if and only if $\prod_{2N}(a_k) = 0$

Lemma (61.3)[205]. If $\mu'(x) > 0$ a.e. then $\prod_{2N}(x) = 2^N T_N(x)\prod_{k=1}^{K}(x-a_k)^{N_k^*}$.

Proof. For a given $k \in \{1,\dots,K\}$, let $\psi_{i,\varepsilon}(x), \varepsilon > 0$ and $i = 1,\dots,N_k^*$ be the functions given by $\psi_{i,\varepsilon}(x) = \begin{cases} \frac{1}{(x-a_k)^i} & \text{if } |x-a_k| > \varepsilon, \\ 0 & \text{if } |x-a_k| \le \varepsilon. \end{cases}$

Using Lemma (6.1.2), write $\sum_{v=0}^{2N} A_v T_v = \prod_{k=1}^{K}(x-a_k)^{N_k^*}T_N(x)$ where $R_N(x)$ polynomial of degree N . From the boundedness of $\psi_{i,\varepsilon}(x)$ we get

$$\lim_{\varepsilon\to 0}\lim_{n\to\Lambda}\int_{-1}^{1} W_N(x)\widehat{B}_n(x)p_{n+N}(x)\psi_{i,\varepsilon}(x)d\mu(x) = \lim_{\varepsilon\to 0}\int_{-1}^{1}\sum_{v=0}^{2N} A_v T_v(x)\psi_{i,\varepsilon}(x)\frac{dx}{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1}\frac{\prod_{j=1}^{K}(x-a_j))^{N_j^*}}{(x-a_k)^i}R_N(x)\frac{dx}{\sqrt{1-x^2}} \tag{3}$$

Because $\prod_{j=1}^{K}\left(x-a_j)\right)^{N_j^*}T_N(x)$ are bounded and as a consequence of the Lebesgue dominated convergence Theorem. Moreover

$$\left|\int_{-1}^{1}\widehat{B}_n(x)p_{n+N}(x)\frac{W_N(x)}{(x-a_k)^i}d\mu(x) - \frac{1}{\pi}\int_{-1}^{1}\sum_{v=0}^{2N}A_vT_v(x)\frac{1}{(x-a_k)^i}\frac{dx}{\sqrt{1-x^2}}\right|$$

$$\le \left|\int_{-1}^{1}\widehat{B}_n(x)p_{n+N}(x)\frac{W_N(x)}{(x-a_k)^i}d\mu(x) - \int_{-1}^{1}\widehat{B}_n(x)p_{n+N}(x)W_N(x)\psi_{i,\varepsilon}(x)d\mu(x)\right|$$

$$+\left|\int_{-1}^{1}\widehat{B}_n(x)p_{n+N}(x)W_N(x)\psi_{i,\varepsilon}(x)d\mu(x) - \frac{1}{\pi}\int_{-1}^{1}\sum_{v=0}^{2N}A_vT_v(x)\frac{1}{(x-a_k)^i}\frac{dx}{\sqrt{1-x^2}}\right| = I_{n,\varepsilon}^{(1)} + I_{n,\varepsilon}^{(2)}.$$

Given > 0 , from (3), $\lim_{n\to\Lambda} I_{n,\varepsilon}^{(2)} < \delta$ for $\varepsilon > 0$ small enough. On the other hand, $I_{n,\varepsilon}^{(1)} = \left|\int_{-1}^{1}\widehat{B}_n(x)p_{n+N}(x)\frac{W_N(x)}{(x-a_k)^i}d\mu(x)\right|$and, since there is a constant C , independent from e and i; such that $\left|\frac{W_N(x)}{(x-a_k)^i}\right| \le C$,from the Cauchy–Schwarz inequality, $I_{n,\varepsilon}^{(1)} \le C\left(\int_{a_k-\varepsilon}^{a_k+\varepsilon}\widehat{B}_n(x)p_{n+N}^2(x)d\mu(x)\right)^{1/2}$.

But $p_{n+N}^2(x)d\mu(x) \overset{*}{\to} \frac{1}{\pi}\frac{dx}{\sqrt{1-x^2}}$ and it means that, for ε small enough, $\lim\ \sup_{n\to\infty} I_{n,\varepsilon}^{(1)} < \delta$.

As a consequence,

$$\limsup_{n\in\Lambda}\left|\int_{-1}^{1}\widehat{B}_n(x)p_{n+N}(x)\frac{W_N(x)}{(x-a_k)^i}d\mu(x) - \frac{1}{\pi}\int_{-1}^{1}\sum_{v=0}^{2N}A_vT_v(x)\frac{1}{(x-a_k)^i}\frac{dx}{\sqrt{1-x^2}}\right| < 2\delta \text{for } \delta > 0.$$

By orthogonality, $\int_{-1}^{1} \widehat{B}_n(x) p_{n+N}(x) \frac{W_N(x)}{(x-a_k)^i} d\mu(x) = 0$, $i = 1, \dots, N_k^*$, $k = 1, \dots, K$ and $R_N(x)$ satisfies

Since $\int_{-1}^{1} \prod_{k=1}^{K} (x - a_k)^{N_k^*} \frac{1}{(x-a_k)^i} R_N(x) \frac{dx}{\sqrt{1-x^2}} = 0$, $i = 1, \dots, N_k^*$, $k = 1, \dots, K$

Since $\left\{\frac{1}{(x-a_k)^i} \prod_{k=1}^{K} (x - a_k)^{N_k^*}\right\} : i = 1, \dots, N_k^*$, $k = 1, \dots, K$ is a basis of the space of polynomials of degree less than or equal to $N - 1$, $R_N(x)$ is the Chebyshev polynomial of the first kind and degree N up to a constant factor. If we compare the leading coefficients,

$\sum_{v=0}^{2N} \frac{A_v}{A_{2N}} T_v(x) = 2^N T_N(x) \prod_{k=1}^{K} (x - a_k)^{N_k^*}$ and the proof is complete.

Lemma (6..1.4)[205]. If $\mu' > 0$ a.e., the coefficients $A_{n,v}$ satisfy

(i) $\lim_{n\to\infty} A_{n,v} = A_v$, $v = 0, \dots, 2N$ where $\sum_{v=0}^{2N} A_v T_v = \prod_{k=1}^{K} (x - ak))^{N_k^*} T_N(x)$.

(ii) $\sum_{v=0}^{2N} A_v (\varphi^-(x))^v = \frac{1}{2^N} \prod_{k=1}^{K} ((\varphi^-(x))^2 - 2a_k \varphi^-(x) + 1)^{N_k^*}$.

Proof. From Lemma (6.1.1) and the ratio a symptotics of p_n with $\mu'(x) > 0$ a.e., we get

$$\lim_{n\in\Lambda} \frac{W_N \widehat{B}_n(x)}{p_{n+N}(x)} = \lim_{n\in\Lambda} \sum_{v=0}^{2N} A_{n,v} \frac{p_{n+N-v}(x)}{p_{n+N}(x)} = \sum_{v=0}^{2N} A_v [\varphi^-(x)]^v$$

uniformly in compact sets of $\mathbb{C}/[-1,1]$. Denoting again $\prod_{2N}(x) = \sum_{v=0}^{2N} \frac{A_v}{A_N} T_v(x)$, since

$$\sum_{v=0}^{2N} \frac{A_v}{A_N} (\varphi^-(x))^v = \frac{\sqrt{x^2-1}}{\pi} \int_{-1}^{1} \frac{\prod_{2N}(t)}{x-t} \frac{dt}{\sqrt{1-t^2}}$$

as it can be deduced from the residue theorem after the change $t = \cos\theta$ the expression of $\prod_{2N}$.

Lemma (6.1.3) gives

$$\sum_{v=0}^{2N} \frac{A_v}{A_N} (\varphi^-(x))^v = \frac{-\sqrt{x^2-1}}{2\pi i} \int_{|\xi|=1} \frac{(1+\xi^{-2N}) \prod_{k=1}^{K} (\xi^2 - 2a_k\xi + 1)^{N_k^*}}{(\xi - \varphi^-(x))(\xi - \varphi^+(x))} d\xi$$

$$= \frac{-\sqrt{x^2-1}}{\pi i} \int_{|\xi|=1} \frac{(1+\xi^{-2N}) \prod_{k=1}^{K} (\xi^2 - 2a_k\xi + 1)^{N_k^*}}{(\xi - \varphi^-(x))(\xi - \varphi^+(x))} d\xi = \prod_{k=1}^{K} ((\varphi^-(x))^2 - 2a_k \varphi^-(x) + 1)^{N_k^*}.$$

In particular, this means that $\frac{A_0}{A_{2N}} = \lim_{n\to\infty} \sum_{v=0}^{2N} \frac{A_v}{A_N} (\varphi^-(x))^v = 1$; but, from Lemma (6.1.1) , $A_{2N} = \lim_{n\to\infty} \frac{k(p_{n-N})}{k(p_{n+N})} \frac{1}{A_0} = \frac{1}{2^{2N} A_0}$ and $A_0 = A_{2N} = \frac{1}{2^N}$ follows.

Now the coefficients A_j are completely determined for any subsequence Λ and we can assert that $\lim_{n\to\infty} A_{n,v} = A_v$, $v = 0, \dots, 2N$ with $\sum_{v=0}^{2N} A_v T_v = \prod_{k=1}^{K} (x - a_k)^{N_k^*} T_N(x)$

$$\sum_{v=0}^{2N} A_v (\varphi^-(x))^v = \frac{1}{2^N} \prod_{k=1}^{K} ((\varphi^-(x))^2 - 2a_k \varphi^-(x) + 1)^{N_k^*}.$$

Theorem (6.1.5)[205]. If $\mu'(x) > 0$ a.e. then

(i) $\lim_{n\to\infty} \frac{\widehat{B}_n(x)}{p_n(x)} = 1$ uniformly on compact subsets of $\mathbb{C}/[-1,1]$.

(ii) $n - N$ zeros of $\widehat{B}_n(x)$ belong to $[-1,1]$ and the other N zeros accumulate in [-1,1].

(iii) $\lim_{n\to\infty} \frac{\widehat{B}_{n+1}(x)}{\widehat{B}_n(x)} = x + \sqrt{x^2 - 1}$ uniformly on compact subsets of $\mathbb{C}/[-1,1]$.

(iv) If $\int_{-1}^{1} \log \mu'(x) \frac{dx}{\sqrt{1-x^2}} > -\infty$ then $\lim_{n\to\infty} \frac{\widehat{B}_n(x)}{(x+\sqrt{x^2-1})^n} = S_{\mu'}(x)$

uniformly on compact subsets of $\mathbb{C}/[-1,1]$. Here $S_{\mu'}(x)$ denotes the Szegö function of $\mu'(x)$.see [207]

Proof. Item (ii) follows from $\int_{-1}^{1} x^k \widehat{B}_n(x) W_n(x) d\mu(x) = 0$ for $k + N < n$ and formula (i). Items (iii) and (iv) are consequences of (i) and the well-known ratio and strong asymptotes of p_n So, we only needto prove (i). From Lemma (6.1.4) we have

$\lim_{n\to\infty} W_N(x)\frac{\widehat{B}_n(x)}{p_{n+N}(x)} = \sum_{v=0}^{2N} A_j(\varphi^-(x))^j = \frac{1}{2^N}\prod_{k=1}^{K}((\varphi^-(x))^2 - 2a_k\varphi^-(x) + 1)^{N_k^*}$, which yields

$\lim_{n\to\infty} W_N(x)\frac{\widehat{B}_n(x)}{p_n(x)} = (\varphi^+(x))^N \frac{1}{2^N}\prod_{k=1}^{K}((\varphi^-(x))^2 - 2a_k\varphi^-(x) + 1)^{N_k^*} = W_N(x)$.

Lemma (6.1.6)[205]. With the previous notation, if $\mu'(x) > 0$ a.e., we get

$$w_N^2(x) = \sum_{j=0}^{2N} A_j^2 T_0(x) + \sum_{j=0}^{2N} A_j A_{j+v} T_v(x)$$

Proof. From Lemma (6.1.5), $\sum_{j=0}^{2N} A_j\ T_j(x) = T_N(x)\prod_{k=0}^{K}(x-a_k)^{N_k^*}$

Besides, as it was proved, $A_{2N} = A_0$ and for $j = 1, \dots, N-1$ we get

$$\frac{1}{2}A_{N+j} = \frac{1}{\pi}\int_{-1}^{1}\prod_{k=1}^{K}(x-a_k)^{N_k^*}T_N(x)T_{N+j}(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{2\pi}\int_{-1}^{1}\prod_{k=1}^{K}(x-a_k)^{N_k^*}T_{2N+j}(x)T_{N+j}(x)\frac{dx}{\sqrt{1-x^2}} \text{ and,}$$
$$\frac{1}{2}A_{N-j} = \frac{1}{\pi}\int_{-1}^{1}\prod_{k=1}^{K}(x-a_k)^{N_k^*}T_N(x)T_{N-j}(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{2\pi}\int_{-1}^{1}\prod_{k=1}^{K}(x-a_k)^{N_k^*}(T_{2N-j}(x) + T_j(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{2\pi}\int_{-1}^{1}\prod_{k=1}^{K}(x-a_k)^{N_k^*}T_j(x)\frac{dx}{\sqrt{1-x^2}}$$

which yields $A_{N+j} = A_{N-j}$ for $j = 1, \dots, N-1$.

As a consequence

$$T_N(x)\prod_{k=1}^{K}(x-a_k)^{N_k^*} = A_N T_N(x) + \sum_{j=1}^{N} A_{N+j}(T_{N+j}(x)T_{N-j}(x))$$

And thus $\prod_{k=1}^{K}(x-a_k)^{N_k^*} = A_N T_0(x) + 2\sum_{j=1}^{N} A_{N+j}T_j(x)$.

Now, if we work out the coefficients of $w_N^2(x) = \left(A_N T_0(x) + 2\sum_{j=1}^{N} A_{N+j}T_j(x)\right)^2$ in terms of the polynomials $(T_v)_{v=0}^{2N}$ the statement of the Lemma follows.

Lemma (6.1.7)[205]. If $\mu'(x) > 0$ a.e. and f is a Borel measurable function bounded on [-1,1] then

$$\lim_{n\to\infty}\int_{-1}^{1} f(x)W_N^2(x)\widehat{B}_n(x)\widehat{B}_{n+k}(x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} f(x)W_N^2(x)\ T_k(x)\frac{dx}{\sqrt{1-x^2}},\ k = 0,1,\dots$$

Proof. Let f be a Borel measurable function bounded on [-1,1] Writing the polynomials $W_n(x)\widehat{B}_n$ in terms of $(p_n)_{n=0}^{\infty}$ as in Lemma (6.1.1), from the a symptotics of the polynomials p_n we get

$$\lim_{n\to\infty}\int_{-1}^{1} f(x)W_N(x)\widehat{B}_n(x)W_N(x)\widehat{B}_{n+N}(x)d\mu(x)$$
$$= \lim_{n\to\infty}\int_{-1}^{1} f(x)\sum_{j=0}^{2N} A_{n,j}p_{n+N-j}(x)\sum_{j=0}^{2N} A_{n+k,v}p_{n+k+N-v}(x)\,d\mu(x)$$
$$= \lim_{n\to\infty}\int_{-1}^{1} f(x)\left(\sum_{j=v} + \sum_{j>v} + \sum_{j<v}\right)(A_{n,j}A_{n+k,v}p_{n+N-j}(x))d\mu(x)$$
$$= \frac{1}{\pi}\int_{-1}^{1} f(x)\left\{\sum_{j=0}^{2N} A_j^2 T_k(x) + \sum_{j>v} A_j A_v(T_{k+j-v}(x) + T_{k-(j-v)}(x)\right\}\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{\pi}\int_{-1}^{1} f(x)\left\{\sum_{j=0}^{2N} A_j^2 + 2\sum_{j>v} A_j A_v T_{j-v}(x)\right\}T_k(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{\pi}\int_{-1}^{1} f(x)\left\{\sum_{j=0}^{2N} A_j^2 + 2\sum_{v=1}^{2N}\sum_{j=0}^{2N-v} A_j A_{j+v}T_v(x)\right\}T_k(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{\pi}\int_{-1}^{1} f(x)W_N^2 T_k(x)\frac{dx}{\sqrt{1-x^2}} \text{ according to Lemma (6.1.6)}$$

Lemma (6.1.8)[205]. If $\mu'(x) > 0$ a.e. then $\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{a_k-\varepsilon}^{a_k+\varepsilon}\widehat{B}_n^2(x)d\mu(x) = 0$, $k = 1, \dots, K$.

Proof . Denoting by $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$ the Sobolev norm, for $k = 1, \dots, K$ and $i = 0, \dots, N_k$, we have

$$M_{k,i} \le \inf\{\|\pi_n\|^2 : \deg\pi_n \le n,\ \pi^{(i)}(a_k) = 1\} = \frac{1}{\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2}$$

because $1=\sum_{v=0}^{n} C_v \widehat{B}_v^{(i)}(a_k) \le C_v^2 \sum_{v=0}^{n}\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2$ and $\left\|\frac{\sum_{v=0}^{n}\widehat{B}_v^{(i)}(a_k)\widehat{B}_v^{(i)}(x)}{\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2}\right\|^2=\frac{1}{\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2}$.

Then, for any (k,i) such that $M_{k,i}>0$, $\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2 \le \frac{1}{M_{k,i}}$ and, in particular, $\widehat{B}_n^{(i)}(a_k)\to 0$ for $0\le k\le K$ provided that the corresponding coefficient satisfies the condition $M_{k,i}>0$. As a consequence,

$$\lim_{\varepsilon\to 0}\int_{-1}^{1}\widehat{B}_n^2(x)d\mu(x)=1 \tag{4}$$

For $\varepsilon>0$ let ψ_ε be the function defined by

$$\psi_\varepsilon=\begin{cases}\frac{1}{w_N(x)} & \text{if } x\in[-1,1]\setminus\bigcup_{k=1}^{K}[a_k-\varepsilon,a_k+\varepsilon],\\ 0 & \text{if } \bigcup_{k=1}^{K}[a_k-\varepsilon,a_k+\varepsilon].\end{cases}$$

Then, using Eq. (4), Lemma (6.1.8) and dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{a_k-\varepsilon}^{a_k+\varepsilon}\widehat{B}_n^2(x)d\mu(x)=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{-1}^{1}\left(1-\psi_\varepsilon^2(x)w_N^2(x)\right)\widehat{B}_n^2(x)d\mu(x)$$
$$=\lim_{\varepsilon\to 0}\left(1-\frac{1}{\pi}\int_{-1}^{1}\psi_\varepsilon^2(x)w_N^2(x)\frac{1}{\sqrt{1-x^2}}\right)=0,$$

which gives the lemma.

Now we can prove the weak convergence for the Sobolev orthonormal polynomials.

Theorem (6.1.9)[205]. If $\mu'>0$ a.e. and f is a Borel measurable function bounded on [-1,1] then $\lim_{n\to\infty}\int_{-1}^{1}f(x)\widehat{B}_n(x)\widehat{B}_{n+k}(x)d\mu(x)=\frac{1}{\pi}\int_{-1}^{1}f(x)T_k(x)\frac{dx}{\sqrt{1-x^2}}$, $k=0,1,\ldots$.

Proof. For $\varepsilon>0$, let ψ_ε be the function defined in the previous lemma. Let f be a Borel measurable function bounded on [-1,1] Since $f(x)\psi_\varepsilon^2(x)$ is also bounded, according to Lemma (6.1.7),

$$\lim_{n\to\infty}\int_{-1}^{1}f(x)\psi_\varepsilon^2(x)W_N^2(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)=\frac{1}{\pi}\int_{-1}^{1}f(x)\psi_\varepsilon^2(x)W_N^2(x)\,T_v(x)\frac{dx}{\sqrt{1-x^2}}$$

and, by the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{-1}^{1}f(x)\psi_\varepsilon^2(x)W_N^2(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)=\frac{1}{\pi}\int_{-1}^{1}f(x)\,T_v(x)\frac{dx}{\sqrt{1-x^2}} \tag{5}$$

Moreover

$$\left|\int_{-1}^{1}f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)=\frac{1}{\pi}\int_{-1}^{1}f(x)\,T_v(x)\frac{dx}{\sqrt{1-x^2}}\right|$$
$$\le\left|\int_{-1}^{1}f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)-\int_{-1}^{1}f(x)\psi_\varepsilon^2(x)W_N^2(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)\right|$$
$$+\left|\int_{-1}^{1}f(x)\psi_\varepsilon^2(x)W_N^2(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)-\frac{1}{\pi}\int_{-1}^{1}f(x)T_v(x)\frac{dx}{\sqrt{1-x^2}}\right|$$
$$=I_{n,\varepsilon}^{(1)}+I_{n,\varepsilon}^{(2)}.$$

Given $\delta>0$ from (5), $\lim_{n\to\infty}I_{n,\varepsilon}^{(2)}<\delta$ for $\varepsilon>0$ small enough. On the other hand,

$$I_{n,\varepsilon}^{(1)}\le\sum_{K=1}^{K}\left|\int_{a_k-\varepsilon}^{a_k+\varepsilon}f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)\right|.$$

Since f is bounded on $[-1,1]$ there exists a constant C such that $|f(x)|\le C$, $x\in[-1,1]$ and we get

$$I_{n,\varepsilon}^{(1)}\le\sum_{K=1}^{K}\left|\int_{a_k-\varepsilon}^{a_k+\varepsilon}f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x)\right|\le C\sum_{K=1}^{K}\left(\int_{a_k-\varepsilon}^{a_k+\varepsilon}\widehat{B}_n^2(x)d\mu(x)\right)^{\frac{1}{2}}$$

By Lemma (6.1.8) $\lim_{n\to\infty} \sup I_{n,\varepsilon}^{(1)} < \delta$ for ε small enough then

$$\lim\ \sup_{n\to\infty} \left| \int_{-1}^{1} f(x)\widehat{B}_n(x)\widehat{B}_{n+v}(x)d\mu(x) - \int_{-1}^{1} f(x)T_v(x)\frac{dx}{\sqrt{1-x^2}} \right| < 2\delta$$

And the proof is complete.

Theorem (6.1.10)[205]. The polynomials $\widehat{B}_n$ satisfy the recurrence relation

$$W_N(x)\widehat{B}_n(x) = \sum_{j=1}^{N} \alpha_{n,j}\widehat{B}_{n+j}(x)\ ,\ \alpha_{n,-j} = \alpha_{n-j}\ , j = 1,2,\dots,N\ , \alpha_{n,N} \neq 0.$$

Furthermore if $\mu'(x) > 0$ a.e . then $\lim_{n\to\infty} \alpha_{n,j} = \alpha_j\ , j = 0,\dots,N$ where $W_N(x) = \alpha_0 + 2\sum_{j=1}^{N} \alpha_j\widehat{B}_j(x)$. and are given by $\alpha_j = \frac{1}{2^N}\frac{W_{2N}^{(N-1)}(0)}{(N-j)!}\ , j = 0\ ,\dots, N$ with $W_{2N}(\xi) = \prod_{k=1}^{K}(\xi^2 - 2a_k\xi + 1)^{N_k^*}$.

Proof. We can write $W_N(x)\widehat{B}_n(x) = \sum_{j=0}^{n+N} \lambda_{n,j}\widehat{B}_j(x)$ where

$\lambda_{n,j} = \langle W_N\widehat{B}_n\ ,\widehat{B}_j\rangle = \int_{-1}^{1} W_N(x)\widehat{B}_n(x)\widehat{B}_j(x)d\mu(x) = \langle\widehat{B}_n\ , W_N\widehat{B}_j\rangle = 0$ for $j < n - N$.

Thus we get the recurrence relation with $\alpha_{n,j} = \lambda_{n,n+j}\ , j = -N,\dots,N$. Moreover, for

$$j = 1\ ,\dots, N\ , \alpha_{n,-j} = \langle W_N\widehat{B}_n\ ,\widehat{B}_{n-j}\rangle = \langle W_N\widehat{B}_{n-j}\ ,\widehat{B}_n\rangle = \alpha_{n-j,j}\ .$$

On the other hand, if $\mu'(x) > 0$ a.e., for $j = 0\ ,\dots, N$, from Theorem (6.1.10)

$$\lim_{n\to\infty} \alpha_{n,j} = \lim_{n\to\infty} \int_{-1}^{1} W_N(x)\widehat{B}_n(x)\widehat{B}_{n+j}(x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} f(x)T_v(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{\pi i}\int_{|\xi|=1} \prod_{k=1}^{K}(\xi^2 - 2a_k\xi + 1)^{N_k^*} = \frac{1}{2^N(N-1)!}W_{2N}^{(N-1)}(0).$$

In terms of linear operator theory, the recurrence relation may be more useful in the form given in the following theorem.

Theorem (6.1.11)[205]. If $\mu'(x) > 0$ a.e., the Sobolev polynomials satisfy the recurrence relation $x\widehat{B}_n(x) = h_n\widehat{B}_{n+1}(x) + v_n\widehat{B}_n(x) + h_{n-1}\widehat{B}_{n-1}(x) + F_n(x)$, where h_n and v_n are the coefficients of the recurrence relation $xp_n(x) = h_np_{n+1}(x) + v_np_n(x) + h_{n-1}p_{n-1}(x)$ and $F_n(x)$ are functions such that $\lim_{n\to\infty}\frac{F_n(x)}{\widehat{B}_n(x)} = 0$ uniformly on compact subsets of $\mathbb{C}/[-1\ ,\ 1]$.

Proof. From Lemma (6.1.1),

$$\begin{aligned} Xw_N\widehat{B}_n(x) &= \sum_{j=0}^{2N} A_{n,j}xp_{n+N-j}(x) \\ &= \sum_{j=0}^{2N} A_{n,j}(h_{n+N-j}(x)p_{n+N-j}(x) + v_{n+N-j}(x)p_{n+N-j}(x) + h_{n+N-j}(x)p_{n+N-j}(x)) \\ &= w_N(h_n\widehat{B}_{n+1}(x) + v_n\widehat{B}_n(x) + h_{n-1}\widehat{B}_{n-1}(x)) + \sum_{j=0}^{2N}(A_{n,j}h_{n+N-j} - h_nA_{n+1})p_{n+N-j}(x) \\ &\quad + \sum_{j=0}^{2N} A_{n,j}(v_{n+N-j} - v_n)A_{n+1})p_{n+N-j}(x) + \sum_{j=0}^{2N}(A_{n,j}h_{n+N-j} - h_nA_{n+1})p_{n+N-j}(x). \end{aligned}$$

And the lemma follows from Lemma (6.1.4), Theorem (6.1.5) (i) and the a symptotics of the sequence $(p(x))_{n=0}^{\infty}$. In this section we are focused on the study of the point wise convergence of the Fourier series expansions in terms of the sequence of polynomials $\left(\widehat{B}_n\right)_{n=0}^{\infty}$ orthonormal with respect to the inner product (1) provided that μ is the Jacobi measure. In order to do this we need some previous results and, in what follows, we will denote by $\|f\| = \langle f\ , f\rangle^{\frac{1}{2}}$ the Sobolev norm of a function f .

Lemma (6.1.12)[205]. Given a positive Borel measure m supported on $[-1\ ,1]$ with infinitely many points at the support, the polynomials $\widehat{B}_n(x)$ satisfy

(i) if $M_{k,i} > 0$ *then* $\sum_{n=0}^{\infty}\left(\widehat{B}_n^{(i)}(a_k)\right)^2 = \frac{1}{M_{k,j}}$.

(ii) if $M_{k,i} > 0$ *then* $\sum_{n=0}^{\infty}\widehat{B}_n^{(i)}(a_k)\widehat{B}_n^{(j)}(a_t) = 0$ for $(\ t\ , g\ ) \neq (\ k\ , i\ )$ such that $M_{k,i} > 0$.

(iii)if $M_{k,i} > 0$ *then* $\lim_{n\to\infty}\int_{-1}^{1}\sum_{v=0}^{n}\left(\widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)\right)^2 d\mu = 0$

Proof. For $i = 0, \dots, N_k$ and $k = 1, \dots, K$, let $\ell_{n,k}^{(i)} = \inf\{\|\pi_n\|^2 : \deg \pi_n \le n,\ \pi^{(i)}(a_k) = 1\}$.

It is clear that for all $M_{k,i} \le \ell_{n,k}^{(i)}$ and, as it was proved at the beginning of the proof of Lemma (6.1.8). $\ell_{n,k}^{(i)} = \frac{1}{\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2}$ let $N^* = \sum_{k=0}^{K}(N_k + 1)$ and introduce the function

$$\varphi(x) = \begin{cases} \exp\left\{\frac{x^{2N^*}}{1-x^2}\right\}, & |x| < 1, \\ 0, & |x| \ge 1. \end{cases}$$ then $\varphi \in \ell^{\infty}(\mathbb{R}), \varphi(0) = 1, |\varphi(x)| \le 1$ for every $x \in \mathbb{R}$ and $\varphi(0) = 0$

for $i = 1, \dots, N_k, k = 1, \dots, K$. For a fixed $k \in \{1, \dots, K\}$ and $\varepsilon > 0$ such that,

$a_t \notin [a_k - \varepsilon, a_k + \varepsilon]$ for $t \ne k$, let us consider the function $\varphi_{k,\varepsilon} = \varphi\left(\frac{x-a_k}{\varepsilon}\right)$,

for $i \in \{0,1, \dots, N_k\}$ let $w_{k,i}(x) = \left(\frac{x-a_k}{i!}\right)^i$ and consider a polynomial $\prod(x)$ such that $\prod^{(i)}(a_k) = 1$ and satisfies $\max_{x\in[-1,1]}\left|\prod^{(j)}(x) - \left(w_{k,i}\varphi_{k,\varepsilon}\right)^{(j)}(x)\right| < \varepsilon$, $j = 0,1, \dots, N^*$.

Since $\left(w_{k,i}\varphi_{k,\varepsilon}\right)^{(j)}(a_t) = 0$ for $t \ne k$ and $= 0,1, \dots, N_t$, when $0 \le j \le N_k$ we have

$$\|\Pi\| \le \|w_{k,i}(x)\varphi_{k,\varepsilon}(x)\| + \|\Pi(x) - w_{k,i}(x)\varphi_{k,\varepsilon}(x)\|$$

$$\le \left\{\mu([a_k - \varepsilon, a_k + \varepsilon])\max_{x\in[-1,1]} w_{k,i}^2(x) + M_{k,i} + \varepsilon^2\right\}^{\frac{1}{2}} = \left(M_{k,i} + h(\varepsilon)\right)^{\frac{1}{2}}$$

where $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$ because $\mu(\{a_k\}) = 0$ a consequence,

$$M_{k,i} \le \lim_{n\to\infty}\ell_{n,k}^{(i)} = \frac{1}{\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2} \le M_{k,i} + h(\varepsilon)$$ and thus $M_{k,i} = \frac{1}{\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2}$, moreover,

for (f, i) such that $M_{k,i} > 0$

$$\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2 = \left\|\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2\widehat{B}_v(x)\right\|^2$$

$$= \int_{-1}^{1}\left(\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2\widehat{B}_v(x)\right)^2 d\mu(x) + M_{k,i}\left\{\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2\right\}^2$$

$$+\sum_{j\ne i}M_{k,j}\left(\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2\widehat{B}_v^{(j)}(a_k)\right)^2 + \sum_{t\ne k}\sum_{j=0}M_{t,j}\left(\sum_{v=0}^{\infty}\left(\widehat{B}_v^{(i)}(a_k)\right)^2\widehat{B}_v^{(j)}(a_t)\right)^2.$$

Multiplying this equality by $\ell_{n,k}^{(i)}$ and taking limit when $n \to \infty$ (ii) and (iii) follows from (i) and the proof is complete.

Corollary (6.1.13)[205]. Let μ be a positive Borel measure supported on $[-1,1]$ with infinitely many points at the support and let f be a function of $L^2(\mu)$ such that there exist the derivatives $f^{(i)}(a_k)$ for $i = 0,1, \dots, N_k$ and $k = 1, \dots, K$. If $M_{k,i} > 0$, then $\sum_{n=0}^{\infty}\langle f, \widehat{B}_n\rangle\widehat{B}_n^{(i)}(a_k) = f^{(i)}(a_k)$ $i = 0,1, \dots, N_k$, $K = 1, \dots, K$

Proof.

$$\sum_{v=0}^{n}\langle f, \widehat{B}_v\rangle\widehat{B}_v^{(i)}(a_k) = \int_{-1}^{1} f(x)\sum_{v=0}^{n}\widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)d\mu(x)$$

$$+M_{k,i}f^{(i)}(a_k)\sum_{v=0}^{n}\left(\widehat{B}_v^{(i)}(a_k)\right)^2 + \sum_{j\ne i}M_{k,i}f^{(i)}(a_k)\sum_{v=0}^{n}\widehat{B}_v^{(i)}(a_k)\widehat{B}_v^{(j)}(a_k)$$

$$+\sum_{t\ne k}\sum_{j\ne i}M_{k,i}f^{(i)}(a_t)\sum_{v=0}^{n}\widehat{B}_v^{(i)}(a_k)\widehat{B}_v^{(j)}(a_t).$$

Since $\left|\int_{-1}^{1} f(x)\sum_{v=0}^{n}\widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)d\mu(x)\right| \le \left(\int_{-1}^{1} f^2 d\mu(x)\right)^{\frac{1}{2}}\left\{\int_{-1}^{1}\left(\sum_{v=0}^{n}\widehat{B}_v^{(i)}(a_k)\widehat{B}_v(x)\right)^2 d\mu(x)\right\}^{\frac{1}{2}}$
taking limit in n the statement follows from Lemma (6.1.12).
So, we have convergence at the mass points for any function belonging to $L^2(\mu)$
and with derivatives at such points. But for the convergence at other points, more
conditions are needed and, in order to study this problem, we start with some
straight forward estimates for the polynomials $\widehat{B}_n$.

Lemma (6.1.14)[205]. Let $(p_n)_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the measure . Then there exists a positive constant C such that

$$|w_N(x)\widehat{B}_n(x)| \le \sum_{j=-N}^{N}|p_{n+j}(x)| \quad \text{for every} \quad x \in \mathbb{R}$$

This lemma is an obvious consequence of Lemma (6.1.1).
When $d_\mu(x) = (1-x)^\alpha(1+x)^\beta dx\,,\alpha > -1\,,\beta > -1$, i.e. the Jacobi measure, as it
is well known (see [208]), the orthonormal polynomials $p_n$satisfy

$$(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}|p_n(x)| \le C\,, \qquad \alpha > -\frac{1}{2}\ ,\beta > -\frac{1}{2}\,, \tag{6}$$

$|p_n(x)| \le C\,, -1 < \alpha \le -\frac{1}{2}$, $-1 < \beta \le -\frac{1}{2}$, for $x \in [-1,1]$ and, as a consequence of the previous lemma, the corresponding Jacobi–Sobolev polynomials $\widehat{B}_n$ satisfy the condition

$$|\widehat{B}_n(x)| \le Ch(x) \tag{7}$$

for $x \in [-1,1]\setminus \bigcup_{k=1}^{K}\{a_k\}$, and for all n; where h(x) is the function which depends on α and β deduced from (6) and Lemma (6.1.14). Lemma (6.1.12) gives some properties of the Dirichlet kernels $\sum_{v=0}^{n}\widehat{B}_v(x)\widehat{B}_v(t)$ and, as it was proved in [279] for the case $|a_k| > 1$ they satisfy a Christoffel-Darboux formula deduced from the recurrence relation. If $x_0 \in [-1,1]$ the polynomial $w_N(x) - w_N(x_0)$ may have more than one zero at $[-1,1]$ and this is not convenient for the representation of the Dirichlet kernel. Instead of $w_N(x)$ we will consider the polynomial $w_{N+1}(x) = \int_{-1}^{x} w_N(t)dt$ and, from the positivity of $w_N(x)$ when $x_0 \ne a_k\,, k = 1,\dots,K, x_0$ is the only zero of $w_{N+1}(x) - w_{N+1}(x_0)$ in $[-1,1]$ Because the derivatives of $w_{N+1}(x)$ vanish at the $a_k'S$ we have
$\langle W_{N+1}\widehat{B}_n\,,\widehat{B}_m\rangle = \langle \widehat{B}_n\,,W_{N+1}\widehat{B}_m\rangle$ and this means that the Sobolev polynomials $\widehat{B}_n$
satisfy the recurrence relation

$$W_{N+1}(x)\widehat{B}_n(x) = \sum_{v=0}^{N+1}\alpha_{n,v}(x)\widehat{B}_{n+v}(x) + \sum_{v=0}^{N+1}\alpha_{n-v}(x)\widehat{B}_{n-v}(x) \tag{8}$$

Moreover, the coefficients $\alpha_{n,v}$ are bounded because

$$\begin{aligned}|\alpha_{n,v}| = |\langle W_{N+1}\widehat{B}_n\,,\widehat{B}_{n+v}\rangle| &\le \left|\int_{-1}^{1}\widehat{B}_n(x),\widehat{B}_{n+v}(x)\,w_{N+1}(x)d\mu(x)\right| \\ &\quad + \sum_{k=1}^{K}\sum_{i=0}^{N_k} M_{k,i}|w_{N+1}(a_k)|\left|\widehat{B}_n^{(i)}(a_k)\widehat{B}_{n+v}^{(i)}(a_k)\right| \\ &\le \max_{x\in[-1,1]}|w_{N+1}(x)|\left(1+\sum_{k=1}^{K}\sum_{i=0}^{N_k} M_{k,i}\left|\widehat{B}_n^{(i)}(a_k)\widehat{B}_{n+v}^{(i)}(a_k)\right|\right),\end{aligned}$$

and, from Lemma (6.1.12), $\widehat{B}_n^{(i)}(a_k)$ are bounded when $M_{k,i} > 0$.
Christoffel-Darboux formula now takes the following form.

Lemma (6.1.15)[205]. The orthonormal polynomials with respect to the inner product (1) satisfy the following Christoffel–Darboux type formula:

$$\begin{aligned}\{w_{N+1}(x)-w_{N+1}(y)\}\sum_{n=0}^{v}\widehat{B}_n(x)\widehat{B}_n(y) &= \alpha_{v,1}(\widehat{B}_{v+1}(x)\widehat{B}_v(y))-\widehat{B}_{v+1}(y)\widehat{B}_v(x))\\ &+\alpha_{v,2}(\widehat{B}_{v+2}(x)\widehat{B}_v(y))-\widehat{B}_{v+2}(y)\widehat{B}_v(x))\\ &+\alpha_{v-1,2}(\widehat{B}_{v+1}(x)\widehat{B}_{v-1}(y))-\widehat{B}_{v+1}(y)\widehat{B}_{v-1}(x))\\ &+\cdots+\alpha_{v,N+1}(\widehat{B}_{v+N+1}(x)\widehat{B}_v(y))-\widehat{B}_{v+N+1}(y)\widehat{B}_v(x))\\ &+\cdots+\alpha_{v-N,N+1}(\widehat{B}_{v+1}(x)\widehat{B}_{v-N}(y))-\widehat{B}_{v+1}(y)\widehat{B}_{v-N}(x))\end{aligned}$$

with bounded coefficients.

Corollary (6.1.16)[205]. Let $x\in(-1,1)\setminus\cup_{k=1}^{K}\{a_k\}$ and μ the Jacobi measure. If $M_{k,i}>0$ then $\sum_{n=0}^{\infty}\widehat{B}_n^{(i)}(a_k)\widehat{B}_n(x)=0$

and the convergence is uniform in compact subsets of $(-1,1)\setminus\cup_{k=1}^{K}\{a_k\}$.

Proof. Let (k,i) be such that $M_{k,i}>0$ From the Christoffel–Darboux formula of Lemma (6.1.15)it is clear that $\sum_{v=0}^{n}\widehat{B}_v(a_k)\widehat{B}_v(x)$ is a sum of a finite-depending on N-number of terms of the following type: $\alpha_{n-v,j}\frac{\widehat{B}_{n-v+j}(x)\widehat{B}_{n-v}^{(i)}(a_k)}{w_{N+1}(x)-w_{N+1}(a_k)}$. Since the coefficients $\alpha_{n-v,j}$ are bounded, $|B_n(x)|\le h(x)$ with $h(x)$ a continuous function in compact subsets of $(-1,1)\setminus\cup_{k=1}^{K}\{a_k\}$ an $\lim_{n\to\infty}\widehat{B}_n^{(i)}(a_k)=0$ lemma is proved.

Theorem (6.1.17)[205]. Let $x_0\in[-1,1]\setminus\cup_{k=1}^{K}\{a_i\}$ and let f be a function with derivatives at the points a_k such that $\frac{f(x_0)-f(t)}{x_0-t}$ belongs to $L^2(\mu)$ when μ is the Jacobi measure. Then

(i) $\sum_{n=0}^{\infty}\langle f,\widehat{B}_n\rangle\widehat{B}_n(x_0)=f(x_0)$.

(ii)$M_{k,i}>0$ *then* $\sum_{n=0}^{\infty}\langle f,\widehat{B}_n\rangle\widehat{B}_n^{(i)}(a_k)=f^{(i)}(a_k)$.

Proof. Because of $f\in L^2(\mu)$ when $\frac{f(x_0)-f(t)}{x_0-t}\in L^2(\mu)$, Corollary (6.1.15) yields (ii).

Now, we denote by $S_n(x_0;f)$ the nth partial sum of the Fourier Sobolev expansion and by $D_n(x,t)$ the Dirichlet kernel $\sum_{v=0}^{n}\widehat{B}_v(x)\widehat{B}_v(t)$. Then

$$\begin{aligned}f(x_0)-S_n(x_0;f)&=\langle f(x_0)-f(t),D_n(x_0,t)\rangle\\ &=\int_{-1}^{1}f(x_0)-f(t)D_n(x_0,t)\,d\mu(t)+\sum_{k=1}^{K}M_{k,0}\big(f(x_0)-f(a_k)\big)D_n(x_0,a_k)\\ &\quad+\sum_{j=1}^{K}\sum_{i=1}^{N_k}M_{k,0}f^{(i)}(a_k)\frac{\partial^iD_n}{\partial t^i}(x_0,a_k)\end{aligned}$$

From Corollary (6.1.16) we get

$$\lim_{n\to\infty}f(x_0)-S_n(x_0;f)=\lim_{n\to\infty}\int_{-1}^{1}f(x_0)-f(t)D_n(x_0,t)\,d\mu(t)$$

Using the Christoffel-Darboux type formula, the above expression is the limit of a sum of a finite depending on N number of terms

$\int_{-1}^{1}f(x_0)-f(t)\alpha_{n-i,j}\frac{\widehat{B}_{n-i+j}(x)\widehat{B}_{n-i}(t)}{w_{N+1}(x_0)-w_{N+1}(t)}d\mu(t)$ but ,

$$\left|\int_{-1}^{1}f(x_0)-f(t)\alpha_{n-i,j}\frac{\widehat{B}_{n-i+j}(x)\widehat{B}_{n-i}(t)}{w_{N+1}(x_0)-w_{N+1}(t)}d\mu(t)\right|=\left|\alpha_{n-i,j}\right|\left|\widehat{B}_{n-i+j}(x_0)\right|\left|\int_{-1}^{1}\frac{f(x_0)-f(t)}{x_0-t}\frac{x_0-t}{w_{N+1}(x_0)-w_{N+1}(t)}\widehat{B}_{n-i}(t)\right|$$

where the coefficients $\left|\alpha_{n-i,j}\right|$ are bounded and $\left|\widehat{B}_{n-i+j}(x_0)\right|\le h(x_0)$ from Lemma (6.1.13) and the comments after the lemma.

Since the function $g_{x_0}(t)=\frac{f(x_0)-f(t)}{x_0-t}\frac{x_0-t}{w_{N+1}(x_0)-w_{N+1}(t)}$

belongs to $L^2(\mu)$ and there exist the derivatives $g_{x_0}^{(i)}(a_k)$ then

$\sum_{n=0}^{\infty}\langle g_{x_0},\widehat{B}_n\rangle = \sum_{n=0}^{\infty}\left(\int_{-1}^{1} g_{x_0}(t)\widehat{B}_n(t)d\mu(t) + \sum_{j=1}^{K}\sum_{i=0}^{N_k} M_{k,0}g_{x_0}^{(i)}(a_k)\widehat{B}_n^{(i)}(a_k)\right)^2 \le \|g_{x_0}\|^2$

and , as a consequence, $\lim_{n\to\infty}\langle g_{x_0},\widehat{B}_n\rangle = 0$. Taking into account that, when $M_{k,i} > 0$, $\lim_{n\to\infty}\widehat{B}_n^{(i)}(a_k) = 0$ we have $\lim_{n\to\infty}\int_{-1}^{1} g_{x_0}(t)\widehat{B}_n(t)d\mu(t) = 0$

This means that $\lim_{n\to\infty} f(x_0) - S_n(x_0; f) = 0$.

Section(6.2): Orthogonal Polynomials With a Non-Discrete Gegenbauer-Sobolev Inner Product:

Let $d_\mu = (1-x^2)^{\alpha-1/2}dx$ with $-1/2$, be the Gegenbauer measure supported on the interval $[-1,1]$. We shall say that $f \in L^p(d_\mu)$ if f is measurable on $[-1,1]$ and $\|f\|_{L^p(d_\mu)} < \infty$ where

$$\|f\|_{L^p(d_\mu)} = \begin{cases} \left(\int_{-1}^{1}|f(x)|^p\, d_\mu(x)\right)^{\frac{1}{p}} \text{ if } 1 \le p < \infty, \\ \text{ess}\sup_{-1<x<1}|f(x)| \text{ if } p = \infty. \end{cases}$$

Let us now introduce the Sobolev-type spaces (see [209])

$$S_p^\alpha = \left\{f: \|f\|_{S_p^\alpha}^p = \|f\|_{L^p(d_\mu)}^p + \lambda\|f'\|_{L^p(d_\mu)}^p < \infty\right\}, 1 \le p < \infty,$$

$$S_\infty^\alpha = \left\{f: \|f\|_{S_\infty^\alpha} = \max\|f\|_{L^\infty(d_\mu)}, \lambda\|f'\|_{L^\infty(d_\mu)} < \infty\right\} < \infty, \quad \text{where } \lambda > 0.$$

Let f and g in S_2^α. We can introduce the Sobolev-type inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)d_\mu(x) + \lambda\int_{-1}^{1} f'(x)g'(x)d_\mu(x) \tag{9}$$

where $\lambda > 0$ and $d_\mu = (1-x^2)^{\alpha-1/2}dx$ with $\alpha > -1/2$. Let $\left\{Q_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ denote the sequence of monic polynomials orthogonal with respect to (9). We call these polynomials the Gegenbauer–Sobolev polynomials. These polynomials constitute a particular case of the so called coherent pairs of measures, studied in [210]. In [211] the authors established the asymptotics of the zeros of the Gegenbauer–Sobolev polynomials.

Let $q_n^{(\alpha)}$ be the Gegenbauer–Sobolev orthonormal polynomials i.e.

$$q_n^{(\alpha)} = \left(\left\|Q_k^{(\alpha)}\right\|_{S_2^\alpha}\right)^{-1} Q_n^{(\alpha)}(x), \tag{10}$$

Where $\left\|Q_k^{(\alpha)}\right\|_{S_2^\alpha} \cong \sqrt{\pi\lambda}2^{3/2-\alpha-n}n$.

For $f \in S_p^\alpha$ the Fourier expansion in terms of Gegenbauer–Sobolev orthonormal polynomials is

$$\sum_{k=0}^{\infty}\hat{f}(k)q_k^{(\alpha)}(x), \tag{11}$$

Where $\hat{f}(k) = \left\langle f, q_k^{(\alpha)}\right\rangle$, $k = 0,1,\ldots$

The main goal of this contribution is to study the necessary conditions for S_p^α –norm convergence of the Fourier expansion(11),Theorem(6.2.8) . Also, we will prove that, for $\lambda > 0$ and $\alpha > 0$ there are functions $f \in S_p^\alpha, 1 \leq p \leq \frac{2\alpha+1}{\alpha+1}$, whose Fourier expansions (11) are divergent almost everywhere o [-1,1] in the norm of S_∞^α Theorem(6.2.9) . In particular, for $\lambda = 0$, the Theorem(6.2.9) agrees with the result in [212]. In order to prove the main results, we need some estimates for the polynomials $Q_n^{(\alpha)}(x)$ as well as for the polynomials $Q'^{(\alpha)}_k(x)$ such as the Mehler–Heine type formula, a symptotics on compact subsets of (-1,1), and S_p^α norms of Gegenbauer–Sobolev orthonormal polynomials.

For $\alpha > -1/2$ we denote by $\left\{\hat{C}_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ the sequence of the Gegenbauer polynomials, orthogonal on [-1 , 1] with respect to the measure $d_\mu(x)$ see [214],[215]. They are normalized in such a way that $\hat{C}_n^{(\alpha)}(x) = \frac{\Gamma(n+2\alpha)}{\Gamma(2\alpha)\Gamma(n+1)}$.

Note that this normalization does not allow α to be zero or a negative integer. Nevertheless, the limits see([214]) $\lim_{\alpha\to 0} \hat{C}_0^{(\alpha)}(x) = T_0(x)$, $\lim_{\alpha\to 0} \frac{\hat{C}_n^{(\alpha)}(x)}{\alpha} = \frac{2}{n} T_n(x)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind, exist for every $x \in [-1, 1]$. To avoid confusing notation, we define the polynomials $\hat{C}_n^{(0)}$ to be the Chebyshev polynomials of first kind $T_0(x)$ as obtained by limits, i.e. $\hat{C}_n^{(0)}(x) = 1$, $\hat{C}_n^{(0)}(x) = \frac{2}{n}$, $\hat{C}_n^{(\alpha)}(x) = \frac{2}{n} T_n(x), n = 1,2, \dots$.

We denote the monic Gegenbauer orthogonal polynomial by $C_n^{(\alpha)}(x) = (h_n^\alpha)^{-1} \hat{C}_n^{(\alpha)}(x)$ where (see[214])

$$h_n^\alpha = \frac{2^n \Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \alpha \neq 0 \tag{12}$$

$$h_n^0 = \lim_{\alpha\to 0} \frac{h_n^\alpha}{\alpha} = \frac{2^n}{n}, n \geq 1 .$$

Clearly, $C_n^0(x) = \lim\ (h_n^\alpha)^{-1} \hat{C}_n^{(\alpha)}(x) = 2^{1-n} T_n(x), n \geq 1$.

Now we list some properties of the monic Gegenbauer polynomials which we will use in the sequel. The following integral formula for Gegenbauer polynomials holds see [215]

$$\int_{-1}^{1} \left[C_n^{(\alpha)}(x)\right]^2 = \pi 2^{1-2\alpha-2n} \frac{\Gamma(n+1)\Gamma(n+2\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha)}, n \geq 1 . \tag{13}$$

They satisfy a structure relation

$$C_n^{(\alpha-1)}(x) = C_n^{(\alpha)}(x) - \xi_{n-2}^{(\alpha)} C_{n-2}^{(\alpha)}(x), n \geq 2 , \tag{14}$$

where

$$\xi_n^{(\alpha)} = \frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)}, n \geq 0 . \tag{15}$$

as well as the following relation for the derivatives see [290]

$$\frac{d}{dx} C_n^{(\alpha)}(x) = n C_{n-1}^{(\alpha+1)}(x). \tag{16}$$

The formula of Mehler–Heine type for Gegenbauer orthogonal polynomials is in [214]

$$\lim_{\alpha\to\infty} 2^n n^{-\alpha} C_n^{(\alpha)}\left(\cos\frac{z}{n}\right) = 2^{1/2-\alpha}\sqrt{\pi}\, z^{-\alpha+1/2} J_{\alpha-1/2}(z), \tag{17}$$

where α is a real number and $\alpha \neq -1, -2, \dots$, and $J_\alpha(z)$ is the Bessel function of the first kind. This formula holds uniformly for $|z| \leq R$, for R a given positive real number.

Let α be a real number and $\neq -1, -2, \dots$.

The inner strong asymptotic behavior of $C_n^{(\alpha)}$
for $\theta \in [\epsilon, \pi - \epsilon]$ and $\epsilon > 0$, is given by

$$C_n^{(\alpha)}(\cos\theta) = \begin{cases} C_n(\alpha)\left[\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\alpha}\cos(k\theta+\gamma) + O(n^{-1})\right] & \text{if } \alpha \neq 0, \\ C_n(\alpha)\cos n\theta & \text{if } \alpha = 0 \end{cases} \tag{18}$$

where $= n + \alpha$, $\gamma = {}^{-\alpha\pi}/_2$, and

$$C_n(\alpha) = \begin{cases} 2^{1-2\alpha-n}2^{-1/2}\dfrac{\Gamma(n+1)\Gamma(n+2\alpha)}{\Gamma(n+\alpha)\Gamma(n+\alpha+{}^1/_2)} & \text{if } \alpha \neq 0, \\ 2^{1-n} & \text{if } \alpha = 0 \end{cases} \tag{19}$$

Moreover, for $\alpha = 0$, the relation (18) holds for any $\theta \in [0, \pi]$.
For $\alpha > 0$ and $1 \le p \le \infty$ see [216]

$$\left\|C_n^{(\alpha)}\right\|_{L^p(d\mu)} \sim (h_n^{\alpha})^{-1} \begin{cases} n^{\alpha-1} & \text{if } {}^{(2\alpha+1)}/_{\alpha} > p, \\ n^{\alpha-1}(\log n)^{1/p} & \text{if } {}^{(2\alpha+1)}/_{\alpha} = p, \\ n^{2\alpha-1-\frac{2\alpha+1}{p}} & \text{if } {}^{(2\alpha+1)}/_{\alpha} < p. \end{cases} \tag{20}$$

From (14) and [210] (see also [217] and [218] in a more general framework) we have the following relation between the Gegenbauer–Sobolev and Gegenbauer monic orthogonal polynomials:

Proposition (6.2.1)[213]. For $\alpha > {}^{-1}/_2$

$$C_n^{(\alpha-1)}(x) = Q_n^{(\alpha)}(x) - d_{n-2}Q_{n-2}^{(\alpha)}(x), \qquad n \ge 2, \tag{21}$$

where

$$d_n(\alpha) = \hat{\xi}_n(\alpha)\frac{\left\|C_n^{(\alpha)}\right\|^2_{L^2(d\mu)}}{\left\|Q_n^{(\alpha)}\right\|^2_{S_2^{\alpha}}} \tag{22}$$

Moreover, by using (10), (13)and (15) we find from (22) that

$$d_n(\alpha) = \frac{1}{16\lambda n^2} \tag{23}$$

Using (21) in a recursive way and taking into account (14) we get the representation of the polynomials $Q_n^{(\alpha)}$ in terms of the $C_n^{(\alpha)}$

$$\begin{aligned} Q_n^{(\alpha)}(x) &= C_n^{(\alpha-1)}(x) + \sum_{i=1}^{\left[\frac{n}{2}\right]} a_i^{(n)}C_{n-2i}^{(\alpha-1)}(x)(x) + \sum_{i=0}^{\left[\frac{n}{2}\right]} a_i^{(n)}C_{n-2i}^{(\alpha-1)}(x) \\ &= \sum_{i=1}^{\left[\frac{n}{2}\right]} a_i^{(n)}C_{n-2i}^{(\alpha-1)}(x) - \xi_{n-2i-2}^{(\alpha)}C_{n-2i}^{(\alpha)}(x), \; n \ge 3 \end{aligned} \tag{24}$$

where $a_i^{(n)}(\alpha) = \prod_j^i d_{n-2j}(\alpha)$, $a_0^{(n)}(\alpha) = 1$.
Note that $Q_n^{(\alpha)}(x) = C_n^{(\alpha)}(x)$ for n = 1, 2.
Now, using a technique similar to the one used in [219] we obtain:

Proposition (6.2.2)[213]. For any fixed real number r > 0 there exists a constant c > 1 such that the coefficients $a_i^{(n)}(\alpha)$ in (24) verify $a_i^{(n)}(\alpha) < cr^i$, for all $n \geq 3$ and $1 \leq i \leq [n/2]$

Next, we deduce a Mehler–Heine type formula for $Q_n^{(\alpha)}(x)$ and $\left(Q_n^{(\alpha)}(x)\right)'$.

Proposition (6.2.3) [213]. Let $\alpha > -1/2$ and $\alpha \neq 0$.Then, uniformly on compact subsets of C.

(i) $\lim_{\alpha\to\infty} 2^n n^{-\alpha+1} Q_n^{(\alpha)}\left(\cos\frac{z}{n}\right) = 2^{3/2-\alpha}\sqrt{\pi}\, z^{-\alpha+3/2} J_{\alpha-3/2}(z),$ (25)

(ii) $\lim_{\alpha\to\infty} 2^n n^{-\alpha+1} Q'(\alpha)_n\left(\cos\frac{z}{n}\right) = 2^{3/2-\alpha}\sqrt{\pi}\, z^{-\alpha+1/2} J_{\alpha-1/2}(z),$ (26)

Proof. (i) Multiplying in (21) by $2^n n^{-\alpha+1}$, we obtain

$$Y_n(z) = 2^n n^{-\alpha+1} C_n^{(\alpha-1)}\left(\cos\frac{z}{n}\right) + 4d_{n-2}(\alpha)\left(\frac{n-2}{n}\right)^{\alpha-1} Y_{n-2}(z)$$

Where $Y_n(z) = 2^n n^{-\alpha+1} Q_n^{(\alpha)}\left(\cos\frac{z}{n}\right)$ From (17), for $\alpha > -1/2$ and $\alpha \neq 0$,

we have that $\left\{2^n n^{-\alpha+1} C_n^{(\alpha-1)}\left(\cos\frac{z}{n}\right)\right\}_{n=1}^{\infty}$ is uniformly bounded on compact subsets of C. Thus, for a fixed compact set $K \subset C$ there exists a constant B, depending only on K, such that when $z \in K$

$$\left|2^n n^{-\alpha+1} C_n^{(\alpha-1)}\left(\cos\frac{z}{n}\right)\right| < B\,, n \geq 1\,.$$

On the other hand $4d_{n-2}(\alpha)\left(\frac{n-2}{n}\right)^{\alpha-1} = O(n^{-2})$.

Therefore, there exists $n_1 \in N$ such that $4d_{n-2}(\alpha)\left(\frac{n-2}{n}\right)^{\alpha-1} < 1/2$, $n \geq n_1$

Thus, for $z \in K |Y_n(z)| \leq B + 1/2 |Y_{n-2}(z)|$, $n \geq n_1$.

Now we have that $Y_n(x)$ is uniformly bounded on $K \subset C$. As conclusion

$$Y_n(z) = 2^n n^{-\alpha+1} C_n^{(\alpha-1)}\left(\cos\frac{z}{n}\right) + 4d_{n-2}(\alpha)\left(\frac{n-2}{n}\right)^{\alpha-1} + O(n^{-2}),$$

and by using (17) we obtain the result.

(ii) Since we have uniform convergence in (25), and taking derivatives and using properties of Bessel functions we obtain (26).

Now we give the strong asymptotics of $Q_n^{(\alpha)}$ on $(-1\,,1)$.

Proposition (6.2.4)[213]. Let $\alpha > -1/2$ and $\alpha \neq 0$. For $\theta \in [\epsilon\,, \pi-\epsilon]$ and $\epsilon > 0$

$$Q_n^{(\alpha)}(\cos\theta) = C_n(\alpha-1)\left[\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\alpha+1}\cos(k\theta+\gamma_1) + O(n^{-1})\right], \quad (27)$$

$$Q'^{(\alpha)}_n(\cos\theta) = nC_n(\alpha-1)\left[\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\alpha}\cos(k\theta+\gamma_2) + O(n^{-1})\right], \quad (28)$$

where $K = n+\alpha-1\,, \gamma_1 = -(\alpha-1)\pi/2\,, \gamma_2 = -\alpha\pi/2$ and $c_n(\alpha)$ is given by (19).

Proof. From (18) we have $\left\{Q_n^{(\alpha)}(x)/c_n(\alpha)\right\}_n$ is uniformly bounded on compact sets of(-1 , 1).

Dividing in (21) by $C_n(\alpha-1)$, we get $\frac{Q_n^{(\alpha)}(x)}{C_n(\alpha-1)} = \frac{C_n^{(\alpha-1)}(x)}{C_n(\alpha-1)} + d_{n-2}(\alpha)\frac{C_n(\alpha-1)}{C_n(\alpha-1)}\frac{Q_{n-2}^{(\alpha)}(x)}{C_{n-2}(\alpha-1)}$.

Since

$$d_{n-2}(\alpha)\frac{C_n(\alpha-1)}{C_n(\alpha-1)} = d_{n-2}(\alpha)\frac{4(n+\alpha-1)(n+\alpha-3)(n+\alpha-3/2)(n+\alpha-5/2)}{n(n-1)(n+2\alpha-3)(n+2\alpha-4)} \times \sqrt{\frac{n}{n-2}} = O\left(\frac{1}{n^2}\right)$$

standard arguments yield that $\left\{ Q_n^{(\alpha)}(x) / c_n(\alpha) \right\}_n$ is uniformly bounded on compact sets of

(-1 , 1). Thus $\frac{Q_n^{(\alpha)}(x)}{C_n(\alpha-1)} = \frac{C_n^{(\alpha-1)}(x)}{C_n(\alpha-1)} + O(n^{-2})$

Using (18), the relation (27) follows.Concerning (28), it can be obtained in a similar way by using (16). With the next proposition we establish the Sobolev norms of the Gegenbauer–Sobolev polynomials.

Proposition (6.2.5)[213]. For $\alpha > 0$ and $1 \le p \le \infty$

$$\left\| Q_n^{(\alpha)} \right\|_{S_p^\alpha} \sim (h_n^{\alpha-1})^{-1} \begin{cases} n^{\alpha-1} & \text{if } (2\alpha+1)/\alpha > p, \\ n^{\alpha-1}(\log n)^{1/p} & \text{if } (2\alpha+1)/\alpha = p, \\ n^{2\alpha-1-\frac{2\alpha+1}{p}} & \text{if } (2\alpha+1)/\alpha < p. \end{cases} \tag{29}$$

Proof. In order to prove the upper bound of (29) it is enough to prove $\left\| Q_n^{(\alpha)} \right\|_{S_p^\alpha} \le cn \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)}$

Using (24), Minkowski's inequality and (15) we have

$$\left\| Q_n^{(\alpha)} \right\|_{L^p(d\mu)} \le \sum_{i=0}^{[n/2]} a_i^{(n)} \left\| C_{n-2i}^{(\alpha)} \right\|_{L^p(d\mu)}$$

$$\le \sum_{i=0}^{[n/2]} a_i^{(n)} \left\| C_{n-2i}^{(\alpha)} \right\|_{L^p(d\mu)} + \sum_{i=0}^{[n/2]} a_i^{(n)} \left\| C_{n-2i-2}^{(\alpha)} \right\|_{L^p(d\mu)}, n \ge 3. \tag{30}$$

It is easy to prove that, for $\alpha > 0$ and $i = 0,1, \dots, [n/2]$, by (20)

$2^{n-2i} \left\| C_{n-2i}^{(\alpha)} \right\|_{L^p(d\mu)} \le c_1 n^2 \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)}$. Thus $\left\| C_{n-2i}^{(\alpha)} \right\|_{L^p(d\mu)} \le c_1 4^i \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)}$.

Using this and (6.2.4) for r < ¼ , we get

$$\sum_{i=0}^{[n/2]} a_i^{(n)} \left\| C_{n-2i-2}^{(\alpha)} \right\|_{L^p(d\mu)} \le c_2 \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)} \sum_{i=0}^{[n/2]} (4r)^i \le c_3 \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)}$$

In a similar way we can prove that

$$\sum_{i=0}^{[n/2]} a_i^{(n)} \left\| C_{n-2i-2}^{(\alpha)} \right\|_{L^p(d\mu)} \le c_4 \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)}$$

Thus

$$\left\| Q_n^{(\alpha)} \right\|_{L^p(d\mu)} \le c_5 \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)} \tag{31}$$

On the other hand, from (8), (16) and Minkowski's inequality

$$\left\| {Q'}_n^{(\alpha)} \right\|_{L^p(d\mu)} \le c_6 n \sum_{i=0}^{[n/2]} a_i^{(n)} \left\| C_{n-2i-1}^{(\alpha)} \right\|_{L^p(d\mu)} \le c_7 n \left\| C_n^{(\alpha)} \right\|_{L^p(d\mu)}. \tag{32}$$

Thus, from (31) and (32) we get (30).

In order to prove the lower bound in relation (29) we will need the following:
Proposition (6.2.6)[213]. Let $\alpha > 0$ and $1 \le p \le \infty$. Then, for sufficiently large n

$$\|Q'^{(\alpha)}_n\|_{S_p^\alpha} \ge c(h_n^{\alpha-1})^{-1}\begin{cases} n^{\alpha-1} & \text{if } (2\alpha+1)/\alpha > p, \\ n^{\alpha-1}(\log n)^{1/p} & \text{if } (2\alpha+1)/\alpha = p, \\ n^{2\alpha-1-\frac{2\alpha+1}{p}} & \text{if } (2\alpha+1)/\alpha < p. \end{cases} \tag{33}$$

Proof. Let α> 0 and n large enough. From (16) and (21) it follows that
$nC^{(\alpha)}_{n-1}(x) = Q'^{(\alpha)}_n(x) - d_{n-2}(\alpha)Q'^{(\alpha)}_{n-2}(x)$,and by using Minkowski's inequality
$1 \le \frac{\|Q'^{(\alpha)}_n\|_{L^p(d\mu)}}{n\|C^{(\alpha)}_{n-1}\|_{L^p(d\mu)}} + d_{n-2}(\alpha)\frac{\|Q'^{(\alpha)}_{n-2}\|_{L^p(d\mu)}}{n\|C^{(\alpha)}_{n-1}\|_{L^p(d\mu)}}$. On the other hand, from (20), (23) and (32)

$$d_{n-2}(\alpha)\frac{\|Q'^{(\alpha)}_{n-2}\|_{L^p(d\mu)}}{n\|C^{(\alpha)}_{n-1}\|_{L^p(d\mu)}} = d_{n-2}(\alpha)\frac{(n-2)\|Q'^{(\alpha)}_{n-2}\|_{L^p(d\mu)}}{n\|C^{(\alpha)}_{n-1}\|_{L^p(d\mu)}}\frac{\|Q'^{(\alpha)}_n\|_{L^p(d\mu)}}{(n-2)\|C^{(\alpha)}_{n-2}\|_{L^p(d\mu)}} = O(n^{-2}),$$

which implies that $1 \le \frac{\|Q'^{(\alpha)}_n\|_{L^p(d\mu)}}{n\|C^{(\alpha)}_{n-1}\|_{L^p(d\mu)}} + O(n^{-2})$.

Thus, there exists a positive constant c and $n_0 \in N$ such that $c \le \frac{\|Q'^{(\alpha)}_n\|_{L^p(d\mu)}}{n\|C^{(\alpha)}_{n-1}\|_{L^p(d\mu)}}$, $n \ge n_0$

The proof of Proposition (6.2.6) is complete. From (33), or $\alpha > 0$, $1 \le p \le \infty$
and sufficiently large n

$$\|Q^{(\alpha)}_n\|_{S_p^\alpha} \ge c(h_n^{\alpha-1})^{-1}\begin{cases} n^{\alpha-1} & \text{if } (2\alpha+1)/\alpha > p, \\ n^{\alpha-1}(\log n)^{1/p} & \text{if } (2\alpha+1)/\alpha = p, \\ n^{2\alpha-1-\frac{2\alpha+1}{p}} & \text{if } (2\alpha+1)/\alpha < p. \end{cases}$$

Now, using this and (30) and the relation (29) follows.
The problem of the norm convergence of partial sums of the Fourier expansions in terms of Gegenbauer polynomials has been discussed by [220].Taking into account (9) and Proposition (6.2.5), we obtain the Sobolev norms of Gegenbauer –Sobolev orthonormal polynomials.
Proposition (6.2.7)[213]. For $\alpha > 0$, $1 \le p \le \infty$

$$\|q^{(\alpha)}_n\|_{S_p^\alpha} \sim \begin{cases} c & \text{if } (2\alpha+1)/\alpha > p, \\ n^{\alpha-1}(\log n)^{1/p} & \text{if } (2\alpha+1)/\alpha = p, \\ n^{2\alpha-1-\frac{2\alpha+1}{p}} & \text{if } (2\alpha+1)/\alpha < p. \end{cases}$$

Let $S_n f$ be the n-th partial sum of the expansion (10) $S_n(f,x) = \sum_{k=0}^{n}\hat{f}(k)q^{(\alpha)}_k(x)$.

Theorem (6.2.8)[213]. Let $\alpha > 0$ and $1 < p < \infty$. If there exists a constant $c > 0$ such that

$$\|S_n f\|_{S_p^\alpha} \le c\|f\|_{S_p^\alpha} \tag{34}$$

for every $f \in S_p^\alpha$ then $p \in (p_0, q_0)$, where $p_0 = \frac{2\alpha+1}{\alpha+1}, q_0 = \frac{2\alpha+}{\alpha}$.

Proof. We apply the same argument as in [221]. Assume that (26) holds.

Then $\left\|\left\langle f, q_n^{(\alpha)}\right\rangle q_n^{(\alpha)}\right\|_{S_p^\alpha} = \|S_n f - S_{n-1} f\|_{S_p^\alpha} \le 2c\|f\|_{S_p^\alpha}$.

Consider the functionals $L_n(f) = \left\langle f, q_n^{(\alpha)}\right\rangle \left\|q_n^{(\alpha)}\right\|_{S_p^\alpha}$ on S_p^α. Hence, for every f in S_p^α

we have $\sup_n |L_n(f)| < \infty$ This implies, by the Banach-Steinhaus theorem,

that $\sup_n \|L_n\| < \infty$. On the other hand, by duality see [209] we have

$\|L_n\| = \left\|q_n^{(\alpha)}\right\|_{S_p^\alpha} \left\|q_n^{(\alpha)}\right\|_{S_p^\alpha}$ where p is the conjugate of q. Therefore $\left\|q_n^{(\alpha)}\right\|_{S_p^\alpha} \left\|q_n^{(\alpha)}\right\|_{S_p^\alpha} < \infty$.

From Proposition (6.2.7), it follows that the last inequality holds if and only if $p \in (p_0, q_0)$.The proof of Theorem (6.2.8) is complete.

If $\|S_n f\|_{S_\infty^\alpha}$ is uniformly bounded on a set, say E, of positive measure in [-1, 1] then $\left\|\hat{f}(n) q_n^{(\alpha)}\right\|_{S_p^\alpha} < c, n \in N$. Therefore $\left\|\hat{f}(n) {q'}_n^{(\alpha)}(x)\right\|_{S_p^\alpha} < c, n \in N$

a.e. on E. From Egorov's Theorem see [222] it follows that there is a subset $E_1 \subset E$ of positive measure such that $\left\|\hat{f}(n) {q'}_n^{(\alpha)}(x)\right\|_{S_p^\alpha} < c$ uniformly for $x \in E_1$ On the other hand, from (9) and (28)

${q'}_n^{(\alpha)}(\cos\theta) = A_n\left[\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\alpha}\cos(k\theta + \gamma_2) + O(n^{-1})\right]$ where $A_n =\cong \frac{2^{1/2-\alpha}}{\sqrt{\pi\lambda}}$.

Thus $\left|\hat{f}(n)(\cos(k\theta + \gamma_2) + O(n^{-1}))\right| < c$

uniformly for $\cos\theta \in E_1$. Using the Cantor–Lebesgue Theorem ,see [223] we obtain

$$\left|\hat{f}(n)\right| < c. \tag{35}$$

Theorem (6.2.9)[213]. Let $\alpha > 0$ There is an $f \in S_p^\alpha$, $1 \le p \le p_0$, whose Fourier expansion (11) diverges almost everywhere on [-1 , 1] in the norm of S_∞^α.

Proof. The uniform boundedness principle and Proposition (6.2.10) yield the existence of functions $f \in S_p^\alpha$, $1 \le p \le p_0$, such that the linear functional $\hat{f}(n)$ satisfies $\hat{f}(n) \to \infty$ when $\to \infty$. Since this result contradicts (35) then Fourier series (11) diverges almost everywhere on [-1 , 1] in the norm of S_∞^α.

Corollary (6.2.10)[237]. Let $\alpha^2 > 1/2$ and $\alpha^2 \ne 1$. Then uniformly on compact subsets of C.

(i) $\lim_{\alpha\to\infty} 2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2+2} Q_{1+\alpha_1}^{(\alpha^2-1)}\left(\cos\frac{z}{1+\alpha_1}\right) = 2^{3/1-\alpha^2}\sqrt{\pi} z^{-\alpha^2+5/2} J_{\alpha^2-5/2}(z)$,

(ii) $\lim_{\alpha\to\infty} 2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2+2} Q'(\alpha^2-1)_{1+\alpha_1}\left(\cos\frac{z}{1+\alpha_1}\right) = 2^{3/1-\alpha^2}\sqrt{\pi} z^{-\alpha^2+3/2} J_{\alpha^2-3/2}(z)$,

Proof.(i) Multiplying in (21) by $2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2+2}$, we obtain

$$Y_{1+\alpha_1}(z) = 2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2+2} C_{1+\alpha_1}^{(\alpha^2-2)}\left(cos\frac{z}{1+\alpha_1}\right) + 4d_{\alpha_1-1}(\alpha^2-1)\left(\frac{\alpha_1-1}{1+\alpha_1}\right)^{\alpha^2-2} Y_{\alpha_1-1}(z)$$

Where $Y_{1+\alpha_1}(z) = 2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2+2} Q_{1+\alpha_1}^{(\alpha^2-1)}\left(cos\frac{z}{1+\alpha_1}\right)$.

From (17), for $\alpha^2 > 1/2$ and $\alpha^2 \neq 1$,we have that $\left\{2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2} C_{1+\alpha_1}^{(\alpha^2-2)}\left(\cos\frac{z}{1+\alpha_1}\right)\right\}_{\alpha_1=0}^{\infty}$ is uniformly bounded on compact subsets of C. Thus, for a fixed compact set $K \subset C$ there exists a constant B, depending only on K, such that when $z \in K$,

$\left|2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2+2} C_{1+\alpha_1}^{(\alpha^2-2)}\left(\cos\frac{z}{1+\alpha_1}\right)\right| < B\ ,\alpha_1 \geq 0$. On the other hand

$4d_{\alpha_1-1}\left(\frac{\alpha_1-1}{1+\alpha_1}\right)^{\alpha^2-2} = O((1+\alpha_1)^{-2})$. Therefore, there exists $n_1 \in N$ such that ,

$4d_{\alpha_1-1}\left(\frac{\alpha_1-1}{1+\alpha_1}\right)^{\alpha^2-2} < 1/2\ ,n = n_1 + \alpha_3$.

Thus, for $\in K$, $|Y_{1+\alpha_1}(z)| \leq B + 1/2\,|Y_{\alpha_1-1}(z)|$, $n = n_1 + \alpha_3$. Now, we have that $Y_{1+\alpha_1}(x)$ is uniformly bounded on $K \subset C$. As conclusion

$$Y_{1+\alpha_1}(z) = 2^{1+\alpha_1}(1+\alpha_1)^{-\alpha^2+2} C_{1+\alpha_1}^{(\alpha^2-2)}\left(\cos\frac{z}{1+\alpha_1}\right) + 4d_{\alpha_1-1}(\alpha^2-1)\left(\frac{\alpha_1-1}{1+\alpha_1}\right)^{\alpha^2-2} + O((1+\alpha_1)^{-2})$$

and by using (17) we obtain the result.(ii) Since we have uniform convergence in (25), and taking derivatives and using properties of Bessel functions we obtain (26). Now we give the strong asymptotics of $Q_{1+\alpha_1}^{(\alpha^2-1)} O(1+\alpha_1)\ (-1\,,1)$.

Corollary (6.2.11)[237]. Let $\alpha^2 > 1/2$ and $\alpha^2 \neq 1$. For $\theta \in [\epsilon\,,\pi-\epsilon]$ and $\epsilon > 0$

$$Q_{1+\alpha_1}^{(\alpha^2-1)}(\cos\theta) = C_{1+\alpha_1}(\alpha^2-2)\left[\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\alpha^2+2}\cos(k\theta+\gamma_1) + O((1+\alpha_1)^{-1})\right],$$

$$Q'^{(\alpha^2-1)}_{1+\alpha_1}(\cos\theta) = 1 + \alpha_1 C_{1+\alpha_1}(\alpha^2-1)\left[\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-\alpha^2-1}\cos(k\theta+\gamma_2) + O((1+\alpha_1)^{-1})\right]$$

Where $K = \alpha_1 + \alpha^2 - 1\,,\gamma_1 = -(\alpha^2-2)\pi/2\,,\gamma_2 = -(\alpha^2-1)\pi/2$ and $c_{1+\alpha_1}(\alpha^2-1)$ is given by (19).

Proof. From (18) we have $\left\{Q_{1+\alpha_1}^{(\alpha^2-1)}(x)/c_{1+\alpha_1}(\alpha^2-1)\right\}_{1+\alpha_1}$ is uniformly bounded on compact sets of $(-1\,,1)$. Dividing in (21) by $C_{1+\alpha_1}(\alpha^2-2)$, we get

$\frac{Q_{1+\alpha_1}^{(\alpha^2-1)}(x)}{C_{1+\alpha_1}(\alpha^2-2)} = \frac{C_{1+\alpha_1}^{(\alpha^2-2)}(x)}{C_{1+\alpha_1}(\alpha^2-2)} + d_{\alpha_1-1}(\alpha^2-1)\frac{C_{1+\alpha_1}(\alpha^2-2)}{C_{1+\alpha_1}(\alpha^2-2)}\frac{Q_{\alpha_1-1}^{(\alpha^2-1)}(x)}{C_{\alpha_1-1}(\alpha^2-2)}$. Since

$$\mathrm{d}_{\alpha_1-1}(\alpha^2-1)\frac{\mathrm{C}_{1+\alpha_1}(\alpha^2-2)}{\mathrm{C}_{1+\alpha_1}(\alpha^2-2)} = \mathrm{d}_{\alpha_1-1}(\alpha^2-1)\ \frac{4(\alpha_1+\alpha^2-1)(\alpha_1+\alpha^2-3)(\alpha_1+\alpha^2-3/2)(\alpha_1+\alpha^2-5/2)}{(\alpha_1+\alpha_1^2)(\alpha_1+2\alpha^2-4)(\alpha_1+2\alpha^2-5)} \times \sqrt{\frac{1+\alpha_1}{\alpha_1-1}}$$

$$= O\left(\frac{1}{(1+\alpha_1)^2}\right)$$

standard arguments yield that $\left\{Q_{1+\alpha_1}^{(\alpha^2-1)}(x)/c_{1+\alpha_1}(\alpha^2-1)\right\}_{1+\alpha_1}$ is uniformly bounded on

compact sets of $(-1\,,1)$. Thus $\frac{Q_{1+\alpha_1}^{(\alpha^2-1)}(x)}{C_{1+\alpha_1}(\alpha^2-2)} = \frac{C_{1+\alpha_1}^{(\alpha^2-2)}(x)}{C_{1+\alpha_1}(\alpha^2-2)} + O((1+\alpha_1)^{-2})$.

Using (18), the relation (27) follows.

Concerning (28), it can be obtained in a similar way by using (16).

With the next proposition we establish the Sobolev norms of the Gegenbauer-Sobolev polynomials.

Corollary (6.2.12)[237]. For $\alpha^2 > 1$ and $\varepsilon_1 > 0$

$$\left\|Q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{S_{2+\varepsilon_1}^{\alpha^2-1}} \sim \left(h_{1+\alpha_1}^{\alpha^2-2}\right)^{-1} \begin{cases} (1+\alpha_1)^{\alpha^2-2} if \alpha < \pm\sqrt{1+1/\varepsilon_1}\ , \\ (1+\alpha_1)^{\alpha^2-2}(\log 1+\alpha_1)^{1/2+\varepsilon_1} if \alpha = \pm\sqrt{1+1/\varepsilon_1}, \\ (1+\alpha_1)^{2\alpha^2-3-\frac{2\alpha^2-1}{2+\varepsilon_1}} if \alpha > \pm\sqrt{1+1/\varepsilon_1}. \end{cases}$$

Proof. In order to prove the upper bound of (29) it is enough to prove

$$\left\|Q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{S_{2+\varepsilon_1}^{\alpha^2-1}} \le c(1+\alpha_1)\left\|C_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)}$$

Using (24), Minkowski's inequality and (15) we have

$$\left\|Q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} \le \sum_{i=0}^{[1+\alpha_1/2]} a_i^{(1+\alpha_1)}\left\|C_{1+\alpha_1-2i}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)}$$

$$\le \sum_{i=0}^{[1+\alpha_1/2]} a_i^{(1+\alpha_1)}\left\|C_{1+\alpha_1-2i}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} + \sum_{i=0}^{[1+\alpha_1/2]} a_i^{(1+\alpha_1)}\left\|C_{\alpha_1-2i-1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)},$$

$\alpha_1 \ge 2$. It is easy to prove that, for $\alpha > 1$ and $i = 0,1,\ldots,[(1+\alpha_1)/2]$, by (20)

$$2^{1+\alpha_1-2i}\left\|C_{1+\alpha_1-2i}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} \le c_1(1+\alpha_1)^2\left\|C_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)}$$

Thus $\left\|C_{1+\alpha_1-2i}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} \le c_1 4^i \left\|C_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)}$

Using this and Proposition (6.2.2), for $< 1/4$, we get

$$\sum_{i=0}^{[1+\alpha_1/2]} a_i^{(1+\alpha_1)}\left\|C_{\alpha_1-2i-1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} \le c_2\left\|C_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)}\sum_{i=0}^{[1+\alpha_1/2]}(4r)^i$$

$$\le c_3\left\|C_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} .$$

In a similar way we can prove that

$$\sum_{i=0}^{[1+\alpha_1/2]} a_i^{(1+\alpha_1)}\left\|C_{\alpha_1-2i-1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} \le c_4\left\|C_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} .$$

Thus $\left\|Q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)} \le c_5\left\|C_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)}$

On the other hand, from (8), (16) and Minkowski's inequality

$$\left\|Q'^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)} \le c_6(1+\alpha_1)\sum_{i=0}^{[1+\alpha_1/2]} a_i^{(1+\alpha_1)}\left\|C_{\alpha_1-2i}^{(\alpha^2-1)}\right\|_{L^{2+\varepsilon_1}(d\mu)}$$

$$\le c_7(1+\alpha_1)\left\|C_{1+\alpha_1}^{\alpha^2-1}\right\|_{L^{2+\varepsilon_1}(d\mu)}.$$

Thus, from (31) and (32) we get (30).

In order to prove the lower bound in relation (29) we will need the following .

Corollary (6.2.13)[237]. Let $\alpha^2 > 1$ and $\varepsilon_1 > 0$. Then, for sufficiently large $(1+\alpha_1)$

$$\left\|Q'^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{S^{\alpha^2-1}_{2+\varepsilon_1}} \geq c\left(h^{\alpha^2-2}_{1+\alpha_1}\right)^{-1}\begin{cases}(1+\alpha_1)^{\alpha^2-2} if \alpha < \pm\sqrt{1+1/\varepsilon_1}\ , \\ (1+\alpha_1)^{\alpha^2-2}(log\, 1+\alpha_1)^{1/2+\varepsilon_1} if \alpha = \pm\sqrt{1+1/\varepsilon_1} \\ (1+\alpha_1)^{2\alpha^2-3-\frac{2\alpha^2-1}{2+\varepsilon_1}} if \alpha > \pm\sqrt{1+1/\varepsilon_1}.\end{cases}$$

Proof. Let $\alpha^2 > 1$ and $1+\alpha_1$ large enough. From (16) and (21) it follows that $1+\alpha_1 C^{(\alpha^2-1)}_{\alpha_1}(x) = Q'^{(\alpha^2-1)}_{1+\alpha_1}(x) - d_{\alpha_1-1}(\alpha^2-1)Q'^{(\alpha^2-1)}_{\alpha_1-1}(x)$, and by using Minkowski's inequality

$$1 \leq \frac{\left\|Q'^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}{1+\alpha_1\left\|C^{(\alpha^2-1)}_{\alpha_1-1}\right\|_{L^{2+\varepsilon_1}(d\mu)}} + d_{\alpha_1-1}(\alpha^2-1)\frac{\left\|Q'^{(\alpha^2-1)}_{\alpha_1-1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}{1+\alpha_1\left\|C^{(\alpha^2-1)}_{\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}.$$

On the other hand, from (20), (23) and (32)

$$d_{\alpha_1-1}(\alpha^2-1)\frac{\left\|Q'^{(\alpha^2-1)}_{\alpha_1-1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}{1+\alpha_1\left\|C^{(\alpha^2-1)}_{\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}} = d_{\alpha_1-1}(\alpha^2-1)\frac{(\alpha_1-1)\left\|Q'^{(\alpha^2-1)}_{\alpha_1-1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}{1+\alpha_1\left\|C^{(\alpha^2-1)}_{\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}\frac{\left\|Q'^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}{(\alpha_1-1)\left\|C^{(\alpha^2-1)}_{\alpha_1-1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}$$
$$= O((1+\alpha_1)^{-2}),$$

which implies that $1 \leq \frac{\left\|Q'^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}{1+\alpha_1\left\|C^{(\alpha^2-1)}_{\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}} + O((1+\alpha_1)^{-2})$.

Thus, there exists a positive constant c and $n_0 \in N$ such that $c \leq \frac{\left\|Q'^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}{1+\alpha_1\left\|C^{(\alpha^2-1)}_{\alpha_1}\right\|_{L^{2+\varepsilon_1}(d\mu)}}, \alpha_1 \geq n_0 - 1$,

the proof of Proposition (6.2.6) is complete.
From (33), for $\alpha^2 > 1$, $\varepsilon_1 > 0$ and sufficiently large $(1+\alpha_1)$

$$\left\|Q^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{S^{\alpha^2-1}_{2+\varepsilon_1}} \geq c\left(h^{\alpha^2-2}_{1+\alpha_1}\right)^{-1}\begin{cases}(1+\alpha_1)^{\alpha^2-2} if \alpha < \pm\sqrt{1+1/\varepsilon_1}\ , \\ (1+\alpha_1)^{\alpha^2-2}(log\, 1+\alpha_1)^{1/2+\varepsilon_1} if \alpha = \pm\sqrt{1+1/\varepsilon_1}, \\ (1+\alpha_1)^{2\alpha^2-3-\frac{2\alpha^2-1}{2+\varepsilon_1}} if \alpha > \pm\sqrt{1+1/\varepsilon_1}.\end{cases}$$

Now, using this and (30) and the relation (29) follows.

Corollary (6.2.14)[237]. Let $\alpha^2 > 1$ and $\varepsilon_1 > 0$. If there exists a constant $c > 0$ then $\|S_{1+\alpha_1}f\|_{S^{\alpha^2-1}_{2+\varepsilon_1}} \leq c\|f\|_{S^{\alpha^2-1}_{2+\varepsilon_1}}$ for every $f \in S^{\alpha^2-1}_{2+\varepsilon_1}$ then $(2+\varepsilon_1) \in (\frac{2\alpha^2-1}{\alpha^2}, \frac{2\alpha^2-1}{\alpha^2-1})$.

Proof. We apply the same argument as in [221] **.**Assume that (26) holds.

Then $\left\|\left\langle f, q^{(\alpha^2-1)}_{1+\alpha_1}\right\rangle q^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{S^{\alpha^2-1}_{2+\varepsilon_1}} = \|S_{1+\alpha_1}f - S_{\alpha_1}f\|_{S^{\alpha^2-1}_{2+\varepsilon_1}} \leq 2c\|f\|_{S^{\alpha^2-1}_{2+\varepsilon_1}}$. Consider the functional $L_{1+\alpha_1}(f) = \left\langle f, q^{(\alpha^2-1)}_{1+\alpha_1}\right\rangle \left\|q^{(\alpha^2-1)}_{1+\alpha_1}\right\|_{S^{\alpha^2-1}_{2+\varepsilon_1}}$ on $S^{\alpha^2-1}_{2+\varepsilon_1}$.

Hence, for every f in $S_{2+\varepsilon_1}^{\alpha^2-1}$ we have $sup_{1+\alpha_1}|L_{1+\alpha_1}(f)| < \infty$. This implies, by the Banach-Steinhaus theorem, that $sup_{1+\alpha_1}\|L_{1+\alpha_1}\| < \infty$.

On the other hand, by duality see [209] we have

$$\|L_{1+\alpha_1}\| = \left\|q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{S_{2+\varepsilon_1}^{\alpha^2-1}} \left\|q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{S_{2+\varepsilon_1}^{\alpha^2-1}}.$$

Where $(2+\varepsilon_1)$ is the conjugate of $(2+\varepsilon_1/1+\varepsilon_1)$.

Therefore

$$\left\|q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{S_{2+\varepsilon_1}^{\alpha^2-1}} \left\|q_{1+\alpha_1}^{(\alpha^2-1)}\right\|_{S_{2+\varepsilon_1}^{\alpha^2-1}} < \infty\ .$$

From Proposition (6.2.7), it follows that the last inequality holds iff $\varepsilon_1 \in (\frac{2\alpha^2-1}{\alpha^2}-2\,,\frac{2\alpha^2-1}{\alpha^2-1}-2)$.

The proof of Theorem (6.2.8) is complete.

In general way, with higher exponents ,we extend the following corollary.

Corollary (6.2.15)[237]. Let $\alpha^2 > 1$ and $\varepsilon_2 > 0$. If there exists a positive constant $\tilde{C} > 0$ such that ,

$$\sum_{j=1}^{m}\|S_{1+\alpha_1}f_j\|_{S_{2+\varepsilon_2}^{\alpha^2-1}} \leq \tilde{C}\sum_{j=1}^{m}\|f_j\|_{S_{2+\varepsilon_2}^{\alpha^2-1}}$$

for every $\sum_{j=1}^{m} f_j \in S_{2+\varepsilon_2}^{\alpha^2-1}$ then $\varepsilon_2 \in (\frac{\sqrt{2\alpha^2-1}}{\alpha}-2\,,\sqrt{\frac{2\alpha^2-1}{\alpha^2-1}}-2)$.

Proof. From [221] and (26) we have

$$\left\|\left\langle f_j,((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\rangle((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\|_{S_{2+\varepsilon_2}^{\alpha^2-1}} = \sum_{j=1}^{m}\|S_{1+\alpha_1}f_j - S_{\alpha_1}f_j\|_{S_{2+\varepsilon_2}^{\alpha^2-1}}$$

$$\leq 2\tilde{C}\sum_{j=1}^{m}\|f_j\|_{S_{2+\varepsilon_2}^{\alpha^2-1}}\ .$$

Take the functional

$$\sum_{j=1}^{m} L_{1+\alpha_1}(f_j) = \left\langle f_j,((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\rangle\left\|((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\|_{S_{2+\varepsilon_2}^{\alpha^2-1}} \text{ on } S_{2+\varepsilon_2}^{\alpha^2-1}.$$

Hence

$$\sum_{j=1}^{m} f_j \in S_{2+\varepsilon_2}^{\alpha^2-1} \text{ in } S_{2+\varepsilon_1}^{\alpha^2-1}$$

therefore

$$sup_{1+\alpha_1}\left|\sum_{j=1}^{m} L_{1+\alpha_1}(f_j)\right| \leq sup_{1+\alpha_1}\sum_{j=1}^{m}\left|L_{1+\alpha_1}(f_j)\right| < \infty$$

and $sup_{1+\alpha_1}\|L_{1+\alpha_1}\| < \infty$,

by the Banach-Steinhaus theorem, gives that

$$\|L_{1+\alpha_1}\| = \left\|((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\|_{S_{2+\varepsilon_2}^{\alpha^2-1}} \left\|((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\|_{S_{(2+\varepsilon_2)^2+\varepsilon_3)}^{\alpha^2-1}}\ .$$

Hence ,

$$\left\|((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\|_{S_{2+\varepsilon_2}^{\alpha^2-1}}\left\|((2+\varepsilon_2)^2+\varepsilon_3)_{1+\alpha_1}^{\frac{(\alpha^2-1)}{2}}\right\|_{S_{(2+\varepsilon_2)^2+\varepsilon_3}^{\alpha^2-1}}<\infty .$$

The last inequality holds iff $\varepsilon_2 \geq 0$ be considering by rearrangement of Proposition (6.2.7).

Section (6.3): Fourier-Sobolev Expansions:

Given $1 \leq p < \infty$, let $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ be the following weighted Sobolev space $\mathbb{W}^{1,p}([-1,1],(w_0,w_1)) := \{f : [-1,1] \to \mathbb{R} : f \in L^p([-1,1],w_0), f' \in Lp([-1,1],w_1)\}$, with the norm $\|f\|^p_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} := \|f\|^p_{L^p([-1,1],w_0)} + \|f'\|^p_{L^p([-1,1],w_1)}$, where $w_0 \in L^\infty([-1,1])$ and w_1 is a Kufner–Opic type weight .

For $f, g \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ we introduce the weighted Sobolev inner product

$$\langle f,g\rangle_s := \int_{-1}^{1} f(x)g(x)w_0(x)dx + \int_{-1}^{1} f'(x)g'(x)w_1(x)dx \tag{36}$$

Let $\mathbb{p}$ be the space of the polynomials with real coefficients. In general it is not true that $\mathbb{p} \subseteq \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, but when it holds we can consider the sequence $\{q_n\}_n \geq 0$ of orthonormal polynomials with respect to (36) and for $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ its Fourier

$$\mathcal{s}f \sim \sum_{k=0}^{\infty} \tilde{f}(k)q_k, \tag{37}$$

Where $\tilde{f}(k) = \langle f,g\rangle_s$, for $k \geq 0$.

This definition of the Fourier–Sobolev expansion of f is purely formal and it is not obvious whether it converges to . In fact, the solution of this problem can be very hard, or relatively easy, depending on either the sense of the convergence, or in terms of additional restrictions on f and the pair of weights (w_0,w_1) see [224].The main goal is to study necessary and/or sufficient conditions for the $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ – norm convergence of the Fourier–Sobolev expansion (37). The structure of the section is as follows. We study of necessary and sufficient conditions for the $\mathbb{W}^{1,p}([-1,1]),(w_0,w_1)$ –norm convergence of the Fourier–Sobolev expansion (37). Following the ideas of [226],see [227] introduced general classes of Sobolev spaces appearing in the context of orthogonal polynomials on the real line. We will use the approach given in these section to establish the Kufner-Opic type property as follows.

Definition (6.3.1)[225]. Let $1 \leq p \leq \infty$. A weight function w on $[a,b]$ is said to satisfy the Kufner - Opic type property (or belongs to $B_p([a,b])$ if and only if

$$w^{-1} \in L^{1/(p-1)}([a,b]), \text{for } 1 \leq p < \infty,$$
$$w^{-1} \in L^{1}([a,b]), \text{for } p = \infty.$$

Also, if J is any interval we say that $w \in B_p(J)$ if $w \in B_p(I)$ for every compact interval $I \subseteq J$.

We say that a weight belongs to $B_p(J)$, where J is a union of disjoint intervals $\cup_{i\in A} J_i$, if it belongs to $B_p(J_i)$, for $i \in A$.

Notice if $v \geq w$ in J and $w \in B_p(J)$, then $v \in B_p(J)$.

This class contains the classical Muckenhoupt A_p weights appearing in Harmonic Analysis see [228].Other properties of the class of weights of the Kufner–Opic type we will need in the sequel are contained in the following result.

Lemma (6.3.2)[225],[229]. Let us consider $1 \leq p < \infty$ and $w \in B_p((a,b))$.
For any compact interval $I \subseteq (a,b)$, there is a positive constant C_1 , which only depends on $p, w,$ and I , such that $\|g\|_{L^1(I)} \leq C_1\|g\|_{L^p(I,w)} \leq C_1\|g\|_{L^p([a,b])}$, for any $g \in L^p([a,b],w)$.
Furthermore, if $w \in B_p([a,b])$, then there is a positive constant C_2, which only depends on p and w , such that $\|g\|_{L^1(a,b)} \leq C_2\|g\|_{L^p([a,b])}$, for any $g \in L^p([a,b],w)$.
As a consequence, if $w \in B_p([a,b])$ and $f' \in L^p([a,b],w)$, then $f \in AC([a,b])$.
The proof of this lemma will be not included here, however can check it in [229].
Definition (6.3.3)[225]. We denote by $AC([a,b])$ the set of absolutely continuous functions in [a, b], i.e. the functions $f \in C([a,b])$ such that $f(x) - f(a) = \int_a^x f'(t)dt$ for every $x \in [a,b]$. If J is any interval, AC_{loc} (J) denotes the set of absolutely continuous functions in every compact subinterval of J.
For $1 \leq p < \infty$, let us consider the weighted Sobolev space
$W^{1,p}([-1,1],(w_0,w_1))$, given by

$$\mathbb{W}^{1,p}\big([-1,1],(w_0,w_1)\big) := \{f : [-1,1] \to \mathbb{R} : f \in L^p([-1,1],w_0), f' \in Lp([-1,1],w_1)\}$$

where $w_0 \in L^\infty([-1,1])$ and $w_1 \in B_p([a,b])$. In [229] it is shown that $\mathbb{W}^{1,p}\big([-1,1],(w_0,w_1)\big)$ with the norm $\|f\|^p_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} := \left(\|f\|^p_{L^p([-1,1],w_0)} + \|f'\|^p_{L^p([-1,1],w_1)}\right)^{1/p}$ is a Banach space.
Let $\mathbb{p}$ be the space of polynomials with real coefficients. In general it is not true that
$\mathbb{p} \subset \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, however if we denote by $\mathbb{p}^{1,p}([-1,1],(w_0,w_1))$
the subset $\mathbb{p}^{1,p} \cap ([-1,1],(w_0,w_1))$, then as a consequence of [230] $\mathbb{p}^{1,p}([-1,1],(w_0,w_1))$ is dense in $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ and from Lemma (6.3.2) it follows in a straightforward way that $\mathbb{W}^{1,p}([-1,1],(w_0,w_1)) \subseteq AC([-1,1])$.
Notice that if $\mathbb{p} \subset L^p([-1,1],w_1)$, since $w_1 \in B_p([-1,1])$, then $w_1 \in A_p([-1,1])$. Also, when $p = 2$ and $w_1 \in L^1([-1,1]), \mathbb{W}^{1,2}([-1,1],(w_0,w_1))$ is a Hilbert space and we can consider the sequence of orthonormal polynomials $\{q_n\}_n \geq 0$ associated with the inner Sobolev inner product

$$\langle f,g\rangle_s := \int_{-1}^{1} f(x)g(x)w_0(x)dx + \int_{-1}^{1} f'(x)g'(x)w_1(x)dx \qquad (38)$$

With these remarks in mind, we can give the following definition.
Definition (6.3.4)[225]. Let $\{q_n\}_n \geq 0$ be the sequence of orthonormal polynomials with respect to Sobolev inner product (37). For $1 < p < \infty$ let us consider (w_0,w_1) a vector of weights such that $w_0 \in L^\infty([-1,1])$ and $w_1 \in A_p([-1,1])$. Let $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ and $x \in [-1,1]$, for each $n \geq 0$, we define the n-th Fourier–Sobolev partial sum

$$\mathcal{S}_n(f,x) = \sum_{k=0}^{n} \hat{f}(k)q_k(x), \quad \text{where } \hat{f}(k) = \langle f,q_k\rangle_s \qquad (39)$$

as well as the Fourier–Sobolev expansion of f by means the formal expression

$$\mathcal{S}f \sim \sum_{k=0}^{\infty} \tilde{f}(n)q_n, \qquad (40)$$

In a similar way to the classical case, for each $n \geq 0$ the n-th Fourier–Sobolev partial sum (39) induces a linear operator $\mathcal{S}_n: \mathbb{W}^{1,p}([-1,1],(w_0,w_1)) \to \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ given by
$(\mathcal{S}_n f)(x) := \mathcal{S}_n(f,x)$, for $x \in [-1,1]$.

The following result shows that under the conditions of Definition (6.3.4), the convergence in $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ -norm of the Fourier–Sobolev expansion (40) is equivalent to the uniform boundedness of the operator $\mathcal{s}_n$, for each n.

Theorem (6.3.5)[225]. Let $\{q_n\}_n \geq 0$ be the sequence of orthonormal polynomials with respect to (36). Let (w_0,w_1) be a pair of weight functions such that $w_0 \in L^{\infty}([-1,1])$ and $w_1 \in A_p([-1,1])$ for $1 < p < \infty$. Then the following conditions are equivalent.

(i)$\mathcal{s}_n f \to f$ in $\mathbb{W}^{1,p}\big([-1,1],(w_0,w_1)\big)$, for all $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$.

(ii) There exists $C > 0$, independent of n, such that

$\|\mathcal{s}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq C\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \quad \forall f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$.

Proof. (i) ⇒ (ii) Using the Hölder inequality,

$$\left|\hat{f}(k)\right| = \left|\int_{-1}^{1} f(x)q_k(x)w_0(x)dx + \int_{-1}^{1} f'(x)q_k'(x)w_1(x)dx\right|$$
$$\leq \|f\|_{L^p\mathbb{W}^{1,p}([-1,1],w_0)}\|q_k\|_{L^p\mathbb{W}^{1,p}([-1,1],w_0)} + \|f'\|_{L^p\mathbb{W}^{1,p}([-1,1],w_1)}\|q_k'\|_{L^p\mathbb{W}^{1,p}([-1,1],w_1)}$$

Then for each n, we have $\|\mathcal{s}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq \max(A_n,B_n)\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}$,

where $A_n = \sum_{k=0}^{n}\|q_k\|_{L^p\mathbb{W}^{1,p}([-1,1],w_0)}$ and $B_n = \sum_{k=0}^{n}\|q_k'\|_{L^p\mathbb{W}^{1,p}([-1,1],w_1)}$.

Consequently, $\mathcal{s}_n$ is a continuous operator for each n . Furthermore,

$$\|\mathcal{s}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq \|\mathcal{s}_n f - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} + \|\mathcal{s}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \leq \tilde{C}(f),$$

where $\tilde{C}(f)$ is a constant independent of n . Thus, $\sup_{n\in N}\|\mathcal{s}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} < \infty$

and from Banach - Steinhaus theorem we obtain (ii).

(ii) ⇒ (i) Since $w_0 \in L^{\infty}([-1,1])$ and $w_1 \in A_p([-1,1])$, $1 < p < \infty$, then as a consequence of [230] the linear space P is dense in $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$. Then, given $f \in \mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ and $\varepsilon > 0$, let $p(x) = \sum_{k=0}^{m} a_k q_k(x)$ such that

$\|p - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} < \varepsilon$.

Using that $\mathcal{s}_n p = p$, whenever $n \geq m$, we have

$$\begin{aligned}\|\mathcal{s}_n f - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} &\leq \|\mathcal{s}_n f - \mathcal{s}_n p\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} + \|\mathcal{s}_n p - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\\ &= \|\mathcal{s}_n(f-p)\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} + \|p - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\\ &\leq (C+1)\|p - f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\\ &\leq (C+1)\varepsilon ,\end{aligned}$$

and from these last inequalities we can deduce (i).

The advantage of the previous result is that it allows us to work as in the case of $L^p[-1,1]$, where a similar conditionto (ii) is stated for studying necessary conditions for the mean convergence of the Fourier expansions in terms of classical orthogonal polynomials see [231],[232],[233].

When $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ is a Banach space, some of their properties can be easily deduced taking into account that $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ is a closed subspace of the cartesian product

$L^p([-1,1],w_0) \times L^p([-1,1],w_1)$

with the norm

$$\|u\|_{L^p([-1,1],w_0)\times L^p([-1,1],w_1)} = \|u_1,u_2\|_{L^p([-1,1],w_0)\times L^p([-1,1],w_1)}$$
$$= \begin{cases}\left(\|u_1\|^p_{L^p([-1,1],w_0)} + \|u_1\|^p_{L^p([-1,1],w_1)}\right)^{1/p}, 1 \leq p < \infty,\\ \max\left\{\|u_1\|_{L^{\infty}([-1,1],w_0)},\|u_2\|_{L^{\infty}([-1,1],w_1)}\right\}, p = \infty .\end{cases}$$

Lemma (6.3.6)[225]. Let (w_0, w_1) be a pair of weights on [−1, 1] such that $w_j \in L^1([-1,1])$ and $1 \le p < \infty$. If q is the conjugate of p, i.e. $\frac{1}{p}+\frac{1}{q}=1$, then we can associate with every continuous linear functional $L \in (L^p([-1,1],w_0) \times L^p([-1,1],w_1))'$
a unique $v = (v_1, v_2) \in L^q([-1,1],w_0) \times L^q([-1,1],w_1)$ such that for every
$u = (u_1, u_2) \in (L^p([-1,1],w_0) \times L^p([-1,1],w_1)$

$$L(u) = \langle v_1 , v_2\rangle_{w_0} + \langle u_1 , u_2\rangle_{w_1} = \int_{-1}^{1} u_1(x)v_1(x)w_0(x) + \int_{-1}^{1} u_2(x)v_2(x)w_1(x). \quad (41)$$

Moreover, $\|L\| = \|v\|_{L^q([-1,1],w_0)\times L^q([-1,1],w_1)} = \left(\|v_1\|^p_{L^p([-1,1],w_0)} + \|v_1\|^p_{L^p([-1,1],w_1)}\right)^{1/p}$.
Thus $L \in \left(L^p([-1,1],w_0) \times L^p([-1,1],w_1)\right)' \cong L^q([-1,1],w_0) \times L^q([-1,1],w_1)$.

Proposition (6.3.7)[225]. If (w_0, w_1) is a pair of weights on [−1, 1] such that $w_j \in L^1([-1,1])$, $j = 0, 1$, $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$, $\mathbb{W}^{1,q}([-1,1],(w_0,w_1))$ are Banach spaces, with q the conjugate of $p, 1 \le p < \infty$, then $(\mathbb{W}^{1,p}([-1,1],(w_0,w_1)))' = \mathbb{W}^{1,q}([-1,1],(w_0,w_1))$ and

$$\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} = \sup\{|\langle f , g\rangle_s| \, \|g\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} = 1\}. \quad (42)$$

Theorem (6.3.8)[225]. Let $\{q_n\}_{n\ge0}$ be the sequence of orthonormal polynomials with respect to (36), (w_0, w_1) be a pair of weights such that $w_0 \in L^\infty([-1,1])$ and $w_1 \in A_p([-1,1])$ for $1 < p < \infty$.
If there exists C > 0, independent of n, such that

$$\|\mathcal{S}_n f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \le C\|f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \quad (43)$$

for all $f \in \mathbb{W}^{1,p}\left([-1,1],(w_0,w_1)\right)$, then $\|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \le C$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We apply the same argument as in [234],[235]. Assume that (43) holds, then

$$\|\langle f , q_n\rangle_s q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\|\mathcal{S}_n f - \mathcal{S}_{n-1} f\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} \le 2C$$

with $\mathcal{S}_{-1} \equiv 0$. Now, we consider the functionals L_n on $\mathbb{W}^{1,p}([-1,1],(w_0,w_1))$ given by

$$L_n f := \langle f , q_n\rangle_s \|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}.$$

Hence, for every $f \in \mathbb{W}^{1,p}\left([-1,1],(w_0,w_1)\right)$ we have $\sup_n\{|L_n f|\} < \infty$ and from the Banach–Steinhaus theorem we obtain that $\sup_n\{\|L_n\|\} < \infty$.
On the other hand, taking into account Proposition (6.3.7) we get
$\|L_n\| = \|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}$, where q is the conjugate of p.
Therefore $\|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))}\|q_n\|_{\mathbb{W}^{1,p}([-1,1],(w_0,w_1))} < \infty$.
From the above inequality our statement follows.
We include a well-known result of [309], which allows to find necessary conditions for the convergence of the Fourier expansions in terms of orthogonal polynomials in $L^p\left([-1,1], d_\mu\right)$ norm.

Theorem (6.3.9)[225]. Let $\{p_n\}_n \ge 0$ be a orthonormal system with respect to a non-trivial probability measure d_μ in $[-1,1]$, $\mu' > 0$ *a.e. in* $[-1,1]$ *and* $0 < r < \infty$.
If g is a measurable function in $[-1,1]$, then

$$\int_{-1}^{1}|g(x)|^r(1-x^2)^{\frac{-r}{4}}\mu'(x)^{\frac{-r}{2}}dx \le \pi^{\frac{r}{2}}2^{\max\{1-r/2,0\}} \lim_{n\to\infty}\inf\int_{-1}^{1}|g(x)p_n|^r dx.$$

In particular, if the above inferior limit is 0, then g = 0 a.e.
As an immediate consequence of the above theorem we get the following.

Corollary (6.3.10)[225]. Let $\{p_n\}_n \geq 0$ be an orthonormal system with respect to $d\mu$ supported in $[-1, 1]$, such that $\mu' > 0$ a.e. in $[-1, 1]$, and $1 < p < \infty$. If there exists a constant C, independent of n, such that $\|S_n f\|_{L^p([-1,1],d_\mu)} \leq C\|f\|_{L^p([-1,1],d_\mu)}$, for all $f \in L^p([-1,1], d_\mu)$. Then

(i) $\int_{-1}^{1} d_\mu(x) < \infty$.

(ii) $\int_{-1}^{1}(1-x^2)^{\frac{-p}{4}}\mu'(x)^{\frac{1-p}{2}}dx < \infty$.

From Theorems (6.3.8)and (6.3.9) we get the following.

Theorem (6.3.11)[225]. Let $1 < p < \infty$, $\{p_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ be the sequences of orthonormal polynomials with respect to $w_0 dx$ and $w_0 dx$, respectively. If there exists a constant C such that condition (ii) of Theorem (6.3.7) holds, then

(i)$w_j \in L^1([-1,1])$, $j = 0,1$.

(ii) $\int_{-1}^{1}(1-x^2)^{\frac{-p}{4}}(w_0(x))^{\frac{1-p}{2}}dx < \infty$.

(iii) $\lim_{n\to\infty}\inf\frac{1}{(n+1)\|p_n\|_S}\left(\int_{-1}^{1}|p_n(x)|^p w_0(x)dx\right)^{1/p} < \infty$.

(iv) $\lim_{n\to\infty}\inf\frac{1}{(n+1)\|t_n\|_S}\left(\int_{-1}^{1}|t_n'(x)|^p w_1(x)dx\right)^{1/p} < \infty$.

Proof. From Theorem (6.3.8) we deduce that

$\left(\int_{-1}^{1}|q_n(x)|^p w_0(x)dx\right)^{1/p}\left(\int_{-1}^{1}|q_n(x)|^q w_1(x)dx\right)^{1/p} \leq C.$

Therefore, when n = 0 (i) follows in a straightforward way. Let us consider the function $g_k(x) = q_k(x)w_0$, $k \leq 0$. Then, by Theorem (6.3.9) we have

$\int_{-1}^{1}|g(x)|^p(1-x^2)^{\frac{-p}{4}}(w_0(x))^{\frac{1-p}{2}}dx \leq \pi^{\frac{p}{2}}2^{\max\{1-p/2,0\}}\lim_{n\to\infty}\inf\int_{-1}^{1}|q_k(x)p_n(x)|^p w_0(x)dx$,

for each $k \geq 0$. In particular, when $k = 0$ the above equation becomes condition (ii).

Finally, we only need to prove the condition (iii), taking into account similar arguments yield condition (iv). For $x \in [-1, 1]$, we have that $p_n(x) = \sum_{k=0}^{n}\hat{p}_n(k)q_k(x)$ and by the Cauchy–Schwarz inequality $|p_n(x)|^p \leq \|p_n\|_S^p(\sum_{k=0}^{n}|q_k(x)|)^p$. On the other hand, using the Hölder inequality for finite sums we have $(\sum_{k=0}^{n}|q_k(x)|)^p \leq (n+1)^{p-1}\sum_{k=0}^{n}|q_k(x)|^p$, for every $x \in [-1,1]$. Consequently,

$$\left|\frac{p_n(x)}{(n+1)\|p_n\|_S}\right|^p w_0(x) \leq \frac{1}{n+1}\sum_{k=0}^{n}|q_k(x)|^p w_0 \text{ a.e.} \tag{44}$$

From Theorem (6.3.8), we have $\left(\int_{-1}^{1}|q_k(x)|^p w_0(x)dx\right)^{1/p} \leq C$ for each $k \geq 0$. Therefore,

$$\frac{1}{n+1}\left(\int_{-1}^{1}\left(\sum_{k=0}^{n}|q_n(x)|^p\right)w_0(x)dx\right)^{1/p} \leq C. \tag{45}$$

Condition (iii) is deduced from (44) and (45).

Corollary (5.3.12)[238]. Let $\{q_n\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to (1.1).

Let (w_{m-1}, w_m) be a pair of weight functions such that

$w_{m-1} \in L^\infty([-m,m])$ and $w_m \in A_{1+\varepsilon}([-m,m])$ for $\varepsilon > 0$.

Then the following conditions are equivalent.

(i)$\mathcal{s}_n f \to f$ in $\mathbb{W}^{1,1+\varepsilon}\big([-m,m],(w_{m-1},w_m)\big)$, for all $f \in \mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))$.

(ii) There exists $\tilde{C} > 0$, independent of n, such that

$\|\mathcal{s}_n f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} \leq \tilde{C}\|f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$ $\forall\, f \in \mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))$.

Proof.(i) ⇒ (ii) Using the Hölder inequality,

$$\left|\hat{f}(k)\right| = \left|\int_{-m}^{m} f(x)q_k(x)w_{m-1}(x)dx + \int_{-m}^{m} f'(x)q_k'(x)w_m(x)dx\right|$$
$$\leq \|f\|_{L^{1+\varepsilon}\mathbb{W}^{m,1+\varepsilon}([-m,m],\,w_{m-1})}\|q_k\|_{L^{1+\varepsilon}\mathbb{W}^{m,1+\varepsilon}([-m,m],\,w_{m-1})}$$
$$+\left\|f'\right\|_{L^{1+\varepsilon}\mathbb{W}^{m,1+\varepsilon}([-m,m],\,w_m)}\left\|q_k'\right\|_{L^{1+\varepsilon}\mathbb{W}^{m,1+\varepsilon}([-m,m],\,w_m)}$$

Then for each n, we have

$\|\mathcal{s}_n f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} \leq \max(A_n, B_n)\|f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$,

Where $A_n = \sum_{k=0}^{n}\|q_k\|_{L^{1+\varepsilon}\mathbb{W}^{m,1+\varepsilon}([-m,m],\,w_{m-1})}$ and $B_n = \sum_{k=0}^{n}\left\|q_k'\right\|_{L^{1+\varepsilon}\mathbb{W}^{m,1+\varepsilon}([-m,m],\,w_m)}$.

Consequently, $\mathcal{s}_n$ is a continuous operator for each .

Furthermore,

$$\|\mathcal{s}_n f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} \leq \|\mathcal{s}_n f - f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} + \|\mathcal{s}_n f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$$
$$\leq \tilde{C}(f),$$

where $\tilde{C}(f)$ is a constant independent of n.

Thus, $\sup_{n\in N}\|\mathcal{s}_n f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} < \infty$, from Banach-Steinhaus theorem we obtain (ii).

(ii) ⇒(i) Since $w_{m-1} \in L^{\infty}([-m,m])$and $w_m \in A_{1+\varepsilon}([-m,m])$, $\varepsilon > 0$, then as a consequence of [230] the linear space $\mathbb{P}$ is dense in $\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))$.

Then, given $f \in \mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))$ and $\varepsilon > 0$,let $p(x) = \sum_{k=0}^{m} a_k q_k(x)$ such that $\|p(x) - f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} < \varepsilon$.

Using that $\mathcal{s}_n p(x) = p(x)$, when ever $n \geq m$, we have

$$\|\mathcal{s}_n f - f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$$
$$\leq \|\mathcal{s}_n f - \mathcal{s}_n p(x)\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} + \|\mathcal{s}_n p(x) - f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$$
$$= \|\mathcal{s}_n(f - p(x))\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} + \|p(x) - f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$$
$$\leq \left(\tilde{C}+1\right)\|p(x) - f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$$
$$\leq (\tilde{C}+1)\varepsilon,$$

and from these last inequalities we can deduce (i).

The advantage of the previous result is that it allows us to work as in the case of $L^{1+\varepsilon}[-m,m]$, where a similar conditionto (ii) is stated for studying necessary conditions for the mean convergence of the Fourier expansions in terms of classical orthogonal polynomials (see [231], [232]).

When $\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))$, is a Banach space, some of their properties can be easily deduced taking into account that $\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))$ is a closed subspace of the cartesian product $L^{1+\varepsilon}([-m,m],w_{m-1}) \times L^{1+\varepsilon}([-m,m],w_m)$ with the norm

$$\|u\|_{L^{1+\varepsilon}([-m,m],\,w_{m-1})\times L^{1+\varepsilon}([-m,m],w_m)} = \|u_m,u_{m+1}\|_{L^{1+\varepsilon}([-m,m],\,w_{m-1})\times L^{1+\varepsilon}([-m,m],w_m)}$$

$$= \begin{cases} \left(\|u_m\|^{1+\varepsilon}_{L^{1+\varepsilon}([-m,m],w_{m-1})} + \|u_m\|^{1+\varepsilon}_{L^{1+\varepsilon}([-m,m],w_m)}\right)^{1/1+\varepsilon}, \varepsilon > 0, \\ \max\left\{\|u_m\|_{L^{\infty}([-m,m],\,w_{m-1})}, \|u_{m+1}\|_{L^{\infty}([-m,m],\,w_m)}\right\}, . \end{cases}$$

Corollary (5.3.13)[238]. Let $\{q_n\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to (1.1), (w_{m-1},w_m) be a pair of weights such that $w_{m-1} \in L^{\infty}([-m,m])$ and $w_m \in A_{1+\varepsilon}([-m,m])$ for $\varepsilon > 0$. If there exists $\tilde{C} > 0$, independent of n,

such that $\|\mathcal{S}_n f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} \leq \tilde{C}\|f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$

for all $f \in \mathbb{W}^{1,1+\varepsilon}\big([-m,m],(w_{m-1},w_m)\big)$,

Then $\|q_n\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}\|q_n\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} \leq \tilde{C}$,

Proof. We apply the same argument as in [234]. Assume that (43) holds, then

$\|\langle f, q_n\rangle_s q_n\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}\|\mathcal{S}_n f - \mathcal{S}_{n-1} f\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))} \leq 2\tilde{C}$ with $\mathcal{S}_{-1} \equiv 0$.

Now, we consider the functional L_n on $\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))$

given by $L_n f \coloneqq \langle f, q_n\rangle_s \|q_n\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$.

Hence, for every $f \in \mathbb{W}^{m,1+\varepsilon}\big([-m,m],(w_{m-1},w_m)\big)$

we have $\sup_n\{|L_n f|\} < \infty$ and from the Banach-Steinhaus theorem we obtain that $\sup_n\{\|L_n\|\} < \infty$.

On the other hand, taking into account Proposition (6.3.7) we get

$\|L_n\| = \|q_n\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}\|q_n\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}$,

Where $\frac{1+\varepsilon}{\varepsilon}$ is the conjugate of $1+\varepsilon$. Therefore

$\|q_n\|_{\mathbb{W}^{m,1+\varepsilon}([-m,m],(w_{m-1},w_m))}\|q_n\|_{\mathbb{W}^{1,1+\varepsilon}([-m,m],(w_{m-1},w_m))} < \infty$.

From the above inequality our statement follows.

Corollary (5.3.14)[238]. Let $\varepsilon > 0$, $\{p_n\}_{n\geq 0}$ and $\{t_n\}_{n\geq 0}$ be the sequences of orthonormal polynomials with respect to $w_{m-1}dx$ and $w_m dx$, respectively. If there exists a constant $\tilde{C}$ such that condition (ii) of Theorem (6.3.7) holds, then

(i) $w_j \in L^1([-m,m])$, $j = 0,1$.

(ii) $\int_{-m}^{m}(1-x^2)^{\frac{-1-\varepsilon}{4}}(w_{m-1}(x))^{\frac{-\varepsilon}{2}}dx < \infty$.

(iii) $\lim_{n\to\infty}\inf\frac{1}{(n+1)\|p_n\|_S}\left(\int_{-1}^{m}|p_n(x)|^{1+\varepsilon}w_{m-1}(x)dx\right)^{1/1+\varepsilon} < \infty$.

(iv) $\lim_{n\to\infty}\inf \frac{1}{(n+1)\|t_n\|_S}\left(\int_{-m}^{m}\left|t_n'(x)\right|^{1+\varepsilon} w_m(x)dx\right)^{1/1+\varepsilon} < \infty.$

Proof. From Theorem (6,3.8), we deduce that

$$\left(\int_{-m}^{m}\left|q_n(x)\right|^{1+\varepsilon} w_{m-1}(x)dx\right)^{1/1+\varepsilon}\left(\int_{-m}^{m}\left|\left(q_n\right)_n(x)\right|^{\frac{1+\varepsilon}{\varepsilon}} w_m(x)dx\right)^{1/1+\varepsilon} \le \tilde{C}.$$

Therefore , when $n = 0$ (i) follows in a straight forward way.

Let us consider the function $(f_\varepsilon)_k(x) = q_k(x)w_m$, $k \le 0$ .Then by Theorem (6.3.9) we have

$$\int_{-m}^{m}|q_k(x)|^{1+\varepsilon}(1-x^2)^{\frac{-1-\varepsilon}{4}}\left(w_{m-1}(x)\right)^{\frac{-\varepsilon}{2}}dx \le \pi^{\frac{1+\varepsilon}{2}}2^{\max\left\{-\varepsilon/2,0\right\}}\lim_{n\to\infty}\inf\int_{-m}^{m}\left|q_k(x)p_n(x)\right|^{1+\varepsilon}w_{m-1}(x)dx,$$

for each $k \ge 0$. In particular, when $k = 0$ the above equation becomes condition (ii).

Finally, we need to prove the condition (iii), taking into account similar arguments yield condition (iv).

For $x \in [-m, m]$, we have that $p_n(x) = \sum_{k=0}^{n}\hat{p}_n(k)q_k(x)$ and by theCauchy-Schwarz inequality

$$\left|p_n(x)\right|^{1+\varepsilon} \le \left\|p_n(x)\right\|_S^{1+\varepsilon}\left(\sum_{k=0}^{n}\left|q_k(x)\right|\right)^{1+\varepsilon}.$$

On the other hand, using the Hölder inequality for finite sums we have

$$\left(\sum_{K=0}^{n}\left|q_k(x)\right|\right)^{1+\varepsilon} \le (n+1)^{\varepsilon}\sum_{k=0}^{n}\left|q_k(x)\right|^{1+\varepsilon},$$

for every $x \in [-m, m]$.

Consequently,

$$\left|\frac{p_n(x)}{(n+1)\left\|p_n\right\|_S}\right|^{1+\varepsilon} w_{m-1}(x) \le \frac{1}{n+1}\sum_{k=0}^{n}\left|q_k(x)\right|^{1+\varepsilon} w_{m-1} \text{ a.e.}$$

From Theorem (6.3.8), we have

$$\left(\int_{-m}^{m}\left|q_k(x)\right|^{1+\varepsilon} w_{m-1}(x)dx\right)^{1/1+\varepsilon} \le \tilde{C}$$

for each $k \ge 0$.

Therefore, $\frac{1}{n+1}\left(\int_{-m}^{m}\left(\sum_{k=0}^{n}\left|q_n(x)\right|^{1+\varepsilon}\right)w_{m-1}(x)dx\right)^{1/1+\varepsilon} \le \tilde{C}$. Condition (iii) is deduced from (44) and (45).

Corollary (5.3.15)[238]. For $\varepsilon > 0$, $\left\{\left(\sqrt{2+\varepsilon}\right)_n\right\}_{n\ge 0}$ and $\{t_n\}_{n\ge 0}$ be the sequences of orthonormal polynomials with respect to $w_{m-1}dx$ and $w_m dx$, respectively. If there exists a constant $\tilde{C}$ such that condition (ii) of Theorem (6.3.7) holds, then

(i) $w_j \in L^1([-m, m])$, $j = 0,1$.

(ii) $\int_{-m}^{m}(1-x^2)^{\frac{-\sqrt{2+\varepsilon}}{4}}(w_{m-1}(x))^{\frac{1-\sqrt{2+\varepsilon}}{2}}dx < \infty.$

(iii) $\lim_{n\to\infty}\inf\frac{1}{(n+1)\left\|\left(\sqrt{2+\varepsilon}\right)_n\right\|_S}\left(\int_{-1}^{m}\left|\left(\sqrt{2+\varepsilon}\right)_n(x)\right|^{\sqrt{2+\varepsilon}}w_{m-1}(x)dx\right)^{1/\sqrt{2+\varepsilon}} < \infty.$

(iv) $\lim_{n\to\infty}\inf\frac{1}{(n+1)\|t_n\|_S}\left(\int_{-m}^{m}\left|t_n'(x)\right|^{\sqrt{2+\varepsilon}}w_m(x)dx\right)^{1/\sqrt{2+\varepsilon}} < \infty.$

Proof. We deduce , using Theorem 3.4 that

$$\left(\int_{-m}^{m}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_n(x)\right|^{\sqrt{2+\varepsilon}}w_{m-1}(x)dx\right)^{1/\sqrt{2+\varepsilon}}\left(\int_{-m}^{m}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_n(x)\right|^{\sqrt{2+\varepsilon}-\varepsilon_1}w_m(x)dx\right)^{1/\sqrt{2+\varepsilon}-\varepsilon_1}$$

$$\leq \widetilde{C}.$$

So when $n = 0$ (i) follows .Consider the function

$(f_\varepsilon)_k(x) = \left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)w_{m-1}{}^{\frac{1}{\sqrt{2+\varepsilon}}}, k \geq 0$ .Theorem (6.3.9),then show that

$$\int_{-m}^{m}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\right|^{\sqrt{2+\varepsilon}}(1-x^2)^{\frac{-\sqrt{2+\varepsilon}}{4}}\left(w_{m-1}(x)\right)^{\frac{-\sqrt{2+\varepsilon}}{2}}dx$$

$$\leq \pi^{\frac{\sqrt{2+\varepsilon}}{2}}2^{\max\left\{-\sqrt{2+\varepsilon}/2,0\right\}}\lim_{n\to\infty}\inf\int_{-m}^{m}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\left(\sqrt{2+\varepsilon}\right)_n(x)\right|^{\sqrt{2+\varepsilon}}w_{m-1}(x)dx,$$

for any $k \geq 0$, when $k = 0$ satisfy (ii).

To prove the condition (iii) similarly for condition (iv). For $-m \leq x \leq m$ we have given that

$\left(\sqrt{2+\varepsilon}\right)_n(x) = \sum_{k=0}^{n}\widehat{\left(\sqrt{2+\varepsilon}\right)_n}(k)\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x).$

Using Cauchy-Schwarz inequality we get

$$\left|\left(\sqrt{2+\varepsilon}\right)_n(x)\right|^{\sqrt{2+\varepsilon}} \leq \left\|\left(\sqrt{2+\varepsilon}\right)_n(x)\right\|_S^{\sqrt{2+\varepsilon}}\left(\sum_{k=0}^{n}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\right|\right)^{\sqrt{2+\varepsilon}}.$$

Using the Hölder inequality for finite sums we have

$$\left(\sum_{k=0}^{n}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\right|\right)^{\sqrt{2+\varepsilon}} \leq (n+1)^{\sqrt{2+\varepsilon}-1}\sum_{k=0}^{n}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\right|^{\sqrt{2+\varepsilon}},$$

for $-m \leq x \leq m$. Hence, $\left|\frac{\left(\sqrt{2+\varepsilon}\right)_n(x)}{(n+1)\left\|\left(\sqrt{2+\varepsilon}\right)_n\right\|_S}\right|^{\sqrt{2+\varepsilon}}w_{m-1}(x) \leq \frac{1}{n+1}\sum_{k=0}^{n}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\right|^{\sqrt{2+\varepsilon}}w_{m-1}$ a.e.

From Theorem (6.3.8), gives that

$$\left(\int_{-m}^{m}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\right|^{\sqrt{2+\varepsilon}}w_{m-1}(x)dx\right)^{1/\sqrt{2+\varepsilon}} \leq \widetilde{C} \text{ for any } k \geq 0. \quad \text{Hence,}$$

$$\frac{1}{n+1}\left(\int_{-m}^{m}\left(\sum_{k=0}^{n}\left|\left(\sqrt{2+\varepsilon}-\varepsilon_1\right)_k(x)\right|^{\sqrt{2+\varepsilon}}\right)w_{m-1}(x)dx\right)^{1/\sqrt{2+\varepsilon}} \leq \widetilde{C}.$$

Now condition (iii) follows.

List of symbols

Symbol			Page
L^p	:	Lebesgue Space	1
Loc	:	Local	1
Sup	:	Supremum	1
L^1	:	Lebesgue Space on the real line	3
Max	:	maximum	7
cl	:	closure	12
Supp	:	Support	14
deg	:	degree	14
Reg	:	Regular	15
cap	:	logarithmic capacity	20
q.e	:	quasi everywhere	20
inf	:	infimum	21
H^s	:	Fractional Sobolev Space	26
L^2	:	Hilbert Space	26
H_0^1	:	Trace Sobolev Space Embedding	26

$W^{1,p}$:	Sobolev Space	27
a.e :	almost everywhere	28
dist :	distance	37
L^{∞} :	Essential Lebesgue Space Operater	38
op :	operator	44
BV :	Bounded Variation	47
per :	perimeter	52
VMO :	Vanishing Mean Oscillation	53
$F_{p,q}^{s}$:	Triebel-lizorkin Spaces	54
$B_{p,q}^{s}$:	Besov Spaces	55
min :	minimum	64
Lip :	Lipschitz	67
ess :	essential	80
int :	interior	81
L^{q} :	Lebesgue dual Space	137
S_{p}^{α} :	Sobolev-type Space	158
B_{p} :	Kufner-opic type property	169