Chapter 1

Invariant Manifolds of Dynamical System and Some examples

Section (1.1): Invariant Manifolds of Dynamical Systems

Definition (1.1.1):

A dynamical system as an ordinary differential equation

$$\dot{x} \equiv \frac{dx}{dt} = f(x,t)$$

Where $x \in \mathbb{R}^n$ and $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is avector field. We could also define f on a subset$E \subset \mathbb{R}^n$. Either way the domain of f is called the Phase space of the system. For this section, our focus will be on systems where f does not depend on time (called autonomous systems), in which case we can define a flow as seen in class, which we denote $\varphi_t(x)$. We will focus on points where f vanishes, that is for which $x(t) = x(0)$ for all t. These we call equilibrium points of the system. To study these points, we will require the notion of invariant sets which we define now.

Definition (1.1.2):

Aninvariant set is a subset $A \subset E$ of the phase space s.t for any $x \in A$ and $t \in \mathbb{R}, \varphi_t(x) \in A$.

In particular, we will study the invariant sets associated with equilibrium points. If we let x_0 be such a point, we define the stable and unstable sets, W^s and W^u, associated with x_0 as follows

$$W^s = \{x: \varphi_t(x) \to x_0 \text{as } t \to \infty\} \qquad (1.1)$$

$W^u = \{x: \varphi_t(x) \to x_0 \text{as } t \to -\infty\}$ (1.2) We now turn to computing the stable and unstable sets in the case where the system is linear.

Definition (1.1.3): (Linear Systems)

Consider a system given by

$$\dot{x} = Ax \tag{1.3}$$

Where A is an n-by-$n$real matrix. Such a system can be solved directly using a matrix exponential defined as

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i \tag{1.4}$$

We will prove this. First, define the matrix norm by

$$\|A\| = \sup_{x:\, |x|=1} |Ax| \tag{1.5}$$

and notice that the following inequalities hold

$$\max_{j,k} \left|A_{j,k}\right| \leq \|A\| \leq n \max_{j,k} \left|A_{j,k}\right| \tag{1.6}$$

So that the formal sum of the exponential converges in the matrix norm iff every entry converges.

Theorem (1.1.4):

If A is an n-by-n real matrix, show that

$$\left\|A^i\right\| \leq \|A\|^i \tag{1.7}$$

Proof:

Notice that

$$\left\|A^i\right\| = \sup_{x:\, |x|=1} \left|A^i x\right| = \sup_{x:\, |x|=1} \left|A^{i-1}(Ax)\right| \leq \sup_{x:\, |x|=1} |Ax| \sup_{x:\, |x|=1} \left|A^{i-1}x\right|$$
$$\leq \|A\| \left\|A^{i-1}\right\|$$

so the conclusion follows by induction.

Thus we have the convergence of the matrix exponential from the convergence of

$$\sum_{i=0}^{\infty} \frac{1}{i!} \|A\|^i$$

The direct differentiation shows that

$$x(t) = \exp(tA)x(0) = \sum_{i=0}^{\infty} \frac{t^i A^i}{i!} x(0) \tag{1.8}$$

is a solution to the (1.3) with initial condition $x(0)$. Now, notice that the origin is a fixed point of any such system, and so we will study its stable and unstable sets, which turn out to be linear subspaces of $\mathbb{R}^n$. although this could be studied for general real matrices, we will restrict ourselves to the case $n = 2$. Furthermore, we assume the eigenvalues of A have non-zero real part. Any linear system that satisfies the last condition is called hyperbolic and the origin would then be a hyperbolic fixed point. To prove that the invariant sets are the generalized eigenspace of the matrix A:

First, note that the following holds

$$U^{-1}A^jU = (U^{-1}AU)^j \Longrightarrow U^{-1}\exp(A)U = \exp(U^{-1}AU)$$

Now, given the system in (1.3) we can perform a change of coordinates so that A is in Jordan form. This is of course done in $\mathbb{C}^2$ rather than $\mathbb{R}^n$ so some care must be taken in checking that the solutions we get are constrained to vectors with zero imaginary part. Assume first that A has two distinct eigenvalues, then it has Jordan form

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \Longrightarrow \exp(A) = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}$$

In the new coordinates then

$$x(0) = x_{0,1}u_1 + x_{0,2}u_2 \Longrightarrow x(t) = e^{t\alpha_1}x_{0,1}u_1 + e^{t\alpha_2}x_{0,2}u_2 \tag{1.9}$$

Where u_1 and u_2 are the eigenvectors ofα_1 and α_2 respectively. If both eigenvalues are real then the eigenvector can also be chosen to be real so that this solution is indeed an acceptable one for real initial condition. If not, then we have $\alpha_2 = \overline{\alpha_1}$ which then allows us to assume $u_2 = \overline{u_2}$ and since$\overline{x(0)} = x(0)$ for real initial conditions we get $\overline{x_{0,2}} = x_{0,1}$ and (1.9) becomes

$$x(t) = 2Re\left(e^{t\alpha_1} x_{0,1} u_1\right)$$

In both cases, it isa simple calculation to cheek that

$$W^s = E^s \tag{1.10}$$
$$W^u = E^u \tag{1.11}$$

Where E^s (respectively E^u) is the direct sum of the generalized eignspaces of the eigenvalues, we have

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \Longrightarrow \exp(tA) = \begin{pmatrix} e^{t\alpha} & te^{t\alpha} \\ 0 & e^{t\alpha} \end{pmatrix}$$

First, we prove by induction that

$$A^j = \begin{pmatrix} \alpha^j & j\alpha^{j-1} \\ 0 & \alpha^j \end{pmatrix} \tag{1.12}$$

since it clearly holds for $j = 1$, we move to the inductive step

$$A^{j+1} = \begin{pmatrix} \alpha^j & j\alpha^{j-1} \\ 0 & \alpha^j \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^{j+1} & (j+1)\alpha^j \\ 0 & \alpha^{j+1} \end{pmatrix}$$

so the conclusion follows readily from the definition of the matrix exponential.

Then we get that

$$x(0) = x_{0,1} u_1 + x_{0,2} u_2$$

$$\Longrightarrow x(t) = \left(e^{t\alpha} x_{0,1} + te^{t\alpha} x_{0,2}\right) u_1 + e^{t\alpha} x_{0,2} u_2 \tag{1.13}$$

Now, since α is real, we can choose u_1 and u_2 to be real, so that the solution is valid for real initial conditions. Now, clearly $W^s = \mathbb{R}^n$ if $\alpha < 0$ and $W^u = \mathbb{R}^n$if $\alpha > 0$ thus we have the same conclusion as in the first case. This is indeed true in general but we will not go through the proof. The main idea is again to use the Jordan form and compute the exponential of a general Jordan block so that a computation similar to the one here can be done. We will however state the theorem as it will be relevant later.

Theorem (1.1.5):

Give a system of the form (1.3) were the eigenvalues of A have non-zero real parts, then we have

$$W^s = E^s \tag{1.14}$$
$$W^u = E^u \tag{1.15}$$

Where E^s (respectively E^u) is the direct sum of the generalized eignspaces of the eigenvalues with negative (respectively positive) real part. In particular

$$\mathbb{R}^n = W^s \oplus W^u \tag{1.16}$$

Theorem (1.1.6): (Stable Manifold Theorem)

Suppose the origin is a fixed point of $\dot{x} = f(x)$. Let E^s and E^u be the stable and unstable subspace of the linearization $\dot{x} = Ax$ where $A = Df(0)$is the Jacobian of f at the origin. If $|f(x) - Ax| = \mathrm{O}(|x^2|)$ then $\exists$ local stable and unstable manifolds $W^s_{loc}(0), W^u_{loc}(0)$, which have the same dimension as E^s and E^u and are tangent to them at 0, such that for $x \in U$ but $x \neq 0$ for some neighborhood U of 0

$$W^s_{loc}(0) = \{x: \varphi_t(x) \to 0 \text{ as } t \to \infty\}$$

$$W^u_{loc}(0) = \{x: \varphi_t(x) \to 0 \text{ as } t \to -\infty\}$$

These manifolds can be extended to what are called global manifolds by letting

$$W^s(0) = \{x : \exists t \in \mathbb{R} \text{ s.t} \varphi_t(x) \in W^s_{loc}(0)\}$$

$$W^u(0) = \{x : \exists t \in \mathbb{R} \text{ s.t} \varphi_t(x) \in W^u_{loc}(0)\}$$

Consider the following system

$$\dot{x} = -x + y^2, \quad \dot{y} = y - x^2$$

The origin is a fixed point of this system, and the linearization is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

And we have

$$|f(x) - Ax| = \left| \begin{matrix} y^2 \\ -x^2 \end{matrix} \right| = |x^2|$$

So we can apply the theorem. First, notice that

$$E^u = \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$E^s = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we look at the unstable manifold. It is tangent to $x = 0$ so we let $x = p(y)$ and get

$$-p(y) + y^2 = \dot{x} = p'(y)\dot{y} = p'(y)\left(y - \left(p(y)\right)^2\right) \tag{1.17}$$

Which we solve approximately using a power series. Notice that since we need $p(y)$ to be tangent to $x = 0$ so the first two terms vanish.

$$p(y) = a_2 y^2 + a_3 y^3 + \cdots .$$

Plugging this back into (1.17) we can match coefficients and get a series for $p(y)$. For this problem we will only use a third order approximation, which gives $a_2 = \frac{1}{3}$ and $a_3 = 0$ so our approximate solution is

$$W^u_{loc}(0) = \left\{ x : x = \begin{pmatrix} \frac{1}{3}y^2 \\ y \end{pmatrix} \right\} \tag{1.18}$$

We can use the same method to for the stable manifold, which gives

$$W^s_{loc}(0) = \left\{ x : x = \begin{pmatrix} x \\ \frac{1}{3}x^2 \end{pmatrix} \right\} \tag{1.19}$$

This example was used to illustrate how difficulties can arise in computing the manifolds of even simple two-dimensional systems. It is therefore notsurprising that the computation of global invariant manifolds for general systems has proven to extremely difficult.

Section (1.2): Examples of Dynamical Systems

Example (1.2.1): (The Standard Map)

The standard map describes the motion of a "rotator" with one angulardegree of freedom$q \in \mathbb{S}^1$ ($\mathbb{S}^1$denotes the circle$\mathbb{R}/2\pi\mathbb{Z}$), which is periodically kicked by apendulum-likeforce of intensity proportional to $-\sin q$. If$q_n$and $p_n$denotethe position and momentumjust before the nth kick, one has

$$\begin{aligned} q_{n+1} &= q_n + p_{n+1} \pmod{2\pi} \\ p_{n+1} &= p_n - \varepsilon \sin q_n \end{aligned} \qquad (1.20)$$

For$\varepsilon = 0$, the dynamics is very simple and one has explicitly

$$\begin{aligned} q_n &= q_0 + \mathrm{n}p_0 \\ p_n &= p_0 \end{aligned} \qquad (1.21)$$

Let us now analysis the iterated map (1.20) according to the perturbativemethod. Theidea is to look for a change of variables$(q, p) \mapsto (\varphi, I)$ transforming the system into asimilar one, but without the term $\varepsilon \sin q_n$. Let us write

$$\begin{aligned} p &= \varphi + f(\varphi, I) \\ q &= I + g(\varphi, I) \end{aligned} \qquad (1.22)$$

where f and g are unknown functions, which are2π-periodic inφ. Weimpose that thischange of variables transforms the map (1.20) into the map

$$\begin{aligned} \varphi_{n+1} &= \varphi_n + \mathrm{I}_{n+1} (\mathrm{mod}\ 2\pi) \\ \mathrm{I}_{n+1} &= \mathrm{I}_n =: \omega\,. \end{aligned} \qquad (1.23)$$

This is equivalent to requiring that f and g solve the functional equations

$$\begin{aligned} f(\varphi + \omega, \omega) &= f(\varphi, \omega) + g(\varphi + \omega, \omega) \\ g(\varphi + \omega, \omega) &= g(\varphi, \omega) - \varepsilon \sin\big(\varphi + f(\varphi, \omega)\big)\,. \end{aligned} \qquad (1.24)$$

One can try to solve these equations by expanding f and g into Taylor series inε andFourier series inφ:

$$f(\varphi, \omega) = \sum_{j=1}^{\infty} \varepsilon^j f_j(\varphi, \omega) \quad , f_j(\varphi, \omega) = \sum_{k=-\infty}^{\infty} a_{k,j}(\omega) e^{\mathrm{i}k\varphi} \qquad (1.25)$$

$$g(\varphi, \omega) = \sum_{j=1}^{\infty} \varepsilon^j g_j(\varphi, \omega) \quad , g_j(\varphi, \omega) = \sum_{k=-\infty}^{\infty} b_{k,j}(\omega) e^{\mathrm{i}k\varphi}\,. \qquad (1.26)$$

We will use the expansion

$$\sin(\varphi + \varepsilon f) = \sin\varphi + \varepsilon f_1 \cos\varphi + \varepsilon^2 \left(f_2 \cos\varphi - \frac{1}{2} f_1^2 \sin\varphi \right) + \mathcal{O}(\varepsilon^3). \qquad (1.27)$$

At order ε, we have to solve the relations

$$f_1(\varphi+\omega,\omega) = f_1(\varphi,\omega) + g_1(\varphi+\omega,\omega)$$
$$g_1(\varphi+\omega,\omega) = g_1(\varphi,\omega) - \sin\varphi \qquad (1.28)$$

which become, in Fourier components,

$$a_{k,1}e^{\mathrm{i}\,k\omega} = a_{k,1} + b_{k,1}e^{\mathrm{i},k\omega}$$
$$b_{k,1}e^{\mathrm{i}\,k\omega} = b_{k,1} + c_{k,1}, \qquad (1.29)$$

where$c_{k,1}$are the Fourier components of $\sin\varphi$,that is,
$c_{1,1} = -c_{-1,1} = 1/(2i)$ and allother components vanish. We thus get

$$g_1(\varphi,\omega) = \frac{e^{\mathrm{i}\varphi}}{2\mathrm{i}(1-e^{i\omega})} - \frac{e^{-\mathrm{i}\varphi}}{2\mathrm{i}(1-e^{-i\omega})} = \frac{\cos(\varphi-\omega/2)}{2\sin(\omega/2)}$$
$$f_1(\varphi,\omega) = -\frac{e^{\mathrm{i}\omega}e^{\mathrm{i}\varphi}}{2\mathrm{i}(1-e^{i\omega})^2} + \frac{e^{-i\omega}e^{-\mathrm{i}\varphi}}{2\mathrm{i}(1-e^{-i\omega})^2} = \frac{\sin\varphi}{4\sin^2(\omega/2)}. \qquad (1.30)$$

Note that this is only possible for$e^{\mathrm{i}\omega} \neq 1$, that is,$\omega \neq 0(mod\ 2\pi)$. At order$\varepsilon^2$ we obtainsimilar relations as(1.29), but now$c_{k,2}$, denotes the Fourier coefficients of$f_1(\varphi,\omega)\cos\varphi$,which are nonzero for$|k| = 2$. Thus$g_2$ and f_2 only exist if$e^{2\mathrm{i}\omega} \neq 1$, or$\omega \neq 0, \varphi(\mathrm{mod}\ 2\pi)$.Similarly, we will find that$g_j$and $f_j$only exist if$e^{j\mathrm{i}\omega} \neq 1$, so the equations(1.24) canonly be solved for irrational$\omega/(2\pi)$. Even then, the expansions of f and g willcontainsmall terms of the form$1 - e^{\mathrm{i}\,k\omega}$in the denominators, so that theconvergence of the seriesis not clear at all. In fact, the convergence has beenproved by Moser for certain irrationalωcalled Diophantine numbers.

Now let us turn to the geometric approach. We can consider(q,p) ascoordinates inthe plane (or on the cylinder because of the periodicity of$q$). For given(q_0,p_0), the setof points$\{(q_n,p_n)\}_n \geq 0$ is called the orbitwith initial condition(q_0,p_0). We would like toknow what the different orbitslook like. The simplest case is the fixed point: if

$$q_{n+1} = q_n$$
$$p_{n+1} = p_n \qquad (1.31)$$

then the orbit will consist of a single point. The fixed points of the standard map are$(0,k)$ and $(\pi,k)$with $k \in \mathbb{Z}$. We can also have periodic orbits, consisting of m points, if

$$q_{n+m} = q_n$$
$$p_{n+m} = p_n \qquad (1.32)$$

Another possible orbit is the invariant curve. For instance if the equations(1.24) admita solution, we can write

$$q_n = \varphi_n + f(\varphi_n, \omega)(\text{mod } 2\pi)$$
$$p_n = \omega + g(\varphi_n, \omega) \qquad (1.33)$$

where $\varphi_n = \varphi_0 + n\omega$. This is the parametric equation of a curve winding around thecylinder. Sinceω is irrational, the points fill the curve densely. One can also analysis thedynamics in the vicinity of periodic orbits. It turnsout that for this kind of map, mostperiodic orbits are of one of two types: elliptic orbits are surrounded by invariant curves,while hyperbolic orbitsattract other orbits from one direction and expel them into anotherone. There are, however, much more exotic types of orbits. Some live oninvariant Cantorsets, others densely fill regions of phase space with a positive surface.

The aim of the geometrical theory of dynamical systems is to classify the possiblebehaviours and to find ways to determine the most important qualitative features of thesystem. An important advantage is that largeclasses of dynamical systems have a similarqualitative behaviour. This doesnot mean that the perturbative approach is useless. Butit is in general preferable to start by analysing the system from a qualitative point of view,and then, if necessary, use more sophisticated methods in order to obtain more detailedinformation.

Example (1.2.2): (The Lorenz Model)

Convection is an important mechanism in the dynamics of the atmosphere: warm air hasa lower density and therefore rises to higher altitudes, where it cools down and falls again,giving rise to patterns in the atmospheric currents.

This mechanism can be modeled in the laboratory, an experiment known as Rayleigh-Be′nard convection. A fluid is contained between twohorizontal plates, the upper one attemperature$T_0$and the lower one at temperature

$T_1 = T_0 + \Delta T > T_0$. The temperaturedifferenceΔT is thecontrol parameter, which can be modified.

For small values of ΔT, the fluid remains at rest, and the temperature decreaseslinearly in the vertical direction. At slightly larger ΔT, convection rolls appear (theirshape depends on the geometry of the set-up). The flow isstill stationary, that is, thefluid velocity at any given point does not change in time.

For still largerΔT, the spatial arrangement of the rolls remains fixed, but their timedependence becomes more complex. Usually, it starts by getting periodic. Then differentscenarios are observed, depending on the set-up. One of them is the period doublingcascade: the time-dependence of the velocity field

has period$P, 2P, 4P, \ldots, 2^n P, \ldots$, wherethe nth period doubling occurs for a temperature differenceΔT_nsatisfying

$$\lim_{n\to\infty} \frac{\Delta T_n - \Delta T_{n-1}}{\Delta T_{n+1} - \Delta T_n} = \delta \simeq 4.4 \ldots \qquad (1.34)$$

These ΔT_naccumulate at some finite $\Delta T_\infty$for which the behaviour is no longer periodic,but displays temporal chaos. In this situation, the direction of rotation of the rolls changeserratically in time.

For very large ΔT, the behaviour can become turbulent: not only is the time dependenceno periodic, but the spatial arrangement of the velocity field also changes.

RB convection has been modeled in the following way. For simplicity, one considersthe two-dimensional case, with an infinite extension in the horizontalx_1-direction, whilethe vertical x_2-direction is bounded between$-\frac{1}{2}$ and$\frac{1}{2}$. Let$D = \mathbb{R} \times \left[-\frac{1}{2}, \frac{1}{2}\right]$. The stateof the system is described by three fields

$$\begin{aligned} v &: \mathcal{D} \times \mathbb{R} \to \mathbb{R}^2 && \text{velocity,} \\ T &: \mathcal{D} \times \mathbb{R} \to \mathbb{R} && \text{temperature,} \\ P &: \mathcal{D} \times \mathbb{R} \to \mathbb{R} && \text{pressure.} \end{aligned} \qquad (1.35)$$

The deviation$\theta(x,t)$ from the linear temperature profile is defined by

$$T(x,t) = T_0 + \Delta T\left(\frac{1}{2} - x_2\right) + T_1\theta(x,t). \qquad (1.36)$$

The equations of hydrodynamics take the following form:

$$\begin{aligned} \frac{1}{\sigma}\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right] &= \Delta v + \Delta p + (0,\theta)^T \\ \frac{\partial\theta}{\partial t} + (v\cdot\nabla)\theta &= \Delta\theta + Rv_2 \\ v\cdot\nabla &= 0. \end{aligned} \qquad (1.37)$$

Here σ, the Prandtl number, is a constant related to physical properties of the fluid, while$R$, the Reynolds number, is proportional toΔT. Furthermore,

$$\nabla p = \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}\right)^T$$

$$\nabla^2\theta = \frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2}$$

$$\nabla\cdot v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$$

$$(v \cdot \nabla)\theta = v_1 \frac{\partial \theta}{\partial x_1} + v_2 \frac{\partial \theta}{\partial x_2}.$$

The terms containing $(v \cdot \nabla)$ introduce the nonlinearity into the system. The boundaryconditions require thatθ, v_2 and$\partial v_1/\partial x_2$ should vanish for$x_2 = \pm\frac{1}{2}$. We thus have tosolve four coupled nonlinear partial differential equations for the four fieldsv_1, v_2, θ, p. Thecontinuity equation $\nabla \cdot v = 0$ can be satisfied by introducing the vorticity$\psi: \mathcal{D} \times \mathbb{R} \to \mathbb{R}$,such that

$$(v_1, v_2) = \left(-\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}\right). \tag{1.38}$$

It is also possible to eliminate the pressure from the two equations for$\partial v/\partial t$. We are left with two equations for ψandθ. The problem can be further simplified by assuminga periodic dependence onx_1, of period$2\pi/q$. A possible approach (not the best one bymodern standards, but historically important) is to expand the two fields into Fourierseries (or "modes"):

$$\begin{aligned} \psi(x_1, x_2) &= \sum_{k \in \mathbb{Z}^2} a_k e^{\mathrm{i} k_1 q x_1} e^{i\, k_2 \pi x_2} \\ \theta(x_1, x_2) &= \sum_{k \in \mathbb{Z}^2} b_k e^{\mathrm{i} k_1 q x_1} e^{i\, k_2 \pi x_2} \end{aligned} \tag{1.39}$$

(where the boundary conditions impose some relations between Fourier coefficients of thesame $|k_1|$and $|k_2|$). Note that the terms of this sum are eigenfunctions of the linearoperators in (1.37). Plugging these expansions into the equations, we obtain relations ofthe form

$$\frac{d}{dt}\begin{pmatrix} a_k(t) \\ b_k(t) \end{pmatrix} = L_k \begin{pmatrix} a_k(t) \\ b_k(t) \end{pmatrix} + N(\{a_{k'}, b_{k'}\}_{k' \in \mathbb{Z}}), \tag{1.40}$$

where$L_k$are 2×2 matrices and the term $N(\cdot)$ comes from the nonlinear terms in $(v \cdot \nabla)$and may depend on all other k'. Without these nonlinear terms the problem would beeasy to solve.

In 1962, Saltzmann considered approximations of the equations(1.40) with finitelymany terms, and observed that the dynamics seemed tobe dominated by three Fouriermodes. In 1963, Lorenz decided to truncatethe equations to these modes [2], setting

$$\begin{aligned} \psi(x_1, x_2) &= \alpha_1 X(t) \sin q x_1 \cos \pi x_2 \\ \theta(x_1, x_2) &= \alpha_2 Y(t) \cos q x_1 \cos \pi x_2 + \alpha_3 Z(t) \sin 2\pi x_2 \,. \end{aligned} \tag{1.41}$$

Here$\alpha_1 = \sqrt{2}\,(\pi^2 + q^2)/(\pi q)$, $\alpha_2 = \sqrt{2}\alpha_3and\alpha_3 = (\pi^2 + q^2)^3/(\pi q^2)$are constantsintroduced only in order to simplify the resulting equations. All other Fourier modesin the expansion(1.40) are set to be zero, a rather drastic approximation. After scaling timeby afactor$(\pi^2 + q^2)$, one gets the equations

$$\begin{aligned} dX/dt &= \sigma(Y - X) \\ dY/dt &= rX - Y - XZ \qquad (1.42) \\ dZ/dt &= -bZ + XY, \end{aligned}$$

where $b = 4\pi^2/(\pi^2 + q^2)$, and $r = Rq^2/(\pi^2 + q^2)^3$is proportional to the control parameter$R$, and thus toΔT. These so-called Lorenz equations are a very crude approximation ofthe equations (1.37), nevertheless they may exhibit very complicated dynamics.

In fact, for $0 \leq r \leq 1$, all solutions are attracted by the origin $X = Y = Z = 0$,corresponding to the fluid at rest. For $r > 1$, a pair of equilibria with$X \neq 0$ attracts theorbits, they correspond to convection rolls with the two possible directions of rotation. Increasing$r$produces a very complicatedsequence of bifurcations, including period doublingcascades. For certain values of the parameters, a strange attractoris formed, inwhich case the convection rolls change their direction of rotation very erratically,and the dynamics is very sensitive to small changes in the initial conditions (theButterflyeffect). The big surprise was that such a simple approximation,containing only threemodes, could capture such complex behaviors[1].

Example (1.2.3):(The Logistic Map)

Our last example is a famous map inspired by population dynamics. Consider a populationof animals that reproduce once a year. Let$P_n$be thenumber of individuals in the yearnumber n. The offspring being usuallyproportional to the number of adults, the simplestmodel for the evolution ofthe population from one year to the next is the linear equation

$$P_{n+1} = \lambda P_n \qquad (1.43)$$

where λ is the natality rate (minus the mortality rate). This law leads to an exponentialgrowth of the form

$$P_n = \lambda^n P_0 = e^{n \ln \lambda} P_0 \qquad (1.44)$$

(the Malthus law). This model becomes unrealistic when the number of individuals is solarge that the limitation of resources becomes apparent. The simplest possibility to limitthe growth is to introduce a quadratic term $-\beta P_n^2$, leading to the law

$$P_{n+1} = \lambda P_n - \beta P_n^2. \qquad (1.45)$$

The rescaled variable $x = \beta P$ then obeys the equation

$$x_{n+1} = f_\lambda(x_n) := \lambda x_n(1 - x_n). \qquad (1.46)$$

The map $f_\lambda$is called the logistic map. Observe that for $0 \leq \lambda \leq 4, f_\lambda$maps the interval[0,1] into itself. The dynamics of the sequence $x_n$depends drastically on the value ofλ.

For $0 \leq \lambda \leq 1$, all orbits converge to 0, which means that the population becomesextinct. For $1 < \lambda \leq 3$, all orbits starting at $x_0 > 0$ converge to$1 - 1/\lambda$, and thus thepopulation reaches a stable equilibrium. For$3 < \lambda \leq 1 + \sqrt{6}$, the orbits converge to acycle of period 2, so that the population asymptotically jumps back and forth betweentwo values.

For $\lambda > 1 + \sqrt{6}$, the system goes through a whole seqence of period doublings. Similarlyas in RB convection, the values $\lambda_n$of the parameter for which the nth period doublingoccurs obey the law

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta = 4.669 \ldots \qquad (1.47)$$

where δ is called the Feigenbaum constant. In 1978, Feigenbaum as well as Coullet andTresser independently outlined an argument showing that such period doubling cascades should be observable for a large class of systems, and that the constant δ is universal. Forinstance, it also appears in the two-dimensional He′non map

$$\begin{aligned} x_{n+1} &= 1 - \lambda x_n^2 + y_n \\ y_{n+1} &= b x_n \quad 0 < b < 1 \end{aligned} \qquad (1.48)$$

Rigorous proofs of these properties were later worked out by Collet, Eckmann, Koch,Lanford and others[12].

For the logistic map, the period doublings accumulate at a value$\lambda_\infty = 3.56 \ldots$, beyondwhich the orbits become chaotic. For larger λ, there is a complicated interplay of regularand chaotic motion, containing other period doubling cascades. Finally, for$\lambda = 4$, one can prove that the dynamics is as random as coin tossing.

Chapter 2

Stationary and Periodic Solutions

In Chapter (1), we have seen examples of two kinds of dynamical systems: ordinary differential equations (ODEs) and iterated maps. There are other types of dynamical systems, such as partial differential equations or cellular automata. These are in general more difficult to analysis, although some ideas developed for maps and ODEs can be carried over to their study. Here we will concentrate on ODEs and maps, by starting with the simplest kinds of dynamics: stationary and periodic.

Section (2.1): Basic Concepts and Stationary Solutions

Now, we will discuss orbits and flows.

Let $\mathcal{D} \subset \mathbb{R}^n$ be an open domain. One type of dynamical systems we will consider is given by a map $F: \mathcal{D} \to \mathcal{D}$. $\mathcal{D}$is called the phase spaceof the system. It is possible to consider more general differentiable manifolds as phase space, such as the circle, the cylinder or the torus, but we will limit the discussion to Euclidean domains. Generalizations to other manifolds are usually straightforward.

Definition (2.1.1):

The(positive) orbit of$F$through a point $x_0 \in \mathcal{D}$is the sequence$(x_k)_{k\geq 0}$defined by$x_{k+1} = F(x_k)$ for all integers$k > 0$. We have thus

$$x_k = F^k(x) \quad where\ F^k = \underbrace{F \circ F \circ \ldots \circ F}_{k\ times} \quad (2.1)$$

In case F is invertible, we can also define the negative orbit ofx_0 by the relations $F(x_k) = x_{k+1}$for all $k < 0$, which are equivalent to (2.1) if we set$F^{-k} = (F^{-1})^k$for all $k > 0$.The orbit of x_0 is then given by$(x_k)_{k\in\mathbb{Z}}$.

Note the trivial relation

$$F^{k+l}(x_0) = F^k\left(F^l(x_0)\right) \qquad (2.2)$$

for all positive integers k, l (and all integers if F is invertible), which will admit an analoguein the case of ODEs.

As particular cases of maps F, we have homeomorphisms, which are continuous mapsadmitting a continuous inverse, and diffeomorphisms,which are continuously differentiablemaps admitting a continuouslydifferentiable inverse. Similarly, for all $r \geq 1$, aC^r-diffeomorphismis an invertible map F such that both F and F^{-1} admit continuousderivatives up to order r.

The ordinary differential equationswe are going to consider are of the form

$$\dot{x} = f(x) \qquad (2.3)$$

where $f: \mathcal{D} \to \mathbb{R}^n$, and $\dot{x}$ denotes$\frac{dx}{dt}$. Equivalently, we can write (2.3) as a system ofequations for the components of x,

$$\dot{x}_i = f_i(x), \quad i = 1, \dots, n \qquad (2.4)$$

$\mathcal{D} \subset \mathbb{R}^n$is again called phase spaceand f is called a vector field.

To define the orbits of f, we have to treat the problem of existenceand uniquenessa bit more carefully. The following results are assumed to be known from basic analysis[6],[13],[11].

Theorem (2.1.2): (Peano-Cauchy)

Let f be continuous. For every$x_0 \in \mathcal{D}$, there existsat least one local solution of (2.3) throughx_0, that is, there is an open interval$I \ni 0$and afunction $x: I \to \mathcal{D}$ such that $x(0) = x_0$ and $\dot{x}(t) = f(x(t))$ for all $t \in I$.

Theorem (2.1.3):

Every solution $x(t)$ with $x(0) = x_0$ can be continued to a maximal interval of existence $(t_1, t_2) \ni 0$. If $t_2 < \infty$or $t_1 > -\infty$, then for any compact $K \subset \mathcal{D}$, there exists a time$t \in (t_1, t_2)$ with $x(t) \notin K$(this means that solutions will diverge or reach$\partial\mathcal{D}$).

Theorem (2.1.4): (Picard-Lindel¨of)

Assume f is continuous and locally Lipschitzian, that is, for every compact $K \subset \mathcal{D}$, there exists a constant $L_K$such that

$\|F(x) - f(y)\| \leq L_K \|x - y\|$for all $x, y \in K$. Then there is a unique solution $x(t)$ of (2.3) with $x(0) = x_0$for every $x_0 \in \mathcal{D}$.

Note in particular that if f is continuously differentiable, then it islocally Lipschitzian.We will usually consider vector fields which are at least once continuously differentiable.

Example (2.1.5):

It is easy to give counterexamples to global existence and uniqueness.

For instance,

$$\dot{x} = x^2 \qquad \Rightarrow x(t) = \frac{1}{\frac{1}{x_0} - t} \qquad (2.5)$$

has a solution diverging for $t = \frac{1}{x_0}$.A physically interesting counterexampleto uniquenessis the leaky bucket equation

$$\dot{x} = -\sqrt{|x|}. \qquad (2.6)$$

Here x is proportional to the height of water in a bucket with a hole in the bottom, and(2.6) reflects the fact that the kinetic energy (proportional to $\dot{x}^2$) of the water leavingthe bucket is equal to the potential energy of the water inside. For every c, (2.6) admitsthe solution

$$\mathrm{x(t)} = \begin{cases} \frac{1}{4}(\mathrm{t} - \mathrm{c})^2 & \text{for } \mathrm{t} < c \\ 0 & \text{for } \mathrm{t} \geq \mathrm{c} \end{cases} \qquad (2.7)$$

In particular, for any $c \leq 0$, (2.7) is a solution of (2.6) such that $x(0) = 0$. This reflectsthe fact that if the bucket is empty at time 0, we do not know at what time it was full.

For simplicity, we will henceforth assume that the ODE (2.3) admits a unique globalsolution for all $x_0 \in \mathcal{D}$. This allows to introduce the followingdefinitions

Definition (2.1.6):

Let $x_0 \in \mathcal{D}$and let $x(t)$ be the unique solution of (2.3) with initialcondition $x(0) = x_0$.

i. The integral curve through x_0 is the set $\{(x,t) \in \mathcal{D} \times \mathbb{R} : x = x(t)\}$.
ii. The orbit through x_0 is the set $\{x \in \mathcal{D}: x = x(t), t \in \mathbb{R}\}$.
iii. The flow of the equation (2.3) is the map

$$\varphi : \mathcal{D} \times \mathbb{R} \to \mathcal{D}$$
$$(x_0, t) \longmapsto \varphi_t(x_0) = x(t) \qquad (2.8)$$

Geometrically speaking, the orbit is a curve in a phase space containingx_0 such that thevector field $f(x)$ is tangent to the curve at any point x of thecurve.

Uniqueness of thesolution means that there is only one orbit throughany point in phase space.

By definition, we have $\varphi_0(x_0) = x_0$ for all $x_0 \in \mathcal{D}$, and uniqueness of solutions impliesthat $\varphi_t(\varphi_s(x_0)) = \varphi_{t+s}(x_0)$. These properties can berewritten as

$$\varphi_0 = \mathrm{id} \qquad \varphi_t \circ \varphi_s = \varphi_{t+s} \tag{2.9}$$

which means that the family $\{\varphi_t\}_t$forms a group. Note the similaritybetween this relationand the relation (2.2) for iterated maps.

Example (2.1.7):

In the case $f(x) = -x, x \in \mathbb{R}$, we have

$$\varphi_t(x_0) = x_0 e^t. \tag{2.10}$$

The system admits three distinct orbits $(0, \infty), (-\infty, 0)$ and $\{0\}$.

Now we will discuss evolution of volumes.

Let $M \subset \mathcal{D}$be a compact subset of phase space. We can define its volumeby a usualRiemannintegral:

$$Vol\,(\mathcal{M}) = \int_{\mathcal{M}} dx\,, \quad dx = dx_1, \dots, dx_n \tag{2.11}$$

The set $\mathcal{M}$will evolve under the influence of the dynamics: we can definethe sets $\mathcal{M}_k = F^k(\mathcal{M})$or $\mathcal{M}(t) = \varphi_t(\mathcal{M})$. How does their volume evolvewith time? The answer isactually quite simple:

Consider first the case of a map F. We assume that F is continuouslydifferentiableand denote by

$$\frac{\partial F}{\partial x}(x) \tag{2.12}$$

the Jacobian matrix of F at x, which is the $n \times n$ matrix A with elements$a_{ij} = \frac{\partial F_j}{\partial x_j}(x)$.

Proposition (2.1.8):

Assume F is a diffeomorphism and let $V_k = \mathrm{Vol}(\mathcal{M}_k)$. Then

$$V_{k+1} = \int_{\mathcal{M}_k} \left| \det\left(\frac{\partial F}{\partial x}(x)\right)\right| dx. \qquad (2.13)$$

Proof:

This is a simple application of the formula for a change of variables inan integral:

$$V_{k+1} = \int_{\mathcal{M}_{k+1}} dy = \int_{\mathcal{M}_k} \left| \det\left(\frac{\partial y}{\partial x}(x)\right)\right| dx = \int_{\mathcal{M}_k} \left| \det\left(\frac{\partial F}{\partial x}(x)\right)\right| dx.$$

Definition (2.1.9):

The map F is called conservative if

$$\left| \det\left(\frac{\partial F}{\partial x}(x)\right)\right| dx = 1 \quad \forall x \in \mathcal{D} \qquad (2.14)$$

The map F is called dissipative if

$$\left| \det\left(\frac{\partial F}{\partial x}(x)\right)\right| dx < 1 \quad \forall x \in \mathcal{D}. \qquad (2.15)$$

Proposition (2.1.8) implies that $V_{k+1} = V_k$if F is conservative and$V_{k+1} < V_k$if F isdissipative. More generally, if $\left|\det\left(\frac{\partial F}{\partial x}(x)\right)\right| \leq \lambda$for some constant λ and all $x \in \mathcal{D}$, then$V_k \leq \lambda^k V_0$.

For differential equations, the result is the following:

Proposition (2.1.10):

Assume f is continuously differentiable and let $V(t) = \mathrm{Vol}(\mathcal{M}(t))$. Then

$$\frac{d}{dt}V(t) = \int_{\mathcal{M}(t)} \nabla \cdot f(x)\, dx, \qquad (2.16)$$

where $\nabla \cdot f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$is the divergence.

Proof:

We have

$$V(t) = \int_{\mathcal{M}(t)} dy = \int_{\mathcal{M}(t)} \left| \det\left(\frac{\partial}{\partial x}\varphi_t(x)\right)\right| dx.$$

Let us fix $x \in \mathcal{M}$, let $y(t) = \varphi_t(x)$ and set

$$J(t) := \frac{\partial}{\partial x}\varphi_t(x), \quad A(t) := \frac{\partial f}{\partial x}\big(y(t)\big).$$

Note that by definition of $\varphi_t, J(0) = \mathbb{1}$ is the identity matrix. Now we can compute

$$\frac{d}{dt}J(t) = \frac{\partial}{\partial x}\dot{y}(t) = \frac{\partial}{\partial x}f\big(\varphi_t(x)\big) = \frac{\partial f}{\partial x}\big(\varphi_t(x)\big) = \frac{\partial}{\partial x}\varphi_t(x),$$

and thus

$$\frac{d}{dt}J(t) = A(t)J(t), \qquad J(0) = \mathbb{1}.$$

This is a linear, time-dependent differential equation for $J(t)$, which is known to admit aunique global solution. This implies in particular that$\det J(t) \neq 0\ \forall t$, since otherwise $J(t)$would not be surjective, contradicting uniqueness. Since $\det J(0) = 1$, continuity impliesthat $\det J(t) > 0 \forall t$. Now let us determine the evolution of $\det J(t)$. By Taylor's formula,there exists $\theta \in [0,1]$ such that

$$\begin{aligned} J(t+\varepsilon) &= J(t) + \varepsilon\frac{d}{dt}J(t+\theta\varepsilon) \\ &= J(t)[\mathbb{1} + \varepsilon J(t)^{-1}A(t+\theta\varepsilon)J(t+\theta\varepsilon)]. \end{aligned}$$

From linear algebra, we know that for any $n \times n$ matrix B,

$$\det(\mathbb{1} + \varepsilon B) = 1 + \varepsilon TrB + r(\varepsilon)$$

with $\lim_{\varepsilon\to 0} r(\varepsilon)/\varepsilon = 0$(this is a consequence of the definition of the determinant as a sumover permutations). Using $Tr(AB) = Tr(BA)$, this leads to

$$\det J(t+\varepsilon) = \det J(t)\,[1 + \varepsilon Tr(A(t+\theta\varepsilon)J(t+\theta\varepsilon)J(t)^{-1}) + r(\varepsilon)],$$

and thus

$$\frac{d}{dt}\det J(t) = \lim_{\varepsilon\to 0}\frac{\det J(t+\varepsilon) - \det J(t)}{\varepsilon} = Tr\big(A(t)\big)\det J(t).$$

Taking the derivative of $V(t)$ we get

$$\begin{aligned} \frac{d}{dt}V(t) &= \int_{\mathcal{M}} \frac{d}{dt}\det J(t)\,dx \\ &= \int_{\mathcal{M}} Tr\left(\frac{\partial f}{\partial x}\big(y(t)\big)\right)\det J(t)\,dx \\ &= \int_{\mathcal{M}(t)} Tr\left(\frac{\partial f}{\partial x}(y)\right)dy, \end{aligned}$$

and the conclusion follows from the fact that $\mathrm{Tr}\ \frac{\partial \mathrm{f}}{\partial \mathrm{x}} = \nabla \cdot \mathrm{f}$.

Definition (2.1.11):

The vector field f is called conservative if

$$\nabla \cdot f(x) = 0 \qquad \forall x \in \mathcal{D} \qquad (2.17)$$

The vector field f is called dissipative if

$$\nabla \cdot f(x) < 0 \qquad \forall x \in \mathcal{D}. \qquad (2.18)$$

Proposition (2.1.10) implies that $\mathrm{V(t)}$ is constant if f is conservative, and monotonouslydecreasing when f is dissipative. More generally, if $\nabla \cdot f(x) \leq \mathrm{c}$, $\forall x \in \mathcal{D}$, then $\mathrm{V(t)} \leq \mathrm{V(0)e}^{\mathrm{ct}}$.

Of course, one can easily write down dynamical systems which areneither conservativenor dissipative, but the conservative and dissipativesituations are very common inapplications.

Example (2.1.12):

Consider a Hamiltonian system, with Hamiltonian $\mathrm{H} \in \mathrm{C}^2(\mathbb{R}^{2\mathrm{m}}, \mathbb{R})$. Then $x = (q, p) \in \mathbb{R}^{\mathrm{m}} \times \mathbb{R}^{\mathrm{m}}$ and the equations $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$ take the form

$$f_{\mathbf{i}}(\mathbf{x}) = \begin{cases} \dfrac{\partial H}{\partial p_i} & i = 1, \dots m \\ -\dfrac{\partial H}{\partial q_{i-m}} & i = m+1, \dots 2m. \end{cases} \qquad (2.19)$$

This implies that

$$\nabla \cdot f = \sum_{i=1}^{m} \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \sum_{i=1}^{m} \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial q_i}\right) = 0. \qquad (2.20)$$

Thus all (sufficiently smooth) Hamiltonian systems are conservative.
Now, we will discuss stationary solution. A stationary solution of a dynamical system is a solution that does not change in time.We thus define:

Definition (2.1.13):

1- A fixed point of the map F is a point $x^\star \in \mathcal{D}$such that

$$F(x^\star) = x^\star. \qquad (2.21)$$

2- A singular point of the vector field $f$is a point $x^\star \in \mathcal{D}$such that

$$f(x^\star) = 0. \qquad (2.22)$$

In both cases, $x^\star$is also called equilibrium point. Its orbit is simply$\{x^\star\}$and is calleda stationary orbit. Note that a singular point of $f$is also a fixed point of the flow, andtherefore sometimes abusively called a "fixedpoint of f".

We are now interested in the behaviour near an equilibrium point. Inthis section, wewill always assume that f and F are twice continuouslydifferentiable. If$x^\star$is a singularpoint of f, the change of variables $x = x^\star + y$ leads to the equation

$$\begin{aligned} \dot{y} &= f(x^\star + y) \\ &= Ay + g(y), \end{aligned} \qquad (2.23)$$

where we have introduced the Jacobian matrix

$$A = \frac{\partial f}{\partial x}(x^\star) \qquad (2.24)$$

Taylor's formula implies that there exists a neighborhood$\mathcal{N}$of 0 and a constant $M > 0$such that

$$\|g(y)\| \leq M\|y\|^2 \qquad \forall y \in \mathcal{N} \qquad (2.25)$$

Similarly, the change of variables$x_k = x^\star + y_k$transforms an iterated map into

$$\begin{aligned} y_{k+1} &= F(x^\star + y_k) - x^\star \\ &= By_k + G(y_k), \end{aligned} \qquad (2.26)$$

where

$$B = \frac{\partial F}{\partial x}(x^\star), \quad \|G(y)\| \leq M\|y\|^2 \quad \forall y \in \mathcal{N}. \qquad (2.27)$$

Now we will discuss thelinear case.

Let us start by analysing theequations (2.23) and (2.26) in the linear case, that is withoutthe terms $g(x)$and $G(x)$. Consider first the ODE

$$\dot{y} = Ay. \qquad (2.28)$$

The solution can be written as

$$y(t) = e^{At}y(0), \qquad (2.29)$$

where the exponential of$A$ is defined by the absolutely convergent series

$$e^{At} = \exp(At) := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k. \qquad (2.30)$$

In order to understand the behaviour of e^{At}, let us recall some facts from linear algebra.We can write the characteristic polynomial of A as

$$c_A(\lambda) = \det(\lambda \mathbb{I} - A) = \prod_{j=1}^{m} (\lambda - a_j)^{m_j}, \qquad (2.31)$$

where $a_1, \ldots, a_m \in \mathbb{C}$are distinct eigenvalues of A, and $m_j$are theiralgebraic multiplicities.

The geometric multiplicity$g_j$of $a_j$is defined as the number of independenteigenvectors associated with a_j, and satisfies $1 \leq g_j \leq m_j$.

The results on decomposition of matrices leading to the Jordan canonical form can beformulated as follows. The matrix A can be decomposed as

$$A = S + N, \quad SN = NS. \qquad (2.32)$$

Here S, the semi simple part, can be written as

$$S = \sum_{j=1}^{m} a_j P^j, \qquad (2.33)$$

where the $P_j$are projectors on the eigenspaces of A, satisfying

$P_j P_k = \delta_{jk}, \sum_j P_j = \mathbb{I}$and $m_j = \dim(P_j \mathbb{R}^n)$. The nilpotent part$N$ can be written as

$$N = \sum_{j=1}^{m} N_j, \qquad (2.34)$$

where the $N_j$satisfy the relations

$$N_j^{m_j} = 0 \quad N_j N_k = 0 \quad \text{for } j \neq k \quad P_j N_k = N_k P_j = \delta_{jk} N_j \qquad (2.35)$$

In an appropriate basis, each $N_j$is block-diagonal, with $g_j$blocks of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \qquad (3.36)$$

In fact, $N_j = 0$ unless $g_j < m_j$.

Lemma (2.1.14):

With the above notations

$$e^{At} = \sum_{j=1}^{m} e^{a_j t} P_j \left(\mathbb{I} + N_j t + \cdots + \frac{1}{(m_j - 1)!} N_j^{m_j - 1} t^{m_j - 1} \right) \qquad (2.37)$$

Proof:

We use the fact that $e^{At}e^{Bt} = e^{(A+B)t}$whenever $AB = BA$, which can be checkedby a direct calculation. Then $e^{At} = e^{St}e^{Nt}$with

$$e^{St} = \prod_{j=1}^{m} e^{a_j P_j t} = \prod_{j=1}^{m} \left(\mathbb{I} + (e^{a_j t} - 1) P_j \right) = \mathbb{I} + \sum_{j=1}^{m} (e^{a_j t} - 1) P_j = \sum_{j=1}^{m} e^{a_j t} P_j,$$

$$e^{Nt} = \prod_{j=1}^{m} e^{N_j t} = 1 + \sum_{j=1}^{m} (e^{N_j t} - \mathbb{I}).$$

The result follows from the facts that $P_j(e^{N_j t} - \mathbb{I}) = 0$for $j \neq k$, and that$e^{N_j t}$containsonly finitely many terms, being nilpotent.

The expression (2.27) shows that the long-time behaviour isdetermined by the realparts of the eigenvalues a_j, while the nilpotent terms, when present, influence the shorttime behaviour. This motivates thefollowing terminology:

Definition (2.1.15):

The unstable, stable and center subspace of the singular point$x^{\star}$are

defined, respectively, by

$$E^{+} := P^{+}\mathbb{R}^{n} = \left\{ y : \lim_{t \to -\infty} e^{At} y = 0 \right\}, \qquad P^{+} := \sum_{j : Re a_j > 0} P_j ,$$

$$E^{-} := P^{-}\mathbb{R}^{n} = \left\{ y : \lim_{t \to +\infty} e^{At} y = 0 \right\}, \qquad P^{-} := \sum_{j : Re a_j < 0} P_j , \qquad (2.38)$$

$$E^{0} := P^{0}\mathbb{R}^{n} , \qquad P^{0} := \sum_{j : Re a_j = 0} P_j .$$

The subspaces are invariant subspaces ofe^{At}, that is,
$e^{At}E^{+} \subset E^{+}, e^{At}E^{-} \subset E^{-}ande^{At}E^{0} \subset E^{0}$. The fixed point is called
i.asink if$E^{+} = E^{0} = \{0\}$,
ii.asource if$E^{-} = E^{0} = \{0\}$,
iii.ahyperbolic point if$E^{0} = \{0\}$,
iv.anelliptic point if$E^{+} = E^{-} = \{0\}$.

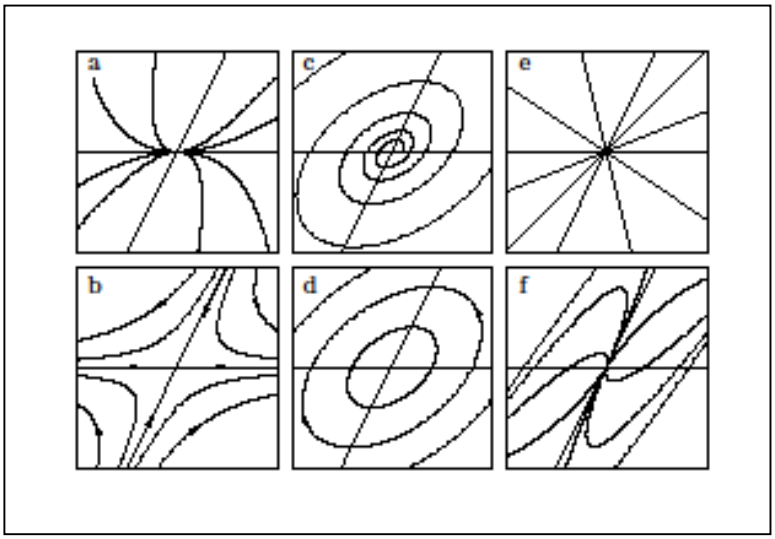

Figure 2.1

Phase portraits of a linear two-dimensional system: (a) node, (b) saddle, (c) focus, (d) center, (e) degenerate node, (f) improper node.

Example (2.1.16):

Letn $= 2$, and let A be in Jordan canonical form, with $\det A \neq 0$. Then we can distinguish between the following behaviours, depending on the eigenvalues $a_1, a_2$of A (see Fig.(2.1)).

1. $a_1 \neq a_2$

(a) If $a_1, a_2 \in \mathbb{R}$, then $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ and

$$e^{At} = \begin{pmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{pmatrix} \Rightarrow \quad \begin{matrix} y_1(t) = e^{a_1 t} y_1(0) \\ y_2(t) = e^{a_2 t} y_2(0) \end{matrix}$$

The orbits are curves of the form $y_2 = c y_1^{a_2/a_1} \cdot x^\star$is called a nodeif$a_1 a_2 > 0$, anda saddleif $a_1 a_2 < 0$.

(b) If $a_1 = \overline{a_2} = a + i\omega \in \mathbb{C}$, then the real canonical form of A is $A = \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix}$ and

$$e^{At} = e^{at} \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} y_1(t) = e^{at}(y_1(0) \cos \omega t - y_2(0) \sin \omega t) \\ y_2(t) = e^{at}(y_1(0) \sin \omega t + y_2(0) \cos \omega t) \end{matrix}$$

$x^\star$is called a focusif $a \neq 0$, and a centerif $a = 0$. The orbits are spirals orellipses.

2. $a_1 = a_2 =: a$

(a) If a has geometric multiplicity 2, then $A = a\mathbb{1}$ and $e^{At} = e^{at}\mathbb{1}$; $x^\star$is called a degenerate node.

(b) If $a$has geometric multiplicity 1, then$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ and

$$e^{At} = e^{at}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \Rightarrow \quad \begin{aligned} y_1(t) &= e^{at}(y_1(0) + y_2(0)t) \\ y_2(t) &= e^{at}y_2(0) \end{aligned}$$

$x^\star$is called an improper node.

Let us now turn to the case of the linear iterated map

$$y_{k+1} = By_k \qquad (2.39)$$

which admits the solution

$$y_k = B^k y_0. \qquad (2.40)$$

Using a similar decomposition $B = S + N$ into the semi simple and nilpotent part.

Lemma (2.1.17):

Let $b_i$be the eigenvalues of B, andP_i, $N_i$the associated projectors andnilpotent matrices. Then

$$B^k = \sum_{i=1}^{m} P_i \sum_{j=0}^{\min\{k, m_i - 1\}} \binom{k}{j} b_i^{k-j} N_i^j. \qquad (2.41)$$

Proof:

The main point is to observe that

$$B^k = \left(\sum_{i=1}^{m} (b_i P_i + N_i)\right)^k = \sum_{i=1}^{m} (b_i P_i + N_i)^k$$

because all cross-terms vanish. Then one applies the binomial formula.

For large k, the behaviour of $B^k$is dictated by the terms $b_i^{k-m_i+1}$.

This leads to thefollowing equivalent of Definition (2.1.15):

Definition (2.1.18):

The unstable, stable and center subspaces of the fixed point $x^\star$are defined, respectively, by

$$E^+ := P^+\mathbb{R}^n = \left\{y: \lim_{t\to-\infty} B^k y = 0\right\}, \quad P^+ := \sum_{j:|b_j|>1} P_j,$$

$$E^- := P^-\mathbb{R}^n = \left\{y: \lim_{t\to+\infty} B^k y = 0\right\}, \quad P^- := \sum_{j:|b_j|<1} P_j, \tag{2.42}$$

$$E^0 := P^0\mathbb{R}^n, \quad P^0 := \sum_{j:|b_j|=1} P_j.$$

These subspaces are invariant under B. The remaining terminology on sinks, sources,hyperbolic and elliptic points is unchanged.

Now we will discuss stability and Liapunov functions.

Definition (2.1.19):

Let x*be an equilibrium point of the system $\dot{x} = f(x)$:

1. $x^\star$is called stable if for any $\varepsilon > 0$, one can find a $\delta_0 = \delta(\varepsilon) > 0$such that whenever $\|x_0 - x^\star\| < \delta$, one has $\|\varphi_t(x_0) - x^\star\| < \varepsilon$ for all$t \geq 0$.
2. $x^\star$ is called asymptotically stable if it is stable, and there is a$\delta_0 > 0$ such that$\lim_{t\to\infty} \varphi_t(x_0) = x^\star$ for allx_0 such that $\|x_0 - x^\star\| < \delta_0$.
3. The basin of attraction of an asymptotically stable equilibrium $x^\star$is the set
$$\left\{x \in \mathcal{D}: \lim_{t\to\infty} \varphi_t(x_0) = x^\star\right\}. \tag{2.43}$$
4. $x^\star$is called unstable if it is not stable.

If $x^\star$isa fixed point of the map F, similar definitions hold with $\varphi_t(\cdot)$ replaced by $F^k(\cdot)$.

The linearization of the system $\dot{x} = f(x)$ around an equilibrium $x^\star$is the equation$\dot{y} = Ay$ with $A = \frac{\partial f}{\partial x} x^\star$. $x^\star$is called linearly stableif $y = 0$ is a stable equilibrium of itslinearization, and similarly in the asymptotically stable and unstable cases. Lemma (2.1.14) shows that

1. $x^\star$islinearly asymptotically stable if and only if all eigenvalues of A have a strictly negative real part;
2. $x^\star$is linearly stable if and only if no eigenvalue of A has positive real part, and all purely imaginary eigenvalues have equal algebraic and geometric multiplicities.

The problem is now to determine relations between linear and nonlinear stability. A usefulmethod to do this is due to Liapunov. Here we will limit the

discussion to differentialequations, although similar results can be obtained for maps.

Theorem (2.1.20): (Liapunov)

Let $x^\star$be a singular point of f, let $\mathcal{U}$be a neighborhoodof $x^\star$and set$\mathcal{U}_0 := \mathcal{U}\backslash\{x^\star\}$. Assume there exists a continuous function$V : \mathcal{U} \to \mathbb{R}$,continuously differentiable on $\mathcal{U}_0$, such that

1. $V(x) > V(x^\star)$ for all $x \in \mathcal{U}_0$;
2. The derivative of V along orbits is negative in $\mathcal{U}_0$, that is,

$$\dot{V}(x) := \frac{d}{dt}V\big(\varphi(x)\big)\Big|_{t=0} = \nabla V(x) \cdot f(x) \leq 0 \quad \forall x \in \mathcal{U}_0 \qquad (2.44)$$

Then $x^\star$is stable. If, furthermore,

3. The derivative of V along orbits is strictly negative,

$$\dot{V}(x) = \nabla V(x) \cdot f(x) < 0 \quad \forall x \in \mathcal{U}_0 \qquad (2.45)$$

then $x^\star$is asymptotically stable.

Proof:

Pick $\varepsilon > 0$ small enough that the closed ball $\bar{B}(x^\star, \varepsilon)$ with center $x^\star$and radiusε, is contained in $\mathcal{U}$. Let $S = \partial\bar{B}(x^\star, \varepsilon)$ be the sphere of radiusε centered in $x^\star$. Sbeingcompact, V admits a minimum on S, that we call β. Consider the open set

$$\mathcal{W} = \{x \in \bar{B}(x^\star, \varepsilon): V(x) < \beta\}.$$

$x^\star \in \mathcal{W}$by Condition1., and thus there exists $\delta > 0$ such that the open ball$B(x^\star, \delta)$ iscontained in $\mathcal{W}$. For any $x_0 \in B(x^\star, \delta)$, we have $V(\varphi_t(x_0)) < \beta$ for all $t \geq 0$, and thus$\varphi_t(x_0) \in \mathcal{W}$for all $t \geq 0$ by Condition 2., which proves that $x^\star$is stable.

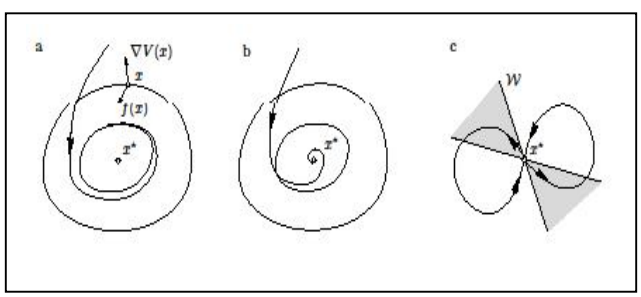

Figure 2.2

(a)Stable fixed point with level curves of a Liapunov function. (b) Asymptotically stable fixed point, here the vector field must cross all level

curves in the same direction. (c) Example of an unstable fixed point with the set W of ˇCetaev's Theorem.

Assume now that (2.45) holds. Since the positive orbit of $x_0 \in \mathcal{W}$is boundedit admits a convergent subsequence

$(x_n)_{n\geq 0} = \left(\varphi_{t_n}(x_0)\right)_{n\geq 0} \to y^\star \in \overline{\mathcal{W}}, t_n \to \infty$. Considerthe function $t \longmapsto V\left(\varphi_t(x_0)\right)$. It is continuouslydifferentiable, monotonously decreasing,and admits a subsequence converging to $V(y^\star)$. Thus $V\left(\varphi_t(x_0)\right)$ must converge to $V(y^\star)$as $t \to \infty$. Let $\delta > 0$ be a small constant and define the compact set

$$\kappa = \{x \in \overline{\mathcal{W}}: V(y^\star) \leq V(x) \leq V(y^\star) + \delta\}.$$

If $y^\star \neq x^\star$, then $x^\star \notin \kappa$, and thus the maximum of $\dot{V}$ on $\kappa$is a strictlynegative constant$c$. Take n large enough that $x_n \in \kappa$. Then $\varphi_t(x_n) \in \kappa$for all $t > 0$. But this implies that$V\left(\varphi_t(x_n)\right) \leq V(x_n) + ct$ which becomes smaller than $V(y^\star)$ for t large enough, which isimpossible. Thus $y^\star = x^\star$, and all orbits starting in $\mathcal{W}$converge to $x^\star$.

The interpretation of (2.44) is that the vector field crosses all level sets of V in thesame direction (Fig. (2.2)). V is called a Liapunov function for $x^\star$, and a strict Liapunovfunctionif (2.45) holds. In fact, the proof also shows that if V is a strict Liapunovfunction on $\mathcal{U}$, and $\mathcal{W}$is a set of the form $\mathcal{W} = \{x: V(x) < \beta\}$contained in $\mathcal{U}$, then $\mathcal{W}$is contained in the basin of attraction of $x^\star$. Thus Liapunov functions can be used toestimate suchbasins of attraction.

The advantage of the Liapunov method is that one does not need tosolve the differentialequation. However, the method is not constructive, and the form of V has to beguessed in each case. In the linearly asymptotically stable case, such a V can always beconstructed, which leads to the following result.

Corollary (2.1.21):

Assume $x^\star$is a linearly asymptotically stable equilibrium, that is, alleigenvalues of $A = \frac{\partial f}{\partial x}(x^\star)$have strictly negative real parts. Then $x^\star$is asymptoticallystable.

proof:

There are many different constructions of strict Liapunov functions. We will giveone of them. In orderto satisfy Condition 1. of theorem(2.1.20), we will look for a quadraticform

$$V(x) = (x - x^\star) \cdot Q(x - x^\star)$$

where Q is a symmetric, positive definite matrix. By assumption, there is a constant$a_0 > 0$ such that $Re\ a_i \leq -a_0$for all eigenvalues $a_i$of A. ThusLemma (2.1.14) implies that

$$\|e^{At}y\| \leq p(t)e^{-a_0 t}\|y\| \quad \forall y,$$

where p is a polynomial of degree less than n. This implies that the function

$$V(x) = \int_0^\infty \|e^{As}(x - x^\star)\|^2\, ds$$

exists. $V(x)$ is of the above form with

$$Q = \int_0^\infty e^{A^T s} e^{As}\, ds.$$

Q is clearly symmetric, positive definite and bounded, thus there is a $K > 0$ such that$\|Qy\| \leq K\|y\|$for all y. Now we calculate the followingexpression in two different ways:

$$\int_0^\infty \frac{d}{ds}\left(e^{A^T s} e^{As}\right) ds = \lim_{t\to\infty} e^{A^T t} e^{At} - \mathbb{I} = -\mathbb{I}$$

$$\int_0^\infty \frac{d}{ds}\left(e^{A^T s} e^{As}\right) ds = \int_0^\infty \left(A^T e^{A^T s} e^{As} + e^{A^T s} e^{As} A\right) ds = A^T Q + QA.$$

We have *thus* proved that

$$A^T Q + QA = -\mathbb{I}.$$

Now if $y(t) = \varphi_t(x) - x^\star$, we have

$$\dot{V} = \frac{d}{dt} V(\varphi_t(x)) = \frac{d}{dt}\left(y(t) \cdot Qy(t)\right) = \dot{y}(t) \cdot Qy(t) + y(t) \cdot Q\dot{y}(t).$$

Inserting $\dot{y} = Ay + g(y)$ produces two terms. The first one is

$$Ay \cdot Qy + y \cdot QAy = y \cdot (A^T Q + QA)y = -\|y\|^2,$$

and the second one gives

$$g(y) \cdot Qy + y \cdot Qg(y) = 2g(y) \cdot Qy \leq 2\|g(y)\|\|Qy\| \leq 2MK\|y\|^3.$$

Hence $\dot{V} \leq -\|y\|^2 + 2MK\|y\|^3$, which shows that V is a strict Liapunov function for$\|x - x^\star\| < \frac{1}{2MK}$, and the result is proved.

There exists a characterization of unstable equilibria based on similar ideas:

Theorem (2.1.22): (Četaev)

Let x*be a singular point of f, $\mathcal{U}$a neighborhood of $x^\star$and

$\mathcal{U}_0 = \mathcal{U}\backslash\{x^\star\}$. Assume there exists an open set $\mathcal{W}$, containing $x^\star$in its closure,anda continuous function $V: \mathcal{U} \to \mathbb{R}$, which is continuously differentiable on$\mathcal{U}_0$ andsatisfies

1. $V(x) > 0$ for all $x \in \mathcal{U}_0 \cap W$;
2. $\dot{V}(x) > 0$ for all $x \in \mathcal{U}_0 0 \cap W$;
3. $V(x) = 0$for all $x \in \mathcal{U}_0 \cap \partial W$.

Then $x^\star$is unstable.

Proof:

First observe that the definition of an unstable point can be stated asfollows: there exists $\varepsilon > 0$ such that, for any $\delta > 0$, one can find x_0 with$\|x_0 - x^\star\| < \delta$and $T > 0$such that$\|\varphi_T(x_0) - x^\star\| \geq \varepsilon$.

Now take $\varepsilon > 0$ sufficiently small that $B(x^\star, \varepsilon) \subset \mathcal{U}$. For any

$\delta > 0, B(x^\star, \delta) \cap \mathcal{W} \neq \phi$.We can thus take an $x_0 \in \mathcal{U}_0 \cap \mathcal{W}$such that$\|x_0 - x^\star\| < \delta$, and by Condition 1. $V(x_0) > 0$.Now assume by contradiction that$\|\varphi_t(x_0) - x^\star\| < \delta$ for all $t \geq 0$. $\varphi_t(x_0)$ must stay in

$\mathcal{U}_0 \cap \mathcal{W}$for all t, because it cannot reach the boundary of $\mathcal{W}$where $V = 0$. Thus thereexists a sequence $x_n = \varphi_{t_n}(x_0)$ converging to some

$x_1 \in \mathcal{U} \cup \mathcal{W}$. But this contradictsthe fact that $\dot{V}(x_1) > 0$, as in the proof of Theorem(2.1.20), and thus$\|\varphi_t(x_0) - x^\star\|$mustbecome larger than ε.

Corollary (2.1.23):

Assume $x^\star$is an equilibrium point such that $A = \frac{\partial f}{\partial x}(x^\star)$has at least

one eigenvalue with positive real part. Then $x^\star$is unstable.

Proof:

Consider first the case of A having no purely imaginary eigenvalues. We canchoose a coordinate system along the unstable and stable subspacesof $x^\star$, in which thedynamics is described by the equation

$$\dot{y}_+ = A_+ y_+ + g_+(y)$$
$$\dot{y}_- = A_- y_- + g_-(y),$$

where all eigenvalues of A_+ have a strictly positive real part, all eigenvalues of A_-havea strictly negative real part, $y = (y_+, y_-)$ and the terms$g_\pm$arebounded in norm by apositive constant M times$\|y\|^2$. We can define the matrices

$$Q_- = \int_0^\infty e^{A_-^T s} e^{A_- s}\, ds, \quad Q_+ = \int_0^\infty e^{-A_+^T s} e^{-A_+ s}\, ds,$$

which are bounded, symmetric, positive definite, and satisfy

$A_-^T Q_- + A_-^T Q_- = -\mathbb{I}andA_+^T Q_+ + A_+^T Q_+ = \mathbb{I}$. Define the quadratic form

$$V(y) = y_+ \cdot Q_+ y_+ - y_- \cdot Q_- y_-.$$

The cone $\mathcal{W} = \{y : V(y) > 0\}$is non-empty, because it contains aneigenvector of Acorresponding to an eigenvalue with positive real part. Proceeding similarly as in theproof of Corollary (2.1.21), we find

$$\dot{V}(y) = \|y_+\|^2 + 2g_+(y) \cdot Q_+ y_+ + \|y_-\|^2 - 2g_-(y) \cdot Q_- y_-$$
$$\geq \|y\|^2 - 2KM\|y\|^3.$$

Thus Četaev's theorem can be applied to show that $x^\star$is unstable. If A also has purelyimaginary eigenvalues, we obtain the additional equation

$$\dot{y}_0 = A_0 y_0 + g_0(y),$$

where all eigenvalues of A_0 are purely imaginary. Let S be an invertiblecomplex matrixof the same dimension as A_0 and consider the function

$$V(y) = y_+ \cdot Q_+ y_+ - y_- \cdot Q_- y_- - \|S y_0\|^2,$$

where $u \cdot v = \sum \bar{u}_i v_i$for complex vectors u, v. Proceeding as above, we obtain that

$$\dot{V}(y) \geq \|y_+\|^2 + \|y_-\|^2 - 2KM\|y\|^2(\|y_+\| + \|y_-\|) - 2Re(S y_0 \cdot A_0 S y_0)$$
$$- 2Re(S g_0(y) \cdot S y_0).$$

We shall prove below that for any $\varepsilon > 0$, one can construct a matrix $S(\varepsilon)$ such that$Re(S y_0 \cdot A_0 S y_0) \leq \varepsilon \|S y_0\|^2$. We now take $\mathcal{U} = \{y : \|y\| < \delta\}$, where $\delta > 0$ has to bedetermined, and $\mathcal{W} = \{y : V(y) > 0\}$. If $y \in \mathcal{W}$, we have

$$Re(S y_0 \cdot A_0 S y_0) \leq \varepsilon \|S y_0\|^2 < \varepsilon y_+ \cdot Q_+ Y_+ < \varepsilon K \|y_+\|^2.$$

We introduce the constants

$$C(\varepsilon) = \sup_{y_0 \neq 0} \frac{\|Sy_0\|}{\|y_0\|}, \quad c(\varepsilon) = \sup_{y_0 \neq 0} \frac{\|y_0\|}{\|Sy_0\|}.$$

Then we have

$$Re(Sg_0(y) \cdot Sy_0) \leq C\|g_0(y)\|\|Sy_0\| \leq CMK^{1/2}\|y\|^2\|y_+\|$$
$$\|y\|^2 \leq \|y_+\|^2 + \|y_-\|^2 + c^2\|Sy_0\|^2 < (1 + c^2K)(\|y_+\|^2 + \|y_-\|^2).$$

Putting everything together, we obtain for all $y \in \mathcal{U} \cap \mathcal{W}$

$$\dot{V}(y) > (1 - \varepsilon K)(\|y_+\|^2 + \|y_-\|^2) - 2KM\|y\|^2\left[\left(1 + CK^{-1/2}\right)\|y_+\| + \|y_-\|\right]$$
$$> (\|y_+\|^2 + \|y_-\|^2)\left[1 - \varepsilon K - 2KM(1 + c(\varepsilon)^2K)\left(2 + C(\varepsilon)K^{-1/2}\right)\delta\right].$$

Taking $\varepsilon < 1/K^2$ and then δ small enough, we can guarantee that this quantity is positivefor all $y \in \mathcal{U}_0 \cap \mathcal{W}$, and the corollary is proved.

In the proof we have used the following result.

Lemma (2.1. 24):

Assume all the eigenvalues of the $m \times m$ matrix A_0 are purely imaginary.For every $\varepsilon > 0$, there exists a complex invertible matrix S such that

$$|Re(Sz \cdot SA_0z)| \leq \varepsilon\|Sz\|^2 \quad \forall z \in \mathbb{C}^m. \qquad (2.46)$$

Proof:

Let S_0 be such that $B = S_0A_0S_0^{-1}$is in complex Jordan canonical form, thatis, the diagonal elements $b_{jj} = i\lambda_j$of B are purely imaginary,$b_{jj+1} = \sigma_j$is either zeroor one, and all other elements of B are zero. Let S_1 be the diagonal matrix with entries$1, \varepsilon^{-1}, \dots, \varepsilon^{1-m}$. Then

$$C := S_1BS_1^{-1} = \begin{pmatrix} \mathrm{i}\lambda_1 & \varepsilon\sigma_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon\sigma_{m-1} \\ 0 & & & \mathrm{i}\lambda_m \end{pmatrix}$$

Let $S = S_1S_0$ and $u = Sz$. Then

$$\left|Re\frac{Sz \cdot SA_0z}{\|Sz\|^2}\right| = \left|Re\frac{u \cdot Cu}{\|u\|^2}\right| = \left|Re\frac{\sum \mathrm{i}\lambda_j|u_j|^2 + \varepsilon\sum \sigma_j\bar{u}_ju_{j+1}}{\sum|u_j|^2}\right| \leq \varepsilon\frac{\sum|u_j||u_{j+1}|}{\sum|u_j|^2}$$
$$\leq \varepsilon,$$

where we used the fact that the upper sum has $m - 1$ terms and the lower one m terms.

The two corollaries of this section can be stated as follows: if $x^\star$is a hyperbolicequilibrium, then it has the same type of stability as the linearized

system. The sameproperties are valid for maps (see Table (2.1)). This is in general not true for non-hyperbolicequilibria. This situation will be studied in more detail in Chapter (3).

	ODE $\dot{x} = f(x)$	Map$x_{k+1} = F(x_k)$
Conservative	$\nabla \cdot f = 0$	$\left\lvert \det \frac{\partial F}{\partial x} \right\rvert = 1$
Dissipative	$\nabla \cdot f < 0$	$\left\lvert \det \frac{\partial F}{\partial x} \right\rvert < 1$
Equilibrium	$f(x^\star) = 0$	$F(x^\star) = x^\star$
Asympt. Stable if Unstable if	$Re\ a_i < 0\ \ \forall i$ $\exists i : Re\ a_i > 0$	$\lvert b_i \rvert < 1\ \ \forall i$ $\exists i : \lvert b_i \rvert > 1$

Table (2.1). Comparison of some properties of ordinary differentialequations and iteratedmaps. Here a_i and b_i are eigenvalues of the matrices$A = \frac{\partial f}{\partial x}(x^\star)$ and $B = \frac{\partial F}{\partial x}(x^\star)$.

Now we will discuss theinvariant manifolds.

For hyperbolic equilibrium points, the analogies between nonlinear and linear systems canbe pushed further.One of them has to do with invariant manifolds, which generalize theinvariant subspaces of the linear case. We assume in this illustration that f is of class $C^r$with$r > 2$.

Definition (2.1.25):

Let $x^\star$be a singular point of the system $\dot{x} = f(x)$, and let $\mathcal{U}$be aneighborhood of $x^\star$. The local stable and unstable manifolds of $x^\star$in $\mathcal{U}$are defined, respectively, by

$$W^s_{\text{loc}}(x^\star) := \left\{x \in \mathcal{U}: \lim_{t\to\infty} \varphi_t(x) \text{and} \varphi_t(x) \in \mathcal{U}\ \forall t \geq 0\right\}$$
$$W^{\text{u}}_{\text{loc}}(x^\star) := \left\{x \in \mathcal{U}: \lim_{t\to-\infty} \varphi_t(x) \text{and} \varphi_t(x) \in \mathcal{U}\ \forall t \leq 0\right\}. \quad (2.47)$$

The global stable and unstable manifolds of $x^\star$are defined by

$$W^s(x^\star) = \bigcup_{t\leq 0} \varphi_t\left(W^s_{\text{loc}}(x^\star)\right),$$
$$W^{\text{u}}(x^\star) = \bigcup_{t\geq 0} \varphi_t\left(W^{\text{u}}_{\text{loc}}(x^\star)\right). \quad (2.48)$$

Similar definitions can be made for maps. Global invariant manifolds can have avery complicated structure, and may return infinitely often to aneighborhood of theequilibrium point. This is why one prefers to defineseparately local and global invariantmanifolds. The following theoremstates that local invariant manifolds have a nicestructure.

Theorem (2.1.26): (Stable manifold theorem)

Let $x^\star$be a hyperbolic equilibrium pointof the system $\dot{x} = f(x)$, such that the matrix$\frac{\partial f}{\partial x}(x^\star)$has n_+ eigenvalues with positive realparts and n_-eigenvalues with negative real parts, with $n_+, n_- > 1$. Then $x^\star$admits, in a

neighborhood$\mathcal{U}$,

i. a local stable manifold $W^s_{\text{loc}}(x^\star)$, which is a differentiable manifold of class $C^r$and dimension n_-, tangent to the stable subspace E^-at $x^\star$, and which can be represented as a graph;
ii. a local unstable manifold $W^{\text{u}}_{\text{loc}}(x^\star)$, which is a differentiable manifold of class $C^r$and dimension n_+, tangent to the unstable subspace E^+at $x^\star$, and which can be represented as a graph.

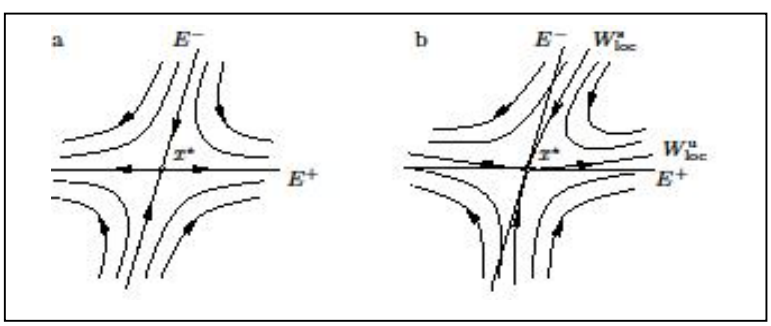

Figure 2.3

Orbits near a hyperbolic fixed point: (a) orbits of the linear zed system, (b)orbits of the nonlinear system with local stable and unstable manifolds.

We will omit the proof of this result, but we will give in Chapter(3) the proof of the centermanifold theorem, which relies on similar ideas.Let us now explain a bit more preciselywhat this result means. Thegeometric interpretation is shown in Fig. (2.3). To explainthe meaning of "a differentiable manifold of class C^rtangent to $E^{\pm}$and representableas agraph", let us introduce a coordinate system along the invariant subspaces of thelinearization. The vector field near $x^{\star}$can be written as

$$\begin{aligned} \dot{y}_+ &= A_+ y_+ + g_+(y_+, y_-) \\ \dot{y}_- &= A_- y_- + g_-(y_+, y_-). \end{aligned} \qquad (2.49)$$

where A_+ is a $n_+ \times n_+$ matrix, which has only eigenvalues with positive realparts, and$A_-$is a $n_- \times n_-$matrix, which has only eigenvalues with negative real parts.The terms$g_{\pm}$are nonlinear and satisfy$\|g_{\pm}(y_+, y_-)\| \leq M\|y\|^2in\mathcal{U}$, where M is a positive constant.The theorem implies the existence of a function of class C^r

$$h^{\mathrm{u}}: \mathcal{U}_+ \to \mathbb{R}^n, \quad h^{\mathrm{u}}(0) = 0, \quad \frac{\partial h^{\mathrm{u}}}{\partial y_+}(0) = 0, \qquad (2.50)$$

where $\mathcal{U}_+$ is a neighbourhood of the origin in $\mathbb{R}^{n_+}$, such that the local unstable manifoldis given by the equation

$$y_- = h^{\mathrm{u}}(y_+). \qquad (2.51)$$

Similar relations hold for the stable manifold.

In order to determine the functionh^{u}, let us compute $\dot{y}_-$in two ways, for a given orbiton the unstable manifold:

$$\dot{y}_- = A_- h^{\mathrm{u}}(y_+) + g_-\big(y_+, h^{\mathrm{u}}(y_+)\big)$$

$$\dot{y}_- = \frac{\partial h^{\mathrm{u}}}{\partial y_+}(y_+)\dot{y}_+ = \frac{\partial h^{\mathrm{u}}}{\partial y_+}(y_+)\big[A_+ y_+ + g_+\big(y_+, h^{\mathrm{u}}(y_+)\big)\big]. \qquad (2.52)$$

Since both expressions must be equal, we obtain that h^{u} must satisfy the partial differentialequation

$$\begin{aligned} &A_- h^{\mathrm{u}}(y_+) + g_-\big(y_+, h^{\mathrm{u}}(y_+)\big) \\ &\qquad = \frac{\partial h^{\mathrm{u}}}{\partial y_+}(y_+)\big[A_+ y_+ + g_+\big(y_+, h^{\mathrm{u}}(y_+)\big)\big]. \end{aligned} \qquad (2.53)$$

This equation is difficult to solve in general. However, since we know by thetheorem thath^{u} is of class C^r, we can compute h^u perturbatively, by inserting its Taylor expansion into(2.53) and solving order by order.

Example (2.1.27):

Consider, for $n = 2$, the system

$$\dot{y}_1 = y_1$$
$$\dot{y}_2 = -y_2 + y_1^2. \qquad (2.54)$$

Then the equation (2.53) reduces to

$$-h^{\mathrm{u}}(y_1) + y_1^2 = h^{\mathrm{u}'}(y_1)y_1, \qquad (2.55)$$

which admits the solution

$$h^{\mathrm{u}}(y_1) = \frac{1}{3}y_1^2. \qquad (2.56)$$

This is confirmed by the explicit solution of (2.54),

$$y_1(t) = y_1(0)e^t$$
$$y_2(t) = \left(y_2(0) - \frac{1}{3}y_1(0)^2\right)e^{-t} + \frac{1}{3}y_1(0)^2e^{2t}. \qquad (2.57)$$

Now we will discuss thenormal forms.

In this part we will examinesome further connections between the flow near an equilibriumpoint and itslinearization. We will assume that $x^\star$is a singular point of the system

$\dot{x} = f(x)$, and that f is of class$C^r, r > 2$, in a neighbourhood of $x^\star$.

Definition (2.1.28):

Let $\mathcal{U}$and $\mathcal{W}$be open sets in $\mathbb{R}^n$, and let $f : \mathcal{U} \to \mathbb{R}^n$and

$g : \mathcal{W} \to \mathbb{R}^n$be two vector fields of class C^r. These vector fields are called

i. topologically equivalent if there exists a homeomorphism $h : \mathcal{U} \to \mathcal{W}$taking the orbitsof $\dot{x} = f(x)$ to the orbits of$\dot{y} = g(y)$andpreserving the sense of time;
ii. differentiably equivalent if there exists a diffeomorphism $h : \mathcal{U} \to \mathcal{W}$taking the orbitsof $\dot{x} = f(x)$ to the orbits of $\dot{y} = g(y)$ and preserving the sense of time.

If, in addition, h preserves parametrization of the orbits by time, the vector fields are calledconjugate.

Equivalence means that if $\varphi_t$and $\psi_t$are the flows of the two systems, then

$$\psi_t \circ h = h \circ \varphi_{\tau(t)} \qquad (2.58)$$

on $\mathcal{U}$, where τ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. If $\tau(t) = t$ for all t, the systems areconjugate.

Theorem (2.1.29): (Hartman-Grobman)

Let $x^\star$be a hyperbolic equilibrium point of$\dot{x} = f(x)$, that is, the matrix$A = \frac{\partial f}{\partial x}(x^\star)$has no eigenvalue with zero real part. Then, ina sufficiently small neighborhood of $x^\star$, f is topologically conjugate to the linearization$\dot{y} = Ay$.

Note, however, that topological equivalence is not a very strongproperty, since $h$need not be differentiable. In fact, one can show that all linear systems with the samenumber of eigenvalues with positive andnegative real parts are topologically equivalent. So for instance, the node andfocus in Fig. (2.1) are topologicallyequivalent. On the other hand,differentiable equivalence is harder to achieve as shows thefollowingexample.

Example (2.1.30)

Consider the following vector field and its linearization:

$$\dot{y}_1 = 2y_1 + y_2^2 \dot{Z}_1 = 2z_1$$
$$\dot{y}_2 = y_2 \dot{z}_2 = z_2. \qquad (2.59)$$

The orbits can be found by solving the differential equations

$$\frac{dy_1}{dy_2} = 2\frac{y_1}{y_2} + y_2, \quad \frac{dz_1}{dz_2} = 2\frac{z_1}{z_2}, \qquad (2.60)$$

which admit the solutions

$$y_1 = [c + \log|y_2|]y_2^2, \qquad z_2 = cz_2^2. \qquad (2.61)$$

Because of the logarithm, the two flows are C^1-but not C^2-conjugate.

The theory of normal forms allows to explain these phenomena, and to obtain conditionsfor the existence of such C^r-conjugacies. Consider the system for $y = x - x^\star$,

$$\dot{y} = Ay + g(y). \qquad (2.62)$$

We can try to simplify the nonlinear term by a change of coordinates$y = z + h(z)$, whichleads to

$$\dot{z} + \frac{\partial h}{\partial z}(z)\dot{z} = Az + Ah(z) + g\big(z + h(z)\big). \qquad (2.63)$$

Assume that $h(z)$ solves the partial differential equation

$$\frac{\partial h}{\partial z}(z)Az - Ah(z) = g\big(z + h(z)\big). \qquad (2.64)$$

Then we obtain for z the linear equation

$$\dot{z} = Az. \qquad (2.65)$$

Unfortunately, we do not know how to solve the equation (2.64) in general. One can,however, work with Taylor series. To this end, we rewrite thesystem (2.62) as

$$\dot{y} = Ay + g_2(y) + g_3(y) + \cdots + g_{r-1}(y) + \mathcal{O}(\|y\|^r). \qquad (2.66)$$

Here the last term is bounded in norm by a constant times$\|y\|^r$, and the terms $g_k(y)$ arehomogeneous polynomial maps of degree k from $\mathbb{R}^n$to $\mathbb{R}^n$($g_k(\lambda_y) = \lambda^k g_k(y)\ \forall\lambda \in \mathbb{R}$).Let us denote by $\mathcal{H}_k$the set of all such maps.$\mathcal{H}_k$is a vector space for the usual additionand multiplication by scalars. For instance, when $n = 2$, $\mathcal{H}_2$ admits the basis vectors

$$\begin{pmatrix} y_1^2 \\ 0 \end{pmatrix},\ \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix},\ \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix},\ \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix},\ \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix}. \qquad (2.67)$$

We now define a linear map from $\mathcal{H}_k$to itself given by

$$\mathrm{ad}_k A: \mathcal{H}_k \to \mathcal{H}_k$$

$$h(y) \longmapsto \frac{\partial h}{\partial y}(y)Ay - Ah(y). \qquad (2.68)$$

The fundamental result of normal form theory is the following:

Proposition (2.1.31):

For each k, $2 \le k < r$, choose a complementary space $\mathcal{G}_k$of theimage of $\mathrm{ad}_k A$, that is, such that $\mathcal{G}_k \oplus \mathrm{ad}_k A(\mathcal{H}_k) = \mathcal{H}_k$. Then there exists, in a neighborhoodof the origin, an analytic (polynomial) change of variables $y = z + h(z)$ transforming(2.66) into

$$\dot{z} = Az + g_2^{\mathrm{res}}(z) + g_3^{\mathrm{res}}(z) + \cdots + g_{r-1}^{\mathrm{res}}(z) + \mathcal{O}(\|r\|^2), \qquad (2.69)$$

where $g_k^{\mathrm{res}} \in \mathcal{G}_k, 2 \le k < r$.

Proof:

The proof proceeds by induction. Assume that for some k,

$2 \le k \le r - 1$, wehave obtained an equation of the form

$$\dot{y} = Ay + \sum_{j=2}^{k-1} g_j^{\mathrm{res}}(y) + g_k(y) + \mathcal{O}(\|y\|^{k+1}).$$

We decompose the term $g_k(y)$ into a resonant and a non-resonant part,

$$g_k(y) = g_k^{\mathrm{res}}(y) + g_k^0(y),\ \ g_k^0(y) \in \mathrm{ad}_k A(\mathcal{H}_k),\ \ g_k^{\mathrm{res}}(y) \in \mathcal{G}_k.$$

There exists $h_k \in \mathcal{H}_k$satisfying

$$\mathrm{ad}_k A\big(h_k(z)\big) := \frac{\partial h_k}{\partial z}(z)Az - Ah_k(z) = g_k^0(z).$$

Observe that$g_j^{\text{res}}\big(z+h_k(z)\big)=g_j^{\text{res}}(z)+\mathcal{O}(\|z\|^{k+1})$for all j, and a similar relation holdsfor g_k. Thus the change of variables $y=z+h(z)$ yields the equation

$$\dot{y}=\left[\mathbb{I}+\frac{\partial h_k}{\partial z}(z)\right]\dot{z}=Az+Ah_k(z)+\sum_{j=2}^{k-1}g_j^{\text{res}}(z)+g_k(z)+\mathcal{O}(\|z\|^{k+1})$$

$$=\left[\mathbb{I}+\frac{\partial h_k}{\partial z}(z)\right]Az+\sum_{j=2}^{k-1}g_j^{\text{res}}(z)+\mathcal{O}(\|z\|^{k+1}),$$

where we have used the definition of h to get the second line. Now for sufficiently small z,the matrix $\mathbb{I}+\frac{\partial h_k}{\partial z}$admits an inverse $\mathbb{I}+\mathcal{O}(\|z\|^{k-1})$. Multiplying the above identity onthe left by this inverse, we have proved the induction step.

The terms$g_k^{\text{res}}(z)$are called resonant and (2.69) is called the normal form of (2.66).The equation$a\mathrm{d}_kA\big(h_k(z)\big)=g_k^0(z)$that has to be satisfied to eliminate the non-resonantterms of order k is called the homologicalequation. Whether a term is resonant or not is aproblem of linear algebra,which depends only on the matrix A. While it can be difficultto determinethe coefficients of the resonant terms in a particular case, it is in generalquite easy to find which terms can be eliminated. In particular, the followingresult holds:

Lemma (2.1.32):

Let $(a_1,\dots,a_n)$ be the eigenvalues of A, counting multiplicity. Assumethat for each $j, 1\le j\le n$, we have

$$p_1a_1+\cdots+p_na_n\neq a_j \quad (2.70)$$

for all n-tuples of non-negative integers $(p_1,\dots,p_n)$ satisfying

$p_1+\cdots+p_n=k$. Then$a\mathrm{d}_kA$is invertible, and thus there are no resonant terms oforder k.

Proof:

We can assume that A is in Jordan canonical form. Consider first the case of adiagonal A. Let $(e_1,\dots,e_n)$ be the canonical basis of $\mathbb{R}^n$. For $\mathcal{H}_k$we choose the basisvectors$z_1^{p_1}\dots z_n^{p_n}e_j$, where $p_1+\cdots+p_n=k$. Then an explicit calculation shows that

$$a\mathrm{d}_kA\big(z_1^{p_1}\dots z_n^{p_n}e_j\big)=\big(p_1a_1+\cdots+p_na_n-a_j\big)z_1^{p_1}\dots z_n^{p_n}e_j.$$

By assumption, the term in brackets is different from zero. Thus the linear operator $a\mathrm{d}_k A$is diagonal in the chosen basis, with nonzero elements on the diagonal, which shows thatit is invertible.

Consider now the case of a matrix A that is not diagonal, but haselements of theform $a_{jj+1} = 1$. Then $a\mathrm{d}_k A$ applied to a basis vector will contain additional, off-diagonalterms. One of them is proportional to $z_1^{p_1} \dots z_n^{p_n} e_{j-1}$, while the others are of the form$z_{k+1}\partial_{zk}\left(z_1^{p_1} \dots z_n^{p_n}\right)e_j$. One can show that the basis vector of $\mathcal{H}_k$can be ordered in such away that $a\mathrm{d}_k A$is represented by a triangular matrix, with the same diagonal elements asin the case of a diagonal A, thus the conclusion is unchanged.

The non-resonance condition (2.70) is called a Diophantine condition,since it involvesinteger coefficients. Thus resonant terms can exist onlywhen the eigenvalues of A satisfy arelation of the form

$p_1 a_1 + \cdots + p_n a_n = a_j$, which is called a resonance of order$p_1 + \cdots + p_n$.In Example (2.1.30),the relation $2a_2 = a_1$ induces a resonance of order 2, which makes itimpossible to eliminate the term $y_2^2$by a polynomial change of coordinates.

In order to solve the question of differentiable equivalence, the reallydifficult problem isto eliminate the remainder $\mathcal{O}(\|z\|^r)$ in the normal form(2.69). This problem was solvedby Poincare' for sources and sinks, and bySternberg and Chen for general hyperbolic. We state here the main result without proof.

Theorem (2.1.33): (Poincare'-Sternberg-Chen)

Let A be a $n \times n$ matrix with no eigenvalueson the imaginary axis. Consider the two equations

$$\begin{aligned} \dot{x} &= Ax + b_1(x) \\ \dot{y} &= Ay + b_2(y). \end{aligned} \tag{2.71}$$

We assume that b_1 and b_2 are of class $C^r, r > 2$, in a neighbourhood of the origin, that$b_1(x) = \mathcal{O}(\|x\|^2), b_2(x) = \mathcal{O}(\|x\|^2)$, and

$b_1(x) - b_2(x) = \mathcal{O}(\|x\|^2)$. Then for every $k > 2$, there is an integer$N = N(k, n, A) \geq k$ such that, if $r \geq N$, there exists a map h of classC^ksuch that the

two systems can be transformed into one another by the transformation$x = y + h(y)$.

This result implies that for the system $\dot{y} = Ay + g(y)$ to be C^kconjugate to its linearization$\dot{z} = Az$, it must be sufficiently smooth and satisfy non-resonance conditionsup to sufficiently high order N, where this order and the conditions depend only on k, n and the eigenvalues of A. In the special case of all eigenvalues of A having real partswith the same sign (source or sink), Poincare' showed that $N = k$. For general hyperbolicequilibria, N can be much larger than k.

Another, even more important consequence, is that vector fields near singular pointscan be classified by their normal forms, each normal form being a representative of anequivalence class (with respect to C^kconjugacy). This property plays an important rolein bifurcation theory, as we shall see inChapter (3).

Let us finally remark that in the much more difficult case of non-hyperbolic equilibria,certain results on C^k-conjugacy have been obtained by Siegel, Moser, Takens and others[8].

Section (2.2): Periodic Solutions

Definition (2.2.1):

Let $p > 1$ be an integer. A periodic orbit of period p of the map F is

a set of points $\{x_1^\star, \dots, x_p^\star\}$such that

$$F(x_1^\star) = x_2^\star, \;\; F(x_{p-1}^\star) = x_p^\star, \quad F(x_p^\star) = x_1^\star. \qquad (2.72)$$

Each point of the orbit is called a periodic point of period$p$. Thus a periodicpoint $x^\star$ofperiod p is also a fixed point ofF^p. p is called the least period of$x^\star$if $F^j(x^\star) \neq x^\star$for$1 \leq j < p$.

To find periodic orbits of an iterated map, it is thus sufficient to findthe fixed points of$F^p, p = 1,2, \dots$. Unfortunately, this becomes usuallyextremely difficult with increasing$p$. Moreover, the number of periodic orbits of period $p$often grows very quickly with p.Methods that simplify the search for periodic orbits are known for special classes of maps.Forinstance, for two-dimensional conservative maps, there exists a variationalmethod:periodic orbits of period p correspond to stationary points of some function of $\mathbb{R}^p$to $\mathbb{R}$.

Once a periodic orbit has been found, the problem of its linearstability is rather easilysolved. Indeed, it is sufficient to find the eigenvalues of the matrix

$$\begin{aligned}\frac{\partial F^p}{\partial x}(x_1^\star) &= \frac{\partial F}{\partial x}\left(F^{p-1}(x_1^\star)\right)\frac{\partial F}{\partial x}\left(F^{p-2}(x_2^\star)\right)\dots\frac{\partial F}{\partial x}(x_1^\star)\\ &= \frac{\partial F}{\partial x}(x_p^\star)\frac{\partial F}{\partial x}(x_{p-1}^\star)\dots\frac{\partial F}{\partial x}(x_1^\star).\end{aligned} \tag{2.73}$$

Note that the result has to be invariant under cyclic permutations of thematrices.

The dynamics near any point of the periodic orbit can be inferred from the dynamicsnear one of them, considered as a fixed point of F^p. Thus periodic orbits can also beclassified into sinks, sources, hyperbolic and elliptic orbits, and the concepts of nonlinearstability, invariant manifolds and normal forms can be carried over from fixed points toperiodic orbits.

Now we will discuss the periodic orbits of flows and poincare' sections.

Definition (2.2.2):

Let f be a vector field, $\varphi_t$its flow and $T > 0$ a constant. A periodic

solution of period T of f is a function such that

$$\dot{\gamma}(t) = f\big(\gamma(t)\big)\text{and}\quad \gamma(t+T) = \gamma(t)\quad \forall t. \tag{2.74}$$

The corresponding closed curve $\Gamma = \{\gamma(t) : 0 \le t \le T\}$is called a periodic orbit of period$T$. Thus each point x of this orbit is a fixed point of φ_T. T is called the least period ofthe orbit if $\varphi_t(x) \neq x$ for $0 < t < T$.

Finding periodic orbits of differential equations is even more difficult than for maps.There exist methods which help to find periodic orbits in a number of particular cases,such as two-dimensional flows, or systemsadmitting constants of the motion and smallperturbations of them.

Let us now assume that we have found a periodic solution $\gamma(t)$. We would like todiscuss its stability. The difference $y(t) = x(t) - \gamma(t)$ between an arbitrary solution andthe periodic solution satisfies the equation

$$\dot{y} = f(\gamma(t) + y) - f\big(\gamma(t)\big). \tag{2.75}$$

If f is twice continuously differentiable and y is small, we may expand f into Taylor series,which yields

$$\dot{y} = A(t)y + g(y,t), \;\; A(t) = \frac{\partial f}{\partial x}\big(\gamma(t)\big), \;\; \|g(y,t)\| \le M\|y\|^2. \qquad (2.76)$$

Let us examine the linearization of this equation, given by

$$\dot{y} = A(t)y. \qquad (2.77)$$

Note that a similar equation already appeared in the proof of Proposition(2.1.10). Thisequation admits a unique global solution, which can be represented, because of linearity,as

$$y(t) = U(t)y(0). \qquad (2.78)$$

where $U(t)$ is an $n \times n$ matrix-valued function solving the equation

$$\dot{U}(t) = A(t)U(t), \;\; U(0) = \mathbb{1}. \quad (2.79)$$

The function $U(t)$ is called the principal solution of the equation (2.77). It should be clearthat the linear stability of Γ is related to the asymptotic behaviour of the eigenvalues of$U(t)$. Unfortunately, there is no general method to determine these eigenvalues. Note,however, that $A(t)$ is periodic in t, and in this case we can say more.

Theorem (2.2.3): (Floquet)

Let $A(t) = A(t+T)$ for all t. Then the principal solutionof $\dot{y} = A(t)y$ can be written as

$$U(t) = P(t)e^{Bt}, \qquad (2.80)$$

where $P(t+T) = P(t)$ for all $t, P(0) = \mathbb{1}$, and B is a constant matrix.

Proof:

The matrix $V(t) = U(t\ +T)$ satisfies the equation

$$\dot{V}(t) = \dot{U}(t+T) = A(t+T)U(t+T) = A(t)V(t),$$

which is the same as (2.79), except for the initial value $V(0) = U(T)$. Wealready sawin Proposition (2.1.10) that $\det U(t) \neq 0$ for all t. Thus the matrix $V(t)U(T)^{-1}$ exists andsatisfies (2.79), including the initialcondition. By uniqueness of the solution, it must beequal to $U(t)$:

$$V\ (t)U(T)^{-1} = \ U(t) \ \Rightarrow U(t+T) = U(t)U(T).$$

We claim that there exists a matrix B such that $(T) = e^{BT}$. To see this, let$\lambda_i \neq 0$and $m_i, i = 1, \dots, m$ be the eigenvalues of $U(T)$ and their algebraic multiplicities. Let$U(T) = \sum_{i=1}^{m}(\lambda_i P_i + N_i)$be the decomposition of $U(T)$ into its semi simple and nilpotentparts. Then, using Lemma (2.1.14), it is easy to check that

$$B = \frac{1}{\mathrm{T}} \sum_{i=1}^{m} \left(\log(\lambda_i)\, P_i - \sum_{j=1}^{m_i} \frac{(-N_i)^j}{j\lambda_i^j} \right)$$

satisfies $e^{BT} = U(T)$. Here the sum over j is simply the Taylor expansion of$\log(\mathbb{1} + N_i/\lambda_i)$ .B is unique up to the determination of the logarithms. Wenow define

$$P(t) = U(t)e^{-Bt}.$$

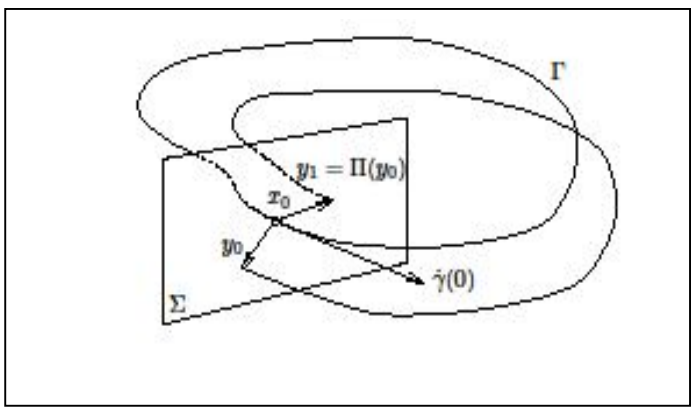

Figure 2.4
Definition of the Poincare' map associated with the periodicorbit$\gamma(t)$.

Then we have for all t

$$P(t+T) = U(t+T)e^{-B(t+T)} = U(t)e^{BT}e^{-B(t+T)} = P(t).$$

Finally, $P(0) = U(0) = \mathbb{1}$, which completes the proof.

Floquet's theorem shows that the solution of (2.77) can be written as

$$y(t) = P(t)\, e^{Bt} y(0). \qquad (2.81)$$

Since $P(t)$ is periodic, the long-time behaviour depends only on B. The eigenvalues of$B$ are called the characteristic exponents of the equation. $e^{Bt}$is called the monodromymatrix, and its eigenvalues, called thecharacteristic multipliers, are exponentials of thecharacteristic exponents times T. Computing the characteristic exponents is difficult ingeneral, but the existence of the

representation (2.81) is already useful to classify thepossible behaviours near a periodic orbit.

Once we have determined the linear stability of the periodic orbit, we could proceedin a similar way as in the case of a stationary point, in order to determine the nonlinearstability, the existence of invariant manifolds, and similar properties. However, Poincare' invented a remarkable method, which allows to shortcut all these steps by reducing theproblem to a simpler one,which has already been studied. Appropriately enough, thismethod is calledthe Poincare' section.

Definition (2.2.4):

Let $\Upsilon(t)$ be a periodic solution of period $T, x_0 = \gamma(0)$, and let $\sum$ be ahyper plane transverse to the orbit at x_0 (see Fig. (2.4)). By continuity of the flow, there isa neighborhood$\mathcal{U}$of x_0 in $\sum$ such that for all $x = x_0 + y \in \mathcal{U}$, we can define a continuousmap $\tau(y), \tau(0) = T$, such that $\varphi_t(x)$ returns for the first time to $\sum$ in a vicinity of x_0 at$t = \tau(y)$. The Poincare' map $\prod$ associated with the periodic orbit is defined by

$$x_0 + \prod(y) := \varphi_{\tau(y)}(x_0 + y). \qquad (2.82)$$

Proposition (2.2.5):

The Poincare' map is as smooth as the vector field in a neighbourhood

of the origin. The characteristic multipliers of the periodic orbit are given by1 and the$n - 1$ eigenvalues of the Jacobian matrix$\frac{\partial \prod}{\partial y}(0)$.

Proof:

The smoothness of $\prod$ follows directly from the smoothness of the flow and theimplicit function theorem. Let us now observe that

$$\frac{d}{dt}\dot{\gamma}(t) = \frac{d}{dt}f\big(\gamma(t)\big) = \frac{\partial f}{\partial x}\big(\gamma(t)\big)\dot{\gamma}(t) = A(t)\dot{\gamma}(t).$$

Thus by Floquet's theorem, we can write

$$\dot{\gamma}(t) = P(t)e^{Bt}\dot{\gamma}(0),$$

and, in particular,

$$\dot{\gamma}(0) = \dot{\gamma}(T) = P(T)e^{BT}\dot{\gamma}(0) = e^{BT}\dot{\gamma}(0).$$

This shows that $\dot{\gamma}(0)$ is an eigenvector of $e^{BT}$with eigenvalue 1. Let $(e_1, \dots, e_{n-1}, \dot{\gamma}(0))$,$e_1, \dots, e_{n-1} \in \sum$, be a basis of $\mathbb{R}^n$. In this basis,

$$e^{BT} = \begin{pmatrix} e^{B_\Sigma T} & 0 \\ \dots & 1 \end{pmatrix},$$

where B_Σ is the restriction of B to $\sum$, and the dots denote arbitrary entries. Now, if weconsider momentarily y as a vector in $\mathbb{R}^n$instead of $\sum$,linearization of (2.82) gives

$$\frac{\partial \prod}{\partial y}(0) = \frac{\partial}{\partial y}\left(\varphi_{\tau(y)}(x_0 + y)\right)\Big|_{y=0} = \frac{\partial \varphi_T}{\partial t}(x_0)\frac{\partial \tau}{\partial y}(0) + \frac{\partial \varphi_T}{\partial x}(x_0)$$
$$= \dot{\gamma}(0)\frac{\partial \tau}{\partial y}(0) + e^{BT}.$$

The first term is a matrix with zero entries except on the last line, so that$\partial_y \prod(0)$ has thesame representation as e^{BT}, save for the entries marked by dots. In particular, when y isrestricted to $\sum$,$\frac{\partial \prod}{\partial y}(0)$has the same eigenvalues as$e^{B_\Sigma T}$.

The consequence of this result is that, by studying the Poincare' map,we obtain acomplete characterization of the dynamics in a neighborhood of the periodic orbit. In particular, if $y = 0$, considered as a fixed point of the Poincare' map, admits invariantmanifolds, they can be interpreted as the intersection of $\sum$ and invariant manifolds of theperiodic orbit.

Chapter 3

Local Bifurcations

Section (3.1): Center Manifold

Up to now, we have obtained quite a precise picture of the dynamics near hyperbolicequilibrium solutions. One might wonder whether it is of any interest to examine thecase of nonhyperbolic equilibria, since a matrix chosen at random will have eigenvalues onthe imaginary axis with probability zero. This argument no longer works, however, if thedynamical system depends on a parameter:

$$\dot{x} = f(x, \lambda) \text{or} x_{k+1} = F(x_k, \lambda), \qquad (3.1)$$

where $\lambda \in R$ (or $\mathbb{R}^p$). By changing λ, it is quite possible to encounter nonhyperbolicequilibria.

An important result in this connection is the implicit function theorem:

Theorem (3.1.1):

Let $\mathcal{N}$be a neighborhood of $(x^\star, y^\star)$ in $\mathbb{R}^n \times \mathbb{R}^m$. Let $f: \mathbb{N} \to \mathbb{R}^n$beof class $C^r, r \geq 1$, and satisfy

$$f(x^\star, y^\star) = 0, \qquad (3.2)$$

$$\det \frac{\partial f}{\partial x}(x^\star, y^\star) \neq 0 \qquad (3.3)$$

Then there exists a neighborhood$\mathcal{U}$of $y^\star$in $\mathbb{R}^m$and a unique function$\emptyset: \mathcal{U} \to \mathbb{R}^n$ofclass C^rsuch that

$$\varphi(\mathrm{y}^\star) = y^\star \qquad (3.4)$$

$$f(\varphi(y), y) = 0 \quad \text{for all } y \in \mathcal{U} \qquad (3.5)$$

This result tells us under which conditions the equation $f(x, y) = 0$ "can be solved withrespect to x". Assume $x^\star$is an equilibrium point of $f(x, \lambda_0)$ and let A be the linearization$\frac{\partial f}{\partial x}(x^\star, \lambda_0)$.Then the following situations can occur:

i. If A has no eigenvalues with zero real part, then f will admit equilibrium points $x^\star(\lambda)$ for all λ in a neighbourhood of λ_0. By continuity of the

eigenvalues of a matrixvalued function, $x^\star(\lambda)$ will be hyperbolic near λ_0. The curve $x^\star(\lambda)$ is usually called an equilibrium branch of f.

ii. If A has one or several eigenvalues equal to zero, then the implicit function theorem can no longer be applied, and various interesting phenomena can occur. For instance, the number of equilibrium points of f may change at $\lambda = \lambda_0$. Such a situation is called a bifurcation, and $(x^\star, \lambda)$ is called a bifurcation point of f.

iii. If A has purely imaginary, nonzero eigenvalues, the implicit function theorem can still be applied to show the existence of an equilibrium branch,$x^\star(\lambda)$ but its stability may change at $\lambda = \lambda_0$. This situation is also called a bifurcation.

Let us point out that we consider here localbifurcations, that is, changes of the orbitstructure in a small neighborhood of equilibria. We will not discuss in any depth globalbifurcations, which involve simultaneous changes in a larger region of phase space.

Now we will discuss the existence of center manifolds.

One of the most useful methods to study the flow near a bifurcation point is the centermanifold theorem, which generalizes the stable manifold theorem (Theorem (2.1.26)) tononhyperbolic equilibrium points.

Definition (3.1.2):

Let $\mathcal{U} \subset \mathbb{R}^n$be an open set. Let $S \subset \mathbb{R}^n$have the structure of differentiable manifold. For $x \in \mathcal{U}$, let $0 \in (t_1^x, t_2^x)$ be the maximal interval such that$\varphi_t(x) \in \mathcal{U}$for all $t \in (t_1^x, t_2^x) \cdot S$is called a local invariant manifold if $\varphi_t(x) \in S$for all$x \in S \cap \mathcal{U}$and all $t \in (t_1^x, t_2^x)$.

Theorem (3.1.3):

Let $x^\star$be a singular point of f, where f is of class $C^r, r \geq 2$, in aneighbourhood of $x^\star$. Let $A = \frac{\partial f}{\partial x}(x^\star)$ have, respectively, n_+, n_0 and n_-eigenvalues withpositive, zero and negative real parts, where $n_0 > 0$. Then there exist, in a neighbourhoodof $x^\star$, local invariant C^rmanifolds $W^u_{\text{loc}}, W^c_{\text{loc}}$and W^s_{loc}, of respective dimension $n_+, n_0$and n_-, and such that

1. W^u_{loc} is the unique local invariant manifold tangent to E_+ at $x^\star$, and $\varphi_t(x) \to x^\star$ as $t \to -\infty$for all $x \in W^u_{\text{loc}}$
2. W^s_{loc} is the unique local invariant manifold tangent to E_-at $x^\star$, and $\varphi_t(x) \to x^\star$ as $t \to \infty$ for all $x \in W^s_{\text{loc}}$.
3. W^c_{loc} is tangent to E_0, but not necessarily unique.

Before giving a (partial) proof of this result, we shall introduce a useful lemma fromthe theory of differential inequalities.

Lemma (3.1.4): (Gronwall's inequality)

Let φ, α and β be continuous and real-valued on $[a, b]$, with β non-negative, and assume that

$$\varphi(t) \leq \alpha(t) + \int_a^t \beta(s)\varphi(s)\, ds \quad \forall t \in [a, b] \tag{3.6}$$

Then

$$\varphi(t) \leq \alpha(t) + \int_a^t \beta(s)\alpha(s)e^{\int_s^t \beta(u)du}\, ds \quad \forall t \in [a, b] \tag{3.7}$$

Proof:

Let

$$R(t) = \int_a^t \beta(s)\varphi(s)\, ds.$$

Then $\varphi(t) \leq \alpha(t) + R(t)$ for all $t \in [a, b]$ and, since $\beta(t) \geq 0$,

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = \beta(t)\varphi(t) \leq \beta(t)\alpha(t) + \beta(t)R(t).$$

Let $K(s) = \int_a^s \beta(u)\, du$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{-K}R(s) = [R'(s) - \beta(s)]e^{-K(s)} \leq \beta(s)\alpha(s)e^{-K(s)}$$

and thus, integrating from a to t,

$$e^{-K(t)}R(t) \leq \int_a^t \beta(s)\alpha(s)e^{-K(s)}\, ds.$$

We obtain the conclusion by multiplying this expression by $e^{K(t)}$ and inserting the result into (3.6).

There exist various generalizations of this result, for instance to functions $\beta(t)$ thatare only integrable.

Let us now proceed to the proof of the center manifold theorem. There exist moreor less sophisticated proofs. We will give here a rather straightforward one taken. For simplicity, we consider the case $n_+ = 0$, and we will only prove the existenceof a Lipschitz continuous center manifold.

Proof of Theorem (3.1.3):

We write the system near the equilibrium point as

$$\dot{y} = By + g_-(y,z)$$
$$\dot{z} = Cz + g_0(y,z),$$

where all eigenvalues of B have strictly negative real parts, all eigenvalues of C havezero real parts, and $\|g_-(y,z)\|, \|g_0(y,z)\| \leq M(\|y\|^2 + \|z\|^2)$ in a neighbourhood of theorigin. We shall prove the existence of a local center manifold for a modified equation,that agrees with the present equation in a small neighbourhood of the equilibrium. Let $\psi: \mathbb{R}^{n_0} \to [0,1]$ be a C^∞function such that $\psi(x) = 1$ when $\|x\| \leq 1$ and $\psi(x) = 0$ when$\|x\| \geq 2$. We introduce the functions

$$G_-(y,z) = g_-\left(y, z\psi\left(\frac{z}{\varepsilon}\right)\right), \quad G_0(y,z) = g_0\left(y, z\psi\left(\frac{z}{\varepsilon}\right)\right),$$

Then the system

$$\dot{y} = By + G_-(y,z)$$
$$\dot{z} = Cz + G_0(y,z),$$

agrees with the original system for $\|z\| \leq \varepsilon$. We look for a center manifold with equation$y = h(z)$, where h is in a well-chosen function space, in which we want to apply Banach's fixed point theorem. Let $\rho > 0$ and $\kappa > 0$ be constants, and let χbe the set of Lipschitz continuous functions

$h: \mathbb{R}^{n_0} \to \mathbb{R}^{n-}$with Lipschitz constant $\kappa, \|h(z)\| \leq \rho$for all $z \in \mathbb{R}^{n_0}$ and $h(0) = 0$. $\chi$is a complete space with the supremum norm $|\cdot|$.

For $h \in \chi$, we denote by $\varphi_t(\cdot, h)$ the flow of the differential equation

$$\dot{z} = Cz + G_0(h(z), z).$$

For any solution $\big(y(s), z(s)\big)_{t_0 \le s \le t}$, the relation

$$y(t) = e^{B(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{B(t-s)} G_-\big(y(s), z(s)\big)\, ds$$

is satisfied, as is easily checked by differentiation. Among all possible solutions, we wantto select a class of solutions that are tangent to the center subspace $y = 0$ at the origin.In the linear case, the above solutions satisfy this property only if $e^{-Bt_0}y(t_0) = 0$. Wewill thus consider the class of particular solutions satisfying

$$y(t) = \int_{-\infty}^{t} e^{B(t-s)} G_-\big(y(s), z(s)\big)\, ds$$

If we set $t = 0$ and require that $y(s) = h(z(s))$ for all s, we arrive at the equality

$$h\big(z(0)\big) = \int_{-\infty}^{t} e^{-Bs} G_-\big(h(z(s)), z(s)\big)\, ds,$$

where $z(s) = \varphi_s(z(0), h)$. Thus we conclude that if h is a fixed point of the operator$T: \chi \to \chi$, defined by

$$(Th)(z) = \int_{-\infty}^{0} e^{Bs} G_- \Big(h\big(\varphi_s(z,h)\big), \varphi_s(z,h)\Big)\, ds,$$

then h is a center manifold of the equation. Note that there may be center manifolds thatdo not satisfy this equation, and thus we will not be proving uniqueness.

Now we want to show that T is a contraction on $\chi$for an appropriate choice of ε, κ, and ρ. Observe first that since all eigenvalues of B have a strictly negative real part,

Lemma (2.1.14) implies the existence of positive constants β and K such that

$$\|e^{-Bs}y\| \le Ke^{\beta s}\|y\| \qquad \forall y \in \mathbb{R}^{n_-}, \forall s \le 0$$

Since the eigenvalues of C have zero real parts, the same lemma implies that $\|e^{Cs}z\|$is apolynomial in s. Hence, for every $\nu > 0$, there exists a constant $Q(\nu)$ such that

$$\|e^{Cs}z\| \leq Q(\nu)e^{\nu|s|}\|z\| \qquad \forall z \in \mathbb{R}^{n_0}, \forall s \in \mathbb{R}.$$

It is possible that $Q(\nu) \to \infty$as $\nu \to 0$.

We first need to show that $T\chi \subset \chi$. Assume from now on that $\leq \varepsilon$, so that$\|h(z)\| \leq \varepsilon$. The definition of G_-implies that

$$\|G_-(h(z), z)\| \leq M'\varepsilon^2$$

for a constant $M' > 0$ depending on M. Moreover, the derivatives of G_0 are of order ε, sothat there is a constant $M'' > 0$ depending on M with

$$\begin{aligned}\|G_0(h(z_1), z_1) - G_0(h(z_2), z_2)\| &\leq M''\varepsilon[\|h(z_1) - h(z_2)\| + \|z_1 - z_2\|] \\ &\leq M''\varepsilon(1 + \kappa)\|z_1 - z_2\|.\end{aligned}$$

A similar relation holds for G_-. The bound on $\|G_-\|$implies that

$$\|Th(z)\| \leq \int_{-\infty}^{0} Ke^{\beta s}\, M'\varepsilon^2 ds \leq \varepsilon^2 \frac{KM'}{\beta},$$

and hence$\|Th(z)\| \leq \rho$ provided $\varepsilon \leq (\beta/KM')(\rho/\varepsilon)$. Next we want to estimate theLipschitz constant of T. Let $z_1, z_2 \in \mathbb{R}^{n_0}$. By definition of the flow $\varphi_t(\cdot, h)$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi_t(z_1, h) - \varphi_t(z_2, h)\big) \\ &\qquad = C\big(\varphi_t(z_1, h) - \varphi_t(z_2, h)\big) + G_0\Big(h\big(\varphi_t(z_1, h)\big), \varphi_t(z_1, h)\Big) \\ &\qquad - G_0\Big(h\big(\varphi_t(z_2, h)\big), \varphi_t(z_2, h)\Big).\end{aligned}$$

Taking into account the fact that $\varphi_0(z, h) = z$, we get

$$\begin{aligned}\varphi_t(z_1, h) - \varphi_t(z_2, h) &= e^{Ct}(z_1 - z_2) \\ &+ \int_0^t e^{C(t-s)}\Big[G_0\Big(h\big(\varphi_s(z_1, h)\big), \varphi_s(z_1, h)\Big) \\ &- G_0\Big(h\big(\varphi_s(z_2, h)\big), \varphi_s(z_2, h)\Big)\Big]\, ds.\end{aligned}$$

For $t \leq 0$, we obtain with the properties of G_0

$$\|\varphi_t(z_1,h)-\varphi_t(z_2,h)\| \leq \mathrm{Q}(\nu)e^{-\nu t}\|z_1-z_2\| + \int_0^t \mathrm{Q}(\nu)e^{-\nu(t-s)}M''\varepsilon(1+\kappa)\|\varphi_s(z_1,h)-\varphi_s(z_2,h)\|\,ds.$$

We can now apply Gronwall's inequality to

$\psi(t)=e^{-\nu t}\left\|\varphi_{-t}\big((z_1,h)-\varphi_{-t}(z_2,h)\big)\right\|$, with theresult

$$\|\varphi_t(z_1,h)-\varphi_t(z_2,h)\| \leq \mathrm{Q}(\nu)\mathrm{e}^{-\gamma t}\|\mathrm{z}_1-\mathrm{z}_2\|,$$

where $\gamma=\nu+Q(\nu)M''\varepsilon(1+\kappa)$. We can arrange that $\gamma<\beta$, taking for instance $\nu=\beta/2$and ε small enough. We thus obtain

$$\|Th(z_1)-Th(z_2)\| \leq \int_{-\infty}^{0} Ke^{\beta s}M''\varepsilon(1+\kappa)\,\|\varphi_s(z_1,h)-\varphi_s(z_2,h)\|ds$$

$$\leq \frac{KM''\varepsilon(1+\kappa)\mathrm{Q}(\nu)}{\beta-\gamma}\|z_1-z_2\|.$$

For any given κ and ν, we can find ε small enough that

$\|Th(z_1)-Th(z_2)\| \leq \kappa\|z_1-z_2\|$.This completes the proof that $T\chi\subset\chi$.

Finally, we want to show that T is a contraction. For $h_1,h_2\in\chi$, using

$$\left\|G_0\left(h_1\big(\varphi_s(z,h_1)\big),\varphi_s(z,h_1)\right)-G_0\left(h_2\big(\varphi_s(z,h_2)\big),\varphi_s(z,h_2)\right)\right\| \leq M''\varepsilon[(1+\kappa)\|\varphi_s(z,h_1)-\varphi_s(z,h_2)\|+|h_2-h_1|],$$

we obtain in a similar way by Gronwall's inequality that

$$\|\varphi_t(z,h_1)-\varphi_t(z,h_2)\| \leq 2\,\frac{\mathrm{Q}(\nu)}{\nu}\varepsilon M''|h_2-h_1|$$

(this is a rough estimate, where we have thrown away some t-dependent terms). Thisleads to the bound

$$\|Th_1(z)-Th_2(z)\| \leq \frac{K}{\beta}M''\varepsilon\left[1+2M''\varepsilon(1+\kappa)\frac{\mathrm{Q}(\nu)}{\nu}\right]|h_2-h_1|.$$

Again, taking ε small enough, we can achieve that

$\|Th_1(z) - Th_2(z)\| \leq \lambda|h_2 - h_1|$for some $\lambda < 1$ and for all $z \in \mathbb{R}^{n_0}$. This shows that T is a contraction, and we have provedthe existence of a Lipschitz continuous center manifold by Banach's fixed point theorem.One can proceed in a similar way to show that T is a contraction in a space of Lipschitzdifferentiable functions.

One can also prove that if f is of class $C^r, r > 2$, then h is also of class C^r. If f isanalytic or C^∞, however, then h will be $C^r$for all $r > 1$, but in general it will not beanalytic, and not even C^∞. In fact, the size of the domain in which h is C^rmay becomesmaller and smaller as r goes to infinity.

As pointed out in the proof of the theorem, the center manifold is not necessarilyunique. It is easy to give examples of systems admitting a continuous family of centermanifolds. However, as we shall see, these manifolds have to approach each other extremelyfast near the equilibrium point, and thus the dynamics will be qualitatively the same onall center manifolds.

Example (3.1.5):

The system

$$\begin{aligned} \dot{y} &= -y \\ \dot{z} &= -z^3 \end{aligned} \tag{3.8}$$

admits a two-parameter family of center manifolds

$$y = h(z, c_1, c_2) = \begin{cases} c_1 e^{-1/2z^2} & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ c_2 e^{-1/2z^2} & \text{for } z < 0. \end{cases} \tag{3.9}$$

The operator T in the proof of Theorem (3.1.3) admits a unique fixed point $h(z) \equiv 0$, butthere exist other center manifolds which are not fixed points of T. Note, however, that all functions $h(z, c_1, c_2)$ have identically zero Taylor expansions at $z = 0$.

Now we willdiscuss the properties of center manifolds.

We assume in thisdiscussionthat $x^\star$is a non-hyperbolic equilibrium point of f, such that$A = \frac{\partial f}{\partial x}(x^\star)$ has $n_0 > 1$eigenvalues with zero real partsand $n_- > 1$ eigenvalues withnegative real parts. In appropriate coordinates, we can write

$$\begin{aligned} \dot{y} &= By + g_{-}(y,z) \\ \dot{z} &= Cz + g_0(y,z), \end{aligned} \qquad (3.10)$$

where all eigenvalues of B have strictly negative real parts, all eigenvalues of C have zeroreal parts, and $g_{-}(y,z), g_0(y,z)$ are nonlinear terms. Theorem (3.1.2) shows the existence ofa local center manifold with parametric equation $y = h(z)$. The dynamics on this locallyinvariant manifold is governed by the equation

$$\dot{u} = Cu + g_0(h(u),u). \qquad (3.11)$$

Equation (3.11) has the advantage to be of lower dimension than (3.10), and thus easierto analyse. The following result shows that (3.11) is a good approximation of (3.10).

Theorem (3.1.6):

If the origin of (3.11) is stable (asymptotically stable, unstable), thenthe origin of (3.10) is stable (asymptotically stable, unstable).

Assume that the origin of (3.11) is stable. For any solution $\big(y(t), z(t)\big)$of (3.10) with$\big(y(0), z(0)\big)$sufficiently small, there exist a solution $u(t)$ of (3.11) and a constant $\gamma > 0$such that

$$\begin{aligned} y(t) &= h\big(u(t)\big) + \mathcal{O}(e^{-\gamma t}) \\ z(t) &= u(t) + \mathcal{O}(e^{-\gamma t}) \end{aligned} \qquad (3.12)$$

as $t \to \infty$.

The center manifold is said to be locallyattractive. Note thatif the center manifold is not unique, then (3.12) holds for any center manifold.

Let us now discuss how to compute center manifolds. In the proof of Theorem (3.1.3),we used the fact that h is a fixed point of a functional operator T. While being useful toprove existence of a center manifold, T is not very helpful for the computation of h, butthere exists another operator for this purpose. Replacing y by $h(z)$ in (3.10), we obtain

$$\frac{\partial h}{\partial z}(z)\big[Cz + g_0\big(h(z)\big), z\big] = Bh(z) + g_{-}(h(z),z) \qquad (3.13)$$

For functions $\phi: \mathbb{R}^{n_0} \to \mathbb{R}^{n_-}$which are continuously differentiable in a neighborhood ofthe origin, let us define

$$(L\phi)(z) = \frac{\partial \phi}{\partial z}[Cz + g_0(\phi(z), z)] - B\phi(z) - g_-(\phi(z), z) \qquad (3.14)$$

Then (3.13) implies that $(Lh)(z) = 0$ for any center manifold of (3.10). This equationis impossible to solve in general. However, its solutions can be computed perturbatively,and the approximation procedure is justified by the following result, which is also proved.

Theorem (3.1.7):

Let $\mathcal{U}$be a neighborhood of the origin in $\mathbb{R}^{n_0}$ and let$\phi \in \mathcal{C}^1(\mathcal{U}, \mathbb{R}^{n_-})$satisfy $\phi(0) = 0$ and $\frac{\partial \varphi}{\partial z}(0) = 0$. If there is$a q > 1$such that

$(L\phi)(z) = \mathcal{O}(\|z\|^q)$ as $z \to 0$,then $\|h(z) - \phi(z)\| = \mathcal{O}(\|z\|^q)$ as $z \to 0$ for any center manifold h.

An important consequence of this result is that if h_1 and h_2 are two different centermanifolds of $x^\star$, then one must have

$h_1(z) - h_2(z) = \mathcal{O}(\|z\|^q)$for all $q > 1$, i.e., all centermanifolds have the same Taylor expansion at $z = 0$.

Example (3.1.8):

Consider the two-dimensional system

$$\begin{aligned} \dot{y} &= -y + cz^2 \\ \dot{z} &= yz - z^3, \end{aligned} \qquad (3.15)$$

where c is a real parameter. We want to determine the stability of the origin. Naively, onemight think that since the first equation suggests that $y(t)$ converges to zero as $t \to \infty$, the dynamics can be approximated by projecting on the line $y = 0$. This would lead tothe conclusion that the origin is asymptotically stable, because $\dot{z} = -z_3$ when $y = 0$. Wewill now compute the center manifold in order to find the correct answer to the questionof stability. The operator (3.14) has the form

$$(L\phi)(z) = \phi'(z)[z\phi(z) - z^3] + \phi(z) - cz^2. \qquad (3.16)$$

Theorem (3.1.7) allows us to solve the equation $(Lh)(z) = 0$perturbatively, by an Ansatzof the form

$$h(z) = h_2 z^2 + h_3 z^3 + h_4 z^4 + \mathcal{O}(z^5). \tag{3.17}$$

the equation $(Lh)(z) = 0$, becomes

$$(Lh)(z) = (h_2 - c)z^2 + h_3 z^3 + [h_4 + 2h_2(h_2 - 1)]z^4 + \mathcal{O}(z^5) = 0, \tag{3.18}$$

which requires $h_2 = c, h_3 = 0$ and $h_4 = -2c(c - 1)$. Hence the center manifold has aTaylor expansion of the form

$$h(z) = cz^2 - 2c(c - 1)z^4 + \mathcal{O}(z^5), \tag{3.19}$$

and the motion on the center manifold is governed by the equation

$$\dot{u} = uh(u) - u^3 = (c - 1)u^3 - 2c(c - 1)u^5 + \mathcal{O}(u^6), \tag{3.20}$$

It is easy to show (using, for instance, u^2 as a Liapunov function) that the equilibrium point $u = 0$is asymptotically stable if $c < 1$ and unstable if $c > 1$. By Theorem (3.1.6),we conclude that the origin of the system (3.15) is asymptotically stable if $c < 1$ and unstable if $c > 1$, which contradicts the naive approach when $c > 1$.

The case $c = 1$ is special. In this case, the function $h(z) = z^2$ is an exact solution of the equation $(Lh)(z) = 0$, and the curve $y = z^2$ is the unique center manifold of (3.15),which has the particularity to consist only of equilibrium points. The origin is thus stable.

Section (3.2): Bifurcations of Differential Equations

We consider in this section parameter-dependent ordinary differential equations of theform

$$\dot{x} = f(x, \lambda) \tag{3.21}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n, \lambda \in \Lambda \subset \mathbb{R}^p$and f is of class $C^r$for some $r \geq 2$. We assume that$(x^\star, 0)$ is a bifurcation point of (3.21), which means that

$$\begin{aligned} f(x^\star, 0) &= 0 \\ \frac{\partial f}{\partial x}(x^\star, 0) &= A \end{aligned} \tag{3.22}$$

where the matrix A has $n_0 > 1$ eigenvalues on the imaginary axis. When $\lambda = 0$, theequilibrium point $x^\star$admits a center manifold.

We would like, however, to examine the dynamics of (3.21) for all λ in a neighbourhood of 0, where the center manifold theoremcannot beapplieddirectly. There is, however, an elegant trick to solve this problem. Inthe enlarged phase space $\mathcal{D} \times \Lambda$, consider the system

$$\begin{aligned} \dot{x} &= f(x, \lambda) \\ \dot{\lambda} &= 0. \end{aligned} \tag{3.23}$$

It admits $(x^\star, 0)$ as a non-hyperbolic equilibrium point. The linearization of (3.23) aroundthis point is a matrix of the form

$$\begin{pmatrix} A & \frac{\partial f}{\partial x}(x^\star, 0) \\ 0 & 0 \end{pmatrix} \tag{3.24}$$

which has $n_0 + p$ eigenvalues on the imaginary axis, including the (possibly multiple)eigenvalue zero. This matrix can be made block-diagonal by a linear change of variables,where one of the blocks contains all eigenvalues with zero real part. In these variables, the system (3.23) becomes

$$\begin{aligned} \dot{y} &= By + g_-(y, z, \lambda) \\ \dot{z} &= Cz + D\lambda + g_0(y, z, \lambda) \\ \dot{\lambda} &= 0. \end{aligned} \tag{3.25}$$

Here the $(n - n_0) \times (n - n_0)$ matrix B has only eigenvalues with nonzero real parts, the$n_0 \times n_0$ matrix C has all eigenvalues on the imaginary axis, D is a matrix of size $n_0 \times p$,and

$$\|g_-(y, z, \lambda)\|, \|g_0(y, z, \lambda)\| \leq M(\|y\|^2 + \|z\|^2 + \|\lambda\|^2) \tag{3.26}$$

in a neighbourhood of the origin, for some positive constant M. We can thus applyTheorem (3.1.3), which shows the existence of a local invariant center manifold of the form$y = h(z, \lambda)$. The dynamics on this manifold is governed by the n_0-dimensional equation

$$\dot{u} = Cu + D\lambda + g_0(h(u, \lambda), u, \lambda). \tag{3.27}$$

If A has no eigenvalues with positive real part, Theorem (3.1.6) shows that this equationgives a good approximation to the dynamics of (3.21) for small λ and

near $x = x^{\star}$. Thebig advantage is that generically, the number of eigenvalues on the imaginary axis at thebifurcation point is small, and thus the reduced equation (3.27) is of low dimension.

With these preliminaries, it becomes possible to investigate bifurcations in a systematicway. Recall that A is a real matrix, and thus its eigenvalues are either real or appear incomplex conjugate pairs. Thus the two simplest bifurcations, which we will consider below,involve either a single zero eigenvalue, or a pair of conjugate imaginary eigenvalues. Morecomplicated bifurcations correspond to a double zero eigenvalue, a zero eigenvalue andtwo conjugate imaginary ones, and so on. These cases are, however, less "generic", andwe will not discuss them here.

Now, we will discuss the one-dimensional center manifold.

We first discuss bifurcations involving a single eigenvalue equal to zero, i.e. $n_0 = 1$ and$C = 0$. The dynamics on the one-dimensional center manifold is governed by an equationof the form

$$\dot{u} = F(u, \lambda), \;\; u \in \mathbb{R} \qquad (3.28)$$

We will also assume that $\lambda \in R$. The origin $(0,0)$ is a bifurcation point of (3.28), meaning

$$F(0,0) = 0, \quad \frac{\partial f}{\partial u}(0,0) = 0 \qquad (3.29)$$

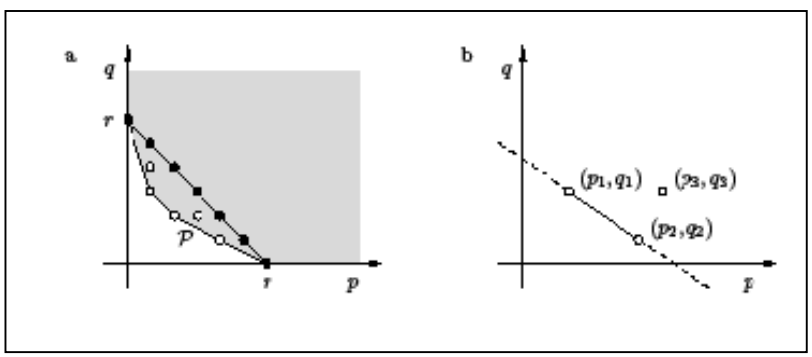

Figure 3.1

(a) Definition of the Newton polygon. White circles correspond to points(p, q) such that$c_{pq} \neq 0$, black circles to points (p, q) such that$p + q = r$. The slopes offull lines correspond to possible exponents of equilibrium branches. (b) The setting of theproof of Proposition (3.2.2).

In order to understand the dynamics for small u and λ, we need in particular to determinethe singular points of F, that is, we have to solve the equation $F(u,\lambda) = 0$ in a neighbourhoodof the origin. Note that the second condition in (3.29) implies that we cannotapply the implicit function theorem.

Let us start by expanding F in Taylor series,

$$F(u,\lambda) = \sum_{\substack{p+q\leq r\\ p,q\geq 0}} c_{pq} u^p \lambda^q + \sum_{\substack{p+q=r\\ p,q\geq 0}} u^p \lambda^q R_{pq}(u,\lambda), \tag{3.30}$$

where the functions $R_{pq}$are continuous near the origin, $R_{pq}(0,0) = 0$,and

$$c_{pq} = \frac{1}{p!\,q!}\frac{\partial^{p+q}F}{\partial u^p \partial \lambda^q}(0,0). \tag{3.31}$$

The bifurcation conditions (3.29) amount to $c_{00} = c_{10} = 0$. An elegant way to describe the solutions of $F(u,\lambda) = 0$ is based on Newton's polygon.

Definition (3.2.1):

Consider the set

$$\mathcal{A} = \{(\mathrm{p},\mathrm{q}) \in \mathbb{N}^2 : \mathrm{p}+\mathrm{q} \leq \mathrm{r} \ \text{ and } \ \mathrm{c_{pq}} \neq 0\} \tag{3.32}$$

(In case F is analytic, we simply drop the condition $p+q \leq r$). For each $(p,q) \in \mathcal{A}$, weconstruct the sector $\{(x,y) \in \mathbb{R}^2 : x \geq p \text{ and } y \geq q\}$. The Newton polygon $\mathcal{P}$of (3.30)is the broken line in $\mathbb{R}^2$ defined by the convex envelope of the union of all these sectors,see Fig.(3.1) a.

Proposition (3.2.2):

Assume for simplicity that $c_{pq} \neq 0$ whenever $p+q = r$. Assumefurther that the equation $F(u,\lambda) = 0$ admits a solution of the form

$u = C|\lambda|^{\mu}(1+\rho(\lambda))$for small λ, where $C \neq 0$ and $\rho(\lambda) \to 0$ continuously as $\lambda \to 0$. Then Newton's polygonmust have a segment of slope $-\mu$.

Proof:

It is sufficient to consider a function F of the form

$$F(u,\lambda) = \sum_{i=1}^{3} c_i u^{p_i} \lambda^{q_i}.$$

Indeed, if the expansion (3.30) contains only two terms, the result is immediate, and ifit has more than three terms, one can proceed by induction. The hypothesis implies

$$\sum_{i=1}^{3} \sigma_i c_i C^{p_i} |\lambda|^{\mu p_i + q_i} \left(1 + \rho(\lambda)\right)^{p_i} = 0,$$

where $\sigma_i = \pm 1$. Assume for definiteness that

$p_1\mu + q_1 \leq p_2\mu + q_2 \leq \ p_3\mu + q_3$. Considerfirst the case

$p_1\mu + q_1 < p_2\mu + q_2$. Then division be $|\lambda|^{\mu p_1 + q_1}$ gives

$$\sigma_1 c_1 C^{p_1}\left(1 + \rho(\lambda)\right)^{p_1} + \sum_{i=2}^{3} \sigma_i c_i C^{p_i} |\lambda|^{\mu p_i + q_i - \mu p_1 - q_1}\left(1 + \rho(\lambda)\right)^{p_i} = 0$$

The exponent of $|\lambda|$is strictly positive. Thus taking the limit $\lambda \to 0$, we obtain $C^{p_1} = 0$,a contradiction. We conclude that we must have

$p_1\mu + q_1 = p_2\mu + q_2$. Graphically, the relation

$p_1\mu + q_1 = p_2\mu + q_2 \leq p_3\mu + q_3$ means that $\mu = \frac{q_2 - q_1}{p_1 - p_2}$is minus the slope of the segment from (p_1, q_1) to (p_2, q_2), and that (p_3, q_3) lies above, see Fig.(3.1) b.

This result does not prove the existence of equilibrium branches $u = u^\star(\lambda)$, but it tellsus where to look. By drawing Newton's polygon, we obtain the possible values of μ. Byinserting the Ansatz$u^\star(\lambda) = C|\lambda|^\mu(1 + \rho)$ into the equation $F(u,\lambda) = 0$, we can determinewhether or not such a branch exists. This is mainly a matter of signs of the coefficientsc_{pq}. For instance, the equation $u^2 + \lambda^2 = 0$ admits no solution other than (0,0), while theequation $u^2 - \lambda^2 = 0$does admits solutions $u \ = \ \pm\lambda$.

Now we will discuss the saddle-node bifurcation.

We now illustrate the procedure of computing equilibrium branches in the generic case$r = 2, c_{20} \neq 0, c_{01} \neq 0$. Then

$$F(u,\lambda) = c_{01}\lambda + c_{20}u^2 + c_{11}u\lambda + c_{02}\lambda^2 + \sum_{p+q=2} u^p\lambda^q R_{pq}(u,\lambda) \qquad (3.33)$$

and Newton's polygon has two vertices $(0,1)$ and $(2,0)$, connected by a segment with slope$-1/2$. Proposition (3.2.2) tells us that if there is an equilibrium branch, then it must be of the form $u = C|\lambda|^{1/2}(1+\rho(\lambda))$, where $\lim_{\lambda\to 0}\rho(\lambda) = 0$. In fact, it turns out to be easierto express λ as a function of u:

Lemma (3.2.3):

In a neighbourhood of the origin, there exists a continuous function$\overline{\rho}(u)$with $\overline{\rho}(0) = 0$ such that $F(u,\lambda) = 0$ if and only if

$$\lambda = -\frac{c_{20}}{c_{01}}u^2\big(1+\overline{\rho}(u)\big). \qquad (3.34)$$

Proof:

Proposition (3.2.2) indicates that any equilibrium branch must be of the form$\lambda = \overline{C}u^2\big(1+\overline{\rho}(u)\big)$. Define the function

$$G(\overline{\rho},u) = \frac{1}{u^2}F\left(u,\overline{C}u^2(1+\overline{\rho})\right).$$

Using the expansion (3.33) of F, it is easy to see that

$$\lim_{u\to 0} G(0,u) = c_{10}\overline{C} + c_{20},$$

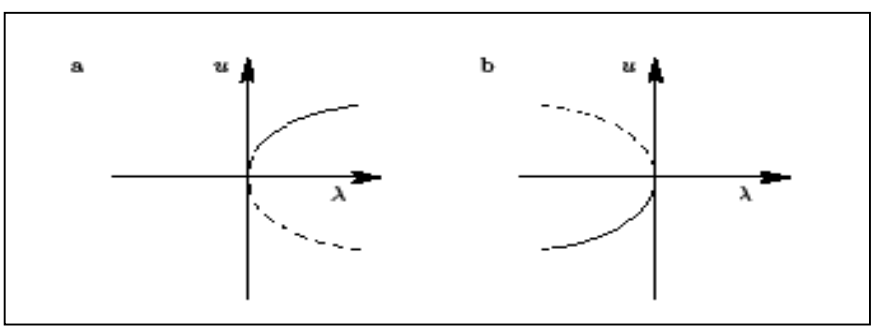

Figure 3.2

Saddle-node bifurcation, (a) in the case $c_{01} > 0, c_{20} < 0$ (direct bifurcation),and (b) in the case $c_{01} > 0, c_{20} > 0$ (indirect bifurcation).The other cases are similar,with stable and unstable branches interchanged.

Full curves indicate stable equilibriumbranches, while dashed curves indicate unstable equilibrium branches.

and thus $G(0,0) = 0$ if and only if $\overline{C} = -c_{20}/c_{01}$. Moreover, using a Taylor expansion tofirst order of $\frac{\partial F}{\partial \lambda}$, one obtains

$$\lim_{u \to 0} \frac{\partial G}{\partial \overline{\rho}}(0, u) = c_{01}\overline{C} \neq 0.$$

Thus, by the implicit function theorem, there exists, for small u, a unique function $\overline{\rho}(u)$ such that $\overline{\rho}(0) = 0$ and $G(\overline{\rho}(u), u) = 0$.

Expressing u as a function of λ, we find the existence of two equilibrium branches

$$u = u_{\pm}^{\star}(\lambda) = \pm\sqrt{\frac{c_{01}}{c_{20}}\lambda}[1 + \rho(\lambda)], \tag{3.35}$$

which exist only for sign $\lambda = -\operatorname{sign}(c_{01}/c_{20})$. Their stability can be determined by usingthe Taylor expansion

$$\frac{\partial F}{\partial u}(u, \lambda) = 2c_{20}u + c_{11}\lambda + \sum_{p+q=1} u^p \lambda^q \overline{R}_{pq}(u, \lambda) \tag{3.36}$$

where $\overline{R}_{pq}$are some continuous functions vanishing at the origin. Inserting (3.35), weget

$$\frac{\partial F}{\partial u}\left(u_{\pm}^{\star}(\lambda), \lambda\right) = \pm 2c_{20}\sqrt{-\frac{c_{01}}{c_{20}}\lambda} + \mathcal{O}\left(\sqrt{|\lambda|}\right). \tag{3.37}$$

We thus obtain the following cases, depending on the signs of the coefficients:

1. If $c_{01} > 0$ and $c_{20} < 0$, the branches exist for $\lambda > 0, u_{+}^{\star}$is stable and $u_{-}^{\star}$is unstable;
2. if $c_{01} < 0$ and $c_{20} > 0$, the branches exist for $\lambda > 0, u_{+}^{\star}$is unstable and $u_{-}^{\star}$is stable;
3. if $c_{01} < 0$ and $c_{20} < 0$, the branches exist for $\lambda < 0, u_{+}^{\star}$is stable and $u_{-}^{\star}$is unstable;
4. if $c_{01} > 0$ and $c_{20} > 0$, the branches exist for $\lambda < 0, u_{+}^{\star}$ is unstable and $u_{-}^{\star}$ is stable.

These bifurcations are called saddle-node bifurcations, because when considering the fullsystem instead of its restriction to the center manifold, they

involve a saddle and a node. If the branches exist for $\lambda > 0$, the bifurcation is called direct, and if they exist for $\lambda < 0$,it is called indirect (Fig. 3.2).

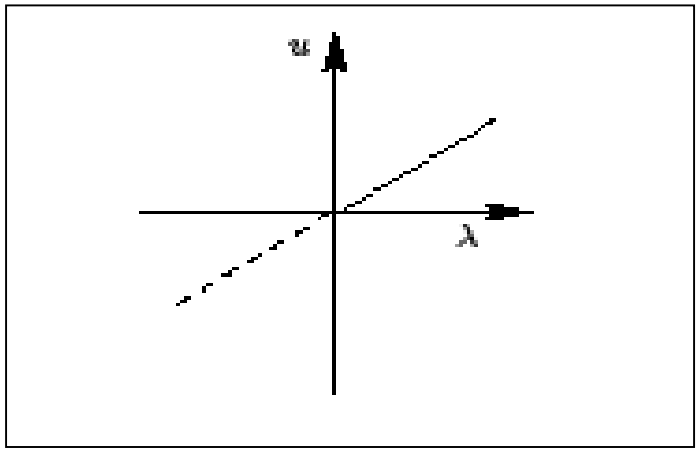

Figure 3.3

Transcritical bifurcation, in the case of equation (3.41) with $c_{20} < 0$ and$c_{11} > 0$.

It is important to observe that the qualitative behaviour depends only on those coefficientsin the Taylor series which correspond to vertices of Newton's polygon. Thuswe could have thrown away all other terms, to consider only the truncated equation, ornormal form,

$$\dot{u} = c_{20}u^2 + c_{01}\lambda \qquad (3.38)$$

Now we will discuss transcritical bifurcation.

let us consider next the slightly less generic case where $c_{01} = 0$, but $c_{20} \neq 0, c_{11} \neq 0, c_{02} \neq 0$. Then the Taylor expansion of F takes the form

$$F(u,\lambda) = c_{20}u^2 + c_{11}u\lambda + c_{02}\lambda^2 + \sum_{p+q=2} u^p\lambda^q R_{pq}(u,\lambda), \qquad (3.39)$$

and Newton's polygon has three vertices $(0,2), (1,1)$ and $(2,0)$, connected by segmentsof slope -1. Proposition (3.2.2) tells us to look for equilibrium branches of the form $u = C\lambda\big(1 + \rho(\lambda)\big)$. Proceeding in a similar way as in Lemma (3.2.3), we obtain the conditions

$$c_{20}C^2 + c_{11}C + c_{02} = 0$$
$$2c_{20}C^2 + c_{11}C \neq 0 \qquad (3.40)$$

for the existence of a unique equilibrium branch of this form. We thus conclude that if$c_{11}^2 - 4c_{20}c_{02} > 0$, there are two intersecting equilibrium branches. It is easy to see thatthe linearization of $f$around such a branch is$(2c_{20}C + c_{11})\lambda + \mathcal{O}(\lambda)$, and thus one of thebranches is stable, the other is unstable, and they exchange stability at the bifurcationpoint. This bifurcation is called transcritical.

If $c_{11}^2 - 4c_{20}c_{20} < 0$, there are no equilibrium branches near the origin. Finally, if$c_{11}^2 - 4c_{20}c_{02} = 0$there may be several branches with the same slope through the origin.

Let us point out that the condition $c_{02} \neq 0$ is not essential. In fact, one often carries outa change of variables taking one of the equilibrium branches to the λ-axis. The resultingnormal form is

$$\dot{u} = c_{20}u^2 + c_{11}u\lambda \qquad (3.41)$$

One of the equilibrium branches is $u \equiv 0$. The slope of the other branch and the stabilitydepend on the signs of c_{20} and c_{11} (Fig. 3.3)

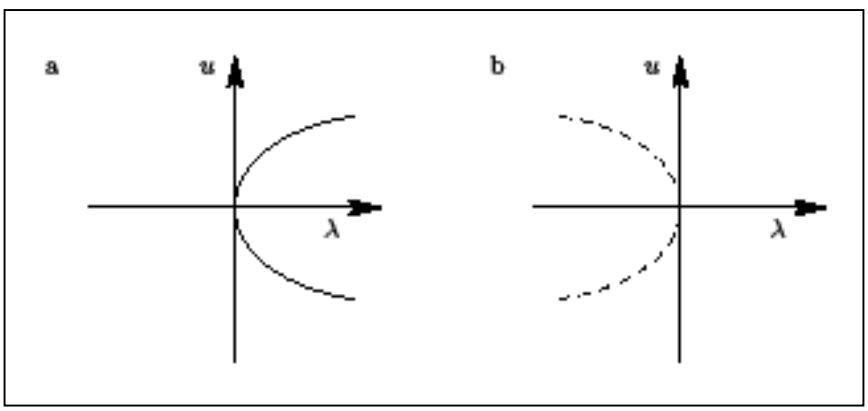

Figure 3.4

Pitchfork bifurcation, (a) in the case $c_{11} > 0, c_{30} < 0$ (supercritical bifurcation),and (b) in the case $c_{11} > 0, c_{30} > 0$ (subcritical bifurcation).

Now we will discuss thepitchfork bifurcation.

One can go on like that for ever, considering cases with more coefficients in the Taylorseries equal to zero, which is not especially interesting unless one has to do with a concreteproblem. However, sometimes symmetries of the differential

equation may cause manyterms in the Taylor expansion to vanish. Consider the case $F \in C^3$ satisfying

$$F(-u,\lambda) = -F(u,\lambda)\forall(u,\lambda). \qquad (3.42)$$

Then $c_{pq} = 0$ for even p. In fact, $F(u,\lambda)/u$ is C_2 near the origin and thus we can write

$$F(u,\lambda) = u\left[c_{11}\lambda + c_{30}u^2 + c_{12}\lambda^2 + \sum_{p+q=2} u^p\lambda^q\overline{R}_{pq}(u,\lambda)\right]. \qquad (3.43)$$

The term in brackets is similar to the expansion for the saddle-node bifurcation, and thuswe obtain similar equilibrium branches. In addition, there is the equilibrium branch $u \equiv 0$.Depending on the signs of the coefficients, we have the following cases:

1. If $c_{11} > 0$ and $c_{30} < 0$, the branch $u = 0$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$, and two additional stable branches exist for $\lambda > 0$;
2. if $c_{11} < 0$ and $c_{30} > 0$, the branch $u = 0$ is unstable for $\lambda < 0$ and stable for $\lambda > 0$,and two additional unstable branches exist for $\lambda > 0$;
3. if $c_{11} > 0$ and $c_{30} > 0$, the branch $u = 0$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$,and two additional unstable branches exist for $\lambda < 0$;
4. if $c_{11} < 0$ and $c_{30} < 0$, the branch $u = 0$ is unstable for $\lambda < 0$ and stable for $\lambda > 0$, and two additional stable branches exist for $\lambda < 0$.

This situation is called a pitchfork bifurcation, which is said to be supercritical ifstable equilibrium branches are created, and subcriticalis unstable equilibrium branchesare destroyed (Fig. 3.4). The normal form of the pitchfork bifurcation is

$$\dot{u} = c_{11}\lambda u + c_{30}u^3. \qquad (3.44)$$

In Physics, this equation is often written in the form

$$\dot{u} = -\frac{\partial V}{\partial u}(u,\lambda), \qquad V(u,\lambda) = -\frac{c_{11}}{2}\lambda u^2 - \frac{c_{30}}{4}u^4. \qquad (3.45)$$

If $c_{30} < 0$, the function $V(u,\lambda)$ has one or two minima, and in the latter case it is calleda double-well potential.

Now we will discuss the two-dimensional center manifold Hopf bifurcation.

We consider now the case $n_0 = 2$, with C having eigenvalues $\pm i\omega_0$, where $\omega_0 \neq 0$. Thedynamics on the center manifold is governed by a two-dimensional system of the form

$$\dot{u} = F(u, \lambda), \qquad (3.46)$$

where we shall assume that $\lambda \in \mathbb{R}$ and $F \in C^3$. Since $\frac{\partial F}{\partial u}(0,0) = C$, is invertible, theimplicit function theorem (Theorem (3.1.1)) shows the existence, near $\lambda = 0$, of a uniqueequilibrium branch $u^\star(\lambda)$, with $u(0) = 0$ and $F(u^\star(\lambda), \lambda) = 0$. By continuity of theeigenvalues of a matrix-valued function, the linearization of F around this branch haseigenvalues $a(\lambda) \pm i\omega(\lambda)$, where $\omega(0) = \omega_0$ and $a(0) = 0$. A translation of $-u^\star(\lambda)$,followed by a linear change of variables, puts the system (3.46) into the form

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} a(\lambda) & -\omega(\lambda) \\ \omega(\lambda) & a(\lambda) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} g_1(u_1, u_2, \lambda) \\ g_2(u_1, u_2, \lambda) \end{pmatrix}, \qquad (3.47)$$

where the $g_i$are nonlinear terms satisfying $\|g_i(u_1, u_2, \lambda)\| \leq M(u_1^2 + u_2^2)$ for small u_1 andu_2 and some constant $M > 0$.

Our strategy is now going to be to simplify the nonlinear terms as much as possible,following the theory of normal forms developed . It turns out to beuseful to introduce the complex variable $z = u_1 + iu_2$(an idea going back to Poincaré),which satisfies an equation of the form

$$\dot{z} = [a(\lambda) + i\omega(\lambda)]z + g(z, \bar{z}, \lambda), \qquad (3.48)$$

where $\bar{z}$ is the complex conjugate of z. This should actually be considered as a two-dimensionalsystem for the independent variables z and $\bar{z}$:

$$\begin{aligned} \dot{z} &= [a(\lambda) + i\omega(\lambda)]z + g(z, \bar{z}, \lambda), \\ \dot{\bar{z}} &= [a(\lambda) + i\omega(\lambda)]\bar{z} + \bar{g}(z, \bar{z}, \lambda). \end{aligned} \qquad (3.49)$$

Lemma (2.1.32) shows that monomials of the form $c_p z^{p_1} \bar{z}^{p_2}$ in the nonlinear term g canbe eliminated by a nonlinear change of variables, provided the non-resonance condition (2.70) is satisfied. In the present case, this condition has the form

$$(p_1 + p_2 - 1)a(\lambda) + (p_1 - p_2 \mp 1)i\omega(\lambda) \neq 0, \qquad (3.50)$$

where the signs $\mp$refer, respectively, to the first and second equation in (3.49). Thiscondition can also be checked directly by carrying out the transformation $z = \zeta + h_p \zeta^{p_1} \bar{\zeta}^{p_2}$ in (3.49). Condition (3.50) is hardest to satisfy for $\lambda = 0$, where it becomes

$$(p_1 - p_2 \mp 1) i\omega_0 \neq 0 \qquad (3.51)$$

Since $\omega_0 \neq 0$ by assumption, this relation always holds for $p_1 + p_2 = 2$, so quadraticterms can always be eliminated. The only resonant term of order 3 in g is $z^2\bar{z} = |z|^2 z$, corresponding to $(p_1, p_2) = (2,1)$. (Likewise, the term$z\bar{z}^2 = |z|^2\bar{z}$ is resonant in $\bar{g}$.)We conclude from Proposition (2.1.31) that there exists a polynomial change of variables$z = \zeta + h(\zeta, \bar{\zeta})$, transforming (3.48) into

$$\dot{\zeta} = [a(\lambda) + i\omega(\lambda)]\zeta + c(\lambda)|\zeta|^2\zeta + R(\zeta, \bar{\zeta}, \lambda), \qquad (3.52)$$

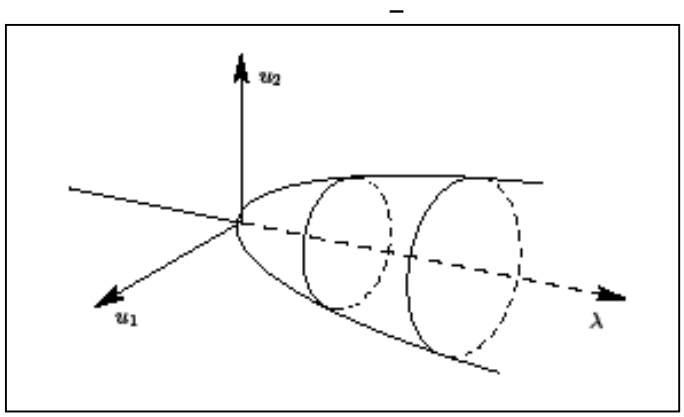

Figure 3.5

Supercritical Hopf bifurcation. The stationary solution $(u_1, u_2) = (0,0)$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$. For positive λ, a stable periodic orbit close to acircle of radius $\sqrt{\lambda}$ appears.

where $c(\lambda) \in \mathbb{C}$ and $R(\zeta, \bar{\zeta}, \lambda) = \mathcal{O}(|\zeta|^3)$(meaning that $\lim_{\zeta \to 0} |\zeta|^3 R(\zeta, \bar{\zeta}, \lambda) = 0$). Equation(3.52) is the normal form of our bifurcation. To analyse it further, we introduce polar coordinates $\zeta = re^{i\varphi}$, in which the system becomes

$$\begin{aligned} \dot{r} &= a(\lambda)r + Re\, c(\lambda)r^3 + R_1(r, \varphi, \lambda) \\ \dot{\varphi} &= \omega(\lambda) + \operatorname{lm} c(\lambda)\, r^2 + R_2(r, \varphi, \lambda), \end{aligned} \qquad (3.53)$$

where $R_1 = \mathcal{O}(r^3)$ and $R_2 = \mathcal{O}(r^2)$. We henceforth assume that $a'(0) \neq 0$, and changing λinto $-\lambda$ if necessary, we may assume that $a'(0) > 0$. If we discard

the remainder R_1, thefirst equation in (3.53) describes a pitchfork bifurcation for r, which is supercritical if$Re\ c(0) < 0$, and subcritical if $Re\ c(0) > 0$. The first case corresponds to the appearanceof a stable periodic orbit, of amplitude$\sqrt{-a(\lambda)/Re\ c(\lambda)}$ (Fig. 3.5), the second to thedestruction of an unstable periodic orbit. The rotation frequency on this orbit is $\omega_0 + \mathcal{O}(\lambda)$.

It remains to show that this picture is not destroyed by the remainders R_1 and R_2.Note that in the present case we cannot apply the Poincaré-Sternberg-Chen theorembecause the linear part is not hyperbolic.

Theorem (3.2.4): (Andronov-Hopf)

Assume that the system $\dot{x} = f(x, \lambda)$ admits anequilibrium branch $x^\star(\lambda)$such that the linearization of f at $x^\star(\lambda)$ has two eigenvalues

$a(\lambda) \pm i\omega(\lambda)$, with $a(0) = 0, a'(0) > 0$ and $\omega(0) \neq 0$, and all other eigenvalues havestrictly negative real parts. If the coefficient $c(\lambda)$ in the normal form (3.52) satisfies$Re\ c(0) \neq 0$, then

i. if Re $c(0) < 0$, the system admits a stable isolated periodic orbit for small positive λ, close to a circle with radius proportional to $\sqrt{\lambda}$ (supercritical case);
ii. if Re $c(0) > 0$, the system admits an unstable isolated periodic orbit for small negative λ, close to a circle with radius proportional to $\sqrt{\lambda}$ (subcritical case).

Proof:

There exist various proofs of this result. One of them is based on the method ofaveraging, another one on the Poincaré-Bendixson theorem. Since we did not introducethese methods, we will give a straightforward geometrical proof. The main idea is toconsider the set

$\{(r, \varphi): \varphi = 0, r > 0\}$as a Poincaré section, and to examine the associated Poincaré map. Consider the case $Re\ c(0) < 0$. If we assume that $0 < \lambda \ll 1$, set $r = \sqrt{\lambda}\rho$ in (3.53), and expand the λ-dependent terms, we arrive at the equivalent system

$$\dot{\rho} = \rho[a'(0)\lambda + Re\ c(0)\lambda\rho^2 + \mathcal{O}(\lambda)]$$
$$\dot{\varphi} = \omega_0 + \lambda[\mathrm{lm}\, c(0)\, \rho^2 + \omega'(0)] + \mathcal{O}(\lambda).$$

Let $\rho_0 = \sqrt{-a'(0)/Re\ c(0)}$. We shall consider this system in the disc $\rho \leq 2\rho_0$. The secondequation tells us that $\dot{\varphi} \neq 0$ (say, $\dot{\varphi} > 0$) for λ small enough. Under this condition, we canuse φ instead of t as new time variable, and write the system in the form

$$\dot{\rho} = \frac{a'(0)}{\omega_0}\rho\left[\lambda - \frac{\rho^2}{\rho_0^2}\lambda + \mathcal{O}(\lambda)\right]$$
$$\dot{\varphi} = 1.$$

We define a Poincaré map P by the fact that the solution starting at $(\rho, 0)$ passes, after onerevolution, through the point $(P(\rho), 2\pi)$. $P(\rho)$ is monotonous by uniqueness of solutions.Fix a constant $\delta \in (0,1)$. The term $\mathcal{O}(\lambda)$ may depend on φ, but by taking λ small enough,it can be made smaller in absolute value than $(\delta/2)\lambda$ (uniformly in ρ for $\rho \leq 2\rho_0$). Itfollows that

$$\text{for } 0 < \rho^2 \leq (1-\delta)\rho_0^2: \qquad \dot{\rho} \geq \frac{a'(0)}{\omega_0}\rho\lambda\left(1 - (1-\delta) - \frac{\delta}{2}\right) > 0$$
$$\text{for} (1+\delta)\rho_0^2 \leq \rho^2 \leq (2\rho_0)^2: \quad \dot{\rho} \leq \frac{a'(0)}{\omega_0}\rho\lambda\left(1 - (1+\delta) + \frac{\delta}{2}\right) < 0$$

This means that $\rho(t)$ is monotonous outside the annulus

$(1-\delta)\rho_0^2 \leq \rho^2 \leq (1+\delta)\rho_0^2$, and thus$P$ cannot have fixed points outside the interval $I = \left[\sqrt{1-\delta}\rho_0, \sqrt{1+\delta}\rho_0\right]$. Furthermore,since the vector field enters the annulus, P maps the interval I into itself, and must admit afixed point $\rho^\star$. Returning to the original variables, we see that this fixed point correspondsto the desired periodic orbit.

Finally, it is also possible to show that $\rho^\star$is the only fixed point of P in $(0,2\rho_0)$. Weknow that there can be no fixed points outside I (except 0). By examining the error term$\mathcal{O}(\lambda)$a bit more carefully, one finds that$\dot{\rho}$ is a decreasing function of ρ near $\rho = \rho_0$. Thus,if $\rho_1 < \rho_2$belong to I, with δ small enough, the orbits starting in ρ_1 and ρ_2 must approacheach other as time increases. This means that P is contracting in I, and thus its fixedpoint is unique.

This bifurcation is called Poincaré-Andronov-Hopf bifurcation. One should note thatit is the first time we prove the existence of periodic orbits in any generality. Bifurcationswith eigenvalues crossing the imaginary axis are common in parameter-dependentdifferential equations, and thus periodic orbits are frequent in these systems.

The nature of the bifurcation depends crucially on the sign of $Re\ c(0)$. This quantitycan be determined by a straightforward, though rather tedious computation, and is given, for instance.

Section (3.3): Bifurcations of Maps

We turn now to parameter-dependent iterated maps of the form

$$x_{k+1} = F(x_k, \lambda) \qquad (3.54)$$

with $x \in \mathcal{D} \subset \mathbb{R}^n, \lambda \in \Lambda \subset \mathbb{R}^p$and $F \in C^r$for some $r \geq 2$. We assume again that $(x^\star, 0)$ is a bifurcation point of (3.54), which means that

$$\begin{aligned} F(x^\star, 0) &= x^\star \\ \frac{\partial F}{\partial x}(x^\star, 0) &= A \end{aligned} \qquad (3.55)$$

where A has $n_0 \geq 0$ eigenvalues of module 1. For simplicity, we assume that all othereigenvalues of A have a module strictly smaller than 1. In appropriate coordinates, wecan thus write this system as

$$\begin{aligned} y_{k+1} &= By_k + g_-(y_k, z_k, \lambda_k) \\ z_{k+1} &= Cz_k + D\lambda_k + g_0(y_k, z_k, \lambda_k) \\ \lambda_{k+1} &= \lambda_k \end{aligned} \qquad (3.56)$$

where all eigenvalues of B are inside the unit circle, all eigenvalues of C are on the unitcircle, and $g_-$and g_0 are nonlinear terms. One can prove, in much the same way we usedfor differential equations, the existence of a local invariant center manifold $y = h(z, \lambda)$.This manifold has similar properties as in the ODE case: it is locally attractive, and canbe computed by solving approximately the equation

$$h\big(Cz + D\lambda + g_0(h(z, \lambda), z, \lambda)\big) = Bh(z, \lambda) + g_-(h(z, \lambda), z, \lambda). \qquad (3.57)$$

The dynamics on this manifold is governed by the n_0-dimensional map

$$u_{k+1} = Cu_k + D\lambda + g_0(h(u_k, \lambda), u_k, \lambda). \qquad (3.58)$$

Since C is a real matrix with all eigenvalues on the unit circle, the most generic cases arethe following:

1. one eigenvalue equal to $1: n_0 = 1$ and $C = 1$;
2. one eigenvalue equal to $-1: n_0 = 1$ and $C = -1$;
3. two complex conjugate eigenvalues of module $1: n_0 = 2$ and C having eigenvalues $e^{\pm 2\pi\, i\, \theta_0}$ with $2\theta_0 \notin \mathbb{Z}$.

The first case is easily dealt with, because extremely similar to the case of differentialequations. Indeed, the dynamics on the one-dimensional center manifold is governed bythe equation

$$u_{k+1} = u_k + G(u_k, \lambda), \;\; G(0,0) = 0, \;\; \frac{\partial G}{\partial u}(0,0) = 0. \qquad (3.59)$$

The fixed points are obtained by solving the equation $G(u, \lambda) = 0$, which behaves exactly as the equation $F(u, \lambda) = 0$. Let us simply indicate the normal formsof the most common bifurcations. For the saddle-node bifurcation, we have

$$u_{k+1} = u_k + c_{01}\lambda + c_{20}u_k^2 \quad (3.60)$$

for the transcritical bifurcation, one can reduce the equation to

$$u_{k+1} = u_k + c_{11}\lambda u_k + c_{20}u_k^2 \qquad (3.61)$$

and for the pitchfork bifurcation, the normal form is given by

$$u_{k+1} = u_k + c_{11}\lambda u_k + c_{30}u_k^3 \qquad (3.62)$$

Now we will discuss the period-doubling bifurcation.

We turn now to the case $n_0 = 1, C = -1$, and assume that the map is of class C^3. Themap restricted to the center manifold has the form

$$\begin{gathered} u_{k+1} = -u_k + G(u_k, \lambda), \\ G(u, \lambda) = c_{01}\lambda + c_{02}\lambda^2 + c_{11}u\lambda + c_{20}u^2 + c_{30}u^3 + \cdots \end{gathered} \qquad (3.63)$$

We first note that the implicit function theorem can be applied to the equation $u_{k+1} = u_k$,and yields the existence of a unique equilibrium branch through the origin. It has the form

$$u = u^{\star}(\lambda) = \frac{c_{01}}{2}\lambda + \mathcal{O}(\lambda) \qquad (3.64)$$

and changes stability as λ passes through 0 if $c_{11} + c_{20}c_{01} \neq 0$. Thus something musthappen to nearby orbits. To understand what is going on, it is useful to determine thesecond iterates. A straightforward computation gives

$$\begin{aligned} u_{k+2} &= -u_{k+1} + G(u_{k+1}, \lambda) \\ &= u_k - G(u_k, \lambda) + G(-u_k + G(u_k, \lambda), \lambda) \\ &= u_k + c_{01}(c_{11} + c_{01}c_{20})\lambda^2 - 2(c_{11} + c_{01}c_{20})u_k\lambda - 2(c_{30} + c_{20}^2)u_k^3 + \cdots \end{aligned} \qquad (3.65)$$

Now let us consider the equation $u_{k+2} = u_k$, the solutions of which yield orbits of period 2.Using the method of Newton's polygon, we find that in addition to the solution $u = u^{\star}(\lambda)$,there exist solutions of the form

$$u^2 = \frac{-c_{11} + c_{20}c_{01}}{c_{30} + c_{20}^2}\lambda + \mathcal{O}(\lambda), \qquad (3.66)$$

provided $c_{30} + c_{20}^2 \neq 0$and $c_{11} + c_{20}c_{01} \neq 0$. In other words, the map

$u_k \mapsto u_{k+2}$undergoesa pitchfork bifurcation. The equilibrium branch (3.66) does not correspond to fixed points of (3.63), but to an orbit of least period 2. We have thus obtained the followingresult:

Theorem (3.3.1):

Let $F(\cdot, \lambda): \mathbb{R} \to \mathbb{R}$ be a one-parameter family of maps of the form (3.63), i.e., such that $F(\cdot\,,0)$ admits 0 as a fixed point with linearization -1. Assume

$$\begin{aligned} &c_{11} + c_{01}c_{20} \neq 0 \\ &c_{30} + c_{20}^2 \neq 0 \end{aligned} \qquad (3.67)$$

Then there exists a curve of fixed points $u = u^{\star}(\lambda)$ passing through the origin, whichchanges stability at $\lambda = 0$. In addition, there is a curve of the form (3.66), tangent tothe u-axis, consisting of points of period 2. The orbit of period 2 is stable if $c_{30} + c_{20}^2 > 0$and unstable if $c_{30} + c_{20}^2 < 0$.

This bifurcation is called a period doubling, flip or subharmonic bifurcation. It is calledsupercritical if a stable cycle of period 2 is created, and subcritical if an unstable orbit ofperiod 2 is destroyed.

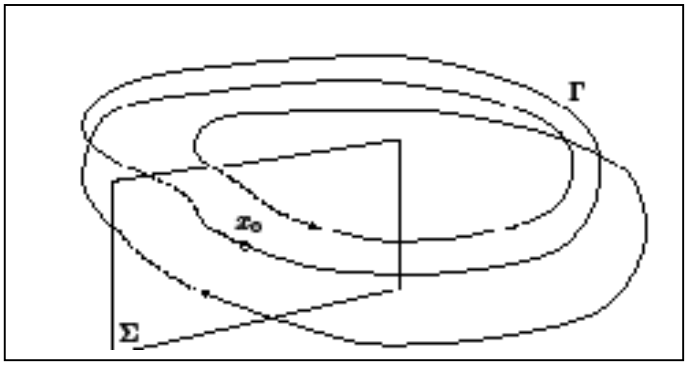

Figure 3.6

The supercritical flip bifurcation of a Poincar´e map, associated with a periodicΓ, corresponds to the appearance of a new periodic orbit with twice the period.

Remark (3.3.2):

If f is a function of classC^3, its Schwartzian derivative is defined as

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \qquad (3.68)$$

The bifurcation is supercritical if the Schwartzian derivative of the map at (0,0) withrespect to u is negative, and subcritical if this derivative is positive.

The period doubling bifurcation has an interesting consequence if F is the Poincaré map associated with a periodic orbit Γ. Consider for instance the supercritical case. For$\lambda < 0$, the Poincaré map has a stable fixed point, which means that the periodic orbit Γ is stable. For $\lambda > 0$, the fixed point is unstable and there exists a stable orbit of periodtwo. In phase space, this means that the periodic orbit Γ has become unstable, but a newstable periodic orbit has appeared (Fig. 3.6). If Γ has period T for $\lambda = 0$, the new orbithas a period close to 2T, or half the frequency (which accounts for the name subharmonicgiven to the bifurcation). Note that this bifurcation requires a phase space with dimensionat least 3.

Now we will discuss Hopf bifurcation and invariant tori,We finally consider what happens when two eigenvalues cross the unit circle in complexplane. Then we have to study the map

$$u_{k+1} = Cu_k + G(u_k, \lambda) \qquad (3.69)$$

where $u \in R^2$, and C has eigenvalues $e^{\pm 2\pi i\,\theta_0}$ with $2\theta_0 \notin \mathbb{Z}$. We shall assume that $\lambda \in \mathbb{R}$and $G \in C^3$. As in the case of differential equations, the implicit function theorem showsthe existence of an equilibrium branch $u^\star(\lambda)$ through the origin. The linearization aroundthis branch has eigenvalues

$\mu(\lambda) = \rho(\lambda)e^{2\pi\, i\, \theta(\lambda)}$ and $\bar{\mu}(\lambda)$, with $\rho(0) = 1$ and

$\theta(0) = \theta_0$.

An appropriate linear change of variables casts the system (3.69) into the form

$$z_{k+1} = \mu(\lambda)z_k + g(z_k, \bar{z}_k, \lambda), \qquad (3.70)$$

where g is a nonlinear term, which we will try to simplify by normal form theory. We canassume that

$$g(z, \bar{z}, \lambda) = g_2(z, \bar{z}, \lambda) + g_3(z, \bar{z}, \lambda) + \mathcal{O}(|z|^3), \qquad (3.71)$$

where g_2 is a homogeneous polynomial of degree 2 in $z, \bar{z}$, and g_3 is of degree 3. Normalform theory for maps is very similar to normal form theory for differential equations. In fact, if $h_2(z, \bar{z}, \lambda)$ satisfies the homological equation

$$h_2(\mu(\lambda)z, \bar{\mu}(\lambda)\bar{z}, \lambda) - \mu(\lambda)h_2(z, \bar{z}, \lambda) = g_2(z, \bar{z}, \lambda), \qquad (3.72)$$

then it is easy to see that the transformation $z = \zeta + h_2(\zeta, \bar{\zeta}, \lambda)$ eliminates terms of order 2 from (3.70). A similar transformation can be used to eliminate terms of order 3. Letus now try to solve the homological equation, assuming

$$h_2(z, \bar{z}, \lambda) = \sum_{p+q=2} h_{pq}(\lambda)z^p\bar{z}^q, \quad g_2(z, \bar{z}, \lambda) = \sum_{p+q=2} c_{pq}(\lambda)z^p\bar{z}^q. \qquad (3.73)$$

Substitution into (3.72) shows that

$$h_{pq}(\lambda) = \frac{c_{pq}(\lambda)}{\mu(\lambda)^p\bar{\mu}(\lambda)^q - \mu(\lambda)}. \qquad (3.74)$$

In particular, for $\lambda = 0$, we obtain

$$h_{pq}(0) = \frac{c_{pq}(0)}{e^{2\pi\, i\theta_0}[e^{2\pi\, i\theta_0(p-q-1)} - 1]}. \qquad (3.75)$$

This equation can only be solved under the non-resonance condition

$$e^{2\pi\, i\theta_0(p-q-1)} \neq 1. \qquad (3.76)$$

Since $2\theta_0 \notin \mathbb{Z}$, this condition is always satisfied for $(p,q) = (2,0)$ and $(1,1)$. Thus terms ofthe form z^2 and $z\bar{z}$ can always be eliminated. Terms of the form $\bar{z}^2$, however, can only be eliminated if $3\theta_0$ is not an integer. Similarly, third order terms of the form z^3 and $z\bar{z}^2$ canalways be eliminated, terms of the form $\bar{z}^3$ can only be eliminated if $4\theta_0$ is not an integer,while the term $z^2\bar{z}$ can never be removed. We conclude that if $e^{2\pi\, i\theta_0}$ is neither a cubic nora quartic root of unity, there exists a polynomial change of variables transforming (3.70)into its normal form

$$\zeta_{k+1} = \mu(\lambda)\zeta_k + c(\lambda)|\zeta_k|^2\zeta_k + \mathcal{O}(|\zeta_k|^3). \qquad (3.77)$$

Using polar coordinates $\zeta_k = r_k e^{2\pi\, i\varphi_k}$, one obtains a map of the form

$$\begin{aligned} r_{k+1} &= \rho(\lambda)r_k + \alpha(\lambda)r_k^3 + \mathcal{O}(r_k^3) \\ \varphi_{k+1} &= \varphi_k + 2\pi\theta(\lambda) + \beta(\lambda)r_k^2 + \mathcal{O}(r_k^2) \end{aligned} \qquad (3.78)$$

An explicit calculation shows that if$c(\lambda) = |c(\lambda)|e^{2\pi\, i\psi(\lambda)}$, then the coefficients α and $\beta$are given by

$$\begin{aligned} \alpha(\lambda) &= |c(\lambda)| \cos 2\pi\big(\psi(\lambda) - \theta(\lambda)\big), \quad \beta(\lambda) \\ &= \frac{|c(\lambda)|}{\rho(\lambda)} \sin 2\pi\,\big(\psi(\lambda) - \theta(\lambda)\big) \qquad (3.79) \end{aligned}$$

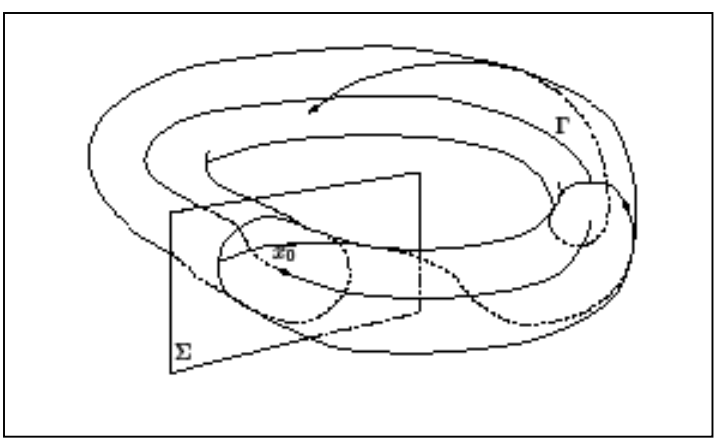

Figure 3.7

If the Poincaré map of a periodic orbit Γ undergoes supercritical Hopf bifurcation,an attracting invariant torus is created.

If we neglect the remainders, the map (3.78) describes a pitchfork bifurcation for theradial variable r. Depending on the signs of $\rho'(0)$ and $\alpha(0)$, there will be creation ordestruction of an invariant circle. This can be proved to remain true when the $\mathcal{O}(\cdot)$ termsare present, but the proof is more difficult than in the case of differential equations.

Theorem (3.3.3): (Ruelle)

Let $F(\cdot,\lambda): \mathbb{R}^2 \to \mathbb{R}^2$ be a one-parameter family of maps,admitting a smooth curve of fixed points $u^\star(\lambda)$. Assume the linearization around the fixedpoints has complex conjugate eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ such that

$$|\mu(0)| = 1 \quad \text{but}\, \mu(0)^j \neq 1 \quad \text{for}\, j = 1,2,3,4, \tag{3.80}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}|\mu(\lambda)|_{\lambda=0} \neq 0. \tag{3.81}$$

Then there is a smooth change of coordinates transforming the map F into (3.78). If$\alpha(0) \neq 0$, then the map admits, either for small positive λ or for small negative λ, aninvariant curve close to a circle of radius$\sqrt{|\lambda|}$.

In the strongly resonant cases $\mu(0)^j = 1$ for $j = 1,2,3$ or 4, the situation is morecomplicated, and there is no invariant curve in general. If such an invariant curve exists,the dynamics on this curve is described by a circle map. The theory of circle maps is ahuge subject in itself. Roughly speaking, they can be characterized by a rotation number,measuring the average angle of rotation per iteration. Two cases can occur:

i. if the rotation number is rational, there exists a periodic orbit, which usually attracts most orbits;
ii. if the rotation number is irrational, all orbits are dense, and under suitable smoothness

assumptions, the map is conjugate to a rotation.

If the bifurcating map is the Poincaré map of a periodic orbit Γ, the invariant circle will bethe intersection of an invariant toruswith the Poincaré section $\sum$ (Fig. 3.7). If the rotationnumber is irrational, orbits fill this torus in a dense way.

They are called quasiperiodic(with two frequencies), which means that any solution $x(t)$ on the invariant torus can bewritten as

$$x(t) = H(\omega t, \omega' t), \quad H(u+1, v) = H(u, v+1) = H(u, v), \qquad (3.82)$$

where $\omega/\omega'$is equal to the irrational rotation number. Besides invariant points, curves,and manifolds, we have thus found a new kind of invariant set appearing quite commonlyin dynamical systems.

Chapter 4

Introduction to Chaotic Dynamics

The characterization of chaotic dynamics is a large subject, with many recent developments,and ramifications in several domains of Mathematics. We will discuss here only a few selected topics, the major objective being to provide an idea of what we call chaoticmotion.

The definition of chaos may vary from system to system, but usually at least one ofthe following elements is present:

i. The time dependence of solutions is more complicated than stationary, periodic orquasiperiodic.
ii. The motion is very sensitive to variations in the initial conditions: nearby solutionsdiverge exponentially fast.
iii. The asymptotic motion takes place on a geometrically complicated object (often afractal), called a strange attractor.
iv. Chaotic orbits coexist with a (countable) infinity of unstable periodic orbits; the numberof orbits of period less or equal T grows exponentially fast with T.
v. As time goes by, the images under the flow of any two subsets of phase space getentangled in a complicated way.

These properties do not necessarily all occur at once: for instance, strange attractorscannot occur in conservative systems, but are quite typical in chaotic dissipative systems.Often, there are subtle relations between the above properties, many of which can becharacterized quantitatively (by Liapunov exponents, topological entropy, …)[7],[14].

Our approach in this introduction to chaotic dynamics will be to start with some verysimple examples, where many of these chaotic properties can be proven to hold. Then we will show how similar properties can be proven to exist for more realistic systems.

Section (4.1): Symbolic dynamics

Symbolic dynamics is a very useful technique for characterizing dynamical systems. Itconsists in associating with every orbit a sequence of symbols, related to a partition ofphase space, and describing in which order the orbit of x visits the elements of the partition. In this way, one can sometimes

reduce the problem to a combinatorial one, by transposingthe dynamics to the space of allowed symbolic sequences.

We will start by illustrating the method on a very simple one dimensional map calledthe tent map, and later discuss a less trivial two dimensional map which is already useful in proving existence of chaotic orbits for a more general class of systems.

Now we will discuss the tent map, we claimed that the logistic map

$$f_\lambda(y) = \lambda y(1-y) \qquad (4.1)$$

is as random as coin tossing when $\lambda = 4$. Let us now explain what we meant. It is easyto check that the transformation $y = h(x) = \frac{1}{2}(1-\cos \pi x)$transforms the logistic map $f_4$into the map

$$g_2(x) = h^{-1} \circ f_4 \circ h(x) = \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2} \\ 2-2x & \text{for } \frac{1}{2} \le x \le 1 \end{cases} \qquad (4.2)$$

This map is called the tent mapbecause of its triangular shape (one can define moregeneral $g_\mu$with slope $\mu \neq 2$). In the transformation, we have lost differentiability at$x = \frac{1}{2}$, but the piecewise linearity will be useful to classify the orbits.

First of all, we note that the map (4.2) has two unstable fixed points, at 0 and $2/3$.

We also observe that 1 is mapped to 0 and $1/2$ is mapped to 1, so there exist "transient"orbits, ending at 0 after finitely many iterations.

In order to understand the other orbits, it turns out to be a good idea to write x inbinary expansion. We will write

$$x = \sum_{i=0}^{\infty} b_i 2^{-i}, \quad b_i \in \{0,1\}, \qquad \Rightarrow x = b_0 \cdot b_1 b_2 b_3 \qquad (4.3)$$

This decomposition is not unique. Indeed,

$$\sum_{i=1}^{\infty} 2^{-i} \quad = 1 \quad \Rightarrow 0.1^\infty = 1.0^\infty, \qquad (4.4)$$

where the superscript ∞ means that the symbol is repeated indefinitely. This, however, isthe only kind of degeneracy. For our purposes, the following

convention will be useful. Wedenote by $\beta$the set of symbolic sequences $\underline{b} = 0.b_1b_2 \ldots$ which are notterminated by 0^∞, except that we include the sequence $0 \cdot 0^\infty$. Then (4.3) defines a bijection $\underline{b}: [0,1] \to \beta$.In fact, $\beta$is a metric space with the distance

$$d(\underline{b}, \underline{b'}) = \sum_{i=1}^{\infty} \left(1 - \delta_{b_i b_i'}\right) 2^{-i} \qquad (4.5)$$

where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise, and $\underline{b}$ is continuous in the resulting topology.

How does the tent map act on binary expansions? We first observe that $x \leq \frac{1}{2}$ ifand only if $b_1 = 0$. In this case,

$$g_2(0.0b_2b_3) = 2\sum_{i=2}^{\infty} b_i 2^{-i} = \sum_{i=1}^{\infty} b_{i+1} 2^{-i} = 0.b_2b_3 \ldots \qquad (4.6)$$

Otherwise, we have $x > 1/2, b_1 = 1$ and

$$g_2(0.1b_2b_3) = 2 - 2\left[\frac{1}{2} + \sum_{i=2}^{\infty} b_i 2^{-i}\right] = 1 - \sum_{i=1}^{\infty} b_{i+1} 2^{-i}$$

$$= \sum_{i=1}^{\infty} (1 - b_{i+1}) 2^{-i} = 0.(1 - b_2)(1 - b_3) \ldots \qquad (4.7)$$

Hence g_2 induces a map $\mathcal{T}$ from $\beta$to itself, defined by the following rules:

i. shift all digits of $\underline{b}$ one unit to the left, discarding b_0;
ii. if the first digit is 1, reverse all digits;
iii. replace the sequence 10^∞, if present, by 01^∞.

We can now analyse the dynamics in β instead of [0,1], which is easier due to the relativelysimple form of $\mathcal{T}$. In fact$g_2$has an even simpler representation. With every $\underline{b} \in \mathcal{B}$, weassociate a sequence $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots)$ defined by

$$\varepsilon_j = (-1)^{b_{j-1} + b_j}, \quad j = 1,2, \ldots \qquad (4.8)$$

The elements of ε are -1 if adjacent digits of $\underline{b}$ are different, and $+1$ if they are equal. Theset $\sum$ of sequences $\underline{\varepsilon}$ constructed in this way consists of $\{-1, +1\}^{\mathbb{N}}$, from which we excludesequences ending with $(+1)^\infty$and containing an even number of -1. The correspondence(4.8) admits an inverse defined by

$$b_0 = 0, (-1)^{b_j} = \varepsilon_j (-1)^{b_{j-1}} = \prod_{i=1}^{j} \varepsilon_i \ , \ \ b_j \in \{0,1\}, \quad j \geq 1 \qquad (4.9)$$

$\sum$ can be endowed with a similar distance as (4.5), and then (4.8)defines a homeomorphismbetween $\mathcal{B}$and $\sum$, and by composition with $\underline{b}$, a homeomorphism between $[0,1]$and $\sum$.

Treating separately the cases $b_1 = 0$ and $b_1 = 1$, it is easy to see that the tent mapinduces a dynamics in $\sum$ given by

$$\sigma = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots) \longmapsto (\varepsilon_2, \varepsilon_3, \dots) \qquad (4.10)$$

which is called the shift map. The sequence $\underline{\varepsilon}$ has a very simple interpretation. Indeed,

$$x \in \left[0, \tfrac{1}{2}\right] \ \Leftrightarrow \ b_1 = 0 \ \Leftrightarrow \ \varepsilon_1 = +1, \qquad (4.11)$$

so that

$$\varepsilon_j = +1 \quad \Leftrightarrow \quad \sigma^j(\underline{\varepsilon})_1 = +1 \quad \Leftrightarrow g_2^j(x) \in \left[0, \tfrac{1}{2}\right] \qquad (4.12)$$

We have thus shown that the sequence of $\varepsilon_j$indicates whether the jthiterate of x is to theleft or to the right of $1/2$, and that this information is encoded in the binary expansion of$\underline{b}(x)$. The sequence $\underline{\varepsilon}$ associated with x is called its itinerary.

Let us now examine the different possible orbits.

1. Periodic orbits: The itinerary of a periodic orbit must be periodic. Conversely, if an itinerary $\underline{\varepsilon}$ is periodic with period$p \geq 1$, then $\sigma^p(\underline{\varepsilon}) = \underline{\varepsilon}$, and by bijectivity the corresponding x is a fixed point of p_2^g. Thus x is a point of period p if and only ifits itinerary is of the form $\underline{\varepsilon} = A^\infty$, with A a finite sequence of length p. There are exactly 2^psuch sequences, and thus $2^p/p$ orbits of period p (the number of orbits of least period p can be a bit smaller). For instance, we obtain again the fixed points of g_2:

$$\underline{\varepsilon} = (+1)^\infty \quad \Leftrightarrow \underline{b} = 0.0^\infty \qquad \Leftrightarrow \quad x = 0$$
$$\underline{\varepsilon} = (-1)^\infty \quad \Leftrightarrow \underline{b} = 0.(10)^\infty \quad \Leftrightarrow \quad x = \tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{32} + \cdots = \tfrac{2}{3} \qquad (4.13)$$

2. Transient orbits: Itineraries of the form $\underline{\varepsilon} = BA^\infty$with B a finite sequence of length q correspond to orbits that reach a periodic orbit after a finite number of steps.

Let us call eventually periodicorbits which are either periodic, or reach a periodic orbit after a finite number of iterations. Their itineraries are of the form $\underline{\varepsilon} = A^{\infty}$or $\underline{\varepsilon} = BA^{\infty}$. It is easy to see that the orbit of $x \in [0,1]$ is eventually periodic if and only if$x$ is rational.$\mathbb{Q}$ being dense in $\mathbb{R}$, the union of all eventually periodic orbits is dense in $[0,1]$. However, this set is countable and has zero Lebesgue measure.

3. Chaotic orbits: All irrational initial conditions, by contrast, admit itineraries which are not periodic. The corresponding orbits will typically look quite random.

The following properties are direct consequences of the symbolic representation:

i. For every sequence $\underline{\varepsilon} \in \sum$, there exists an $x \in [0,1]$ with itinerary $\underline{\varepsilon}$. We may thus choose the initial condition in such a way that the orbit passes left and right of $1/2$ in any prescribed order.
ii. The dynamics is sensitive to initial conditions. If we only know x with finite precisionδ, we are incapable of making any prediction on its orbit after n iterations, whenever$2^n \geq 1/\delta$.
iii. However, simply by looking whether successive iterates of x lie to the left or rightof $1/2$, we are able to determine the binary expansion of x (even though g_2 is notinjective).
iv. There exists a dense orbit. Indeed, choose x in such a way that its itinerary

$$\underline{\varepsilon} = (\underbrace{+1, -1}_{\text{period 1}}, \underbrace{+1, +1, +1, -1, -1, +1, -1, -1}_{\text{period 2}}, \underbrace{+1, +1, +1}_{\text{period 3}}, \dots, \dots) \quad (4.14)$$

contains all possible finite sequences, ordered by increasing length. Given any $\delta > 0$and $y \in [0,1]$, one can find $n \in \mathbb{N}$ such that $|g_2^n(x) - y| < \delta$.It suffices to take $n$in such a way that the shifted sequence $\sigma^n \underline{\varepsilon}$ and the itinerary of y agree for the first$\lceil \log \delta / \log 2 \rceil$bits.

Remark (4.1.1):

The tent map g_2 has other interesting properties, from the point of viewof measure theory. The main property is that its unique invariant measure which isabsolutely continuous with respect to the Lebesgue measure is the uniform measure. Thismeasure is ergodic, meaning that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi\left(g_2^n(x)\right) = \int_0^1 \varphi(x)\, dx \qquad (4.15)$$

for every continuous test function φ, and for Lebesgue-almost all $x \in [0,1]$. Thus from aprobabilistic perspective, almost all orbits will be uniformly distributed over the interval.

Itineraries can be defined for more general maps, just by choosing some partition ofphase space. In general, however, there will be no simple relation between a point andits itinerary, and the correspondence need not be one-to-one. Some symbolic sequences

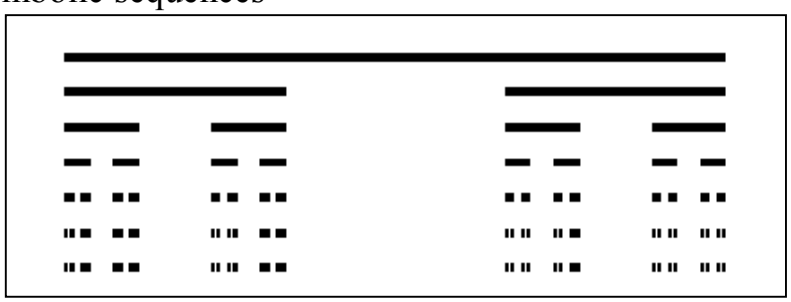

Figure 4.1

The first steps of the construction of the Cantor set Λ.

may never occur, and several initial conditions may have the same itinerary, for instanceif they are in the basin of attraction of a stable equilibrium.

Let us now consider the following variant of the tent map:

$$g_3(x) = \begin{cases} 3x & \text{for } x \le \frac{1}{2} \\ 3 - 3x & \text{for } x \ge \frac{1}{2} \end{cases} \qquad (4.16)$$

Since the interval [0,1] is not invariant, we define g_3 on all of R. We first observe that

i. if $x_0 < 0$, the orbit of x_0 converges to $-\infty$;
ii. if $x_0 > 1$, then $x_1 = g_3(x_0) < 0$;
iii. if $x_0 \in (1/3\,, 2/3)$, then $x_1 = g_3(x_0) > 1$ and thus $x_2 < 0$.

Hence all orbits which leave the interval [0,1] eventually converge to $-\infty$. One couldsuspect that all orbits leave the interval after a certain number of iterations. This is notthe case since, for instance, orbits starting in multiples of 3^{-n}reach the fixed point 0 afterfinitely many iterations. But there exists a more subtle nontrivial invariant set. In orderto describe it, we use a ternary (base 3) representation of $x \in [0,1]$:

$$x = \sum_{i=0}^{\infty} b_i 3^{-i}, \quad b_i \in \{0,1,2\} \quad \Rightarrow \quad x = b_0 \cdot b_1 b_2 b_3 \ldots \tag{4.17}$$

Again, this representation is not unique. We can make it unique by replacing $10^{\infty}$with$02^{\infty}$and$12^{\infty}$with 20^{∞}if applicable. Now we observe that

$$g_3(0.0b_2b_3 \ldots) = 0.b_2b_3 \ldots$$

$$g_3(0.2b_2b_3 \ldots) = 0.(2-b_2)(2-b_3) \ldots \tag{4.18}$$

Points of the form$0.1b_2b_3 \ldots$ belong to $(1/3\,, 2/3)$, and we already know that these leave the interval [0,1]. It is thus immediate that the orbit of x never leaves the interval [0,1] ifand only if its ternary expansion b(x) does not contain the symbol 1. The largest invariant subset of [0,1] is thus

$$\Lambda = \{x \in [0,1]: b_i(x) \neq 1 \forall i \geq 1\} \tag{4.19}$$

Λ is obtained by removing from [0,1] the open intervals

$(1/3, 2/3), (1/9, 2/9), (7/9, 8/9), (1/27, 2/27)$ and so on (Fig. 4.1). The resulting set is called a Cantor set:

Definition (4.1.2):

A set Λ is called a Cantor set if it is closed, its interior is empty, andall its points are accumulation points.

Definition (4.1.3):

Let $\mathcal{M}$be a subset of$\mathbb{R}^d$. Assume that for any $\varepsilon > 0$, M can becovered by a finite number of hypercubes of side lengthε. Let $N(\varepsilon)$ be the smallest possiblenumber of such cubes. Then the box-counting dimension of $\mathcal{M}$is defined by

$$D_0(\mathcal{M}) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}. \qquad (4.20)$$

One easily shows that for "usual" sets $\mathcal{M} \subset \mathbb{R}^d$, such as a d-dimensional hypercube,$D_0 = d$. For the Cantor set (4.19), however, we find that $N(\varepsilon) = 2^n$if $3^{-n} \leq \varepsilon < 3^{-n+1}$,and thus

$$D_0(\Lambda) = \frac{\log 2}{\log 3}. \qquad (4.21)$$

The map g_3 restricted to Λ is conjugate to the tent map g_2. Indeed, with any point$x = 0 \cdot b_1 b_2 b_3$ ...in Λ, we can associate

$h(x) = 0.(b1/2)(b2/2)(b3/2)\ldots$ in [0,1], and the relations (4.18) and (4.6), (4.7) show that

$$h \circ g_{3\ |\Lambda} \circ h^{-1} = g_2. \qquad (4.22)$$

Thus we can easily compute itineraries of points $x \in \Lambda$, and the conclusions on thebehaviour of orbits of g_2 can be transposed to $g_{3|\Lambda} \cdot \Lambda$is the simplest example of what iscalled a hyperbolic invariant set.

Similar properties can be seen to hold for nonlinear perturbations of the tent map.Indeed, the ternary representation is a useful tool, but not essential for the existence oforbits with all possible symbolic representations. It is, in fact, sufficient that f maps twodisjoint intervals $I_1, I_2 \subset [0,1]$ onto [0,1], and maps all other points outside [0,1]. Thenthe symbolic representation corresponds to the sequence of intervals I_1 or I_2visited bythe orbit.

Now we will discuss the homoclinic tangles and smale's Horseshoe Map. One-dimensional maps are rather special cases of dynamical systems, but it turns outthat some of their properties can often be transposed to more "realistic" systems. As amotivating example, let us consider the equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 \end{aligned} \qquad (4.23)$$

called the (undamped) unforced Duffing oscillator. This is a Hamiltonian system, withHamiltonian

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4 - \frac{1}{2}x_1^2. \tag{2.24}$$

H is a constant of the motion, and thus orbits of (4.23) belong to level curves of H.The point (0,0) is a hyperbolic equilibrium, with the particularity that itsunstable and stable manifolds W^u and W^s are identical: they form the curve $H = 0$, andare called homoclinic loops.

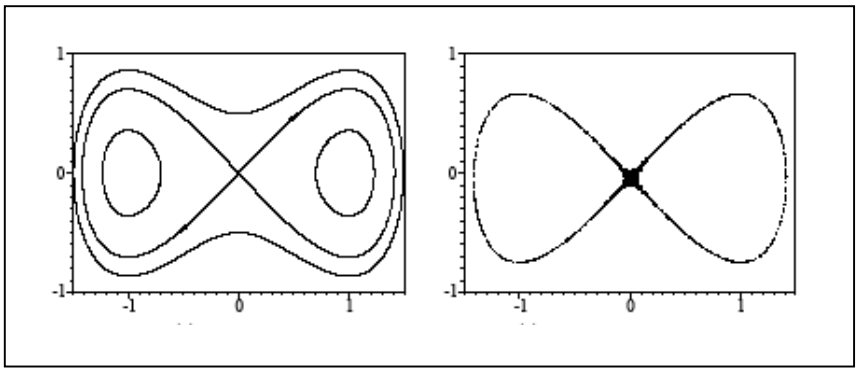

Figure 4.2

(a) Orbits of Duffing's equation for $\varepsilon = 0$, (b) one orbit of the Poincaré map obtained for $\varepsilon = 0.25$.

Let us now perturb (4.23) by a periodic forcing,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 + \varepsilon \sin(2\pi t) \end{aligned} \tag{4.25}$$

We can introduce $x_3 = t$ as additional variable to obtain the autonomous system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 + \varepsilon \sin(2\pi x_3) \\ \dot{x}_3 &= 1 \end{aligned} \tag{4.26}$$

Here $x_3 \in \mathbb{S}^1$ should be considered as a periodic variable. We can thus use the surface$x_3 = 0$ as a Poincar´e section. Fig.(4.2) b shows an orbit of the Poincar´e map, which has acomplicated structure: it seems to fill a two-dimensional region, in which its dynamics isquite random.

This phenomenon can be explained at least partially, by analysing the properties of the Poincaré map P_ε. First observe that the system (4.26) is conservative. The section beingperpendicular to the flow, the Poincaré map is also conservative. When $\varepsilon = 0, P_0 = \varphi_1$ is the time-one flow of (4.23), and thus P_0 admits the

origin as hyperbolic fixed point.The implicit function theorem implies that for small $\varepsilon, P_\varepsilon$admits an isolated hyperbolicfixed point $x^*(\varepsilon)$ near (0,0). It is unlikely that the stable and unstable manifolds of $x^*(\varepsilon)$still form a loop when $\varepsilon > 0$, and if not, they must intersect transversally at some point x_0 because of area conservation. x_0 is called a homoclinic point. A method due to Melnikovallows to prove that such a transverse intersection indeed exists.

Consider now the successive images $x_n = P_\varepsilon^n(x_0)$. Since x_0 belongs to the stablemanifold W^s of $x^*(\varepsilon)$, all $x_n$must also belong to W^s, and they must accumulate at$x^*$for $n \to \infty$. Similarly, since x_0 belongs to the unstable manifold W^u, all $x_n$mustbelong to W^u and accumulate at $x^*$for $n \to -\infty$. Thus W^s and W^u must intersectinfinitely often. Because the map is area preserving, the area between W^s, W^u, andany consecutive intersection points must be the same, and thus W^u has to oscillate withincreasing amplitude when approaching x^*(Fig. 4.3). Ws has a similar behaviour. Thiscomplicated geometrical structure, which was first described by Poincaré, is called thehomoclinic tangle.

The dynamics near the homoclinic tangle can be described as follows: Let Q be a smallrectangle, containing a piece of unstable manifold W^u and two points x_{-m}, x_{-m+1} of thehomoclinic orbit (Fig. 4.4).

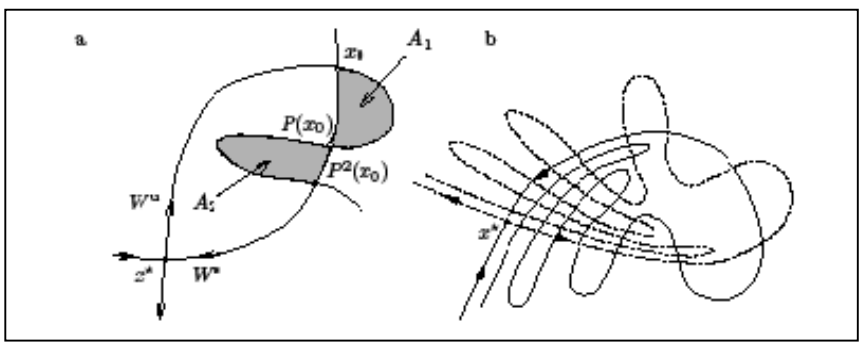

Figure 4.3

If the stable and unstable manifolds of a conservative map intersect transversally,they must intersect infinitely often, forming a homoclinic tangle.

At least during the first iterations of P_ε, Q will be stretchedin the unstable direction, and contracted in the stable one. After m iterations, the image$P_\varepsilon^m(Q)$ contains x_0 and x_1. Let n be sufficiently large that the points $x_n$and x_{n+1} of thehomoclinic orbit are close to x^*. Due to area conservation, one can arrange

that the pieceof W^u between $x_n$and x_{n+1} crosses Q at least twice, and so does the image $P_{\varepsilon}^{m+n}(Q)$.

This behaviour is reproduced qualitatively by the Smale horseshoe map(Fig. 4.5). Thismap T takes a square $[0,1] \times [0,1]$, stretches it vertically by a factor $\mu > 2$, and contracts it horizontally by a factor $\lambda < 1/2$. Then it bends the resulting rectangle in the shape of ahorseshoe, and superimposes it with the initial square. Two horizontal strips H_+ and H_-, of size $1 \times \mu^{-1}$, are mapped, respectively, to two vertical strips V_+ and V_-, of size $\lambda \times 1$.For simplicity, the map restricted to $V_+ \cup V_-$is assumed to be linear, but its qualitativefeatures can be shown to remain unchanged by small nonlinear perturbations.

The aim is now to construct an invariant set Λ of T. To this end, we observe that$T(x) \in Q$ if and only if $x \in H_- \cup H_+$. Since

$T(x) \in V_- \cup V_+, T^2(x) \in Q$ if and only if$T(x)$ belongs to one of the four rectangles $\Gamma_{\varepsilon_0\varepsilon_1} = V_{\varepsilon_0} \cap H_{\varepsilon_1}$, where $\varepsilon_0, \varepsilon_1 \in \{-,+\}$. Theserectangles have size $\lambda \times \mu^{-1}$. The preimage of each $\Gamma_{\varepsilon_0\varepsilon_1}$ is a rectangle $H_{\varepsilon_0\varepsilon_1} \subset H_{\varepsilon_0}$, of size $1 \times \mu^{-2}$. The image of $\Gamma_{\varepsilon_0\varepsilon_1}$ is a rectangle $V_{\varepsilon_0\varepsilon_1} \subset V_{\varepsilon_1}$, of size $\lambda^2 \times 1$. Thus T^2 maps$H_{\varepsilon_0\varepsilon_1}$to $V_{\varepsilon_0\varepsilon_1}$.

More generally, we can define

$$H_{\varepsilon_0\varepsilon_1\ldots\varepsilon_n} = \{x \in Q: x \in H_{\varepsilon_0}, T(x) \in H_{\varepsilon_1}, \ldots, T^n(x) \in H_{\varepsilon_n}\}. \qquad (4.27)$$

Note that $H_{\varepsilon_0\varepsilon_1\ldots\varepsilon_n} \subset H_{\varepsilon_0\ldots\varepsilon_{n-1}}$, and that

$$H_{\varepsilon_0\varepsilon_1\ldots\varepsilon_n} = \{x \in H_{\varepsilon_0}: T(x) \in H_{\varepsilon_1\ldots\varepsilon_n}\}. \qquad (4.28)$$

By induction, it is straightforward to show that $H_{\varepsilon_0\varepsilon_1\ldots\varepsilon_n}$ is a rectangle of size $1 \times \mu^{-n}$.Similarly, we introduce

$$\begin{aligned} V_{\varepsilon_{-m}\ldots\varepsilon_{-1}} &= \{x \in Q: T^{-1}(x) \in H_{\varepsilon-1}, \ldots, T^{-m}(x) \in H_{\varepsilon-m}\} \\ &= \{x \in V_{\varepsilon-1}: T^{-1}(x) \in V_{\varepsilon-m\ldots\varepsilon-2}\}, \end{aligned} \qquad (4.29)$$

which is a rectangle of size $\lambda^m \times 1$ contained in $V_{\varepsilon-m+1\ldots\varepsilon-1}$. It follows that

$$\begin{aligned} \Gamma_{\varepsilon-m\ldots\varepsilon-1,\varepsilon_0\varepsilon_1\ldots\varepsilon_n} &= V_{\varepsilon-m\ldots\varepsilon-1} \cap H_{\varepsilon_0\varepsilon_1\ldots\varepsilon_n} \\ &= \{\, x \in Q: T^j(x) \in H_{\varepsilon_j}, -m \le j \le n \,\} \end{aligned} \qquad (4.30)$$

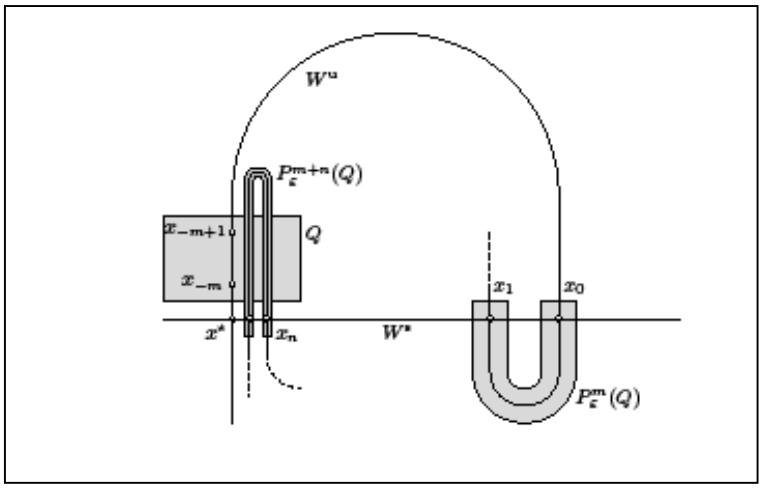

Figure 4.4

Schematic representation of the homoclinic tangle.

The homoclinic orbit$\{x_n\}_{n\in\mathbb{Z}}$ is asymptotic to x^* for $n \to \pm\infty$. One can choose m and n in such a way thatthe image of a rectangle Q after $m+n$ iterationsintersects Q twice.

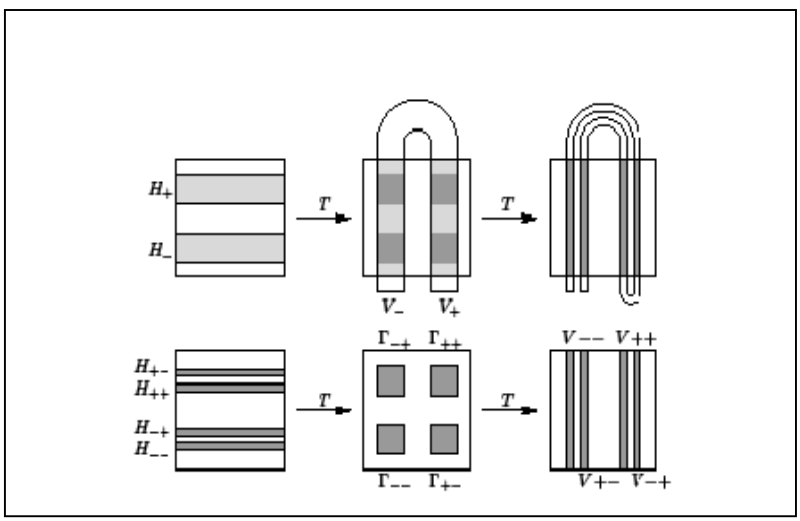

Figure 4.5

The horseshoe map T maps the rectangles $H_\pm$of the unit square Q to therectangles $V_\pm$.

If$\Gamma_{\varepsilon_0\varepsilon_1} = V_{\varepsilon_0} \cap H_{\varepsilon_1}$, one also has $T\left(H_{\varepsilon_0\varepsilon_1}\right) = \Gamma_{\varepsilon_0\varepsilon_1}$ and $T\left(\Gamma_{\varepsilon_0\varepsilon_1}\right) = V_{\varepsilon_0\varepsilon_1}$.

is a rectangle of size $\lambda^m \times \mu^{-n}$. By construction, $T_j(x) \in Q$ for

$-m \leq j \leq n+1$ if andonly if $x \in \Gamma_{\varepsilon_{-m},\dots,\varepsilon_0 \dots \varepsilon_n}$for some sequence $\varepsilon_{-m}, \dots, \varepsilon_0 \dots \varepsilon_n$ of symbols -1 and $+1$.

Now let n, m go to ∞. Let $\sum$ be the set of b_i-infinite sequences

$\underline{\varepsilon} = \varepsilon_{-1}, \varepsilon_0 \varepsilon_1 \dots$ ofsymbols -1 and $+1$. It is a metric space for the distance

$$d(\underline{\varepsilon}, \underline{\varepsilon'}) = \sum_{i=-\infty}^{\infty} (1 - \delta_{\varepsilon_i \varepsilon_i'}) 2^{-|i|}. \tag{4.31}$$

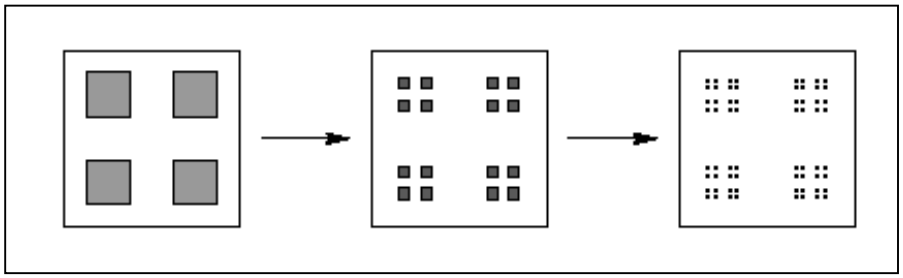

Figure 4.6

Successive approximations of the invariant set Λ.

From (4.30) we conclude that the largest subset of Q invariant under T is

$$\Lambda = \bigcap_{n=-\infty}^{+\infty} T^n(Q) = \{\Gamma_{\underline{\varepsilon}} : \underline{\varepsilon} \in \sum\}, \tag{4.32}$$

where $\Gamma_{\underline{\varepsilon}}$, defined as the limit of (4.30) when the finite sequence $\varepsilon_{-m}, \dots, \varepsilon_0 \dots \varepsilon_n$ convergesto $\underline{\varepsilon}$, is a single point. Λ is a Cantor set, obtained by taking the product of twoone-dimensional Cantor sets. The map

$\phi: \underline{\varepsilon} \mapsto \Gamma_{\underline{\varepsilon}}$is a bijection from $\sum$ to Λ, which is continuous in the topology defined by (4.31). It follows from (4.30) that T is conjugatedto the shift map

$$\sigma = \phi^{-1} \circ T_{|\Lambda} \circ \phi : \dots \varepsilon_{-2}\varepsilon_{-1}, \varepsilon_0 \varepsilon_1 \dots \mapsto \varepsilon_{-1}\varepsilon_0, \varepsilon_1 \varepsilon_2 \dots \tag{4.33}$$

This conjugacy can be used to describe the various orbits of $T|_\Lambda$ in a similar way as we didfor the tent map. In particular, periodic itineraries $\underline{\varepsilon}$ correspond to periodic orbits of T,aperiodic itineraries correspond to chaotic orbits. Smale has shown that these qualitativeproperties are robust under small perturbations of T[11].

Theorem (4.1.4) (Smale):

The horseshoe map T has an invariant Cantor set Λ suchthat

i. Λ contains a countable set of periodic orbits of arbitrarily long periods;
ii. Λ contains an uncountable set of bounded nonperiodic orbits;
iii. Λ contains a dense orbit.

Moreover, any map $\tilde{T}$ sufficiently close to T in the C^1 topology has an invariant Cantorset $\tilde{\Lambda}$ with $\tilde{T}|_{\tilde{\Lambda}}$topologically equivalent to $T|_{\Lambda}$.

The example of the horseshoe map can be generalised to a class of so-called axiom Asystems with similar chaotic properties. These systems can be used to show the existenceof chaotic orbits in a large class of dynamical systems, including (but not limited to) thosewith a transverse homoclinic intersection.

Note that the examples of hyperbolic invariant sets that we have encountered havemeasure zero and are not attracting. Showing the existence of invariant sets containingchaotic orbits and attracting nearby orbits is a much more difficult task.

Section (4.2): Strange Attractors

Now we will discuss theattracting sets and attractors.

We consider in this section a dynamical system on $\mathcal{D} \subset \mathbb{R}^n$, defined by a flow$\varphi_t$. One caninclude the case of iterated maps by restricting t to integer values and setting $\varphi_t = F^t$.Various subsets of D can be associated with the flow.

Definition (4.2.1):

i. A subset $S \subset \mathcal{D}$is called invariant if $\varphi_t(S) = S$for all t.
ii. A subset $\mathcal{A} \subset \mathcal{D}$is called an attracting set if there exists a neighborhood$\mathcal{U}$of $\mathcal{A}$such that for all $x \in \mathcal{U}, \varphi_t(x) \in \mathcal{U}$for all $t > 0$ and $\varphi_t(x) \to A$as $t \to \infty$.
iii. A point x is nonwandering for $\varphi_t$if for every neighborhood$\mathcal{U}$of x and every $T > 0$, there exists a $t > T$ such that $\varphi_t(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. The set of all nonwandering points is the nonwandering set Ω.

These sets can be partly determined by looking at the asymptotic behaviour of variousorbits of the flow.

Definition (4.2.2):

The ω-limit set of x for $\varphi_t$is the set of $y \in \mathcal{D}$such that there exists asequence $t_n \to \infty$with $\varphi_{t_n}(x) \to y$. The α-limit set of x for $\varphi_t$is the set of $y \in \mathcal{D}$suchthat there exists a sequence $t_n \to -\infty$with $\varphi_{t_n}(x) \to y$.

The α- and ω-limit sets of any x are invariant sets, and the ω-limit set is included in thenonwandering set. One can show that asymptotically stable equilibrium points, periodicorbits, and invariant tori are all ω-limit sets of the orbits in their basin of attraction,nonwandering sets and attracting sets. There is, however, a problem with the definitionof attracting set, as shows the following example.

Example (4.2.3):

Consider the differential equation

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 \\ \dot{x}_2 &= -x_2 \end{aligned} \qquad (4.34)$$

The ω-limit set of x is $(-1,0)$ if $x_1 < 0, (0,0)$ if $x_1 = 0$ and $(1,0)$ if $x_1 > 0$. $(0,0)$ isalso the α-limit set of all points in $(-1,1) \times \{0\}$. The nonwandering set is composed ofthe three equilibrium points of the flow, while $(\pm 1,0)$ are attracting sets. However, thesegment $[-1,1] \times \{0\}$is also an attracting set.

One would like to exclude attracting sets such as the segment$[-1,1] \times \{0\}$in theexample, which contains wandering points. This is generally solved in the following way.

Definition (4.2.4):

A closed invariant set Λ is topologically transitive if $\varphi_t$has an orbitwhich is dense in Λ. An attractor is a topologically transitive attracting set.

Remark (4.2.5):

Sometimes, one uses a weaker notion of in decomposability than topologicaltransitivity, based on the notion of chain recurrence. In this case, one requires that for any pair of points x, y and any $\varepsilon > 0$, there exist points$x_0 = x, x_1, \dots, x_n = y$ andtimes $t_1, \dots, t_n$such that $\left\| \varphi_{t_j}(x_{j-1}) - x_j \right\| \leq \varepsilon$for all j.

Remark (4.2.6):

A slightly weaker definition of attractor is proposed in [4]. An attractoris an indecomposable closed invariant set Λ with the property that, given $\varepsilon > 0$,there is a set $\mathcal{U}$of positive Lebesgue measure in the ε-neighbourhood of Λ such that, if$x \in \mathcal{U}$, the forward orbit of x is contained in $\mathcal{U}$and the ω-limit set of x is contained in Λ.

Now we will discuss sensitive dependence on initial conditions .

a strange attractor is, basically, an attractor on which the dynamics is chaotic. Obviously,this requires that we define what we mean by "chaotic". For instance, a strangeattractor is defined as an attractor containing a transversal homoclinic orbit. As we saw, the existence of such an orbit implies various chaotic properties (it is notnecessary to assume that the system is conservative). Modern definitions are a bit lessspecific, they require that the dynamics be sensitive to initial conditions:

Definition (4.2.7):

Let Λ be a compact set such that $\varphi_t(\Lambda) \subset \Lambda$ for all $t \geq 0$.

i. The flow $\varphi_t$is said to have sensitive dependence on initial conditions on Λ if there exists $\varepsilon > 0$ with the following property: For any $x \in \Lambda$ and any neighborhood$\mathcal{U}$of x, there exists $y \in U$and $t > 0$ such that$\|\varphi_t(x) - \varphi_t(y)\| \geq \varepsilon$.
ii. Λ is a strange attractor if it is an attractor and $\varphi_t$has sensitive dependence on initial conditions on Λ.

Sensitive dependence on initial conditions requires that for any point$x$, one can find an arbitrarily close point y such that the orbits of x and$y$ diverge from each other (the quantity ε should not depend on $\|x - y\|$, though t may depend on it). This is a rather weak property, and one often requires that the divergence occur at an exponential rate.

Definition (4.2.8):

Assume $\varphi_t$is the flow of a differential equation $\dot{x} = f(x)$. Let $x \in \mathcal{D}$and let $U(t)$ be the principal solution of the equation linearized around the orbit of x,

$$\dot{y} = \frac{\partial f}{\partial x}\big(\varphi_t(x)\big)y \qquad (4.35)$$

proved that under quite weak assumptions on f, the limit

$$L = \lim_{t\to\infty} \frac{1}{2t}\log\big(U(t)^T U(t)\big) \qquad (4.36)$$

exists. The eigenvalues of L are called the Liapunov exponents of the orbit of x.

If F is an iterated map and $\{x_k\}$is a given orbit of F, consider the linear equation

$$y_{\kappa+1} = \frac{\partial F}{\partial x}(x_\kappa)_{y\kappa} \;\Rightarrow\; y_\kappa = U_\kappa y_0, \qquad U_\kappa = \frac{\partial F}{\partial x}(x_{\kappa-1}) \ldots \frac{\partial F}{\partial x}(x_0) \qquad (4.37)$$

The Liapunov exponents of the orbit $\{x_k\}$are the eigenvalues of

$$L = \lim_{\kappa\to\infty} \frac{1}{2\kappa}\log(U_\kappa^T U_\kappa). \qquad (4.38)$$

Note that $U(t)^T U(t)$ is a symmetric positive definite matrix, and hence it is always iagonalizable and has real eigenvalues. This definition implies that the solution $y(t)$ ofthe equation linearized around the particular solution $\varphi_t(x)$ satisfies

$$\begin{aligned} \|y(t)\|^2 &= \|U(t)y(0)\|^2 = y(0)\cdot U(t)^T U(t) y(0) \\ &\simeq y(0)\cdot e^{2tL} y(0). \end{aligned} \qquad (4.39)$$

Let $\lambda_1$be the largest eigenvalue of L. Unless the projection of $y(0)$ on the eigenspace of L associated with λ_1 is zero, $\|y(t)\|$will grow asymptotically like $e^{\lambda_1 t}$. We thus say that the flow has exponentially sensitive dependence on initial conditions in Λ if the largestLiapunov exponent of all orbits in Λ is positive.

Proposition (4.2.9):

i. If $\varphi_t$is conservative, then the sum of all Liapunov exponents of any orbit is zero.

ii. If the flow is dissipative, then this sum is negative.

iii. Assume $\{\varphi_t(x)\}$is an orbit of the differential equation $\dot{x} = f(x)$, such that $\|f\|$is bounded below and above by strictly positive constants on $\{\varphi_t(x)\}$. Then this orbit has at least one Liapunov exponent equal to zero.

Proof:

We saw in the proof of Proposition (2.1.10) that the determinant of $U(t)$ is constantif the system is conservative, and decreasing if the system is dissipative. By definition,$\det U(0) = 1$and by uniqueness of solutions, $\det U(t) > 0$ for all t. Thus for $t > 0$, theproduct of all eigenvalues of $U(t)$ is equal to 1 in the conservative case, and belongs to$(0,1)$ in the dissipative case. The same is true for $\det U(t)^T\, U(t)$. But for any matrix B,

$\det e^B = e^{T_r\, B}$,because the eigenvalues of $e^B$are exponentials of the eigenvalues of B.

Assume now $x(t) = \varphi_t(x)$ is a solution of $\dot{x} = f(x)$. Then

$$\frac{d}{dt}\dot{x}(t) = \frac{d}{dt}f\big(\varphi_t(x)\big) = \frac{\partial F}{\partial x}\big(\varphi_t(x)\big)\dot{x}(t),$$

and thus $\dot{x}(t) = U(t)\dot{x}(0)$ by definition of $U(t)$. Hence,

$$\dot{x}(0) \cdot U(t)^T U(T)\dot{x}(0) = \|U(t)\dot{x}(0)\|^2 = \|\dot{x}(t)\|^2 = \left\|f\big(\varphi_t(x)\big)\right\|^2$$

By the assumption on $\|f\|$, it follows that as $t \to \infty$, the function

$$\dot{x}(0) \cdot e^{2tL}\dot{x}(0)$$

is bounded above and below by strictly positive constants. Being symmetric, L admits anorthonormal set of eigenvectors $u_1, \ldots, u_n$, that is they satisfy$u_i \cdot u_j = \delta_{ij}$. If $c_j = u_j \cdot \dot{x}(0)$,then $\dot{x}(0) = \sum_j c_j u_j$and thus

$$\dot{x}(0) \cdot e^{2tL}\dot{x}(0) = \sum_{j=1}^{n} c_j^2 e^{2\lambda_j t}.$$

Since $\dot{x}(0) \neq 0$, at least one of the coefficients, say $c_k$is different from zero, and thus λ_k must be equal to zero.

It is easy to find systems with positive Liapunov exponents. Consider for instance the linear system $\dot{x} = Ax$. In this case, $U(t) = e^{At}$and one shows

(using, for instance, theJordan canonical form of A), that the Liapunov exponents are exactly the real parts ofthe eigenvalues of A. Thus if A has an eigenvalue with positive real part, all orbits haveexponentially sensitive dependence on initial conditions.

More generally, if $x^*$is a linearly unstable equilibrium point, its largest Liapunovexponent will be positive. Similarly, if Γ is a periodic orbit, then we have seen in Theorem (2.2.3) that $U(t) = P(t)\, e^{Bt}$, where $P(t)$ is periodic. The Liapunov exponents are the realparts of the eigenvalues of B, that is, the real parts of the characteristic exponents. e^{Bt}has at least one eigenvalue equal to 1, corresponding to translations along the periodicorbit (seen in Proposition (2.2.5)). Thus one of the Liapunov exponents is equal to zero.More generally, orbits on an invariant torus of dimension m will have at least m Liapunovexponents equal to zero. If one or more of the remaining exponents are positive, then theperiodic or quasiperiodic orbits on the torus will depend sensitively on initial conditions.

Attractor	Sign of Liapunov exponents	Asymptotic dynamics
Stable equilibrium	$(-,-,-)$	Stationary
Stable periodic orbit	$(0,-,-)$	Periodic
Attracting torus	$(0,0,-)$	Quasiperiodic
Strange attractor	$(+,0,-)$	Chaotic

Table 4.1. Examples of attractors of a three-dimensional flow.

However, all the above invariant sets (equilibrium, periodic orbit or torus) must be unstablein order to have sensitive dependence on initial conditions, and thus they have zero measureand are not attractors. Orbits starting near these sets will not necessarily have positiveLiapunov exponents.

By contrast, orbits of a strange attractor must attract all nearby orbits that do notbelong to the attractor, while they repel nearby orbits that dobelong to it. In order toattract nearby orbits, the flow must be dissipative in a neighborhood of the attractor, sothat by Proposition(4.2.9), the sum of all Liapunov exponents must be negative. Since orbitson the attractor are bounded and not attracted by an equilibrium point, one Liapunovexponent is equal to zero. Hence a two-dimensional flow cannot admit a strange attractor. If a three-dimensional flow has a strange attractor, its Liapunov exponents must satisfy$\lambda_1 > \lambda_2 = 0 > \lambda_3$ and $|\lambda_3| > \lambda_1$ (see Table 4.1). Because of volume contraction, theattractor must

have zero volume, but it cannot be a surface and have positive Liapunovexponents. This accounts for the fractal nature of many observed attractors.

For iterated maps, the situation is less restrictive, since their orbits need not have oneLiapunov exponent equal to zero. Thus two-dimensional dissipative maps may admit astrange attractor, with Liapunov exponents satisfying $|\lambda_2| > \lambda_1 > 0 > \lambda_2$. For instance,the intersection of the strange attractor of a three-dimensional flow with a Poincaré sectionis also a strange attractor of the associated two-dimensional Poincaré map. One-dimensional iterated maps may have strange attractors if they are non-invertible.

Now we will discuss the Hénon and Lorenz attractors.

The first dynamical system for which the existence of a strange attractor was proved isthe Hénon map

$$\begin{aligned} x_{k+1} &= 1 - \lambda x_k^2 + y_k \\ y_{k+1} &= b x_k. \end{aligned} \tag{4.40}$$

This map does not describe a physical system. It has been introduced as a two-dimensionalgeneralization of the one-dimensional map

$x_{k+1} = 1 - \lambda x_k^2$, which is equivalent to the logistic map. When $b = 0$, (4.40) is reduced to this one-dimensional map. If $b \neq 0$, the Hénon map is invertible, and it is dissipative if$|b| < 1$. Numerical simulations indicatethat for some parameter values, the Hénon map has indeed a strange attractor with aself-similar structure (Fig. 4.7) [9].

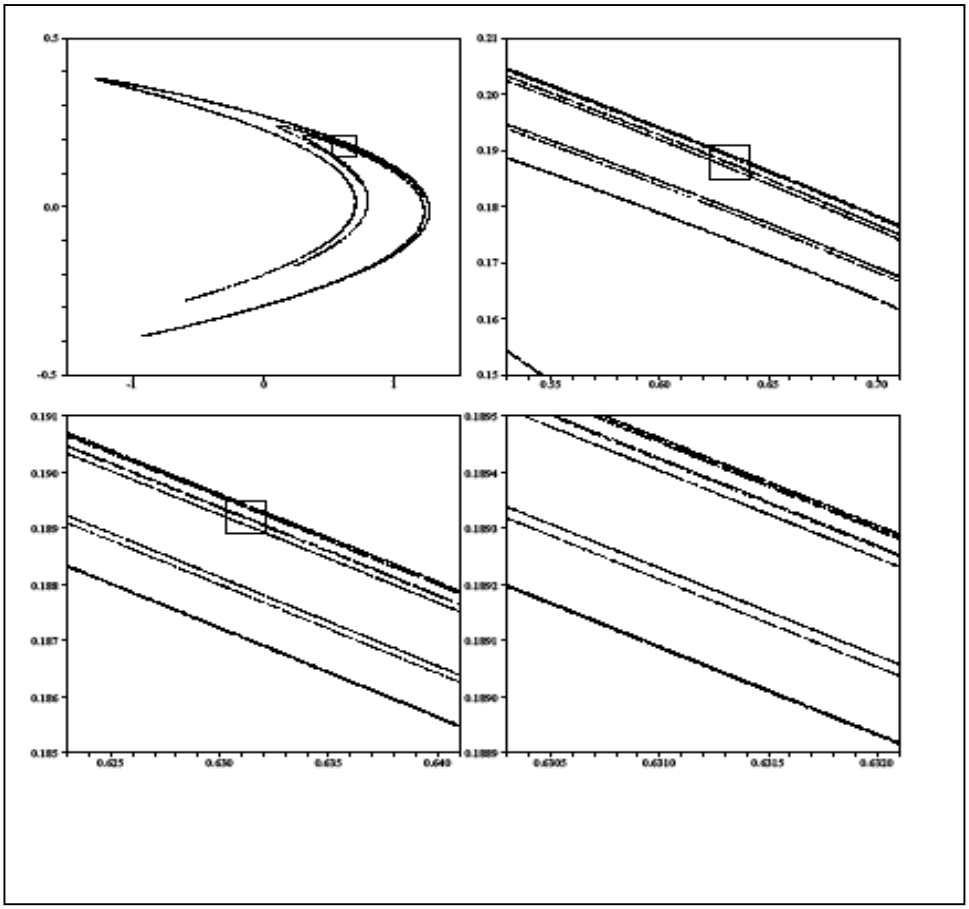

Figure 4.7

Hénon attractor for $\lambda = 1.4$ and $b = 0.3$. Successive magnifications of details, by a factor 10, show its self-similar structure**.**

Theorem (4.2.10): (Benedicks, Carleson [9])

Let $z^\star = (x^\star, y^\star)$ be the fixed pointof (4.40) with $x^\star, y^\star > 0$, and let W^u be its unstable manifold. For all $c < \log 2$, there isa set of positive Lebesgue measure of parameters (b, λ) for which:

1. there is an open set $\mathcal{U} \subset \mathbb{R}^2$, depending on$\lambda$ and b, such that for all$z \in \mathcal{U}$,
$$T^\kappa(z) \to \overline{W^u} \text{as} \quad \kappa \to \infty \qquad ; \qquad (4.41)$$
2. there is a point $z_0 \in W^u$ such that

(a) the positive orbit of z_0 is dense in W^u;

(b) the largest Liapunov exponent of the orbit of z_0 is larger than c.

Hence the closure of the unstable manifold W^u is a strange attractor. Locally, thestrange attractor is smooth in the unstable direction (with positive Liapunov

exponent),and has the structure of a Cantor set in the transverse, stable direction (with negativeLiapunov exponent) [14].

As a final illustration, we return to the Lorenz equations

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= -bx_3 + x_1x_2. \end{aligned} \tag{4.42}$$

The existence of strange attractor for this system has not been proved to our know 1-edge, although there is strong numerical evidence that such an attractor exists for certain parameter values, including in particular

$\sigma = 10, b = {}^{8}/_{3}$ and $r = 28$. The qualitative properties of dynamics nonetheless quite well understood. The strange attractor seems to appear after a rather subtle sequence of bifurcations. Let us consider the case

$\sigma = 10, b = {}^{8}/_{3}$, and take r as bifurcation parameter .

1. For $0 \leq r \leq 1$, the origin is globally asymptotically stable, that is all orbits converge to the origin.
2. At $r = 1$,the origin undergoes a pitchfork bifurcation , and two new stable equilibriums $C_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ appear. These points become unstable in Hope bifurcation at $r = \frac{470}{19} \approx 24.74$. The Hopebifurcation is subcritical, and thus corresponds to the destruction of an unstable periodic orbit, that must have been created somehow for a smaller value of r.
   The dynamics for $r > 1$ can be described by taking a Poincare section on the surface $\Sigma : x_3 = r - 1$ (we only take into account intersection with $\dot{x}_3 < 0$). Consider arectangle containing the segment C_-C_+ and the intersection S the two-dimensional stable manifold of the origin with Σ. S is attracted by the origin, which has the effect to pinch the rectangle and map it to two pieces of triangular shape. One vertice of each triangle belongs to a piece of the one- dimensional unstable manifold W^u of the origin .Due to the dissipation, the angle at this vertice is quite small. In first approximation, the dynamics can thus be described by a one-dimensional map for a coordinate transverse to S, parameterizing the long

side of the triangles. For r sufficiently small, this map is increasing, discontinuous at 0, and admits two stable fixed points $c_{\pm}$corresponding to$C_{\pm}$ Fig.(4.8) a.

3. At r= $r_1 \approx 13.296$, a homoclinic bifurcation occurs Fig.(4.8) b the unstable manifold W^ubelongs to the stable manifoldW^s, and thus the sharpest vertices of both triangles belong to S .The one-dimensional approximation of the Poincaré map is continuous, but still monotonously increasing.
4. For r slightly larger than $r_1$each piece of W^u hitsΣ on opposite side of S Fig.(4.8) c . The one-dimensional map becomes non-invertible, and has two new unstable equilibria $b_- \; and \; b_+$ .The map does not leave the interval$[b_-, b_+]$ invariant but maps tow of its subintervals onto $[b_-, b_+]$.The situation is thus similar to that of the tent map g_3 there exists an invariant cantor set containing chaotic orbits. The same can be seen to hold for the two-dimensional Poincaré map, which resembles the horseshoe map. ThusThere exists astrange nonwandering set A, but it is repelling and has zero measure (thus it is difficult to observe numerically).
5. As r increases, the fixed points $b_{\pm}$ move towards the fixed points$c_{\pm}$. It appears that for $r \geq r_2 \approx 24.06$,theinterval $[b_-, b_+]$ .is mapped into itselfFig.(4.8) d .the strange nonwandering set A becomes attracting, although it does not attract a full neighborhood, since orbit in $b_{\pm}$may converge$c_{\pm}$.
6. The subcritical Hopf bifurcation at $r = r_3 \approx 24.74$ makes the fixed points$C_{\pm}$ unstable, so that finally all orbits can converge to the neighborhood set A which has become an attractor.

   The method used her to describe the dynamics near a monoclinic bifurcation by Poincare map has been applied to other systems. Transitions to chaotic behavior arequite frequently associated with such homo clinic bifurcations. The Poincare map in a vicinity of the unstable manifold is described as the composition of an almost linear map, reflecting the motion near the equilibrium point, and a nonlinear, but usually rather simple map, reflecting the motion near W^uaway from theequilibrium. The composition of two rather innocent-looking maps contains all the ingredients necessary for the existence of chaotic dynamics [2].

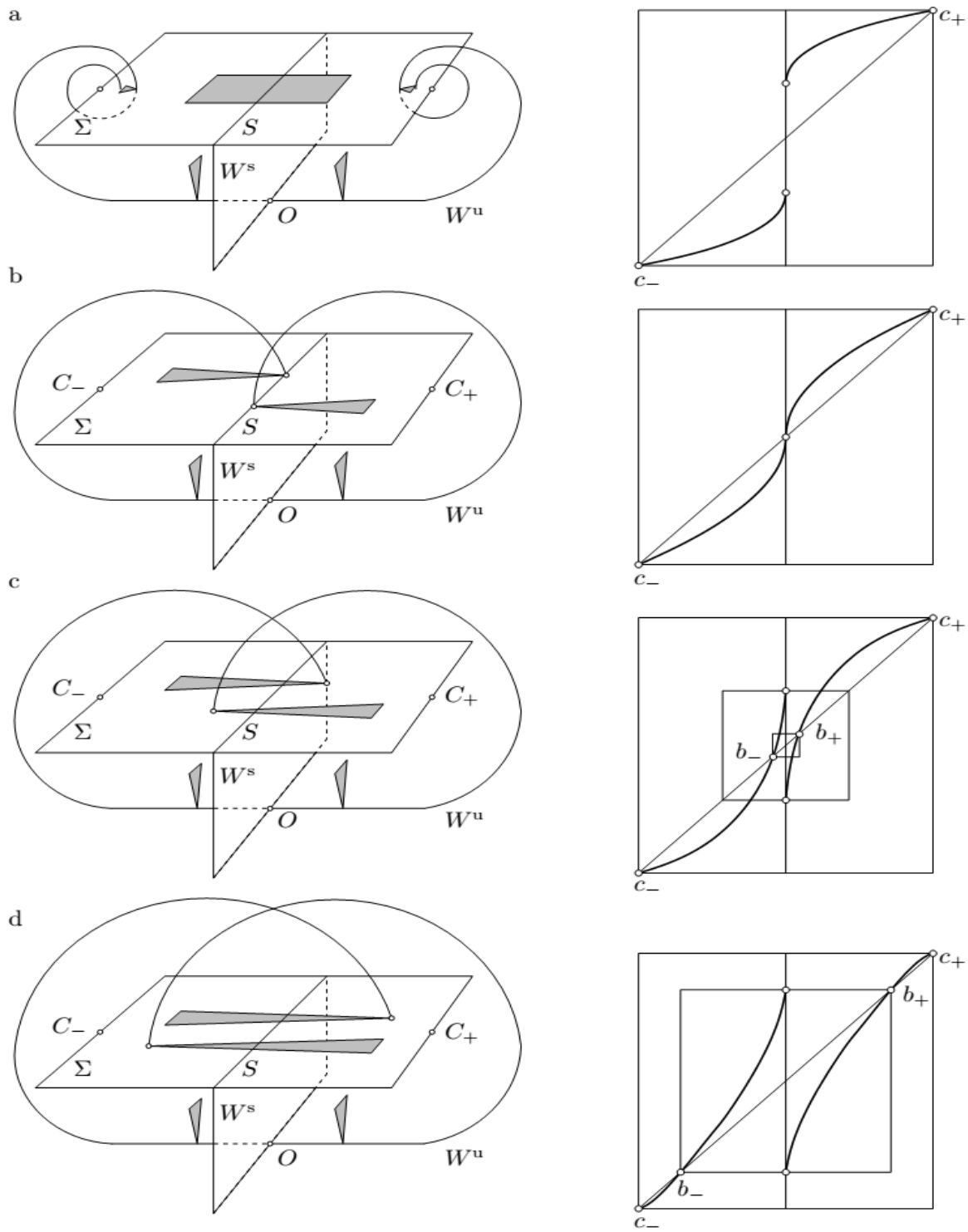

FIGURE 4.8. Schematic representation of the stable and unstable manifolds of the Lorenz equations (left), and one-dimensional approximation of the Poincaré map for the coordinate perpendicular to W^s (right). (a) $1 < r < r_1$, (b) homoclinic bifurcation at $r = r_1$, (c) existence of a nonwandering repeller for $r_1 < r < r_2$, and (d) bifurcation to a strange attractor at $r = r_2$.

Bibliography

[1] C. Sparrow, The Lorenz equations: bifurcations, chaos and strange attractors(Springer-Verlag, New York, 1982).

[2] E.N. Lorenz, Deterministic non-periodic flows, J. Atmos. Sci 20:130–141 (1963).

[3] J. Carr, Applications of Centre Manifold Theory (Springer–Verlag, New York,1981).

[4] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, andBifurcations of Vector Fields (Springer-Verlag, New York, 1983).

[5] J. Hale, H. Ko¸cak, Dynamics and Bifurcations (Springer–Verlag, New York,1991).

[6] J.K. Hale, Ordinary differential equations (J. Wiley & sons, New York, 1969).

[7] J. Laskar, A numerical experiment on the chaotic behaviour of the Solar System, Nature 338:237–238 (1989).

[8] J. Moser, Stable and Random Motions in Dynamical Systems (Princeton University Press, Princeton, New Jersey, 1973).

[9] M. Benedicks, L. Carleson, The dynamics of the H´enon map, Ann. of Math. (2)133:73–169 (1991).

[10] M.C. Gutzwiller, Moon-Earth-Sun: The oldest three-body problem, Rev. Mod.Phys. 70:589–639 (1998).

[11] M.W. Hirsch, S. Smale, Differential Equations, Dynamical Systems, and LinearAlgebra (Academic Press, New York, 1974).

[12] P. Collet, J.-P. Eckmann, Iterated maps on the interval as dynamical systems(Birkh¨auser, Boston, 1980).

[13] P. Hartman, Ordinary differential equations (J. Wiley & sons, New York, 1964).

[14] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos(Springer–Verlag, New York, 1990).

[15] V.I. Oseledec, A multiplicative ergodic theorem. Liapunov characteristic numbersfor dynamical systems, Trans. Moscow Math. Soc. 19:197–231 (1968).