Chapter One

Introduction

1.1 History of Macroscopic and Microscopic World:

The physics starts from the trials of scientists to describe the physical behavior of the objects that can be seen and known by our common sense. These objects are known as macro systems. The macro systems are described physically by the classical laws of physics like Newton's law. These objects are related to the so called matter. Our world also consists of the so called energy, like electromagnetic waves (e.m.w) [1, 2]. This e.m.w can be described by Maxwell's equations. Maxwell's equations are one of the biggest achievements that describe the behavior of electromagnetic waves (e.m.w) they describe interference, diffraction of light, as well as generation, reflection, transmittance and interaction of electromagnetic waves with matter[3, 4, 5].

The light was accepted as having a wave nature for long time. But, unfortunately, this nature was unable to describe black body radiation phenomenon. This forces Max Plank to propose that light and electromagnetic waves behave as discrete particles known later as photons. This particle nature succeeded in describing a number of physical phenomena, like atomic radiation, photoelectric, Compton and pair production effects. The pair production effect needs particle nature of light as well as special

relativity (SR) to be explained [6, 7]. This dual nature of light encourages De Broglie to propose that particles like electrons can behave some times as waves. The experimental confirmation of this hypothesis leads to formation of new physical laws known as quantum mechanics. Quantum mechanics (QM) is formulated by Heisenberg first and independently by Schrödinger, to describe the dual nature of the atomic world [8].

Quantum theory starts from the discovery of Max Plank, that light can be treated discrete quanta, known recently as photons. This means that waves can behave sometimes like particles. This encourages De Broglie to propose that particles can also behave like waves. This dual nature of microscopic particles, leads to proposing a new physical framework known as quantum mechanics (QM) [9, 10, 11].

The laws of quantum mechanics are now widely used to describe the behavior of atomic and subatomic particles beside Nano particles [12]. The spectrum of any atom beside some electrical and magnetic properties can be easily described by the laws of quantum mechanic [13, 14].

Despite these remarkable successes of quantum mechanic, it suffers from noticeable set backs. For instance, there is no full quantum theory that can describe the behavior of superconductors (SC). The behavior of Nano systems are now far from being described fully by quantum mechanic. The situation for elementary

particles, fields is even worse. There is no theoretical model that can put gravity under the umbrella of quantum mechanic [15].

The dream of unification of forces is too difficult to be achieved within the present physical theories including quantum mechanics [16]. These failures may be related to mathematical and physical laws are based on the dual nature of wave packets beside the energy expression in classical mechanics and relativity [17]. Unfortunately the energy expression take care of the effect of the field potentials only, without accounting other effects that can change the behavior of the particle under study.

1.2 Research Problem:

The lack of quantum gravity theory and the failure to explain some superconductor's behaviors indicates the need for new laws of quantum mechanics; these new laws are needed also to explain the behavior of Nano particles.

1.3 Literature Review:

Different attempts were made to construct new quantum laws [17], one of them is proposed by Khalid Haroon [18]. It is based on the form of the electric field intensity in a damping media. Another attempt was also made by Kamil Elsaid Algailani to construct Klein-Gordon equations [19]. But no one of them directly uses Maxwell's equations to construct directly and simply a quantum equation that accounts for the effect of friction and the bulk matter.

1.4 Aim of the Work:

The aim of this work is to relate microscopic atomic and subatomic word to macroscopic world by deriving a quantum equation form one of the classical equations.

1.5 Presentation of the Thesis:

This thesis composed of five chapters; chapter one is an introduction, chapter two contains a derivation of Maxwell's equation by using ordinary laws of electricity and magnetism. Chapter three is devoted to derive Schrödinger and Klein-Gordon equation, chapter four is concerned for the literature review, while the contribution is given in chapter five.

Chapter Two

Maxwell's Equations

2.1 Introduction:

The basic electrodynamics equations are usually deriven from laws of a general course of electricity and magnetism [20]. These laws can describe the behavior of electromagnetic fields inside matter as well as free space. In this chapter Maxwell Equations (M.E) are derived by utilizing the basic laws of electricity and magnetism [21].

2.2 Electric and Magnetic Field Intensity:

It is well Know that, the electromagnetic field in a medium is described by four vectors quantities the electromagnetic field, the electric induction, the magnetic field and the magnetic induction. The force acting on unit electric charge at a given point in space is called the electric field intensity [22].

In future, instead of the field intensity one can simply speak of the field at a given point in space. The magnetic field intensity or, for short the magnetic field is defined analogously, separate magnetic charge, unlike electric charges, don't exists in nature, however if we make a long permanent magnet in the from of a needle, then the magnetic force acting at it's ends will be the same as if there existed point charges at the end [23].

A rigorous definition of the electric and magnetic induction vectors where the field equations in a medium will be derived from the equations for point charges in free space. It need only be recalled that in free space. There is no need to use four vectors for a description of the electromagnetic field, only two vectors being sufficient: the electric and magnetic fields [24].

2.3 Electromotive Force:

One can recall the definition for electromotive force in circuit this is the work performed by the forces of the electric field when unit charge is taken along the given closed circuit [23]. It is absolutely immaterial what the given circuit represents: whether it is filled with a conductor or whether it is merely a closed line drown in space (e.m.f). The force acting on unit charge at a given point is the electric field E. The work done by this force on an element of path dl is the scalar product E.dl. Then the work done on the whole closed circuit. Or the e.m.f is equal to the integral [25]:

$$V = e.m.f = \int E\,dl \qquad (2.3.1)$$

Where V is the induction potential

2.4 Magnetic Field Flux Across a Surface:

Let us suppose that some surface is bounded by the given circuit. We shall denote the magnetic field by the letter H. The

magnetic field flux through an element of the chosen surface dS is given by [26] $\int d\Phi = \int H\, dS$

The magnetic field flux through the whole surface, bounded by the circuit, is given by

$$\Phi = \int H.dS = \int B\, d\, S \qquad (2.4.1)$$

Where B is the magnetic field density

One can consider a section of the surface through which unit flux $\Delta\Phi = 1$ passes. If one draws through this section of the surface a line tangential to the direction of the field at some point on the surface. A line which is tangential to the direction of the field at its point is called a magnetic line of force. For this reason the total flux Φ is equal, by definition, to number of magnetic line of force crossing the surface [27].

Magnetic line force are either closed or extended to infinity. Indeed a magnetic line force may being or end only at a single charge, but separate magnetic charges do not exist in nature. In a permanent magnet the lines of force are completed inside the magnet. From this it follows that a magnet flux through any surface, bounded by circuit, is the same at a given instant. Other wise, a number of the magnetic lines of force would have to begin or end in the space between the surfaces through which different fluxes pass. Consequently, at a given instant a constant a number of magnetic lines force, i.e. a constant magnetic field flux passes

across any surface bounded by the circuit. Therefore, the flux can be a scribed to the circuit it self, Irrespective of the surface for which it is calculated [28].

2.5 Faraday's Induction Law:

Faraday's induction law is written in the form of the following equation

$$V = e.m.f = -\frac{1}{c}\frac{\partial \Phi}{\partial t} \qquad (2.5.1)$$

The constant of proportionality c is a universal constant with the dimensions of velocity equals to $3\times10^{8}\ ms^{-1}$. Usually, Faraday's law is applied to circuits of conductors, however, e.m.f is simply the quantity of work performed by unit charge in moving a long the circuit, and for a given field value through the circuit, cannot depend upon the form of the circuit. The e.m.f is simply equal to the integral $\int E\,dl$.

In a conducting circuit, this work can be dissipated in the generation of Joule heat (an ohmic load). However it is completely justifiable to consider the circuit in a vacuum also. In this case, the work performed on the charge is spent in increasing the kinetic energy of the charged particles, as for instance in the case in an induction accelerator, the betatron [29].

2.6 Maxwell's Equations:

Equation (2.5.1) refers to any arbitrary closed circuit. We substitute the definitions (2.3.1) and (2.4.1) into this equation we get [30]:

$$\int E\,dl = \frac{1}{c}\frac{\partial}{\partial t}\int B\,dS \qquad (2.6.1)$$

The left hand side of the equation can be transformed by the stokes theorem: which state that; the line integral of the tangential component of a vector A taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of A taken over any surface S having C as it's boundary [31] i.e.

$$\int A dl = \int (\nabla \times A)\,dS \qquad (2.6.2)$$

When one takes A = E, Then

$$\int E\,dl = \int (\nabla \times E)\,dS \qquad (2.6.3)$$

Thus equation (2.6.1) becomes

$$\int (\nabla \times E)\,dS = -\frac{1}{c}\frac{\partial}{\partial t}\int B\,dS = -\frac{1}{c}\int \frac{\partial B}{\partial t}\,dS \qquad (2.6.4)$$

Where, on the right hand side, the order of time differentiation and surface integration is interchanged. Thus taking this integral over to the left hand side, one obtains

$$\int ((\nabla \times E) + \frac{1}{c}\frac{\partial B}{\partial t})\,dS = 0 \qquad (2.6.5)$$

But initial circuit is completely arbitrary. I.e. it can have arbitrary magnitude and shape. Let us assume that the integrand, in parentheses, of equation (2.6.5) is not equal to zero. Then one can chose the surface and the circuit that bounds it so that the integral (2.6.5) does not become zero. Thus, in all cases the following equation must be satisfied

$$\nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \qquad (2.6.6)$$

Which is one of the Maxwell's equations relating electric and magnetic fields in differential form? In many applications the differential form is more convenient than the integral form. Magnetic field lines of force are either closed or go off to infinity. Hence, in any closed surface, the same number of magnetic field lines enters as leave. The magnetic field flux, in free space, across any closed surface [32], is equal to

$$\Phi = \int B\, dS = 0 \qquad (2.6.7)$$

Transforming this integral to a volume integral according to the Gauss-Ostrogradsky theorem [33]:

$$\int A\, dS = \int \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dv = \int (\nabla . A) dv \qquad (2.6.8)$$

One obtains

$$\Phi = \int B\, dS = \int (\nabla . B)\, dv = 0 \qquad (2.6.9)$$

Due to the fact that the surface bonding the volume is completely arbitrary, we can always choose this volume to be so small that the integral is taken over the region in witch $\nabla . B$ has constant sign if it is not equal to zero. But then in spite of (2.6.7) and (2.6.9) $\int \nabla . B\, dS$ will not be equal to zero.

For this reason, the divergence of B must become zero. Thus

$$\nabla . B = 0 \qquad (2.6.10)$$

This is the differential form of (2.6.7) for an infinitely small volume. Since the divergence of a vector is the density of source of a vector field. The sources of the field are free charges from which the vector (force) magnetic field lines originate. Thus (2.6.10) indicates the absences of free magnetic charges.

The equations (2.6.6) and (2.6.10) are together called the first pair of Maxwell's equations. The electric field flux through a closed surface is not equal to zero. But to the total electric charge q in side the surface multiplied by 4π [34]

$$\int D . dS = 4\pi q \qquad (2.6.11)$$

Where, D is the electric flux density, the field due to a point charge q is expressed by the following equation.

$$E = \frac{q}{r^2} \qquad (2.6.12)$$

Then the field is inversely proportional to r^2 if one surrounds the charge by a spherical surface centered on the charge. The element

of the surface for the sphere dS is $r^2 d\Omega$ where $d\Omega$ an elementary solid angle.

The flux of the field across the surface element is given by [35]

$$D.dS = \frac{q}{r^2}.r^2\, d\Omega = q\, d\Omega \qquad (2.6.13)$$

The flux across the whole surface of the sphere is thus given by

$$\int q.d\Omega = q\int d\Omega = 4\pi q \qquad (2.6.14)$$

But since lines of force begin only at a charge the flux will be the same through the sphere as through any closed surface around the charge .Therefore if there is an arbitrary charge ,distribution q inside a closed surface, then equation (2.6.11)holds. In order to rewrite this equation, in differential form, we introduce the concept of charge density. The charge density ρ is the charge contained in unit volume, so that the total charge in the volume is related to the density by the following equation

$$q = \int \rho\, dv \qquad (2.6.15)$$

Introducing the charge density in (2.6.11), and utilizing the relation

$$\int D\,dS = \int \nabla.D\,dv$$

$$\int (\nabla.D - 4\pi\rho)\, dv = 0 \qquad (2.6.16)$$

Repeating the same argument for this integral as used (2.6.9) one have

$$\nabla . D = 4\pi\rho \qquad (2.6.17)$$

According to (2.6.9) one can say that density of sources of an electric field is equal to the electric charge density multiplied by 4π. [36]

2.7 Electromagnetic Potentials:

One can introduce new unknown quantities such that each equation will contain only one unknown .In this way overall number of equations is reduced. These new quantities are called electromagnetic potential .Thus for the magnetic field one can define by $B = \nabla \times A$ where A is a vector called the vector potential and for the electric field the electric potential is defined to satisfy

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\phi$$

Where ϕ is also called the scalar potential

2.8 Magnetomotive Force:

By analogy with electromotive Force $\int E\, dl$ one can define the magnetomotive force $\int H\, dl$, where the integration is performed over a closed circuit. Using Ampere's law, it may be shown that the integration of H in a closed circuit is equal to the summation of the Electric current I surrounded by the magnetic loop. In other word

$$\int H\,dl = \sum I$$

$$\sum J\,dS = \int J.dS \qquad (2.8.1)$$

But according to vector algebra

$$\int H\,dl = \int (\nabla \times H)\,dS$$

Hence

$$\int (\nabla \times H)\,dS = \int J\,dS$$

$$\int (\nabla \times H - J)\,dS = 0$$

This relation can be satisfied if

$$(\nabla \times H - J) = 0$$

$$\nabla \times H = J \qquad (2.8.2)$$

This relation holds for static magnetic field and constant current which doesn't vary with time. But it is no larger hold for time dependant, current and field. To verify this take divergence of both sides of equation (2.8.2), one gets

$$\nabla.(\nabla \times H) = \nabla.J \qquad (2.8.3)$$

But for vector algebra

$$\nabla.(\nabla \times H) = |\nabla||\nabla \times H|\cos 90 = 0 \qquad (2.8.4)$$

Hence

$$\nabla.J = 0 \qquad (2.8.5)$$

where J is the current density in infinitesimal area, mean while if

the electric field E is not stable, i.e .,varying with respect to time, and the variation frequency is high enough and extends into the radar frequency, there will be another current in the medium known as the displacement current and is proportional to the variation of the electric field E, and the proportionality factor is the dielectric permittivity ε.Thus, there will be another contributor, $\partial D/\partial t$, to induce the magnetic field H. The displacement current works exactly the same way as the conductive current J, so that the total current works is $J+\partial D/\partial t$. Put both contributors into the above equation ends up with other Maxwell's equation.

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad (2.8.6)$$

This equation can be derived mathematically by using continuity equation

$$\nabla . J + \frac{\partial \rho}{\partial t} = 0$$

Thus

$$\nabla . J = -\frac{\partial \rho}{\partial t} \qquad (2.8.7)$$

In view of equation (2.8.2) let:

$$\nabla \times H = J + G \qquad (2.8.8)$$

Taking the divergence for both sides yields [see (2.8.4)]

$$\nabla . \nabla \times H = \nabla . J + \nabla . G = 0$$

$$\nabla . J = -\nabla . G \tag{2.8.9}$$

But since

$$\nabla . D = \rho \tag{2.8.10}$$

Thus equation (2.8.7) reads

$$\nabla . J = -\nabla . \frac{\partial D}{\partial t} \tag{2.8.11}$$

Comparing (2.8.9) and (2.8.11)

$$G = \frac{\partial D}{\partial t} \tag{2.8.12}$$

In view of equation (2.8.8)

$$H = J + \frac{\partial D}{\partial t} \tag{2.8.13}$$

Chapter Three

Schrödinger and Klein-Gordon Equations

3.1 Introduction:

The Schrödinger equation is the fundamental of quantum mechanics and the starting point for any improvement to the description of submicroscopic physical systems [37]. Although it cannot be proved or derived strictly, it has associated with it various formulations and derivations [38].

The Klein- Gordon equation is analog of the Schrödinger equation which tries to make quantum mechanics compatible with special relativity unlike the Schrödinger equation which is compatible only with Galilean relativity. Historically, the Klein-Gordon equation invented by Schrödinger even be for Klein and Gordon in the context of understanding the fine structure of the hydrogen spectrum but was abounded by him as it did not give him the right results[39].

In this chapter we try to obtain the Schrödinger equation for a particle with energy E and momentum p traveling in the x direction, and then we apply the relativistic energy E to obtain the Klein-Gordon equation [40].

3.2 Derivation of Schrödinger Equation:

Suppose the wave function for plane wave travelling in the x direction with a well defined energy and momentum that is [41]:

$$\psi = Ae^{i/\hbar(px-Et)} \qquad (3.2.1)$$

Where

$$E = \hbar\omega$$

$$p = \hbar k$$

For a particle moving in a potential energy field we write the energy according to the relation

$$E = \frac{p^2}{2m} + V(x) \qquad (3.2.2)$$

Multiplying the both sides of equation (3-2-2) by ψ, one gets

$$E\psi = \frac{p^2}{2m}\psi + V(x)\psi \qquad (3.2.3)$$

From equation (3-2-1) we see that for the equality to hold the product of energy times the wave function $E\psi(x,t)$ must be equal to the first derivation of the wave function with respect to time multiplied by $i\hbar$, that is [42]:

$$\frac{\partial\psi}{\partial t} = \frac{-iE}{\hbar} Ae^{\frac{i}{\hbar}(px-Et)}$$

$$= \frac{-iE}{\hbar}\psi$$

$$E\psi = \frac{-\hbar}{i}\frac{\partial\psi}{\partial t}$$

$$E\psi = \frac{-i\hbar}{i^2}\frac{\partial\psi}{\partial t}$$

$$E\psi = +i\hbar\frac{\partial\psi}{\partial t} \qquad (3.2.4)$$

Similarly by examining equation (3-2-1) we see that:

$$\frac{\partial\psi}{\partial x} = \frac{i}{\hbar}pAe^{\frac{i}{\hbar}(px-Et)}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{i^2}{\hbar^2}p^2Ae^{\frac{i}{\hbar}(px-Et)}$$

$$= \frac{-1}{\hbar^2}p^2\psi$$

$$p^2\psi = -\hbar^2\frac{\partial^2\psi}{\partial x^2} \qquad (3.2.5)$$

Inserting equations (3-2-4) and (3-2-5) in equation (3-2-3) hence one get

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi \qquad (3.2.6)$$

Which is the famous Schrödinger equation.

3.3 Derivation of Klein-Gordon Equation:

The Schrödinger equation is motivated by taking a look at the classical relation between energy and momentum of particle, quantization is done by replacing the physical quantities by operators corresponding to them and state or wave function on which they operate. These corresponding operators for the energy and momentum are given by [43, 44, 45]:

$$p \rightarrow -i\hbar \vec{\nabla} \qquad (3.3.1)$$

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \qquad (3.3.2)$$

Assuming the case of a free particle one get the following relation between momentum and energy

$$E = \frac{p^2}{2m} \qquad (3-3-3)$$

Multiplying both sides of equation (3-3-3) by ψ , one gets

$$E\psi = \frac{p^2}{2m}\psi \qquad (3-3-4)$$

Substituting the operators in (3-3-1) and (3-3-2) to equation (3-3-4), one gets

$$E\psi = i\hbar \frac{\partial \psi}{\partial t} \qquad (3-3-5)$$

$$\frac{p^2}{2m}\psi = (-i\hbar\nabla)^2\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

$$\frac{p^2}{2m}\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \qquad (3-3-6)$$

Inserting equations (3-3-5) and (3-3-6) in equation (3-3-4), one gets

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \qquad (3-3-7)$$

Equation (3-3-7) is a Schrödinger equation for a free particle. We could now assume that we could obtain the relativistic version of Schrödinger equation by simply repeating the same procedure with relativistic correlation between momentum and energy [46, 47].

$$E = \sqrt{p^2c^2 + m_0{}^2c^4} \qquad (3-3-8)$$

$$E^2 = p^2 c^2 + m_0{}^2 c^4 \qquad (3-3-9)$$

Where

$E \equiv$ *energy of particle*

$p \equiv$ *particle momentum*

$c \equiv$ *speed of light*

$m_0 \equiv$ *mass of rest electron*

Suppose the wave function for plane wave travelling in x direction is given by:

$$\psi = Ae^{\frac{i}{\hbar}(px-Et)} \qquad (3-3-10)$$

Where

$$p = \hbar k$$

$$E = \hbar \omega$$

Multiplying both sides of equation (3-3-9) by ψ, one gets

$$E^2\psi = p^2 c^2 \psi + m_0{}^2 c^4 \psi \qquad (3-3-11)$$

From equation (3-3-10)

$$\frac{\partial \psi}{\partial t} = \frac{-i E}{\hbar} Ae^{\frac{i}{\hbar}(px-Et)}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{+i^2 E}{\hbar^2} Ae^{\frac{i}{\hbar}(px-Et)}$$

$$= \frac{-E^2}{\hbar^2} A e^{\frac{i}{\hbar}(px-Et)}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{-E^2}{\hbar^2} \psi$$

$$\Rightarrow \quad E^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial t^2} \qquad (3-3-12)$$

Also from equation (3-3-10)

$$\frac{\partial \psi}{\partial x} = \frac{i\,p}{\hbar} A\, e^{\frac{i}{\hbar}(px-Et)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{i^2\, p^2}{\hbar^2} A\, e^{\frac{i}{\hbar}(px-Et)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{-p^2}{\hbar^2} \psi$$

$$\Rightarrow \quad p^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} \qquad (3-3-13)$$

Inserting equations (3-3-12) and (3-3-13) in equation (3-3-11), one gets

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -c^2 \hbar^2 \frac{\partial^2 \psi}{\partial x^2} + m_0{}^2 c^4 \psi$$

$$\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = c^2 \hbar^2 \frac{\partial^2 \psi}{\partial x^2} - m_0{}^2 c^4 \psi \qquad (3-3-14)$$

Which is the Klein-Gordon equation

The Klein-Gordon equation describes a wide variety of physical phenomena such as in wave propagation in continuum mechanics and in the theoretical description of spin less particles in relativistic quantum mechanics [48, 49, 50].

Chapter Four

Literature Review

4.1 Introduction:

Different attempts were made to construct new quantum laws [51, 52], some of them based on the form of the electric field intensity in a damping media, and others to construct Klein-Gordon equation. But none of them directly uses Maxwell's equations to construct directly and simply a quantum equation that accounts for the effect of friction and the bulk matter [53]. Here in this chapter one tries to mention some of them.

4.2 Derivation of Schrödinger Equation from Variational Principle:

Sami.H.Altoum derived Schrödinger equation by using variational principle. The calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functional. Which are mappings from a set function to the real numbers. Functional are often expressed as definite integrals involving functions and their derivatives. The interest is in external functions that make the functional attain a maximum or minimum value or stationary functions those where the rate of change of the functional is zero [54].

In several problems of physics and mechanics it is convenient to recast Euler's equations in canonical form, which

make possible a general approach to variational problems. Further, the new variable introduced in the process admits of a simple physical interpretation [55].

Consider the extremum of the functional

$$I[y_1, y_2,....., y_n] = \int_{x_1}^{x_2} F(x.y_1, y_2,....., y_n, y_1', y_2',...., y_n')\, dx \qquad (4.2.1)$$

Where $y_1(x),...., y_n(x)$ satisfy certain boundary conditions at x_1 and x_2. The Euler equations are

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0 \quad , i = 1,2,....., n \qquad (4.2.2)$$

Which constitute a system of n ordinary differential equations in $y_1(x),...., y_n(x)$ we introduce

$$p_i = F_{y'_i}(x, y_1,.....y_1',....., y_n') \quad , i = 1,2,....., n \qquad (4.2.3)$$

Which together with $y_i (i = 1,2,...., n)$ are called canonical variables for the above functional. The variables y_i and p_i are known as canonically conjugate variables. Then (4.2.2) gives

$$\frac{dp_i}{dx} = \frac{\partial F}{\partial y_i} \ , \ i = 1,2,....., n \qquad (4.2.4)$$

Now if the Jacobian

$$\frac{D(F_{y_1'}, F_{y_2'},...., F_{y_n'})}{D(y_1', y_2',.....y_n')} \neq 0 \ ,$$

The system of equation (4.2.3) can be solved as

$$y_i' = \omega_i(x, y_1, ..., y_n, p_1, ..., p_n)$$

When these are substituted into (4.2.4), we get a system of first order equation as

$$\frac{dy_i}{dx} = \omega_i(x, y_1, ..., y_n, p_1, ...p_n), \frac{dp_i}{dx} = \frac{dF}{dy_i} \quad with \ i = 1,2,.....n \tag{4.2.5}$$

Henceforward the parentheses in the second equation of (4.2.5) signify that y_i' in F are replaced by ω_i. We now introduce the Hamiltonian function

$$H(x, y_1, ..., y_n, p_1, ...p_n) = \sum_{i=1}^{n} \omega_i \ p_i - F \tag{4.2.6}$$

Then the system can be written as

$$\frac{dy_i}{dx} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dx} = -\frac{\partial H}{\partial y_i} \quad , \ i = 1,2,....., n \tag{4.2.7}$$

This system is referred to as the Hamiltonian (canonical) system of Euler's equation and second ordinary equations in second unknown functions $y_i(x) \ and \ p_i$

The fundamental equation of quantum mechanics (Schrödinger equation) can be derived from a variational principle.

First we define an operator known as the Hamiltonian operator as follows:

$$H = -k\nabla^2 + V(x, y, z) \tag{4.2.8}$$

Here $k = h^2 / (8\pi^2 m)$, where h and m stand for the Plank's constant the mass of principle whose motion is considered in a field of potential energy V. We now seek a wave function ψ

Possibly complex extremize the functional

$$\iiint \psi^* (H\psi)\, dxdydz \qquad (4.2.9)$$

Subject to constraint

$$\iiint \psi^* \psi\, dxdydz = 1 \qquad (4.2.10)$$

Where ψ^* is the complex conjugate of ψ.The integration is over a fixed domain of x, y and z, we further assume that the admissible function ψ and ψ^* either vanish at corresponding points on opposite boundaries. As a consequence

$$\iiint \psi^* \nabla^2 \psi\, dxdydz = -\iiint \nabla\psi^* . \nabla\psi\, dxdydz$$

Introducing Lagrange multiplier λ, we then find the extremum of the functional

$$\iiint K\, dxdydz = -\iiint [k(\nabla\psi_x^* \psi_x + \psi_y^* + \psi_z^* \psi_z) + V\psi^*\psi - \lambda\psi^*\psi]dxdydz$$

The Euler equation is

$$\frac{\partial K}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial K}{\partial \psi_X}\right) - \frac{\partial}{\partial y}\left(\frac{\partial K}{\partial \psi_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial K}{\partial \psi_z}\right) = 0 \quad ,$$

$$\frac{\partial K}{\partial \psi^*} - \frac{\partial}{\partial x}\left(\frac{\partial K}{\partial \psi_x^*}\right) - \frac{\partial}{\partial y}\left(\frac{\partial K}{\partial \psi_y^*}\right) - \frac{\partial}{\partial z}\left(\frac{\partial K}{\partial \psi_z^*}\right) = 0 \quad ,$$

Which reduce to

$$-k\nabla^2\psi + V\psi = \lambda\psi \qquad (4.2.11)$$

This is written as $H\psi = \lambda\psi$

If we multiply this by ψ^* and integrate over the domain of x, y, z the left side becomes the stationary integral (4.2.9) which depends on E which is the energy of the system. Hence by (4.2.10) we have $\lambda = E$, so (4.2.11) reduces to Schrödinger equation. It is worth pointing out here that there is an interesting and important connection between Hamiltonian-Jacobi equation for classical system and the Schrödinger equation for a quantum mechanical system. In fact, if we put the wave function $\psi = e^{(i/\hbar)S}$, where S is action function of the classical system, then the Schrödinger equation reduces to the Hamiltonian-Jacobi equation provided S is much larger than Plank's constant $\hbar$. Thus in the limit of large values of action and energy, the surfaces of constant phase for the wave function ψ reduce to surfaces of constant action S for the corresponding classical system. In this case, wave mechanics reduces to classical mechanics just as wave optics reduces to geometrical optics in the limit of very small wavelength. It may be noted that the Klein-Gordon equation [56]

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \left(\frac{mc}{\hbar}\right)^2\psi = 0$$

(c =velocity of light) representing a possible wave equation for relativistic particle (though it is not correct for an electron or proton) can be constructed in

$$L = -\frac{h^2}{2m}[\nabla\psi^*.\nabla\psi - \frac{1}{c^2}(\frac{\partial\psi}{\partial t})^2 + [\frac{mc}{\hbar}]^2\psi^*\psi]$$

4.3 Derivation of Klein-Gordon Equation from Maxwell's Electric Wave Equation:

Kamil.E.Algailani derived Klein-Gordon equation from Maxwell's equation. The behaviors of electromagnetic waves are described by Maxwell's equations according to the relations: [57]

$$\nabla.D = \rho, \nabla.B = 0, \nabla\times E = -\frac{\partial B}{\partial t}, \ \nabla\times H = J + \frac{\partial D}{\partial t} \quad (4.3.1)$$

where $D, B, E, H \text{ and } J$ represent the electric flux density, the magnetic flux density, the electric field and the current density respectively. Satisfying the following relations, we have

$$B = \mu_\circ H \quad , J = \sigma E \quad , D = \varepsilon_\circ E + P \quad (4.3.2)$$

Where $P, \mu_\circ$ and $\varepsilon_\circ$ are the macroscopic polarization of the medium, the permittivity of free space and the permeability of free space, respectively. Appling the curl operator to both sides of the third equation in (4.3.1) the following equation is obtained:

$$\nabla\times(\nabla\times E) = -\nabla\times\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times B) \quad (4.3.3)$$

Using the identity [58]:

$$\nabla\times(\nabla\times E) = \nabla(\nabla.E) - \nabla^2 E \quad (4.3.4)$$

Equation (4.3.3) gives:

$$\nabla(\nabla.E)-\nabla^2E=-\frac{\partial}{\partial t}(\nabla\times B) \tag{4.3.5}$$

From (4.3.2) since:

$$B=\mu_\circ H \tag{4.3.6}$$

Then (4.3.5) becomes:

$$\nabla(\nabla.E)-\nabla^2E=-\frac{\partial}{\partial t}(\nabla\times \mu_\circ H) \tag{4.3.7}$$

From equation (4.3.7), since:

$$\nabla\times H=J+\frac{\partial D}{\partial t} \tag{4.3.8}$$

From (4.3.1) we have:

$$\nabla(\nabla.E)-\nabla^2E=-\frac{\partial}{\partial t}(\mu_\circ J+\mu_\circ\frac{\partial D}{\partial t}) \tag{4.3.9}$$

But:

$$D=\varepsilon_\circ E \quad + \quad P \tag{4.3.10}$$

Therefore:

$$\nabla(\nabla.E)-\nabla^2E=-\mu_\circ\frac{\partial J}{\partial t}-\varepsilon_\circ\mu_\circ\frac{\partial^2E}{\partial t^2}-\mu_\circ\frac{\partial^2P}{\partial t^2} \tag{4.3.11}$$

Also:

$$J=\sigma E \tag{4.3.12}$$

$$\nabla(\nabla.E)-\nabla^2E+\mu_\circ\frac{\partial J}{\partial t}+\varepsilon_\circ\mu_\circ\frac{\partial^2E}{\partial t^2}=-\mu_\circ\frac{\partial^2P}{\partial t^2} \tag{4.3.13}$$

The polarization, P, thus acts as a source term in the equation for radiation field [59].since:

$$D = \varepsilon_\circ E \quad , \nabla . D = \rho \quad , \rho = 0 \tag{4.3.14}$$

$$\varepsilon \nabla . E = \rho = 0 \qquad , \nabla . E = 0 \tag{4.3.15}$$

Therefore equation (4.3.13) becomes:

$$-\nabla^2 E + \mu_\circ \frac{\partial J}{\partial t} + \varepsilon_\circ \mu_\circ \frac{\partial^2 E}{\partial t^2} = -\mu_\circ \frac{\partial^2 P}{\partial t^2} \tag{4.3.16}$$

This represents the wave equation for the electric field.

Klein-Gordon equation for free particles is usually derived by using Einstein relativistic energy equation:

$$E^2 = p^2 c^2 + m_\circ^2 c^4 \tag{4.3.17}$$

The Klein-Gordon equation can be obtained by replacing the electric dipole moment term in equation (4.3.17) by the term standing for photon rest mass to gets

$$-\nabla^2 E + \varepsilon_\circ \mu_\circ \frac{\partial^2 E}{\partial t^2} = -k_m^2 E \tag{4.3.18}$$

Multiplying both sides by $c^2\hbar^2$, the following equation is obtained:

$$-c^2\hbar^2 \nabla^2 E + \hbar^2 \frac{\partial^2 E}{\partial t^2} = -c^2\hbar^2 k_m^2 E \ , \quad \textit{where } \frac{1}{c^2} = \mu_\circ \varepsilon_\circ \tag{4.3.19}$$

We have

$$p_m^2 = \hbar^2 k_m^2 = m_\circ^2 c^2$$

Thus,

$$-c^2\hbar^2\nabla^2 E + m_\circ^2 c^4 E = -\hbar^2\frac{\partial^2 E}{\partial t^2} \qquad (4.3.20)$$

The incorporation of mass for photon in Maxwell's equations corresponds to adding the term $m_\circ A^\mu A_\mu$ to the electromagnetic field Lagrangian.

Since in the electromagnetic (e.m) theory the oscillating electric wave E is related to its e.m, the energy or intensity is obtained according to the relation:

$$I \propto c\,\varepsilon_\circ E^2 \qquad (4.3.21)$$

And since the e.m intensity, when treated as a stream of photons of density n is given by:

$$I \propto n h f \propto |\psi|^2 h f \qquad (4.3.22)$$

Where the photon density is related to the wave function ψ according to the relation:

$$n = |\psi|^2 \qquad (4.3.23)$$

Comparing (4.3.21) and (4.3.22) it follows that:

$$E^2 \Leftrightarrow |\psi|^2 \ , \quad E \Leftrightarrow \psi \qquad (4.3.24)$$

Thus the correspondence between E and ψ secure the replacement of E by ψ in equation (4.3.20)

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = -c^2\hbar^2\nabla^2\psi + m_\circ^2 c^4\psi \qquad (4.3.25)$$

This represents the Klein-Gordon equation.

4.4 Direct Derivation of Schrödinger Equation from Hamiltonian-Jacobi Equation Using Uncertainty Principle:

In the work of Pranab Rudra Sarma the similarly between the Schrödinger equation and the Hamiltonian-Jacobi (H-J) equation of classical mechanics was used to derive Schrödinger equation. The Hamiltonian-Jacobi equation in one dimension for a particle of mass m and momentum p can be written as [60]

$$-\frac{\partial S}{\partial t^2}=\frac{P^2}{2m}+V(x,t) \quad =\frac{1}{2m}(\frac{\partial S}{\partial x})^2+V(x,t) \tag{4.4.1}$$

Where $S\equiv S(x,t)$ is the generating function. Here the momentum p of the particles has been, by definition, replaced by

$$p=\frac{\partial S}{\partial x} \tag{4.4.2}$$

One can attempt to derive Schrödinger equation from H-J equation by substituting

$$\psi(x,t)=\psi_{\circ}\exp(iS/\hbar) \quad or \quad S=-i\hbar(\ln\psi-\ln\psi_{\circ}) \tag{4.4.3}$$

Where $\psi_{\circ}$ is constant. The substituting yields

$$p=\frac{\partial S}{\partial x}=-i\hbar\frac{1}{\psi}\frac{\partial\psi}{\partial x} \tag{4.4.4}$$

$$\frac{\partial S}{\partial x}=-i\hbar\frac{1}{\psi}\frac{\partial\psi}{\partial x} \tag{4.4.5}$$

Substituting equations (4.4.4) and (4.4.5) in equation (4.4.1) one gets

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{1}{\psi} \left(\frac{\partial \psi}{\partial x}\right)^2 + V(x,t)\psi \qquad (4.4.6)$$

This differs from the Schrödinger equation because of the presence of the $(\partial\psi/\partial x)^2/\psi$ term in place of the term $\partial^2\psi/\partial x^2$. For obtaining a second order equation instead of a second degree equation, one can proceed as follows. One can write equation (4.4.4) in the form

$$p\psi = -i\hbar \frac{\partial \psi}{\partial x} \qquad (4.4.7)$$

And proclaim p as an operator

$$p = -i\hbar \frac{\partial}{\partial x} \qquad (4.4.8)$$

Substitution of this in equation (4.4.1) gives the fundamental Schrödinger equation.

In this work one have made an attempt to derive Schrödinger equation without invoking the concept of operators. Instead, we have used the concept of uncertainty principle for deriving the equation. This can be done by assuming that there is a basic uncertainty in the momentum in equation (4.4.1). If the root-mean square (RMS) uncertainty in p is Δp, then the average value of momentum-square is

$$< p^2 > = < p >^2 + (\Delta p)^2 = p^2 + (\Delta p)^2 \quad (4.4.9)$$

Therefore, for a particle with average momentum p, the average kinetic energy is not $p^2/2m$ but $[p^2 + (\Delta p)^2]/2m$ because of the uncertainty. The term $(\Delta p)^2$ is related to the uncertainty $(\Delta x)^2$ in x by the uncertainty principle. $(\Delta p)^2$ Can be written as

$$(\Delta p)^2 = \frac{\Delta p}{\Delta x} \Delta p \, \Delta x \quad (4.4.10)$$

Now we can estimate $\frac{\Delta p}{\Delta x}$ from $\frac{\partial p}{\partial x}$ and assume that $(\frac{\Delta p}{\Delta x})^2$ can be replaced by $(\frac{\partial p}{\partial x})(\frac{\partial p}{\partial x})^*$. From equation (4.4.4) we have

$$\frac{\partial p}{\partial x} = -i \frac{\partial}{\partial x}\left(\frac{1}{\psi}\frac{\partial \psi}{\partial x}\right) = i\hbar\left[\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 - \frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2}\right] \quad (4.4.11)$$

$$\frac{\Delta p}{\Delta x} = \hbar \frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 - \hbar \frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2} \quad (4.4.12)$$

We have to find an expression for $\Delta p \, \Delta x$ the minimum value of $\Delta p \, \Delta x$ is considered to be $\frac{\hbar}{2}$. The average value of this should be higher. Considering a Gaussian error function uncertainties can be shown to be related by [61]

$$\Delta p . \Delta x \quad \hbar \quad (4.4.13)$$

Using equations (4.4.10), (4.4.12) and (4.4.13), the expression for the average value of the square of the momentum becomes

$$p^2 + (\Delta p)^2 = -\hbar^2 \frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 + \left[\hbar \frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 - \hbar \frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2}\right]\hbar = -\hbar^2 \frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2} \quad (4.4.14)$$

Replacing p^2 by $< p^2 >$ i.e. b y $p^2 + (\Delta p)^2$ in the H-J equation and using equation (4.4.14) one gets the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial x}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x,t)\psi \qquad (4.4.15)$$

4.5 Derivation of Schrödinger Equation from Maxwell's Solution of Electric Field Intensity:

Khalid Haroon and others tried to use the expression for the electric field in a damping medium in the form:

$$E = E_{\circ}\, e^{-\beta t}\, e^{-\alpha x}\, e^{i(kx-\omega t)} \qquad (4.5.1)$$

To derive new Schrödinger equation that accounts for the effect of friction of the medium, they use the fact that

$$E=\hbar\omega \quad , \qquad p=\hbar k$$

And by replacing E by ψ they found that

$$\psi = Ae^{-\beta t}\, e^{-\alpha x}\, e^{\frac{i}{\hbar}(px-Et)}$$

Where

$$\alpha=\frac{\sigma}{\varepsilon} = \frac{2\mu c}{n_1}\sigma \qquad (4.5.2)$$

$\varepsilon\equiv$ Electric permittivity $\mu\equiv$ magnetic permittivity

$\sigma\equiv$ Electrical conductivity $n_1\equiv$ refractive index

By using this expression they found that the energy and momentum operators becomes

$$i\hbar\frac{\partial\psi}{\partial t} = (-i\hbar\beta+E)\psi$$

$$\frac{\hbar}{i}\nabla\psi = (-i\hbar\alpha + p)\psi \quad (4.5.3)$$

By using the energy expression for classical system multiplied by ψ they get [62]

$$E\psi = \frac{p^2}{2m}\psi + V\psi \quad (4.5.4)$$

By inserting (4.5.2) and (4.5.3) in (4.5.4) one gets the new Schrödinger quantum equation

$$i\hbar\frac{\partial\psi}{\partial t} + i\hbar\beta\psi = -\frac{\hbar^2 c_1}{2m n_1^2}\nabla^2\psi - \frac{\hbar^2\alpha c_1}{m n_1^2}\nabla\psi - \frac{\hbar^2\alpha c_1}{m n_1^2}\psi \quad (4.5.5)$$

4.6 Summary and Critique:

The first attempt in section (4.2) uses variational principle to derive Schrödinger equation; this derivation is complex and does not account for rest mass energy and friction.

In the attempt of Kamil. Algailani in section (4.3), the Klein-Gordon equation was derived from Maxwell's equations. But this attempt does not derive Schrödinger equation and does not account for friction.

In work of Pranab Rudra Sarma in section (4.4) Schrödinger equation was derived from Hamiltonian-Jacobi equation using uncertainty principle, but this attempt does not account for a relativistic energy.

In Khalid Haroon model in section (4.5) Schrödinger equation was derived from the electric field intensity wave

expression in a resistive medium, unfortunately this model does not account for the friction for the relativistic particles.

Chapter Five

Unification of Macroscopic and Microscopic World on the Basis of Maxwell's and Quantum Equations

(5.1)Introduction:

The unification of macroscopic and microscopic world needs a sort of link between classical and quantum laws. This is done in this chapter by deriving Schrödinger and Klein-Gordon equations for frictional medium from Maxwell's equations.

5.2 Maxwell's Electric Wave Equation:

The magnetic field intensity and the current density are related according to equation (2.8.6) by

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad (5.2.1)$$

Let $\frac{\partial D}{\partial t}$ is equal G

The equation of continuity takes the form

$$\nabla . J + \frac{\partial \rho}{\partial t} - \frac{\partial \rho_b}{\partial t} + c_d \nabla^2 \rho = 0 \qquad (5.2.2)$$

The current density J is assumed to result from external ohmic field J_0, beside bounded charge J_b and diffusion process J_d

$$J = J_0 + J_b + J_d \qquad (5.2.3)$$

Where

$$J_0 = \frac{-\partial D}{\partial t}$$

$$\Rightarrow \quad \nabla . J_0 = \frac{-\partial}{\partial t}(\nabla . D) = -\frac{\partial \rho}{\partial t} \tag{5.2.4}$$

$$J_b = \frac{-\partial P}{\partial t}$$

$$\Rightarrow \nabla . J_b = -\frac{\partial}{\partial t}(\nabla . P) = \frac{-\partial \rho_b}{\partial t} \tag{5.2.5}$$

$$J_d = -c_d \nabla \rho$$

$$\Rightarrow \quad \nabla . J_d = -c_d \nabla^2 \rho \tag{5.2.6}$$

Thus the divergence of both sides of equation (5.2.3) gives

$$\nabla . J = \nabla . J_0 + \nabla . J_b + \nabla . J_d \tag{5.2.7}$$

In view of equations (5.2.4), (5.2.5) and (5.2.6)

$$\nabla . J = -\frac{\partial \rho}{\partial t} + \frac{\partial \rho_b}{\partial t} - c_d \nabla^2 \rho \tag{5.2.8}$$

By rearranging the above equation

$$\nabla . J + \frac{\partial \rho}{\partial t} - \frac{\partial \rho_b}{\partial t} - c_d \nabla^2 \rho = 0 \tag{5.2.9}$$

To find the unknown G, one uses

$$\rho = \nabla . D = \varepsilon . \nabla . E \tag{5.2.10}$$

$$\rho_b = -\nabla . P \tag{5.2.11}$$

Taking the divergence of equation (5.2.1), one have

$$\nabla . \nabla \times H = 0$$

$$\nabla . \nabla \times H = \nabla . J + \nabla . G = 0 \tag{5.2.12}$$

Insert equation (5.2.12) in (5.2.8) yields

$$-\frac{\partial \rho}{\partial t} + \frac{\partial \rho_b}{\partial t} - c_d \nabla^2 \rho = -\nabla . G \qquad (5.2.13)$$

Using equation (5.2.10) and (5.2.11) yields

$$-\frac{\partial}{\partial t}(\nabla . D) + \frac{\partial}{\partial t}(-\nabla . P) - c_d \nabla . (\nabla \rho) = -\nabla . G \qquad (5.2.14)$$

But $\nabla . D = \rho$

Thus

$$\nabla \rho = \nabla . (\nabla . D) \qquad (5.2.15)$$

Using relations (5.2.10) and (5.2.15) yields

$$-\frac{\partial}{\partial t}(\nabla . \varepsilon E) + \frac{\partial}{\partial t}(-\nabla . P) - c_d \nabla . (\nabla (\nabla . D)) = -\nabla . G$$

$$-\frac{\partial}{\partial t}(\nabla . \varepsilon E) + \frac{\partial}{\partial t}(-\nabla . P) - c_d \nabla . (\nabla (\nabla . \varepsilon E)) = -\nabla . G$$

$$-\varepsilon \nabla . \frac{\partial E}{\partial t} - \nabla . \frac{\partial P}{\partial t} - \varepsilon c_d \nabla . (\nabla (\nabla . E)) = -\nabla . G$$

Comparing both sides of above equations yields

$$\varepsilon \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + \varepsilon c_d \nabla (\nabla . E) = G$$

$$G = \varepsilon \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + \varepsilon c_d \nabla (\nabla . E) \qquad (5.2.16)$$

Thus from equation (5.2.1) and the fact that $J = \sigma_0 E$

$$\nabla \times H = J + G$$

$$\nabla \times H = \sigma_0 E + \varepsilon \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + \varepsilon c_d \nabla(\nabla . E) \quad (5.2.17)$$

Also from Maxwell's equations we have

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times \nabla \times E = -\mu \frac{\partial (\nabla \times H)}{\partial t} \quad (5.2.18)$$

From equation (5.2.16) and (5.2.1) one found that

$$\nabla \times H = J + \varepsilon \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + \varepsilon c_d \nabla(\nabla . E) \quad (5.2.19)$$

Multiplying both sides of equation (5.2.19) by μ and differentiate over time t yields

$$\mu \frac{\partial}{\partial t}(\nabla \times H) = \mu \frac{\partial J}{\partial t} + \mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \frac{\partial^2 P}{\partial t^2} + \varepsilon \mu c_d \nabla(\nabla . \frac{\partial E}{\partial t}) \quad (5.2.20)$$

But

$$J = \sigma E \quad (5.2.21)$$

$$\mu \frac{\partial}{\partial t}(\nabla \times H) = \mu \sigma \frac{\partial E}{\partial t} + \mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \frac{\partial^2 P}{\partial t^2} + \varepsilon \mu c_d \nabla(\nabla . \frac{\partial E}{\partial t}) \quad (5.2.22)$$

Also we have

$$\nabla \times \nabla \times E = -\nabla^2 E + \nabla(\nabla . E) \quad (5.2.23)$$

From equations (5.2.23), (5.2.22) and (5.2.18) yields

$$-\nabla^2 E + \nabla(\nabla . E) = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \sigma \frac{\partial E}{\partial t} + \mu \frac{\partial^2 P}{\partial t^2} + \varepsilon \mu c_d \nabla(\nabla . \frac{\partial E}{\partial t}) \quad (5.2.24)$$

5.3 Derivation of Klein-Gordon Equation from Maxwell's Equation for a Massive Photon:

From Maxwell's equation

$$-\nabla^2 E + \mu\sigma\frac{\partial E}{\partial t} + \mu\varepsilon\frac{\partial^2 E}{\partial t^2} + \mu\frac{\partial^2 P}{\partial t^2} + \frac{m^2c^2}{\hbar^2}E = \quad 0 \qquad (5.3.1)$$

Neglecting polarization effect and considering the propagation in free space where

$$\left.\begin{array}{ll} \sigma = & 0 \\ \mu = & \mu_0 \\ \varepsilon = & \varepsilon_0 \end{array}\right] \qquad (5.3.2)$$

$$\mu_0\ \varepsilon_0 \qquad = \qquad \frac{1}{c^2} \qquad (5.3.2.1)$$

Where c is speed of light

Equation (5.3.1) reduce to

$$-\nabla^2 E + zero + \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} + zero + \frac{m^2c^2}{\hbar^2}E = \quad 0 \qquad (5.3.3)$$

$$-\nabla^2 E + \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} + \frac{m^2c^2}{\hbar^2}E = 0$$

$$\hbar^2(-\nabla^2 E + \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}) + m^2c^2 = 0 \qquad (5.3.4)$$

Inserting equation (5.3.2.1) in (5.3.4), one gets

$$-\hbar^2\nabla^2 E + \hbar^2\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + m^2c^2 = 0$$

Multiplying both sides of above equation by c^2

$$-\hbar^2c^2\nabla^2 E + \hbar^2\frac{\partial^2 E}{\partial t^2} + m^2c^4E = 0 \qquad (5.3.5)$$

If the rest mass equals the relativistic mass, when no potential exist then,

$$m = m_0(1 - \frac{v^2}{c^2} + \frac{2\varphi}{c^2})$$

$$= m_0(1 - \frac{v^2}{c^2})$$

When $v <<< c$

Thus equation (5.3.5) reduces to

$$m = m_0 \qquad (5.3.6)$$

$$-\hbar^2 \frac{\partial^2 E}{\partial t^2} = -c^2\hbar^2\nabla^2 E + {m_0}^2 c^4 E \qquad (5.3.7)$$

Replacing E by ψ in equation (5.3.7), one gets

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -c^2\hbar^2\nabla^2 \psi + {m_0}^2 c^4 \psi \qquad (5.3.8)$$

Which is the ordinary Klein-Gordon Equation

5.4 Derivation of Schrödinger Equation from Maxwell's Equations:

Equation (5.2.1) can be rewritten as:

$$-\hbar^2 c^2\nabla^2 E + \hbar^2 c^2 \mu\sigma \frac{\partial E}{\partial t} + \hbar^2 c^2 \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \hbar^2 c^2 \mu \frac{\partial^2 P}{\partial t^2} + \frac{m^2 c^4}{\hbar^2}\hbar^2 E = 0$$

$$-\hbar^2 c^2\nabla^2 E + \hbar^2 c^2 \mu\sigma \frac{\partial E}{\partial t} + \hbar^2 \frac{\partial^2 E}{\partial t^2} + \hbar^2 \mu c^2 \frac{\partial^2 P}{\partial t^2} + m^2 c^4 E = 0 \qquad (5.4.1)$$

Where

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

Neglecting the dipole moment contribution and taking into account the fact that:

$$c >> 1$$

Thus the terms that do not consist of c can be neglected to get:

$$-\hbar^2 c^2 \nabla^2 E + \hbar^2 c^2 \mu\sigma \frac{\partial E}{\partial t} + zero + zero + m^2 c^4 E = 0 \qquad (5.4.2)$$

Dividing both sides of equation (5.4.2) by $2mc^2$, yields

$$\frac{-\hbar^2 c^2 \nabla^2 E}{2mc^2} + \frac{\hbar^2 c^2 \mu\sigma}{2mc^2} \frac{\partial E}{\partial t} + \frac{m^2 c^4}{2mc^2} E = 0$$

$$\frac{-\hbar^2}{2m} \nabla^2 E + \frac{\hbar^2}{2m} \mu\sigma \frac{\partial E}{\partial t} + \frac{1}{2} mc^2 E = 0 \qquad (5.4.3)$$

To find conductivity consider the electron equation for oscillatory system, where the electron velocity is given by:

$$v = v_0 e^{i\omega t} \qquad (5.4.4)$$

And its equation of motion takes the form

$$m \frac{dv}{dt} = eE \qquad (5.4.5)$$

Differentiate equation (5.4.4) over dt hence one get

$$\frac{dv}{dt} = i\ \omega\ v_0 e^{iwt}$$

$$\frac{dv}{dt} = i\ \omega\ v \qquad (5.4.6)$$

Inserting equation (5.4.6) in (5.4.5)

$$i\omega m v = eE$$

$$v = \frac{e}{i\omega m} E \qquad (5.4.7)$$

But for electron, the current J is given by

$$J = nev$$

$$J = \frac{ne^2 E}{i m \omega} = \frac{ine^2 E}{i^2 \omega m}$$

$$J = \frac{-ine^2 E}{\omega m} \qquad (5.4.8)$$

Also we know that

$$J = \sigma E \qquad (5.4.9)$$

Comparing equation (5.4.8) and (5.4.9) hence one get

$$\sigma = \frac{-ine^2}{m\omega} \qquad (5.4.10)$$

The coefficient of the first order differentiation of E with respect to time is given with the aid of equations (5.4.3) and (5.4.10)

$$\frac{\hbar^2 \mu \sigma}{2m} = \frac{-i\hbar^2 \mu n e^2}{2m^2 \omega} \qquad (5.4.11)$$

Using Gauss law

$$\varepsilon E A = Q = neAx \qquad (5.4.12)$$

Where x is the average distance of oscillator and is related to the maximum displacement according to the relation

$$x = \frac{1}{2}x_0 \qquad (5.4.13)$$

$$\frac{\hbar^2\mu\sigma}{2m} = \frac{-i\hbar^2\mu n e^2}{2m^2\omega} = \frac{-i\hbar(\hbar\omega)\mu(ne^2 A{x_0}^2)}{4(\frac{1}{2}m\omega^2{x_0}^2)mA} \qquad (5.4.14)$$

By using equation (5.4.13)

$${x_0}^2 = 4x^2$$

Thus equation (5.4.14) becomes

$$\frac{\hbar^2\mu\sigma}{2m} = \frac{-i\hbar(\hbar\omega)\mu(ne^2 A4x^2)}{4(\frac{1}{2}m\omega^2{x_0}^2)mA}$$

$$= \frac{-i\hbar(\hbar\omega)(ne^2 A x)(x)\mu}{(\frac{1}{2}m\omega^2{x_0}^2)mA}$$

$$= \frac{-i\hbar(\hbar\omega)(neA x)e(x)\mu}{(\frac{1}{2}m\omega^2{x_0}^2)mA} \qquad (5.4.15)$$

By using equation (5.4.12), one gets

$$= \frac{-i\hbar(\hbar\omega)(\varepsilon EA)e(x)\mu}{(\frac{1}{2}m\omega^2{x_0}^2)mA} = \frac{-i\hbar(\hbar\omega)c^2(\mu\varepsilon)eEAx}{(mc^2)A(\frac{1}{2}m\omega^2{x_0}^2)}$$

$$= \frac{-i\hbar(\hbar\omega)c^2(\frac{1}{c^2})(Fx)}{(mc^2)(\frac{1}{2}m\omega{x_0}^2)} \qquad (5.4.16)$$

But according to quantum mechanical and classical energy formula

$$\hbar\omega \;=\; mc^2 \;=\; E \qquad (quantum)$$

$$F\,x = \frac{1}{2} m\omega^2 {x_0}^2 \;=\; E \qquad (classical)$$

Therefore equation (5.4.16) reduce to

$$\frac{\hbar^2 \mu\sigma}{2m} \;=\; -i\hbar \qquad (5.4.17)$$

As a result equation (5.4.3) becomes

$$\frac{-\hbar^2}{2m}\nabla^2 E - i\hbar\frac{\partial E}{\partial t} + \frac{1}{2} mc^2 E \;=\; 0 \qquad (5.4.18)$$

We have

$$m \;=\; m_0 (1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2})^{\frac{1}{2}}$$

Since Schrödinger deals with low speed therefore

$$\frac{v}{c} << 1$$

Thus one can neglect the speed term to get

$$m \;=\; m_0 (1 + \frac{2\varphi}{c^2})^{\frac{1}{2}} \qquad (5.4.19)$$

Taking c as a maximum value of light speed, such that the average light speed c_e is given by

$$c_e \;=\; \frac{1}{\sqrt{2}}\, c$$

$$\Rightarrow \qquad c_e \qquad = \qquad \frac{c^2}{2} \qquad (5.4.20)$$

And assuming

$$m \;=\; m_0 (1+\frac{2\varphi}{2c_e{}^2})^{\frac{1}{2}}$$

$$\cong \; m_0 (1+\frac{\varphi}{c_e{}^2}) \qquad (5.4.21)$$

Thus

$$\frac{1}{2}mc^2 \;=\; mc_e^2 = \; m_0 c_e{}^2 (1+\frac{\varphi}{c_e{}^2})$$

$$= \; m_0 \; + \; m_0 \varphi$$

$$= \; m_0 \; + \; V \qquad (5.4.22)$$

Since atomic particles which are describes by quantum laws are very small, thus one can neglect m_0 compared to the potential V to get

$$m_0 \; + \; V \quad \cong \quad V \qquad (5.4.23)$$

Hence from equations (5.4.22) and (5.4.23)

$$\frac{1}{2}mc^2 \;=\; m_0 \; + \; V \quad \cong \quad V \qquad (5.4.24)$$

Thus equation (5.4.18) reduce to

$$\frac{-\hbar^2}{2m}\nabla^2 E - i\hbar\frac{\partial E}{\partial t} + V E = 0$$

Taking into account that the electromagnetic energy density is proportional to E^2 , and since $|\psi|^2$ is also reflects photon density. Thus one cans easily replace E by ψ, in the above equation, to get

$$\frac{-\hbar^2}{2m}\nabla^2\psi - i\hbar\frac{\partial\psi}{\partial t} + V\psi = 0$$

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad (5.4.25)$$

Which is Schrödinger equation

5.5 The Electric Polarization and Special Relativity:

We have

$$P = -nex \qquad (5.5.1)$$

$$\mu\frac{\partial^2 P}{\partial t^2} = -n\,\mu e\ddot{x} \qquad (5.5.2)$$

Where $x = x_0 e^{i\omega_0 t}$

$$\dot{x} = i\omega_0\, x_0 e^{i\omega_0 t}$$

$$\ddot{x} = i^2\omega_0{}^2\, x_0 e^{i\omega_0 t} = -\omega_0{}^2\, x \qquad (5.5.3)$$

$$\mu\frac{\partial^2 P}{\partial t^2} = +\mu\omega_0{}^2 n\ ex \qquad (5.5.4)$$

From equation (5.4.12) we have

$$\varepsilon E A = Q = neAx$$

$$\varepsilon E = nex \qquad (5.5.5)$$

Inserting equation (5.5.5) in equation (5.5.4), one gets

$$\mu \frac{\partial^2 P}{\partial t^2} = \mu \omega_0^{\,2} \varepsilon E = \mu \varepsilon \omega_0^{\,2} E$$

$$= \frac{\omega_0^{\,2}}{c^2} E \qquad (5.5.6)$$

Equation (5.2.1) can be written as

$$-\nabla^2 E + \mu\varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \frac{\partial^2 P}{\partial t^2} = 0 \qquad (5.5.7)$$

From equations (5.5.6) and (5.5.7), one gets

$$-\nabla^2 E + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \frac{\omega_0^{\,2}}{c^2} E = 0 \qquad (5.5.8)$$

Consider

$$E = E_0 e^{i(kx-\omega t)} \qquad (5.5.9)$$

$$\nabla^2 E = -k^2 E \qquad (5.5.10)$$

$$\frac{\partial E}{\partial t} = -i\omega E_0 e^{i(kx-\omega t)}$$

$$\frac{\partial^2 E}{\partial t^2} = -i \times -i\omega^2 E_0 e^{i(kx-\omega t)}$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E \qquad (5.5.11)$$

Inserting equations (5.5.10) and (5.5.11) in (5.5.8)

$$+k^2 E - \frac{1}{c^2} \omega^2 E + \frac{\omega_0^{\,2}}{c^2} E = 0$$

Multiplying both sides of above equation by $\frac{c^2}{E}$

$$\frac{k^2 E c^2}{E} - \frac{1}{c^2}\omega^2 \frac{E c^2}{E} + \frac{\omega_0^{\,2}}{c^2}\frac{E c^2}{E} = \frac{0 c^2}{E}$$

$$k^2 c^2 - \omega^2 + \omega_0^{\,2} = 0 \qquad (5.5.12)$$

Multiplying both sides of equation (5.5.12) by $\hbar^2$

$$\hbar^2 k^2 c^2 - \hbar^2 \omega^2 + \hbar^2 \omega_0^{\,2} = 0\hbar^2$$

$$\hbar^2 k^2 c^2 + \hbar^2 \omega_0^{\,2} = \hbar^2 \omega^2 \qquad (5.5.13)$$

For a photon the energy and momentum are given by Plank and De Broglie hypothesis to be

$$p = \frac{h}{\lambda} \quad , \quad E = h f$$

$$E = h\frac{c}{\lambda} = p c \qquad (5.5.14)$$

But the photon momentum is given by

$$p = mc$$

Thus

$$E = pc = mc^2 \qquad (5.5.15)$$

Therefore

$$\left.\begin{aligned} \hbar\omega = h f = E = mc^2 \\ \hbar\omega_0 = h f_0 = E_0 = m_0 c^2 \end{aligned}\right] \qquad (5.5.16)$$

Substituting equations (5.5.16), (5.5.15) and (5.5.14) in equation (5.5.13), one gets

$$p^2 c^2 + m_0{}^2 c^4 = m_0{}^2 c^4 \qquad (5.5.17)$$

5.6 New Generalized Quantum Equation:

Schrödinger equation deals only with non relativistic particles, thus it does not take into account the rest mass energy. On contrary Klein-Gordon equation can account for rest mass energy but does not have potential energy term for fields other than electromagnetic fields [63]. Thus there is a need to find a new quantum equation that accounts for rest mass energy, beside potential energy. This can be done with the aid of equation (5.3.1), where one uses the mass expression of the generalized special relativity which is given by [64]:

$$m = m_0 (1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2})^{\frac{1}{2}} \qquad (5.6.1)$$

$$m^2 = m_0{}^2 (1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2})$$

$$m^2 = m_0{}^2 + 2 m_0 (\frac{m_0 \varphi}{c^2}) - \frac{m_0{}^2 v^2}{c^2} \qquad (5.6.2)$$

But we have

$$m_0 \varphi = V \qquad (5.6.3)$$

$$m_0 v = p \qquad (5.6.4)$$

Substituting equation (5.6.4) and (5.6.3) in (5.6.2), one gets

$$m^2 = m_0{}^2 + 2 m_0 \frac{V}{c^2} - \frac{p^2}{c^2} \qquad (5.6.5)$$

Multiplying both sides of equation (5.6.5) by $E c^4$

$$m^2 c^4 E = m_0^2 c^4 E + 2 m_0 c^2 V E - p^2 c^2 E \quad (5.6.6)$$

But for oscillating electric field

$$E = E_0 e^{i(kx-\omega t)}$$

$$\frac{\partial E}{\partial x} = i k E_0 e^{i(kx-\omega t)}$$

$$\nabla^2 E = \frac{\partial^2 E}{\partial x^2} = i^2 k^2 E_0 e^{i(kx-\omega t)}$$

$$\nabla^2 E = -k^2 E$$

$$\hbar^2 \nabla^2 E = -\hbar^2 k^2 E$$

$$\hbar^2 \nabla^2 E = - p^2 E \quad (5.6.7)$$

Thus equation (5.6.6) becomes

$$m^2 c^4 E = m_0^2 c^4 E + 2 m_0 c^2 V E - c^2 \hbar^2 \nabla^2 E \quad (5.6.8)$$

By using the identity $\mu \varepsilon = \frac{1}{c^2}$ and inserting equation (5.6.8) in equation (5.6.2)

$$-c^2 \hbar^2 \nabla^2 E + c^2 \hbar^2 \mu \sigma \frac{\partial E}{\partial t} + \hbar^2 \frac{\partial^2 E}{\partial t^2} + m_0^2 c^4 E + 2 m_0 c^2 V E - c^2 \hbar^2 \nabla^2 E = 0$$

Replacing E by ψ and collecting similar terms leads to the new quantum equation of the form

$$-2 c^2 \hbar^2 \nabla^2 \psi + c^2 \hbar^2 \mu \sigma \frac{\partial \psi}{\partial t} + \hbar^2 \frac{\partial^2 \psi}{\partial t^2} + m_0^2 c^4 \psi + 2 m_0 c^2 V \psi = 0 \quad (5.6.9)$$

5.7 Discussion:

The fact that Maxwell's equation is used to derive Klein-Gordon equation is related to the fact that quantum mechanical laws are based on Plank quantum light equation. The replacement of the electric field intensity vector E by the wave function ψ is reasonable as far as the electromagnetic energy density which is related to the number of photons is proportional to E^2 , i.e.

$$n \quad \propto \quad E^2$$

While it is also related to $|\psi|^2$

I.e. $n \quad \propto \quad |\psi|^2$

Thus

$$E \rightarrow \psi$$

Since Schrödinger equation is first order in time, thus the second order time term should disappear in equation (5.4.1). This is achieved by taking into account that all terms that consist of c are larger compared to terms free of c. this is since the speed of light is very large ($c \approx 10^8$). The dipole term in(5.4.1) is neglected, which is also natural as well as Schrödinger equation deals only with particles moving in a field potential through the term v which is embedded in the mass term according to GSR [see equation (5.4.1)].

In deriving the conductivity term the effect on the particle is only the electric field, while the effect of friction is neglected. This is also compatible with Schrödinger hypothesis which considers the

effect of the medium is only through the potential according to the energy wave equation.

$$E\psi = \frac{p^2}{2m}\psi + V\psi \qquad (43)$$

$$\psi = A\, e^{\frac{i}{\hbar}(px - Et)} \quad , \quad i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$, \quad -\hbar^2\nabla^2\psi = p^2\psi \qquad (44)$$

The fact that the velocity in equation (5.4.4) represents oscillating reflects the wave nature of particles, on which one of the main quantum hypotheses is based. By using this hypothesis together with plank expression of energy, beside classical energy of an oscillating system, the coefficient of the first time derivative of E is found to be equal to $(-i\hbar)$.

In view of equation (5.4.18) and the GSR expression of mass (5.4.19) the potential term in Schrödinger equation is clearly stems from the mass term. Again the wave nature of particles relates the maximum light speed to its average speed according to equation (5.4.20). Neglecting the rest mass, in the third term in equation (5.4.18) the coefficient of E is equal to the potential. The final Schrödinger equation was found by the replacing E by ψ. This is not surprising since number of photons $\propto |\psi|^2 \propto E^2$.

The relation of energy and momentum in SR, by assuming oscillating atoms in the media with frequency ω_0 as representing the background rest energy as shown by equations (5.5.3) and (5.5.6).

The energy gained by the system is the electromagnetic energy of frequency ω [see equations (5.5.9) and (5.5.11)]. Using Plank hypothesis for a photon, beside momentum mass relation in equations (5.5.14), (5.5.15) and (5.5.16) the special relativity momentum energy relation was found.

The new quantum mechanical law shown in equation (5.6.9) is more general than Schrödinger and Klein-Gordon equations. It consists of conductivity of the medium, which is related to the friction of the system. The conductivity term can also feels the existence of the bulk matter through the particle density term n, where

$$\sigma = \frac{n e^2 \tau}{m}$$

Unlike Schrödinger equation the new quantum equation consists of a term representing rest mass energy. This equation is also more general than Klein-Gordon equation by having terms accounting for the effect of friction, collision through conductivity, besides having a potential term accounting for all fields other than electromagnetic field.

5.8 Conclusion:

The derivation of Schrödinger quantum equation and SR energy-momentum relation from Maxwell electric equation shows the possibility of unifying the wave and particle nature of electromagnetic waves. It shows also of unifying Maxwell's equations, SR and quantum equations**.**

Quantum equations derived from Maxwell's equations are very promising, since they reduce to Klein-Gordon equation. It also accounts for collision, friction and scattering processes.

5.9 References:

[1]. Dente, G. C. Classical limit of quantum electrodynamics. Physical Review D 12: 1733-8. . 1975

[2]. Becker, R. Electromagnetic fields and interactions. New York: Dover Publications. 1964.

[3]. Dario, O. Electrodynamics from Ampere to Einstein. Oxford: Oxford University Press. 2003

[4]. Hand LN, Finch JD. Analytical Mechanics, Cambridge University press, Cambridge. (1998)

[5]. Taylor JR, Classical Mechanics, University Science Books (2005).

[6]. Char. K. Relativistic mechanics," Eur. J. Phys. 25, 385-397 (2004).

[7]. Bjorken, J. D., and Drell, S. D., Relativistic Quantum Mechanics, New York, USA (1964).

[8]. Field, J.H. "Relationship of quantum mechanics to classical electromagnetism and classical (1981).

[9]. Schiff, L.I, quantum mechanics, New York, USA, 1968.

[10]. Steven Weinberg, the quantum theory of fields, Cambridge university press, 1975.

[11]. Sakurai, J. J, Fu Tuan and son, Modern quantum mechanics revised edition, Addison Wesley, California, 1994.

[12]. Hand LN, Finch JD. Analytical Mechanics, Cambridge University press, Cambridge. (1998)

[13]. Nikon M, Parker P ,Advanced Level Physics, fourth edition, Heinemann Educational Book-London(1978).

[14]. Ballentine, L. E. Quantum mechanics: a modern development. Singapore: World Scientific. 1998.

[15]. Bohm, D. Quantum theory. New York: Dover Publications. 1951.

[16]. Fermi, E. Quantum theory of radiation. Reviews of Modern Physics 4: 87-132. 1932.

[17]. Landau, L. and Lifshitz, E. Quantum mechanics (non-relativistic theory): course of theoretical1974.

[18]. Haroon .K, Derivation of Schrödinger equation from Maxwell's electric field intensity, IJPS, 2012

[19]. Algeilani. K, Derivation of Klein-Gordon equation from Maxwell's electric wave equation, International Journal of Physical Sciences, USA (2014).

[20]. Björken, J. D., and Drell, S. D. Relativistic quantum fields. New York: McGraw-Hill. 1965.

[21]. Berestetskii, V. B., Lifshitz, E. M., and Pitaevskii, L. P. Quantum electrodynamics. Oxford: 1982

[22]. Darrigol, O. The modular structure of physical theories. Synthese 162: 195-223. 2008.

[23]. Fleischer D, A student's Guide to Maxwell's Equations, Cambridge University Press, Cambridge (2008).

[24]. Griffiths, D., Introduction to Electrodynamics, (Prentice-Hall) 1999.

[25]. Bansi Lal, Mathematical Theory of Electromagnetism, Asia Publishing House, Dalhi (1965).

[26]. Arbab, A. I., Derivation of Dirac, Klein-Gordon, Schrödinger, diffusion and quantum heat transport equations from a universal quantum wave equation, EPL, 92 40001 . (2010)

[27]. Maxwell James Clark, A dynamical Theory of The Electrodynamics Field, Royal Society, Transactions 155454-512. (1965)

[28]. Arbab, A. I., The analogy between matter and electromagnetic waves, EPL, 94 50005. (2011)

[29]. Dirac, P. A. M., The complex Maxwell's equations, Proc. R. Soc. London, 133 60. (1931)

[30]. Daniel Fleisch, A student's Guide to Maxwell's Equations, Cambridge University Press, Cambridge (2008).

[31]. Matthew N.O .Sadiku, Elements of electromagnetic, third edition, Oxford University Press, Oxford, (2001).

[32]. John D. Kraus, Daniel A. Fleisch, Electromagnetic with Application, Fifth edition, Mc Grew-Hill Book Co. New York, (1999).

[33]. Jack Vanderlinde, Classical Electromagnetic Theory, second edition, Kluwer Academic Publishers, (2005)

[34]. Adler, S. L., Quaternionic Quantum Mechanics and Quantum Fields, Oxford University Press, New York, (1995).

[35]. Moscow: Mir. Quantum theory. New York: Dover Publications. Physics, Vol. 3 (Portuguese edition). 1969

[36]. Arbab, A. I., The quaternioic quantum mechanics, Appl. Phys. Res., 3 160. (2011)

[37]. Kirchanov VS, Dual Klein-Gordon and Dirac Equations. Russian Phys. J., 55: 6, (2012).

[38]. Bruce S, Mining P, The Klein-Gordon Oscillator, IL Nuovo Cimento, 106 A: 5, (1993).

[39]. David. J, and Drell, S. D., Relativistic quantum mechanics, (McGraw-Hill,) 1964.

[40]. Gatehouse DC, Geometrical Derivation of the Klein-Gordon Equation. Int. J. Theory. Phys., 20: 6(1981).

[41]. Thirring, W. E. Principles of Quantum Electrodynamics. New York: Academic Press. . (1958)

[42]. Wachter. A, Relativistic Quantum Mechanics, Springer, 2011

[43]. Dirac. P. M, The Quantum Theory of the Electron, Proceedings of the Royal Society of London, Series A, Vol. 117, No. 778, 610-624, 1928.

[44]. Feshbach. H and Villars. F, Elementary Relativistic Wave Mechanics of Spin 0 and Spin 1/2 Particles, Reviews of Modern Physics, Vol. 30, No. 1, 24-45, 1958.

[45]. Einstein. A, On the Motion of Small Particles Suspended in Liquids at Rest Required by the Molecular-Kinetic Theory of Heat, Annalen Der Physics, 17, 549-560, 1905.

[46]. Proca. A, Fundamental Equations of Elementary Particles, C. R. Acad. Sci. Paris, 202, 1490, 1936

[47]. Laskin. N, Fractals and Quantum Mechanics. Chaos 10, 780-790, 2000.

[48]. Dong. J and M. Xu, Space-time Fractional Schrodinger Equation with Time-Independent Potentials, Journal of Mathematical Analysis and Ap- plications Vol. 344, Issue 2, 10051017, 2008.

[49]. Iomin, Fractional-Time Quantum Dynamics, Phys. Rev. E 80, 2009, 022103, 2009.

[50]. Dirac P.A.M. The principles of Quantum Mechanics, Oxford University press. [This is a classic book by a father of quan tum mechanics that presents the basic pr inciples and the mathematical structure of the quantum theory] (1958)

[51]. Fano U. Quantum Mechanics: Then and Since, La Rivista del Nuovo Cimento, vol.11, Editrice Compositori, Bologna. [A critical analysis of basic concepts and definitions in quantum mechanics] (1995)

[52]. Levine I.N. Quantum Chemistry, prentice-Hall International, New Jersey. (1991)

[53]. Messiah A. Quantum Mechanics, North-Holland Publishing Company, Amsterdam. [A classic textbook on quantum mechanics that illustrates basic principles, explains mathematical structure and analyses in detail several topics] (1970)

[54]. Sami .H Derivation of Schrödinger Equation from Variational Principle, pure and applied mathematical journal, 2:4-146-148,2013

[55]. David Halliday, Robert Resnick, Physics(parts 1 & 2 combined),third edition, John Wiley and sons, New York (1978).

[56]. Mc. Weeny R. Coulson's Valence, Oxford University Press. [This is an enlarged and critical edition of the classical book (1979)

[57]. Nelkon. M and Parker. P, Advanced Level Physics, fourth edition, Heinemann Educational Book-London (1978)

[58]. Abbot. A. F, Ordinary Level Physics, third edition, Heinemann Educational Book-London (1977)

[59]. Sarma. P. R, Direct Derivation of Schrödinger Equation from Hamiltonian-Jacobi Equation Using Uncertainty Principle,variable energy cyclotron center, 2010

[60]. Meluin Schwarts, Principles of Electrodynamics, Mc Graw-Hill Book Company, New York (1972)

[61]. E.Wolf, New Theory of Radiative Energy Transfer in Free Electromagnetic Fields, Phys. REv. D., Vol.13, PP.869 886(1976)

[62]. David J. Griffths, Introduction to Electrodynamics, Third edition, Prentice Hall New Jersey (1999)

[63]. Saleh. B. E and Teich. M. C., Fundamentals of Photonics, second edition, John wiley and sons, New York (2007)

[64]. John Robert Taylor, classical mechanics, university science books (2005)