Chapter 1

C^*-Algebras and Semigroupoid

In this chapter we show that a semigroupoid is a set equipped with a partially defined associative operation. Given a semigroupoid Λ we construct a C*-algebra $\mathcal{O}(\Lambda)$ from it. We then present two main examples of semigroupoids, namely the Markov semigroupoid associated to an infinite 0-1 matrix, and the semigroupoid associated to a row-finite higher- rank graph without sources [1].

Section (1.1): Representations of Semigroupoid and Springs

The theory of C^*-algebras has greatly benefited from Combinatorics in the sense that some of the most interesting examples of C^*-algebras arise from combinatorial objects, such as the case of graph C^*-algebras. More recently Kumjian and Pask have introduced the notion of higher-rank graphs, inspired by Robertson and Steger's work on buildings, which turns out to be another combinatorial object with which an interesting new class of C^*-algebras may be constructed.

The crucial insight leading to the notion of higher-rank graphs lies in viewing ordinary graphs as categories (in which the morphisms are finite paths) equipped with a length function possessing a certain unique factorization property.

Kumjian and Pask's interesting idea of viewing graphs as categories suggests that one could construct C^*-algebras for more general categories.

Since Eilenberg and Mac Lane introduced the notion of categories in the early 40's, the archetypal idea of composition of functions has been mathematically formulated in terms of categories, whereby a great emphasis is put on the domain and co-domain of a function. However one may argue that, while the domain is an intrinsic part of a function, co-domains are not so vital. If one imagines a very elementary function f with domain, say $X = \{1,2\}$, defined by $f(x) = x^2$, one does not really need to worry about its co-domain. But if f is to be seen as a morphism in the category of sets, one needs to first choose a set Y containing the image of f, and only then f becomes an element of $\mathrm{Hom}(X, Y)$. Regardless of the very innocent nature of our function f, it suddenly is made to evoke an enormous

amount of morphisms, all of them having the same domain X, but with the wildest possible collection of co-domains.

Addressing this concern one could replace the idea of categories with the following: a big set (or perhaps a class) would represent the collection of all morphisms, regardless of domains, ranges or co-domains. A set of composable pairs (f, g) of morphisms would be given in advance and for each such pair one would define a composition $f\mathsf{g}$. Assuming the appropriate associativity axiom one arrives as the notion of a semigroupoid, precisely defined below. Should our morphisms be actual functions a sensible condition for a pair (f, g) to be composable would be to require the range of g to be contained in the domain of f, but we might also think of more abstract situations in which the morphisms are not necessarily functions.

For example, let $A = \{A(i,j)\}_{i,j\in\mathcal{G}}$, be an infinite 0-1 matrix, where $\mathcal{G}$ is an arbitrary set, and let Λ_A be the set of all finite admissible words in $\mathcal{G}$, meaning finite sequences $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ of elements $\alpha_i \in \mathcal{G}$, such that $A(\alpha_i, \alpha_{i+1}) = 1$. Given $\alpha, \beta \in \Lambda_A$ write

$$\alpha = \alpha_1\alpha_2 \dots \alpha_n \quad \text{and} \quad \beta = \beta_1, \beta_2, \dots, \beta_m,$$

and let us say that α and β are composable if $A(\alpha_n, \beta_1) = 1$, in which case we let β be the concatenated word

$$\alpha\beta = \alpha_1\alpha_2 \dots \alpha_n\beta_1, \beta_2, \dots, \beta_m.$$

We shall refer to Λ_A as the Markov semigroupoid. This category-like structure lacks a notion of objects and in fact it cannot always be made into a category. Consider for instance the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we let the index set of A be $\mathcal{G} = \{\alpha_1, \alpha_2\}$, notice that the words α_1 and α_2 may be legally composed to form the words $\alpha_1\alpha_1, \alpha_2\alpha_2$, and $\alpha_2\alpha_1$, but $\alpha_2\alpha_2$ is forbidden, precisely because $A(\alpha_2, \alpha_2) = 0$.

Should there exist an underlying category, the fact that, say, $\alpha_1\alpha_2$ is a legal composition would lead one to believe that $s(\alpha_1)$, the domain, or source of α_1

coincides with $r(\alpha_2)$, the co-domain of α_2. But then for similar reasons one would have

$$s(\alpha_2) = r(\alpha_1) = s(\alpha_1) = r(\alpha_2),$$

which would imply that $\alpha_2\alpha_2$ is a valid composition, but it is clearly not. This example was in fact already noticed by Tomforde with the purpose of showing that a 0-1 matrix A is not always the edge matrix of a graph. Although the above matrix may be replaced by another one which is the edge matrix of a graph and gives the same Cuntz-Krieger algebra, the same trick does not work for infinite matrices.

This is perhaps an indication that we should learn to live with semigroupoids which are not true categories. Given the sheer simplicity of the notion of semigroupoid, one can easily fit all of the combinatorial objects so far referred to within the framework of semigroupoids.

The goal of this work is therefore to introduce a notion of representation of semigroupoids, with its accompanying universal C^*-algebra, which in turn generalizes earlier constructions such as the Cuntz-Krieger algebras for arbitrary matrices and the higher-rank graph C^*-algebras [5]: (A k graph (rank k graph or higher rank graph) (Λ, d) consists of a countable small category Λ (with range and source maps r and s respectively) together with a functor $d: \Lambda \to \boldsymbol{N}^k$ satisfying the factorization property: for every $\lambda \in \Lambda$ and $m, n \in \boldsymbol{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m, d(\nu) = n$. For $n \in \boldsymbol{N}^k$ we write $\Lambda^n := d^{-1}(n)$. A morphism between k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f: \Lambda_1 \to \Lambda_2$ compatible with the degree maps.), and hence ordinary graph C^*-algebras as well.

We devoted to comparing semigroupoid C^*-algebras with Cuntz-Krieger and higher-rank graph C^*-algebras. A deeper study is made of the structure of semigroupoid C^*-algebras, including describing them as groupoid C^*-algebras.

The definition of a representation of a semigroupoid Λ given, and consequently of the C^*-algebra of Λ, here denoted $\mathcal{O}(\Lambda)$, is strongly inuenced, and hence it is capable of smoothly dealing with the troubles usually caused by non-row-finiteness.

Speaking of another phenomenon that requires special attention in graph C^*-algebra theory, the presence of sources, once cast in the perspective of semigroupoids, becomes much easier to deal with.

To avoid confusion we use a different term and define a spring (rather than source) to be an element f of a semigroupoid Λ for which $f\mathfrak{g}$ is not a legal multiplication for any $\mathfrak{g} \in \Lambda$. The sources of graph theory are much the same as our springs, and they cause the same sort of problems, but there are some subtle, albeit important differences. For example, in a category any element f may be right-multiplied by the identity morphism on its domain, and hence categories never have any springs. On the other hand, even though higher-rank graphs are defined as categories, sources may still be present and require a special treatment.

While springs are irremediably killed when considered within the associated semigroupoid C^*-algebra, it is rather easy to get rid of them by replacing the given semigroupoid by a somewhat canonical spring-less. This is specially interesting because a slight correction performed on the ingredient semigroupoid is seen to avoid the need to redesign the whole theoretical apparatus.

As already mentioned, the C^*-algebra $C^*(\Lambda)$ associated to a higher-rank graph (Λ, d) in turns out to be a special case of our construction: since Λ is defined to be a category, it is obviously a semigroupoid, so we may consider its semigroupoid C^*-algebra $\mathcal{O}(\Lambda)$, which we prove to be isomorphic to $C^*(\Lambda)$.

One of the most interesting aspects of this is that the construction of $\mathcal{O}(\Lambda)$ does not use the dimension function "d" at all, relying exclusively on the algebraic structure of the subjacent category. In other words, this shows that the dimension function is superuous in the definition of $C^*(\Lambda)$.

It should be stressed that our proof of the isomorphism between $\mathcal{O}(\Lambda)$ and $C^*(\Lambda)$ is done under the standing hypotheses, namely that (Λ, d) is row-finite and has no sources. We will not find here a comparison between our construction and the more recent treatment of Farthing, Muhly and Yeend for general finitely aligned higher rank graphs.

It is a consequence of Definition (1.1.9), describing our notion of a representation S of a semigroupoid Λ, that if Λ contains elements f, g and h such that $f\mathsf{g} = fh$, then

$$S_{\mathsf{g}} = S_h.$$

Therefore, even if g and h are different, that difference is blurred when these elements are seen in $\mathcal{O}(\Lambda)$ via the universal representation. This should probably be interpreted as saying that our representation theory is not really well suited to deal with general semigroupoids in which non monic elements are present. An element f is said to be monic if

$$f\mathsf{g} = fh \quad \Longrightarrow \quad \mathsf{g} = h.$$

Fortunately all of our examples consist of semigroupoids containing only monic elements.

No attempt has been made to consider topological semigroupoids although we believe this is a worthwhile program to be pursued. Among a few indications that this can be done is Katsura's topological graphs and Yeend's topological higher-rank graphs, not to mention Renault's pioneering work on groupoids.

After recognizing the precise obstruction for interpreting Cuntz-Krieger algebras from the point of view of categories or graphs, one can hardly help but to think of the obvious generalization of higher-rank graphs to semigroupoids based on the unique factorization property. Even though we do not do anything useful based on this concept we spell out the precise Definition. As an example, the Markov semigroupoid for the above 2×2 matrix is a rank 1 semigroupoid which is not a rank 1 graph.

We would also like to mention that although we have not seriously considered the ultra-graph C^*-algebras of Tomforde from a semigroupoid point of view, we believe that these may also be described in terms of naturally occurring semigroupoids.

In this section we introduce the basic algebraic ingredient of our construction.

Definition (1.1.1) [1]:

A semigroupoid is a triple $(\Lambda, \Lambda^{(2)}, .)$ such that Λ is a set, $\Lambda^{(2)}$ is a subset of $\Lambda \times \Lambda$, and

$$\cdot : \Lambda^{(2)} \to \Lambda$$

is an operation which is associative in the following sense: if $f, \mathsf{g}, h \in \Lambda$ are such that either

(i) $(f, \mathsf{g}) \in \Lambda^{(2)}$ and $(\mathsf{g}, h) \in \Lambda^{(2)}$ or

(ii) $(f, \mathsf{g}) \in \Lambda^{(2)}$ and $(f\mathsf{g}, h) \in \Lambda^{(2)}$ or

(iii) $(\mathsf{g}, h) \in \Lambda^{(2)}$ and $(f, \mathsf{g}h) \in \Lambda^{(2)}$,

then all of $(f, \mathsf{g}), (\mathsf{g}, h), (f\mathsf{g}, h)$ and $(f, \mathsf{g}h)$ lie in $\Lambda^{(2)}$, and

$$(f\mathsf{g})h = f(\mathsf{g}h).$$

Moreover, for every $f \in \Lambda$, we will let

$$\Lambda^f = \{\mathsf{g} \in \Lambda : (f, \mathsf{g}) \in \Lambda^{(2)}\}.$$

From now on we fix a semigroupoid Λ.

Definition (1.1.2) [1]:

Let $\mathsf{f}, \mathsf{g} \in \Lambda$. We shall say that f divides g, or that g is a multiple of f, in symbols $f|\mathsf{g}$, if either

(i) $f = \mathsf{g}$, or

(ii) There exists $h \in \Lambda$ such that $fh = \mathsf{g}$.

When $f|\mathsf{g}$, and $\mathsf{g}|f$, we shall say that f and g are equivalent, in symbols $f \simeq g$.

Perhaps the correct way to write up the above definition is to require that $(f, h) \in \Lambda^{(2)}$ before referring to the product "fh". However we will adopt the convention that, when a statement is made about a freshly introduced element which involves a multiplication, then the statement is implicitly supposed to include the requirement that the multiplication involved is allowed.

Notice that in the absence of anything resembling a unit in Λ, it is conceivable that for some element $f \in \Lambda$ there exists no $u \in \Lambda$ such that $f = fu$. Had we not explicitly included (1.1.2) (i), it would not always be the case that $f|f$.

A useful artifice is to introduce a unit for Λ, that is, pick some element in the universe outside Λ, call it 1, and set $\widetilde{\Lambda} = \Lambda \dot{\cup} \{1\}$. For every $f \in \Lambda$ put

$$1f = f1 = f.$$

Then, whenever $f|g$, regardless of whether $f = g$ or not, there always exists $x \in \widetilde{\Lambda}$ such that $g = fx$.

We will find it useful to extend the definition of Λ^f , for $f \in \widetilde{\Lambda}$ by putting

$$\Lambda^1 = \Lambda.$$

Nevertheless, even if $f1$ is a meaningful product for every $f \in \Lambda$, we will not include 1 in Λ^f.

We should be aware that $\widetilde{\Lambda}$ is not a semigroupoid. Otherwise, since $f1$ and $1g$ are meaningful products, axiom (1.1.1) (i) would imply that $(f1)g$ is also a meaningful product, but this is clearly not always the case.

It is interesting to understand the extent to which the associativity property fails for $\widetilde{\Lambda}$. As already observed, (1.1.1) (i) does fail irremediably whe $g = 1$. Nevertheless it is easy to see that (1.1.1) generalizes to $\widetilde{\Lambda}$ in all other cases. This is quite useful, since when we are developing a computation, having arrived at an expression of the form $(fg)h$, and therefore having already checked that all products involved are meaningful, we most often want to proceed by writing

$$\cdots = (fg)h = f(gh).$$

The axiom to be invoked here is (1.1.1) (i) (or (1.1.1) (iii) in a similar situation), and fortunately not (1.1.1) (i)!

Proposition (1.1.3) [1]:

Division is a reflexive and transitive relation.

Proof:

That division is reflexive follows from the definition. In order to prove transitivity let $f, \mathsf{g} \in \Lambda$ be such that $f|\mathsf{g}$ and $\mathsf{g}|h$. We must prove that $f|h$.

The case in which $f = \mathsf{g}$, or $\mathsf{g} = h$ is obvious. Otherwise there are u, v in Λ (rather than in $\widetilde{\Lambda}$) such that $fu = \mathsf{g}$, and $\mathsf{g}v = h$. As observed above, it is implicit that $(f, u), (g, v) \in \Lambda^{(2)}$, which implies that

$$(f, u), (fu, v) \in \Lambda^{(2)}.$$

By (1.1.1) (ii) we deduce that $(u, v) \in \Lambda^{(2)}$ and that

$$f(uv) = (fu)v = \mathsf{g}v = h,$$

and hence $f|h$.

Division is also invariant under multiplication on the left:

Proposition (1.1.4) [1]:

If $k, f, \mathsf{g} \in \Lambda$ are such that $f|\mathsf{g}$, and $(k, f) \in \Lambda^{(2)}$, then $(k, \mathsf{g}) \in \Lambda^{(2)}$ and $kf|k\mathsf{g}$.

Proof:

The case in which $f = \mathsf{g}$ being obvious we assume that there is $u \in \Lambda$ such that $fu = \mathsf{g}$. Since $(k, f), (f, u) \in \Lambda^{(2)}$ we conclude from (1.1.1) (i) that (kf, u) and $(k, g) = (k, fu)$ lie in $\Lambda^{(2)}$, and that

$$(kf)u = k(fu) = k\mathsf{g},$$

so $kf|k\mathsf{g}$.

The next concept will be crucial to the analysis of the structure of semigroupoids.

Definition (1.1.5) [1]:

Let $f, \mathsf{g} \in \Lambda$. We shall say that f and g intersect if they admit a common multiple, that is, an element $m \in \Lambda$ such that $f|m$ and $\mathsf{g}|m$. Otherwise we will say

that f and g are disjoint. We shall indicate the fact that f and g intersect by writing $f \Cap \mathsf{g}$, and when they are disjoint we will write $f \perp g$.

If there exists a right-zero element, that is, an element $0 \in \Lambda$ such that $(f, 0) \in \Lambda^{(2)}$ and $f0 = 0$, for all $f \in \Lambda$, then obviously $f|0$, and hence any two elements intersect. We shall be mostly interested in semigroupoids without a right-zero element.

Employing the unitization $\widetilde{\Lambda}$ notice that $f \Cap \mathsf{g}$ if and only if there are $x, y \in \widetilde{\Lambda}$ such that $fx = \mathsf{g}y$.

The last important concept, borrowed from the Theory of Categories, is as follows:

Definition (1.1.6) [1]:

We shall say that an element $f \in \Lambda$ is monic if for every $\mathsf{g}, h \in \Lambda$ we have

$$f\mathsf{g} = fh \quad \Longrightarrow \quad \mathsf{g} = h.$$

We would now like to discuss certain special properties of elements $f \in \Lambda$ for which $\Lambda^f = \phi$. It would be sensible to call these elements sources, following the terminology adopted in Graph Theory, but given some subtle differences we'd rather use another term:

Definition (1.1.7) [1]:

We will say that an element f of a semigroupoid Λ is a spring when $\Lambda^f = \phi$.

Springs are sometimes annoying, so we shall now discuss a way of getting rid of springs. Let us therefore fix a semigroupoid Λ which has springs.

Denote by Λ_0 the subset of Λ formed by all springs and let E' be a set containing a distinct element e'_{g}, for every $\mathsf{g} \in \Lambda_0$. Consider any equivalence relation " $\sim$ " on E' according to which

$$e'_{\mathsf{g}} \sim e'_{f\mathsf{g}}, \tag{1}$$

for any spring g, and any f such that $\mathsf{g} \in \Lambda^f$. Observe that $f\mathsf{g}$ is necessarily also a spring since $\Lambda^{f\mathsf{g}} = \Lambda^{\mathsf{g}}$, by (1.1.1) (i)-(ii). For example, one can take the

equivalence relation according to which any two elements are related. Alternatively we could use the smallest equivalence relation satisfying (1).

We shall denote the quotient space $E'/\sim$ by E, and for every spring g we will denote the equivalence class of e'_{g} by e_{g}. Unlike the e'_{g}, the e_{g} are obviously no longer distinct elements. In particular we have

$$e_{\mathrm{g}} = e_{f\mathrm{g}}, \quad \forall f \in \Lambda, \quad \forall g \in \Lambda_0 \cap \Lambda^f.$$

We shall now construct a semigroupoid Γ as follows: set $\Gamma = \Lambda \dot{\cup} E$, and put

$$\Gamma^{(2)} = \Lambda^{(2)} \cup \{(\mathrm{g}, e_{\mathrm{g}}) : \mathrm{g} \in \Lambda_0\} \cup \{(e_{\mathrm{g}}, e_{\mathrm{g}}) : \mathrm{g} \in \Lambda_0\}.$$

Define the multiplication

$$\cdot : \Gamma^{(2)} \to \Gamma,$$

to coincide with the multiplication of Λ when restricted to $\Lambda^{(2)}$, and moreover set

$$\mathrm{g} \cdot e_{\mathrm{g}} = \mathrm{g} \quad \text{and} \quad e_{\mathrm{g}} \cdot e_{\mathrm{g}} = e_{\mathrm{g}}, \quad \forall \mathrm{g} \in \Lambda_0.$$

It is rather tedious, but entirely elementary, to show that Γ is a semigroupoid without any springs containing Λ. To summarize the conclusions of this section we state the following:

Theorem (1.1.8) [1]:

For any semigroupoid Λ there exists a spring-less semigroupoid Γ containing Λ.

Given a certain freedom in the choice of the equivalence relation "$\sim$" above, there seems not to be a canonical way to embed Λ in a spring-less semigroupoid. The user might therefore have to make a case by case choice according to his or her preference.

In this section we begin the study of the central notion bridging semigroupoids and operator algebras.

Definition (1.1.9) [1]:

Let Λ be a semigroupoid and let B be a unital C^*-algebra. A mapping $S : \Lambda \to B$ will be called a representation of Λ in B, if for every $f\mathrm{g} \in \Lambda$, one has that:

(i) S_f is a partial isometry,

(ii) $S_f\, S_{\mathrm{g}} = \begin{cases} S_{f\mathrm{g}} & \text{if } (f,\mathrm{g}) \in \Lambda^{(2)} \\ 0, & \text{otherwise.} \end{cases}$

Moreover the initial projections $Q_f = S_f^* S_f$, and the final projections $P_{\mathrm{g}} = S_{\mathrm{g}} S_{\mathrm{g}}^*$, are required to commute amongst themselves and to satisfy

(iii) $P_f P_{\mathrm{g}} = 0$, if $f \perp \mathrm{g}$,

(iv) $Q_f P_{\mathrm{g}} = P_{\mathrm{g}}$, if $(f,\mathrm{g}) \in \Lambda^{(2)}$.

Notice that if $(f,\mathrm{g}) \notin \Lambda^{(2)}$, then $Q_f P_{\mathrm{g}} = S_f^* S_f S_{\mathrm{g}}^* = 0$; by (ii). Complementing (iv) above we could therefore add:

(v) $Q_f P_{\mathrm{g}} = 0$, if $(f,\mathrm{g}) \notin \Lambda^{(2)}$.

We will automatically extend any representation S to the unitization $\widetilde{\Lambda}$ by setting $S_1 = 1$. Likewise we put $Q_1 = P_1 = 1$.

Notice that in case Λ contains an element f which is not monic, say $f\mathrm{g} = fh$, for a pair of distinct elements $\mathrm{g}, h \in \Lambda$, one necessarily has $S_{\mathrm{g}} = S_h$, for every representation S. In fact

$$S_{\mathrm{g}} = S_{\mathrm{g}} S_{\mathrm{g}}^* S_{\mathrm{g}} = P_{\mathrm{g}} S_{\mathrm{g}} = Q_f P_{\mathrm{g}} S_{\mathrm{g}} = S_f^* S_f S_{\mathrm{g}} S_{\mathrm{g}}^* S_{\mathrm{g}} = S_f^* S_f S_{\mathrm{g}} = S_f^* S_{f\mathrm{g}}$$

and similarly $S_h = S_f^* S_{fh}$, so it follows that $S_{\mathrm{g}} = S_h$, as claimed.

This should probably be interpreted as saying that our representation theory is not really well suited to deal with general semigroupoids in which non monic elements are present. In fact, all of our examples consist of semigroupoids containing only monic elements.

From now on we will fix a representation S of a given semigroupoid Λ in a unital C^*-algebra B. By (1.1.9) (iv) we have that $P_h \leq Q_f$, for all $h \in \Lambda^f$, so if $h_1, h_2 \in \Lambda^f$ we deduce that

$$P_{h_1} \vee P_{h_2} := P_{h_1} + P_{h_2} - P_{h_1} P_{h_2} \leq Q_f.$$

More generally, if H is a finite subset of Λ^f we will have

$$\bigvee_{h\in H} P_h \leq Q_f.$$

We now wish to discuss whether or not the above inequality becomes an identity under circumstances which we now make explicit:

Definition (1.1.10) [1]:

Let X be any subset of Λ. A subset $H \subseteq X$ will be called a covering of X if for every $f \in X$ there exists $h \in H$ such that $h \Cap f$. If moreover the elements of H are mutually disjoint then H will be called a partition of X.

The following elementary fact is noted for further reference:

Proposition (1.1.11) [1]:

A subset $H \subseteq X$ is a partition of X if and only if H is a maximal subset of X consisting of pairwise disjoint elements.

Returning to our discussion above we wish to require that

$$\bigvee_{h\in H} P_h \overset{?}{=} Q_f, \tag{2}$$

whenever H is a covering of Λ^f . The trouble with this equation is that when H is infinite there is no reasonable topology available on B under which one can make sense of the supremum of infinitely many commuting projections.

Before we try to attach any sense to (2) notice that if $\mathrm{g} \in \widetilde{\Lambda}$ and $h \in \Lambda/\Lambda^{\mathrm{g}}$, then $P_h \leq 1 - Q_{\mathrm{g}}$, by (1.1.9) (v), and hence also

$$\bigvee_{h\in H} P_h \leq 1 - Q_{\mathrm{g}},$$

for every finite set $H \subseteq \Lambda/\Lambda^{\mathrm{g}}$. More generally, given finite subsets $F, G \subseteq \widetilde{\Lambda}$, denote

$$\Lambda^{F,G} = \left(\bigcap_{f\in F} \Lambda^f\right) \cap \left(\bigcap_{\mathrm{g}\in G} \Lambda/\Lambda^{\mathrm{g}}\right),$$

and let $h \in \Lambda^{F,G}$. By (1.1.9) (iv)-(v), we have that

$$P_h \leq \prod_{f \in F} Q_f \prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}).$$

As in the above cases we deduce that

$$\bigvee_{h \in H} P_h \leq \prod_{f \in F} Q_f \prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}),$$

for every finite subset $H \subseteq \Lambda^{F,G}$.

Definition (1.1.12) [1]:

A representation S of Λ in a unital C^*-algebra B is said to be tight if for every finite subsets $F, G \subseteq \widetilde{\Lambda}$, and for every finite covering H of $\Lambda^{F,G}$ one has that

$$\bigvee_{h \in H} P_h = \prod_{f \in F} Q_f \prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}),$$

Observe that if for every pair of finite subset $F, G \subseteq \widetilde{\Lambda}$, one has that $\Lambda^{F,G}$ admits no finite covering then any representation is tight by default.

For many representation theories there is a C^*-algebra whose representations are in one-to-one correspondence with the representations in the given theory. Semigroupoid representations are no exception:

Definition (1.1.13) [1]:

Given a semigroupoid Λ we shall let $\tilde{\mathcal{O}}(\Lambda)$ be the universal unital C^*-algebra generated by a family of partial isometries $\{S_f\}_{f \in \Lambda}$ subject to the relations that the correspondence $f \longmapsto S_f$ is a tight representation of Λ. That representation will be called the universal representation and the closed $*$-subalgebra of $\tilde{\mathcal{O}}(\Lambda)$ generated by its range will be denoted $\mathcal{O}(\Lambda)$.

It is clear that $\tilde{\mathcal{O}}(\Lambda)$ is either equal to $\mathcal{O}(\Lambda)$ or to its unitization. Observe also that the relations we are referring to in the above definition are all expressable in

the form described. Moreover these relations are admissible, since any partial isometry has norm one. It therefore follows that $\tilde{\mathcal{O}}(\Lambda)$ exists.

The universal property of $\tilde{\mathcal{O}}(\Lambda)$ may be expressed as follows:

Proposition (1.1.14) [1]:

For every tight representation T of Λ in a unital C^*-algebra B there exists a unique $*$-homomorphism

$$\varphi : \tilde{\mathcal{O}}(\Lambda) \to B,$$

such that $\varphi(\mathsf{S}_f) = \mathsf{T}_f$, for every $f \in \Lambda$.

It might also be interesting to define a "Toeplitz" extension of $\tilde{\mathcal{O}}(\Lambda)$, as the universal unital C^*-algebra generated by a family of partial isometries $\{S_f\}_{f\in\Lambda}$ subject to the relations that the correspondence $f \longmapsto S_f$ is a (not necessarily tight) representation of Λ. If such an algebra is denoted $\mathcal{T}(\Lambda)$, it is immediate that $\tilde{\mathcal{O}}(\Lambda)$ is a quotient of $\mathcal{T}(\Lambda)$.

As already observed the usefulness of these constructions is probably limited to the case in which every element of Λ is monic.

Tight representations and springs do not go together well, as explained below:

Proposition (1.1.15) [1]:

Let S be a tight representation of a semigroupoid Λ and let $f \in \Lambda$ be a spring (as defined in (1.1.7)). Then $S_f = 0$.

Proof:

Under the assumption that $\Lambda^f = \phi$, notice that the empty set is a covering of Λ^f and hence $Q_f = 0$, by (1.1.12). Since $Q_f = S_f^* S_f$, one has that $S_f = 0$, as well.

We thus see that springs do not play any role with respect to tight representations. There are in fact some other non-spring elements on which every tight representation vanishes. Consider for instance the situation in which Λ^f consists of a finite number of elements, say $\Lambda^f = \{h_1, \dots, h_n\}$, each h_i being a spring. Then Λ^f is a finite cover of itself and hence by (1.1.12) we have

$$Q_f = \bigvee_{i=1}^{n} P_{h_i} = 0,$$

which clearly implies that $S_f = 0$.

One might feel tempted to redesign the whole concept of tight representations especially if one is bothered by the fact that springs are killed by them. However we strongly feel that the right thing to do is to redesign the semigroupoid instead, using (1.1.8) to replace Λ by a spring-less semigroupoid containing it.

In this case it might be useful to understand the following situation:

Proposition (1.1.16) [1]:

Let S be a tight representation of a semigroupoid Λ and suppose that $f \in \Lambda$ is such that Λ^f contains a single element e such that $e^2 = e$. Then S_e is a projection and moreover $S_e = Q_f$.

Proof:

Since $e^2 = e$, we have that $S_e^2 = S_e$. But since S_e is also a partial isometry, it must necessarily be a projection. By assumption we have that $\{e\}$ is a finite covering for Λ^f so

$$Q_f = P_e = S_e S_e^* = S_e.$$

With this in mind we will occasionally work under the assumption that our semigroupoid has no springs.

Section (1.2): Markov Semigroupoid with Categories and Higher-rank Graphs

In this section we present a semigroupoid whose C^*-algebra which is isomorphic to the Cuntz-Krieger algebra introduced. For this let G be any set and let $A = \{A(i,j)\}_{i,j\in\mathcal{G}}$ be an arbitrary matrix with entries in $\{0,1\}$. We consider the set $\Lambda = \Lambda_A$ of all finite admissible words

$$\alpha = \alpha_1\alpha_2 \dots \alpha_n,$$

i.e., finite sequences of elements $\alpha_i \in \mathcal{G}$, such that $A(\alpha_i, \alpha_{i+1}) = 1$. Even though it is sometimes interesting to consider the empty word as valid, we shall not do so. If allowed, the empty word would duplicate the role of the extra element $1 \in \widetilde{\Lambda}$. Our words are therefore assumed to have strictly positive length $(n \geq 1)$.

Given another admissible word, say $\beta = \beta_1\beta_2 \dots \beta_m$, the concatenated word

$$\alpha\beta := \alpha_1 \dots \alpha_n\beta_1 \dots \beta_m$$

is admissible as long as $A(\alpha_n, \beta_1) = 1$. Thus, if we set

$$\Lambda^{(2)} = \{(\alpha, \beta) = (\alpha_1\alpha_2 \dots \alpha_n, \beta_1\beta_2 \dots \beta_m) \in \Lambda \times \Lambda : A(\alpha_n, \beta_1) = 1\},$$

we get a semigroupoid with concatenation as product.

Definition (2.1.1) [1]:

The semigroupoid $\Lambda = \Lambda_A$ defined above will be called the Markov semigroupoid.

Observe that the springs in Λ are precisely the words $\alpha = (\alpha_1\alpha_2 \dots \alpha_n)$ for which $A(\alpha_n, j) = 0$, for every $j \in \mathcal{G}$, that is, for which the α_nth row of A is zero. To avoid springs we will assume that no row of A is zero.

Theorem (1.2.2) [1]:

Suppose that A has no zero rows. Then $\tilde{\mathcal{O}}(\Lambda)$ is $*$-isomorphic to the Exel-Laca algebra $\tilde{\mathcal{O}}_A$.

Proof:

Throughout this proof we will denote the standard generators of $\tilde{\mathcal{O}}_A$ by $\{\check{S}_x\}_{x \in \mathcal{G}}$, together with their initial and final projections $\check{Q}_x = \check{S}_x^*\check{S}_x$ and $\check{P}_x = \check{S}_x\check{S}_x^*$, respectively. Likewise the standard generators of $\tilde{\mathcal{O}}(\Lambda)$ will be denoted by $\{\hat{S}_f\}_{f \in \Lambda}$, along with their initial and final projections $\hat{Q}_f = \hat{S}_f^*\hat{S}_f$ and $\hat{P}_f = \hat{S}_f\hat{S}_f^*$. In addition, for every $x \in \mathcal{G}$ we will identify the one-letter word "x" with the element x itself, so we may think of $\mathcal{G}$ as a subset of Λ.

We begin by claiming that the set of partial isometries

$$\{\hat{S}_x\}_{x\in\mathcal{G}} \subseteq \tilde{\mathcal{O}}(\Lambda)$$

satisfies the defining relations of $\tilde{\mathcal{O}}_A$, namely TCK_1, TCK_2, and TCK_3.

Conditions TCK_1 and TCK_2 follow immediately from (1.1.9), and the observation that if x and y are distinct elements of $\mathcal{G}$, then $x \perp y$ as elements of Λ.

When $A(i,j) = 1$ we have that $(i,j) \in \Lambda^{(2)}$ and hence $\hat{P}_i\hat{Q}_j = \hat{P}_j = A(i,j)\hat{P}_j$, by (1.1.9) (iv). Otherwise, if $A(i,j) = 0$, we have that $(i,j) \notin \Lambda^{(2)}$ and hence $\hat{P}_i\hat{Q}_j = 0 = A(i,j)\hat{P}_j$, by (1.1.9) (v). This proves TCK_3.

In order let X, Y be finite subsets of $\mathcal{G}$ such that

$$A(X,Y,j) := \prod_{x\in X} A(x,j) \prod_{y\in Y} \big(1 - A(y,j)\big) \tag{3}$$

equals zero for all but finitely many j's. It is then easy to see that

$$Z := \{j \in \mathcal{G} : A(X,Y,j) \neq 0\}$$

is a finite partition of $\Lambda^{X,Y}$, so

$$\prod_{x\in X} \hat{Q}_x \prod_{y\in Y} \big(1 - \hat{Q}_y\big) = \bigvee_{j\in Z} \hat{P}_j = \sum_{j\in Z} \hat{P}_j \sum_{j\in\mathcal{G}} A(X,Y,j)\, \hat{P}_j,$$

because the canonical representation $f \in \Lambda \longmapsto \hat{S}_f \in \tilde{\mathcal{O}}(\Lambda)$ is tight by definition. It then follows from the universal property of $\tilde{\mathcal{O}}_A$ that there exists a $*$ homomorphism

$$\Phi : \tilde{\mathcal{O}}_A \to \tilde{\mathcal{O}}(\Lambda), \tag{4}$$

such that $\Phi(\check{S}_x) = \hat{S}_x$, for every $x \in \mathcal{G}$.

Next consider the map $\check{S} : \Lambda \to \tilde{\mathcal{O}}_A$ defined as follows: given $\alpha \in \Lambda$, write $\alpha_1\alpha_2 \dots \alpha_n$, with $\alpha_i \in \mathcal{G}$, and put

$$\check{S}_\alpha = \check{S}_{\alpha_1}\check{S}_{\alpha_2} \dots \check{S}_{\alpha_n},$$

We claim that $\check{S}$ is a tight representation of Λ in $\tilde{\mathcal{O}}_A$. The first two axioms of (1.1.9) are immediate, while the commutativity of the $\check{P}_f$, and $\check{Q}_g$ follow. Next suppose that $\alpha, \beta \in \Lambda$ are such that $\alpha \perp \beta$. One may then prove that

$$\alpha = \alpha_1 \dots \alpha_p \dots \alpha_n \quad \text{and} \quad \beta = \beta_1 \dots \beta_p \dots \beta_m,$$

with $1 \leq p \leq n, m$, and such that $\alpha_i = \beta_i$ for $i < p$, and $\alpha_p \neq \beta_p$. Denoting by $\gamma = \alpha_1 \dots \alpha_{p-1}$ (possibly the empty word), we have that

$$\check{P}_\alpha \leq \check{S}_\gamma \check{S}_{\alpha_p} \check{S}^*_{\alpha_p} \check{S}^*_\gamma = \check{S}_\gamma \check{P}_{\alpha_p} \check{S}^*_\gamma,$$

and similarly $\check{P}_\beta \leq \check{S}_\gamma \check{P}_{\beta_p} \check{S}^*_\gamma$. It follows that

$$\check{P}_\alpha \check{P}_\beta \leq \check{S}_\gamma \check{P}_{\alpha_p} \check{S}^*_\gamma \check{P}_{\beta_p} \check{S}^*_\gamma = \check{S}_\gamma \check{P}_{\alpha_p} \check{Q}_\gamma \check{P}_{\beta_p} \check{S}^*_\gamma = \check{S}_\gamma \check{P}_{\beta_p} \check{P}_{\beta_p} \check{Q}_\gamma \check{S}^*_\gamma = 0,$$

Hence proving (1.1.9) (iii). In order to verify (1.1.9) (iv) let $(\alpha, \beta) \in \Lambda^{(2)}$, so that $A(\alpha_n, \beta_1) = 1$, where n is the length of α. As shown in "Claim (1)", we have that $\check{Q}_\alpha = \check{Q}_{\alpha_n}$, so

$$\check{Q}_\alpha \check{P}_\beta = \check{Q}_{\alpha_n} \check{P}_{\beta_1} \check{P}_\beta = \check{P}_{\beta_1} \check{P}_\beta = \check{P}_\beta,$$

where we have used TCK_3 in the second equality.

We are then left with the task of proving $\check{S}$ to be tight. For this let X and Y be finite subsets of Λ and let Z be a finite covering of $\Lambda^{X,Y}$. We must prove that

$$\bigvee_{h \in Z} \check{P}_h = \prod_{f \in X} \check{Q}_f \prod_{g \in Y} (1 - \check{Q}_g). \tag{5}$$

Using TCK_3 it is easy to check the inequality "$\leq$" in (5) so it suffices to verify the opposite inequality.

Let $h_1, h_2 \in Z$ be such that $h_1 \Cap h_2$, and write $h_1 x_1 = h_2 x_2$, where $x_1 x_2 \in \Lambda$. Assuming that the length of h_1 does not exceed that of h_2, one sees that h_1 is an initial segment of h_2, and hence $h_1 | h_2$. Any element of $\Lambda^{X,Y}$ which intersects h_2 must therefore also intersect h_1. This said we see that $Z' := Z/\{h_2\}$ is also a covering of $\Lambda^{X,Y}$. Since the left-hand-side of (5) decreases upon replacing Z by Z', it is clearly enough to prove the remaining inequality "$\geq$" with Z' in place of Z.

Proceeding in such a way every time we find pairs of intersecting elements in Z we may then suppose that Z consists of pairwise disjoint elements, and hence that Z is a partition.

Given $f \in \Lambda$, write $f = \alpha_1 \dots \alpha_n$, with $\alpha_i \in \mathcal{G}$, and observe that $\check{Q}_f = \check{Q}_{\alpha_n}$, as already mentioned. Since $\Lambda^f = \Lambda^{\alpha_n}$, as well, we may assume without loss of generality that X and Y consist of words of length one, or equivalently that $X, Y \in \mathcal{G}$. Let

$$J = \{j \in \mathcal{G} : A(X, Y, j) \neq 0\},$$

where $A(X, Y, j)$ is as in (5). Notice that $j \in J$ if and only if $A(x, j) = 1$, and $A(y, j) = 0$, for all $x \in X$ and $y \in Y$, which is precisely to say that $j \in \Lambda^{X,Y}$. In other words

$$J = \Lambda^{X,Y} \cap \mathcal{G}.$$

It is clear that J shares with Z the property of being maximal among the subsets of pairwise disjoint elements of $\Lambda^{X,Y}$ (see (1.1.11)).

Suppose for the moment that Z is formed by words of length one, i.e, that $Z \subseteq \mathcal{G}$. Then $Z \subseteq J$, and so $Z = J$, by maximality. This implies that J is finite and

$$\bigvee_{z \in Z} \check{P}_z = \bigvee_{j \in J} \check{P}_j = \sum_{j \in J} \check{P}_j = \sum_{j \in \mathcal{G}} A(X, Y, j) \check{P}_j = \prod_{x \in X} \check{Q}_f \prod_{y \in Y} (1 - \check{Q}_y),$$

Thus proving (5). Addressing the situation in which Z is not necessarily contained in G, let

$$Z_j = \{\alpha \in Z : \alpha_1 = j\}, \quad \forall j \in J.$$

Since $Z \subseteq \Lambda^{X,Y}$ it is evident that

$$Z = \bigcup_{j \in J} Z_j.$$

Moreover notice that each Z_j is nonempty since otherwise $Z \cup \{j\}$ will be a subset of $\Lambda^{X,Y}$ formed by mutually disjoint elements, contradicting the maximality of Z. In particular this shows that J is finite and hence, so that

$$\prod_{x \in X} \check{Q}_x \prod_{y \in Y} (1 - \check{Q}_y) = \sum_{j \in \mathcal{G}} A(X, Y, j) \check{P}_j \sum_{j \in J} \check{P}_j \tag{6}$$

We claim that for every $j \in J$ one has that

$$\breve{P}_j = \sum_{z \in Z_j} \breve{P}_z.$$

Before proving the claim lets us notice that it does implies our goal, for then

$$\sum_{z \in Z} \breve{P}_z = \sum_{j \in J} \sum_{z \in Z_j} \breve{P}_z = \sum_{j \in J} \breve{P}_j \overset{(6)}{=} \prod_{x \in X} \breve{Q}_f \prod_{y \in Y} (1 - \breve{Q}_y),$$

proving (5).

Noticing that each Z_j is maximal among subsets of mutually disjoint elements beginning in j, the claim follows from the following:

Lemma (1.2.3) [1]:

Given $x \in \mathcal{G}$, let $\Lambda(x) = \{\alpha \in \Lambda : \alpha_1 = x\}$, and let H be a finite partition of $\Lambda(x)$. Then

$$\sum_{h \in H} \breve{P}_h = \breve{P}_x.$$

Proof:

Let n be the maximum length of the elements of H. We will prove the statement by induction on n. If $n = 1$ it is clear that $H = \{x\}$ and the conclusion follows by obvious reasons. Supposing that $n > 1$ observe that $x \notin H$, or else any element in H with length n will intersect x, violating the hypothesis that H consists of mutually disjoint elements. Therefore every element of H has length at least two.

Let $J = \{j \in \mathcal{G} : A(X, j) = 1\}$ and set $H_j = \{\alpha \in H : \alpha_2 = j\}$. It is clear that

$$H = \bigcup_{j \in J} H_j.$$

Moreover notice that every H_j is nonempty, since otherwise $H \cup \{xj\}$ consists of mutually disjoint elements and properly contains H, contradicting maximality. In particular this implies that J is finite and hence we have

$$\breve{Q}_x = \sum_{j \in \mathcal{G}} A(X, Y, j) \breve{P}_j = \sum_{j \in J} \breve{P}_j. \tag{7}$$

For every $j \in J$, let H_j' be the set obtained by deleting the first letter from all words in H_j, so that $H_j' \subseteq \Lambda(j)$, and $H_j = xH_j'$. One moment of reffexion will convince that H_j' is maximal among the subsets of mutually disjoint elements of $\Lambda(j)$. Since the maximum length of elements in H_j' is no bigger than $n-1$, we may use induction to conclude that

$$\check{P}_j = \sum_{k \in H_j'} \check{P}_k.$$

Therefore

$$P_x = \check{S}_x \check{S}_x^* \check{S}_x \check{S}_x^* = \check{S}_x \check{Q}_x \check{S}_x^* \overset{(7)}{=} \sum_{j \in J} \check{S}_x \check{P}_j \check{S}_x^* = \sum_{j \in J} \sum_{k \in H_j'} \check{S}_x \check{P}_k \check{S}_x^* = \sum_{j \in J} \sum_{k \in H_j'} \check{P}_{xk}$$

$$= \sum_{j \in J} \sum_{h \in H_j} \check{P}_h = \sum_{h \in H} \check{P}_h.$$

Returning to the proof of (1.2.2), now in possession of the information that $\check{S}$ is a tight representation of Λ, we conclude by the universal property of $\tilde{\mathcal{O}}(\Lambda)$ that there exists a $*$-homomorphism

$$\Psi \colon \tilde{\mathcal{O}}(\Lambda) \to \tilde{\mathcal{O}}_A,$$

such that $\Psi(\hat{S}_\alpha) = \hat{S}_\alpha$, for all $\alpha \in \Lambda$. It is then clear that Ψ is the inverse of the homomorphism Φ of (4), and hence both Φ and Ψ are isomorphisms.

In this section we fix a small category Λ. Notice that the collection of all morphisms of Λ (which we identify with Λ itself) is a semigroupoid under composition. We shall now study Λ from the point of view of the theory introduced in the previous section.

Given $v \in \mathrm{obj}(\Lambda)$ (meaning the set of objects of Λ) we will identify v with the identity morphism on v, so that we will see $\mathrm{obj}(\Lambda)$ as a subset of the set of all morphisms.

Given $f \in \Lambda$ we will denote by $s(f)$ and $r(f)$ the domain and co-domain of f, respectively. Thus the set of all composable pairs may be described as

$$\Lambda^{(2)} = \{(f, \mathrm{g}) \in \Lambda \times \Lambda : s(f) = r(\mathrm{g})\}.$$

Given $f \in \Lambda$ notice that $\Lambda^f = \{\mathrm{g} \in \Lambda : s(f) = r(\mathrm{g})\}$. In particular, if $v \in \mathrm{obj}(\Lambda)$ then $s(v) = r(v) = v$, so

$$\Lambda^v = \{\mathrm{g} \in \Lambda : r(\mathrm{g}) = v\}.$$

A category is a special sort of semigroupoid in several ways. For example, if f_1, f_2, g_1 and $\mathrm{g}_2 \in \Lambda$ are such that $(f_i, \mathrm{g}_i) \in \Lambda^{(2)}$ for all i, j, except perhaps for $(i, j) = (2,2)$, then necessarily $(f_2, \mathrm{g}_2) \in \Lambda^{(2)}$, because

$$s(f_2) = r(\mathrm{g}_1) = s(f_1) = r(\mathrm{g}_2).$$

Another special property of a category among semigroupoids is the fact that for every $f \in \Lambda$ there exists $u \in \Lambda^f$ such that $f = fu$, namely one may take u to be (the identity on) $s(f)$. Thus $f|f$ even if we had omitted (1.1.2) (i) in the definition of division. Clearly this also implies that Λ has no springs.

From now on we fix a representation S of Λ in a unital C^*-algebra B and denote by Q_f and P_f , the initial and final projections of each Sf , respectively. A few elementary facts are in order:

Proposition (1.2.4) [1]:

(i) For every $v \in \mathrm{obj}(\Lambda)$ one has that S_v is a projection, and hence $S_v = P_v = Q_v$.

(ii) If u and v are distinct objects then $P_u \perp P_v$.

(iii) For every $f \in \Lambda$ one has that $Q_f = P_{s(f)}$.

Proof:

We leave the elementary proof of (i). Given distinct objects u and v it is clear that $u \perp v$, so $P_u \perp P_v$, by (1.1.9) (iii). With respect to (iii) we have

$$Q_f = S_f^* S_f = S_f^* S_{fs(f)} = S_f^* S_f S_{s(f)} = Q_f P_{s(f)} = P_{s(f)},$$

where the last equality follows from (1.1.9) (iv).

Definition (1.2.5) [1]:

Let H be a Hilbert space and let $S: \Lambda \to \mathcal{B}(H)$ be a representation. We will say that Λ is nondegenerated if the closed $*$-subalgebra of $\mathcal{B}(H)$ generated by the range of S is nondegenerated.

Nondegenerated Hilbert space representations are partly tight in the following sense:

Proposition (1.2.6) [1]:

Let $S: \Lambda \to \mathcal{B}(H)$ be a representation. If either

(i) S is nondegenerated, or

(ii) $\mathrm{obj}(\Lambda)$ is infinite,

then for every finite subsets $F, G \subseteq \Lambda$ such that $\Lambda^{F,G} = \phi$, one has that

$$\prod_{f \in F} Q_f \prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}) = 0.$$

Proof:

Notice that $\Lambda^{\mathrm{F,G}} = \phi$, implies that

$$\left(\bigcap_{f \in F} \Lambda^f \right) \subseteq \Lambda \setminus \left(\bigcap_{\mathrm{g} \in G} \Lambda \setminus \Lambda^{\mathrm{g}} \right) = \bigcup_{\mathrm{g} \in G} \Lambda^{\mathrm{g}}. \tag{8}$$

Case (1):

Assuming that $\mathrm{F} \neq \phi$, let $f_0 \in \mathrm{F}$. Then either there is some $f \in \mathrm{F}$, with $s(f) \neq s(f_0)$, in which case

$$Q_{f_0} Q_f = P_{s(f_0)} P_{s(f)} = 0,$$

proving the statement; or $s(f) = s(f_0)$, for all $f \in F$. Therefore we may suppose that $s(f_0)$ belongs to Λ^f for every $f \in F$, and hence by (8) there exists $\mathrm{g}_0 \in G$ such that $s(f_0) \in \Lambda^{\mathrm{g}_0}$. But this is only possible if $s(f_0) = s(\mathrm{g}_0)$ and hence

$$Q_{f_0}(1 - Q_{\mathrm{g}_0}) = P_{s(f_0)}(1 - P_{s(\mathrm{g}_0)}) = 0,$$

concluding the proof in case (1).

Case (2):

Assuming next that ϕ, we claim that

$$\mathrm{obj}(\Lambda) = \{s(\mathrm{g}) : \mathrm{g} \in G\}.$$

In fact, arguing as in (8) one has that $\bigcup_{\mathrm{g}\in G} \Lambda^{\mathrm{g}} = \Lambda$, so for every $v \in \mathrm{obj}(\Lambda)$ there exists g in G such that $v \in \Lambda^{\mathrm{g}}$, whence $v = s(\mathrm{g})$, proving our claim.

Under the assumption that $\mathrm{obj}(\Lambda)$ is infinite we have reached a contradiction, meaning that case (2) is impossible and the proof is concluded. We thus proceed supposing nondegeneracy. Let

$$R = \prod_{\mathrm{g}\in G} (1 - Q_{\mathrm{g}}),$$

so, proving the statement is equivalent to proving that $R = 0$. Given $v \in \mathrm{obj}(\Lambda)$, let $\mathrm{g} \in G$ be such that $v = s(\mathrm{g})$. Then

$$Q_{\mathrm{g}} S_v = P_{s(\mathrm{g})} S_v = S_v,$$

from where we deduce that

$$R S_v = R(1 - Q_{\mathrm{g}}) S_v = 0.$$

Given any $f \in \Lambda$ we then have that

$$R S_f = R S_{r(f)} S_f = 0 \quad \text{and} \quad R S_f^* = R S_{s(f)} S_f^* = 0,$$

so $R = 0$, by nondegeneracy.

We next present a greatly simplified way to check that a representation of Λ is tight.

Proposition (1.2.7) [1]:

Given a representation $S: \Lambda \to \mathcal{B}(H)$, consider the following two statements:

a. S is right.

b. For every $v \in \mathrm{obj}(\Lambda)$ and every finite covering H of Λ^v one has that $\bigvee_{h \in H} P_h = P_v$.

Then

(i) (a) implies (b).

(ii) If S is nondegenerated,

Proof:

(i) Assume that S is tight and that H is a finite covering of Λ^v. Setting $F = \{v\}$ and $G = \phi$, notice that

$$\Lambda^{F,G} = \Lambda^v,$$

so H is a finite covering of $\Lambda^{F,G}$, and hence we have by definition that

$$\bigvee_{h \in H} P_h = \prod_{f \in F} Q_f \prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}) = Q_v = P_v.$$

(ii) Assuming S nondegenerated, or $\mathrm{obj}(\Lambda)$ infinite, we next prove that (b) implies (a). So let F and G be finite subsets of Λ and let H be a finite covering of $\Lambda^{F,G}$. We must prove that the identity in (1.1.12) holds. If $\Lambda^{F,G} = \phi$, the conclusion follows from (1.2.6). So we assume that $\Lambda^{F,G} \neq \phi$.

Case (1): $F \neq 0$:

Pick $h \in \Lambda^{F,G}$ and notice that for every $f \in F$ one has that $s(f) = r(h)$, and for every $\mathrm{g} \in G$, it is the case that $s(\mathrm{g}) \neq r(h)$. It therefore follows that

$$\Lambda^{F,G} = \Lambda^v,$$

where $v = r(h)$, so H is in fact a covering of Λ^v. By hypothesis we then have that

$$\bigvee_{h \in H} P_h = P_v. \tag{9}$$

On the other hand observe that for every $\mathrm{g} \in G$, we have that

$$Q_{\mathrm{g}} \overset{(1.2.4)(\mathrm{iii})}{=} P_{s(\mathrm{g})} \perp P_v,$$

given that $s(\mathrm{g}) \neq v$. Noticing that for $f \in F$, we have $Q_f = P_{s(f)} = P_v$, we deduce that

$$\prod_{f \in F} Q_f \prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}) = P_v \overset{(9)}{=} \bigvee_{h \in H} P_h,$$

proving that the identity in (1.1.12) indeed holds in case $F \neq \phi$.

Case (2): $F = \phi$:

Let

$$V = \mathrm{obj}(\Lambda) \backslash \{s(\mathrm{g}) : \mathrm{g} \in G\},$$

so that

$$\Lambda^{F,G} = \bigcup_{v \in V} \Lambda^v.$$

Given that H is a finite covering of $\Lambda^{F,G}$, we have that for each $v \in V$ there exists $h \in H$ such that $v \Cap h$, which in turn implies that $r(h) = v$. Therefore V is finite and hence so is $\mathrm{obj}(\Lambda)$.

Thus, case (2) is impossible under the hypothesis that $\mathrm{obj}(\Lambda)$ is infinite, and hence the proof is finished under that hypothesis. We therefore proceed supposing nondegeneracy. It is then easy to show that

$$\sum_{v \in \mathrm{obj}(\Lambda)} P_v = 1,$$

and hence

$$\prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}) \overset{(1.2.4)(\mathrm{iii})}{=} \prod_{\mathrm{g} \in G} (P_{s(\mathrm{g})}) = \sum_{v \in V} P_v. \tag{10}$$

By assumption H is contained in $\Lambda^{F,G}$, and hence the range of each $h \in H$ belongs to V. Thus

$$H = \bigcup_{v \in V} H_v,$$

where $H_v = \{h \in H : r(h) = v\}$. Observe that H_v is a covering for Λ^v, since if $f \in \Lambda^v$, there exists some $h \in H$ with $h \Cap f$, but this implies that $r(h) = r(f) = v$, and hence $h \in H_v$. Thus

$$\bigvee_{h \in H} P_h = \bigvee_{v \in V} \left(\bigvee_{h \in H_v} P_h \right) = \bigvee_{v \in V} P_h = \sum_{v \in V} P_v \overset{(10)}{=} \prod_{\mathrm{g} \in G} (1 - Q_{\mathrm{g}}).$$

We shall now apply the conclusions above to show that higher-rank graph C^*-algebras may be seen as special cases of our construction. See the definitions and a detailed treatment of higher-rank graph C^*-algebras.

Before we embark on the study of k-graphs from the point of view of semigroupoids let us propose a generalization of the notion of higher-rank graphs to semigroupoids which are not necessarily categories. We will not draw any conclusions based on this notion, limiting ourselves to note that it is a natural extension of Kumjian and Pask's interesting idea.

Definition (1.2.9) [1]:

Let k be a natural number. A rank k semigroupoid, or a k-semigroupoid, is a pair (Λ, d), where Λ is a semigroupoid and

$$d : \Lambda \to \boldsymbol{N}^k,$$

is a function such that

(i) For every $(f, \mathrm{g}) \in \Lambda^{(2)}$, one has that $d(f\mathrm{g}) = d(f) + d(\mathrm{g})$,

(ii) If $f \in \Lambda$, and $m, n \in \boldsymbol{N}^k$ are such that $d(f) = n + m$, there exists a unique pair $(\mathrm{g}, h) \in \Lambda^{(2)}$ such that $d(\mathrm{g}) = n$, $d(h) = m$ and $\mathrm{g}h = f$.

For example, the Markov semigroupoid is a 1-semigroupoid, if equipped with the word length function.

Let (Λ, d) be a k-graph. In particular Λ is a category and hence a semigroupoid. Under suitable hypothesis we shall now prove that the C^*-algebra of the subjacent

semigroupoid is isomorphic to the C^*-algebra of Λ, as defined by Kumjian and Pask. In particular it will follow that the dimension function d is superuous for the definition of the corresponding C^*-algebra.

As before, if $v \in \mathrm{obj}(\Lambda)$ we will denote by Λ^v the set of elements $f \in \Lambda$ for which $r(f) = v$. For every $n \in \boldsymbol{N}^k$ we will moreover let

$$\Lambda_n^v = \{f \in \Lambda^v : d(f) = n\}.$$

We should observe that Λ_n^v is denoted $\Lambda^n(v)$.

According, Λ is said to have no sources if Λ_n^v is never empty. In case Λ_n^v is finite for every v and n one says that Λ is row-finite.

Notice that the absence of sources is a much more stringent condition than to require that Λ has no springs, according to Definition (1.1.7). In fact, since Λ is a category, and hence $s(f) \in \Lambda^f$, for every $f \in \Lambda$, we see that $\Lambda^f \neq \phi$, and hence higher-rank graphs automatically have no springs!

Below we will work under the standing hypotheses, but we note that our construction is meaningful regardless of these requirements, so it would be interesting to compare our construction where these hypotheses are not required. This said, we suppose throughout that Λ is a k-graph for which

$$0 < |\Lambda_n^v| < \infty, \quad \forall v \in \mathrm{obj}(\Lambda), \quad \forall n \in \boldsymbol{N}^k. \tag{11}$$

Lemma (1.2.10) [1]:

For every object v of Λ and every $n \in \boldsymbol{N}^k$ one has that Λ_n^v is partition of Λ^v.

Proof:

Suppose that $f, \mathrm{g} \in \Lambda_n^v$ are such that $f \Cap \mathrm{g}$. So there are $p, q \in \Lambda$ such that $fp = \mathrm{g}q$. Since $d(f) = n = d(\mathrm{g})$ we have that $f = g$, by the uniqueness of the factorization. This shows that the elements of Λ_n^v are pairwise disjoint.

In order to show that Λ_n^v is a covering of Λ^v, let $\mathrm{g} \in \Lambda^v$. By (11) pick any $h \in \Lambda_n^{s(\mathrm{g})}$. Since

$$d(\mathrm{g}h) = d(\mathrm{g}) + d(h) = d(\mathrm{g}) + n = n + d(\mathrm{g}),$$

we may write $\mathsf{g}h = fk$, with $d(f) = n$, and $d(k) = d(\mathsf{g})$. It follows that $f \in \Lambda_n^v$ and $\mathsf{g} \Cap f$.

We now reproduce the definition of the C^*-algebra of a k-graph.

Definition (1.2.11) [1]:

Given a k-graph Λ satisfying (11), the C^*-algebra of Λ, denoted by $C^*(\Lambda)$, is defined to be the universal C^*-algebra generated by a family $\{S_f : f \in \Lambda\}$ of partial isometries satisfying:

(i) $\{S_v : v \in \mathsf{obj}(\Lambda)\}$ is a family of mutually orthogonal projections,

(ii) $S_{f\mathsf{g}} = S_f S_{\mathsf{g}}$ for all $f, \mathsf{g} \in \Lambda$ such that $s(f) = r(\mathsf{g})$,

(iii) $S_f^* S_f = S_{s(f)}$ for all $f \in \Lambda$,

(iv) For every object v and every $n \in \boldsymbol{N}^k$ one has $S_v = \sum_{f \in \Lambda_n^v} S_f S_f^*$.

The following is certainly well known to specialists in higher-rank graph C^*-algebras:

Proposition (1.2.12) [1]:

For every f and g in Λ one has that

(i) If $f \perp \mathsf{g}$ then $S_{\mathsf{g}} S_{\mathsf{g}}^*$,

(ii) $S_f S_f^*$ commutes with $S_{\mathsf{g}} S_{\mathsf{g}}^*$.

Proof:

Recall that whenever $n \in \boldsymbol{N}^k$ is such that $d(f), d(g) \leq n$, we have

$$S_f^* S_{\mathsf{g}} = \sum S_p S_q^*,$$

where the sum extends over all pair (p, q) of elements in Λ such that $fp = \mathsf{g}q$, and $d(fp) = n$. So

$$S_f S_f^* S_{\mathsf{g}} S_{\mathsf{g}}^* = \sum_{p,q} S_f S_p S_q^* S_{\mathsf{g}}^* = \sum_{p,q} S_{fp} S_{\mathsf{g}q}^*.$$

Since the last expression is symmetric with respect to f and $\mathfrak{g}$, we see that (ii) is proved. Moreover, when $f \perp \mathfrak{g}$, it is clear that there exist no pairs (p, q) for which $fp = \mathfrak{g}q$, and hence (i) is proved as well.

We shall now prove that the crucial axiom (1.2.11) (iv) generalizes to coverings:

Lemma (1.2.13) [1]:

Let v be an object of Λ. If H is a finite covering of Λ^v then

$$S_v = \bigvee_{h \in H} S_h S_h^*.$$

Proof:

Let $n \in \boldsymbol{N}^k$ with $n \geq d(h)$, for every $h \in H$. For all $f \in \Lambda_n^v$ we know that there is some $h \in H$ such that $f \Cap h$, so we may write $fx = hy$, for suitable x and y. Since $d(f) = n \geq d(h)$, we may write $f = f_1 f_2$, with $d(f_1) = d(h)$. Noticing that

$$f_1 f_2 x = hy,$$

we deduce from the unique factorization property that $f_1 = h$, which amounts to saying that $h|f$. We claim that this implies that $S_f S_f^* \leq S_h S_h^*$. In fact

$$S_h S_h^* S_f S_f^* = S_{f_1} S_{f_1}^* S_{f_1} S_{f_2} S_f^* = S_{f_1} S_{f_2} S_f^* = S_f S_f^*.$$

Summarizing, we have proved that for every $f \in \Lambda_n^v$, there exists $h \in H$, such that $S_f S_f^* \leq S_h S_h^*$. Therefore

$$S_v \overset{(1.2.11)(\mathrm{iv})}{=} \sum_{f \in \Lambda_n^v} S_f S_f^* = \bigvee_{f \in \Lambda_n^v} S_f S_f^* \leq \bigvee_{h \in H} S_h S_h^* \leq S_v,$$

from where the conclusion follows.

Theorem (1.2.14) [1]:

If Λ is a k-graph satisfying (11) then $C^*(\Lambda)$ is $*$-isomorphic to $\mathcal{O}(\Lambda)$.

Proof:

Throughout this proof we denote the standard generators of $C^*(\Lambda)$ by $\{\check{S}_f\}_{f\in\Lambda}$, together with their initial and final projections $\check{Q}_f$ and $\check{P}_f$, respectively. Meanwhile the standard generators of $\tilde{\mathcal{O}}(\Lambda)$ will be denoted by $\{\check{S}_f\}_{f\in\Lambda}$, along with their initial and final projections $\check{Q}_f$ and $\check{P}_f$. In particular the $\check{S}_f$ are known to satisfy (1.2.11) (i)-(iv), while the $\check{S}_f$ are known to give a tight representation of the semigroupoid Λ.

Working within the semigroupoid C^*-algebra $\tilde{\mathcal{O}}(\Lambda)$, we begin by arguing that the $\hat{S}_f$ also satisfy (1.2.11) (i)-(iv). In fact (1.2.11) (i) follows from (1.2.4) (i)-(ii), while (1.2.11) (ii) is a consequence of (1.1.9) (ii). With respect to (1.2.11) (iii) it was proved in (1.2.4) (iii). Finally (1.2.11) (iv) results from the combination of (1.2.10) and (1.2.7) (i).

Therefore, by the universal property of $C^*(\Lambda)$, there exists a *-homomorphism

$$\Phi: C^*(\Lambda) \to \tilde{\mathcal{O}}(\Lambda),$$

such that $\Phi(\hat{S}_f) = \hat{S}_f$, for every $f \in \Lambda$. Evidently the range of Λ is contained in the closed $*$-subalgebra of $\tilde{\mathcal{O}}(\Lambda)$ generated by the $\hat{S}_f$, also known as $\mathcal{O}(\Lambda)$.

We next move our focus to the higher-rank graph algebra $C^*(\Lambda)$, and prove that the correspondence

$$f \in \Lambda \longmapsto \check{S}_f \in C^*(\Lambda)$$

is a tight representation. Skipping the obvious (1.1.9) (i) we notice that (1.1.9) (ii) follows from (1.2.11) (ii) when $s(f) = r(\mathrm{g})$. On the other hand, if $s(f) \neq r(\mathrm{g})$ we have

$$\check{S}_f\check{S}_{\mathrm{g}} = \check{S}_f\check{S}_{s(f)}\check{S}_{r(\mathrm{g})}\check{S}_{\mathrm{g}} = 0,$$

by (1.2.11) (i).

We next claim that the initial and final projections of the $\hat{S}_f$ commute among themselves. That two initial projections commute follows from (1.2.11) (i) and (iii). Speaking of the commutativity between an initial projection $\check{Q}_f$ and a final projection $\check{P}_{\mathrm{g}}$, we have that $\check{Q}_f = \hat{S}_{s(f)}$, by (1.2.11) (iii) and $P_{\mathrm{g}} \leq S_{r(\mathrm{g})}$ by (1.2.11) (iv). So either $\check{P}_{\mathrm{g}} \leq \check{Q}_f$, if $s(f)r(\mathrm{g})$, or $\check{P}_{\mathrm{g}} \perp \check{Q}_f$, if $s(f) \neq r(\mathrm{g})$, by (1.2.11) (i). In any case it is clear that $\check{Q}_f$ and $\check{P}_{\mathrm{g}}$ commute. That two final projections commute is precisely the content of (1.2.12) (ii).

Clearly (1.1.9) (iii) is granted by (1.2.12) (i). In order to prove (1.1.9) (iv) let $f, \mathrm{g} \in \Lambda$ with $s(f) = r(\mathrm{g})$. We then have that

$$\check{Q}_f \check{P}_{\mathrm{g}} = \check{S}_f^* \check{S}_f \check{P}_{\mathrm{g}} = \check{S}_{s(f)} \check{P}_{\mathrm{g}} = \check{S}_{r(\mathrm{g})} \check{S}_{\mathrm{g}} \check{S}_{\mathrm{g}}^* = \check{S}_{\mathrm{g}} \check{S}_{\mathrm{g}}^* = \check{P}_{\mathrm{g}}.$$

This shows that $\check{S}$ is a representation of Λ in $C^*(\Lambda)$, which we will now prove to be tight. For this let

$$\pi : C^*(\Lambda) \longrightarrow \mathcal{B}(H)$$

be a faithful nondegenerated representation of $C^*(\Lambda)$. Through π we will view $C^*(\Lambda)$ as a subalgebra of $\mathcal{B}(H)$, and hence we may consider $\check{S}$ as a representation of Λ on H. It is clear that $\check{S}$ is nondegenerated, according to definition (1.2.5). By (1.2.7) (ii) it is then enough to show that for every $v \in \mathrm{obj}(\Lambda)$ and every finite partition H of v one has that

$$\bigvee_{h \in H} \check{P}_h = \check{P}_v,$$

but this is precisely what was proved in (1.2.13).

By the universal property of $\tilde{\mathcal{O}}(\Lambda)$ there is a *-homomorphism

$$\Psi : \tilde{\mathcal{O}}(\Lambda) \to B(H)$$

such that $\Psi(\hat{S}_f) = \check{S}_f$, for every $f \in \Lambda$. Clearly $\Psi(\mathcal{O}(\Lambda)) \subseteq C^*(\Lambda)$, so we may then view Ψ and Φ as maps

$$\Psi : \tilde{\mathcal{O}}(\Lambda) \to C^*(\Lambda) \quad \text{and} \quad \Phi : C^*(\Lambda) \to \mathcal{O}(\Lambda),$$

which are obviously each others inverses.

Chapter 2

C^*-Algebras and Tracially Z-Absorbing

In this chapter we show that if $\mathcal{A}$ is a C^*-algebra for which the a tracial notion of $\mathbb{Z}$-absorption for simple holds then $\mathcal{A}$ has almost unperforated Cuntz semigroup, and if in addition $\mathcal{A}$ is nuclear and separable we show this property is equivalent to having $\mathcal{A} \cong \mathcal{A} \otimes \mathbb{Z}$. We furthermore show that this property is preserved under forming certain crossed products by actions satisfying a tracial Rokhlin type property [2].

Section (2.1): Tracially Z-Absorbing C^*-Algebras and Absorption in The Unclear Case

The purpose of this section is to introduce and study a property of C^*-algebras which can be thought of as a tracial version of Z-absorption, in a way reminiscent of the definition of tracially AF algebras and C^*-algebras of higher tracial rank. There are several motivations for looking at this property. A property closely related to ours (the two coincide in certain cases) appeared as a technical step in Winter's work concerning Z-absorption for C^*-algebras of finite nuclear dimension and the recent work of Matui-Sato on strict comparison and Z-absorption, and it may thus be profitable to isolate this property for further study. Additionally, this property may be easier to establish in certain instances, in particular when considering crossed products by an action which satisfies a tracial version of the Rokhlin property. Indeed, we study a generalization of the tracial Rokhlin property which does not require projections and show that under certain conditions, crossed products of tracially Z-absorbing C^*-algebras by such actions are again tracially Z-absorbing. The Rokhlin property has been instrumental in the study of group actions on C^*-algebras. However, the Rokhlin property might be harder to establish in certain cases, and might not exist in others. In the finite group case, the Rokhlin property is uncommon and its existence requires restrictive K theoretic constraints on the algebra and the action. While for the single automorphism case the Rokhlin property is much more common (and even generic in certain cases), it still requires the existence of many projections. The Rokhlin property has been generalized to a tracial version, although this generalization still requires the existence of projections. Further tracial-type generalizations in which

the projections are replaced by positive elements were considered, and we will study a closely related variant. We note that a different generalization which does not require projections has been to view the Rokhlin property as a zero-dimensional level of what can be called Rokhlin dimension.

A unital C^*-algebra $\mathcal{A}$ is Z-absorbing if and only if for any $n \in \mathbb{N}$, any finite subset $F \subseteq A$ and for any $\varepsilon > 0$ there exist c.p.c. order zero maps $\varphi : M_n \to \mathcal{A}$ and $\psi : M_2 : \to \mathcal{A}$ such that the image of any normalized matrix under φ or ψ commutes up to ε with the elements of F, and such that $\psi(e_{1,1})\varphi(e_{1,1}) = \psi(e_{1,1})$ and $\psi(e_{2,2}) = 1_{\mathcal{A}} - \varphi(1)$. In this characterization, $1 - \varphi(1)$ must be "small" in a tracial sense. We introduce a property called tracial Z-absorption which basically amounts to dropping the requirement for an order zero map from M_2 into $\mathcal{A}$ and instead asking that $1 - \varphi(1)$ is arbitrarily small in the sense of Cuntz comparison. We show that tracially Z-absorbing C^*-algebras have almost unperforated Cuntz semigroups. This generalizes results of Rørdam concerning Z-absorption. We then use ideas to show that tracial Z-absorption implies Z absorption for simple, unital, separable, nuclear C^*-algebras.

Recall that a c.p.c. map $\varphi : \mathcal{A} \to \mathcal{B}$ is said to have order zero if $\varphi(a)\varphi(b) = 0$ whenever $a, b \in \mathcal{A}_+$ satisfy $ab = 0$. For positive elements a, $b \in \mathcal{A}$ we say that a is Cuntz-subequivalent to b (written $a \precsim b$) if there is a sequence $x_n \in \mathcal{A}$ such that $\lim_{n\to\infty} \|a - x_n b x_n^*\| = 0$.

Definition (2.1.1) [2]:

We say that a unital C^*-algebra $\mathcal{A}$ is tracially Z-absorbing if $\mathcal{A} \ncong \mathbb{C}$ and for any finite set $F \subset \mathcal{A}, \varepsilon > 0$ and non-zero positive element $a \in \mathcal{A}$ and $n \in \mathbb{N}$ there is an order zero contraction $\psi : M_n \to \mathcal{A}$ such that the following hold:

(I) $1 - \varphi(1) \precsim a$.

(II) For any normalized element $x \in M_n$ and any $y \in F$ we have $\|\varphi(x), y\| < \varepsilon$.

Proposition (2.1.2) [2]:

Let $\mathcal{A}$ be a simple unital C^*-algebra. If $\mathcal{A}$ is Z-absorbing then $\mathcal{A}$ is tracially Z-absorbing.

Proof:

First assume $\mathcal{A}$ is stably finite. We know that $\mathcal{A}$ has strict comparison. Let a, ε, n, F be given. Since $\mathcal{A}$ is simple there exist $\ell \in N$ and $c_1, \dots, c_\ell \in A$ such that

$$\sum_{k=1}^{\ell} c_k^* a c_k = 1.$$

This clearly implies that $d_\tau(a) \geq 1/\ell$ for all $\tau \in T(A)$. Let m be some number such that $m > l$ and n divides m. Since $\mathcal{A}$ is Z-absorbing there is a unital homomorphism $\psi : Z_{m,m+1} \to A$ such that for any normalized element $x \in Z_{m,m+1}$ and for any $y \in F$ we have

$$\|\psi(x), y\| < \varepsilon.$$

We have a c.p.c. order zero map $\tilde{\varphi} : M_m \to x \in Z_{m,m+1}$ such that

$$1 - \tilde{\varphi}(1) \precsim \tilde{\varphi}(e_{1,1}).$$

Let $\varphi = \psi \circ \tilde{\varphi} : M_m \to \mathcal{A}$. For any $\tau \in T(\mathcal{A})$ we have that

$$d_\tau(1 - \varphi(1)) \leq d_\tau\left(\varphi(e_{1,1})\right) \leq 1/m \leq d_\tau(a)$$

which entails that $1 - \varphi(1) \precsim a$. Since M_n embeds unitally into M_m we restrict φ to obtain the map we are looking for.

If $\mathcal{A}$ is not stably finite then, $\mathcal{A}$ is purely infinite. Set $m = n$ and continue as before. The condition $1 - \varphi(1) \precsim a$ is automatically satisfied.

In what follows we identify $M_n(\mathcal{A})$ with $M_n \otimes \mathcal{A}$ in the usual way.

Lemma (2.1.3) [2]:

Let $\mathcal{A}$ be a simple, unital, non type IC^*-algebra and let $n \in \mathbb{N}$. For every non-zero positive element $a \in M_n(\mathcal{A})$ there exists a non-zero positive element $b \in \mathcal{A}$ such that $a \succsim 1 \otimes b$.

Proof:

Since a is positive and non-zero, there exists i such that $(e_{ii} \otimes 1_{\mathcal{A}})a(e_{ii} \otimes 1_{\mathcal{A}}) = 0$. Assume without loss of generality that $i = 1$. Furthermore, by replacing a with the element $(e_{1,1} \otimes 1_{\mathcal{A}})a(e_{1,1} \otimes 1_{\mathcal{A}})$ we may assume that a is of the form $e_{1,1} \otimes c$ for some non-zero positive $c \in \mathcal{A}$.

Note that both $\mathcal{A}$ and $c\mathcal{A}c$ have a strictly positive element and hence, by Brown's Theorem, we have that $\overline{c\mathcal{A}c}$ is stably isomorphic to $\mathcal{A}$, this implies that the former is a simple, infinite dimensional, C^*-algebra not isomorphic to the compact operators. In particular $\overline{c\mathcal{A}c}$ is non-type I. Therefore, there is a non-zero homomorphism $\theta : CM_n \to c\mathcal{A}c$. Let $z \in C_0((0,1])$ denote the identity function. Using the picture $CM_n = C_0((0,1]) \otimes M_n$ we denote $h = \theta(z \otimes 1)$ and $b = \theta(z \otimes e_{1,1})$. Observe that $e_{1,1} \otimes h \sim 1 \otimes b$ in $M_n(\mathcal{A})$. Consequently we have that $a \gtrsim 1 \otimes b$ as needed.

Lemma (2.1.4) [2]:

Let $\mathcal{A}$ be a simple unital C^*-algebra. If $\mathcal{A}$ is tracially Z-absorbing then so is $M(\mathcal{A})$ for any n.

Proof:

It is clear that $M(\mathcal{A}) \cong M_n \otimes \mathcal{A}$ satisfies the conditions of Definition (2.1.1) provided the positive element a is of the form $1 \otimes b, b \in \mathcal{A}$. Thus, it suffices to show that every non-zero positive element $a \in M_n(\mathcal{A})$ Cuntz-dominates a non-zero positive element of the form $1 \otimes b$. But this follows from the previous lemma.

Notation (2.1.5) [2]:

Let $\mathcal{A}$ be a separable C^*-algebra. We denote

$$\mathcal{A}_\infty = \prod_{\mathbb{N}} \mathcal{A} \Big/ \bigoplus_{\mathbb{N}} \mathcal{A}$$

We view $\mathcal{A}$ as embedded into $\mathcal{A}_\infty$ as equivalence classes of constant sequences and we denote by

$$\mathcal{A}_\infty \cap \mathcal{A}'$$

the relative commutant of $\mathcal{A}$ in $\mathcal{A}_\infty$.

The following result can help simplify proofs.

Lemma (2.1.6) [2]:

Let $\mathcal{A}$ be a separable, unital C^*-algebra. If $\mathcal{A}$ is tracially Z-absorbing then for any $n \in \mathbb{N}$ and any non-zero positive contraction $a \in \mathcal{A}$ there exists a c.p.c. order zero map $\varphi : M_n \to \mathcal{A}_\infty \cap \mathcal{A}'$ such that $1_{\mathcal{A}_\infty} - \varphi(1) \precsim a$ in $\mathcal{A}_\infty$.

Proof:

Let $(b_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ be a dense sequence and denote $F_k = \{b_1, \ldots, b_k\}$. For each $k \in \mathbb{N}$ find a c.p.c. order zero map $\psi : M_n \to \mathcal{A}$ such that $1 - \psi_k(1) \precsim a$ and $\|[\psi(x), y]\| < \frac{1}{k}$ for all normalized $x \in M_n$ and for all $y \in F_k$. Let φ be the composition of the map

$$(\psi_k)_{k\in\mathbb{N}} : M_n \to \prod_{\mathbb{N}} \mathcal{A}$$

with the quotient map into $\mathcal{A}_\infty$. It is easy to see that φ has the desired properties.

The following is a restatement along with some additional notation:

Notation (2.1.7) [2]:

Let $\varphi : F \to \mathcal{A}$ be an order zero c.p.c. map, where F is finite dimensional. Denote $h = \varphi(1)$. Recall that there is a homomorphism $\pi : F \to \mathcal{A}^{**} \cap \{h\}'$ such that $\varphi(x) = \pi(x)h$ where $h = \varphi(1)$.

If $f \in C_0\big((0,1]\big)_+$, we denote by

$$f[\varphi]$$

the c.p. order zero map given by

$$f[\varphi](x) = \pi(x)f(h).$$

If f is the square root function, we may use the notation $\sqrt{[\varphi]}$ for $f[\varphi]$.

For a c.p.c. map $\varphi: F \to \mathcal{A}$ from a finite dimensional algebra F, and an element $b \in \mathcal{A}$, we write

$$\|[\varphi, b]\| < \varepsilon$$

to mean $\|\varphi(x), b\| < \varepsilon$ for all normalized elements $x \in M_n$. If $X \subseteq \mathcal{A}$ is some subset, we shall use the notation

$$\|[\varphi, X]\| < \varepsilon$$

to mean $\|\varphi, b\| < \varepsilon$ for all $b \in X$.

We have

Lemma (2.1.8) [2]:

For any $f \in C_0((0,1])_+$ and any $\varepsilon > 0$ there is an $\eta > 0$ such that whenever $\varphi: M_n \to \mathcal{A}$ is a c.p.c. order zero map and b is a contraction satisfying $\|\varphi, b\| < \eta$, we have that $\|f[\varphi], b\| < \varepsilon$.

Proof:

Let f and ε as above be given. We can find $g \in C_0((0,1])$ such that $\|gz - f\| < \frac{\varepsilon}{3}$ where z is the identity function on $(0,1]$. Find $\eta > 0$ such that $\|[a, b]\| < \eta$ implies $\|[g(a), b]\| < \frac{\varepsilon}{6}$ for any normalized elements $a, b \in \mathcal{A}$ with a positive (this is easily done by approximating g uniformly by polynomials). We may of course assume that $\eta < \frac{\varepsilon}{6}$. Let $\varphi : M_n \to \mathcal{A}$ be a c.p.c. order zero map and $b \in \mathcal{A}$ a contraction such that $\|\varphi, b\| < \eta$. Let π, h be as in the notation above. For any normalized $x \in M_n$ we have the following:

$$\begin{aligned} f[\varphi](x)b &= \pi(x)f(h)b \\ &\approx \varepsilon/3\, \pi(x)hg(h)b \\ &= \varphi(x)g(h)b \\ &\approx \varepsilon/3\, b\varphi(x)g(h) \\ &\approx \varepsilon/3\, bf[\varphi](x). \end{aligned}$$

We will also be needing a slight variation of this previous lemma. We omit the proof since it is essentially the same as that of the preceding lemma.

Lemma (2.1.9) [2]:

For any $f \in C_0((0,1])_+$ and any $\varepsilon > 0$ there is an $\eta > 0$ such that whenever $\varphi : M_n \to \mathcal{A}$ is a c.p.c. order zero map and $\alpha \in \mathrm{Aut}(\mathcal{A})$ is an automorphism satisfying $\|\alpha(\varphi(x)) - \varphi(x)\| < \eta$ for all normalized elements $x \in M_n$, we have that $\|\alpha(f[\varphi](x)) - f[\varphi](x)\| < \varepsilon$ for all normalized $x \in M_n$.

In this section we will prove that tracially Z-absorbing C^*-algebras have almost unperforated Cuntz semigroup and therefore strict comparison. The proof mixes ideas with ideas originating from the study of tracially AF algebras.

Recall that a positive element $a \in \mathcal{A}$ is called purely positive if a is not Cuntz-equivalent to a projection. This is equivalent to saying that 0 is an accumulation point of $\sigma(a)$.

Lemma (2.1.10) [2]:

Let $\mathcal{A}$ be a simple non type IC^*-algebra. For any projection $p \in \mathcal{A}$ and $k > 0$ there is a purely positive element $a \leq p$ such that $(k-1)\langle p\rangle \leq k\langle a\rangle$.

Proof:

Since $\mathcal{A}$ is simple and non type I, the algebra $\mathcal{B} = \overline{p\mathcal{A}p}$ is also non type I (as in the proof of Lemma (2.1.1)). Hence, there is an injective homomorphism $\theta : CM_k \to \mathcal{B}$. Let z denote the identity function on $(0,1]$ and let $c_i = \theta(e_{ii} \otimes z)$. Let $a = p - c_1$. It is clear by functional calculus (and because θ is injective) that a is purely positive.

Additionally, we have that $k\langle a\rangle$ is represented by $1_k \otimes a \in M_k \otimes \mathcal{B}$, and notice that

$$1_k \otimes a \geq (1_k - e_{11}) \otimes a + e_{11} \otimes \sum_{j=2}^{k} c_i.$$

The Cuntz class of the right hand side is $(k-1)\langle a\rangle + (k-1)\langle c_1\rangle$, which dominates $(k-1)\langle a + c_1\rangle = (k-1)\langle p\rangle$.

Lemma (2.1.11) [2]:

Let $\mathcal{A}$ be a unital tracially Z-absorbing C^*-algebra. If $a, b \in \mathcal{A}$ such that $k\langle a\rangle \leq k\langle b\rangle$ in $W(\mathcal{A})$ for some $k \in \mathbb{N}$ and b is purely positive then $a \precsim b$.

Proof:

Fix $\varepsilon > 0$. Without loss of generality we may assume that $\|a\| = \|b\| = 1$. We choose $c = (c_{ij}) \in M_k(\mathcal{A})$ and $\delta > 0$ such that $c[(b-\delta)_+ \otimes 1_k]c^* = (a-\varepsilon)_+ \otimes 1_k$. Let $f \in C_0\big((0,1]\big)$ be a non-negative function such that $f = 0$ on $(\delta/2, 1], f > 0$ on $(0, \delta/2)$ and $\|f\| = 1$ and denote $d = f(b)$. Note that $d \neq 0$ because b is purely positive. We now fix $\mu > 0$. We shall find $z \in \mathcal{A}$ such that $\|z[(b-\delta)_+ + d]z^* - (a-\varepsilon)_+\| < \mu$. Since μ is arbitrary this will show that $(a-\varepsilon)_+ \precsim (b-\delta)_+ + d \precsim b$. By replacing each c_{ij} with $c_{ij}q(b)$ for some function $q \in C_0\big((0,1]\big)$ that vanishes on $(0, \delta/2]$ and $q = 1$ on $[\delta, 1]$ we may assume that $c_{ij}d = 0$. Note that after doing this we still have $c[(b-\delta)_+ \otimes 1]c^* = (a-\varepsilon)_+ \otimes 1_k$ which yields

$$\sum_{\ell=1}^{k} c_{i\ell}\,(b-\delta)_+ c_{j\ell}^* = \begin{cases} (a-\varepsilon)_+, & i = j \\ 0, & i \neq j \end{cases} \tag{1}$$

Let $g, h \in C_0\big((0,1]\big)$ be defined by

$$g(t) = \begin{cases} \sqrt{\dfrac{7}{\mu}}\,t, & t \leq \dfrac{\mu}{7}, \\ 1, & t \geq \dfrac{\mu}{7} \end{cases} \qquad h(t) = 1 - \sqrt{1-t}$$

We observe that those functions satisfy the following.

$$|g(t)^2 t - t| < \frac{\mu}{6}, \quad 1 - h(t) = \sqrt{1-t} \tag{2}$$

in the algebra $C_0\big((0,1]\big)$.

Find a c.p.c order zero map $\varphi : M_k \to \mathcal{A}$ such that $1 - \varphi(1) \precsim d$ and $\|[\varphi, F]\| < \eta$ where $F = \{(a-\varepsilon)_+, (b-\delta)_+\} \cup \{c_{ij}\}$, and where $\eta > 0$ is chosen using Lemma (2.1.8) such that

$$\|[g[\varphi],F]\| < \frac{\mu}{36k^2}, \quad \left\|\left[\sqrt{[\varphi]},F\right]\right\| < \frac{\mu}{36k^2}, \quad \|[h[\varphi],F]\| < \frac{\mu}{3}.$$

Let $a_1 = \varphi(1)(a-\varepsilon)_+$. Denote $r = 1-\varphi(1)$ and set $a = r^{1/2}(a-\varepsilon)_+ r^{1/2}$. We have

$$\|(a-\varepsilon)_+ - (a_1 + a_2)\| < \frac{\mu}{3}$$

since $r^{1/2} = 1 - h[\varphi](1)$. We denote $g_{ij} = g([\varphi])(e_{ij})$. We now define $\hat{c}_{ij} = \sqrt{[\varphi]}(1) g_{ij} c_{ij}$ and $\hat{c} = \sum_{ij=1}^{k} \hat{c}_{ij}$.

Our goal is to first show that $\|\hat{c}(b-\delta)_+\hat{c}^* - a_1\| < \frac{\mu}{3}$:

$$\hat{c}(b-\delta)_+\hat{c}^* = \sum_{i,j,m,\ell=1}^{k} \hat{c}_{ij}\,(b-\delta)_+\hat{c}^*_{m\ell}$$

$$= \sqrt{[\varphi]}(1)\left(\sum_{i,j,m,\ell=1}^{k} g_{ij}c_{ij}(b-\delta)_+c^*_{m\ell g\ell m}\right)\sqrt{[\varphi]}(1)$$

$$\approx \mu/6 \sum_{i,j,m,\ell=1}^{k} g_{ij}g_{\ell m}c_{ij}(b-\delta)_+c^*_{m\ell}$$

$$= \varphi(1)g[\varphi](1) \sum_{i,j,m=1}^{k} g_{im}c_{ij}(b-\delta)_+c^*_{m\ell}$$

$$= \varphi(1)g[\varphi](1) \sum_{i,m=1}^{k} g_{im} \sum_{j=1}^{k} c_{ij}(b-\delta)_+c^*_{m\ell}$$

$$= \varphi(1)g[\varphi](1) \sum_{i=1}^{k} g_{ii}(a-\varepsilon)_+$$

$$= \varphi(1)g^2[\varphi](1)(a-\varepsilon)_+$$

$$\approx \mu/6\,\varphi(1)(a-\varepsilon)_+$$

$$= a_1$$

where the first approximation is due to our choice of η and the second follows from (2).

We now deal with a_2. Since $a_2 \precsim r \precsim d$, we may find $s \in \mathcal{A}$ such that $\|sds^* - a_2\| < \mu/3$. Furthermore, by replacing s with $sp(b)$ where $p \in C_0((0,1])$ is some function that is 1 on $[0, \delta/2]$ and vanishes on $[\delta, 1]$, we may assume that $s(b-\delta)_+ = 0$. We take $z = \hat{c} + s$ and calculate:

$$\|[z(b-\delta)_+ + d]z^* - (a-\varepsilon)_+\| = \|\hat{c}(b-\delta)_+\hat{c}^* + sds^* - (a-\varepsilon)_+\|$$

$$\leq \|\hat{c}(b-\delta)_+\hat{c}^* - a_1\| + \|sds^* - a_2\| + \|a_1 + a_2 - (a-\varepsilon)_+\| < \mu.$$

Since this holds for every $\mu > 0$ we have that $(a-\varepsilon)_+ \precsim b$. Finally, since ε is arbitrary, we have $a \precsim b$.

Theorem (2.1.12) [2]:

Let $\mathcal{A}$ be a simple unital C^*-algebra. If $\mathcal{A}$ is tracially Z-absorbing then $W(\mathcal{A})$ is almost unperforated, and therefore $\mathcal{A}$ has strict comparison.

Proof:

Let $a, b \in M_n(\mathcal{A})_+$ such that $k\langle a\rangle \leq (k-1)\langle b\rangle$ for some k. We want to show that $\langle a\rangle \leq \langle b\rangle$. Since $M_n(\mathcal{A})$ is also tracially Z-absorbing, we may assume without loss of generality that $a, b \in \mathcal{A}$. If b is purely positive then we are through because in particular $k\langle a\rangle \leq k\langle b\rangle$ and now we can apply Lemma (2.1.11).

If b is not purely positive then b is Cuntz equivalent to a projection $p \in \mathcal{A}$ and then by Lemma (2.1.10) we may find a purely positive element $b^l \in \mathcal{A}$ such that $(k-1)\langle p\rangle \leq k\langle b'\rangle$ and $b' \leq p$. We may now apply Lemma (2.1.11) to get $a \precsim b' \leq p \sim b$

The main result of this section is the following theorem.

Theorem (2.1.13) [2]:

Let $\mathcal{A}$ be a simple, separable, unital, nuclear C^*-algebra. If $\mathcal{A}$ is tracially Z-absorbing then $\mathcal{A} \cong \mathcal{A} \otimes Z$.

Proof:

By Theorem (2.1.13) we know that $\mathcal{A}$ has strict comparison. Thus, if $\mathcal{A}$ is traceless then $\mathcal{A}$ is purely infinite. By Kirchberg's $\mathcal{O}_\infty$ absorption Theorem $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{O}_\infty$ so we are done.

Let us then assume that $T(\mathcal{A}) \neq \emptyset$. This implies that $\mathcal{A}$ is stably finite so the conclusion of Lemma (2.1.16) holds. We now apply Theorem (2.1.14) to complete the proof.

We begin this section with a general lemma that we will need in this section and the following. We will only use it for actions of finite groups or of the integers.

We have the following theorem.

Theorem (2.1.14) [2]:

Let $\mathcal{A}$ be a unital, separable, simple, nuclear C^*-algebra such that $\mathcal{A}$ has strict comparison, $T(\mathcal{A}) \neq \emptyset$, and the following condition holds:

For any $k \in \mathbb{N}$ there exists a c.p.c. order zero map $\psi : M_k \to \mathcal{A}_\infty \cap \mathcal{A}'$ and a representative

$$(c_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{A}$$

of $\psi(e_{1,1}) \in \mathcal{A}_\infty$ such that $c_n \in \mathcal{A}$ is a positive contraction for all n and

$$\lim_{n\to\infty} \max_{\tau\in T(\mathcal{A})} |\tau(c_n^m) - 1/k| = 0.$$

Then $\mathcal{A} \cong \mathcal{A} \otimes Z$.

Lemma (2.1.15) [2]:

Let $\mathcal{A}$ be a simple, separable, unital, infinite dimensional C^*-algebra. For any n there exists a positive contraction $c \in \mathcal{A}$ such that $c \neq 0$ and $d_\tau(c) \leq 1/n$ for all $\tau \in T(\mathcal{A})$.

Lemma (2.1.16) [2]:

Let $\mathcal{A}$ be a simple, separable, stably finite, unital C^*-algebra. If $\mathcal{A}$ is tracially Z-absorbing then for any $k \in \mathbb{N}$, we can find a sequence of c.p.c. order zero maps $\varphi_n : M_k \to \mathcal{A}$ such that

(i). $\lim_{n\to\infty} \max_{\tau\in T} \left|\tau\left(\varphi_n\left(e_{1,1}\right)^m\right) - 1/k\right| = 0$ for all $m \in \mathbb{N}$.

(ii). $\lim_{n\to\infty} \|[a, \varphi_n(x)]\| = 0$ for all $a \in \mathcal{A}$ and for all $x \in M_k$.

Proof:

Using Lemma (2.1.15) we can find a sequence of non-zero positive contractions $c \in \mathcal{A}$ such that

$$\lim_{n\to\infty} \max_{\tau\in T(\mathcal{A})} d_\tau(c_n) = 0.$$

Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a dense sequence. We will denote $F_n = \{a_1, \dots, a_n\}$. For each n we can now find a c.p.c. order zero map $\varphi_n : M_k \to \mathcal{A}$ such that

$$\|[F_n, \varphi_n]\| \leq 1/n$$

and

$$1 - \varphi_n(1) \precsim c_n.$$

Clearly property (ii) holds, so it remains to see that the maps φ_n satisfy (i). It is clear that that $\tau\left(\varphi_n\left(e_{1,1}\right)^m\right) = \tau\left(\varphi_n\left(e_{i,i}\right)^m\right)$ for any trace τ, so it suffices to show that

$$1 - \max_{\tau\in T(\mathcal{A})} \tau\left(\sum_{i=1}^{k} \varphi_n\left(e_{i,i}\right)^m\right) = \min_{\tau\in T(\mathcal{A})} \tau\left(\sum_{i=1}^{k} \varphi_n\left(e_{i,i}\right)^m\right) \xrightarrow[n\to\infty]{} 0.$$

We claim that

$$1 - \sum_{i=1}^{k} \varphi_n\left(e_{i,i}\right)^m \sim 1 - \varphi_n(1)$$

for all n. To see this, fix n, and note that by functional calculus we have a homomorphism $\pi : C([0,1]) \to \mathcal{A}$ given by

$$\pi(f) = f\big(\varphi_n(1)\big).$$

Letting z denote the identity function on $[0,1]$, a simple calculation yields:

$$\pi(1 - z^m) = 1 - \sum_{i=1}^{k} \varphi_n\big(e_{i,i}\big)^m, \quad \pi(1 - z) = 1 - \varphi_n(1)$$

Since homomorphisms of C^*-algebras induce maps on the level of the Cuntz semigroup, our claim now follows from the simple fact that $1 - z^m \sim 1 - z$ in $C([0,1])$.

We thus have

$$0 \leq \tau\left(1 - \sum_{i=1}^{k} \varphi_n\big(e_{i,i}\big)^m\right) \leq d_\tau\left(1 - \sum_{i=1}^{k} \varphi_n\big(e_{i,i}\big)^m\right)$$

$$= d_\tau\big(1 - \varphi(1)\big)$$

$$\leq d_\tau(c_n)$$

and since $\lim_{n\to\infty} \max_{\tau\in T(\mathcal{A})} d_\tau(c_n) = 0$, we have that (i) holds.

Lemma (2.1.18) [2]:

Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a discrete group G on a simple, unital C^*-algebra $\mathcal{A}$. Suppose that α_g is outer for all $g \in G\backslash\{1\}$. Then for every non-zero positive element $a \in \mathcal{A} \rtimes_{\alpha,r} G$ in the reduced crossed product there exists a non-zero positive element $b \in \mathcal{A}$ such that $b \precsim a$.

Proof:

Let $E : \mathcal{A} \rtimes_{\alpha,r} G \to \mathcal{A}$ be the canonical faithful conditional expectation. By replacing a with $\|E(a)\|^{-1}a$ we may assume that $\|E(a)\| = 1$. We can find a finite sum $a' = \sum_{k=1}^{n} a_k u_{g_k} \in \mathcal{A} \rtimes_{\alpha,r} G$ such that $\|a - a'\| < \frac{1}{3}$ where u_g is the canonical unitary implementing α_g for $g \in G$. We can assume that a' is positive and non-zero, and we label the indices such that $g_1 = 1_G$. Note that a_1 is a non-zero positive element in $\mathcal{A}$ because $a_1 = E(a')$. Since E is a contractive map, we have

$$\|a_1\| \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

Let $0 < \varepsilon < \frac{1}{3(n+1)}$. By Lemma (2.1.11) we can find a positive element $x \in \mathcal{A}$ with $\|x\| = 1$ such that

$$\|xa_1x\| > \|a_1\| - \varepsilon \geq \frac{2}{3} - \varepsilon, \qquad \left\|xa_{g_k}\alpha_{g_k}(x)\right\| < \varepsilon, \ \ \forall k \neq 1.$$

This implies that for $k \neq 1$ we have

$$\left\|xa_{g_k}u_{g_k}x\right\| = \left\|xa_{g_k}\alpha_{g_k}(x)u_{g_k}\right\| = \left\|xa_{g_k}\alpha_{g_k}(x)\right\| < \varepsilon.$$

We deduce that $\|xa'x - xa_1x\| < n\varepsilon$ and thus

$$\|xax - xa_1x\| < n\varepsilon + \frac{1}{3}.$$

Let $b = \left(xa_1x - \left(n\varepsilon + \frac{1}{3}\right)\right)_+$, note that b is a non-zero positive element by choice of ε and we have

$$b \precsim xax \precsim a$$

where the first Cuntz subequivalence follows.

Definition (2.1.19) [2]:

If $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is an action of a finite group G on a simple, unital C^*-algebra $\mathcal{A}$, then α is said to have the generalized tracial Rokhlin property if for any $\varepsilon > 0$, any finite subset $F \subseteq \mathcal{A}$, and any non-zero positive element $a \in \mathcal{A}$ there exist normalized positive contractions $\{e_g\}_{g\in G} \subseteq \mathcal{A}$ such that:

(i) $e_g \perp e_g$ when $g \neq h$.

(ii) $1 - \sum_{g\in G} e_g \precsim a$.

(iii) $\|[e_g, y]\| < \varepsilon$ for all $g \in G, y \in F$

(iv) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$

We first collect a few basic properties of such actions.

Proposition (2.1.20) [2]:

Let $\alpha : \mathrm{G} \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on a simple, unital C^*-algebra $\mathcal{A}$. If α has the generalized tracial Rokhlin property then α_g is outer for all $g \in G \backslash \{1\}$.

Proof:

Let $g \in G$ and assume that $\alpha_g = \mathrm{Ad}(u)$ for some $u \in \mathcal{A}$. Let $\{e_g\}_{g \in G}$ be elements as in Definition (2.1.19) taking $\varepsilon = 1/2$, $a = 1_{\mathcal{A}}$, and $F = \{u\}$. We have

$$\begin{aligned} 1 &= \|e_1 - e_g\| \\ &< \|u e_1 u^* - e_g\| + \frac{1}{2} \\ &= \|\alpha_g(e_1) - e_g\| + \frac{1}{2} \\ &< \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned}$$

which is a contradiction.

We state that if $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is an action of a discrete group on a simple C^*-algebra such that α_g is outer for all $g \neq 1$ then $\mathcal{A} \rtimes_{\alpha,r} G$ is simple.

Corollary (2.1.21) [2]:

Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on a simple, unital C^*-algebra $\mathcal{A}$. If α has the generalized tracial Rokhlin property then $\mathcal{A} \rtimes_{\alpha} G$ is simple.

Lemma (2.1.22) [2]:

Let $\mathcal{A}$ be a simple, separable, unital tracially Z-absorbing C^*-algebra and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$. Assume that α has the generalized tracial Rokhlin property. Then for any finite set $F \subset \mathcal{A}, \varepsilon > 0$ and

non-zero positive element $a \in \mathcal{A}$ and $n \in \mathbb{N}$ there is a c.p.c. order zero map $\psi : M_n \to \mathcal{A}$ such that:

(i) $1 - \psi(1) \precsim a$.

(ii) For any normalized element $x \in M_n$ and any $y \in F$ we have $\|\psi(x), y\| < \varepsilon$.

(iii) For any normalized element $x \in M_n$ and any $g \in G$ we have $\left\|\alpha_g\left(\psi(x)\right) - \psi(x)\right\| < \varepsilon$.

Proof:

Let F, a, ε, n be given as in the statement of the lemma. Since $\mathcal{A}$ is simple and unital, we can find $c > 0$ such that $d_\tau(a) \geq c$ for all $\tau \in T(\mathcal{A})$. Use Lemma (2.1.15) to find a positive element $b \in \mathcal{A}$ such that $d_\tau(b) < \frac{c}{|G|+2}$ for all $\tau \in T(\mathcal{A})$. Since $\tau \circ \alpha_g$ is also a trace, we have that

$$d_\tau\left(\alpha_g(b)\right) < \frac{c}{|G| + 2} \tag{3}$$

for all $g \in G$ and $\tau \in T(\mathcal{A})$.

Let $\eta > 0$ be as in Lemma (2.1.8) such that $\|[h(\psi), y]\| < \varepsilon/2$ whenever $\psi : M_k \to \mathcal{A}$ is a c.p.c order zero map and $y \in \mathcal{A}$ is a contraction such that $\|[\psi, y]\| < \eta$ where $h \in C_0((0,1])$ is given by

$$h(x) = \begin{cases} 2t, & t < 1/2 \\ 1, & t \geq 1/2 \end{cases}.$$

Let $\left(e_g\right)_{g \in G}$ be positive contractions such that

(i) $\left\|\left[e_g, y\right]\right\| < \eta$ for all $g \in G$ and for all $y \in F$.

(ii) $\left\|\alpha_g(e_h) - e_{gh}\right\| < \frac{\eta}{|G|}$.

(iii) $1 - \sum_{g \in G} e_g \precsim b$

as guaranteed by the generalized tracial Rokhlin property.

By Lemma (2.1.6) we can find a c.p.c. order zero map $\varphi : M_n \to \mathcal{A}_\infty \cap \mathcal{A}'$ such that $1 - \varphi(b) \precsim b$. We denote by $\bar{\alpha}_g$ the automorphism of $\mathcal{A}_\infty$ induced by α_g, and define $\hat{\psi} : M_n \to \mathcal{A}_\infty$ by

$$\hat{\psi}(x) = \sum_{g \in G} e_g \bar{\alpha}_g \left(\varphi(x)\right).$$

$\hat{\psi}$ is clearly an order zero c.p.c. map.

First we remark that $\mathcal{A}_\infty \cap \mathcal{A}'$ is invariant under $\bar{\alpha}$. Thus we have

$$\hat{\psi}(x) = \sum_{g \in G} e_g \alpha_g \left(\varphi(x)\right) y \approx \eta \sum_{g \in G} y e_g \alpha_g \left(\varphi(x)\right) = y \hat{\psi}(x)$$

for all $y \in F$. This is the first property we will be needing.

We now consider $\bar{\alpha}_g \left(\hat{\psi}(x)\right)$ for normalized $x \in M_n$:

$$\begin{aligned} \bar{\alpha}_g \left(\hat{\psi}(x)\right) &= \bar{\alpha}_g \left(\sum_{h \in G} e_h \bar{\alpha}_h \left(\varphi(x)\right) \right) \\ &\approx \varepsilon/2 \sum_{h \in G} e_{gh} \bar{\alpha}_{gh} \left(\varphi(x)\right) \\ &= \sum_{g \in G} e_g \bar{\alpha}_g \left(\varphi(x)\right) = \hat{\psi}(x) \end{aligned}$$

The third and last property we will be needing of $\hat{\psi}$ is:

$$1 - \hat{\psi}(1) = \left(1 - \sum_{g \in G} e_g\right) + \left(\sum_{g \in G} e_g - \hat{\psi}(1)\right)$$

$$= \left(1 - \sum_{g \in G} e_g\right) + \sum_{g \in G} e_g^{1/2} \bar{\alpha}_g \left(1 - \varphi(1)\right) e_g^{1/2}$$

$$\precsim b \oplus \bigoplus_{g \in G} \alpha_g(b) \precsim a$$

where the last Cuntz-subequivalence follows from (iii) and Theorem (2.1.12). By Proposition (2.1.4), we can find $v \in \mathcal{A}_\infty$ such that $v^* a v = \left(1 - \hat{\psi}(1) - 1/4\right)_+$.

We now use liftability of order zero maps with finite dimensional domains to lift $\hat{\psi}$ to a sequence of order zero maps $\psi_\ell : M_k \to \mathcal{A}$. Let $(v_\ell)_{\ell \in \mathbb{N}} \in \prod_{\mathbb{N}} \mathcal{A}$ be a representative of v. We have

$$\|v_\ell^* a v_\ell - (1 - \psi_\ell(1) - 1/4)_+\| \xrightarrow{\ell \to \infty} 0.$$

The properties of $\hat{\psi}$ that we have established imply that for $\ell_0 \in \mathbb{N}$ large enough, the following properties hold:

(i) $\|[\psi_{\ell_0}, y]\| < \eta$ for all $y \in K$,

(ii) $\left\|\alpha_g\left(\psi_{\ell_0}(x)\right) - \psi_{\ell_0}(x)\right\| < \eta$ for all normalized elements $x \in M_k$,

(iii) $\left\|v_{\ell_0}^* a v_{\ell_0} - \left(1 - \psi_{\ell_0}(1) - 1/4\right)_+\right\| < 1/4$,

By Proposition (2.1.2), the third property above implies that

$$\left(1 - \psi_{\ell_0}(1) - 1/2\right)_+ = \left(\left(1 - \psi_{\ell_0}(1) - 1/4\right)_+ - 1/4\right)_+ \precsim v_{\ell_0}^* a v_{\ell_0} \precsim a.$$

Define $\psi = h[\psi_{\ell_0}]$. By functional calculus, one checks that

$$1 - \psi(1) = 2\left(1 - \psi_{\ell_0}(1) - 1/2\right)_+$$

which gives

$$1 - \psi(1) \precsim a.$$

Property (ii) above together with (2.1.9) ensures that $\left\|\alpha_g\left(\psi(x)\right) - \psi(x)\right\| < \varepsilon/2 < \varepsilon$ for all normalized $x \in M_k$ and $g \in G$. This shows that ψ meets our requirements.

We are now ready to prove the following theorem.

Theorem (2.1.23) [2]:

Let $\mathcal{A}$ be a simple unital C^*-algebra and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group on $\mathcal{A}$. Assume that α has the generalized tracial Rokhlin property. If $\mathcal{A}$ is tracially Z-absorbing then $\mathcal{A} \rtimes_\alpha G$ is also tracially Z-absorbing.

Proof:

Lemma (2.1.22) implies that we can always find c.p.c. order zero maps as in Definition (2.1.2) provided that the positive element a is taken from $\mathcal{A}$. By Lemma (2.1.18) we may always assume without loss of generality that $a \in \mathcal{A}$ (otherwise replace it by a non-zero positive element from $\mathcal{A}$ which it dominates).

Putting together Theorems (2.1.13) and (2.1.23) we obtain the following partial generalization.

Corollary (2.1.24) [2]:

Let $\mathcal{A}$ be a simple, separable, unital, nuclear C^*-algebra and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G with the generalized tracial Rokhlin property. If $\mathcal{A}$ is Z absorbing then so is $\mathcal{A} \rtimes_\alpha G$.

As a non-trivial example of an action satisfying the generalized tracial Rokhlin property, we consider the symmetric group acting by permutation on the n-fold tensor power of Z. This example was studied, Corollary (2.1.24) gives a different proof of those results.

We recall the following special case.

Theorem (2.1.25) [2]:

Let τ denote the unique tracial state on Z. There exists a unital embedding $\psi : C([0,1]) \to Z$ such that

$$\tau(\psi(f)) = \int_0^1 f(t)\, dt$$

Corollary (2.1.26) [2]:

Let τ denote the unique tracial state on Z, let $F \subseteq Z$ be a finite subset, and let $\varepsilon > 0$. There exists a unital embedding $\psi : C([0,1]) \to Z$ such that

$$\tau(\psi(f)) = \int_0^1 f(t)\, dt$$

and $\|[y, \psi(f)]\| < \varepsilon$ for all $y \in F$ and normalized $f \in C([0,1])$.

Proof:

We may assume that all elements of F are of norm at most 1. Now, we may decompose Z as $Z \cong Z \otimes Z$ such that each element $y \in F$ is close to within $\varepsilon/2$ to an element y' in the unit ball of $Z \otimes 1$. Now choose a unital embedding $\psi_0 : C([0,1]) \to Z$ as in Theorem (2.1.25). Define $\psi : C([0,1]) \to Z \otimes Z \cong Z$ by $\psi(f) = 1 \otimes \psi_0(f)$. ψ clearly satisfies the commutation requirement, and since the restriction of the unique trace τ on $Z \otimes Z$ to the second component $1 \otimes Z$ is the unique trace on that C^* -algebra, we see that $\tau(\psi(f))$ is indeed given by integrating f against Lebesgue measure, as required.

Example (2.1.27) [2]:

Let $G = S_n$ be the symmetric group on the set $\{1, \dots, n\}$, and let $\mathcal{A} = Z^{\otimes n} \cong Z$. Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be the action defined on elementary tensors by the formula

$$\alpha_\sigma(z_1 \otimes \dots \otimes z_n) = z_{\sigma^{-1}(1)} \otimes \dots \otimes z_{\sigma^{-1}(n)}$$

Let ε, F, a be as in Definition (2.1.19). We denote $\delta = d_\tau(a)$ where τ is the unique trace on $\mathcal{A}$. Let

$$f_1 \in C([0,1]^n)$$

be a continuous function of norm one, such that:

(i) f_1 is supported on the set $X := \{(x_1, \dots, x_n) \in [0,1]^n | x_1 < \dots < x_n\}$,

(ii) $\lambda(\{x \in X | f_{id}(x) \neq 1\}) < \frac{\delta}{n!}$

where λ is Lebesgue measure on $[0,1]^n$. For $\sigma \in G$ we now define $f_\sigma \in C([0,1]^n)$ by

$$f_\sigma(x_1, \dots, x_n) = f_1(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

The f_σ are pairwise orthogonal and sum up to an element of norm 1 that equals 1 on a subset of $[0,1]^n$ of measure larger than $1-\delta$.

Our next step will be to define an embedding of $C([0,1]^n)$ into $\mathcal{A}$. We may assume that the finite set F consists of elementary tensors. Denote $F = \left\{z_1^{(i)} \otimes \dots \otimes z_n^{(i)}\right\}_{i\in I}$ where I is some finite index set. For $k = 1, \dots, n$ we take unital embeddings $\psi_k : C([0,1]) \to Z$ such that

$$\tau(\psi_k(g)) = \int_0^1 g(t)\, dt$$

and $\left\|\left[z_k^{(i)}, \psi_k(g)\right]\right\| < \varepsilon^{\frac{1}{n}}$ for all $i \in I$ and normalized $g \in C([0,1])$. Using the identification $C([0,1]^k) \cong C([0,1])^{\otimes n}$ we define $\psi : C([0,1]^n) \to \mathcal{A}$ by

$$\psi(g_1 \otimes \dots \otimes g_n) = \psi_1(g_1) \otimes \dots \otimes \psi_n(g_n)$$

We define $e_\sigma = \psi(f_\sigma)$. Clearly we have that $\|[e_\sigma, y]\| < \varepsilon$ for all $y \in F$. It is also not hard to see that

$$\tau(\psi(f)) = \int_{[0,1]^n} f\, d\lambda$$

for any $f \in C([0,1]^n)$ which implies that for the elements $e_\sigma = \psi(f_\sigma)$ we have $d_\tau(1 - \sum_{\sigma\in G} e_\sigma) < \delta$. Since $\mathcal{A}$ has strict comparison and τ is the only trace on $\mathcal{A}$ this entails $1 - \sum_{\sigma\in G} e_\sigma \precsim a$.

section (2.2): Actions of $\mathbb{Z}$

Definition (2.2.1) [2]:

Let α be an automorphism of a simple, unital C^*-algebra $\mathcal{A}$. We say that α has the generalized tracial Rokhlin property if for any finite set $F \subseteq \mathcal{A}$, any $\varepsilon > 0$, and $k \in \mathbb{N}$, and any non-zero positive element $a \in \mathcal{A}$ there exist normalized orthogonal positive contractions $e_1, \dots, e_k \in \mathcal{A}$ such that the following holds:

(i) $1 - \sum_{i=1}^k e_i \precsim a$.

(ii) $\|[e_i, y]\| < \varepsilon$ for all i and for all $y \in F$.

(iii) $\|\alpha(e_i) - e_{i+1}\| < \varepsilon$ for all $i \le i \le n-1$.

Note that in the definition we do not require that $\alpha(e_k)$ be close to e_1.

Notation (2.2.2) [2]:

Elements $e_1, \ldots, e_k$ as in Definition (2.2.1) are said to satisfy the relations $\mathcal{R}(k, \varepsilon, a, F)$.

We begin as we did in the previous section with some basic properties.

Proposition (2.2.3) [2]:

Let $\mathcal{A}$ be a simple, unital C^*-algebra and let $\alpha \in \mathrm{Aut}(\mathcal{A})$ have the generalized tracial Rokhlin property. Then α_m is outer for all $m \in \mathbb{Z}\backslash\{0\}$.

Proof:

Suppose α^m is implemented by some unitary $u \in \mathcal{A}$. Let $e_1, \ldots, e_{m+1} \in \mathcal{A}$ be elements satisfying $\mathcal{R}\left(m+1, \frac{1}{2m+2}, 1_{\mathcal{A}}, \{u\}\right)$. We have

$$\begin{aligned} 1 &= \|e_1 - e_{m+1}\| \\ &< \|ue_1u^* - e_{m+1}\| + \frac{1}{2m+2} \\ &= \|\alpha^m(e_1) - e_{m+1}\| + \frac{1}{2m+2} \\ &< +\frac{1}{2} + +\frac{1}{2m+2} \\ &< 1 \end{aligned}$$

Corollary (2.2.4) [2]:

Let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism on a simple, unital C^*-algebra $\mathcal{A}$. If α has the generalized tracial Rokhlin property then $\mathcal{A} \rtimes_\alpha Z$ is simple.

Lemma (2.2.5) [2]:

Let $\mathcal{A}$ be a simple, separable, unital, C^*-algebra and let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism with the generalized tracial Rokhlin property. Recall that α acts naturally on $T(\mathcal{A})$ via $\tau \longmapsto \tau \circ \alpha$. Assume that there exists $m \in \mathbb{N}$ such that the

action of α^m on $T(\mathcal{A})$ is trivial. Then for any $c > 0$ there exist $k_0 \in \mathbb{N}$, such that whenever $k \geq k_0, 0 < \varepsilon < \frac{1}{k}, b \in \mathcal{A}_+$ satisfies $k\langle b\rangle \leq \langle 1\rangle$, $F \subseteq \mathcal{A}$ is some finite subset and $e_1, \dots, e_k \in \mathcal{A}$ are elements satisfying the relations $\mathcal{R}(k, \varepsilon, b, F)$ then $d_\tau(e_i) < c$ for all $\tau \in T(\mathcal{A})$ and for all i.

Proof:

Choose k_0 such that $\frac{2m}{k_0} < c$. Let $k, \varepsilon > 0, b \in \mathcal{A}_+$ be as in the statement of the lemma and let $e_1, \dots, e_k$ be elements satisfying $\mathcal{R}(k, \varepsilon, b, F)$. We partition the set $I = \{1, \dots, k\}$ into m sets

$$I = \coprod_{j=1}^{m} I_j$$

where $I = I_j \cap (j + m\mathbb{Z})$. We have that

$$\|\alpha^m(e_i) - e_{i+m}\| < m\varepsilon$$

which implies that

$$(\alpha^m(e_i) - m\varepsilon)_+ = \alpha^m((e_i - m\varepsilon)_+) \precsim e_{i+m}$$

whenever $1 \leq i \leq k - m$. Since α^m acts trivially on $T(\mathcal{A})$, this entails that

$$d_\tau((e_i - \mathrm{m}\varepsilon)_+) \leq d_\tau(e_{i+m})$$

for all $1 \leq i \leq k - m$. Similarly we have that

$$d_\tau((e_i - m\varepsilon)_+) \leq d_\tau(e_{i-m})$$

whenever $m + 1 \leq i \leq k$. By iterating this argument we get that

$$d_\tau((e_i - k\varepsilon)_+) \leq d_\tau(e_{i'})$$

for all $1 \leq j \leq m$ and for all $i, i' \in I_j$. Note that $|I_j| \leq \left\lceil \frac{k}{m} \right\rceil$ for all j. This implies that, if $i \in I_j$ we have

$$d_\tau((e_i - k\varepsilon)_+) \leq \frac{1}{|I_j|} \sum_{i'} d_\tau(e_{i'}) \leq \frac{m}{k} d_\tau(1) < \frac{c}{2}.$$

Denote $\eta = k\varepsilon$ and let $g, f \in C_0((0,1])$ be defined by

$$g(t) = \begin{cases} 0, & t \leq \eta \\ \dfrac{1}{1-\eta}(t-\eta), & t > \eta \end{cases}, \qquad f(t) = t - g(t) \begin{cases} t, & t \leq \eta \\ \dfrac{\eta}{1-\eta}(1-t), & t > \eta \end{cases}.$$

Notice that $g(x) \sim (x-\eta)_+$ for all $x \in \mathcal{A}_+$. Together with the previous observations this implies that

$$d_\tau\big(g(e_i)\big) < \frac{c}{2}$$

for all i and for all $\tau \in T(\mathcal{A})$. Now define another function $h \in C_0((0,1])$ by

$$h(t) = \begin{cases} \dfrac{t}{1-t}, & t \leq \eta \\ \dfrac{\eta}{1-\eta}, & t > \eta \end{cases}.$$

We have that

$$h(t)^{\frac{1}{2}}(1-t)h(t)^{\frac{1}{2}} = h(t)(1-t) = f(t),$$

thus

$$h(e_i)^{\frac{1}{2}}(1-e_i)h(e_i)^{\frac{1}{2}} = f(e_i)$$

for all i. Since e_j is orthogonal to e_j for $i \neq j$, this implies that

$$h(e_i)^{\frac{1}{2}}\left(1 - \sum_{j=1}^{k} e_j\right)h(e_i)^{\frac{1}{2}} = f(e_i)$$

entailing that

$$f(e_i) \precsim 1 - \sum_{j=1}^{k} e_j \precsim b$$

for all i. Finally we have that

$$e_i = f(e_i) + g(e_i)$$

$$\precsim f(e_i) \oplus g(e_i)$$

$$\precsim b \oplus g(e_i)$$

In particular we have

$$d_\tau(e_i) \leq d_\tau(b) + d_\tau\big(g(e_i)\big) < c$$

Lemma (2.2.6) [2]:

Let $\mathcal{A}$ be a simple, separable, unital, tracially Z-absorbing C^*-algebra and let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism with the generalized tracial Rokhlin property. If α^m acts trivially on $T(\mathcal{A})$ for some m, then for any finite set $F \subset \mathcal{A}, \varepsilon > 0$ and non-zero positive element $a \in \mathcal{A}$ and $n \in \mathbb{N}$ there is a c.p.c. order zero map $\psi : M_n \to \mathcal{A}$ such that:

(i) $1 - \psi(1) \precsim a$.

(ii) For any normalized element $x \in M_n$ and any $y \in F$ we have $\|\psi(x), y\| < \varepsilon$.

(iii) For any normalized element $x \in M_n$ we have $\big\|\alpha\big(\psi(x)\big) - \psi(x)\big\| < \varepsilon$.

Proof:

Let F, ε, a, n be given as in the statement of the lemma. We assume throughout that $\varepsilon < 1$. As in the proof of Lemma (2.1.22) we can find $c > 0$ such that $d_\tau(a) > c$ for all $\tau \in T(A)$. Let $M \in \mathbb{N}$ be such that $\sqrt{\frac{4}{M}} < \frac{\varepsilon}{2}$. Denote $c' = \frac{c}{2M+1}$. By Lemma (2.2.5) we can find k_0 such that whenever $k \geq k_0, 0 < \delta < \frac{1}{k}, b \in \mathcal{A}_+$ with $k\langle b\rangle \leq \langle 1\rangle, K \subseteq \mathcal{A}$ is some finite subset, and $e_1, \dots, e_k \in \mathcal{A}$ are elements satisfying $\mathcal{R}(k, \delta, b, K)$ then we have $d_\tau(e_i) < c'$ for all i and for all $\tau \in T(\mathcal{A})$. We may furthermore assume $k_0 > 2M$. Use Lemma (2.1.15) to find a positive element $b \in \mathcal{A}$ such that $d_\tau(b) < \min\left\{\frac{1}{k}, \frac{c}{2M}\right\}$ for all $\tau \in T(\mathcal{A})$. Choose $k \geq k_0$ and choose $\delta > 0$ such that $\left(2k^2\delta + k\delta^2 + \frac{4}{M}\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}$. In particular $\delta < \frac{1}{k}$. Let $K = \left\{\alpha^{-i}(y) | y \in F, 1 \leq i \leq k\right\}$

Let $e_1, \dots, e_k \in \mathcal{A}$ be elements satisfying $\mathcal{R}(k, \delta, b, K)$. We now modify the elements e_i to obtain new elements f_i as follows:

$$f_i = \begin{cases} e_i, & M \leq i \leq k - M \\ \dfrac{i}{M} e_i, & 1 \leq i < M \\ \dfrac{k-i}{M} e_i, & k - M < i \leq k \end{cases}$$

By Lemma (2.1.6) we can find a c.p.c. order zero map $\varphi : \mathrm{M_n} \to \mathcal{A}_\infty \cap \mathcal{A}'$ such that $1 - \varphi(1) \precsim b$. We define $\hat{\psi} : M_n \to \mathcal{A}$ by

$$\hat{\psi}(x) = \sum_{i=1}^{k} f_i \bar{\alpha}^i \left(\varphi(x)\right).$$

$\hat{\psi}$ is clearly a c.p.c. order zero map. Our first goal is now to find a nice bound on $\left\|\hat{\psi}(x) - \bar{\alpha}\left(\hat{\psi}(x)\right)\right\|$ for normalized $x \in M_n$. First, note that $\|\alpha(f_i) - f_{i+1}\| < \frac{2}{M}$ for all $i < k$ and also $\|f_1\|, \|f_k\| < \frac{1}{M}$. Now assume $i \neq j$, we have

$$\begin{aligned} \left\|(\alpha(f_i) - f_{i+1})(\alpha(f_j) - f_{j+1})\right\| &= \left\|-\alpha(f_i) f_{j+1} - f_{i+1}\alpha(f_j)\right\| \\ &\leq \left\|\alpha(f_i) f_{j+1}\right\| + \left\|f_{i+1}\alpha(f_j)\right\| \\ &\leq \left\|\alpha(e_i) e_{j+1}\right\| + \left\|e_{i+1}\alpha(e_j)\right\| \\ &\leq \left\|e_{i+1} e_{j+1}\right\| + \left\|e_{i+1} e_{j+1}\right\| + 2\delta \\ &= 2\delta. \end{aligned}$$

We use this estimate to obtain that for normalized $x \in M_n$

$$\left\|\bar{\alpha}\left(\sum_{i=1}^{k} f_i \bar{\alpha}^i \left(\varphi(x)\right)\right) - \sum_{i=1}^{k} f_{i+1} \bar{\alpha}^i \left(\varphi(x)\right)\right\|^2$$

$$= \left\|\sum_{i=1}^{k} (\alpha(f_i) - f_{i+1}) \bar{\alpha}^{i+1} \left(\varphi(x)\right)\right\|^2$$

$$\leq \left\| \sum_{1\leq i,j\leq k-1} (\alpha(f_i) - f_{i+1})(\alpha(f_j) - f_{j+1})\bar{\alpha}^{i+1}(\varphi(x))\bar{\alpha}^{j+1}(\varphi(x)) \right\|$$

$$+ \left\| \sum_{i=1}^{k} (\alpha(f_i) - f_{i+1})^2 \left(\bar{\alpha}^{i+1}(\varphi(x))\right)^2 \right\|$$

$$\leq 2k^2\delta + k\delta^2 + \frac{4}{M}.$$

Now we are ready to consider the expression we are interested in. Let $x \in M_n$ be a normalized element. We have

$$\left\| \bar{\alpha}\left(\hat{\psi}(x)\right) - \hat{\psi}(x) \right\|$$

$$\leq \left\| \bar{\alpha}\left(\sum_{i=1}^{k} f_i \bar{\alpha}^i\left(\varphi(x)\right) \right) - \sum_{i=1}^{k} f_{i+1}\bar{\alpha}^i\left(\varphi(x)\right) \right\| + \left\| f_1 \bar{\alpha}(\varphi(1)) \right\|$$

$$+ \left\| \bar{\alpha}\left(f_k \bar{\alpha}^k(\varphi(1))\right) \right\|$$

$$\leq \left(2k^2\delta + k\delta^2 + \frac{4}{M}\right)^{1/2} + \frac{2}{M}$$

$$\leq \left(2k^2\delta + k\delta^2 + \frac{4}{M}\right)^{1/2} + \frac{2}{M}$$

$$< \varepsilon.$$

Next, we claim that $1 - \hat{\psi}(1) \precsim a$. To see this, first note that

$$\left(1 - \sum_{i=1}^{k} f_i\right) \precsim 1 - \sum_{i=M}^{k-M} e_i$$

$$\precsim b^{\oplus 2M-1}$$

Using this, we may write

$$1-\hat{\psi}(1) = 1-\varphi(1) + \varphi(1)\left(1-\sum_{i=1}^{k} f_i\right)$$

$$\precsim 1-\varphi(1) \oplus \left(1-\sum_{i=1}^{k} f_i\right)$$

$$\precsim b^{\oplus 2M-}$$

Since $\mathcal{A}$ has strict comparison and $d_\tau(b) < \frac{c}{2M}$ for all $\tau \in T(\mathcal{A})$, we have that $b^{\oplus 2M} \precsim a$ which proves our claim.

The final condition we need on $\hat{\psi}$ is that $\|[\hat{\psi}(x), y]\| < \varepsilon$ for all normalized $x \in M_n$ and for all $y \in K$. This follows immediately from our construction.

We now continue as in the proof of Lemma (2.1.22) and use projectivity of order zero maps to lift $\hat{\psi}$ to a sequence of c.p.c. order zero maps from M_n into $\mathcal{A}$. Going far enough along this sequence we obtain the map ψ which we need.

Combining this with Theorem (2.1.13), we obtain the following.

Theorem (2.2.7) [2]:

Let $\mathcal{A}$ be a simple, separable, unital, tracially Z-absorbing C^*-algebra and let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism with the generalized tracial Rokhlin property. If α^m acts trivially on $T(\mathcal{A})$ for some m then $\mathcal{A} \rtimes_\alpha Z$ is also tracially Z-absorbing.

Example (2.2.8) [2]:

It is shown that the bilateral tensor shift automorphism α on $Z^{\otimes\infty} \cong Z$ satisfies what is there defined as the weak Rokhlin property. Since Z has strict comparison, this implies that α has the generalized tracial Rokhlin property. Since Z has unique trace this shows that α satisfies the conditions of Theorem (2.2.7).

Chapter 3

C^*-Algebras and Non-Stable K-Theory

In this chapter we consider the properties weak cancellation, K_1−surjectivity, good index theory, and K_1−injectivity, for the class of extremally rich C^*−algebras, and for the smaller class of isometrically rich C^*−algebras. We establish all four properties for isometrically rich C^*−algebras and for extremally rich C^*−algebras that are either purely infinite or of real rank zero, K_1−injectivity in the real rank zero case following from a prior result of H. Lin [3].

Section (3.1): Weak cancellation

We defined the concept of extremal richness. One of several equivalent criteria for the C^*-algebra A to be extremally rich is that the closed unit ball of $\tilde{A}$, the unitization, is the convex hull of $\mathcal{E}(\tilde{A})$, the set of its extreme points. Further review of the concept is given in the next section. A simple C^*-algebra is extremally rich if and only if it is either of stable rank one or purely infinite, and a theme of our work has been that extremal richness is a generalization of the stable rank one property which is suitable for infinite algebras. Since much of the success in the classification of simple C^*-algebras has been for algebras that are either purely infinite or of stable rank one, it seems worthwhile to study non-simple extremally rich C^*-algebras.

J. Cuntz defined purely infinite simple C^*-algebras and showed that they have many good non-stable K-theoretic properties. And M. Rieffel, motivated by algebraic results of H. Bass, defined topological stable rank and showed that C^*-algebras of (topological) stable rank one have similarly good properties. We therefore investigated whether extremally rich C^*-algebras also have the good properties. Although we haven't proved that all extremally rich algebras have good non-stable K-theoretic properties, we have found large subclasses that do. In particular, the summary includes all four properties listed in the abstract for isometrically rich C^*-algebras and for extremally rich C^*-algebras which are either purely infinite (in the sense of E. Kirchberg and M. Rørdam) or of real rank zero. All three cases of cover purely infinite simple C^*-algebras.

Murray-von Neumann [6]: Two subspaces belong to M are called Murray von Neumann if there is a partial isometry mapping the first isomorphically onto the other that is an element of the von Neumann algebra (informly, if M " Knows " that the subspace are isomorphic).

A C^*-algebra A has weak cancellation if whenever p and q are projections in A which generate the same (closed, two-sided) ideal I and have the same class in $K_0(I)$, then p is Murray-von Neumann equivalent to $q(p \sim q)$. Of course, Typeset $p \sim q$ implies that they generate the same ideal I and that $[p] = [q]$ in $K_0(I)$. Nevertheless, it was observed by Rieffel, that C^*-algebras of stable rank one ($\mathrm{tsr}(A) = 1$) satisfy a stronger property: If $[p] = [q]$ in $K_0(A)$, then $p \sim q$. Of course this stronger property can't hold in infinite algebras. Cuntz showed that if A is purely infinite simple, if p and q are both non-zero, and if $[p] = [q]$ in $K_0(A)$, then $p \sim q$; and the concept of weak cancellation was designed to specialize to this property in the simple case. More about weak cancellation and its history is given in the next section.

We say that A has K_1-surjectivity if the map from $\mathcal{U}(\tilde{A})/\mathcal{U}_0(\tilde{A})$ to $K_1(A)$ is surjective, K_1-injectivity if this map is injective, and K_1-bijectivity if it is bijective. Here $\mathcal{U}(\tilde{A})$ is the unitary group and $\mathcal{U}_0(\tilde{A})$ the connected component of the identity (so that $\mathcal{U}(\tilde{A})/\mathcal{U}_0(\tilde{A})$ is the set of homotopy classes). Rieffel showed that $\mathrm{tsr}(A) = 1$ implies that A has K_1-bijectivity and Cuntz showed the same for A purely infinite simple. A result of P. Ara, states that every quotient C^*-algebra of a Rickart C^*-algebra has K_1-surjectivity. Below states that every extremally rich C^*-algebra with weak cancellation has K_1-surjectivity. This should be compared, which states that every C^*-algebra with real rank zero and stable weak cancellation has K_1-surjectivity.

We defined the extremal K-set, $K_e(A)$, which is roughly analogous to $K_1(A)$ with extremal partial isometries used in place of unitaries. The equivalence relation for K_e is more complicated than that for K_1; we show that if A is extremally rich with weak cancellation, then $K_e(A) = \xrightarrow{\lim} n\left(\mathcal{E}\left(\mathbb{M}_n(\tilde{A})\right)/\mathrm{homotopy}\right)$), in exact analogy with the K_1-case. We say that A has K_e-surjectivity, K_e-injectivity, or K_e-bijectivity if the map from

$\mathcal{E}(\tilde{A})/\mathrm{homoopy}$ to $K_e(A)$ is respectively surjective, injective, or bijective. These properties actually imply the corresponding K_1-properties. We show that every extremally rich C^*-algebra with weak cancellation has K_e-surjectivity.

We say that a (non-unital) C^*-algebra K has good index theory if whenever K is embedded as an ideal in a unital C^*-algebra A and u is a unitary in A/K such that $\partial_1([u]\kappa_1) = 0$ in $K_0(K)$, there is a unitary in A which lifts u. Of course, the boundary map, $\partial_1 \colon K(A/K) \to K_0(K)$, from the K-theory long exact sequence is often regarded as an abstract index map. Using that long exact sequence, we can reformulate the good index theory property: If $u \in \mathcal{U}(A/K), \alpha \in K_1(A)$, and α lifts $[u]\kappa_1$, then u lifts to $\mathcal{U}(A)$. This suggests a stronger property–require that the lift of u lie in the class α. However, there is no need to name this stronger property (still demanded for all choices of A), since it is equivalent to requiring that K have both K_1-surjectivity and good index theory.

Good index theory has been considered, so far as we know, no one previously proposed a name for it. M. Pimsner, S. Popa, and D. Voiculescu proved that $C(X) \otimes \mathbb{K}$ has good index theory when X is compact, where K denotes the algebra of compact operators on a separable, infinite dimensional Hilbert space. J. Mingo showed that $C(X)$ can be replaced by an arbitrary unital C^*-algebra and asked whether every stable C^*-algebra has good index theory. (In both of these results the property is considered only for $A = M(K)$, the multiplier algebra, but this is sufficient to imply that it holds for all A.) Shortly after, G. Nagy proved, that $\mathrm{csr}(K) \leq 2$ implies that K has good index theory. Here csr denotes connected stable rank (Rieffel), and $\mathrm{csr}(K) \leq 2$ also implies K_1-surjectivity for K. It had already been proved by A. Sheu, and V. Nistor, that $\mathrm{csr}(K) \leq 2$ for all stable K. So Nagy completed the affirmative answer to Mingo's question (but was apparently unaware of Mingo's work). Since Rieffel proved that $\mathrm{csr}(K) \leq \mathrm{tsr}(K) + 1$, Nagy also established good index theory for C^*-algebras of stable rank one. The fact that purely infinite simple C^*-algebras have good index theory should also be considered previously known.

A unital C^*-algebra A has stable rank one if A^{-1}, the set of invertible elements, is dense in A, is isometrically rich if $A_l^{-1} \cup A_r^{-1}$, the set of one-sided invertible elements, is dense, and is extremally rich if A_q^{-1}, the set of quasi-invertible

elements is dense. If A is non-unital, we say that A has one of these properties if $\tilde{A}$ has the property. All three properties pass to (closed, two-sided) ideals and hereditary C^*-subalgebras, quotient algebras, and matrix algebras and stabilizations.

Before reviewing the definition of quasi-invertibility, we recall R. Kadison's criterion for extreme points of the unit ball of a C^*-algebra A. Extreme points exist if and only if A is unital, and u is extremal if and only if

$$(1 - uu^*)A(1 - u^*u) = 0.$$

Equivalently, u is a partial isometry and $I \cap J = 0$, where $I = \mathrm{id}(1 - uu^*)$ and $J = \mathrm{id}(1 - u^*u)$. Here $\mathrm{id}(\cdot)$ denotes the ideal generated by $\cdot$, $1 - uu^*$ and $1 - u^*u$ are called the left and right defect projections of u, and I and J are the left and right defect ideals of u.

We showed that seven conditions on an element t of a unital C^*-algebra A are equivalent, and these conditions are the definition of quasi-invertible. (Non-unital algebras have no quasi-invertibles.) One of these conditions amounts to saying that t has closed range and that if $t = u|t|$ is its canonical polar decomposition (the closed range condition implies $u \in A$), then $u \in \mathcal{E}(A)$. Another is that there are ideals I and J with $I \cap J = 0$ such that $t + J$ is left invertible in A/J and $t + I$ is right invertible in A/I. Clearly the minimal choices for I and J are the defect ideals of u, and we also call these the defect ideals of t. Of course, if A is prime, one of I and J must be 0; thus every quasi-invertible element is one-sided invertible and every extremal partial isometry is an isometry or co-isometry.

In general, there is an analogy in the two-step progressions from stable rank one (through isometrically rich) to extremally rich, from invertibility to quasi-invertibility, and from unitary to extremal partial isometry. It is not always necessary to pursue all three concepts in parallel because A (unital) has stable rank one if and only if it is extremally rich and every extremal is unitary, and A is isometrically rich if and only if it is extremally rich and every extremal is an isometry or co-isometry. However, the most general results are usually not proved first. Rørdam proved Robertson's conjecture: For A unital, $\mathrm{tsr}(A) = 1$ if and only if the closed unit ball is the convex hull of $\mathcal{U}(A)$.

For technical reasons it is sometimes necessary to consider extremals, quasi-invertibility, and extremal richness for objects other than C^*-algebras, namely bimodules of the form pAq, where p and q are projections in A. S. Sakai's criterion for u to be an extreme point of the unit ball of pAq is:

$$(p - uu^*)A(q - u^*u) = 0 \text{ or equivalently,} \qquad (1 - uu^*)qAq(1 - u^*u) = 0.$$

Quasi-invertibility and extremal richness for bimodules are treated analogously to the treatments for C^*-algebras with no difficulty. It is not necessary to be explicitly aware of the fact that pAq is a bimodule or even to know what "bimodule" means. Nevertheless, the abstract setting was discussed. One warning: There are no concepts of unitality or unitization for bimodules. It is required that $\mathcal{E}(pAq) \neq \emptyset$ in order for pAq to have a chance to be extremally rich. (When $pAq = 0$, it is automatically extremally rich.) The extremal richness of A does not imply that of pAq, but it is important to our main results that sometimes pAq is extremally rich . Since $\mathrm{tsr}(A) \leq n$ if and only if "left invertibles" are dense in the bimodule $\tilde{A}^n$ $(= \mathbf{1}_n \mathbb{M}_n(\tilde{A})\mathbf{1}_1)$, we once looked at extremal richness for $\tilde{A}^n$; but it turned out that A extremally rich does not imply $\tilde{A}^n$ extremally rich.

An example that does invoke the abstract setting may be interesting. The right module called $\mathcal{H}_A$ by G. Kasparov is also an $A \otimes \mathbb{K}$–A-imprimitivity bimodule. Regardless of whether or not A is extremally rich, $\mathcal{H}_A$ is an extremally rich bimodule if and only if A is unital.

Definitions (3.1.1) [3]:

If p and q are projections in a C^*-algebra, we write $p - q$ to mean $p \sim q' \leq q$ for some projection q'. The relations $\sim$ and $\precsim$ can be extended to projections in $\bigcup_n \mathbb{M}_n(A)$ either by replacing p and q by $p \oplus 0_k$ and $q \oplus 0_k$ for suitable k and l or by allowing the partial isometries to be non-square matrices. The projection p is infinite if it is equivalent to a proper subprojection of itself, otherwise finite, and a unital C^*-algebra A is finite or infinite according as $\mathbf{1}_A$ is, and stably finite if all the matrix algebras $\mathbb{M}_n(A)$ are finite. If p and q are projections in $\mathbb{M}_m(A)$ and $\mathbb{M}_n(A)$, respectively, then $p \oplus q$ denotes the projection $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ in $\mathbb{M}_{m+n}(A)$, and $p_1 \oplus \ldots \oplus p_k$ is defined similarly. In this context kp is used for the k-fold sum, $p \oplus \ldots \oplus p$. A projection p is called properly infinite if $2p \precsim p$.

It is possible to reformulate weak cancellation in a way that does not mention K-theory. Note that if p, q, and r are projections in an ideal I and if p is full in I (i.e., $I = \mathrm{id}(p)$), then $[q]_{\kappa_0(I)} = [r]_{\kappa_0(I)}$ if and only if $q \oplus np \sim r \oplus np$ for sufficiently large n. Thus the hypotheses, $\mathrm{id}(p) = \mathrm{id}(q) = I$ and $[p]_{\kappa_0(I)} = [q]_{\kappa_0(I)}$, can be replaced by, $p \oplus nq \sim (n+1)q$ and $q \oplus np \sim (n+1)p$ for sufficiently large n. It was pointed out to us by K. Goodearl that the concept can be simplified further if we demand weak cancellation for the stabilization of A: $A \otimes \mathbb{K}$ has weak cancellation if and only if $2p \sim p \oplus q \sim 2q$ implies $p \sim q$ for all projections p and q in $A \otimes \mathbb{K}$. Moreover, this is equivalent to "separativity," a term which was introduced into semigroup theory by A. Clifford and G. Preston. The set of Murray-von Neumann equivalence classes of projections in A is not in general a semigroup, and it is only the stable version of weak cancellation that is literally equivalent to separativity. However, can be used to show that weak cancellation is a stable property in the real rank zero case.

Defect ideals were treated. The defect ideal of A, denoted $\mathcal{D}(A)$, is the ideal generated by all defect projections of elements of $\mathcal{E}(\tilde{A})$. (All of these defect projections are in A.) If A is extremally rich, then $\mathcal{D}(A)$ is the smallest ideal I such that $\mathrm{tsr}(A/I) = 1$. It follows that defect ideals are compatible with Rieffel-Morita equivalence among extremally rich C^*-algebras. In particular, if B is a full hereditary C^*-subalgebra, then $\mathcal{D}(B) = B \cap \mathcal{D}(A)$. The symbol $\mathcal{D}^n(A)$ denotes the n-fold iteration of $\mathcal{D}$.

The primitive ideal space of A will be denoted by $A^\vee$. If I is an ideal of A, then $I^\vee$ is identified with an open subset of $A^\vee$ (namely, the complement of $\mathrm{hull}(I)$), and if B is a hereditary C^*-subalgebra, then $B^\vee$ is identified with $\mathrm{id}(B)^\vee$. For B hereditary, $B^\perp$ denotes the two-sided annihilator of B, which is again hereditary, and $I^\perp$ is an ideal if I is an ideal.

Also, $=$ denotes norm closure and $\mathcal{T}_e$ denotes the extended Toeplitz algebra, which was discussed.

Definitions (3.1.2) [3]:

Recall that a C^*-algebra A has weak cancellation if any pair of projections p, q in A that generate the same closed ideal I of A and have the same image in $K_0(I)$

must be Murray–von Neumann equivalent in A (hence in I). If $\mathbb{M}_n(A)$ has weak cancellation for every n, equivalently, if $A \otimes \mathbb{K}$ has weak cancellation, we say that A has stable weak cancellation. We shall show below that weak cancellation implies stable weak cancellation if A is extremally rich, but for now we need the distinction.

Proposition (3.1.3) [3]:

If p and q are projections in an extremally rich C^*-algebra A such that $[p] = [q]$ in $K_0(A)$ then pAq is extremally rich.

Proof:

Since $K_0(\mathrm{A}) \subset K_0(\tilde{\mathrm{A}})$ and $\mathrm{pAq} = p\tilde{A}q$ we may assume that A is unital. Then also $[\mathbf{1} - p] = [\mathbf{1} - q]$ in $K_0(A)$, so

$$(\mathbf{1} - p) \oplus n\mathbf{1} \sim (\mathbf{1} - q) \oplus n\mathbf{1} \quad (\text{in } \mathbb{M}_{n+1}(A))$$

for n sufficiently large. Since $\mathbb{M}_{n+1}(A)$ is extremally rich conclude that

$$pAq = (p \oplus 0)\mathbb{M}_{n+1}(A)(q \oplus 0)$$

is extremally rich.

Lemma (3.1.4) [3]:

Let p and q be projections in a C^*-algebra A and for each element x in A let $\mathrm{id}(x)$ denote the closed ideal generated by x. If now $v \in \mathcal{E}(pAq)$ then

$$id(v) = id(p) \cap id(q).$$

Proof:

Since $vv^* \leq p$ we have $vv^* \in \mathrm{id}(p)$, whence $v \in \mathrm{id}(p)$. Similarly $v^*v \leq q$, so $v \in \mathrm{id}(q)$.

Conversely, let $\pi : A \to A/id(v)$ denote the quotient map. Then the extremality equation gives

$$\pi(p)\pi(A)\pi(q) = 0,$$

whence $\pi(id(p) \cap id(q)) = 0$; so $id(p) \cap id(q) \subset id(v)$, as desired.

Although we will prove later that every extremally rich C^*-algebra with weak cancellation also has K_1-surjectivity, it will facilitate some of the following arguments to impose it as a condition on the algebras.

Lemma (3.1.5) [3]:

Let p and q be full projections in an extremally rich C^*-algebra A such that $[p] = [q]$ in $K_0(A)$, and assume that we have found an extreme partial isometry u in $\mathcal{E}(pAq)$ such that

$$p_1 = p - uu^* \quad \text{and} \quad q_1 = q - u^*u$$

are projections in an ideal I of A. Assume further that eAe/eIe has K_1-surjectivity for every full projection e in A. Then

$$p \sim p_2 \oplus e_2 \quad \text{and} \quad q = q_2 + e_2$$

for some full projection e_2 in A and projections p_2 and q_2 in I, with $[p_2] = [q_2]$ in $K_0(I)$.

Proof:

Note first that the element $[p_1] - [q_1]$ of $K_0(I)$ belongs to the kernel of the natural map from $K_0(I)$ into $K_0(A)$. By the six–term exact sequence in K-theory there is therefore an α in $K_1(A/I)$ such that $\partial_1\alpha = [p_1] - [q_1]$. Since $e_1 = u^*u$ is a full projection in A by Lemma (3.1.4), we may identify $K_1(A/I)$ with $K_1(e_1Ae_1/e_1Ie_1)$, and by K_1-surjectivity we can therefore find v in e_1Ae_1 such that $v + e_1Ie_1$ is unitary with $[v + e_1Ie_1] = \alpha$. Since e_1Ae_1 is extremally rich, extreme points lift from quotients, so we may assume that $v \in \mathcal{E}(e_1Ae_1)$. Computing in $K_0(I)$ we find that

$$[p_1] - [q_1] = \partial_1\alpha = \text{index}\, v = [e_1 - v^*v] - [e_1 - vv^*].$$

Let

$$p_2 = p_1 + u(e_1 - vv^*)u^*, \;\; q_2 = q_1 + (e_1 - v^*v), \;\; e_2 = v^*v.$$

Since $v \in \mathcal{E}(e_1Ae_1)$ we see from Lemma (3.1.4) that e_2 is full in e_1Ae_1, and since e_1 is full in A, we have also that e_2 is a full projection in A. By construction $e_1 - vv^*$ and $e_1 - v^*v$ belong to I, so p_2 and q_2 belong to I; and evidently $[p_2] = [q_2]$ in $K_0(I)$. Finally,

$$p = p_1 + uu^* = p_2 + uvv^*u^* \sim p_2 \oplus e_2,$$

$$q = q_1 + u^*u = q_2 + e_2.$$

Theorem (3.1.6) [3]:

For an extremally rich C^*-algebra A the following conditions are equivalent:

(i) A has weak cancellation;

(ii) If $B = pAp$ for some projection p in A and $u \in \mathcal{E}(\mathbb{M}_2(B))$, there is a projection q in B such that

$$q \oplus 0 \sim (p \oplus p) - uu^* \quad (in\ \mathbb{M}_2(B));$$

(iii) If $B = pAp$ for some projection p in A, and $\{u_1, \dots, u_n\}$ is a finite subset of $\mathcal{E}(B)$ there is a projection q in B such that

$$q \oplus 0 \sim \bigoplus_{i=1}^{n} (p - u_i u_i^*) \quad (in\ \mathbb{M}_2(B)).$$

Proof:

(i) ⇒(ii) Since homotopic projections are equivalent we may assume that

$$(p \oplus p) - uu^* = (p - v_1 v_1^* - w_{21} w_{21}^*) \oplus (p - v_2 v_2^* - w_{21} w_{21}^*),$$

where $v_i \in \mathcal{E}(B)$ and $w_{ij} \in \mathcal{E}(p_i B q_j)$, and where we set $p_i = p - v_i v_i^*$ and $q_i = p - v_i^* v_i$ for $i, j = 1,2$. Evidently the two projections $p - w_{21} w_{21}^*$ and $p - w_{21}^* w_{21}$ have the same image in $K_0(I)$ for any ideal I containing p. To show that they also generate the same ideal in A it suffices to show that they both generate B as an ideal (inside B). However,

$$p - w_{21} w_{21}^* \geq v_2 v_2^* \text{ and } p - w_{21}^* w_{21} \geq v_1^* v_1,$$

and both v_1 and v_2 generate B as an ideal by Lemma (3.1.4).

By assumption there is therefore a partial isometry v in B such that

$$v^* v = p - w_{21} w_{21}^* \text{ and } vv^* = p w_{21}^* w_{21}.$$

Thus

$$e = v(p_2 - w_{21}w_{21}^*)v^*$$

is a projection equivalent to $p_2 - w_{21}w_{21}^*$, and since $e \leq vv^* = p - w_{21}^*w_{21}$ and e is centrally orthogonal to $q_1 - w_{21}^*w_{21}$ (because $w_{21} \in \mathcal{E}(p_2Bq_1)$) it follows that actually

$$e \leq p - q_1 = v_1v_1^*.$$

But then $v_1ev_1^*$ is a projection in B equivalent to $p_2 - w_{21}w_{21}^*$ and orthogonal to $p - v_1v_1^* = p_1$. We may therefore take

$$q = p_1 - w_{21}w_{21}^* + v_1ev_1^*,$$

which is a projection in B equivalent to $(p \oplus p) - uu^*$.

(ii) ⇒(iii) We use induction on n, the case $n = 1$ being trivial. By assumption, used on $\{u_1, \dots, u_{n-1}\}$, we can therefore find a projection q_1 in B such that

$$q_1 \sim \bigoplus_{i=1}^{n} (p - u_i^2u_i^{*2}) \sim 2\bigoplus_{i=1}^{n-1} (p - u_iu_i^*).$$

This means that we can write $q_1 = q_2 + (q_1 - q_2)$, where

$$q_2 \sim q_1 - q_2 \bigoplus_{i=1}^{n-1} (p - u_iu_i^*).$$

In particular, $q_2 \precsim p - q_2 = p_1$.

Set $B_1 = p_1Ap_1$. Since

$$p - p_1 + q_2 \precsim 2p_1$$

there is a partial isometry w in $\mathbb{M}_2(B)$ such that $w^*w = p \oplus 0$ and $ww^* = e \leq p_1 \oplus p_1$. It follows that if we define

$$v = p_1 \oplus p_1 - e + w(u_n \oplus 0)w^*$$

then $v \in \mathcal{E}(\mathbb{M}_2(B_1))$. Moreover,

$$p_1 \oplus p_1 - vv^* = (w(p - u_n u_n^*) \oplus 0)w^* \sim p - u_n u_n^*.$$

Applying (ii) to B_1 and v we find a projection p_2 in B_1 such that $p_2 \sim p - u_n u_n^*$. Thus we may take $q = q_2 + p_2$ to complete the induction step.

(iii) ⇒ (i) Let p and q be projections in A that generate the same closed ideal I and for which $[p] = [q]$ in $K_0(I)$. Replacing A by I does not effect the conditions in (iii) so we may assume that $I = A$; i.e., p and q are full projections in A. Since $[p] = [q]$ in $K_0(A)$ it follows from Proposition (3.1.3) that pAq is extremally rich (and non–zero). Take therefore u in $\mathcal{E}(pAq)$ and define

$$p_1 = p - uu^*, \quad q_1 = p - u^*u.$$

Then $[p_1] = [q_1]$ in $K_0(A)$.

If $\pi : A \to A/D(A)$ denotes the quotient map, we have $[\pi(p)] = [\pi(q)]$ in $K_0(\pi(A))$. Since $\mathrm{tsr}(\pi(A)) = 1$, this implies that $\pi(p) \sim \pi(q)$. Thus $\pi(pAq)$ is isometrically isomorphic to $\pi(pAp)$, which has stable rank one. It follows that $\pi(u)$ is "unitary" so that $\pi(p_1) = \pi(q_1) = 0$. Thus both p_1 and q_1 belong to $\mathcal{D}(A)$.

Since $\mathrm{tsr}(eAe/eD(A)e) = 1$ for every projection e in A we may apply Lemma (3.1.5) with $I = \mathcal{D}(A)$ to obtain a full projection e_2 in A and projections p_2 and q_2 in $\mathcal{D}(A)$ such that $[p_2] = [q_2]$ in $K_0(\mathcal{D}(A))$ and

$$p \sim p_2 \oplus e_2, \quad q = q_2 + e_2.$$

Let $B = e_2 A e_2$. Then B is a full corner of A, so $e_2 \mathcal{D}(A) e_2 = \mathcal{D}(B)$ is a full hereditary C^*-subalgebra of $\mathcal{D}(A)$. Consequently the set of projections

$$\mathcal{D} = \{e_2 - ww^* | w \in \mathcal{E}(B)\}$$

generates $\mathcal{D}(A)$ as an ideal. For some finite subset $\{w_i\}$ of $\mathcal{D}$ we therefore have

$$p_2 \precsim \bigoplus (e_2 - w_i w_i^*), \quad q_2 \precsim \bigoplus (e_2 - w_i w_i^*);$$

and since $[p_2] = [q_2]$ in $K_0(\mathcal{D}(A))$ we can assume, possibly after enlarging the subset, that

$$p_2 \oplus \left(\bigoplus (e_2 - w_i w_i^*)\right) \sim q_2 \oplus \left(\bigoplus (e_2 - w_i w_i^*)\right).$$

Applying condition (iii) we find a projection q_0 in B such that $q_0 \sim \oplus (e_2 - w_i w_i^*)$. Thus $p_2 \oplus q_0 \sim q_2 + q_0$ with $q_0 \leq e_2$, whence

$$p \sim p_2 \oplus e_2 \sim q_2 + e_2 = q.$$

Corollary (3.1.7) [3]:

Every isometrically rich C^*-algebra has stable weak cancellation.

The term "purely infinite" will be used in the Kirchberg- Rørdam sense. Every projection in a purely infinite C^*-algebra is properly infinite. We made the definition that A is purely properly infinite if every non-zero hereditary C^*-subalgebra is generated as an ideal by its properly infinite projections. This is equivalent to purely infinite when A is simple but stronger in general. However, the two concepts are equivalent if A has enough projections. We showed that an extremally rich C^*-algebra A is purely properly infinite if and only if $\mathcal{D}(I) = I$ for every ideal I of A. We also showed that for A extremally rich every non-zero projection in A is properly infinite if and only if $\mathcal{D}(I) = I$ for every ideal I which is generated (as an ideal) by a projection. The next result has a still weaker hypothesis.

Corollary (3.1.8) [3]:

If A is an extremally rich C^*-algebra such that $\mathcal{D}(I) = I$ whenever I is the left defect ideal of an element of $\mathcal{E}(\tilde{A})$, then A has weak cancellation. In particular, this applies if every defect projection is properly infinite or if A is purely infinite.

Proof:

To show that A satisfies condition (iii) let $u_1, \dots, u_n$ be in $\mathcal{E}(B)$, where $B = pAp$ for a projection p in A, and let $q_i = p - u_i u_i^*$. Since q_i is also a defect projection for A, the hypothesis implies that $\mathcal{D}(\mathrm{id}(q_i)) = \mathrm{id}(q_i)$; and since $q_i B q_i$ is Rieffel- Morita equivalent to $\mathrm{id}(q_i)$, this implies $\mathcal{D}(q_i B q_i) = q_i B q_i$. Then it implies that mq_i is properly infinite for some m. But mq_i is equivalent to a projection r_i in B, namely the left defect projection of u_i^m. Now the proper infiniteness of r_i implies that there is an isometry v_i in $r_i B r_i$ such that $r_i \precsim r_i - v_i v_i^*$. So if $w_i = p - r_i + v_i$, then w_i is an isometry in B and $q_i \precsim p - w_i w_i^*$. Finally, if $w = \prod_{i=1}^n w_i$, then w is an isometry in B and $\bigoplus_{i=1}^n q_i \precsim (p - w_i w_i^*)$.

Lemma (3.1.9) [3]:

Let p, q and q_0 be projections in a unital C^*-algebra A such that

$$q \sim q_0 \leq p \quad \text{and} \quad \mathbf{1} - q \sim \mathbf{1} - p \sim \mathbf{1}.$$

Then also $\mathbf{1} - q_0 \sim \mathbf{1}$.

Proof:

We compute (in A and in $\mathbb{M}_2(A)$)

$$\mathbf{1} - q_0 = \mathbf{1} - p + p - q_0 \sim (\mathbf{1} - p) \oplus (p - q_0)$$

$$\sim \mathbf{1} \oplus (p - q_0) = (\mathbf{1} - q + q) \oplus (p - q_0)$$

$$\sim (\mathbf{1} - p + q_0) \oplus p - q_0 \sim \mathbf{1} - p + q_0 + p - q_0 = \mathbf{1}.$$

Lemma (3.1.10) [3]:

Let q and q_0 be projections in a unital C^*-algebra A of real rank zero such that $q \sim q_0$ and $\mathbf{1} - q \sim \mathbf{1} - q_0 \sim \mathbf{1}$. If I denotes the closed ideal of A generated by q (and q_0) then $\mathbf{1} - q \sim 1 - q_0$ in $\tilde{I} = I + \mathbb{C}\mathbf{1}$.

Proof:

By assumption there are u and v in A such that

$$u^*u = v^*v = \mathbf{1}\,, \quad uu^* = \mathbf{1} - q\,, \quad vv^* = \mathbf{1} - q_0\,.$$

Let $\pi : A \to A/I$ be the quotient map and note that $\pi(u)$ and $\pi(v)$ are unitaries. If w is the partial isometry in A for which $w^*w = q_0$ and $ww^* = q$ (so that $w \in I$) then $uv^* + w$ is unitary in A.

Since $\pi(A)$ has K_1-injectivity by Lin, and since $[\pi(v^*uvu^*)] = 0$ in $K_1(\pi(A))$, we must have

$$\pi(v^*uvu^*) = \pi(u_0)$$

for some unitary $\pi(u_0)$ in the identity component of $\pi(\mathcal{U}(A))$. However, $\mathcal{U}_0(\pi(A)) = \pi(\mathcal{U}_0(A))$ so we may assume that $u \in \mathcal{U}_0(A)$. Consider now the unitary $w_1 = u_0(uv^* + w)$ and note that

$$\pi(w_1) = \pi(v^*uvu^*) = \pi(v^*u).$$

Therefore $v_1 = vw_1u^*$ is a partial isometry in A, and by computation

$$v_1^*v_1 = uu^* = \mathbf{1} - q \quad \text{and} \quad v_1v_1^* = vv^* = \mathbf{1} - q_0 .$$

Finally,

$$\pi(v_1) = \pi(v(v^*u)u^*) = \pi(\mathbf{1}) ,$$

so that $v_1 \in \tilde{I}$, as desired.

Theorem (3.1.11) [3]:

Every extremally rich C^*-algebra A of real rank zero has stable weak cancellation.

Proof:

The given data are stable so it suffices to show that A has weak cancellation.

We do this by verifying condition (ii) in Theorem (3.1.6), and since the given data are also hereditary it suffices to verify the condition for A alone, assuming that A is unital. Finally, using we may assume that the defect projection in $\mathbb{M}_2(A)$ has the form

$$(\mathbf{1} - v_1v_1^* - w_{12}w_{12}^*) \oplus (\mathbf{1} - v_2v_2^* - w_{21}w_{21}^*) ,$$

where $v_i \in \mathcal{E}(A)$ and $w_{ij} \in (p_iAq_j)$, for $p_i = \mathbf{1} - v_iv_i^*$ and $q_i = \mathbf{1} - v_i^*v_i, i, j = 1,2$.

Let J be the closed ideal of A generated by the two interesting projections $p_1 - w_{12}w_{12}^*$ and $p_2 - w_{21}w_{21}^*$. We have

$$(p_2 - w_{21}w_{21}^*)A(q_1 - w_{21}^*w_{21}) = 0$$

since $w_{21} \in \mathcal{E}(p_2Aq_1)$. Moreover,

$$(p_1 - w_{12}w_{12}^*)A(q_1 - w_{21}^*w_{21}) = 0$$

since already $p_1Aq_1 = 0$. It follows that $q_1 - w_{21}^*w_{21} \in J^\perp$. Since J is isomorphic to its image in $A/J^\perp$ we may replace A by $A/J^\perp$ without changing notation. In

other words we may assume that $J^{\perp} = 0$. In that case $q_1 - w_{21}^* w_{21} = 0$, so if we put $q_0 = w_{21} w_{21}^*$ we have $q_1 \sim q_0 \leq p_2$.

Let I be the closed ideal of A generated by q_1 (and q_0). Since $p_1 A q_1 = 0$ we see that $p_1 \in I^{\perp}$. But since $q_1 \lesssim p_2$ and $q_2 A p_2 = 0$ also $q_2 \in I^{\perp}$. With $\pi : A \to A/I^{\perp}$ the quotient map this means that the three projections

$$\pi(p_2)\,, \quad \pi(q_1)\,, \quad \pi(q_0)$$

satisfy the assumptions of Lemma (3.1.9). Consequently $\pi(\mathbf{1} - q_0) \sim \pi(\mathbf{1}) \sim \pi(\mathbf{1} - q_1)$. But now Lemma (3.1.10) applies to show that $\pi(\mathbf{1} - q_1) = u^* u$ and $\pi(\mathbf{1} - q_0) = u u^*$ for some partial isometry u in $\pi(\tilde{I})$. Since $\pi(\tilde{I})$ is isomorphic to $\tilde{I}$ this means that $\mathbf{1} - q_1 \sim \mathbf{1} - q_0$ in $\tilde{I}$.

Since we already had

$$\mathbf{1} - q_1 = v_1^* v_1 \sim v_1 v_1^* = \mathbf{1} - p_1\,,$$

and since $p_2 - q_0 \leq 1 - q_0$, we conclude that $p_2 - q_0 \sim p_0$ for some projection $p_0 \leq \mathbf{1} - p_1$. But then

$$p = p_1 - w_{12} w_{12}^* + p_0$$

is a projection in A and

$$p \sim (p_1 - w_{12} w_{12}^*) \oplus (p_2 - w_{21} w_{21}^*)\,.$$

Since we will prove later that all extremally rich C^*-algebras with weak cancellation also have K_1-surjectivity, the use of K_1-surjectivity in the hypothesis of the next lemma is just a temporary expedient.

Lemma (3.1.12) [3]:

Let A be an extremally rich C^*-algebra and I a closed ideal of A. Assume that both I and A/I have weak cancellation and that eAe/eIe has K_1-surjectivity for every projection e in A. Then A has weak cancellation.

Proof:

Let p and q be projections in A which generate the same closed ideal J in A and have the same image in $K_0(J)$. Since weak cancellation passes to ideals we may

replace A by J, i.e. we may assume that p and q are full projections. If $\pi : A \to A/I$ denotes the quotient map then the conditions above are also satisfied for $\pi(p)$ and $\pi(q)$ relative to $\pi(A)$ and $K_0(\pi(A))$. By hypothesis there is therefore an element u in pAq such that

$$\pi(uu^*) = \pi(p) \quad \text{and} \quad \pi(u^*u) = \pi(q) .$$

Since pAq is extremally rich by Proposition (3.1.3), and $\pi(u) \in \mathcal{E}(\pi(pAq))$, we can choose u in $\mathcal{E}(pAq)$. Let

$$p_1 = p - uu^* , \quad q_1 = q - u^*u , \quad e_1 = u^*u .$$

Then p_1 and q_1 belong to I and we can apply Lemma (3.1.5) to obtain a full projection e_2 in A, projections p_2 and q_2 in I such that $[p_2] = [q_2]$ in $K_0(I)$, and such that

$$p \sim p_2 \oplus e_2 , \quad q = q_2 + e_2 .$$

Next, an argument similar to part of the proof of (iii) ⇒ (i) in Theorem (3.1.6) yields

$$p \sim p_3 \oplus e_3 , \qquad q = q_3 + e_3 ,$$

where e_3 is a full projection in A and p_3, q_3 are projections in $\mathcal{D}(I)$ such that $[p_3] = [q_3]$ in $K_0(\mathcal{D}(I))$, as follows:

Let $\rho : I \to I/\mathcal{D}(I)$ denote the quotient map. From $[\rho(p_2)] = [\rho(q_2)]$ and $\mathrm{tsr}(\rho(I)) = 1$ we conclude that $\rho(p_2) \sim \rho(q_2)$. Then there is v in $\mathcal{E}(p_2 I q_2)$ such that $\rho(v)$ implements this equivalence. Thus

$$p_2 \sim p_3' \oplus e_3' , \quad q_2 = q_3' + e_3' ,$$

where $e_3' = v^*v$, $p_3' = p_2 - vv^*$, $q_3' = q_2 - v^*v$, and $p_3', q_3' \in \mathcal{D}(I)$. Since $[p_3'] - [q_3']$ is in the kernel of $\iota_* *: K_0(\mathcal{D}(I)) \to K_0(I)$, there is α in $K_1(I/\mathcal{D}(I))$ with $\partial_1 \alpha = [p_3'] - [q_3']$. Since e_2 is full in A, $e_2 I e_2$ is a full hereditary C^*-subalgebra of I and $\rho(e_2 I e_2)$ is full in $I/\mathcal{D}(I)$. Thus $\alpha = [\rho(v_0)]$, where $\rho(v_0)$ is unitary in $\rho(e_2 \tilde{I} e_2)$, and v_0 may be taken in $\mathcal{E}(e_2 \tilde{I} e_2)$. Hence

$$p_3' \oplus e_2 \sim p_3 \oplus e_3'' , \quad q_3' + e_2 = q_3 + e_3'' ,$$

where $e_3'' = v_0^* v_0, p_3 \sim p_3' \oplus (e_2 - v_0 v_0^*), q_3 = q_3' + e_2 - v_0^* v$, and e_3'' is full in A. Now let $e_3 = e_3' + e_3''$.

Finally we note that $\mathcal{D}(I) = (\cup K_j)^=$, where $\{K_j\}$ is an upward directed family of ideals each of which is generated by finitely many defect projections $e_3 - ww^*, w \in \mathcal{E}(e_3 \tilde{I} e_3)$. Here we are identifying $\tilde{I}$ with $I + \mathbb{C}\mathbf{1}_{\tilde{A}}$. Since K-theory is compatible with direct limits, and since every projection in $\mathcal{D}(I)$ is contained in some K_j, there are j_0 and a finite collection, $w_1, \dots w_n$, in $\mathcal{E}(e_3 \tilde{I} e_3)$ such that $e_3 - w_1 w_1^*, \dots, e_3 - w_n w_n^*$ generate $K_{j_0}, p_3, q_3 \in K_{j_0}$, and $[p_3] = [q_3]$ in $K_0(K_{j_0})$. Now an argument similar to part of the proof shows that there is a projection e_4 in $e_3 \mathcal{D}(I) e_3$ which generates K_{j_0} (as an ideal). Since I has weak cancellation, it follows that $p_3 \oplus e_4 \sim q_3 + e_4$, whence $p \sim p_3 \oplus e_3 \sim q_3 + e_3 = q$. (Since $p_3 \oplus e_4 \precsim p$, we are not here assuming stable weak cancellation for I).

Theorem (3.1.6) (iii) implies that the extremally rich C^*-algebra A has weak cancellation if the set of equivalence classes of projections in B is "convex" in a suitable sense for all B of the form pAp. Any extra hypotheses that guarantee this convexity imply additional positive results.

Corollary (3.1.13) [3]:

If A is an extremally rich C^*-algebra with weak cancellation, then $\tilde{A}$ has weak cancellation.

Proposition (3.1.14) [3]:

If I is a closed ideal in an extremally rich C^*-algebra A and if A has weak cancellation, then A/I has weak cancellation.

Proof:

Let p be a projection in A/I and put $B = p(A/I)p$. Then let C be the inverse image of B in A. Since C is a hereditary C^*-subalgebra of A it has weak cancellation and is extremally rich. (But if the projection p does not lift we may never find a unital substitute for C). It follows that extreme points lift from B to $\tilde{C}$, so if $\{u_1, \dots, u_n\}$ is a finite subset of $\mathcal{E}(B)$ it can be lifted to a finite subset $\{w_1, \dots, w_n\}$ in $\mathcal{E}(\tilde{C})$. Since $\tilde{C}$ has weak cancellation by Corollary (3.1.14) we can

find a projection q in $\tilde{C}$ (actually in $\mathcal{D}(\tilde{C}) = \mathcal{D}(C)$) such that $q \sim \bigoplus (1 - w_i w_i^*)$. It follows that

$$\pi(q) \sim \bigoplus (p - u_i u_i^*),$$

where $\pi : \tilde{C} \to B$ is the quotient map, whence A/I has weak cancellation by Theorem (3.1.6).

Proposition (3.1.15) [3]:

If A is an extremally rich C^*-algebra with weak cancellation, then:

(i) There is for every projection p in $\mathcal{D}(A \otimes \mathbb{K})$ $(= \mathcal{D}(A) \otimes \mathbb{K})$ a projection q in $\mathcal{D}(A)$ such that $p \sim q$.

(ii) If p is a projection in $\mathcal{D}(A) \otimes \mathbb{K}$, then there is an infinite set $\{p_n\}$ of mutually orthogonal projections in $\mathcal{D}(A)$ such that $p_n \sim p, \forall n$.

(iii) If $\mathcal{D}(A)$ is σ-unital, then $\mathcal{D}(A)$ has a full, hereditary, stable, σ-unital C^*-subalgebra B.

Proof:

(i) By Corollary (3.1.14) we may assume that A is unital. Since $\mathcal{D}(A)$ is generated as an ideal by the set

$$\mathcal{D} = \{\mathbf{1} - uu^* | u \in \mathcal{E}(A)\}$$

it follows that $\mathcal{D}(A \otimes \mathbb{K})$ is generated by the set $\mathcal{D} \otimes e_{11}$. There is therefore a finite subset $\{u_i\}$ in $\mathcal{E}(A)$ such that

$$p \precsim \bigoplus (\mathbf{1} - uu^*).$$

Applying condition (iii) of Theorem (3.1.6) with $B = A$ we find a projection q_0 in A with $q_0 \sim \bigoplus (\mathbf{1} - u_i u_i^*)$. Thus $p \sim q \leq q_0$ and evidently

$$q \in \mathcal{D}(A \otimes \mathbb{K}) \cap A = \mathcal{D}(A).$$

(ii) We do a recursive construction. At step n we construct $n + 1$ mutually orthogonal and equivalent projections, $p_1, \dots, p_n, q_n$. The first step is done by applying part (i) with $2p$ in place of p. At step $n + 1$ we apply part (i) to $B = \mathrm{her}(\mathbf{1} - s_n)$, where $s_n = p_1 + \cdots + p_n$. Since B is full in $A, \mathcal{D}(B) = B \cap \mathcal{D}(A)$, and thus $q_n \in \mathcal{D}(B)$.

Hence, we can obtain $p_n + 1$ and $q_n + 1$ by applying part (i) with $2q_n$ in place of p.

(iii) The hypothesis implies that there is a countable set, $\{p_n\}$, of projections in $\mathcal{D}(A)$ such that $\mathcal{D}(A) = \mathrm{id}(\{p_n\})$. Then the same technique as in part (ii) produces a countable set, $\{q_m\}$, of mutually orthogonal projections which consists of infinitely many equivalent copies of each p_n. Then take $B = \mathrm{her}(\{q_m\})$.

Corollary (3.1.16) [3]:

If A is an extremally rich C^*-algebra with weak cancellation, then A also has stable weak cancellation.

Proof:

We apply Theorem (3.1.6) to prove that $\mathcal{D}(A) \otimes \mathbb{K}$ has weak cancellation. This is immediate since each subalgebra of the form $p(\mathcal{D}(A) \otimes \mathbb{K})p$ is isomorphic to an algebra $q\mathcal{D}(A)q$. Then Lemma (3.1.12) implies $A \otimes \mathbb{K}$ has weak cancellation, since $\mathrm{tsr}((A \otimes \mathbb{K})/(\mathcal{D}(A) \otimes \mathbb{K})) = 1$.

We have

Corollary (3.1.17) [3]:

If A is an extremally rich C^*-algebra with weak cancellation and if p is a projection in A such that $\mathcal{D}(pAp) = pAp$ (equivalently $\mathcal{D}(I) = I$ where $I = \mathrm{id}(p)$), then p is properly infinite. In particular, the hypotheses of Corollary (3.1.8) imply that every defect projection is properly infinite.

Proposition (3.1.18) [3]:

If A is a unital C^*-algebra which is extremally rich and has weak cancellation there is for each n and every u in $\mathcal{E}(\mathbb{M}_n(A))$ a v in $\mathcal{E}(A)$ such that

$$\mathbf{1} - vv^* \sim \mathbf{1}_n - uu^* \quad \text{and} \quad \mathbf{1} - v^*v \sim \mathbf{1}_n - u^*u.$$

Proof:

By Proposition (3.1.16) we can find projections p_1 and q_1 in $\mathcal{D}(A)$ such that

$$2(\mathbf{1}_n uu^*) \sim p_1 \quad \text{and} \quad 2(\mathbf{1} - uu^*) \sim q_1.$$

Thus for subprojections $p \leq p_1$ and $q \leq q_1$ we have

$$\mathbf{1}_n - uu^* \sim p \precsim \mathbf{1} - p \text{ and } \mathbf{1}_n - u^*u \sim q \precsim \mathbf{1} - q.$$

But then both $1 - p$ and $1 - q$ generate A as an ideal, and evidently $[1 - p] = [1 - q]$ in $K_0(A)$ (since $[p] = [q]$). By weak cancellation $\mathbf{1} - p = vv^*$ and $\mathbf{1} - q = v^*v$ for some partial isometry v in A. But since $\mathbf{1}_n - uu^*$ and $\mathbf{1}_n - u^*u$ are centrally orthogonal in $\mathbb{M}_n(A)$ we see that p and q are centrally orthogonal in A, whence $v \in \mathcal{E}(A)$.

Corollary (3.1.19) [3]:

With A as above, if $\mathbb{M}_n(A)$ contains a proper isometry so does A. (Finiteness implies stable finiteness).

Section (3.2): K_1-Surjectivity and Good Index Theory K_1-Injectivity and K_0-Surjectivity

In the next three lemmas we shall be concerned with a closed ideal I in a unital C^*-algebra A and $\tilde{I}$ will denote $I + \mathbb{C}1$. (The possibility $I = A$ is not excluded). In $\mathbb{M}_2(A)$ we consider the unital C^*-subalgebra B consisting of matrices of the form

$$\begin{pmatrix} a & x_{12} \\ x_{21} & \lambda\mathbf{1} + x_{24} \end{pmatrix}, \lambda \in \mathbb{C}, x_{ij} \in I, a \in A.$$

Note that the subset of B determined by $\lambda = 0$ is an ideal of B which is Rieffel-Morita equivalent to A. (If $I = A$ this ideal is the whole of B). Thus every ideal J of A gives rise to an ideal of B (which is determined by $a \in J, x_{ij} \in I \cap J$, and $\lambda = 0$). We shall commit a slight abuse of notation and denote both ideals by the same symbol. We shall denote by B_{00}^{-1} the connected component containing 1 of the group of invertible elements in $\mathbb{M}_2(\tilde{I}) \cap B$.

Lemma (3.2.1) [3]:

Assume $\tilde{I}$ has weak cancellation and $v, w \in \tilde{I}$. If $v \in \mathcal{E}(\tilde{I})$ and $v^*v + w^*w = \mathbf{1}$, and if furthermore $\begin{pmatrix} w \\ v \end{pmatrix}$ is the second column of a left invertible element of B, then $\mathbf{1} - ww^* \sim vv^*$ in $\tilde{I}$.

Proof:

Since $v \in \mathcal{E}(\tilde{I})$ we know that w is a partial isometry $(w^*w = \mathbf{1} - v^*v)$ and

$$[\mathbf{1} - ww^*] = [\mathbf{1}] - [w^*w] = [v^*v] = [vv^*]$$

in $K_0(\tilde{I})$. Thus the lemma follows by weak cancellation if we can show that both projections generate $\tilde{I}$ as an ideal.

In fact it is equivalent to show that they generate A as an ideal. The condition for a projection p in $\tilde{I}$ to generate $\tilde{I}$ as an ideal is twofold:

(i) Either $p \notin I$ or $I = A$.
(ii) The algebra pIp generates I as an ideal.

But if p generates A as an ideal, then (i) is obviously true, and also pAp is a full hereditary C^*-subalgebra of A. And from this we easily deduce that pIp $(= pAp \cap I)$ is full in I.

Now the fullness of vv^* follows from Lemma (3.1.4). Let J denote the closed ideal of A generated by $\mathbf{1} - ww^*$. If $J \neq A$ we pass to A/J without changing notation. The conditions on v and w are unchanged, but now $ww^* = \mathbf{1}$. Since $v \in \mathcal{E}(A)$ we have $\mathbf{1} - vv^*$ centrally orthogonal to w^*w $(= \mathbf{1} - v^*v)$, and this now forces $\mathbf{1} - vv^* = 0$. Thus both v and w are co–isometries. But this contradicts the last part of the hypothesis.

Lemma (3.2.2) [3]:

Suppose that b is a left invertible element of B and that $\tilde{I}$ is extremally rich with weak cancellation. There is then an element b_0 in B_{00}^{-1} such that $b_0 b = \begin{pmatrix} a & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ for some left invertible element a in A.

Proof:

Write $b = \begin{pmatrix} \star & y \\ \star & x \end{pmatrix}$. By assumption $b^* b$ is invertible so $y^* y + x^* x \in (\tilde{I})_+^{-1}$.If $y^* = u_1 |y^*|$ is the polar decomposition of y^* and f is a function in $C_0\big((0, \|y\|]\big)$ such that $|f(t)t - t|$ is small for all t in $[0, \|y\|]$ then with $y_1 = f(|y^*|)|y^*|u_1^*$ we still have $y_1^* y_1 + x^* x \in (\tilde{I})_+^{-1}$. There is an element v_1 in $\mathcal{E}(\tilde{I})$ such that $v_1^* y_1 + x$ is quasi–invertible. Thus $v_1^* y_1 + x = ev$ for some v in $\mathcal{E}(\tilde{I})$ and e in $(\tilde{I})_+^{-1}$. Multiplying b from the left by the element

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ v_1^* f|y^*| & \mathbf{1} \end{pmatrix}$$

in B_{00}^{-1} gives a matrix $\begin{pmatrix} \star & y \\ \star & v \end{pmatrix}$. Left multiplication by $\begin{pmatrix} \mathbf{1} & -yv^* \\ 0 & \mathbf{1} \end{pmatrix}$, also in B_{00}^{-1}, gives the matrix $\begin{pmatrix} \star & z \\ \star & v \end{pmatrix}$, where $z = y(\mathbf{1} - v^* v)$. Since we still have $z^* z + v^* v$ invertible and $v^* v$ is a projection this implies that $z^* z \geq \varepsilon(\mathbf{1} - v^* v)$ for some $\varepsilon > 0$. If therefore $z = w|z|$ is the polar decomposition of z then $w \in \tilde{I}$ (actually $w \in I$) and we consider $wh(|z|)$ in I for some h in $C_0\big((0, \|z\|]\big)$ such that $h(|z|)|z| = \mathbf{1} - v^* v$. Left multiplication with

$$\begin{pmatrix} \mathbf{1} - ww^* + h(|z^*|) & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

which belongs to B_{00}^{-1}, transforms the matrix $\begin{pmatrix} \star & z \\ \star & v \end{pmatrix}$ into $\begin{pmatrix} \star & w \\ \star & v \end{pmatrix}$, where now $w^* w = \mathbf{1} - v^* v$.

The elements v, w now satisfy the conditions in Lemma (3.2.1) since $\begin{pmatrix} \star & w \\ \star & v \end{pmatrix}$ is left invertible. Thus $v^* v = w_1^* w_1$ and $\mathbf{1} - ww^* = w_1 w_1^*$ for some partial isometry w_1 in $\tilde{I}$. Since $v \in \mathcal{E}(\tilde{I})$ it has the form $v = \lambda \mathbf{1} + c$ for some c in I and λ in C with

$|\lambda| = 1$. After left multiplication with $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$ we can assume that $\lambda = 1$ without any other changes. Since $w^*w + w_1^*w_1 = \mathbf{1}$ and $ww^* + w_1w_1^* = \mathbf{1}$ the element $u = w + w_1$ is unitary in $\tilde{I}$. For $0 \leq t \leq 1$ define

$$b_t = \begin{pmatrix} \mathbf{1} & -u \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ t(\mathbf{1} - v)u^* & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & w_1v^* \\ 0 & \mathbf{1} \end{pmatrix}.$$

Since $\mathbf{1} - v \in I$ it follows by routine calculations that $b_t \in B_{00}^{-1}$ for all t. However,

$$b_1 = \begin{pmatrix} \star & w \\ \star & v \end{pmatrix} = \begin{pmatrix} a & 0 \\ d & \mathbf{1} \end{pmatrix}$$

for some elements a in A and d in I. Since the original element b was left invertible it now follows that a must be left invertible (in A); so if $a1a = \mathbf{1}$ we perform the final left multiplication by the element

$$\begin{pmatrix} \mathbf{1} & 0 \\ -da_1 & \mathbf{1} \end{pmatrix} \text{ in } B_{00}^{-1}$$

to obtain the desired solution $\begin{pmatrix} a & 0 \\ 0 & \mathbf{1} \end{pmatrix}$.

Lemma (3.2.3) [3]:

Suppose that b is a quasi–invertible element in B and that $\tilde{I}$ is extremally rich with weak cancellation. We can then find b_1 and b_2 in B_{00}^{-1} such that $b_1bb_2 = \begin{pmatrix} a & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ for some quasi–invertible element a in A.

Proof:

Without changing notation, we replace A with $A \oplus \mathbb{C}, I$ with $I \oplus 0$, and b with $b \oplus \lambda\mathbf{1}$. This avoids an annoying but essentially trivial complication that would occur if $I \neq A$ but $(I + J_2)/J_2 = A/J_2$.

There are orthogonal closed ideals J_1 and J_2 of B with corresponding quotient morphisms π_1 and π_2 such that $\pi_1(b)$ is left invertible and $\pi_2(b)$ is right invertible. Using Lemma (3.2.2) with $\pi_2(B), \pi_2(I)$ and $\pi_2(b^*)$ in place of B, I and b, which is legitimate by Proposition (3.1.15), we find an element $\pi_2(b_2)$ in $\pi_2 B_{00}^{-1}$ such that

$$\pi_2(b_2)\pi_2(b) = \begin{pmatrix} \pi_2(a_2) & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Since invertible elements in the connected component of the identity are always liftable we may assume that $b_2 \in B_{00}^{-1}$, and we can write

$$bb_2 = \begin{pmatrix} a_2 & x_{12} \\ x_{21} & \mathbf{1} + x_{22} \end{pmatrix}, \qquad \text{with } x_{ij} \text{ in } I_2 = I \cap J_2 .$$

Define the C^*-algebra

$$B_2 = \begin{pmatrix} A & I_2 \\ I_2 & \tilde{I}_2 \end{pmatrix} \subset B$$

and note that the restriction of the morphism π_1 to B_2 is an isomorphism except possibly at the $(1,1)$–corner since $J_1 \cap J_2 = \{0\}$. As $\pi_1(I_2)$ $(= I_2)$ has weak cancellation, we can apply Lemma (3.2.2) with $\pi_1(B_2), \pi_1(I_2)$ and $\pi_1(bb_2)$ in place of B, I and b; and find an element $\pi_1(b_1)$ in $\pi_1(B_2)_{00}^{-1}$ such that

$$\pi_2(b_1)\pi_1(bb_2) = \begin{pmatrix} \pi_1(a_1) & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Again we may assume that $b_1 \in (B_2)_{00}^{-1}$, but since $\pi_1|B_2$ is an isomorphism except at the $(1,1)$–corner this implies that

$$b_1 b b_2 = \begin{pmatrix} a & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

for some a in A. Necessarily then $a \in A_q^{-1}$, as desired.

Theorem (3.2.4) [3]:

Every extremally rich C^*-algebra with weak cancellation has K_1-surjectivity.

Theorem (3.2.5) [3]:

In the category of extremally rich C^*-algebras the subcategory of algebras that also have weak cancellation is stable under the formation of quotients, hereditary C^*-subalgebras (in particular ideals), matrix tensoring, Rieffel-Morita equivalence, arbitrary extensions, and inductive limits. Also if the extremally rich C^*-algebra A has a composition series of ideals, $\{I_\alpha | 0 \leq \alpha \leq \lambda\}$, such that $I_0 = 0, I_\lambda = A$, and $I_{\alpha+1}/I_\alpha$ has weak cancellation for each $\alpha < \lambda$, then A has weak cancellation.

Proof:

The wording of the result reflects the fact that the category of extremally rich C^*-algebras is not itself stable under (arbitrary) extensions, and stable only under extreme point preserving inductive limits.

We defined two analogues of K_1 which use extremal partial isometries in place of unitaries, or equivalently, quasi-invertibles in place of invertibles. One of these, denoted $\mathcal{E}_\infty(A)$, takes two extremals, each in some matrix algebra over $\tilde{A}$, to be equivalent if $u \oplus \mathbf{1}_k$ is homotopic to $v \oplus \mathbf{1}_t$ in $\mathcal{E}(\mathbb{M}_n(\tilde{A}))$ for some (large) n and suitable k, l. The equivalence relation for $K_e(A)$ is coarser. For example, the defect ideals, $I = \mathrm{id}(1 - uu^*)$ and $J = \mathrm{id}(1 - u^*u)$, are invariants of the K_e−class of u, as are also the classes of $\mathbf{1} - uu^*$ and $\mathbf{1} - u^*u$ in $K_0(I)$ and $K_0(J)$, respectively. But even the Murray-von Neumann equivalence classes of $\mathbf{1} - uu^*$ and $\mathbf{1} - u^*u$ are invariants of the $\mathcal{E}_\infty$− class of u. If this were the only difference between K_e and $\mathcal{E}_\infty$, then obviously weak cancellation would imply $K_e = \mathcal{E}_\infty$. Although the difference is more extensive, we shall prove in the next section that $K_e(A) = \mathcal{E}_\infty(A)$ when A is extremally rich with weak cancellation. Neither K_e nor $\mathcal{E}_\infty$ is a group, but both contain K_1 and the group K_1 acts on both. The next result is that extremal richness with weak cancellation implies K_e-surjectivity. The same proof shows "$\mathcal{E}_\infty$-surjectivity", a property which is formally stronger, but equivalent in this situation.

Theorem (3.2.6) [3]:

Every extremally rich C^*-algebra with weak cancellation has K_e-surjectivity.

Theorem (3.2.7) [3]:

Every extremally rich C^*-algebra with weak cancellation has good index theory.

Proof:

We are given a unital C^*-algebra A, an ideal I which is extremally rich with weak cancellation, the quotient map $\pi : A \to A/I$, a unitary $\overline{u}$ in A/I, and α in $K_1(A)$ such that $\pi^*(\alpha) = [u]$ in $K_1(A/I)$. We wish to find a unitary u in A such that $\pi(u) = \overline{u}$. Let π_n denote the natural map from $\mathbb{M}_n(A)$ to $\mathbb{M}_n(A/I)$. We may choose n, a power of 2, so that there is a unitary v in $\mathbb{M}_n(A)$ which belongs to the class α such that $\pi_n(v)$ is homotopic to $\overline{u} \oplus \mathbf{1}_n - \mathbf{1}$ in $\mathcal{U}(\mathbb{M}_n(A/I))$. Then

$(\overline{u} \oplus \mathbf{1}_{n-1})(\pi_n(v))^{-1}$ can be lifted to w in $\mathcal{U}_0(\mathbb{M}_n(A))$. We replace v with wv, without changing notation, and thus achieve that $\pi_n(v) = \overline{u} \oplus \mathbf{1}_{n-1}$.

Thus v belongs to the algebra called B in connection with Lemma (3.2.2), with $\mathbb{M}_{n/2}(A)$ in place of A, and that Lemma provides b_0 in B_{00}^{-1} such that $b_0 b$ has the form $v' \oplus \mathbf{1}_{n/2}$. The $(1,1)-$ corner of b_0 is congruent to a scalar modulo I, and clearly we may assume this scalar is 1. Thus v' satisfies the same properties as v, relative to $n/2$, except for the inconsequential fact that it is only invertible instead of unitary. We may remedy this, if desired, with a polar decomposition. Continuing in this way, we attain our goal.

Proposition (3.2.8) [3]:

Assume u and v are extremal partial isometries in matrix algebras over the unital C^*-algebra A which lie in the same K_e-class. If the defect ideals are extremally rich with weak cancellation, then u and v lie in the same $\mathcal{E}_\infty$-class.

Proof:

Let I and J be the left and right defect ideals, which are the same for u and v. (Here we regard the defect ideals as ideals of A, using the identification of ideals of A with ideals of $\mathbb{M}_n(A)$). By replacing A with $\mathbb{M}_n(A)$ for suitable n, u with $u \oplus \mathbf{1}_{n-k}, v$ with $v \oplus \mathbf{1}_{n-l}$, and changing notation, we may assume u and v are in A. Let $\pi : A \to A/(I+J), \rho : A \to A/J$, and $\lambda : A \to A/I$ be the quotient maps.

Since $\pi(w) = \pi(v)\pi(u)^{-1}$ is a unitary whose class in $K_1(A/(I+J))$ is 0, it follows from Theorems (3.2.4) and (3.2.6) that we may take w to be a unitary whose class in $K_1(A)$ is 0. Thus $\pi(v) = \pi(u')$, where $u' = wu$. We now construct unitaries $w_1 \in \mathbf{1} + I$ and $w_2 \in \mathbf{1} + J$ such that the classes of w_1 and w_2 in $K_1(I)$ and $K_1(J)$, respectively, are trivial, $\rho(v) = \rho(w_1 u')$, and $\lambda(v) = \lambda(u' w_2)$. Once this is done, we have that $v = w_1 u' w_2 = w_1 w u w_2$, since $I \cap J = 0$. Since all of the w's are trivial in $K_1(A)$, it follows that u and v are equivalent in $\mathcal{E}_\infty(A)$.

To construct w_1, we may replace A with A/J, since ρ is an isomorphism on I. Then u' and v are isometries which agree modulo I. The defect projections $p = \mathbf{1} - u'u'^*$ and $q = \mathbf{1} - vv^*$ each generate the ideal I and have the same class in $K_0(I)$. Thus there is x such that $x^*x = p$ and $x^*x = q$. Then $w' = x + vu'^*$ is a unitary in $\mathbf{1} + I$, and $v = w'u'$. Now since $K_1(pIp) = K_1(I)$ and pIp has K_1-

surjectivity, there is y in $\mathcal{U}(pIp)$ which induces the same class as w' in $K_1(I)$. So we can take $w = w'(\mathbf{1} - p + y^*)$. The construction of w_2 is similar.

Corollary (3.2.9) [3]:

If A is extremally rich with weak cancellation, then $K_e(A) = \mathcal{E}_\infty(A)$.

Extremal Analogues of Good Index Theory (3.2.10) [3]:

Four different properties are listed below. In all cases K is an ideal in a unital C^*-algebra A and $\pi_n : \mathbb{M}_n(A) \to \mathbb{M}_n(A/K)$ are the quotient maps.

(i) If $\overline{u} \in \mathcal{E}(A/K), v \in E(\mathbb{M}_n(A))$, and $[\pi_n(v)]_{K_e} = [\overline{u}]_{K_e}$, then there is $u \in \mathcal{E}(A)$ such that $\pi(u) = \overline{u}$ and $[u]_{K_e} = [v]_{K_e}$.

(ii) If $\overline{u} \in \mathcal{E}(A/K), v \in \mathcal{E}(\mathbb{M}_n(A))$, and $[\pi_n(v)]_{\mathcal{E}_\infty} = [\overline{u}]_{\mathcal{E}_\infty}$, then there is $u \in \mathcal{E}(A)$ such that $\pi(u) = \overline{u}$ and $[u]_{\mathcal{E}_\infty} = [v]_{\mathcal{E}_\infty}$.

(iii) If $\overline{u} \in \mathcal{E}(A/K), v \in \mathcal{E}(\mathbb{M}_n(A))$, and $[\pi_n(v)]_{K_e} = [\overline{u}]_{K_e}$, then there is $u \in \mathcal{E}(A)$ such that $\pi(u) = \overline{u}$.

(iv) If $\overline{u} \in \mathcal{E}(A/K), v \in E(\mathbb{M}_n(A))$, and $[\pi(v)]_{\mathcal{E}_\infty} = [\overline{u}]_{\mathcal{E}_\infty}$, then there is $u \in \mathcal{E}(A)$ such that $\pi(u) = \overline{u}$.

Theorem (3.2.11) [3]:

Let K be a closed ideal in a unital C^*-algebra A and assume that K is extremally rich with weak cancellation. If $\overline{u} \in \mathcal{E}(A/K), v \in E\big(\mathbb{M}_n(A)\big)$, and $[\pi(v)]_{\mathcal{E}_\infty} = [\overline{u}]_{\mathcal{E}_\infty}$, where $\pi_n : \mathbb{M}_n(A) \to \mathbb{M}_n(A/K)$ is the quotient map, then there is $u \in \mathcal{E}(A)$ such that $\pi_1(u) = \overline{u}$ and $[u]_{\mathcal{E}_\infty} = [v]_{\mathcal{E}_\infty}$.

Proof:

The proof is almost identical to that of Theorem (3.2.7), except that we use Lemma (3.2.3) instead of (3.2.2) We may assume that n is a power of 2 and that $\pi_n(v)$ is homotopic to $\overline{u} \oplus \mathbf{1}_{n-1}$ in $\mathcal{E}(\mathbb{M}_n(A/K))$. There are $\pi_n(w_1)$ and $\pi_n(w_2)$ in $\mathcal{U}_0(\mathbb{M}_n(A/K))$ such that $\overline{u} \oplus \mathbf{1}_{n-1} = \pi_n(w_1)\pi_n(v)\pi_n(w_2)$. We may assume $w_1, w_2 \in \mathcal{U}_0(\mathbb{M}_n(A))$. Then without changing notation, we replace v with $w_1 v w_2$ to achieve $\pi_n(v) = \overline{u} \oplus \mathbf{1}_{n-1}$. Then coninue as in (3.2.7).

Corollary (3.2.12) [3]:

Let K be a closed ideal in a unital C^*-algebra A and assume that K is extremally rich with weak cancellation. If $\overline{u} \in \mathcal{E}(A/K), v \in \mathcal{E}(\mathbb{M}_n(A))$, and $[\pi_n(v)]_{K_e} = [u]_{K_e}$, where $\pi : \mathbb{M}_n(A) \to \mathbb{M}_n(A/K)$ is the quotient map, and if the defect ideals of u are extremally rich with weak cancellation, then there is u in $\mathcal{E}(A)$ such that $\pi_1(u) = \overline{u}$ and $[u]_{\mathcal{E}_\infty} = [v]_{\mathcal{E}_\infty}$.

Corollary (3.2.13) [3]:

Let K be a closed ideal in a unital C^*-algebra A and assume that K is extremally rich with weak cancellation. If $\overline{u} \in \mathcal{U}(A/K), v \in \mathcal{E}(\mathbb{M}_n(A))$, the defect ideals of v are in K, and $[\pi_n(v)]_{K_1} = [\overline{u}]_{K_1}$, where $\pi : \mathbb{M}_n(A) \to \mathbb{M}_n(A/K)$ is the quotient map, then there is $u \in \mathcal{E}(A)$ such that $\pi_1(u) = \overline{u}$ and $[u]_{K_e} = [v]_{K_e}$.

Corollary (3.2.14) [3]:

Let K be a closed ideal in a unital C^*-algebra A and assume that K is extremally rich with weak cancellation. If $\overline{u} \in \mathcal{U}(A/K)$, and if $[u]_{K_1}$ is in the image of $K_e(A)$, then u can be lifted to $\mathcal{E}(A)$.

Proposition (3.2.15) [3]:

Let K be a closed ideal in a unital C^*-algebra A and assume that K is extremally rich with weak cancellation. If $\overline{u} \in \mathcal{E}(A/K), v \in \mathcal{E}(\mathbb{M}_n(A))$, and $[\pi_n(v)]_{K_e} = [\overline{u}]_{K_e}$, where $\pi : \mathbb{M}_n(A) \to \mathbb{M}_n(A/K)$ is the quotient map, and if $(I + J) \cap K = 0$, where I and J are the defect ideals of v, then there is $u \in \mathcal{E}(A)$ such that $\pi_1(u) = \overline{u}$ and $[u]_{K_e} = [v]_{K_e}$.

Proof:

Consider the pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & A/(I+J) \\ \pi \downarrow & & \downarrow \mathcal{T} \\ A/K & \xrightarrow[\sigma]{} & A/(I+J+K) \end{array}$$

where the maps are the obvious ones and $\pi = \pi_1$. Then $\sigma(u)$ is a unitary whose K_1-class is lifted by $[\rho_n(v)] \in K_1(A/(I+J))$. By Theorems (3.2.7) and (3.2.4)

there is w in $\mathcal{U}(A/(I+J))$ such that $\tau(w) = \sigma(\overline{u})$ and $[w] = [\rho_n(v)]$ in $K_1(A/(I+J))$. Thus there is u in A such that $\pi(u) = \overline{u}$ and $\rho(u) = w$, whence $u \in \mathcal{E}(A)$. Now the defect ideals of u are contained in $I + J$ and map under π to the defect ideals of u, the defect ideals of u are the same as those of $\pi_n(v)$, and $\pi_{|I+J}$ is an isomorphism. Thus u also has defect ideals I and J. Then it follows that $[u]_{K_e} = [v]_{K_e}$.

Remarks (3.2.16) [3]:

(i) The relations of the results in this section to classical index theory become clearer if reformulated in an equivalent way. Thus for good index theory we would start with x in A such that $\pi(x)$ is invertible in A/K (such an x is called a Fredholm element relative to K) and seek a K-perturbation of x which is invertible. Similarly, for the extremal analogues of good index theory we would start with x such that $\pi(x) \in (A/K)_q^{-1}$ (such an x was called quasi-Fredholm) and seek a K-perturbation in A_q^{-1}. Quasi-Fredholm elements are meant to be analogous to classical semi-Fredholm operators (but we have used the name quasi-invertible, not semi-invertible). Also, instead of hypothesizing v in $\mathcal{E}(\mathbb{M}_n(A))$, we would hypothesize a class α in $K_e(A)$. Furthermore, we also defined an index space, $\mathsf{Inde}(K)$, which is the orbit space of $K_e(A/K)$ under the image of $K_1(A)$. The existence of α could be reformulated in terms of the index in this sense of $[u]_{K_e}$ (in (3.2.11) (i) or (iii)). Now every element $\overline{\alpha}$ of $\mathsf{Ind}_e(K)$ has built into it a pair (I,J) of defect ideals and a class β in $K_0(D)$, where $D = \pi^{-1}(I+J)$, and β is obtained from the boundary map just as in the Fredholm case. Moreover, for given (I,J), $\overline{\alpha}$ is determined by β. (However, it is awkward to describe which classes β arise in this way.) Since β lives in $K_0(D)$ instead of $K_0(K)$, it seems reasonable that we should use hypotheses on the defect ideals as well as on K to prove (3.2.11) (i) or (iii).

(ii) Corollaries (3.2.14) and (3.2.15) should be compared to a result of G. Nagy. This implies a fortiori.

(N) If K has general stable rank (gsr) at most 2, and if u is an element of $\mathcal{U}(A/K)$ such that $\partial_1([\overline{u}]_{K_1}) \leq 0$, then u can be lifted to an isometry in A.

Now the hypothesis $\mathrm{gsr}(K) \leq 2$ is not comparable with our hypothesis on K, but it is implied by $\mathrm{tsr}(K) = 1$ or even $\mathrm{csr}(K) \leq 2$. Aside from this difference, (N) is intermediate in strength between (3.2.14) and (3.2.15). It is fairly routine to deduce from $\partial_1([\overline{u}]_{K_1}) \leq 0$ the existence of an isometry v in some $\mathbb{M}_n(A)$ such that $[v]_{K_e}$ lifts $[\overline{u}]_{K_1}$. Thus (3.2.14) gives the conclusion of (N) and also allows us to control the K_e-class of the lift if $[v]_{K_e}$ is given, whereas (3.2.15) states only that u can be lifted to $\mathcal{E}(A)$ and doesn't require that the lift be an isometry. It is also interesting that even though our hypothesis on K doesn't imply Nagy's, nevertheless Nagy's proof will work with our hypothesis.

(iii) If A/K is extremally rich with weak cancellation, then the hypothesis on defect ideals in Corollary (3.2.13) is automatically satisfied. Also, by the last remark in (3.2.11), if the Corona algebra $C(K) = M(K)/K$ is extremally rich with weak cancellation, then (3.2.11) (iii) is true for all A with no hypotheses other than those on K. If the Corona algebra hypothesis is satisfied also for all ideals of K, then we even get (3.2.11) (1). To see this we first use Proposition (3.2.16), applied to $A/\big(K \cap (I + J)\big)$, to reduce to the case $K \subset I + J$. Then the argument based applies not just to give a lift u but to show that u has the same defect ideals as v. But implies that $[u]_{K_e} = [v]_{K_e}$.

Example (3.2.17) [3]:

For ease of notation set $\mathbb{B} = \mathbb{B}(H)$ for some infinite dimensional separable Hilbert space H and choose a projection p in $\mathbb{B}$ such that both spaces $p(H)$ and $(1-p)(H)$ are infinite-dimensional. Let A be the C^*-subalgebra of $\mathbb{B} \otimes c$ consisting of convergent sequences $x = (x_n)$ such that

$$\lim(\mathbf{1} - p)\, x_n p = \lim p x_n (\mathbf{1} - p) = 0\,.$$

This algebra was considered to give an example of a C^*-algebra which is the inductive limit of extremally rich C^*-algebras (actually von Neumann algebras) without itself being extremally rich.

Let $I = \mathbb{B} \otimes c_0$, which is clearly a closed ideal in A, and consider the (split) extension

$$0 \to I \to A \to \mathbb{B} \oplus \mathbb{B} \to 0\,.$$

In this piquant situation all K-groups vanish; but the extremal K-sets do not, and they control the quasi-Fredholm elements in A since I and A/I are extremally rich with weak cancellation. The Fredholm theory is trivial in this example: Every invertible element in A/I lifts to a invertible element in A – as it must by Theorem (3.2.7).

Writing $\mathbb{Z}^e = \mathbb{Z} \cup \{\pm\infty\}$ we find that $K_e(I)$ is the set of sequences in $\mathbb{Z}^e$ that are eventually zero, whereas $K_e(A/I) = (\mathbb{Z}^e)^2$. An element $x = (x_n)$ belongs to A_q^{-1} if and only if every x_n is either left or right invertible and there is an $\varepsilon > 0$ such that for all $n, |x_n|$ (and $|x_n^*|$) has a gap $(0, \varepsilon)$ in its spectra. Since (x_n) converges to a block diagonal operator in $\mathbb{B}^2$ we can describe $K_e(A)$ as eventually constant sequences (α_n) in $\mathbb{Z}^e$ together with an element $(\alpha_\infty^1, \alpha_\infty^2)$ in the first or third quadrant of $(\mathbb{Z}^e)^2$ such that $\alpha_\infty^1 + \alpha_\infty^2 = \lim \alpha_n$. Elements (α_n) and (β_n) in $K_e(A)$ are composable if and only if α_n and β_n have the same sign for $1 \le n \le \infty$, in which case $(\alpha_n) + (\beta_n) = (\alpha_n + \beta_n)$.

Thus the image of $K_e(A)$ in $K_e(A/I)$ is $(\mathbb{Z}_+^e)^2 \cup (\mathbb{Z}_-^e)^2$, and a quasi-Fredholm element can be perturbed to an element of A_q^{-1} if and only if its index is in the union of the first and third quadrants. Equivalently, an element of $\mathcal{E}(A/I)$ can be lifted to $\mathcal{E}(A)$ if and only if both components of its K_e-class have the same sign.

It is also interesting to consider $K = \mathbb{K} \otimes c_0$ and $K = \{x \in A | x_n \in \mathbb{K}, \forall n\}$. Then $K = \overline{Soc(A)}$, and $K_1/K = \overline{Soc(A/K)}$. Here the ordinary K-groups do not all vanish, and it can be seen that $K_e(A/K) = K_e(A/K_1) = \{((\alpha_n), \alpha_\infty^1, \alpha_\infty^2) : \alpha_n, \alpha_\infty^1, \alpha_\infty^2 \in \mathbb{Z}^e, (\alpha_\infty^1, \alpha_\infty^2) \neq (\infty, -\infty), (-\infty, \infty) \alpha_n = \alpha_\infty^1 + \alpha_\infty^2, \text{eventually}\}$.

Proposition (3.2.18) [3]:

Suppose that A is an extremally rich C^*-algebra with weak cancellation such that the natural map

$$\mathcal{U}\left(M(A)\right)/\mathcal{U}_0\left(M(A)\right) \to K_1\left(M(A)\right)$$

is injective. Then also the following map is injective:

$$\mathcal{U}\left(C(A)\right)/\mathcal{U}_0\left(C(A)\right) \to K_1\left(C(A)\right).$$

Proof:

Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{U}\left(M(A)\right) & \longrightarrow & K_1\left(M(A)\right) \\ \downarrow \pi & & \downarrow \pi_1 \\ \mathcal{U}\left(C(A)\right) & \longrightarrow & K_1\left(C(A)\right) \end{array}$$

If u is in $\mathcal{U}(C(A))$ such that $[u] = 0$ then by Theorems (3.2.7) and (3.2.4) we have $u = \pi(v)$ for some v in $\mathcal{U}(M(A))$ with $[v] = 0$. By assumption this means that $v \in \mathcal{U}_0(M(A))$, whence $u \in \mathcal{U}_0(C(A))$.

The next result does not mention extremal richness, but it illustrates an application of good index theory.

Corollary (3.2.19) [3]:

If A is σ-unital and stable, then we have a short exact sequence of groups

$$0 \to \mathcal{U}_0\left(C(A)\right) \to \mathcal{U}\left(C(A)\right) \to K_0(A) \to 0\,.$$

The main goal of this section is to prove that extremal richness plus weak cancellation implies K_1-injectivity. This is accomplished in Theorem (3.2.28), the main step being Lemma (3.2.26), which already includes all the extremally rich C^*-algebras which are purely properly infinite. Our proof is partly modeled on Cuntz's K_1-injectivity, but we need some additional ideas, in particular the

introduction of K_0-surjectivity. We also use a technique similar to one used by Zhang.

The map (3.2.20) [3]:

$\partial_0 : K(A/I) \to K_1(I)$. Since Bott periodicity identifies $K_0(A/I)$ with $K(S(A/I))$, where S denotes suspension, we may consider ∂_0 to be defined on this latter group. We need to know the form of $\partial_0\beta$ in the special case $\beta = [u]$, $u \in \mathcal{U}((\widetilde{A/I}))$. Thus, u is given by a continuous function $f : [0,1] \to \mathcal{U}(\widetilde{A/I})$ such that $f(0) = f(1) = 1$. Then f can be lifted to $g : [0,1] \to \mathcal{U}(\tilde{A})$ such that $g(0) = 1$ and $g(1) \in (1 + I) \cap \mathcal{U}(\tilde{I})$, and $\partial_0\beta = [g(1)]$. It is important to note that $g(1)$ is null-homotopic in $\mathcal{U}(\tilde{A})$.

Definitions (3.2.21) [3]:

We say that A has (strong) K_0-surjectivity if the group $K_0(A)$ is generated by $\{[p] | p$ is a projection in $A\}$. Thus Zhang's result shows that C^*-algebras of real rank zero have strong K_0-surjectivity. Cuntz showed that purely infinite simple C^*-algebras satisfy a still stronger property, which, however, is too strong for our purposes below. We say that A has weak K_0-surjectivity if SA has K_1-surjectivity. Then strong K_0-surjectivity implies weak K_0-surjectivity because the function defined by $f(t) = \exp(2\pi i t p)$ is a unitary in $(SA)^{\sim}$ which corresponds to p under Bott periodicity, for each projection p in A. Since Rieffel showed that $\mathrm{csr}(A) \leq 2$ implies K_1-surjectivity for A and also that $\mathrm{tsr}(A) \leq 1$ implies $\mathrm{csr}(SA) \leq 2$, we see that all C^*-algebras of stable rank one have weak K_0-surjectivity.

Proposition (3.2.22) [3]:

If A is extremally rich with weak cancellation, then $\mathcal{D}(A)$ has strong K_0-surjectivity.

Proof:

Since $\mathcal{D}(A)$ is generated as an ideal by projections, $K_0(\mathcal{D}(A))$ is generated by $\{[p] | p$ is a projection in $\mathcal{D}(A) \otimes \mathbb{K}\}$. But by Proposition (3.1.16), each such p is equivalent to a projection in $\mathcal{D}(A)$.

Lemma (3.2.23) [3]:

If A is a C^*-algebra and B is a σ-unital hereditary C^*-subalgebra such that $B^{\perp}$ contains an infinite set $\{p_n\}$ of mutually orthogonal and equivalent projections which are full (in A), then there is a full hereditary C^*-subalgebra $B' \supset B$ such that $B' \cong B'' \otimes \mathbb{K}$, where B'' is unital. In particular, B' has an approximate identity, (e_n), consisting of full projections, such that for each m and n, men $me_n \lesssim \mathbf{1}_{\tilde{A}}$.

Lemma (3.2.24) [3]:

If A is an extremally rich C^*-algebra with weak cancellation, then $\mathcal{D}(A)$ has K_1-injectivity.

Proof:

Let $D = \mathcal{D}(A)$ and let $\widetilde{D}$ be the forced unitization. Assume there is u in $(1 + D) \cap \mathcal{U}(\widetilde{D})$ whose K_1 -class is trivial but u is not null-homotopic in $\mathcal{U}(D)$. We claim then that there is an ideal J of D which is maximal with respect to the property that $u + J$ fails to be null-homotopic in $\mathcal{U}(\widetilde{D}/J)$. To prove this by Zorn's Lemma, we may assume a totally ordered collection $\{J_i\}$ of ideals such that, with $J = (\cup J_i)$, $u + J$ is null-homotopic and prove that for some i, $u + J_i$ is null-homotopic in $\mathcal{U}(\widetilde{D}/J_i)$. Since $\mathcal{U}_0(\widetilde{D}/J)$ is the image of $\mathcal{U}_0(\widetilde{D})$, u is homotopic in $\mathcal{U}(\widetilde{D})$ to some $v \in (\mathbf{1} + J) \cap \mathcal{U}(\tilde{J})$. Writing $v = \mathbf{1} + x, x \in J$, we see that for some i, $\|x + J_i\| < 1$, whence $v + J_i$ is null-homotopic. Now, extremal partial isometries lift from $\tilde{A}/J$ to $\tilde{A}$, and thus $\mathcal{D}(A/J) = D/J$. Therefore we may replace A with A/J and D with D/J, without changing notation, and seek to obtain a contradiction.

Next choose a continuous function $f : \mathbb{T} \to [0, \infty)$, where $\mathbb{T}$ is the unit circle, such that $\{z | f(z) \neq 0\} = \{e^{i\theta} | 2\pi/3 < \theta < 4\pi/3\}$. Let $B_1 = \mathrm{her}(f(u))$, and note that $B_1 \subset D$, since $f(1) = 0$. If $B_1 = 0$, then the spectrum of u omits -1, a contradiction.

Case (i): If $\mathrm{tsr}(B_1) = 1$, let $I = \mathrm{id}(B_1)$. By construction, $u + I$ is null-homotopic. Hence we see as above that u is homotopic to some $v \in \mathcal{U}(\tilde{I})$. If α is the class of v in $K_1(I)$, then α maps to 0 in $K_1(D)$. It follows that $\alpha = \partial_0 \beta$ for some β in $K_0(D/I)$. Now Proposition (3.2.24) implies that D/I has K_0-surjectivity (since $D(A/I) = D/I$). Thus (3.2.22) applies, and we see that α is represented by a

unitary w in $\tilde{I}$ which is null-homotopic in $\mathcal{U}(\widetilde{D})$. But now w^*v is a unitary in $\tilde{I}$ whose K_1-class vanishes and $\mathrm{tsr}(I) = 1$. Thus Rieffel's K_1-injectivity result, now applies to give our contradiction.

Case (ii): If $\mathrm{tsr}(B_1) > 1$, then let p be a non-zero defect projection for B_1 and note that $1 - p$ is full (by Lemma (3.1.4), for example). Let $I = \mathrm{id}(p)$, and let $\pi : \widetilde{D} \to \widetilde{D}/I$ be the quotient map. Since $\mathrm{Re}(pup) \leq -\frac{1}{2}p$, pup is invertible in pDp and is homotopic to p within $(pDp)^{-1}$. It follows that u is homotopic to $p + u_1$, where u_1 is a unitary element of $(\mathbf{1} - p)\widetilde{D}(\mathbf{1} - p)$ such that $\pi(u_1) = \pi(u)$. To see this, first homotop u within $\widetilde{D}^{-1}$ to

$$(\mathbf{1} - (\mathbf{1} - p)u(pup)^{-1})u\big(\mathbf{1} - (pup)^{-1}u(\mathbf{1} - p)\big),$$

which has the form $pup + u'$, u' invertible in $(\mathbf{1} - p)\widetilde{D}(\mathbf{1} - p)$.

By construction $\pi(u_1)$ is null-homotopic. Since $\pi(\widetilde{D}) = \pi((\mathbf{1} - p)\widetilde{D}(\mathbf{1} - p))$, every element of $\mathcal{U}_0\big(\widetilde{D}/I\big)$ lifts to $\mathcal{U}_0((\mathbf{1} - p)\widetilde{D}(\mathbf{1} - p))$. It follows that u_1 is homotopic (within $\mathcal{U}((\mathbf{1} - p)\widetilde{D}(\mathbf{1} - p))$) to a unitary u_2 in $(\mathbf{1} - p)\tilde{I}(\mathbf{1} - p)$.

If α is the class of u_2 in $K_1(I)$, then α maps to 0 in $K_1(D)$. It follows that $\alpha = \partial_0\beta$ for some β in $K_0(D/I)$. As above we use the K_0-surjectivity of D/I and (3.2.22) to get a special form for $\partial_0\beta$, but now we use $(\mathbf{1} - p)\widetilde{D}(\mathbf{1} - p)$ in place of $\widetilde{D}$. So we find a unitary v in $(\mathbf{1} - p)\tilde{I}(\mathbf{1} - p)$ such that v is null-homotopic in $\mathcal{U}((\mathbf{1} - p)\widetilde{D}(\mathbf{1} - p))$ and $[v] = \alpha$ in $K_1(I)$. Then if $u_3 = u_2v^*$, we see that $p + u_3$ is homotopic to u in $\mathcal{U}(\widetilde{D})$ and $[p + u_3] = 0$ in $K_1(I)$. From now on all the action takes place in $\tilde{I}$, and we will make no further use of the assumption that u has a null-homotopic image in any non-trivial quotient.

Next let $\rho : \tilde{I} \to \tilde{I}/\mathcal{D}(I)$ be the quotient map. By which we apply within $\rho((\mathbf{1} - p)\tilde{I}(\mathbf{1} - p))$, $\rho(u_3)$ is null-homotopic. (Here we are using the fullness of $\mathbf{1} - p$, which implies that $[u_3] = 0$ in $K_1((\mathbf{1} - p)I(\mathbf{1} - p))$). Since $\rho((\mathbf{1} - p)I(\mathbf{1} - p))$ has weak K_0-surjectivity, we may use the above argument to homotop u_3 to an element u_4 in $\mathcal{U}\left((\mathbf{1} - p)\widetilde{\mathcal{D}(I)}(\mathbf{1} - p)\right)$ such that $[u_4] = 0$ in $K_1(\mathcal{D}(I))$.

Then we write $\mathcal{D}(I) = \left(\cup I_j\right)^{=}$, where each I_j is the ideal generated by finitely many defect projections, and $\{I_j\}$ is directed upward. Then u_4 is homotopic to u_5 in some $\mathcal{U}((\mathbf{1}-p)\tilde{I}_j(\mathbf{1}-p))$, and because of the compatibility of K_1 with direct limits, we may assume $[u_5] = 0$ in $K_1(I_j)$. Clearly we may also assume $u_5 \in \mathbf{1} - p + (\mathbf{1}-p)I_j(\mathbf{1}-p)$. Since p is full in I, $\mathcal{D}(pIp) = pIp \cap \mathcal{D}(I)$, a full hereditary C^*-subalgebra of $\mathcal{D}(I)$. Hence every projection in $\mathcal{D}(I)$ is equivalent to a projection in $\mathcal{D}(pIp) \otimes \mathbb{K}$. So Proposition (3.1.16) (ii) can be applied to pIp to find an infinite set, $\{q_n\}$, of mutually orthogonal and mutually equivalent projections in pIp, which are full in I_j. (So that actually $q_n \in pI_jp$).

Finally we apply Lemma (3.2.25) with I_j in place of A and $B = \mathrm{her}(u_5 - \mathbf{1} + p)$. If B' and $\{e_n\}$ are as in the Lemma, let $t_n = \mathbf{1} + e_n(u_5 - \mathbf{1} + p)e_n$. Since $t_n \to p + u_5$, for large n there is a unitary w_n in $e_nI_je_n$ such that $\mathbf{1} - e_n + w_n$ is homotopic to $p + u_5$ in $\mathcal{U}(\tilde{I}_j)$. Since e_n is full in I_j, $K_1(e_nI_je_n)$ is naturally isomorphic to $K_1(I_j)$, and hence $[w_n] = 0$ in $K_1(e_nI_je_n)$. It follows that for some m, $w_n \oplus me_n$ is null-homotopic in $\mathcal{U}(\mathbb{M}_{m+1}(e_nI_je_n))$. But the conclusion of Lemma (3.2.25) states that men $me_n \precsim \mathbf{1} - e_n$. Thus $\mathbf{1} - e_n + w_n$ is null-homotopic in $\mathcal{U}(eI_j)$, and u is null-homotopic in $\mathcal{U}(\widetilde{D})$.

Proposition (3.2.25) [3]:

Let I be an ideal of a C^*-algebra A. Then:

(i) If I and A/I have K_1-injectivity and A/I has weak K_0-surjectivity, then A has K_1-injectivity.

(ii) If I and A/I have weak K_0-surjectivity and I has K_1-injectivity, then A has weak K_0-surjectivity.

Proof:

We may assume A unital. Let $\pi : A \to A/I$ be the quotient map.

The argument for part (i) already occurred several times in the proof of Lemma (3.2.26). If $[u]K_1 = 0, u \in \mathcal{U}(A)$, then $\pi(u)$ is null-homotopic, whence there is v in $\mathcal{U}(\tilde{I})$ which is homotopic to u. If $\alpha = [v]K_1(I)$, then $\alpha = \partial_0\beta$, and (3.2.22)

applies to give w in $\mathcal{U}(\tilde{I})$ which is null-homotopic in $U(A)$ and which represents the class α. Then the K_1-injectivity of I is applied to w^*v.

For part (ii) we are given α in $K_0(A)$, which is identified with $K_1(SA)$. Then $K_0(\pi)(\alpha)$ is represented by u in $\mathcal{U}\big(\widetilde{S(A/I)}\big)$, and u is represented by a function $f : [0,1] \to \mathcal{U}(A/I)$ such that $f(0) = f(1) = \mathbf{1}$. Lift f to $g : [0,1] \to \mathcal{U}(A)$ such that $g(0) = \mathbf{1}$. Then, since $\partial_0(K_0(\pi)(\alpha)) = 0$, we see from (3.2.22) that $[g(1)]_{K_1(I)} = 0$ and so $g(1)$ is null-homotopic in $\mathcal{U}(\tilde{I})$. Thus there is $h : [0,1] \to \mathcal{U}(\tilde{I})$ such that $h(0) = \mathbf{1}$ and $h(1) = g(1)$, so that h^*g gives an element of $\mathcal{U}(\widetilde{SA})$ which represents a class β in $K_0(A)$. Finally, $\alpha - \beta$ is in the image of $K_0(I)$ (which is the kernel of $K_0(\pi)$) and is therefore represented by a unitary.

Theorem (3.2.26) [3]:

If A is an extremally rich C^*-algebra with weak cancellation, then A has K_1-injectivity and weak K_0-surjectivity.

Theorem (3.2.27) [3]:

If A is an extremally rich C^*-algebra with weak cancellation, then A also has K_e-injectivity.

Proof:

Let u and v be elements of $\mathcal{E}(\tilde{A})$ which lie in the same K_e-class. Exactly as in the proof of Proposition (3.2.9), we show that $v = w_1 w u w_2$, where all the w's are unitaries in $\tilde{A}$ whose K_1-classes vanish. Thus, by Theorem (3.2.28), all the w's are in $\mathcal{U}_0(\tilde{A})$, and hence u is homotopic to v in $\mathcal{E}(\tilde{A})$.

The next result should be regarded as an example.

Corollary (3.2.28) [3]:

If A is a purely infinite simple C^*-algebra, then any two proper isometries in $\tilde{A}$ are homotopic within the set of isometries.

The next result is a summary theorem which includes the results we have proved about three classes of extremally rich C^*-algebras, except that it omits the results related to the extremal analogues of good index theory.

Theorem (3.2.29) [3]:

If A is an extremally rich C^*-algebra, then, under any one of the hypotheses listed below, A has weak cancellation, K_1-bijectivity (i.e., $K_1(\mathsf{A}) = \mathcal{U}(\widetilde{\mathsf{A}})/\mathcal{U}_0(\widetilde{\mathsf{A}})$), good index theory, K_e-bijectivity (i.e., $\mathsf{K}_e(\mathsf{A}) = \mathcal{E}(\widetilde{\mathsf{A}})/\mathsf{homotopy}$), and weak K_0-surjectivity. Moreover, $\mathcal{D}(\mathsf{A})$ has strong K_0-surjectivity.

(i) A has real rank zero.

(ii) If I is the left defect ideal of an element of $\mathcal{E}(\tilde{A})$, then $\mathcal{D}(I) = I$. In particular, this applies if A is purely infinite or if every defect projection is properly infinite.

(iii) A is isometrically rich.

Lemma (3.2.30) [3]:

If A is an extremally rich C^*-algebra whose primitive ideal space is Hausdorff, then A has weak cancellation.

Proof:

We verify condition (iii) in Theorem (3.1.5). Thus let $B = pAp$ for a projection p in A, and consider $u \in \mathcal{E}(B)$ with left and right defect ideals I and J. Since $I^{\vee}$ is a compact-open subset of $B^{\vee}$, which is Hausdorff, we see that $I^{\vee}$ is closed. Thus $B = I \oplus I^{\perp}$. Let $u = v \oplus w$. From $J \subset I^{\perp}$ it follows that v is an isometry and w a co-isometry. Hence if $u = v \oplus \mathbf{1}_{I^{\perp}}$, then u' is an isometry with the same left defect projection as u. Now if $u_1, u_2, \dots, u_n$ are in $\mathcal{E}(B)$, then $s = \prod_1^n u_i'$ is an isometry in B and

$$p - ss^* \sim \bigoplus_{i=1}^{n} (p - u_i u_i^*)$$

Proposition (3.2.31) [3]:

If A is an extremally rich C^*-algebra whose primitive ideal space is almost Hausdorff, then A has weak cancellation.

Proof:

The hypothesis implies that A has a composition series of ideals, $\{I_\alpha | 0 \leq \alpha \leq \lambda\}$, such that $I_0 = 0, I_\lambda = A$, and $(I_{\alpha+1}/I_\alpha)^\vee$ is Hausdorff for each $\alpha < \lambda$. Thus the result follows from the Lemma and the last sentence of Theorem (3.2.5).

Corollary (3.2.32) [3]:

If A is an extremally rich C^*-algebra which is of type I, then A has weak cancellation.

We state with at most minimal indications of proof some results which are relevant mainly to non-extremally rich C^*-algebras.

Theorem (3.2.33) [3]:

Assume I is an ideal of a C^*-algebra A, I and A/I have weak cancellation, I has real rank zero, and pAp/pIp has K_1-surjectivity for each projection p in A. Then A has weak cancellation.

Theorem (3.2.34) [3]:

Assume I is an ideal of a C^*-algebra $A, A/I$ has weak cancellation, $\tilde{B}$ has stable weak cancellation for every hereditary C^*-subalgebra of I, and pAp/pIp has K_1-surjectivity for each projection p in A. Then A has weak cancellation.

Theorem (3.2.35) which has a similar conclusion when A has real rank zero. The proof of (3.2.35) is somewhat similar to that of Lemma (3.1.11), one difference being that a different method is used for lifting partial isometries. The proof of (3.2.36) is also somewhat similar. Here a key difference is that the boundary map, $\partial_1 : K(A/I) \to K(I)$, is dealt with by the method applicable to general C^*-algebras –i.e., the unitary in A/I need not be liftable to a partial isometry in A.

The proof of the next theorem is somewhat similar to parts of the proof of Lemma (3.2.26).

Theorem (3.2.35) [3]:

Every purely properly infinite C^*-algebra has K_1-injectivity. Just as we found it necessary to link weak cancellation with K_1-surjectivity and K_1-injectivity with K_0-surjectivity to facilitate several proofs, so the following more obvious linkage can be useful.

Proposition (3.2.36) [3]:

Let I be an ideal of a C^*-algebra A. Then:

(i) If I and A/I have K_1-surjectivity and I has good index theory, then A has K_1-surjectivity.

(ii) If I and A/I have good index theory and A/I has K_1-surjectivity, then A has good index theory.

Propositions (3.2.27) and (3.2.38), and Theorem (3.2.36), combined with easy direct limit arguments, can be used to derive results for C^*-algebras that have composition series with well-behaved quotients. Here is one example. Note that A has generalized stable rank one if A has a composition series of ideals$\{I_\alpha | 0 \leq \alpha \leq \lambda\}$, such that $I_0 = 0, I_\lambda = A$, and $\mathrm{tsr}(I_{\alpha+1}/I_\alpha) = 1$ for each $\alpha < \lambda$. Since implies that every type I extremally rich C^*-algebra has generalized stable rank one, the following result includes Corollary (3.2.34).

Proposition (3.2.37) [3]:

Every C^*-algebra of generalized stable rank one has stable weak cancellation, K_1-bijectivity, good index theory, and weak K_0-surjectivity.

Of the three parts of Theorem (3.2.31), case (iii) is the widest in scope, and it is this case which most justifies the belief that extremal richness is a useful hypothesis for proving weak cancellation. Note that it is also unknown whether real rank zero implies weak cancellation.

Chapter 4

Nuclear C^*-algebras A noncommutative Amir- Cambern Theorem for von Neumann algebras

In this chapter we show that an intermediate result, we compare the Banach-Mazur cb-distance and the Kadison-Kastler distance. Finally, we show that if two C^*-algebras are close enough for the cb-distance, then they have at most the same length [4].

Section (4.1): Almost Completely Isometric Maps

We concern perturbations of operator algebras as operator spaces, more precisely perturbations relative to the Banach-Mazur cb-distance. G. Pisier introduced the Banach-Mazur cb-distance (or cb-distance in short) between two operator spaces $\mathcal{X}, \mathcal{Y}$:

$$d_{cb}(\mathcal{X}, \mathcal{Y}) = \inf\{\|T\|_{cb}\|T^{-1}\|_{cb}\},$$

where the infimum runs over all possible linear completely bounded isomorphisms $T: \mathcal{X} \to \mathcal{Y}$. This extends naturally the classical Banach-Mazur distance for Banach spaces when these are endowed

With their minimal operator space structure (in particular, the Banach-Mazur distance and the cb-distance between two $C(K)$-spaces coincide).

For $C(K)$-spaces, being isomorphic (or equivalently cb-isomorphic) is a very flexible relation; Milutin's theorem states that $C(K)$ is isomorphic to $C([0,1])$ for any uncountable compact metric space K. this theorem has some noncommutative generalization for separably acting injective von Neumann algebras [7]: (Avon Neumann algebra is a $*$-algebra of bounded operators on a Hillbert space that is closed in the weak operator topology and contains the identity operator) and for separable non-type I unclear C^*-algebras. But here, we are interested in perturbation results when the cb-distance is small. Let us recall the generalization of Banach-Stone theorem obtained independently by D. Amir and M. Cambern: if the Banach-Mazur distance between two $C(K)$-spaces is strictly smaller than 2, then they are $*$-isomorphic (as C^*-algebras). Actually, this is also true for spaces of continuous functions vanishing at infinity on locally compact Hausdorff spaces.

One is tempted to extend the Amir-Cambern Theorem to noncommutative C^*-algebras. R. Kadison described isometries between C^*-algebras, in particular the isometric structure of a C^*-algebra only determines its Jordan structure, hence to recover the C^*-structure we need a priori assumption on the cb-distance (not only on the classical Banach-Mazur distance). Here, we show:

Theorem (4.1.1) [4]:

Let $\mathcal{A}$ be a separable nuclear C^*-algebra or a von Neumann algebra, then there exists an $\varepsilon_0 > 0$ such that for any C^*-algebra $\mathcal{B}$, the inequality $d_{cb}(\mathcal{A},\mathcal{B}) < 1 + \varepsilon_0$ implies that $\mathcal{A}$ and $\mathcal{B}$ are $*$ -isomorphic.

When $\mathcal{A}$ is a separable nuclear C^*-algebra, one can take $\varepsilon_0 = 3.10^{-19}$. When $\mathcal{A}$ is a von Neumann algebra, $\varepsilon_0 = 4.10^{-6}$ is sufficient.

Such a result can not be extended to all unital C^*-algebras, see Corollary (4.2.9) below for a counter-example involving nonseparable C^*-algebras.

The proof of Theorem (4.1.1) is totally different from the commutative case, the cb-distance concerns only the operator space structure, hence the basic idea is to gain the algebraic structure. It is crucial to work with the completely bounded cohomology here, because we can exploit the deep result that every completely bounded cohomology group of a von Neumann algebra over itself vanishes, this is unknown for the bounded cohomology. This allows us to conclude for von Neumann algebras.

For separable nuclear C^*-algebras, the strategy is different, because vanishing of completely bounded cohomology groups is not available. First, we compare the cb-distance d_{cb} and the completely bounded Kadison- Kastler distance $d_{H,cb}$ (see the definition below):

Proof:

The proof of Theorem (4.1.1) is a variant of the proof of Theorem (4.1.2). For clarity we postpone the quantitative estimate to the next Remark.

When $\mathcal{A}$ is a separable nuclear C^*-algebra, this follows directly from Theorem (4.1.2).

When $\mathcal{A}$ is a von Neumann algebra, one does not need to go to the bidual $\mathcal{A}^{**}$ in the preceding proof (to apply Proposition (4.2.2)), so that we directly conclude that $\pi : \mathcal{A} \to \mathcal{B}$ is a $*$-is omorphism. But we should mention another way (which improves theoretically our bound in the von Neumann algebras case): the second step in the proof is not necessary, just define directly a new multiplication using the cb-isomorphism S (instead of T). Then π is just an algebra isomorphism (not necessarily selfadjoint).

Theorem (4.1.2) [4]:

There exists a constant $K > 0$, such that for any C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, there exist faithful $*$-representations $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$ on the same Hillerbt space H such that

$$d_{H,cb}\big(\pi_{\mathcal{A}}(\mathcal{A}), \pi_{\mathcal{B}}(\mathcal{B})\big) \leq K\sqrt{\ln d_{cb}(\mathcal{A}, \mathcal{B})}.$$

One can choose $K = 3620$, when $d(\mathcal{A}, \mathcal{B}) < 1 + 10^{-7}$.

In order to prove this theorem, we need to control explicitly the defect of selfadjointness of a unital almost completely isometric map. Then we will use stability of separable nuclear C^*-algebras for the Kadison- Kastler distance, this is a major result in perturbation theory. Let us recall this result more precisely. As usual H denotes a Hilbert space and $\mathbb{B}(H)$ its bounded linear endomorphisms. Let $\mathcal{A}, \mathcal{B}$ be subalgebras of $\mathbb{B}(H)$, the Kadison-Kastler distance between $\mathcal{A}$ and $\mathcal{B}$ inside $\mathbb{B}(H)$ is

$$d_H(\mathcal{A}, \mathcal{B}) = \delta_H\big(Ball(\mathcal{A}), Ball(\mathcal{B})\big),$$

where δ_H denotes the Hausdorff distance and $Ball(\mathcal{A})$ (respectively $Ball(\mathcal{B})$) denotes the unit ball of $\mathcal{A}$ ($\mathcal{B}$ respectively). We also recall its completely bounded version defined by:

$$d_{H,cb}(\mathcal{A}, \mathcal{B}) = \sup_n \big\{d_{\ell_n^2 \otimes H}\big(\mathbb{M}_n(\mathcal{A}), \mathbb{M}_n(\mathcal{B})\big)\big\}$$

(using the well-known identification $\mathbb{M}_n\big(\mathbb{B}(H)\big) = \mathbb{B}(\ell_n^2 \otimes H)$). Then we have the main result: for any $\gamma < 42^{-1}.10^{-4}$, if $\mathcal{A} \subset \mathbb{B}(H)$ is a separable nuclear sub-C^*-agebra and $\mathcal{B} \subset \mathbb{B}(H)$ is another sub-C^*-agebra if $d_H(\mathcal{A}, \mathcal{B}) \leq \gamma$ then $\mathcal{A}$ and $\mathcal{B}$ are

$*$-isomorphic. Therefore, it is clear that C^*-case of Theorem (4.1.1) is a corollary of this last result and our Theorem (4.1.2).

We already mentioned that an Amir- Cambern type theorem is false for any C^*-algebras, however we can try to prove that some C^*-algebraic invariants are preserved under perturbation relative to the cb-distance. The notion of length of an operator algebra has been defined by G. Pisier in order to attack the Kadison similarity problem (he proved that a C^*-algebra has finite length if and only if it has the Kadison similarity property). It has been proved that having finite length is a property which is stable under perturbation for the Kadison- Kastler distance. Here, we prove that if two C^*-algebras are close enough for the cb-distance, then they have at most the same length.

Proof:

As $d_{H,cb}$ is bounded, the statement is only interesting when $d_{cb}(\mathcal{A},\mathcal{B})$ is close to 1. The proof uses ideas.

Let $L : \mathcal{A} \to \mathcal{B}$ be a cb-isomorphism with $\|L\|_{cb} \leq 1$ and $\|L^{-1}\|_{cb}$ sufficiently closed to 1.

Consider the bidual extension still denoted by $L : \mathcal{A}^{**} \to \mathcal{B}^{**}$, it remains a cb-isomorphism and satisfies the same norm estimates.

We suppose $\mathcal{A}$ unital, we will treat the non-unital case afterwards. The first step is to unitize L. By Proposition (4.1.7), $L(1)$ is invertible in $\mathcal{B}$. If $\|L^{-1}\|_{cb} < \sqrt{2}$. Let $S = L(1)^{-1}L$, then S is unital and

$$\|S\|_{cb} \leq \frac{\|L^{-1}\|_{cb}}{\sqrt{2 - \|L^{-1}\|_{cb}^2}}$$

and $\|S^{-1}\|_{cb} \leq \|L^{-1}\|_{cb}$. Note also that $S(\mathcal{A}) = \mathcal{B}$.

The second step is to make our cb-isomorphism selfadjoint. By Theorem (4.1.10), we have $\|S - S^\star\|_{cb} \leq 2f_1(\|L^{-1}\|_{cb})$ for some continuous function with $f_1(1) = 0$. Let $T = \frac{1}{2}(S + S^\star)$. Then $T: \mathcal{A}^{**} \to \mathcal{B}^{**}$ is unital $*$-preserving, $\|T - S\|_{cb} \leq f_1(\|L^{-1}\|_{cb})$ and $T(\mathcal{A}) \subset \mathcal{B}$. So if $\|L^{-1}\|_{cb}$ is close enough to 1, T is also a cb-isomorphism such that $T(\mathcal{A}) = \mathcal{B}$ with norm estimates $\|T\|_{cb} \leq$

$f_2(\|L^{-1}\|_{cb})$ and $\|T^{-1}\|_{cb} \leq f_2(\|L^{-1}\|_{cb})$, for some continuous function at 1 with $f_2(1) = 1$.

Define on $\mathcal{A}^{**}$ a new multiplication by, for $x, y \in \mathcal{A}^{**}$

$$\mathrm{m}(x, y) = T^{-1}\big(T(x)T(y)\big).$$

The multiplication m is associative and $*$-preserving. It is obviously completely bounded and clearly

$$\|\mathrm{m} - \mathrm{m}_{\mathcal{A}}\|_{cb} \leq \|T^{-1}\|_{cb}\|T^{\vee}\|_{cb}.$$

Thus, the estimate in Theorem (4.1.10) gives that $\|\mathrm{m} - \mathrm{m}_{\mathcal{A}^{**}}\|_{cb} \leq f_3(\|L^{-1}\|_{cb})$ for some continuous function f_3 with $f_3(1) = 0$. If $\|L^{-1}\|_{cb}$ is close enough to 1, we get from Proposition (4.2.2), that there is a completely bounded $*$-preserving linear isomorphism $\Phi : \mathcal{A} \to \mathcal{A}$ with

$$\|\Phi - \mathrm{id}_{\mathcal{A}^{**}}\|_{\mathrm{cb}} \leq f_4(\|L^{-1}\|_{cb})$$

for some continuous function f_4 with $f_4(1) = 0$ and for $x, y \in \mathcal{A}^{**}$

$$\Phi^{-1}\big(\Phi(x)\Phi(y)\big) = \mathrm{m}(x, y) = T^{-1}\big(T(x)T(y)\big).$$

Note that necessarily $\Phi(1) = 1$, and Φ is $*$-preserving.

Let $\pi = T\Phi^{-1} : \mathcal{A}^{**} \to \mathcal{B}^{**}$, it is a $*$-preserving cb-isomorphism. Moreover, for $x, y \in \mathcal{A}^{**}, \pi(xy) = \pi(x)\pi(y)$, hence π is actually a $*$-isomorphism. Now represent faithfully $\mathcal{B}^{**}$ inside $\mathbb{B}(H)$ for some Hilbert space H and let us check that the C^*-algebras $\pi(\mathcal{A})$ and $\mathcal{B}$ are close for the Kadison-Kastler distance inside $\mathbb{B}(H)$.

Actually we just need to check this for the Kadison- Kastler distance, as Φ above is close to $\mathrm{id}_{\mathcal{A}^{**}}$ in cb-norm. For any $a \in Ball\ (\mathcal{A})$, we have:

$$\|\pi(\mathcal{A}) - T(a)\| \leq f_4(\|L^{-1}\|_{cb}), \tag{1}$$

and as $T(\mathcal{A}) = \mathcal{B}$, for $b \in Ball(\mathcal{B})$, we have

$$\big\|b - \pi\big(T^{-1}(b)\big)\big\| \leq \|T^{-1}\|_{cb} f_4(\|L^{-1}\|_{cb}) = f_5(\|L^{-1}\|_{cb}).$$

From which one easily deduces $d_{H,cb}(\pi(\mathcal{A}), B) \leq f_5(\|L^{-1}\|_{cb})$, for some continuous function f_5 with $f_5(1) = 0$.

Now if $\mathcal{A}$ is non-unital, in the preceding proof, $L(1)$ is now invertible in $\mathcal{B}^{**}$ (here 1 denotes the unit of $\mathcal{A}^{**}$), so $S(\mathcal{A}) = \mathcal{B}$ is not valid anymore. But the inequality (1) above still holds and we deduce that for $a \in Ball(\mathcal{A})$

$$\|\pi(a) - S(a)\| \leq (f_4 + f_1)(\|L^{-1}\|_{cb}).$$

Now from Lemma (4.1.8), there is a unitary u in $\mathcal{B}^{**}$ such that $\|u - L(1)^{-1}\| \leq f_6(\|L^{-1}\|_{cb})$ for some continuous function f_6 with $f_6(1) = 0$. Therefore

$$\|\pi(a) - uL(a)\| \leq (f_4 + f_1 + f_6)(\|L^{-1}\|_{cb}).$$

Taking the adjoints we obtain

$$\|\pi(a) - L(a^*)^* u^*\| \leq (f_4 + f_1 + f_6)(\|L^{-1}\|_{cb}).$$

Write $a = xy$, for some x and y in the $Ball(\mathcal{A})$, then

$$\|\pi(a) - uL(x)\| \leq 2(f_4 + f_1 + f_6)(\|L^{-1}\|_{cb}). \qquad (2)$$

As $L(x)L(y^*)^*$ belongs to $Ball(\mathcal{B})$, we conclude that the C^*-algebra $\pi(\mathcal{A})$ is nearly included in the C^*-algebra $u\mathcal{B}u^*$. Let us prove the converse near inclusion. Let $b \in Ball(\mathcal{B})$, we can factorize $b = L(x)L(y^*)^*$ with $x, y \in \mathcal{A}$ such that $\|x\| \leq \|L^{-1}\|$ and $\|y\| \leq \|L^{-1}\|$. From inequality (2), we get

$$\|\pi(xy) - uL(x)L(y^*)^* u^*\| \leq 2\|L^{-1}\|^2 (f_4 + f_1 + f_6)(\|L^{-1}\|_{cb}).$$

Finally, $d_{H,cb}(\pi(\mathcal{A}), u\mathcal{B}u^*) \leq \|L^{-1}\|^2 (f_4 + f_1 + f_6)(\|L^{-1}\|_{cb})$.

Theorem (4.1.3) [4]:

Let $K \geq 1$ and $\ell \in \mathbb{N}\backslash\{0\}$ fixed but arbitrary constants. If $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras with $d_{cb}(\mathcal{A}, \mathcal{B}) < 1 + 10^{-4} 4^{-\ell} K^{-2}$ and $\mathcal{A}$ has length at most ℓ and length constant at most K, then $\mathcal{B}$ has length at most ℓ.

This section starts with few technical lemmas relating algebraic properties to norm estimates in operator algebras. We next use them to study almost completely isometric isomorphisms.

It is well known that unital completely isometric isomorphisms between operator algebras are necessarily multiplicative. Hence one can hope that unital almost completely isometric bijections are almost multiplicative and almost selfadjoint. This can be checked easily by an ultraproduct argument, but the important point is to control explicitly the defect of multiplicativity and the defect of selfadjointness.

When $T: \mathcal{A} \to \mathcal{B}$ is a map between two operator algebras, the defect of multiplicativity of T is denoted by $T^{\vee}$. It consists of the bilinear map $T^{\vee}: \mathcal{A} \to \mathcal{B}$ given by $T^{\vee}(a, b) = T(ab) - T(a)T(b)$. As usual when dealing with bilinear maps, the completely bounded norm of $T^{\vee}$ is the cb-norm of the induced linear map $T^{\vee}: \mathcal{A}_{\otimes h}\mathcal{A} \to \mathcal{B}$ on the Haagerup tensor product.

Given a cb map $T : S \to \mathcal{T}$ between two operator systems, we use the notation $T^{\star}$ defined on S by $T^{\star}(x) = T(x^{*})^{*}$. The linear map $T - T^{\star}$ measures the defectness of selfadjointness of T. We recall the following proof.

Proof:

As in the proof of Theorem (4.1.2), let $L : \mathcal{A} \to \mathcal{B}$ a linear cb-isomorphism with $\|L\|_{cb}\|L^{-1}\|_{cb} < 1 + \epsilon$. We consider $S = L(1)^{-1}L$ and the multiplication m on $\mathcal{A}$ defined by

$$\mathrm{m}(x, y) = S^{-1}\big(S(x)S(y)\big).$$

Hence

$$\left\|\mathrm{m}^{\ell} - \mathrm{m}_{\mathcal{A}}^{\ell}\right\|_{cb} \leq \|S^{-1}\|_{cb}\left\|S^{\vee\ell}\right\|_{cb}.$$

Since $\|S^{-1}\|_{cb} \leq 1 + f(\epsilon)$ and $\|S\|_{cb} \leq 1 + f(\epsilon)$ with $f(\epsilon)$ tending to 0 when ϵ tends to 0, by Theorem (4.1.10), $\|S^{-1}\|_{cb}\left\|S^{\vee\ell}\right\|_{cb} < 1/K$, if ϵ is small enough. (Naturally here, $S^{\vee\ell}$ denotes the ℓ-linear map defined on Al by $S^{\vee\ell}(x_1, \dots, x_{\ell}) = S(x_1 \dots x_{\ell}) - S(x_1) \dots S(x_{\ell})$.) Therefore by Proposition (4.2.7), $\mathcal{A}$ equipped with m has also length at most ℓ. But S is a cb-isomorphic algebra isomorphism from $\mathcal{A}$ equipped with m onto $\mathcal{B}$, so $\mathcal{B}$ has length at most ℓ as well.

For the quantitative estimates, clearly $\left\|S^{\vee\ell}\right\|_{cb} \leq \|S^{\vee}\|_{cb} \sum_{k=0}^{\ell-2} \|S\|_{cb}^{k}$, hence to apply Proposition (4.2.7) we need $\|S^{-1}\|_{cb} \leq 88\sqrt{\delta}2^{\ell-1} < 1/K$, which gives the explicit bound.

Proposition (4.1.4) [4]:

For any $\eta > 0$, there exists $\rho \in (0,1)$ such that for any unital operator algebras $\mathcal{A}, \mathcal{B}$, for any unital cb-isomorphism $T: \mathcal{A} \to \mathcal{B}$, $\|T\|_{cb} \leq 1 + \rho$ and $\|T^{-1}\|_{cb} \leq 1 + \rho$ imply $\|T^{\vee}\|_{cb} < \eta$.

Proof:

Suppose the assertion is false. Then there exists $\eta_0 > 0$ such that for every positive integer $n \in \mathbb{N}\backslash\{0\}$, there is a unital cb-isomorphism $T: \mathcal{A}_n \to \mathcal{B}_n$ between some unital operator algebras satisfying

$$\|T_n\|_{cb} \leq \frac{1}{n}, \quad \|T_n^{-1}\|_{cb} \leq 1 + \frac{1}{n} \quad \text{and} \quad \left\|T_n^{\vee}\right\|_{cb} \geq \eta_0.$$

Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$, let us denote $\mathcal{A}_{\mathcal{U}}$ (resp. $\mathcal{B}_{\mathcal{U}}$) the ultraproduct $\prod_n (\mathbb{K}^1{}_{\otimes\min}\mathcal{A}_n)/\mathcal{U}$ (resp. $\prod_n (\mathbb{K}^1{}_{\otimes\min}\mathcal{B}_n)/\mathcal{U}$), here $\mathbb{K}^1$ denotes the unitization of the C^*-algebra of all compact operators on ℓ_2. Then $\mathcal{A}_{\mathcal{U}}$ (resp. $\mathcal{B}_{\mathcal{U}}$) is a unital operator algebra. Now consider $T_{\mathcal{U}}: \mathcal{A}_{\mathcal{U}} \to \mathcal{B}_{\mathcal{U}}$ the ultraproduct map obtained from the $\mathrm{id}_{\mathbb{K}^1} \otimes T_n$'s. Hence $T_{\mathcal{U}}$ is a unital surjective linear complete isometry between operator algebras, so $T_{\mathcal{U}}$ is multiplicative hence $T_{\mathcal{U}}^{\vee} = 0$.

This contradicts the hypothesis for all n, $\|T_{\mathcal{U}}^{\vee}\|_{cb} \geq \eta_0$. Indeed $\|T_{\mathcal{U}}^{\vee}\|_{cb} = \|(\mathrm{id}_{\mathbb{K}^1} \otimes T_n)^{\vee}\|$, so there are u_n, v_n in the closed unit ball of $\mathbb{K}^1 \otimes_{\min} \mathcal{A}_n$ such that

$$\|(\mathrm{id}_{\mathbb{K}^1} \otimes T_n)(u_n, v_n) - (\mathrm{id}_{\mathbb{K}^1} \otimes T_n)(u_n)(\mathrm{id}_{\mathbb{K}^1} \otimes T_n)(v_n)\| \geq \eta_0,$$

which implies that

$$\|T_{\mathcal{U}}(\dot{u}\dot{v}) - T_{\mathcal{U}}(\dot{u})T_{\mathcal{U}}(\dot{v})\|\eta_0$$

(where $\dot{x}$ denotes the equivalence class of $(x_n)_n$ in $\mathcal{A}_{\mathcal{U}}$).

A similar proof gives.

Proposition (4.1.5) [4]:

For any $\eta > 0$, there exists $\rho \in (0,1)]$ such that for any operator systems $S, \mathcal{T}$, for any unital cb-isomorphism $T : S \to \mathcal{T}, \|T\|_{cb} \leq 1 + \rho$ and $\|T^{-1}\|_{cb} \leq 1 + \rho$ imply $\|T - T^{\star}\|_{cb} < \eta$.

We turn to quantitative versions of the previous Propositions for C^*-algebras.

The next Lemma is interesting because it gives an operator space characterization (it involves computations on 2×2 matrices) of invertibility inside a von Neumann algebra.

Lemma (4.1.6) [4]:

Let M be a von Neumann algebra and $x \in \mathcal{M}, \|x\| \leq 1$. Then, x is invertible if and only if there exists $\alpha > 0$ such that for any projection $y \in \mathcal{M}$,

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 \geq \alpha + \|y\|^2 \quad \text{and} \quad \|[x \quad y]\|^2 \geq \alpha + \|y\|^2 \tag{3}$$

If this holds, the maximum of the α's satisfying (3) equals $\|x^{-1}\|^{-2}$ and (3) holds for any $y \in \mathcal{M}$.

Proof:

If $x \in \mathcal{M}$ is invertible and $y \in \mathcal{M}$ is arbitrary, by functional calculus, $x^*x \geq \|x^{-1}\|^{-2}$, thus

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 = \|x^*x + y^*y\| \geq \|\|x^{-1}\|^{-2} + y^*y\| = \|x^{-1}\|^{-2} + \|y\|^2.$$

Thanks to a similar argument for the row estimate, we get that (3) holds with $\alpha = \|x^{-1}\|^{-2}$.

Assume that (3) is satisfied. Fix $\lambda \geq 0$ and let $p_\lambda = \mathcal{X}_{[0,\lambda]}(x^*x) \in \mathcal{M}$ be the spectral projection of $|x|^2$ corresponding to $[0, \lambda]$. By the functional calculus $1 + \lambda \geq x^*x + p_\lambda$ as $\|x\| \leq 1$. Taking $y = p_\lambda$ in the first part of (3) gives,

$$1 + \lambda \geq \left\| \begin{bmatrix} x \\ p_\lambda \end{bmatrix} \right\|^2 \geq \alpha + \|p_\lambda\|^2.$$

Hence $p_\lambda = 0$ for $\lambda < \alpha$. Thus x^*x is invertible and $\|(x^*x)^{-1}\| \leq \alpha^{-1}$. Similarly xx^* must have the same property and x is left and right invertible hence invertible.

In the polar decomposition of $x = u|x|, u$ must be a unitary so that we finally get $\alpha \leq \|x^{-1}\|^{-2}$ and the proof is complete.

The next Proposition generalizes the well-known fact that a complete isometry between C^*-algebras sends unitaries to unitaries.

Proposition (4.1.7) [4]:

Let $\mathcal{A}, \mathcal{B}$ be two C^*-algebras. Let $T : \mathcal{A} \to \mathcal{B}$ be a cb-isomorphism such that $\|T\|_{cb}\|T^{-1}\|_{cb} < \sqrt{2}$. Then, for any unitary $u \in \mathcal{A}, T(\mathcal{U})$ is invertible and

$$\|T(u)^{-1}\| \leq \frac{\|T^{-1}\|_{cb}}{\sqrt{2 - \|T\|_{cb}^2\|T^{-1}\|_{cb}^2}}$$

Proof:

Passing to biduals, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras as $x \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if it is invertible in $\mathcal{B}^{**}$. Replacing T by $T/\|T\|_{cb}$, we may assume that $\|T\|_{cb} = 1$ and $\|T^{-1}\|_{cb} < \sqrt{2}$.

Let $y \in \mathcal{B}$ with $\|y\| = 1$, as $\|T(u)\| \leq 1$:

$$\left\|\begin{bmatrix} T(u) \\ y \end{bmatrix}\right\|^2 \geq \frac{1}{\|T\|_{cb}^2}\left\|\begin{bmatrix} u \\ T^{-1}(y) \end{bmatrix}\right\|^2 \geq \frac{1 + \|T^{-1}(y)\|^2}{\|T^{-1}\|_{cb}^2} \geq \frac{2}{\|T^{-1}\|_{cb}^2}.$$

Hence $T(u)$ satisfies (1) with $\alpha = \frac{2}{\|T^{-1}\|_{cb}^2} - 1 > 0$. Finally applying Lemma (4.1.3), we obtain

$$\|T(u)^{-1}\|^2 \leq \frac{1}{\alpha} \leq \frac{\|T^{-1}\|_{cb}^2}{2 - \|T^{-1}\|_{cb}^2}.$$

We have the Lemma.

Lemma (4.1.8) [4]:

Let $\mathcal{A}$ be a unital C^*-algebra and $x \in \mathcal{A}$ invertible. Then there exists a unitary $u \in \mathcal{A}$ such that $\|x - u\| = \max\left\{\|x\| - 1, 1 - \frac{1}{\|x^{-1}\|}\right\}$.

Proof:

Write the polar decomposition of $x = u|x|$. As x is invertible, $|x|$ is strictly positive element of $\mathcal{A}$, so u is a unitary of $\mathcal{A}$. Obviously, $\|x - u\| = \||x| - 1\|$. Seeing $|x|$ as a strictly positive function thanks to the functional calculus, it is not difficult to conclude.

The next Lemma is the key result to compute explicitly the defect of multiplicativity. As in Lemma (4.1.6), operator space structure is needed.

Lemma (4.1.9) [4]:

Let u, v be two unitaries in $\mathbb{B}(H)$. Let $x \in \mathbb{B}(H)$ and $c \geq 1$ such that

$$\left\|\begin{bmatrix} u & x \\ -1 & v \end{bmatrix}\right\| \leq c\sqrt{2},$$

then $\|x - uv\| \leq 2\sqrt{c^2 - 1}$.

Proof:

Note first that

$$\begin{bmatrix} u^* & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u & x \\ -1 & v \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & v^* \end{bmatrix} = \begin{bmatrix} 1 & u^*xv^* \\ -1 & 1 \end{bmatrix},$$

hence without loss of generality we can assume that $u = v = 1$. Take $h \in H$, then

$$\left\|\begin{bmatrix} 1 & x \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -h \\ h \end{bmatrix}\right\| \leq c\sqrt{2}\left\|\begin{bmatrix} -h \\ h \end{bmatrix}\right\|.$$

Therefore $\|x(h) - h\|^2 + 4\|h\|^2 \leq 4c^2\|h\|^2$, which implies $\|x - 1\| \leq 2\sqrt{c^2 - 1}$.

We are now ready to prove the main result.

Theorem (4.1.10) [4]:

Let $\mathcal{A}, \mathcal{B}$ be two unital C^*-algebras. Let $T : \mathcal{A} \to \mathcal{B}$ be a cb-isomorphism with $T(1) = 1$ and $\|T\|_{cb}\|T^{-1}\|_{cb} < \sqrt{2}$, then

$$\|T^{\vee}\|_{cb} \leq 2\sqrt{(\|T\|_{cb})^2 - 1} + \mu(T)(1 + \|T\|_{cb}).$$

$$\|T - T^{\star}\|_{cb} \leq 2\sqrt{\left(\|T\|_{cb} + \frac{\mu(T)}{\sqrt{2}}\right)^2 - 1} + 2\mu(T),$$

where $\mu(T) = \max\left\{\|T\|_{cb} - 1, 1 - \sqrt{\frac{2}{\|T\|_{cb}^2} - \|T\|_{cb}^2}\right\}$.

Proof:

We start with the defect of multiplicativity. From the definition of the Haagerup tensor norm and the Russo-Dye Theorem, it suffices to show that for any unitaries $u, v \in \mathcal{M}_n(\mathcal{A})$ we have

$$\|T_n(uv) - T_n(u)T_n(v)\|_{\mathbb{M}_n(\mathcal{B})} \leq \sqrt{\left(\|T\|_{cb} + \frac{\mu(T)}{\sqrt{2}}\right)^2 - 1} + \mu(T)(1 + \|T\|_{cb}) \qquad (4)$$

where $T_n = Id_{\mathbb{M}_n} \otimes T$. Without loss of generality, we can assume $n = 1$.

Let $u, v \in \mathcal{A}$ unitaries, as $\left\|\begin{bmatrix} u & uv \\ -1 & v \end{bmatrix}\right\| = \sqrt{2}$, we get

$$\left\|\begin{bmatrix} T(u) & T(uv) \\ -1 & T(v) \end{bmatrix}\right\| \leq \|T\|_{cb}\sqrt{2}.$$

From Lemma (4.1.8) and Proposition (4.1.7) we deduce that there are unitaries $u', v' \in \mathcal{B}$ with $\|T(u) - u'\| \leq \mu(T)$ and $\|T(v) - v'\| \leq \mu(T)$. The triangular inequality gives

$$\left\|\begin{bmatrix} u' & T(uv) \\ -1 & v' \end{bmatrix}\right\| \leq \|T\|_{cb}\sqrt{2} + \mu(T).$$

Lemma (4.1.9) implies that $\|T(uv) - u'v'\| \leq 2\sqrt{\left(\|T\|_{cb} + \frac{\mu(T)}{\sqrt{2}}\right)^2 - 1}$, so that we get the estimate using the triangular inequality once more.

For the second estimate, as $T_n^\star = (T^\star)_n$ we may also assume $n = 1$. Thanks to the Russo-Dye Theorem, we just need to check that for any $u \in \mathcal{A}$ unitary

$$\|T(u) - T(u^*)^*\| \leq 2\sqrt{\left(\|T\|_{cb} + \frac{\mu(T)}{\sqrt{2}}\right)^2 - 1} + 2\mu(T).$$

Taking $v = u^*$ in the above arguments leads to $\|T(uu^*) - u'v'\| \leq 2\sqrt{\left(\|T\|_{cb} + \frac{\mu(T)}{\sqrt{2}}\right)^2 - 1}$. Hence $\|u' - v'^*\| \leq 2\sqrt{\left(\|T\|_{cb} + \frac{\mu(T)}{\sqrt{2}}\right)^2 - 1}$ and we conclude using the triangular inequality (4) above using the triangle inequality.

Section (4.2): A noncommutative Amir-Cambern Theorem

Definition (4.2.1) [4]:

Let $\mathcal{X}$ be an operator space. A bilinear map $m : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is called a multiplication on $\mathcal{X}$ if it is associative and extends to the Haagerup tensor product $\mathcal{X} \otimes_h \mathcal{X}$.

We denote by $\mathsf{m}_{\mathcal{A}}$ the original multiplication on an operator algebra $\mathcal{A}$.

In the following, $H_{cb}^k(\mathcal{A}, \mathcal{A})$ denotes the kth completely bounded cohomology group of $\mathcal{A}$ over itself.

The next proposition is the operator space version. It gives a precise form of small perturbations of the product on anoperator algebra under cohomological conditions.

As before, the quantity $\|\mathsf{m} - \mathsf{m}_{\mathcal{A}}\|_{cb}$ is the cb-norm of $\mathsf{m} - \mathsf{m}_{\mathcal{A}}$ as a linear map from $\mathcal{A} \otimes_h \mathcal{A}$ into $\mathcal{A}$.

Proposition (4.2.2) [4]:

Let $\mathcal{A}$ be an operator algebra satisfying

$$H_{cb}^2(\mathcal{A}, \mathcal{A}) = H_{cb}^3(\mathcal{A}, \mathcal{A}) = 0. \tag{5}$$

Then there exist $\delta, C > 0$ such that for every multiplication m on $\mathcal{A}$ satisfying $\|\mathsf{m} - \mathsf{m}_{\mathcal{A}}\| \leq \delta$, there is a completely bounded linear isomorphism $\Phi : \mathcal{A} \to \mathcal{A}$ such that

$$\|\Phi - \mathrm{id}_{\mathcal{A}}\|_{cb} \leq C\|\mathrm{m} - \mathrm{m}_{\mathcal{A}}\|_{cb} \quad \text{and} \quad \Phi\big(\mathrm{m}(x,y)\big) = \Phi(x)\Phi(y).$$

If $\mathcal{A}$ is a von Neumann algebra, then (5) is automatically satisfied with values $\delta = 1/11$ and $C = 10$. Moreover if m satisfies $\mathrm{m}(x^*, y^*) = \mathrm{m}(y,x)^*$ for all $x, y \in \mathcal{A}$, then Φ can be taken to satisfy $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathcal{A}$.

In the bounded situation, one has to apply an implicit function theorem to the right spaces of multilinear maps. This is done in details. With the notation there (taking $\mathcal{M} = \mathcal{A}$) one simply need to replace $\mathcal{L}^k(\mathcal{M}, \mathcal{M})$by their cb-version $\mathcal{L}^k_{cb}(\mathcal{A}, \mathcal{A})$ which are obviously Banach spaces. The statement about selfadjointness of Φ is justified.

If $\mathcal{A}$ is a von Neumann algebra, all completely bounded cohomology groups of $\mathcal{A}$ over itself vanish and we can choose $K = L = 1$, which gives $\delta = 11^{-1}$. Now let us compute C, the rational function p satisfies $p(x) \leq 9.75x^2$, for x small enough. Hence the sequence (ε_i) defined by $\varepsilon_{i+1} = p(\varepsilon_i)$ (and $\varepsilon = \|\mathrm{m} - \mathrm{m}_{\mathcal{A}}\|_{cb}$) verifies $\varepsilon \leq 9.75^{2^i - 1}\varepsilon_0^{2^i}$. As $K = L = 1$, we have $\|S_i\|_{cb} \leq \varepsilon_{i-1} + 2\varepsilon_{i-1}^2$. Then, using the previous estimates of the ε_i's we get $\|W_n - \mathrm{id}_{\mathcal{A}}\|_{cb} \leq \exp(\sum_{i=1}^n \|S_i\|_{cb}) \leq 10\varepsilon_0$.

Remark (4.2.3) [4]:

We give a rough estimate for the constants in Theorems (4.1.1) and (4.1.2). With notation from the proof of Theorem (4.1.2), we start with $\|L\|_{cb} \leq 1$ and $\delta = \|L^{-1}\|_{cb} - 1$, then

$$\|S\|_{cb} \leq \frac{1 + \delta}{(2 - (1 + \delta)^2)^{1/2}}$$

and $\|S^{-1}\|_{cb} \leq 1 + \delta$. From now, we assume that $\delta \leq \frac{1}{10}$. One easily checks, computing derivative that

$$\mu(S) = \max\left\{\|S\|_{cb} - 1, 1 - \sqrt{\frac{2}{\|S^{-1}\|_{cb}^2} - \|S\|_{cb}^2}\right\} \leq 2\delta,$$

$$\|S\|_{cb} < 1 + 3\delta.$$

Then

$$\|S - S^*\|_{cb} \leq 2\sqrt{\left(1 + 3\delta + \sqrt{2}\delta\right)^2 - 1} + 4\delta \leq 10\sqrt{\delta} = 2f_1(\delta).$$

We get that T is invertible as soon as $5\sqrt{\delta} < \frac{1}{1+\delta} + \delta$. Let us now assume that $\delta < \frac{1}{200}$ so that

$$\|T\|_{cb} \leq \|S\|_{cb} + \|T + S\|_{cb} \leq 1 + 6\sqrt{\delta},$$

$$\|T^{-1}\|_{cb} \leq \frac{\|S^{-1}\|_{cb}}{\|S^{-1}\|_{cb}\|S - T\|_{cb}} \leq \frac{1 + \delta}{1 - 5(1 + \delta)\sqrt{\delta}} \leq 1 + 8\sqrt{\delta}.$$

Now we get $\mu(T) \leq 40\sqrt{\delta}$. Thus basic estimates lead to $\|T^{-1}\|_{cb}\left\|T^{\vee}\right\|_{cb} \leq 180\sqrt{\delta}$, so we can choose $f_3(1 + \delta) = 180\sqrt{\delta}$ for $\delta < 1$. We need $f_3(1 + \delta) < 1/11$ to apply Proposition (4.2.2), so we assume $\delta < 2.10^{-7}$. Hence for the von Neumann algebras case of Theorem (4.1.1), we could choose $\varepsilon_0 = 2.10^{-7}$, but we will improve this bound later.

As $\mathrm{C} = 10$ in Proposition (4.2.2), $\mathrm{f}_4 = 10\mathrm{f}_3$. Moreover $\mathrm{f}_6(1 + \delta) = \|L(1)\| - 1 = 3\delta$. Finally we obtain $d_{H,cb}(\pi(\mathcal{A}), u\mathcal{B}u^*) \leq 3620\sqrt{\delta}$.

We need $d_{H,cb}(\pi(\mathcal{A}), u\mathcal{B}u^*) < 1/420000$ to conclude for separable nuclear C^*-algebras. Finally, Theorem (4.1.1) for separable nuclear C^*-algebras is true with $\varepsilon_0 = 3.10^{-19}$.

For von Neumann algebras, as explained in the proof of Theorem (4.1.1), we only need to deal with S. Hence $\varepsilon_0 = 4.10^{-6}$ is enough to ensure $\|S^{-1}\|_{cb}\|S^{\vee}\|_{cb} \leq 88\sqrt{\delta} < 1/11$ and to get the conclusion.

We now give a counter-example to Amir-Cambern theorem for general (nuclear) C^* -algebras. We have the following:

Theorem (4.2.4) [4]:

There exist a family $(C_\theta)_{\theta\in[0,\pi[}$ of non separable C^* -subalgebras of $\mathbb{B}(\ell_2)$ containing $\mathbb{K}(\ell_2)$ such that $d_{cb}(C_\theta, C_\tau) \leq C|\theta - \tau|, d_{\ell_2}(C_\theta, C_\tau) \leq C|\theta - \tau|$ for some $C > 0$ but C_0 is not isomorphic to C_θ with $\theta \neq 0$.

Actually the isomorphisms $C_\theta \to C_\tau$ are completely positive. This shows somehow that Theorem (4.1.1) is optimal in full generality. We conclude with the following application.

Corollary (4.2.5) [4]:

There exists two (non separable) non isomorphic C^* -algebras $\mathcal{C}$ and $\mathcal{D}$ such that

$d_{cb}(\mathcal{C},\mathcal{D}) = 1$.

Proof:

Just take $\mathcal{C} = \bigoplus_{\theta\in\mathbb{Q}\cap[0,1[} \mathcal{C}_\theta$ and $\mathcal{D} = \bigoplus_{\theta\in\mathbb{Q}\cap[0,1[} \mathcal{C}_\theta$. For any $\varepsilon > 0$, there is a bijection $f : \mathbb{Q} \cap [0,1[\to \mathbb{Q} \cap]0,1[$ with $|f(x) - x| < \varepsilon$, thus $d_{cb}(\mathcal{C},\mathcal{D}) < C\varepsilon$. We now explain why $\mathcal{C}$ and $\mathcal{D}$ are not isomorphic. First notice that as any $\mathcal{C}_\theta$ contains the compact operators of the space it acts on, we deduce that the minimal centeral projections in $\mathcal{C}$ and $\mathcal{D}$ are exactly the projection onto the $\mathcal{C}_\theta$'s. Hence there is a minimal central projection $p \in \mathcal{C}$ with $p\mathcal{C} = \mathcal{C}$ whereas $q\mathcal{D} \neq \mathcal{C}_0$ for all minimal central projections $q \in \mathcal{D}$.

For details on the notion of length. To prove that the length function is locally constant, the first step is to notice that the length is stable for cb-close multiplications, then an application of Theorem (4.1.10) allows to conclude.

We have the next lemma.

Lemma (4.2.6) [4]:

Let $S, T : \mathcal{X} \to \mathcal{Y}$ be two completely bounded linear maps between operator spaces such that $\tilde{T} : \mathcal{X}/\ker T \to \mathcal{Y}$ is a cb-isomorphism with $\|\tilde{T}^{-1}\|_{cb} \leq K$. If $\|T - S\|_{cb} < 1/K$ then $\tilde{S} : \mathcal{X}/\ker S \to \mathcal{Y}$ is also a cb-isomorphism and

$$\|\tilde{S}^{-1}\|_{cb} \leq \frac{K}{1 - K\|T - S\|_{cb}}.$$

Proof:

Let y be in the unit ball of $\mathbb{M}_n(\mathcal{Y})$. Then there exists x_0 in $\mathbb{M}_n(\mathcal{X}), \|x\| < K$ such that $T(x_0) = y$. Hence

$$\|y - S(x_0)\| < \alpha,$$

where $\alpha = K\|T - S\|_{cb} < 1$. Applying the same procedure to $\frac{1}{\alpha}(y - S(x_0))$, we obtain x_1 in $\mathbb{M}_n(\mathcal{X})$, $\|x_1\| < K$ such that

$$\|y - S(x_0 + \alpha x_1)\| < \alpha^2$$

proceeding by induction, we obtain the result.

We know that an operator space $\mathcal{A}$ endowed with a multiplication m (in the sense of Definition (4.2.1) is cb-isomorphic via an algebra homomorphism to an actual operator algebra. As the length is invariant under algebraic cb-isomorphisms, it makes sense to talk about the length of $\mathcal{A}$ equipped with m. As max is functorial, remain true in the case of a completely bounded multiplication (not necessarily completely contractive). We denote by m^ℓ the ℓ-linear map defined (by associativity) on $\mathcal{A}^\ell$.

Proposition (4.2.7) [4]:

Let $\mathcal{A}$ be a unital operator algebra of length at most ℓ and length constant at most K. Let m be another multiplication on $\mathcal{A}$ such that $\left\|\mathsf{m}^\ell - \mathsf{m}_\infty^\ell\right\|_{cb} < 1/K$. Then $\mathcal{A}$ equipped with m has also length at most ℓ.

Proof:

Let us denote by T_ℓ: $\max(\mathcal{A})^{\otimes_h \ell} \to \mathcal{A}$ (resp. $S_\ell : \max(\mathcal{A})^{\otimes_h \ell} \to \mathcal{A}$) the completely bounded linear map induced by the original multiplication $\mathsf{m}_{\mathcal{A}}$ (resp. by the new multiplication m) on the ℓ-fold Haagerup tensor product of $\max(\mathcal{A})$ (i.e. A endowed with its maximal operator space structure). The hypothesis that $\mathcal{A}$ has length at most ℓ and length constant at most K exactly means that $\tilde{T}_\ell : \max(\mathcal{A})^{\otimes_h \ell}/\ker T_\ell \to \mathcal{A}$ is a cb-isomorphism with $\left\|\tilde{T}_\ell^{-1}\right\|_{cb} \leq K$. But $\left\|\mathsf{m}^\ell - \mathsf{m}_{\mathcal{A}}^\ell\right\|_{cb} < 1/K$ implies that $\|T_\ell - S_\ell\|_{cb} < 1/K$. By Lemma (4.2.6), $\tilde{S}_\ell : \max(\mathcal{A})^{\otimes_h \ell}/\ker S_\ell \to \mathcal{A}$ is also a cb-isomorphism, so the result follows.

We have just proved that the length is stable under cb-isomorphisms with small bounds. More generally, we have the following general principle: any property which is stable under perturbation by cb-close multiplications is also stable under perturbation by cb-isomorphisms with small bounds.

List of symbols

References

[1] R. Exel: Semigroupid c^*- algebras: J.Math Anal . Appl. 377(2011) 303– 318.

[2] Ilan . Hirshberg and Jonvor Orovitz :tracially Z- Absorbing c^*- Algeberas : J. math Anal . Appl. 265 (2013) (1-17).

[3] Lawence G. Brown and Gert K. Pedersen: NON- Stable K- Theory and Externally Rich c^* - Algebras : J, Math Anal. Appl 267 (2014) 1-34.

[4] Eric Ricard , Jean Roydor:Anon commutative Amir-combern theorem for Von Neumann Agebras and nuclear c^* Agebras J. of Functional Anal267(2011) 1121-1136.

[5] Alex Kumjian and David Pask: Hifher Rank Graph c^* - Algebras: New York J . Math 6(2000) 1-20.

[6] Araki, H, Woods : E j(1968): Classification of Factors: Publ . Res Inst. Math. Sci. Ser. A4(1) 51-130.

[7] A. Kumjian. D. Pask. I.Raeburn . cuntz . Krieger algebras of directed graphs . Pacific. J. Math184 (1998) .161-174.

[8] A.Msinclair. R.R.SMITH, Hochschild cohomology of Von Neumann Algebras : London Math ., Vol 203, Cambridge University Press, Cambridge 1995.