Sudan University of Sciences & Technology

College of Graduate Studies

Almost Over Complete and Over Total Sequences with Polynomials and Complex Structures on Banach Space

المتتاليات شبه فوق التامة وفوق الكاملة

مع كثيرات الحدود والبنيات المركبة على فضاء باناخ

A thesis submitted in partial fulfillment for the degree of

M.Sc in Mathematics

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Dedication

To my dear lovely parents..

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Abstract

We introduce an almost over complete sequence in a Banach space and almost overtotal sequence in a dual space. We show that any of such sequences is relatively norm- compact. We study Banach space of traces of real polynomials on the Euclidean space to a compact subsets equipped with supremum norms. We develop a notion of a dimension where a Banach space with a uniformly bounded action of sofic group is a sofic approximation. We also develop a notion of the dimension with an embeddable group and the space of finite- dimensional Schatten p- class operators. We give examples of real Banach spaces with exactly infinite countably many complex structures.

الخلاصــة

تم إدخال متتالية شبه فوق التامة في فضاء باناخ ومتتالية فوق الكاملة في الفضاء المزدوج. أوضحنا أنه لأي من مثل هذه المتتاليات هي نظيم- متراص نسبياً. درسنا فضاء باناخ لآثار كثيرات الحدود الحقيقية على الفضاء الاقليدي إلى الفئات الجزئية المتراصة المتزنة مع نظيم أقل حد أعلى. تم تطوير فكرة البعد حيث فضاء باناخ مع عمل محدود منتظم من زمرة سوفيك هي تقريب سوفيك. أيضاً تم تطوير فكرة البعد مع زمرة التضمين وفضاء من مؤثرات عائلة $p -$ شاتن منتهية البعد. أعطينا أمثلة من فضاءات باناخ الحقيقية مع بنيات مركبة كثيرة قابلة للعد لا نهائية بالضبط.

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Chapter 1
Almost Over Complete and Almost Over Total Sequences in Banach Spaces

A sequence in a Banach space X is said to be over complete in X whenever the linear span of any its subsequence is dense in X. It is well-known facts that overcomplete sequences exist in any separable Banach space. In the spirit of this notion, we introduce the new notion of overtotal sequence and weaken both these notions to that ones of almost overcomplete sequence and almost overtotal sequence.

Section (1.1): Main Results

We show that any bounded almost overcomplete sequence as well as any bounded almost overtotal sequence is relatively norm-compact. We feel that these facts provide useful tools for attacking many questions: several applications are presented to support this feeling.

We use standard Geometry of Banach Spaces. In particular, [S] stands for the closure of the linear span of the set S and by "subspace" we always mean "closed subspace".

Let us start by the following definitions.

Definition (1.1.1) [1]: Let X be a Banach space. A sequence in the dual space X^* is said to be overtotal on X whenever any of its subsequence is total over X.

If X admits a total sequence $\{x_n^*\} \subset X^*$, then there is an overtotal sequence on X. Indeed, put $Y = [\{x_n^*\}]$: Y is a separable Banach space, so it has an overcomplete sequence $\{y_n^*\}$. It is easy to see that $\{y_n^*\}$ is overtotal on X.

As an easy example of an overtotal sequence, consider $X = A(D)$, where $A(D)$ is the usual Banach disk algebra A(D) (also spelled disc algebra) is the set of holomorphic functions ($f: D \to C$), where D is the open unit disk in the complex plane C, f extends to a continous function on the closure of D. That is:

$$A(D) = H^\infty(D) \cap C(\overline{D}),$$

Where $H^\infty(D)$ denotes the Banach space of bounded analytic finction on the unit disc D (i.e. a Hardy space). When endowed with the point wise addition, $(f + g)(z) = f(z) + g(z)$, and point wise multiplication,

$$(fg)(z) = f(z)\, g(z),$$

This set becomes an algebra over C, since if f and g belong to the disk algebra then so do $f+g$ and fg.
Given the uniform norm

$$\|f\| = \sup\{|f(z)|: z \in D\} = \max\{|f(z)|: z \in \overline{D}\},$$

By construction it becomes a uniform algebra and a commutative Banach algebra.

By construction the disc algebra is closed subalgebra of the Hardy space H^{∞}. In contrast to the stronger requirement that is a continuous extension to the circle exists, it is lemma of Fatou that a general element of H^{∞} can be radially extended to the circle almost everywhere [5].

Whose elements are the holomorphic functions on the open unit disk D of the plane that admit continuous extension to ∂D, and $\{x_n^*\} = \{z_n|_{A(D)}\}$ where $\{z_n\}$ is any sequence of points of D converging inside D.

Definition (1.1.2) [1]: A sequence in a Banach space X is said to be almost over complete when-ever the closed linear span of any of its subsequence has finite codimension in X.

Definition (1.1.3) [1]: Let X be a Banach space. A sequence in the dual space X^* is said to be almost overtotal on X whenever the annihilator (in X) of any of its subsequence has finite dimension.

Clearly, any overcomplete < overtotal > sequence is almost overcomplete < almost overtotal > and the converse is not true. It is easy to see that, if $\{(x_n, x_n^*)\}$ is a countable biorthogonal system, then neither $\{x_n\}$ can be almost overcomplete in $[\{x_n\}]$, nor $\{x_n^*\}$ can be almost overtotal on $[\{x_n\}]$. In particular, any almost overcomplete sequence has no basic subsequence.

Theorem (1.1.4) [1]: Each almost overcomplete bounded sequence in a Banach space is relatively norm-compact.

Proof. Let $\{x_n\}$ be an almost overcomplete bounded sequence in a (separable) Banach space $(X, \|\cdot\|)$. Without loss of generality we may assume, possibly passing to an equivalent norm, that the norm $\|\cdot\|$ is locally uniformly rotund (LUR) and that $\{x_n\}$ is normalized under that norm.

First note that $\{x_n\}$ is relatively weakly compact: otherwise, it is known that it should admit some subsequence that is a basic sequence, a contradiction. Hence, by the Eberlein- Smulyan theorem states that the three are equivalent on a banach space. While this equivalence is truein general for a (metric space), the weak topology is not metrizable in infinite dimensional vector spaces, and so the Eberlein- Smulian theorem is needed [6], then $\{x_n\}$ admits some subsequence $\{x_{n_k}\}$ that weakly converges to some point $x_0 \in B_X$. Two cases must now be considered.

(i) $\|x_0\| < 1$. From $\|x_{n_k} - x_0\| \geq 1 - \|x_0\| > 0$, according to a well known result, it follows that some subsequence $\{x_{n_{k_i}} - x_0\}$ is a basic sequence: hence $codim\left[\{x_{n_{k_{2i}}} - x_0\}\right] = codim\left[\{x_{n_{k_{2i}}}\}, x_0\right] = codim\left[\{x_{n_{k_{2i}}}\}\right] = \infty$, a contradiction.

(ii) $\|x_0\| = 1$. Since we are working with a LUR norm, the subsequence $\{x_{n_k}\}$ actually converges to x_0 in the norm too and we are done.

As a first immediate consequence we get the following Corollary.

Corollary (1.1.5) [1]: Let X be a Banach space and $\{x_n\} \subset B_X$ be a sequence that is not relatively norm-compact. Then there exists an infinite-dimensional subspace Y of X^* such that $|\{x_n\} \cap Y^T| = \infty$. For instance this is true for any δ-separated sequence $\{x_n\} \subset B_X(\delta > 0)$.

Theorem (1.1.6) [1]: Let X be a separable Banach space. Any bounded sequence that is almost overtotal on X is relatively norm-compact.

Proof. Let $\{f_n\}_{n=1}^{\infty} = 1 \subset X^*$ be a bounded sequence almost overtotal on X. Without loss of generality, like in the proof of Theorem (1.1.4), we may assume $\{f_n\} \subset S_{X^*}$. Let $\{f_{n_K}\}$ be any subsequence of $\{f_n\}$: since X is separable, without loss of generality we may assume that $\{f_{n_K}\}$ weakly converges, say tof_0.

Let Z be a separable subspace of X^* that is 1-norming for X. Put $Y = [\{f_n\}_{n=0}^{\infty}, Z]$. Clearly X isometrically embeds into Y^* (we isometrically embed X into X^{**} in the usual way) and X is 1-norming for Y. There is an equivalent norm $|||\cdot|||$ on Y such that, for any sequence $\{h_k\}$ and h_0 in Y,

$$h_k(x) \to h_0(x)\ \forall x \in X \ implies\ |||h_0||| \leq \liminf |||h_k||| \qquad (1)$$

and, in addition,

$$|||h_k||| \to |||h_0|||\ implies\ |||h_k - h_0||| \to 0. \qquad (2)$$

Take such an equivalent norm on Y and put $h_k = f_{n_k}$ and $h_0 = f_0$. By (2), we are done if we prove that $|||h_k||| \to |||h_0|||$. Suppose to the contrary that

$$|||f_{n_k}||| \nrightarrow |||f_0|||. \qquad (3)$$

From (1) it follows that there are $\{n_{k_i}\}$ and $\delta > 0$ such that $\left|\left|\left|f_{n_{k_i}}\right|\right|\right| - |||f_0||| > \delta$, that forces $\left|\left|\left|f_{n_{k_i}} - f_0\right|\right|\right| > \delta$ for i big enough. It follows that some subsequence $\left\{f_{n_{k_{i_m}}} - f_0\right\}_{m=0}^{\infty}$ is a w^*-basic sequence (remember that $Y \subset X^*, X$ is separable and $|||\cdot|||$ is equivalent to the original norm on Y). For $m = 1, 2, \dots$ put $g_m = f_{n_{k_{i_m}}}$. Since $\{g_m - f_0\}$ is a w^*–basic sequence, it follows that for some sequence $\{x_m\}_{m=1}^{\infty}$ in X

$$\{(g_m - f_0, x_m)\}_{m=1}^{\infty} \text{ is a biorthogonal sequence.} \qquad (4)$$

Only two cases must be now considered.

(i) For some sequence $\{m_j\}_{j=1}^{\infty}$ we have $f_0\left(x_{m_j}\right) = 0, j = 1, 2, \dots$: in this case $\left\{\left(g_{m_j}, x_{m_j}\right)\right\}$ would be a biorthogonal system, contradicting the fact that $\left\{g_{m_j}\right\}$ is almost overtotal on X.

(ii) There exists q such that for any $m \geq q$ we have $f_0(x_m) \neq 0$. For any $j > q$, from (4) it follows

$$0 = (g_{3j} - f_0)\big(f_0(x_{3j-1}) \cdot x_{3j-2} - f_0(x_{3j-2}) \cdot x_{3j-1}\big)$$
$$= g_{3j}\big(f_0(x_{3j-1}) \cdot x_{3j-2} - f_0(x_{3j-2}) \cdot x_{3j-1}\big).$$

It follows that the almost overtotal sequence $\{g_{3j}\}_{j=q}^{\infty}$ annihilates the subspace $W = \left[\left\{\big(f_0(x_{3j-1}) \cdot x_{3j-2} - f_0(x_{3j-2}) \cdot x_{3j-1}\big)\right\}_{j=q}^{\infty}\right] \subset X$: being $\{x_m\}_{m=1}^{\infty}$ a linearly independent sequence, W is infinite-dimensional, a contradiction.

Hence (3) does not work and we are done.

As an immediate consequence we get the following Corollary.

Corollary (1.1.7) [1]: Let X be an infinite-dimensional Banach space and $\{f_n\} \subset B_{X^*}$ be a sequence that is not relatively norm-compact. Then there is an infinite-dimensional sub-space $Y \subset X$ such that $|\{F_n\} \cap Y^{\perp}| = \infty$. For instance this is true for any δ-separated sequence $\{f_n\} \subset B_{X^*} (\delta > 0)$.

Section (1.2): Applications

The following theorem easily follows from Corollary (1.1.7) [1].

Theorem (1.2.1) [1]: Let $X \subset C(K)$ be an infinite-dimensional subspace of $C(K)$ where $K$is metric compact. Assume that, for $\{t_n\}_{n\in\mathbb{N}} \subset K$, the sequence $\{t_n|_X\} \subset X^*$ is not relatively norm-compact. Then there are an infinite-dimensional subspace $Y \subset X$ and a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ such that $y(t_{n_k}) = 0$ for any $y \in Y$ and for any $k \in \mathbb{N}$.

Remark. Sequences $\{t_n\} \subset K$ as required in the statement of Theorem (1.2.1) always exist: trivially, for any sequence $\{t_n\}$ dense in K, the sequence $\{t_n|_X\}$, being a 1-norming sequence for X, cannot be relatively norm-compact (since X is infinite-dimensional).

For any infinite-dimensional subspace $X \subset C(K)$, there are an infinite-dimensional subspace $Y \subset X$ and a sequence $\{t_k\}_{k\in\mathbb{N}} \subset K$ such that $y(t_k) = 0$ for any $y \in Y$ and any $k \in \mathbb{N}$. Theorem (1.2.1) strengthens this result. In fact actually, for any infinite-dimensional subspace $X \subset C(K)$, we can find such a sequence $\{t_k\}$ as a suitable subsequence $\{w_{n_k}\}$ of any prescribed sequence $\{w_n\} \subset K$ for which $\{w_n|_X\} \subset X^*$ is not relatively norm-compact.

There exist an infinite-dimensional subspace of l_∞ every non-zero element of which has only finitely many zero-coordinates? Let us reformulate this question in the following equivalent way: does there exist an infinite-dimensional subspace $Y \subset l_\infty$ such that the sequence $\{e_n|_X\}$ of the "coordinate functionals" is overtotal on Y?

Since the sequence $\{e_n|_Y\}$is norming for Y, it is not norm-compact (Y is infinite dimensional), hence by Theorem (1.1.6) [1] it cannot be overtotal on Y. So the answer to the Aron- Gurariy's question is negative. Actually we can say much more. In fact, from Theorem (1.1.6) [1] it follows that there exist an infinite-dimensional subspace $Z \subset Y$ and a strictly increasing sequence $\{n_k\}$ of integers such that $\{e_{n_k}(z) = 0\}$ for every $z \in Z$ and $k \in \mathbb{N}$.

The next Theorem generalizes the previous argument.

Theorem (1.2.2) [1]: Let X be a separable infinite-dimensional Banach space and $T: X \to l_\infty$ be a one-to-one bounded non compact linear operator. Then there exist an infinite- dimensional subspace $Y \subset X$ and a strictly increasing sequence $\{n_k\}$ of integers such that $e_{n_k}(Ty) = 0$ for any $y \in Y$ and for any k (e_n the "n-coordinate functional" on l_∞).

Proof. Assume to the contrary that for any sequence of integers $\{n_k\}$ we have $dim\left(\{T^* e_{n_k}\}^{\top}\right) < \infty$. Then the sequence $\{T^*(e_n)\} \subset X^*$ is almost overtotal on X, so $K = \|\cdot\| - cl\{T^*(e_n)\}$ is norm-compact in X^* by Theorem (1.1.6) Clearly we can consider B_X as a subset of $C(K)$ (by putting, for $x \in B_X$ and $t \in K, x(t) = t(x)$). We claim that B_X is relatively norm-compact in $C(K)$. In fact, B_X is clearly bounded in $C(K)$ and its elements are equi-continuous since, for $t_1, t_2 \in K$ and $x \in B_X$, we have

$$|x(t_1) - x(t_2)| \leq \|x\| \cdot \|t_1 - t_2\| \leq \|t_1 - t_2\| :$$

we are done by the Ascoli-Arzela theorem. Since, for $x \in X$ we have $\|x\|_{C(K)} = \|Tx\|_{l_\infty}, T(B_X)$ is relatively norm-compact in l_∞ too. This leads to a contradiction since we assumed that T is not a compact operator.

Let now X be an infinite-dimensional space and $\{f_n\} \subset X^*$ a norming sequence for X. By Theorem (1.1.6), the fact that $\{f_n\}$ is not relatively norm-compact immediately forces $\{f_n\}$ not to be overtotal on X. Since any norming sequence is a total sequence, it follows that any norming sequence for any infinite-dimensional space X admits some subsequence that is not a norming sequence for X. In other words and following our terminology, "overnorming" sequences do not exist.

As one more application of Theorem (1.1.6) [1] we obtain the following Theorem.

We need some preparation. First note that, without loss of generality, from now on we may assume that T has norm one and that the unconditional basis $\{u_i\}_{i=1}^{\infty}$ is normalized and unconditionally monotone (i.e., if $x = \sum_{i=1}^{\infty} \alpha_i u_i$ and $\sigma \subset \mathbb{N}$, then $\|\sum_{i\in\sigma} \alpha_i \beta_i u_i\| < \|x\|$for any choice of β_i with $|\beta_i| \leq 1$.

Lemma (1.2.3) [1]: Let X, Y be infinite-dimensional Banach spaces, Y having an unconditional basis $\{u_i\}_{i=1}^{\infty}$ with $\{e_i\}_{i=1}^{\infty}$ as the sequence of the associated coordinate functionals.

Let $T: X \to Y$ be a one-to-one bounded non compact linear operator. Then there exists $\delta > 0$ such that, for any natural integer m, some point $z \in B_X$ exists (depending on m) such that $\|Tz\| \geq \delta$ and the first m coordinates of Tz are 0.

Proof. Let us start by:

$\exists\{x_k\}_{k=1}^{\infty} \subset B_X, \exists\, 0 < \beta < 1 : e_i(T_{x_k}) \to 0 \ as\ k \to \infty\ \forall i \in \mathbb{N} \wedge$
$\|T_{x_k}\| > \beta\ \forall k \in \mathbb{N}.$ (5)

In fact, let $\{z_n\}_{n=1}^{\infty}$ be any $r-$separated sequence in $T(B_X)$ for some $r > 0$ ($T(B_X)$ is not pre-compact). By a standard diagonal procedure we can select a subsequence $\{z_{n_k}\}$ such that, for any $i \in \mathbb{N}$, the numbers $e_i(z_{n_k})$ converge as $k \to \infty$. Of course, for any i we have $e_i(z_{n_k} - z_{n_{k+1}}) \to 0$ as $k \to \infty$ with $||z_{n_k} - z_{n_{k+1}}|| \geq r$. For each k, put $2y_k = z_{n_{2k}} - z_{n_{2k+1}}$: since $T(B_X)$ is both convex and symmetric with respect to the origin, it is clear that $\{y_k\}_{k=1}^{\infty} \subset T(B_X)$ too; moreover for any k we have $||y_k|| \geq r/2$ and for any i we have $e_i(y_k) \to 0$ as $k \to \infty$. So it is enough to assume $x_k = T^{-1}y_k$ for any k and $\beta = r/2$ and (5) is proved.

Now fix $m \in \mathbb{N}$. Put $L = [\{T^*(e_n)\}_{n=1}^{m}]^{\top}$ and let $x \in X$. Then, denoting by $q: X \to X/L$ the quotient map, for some positive constant C_m independent on x it is true that

$$\begin{aligned} dist(x, L) = \|q(x)\| &= Sup\left\{|f(q(x))| : f \in S_{\left(\frac{X}{L}\right)^*}\right\} \\ &= Sup\left\{|g(x)| : g \in S_{[\{T^*(e_n)\}_{n=1}^{m}]}\right\} \\ &\leq C_m Max\{|e_n(Tx)| : 1 \leq n \leq m\}. \qquad (6) \end{aligned}$$

Take $\{x_k\}_{k=1}^{\infty}$ as in (5): some $\tilde{k} \in \mathbb{N}$ exists such that

$$C_m Max\{|e_n(Tx_{\tilde{k}})| : 1 \leq n \leq m\} < \beta/2$$

that by (6) implies

$$dist(x_{\tilde{k}}, L) < \beta/2.$$

Let $2z \in L$ be such that $\|x_{\tilde{k}} - 2z\| < \beta/2$: clearly $\|z\| < 1$ and $\|Tz\| > (\|T_{x_{\tilde{k}}}\| - \beta/2)/2$, so, since $\|T_{x_{\tilde{k}}}\| > \beta$, we are done by assuming $\delta = \beta/4$.

Lemma (1.2.4) [1]: Let Y be as in the statement of Lemma (1.2.3) [1]. Then for any

$$n \in \mathbb{N}, 0 < \varepsilon \leq \frac{1}{2}, v = \sum_{i=1}^{n} v_i u_i\,, w = \sum_{i=1}^{n} w_i u_i \ with\ \|v\| < \varepsilon^2 \ and\ \|w\| > 1 - \varepsilon, \quad (7)$$

there exists $j, 1 \leq j \leq n,$ *such that* $|v_j| < \varepsilon |w_j|$.

Proof. Recall that, under our assumptions, basis $\{u_i\}$ is unconditionally monotone.

Hence, without loss of generality, we may assume that $w_i \neq 0, i = 1,\ldots,n$. Moreover, for any $n \in \mathbb{N}$, any scalars $\alpha_1,\ldots,\alpha_n$ and $|\beta_1|,\ldots,|\beta_n| \leq 1$, the following is true

$$\left\|\sum_{i=1}^{n} \beta_i \alpha_i u_i\right\| \leq \left\|\sum_{i=1}^{n} \alpha_i u_i\right\|. \tag{8}$$

Assume to the contrary that some v, w exist satisfying (7) for some $\varepsilon, 0 < \varepsilon < 1/2$, for which $|v_i| \geq \varepsilon|w_i|$ (i.e.$\varepsilon|w_i/v_i| \leq 1$) for every $i, 1 \leq i \leq n$. By putting in (8) $\alpha_i = v_i/\varepsilon$ and $\beta_i = \varepsilon w_i/v_i$ for any i, we get

$$1 - \varepsilon < \left\|\sum_{i=1}^{n} w_i u_i\right\| \leq \left\|\sum_{i=1}^{n} v_i u_i/\varepsilon\right\| < \epsilon$$

that gives $\varepsilon > 1/2$, a contradiction.

Theorem (1.2.5) [1]: Let X, Y be infinite-dimensional Banach spaces, Y having an unconditional basis $\{u_i\}_{i=1}^{\infty}$ with $\{e_i\}_{i=1}^{\infty}$ as the sequence of the associated coordinate functionals.

Let $T: X \to Y$ be a one-to-one bounded non compact linear operator. Then there exist an infinite-dimensional subspace $Z \subset X$ and a strictly increasing sequence $\{k_i\}$ of integers such that $e_{k_i}(Tz) = 0$ for any $z \in Z$ and any $l \in \mathbb{N}$.

Proof. By Lemma (1.2.3) [1], a bounded sequence $\{x_n\}_{n=1}^{\infty}, \|x_n\| < R$ for some $R > 0$, can be found in X such that $T_{x_n} \in S_Y$ for every n and $e_j(T_{x_n}) = 0, j = 1,\ldots,n$. For any n put

$$T_{x_n} = y_n = \sum_{i=n+1}^{n} y_n^i u_i.$$

Now we are going to construct a subsequence $\{y_{n_k}\}_{k=0}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ with special properties.

Put in short $1/2^{n+1} = \varepsilon_n, n = 1, 2, \ldots.$

Put $n_0 = 1$ and let $p_0 > n_0$ be such that $y_{n_0}^{p_0} \neq 0$

Take $n_1 \geq p_0$ such that

$$\left\|\sum_{n_1+1}^{\infty} y_{n_0}^i u_i\right\| < \varepsilon_1^2.$$

Let $n_2 > n_1$ such that (remember that our basis is unconditionally monotone)

$$\left\|\sum_{n_2+1}^{\infty}|y_{n_0}^i|u_i\right\|+\left\|\sum_{n_2+1}^{\infty}|y_{n_1}^i|u_i\right\|<\varepsilon_2^2$$

and consider the two vectors

$$v_1=\sum_{n_1+1}^{n_2}y_{n_0}^i u_i\,,w_1=\sum_{n_1+1}^{n_2}y_{n_1}^i u_i:$$

clearly we have $\|v_1\|<\varepsilon_1^2$ and $\|w_1\|>1-\varepsilon_2^2>1-\varepsilon_1$, hence by Lemma (1.2.4) [1] an integer $p_1, n_1, +1\le p_1, n_2$ can be found such that

$$\frac{\left|y_{n_0}^{p_1}\right|}{\left|y_{n_1}^{p_1}\right|}<\varepsilon_1.$$

Now take $n_3>n_2$ such that

$$\sum_{j=0}^{2}\sum_{n_3+1}^{\infty}\left\|\,\left|y_{n_j}^i\right|u_i\right\|<\varepsilon_3^2$$

and consider the two vectors

$$v_2=\sum_{n_2+1}^{n_3}\left(\left|y_{n_0}^i\right|+\left|y_{n_1}^i\right|\right)u_i\,,w_2=\sum_{n_1+1}^{n_2}y_{n_2}^i u_i:$$

clearly we have $\|v_2\|<\varepsilon_2^2$ and $\|w_2\|>1-\varepsilon_3^2>1-\varepsilon_2$, hence by Lemma (1.2.4) [1] an integer $p_2, n_2, +1\le p_1, n_3$, can be found such that

$$\frac{\left|y_{n_0}^{p_2}\right|+\left|y_{n_1}^{p_2}\right|}{\left|y_{n_2}^{p_2}\right|}<\varepsilon_2.$$

It is now clear how to iterate the process, so getting a sequence $\{y_{n_k}\}_{k=0}^{\infty}$ in $S_{T(X)}$, a corresponding subsequence $\{p_k\}_{k=0}^{\infty}$ being determined such that for $k\ge 0$

$$n_k+1\le p_k\ \le n_{k+1}\wedge\frac{\sum_{j=0}^{k-1}\left|y_{n_j}^{p_k}\right|}{\left|y_{n_k}^{p_k}\right|}<\varepsilon_k. \qquad (9)$$

Put

$$E=\left[\left\{y_{n_k}\right\}_{k=0}^{\infty}\right],W=T^{-1}\,(E),\tilde{e}_k=e_{p_k}/y_{n_k}^{p_k},k=0,1,2,\ldots.$$

Clearly we have

$$\tilde{e}_k\left(y_{n_i}\right)=0\ if\ k<i,\tilde{e}_k\left(y_{n_k}\right)=\ 1,k\ =0,1,2,\ldots. \qquad (10)$$

Note that, by our construction, $\{y_{n_k}\}_{k=0}^{\infty}$ is a sufficiently small perturbation of a block basis of the basis $\{u_i\}$. Hence it is an unconditional basis for E. Let B its basis constant.

We claim that $\{T^*\tilde{e}_k|_W\}_{k=1}^{\infty} \subset W^*$ is a bounded sequence. Clearly it is enough to prove that $\{\tilde{e}_k|_E\}_{k=1}^{\infty}$ is bounded. In fact, for any $k \in \mathbb{N}$ and any $y = \sum_{i=1}^{\infty} a_i y_{n_i} \in S_E$, taking into account (10) and (9) we have

$$|\tilde{e}_k(y)| = \left|\tilde{e}_k\left(\sum_{i=0}^{\infty} a_i y_{n_i}\right)\right| = \left|\tilde{e}_k\left(\sum_{i=0}^{k} a_i y_{n_i}\right)\right| \leq \sum_{i=0}^{k} |a_i| \left|\tilde{e}_k\left(y_{n_i}\right)\right|$$
$$\leq 2B(\varepsilon_k + 1) < 4B.$$

Moreover we claim that it is a $1/R$-separated sequence. In fact for any k, m with $k > m \geq 0$, again remembering (10), we have

$$\|T^*\tilde{e}_k - T^*\tilde{e}_m\| \geq \left|(T^*\tilde{e}_k)\left(x_{n_k}/R\right) - (T^*\tilde{e}_m)\left(x_{n_k}/R\right)\right|$$
$$= \left(\frac{1}{R}\right)\left|(\tilde{e}_k)\left(y_{n_k}\right) - (\tilde{e}_m)\left(y_{n_k}\right)\right| = 1/R$$

Hence, by Theorem (1.1.6), the sequence $\{T^*\tilde{e}_k|_W\}_{k=1}^{\infty}$ cannot be almost overtotal on W: it means that there is an infinite-dimensional subspace $Z \subset W$ that annihilates some subsequence of the sequence $\{T^*\tilde{e}_k\}$.

The proof is complete.

Chapter 2
Banach Spaces of Polynomials as "Large" Subspaces of ℓ^{∞}-Spaces

Recall that the Banach-Mazur distance between two k-dimensional real Banach spaces E, F is defined as

$$d_{BM}(E, F) := \inf\{\|u\| \cdot \|u^{-1}\|\},$$

where the infimum is taken over all isomorphisms $u: E \to F$. We say that E and F are equivalent if they are isometrically isomorphic (i.e., $d_{BM}(E, F) = 1$). Then ln d_{BM} determines a metric on the set $\mathcal{B}_k$ of equivalence classes of isometrically isomorphic k-dimensional Banach spaces (called the Banach-Mazur compactum). It is known that $\mathcal{B}_k$is compact of d_{BM}-"diameter" $\sim k$.

Let $C(K)$ be the Banach space of real continuous functions on a compact Hausdorff space K equipped with the supremum norm. Let $F \subset C(K)$ be a filtered subalgebra with filtration $\{0\} \subset F_0 \subseteq F_1 \subseteq \cdots \subseteq F_d \subseteq \cdots \subseteq F$ (that is, $F = \bigcup_{d \in \mathbb{Z}_+} F_i$ and $F_i \cdot F_j \subset F_{i+j}$ for all $i, j \in \mathbb{Z}_+$) such that $n_d := \dim F_d < \infty$ for all d. In what follows we assume that F_0 contains constant functions on K. We have the following:

Theorem (2.1) [2]: Suppose there are $c \in \mathbb{R}$ and $\{p_d\}_{d \in \mathbb{N}} \subset \mathbb{N}$ such that

$$\frac{\ln n_{d.p_d}}{p_d} \leq c \; for \; all \; d \in \mathbb{N}. \qquad (1)$$

Then there exist linear injective maps $i_d: F_d \hookrightarrow \ell^{\infty}_{n_{d.p_d}}$ such that

$$d_{BM}\big(F_d, i_d(F_d)\big) \leq e^c, \qquad d \in \mathbb{N}.$$

As a corollary we obtain:

Corollary (2.2) [2]: Suppose $\{n_d\}_{d \in \mathbb{N}}$ grows at most polynomially in d, that is,

$$\exists \; k, \hat{c} \in \mathbb{R}_+ \; such \; that \; \forall \; d \quad n_d \leq \hat{c} d^k. \qquad (2)$$

Then for each natural number $s \geq 3$ there exist linear injective maps $i_{d,s}: F_d \hookrightarrow \ell^{\infty}_{N_{d,s}}$, where $N_{d,s} := \left\lfloor \hat{c} d^k . s^k . \left(\lfloor \ln(\hat{c} d^k) \rfloor + 1\right)^k \right\rfloor$, such that

$$d_{BM}\left(F_d, i_{d,s}(F_d)\right) \leq (e s^k)^{\frac{1}{s}}, \qquad k \in \mathbb{N}.$$

Let $\mathcal{F}_{\hat{c},k}$be the family of all possible filtered algebras F on compact Hausdorff spaces K satisfying condition (2). By $\mathcal{B}_{\hat{c},k,\bar{n}_d} \subset \mathcal{B}_{\bar{n}_d}$ we denote the closure in $\mathcal{B}_{\bar{n}_d}$ of the set formed by all subspaces F_d of algebras $F \in \mathcal{F}_{\hat{c},k}$ having a fixed dimension $\bar{n}_d \in \mathbb{N}$.

Corollary (2.2) [2] allows to estimate the metric entropy of $\mathcal{B}_{\hat{c},k,\bar{n}_d}$. Recall that for a compact subset $S \subset \mathcal{B}_{\bar{n}_d}$ its ε-entropy ($\varepsilon > 0$) is defined as $H(S,\varepsilon) := \ln N(S, d_{BM}, 1+\varepsilon)$, where $N(S, d_{BM}, 1+\varepsilon)$ is the smallest number of open d_{BM}-"balls" of radius $1+\varepsilon$ that cover S.

Corollary (2.3) [2]: For $k \geq 1$ there exists a numerical constant C such that for each $\varepsilon \in \left(0, \frac{1}{2}\right]$

$$H(\mathcal{B}_{\hat{c},k,\bar{n}_d}, \varepsilon) \leq (Ck.\ln(k+1))^k.(\hat{c}d^k)^2.(\ln(\hat{c}d^k)+1)^{k+1}.\left(\frac{1}{\varepsilon}\right)^k.\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{k+1}$$

Let $\mathcal{F}_d^n$ be the space of real polynomials on $\mathbb{R}^n$ of degree at most d. For a compact subset $K \subset \mathbb{R}^n$ by $\mathcal{P}_d^n|_K$ we denote the trace space of restrictions of polynomials in $\mathcal{P}_d^n$ to K equipped with the supremum norm. Applying Corollary (2.1.2) to algebra $\mathcal{P}^n|_K := \cup_{d\geq 0} \mathcal{P}_d^n|_K$ we obtain:

(A) There exist linear injective maps $i_{d,k}: \mathcal{P}_d^n|_K \hookrightarrow \ell_{N_{d,n}}^\infty$, where

$$N_{d,n} = \lfloor e^{2n}.(n+2)^{2n}.d^n.(2n+1+\lfloor n \ln d \rfloor)^2 \rfloor, \qquad (1)$$

such that

$$d_{BM}\left(\mathcal{P}_d^n|_K, i_{d,K}(\mathcal{P}_d^n|_K)\right) \leq (e.(n+2)^2)^{\frac{1}{n+2}} (< 2.903). \qquad (2)$$

Indeed,

$$\widehat{N}_{d,n} := \dim \mathcal{P}_d^n|_K \leq \binom{d+n}{n} < \left(\frac{e.(d+n)}{n}\right)^n \leq \left(\frac{e.(1+n)}{n}\right)^n.d^n < e^{2n}.d^n. \qquad (3)$$

Hence, Corollary (2.2) [2] with $c = e^{2n}, k = n$ and $s := (n+2)^2$ implies the required result.

If K is $\mathcal{P}^n$-determining (i.e., no nonzero polynomial vanish on K), then $\widehat{N}_{d,n} = \binom{d+n}{n}$ and so for some constant $c(n)$ (depending on n only) we have

$$\widehat{N}_{d,n} < N_{d,n} \leq c(n).\widehat{N}_{d,n}.\left(1+\ln \widehat{N}_{d,n}\right)^n. \qquad (4)$$

Hence, $V_{d,n}(K) := i_{\ d,K}(\mathcal{P}_d^n|_K)$ is a "large" subspace of $\ell_{N_{d,n}}^\infty$. Therefore from (A) applied to $V_{d,n}(K)$ we obtain:

(B) There is a constant $c_1(n)$ (depending on n only) such that for each $\mathcal{P}^n$-determining compact set $K \subset \mathbb{R}^n$ there exists an m-dimensional subspace $F \subset \mathcal{P}_d^n|_K$ with

$$m := \dim F > c_1(n).\left(\widehat{N}_{d,n}\right)^{\frac{1}{2}} \; and \; d_{BM}(F, \ell_m^\infty) \le 3. \qquad (5)$$

In turn, if $\hat{d} \in \mathbb{N}$ is such that $N_{d,n} \le c_1(n).\left(\widetilde{N}_{d,n}\right)^{\frac{1}{2}}$, then due to property (A) for each $\mathcal{P}^n$-determining compact set $K' \subset \mathbb{R}^n$ there exists $\alpha\widetilde{N}_{\tilde{d},n}$-dimensional subspace $F_{\hat{d},n,K'} \subset F$ such that

$$d_{BM}\left(F_{\hat{d},n,K'}, \mathcal{P}_{d'}^n\big|_{K'}\right) < 9. \qquad (6)$$

Further, the dual space $\left(V_d^n\,(K)\right)^*$ of $V_d^n\,(K)$ is the quotient space of $\ell_{N_{d,n}}^1$. In particular, the closed ball of $\left(V_d^n\,(K)\right)^*$ contains at most $c(n) \cdot \widetilde{N}_{d,n} \cdot \left(1 + \ln \widetilde{N}_{d,n}\,\right)^n$ extreme points, see (4). Thus the balls of $\left(V_d^n\,(K)\right)^*$ and $V_d^n\,(K)$ are "quite different" as convex bodies.

This is also expressed in the following property (similar to the celebrated John ellipsoid theorem but with an extra logarithmic factor) which is a consequence of property (A):

(C) There is a constant $c_2(n)$ (depending on n only) such that for all $\mathcal{P}^n$-determining compact sets $K_1, K_2 \subset \mathbb{R}^n$

$$d_{BM}\left(\mathcal{P}_d^n|_{K_1}\left(\mathcal{P}_d^n|_{K_2}\right)^*\right) \le c_2(n) \cdot \left(\widetilde{N}_{d,n} \cdot \left(1 + \ln \widetilde{N}_{d,n}\,\right)\right)^{\frac{1}{2}}. \qquad (7)$$

A stronger inequality is valid if we replace $\left(\mathcal{P}_d^n|_{K_2}\right)^*$ above by $\ell_{N_{d,n}}^1$.

Remark (2.4) [2]: Property (C) has the following geometric interpretation. By definition, $\left(\mathcal{P}_d^n|_{K_2}\right)^*$ is a $\widetilde{N}_{d,n}$-dimensional real Banach space generated by evaluation functional δ_x at points $x \in K_2$ with the closed unit ball being the balanced convex hull of the set $\{\delta_x\}_x \in K_2$. Thus K_2 admits a natural isometric embedding into the unit sphere of $\left(\mathcal{P}_d^n|_{K_2}\right)^*$. Moreover, the Banach space of linear maps $\left(\mathcal{P}_d^n|_{K_2}\right)^* \to \mathcal{P}_d^n|_{K_1}$ equipped with the operator norm is isometrically isomorphic to the Banach space of real polynomial maps $p: \mathbb{R}^n \to \mathcal{P}_d^n|_{K_1}$ of degree at most d (i.e., $f^* \circ p \in \mathcal{P}_d^n$ for all $f^* \in \left(\mathcal{P}_d^n|_{K_1}\right)^*$) with norm $\|p\| := \sup_{x \in K_2} \|p(x)\|_{\mathcal{P}_d^n|_{K_1}}$.Thus property (C) is equivalent to the following one:

(C') There exists a polynomial map $p: \mathbb{R}^n \to \mathcal{P}_d^n|_{K_1}$ of degree at most d such that the balanced convex hull of $p(K_2)$ contains the closed unit ball of $\mathcal{P}_d^n|_{K_1}$ and is contained in the closed ball of radius $c_2(n) \cdot \left(\widetilde{N}_{d,n} \cdot \left(1 + \ln \widetilde{N}_{d,n}\,\right)\right)^{\frac{1}{2}}$ of this space (both centered at 0).

Our next property, a consequence of Corollary (2.3) [2] and equation (3), estimates the metric entropy of the closure of the set $\tilde{\mathcal{P}}_{d,n} \subset \mathcal{B}_{\tilde{N}_{d,n}}$ formed by all $\tilde{N}_{d,n}$-dimensional spaces $\mathcal{P}_d^n|_K$with $\mathcal{P}^n$-determining compact subsets $K \subset \mathbb{R}^n$.

(D) There exists a numerical constant $c > 0$ such that for each $\varepsilon \in \left(0, \frac{1}{2}\right]$,

$$H\left(cl(\tilde{\mathcal{P}}_{d,n}), \varepsilon\right) \le (cn^2 \cdot \ln(n+1))^n \cdot d^{2n} \cdot (1 + \ln d)^{n+1} \cdot \left(\frac{1}{\epsilon}\right)^n \cdot \left(\ln\left(\frac{1}{\epsilon}\right)\right)^{n+1}. \qquad (8)$$

Remark (2.5) [2]: The above estimate shows that $\tilde{\mathcal{P}}_{d,n}$ with sufficiently large d and n is much less massive than $\mathcal{B}_{\tilde{N}_{d,n}}$. Indeed, as follows :

$$H\left(\mathcal{B}_{\tilde{N}_{d,n}}, \epsilon\right) \sim \left(\frac{1}{\epsilon}\right)^{\frac{\tilde{N}_{d,n}-1}{2}} \qquad as\ \epsilon \to 0^+$$

(here the equivalence depends on d and n as well). On the other hand, implies that for any $\varepsilon > 0$,

$$0 < \lim_{\tilde{N}_{d,n}\to\infty} \inf \frac{\ln H\left(\mathcal{B}_{\tilde{N}_{d,n}}, \epsilon\right)}{\tilde{N}_{d,n}} \le \lim_{\tilde{N}_{d,n}\to\infty} \inf \frac{\ln H\left(\mathcal{B}_{\tilde{N}_{d,n}}, \epsilon\right)}{\tilde{N}_{d,n}} < \infty.$$

It might be of interest to find sharp a symptotics of $H\left(cl(\tilde{\mathcal{P}}_{d,n}), \varepsilon\right)$, as $\epsilon \to 0^+$ and $d \to \infty$, and to compute (up to a constant depending on n) d_{BM}-"diameter" of $\tilde{\mathcal{P}}_{d,n}$.

Similar results are valid for K being a compact subset of a real algebraic variety $X \subset \mathbb{R}^n$ of dimension $m < n$ such that if a polynomial vanishes on K, then it vanishes on X as well. In this case there are positive constants $c_X, \tilde{c}_X$ depending on X only such that $\tilde{c}_X d^m \le \dim \mathcal{P}_d^n|_K \le c_X d^m$. For instance, Corollary (2.2) [2] with $c = c_X, k := m$ and $s := (m+2)^2$ implies that $\mathcal{P}_d^n|_K$ is linearly embedded into $\ell_{N_{d,X}}^\infty$, where $N_{d,X} := \lfloor c_X d^m \cdot (m+2)^{2m} \cdot (\lfloor \ln c_X d^m \rfloor + 1)^m \rfloor$, with distortion < 2.903.

Since $\dim F_i = n_i, i \in \mathbb{N}$, and evaluations δ_z at points $z \in K$determine bounded linear functionals on F_i, the Hahn-Banach theorem needs some preparation. Given a real vector space V, a function $f: V \to R$ is called sublinear if

(i) Positive Homogeneity: $f(\gamma x) = \gamma f(x)$ for all $\gamma \in R_+, x \in V$.

(ii) Subadditivity: $f(x+y) \le f(x) + f(y)$ for all $x, y \in V$.

Every seminorm on V (in norm on V) is sublinear. Other functions can be useful as well, especially Minkowski functional of convex sets. If $p: V \to R$

is a sublinear function and $\varphi: U \to R$ is a linear function on a linear subspace $U \subseteq V$ which is dominated by p on U, i.e.

$$\varphi(x) \le p(x) \qquad \forall x \in U$$

Then there exists a linear extension $\psi: V \to R$ of φ to the whole space V, i.e. there exists a linear functional ψ such that:

$$\psi(x) = \varphi(x) \qquad \forall x \in U,$$
$$\psi(x) \le \varphi(x) \qquad \forall x \in V. \text{ [7]}$$

Implies easily that span $\{\delta_z\}_{z\in K} = F_i^*$. Moreover, $\|\delta_z\|_{F_i^*} = 1$ for all $z \in K$ and the closed unit ball of F_i^* is the balanced convex hull of the set $\{\delta_z\}_{z\in K}$. Let $\{f_{1i}, \dots, f_{n_i i}\} \subset F_i$ be an Auerbach basis with the dual basis $\{\delta_{z_{1i}}, \dots, \delta_{z_{n_i}i}\} \subset F_i^*$, that is, $f_{k_i}\left(\delta_{z_{l_i}}\right) := f_{k_i}(z_{l_i}) = \delta_{k_l}$ (the Kroneckerdelta) and $\|f_{k_i}\|_K = 1$ for all k. (Its construction is similar to that of the fundamental Lagrange interpolation polynomials for $F_i = \mathcal{P}_i^n|_K$.

Now, we use a "method of E. Landau". By the definition, for each $g \in F_i$ we have $g(z) = \sum_{k=1}^{n_i} f_{ki}(z) g(z_{ki}), z \in K$. Hence, $\|g\|_K \le n_i \|g\|_{\{z_{1i},\dots,z_{n_i i}\}}$. Applying the latter inequality to $g = f^{p_d}, f \in F_d$, containing in $F_i, i := d \cdot p_d$, , and using condition (1) we get for $A_d := \{z_{1i}, \dots, z_{n_i i}\} \subset K$

$$\|f\|_K = (\|g\|_K)^{\frac{1}{p_d}} \le (n_{d\cdot p_d})^{\frac{1}{p_d}} \cdot (\|g\|_{A_d})^{\frac{1}{p_d}} \le e^c \cdot \|f\|_{A_d}.$$

Thus, restriction $F_d \mapsto F_d|_{A_d}$ determines the required map $i_d: F_d \mapsto \ell^\infty_{n_{d\cdot p_d}}$.

Proof of Corollary (2.2) [2] We set $pd := s \cdot (\lfloor \ln(\hat{c}d^k) \rfloor + 1), d \in \mathbb{N}$. Then the condition of the corollary implies

$$\frac{\ln n_{d\cdot p_d}}{p_d} \le \frac{\ln(\hat{c}d^k) + k \ln p_d}{p_d} \le \frac{1}{S} + \frac{k \ln S}{S} =: c.$$

Thus the result follows from Theorem (2.1) [2].

Proof of Corollary (2.3) [2]. We make use adapted to our setting:

Lemma (2.6) [2]: Let $S_{\bar{n}_d} \subset \mathcal{B}_{\bar{n}_d}$ be the subset formed by all $\bar{n}_d$-dimensional subspaces of $\ell^\infty_{N_{d,s}}$. Consider $0 < \xi < \frac{1}{\bar{n}_d}$ and let $R = \frac{1+\xi\bar{n}_d}{1-\xi\bar{n}_d}$. Then $S_{\bar{n}_d}$ admits an R-net T_R of cardinality at most $(1+\frac{2}{\xi})^{N_{d,s}\cdot\bar{n}_d}$.

Now given $\varepsilon \in \left(0, \frac{1}{2}\right]$ we choose $s = \lfloor s_\varepsilon \rfloor$ with s_ε satisfying $(e s_\varepsilon^k)^{\frac{1}{s_\varepsilon}} = \sqrt[4]{1+\varepsilon}$ and ξ such that $R = R_\varepsilon = \sqrt[4]{1+\varepsilon}$. Then according to

Corollary (2.1.2) and Lemma (2.2.3), $dist_{BM}\left(T_{R_\varepsilon}, \mathcal{B}_{\hat{c},k,\bar{n}_d}\right) < \sqrt{1+\varepsilon}$. For each $p \in T_{R_\varepsilon}$ we choose $q_p \in \mathcal{B}_{\hat{c},k,\bar{n}_d}$ such that $d_{BM}(p, q_p) < \sqrt{1+\varepsilon}$. Then the multiplicative triangle inequality for d_{BM} implies that open d_{BM}-"balls" of radius $1+\varepsilon$ centered at points $q_p, p \in T_{R_\varepsilon}$, cover $\mathcal{B}_{\hat{c},k,\bar{n}_d}$. Hence,

$$N\left(\mathcal{B}_{\hat{c},k,\bar{n}_d}, d_{BM}, 1+\varepsilon\right) \le card\ T_{R_\varepsilon} \le \left(1+\frac{2}{\xi}\right)^{N_{d,s}\cdot\bar{n}_d}. \tag{9}$$

Next, the function $\varphi(x) = \ln(ex^k)^{\frac{1}{x}}$decreases for $x \in \left[e^{\frac{k-1}{k}}, \infty\right)$ and $\lim_{x\to\infty} \varphi(x) = 0$. Its inverse φ^{-1} on this interval has domain $\left(0, e^{-\frac{k-1}{k}}\right]$, increases and is easily seen (using that $\varphi \circ \varphi^{-1} = id$) to satisfy

$$\varphi^{-1}(x) \le \frac{3k}{x} \cdot \ln\left(\frac{3k}{x}\right), x \in \left(0, e^{-\frac{k-1}{k}}\right].$$

Since $\frac{1}{4}\ln(1+\varepsilon) < e^{-\frac{k-1}{k}}$ for $\varepsilon \in \left(0, \frac{1}{2}\right]$, the required s_ε exists and the previous inequality implies that

$$s_\varepsilon \le \frac{12k}{\ln(1+\varepsilon)} \cdot \ln\left(\frac{12k}{\ln(1+\varepsilon)}\right). \tag{10}$$

Further, we have

$$\frac{1}{\xi} = \frac{\bar{n}_d(1+R_\varepsilon)}{R_\varepsilon - 1} = \frac{\bar{n}_d(\sqrt[4]{1+\varepsilon}+1)}{\sqrt[4]{1+\varepsilon}-1}$$

$$= \frac{\bar{n}_d(\sqrt[4]{1+\varepsilon}+1)^2 \cdot \left(\sqrt{1+\varepsilon}+1\right)}{\varepsilon}. \tag{11}$$

From (9), (10), (11) invoking the definition of $N_{d,s}$ we obtain

$$\ln N\left(\mathcal{B}_{\hat{c},k,\bar{n}_d}, d_{BM}, 1+\varepsilon\right)$$

$$\le \bar{n}_d \hat{c} d^k (\ln(\hat{c}d^k)+1)^k \ln\left(\frac{21\bar{n}_d}{\varepsilon}\right)\left(\frac{12k}{\ln(1+\varepsilon)}\ln\left(\frac{12k}{\ln(1+\varepsilon)}\right)\right)^k.$$

Using that $\bar{n}_d \le \hat{c}d^k$ and the inequality $\frac{2}{3}.\varepsilon < \ln(1+\varepsilon), \varepsilon \in \left(0, \frac{1}{2}\right]$, we get the required estimate.

Chapter 3
An l^p- Version of Von Neumann Dimension for Banach Space Representations of Sofic Groups

A theory of entropy for actions of a sofic group on a probability space or a compact metrizable space has been developed. Using this theory, it was shown for sofic groups Γ that probability measure preserving Bernoulli actions $\Gamma \curvearrowright (X,\mu)^{\Gamma}, \Gamma \curvearrowright (Y,\nu)$ are not isomorphic if the entropy of (X,μ) does not equal the entropy of (Y,ν), if Γ, and that Bernoulli actions $\Gamma \curvearrowright X^{\Gamma}, \Gamma \curvearrowright Y^{\Gamma}$ are not isomorphic as actions on compact metrizable spaces if $|X| \neq |Y|$ (here X and Y are finite). We can think of the action of Γ on $l^p(\Gamma, V)$ as analogous to a Bernoulli action, since both actions are given by translating functions on the group.

Section (3.1): Definition of the Invariants

Let Γ be a countable discrete group. Suppose that H is a closed Γ-invariant subspace of $l^2(\Gamma \times N)$, and let P_H be the projection onto H, then it is known that the number

$$dim_{L(\Gamma)}(H) = \sum_{n\in\mathbb{N}} \langle P_H \delta_{(e,n)}, \delta_{(e,n)} \rangle$$

obeys the usual properties of dimension,

Property 1: $dim_{L(\Gamma)}(H) = dim_{L(\Gamma)}(K)$, if there is a Γ -equivariant bounded linear bijection from H to K,

Property 2: $dim_{L(\Gamma)}(H \oplus K) = dim_{L(\Gamma)}(H) + dim_{L(\Gamma)}(K)$,

Property 3: $dim_{L(\Gamma)}(H) = 0$ if and only if $H = 0$,

Property 4: $dim_{L(\Gamma)}(\bigcap_{n=1}^{\infty} H_n) = \lim_{n\to\infty} dim_{L(\Gamma)}(H_n)$, if $dim_{L(\Gamma)}(H_n) < \infty$, and also $H_{n+1} \subseteq H_n$,

Property 5: $dim_{L(\Gamma)}\left(\overline{\bigcup_{n=1}^{\infty} H_n}\right) = \lim_{n\to\infty} dim_{L(\Gamma)}(H_n)$, if $H_n \subseteq H_{n+1}$.

We also have

$$dim_{L(\Gamma)}\left(l^2(\Gamma)^{\oplus n}\right) = n,$$

Voiculescu and Gournay noticed that for amenable groups Γ, we can define this dimension as a limit of normalized approximate dimensions of $F_n\Omega$, with F_n a Følner sequence, and $\Omega \subseteq H$. This formula is analogous to the definition of entropy for actions of an amenable group on a compact metrizable space or measure space. Gournay noted that a formula for von Neumann dimension similar to Voiculescu's makes senses for subspaces of $l^p(\Gamma, V)$, with Γ amenable. Using this, he defined an isomorphism invariant for subspaces of $l^p(\Gamma, V)$ agreeing with von Neumann dimension in the case $p = 2$. In particular, Gournay shows that if Γ is amenable, and there is an injective Γ-equivariant linear map of finite type with closed image from $l^p(\Gamma, V) \to l^p(\Gamma, W)$ then dim $V \leq dimW$.

Combining ideas of Kerr and Li and Voiculescu, we define an isomorphism invariant

$$\dim \Sigma, l^p(Y, \Gamma)$$

for a uniformly bounded action of a sofic group on a separable Banach space Y.

A sofic group is a group whose Cayley graph is an initially subamenable graph, or equivalently a subgroup of an ultraproduct of finite- rank symmetric groups such that every two elements of the group have distance1. They were introduced by Gromov as a common generalization of amenable and residually finite groups. The name "sofic", from the Hebrew word meaning "finite", was later applied by Weiss, following Weiss's earlier use of the same word to indicate a generalization of finiteness in sofic subshifts.

The class of sofic groups is closed under the operations of taking subgroups, extensions by amenable groups, and free products. A finitely generated group is a sofic if it is the limit of a sequence of sofic groups. Te limit of a sequence amenable groups (that is, an initially subamenable group) is necessarily sofic, but there exist sofic groups that are not initially subamenable groups [8].

This definition of dimension has the following properties:

Property 1: $\dim \Sigma, l^p(Y, \Gamma) \leq \dim \Sigma, l^p(X, \Gamma)$ if there is an equivariant bounded linear map $X \to Y$ with dense image,

Property 2: $\dim \Sigma, l^p(V, \Gamma) \leq \dim \Sigma, l^p(W, \Gamma) + \dim \Sigma, l^p(V/W, \Gamma)$, if $W \subseteq V$ is a closed Γ-invariant subspace,

Property 3: $\dim \Sigma, l^p(Y \oplus W, \Gamma) \geq \dim \Sigma, l^p(Y, \Gamma) + \underline{\dim} \Sigma, l^p(W, \Gamma)$, for $2 \leq p < \infty$, where $\underline{\dim}$ is a "lower dimension," and is also an invariant,

Property 4: $\dim \Sigma, l^p\big(l^p(\Gamma, V)\big) = \underline{\dim} \Sigma, l^p\big(l^p(\Gamma, V)\big) = \dim(V)$, for $1 \leq p \leq 2$,

Property 5: $\dim \Sigma, l^p(X, \Gamma) \geq \dim_{L(\Gamma)}\big(\bar{X}^{\|\cdot\|_2}\big)$,, when $X \subseteq l^p\big(\mathbb{N}, l^p(\Gamma)\big)$ and $1 \leq p \leq 2$.

We also note that for defining $\dim_{l^p}\ (Y,\Gamma)$, little about soficity of Γ is used, and we can more generally define our invariants associated to a sequence of maps $\sigma_i\colon \Gamma \to Isom(V_i)$ where V_i are finite-dimensional Banach spaces.

In particular, we can show that $dim_{\Sigma,l^p}(Y,\Gamma)$ can be defined for $\mathcal{R}^\omega$-embeddable groups Γ. Because unitaries also act isometrically on the space of Schatten p-class operators, we can also define an invariant

$$dim_{\Sigma,S^2}(Y,\Gamma),$$

S^p -dimension has properties analogous to l^p -dimension.

Property 1: $dim_{\Sigma,S^p}(Y,\Gamma) \leq dim_{\Sigma,S^P}(X,\Gamma)$,, if there is a Γ-equivariant bounded linear bijection $X \to Y$,

Property 2: $dim_{\Sigma,S^p}(V,\Gamma) \leq dim_{\Sigma,S^P}(W,\Gamma) + dim_{\Sigma,S^p}(V/W,\Gamma)$, if $W \subseteq V$ is a closed Γ-invariant subspace,

Property 3: $dim_{\Sigma,S^p}(Y \oplus W,\Gamma) \geq dim_{\Sigma,S^P}(Y,\Gamma) + \underline{dim}_{\Sigma,S^p}(W,\Gamma)$ for $2 \leq p < \infty$,

Property 4: $\underline{dim}_{\Sigma,S^p}(l^p(\Gamma,V) = dim(V)$ for $1 \leq p \leq 2$,

Property 5: $\underline{dim}_{\Sigma,S^p}(W,\Gamma) \geq dim_{\mathrm{L}(\Gamma)}\left(\overline{W}^{\|\cdot\|_2}\right)$ if $W \subseteq l^p\left(\mathbb{N}, l^p(\Gamma)\right)$ is a nonzero closed invariant subspace and $1 \leq p \leq 2$,

Property 6: $\underline{dim}_{\Sigma,l^2}(H,\Gamma) = dim_{\Sigma,l^2}(H,\Gamma) = dim_{\mathrm{L}(\Gamma)} H$if $H \subseteq l^2(\mathbb{N}, l^2(\Gamma))$ is Γ invariant.

In particular $l^p(\Gamma,V)$ is not isomorphic to $l^p(\Gamma,W)$ as a representation of Γ, if Γ is $\mathcal{R}^\omega$-embeddable and $1 \leq p < \infty$. This extends a result from amenable groups to $\mathcal{R}^\omega$-embeddable groups, and answers a question of Gromov in the case of $\mathcal{R}^\omega$-embeddable groups.

We recall the definition of sofic and $\mathcal{R}^\omega$-embeddable groups. To fix notation we use $Sym\ (A)$ for the group of bijections of the set A, and we let $S_n = Sym(\{1,\cdots,n\})$, finally we let $U(n)$ denote the unitary group of $\mathbb{C}^n$, where $\mathbb{C}^n$ has the usual inner product. It is useful to introduce metrics on the symmetric and unitary groups. For $\sigma,\tau \in S_n$, we define the Hamming distance by

$$d_{Hamm}(\sigma,\tau) = \frac{1}{n}|\{j\colon \sigma(j) \neq \tau(j)\}|.$$

If $A,B \in M_n(\mathbb{C})$ we let

$$\langle A,B\rangle = \frac{1}{n}Tr(B^*A),$$

note that $\langle A, B\rangle$ is indeed an inner product. We let $\|\cdot\|_2$ denote the Hilbert space norm induced by this inner product.

Definition (3.1.1) [3]: Let Γ be a countable group. A sofic approximation for Γ is a sequence of maps $\sigma_i: \Gamma \to S_{d_i}$ with $d_i \to \infty$, (not assumed to be homomorphisms) which is approximately multiplicatively and approximately free in the sense that

$$d_{Hamm}\big(\sigma_i(st), \sigma_i(s)\sigma_i(t)\big) \to 0, \quad for\ all\ s, t \in \Gamma,$$
$$d_{Hamm}\big(\sigma_i(st), \sigma_i(s')\big) \to 1, \quad for\ all\ s \neq s' \in \Gamma.$$

We say that Γ is sofic if it has a sofic approximation.

One can think of a sofic approximation σ_i as above as maps so that if

$$x_1, \ldots, x_n, y_1, \ldots, y_m \in \Gamma,$$

and $a_1, \ldots, a_n, b_1, \ldots, b_m \in \{-1, 1\}$, then with high probability,

$$\sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(j) = \sigma_i(y_1)^{a_1} \cdots \sigma_i(y_n)^{a_n}(j) if\ x_1^{a_1} \ldots x_n^{a_n} = y_1^{a_1} \ldots y_n^{a_n},$$
$$\sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(j) \neq \sigma_i(y_1)^{a_1} \cdots \sigma_i(y_n)^{a_n}(j) if\ x_1^{a_1} \ldots x_n^{a_n} \neq y_1^{a_1} \ldots y_n^{a_n},$$

The requirement $d_i \to \infty$ is not necessary since one can replace σ_i with $\sigma_i^{\otimes k_i}$ where $\sigma_i^{\otimes k_i}: \Gamma \to Sym\ (\{1, \ldots, d_i\}^{k_i})$ is given by

$$\sigma_i^{\otimes k_i}(s)\big(a_1, \ldots, a_{k_i}\big) = \Big(\sigma_i(s)(a_1) \ldots \sigma_i(s)\big(a_{k_i}\big)\Big).$$

We require that $d_i \to \infty$ simply for our properties of l^p-dimension to behave appropriately. Note that $d_i \to \infty$ is automatic when the group is infinite by our approximate freeness assumption.

A related notion is that of being $\mathcal{R}^{\omega}$-embeddable.

Definition (3.1.2) [3]: Let Γ be a countable group. An embedding sequence for Γ is a sequence of maps $\sigma_i: \Gamma \to U(d_i)$, with $d_i \to \infty$, (not assumed to be homomorphisms) such that

$$\|\sigma_i(st) - \sigma_i(s)\sigma_i(t)\|_2 \to 0 \quad for\ all\ s, t \in \Gamma,$$
$$\frac{1}{d_i} Tr\big(\sigma_i(s')^* \sigma_i(s)\big) \to 0 \quad for\ all\ s \neq s'\ in\ \Gamma.$$

A group is said to be $\mathcal{R}^{\omega}$-embeddable if it has a embedding sequence.

The second condition says that if $s \neq s'$, then asymptotically $\sigma_i(s), \sigma_i(s')$ become orthogonal under the inner product which induces $\|\cdot\|_2$. One can formulate a probabilistic interpretation of an embedding sequence analogous to that of a sofic approximation: for any $\varepsilon > 0$, if

$x_1, \dots, x_n, y_1, \dots, y_m \in \Gamma$, and $a_1, \dots, a_n, b_1, \dots, b_m \in \{-1,1\}$, then if $x_1^{a_1} \dots x_n^{a_n} = y_1^{a_1} \dots y_n^{a_n}$,

$$\mathbb{P}(\{\xi \in S^{2d_i-1}: \|\sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(\xi) - \sigma_i(y_1)^{a_1} \cdots \sigma_i(y_n)^{a_n}(\xi)\| < \varepsilon\}) \to 1,$$

and if $x_1^{a_1} \dots x_n^{a_n} \neq y_1^{a_1} \dots y_n^{a_n}$,

$$\mathbb{P}(\{\xi \in S^{2d_i-1}: |\langle \sigma_i(x_1)^{a_1} \cdots \sigma_i(x_n)^{a_n}(\xi), \sigma_i(y_1)^{a_1} \cdots \sigma_i(y_n)^{a_n}(\xi)\rangle| < \varepsilon\}) \to 1,$$

This equivalence follows by concentration of measure.

Note that if $\sigma \in S_n$ and U_σ is the unitary on $\mathbb{C}^n$ which σ induces, we have that

$$d_{Hamm}(\sigma, \tau) = d_{Hamm}(\tau^{-1}\sigma, Id) = 1 - \frac{1}{n} Tr(U_{\tau^{-1}\sigma}) = 1 - \frac{1}{2} Tr(U_\tau^* U_\sigma),$$

$$\|U_\sigma - U_\tau\|_2^2 = 2 - 2\big(1 - d_{Hamm}(\tau^{-1}\sigma, Id)\big) = 2 d_{Hamm}(\sigma, \tau)$$

thus all sofic groups are $\mathcal{R}^\omega$-embeddable.

We will sometimes use an alternate definition of $\mathcal{R}^\omega$-embeddable: a group is $\mathcal{R}^\omega$-embeddable if its group von Neumann algebra embeds into an ultraproduct of matrix algebras. For a good introduction to sofic and $\mathcal{R}^\omega$-embeddable groups.

We now give examples of sofic groups, and thus $\mathcal{R}^\omega$-embeddable groups, although most of these can be shown directly).

Example (3.1.3) [3]: All countable amenable groups are sofic. To prove this, let F_n is a Følner sequence for Γ. For $g \in \Gamma$, let $\tau_i(g): F_i \setminus g^{-1}F_i \to F_i \setminus gF_i$ be an arbitrary bijection. Define $\sigma_i: \Gamma \to Sym(F_i)$ by

$$\sigma_i(s)(x) = \begin{cases} sx & if\ x \in F_i \cap s^{-1}F_i \\ \tau_i(s)(x) & otherwise. \end{cases}$$

It follows directly from the definition of a Følner sequence that σ_i is a sofic approximation.

Example (3.1.4) [3]: All countable residually sofic groups are sofic. In particular, this includes all free groups and residually amenable groups.

Example (3.1.5) [3]: Countable locally sofic groups are sofic.

Example (3.1.6) [3]: By Malcev's Theorem all finitely generated linear groups are residually finite, hence sofic. By the preceding example all countable linear groups are sofic.

It is shown that sofic groups are closed under direct products, taking subgroups, inverse limits, direct limits, free products, and extensions by amenable groups: if $\Lambda \lhd \Gamma, \Lambda$ is sofic, and Γ/Λ is amenable, then Γ is sofic. It is also known that $\mathcal{R}^{\omega}$-embeddable groups are closed under these operations as well. It is unknown whether all countable groups are sofic. As mentioned earlier, a group is $\mathcal{R}^{\omega}$-embeddable if and only if its group von Neumann algebra embeds into an ultrapower of the hyperfinite II1 factor. It follows that if the Connes Embedding Conjecture is true, then all countable discrete groups are $\mathcal{R}^{\omega}$-embeddable. Even without the Connes Embedding conjecture we still have many examples of $\mathcal{R}^{\omega}$- embeddable groups.

Definition (3.1.7) [3]: Let X be a Banach space. An action Γ on X by is said to be uniformly bounded if there is a constant $C > 0$ such that

$$\|sx\| \leq C\|x\| \quad for\ all\ x \in X, s \in \Gamma.$$

We say that a sequence $S = (x_j)_{j=1}^{\infty}$ in X is dynamically generating, if S is bounded and $Span\ \{sx_j : s \in \Gamma, j \in \mathbb{N}\}$ is dense.

If X is a Banach space we shall write $Isom(X)$ for the group of all linear isometries from X to itself.

Definition (3.1.8) [3]: Let V be a vector space with a pseudonorm ρ. If $A \subseteq V$, a linear subspace $W \subseteq V$ is said to be ε-contain A, denoted $A \subseteq_{\varepsilon} W$, if for every $v \in A$, there is a $w \in W$ such that $\rho(v - w) < \varepsilon$. We let $d_{\varepsilon}(A, \rho)$ be the minimal dimension of a subspace which ε-contains A.

Definition (3.1.9) [3]: A dimension triple is a triple $\left(X, \Gamma, \Sigma = \left(\sigma_i : \Gamma \to Isom(V_i)\right)\right)$, where X is a separable Banach space, Γ is a countable discrete group with a uniformly bounded action on X, each V_i is finite-dimensional, and the σ_i are functions with no structure assumed on them.

Definition (3.1.10) [3]: Let $\left(X, \Gamma, \Sigma = \left(\sigma_i : \Gamma \to Isom(V_i)\right)\right)$ be a dimension triple. Fix $S = (x_j)_{j=1}^{\infty}$ a dynamically generating sequence in X.

For $e \in E \subseteq \Gamma$ finite, $l \in \mathbb{N}$ let

$$X_{E,l} = Span\{sx_j : s \in E^l, 1 \leq j \leq l\}.$$

If $e \in E \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $C, \delta > 0$, let $Hom_{\Gamma}(S, F, m, \delta, \sigma_i)_C$ be the set of all linear maps T: $X_{F,m} \to V_i$ such that $\|T\| \leq C$ and

$$\left\|T\left(s_1 \dots s_k x_j\right) - \sigma_i(s_1) \dots \sigma_i(s_k) T\left(x_j\right)\right\| < \delta$$

if $1 \leq j, k \leq m, s_1, \cdots, s_k \in F$. If $C = 1$ we shall use $Hom_{\Gamma}(S, F, m, \delta, \sigma_i)$ instead of $Hom_{\Gamma}(S, F, m, \delta, \sigma_i)_1$.

We shall frequently deal with inducing pseudonorms on $l^{\infty}(\mathbb{N}, V)$ from pseudornoms on $l^{\infty}(\mathbb{N})$. For this, we use the following notation: if ρ is a pseudonorm on $l^{\infty}(\mathbb{N})$ and V is a Banach space, we let ρ_V be the pseudonorm on $l^{\infty}(\mathbb{N}, V)$ defined by $\rho_V(f) = \rho(j \mapsto \|f(j)\|)$.

Definition (3.1.11) [3]: Let Σ, S be as in the proceeding definition and let ρ be a pseudonorm on $l^{\infty}(\mathbb{N})$. Let $\alpha_S: B\left(X_{F,m}, V_i\right) \to l^{\infty}(\mathbb{N}, V_i)$ be given by $\alpha_S(T)(j) = \chi_{\{k \le m\}}(j) T\left(x_j\right)$. We let

$$\hat{d}_{\varepsilon}(Hom_{\Gamma}(S, F, m, \delta, \sigma_i), \rho) = d_{\varepsilon}\left(\alpha_S\left(Hom_{\Gamma}(S, F, m, \delta, \sigma_i)\right), \rho V_i\right)$$

define the dimension of S with respect to ρ by

$$f.dim_{\Sigma}(S, F, m, \delta, \varepsilon, \rho) = \limsup_{i \to \infty} \frac{1}{\dim V_i} \hat{d}_{\varepsilon}(Hom_{\Gamma}(S, F, m, \delta, \sigma_i), \rho),$$

$$f.dim_{\Sigma}(S, \varepsilon, \rho) = \limsup_{\substack{e \in F \subseteq \Gamma \, finite \\ m \in \mathbb{N} \\ \delta > 0}} f.dim_{\Sigma}(S, F, m, \delta, \varepsilon, \rho),$$

$$f.dim_{\Sigma}(S, \rho) = \sup_{\varepsilon > 0} f.dim_{\Sigma}(S, \varepsilon, \rho),$$

where the pairs (F, m, δ) are ordered as follows $(F, m, \delta) \le (F', m', \delta')$ if $F \subseteq F', m \le m', \delta \ge \delta'$.

We also use

$$\underline{f.dim}_{\Sigma}(S, F, m, \delta, \varepsilon, \rho) = \liminf_{i \to \infty} \frac{1}{\dim V_i} \hat{d}_{\varepsilon}(Hom_{\Gamma}(S, F, m, \delta, \sigma_i), \rho),$$

$$\underline{f.dim}_{\Sigma}(S, \varepsilon, \rho) = \liminf_{\substack{e \in F \subseteq \Gamma \, finite \\ m \in \mathbb{N} \\ \delta > 0}} f.dim_{\Sigma}(S, F, m, \delta, \varepsilon, \rho),$$

$$\underline{f.dim}_{\Sigma}(S, \rho) = \sup_{\varepsilon > 0} f.dim_{\Sigma}(S, \varepsilon, \rho).$$

We will show that

$$f.dim_{\Sigma}(S, \rho) = \sup_{\varepsilon > 0} \liminf_{(F, m, \delta)} \limsup_{i \to \infty} \frac{1}{\dim V_i} \hat{d}_{\varepsilon}(Hom_{\Gamma}(S, F, m, \delta, \sigma_i), \rho),$$

$$\underline{f.dim}_{\Sigma}(S, \rho) = \sup_{\varepsilon > 0} \liminf_{(F, m, \delta)} \liminf_{i \to \infty} \frac{1}{\dim V_i} \hat{d}_{\varepsilon}(Hom_{\Gamma}(S, F, m, \delta, \sigma_i), \rho).$$

We introduce two other versions of dimension, which will be used to prove that the above notion of dimension does not depend on the generating sequence.

Definition (3.1.12) [3]: Let X be a separable Banach space, we say that X has the C-bounded approximation property if there is a sequence $\theta_n: X \to X$ of finite rank maps such that $\|\theta_n\| \le C$ and

$$\|\theta_n(x) - x\| \to 0, \quad for\ all\ x \in X.$$

We say that X has the bounded approximation property if it has the C-bounded approximation property for some $C > 0$.

Definition (3.1.13) [3]: Let X be a separable Banach space with a uniformly bounded action of a countable discrete group Γ. Let $q: Y \to X$ be a bounded linear surjective map, where Y is a separable Banach space with the bounded approximation property. A q-dynamical filtration is a pair $\mathcal{F} = \left((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, (Y_{E,l})_{e\in E\subseteq\Gamma\, finite, l\in\mathbb{N}}\right)$ where $a_{sj} \in Y, Y_{E,l} \subseteq Y$ is a finite dimensional linear subspace such that

(i) $sup_{(s,j)} \|a_{sj}\| < \infty$,

(ii) $q(a_{sj}) = sq(a_{ej})$,

(iii) $\left(q(a_{ej})\right)_{j=1}^{\infty}$ is dynamically generating,

(iv) $Y_{E,l} \subseteq Y_{E',l'}\ if\ E \subseteq E', l \leq l'$,

(v) $ker(q) = \overline{\bigcup_{(E,l)} Y_{E,l} \cap\ ker(q)}$,

(vi) $Y_{E,l} = Span\{a_{sj}: s \in E^l, 1 \leq j \leq l\} +\ ker(q) \cap Y_{E,l}$.

Note that if X has the bounded approximation property and $Y = X$ with q the identity, then a dynamical filtration simply corresponds to a choice of a dynamically generating sequence. In general, if $S = (x_j)_{j=1}^{\infty}$ is a dynamically generating sequence, then there is always a q-dynamical filtration $\mathcal{F} = \left((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{F,l}\right)$ such that $q(a_{ej}) = x_j$. Simply choose a_{sj} such that $\|a_{sj}\| \leq C\|x_j\|$ and $q(a_{sj}) = sx_j$ for some $C > 0$. If $(y_j)_{j=1}^{\infty}$ is a dense sequence in $ker(q)$, we can set

$$Y_{E,l} = Span\left\{a_{sj}: (s,j) \in E^l \times \{1, \dots, l\}\right\} + \sum_{j=1}^{l} \mathbb{C}y_j.$$

We can always find a Banach space Y with the bounded approximation property and a quotient map $q: Y \to X$, in fact we can choose $Y = l^1(\mathbb{N})$.

Definition (3.1.14) [3]: A quotient dimension tuple is a tuple $\left(Y, q, X, \Gamma, \sigma_i: \Gamma \to\ Isom(V_i)\right)$ where (X, Γ, σ_i) is a dimension triple, Y is a separable Banach space with the bounded approximation property and $q: Y \to X$ is a bounded linear surjection.

Definition (3.1.15) [3]: Let $\left(Y, q, X, \Gamma, \sigma_i: \Gamma \to\ Isom(V_i)\right)$ be a quotient dimension triple, and let $\mathcal{F} = \left((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{F,l}\right)$ be a q-dynamical filtration. For $e \in F \subseteq \Gamma$ finite, $m \in \mathbb{N}, \delta, C > 0$ we let $Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_C$ be the set of all bounded linear maps $T: Y \to V_i$ such that $\|T\| \leq C$ and

$$\left\|T\left(a_{s_1} \dots s_k j\right) - \sigma_i(s_1) \dots \sigma_i(s_k) T\left(a_{ej}\right)\right\| < \delta,$$

$$\left\|T|_{\ker(q) \cap Y_{F,l}}\right\| < \delta.$$

As before, if $C = 1$ we will use $Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)$ instead of $Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_C$.

Again, in the case X has the bounded approximation property, we are simply looking at almost equivariant maps from Γ to V_i, and this is similar in spirit to the definition of topological entropy. In the general case, note that genuine equivariant maps from X to V_i would correspond to maps on Y which vanish on the kernel of q, and so that

$$T\left(a_{s_1} \dots s_{k_j}\right) = \sigma_i(s_1) \dots \sigma_i(s_k) T\left(a_{ej}\right).$$

so we are still looking at almost equivariant maps on X, in a certain sense.

Definition (3.1.16) [3]: Fix a pseudonorm ρ on $l^\infty(\mathbb{N})$, let $\left(Y, q, X, \Gamma, \Sigma = (\sigma_i: \Gamma \to Isom(V_i)\right))$ be a quotient dimension tuple, and $\mathcal{F}$ a q-dynamical filtration. Let $\alpha_\mathcal{F}: B(Y, V_i) \to l^\infty(\mathbb{N}, V_i)$ be given by $\alpha\mathcal{F}(\phi) = (\phi(a_{ej}))_{j=1}^\infty$ we again use $\hat{d}_\varepsilon(A, \rho) = d_\varepsilon(\alpha_\mathcal{F}(A), \rho V_i)$. We define the dimension of $\mathcal{F}$ with respect to ρ, Σ as follows:

$$f.dim_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho) = \limsup_{i \to \infty} \frac{1}{\dim V_i} \hat{d}_\varepsilon(Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), \rho),$$

$$f.dim_\Sigma(\mathcal{F}, \varepsilon, \rho) = \lim_{\substack{e \in F \subseteq \Gamma \, finite \\ m \in \mathbb{N} \\ \delta > 0}} f.dim_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho),$$

$$f.dim_\Sigma(\mathcal{F}, \rho) = \sup_{\varepsilon > 0} f.dim_\Sigma(\mathcal{F}, \varepsilon, \rho).$$

Note that unlike $f.dim_\Sigma(S, F, m, \delta, \varepsilon, \rho)$ we know that $f.dim_\Sigma(\mathcal{F}, F, m, \delta, \varepsilon, \rho)$ is smaller when we enlarge F and m and shrink δ, thus the infimum is a limit and there are no issues between equality of limit supremums and limit infimums for this definition.

Definition (3.1.17) [3]: Let Y, X be Banach spaces, and let ρ be a pseduonorm on $B(X, Y)$. For $\varepsilon > 0, 0 < M \leq \infty$, and $A, C \subseteq B(X, Y)$, the set C is said to (ε, M) contain A if for every $T \in A$, there is an $S \in C$ such that $\|S\| \leq M$ and $\rho(S - T) < \varepsilon$. In this case we shall write $A \subseteq_{\varepsilon,M} C$. We let $d_{\varepsilon,M}(A, \rho)$ be the smallest dimension of a linear subspace which (ε, M) contains A.

Definition (3.1.18) [3]: Let $\big(Y, q, X, \Gamma, \sigma_i : \Gamma \to Isom(V_i)\big)$ be a quotient dimension tuple. Let $\mathcal{F} = \big(a_{sj}, Y_{F,l}\big)$ be a q-dynamical filtration. Fix a sequence of pseudonorms of ρ_i on $B(Y, V_i)$ and $0 < M \leq \infty$, set

$$opdim_{\Sigma,\mathrm{M}}(\mathcal{F}, F, m, \delta, \varepsilon, \rho_i) = \limsup_{i\to\infty} \frac{1}{\dim V_i}\, d_{\varepsilon,M}(Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), \rho_i),$$

$$opdim_{\Sigma,\mathrm{M}}(\mathcal{F}, \varepsilon, \rho_i) = \inf_{\substack{e\in F\subseteq \Gamma\, finite \\ m\in\mathbb{N} \\ \delta>0}} opdim_{\Sigma,\mathrm{M}}(\mathcal{F}, F, m, \delta, \varepsilon, \rho),$$

$$opdim_{\Sigma,\mathrm{M}}(\mathcal{F}, \rho_i) = \sup_{\varepsilon>0} opdim_{\Sigma,M}(\mathcal{F}, \varepsilon, \rho)\,.$$

As before, we shall use

$$\underline{opdim}_{\Sigma,\mathrm{M}}(\mathcal{F}, \rho_i), \underline{f.dim}_{\Sigma,\mathrm{M}}(\mathcal{F}, \rho\,)$$

for the same definitions as above, but replacing the limit supremum with the limit infimum.

By scaling,

$$\inf_{0<M<\infty} opdim_{\Sigma,\mathrm{M}}(\mathcal{F}, \rho_i)\,, opdim_{\Sigma,\infty}(\mathcal{F}, \rho_i), f.dim_\Sigma(S, \rho\,), f.dim_\Sigma(\mathcal{F}, \rho\,)$$

remain the same when we replace $Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), Hom_\Gamma(S, F, m, \delta, \sigma_i)$, by $Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_C, Hom_\Gamma(S, F, m, \delta, \sigma_i)_C$, for C a fixed constant. This will be useful in several proofs.

Note that if ρ is a pseudonorm on $l^\infty(\mathbb{N})$, then we get a pseudonorm $\rho_{\mathcal{F},\mathrm{i}}$ on $B(Y, V_i)$ by

$$\rho_{\mathcal{F},\mathrm{i}} = \rho\big(j \longmapsto \|T\,(a_{ej})\|\big).$$

Further, for $0 < M \leq \infty$

$$opdim_{\Sigma,\mathrm{M}}\big(\mathcal{F}, \rho_{\mathcal{F},\mathrm{i}}\big) \geq f.dim_\Sigma(\mathcal{F}, \rho\,).$$

Definition (3.1.19) [3]: A product norm ρ is a norm on $l^\infty(\mathbb{N})$ such that

(i) ρ induces a topology stronger than the product topology,

(ii) ρ induces a topology which agrees with the product topology on $\{f \in l^\infty(\mathbb{N}) : \|f\|_\infty \leq 1\}$.

Typical examples are the l^p-norms:

$$\rho(f)^p = \sum_{j=1}^{\infty} \frac{1}{2j} |f(j)|^p.$$

We shall show that there is constant $M > 0$, depending only on Y, so that if $\mathcal{F}, \mathcal{F}'$ are dynamical filtrations of q and S is a dynamically generating sequence, then for any two product norms ρ, ρ',

$$opdim_{\Sigma,\mathrm{M}}(\mathcal{F},\rho'_{\mathcal{F},i}) = opdim_{\Sigma,\mathrm{M}}(\mathcal{F},\rho_{\mathcal{F},i}) = f.dim_{\Sigma}(\mathcal{F},\rho)$$
$$= f.dim_{\Sigma}(\mathcal{F}',\rho) = dim_{\Sigma}(S,\rho)$$
and the same with dim replaced by $\underline{dim}$. In particular all these dimension only depend of the action of Γ on X, and give an isomorphism invariant. When we show all these equalities we let

$$dim_{\Sigma}(X,\Gamma)$$

denote any of these common numbers.

The equality between these dimensions is easier to understand in the case when X has the bounded approximation property. When X has the bounded approximation property, we can take $Y = X, q = Id$ and then the equality

$$opdim_{\Sigma,\mathrm{M}}(\mathcal{F},\rho_{\mathcal{F},i}) = f.dim_{\Sigma}(S,\rho),$$

says the data of local almost equivariant maps on X is the same as the data of global almost equivariant maps on X. This is essentially because if we take $\theta_{E,l}: X \to X_{E,l}$ which tend pointwise to the identity, then any almost equivariant map on $X_{E,l}$ gives an almost equivariant map on X by composing with $\theta_{E,l}$.

Since the maps $\sigma_i: \Gamma \to Isom(V_i)$ are not assumed to have any structure, this invariant is uninteresting unless the maps σ_i model the action of Γ on X in some manner. Thus we note that if Γ is a sofic group, then the maps $\sigma_i : \Gamma \to S_{d_i}$ model at least the group Γ in a reasonable manner.

Because S_n acts naturally on $l^p(n)$ we get an induced sequence of maps $\sigma_i : \Gamma \to Isom(l^p(d_i))$ and the above invariant measures how closely the action of Γ on X is modeled by these maps. When Γ is sofic, and $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$ is a sofic approximation and $\Sigma^{\mathrm{p}} = \left(\sigma_i : \Gamma \to Isom(l^p(d_i))\right)$ are the maps induced by the action of S_n on $l^p(n)$, we let

$$dim_{\Sigma,l^p}(X,\Gamma) = dim_{\Sigma^{(p)}}(X,\Gamma),$$
$$\underline{dim}_{\Sigma,l^p}(X,\Gamma) = \underline{dim}_{\Sigma^{(p)}}(X,\Gamma).$$

Similarly, if Γ is $\mathcal{R}^{\omega}$-embeddable, and $\sigma_i : \Gamma \to U(d_i)$ is a embedding sequence, then since $U(d_i)$ is the isometry group of $l^2(d_i)$ we shall let

$$dim_{\Sigma,l^2}(X,\Gamma) = dim_{\Sigma}(X,\Gamma),$$
$$\underline{dim}_{\Sigma,l^2}(X,\Gamma) = \underline{dim}_{\Sigma}(X,\Gamma).$$

Just as S_n acts on commutative l^p-Spaces, we have two natural actions of $U(n)$ on non-commutative L^p-spaces. Let $S^p(n)$ be $M_n(\mathbb{C})$ with the norm

$$\|A\|_{S^p} = Tr(|A|^p)$$

where $|A| = (A * A)^{1/2}$. Then $U(n)$ acts isometrically on $S^p(n)$ by conjugation and by left multiplication. We shall use

$$dim_{\Sigma,S^p,conj}(X,\Gamma)$$

for our dimension defined above, thinking of σ_i as a map into $Isom\big(S^p(n)\big)$ by conjugation and

$$dim_{\Sigma,S^p,multi}(X,\Gamma)$$

thinking of σ_i as a map into $Isom\big(S^p(n)\big)$ by left multiplication.

One of our main applications will be showing that when Γ is $\mathcal{R}^\omega$-embeddable

$$\underline{dim}_{\Sigma,S^p,conj}\big(l^p(\Gamma)^{\oplus \mathrm{n}},\Gamma\big) = dim_{\Sigma,S^p,conj}\big(l^p(\Gamma)^{\oplus \mathrm{n}},\Gamma\big) = n,$$

if $1 \le p \le 2$, and

$$\underline{dim}_{\Sigma,l^p}\big(l^p(\Gamma)^{\oplus \mathrm{n}},\Gamma\big) = dim_{\Sigma,l^p}\big(l^p(\Gamma)^{\oplus \mathrm{n}},\Gamma\big) = n,$$

if $1 \le p \le 2$, In particular the representations $l^p(\Gamma)^{\oplus \mathrm{n}}$ are not isomorphic for different values of n, if Γ is $\mathcal{R}^\omega$-embeddable.

We show that our various notions of dimension agree. Here is the main strategy of the proof. First we show that there is an $M > 0$, independent of $\mathcal{F}$ so that

$$opdim_{\Sigma,\mathrm{M}}\big(\mathcal{F},\rho_{\mathcal{F},i}\big) = f.dim_\Sigma(\mathcal{F},\rho),$$

the constant M comes from the constant in the definition of bounded approximation property. A compactness argument shows that

$$opdim_{\Sigma,\mathrm{M}}\big(\mathcal{F},\rho_{\mathcal{F},i}\big)$$

does not depend on the choice of pseudonorm. We then show that

$$opdim_{\Sigma,\infty}\big(\mathcal{F},\rho_{\mathcal{F},i}\big)$$

does not depend on the choice of $\mathcal{F}$, this is easier than trying to show that

$$f.dim_\Sigma(S,\rho)$$

does not depend on the choice of S. This is because the maps used to define

$$opdim_{\Sigma,\infty}\big(\mathcal{F},\rho_{\mathcal{F},i}\big)$$

all have the same domain, which makes it easy to switch from one generating set to another, since we can use that generators for $\mathcal{F}$ have to be close to linear combinations of generators for $\mathcal{F}'$. Then we show that

$$f.dim_\Sigma(\mathcal{F},\rho) = f.dim_\Sigma(S,\rho),$$
this will reduce to showing that if we are given an almost equivariant map $\phi: Y \to V_i$ which is small on the kernel of q, then there is a $T: X' \to V$ with $X' \subseteq X$ finite dimensional such that $T \circ q$ is close to ϕ on a prescribed finite set.

First we need a simple fact about spaces with the bounded approximation property.

Proposition (3.1.20) [3]: Let Y be a separable Banach space with the C-bounded approximation property, and let I be a countable directed set. Let $(Y_\alpha)_{\alpha\in I}$ be an increasing net of subspaces of Y such that
$$Y = \overline{\bigcup_\alpha Y_\alpha}.$$
Then there are finite-rank maps $\theta_\alpha: Y \to Y_\alpha$ such that $\|\theta_\alpha\| \le C$ and
$$\lim_\alpha \|\theta_\alpha(y) - y\| = 0$$
for all $\in Y$.

Proof. Fix $Y_1, \cdots, Y_k \in Y$ and $\varepsilon > 0$. Then there is a finite rank $\theta: Y \to Y$ such that
$$\|\theta(y_j) - y_j\| < \varepsilon,$$
$$\|\theta\| \le C.$$
Write
$$\theta = \sum_{j=1}^{n} \phi_j \otimes x_j$$
with $\phi_j \in Y^*$ and $x_j \in Y$. If α is sufficiently large, then we can find $x_j' \in Y_\alpha$ close enough to x_j so that if we let
$$\theta_0 = \sum_{j=1}^{n} \phi_j \otimes x_j',$$
$$\tilde{\theta} = \begin{cases} \theta_0 & if\ \|\theta_0\| \le C, \\ C\dfrac{\theta_0}{\|\theta_0\|} & otherwise. \end{cases}$$
then
$$\|\tilde{\theta}(y_j) - y_j\| < 2\varepsilon.$$
Now let $(y_j)_{j=1}^{\infty}$ be a dense sequence in Y, and let
$$\alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots$$
with $\alpha_j \in I$ be such that for all $\beta \in I$, there is a j such that $\beta \le \alpha_j$. By the

preceding paragraph, we can inductively construct an increasing sequence n_k of integers and finite-rank maps

$$\theta_k: Y \to Y_{\alpha_{n_k}}$$

such that

$$\|\theta_k\| < C,$$
$$\|\theta_k(y_j) - y_j\| \leq 2^{-k} \ \ if\ j \leq k.$$

Set $\theta_\alpha = \theta_{\alpha_{n_k}}$ if k is the largest integer such that α_{n_k} is not bigger than α. Let $\theta_\alpha = 0$ if $\alpha < \alpha_1$. Then θ_α has the desired properties.

Lemma (3.1.21) [3]: Let $\left(Y, q, X, \Gamma, \Sigma = \left(\sigma_i: \Gamma \to Isom(V_i)\right)\right)$ be a quotient dimension tuple. Let $\mathcal{F} = \left((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{F,l}\right)$ be a q-dynamical filtration and ρ a product norm, and let $C > 0$ be such that Y has the C-bounded approximation property. Fix $M > C$. Then for any $V \subseteq Y$ finite-dimensional, and $\kappa > 0$, there is a $F \subseteq \Gamma\ finite\ m \in \mathbb{N}, \delta, \varepsilon > 0$ and linear maps

$$L_i: l^\infty(\mathbb{N}, V_i) \ \to \ B(Y, V_i)$$

so that if $\phi \in Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i), f \in \ l^\infty(\mathbb{N}, V_i)$ satisfy $\rho_{V_i}(\alpha_{\mathcal{F}}(\phi) - f) < \varepsilon$, then

$$\|L_i(f)\| \leq M,$$
$$\|L_i(f)|_V - \phi|_V\| \leq \kappa.$$

Proof. Note that for every V finite-dimensional there are an $E \subseteq \Gamma$ finite, $l \in \mathbb{N}$, such that

$$\max_{\substack{v\in V\\ \|v\|=1}} \inf_{\substack{w\in Y_{E,l}\\ \|w\|=1}} \|v - w\| < \kappa,$$

so we may assume that $V = \ Y_{F,l}$ for some E, l.

Fix $\eta > 0$ to be determined later. By the preceding proposition let $\theta_{F,k}: Y \to Y_{F,k}$ be such that

$$\|\theta_{F,k}\| \leq C,$$
$$\lim_{(F,k)} \|\theta_{F,k}(y) - y\| = 0 \ \ for\ all\ y \in Y.$$

Choose F, m sufficiently large such that

$$\left\|\theta_{F,k}|_{Y_{E,l}} - Id|_{Y_{E,l}}\right\| \leq \eta.$$

Let $\mathcal{B}_{F,m} \subseteq F^m \times \{1,\cdot\cdot\cdot, m\}$ be such that $\{q(a_{sj}): (s,j) \in \mathcal{B}_{F,m}\}$ is a basis for $X_{F,m} = span\ \{q(a_{sj}): (s,j) \in F^m \times \{1,\cdot\cdot\cdot, m\}\}$. Define

$$\widetilde{L_i}: l^\infty(\mathbb{N}, V_i) \ \to \ B(X_{F,m}, V_i)$$

by

$$\widetilde{L_\iota}(f)\left(q\left(a_{sj}\right)\right) = \sigma_i(s)f(j)\ for\ (s,j)\ \in \mathcal{B}_{F,m}.$$

We claim that if $\delta > 0, \varepsilon' > 0$ are sufficently small, $\phi \in Hom_\Gamma(\mathcal{F}, F^m, m, \delta, \sigma_i)\ and\ f \in\ l^\infty(\mathbb{N}, V_i)$ satisfy

$$\rho_{V_i}\big(f - \alpha_{\mathcal{F}}(\phi)\big) < \varepsilon',$$

Then

$$\left\| \widetilde{L_\iota}(f) \circ q\big|_{Y_{F,m}} - \phi|_{Y_{F,m}} \right\| \le \eta. \qquad (1)$$

By finite-dimensionality, there is a $D(F,m) > 0$ such that if $v \in ker(q) \cap Y_{F,m}, (d_{tr}) \in \mathbb{C}^{\mathcal{B}_{F,m}}$, then

$$\sup(\|v\|, |d_{tr}|) \le D(F,m) \left\| v + \sum_{(t,r)\in\mathcal{B}_{F,m}} d_{tr}a_{tr} \right\|.$$

Thus if $x = v + \sum_{(t,r)\in\mathcal{B}_{F,m}} d_{tr}a_{tr}$ with $v \in ker(q) \cap Y_{F,m}$ has $\|x\| = 1$, then

$$\begin{aligned}
&\left\|\widetilde{L_\iota}(f)\big(q(x)\big) - \phi(x)\right\| \\
&\qquad \le D(F,m)\delta + D(F,m) \sum_{(t,r)\in\mathcal{B}_{F,m}} \|\phi(a_{tr}) - \sigma_i(t)f(r)\| \\
&\qquad \le D(F,m)\delta + D(F,m)|F|^m m\delta \\
&\qquad + \sum_{(t,r)\in\mathcal{B}_{F,m}} \|\phi(a_{er}) - f(r)\|,
\end{aligned}$$

if $\delta < \frac{\eta}{2D(F,m)(1+|F|^m m)}$, and $\varepsilon' > 0$ is small enough so that $\rho(g) < \varepsilon'$ implies

$$\sum_{(t,r)\in\mathcal{B}_{F,m}} |g(r)| < \frac{\eta}{2},$$

then our claim holds.

So assume that $\delta, \varepsilon' > 0$ are small enough so that (1) holds, and set $L_i(f) = \widetilde{L_\iota}(f) \circ q\big|_{Y_{F,m}} \circ \theta_{F,m}$. Then

$$\|L_i(f)\| \le C(1+\eta)$$

and for ϕ, f as above and $y \in Y_{E,l}$

$$\begin{aligned}
&\|L_i(f)(y) - \phi(y)\| \\
&\qquad \le (1+\eta)\left\|\theta_{F,m}(y) - (y)\right\| + \left\|\widetilde{L_\iota}(f) \circ q(y) - \phi(y)\right\| \\
&\qquad \le (2+\eta)\eta\|y\|.
\end{aligned}$$

So we force η to be small enough so that $(2+\eta)\eta < \kappa, C(1+\eta) < M$.

Lemma (3.1.22) [3]: Let $\left(Y, q, X, \Gamma, \Sigma = \left(\sigma_i : \Gamma \to Isom(V_i)\right)\right)$ be a quotient dimension tuple.

Let $\mathcal{F} = \left(\left(a_{sj}\right)_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{E,l}\right)$ be a q-dynamical filtration, and ρ a product norm, suppose that Y has the C-bounded approximation property.

(a) If $\infty \geq M > C$, then

$$f.dim_{\Sigma}(\mathcal{F},\rho) = opdim_{\Sigma,\mathrm{M}}(\mathcal{F},\rho),$$
$$\underline{f.dim}_{\Sigma}(\mathcal{F},\rho) = \underline{opdim}_{\Sigma,\mathrm{M}}(\mathcal{F},\rho).$$

(b) If ρ' is another product norm then for all $0 < M < \infty$,

$$opdim_{\Sigma,M}\left(\mathcal{F},\rho_{\mathcal{F},i}\right) = opdim_{\Sigma,\mathrm{M}}\left(\mathcal{F},\rho'_{\mathcal{F},i}\right),$$
$$\underline{opdim}_{\Sigma,M}\left(\mathcal{F},\rho_{\mathcal{F},i}\right) = \underline{opdim}_{\Sigma,\mathrm{M}}\left(\mathcal{F},\rho'_{\mathcal{F},i}\right).$$

Proof. (a) First note that

$$opdim_{\Sigma,\mathrm{M}}(\mathcal{F},\rho) \geq opdim_{\Sigma,\infty}(\mathcal{F},\rho) \geq f.dim_{\Sigma}(\mathcal{F},\rho)$$

so it suffices to handle the case that $M < \infty$.

Let $A > 0$ be such that

$$\left\|a_{sj}\right\| \leq A \quad for\ all\ (s,j) \in \Gamma \times \mathbb{N}.$$

Take $1 > \varepsilon > 0$. Let k be such that if $f \in l^{\infty}(\mathbb{N})$, and $\|f\|_{\infty} \leq 1$, and f is supported on $\{n : n \geq k\}$, then $\rho(f) < \varepsilon$. Since ρ induces a topology weaker than the norm topology, we can find an $\varepsilon > \kappa > 0$ such that

$$\rho(f) < \varepsilon$$

if

$$\|f\|_{\infty} \leq \kappa.$$

By Lemma (3.1.21), let $e \in F \subseteq \Gamma$ be finite, $m \in \mathbb{N}, \varepsilon > \varepsilon' > 0, \kappa > \delta > 0$ and $L_i : l^{\infty}(\mathbb{N}, V_i) \to B(Y, V_i)$be such that if $\phi \in Hom_{\Gamma}(\mathcal{F}, F, m, \delta, \sigma_i)$ and $f \in l^{\infty}(\mathbb{N}, V_i)$ has $(\alpha\mathcal{F}(\phi) - f) < \varepsilon'$, then

$$\left\|L_i(f)|_{Y_{\{e\},k}} - \phi|_{Y_{\{e\},k}}\right\| < \kappa,$$
$$\|L_i(f)\| \leq M.$$

Then if ϕ, f are as above we have

$$\rho_{\mathcal{F},i}\left(\phi - L_i(f)\right)$$
$$\leq (M+1)A\varepsilon + \rho\left(\chi_{l\leq k}(j)\left(\left\|\phi\left(a_{ej}\right) - L_i(f)\left(a_{ej}\right)\right\|\right)_{j=1}^{\infty}\right)$$

and for $j \leq k$

$$\left\|\phi\left(a_{ej}\right) - L_i(f)\left(a_{ej}\right)\right\| \leq A(M+1)\kappa.$$

Thus

$$\rho_{\mathcal{F},i}\big(\phi - L_i(f)\big) \le (m+1)(A+1)\varepsilon.$$

This implies that

$$d_{(M+1)(A+1)\varepsilon,M}\left(Hom_\Gamma(\mathcal{F},F',m',\delta',\sigma_i),\rho_{\mathcal{F},i}\right)$$
$$\le \hat{d}_{\varepsilon'}\left(Hom_\Gamma(\mathcal{F},F',m',\delta',\sigma_i),\rho_{\mathcal{F},i}\right)$$

for all $F' \supseteq F, m' \ge m$, and all $\delta' < \delta$. This completes the proof.

(b) This is a simple consequence of the compactness of the $\|\cdot\|_\infty$ unit ball of $l^\infty(\mathbb{N})$ in the product topology.

Lemma (3.1.23) [3]: Let $\big(Y,q,X,\Gamma,\sigma_i\colon \Gamma \to Isom(V_i)\big)$ be a quotient dimension tuple. Let $\mathcal{F},\mathcal{F}'$ be two q-dynamical filtrations. If ρ_i is any fixed sequence of pseudonorms on $B(Y,V_i)$, then for all $0 < M \le \infty$,

$$opdim_{\Sigma,M}(\mathcal{F},\rho_i) = opdim_{\Sigma,\mathrm{M}}\left(\mathcal{F}',\rho_i\right),$$
$$\underline{opdim}_{\Sigma,M}(\mathcal{F},\rho_i) = \underline{opdim}_{\Sigma,\mathrm{M}}\left(\mathcal{F}',\rho_i\right).$$

Proof. Let $\mathcal{F}' = \left(\left(a'_{sj}\right)_{(s,j)\in\Gamma\times\mathbb{N}}, Y'_{E.l}\right), \mathcal{F} = \left(\left(a_{sj}\right)_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{E,l}\right)$. We do the proof for $opdim_\Sigma$, the other case is proved in the same manner. Let $C > 0$ be such that $\|sx\| \le C\|x\|$ for all $s \in \Gamma, x \in X$ and such that $\|a_{sj}\|, \|a'_{sj}\| \le C$. Fix $F \subseteq \Gamma$ finite, and $m \in \mathbb{N}, \delta > 0$. Fix $\eta > 0$ which will depend upon F, m, δ in a manner to be determined later.

Choose $E \subseteq \Gamma$ finite $l \in \mathbb{N}$, such that for $1 \le j \le m, s \in F^m$ there are $c_{j,t,k}$ with $(t,k) \in E \times \{1,\cdots,l\}$ and $v_{sj} \in Y'_{E.l} \cap ker(q)$ such that

$$\left\| a_{sj} - v_{sj} - \sum_{(t,k)\in E\times\{1,\cdots,l\}} c_{j,t,k}\, a'_{stk} \right\| < \eta,$$

and so that for every $w \in Y_{F,m} \cap ker(q)$ there is a $v \in Y'_{E.l} \cap ker(q)$ such that $\|v - w\| \le \eta\|w\|$. Let $A(\eta) = \sup\left(|c_{j,t,k}|, \sup\|v_{sj}\|\right)$.

Set $m' = 2\max(m,l) + 1, F' = [(F \cup F^{-1} \cup \{e\})(E \cup E^{-1} \cup \{e\})]^{2m'+1}$, we claim that we can choose $\delta' > 0, \eta > 0$ small so that

$$Hom_\Gamma(\mathcal{F},F',m',\delta',\sigma_i) \subseteq Hom_\Gamma(\mathcal{F},F,m,\delta,\sigma_i).$$

If $T \in Hom_\Gamma(\mathcal{F}',F',m',\delta',\sigma_i), 1 \le j,r \le m,$ and $s_1,\cdots,s_r \in F$ then

$$\begin{aligned}\|T(a_{s_1} \dots s_r j) - \sigma_i(s_1) \dots \sigma_i(s_r)T(a_{ej})\| \\ &\le 2\eta + \|T(v_s j)\| + \|\sigma_i(s_1) \dots \sigma_i(s_r)T(v_{ej})\| \\ &+ \left\| \sum_{(t,k)\in E\times\{1,\cdots,l\}} c_{j,t,k} \left[T(a'_{s_1\dots s_r tk}) \right.\right. \\ &\left.\left. - \sigma_i(s_1) \dots \sigma_i(s_r)T(a'_{tk})\right] \right\| \\ &\le 2\eta + \delta' A(\eta) + \delta' A(\eta) + 2|E|lA(\eta)\delta'.\end{aligned}$$

By choosing $\eta < \delta/2$, and then choosing δ' very small we can make the above expression less than δ. If we also force $\delta' < \delta/2$ our choice of η implies that

$$\|T(w)\| \le \delta\|w\|$$

for T as above and $w \in Y_{F,m} \cap ker(q)$. This completes the proof.

Because of the above lemma, the only difficulty in proving that $opdim_\Sigma(\mathcal{F}, \rho\mathcal{F}, i)$ does not depend on the choice of $\mathcal{F}$ is switching the pseudonorm from $\rho_{\mathcal{F},i}$ to $\rho_{\mathcal{F}',i}$. Because of this we will investigate how the dimension changes when we switch pseudonorms.

Definition (3.1.24) [3]: Let $\left(Y, q, X, \Gamma, \Sigma = (\sigma_i\colon \Gamma \to Isom(V_i))\right)$ be a quotient dimension tuple, and fix a q-dynamical filtration F. If ρ_i, q_i are pseduornoms on $B(Y, V_i)$ we say that ρ_i is $(\mathcal{F}, \Sigma)$-weaker than q_i and write $\rho_i \preccurlyeq \mathcal{F}, \Sigma, q_i$ if the following holds. For every $\varepsilon > 0$, there are $F \subseteq \Gamma$ finite, $\delta, \varepsilon' > 0$, $m, i_0 \in \mathbb{N}$, and linear maps $L_i\colon B(Y, V_i) \to B(Y, V_i)$ for $i \ge i_0$ such that if $\phi \in Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)$ and$\psi \in B(Y, V_i)$ satisfy $q_i(\phi - \psi) < \varepsilon'$, then $\rho_i(\phi - L_i(\psi)) < \varepsilon$. We say that ρ_i is $(\mathcal{F}, \Sigma)$ equivalent to q_i, and write $\rho_i \sim \mathcal{F}, \Sigma q_i$, if $\rho_i \preccurlyeq \mathcal{F}, \Sigma\, q_i$ and $q_i \preccurlyeq \mathcal{F}, \Sigma\, \rho_i$.

Lemma (3.1.25) [3]: Let $(Y, X, q, \Gamma, \Sigma)$ be a quotient dimension tuple and $\mathcal{F}$ a q-dynamical filtration.

(a) If ρ_i, q_i are pseudonorms with $\rho_i \preccurlyeq \mathcal{F}, \Sigma\, q_i$, then

$$opdim_{\Sigma,\infty}(\mathcal{F}, \rho_{\,i}) = opdim_{\Sigma,\infty}(\mathcal{F}, q_{\,i}),$$

$$\underline{opdim}_{\Sigma,\infty}(\mathcal{F}, \rho_{\,i}) = \underline{opdim}_{\Sigma,\infty}(\mathcal{F}, q_{\,i}).$$

(b) Let $\mathcal{F}' = \left((a'_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y'_{E.l}\right), \mathcal{F} = \left((a_{sj})_{(s,j)\in\Gamma\times\mathbb{N}}, Y_{E,l}\right)$ be q-dynamical filtrations. Let ρ be any product norm.

Define a pseudonorm on $B(Y,V_i)$ by$\rho_{\mathcal{F},i}(\phi)=\rho\left(\left(\|\phi(a_{ej})\|\right)_{j=1}^{\infty}\right)$, and similarly define $\rho_{\mathcal{F}',i}$. Then

$$\rho_{\mathcal{F}',i} \preccurlyeq \mathcal{F},\Sigma\, \rho_{\mathcal{F},i}.$$

Proof. Let $\Sigma=\left(\sigma_i\colon \Gamma\to\ Isom(V_i)\right)$.

(a) This follows directly follow the definitions.

(b) Let $C>0$ be such that Y has the C-bounded approximation property and

$$\|a_{sj}\|\leq C,$$
$$\|a'_{sj}\|\leq C.$$

Choose $m\in\mathbb{N}$ such that $\rho(f)<\varepsilon$ if $\|f\|_\infty\leq 1$ and f is supported on $\{n\colon n\geq m\}$, and let $\kappa>0$ be such that $\rho(f)<\varepsilon$ if $\|f\|_\infty\leq\kappa$.

By Lemma (3.1.21) choose $F'\supseteq F$ finite $m\leq m'\in\mathbb{N}$, and $\delta,\varepsilon>0$ and

$$\tilde{L}_i\colon l^\infty(\mathbb{N},V_i)\ \to\ B(Y,V_i)$$

so that if $f\in l^\infty(\mathbb{N},V_i)$ and $\phi\in Hom_\Gamma(\mathcal{F},F',m',\delta,\sigma_i)$ has $\rho V_i(\alpha_{\mathcal{F}}(\phi)-f)<\varepsilon'$ then

$$\left\|\tilde{L}_i(f)|_{Y'_{\{e\},m}}-\phi|_{Y'_{\{e\},m}}\right\|\leq\kappa,$$
$$\|\tilde{L}_i(f)\|\leq 2C.$$

Let $L_i\colon B(Y,V_i)\to\ B(Y,V_i)$ be given by $L_i(\psi)=\tilde{L}_i\left(\alpha_{\mathcal{F}}(\psi)\right)$.

Suppose $\phi\in Hom_\Gamma(\mathcal{F},F',m',\delta,\sigma_i)$ and $\psi\in B(Y,V_i)$ satisfy $\rho_{\mathcal{F},i}(\phi-\psi)<\varepsilon'$. Then, for $1\leq j\leq m$ we have

$$\|\phi(a'_{ej})-L_i(\psi)(a'_{ej})\|\leq C\kappa$$

Our choice of m,κ then imply that $\rho_{\mathcal{F}',i}\left(\phi-L_i(\psi)\right)<2C(C+1)\varepsilon$. This completes the proof.

Corollary (3.1.26) [3]: Let $\left(Y,q,X,\Gamma,\sigma_i\colon\Gamma\to\ Isom(V_i)\right)$ be a quotient dimension tuple. Let ρ,ρ' be two product norms. For any two q-dynamical filtrations $\mathcal{F},\mathcal{F}'$ we have

$$opdim_{\Sigma,\infty}(\mathcal{F},\rho\,\mathcal{F},i)=opdim_{\Sigma,\infty}(\mathcal{F}',\rho\,\mathcal{F}',i)=opdim_{\Sigma,\infty}(\mathcal{F}',\rho'\mathcal{F}',i),$$
$$\underline{opdim}_{\Sigma,\infty}(\mathcal{F},\rho\,\mathcal{F},i)=\underline{opdim}_{\Sigma,\infty}(\mathcal{F}',\rho\,\mathcal{F}',i)=\underline{opdim}_{\Sigma}(\mathcal{F}',\rho'\mathcal{F}',i).$$

Proof. Combining Lemmas (3.1.21), (3.1.25), and (3.1.23) we have

$$opdim_{\Sigma,\infty}(\mathcal{F}',\rho'\mathcal{F}',i)=opdim_{\Sigma,\infty}(\mathcal{F}',\rho\,\mathcal{F}',i)\leq opdim_{\Sigma,\infty}(\mathcal{F},\rho\,\mathcal{F},i).$$

The opposite inequality follows by symmetry.

Because of the preceding corollary $f.dim_\Sigma(\mathcal{F},\rho)$ only depends on the action of Γ and the quotient map $q\colon Y\to X$. Thus we can define

$$dim_{\Sigma}(q,\Gamma) = opdim_{\Sigma,\infty}\left(\mathcal{F}, \rho_{\mathcal{F},i}\right) = f.dim_{\Sigma}(\mathcal{F}, \rho)$$

where $\mathcal{F}$ is any q-dynamical filtration and ρ is any product norm.

We now proceed to show that $dim_{\Sigma}(q,\Gamma)$ does not depend on q, as stated before the idea is to prove that

$$dim_{\Sigma}(q,\Gamma) = f.dim_{\Sigma}(S,\rho)$$

where S is any dynamically generating sequence for X.

For this, we will prove that we can approximate maps T on Y which almost vanish on the kernel of q, by maps on X. For the proof, we need the construction of ultraproducts of Banach spaces.

Let X_n be a sequence of Banach spaces and $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ a free ultrafilter. We define the ultraproduct of the X_n, written $\prod^{\omega} X_n$ by

$$\prod^{\omega} X_n = \left\{(x_n)_{n=1}^{\infty} : x_n \in X_n, \sup_n \|x_n\| < \infty\right\}$$
$$/\left\{(x_n)_{n=1}^{\infty} : x_n \in X_n, \lim_{n\to\omega} \|x_n\| = 0\right\}.$$

We use $(x_n)_{n\to\omega}$ for the image of $(x_n)_{n=1}^{\infty}$ under the canonical quotient map to

$$\prod^{\omega} X_n$$

If a set $A \subseteq \mathbb{N}$ is in ω, we will say that A is ω-large.

Lemma (3.1.27) [3]: Let X, Y be Banach spaces with X and $q: Y \to X$ a bounded linear surjective map. Let $F \subseteq X$ be finite and Z a finite-dimensional subspace of Y with $q(F) \subseteq Z$. Let $C > 0$ be such that for all $x \in X$, there is a $y \in Y$ with $\|y\| \le C\|x\|$ such that $q(y) = x$, and fix $A > C$. Let I be a countable directed set, and $(Y_{\alpha})_{\alpha\in I}$ a net of subspaces of Y such that $Y_{\alpha} \subseteq Y_{\beta}$ if $\alpha \le \beta$, and

$$q(Y_{\alpha}) \supseteq Z,$$

$$ker(q) = \overline{\bigcup_{\alpha} Y_{\alpha} \cap \ker(q)},$$

$$F \subseteq \bigcup_{\alpha} Y_{\alpha}.$$

Then for all $\varepsilon > 0$, there are a $\delta > 0$ and α_0 with the following property. If $\alpha \ge \alpha_0$ and W is a Banach space with $T: Y_{\alpha} \to W$ a linear contraction such that

$$\left\|T|_{\ker(q)\cap Y_{\alpha}}\right\| \le \delta,$$

then there is a $S: Z \to W$ such that $\|S\| \leq A$ and

$$\|T(x) - S \circ q(x)\| \leq \varepsilon,$$

for all $x \in F$.

Proof. Note that our assumptions imply

$$Y = \overline{\bigcup_{\alpha} Y_{\alpha}}.$$

Fix a countable increasing sequence α_n in I, such that for every $\beta \in I$ there is an n such that $\beta \leq \alpha_n$. Assume also that $F \subseteq Y_{\alpha 1}$. Since I is directed, if the claim is false, then we can find an $\varepsilon > 0$ and an increasing sequence β_n with $\beta_n \geq \alpha_n$ and a $T_n: Y_{\beta_n} \to W_n$ such that $\|T_n\| \leq 1$,

$$\left\| T|_{\ker(q) \cap Y_{\beta_n}} \right\| \leq 2^{-n},$$

and for every $S: X \to W_n$ with $\|S\| \leq A$,

$$\|T_n(x) - S \circ q(x)\| \geq \varepsilon, \ \ for\ some\ x \in F.$$

Fix $\omega \in \beta\mathbb{N} \backslash \mathbb{N}$ and let

$$W = \prod^{\omega} W_n.$$

Define

$$T: \overline{\bigcup_{n} Y_{\beta_n}} \to W$$

by

$$T\,(x) = \left(T_n(x)\right)_{n \to \omega},$$

note that for any k, the map T_n is defined on Y_{β_k} for $n \geq k$, so T is well-defined. Also

$$\|T(x)\| \leq \|x\|,$$

$$T(x) = 0\ on \bigcup_{n} Y_{\beta_n} \cap \ker(q).$$

Our density assumptions imply that T extends uniquely to a bounded linear map, still denoted T, from Y to W, which vanishes on the kernel of q. Thus there is $S: Z \to W$ such that $T = S \circ q$, and our hypothesis on C implies that $\|S\| \leq C$.

Since Z is finite dimensional, we can find $S_n: X \to W_n$ such that $S(x) = \left(S_n(x)\right)_{n \to \omega}$. Compactness of the unit sphere of Z and a simple diagonal argument show that

$$C \geq \|S\| = \lim_{n \to \omega} \|S_n\|.$$

Thus $B = \{n: \|S_n\| < A\}$ is an ω-large set, and by hypothesis

$$B = \bigcup_{x \in F} \{n \in B: \|T_n(x) - S_n(q(x))\| \geq \varepsilon\}$$

Since B is ω-large, there is some $x \in F$ such that

$$\{n \in B: \|T_n(x) - S_n(q(x))\| \geq \varepsilon\}$$

is ω-large. But then $T(x) \neq S \circ q(x)$, a contradiction.

Lemma (3.1.28) [3]: Let $\left(Y, q, X, \Gamma, \Sigma = \left(\sigma_i: \Gamma \to Isom(V_i)\right)\right)$ be a quotient dimension tuple. Fix a dynamically generating sequence S in X, and ρ a product norm. Then

$$dim_\Sigma(q, \Gamma) = f.dim_\Sigma(S, \rho),$$
$$\underline{dim}_\Sigma(q, \Gamma) = \underline{f.dim}_\Sigma(S, \rho).$$

Proof. We will only do the proof for dim .

Let $S = (x_j)_{j=1}^{\infty}$ and let $\mathcal{F} = \left((a_{sj})_{(s,j) \in \Gamma \times \mathbb{N}}, Y_{E,l}\right)$ be a dynamical filtration such that $q(a_{ej}) = xj$. Let $C > 0$ be such that

$$\sup_{(s,j)} \|a_{sj}\| \leq C,$$
$$\sup_j \|x_j\| \leq C,$$
$$\|q\| \leq C,$$

for every $x \in X$, there is a $y \in Y$ such that $q(y) = x$ and $\|y\| \leq C\|x\|$, and so that Y has the C-bounded approximation property. By Proposition, (3.1.20), we may find $\|\theta_{E.l}\|: Y \to Y_{E.l}$ such that $\|\theta_{E.l}\| \leq C$ and

$$\lim_{(E,l)} \|\theta_{E.l}(y) - y\| = 0 \quad for\ all\ y \in Y.$$

We first show that

$$dim_\Sigma(q, \Gamma) \geq f.dim_\Sigma(S, \rho).$$

For this, fix $\varepsilon > 0$, and choose $r \in \mathbb{N}$ such that

$$\rho(f) < \varepsilon, \quad \text{if } f \text{ is supported on } \{n: n \geq r\}\ and\ \|f\|_\infty \leq 1,$$

as before choose $\varepsilon \geq \kappa > 0$ such that if $\|f\|_\infty \leq \kappa$, then

$$\rho(f) < \varepsilon.$$

Let $e \in E \subseteq \Gamma$ finite and $l \in \mathbb{N}$ be such that if $E \subseteq F \subseteq \Gamma$ is finite, and $k \geq l$ then

$$\|\theta_{F,k}(a_{ej}) - a_{ej}\| < \kappa$$

for $1 \leq j \leq r$.

Now fix $E \subseteq F \subseteq \Gamma$ finite, $l \leq m \in \mathbb{N}, \delta > 0$. We claim that we can find $F \subseteq F' \subseteq \Gamma$ finite $m \leq m'$ in $\mathbb{N}, \delta > \delta' > 0$ such that

$$Hom_\Gamma(S, F', m', \delta', \sigma_i) \circ q|_{Y_{F',m'}} \circ \theta_{F',m'} \subseteq Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_{C^2}.$$

For $T \in Hom_\Gamma(S, F', m', \delta', \sigma_i)$, for $1 \leq j, k \leq m$ and $s_1, \cdots, s_k \in F$,

$$\begin{aligned} \left\|T \circ q \circ \theta_{F',m'}(a_{s_1} \dots s_k j) - \sigma_i(s_1) \dots \sigma_i(s_k) T \circ q \circ \theta_{F',m'}(a_{ej})\right\| \\ \leq C\left\|\theta_{F',m'}(\alpha_{s_1} \dots s_k j) - \alpha_{s_1} \dots s_k j\right\| \\ + C\left\|\theta_{F',m'}(a_{ej}) - a_{ej}\right\| \\ + \left\|T(\alpha_{s_1} \dots s_k x_j) - \sigma_i(s_1) \dots \sigma_i(s_k) T(x_j)\right\| \\ < C\left\|\theta_{F',m'}(\alpha_{s_1} \dots s_k x_j) - \alpha_{s_1} \dots s_k j\right\| \\ + C\left\|\theta_{F',m'}(a_{ej}) - a_{ej}\right\| + \delta'. \end{aligned}$$

Also for $y \in ker(q) \cap Y_{F,m}$ we have

$$\left\|T \circ q \circ \theta_{F',m'}(y)\right\| \leq C\left\|\theta_{F',m'}(y) - y\right\|.$$

So it suffices to choose $\delta' < \min(\delta, \kappa)$ and then $F' \supseteq F, m' \geq \max(m, l, r)$ such that

$$C\left\|\theta_{F',m'}(\alpha_{s_1} \dots s_k j) - \alpha_{s_1} \dots s_k j\right\| + C\left\|\theta_{F',m'}(a_{ej}) - a_{ej}\right\| < \delta - \delta',$$

$$C\left\|\theta_{F',m'}|_{Y_{F,m}} - Id|_{Y_{F,m}}\right\| < \delta,$$

for $1 \leq j, k \leq m$ and $s_1, \cdots, s_k \in F$.

Suppose that δ', F', m' are so chosen. If $T \in Hom_\Gamma(S, F', m', \delta', \sigma_i)$ and $\phi = T \circ q|_{Y_{F',m'}} \circ \theta_{F',m'}$ then,

$$\rho V_i(\alpha_S(T) - \alpha_{\mathcal{F}}(\phi)) \leq C(C^2 + 1)\varepsilon + \rho V_i\left(\chi_{\{j: j \leq r\}}(\alpha_S(T) - \alpha_{\mathcal{F}}(\phi))\right)$$

and if $j \leq r$,

$$\begin{aligned} \|\alpha_S(T)(j) - \alpha_{\mathcal{F}}(\phi)(j)\| = \left\|T(x_j) - T \circ q \circ \theta_{F,m}(a_{ej})\right\| \\ \leq C\kappa + \left\|T(x_j) - T \circ q(a_{ej})\right\| = C\kappa. \end{aligned}$$

Thus

$$\rho V_i(\alpha_S(T) - \alpha_{\mathcal{F}}(\phi)) \leq (C^2 + C + 1)\varepsilon.$$

Therefore

$$\hat{d}_{(C^2+C+1)\varepsilon}\left(Hom_\Gamma(S, F', m', \delta', \sigma_i), \rho\right) \leq \hat{d}_\varepsilon(Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_{C^2}, \rho).$$

Since F', m' can be made arbitrary large and δ' arbitrarily small, this implies

$$\begin{aligned} f.dim_\Sigma(S, \rho(C^2 + C + 1)\varepsilon) \\ \leq \limsup_i \frac{1}{\dim V_i} \hat{d}_\varepsilon(Hom_\Gamma(\mathcal{F}, F, m, \delta, \sigma_i)_{C^2}, \rho), \end{aligned}$$

taking the limit supremum over (F, m, δ) and then the supremum over $\varepsilon > 0$,

$$f.dim_{\Sigma}(S, \rho) \leq f.dim_{\Sigma}(q, \Gamma).$$

For the opposite inequality, fix $1 > \varepsilon > 0$ and let r, κ, E, l be as before. Fix $E \subseteq F \subseteq \Gamma$ finite, $m \geq \max(r, l)$ and $\delta < \min(\kappa, \varepsilon)$.
By Lemma (3.1.27) we can find $\delta' < \delta$, and $F \subseteq F' \subseteq \Gamma$ finite and $m \leq m' \in \mathbb{N}$ such that if W is a Banach space and

$$T: Y_{F', m'} \to W$$

has

$$\|T\| \leq 1,$$
$$\left\|T|_{\ker(q) \cap Y'_{F', m'}}\right\| \leq \delta',$$

then there is a $\phi: X_{F,m} \to W$ such that

$$\left\|T\left(a_{s_1} \dots s_k j\right) - \phi\left(s_1 \dots s_k x_j\right)\right\| \leq \delta, \quad for\ 1 \leq \mathrm{j}, \mathrm{k} \leq \mathrm{m}, s_1 \dots s_k \in \mathrm{F}$$

and $\|\phi\| \leq 2C$.

Fix $T \in Hom_{\Gamma}(\mathcal{F}, F', m', \delta', \sigma_i)$, and choose $\phi: X_{F,m} \to V_i$ such that $\|\phi\| \leq 2C$ and

$$\left\|T\left(a_{s_1} \dots s_k j\right) - \phi \circ q\left(a_{s_1} \dots s_k j\right)\right\| \leq \delta, \quad for\ 1 \leq \mathrm{j}, \mathrm{k} \leq \mathrm{m}, s_1 \dots s_k \in \mathrm{F}.$$

Thus for $1 \leq \mathrm{j}, \mathrm{k} \leq \mathrm{m}, s_1 \dots s_k \in \mathrm{F}$ we have

$$\begin{aligned} &\left\|\phi\left(s_1 \dots s_k x_j\right) - \sigma_i(s_1) \dots \sigma_i(s_k) \phi\left(x_j\right)\right\| \\ &\qquad \leq 2\delta \left\|T\left(\alpha_{s_1} \dots s_k j\right) - \sigma_i(s_1) \dots \sigma_i(s_k) T\left(a_{ej}\right)\right\| < 2\delta + \delta' \\ &\qquad < 3\delta. \end{aligned}$$

Thus $\phi \in Hom_{\Gamma}(S, F, m, 3\delta, \sigma_i)_{2C}$. Furthermore, for $1 \leq j \leq r$

$$\|\alpha_S(T)(j) - \alpha_{\mathcal{F}}(\phi)(j)\| = \left\|T\left(a_{ej}\right) - \phi \circ q(a_{ej})\right\| \leq \kappa,$$

so

$$\rho V_i\left(\alpha_{\mathcal{F}}(T) - \alpha_S(\phi)\right) \leq \varepsilon + (2C^2 + C)_{\varepsilon} = (2C^2 + C + 1)_{\varepsilon}.$$

Thus

$$\begin{aligned} &f.dim_{\Sigma}(\mathcal{F}, (C^2 + C + 2)\varepsilon, \rho) \\ &\leq \limsup_i \frac{1}{\dim V_i} \hat{d}_{\varepsilon}(Hom_{\Gamma}(S, F, m, 3\delta, \sigma_i)_{2C}, \rho), \end{aligned}$$

and since $F, m, \delta, \varepsilon$ are arbitrary this completes the proof.

Because of the preceding Lemma and Corollary (3.1.26), we know that

$$f.dim_{\Sigma}(S, \rho), dim_{\Sigma}(q, \Gamma)$$

only depend upon the action of Γ on X, and are equal. Because of this we will use

$$dim_{\Sigma}(X,\Gamma) = f.dim_{\Sigma}(S,\rho) = dim_{\Sigma}(q,\Gamma)$$

for any dynamically generating sequence S, and any bounded linear surjective map $q: Y \to X$, where Y has the bounded approximation property. We similarly define $\underline{dim}_{\Sigma}(X,\Gamma)$.

We now prove a lemma which allows us to treat the limit supremum over (F,m,δ) in the definition of f. $f.dim_{\Sigma}(S,\rho)$ as a limit.

Lemma (3.1.29) [3]: Let $(X,\Gamma,\Sigma) = \left(\sigma_i: \Gamma \to Isom(V_i)\right)$ be a dimension triple, fix a dynamically generating sequence S in X and ρ a product norm. Then

$$f.dim_{\Sigma}(S,\rho) = \sup_{\varepsilon>0}\liminf_{(F,m,\delta)} \limsup_{i} \frac{1}{\dim V_i}\ \hat{d}_{\varepsilon}(Hom_{\Gamma}(S,F,m,\delta,\sigma_i),\rho),$$

$$\underline{f.dim}_{\Sigma}(S,\rho) = \sup_{\varepsilon>0}\limsup_{(F,m,\delta)} \liminf_{i} \frac{1}{\dim V_i}\ \hat{d}_{\varepsilon}(Hom_{\Gamma}(S,F,m,\delta,\sigma_i),\rho).$$

Proof. Let $S = \left(x_j\right)_{j=1}^{\infty}$. We do the proof for dim only, the proof for $\underline{dim}$ is the same. Fix $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that if $\|f\|_{\infty} \leq 1 + \sup_{j\in\mathbb{N}}\|x_j\|$ and f is supported on $\{n: n \geq k\}$, then $\rho(f) < \varepsilon$. It suffices to show that

$$f.dim_{\Sigma}(S,\rho) \leq \sup_{\varepsilon}\liminf_{(F,m,\delta)} \limsup_{i} \frac{1}{\dim V_i}\ \hat{d}_{\varepsilon}(Hom_{\Gamma}(S,F,m,\delta,\sigma_i),\rho).$$

Fix $F \subseteq \Gamma$ finite $m \geq k, \delta > 0$. Then for any $F \subseteq F' \subseteq \Gamma$ finite, $m' \geq m, \delta' <$ and $\psi \in Hom_{\Gamma}\left(S,F',m',\delta',\sigma_i\right)$ we have $\psi \in Hom_{\Gamma}(S,F,m,\delta,\sigma_i)$.

Furthermore if $f,g \in l^{\infty}(\mathbb{N},V_i)$ are defined by

$$f(j) = \chi_{\{n\leq m\}}(j)\psi(x_j), \qquad g(j) = \chi_{\{n\leq m'\}}\psi(x_j)$$

then

$$\rho(j \mapsto \|f(j) - g(j)\|) < \varepsilon.$$

Thus

$$\hat{d}_{2\varepsilon}\left(Hom_{\Gamma}\left(S,F',m',\delta',\sigma_i\right),\rho\right) \leq \hat{d}_{\varepsilon}(Hom_{\Gamma}(S,F,m,\delta,\sigma_i),\rho).$$

Therefore

$$f.dim_{\Sigma}(S,2\varepsilon,\rho) \leq \limsup_{i} \frac{1}{\dim V_i}\ \hat{d}_{\varepsilon}(Hom_{\Gamma}(S,F,m,\delta,\sigma_i),\rho).$$

Since F,m,δ were arbitrary

$$f.dim_{\Sigma}(S,2\varepsilon,\rho) \leq \liminf_{(F,m,\delta)} \limsup_{i} \frac{1}{\dim V_i}\ \hat{d}_{\varepsilon}(Hom_{\Gamma}(S,F,m,\delta,\sigma_i),\rho),$$

and taking the supremum over $\varepsilon > 0$ completes the proof.

Section (3.2): Main Properties of $\boldsymbol{dim_\Sigma(X,\Gamma)}$ and Computation of $\boldsymbol{dim_{\Sigma,l^p}(l^p(\Gamma,V),\Gamma)}$, and $\boldsymbol{dim_{\Sigma,S^p,conk}(l^p(\Gamma,V),\Gamma)}$

The first property that we prove is that dimension is decreasing under surjective maps, as in the usual case of finite-dimensional vector spaces.

Proposition (3.2.1) [3]: Let $\left(Y,\Gamma,\Sigma=\left(\sigma_i:\Gamma\to Isom(V_i)\right)\right)$, (X,Γ,Σ) be two dimension triples. Suppose that there is a Γ -equivariant bounded linear map $T:Y\to X$, with dense image. Then

$$dim_\Sigma(X,\Gamma)\le dim_\Sigma(Y,\Gamma),$$
$$\underline{dim}_\Sigma(X,\Gamma)\le \underline{dim}_\Sigma(Y,\Gamma).$$

Proof. Let $S'=\left(y_j\right)_{j=1}^{\infty}1$ be a dynamically generating sequence for Y. Let $S=\left(T\left(x_j\right)\right)_{j=1}^{\infty}$, then S is dynamically generating for X. Then

$$Hom_\Gamma(S,F,m,\delta,\sigma_i)\circ T\subseteq Hom_\Gamma(S',F,m,\delta,\sigma_i)_{\|T\|},$$

and

$$\alpha_{S'}\,(\phi\circ T)=\alpha_S(\phi),$$

so the proposition follows.

We next show that dimension is subadditive under exact sequences. It turns out to be strong of a condition to require that dimension be additive under exact sequences. As noted if $\dim_{\Sigma,l^p}$ is additive under exact sequences and

$$\dim_{\Sigma,l^p}\left(l^p(\Gamma)^{\oplus n},\Gamma\right)=n,$$

then we can write the Euler characteristic of a group as an alternating sum of dimensions of l^p cohomology spaces. But torsion-free cocompact lattices in $SO(4,1)$ have positive Euler characteristic and their l^p cohomology vanishes when p is sufficiently large, so this would give a contradiction.

Proposition (3.2.2) [3]: Let $\left(V,\Gamma,\Sigma=\left(\sigma_i:\Gamma\to Isom(V_i)\right)\right)$ be a dimension triple. Let $W\subseteq V$ be a closed Γ-invariant subspace. Then

$$dim_\Sigma(V,\Gamma)\le dim_\Sigma(V/W,\Gamma)+dim_\Sigma(W,\Gamma),$$
$$\underline{dim}_\Sigma(V,\Gamma)\le \underline{dim}_\Sigma(V/W,\Gamma)+dim_\Sigma(W,\Gamma),$$
$$\underline{dim}_\Sigma\left(V^{\oplus n},\Gamma\right)\le n\,\underline{dim}_\Sigma(V,\Gamma).$$

Proof. Let $S_2 = (w_j)_{j=1}^{\infty}$ be a dynamically generating sequence for W, and let $S_1 = (a_j)_{j=1}^{\infty}$ be a dynamically generating sequence for V/W. Let $x_j \in V$, be such that $x_j + W = a_j$, and $\|x_j\| \le 2\|a_j\|$. Let S be the sequence

$$x_1, w_1, x_2, w_2, \cdots.$$

We shall use the product norm on $l^{\infty}(\mathbb{N})$ given by

$$\rho_1(f) = \sum_{j=1}^{\infty} \frac{1}{2^j} |f(j)|,$$

$$\rho_2(f) = \sum_{j=1}^{\infty} \frac{1}{2^j} |f(2j)| + \sum_{j=1}^{\infty} \frac{1}{2^j} |f(2j-1)|.$$

Let $\varepsilon > 0$, and choose m such that $2^{-m} < \varepsilon$. Let $e \in F_1 \subseteq \Gamma$ be finite, $m \le m_1 \in \mathbb{N}$, and $\delta_1 > 0$. Let $\eta > 0$ to be determined later. By Lemma (3.1.27), we can find a $\delta_1 > \delta > 0$, a $F_1 \in E \subseteq \Gamma$ finite, and a $m \le k \in \mathbb{N}$, so that if X is a Banach space, and

$$T: V_{E,2k} \to X$$

has $\|T\| \le 2$, and

$$\|T|_{W \cap V_{E,2k}}\| \le \delta,$$

then there is a ϕ: $(V/W)_{F_1,m_1} \to X$ with $\|\phi\| \le 3$, and

$$\|\phi(s_1, \cdots s_k a_j) - T(s_1, \cdots s_k x_j)\| < \delta_1,$$

for all $1 \le j, k \le m_1, and\ s_1, \cdots s_k \in F_1$.

By finite-dimensionality, we can find a finite set $F' \supseteq E, m' \ge 2k$, and a $0 < \delta' < \delta_1$, so that if $T: V_{F',m'} \to X$, satisfies

$$\|T(s_1, \cdots s_k x_j)\| < \delta',$$

for all $1 \le j, k \le m', and\ s_1, \cdots s_k \in F'$, then

$$\|T|_{W \cap V_{E,2k}}\| \le \delta.$$

Define

$$R: Hom_{\Gamma}(S, F', 2m', \delta', \sigma_i) \to Hom_{\Gamma}(S_2, F', m', \delta', \sigma_i)$$

by

$$R(T) = T|_{W_{F',m'}}.$$

Find

$$\Theta: im(R) \to Hom_{\Gamma}(S, F', 2m', \delta', \sigma_i)$$

so that $R \circ \Theta = Id$.

Then

$$(T - \theta(R(T))(s_1,\cdots s_k w_j) = 0,$$

for all $1 \leq j,k \leq m', and\ s_1,\cdots s_k \in F'$. Thus by assumption, we can find a

$$\phi\colon (V/W)_{F_1,m_1} \to V_i,$$

so that $\|\phi\| \leq 3$, and

$$\left\|\phi(s_1,\cdots s_k a_j) - \left(T - \theta(R(T))\right)\ (s_1,\cdots s_k x_j)\right\| < \delta_1,$$

for all $1 \leq j,k \leq m_1, and\ s_1,\cdots s_k \in F_1$, in particular,

$$\left\|\phi(a_j) - \left(T - \theta(R(T))\right)\ (x_j)\right\| < \delta_1,$$

for $1 \leq j \leq m$.

Thus whenever $1 \leq j,k \leq m_1, s_1,\cdots s_k \in F_1$,

$$\|\phi(s_1,\cdots s_k a_j) - \sigma_i\ (s_1) \ldots \sigma_i\ (s_k)\phi(a_j)\| < 2\delta_1 + 2\delta' < 4\delta_1.$$

Now suppose that

$$\alpha_{S_2}(Hom_\Gamma(S_2, F_1, m_1, \delta_1, \sigma_i)) \subseteq_{\varepsilon,\rho_1,V_i} G,$$
$$\alpha_{S_1}(Hom_\Gamma(S_1, F, m, 4\delta_1, \sigma_i)_3) \subseteq_{\varepsilon,\rho_1,V_i} F.$$

Let $E \subseteq l^\infty(\mathbb{N}, V_i)$ be the subspace consisting of all h so that there are $f \in F, g \in G$ so that

$$h(2k) = g(k), h(2k-1) = f(k).$$

Then $dim(E) = dim(F) + dim(G)$. It easy to see that

$$\alpha_S(Hom_\Gamma(S, F', m', \delta', \sigma_i)) \subseteq_{3\varepsilon+\delta_1,\rho_2,V_i} E.$$

So if $\delta_1 < \varepsilon$, we find that

$$\alpha_S(Hom_\Gamma(S, F_1, m_1, \delta', \sigma_i)) \subseteq_{3\varepsilon} E.$$

From this the first two inequalities follow.

The last inequality is easier and its proof will only be sketched. Let $S = (x_j)_{j=1}^\infty$ be a dynamically generating sequence for X, and $y_j = x_q \otimes e_r$ if $j = nq + r$, with $1 \leq r \leq n$, and $x_q \otimes e_r$ is the element of $X^{\oplus n}$ which is zero in all coordinates except for the rth, where it is x_q. If $F \subseteq \Gamma$ is finite $m \in \mathbb{N}, \delta > 0$, then

$$Hom_\Gamma(S, F, nm, \delta, \sigma_i) \subseteq Hom_\Gamma(S, F, m, \delta, \sigma_i)^{\oplus n}.$$

The rest of the proof proceeds as above.

We note here that subadditivity is not true for weakly exact sequences, that is sequences

$$0 \to X \to Y \to Z \to 0,$$

where $X \to Y$ is injective, $\overline{im(X)} = ker(Y \to Z)$, and the image of Y is dense in Z. In fact, using $\mathbb{F}_n$ for the free group on n letters $a_1, \cdots, a_n$, it is known that the map

$$\partial : l^1(\mathbb{F}_n)^{\oplus n} \to l^1(\mathbb{F}_n),$$

given by

$$\partial(f_1, \cdots, f_n)(x) = \sum_{j=1}^{n} f_j(x) - \sum_{j=1}^{n} f_j(x_j^{-1})$$

has dense image and is injective. We will show that

$$\underline{dim}_{\Sigma,l^1}(l^1(\mathbb{F}_n)^{\oplus n}, \mathbb{F}_n) = dim_{\Sigma,l^1}(l^1(\mathbb{F}_n)^{\oplus n}, \mathbb{F}_n) = n,$$
$$\underline{dim}_{\Sigma,l^1}(l^1(\mathbb{F}_n), \mathbb{F}_n) = dim_{\Sigma,l^1}(l^1(\mathbb{F}_n), \mathbb{F}_n) = 1.$$

this gives a counterexample to subadditivity under weakly exact sequences. This also gives a counterexample to monotonicity under injective maps, though once should note in this case that the map defined above does not have closed image.

For $2 \leq p \leq \infty$, we have a lower bound for direct sums, whose proof requires a few more lemmas.

Lemma (3.2.3) [3]: Let H_1, H_2 be Hilbert spaces and let $H = H_1 \oplus H_2$ and let $\Omega_j \subseteq H_j$ and suppose $C_1, C_2 > 0$ are such that $C_1 \leq \|\xi\| \leq C_2$, for all $\xi \in \Omega_j$. If $0 < \delta < C_1$, then

$$d_{C_2^{-1}\delta}(\Omega_1 \oplus 0 \cup 0 \oplus \Omega_2) \geq d_{C_1^{-1}\sqrt{5\delta}}(\Omega_1) + d_{C_1^{-1}\sqrt{5\delta}}(\Omega_2).$$

Proof. By replacing Ω_j with

$$\left\{\frac{\xi}{\|\xi\|} : \xi \in \Omega_j\right\}$$

we may assume $C_1 = C_2 = 1$. Let P_i be the projection onto each H_i, and set $\Omega = (\Omega_1 \oplus 0) \cup (0 \oplus \Omega_2)$. Suppose that V is a subspace such that $\Omega \subseteq_\delta V$, and let Q be the projection onto V and $T = QP_1Q|_V$. Define

$$\Omega_1' = Q(\Omega_1 \oplus 0), \quad \Omega_2' = Q(0 \oplus \Omega_2).$$

For $\xi \in \Omega$ we have

$$\|(1-Q)\xi\| \leq \delta$$

thus for $\xi \in \Omega_1 \oplus \{0\}$

$$\langle TQ\xi, Q\xi\rangle = \langle QP_1Q\xi, Q\xi\rangle = \|P_1Q\xi\|^2 \geq (\|\xi\| - \|P_1(1-Q)\xi\|)^2$$
$$\geq (1-\delta)^2.$$

So if $T = \int_{[0,1]} t dE(t)$ we have with $\eta = Q\xi$

$$\left(\sqrt{1-\delta^2} - \delta\right)^2 \leq \langle\left(1 - \frac{1}{2}E([0,1/2])\right)\eta, \eta\rangle \leq 1 - \frac{1}{2}\|E([0,1/2])\eta\|^2.$$

Thus

$$\|E([0,1/2])\eta\|^2 \leq 2(1-\ (1-\ \delta)^2) \leq 4\delta$$

i.e.

$$\|\eta - E((1/2,1])\,\eta\|^2 \leq\ 4\delta.$$

Thus

$$\Omega_1' \subseteq_{2\sqrt{\delta}} E((1/2,1])\,V.$$

Similarly, because $QP_2Q|_V = 1 - T$ we have

$$\Omega_2' \subseteq_{2\sqrt{\delta}} E([0,1/2])\,V.$$

For any projection P' and any $x \in H$ we have $\|x - P'x\|^2 = \|x\|^2 - \|P'x\|^2$. So for all $\xi \in \Omega_1 \oplus 0$ we have since, $QE((1/2,1]\,) = E((1/2,1])$ (and $E((1/2,1]Q = E((1/2,1])$ by taking adjoints), that

$$\begin{aligned}\|\xi - E((1/2,1])Q\xi\|^2 &= \|\xi - E((1/2,1])\xi\|^2\\ &= \|\xi\|^2 - \|E((1/2,1])\xi\|^2\\ &= \|\xi\|^2 - \|Q\xi\|^2\ + \|Q\xi\|^2 - \|E((1/2,1])\xi\|^2\\ &= \|\xi - Q\xi\|^{\,2} + \|Q\xi - E((1/2,1])Q\xi\|^2\ \leq \delta^2 + 4\delta < 5\delta.\end{aligned}$$

Thus with a similar proof for Ω_2 we have

$$\Omega_1 \oplus 0 \subseteq_{\sqrt{5\delta}} E((1/2,1])\,V,$$
$$0 \oplus \Omega_2 \subseteq_{\sqrt{5\delta}} E((0/1,2])\,V$$

since

$$V = E([0,1/2])V\ \oplus\ E\ ((1/2,1])\,V$$

the desired claim follows.

Lemma (3.2.4) [3]: Let $(X, \Gamma\,, \Sigma)$ be a dimension triple. Let S be a dynamically generating sequence in X, and ρ a product norm such that $\rho(f) \leq \rho(g)$ if $|f| \leq |g|$. Set

$$\rho^{(N)}(f) = \rho(\chi_{j\leq N}f).$$

Then

$$f.dim_\Sigma(S,\rho) = \lim_{N\to\infty} f.dim_\Sigma\big(S,\rho^{(N)}\big),$$
$$\underline{f.dim}_\Sigma(S,\rho) = \lim_{N\to\infty} \underline{f.dim}_\Sigma\big(S,\rho^{(N)}\big).$$

Proof. Let $\Sigma = \big(\sigma_i\colon \Gamma \to Isom(V_i)\big)$. Let $S = \big(x_j\big)_{j=1}^{\infty}$, $C = \sup_j\|x_j\|$.

Since $\rho^{(N)} \leq \rho$, for any $\varepsilon > 0$

$$f.dim_\Sigma\big(S,\varepsilon,\rho^{(N)}\big) \leq f.dim_\Sigma(S,\varepsilon,\rho) \leq f.dim_\Sigma(S,\rho).$$

thus

$$\limsup_{n\to\infty} f.dim_\Sigma\big(S,\rho^{(N)}\big) \leq f.dim_\Sigma(S,\rho).$$

For the opposite inequality, fix $\varepsilon > 0$. and choose N such that $\rho(f) < \varepsilon$ if $f \in l^{\infty}(\mathbb{N}, V_i)$ is supported on $\{k: k \geq N\}$ and $\|f\|_{\infty} \leq C$. Thus for $T \in B(X, V_i)$, and $f \in l^{\infty}(\mathbb{N}, V_i)$ with $\|T\| \leq 1$, and $n \geq N$ we have

$$\left|\rho_{V_i}\left(\alpha_S(T) - \chi_{\{j \leq N\}}\right) - \left(\rho_{V_i}^{(n)}\left(\alpha_S(T) - \chi_{\{j \leq N\}} f\right)\right)\right|$$
$$\leq \left|\rho_{V_i}\left(\chi_{\{k>N\}}\alpha_S(T)\right)\right| \leq \varepsilon.$$

Thus for $n \geq N$,

$$f.dim_{\Sigma}(S, 2\varepsilon, \rho) \leq f.dim_{\Sigma}\left(S, \varepsilon, \rho^{(n)}\right) \leq f.dim_{\Sigma}\left(S, \rho^{(n)}\right),$$

so

$$f.dim_{\Sigma}(S, 2\varepsilon, \rho) \leq \liminf_{n\to\infty} f.dim_{\Sigma}\left(S, \rho^{(N)}\right).$$

For the next lemma, we recall the notion of the volume ratio of a finite-dimensional Banach space. Let X be an n-dimensional real Banach space, which we will identify with $\mathbb{R}^n$ with a certain norm. By an ellipsoid in $\mathbb{R}^n$ we mean a set which is the unit ball for some Hilbert space norm on $\mathbb{R}^n$. Let $B \subseteq \mathbb{R}^n$ be the unit ball of X. We define the volume ratio of B, denoted $vr(B)$ by

$$vr(B) = \inf\left(\frac{vol\,(B)}{vol\,(D)}\right)^{1/n},$$

where the infimum runs over all ellipsoids $D \subseteq B$. It is know that for any unit ball B of a Banach space norm on $\mathbb{R}^n$, there is an ellipsoid D^{max} such that $D^{max} \subseteq B$, and D^{max}has the largest volume of all such ellipsoids. So we have

$$vr(B) = \left(\frac{vol\,(B)}{vol\,(D^{max})}\right)^{1/n}.$$

The main property we will need to know about volume ratio is the following theorem.

Theorem (3.2.5) [3]: Let $B \subseteq \mathbb{R}^n$ be the unit ball for a norm $\|\cdot\|$ on $\mathbb{R}^n$. Let $D \subseteq B$ be an ellipsoid. Set

$$A = \left(\frac{vol\,(B)}{vol\,(D)}\right)^{1/n}.$$

Let $|\cdot|$ be a norm such that D is the unit ball of $(\mathbb{R}^n, |\cdot|)$, in particular $\|\cdot\| \leq |\cdot|$. Then for all $k = 1, \cdots, n-1$ there is a subspace $F \subseteq \mathbb{R}^n$ such that $dim\, F = k$ and for every $x \in F$

$$|x| \leq (4\pi A)^{\frac{n}{n-k}}\|x\|. \qquad (2)$$

Further if we let G_{nk} be the Grassmanian manifold of k-dimensional subspaces of $\mathbb{R}^n$, then

$$\mathbb{P}(\{F \in G_{nk}: for\ all\ x \in F, Eq.(2) holds\}) > 1 - 2^{-n},$$

for the unique $O(n)$-invariant probability measure on G_{nk}.

What we will actually use is the following corollary.

Corollary (3.2.6) [3]: Let $B \subseteq \mathbb{R}^n$ be the unit ball for a norm $\|\cdot\|$ on $\mathbb{R}^n$, and let B^0 be its polar. Let $D \subseteq B^{\text{o}}$ be an ellipsiod. Set

$$A = \left(\frac{vol\ (B^{\text{o}})}{vol\ (D^{\text{o}})}\right)^{1/n}.$$

Let $|\cdot|$ be a norm such that D is the unit ball of $(\mathbb{R}^n, |\cdot|)$, in particular $|\cdot| \leq \|\cdot\|$. Then for all $k = 1, \cdots, n - 1$ there is a subspace $F \subseteq \mathbb{R}^n$ such that $dim\ F = k$ and for every $x \in \mathbb{R}^n/F^{\perp}$

$$\|x\|_{(\mathbb{R}^n/F^{\perp}, \|\cdot\|)} \leq (4\pi A)^{\frac{n}{n-k}} |x|_{(\mathbb{R}^n/F^{\perp}, \|\cdot\|)}, \quad (3)$$

where we use $\|\cdot\|_{(\mathbb{R}^n/F^{\perp}, \|\cdot\|)}$ for the quotient norm induced by $\|\cdot\|$ and similarly for $|\cdot|$. Further,

$$\mathbb{P}(\{F \in G_{nk}: for\ all\ x \in F, Eq.(3) holds\}) > 1 - 2^{-n}.$$

Here is the main application of the above corollary to dimension theory.

Theorem (3.2.7) [3]: Let Γ be a countable group with a uniformly bounded action on separable Banach spaces X, Y. Let $\Sigma = \left(\sigma_i: \Gamma \to Isom(V_i)\right)$ with dim $V_i < \infty$. Suppose that V_i is the complexification of a real Banach space V_i' such that

$$\sup_i vr\ ((V_i'\ )^*) < \infty,$$

and there are constants $C_1, C_2 > 0$ so that

$$C_1 \left(\|x\|_{V_i'} + \|y\|_{V_i'}\right) \leq \|x + iy\| \leq C_2 \left(\|x\|_{V_i'} + \|y\|_{V_i'}\right),$$

for all $x, y \in V_i$. Then the following inequalities hold,

$$\underline{dim}_{\Sigma}(X \oplus Y, \Gamma) \geq \underline{dim}_{\Sigma}(X, \Gamma) + \underline{dim}_{\Sigma}(Y, \Gamma),$$

$$dim_{\Sigma}(Y_1 \oplus Y_2, \Gamma) \geq dim_{\Sigma}(X, \Gamma) + \underline{dim}_{\Sigma}(Y, \Gamma),$$

$$dim_{\Sigma}\left(Y^{\oplus n}, \Gamma\right) \geq n\ dim_{\Sigma}(Y, \Gamma).$$

Proof. We will do the proof for dim only, the proof of the other claims are the same. Let $S = (x_n)_{n=1}^{\infty}, T = (y_n)_{n=1}^{\infty}$ be dynamically generating sequences, enumerate $S \oplus \{0\} \cup \{0\} \oplus T$ by $x_1, y_1, x_2, y_2, \cdots$, and fix integers k, m. By Lemma (3.2.4), it suffices to show that for fixed $m, k \in \mathbb{N}$, and for the pseudonorms ρ, ρ_1, ρ_2 on $l^{\infty}(\mathbb{N})$ given by

$$\rho(f) = \left(\sum_{j=1}^{m+k} |f(j)|^2\right)^{1/2},$$

$$\rho_1(f) = \left(\sum_{j=1}^{m} |f(j)|^2\right)^{1/2},$$

$$\rho_2(f) = \left(\sum_{j=1}^{k} |f(j)|^2\right)^{1/2}.$$

we have

$$f.dim_\Sigma(S \oplus 0 \cup 0 \oplus T, \rho) \geq \underline{f.dim}_\Sigma(S, \rho_1) + f.dim_\Sigma(T, \rho_2),$$

Fix $\kappa, \varepsilon > 0$ and fix $\eta > 0$ which will depend upon κ, ε in a manner to be determined later. By Corollary (3.2.6) there is a constant A, which depends only on κ, C_1, C_2 Hilbert space norms $|\cdot|_i$ on X_i, and finite dimensional complex subspaces $F_i \subseteq V_i^*$ of complex dimension $\lfloor(1-\kappa)(dim\ V_i)\rfloor$ such that

$$\frac{1}{A}|x|_i \leq \|x\| \leq A|x|_i$$

for all $x \in V_i/F_i^\perp$. Here, as in the Corollary (3.2.6), we abuse notation by using $\|x\|$ for the norm on $X_i/F_i^\perp$ induced by $\|\cdot\|$, and similarly for $|\cdot|$.

For $m' \geq m \in \mathbb{N}, \delta > 0$ and $F \subseteq \Gamma$ finite we have

$$Hom_\Gamma(S, F, 2m', \delta, \sigma_i) \oplus Hom_\Gamma(T, F, 2m', \delta, \sigma_i)_2$$
$$\subseteq Hom_\Gamma\big((S \oplus \{0\}) \cup (\{0\} \oplus T), F, m', 2\delta, \sigma_i\big).$$

Thus

$$\hat{d}_\eta\left(Hom_\Gamma\big((S \oplus \{0\}) \cup (\{0\} \oplus T), F, 2m', 2\delta, \sigma_i\big)_2, \rho\right)$$
$$\geq \hat{d}_\eta(Hom_\Gamma(S, F, 2m', \delta, \sigma_i) \oplus Hom_\Gamma(T, F, 2m', \delta, \sigma_i)_2, \rho).$$

Let

$$K_1 = \{\big(T(x_1), \cdots, T(x_m)\big) : T \in Hom_\Gamma(S, F, 2m', \delta, \sigma_i)\},$$
$$K_2 = \{\big(S(y_1), \cdots, S(y_k)\big) : S \in Hom_\Gamma(S, F, 2m', \delta, \sigma_i)\}.$$

Then, by definition,

$$\hat{d}_\eta(Hom_\Gamma(S, F, 2m', \delta, \sigma_i) \oplus Hom_\Gamma(T, F, 2m', \delta, \sigma_i), \rho)$$
$$= d_\eta\big(K_1 \oplus K_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}\big)$$

where we use the l^2-direct sum.

Let $\pi_i: V_i \to Vi/F_i^\perp$ be the quotient map and let

$$G_j = \pi_i^{\oplus l}(K_j),$$

where $l = m$ if $j = 1$, and $l = k$ if $j = 2$.

Then

$$d_\eta\left(K_1 \oplus K_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}\right) \geq d_\eta\left(G_1 \oplus G_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}\right)$$
$$\geq d_{A\eta}\left(G_1 \oplus G_2, |\cdot|_i^{\oplus m} \oplus |\cdot|_i^{\oplus k}\right).$$

Set

$$B_i = \left\{x \in G_i : lA \geq |x| \geq A\,\frac{\varepsilon}{4}\right\},$$

where $l = m$ if $i = 1$, and $l = k$ if $i = 2$.

Then

$$d_{A\eta}\left(G_1 \oplus G_2, |\cdot|_i^{\oplus m} \oplus |\cdot|_i^{\oplus k}\right)$$
$$\geq d_{\max(l,m)(\varepsilon/4)^{-1}\sqrt{5\eta A \max(l,m)}}\left(B_1, |\cdot|^{\oplus m}\right)$$
$$+ d_{\max(l,m)(\varepsilon/4)^{-1}\sqrt{5A\eta \max(l,m)}}\left(B_2, |\cdot|^{\oplus k}\right).$$

Setting $\eta = \frac{\varepsilon^{4/3}}{A \max(l,m).5^{1/3}}$ we have

$$d_\eta\left(K_1 \oplus K_2, \|\cdot\|^{\oplus m} \oplus \|\cdot\|^{\oplus k}\right) \geq d_{\frac{\varepsilon}{A}}\left(B_1, |\cdot|^{\oplus m}\right) + d_{\frac{\varepsilon}{A}}\left(B_2, |\cdot|^{\oplus k}\right)$$
$$\geq d_\varepsilon\left(B_1, \|\cdot\|^{\oplus m}\right) + d_\varepsilon\left(B_2, \|\cdot\|^{\oplus k}\right).$$

Since $B_i \supseteq \left\{x \in C_i : \|x\| \geq \frac{\varepsilon}{4}\right\}$ we have

$$d_\varepsilon\left(B_1, \|\cdot\|^{\oplus k}\right) + d_\varepsilon\left(B_2, \|\cdot\|^{\oplus k}\right) = d_\varepsilon\left(G_1, \|\cdot\|^{\oplus k}\right) + d_\varepsilon\left(G_2, \|\cdot\|^{\oplus k}\right)$$

Let $E_i \subseteq (V_i/F_i^{\perp})^{\oplus l}$ be a linear subspace of minimal dimension which ε-contains C_i with respect to $\|\cdot\|^{\oplus l}$ $(l = k, if\ i = 1, and\ l = m\ if\ i = 2.)$ Let $\tilde{E}_i \subseteq V_i$ be a linear subspace such that $\dim E_i = \dim \tilde{E}_i$ and $\pi_i^{\oplus l}(\tilde{E}_i) = E_i$. Set $W_i = \tilde{E}_i + F_i^{\oplus l}$. Then W_i has dimension at most $\dim E_i + lc_i$ with $\lim_{i\to\infty} \frac{c_i}{\dim V_i} = \kappa$, since $\dim V_i \to \infty$, and $K_i \subseteq_{\varepsilon,\|\cdot\|} V_i$.

Thus

$$d_\varepsilon\left(G_i, \|\cdot\|^{\oplus l}\right) \geq \hat{d}_\varepsilon\left(K_i, \|\cdot\|^{\oplus l}\right) - lc_i.$$

Since $\varepsilon \to 0$ as $\eta \to 0$ (and vice versa) we conclude that

$$dim_\Sigma\left(S_1 \oplus S_2, \Gamma, \|\cdot\|_{S,T,i}\right)$$
$$\geq -\kappa(k+m) + dim_\Sigma\left(S_1, \Gamma, \|\cdot\|_{S,i}\right) + \underline{dim}_\Sigma\left(Y_2, \Gamma, \|\cdot\|_{T,i}\right).$$

Since κ is arbitrary this proves the desired inequality.

Corollary (3.2.8) [3]: Let $2 \leq p < \infty$.

(a) Let Γ be a sofic group with uniformly bounded actions on separable Banach spaces X, Y and let Σ be a sofic approximation. Then

$$\underline{dim}_{\Sigma,l^p}(X \oplus Y, \Gamma) \geq \underline{dim}_{\Sigma,l^p}(X, \Gamma) + \underline{dim}_{\Sigma,l^p}(Y, \Gamma),$$
$$\underline{dim}_{\Sigma,l^p}(X \oplus Y, \Gamma) \geq \underline{dim}_{\Sigma,l^p}(X, \Gamma) + \underline{dim}_{\Sigma,l^p}(Y, \Gamma).$$

(b) Let Γ be an $\mathcal{R}^{\omega}$-embeddable group with uniformly bounded actions on separable Banach spaces X, Y and let Σ be an embedding sequence. Then
$$dim_{\Sigma,S^p}(X \oplus Y, \Gamma) \geq \underline{dim}_{\Sigma,S^p}(X, \Gamma) + \underline{dim}_{\Sigma,S^p}(Y, \Gamma),$$
$$\underline{dim}_{\Sigma,S^p}(X \oplus Y, \Gamma) \geq \underline{dim}_{\Sigma,S^p}(X, \Gamma) + \underline{dim}_{\Sigma,S^p}(Y, \Gamma).$$

Proof. For $1 \leq q \leq \infty$, let B_q be the unit ball of $L^q(\{1,\cdot\cdot\cdot,n\}, \mu_n)$ where μ_n is the uniform measure.

It is known that for all q,
$$\inf_n \left(\frac{vol\,(B_q)}{vol\,(B_2)}\right)^{1/n} > 0,$$
$$\sup_n \left(\frac{vol\,(B_q)}{vol\,(B_2)}\right)^{1/n} < \infty$$
Similarly if we let C_q be the unit ball of $\{A \in M_n(\mathbb{C}): A = A^*\}$ in the norm $\|\cdot\|_{L^p\left(\frac{1}{n}Tr\right)}$, it is known that for all q,
$$\inf_n \left(\frac{vol\,(C_q)}{vol\,(C_2)}\right)^{1/n} > 0,$$
$$\sup_n \left(\frac{vol\,(C_q)}{vol\,(C_2)}\right)^{1/n} < \infty$$
Apply the preceding theorem.

We note one last property of l^2-dimension for representations, to show that our dimension agrees with von Neumann dimension in the l^2-case.

Proposition (3.2.9) [3]: Let H be a separable unitary representation of a $\mathcal{R}^{\omega}$-embeddable group Γ. Let Σ be an embedding sequence of Γ. Suppose that $H = \overline{\bigcup_{k=1} H_k}$ with H_k increasing, closed invariant subspaces, and that each H_k has a finite dynamically generating sequence. Then
$$dim_{\Sigma,l^2}(H, \Gamma) = \sup_k dim_{\Sigma,l^2}(H_k, \Gamma),$$
$$dim_{\Sigma,l^2}(H, \Gamma) = \sup_k \underline{dim}_{\Sigma,l^2}(H_k, \Gamma).$$

Proof. We will do the proof for dim only, the other cases are the same. By Proposition (3.2.2) we know that dim_{Σ,l^2} is monotone for unitary representations, so we only need to show

$$dim_{\Sigma,l^2}(H,\Gamma) \geq \sup_k dim_{\Sigma,l^2}(H_k,\Gamma).$$

Let $\{\xi_1^{(k)},\cdot\cdot\cdot,\xi_{r_k}^{(k)}\}$ be unit vectors which dynamically generate H_k. Let S_N be the sequence

$$\xi_1^{(1)},\dots,\xi_{r_1}^{(1)},\xi_1^{(2)},\dots,\xi_{r_2}^{(2)},\xi_1^{(N)},\dots,\xi_{r_N}^{(N)},$$

i.e. the lth term of S_N is

$$\xi_{q_l}^{(i)}$$

if i is the largest integer such that

$$C_i = \sum_{j\leq i} r_j < l,$$

and

$$q_l = l - \sum_{j\leq i} r_j.$$

Let S be the sequence obtained by the infinite concatenation of the S_N's. We will use S_N to compute $dim_{\Sigma,l^2}(H_N,\Gamma)$ and S to compute $dim_{\Sigma,l^2}(H,\Gamma)$, we also use the pseudornorms

$$\|T\|_{S,i} = \sum_{j=1}^{\infty}\frac{1}{2^j}\|T(\xi_j)\|, \quad \|T\|_{S_N,i} = \sum_{j=1}^{\infty}\frac{1}{2^j}\|T(\xi_j)\|.$$

Fix $\varepsilon > 0$, and let M be such that $2^{-M} < \varepsilon$. Suppose $F \subseteq \Gamma$ is finite, $\delta > 0$ and $m \in \mathbb{N}$ with $m > C_M$. Let $P_M \in B(H)$ be the projection onto H_M. Suppose V is a subspace of $B(H_M,\mathbb{C}^{d_i})$ of minimal dimension such that

$$Hom_\Gamma(S_M,F,m,\delta,\sigma_i) \subseteq_{\varepsilon,\|\cdot\|_{S,i}} V,$$

let $\tilde{V} \subseteq B(H,\mathbb{C}^{d_i})$ be the image of V under the map $T \to T \circ P_M$. If $T \in Hom_{\Gamma,l^2,(d_i)}(S,F,m,\delta,\sigma_i)$ then $\tilde{T} = T|_{H_M}$ is in $Hom_\Gamma(S_M,F,m,\delta,\sigma_i)$, and there exists $\phi \in V$ such that $\|\phi - \tilde{T}\|_{S_M,i} < \varepsilon$. Then

$$\|\phi \circ P - T\|_{S,i} \leq 2\sum_{n=C_M+1}^{\infty}\frac{1}{2^n}\|\phi - \tilde{T}\|_{S_M,i} \leq 2^{-m+1} + \varepsilon \leq 3\varepsilon.$$

Thus

$$Hom_\Gamma(S,F,m,\delta,\sigma_i) \subseteq_{3\varepsilon,\|\cdot\|_{S,i}} \tilde{V},$$

So

$$d_{3\varepsilon}\big(Hom_\Gamma(S_M,F,m,\delta,\sigma_i),\|\cdot\|_{S,i}\big) \leq d_\varepsilon\big(Hom_\Gamma(S_M,F,m,\delta,\sigma_i),\|\cdot\|_{S_M,i}\big).$$

Thus

$$dim_{\Sigma,l^2}(S,\Gamma,3\varepsilon,\|\cdot\|_{S,i,2}) \le dim_{\Sigma,l^2}(S_M,3\varepsilon,\|\cdot\|_{S,i,2}) \le \sup_M dim_{\Sigma,l^2}(\pi_M)$$

and similarly for $\underline{dim}$. Taking the supremum over $\varepsilon > 0$ completes the proof.

Corollary (3.2.10) [3]: Let Γ be a $\mathcal{R}^\omega$-embeddable group, and let $\Sigma = (\sigma_i: \Gamma \to U(d_i))$ be an embedding sequence. Let $\pi_k: \Gamma \to U(H_k)$ be a representations of Γ such that each π_k has a finite dynamically generating sequence. Then

$$dim_{\Sigma,l^2}\left(\bigoplus_{k=1}^{\infty} \pi_k\right) \le \sum_{k=1}^{\infty} dim_{\Sigma,l^2}(\pi_k),$$

$$\underline{dim}_{\Sigma,l^2}\left(\bigoplus_{k=1}^{\infty} \pi_k\right) \ge \sum_{k=1}^{\infty} \underline{dim}_{\Sigma,l^2}(\pi_k).$$

We show that if Σ is a sofic approximation of Γ and $1 \le p \le 2$, then

$$dim_{\Sigma,l^p}(l^p(\Gamma,V),\Gamma) = \dim V,$$

for V finite dimensional. Similarly if Σ is a embedding sequence of Γ and $1 \le p \le 2$, we show that

$$dim_{\Sigma,S^p,conj}(l^p(\Gamma,V),\Gamma) = \dim V,$$

$$dim_{\Sigma,l^2}(l^2(\Gamma,l^2(n)),\Gamma) = n,$$

again for V finite dimensional.

The proof for sofic groups will be relatively simple, but the proof for $\mathcal{R}^\omega$- embeddable groups requires a few more lemmas.

Let ν be the unique $U(n)$ invariant Borel probability measure on S^{2n-1}, for the next lemma we need that if $T: \mathbb{C}^n \to \mathbb{C}^n$ is linear, then

$$\frac{1}{2}Tr(T) = \int_{S^{2n-1}} \langle T\xi,\xi\rangle d\nu(\xi).$$

This follows from the fact that Tr is, up to scaling, the unique linear functional on $M_n(\mathbb{C})$ invariant under conjugation by $U(n)$.

Additionally, we will use the following concentration of measure fact, if f is a Lipschitz function on S^{n-1}, then

$$\mathbb{P}(|f - \mathbb{E}f| > t) \le 4e^{\frac{-nt^2}{\|f\|_{Lip}^2 72\pi^2}}.$$

Lemma (3.2.11) [3]: Let Γ be a $\mathcal{R}^\omega$-embeddable group, let $\sigma_i: \Gamma \to U(d_i)$ be an embedding sequence, and fix $E \subseteq \Gamma\ finite, m \in \mathbb{N}. For\ j \in \{1,\cdot\cdot,m\}, \xi,\eta \in S^{2d_i-1}$ define

$$T_{\xi,j}: l^2(\Gamma \times \{1,\cdots,m\}) \to l^2(d_i),$$
$$T_{\xi,\eta,j}: l^p(\Gamma \times \{1,\cdots,m\}) \to S^p(d_i)$$

By

$$T_{\xi,j}(f) = \sum_{s\in E} f(s,j)\sigma_i(s)\xi,$$
$$T_{\xi,\eta,j}(f) = \sum_{s\in E} f(s,j)\sigma_i(s)\xi \otimes \overline{\sigma_i(s)\eta}.$$

Then for any $\delta > 0$ and $1 \le p < \infty$,

(a)

$$\lim_{i\to\infty} \mathbb{P}\left(\left\{\xi \in S^{2d_i-1}: \left\|T_{\xi,j}: l^2(\Gamma \times \{1,\cdots,m\}) \to l^2(d_i)\right\| < 1+\delta\right\}\right) = 1,$$

(b)

$$\left\{(\xi,\eta) \in (S^{2d_i-1})^2: \left\|T_{\xi,\eta,j}: l^p(\Gamma \times \{1,\cdots,m\}) \to S^p(d_i)\right\| < 1+\delta\right\}$$
$$\supseteq A_i \times A_i,$$

Where $A_i \subseteq S^{2d_i-1}$ has $\nu(A_i) \to 1$.

Proof. Let $\kappa > 0$ which will depend upon $\delta > 0, p$ in a manner to be determined later. Let

$$A = \bigcap_{s\neq t, t\in E} \{\xi \in S^{2d_i-1}: |\langle\sigma_i(s)\xi, \sigma_i(t)\xi\rangle| < \kappa\},$$

since

$$\int_{S^{2d_i-1}} \langle\sigma_i(s)\xi, \sigma_i(t)\xi\rangle d\nu(\xi) = \frac{1}{d_i} Tr\left(\sigma_i(t)^{-1}\sigma_i(s)\right) \to 0$$

for $s \neq t$, the concentration of measure estimate mentioned before the Lemma implies that

$$\nu(A) \to 1.$$

For the proof of (a), (b) we prove that if $\xi, \eta \in A$ then

$$\left\|T_{\xi,j}\right\|_{l^2\to l^2} \le 1+\delta,$$
$$\left\|T_{\xi,\eta,j}\right\|_{l^p\to S^p} \le 1+\delta,$$

if $\kappa > 0$ is sufficiently small.

(a) For $f \in l^2(\Gamma \times \{1,\cdots,m\}), \xi \in A$ we have

$$\left\|T_{\xi,j}(f)\right\|_2^2 = \sum_{s,t\in E} f(s,j)\overline{f(t,j)}\langle\sigma_i(s)\xi,\sigma_i(t)\xi\rangle$$

$$\leq \|f\chi_E\|_2^2 + \sum_{s\neq t,s,t\in E} \|f\|_2^2\kappa \leq \|f\|_2^2(1+\kappa|E|^2)$$

$$\leq (1+\delta)\|f\|_2^2$$

if $\kappa < \frac{\delta}{|E|^2}$.

(b) Fix $\varepsilon > 0$ to be determined later. If κ is sufficently small, then for any $(\xi,\eta)\in A^2$ we can find $(\xi_s)_{s\in E}(\eta_s)_{s\in E}$ such that $\langle\xi_s,\xi_t\rangle = \delta_{s=t}, \langle\eta_s,\eta_t\rangle = \delta_{s=t}$ and

$$\|\xi_s - \sigma_i(s)\xi\| < \varepsilon, \|\eta_s - \sigma_i(s)\eta\| < \varepsilon.$$

Then

$$\left\|T_{\xi,\eta,j}(f) - \sum_{s\in E} f(s)\xi_s\otimes\overline{\eta_s}\right\|_p$$

$$\leq \|f\|_p \sum_{s\in E}(\|\xi_s - \sigma_i(s)\xi\| + \|\sigma_i(s)\eta - \eta_s\|) \leq 2|E|\varepsilon\|f\|_p.$$

Note that

$$\left|\sum_{s\in E} f(s)\xi_s\otimes\overline{\eta_s}\right|^2 = \sum_{s,t\in E}\overline{f(s)}f(t)\langle\xi_t,\xi_s\rangle\eta_s\otimes\overline{\eta_t} = \sum_{s\in E}|f(s)|^2\eta_s\otimes\overline{\eta_s}.$$

Thus

$$\left\|\sum_{s\in E} f(s)\xi_s\otimes\overline{\eta_s}\right\|_p^p = \|f\chi_E\|_p^p \leq \|f\|_p^p.$$

So if $\varepsilon < \frac{\delta}{2|E|}$ the claim follows.

Lemma (3.2.12) [3]: Let H be a Hilbert space, and $\eta_1,\cdots,\eta_k$ an orthonormal system in H, and $V = Span\left\{\eta_j : 1\leq j\leq k\right\}$ and P_V the projection onto V. Let K be a Hilbert space and $T\in B(H,K)$ with $\|T\| \leq 1$. Then

$$d_\varepsilon(\{T(\eta_1),\cdots,T(\eta_k)\}) \geq -k\varepsilon + Tr(P_V T^* T P_V).$$

Proof. For a subspace $E\subseteq H$ we let P_E be the projection onto E. Let W be a subspace of minimal dimension which ε-contains $\{T(\eta_1),\cdots,T(\eta_k)\}$. Then

$$Tr(P_W T T^*) = Tr(P_W T T^* P_W) \leq Tr(P_W),$$

similarly

$$Tr(P_W T\, T^*) \geq \; Tr(P_V\, T^* P_W T P_V) = \sum_{j=1}^{k} \langle P_W T(\eta_j), T(\eta_j)\rangle$$

$$\geq -\varepsilon k + \sum_{j=1}^{k} \langle T(\eta_j), T(\eta_j)\rangle = -\varepsilon k + Tr(P_V\, T^* T P_V).$$

For convenience, we shall identify $L(\Gamma)$ as a set of vectors in $l^2(\Gamma)$. That is, we shall consider $L(\Gamma)$ to be all $\xi \in l^2(\Gamma)$ so that

$$\|\xi\|_{L(\Gamma)} = \sup_{\substack{f \in c_c(\Gamma) \\ \|f\|_2 \leq 1}} \|\xi * f\|_2 < \infty.$$

Here $\xi * f$ is the usual convolution product. By standard arguments, if $\xi \in L(\Gamma)$, then for all $f \in l^2(\Gamma), \xi * f \in l^2(\Gamma)$ and

$$\|\xi * f\|_2 \leq \|\xi\|_{L(\Gamma)} \|f\|_2.$$

By general theory, $L(\Gamma)$ is closed under convolution and

$$(\xi * \eta) * \zeta = \xi * (\eta * \zeta)$$

for $\xi, \eta, \zeta \in L(\Gamma)$. Finally for $\xi \in L(\Gamma)$, we set

$$\xi^*(x) = \overline{\xi(x^{-1})}.$$

If $\xi \in L(\Gamma), \zeta, \eta \in l^2(\Gamma)$, then

$$\langle \xi * \eta, \zeta\rangle = \langle \eta, \xi^* * \zeta\rangle.$$

Finally, for $\xi \in L(\Gamma), f \in c_c(\Gamma)$,

$$\|f * \xi\|_2 = \|\xi^* * f^*\|_2 \leq \|f^*\|_2 \|\xi^*\|_{L(\Gamma)} = \|f\|_2 \|\xi\|_{L(\Gamma)}.$$

Hence every element of $L(\Gamma)$ is bounded as a right convolution operator.

We shall need a few more lemmas, for the first we require the following definitions.

Definition (3.2.13) [3]: We let $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ be the free $*$-algebra in n noncommuting variables. That is $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ is the universal $\mathbb{C}$ -algebra generated by elements $X_1, \cdots, X_n, X_1^*, \cdots, X_n^*$, and we equip $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ with a $*$-algebra structure defined on words (and extended by conjugate linearity) by

$$(Y_1 \cdots Y_l)^* = Y_l^* \cdots Y_1^*, Y_j \in \{X_1, \cdots, X_n, X_1^*, \cdots, X_n^*\},$$

here $(X_j^*)^* = X_j$. We call elements of $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ $*$-polynomials in n noncommuting variables. Note that if A is a $*$-algebra, and $a_1, \cdots, a_n \in A$, then there is a unique $*$-homomorphism $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle \to A$ sending X_j to a_j. For $P \in \mathbb{C}^*\langle X_1, \cdots, X_n\rangle$, we denote the image under this homomorphism by $P(a_1, \cdots, a_n)$.

Definition (3.2.14) [3]: A tracial $*$-algebra is a pair (A, τ) where A is a unital $*$-algebra, $\tau: A \to \mathbb{C}$ is a linear map so that $\tau(1) = 1, \tau(x^*x) \geq 0$, with $\tau(x^*x) = 0$ if and only if $x = 0$, and $\tau(xy) = \tau(yx)$ for all $x, y \in A$, and for all $x \in A$, there is a $M > 0$ so that $\tau(y^*x^*xy) \leq M\tau(y^*y)$ for all $y \in A$. An embedding sequence of (A, τ) is a sequence of maps $\sigma_i: A \to M_{d_i}(\mathbb{C})$ such that

$$\sup_i \|\sigma_i(x)\|_\infty < \infty,$$

$$where\ \|\cdot\|_\infty\ is\ the\ operator\ norm, for\ all\ x \in A,$$

$$\sigma_i(1) = 1,$$

$$\frac{1}{n} Tr\big(\sigma_i(x)\big) \to \tau(x),$$

$$\big\|\sigma_i\big(P(x_1, \cdots, x_n)\big) - P\big(\sigma_i(x_i), \dots, \sigma_i(x_n)\big)\big\|_2 \to 0$$

for all $x_1, \cdots, x_n \in A$, and $*$-polynomials P in n noncomuting variables. Here $\|x\|_2 = \tau(x^*x)^{1/2}$ for $x \in A$. We let $L^2(A, \tau)$ be the completion of A in $\|\cdot\|_2$. We also let $\pi_\tau: A \to B\big(L^2(A, \tau)\big)$ be given by $\pi_\tau(x)a = xa$, for $x, a \in A$.

The main example which will be relevant for us is $A = c_c(\Gamma)$ with the product being convolution and the $*$-being defined by consider $c_c(\Gamma) \subseteq L(\Gamma)$, and $\tau(f) = f(e)$. Then an embedding sequence of Γ extends to one of $c_c(\Gamma)$ by

$$\sigma_i(f) = f(e)Id + \sum_{g \in \Gamma \setminus \{e\}} f(g)\sigma_i(g).$$

We note that for the next Lemma, we will use measure theoretic notation for certain norms on tracial von Neumann algebras (M, τ). Thus $\|x\|_\infty$ will be the operator norm of x, and $\|x\|_p = \tau((x^*x)^{p/2})^{1/p}$.

Lemma (3.2.15) [3]: Let (A, τ) be a tracial $*$-algebra. And let M be the weak operator topology closure of $\pi_\tau(A)$ equipped with the trace $\tau(x) = \langle x1, 1 \rangle$ for all $x \in M$. Then any embedding sequence of A extends to one of M.

Proof. By standard arguments, τ is indeed a trace, since A is $\|\cdot\|$-dense in $L^2(A, \tau)$, and elements of M commute with right multiplication it follows that $\tau(x^*x) = 0$ for $x \in M$ if and only if $x = 0$. If $x \in M \setminus A$, by the Kaplansky Density Theorem we may choose a sequence $a_{n,x}$ so that $\big\|\pi_\tau(a_{n,x})\big\|_\infty \leq \|x\|_\infty$ and

$$\big\|a_{n,x} - x\big\|_2 < 2^{-n}.$$

Note that if ω is a free ultrafilter on $\mathbb{N}$, then σ gives a trace-preserving embedding of A into

$$N = \left\{(x_i): x_i \in M_{d_i}(\mathbb{C}), \sup_n \|x_i\|_\infty < \infty\right\} \Big/ \left\{(x_i): \lim_{n\to\omega} \frac{1}{n} Tr(x_i^* x_i) = 0\right\},$$

where N has the trace

$$\tau_\omega(x) = \lim_{n\to\omega} \frac{1}{d_i} Tr(x_i),$$

if $x = (x_i)$. Thus $\sigma|_{\{a\in A:\, \|a\|_\infty \le 1\}}$is strong operator topology-strong operator topology continuous, and hence has an extension

$$\tau: \{a \in M: \|a\|_\infty \le 1\} \to N.$$

If we define

$$\rho(a) = \tau\left(\frac{a}{\|a\|_\infty}\right)\|a\|_\infty,$$

it follows that ρ is a trace-preserving $*$-homomorphism $M \to N$.

So by a standard contradiction and ultrafilter argument, for all $a \in A$, we may find $a_i \in M_{d_i}(\mathbb{C})$ so that $\|a_i\|_\infty \le \|\pi_\tau(a)\|_\infty$ and $\|a_i - \sigma_i(a)\|_2 \to 0$.

For $x \in M$, choose integers $1 \le i_1 < i_2 < i_3 < \cdots$, and elements $b_{n,x,i} \in M_{d_i}(\mathbb{C})$ so that $\|b_{n,x,i}\|_\infty \le \|x\|_\infty$ and

$$\|b_{j,x,i} - \sigma_i(a_{j,x})\|_2 < 2^{-n} \ for\ 1 \le j \le n, i \ge i_n,$$

$$\|\sigma_i(a_{j,x}) - \sigma_i(a_{k,x})\|_2 < 2^{-n} + \|a_{j,x} - a_{k,x}\|_2 for\ 1 \le j,k \le n, i \ge i_n,$$

the last inequality being possibile since σ_i is an embedding sequence on A.

For $x \in M\backslash A$, define $\sigma_i(x) = b_{n,x,i}$ where n is such that $i_n \le i < i_{n+1}$. If $x \in M\backslash A$, and $i \ge i_n$ and N is such that $i_N \le i \le i_{N+1}$, then

$$\begin{aligned}\|\sigma_i(x) - \sigma_i(a_{n,x})\|_2 &< 2^{-n} + \|\sigma_i(a_{N,x}) - \sigma_i(a_{n,x})\| \\ &\le 2.2^{-n} + \|a_{N,x} - a_{i,x}\|_2 \le 4.2^{-n}, \\ \|\sigma_i(x)\|_\infty &\le \|x\|_\infty.\end{aligned}$$

From this estimate it is not hard to see that σ_i is an embedding sequence of M.

Lemma (3.2.16) [3]: Let Γ be a countable sofic group, and $\Sigma = (\sigma_i: \Gamma \to S_{d_i})$ a sofic approximation of Γ. Extend σ_i to a embedding sequence, still denoted σ_i, of $(L(\Gamma), \tau)$ with τ the group trace. For $r, s \in \mathbb{N}$ define $\sigma_i: M_{h,s}(L(\Gamma)) \to M_{h,s}\left(M_{d_i}(\mathbb{C})\right)$ by $\sigma_i(A) = [\sigma_i(a_{lr})]_{1\le l\le h, 1\le r\le s}$. Fix $n \in$

$\mathbb{N}$. For $1 \leq j \leq d_i, 1 \leq k \leq n$ and $E \subseteq \Gamma$ finite define $T_{j,k}^{(E)}: l^p(\Gamma)^{\oplus n} \to l^p(d_i)$ by

$$T_{j,k}^{(E)}(f) = \sum_{g \in E} f_k(g)\sigma_i(g)e_j.$$

Then

(a) For all E and $(1 - o(1))nd_i$ of the j,k we have $\left\|T_{j,k}^{(E)}\right\|_{l^p \to l^p} \leq 1$ as $i \to \infty$.

(b) For $1 \leq p \leq \infty$, for all $\varepsilon > 0$, for all $f \in c_c(\Gamma), g \in l^p(\Gamma)^{\oplus n}$, there is a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then the set of (j, k) so that

$$\left\|T_{j,k}^{(E')}(f * g) - \sigma_i(f)T_{j,k}^{(E)}(g)\right\|_p \leq \varepsilon \|g\|_p$$

has cardinality at least $(1 - \varepsilon))nd_i$ for all large i.

(c) For all $\varepsilon > 0$, for all $\xi \in M_{1,n}(L(\Gamma))$, (identifying $M_{1,n}(L(\Gamma))$ as a subset of $l^2(\Gamma)^{\oplus n}$ there is a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then the set of (j, k) so that

$$\left\|T_{j,k}^{(E')}(\xi) - \sigma_i(\xi)(e_j \oplus e_k)\right\|_2 \leq \varepsilon$$

(here $e_j \oplus e_k \in l^p(d_i)^{\oplus n}$ is e_j in the kth coordinate and zero otherwise). Has cardinality at least $(1 - \varepsilon))nd_i$ for all large i.

Proof. (a) We have

$$\left\|T_{j,k}^{(E)}(f)\right\|_p^p = \sum_{r=1}^{d_i} \left| \sum_{\substack{g \in E \\ \sigma_i(g)(j) = r}} f_k(g) \right|^p.$$

Let $C_i = \{j \in \{1, \cdots, d_i\}: \sigma_i(g)(j) \neq \sigma_i(h)(j)\ for\ g \neq h\ in\ E\}$. By soficity, we have $\frac{|C_i|}{d_i} \to 1$, and if $j \in C_i$ we have

$$\left\|T_{j,k}^{(E)}(f)\right\|_p^p \leq \|f_k\|_p^p \leq \|f\|_p^p.$$

(b) For $A \in M_{d_i}(\mathbb{C})$,

$$\|A\|_2^2 = \frac{1}{d_i}\sum_{j=1}^{d_i} \left\|Ae_j\right\|_2^2,$$

where e_j is the vector which has jth coordinate equal to 1, and all other coordinates zero. Hence by Chebyshev's inequality, the fact that

$\left\|T_{j,k}^{(E)}(f)\right\|_p \leq 1$, and the definition of embedding sequences, it is enough to verify this for $f = \delta_x, g = \delta_y$ for some $x, y \in \Gamma$. But this is trivial from the definition of soficity.

(c) Let us first verfiy this when $\xi \in M_{1,n}(c_c(\Gamma))$. In this case, we may again reduce to $\xi = (\delta_{a_1}, \cdots, \delta_{a_k})$ for some $a_1, \cdots, a_k \in \Gamma$. Then if $E \supseteq \{a_1, \cdots, a_k\}$ we have

$$T_{j,k}^{(E)}(\xi) = \sigma_i(a_k)e_j = \sigma_i(\xi)(e_j \oplus e_k).$$

In the general case let $\varepsilon > 0$, given $\xi \in M_{1,n}(L(\Gamma))$ choose $f \in M_{1,n}(c_c(\Gamma))$ so that $\|f - \xi\|_2 < \varepsilon$. Thus for $\left(1 - (\varepsilon + o(1))\right) kd_i$ of the (j, k) we have

$$\left\|T_{j,k}^{(E')}(\xi) - \sigma_i(\xi)(e_j \oplus e_k)\right\|_2 \leq 2\varepsilon + \left\|\left(\sigma_i(\xi) - \sigma_i(f)\right)(e_j \oplus e_k)\right\|.$$

By the definition of embedding sequence for all large i we have

$$\frac{1}{d_i}\sum_{j=1}^{d_i}\sum_{k=1}^{n}\left\|\left(\sigma_i(\xi) - \sigma_i(f)\right)(e_j \oplus e_k)\right\|_2^2 < \varepsilon^2,$$

thus for at least $(1 - \sqrt{\varepsilon})nd_i$ of the (j, k) we have

$$\left\|\left(\sigma_i(\xi) - \sigma_i(f)\right)(e_j \oplus e_k)\right\|_2 < \sqrt{\varepsilon},$$

combining these estimates completes the proof.

We need a similar lemma for $\mathcal{R}^\omega$-embeddable groups.

Lemma (3.2.17) [3]: Let Γ be a countable $\mathcal{R}^\omega$-embeddable group, and $\Sigma = (\sigma_i: \Gamma \to U(d_i))$ an embedding sequence. Define $\rho_i: \Gamma \to U(S^2(d_i))$ by $\rho_i(g)A = \sigma_i(g)A\sigma_i(g)^{-1}$. Extend σ_i, ρ_i to embedding sequences, still denoted σ_i, ρ_i of $(L(\Gamma), \tau)$ with τ the group trace. For $h, s \in \mathbb{N}$ define $\sigma_i: M_{h,s}(L(\Gamma)) \to M_{h,s}\left(M_{d_i}(\mathbb{C})\right)$ by $\sigma_i(A) = [\sigma_i(a_{lr})]_{1\leq l\leq h, 1\leq r\leq s}$. Fix $n \in \mathbb{N}$. For $\xi, \eta \in l^2(d_i), 1 \leq k \leq d_i$ and $E \subseteq \Gamma$ finite define $T_{\xi,\eta,k}^{(E)}: l^p(\Gamma)^{\oplus n} \to S^p(d_i)$ by

$$T_{\xi,\eta,k}^{(E)}(f) = \sum_{g\in E} f_k(g)\sigma_i(g)\xi \otimes \overline{\sigma_i(g)\eta}.$$

Then

(a) There exists measurable $A_i \subseteq S^{2d_i-1}$ with $\mathbb{P}(A_i) \to 1$, so that

$$\left\{(\xi, \eta) \in (S^{2d_i-1})^2 : \left\|T_{\xi,\eta,k}^{(E)}\right\|_{l^p\to S^p} \leq 2\right\} \supseteq A_i \times A_i,$$

for $\left(1- o(1)\right)d_i$ of the k.

(b) For all $\varepsilon > 0$, for all $f \in c_c(\Gamma), g \in l^p(\Gamma)^{\oplus n}$, there exists measurable $B_i \subseteq S^{2d_i-1}$, with $\mathbb{P}(B_i) \geq 1-\varepsilon$, for all large i, a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then for $(1- \varepsilon)d_i$ of the k and for all large i,

$$\left\{(\xi,\eta) \in (S^{2d_i-1})^2 : \left\|T^{(E')}_{\xi,\eta,k}(f * g) - \rho_i(f)T^{(E)}_{\xi,\eta,k}(g)\right\|_p \leq \varepsilon\right\} \supseteq B_i \times B_i,$$

(c) For all $\varepsilon > 0$, for all $\zeta \in M_{1,n}\left(L(\Gamma)\right)$, (identifying $M_{1,n}\left(L(\Gamma)\right)$ as a subset of $l^2(\Gamma)^{\oplus n}$) there are measurable $C_i \subseteq S^{2d_i-1}$, with $\mathbb{P}(C_i) \geq 1-\varepsilon$ for all large i, a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, so that for at least $(1-\varepsilon)d_i$ of the k and for all large i,

$$\left\{(\xi,\eta) \in (S^{2d_i-1})^2 : \left\|T^{(E')}_{\xi,\eta,k}(\zeta) - \rho_i(\zeta)\xi \otimes \overline{\eta}\right\|_2 \leq \varepsilon\right\} \supseteq C_i \times C_i,$$

has cardinality at least $(1-\varepsilon)nd_i$ for all large i.

Finally we need one last lemma, which allows us to reduce to considering subspaces of finite direct sums of $l^p(\Gamma)$.

Lemma (3.2.18) [3]: Let Γ be a countable discrete group. Let $H \subseteq l^2\left(\mathbb{N}, l^2(\Gamma)\right)$ be a closed Γ-invariant subspace.

(a) Define $\pi_k : l^2\left(\mathbb{N}, l^2(\Gamma)\right) \to l^2(\Gamma)^{\oplus k}$ *by* $\pi_k f(j) = f(j)$ for $1 \leq j \leq k$. Then

$$dim_{L(\Gamma)}(H) = \sup_k dim_{L(\Gamma)}\left(\overline{\pi_k(H)}^{\|\cdot\|_2}\right).$$

(b) The representation H is isomorphic to a direct sum of representations of the form $l^2(\Gamma)p$ with $p \in L(\Gamma)$ (by the remarks preceding definition (3.2.13) each element of $L(\Gamma)$ is a bounded right convolution operator) an orthogonal projection.

Proof. (a) Since $\pi_k(H)$ is dense in $\overline{\pi_k(H)}$ we have

$$dim_{L(\Gamma)}(H) \geq \sup_k dim_{L(\Gamma)}\left(\overline{\pi_k(H)}^{\|\cdot\|_2}\right).$$

Let us first handle the case when $dim_{L(\Gamma)}(H) < \infty$, let P be the projection onto H.

Then

$$\begin{aligned} dim_{L(\Gamma)}\left(\overline{\pi_k(H)}\right) &= dim_{L(\Gamma)}(ker(\pi_k(P)^{\perp}) \\ &= dim_{L(\Gamma)}\left(H \cap \overline{\left(H^{\perp} + l^2(\Gamma)^{\oplus k}\right)}\right) \\ &= dim_{L(\Gamma)}\left(H \cap \left(H \cap l^2(\mathbb{N}\{1, \ldots k\}, \Gamma)\right)^{\perp}\right). \end{aligned}$$

Let Q_k be the projection onto $H \cap l^2(\mathbb{N}\{1, \dots k\}, \Gamma)$. Then

$$dim_{L(\Gamma)}\left(H \cap l^2(\mathbb{N}\{1, \dots k\}, \Gamma)\right) = \sum_{n=1}^{\infty} \langle Q_k(\delta_e \otimes e_n), \delta_e \otimes e_n \rangle$$

$$= \sum_{n=k}^{\infty} \langle Q_k(\delta_e \otimes e_n), \delta_e \otimes e_n \rangle \leq \sum_{n=k}^{\infty} \langle P(\delta_e \otimes e_n), \delta_e \otimes e_n \rangle$$

$$\to 0,$$

as $dim_{L(\Gamma)}(H) < \infty$.

In the general case, it suffices to show that we may write H as a direct sum of representations with finite von Neumann dimension. Zorn's Lemma implies that every representation is a direct sum of cyclic representations which are contained in $l^2(\mathbb{N}, l^2(\Gamma))$, so it suffices to show every cyclic representation contained in $l^2(\mathbb{N}, l^2(\Gamma))$ has finite von Neumann dimension.

For this, let $\xi \in H$ be a cyclic vector, then there is vector $\zeta \in l^2(\Gamma)$ so that

$$\langle g\xi, \xi \rangle = \langle g\zeta, \zeta \rangle$$

for all $g \in \Gamma$. Thus H is isomorphic to $\overline{Span}^{\|\cdot\|_2}(\Gamma\xi)$ via the unitary sending $g\xi \to g\zeta$. From this it clear that H has dimension at most 1.

(b) As in part (a), we may assume that H is a cyclic representation contained in $l^2(\Gamma)$. Let p be the projection onto H, then p commutes with $L(\Gamma)$. Set $\xi = p(\delta_e)$, since p commutes with $L(\Gamma)$, it is not hard to see that $p(f) = f * \xi$ for $f \in c_c(\Gamma)$. Arguments entirely similar to those before Definition (3.2.13) prove that ξ is a bouned left convolution operator. Hence ξ is an orthogonal projection in $L(\Gamma)$, and $H = l^2(\Gamma)\xi$.

Theorem (3.2.19) [3]: Let Γ be a countable discrete group, let $1 \leq p \leq 2$, and Y a closed Γ -invariant subspace of $l^p(\mathbb{N}, l^p(\Gamma))$, with Γ acting by $gf(x) = f(g^{-1}x)$. Set $H = \overline{Y}^{\|\cdot\|_2}$.

(a) Suppose Σ is a sofic approximation of Γ, then

$$\underline{dim}_{\Sigma, l^p}(Y, \Gamma) \geq dim_{L(\Gamma)}(H).$$

(b) Suppose Σ is an embedding sequence of Γ, then

$$\underline{dim}_{\Sigma, S^p, conj}(Y, \Gamma) \geq dim_{L(\Gamma)}(H)$$

(c) Suppose Σ is an embedding sequence of Γ, and $H \subseteq l^2\left(\mathbb{N}, l^2(\Gamma)\right)$ is Γ invariant, then

$$\underline{dim}_{\Sigma, l^2}(H, \Gamma) \geq dim_{L(\Gamma)}(H)$$

Proof. We first reduce to the case that $Y \subseteq l^p(\Gamma)^{\oplus h}$ with h finite.

Consider the projection

$$\pi_h: l^p(\mathbb{N}, \Gamma) \to l^p(\{1,\cdots,h\}, l^p(\Gamma))$$

given by

$$\pi_h f(j) = f(j),$$

assume we know the result for $Y \subseteq l^p(\Gamma)^{\oplus h}$ for each h.

Then,

$$dim_{\Sigma,l^p}(Y,\Gamma) \geq dim_{\Sigma,l^p}\left(\overline{\pi_h(Y)}^{\|\cdot\|_p}, \Gamma\right) \geq dim_{L(\Gamma)}\left(\overline{\pi_h(H)}^{\|\cdot\|_2}\right),$$

letting $h \to \infty$ and applying the preceding Lemma proves the claim. Thus, we shall assume that $Y \subseteq l^p(\Gamma)^{\oplus n}$ with $n \in \mathbb{N}$.

By part (b) of the preceding Lemma, we can find vectors $\left(\xi^{(q)}\right)_{q=1}^{\infty} \in H$, so that

$$\langle \lambda(g)\xi^{(s)}, \xi^{(s)} \rangle = \langle \lambda(g)q_S, q_S \rangle$$

$$= q_S(g^{-1}), where\ q_S\ is\ a\ projection\ in\ L(\Gamma),$$

$$\sum_{s=1}^{\infty} \tau(q_S) = dim_{L(\Gamma)}(H),$$

$$\langle \lambda(g)\xi^{(j)}, \xi^{(l)} \rangle = 0\ \ for\ j \neq l, g \in \Gamma,$$

$$H = \bigoplus_{j=1}^{\infty} \overline{L(\Gamma)\xi^{(j)}}.$$

These equations can be rewritten as

$$\sum_{i=1}^{n} \xi^{(j)} * \left(\xi^{(j)}\right)^* = q_j\,, for\ 1 \leq j \leq \infty,$$

$$\sum_{i=1}^{n} \xi^{(j)} * \left(\xi^{(l)}\right)^* = 0\,, if\ j \neq l.$$

Let us illuminate these equations a little. Regard a vector $\xi \in l^2(\Gamma)^{\oplus n}$ as a element in $M_{1,n}(l^2(\Gamma))$ with the product of two matrices induced from convolution of vectors. Then the product of elements of $M_{1,n}(l^2(\Gamma))$, $M_{1,n}(L(\Gamma))$ makes sense, but may not land back in $l^2(\Gamma)$. The above equations then read

$$\xi^{(j)}\left(\xi^{(j)}\right)^* = q_j, for\ 1 \leq j < \infty,$$

$$\xi^{(j)}\left(\xi^{(l)}\right) = 0\ for\ j \neq l.$$

In particular, the above equations imply that

$$\left\|\xi_r^{(j)}\right\|_{L(\Gamma)} \leq 1.$$

So that $\xi^{(j)} \in M_{1,n}(L(\Gamma))$. Extend σ_i to a embedding sequence of $M_{n,m}(L(\Gamma))$ for all n, m and such that

$$\|\sigma_i(\xi^{(j)})\| \leq 1, for\ all\ j,$$

$$\left\|\sigma_i\left(\xi_r^{(j)}\right)\right\| \leq 1, for\ all\ j, r,$$

$$\sigma_i(\xi^{(j)})\sigma_i(\xi^{(l)})^* = 0, for\ all\ j \neq l.$$

for all j, r.

(a) Let $S = (x_j)_{j=1}^n$ be a dynamical generating sequence for Y.

Fix $\eta > 0, t \in \mathbb{N}$ and choose a finite subset $F_1 \subseteq \Gamma, m_1 \in \mathbb{N}$, and $c_{gj}^{(s)}$ for $1 \leq s \leq t, (g, j) \in F_1 \times \{1, \cdots, m_1\}$ so that for all $1 \leq s \leq t$

$$\left\|\xi^{(s)} - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(s)} g x_j\right\|_2 < \eta.$$

Choose finitely supported functions x_j' so that $\|x_j - x_j'\|_p < \eta'$. Since $p \leq 2$, it is easy to see that if we force η' to be sufficiently small then,

$$\left\|\xi^{(s)} - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(s)} g x_j'\right\|_2 < \eta.$$

Let $S = (x_j)_{j=1}^n$ be a dynamically generating sequence for Y. Fix $F_1 \subset \Gamma\ finite\ m \in \mathbb{N}, \delta > 0$. Let $E \subseteq \Gamma$ be finite, let $T_{j,k}^{(E)}$ be defined as Lemma (3.2.16) [3].

It is easy to see that if E is sufficently large, then $T_{j,k}^{(E)}\Big|_{Y_{F,m}} \in Hom_\Gamma(S, F, m, \delta, \sigma_i)_2$ for $(1 - o(1))nd_i$ of the j, k, and in fact $\left\|T_{j,k}^{(E)}\right\|_{l^p \to l^p} \leq 2$ for $1 \leq p \leq 2$. For such (j, k), and for all small δ, for $1 \leq s \leq t + 1$

$$\left\|T_{j,k}^{(E)}(\xi^{(s)}) - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(p)} \sigma_i(g) T_{j,k}^{(E)}(x_j)\right\|_2 < 2\eta,$$

$$\left\| T_{j,k}^{(E)}(gx_j') - T_{j,k}^{(E)}(gx_j) \right\|_2 < \eta.$$

Thus by Lemma (3.2.16) [3] for at least $(1-(2013)!\varepsilon)\, nd_i$ of the j,k we have

$$\left\| \sigma_i(\xi^{(s)})(e_j \otimes e_k) - \sum_{\substack{g\in F_1\\ 1\le j\le m_1}} c_{gj}^{(p)} \sigma_i(g) T_{j,k}^{(E)}(x_j) \right\|_2 < \varepsilon + \eta.$$

Now consider the linear map $A\colon l^{\infty}(\mathbb{N}, l^p(d_i)) \to l^2(d_i)^{\oplus t}$ given by

$$S(f) = \left(\sum_{\substack{g\in F_1\\ 1\le j\le m_1}} c_{gj}^{(p)} \sigma_i(g) f(j) \right)_{p=1}^{t},$$

from the above it is easy to see that if $\alpha_S(Hom_{\Gamma}(S,F,m,\delta,\sigma_i)) \subseteq_{\varepsilon'} V$ and ε' is sufficiently small,

$$A(V) \supseteq_{\varepsilon,\|\cdot\|_2} \{\phi_i(e_j \otimes e_k) : (j,k) \in A_i\},$$

with

$$\frac{|A_i|}{d_i} \to (1-(2013)!\,\varepsilon) nd_i,$$

$$\phi_i(f) = \left(\sigma_i(\xi^{(1)})(f), \sigma_i(\xi^{(2)})(f), \cdots, \sigma i(\xi^{(t)})(f)\right).$$

Thus ϕ_i is given in matrix form by

$$\phi_i = \begin{bmatrix} \sigma_i(\xi^{(1)}) & 0 & \cdots & 0 \\ 0 & \sigma_i(\xi^{(2)}) & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_i(\xi^{(t)}) \end{bmatrix}$$

As

$$\phi_i\phi_i^* = \begin{bmatrix} \sigma_i(\xi^{(1)})\sigma_i(\xi^{(1)})^* & 0 & \cdots & 0 \\ 0 & \sigma_i(\xi^{(2)})\sigma_i(\xi^{(2)})^* & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_i(\xi^{(t)})\sigma_i(\xi^{(t)})^* \end{bmatrix}.$$

By our choice of σ_i we have

$$\|\phi_i\| \le 1.$$

By Lemma (3.2.12) [3], we find that

$$dim_{\Sigma,l^p}(V,\Gamma) \ge (1-(2013)!\,\varepsilon) n\, dim_{L(\Gamma)}(H).$$

Letting $\varepsilon \to 0, t \to \infty$ completes the proof.

(b), (c) Same proof as in (a), one instead uses Lemma (3.2.17) [3], Lemma (3.2.11) [3], and the formula

$$\mathbb{P}(A) = \int_{U(d_i)} \underline{\left|\{j: Ue_j \in A\}\right| |d_i}\ dU,$$

for $A \subseteq S^{2d_i - 1}$, to find an orthonormal system $\zeta_1, \cdots, \zeta_q$ with $q \geq (1 - \varepsilon)d_i$, so that $T^{(E)}_{\zeta_1, \zeta_q, k} \in Hom_\Gamma(\cdots)$ for most k and all j, p.

Corollary (3.2.20) [3]: Let $1 \leq p \leq 2, V$ a finite-dimensional normed vector space, and Γ a countable discrete group.

(a) If Γ is sofic and Σ is a sofic approximation of Γ, then

$$\underline{dim}_{\Sigma, l^p}(l^p(\Gamma, V), \Gamma) = dim_{\Sigma, l^p}(l^p(\Gamma, V), \Gamma) = \dim V.$$

(b) If Γ is $\mathcal{R}^\omega$-embeddable and Σ is an embedding sequence of Γ, then

$$\underline{dim}_{\Sigma, l^2}\left(l^2\left(\Gamma, l^2(n)\right), \Gamma\right) = dim_{\Sigma, l^2}\left(l^2\left(\Gamma, l^2(n)\right), \Gamma\right) = n,$$

$$\underline{dim}_{\Sigma, s^p, conj}(l^p(\Gamma, V), \Gamma) = dim_{\Sigma, S^p, conj}(l^p(\Gamma, V), \Gamma) = \dim V.$$

Corollary (3.2.21) [3]: Let Γ be a $\mathcal{R}^\omega$-embeddable group $1 \leq p \leq 2$. If V, W are finite dimensional vector spaces with $\dim V < \dim W$, then there are no Γ–equivariant bounded linear maps from $l^p(\Gamma, V)$ to $l^p(\Gamma, W)$ with dense image. Consequently if $2 \leq p < \infty$, then there are no Γ -equivariant bounded linear injections from $l^p(\Gamma, W)$ to $l^p(\Gamma, V)$.

Theorem (3.2.22) [3]: Let Γ be an $\mathcal{R}^\omega$-embeddable group, and $\pi: \Gamma \to U(H)$ a representation, such that $\pi \leq \lambda^{\oplus\infty}$. Then for every embedding sequence Σ,

$$dim_{\Sigma, l^2}(\pi) = \underline{dim}_{\Sigma, l^2}(\pi) = dim_{L(\Gamma)}(\pi).$$

Proof. Let $\lambda: \Gamma \to U(l^2(\Gamma))$ be given by $\lambda(g)\, f(x) = f(g^{-1}x)$. We already know from Theorem (3.2.20) that

$$dim_{\Sigma, l^2}\lambda^{\oplus\infty} = \underline{dim}_{\Sigma, l^2}\lambda^{\oplus\infty} = n.$$

Let us first assume that π is cyclic with cyclic vector ξ, then as in Lemma (3.2.18) [3] we may find a $\zeta \in l^2(\Gamma)$ so that

$$\langle \pi(x)\xi, \xi \rangle = \langle \lambda(x)\zeta, \zeta \rangle,$$

so $\pi \leq \lambda$. Let π' be a representation such that $\lambda = \pi \oplus \pi'$, then by Theorem (3.2.19) [3] we have

$$1 = dim_{\Sigma, l^2}\lambda \geq dim_{\Sigma, l^2}\pi + dim_{\Sigma, l^2}\pi' \geq dim_{\Sigma, l^2}\pi + \underline{dim}_{\Sigma, l^2}\pi'$$

$$\geq dim_{L(\Gamma)}\pi + dim_{L(\Gamma)}\pi' = 1.$$

Thus all the above inequalities must be equalities, in particular

$$dim_{\Sigma, l^2}\pi = \underline{dim}_{\Sigma, l^2}\pi = dim_{L(\Gamma)}\pi.$$

In the general case, apply Zorn's Lemma to write $\pi = \oplus_{n=1}^{\infty} \pi_n$ with π_n cycle. Then by Corollary (3.2.10) [3]

$$\underline{dim}_{\Sigma,l^2}(\pi) \geq \sum_{n=1}^{\infty} \underline{dim}_{\Sigma,l^2}(\pi_n) = \sum_{n=1}^{\infty} dim_{L(\Gamma)}(\pi_n) = dim_{\,L(\Gamma)}\pi,$$

$$dim_{\Sigma,l^2}(\pi) \leq \sum_{n=1}^{\infty} dim_{\Sigma,l^2}(\pi_n) = \sum_{n=1}^{\infty} dim_{L(\Gamma)}(\pi_n) = dim_{\,L(\Gamma)}\pi.$$

This completes the proof of the theorem.

Chapter 4
A Banach Space with a Countable Infinite Number of Complex Structures

We show the question remains about finding examples of Banach spaces with exactly infinite countably many different complex structures. A first natural approach to solve this problem is to construct an infinite sum of copies of $X(\mathbb{C})$, and in order to control the number of complex structures to take a regular sum, for instance, $\ell_1\big(X(\mathbb{C})\big)$.

Section (4.1): Construction and Complex Structure of the Space $\mathfrak{X}_{\omega_1}(\mathbb{C})$

A real Banach space X is said to admit a complex structure when there exists a linear operator I on X such that $I^2 = -Id$. This turns X into a $\mathbb{C}$-linear space by declaring anew law for the scalar multiplication:

$$(\lambda + i\mu).x = \lambda x + \mu I(x) \qquad (\lambda, \mu \in \mathbb{R}).$$

Equipped with the equivalent norm

$$\|x\| = \sup_{0\leq\theta\leq 2\pi} \|\cos\theta x + \sin\theta I x\|$$

we obtain a complex Banach space which will be denoted by X^I. The space X^1 is the complex structure of X associated to the operator I, which is often itself referred to as a complex structure for X.

When the space X is already a complex Banach space, the operator $Ix = ix$ is a complex structure on $X_{\mathbb{R}}$ (i.e., X seen as a real space) which generates X. Recall that for a complex Banach space X its complex conjugate $\bar{X}$ is defined to be the space X equipped with the new scalar multiplication $\lambda.x = \bar{\lambda}x$.

Two complex structures I and J on a real Banach space X are equivalent if there exists a real automorphism T on X such that $TI = JT$. This is equivalent to saying that the spaces X^I and X^J are $\mathbb{C}$-linearly isomorphic. To see this, simply observe that the relation $TI = JT$ actually means that the operator T is $\mathbb{C}$-linear as defined from X^I to X^J.

We note that a complex structure I on a real Banach space X is an automorphism whose inverse is $-I$, which is itself another complex structure on X. In fact, the complex space X^{-I} is the complex conjugate space of X^I. Clearly the spaces X^I and X^{-I} are always $\mathbb{R}$-linearly isometric. On the other hand, J. Bourgain and N. J. Kalton constructed examples of complex Banach spaces not isomorphic to their corresponding complex conjugates, hence these spaces admit at least two different complex structures. The Bourgain example is an $\ell_2$sum of finite dimensional spaces whose distance to their conjugates tends to infinity. The Kalton example is a twisted sum of two Hilbert spaces, i.e., X has a closed subspace E such that E and X/E are Hilbertian, while X itself is not isomorphic to a Hilbert space. More recently R. Anisca constructed a complex weak Hilbert space not isomorphic to its complex conjugate.

Complex structures do not always exist on Banach spaces. The first example in the literature was the James space, proved by J. Dieudonné. Other examples of spaces with-out complex structures are the uniformly convex space constructed by S. Szarek and the hereditary indecomposable space of W. T. Gowers and B. Maurey. W. T. Gowers and B. Maurey and S. A. Argyros, K. Beanland and T. Raikoftsalis also constructed a space with unconditional basis but without complex structures, the second is a weak Hilbert space. In general these spaces have few operators. For example, every operator on the Gowers– Maurey space is a strictly singular perturbation of a multiple of the identity and this forbids complex structures: suppose that T is an operator on this space such that $T^2 = -Id$ and write $T = \lambda Id + S$ with S a strictly singular operator. It follows that $(\lambda^2 + 1)Id$ is strictly singular and of course this is impossible.

More examples of Banach spaces without complex structures were constructed by P. Koszmider, M. Martín and J. Merí. In fact, they introduced the notion of *extremely non-complex Banach space*: A real Banach space X is extremely non-complex if every bounded linear operator $T: X \to X$ satisfies the norm equality $\|Id + T^2\| = 1 + \|T\|^2$. Among their examples of extremely non-complex spaces are $C(K)$ spaces with few operators (e.g. when every bounded linear operator T on $C(K)$ is of the form $T = gId + S$ where $g \in C(K)$ and S is a weakly compact operator on $C(K)$), a $C(K)$ space containing a complemented isomorphic copy of ℓ_∞ (thus having a richer space of operators than the first one mentioned) and an extremely non-complex space not isomorphic to any $C(K)$ space.

Going back to the problem of uniqueness of complex structures, Kalton proved that spaces whose complexification is a primary space have at most one complex structure (this result may be found in V. Ferenczi and E. Galego). In particular, the classical spaces $c_0, \ell_p(1 \leq p \leq \infty), L_p[0,1](1 \leq p \leq \infty)$, and $C[0,1]$ have a unique complex structure.

We have mentioned before examples of Banach spaces with at least two different complex structures. In fact, V. Ferenczi constructed a space $X(\mathbb{C})$ such that the complex structure $X(\mathbb{C})^J$ associated to some operator J and its conjugate are the only complex structures on $X(\mathbb{C})$up to isomorphism. Furthermore, every $\mathbb{R}$-linear operator T on $X(\mathbb{C})$ is of the form $T = \lambda Id + \mu J + S$, where λ, μ are reals and S is strictly singular. Ferenczi also proved that the space $X(\mathbb{C})^n$ has exactly $n+1$complex structures for every positive integer n. Going to the extreme, R.Anisca [1] gave examples of subspaces of $L_p(1 \leq p < 2)$ which admit continuum many non-isomorphic complex structures.

It follows that every $\mathbb{R}$-linear bounded operator T on $\ell_1(X(\mathbb{C}))$ is of the form $T = \lambda(T) + S$, where $\lambda(T)$ is the scalar part of T, i.e., an infinite matrix of operators on $X(\mathbb{C})$ of the form $\lambda_{i,j} Id + \mu_{i,j} J$, and S is an infinite matrix of strictly singular operators on $X(\mathbb{C})$. It is easy to prove that if T is a complex structure then $\lambda(T)$ is also a complex structure. Recall that two complex structures whose difference is strictly singular must be equivalent. Unfortunately, the operator S in the representation of T is not necessarily strictly singular, and this makes very difficult to understand the complex structures on $\ell_1(X(\mathbb{C}))$.

It is necessary to consider a more "rigid" sum of copies of spaces like $X(\mathbb{C})$. We found this interesting property in the space $\mathfrak{X}_{\omega_1}$ constructed by S.A.Argyros, J.Lopez-Abad and S.Todorcevic. Based on that construction we present a separable reflexive Banach space $\mathfrak{X}_{\omega^2}(\mathbb{C})$ with exactly infinite countably many different complex structures which admits an infinite dimensional Schauder decomposition $\mathfrak{X}_{\omega^2}(\mathbb{C}) = \oplus_k \mathfrak{X}_k$ for which every $\mathbb{R}$-linear operator T on $\mathfrak{X}_{\omega^2}(\mathbb{C})$ can be written as $T = D_T + S$, where S is strictly singular, $D_T|_{\mathfrak{X}_k} = \lambda_k Id_{\mathfrak{X}_k}(\lambda k \in \mathbb{C})$ and $(\lambda_k)_k$ k is a convergent sequence.

This construction also shows the existence of continuum many examples of Banach spaces with the property of having exactly ω complex structures and the existence of a Banach space with exactly ω_1 complex structures.

We construct a complex Banach space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ with a bimonotone transfinite Schauder basis $(e_\alpha)_{\alpha<\omega_1}$, such that every complex structure I on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is of the form $I = D + S$, where D is a suitable diagonal operator and S is strictly singular.

By a bimonotone transfinite Schauder basis we mean that $\mathfrak{X}_{\omega_1}(\mathbb{C}) = \overline{\text{span}}\,(e_\alpha)_{\alpha<\omega_1}$ and such that for every interval I of ω_1 the naturally defined map on the linear span of $(e_\alpha)_{\alpha<\omega_1}$

$$\sum_{\alpha<\omega_1} \lambda_\alpha e_\alpha \mapsto \sum_{\alpha\in I} \lambda_\alpha e_\alpha$$

extends to a bounded projection $P_I: \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_I = \overline{\text{span}}_{\mathbb{C}}(e_\alpha)_{\alpha\in I}$ with norm equal to1.

Basically $\mathfrak{X}_{\omega_1}(\mathbb{C})$ corresponds to the complex version of the space $\mathfrak{X}_{\omega_1}$ constructed in this section modifying the construction in a way that its $\mathbb{R}$-linear operators have similar structural properties to the operators in the original space $\mathfrak{X}_{\omega_1}$ (i.e., the operators are strictly singular perturbation of a complex diagonal operator).

Recall that ω and ω_1 denotes the least infinite cardinal number and the least uncountable cardinal number, respectively. Given ordinals γ, ξ we write $\gamma + \xi, \gamma \cdot \xi, \gamma^\xi$ for the usual arithmetic operations. For an ordinal γ we denote by $\Lambda(\gamma)$ the set of limit ordinals $< \gamma$. Denote by $c_{00}(\omega_1, \mathbb{C})$ the vector space of all functions $x : \omega_1 \to \mathbb{C}$ such that the set $supp\, x = \{\alpha < \omega_1 : x(\alpha) \neq 0\}$is finite and by $((e_\alpha)_{\alpha\in\omega_1}$ its canonical Hamel basis. For a vector $x \in c_{00}(\omega_1, \mathbb{C})$ ran x will denote the minimal interval containing $supp\, x$. Given two subsets E_1, E_2 of ω_1 we say that $E_1 < E_2$ if $\max E_1 < \max E_2$. Then for $x, y \in c_{00}(\omega_1, \mathbb{C})$ $x < y$ means that $supp\, x < supp\, y$. For a vector $x \in c_{00}(\omega_1, \mathbb{C})$ and a subset E of ω_1 we denote by Ex (or P_{E_x}) the restriction of x on E or simply the function $x\chi_E$. Finally in some cases we shall denote elements of $c_{00}(\omega_1, \mathbb{C})$ as $f, g, h, \ldots$ and its canonical Hamel basis as $(e_\alpha^*)_{\alpha\in\omega_1}$ meaning that we refer to these elements as being functionals in the norming set.

Definition (4.1.1) [4]: The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ shall be defined as the completion of $c_{00}(\omega_1, \mathbb{C})$ equipped with a norm given by anorming set $\kappa_{\omega_1}(\mathbb{C}) \subseteq c_{00}(\omega_1, \mathbb{C})$. This means that the norm for every $x \in c_{00}(\omega_1, \mathbb{C})$

is defined as $\sup\{|\phi(x)| = |\sum_{\alpha<\omega_1} \phi(\alpha)x(\alpha)| : \phi \in \kappa_{\omega_1}(\mathbb{C})\}$. The norm of this space can also be defined inductively.

We start by fixing two fast increasing sequences (m_j) and (n_j) that are going to be used in the rest of this work. The sequences are defined recursively as follows:

(i) $m_1 = 2$ and $m_{j+1} = m_j^4$;

(ii) $n_1 = 4$ and $n_{j+1} = (4n_j)^{s_j}$, where $s_j = \log_2 m_{j+1}^3$.

Let $\kappa_{\omega_1}(\mathbb{C})$ be the minimal subset of $c_{00}(\omega_1, \mathbb{C})$ such that

(a) It contains every $(e_\alpha^*)_{\alpha<\omega_1}$. It satisfies that for every $\phi \in \kappa_{\omega_1}(\mathbb{C})$ and for every complex number $\theta = \lambda + i\mu$ with λ and μ rationals and $|\theta| \leq 1, \theta\phi \in \kappa_{\omega_1}(\mathbb{C})$. It is closed under restriction to intervals of ω_1.

(b) For every $\{\phi_i : i = 1, \ldots, n_{2j}\} \subseteq \kappa_{\omega_1}(\mathbb{C})$ such that $\phi_1 < \cdots < \phi_{n_{2j}}$, the combination

$$\phi = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \phi_i \in \kappa_{\omega_1}(\mathbb{C}).$$

In this case we say that ϕ is the result of an (m_{2j}^{-1}, n_{2j})-operation.

(c) For every special sequence $\left(\phi_1 < \cdots < \phi_{n_{2j+1}}\right)$ (see Definition (4.2.4), the combination

$$\phi = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} \phi_i \in \kappa_{\omega_1}(\mathbb{C}).$$

In this case we say that ϕ is a special functional and that ϕ is the result of an $(m_{2j+1}^{-1}, n_{2j+1})$-operation.

(d) It is rationally convex.

Define a norm on c$c_{00}(\omega_1, \mathbb{C})$ by setting

$$\|x\| = \sup\left\{\left|\sum_{\alpha<\omega_1} \phi(\alpha)x(\alpha)\right| : \phi \in \kappa_{\omega_1}(\mathbb{C})\right\}.$$

The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is defined as the completion of $(c_{00}(\omega_1, \mathbb{C}), \|\cdot\|)$.

This definition of the norming set $\kappa_{\omega_1}(\mathbb{C})$ is similar to other (c). We add the property of being closed under products with rational complex numbers of the unit ball. This, together with property (b) above, guarantees the existence of some type of sequences in the same way they are constructed for $\mathfrak{X}_{\omega_1}$. It follows that the norm is also defined by

$$\|x\| = \sup\left\{\phi(x) = \sum_{\alpha<\omega_1} \phi(\alpha)x(\alpha) : \phi \in \kappa_{\omega_1}(\mathbb{C}), \phi(x) \in \mathbb{R}\right\}.$$

We also have the following implicit formula for the norm:

$$\|x\| = \sup\left\{\|x\|_\infty, \sup_j \sup \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \|E_i x\|,\ E_1 < E_2 < \cdots < E_{n_{2j}}\right\}$$

$$\vee \sup\left\{\frac{1}{m_{2j+1}} \left|\sum_{i=1}^{n_{2j}+1} \phi(Ex)\right| : (\phi_i)_{i=1}^{n_{2j}+1} is\ n_{2j+1} special, E\ interval\right\}.$$

It follows from the definition of the norming set that the canonical Hamel basis $(e_\alpha)_{\alpha<\omega_1}$ is a transfinite bimonotone Schauder basis of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. In fact, by property (a) for every interval I of ω_1 the projection P_I has norm 1:

$$\|P_I x\| = \sup_{f\in\kappa_{\omega_1}(\mathbb{C})} |f P_I x| = \sup_{f\in\kappa_{\omega_1}(\mathbb{C})} |P_I f x| \le \|x\|$$

Moreover, we have that the basis $(e_\alpha)_{\alpha\in\omega_1}$ is boundedly complete and shrinking, the proof is the obvious modification to the one for $\mathfrak{X}_{\omega_1}$. In consequence $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is reflexive.

Proposition (4.1.2) [4]: $\overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*} = B_{\mathfrak{X}^*_{\omega_1}}(\mathbb{C})$.

Proof. Recall that the set $\kappa_{\omega_1}(\mathbb{C})$ is by definition rationally convex. We notice that $\overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$ is actually a convex set. Indeed let $f, g \in \overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$ and $t \in (0,1)$. Suppose that $f_n \overset{\omega^*}{\to} f, g_n \overset{\omega^*}{\to} g\ and\ t_n \to t$, where $f_n, g_n \in \kappa_{\omega_1}(\mathbb{C})$ and $t_n \in \mathbb{Q} \cap (0,1)$ for every $n \in \mathbb{N}$. Then $tf + (1-t)g \in \overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$ because

$$t_n f_n + (1-t_n) g_n \overset{\omega^*}{\to} tf + (1-t)g.$$

In the same manner we can prove that $\mathfrak{X}^*_{\omega_1}(\mathbb{C})$ is balanced, i.e., $\lambda\, \mathfrak{X}^*_{\omega_1}(\mathbb{C}) \subseteq \mathfrak{X}^*_{\omega_1}$ for every $|\lambda\,| \le 1$. To prove the proposition suppose that there exists $f \in B_{\mathfrak{X}^*_{\omega_1}(\mathbb{C})} \backslash \overline{\kappa_{\omega_1}(\mathbb{C})}^{\omega^*}$. It follows by a standard separation argument that there exists $x \in \mathfrak{X}_{\omega_1}(\mathbb{C})$ such that

$$|f(x)| > \sup\{|g(x)| : g \in \kappa_{\omega_1}(\mathbb{C})\}$$

which is absurd.

Now we show the complex structures on $\mathfrak{X}_{\omega_1}(\mathbb{C})$

Let $I \subseteq \omega_1$ be an interval of ordinals, we denote by $\mathfrak{X}_1(\mathbb{C})$ the closed subspace of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ generated by $\{e_\alpha\}_{\alpha\in I}$. For every ordinal $\gamma < \omega_1$ we write $\mathfrak{X}_\gamma(\mathbb{C}) = \mathfrak{X}_{[0,\gamma]}(\mathbb{C})$. Notice that $\mathfrak{X}_I(\mathbb{C})$ is a 1-complemented subspace of $\mathfrak{X}_{\omega_1}(\mathbb{C})$: the restriction to coordinates in Iis a projection of norm 1 onto $\mathfrak{X}_I(\mathbb{C})$. We denote this projection by P_I and by $P^I = (Id - P_I)$ the corresponding projection onto the complement space $(Id - P_I)\mathfrak{X}_{\omega_1}(\mathbb{C})$, which we denote by $\mathfrak{X}^I(\mathbb{C})$.

A transfinite sequence $(y_\alpha)_{\alpha<\gamma}$ is called a block sequence when $y_\alpha < y_\beta$ for all $\alpha < \beta < \gamma$. Given a block sequence $(y_\alpha)_{\alpha<\gamma}$ a *block subsequence* of $(y_\alpha)_{\alpha<\gamma}$ is a block sequence $\left(x_\beta\right)_{\beta<\xi}$ in the span of $(y_\alpha)_{\alpha<\gamma}$. A *real block subsequence*of $(y_\alpha)_{\alpha<\gamma}$ is a block subsequence in the *real* span of $(y_\alpha)_{\alpha<\gamma}$. A sequence $(x_n)_{n\in\mathbb{N}}$ is a block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ when it is a block subsequence of $(e_\alpha)_{\alpha<\omega_1}$.

Theorem (4.1.3) [4]: *Let $T: \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$, that is, T is a bounded $\mathbb{R}$-linear operator such that $T^2 = -Id$. Then there exists a bounded diagonal operator $D_T: \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$, which is another complex structure, such that $T - D_T$ is strictly singular. Moreover $D_T = \sum_{j=1}^{k} \epsilon_j i P_{I_j}$ for some signs $\left(\epsilon_j\right)_{j=1}^{k}$ and ordinal intervals $I_1 < I_2 <\ldots< I_k$ whose extremes are limit ordinals and such that $\omega_1 = \bigcup_{j=1}^{k} I_j$.*

The strategy for the proof of Theorem (4.1.4) is for the real case. However here we want to understand bounded $\mathbb{R}$-linear operators in acomplex space. The result is obtained using the following theorems that we explain with more details in Appendix A.

Step I. There exists a family $\mathfrak{F}$ of semi-normalized block subsequences of $(e_\alpha)_{\alpha<\omega_1}$, called *R.I.S.*(*Rapidly Increasing Sequences*), such that every normalized block sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has a real block subsequence in $\mathfrak{F}$.

Recall that a Banach space X is hereditarily indecomposable (or H.I.) if no (closed) subspace of X can be written as the direct sum of infinite-dimensional subspaces. Equivalently, for any two subspaces Y, Z of X and $\epsilon > 0$, there exist $y \in Y, z \in Z$such that $\|y\| = \|z\| = 1$ and $\|y - z\| < \epsilon$.

Step II. For every normalized block sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, the subspace $\overline{span}_{\mathbb{R}}(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is a real H.I. space.

Step III. Let $(x_n)_{n\in\mathbb{N}}$ be a *R.I.S.* and $T: \overline{span}_{\mathbb{C}}(x_n)_{n\in\mathbb{N}} \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$ -linear operator. Then $\lim_{n\to\infty}(Tx_n, \mathbb{C}x_n) = 0$.

The proofs of Steps I, II and III are given in Appendix A.

Step IV. Let $((x_n)_{n\in\mathbb{N}}$ be a *R.I.S.* and $T: \overline{span}_{\mathbb{C}}(x_n)_{n\in\mathbb{N}} \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$ -linear operator. Then the sequence $\lambda_T: \mathbb{N} \to \mathbb{C}$ defined by $d(Tx_n, \mathbb{C}x_n) = \|Tx_n - \lambda_T(n)x_n\|$ is convergent.

Proof of Step IV. First we note that the sequence $\left(\lambda_T(n)\right)_n$ is bounded. Then consider $(\alpha_n)_n$ and $(\beta_n)_n$ as two strictly increasing sequences of positive integers and suppose that $\lambda_T(\alpha_n) \to \lambda_1$ and $\lambda_T(\beta_n) \to \lambda_2$, when $n \to \infty$. Going to a subsequence we can assume that $x_{\alpha_n} < x_{\beta_n} < x_{\alpha_n+1}$ for every $n \in \mathbb{N}$.

Fix $\epsilon > 0$. Using the result of Step III, we have that $\lim_{n\to\infty}\left\|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\right\| = 0$. By passing to a subsequence if necessary, assume

$$\left\|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\right\| \leq \frac{\epsilon}{2^n 6},$$

for every $n \in \mathbb{N}$. Hence, for every $w = \sum_n \alpha_n x_n \in span_{\mathbb{R}}\left(x_{\alpha_n}\right)_n$ with $\|w\| \leq 1$ we have

$$\|Tw - \lambda_1 w\| \leq \sum_n |\alpha_n| \left\|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\right\| \leq \epsilon/3,$$

because $(e_\alpha)_{\alpha<\omega_1}$ is a bimonotone transfinite basis. In the same way, we can assume that for every $w \in span_{\mathbb{R}}\left(x_{\beta_m}\right)_m$ with $\|w\| \leq 1$, $\|Tw - \lambda_2 w\| \leq \epsilon/3$. By Step II we have that $\overline{span}_{\mathbb{R}}\left(x_{\alpha_n}\right)_n \cup \left(x_{\beta_n}\right)_n$ is real- H.I. Then there exist unit vectors $w_1 \in span_{\mathbb{R}}\left(x_{\alpha_n}\right)_n$ and $w_2 \in span_{\mathbb{R}}\left(x_{\beta_m}\right)_m$, such that $\|w_1 - w_2\| \leq \frac{\epsilon}{3}\|T\|$. Therefore,

$$\|\lambda_1 w_1 - \lambda_2 w_2\| \leq \|Tw_1 - \lambda_1 w_1\| + \|Tw_1 - Tw_2\| + \|Tw_2 - \lambda_2 w_2\| \leq \epsilon$$

By other side

$$\begin{aligned}\|\lambda_1 w_1 - \lambda_2 w_2\| &\geq \|(\lambda_1 - \lambda_2)w_1\| - \|\lambda_2(w_1 - w_2)\| \\ &= |\lambda_1 - \lambda_2| - |\lambda_2|\epsilon\end{aligned}$$

In consequence, $|\lambda_1 - \lambda_2| \leq (1 + |\lambda_2|)\epsilon$. Since ϵ was arbitrary, it follows that $\lambda_1 = \lambda_2$.

Let $T: \mathfrak{X}_{\omega_1}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. We define a bounded diagonal operator D_T (with respect to the basis $(e_\gamma)_{\gamma<\omega_1}$) such that $T - D_T$ is strictly singular: Let $\alpha \in \Lambda(\omega_1)$ be a limit ordinal, and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be two *R.I.S.* such that $\sup_n \max supp\ x_n = \sup_n \max supp\ y_n = \alpha + \omega$. By a property of $\mathfrak{F}$ we can mix the sequences $(x_n)_n$, $(y_n)_n$ in order to form a new *R.I.S.* $(z_n)_{n\in\mathbb{N}}$, such that $z_{2k} \in \{x_n\}_{n\in\mathbb{N}}$ and $z_{2k-1} \in \{y_n\}_{n\in\mathbb{N}}$ for all $k \in \mathbb{N}$. Then it follows from Step IV that the sequences defined by the formulas $d(Tx_n, \mathbb{C}_{x_n}) = \|Tx_n - \lambda_T(n)_{x_n}\|$ and $d(Ty_n, \mathbb{C}_{y_n}) = \|Ty_n - \mu(n)_{y_n}\|$ are convergent, and by the mixing argument, they must have the same limit. Hence for each $\alpha \in \Lambda(\omega_1)$ there exists a unique complex number $\xi_T(\alpha)$ such that

$$\lim_{n\to\infty} \|Tw_n - \xi_T(\alpha)_{w_n}\| = 0$$

for every *R.I.S.* $(w_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}_{I_\alpha}$, where we write I_α to denote the ordinal interval $[\alpha, \alpha + \omega)$. We proceed to defining a diagonal linear operator D_T on the (linear) decomposition of span $(e_\alpha)_{\alpha<\omega_1}$

$$span\ (e_\alpha)_{\alpha<\omega_1} = \bigoplus_{\alpha\in\Lambda(\omega_1)} span\ (x_\beta)_{\beta\in I_\alpha}$$

by setting $D_T(e_\beta) = \xi_T(\alpha)_{e_\beta}$ when $\beta \in I_\alpha$.

Observe in addition that this sequence $(\xi_T(\alpha))_{\alpha\in\Lambda(\omega_1)}$is convergent. That is, for every strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\Lambda(\omega_1)$, the corresponding subsequence $(\xi_T(\alpha))_{n\in\mathbb{N}}$ is convergent. In fact, for every $n \in \mathbb{N}$, let $(y_n^k)_{n\in\mathbb{N}}$ be a *R.I.S.* in $\mathfrak{X}_{I_{\alpha_n}}$.
Then we can take a *R.I.S.* $(y_n^{k_n})_{n\in\mathbb{N}}$ such that $\|Ty_n^{k_n} - \xi_T(\alpha_n + \omega)y_n^{k_n}\| < 1/n$. It follows by Step IVthere exists $\lambda \in \mathbb{C}$ such that $\lim_n \|Ty_n^{k_n} - \lambda y_n^{k_n}\| = 0$. This implies that $\lim_n \xi_T(\alpha_n + \omega) = \lambda$.

In general this operator D_T defines a bounded operator on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. The proof is the same as that uses certain James like space of a mixed Tsirelson space is finitely interval representable in every normalized transfinite block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. For the case of complex structures we have a simpler proof (see Proposition (4.1.8)).

Proposition (4.1.4) [4]: *Let A be a subset of ordinals contained in ω_1 and $X = \overline{span}_{\mathbb{C}}(e_\alpha)_{\alpha\in A}$. Let $T: X \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear* *operator. Then T is strictly singular if and only if for every R.I.S. $(y_n)_{n\in\mathbb{N}}$ on $X, \lim_n Ty_n = 0$.*

Proof. The proposition is trivial when the set A is finite, then we assume that A is infinite. Suppose that T is strictly singular. Let $(y_n)_{n\in\mathbb{N}}$ be a *R.I.S.* on X such that $\lim_n Ty_n = 0$, then by Step IV there is $\lambda \neq 0$ with $\lim_n \|Ty_n - \lambda y_n\| = 0$. Take $0 < \epsilon < |\lambda|$. By passing to a subsequence if necessary, we assume that $\left\|(T - \lambda Id)|_{\overline{span}_{\mathbb{C}}(y_n)_n}\right\| < \epsilon$. This implies that $T|_{\overline{span}_{\mathbb{C}}(y_n)_n}$ is an isomorphism, which is a contradiction.

Conversely, suppose that for every *R.I.S.* $(y_n)_n$on X, $\lim_n Ty_n = 0$. Assume that T is not strictly singular. Then there is a block sequence subspace $Y = \overline{span}_{\mathbb{C}}(y_n)_{n\in\mathbb{N}}$ of X such that T restricted to Y is an isomorphism. By Step I we can assume that the sequence $(y_n)_n$ is already a *R.I.S.* on X. Then $\inf_n \|Ty_n\| > 0$. And we obtain acontradiction.

Given $Y \subseteq \mathfrak{X}_{\omega_1}(\mathbb{C})$ we denote by i_Y the canonical inclusion of Y into $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

Corollary (4.1.5) [4]: *Let $\alpha \in \Lambda(\omega_1)$ and $T: \mathfrak{X}_{I_\alpha}(\mathbb{C}) \to \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. Then there exists (unique) $\xi_T(\alpha) \in \mathbb{C}$ such that $T - \xi_T(\alpha) i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$ is strictly singular.*

Proof. Let $\xi_T(\alpha)$ be the (unique) complex number such that $\lim_n \|Ty_n - \xi_T(\alpha) y_n\| = 0$ for every *R.I.S.* $(y_n)_n$ on $\mathfrak{X}_{I_\alpha}(\mathbb{C})$. Then by the previous proposition $T - \xi_T(\alpha) i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$ is strictly singular.

Corollary (4.1.6) [4]: *Let $\alpha \in \Lambda(\omega_1)$ and $R: \mathfrak{X}_{I_\alpha}(\mathbb{C}) \to \mathfrak{X}^{I_\alpha}(\mathbb{C})$ be a bounded $\mathbb{R}$-linear operator. Then R is strictly singular.*

Proof. By the previous result, $i_{\mathfrak{X}^{I_\alpha}(\mathbb{C})} R = \lambda i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S$ with S strictly singular. Then projecting by P^{I_α} we obtain $R = P^{I_\alpha} \circ i_{\mathfrak{X}^{I_\alpha}(\mathbb{C})} R = P^{I_\alpha} S$ which is strictly singular.

Proposition (4.1.7) [4]: *Let T be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then the linear operator D_T is a bounded complex structure.*

Proof. Let T be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and D_T the corresponding diagonal operator defined above. Fix $\alpha \in \Lambda(\omega_1)$. We shall prove that $\xi_T(\alpha)^2 = -1$. In fact,

$$T \circ i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} = P_{I_\alpha} T \circ i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + P^{I_\alpha} T \circ i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} = P_{I_\alpha} T \circ i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_1$$

where S_1 is strictly singular. This implies $P_{I_\alpha} T \circ i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} = \xi_T(\alpha)\, Id_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_2: \mathfrak{X}_{I_\alpha}(\mathbb{C}) \to \mathfrak{X}_{I_\alpha}(\mathbb{C})$ with S_2 strictly singular. Now computing:

$$\left(P_{I_\alpha}Ti_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}\right)\circ\left(P_{I_\alpha}Ti_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}\right) = P_{I_\alpha}T\circ P_{I_\alpha}Ti_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$$
$$= P_{I_\alpha}T\circ(Id - P^{I_\alpha})Ti_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$$
$$= P_{I_\alpha}T^2 i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} - P_{I_\alpha}T\underline{P^{I_\alpha}Ti_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}} = -Id_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_3$$

where S_3 is strictly singular because the underlined operator is strictly singular. Hence we have that $(\xi_T(\alpha)^2+1)Id_{\mathfrak{X}_{I_\alpha}}$ is strictly singular, which allows us to conclude that $\xi_T(\alpha)^2 = -1$. The continuity of D_T is then guaranteed by the convergence of $(\xi_T(\alpha))_{\alpha\in\Lambda(\omega_1)}$.
Indeed, $\xi_T(\alpha) = \pm i$ for every $\alpha\in\Lambda(\omega_1)$ and by convergence we have that the variation of signals is finite, then there exist ordinal intervals $I_1 < I_2 < \cdots I_k$ with $\omega_1 = \bigcup_{j=1}^{k} I^*{}_j$ and such that $D_T = \sum_{j=1}^{k}\epsilon_j i\, P_{I_j}$ for some signs $\left(\epsilon_j\right)_{j=1}^{n}$.

Remark (4.1.7) [4]: More generally, the proof of Proposition (4.1.8) actually shows that if T is an $\mathbb{R}$-linear bounded operator on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that $T^2 + Id = S$ for some S strictly singular, then D_T is bounded and $D_T^2 = -Id$.

Let $T:\mathfrak{X}_{\omega_1}(\mathbb{C})\to\mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded $\mathbb{R}$ -linear operator which is a complex structure and D_T be the diagonal bounded operator associated to it. It only remains to prove that $T - D_T$ is strictly singular. And this follows directly from Proposition (4.1.5), because by definition $\lim_n (T - D_T)_{y_n} = 0$ for every *R.I.S.* $(y_n)_n$ on $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

We come back to the study of the complex structures on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Denote by $\mathfrak{D}$ the family of complex structures D_T on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ as in Theorem (4.1.4), i.e., $D_T = \sum_{j=1}^{k}\epsilon_j i\, P_{I_j}$ where $\left(\epsilon_j\right)_{j=1}^{n}$ are signs and $I_1 < I_2 < \cdots I_k$ are ordinal intervals whose extremes are limit ordinals and such that $\omega_1 = \bigcup_{j=1}^{k} I_j$. Notice that $\mathfrak{D}$ has cardinality ω_1.

Recall that two spaces are said to be incomparable if neither of them embed into the other.

Corollary (4.1.8) [4]: *The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has ω_1 many complex structures up to isomorphism. Moreover any two non-isomorphic complex structures are incomparable.*

Proof. Let J be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. By Theorem (4.1.4) we have that $J - D_J$ is a strictly singular operator and $D_J\in\mathfrak{D}$. Recall that two complex structures whose difference is strictly singular must be equivalent. Then J is equivalent to D_J.

To complete the proof it is enough to show that given two different elements of $\mathfrak{D}$ they define non-equivalent complex structures. Moreover, we prove that one structure does not embed into the other. Fix $J \neq K \in \mathfrak{D}$. Then there exists an ordinal interval $I_\alpha = [\alpha, \alpha + \omega)$ such that, without loss of generality, $J|_{\mathfrak{X}_{I_\alpha}} = iId|_{\mathfrak{X}_{I_\alpha}}$ and $K|_{\mathfrak{X}_{I_\alpha}} = -iId|_{\mathfrak{X}_{I_\alpha}}$. Suppose that there exists $T: \mathfrak{X}_{\omega_1}(\mathbb{C})^J \to \mathfrak{X}_{\omega_1}(\mathbb{C})^K$ an isomorphic embedding. Then T is in particular an $\mathbb{R}$-linear operator such that $TJ = KT$. We write using Corollary (4.1.6), $T|_{\mathfrak{X}_{I_\alpha}} = \xi_T(\alpha)i_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S$ with S strictly singular. Then $\xi_T(\alpha)J|_{\mathfrak{X}_{I_\alpha}} - \xi_T(\alpha)K|_{\mathfrak{X}_{I_\alpha}} = S_1$ where S_1 is strictly singular. In particular for each $x \in \mathfrak{X}_{I_\alpha}, S_1 x = 2\,\xi_T(\alpha)ix$. It follows from the fact that $\mathfrak{X}_{I_\alpha}$ is infinite dimensional that $\xi_T(\alpha) = 0$. Hence $T|_{\mathfrak{X}_{I_\alpha}} = S$, but this is a contradiction because T is an isomorphic embedding.

The next corollary offers uncountably many examples of Banach spaces with exactly countably many complex structures.

Corollary (4.1.9) [4]: *The space $\mathfrak{X}_\gamma(\mathbb{C})$ has ω complex structures up to isomorphism for every limit ordinal $\omega^2 \leq \gamma < \omega_1$.*

Proof. Let J be a complex structure on $\mathfrak{X}_\gamma(\mathbb{C})$. We extend J to a complex structure defined in the whole space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ by setting $T = JP_I + iP^I$, where $I = [0, \gamma)$. It follows that $T = D_T + S$ for a strictly singular operator S and a diagonal operator D_T like in Theorem (4.1.4). Notice that $D_T x = ix$ for every $x \in \mathfrak{X}^I$, otherwise there would be a limit ordinal α such that $S|_{\mathfrak{X}_{I_\alpha}} = 2iId|_{\mathfrak{X}_{I_\alpha}}$. Hence $JP_I = D_T P_I + S$. Which implies that J has the form $J = \sum_{j=1}^{k} \epsilon_j i\, P_{I_j} + S_1$ where S1is strictly singular on $\mathfrak{X}_{\omega_1}(\mathbb{C})$, $(\epsilon_j)_{j=1}^{k}$are signs and $I_1 < I_2 < \cdots < I_k$are ordinal intervals whose extremes are limit ordinals and such that $\gamma = \bigcup_{j=1}^{k} I_j$. Now the rest of the proof is identical to the proof of the previous corollary. In particular, all the non-isomorphic complex structures on $\mathfrak{X}_\gamma(\mathbb{C})$ are incomparable.

We also have, using the same proof of the previous corollary, that for every in-creasing sequence of limit ordinals $A = (\alpha_n)_n$, the space $\mathfrak{X}_A = \oplus_n \mathfrak{X}_{I_{\alpha_n}}(\mathbb{C})$, where $I_{\alpha_n} = [\alpha_n, \alpha_n + \omega)$, has exactly infinite countably many different complex structures. Hence there exists a family, with the cardinality of the continuum, of Banach spaces such that every space in it has exactly ω complex structures.

Section (4.2): Observations

It is easy to check that subspaces of even codimension of a real Banach space with complex structure also admit complex structure. An interesting property of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is that none of its real hyperplanes (and thus every real subspace of odd codimension) admit complex structure.

Proposition (4.2.1) [4]: *The real hyperplanes of* $\mathfrak{X}_{\omega_1}(\mathbb{C})$ *do not admit complex structure.*

Proof. By the results of V. Ferenczi and E. Galego it is sufficient to prove that the ideal of all $\mathbb{R}$-linear strictly singular operators on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has the lifting property, that is, for any $\mathbb{R}$ -linear isomorphism on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that $T^2 + Id$ is strictly singular, there exists a strictly singular operator S such that $(T - S)^2 = -Id$. The proof now follows easily from Remark (4.1.7) [4].

Appendix A

The purpose of this section is to give a proof for the results in Steps I, II and III. Several proofs are very similar to the corresponding ones in [3]. In order to make this section as self contained as possible, we reproduce them in detail.

First we clarify the definition of the norming set by defining what being a special sequence means. All the definitions we present in this part are the corresponding translation of [3] for the complex case.

A.1. Coding and special sequences

Recall that $[\omega_1]^2 = \{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\}$.

Definition (4.2.2) [4]: A function $\varrho: [\omega_1]^2 \to \omega$ such that

(1) $\varrho(\alpha, \gamma) \le \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.

(2) $\varrho(\alpha, \beta) \le \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.

(3) The set $\{\alpha < \beta : \varrho(\alpha, \beta) \le n\}$ is finite for all $\beta < \omega_1$ and $n \in \mathbb{N}$

is called a ϱ-function.

The existence of ϱ-functions is due to S. Todorcevic. Let us fix a ϱ-function $\varrho: [\omega_1]^2 \to \omega$, and then all the following work relies on that particular choice of ϱ.

Definition (4.2.3) [4]: Let F be a finite subset of ω_1 and $p \in \mathbb{N}$, then we write

$$\rho F = \rho_\varrho(F) = \max_{\alpha,\beta \in F} \varrho(\alpha, \beta).$$

$$\bar{F}^p = \{\alpha \le \max F : there\ is\ \beta \in F\ such\ that\ \alpha \le \beta\ and\ \varrho(\alpha, \beta) \le p\}$$

A.1.1. σ_ϱ-coding and the special sequences

We denote by $\mathbb{Q}_s(\omega_1,\mathbb{C})$ the set of finite sequences $(\phi_1,w_1,p_1,\dots,\phi_d,w_d,p_d)$ such that

(i) For all $i\le d$, $\phi_i\in c_{00}(\omega_1,\mathbb{C})$ and for all $\alpha<\omega_1$ the real and the imaginary part of $\phi(\alpha)$ are rationals.

(ii) $(w_i)_{i=1}^d,(p_i)_{i=1}^d\in\mathbb{N}^d$ are strictly increasing sequences.

(iii) $p_i\ge\rho_{(\cup_{k=1}^i supp\ \phi_k)}$ for every $i\le d$.

Let $\mathbb{Q}_s(\mathbb{C})$ be the set of finite sequences $(\phi_1,w_1,p_1,\phi_2,w_2,p_2,\dots,\phi_d,w_d,p_d)$ satisfying properties (1), (2) above and for every $i\le d$, $\phi_i\in c_{00}(\omega_1,\mathbb{C})$. Then $\mathbb{Q}_s(\mathbb{C})$ is a countable set while $\mathbb{Q}_s(\omega_1,\mathbb{C})$ has cardinality ω_1. Fix a one to one function $\sigma:\mathbb{Q}_s(\mathbb{C})\to\{2j: j\ is\ odd\}$ such that

$$\sigma(\phi_1,w_1,p_1,\dots,\phi_d,w_d,p_d)>\max\left\{p_d^2,\frac{1}{\epsilon^2},\max supp\ \phi_d\right\}$$

where $\epsilon=\min\{|\phi_k(e_\alpha)|:\alpha\in supp\ \phi_k, k=1,\dots,d\}$. Given a finite subset F of ω_1, we denote by $\pi_F:\{1,2,\dots,\#F\}\to F$ the natural order preserving map, i.e., π_F is the increasing numeration of F.

Given $\Phi=(\phi_1,w_1,p_1,\dots,\phi_d,w_d,p_d)\in\mathbb{Q}_s(\mathbb{C})$, we set

$$G_\Phi=\overline{\bigcup_{\iota=1}^{d} supp\ \phi_\iota}^{\,p_d}\ .$$

Consider the family

$\pi_{G_\Phi}(\Phi)=(\pi_G(\phi_1),w_1,p_1,\pi_G(\phi_2),w_2,p_2,\dots,\pi_G(\phi_d),w_d,p_d)$ where

$$\pi_G(\phi_k)(n)=\begin{cases}\phi_k\left(\pi_{G_\Phi}(n)\right) & if\ n\in G_\Phi,\\ 0, & othersise.\end{cases}$$

Finally $\sigma_p:\mathbb{Q}_s(\omega_1,\mathbb{C})\to\{2j: j\ odd\}$ is defined by $\sigma_p(\Phi)=\sigma(\pi_G(\Phi))$.

Definition (4.2.4)[4]: A sequence $\Phi=(\phi_1,\phi_2,\dots,\phi_{n_{2j+1}})$ of functionals of $\mathcal{K}_{\omega_1}(\mathbb{C})$ is called a $2j+1$ special sequence if

(SS.1) $supp\phi_1<supp\phi_2<\dots<supp\phi_{n_{2j+1}}$. For each $k\le n_{2j+1},\phi_k$ is of type I, $w(\phi_k)=m_{2jk}$ with j_1 even and $m_{2j_1}>n_{2j+1}^2$.

(SS.2) There exists a strictly increasing sequence $\left(p_1^\Phi,p_2^\Phi,\dots,p_{n_{2j+1}-1}^\Phi\right)$ of natural numbers such that for all $1\le i\le n_{2j+1}-1$ we have that $w(\phi_{i+1})\ =m_{\sigma_\varrho}(\Phi_i)$ where

$$\Phi_i=\left(\phi_1,w(\phi_1),p_1^\Phi,\phi_2,w(\phi_2),p_2^\Phi,\dots,\phi_i,w(\phi_i),p_i^\Phi\right)$$

Special sequences in separable examples with one to one codings are in general simpler: they are of the form $(\phi_1, w(\phi_1), \dots, \phi_k, w(\phi_k))$. Their main feature is that if $(\phi_1, w(\phi_1), \dots, \phi_k, w(\phi_k))$ and $(\psi_1, w(\psi_1), \dots, \psi_l, w(\psi_l))$ are two of them, there exists $i_0 \le \min\{k, l\}$ with the property that

$$(\phi_i, w(\phi_i)) = (\psi_i, w(\psi_i)) \text{ for all } i \le i_0 \qquad (1)$$

$$\{w(\phi_i): i_0 \le i \le k\} \cap = \{w(\psi_i): i_0 \le i \le l\} \ = \emptyset \qquad (2)$$

In non-separable spaces, one to one codings are obviously impossible, and (1), (2) are no longer true. Fortunately, there is a similar feature to (1), (2) called the tree-like interference of a pair of special sequences: Let $\Phi = (\phi_1, \dots, \phi_{2j+1})$ and $\psi = (\psi_1, \dots, \psi_{2j+1})$ be two $2j+1$-special sequences, then there exist two numbers $0 \le \kappa_{\Phi,\psi} \le \lambda_{\Phi,\psi} \le n_{2j+1}$ such that the following conditions hold:

(TP.1) For all $i \le \lambda_{\Phi,\psi}, w(\phi_i) = w(\psi_i)$ and $p_i^{\Phi} = p_i^{\psi}$.

(TP.2) For all $i < \kappa_{\Phi,\psi}, \phi_i = \psi_i$.

(TP.3) For all $\kappa_{\Phi,\psi} < i < \lambda_{\Phi,\psi}$

$$\text{supp } \phi_i \cap \overline{\text{supp } \psi_1 \cup \dots \cup \text{supp } \psi_{\lambda_{\Phi,\psi}-1}}^{\,p_{\lambda_{\Phi,\psi}-1}} = \emptyset$$

And

$$\text{supp } \psi_i \cap \overline{\text{supp } \phi_\iota \cup \dots \cup \text{supp } \phi_{\lambda_{\Phi,\psi}-1}}^{\,p_{\lambda_{\Phi,\psi}-1}} = \emptyset$$

(TP.4) $\{w(\phi_i) : \lambda_{\Phi,\psi} < i \le n_{2j+1}\} \cap \{w(\psi_i) : i \le n_{2j+1}\} = \emptyset$ and $\{w(\psi_i) : \lambda_{\Phi,\psi} < i \le n_{2j+1}\} \cap \{w(\phi_i) : i \le n_{2j+1}\} = \emptyset$.

A.2. Rapidly increasing sequences (R.I.S.)

For the proof of Step I we shall construct a family of block sequences on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ commonly called *rapidly increasing sequences* (*R.I.S.*). These sequences are very useful because one has good estimates of upper bounds on $|f(x)|$ for $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and x averages of *R.I.S.*

For the construction of the family $\mathfrak{F}$ the only difference from the general theory is that our interest now is to study bounded $\mathbb{R}$-linear operators on the complex space $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Hence, all the construction of *R.I.S.* in a particular block sequence $(x_n)_{n\in\mathbb{N}}$ must be on its *real* linear span. We point out here that there are no problems with this, because all the combinations of the vectors $(x_n)_{n\in\mathbb{N}}$ to obtain *R.I.S.* use rational scalars.

Definition (4.2.5) [4]: *(R.I.S.).* We say that a block sequence $(x_k)_k$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is a (C,ϵ)-*R.I.S.*, $C,\epsilon > 0$, when there exists a strictly increasing sequence of natural numbers $(j_k)_k$ such that:

(i) $\|x_k\| \leq C$;

(ii) $|supp\ x_k| \leq m_{j_{k+1}}\epsilon$;

(iii) For all the functionals ϕ of $\mathcal{K}_{\omega_1}(\mathbb{C})$ of type I, with $\omega(\phi) < m_{j_k}$, $|\phi(x_k)| \leq \frac{C}{\omega(\phi)}$.

The following remark is an immediate consequence of this definition.

Remark (4.2.6) [4]: Let $\epsilon' < \epsilon$. Every (C,ϵ) -*R.I.S.* has a subsequence which is a (C,ϵ')-*R.I.S.* And for every strictly increasing sequence of ordinals $(\alpha_n)_n$ and every$\epsilon > 0$, $\left(e_{\alpha_n}\right)_n$ is a $(1,\epsilon)$-*R.I.S.*

Remark (4.2.7) [4]: Let $(x_n)_n$ and $(y_n)_n$ be two (C,ϵ)-*R.I.S.* such that $\sup_n \max supp\ x_n = \sup_n \max supp\ y_n$. Then there exists a (C,ϵ)-*R.I.S.* $(z_n)_n$ such that $z_{2n-1} \in \{x_k\}_{k\in\mathbb{N}}$ and $z_{2n} \in \{y_k\}_{k\in\mathbb{N}}$.

Proof. Suppose that $(t_k)_k$ and $(s_k)_k$ are increasing sequences of positive integers satisfying the definition of *R.I.S.* for $(x_k)_k$ and $(y_k)_k$ respectively. We construct $(z_k)_k$ as follows. Let $z_1 = x_1$ and $j_1 = t_1$. Pick s_{k_1} such that $x_1 < y_{s_{k_1}}$and $t_2 < s_{k_1}$. Then we define $j_2 = s_{k_1}$ and $z_2 < y_{s_{k_1}}$. Notice that

(i) $\|z_1\| \leq C$;

(ii) $|supp\ z_1| \leq m_{t_2}\epsilon < m_{s_{k_1}}\epsilon = m_{j_2}\epsilon$;

(iii) For all the functionals ϕ of $\mathcal{K}_{\omega_1}(\mathbb{C})$ of type I, with $\omega(\phi) < m_{j_1}$, $|\phi(z_1)| \leq \frac{C}{\omega(\phi)}$.

Continuing with this process we obtain the desired sequence.

Theorem (4.2.8) [4]: *Let* $(x_k)_k$ *be a normalized block sequence of* $\mathfrak{X}_{\omega_1}$ *and* $\epsilon > 0$. *Then there exists a normalized block subsequence* $(y_n)_n$ *in* $span_{\mathbb{R}}\ \{x_k\}$ *which is a* $(3,\epsilon)$-*R.I.S.*

For the proof of Theorem (4.2.8) [4] we first construct a simpler type of sequence.

Definition (4.2.9) [4]: Let X be a Banach space, $C \geq 1$ and $k \in \mathbb{N}$. A normalized vector y is called a $C - \ell_1^k$-average of X, when there exists a block sequence $(x_1,\ldots,x_k)$ such that

(i) $y = (x_1 + \ldots + x_k)/k$;

(ii) $\|x_i\| \leq C$, for all $i = 1,\ldots,k$.

In the next result we want to emphasize that this special type of sequences is really constructed on the real structure of the space $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

Theorem (4.2.10) [4]: *For every normalized block sequence (x_n) of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every integer k, there exist $z_1 <\ldots< z_k$ in $span_{\mathbb{R}}(x_n)$, such that $z_1+\ldots+z_k/k$ is a $2-\ell_1^k$-average.*

Proof. The proof is standard. Suppose that the result is false. Let j and n be natural numbers with

$$2^n > m_{2j}$$
$$n_{2j} > k^n.$$

Let $N = k^n$ and $x = \sum_{i=1}^{N} x_i$. For each $1 \le i \le n$ and every $1 \le j \le k^{n-i}$, we define

$$x(i,j) = \sum_{t=(j-1)k^i+1}^{jk^i} x_t.$$

Hence, $x(0,j) = x_j$ and $x(n,1) = x$.

It is proved by induction on i that $\|x(i,j)\| \ge 2^{-i}k^i$, for all i,j. In particular, $\|x\| = \|x(n,1)\| \le 2^{-n}k^n = 2^{-n}N$. Then by property (i) of the definition in the norming set

$$\|x\| \ge \frac{1}{m_{2j}}\sum_{t=1}^{n_{2j}}\|x_t\| = \frac{n_{2j}}{m_{2j}} > \frac{N}{m_{2j}}.$$

Hence,

$$2^{-n}N > \frac{N}{m_{2j}}$$
$$m_{2j} > 2^n,$$

which is a contradiction.

Finally, for the construction of *R.I.S.* we observe these simple facts:

(a) If y is a $C-\ell_1^{n_j}$-average of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $\phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ has weight $\omega(\phi) < m_j$, then $|\phi(y)| \le \frac{3C}{2\omega(\phi)}$;

(b) Let $(x_k)_k$ be a block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that there exists a strictly increasing sequence of positive integers $(j_k)_k$ and $\epsilon > 0$ satisfying:

(a) Each x_k is a $2-\ell_1^{n_{j_k}}$-average;

(b) $|supp\ x_k| \le \epsilon m_{j_{k+1}}$.

Then $(x_k)_k$ is a $(3,\epsilon)$-*R.I.S.*

A.3. Basic inequality

To prove Steps II and III we need a crucial result called *the basic inequality* which is very important to find good estimations for the norm of certain combinations of *R.I.S.* in $\mathfrak{X}_{\omega_1}(\mathbb{C})$. First we need to introduce the *mixed Tsirelson spaces*.

The mixed Tsirelson space $T\left[\left(m_j^{-1}, n_j\right)_j\right]$ is defined by considering the completion of $c_{00}(\omega_1, \mathbb{C})$ under the norm $\|\cdot\|_0$ given by the following implicit formula

$$\|x\|_0 = \max\left\{\|x\|_\infty, \sup_j \sup \frac{1}{m_j} \sum_{i=1}^{n_j} \|E_i x\|_0\right\}.$$

The supremum inside the formula is taken over all the sequences $E_1 <\ldots< E_{n_j}$ of subsets of ω. Notice that in this space the canonical Hamel basis $(e_n)_{n<\omega}$ of $c_{00}(\omega_1, \mathbb{C})$ is 1-subsymmetric and 1-unconditional basis.

We can give an alternative definition for the norm of $T\left[\left(m_j^{-1}, n_j\right)_j\right]$ by defining the following norming set. Let $W\left[\left(m_j^{-1}, n_j\right)_j\right] \subseteq c_{00}(\omega_1, \mathbb{C})$ be the minimal set of $c_{00}(\omega_1, \mathbb{C})$ satisfying the following properties:

(i) For every $\alpha < \omega, e_\alpha^* \in W\left[\left(m_j^{-1}, n_j\right)\right]$. If $\phi \in W\left[\left(m_j^{-1}, n_j\right)\right]$ and $\theta = \lambda + i\mu$ is a complex number with λ and μ rationals and $|\theta| \leq 1, \theta\phi \in W\left[\left(m_j^{-1}, n_j\right)\right]$;

(ii) For every $\phi \in W\left[\left(m_j^{-1}, n_j\right)\right]$ and $E \subseteq \omega$, $E\phi \in W\left[\left(m_j^{-1}, n_j\right)\right]$;

(iii) For every $j \in \mathbb{N}$ and $\phi_1 <\ldots< \phi_{n_j}$ in $W\left[\left(m_j^{-1}, n_j\right)\right]$, $(1/m_j) \sum_{i=1}^{n_j} \phi_i \in W\left[\left(m_j^{-1}, n_j\right)\right]$;

(iv) $W\left[\left(m_j^{-1}, n_j\right)\right]$ is closed under convex rational combinations.

Theorem (4.2.11) [4]: *(Basic inequality for R.I.S.). Let* $(x_n)_n$ *be a* (C, ϵ)*-R.I.S. of* $\mathfrak{X}_{\omega_1}(\mathbb{C})$ *and* $(b_k)_k \in c_{00}(\mathbb{C}, \mathbb{N})$*. Suppose that for some* $j_0 \in \mathbb{N}$ *we have that for every* $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ *with weight* $w(f) = m_{j_0}$ *and for every interval E of* ω_1,

$$\left|f\left(\sum_{k\in E} b_x x_k\right)\right| \leq C\left(\max_{k\in E}|b_k| + \epsilon \sum_{k\in E}|b_k|\right).$$

Then for every $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ *of type I, there exist* $g_1, g_2 \in c_{00}(\mathbb{C}, \mathbb{N})$ *such that*

$$\left|f\left(\sum_{k\in E} b_x x_k\right)\right| \leq C(g_1+g_2)\left(\sum_{k\in E} |b_k|\, e_k\right),$$

Where $g_1 = h_1$ *or* $g_1 = e_t^* + h_1, t \notin supp\, h_1$, *and* $h_1 \in W\left[\left(m_j^{-1}, 4n_j\right)\right]$ *such that* $h_1 \in \mathrm{conv}_{\mathbb{Q}}\left\{h \in W\left[\left(m_j^{-1}, 4n_j\right)\right]: w(f) = w\{f\}\right\}$ *and* m_j *does not appear as a weight of a node in the tree analysis of* h_1, *and* $\|g_2\|_\infty \leq \epsilon$.

The following results are consequences of the basic inequality.

Proposition (4.2.12) [4]: *Let* $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ *or* $f \in W\left[\left(m_j^{-1}, 4n_j\right)\right]$ *be of type I. Consider* $j \in \mathbb{N}$ *and* $l \in \left[\frac{n_j}{m_j}, n_j\right]$. *Then for every set* $F \in c_{00}(\omega_1, \mathbb{C})$ *of cardinality l,*

$$\left|f\left(\frac{1}{l}\sum_{\alpha\in F} e_\alpha\right)\right| \leq \begin{cases} \dfrac{2}{w(f)m_j}, & if\ w(f) < m_j, \\ \dfrac{1}{w(f)} & if\ w(f) \geq m_j. \end{cases}$$

If the tree analysis of f *does not contain nodes of weight* m_j, *then*

$$\left|f\left(\frac{1}{l}\sum_{\alpha\in F} e_\alpha\right)\right| \leq \frac{2}{m_j^3}$$

Proposition (4.2.13) [4]: *Let* $(x_k)_k$ *be a* (C,ϵ)*-R.I.S. of* $\mathfrak{X}_{\omega_1}(\mathbb{C})$ *with* $\epsilon \leq \frac{1}{n_j}, l \in \left[\frac{n_j}{m_j}, n_j\right]$ *and let* $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ *be of type I. Then,*

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \leq \begin{cases} \dfrac{3C}{w(f)m_j}, & if\ w(f) < m_j, \\ \dfrac{C}{w(f)} + \dfrac{2C}{n_j}, & if\ w(f) \geq m_j. \end{cases}$$

Consequently, if $(x_k)_{k=1}^{l}$ *is a normalized* (C,ϵ)*-R.I.S. with* $\epsilon \leq \frac{1}{n_{2j}}, l \in \left[\frac{n_{2j}}{m2_j}, n_{2j}\right]$, *then*

$$\frac{1}{m_{2j}} \leq \left\|\frac{1}{l}\sum_{k=1}^{l} x_k\right\| \leq \frac{2C}{m_{2j}}.$$

Proof. Let $(x_k)_k$ *be a* (C,ϵ)*-R.I.S.* and take $b = \left(\frac{1}{l}, \dots, \frac{1}{l}, 0, 0, \dots\right) \in c_{00}(\mathbb{C}, \mathbb{N})$. It follows from the basic inequality that for every $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$

of type I, there exist $h_1 \in W[(m_j^{-1}, 4n_j)]$ with $\omega(h_1) = \omega(f), t \in \mathbb{N}$ and $g_2 \in c_{00}(\mathbb{N})$ with $\|g\|_\infty \leq \epsilon$ such that

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \leq C(e_t^* + h_1 + g_2)\left(\frac{1}{l}\sum_{k=1}^{l} e_k\right).$$

Moreover,

$$\left|g_2\left(\frac{1}{l}\sum_{k=1}^{l} e_k\right)\right| \leq \|g_2\|_\infty \left\|\frac{1}{l}\sum_{k=1}^{l} e_k\right\|_1 \leq \epsilon \leq \frac{1}{n_j}.$$

Now by the estimates on the auxiliary space $T\left[(m_j^{-1}, 4n_j)_j\right]$ of Proposition (4.2.12), we have

(a) If $\omega(f) < m_j$,

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \leq C\left(\frac{1}{l} + \frac{2}{\omega(f)m_j} + \frac{1}{n_j}\right) \leq C\left(\frac{m_j}{n_j} + \frac{2}{\omega(f)m_j} + \frac{1}{n_j}\right) \leq \frac{3C}{\omega(f)m_j}$$

(b) If $\omega(f) \geq m_j$,

$$\left|f\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right)\right| \leq C\left(\frac{1}{l} + \frac{C}{\omega(f)} + \frac{1}{n_j}\right) \leq \frac{C}{\omega(f)} + \frac{2C}{n_j}$$

And notice

(c) $\frac{3C}{\omega(f)m_{2j}} \leq \frac{2C}{m_{2j}}$, if $\omega(f) < m_{2j}$,

(d) $\frac{C}{\omega(f)} + \frac{2C}{n_{2j}} \leq \frac{C}{m_{2j}} + \frac{C}{m_{2j}} = \frac{2C}{m_{2j}}$, if $\omega(f) \geq m_{2j}$.

We conclude from the fact that $\mathcal{K}_{\omega_1}(\mathbb{C})$ is the norming set:

$$\left\|\frac{1}{l}\sum_{k=1}^{l} x_k\right\| \leq 2C/m_{2j}.$$

For the proof of the second part of the theorem, let $(x_k)_{k=1}^{l}$ be a normalized (C,ϵ)-*R.I.S.* with $\epsilon \leq \frac{1}{n_{2j}}, l \in \left[\frac{n_{2j}}{m2_j}, n_{2j}\right]$. For every $k \leq l$, we consider $x_k^* \in \mathcal{K}_{\omega_1}(\mathbb{C})$, such that $x_k^*(x_k) = 1$ and $ran\ x_k^* \subseteq ran\ x_k$, then $x^* = \frac{1}{m2_j}\sum_{k=1}^{l} x_k^* \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and $x^*\left(\frac{1}{l}\sum_{k=1}^{l} x_k\right) = \frac{1}{m2_j}$. Hence, $\frac{1}{m2_j} \leq \left\|\frac{1}{l}\sum_{k=1}^{l} x_k\right\|$.

A.4. Proof of Step II

Now we introduce another type of sequences in order to construct the conditional frame in $\mathfrak{X}_{\omega_1}(\mathbb{C})$. In fact, this space has no unconditional basic sequence.

Definition (4.2.14) [4]: A pair (x, ϕ) with $x \in \mathfrak{X}_{\omega_1}(\mathbb{C})$ and $\phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ is called a (C, j)-exact pair when:

(a) $\|x\| \le C, \omega(\phi) = m_j$ and $\phi(x) = 1$.

(b) For each $\psi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ of type I and $\omega(x) = m_{i,} i \ne j$, we have

$$|\psi(x)| \le \begin{cases} \dfrac{2C}{m_i}, & if\ i < j, \\ \dfrac{C}{m_j^2}, & if\ i > j. \end{cases}$$

Proposition (4.2.15) [4]: *Let $(x_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then for every $j \in \mathbb{N}$, there exists (x, ϕ) such that $x \in span_{\mathbb{R}}(x_n), \phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and (x, ϕ) is a$(6, 2j)$-exact pair.*

Proof. Fix a normalized block sequence $(x_n)_n$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and a positive integer j. By Proposition (4.2.8) [4] there exists $(y_n)_n$, a normalized $(3, 1/n_{2j})$-*R.I.S.*, in $span_{\mathbb{R}}(x_n)$. For every $1 \le i \le n_{2j}$ and $\epsilon > 0$, we take $\phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $\phi_i(y_i) > 1 - \epsilon$, and $\phi_i < \phi_{i+1}$.

Let $x = (m_{2j}/n_{2j}) \sum_{i=1}^{n_{2j}} y_i$ and $\phi = (1/m_{2j}) \sum_{i=1}^{n_{2j}} \phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C})$. By perturbating x by a rational coefficient on the support of some y_i we may assume that then $\phi(x) = 1$ and using Proposition (4.2.13) [4] we conclude that (x, ϕ) is a $(6, 2j)$-exact pair.

Definition (4.2.16) [4]: Let $j \in \mathbb{N}$. A sequence $\left(x_1, \phi_1, \ldots, x_{n_{2j+1}}, \phi_{n_{2j+1}}\right)$ is called a $(1, j)$-dependent sequence when:

(DS.1) $supp\ x_1 \cup supp\ \phi_1 < \ldots < supp\ x_{2_{nj+1}} \cup supp\ \phi_{2_{nj+1}}$.

(DS.2) The sequence $\Phi = (\phi_1, \ldots, \phi_{n_{2j+1}})$ is a $2j + 1$-special sequence.

(DS.3) (x_i, ϕ_i) is a $(6, 2j_i)$-exact pair. $\#\ supp\ x_i \le m_{2j+1}/n_{2j+1}^2$ for every $1 \le i \ \le n_{2j} + 1$.

(DS.4) For every $(2j + 1)$-special sequence $\Psi = \left(\psi_1, \ldots, \psi_{n_{2j}+1}\right)$ we have that

$$\bigcup_{\kappa_{\Phi,\Psi} < i < \lambda_{\Phi,\Psi}} supp\ x_i \cap \bigcup_{\kappa_{\Phi,\Psi} < i < \lambda_{\Phi,\Psi}} supp\ \psi_i = \emptyset,$$

where $\kappa_{\Phi,\Psi}, \lambda_{\Phi,\Psi}$ are numbers introduced in Definition (4.2.4).

Proposition (4.2.17) [4]: *For every normalized block sequence $(y_n)_n$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every natural number j there exists a $(1,j)$-dependent sequence $\left(x_1,\phi_1,\ldots,x_{n_{2j+1}},\phi_{n_{2j+1}}\right)$ such that x_i is in the $\mathbb{R}$-span of $(y_n)_n$ for every $i=1,\ldots,n_{2j+1}$.*

Proof. Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $j \in \mathbb{N}$. We construct the sequence $\left(x_1,\phi_1,\ldots,x_{n_{2j+1}},\phi_{n_{2j+1}}\right)$ inductively. First using Proposition (4.2.15) we choose a $(6,2j_1)$-exact pair (x_1,ϕ_1) such that j_1 is even, $m_{2j_1} > n_{2j+1}^2$ and $x_1 \in span_{\mathbb{R}}(y_n)_n$. Assume that we have constructed $(x_1,\phi_1,\ldots,x_{l-1},\phi_{l-1})$ such that there exists $(p_1,\ldots,p_{l-1})$ satisfying

(i) $supp\, x_1 \cup supp\, \phi_1 <\ldots< supp\, x_{l-1} \cup supp\, \phi_{l-1}$, where $x_i \in span_{\mathbb{R}}(y_n)_n$ and (x_i,ϕ_i) is a $(6,2j_1)$-exact pair.

(ii) For $1< i \le l-1, w(\phi_i) = \sigma_{\varrho}(\phi_1,w(\phi_1),p_1,\ldots,\phi_{i-1},w(\phi_{i-1}),p_{i-1})$.

(iii) For $1\le i < l-1, p_i \ge \max\{p_{i-1},p_{F_i}\}$, where $F_i = \bigcup_{k=1}^{i} supp\phi_k \cup supp\, x_k$.

To complete the inductive construction choose

$$p_{l-1} \ge \max\left\{p_{l-2},p_{F_{i-1}},n_{2j+1}^2\#supp\, x_{l-1}\right\}$$

and $2j_l = \sigma_{\varrho}\left(\phi_1,w(\phi_1),p_1,\ldots,\phi_{l-1},w(\phi_{l-1}),p_{l-1}\right)$. Hence take a $(6,2j_l)$-exact pair (x_l,ϕ_l) such that $x_1 \in span_{\mathbb{R}}(y_n)_n$ and $suppx_{l-1} \cup suppx_l supp\, \phi_l$. Notice that properties (DS.1), (DS.2) and (DS.3) are clear by definition of the sequence and (DS.4) follows from (iii) and (TP.3).

Modifying a little the previous argument we obtain the following:

Proposition (4.2.18) [4]: *For every two normalized block sequences $(y_n)_n$ and $(z_n)_n$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every $j \in \mathbb{N}$ there exists a $(1,j)$-dependent sequence $\left(x_1,\phi_1,\ldots,x_{n_{2j+1}},\phi_{n_{2j+1}}\right)$ such that $x_{2l-1} \in span_{\mathbb{R}}(y_n)$ and $x_{2l} \in span_{\mathbb{R}}(z_n)$ for every $l=1,\ldots,n_{2j+1}$.*

Another consequence of the basic inequality is the following proposition.

Proposition (4.2.19) [4]: *Let* $\left(x_1,\phi_1,\ldots,x_{n_{2j+1}},\phi_{n_{2j+1}}\right)$ *be a* $(1,j)$-*dependent sequence. Then:*

(i) $\left\|\frac{1}{n_{2j+1}}\sum_{i=1}^{n_{2j+1}} x_i\right\| \geq \frac{1}{m_{2j+1}}$;

(ii) $\left\|\frac{1}{n_{2j+1}}\sum_{i=1}^{n_{2j+1}}(-1)^{i+1}x_i\right\| \leq \frac{1}{m_{2j+1}^3}$.

Proof. The first inequality is clear since the functional $\phi = 1/m_{2j+1}\sum_{i=1}^{n_{2j+1}}\phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and $\phi\left(\sum_{i=1}^{n_{2j+1}} x_i\right) = n_{2j+1}/m_{2j+1}$. The second is obtained by the basic inequality.

Now we can give a proof of Step II.

Proposition (4.2.20) [4]: *Let* $(y_n)_n$ *be a normalized block sequence of* $\mathfrak{X}_{\omega_1}(\mathbb{C})$. *Then the closure of the* real *span of* $(y_n)_n$ *is H.I.*

Proof. Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Fix $\epsilon > 0$ and two block subsequences $(z_n)_n$ and $(w_n)_n$ in $span_{\mathbb{R}}(y_n)_n$. Take an integer j such that $m_{2j+1}\epsilon > 1$. By Proposition (4.2.18) there exists a $(1,j)$-dependent sequence $\left(x_1,\phi_1,\ldots,x_{n_{2j+1}},\phi_{n_{2j+1}}\right)$ such that $x_{2i-1} \in span_{\mathbb{R}}(z_n)$ and $x_{2i} \in span_{\mathbb{R}}(w_n)$.

We define $z = (1/n_{2j+1})\sum_{i=1(odd)}^{n_{2j+1}} x_i$ and $w = 1/n_{2j+1}\sum_{i=1(even)}^{n_{2j+1}} x_i$. Notice that $z \in span_{\mathbb{R}}(z_n)$ and $w \in span_{\mathbb{R}}(w_n)$. Then by Proposition (4.2.19) we get $\|z+w\| \geq 1/m_{2j+1}$ and $\|z-w\| \geq 1/m_{2j+1}^2$. Hence $\|z-w\| \leq \epsilon\|z+w\|$.

A.5. Proof of Step III

Definition (4.2.21) [4]: A sequence $\left(z_1,\phi_1,\ldots,z_{n_{2j+1}},\phi_{n_{2j+1}}\right)$ is called a $(0,j)$-dependent sequence when it satisfies the following conditions:

- (0DS.1) The sequence $\Phi = \left(\phi_1,\ldots,\phi_{n_{2j+1}}\right)$ is a $2j+1$-special sequence and $\phi_i(z_k) = 0$ for every $1 \leq i,k \leq n_{2j+1}$.
- (0DS.2) There exists $\left\{\psi_1,\ldots,\psi_{n_{2j+1}}\right\} \subseteq \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $w(\psi_1) = w(\phi_i), \#\, supp\, z_i \leq w(\phi_i+1)/n_{2j+1}^2$ and $\left(z_i,\psi_i\right)$ is a $(6,2j_1)$-exact pair for every $1 \leq i \leq n_{2j+1}$.
- (0DS.3) If $H = \left(h_1,\ldots,h_{n_{2j+1}}\right)$ is an arbitrary $2j+1$-special sequence, then

$$\left(\bigcup_{\kappa_{\Phi,H}<i<\lambda_{\Phi,H}} supp\ z_i\right)\cap\left(\bigcup_{\kappa_{\Phi,H}<i<\lambda_{\Phi,H}} supp\ h_i\right)=\emptyset.$$

Proposition (4.2.22) [4]: *For every* $(0,j)$*-dependent sequence* $\left(x_1,\phi_1,\ldots,x_{n_{2j+1}},\phi_{n_{2j+1}}\right)$ *we have that*

$$\left\|\frac{1}{n_{2j+1}}\sum_{k=1}^{n_{2j+1}} x_k\right\|\leq\frac{1}{m_{2j+1}^3}$$

Proposition (4.2.23) [4]: *Let* $(y_n)_n$ *be a* (C,ϵ)*-R.I.S.,* $Y=\overline{span}_{\mathbb{C}}(y_n)$*, and*
$T:Y\to\mathfrak{X}_{\omega_1}(\mathbb{C})$ *an* $\mathbb{R}$*-linear bounded operator.*
Then $\lim_{n\to\infty} d(Ty_n,\mathbb{C}y_n)=0$.

Proof. Suppose that $\lim_{n\to\infty} d(Ty_n,\mathbb{C}y_n)\neq 0$. Then there exists an infinite subset $B\subseteq\mathbb{N}$ such that $\inf_{n\in B} d(Ty_n,\mathbb{C}y_n)>0$. We shall show that for every $\epsilon>0$ there exists $y\in Y$ such that $\|y\|<\epsilon\|Ty\|$, and this is a contradiction.

Claim 1. *There exists a limit ordinal* $\gamma_0, A\subseteq\mathbb{N}$ *infinite and* $\delta>0$ *such that*

$$\inf_{n\in A} d(P\gamma_0 Ty_n,\mathbb{C}y_n)>\delta$$

To prove this claim we observe that

$$\gamma_0=\min\left\{\gamma<\omega_1:\exists A\in[\mathbb{N}]^\infty \inf_{n\in A} d\left(P_\gamma Ty_n,\mathbb{C}y_n\right)>0\right\}$$

is a limit ordinal. In fact, by the assumption the set on the right hand side is not empty. And if γ_0 is not limit, then we have $\gamma_0=\beta+1$. The sequence $(y_n)_n$ is weakly null (because $(e_\alpha)_\alpha$ is shrinking) and then

$$\lim_{n\to\infty} e_{\beta+1}^* Ty_n=0$$

And for large n and every $\lambda\in\mathbb{C}$

$$\|P_\beta Ty_n-\lambda y_n\|\geq\|P_{\beta+1}Ty_n-\lambda y_n\|-\|e_{\beta+1}^*Ty_n\|\geq\delta-|e_{\beta+1}^*Ty_n|\geq\delta/2,$$

which is a contradiction.

Claim 2. *Fix* γ_0 *and* $A\subseteq\mathbb{N}$ *as in Claim 1. Then there exist a sequence* $n_2<n_3<\ldots$ *in A, a sequence of functionals* $f_2,,f_3,\ldots$ *in* $\mathcal{K}_{\omega_1}(\mathbb{C})$ *and a sequence of ordinals* $\gamma_1<\gamma_2<\ldots<\gamma_0$ *such that*

(i) $d(P_{[\gamma_k,\gamma_{k+1}]}Ty_{n_k},\mathbb{C}y_{n_k})\geq\delta/2;$

(ii) $f_kTy_{n_k}\geq\delta/2;$

(iii) $f_k(y_{n_k}) = 0$;

(iv) $ranf_k \subseteq ranTy_{n_k}$;

(v) $suppf_k \cap supp\ y_{n_m} = \emptyset\ when\ m \neq k$.

To prove this claim, let $\xi = \sup \max y_n$. We analyze the three possibilities for ξ:

Case (a): $\xi < \gamma_0$. Let $n_1 = \min A$ and choose $\xi < \gamma_1 < \gamma_0$ such that

$$\|P\gamma_0 Ty_{n_1} - P_{\gamma_1}Ty_{n_1}\| < \delta/2,$$

hence, $d(P\gamma_1 Ty_{n_1}, \mathbb{C}y_{n_1}) > \delta/2$. By minimality of γ_0 we have

$$\inf_{n\in A} d(P\gamma_1 Ty_n, \mathbb{C}y_n) = 0,$$

then we can choose $n_2 > n_1$ in A such that $d(P\gamma_1 Ty_{n_2}, \mathbb{C}y_{n_2}) < \delta/2$ and this implies that

$$d\left((P_{\gamma_0} - P_{\gamma_1})Ty_{n_2}\mathbb{C}y_{n_2}\right) > \delta/2$$

Approximating the vector $(P_{\gamma_0} - P_{\gamma_1})Ty_{n_2}$ choose $\gamma_0 > \gamma_2 > \gamma_1$ such that $\|(P_{\gamma_0} - P_{\gamma_1}) \times Ty_{n_2}\|$ is small in order to guarantee that

$$d(P_{[\gamma_1,\gamma_2]}Ty_{n_2}, \mathbb{C}y_{n_2}) \ \geq \delta/2.$$

Using the complex Hahn–Banach theorem, there exists $g_2 \in B_{\mathfrak{X}^*_{\omega_1}(\mathbb{C})}$ such that

(a) $g_2(P_{[\gamma_1,\gamma_2]}Ty_{n_2}) \ > \delta/2$;

(b) $g_2(y_{n_2}) = 0$,

and by Proposition (4.1.3) [4] we can choose $h_2 \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $h_2(P_{[\gamma_1,\gamma_2]}Ty_{n_2}) \ > \delta/2$ and $h_2(y_{n_2})$ is arbitrarily small. Replacing h_2 by $\alpha h_2 + \beta k_2$ where $|\alpha| + |\beta| = 1, k_2(y_{n_2})$ is close enough to 1, and $k_2 \in \mathcal{K}_{\omega_1}(\mathbb{C})$ we may assume that $h_2(y_{n_2}) = 0$.

Let $f_2 = h_2 P_{[\gamma_1,\gamma_2]\cap ran\ Ty_{n_2}} \in \mathcal{K}_{\omega_1}(\mathbb{C})$. Again by minimality of γ_0, there exists $n_3 > n_2$ in A such that $d(P_{\gamma_2}Ty_{n_3}, \mathbb{C}y_{n_3}) < \delta/2$ and we can choose $\gamma_0 > \gamma_3 > \gamma_2$ satisfying

$$d(P_{[\gamma_2,\gamma_3]}Ty_{n_3}, \mathbb{C}y_{n_3}) \ > \delta/2.$$

Again by Hahn–Banach theorem and by Proposition (4.1.3) [4] there exists a functional $h_3 \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that

(i) $h_3(P_{[\gamma_2,\gamma_3]}Ty_{n_3}) \ > \delta/2$;

(ii) $h_3(y_{n_3}) = 0$,

then we define $f_3 = h_3 P_{[\gamma_2,\gamma_2]\cap ran\ Ty_{n_3}} \in \mathcal{K}_{\omega_1}(\mathbb{C})$. The previous argument gives us the way to construct the sequences of Claim 2. Properties (i)–(v) are easy to check, in particular property (v) is true because $\min supp\ f_k > \xi > \max supp\ y_{n_l}$ for every positive integers k, l.

Case (b): $\xi > \gamma_0$. In this case we start by picking $n_1 \in A$ such that $\min supp\ y_{n_1} > \gamma_0$. Then we repeat exactly the same argument as in Case (a).

Case (c): $\xi = \gamma_0$. We basically repeat the same argument of Case (a) with the additional care of maintaining property (v) true. That is, each time we choose the ordinal γ_{k+1} (with $\gamma_0 > \gamma_{k+1} > \gamma_k$) we take it such that $\gamma_{k+1} > \max supp\ y_{n_{k+1}}$.

Claim 3. *There exists a* $(0, j)$*-dependent sequence* $\left(z_1, \phi_1, \ldots, z_{n_{2j+1}}\right)$ *such that*

(i) $z_i \in X$ *for every* $1 \leq i \leq n_{2j+1}$;

(ii) $ran\ \phi_k \subseteq ranTy_k$ *and* $\phi_k(Tz_k) > \delta/2$.

Let j with $m_{2j+1} > 24/\epsilon\delta$. Choose j_1 even such that $m_{2j_1} > n_{2j+1}^2$ (see definition of special sequence) and $F_1 \subseteq A$ with $\#F_1 = n_{2j_1}$ such that $(y_{n_k})_{k \in F_1}$ is a $(3, 1/n_{2j_1}^2)$ *-R.I.S.* Then define

$$\phi_1 = \frac{1}{m_{2j_1}} \sum_{i \in F_1} f_i \in \mathcal{K}_{\omega_1}(\mathbb{C}) \quad \text{and} \quad z_1 = \frac{m_{2j_1}}{n_{2j_1}} \sum_{i \in F_1} y_k$$

observe that $w(\phi_1) = m_{2j_1}$, $\phi_1(Tz_1) = \frac{1}{n_{2j_1}} \sum_{i \in F_1} f_i\left(\sum_{k \in F_1} Ty_k\right) > \delta/2$ and $\phi_1(Tz_1) = \frac{1}{n_{2j_1}} \sum_{i \in F_1} f_i\left(\sum_{k \in F_1}\ \right) = 0$. Select

$$p_1 \geq \max\{p_i, p_\varrho(supp\ z_1 \cup supp\ Tz_1 \cup supp\ \phi_1), n_{2j+1}^2 \#\ supp\ z_1\},$$

denote $2j_2 = \sigma_\varrho(\phi_1, m_{2j_1}, p_1)$. Then take $F_2 \subseteq A$ with $\#F_2 = n_{2j_2}$ and $F_2 > F_1$ such that $(y_k)_k \in F_2$ is $(3, 1/n_{2j_2}^2)$-*R.I.S.* and define

$$\phi_2 = \frac{1}{m_{2j_2}} \sum_{i = F_2} f_i \in \mathcal{K}_{\omega_1}(\mathbb{C}) \quad \text{and} \quad z_2 = \frac{n_{2j_2}}{n_{2j_1}} \sum_{k = F_2} y_k$$

So we have $\phi_1 < \phi_2, \phi_2(Tz_2) > \delta$ and $\phi_2(z_1) = \phi_2(z_2) = 0$. Pick

$$p_2 \geq \max\{p_1, p_\varrho(supp\ z_1 \cup supp\ z_2 \cup supp\ Tz_1 \cup supp\ Tz_2 \cup\ supp\ \phi_1 \cup\ supp\ \phi_2) n_{2j+1}^2 \#\ supp\ z_2\}$$

and set $2j_3 = \sigma_\varrho(\phi_1, m_{2j_1}, p_1, \phi_2, m_{2j_2}, p_2)$. Continuing with this procedure we form a sequence $\left(z_1, \phi_1, \ldots, z_{n_{2j+1}}, \phi_{n_{2j+1}}\right)$. Now we check that this is a $(0, j)$-dependent sequence.

Property (0DS.1) is clear, because of the construction of the functionals their weights satisfy $w(\phi_i + 1) = m_{\sigma_\varrho(\Phi_i)}$ where $\Phi_i = (\phi_1, w(\phi_1), p_1, \dots, \phi_i, w(\phi_i), p_i)$.

Property (0DS.2).We proceed to the construction of the sequence $\{\psi_1, \dots, \psi_{n_{2j+1}}\}$ in $\mathcal{K}_{\omega_1}(\mathbb{C})$ such that (z_i, ψ_i) is a $(6,2j_i)$-exact pair and $w(\psi_i) = w(\phi_i)$ for every $1 \leq i \leq n_{2j+1}$. The other condition $\#\, supp\, z_i \leq w(\phi_{i+1})/n_{2j+1}^2$ is already obtained by the construction of the weights. For each z_i there exists a subset $F_i \subseteq A$ with $\#F_i = n_{2j_i}$ such that $z_i = (m_{2j_i}/n_{2j_i}) \sum_{k \in F_i} y_{n_k}$ where $(y_{n_k})_k \in F_i$ is a $(3,1/n_{2j_i}^2)R.I.S.$ Now we follow the same arguments as in Proposition (4.2.15). For every $k \in F_i$ we take $f_{n_k} \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $f_{n_k}(y_{n_k}) = 1$ and $f_{n_k} < f_{n_k+1}$. Then $\psi_i = (1/m_{2j_i}) \sum_{k \in F_i} f_{n_k} \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and (z_i, ϕ_i) is a $(6,2j_i)$-exact pair.

Property (0DS.3). Let $H = (h_1, \dots, h_{n_{2j+1}})$ be an arbitrary $2j+1$-special sequence. We consider two cases: (a) Suppose that $\max supp\, z_k \leq \max supp\, \phi_k$ for every $1 \leq k \leq n_{2j+1}$. Then $supp\, z_k \subseteq supp\, \overline{\phi_{\lambda_{\Phi,H^{-1}}}}^{p_{\lambda_{\Phi,H^{-1}}}}$ for every $\kappa_{\Phi,H} < k < \lambda_{\Phi,H}$. Then for the second part of (TP.3) we obtain the desired result. (b) Suppose that $\max supp\, \phi_k \leq \max supp\, z_k$ for every $1 \leq k \leq n_{2j+1}$. Then $supp\, \phi_k \subseteq supp\, \overline{z_{\lambda_{\Phi,H^{-1}}}}^{p_{\lambda_{\Phi,H^{-1}}}}$ for every $\kappa_{\Phi,H} < k < \lambda_{\Phi,H}$, and the result follows from the first part of (TP.3).

Fix a $(0,j)$-dependent sequence as obtained in the previous claim, and define

$$z = (1/n_{2j+1}) \sum_{k=1}^{n_{2j+1}} z_k \quad and \quad \phi = (1/m_{2j+1}) \sum_{k=1}^{n_{2j+1}} \phi_k$$

Then $\phi(Tz) = (1/n_{2j+1}) \sum_{k=1}^{n_{2j+1}} \phi_k(Tz) \geq \delta/m_{2j+1}$ and $\|z\| < 12/m_{2j+1}^2$. Hence, $\|Tz\| \geq \delta/m_{2j+1} \geq \delta m_{2j+1} \|z\|/12 > \epsilon\|z\|$, and this completes the proof.

List Of Symbols

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