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Measures of Convex Bodies with Caffarelli Log-Concave Perturbation Theorem and Brunn–Minkowski Inequalities

قياسات الاجسام المحدبة مع مبرهنة ارتجاج كافاريلي المقعرة –اللوغريثمية ومتباينات براين-منكوفسكاي

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Dedication

To my Family

Acknowledgements

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Abstract

We study the sections , eestimates for the affine , dual affine quermassintegrals, slicing inequalities for measures and estimates for measures of lower dimensional sections of convex bodies in addition the boundary regularity of maps with convex potentials . The centroid bodies, logarithmic Laplace transform, monotonicity properties of optimal transportation ,rigidity , stability of caffarellis log-concave perturbation theorem and related inequalities examined and characterized . The behavior of the extensions of the Brunn-Minkowski and Prbkopa-Leindler theorems, including inequalities for log concave functions, and application to the diffusion equation are obtained . We give the relations form Brunn Minkowski to brascamp and to sharp and logarithmic sobolev inequalities. We conclude the study by the stability ,Gaussian and logarithmic Brunn–Minkowski type inequalities.

الخلاصة

قمنا بدراسة الأقسام والتقديرات لتكاملات كتلة كوير الافيقية والافيقية المزدوجة ومتباينات التقطيع لاجل القياسات والتقديرات لاجل قياسات اقسام البعد الأسفل للاجسام المحدبة واضافة انتظامية الحدود للرواسم مع الجهد المحدب . تم اختبار وتشخيص اجسام النقطة الوسطي وتحويل لابلاس اللوغريثمي وخصائص الرتيبية الي التنقل الأمثل والصلابة واستقرارية مبرهنة اضطراب كافاريلي اللوغريثمي –المقعر والمتباينات ذات العلاقة . تم الحصول علي السلوك لتمديدات مبرهنات بروم-مينكوفسكي بروكوبان ليندلر المحتوية علي المتباينات لاجل الدوال المقعرة اللوغريثمية والتطبيق الي معادلة الانتشار . قمنا باعطاء العلاقات من بروم-مينكو فسكي الي براسكامب والي متباينات سوبوليف القاطعة اللوغريثمية . خلصت الدراسة بواسطة الاستقرارية والمتباينات نوع جاوسيان وبروم-مينكوفسكي اللوغريثمية .

Introduction

The generalized Busemann-Petty problem asks: If the volume of i-dimensional central section of a centrally symmetric convex body in R^{n} is smaller than that of another such body, is the volume of the body also smaller? It is proved that the answer is negative if $2 < \mathrm{i} < \mathrm{n}$. The case of a 2-dimensional section remains open. The proof uses techniques in functional analysis and Radon transforms on Grassmannians. We provide estimates for suitable normalizations of the affine and dual affine quermassintegrals of a convex body K in R^n.

We extend the Prtkopa-Leindler theorem to other types of convex combinations of two positive functions and we strengthen the PrCkopa-Leindler and Brunn-Minkowski theorems by introducing the notion of essential addition. Our proof of the Prekopa-Leindler theorem is simpler than the original one. We show C^1," regularity to the boundary under the assumptions that both Ω_I, Ω_2 be convex.

We develop several applications of the Brunn{Minkowski inequality in the Prekopa Leindler form. We show that an argument of B. Maurey may be adapted to deduce from the Prekopa Leindler theorem the Brascamp Lieb inequality for strictly convex potentials.

We unify and slightly improve several bounds on the isotropic constant of high-dimensional convex bodies in particular, a linear dependence on the body's ψ_2 constant is obtained. We present an alternative approach to some results of Koldobsky on measures of sections of symmetric convex bodies, which allows us to extend them to the not necessarily symmetric setting.

Optimal transportation between densities $f\ (X), g(Y)$ can be interpreted as a joint probability distribution with marginally $f\ (X), and\ g(Y)$. We prove monotonicity and concavity properties of optimal transportation *(Y(X))* under suitable assumptions on f and g. We establish some rigidity and stability results for Caffarelli' s log-concave perturbation theorem.

A detailed investigation is undertaken into Brunn-Minkowski-type inequalities for Gauss measure. A Gaussian dual Brunn-Minkowski inequality, the first of its type, is proved, together with precise equality conditions, and is shown to be the best possible from several points of view. A new Gaussian Brunn-Minkowski inequality is proposed and proved to be true in some significant special cases. For origin-symmetric convex bodies (i.e., the unit balls of finite dimensional Banach spaces) it is conjectured that there exist a family of inequalities each of which is stronger than the classical Brunn-Minkowski inequality and a family of inequalities each of which is stronger than the classical Minkowski mixed-volume inequality.

The Contents

Chapter 1

Convex Bodies

We require the notion of an *i*-intersection body which generalizes the notion of an intersection body. Inequalities among the volumes of projection bodies, polar projection bodies and their central sections are proved. They are related to the maximal slice problem. We show a convex a more general study of normalized p-means of projection and section functions of K.

Section (1.1): Sections of Convex Bodies

The starting point is an integral formula of Furstenberg and Tzkoni [5] about the volume of k-dimensional of ellipsoids: for every ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$ and every $1 \le k \le n$ one has

$$\int_{G_{n,k}} |\mathcal{E} \cap F|^n dv_{n,k}(F) = c_{n,k}|\mathcal{E}|^k, \quad (1)$$

where $v_{n,k}$ is the Haar measure on the Grassmannian $G_{n,k}$ and $c_{n,k}$ is a constant depending only on n and k; more precisely, $c_{n,k} = \Gamma\left(\frac{n}{2}+1\right)^k / \Gamma\left(\frac{k}{2}+1\right)^n$. It was proved by Miles [16] that this formula can be obtained in a simpler way as a consequence of classical formulas of Blaschke and Petkantschin.

Later, analogous quantities were considered by Lutwak and Grinberg in the setting of convex bodies. Lutwak introduced in [11] – for every convex body K in $\mathbb{R}^n$ and every $1 \le k \le n - 1$ – the quantities

$$\Phi_{n-k}(K) = \frac{\omega_n}{\omega_k}\left(\int_{G_{n,k}} |P_F(K)|^{-n} dv_{n,k}(F)\right)^{-\frac{1}{n}} \quad (2)$$

where $P_F(K)$ is the orthogonal projection onto F and ω_k is the volume of the Euclidean unit ball in $\mathbb{R}^k$. For $k = 0$ and $k = n$ one sets $\Phi_0(K) = |K|$ and $\Phi_n(K) = \omega_n$ respectively. Grinberg [8] proved that these quantities are invariant under volume preserving affine transformations; this justifies the terminology "affine quermassintegrals" for $\Phi_{n-k}(K)$. From the definition of $\Phi_{n-k}(K)$ it is clear that

$$\Phi_{n-k}(K) \le \frac{\omega_n}{\omega_k}\int_{G_{n,k}} |P_F(K)| dv_{n,k}(F) = W_{n-k}(K) \quad (3)$$

where $W_{n-k}(K) = V(K, [k]B_2^n, [n-k])$ are the Quermassintegrals of K. Lutwak conjectured in [12] that the affine quermassintegrals satisfy the inequalities

$$\omega_n^j \Phi_i^{n-j} \le \omega_n^i \Phi_j(K)^{n-i} \quad (4)$$

for all $0 \le i < j < n$. For example, Lutwak asks if

$$\Phi_{n-k}(K) \ge \omega_n^{(n-k)/n}|K|^{k/n} \quad (5)$$

with equality if and only if K is an ellipsoid; note that the weaker inequality $W_{n-k}(K) \ge \omega_n^{(n-k)/n}|K|^{k/n}$ holds true by the isoperimetric inequality. Most of these questions remain

open (see [6, Chapter 9]); two cases of (5) follow from classical results: when $k = n - 1$ this inequality is the Petty projection inequality and when $k = 1$ and K is symmetric then (5) is the Blaschke-Santal´o inequality.
Lutwak proposed in [13] to study the dual affine quermassintegrals $\widetilde{\Phi}_{n-k}(K)$. For every convex body K in $\mathbb{R}^n$ and every $1 \leq k \leq n - 1$ one defines

$$\widetilde{\Phi}_{n-k}(K) = \frac{\omega_n}{\omega_k}\left(\int_{G_{n,k}} |K \cap F|^n dv_{n,k}(F)\right)^{\frac{1}{n}} \qquad (6)$$

For $k = 0$ and $k = n$ one sets $\widetilde{\Phi}_0(K) = |K|$ and $\widetilde{\Phi}_n(K) = \omega_n$ respectively. Grinberg proved in [8] that these quantities are also invariant under volume preserving linear transformations, and he established the inequality

$$\widetilde{\Phi}_{n-k}(K) \leq \omega_n^{(n-k)/n} |K|^{k/n} \qquad (7)$$

for all $1 \leq k \leq n - 1$, with equality if and only if K is a centered ellipsoid. The case $k = n - 1$ of this inequality is the Busemann intersection inequality (while the case $k = 1$ becomes an identity for symmetric convex bodies).
Being affinely invariant, affine and dual affine quermassintegrals appear to be useful in asymptotic convex geometry. So, one of the purposes is to give upper and lower bounds for $\Phi_{n-k}(K)$ and $\widetilde{\Phi}_{n-k}(K)$ in the remaining cases. We introduce a different notation and normalization which is better adapted to our needs. The question we study is equivalent to e.g. [6, Problem 9.7].
Definition (1.1.1)[1]: (normalized affine quermassintegrals). For every convex body K in $\mathbb{R}^n$ and every $1 \leq k \leq n - 1$ we define

$$\Phi_{[k]}(K) = \left(\int_{G_{n,k}} |P_F(K)|^{-n} dv_{n,k}(F)\right)^{-\frac{1}{kn}} \qquad (8)$$

We also set $\Phi_{[n]}(K) = |K|^{1/n}$. Lutwak's conjectures about affine quermassintegrals can now be restated as follows:
(i) For every (symmetric) convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leq k \leq n - 1$,

$$\Phi_{[k]}(K) \geq \Phi_{[k]}(D_n), \qquad (9)$$

where D_n is the Euclidean ball of volume 1.
(ii) For every convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leq k \leq n - 1$,

$$\Phi_{[k]}(K) \leq \Phi_{[k]}(Sn) \qquad (10)$$

where S_n is the regular Simplex of volume 1.
In view of these conjectures, in the asymptotic setting it is reasonable to ask if the following holds true: There exist absolute constants $c_1, c_2 > 0$ such that for every convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leq k \leq n - 1$,

$$c_1\sqrt{n/k} \leq \Phi_{[k]}(K) \leq c_2\sqrt{\frac{n}{k}} \qquad (11)$$

For $k = 1$ the Blaschke-Santal´o inequality shows that (9) holds true. Proving (10) for $k = 1$ corresponds to Malher's conjecture. Clearly, (11) for $k = 1$ follows from the Blaschke-Santal´o and the reverse Santal´o inequality of Bourgain-Milman [3].
Note that for $k = n - 1$ we have

$$\Phi_{[n-1]}(K) = \left(\frac{|B_2^n|}{|\Pi^*(K)|}\right)^{\frac{1}{n(n-1)}} \tag{12}$$

where (K) is the polar projection body of K. Then, Holder's inequality and the isoperimetric inequality show that (9) holds true. The same is true for (10): this follows from Zhang's inequality; see [30].

Definition (1.1.2)[1]: (normalized dual affine quermassintegrals). For every convex body K in $\mathbb{R}^n$ and every $1 \le k \le n-1$ we define

$$\widetilde{\Phi}_{[k]}(K) = \left(\int_{G_{n,k}} |K \cap F^{\perp}|^n d\nu_{n,k}(F)\right)^{\frac{1}{kn}} \tag{13}$$

Grinberg's theorem about dual affine quermassintegrals states that if K has volume 1 then

$$\widetilde{\Phi}_{[k]}(K) \le \widetilde{\Phi}_{[k]}(D_n) \le c_2, \tag{14}$$

where $c_n > 0$ is an absolute constant. As we will see, if the hyperplane conjecture has an affirmative answer then

$$\widetilde{\Phi}_{[k]}(K) \ge c_1 \tag{15}$$

for every centered convex body of volume 1, where $c_1 > 0$ is an absolute constant.

In view of the above, here one asks if the following holds true: There exist absolute constants $c_1, c_2 > 0$ such that for every centered convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \le k \le n - 1$,

$$c_1 \le \widetilde{\Phi}_{[k]}(K) \le c_2 \tag{16}$$

Our estimates on the normalized affine and dual affine quermassintegrals are summarized in the following:

Theorem (1.1.3)[1]:. Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \le k \le n-1$,

$$\Phi_{[k]}(K) \le c_1\sqrt{\frac{n}{k}}\, log\, n \tag{17}$$

and, if K is also centered,

$$\widetilde{\Phi}_{[k]}(K) \ge \frac{c_2}{L_K} \tag{18}$$

where L_K is the isotropic constant of K. In particular, assuming the hyperplane conjecture we have that $\widetilde{\Phi}_{[k]}(K) \simeq 1$ for all $1 \le k \le n-1$. We also have the bounds

$$\Phi_{[k]}(K) \le c_3 \left(\frac{n}{k}\right)^{3/2} \sqrt{\log\left(\frac{en}{k}\right)} \tag{19}$$

and

$$\widetilde{\Phi}_{[k]}(K) \ge \frac{c_4}{\sqrt{\frac{n}{k}}\sqrt{\log\left(\frac{en}{k}\right)}} \tag{20}$$

which are sharp when k is proportional to n.

For the proofs of these estimates, we attempt a more general study of normalized p-means of projection functions of K, which we introduce for every $1 \le k \le n-1$ and every $p \ne 0$ by setting

$$W_{[k,p]}(K) := \left(\int_{G_{n,k}} |P_F(K)|^p d\nu_{n,k}(F)\right)^{\frac{1}{kp}}. \tag{21}$$

and

$$\widetilde{W}_{[k,p]}(K) := \left(\int_{G_{n,k}} |K \cap F^{\perp}|^{p} d\nu_{n,k}(F)\right)^{\frac{1}{kp}} \tag{22}$$

respectively. The k-th normalized affine and dual affine quermassintegrals of K correspond to the cases $p = -n$ and $p = n$ respectively:

$$\Phi_{[k]}(K) = W_{[k,-n]}(K) \; and \; \widetilde{\Phi}_{[k]}(K) = \widetilde{W}_{[k,-n]}(K). \tag{23}$$

We list several properties of the p-means and prove some related inequalities.

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write $\omega_n$for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. We also write $\bar{A}$ for the homothetic image of volume 1 of a compact set $A \subseteq \mathbb{R}^n$ of positive volume, i.e. $\bar{A} := |A|^{-\frac{1}{n}}A$. If A and B are compact sets in $\mathbb{R}^n$, then the covering number $N(A,B)$ of A by B is the smallest number of translates of B whose union covers A.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$.

A star-shaped body C with respect to the origin is a compact set that satisfies $tC \subseteq C$ for all $t \in [0,1]$. We denote by $\|\cdot\|_C$the gauge function of C:

$$\|x\|_C = \inf\{\lambda > 0 : x \in \lambda C\} \tag{24}$$

A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if it has centre of mass at the origin: $\int_C \langle x, \theta\rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C: \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x,y\rangle : y \in C\}$. The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^{\circ} := \{y \in \mathbb{R}^n : \langle x,y\rangle \leq 1 \; for \; all \; x \in C\}. \tag{25}$$

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, the Blaschke–Santal´o inequality and the Bourgain–Milman inequality imply that

$$|K^{\circ}|^{\frac{1}{n}} \simeq \frac{1}{n} \tag{26}$$

Let K be a centered convex body in $\mathbb{R}^n$. For every $F \in G_{n,k}, 1 \leq k \leq n-1$, we have that $P_F(K^{\circ}) = (K \cap F)^{\circ}$, and hence,

$$|K \cap F|^{1/k} |P_F K^{\circ}|^{1/k} \simeq \frac{1}{k} \tag{27}$$

The Rogers-Shephard inequality [26] states that

$$1 \leq |P_F K|^{1/k} |K \cap F^{\perp}|^{1/k} \leq \binom{n}{k}^{1/k} \leq \frac{en}{k}. \tag{28}$$

See [28], [21] and [25] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $q \geq 1$ and $\theta \in S^{n-1}$ we define

$$h_{Z_q(K)}(\theta) := \left(\int_K |\langle x,\theta\rangle|^q dx\right)^{1/q} \quad (29).$$
We define the L_q-centroid body $Z_q(K)$ of K to be the centrally symmetric convex set with support function $h_{Z_q}(K)$. L_q–centroid bodies were introduced in [14]. Here we follow the normalization (and notation) that appeared in [23].
It is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \le p \le q \le \infty$, where $Z_\infty(K) = conv\{K,-K\}$. Note that if $T \in SL(n)$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover, as a consequence of the Brunn–Minkowski inequality (see [23]), one can check that
$$Z_q(K) \subseteq c\frac{q}{p} Z_p(K) \quad (30)$$
for all $1 \le p < q$, where $c > 1$ is an absolute constant, and
$$Z_q(K) \supseteq cK$$
for all $q \ge n$, where $c > 0$ is an absolute constant.
A centered convex body K of volume 1 in $\mathbb{R}^n$ is called isotropic if $Z_2(K)$ is a multiple of B_2^n. Then, we define the isotropic constant of K by
$$L_K := \left(\frac{|Z_2(K)|}{|B_2^n|}\right)^{1/n}.$$
It is known that $L_K \ge L_{B_2^n} \ge c > 0$ for every convex body K in $\mathbb{R}^n$. Bourgain proved in [2] that $L_K \le c\sqrt[4]{n}\, log\, n$ and, a few years ago, Klartag [9] obtained the estimate $L_K \le c\sqrt[4]{n}$ (see also [10]). The hyperplane conjecture asks if $L_K \le C$, where $C > 0$ is an absolute constant. See [19], [7] and [23] for additional information on isotropic convex bodies.
Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every star shaped body C in $\mathbb{R}^n$ and any $-n < p \le \infty, p \ne 0$, we set
$$I_p(K,C) := \left(\int_K \|x\|_C^p dx\right)^{1/p}.$$
If $C = B_2^n$ we simply write $I_p(K)$ instead of $I_p(K, B_2^n)$.
We first consider the question whether there exist absolute constants $c_1, c_2 > 0$ such that for every convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \le k \le n-1$,
$$c_1\sqrt{n/k} \le \Phi_{[k]}(K) \le c_2\sqrt{n/k}.$$
We can prove that the right-hand side inequality holds true up to a log n term.
Theorem (1.1.4)[1]: Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \le k \le n-1$,
$$\Phi_{[k]}(K) \le c\sqrt{n/k}\, log\, n.$$
we introduce a normalized version of the quermassintegrals of a convex body.
Let K be a convex body in $\mathbb{R}^n$. For every $1 \le k \le n-1$ we define the normalized k-quermassintegral of K by
$$W_{[k]}(K) := \left(\int_{G_{n,k}} |P_F(K)| d\nu_{n,k}(F)\right)^{1/k}.$$
We also set $W_{[n]}(K) = |K|^{1/n}$ and $W_{[0]}(K) = 1$. Note that

$$W_{[1]}(K) = \int_{S^{n-1}} [h_K(\theta) + h_K(-\theta)]\, d\sigma(\theta) = 2w(K).$$

From the definition and Kubota's formula (see [28]) it is clear that, for every $1 \leq k \leq n-1$ one has

$$W_{[k]}(K) = \left(\frac{\omega_k}{\omega_n} V(K, [k]; B_2^n, [n-k])\right)^{1/k}.$$

Applying the Aleksandrov-Fenchel inequality (see [28, Chapter 6]) one can check the following:

(i) If K and L are convex bodies in $\mathbb{R}^n$, then, for all $1 \leq k \leq n$,

$$W_{[k]}(K + L) \geq W_{[k]}(K) + W_{[k]}(L).$$

(ii) For all $0 \leq k_1 < k_2 < k_3 \leq n$,

$$\frac{W_{[k_2]}(K) W_{[k_1]}(B_2^n)}{W_{[k_1]}(K) W_{[k_2]}(B_2^n)} \geq \left(\frac{W_{[k_2]}(K) W_{[k_1]}(B_2^n)}{W_{[k_1]}(K) W_{[k_2]}(B_2^n)}\right)^{\frac{(k_2-k_1)k_3}{k_2(k_3-k_1)}}.$$

(iii) For all $1 \leq k_1 \leq k_2 \leq n$,

$$\frac{W_{[k_2]}(K)}{W[k_2](B_2^n)} \leq \frac{W_{[k_1]}(K)}{W_{[k_1]}(B_2^n)}.$$

Since $\Phi_{[k]}(K)$ is affine invariant we may assume that K is centered. It is well-known that Pisier's inequality (see [25, Chapter 2]) on the norm of the Rademacher projection implies that there exists $T \in SL(n)$ such that

$$W_{[1]}(T(K)) = 2w(T(K)) \leq c\sqrt{n} \log n.$$

More precisely, follows from Pisier's inequality in the case where K is symmetric. However, it is not difficult to extend the inequality to the non necessarily symmetric case (see e.g. [22, Lemma3]). Then, using the affine invariance of $\Phi_{[k]}$ and the fact that $\Phi_{[k]}(K) \leq W_{[k]}(K)$, we write

$$\Phi_{[k]}(K) = \Phi_{[k]}(T(K)) \leq W_{[k]}(T(K)).$$

Since $W_{[k]}(B_2^n) = \omega_k^{1/k} \simeq \frac{1}{\sqrt{k}}$, it follows from that

$$W_{[k]}(T(K)) \leq \frac{W_{[k]}(B_2^n)}{W_{[1]}(B_2^n)} W_{[1]}(T(K)) \leq c\sqrt{n/k} \log n.$$

This completes the proof.

Next, we introduce the p-mean projection function $W_{[k,p]}(K)$ and the p-mean width $w_p(K)$ of a convex body K and prove a weak lower bound in the direction of the left hand side inequality.

p-mean projection function. Let K be a convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$ and for every $p \neq 0$ we define the p-mean projection function $W_{[k,p]}(K)$ by

$$W_{[k,p]}(K) := \left(\int_{G_{n,k}} |P_F(K)|^p \, d\nu_{n,k}(F)\right)^{\frac{1}{kp}}.$$

We also set $W_{[n]}(K) := |K|^{1/n}$. Observe that the k-th normalized affine quermassintegral of K corresponds to the case $p = -n$:

$$\Phi_{[k]}(K) := W_{[k,-n]}(K).$$

It is clear that $W_{[k,p]}(K)$ is an increasing function of p, $W_{[s,p]}(\lambda K) = \lambda W_{[s,p]}(K)$ for every $\lambda > 0$ and $W_{[s,p]}(K) \leq W_{[s,p]}(L)$ whenever $K \subseteq L$. Moreover, for every $1 \leq k < m \leq n - 1$ and every $p \neq 0$, one has

$$W_{[k,p]}(K) := \left(\int_{G_{n,m}} W_{[k,p]}^{kp}\,(P_E(K))dv_{n,m}(E)\right)^{\frac{1}{kp}}.$$

In particular,

$$W_{[k,-m]}(K) := \left(\int_{G_{n,m}} \Phi_{[k]}^{kp}\,(P_E(K))dv_{n,m}(E)\right)^{-\frac{1}{km}}.$$

mean width. The p-mean width of K is defined for every $p \neq 0$ by

$$w_p(K) = \left(\int_{S^{n-1}} h_K^p(\theta)d\sigma(\theta)\right)^{1/p}.$$

It is clear that $w_p(K)$ is an increasing function of $p, w_p(\lambda K) = \lambda w_p(K)$ for every $\lambda > 0$ and $w_p(K) \leq w_p(L)$ whenever $K \subseteq L$. Note that, if K° is the polar body of K, then

$$w_{-n}(K) = \left(\frac{|B_2^n|}{|K^\circ|}\right)^{\frac{1}{n}}.$$

Also, for every $1 \leq k \leq n - 1$,

$$w_p(K) = \left(\int_{G_{n,k}} w_p^p(P_E(K))dv_{n,k}(E)\right)^{1/p}$$

and, in particular,

$$w_{-k}(K) = \omega_k^{1/k}\left(\int_{G_{n,k}} |(P_E(K))^\circ| dv_{n,k}(E)\right)^{-1/k}.$$

Using the above we are able to prove that, in the symmetric case, $W_{[k,-q]}(K) > c\sqrt{n/k}$ as far as $q \leq n/k$; recall that $\Phi_{[k]}(K) = W_{[k,-n]}(K)$.

Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leq k \leq n - 1$,

$$W_{[k,-n/k]}(K) \geq c\sqrt{n/k}.$$

Proof. Using Holder's inequality, the Blaschke-Santal´o and the reverse Santal´o inequality, for every $p \geq 1$ we can write

$$\left(\int_{G_{n,k}} |P_F(K)|^{-p} dv_{n,k}(F)\right)^{\frac{1}{kp}} \simeq \left(\int_{G_{n,k}} \frac{|(P_F(K))^\circ|^p}{\omega_k^{2p}} dv_{n,k}(F)\right)^{\frac{1}{kp}}$$

$$\simeq \sqrt{k}\left(\int_{G_{n,k}} \left(\int_{S_F} \frac{1}{h_K^k(\theta)} d\sigma_F(\theta)\right)^p dv_{n,k}(F)\right)^{\frac{1}{kp}}$$

$$\leq c\sqrt{k}\left(\int_{G_{n,k}} \int_{S_F} \frac{1}{h_K^{kp}(\theta)} d\sigma_F(\theta)\, dv_{n,k}(F)\right)^{\frac{1}{kp}}$$

$$= c\sqrt{k}\ \left(\int_{S^{n-1}} \frac{1}{h_K^{kp}(\theta)} d\sigma(\theta)\right)^{\frac{1}{kp}}$$
$$= c\sqrt{k} w_{-kp}^{-1}(K).$$

We set $p := n/k \geq 1$. Then, we get

$W_{[k,-n/k]}(K) \geq \frac{w_{-n}(K)}{c\sqrt{k}} \simeq \frac{1}{c\sqrt{k}} \frac{\omega_n^{1/n}}{|K^\circ|^{1/n}} \simeq \sqrt{n/k}.$

This completes the proof.

Note. What we have actually shown in the proof of Theorem (4.1.2) is that

$$W_{[k,-p]}(K) \simeq \sqrt{k}\left(\int_{G_{n,k}} \left(\int_{S_F} \frac{1}{h_K^k(\theta)} d\sigma_F(\theta)\right)^p d\nu_{n,k}(F)\right)^{-\frac{1}{kp}} \geq c\,\frac{w_{-kp}(K)}{\sqrt{k}}$$

for all $1 \leq k \leq n-1$ and $p \geq 1$.

Next, we consider the dual affine quermassintegrals. We first provide a lower bound which is sharp up to the isotropic constant of the body.

Theorem (1.1.5)[1]: Let K be a centered convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leq k \leq n-1$. Then,

$$\widetilde{\Phi}_{[k]}(K) \geq \frac{c}{LK}.$$

Proof. By the linear invariance of $\widetilde{\Phi}_{[k]}(K)$, we may assume that K is in the isotropic position. Let F be a k-dimensional subspace of $\mathbb{R}^n$. We denote by E the orthogonal subspace of F and for every $\phi \in F \setminus \{0\}$ we define $E^+(\phi) = \{x \in span\{E,\phi\}: \langle x,\phi\rangle \geq 0\}$. K. Ball (see [2] and [19]) proved that, for every $q \geq 0$, the function

$$\phi \longmapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left(\int_{K\cap E^+(\phi)} \langle x,\phi\rangle^q dx\right)^{-\frac{1}{q+1}}$$

is the gauge function of a convex body $B_q(K,F)$ on F. We will make use of the fact that, if K is isotropic then

$$|K \cap F^\perp|^{1/k} \simeq \frac{L_{B_{k+1}(K,F)}}{LK}.$$

See [19] and [23] for a proof. Therefore,

$$\widetilde{\Phi}_{[k]}(K)L_K \simeq \left(\int_{G_{n,k}} L_{B_{k+1}(K,F)}^{kn} d\nu_{n,k}(F)\right)^{\frac{1}{kn}}.$$

Recall that the isotropic constant is uniformly bounded from below: we know that $L_{B_{k+1}(K,F)} \geq c$, where $c > 0$ is an absolute constant. It follows that

$$\widetilde{\Phi}_k(K)L_K \simeq \left(\int_{G_{n,k}} L_{B_{k+1}(K,F)}^{kn} d\nu_{n,k}(F)\right)^{\frac{1}{kn}} \geq c,$$

and the result follows. Note. shows that if the hyperplane conjecture is correct then (if we also take into account Grinberg's theorem), for every centered convex body K of volume 1 in $\mathbb{R}^n$ and for every $1 \leq k \leq n-1$,

$$c_1 \leq \widetilde{\Phi}_{[k]}(K) \leq c_2$$

where $c_1, c_2 > 0$ are absolute constants. This would answer completely the asymptotic version of our original problems about the dual affine quermassintegrals.
The proof of Theorem (1.1.4) has some interesting consequences:
Corollary (1.1.6)[1]:. Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$ we have

$$v_{n,k}\{F \in G_{n,k}: L_{B_{k+1}(K,F)} \geq cL_K\} \leq e^{-kn},$$

where $c > 0$ is an absolute constant.
Proof. From Grinberg's theorem – we know that $\tilde{\Phi}_{[k]}(K) \leq \tilde{\Phi}_{[k]}(D_n) \leq c_2$, where $c_2 > 0$ is an absolute constant. From (4.5) we get

$$\left(\int_{G_{n,k}} L_{B_{k+1}(K,F)}^{kn} dv_{n,k}(F)\right)^{\frac{1}{kn}} \leq c_3 L_K,$$

and the result follows from Markov's inequality.
We complement with a second lower bound for $\tilde{\Phi}_{[k]}(K)$, which is sharp when k is proportional to n.
Theorem (1.1.7)[1]: Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$ we have that

$$\tilde{\Phi}_{[k]}(K) \geq \frac{c}{\sqrt{n/k}\sqrt{\log(en/k)}}.$$

For the proof of this bound, we introduce the p-mean function $\tilde{W}_{[k,p]}(K)$ of a convex body K.
Let K be a convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$ and for every $p \neq 0$ we define the p-mean $\tilde{W}_{[k,p]}(K)$ by

$$\tilde{W}_{[k,p]}(K) = \left(\int_{G_{n,k}} |K \cap F^{\perp}|^p dv_{n,k}(F)\right)^{\frac{1}{kp}}.$$

The normalized dual k-quermassintegral of K is $\tilde{W}_{[k]}(K) := \tilde{W}_{[k,1]}(K)$. Observe that the k-th normalized dual affine quermassintegral of K corresponds to the case $p = n$:

$$\tilde{\Phi}_{[k]}(K) = \tilde{W}_{[k,n]}(K).$$

Hölder's inequality implies that, for a fixed value of k, $\tilde{W}_{[k,n]}(K)$ is an increasing function of p.
The next Proposition shows that the normalized dual quermassintegrals $\tilde{W}_{[k]}(K)$ are strongly related to the quantities $I_p(K)$.
Proposition (1.1.8)[1]:. Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leq k \leq n-1$. Then,

$$\tilde{W}_{[k]}(K) I_{-k}(K) = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} = \tilde{W}_{[k]}(D_n) I_{-k}(D_n).$$

Note. It is easy to check that $\left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} \simeq \sqrt{n}$.
Proof. We integrate in polar coordinates:

$$\Gamma_{-k}^{-k}(K) = \frac{n\omega_n}{n-k}\frac{1}{\|x\|_K^{n-k}}\,d\sigma(x)$$

$$= \frac{n\omega_n}{n-k\omega_{n-k}}\int_{G_{n,n-k}} \omega_{n-k}\int_{S_F}\frac{1}{\|\theta\|_{K\cap F}^{n-k}}\,d\sigma(x)\,dv_{n,n-k}(F)$$

$$= \frac{n\omega_n}{n-k\omega_{n-k}}\int_{G_{n,n-k}} |K\cap F|\,dv_{n,n-k}(F)$$

$$= \frac{n\omega_n}{n-k\omega_{n-k}}\int_{G_{n,k}} |K\cap F^{\perp}|\,dv_{n,k}(F)\,.$$

The definition of $\widetilde{W}_{[k]}(K)$ completes the proof.

Proposition (1.1.9)[1]:. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \le s \le m \le n-1$,

$$\widetilde{W}_{[s]}(K) \le \widetilde{W}_{[s]}(D_n)$$

and

$$\frac{\widetilde{W}_{[m]}(K)}{\widetilde{W}_{[s]}(K)} \ge \frac{\widetilde{W}_{[m]}(D_n)}{\widetilde{W}_{[s]}(D_n)}.$$

Proof. It is known (see [24]) that for any $q \ge p \ge -n$ we have

$$I_p(K) \ge I_p(D_n)$$

and

$$\frac{I_q(K)}{I_p(K)} \ge \frac{I_q(D_n)}{I_p(D_n)}.$$

Note. It is easy to check that

$$\widetilde{W}_{[k]}(D_n) = \widetilde{W}_{[k,p]}(D_n) = \widetilde{\Phi}_{[k]}(D_n) \simeq 1.$$

Hölder's inequality and imply that

$$\widetilde{\Phi}_{[k]}(K) \ge \widetilde{W}_{[k]}(K) \ge \frac{c\sqrt{n}}{I_{-k}(K)}\ .$$

Now, we use the fact (see Theorem 5.2 and Lemma 5.6 in [4]) that there exists $T \in SL(n)$ such that

$$I_{-k}(T(K)) \le c\sqrt{n}\sqrt{n/k}\sqrt{log\ en/k}.$$

By the affine invariance of $\widetilde{\Phi}_{[k]}(K)$ we have

$$\widetilde{\Phi}_{[k]}(K) = \widetilde{\Phi}_{[k]}(T(K)) \ge \frac{c\sqrt{n}}{I_{-k}(T(K))},$$

and this completes the proof.

we prove some inequalities involving the p-means of projection functions of a convex body. In particular, we obtain duality relations between $\Phi_{[n/2]}(K)$ and $\widetilde{\Phi}_{[n/2]}(\overline{K^{\circ}})$. These will allow us to obtain a second upper bound for $\Phi_{[k]}(K)$ which is sharp when k is proportional to n.

One source of such inequalities is the following "L_q–version of the Rogers- Shephard inequality" which was proved in [24].

Lemma (1.1.10)[1]:. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \le k \le n-1$ and every $F \in G_{n,k}$ we have that

$$c_1 \leq |K \cap F^{\perp}|^{1/k} |P_F(Z_k(K))|^{1/k} \leq c_2,$$
where $c_1, c_2 > 0$ are universal constants.
A direct application of Lemma (1.1.10) leads to the following:
Proposition (1.1.11)[1]:. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$ and $p \neq 0$ we have that
(i) $c_1 \leq \widetilde{W}_{[k,p]}(K) W_{[k,-p]}(Z_k(K)) \leq c_2$,
(ii) $c_3 \leq \widetilde{\Phi}_{[k]}(K) \Phi_{[k]}(Z_k(K)) \leq c_4$,
(iii) $c_5 \leq \widetilde{\Phi}_{[k]}(K) \Phi_{[k]}(K) \leq c_6 n/k$,
where $c_i > 0, i = 1, \ldots, 6$ are absolute constants.
Proof. From the definitions we readily see that
$$\widetilde{W}_{[k,p]}(K) = \left(\int_{G_{n,k}} |K \cap F^{\perp}|^p dv_{n,k}(F)\right)^{1/(kp)},$$
$$\simeq \left(\int_{G_{n,k}} |P_F(Z_k(K))|^{-p} dv_{n,k}(F)\right)^{1/(kp)}$$
$$= W_{[k,-p]}^{-1}(Z_k(K)).$$
This proves (i). Then, (ii) corresponds to the special case $p = n$. Since $K \subseteq \frac{cn}{k} Z_k(K)$, (iii) follows.
A second source of inequalities is the Blaschke-Santalo and the reverse Santalo inequality. Since $(K \cap F^{\perp})^{\circ} = P_{F^{\perp}}(K^{\circ})$, for every $1 \leq k \leq n-1$ and $F \in G_{n,k}$ we have
$$c^{n-k} \omega_{n-k}^2 \leq |P_{F^{\perp}}(K^{\circ})|\, |K \cap F^{\perp}| \leq \omega_{n-k}^2.$$
Therefore,
$$\widetilde{W}_{[k,p]}(K) = \left(\int_{G_{n,k}} |K \cap F^{\perp}|^p dv_{n,k}(F)\right)^{1/(kp)}$$
$$\leq \omega_{n-k}^{2/k} \left(\int_{G_{n,k}} |P_{F^{\perp}}(K^{\circ})|^{-p} dv_{n,k}(F)\right)^{1/(kp)}$$
$$= \omega_{n-k}^{2/k} \left(\int_{G_{n,n-k}} |P_F(K^{\circ})|^{-p} dv_{n,n-k}(F)\right)^{1/(kp)}$$
$$= \omega_{n-k}^{2/k} W_{[n-k,p]}^{-(n-k)/k}(K^{\circ}).$$
Working in the same way we check that
$$\widetilde{W}_{[k,p]}(K) W_{[k,p]}^{-(n-k)/k}(K^{\circ}) \geq c^{(n-k)/k} \omega_{n-k}^{2/k}.$$
We summarize in the following Proposition.
Proposition (1.1.12)[1]:. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$ and $p \neq 0$ we have:
(i) $c^{(n-k)/k} \omega_{n-k}^{2/k} \leq \widetilde{W}_{[k,p]}(K) W_{[k,p]}^{-(n-k)/k}(K^{\circ}) \leq \omega_{n-k}^{2/k}$.
(ii) If n is even, then $\widetilde{W}_{[n/2,p]}(K) W_{[n/2,p]}(K^{\circ}) \simeq \frac{1}{n}$.
(iii) If n is even, then $\widetilde{\Phi}_{[n/2]}(K) \Phi_{[n/2]}(\overline{K^{\circ}}) \simeq 1$.

Taking into account Proposition (1.1.12)(iii) we have the following:
Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,

$$\widetilde{\Phi}_{[n/2]}(K) \simeq \widetilde{\Phi}_{[n/2]}(\overline{K^\circ}) \; and \; \Phi_{[n/2]}(K) \simeq \Phi_{[n/2]}(\overline{K^\circ}).$$

We can get more precise information if we use the M-ellipsoid of K. Let K be a convex body of volume 1 in $\mathbb{R}^n$. Milman (see [17], [18] and also [20] for the not necessarily symmetric case) proved that there exists an ellipsoid $\mathcal{E}$ with $|\mathcal{E}| = 1$, such that

$$log\, N(K, \mathcal{E}) \leq vn,$$

where $v > 0$ is an absolute constant. In other words, for any centered convex body K of volume 1 in $\mathbb{R}^n$ there exists $T \in SL(n)$ such that

$$N(T(K), D_n) \leq e^{vn}.$$

Theorem (1.1.13)[1]:. Let n be even and let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,

$$c_1 \leq \widetilde{\Phi}_{[n/2]}(K) \leq c_2,$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. We will use the following inequality of Rogers and Shephard [27]. If K is a centered convex body of volume 1 in $\mathbb{R}^n$ then

$$|K - K| \leq 4^n.$$

We choose $T \in SL(n)$ so that

$$N(T(\overline{K - K}), D_n) \leq e^{vn}$$

Then, for any $F \in G_{n,\frac{n}{2}}$,

$$|P_F(T(\overline{K - K}))| \leq N(T(\overline{K - K})), D_n)|P_F\,(D_n)| \leq e^{vn} c^n.$$

Moreover, using (5) we have that

$$\left|P_F\left(Z_{\frac{n}{2}}\big(T(K)\big)\right)\right| \leq \left|P_F\left(conv\big(T(K), -T(K)\big)\right)\right| \leq \left|P_F\big(T(K - K)\big)\right|$$
$$\leq\; 4^n |P_F(T(\overline{K - K}))|.$$

Combining the above with (5.10) and (5.1) we have that

$$|T(K) \cap F^\perp| \geq \frac{c_0^{\frac{n}{2}}}{|P_F(Z_{\frac{n}{2}}(T(K)))|} \geq \frac{c_0^{\frac{n}{2}}}{e^{vn} c^n} =: c_1^{\frac{n}{2}}.$$

So, we have shown that for any $F \in G_{n,\frac{n}{2}}$,

$$|T(K) \cap F| \geq c_1^{\frac{n}{2}}.$$

This implies that

$$\widetilde{\Phi}_{\left[\frac{n}{2}\right]}(K) = \widetilde{\Phi}_{\left[\frac{n}{2}\right]}(T(K)) \geq \min_{F \in G_{n,\frac{n}{2}}} |T(K) \cap F|^{\frac{2}{n}} \geq c_2.$$

This shows the left hand side inequality in (7). The right hand side inequality follows.

Corollary (1.1.14)[1]: Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,

$$\widetilde{\Phi}_{[n/2]}(K) \simeq \widetilde{\Phi}_{[n/2]}(\overline{K^\circ}) \simeq \Phi_{[n/2]}(K) \simeq \Phi_{[n/2]}(\overline{K^\circ}) \simeq 1.$$

Note. In view of Corollary (1.1.6), if n is even and $k = n/2$, becomes a formula:

Corollary (1.1.15)[1]: Let K be an isotropic convex body in $\mathbb{R}^n$. Then,

$$L_K \simeq \left(\int_{G_{n,n/2}} L_{B_{\frac{n}{2}+1}(K,F)}^{n^2/2} d\nu_{n,n/2}(F) \right)^{2/n^2}.$$

In particular, there exists $F \in G_{n,n/2}$ such that

$$L_K \leq c L_{B_{\frac{n}{2}+1}(K,F)}.$$

we can now give a second upper bound for $\Phi_{[k]}(K)$, which sharpens the estimate in

Theorem (1.1.16)[1]:. Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leq k \leq n-1$. Then,

$$\Phi_{[k]}(K) \leq c(n/k)^{3/2} \sqrt{\log en/k}.$$

Proof. We may assume that K is also centered. we have that

$$\Phi_{[k]}(K) = \frac{\Phi_{[k]}(K)\tilde{\Phi}_{[k]}(K)}{\tilde{\Phi}_{[k]}(K)} \leq \frac{cn/k}{\tilde{\Phi}_{[k]}(K)}.$$

Then, we use the lower bound for $\tilde{\Phi}_{[k]}(K)$.

Section (1.2): Estimates for the Affine and Dual Affine Quermassintegrals

For K_e^n be the class of origin-symmetric convex bodies in the Euclidean space $\mathbb{R}^n$. Denote by $vol_i\,(\cdot)$ the i-dimensional Lebesgue measure. We will discuss the following generalized Busemann-Petty problem and its variations:

(GBP). If $K, L \in K_e^n$ and for every i-dimensional subspace H

$$vol_i(K \cap H) \leq vol_i\,(L \cap H),$$

does it follow that

$$vol_n(K) \leq vol_n(L)?$$

When $i = n - 1$ this problem was posed by Busemann and Petty [36] in 1956. The Busemann-Petty problem has received considerable attention (see, M. Berger [33], V. L. Klee [34], and [35] [37] [39]). Many contributed to the solution of this problem (see [40]). For the history of Busemann-Petty problem, see [41] [42]. It is now known that the Busemann-Petty problem has a negative answer for $n \geq 4$ (see [43]), and has a positive answer for $n = 3$ (see [44]).

As observed by K. Ball, one can construct counterexamples to the generalized Busemann-Petty problem by using the techniques in [45] and letting $K =$ the unit cube, $L = a$ ball of appropriate radius, when n is sufficiently large and $i > n/2$. What are the dimensions of and ambient spaces so that the generalized Busemann-Petty problem has a positive answer? One of the objectives is to prove that the generalized Busemann-Petty problem has a negative answer for $2 < i < n$. Therefore, only the 2-dimensional case might have a positive answer. This remains open in $\mathbb{R}^n (n > 3)$.

The notion body, introduced by Lutwak [46], plays an important role in the solution of the Busemann-Petty problem. An origin-symmetric convex body K is called an intersection body if the inverse spherical Radon transform of the radial function of K is a nonnegative measure. Based on the work of Lutwak [47], it was shown in [48] that the existence of origin-symmetric convex non-intersection bodies is equivalent to a negative answer to the

Busemann-Petty problem. Then the negative answer to the Busemann-Petty problem in $\mathbb{R}^n (n \geq 4)$ comes from the fact that every polytope in $\mathbb{R}^n (n \geq 4)$ is not an intersection body ([49] [50]); the positive answer to the problem in $\mathbb{R}^3$ comes from the fact that every origin-symmetric convex body in $\mathbb{R}^3$ is an intersection body ([32]). The methods employed in [33] and [34] depend on the bijectivity of the spherical Radon transform in the space of C_1 even functions on the sphere S^{n-1}. For the generalized Busemann-Petty problem GBP, though the volume of central of convex bodies can be expressed as a Radon transform from the sphere S^{n-1} to the Grassmannian $Gr(n, i)$, we cannot expect any surjectivity of the Radon transform except the hyperplane case. One of the reasons is that the rank of the Grassmannian $Gr(n, i)$ is different from that of the sphere S^{n-1} except $i = 1, n - 1$. Consequently, the arguments in [36] and [37] cannot be generalized directly. Moreover, to deal with the generalized Busemann-Petty problem, one needs to extend the notion of intersection body.

We deal with problem GBP by a different approach from the point of view of functional analysis. This approach shows that the answer to problem GBP is equivalent to asking the positivity of inverse Radon transforms on Grassmannians. It enables one to relate problem GBP to certain new classes of centered bodies. They are extensions of the class of intersection bodies. A body is called centered if it is star-shaped and symmetric with respect to the origin. Let S_e^n be the class of centered bodies with continuous radial functions. Then K_e^n is a subclass of S_e^n . For each $2 \leq i \leq n - 1$, we introduce a class of centered bodies $I_i^n \subseteq S_e^n$ by using Radon transforms on Grassmannians. In the hyperplane case, I_{n-1}^n is exactly the class of intersection bodies. We show that problem GBP has a positive answer if $K \in I_i^n$, and that problem GBP is equivalent to whether there is the inclusion $K_e^n \subseteq I_i^n$. we prove that there is no polytope in I_i^n for $3 \leq i \leq n - 1$. This yields a negative answer to problem GBP for $3 \leq i \leq n - 1$. The case of 2-dimensional remains open for $n > 3$. It might have a positive answer which would depend on a better understanding about the geometry of Grassmannians. Note that the convexity is a 2-dimensional notion.

M. Meyer [32] showed that if K is the cross-polytope (octahedron) then the Busemann-Petty problem has a positive answer. He asked whether this could be generalized to polar projection bodies (see [32] p. 423). Analytically, polar projection bodies are finite dimensional sections of the unit ball of the Banach space L^1. we give a strong negative answer to this question by proving the following theorem:

For $3 \leq i \leq n - 1$ there exist polar projection bodies K and L in $\mathbb{R}^n (n \geq 4)$ so that

$$vol_i(K \cap H) < vol_i (L \cap H), \qquad for\ all\ H \in Gr(n, i),$$

but

$$vol_n(K) > vol_n(L).$$

Projection bodies and their polars arise in a number of disciplines, including functional analysis, crystallography, stereology, geometric tomography, and stochastic and convex geometry (see [35]). We will consider their central and establish inequalities related to them .

The following variation of the Busemann-Petty problem is considered to be one of the main problems in the local theory of Banach spaces (see [34]). For $K, L \in K_e^n$, if for every hyperplane H through the origin

$$vol_{n-1}(K \cap H) \leq vol_{n-1}(L \cap H),$$
does there exist a numerical constant c (not depending on the dimension n) so that
$$vol_n(K) \leq c\, vol_n(L)?$$
The result above shows that $c > 1$ for the class of polar projection bodies. We will show that $c \leq 2$ in this case. See [35] for related results. The above question has many equivalent formulations (see [36]). One of them is the maximal slice problem: Does there exist a numerical constant c_1 so that
$$vol_n(K)^{\frac{n-1}{n}} \leq c_1 \max_{H \in Gr(n,n-1)} vol_{n-1}(K \cap H)?$$
See [38] for a detailed discussion. When K is restricted to the class of projection bodies or to the class of polar projection bodies, the question has a positive answer (see Ball [40], Milman and Pajor [39], and Lindenstrauss and Milman [36]).
The proof involves finite dimensional Banach space theory. We give a geometric proof and give a specific value for the constant so that the results are useful in lower dimensional spaces. It will be shown that one can choose $c_1 < 1$ for any polar projection body K. Similar results are proved for projection bodies.
Let $C_e(S^{n-1})$ be the space of continuous even functions on the unit sphere S^{n-1}. Denote by $Gr(n,i)$ the Grassmann manifold of i-dimensional subspaces in $\mathbb{R}^n$, and denote by $C(Gr(n,i))$ the space of continuous functions on $Gr(n,i)$. The Radon transform, for $2 \leq i \leq n-1$,
$$R_i: C_e(S^{n-1}) \to C(Gr(n,i))$$
is defined by
$$(R_i f)(H) = \frac{1}{i\kappa_i} {}_{u \in S^{n-1} \cap H} f(u)\, du, \qquad H \in Gr(n,i), f \in C_e(S^{n-1}),$$
whereκ_i and du are the volume and the surface area element of the i-dimensional unit ball, respectively.
Let ρ_K be the radial function of a centered body $K \in S_e^n$ given by
$$\rho_K(u) = \max\{\lambda \geq 0: \lambda u \in K\}, \qquad u \in S^{n-1}.$$
The Radon transform R_i is closely connected with the central of centered bodies by the following formula
$$\left(R_i \rho_K^i\right)(H) = \frac{1}{\kappa_i} vol_i(K \cap H), \qquad H \in Gr(n,i) \tag{31}$$
The dual transform R_i^t of R_i is given by
$$R_i^t: C(Gr(n,i)) \to C_e(S^{n1})$$
$$(R_i^t g)(u) = {}_{u \in H \in Gr(n,i)} g(H) dH, \qquad u \in S^{n-1}, g \in C(Gr(n,i)).$$
We have the following duality (see [40], p. 144, p. 161)
$$\langle R_i f, g \rangle = \langle f, R_i^t g \rangle, \qquad f \in C_e(S^{n-1}), \quad g \in C\big(Gr(n,i)\big), \tag{32}$$
where $\langle\ ,\rangle$ is the usual inner product of functions on homogeneous spaces.
Let $X = R_i(C_e(S^{n-1}))$, the range of R_i. Then X is a subspace of $C(Gr(n,i))$.
For a positive linear functional μ on X, we can define the dual transform $R_i^t \mu$ as an even positive measure on S^{n-1} by

$$\langle R_i^t \mu, f\rangle = \langle \mu, R_i f\rangle, \qquad f \in C_e(S^{n-1}),$$
where $\langle , \rangle$denotes the pairing of a linear functional and an element of the vector space X. Let $M^+(X)$ be the set of positive linear functionals on X. We consider the convex cone
$$N_i = \{R_i^t \mu : \mu \in M^+(X)\}$$
In $M(S^{n-1})$, the space of signed measures on S^{n-1}. This convex cone N_i is closed under the weak* topology of $M(S^{n-1})$. Indeed, for a net $\sigma_m \to \sigma, \sigma_m \in N_i, \sigma \in M(S^{n-1})$, and $f \in C_e(S^{n-1})$, there exists $\mu_m \in M^+(X)$ so that $\sigma_m = R_i^t \mu_m$.
We have
$$\langle \sigma, f\rangle = lim\langle \sigma_m, f\rangle = lim\langle R_i^t \mu_m, f\rangle = lim\langle \mu_m, R_i f\rangle.$$
This shows that there exists $\mu \in M^+(X)$ so that
$$\langle \mu, R_i f\rangle = lim\langle \mu_m, R_i f\rangle.$$
Hence,
$$\langle \sigma, f\rangle = \langle R_i^t \mu, f\rangle,$$
that is,$\sigma \in N_i$.

Lemma (1.2.1)[32]: Let $\rho \in M(S^{n-1})$. If $\rho \notin N_i$, then there exists $g \in C(S^{n-1})$ so that
$$\langle \rho, g\rangle > 0, \qquad \langle \sigma, g\rangle \leq 0 \qquad for\ all\ \sigma \in N_i.$$
Proof. Since $M(S^{n-1})$ is a locally convex Hausdorff space under the weak* topology and N_i is a closed convex cone, we can apply the separation theorem.
If $\rho \notin N_i$, there exist $g \in C(S^{n-1})$, a constant c and $\varepsilon > 0$ so that
$$\langle \rho, g\rangle \geq c + \varepsilon > c - \varepsilon \ \geq \langle \sigma, g\rangle, \qquad for\ all\ \sigma \in N_i.$$
Since $0 \in N_i$, we have $c - \varepsilon \geq 0$ and $\langle \rho, g\rangle > 0$. Since N_i is a cone, we have $\langle \sigma, g\rangle \leq 0$ for all $\sigma \in N_i$. Otherwise, there is σ_1 so that $\langle \sigma_1, g\rangle > 0$. For $r > 0, r\sigma_1 \in N_i$ and
$$\langle r\sigma_1, g\rangle = r\langle \sigma_1, g\rangle > c - \varepsilon$$
for r large. This is impossible.

Lemma (1.2.2)[32]:. Let $\rho \in M(S^{n-1})$. If $\rho \notin N_i$, then there exists $g \in C^\infty(S^{n-1})$ so that
$$\langle \rho, g\rangle > 0, \qquad Rig < 0.$$
Proof. By Lemma (1.2.1), there exists $g \in C(S^{n-1})$ so that
$$\langle \rho, g\rangle > 0, \qquad \langle \sigma, g\rangle \leq 0 \qquad for\ all \sigma \in N_i.$$
Choose a sequence $g_m \in C^\infty(S^{n-1})$ such that $g_m \leq g$ and $g_m \to g$ uniformly. Then $\langle \rho, g_m\rangle > 0$ when m is large. Since $N_i \subset M^+(S^{n-1})$, for $\sigma \in N_i$, we have
$$\langle \sigma, g_m\rangle \leq \langle \sigma, g\rangle \leq 0.$$
Therefore, for $\sigma = R_i^t \mu, \mu \in M^+(X)$,
$$0 \geq \langle \sigma, g_m\rangle = \langle R_i^t \mu, g_m\rangle = \langle \mu, R_i g_m\rangle.$$
This implies that $R_i g_m \leq 0$. Then $g_m - \varepsilon$ satisfies the requirement for small $\varepsilon > 0$.
If the Radon-Nikodym derivative of the measureρ with respect to the Lebesgue measure on S^{n-1} is an even continuous function and $\rho \notin N_i$, then the function g in Lemma (1.2.2) can be chosen in $C_e^\infty(S^{n-1})$. Furthermore, if ρ is invariant under a subgroup of $SO(n)$, then g can be chosen as invariant under the same subgroup.
We are ready to introduce certain new classes of centered bodies. Let $\lambda_{S^{n-1}}$ be the Lebesgue measure on S^{n-1}. As usual, one can view a continuous function ρ on S^{n-1} as a measure by identifying ρ with $\rho\lambda_{S^{n-1}}$. Define

$$I_i^n = \{K \in S_e^n : \rho_K^{n-i} \in N_i\}, \qquad 2 \le i \le n-1,$$

where the continuous function ρ_K^{n-i} is viewed as a measure on S^{n-1}. Then $I_i^n \subseteq S_e^n$ and I_{n-1}^n is exactly the class of intersection bodies. These classes of centered bodies are crucial for the generalized Busemann-Petty problem. They are generalizations of the class of intersection bodies. It can be shown that the class of centered bodies I_i^n is affine invariant and contains all the intersection bodies, i.e.,

$$I_{n-1}^n \subseteq I_i^n, \qquad 2 \le i \le n-1.$$

Elements in I_i^n are called i-intersection bodies.

Lemma (1.2.3)[32]:. Let $K \in S_e^n$. Then $K \in I_i^n$ if and only if

$$R_i g \le 0 \Longrightarrow \langle \rho_K^{n-i}, g\rangle \le 0$$

for any $g \in C_e^\infty(S^{n-1})$.

Proof. If $K \in I_i^n$, then there exists $\mu \in M^+(X)$ so that $\rho_K^{n-i} = R_i^t \mu$. We have

$$\langle \mu, R_i g\rangle \le 0 \ whenever\ R_i g \le 0.$$

By $(1.2\frac{1}{2})$, the necessity is clear. The sufficiency follows from Lemma (1.2.1).

The above lemma is an analytic characterization of the classes of centered bodies $I_i^n, i = 2, \dots, n-1$. We give a geometric characterization by using dual mixed volumes. For $K, L \in S_e^n$ and $r \in \mathbb{R}$, the rth dual mixed volume of K and L , $\tilde{V}_r(K, L)$, is defined as

$$\tilde{V}_r(K, L) = \frac{1}{n} {}_{u \in S^{n-1}} \rho_K^{n-r}(u) \rho_L^r(u) du. \tag{33}$$

By the Hölder inequality, there are inequalities

$$\tilde{V}_r(K, L)^n \le vol_n(K)^{n-r}\ vol_n(L)^r\ r\ >\ 0 \tag{34}$$

$$\tilde{V}_r(K, L)^n \ge vol_n(K)^{n-r}\ vol_n(L)^r\ r < 0 \tag{35}$$

with equality in each of the inequalities if and only if K and L are dilations of each other. Dual mixed volumes were introduced by Lutwak [33] ,[35]. Inequalities (34) and (35) are from [36], [34].

Lemma (1.2.4)[32]:. *If* $K \in I_i^n$, then

$$vol_i(M \cap H) \le vol_i(L \cap H), for\ all\ H \in Gr(n, i) \Longrightarrow \tilde{V}_i(K, M) \le \tilde{V}_i(K, L) \tag{36}$$

for all $M, L \in S_e^n$. Conversely, let $L \in K_e^n$ be a fixed body with C^2 boundary and positive curvature. If the implication (36) holds for all $M \in K_e^n$, then $K \in I_i^n$.

Proof. Assume $K \in I_i^n$. Then there exists $\mu \in M^+(X)$ such that $\rho_K^{n-i} = R_i^t \mu$.

From (32), it follows that

$$\langle \mu, R_i \rho_M^i\rangle \le \langle \mu, R_i \rho_L^i\rangle.$$

By (32), this can be written as

$$\langle \rho_K^{n-i}, \rho_M^i\rangle \le \langle \rho_K^{n-i}, \rho_L^i\rangle.$$

In view of (33), the last inequality is the right-hand side of the implication (36). Conversely, for any $g \in C_e^\infty(S^{n-1})$ satisfying $R_i g \le 0$, define a centered convex body L_ε by

$$\rho_{L_\varepsilon}^i = \rho_L^i + \varepsilon g$$

for $\varepsilon > 0$ sufficiently small. Since L has C^2 boundary and positive curvature, this is possible. Let $M = L_\varepsilon$. Then the left-hand side of the implication (36) is equivalent to $R_i g \le 0$. The right-hand side of (36) becomes $\langle \rho_K^{n-i}, g\rangle \le 0$. From Lemma (1.2.3), this shows that $K \in I_i^n$.

Theorem (1.2.5)[32]:. If $K \in I_i^n$, then

$$vol_i\ (K \cap H) \le vol_i(L \cap H), for\ all\ H \in Gr(n,i) \Longrightarrow vol_n(K) \le vol_n(L)$$

for all $L \in S_e^n$.

Proof. Let $M = K$. From the necessity part of Lemma (1.2.4) and inequality (34), we obtain

$$vol_n(K) \le \tilde{V}_i(K,L) \le vol_n(K)^{\frac{n-i}{n}}\ vol_n(L)^{\frac{i}{n}}.$$

This gives the required inequality.

The case of $i = n - 1$ was proved by Lutwak [36].

Theorem (1.2.6)[32]:. Let $K\ \in K_e^n$ have C^2 boundary and positive curvature. If $K \notin I_i^n$, then there exists $L\ \in K_e^n$ so that

$$vol_i(L \cap H) < vol_i\ (K \cap H), for\ all\ H \in Gr(n,i),$$

but

$$vol_n\ (L) > vol_n(K).$$

Proof. We can apply either Lemma (1.2.2) or Lemma (1.2.4). By Lemma (1.2.2), there is $g \in C_e^\infty\ (S^{n-1})$ so that

$$\langle \rho_K^{n-i}, g\rangle >\ 0,\ \ R_i g < 0. \tag{37}$$

Define $K_\varepsilon \in K_e^n$ by

$$\rho_{K_\varepsilon}^i = \rho_K^i + \varepsilon g \tag{38}$$

for $\varepsilon > 0$ sufficiently small. Substituting (38) into (37) and using (31) and (33), we have

$$vol_n(K_\varepsilon) > vol_n(K),$$
$$vol_i(K_\varepsilon \cap H) < vol_i(K \cap H), for\ all\ H \in Gr(n,i).$$

The case of $i = n - 1$ in Theorem (1.2.6) was proved in [33], [34] ,[35]. It was first proved in [36] without the requirement of convexity. From Theorems (1.2.5) and (1.2.6), we obtain the following

Theorem (1.2.7)[32]:. In a given dimension, the problem GBP has a positive answer if and only if $K_e^n \subseteq\ I_i^n$.

The following lemma is elementary. Its proof is similar to that of Lemma 2.1 in [36].

Lemma (1.2.8)[32]:. Let $K \in S_e^n$ be a centered body of revolution about the x_n-axis. If $\phi$is the angle between $H \in Gr(n,i)$ and the x_n-axis, then the volume $K \cap H$ has the expression

$$vol_i(K \cap H) = \frac{2(i-1)\kappa_{i-1}}{i cos\phi} \,{}_{\phi}^{\frac{\pi}{2}} \rho(\psi)^i\ \ 1 - \frac{cos^2\psi}{cos^2\phi}^{\frac{i-3}{2}} sin\,\psi\, d\psi.$$

Let $u \in S^{n-1}, u = u(u_1,\psi) = (u_1 \sin\psi\,, \cos\psi), u_1 \in S^{n-2}, 0 \le \psi \le \pi.$ For any $f \in C(S^{n-1})$, define

$$\bar{f}(\psi) = \frac{1}{(n-1)\kappa_{n1}}\, {}_{S^{n-2}} f\big(u(u_1,\psi)\big)du_1.\ u \in S^{n-1}.$$

The function $\bar{f}$ is obtained by averaging f over subspheres parallel to the equator of S^{n-1}. It can be viewed as a function on S^{n-1}. In fact, we have

$$\bar{f}(u) = {}_{\alpha\in SO(n-1)} f(\alpha u)d\alpha, \qquad u \in S^{n-1},$$

where $d\alpha$ is the normalized Haar measure on $SO(n-1)$.

Lemma (1.2.9)[32]:. If $K \in I_i^n$,then

$$\int_{\phi}^{\frac{\pi}{2}} g(\psi)\left(1-\frac{cos^2\psi}{cos^2\phi}\right)^{\frac{i-3}{2}}\sin\psi\ d\psi \le 0, \qquad 0 \le \phi \le \frac{\pi}{2} \tag{39}$$

$$\Downarrow$$

$$\int_{0}^{\frac{\pi}{2}} g(\psi)\overline{\rho_K^{n-i}}(\psi) sin^{n-2}\psi\, d\psi \le 0$$

for all $g \in C^\infty([0,\frac{\pi}{2}])$. The converse is true if $K \in S_e^n$ is a centered body of revolution about the x_n-axis.

Proof. If $K \in I_i^n$, by Lemma (1.2.3) we have

$$R_i g \le 0 \Longrightarrow \langle \rho_K^{n-i}, g\rangle \le 0 \tag{40}$$

for any $g \in C_e^\infty(S^{n-1})$. If g is $SO(n-1)$ invariant, this gives the implication (39) by using (31) and Lemma (1.2.8). Conversely, assume K is a convex body of revolution about the x_n-axis. Then $\rho_K$is $SO(n-1)$ invariant. The $K \in I_i^n$ if there is the implication (40) for every g which is $SO(n-1)$ invariant. Since (39) is equivalent to (40) in this case, we conclude the proof.

The above lemma is an analytic characterization of I_i^n . We do not know if the converse in the lemma is true without the assumption of revolution. However, when $i = n-1$, the converse is true for any centered bodies (see [33]). In this case, Lemma (1.2.9) provides a characterization for the positivity of the inverse spherical Radon transform.

We use some techniques used in [33] to prove that there are no polytopes in $I_i^n, 3 \le i \le n-1$.

Lemma (1.2.10)[32]:. If $K \in S_e^n$ is a polytope and $k > 0$, then there is $\alpha \in SO(n)$ so that $\overline{\rho_{\alpha K}^k}(\psi)\ sin^k\psi$ is strictly decreasing on $[\psi_1,\frac{\pi}{2}]$ for some $0 < \psi_1 < \frac{\pi}{2}$.

Proof. Let $K \in S_e^n$ be a polytope. We can rotate K to a general position αK for some $\alpha \in SO(n)$ such that no $(n-2)$-face of αK is in the subspace

$$H_1 = \{(x_1, \dots , x_n) \in \mathbb{R}^n: x_n = 0\};$$

and no $(n-1)$-face of αK is parallel to the x_n-axis. For simplicity, assume that K is already in such a position.

Let p_{u_1} be the plane spanned by the x_n-axis and $u_1 \in S^{n-2} \subset H_1$. Then $\partial K \cap p_{u_1}$ is a centered polygon, denoted by $C(u_1)$. The intersection $C(u_1) \cap H_1$ has two points, p_1, p_2, which are possibly vertices of $C(u_1)$. The point p_i is a vertex of the polygon $C(u_1)$ only if p_i lies on the intersection of two $(n-1)$-faces of K, i.e., on an $(n-2)$-face of K. But if no $(n-2)$-face of K is contained in H_1, then the intersection of H_1 with an $(n-2)$-face of K is at most of dimension $n-3$. Thus, the set

$$\omega = \{u_1 \in S^{n-1}: C(u_1) \cap H_1\ are\ vertices\ of\ C(u_1)\}$$

has measure zero in S^{n-1}. We consider those H_1 intersecting only with two parallel sides of $C(u_1)$. Denote by l_{u_1} the pair of parallel sides. Let $u \in S^{n-1}, u = u(u_1,\psi\ ) = (u_1 \sin\psi\ , \cos\psi\ ), u_1 \in S^{n-2}, 0 \le \psi \le \pi$. Let $\rho_K(u) = \rho(\psi, u_1)$, and let θ be the angle between l_{u_1} and $x_n = 0$, and $2b$ be the length of

$$K \cap p_{u_1} \cap \{x \in \mathbb{R}^n: x_n = 0\}.$$

Then for ψ near to $\frac{\pi}{2}$,

$$\rho(\psi, u_1) = \frac{bsin\theta}{-cos(\psi+\theta)}, \qquad \rho(\psi, -u_1) = \frac{bsin\theta}{cos(\psi-\theta)}.$$

Since no $(n-1)$-face of K is parallel to the x_n-axis, we conclude that l_{u_1} is not parallel to the x_n-axis. Hence, we have $0 < \theta < \frac{\pi}{2}$.

Let

$$f(\psi) = [-\cos(\psi+\theta)]^{-k} + [\, cos\,(\psi-\theta)]^{-k} sin^k\psi\,.$$

By an elementary computation, we have

$$f\left(\frac{\pi}{2}\right) = 0, \qquad f''(\frac{\pi}{2}) = 2k(k+1)(\,sin\theta)^{-k-2}\, cos^2\theta > 0.$$

From the following identity,

$$\overline{\rho_K^k}(\psi\,)sin^k\psi \;= \frac{1}{2(n-1)\kappa_{n-1}} \, _{S^{n-2}} b^k\, sin^k\theta f(\psi) du_1.$$

It is easy to see that $\overline{\rho_K^k}(\psi\,)\, sin^k\psi$ is strictly decreasing on $[\psi_1, \frac{\pi}{2}]$ for some $0 < \psi_1 < \frac{\pi}{2}$.

The following lemma was proved in [33].

Lemma (1.2.11)[32]:. Suppose that $g(t)$ is continuous on $[a,b]$, $g_1(t,x) > 0$ is continuous and increasing for $t \in [a,x)$, and $g_2(t,x) > 0$ is continuous and decreasing for $t \in [a,x)$. For $x \in [a,b]$, let

$$I_k(x) = {}_a^x g(t) g_k(t,x) dt \qquad (k = 1,2),$$
$$I(x) = {}_a^x g(t) dt.$$

Then

$$I_1(x) \geq 0 \implies I(x) \geq 0 \implies I_2(x) \geq 0.$$

Theorem (1.2.12)[32]:. There are no polytopes in $I_i^n, 2 < i < n$.

Proof. Let $K \in S_e^n$ be a polytope in general position as in Lemma (1.2.10). We want to show that there exists $g(\psi)$ on $[0, \frac{\pi}{2}]$ so that for $3 \leq i \leq n-1$

$$_\phi^{\frac{\pi}{2}} g(\psi) \;\; 1 - \frac{cos^2\psi}{cos^2\phi}^{\frac{i-3}{2}} \sin\psi \, d\psi \;\leq 0, \quad 0 \leq \phi \leq \frac{\pi}{2}, \tag{41}$$

but

$$_0^{\frac{\pi}{2}} g(\psi) \overline{\rho_K^{n-i}}(\psi) sin^{n-2}\psi \, d\psi > 0 \tag{42}$$

Then by Lemma (1.2.9), $K \notin I_i^n$ for $3 \leq i \leq n-1$.

From Lemma (1.2.10), $\overline{\rho_K^{n-i}}(\psi) sin^{n-i}\psi$ is strictly decreasing on $[\psi_1, \frac{\pi}{2}]$. It is quite straightforward to show that there exists $g \in C^\infty([0, \frac{\pi}{2}])$ so that

$$_\phi^{\frac{\pi}{2}} g(\psi) sin^{i-2}\psi \, d\psi \leq 0, \quad 0 \leq \phi \leq \frac{\pi}{2} \tag{43}$$

and (42) holds. In fact, a function g satisfying the following conditions does the job,

$$g(\psi) \begin{cases} = 0, & 0 \leq \psi \leq \psi_1, \\ > 0, & \psi_1 < \psi < \psi_2 \\ < 0, & \psi_2 < \psi < \frac{\pi}{2} \end{cases}$$

$$\int_0^{\frac{\pi}{2}} g(\psi)\, sin^{i-2}\psi\, d\psi = 0.$$

Hence, it suffices to show that (43) implies (40).
Let $t = cos\psi$, $x = cos\phi$, then (40) and (43) become

$$\int_0^x g\big(\psi(t)\big)\left(1-\frac{t^2}{x^2}\right)^{\frac{i-3}{2}} dt \le 0 \tag{43}$$

and

$$\int_0^x g\big(\psi(t)\big)(1-t^2)^{\frac{i-3}{2}} dt \le 0, \tag{44}$$

respectively. Since the function

$$(1-t^2)^{\frac{3-i}{2}}\left(1-\frac{t^2}{x^2}\right)^{\frac{i-3}{2}}$$

is decreasing with respect to t for $i \ge 3, 0 \le t < x \le 1$, the inequality (44) implies (43) by Lemma (1.2.11), that is, (43) implies (41).
From Theorem (1.2.7) and Theorem (1.2.12), we have the following:
Theorem (1.2.13)[32]:. The generalized Busemann-Petty problem has a negative answer for $2 < i < n$.
The case of 2-dimensional remains open for $n > 3$. In $\mathbb{R}^3$, the answer is positive (see [34]).
Proposition (1.2.14)[32]:. If K is a centered convex body of revolution in $\mathbb{R}^n$, then $K \in I_2^n, I_3^n$, and hence the generalized Busemann-Petty problem has a positive answer for $i = 2,3$.
Proof. Without loss of generality, assume that the axis of revolution is the x_n-axis. Let

$$t = cos\psi , \qquad x = cos\phi, \qquad g_1(t) = \rho_K^{n-i}(\psi(t))\, sin^{n-i}\psi(t).$$

Then (40) becomes

$$\int_0^x g(t) g_2(t,x) dt \le 0 \implies \int_0^1 g(t) g_1(t) dt \le 0 \tag{45}$$

where

$$g_2(t,x) = \left(\frac{1-\frac{t^2}{x^2}}{1-t^2}\right)^{\frac{i-3}{2}} \qquad ,0 \ge t < x \le 1.$$

Since $g_1(t)$ is decreasing and $g_2(t,x)$ is increasing with respect to t for $i = 2,3$, the implication (36) holds by Lemma (1.2.11). Thus, $K \in I_2^n$, I_3^n by Lemma (1.2.9).
In the cases of $\mathbb{R}^3$ and $\mathbb{R}^4$, Proposition (1.2.14) was proved in [33] ,[34]. We remark that the class $I_i^n (i > 3)$ does not contain all the centered convex bodies of revolution. In fact, it does not contain any cylinder. This was shown in [35], [36] when $3 < i = n - 1$. The general case $3 < i \le n - 1$ can be proved similarly.
If K is a cross-polytope (octahedron), Meyer [37] showed that the Busemann-Petty problem has a positive answer. He also asked if it is true for any polar projection bodies. We give a negative answer to Meyer's question. It will be seen that the counterexample is even very close to the cross-polytope.
For $f \in C(S^{n-1})$, the cosine transform Cf of f is defined by

$$(Cf)(u) = \frac{1}{2}\int_{S^{n-1}} |\langle u, v\rangle| f(v) dv, \qquad u \in S^{n-1}.$$

The cosine transform is a bijection of $C_e^\infty(S^{n-1})$ to itself. Denote by h_K the support function of a convex body K. Recall that K is a projection body (centered zonoid) if and only if there is a (positive) measure μ on S^{n-1} so that

$$h_K(u) = \frac{1}{2}{}_{S^{n-1}}|\langle u, v\rangle| d\mu(v), \qquad u \in S^{n-1}.$$

Denote by K^* the polar of K. We need the following lemma which was proved in [38] by using convolutions on $SO(n)$.

Lemma (1.2.15)[32]:. Let Z be a projection body in $\mathbb{R}^n$. Then there exist C^∞ projection bodies of positive curvature, $Z_m, m = 1,2, \ldots,$ so that $Z_m \to Z$ uniformly and the inverse cosine transforms $C^{-1}h_{Z_m} > 0, m = 1,2, \ldots.$

Theorem (1.2.16)[32]:. Let $3 \le i \le n-1$. There exist polar projection bodies K and L in $\mathbb{R}^n (n \ge 4)$ so that

$$vol_i(K \cap H) < \ vol_i(L \cap H), H \ \in Gr(n, i),$$

but

$$vol_n(K) > vol_n(L).$$

Proof. Let Z be a zonotope, e.g., the unit cube. Then the polar Z^* is the cross-polytope. Since every polytope in $\mathbb{R}^n (n \ge 4)$ is not in $I_i^n, 3 \le i \le n-1, Z^*$ is not in I_i^n. By Lemma (1.2.15), there are polar projection bodies $Z_m^* \to Z^*$ uniformly. In view of the openness of the complement of I_i^n with respect to the Hausdorff metric, Z_m^* is not in I_i^n when m is sufficiently large. Therefore, there exists a C^∞ projection body $\tilde{Z}$ of positive curvature such that $\tilde{Z}^*$ is not in I_i^n and the inverse cosine transform $C^{-1}h_{\tilde{Z}} > 0$.

Let $L = \tilde{Z}^*$. Since L is not in I_i^n, by Lemma (1.2.2) there exists $g \in C_e^\infty(S^{n-1})$ so that

$$\langle \rho_L^{n-i}, g\rangle > 0, \qquad R_i g < 0. \tag{46}$$

Consider the deformation of L, L_ε, defined by

$$\rho_L^{-1} = \rho_L^{-1} - \varepsilon \frac{g}{i\rho_L^{i+1}}, \qquad \varepsilon > \ 0 \tag{47}$$

Since $\tilde{Z}$ has positive curvature and $C^{-1}\rho_L^{-1} > \ 0, L_\varepsilon$ is a polar projection body when ε is small. From (48) we have

$$\tfrac{1}{\varepsilon}\left(\rho_{L_\varepsilon}^i - \rho_L^i\right) \to g \tag{48}$$

uniformly as $\varepsilon \to 0$.

On the other hand, from (46) we deduce that there exists $\delta > 0$ so that

$$\langle \rho_L^{n-i}, g_1\rangle > 0, \qquad R_i g_1 < 0 \tag{49}$$

whenever $|g_1 - g| < \delta$. Therefore, (43) and (44) give that

$$\rho_L^{n-i}, \rho_{L_\varepsilon}^i - \rho_L^i > 0, \quad R_i(\rho_{L_\varepsilon}^i - \rho_L^i) < 0$$

when ε is small. By applying (31), (33) and (34), we obtain

$$vol_n(L_\varepsilon) > vol_n(L),$$
$$vol_i(L_\varepsilon \cap H) < vol_i(L \cap H), \qquad H \in Gr(n, i).$$

We have seen that the Busemann-Petty problem has a negative answer in the class of polar projection bodies in $\mathbb{R}^n (n > 3)$. Concerning the shadows of convex bodies, Petty [35] constructed the following example: there exist a double cone K and a ball L in $\mathbb{R}^3$ so that

$$vol_2(K|\, u^{\perp}) < vol_2(L|u^{\perp}), \qquad u \in S^2,$$

but

$$vol_3(K) > vol_3(L),$$

where $K|u^{\perp}$ is the projection of K onto the space $u^{\perp}$ orthogonal to u.

Double cones and balls are polar projection bodies. One can use an argument similar to the proof of Theorem (1.2.16) to show that there are polar projection bodies K and L so that

$$vol_{n-1}(K|u^{\perp}) < vol_{n-1}(L|u^{\perp}), \qquad u \in S^{n-1},$$

but

$$vol_n(K) > vol_n\,(L).$$

It is natural to ask how far the volumes go when we compare or shadows of polar projection bodies. The following is a quantitative answer.

Theorem (1.2.17)[32]:. If K is a polar projection body and $L \in K_e^n$, then

$$vol_{n-1}\,(K\ \cap u^{\perp}) \le vol_{n-1}(L \cap u^{\perp}), u \in S^{n-1} \Rightarrow vol_n(K) < 2\, vol_n(L),$$

$$vol_{n-1}(K|u^{\perp}) \ge vol_{n-1}(L|u^{\perp}), u \in S^{n-1} \Rightarrow vol_n(K) > \frac{3}{4} vol_n(L).$$

The case of projection is an easy consequence of Ball's results on the volume ratio. We need several lemmas to treat the case of intersection. The first lemma is from [36]. Let $\beta(\cdot,\cdot)$ be the beta function.

Lemma (1.2.18)[32]:. If $K \in K_e^n$,then for $p \ge 1, u \in S^{n-1}$,

$$c_1 \frac{vol_n(K)}{vol_{n-1}(K \cap u^{\perp})} \le \frac{1}{vol_n(K)} \ {}_K|\langle u,x\rangle|^p dx\ ^{\frac{1}{p}} \le c_2 \frac{vol_n(K)}{vol_{n-1}\,(K \cap u^{\perp})},$$

where $c_1 = \frac{1}{2}(p+1)^{-\frac{1}{p}}, c_2 = \frac{1}{2} n^{\frac{p+1}{p}} \beta(p+1,n)^{\frac{1}{p}}$.

As noted above, a polar projection body is the unit ball of a finite dimensional subspace of L^1. For generality, we consider finite dimensional subspaces of L^p.

Lemma (1.2.19)[32]:. If M is the unit ball of an n-dimensional subspace of $L^p, p \ge 1$, then

$$\min_{u \in S^{n-1}} \frac{{}_{x\in K}|\langle u,x\rangle|^p dx}{{}_{x\in L}|\langle u,x\rangle|^p dx} \le \frac{\tilde{V}_{-p}(K,M)}{\tilde{V}_{-p}(L,M)}. \tag{50}$$

Proof. Since M is the unit ball of an n-dimensional subspace of L^p, there exists a nonnegative measure μ on S^{n-1} so that the radial function ρ_M is given by

$$\rho_M^{-p}(u) = {}_{S^{n-1}}|\langle u,v\rangle|^p\, d\mu(v).$$

Integrating $|\langle v,x\rangle|^p$ over K and L by polar coordinates and using (33), we have

$$\frac{\tilde{V}_{-p}(K,M)}{\tilde{V}_{-p}(L,M)} = \frac{{}_{u\in S^{n-1}}\rho_K^{n+p}(u)\rho_M^{-p}(u)du}{{}_{u\in S^{n-1}}\rho_L^{n+p}(u)\rho_M^{-p}(u)du}$$

$$= \frac{{}_{u\in S^{n-1}}\ {}_{x\in K}|\langle u,x\rangle|^p dx d\mu(v)}{{}_{u\in S^{n-1}}\ {}_{x\in L}|\langle u,x\rangle|^p dx d\mu(v)}$$

$$\ge \min_{u\in S^{n-1}} \frac{{}_{x\in K}|\langle u,x\rangle|^p dx}{{}_{x\in L}|\langle u,x\rangle|^p dx}.$$

Lemma (1.2.20)[32]:. If M is the unit ball of an n-dimensional subspace of L^p containing $K \in K_e^n$, then

$$\min_{u\in S^{n-1}} \frac{vol_{n-1}(L\cap u^{\perp})}{vol_{n-1}(K\cap u^{\perp})} \le c_3 \; \frac{vol_n(M)}{vol_n(K)}^{\frac{1}{n}} \frac{vol_n(L)}{vol_n(K)}^{\frac{n-1}{n}},$$

where $c_3 = ((p+1)n^{p+1}\beta(p+1,n))^{\frac{1}{p}}$.

Proof. From Lemmas (1.2.18) and (1.2.19), we have

$$\min_{u\in S^{n-1}} \frac{vol_{n-1}(L\cap u^{\perp})}{vol_{n-1}(K\cap u^{\perp})} \le \frac{c_2 vol_n(M)}{c_1 vol_n(K)} \min_{u\in S^{n-1}} \frac{\frac{1}{Vol_n(K)}\,{}_K|\langle u,x\rangle|^p dx^{\frac{1}{p}}}{\frac{1}{Vol_n(L)}\,{}_L|\langle u,x\rangle|^p dx^{\frac{1}{p}}}$$

$$\le \frac{c_2}{c_1}\frac{vol_n(L)}{vol_n(K)}^{\frac{p+1}{p}} \; \frac{\tilde{V}_{-p}\,(K,M)}{\tilde{V}_{-p}\,(L,M)}^{\frac{1}{p}}$$

$$\le \frac{c_2}{c_1}\frac{vol_n(L)}{vol_n(K)}^{\frac{p+1}{p}} \; \frac{vol_n(K)}{vol_n(L)^{\frac{n+p}{n}}\, vol_n(M)^{-\frac{p}{n}}}^{\frac{1}{p}}$$

$$= c_3 \; \frac{vol_n(M)^{\frac{1}{n}}}{vol_n(K)} \frac{vol_n(L)}{vol_n(K)}^{\frac{n-1}{n}}.$$

Let us turn to the proof of Theorem (1.2.17). From $p = 1$ and $K = M$ in Lemma (1.2.20), we obtain

$$vol_{n-1}(K\cap u^{\perp}) \le vol_{n-1}(L\;\cap u^{\perp}), u\in S^{n-1} \;\Rightarrow vol_n(K) \le \frac{2n}{n+1}^{\frac{n}{n-1}} vol_n(L)$$
$$\Longrightarrow vol_n(K) < 2vol_n(L).$$

For the case of projection, Ball [33] showed the following fact:

$$vol_{n-1}(K|u^{\perp}) \ge vol_{n-1}\,(L|u^{\perp}), u\in S^{n-1} \Longrightarrow$$
$$vol_n(K) \ge \frac{vol_n(E)}{vol_n(K)}^{\frac{1}{n-1}} vol_n(L),$$

where E is the ellipsoid of maximal volume contained in K. Ball also showed the volume ratio inequality (see [34], Theorem (1.2.6))

$$\frac{vol_n(E)}{vol_n(K)} \ge \frac{n!\,\kappa_n}{2^n n^{n/2}}.$$

It is an exercise to check that

$$\frac{n!\,\kappa_n}{2^n n^{n/2}}^{\frac{1}{n-1}} > \frac{\pi}{2e}^{\frac{1}{2}} > \frac{3}{4}.$$

Theorem (1.2.21)[32]:. If K is a polar projection body in $\mathbb{R}^n$, then there exists a constant $c < 0.92$ so that

$$vol_n(K)^{\frac{n-1}{n}} \le c \;\max_{u\in S^{n-1}} vol_{n-1}(K\cap u^{\perp}).$$

Proof. From Lemma (1.2.18), we have

$$\frac{vol_n(K)}{4vol_{n-1}(K\cap u^{\perp})} \le \frac{1}{vol_n(K)}\, {}_K|\langle u,x\rangle|\,dx, \qquad u\in S^{n-1}.$$

By Lemma (1.1.19), for the unit ball B we have

$$\min_{u\in S^{n-1}} \frac{1}{vol_n(K)}\, {}_K|\langle u,x\rangle|\,dx \le \tilde{V}_{1-}(B,K)^{-1}\, {}_B|\langle u,x\rangle|\,dx$$

$$\le \frac{1}{n+1}\kappa_n^{-\frac{n+1}{n}} vol_n(K)^{\frac{1}{n}}\, {}_{S^{n-1}}|\langle u,v\rangle|\,dv$$

$$= \frac{2\kappa_{n-1}}{(n+1)\kappa_n^{\frac{n+1}{n}}} vol_n(K)^{\frac{1}{n}}.$$

It follows that

$$vol_n(K)^{\frac{n-1}{n}} \le \frac{4}{\pi}\ \frac{\kappa_{n-1}}{\kappa_n^{\frac{n+1}{n}}} \max_{u\in S^{n-1}} vol_{n-1}(K\cap u^{\perp})\,.$$

It can be shown that $\frac{\kappa_{n+1}}{\kappa_n^{\frac{n+1}{n}}}$ is decreasing, for example, by using an argument similar to that in [36]. It is known that $c = \frac{\kappa_3^{\frac{2}{3}}}{\kappa_2} = \frac{16}{9\pi}^{\frac{1}{3}} = 0.827\ ...$ is the best constant for all convex bodies in $\mathbb{R}^n$ (see [37], Theorem 9.4.11). Hence, the case of $n = 4$ gives that $c < 0.92$.

The above theorem was proved by Ball [38] with a bigger constant. He used the complementary Blaschke-Santal´o inequality of Bourgain and Milman [39] and the local theory of Banach spaces. The main interest is that $c < 1$. This implies that

$$vol_n(K) < \max_{H\in Gr(n,i)} vol_i(K\cap H)\,, 2\le i\le n-1,$$

for polar projection bodies. One suspects that the last inequality is true for all centered convex bodies.

We consider of projection bodies (centered zonoids). Let K be a convex body, and let E be the ellipsoid of minimal volume containing K. The following lemma is a variation of a result of Ball [33].

Lemma (1.2.22)[32]:. If K is a projection body, then the outer volume ratio of K satisfies the inequality

$$\frac{vol_n(E)^{\frac{1}{n}}}{vol_n(K)} \le \frac{\sqrt{n}\kappa_n^{\frac{1}{n}}}{2} \tag{51}$$

with equality if K is a cube.

Proof. Since the volume ratio is affine invariant, it suffices to consider convex bodies K defined by

$$h_K(u) = {}_{j=1}^{m} c_j\left|\langle u_j,u\rangle\right|, \quad u,u_j\in S^{n-1}, \tag{52}$$

$${}_{j=1}^{m} c_j u_j \otimes u_j = I_n, \qquad c_j > 0,$$

where $u_j\otimes u_j$ is the rank-1 orthogonal projection onto the span of u_j and $I_n$is the identity operator on $\mathbb{R}^n$ (see, for example, [34]). The last equality implies that

$${}_{j=1}^{m} c_j = n,\ {}_{j=1}^{m} c_j\left|\langle u_j,u\rangle\right|^2 = 1.$$

By the Holder inequality, we obtain

$$h_K(u) \le n^{\frac{1}{2}} \sum_{j=1}^{m} c_j \left|\langle u_j, u\rangle\right|^2 {}^{\frac{1}{2}} = n^{\frac{1}{2}}. \tag{53}$$

By a result of Ball [35], the volume of the projection body Z with support function

$$h_Z(u) = \sum_{j=1}^{m} c_j \left|\langle u_j, u\rangle\right|$$

is at least 2^n, that is, $vol_n(K) \ge 2^n$. From (45), K is contained in a ball of radius $n^{\frac{1}{2}}$, and hence $vol_n(E) \le n^{\frac{n}{2}}\kappa_n$. Inequality (46) follows.

Theorem (1.2.23)[32]:. If K is a projection body, then

$$vol_{n-1}(K \cap u^{\perp}) \le vol_{n-1}(L \cap u^{\perp}), u \in S^{n-1} \Rightarrow vol_n(K) < 2.07\, vol_n(L),$$

for all $L \in K_e^n$.

Proof. Let I be an intersection body containing K. The Radon inverse $R_{n-1}^{-1}\rho_I$ is a positive measure on S^{n-1}, denoted by μ. By the self-adjointness of R_{n-1} and formula (31), we have

$$\max_{u \in S^{n-1}} \frac{vol_{n-1}(K \cap u^{\perp})}{vol_{n-1}(L \cap u^{\perp})} \ge \frac{\int_{S^{n-1}} vol_{n-1}(K \cap u^{\perp}) d\mu(u)}{\int_{S^{n-1}} vol_{n-1}(L \cap u^{\perp}) d\mu(u)}$$
$$= \frac{\langle R_{n-1}\rho_K^{n-1}, R_{n-1}^{-1}\rho_I\rangle}{\langle R_{n-1}\rho_L^{n-1}, R_{n-1}^{-1}\rho_I\rangle} = \frac{\langle \rho_K^{n-1}, \rho_I\rangle}{\langle \rho_L^{n-1}, \rho_I\rangle}$$
$$\ge \frac{vol_n(K)}{vol_n(L)^{\frac{n-1}{n}} vol_n(I)^{\frac{1}{n}}}.$$

Therefore, we obtain

$$vol_{n-1}(K \cap u^{\perp}) \le vol_{n-1}(L \cap u^{\perp}), \qquad u \in S^{n-1} \Rightarrow$$
$$vol_n(K) \le \left(\frac{vol_n(I)}{vol_n(K)}\right)^{\frac{1}{n-1}} vol_n(L),$$

which holds for all $K, L \in K_e^n$. In particular, If K is a projection body, then Lemma (1.2.22) shows that I can be chosen so that

$$\left(\frac{vol_n(I)}{vol_n(K)}\right)^{\frac{1}{n-1}} \le \left(\frac{\sqrt{n}\kappa_n^{\frac{1}{n}}}{2}\right)^{\frac{n}{n-1}}$$
$$< \left(\frac{\pi e}{2}\right)^{\frac{1}{2}} < 2.07.$$

From the above proof, we have seen that if a class of convex bodies has uniformly bounded outer volume ratio then the maximal slice problem has a positive answer in that class. This was clear in [33]. More generally, in view of Lemma (1.2.20), this is still true if the minimal ellipsoid is replaced by a minimal unit ball of subspaces of $L^p, 1 \le p \le 1000$. However, p cannot be arbitrarily large.

Chapter 2
Extensions of the Brunn-Minkowski and Boundary Regularity

We sharpen the inequality that the marginal of a log concave function is log concave, and we show various moment inequalities for such functions. Finally, we use these results to derive inequalities for the fundamental solution of the diffusion equation with a convex potential.

Section (2.1): Prèkopa- -Leindler Theorems Including Inequalities for Log Concave Functions with Application to the Diffusion Equation

We give various extensions of the Brunn-Minkowski and PrCkopa-Leindler theorems. The Brunn-Minkowski theorem for the convex addition $D = \lambda A + (1 - \lambda)B = \{x \in R^n | x = \lambda y + (1 - \lambda)z, y \in A, \mathcal{Z} \in B)\}$ of two nonempty, measurable sets $A, B\ C\ R^n$ reads [1,2]

$$\mu_n(D)^{\,1/n} \geq \lambda_{\mu_n}(A)^{\frac{1}{n}} + (1 - \lambda)\mu_n(B)^{\frac{1}{n}}, \tag{1}$$

where μ_n means Lebesgue measure in R^n. The requirement that A and B are nonempty is crucial.

The Prèkopa -Leindler theorem [65] reads

$$\|\mathcal{R}\|_1 \geq \|f\|_1^{\lambda} \|g\|_1^{1-\lambda}, \tag{2}$$

where

$$\mathcal{R}(x \setminus f, g) = \sup\ f\ \left(\frac{x-y}{\lambda}\right)^{\lambda}\ g\ \left(\frac{\lambda}{1-\lambda}\right)^{1-\lambda} \tag{3}$$

and f, g are nonnegative, measurable functions on R^n. If f and g are the characteristic functions of A and B, respectively, $\mathcal{R}$ is the characteristic function of D. Thus, $Eq.$ (2) states that $\mu_n(\lambda A) => 1$ if $\mu_n(A) = \mu_n(B) = 1$. By the scaling property$\mu_n(\lambda A) = \lambda^n \mu_n(A)$.

Thus $Eq.$ (2) implies $Eq.$ (1). In that sense, the Prèkopa -Leindler theorem can be viewed as an extension of the Brunn-Minkowski theorem.

These theorems are extended here in the following ways. The sup in Eq. (3) is replaced by ess sup:

$$h(x\ |f, g) = \operatorname*{ess\,sup}_{y \in R^n} f\ \left(\frac{x-y}{\lambda}\right)^{\lambda} g\ \left(\frac{y}{1-\lambda}\right)^{1-\lambda}. \tag{4}$$

The Prèkopa -Leindler theorem strengthened in this way is contained in Theorems (2.1.2) and (2.1.3).

Our new version really is stronger than the old; generally, $\|h\|_1 \leq \|R\|_1$and there are functions f and g such that h differs greatly from R. It is a fact, however, established that f and g can always be replaced by functions f^* and g^* which differ only by null functions from f and g such that

$$h(x\,|f, g) = h(x\,|f^*, g^*) = \mathcal{R}\,(x\,|f^*, g^*)\,.$$

Thus, once one knows how to construct f^* and g^*, the strengthened Prèkopa -Leindler theorem follows from the known one.

However, we prefer to work with the essential supremum h, because (1) $h(\ x\ )$ is unaltered if null functions are added to f and g, and (2) $h(x)$ is lower semicontinuous for any measurable f and g.
The supremum $\mathcal{R}$has neither property.

By taking characteristic functions for f and g, a stronger form of the Brunn-Minkowski theorem results; as above, it can be derived from the known theorem. The proof given here of the PrCkopa-Leindler theorem is based on the Brunn-Minkowski theorem; it is simpler than the original proof by Prèkopa and Leindler.
The idea of our proof is already contained in [66]. Another (rather involved) proof of the strengthened Prèkopa -Leindler theorem is given [67].
Other types of convex combinations, h_α , of two functions, f and g are defined for $\alpha \in [-\infty, \infty]$; see Eqs. $(5)-(7)$.
The convex combination in Eq. (4) is the case $\alpha = 0$. theorems of the PrCkopa-Leindler type are given for general α (Theorems (2.1.1)-(2.1.3)). A Brunn-Minkowski-like version of these theorems is contained in Corollary (2.1.4). For the case $\alpha = 0$ and with sup instead of ess sup, it was first given by Prèkopa [68]A much simpler proof for that case was found by Rinott [68]; his proof is completely different. Rinott also found the case $\alpha = -1/n$ in Corollary (2.1.4). Moreover, he found a converse of Corollary (2.1.4), saying that Eq. (18) f or all A, B implies the existence of a log concave density function. we consider log concave functions. A corollary of the Prèkopa -Leindler theorem is that $\int F\,(x, y)$ dy is log concave in x if $F(x, y)$ is log concave in (x, y). This result is sharpened in Theorem (2.1.7). In Theorem (2.1.6) a Sobolev-type inequality for log concave measures is given. Some theorems on log concave functions have counterparts for log convex functions (Theorems (2.1.9), (2.1.10), and (2.1.14)). However, these counterparts are comparatively trivial; they essentially follow from the usual convexity arguments (Hölder’s inequality). We stress that the log concave theorems and other Brunn-Minkowski and Prèkopa -Leindler-like theorems do not follow trivially from Holder’s inequality. we give inequalities for the moments of a Gaussian distribution, compared with the moments of the same distribution perturbed by a log concave (or log convex) function (Theorem (2.1.10)). We give an application to the diffusion equation in R^n with convex potential. More applications (the Ising model, the one dimensional Coulomb plasma) are given in [66].
Given nonnegative measurable functions $f(x), g(x)$ on R^n, we shall introduce various convex combinations of them, parametrized by the real number $\alpha \in [-\infty, \infty]$. With $0 < \lambda < 1$, we define

$$h_\alpha(x\ |f, g) = \underset{y \in R^n}{ess\ sup} \left\{ \left[\lambda f\left(\frac{x-y}{\lambda}\right)^\alpha \oplus (1-\lambda) g\left(\frac{y}{1-\lambda}\right)^\alpha\right\}^{\frac{1}{\alpha}}. \qquad (5)$$

The symbol $\oplus$differs from the ordinary addition $+$ in that for

$$f = O \quad or \quad g = O, \quad \{\lambda f^\alpha \oplus (1-\lambda)\, g^\alpha\}^{1/\alpha} = 0. \qquad (6)$$

Otherwise, $\oplus$and $+$ are the same: For $f > 0$ and $g > 0$,

$$\{\lambda f^\alpha \oplus (1-\lambda)\, g^\alpha\}^{1/\alpha}$$

$$\{\lambda f^{\alpha} + (1-\lambda) g^{\alpha}\}^{\frac{1}{\alpha}}, \; if \quad -\infty < \alpha < 0, 0 < \alpha < \infty;$$
$$= min(f,g), \quad if\ \alpha = -\infty; \qquad (7)$$
$$= max(f,g), \quad if\ \alpha = \infty;$$
$$= f^{\lambda} g^{1-\lambda}, \quad if\ \alpha = \infty.$$
Note, that $\oplus$ and $+$ are completely identical for $\alpha < 0$; however, for $\alpha > 0$ E_q (6) makes them essentially different. Note further that

$$h_{\alpha}(x) \le h_{\beta}(x) if \ \alpha < \beta .$$

We shall often write $h_{\alpha}(f,g)$ $h_{\alpha}(x)$ or h_{α} if the dependence of $h_{\alpha}(x \mid f,g)$ on X, f and g, or both is obvious. The dependence on h is not displayed, λ being held fixed.

As a particular case, take for f and g characteristic functions of measurable sets $A, B \subset R^n$: $f = x_A, g = X_B$. Then by $Eqs.$ (6), (7),

$$\{\lambda f^{\alpha} \oplus (1-\lambda) g^{\alpha}\}^{1/\alpha} = 0 \quad or \quad 0,$$

independent of LY. Hence, there is a set C such that

$$h_{\alpha}(x_A, x_B) = x_c, \ \forall_{\alpha}$$

We shall use the notation

$$C = ess\{\lambda A + (1-\lambda)B\}.$$

To stress the difference with the ordinary Brunn-Minkowski addition we give appropriate definitions:

$$\lambda A + (1-\lambda)B = \{x \in R^n | (x - \lambda A) \cap (1-\lambda)B \neq \phi\};$$
$$ess\{\lambda A + (1-\lambda)B = \{x \in R^n | \ _{\mu_n}[(x-\lambda A) \cap (1-\lambda)B] > 0\}. \qquad (8)$$

The ordinary addition results, if ess sup in $Eq.$ (5) is replaced by sup, The ordinary and the essential additions may differ considerably, as can be seen by taking for A a single point. However, there always 5^{8_0} $/22/4-4$ exist sets A^* and B^* which differ from A and B by null sets and such that

$$A^* + B^* = ess(A^* + B^*) = ess(A + B) \qquad (9)$$

Equation (9) and the Brunn-Minkowski theorem, $Eq.$ (1), immediately imply the strengthened Brunn- Minkowski theorem

$$_{\mu_n}(C)^{1/n} \ge \lambda_{\mu_n}(A)^{\frac{1}{n}} + (1-\lambda)\mu_n(B)^{\frac{1}{n}}, \qquad (10)$$

If $\mu_n(A) > 0, \mu_n(B) > 0$.

We show how $Eq.$ (10) extends to inequalities for $\|h_{\alpha}\|_1$ in terms of $\|f\|_1$ and $\|g\|_1$. The following theorem is basic.

Theorem (2.1.1)[63]: Let f, g be nonnegative, measurable functions on $\boldsymbol{R}$ and define $h_{-\infty}$ as in $Eqs.$ (5) $-$ (7):

$$h_{-\infty}(x) = \underset{y \in \boldsymbol{R}}{ess\ sup\ min} \left\{ f\left(\frac{x-y}{1-\lambda}\right), g\left(\frac{y}{1-\lambda}\right)\right\}.$$

Let $\|f\|_{\infty} = \|g\|_{\infty} \equiv m$. Then

$$\|h_{-\infty}\|_1 \ge \lambda \|f\|_1 + (1-\lambda)\|g\|_1 .$$

Proof: For $z > 0$, define the sets

$$A(z) = \{x \in \boldsymbol{R} \mid f(x) > z\},$$

$$B(z) = \{x \in \boldsymbol{R}\ |g(x) > z\},$$
$$D(z) = \{x \in \boldsymbol{R}\ |h_{-\infty}(x) > z\},$$
Then
$$D(z)\ = ess\{\lambda A(z) + (1-\lambda)B(z)\},$$
by the definitions of h-, and of the essential addition.
$If\ z > m, \mu_n(\mathrm{A}(z))\ >\ 0\ and\ \mu_n(\mathrm{A}(z)) >\ 0. Thus, by\ Eq.$ (10)
$$\mu_n\big(\mathrm{A}(z)\big) >\ \lambda_{\mu_1}\big(\mathrm{A}(z)\big) + (1-\lambda)\ _{\mu_1}\big(\mathrm{B}(z)\big).$$
Note, further, that $_{\mu_1}(D(z))\ =\ _{\mu_1}(A(z))\ =\ _{\mu_1}(B(x))\ =\ 0\ for\ x\ \geq\ m$, and that
$$\|f\|_1 = \int_0^{\infty}\ _{\mu_1}\big(A(z)\big)dx, \quad \text{etc.}$$
This gives the desired result.
Theorem (2.1.1) immediately leads to
Theorem (2.1.2) [63]: Let f, g be nonnegative measurable functions on R and define h_α, as in $Eqs.$ (5)-(7). Let 1f) $\|f\|_1 >\ 0, \|g\|_1 >\ 0$. Then, for $\alpha \geq -1$,
$$\|h_\alpha\|_1 \geq \left\{\lambda\|f\|_1^{\beta} + (1-\lambda)\|g\|_1^{\beta}\right\}^{1/\beta}, \tag{11}$$
with $\beta =\ \alpha/(1+\alpha)$. In particular,
$$\|h_0\|_1 \geq \|f\|_1^{\lambda}\|g\|_1^{1-\lambda}. \tag{12}$$
Proof: It is sufficient to consider bounded functions f and g, since any f, g can be approximated from below in L^1 by bounded functions. Now define
$$F(x) = f(x)/\|f\|_\infty; \quad G(x) = g(x)/\|g\|_\infty.$$
Let us first consider the case $\alpha \neq\ 0$. Then
$$h_\alpha(x|f,g)\ \underset{v\in R}{\text{ess sup}}\{\lambda\|f\|_\infty^\alpha F\left(\frac{x-y}{\lambda}\right)^\alpha \oplus(1-\lambda)\|g\|_\infty^\alpha G\left(\frac{y}{1-\lambda}\right)^\alpha\}^{1/\alpha}$$
$$= [\lambda\|f\|_\infty^\alpha(1-\lambda)\|g\|_\infty^\alpha]^{1/\alpha}$$
$$+\ \underset{v\in R}{\text{ess sup}}\{\theta F\left(\frac{x-y}{\lambda}\right)^\alpha \oplus(1-\theta)G\left(\frac{y}{1-\lambda}\right)^\alpha\}^{1/\alpha},$$
with the obvious meaning of $\theta, 0\ <\ \theta\ <\ 1$. Thus
$$h_\alpha(x|f,g) \geq [\lambda\|f\|_\infty^\alpha + (1-\lambda)\|g\|_\infty^\alpha]^{1/\alpha}\ h_{-\infty}(x|F,G),$$
and by Theorem 1
$$\|h_\alpha\|_1 \geq [\lambda\|f\|_\infty^\lambda + (1-\lambda)\|g\|_\infty^\alpha]^{1/\alpha}\left[\lambda\frac{\|f\|_1}{\|f\|_\infty} + (1-\lambda)\frac{\|g\|_1}{\|g\|_\infty}\right] \tag{13}$$
Now Eq. (11) for $-1 \leq \alpha < 0\ or\ 0 < \alpha \leq \infty$ follows by Hölder's inequality.
For $\alpha =\ 0$,
$h_0(f,g)\ = \|f\|_\infty^\lambda\|g\|_\infty^{1-\lambda}h_0(F,G) \geq \|f\|_\infty^\lambda\|g\|_\infty^{1-\lambda}h_{-\infty}(F,G)$.
Then Theorem (2.1.1) gives
$$\|h_0\|_1 \geq \|f\|_\infty^\lambda\|g\|_\infty^{1-\lambda}\left[\lambda\frac{\|f\|_1}{\|f\|_\infty} + (1-\lambda)\frac{\|g\|_1}{\|g\|_\infty}\right] \tag{14}.$$
and Eq. (12) follows by the arithmetic-geometric mean inequality.
Theorem (2.1.3) [63]: Let f, g be nonnegative measurable functions on R^n and define h_α as in Eqs. (5)-(7). Let $\|f\|_1\ =\ 0, \|g\|_1 >\ 0$. Then for $\alpha\ \geq -1/n$,
$$\|h_\alpha\|_1 \geq \{\lambda\|f\|_1^\gamma + (1-\lambda)\|g\|_1^\gamma\}^{1/\gamma} \tag{15},$$

with $\gamma = \alpha/(1 + n\alpha)$. In particular,

$$\|h_0\|_1 \geq \|f\|_1^{\lambda}\|g\|_1^{1-\lambda}.$$

Proof: Write $R^n \ni x = (y,z)$, with$y \in R, z \in R^{n-1}$. Define

$$F(z) = \int dyf(y,z)\,;\ G(z) = \int dyg(y,z). \tag{16}$$

Since

$$h_\alpha(y,z|f,g) = \operatorname*{ess\,sup}_{w\in R^{n-1}} \operatorname*{ess\,sup}_{v\in R}\{\lambda f\left(\frac{y-v}{\lambda},\frac{z-w}{\lambda}\right)^{\alpha} \oplus (1-\lambda)g\left(\frac{v}{1-\lambda},\frac{w}{1-\lambda}\right)^{\alpha}\}^{1/\alpha},$$

it follows from Theorem (2.1.2) that

$$\int dy\, h_\alpha(y,z|f,g) \geq h_\beta(z|F,G), \tag{17}$$

with $\beta = a/(a + 1)$. Note, that we used that

$$\int dy\, ess_{\,w}\, sup\ \geq\ ess_{\,w}\, sup \int dy.$$

Note further, that Theorem (2.1.2) does not apply, if x and w are such that $F((z - w)/\lambda) = 0\ or\ G(x/(1 - \lambda)) = 0$. However, Eq. (17) is saved by the $\oplus$ sign in the definition of h_β [cf. Eq. (16)].

If we assume Theorem (2.1.3) to be true for $n - 1$, we have that

$$\left\|h_\beta(F,G)\right\|_1 \geq \left\{\lambda\|F\|_1^{\gamma} + (1-\lambda)\|G\|_1^{\gamma}\right\}^{1/\gamma},$$

with $\gamma = \beta/[1 + (n - I)\beta] = \alpha/(1 + n\alpha)$. With Eqs. (16),(17) and Fubini's theorem, this leads to Eq. (15). Thus Theorem (2.1.3) is proved by induction.

As an introduction to two corollaries of Theorem (2.1.3), let us define the classes of functions $K_\alpha(R^n)$. $K_\alpha(R^n)$ consists of the nonnegative, measurable functions F on R^n such that for all $\lambda \in (0,1)$

$$F = h_\alpha(F,F) \quad a.e.$$

In more pedestrian terms, this means that F has the following convexity properties (apart from null functions).

$\alpha = -\infty :$ F is unimodal, $i.e.$, the sets $\{z\ |F(x) > z]$ are convex.

$-\infty < \alpha < O$: $F^\alpha$is convex.

$\alpha = 0 :$ F is logarithmically concave, i.e.,

$$F(\lambda x + (1 - \lambda)\,y) \geq F(x)^{\lambda}F(y)^{I-\lambda}.$$

$0 < \alpha < \infty :$ F^α is concave on a convex set, and $F(x) = 0$ outside this set.

$\alpha = \infty :$ $F(x)$ = const. on a convex set, and $F(x) = 0$ outside this set.

Note, that $K_\alpha \subset K_\beta$ if $\alpha > \beta$. This follows from Jensen's inequality.

Corollary (2.1.4) [63]: Let A, B be measurable sets in R^n of positive measure, and let

$$C = ess(\lambda A + (1 - \lambda)B\}.$$

Let $F \in K_\alpha(R^n), \alpha \geq -1/n$, and let

$$\mu_F(A) = \int_A F(x)dx.$$

Then, with $\gamma = \alpha/(1 + n\alpha)$,

$$\mu_F(C) \geq \left\{\lambda_{\mu_F}(A)^{\gamma} + (1-\lambda)\mu_F(B)^{\gamma}\right\}^{1/\gamma}.$$

In. particular, if F is log concave,

$$\mu_F(C) \geq \mu_F(A)^{\lambda}\mu_F(B)^{1/\lambda}. \quad (18)$$

Proof: Let $f = F_{\chi_A}$ and $g = F_{\chi_B}$. Then $h_\alpha(f,g) \leq \chi ch_\alpha(F,F) = \chi cF$.Apply Theorem (2.1.3) to complete the proof

(i) Let $F(x) \equiv 1 \in K_\infty$. Then $\gamma = 1/n$ and we recover the Brunn-Minkowski theorem, Eq. (10).

(ii) Let $G(x) = exp(-x^2) \in K_0$. Then in any R^n

$$\mu_G(C) \geq \mu_G(A)^{\lambda}\mu_G(B)^{1/\lambda}.$$

(iii) Let $L(x) = (1 + x^2)^{-1} \in K_{-1/2}$. Then

$$\mu_L(C) \geq \left\{\lambda_{\mu_L}(A)^{-1} + (1-\lambda)\mu_L(B)^{-1}\right\}^{-1}, \text{ in R},$$

$$\mu_L(C) \geq \min\{\mu_L(A), \mu_L(B)\}, \text{in R2}.$$

Corollary (2.1.5) [63]: Let $F(x,y) \in K_\alpha(R^{m+n}), x \in R^m, y \in R^n$. Let

$$G(x) = \int_{R^n} F(x,y)\,dy.$$

Then $G \in K_\gamma(R^m), \gamma = \alpha/(1 + n\alpha)$. In particular, if F is log concawe, so is G.

Proof: Since $F(x,y) > 0$ on a convex set in R^{m+n}, $G(x) > 0$ on a convex set in R^m.Now fix points x_0, x_1 in this set, and define $f\ (y) = F(x_1,y), g(y) = F(x_0,y)$. Then

$$F(\lambda_{x_1} + (1-\lambda)\,x_0, y) \geq h_\alpha(y\,|f,g).$$

apply Theorem (2.1.3) to $h_\alpha(y\,|f,g)$.

We prove a Sobolev-type inequality (Theorem (2.1.6)) for log concave measures (i.e., measures given by a log concave density function). We shall write $F(x) = exp[-f(x)], x \in R^n$; $F(x)$ is log concave iff $f(x)$ is convex. If $f(x)$ is twice continuously differentiable, this means that the second derivatives matrix, f_{xx} , is nonnegative.

It is often convenient to write $R^{n+m} \ni x = (y,z), y \in R^m, z \in R^n$.

The matrix f_{xx} is then partitioned in an obvious way as

$$f_{xx} = \begin{pmatrix} f_{yy} & f_{yz} \\ f_{zy} & f_{zz} \end{pmatrix}. \quad (19)$$

We shall often encounter

$$G(y) = \exp[-g(y) \equiv \int_{R^n} F(y,z)\,dz. \quad (20)$$

Then $G(y)$ is log concave by Corollary (2.1.5). A sharper form of this result will be given in Theorem (2.1.7).

With F as a density function, define

$$\langle A \rangle = \int_{R^n} A(x)F(x)\,dx \Big/ \int_{R^n} F(x)dx,$$

$$\text{varA} = \langle |A - \langle A \rangle|^2 \rangle,$$

$$\text{var} \quad (A,B) = \langle (\overline{A} - \langle \overline{A} \rangle)(B - \langle B \rangle\,) \rangle. \quad (21)$$

If $x = (y,x), y \in R^m, z \in R^n$, we write

$$\langle A \rangle_z(y) = \int_{R^n} A(y,z)F(y,z)\,dz \Big/ \int_{R^n} F(y,z)dz,$$

$$\langle B\rangle_y = \int_{R^m} B(y)G(y)\,dy / \int_{R^m} G(y)dy, \tag{22}$$

so that $\langle A\rangle = \langle\langle A\rangle_Z\rangle_y$. In analogy with $Eq.$ (21), var_y, , $covy_y$, cov_z, and cov_z, are defined.

Theorem (2.1.6) [63]: Let $F(x) = exp[-f(x)], x \in R^n$, let f be twice continuously dajfferentiable and let f be strictly convex. Let f have a minimum, so that F decreases exponentially in all directions; then

$$\int_{R^n} F(x)\,dx < \infty.$$

Let $h \in C^1(R^n)$, and let var $h < \infty$. Then

$$var\ h \le \langle (h_x , (f_{xx})^{-1} h_x)\rangle, \tag{23}$$

where the inner product is with respect to C^n, and h, denotes the gradient of h.

It is convenient to postpone the proof of Theorem (2.1.6) a moment.

We prefer to give an immediate corollary first.

Theorem (2.1.7) [63]: Let $F(x) = F(y, x) = exp[-f\ (y, z)], y \in R^m$, $z \in R^n$, satisfy the assumptions of Theorem (2.1.6). Moreover, let the Integrals

$$\int_{R^n} (\phi, f_{yy}\phi) F dz, \int_{R^n} (\phi, f_y)^2 F dz, \tag{24}$$

converge uniformly in y in a neighborhood of a given point $y_0 \in R^m$, for all vectors $\phi \in R^m$.Then, with the notation of Eqs. (19, 20, 22), $g(y)$ is twice continuously differentiable near y_0 , and

$$g_{yy} \ge \langle f_{yy} - f_{yz}(f_{zz})^{-1} f_{zy}\rangle_z \tag{25}$$

as a matrix inequality.

Proof: We denote differentiation in a direction t at $y_0$by a subscript t. Then Eq. (25) is equivalent to saying that for all directions t

$$g_{tt} \ge \langle f_{tt} - f_{tz}(f_{zz})^{-1} f_{zt}\rangle_z.$$

By differentiating $g(y) = \log \int F(y, z)\,dz$, one gets

$$g_{tt} = \langle f_{tt}\rangle_z - \mathrm{var}_z\, f_t \tag{26}.$$

The differentiation can be done under the integral sign by the uniform convergence of the integrals (24), which also ensures the continuity of g_{tt}.

The result (25) follows by applying Theorem (2.1.6) with $h(z) = f_t(y_0, z)$.

Q.E.D.

Remark(2.1.8) [63]: Even though F is assumed to be a log concave function, decreasing exponentially in all directions, the convergence of the integrals (24) does not follow automatically. For example, define the convex function $\phi(x), x \in R$, by $\phi(0) = \phi'(0) = 0$, and

$$\phi''(x) = \sum_{n\ne 0} a_n \delta(x - n),\ a_n > 0,\ a_n = a_{-n}.$$

Then

$$\int \phi''(x) \exp[-\phi(x)]dx = 2\sum_{n=1}^{\infty} a_n \exp[-\sum_{k=1}^{n-1}(n-k)a_k],$$

which can be made divergent by an appropriate recursive definition of a_n . If we take

$$f(y, z) = y^2 + \phi(y + z)\ ,\quad y, z \in R,$$

the integrals (24) obviously diverge for all y.
The function ϕ can be approximated by a C^2 function without changing the conclusion.
We can obviously restrict h to be real valued. Let us first give the proof for R^1. If $f(x)$ has its unique minimum at $x = a$, write

$$h(x) - h(a) = f'(x)\, k(x).$$

Then $k(x)$ is continuously differentiable, except possibly at $x = a$. However, if we set $k(a) = h'(a)/f''(a), k$ is continuous at $x = a$.
Now

$$\int (h')^2/f''F\, dx = \int [(k'f')^2/f'' + 2kk'f' + k^2 f''] \,F\, dx$$
$$= \int [(k'f')^2/f'' + (kf')^2]F\, dx + [k^2 f'F]_{-\infty}^{a} + [k^2 f'F]_{a}^{\infty}$$
$$\geq \int [h(x) - h(a)]^2 F(x) dx.$$

Equation (23) follows by noting that

$$\operatorname{var} h \leq \langle [h - h(a)]^2 \rangle.$$

Now assume that Theorem (2.1.6) has been proved for $x \in R^{n-1}$. Hence we also have Theorem (2.1.7) for $z \in R^{n-1}$ at our disposition. Write $R^n \ni x = (y, z), y \in R\,, z \in R^{n-1}$.
Then

$$\operatorname{var} h = \langle \operatorname*{var}_z h \rangle_y + \operatorname*{var}_y \langle h \rangle_z\,,$$

with the notation of Eqs. (21, 22).
Let us first restrict ourselves to functions h with compact support.
This has the advantage that F can be modified outside the support of h in such a way, that it satisfies all the assumptions of Theorem (2.1.7) for all y. Then $G(y) = \int F(y, z)\, dz$ satisfies the assumptions of Theorem (2.1.6), so that

$$\operatorname*{var}_y \langle h \rangle_z \leq \langle \left(\left(\frac{d}{dy} \right) \langle h \rangle_z \right)^2 /g'' \rangle_y.$$

Now all differentiations can be carried out under the integral signs, since h has compact support and F has been appropriately modified. Thus we find (cf. Eq. (26))

$$\operatorname{var} h \leq \langle B \rangle_y,$$
$$B = \operatorname*{var}_z\; h + \frac{[\langle h_y \rangle_z - \operatorname*{cov}_z\,(\mathrm{h}, \mathrm{f}_y)\,]^2}{\langle \mathrm{f}_{yy} \rangle_z - \operatorname*{var}_z \mathrm{f}_y}. \qquad (27)$$

Applying Theorem (2.1.6) for $z \in R^{n-1}$, with fixed $y \in R$, we have

$$\operatorname*{var}_z H \leq \langle H_z, f_{zz}^{-1} H_z \rangle_z.$$

Since this is true for

$$H = \lambda h + \mu \mathrm{f}_y$$

with arbitrary λ and μ, we get

$$H \leq \langle (h_z, f_{zz}^{-1} h_z) \rangle_z + \frac{\langle h_y - (h_z, f_{zz}^{-1} f_{zy}) \rangle_z^2}{\langle f_{yy} - (f_{yz}, f_{zz}^{-1} f_{zy}) \rangle_z}.$$

Since f is convex, the denominator above is positive and we can use Schwarz's inequality to obtain

$$H \leq \langle (h_z, f_{zz}^{-1} h_z) + \frac{\langle h_y - (h_z, f_{zz}^{-1} f_{zy}) \rangle_z^2}{f_{yy} - (f_{yz}, f_{zz}^{-1} f_{zy}} \rangle_z$$
$$= \langle (h_x, f_{xx}^{-1} h_x) \rangle_z. \tag{28}$$

Eq. (23) follows by combining Eqs. (27) and (28).
Now only the restriction that h has compact support remains to be removed. As an intermediate step, let us show that for all h and F satisfying the assumptions of Theorem (2.1.6)

$$\underset{S}{\text{var}}\, h \leq \langle h_x, f_{xx}^{-1} h_x \rangle_S, \tag{29}$$

where the averages are taken over a ball with radius S centered at the origin, instead of over all R^n.

Modify h outside the ball smoothly to a function k with compact support, and let

$$f^{(N)}(x) = f(x), \qquad if \quad |x| \leq S;$$
$$f^{(N)}(x) = f(x)N(|x| - S)^4, \quad if \quad |x| \geq S.$$

By our results until now, we have that

$$var_N\, R \leq \langle \left(k_x, \left(f_{xx}^{(N)} \right)^{-1} R_x \right) \rangle_N,$$

with averages with respect to the weight $\exp[-f^{(N)}(x)]$. Equation (29) is proved by taking the limit $N \to \infty$ and using the monotone convergence theorem.
Now let $S \to \infty$ in $Eq.$ (29). Then $var_S h + var\, h$, and

$$\int_S (h_x\,, f_{xx}^{-1} h_x) F\, dx$$

increases (it may actually increase to ∞). This concludes the proof.
Let $M_{ij} = cov(x_i\,, x_j)$. Then we have the matrix inequality

$$M \leq \langle (f_{xx})^{-1} \rangle, \tag{30}$$

as can be seen by taking $h(x) = (\phi, x)$ for any $\phi \in R^n$ in Theorem (2.1.6).
As a curiosity, compare (30) with the one dimensional inequality

$$var\, x \geq \langle f'' \rangle^{-1}, \tag{31}$$

which holds for general weights F. The proof is

(i) $[cov(x, f')]^2 \leq var f'\, var\, x = \langle f'' \rangle\, var\, x,$

with Schwarz's inequality and two integrations by parts.

(ii) . For the Gaussian weight $F(x) = exp[-(x, Ax)]$,

$$var\, h \leq \langle (h_x, (2A)^{-1} h_x) \rangle. \tag{32}$$

In particular, if $F(x) = exp[-(x, x)/2]$,

$$var\, h \leq \langle |h_x|^2 \rangle \tag{33}.$$

(iii) . If $F(x) = exp[-(x, Ax)], M = (2A)^{-1}$, and thus the inequality in (30) holds as an equality.

(iv). The analog in the setting of Theorem (2.1.7) concerns the Gaussian

$$\Phi(x,y) = exp\left[-(x,y)\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right], (x,y) \in R^m \times R^n, \tag{34}$$

with a real, positive matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$.Then

$$\int \Phi(x,y)dy = const.\exp[-(x,D_x)], \tag{35}$$

with

$$D = B - BC^{-1}B^*. \tag{36}$$

Thus for Gaussians the equality sign in $Eq.$ (25) holds,

Theorem (2.1.9) [63]: With the notation of $Eqs.$ $(34-36)$, let $G(x)$ be defined by

$$\int \Phi(x,y)\, F(x,y)\, dy = G(x)\, exp[-x, D_x)].$$

Then, if $F(x,y)$ is log concave, $G(x)$ is log concave; if $F(x,y)$ is log convex, $G(x)$ is log convex.

Proof: Write

$$\Phi(x,y) = exp\big[-(x,D_x) - \big(y', C_{y'}\big)\big],$$
$$y' = y + C^{-1}B^*x.$$

Then

$$G(x) = \int \exp[-(y,Cy)]\, F(x, y - C^{-1}B^*x)dy. \tag{37}$$

If $F(x,y)$ is log concave, the integrand in $Eq.$ (37) is log concave.

Then $G(x)$ is log concave by Corollary (2.1.5). If $F(x,y)$ is log convex, the integrand is log convex in x for all fixed y. Then $G(x)$ is log convex by Hölder's inequality.

Note, that the log concave part of Theorem (2.1.9) also follows from

Theorem (2.1.7).

Theorem (2.1.10) [63]: Let $F(x)$ be a nonnegative function on R^n, and let A be a real, positive defnite, $n \times n$ matrix. Assume $exp[-(x,Ax)]\, F(x) \in L^1$ and define

$$\langle R\rangle_F = \int R(x)\, exp[-(x,Ax)]\, F(x)\, dx / \int exp[-(x,Ax)]\, F(x)\, dx\,.$$

If $F(x) \equiv 1$ we write $(\cdot)_1$. Let $\phi \in R^n, a \in R$. Then

$$\langle|(\phi,x) - \langle(\phi,x)\rangle_F|^\alpha\rangle_F \leq \langle|(\phi,x)|^\alpha\rangle_1,$$

when F is log concave and $\alpha \geq 1$;

$$\langle|(\phi,x) - a|^\alpha\rangle_F \leq \langle|(\phi,x)|^\alpha\rangle_1, \qquad if \alpha > 0,$$
$$\langle|(\phi,x) - a|^\alpha\rangle_F \geq \langle|(\phi,x)|^\alpha\rangle_1, \qquad if -1 < \alpha > 0,$$

when F is log convex.

Proof: By a linear transformation such that $(\phi,x) \to x_1$ and by

Theorem (2.1.9) it suffices to prove Theorem (2.1.10) for the one-dimensional case. This wll be done in Lemmas (2.1.11) and (2.1.12).

Lemma (2.1.11) [63]: Let $F(x)$ be a log convex function on R, and let the averages $(\cdot)_F$ and $(\cdot)_1$ be computed with the weights $exp(-x^2)\, F(x)$ and $exp(-x^2)$, respectively. Let $a \in R$. Then

$$\langle|x-a|^\alpha\rangle_F \geq \langle|x|^\alpha\rangle_1, \qquad if \alpha > o; \tag{38}$$
$$\langle|x-a|^\alpha\rangle_F \leq \langle|x|^\alpha\rangle_1, \qquad if -1 < \alpha < o. \tag{39}$$

Proof: Note that

$$\langle |x-a|^{\alpha}\rangle_F = \langle |x|^{\alpha}\rangle_G = \langle |x|^{\alpha}\rangle_H,$$

where

$$G(x) = F(x + a)\, exp(-2ax),$$
$$H(x) = G(x) + G(-x).$$

Since F is log convex, G and H are log convex; moreover, H is even.
Thus, for $\alpha > 0$, it has to be shown that

$$\langle x^{\alpha} H(x)\rangle \geq \langle x^{\alpha}\rangle \langle H(x)\rangle, \tag{40}$$

with the averages computed over $x > 0$ with the weight $exp(-x^2)$. But this is equivalent to the inequality

$$\int_0^{\infty}\int_0^{\infty} dx\, dy\, exp(-x^2 - y^2)[H(x) - H(y)](x^{\alpha} - y^{\alpha}) \geq 0, \tag{41}$$

which is obvious, since $H(x)$ and x^{α} are increasing functions for
$x > 0$.
If $-1 < \alpha < 0, x^{\alpha}$ is decreasing for $x > 0$, and hence

$$\langle x^{\alpha} H(x)\rangle \leq \langle x^{\alpha}\rangle \langle H(x)\rangle.$$

This proves Eq. (39).
Lemma (2.1.12) [63]: Let $F(x)$ be a log concave function on R. Then, with the notation of Lemma (2.1.11),

$$\langle |x - \langle x\rangle_F|^{\alpha}\rangle_F \leq \langle |x|^{\alpha}\rangle_1, \qquad if \alpha \geq 1 \tag{42}.$$

Proof: Write

$$\langle |x - \langle x\rangle_F|^{\alpha}\rangle_F = \langle |x|^{\alpha}\rangle_G,$$

with

$$G(x) = F(x + \langle x\rangle_F)\, exp(-2x\langle x\rangle_F).$$

Then $G(x)$ is log concave, and $\langle x\rangle_G = 0$. By approximation, it is sufficient to assume $G \in C^1$. Hence

$$\int dx\, exp(-x^2) G'(x) = 2 \int dx\, x\, \exp(- - x^2)\, G(x) = 0. \tag{43}$$

Moreover, there must exist a number K such that $G(x)$ is increasing for $x < K$; decreasing for $x > K. By\, Eq.$ (43) K must be finite and we can assume that $K \geq 0$, say. Then $G'(x) \geq 0$ for $x < 0$, and
$Eq.$ (43) implies that

$$\int_0^{\infty} dx\, exp(-x^2)\, G'(x) \leq 0. \tag{44}$$

It has to be shown that

$$\langle x^{\alpha}[G(x) + G(-x)]\rangle \leq \langle x^{\alpha}\rangle\langle G(x) + G(-x)\rangle, \tag{45}$$

where the averages are with respect to $exp(-x^2), x > 0$.
We assumed, that $G'(x) \geq 0$ for $x < 0$, and thus (cf. Eqs. (40, 41)]

$$\langle x^{\alpha} G(-x)\rangle \leq \langle x^{\alpha}\rangle\langle G(-x)\rangle.$$

We wish to show the same inequality for the $G(x)$ part in Eq. (45), which is equivalent to

$$\int_0^{\infty} dx \int_0^{x} dy\, exp(-x^2 - y^2)[G(x) - G(y)](x^{\alpha} - y^{\alpha}) \leq 0. \tag{46}$$

If we write

$$G(x) - G(y) = \int_y^x G'(z)\, dz\,,$$

Eq. (46) becomes

$$\int_0^\infty dz\, \psi(z) exp(-z^2) G'(z) \leq 0, \tag{47}$$

$$\psi(z) = exp(z^2) \int_z^\infty dx \int_0^z dy\, exp(-x^2 - y^2)(x^\alpha - y^\alpha). \tag{48}$$

If we manage to show that $\psi(z)$is an increasing function for $z > 0$,
Eq. (47) follows from Eq. (44) and the fact that $G'(x) \geq 0$ for
$0 < x < K$; $G'(x) \leq 0$ for $x > K$, and Lemma (2.1.12) is proved.
After some manipulation, we find that

$$\psi'(z) = \int_z^\infty dx\, exp(-x^2)(x^\alpha - z^\alpha)$$

$$+ z\, exp(z^2) \int_z^\infty dx \int_0^\infty dy\, exp(-x^2 - y^2)[(\alpha - 1)\, x^{\alpha-2} + y^\alpha x^{-2}].$$

Thus, if $\alpha \geq 1, \psi'(z) > 0$.

Theorem (2.1.13) [63]: Under the assumptions of Theorem (2.1.10), let M be the covariance matrix

$$M_{ij} = \langle x_i x_j \rangle_F - \langle x_i \rangle_F \langle x_j \rangle_F.$$

Then
$M \leq \langle (2A + f_{xx})^{-1} \rangle_F \leq (2A)^{-1}, if\ F \equiv exp(-f)$ is log concave;

$$M \geq (2A)^{-1}, \tag{49}$$ if F is log convex.

Proof: Setting $\alpha = 2$ in Theorem (2.1.10) leads to $M \leq (2A)^{-1}$
resp. $M \geq (2A)^{-1}$.The stronger inequality (49) is obtained from
Theorem (2.1.6) by taking $h(x) = (\phi, x)$ and replacing the weight $F(x)$ y
$exp[-(x, Ax)]\, F(x)$. Q.E.D.

Consider the diffusion equation in R^n

$$\partial\psi/\partial t = -H_{A\psi} \tag{50}$$

with the Hamiltonian

$$(H_{A\psi})(x) = -\tfrac{1}{2}(\Delta\psi)(x) + V(x)\, \psi(x), \tag{51}$$

defined on an open, connected region $A\ C\ R^n$, with zero boundary conditions. The potential $V(x)$ is assumed to be convex; in particular, $V(x)$ may be ∞ outside a convex set D.
Further we assume the region A to be such that

$$\int_A exp[-tV(x)] dx < \infty, \qquad \forall t > 0. \tag{52}$$

(This means that A is bounded in the directions, for which $V(x)$ does not go to ∞ as $|x| \to \infty$.)

The fundamental solution $G_A(x, y;\ t)$ of Eq. (50) is defined by

$$\left((\partial/\partial t) - H_{A,x}\right) G_A(x, y; t) = 0, \qquad x, y \in A \cap D, t > O;$$

$$G_A(x,y;t) = \delta(x - y), \qquad x,y \in A \cap D;$$
$$G_A(x,y;t) = 0, \qquad x \in \partial(A \cap D);$$
$$G_A(x,y;t) = 0, \qquad x \notin A \cap D \; or \; y \notin A \cap D.$$

We could, of course, replace A by $A \cap D$ without changing G_A , but the point is that in Theorem (2.1.15) we want to vary A while keeping D fixed.
Using the Trotter product formula, we can write

$$G_A(x,y;\ t) = \lim_{M\to\infty} \left(\frac{2\pi t}{M}\right)^{-nm/2} \int_A dx_1 \cdots \int_A dx_{M-1}$$
$$x \prod_{j=1}^{M} fi \; exp\,[-\frac{M}{2_t}\left(x_i - x_{j-1}\right)^2 - \frac{t}{M} V(x_j)], \qquad (53)$$

where $x_0 = x,\ x_M = y$.
Define the partition function by

$$Z_A(t) = Tr\, exp(-tH_A) = \int_A G_A(x,x;\ t)dx. \qquad (54)$$

Then Eq. (52) guarantees, that $Z_A(t) < \infty$ for all $t > 0$, so that HA has a pure point spectrum. In fact, Hölder's inequality applied to Eqs. (53, 54) gives that

$$Z_A(t) \le \int_A C^0(x,x;\ t)\, exp[-tV(x)]\, dx$$
$$= (2_\pi t)^{-n/2} \int_A exp[-tV(x)]\, dx,$$

where G^0 is the fundamental solution of Eq. (50) with $V(x) = 0$. Moreover the ground state is nondegenerate and the corresponding eigenfunction is nonnegative [69].

Theorem (2.1.14) [63]: Let $A = R^n$, and let the potential be of the form

$$V(x) = \tfrac{1}{2}w^2x^2 + W(x), \qquad w \ge 0, \qquad (55)$$

with a convex function W(x). Then the ground state wave function $\psi_0(x)$ is of the form

$$\psi_0(x) = exp\left(-\tfrac{1}{2}wx^2\right)\psi(x),$$

where $\phi(x)$ is log concave.
Proof: Let $G_w(x,y;\ t)$ be the fundamental solution of Eq. (50) for $V(x) = \frac{1}{2}w^2x^2$. Then the fundamental solution for the potential (55) is of the form

$$G(x,y;\ t) = G_w(x,y;\ t)\, H(x,y;\ t),$$

where $H(x,y;\ t)$ is log concave in (x,y) for all t. This follows directly from Theorem (2.1.9) applied to Eq. (53).
If ϵ is the ground state energy,

$$\psi_0(x)\psi_0(y) = \lim_{t\to\infty} G(x,y;t)\exp(\epsilon t)$$

Since the pointwise Iimit of log concave functions is log concave, the theorem follows.

Theorem (2.1.15) [63]: Let A and B be open, connected regions, let $C = \lambda A + (1 - \lambda)B$, and let $V(x)$ be convex. Then

$$Z_c(t) \geq Z_A(t)^{\lambda} Z_B(t)^{1-\lambda}; \quad (56)$$

$$\epsilon_C \leq \lambda\epsilon_A + (1-\lambda)\epsilon_B, \quad (57)$$

where $\epsilon_A(\epsilon_B, \epsilon_C)$ iS the ground State energy of $H_A, (H_B, H_C)$.

Proof: Equations (53, 54) together give an expression for the partition function. We note, that we can apply Corollary (2.1.4) to the sets A^M, B^M, and C^M. This proves Eq. (56). Further

$$\epsilon_A = -\lim_{t\to\infty} t^{-1} \log Z_A(t),$$

which gives Eq. (57). Q.E.D.

Theorem (2.1.16) [63]: For measurable sets A and $B \in R^n$, dejne the essential sum $C = ess\{A + B\}$ as in Eq. (8). Then C is open, and

$$\mu_n(C)^{1/n} \geq \mu_n(A)^{1/n} + \mu_n(B)^{1/n}. \quad (58)$$

Theorem (2.1.17) [63]: For nonnegative, measurable functions $f(x)$ and $g(x)$ on R^n, dejke

$$H_\alpha(x \mid f, g) = ess_{y\in R^n} sup\{f(x-y)^\alpha \oplus g(y)^\alpha\}^{1/\alpha} \quad (59)$$

cf. Eqs. (5-7). Then $H_\alpha(x)$ IS ower semicontinuous in x for all α,

All the above facts are based on the following observation: For an arbitrary measurable set $A\ C\ R^n$,define

$$A^* = (x \in R^n | \mu_n[A \cap V(\epsilon, x)]/W_n(\epsilon) \to 1 \;\; for\ \epsilon \downarrow 0\}, \quad (60)$$

where $V(c, x)$ is the open ball of radius ϵ centered at x, and $W_n(\epsilon)$ is its volume. Then A^* is measurable and $\mu_n(A^* \varDelta A) = 0$, where $\varDelta$ means symmetric difference [65, Theorem 2.9.111]. Hence

$$ess(A + B) = ess(A^* + B^*), \quad (61)$$

and it is sufficient to prove the theorem when A and B are replaced by $A^*$and B^*.

Let $x \in A^* + B^*$, i.e., there is a point $y \in A^* \cap (x - B^*)$. Notice, that $A^{**} = A^*$; thus for some $\epsilon > 0$,

$$\mu_n[A^* \cap V(\epsilon, y)] \geq \tfrac{3}{4} W_n(\epsilon);$$

$$\mu_n[x - B^*) \cap V(\epsilon, y)] \geq \tfrac{3}{4} W_n(\epsilon).$$

Hence, $\mu_n[A^* \cap (v, B^*)] > 0$ for all v in some neighborhood $V(\delta, x)$, which implies that $A^* + B^*$ is open, and that

$$A^* + B^* = ess(A^* + B^*). \quad (62)$$

Equation (58) now follows from Eqs. (61, 62) and the Brunn-Minkowski theorem, Eq. (1).

For a nonnegative, measurable function f, let

$$A_f = \{(x, z) \in R^{n+1} | O < z < f(x)\}. \quad (63)$$

Define $A_f *$ as in (60). If $(x, x) \in A_f *, (x, t) \in A_f *$ for all $t, 0 < t < z$. Thus it makes sense to define

$$f^*(x) = \sup\{z \,|(x, y) \in A_f *\}. \quad (64)$$

The supremum over the empty set is taken to be zero. Given f^*, define A_{f^*} according to definition (63). Clearly A_f, A_{f^*} and f^* are all measurable. By (63) and (64), $A_{f^*} \supset A_{f^*}$. Since

$$A_{f^*} \backslash A_{f^*} \subset G \equiv \{(x, f^*(x)) \ |x \in R^x\},$$

and since $\mu_{n+1}(G) = 0$, it follows that $\mu_{n+1}\big(A_f * \backslash A_{f^*}\big) = 0$.

$\int p = \mu_{n+1}\big(A_p\big)$Therefore

$$\int s \ |f^* - f|dx = \mu_{n+1}\big(A_{f^*} \varDelta A_f\big)$$
$$= \mu_{n+1}\big(A_f * \varDelta A_f\big) = 0. \tag{65}$$

As a consequence of (65),

$$H_\alpha(f, g) = H_\alpha(f^*, g^*). \tag{66}$$

Now consider the function

$$H_\alpha(x\,|f, g) = \sup_{y \in R^n}\{f(x - y)^\alpha \oplus g(y)^\alpha\}^{1/\alpha}. \tag{67}$$

Note that generally $K_\alpha(x) \geq K_\alpha(x)$. Let

$$D(z) = \{x \in R^n | K_\alpha(x\,|f^*, g^*) > z\} \quad z \geq 0 \tag{68}.$$

Choose $z \geq 0$, $x \in D(z)$. By definitions (67) and (68), there is $y \in R^n$, and numbers $b, c > 0$ such that

$$z \leq (b^\alpha, c^\alpha)^{1/\alpha},$$
$$f^*(x - y) > b, g^*(y) > c.$$

In other words

$$\beta \equiv (x - y, b) \in A_{,}., \ y \equiv (y, c) \in A_{,}. .$$

Then for all $\delta > 0$ there exist balls $V(\epsilon, \beta)$ and $V(\epsilon, \gamma)$in R^{n+1} such that, in the notation of (60),

$$\mu_{n+1}(A_{J*} \cap V(\epsilon, \beta)) \geq (1 - \delta)W_{n+1}(\epsilon),$$
$$\mu_{n+1}(A_{g*} \cap V(\epsilon, \gamma)) \geq (1 - \delta)W_{n+1}(\epsilon),$$

If δ is small enough, it follows that the sets

$$\{v \in V(\epsilon, x - y) | f^*(v) > b\},$$
$$\{w \in V(\epsilon, y) | g^*(w) > c\},$$

have measure at least equal to $\frac{3}{4}W_n(\epsilon)$. This implies (1) that $H_\alpha(x\,|f^*, g^*) > z$, so that in fact

$$H_\alpha(f^*, g^*) = K_\alpha(f^*, g^*), \tag{69}$$

and (2) that $D(z)$ contains a neighborhood of x, such that $D(z)$ is open. Hence $K_\alpha(f^*, g^*)$ is lower semicontinuous. By Eqs. (66,69), so is $H_\alpha(f, g)$.

Section (2.2): Maps with Convex Potentials

To recapitulate the existence theory of [74] given Ω_1, Ω_2 bounded domains, with $|\partial\Omega_i| = 0$, and non-negative functions f, g defined in Ω_1 (resp. Ω_2) and bounded away from zero and infinity, with $f_{\Omega_1} f = f_{\Omega_2} g$ one may construct convex potentials ψ, φ such that $\nabla\psi: \Omega_1 \to \Omega_2$ and $\nabla\varphi: \Omega_2 \to \Omega_1$ (in an a.e. sense) and satisfying

$$g\ (\nabla\psi)\ det\ D_{i,j}\psi = \ f(x)$$

in the integral sense

$$\int_{\Omega_2} \eta(Y)g(Y)dY = \int_{\Omega_1} \eta(\nabla\psi)f(X)dX$$

for any continuous η.

A similar equation holds for φ since in fact ψ and φ are constructed among those pairs of continuous simultaneously by minimizing

$$\int_{\Omega_1} \psi(X)f(X)dX + \int_{\Omega_2} \varphi(\mathrm{Y})g(Y)dY$$

among those pairs of continuous functions ψ, φ such that

$$\psi(X) + \varphi(\mathrm{Y}) \geqq \ (X, Y\ )$$

For any $X \ \in \Omega_1$, and $Y \in \Omega_2$. (This approach, slightly different than Brenier's ,was proposed by Varadhan.)

It is easy to see that ψ, φ can be taken Lipschitz and bounded (up to a normalization constant), since given the pair ψ, φ one may substitute ψ by

$$\psi^*(X) = \sup_{Y\in\overline{\Omega_2}} \langle X, Y\rangle - \varphi(\mathrm{Y}).$$

If we note that for a Lipschitz convex function points of Lebesgue differentiability for $\boldsymbol{\nabla\psi}$ must actually be points of continuity (see **[74]),** one can see that $\boldsymbol{\psi}$**,** $\boldsymbol{\varphi}$ are unique, inverse to each other, and satisfy the weak equation.
Without entering into the details of the proof, the weak equation is obtained as the Euler equation by making a variation $\boldsymbol{\varphi_\varepsilon} = \boldsymbol{\varphi} + \varepsilon\eta$ and

$$\boldsymbol{\varphi_\varepsilon(X)} = \boldsymbol{inf}\langle X, Y\rangle - \boldsymbol{\varphi_\varepsilon(Y)}$$

And computing

$$\lim_{\boldsymbol{\varepsilon\to 0}} \frac{1}{\boldsymbol{\varepsilon}} [\boldsymbol{\varphi_\varepsilon(X)} - \boldsymbol{\psi}(X)]f(X)dX$$

at the differentiability points of $\boldsymbol{\nabla\psi}\ (\ X\ )$.
That $\boldsymbol{\nabla\varphi}$ is the inverse of $\boldsymbol{\nabla\psi}$ has to be given careful interpretation at this point.
By the minimization property, given X_0 in $\boldsymbol{\Omega_1}$, there exists a Y_0 in $\boldsymbol{\overline{\Omega}_1}$ such that
$\boldsymbol{\psi}(X_0\ ) + \boldsymbol{\varphi}(Y_0\ ) = \langle X_0\ , Y_0\ \rangle$**.**
By symmetry Y_0 is the slope of a supporting plane to $\boldsymbol{\psi}$ at X_0 and vice versa.
Uniqueness is seen by noting that if $\boldsymbol{\psi}, \boldsymbol{\varphi}$ and $\overline{\boldsymbol{\psi}, \boldsymbol{\bar\varphi}}$ are minimizing pairs, so does any convex combination.
Hence, if X_0 is a point of joint differentiability for $\boldsymbol{\psi}$ and $\boldsymbol{\bar\psi}$**,** then $\boldsymbol{\nabla\psi}$ and
$\boldsymbol{\nabla\ \overline{\psi}}$ must be the same.

(If not
$\frac{1}{2}(\boldsymbol{\psi} + \overline{\boldsymbol{\psi}})(X_0) + \frac{1}{2}(\boldsymbol{\varphi} + \overline{\boldsymbol{\varphi}})(Y) > \langle X_0, Y\rangle$
for any Y in $\overline{\boldsymbol{\Omega}}_{\boldsymbol{2}}$.)

The regularity results of **[74]** are as follows: If $\boldsymbol{\Omega_2}$ is convex, $\boldsymbol{\psi}$ can be extended to a global ***(R^n)***viscosity solution of $C_1 x_{\Omega_1} \leqq \, det\, \boldsymbol{D_{i,j}\psi} \leqq C_2 x_{\Omega_2}$. Further, $\boldsymbol{\psi}$ is strictly convex on $\boldsymbol{\Omega_1}$.

This puts us under the framework of the local regularity theory developed in [75] and hence it follows that $\boldsymbol{\psi}$ is locally $\boldsymbol{C^{1,\alpha}}$**.**

From the discussion above, now $\boldsymbol{\nabla\varphi}$ is continuous in the image of $\boldsymbol{\Omega_1}$by $\boldsymbol{\nabla\psi}$ and

$$\boldsymbol{\nabla\varphi}(\boldsymbol{\nabla\psi}) = \boldsymbol{Id}$$

as continuous functions. If further f and $\boldsymbol{g}$ are $\boldsymbol{C^\alpha}$ functions, $\boldsymbol{\psi}$ is locally a $\boldsymbol{C^{2,\alpha}}$classical solution.

Let us point out that the problem being compact on $\boldsymbol{\Omega_i}, f, \boldsymbol{g}$ **,** the estimates remain uniform on those parameters as long as they remain in a closed family.

If now we assume both $\boldsymbol{\Omega_1}$**,** $\boldsymbol{\Omega_2}$ to be convex, both $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$**,** are locally $\boldsymbol{C^{1,\alpha}}$ for some $\alpha > 0$, and $\boldsymbol{\nabla\psi}$**,** $\boldsymbol{\nabla\varphi}$, are Hölder continuous inverses of each other.

If further f , $\boldsymbol{g}$ are Hölder continuous, , $\boldsymbol{\varphi}$ are locally $\boldsymbol{C^{2,\alpha}}$ and hence $\boldsymbol{\nabla\psi}$**,** $\boldsymbol{\nabla\varphi}$, become Hölder differentiable maps.

We now pass to study the boundary regularity of $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$.

The main theorem is the following.

Theorem(2.2.1)[73]: ***If*** *both* $\boldsymbol{\Omega_i}$ *are convex and* f, g *bounded away from zero and infinity, then* $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ *are* $\boldsymbol{C^{1,\alpha}}$ *up to*$\boldsymbol{\partial\Omega_i}$ ***for*** *some* $\alpha > 0s$. *Both a and* $\|\boldsymbol{\psi}\|_{C^{1,\alpha}}$, $\|\boldsymbol{\varphi}\|_{C^{1,\alpha}}$ *depend only on the maximum and minimum diameter* of $\boldsymbol{\Omega_i}$ ***,****and the* bounds on f, g.

The proof of the theorem is based on an iteration of a strict convexity property of functions $\boldsymbol{\psi}$ that satisfy an equation of the form

$$\det \boldsymbol{D_{i,j}\psi} = \boldsymbol{d\mu}$$

on suitable points for $\boldsymbol{d\mu}$.

We start by constructing adequate of such a $\boldsymbol{\psi}$.

Let $\boldsymbol{\psi}$ define a global convex graph ($\boldsymbol{\psi}: \boldsymbol{R^n} + (\overline{\boldsymbol{R}})$).

Assume that:

(a) $\boldsymbol{\psi}$ is finite in a neighborhood of zero,

(b) $\boldsymbol{\psi}$ is non-negative and $\boldsymbol{\psi}(0) = 0$.

Then, the set of slopes of all supporting planes, $\boldsymbol{S} = \{Y: (\boldsymbol{Y}, \boldsymbol{X}) + \lambda$, is a supporting plane to $\boldsymbol{\psi}\}$ is convex, and it has nonempty interior if and only if graph $\boldsymbol{\psi}$ contains no lines.

Consider now, for any $\boldsymbol{Y}$ in $\boldsymbol{S}$**,**

$$\sum\nolimits_Y = \{\boldsymbol{X}: \boldsymbol{\varphi}(\boldsymbol{X}) \leqq \langle \boldsymbol{X}, Y\rangle + 1\}.$$

Then, we prove

Lemma (2.2.2) [73]: *If* $\boldsymbol{S}$ *has nonempty interior, there exists a* Y *in* S^0 *such that the center* of *mass* of $\sum_Y$ *is zero.*

We call such a $\boldsymbol{\Sigma_Y}$ the centered at zero. If we replace $\langle \boldsymbol{X}, Y\rangle + 1$ by $\langle \boldsymbol{X}, Y\rangle + \varepsilon$ **,** we call it the entered at zero.

In order to do that we start with a lemma of independent interest, a simple variant of some theorems of Fritz John.

Lemma(2.2.3) [73]: *Let* $\boldsymbol{\Omega}$ *be a bounded convex set in* $\boldsymbol{R^n}$ *with the origin* ***for*** *center* of *mass. Let* $\boldsymbol{E}$ *be the symmetric ellipsoid* of *minimum volume containing* $\boldsymbol{\Omega}$. *Then, there exists a* $\boldsymbol{\lambda}$, *depending only on dimension, such that* $\boldsymbol{\lambda E}$ *is contained in* $\boldsymbol{\Omega}$.

Proof: By a linear transformation, we may assume that $\boldsymbol{E}$ is the unit ball. Then, if σe_1 is the closest point to the origin, we must show that $\boldsymbol{\sigma} \geqq \boldsymbol{\lambda}(n)$.

We first point out that $\boldsymbol{\Omega}$ is contained between two hyperplanes $-c\sigma \leqq \langle \boldsymbol{X}, e_1 \rangle \leqq \sigma$.

Indeed, the inequality on the right is just the definition **of** σ.

If $S \ S = \{ Y : \langle Y, e_1 \rangle = O \} \cap \boldsymbol{\Omega}$

and X satisfies

$$\langle X, e_1 \rangle < 0$$

the cone generated by X and $\boldsymbol{S}$ **is** contained in $\boldsymbol{\Omega}$ "to the left" of $\boldsymbol{S}$ (i.e., for $\langle \boldsymbol{Y}, e_1 \rangle < 0$) and contains $\boldsymbol{\Omega}$ to the right of $\boldsymbol{S}$.

If $\langle X, e_1 \rangle = -\boldsymbol{\mu}$, this allows us to estimate the e_1 component of the center of mass as

$$C_{(e_1)} \leqq \frac{|S|}{|\boldsymbol{\Omega}|} \int_{-\boldsymbol{\mu}}^{\sigma} t \left[\frac{t + \boldsymbol{\mu}}{\boldsymbol{\mu}} \right]^{n-1} dt \,,$$

a negative quantity for $\sigma/\boldsymbol{\mu}$ small.

Hence, we must have $\boldsymbol{\mu} \leqq \boldsymbol{C}\sigma$ and if $\sigma << 1$ since

$$\boldsymbol{\Omega} \subset B_1 \cap \{ X : |\langle \boldsymbol{X}, \boldsymbol{e_1} \rangle| \leqq \boldsymbol{C}\sigma \}$$

we may cover $\boldsymbol{\Omega}$ by an ellipsoid $\tilde{E}$ with

$$\left|\tilde{E}\right| < |B_1|$$

contradicting the definition of $\boldsymbol{B_1}$ .Let us now go back to the proof of Lemma (2.2.2).

For any $\boldsymbol{Y}$ in S^0, the $\sum_{\boldsymbol{Y}}$ **is** bounded, since $\sum_Y \subset \{ X : \langle \boldsymbol{Y}, \boldsymbol{X} \rangle + 1 \geqq \boldsymbol{L} \}$ for any supporting plane $\boldsymbol{L}$, and such a family of $\boldsymbol{L's}$ contains as slopes a neighborhood of $\boldsymbol{Y}$.

Hence, the functions "center of mass" $\boldsymbol{c}(\boldsymbol{Y}) = \boldsymbol{c}(\sum_{\boldsymbol{Y}})$ and momentum $\boldsymbol{m}(\boldsymbol{Y}) = c(Y)|\sum_{\boldsymbol{Y}}|$ are well defined for $Y \in S^0$. Assume first that $\boldsymbol{S}$ is bounded, i.e., ψ is globally Lipschitz.

We shall prove that $\boldsymbol{c}(\boldsymbol{Y})$ **is** locally Lipschitz in S^0, goes to infinity when $\boldsymbol{y}$ goes to ∂S, and has a local "transversality" property that forces, for $\boldsymbol{m}$ the momentum of $\sum_{\boldsymbol{Y}}$

$\min\limits_{S} |\boldsymbol{m}(\boldsymbol{Y})| = 0$.

Note first that when $\boldsymbol{Y_n}$ converges to $\mathbf{Y} \in \partial S$, the $\sum_{\boldsymbol{Y_n}}$ and its center of mass cannot remain bounded.

If not $\sum_{\boldsymbol{Y}}$ would be bounded, ψwould be transversal to $\boldsymbol{L_Y} = \langle \boldsymbol{X}, \boldsymbol{Y} \rangle + 1$ on $\boldsymbol{\partial} \sum_{\boldsymbol{Y}}$ **,**and hence $\boldsymbol{Y}$ would be interior to $\boldsymbol{S}$.

But, then if $c(Y_n)$ remains bounded, from Lemma $(\mathbf{2.2.3})$ we would have a sequence of ellipsoids $E_n(c(Y_n))$ centered on $c(Y_n)$ with maximum diameter going to infinity and contained in $\sum_{Y_n}$.

It follows that graph ψ contains a line, a contradiction. Hence

$$\lim_{Y\to\partial S}|c(Y)| = +\infty$$

And

$$\lim_{Y\to\partial S}|m(Y)| = +\infty$$

since $|\sum_Y| > \frac{1}{2}|\boldsymbol{B_\rho}|$, for $\boldsymbol{B_{\rho_{(o)}}}$ a small ball satisfying $\psi|_{B_p} < 1$.
The second observation is that, arguing as above, if Y remains in a compact subset of S^0 both $c(Y)$ and diam $(\sum_Y)$ remain uniformly bounded.
In particular, ψ and $L_Y = \langle X, Y\rangle + 1$, remain uniformly transversal along $\partial\sum_Y$ (i.e., $(\boldsymbol{\psi} - L_Y)(x) \geqq \boldsymbol{C}$ dist $(X, \sum_Y)$ with $\boldsymbol{C}$ independent of Y. It follows that if Y_1, Y_2, are both in such a compact subset, the Hausdorf distance

$$d\left(\sum_{Y_1}, \sum_{Y_2}\right)$$

Satisfies

$$d\left(\sum_{Y_1}, \sum_{Y_2}\right) \leqq C|Y_1 - Y_2|$$

and

$$|c(Y_1) - c(Y_2)| \leqq C|Y_1 - Y_2|$$
$$|m(Y_1) - m(Y_2)| \leqq C|Y_1 - Y_2|.$$

The third and final observation is that, always for Y in a compact subset of S^0,

$$\langle m(Y + \boldsymbol{\varepsilon e}) - m(Y), e\rangle \geqq \boldsymbol{K\varepsilon}.$$

Indeed, if we look at both half spaces $\boldsymbol{H}^+ = \{\langle X, e\rangle > \boldsymbol{O}\}$ and $\boldsymbol{H}^- = \{\langle \boldsymbol{X}, \boldsymbol{e}\rangle < 0\}$,

$$\Sigma_{Y+\boldsymbol{\varepsilon e}} \cap \boldsymbol{H}^+ \supset \boldsymbol{\Sigma_Y} \cap \boldsymbol{H}^+$$

And vice versa

$$\boldsymbol{\Sigma_Y} \cap \boldsymbol{H}^- \Sigma_{Y+\boldsymbol{\varepsilon e}} \cap \boldsymbol{H}^- ,$$

There fore

$$\langle m(Y + \boldsymbol{\varepsilon e}), \boldsymbol{e}\rangle \geqq \langle m(X), \boldsymbol{e}\rangle.$$

To see that there is effectively a gain of order ε, we recall first that ψ is Lipschitz (with norm A) and hence if $X \in \Sigma_Y \cap +\boldsymbol{H}^+$, then $X + \frac{\varepsilon}{\Lambda}\langle X, e\rangle e \in \Sigma_{Y+\boldsymbol{\varepsilon e}}$
for μ small enough (if $\psi(X) \leqq \langle X, Y\rangle + 1$

$$\psi\left(X + \frac{\varepsilon}{\Lambda}\langle X, e\rangle e\right) \leqq \langle X, Y\rangle$$
$$+\boldsymbol{\varepsilon}\langle \boldsymbol{X}, \boldsymbol{e}\rangle \leqq L_{Y+\boldsymbol{\varepsilon e}}$$

and hence, since for bounded Y, $\boldsymbol{\Sigma_Y}$ contains a fixed neighborhood of zero, say $\boldsymbol{B\tilde{\rho}}$,

$$(\boldsymbol{\Sigma_{Y+e\varepsilon}} \setminus \boldsymbol{\Sigma_Y}) \cap \{\langle \boldsymbol{X}, \boldsymbol{e}\rangle \geqq \boldsymbol{\tilde{\rho}}/\boldsymbol{2}\}$$

has measure of order $\boldsymbol{C}(\boldsymbol{\tilde{\rho}}, \boldsymbol{\Lambda})\,\boldsymbol{\varepsilon}$.
With these three remarks, it now follows (always for $\boldsymbol{\psi}$ Lipschitz) that

$$\min_{Y\in S^0}|m(Y)|^2 = 0.$$

Indeed, if not, let Y_0 be the point where such a minimum is attained.
Let $e = m(Y) / |m(Y)|$ and compute

$$|m(Y-\boldsymbol{e\varepsilon})|^2 = \langle \boldsymbol{m}\,(Y-\boldsymbol{e\varepsilon})\,,\boldsymbol{e}\,\rangle^{\mathbf{2}} + \langle m\,(Y-\boldsymbol{e\varepsilon},\boldsymbol{\tau}\rangle^2,$$

for some unit vector τ, with $(\,\boldsymbol{\tau}\,,\boldsymbol{e}\,)\;=\;0$.

Adding and subtracting $m(\,\boldsymbol{Y}\,)$ to each term we get

$$|m(Y)-\boldsymbol{e\varepsilon}|^2 \leqq (|m(Y)|-C_1\varepsilon)^2 + C_1\varepsilon^2 < |m(Y)|^2$$

for ε small.

This completes the proof of the lemma for $\boldsymbol{\psi}$ Lipschitz.

For a general graph $\boldsymbol{\psi}$**,** as in the hypothesis of the lemma, consider the increasing family of Lipschitz functions

$$\psi_M = \sup\ L_Y$$

with $\boldsymbol{L_Y}$ a supporting plane for $\boldsymbol{\psi}$ with $Y\;=\;\boldsymbol{\nabla L}$ satisfying $|Y| \leqq \boldsymbol{M}$**.** For M large enough $\boldsymbol{S_\psi}\ \cap M$ has nonempty interior and hence we may find a centered $\boldsymbol{\Sigma_{Y_M}}, of\ \boldsymbol{\psi_M}$ **.**

We show that for M going to infinity:

(a) $|\boldsymbol{Y_M}|$ remains bounded. Indeed $\boldsymbol{\psi}$ was finite (and hence $\boldsymbol{\psi} <\ 1/2$) in a neighborhood $\boldsymbol{B_\rho}$ of zero. Recall from Lemma (2.2.3) that $\boldsymbol{\Sigma_{Y_M}}$ is equivalent to a centered ellipsoid. Hence, since for any $\boldsymbol{\varepsilon}, (-\boldsymbol{Y_{\mathrm{M}}}-\boldsymbol{\varepsilon})\;/\;(\;|\boldsymbol{Y_{\mathrm{M}}}|^{\;\mathbf{2}}) \notin \boldsymbol{\Sigma_{Y_M}}$ (because $\boldsymbol{\psi_M} \geqq\ \mathbf{0}$), we get that $\boldsymbol{\Lambda}(\,\boldsymbol{Y_{\mathrm{M}}}+\ \boldsymbol{\varepsilon})\;/|\;\boldsymbol{Y_{\mathrm{M}}}|^{\mathbf{2}} \notin \boldsymbol{\Sigma_{Y_M}}$ either for $\boldsymbol{\Lambda}$ large ($\boldsymbol{\Lambda}\ >\ 1\;/\;\lambda$ as in Lemma (2.2.3)). That is

$$\boldsymbol{\psi_M}\left(\Lambda\frac{\boldsymbol{Y_{\mathrm{M}}}}{|\,\boldsymbol{Y_{\mathrm{M}}}|^2}\right) \geqq \langle \boldsymbol{Y_{\mathrm{M}}}\,,\frac{\Lambda \boldsymbol{Y_{\mathrm{M}}}}{|\,\boldsymbol{Y_{\mathrm{M}}}|^2}\rangle + 1 \geqq \Lambda + 1,$$

a contradiction if $|\boldsymbol{Y_{\mathrm{M}}}| >\ \boldsymbol{\Lambda}/\,\boldsymbol{\rho}$.

(b) The minimum and maximum diameters of $\boldsymbol{\Sigma_{Y_M}}$ (understood as those of the equivalent ellipsoid) are bounded away from zero (since $\boldsymbol{\psi}$ is close to zero near zero) and infinity (if not graph $\boldsymbol{\psi}$ would contain **a** line).

(c) For an appropriate subsequence $\boldsymbol{\Sigma_{Y_M}}$ converges in Hausdorff metric t $\boldsymbol{\Sigma_Y}$ of ψ with $c(\,Y\,)\;=\;0$.

Indeed, choose $\boldsymbol{\Sigma_Y}$ converging to , and $\boldsymbol{\Sigma_{Y_M}}$ converging to $\overline{\Sigma}$ in Hausdorf metric. Since ψ_M is increasing,

$\boldsymbol{\Sigma_Y}\,(\boldsymbol{\psi}) \subset \underline{\boldsymbol{lim}}\,\boldsymbol{\Sigma_{Y_M}}(\boldsymbol{\psi_M}) \subset\ \overline{\Sigma}$.

On the other hand, since $|\mathrm{Y}_M|$a nd diam $\boldsymbol{\Sigma_{Y_M}}$ remain bounded, $\boldsymbol{\psi_M}$ remain uniformly bounded in $\boldsymbol{\Sigma_{Y_M}}$ and uniformly transversal, that is

$$(\boldsymbol{\psi_M}(\boldsymbol{X}) - [\langle \boldsymbol{X}, \mathrm{Y}_M\,\rangle + \mathbf{1}]) \leqq -\boldsymbol{Cd}\big(\boldsymbol{X}, \boldsymbol{\partial\Sigma_{Y_M}}\big).$$

(Note that $g_M - 1$ is an upper barrier for

$$\boldsymbol{\psi_M} - [\langle \boldsymbol{X}, \mathrm{Y}_M\,\rangle + \mathbf{1}]\,,$$

with g_M the function, homogeneous of degree one, satisfying

$$g_M(0) = 0$$

And

$$g_M|_{\boldsymbol{\partial\Sigma_{Y_M}}} = 1\,.)$$

Hence, if $X\ \in (\overline{\Sigma})^0$, we have for M large that

$$d(X, \boldsymbol{\partial\Sigma_{Y_M}})\ >\ \delta$$

and hence

$$\psi(X) = \lim \boldsymbol{\psi_M}(X) \leqq \langle X, Y \rangle + 1 - C\delta .$$

Hence $\bar{\Sigma} \subset \Sigma_Y(\boldsymbol{\psi})$.
The proof of the lemma is now complete.
The following lemma can be found in [75].

Lemma (2.2.4) [73]: Let u be a convex solution of

$$\det D_{ij} u = d\mu$$

in the convex domain Ω in the Alexandrov sense with $B_1 \subset \Omega \ B_k$ and $u \equiv 0$ on $\partial\Omega$. Assume that for some $\lambda < 1$
$\mu(\lambda \Omega) \geqq \theta\mu(\Omega)$.
Then for $C_i = C_i(\theta, \lambda, K)$

$$C_1|\inf u| \leqq \mu(\Omega) \leqq C_2|\inf u| .$$

Further, for some λ', with $\lambda < \lambda' < 1$ and

$$C_i = C_i(\lambda, \lambda', \theta, K)$$

$B_{C_1\mu(\Omega)^{1/n}} \subset \nabla u(\lambda'\Omega) \subset B_{C_2\mu(\Omega)^{1/n}}$.
Proof: From the classic Alexandrov estimate

$$|u(x)|^n \leqq C\, vol(\nabla u(\Omega)) \cdot d(X, \partial\Omega)$$
$$= C\mu(\Omega)\, d(X, \partial\Omega) .$$

On the other hand, for any $\lambda < \lambda' < 1$

$$|\nabla u|_{\lambda\Omega} \leqq C(\lambda, \lambda')\left| \inf_{\lambda'\Omega} u \right|.$$

That is

$$\nabla u(\lambda\Omega) \subset B_{C(\lambda,\lambda')|\inf_{\lambda'\Omega} u|}$$

and hence

$$\mu(\lambda\Omega) \leqq C(\lambda, \lambda')\left| \inf_{\lambda'\Omega} u \right|^n.$$

In our case, since we are assuming

$$\theta\mu(\Omega) \leqq \mu(\lambda\Omega) ,$$

the first set of inequalities is proven.
To complete the second set of inequalities, we note that, from the Alexandrov estimate above, for λ' close enough to one.

$$\left| \inf_{\partial\lambda'\Omega} u \right| \leqq \frac{1}{2}\left| \inf_{\lambda'\Omega} u \right|$$

And therefore any linear function L with slope $s(L)$ smaller than C inf u is a supporting plane for u in $\lambda'\Omega$.
The proof of the lemma is now complete.

We are now ready to prove strict convexity of ψ up to $\partial\Omega$. This is due to the fact that $d\mu = det\, D_{ij}\psi$ satisfies the hypothesis of the previous lemma for any centered at a point of Ω .We may as well consider such a class of measure μ, that is Let $\Gamma = \Gamma(\theta, \lambda)$ be the class of non-negative measures μ with convex support $\Omega(\mu)$, such that for any convex set S with center of mass 0 in $\Omega(\mu)$, satisfies $\mu(\lambda S) \geqq \theta\mu(S)$.
Then we may prove the following lemma.

Lemma (2.2.5) [73]: Let u be a solution in S of

$$\det D_{ij}\, u = d\mu$$

with μ in Γ.

Assume that S is centered (ie., $C(S) = 0$), that $B_1 \subset S \subset B_K$, and that $\bar{X} \in \Omega(\mu) \cap \frac{1}{2}S$.

Extend u to $\ell S\ as + \infty$, and normalize μ so that $\mu(S) = 1$. Then there is $a\delta < 1$ so that the δ −centered of u at $\bar{X}$ (i.e., $\Sigma(\delta) = \{ X: \langle X - \bar{X}, Y\rangle + \delta > u(X) - u(\bar{X}) \}$, that has $\bar{X}$ as center of mass) is strictIy contained in $\lambda' S$ (for some '$(\lambda, \theta < 1)$.

Further, if $\tilde{X} \in \partial\lambda' \cap \Omega$, then $(\nabla u(\tilde{X}) - \nabla u(\bar{X}), \tilde{X} - \bar{X}) > \tau_0 > 0$.

Proof: From Lemma (2.2.4), u and ∇u are bounded in λS, and such exists. Assume that there is a sequence of fucntions u_k for which $\Sigma(1/k)$ always reaches $\partial\lambda' S$. Then, taking limits in the subsequence of solutions u_k the centers Σ_K and the corresponding linear functions $\langle X, Y_k\rangle$that define the, we find that the convex contact set D (we take from now on $X_0 = \lim \bar{X}_k$ as center of coordinates)

$$D = \{ X: u(X) - u(X_0) = \langle X, Y_\infty\rangle\}$$

has the following properties:

(a) There is a segment, $[-\alpha X_1, X_1]$in D with $X_1 \in \partial\lambda S$ and $\alpha \sim 1$ (since the ellipsoids Σ are centered and they all touch $\partial\lambda' S$. Therefore

(b) $|\min_D u| \geqq (1+t)|u(0)|$ since $|u(0)| \sim 1$ and, X_1 being in $\partial\lambda' S$, $|u(X_1)| \leqq \frac{1}{2}|u(0)|$(for this, λ' must be close to one).

(c) If $\widetilde{\Omega} = \lim \Omega_k$(the support of μ_k) then the extremal points of D in S^0 are contained in (the convex set) $\widetilde{\Omega}$.

Indeed (see, for instance, [76]), given the convex contact set $K = \{u = L\}$ of any convex function u with a supporting plane L, and X an extremal point of , one may find $\{ Y: u(Y) < \tilde{L}\}$ of diameter as small as one wishes containing X.

Therefore the approximating functions u_k have nontrivial of small diameter as close as we want to X and hence $X \in \widetilde{\Omega}$.

It follows from (a), (b), and (c) that both X_0 and the set $\widetilde{D} = \{X \in D / u(X) = \min_{Y\in D} u(Y)\}$ are in $\widetilde{\Omega}$, X_0 by hypothesis, and $\widetilde{D}$ because the extremal points of $\widetilde{D}$ are extremal points of D, obviously in S_0. Hence its convex envelope is also in $\widetilde{\Omega}$. Let now X_2 be the closest point in $\widetilde{D}$ to the origin, $X_1 = \mu X_2$ with $\mu < 1$ to be chosen and Σ the ε of u at X_1 (i.e., X_1 the center of mass of Σ), and $u(X_1) - L = -\varepsilon$ for L the linear function defining Σ.

We first point out that for μ close to one and ε close to zero, Σ must be strictly contained in S. This is because, once more Σ being centered at X_1 it is equivalent to a centered ellipsoid (Lemma (2.2.3)) and therefore if it has a segment joining $X_1$with ∂S_1 (note that $d(\widetilde{D}, \partial S_1)$ is strictly positive from Lemma (2.2.4)), it has a segment in the opposite direction.

Taking limits μ going to 1 and ε going to zero, we find that the graph of u has a nontrivial segment through X_2,_along which u is linear and nonconstant, a contradiction to the definition of $\widetilde{D}$.

Now fix μ close to one. Then $\Sigma(\varepsilon)$ contains a segment $[\alpha X_2, \beta X_2]$ through $X_1 = \mu X_2$ and since u is linear between 0 and X_2, we must have $\alpha < 0$ or $\beta > 1$. If μ is close enough to one, $\beta < 1$ will contradict the fact that Σ is centered since the segment $[0, X_1]$is much larger than $[X_1, X_2]$ .Thus, $\beta > 1$ and we must have $\lim_{\varepsilon \to 0} \beta = 1$ in order not to contradict the definition of $\widetilde{D}$. This makes $\alpha > 0$ since Σ is centered at X_1.
At this point we fix ε, so that β is very close to one, in order to make

$$\frac{\beta - 1}{(\beta - \mu)}$$

very small. We point out that, if L defines Σ
$(L - U)(X_1) < (L - u)(X_2)$, since $L - u$ is a linear function in $[O, X_2]$, positive at X_2, and zero at αX_2 (recall that $\alpha > 0$).
Let us now normalize u to the situation of Lemma (2.2.4), that is, by an affine transformation we transform Σ into Σ^*, X_1 into $X_1^* = 0$, and X_2 into X_2^* with
$B_1 \subset \Sigma^* \subset B_K$.
Since ratios along a ray are preserved by linear transformations and $BX_2^* \in \partial\Sigma^*$ we get that X_2^* is as close as we want to $\partial\Sigma^*$ (recall that $(\beta - 1)/(\beta - \mu)$ was as small as we wished and hence

$$\frac{|\beta X_0^* - X_0^*|}{|\beta X_0^* - X_1^*|} = \frac{\beta - 1}{\beta - \mu}$$

is small). Then $u - L$ gets renormalized to a function $u^*$and we would complete the proof of the lemma if we could say that
$u^*(O) = (L - u)(X_1) \sim \inf u^*$, and
$u^*(X_2^*) = (L - U)(X_2) \sim \inf u^* d(X_2^*, \partial\Sigma^*)$,
but this follows from the fact that u^* is on Σ^* the uniform limit of u_k^* (the renormalization of $L - u_k$) and $0 = X_1^*$ being in $\widetilde{\Omega}^*$ (the renormalization of $\widetilde{\Omega}$).
(Notice that the elements μ of $\Gamma(\lambda, \theta)$ are invariant under affine transformations.
This proves (i).) The second assertion follows similar lines (we again find a segment in $\overline{\Omega}$ where u is linear).
It is now easy to prove Theorem (2.2.1).
Let ψ be a global solution of $\det \boldsymbol{D_{i,j}\psi = d\mu} -$, with
μ in Γ and 0 in $\Omega\mu$. Let Σ_k be the ε^k centered at zero (k an integer).
The size of Σ_0 (i.e., maximum and minimum diameters) is, by compactness, controlled by the maximum and minimum diameters of Ω_i. By iteration of

(i) in the previous lemma we have that
$\Sigma_k \subset {\lambda'}^k \Sigma_0$,

and from part (ii) (and Lemma (2.2.4)), it follows that if $X_0 \in \Sigma_k \setminus \Sigma_{k+1} \cap \Omega$,

$$|\nabla\psi(X_0) - \nabla\psi(0)| \geqq C|X_0|^M$$

for some M.
This implies the Holder continuity of $\nabla\varphi$. The proof of the theorem is thus complete.

Chapter 3
From Brunn Minkowski to Brascamp Lieb

We deduce similarly the logarithmic Sobolev inequality for uniformly convex potentials for which we deal more generally with arbitrary norms and obtain some new results. Applications to transportation cost and to concentration on uniformly convex bodies complete the exposition. We present a simple direct proof of the classical Sobolev inequality in R^n with best constant from the geometric Brunn–Minkowski–Lusternik inequality.

Section (3.1): Logarithmic Sobolev Inequalities

After the first complete proof of the classical isoperimetric inequality was found, Minkowski proved the following inequality:

$$V\left((1-\lambda)K+\lambda L\right)^{1/n} \geq (1-\lambda)V(K)^{1/n}+\lambda V(L)^{1/n}. \qquad (1)$$

Here K and L are convex bodies (compact convex sets with nonempty interiors) in $\mathbb{R}^n, 0 < \lambda < 1, V$ denotes volume, and $+$ denotes vector or Minkowski sum. The inequality (1) had been proved for $n = 3$ earlier by Brunn, and now it is known as the Brunn-Minkowski inequality. It is a sharp inequality, equality holding if and only if K and L are homothetic.

The Brunn-Minkowski inequality was inspired by issues around the isoperimetric problem, and was for a long time considered to belong to geometry, where its significance is widely recognized. It implies, but is much stronger than, the intuitively clear fact that the function that gives the volumes of parallel hyperplane of a convex body is unimodal. It can be proved on a single , yet it quickly yields the classical isoperimetric inequality (21) for convex bodies and other important classes of sets. The fundamental geometric content of the Brunn-Minkowski inequality makes it a cornerstone of the Brunn-Minkowski theory, a beautiful and powerful apparatus for conquering all sorts of problems involving metric quantities such as volume, surface area, and mean width.

By the mid-twentieth century, however, when Lusternik, Hadwiger and Ohmann, and Henstock and Macbeath had established a satisfactory generalization of (1) and its equality conditions to Lebesgue measurable sets, the inequality had begun its move into the realm of analysis. The last twenty years have seen the Brunn Minkowski inequality consolidate its role as an analytical tool, and a compelling picture (see Figure 1) has emerged of its relations to other analytical inequalities. In an integral version of the Brunn-Minkowski inequality often called the Prèkopa -Leindler inequality (12), a reverse form of Hölder 's inequality, the geometry seems to have evaporated. Largely through the efforts of Brascamp and Lieb, this can be viewed as a special case of a sharp reverse form (32) of Young's inequality for convolution norms. A remarkable sharp inequality (36) proved by Barthe, closely related to (32), takes us up to the present time. The modern viewpoint entails an interaction between analysis and convex geometry so potent that whole conferences and books are devoted to “analytical convex geometry" or “convex geometric analysis." The main development of this includes historical remarks and several detailed proofs that amplify the previous paragraph and show that even the latest developments are accessible to graduate students. Several applications are also discussed at some length. Extensions of the Prèkopa-Leindler inequality can be used to obtain concavity properties of probability measures generated by densities of well-known distributions. Such results are related to Anderson's theorem on multivariate unimodality, an application of the Brunn-Minkowski inequality that in turn is useful in statistics. The entropy power inequality (48) of information theory has a form similar to that of the Brunn-Minkowski inequality. To some extent this is explained by Lieb's proof that the entropy power inequality is a special case of a sharp form of Young's inequality (31). This is given in detail along with some brief comments on the role of Fisher information and applications to physics. We come full circle with consequences of the later inequalities in convex geometry. Ball started these rolling with his elegant application of the Brascamp-Lieb inequality (35) to the volume of central of the cube and to a reverse isoperimetric inequality (45).

The whole story extends far beyond Figure 1 and the previous paragraph. The final is a survey of the many other extensions, analogues, variants, and applications of the Brunn-Minkowski inequality. Essentially the strongest inequality for compact convex sets in the direction of the Brunn-Minkowski inequality is the Aleksandrov-Fenchel inequality (51). Here there is a remarkable link with algebraic geometry: Khovanskii and Teissier independently discovered that the Aleksandrov-Fenchel inequality can be deduced from the Hodge index theorem. Analogues and variants of the Brunn-Minkowski inequality include Borell's inequality (57) for capacity, employed in the recent solution of the Minkowski problem for capacity; Milman's reverse Brunn-Minkowski inequality (64), which features prominently in the local theory of Banach spaces; a discrete Brunn-Minkowski inequality (65) due to Gronchi, closely related to a rich area of discrete mathematics, combinatorics, and graph theory concerning discrete isoperimetric inequalities; and inequalities (67), (68) originating in Busemann's theorem, motivated by his theory of area in Finsler spaces and used in Minkowski geometry and geometric tomography. Around the corner from the Brunn-Minkowski inequality lies a slew of related affine isoperimetric inequalities, such as the Petty projection inequality (62) and Zhang's affine Sobolev inequality (63), much more powerful than the isoperimetric inequality and the classical Sobolev inequality (24), respectively.

There are versions of the Brunn-Minkowski inequality in the sphere, hyperbolic space, Minkowski spacetime, and Gauss space, and there is a Riemannian version of the Prѐkopa - Leindler inequality, obtained very recently by Cordero-Erausquin, McCann, and Schmuckensch lÄager. Finally, pointers are given to other applications of the Brunn-Minkowski inequality. Worthy of special mention here is the derivation of logarithmic Sobolev inequalities from the Prekopa-Leindler inequality by Bobkov and Ledoux, and work of Brascamp and Lieb, Borell, McCann, and others on diffusion equations. Measure-preserving convex gradients and transportation of mass, utlilized by McCann in applications to shapes of crystals and interacting gases, were also employed by Barthe in the proof of his inequality.

In a sea of mathematics, the Brunn-Minkowski inequality appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams from one area to the next. It is quite clear that research opportunities abound. For example, what is the relationship between the Aleksandrov-Fenchel inequality and Barthe's inequality? Do even stronger inequalities await discovery in the region above Figure 1? Are there any hidden links between the various inequalities in? Perhaps, as more connections and relations are discovered, an underlying comprehensive theory will surface, one in which the classical Brunn-Minkowski theory represents just one particularly attractive piece of coral in a whole reef. Within geometry, the work of Lutwak and others in developing the dual Brunn-Minkowski and L^p-Brunn-Minkowski theories strongly suggests that this might well be the case.

We show the following easy result (for definitions and notation).

Theorem (3.1.1)[78]: (Brunn-Minkowski inequality in $\mathbb{R}$.) Let $0 < \lambda < 1$ and let X and Y be nonempty bounded measurable sets in $\mathbb{R}$ such that $(1-\lambda)X + \lambda Y$ is also measurable. Then

$$V_1\big((1-\lambda)X + \lambda Y\big) \geq (1-\lambda)V_1(X) + \lambda V_1(Y). \qquad (2)$$

Proof: Suppose that X and Y are compact sets. It is straightforward to prove that $X + Y$ is also compact. Since the measures do not change, we can translate X and Y so that $X \cap Y = \{o\}$, $X \subset \{x : x \leq 0\}$, and $Y \subset \{x : x \leq 0\}$. Then $+Y \supset X \cup Y$, so

$$V_1(X + Y) \geq V_1(X \cup Y) = V_1(X) + V_1(Y).$$

If we replace X by $(1-\lambda)X$ and Y by λY , we obtain (2) for compact X and Y. The general case follows easily by approximation from within by compact sets.

Simple though it is, Theorem (3.1.1) already raises two important matters.

Firstly, observe that it was enough to prove the theorem when the factors $(1-\lambda)$ and λ are omitted. This is due to the positive homogeneity (of degree 1) of Lebesgue measure in $\mathbb{R}$: $V_1(rX) = rV_1(X)$ for $r \geq 0$. In fact, this property allows these factors to be replaced by arbitrary nonnegative real numbers. For reasons that will become clear, it will be convenient for most to incorporate the factors $(1-\lambda)$ and λ.

Secondly, the set $(1-\lambda)X + \lambda Y$ may not be measurable, even when X and Y are measurable. We discuss this point in more detail.

The assumption in Theorem (3.1.1) and its n-dimensional forms, Theorem (3.1.4) and Corollary (3.1.6) below, that the sets are bounded is easily removed and is retained simply for convenience.

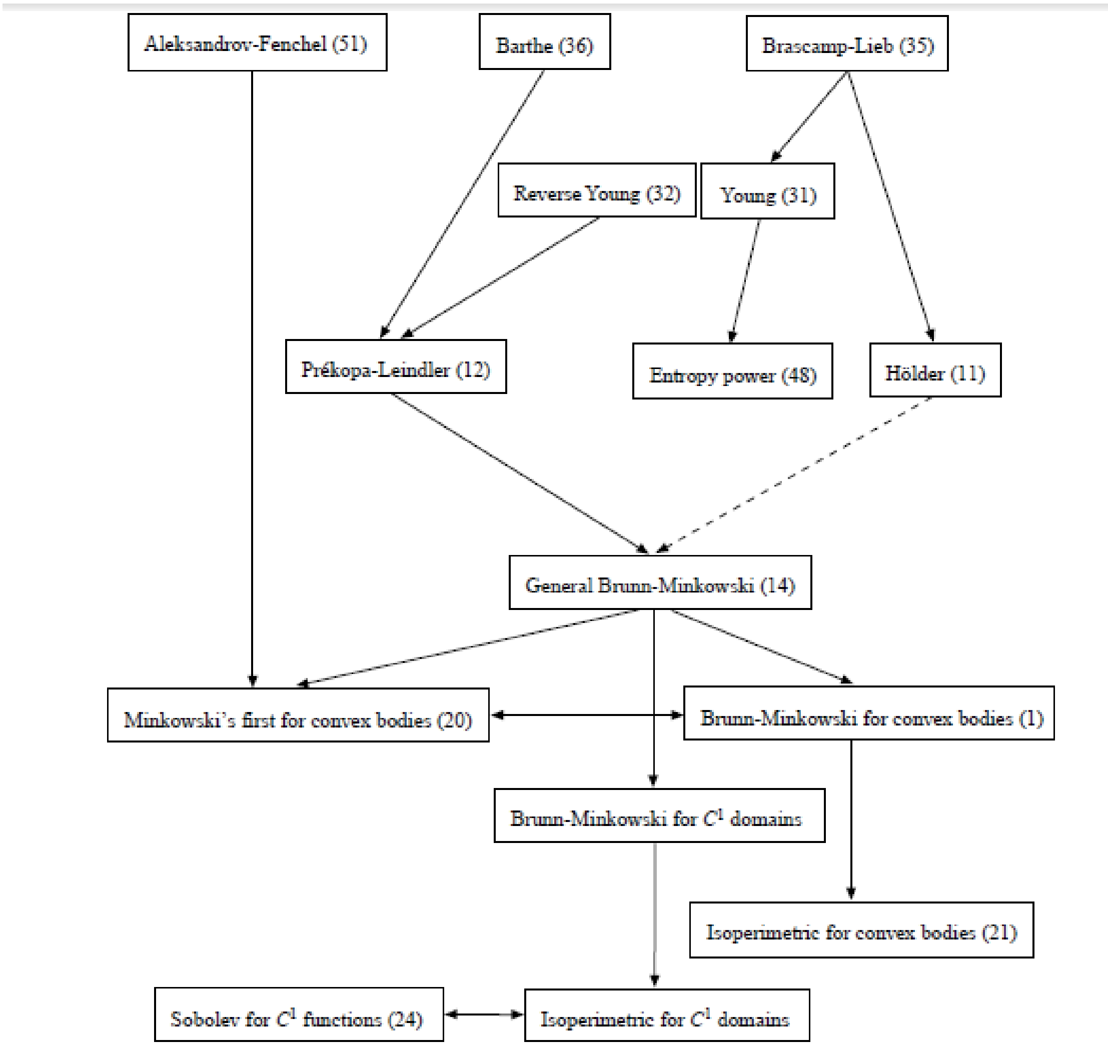

Figure 1[78]: Relations between inequalities.

We denote the origin, unit sphere, and closed unit ball in n-dimensional Euclidean space $\mathbb{R}^n$ by o, S^{n-1}, and B, respectively. The Euclidean scalar product of x and y will be written $x \cdot y$, and $\|x\|$ denotes the Euclidean norm of x. If $u \in S^{n-1}$, then $u^{\perp}$ is the hyperplane containing o and orthogonal to u.

Lebesgue k-dimensional measure V_k in $\mathbb{R}^n, k = 1, \ldots, n,$ can be identified with k-dimensional Hausdorff measure in $\mathbb{R}^n$. Then spherical Lebesgue measure in S^{n-1} can be identified with V_{n-1} in S^{n-1}. dx will denote integration with respect to V_k for the appropriate k and integration over S^{n-1} with respect to V_{n-1} will be denoted by du.

The term “measurable" applied to a set in $\mathbb{R}^n$ will mean V_n-measurable unless stated otherwise. If X is a compact set in $\mathbb{R}^n$ with nonempty interior, we often write $V(X) = V_n(X)$ for its volume. We shall do this in particular when X is a convex body, a compact convex set with nonempty interior. We also write $\kappa_n = V\,(B)$. In geometry, it is customary to use the term volume, more generally, to mean the k-dimensional Lebesgue measure of a k-dimensional compact body X (equal to the closure of its relative interior), i.e. to write $V(X) = V_k(X)$ in this case.

Let X and Y be sets in $\mathbb{R}^n$. We define their vector or Minkowski sum by

$$X + Y = \{x + y : x \in X, y \in Y\}.$$
If $r \in \mathbb{R}$, let
$$rX = \{rx : x \in X\}.$$
If $r > 0$, then rX is the dilatation of X with factor r, and if $r < 0$, it is the reflection of this dilatation in the origin. If $0 < \lambda < 1$, the set $(1-\lambda)X + \lambda Y$ is called a convex combination of X and Y.
Minkowski's definition of the surface area $S(M)$ of a suitable set M in $\mathbb{R}^n$ is
$$S(M) = \lim_{\varepsilon \to 0+} \frac{V_n(M+\varepsilon B) - V_n(M)}{\varepsilon}. \qquad (3)$$
we will use this definition when M is a convex body or a compact domain with piecewise C^1 boundary.
A function f on $\mathbb{R}^n$ is concave on a convex set C if
$$f\left((1-\lambda)x + \lambda y\right) \geq (1-\lambda)f(x) + \lambda f(y),$$
for all $x, y \in C$ and $0 < \lambda < 1$, and a function f is convex if $-f$ is concave. A nonnegative function f is log concave if $log\, f$ is concave. Since the latter condition is equivalent to
$$f\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda} f(y)^{\lambda}.$$
the arithmetic-geometric mean inequality implies that each concave function is log concave.
If f is a nonnegative measurable function on $\mathbb{R}^n$ and $t \geq 0$, the level set $L(f,t)$ is defined by
$$L(f,t) = \{x : f(x) \geq t\}. \qquad (4)$$
By Fubini's theorem,
$$\int_{\mathbb{R}^n} f(x)\, dx = \int_{\mathbb{R}^n} \int_0^{f(x)} 1\, dt\ dx = \int_0^{\infty} \int_{L(f,t)} 1 dx\, dt = \int_0^{\infty} V_n\left(L(f,t)\right) dt. \qquad (5)$$
If E is a set, 1_E denotes the characteristic function of E. The formula
$$f(x) = \int_0^{\infty} 1_{L(f,t)}(x) dt \qquad (6)$$
follows easily from $f(x) = \int_0^{f(x)} dt$. In [79, Theorem 1.13], equation (6) is called the layer cake representation of f.
Theorem (3.1.2) [78]: (Prèkopa -Leindler inequality in $\mathbb{R}$.) Let $0 < \lambda < 1$ and let f, g, and h be nonnegative integrable functions on $\mathbb{R}$ satisfying
$$h\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda} g(y)^{\lambda}, \qquad (7)$$
for all $x, y \in \mathbb{R}$. Then
$$\int_{\mathbb{R}} h(x)\, dx \geq \left(\int_{\mathbb{R}} f(x) dx\right)^{1-\lambda} \left(\int_{\mathbb{R}} g(x) dx\right)^{\lambda}.$$
Two proofs of this fundamental result will be presented after a comment about the strange-looking assumption (7) that ensures h is not too small. Fix a $z \in \mathbb{R}$ and choose $0 < \lambda < 1$ and any $x, y \in \mathbb{R}$ such that $z = (1-\lambda)x + \lambda y$. Then the value of h at z must be at least the weighted geometric mean (it is the geometric mean if $\lambda = 1/2$) of the values of f at x and g at y. Note also that the logarithm of (7) yields the equivalent condition
$$logh\left((1-\lambda)x + \lambda y\right) \geq (1-\lambda) log\, f(x) + \lambda\, log\, g(y).$$
If $f = g = h$, we would have

$$log\ f\left((1-\lambda)x+\lambda y\right) \geq (1-\lambda)log\ f(x)+\lambda\ log\ f(y),$$

which just says that f is log concave. Of course, the previous theorem does not say anything when $f=g=h$.

First proof: We can assume without loss of generality that f and g are bounded with

$$sup_{x\in\mathbb{R}}f(x) = sup_{x\in\mathbb{R}}g(x) = 1.$$

If $t \geq 0, f(x) \geq t$, and $g(y) \geq t$, then by (7), $h\ ((1-\lambda)x+\lambda y) \geq t$. With the notation (4) for level sets,

$$L(h,t) \supset (1-\lambda)L(f,t)+\lambda L(g,t),$$

for $0 \leq t < 1$. The sets on the right-hand side are nonempty, so by (5), the Brunn Minkowski inequality (2) in $\mathbb{R}$, and the arithmetic-geometric mean inequality, we obtain

$$\int_{\mathbb{R}} h(x)dx \geq \int_0^1 V_1(L(h,t))\,dt$$

$$\geq \int_0^1 V_1((1-\lambda)L(f,t)+\lambda L(g,t))\,dt$$

$$\geq (1-\lambda)\int_0^1 V_1\big(L(f,t)\big)dt + \lambda\int_0^\infty V_1(L(g,t))\,dt$$

$$= (1-\lambda)\int_{\mathbb{R}} f(x)\,dx + \lambda\int_{\mathbb{R}} g(x)\,dx$$

$$\geq \left(\int_{\mathbb{R}} f(x)\,dx\right)^{1-\lambda}\left(\int_{\mathbb{R}} g(x)\,dx\right)^{\lambda}.$$

Second proof. We can assume without loss of generality that

$$\int_{\mathbb{R}} f(x)\,dx = F > 0\ and\ \int_{\mathbb{R}} g(x)\,dx = G > 0.$$

Define $u, v: [0,1] \to \mathbb{R}$ such that $u(t)$ and $v(t)$ are the smallest numbers satisfying

$$\frac{1}{F}\int_{-\infty}^{u(t)} f(x)\,dx = \frac{1}{G}\int_{-\infty}^{v(t)} g(x)dx = t. \qquad (8)$$

Then u and v may be discontinuous, but they are strictly increasing functions and so are differentiable almost everywhere. Let

$$w(t) = (1-\lambda)u(t)+\lambda v(t).$$

Take the derivative of (8) with respect to t to obtain

$$\frac{f(u(t))u'(t)}{F} = \frac{g(v(t))\,v'(t)}{G} = 1.$$

Using this and the arithmetic-geometric mean inequality, we obtain (when $f\ (u(t)) \neq 0$ and $g(u(t)) \neq 0$)

$$w'(t) = (1-\lambda)u'(t)+\lambda v'(t)$$

$$\geq u'(t)^{1-\lambda}v'(t)$$

$$= \left(\frac{F}{f\ (u(t))}\right)^{1-\lambda}\left(\frac{G}{g(v(t))}\right)^{\lambda}.$$

Therefore

$$\int_{\mathbb{R}} h(x)dx \geq \int_0^1 h\,(w(t))w'(t)\,dt$$
$$\geq f(u(t))^{1-\lambda}\, g(v(t))^{\lambda} \left(\frac{F}{f(u(t))}\right)^{1-\lambda} \left(\frac{G}{g\,(v(t))}\right)^{\lambda} dt = F^{1-\lambda}G^{\lambda}.$$

There are two basic ingredients in the second proof of Theorem (3.1.2): the introduction in (8) of the volume parameter t, and use of the arithmetic-geometric mean inequality in estimating $w'(t)$.

The same ingredients appear in the first proof, though the parametrization is somewhat disguised in the use of the level sets.

Theorem (3.1.3) [78]: (Prèkopa -Leindler inequality in $\mathbb{R}^n$.) Let $0 < \lambda < 1$ and let $f, g,$ and h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda}, \qquad (9)$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda}.$$

Proof*:* The proof is by induction on n. It is true for $n = 1$, by Theorem (3.1.2). Suppose that it is true for all natural numbers less than n.

For each $s \in \mathbb{R}$, define a nonnegative function $h_s$on $\mathbb{R}^{n-1}$ by $h_s(z) = h(z, s)$ for $z \in \mathbb{R}^{n-1}$, and define f_s and g_s analogously. Let $x, y \in \mathbb{R}^{n-1}$, let $a, b \in \mathbb{R}$, and let $c = (1-\lambda)a + \lambda b$. Then

$$\begin{aligned} h_c\,((1-\lambda)x + \lambda y) &= h((1-\lambda)x + \lambda y, (1-\lambda)a + \lambda b) \\ &= h((1-\lambda)(x, a) + \lambda(y, b)) \\ &\quad f(x, a)^{1-\lambda} g(y, b)^{\lambda} \\ &= f_a(x)^{1-\lambda} g_b(y)^{\lambda}. \end{aligned}$$

By the inductive hypothesis,

$$\int_{\mathbb{R}^{n-1}} h_c(x)\,dx \;\geq \left(\int_{\mathbb{R}^{n-1}} f_a(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_b(x)\,dx\right)^{\lambda}.$$

Let

$$H(c) = \int_{\mathbb{R}^{n-1}} h_c(x)dx\,, F(a) = \int_{\mathbb{R}^{n-1}} f_a(x)dx\,, and\;\; G(b) = \int_{\mathbb{R}^{n-1}} g_b(x)dx\,.$$

Then

$$H(c) = H\,((1-\lambda)a + \lambda b) \geq F(a)^{1-\lambda} G(b)^{\lambda}$$

So, by Fubini's theorem and Theorem (3.1.2),

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} h_c(z)\,dz\;\,dc \\ &= \int_{\mathbb{R}} H(c)\,dc \\ &\geq \left(\int_{\mathbb{R}} F(a)\,da\right)^{1-\lambda} \left(\int_{\mathbb{R}} G(b)\,db\right)^{\lambda} \end{aligned}$$

$$= \left(\int_{\mathbb{R}^n} f(x)\,dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)\,dx\right)^{\lambda}.$$

Suppose that $f_i \in L^{p_i}(\mathbb{R}^n), p_i \geq 1, i = 1, \dots, m$ are nonnegative functions, where

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1. \qquad (10)$$

Holder's inequality in $\mathbb{R}^n$ states that

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(x)\,dx \leq \prod_{i=1}^m \|f_i\|_{p_i} = mYi = \prod_{i=1}^m \left(\int_{\mathbb{R}^n} f_i(x)^{p_i}\,dx\right)^{p_i}. \qquad (11)$$

Let $0 < \lambda < 1$. If $m = 2, 1/p_1 = 1 - \lambda, 1/p_2 = \lambda$, and we let $f = f_1^{p_1}$ and $g = f_2^{p_2}$, we get

$$\int_{\mathbb{R}^n} f(x)^{1-\lambda} g(x)^{\lambda}\,dx \leq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda},$$

The Prekopa-Leindler inequality in $\mathbb{R}^n$ can be written in the form

$$\overline{\int_{\mathbb{R}^n}} \sup\{f(x)^{1-\lambda} g(y)^{\lambda} : (1-\lambda)x + \lambda y = z\}\,dz \geq$$
$$\left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda} \qquad (12),$$

because we can use the supremum for h in (9). A straightforward generalization is

$$\overline{\int_{\mathbb{R}^n}} sup\left\{\prod_{i=1}^m f_i(x_i) : \sum_{i=1}^m \frac{x_i}{p_i} = z\right\} dz \geq \prod_{i=1}^m \|f_i\|_{p_i}, \qquad (13)$$

where $p_i \geq 1$ for each i and (10) holds. So we see that the Prèkopa -Leindler inequality is a reverse form of Hölder's inequality and that some condition such as (7) is therefore necessary for it to hold.

Notice that the upper Lebesgue integral is used on the left in (12) and (13). This is because the integrands there are generally not measurable. We shall return to this point the Brunn-Minkowski inequality is derived from the Prekopa-Leindler inequality.

A different and self-contained short proof can be found.

Theorem (3.1.4) [78]: (General Brunn-Minkowski inequality in $\mathbb{R}^n$, first form.) Let $0 < \lambda < 1$ and let X and Y be bounded measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)X + \lambda Y$ is also measurable. Then

$$V_n((1-\lambda)X + \lambda Y) \geq V_n(X)^{1-\lambda} V_n(Y)^{\lambda}. \qquad (14)$$

Theorem (3.1.5) [78]: The Prèkopa -Leindler inequality in $\mathbb{R}^n$ implies the general Brunn-Minkowski inequality in $\mathbb{R}^n$.

Proof: Let $h = 1_{(1-\lambda)X+\lambda Y}, f = 1_X$, and $g = 1_Y$. If $x, y \in \mathbb{R}^n$, then $f(x)^{1-\lambda} g(y)^{\lambda} > 0$ (and in fact equals 1) if and only if $x \in X$ and $y \in Y$. The latter implies $(1-\lambda)x + \lambda y \in (1-\lambda)X + \lambda Y$, which is true if and only if $h((1-\lambda)x + \lambda y) = 1$. Therefore (9) holds. We conclude by Theorem (3.1.3) that

$$V_n((1-\lambda)X + \lambda Y) = 1_{(1-\lambda)X+\lambda Y}(x)dx$$
$$\geq \left(\int_{\mathbb{R}^n} 1_X(x)\,dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} 1_Y(x)\,dx\right)^{\lambda} = V_n(X)^{1-\lambda} V_n(Y)^{\lambda}.$$

Corollary (3.1.6) [78]: (General Brunn-Minkowski inequality in $\mathbb{R}^n$, standard form.) Let $0 < \lambda < 1$ and let X and Y be nonempty bounded measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)X + \lambda Y$ is also measurable. Then

$$V_n((1-\lambda)X + \lambda Y)^{1/n} \geq (1-\lambda)V_n(X)^{1/n} + \lambda V_n(Y)^{1/n}. \qquad (15)$$

Proof: Let

$$\lambda' = \frac{V_n(Y)^{1/n}}{V_n(X)^{1/n} + V_n(Y)^{1/n}}$$

and let $X' = V_n(X)^{-1/n}X$ and $Y' = V_n(Y)^{-1/n}Y$. Then

$$1 - \lambda' = \frac{V_n(X)^{1/n}}{V_n(X)^{1/n} + V_n(Y)^{1/n}}$$

and $V_n(X') = V_n(Y') = 1$, by the positive homogeneity (of degree n) of Lebesgue measure in $\mathbb{R}^n$ ($V_n(rA) = r^n V_n(A)$ for $r \geq 0$). Therefore (14), applied to $X, Y',$ and λ', yields

$$V_n\big((1-\lambda)X' + \lambda' Y'\big) \geq 1.$$

But

$$V_n((1-\lambda)X' + \lambda' Y') = V_n\left(\frac{X+Y}{V_n(X)^{1/n} + V_n(Y)^{1/n}}\right) = \frac{V_n(X+Y\ )}{(V_n(X)^{1/n} + V_n(Y)^{1/n})^n}.$$

This gives

$$V_n(X+Y)^{1/n} \geq V_n(X)^{1/n} + V_n(Y)^{1/n}.$$

To obtain (15), just replace X and Y by $(1-\lambda)X$ and λY, respectively.

Remark (3.1.7) [78]: Using the homogeneity of volume, it follows that for all $s, t > 0$,

$$V_n(sX + tY\ )^{1/n} \geq sV_n(X)^{1/n} + tV_n(Y)^{1/n}. \quad (16)$$

Note the advantages of the first form (14) of the general Brunn-Minkowski inequality. One need not assume that X and Y are nonempty, and the inequality is independent of the dimension n. The two forms are equivalent, however; to get from the standard to the first form, just use Jensen's inequality for means (see (28) below with $p = 0$ and $q = 1/n$).

For detailed remarks and references concerning the early history of the Brunn-Minkowski inequality for convex bodies, see [80, p. 314]. Briefly, the inequality for convex bodies in $\mathbb{R}^n$ was discovered by Brunn around 1887. Minkowski pointed out an error in the proof, which Brunn corrected, and found a different proof himself. Both Brunn and Minkowski showed that equality holds if and only if K and L are homothetic (i.e., K and L are equal up to translation and dilatation). The proof presented in [80, Section 6.1], due to Kneser and Suss in 1932, is very similar to the proof we gave above of the Prèkopa -Leindler inequality, restricted to characteristic functions of convex bodies; note that the case $n = 1$ is trivial, and the equality condition vacuous, in this case. This is perhaps the simplest approach for the equality conditions for convex bodies.

Another quite different proof, due to Blaschke in 1917, is worth mentioning. This uses Steiner symmetrization. Let K be a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. The Steiner symmetral S_uK of K in the direction u is the convex body obtained from K by sliding each of its chords parallel to u so that they are bisected by the hyperplane $u^{\perp}$, and taking the union of the resulting chords. Then $V(S_uK) = V(K)$ by Cavalieri's principle, and it is not hard to show that if K and L are convex bodies in $\mathbb{R}^n$, then

$$S_u(K+L) \supset S_uK + S_uL. \quad (17)$$

One can also prove that there is a sequence of directions $u_m \in S^{n-1}$ such that if K is any convex body and $K_m = S_{u_m} K_{m-1}$, then $K_m \to r_K B$ as $m \to \infty$, where r_K is the constant such that $V(K) = V(r_K B)$. Repeated application of (17) now gives

$$V(K+L)^{1/n} \geq V(r_k B + r_L B)^{1/n} = (r_R + r_L) V(B)^{1/n}$$
$$= V(r_K B)^{1/n} + V(r_L B)^{1/n} = V(K)^{1/n} + V(L)^{1/n}.$$

See [81, Chapter 5, Section 5] or [150, pp. 310{314].

The general Brunn-Minkowski inequality and its equality conditions were first proved by Lusternik [82]. The equality conditions he gave were corrected by Henstock and Macbeath [79], who basically used the method in the second proof of Theorem (3.1.2) to derive the inequality. Another method, found by Hadwiger and Ohmann [79], is so beautiful that we cannot resist reproducing it in full (see also [95, Section 8], [93, Section 6.6], [58, Theorem 3.2.41], or [96, Section 6.5]).

The idea is to prove the result first for boxes, rectangular parallelepipeds whose sides are parallel to the coordinate hyperplanes. If X and Y are boxes with sides of length x_i and y_i, respectively, in the ith coordinate directions, then

$$V(X) = \prod_{i=1}^{n} x_i \,, V(Y) = \prod_{i=1}^{n} y_i \,, and\ V(X+Y) = \prod_{i=1}^{n} (x_i + y_i)\,.$$

Now

$$\left(\frac{x_i}{x_i + y_i}\right)^{1/n} + \left(\frac{y_i}{x_i + y_i}\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_i + y_i} + \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i + y_i} = 1,$$

by the arithmetic-geometric mean inequality. This gives the Brunn-Minkowski inequality for boxes. One then uses a trick sometimes called a Hadwiger-Ohmann cut to obtain the inequality for finite unions X and Y of boxes, as follows. By translating X, if necessary, we can assume that a coordinate hyperplane, $\{x_n = 0\}$ say, separates two boxes in X. Let X_+ (or X_-) denote the union of the boxes formed by intersecting the boxes in X with $\{x_n \geq 0\}$ (or $\{x_n \leq 0\}$, respectively). Now translate Y so that

$$\frac{V(X_\pm)}{V(X)} = \frac{V(Y_\pm)}{V(Y)}\ , \qquad (18)$$

where Y_+ and Y_- are defined analogously to X_+ and X_-. Note that $X_+ + Y_+ + \subset \{x_n \geq 0\}, X_- + Y_- \subset \{x_n \leq 0\}$, and that the numbers of boxes in $X_+ \cup Y_+$ and $X_- \cup Y_-$ are both smaller than the number of boxes in $X \cup Y$. By induction on the latter number and (18), we have

$$V(X+Y) \geq V(X_+ + Y_+) + V(X_- + Y_-)$$
$$\geq \left(V(X_+)^{1/n} + V(Y_+)^{1/n}\right)^n + \left(V(X_-)^{1/n} + V(Y_-)^{1/n}\right)^n$$
$$= V(X_+)\left(1 + \frac{V(Y)^{1/n}}{V(X)^{1/n}}\right)^n + V(X_-)\left(1 + \frac{V(Y)^{1/n}}{V(X)^{1/n}}\right)^n$$
$$= V(X)\left(1 + \frac{V(Y)^{1/n}}{V(X)^{1/n}}\right)^n = \left(V(X)^{1/n} + V(Y)^{1/n}\right)^n.$$

Now that the inequality is established for finite unions of boxes, the proof is completed by using them to approximate bounded measurable sets. A careful examination of this proof allows one to conclude that if $V_n(X)V_n(Y) > 0$, equality holds only when

$$V_n((conv\,X)\backslash X) = V_n((conv\,Y\,)\backslash Y) = 0,$$

where $convX$ denotes the convex hull of X. Putting the equality conditions above together, we see that if $V_n(X)V_n(Y) > 0$, equality holds in the general Brunn-Minkowski inequality if and only if X and Y are homothetic convex bodies from which sets of measure zero have been removed. See [37, Section 8] and [150, Section 6.5] for more details and further comments about the case when X or Y has measure zero.

Since Holder's inequality (11) in its discrete form implies the arithmetic geometric mean inequality, there is a sense in which Hölder's inequality implies the Brunn-Minkowski inequality.

by

$$nV_1(K,L) = \lim_{\varepsilon\to 0_+} \frac{V(K+\varepsilon L)-V(K)}{\varepsilon}. \qquad (19)$$

Note that if $L = B$, then $S(K) = nV_1(K,B)$; it is this relationship that will quickly lead us to the isoperimetric inequality and its equality condition. An even shorter path (see [80, Theorem B.2.1]) yields the inequality but without the equality condition.

The quantity $V_1(K,L)$ is a special mixed volume, and its existence requires just a little of the theory of mixed volumes to establish; see [80, Section 6.4]. In fact, Minkowski showed that if $K_1, \dots, K_m$ are compact convex sets in $\mathbb{R}^n$, and $t_1, \dots, t_m \geq 0$, the volume $V(\sum\{t_iK_i : i = 1, \dots, m\})$ is a polynomial of degree n in the variables $t_1, \dots, t_m$. The coefficient $V\left(K_{j_1}, \dots, K_{j_n}\right)$ of $t_{j_1} \cdots t_{j_n}$ in this polynomial is called a mixed volume. Then $V_1(K,L) = V(K, n-1, L)$, where the notation means that K appears $(n-1)$ times and L appears once. See [81, Appendix A] for a gentle introduction to mixed volumes.

Theorem (3.1.8) [78]: (Minkowski's first inequality for convex bodies in $\mathbb{R}^n$.) Let K and L be convex bodies in $\mathbb{R}^n$. Then

$$V_1(K,L) \geq V(K)^{\frac{(n-1)}{n}} V(L)^{1/n}, \qquad (20)$$

with equality if and only if K and L are homothetic.

Minkowski's first inequality plays a role in the solution of Shephard's problem: If the projection of a centrally symmetric (i.e., $-K$ is a translate of K) convex body onto any given hyperplane is always smaller in volume than that of another such body, is its volume also smaller? The answer is no in general in three or more dimensions; see [87, Chapter 4] and [99, p. 255].

Theorem (3.1.9) [78]: The Brunn-Minkowski inequality for convex bodies in $\mathbb{R}^n$ (and its equality condition) implies Minkowski's first inequality for convex bodies in $\mathbb{R}^n$ (and its equality condition).

Proof: Substituting $\varepsilon = t/(1-t)$ in (19) and using the homogeneity of volume, we obtain

$$nV_1(K,L) = \lim_{\varepsilon\to 0_+} \frac{V(K+\varepsilon L)-V(K)}{\varepsilon}$$

$$= lim_{t\to 0_+} \frac{V\big((1-t)K+tL\big)-(1-t)^n V(K)}{t(1-t)^{n-1}}$$
$$= lim_{t\to 0_+} \frac{V\big((1-t)K+tL\big)-V(K)}{t} + lim_{t\to 0_+} \frac{(1-(1-t)^n))V(K)}{t}$$
$$= lim_{t\to 0_+} \frac{V\big((1-t)K+tL\big)-V(K)}{t} + nV(K).$$
Using this new expression for $V_1(K,L)$ (see [107, p. 7]) and letting $f(t) = V((1-t)K+tL)^{1/n}$, for $0 \le t \le 1$, we see that
$$f'(0) = \frac{V_1(K,L)-V(K)}{V(K)^{(n-1)/n}}.$$
Therefore (20) is equivalent to $f'(0) \ge f(1)-f(0)$. Since the Brunn-Minkowski inequality says that *f* is concave, Minkowski's first inequality follows.
Suppose that equality holds in (20). Then $f'(0) = f(1)-f(0)$. Since f is concave, we have
$$\frac{f(t)-\ f(0)}{t} = f(1)-f(0)$$
for $0 < t\ \le 1$, and this is just equality in the Brunn-Minkowski inequality. The equality condition for (20) follows immediately.
The following corollary is obtained by taking $L = B$ in Theorem (3.1.8).
Corollary (3.1.10) [78]: (Isoperimetric inequality for convex bodies in $\mathbb{R}^n$.) Let K be a convex body in $\mathbb{R}^n$. Then
$$\left(\frac{V(K)}{V(B)}\right)^{1/n} \le \left(\frac{S(K)}{S(B)}\right)^{1/(n-1)}, \qquad (21)$$
with equality if and only if K is a ball.
It can be shown (see [85]) that if M is a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary and L is a convex body in $\mathbb{R}^n$, the quantity $V_1(M,L)$ defined by (19) with K replaced by M exists.
From the Brunn-Minkowski inequality for compact domains in $\mathbb{R}^n$ with piecewise C^1 boundary and the above argument, one obtains Minkowski's first inequality when the convex body K is replaced by such a domain. Taking $L = B$, this immediately gives the isoperimetric inequality for compact domains in $\mathbb{R}^n$ with piecewise C^1 boundary.
Essentially the most general class of sets for which the isoperimetric inequality in $\mathbb{R}^n$ is known to hold comprises the sets of finite perimeter; see, for example, the book of Evans and Gariepy [87, p. 190], where the rather technical setting, sometimes called the BV theory, is expounded. It is still possible to base the proof on the Brunn-Minkowski
$$V_1(M,L) = \frac{1}{n}\int_{\partial M} h_L(u_X)dx, \qquad (22)$$
where h_L is the support function of L and u_x is the outer unit normal vector to ∂M at x. (If we replace h_L by an arbitrary function f on S^{n-1}, then up to a constant, this integral represents the surface energy of a crystal with shape M, where f is the surface tensionWhen $M = K$ is a sufficiently smooth convex body, (22) can be written
$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u)f_K(u)du, \qquad (23)$$

where f_K is the reciprocal of the Gauss curvature of K at the point on ∂K where the outer unit normal is u; for general convex bodies, $f_K(u)du$ must be replaced by $dS(K,u)$, where $S(K,\cdot)$ is the surface area measure of K. Minkowski's existence theorem gives necessary and sufficient conditions for a measure μ in S^{n-1} to be the surface area measure of some convex body. Now (20) and (23) imply that if $S(K,\cdot) = \mu$, then K minimizes the functional

$$L \to \int_{S^{n-1}} h_L(u)d\mu$$

under the condition that $V(L) = 1$, and this fact motivates the proof of Minkowski's existence theorem. See [96, Section 7.1], where pointers can also be found to the vast literature surrounding the so-called Minkowski problem, which deals with existence, uniqueness, regularity, and stability of a closed convex hypersurface whose Gauss curvature is prescribed as a function of its outer normals.

Theorem (3.1.11) [78]: (Sobolev inequality.) Let f be a C^1 function on $\mathbb{R}^n$ with compact support. Then

$$\int_{\mathbb{R}^n} \|\nabla f(x)\| dx \geq n\kappa_n^{1/n} \|f\|_{n/(n-1)}. \tag{24}$$

The previous inequality is only one of a family, all called Sobolev inequalities. See [91, Chapter 8], where it is pointed out that such inequalities bound averages of gradients from below by weighted averages of the function, and can thus be considered as uncertainty principles.

Theorem (3.1.12) [78]: The Sobolev inequality is equivalent to the isoperimetric inequality for compact domains with C^1 boundaries.

Proof: Suppose that the isoperimetric inequality holds, and let f be a C^1 function on $\mathbb{R}^n$ with compact support. The coarea formula (a sort of curvilinear Fubini theorem; see [85, p. 112]) implies that

$$\int_{\mathbb{R}^n} \|\nabla f(x)\| dx = \int_{\mathbb{R}} V_{n-1}(f^{-1}\{t\})\, dt$$
$$= \int_0^\infty S\big(L(|f|,t)\big)dt\,,$$

where $L(|f|,t)$ is a level set of $|f|$, as in (4). Applying the the isoperimetric inequality for compact domains with C^1 boundaries to these level sets, we obtain

$$\int_{\mathbb{R}^n} \|\nabla f(x)\| dx \geq n\kappa_n^{1/n} \int_0^\infty V\big(L(|f|,t)\big)^{(n-1)/n} dt\,.$$

On the other hand, by (6) and Minkowski's inequality for integrals (see [77, (6.13.9), p. 148]), we have

$$\left(\int_{\mathbb{R}^n} |f(x)|^{n/(n-1)} dx\right)^{(n-1)/n} = \left(\int_{\mathbb{R}^n} \left(\int_0^\infty 1_{L(|f|,t)}(x)dt\right)^{n/(n-1)} dx\right)^{(n-1)/n}$$
$$\leq \int_0^\infty \left(\int_{\mathbb{R}^n} 1_{L(|f|,t)}(x)^{n/(n-1)} dx\right)^{(n-1)/n} dt$$
$$= V(L(|f|,t))^{(n-1)/n}\, dt.$$

Therefore (24) is true.
Suppose that (24) holds, let M be a compact domain in $\mathbb{R}^n$ with C^1 boundary ∂M, and let $\varepsilon > 0$. Define $f_\varepsilon(x) = 1$ if $x \in M, f_\varepsilon(x) = 0$ if $x \notin M + \varepsilon B$, and $f_\varepsilon(x) = 1 - d(x, M)/\varepsilon$ if $x \in (M + \varepsilon B)\backslash M$, where $d(x, M)$ is the distance from x to M. Since f_ε can be approximated by C^1 functions on $\mathbb{R}^n$ with compact support, we can assume that (24) holds for f_ε. Note that $f_\varepsilon \to 1_M$ as $\varepsilon \to 0$. Also, $\|\nabla f_\varepsilon(x)\| = 1/\varepsilon \; if \; x \in (M + \varepsilon B)\backslash M$ and is zero otherwise. Therefore, by (3),

$$S(M) = lim_{\varepsilon \to 0_+} \frac{V(M + \varepsilon B) - V(M)}{\varepsilon} = lim_{\varepsilon \to 0_+} \|\nabla f_\varepsilon(x)\| dx$$
$$\geq \lim_{\varepsilon \to 0_+} n\kappa_n^{1/n} \left(\int_{\mathbb{R}^n} |f_\varepsilon(x)|^{n/(n-1)} dx \right)^{(n-1)/n}$$
$$= n\kappa_n^{1/n} \left(\int_{\mathbb{R}^n} 1_M(x) dx \right)^{(n-1)/n}$$
$$= n\kappa_n^{1/n} \, V(M)^{(n-1)/n},$$

which is just a reorganization of the isoperimetric inequality (21) .
As for the isoperimetric inequality, there is a more general version of the Sobolev inequality in the BV theory. This is called the Gagliardo-Nirenberg-Sobolev inequality and it is equivalent to the isoperimetric inequality for sets of finite perimeter; see [87, pp. 138 and 192].
If X and Y are Borel sets, then $(1 - \lambda)X + \lambda Y$, being a continuous image of their product, is analytic and hence measurable. (Erdos and Stone [89] proved that this set need not itself be Borel.) However, an old example of Sierpinski [96] shows that the set $(1 - \lambda)X + \lambda Y$ may not be measurable when X and Y are measurable.
There are a couple of ways around the measurability problem. One can simply replace the measure on the left of the Brunn-Minkowski inequality by inner Lebesgue measure V_{n*}, the supremum of the measures of compact subsets, thus:

$$V_{n*}((1 - \lambda)X + \lambda Y)^{1/n} \geq (1 - \lambda)V_n(X)^{1/n} + \lambda V_n(Y)^{1/n}.$$

A better solution is to obtain a slightly improved version of the Prȅkopa -Leindler inequality, and then deduce a corresponding improved Brunn-Minkowski inequality, as follows.
Recall that the essential supremum of a measurable function f on $\mathbb{R}^n$ is defined by

$$ess\ sup_{x \in \mathbb{R}^n} f(x) = \inf\{t: f(x) \leq t \ for\ almost\ all\ x \in \mathbb{R}^n\}.$$

Brascamp and Lieb [95] proved the following result. (According to Uhrin [146], the idea of using the essential supremum in connection with our topic occurred independently to S. Dancs.)
Theorem (3.1.13) [78]: (Prekopa-Leindler inequality in $\mathbb{R}^n$, essential form.) Let $0 < \lambda < 1$ and let $f, g \in L^1(\mathbb{R}^n)$ be nonnegative. Let

$$s(x) = ess\ sup_y f\left(\frac{x-y}{1-\lambda}\right)^{1-\lambda} g\left(\frac{y}{\lambda}\right)^{\lambda}. \qquad (25)$$

Then s is measurable and

$$\|s\|_1 \geq \|f\|_1^{1-\lambda} \|g\|_1^{\lambda}.$$

Proof*:* First note that s is measurable. Indeed,

$$s(x) = sup_{\phi\in D}\int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)^{1-\lambda} g\left(\frac{y}{\lambda}\right)^{\lambda}\phi(y)dy\,,$$

where D is a countable dense subset of the unit ball of $L^1(\mathbb{R}^n)$. Therefore s is the supremum of a countable family of measurable functions.

With the measurability of s in hand, the proof follows that of the usual Prèkopa -Leindler inequality presented.

The essential form of the Prèkopa -Leindler inequality in $\mathbb{R}^n$ implies the usual form, Theorem (3.1.3).

To see this, replace x by z and y by $\lambda y'$ in (25) and then let $x = (z-\lambda y')/(1-\lambda)$ to obtain

$$s(z) = ess\ sup_{y'} f\left(\frac{z-\lambda y'}{1-\lambda}\right)^{1-\lambda} g(y')^{\lambda}$$
$$= ess\sup\{f(x)^{1-\lambda}g(y)^{\lambda}: z=(1-\lambda)x+\lambda y\}.$$

Now if h is any integrable function satisfying

$$h\big((1-\lambda)x+\lambda y\big) \geq f(x)^{1-\lambda}g(y)^{\lambda},$$

we must have $h \geq s$ almost everywhere. It follows from Theorem (3.1.13) that

$$\|h\|_1 \geq \|s\|_1 \ \geq \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}.$$

The corresponding improvement of the Brunn-Minkowski inequality requires one new concept. Note that the usual Minkowski sum of X and Y can be written

$$X+Y = \{z: X\cap(z-Y\,)\} \neq \emptyset.$$

Adjust this by defining the essential sum of X and Y by

$$X +_e Y = \{z: V_n\big(X\cap(z-Y\,)\big) > 0\}.$$

While

$$1_{X+Y}\,(z) = sup_{x\in\mathbb{R}^n} 1_X(x)1_Y\,(z-x),$$

it is easy to see that

$$1_X +_e Y(z) = ess\ sup_{x\in\mathbb{R}^n} 1_X(x)1_Y(z-x). \qquad (26)$$

Theorem (3.1.14) [78]: (General Brunn-Minkowski inequality in $\mathbb{R}^n$, essential form.) Let $0<\lambda<1$ and let X and Y be nonempty bounded measurable sets in $\mathbb{R}^n$. Then

$$V_n((1-\lambda)X +_e \lambda Y)^{1/n} \geq (1-\lambda)V_n(X)^{1/n} + \lambda V_n(Y)^{1/n}. \qquad (27)$$

Proof*:* In Theorem (3.1.13), let $f = 1_{(1-\lambda)X}$ and $g = 1_{\lambda Y}$. Then, by (26),

$$1_{(1-\lambda)X+_e\lambda Y}(z) = ess\ sup_{x\in\mathbb{R}^n} 1_{(1-\lambda)X}(x)1_{\lambda Y}(z-x)$$
$$= ess\ sup_{x\in\mathbb{R}^n} 1_X\left(\frac{x}{1-\lambda}\right)1_Y\left(\frac{z-x}{\lambda}\right)$$
$$= ess\ sup_{y\in\mathbb{R}^n} 1_X\left(\frac{z-y}{1-\lambda}\right)1_Y\ \left(\frac{y}{\lambda}\right) = s(z).$$

The inequality

$$V_n\big((1-\lambda)X +_e \lambda Y\,\big) \geq V_n(X)^{1-\lambda}V_n(Y)^{\lambda},$$

and hence (27), now follow exactly.

A direct proof of the previous theorem is given in [95, Appendix]. Here is a sketch. One first shows that $X +_e Y$ is measurable (indeed, open). This is proved using the set A^* of density points of a measurable set A, that is,

$$A^* = \left\{x \in \mathbb{R}^n: lim_{\varepsilon\to 0_+} \frac{V_n(A \cap B(x,\varepsilon))}{V_n(B(x,\varepsilon))} = 1\right\},$$

where $B(x,\varepsilon)$ is a ball with center at x and radius ε. Then $V_n(A\Delta A^*) = 0$, where Δ denotes symmetric difference, and this implies that

$$X +_e Y = X^* +_e Y^*.$$

Now it can be shown that $X^* +_e Y^*$ is open and

$$X^* +_e Y^* = X^* + Y^*.$$

The Brunn-Minkowski inequality (15) in $\mathbb{R}^n$ then implies (27).

If f is a nonnegative integrable function defined on a measurable subset A of $\mathbb{R}^n$, and μ is defined by

$$\mu(X) = \int_{A\cap X} f(x)dx\,,$$

for all measurable subsets X of $\mathbb{R}^n$, we say that μ is generated by f and A.

The Prekopa-Leindler inequality implies that if f is log concave and C is an open convex subset of its support, then the measure μ generated by f and C is also log concave. Indeed, if $0 < \lambda < 1$, X and Y are measurable sets, and $z = (1-\lambda)x + \lambda y$, then the log concavity of f implies

$$f(z)1_{C\cap((1-\lambda)X+\lambda Y)}(z) \geq \left(f(x)1_{C\cap X}(x)\right)^{1-\lambda}\left(f(y)1_{C\cap Y}(y)\right)^{\lambda},$$

so we can apply Theorem (3.1.3) to obtain

$$\begin{aligned}\mu((1-\lambda)X + \lambda Y\,) &= \int_{C\cap((1-\lambda)X+\lambda Y)} f(z)dz \\ &= \int_{\mathbb{R}^n} f(z)1_{C\cap((1-\lambda)X+\lambda Y)}(z)\,dz \\ &\geq \left(\int_{\mathbb{R}^n} f(x)1_{C\cap X}(x)dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} f(x)1_{C\cap Y}(x)dx\right)^{\lambda} \\ &= \left(\int_{C\cap X} f(x)dx\right)^{1-\lambda}\left(\int_{C\cap Y} f(x)dx\right)^{\lambda} \\ &= \mu(X)^{1-\lambda}\mu(Y)^{\lambda}.\end{aligned}$$

This observation has been generalized considerably, as follows. If $0 < \lambda < 1$ and $p \neq 0$, we define

$$M_p(a,b,\lambda) = ((1-\lambda)a^p + \lambda b^p)^{1/p}$$

if $ab \neq 0$ and $M_p(a,b,\lambda) = 0$ if $ab = 0$; we also define

$$M_0(a,b,\lambda) = a^{1-\lambda}b^{\lambda},$$

$M_{-\infty}(a,b,\lambda) = min\{a,b\}$, and $M_{\infty}(a,b,\lambda) = max\{a,b\}$. These quantities and their natural generalizations for more than two numbers are called pth means. The classic text of Hardy, Littlewood, and Polya [97] is still the best general reference. (Note, however, the different convention here when $p > 0$ and $ab = 0$.) Jensen's inequality for means (see [97, Section 2.9]) implies that if $-\infty \le p < q \le \infty$, then

$$M_p(a,b,\lambda) \le M_q(a,b,\lambda), \tag{28}$$

with equality if and only if $a = b$ or $ab = 0$.

A nonnegative function f on $\mathbb{R}^n$ is called p-concave on a convex set C if

$$f\big((1-\lambda)x + \lambda y\big) \ge M_p(f(x), f(y), \lambda),$$

for all $x, y \in C$ and $0 < \lambda < 1$. Analogously, we say that a finite (nonnegative) measure μ defined on (Lebesgue) measurable subsets of $\mathbb{R}^n$ is p-concave if

$$\mu\big((1-\lambda)X + \lambda Y\big) \ge M_p(\mu(X), \mu(Y), \lambda),$$

for all measurable sets X and Y in $\mathbb{R}^n$ and $0 < \lambda < 1$.

Thus 1-concave is just concave in the usual sense and 0-concave is log concave. The term quasiconcave is sometimes used for $-\infty$-concave. Also, if $p > 0$ (or $p < 0$), then f is p-concave if and only if f^p is concave (or convex, respectively). It follows from Jensen's inequality (28) that a p-concave function or measure is q-concave for all $q \le p$.

Probability density functions of some important probability distributions are p-concave for some p. Consider, for example, the multivariate normal distribution on $\mathbb{R}^n$ with mean $m \in \mathbb{R}^n$ and $n \times n$ positive definite symmetric covariance matrix A. This has probability density

$$f(x) = c\, exp\left(-\frac{(x-m)\cdot A^{-1}(x-m)}{2}\right),$$

where $c = (2\pi)^{-n/2}(detA)^{-1/2}$. Since A is positive definite, the function $(x-m)\cdot A^{-1}(x-m)$ is convex and so f is log concave. The probability density functions of the Wishart, multivariate β, and Dirichlet distributions are also log concave; see [82]. The argument above then shows that the corresponding probability measures are log concave. Prekopa [183] explains how a problem from stochastic programming motivates this result.

However, Borell [88] noted that the density functions of the multivariate Pareto (the Cauchy distribution is a special case), t, and F distributions are not log concave, but are p-concave for some $p < 0$. To obtain similar concavity conditions for the corresponding probability measures, a technical lemma is required.

Lemma (3.1.15) [78]: Let $0 < \lambda < 1$ and let $a, b, c,$ and d be nonnegative real numbers. If $p + q \ge 0$, then

$$M_p(a,b,\lambda)M_q(c,d,\lambda) \ge M_s(ac,bd,\lambda),$$

where $s = pq/(p+q)$ if p and q are not both zero, and $s = 0$ if $p = q = 0$.

Proof: A general form of Holder's inequality (see [97, p. 24]) states that when $0 < \lambda < 1$, $p_1, p_2, r > 0$ with $1/p_1 + 1/p_2 = 1$, and $a, b, c,$ and d are nonnegative real numbers, then

$$M_r(ac,bd,\lambda) \le M_{rp1}(a,b,\lambda)M_{rp2}(c,d,\lambda),$$

and that the inequality reverses when $r < 0$. Suppose that $p + q > 0$. If $p, q > 0$, we can let $r = s$, $p_1 = p/s$, and $p_2 = q/s$, and the desired inequality follows immediately. If $p < 0$, then $q > 0$ and we let $r = p, p_1 = s/p$, and $p_2 = -q/p$; then replace $a, b, c,$ and d, by $ac, bd, 1/c,$ and $1/d$, respectively. The remaining cases follow by continuity.

The following theorem generalizes the Prekopa-Leindler inequality in $\mathbb{R}^n$, which is just the case $p = 0$. The number $p/(np+1)$ is interpreted in the obvious way; it is equal to $-\infty$ when $p = -1/n$ and to $1/n$ when $p = \infty$.

Theorem (3.1.16) [78]: (Borell-Brascamp-Lieb inequality.) Let $0 < \lambda < 1$, let $-1/n \leq p \leq \infty$, and let f, g, and h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h\big((1-\lambda)x + \lambda y\big) \geq M_p(f(x), g(y), \lambda),$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)dx \geq M_{p/(np+1)}\left(\int_{\mathbb{R}^n} f(x)dx\,, \int_{\mathbb{R}^n} g(x)dx\,, \lambda\right).$$

Proof: This is very similar to the proof of the Prekopa-Leindler inequality. To deal with the case $n = 1$, follow the second proof of Theorem (3.1.2), defining F, G, u, v, and w as in that theorem.

Then, by Lemma (3.1.15) with $q = 1$,

$$\int_{\mathbb{R}} h(x)dx \geq \int_0^1 h(w(t))w'(t)\, dt$$

$$\geq \int_0^1 M_p\big(f\big(u(t)\big), g\big(v(t)\big), \lambda\big) M_1\left(\frac{F}{f\big(u(t)\big)}\,, \frac{G}{g\big(v(t)\big)}, \lambda\right) dt$$

$$M_{p/(p+1)}(F, G, \lambda)dt = M_{p/(p+1)}(F, G, \lambda).$$

The general case follows as in Theorem (3.1.3) by induction on n.

Theorem (3.1.16) was proved (in slightly modified form) for $p > 0$ by Henstock and In calling Theorem (3.1.16) the Borell-Brascamp-Lieb inequality we are following the authors of [92] (who also generalize it to a Riemannian manifold setting; see Section 19.13) and placing the emphasis on the negative values of p. In fact, the proof of [92, Corollary 1.1] shows that the strongest inequality in this family is that for $= -1/n$; that is, Theorem (3.1.16) for $p = -1/n$ implies

Theorem (3.1.16) for all $p > -1/n$. This follows from a suitable rescaling of the functions f, g, and h, Lemma (3.1.15) with $q = -p/(np+1)$, and the observation that $M_p(a, b, \lambda)^{-1} = M_{-p}(1/a, 1/b, \lambda)$.

The approach of Brascamp and Lieb [95], incidentally, was to observe that Theorem (3.1.16) also holds for $n = 1$ and $p = -\infty$ (the argument is contained in the first proof of Theorem (3.1.2)), and then to derive Theorem (3.1.16) for $n = 1$ and $p \geq -1$ from this and Lemma (3.1.15).

Corollary (3.1.17) [78]: Let $-1/n \leq p \leq 1$ and let f be an integrable function that is p-concave on an open convex set C in $\mathbb{R}^n$ contained in its support. Then the measure generated by f and C is $p/(np+1)$-concave.

Proof: This follows from Theorem (3.1.16) in exactly the same way as the special case $p = 0$ follows from the Prekopa-Leindler inequality (see the beginning of this section).

The Brunn-Minkowski inequality says that Lebesgue measure in $\mathbb{R}^n$ is $1/n$-concave, and Theorem (3.1.16) supplies plenty of measures that are p-concave for $-1/n \leq p \leq \infty$. Borell [88] (see also [79, Theorem 3.17]) proves a sort of converse to Corollary (3.1.17):

Given $-\infty \leq p \leq 1/n$ and a p-concave measure μ with n-dimensional support S, there is a $p/(1-np)$-concave function on S that generates μ. Borell also observed that when $p > 1/n$, no nontrivial p-concave measures exist in $\mathbb{R}^n$, and that any $1/n$-concave measure is a multiple of Lebesgue measure; see [89, Theorem 3.14]. Dancs and Uhrin [944, Theorem 3.4] find a generalization of Theorem (3.1.16) in which Lebesgue measure is replaced by a q-concave measure for some $-\infty \leq q \leq 1/n$.

It is convenient to mention here a sharpening of the Brunn-Minkowski theorem proved by Bonnesen in 1929 (see [94] and [84, p. 314]). If X is a bounded measurable set in $\mathbb{R}^n$, the inner function m_X of X is defined by

$$m_X(u) = sup_{t\in\mathbb{R}} V_{n-1}\big(X \cap (u^{\perp} + tu)\big),$$

for $u \in S^{n-1}$. (In 1926, Bonnesen asked if this function determines a convex body in $\mathbb{R}^n, n \geq 3$, up to translation and re°ection in the origin, a question that remains unanswered; see [97, Problem 8.10].) Bonnesen proved that if $0 < \lambda < 1$ and $u \in S^{n-1}$, then

$$V_n\big((1-\lambda)X + \lambda Y\big) \geq M_{1/(n-1)}(m_X(u), m_Y(u), \lambda)\left((1-\lambda)\frac{V_n(X)}{m_X(u)} + \lambda\frac{V_n(Y)}{m_Y(u)}\right). \quad (29)$$

Lemma (3.1.15) with $p = 1/(n-1)$ and $q = 1$ shows that this is indeed stronger than (15). As Dancs and Uhrin [94, Theorem 3.2] show, an integral version of (29), in a general form similar to

Theorem (3.1.16), can be constructed from the ideas already presented here.

At present the most general results in this direction are contained in the papers of Uhrin; see [196], [97]. In particular, Uhrin states in [87, p. 306] that all previous results of this type are contained in [97, (3.42)]. The latter inequality has as an ingredient a "curvilinear Minkowski addition," and its proof reintroduces geometrical methods.

The convolution of measurable functions f and g on $\mathbb{R}^n$ is

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

The next two theorems, on concavity of products of functions, are useful in obtaining a result on the concavity of convolutions.

Theorem(3.1.18) [78]: Let $p_1 + p_2 \geq 0$, and let $p = p_1p_2/(p_1+p_2)$ if p_1 and p_2 are not both zero, and $p = 0$ if $p_1 = p_2 = 0$. For $i = 1,2$, let f_i be a p_i-concave function on a convex set C_i in $\mathbb{R}^n$. Then the function $f(x,y) = f_1(x)f_2(y)$ is p-concave on $C_1 \times C_2$.

Proof: Suppose that $0 < \lambda < 1$, and let $x_i \in C_1$ and $y_i \in C_2$ for $i = 0, 1$. By Lemma (3.1.15),

$$\begin{aligned} f((1-\lambda)(x_0,y_0) + \lambda(x_1,y_1)) &= f_1((1-\lambda)x_0 + \lambda x_1)f_2((1-\lambda)y_0 + \lambda y_1) \\ &\geq M_{p_1}(f_1(x_0), f_1(x_1), \lambda)M_{p_2}(f_2(y_0), f_2(y_1), \lambda) \\ &\geq M_p(f_1(x_0)f_2(y_0), f_1(x_1)f_2(y_1), \lambda) \\ &= M_p(f(x_0,y_0), f(x_1,y_1), \lambda). \end{aligned}$$

Theorem (3.1.19) [78]: Let $p \geq -1/n$ and let f be an integrable p-concave function on an open convex set C in $\mathbb{R}^{m+n}$. For each x in the projection $C|\mathbb{R}^m$ of C onto $\mathbb{R}^m$, let $C(x) = \{y \in \mathbb{R}^n : (x,y) \in C\}$. Then

$$F(x) = \int_{C(x)} f(x, y)dy$$
is $p/(np + 1)$-concave on $C|\mathbb{R}^m$.

Proof: For $i = 0, 1$, let $x_i \in C|\mathbb{R}^m$ and let $g_i(y) = f(x_i, y)$ for $y \in C(x_i)$. Suppose that $0 < \lambda < 1$ and that $x = (1 - \lambda)x_0 + \lambda x_1$, and let $g(y) = f(x, y)$ for $y \in C(x)$. The p-concavity of f implies that
$$g((1 - \lambda)y_0 + \lambda y_1) \geq M_p(g_0(y_0), g_1(y_1), \lambda)$$
whenever $y_i \in C(x_i), i = 0, 1$. Also,
$$C(x) \supset (1 - \lambda)C(x_0) + \lambda C(x_1).$$
Then Theorem(3.1.16) yields
$$\int_{C(x)} g(y)dy \geq M_{p/(np+1)}\left(\int_{C(x_0)} g_0(y)dy, \int_{C(x_1)} g_1(y)dy, \lambda\right).$$
This shows that F is $p/(np + 1)$-concave on $C|\mathbb{R}^m$.

If we apply the previous theorem with $n = 1$ and $f = 1_C$ when C is the interior of a convex body K in $\mathbb{R}^{m+1}$, and let $p \to \infty$, we see that the function giving volumes of parallel hyperplane of K is $1/n$-concave. This statement is equivalent to the Brunn-Minkowski inequality for convex bodies.

Theorem (3.1.20) [78]: Let $p_1 + p_2 \geq 0$, and let $p = p_1p_2/(p_1 + p_2)$ if p_1 and p_2 are not both zero, and $p = 0$ if $p_1 = p_2 = 0$. Suppose further that $p \geq -1/n$. For $i = 1, 2$, let f_i be an integrable p_i-concave function on an open convex set C_i in $\mathbb{R}^n$. Then $f_1 * f_2$ is $p/(np + 1)$-concave on $C_1 + C_2$.

Proof: By Theorem (3.1.18), the function $f_1(x - y)f_2(y)$ is p-concave for $(x - y, y) \in C_1 \times C_2 \subset \mathbb{R}^{2n}$, that is, for $x \in C_1 + C_2$. The result follows from Theorem (3.1.19).

For extensions to measures and some examples that limit the possibility of weakening the conditions on p_1, p_2, and p in Theorem (3.1.20), see [99, Section 3.3], whose general approach we have followed in. Theorem (3.1.19) can be found in [98] and [95]. The early history of

Theorem (3.1.20) (when $p = 0$, this says that the convolution of two log concave functions is also log concave) is discussed by Das Gupta [47, p. 313].

12. The covariogram

Theorem (3.1.21) [78]: Let K and L be convex bodies in $\mathbb{R}^n$. Then the function
$$g_{K,L}(x) = V\big(K \cap (L + x)\big)^{1/n},$$
for $x \in \mathbb{R}^n$, is concave on its support.

Proof: For $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$, we have
$$\begin{aligned} K \cap (L + (1 - \lambda)x + \lambda y) &= K \cap ((1 - \lambda)(L + x) + \lambda(L + y)) \\ &\supset (1 - \lambda)\big(K \cap (L + x)\big) + \lambda(K \cap (L + y)) : \end{aligned}$$
Using the Brunn-Minkowski inequality (15), we obtain
$$\begin{aligned} g_{K,L}((1 - \lambda)x + \lambda y) &\geq V((1 - \lambda)\big(K \cap (L + x)\big) + (K \cap (L + y)))^{1/n} \\ &\geq (1 - \lambda)V(K \cap (L + x))^{1/n} + \lambda V(K \cap (L + y))^{1/n} \\ &= (1 - \lambda)g_{K,L}(x) + \lambda g_{K,L}(y), \end{aligned}$$

as required.

As a corollary, we conclude that the covariogram g_K of a convex body K in $\mathbb{R}^n$, defined for $x \in \mathbb{R}^n$ by

$$g_K(x) = V\big(K \cap (K + x)\big),$$

is $1/n$-concave (and hence log concave) on its support, which, it is easy to check, is the difference body $DK = K + (-K)$ of K. Obviously g_K is unchanged when K is translated or replaced by its reflection $-K$ in the origin. Note that

$$\begin{aligned} g_K(x) &= \int_{\mathbb{R}^n} 1_{K\cap(K+x)}(y)\, dy \\ &= \int_{\mathbb{R}^n} 1_K(y) 1_{K+x}(y) dy \\ &= 1_K(y) 1_K(y - x) dy = 1_{-K} * 1_K(x). \end{aligned}$$

The name “covariogram” stems from the theory of random sets, where the covariance is defined for $x \in \mathbb{R}^n$ as the probability that both o and x lie in the random set. The covariogram is also useful in mathematical morphology. See [95, Chapter 9]) and [90, Section 6.2]. In 1986, G. Matheron (see [92]) asked if the covariogram determines convex bodies, up to translation and reflection in the origin. Remarkably, this question is open even for $n = 2$! Nagel [91] proved that the answer is affirmative when K and L are convex polygons in the plane. Bianchi [93] has shown that the answer is affirmative for much larger class of planar convex bodies. He has also found pairs of convex polyhedra that represent counterexamples in $\mathbb{R}^4$, but these are still reflections of each other in a plane. See also [96, Section 6], and the references given in connection with chord-power integrals in [99, p. 267].

Anderson [102] used the Brunn-Minkowski theorem in his work on multivariate unimodality. He began with the following simple observation. If f is a (i) symmetric $(f(x) = f(-x))$ and (ii) unimodal $(f(cx) \geq f(x)$ for $0 \leq c \leq 1)$ function on $\mathbb{R}$, and I is an interval centered at the origin, then

$$\int_{I+y} f(x)\, dx$$

is maximized when $y = 0$. In probability language, if a random variable X has probability density f and Y is an independent random variable, then

$$Prob\,\{X \in I\} \geq Prob\{X + Y \in I\}.$$

To see this, recall that if g is the probability density of , then $f * g$ is the probability density of $X + Y$; see [82, Section 11.5]. So, by Fubini's theorem,

$$\begin{aligned} Prob\,\{X + Y \in I\} &= \int_I \int_{\mathbb{R}} f(z - y) g(y)\, dy\, dz \\ &= \int_{\mathbb{R}} \int_I f(z - y) g(y) dz\, dy \\ &= \int_{\mathbb{R}} \int_{I-y} f(x) g(y) dx dy \end{aligned}$$

$$\leq \int_{\mathbb{R}} \int_I f(x)g(y)dx\,dy$$
$$= \int_I f(x)dx = Prob\,\{X \in I\}.$$
Anderson generalized this, as follows. If f is a nonnegative function on $\mathbb{R}^n$, call f unimodal if the level sets $L(f,t)$ (see (24)) are convex for each $t \geq 0$. Note that every quasiconcave function and hence all p-concave functions are unimodal.

Theorem (3.1.22) [78] : (Anderson's theorem.) Let K be an origin-symmetric (i.e., $K = -K$) convex body in $\mathbb{R}^n$ and let f be a nonnegative, symmetric, and unimodal function integrable on $\mathbb{R}^n$. Then
$$\int_K f(x+cy)dx \geq \int_K f(x+y)dx\,,$$
for $0 \leq c \leq 1$ and $y \in \mathbb{R}^n$.

Proof: Suppose initially that $f(x) = 1_L(x)$, where L is an origin-symmetric convex body in $\mathbb{R}^n$. Then $f(x+y) = 1_L(x+y) = 1_{L-y}(x)$ and
$$\int_K f(x+y)dx = \int_K 1_{L-y}(x)dx = V\big(K \cap (L-y)\big) = g_{K,L}(-y) = g_{K,L}(y).$$
Theorem (3.1.21) implies that $g_{K,L}$ is log concave. Let $\lambda = (1-c)/2$. Since
$$g_{K,L}(cy) = g_{K,L}((1-2\lambda)y)$$
$$= g_{K,L}((1-\lambda)y + \lambda(-y))$$
$$\geq g_{K,L}(y)^{1-\lambda} g_{K,L}(-y)^{\lambda}$$
$$= g_{K,L}(y)^{1-\lambda} g_{K,L}(y)^{\lambda} = g_{K,L}(y),$$
the theorem follows. In the general case, $L(f,t)$ is an origin-symmetric convex body, so by (6), Fubini's theorem, and the special case just proved,
$$\int_K f(x+cy)dx \;=\; \int_K \int_0^\infty 1_{L(f,t)}(x+cy)dt\;\,dx$$
$$= \int_0^\infty \int_K 1_{L(f,t)}(x+cy)\;dx\;\,dt$$
$$\geq \int_0^\infty \int_K 1_{L(f,t)}(x+y)\;dx\;\,dt$$
$$= \int_K f(x+y)dx\,.$$
Anderson's theorem says that the integral of a symmetric unimodal function f over an n-dimensional centrally symmetric convex body K does not decrease when K is translated towards the origin. Since the graph of f forms a hill whose peak is over the origin, this is intuitively clear.

However, it is no longer obvious, as it was in the 1-dimensional case! There may be points $x \in K$ at which the value of f is larger than it is at the corresponding translate of x.

As above, we can conclude from Anderson's theorem that if a random variable X has probability density f on $\mathbb{R}^n$ and Y is an independent random variable, then

$$Prob\,\{X \in K\} \geq Prob\{X + Y \in K\},$$

where K is any origin-symmetric convex body in $\mathbb{R}^n$. We noted above that density functions of some well-known probability distributions are p-concave for some p, and hence unimodal. If they are also symmetric, Anderson's theorem applies.

Suppose K is a convex body in $\mathbb{R}^n, y \in \mathbb{R}^n, p \geq -1/n$, and f is an integrable p-concave function on $\mathbb{R}^n$. Corollary (3.1.17) implies that the measure μ generated by f and $\mathbb{R}^n$ is $p/(np+1)$-concave on $\mathbb{R}^n$. Let

$$h(y) = \mu(K - y) = \int_{K-y} f(x)dx \;= \int_K f(x+y)dx\,.$$

Since

$$K - (1-\lambda)y_0 - \lambda y_1 = (1-\lambda)(K - y_0) + \lambda(K - y_1),$$

we have

$$\begin{aligned} h((1-\lambda)y_0 + \lambda y_1) &= \mu(K - (1-\lambda)y_0 - \lambda y_1) \\ &= \mu((1-\lambda)(K - y_0) + \lambda(K - y_1)) \\ &\geq M_{p/(np+1)}\,(\mu(K - y_0), \mu(K - y_1), \lambda) \\ &= M_{p/(np+1)}\,(h(y_0), h(y_1), \lambda). \end{aligned}$$

Therefore h is $p/(np+1)$-concave on $\mathbb{R}^n$ and hence unimodal. In particular, $h(cy)$ is unimodal in c for a fixed y. This shows that Corollary (3.1.17) and Anderson's theorem are related. Anderson's theorem replaces the restriction $p \geq -1/n$ with a much weaker condition, but requires in exchange the symmetry of f and K.

Anderson's theorem has many applications in probability and statistics, where, for example, it can be applied to show that certain statistical tests are unbiased. See [102], [106], [99], and [100].

We saw in the previous how the Brunn-Minkowski inequality and convolutions come together naturally. The next theorem provides two convolution inequalities with sharp constants, the first proved independently by Beckner [101] and Brascamp and Lieb [104], and the second by Brascamp and Lieb [104]. We shall soon see that the second inequality actually implies the Brunn-Minkowski inequality.

Theorem (3.1.23) [78]: Let $0 < p, q, r$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{30}$$

and let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ be nonnegative. Then

(Young 's inequality) $\quad \|f * g\|_r \leq C^n \|f\|_p \|g\|_q, \quad for\; p, q, r \geq 1,$ (31)

and

(Reverse Young inequality) $\|f * g\|_r \geq C^n \|f\|_p \|g\|_q, \quad for\; p, q, r \leq 1.$ (32)

Here $C = C_p C_q / C_r$, where

$$C_s^2 = \frac{|s|^{1/s}}{|s'|^{1/s'}} \tag{33}$$

for $1/s + 1/s' = 1$ (that is, s and s' are Hölder conjugates).

The inequality (31), when expanded, reads as follows:

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x-y)g(y)dy\right)^r dx\right)^{1/r} \le C^n\left(\int_{\mathbb{R}^n} f(x)^p dx\right)^{1/p}\left(\int_{\mathbb{R}^n} g(x)^q dx\right)^{1/q}.$$

Inequalities (31) and (32) together show that equality holds in both when $p = q = r = 1$. In fact, since $C_p \to 1$ as $p \to 1$, when $p = q = r = 1$ we have $C = 1$, and substituting $u = x - y, v = y$ in the left-hand side of (31), we obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v)\,dv\,du \le \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(x)dx.$$

But equality holds here and therefore also in (31), and similarly in (32).

Theorem (3.1.24) [78]: The limiting case $r \to 0$ of the reverse Young inequality is the essential form of the Prekopa-Leindler inequality in $\mathbb{R}^n$ (Theorem (3.1.13)).

Proof: Let f_m and g_m be sequences of bounded measurable functions with compact support converging in $L^1(\mathbb{R}^n)$ to f and g, respectively, as $m \to \infty$ and satisfying $f_m \le f$ and $g_m \le g$. Let

$$s_m(x) = ess\ sup_y f_m\left(\frac{x-y}{1-\lambda}\right)^{1-\lambda} g_m\left(\frac{y}{\lambda}\right)^{\lambda} \tag{34}$$

Let $s(x)$ be defined by replacing f_m by f and g_m by g in (34). As in the proof of Theorem (3.1.13). s and each s_m is measurable. Also, $\|s\|_1 \ge \|s_m\|_1$, so if

$$\|s_m\|_1 \ge \|f_m\|_1^{1-\lambda}\|g_m\|_1^{\lambda}$$

for each m we have

$$\|s\|_1 \ge \|f\|_1^{1-\lambda}\|g\|_1^{\lambda}.$$

Therefore it suffices to prove the theorem when f and g are bounded measurable functions with compact support.

Assuming this, note that $s(x) = lim_{m\to\infty} S_m(x)$, where

$$S_m(x) = \left(\int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)^{(1-\lambda)m} g\left(\frac{y}{\lambda}\right)^{\lambda m} dy\right)^{1/(m-1)}.$$

(If we replaced the exponent, $1/(m-1)$ by $1/m$, this would follow from the fact that the mth mean tends to the supremum as $m \to \infty$; compare [97, p. 143]. But this replacement is irrelevant in the limit.) Note also that $\|s_m\|_1 = lim_{m\to\infty}\|S_m\|_1$ (we can interchange the limit and integral because the S_m's are uniformly bounded and have supports lying in some common compact set).

Applying the reverse Young inequality to S_m with $m > \max\{(1-\lambda)^{-1}, \lambda^{-1}\}, p = 1/((1-\lambda)m), q = 1 = (\lambda m)$, and $r = 1/(m-1)$, we obtain

$$\|S_m\|_1 = \int_{\mathbb{R}^n} S_m(x)dx$$

$$= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)^{(1-\lambda)m} g\left(\frac{y}{\lambda}\right)^{\lambda m} dy\right)^{1/(m-1)} dx$$

$$\ge \left(C^n\left(\int_{\mathbb{R}^n} f\left(\frac{x-y}{1-\lambda}\right)dx\right)^{(1-\lambda m)}\left(\int_{\mathbb{R}^n} g\left(\frac{y}{\lambda}\right)^{\lambda m} dy\right)^{\lambda m}\right)^{1/(m-1)}$$

$$= C^{n/(m-1)}\left((1-\lambda)^n\|f\|_1\right)^{(1-\lambda)m/(m-1)}\left(\lambda^n\|g\|_1\right)^{\lambda m/(m-1)}.$$

Therefore

$$\|s\|_1 = lim_{m\to\infty}\|S_m\|_1 \geq \left((1-\lambda)^{1-\lambda}\lambda^\lambda\, lim_{m\to\infty}C^{1/(m-1)}\right)^n\|f\|_1^{1-\lambda}\|g\|_1^\lambda.$$

It remains only to check that

$$\lim_{m\to\infty} C^{1/(m-1)} = (1-\lambda)^{-(1-\lambda)}\lambda^{-\lambda}.$$

The inequalities presented approach the most general known in the direction of Young's inequality and its reverse form, and represent a research frontier that can be expected to move before too long.

Each $m\times n$ matrix A defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and this linear map can also be denoted by A. The Euclidean adjoint A^* of A is then an $n\times m$ matrix or linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ satisfying $Ax\cdot y = x\cdot A^*y$ for each $y\in\mathbb{R}^m$ and $x\in\mathbb{R}^n$.

Theorem (3.1.25) [78]: Let $c_i > 0$ and $n_i\in\mathbb{N}, i = 1,\dots,m,$ with $\sum_i c_i n_i = n.$ Let $f_i\in L^1(\mathbb{R}^{n_i})$ be nonnegative and let $B_i\colon\mathbb{R}^n\to\mathbb{R}^{n_i}$ be a linear surjection, $i = 1,\dots,m$. Then (Brascamp-Lieb inequality)

$$\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(B_i x)^{c_i}\, dx \leq D^{-1/2}\prod_{i=1}^m\left(\int_{\mathbb{R}^{n_i}} f_i(x)dx\right)^{c_i} \qquad (35)$$

and

(Barthe's inequality)

$$\overline{\int_{\mathbb{R}^n}} sup\{\prod_{i=1}^m f_i(z_i)^{c_i} : x = \sum_i c_i B_i^* z_i\,, z_i\in\mathbb{R}^{n_i}\}dx \geq D^{1/2}\prod_{i=1}^m\left(\int_{\mathbb{R}^{n_i}} f_i(x)dx\right)^{c_i} \qquad (36)$$

where

$$D = inf\left\{\frac{det(\sum_{i=1}^m c_i B_i^* A_i B_i)}{\prod_{i=1}^m (det A_i)^{c_i}} : A_i\ is\ a\ positive\ definite\ n_i\times n_i\ matrix\right\} \qquad (37)$$

For comments on equality conditions and ideas of proof, including a proof of an important special case of (36),.

We can begin to understand (35) by taking $n_i = n, B_i = I_n$, the identity map on $\mathbb{R}^n$, replacing f_i by f_i^{1/c_i}, and letting $c_i = 1/p_i, i = 1,\dots,m$. Then $\sum_i 1/p_i = 1$ and the log concavity of the determinant of a positive definite matrix (see, for example, [80, p. 63]) yields $D = 1$. Therefore

$$\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(x)\,dx \leq \prod_{i=1}^m\|f_i\|_{p_i},$$

Holder's inequality in $\mathbb{R}^n$.

Next, take $m = 2, n_1 = n_2 = n, B_1 = B_2 = I_n, c_1 = 1-\lambda$, and $c_2 = \lambda$ in (36). Again we have $D = 1$, so

$$\overline{\int_{\mathbb{R}^n}} sup\{f_1(z_1)^{1-\lambda}f_2(z_2)^\lambda : x = (1-\lambda)z_1 + z_2\}dx$$
$$\geq \left(\int_{\mathbb{R}^n} f_1(x)dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} f_2(x)dx\right)^{\lambda},$$

the Prèkopa -Leindler inequality (12) in $\mathbb{R}^n$.

Theorem (3.1.26) [78]: (Young's inequality in $\mathbb{R}^n$, second form.) Let $0 < p, q, r$ satisfy

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}= 2,$$

and let $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$, and $h \in L^r(\mathbb{R}^n)$ be nonnegative. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)dy\ dx \le \bar{C}^n \|f\|_p\|g\|_q\|h\|_r, \qquad (38)$$

where $\bar{C} = C_pC_qC_r$ is defined using (33).

Theorem (3.1.27) [78]: The second form of Young's inequality in $\mathbb{R}^n$ is equivalent to the first (31).

Proof: Let $p, q, r \ge 1$ satisfy (30). By Holder's inequality (11),

$$sup\left\{\frac{\|f * g\|_r}{\|f\|_p\|g\|_q} : f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)\right\} =$$

$$= sup\left\{\frac{\int_{\mathbb{R}^n}(f * g)(x)h(x)\ dx}{\|f\|_p\|g\|_q\|h\|_{r'}} :\ f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n), h \in L^{r'}(\mathbb{R}^n)\right\}$$

$$= \ sup\left\{\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)h(x)dx\, dy}{\|f\|_p\|g\|_q\|h\|_{r'}} : f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n), h \in L^{r'}(\mathbb{R}^n)\right\}$$

$$= sup\left\{\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(x)dy\, dx}{\|f\|_{\bar{p}}\|g\|_{\bar{q}}\|h\|_{\bar{r}}} : f \in L^{\bar{p}}(\mathbb{R}^n), g \in L^{\bar{q}}(\mathbb{R}^n), h \in L^{\bar{r}}(\mathbb{R}^n)\right\}$$

where the last equality is obtained by replacing $f, g, h, p, q,$ and r', by $g, h, f, \bar{q}, \bar{r},$ and $\bar{p}$, respectively, so that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}= 2.$$

Theorem (3.1.28) [78]: The Brascamp-Lieb inequality (35) implies Young's inequality in $\mathbb{R}^n$.

Proof: In (35), let $m = 3, n_1 = n_2 = n_3 = n$, and let $B_i: \mathbb{R}^{2n} \to \mathbb{R}^n, i = 1,2,3$ be the linear maps taking $(z_1, \dots, z_{2n})$ to $(z_1, \dots, z_n), (z_1 - z_{n+1}, \dots, z_n - z_{2n})$, and $(z_{n+1}, \dots, z_{2n})$, respectively; then replace f_i by f_i^{1/c_i}, $i = 1, 2, 3$ and let $c_1 = 1/p, c_2 = 1/q$, and $c_3 = 1/r$. In this case $D = C^{-2}$, where C is as in Theorem (3.1.23); see [34, Theorem 5]. This gives

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1(x)f_2(x-y)f_3(y)dy\ dx \le C\|f_1\|_p\|f_2\|_q\|f_3\|_r,$$

which is (38).

As a side remark, we note that there is a version of Young's inequality in its second form (38), called the weak Young inequality, which only requires that $g \in L_w^q(\mathbb{R}^n)$, the weak L^q space. See [91, Section 4.3] for details. This allows one to conclude in particular that under the (slightly weakened) hypotheses of Theorem (3.1.26), with $q = n/\lambda$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\|x-y\|^{-\lambda}h(y)dy\ dx \le k(n, \lambda, p)\|f\|_p\|h\|_r. \qquad (39)$$

This was proved in Lieb [89] with a sharp constant $k(n, \lambda, p)$. The classical form without the sharp constant is called the Hardy-Littlewood-Sobolev inequality. The case $\lambda = n - 2$ is of particular interest in potential theory, as is explained in [91, Chapter 9].

As Ball [103] remarks, some geometry comes back into view if we replace $f(x)$ in Young's inequality (38) by $f(-x)$:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(-x_1)g(x_1-x_2)h(x_2)dx_2\,dx_1 \le \bar{C}\|f\|_p\|g\|_q\|h\|_r. \quad (40)$$

Define ϕ: $\mathbb{R}^2 \to \mathbb{R}^3$ by $\phi(x_1,x_2) = z = (z_1,z_2,z_3)$, where $z_1 = -x_1, z_2 = x_1 - x_2$, and $z_3 = x_2$.
Then $\phi(\mathbb{R}^2) = S$, where S is the plane $\{(z_1,z_2,z_3): z_1+z_2+z_3 = 0\}$ through the origin. Let $f = g = h = 1_{[-1,1]}$ and $C_0 = [-1,1]^3$. By (40),

$$V_2(C_0 \cap S) = \int_S 1_{C_0}(z)\,dz$$
$$= \int_S f(z_1)g(z_2)h(z_3)dz$$
$$= J(\phi)^{-1} f(-x_1)g(x_1-x_2)h(x_2)dx_2\,dx_1,$$

where $J(\phi)$ is the Jacobian of ϕ. So Young's inequality might be used to provide upper bounds for volumes of central of cubes. In fact, Ball [109] used the following special case of the Brascamp-Lieb inequality to do just this.
Suppose that $c_i > 0$ and $u_i S^{n-1}, i = 1,\dots,m$ satisfy

$$x = \sum_{i=1}^{m} c_i(x\cdot u_i)u_i\,,$$

for all $x \in \mathbb{R}^n$. This says that the u_i's are acting like an orthonormal basis for $\mathbb{R}^n$. The condition is often written

$$\textstyle\sum_{i=1}^m c_i u_i \otimes u_i = I_n, \quad (41)$$

where $u \otimes u$ denotes the rank one orthogonal projection onto the span of u, the map that sends x to $(x\cdot u)u$. Taking traces in (41), we see that

$$\textstyle\sum_{i=1}^m c_i = n. \quad (42)$$

Theorem (3.1.29) [78]: Let $c_i > 0$ and $u_i \in S^{n-1}, i = 1,\dots,m$ be such that (41) and hence (42) holds.
If $f_i \in L^1(\mathbb{R})$ is nonnegative, $i = 1,\dots,m$, then
(Geometric Brascamp-Lieb inequality)

$$\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(x\cdot u_i)^{c_i}\,dx \le \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(x)dx\right)^{c_i} \quad (43)$$

and
(Geometric Barthe inequality)

$$\int_{\mathbb{R}^n} sup\{\prod_{i=1}^m f_i(z_i)^{c_i} : x = \textstyle\sum_i c_i z_i u_i\,, z_i \in \mathbb{R}\}dx \ge \prod_{i=1}^m\left(\int_{\mathbb{R}} f_i(x)dx\right)^{c_i}. \quad (44)$$

Proof: Let $n_i = 1$ and for $x \in \mathbb{R}^n$, let $B_i x = x\cdot u_i, i = 1,\dots,m$. Then $B_i^* z_i = z_i u_i \in \mathbb{R}^n$ for $z_i \in \mathbb{R}$. The inequalities (35) and (36) become (43) and (44), respectively, because the hypotheses of the theorem and (37) imply that $D = 1$ (see [107, Proposition 9] for the details).
Note that the geometric Barthe inequality (44) still implies the Prekopa-Leindler inequality in $\mathbb{R}$, with the geometric consequences explained above.
Ball [109] used (43) to obtain the best-possible upper bound

$$V_k(C_0 \cap S) \le \left(\sqrt{2}\right)^{n-k}$$

for of the cube $C_0 = [-1,1]^n$ by k-dimensional subspaces $S, 1 \le k \le n-1$, when $2k \ge n$. (For smaller values of k, the best-possible bound is not known except for some special cases; see [109].) He also showed that (43) provides best-possible upper bounds for the volume ratio $vr(K)$ of a convex body K in $\mathbb{R}^n$, defined by

$$vr(K) = \left(\frac{V(K)}{V(E)}\right)^{1/n},$$

where E is the ellipsoid of maximal volume contained in K. The ellipsoid E is called the John ellipsoid of K. The following theorem is a refinement of Ball [102] of a theorem proved by Fritz John.

Theorem (3.1.30) [78]: The John ellipsoid of a convex body K in $\mathbb{R}^n$ is B if and only if $B \subset K$ and there is an $m \ge n, c_i > 0$ and $u_i \in S^{n-1} \cap \partial K, i = 1, \dots, m$ such that (41) holds and $\sum_i c_i u_i = o$.

Ball's argument is as follows. Let K be a convex body in $\mathbb{R}^n$. Since $vr(K)$ is affine invariant, we may assume that the John ellipsoid of K is B. If we can show that $V(K) \le 2^n$, then $vr(K) \le vr(C_0)$, where $C_0 = [-1,1]^n$. Let c_i and u_i be as in John's theorem, and note that the points u_i are contact points, points where the boundaries of K and B meet. If K is origin-symmetric and u_i is a contact point, then so is $-u_i$; therefore $K \subset L$, where

$$L = \{x \in \mathbb{R}^n : |x \cdot u_i| \le 1, i = 1, \dots, m\}$$

is the closed slab bounded by the hyperplanes $\{x : x \cdot u_i = \pm 1\}$. Also, if $f_i = 1_{[-1,1]}$, then

$$1_L(x) = \prod_{i=1}^{m} f_i(x \cdot u_i)^{c_i}.$$

By (43) and (42),

$$V(K) \le V(L) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x \cdot u_i)^{c_i}\, dx$$
$$\le \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i(x) dx\right)^{c_i} = \prod_{i=1}^{m} 2^{c_i} = 2^n.$$

This argument shows that $vr(K)$ is maximal for centrally symmetric K when K is a parallelotope.

One consequence of this estimate is the following result of Ball [101] (Behrend [102] proved the result for $n = 2$).

Theorem (3.1.31) [78]: (Reverse isoperimetric inequality for centrally symmetric convex bodies in $\mathbb{R}^n$.) Let K be a centrally symmetric convex body in $\mathbb{R}^n$ and let $C_0 = [-1,1]^n$. There is an affine

transformation ϕ such that

$$\left(\frac{S(\phi K)}{S(C_0)}\right)^{1/(n-1)} \le \left(\frac{V(\phi K)}{V(C_0)}\right)^{1/n}. \tag{45}$$

Proof: Choose ϕ so that the John ellipsoid of ϕK is B. The above argument shows that $V(\phi K) \leq 2^n$. Since $B \subset \phi K$, we have, by (3),

$$S(\phi K) = lim_{\varepsilon \to 0_+} \frac{V(\phi K + \varepsilon B) - V(\phi K)}{\varepsilon}$$
$$\leq lim_{\varepsilon \to 0_+} \frac{V(\phi K + \varepsilon \phi K) - V(\phi K)}{\varepsilon}$$
$$= V(\phi K)\, lim_{\varepsilon \to 0_+} \frac{(1 + \varepsilon)^{n-1}}{\varepsilon}$$
$$= nV(\phi K) = nV(\phi K)^{(n-1)/n} V(\phi K)^{1/n} \leq 2nV\,(\phi K)^{(n-1)/n}.$$

This is equivalent to (45).

One cannot expect a reverse isoperimetric inequality without use of an affine transformation, since we can find convex bodies of any prescribed volume that are very flat and so have large surface area.

In [101], Ball used the same methods to show that for arbitrary convex bodies, the volume ratio is maximal for simplices, and to obtain a corresponding reverse isoperimetric inequality. The fact that the volume ratio is only maximal for parallelotopes (in the centrally symmetric case) or simplices was shown by Barthe [107] as a corollary of his study of the equality conditions in the Brascamp-Lieb inequality.

For other results of this type that employ Theorem (3.1.29), see [100], [106], and [103]. Barthe [107] states a multidimensional generalization of Theorem (3.1.29), also derived from Theorem (3.1.25), that leads to a multidimensional Brunn-Minkowski-type theorem.

The classical Young inequality is

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \qquad for\ p, q, r \geq 1,$$

that is, (31) with the better constant C^n there replaced by 1, under the same assumptions. This can be proved in a few lines using Holder's inequality (11); see [91, p. 99]. It was proved by $W.H.$ Young in 1912-13 (see [107, Sections 8.3 and 8.4]), and is related to the classical Hausdorff-Young inequality: If $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R}^n)$, then

$$\|\hat{f}\|_{p'} \leq \|f\|_p, \tag{46}$$

where $\hat{f}$ denotes the Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{2\pi i x \cdot y}\, dy$$

of f, and p and p' are Holder conjugates. This was proved by Hausdorff and Young for Fourier series, and extended to integrals by Titchmarsh in 1924. Beckner [21], improving earlier partial results of Babenko, showed that when $1 \leq p \leq 2$,

$$\|\hat{f}\|_{p'} \leq C_p^n \|f\|_p, \tag{47}$$

where C_p is given by (33). (Lieb [90] proved that equality holds only for Gaussians.) This improvement on (46) is related to Young's inequality (31); in fact, the classical Young inequality was motivated by (46). To see the connection, suppose that (47) holds, $n = 1$, and $1 \leq p, q, r' \leq 2$. If p, q, r satisfy (30), then their Holder conjugates satisfy $1/p' + 1/q' = 1/r'$. Using this and Holder's inequality (11), we obtain

$$\|f * g\|_r \leq C_{r'} \|\hat{f}\hat{g}\|_{r'},$$

$$\leq C_{r'} \left\|\hat{f}\right\|_{p'} \|\hat{g}\|_{q'}$$
$$\leq C_{r'}\big(C_p\|f\|_p\big)\big(C_q\|g\|_q\big) = C\|f\|_p\|g\|_q.$$
A similarly easy argument (see [92, pp.169-70]) shows that Young's inequality (31) yields (46) when p' is an even integer.
Young's inequality in the sharp form (31) was proved independently by Beckner [92] and Bras-camp and Lieb [104]. The reverse Young inequality without the sharp constant (that is, with C replaced by 1) is due to Leindler [87]; the sharp version was obtained by Brascamp and Lieb [94]. The latter also found the connection to the Prekopa-Leindler inequality, Theorem (3.1.24), and established the following equality conditions: When $n = 1$ and $p, q \neq 1$, equality holds in (31) or (32) if and only if f and g are Gaussians:
$$f(x) = ae^{-c|p'|(x-\alpha)^2}, g(x) = be^{-c|q'|(x-\beta)^2},$$
for some a, b, c, α, β with $a, b \geq 0$ and $c > 0$.
The simplest known proof of Young's inequality and its reverse form, with the above equality conditions, was found by Barthe [98].
The Brascamp-Lieb inequality in the general form (35), with equality conditions, was proved by Lieb [90]. The special case $n_i = 1$ and $B_i x = x \cdot v_i$, where $x \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n, i = 1, \ldots, m$ is the main result of Brascamp and Lieb [94].
Let A be an $n \times n$ positive definite symmetric matrix, and let
$$G_A(x) = exp(-Ax \cdot x),$$
for $x \in \mathbb{R}^n$. The function G_A is called a centered Gaussian. Lieb [90] proved that the supremum of the left-hand side of (35) for functions f_i of norm one is the same as the supremum of the left-hand side of (35) for centered Gaussians of norm one; in other words, the constant D can be computed using centered Gaussians.
There is also a version of (35) in which a fixed centered Gaussian appears in the integral on the left-hand side and the constant is again determined by taking the functions f_i to be Gaussians; see [94, Theorem 6], where an application to statistical mechanics is given, and [90, Theorem 6.2].
Barthe [97] proved (36), giving at the same time a simpler approach to (35) and its equality conditions.
The fact that the constant D in the geometric Brascamp-Lieb inequality (43) becomes 1 was observed by Ball [99]. Inequality (44) was first proved by Barthe [94]. As in the general case, equality holds in (43) and (44) for centered Gaussians.
The main idea behind Barthe's approach is the use of a familiar construction from measure theory. Let μ be a finite Borel measure in $\mathbb{R}^n$ and $T: \mathbb{R}^n \to \mathbb{R}^n$ a Borel-measurable map defined μ-almost everywhere. For Borel sets M in $\mathbb{R}^n$, let
$$v(M) = (T\mu)(M) = \mu\big(T^{-1}(M)\big).$$
The Borel measure $v = T\mu$ is sometimes called the push-forward of μ by T, and T is said to push forward or transport the measure μ to v. Suppose for simplicity that μ and v are absolutely continuous with respect to Lebesgue measure, so that
$$\mu(M) = \int_M f(x)dx \; and \; v(M) = \int_M g(x)\, dx$$

for Borel sets M in $\mathbb{R}^n$, and T is a differentiable bijection. Then

$$f(x) = g\big(T(x)\big)J(T)(x),$$

where $J(T)$ is the Jacobian of T, and we can talk of T transporting f to g. If μ and v are measures on $\mathbb{R}$, absolutely continuous with respect to Lebesgue measure and with $\mu(\mathbb{R}) = v(\mathbb{R})$, then we can always find a T that transports μ to v, by defining $T(t)$ to be the smallest number such that

$$\int_{-\infty}^{t} f(x)dx = \int_{-\infty}^{T(t)} g(x)\, dx.$$

Moreover, if f and g are continuous and positive, then T is strictly increasing and C^1, and

$$f(x) = g\big(T(x)\big)T'(x).$$

In fact, the same parametrization was used in proving the Prekopa-Leindler inequality in $\mathbb{R}$. To see this, replace the functions f and g in the second proof of Theorem (3.1.2) with g_1 and g_2, respectively. If $f_i = F_i 1_{[0,1]}, i = 1,2,$ then

$$\frac{1}{G_i}\int_{-\infty}^{T_i(t)} g_i(x)\, dx = \int_{-\infty}^{t} 1_{[0,1]}(x)dx = t,$$

so the functions u and v in the second proof of Theorem (3.1.2) are just T_1 and T_2, respectively. In other words, u and v transport a suitable multiple of the characteristic function of the unit interval to g_1 and g_2, respectively.

Barthe saw that this is all that is needed to prove (35) and (36) simultaneously in the special case $n_i = 1$ and $B_i x = x \cdot v_i$, where $x \in \mathbb{R}^n$ and $v_i \in \mathbb{R}^n, i = 1, \dots, m$. To see this, let $c_i > 0$ satisfy $\sum_i c_i = n$ and let f_i and g_i be nonnegative functions in $L^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} f_i(x)\, dx = F_i \ and \ \int_{\mathbb{R}} g_i(x)\, dx = G_i,$$

for $i = 1, \dots, m$. Standard approximation arguments show that there is no loss of generality in assuming f_i and g_i are positive and continuous. Define strictly increasing maps T_i as above, so that

$$\tfrac{1}{F_i}\int_{-\infty}^{t} f_i(x)dx = \tfrac{1}{G_i}\int_{-\infty}^{T_i(t)} g_i(x)\, dx$$

and hence

$$\frac{f_i(x)}{F_i} = g(Ti(x))T0i\ (x)Gi;$$

for $i = 1, \dots, m$. For $x \in \mathbb{R}^n$, let

$$V(x) = \sum_{i=1}^{m} c_i T_i(x \cdot v_i)v_i\,,$$

so that

$$dV(x) = \sum_{i=1}^{m} c_i T_i'\,(x \cdot v_i)(v_i \otimes v_i)(dx).$$

Finally, note that if $B_i x = x \cdot v_i$ for $x \in \mathbb{R}$, then $B_i^* = x v_i$, so $B_i^* B x = v_i \otimes v_i(x)$, and the constant D in (37) becomes

$$D = inf\left\{\frac{det\,(\sum_{i=1}^{m} c_i a_i v_i \otimes v_i)}{\prod_{i=1}^{m} a_i^{c_i}} : a_i > 0\right\}.$$

In the following, we can assume that $D \neq 0$. Using the expression for D with $a_i = T_i'(x \cdot v_i), i = 1, \ldots, m$ to provide a lower bound for the Jacobian of the injective map , we obtain

$$D\left(\prod_{i=1}^{m}\left(\frac{G_i}{F_i}\right)^{c_i}\right)\int_{\mathbb{R}^n}\prod_{i=1}^{m} f_i(x\cdot v_i)^{c_i}\,dx = D\int_{\mathbb{R}^n}\prod_{i=1}^{m}(g_i(T_i(x\cdot v_i))T_i'(x\cdot v_i))^{c_i}\ dx$$

$$\leq \int_{\mathbb{R}^n}\prod_{i=1}^{m} g_i(T_i(x\cdot v_i))^{c_i}\ det\left(\sum_{i=1}^{m} c_i T_i'\,(x\cdot v_i)(v_i\otimes v_i)\right)dx$$

$$\leq \int_{\mathbb{R}^n} sup\left\{\prod_{i=1}^{m}(g_i(z_i)^{c_i} : V = \sum_i c_i z_i v_i, z_i \in \mathbb{R})\right\}dV$$

$$\leq \int_{\mathbb{R}^n} sup\left\{\prod_{i=1}^{m}(g_i(z_i)^{c_i} : x = \sum_i c_i z_i v_i, z_i \in \mathbb{R})\right\}dx.$$

To see how centered Gaussians play a role in the equality conditions, note that if $f_i(x) = exp(-a_i x^2)$, then since $\sum_i c_i = n$,

$$\prod_{i=1}^{m}\left(\int_{\mathbb{R}} f_i(x)dx\right)^{c_i} = \prod_{i=1}^{m}\left(\int_{\mathbb{R}} e^{-a,x^2}dx\right)^{c_i}$$

$$= \prod_{i=1}^{m} a_i^{-c_i/2}\left(\int_{\mathbb{R}} e^{-x^2}dx\right)^{c_i}$$

$$= \prod_{i=1}^{m}\left(\frac{\pi}{a_i}\right)^{c_i/2} = \left(\frac{\pi^n}{\prod_{i=1}^{m} a_i^{c_i}}\right)^{1/2},$$

while

$$\int_{\mathbb{R}^n}\prod_{i=1}^{m} f_i(x\cdot v_i)^{c_i}\,dx = \int_{\mathbb{R}^n}\prod_{i=1}^{m}\left(e^{-a_i(xv_i)^2}\right)^{c_i}dx$$

$$= \int_{\mathbb{R}^n} e^{-(\sum_{i=1}^{m} c_i a_i (x\cdot v_i) v_i)\cdot x}\ dx$$

$$= \left(\frac{\pi^n}{det(\sum_{i=1}^{m} c_i a_i v_i \otimes v_i)}\right)^{1/2}.$$

(The last equality follows from

$$\int_{\mathbb{R}^n} e^{-Ax\cdot x}\,dx = \left(\frac{\pi^n}{detA}\right)^{1/2},$$

where A is a positive definite symmetric $n \times n$ matrix.)

To summarize, we have shown that in the special case under consideration, the left-hand side of (36) is greater than or equal to the left-hand side of (35), and that equality holds in (35) for centered Gaussians. This is already enough to prove (36). One more computation is needed to prove (35), but we shall omit it, since it needs some (quite basic) tools of geometry, see [104].

If one wants to apply the same sort of argument in the general situation of Theorem (3.1.25), one needs an answer to the following question: If μ and v are measures on $\mathbb{R}^n$, absolutely continuous with respect to Lebesgue measure and with $\mu(\mathbb{R}^n) = v(\mathbb{R}^n)$, can we find a T with some suitable monotonicity property that transports μ to v? It turns out that an ideal answer has recently been found, called the Brenier map: Providing μ vanishes on Borel sets of $\mathbb{R}^n$ with Hausdorff dimension $n-1$, there is a convex map $\psi : \mathbb{R}^n \to \mathbb{R}$ such that if $T = \nabla\psi$, then T transports μ to v. See [107]. It is appropriate to highlight the contribution of McCann, whose 1994 PhD thesis [113] shows the relevance of measure-preserving convex gradients to geometric inequalities and helped attract the attention of the convex geometry community to Brenier's result. In [113] and [114], the Brenier map is exploited as a localization technique to derive new global convexity inequalities which imply the Brunn-Minkowski and Prèkopa -Leindler inequalities as special cases.

Barthe [115, Section 2.4] also discovered a generalization of Young's inequality in $\mathbb{R}^n$ that contains the geometric Brascamp-Lieb and geometric Barthe inequalities as limiting cases.

Suppose that X is a discrete random variable taking possible values $x_1, \dots, x_m$ with probabilities $p_1, \dots, p_m$, respectively, where $\sum_i p_i = 1$. Shannon [136] introduced a measure of the average uncertainty removed by revealing the value of X. This quantity,

$$H_m(p_1, \dots, p_m) = -\sum_{i=1}^{m} p_i \log p_i ,$$

is called the entropy of X. It can also be regarded as a measure of the missing information; indeed, the function H_m is concave and achieves its maximum when $p_1 = \cdots = p_m = 1/m$, that is, when all outcomes are equally likely. The words "uncertainty" and "information" already suggest a connection with physics, and a derivation of the function H_m from a few natural assumptions can be found in textbooks on statistical mechanics; see, for example, [106, Chapter 3].

If X is a random vector in $\mathbb{R}^n$ with probability density f, the entropy $h_1(X)$ of X is defined analogously:

$$h_1(X) = h_1(f) = -\int_{\mathbb{R}^n} f(x) \log f(x)\, dx .$$

The notation we use is convenient when $h_1(X)$ is regarded as a limit as $p \to 1$ of the pth Renyi entropy $h_p(X)$ of X, defined for $p > 1$ by

$$h_p(X) = h_p(f) = \frac{p}{1-p} \log \|f\|_p .$$

The entropy of X may not be well defined. However, if $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p > 1$, then $h_1(X) = h_1(f)$ is well defined, though its value may be $+\infty$.

The entropy power $N(X)$ of X is

$$N(X) = \frac{1}{2\pi e} exp\left(\frac{2}{n} h_1(X)\right).$$

Theorem (3.1.32) [78]: (Entropy power inequality.) Let X and Y be independent random vectors in $\mathbb{R}^n$ with probability densities in $L^p(\mathbb{R}^n)$ for some $p > 1$. Then

$$N(X + Y) \geq N(X) + N(Y). \tag{48}$$

The entropy power inequality was proved by Shannon [136, Theorem 15 and Appendix 6] and applied by him to obtain a lower bound [136, Theorem 18] for the capacity of a channel.

Lemma (3.1.33) [78]: Let f and g be nonnegative functions in $L^s(\mathbb{R}^n)$ for some $s > 1$, such that

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} g(x)dx = 1.$$

Then for $0 < \lambda < 1$,

$$h_1(f * g) - (1-\lambda)h_1(f) - \lambda h_1(g) \geq -\frac{n}{2}\big((1-\lambda)log(1-\lambda) + \lambda log\lambda\big). \tag{49}$$

Proof: For $r \geq 1$, let

$$p = p(r) = \frac{r}{(1-\lambda)+\lambda r} \; and \; q = q(r) = \frac{r}{\lambda+(1-\lambda)r}. \tag{50}$$

Then $p, q \geq 1$,

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

and $p(1) = q(1) = 1$. If $r < s$ is close to 1, then $p, q < s$, and since $f, g \in L^1(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$, we have $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Let

$$F(r) = \frac{\|f * g\|_r}{\|f\|_p \|g\|_q} \; and \; G(r) = C^n,$$

where C is as Theorem (3.1.23). By Young's inequality (31), $f * g \in L^1(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ and $F(r) \leq G(r)$ for r close to 1. As we noted after Theorem (3.1.23), the equation $F(1) = G(1+)$ holds. Therefore

$$\frac{F(r) - F(1)}{r-1} \leq \frac{G(r) - G(1+)}{r-1},$$

for r close to 1, which implies that $F'(1+) \leq G'(1+)$. We can assume that $h_1(f * g) < 1$ and therefore that $h_1(f) < \infty$ and $h_1(g) < 1$. Now if $\phi \in L^r(\mathbb{R}^n), \|\phi\|_1 = 1$, and $h_1(\phi) < \infty$, then

$$\begin{aligned}\frac{d}{dr}\|\phi\|_r &= \frac{1}{r}\|\phi\|^{1-r}\frac{d}{dr}\int_{\mathbb{R}^n}\phi(x)^r\,dx \\ &= \frac{1}{r}\|\phi\|^{1-r}\int_{\mathbb{R}^n}\phi(x)^r\,log\,\phi(x)dx \\ &\to \int_{\mathbb{R}^n}\phi(x)log\,\phi(x)dx = -h_1(\phi)\end{aligned}$$

as $r \to 1$. Using this and (50), we see that

$$F'(1+) = -h_1(f * g) + (1-\lambda)h_1(f) + \lambda h_1(g).$$

A calculation, helped by the fact that $p' = r'/(1-\lambda)$ and $q' = r'/\lambda$, where p', q', r' denote as usual the Holder conjugates of p, q, r, respectively, shows that

$$G'(1+) = \frac{n}{2}\big((1-\lambda)log(1-\lambda) + \lambda log\lambda\big).$$

Finally, (49) follows from the inequality $F'(1+) \leq G'(1+)$.

Corollary (3.1.34) [78]: Young's inequality (31) implies the entropy power inequality (48).

Proof. In (49), put

$$\lambda = \frac{N(Y)}{N(X)+N(Y)} \, .$$

Simplification of the resulting inequality leads directly to (48).

Presumably Lieb, via his [104] and [88], first saw the connection between the entropy power inequality (48) and the Brunn-Minkowski inequality (15), the former being a limiting case of Young's inequality (31) as $r \to 1$ and the latter a limiting case of the reverse Young inequality (32) as $r \to 0$. Later, Costa and Cover [103] specifically drew attention to the analogy between the two inequalities, apparently unaware of the work of Brascamp and Lieb. Dembo, Cover, and Thomas [108] explore further connections with other inequalities. These include some involving Fisher information and various uncertainty inequalities.

Fisher information was employed by Stam [108] in his proof of (48). Named after the statistician $R.A.$ Fisher, Fisher information is claimed in [104] by Frieden to be at the heart of a unifying principle for all of physics! If X is a random variable with probability density f on $\mathbb{R}$, the Fisher information $I(X)$ of X is

$$I(X) = I(f) = -\int_{\mathbb{R}} f(x)(\log f(x))'' dx \; = \int_{\mathbb{R}} \frac{f'(x)^2}{f(x)} dx \, ,$$

assuming these integrals exist. The multivariable form of I is a matrix, the natural extension of this definition. The quantity I is another measure of the "sharpness" of f or the missing information in X; see [64, Section 1.3] for a comparison of I and h_1. Stam

Theorem (3.1.35) [78]: (Aleksandrov-Fenchel inequality.) Let $K_1, \dots, K_n$ be compact convex sets in $\mathbb{R}^n$ and let $1 \leq \; i \leq n$. Then

$$V(K_1, K_2, \dots, K_n)^i \; \geq \prod_{j=1}^{i} V\big(K_j, i; K_{i+1}, \dots, K_n\big) \, . \qquad (51)$$

See [107, p. 143] and [114, (6.8.7)]. The quantities $V(K_1, K_2, \dots, K_n)$ and $V(K_j\,, i; K_{i+1}, \dots, K_n)$ (where the notation means that K_j appears i times) are mixed volumes, like the quantity $V_1(K, L)$ we met. In fact, if we put $i = n$ in (51) and then let $K_1 = L$ and $K_2 = \cdots = K_n = K$, we retrieve Minkowski's first inequality (20) for compact convex sets. Therefore the Aleksandrov-Fenchel inequality implies the Brunn-Minkowski inequality for compact convex sets. In fact, there is a more general version of the latter that is equivalent to (51):

Theorem (3.1.36) [78]: (Generalized Brunn-Minkowski inequality for compact convex sets.) Let $K_1, \dots, K_n$ be compact convex sets in $\mathbb{R}^n$ and let $1 \leq i \leq n$. For $0 \leq \lambda \leq 1$, let

$$f(\lambda) = V\big((1-\lambda)K_0 + \lambda K_1, i; K_{i+1}, \dots, K_n\big)^{1/i}.$$

Then f is a concave function on [0,1].

Using the above observations, this can be translated into

$$V(P_1, P_2, P_3, \dots, P_n)^2 \ \geq V(P_1 P_1, P_3, \dots, P_n) V(P_2, P_2\ , P_3, \dots, P_n).$$

The case $i = 2$ of (19.1) (and hence, by induction, (19.1) itself) can be shown to follow by approximation by polytopes with rational coordinates. See [85, Section 27] for many more details and also [82] and [123] for more recent advances in this direction.

Alesker, Dar, and Milman [91] are able to use the Brenier map (see the end of Section 17) to prove some of the inequalities that follow from the Aleksandrov-Fenchel inequality, but the method does not seem to yield a new proof of (51) itself.

In contrast to the Brunn-Minkowski inequality, the Aleksandrov-Fenchel inequality and some of its weaker forms, and indeed mixed volumes themselves, have only partially $\mathbb{R}^n$ closed under Minkowski addition and dilatation is called Minkowski concave if

$$\phi\big((1-\lambda)X + \lambda Y\ \big) \geq (1-\lambda)\phi(X) + \lambda\phi(Y\ ), \qquad (52)$$

for $0 < \lambda < 1$ and sets X, Y in the class. For example, the Brunn-Minkowski inequality implies that $V_n^{1/n}$ is Minkowski concave on the class of convex bodies. When Hadwiger published his extraordinary book [79] in 1957, many other Minkowski-concave functions had already been found, and several more have been discovered since. We shall present some of these; all the functions have the required degree of positive homogeneity to allow the coefficients $(1-\lambda)$ and ¸ to be deleted in (52). Other examples can be found in [79, Section 6.4] and in Lutwak's [96] and [102].

Knothe [83] gave a proof of the Brunn-Minkowski inequality for convex bodies, sketched in [104, pp. 312-314], and the following generalization. For each convex body K in $\mathbb{R}^n$, let $F(K, x)$, $x \in K$, be a nonnegative real-valued function continuous in K and x. Suppose also that for some $m > 0$,

$$F(\lambda K + a, \lambda x + a\ ) = \lambda^m F(K, x)$$

for all $\lambda > 0$ and $a \in \mathbb{R}^n$, and that

$$log\ F\big((1-\lambda)K + \lambda L, (1-\lambda)x + \lambda y\big) \geq (1-\lambda) log\ F(K, x) + \lambda log\ F(L, y)$$

whenever $x \in K, y \in L$, and $0 \leq \lambda \leq 1$. For each convex body K in $\mathbb{R}^n$, define

$$G(K) = \int_K F(K, x) dx\,.$$

Then

$$G(K+L)^{1/(n+m)} \geq G(K)^{1/(n+m)} + G(L)^{1/(n+m)}, \qquad (53)$$

for all convex bodies K and L in $\mathbb{R}^n$ and $0 < \lambda < 1$. This is a consequence of the Prekopa-Leindler inequality. Indeed, taking $f = F((1-\lambda)K + \lambda L, \cdot), g = \ F(K, \cdot)$, and $h = F(L, \cdot)$, Theorem (3.1.3) implies that G is log concave. The method can then be used to derive the $1/(n+m)$-concavity (53) of G from its log concavity. The Brunn-Minkowski inequality for convex bodies is obtained by taking $F(K, x) = 1$ for $x \in K$. Dinghas [80] found further results of this type.

Let $0 \leq i \leq n$. The mixed volume $V(K, n-i, B, i)$ is denoted by $W_i(K)$, and called the ith quermassintegral of a compact convex set K in $\mathbb{R}^n$. Then $W_0(K) = \ V_n(K)$. It can be shown (see [134, (5.3.27), p. 295]) that if K is a convex body and $1 \leq i \leq n-1$, then

$$W_i(K) = \frac{\kappa_n}{\kappa_{n-i}} \int_{G(n,n-i)} V(K|S) dS, \qquad (54)$$

where dS denotes integration with respect to the usual rotation-invariant probability measure on the Grassmannian $G(n, n-i)$ of $(n-i)$-dimensional subspaces of $\mathbb{R}^n$. Thus the quermassintegrals are averages of volumes of projections on subspaces.
Letting $K_{i+1} = \cdots = K_n = B$ in Theorem (3.1.36) yields:
Theorem (3.1.37) [78]: (Brunn-Minkowski inequality for quermassintegrals.) Let K and L be convex bodies in $\mathbb{R}^n$ and let $0 \le i \le n-1$. Then

$$W_i(K+L)^{1/(n-i)} \ge W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \quad (55)$$

with equality for $0 < i < n-1$ if and only if K and L are homothetic.
See [104, (6.8.10), p. 385]. The special case $i = 0$ is the usual Brunn-Minkowski inequality for convex bodies. The quermassintegral $W_1(K)$ equals the surface area $S(K)$, up to a constant, so the case $i = 1$ of (55) is a Brunn-Minkowski-type inequality for surface area. When $i = n-1$, (55) becomes an identity. The equality conditions for $0 < i < n-1$ follow from those known for the corresponding special case of Theorem (3.1.36).
Let K be a convex body in $\mathbb{R}^n$, define $\widehat{W}_0(K) = V(K)$ and for $1 \le i \le n-1$, define

$$\widehat{W}_i(K) = \frac{\kappa_n}{\kappa_{n-i}}\left(\int_{G(n,n-i)} V\,(K|S)^{-n}\, dS\right)^{-1/n},$$

the ith harmonic quermassintegral of K. Similarly, define $\Phi_0(K) = V(K)$ and for $1 \le i \le n-1$, define

$$\Phi_i(K) = \frac{\kappa_n}{\kappa_{n-i}}\left(\int_{G(n,n-i)} V\,(K|S)^{-n}\, dS\right)^{-1/n},$$

the ith affine quermassintegral of K. Note the similarity to (54); the ordinary mean has been replaced by the -1- and $-n$-means, respectively. As its name suggests, $\Phi_i(K)$ is invariant under volume-preserving affine transformations. Hadwiger [79, p. 268] proved the following inequality.
Theorem (3.1.38) [78]: (Hadwiger's inequality for harmonic quermassintegrals.) If K and L are convex bodies in $\mathbb{R}^n$ and $0 \le i \le n-1$, then

$$\widehat{W}_i(K+L)^{1/(n-i)} \ge \widehat{W}_i(K)^{1/(n-i)} + \widehat{W}_i(L)^{1/(n-i)}.$$

Lutwak [97] showed that the same inequality holds for affine quermassintegrals.
Theorem (3.1.39) [78]: (Lutwak's inequality for affine quermassintegrals.) If K and L are convex bodies in $\mathbb{R}^n$ and $0 \le i \le n-1$, then

$$\Phi_i(K+L)^{1/(n-i)} \ge \Phi_i(K)^{1/(n-i)} + \Phi_i(L)^{1/(n-i)}. \quad (56)$$

Let K be a convex body in $\mathbb{R}^n, n \ge 3$. The capacity $Cap(K)$ of K is defined by

$$Cap(K) = inf\left\{\int_{\mathbb{R}^n} \|\nabla f\|^2 dx : f \in C_c^\infty(\mathbb{R}^n), f \ge 1_K\right\},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Here we are following Evans and Gariepy [57, p. 147], where $Cap(K) = Cap_{n-2}(K)$ in their notation.
Several definitions are possible; see [79] and [111, pp. 110-116]. The notion of capacity has its roots in electrostatics and is fundamental in potential theory. Note that capacity is an outer

measure but is not a Borel measure, though it enjoys some convenient properties listed in [97, p. 151].
Borell [99] proved the following theorem.

Theorem (3.1.40) [78]: (Borell's inequality for capacity.) If K and L are convex bodies in $\mathbb{R}^n, n \geq 3$, then

$$Cap(K+L)^{1/(n-2)} \geq Cap(K)^{1/(n-2)} + Cap(L)^{1/(n-2)}. \qquad (57)$$

Caffarelli, Jerison, and Lieb [39] showed that equality holds if and only if K and L are homothetic. Jerison [79] employed the inequality and its equality conditions in solving the corresponding Minkowski problem.

If K and L are convex bodies in $\mathbb{R}^n$, then there is a convex body $K \dotplus L$ such that

$$S(K \dotplus L,\cdot) = S(K,\cdot) + S(L,\cdot),$$

where $S(K,\cdot)$ denotes the surface area measure of K. This is a consequence of Minkowski's existence theorem; see [97, Theorem A.3.2] or [104, Section 7.1]. The operation $\dotplus$ is called Blaschke addition.

Theorem (3.1.41) [78]: (Kneser-Suss inequality.) If K and L are convex bodies in $\mathbb{R}^n$, then

$$V\,(K \dotplus L)^{(n-1)/n} \geq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \qquad (58)$$

with equality if and only if K and L are homothetic.

See [104, Theorem 7.1.3] for a proof.

Using Blaschke addition, a convex body called a mixed body can be defined from $(n-1)$ other convex bodies in $\mathbb{R}^n$. Lutwak [98, Theorem 4.2] exploits this idea, due to Blaschke and Firey, to produce another strengthening of the Brunn-Minkowski inequality for convex bodies.

For convex bodies K and L in $\mathbb{R}^n$, Minkowski addition can be defined by

$$h_{K+L}(u) = h_K(u) + h_L(u),$$

for $u \in S^{n-1}$, where h_K denotes the support function of K. If $p \geq 1$ and K and L contain the origin in their interiors, a convex body $K +_p L$ can be defined by

$$h_{K+_pL}(u)^p = h_K(u)^p + h_L(u)^p,$$

for $u \in S^{n-1}$. The operation $+_p$ is called p-Minkowski addition. Firey [60] proved the following inequality. (Both the definition of p-Minkowski addition and the case $i = 0$ of Firey's inequality are extended to nonconvex sets by Lutwak, Yang, and Zhang [105].)

Theorem (3.1.42) [78]: (Firey's inequality.) If K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, $0 \leq i \leq n-1$ and $p \leq 1$, then

$$W_i(K+_pL)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \qquad (59)$$

with equality when $p > 1$ if and only if K and L are equivalent by dilatation.

The Brunn-Minkowski inequality for quermassintegrals (55) is the case $p = 1$. Note that translation invariance is lost for $p > 1$.

Firey's ideas were transformed into a remarkable extension of the Brunn-Minkowski theory by Lutwak [101], [104], who also calls it the Brunn-Minkowski-Firey theory. Lutwak found the appropriate p-analog $S_p(K,\cdot), p \geq 1$, of the surface area measure of a convex body K in $\mathbb{R}^n$ containing the origin in its interior. In [101], Lutwak generalized Firey's inequality (59).

He also generalized Minkowski's existence theorem, deduced the existence of a convex body $K \dot{+}_p L$ for which

$$S_p(K \dot{+}_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot)$$

(when K and L are origin-symmetric convex bodies), and proved the following result.

Theorem (3.1.43) [78]: (Lutwak's p-surface area measure inequality.) If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ and $n \neq p \geq 1$, then

$$V(K \dot{+}_p L)^{(n-p)/n} \geq V(K)^{(n-p)/n} + V(L)^{(n-p)/n},$$

with equality when $p > 1$ if and only if K and L are equivalent by dilatation.

Note that the Kneser-Suss inequality (58) corresponds to $p = 1$.

Lutwak, Yang, and Zhang [107] study the L^p version of the Minkowski problem . A version corresponding to $p = 0$ is treated by Stancu [109].

Let χ be a random set in $\mathbb{R}^n$, that is, a Borel measurable map from a probability space Ω to the space of nonempty compact sets in $\mathbb{R}^n$ with the Hausdorff metric. A random vector $X: \Omega \to \mathbb{R}^n$ is called a selection of χ if $Prob\ (X \in \chi) = 1$. If C is a nonempty compact set in $\mathbb{R}^n$, let $\|C\| = \max\{\|x\|: x \in C\}$. Then the expectation $E\chi$ of X is defined by

$$E\chi = \{EX : X\ is\ a\ selection\ of\ \chi\ and\ E\|X\| < \infty\}.$$

It turns out that if $E\|\chi\| < \infty$, then $E\chi$ is a nonempty compact set.

Theorem (3.1.44) [78]: (Vitale's random Brunn-Minkowski inequality.) Let χ be a random set in $\mathbb{R}^n$ with $E\|\chi\| < \infty$. Then

$$V_n(E\chi)^{1/n} \geq EV_n(\chi)^{1/n}. \quad (60)$$

See [108] (and [109] for a stronger version). By taking χ to be a random set that realizes values (nonempty compact sets) K and L with probabilities $(1 - \lambda)$ and λ¸ respectively, we see that Theorem (3.1.44) generalizes the Brunn-Minkowski inequality for compact sets.

A version of (60) for intrinsic volumes (weighted quermassintegrals) of random convex bodies, and applications to stationary random hyperplane processes, are given by Mecke and Schwella [107].

Earlier integral forms of the Brunn-Minkowski inequality, using a Riemann approach to pass from a Minkowski sum to a "Minkowski integral," were formulated by A. Dinghas;

$$V(K+L)^{1/n} \geq m^{1/n} + \left(\frac{V(K)V(L)}{m}\right)^{1/n}. \quad (61)$$

He shows that (61) implies the Brunn-Minkowski inequality for convex bodies and proves that it holds in some special cases.

A wide variety of fascinating inequalities lie (for the present) one step removed from the Brunn-Minkowski inequality. The survey [114] of Osserman indicates connections between the isoperimetric inequality and inequalities of Bonnesen, Poincare, and Wirtinger, and since then many other inequalities have been found that lie in a complicated web around the Brunn-Minkowski inequality.

Some of these related inequalities are affine inequalities in the sense that they are unchanged under a volume-preserving linear transformation. The Brunn-Minkowski and Prekopa-Leindler inequalities are clearly affine inequalities. Young's inequality and its reverse are affine inequalities, since if $\phi \in SL(n)$, we have

$$\phi(f * g) = (\phi f) * (\phi g) and \; \|\phi f\|_p = \|f\|_p.$$

The Brascamp-Lieb and Barthe inequalities are also affine inequalities.

The sharp Hardy-Littlewood-Sobolev inequality (39) is not affine invariant, but it is invariant under conformal transformations; see [91, Theorem 4.5]. The isoperimetric inequality is also not an affine inequality (if it were, the equality for balls would imply that equality also held for ellipsoids), and neither is the Sobolev inequality (24).

There is a remarkable affine inequality that is much stronger than the isoperimetric inequality for convex bodies. The Petty projection inequality states that

$$V(K)^{n-1} V(\Pi * K) \leq \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n, \qquad (62)$$

where K is a convex body in $\mathbb{R}^n$, and $\Pi * K$ denotes the polar body of the projection body ΠK of K. (The support function of ΠK at $u \in S^{n-1}$ equals $V(K|u^{\perp})$.) Equality holds if and only if K is an ellipsoid. See [67, Chapter 9] for background information, a proof, several other related inequalities, and a reverse form due to Zhang. Zhang [102] has also recently found an astounding affine Sobolev inequality, a common generalization of the Sobolev inequality (24) and the Petty projection inequality (62): If $f \in C^1(\mathbb{R}^n)$ has compact support, then

$$\left(\int_{S^{n-1}} \|D_u f\|_1^{-n} du\right)^{-1/n} \geq \frac{2\kappa_{n-1}}{n^{1/n}\kappa_n} \|f\|_{n/(n-1)}, \qquad (63)$$

where $D_u f$ is the directional derivative of f in the direction u.

This is only a taste of a banquet of known affine isoperimetric inequalities. Lutwak [103] wrote an excellent survey. For still more recent progress, can do no better than consult the work of Lutwak, Yang, and Zhang, for example, [109] and [110].

Let X and Y be measurable sets in $\mathbb{R}^n$, and let E be a measurable subset of $X \times Y$. Define the restricted Minkowski sum of X and Y by

$$X +_E Y = \{x + y : (x, y) \in E\}.$$

Theorem (3.1.45) [78]: (Restricted Brunn-Minkowski inequality.) There is a $c > 0$ such that if X and Y are nonempty measurable subsets of $\mathbb{R}^n, 0 < t < 1$,

$$t \leq \left(\frac{V_n(X)}{V_n(Y)}\right)^{1/n} \leq \frac{1}{t}, and \; \frac{V_n(E)}{V_n(X)V_n(Y)} \geq 1 - c\min\{t\sqrt{n}, 1\},$$

then

$$V_n(X +_E Y)^{2/n} \geq V_n(X)^{2/n} + V_n(Y)^{2/n}.$$

Szarek and Voiculescu [112] proved Theorem (3.1.45) in the course of establishing an analog of the entropy power inequality in Voiculescu's free probability theory. (Voiculescu has also found analogs of Fisher information within this noncommutative probability theory with applications to physics.) Barthe [109] also gives a proof via restricted versions of Young's inequality and the Prekopa-Leindler inequality.

At first such an inequality seems impossible, since if K and L are convex bodies in $\mathbb{R}^n$ of volume 1, the volume of $K + L$ can be arbitrarily large. As with the reverse isoperimetric inequality (45), however, linear transformations come to the rescue.

Theorem (3.1.46) [78]: (Milman's reverse Brunn-Minkowski inequality.) There is a constant c independent of n such that if K and L are centrally symmetric convex bodies in $\mathbb{R}^n$, there are volume-preserving linear transformations ϕ and ψ for which

$$V(\phi K + \psi L)^{1/n} \leq c\big(V(\phi K)^{1/n} + V(\psi L)^{1/n}\big). \quad (64)$$

First proved by V. Milman in 1986, this result is important in the local theory of Banach spaces. See [92, Section 4.3] and [127, Chapter 7]. The Cauchy-Davenport theorem, proved by Cauchy in 1813 and rediscovered by Davenport in 1935, states that if p is prime and X and Y are nonempty finite subsets of $\mathbb{Z}/p\mathbb{Z}$, then

$$|X+Y| \geq \min\{p, |X|+|Y|-1\}.$$

Here $|X|$ is the cardinality of X. Many generalizations of this result, including Kneser's extension to Abelian groups, are surveyed in [102]. The lower bound for a vector sum is in the spirit of the Brunn-Minkowski inequality. We now describe a closer analog.

Let Y be a finite subset of $\mathbb{Z}^n$ with $|Y| \geq n+1$. For $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, let

$$w_Y(x) = \frac{x_1}{|Y|-n} + \sum_{i=2}^{n} x_i \,.$$

Define the Y -order on $\mathbb{Z}^n$ by setting $x <_Y y$ if either $w_Y(x) < w_Y(y)$ or $w_Y(x) = w_Y(y)$ and for some j we have $x_j > y_j$ and $x_i = y_i$ for all $i < j$. For $m \in \mathbb{N}$, let D_m^Y be the union of the first m points in $\mathbb{Z}_+^n$ (the points in $\mathbb{Z}^n$ with nonnegative coordinates) in the Y -order. The set D_m^Y is called a Y -initial segment. The points of $D_{|Y|}^Y$ are

$$o <_Y e_1 <_Y 2e_1 <_Y \cdots <_Y (|Y|-n)e_1 <_Y e_2 <_Y \cdots <_Y e_n,$$

where $e_1, \dots, e_n$ is the standard orthonormal basis for $\mathbb{R}^n$. Note that the convex hull of $D_{|Y|}^Y$ is a simplex. Roughly speaking, Y -initial segments are as close as possible to being the set of points in $\mathbb{Z}_+^n$ that are contained in a dilatate of this simplex.

Theorem (3.1.47) [78]: (Brunn-Minkowski inequality for the integer lattice.) Let X and Y be finite subsets of $\mathbb{Z}^n$ with $dim\, Y = n$. Then

$$|X+Y| \geq \left|D_{|X|}^Y + D_{|Y|}^Y\right|. \quad (65)$$

See [68], and also [26] for a similar result in finite subgrids of $\mathbb{Z}^n$. That (65) is indeed a Brunn-Minkowski-type inequality is clear by comparing

$$V(K+L) \geq V(r_K B + r_L B),$$

the consequence of (17) given above. Indeed, (65) is proved by means of a discrete version, called compression, of an anti-symmetrization process related to Steiner symmetrization.

Let M be a body in $\mathbb{R}^n$ containing the origin in its interior and star-shaped with respect to the origin. The radial function of M is defined by

$$\rho_M(u) = \max\{c : cu \in M\},$$

for $u \in S^{n-1}$. Call M a star body if ρ_M is positive and continuous on S^{n-1}.

Let M and N be star bodies in $\mathbb{R}^n$, let $p \neq 0$, and define a star body $M \tilde{+}_p N$ by

$$\rho_{M\tilde{+}_p N}(u)^p = \rho_M(u)^p + \rho_N(u)^p.$$

The operation $\tilde{+}_p$ is called p-radial addition.

Theorem (3.1.48) [78]: (p-dual Brunn-Minkowski inequality.) If M and N are star bodies in $\mathbb{R}^n$, and $0 < p \le n$, then

$$V(M\widetilde{+}_p N)^{p/n} \le V(M)^{p/n} + V(N)^{p/n}. \qquad (66)$$

The reverse inequality holds when $p > n$ or when $p < 0$. Equality holds when $p \neq n$ if and only if M and N are equivalent by dilatation.

The inequality (66) follows from the polar coordinate formula for volume and Minkowski's integral inequality (see [97, Section 6.13]). It was found by Firey [99] for convex bodies and $p \le -1$. The general inequality forms part of Lutwak's highly successful dual Brunn-Minkowski theory, in which the intersections of star bodies with subspaces replace the projections of convex bodies onto subspaces in the classical theory; see, for example, [97]. The cases $p = 1$ and $p = n - 1$ are called the dual Brunn-Minkowski inequality and dual Kneser-Suss inequality, respectively. A renormalized version of the case $p = n + 1$ of (66) was used by Lutwak [100] in his work on centroid bodies (see also [97, Section 9.1]).

There is an inequality equivalent to the dual Brunn-Minkowski inequality called the dual Minkowski inequality, the analog of Minkowski's first inequality (20); see [97, p. 373]. This plays a role in the solution of the Busemann-Petty problem (the analog of Shephard's problem mentioned after Theorem (3.1.8)): If the intersection of an origin-symmetric convex body with any given hyperplane containing the origin is always smaller in volume than that of another such body, is its volume also smaller? The answer is no in general in five or more dimensions, but yes in less than five dimensions.

Lutwak [95] also discovered that integrals over S^{n-1} of products of radial functions behave like mixed volumes, and called them dual mixed volumes. He showed that a suitable version of Holder's inequality in S^{n-1} then becomes a dual form of the Aleksandrov-Fenchel inequality (51), in which mixed volumes are replaced by dual mixed volumes (and the inequality is reversed). Special cases of dual mixed volumes analogous to the quermassintegrals are called dual quermassintegrals, and it can be shown that an expression similar to (54) holds for these; instead of averaging volumes of projections, this involves averaging volumes of intersections with subspaces. Dual affine quermassintegrals can also be defined (see [97, p. 332]), but apparently an inequality for these corresponding to (56) is not known.

Let S be an $(n-2)$-dimensional subspace of $\mathbb{R}^n$, let $u \in S^{n-1} \cap S^{\perp}$, and let S_u denote the closed $(n-1)$-dimensional half-subspace containing u and with S as boundary. Let $u, v \in S^{n-1} \cap S^{\perp}$, and let X and Y be subsets of S_u and S_v, respectively. If $0 < \lambda < 1$, let $u(\lambda)$ be the unit vector in the direction $(1-\lambda)u + \lambda v$, and let $(1-\lambda)X +_h \lambda Y$ be the set of points in $S_{u(\lambda)}$ lying on a line segment with one endpoint in X and the other in Y . We call the operation $+_h$ harmonic addition.

Theorem (3.1.48) [78]: (Busemann-Barthel-Franz inequality.) In the notation introduced above, let X and Y be compact subsets of S_u and S_v, respectively, of positive V_{n-1} −measure. If $0 < \lambda < 1$, then

$$\frac{V_{n-1}\,((1-\lambda)X +_h \lambda Y)}{\|u(\lambda)\|} \ge M_{-1}(V_{n-1}(X), V_{n-1}(Y), \lambda). \qquad (67)$$

Though Theorem (3.1.49) looks strange, it has the following nice geometrical consequence called Busemann's theorem. If K is a convex body in $\mathbb{R}^n$ containing the origin in its interior and S is an $(n-2)$-dimensional subspace, the curve $r = r(\theta)$ in $S^\perp$ such that $r(\theta)$ is the $(n-1)$-dimensional volume of the intersection of K with the half-space S_θ forms the boundary of a convex body in $S^\perp$. Proved in this form by H. Busemann in 1949 and motivated by his theory of area in Finsler spaces, it is also important in geometric tomography (see [97, Theorem 8.1.10]). As stated, Theorem (3.1.49) and precise equality conditions were proved by W. Barthel and G. Franz in 1961; see [97, Note 8.1] Milman and Pajor [119, Theorem 3.9] found a proof of Busemann's theorem similar to the second proof of Theorem (3.1.2) given above. Generalizations along the lines of Theorem (3.1.16) are possible, such as the following (stated and proved in [105, p. 9]).

Theorem (3.1.50) [78]: Let $0 < \lambda < 1$, let $p > 0$, and let $f, g,$ and h be nonnegative integrable functions on $[0, \lambda)$ satisfying

$$h\left(M_{-p}(x, y, \lambda)\right) \geq f(x)^{\frac{(1-\lambda)y^p}{(1-\lambda)y^p+\lambda x^p}} g(y)^{\frac{\lambda x^p}{(1-\lambda)y^p+\lambda x^p}}, \tag{68}$$

for all nonnegative $x, y \in \mathbb{R}$. Then

$$\int_0^\infty h(x)dx \geq M_{-p}\left(\int_0^\infty f(x)dx\,, \int_0^\infty g(x)dx\,, \lambda\right).$$

The previous inequality is very closely related to one found earlier by Ball [108]. For other associated inequalities, see [90, Theorem 4.1] and [118, Lemma 1].

Let X be a measurable subset of $\mathbb{R}^n$ and let r_X be the radius of a ball of the same volume as X. If $\varepsilon > 0$, the Brunn-Minkowski inequality (16) implies that

$$V_n(X + \varepsilon B) \geq \left(V_n(X)^{1/n} + \varepsilon V_n(B)^{1/n}\right)^n = \left(V_n(r_X B)^{1/n} + \varepsilon V_n(B)^{1/n}\right)^n = V_n(r_X B + \varepsilon B). \tag{69}$$

For any set A, write

$$A_\varepsilon = A + \varepsilon B = \{x : d(x, A) \leq \varepsilon\}. \tag{70}$$

Then we can rewrite (69) as

$$V_n(X\varepsilon) \geq V_n\left((r_X B)\varepsilon\right). \tag{71}$$

Notice that (71), by virtue of (70), is now free of the addition and involves only a measure and a metric.

With the appropriate measure and metric replacing V_n and the Euclidean metric, (71) remains true in the sphere S^{n-1} and hyperbolic space, equality holding if and only if X is a ball of radius r_X. (Of course, in these spaces, the ball $B(x, r)$ centered at x and with radius $r > 0$ is the set of all points whose distance from x is at most r. In S^{n-1}, balls are just spherical caps.)

Though in $\mathbb{R}^n$ (71) is only a special case of (16), in S^{n-1} and hyperbolic Perhaps more significant than (71) for recent developments is a surprising result that holds in $S^{n-1}, n \geq 3$, with the chordal metric. It can be shown that if $V_{n-1}(X)/V_{n-1}(B) \geq 1/2$ and $0 < \varepsilon < 1$, then

$$\frac{V_{n-1}(X_\varepsilon)}{V_{n-1}(B)} \geq 1 - \left(\frac{\pi}{8}\right)^{1/2} e^{-\frac{(n-2)\varepsilon^2}{2}}. \tag{72}$$

Results of the form (72) are called approximate isoperimetric inequalities, and can be derived from the general Brunn-Minkowski inequality in $\mathbb{R}^n$, as in [84, Theorem 2]. In particular, by taking X to be a hemisphere, we see that for large n, almost all the measure is concentrated near the equator! This result, which again goes back to P. Levy, is proved in

$$d\gamma_n(x) = (2\pi)^{-n/2} e^{-\|x\|^2/2}\, dx.$$

Indeed, for bounded Lebesgue measurable sets X and Y in $\mathbb{R}^n$ for which $(1-\lambda)X+\lambda Y$ is Lebesgue measurable, we have the inequality

$$\gamma_n\big((1-\lambda)X+\lambda Y\big) \geq \gamma_n(X)^{1-\lambda}\gamma_n(Y)^{\lambda} \qquad (73)$$

corresponding to (14). This follows from the Prekopa-Leindler inequality (because the

$$\Phi^{-1}\Big(\gamma_n\big((1-\lambda)K+\lambda L\big)\Big) \geq (1-\lambda)\Phi^{-1}(\gamma_n(K)) + \lambda\Phi^{-1}(\gamma_n(L)) \qquad (74)$$

While (74) is stronger than (73) for convex bodies, it is unknown whether it holds for Borel sets; see [84] and [86, Problem 1]. An approximate isoperimetric inequality similar to (72) also holds in Gauss space; Maurey [112] (see also see [113, Theorem 8.1]) found a simple proof employing the Prekopa-Leindler inequality. As in S^{n-1}, there is a concentration of measure in Gauss space, this time in spherical shells of thickness approximately 1 and radius approximately $\sqrt{n}$. Closelyrelated work on logarithmic Sobolev inequalities is outlined .

Bahn and Ehrlich [115] find an inequality that can be interpreted as a reversed form of the Brunn-Minkowski inequality in Minkowski spacetime, that is, $\mathbb{R}^{n+1}$ with a scalar product of index 1.

Cordero-Erausquin [111] utilizes results of R. McCann to prove a version of the Prekopa-Leindler inequality on the sphere, remarking that a similar version can be obtained for hyperbolic space. These results are generalized in a remarkable [82] by Cordero-Erausquin, McCann, and Schmuckenschlager, who establish a beautiful Riemannian version of Theorem (3.1.16).

A crystal in contact with its melt (or a liquid in contact with its vapor) is modeled by a bounded Borel subset M of $\mathbb{R}^n$ of finite surface area and fixed volume. (We shall ignore measure-theoretic subtleties in this description.) The surface energy is given by

$$F(M) = \int_{\partial M} f(u_x)\,dx,$$

where u_x is the outer unit normal to M at x and f is a nonnegative function on S^{n-1} representing the surface tension, assumed known by experiment or theory. By the Gibbs-Curie principle, the equilibrium shape of such a crystal minimizes this surface energy among all sets of the same volume. This shape is called the Wulff shape. For a soapy liquid drop in air, f is a constant (we are neglecting external potentials such as gravity) and the Wulff shape is a ball, by the isoperimetric inequality. For crystals, however, f will generally reflect certain preferred directions. In 1901, Wulff gave a construction of the Wulff shape W:

$$W = \cap_{u\in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq f(u)\},$$

each set in the intersection is a half-space containing the origin with bounding hyperplane orthogonal to u and containing the point $f(u)u$ at distance $f(u)$ from the origin. The Brunn-Minkowski inequality can be used to prove that, up to translation, W is the unique shape among all with the same volume for which F is minimum; see, for example, [113, Theorem

1.1]. This was done first by A. Dinghas in 1943 for convex polygons and polyhedra and then by various people in greater generality. In particular, Busemann [118] solved the problem when f is continuous, and Fonseca [62] and Fonseca and Muller [113] extend the results to include sets M of finite perimeter in $\mathbb{R}^n$. Good introductions are provided by Taylor [113] and McCann [115].

In fact, McCann [115] also proves more general results that incorporate a convex external potential, by a technique developed [114] on interacting gases. A gas of particles in $\mathbb{R}^n$ is modeled by a nonnegative mass density $\rho(x)$ of total integral 1, that is, a probability density on $\mathbb{R}^n$, or, equivalently, by an absolutely continuous probability measure in $\mathbb{R}^n$. To each state corresponds an energy

$$E(\rho) = U(\rho) + \frac{G(\rho)}{2}$$
$$= \int_{\mathbb{R}^n} A(\rho(x))\, dx \ + \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} V(x-y)\, d\rho(x)\ d\rho(y)\,.$$

Here U represents the internal energy with A a convex function defied in terms of the pressure, and $G(\rho)/2$ is the potential energy defined by a strictly convex interaction potential . The problem is that $E(\rho)$ is not generally convex, making it nontrivial to prove the uniqueness of an energy minimizer. McCann gets around this by defining for each pair ρ, ρ' of probability densities on $\mathbb{R}^n$ and $0 < t < 1$ an interpolant probability density ρ_t such that

$$U(\rho_t) \leq (1-t)U(\rho) + tU(\rho') \qquad (75)$$

(and similarly for G and hence for E). McCann calls (75) the displacement convexity of U; ρ_t is not $(1-t)\rho + t\rho'$, but rather is defined in the natural way by means of the Brenier map that transports ρ to ρ' (see the last paragraph). McCann is also able to recover the Brunn-Minkowski inequality from (75) by taking $A(\rho) = -\rho^{(n-1)/n}$ and ρ and ρ' to be the densities corresponding to the uniform probability measures on the two sets.

Next we turn to applications to diffusion equations. Let V be a nonnegative continuous potential defined on a convex domain C in $\mathbb{R}^n$ and consider the diffusion equation

$$\frac{\partial\psi}{\partial t} = \frac{1}{2}\Delta\psi - V(x)\psi(x,t) \qquad (76)$$

with zero Dirichlet boundary condition (i.e., ψ tends to zero as x approaches the boundary of C for each fixed t). Denote by $f(t,x,y)$ the fundamental solution of (76); that is,$\psi(t,x) = f(t,x,y)$ satisfies (76) and its boundary condition, and

$$lim_{t\to 0_+} f(t,x,y) = \delta(x-y).$$

For example, if $V = 0$ and $C = \mathbb{R}^n$, then

$$f(t,x,y) = (2\pi t)^{-n/2} e^{-|x-y|^2/2t}.$$

Brascamp and Lieb [115] proved that if V is convex, then $f(t,x,y)$ is log concave on C^2. This is an application of the Prekopa-Leindler inequality, via Theorem (3.1.20)with $p = 0$; basically, it is shown that f is given as a pointwise limit of convolutions of log concave functions (Gaussians or $exp(-tV(x))$). Borell [30] uses a version of Theorem (3.1.16) to show that the stronger assumption that V is $-1/2$-concave implies

that $tlog(t^n f(t^2, x, y))$ is concave on $\mathbb{R}_+ \times C^2$. In a further study, Borell [112] generalizes all of these results (and the Prekopa-Leindler inequality) by considering potentials $V(\sigma, x)$ that depend also on a parameter σ.
Another rich area of applications surrounds the logarithmic Sobolev inequality proved by Gross [113]:

$$Ent_{\gamma_n}(f) \le \frac{1}{2} I_{\gamma_n}(f), \tag{77}$$

where f is a suitably smooth nonnegative function on $\mathbb{R}^n, \gamma_n$ is the Gauss measure defined in the previous,

$$Ent_{\gamma_n}(f) = \int_{\mathbb{R}^n} f \log f \, d\gamma_n - \left(\int_{\mathbb{R}^n} f \, d\gamma_n\right)\left(\int_{\mathbb{R}^n} log f \gamma_n\right),$$

and

$$I_{\gamma_n}(f) = \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} d\gamma_n \, .$$

Note that $Ent_{\gamma_n}(f)$ and $I_{\gamma_n}(f)$ are essentially the negative entropy $-h_1(f)$ and Fisher information, respectively, of f, defined with respect to Gauss measure. McCann's displacement convexity (75) plays an essential role in very recent work involving several of the above topics. Otto [120] observed that various diffusion equations can be viewed as gradient flows in the space of probability measures with the Wasserstein metric (formally, at least, an infinite-dimensional Riemannian structure). McCann's interpolation using the Brenier map gives the geodesics in this space, and Otto uses the displacement convexity to derive rates of convergence to equilibrium. The same ideas are utilized by Otto and Villani [116], who find a new proof of an inequality of Talagrand for the Wasserstein distance between two probability measures in an n-dimensional Riemannian manifold, and show that Talagrand's inequality is very closely related to the logarithmic Sobolev inequality (77). See also consult Ledoux's survey [85].

Section (3.2): Sharp Sobolev Inequalities

The classical Sobolev inequality in $\mathbb{R}^n, n \ge 3$, indicates that there is a constant $C_n > 0$ such that for all smooth enough (locally Lipschitz) functions $f: \mathbb{R}^n \to \mathbb{R}$ vanishing at infinity,

$$\|f\|_q \le C_n \|\nabla f\|_2 \tag{78}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$. Here $\|f\|_q$ denotes the usual L^q-norm of f with respect to Lebesgue measure on $\mathbb{R}^n$, and, for $p \ge 1$,

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f|^p dx\right)^{1/p}$$

where $|\nabla f|$ is the Euclidean norm of the gradient ∇f of f.
Inequality (78) goes back to Sobolev [131], as a consequence of a Riesz type rearrangement inequality and the Hardy–Littlewood–Sobolev fractional-integral convolution inequality. Other approaches, including the elementary Gagliardo–Nirenberg argument [130,135], are discussed in classical textbooks (cf. e.g. [123] . . .). The best possible constant in the Sobolev

inequality (78) was established independently by Aubin [124] and Talenti [142] in 1976 using symmetrization methods of isoperimetric flavor, together with the study of the one-dimensional extremal problem. Rearrangements arguments have been developed extensively in (cf. [151,129] . . .). The optimal constant C_n is achieved on the extremal functions $f(x) = (\sigma + |x|^2)^{(2-n)/2}, x \in \mathbb{R}^n, \sigma > 0$. Building on early ideas by Rosen [128], Lieb [128] determined the best constant and the extremal functions in dimension 3. According to [129], the result seems to have been known before, at least back to the early sixties, in unpublished notes by Rodemich.

The geometric Brunn–Minkowski inequality, and its isoperimetric consequence, is a well-known argument to reach Sobolev type inequalities. It states that for every non-empty Borel measurable bounded sets A, B in $\mathbb{R}^n$,

$$vol_n(A+B)^{1/n} \geq vol_n(A)^{1/n} + vol_n(B)^{1/n} \tag{79}$$

where $vol_n(\cdot)$ denotes Euclidean volume. The Brunn–Minkowski inequality classically implies the isoperimetric inequality in $\mathbb{R}^n$. Choose namely for B a ball with radius $\varepsilon > 0$ and let then $\varepsilon \to 0$ to get that for any bounded measurable set A in $\mathbb{R}^n$,

$$vol_{n-1}(\partial A) \geq n\omega_n^{1/n}\, vol_n(A)^{(n-1)/n}$$

where $vol_{n-1}(\partial A)$ is understood as the outer-Minkowski content of the boundary of A and ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$. Bymeans of the co-area formula [129,133], the isoperimetric inequality may then be stated equivalently on functions as the L^1-Sobolev inequality

$$\|f\|_q \leq \frac{1}{n\omega_n^{1/n}} \|\nabla f\|_1 \tag{80}$$

where $\frac{1}{q} = 1 - \frac{1}{n}$. Changing $f \geq 0$ into f^r for some suitable r and applying Hölder's inequality yields the L^2- Sobolev inequality (78), however not with its best constant. In the same way, the argument describes the full scale of Sobolev inequalities

$$\|f\|_q \leq C_n(p)\|\nabla f\|_p, \tag{81}$$

$1 \leq p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}, f\colon \mathbb{R}^n \to \mathbb{R}$ smooth and vanishing at infinity. According to Gromov [34], the L^1-case of the Sobolev inequality appears in Brunn's work from 1887.

We show that the Brunn–Minkowski inequality may actually be used to also reach the optimal constants in the Sobolev inequalities (78) and (81). This new approach thus completely bridges the geometric Brunn–Minkowski inequalities and the functional Sobolev inequalities.

Inequality (79) was first proved by Brunn in 1887 for convex sets in dimension 3, then extended byMinkowski (cf. [130]). Lusternik [130] generalized the result in 1935 to arbitrary measurable sets. Lusternik's proof was further analyzed and extended in the works of Hadwiger and Ohmann [24] and Henstock and Macbeath [125] in the fifties. Note in particular that the one-dimensional case is immediate: assume that A and B are non-empty compact sets in $\mathbb{R}$, and after a suitable shift, that $supA = 0 = inf\ B$. Then $A \cap B = \{0\}$ and $A + B \supset A \cup B$.

Starting with the contribution [125], integral inequalities have been developed throughout the last century in the investigation of the geometric Brunn–Minkowski–Lusternik theorem. The idea of the following elementary, but fundamental, lemma goes back to Bonnesen's proof of the Brunn–Minkowski inequality (cf. [130]) and may be found already by Henstock and Macbeath [125]. The result appears in this form independently in the works of Dancs and Uhrin [124] and Das Gupta [125].We enclose a proof for completeness. As a result, the proof below only relies on the one-dimensional Brunn–Minkowski–Lusternik inequality, which is the only basic ingredient in the argument. All the further developments and applications to Sobolev inequalities are consequences of this elementary lemma.

Lemma(3.2.1)[121]: Let $\theta \in [0,1]$ and u, v, w be non-negative measurable functions on $\mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$w(\theta x + (1-\theta)y) \geq \min(u(x), v(y)).$$

Then, if $sup_{x\in\mathbb{R}}\, u(x) = sup_{x\in\mathbb{R}}\, v(x) = 1$,

$$\int w dx \geq \theta \int u dx + (1-\theta) \int v dx\,.$$

Proof: Define, for $t > 0, E_u(t) = \{x \in \mathbb{R};\ u(x) > t\}$ and similarly $E_v(t), E_w(t)$. Since $sup_{x\in\mathbb{R}}\, u(x) = sup_{x\in\mathbb{R}}\, v(x) = 1$, for $0 < t < 1$, both $E_u(t)$ and $E_v(t)$ are non-empty, and $E_w(t) \supset \theta E_u(t) + (1-\theta)E_v(t)$. By the one-dimensional Brunn–Minkowski–Lusternik inequality (79), for every $0 < t < 1$,

$$\lambda(E_w(t)) \geq \theta\lambda(E_u(t)) + (1-\theta)\lambda(E_v(t))$$

where λ denotes Lebesgue measure on $\mathbb{R}$. Hence,

$$\int w dx \geq \int_0^1 \lambda(E_w(t)) dt$$

$$\geq \theta \int_0^1 \lambda(E_u(t)) dt\ + (1-\ \theta) \int_0^1 \lambda E_v(t) dt$$

$$= \theta \int u dx + (1-\theta) \int v dx$$

which is the conclusion.

As discussed in [124], the preceding lemma may be extended tomore general means by elementary changes of variables. For $\alpha \in [-\infty, +\infty]$, denote by $M_\alpha^{(\theta)}(a,b)$ the α-mean of the non-negative numbers a, b with weights $\theta, 1-\theta \in\ [0,1]$ defined as

$$M_\alpha^{(\theta)}(a,b) = (\theta a^\alpha + (1-\theta)b^\alpha)^{1/\alpha}$$

(with the convention that $M_\alpha^{(\theta)}(a,b) = max(a,b)$ if $\alpha = +\infty, M_\alpha^{(\theta)}(a,b) =\ min(a,b)$ if $\alpha = -\infty$ and $M_\alpha^{(\theta)}(a,b) = a^\theta b^{1-\theta}$ if $\alpha = 0$) if $ab > 0$, and $M_\alpha^{(\theta)}(a,b) = 0$ if $ab = 0$.

Note the extension of the usual arithmetic-geometric mean inequality as

$$M_{\alpha_1}^{(\theta)}(a_1,b_1) M_{\alpha_2}^{(\theta)}(a_2,b_2) \geq M_\alpha^{(\theta)}(a_1a_2, b_1b_2) \qquad (82)$$

if $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}, \alpha_1 + \alpha_2 > 0$.

Corollary (3.2.2) [121]: Let $-\infty \leq \alpha \leq\ +\infty, \theta \in [0,1]$ and u, v, w be non-negative measurable functions on $\mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$w(\theta x+(1-\theta)y)\geq M_\alpha^{(\theta)}(u(x),v(y)).$$
Then, if $a=sup_{x\in\mathbb{R}}\,u(x)<\infty, b=sup_{x\in\mathbb{R}}\,v(x)<\infty$,
$$\int wdx\geq M_\alpha^{(\theta)}(a,b)M_1^{(\theta)}\left(\frac{1}{a}\int udx\,,\frac{1}{b}\int vdx\right).$$
The statement still holds if a or $b=+\infty$ with the convention that $0\times\infty=0$.

Proof: Assume first that $-\infty<\alpha<+\infty$. For $\rho=M_\alpha^{(\theta)}(a,b)>0$, set
$$U(x)=\frac{1}{a}u\left(\frac{a^\alpha x}{\rho^\alpha}\right)\qquad\text{and } V(y)=\frac{1}{b}v\left(\frac{b^\alpha y}{\rho^\alpha}\right).$$
Then, if $\eta=\theta a^\alpha/\rho^\alpha(\in\ [0,1])$,
$$w(\eta x+(1-\eta)y)\geq M_\alpha^{(\theta)}(a,b)\,min(U(x),V(y))$$
for all $x,y\in\mathbb{R}$. Since $sup_{x\in\mathbb{R}}U(x)=sup_{x\in\mathbb{R}}V(x)=1$, by the lemma,
$$\int wdx\ \geq M_\alpha^{(\theta)}\,(a,b)\left(\eta\int Udx+(1-\eta)\int Vdx\right)$$
$$=M_\alpha^{(\theta)}\,(a,b)\left(\frac{\theta}{a}\int udx+\frac{1-\theta}{b}\int vdx\right)$$
by definition of η. The cases $\alpha=-\infty$ and $\alpha=+\infty$ may be proved by standard limit considerations. The corollary is thus established.

By the Hölder inequality (82), the preceding corollary implies the more classical Prékopa–Leindler theorem [127,36,37], as well as its generalized form put forward by Borell [128] and Brascamp and Lieb [129], in which the supremum norms of u and v do not appear. Namely, under the assumption of Corollary (3.2.2) and provided that $-1\leq\alpha\leq+\infty$,
$$\int wdx\geq M_\alpha^{(\theta)}(a,b)M_1^{(\theta)}\left(\frac{1}{a}\int udx\,,\frac{1}{b}\int vdx\right)$$
$$\geq M_\beta^{(\theta)}\left(\int udx\,,\int vdx\right)$$
where $\beta=\alpha/(1+\alpha)$.

The preceding generalized Prékopa–Leindler theorem is easily tensorisable in $\mathbb{R}^n$ by induction on the dimension to yield that whenever $-\frac{1}{n}\leq\alpha\leq+\infty,\theta\in[0,1]$ and $u,v,w\colon\mathbb{R}^n\to\mathbb{R}_+$ are measurable such that
$$w(\theta x+(1-\theta)y)\geq M_\alpha^{(\theta)}(u(x),v(y))$$
for all $x,y\in\mathbb{R}^n$, then
$$\int wdx\geq M_\beta^{(\theta)}\left(\int udx\,,\int vdx\right)$$
where $\beta=\alpha/(1+\alpha n)$. Namely, assuming the result in dimension $n-1$, for $x_1,y_1,z_1=\theta x_1+(1-\theta)y_1\in\mathbb{R}$ fixed,
$$\int_{\mathbb{R}^{n-1}}w(z_1,t)dt\geq M_{\alpha/(1+\alpha(n-1))}^{(\theta)}\left(\int_{\mathbb{R}^{n-1}}u(x_1,t)dt\,,\int_{\mathbb{R}^{n-1}}v(y_1,t)dt\right).$$
Since $\alpha\geq-\frac{1}{n}$ implies that $\tilde{\alpha}=\alpha/(1+\alpha(n-1))\geq-1$, the one-dimensional result applied to $\int_{\mathbb{R}^{n-1}}u(x_1,t)dt\,,\int_{\mathbb{R}^{n-1}}v(y_1,t)dt\,,\int_{\mathbb{R}^{n-1}}w(z_1,t)dt$ yields the conclusion since $\tilde{\alpha}/(1+\tilde{\alpha})=\beta$. The case $\alpha=0$ corresponds to the Prékopa–Leindler theorem. When

applied to the characteristic functions $u = \chi_A, v = \chi_B$ of the bounded non-empty sets A, B in $\mathbb{R}^n$ with $\alpha = +\infty$, we immediately recover the Brunn–Minkowski–Lusternik inequality (79).

Most of the proofs of the preceding integral inequalities rely in one way or another on integral parametrizations. They may be proved either first in dimension one together with induction on the dimension as above, or by suitable versions of the parametrizations by multidimensional measure transportation. See [132] for complete accounts on these various approaches and precise historical developments.

As presented in [124], Corollary (3.2.2) may also be turned in dimension n, as a consequence of the generalized Prèkopa –Leindler theorem. The resulting statement will be the essential step in the proof of the sharp Sobolev inequalities. In particular, the possibility to use α up to $-\frac{1}{n-1}$ will turn out to be crucial.

For a non-negative function $f: \mathbb{R}^n \to \mathbb{R}$, and $i = 1, \ldots, n$, set

$$m_i(f) = sup_{x_i \in \mathbb{R}} \int_{\mathbb{R}^{n-1}} f\,(x) dx_1 \,\cdots\, dx_{i-1} dx_{i+1} \,\cdots\, dx_n\,.$$

Corollary (3.2.3) [121]: Let $-\frac{1}{n-1} \le \alpha \le +\infty, \theta \in [0, 1]$ and u, v, w be non-negative measurable functions on $\mathbb{R}^{n-1}$ such that for all $x, y \in \mathbb{R}^n$,

$$w(\theta x + (1-\theta)y) \ge M_\alpha^{(\theta)}(u(x), v(y)).$$

If, for some $i = 1, \ldots, n, m_i(u) = m_i(v) < \infty$, then

$$\int w dx \ge \theta \int u dx + (1-\theta) \int v dx\,.$$

Proof: Apply the generalized Prékopa–Leindler theorem in $\mathbb{R}^{n-1}$ (thus with $-\frac{1}{n-1} \le \alpha \le +\infty$) to the functions $u(x), v(y), w(z)$ with $x_i, y_i, z_i = \theta x_i + (1-\theta) y_i$ fixed, and conclude with the lemma applied to $\tilde{u}(x_i) = \int_{\mathbb{R}^{n-1}} u(x) dx_1 \,\cdots\, dx_{i-1} dx_{i+1} \,\cdots\, dx_n$, $\tilde{v}(y_i)$ and $\widetilde{w}(z_i)$ being defined similarly.

Under the assumption $m_i(u) = m_i(v)$, the conclusion of Corollary (3.2.3) does not depend on α and is thus sharpest for $\alpha = -\frac{1}{n-1}$ (the statement for $-\frac{1}{n-1} < \alpha \le +\infty$ being actually a consequence of this case). Following the proof of Corollary (3.2.2), the complete form of Corollary (3.2.3) actually states that (cf. [124]), for every $i = 1, \ldots, n$,

$$\int w dx \ge M_\beta^{(\theta)}(m_i(u), m_i(v)) M_1^{(\theta)}\left(\frac{1}{m_i(u)} \int u dx\,, \frac{1}{m_i(v)} \int v dx\right)$$

with $\beta = \alpha/(1 + \alpha(n-1))$.

Recently, mass transportation arguments have been developed to simultaneously reach the Brunn–Minkowski–Lusternik inequality and the sharp Sobolev inequalities (cf. [122] [125] . . .). In particular, Cordero-Erausquin et al. [126] provide a complete treatment of the classical Sobolev inequalities with their best constants by this tool (see also [132]). Their approach covers in the same way the family of Gagliardo– Nirenberg inequalities put forward by Del Pino and Dolbeault [136] in the context of non-linear diffusion equations (see also [127]). More precisely, by means of Hölder's inequality, the Sobolev inequality (78) implies the family of so-called Gagliardo–Nirenberg inequalities [135],

$$\|f\|_r \leq C\|\nabla f\|_2^{\lambda}\|f\|_s^{1-\lambda} \tag{83}$$

for some constant $C > 0$ and all smooth enough functions $f: \mathbb{R}^n \to \mathbb{R}$ where $r, s > 0$ and $\frac{1}{r} = \frac{\lambda}{q} + \frac{1-\lambda}{s}$, $\lambda \in [0, 1]$. The optimal constants are not preserved through Hölder's inequality. However, it was shown by Del Pino and Dolbeault [126] that optimal constants and extremal functions may be described for a sub-family of Gagliardo–Nirenberg inequalities, namely the one for which $r = 2(s-1)$ when $r, s > 2$ and $s = 2(r-1)$ when $r, s < 2$. The extremal functions turn out to be of the form $f(x) = (\sigma + |x|^2)^{2/(2-r)}$ in the first case, whereas in the second case they are given by $f(x) = ([\sigma - |x|^2]_+)^{1/(2-r)}$ (being thus compactly supported). The limiting case $r, s \to 2$ gives rise to the logarithmic Sobolev inequality (in its Euclidean formulation) with the Gaussian kernels as extremals.

While mass transport arguments may be offered to directly reach the n-dimensional Prékopa–Leindler theorem (cf. [127] . . .), we do not know if Corollary (3.2.3) admits an n-dimensional optimal transportation proof.

On the other hand, the Prékopa–Leindler theorem was shown in [127], following the early ideas by Maurey [131] (cf. [126]), to imply the logarithmic Sobolev inequality for Gaussian measures [123] which, in its Euclidean version [132], corresponds to the limiting case $r, s \to 2$ in the scale ofGagliardo–Nirenberg inequalities. We demonstrate that the extended Prékopa–Leindler theorem in the form of Corollary (3.2.3) above may be used to prove in a simple direct way the classical Sobolev inequality (78) with sharp constant. The argument only relies on a suitable choice of functions u, v, w. The varying parameter α in Corollary (3.2.3) allows us to cover in the same way precisely the preceding sub-family of Gagliardo–Nirenberg inequalities with optimal constants, justifying thus this particular subset of functional inequalities. As in [133], we may deal as simply with the L^p-versions of the Sobolev and Gagliardo–Nirenberg inequalities (cf. (81)), and even replace the Euclidean norm on $\mathbb{R}^n$ by some arbitrary norm. The extension of the Sobolev inequalities to arbitrary norms on $\mathbb{R}^n$ was known previously [133] by symmetrization methods. With respect to earlier developments (notably the recent [133], which provides a new and complete treatment in this respect), the approach presented here does not provide any type of characterization of extremal functions and their uniqueness, which have to be hinted in the choice of the functions u, v, w.

The presents an outline of the direct proof of the sharp Sobolev inequality (78) from Corollary (3.2.3). We then discuss variations on the basic principle which lead to the sharp Sobolev and Gagliardo–Nirenberg inequalities (81) and (83).

The describes, with standard technical arguments, the rigorous and detailed proof of the Sobolev inequality.

We follow the strategy put forward in [137] (see also [132]) on the basis of Corollary (3.2.3) rather than the more classical Prékopa–Leindler theorem. For $g: \mathbb{R}^n \to \mathbb{R}$ and $t > 0$, recall the infimum-convolution of g with the quadratic cost defined by

$$Q_t g(x) = inf_{y\in\mathbb{R}}\left\{g(y) + \frac{1}{2t}|x-y|^2\right\}, \qquad x \in \mathbb{R}^n$$

(with $Q_0 g = g$). It is a standard fact (cf. e.g. [128] . . .) that, for suitable C^1 functions g,

$$\partial_t Q_t g|_{t=0} = -\frac{1}{2}|\nabla g|^2. \qquad (84)$$

Actually, if g is Lipschitz continuous, the family $\rho = \rho(x,t) = Q_t g(x), t > 0, x \in \mathbb{R}^n$, represents the solution of the Hamilton–Jacobi initial value problem $\partial_t \rho + \frac{1}{2}|\nabla\rho|^2 = 0$ in $\mathbb{R}^n \times (0,\infty), \rho = g$ on $\mathbb{R}^n \times \{t = 0\}$.

For $\sigma > 0$, set

$$v_\sigma(x) = \sigma + \frac{|x|^2}{2}, x \in \mathbb{R}^n.$$

Let $\sigma > 0$ to be determined and let $g: \mathbb{R}^n \to \mathbb{R}_+$ be smooth and such that $m_1(g^{1-n}) < \infty$. In order not to obscure the main idea, we refer for a precise description of the class of functions g that should be considered in order to justify the technical differential arguments freely used below.

By definition of the infimum-convolution operator, we may apply Corollary (3.2.3) with $\alpha = -\frac{1}{n-1}$ to the set of (positive) functions

$$u(x) = g(\theta x)^{1-n},$$
$$v(y) = v_\sigma\left(\sqrt{\theta}\, y\right)^{1-n},$$
$$w(z) = [(1-\theta)\sigma + \theta Q_{1-\theta}\, g(z)]^{1-n}.$$

Note that$m_1(u) = \theta^{1-n} m_1(g^{1-n})$ and $m_1(v) = (\sigma\,\theta)^{(1-n)/2} m_1(v_1^{1-n}) < \infty$.Choose thus $\sigma = \kappa\,\theta > 0$ such that $m_1(u) = m_1(v)$ where $\kappa = \kappa(n,g) = (m_1(v_1^{1-n})/ m_1(g^{1-n}))^{2/(n-1)}$.

Set $s = 1 - \theta \in (0,1)$. Hence, by Corollary (3.2.3), for every $s \in (0,1)$,

$$\int (\kappa s + Q_s g)^{1-n} dx \geq \int g^{1-n} dx + s\,\kappa^{(2-n)/2} \int v_1^{1-n}\, dx\,.$$

Taking the derivative at $s = 0$ yields, by (84),

$$(1-n)\int g^{-n}\left(\kappa - \frac{1}{2}|\nabla g|^2\right) dx \geq \kappa^{\frac{2-n}{2}} \int v_1^{1-n}\, dx. \qquad (85)$$

Set $g = f^{2/(2-n)}$ so that

$$\frac{2}{(n-2)^2}\int |\nabla f|^2 dx \geq \kappa \int f^q dx + \frac{1}{(n-1)\kappa^{(n-2)/2}} \int v_1^{1-n} dx$$

where we recall that $q = 2n/(n-2)$. In particular,

$$\int |\nabla f|^2 dx \geq inf_{\kappa>0} \frac{(n-2)^2}{2}\left(\kappa \int f^q dx + \frac{1}{(n-1)\kappa^{\frac{n-2}{2}}} \int v_1^{1-n} dx\,.\right) \qquad (86)$$

This infimum is precisely $C_n^{-2}\|f\|_q^2$ where C_n is the optimal constant in the Sobolev inequality (78). Actually, if $(x) = v_1(x) = 1 + \frac{|x|^2}{2}$, the preceding argument develops with equalities at each step with $\kappa = \kappa(n,g) = 1$. Moreover, the infimum on the right-hand side of (86) is attained at $\kappa = 1$ if and only if

$$\int f^q dx = \int v_1^{-n} dx = \frac{n-2}{2(n-1)} \int v_1^{1-n} dx$$
which is easily checked by elementary calculus. Thus (86) is an equality in this case and the conclusion follows.

As emphasized, the same proof, with the varying parameter α in Corollary (3.2.3), yields the sub-family of Gagliardo–Nirenberg inequalities recently put forward in [136]. Let us briefly emphasize the modifications in the argument. (It is somewhat surprising that these optimal Gagliardo–Nirenberg inequalities follow from Corollary (3.2.3) with $-\frac{1}{n-1} < \alpha \leq +\infty$which is a consequence of the $\alpha = -\frac{1}{n-1}$ case, whereas they are not direct consequences of the sharp Sobolev inequality.)

For $-\frac{1}{n-1} \leq \alpha < 0$, apply Corollary (3.2.3) to
$$u(x) = g(\theta x)^{1/\alpha},$$
$$v(y) = v_\sigma\left(\sqrt{\theta}\, y\right)^{1/\alpha},$$
$$w(z) = [(1-\theta)\sigma + \theta Q_{1-\theta}\, g(z)]^{1/\alpha}$$
to get that for all $s \in (0,1)$,
$$\int [\kappa s(1-s)^a + (1-s)Q_s g]^{1/\alpha}\, dx$$
$$\geq (1-s)^{1-n} \int g^{1/\alpha} dx + \kappa^c s(1-s)^b \int v_1^{1/\alpha} dx.$$
Here $a > 0, b, c < 0, \kappa > 0$ depending on n and α (and g), are such that $m_1(u) = m_1(v)$ for some suitable choice of σ. Taking the derivative at $s = 0$,
$$\frac{1}{\alpha} \int g^{(1/\alpha)-1} \left(\kappa - g - \frac{1}{2}|\nabla g|^2\right) dx \geq (n-1) \int g^{1/\alpha} dx + \kappa^c \int v_1^{1/\alpha}\, dx.$$
Set $f = g^p, 2p - 2 = \frac{1}{\alpha} - 1$, so that
$$-\frac{1}{2\alpha p^2} \int |\nabla f|^2 dx - \left[(n-1) + \frac{1}{\alpha}\right] \int f^r dx \geq -\frac{\kappa}{\alpha} \int f^s dx + \kappa^c v_1^{1/\alpha} dx$$
where $r = 2(1-\alpha)/(1+\alpha)$ and $s = 2/(1+\alpha)$. Note that $r, s > 2, r = 2(s-1)$. Take the infimum over $\kappa > 0$ on the right-hand side, and rewrite then the inequality by homogeneity to get the Gagliardo–Nirenberg inequality
$$\|f\|_r \leq C \|\nabla f\|_2^\lambda \|f\|_s^{1-\lambda},$$
$\frac{1}{r} = \frac{\lambda}{q} + \frac{1-\lambda}{s}$, with optimal constant C.

To reach the sub-family $r, s < 2, s = 2(r-1)$, work now with $0 < \alpha < +\infty$ and replace v_σ by the compactly supported function $[\sigma - \frac{|x|^2}{2}]+, |x| < \sqrt{2\sigma}$. Actually, only the values $0 < \alpha < 1$ are concerned in the argument. We do not know what type of functional information is contained in the interval $\alpha \geq 1$. The case $\alpha = 0$ leading to the logarithmic Sobolev inequality has been studied in [127] and follows here as a limiting case.

We can work more generally with the L^p-Sobolev inequalities (81), $1 < p < n$, and similarly with the corresponding sub-family of Gagliardo–Nirenberg inequalities. It is also possible to equip $\mathbb{R}^n$ with an arbitrary norm $\|\cdot\|$ instead of the Euclidean one $|\cdot|$, and to consider

$$\|\nabla f\|_p^p = \|\nabla f(x)\|_*^p dx$$

where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$. To these tasks, consider as in [124],

$$Q_t g(x) = inf_{y\in\mathbb{R}^n}\left\{g(y) + t\,V^*\left(\frac{x-y}{t}\right)\right\}, \qquad t > 0, x \in \mathbb{R}^n,$$

where $V^*(x) = \frac{1}{p^*}\,\|x\|^{p^*}$ with p^* is the Hölder conjugate of p, i.e. $(1/p) + (1/p^*) = 1$. Then $\rho = \rho(x,t) = Q_t g(x)$ is the solution of the Hamilton–Jacobi equation $\partial_t \rho + V(\nabla\rho) = 0$ with initial condition g, where $V(x) = \frac{1}{p}\|x\|_*^p$ is the Legendre transform of V^* (cf. [18]). The proof then follows along the same lines as before . The general statement obtained in this way is the following (cf. [124,125]). For $1 < p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}, s < r \leq q, \lambda \in [0,1]$,

$$\|f\|_r \leq C_n(p,r)\|\nabla f\|_p^\lambda\,\|f\|_s^{1-\lambda}$$

with $\frac{1}{r} = \frac{\lambda}{q} + \frac{1-\lambda}{s}, p(s-1) = r(p-1)$ if $r, s > p, p(r-1) = s(p-1)$ if $r, s < p$, and the optimal constant $C_n(r,p)$ is achieved on the extremal functions $(\sigma + \|x\|^{p^*})^{p/(p-r)}, x \in \mathbb{R}^n, \sigma > 0$, in the first case and $([\sigma - \|x\|^{p^*}]_+)^{(p-1)/(p-r)}, x \in \mathbb{R}^n, \sigma > 0$, in the second case. The optimal Sobolev inequality (81) corresponds to the limiting case $\lambda \to 1, r \to q, s \to r$.

We collect the technical details necessary to fully justify the proof of the Sobolev inequality outlined. Although the case $p = 2$ is a bitmore simple, we can actually easily handle in the same way the more general case of $1 < p < n$ and of an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$. The arguments are easily modified so to deal similarly with the Gagliardo–Nirenberg inequalities discussed.

Consider thus on $\mathbb{R}^n$ the Sobolev inequality

$$\|f\|_q \leq C_n(p)\|\nabla f\|_p \tag{87}$$

in the class of all locally Lipschitz functions f vanishing at infinity, with parameters p, q satisfying $1 < p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. The right-hand side in (87) is understood with respect to the given norm $\|\cdot\|$ on $\mathbb{R}^n$. More precisely,

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|_*^p\,dx$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$. We show that the best constant $C_n(p)$ in (87) corresponds to the family of extremal functions

$$f(x) = \left(\sigma + \|x\|^{p^*}\right)^{(p-n)/p}, x \in \mathbb{R}^n, \sigma > 0,$$

where p^* is the conjugate of p.We may assume that the norm $x \longmapsto \|x\|$ is continuously differentiable in the region $x \neq 0$. In this case, $\|\nabla\|x\|\,\|_* = 1$ for all $x \longmapsto 0$, and all the extremal functions belong to the class $C^1(\mathbb{R}^n)$.

The associated infimum-convolution operator is constructed for the cost function

$V^*(x) = \frac{1}{p^*}\|x\|^{p^*}$, that is,

$$Q_t g(x) = inf_{y\in\mathbb{R}^n} g(y) + t\,V^*\left(\frac{x-y}{t}\right), t > 0, x \in \mathbb{R}^n.$$

The dual (Legendre transform) of V^* is $V(x) = sup_{y\in\mathbb{R}^n}\,[\langle x, y\rangle - V^*(y)] = \frac{1}{p}\|x\|_*^p$ (and conversely).

See [128] for general facts about infimum-convolution operators and solutions to Hamilton–Jacobi equations, and only concentrate below on the aspects relevant to the proof of the Sobolev inequality.

What follows is certainly classical.

Lemma (3.2.4) [121]: If a function g on $\mathbb{R}^n$ is bounded from below and is differentiable at the
point $x \in \mathbb{R}^n$, then

$$\lim_{t\to 0}\frac{1}{t}\,[Q_t g(x) - g(x)] = -V\nabla g(x) = -\frac{1}{p}\|\nabla g(x)\|_*^p.$$

Proof: Fix $x \in \mathbb{R}^n$. By Taylor's expansion, $g(x-h) = g(x) - \langle\nabla g(x), h\rangle + |h|\varepsilon(h)$ with $\varepsilon(h) = \varepsilon_x(h) \to 0$ as $|h| \to 0$. Hence, for vectors $h_t = th$ with fixed $h \in \mathbb{R}^n$,

$$\lim_{t\to 0}\frac{1}{t}[g(x-h_t) - g(x)] = -\langle\nabla g(x), h\rangle.$$

Since we always have $Q_t g(x) \le g(x-h_t) + tV^*(h)$,

$$\lim_{t\to 0} sup\frac{1}{t}\,[Q_t g(x) - g(x)] \le \lim_{t\to 0}\frac{1}{t}[g(x-h_t) - g(x)] + V^*(h)$$
$$= -\langle\nabla g(x), h\rangle + V^*(h).$$

The left-hand side of the preceding does not depend on h. Hence, taking the infimum on the right-hand side over all $h \in \mathbb{R}^n$, we get

$$\lim_{t\to 0} sup\frac{1}{t}[Q_t g(x) - g(x)] \le -(V\nabla g(x)).$$

Now, we need an opposite inequality for the liminf. Assume without loss of generality that $g \ge 0$. Since $Q_t g(x) \le g(x)$, it is easy to see that for any $t > 0$,

$$Q_t g(x) = inf_{tV^*(h)\le g(x)}\{g(x-h_t) + tV^*(h)\}.$$

Hence, recalling Taylor's expansion,

$$\frac{1}{t}[Q_t g(x) - g(x)] = inf_{tV^*(h)\le g(x)}\{-\langle\nabla g(x), h\rangle + |h|\varepsilon(th) + V^*(h)\}. \quad (88)$$

Note first that the argument in $\varepsilon(\cdot) = \varepsilon_x(\cdot)$ is small uniformly over all admissible h since, as is immediate,

$$\sup\{t|h|;\ tV^*(h) \le g(x)\} \to 0 \quad as \quad t \to 0.$$

Thus removing the condition $tV^*(h) \le g(x)$ in (88), we get that, given $\eta > 0$, for all t small enough,

$$\frac{1}{t}[Q_t g(x) - g(x)] \ge inf_h\{-\langle\,\nabla g(x), h\rangle - |h|\eta + V^*(h)\}. \qquad (89)$$

Now, to get rid of η on the right-hand side for t approaching zero, note that the infimum in (89) may be restricted to the ball $|h| \le r$ for some large r. Indeed, the left-hand side in (89) is non-positive. But if $|h|$ is large enough and $0 < \eta < 1$, the quantity for which we take the infimum will be positive for $V^*(h \geq C|h| > \langle \nabla g(x), h\rangle + |h|\eta$ with C taken in advance to be as large as we want. Finally, restricting the infimum to $|h| \leq r$, we get that

$$\frac{1}{t}[Q_t g(x) - g(x)] \geq inf_{|h|\leq r}\{-\langle \nabla g(x), h\rangle + V^*(h)\} - r\eta = -V(\nabla g(x)) - r\eta.$$

It remains to take the liminf on the left for $t \to 0$, and then to send η to 0. The proof of Lemma (3.2.4) is complete.

Our next step is to complement the above convergence with a bound on $|Q_t g(x) - g(x)|/t$ in terms of $\|\nabla g(y)\|_*$ with vectors y that are not far from x. So, given a C^1 function g on $\mathbb{R}^n$, for every point $x \in \mathbb{R}^n$ and $r > 0$, define $Dg(x,r) = sup_{\|x-y\|\leq r}\|\nabla g(y)\|_*$. Note that $Dg(x,r) \to \|\nabla g(x)\|_*$ as $r \to 0$. Assume $g \geq 0$ and write once more

$$Q_t g(x) = inf_{h\in\mathbb{R}^n} g(x-h) + \frac{\|h\|^{p^*}}{p^* t p^{*-1}}, \quad t > 0.$$

Again, since $Q_t g(x) \leq g(x)$, the infimum may be restricted to the ball $(\|h\|^{p^*}/p^* t p^{*-1}) \leq g(x)$. Hence, replacing h with th and applying the Taylor formula in integral form, we get that with $r = (p^* g(x))^{1/p^*}$, for any $t > 0$,

$$\begin{aligned}\frac{1}{t}[g(x) - Q_t g(x)] &\leq sup_{t\|h\|\leq r}\left\{\frac{1}{t}[g(x) - g(x - th)] - \left(\|h\|^{p^*}/p^*\right)\right\} \\ &\leq sup_{t\|h\|\leq r}\left\{Dg(x, t\|h\|)\|h\| - \left(\|h\|^{p^*}/p^*\right)\right\} \\ &\leq sup_h\left\{Dg(x,r)\|h\| - \left(\|h\|^{p^*}/p^*\right)\right\} \\ &= \frac{1}{p}Dg(x,r)^p \quad . \qquad (90)\end{aligned}$$

In applications, we need to work with functions $g(x) = O(|x|^{p^*})$ as $|x| \to \infty$. So, let us define the class $F_{p^*}, p^* > 1$, of all C^1 functions g on $\mathbb{R}^n$ such that

$$\lim_{|x|\to\infty} sup \frac{|\nabla g(x)|}{|x|^{p^*-1}} < \infty.$$

If $\in \mathcal{F}_{p^*}$, then, for some $C, |\nabla g(x)| \leq C|x|^{p^*-1}$ as long as $|x|$ is large enough, and hence $|g(x)|^{1/p^*} \leq C'|x|$ for $|x|$ large. It easily follows that $Dg(x, \left(p^* g(x)\right)^{1/p^*}) \leq C''(1 + |x|^{p^*-1})$ for all x.As a consequence of (90), we may conclude that for any $g \geq 0$ in $\mathcal{F}_{p^*}, p^* > 1$, there is a constant $C > 0$ such that

$$sup_{t>0}\frac{1}{t}[g(x) - Q_t g(x)] \leq C\left(1 + |x|^{p^*}\right), \quad x \in \mathbb{R}^n. \qquad (91)$$

We may now start the proof of the Sobolev inequality according to the scheme outlined in Given a parameter $\sigma > 0$, define

$$v_\sigma(x) = \sigma + \frac{\|x\|^{p^*}}{p^*}, x \in \mathbb{R}^n.$$

For a positive C^1 function g on $\mathbb{R}^n$, and $\theta \in (0,1)$, define the three (positive, continuous) functions

$$u(x) = g(\theta x)^{1-n},$$
$$v(y) = v_\sigma(\theta^{1/p^*} y)^{1-n},$$
$$w(z) = [(1-\theta)\sigma + \theta Q_{1-\theta}\, g(z)]^{1-n}.$$

The function w is chosen as the optimal one satisfying

$$w(\theta x + (1 - \theta)y)^{\alpha} \leq \theta u(x)^{\alpha} + (1-\theta)v(y)^{\alpha}$$

for $\alpha = -\frac{1}{n-1}$ and all $x, y \in \mathbb{R}^n$. Assume that

$$m_1(g^{1-n}) = sup_{x_1\in\mathbb{R}} \int_{\mathbb{R}^{n-1}} g(x_1, x_2, \cdots, x_n)^{1-n}\, dx_2 \dots dx_n < \infty.$$

By homogeneity, $m_1(u) = \theta^{1-n} m_1(g^{1-n})$ and $m_1(v) = \theta^{(1-n)/p^*}\sigma^{(1-n)/p}\, m_1(v_1^{1-n})$. Note that $m_1(v_1^{1-n}) < \infty$. Hence, we may choose σ such that $m_1(u) = m_1(v)$, that is,

$$\sigma = \kappa\,\theta, \text{ where } \kappa = \kappa(n,g) = \left(\frac{m_1(v_1^{1-n})}{m_1(g^{1-n})}\right)^{p/(n-1)}.$$

By Corollary (3.2.3) (with $= -\frac{1}{n-1}$), we have $\int w dx \geq \theta \int u dx + (1-\theta)\int v dx$, that is,

$$\int [(1-\theta)\sigma + \theta Q_{1-\theta}\, g(x)]^{1-n} dx \ \geq \theta \int g(\theta x)^{1-n} dx + (1-\theta)\int v_{\sigma}\big(\theta^{1/p^*} x\big)^{1-n} dx\,.$$

After a change of variable in the last two integrals, and since $\sigma = \kappa\,\theta$, we get, setting $s = 1-\theta$,

$$\int (\kappa s + Q_s g)^{1-n} dx \geq \int g^{1-n} dx + s\,\kappa^{\frac{p-n}{p}} \int v_1^{1-n}\, dx. \qquad (92)$$

Inequality (92) holds true for all $0 < s < 1$, and formally there is equality at $s = 0$.
The next step is to compare the derivatives of both sides at this point. To do this, assume $g \in \mathcal{F}_{p^*}$ and

$$g(x) \geq c 1 + \|x\|^{p^*} \qquad (93)$$

for some constant $c > 0$. (Recall that the functions in $\mathcal{F}_{p^*}$ satisfy an opposite bound $g(x) \leq C(1 + \|x\|^{p^*})$ which will not be used.) Due to (93), $Q_s g(x) \geq c'(1 + \|x\|^{p^*})$ (where $c' > 0$ is independent of s). In particular, $m_1(g^{1-n}) < \infty$, and the first and second integrals in (92) are finite and uniformly bounded over all $s \in (0,1)$. Rewrite (92) as

$$\kappa^{(p-n)/p} \int v_1^{1-n}\, dx \leq \int \frac{1}{s}[(\kappa s + Q_s g)^{1-n} - g^{1-n}] dx. \qquad (94)$$

Now we can use a general inequality

$$|a^{1-n} - b^{1-n}| \leq (n-1)|a-b|(a^{-n} + b^{-n}), a, b > 0,$$

to see that, uniformly in s,

$$\frac{1}{s}[(\kappa s + Q_s g)^{1-n} - g^{1-n}] \leq 2(n-1)\left(\kappa + \frac{1}{s}[g - Q_s g]\right)(Q_s g)^{-n} \leq C'\big(1 + \|x\|^{p^*}\big)^{1-n}$$

for some constant $C' > 0$.Onthe last step, we used that$Q_s g(x) \geq c(1 + \|x\|^{p^*})$ together with the bound (91) for functions from the class $\mathcal{F}_{p^*}$. Since the function $(1 + \|x\|^{p^*})^{1-n}$ is integrable (for $p < n$), we can apply the Lebesgue dominated convergence theorem in order to insert the limit $\lim s \to 0$ inside the integral in (94), and to thus get together with Lemma (3.2.4),

$$\kappa^{(p-n)/p} \int v_1^{1-n}\, dx \leq (1-n)\int g^{-n}\left(\kappa - \frac{\|\nabla g\|_*^p}{p}\right) dx\,,$$

or equivalently,

$$\frac{1}{p}\int g^{-n}\|\nabla g\|_*^p dx \geq \kappa \int g^{-n} dx \ + \frac{1}{(n-1)\kappa^{\frac{n-p}{p}}}\int v_1^{1n}\, dx. \tag{95}$$

Now, let us take a non-negative, compactly supported C^1 function f on $\mathbb{R}^n$, and for $\varepsilon > 0$, define C^1 functions

$$g_\varepsilon(x) = (f(x) + \varepsilon\varphi(x))^{p/(p-n)} + \varepsilon\left(1 + \|x\|^{p^*}\right)$$

where $\varphi(x) = (1 + \|x\|^{p^*})^{(p-n)/p}$. Clearly, all g_ε satisfy (93). The first partial derivatives of f are continuous and vanishing for large values of $|x|$. More precisely, $g_\varepsilon(x) = c_\varepsilon(1 + \|x\|^{p^*})$ for $|x|$ large enough, so all g_ε belong to the class $\mathcal{F}_{p^*}$. Thus, we can apply (95) to get

$$\frac{1}{p}\int g_\varepsilon^{-n}\|\nabla g_\varepsilon\|_*^p dx \geq \kappa \int g_\varepsilon^{-n} dx + \frac{1}{(n-1)\kappa^{\frac{n-p}{p}}} \int v_1^{1-n}\, dx. \tag{96}$$

Note that $g_\varepsilon^{-n} \leq (f + \varepsilon\varphi)^q$ and $\int \varphi^q dx < \infty$ (where we recall that $q = pn/(n-p)$). Hence, by the Lebesgue dominated convergence theorem again, $\int g_\varepsilon^{-n} dx$ is convergent, as $\varepsilon \to 0$, to $\int f^q dx$.By a similar argument, recalling that $\left\|\nabla \|x\|^{p^*}\right\|_* = p^*\|x\|^{p^*-1}, x \in \mathbb{R}^n$, we see that there is a finite limit for the left integral in (96). As a result, we arrive at

$$\frac{p^{p-1}}{(n-p)^p}\int \|\nabla f\|_*^p\, dx \geq \kappa \int f^q dx + \frac{1}{(n-1)\kappa^{\frac{n-p}{p}}}\int v_1^{1n} dx, \tag{97}$$

which implies

$$\frac{p^{p-1}}{(n-p)^p}\int \|\nabla f\|_*^p dx \geq inf_{\kappa>0}\kappa \int f^q dx + \frac{1}{(n-1)\kappa^{\frac{n-p}{p}}}\int v_1^{1-n} dx. \tag{98}$$

As we will see with the case of equality below, this is precisely the desired Sobolev inequality (87) with optimal constant. It is now easy to remove the assumption on the compact support of f and thus to extend (98) to all C^1 and furthermore locally Lipschitz functions f (≥ 0) on $\mathbb{R}^n$ vanishing at infinity. To conclude the argument, we investigate the case of equality. To this task, let us return to the beginning of the argument and check the steps where equality holds true. Take $g = v_1$ so that $\kappa = \kappa(n, g) = 1$ and $\sigma = \theta$. In addition, the right-hand side of (92) automatically turns into $(1 + s)v_1^{1-n} dx$. By direct computation,

$$Q_s v_1(x) = 1 + \frac{\|x\|^{p^*}}{p^*(1+s)^{p^*-1}},$$

so the left-hand side of (92) is

$$\int (\kappa s + Q_s g)^{1-n} dx = \int \left((1+s) + \frac{\|x\|^{p^*}}{p^*(1+s)^{p^*-1}}\right)^{1-n} dx$$

$$= (1+s)\int \left(1 + \frac{\|y\|^{p^*}}{p^*}\right)^{1-n} dy$$

$$= (1+s)\int v_1^{1-n} dy$$

where we used the change of the variable $x = (1+s)y$. Thus, for $g = v_1$ there is equality in (92), and hence in (95) and (97) as well.

As for (98), first note that, given parameters $A, B > 0$, the function $A\kappa + B\kappa^{(pn)/p}, \kappa > 0$, attains its minimum on the positive half-axis at $\kappa = 1$ if and only if $A = B(n-p)/p$. In the situation of the particular functions $g = v_1, f^q = g^{-n} = v_1^{-n}$, we have

$$A = \int v_1^{-n}\, dx\,, B = \frac{1}{n-1}\int v_1^{1-n}\, dx\,.$$

Hence, the infimum in (97) is attained at $\kappa = 1$ if and only if

$$\int v_1^{-n} dx = \frac{n-p}{p(n-1)}\int v_1^{1-n} dx\,.$$

But this equality is easily checked by elementary calculus.

We may thus summarize our conclusions. In the class of all locally Lipschitz functions f on $\mathbb{R}^n$, vanishing at infinity and such that $0 < \|f\|_q < \infty$, the quantity

$$\frac{\|\nabla f\|_p}{\|f\|_q},$$

$1 < p < n, \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, is minimized for the functions

$$f(x) = \left(\sigma + \|x\|^{p^*}\right)^{(p-n)/p}, x \in \mathbb{R}^n, \sigma > 0.$$

Here $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\|\cdot\|$ is a given norm on $\mathbb{R}^n$, and

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} \|\nabla f(x)\|_*^p dx$$

where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$.

Chapter 4
Centroid Bodies and Slicing Inequalities with Estimates for Measures

We present some new bounds on the volume of L_p-centroid bodies and yet another equivalent formulation of Bourgain's hyperplane conjecture. The method is a combination of the L_p-centroid body technique of Paouris and the logarithmic Laplace transform technique . We show that if Kis a convex body in R^n with 0 ∈int(K)and μis a measure on Rnwith a locally integrable non-negative density g on R^n.

Section (4.1): Logarithmic Laplace Transform

We combine two recent techniques in the study of volumes of high dimensional convex bodies. The first technique is due to Paouris [176], and it relies on properties of the L_p-centroid bodies. The second technique was developed by [174], and it uses the logarithmic Laplace transform.

Suppose that μ is a Borel probability measure on $\mathbb{R}^n$ endowed with a Euclidean structure $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. We say that μ is a ψ_α-measure ($\alpha > 0$) with constant $b_\alpha$if:

$$\left(\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^p d\mu(x)\right)^{\frac{1}{p}} \le b_\alpha p^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^2 d\mu(x)\right)^{\frac{1}{2}}, \quad \forall p \ge 2, \forall \theta \in \mathbb{R}^n. \quad (1)$$

It is well known that the uniform probability measure μ_K on any convex body $K \subset \mathbb{R}^n$ is a ψ_1-measure with constant C, where $C > 0$ is a universal constant (this follows from Berwald's inequality [173], see also [171]). Here, as usual, a convex body in $\mathbb{R}^n$ means a compact, convex set with a non-empty interior. The isotropic constant L_K of a convex body $K \subset \mathbb{R}^n$ is the following affine invariant parameter:

$$L_K := Vol_n(K)^{-\frac{1}{n}} (det Cov(\mu_K))^{\frac{1}{2n}},$$

where Vol_n denotes the Lebesgue measure and $Cov(\mu_k)$ denotes the covariance matrix of μ_K. The next theorem unifies and slightly improves several known bounds on the isotropic constant.

Theorem (4.1.1)[168]: Let $K \subset \mathbb{R}^n$ denote a convex body whose barycenter lies at the origin, and suppose that μ_K is a ψ_α-measure ($1 \le \alpha \le 2$) with constant b_α. Then:

$$L_K \le C\sqrt{b_\alpha^\alpha n^{1-\alpha/2}},$$

where $C > 0$ is a universal constant.

A central question raised by Bourgain [169] is whether $L_K \le C$ for some universal constant $C > 0$, for any convex body $K \subset \mathbb{R}^n$ (it is well known that $L_K \le c$ for a universal constant $c > 0$). This question is usually referred to as the slicing problem or hyperplane conjecture, see Milman and Pajor [181] for many of its equivalent formulations. Plugging $\alpha = 1$ in Theorem (4.1.1), we match the best known bound on the isotropic constant, which is $L_K \le Cn^{1/4}$ for any convex body $K \subset \mathbb{R}^n$ (see Bourgain [178] and Klartag [174]). In the case $\alpha = 2$, Theorem (4.1.1) yields $L_K \le Cb_2$. This slightly improves upon the previously known bound, which is:

$$L_K \le Cb_2\sqrt{log\, b_2}, \quad (2)$$

due to Dafnis and Paouris [171] in the precise form (2) and to Bourgain [179] (with a different power of the logarithmic factor). Here, as elsewhere, we use the letters $c, \tilde{c}, C, \tilde{C}, \bar{C}$, etc. to denote positive universal constants, whose value may not necessarily be the same in different occurrences.

We proceed by recalling the definition of the L_p-centroid bodies $Z_p(\mu)$, originally introduced by Lutwak and Zhang in [179] (under different normalization), which lie at the heart of Paouris' remarkable work [176]. Given a Borel probability measure μ on $\mathbb{R}^n$ and $p \geq 1$, denote:

$$h_{Z_p(\mu)}(\theta) = \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^p d\mu(x)\right)^{\frac{1}{p}}, \qquad \theta \in \mathbb{R}^n.$$

The function $h_{Z_p(\mu)}$ is a norm on $\mathbb{R}^n$, and it is the supporting functional of a convex body $Z_p(\mu) \subseteq \mathbb{R}^n$ (see e.g. Schneider [171] for information on supporting functionals). Clearly $Z_p(\mu) \subseteq Z_q(\mu)$ for $p \leq q$.

Now suppose that $K \subset \mathbb{R}^n$ is a convex body whose barycenter lies at the origin, and denote $Z_p(K) = Z_p(\mu_K)$, where μ_K is as before the uniform probability measure on K. As realized by Paouris, obtaining volumetric and other information on $Z_p(K)$ is very useful for understanding the volumetric properties of K itself. For instance, note that:

$$V.Rad.(Z_2(K)) = (detCov(\mu_K))^{\frac{1}{2n}}, \tag{3}$$

where the volume-radius of a compact set $T \subset \mathbb{R}^n$ is defined as:

$$V.Rad.(T) = \left(\frac{Vol_n(T)}{Vol_n(B_n)}\right)^{\frac{1}{n}},$$

measuring the radius of the Euclidean ball whose volume equals the volume of T. Here, $B_n = \{x \in \mathbb{R}^n; |x| \leq 1\}$; note that $cn^{-\frac{1}{2}} \leq Vol_n(B_n)^{\frac{1}{n}} \leq Cn^{-\frac{1}{2}}$, as verified by a direct calculation. Furthermore, it is known (e.g. [178, Lemma 3.6]) that:

$$c \cdot Z_\infty(K) \subseteq Z_n(K) \subseteq Z_\infty(K) := conv(K, -K), \tag{4}$$

where $conv(K, -K)$ denotes the convex hull of K and $-K$.

A sharp lower bound on the volume of $Z_p(K)$ due to Lutwak, Yang and Zhang [18] states that ellipsoids minimize $V.Rad.(Z_p(K))/V.Rad.(K)$ among all convex bodies $K \subset \mathbb{R}^n$, for all $p \geq 1$. An elementary calculation yields:

$$V.Rad.\left(Z_p(K)\right) \geq c\sqrt{\frac{p}{n}}V.Rad.(K) for\ 1 \leq p \leq n, \tag{5}$$

which is the best possible bound (up to the value of the constant $c > 0$) in terms of $Vol_n(K)$. However, in view of the slicing problem and (3), one may try to strengthen (5) by replacing its right-hand side by $c\sqrt{p}V.Rad.(Z_2(K))$. The next two theorems are a step in this direction.

It was realized by Ball [172] that many questions regarding the volume of convex bodies are better formulated in the broader class of logarithmically-concave measures. A function $\rho: \mathbb{R}^n \to [0, \infty)$ is called log-concave if $-log\,\rho: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a convex function. A probability measure on $\mathbb{R}^n$ is log-concave if its density is log-concave. For example, the uniform probability measure on a convex body and its marginals are all log-concave measures (see Borell [175] for a characterization).

Theorem (4.1.2) [168]: Let μ be a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin. Let $1 \leq \alpha \leq 2$, and assume that μ is a ψ_α-measure with constant b_α. Then:

$$V.Rad.\left(Z_p(\mu)\right) \geq c\sqrt{p}V.Rad.(Z_2(\mu)),$$

for all $2 \leq p \leq Cn^{\alpha/2}/b_\alpha^\alpha$. Here $c, C > 0$ denote universal constants.

Theorem (4.1.1) follows immediately from Theorem (4.1.2). Indeed, simply observe that for p in the specified range:

$$c\sqrt{p} \leq \frac{V.Rad.(Z_p(K))}{V.Rad.\left(Z_2(K)\right)} \leq \frac{V.Rad.(conv(K,-K))}{V.Rad.(Z_2(K)} \leq C\sqrt{n}\frac{Vol_n(K)^{1/n}}{V.Rad.(Z_2(K))} = \frac{C\sqrt{n}}{L_K},$$

where the last inequality follows from the Rogers–Shephard inequality [180]. This completes the proof of Theorem (4.1.1), reducing it to that of Theorem (4.1.2). We remark here that the proof (of both theorems) only requires that the ψ_α condition (1) holds in an average sense.

Our next theorem contains an additional lower bound on the volume of $Z_p(\mu)$ which complements that of Theorem (4.1.2) in some sense. A Borel probability measure μ on $(\mathbb{R}^n, |\cdot|)$ is called isotropic when its barycenter lies at the origin, and its covariance matrix equals the identity matrix (i.e. $Z_2(\mu) = B_n$). Any measure with finite second moments and full-dimensional support may be brought into isotropic "position" by means of an affine transformation.

Theorem (4.1.3) [168]: Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then:

$$V.Rad.\left(Z_p(\mu)\right) \geq c\sqrt{p},$$

for all $p \geq 2$ for which:

$$diam\left(Z_p(\mu)\right)\sqrt{logp} \leq C\sqrt{n}. \tag{6}$$

Here, $diam\,(T) = sup_{x,y\in T}|x-y|$ stands for the diameter of $T \subset \mathbb{R}^n$, and $c, C > 0$ are universal constants.

Note that the ψ_α-condition (1) is precisely the requirement that $Z_p(\mu) \subseteq b_\alpha p^{\frac{1}{\alpha}} Z_2(\mu)$ for all $p \leq 2$, and so the conclusion of Theorem (4.1.3) agrees with that of Theorem (4.1.2), up to the logarithmic factor in (6). This discrepancy is explained by the fact that in Theorem (4.1.2), we actually make full use of the growth of $diam(Z_p(\mu))$ for all $p \geq 2$, whereas in Theorem (4.1.3) we only assumed this control for the end value of p. We emphasize that this constitutes a genuine difference in assumptions, and that the logarithmic factor in (6) is not just a technical artifact of the proof: we show that removing this logarithmic factor is actually equivalent to Bourgain's original hyperplane conjecture.

We find condition (6) quite interesting from other respects as well. It is very much related to Paouris' parameter $q^*(\mu)$, to be discussed. In fact, we show there that the parameter:

$$q^\#(\mu) := \sup\{q \geq 1; diam\left(Z_q(\mu)\right) \leq c^\#\sqrt{n}\left(detCov(\mu)\right)^{\frac{1}{2n}}\},$$

for a small-enough universal constant $c^{\#} > 0$, is essentially equivalent to and has the same functionality as Paouris' $q^*(\mu)$ parameter, in addition to being rather convenient to work with. The lower bounds in Theorem (4.1.2) and Theorem (4.1.3) compare with the matching upper bounds on $V.Rad.(Z_p(\mu))$, obtained by Paouris [186, Theorem 6.2], which are valid for all $2 \le p \le n$:

$$V.Rad.\big(Z_p(\mu)\big) \le C\sqrt{p}V.Rad.\big(Z_2(\mu)\big). \qquad (7)$$

This implies that the lower bounds in both theorems above are sharp, up to constants, and so the only pertinent question is the optimality of the range of p's for which their conclusion is valid. In this direction, Paouris obtained a partial converse to (7) in the following range of p's:

$$W\big(Z_p(\mu)\big) \ge c\sqrt{p}V.Rad.\big(Z_2(\mu)\big), \quad \forall 2 \le p \le q^{\#}(\mu). \qquad (8)$$

Here $W(K) = \int_{S^{n-1}} h_K(\theta)d\sigma(\theta)$ denotes half the mean width of K, σ is the Haar probability measure on the Euclidean unit sphere S^{n-1}, and $h_K(\theta) = sup_{x\in K}\langle x, \theta\rangle$ is the supporting functional of K. Note that according to the Urysohn inequality, $W(K) \ge V.Rad.(K)$ (see e.g. [172]), and so Theorem (4.1.3) should be thought of as a formal strengthening of (8), if it were not for the logarithmic factor in (6).

We deduce a new formula for $V.Rad.(Z_p(\mu))$ involving the "tilts" of the measure μ from [174,175], and we relate between the Z_p-bodies of the original measure and its tilts. we deviate from our discussion to review Paouris' q^*-parameter, and compare it with $q^{\#}$; may be read independently. we use projections and the $q^{\#}$-parameter to relate between the determinant of the covariance matrix of μ and its tilts, and conclude the proofs of Theorems (4.1.2) (in fact, a more general version) and (4.1.3)., we show that removing the log-factor from Theorem (4.1.3) is equivalent to the slicing problem.

Given $1 \le k \le n$, the Grassmann manifold of all k-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $G_{n,k}$. Given $E \in Gn, k$, the orthogonal projection onto E is denoted by $Proj_E$, and given a Borel probability measure μ on $\mathbb{R}^n$, we denote by $\pi_{E\mu} := \ (Proj_E)_*(\mu)$ the push-forward of μ via $Proj_E$. For a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, its polar body is denoted by:

$$K^{\circ} = \{x \in \mathbb{R}^n;\ \langle x, y\rangle \le 1, \forall y \ \in\ K\}.$$

Finally, we denote by ∇ and Hess the gradient and Hessian, respectively, of a sufficiently differentiable function.

Throughout, $x \simeq y$ is an abbreviation for $cx \le y \le Cx$ for universal constants $c, C\ >\ 0$. Similarly, we write $x \lesssim y (x \gtrsim y)$ when $x \le Cy$ $(x \ge cy)$. Additionally, for two convex sets $K, T \subset \mathbb{R}^n$ we write $K \simeq T$ when:

$$cK \subseteq T \subseteq CK$$

for universal constants $c, C > 0$.

We first recall the well-known extension of the slicing problem from the class of convex bodies to the class of all log-concave measures, due to Ball [172]. Given a log-concave probability measure μ on $\mathbb{R}^n$, define its isotropic constant L_μ by:

$$L_\mu := \|\mu\|_{L_\infty}^{\frac{1}{n}} \left(detCov(\mu)\right)^{\frac{1}{2n}}, \tag{9}$$

where $\|\mu\|_{L_\infty} := \; sup_{x\in\mathbb{R}^n}\, \rho(x)$ and ρ is the log-concave density of μ. It was shown by Ball [172] that given $n \geq 1$:

$$sup_\mu L_\mu \leq C\; sup_K L_K,$$

where the suprema are taken over all log-concave probability measures μ and convex bodies K in $\mathbb{R}^n$, respectively (see e.g. [174] for the non-even case). The following theorem slightly generalizes Theorem (4.1.1):

Theorem (4.1.4) [168]: Let μ denote a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin. Suppose that μ is in addition a ψ_α-measure $(1 \leq \alpha \leq 2)$ with constant b_α. Then:

$$L_\mu \leq C\sqrt{b_\alpha^\alpha n^{1-\alpha/2}}.$$

As was the case with Theorem (4.1.1), deducing Theorem (4.1.4) from Theorem (4.1.2) is equally elementary. We only require the following additional well-known lemma, which will come in handy in other instances in this work as well. This lemma serves as an extension of (4) to the class of log-concave measures.

Lemma (4.1.5) [168]: Let μ denote a log-concave probability measure on $\mathbb{R}^n$ with barycenter at the origin. Then:

$$V.Rad.(Z_n(\mu)) \simeq \frac{\sqrt{n}}{\|\mu\|_{L_\infty}^{\frac{1}{n}}}.$$

Given Lemma (4.1.5), the reduction of Theorem (4.1.4) to Theorem (4.1.2) is indeed immediate, since for $p \leq n$ in the range specified in the latter:

$$c\sqrt{p} \leq \frac{V.Rad.\left(Z_p(\mu)\right)}{V.Rad.\left(Z_2(\mu)\right)} \leq \frac{V.Rad.\left(Z_n(\mu)\right)}{\left(detCov(\mu)\right)^{\frac{1}{2n}}} \simeq \frac{\sqrt{n}}{\|\mu\|_{L_\infty}^{\frac{1}{n}}\left(detCov(\mu)\right)^{\frac{1}{2n}}} = \frac{\sqrt{n}}{L_\mu}.$$

Proof: Denote by ρ the log-concave density of μ. According to [178, Proposition 3.7] (compare with [185, Lemma 2.8] and Lemma (4.1.7) below):

$$V.Rad.(Z_n(\mu)) \simeq \frac{\sqrt{n}}{\rho(0)^{\frac{1}{n}}}.$$

However, according to Fradelizi [172]:

$$e^{-n}M \leq \rho(0) \leq M, \qquad M := \|\mu\|_{L_\infty} = \; sup_{x\in\mathbb{R}^n}\rho(x),$$

and so the assertion immediately follows.

Now suppose that μ is an arbitrary Borel probability measure on $\mathbb{R}^n$. Its logarithmic Laplace transform is defined as:

$$\Lambda_\mu(\xi) := log\int_{\mathbb{R}^n} exp(\langle\xi, x\rangle)d\mu(x), \qquad \xi \in \mathbb{R}^n.$$

The function Λ_μ is always convex (e.g. by Hölder's inequality), and clearly $\Lambda_\mu(0) = 0$. If in addition the barycenter of μ lies at the origin, then Λ_μ is non-negative (by Jensen's inequality). In this case, for any $t \geq 0$ and $\alpha \geq 1$:

$$\frac{1}{\alpha}\{\Lambda_\mu \leq \alpha t\} \subseteq \{\Lambda_\mu \leq t\} \subseteq \{\Lambda_\mu \leq \alpha t\}, \qquad (10)$$

where we abbreviate $\{\Lambda_\mu \leq t\} = \{\xi \in \mathbb{R}^n;\ \Lambda_\mu(\xi) \leq t\}$. When μ is log-concave, the convex function Λ_μ possesses several additional regularity properties. For instance $\{\Lambda_\mu < \infty\}$ is an open set, and Λ_μ is C^∞-smooth and strictly-convex in this open set (see, e.g., [185, Section 2]).

The following lemma describes a certain equivalence, known to specialists, between the L_p-centroid bodies and the level-sets of the logarithmic Laplace Transform Λ_μ. See Latała and Wojtaszczyk [186, Section 3] for a proof of a dual version in the symmetric case (i.e., when $\mu(A) = \mu(-A)$ for all Borel subsets $A \subset \mathbb{R}^n$).

Definition(4.1.6) [168]: The Λ_p-body associated to μ, for $p \geq 0$, is defined as:

$$\Lambda_p(\mu) := \{\Lambda_\mu \leq p\} \cap -\{\Lambda_\mu \leq p\}.$$

Lemma (4.1.7): Suppose μ is a log-concave probability measure on $\mathbb{R}^n$ whose barycenter lies at the origin. Then for any $p \geq 1$:

$$\Lambda_p(\mu) \simeq pZ_p(\mu)^\circ.$$

These two equivalent points of view turn out to complement each other well, and play asynergetic role. Before providing a proof, we illustrate this in the following naïve example. Given a log-concave probability measure μ, a well-known consequence of Berwald's inequality (see e.g. [171]) is that:

$$q \geq p \geq 1 \quad \Rightarrow\ Z_p(\mu) \subset Z_q(\mu) \subset C\frac{q}{p}Z_p(\mu). \qquad (11)$$

In view of Lemma (4.1.7), note that this is nothing else but a reformulation (up to constants) of the trivial set of inclusions in (10).

Proof : First, suppose that $\xi \in \Lambda_p(\mu)$. Then:

$$\int_{\mathbb{R}^n} exp|\langle\xi, x\rangle| d\mu(x) \leq \int_{\mathbb{R}^n} exp(\langle\xi, x\rangle) d\mu(x) + exp(-\langle\xi, x\rangle) d\mu(x) \leq 2e^p.$$

Using the inequality $(et/p)^p \leq e^t$, valid for any $t \geq 0$, we see that:

$$h_{Z_p(\mu)}(\xi) = \left(\int_{\mathbb{R}^n} |\langle\xi, x\rangle|^p d\mu(x)\right)^{\frac{1}{p}} \leq (2p^p)^{\frac{1}{p}} \leq 2p.$$

Since $\xi \in \Lambda_p(\mu)$ was arbitrary, this amounts to $\Lambda_p(\mu) \subseteq 2pZ_p(\mu)^\circ$, the first desired inclusion. For the other inclusion, suppose $\xi \in \mathbb{R}^n$ is such that $h_{Z_p(\mu)}(\xi) \leq p$, that is:

$$\left(\int_{\mathbb{R}^n} |\langle\xi, x\rangle|^p d\mu(x)\right)^{1/p} \leq p. \qquad (12)$$

Write X for the random vector in $\mathbb{R}^n$ that is distributed according to μ. Then the function:

$$\phi(t) = \mathbb{P}(\langle X, \xi\rangle \geq t), \qquad t \in \mathbb{R},$$

is log-concave, according to the Prékopa–Leindler inequality (see, e.g., the first pages of [179]). Furthermore, since the barycenter of μ lies at the origin, we have $1/e \leq \varphi(0) \leq 1\ -$

$1/e$ by Grünbaum's inequality (see e.g. [174, Lemma 3.3]). Using Markov's inequality, (12) implies that:

$$\varphi(3ep) \leq (3e)^{-p}.$$

Since φ is log-concave, then:

$$\mathbb{p}(\langle X, \xi \rangle \geq t) = \varphi(t) \leq \varphi(0) \left(\frac{\varphi(3eP)}{\varphi(0)} \right)^{\frac{1}{3eP}} \leq C\, exp(-t/(3e)), \qquad \forall t \geq 3ep.$$

An identical bound holds for $\mathbb{p}(\langle X, \xi \rangle \leq -t)$, and combining the two, we obtain:

$$\mathbb{p}(|\langle X, \xi \rangle| \geq t) \leq C\, exp(-t/(3e)), \qquad \forall t \geq 3ep.$$

Therefore:

$$\mathbb{E} exp\left(\frac{|\langle \xi, X \rangle|}{6e} \right) = \frac{1}{6e} exp\left(\frac{t}{6e} \right) \mathbb{P}(|\langle X, \xi \rangle| \geq t) dt$$

$$\leq \frac{1}{6e} \int_0^{3eP} exp\left(\frac{t}{6e} \right) dt + C \int_{3eP}^{\infty} exp(-t/(6e)) dt \leq exp(\tilde{C}p).$$

Consequently:

$$max \Lambda_\mu \left(\frac{1}{6e} \xi \right), \Lambda_\mu \left(-\frac{1}{6e} \xi \right) \leq log \mathbb{E} exp\left(\frac{|\langle \xi, X \rangle|}{6e} \right) \leq Cp,$$

for some $C \geq 1$, and using (10), this implies:

$$max \left\{ \Lambda_\mu \left(\frac{1}{6eC} \xi \right), \Lambda_\mu \left(-\frac{1}{6eC} \xi \right) \right\} \leq p,$$

for any $\xi \in \mathbb{R}^n$ with $h_{Z_p(\mu)}(\xi) \leq p$. This is precisely the second desired inclusion $pZ_p(\mu)^{\circ} \subseteq C' \Lambda_p(\mu)$, and the assertion follows.

The last topic we would like to review pertains to some properties of level sets of convex functions and their gradient images. The possibility to use the gradient image of Λ_μ as in [174] is one of the main reasons for additionally employing the logarithmic Laplace transform, rather than working exclusively with the L_p-centroid bodies.

Lemma (4.1.8) [168]: Let $F: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a non-negative convex function, which is C^1-smooth in $\{F < \infty\}$. Let $q, r \geq 0$. Then:

$$\langle z, \nabla F(x) \rangle \leq q + r \quad for\ any \quad z \in \{F \leq r\}, x \in \frac{1}{2}\{F \leq q\}.$$

In other words:

$$\nabla F\left(\frac{1}{2}\{F \leq q\} \right) \subset (q + r)\{F \leq r\}^{\circ}.$$

Proof: Since F is non-negative and its graph lies above any tangent hyperplane, then:

$$\langle \nabla F(x), \frac{z}{2} \rangle \leq F(x) + \langle \nabla F(x), \frac{z}{2} \rangle \leq F(x + z/2) \leq \frac{F(2x) + F(z)}{2} \leq \frac{q + r}{2}.$$

The following lemma was proved in [175, Lemma 2.3] for an even function F.

Lemma (4.1.9) [168]: Let $F: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a non-negative convex function, C^2-smooth and strictlyconvex in $\{F < \infty\}$, with $F(0) = 0$. Let $p > 0$, and set:

$$F_p := \{F \leq p\} \cap -\{F \leq p\}.$$

Assume that:

$$\Psi_p := \left(\frac{1}{Vol_n\left(\frac{1}{2}F_p\right)}\int_{\frac{1}{2}F_p} det\ HessF(x)dx\right)^{\frac{1}{n}} > 0.$$

Then:

$$V.Rad.\left(F_p\right) \le 2\frac{\sqrt{p}}{\sqrt{\Psi_p}}.$$

Proof: Applying Lemma (4.1.8) with $q = r = p$, and using the change of variables $x = \nabla F(y)$, we obtain:

$$Vol_n\left(2p(F_p)^{\circ}\right) \ge Vol_n\left(\nabla F\left(\frac{1}{2}F_p\right)\right) = \int_{\frac{1}{2}F_p} det\ HessF(y)dy = Vol_n\left(\frac{1}{2}F_p\right)\Psi_p^n.$$

Equivalently, we obtain:

$$Vol_n\left((F_p)^{\circ}\right) \ge \left(\frac{\Psi_p}{4p}\right)^n Vol_n(F_p).$$

Note that F_p is a centrally-symmetric convex body, i.e., $F_p = -F_p$. The Blaschke–Santaló inequality (see, e.g., [181]) for a centrally-symmetric convex body K asserts that:

$$V.Rad.(K^{\circ})V.Rad.(K) \le 1.$$

Combining the last two estimates with $K = F_p$, the result immediately follows.

Let μ denote a log-concave probability measure on $\mathbb{R}^n$ with density ρ, and let $\xi \in \{\Lambda_\mu < \infty\}$.

We denote by μ_ξ the "tilt" of μ by , defined via the following procedure. First, define the probability density:

$$\rho_\xi(x) := \frac{1}{Z_\xi}\ \rho(x)exp(\langle\xi, x\rangle) \qquad for\ x \in \mathbb{R}^n,$$

where $Z_\xi > 0$ is a normalizing factor. Denoting by $b_\xi \in \mathbb{R}^n$ the barycenter of ρ_ξ, we set μ_ξ to be the probability measure with density $\rho_\xi(\cdot - b_\xi)$. Note that μ_ξ is a log-concave probability measure, having the origin as its barycenter. Furthermore, as verified in [175, Section 2], we have:

$$b_\xi = \nabla\Lambda_\xi(\xi), \qquad Cov\left(\mu_\xi\right) = Hess\Lambda_\mu(\xi) \qquad (13).$$

The following proposition is one of the main results:

Proposition (4.1.10) [168]: Let μ denote a log-concave probability measure on $\mathbb{R}^n$ whose barycenter lies at the origin. Then, for all $1 \le p \le n$:

$$V.Rad.\left(Z_p(\mu)\right) \simeq \sqrt{P}\quad inf_{x\in\frac{1}{2}\Lambda_p(\mu)}\left(detCov(\mu_x)\right)^{\frac{1}{2n}}. \qquad (14)$$

In the proofs of the theorems stated, we will not use the full force of Proposition (4.1.10), but rather only the lower bound for $V.Rad.(Z_p(\mu))$. This lower bound has a short proof, as will see below. However, the observation that we actually obtain an equivalence seems interesting, hence we provide the arguments for both directions. Before going into the proof, as a testament of its usefulness, we state the following immediate corollary of Proposition (4.1.10):

Corollary (4.1.11) [168]: Let μ be a log-concave probability measure on $\mathbb{R}^n$ whose barycenter lies at the origin. Then:

$$1 \le p \le q \le n \Rightarrow \frac{V.Rad.(Z_p(\mu))}{\sqrt{p}} \ge c\frac{V.Rad.(Z_q(\mu))}{\sqrt{q}}.$$

Remark (4.1.12) [168]: Using $q = n$ above and the fact that $V.Rad.(Z_n(K)) \simeq V.Rad.(K)$ for a convex body K whose barycenter lies at the origin, which follows from (4) as in the Introduction, we immediately verify that:

$$\forall\ 1 \le p \le n, \quad V.Rad.\left(Z_p(K)\right) \ge c\sqrt{\frac{p}{n}}V.Rad.(K). \qquad (15)$$

This recovers up to a constant the lower bound of Lutwak, Yang and Zhang (5). Moreover, recalling that $V.Rad.(Z_n(\mu)) \simeq \sqrt{n}/\|\mu\|_{L_\infty}^{\frac{1}{n}}$ by Lemma (4.1.5) and the definition (9) of L_μ, the same argument yields the following analog of (15):

$$\forall 1 \le p \le n, \qquad V.Rad.\left(Z_p(\mu)\right) \ge c\frac{\sqrt{p}}{L_\mu}(detCov(\mu))^{\frac{1}{2n}} = c\frac{\sqrt{p}}{L_\mu}V.Rad.(Z_2(\mu)).$$

This may also be deduced by only employing the lower-bound in (14).

We now turn to the proof of Proposition (4.1.10), and begin with the lower bound for $V.Rad.(Z_p(\mu))$. In fact, we show a formally stronger statement:

Lemma (4.1.13) [168]: Let μ denote a log-concave probability measure on $\mathbb{R}^n$ whose barycenter lies at the origin. Then, for all $1 \le p \le n$,

$$V.Rad.\left(Z_p(\mu)\right) \ge c\sqrt{p}\sqrt{\Psi_p},$$

where $c > 0$ is a universal constant and:

$$\Psi_p := \left(\frac{1}{Voln\left(\frac{1}{2}\Lambda_p(\mu)\right)}\int_{\frac{1}{2}\Lambda_p(\mu)} detCov(\mu_x)dx\right)^{\frac{1}{n}}.$$

Proof: Apply Lemma (4.1.9) with $F = \Lambda_\mu$. Since $det\ Hess\Lambda_\mu(x) = detCov(\mu_x)$ according to (13), we deduce that:

$$V.Rad.\left(\Lambda_p(\mu)\right) \le 2\frac{\sqrt{p}}{\sqrt{\Psi_p}}. \qquad (16)$$

Applying Lemma (4.1.7) in order to pass from $\Lambda_p(\mu)$ to $Z_p(\mu)$, and the Bourgain–Milman inequality (see, e.g., [179]) for a centrally-symmetric convex set $K \subset \mathbb{R}^n$:

$$V.Rad.(K^\circ)V.Rad.(K) \ge c,$$

we deduce from (16) that:

$$V.Rad.\left(Z_p(\mu)\right) \simeq pV.Rad.\left(\Lambda_p(\mu)^\circ\right) \gtrsim\ pV.Rad.\left(\Lambda_p(\mu)\right)^{-1} \gtrsim \sqrt{p}\sqrt{\Psi_p}.$$

In order to deduce the upper bound of Proposition (4.1.10), and of crucial importance to the main results, is the following elementary observation:

Proposition (4.1.14) [168]: Let μ denote a log-concave probability measure in $\mathbb{R}^n$ with barycenter at the origin. Then:

$$\forall x \in \frac{1}{2}\Lambda_p(\mu), \qquad \Lambda_p(\mu_x) \simeq \Lambda_p(\mu).$$

Indeed, it is clear that the logarithmic Laplace transform should interact nicely with the tilt operation, and the following identity is verified by a direct calculation:

$$\Lambda_{\mu_x}(z) = \Lambda_\mu(z+x) - \Lambda_\mu(x) - \langle z, b_x\rangle, \ b_x = \nabla\Lambda_\mu(x). \qquad (17)$$

Geometrically, this means that the graph of Λ_{μ_x} is obtained from that of Λ_μ by subtracting the tangent plane at x (given by the linear function $z \mapsto \Lambda_\mu(x) + \langle z - x, \nabla\Lambda_\mu(x)\rangle$), and translating everything by $-x$ (so that x gets mapped to the origin). In particular, we verify that $\Lambda_{\mu_x}(0) = 0$ and that $\Lambda_{\mu_x} \geq 0$, as required from the logarithmic Laplace transform of a probability measure with barycenter at the origin.

It remains to manipulate level sets of convex functions, once again. We require the following:

Lemma (4.1.15) [168]: Let F be as in Lemma (4.1.8), and let $y \in \mathbb{R}^n$ and $D, p > 0$. Define a function G by:

$$G(z) := F(z\ + y) - F(y - \langle z, \nabla F(y)\rangle.$$

Then:

$$y \in \frac{1}{2}\{F \leq DP\}, \qquad z \in \{F \leq p\} \cap -\{F \leq p\} \Longrightarrow\ z \in 2\{G \leq (D+1)p\}.$$

Proof: We apply Lemma (4.1.8) with $q = DP$ and $r = P$. Since $-z \in \{F \leq p\}$ and $y \in \frac{1}{2}\{F \leq DP\}$, then by the conclusion of that lemma, $\langle -z, \nabla F(y)\rangle \leq (D+1)p$. Since F is non-negative and convex, we deduce that:

$$G(z/2) \leq F(z/2 + y) + \frac{D+1}{2}p \leq \frac{F(z) + F(2y)}{2} + \frac{D+1}{2}p \leq (D+1)p.$$

(i) If $z \in \Lambda_p(\mu)$, we apply Lemma (4.1.15) with $D = 1$ and $y = x$ to $F = \Lambda_\mu$. By (17), we deduce that $\Lambda_{\mu_x}(z/2) = G(z/2) \leq 2p$. Using (10), we conclude that $\Lambda_{\mu_x}(z/4) \leq p$. The same argument applies to $-z$ by the symmetry of our assumptions, and so we conclude that $z \in 4\Lambda_p(\mu_x)$.

(ii) If $z\ \in \Lambda_p(\mu_x)$, we would like to apply Lemma (4.1.15) with $y = -x$ to $F = \Lambda_{\mu_x}$, since tilting μ_x by $-x$ gives back μ. To this end, we must verify that $\Lambda_{\mu_x}(-2x) \leq DP$ for some $D > 0$. According to (17):

$$\Lambda_{\mu_x}(-2x) =\ \Lambda_\mu(-x) - \Lambda_\mu(x) + 2\langle x, \nabla\Lambda_\mu(x)\rangle.$$

By Lemma (4.1.8), we know that $\langle x, \nabla\Lambda\mu(x)\rangle \leq 2p$, and using that Λ_μ is non-negative, convex and vanishes at the origin, we obtain:

$$\Lambda_{\mu_x}(-2x) \leq \frac{1}{2}\Lambda_\mu(-2x) + 4p \leq 4.5p.$$

We conclude that we may use $D = 4.5$ above, and so Lemma (4.1.15) finally implies that $\Lambda_\mu(z/2) = G(z/2) \leq 5.5p$. As in the first part of the proof, we deduce that $\mu(z/11) \leq p$.

The same argument applies to $-z$ by the symmetry of our assumptions, and so we conclude that $z \in 11\Lambda_p(\mu)$.
Using Lemma (4.1.7), we equivalently reformulate Proposition (4.1.14) as:
Proposition (4.1.16) [168]: Let μ denote a log-concave probability measure in $\mathbb{R}^n$ with barycenter at the origin. Then:

$$\forall x \in \frac{1}{2}\Lambda_p(\mu), \qquad Z_p(\mu_x) \simeq Z_p(\mu).$$

To complete the proof of Proposition (4.1.10), we state again Paouris' upper bound (7) on $V.Rad.(Z_p(\nu))$:
Theorem (4.1.17) [168]: (Paouris). For any log-concave probability measure ν with barycenter at the origin, and $2 \le p \le n$:

$$V.Rad.\left(Z_p(\nu)\right) \le C\sqrt{p}V.Rad.(Z_2(\nu)).$$

Proof: The statement is invariant under linear transformations, so we may assume that ν is isotropic. The claim is then the content of [176, Theorem 6.2].
Lemma (4.1.13) implies the lower bound:

$$V.Rad.\left(Z_p(\mu)\right) \ge c\sqrt{p}\, inf_{x\in\frac{1}{2}\Lambda_p(\mu)}(detCov(\mu_x))^{\frac{1}{2n}}.$$

Since $(detCov(\mu_x))^{\frac{1}{2n}} = V.Rad.(Z_2(\mu_x))$, then applying Theorem (4.1.17), we obtain:

$$inf_{x\in\frac{1}{2}\Lambda_p(\mu)}V.Rad.\left(Z_p(\mu_x)\right) \le C\sqrt{p}\, inf_{x\in\frac{1}{2}\Lambda_p(\mu)}\left(detCov(\mu_x)\right)^{\frac{1}{2n}}. \quad (18)$$

But by Proposition (4.1.16), $Z_p(\mu_x) \simeq Z_p(\mu)$ for all $x \in \frac{1}{2}\Lambda_p(\mu)$, and hence the left-hand side in (18) is equivalent to $V.Rad.(Z_p(\mu))$, completing the proof.
Given a centrally-symmetric convex body $K \subset \mathbb{R}^n$, its "(dual) Dvoretzky-dimension" $k^*(K)$ was defined by Milman and Schechtman [173] as the largest positive integer $k \le n$ so that:

$$\sigma_{n,k}\left\{E \in G_{n,k}; \frac{1}{2}W(K)B_E \subset Proj_E K \subset 2W(K)B_E\right\}\frac{n}{n+k},$$

where $\sigma_{n,k}$ denotes the Haar probability measure on $G_{n,k}$ and B_E denotes the Euclidean unit ball in the subspace E. It was shown in [173], following Milman's seminal work [174], that:

$$k^*(K) \simeq n\left(\frac{W(K)}{diam(K)}\right)^2. \quad (19)$$

Define $W_q(K) = (\int_{S^{n-1}} h_K(\theta)^q\, d\sigma(\theta))^{\frac{1}{q}}$, the q-th moment of the supporting functional of K. According to Litvak, Milman and Schechtman [177]:

$$c_1 W_q(K) \le \max\left\{W(K), \sqrt{\frac{q}{n}}\, diam(K)\right\} \le c_2 W_q(K). \quad (20)$$

The quantity $W_q(Z_q(\mu))$ has a simple equivalent description: a direct calculation as in [174] confirms that for any Borel probability measure μ on $\mathbb{R}^n$ and $q \ge 1$:

$$\left(W_q Z_q(\mu)\right) \simeq \sqrt{\frac{q}{n+q}}\, I_q(\mu), \quad I_q(\mu) := \left(\int_{\mathbb{R}^n}|x|^q d\mu(x)\right)^{\frac{1}{q}}. \quad (21)$$

Finally, observe that when the barycenter of μ is at the origin, then $I_2(\mu)^2 = traceCov(\mu)$.

In [186] (see also [185]), Paouris defines $q^*(\mu)$ as follows:

$$q^*(\mu) := sup\{q \in \mathbb{N};\ k^*\big(Z_q(\mu)\big) \geq q\}.$$

It is straightforward to check that all of Paouris' results involving $q^*(\mu)$ from [178] remain valid when replacing it with $q_c^*(\mu)$ when $c > 0$ is a fixed universal constant, where q_δ^* is defined as follows (see [177]):

$$q_\delta^*(\mu) := \sup\{q \geq 1;\ k^*\left(Z_p(\mu)\right) \leq \delta^{-2}q\}.$$

Although the particular value of $c > 0$ seems insignificant for the results of [188], the definition we require is essentially that of q_c^* for some small enough universal constant $c > 0$. Our preference to work with a variant of q_c^* is motivated by Lemma (4.1.18) below and the subsequent remarks.

We proceed as follows. Given a log-concave probability measure μ on $\mathbb{R}^n, q \geq 1$ and $\delta > 0$, consider the following four related properties:

(i) $P_1(\delta)$ is the property that $k^*(Z_q(\mu)) \geq \delta^{-2}q$.

(ii) $P_1'(\delta)$ is the property that $diam(Z_q(\mu)) \leq \delta\sqrt{n}\frac{W(Z_q(\mu))}{\sqrt{q}}$.

(iii) $P_2(\delta)$ is the property that $diam(Z_q(\mu)) \leq \delta\sqrt{n}(detCov(\mu))^{\frac{1}{2n}}$.

(iv) P_W is the property that $W(Z_q(\mu)) \geq c\sqrt{q}(detCov(\mu))^{\frac{1}{2n}}$, for some specific, appropriately small universal constant $c > 0$, as in the proof of Lemma (4.1.18) below.

According to (19), we have:

$$P_1(\delta) \Rightarrow P_1'(C_1\delta) \Rightarrow P_1(C_2\delta), \qquad (22)$$

for all $\delta > 0$, where $C_1, C_2 > 1$ are universal constants. The next lemma relates between the other properties above (compare with [177, Section 2]):

Lemma (4.1.18) [168]: Suppose μ is a log-concave probability measure in $\mathbb{R}^n$ whose barycenter lies at the origin. Let $q \in [1, n]$ and $\delta \in (0, 1]$. Then:

(i) If μ is isotropic and $P_1(\delta)$ holds, then $P_2(C_3\delta)$ holds.

(i) (a) If $P_1(\delta)$ holds, then so does P_W.

(ii) Suppose $\delta < \delta_0$ for a certain appropriately small universal constant $\delta_0 > 0$. If $P_2(\delta)$ holds, then so does P_W.

(iii) If $P_2(\delta)$ and P_W hold, then so does $P_1'(C_4\delta)$.

Proof:

(i) Clearly $P_1(\delta)$ implies $P_1(1)$. Using (21), Paouris's main result [176, Theorem 8.1] and the isotropicity of μ, we know that:

$$W_q\big(Z_q(\mu)\big) \simeq \frac{\sqrt{q}}{\sqrt{n}} I_q(\mu) \simeq \frac{\sqrt{q}}{\sqrt{n}} I_2(\mu) = \frac{\sqrt{q}}{\sqrt{n}} (traceCov(\mu))^{\frac{1}{2}} = q.$$

In particular, $W(Z_q(\mu)) \leq W_q(Z_q(\mu)) \leq C\sqrt{q}$. Since $P_1(\delta)$ implies $P_1'(C_1\delta)$, then:

$$diam\big(Z_q(\mu)\big) \leq C_1\delta\sqrt{n}\frac{W(Z_q(\mu))}{\sqrt{q}} \leq CC_1\delta\sqrt{n} = C_3\delta\sqrt{n}(detCov(\mu))^{\frac{1}{2n}},$$

and $P_2(C_3\delta)$ holds true.

(ii) Since all properties are invariant under scaling, we may assume that $detCov(\mu) = 1$. Using (21) and the arithmetic-geometric mean inequality:

$$\frac{1}{n} I_2(\mu)^2 = \frac{1}{n} traceCov(\mu) \geq (detCov(\mu))^{\frac{1}{n}},$$

we see that:

$$W_q\big(Z_q(\mu)\big) \geq c_0 \frac{\sqrt{q}}{\sqrt{n}}\, I_q(\mu) \geq c_0 \frac{\sqrt{q}}{\sqrt{n}} I_2(\mu) \geq c_0\sqrt{q}. \qquad (23)$$

(i) Assuming $P_1'(\delta)$, (20) implies that $W(Z_q(\mu)) \geq c_1 W_q(Z_q(\mu))$, and together with (23), P_W follows.
(ii) Set $\delta_0 = c_0 c_1$, where c_0 is the constant from (23) and c_1 is the constant from (20). Using (23), the property $P_2(\delta)$ with $0 < \delta < \delta_0$ implies:
√

$$\frac{\sqrt{q}}{\sqrt{n}} diam\big(Z_q(\mu)\big) \leq \delta\sqrt{q} < c_0 c_1 \sqrt{q} \leq c_1 W_q\big(Z_q(\mu)\big).$$

Therefore by (20), $W(Z_q(\mu)) \geq c_1 W_q(Z_q(\mu)) \geq c_0 c_1 \sqrt{q}$, and P_W follows.
(iii) This is immediate by plugging the estimates on $diam(Z_q(\mu))$ and $W(Z_q(\mu))$ into the definition of $P_1'(\delta)$.
Lemma (4.1.19) [168]: We may choose the numeric constant $c > 0$ small enough so that:
(i) $q^{\#}(\mu) \leq n$.
(ii) $1 \leq q \leq q^{\#}(\mu)$ implies $k^*(Z_q(\mu)) \geq q$ and $W(Z_q(\mu)) \geq c\sqrt{q}(detCov(\mu))^{\frac{1}{2n}}$.
Proof: Assume first that $q^{\#}(\mu) > 1$. The second point follows immediately from Lemma (4.1.18) and (22). The first point follows from (21), since:

$$n \cdot (detCov(\mu))^{\frac{1}{n}} \leq traceCov(\mu) = I_2(\mu)^2 \leq I_n(\mu)^2 \simeq W_n(Z_n(\mu))^2 \leq diam(Z_n(\mu))^2.$$

It remains to deal with the degenerate case $q^{\#}(\mu) = 1$. By definition, $k^*(Z_1(\mu)) \geq 1$, and e.g. by (19):

$$W(Z_1(\mu)) \geq c \frac{diam(Z_1(\mu))}{\sqrt{n}} \geq cc^{\#}(detCov(\mu))^{\frac{1}{2n}},$$

as required.
Consequently $\lfloor q^{\#}(\mu) \rfloor \leq q^*(\mu)$, and all of Paouris' results for $q \leq q^*(\mu)$ continue to hold for $q \leq q^{\#}(\mu)$. Similarly, by Lemma (4.1.18), if μ is isotropic then $q_c^*(\mu) \leq q^{\#}(\mu)$ for some small constant $c > 0$. To conclude, we reiterate the stability of $q^{\#}(\mu)$ under projections in the following corollary, which is one of the key ingredients in the proof of Theorem (4.1.3):
Corollary (4.1.20) [168]: Let μ denote an isotropic log-concave probability measure in $\mathbb{R}^n$, let $1 \leq k \leq n$ and $q \geq 1$. Then for all $E \in G_{n,k}$ with $k \geq (c^{\#})^{-2} diam^2(Z_q(\mu))$, we have $q^{\#}(\pi_{E\mu}) \geq q$. In particular $k^*(Proj_E Z_q(\mu)) \geq q$ and $W(Proj_E Z_q(\mu)) \geq c\sqrt{q}$.
Proof: Since $\pi_E\mu$ remains isotropic, $Z_q(\pi_E\mu) = Proj_E Z_q(\mu)$ and $diam(Proj_E Z_q(\mu)) \leq diam(Z_q(\mu)) \leq c^{\#}\sqrt{k}$, the assertion follows by definition of $q^{\#}(\pi_E\mu)$ and Lemma (4.1.19).

In view of Proposition (4.1.10), our goal now is to bound from below $(detCov(\mu_x))^{\frac{1}{2n}}$ for the tilted measures μ_x , where $x \in \frac{1}{2}\Lambda_p(\mu)$. Our only available information is provided by Proposition (4.1.16), stating that $Z_p(\mu_x) \simeq Z_p(\mu)$, where μ itself is assumed isotropic.
Suppose ν is a log-concave probability measure on $\mathbb{R}^n$ whose barycenter lies at the origin. Recall that its isotropic constant is defined as:

$$L_\nu := \|\nu\|_{L_\infty}^{\frac{1}{n}} (detCov(\nu))^{\frac{1}{2n}}. \qquad (24)$$

Since the isotropic constant L_ν satisfies $L_\nu \geq c > 0$ (see e.g. [178]), then according to Lemma (4.1.5):

$$(detCov(\nu))^{\frac{1}{2n}} \gtrsim \frac{1}{\|\nu\|_{L_\infty}^{\frac{1}{n}}} \simeq \frac{V.Rad.(Z_n(\nu))}{\sqrt{n}}. \qquad (25)$$

Lemma (4.1.21) [168]: Let ν denote a log-concave probability measure in $\mathbb{R}^n$ with barycenter at the origin, and let k denote an integer between 1 and n. Then:

$$\exists \theta \in S^{n-1} \sqrt{\int_{\mathbb{R}^n} \langle x,\theta\rangle^2 \, d\nu(x)} \geq \frac{c}{\sqrt{k}} sup_{E\in G_{n,k}} V.Rad.\left(Proj_E Z_k(\nu)\right). \qquad (26)$$

Proof: Given $E \in G_{n,k}$, apply (25) to $\pi_E\nu$. We get that

$$(detCov(\pi_E\nu))^{\frac{1}{2k}} \gtrsim \frac{V.Rad.(Z_k(\pi_E\nu))}{\sqrt{k}} \qquad (27).$$

Note that $(detCov(\pi E\nu))^{1/k}$ is the geometric average of the eigenvalues of the symmetric, positive semi-definite matrix $Cov(\pi_E\nu)$. Let $\theta \in S^{n-1} \cap E$ be the eigenvector corresponding to the largest eigenvalue of $Cov(\pi_E\nu)$. From (27) we thus see that

$$\sqrt{\int_{\mathbb{R}^n} \langle x,\theta\rangle^2 d\nu(x)} = \sqrt{\langle Cov(\pi_E\nu)\theta,\theta\rangle} \geq (detCov(\pi_E\nu))^{\frac{1}{2k}} \gtrsim \frac{V.Rad.(Z_k(\pi_E\nu))}{\sqrt{k}}.$$

Noting that $Z_k(\pi_E\nu) = Proj_E Z_k(\nu)$, we obtain (26).
The idea now is to compare $V.Rad.(Proj_E Z_k(\mu_x))$ with $V.Rad.(Proj_E Z_k(\mu))$. Note that if $Z_p(\nu) \simeq Z_p(\mu)$, then by (11):

$$1 \leq q \leq p \Rightarrow \qquad c\frac{q}{p} Z_q(\mu) \subset Z_q(\nu) \subset C\frac{p}{q} Z_q(\mu).$$

Therefore, when $Z_p(\nu) \simeq Z_p(\mu)$ and $k \leq p$,

$$V.Rad.(Proj_E Z_k(\nu)) \geq c\frac{k}{p} V.Rad.\left(Proj_E Z_k(\mu)\right) \qquad (28)$$

for all $E \in G_{n,k}$. To control $V.Rad.(Proj_E Z_k(\mu))$, we have:

Lemma (4.1.22) [168]: Let μ denote a log-concave probability measure in $\mathbb{R}^n$ with barycenter at the origin, and let $1 \leq k \leq q^{\#}(\mu)$. Then:

$$\exists E \in G_{n,k} \; V.Rad.(Proj_E Z_k(\mu)) \geq c\sqrt{k}(detCov(\mu))^{\frac{1}{2n}}.$$

Proof: Lemma (4.1.19) asserts that $1 \leq k \leq q^{\#}(\mu)$ implies that $k^*(Z_k(\mu)) \geq k$. Consequently, there exists at least one (in fact, many) $E \in G_{n,k}$ so that:

$$\frac{1}{2} W(Z_k(\mu)) B_E \subset Proj_E Z_k(\mu) \subset 2W(Z_k(\mu)) B_E,$$

and hence $V.Rad.(Proj_E Z_k(\mu)) \geq \frac{1}{2}W(Z_k(\mu))$. It remains to appeal to Lemma (4.1.19) again and deduce from $1 \leq k \leq q^{\#}(\mu)$ that $W(Z_k(\mu)) \geq c\sqrt{k}(detCov(\mu))^{\frac{1}{2n}}$.
We summarize the preceding discussion with the following fundamental:
Proposition (4.1.23) [168]: Let ν, μ denote two log-concave probability measures in $\mathbb{R}^n$with barycenters at the origin, and let $1 \leq p \leq n$. Assume that $Z_p(\nu) \simeq Z_p(\mu)$. Then:

$$\exists\theta \in S^{n-1} \qquad \sqrt{\int_{\mathbb{R}^n} \langle x, \theta\rangle^2 d\nu(x)} \geq c\, min\left\{1, \frac{q^{\#}(\mu)}{p}\right\}(detCov(\mu))^{\frac{1}{2n}}.$$

Proof: Combine Lemma (4.1.21), Lemma (4.1.22) and (28) for $k = min\{p, \lfloor q^{\#}(\mu)\rfloor\}$.
We can now proceed to control the entire $detCov(\nu)$ by projecting onto the flag of subspaces spanned by the eigenvectors of $Cov(\nu)$. To apply Proposition (4.1.23), we require good control over $q^{\#}(\pi_E\mu)$. One way to obtain such control is to make a definition:
The Hereditary-$q^{\#}$ constant of a log-concave probability measure μ on $\mathbb{R}^n$, denoted $q_H^{\#}(\mu)$, is defined as:

$$q_H^{\#}(\mu) := n\, inf_k inf_{E\in G_{n,k}} \frac{q^{\#}(\pi_E\mu)}{k}.$$

Remark (4.1.24) [168]: It is useful to note the following alternative formula for $q_H^{\#}(\mu)$, valid only for an isotropic, log-concave probability measure μ on $\mathbb{R}^n$. Recalling the definitions of $q^{\#}(\nu), \Delta_\nu(q) = diam(Z_q(\nu))$, and using $supE \in G_{n,k} diam(Proj_E Z_q(\mu)) = diam(Z_q(\mu))$, we obtain:

$$q_H^{q}(\mu) = n\, inf_{1\leq k\leq n} \frac{\Delta_\mu^{-1}(c^{\#}\sqrt{k})}{k} \simeq n\, inf_{1\leq q\leq q^{\#}(\mu)} \frac{q}{diam\left(Z_q(\mu)\right)^2}, \qquad (29)$$

where we use (11) and the definition of $q^{\#}(\nu)$ to justify the last equivalence.
Proposition (4.1.25) [168]: Let ν, μ denote two log-concave probability measures in $\mathbb{R}^n$ with barycenters at the origin, and assume that μ is isotropic. Let $1 \leq p \leq Aq_H^{\#}(\mu)$ with $A \geq 1$, and assume that $Z_p(\nu) \simeq Z_p(\mu)$. Then:

$$(detCov(\nu))^{\frac{1}{2n}} \geq \frac{c}{A},$$

where $c > 0$ denotes a universal constant.
Proof: Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of $Cov(\nu)$, and let $E_k \in G_{n,k}$ denote the subspace spanned by the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_k$. Since $Proj_{E_k} Z_p(\nu) \simeq Proj_{E_k} Z_p(\mu)$, Proposition (4.1.23) applied to $\pi_{E_k}\nu$ and $\pi_{E_k}\mu$ implies that:

$$\sqrt{\lambda_k} \geq c\, min\left(1, \frac{q^{\#}\left(\pi_{E_k}\mu\right)}{p}\right) \geq c\, min\left(1, \frac{q_H^{\#}(\mu)}{p}\frac{k}{n}\right) \geq \frac{c}{A}\frac{k}{n}.$$

Taking geometric average over the λ_k 's, the assertion immediately follows.
Remark (4.1.26) [168]: It is clear from the proof that we may actually replace in the definition of $q_H^{\#}(\mu)$ the infimum over k with a geometric-average over the terms. We denote this variant by $q_{GH}^{\#}(\mu)$, and as in Remark (4.1.24), obtain the following expression for it when μ is in addition isotropic:

$$q^{\#}_{GH}(\mu) = n\left(\prod_{k=1}^{n} \frac{\Delta_{\mu}^{-1}\left(c^{\#}\sqrt{k}\right)}{k}\right)^{\frac{1}{n}} \simeq \left(\prod_{k=1}^{n} \Delta_{\mu}^{-1}\left(c^{\#}\sqrt{k}\right)\right)^{\frac{1}{n}}. \qquad (30)$$

Another way to obtain some (partial) control over $q^{\#}(\pi_E\mu)$ is to invoke Corollary (4.1.20):

Proposition (4.1.27) [168]: Let ν, μ denote two log-concave probability measures in $\mathbb{R}^n$ with barycenters at the origin, and assume that μ is isotropic. Let $1 \le p \le n$ and $A \ge 1$. Assume that $Z_p(\nu) \simeq Z_p(\mu)$ and that:

$$diam\left(Z_p(\mu)\right)\sqrt{log(p)} \le A\sqrt{n}. \qquad (31)$$

Then:

$$(detCov(\nu))^{\frac{1}{2n}} \ge exp(-CA^2).$$

Proof: We employ the same notation as in the previous proof. Setting:

$$k_0 := \left\lceil (c^{\#})^{-2}\, diam^2 Z_p(\mu)\right\rceil,$$

Corollary (4.1.20) states that $q^{\#}(\pi_{E_{k_0}}\mu) \ge p$. Consequently, applying Proposition (4.1.23) to $\pi_{E_{k_0}}\nu$ and $\pi_{E_{k_0}}\mu$, we obtain that $\lambda_{k_0} \ge c > 0$, and hence the largest $n - k_0 + 1$ eigenvalues of $Cov(\nu)$ are bounded below by the same $c > 0$. To bound the contribution of the other eigenvalues, we use (11) to obtain the following trivial bound (which may be improved, but ultimately only results in better numeric constants):

$$\left(detCov(\pi_{E_{k_0}}\nu)\right)^{\frac{1}{2k_0}} = V.Rad.\left(Z_2(\pi_{E_{k_0}}\nu)\right) \gtrsim \frac{1}{p} V.Rad.\left(Z_p(\pi_{E_{k_0}}\nu)\right)$$

$$\simeq \frac{1}{p} V.Rad.\left(Z_p(\pi_{Ek_0}\mu)\right) \ge \frac{1}{p} V.Rad.\left(Z_2\left(\pi_{E_{k_0}}\mu\right)\right) = \frac{1}{p}.$$

Using our estimates separately on E_{k_0} and $E_{k_0}^{\perp}$, we obtain:

$$(detCov(\nu))^{\frac{1}{2n}} = \left(detCov(\pi_{E_{k_0}}\nu)\right)\, detCov(\pi_{E_{k_0}^{\perp}}\nu) \ge c\left(\frac{1}{p}\right)^{\frac{k_0}{n}}.$$

Our assumption (31) precisely ensures that $k_0\, log(p) \le C \cdot A^2 n$, and the assertion follows.

Theorem (4.1.3) now follows immediately from Proposition ((4.1.27), combined with Propositions (4.1.10) and (4.1.16). Similarly, Proposition (4.1.25) and Remark (4.1.26), combined with Propositions (4.1.10) and (4.1.16), yield:

Theorem (4.1.28) [168]: Let μ denote an isotropic log-concave probability measure in $\mathbb{R}^n$. Then:

$$V.Rad.\left(Z_p(\mu)\right) \ge c\sqrt{p}, \qquad \forall 2 \le p \le Cq^{\#}_{H}(\mu).$$

Moreover, the same bound remains valid for $2 \le p \le Cq^{\#}_{GH}(\mu)$.

Now if μ is a log-concave isotropic measure on $\mathbb{R}^n$ which is in addition a ψ_α-measure with constant $b_\alpha (for\ \alpha \in [1,2])$, by definition:

$$diam\left(Z_p(\mu)\right) \le 2b_\alpha p^{\frac{1}{\alpha}}.$$

It therefore follows immediately from (29) that:

$$q^{\#}_{H}(\mu) \ge \frac{c}{b_\alpha^\alpha} n^{\alpha/2},$$

and thus Theorem (4.1.2) follows from Theorem (4.1.28).

Lastly, it may be worthwhile to record the following generalization of Theorems (4.1.1) and (4.1.4), which follows immediately , from Theorem (4.1.28) and (30):

Theorem (4.1.29) [168]: Let μ denote a log-concave probability measure in $\mathbb{R}^n$ with barycenter at the origin. Then:

$$L_\mu \le C\left(\prod_{k=1}^{n}\frac{k}{\Delta_\mu^{-1}(c^\#\sqrt{k})}\right)^{\frac{1}{2n}}.$$

Observe that in this formulation, we only require an on-average control over the growth of $\Delta_\mu(p) = diam(Z_p(\mu))$, as opposed to all previously mentioned bounds on L_μ.

Denote:

$$L_n := sup_{K\subseteq\mathbb{R}^n} L_K, \tag{32}$$

where the supremum runs over all convex bodies $K \subset \mathbb{R}^n$. Recall that K is called isotropic if μ_K, the uniform measure on K, is isotropic. Recall also that $Z_p(K) = Z_p(\mu_K)$. In this, we observe that removing the logarithmic factor in Theorem (4.1.3) is in fact equivalent to Bourgain's hyperplane conjecture.

Theorem (4.1.30) [168]: The following statements are equivalent:

(i) There exists $A > 0$ so that $L_n \le A$ for any $n \ge 1$.

(ii) There exists $B > 0$ so that for any $n \ge 1$ and an isotropic convex body $K \subset \mathbb{R}^n$, we have:

$$V.Rad.\left(Z_p(K)\right) \ge \frac{\sqrt{p}}{B},\ \forall 1 \le p \le \frac{q^\#(\mu_K)}{B}. \tag{33}$$

The proof is based on the following construction from Bourgain, Klartag and Milman [170]. Given $m \ge 1$, let K_m denote an isotropic convex body with $L_{K_m} \ge cL_m$. Choosing $c > 0$ appropriately, it is well-known (see, e.g., the last remark in [173]) that we may assume that K_m is centrally-symmetric and satisfies $K_m \subset \sqrt{m}B_m$. We also set $D_m := \sqrt{m+2}B_m$, and note that D_m is isotropic. Given $1/n \le \lambda < 1$, consider the Cartesian product:

$$T_\lambda = K_{\lfloor\lambda n\rfloor} \times D_{\lfloor(1-\lambda)n\rfloor} \subseteq \mathbb{R}^n.$$

Clearly, T_λ is a centrally-symmetric isotropic convex body, and since $L_{D_m} \simeq 1$, it follows that:

$$L_{T_\lambda} \simeq L_{\lfloor\lambda n\rfloor}^{\frac{\lfloor\lambda n\rfloor}{n}} \simeq L_{\lfloor\lambda n\rfloor}^{\lambda}. \tag{34}$$

Lemma (4.1.31) [168]: For any pair of centrally-symmetric convex bodies $K_1 \subset \mathbb{R}^{n_1}, K_2 \subset \mathbb{R}^{n_2}$ and $p \ge 1$, we have:

$$\frac{1}{2}\left(Z_p(K_1) \times Z_p(K_2)\right) \subset Z_p(K_1 \times K_2) \subset Z_p(K_1) \times Z_p(K_2).$$

Proof: Denote $E_1 := \mathbb{R}^{n_1} \times \{0\}$ and $E_2 := \{0\} \times \mathbb{R}^{n_2}$. By definition, $Z_p(K_1 \times K_2) \cap E_1 = Z_p(K_1) \times \{0\}$ and $Z_p(K_1 \times K_2) \cap E_2 = \{0\} \times Z_p(K_2)$. By the symmetries of K_1, K_2 and the convexity of $Z_p(K_1 \times K_2)$, it follows that:

$$Z_p(K_1 \times K_2) \subseteq Z_p(K_1) \times Z_p(K_2).$$

On the other hand, an elementary argument ensures that:

$$Z_p(K_1 \times K_2 \ \supseteq conv\left(Z_p(K_1) \times \{0\}, \{0\} \times Z_p(K_2)\right) \supseteq \frac{1}{2}\left(Z_p(K_1) \times Z_p(K_2)\right).$$

Corollary(4.1.32) [168]: For any $1/n \leq \lambda \leq 1/2$:

$$diam Z_{\lambda n}(T_\lambda) \leq C\sqrt{\lambda n}.$$

Proof: By Lemma (4.1.31) we see that:

$$diam(Z_{\lambda n}(T_\lambda)) \leq diam\left(Z_{\lambda n}(K_{\lfloor \lambda n \rfloor})\right) + diam Z_{\lambda n}(D_{\lceil (1-\lambda)n \rceil}).$$

Observe that $diam(Z_{\lambda n}(K_{\lfloor \lambda n \rfloor})) \leq diam(K_{\lfloor \lambda n \rfloor}) \leq 20\sqrt{\lambda n}$. As for the other summand, a straightforward computation reveals that when $1/n \leq \lambda \leq 1/2$:

$$Z_{\lambda n}(D_{\lceil (1-\lambda)n \rceil}) \simeq \sqrt{\lambda}\sqrt{n} B_{\lceil (1-\lambda)n \rceil}.$$

The assertion now follows.

Recall that for any isotropic convex body $K \subset \mathbb{R}^n$:

$$q^\#(K) = q^\#(\mu_K) := \sup\left\{q \geq 1;\ diam\left(Z_q(K)\right) \leq c^\# \sqrt{n}\right\}, \qquad (35)$$

where $c^\# > 0$ is an appropriate universal constant.

Corollary (4.1.33) [168]: For any $n \geq 1$, there exists a centrally-symmetric isotropic convex body $K \subset \mathbb{R}^n$, such that:

(a) $q^\#(K) \geq cn$; and

(b) $log L_K \geq c\ log L_n$,

where $c > 0$ is a universal constant.

Proof: Take $\lambda_0 := min\{(c^\#/C)^2, 1/2\}$, where C is the constant from Corollary (4.1.32). Then $K = T_{\lambda_0}$satisfies the first assertion in view of the choice of λ_0, and by (34):

$$L_K \simeq L_{\lfloor \lambda_0 n \rfloor}^{\lambda_0} \gtrsim L_n^{\lambda_0},$$

where the inequality $L_{\lfloor \lambda n \rfloor} \gtrsim L_n$ for any $0 < c \leq \lambda \leq 1$ follows from the techniques in [170,Section 3]. Since $L_K \geq c > 0$, the second assertion follows.

If $L_n \leq A$, then $Vol_n(K)^{\frac{1}{n}} \geq 1/A$ for any isotropic convex body $K \subset \mathbb{R}^n$. Consequently, by the Lutwak–Yang–Zhang lower-bound (5), we even have:

$$V.Rad.\left(Z_p(K)\right) \geq \frac{c}{A}\sqrt{p}, \qquad \forall 1 \leq p \leq n.$$

For the other direction, apply our assumption (33) to the isotropic convex body $K \subset \mathbb{R}^n$ from Corollary (4.1.33), and obtain:

$$\frac{\sqrt{p}}{B} \leq V.Rad.\left(Z_p(K)\right) \leq V.Rad.(K) \simeq \frac{\sqrt{n}}{L_K}, \qquad \forall 1\ \leq p \leq q^\#(K)/B.$$

Corollary (4.1.33) then implies that:

$$L_n \leq (L_K)^C \leq \left(C' B^{\frac{3}{2}} \sqrt{\frac{n}{q^\#(K)}}\right)^C \leq C_1 B^{C_2},$$

as required.

Section (4.2): Slicing Inequalities for Measures of Convex Bodies

The slicing problem [204,205,201,232], a major open problem in convex geometry, asks whether there exists an absolute constant Cso that for any origin-symmetric convex body Kin $\mathbb{R}^n$of volume 1 there is a hyperplane of Kwhose $(n-1)$-dimensional volume is greater than $1/C$. In other words, does there exist a constant Cso that for any $n \in N$ and any origin-symmetric convex body Kin $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^{\perp}| \,, \tag{36}$$

where $\xi^{\perp}$ is the central hyperplane in R^n perpendicular to ξ, and $|K|$ stands for volume of proper dimension? The best current result $C \leq O(n^{1/4})$ is due to Klartag [213], who removed the logarithmic term from an earlier estimate of Bourgain [206]. see [208]for the history and partial results.

For certain classes of bodies the question has been answered in affirmative. These classes include unconditional convex bodies (as initially observed by Bour-gain; see also [232,212,203,208]), unit balls of subspaces of L_p[209], intersection bodies [209,Theorem9.4.11], zonoids, duals of bodies with bounded volume ratio [232], the Schatten classes [226], k-intersection bodies [223,220].

Iterating (36) one gets the lower dimensional slicing problem asking whether the inequality

$$|K|^{\frac{n-k}{n}} \leq C^k \max_{H \in Gr_{n-k}} |K \cap H| \tag{37}$$

holds with an absolute constant C where $1 \leq k \leq n-1$ and $Gr_{n-k}$is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$.

We prove (37) in the case where $k \geq \lambda_n, 0 < \lambda < 1$, with the constant $C = C(\lambda)$ dependent only on λ. Moreover, we prove this result in a more general setting of arbitrary measures in place of volume. We consider the following generalization of the slicing problem.

Problem (4.2.1)[200]: Does there exist an absolute constant C so that for every $n \in \mathbb{N}$, every integer $1 \leq k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$, and every measure μ with non-negative even continuous density f in $\mathbb{R}^n$,

$$\mu(L) \leq C^k \max_{H \in Gr_{n-k}} \mu(L \cap H) |L|^{\frac{k}{n}} \,. \tag{38}$$

Here $\mu(B) = \int_B f$for every compact set Bin $\mathbb{R}^n$, and $\mu(B \cap H) = \int_{B \cap H} f$ is the result of integration of the restriction of f to H with respect to Lebesgue measure in H.

In many cases we will write (38) in an equivalent form

$$\mu(L) \leq C^k \frac{n}{n-k} \; c_{n,k} \; \max_{H \in Gr_{n-k}} \mu(L \cap H) |L|^{\frac{k}{n}}, \tag{39}$$

where $c_{n,k} = |B_2^n|^{\frac{n-k}{n}} / |B_2^{n-k}|$, and B_2^n is the unit Euclidean ball in $\mathbb{R}^n$. Note that $c_{n,k} \in (e^{-k/2}, 1)$ (see for example [221, Lemma 2.1]), and

$$1 \leq \frac{n}{n-k} \leq e^{\frac{k}{n-k}} \leq e^k,$$

so these constants can be incorporated in the constant C.

It appears that some results on the original slicing problem can be extended to the case of arbitrary measures. The first result of this kind was established in [217], namely, when L is an intersection body and $k = 1$, inequality (39)holds with the best possible constant $C = 1$. This result was later proved for arbitrary k in [222]. For arbitrary origin-symmetric convex bodies, inequality (39) was proved with $C = \sqrt{n}$ in [218]and [219], for $k = 1$ and for arbitrary k, respectively. When L is the unit ball of a subspace of $L_p, p \geq 2$, the constant C can be improved to $n^{\frac{1}{2}-\frac{1}{p}}$; see [220]. In [220], (38) was also proved for the unit balls of normed spaces that embed in $L_p, -\infty < p \leq 2$ with C depending only on p. In the case where $k = 1$ and the measure μ is log-concave, (38)holds for any origin-symmetric convex body with $C \leq O(n^{1/4})$, as shown in [225]using the estimate of Klartag [213]mentioned above and the technique of Ball [201]relating log-concave measures to convex bodies.

We prove inequality (38)for unconditional convex bodies and duals of bodies with finite volume ratio, with an absolute constant C. We also prove that for every $\lambda \in (0, 1)$ there exists a constant $C = C(\lambda)$ so that inequality (38)holds for every $n \in \mathbb{N}$, arbitrary origin-symmetric convex body L, every measure μ with continuous density and every codimension k satisfying $\lambda n \leq k < n$.

we show that the properties of the minimal measures may be different from the case of volume. We prove that for every $n \geq 5$ there exist a symmetric convex body L in $\mathbb{R}^n$and a measure μ with continuous density so that

$$\mu(L) < \frac{n}{n-1} c_{n,1} \min_{\xi \in S^{n-1}} \mu(L \cap \xi^{\perp}) \, |L|^{1/n}.$$

Note that in the case of volume

$$\int_{S^{n-1}} |K \cap \xi^{\perp}| d\sigma(\xi) \leq c_{n,1} |K|^{\frac{n-1}{n}},$$

where σ is the normalized uniform measure on the sphere; see [228].

The approach to suggested is based on the concept of an inter-section body. We reduce the problem to computing the outer volume ratio distance from an origin-symmetric convex body to the class of generalized intersection bodies.

We need several definitions and facts. A closed bounded set Kin $\mathbb{R}^n$ is called a star body if every straight line passing through the origin crosses the boundary of Kat exactly two points different from the origin, the origin is an interior point of K, and the Minkowski functional of K defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The radial function of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n, x \neq 0.$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of Kin the direction of x.

We use the polar formula for volume of a star body

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \qquad (40)$$

The class of intersection bodies was introduced by Lutwak [229]. Let K, L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that $K$is the intersection body of Lif the radius of Kin every direction is equal to the $(n-1)$-dimensional volumeof $L$by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^{\perp}|$$
$$= \frac{1}{n-1} \int_{S^{n-1} \cap \xi^{\perp}} \|\theta\|_L^{-n+1} \, d\theta \; = \frac{1}{n-1} R \, \|\cdot\|_L^{-n+1}(\xi),$$

where $R: C(S^{n-1}) \to C(S^{n-1})$is the spherical Radon transform

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^{\perp}} f(x) dx \,, \qquad \forall f \in C(S^{n-1}).$$

All bodies K that appear as intersection bodies of different star bodies form the class of intersection bodies of star bodies. A more general class of intersection bodiesis defined as follows. If μ is a finite Borel measure on S^{n-1}, then the spherical Radon transform $R\mu$ of μ is defined as a functional on $C(S^{n-1})$acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x) \,, \qquad \forall f \in C(S^{n-1}).$$

A star body Kin $\mathbb{R}^n$is called an intersection body if $\|\cdot\|_K^{-1} = R\mu$ for some measure μ, as functionals on $C(S^{n-1})$, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} \, f(x) dx \; = \int_{S^{n-1}} Rf(x) d\mu(x) \,, \qquad \forall f \in C(S^{n-1}).$$

Intersection bodies played a crucial role in the solution of the Busemann–Petty problem and its generalizations; see [216, Chapter 5].

A generalization of the concept of an intersection body was introduced by Zhang [234]in connection with the lower dimensional Busemann–Petty problem. For $1 \le k \le n-1$, the $(n-k)$-dimensional spherical Radon transform $R_{n-k}: C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k} g(H) = \int_{S^{n-1} \cap H} g(x) \, dx \,, \qquad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

We say that an origin symmetric star body Kin $\mathbb{R}^n$is a generalized k-intersection body, and write $K \in BP_k^n$, if there exists a finite Borel non-negative measure μ on Gr_{n-k}so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_K^{-k} \, f(x) dx \; = \int_{Gr_{n-k}} R_{n-k} g(H) d\mu(x) \, (H). \qquad (41)$$

When $k = 1$ we get the class of intersection bodies. It was proved by Grinberg and Zhang [210, Lemma 6.1]that every intersection body in $\mathbb{R}^n$is a generalized k-intersection body for every $k < n$. More generally, as proved later by E. Milman [231], if m dividesk, then every generalized m-intersection body is a generalized k-intersection body. Note that in [234,210]generalized k-intersection bodies are called "k-intersection bodies".

We need a stability result for generalized k-intersection bodies proved in [219, Theorem1](see also [217,222]for similar results). Here we present a slightly simpler version.

Theorem (4.2.2) [200]: Suppose that $1 \le k \le n-1$, $K$is a generalized k-intersection body in $\mathbb{R}^n$, f is an even continuous non-negative function on K, and $\varepsilon > 0$. If

$$\int_{K\cap H} f \le \varepsilon, \forall H \in Gr_{n-k},$$

then

$$\int_K f \le \frac{n}{n-k}\, c_{n,k}\, |K|^{k/n}\varepsilon.$$

Recall that $c_{n,k} \in (e^{-k/2}, 1)$.

Proof: Writing integrals in spherical coordinates we get

$$\int_K f = \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta\,,$$

and

$$\int_{K\cap H} f = \int_{S^{n-1}\cap H} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\, dr \right) d\theta$$

$$= R_{n-k} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\ \cdot)\, dr \right)(H),$$

so the condition of the theorem can be written as

$$R_{n-k} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r\ \cdot) dr \right)(H) \le \varepsilon, \qquad \forall H \in Gr_{n-k}.$$

Integrate both sides with respect to the measure μ on Gr_{n-k}that corresponds to Kas a generalized k-intersection body by (41). We get

$$\int_{S^{n-1}} \|\theta\|_K^{-1} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\, dr \right) d\theta \le\ \varepsilon\mu(Gr_{n-k}).$$

Estimate the integral in the left-hand side from below using $f \ge 0$:

$$\int_{S^{n-1}} \|\theta\|_K^{-1} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta)\, dr \right) d\theta$$

$$= \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta)\, dr \right) d\theta$$

$$+ \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} \left(\|\theta\|_K^{-k} - r^k\right) r^{n-k-1} f(r\theta)\, dr \right) d\theta$$

$$\ge\ \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) dr \right) d\theta = \int_K f.$$

Now we estimate $\mu(Gr_{n-k})$from above. We use $1 = R_{n-k}1(H)/|S^{n-k-1}|$for every $H \in Gr_{n-k}$, definition (41), Hölder's inequality and the fact that $n|B_2^n\ = |S^{n-1}|$:

$$\mu(Gr_{n-k}) = \frac{1}{|S^{n-k-1}|} \int_{Gr_{n-k}} R_{n-k}1(H) d\mu(H)$$

$$= \frac{1}{|S^{n-k-1}|}\ |S^{n-1}|\ \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta$$

$$\leq \frac{1}{|S^{n-k-1}|}\,|S^{n-1}|^{\frac{n-k}{n}}\left(\int_{S^{n-1}}\|\theta\|_K^{-n}d\theta\right)^{\frac{k}{n}}$$
$$=\frac{1}{|S^{n-k-1}|}|S^{n-1}|^{\frac{n-k}{n}}\,n^{k/n}|K|^{k/n}=\frac{n}{n-k}\;c_{n,k}|K|^{k/n}.$$
Combining the estimates,
$$\int_K f \leq \frac{n}{n-k}\;c_{n,k}|K|^{k/n}\varepsilon.$$
For a convex body L in $\mathbb{R}^n$ and $1 \leq k < n$, denote by
$$o.v.r.(L, BP_k^n) = \inf\left\{\left(\frac{|K|}{|L|}\right)^{1/n} : L \subset K, K \in BP_k^n\right\}$$
the outer volume ratio distance from a body L to the class BP_k^n.

Corollary (4.2.3) [200]: Let L be an origin-symmetric star body in $\mathbb{R}^n$. Then for any measure μ with even continuous density on Lwe have
$$\mu(L) \leq \left(o.v.r.(L, BP_k^n)\right)^k\,\frac{n}{n-k}c_{n,k}\max_{H\in Gr_{n-k}}\mu(L\cap H)|L|^{k/n}.$$
Proof: Let $C > o.v.r.(L, BP_k^n)$, then there exists a body K in BP_k^nsuch that $L \subset K$ and $|K|^{1/n} \leq C|L|^{1/n}$.

Let g be the density of the measure μ, and define a function fon $K$by $f = g\chi L$, where $\chi L$is the indicator function of L. Clearly, $f \geq 0$everywhere on K. Put
$$\varepsilon = \max_{H\in Gr_{n-k}}\int_{K\cap H} f = \max_{H\in Gr_{n-k}}\int_{L\cap H} g = \max_{H\in Gr_{n-k}}\mu(L\cap H)\ ,$$
and apply Theorem(4.2.1) to f, K, ε(fis not continuous, but we can do an easy approximation). We have
$$\mu(L) = \int_L g\ = \int_K f \leq \frac{n}{n-k}c_{n,k}|K|^{k/n}\max_{H\in Gr_{n-k}}\mu(L\cap H)$$
$$\leq C^k\frac{n}{n-k}c_{n,k}|L|^{k/n}\max_{H\in Gr_{n-k}}\mu(L\cap H).$$
The result follows by sending C to $o.v.r.(L, BP_k^n)$.

Let $e_i, 1 \leq i \leq n$, be the standard basis of $\mathbb{R}^n$. A star body Kin $\mathbb{R}^n$is called unconditional if for every choice of real numbers x_i and $\delta_i = \pm 1, 1 \leq i \leq n$ we have
$$\left\|\sum_{i=1}^{n}\delta_i x_i e_i\right\|K = \left\|\sum_{i=1}^{n}x_i e_i\right\|K.$$
Theorem (4.2.4) [200]: For every $n \in \mathbb{N}$, every $1 \leq k < n$, every unconditional convex body L in $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on L
$$\mu(L) \leq e^k\,\frac{n}{n-k}c_{n,k}\max_{H\in Gr_{n-k}}\mu(L\cap H)\ |L|^{k/n}. \qquad (42)$$
Proof: By a result of Lozanovskii [207](see the proof in [233, Corollary 3.4]), there exists a linear operator $T: \mathbb{R}^n \to \mathbb{R}^n$ so that
$$T(B_\infty^n) \subset L \subset nT(B_1^n),$$

where B_1^n and B_∞^n are the unit balls of the spaces ℓ_1^n and ℓ_∞^n, respectively. Let $K = nT(B_1^n)$. By [214, Theorem 3] and the fact that a linear transformation of an intersection body is an intersection body (see [229] or [214, Theorem 1]), the body K is an intersection body in $\mathbb{R}^n$. By a result of Grinberg and Zhang [210, Lemma 6.1], K is a generalized k-intersection body for every $1 \le k < n$.

Since $|B_1^n| = 2^n/n!$ (see for example [216, Lemma 2.19]), we have $|K|^{1/n} \le 2e|\, detT|^{1/n}$. On the other hand, $|T(B_\infty^n)| = 2^n|\, detT|$, and $T(B_\infty^n) \subset L$, so $|K|^{1/n} \le e\, |L|^{1/n}$. Therefore, $o.v.r(L, BP_k^n) \le e$. Now (42)follows from Corollary(4.2.3).

The volume ratio of a convex body K in $\mathbb{R}^n$ is defined by

$$v.r.(K) = inf_E \ \left\{\left(\frac{|K|}{|E|}\right)^{1/n} : E \subset K, E - ellipsoid\right\}.$$

The following argument is standard and first appeared in [207] and [232]. Let $K^\circ$and E° be polar bodies of K and E, respectively. If E is an ellipsoid, then

$$|E||E^\circ| = |B_2^n|^2.$$

By the reverse Santalo inequality of Bourgain and Milman [207], there exists an absolute constant $c > 0$ such that

$$(|K||K^\circ|)^{1/n} \ge \frac{c}{n}\ .$$

Combining these and using the asymptotics of B_2^n we get that there exists an absolute constant C such that

$$\left(\frac{|E^\circ|}{|K^\circ|}\right)^{1/n} \le C\left(\frac{|K|}{|E|}\right)^{k/n}.$$

Theorem (4.2.5) [200]: There exists an absolute constant C such that for every $n \in \mathbb{N}$, every $1 \le k < n$, every origin-symmetric convex body L in $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on L

$$\mu(L) \le \ (C\ v.r(L^\circ))^k\ \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap\ H)\ |L|^{k/n}.$$

Proof: If E is an ellipsoid, $E \subset L^\circ$, then the ellipsoid E° contains L. Also every ellipsoid is an intersection body as a linear image of the Euclidean ball, so it is also a generalized k-intersection body for every k. By the argument before the statement of the theorem,

$$o.v.r(L, BP_n^k) \le C\ v.r.(L^\circ).$$

The result follows from Corollary(4.2.3).

The outer volume ratio distance from a general convex body to the class of generalized k-intersection bodies was estimated in [224].

Proposition (4.2.6) [200]: (See [224, Theorem 1.1].)Let L be an origin-symmetric convex body in $\mathbb{R}^n$, and let $1 \le k \le n-1$. Then

$$o.v.r.(L, BP_k^n) \le C_0\sqrt{\frac{n}{k}}\left(log\ \left(\frac{en}{k}\right)\right)^{3/2},$$

where C_0 is an absolute constant.

Theorem (4.2.7) [200]: (See [233, p. 120].)For every $\alpha \in (0, 2)$ and every origin-symmetric convex body K in $\mathbb{R}^n$, there exists a linear image $K\alpha$of Ksuch that

$$\max\{N(K\alpha, tB_2^n), N(B_2^n, tK\alpha)\} \le \ exp\left(\frac{cn}{t^\alpha(2-\alpha)}\right),$$

for every $t \ge 1$, where c is an absolute constant.

Theorem(4.2.7) implies a generalization of V. Milman's reverse Brunn–Minkowski inequality; one can find this in [233]as a combination of several results. We present a proof for the sake of completeness.

Corollary (4.2.8) [200]: Let $\alpha \in [1,2)$, let K be an origin-symmetric convex body in $\mathbb{R}^n$, and let $K\alpha$ be the position of K established in Theorem(4.2.7). Then for every $t \ge 1$,

$$|K_\alpha + tB_2^n|^{1/n} \le \ 2e^c\ t|K_\alpha|^{1/n} \frac{1}{2-\alpha}\ exp\left(\frac{c}{t^\alpha(2-\alpha)}\right),$$

where c is the same absolute constant as in Theorem(4.2.7).

Proof: We first use the part of Theorem (4.2.7) estimating $N(B_2^n, tK_\alpha)$. Put $t = (2-\alpha)^{-1/\alpha}$ in Theorem(4.2.7). Then

$$|B_2^n|^{1/n} \le t|K_\alpha|^{1/n}\ (N(B_2^n, tK_\alpha))^{1/n}$$
$$\le (2-\alpha)^{-1/\alpha} e^c |K_\alpha|^{1/n} \le \frac{e^c}{2-\alpha}|K_\alpha|^{1/n}.$$

Now for every $t \ge 1$we use the estimate for $N(K_\alpha, tB_2^n)$ from Theorem(4.2.7). We have $\frac{|K_\alpha + tB_2^n|^{1/n}}{2t|K_\alpha|^{1/n}} \le \frac{e^c}{2-\alpha}\frac{|K_\alpha + tB_2^n|^{1/n}}{2t|B_2^n|^{1/n}}$

$$\le \frac{e^c}{2-\alpha}\ (N(K_\alpha + tB_2^n, 2tB_2^n))^{1/n}$$
$$\le \frac{e^c}{2-\alpha}(N(K_\alpha, tB_2^n))^{1/n} \le \frac{e^c}{2-\alpha}\ exp\ \left(\frac{c}{t^\alpha(2-\alpha)}\right).$$

In the proof of Theorem 1.1in [204, p. 2705], we have $\alpha = 2 - \frac{1}{loge\frac{n}{k}}$ and $t^\alpha(2-\alpha) = \frac{n}{k}$, so $t \sim \sqrt{\frac{n}{k}log(\frac{en}{k})}$. Then Corollary(4.2.8)implies

$$|K_\alpha + tB_2^n|^{1/n} \le c'^{\sqrt{\frac{n}{k}}\left(log\left(\frac{en}{k}\right)\right)^{3/2}|K_\alpha|^{1/n}},$$

where c' is an absolute constant. Using this estimate in place of Corollary 3.2 in [224,p.2705], we get Proposition(4.2.6).

Proposition(4.2.6) in conjunction with Corollary(4.2.3)implies the following slicing inequality.

Theorem(4.2.9) [200]: There exists an absolute constant C_0 such that for every $n \in \mathbb{N}$, every $1 \le k < n$, every origin-symmetric convex body Lin $\mathbb{R}^n$ and every measure μ with even continuous non-negative density on L

$$\mu(L) \le C_0^k \left(\sqrt{\frac{n}{k}}\left(log\ \left(\frac{en}{k}\right)\right)^{3/2}\right)^k \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap H)\ |L|^{1/n}.$$

Corollary (4.2.10) [200]: If the codimension k satisfies $\lambda n \le k < n$, for some $\lambda \in (0,1)$, then for every origin-symmetric convex body L in $\mathbb{R}^{n-1}$and every measure μ with continuous non-negative density in $\mathbb{R}^n$,

$$\mu(L) \leq C_0^k \left(\sqrt{\frac{(1-log\lambda)^3}{\lambda}}\right)^k \frac{n}{n-k} c_{n,k} \max_{H\in Gr_{n-k}} \mu(L\cap H)\,|L|^{1/n}$$

where $C_0$is an absolute constant.
We consider Schwartz distributions, i.e. continuous functionals on the space $S(\mathbb{R}^n)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function $\phi \in S(\mathbb{R}^n)$. For any even distribution f, we have $(\hat{f})^\wedge = (2\pi)^n f$.
Throughout the bodies $K$and L are origin-symmetric. If $K$is a convex body and $0 < p < n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on R^n and represents a distribution acting by integration. Suppose that $K$is infinitely smooth, i.e. $\|\cdot\|_K \in C^\infty(S^{n-1})$ is an infinitely differentiable function on the sphere. Then by [216,Lemma3.16], the Fourier transform of $\|\cdot\|_K^{-p}$is an extension of some func-tion $g \in C^\infty(S^{n-1})$ to a homogeneous function of degree $-n + p$ on $\mathbb{R}^n$. When we write $\left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)$, we mean $g(\xi), \xi \in S^{n-1}$.
For $f \in C^\infty(S^{n-1})$ and $0 < p < n$, we denote by

$$(f\cdot r^{-p})(x) = f(x/|x|_2)|x|_2^{-p}$$

the extension of f to a homogeneous function of degree $-p$ on $\mathbb{R}^n$. Again by [216,Lemma3.16], there exists $g \in C^\infty(S^{n-1})$ such that

$$(f\cdot r^{-p})^\wedge = g\cdot r^{-n+p}.$$

If K, L are infinitely smooth convex bodies, the following spherical version of Parseval's formula was proved in [218](see also [216, Lemma 3.22]): for any $p \in (-n, 0)$

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)\,\|\cdot\|_L^{-n+p}(\xi) = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}dx. \qquad (43)$$

It was proved in [214, Theorem 1]that an origin-symmetric convex body Kin $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$represents a positive definite distribution. If $K$is infinitely smooth, this means that the function $(\|\cdot\|_K^{-1})^\wedge$ is non-negative on the sphere. We also need a result from [215](see also [216, Theorem 3.8]) expressing volume of central hyperplane in terms of the Fourier transform. For any origin-symmetric star body Kin $\mathbb{R}^n$, the distribution $(\|\cdot\|_K^{-n+1})^\wedge$ is a continuous function on the sphere extended to a homogeneous function of degree -1on the whole of $\mathbb{R}^n$, and for every $\xi \in S^{n-1}$,

$$|K\cap\xi^\perp| = \frac{1}{\pi(n-1)}\,(\|\cdot\|_K^{-n+1})^\xi. \qquad (44)$$

In particular, if $K = B_2^n$ and $|\cdot|_2$is the Euclidean norm, then for every $\xi \in S^{n-1}$

$$(|\cdot|_2^{-n+1})^\xi = \pi(n-1)|B_2^{n-1}|. \qquad (45)$$

Lemma (4.2.11) [200]: Let Kbe an origin-symmetric infinitely smooth convex body in $\mathbb{R}^n$. Then

$$\int_{S^{n-1}} (\|\cdot\|_K^{-1})^\wedge(\xi)d\xi \leq \frac{(2\pi)^n}{\pi(n-1)}\, c_{n,1}|K|^{1/n}.$$

Proof: By (45), Parseval's formula, Hölder's inequality, polar formula for volume (40)and $|S^{n-1}| = n|B_2^n|$, we get

$$\int_{S^{n-1}} (\|\cdot\|_K^{-1})^\wedge(\xi)d\xi$$
$$= \frac{1}{\pi(n-1)|B_n^{n-1}|}\int_{S^{n-1}} (\|\cdot\|_K^{-1})^\wedge(\xi)\ (|\cdot|_2^{-n+1})^\wedge(\xi)$$
$$= \frac{(2\pi)^n}{\pi(n-1)|B_2^{n-1}|}\int_{S^{n-1}} \|\theta\|_K^{-1}\,d\theta$$
$$\leq \frac{(2\pi)^n}{\pi(n-1)|B_2^{n-1}|}\ |S^{n-1}|^{\frac{n-1}{n}}\left(\int_{S^{n-1}} \|\theta\|_K^{-n}\,K\,d\theta\right)^{\frac{1}{n}}$$
$$= \frac{(2\pi)^n}{\pi(n-1)|B_2^{n-1}|}\ |S^{n-1}|^{\frac{n-1}{n}} n^{1/n}|K|^{1/n} = \frac{(2\pi)^n n}{\pi(n-1)}\ c_{n,1}|K|^{1/n}.$$

The following theorem provides examples where the minimal measure behaves in a different way from the case of volume. Note that every non-intersection body can be approximated in the radial metric by infinitely smooth non-intersection bodies with strictly positive curvature; see [216, Lemma 4.10]. Different examples of convex bodies that are not intersection bodies (in dimensions five and higher, as in dimensions up to four such examples do not exist) can be found in [216, Chapter 4]. In particular, the unit balls of the spaces $\ell_q^n, q > 2, n \geq 5$ are not intersection bodies.

Theorem (4.2.12) [200]: Suppose that L is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$ with strictly positive curvature that is not an intersection body. Then for small enough $\varepsilon > 0$ there exists an origin-symmetric convex body K in $\mathbb{R}^n, K \subset L,$ such that

$$|K \cap\ \xi^\perp| \leq |L \cap \xi^\perp| - \varepsilon, \qquad \forall\xi \in S^{n-1},$$

but

$$|K|^{\frac{n-1}{n}} > |L|^{\frac{n-1}{n}} - c_{n,1}\varepsilon.$$

Note that $c_{n,1} \in (\frac{1}{\sqrt{e}}, 1)$.

Proof: Since L is infinitely smooth, the Fourier transform of $\|\cdot\|_L^{-1}$is a continuous function on the sphere S^{n-1}. Also, L is not an intersection body, so $(\|\cdot\|_L^{-1})^\wedge < 0$ on an open set $\Omega \subset S^{n-1}$. Let $\phi \in C^\infty(S^{n-1})$ be an even non-negative, not identically zero, infinitely smooth function on S^{n-1} with support in $\Omega \cup -\Omega$. Extend ϕ to an even homogeneous of degree -1function $\phi\ \cdot r^{-1}$ on $\mathbb{R}^n\backslash\{0\}$. The Fourier transform of this function in the sense of distributions is $\psi \cdot r^{-n+1}$ where ψ is an infinitely smooth function on the sphere.

Let ε be a number such that $|B_2^{n-1}|\|\theta\|_L^{-n+1} > \varepsilon > 0$ for every $\theta \in S^{n-1}$. Define a star body $K$by

$$\|\theta\|_K^{-n+1} = \|\theta\|_L^{-n+1} -\ \delta\psi(\theta) - \frac{\varepsilon}{|B_2^{n-1}|}, \qquad \forall\theta \in S^{n-1}, \tag{46}$$

where $\delta > 0$ is small enough so that for every θ

$$|\delta\psi(\theta)| < min\left\{\|\theta\|_L^{-n+1} - \frac{\varepsilon}{|B_2^{n-2}|}\ , \frac{\varepsilon}{|B_2^{n-2}|}\right\}.$$

The latter condition implies that $K \subset L$. Since L has strictly positive curvature, by an argument from [216, p. 96], we can make ε, δ smaller (if necessary) to ensure that the body $K$is convex.

Now we extend the functions in (46) from the sphere to $\mathbb{R}^n\backslash\{0\}$ as homogeneous functions of degree $-n+1$ and apply the Fourier transform. We get that for every $\xi \in S^{n-1}$

$$(\|\cdot\|_K^{-n+1})^{\xi} = (\|\cdot\|_K^{-n+1})^{\xi} - (2\pi)^n\delta\phi(\xi) - \pi(n-1)\varepsilon. \quad (47)$$

Here, we used (45) to compute the last term. By (47), (44) and the fact that the function ϕ is non-negative,

$$|K \cap \xi^{\perp}| = |L \cap \xi^{\perp}| - \frac{(2\pi)^n}{\pi(n-1)}\delta\phi(\xi) - \varepsilon \le |L \cap \xi^{\perp}| - \varepsilon. \quad (48)$$

Multiplying both sides of (47)by $(\|\cdot\|_L^{-1})^\wedge(\xi)$, integrating over S^{n-1} and using Parseval's formula on the sphere, we get

$$(2\pi)^n \int_{S^{n-1}} \|\theta\|_L^{-1}\|\theta\|_K^{-n+1}d\theta$$
$$= (2\pi)^n n|L| - (2\pi)^n\delta \int_{S^{n-1}} \phi(\theta)(\|\cdot\|_L^{-1})^\wedge(\theta)d\theta$$
$$-\pi(n-1)\varepsilon \int_{S^{n-1}}(\|\cdot\|_L^{-1})^\wedge(\theta)d\theta .$$

Since φ is a non-negative function supported in Ω, where $(\|\cdot\|_L^{-1})^\wedge$ is negative, the latter equality implies

$$(2\pi)^n n|L| - \pi(n-1)\varepsilon \int_{S^{n-1}}(\|\cdot\|_L^{-1})^\wedge(\theta)d\theta$$
$$< (2\pi)^n \int_{S^{n-1}} \|\theta\|_L^{-1}\|\theta\|_K^{-n+1}d\theta$$
$$\le (2\pi)^n\left(\int_{S^{n-1}}\|\theta\|_L^{-n}d\theta\right)^{\frac{n-1}{n}}\left(\int_{S^{n-1}}\|\theta\|_L^{-n}d\theta\right)^{\frac{1}{n}}$$
$$= (2\pi)^n n|L|^{\frac{1}{n}}|K|^{\frac{n-1}{n}} .$$

Combining the latter inequality with the estimate of Lemma(4.2.11), we get the result.

Corollary (4.2.13) [200]: Suppose that $L$is an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$ with strictly positive curvature that is not an intersection body. Then there exists an even continuous function $g \ge 0$ on L so that

$$\int_L g < \frac{n}{n-1}c_{n,1}|L|^{1/n} \min_{\xi\in S^{n-1}} \int_{L\cap\xi^{\perp}} g . \quad (49)$$

Proof: By Theorem6there exist $\varepsilon > 0$ and an origin-symmetric convex body $K \subset L$ such that

$$\varepsilon = \min_{\xi\in S^{n-1}}(|L \cap \xi^{\perp}| - |K \cap \xi^{\perp}|) ,$$

but

$$|L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}} < c_{n,1}\varepsilon.$$

Note that the expression for ε follows from (48)and the fact that the function ϕ is non-negative and equal to zero outside of Ω.

Combining these and applying the Mean Value Theorem to the function $t \to t^{\frac{n-1}{n}}$

$$c_{n,1} \min_{\xi\in S^{n-1}}(|L \cap \xi^{\perp}| - |K \cap \xi^{\perp}|) > |L|^{\frac{n-1}{n}} - |K|^{\frac{n-1}{n}}$$

$$\geq \frac{n-1}{n} |L|^{-1/n} (|L| - |K|).$$

The latter shows that $g_0 = \chi_{L\backslash K}$, the indicator function of the set $L \backslash K$, satisfies (49). By simple approximation one can get (49)with a continuous function g.

Section (4.3): Lower Dimensional Sections of Convex Bodies

We discuss lower dimensional versions of the slicing problem and of the Busemann–Petty problem, both in the classical setting and in the generalized setting of arbitrary measures in place of volume, which was put forward by Koldobsky for the slicing problem and by Zvavitch for the Busemann–Petty problem. We introduce an alternative approach which is based on the generalized Blaschke–Petkantschin formula and on asymptotic estimates for the dual affine quermassintegrals.

The classical slicing problem asks if there exists an absolute constant $C_1 > 0$ such that for every $n \geq 1$ and every convex body K in R^n with center of mass at the origin (we call these convex bodies centered) one has

$$|K|^{\frac{n-1}{n}} \leq C_1 \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}| \tag{50}$$

It is well-known that this problem is equivalent to the question if there exists an absolute constant $C_2 > 0$such that

$$L_n := max\{L_k : K\ is\ isotropic\ in\ R^n\} \leq C_2 \tag{51}$$

for all $n \geq 1$(see for background information on isotropic convex bodies and log-concave probability measures). Bourgain proved in [242] that $L_n \leq c\sqrt[4]{n}\log n$, and Klartag [241] improved this bound to $L_n \leq c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [242]. From the equivalence of the two questions it follows that

$$|K|^{\frac{n-1}{n}} \leq c_1 L_n \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}| \leq c_2 \sqrt[4]{n} \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}| \tag{52}$$

for every centered convex body K in R^n.

The natural generalization, the lower dimensional slicing problem, is the following question: Let $1 \leq k \leq n-1$ and let $\alpha_{n,k}$ be the smallest positive constant $\alpha > 0$ with the following property: For every centered convex body K in R^n one has

$$|K|^{\frac{n-k}{n}} \leq \alpha^k \max_{F \in G_{n,n-k}} |K \cap F| \tag{53}$$

Is it true that there exists an absolute constant $C_3 > 0$ such that $\alpha_{n,k} \leq C_3$ for all n and k?

From (52) we have $\alpha_{n,1} \leq cL_n$ for an absolute constant $c > 0$. We also restrict the question to the class of symmetric convex bodies and denote the corresponding constant by $\alpha_{n,k}^{(s)}$.

The problem can be posed for a general measure in place of volume. Let g be a locally integrable non-negative function on R^n. For every Borel subset $B \subseteq R^n$ we define

$$\mu(B) = \int_B g(x)dx, \tag{54}$$

Where, if $B \subseteq F$ for some subspace $F \in G_{n,s}$, $1 \leq s \leq n-1$, integration is understood with respect to the s-dimensional Lebesgue measure on F. Then, for any $1 \leq k \leq n-1$one may define $\alpha_{n,k}(\mu)$as the smallest constant $\alpha > 0$ with the following property: For every centered convex body K in R^n one has

$$\mu(K) \leq \alpha^k \max_{F\in G_{n,n-k}} \mu(K \cap F)\,|K|^{\frac{k}{n}} \tag{55}$$

Koldobsky proved in [245] that if K is a symmetric convex body in R^n and if g is even and continuous on K then

$$\mu(K) \leq \gamma_{n,1} \frac{n}{n-1} \sqrt{n} \max_{\theta \in s^{n-1}} \mu\big(K \cap \theta^{\perp}\ \big)|K|^{\frac{1}{n}} \tag{56}$$

Where, more generally, $\gamma_{n,k} = |B_2^n|^{\frac{n-k}{n}} / \left|B_2^{n-k}\right| \leq 1$ for all $1 \leq k \leq n-1$. In other words, for the symmetric (both with respect to μ and K) analogue $\alpha_{n,1}^{(s)}$ of $\alpha_{n,1}$ one has

$$\sup_{\mu} \alpha_{n,1}^{(s)}(\mu) \leq C_3 \sqrt{n} \tag{57}$$

In[236], Koldobsky obtained estimates for the lower dimensional: if K is a sym-metric convex body in R^n and if g is even and continuous on K then

$$\mu(K) \leq \gamma_{n,k} \frac{n}{n-k} \left(\sqrt{n}\right)^k \max_{F\in G_{n,n-k}} \mu(K \cap F)|K|^{\frac{k}{n}} \tag{58}$$

for every $1 \leq k \leq n-1$. In other words, for the symmetric analogue $\alpha_{n.k}^{(s)}$ of $\alpha_{n,k}$one has

$$\sup_{\mu} \alpha_{n,k}^{(s)}(\mu) \leq C_4 \sqrt{n} \tag{59}$$

We provide a different proof of this fact; our method allows us to drop the symmetry and continuity assumptions.

Theorem (4.3.1)[235]: Let K be a convex body in R^n with$0 \in int(K)$. Let g be a bounded non-negative measurable function on R^n and let μ be the measure on R^n with density g. For every$1 \leq k \leq n-1$,

$$\mu(K) \leq \left(c_5 \sqrt{n-k}\right)^k \max_{F\in G_{n,n-k}} \mu(K \cap F).\,|K|^{\frac{k}{n}} \tag{60}$$

Where $c_5 > 0$ is an absolute constant. In particular,$\alpha_{n,k}(\mu) \leq c_5\sqrt{n-k}$ In fact, the proof of Theorem(4.3.1) leads to the stronger estimate

$$\mu(K) \leq \left(c_5\sqrt{n-k}\right)^k \left(\int_{G_{n,n-k}} \mu(K \cap F)^n dv_{n,n-k}(F)\right)^{\frac{1}{n}} |K|^{\frac{k}{n}} \qquad (61)$$

The classical Busemann–Petty problem is the following question. Let

K and D be two origin-symmetric convex bodies in R^n such that

$$\left|K \cap \theta^{\perp}\right| \leq \left|D \cap \theta^{\perp}\right|^{G_{n,n-k}} \qquad (62)$$

for all $\theta \in S^{n-1}$. Does it follow that$|K| \leq |D|$? The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$, see Koldobsky's monograph[243]). The isomorphic version of the Busemann–Petty problem asks if there exists an absolute constant $C_4 > 0$ such that whenever K and D satisfy (62) we have$|K| \leq C_4|D|$. This question is equivalent to the slicing problem and to the isotropic constant conjecture (asking if $\{L_n\}$is a bounded sequence). It is known that if K and D are two centered convex bodies in R^n such that (62) holds true for all $\theta \in S^{n-1}$, then

$$|K|^{\frac{n-1}{n}} \leq c_6 L_n |D|^{\frac{n-1}{n}} \qquad (63)$$

Where $c_6 > 0$ is an absolute constant.
The natural generalization, the lower dimensional Busemann–Petty problem, is the following question: Let $1 \leq k \leq n-1$ and let $\beta_{n,k}$ be the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and D in R^n that satisfy

$$|K \cap F| \leq |D \cap F| \qquad (64)$$

for all $F \in G_{n,n-k}$, one has

$$|K|^{\frac{n-k}{n}} \leq \beta^k |D|^{\frac{n-k}{n}} \qquad (65)$$

Is it true that there exists an absolute constant $C_5 > 0$ such that $\beta_{n,k} \leq c_5$ for all n and k?

From (63) we have $\beta_{n,1} \leq c_6 L_n \leq c_7 \sqrt[4]{n}$ for some absolute constant $c_7 > 0$. We also consider the same question for the class of symmetric convex bodies and we denote the corresponding constant by $\beta_{n,K}^{(s)}$.

As in the case of the slicing problem, the same question can be posed for a general measure in place of volume. For any $1 \leq k \leq n-1$and any measure μ on R^n with a locally integrable non-negative density gone may define $\beta_{n,k}(\mu)$as the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and D in R^n that satisfy $\mu(K \cap F) \leq \mu(D \cap F)$ for every$F \in G_{n,n-k}$, one has

$$\mu(K) \leq (\beta)^k \mu(D) \qquad (66)$$

Similarly, one may define the “symmetric” constant $\beta_{n,k}^{(s)}(\mu)$ Koldobsky and Zvavitch [241] proved that $\beta_{n,1}^{(s)} \leq \sqrt{n}$ for every measure μ with an even continuous non-negative density. In fact, the study of these questions in the setting of general measures was initiated by Zvavitch in [240], where he proved that the classical Busemann–Petty problem for general measures has an affirmative answer if $n \leq 4$ and a negative one if $n \geq 5$. We study the lower dimensional question and provide a general estimate in the case where μ has an even log-concave density.

Theorem $(\mathbf{4.3.2})[\mathbf{235}]$: Let μ be a measure on R^n with an even log-concave density g and let $1 \leq k \leq n-1$. Let K be a symmetric convex body in R^n and let D be a compact subset of R^n such that

$$\mu(K \cap F) \leq \mu(D \cap F) \quad (67)$$

for all $F \in G_{n,n-k}$. Then,

$$\mu(K) \leq (c_8 K L_{n-k})^k \mu(D) \quad (68)$$

Where $c_8 > 0$ is an absolute constant.

Comparing Theorem (4.3.2) with the estimate $\beta_{n-1}^{(s)}(\mu) \leq \sqrt{n}$ of Koldobsky and Zvavitch, note that the estimate in [241] is true for an arbitrary measure μ, i.e. the log-concavity of μ is not required; on the other hand, Theorem (4.3.2) is valid for any codimension $k < n$ and the convexity of the second body D is not required.

We prove Theorem(4.3.1)and Theorem (4.3.2). The main tools are the gen-eralized Blaschke–Petkantschin formula and the Busemann–Straus–Grinberg inequality for the dual affine quermassintegrals of a convex body. For the proof of Theorem(4.3.2) we also use a functional version of the latter inequality, recently obtained by Dann , Paouris and $\leq$ In we collect some facts for the case of volume; we obtain the following bounds for the constants $\alpha_{n,k}$ and $\beta_{n,k}$.

Theorem $(\mathbf{4.3.3})[\mathbf{235}]$: For every $1 \leq k \leq n-1$we have

$$\alpha_{n,k} \leq \beta_{n,k} \quad (69)$$

Moreover,

$$\beta_{n,k} \leq \bar{c_1} L_n \quad (70)$$

Where $c_1 > 0$ is an absolute constant. Finally, for codimensions k which are proportional to n we have the stronger bound

$$\beta_{n,k} \leq \bar{c_2}\sqrt{n/k}\,(\log(en/k))^{3/2} \quad (71)$$

where $\bar{c_2} > 0$ is an absolute constant

Most of the estimates in Theorem(4.3.3) are probably known to specialists; we just point out alternative ways to justify them. In particular, Koldobsky has proved in [218] that

$$\beta_{n,k}^{(s)} \leq \bar{c_4}\sqrt{n/k}\,(\log(en/k))^{3/2} \tag{72}$$

for all $1 \leq k \leq n-1$, where $c_4 > 0$ is an absolute constant; this is the symmetric analogue of (71).

We close this article with a general stability estimate in the spirit of Koldobsky's stability theorem (see Theorem(4.3.17)).

Theorem (4.3.4)[235]: Let $1 \leq k \leq n-1$ and let K be a compact set in R^n. If g is a locally integrable non-negative function on R^n such that

$$\int_{G_{n,n-k}} \left(\int_{K \cap F} g(x)dx \right)^n dv_{n,n-k}(F) \leq \varepsilon^n \tag{73}$$

for some $\varepsilon > 0$, then

$$\int_k g(x)dx \leq \left(c_0\sqrt{n-k}\right)^k |K|^{\frac{k}{n}} \varepsilon \tag{74}$$

Where $c_0 > 0$ is an absolute constant.

We work in R^n, which is equipped with a Euclidean structure$\langle .,. \rangle$.We denote the corresponding Euclidean norm by$\| . \|_2$,and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted invariant probability measure onS^{n-1}. We also denote the Haar measure on$O(n)$ by ν. The Grassmann manifold $G_{n,k}$of k-dimensional subspaces of R^n is equipped with the Haar probability measure$\nu_{n,k}$. Let $k \leq n$ and $F \in G_{n,k}$. We will denote the orthogonal projection from R^n onto F by P_F. We also define $B_F = B_2^n \cap F$ and$S_F = S^{n-1} \cap F$.

The letters$c$,$\acute{c}$,c_1,$c_2$etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$such that $c_1 a \leq b \leq c_2 a$. Also if K, $L \subseteq R^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.
A convex body inR^n is a compact convex subset K of R^n with nonempty interior. We say that K is symmetric if $K = -K$. We say that K is centered if the center of mass of K is at the origin, i.e. $\int_k \langle x, \theta \rangle \, dx = 0$ for every $\theta \in S^{n-1}$.

The volume radius of K is the quantity vard$(K) = (|K|/|B_2^n|)^{1/n}$. Integration in polar coordinates shows that if the origin is an interior point of K then the volume radius of K can be expressed as

$$\text{vrad}(K) = \left(\int_{S^{n-1}} \|\theta\|_K^{-n} d\sigma(\theta) \right)^{1/n} \tag{75}$$

where$\|\theta\|_K = min\{t > 0 : \theta \in tK\}$. The radial function of K is defined by$\rho K(\theta) = max\{t > 0 : t\theta \in K\}$,$\theta \in S^{n-1}$.The support function of K is defined by $h_K(y) := max\{\langle x, y\rangle : x \in K\}$ and the mean width of K is the average

$$\omega(K) \coloneqq \int_{S^{n-1}} h_K(\theta) d\sigma(\theta)^{1/n} \tag{76}$$

of h_K on S^{n-1}. The radius $R(K)$of K is the smallest $R > 0$ such that $K \subseteq RB_2^n$. For notational convenience we write K for the homothetic image of volume 1of a convex body$K \subseteq R^n$, i.e. $\overline{K} := |K|^{-1/n} K$.

The polar body $K^o$of a convex body K in R^n with $0 \in \text{int}(K)$ is defined by

$$K^o \coloneqq \{y \in R^n : \langle x, y\rangle \leq 1 \; for \; all \; x \in K\} \tag{77}$$

The Blaschke–Santaló inequality states that if K is centered then$|K||K^o| \leq |B_2^n|^2$, with equality if and only if K is an ellipsoid. The reverse Santaló inequality of Bourgain and$V$. Milman states that there exists an absolute constant $c > 0$ such that, conversely,

$$(|K||K^o|)^{1/n} \geq c/n \tag{78}$$

whenever $0 \in \text{int}(K)$. A convex body K in $R^n$is called isotropic if it has volume 1, it is centered, and if its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such t

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{79}$$

for every θ in the Euclidean unit sphere S^{n-1}. For every centered convex body K in R^n there exists an invertible linear transformation $T \in GL(n)$ such that $T(K)$ is isotropic. This isotropic image of K is uniquely determined up to orthogonal transformations.

For basic facts from the Brunn–Minkowski theory and the asymptotic theory of convex bodies see [247] and [241].

We denote by p_n the class of all Borel probability measures on R^nwhich are absolutely continuous with respect to the Lebesgue measure. The density of μ ∈Pnis denoted by fμ. We say that μ ∈P^n is centered and we write bar(μ) =0if, for all θ∈S^{n-1},

$$\int_{R^n} \langle x,\theta\rangle d\mu(x) = \int_{R^n} \langle x,\theta\rangle f_\mu(x)dx = 0 \qquad (80)$$

A measure μ on R^n is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A and B of R^n and any $\lambda \in (0,1)$. A function$f: R^n \to [0,\infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set and the restriction oflog f to it is concave. It is known that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H, then $\mu \in p_n$ and its density f_μ is log-concave. Note that if K is a convex body in R^n then the Brunn–Minkowski inequality implies that the indicator function $\mathbf{1}_{\boldsymbol{K}}$ of K is the density of a log-concave measure.
If μ is a log-concave measure onR^n with density f_μ, we define the isotropic constant of μ by

$$L_\mu = \left(\frac{\sup_{x\in R^n} f_\mu(x)}{\int_{R^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}} \qquad (81)$$

WhereCov(μ)) is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} = \frac{\int_{R^n} x_i x_j f_\mu(x)dx}{\int_{R^n} f_\mu(x)dx} - \frac{\int_{R^n} x_i f_\mu(x)dx \int_{R^n} x_j f_\mu(x)dx}{\int_{R^n} f_\mu(x) \int_{R^n} f_\mu(x)dx} \qquad (82)$$

We say that a log-concave probability measure μ onR^n is isotropic if $\mathrm{bar}(\mu) = 0$ and Cov$(\mu)$is the identity matrix and we write $\mathcal{L}L_n$ for the class of isotropic log-concave probability measures on R^n Note that a centered convex body K of volume 1 in R^n is isotropic, i.e.it satisfies (79), if and only if the log-concave probability measure μ_K with density $x \longmapsto {L_K^n 1_K}/{L_K}\,(x)$is isotropic. We shall use the fact that for every log-concave measure μ on R^n one has

$$L_\mu \leq KL_n \qquad (83)$$

Where $\kappa > 0$ is an absolute constant (a proof can be found in $[243, Proposition\ 2.5.12]$).

Let $\mu \in p_n$. For every $1 \leq k \leq n-1$ and every $E \in G_{n,k}$, the marginal of μ with respect to E is the probability measure $\pi_E(\mu)$ with density

$$f_{\pi E(\mu)}(x) = \int_{x+E^\perp} f_\mu(y)dy \qquad (84)$$

It is easily checked that if μ is centered, isotropic or log-concave, then $\pi_E(\mu)$ is also centered, isotropic or log-concave, respectively.
If μ is a measure onR^n which is absolutely continuous with respect to the Lebesgue measure, and if f_μ is the density of μ and $f_\mu(0) > 0$, then for every $p > 0$ we define

$$K_p(\mu) \coloneqq K_p(f_\mu) = \left\{x: \int_0^\infty r^{p-1} f_\mu(rx) \geq \frac{f_\mu(0)}{p}\right\} \tag{85}$$

$$\rho K_{p(\mu)}(x) = \left(\frac{1}{f_\mu(0)} \int_0^\infty p r^{p-1} f_\mu(rx) dx\right)^{1/p} \tag{86}$$

for $x \neq 0$. The bodies $K_p(\mu)$ were introduced by$K$. Ball who showed that if μ is log-concave then, for every$p > 0$, $K(\mu)_P$ is a convex body.
For more information on isotropic convex bodies and log-concave measures see [243].
Our approach is based on the following generalized Blaschke–Petkantschin formula (see [248, Chapter 7.2]and [249, Lemma 5.1] for the particular case that we need):
Lemma (4.3.5) [235]: Let $1 \leq s \leq n-1$. There exists a constant $p(n,s) > 0$ such that, for every non-negative bounded Borel

$$\int_{R^n} \dots \int_{R^n} (x_1, \dots, x_s) dx_1 \dots dx_s \tag{87}$$

$$= p(n,s) \int_{G_{n,s}} \int_F \dots \int_F f(x_1, \dots, x_s) |\mathrm{Conv}(0, x_1, \dots, x_s)|^{n-s} dx_1 \dots dx_s dv_{n,s}(F)$$

The exact value of the constant $p(n,s)$ is

$$p(n,s) = \frac{(s!)^{n-s} (n w_n) \dots ((n-s+) w_{n-s+1})}{(S w_n) \dots (S w_n) w_1} \tag{88}$$

We will use some basic facts about Sylvester-type functionals. Let D be a convex body in R^m. For every $p > 0$ we consider the normalized p-th moment of the expected volume of the random simplex $\mathrm{conv}(0, x_1, \dots . x_2)$, the convex hull of the origin and m points from$D$, defined by

$$S_p(D) = \left(\frac{1}{|D|^{m+p}} \int_D \int_D |\mathrm{Conv}(0, x_1, \dots, x_s)|^p dx_1 \dots dx_m\right)^{1/p} \tag{89}$$

Also, for any Borel probability measure ν on R_m we define

$$S_p(v) = \left(\int\limits_{R^m} \int\limits_{R^m} |\mathrm{Conv}(0, x_1, \dots, x_s)|^p d(x_1) \dots d(x_m) \right)^{1/p} \qquad (90)$$

Note that $S_p(v)$ is invariant under invertible linear transformations: $S_p(D) = S_p\big(T(D)\big)$ for every$T \in GL(n)$. The next fact is well-known and goes back to Blaschke (see e.g. [243, Proposition3.5.5]).

Lemma$(4.3.6)$[235]: Let v be a centered Borel probability measure on R^n. Then,

$$m!\, S_2^2(D) = \det(\mathrm{Cov}(v)) \qquad (91)$$

In particular, if D is centered then

$$S_2^2(D) = \frac{L_D^{2m}}{m!} \qquad (92)$$

Hölder's inequality shows that the function $p \longmapsto S_p(D)$ is increasing on $(0, \infty)$.We will need the next reverse Hölder inequality.

Lemma $(4.3.7)$[235]: There exists an absolute constant $\delta > 0$ such that, for every log-concave probability measure v on R^m and every $p > 1$.

$$S_p(v) \le \delta p^m S_1(v) \qquad (93)$$

In particular, for every convex body D in R^m and every$p > 1$,

$$S_p(D) \le \delta p^m S_1(D) \qquad (94)$$

Proof: We use the fact that there exists an absolute constant $\delta > 0$ with the following property: if $v \in p_m$ is a log-concave probability measure then, for any seminorm $u : R^m \to R$ and any$q > p \ge 1$,

$$\left(\int\limits_{R^m} |u(x)|^q dv(x) \right)^{1/q} \le \frac{\delta q}{p} \left(\int\limits_{R^m} |u(x)|^{q^p} dv(x) \right)^{1/p} \qquad (95)$$

This is a consequence of Borel'slemma (see e.g. [243, Theorem 2.4.6]). Next, recall that

$$|\mathrm{Conv}(0, x_1, \dots, x_m)| = \frac{1}{m!} |\det(x_1, .., x_m)| \qquad (96)$$

The function $u_i: R^m \to R$ defined by $x_i \longmapsto |\det(x_1, .., x_n)|$ for fixed x_j in R^m, $j \ne i$, is a seminorm, as is the function $v_i: R^m \to R$ defined by

$$x_i \longmapsto \int\limits_{R^m} \dots \int\limits_{R^m} |\det(x_1, .., x_m)| dx_{i+1} \dots dx_m \qquad (97)$$

for fixed $x_j (1 \le j < i)$ in R^m. By consecutive applications of Fubini's theorem and of (95) we obtain(93).

The next lemma gives upper bounds for the constants $p(n, n-k)$ and $\gamma_{n,k} = |B_2^n|^{\frac{n-k}{n}} / |B_2^n|$; both constants appear frequently.

Lemma$(4.3.8)$[235]: For every $1 \le \mathrm{k} \le \mathrm{n} - 1$we have

$$e^{-k/2} < \gamma_{n,k} < 1 \ \ and \ \left[\gamma_{n,k}^{-n} p(n, n-k)\right]^{\frac{1}{k(n-k)}} \simeq \sqrt{n-k} \qquad (98)$$

Proof: Recall that

$$\gamma_{n,k} := \omega_n^{\frac{n-k}{n}} / \omega_{n-k} \qquad (99)$$

Using the log-convexity of the Gamma function one can check that$e^{-k/2} < \gamma_{n,k} < 1$. Aproof appears in [249, Lemma2.1].

In order to give an upper bound for $p(n, n-k)$ we start from the fact that $\omega_s = \pi^{\frac{s}{2}} / \Gamma\left(\frac{s}{2} + 1\right)$ and use Stirling's approximation. Recall that

$$P(n, n-k) = \left((n-k)!\right)^k \frac{(n\omega_n) \dots \left((k+1)\omega_{k+1}\right)}{\left((n-k)\omega_{n-k}\right) \dots (2\omega_2)\omega_1} \qquad (100)$$

$$= \left((n-k)!\right)^k \binom{n}{k} \frac{\prod_{s=k+1}^{n} \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}}{\prod_{s=1}^{n-k} \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}}$$

$$= \left((n-k)!\right)^k \binom{n}{k} \pi^{\frac{k(n-k)}{2}} \frac{\prod_{s=1}^{n-k} \Gamma\left(\frac{s}{2}+1\right)}{\prod_{s=k+1}^{n} \Gamma\left(\frac{s}{2}+1\right)}$$

Where we have used the identity

$$\frac{1}{2} \sum_{s=k+1}^{n} s - \frac{1}{2} \sum_{s=1}^{n-k} s = \frac{1}{4}\left(n(n+1) - k(k+1) - (n-k)(n-k+1)\right)$$

$$= \frac{1}{2} k(n-k) \qquad (101)$$

Using the estimate

$$\left(\frac{s}{2e}\right)^{\frac{s}{2}} \sqrt{2\pi s} \leq \Gamma\left(\frac{s}{2}+1\right) \leq \left(\frac{s}{2e}\right)^{\frac{s}{2}} \sqrt{2\pi s} e^{\frac{1}{6s}} \leq \left(\frac{s}{2e}\right)^{\frac{s}{2}} \sqrt{2\pi s} e^{\frac{1}{6}} \qquad (102)$$

We get

$$P(n, n-k) \leq \left((n-k)!\right)^k (2\pi e)^{\frac{k(n-k)}{2}} e^{\frac{n-k}{6}} \binom{n}{k}^{1/2} \frac{\prod_{s=1}^{k} s^{\frac{s}{2}} \prod_{s=1}^{n-k} s^{\frac{s}{2}}}{\prod_{s=1}^{n} s^{\frac{s}{2}}} \qquad (103)$$

Let

$$t_m = 1.2^2.3^3 \dots . m^m \qquad (104)$$

It is known that

$$t_m \sim A m^{\frac{m^2}{2} + \frac{m}{2} + \frac{1}{12}} e^{\frac{-m^2}{4}} \qquad (105)$$

as $m \to \infty$, where $A > 0$ is an absolute constant (the Glaisher–Kinkelin constant, see e.g. [248]). Note that

$$\gamma_{n,k}^{-n} = \frac{\omega_{n-k}^{n}}{\omega_{n}^{n-k}} = \frac{\Gamma\left(\frac{n}{2}+1\right)^{n-k}}{\Gamma\left(\frac{n-k}{2}+1\right)^{\mathrm{n}}} \leq \left(\frac{n}{n-k}\right)^{\frac{n(n-k)}{2}} \frac{(\pi n)^{\frac{n-k}{2}} e^{\frac{n-k}{6}}}{\left(\pi(n-k)\right)^{\frac{n}{2}}} \qquad (106)$$
$$\leq e^{\frac{n-k}{6}}\left(\frac{n}{n-k}\right)^{\frac{(n+1)(n-k)}{2}}$$

Using the fact that $n^2 = k^2 + (n-k)^2 + 2k(n-k)$ we get

$$\gamma_{n,k}^{\frac{-n}{k(n-k)}}\left(\frac{t_k t_{n-k}}{t_n}\right)^{\frac{1}{2k(n-k)}} \leq \frac{c_1}{\sqrt{n}}\left(\frac{k}{n}\right)^{\frac{k+1}{4(n-k)}}\left(\frac{n-k}{n}\right)^{\frac{n-k+1}{4k}}\left(\frac{n}{n-k}\right)^{\frac{n+1}{2k}}$$
$$\leq \frac{c_1}{\sqrt{n}}\left(\frac{k}{n}\right)^{\frac{k+1}{4(n-k)}}\left(\frac{n}{n-k}\right)^{\frac{n-k+1}{4k}} \qquad (107)$$
$$\leq \frac{c_1}{\sqrt{n}}\left(\frac{k}{n}\right)^{\frac{k+1}{4(n-k)}}\left(\frac{n}{n-k}\right)^{\frac{n-k}{2k}}\left(\frac{n}{n-k}\right)^{\frac{2k+1}{4k}}$$
$$\frac{c_1}{\sqrt{n}}.\frac{\sqrt{n}}{\sqrt{n-k}} = \frac{c_2}{\sqrt{n-k}} \ .$$

Since

$$\left[\left((n-k)!\right)^k (2\pi e)^{\frac{k(n-k)}{2}} e^{\frac{n-k}{6}}\left(\frac{n}{k}\right)^{1/2}\right]^{\frac{1}{k(n-k)}} \leq c_3(n-k) \qquad (108)$$

We see that

$$\left[\gamma_{n,k}^{-n} p(n,n-k)\right]^{\frac{1}{k(n-k)}} \leq c_3\sqrt{n-k} \qquad (109)$$

For every1 $\leq k \leq n-1$, where $c_0 > 0$ is an absolute constant. The reverse inequality can be obtained from similar computations, but we will not need it in the sequel.

Remark(4.3.9)[235]: An alternative way to give an upper bound for $p(n,n-k)$ is to start by rewriting in the form

$$|K|^{n-k} = p(n,n-k)\int_{G_{n,n-k}} |K \cap F|^n [S_k(K \cap F)]^k dv_{n,n-k}(F) \qquad (110)$$

In particular, setting $K = B_2^n$ we see that if $k \geq 2$ then

$$\omega_n^{n-k} = p(n,n-k)\omega_{n-k}^n\left[S_k B_2^{n-k}\right]^k$$
$$\geq p(n,n-k)\omega_{n-k}^n\left[S_2 B_2^{n-k}\right]^k$$

$$\geq p(n,n-k)\omega_{n-k}^n\left(\frac{L_{B_2^{n-k}}}{\sqrt{n-k}}\right)^{k(n-k)}$$
$$\geq p(n,n-k)\omega_{n-k}^n\left(\frac{c_1}{\sqrt{n-k}}\right)^{k(n-k)}$$

Where $c_1 > 0$ is an absolute constant, which implies that

$$p(n,n-k) \leq \gamma_{n,k}^n\left(c_0\sqrt{n-k}\right)^{k(n-k)} \quad (111)$$

Where$c_0 = c_1^{-1}$. For the case $k = 1$ we can use the fact that $S_1(K \cap F) \geq \delta^{-(n-1)}S_2(K \cap F)$for every$F \in G_{n,n-1}$, and then continue as above. The final estimate is exactly the same as in Lemma(4.3.8):

$$\left[\gamma_{n,k}^{-n}p(n,n-k)\right]^{\frac{1}{k(n-k)}} \leq c_0\sqrt{n-k}\ , \quad (112)$$

and this is what we use. However, the proof of Lemma(4.3.8) shows that this estimate is tight for all n and k; one cannot expect something better.

For the proof of Theorem(4.3.2) we will additionally use the next theorem of D an n, Paouris and Pivovarov from [247].

Theorem(4.3.10)[235]: (Dann–Paouris–Pivovarov). Let u be a bounded integrable non-negative function on R^n with$u_1 > 0$. For every $1 \leq k \leq n-1$ we have

$$\int_{G_{n,n-k}} \frac{1}{\|u|f\|_\infty^k}\left(\int_f u(x)dx\right)^n dv_{n,n-k}(F) \quad (113)$$
$$\leq \gamma_{n,k}^{-n}\left(\int_{R^n} u(x)dx\right)^{n-k}$$

The proof of this fact combines Blaschke–Petkantschin formulas with rearrangement inequalities, and develops ideas that started in[245].

Finally, we use the Busemann–Straus–Grinberg inequality for the dual affine quermassintegrals (introduced by Lutwak, see [243] and [244]) of a convex body K in R^n We use the normalization of [246]: we assume that the volume of K is equal to 1and we set

$$\widetilde{\Phi}_{[\mathrm{k}]}(K) = \left(\int_{G_{n,n-k}} |K \cap F|^n dv_{n,n-k}(F)\right)^{\frac{1}{2n}} \quad (114)$$

For every $1 \leq k \leq n-1$. One can extend the definition to bounded Borel subsets of R^n. The following inequality was proved by Busemann and Straus[244], and independently by Grinberg [240].

Theorem(4.3.11)[235]: (Busemann–Straus, Grinberg). Let K be a compact set of volume 1in R^n. For any $1 \le k \le n-1$ and $T \in SL(n)$ we have

$$\widetilde{\Phi}_{[k]}(K) = \widetilde{\Phi}_{[k]}\big(T(K)\big) \quad (115)$$

Moreover,

$$\widetilde{\Phi}_{[k]}(K) \le \widetilde{\Phi}_{[k]}(\bar{B}_2^n) \quad (116)$$

Where B_2^n is the Euclidean ball of volume1.

We can use Theorem(4.3.11) for compact sets; this can be seen by inspection of Grinberg's argument (for this more general form see also [249, Section 7]). Direct computation and Lemma(4.3.8) show that

$$\widetilde{\Phi}_{[k]}(\bar{B}_2^n) = \left(\frac{\omega_{n-k}^{n}}{\omega_{k}^{n-k}}\right)^{\frac{1}{kn}} = \gamma_{n,k}^{-1/k} \le \sqrt{e} \quad (117)$$

The Busemann–Straus–Grinberg inequality has been also used by Paouris and Valet-tas in [246] where it is proved that if μ is an isotropic log-concave probability measure on R^n then, for every $1 \le k \le \sqrt{n}$ and any $\varepsilon > 0$ one has that, for all k-dimensional subspaces F in an ε-net of the Grassmannian $G_{n,k}$,

$$L_{\pi F(F)} \le \frac{C}{\varepsilon} \quad (118)$$

We prove Theorem(4.3.1) and Theorem(4.3.2).

Let μ be a Borel measure with a bounded non-negative density g onR^n. We consider a convex body K in $R^n$with $0 \in int(K)$, and fix $1 \le k \le n-1$. Applying Lemma(4.3.5)with $s = n-k$for the function $f(x_1, \dots, x_{n-k}) = \prod_{i=1}^{n-k} g(x_i) 1K(x_i)$ we get

$$\mu(K)^{n-k} = \prod_{i=1}^{n-k} \int_{k} g(x_i)dx = \int_{R^n} \dots \int_{R^n} f\big(x_{i,\dots,}x_{n-k}\big)dx_1 \dots dx_{n-k}$$

$$= p(n, n-k) \int_{G_{n,n-k}} \int_{K\cap F} \dots \int_{K\cap F} g(x_1) \dots g(x_{n-k})$$

$$\times |\mathrm{conv}(0, x_1, \dots, x_{n-k})|^k dx_1 \dots dx_{n-k} dv_{n,n-k}(F)$$

$$\le |K \cap F|^k dx_1 \dots dx_{n-k} dv_{n,n-k}(F)$$

$$= p(n, n-k) \int_{G_{n,n-k}} |K \cap F|^k \mu(K \cap F)^{n-k} dv_{n,n-k}(F)$$

$$\le p(n, n-k) \left(\int_{G_{n,n-k}} \mu(K \cap F)^n dv_{n,n-k}(F) \right)^{\frac{n-k}{n}}$$

$$\times\left(\int_{G_{n,n-k}} |K\cap F|^n dv_{n,n-k}(F)\right)^{\frac{k}{n}}$$

In order to estimate the last integral, note that if $\overline{K} = |K|^{-\frac{1}{n}}K$ then

$$\int_{G_{n,n-k}} |K\cap F|^n dv_{n,n-k}(F) = |K|^{n-k} \int_{G_{n,n-k}} |K\cap F|^n dv_{n,n-k}(F) \qquad (119)$$

$$\leq |K|^{n-k} \int_{G_{n,n-k}} |B_2^{-n}\cap F|^n dv_{n,n-k}(F)$$

$$= \gamma_{n,k}^{-n}|K|^{n-k}$$

By Theorem(4.3.11) and (117). Taking into account Lemma(4.3.8) we see that

$$\mu(K)^{n-k} \leq \left(c_0\sqrt{n-k}\right)^{k(n-k)}\left(\int_{G_{n,n-k}} \mu(K\cap F)^n dv_{n,n-k}(F)\right)^{\frac{n-k}{n}}$$

$$|K|^{\frac{k(n-k)}{n}} \qquad (120)$$

This proves (61) and the result follows.

We pass to the proof of Theorem(4.3.2). Let μ be a measure on R^n with a bounded density g. For any $1 \leq k \leq n-1$ and any convex body K in R^n we would like to give upper and lower bounds for $\mu(K)$ in terms of the measures$\mu(K\cap F)$, $F \in G_{n,n-k}$. Alower bound can be given without any further assumption on g. At this point we use Theorem(4.3.10).

Proposition $(\mathbf{4.3.12})[\mathbf{235}]$: Let g be a bounded non-negative measurable function on R^n and let μ be the measure on R^n with density g. For every compact

set D in R^n we have

$$\int_{G_{n,n-k}} \mu(D\cap F)^n dv_{n,n-k}(F) \leq \gamma_{n,k}^{-n}\|g\|_\infty^k \mu(D)^{n-k} \qquad (121)$$

Proof: We apply Theorem(4.3.10) to the function $u = g\cdot \mathbf{1}D$. We simply observe that $\|u|f\|_\infty = \|g|D\cap f\|_\infty \leq \|g\|_\infty$ for all$F \in G_{n,n-k}$. Also,

$$\int_F u(x)dx = \mu(D\cap F)\ and\ \int_{R^n} u(x)dx = \mu(D) \qquad (122)$$

Then, the proposition follows from (113).

We can give an upper bound if we assume that g is an even log-concave function and K is a symmetric convex body.

Proposition $(\mathbf{4.3.13})[\mathbf{235}]$: Let μ be a measure on R^n with an even log-concave density g. For every symmetric convex body K in R^n and any $1 \le k \le n-1$we have

$$\mu(K)^{n-k} \le p(n,n-k)\frac{(\kappa\delta k L_{n-k})^{k(n-k)}}{[(n-k)!]^{\frac{k}{2}}}\frac{1}{\|g\|_\infty^k}$$

$$\int_{G_{n,n-k}} \mu(K\cap F)^n dv_{n,n-k}(F) \qquad (123)$$

Where $\kappa > 0$ is the absolute constant in (83) and $\delta > 0$ is the absolute constant in Lemma (4.3.7).

Proof: We start by writing

$$\mu(K)^{n-k} = \prod_{i=1}^{n-k}\int_k g(x_i)dx$$

$$= p(n,n-k)\int_{G_{n,n-k}}\int_{K\cap F}\dots\int_{K\cap F}|\mathrm{conv}(0,x_1,\dots,x_{n-k})|^k$$

$$\times\prod_{i=1}^{n-k} g(x_i)dx_1\dots dx_{n,n-k}(F)$$

$$= p(n,n-k)\int_{G_{n,n-k}}\mu(K\cap F)^{n-k}[S_k(\mu K\cap F)]^k dv_{n,n-k}(F),$$

Where $\mu K\cap F$ is the even log-concave probability measure with density$gK\cap F:= \frac{1}{\mu(K\cap F)}g\cdot\mathbf{1}K\cap F$. From Lemma(4.3.7) and Lemma(4.3.6) we have

$$[S_k(\mu K\cap F)]^k \le (\delta k)^{k(n-k)}[S_k(\mu K\cap F)]^k = (\delta k)^{k(n-k)}$$
$$\left(\frac{\det(\mathrm{Cov}(\mu K\cap F))}{(n-k)!}\right) \qquad (124)$$

Now, since g is even and log-concave we have

$$\|gK\cap F\|_\infty = \frac{g(0)}{\mu(K\cap F)} = \frac{\|g\|_\infty}{\mu(K\cap F)} \qquad (125)$$

Therefore, (81) implies that

$$\det(\mathrm{Cov}(\mu K\cap F)) = \frac{L_{\mu K\cap F}^{2(n-k)}}{\|gK\cap F\|_\infty^2} \le \mu(K\cap F)^2\frac{(\kappa L_{n-k})^2}{\|g\|_\infty^k} \qquad (126)$$

Where $\kappa > 0$ is the absolute constant in (83). It follows that

$$[S_k(\mu K \cap F)]^k \leq \frac{(\kappa\delta k L_{n-k})^{k(n-k)}}{[(n-k)!]^{\frac{k}{2}}} \frac{\mu(K \cap F)^k}{\|g\|_\infty^k} \tag{127}$$

Combining Proposition(4.3.12) and Proposition(4.3.13) we see that

$$\mu(k)^{n-k} \leq p(n,n-k)\frac{(\kappa\delta k L_{n-k})^{k(n-k)}}{[(n-k)!]^{\frac{k}{2}}} \frac{1}{\|g\|_\infty^k} \tag{128}$$

$$\int_{G_{n,n-k}} \mu(K \cap F)^n dv_{n,n-k}(F) \leq p(n,n-k)\frac{(\kappa\delta k L_{n-k})^{k(n-k)}}{[(n-k)!]^{\frac{k}{2}}} \frac{1}{\|g\|_\infty^k}$$

$$\int_{G_{n,n-k}} \mu(K \cap F)^n dv_{n,n-k}(F) \leq p(n,n-k)\frac{(\kappa\delta k L_{n-k})^{k(n-k)}}{[(n-k)!]^{\frac{k}{2}}} \frac{1}{\|g\|_\infty^k}$$

$$\gamma_{n,k}^{-n}\|g\|_\infty^k \mu(D)^{n-k} \leq (c_8 \kappa L_{n-k})^{k(n-k)}\mu(D)^{n-k}$$

For some absolute constant $c_8 > 0$, where in the last step we have used the estimate

$$p(n,n-k) \leq \gamma_{n,k}^n \left(c_0\sqrt{n-k}\right)^{k(n-k)} \tag{129}$$

From Lemma (4.3.8) This completes the proof.

We collect some estimates for the volume version of the slicing problem and of the Busemann–Petty problem. The first observation is that any upper bound for β^n ,kimplies an upper bound for the lower dimensional slicing problem.

Proposition(4.3.14)[235]: There exists an absolute constant $c > 0$ such that

$$\alpha_{n,k} \leq \beta_{n,k} \tag{130}$$

For all $n \geq 2$ and $1 \leq k \leq n-1$.

Proof: Consider a centered convex body K in R^n, fix $1 \leq k \leq n-1$ and choose $r > 0$ such that

$$\max_{F \in G_{n,n-k}} |K \cap F| = \omega_{n-k} r^{n-k} \tag{131}$$

If we set $B(r) = rB_2^n$ then we have $|K \cap F| \leq |B(r) \cap F|$ for all$F \in G_{n,n-k}$, therefore

$$|K|^{\frac{n-k}{n}} \leq \left(\beta_{n,k}\right)^k |B(r)|^{\frac{n-k}{n}} = \left(\beta_{n,k}\right)^k \omega_n^{\frac{n-k}{n}} r^{n-k} \tag{132}$$

It follows that

$$|K|^{\frac{n-k}{n}} \leq \gamma_{n,k}\left(\beta_{n,k}\right)^k \max_{F \in G_{n,n-k}} |K \cap F| \tag{133}$$

Since$\gamma_{n,k} < 1$ we get the result .

Next, we give two upper bounds for $\beta_{n,k}$ these are essentially contained in the works of Dafnis and Paouris [245] and[246].

Proposition(4.3.15)[235]: Let K be a convex body and D be a compact set in R^n that satisfy

$$|K \cap F| \leq |D \cap F| \tag{134}$$

For all $F \in G_{n,n-k}$. Then,

$$|K|^{\frac{n-k}{n}} \leq |\bar{c}_1 L_k|^k |D|^{\frac{n-k}{n}} \tag{135}$$

Where $c_1 > 0$ is an absolute constant. In particular,

$$\beta_{n,k} = c\sqrt[4]{n}, \tag{136}$$

Where $c \ > 0$ is an absolute constant.

Proof: Recall that $\bar{A} = |A|^{-1/n} A$. Using (134) and the definition of$\widetilde{\Phi}_{[\mathrm{k}]}(A)$we write

$$|K|^{n-k}\left[\widetilde{\Phi}_{[\mathrm{k}]}(\bar{K})\right]^{kn} = \int_{G_{n,n-k}} |K \cap F|^n dv_{n,n-k}(F) \tag{137}$$

$$\leq \int_{G_{n,n-k}} |D \cap F|^n dv_{n,n-k}(F)$$

$$= |D|^{n-k}\left[\widetilde{\Phi}_{[\mathrm{k}]}(\bar{D})\right]^{kn} \leq e^{\frac{kn}{2}} |D|^{n-k}$$

By the affine invariance of ˜$\widetilde{\Phi}_{[\mathrm{k}]}(A)$, if $\widetilde{K}$ is an isotropic image of K we have

$$\widetilde{\Phi}_{[\mathrm{k}]}(\widetilde{K}) = \widetilde{\Phi}_{[\mathrm{k}]}(\bar{K}) \tag{138}$$

Now, we use some standard facts from the theory of isotropic convex bodies (see [243, Chapter5]). For every $1 \ \leq k \leq n \ - 1$ and$F \in G_{n,n-k}$,, the body $\overline{K_{k+1}}\left(\pi_{F^\perp}(\mu_{\tilde{k}})\right)$ satisfies

$$\left(\widetilde{K} \cap F\right)^{1/k} \geq c_1 \frac{L_{\overline{K_{k+1}}}\left(\pi_{F^\perp}(\mu_{\tilde{k}})\right)}{L_k}\ , \tag{139}$$

Where $c_1 > 0$ is an absolute constant. It follows that

$$\widetilde{\Phi}_{[\mathrm{k}]}(\widetilde{K}) L_k \geq \left(\int_{G_{n,n-k}} \left(c_1 L_{\overline{K_{k+1}}}\left(\pi_{F^\perp}(\mu_{\tilde{k}})\right)\right)^{kn} dv_{n,n-k}(F) \right)^{1/kn} \tag{140}$$

Since $L_{\overline{K_{k+1}}}\left(\pi_{F^\perp}(\mu_{\tilde{k}})\right) \geq c_2$ for every $F \in G_{n,n-k}$where $c_2 > 0$ is an absolute constant, we get

$$\widetilde{\Phi}_{[\mathrm{k}]}(\bar{K}) L_k \left(\int_{G_{n,n-k}} \left(c_1 L_{\overline{K_{k+1}}}\left(\pi_F(\mu_{\tilde{k}})\right)\right)^{kn} dv_{n,n-k}(F) \right)^{1/kn} \geq c_3, \tag{141}$$

Where $c_3 = c_1 c_2$ Combining the above we obtain (135). The second claim of the proposition follows from Klartag's general upper bound for L_n. The next proposition provides a better bound in the case where the codimension k is "large".

Proposition $(4.3.16)$[235]: Let K be a convex body and D be a compact set in R^n that satisfy

$$|K \cap F| \leq |D \cap F| \tag{142}$$

For all $F \in G_{n,n-k}$ then,

$$|K|^{\frac{n-k}{n}} \leq \left(\bar{c}_2 \sqrt{n/k}\,(\log(en/k))^{\frac{3}{2}}\right)^k |D|^{\frac{n-k}{n}} \tag{143}$$

Where $c_2 > 0$ is an absolute constant. In particular,

$$\beta_{n,k} \leq \bar{c}_2 \sqrt{n/k}\,(\log(en/k))^{\frac{3}{2}} \tag{144}$$

Proof: We may assume that the volume of K is equal to 1. We consider the quantities

$$\widetilde{W}_{[k]}(K) = \left(\int_{G_{n,n-k}} |K \cap F| dv_{n,k}(F)\right)^{\frac{1}{k}} \tag{145}$$

And

$$I_{-k}(K) = \left(\int_k \|x\|_2^{-k} dx\right)^{-\frac{1}{k}} \tag{146}$$

Integration in polar coordinates shows that

$$\widetilde{W}_{[k]}(K) I_{-k}(K) = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} = \widetilde{W}_{[k]}(\bar{B}_2^n) I_{-k}(\bar{B}_2^n) \tag{147}$$

And that $\left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} \simeq \sqrt{n}$. It was proved in [245] that there exists $T \in SL(n)$ such that the body $K_2 = T(K)$ satisfies

$$I_{-k}(K_2) \leq c_1 \sqrt{n} \sqrt{n/k}\,(\log(en/k))^{\frac{3}{2}} \tag{148}$$

By the affine invariance of $\widetilde{\Phi}_{[k]}(K)$ and by Hölder's inequality we have

$$\widetilde{\Phi}_{[k]}(K) = \widetilde{\Phi}_{[k]}(K_2) \geq \widetilde{W}_{[k]}(K_2) \geq \frac{c_{2\sqrt{n}}}{I_{-k}(K_2)}$$
$$\geq \frac{c_3}{\sqrt{n/k}\,(\log(en/k))^{\frac{3}{2}}} \tag{149}$$

On the other hand, in the proof of Proposition(4.3.15) we checked that if K and Dsatisfy (142) then

$$|K|^{n-k} \left[\widetilde{\Phi}_{[k]}(\bar{K})\right]^{kn} \leq e^{\frac{nk}{2}} |D|^{n-k} \tag{150}$$

Inserting the lower bound of (149) into (150) we conclude the proof.

Theorem(4.3.3) clearly summarizes the results.

Theorem(4.3.17)[235]: (Koldobsky). Let $1 \le k \le n-1$ and let K be a generalized k-intersection body in R^n . If f is an even continuous non-negative function on K such that

$$\int_{K\cap F} f(x)dx \le \varepsilon \tag{151}$$

For some $\varepsilon > 0$ and for all $F \in G_{n,n-k}$, then

$$\int_K f(x)dx \le \gamma_{n,k}\frac{n}{n-k}|K|^{\frac{k}{n}}\varepsilon \tag{152}$$

The next theorem is a byproduct of our methods and provides a general stability estimate in the spirit of Theorem(4.3.17).

Theorem (4.3.18) [235]: Let $1 \le k \le n-1$ and let K be a compact set in R^n. If g is a locally integrable non-negative function on R^n such that

$$\int_{G_{n,n-k}}\left(\int_{K\cap F} g(x)dx\right)^n dv_{n,n-k}(f) \le \varepsilon^n \tag{153}$$

For some $\varepsilon > 0$ and for all$F \in G_{n,n-k}$, then

$$\int_K g(x)dx \le \left(c_0\sqrt{n-k}\right)^k|K|^{\frac{k}{n}}\varepsilon \tag{154}$$

Note. Our assumption (153)is weaker than the assumption (151)in Theorem (4.3.17).

Proof: Applying Lemma(4.3.5) with $s = n-k$ for the function

$f(x_1,\dots,x_{n-k}) = \prod_{i=1}^{n-k} g(x_i)1_k(x_i)$ we get

$$\prod_{i=1}^{n-k}\int_K g(x_i)d(x_i) \le p(n,n-k)\int_{G_{n,n-k}}|K\cap F|^k \tag{155}$$

$$\times\int_{K\cap F}\dots\int_{K\cap F} g(x_i)\dots g(x_{n-k})dx_1\dots dx_{n-k}dv_{n,n-k}(F)$$

$$\le p(n,n-k)\int_{G_{n,n-k}}|K\cap F|^k\left(\int_{K\cap F} g(x)dx\right)^{n-k} dv_{n,n-k}(F)$$

From Hölder’s inequality it follows that

$$\left(\int_K g(x)dx\right)^{n-k} \le p(n,n-k)\left(\int_{G_{n,n-k}}|K\cap F|^n dv_{n,n-k}(F)\right)^{\frac{k}{n}} \tag{156}$$

$$\times\left(\int_{G_{n,n-k}}\left(\int_{K\cap F} g(x)dx\right)^{n} dv_{n,n-k}(F)\right)^{\frac{n-k}{n}}$$

$$\leq p(n,n-k)\varepsilon^{n-k}\left(\int_{G_{n,n-k}} |K\cap F|^{n} dv_{n,n-k}(F)\right)^{\frac{k}{n}}$$

$$\leq \gamma_{n,k}^{-k} p(n,n-k)\varepsilon^{n-k}|K|^{\frac{k(n-k)}{n}}$$

$$\leq \left(c_0\sqrt{n-k}\right)^{k(n-k)}\varepsilon^{n-k}|K|^{\frac{k(n-k)}{n}},$$

Using the assumption (153) and the bound

$$\int_{G_{n,n-k}} |K\cap F|^{n} dv_{n,n-k}(F) \leq \gamma_{n,k}^{-k}|K|^{n-k} \tag{157}$$

As well as Lemma(4.3.8) . This shows that

$$\left(\int_{K} g(x)dx\right)^{n-k} = \prod_{i=1}^{n-k}\int_{K} g(x_i)dx \leq \left(c_0\sqrt{n-k}\right)^{k(n-k)} \varepsilon^{n-k}|K|^{\frac{k(n-k)}{n}} \tag{158}$$

And the result follows.

Recall that the class Bp_k^n of generalized k-intersection bodies in R^n, introduced by Zhang in[249], is the closure in the radial metric of radial k-sums of finite collections of origin symmetric ellipsoids. If we define

$$\operatorname{ovr}(K, Bp_k^n) = \inf\left\{\left(\frac{|D|}{|K|}\right)^{1/n} : K\subseteq D, D\in Bp_k^n\right\}, \tag{159}$$

Then Theorem (4.3.17) directly implies the estimate

$$\mu(K) \leq \operatorname{ovr}(K, Bp_k^n)^k \frac{n}{n-k}\gamma_{n,k} \max_{F\in G_{n,n-k}} \mu(K\cap F)|K|^{\frac{k}{n}} \tag{160}$$

For any measure μ with an even continuous density . Using (154) and bounds for the quantities

$$\sup_{K\in c_n} \operatorname{ovr}(K, Bp_k^n), \tag{161}$$

Koldobsky (in some cases with Zvavitch) has obtained sharper estimates on the lower dimensional slicing problem for various classes C_n of symmetric convex bodies in R^n:

(i)If $k \geq \lambda_n$ for some $\lambda \in (0.1)$ then one has (55) for all symmetric convex bodies K and all even measures μ, with a constant α depending only on λ (see [247]; this result employs an estimate of Koldobsky, Paouris and Zymonopoulou for over$(K, Bp_{n,k})$from [242]).

(ii) If K is an intersection body then one has (55) for all even measures μ, with an absolute constantα; this was proved by Koldobsky in [244] for$k = 1$, and by Koldobsky and Ma in [240]for all k.
(iii) If K is the unit ball of an n-dimensional subspace of $L_p, p > 2$ then one has (55) for all even measures μ, with a constant α≤ $cn^{\frac{1}{2}-\frac{1}{p}}$ (see [246]).
(iv) If K is the unit ball of an n-dimensional normed space that embeds in $L_p, p \in (-n, 2]$ then one has (55) for all even measures μ, with a constant depending only on p(see [247]).
(v)If K has bounded outer volume ratio then one has (55) for all even measures μ, with an absolute constant α (see [247]). It would be interesting to see if our method can be used for the study of special classes of convex bodies.
Our proof of Theorem(4.3.2) makes essential use of the log-concavity of the measureμ. It was mentioned that Koldobsky and Zvavitch [241] have obtained the bound $\beta_{n,1}^{(s)}(\mu) \leq \sqrt{n}$ for every measure μ with an even continuous non-negative density. It would be interesting to see if our method can provide this estimate, and possibly be extended to higher codimensions k, for more general classes of measures. It would be also interesting to see if the symmetry assumptions on both K and μ are necessary.

Corollary (4.3.19) [388]: Let K be a convex body in $\mathbb{R}^n$ with $0 \in int(K)$. Let g be a bounded non-negative measurable function on R^n and let μ be the measure on R^n with density g. For every $0 \leq \varepsilon \leq \infty$,

$$\mu(K) \leq \left(c_5\sqrt{4\varepsilon+1}\right)^{1+\varepsilon} \max_{F\in G_{3+\varepsilon,2}} \mu(K \cap F).\,|K|^{\frac{1+\varepsilon}{3+\varepsilon}}$$

Where $c_5 > 0$ is an absolute constant. In particular,$\alpha_{3+\varepsilon,1+\varepsilon}(\mu) \leq c_5\sqrt{4\varepsilon+1}$ In fact, the proof of leads to the stronger estimate

$$\mu(K) \leq \left(c_5\sqrt{4\varepsilon+1}\right)^{1+\varepsilon} \left(\int_{G_{3+\varepsilon,2}} \mu(K \cap F)^{3+\varepsilon} dv_{3+\varepsilon,2}(F)\right)^{\frac{1}{n}} |K|^{\frac{1+\varepsilon}{3+\varepsilon}}$$

The classical Busemann–Petty problem is the following question. Let K and D be two origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\left|K \cap \theta^{\perp}\right| \leq \left|D \cap \theta^{\perp}\right|^{G_{3+\varepsilon,2}}$$

for all $\theta \in S^{2+\varepsilon}$. Does it follow that$|K| \leq |D|$? The answer is affirmative if $\varepsilon \geq 0$ and negative if $\varepsilon \geq 0$, see Koldobsky's monograph[238]). The isomorphic version of the Busemann–Petty problem asks if there exists an absolute constant $C_4 > 0$ such that whenever K and D satisfy we have$|K| \leq C_4|D|$. This question is equivalent to the slicing problem and to the isotropic constant conjecture (asking if $\{L_{3+\varepsilon}\}$is a bounded sequence). It is known that if K and D are two centered convex bodies in $\mathbb{R}^n$ such that(66) holds true for all $\theta \in S^{2+\varepsilon}$, then

$$|K|^{\frac{4\varepsilon+1}{3+\varepsilon}} \leq c_6 L_{3+\varepsilon} |D|^{\frac{4\varepsilon+1}{3+\varepsilon}}$$

Where $c_6 > 0$ is an absolute constant.

The natural generalization, and the lower dimensional Busemann–Petty problem, is the following question: For $0 \leq \varepsilon \leq \infty$ let $\beta_{3+\varepsilon,1+\varepsilon}$ be the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and D in $\mathbb{R}^n$ that satisfy

$$|K \cap F| \leq |D \cap F|$$

for all $F \in G_{3+\varepsilon,2}$, one has

$$|K|^{\frac{2}{3+\varepsilon}} \leq \beta^{1+\varepsilon}|D|^{\frac{2}{3+\varepsilon}}$$

Is it true that there exists an absolute constant $C_5 > 0$ such that $\beta_{3+\varepsilon,1+\varepsilon} \leq c_5$ for all $3+\varepsilon$ and $1+\varepsilon$?

From (68) we have $\beta_{3+\varepsilon,1} \leq c_6 L_{3+\varepsilon} \leq c_7 \sqrt[4]{3+\varepsilon}$ for some absolute constant $c_7 > 0$. We also consider the same question for the class of symmetric convex bodies and we denote the corresponding constant by

$$\beta^{(1+\varepsilon)}_{3+\varepsilon,K} .$$

As in the case of the slicing problem, the same question can be posed for a general measure in place of volume. For any $0 \leq \varepsilon \leq \infty$ and any measure μ on $\mathbb{R}^n$ with a locally integrable non-negative density g one may define $\beta_{3+\varepsilon,1+\varepsilon}(\mu)$as the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and D in $\mathbb{R}^n$ that satisfy $\mu(K \cap F) \leq \mu(D \cap F)$ for every$F \in G_{3+\varepsilon,2}$, one has

$$\mu(K) \leq (\beta)^{1+\varepsilon}\mu(D)$$

Similarly, one may define the "symmetric" constant $\beta^{(1+\varepsilon)}_{3+\varepsilon,1+\varepsilon}(\mu)$ Koldobsky and Zvavitch [239] proved that $\beta^{(1+\varepsilon)}_{3+\varepsilon,1} \leq \sqrt{3+\varepsilon}$ for every measure μ with an even continuous non-negative density. In fact, the study of these questions in the setting of general measures was initiated by Zvavitch in [240], where he proved that the classical Busemann–Petty problem for general measures has an affirmative answer if $\varepsilon \geq 0$ and a negative one if $\varepsilon \geq 0$. We study the lower dimensional question and provide a general estimate in the case where μ has an even log-concave density (see [241]).

Corollary (4.3.20) [388]: Let μ be a measure on $\mathbb{R}^n$ with an even log-concave density g and let $10 \leq \varepsilon \leq \infty$. Let K be a symmetric convex body in $\mathbb{R}^n$ and let D be a compact subset of $\mathbb{R}^n$ such that

$$\mu(K \cap F) \leq \mu(D \cap F)$$

for all $F \in G_{3+\varepsilon,2}$. Then,

$$\mu(K) \leq (c_8 K L_2)^{1+\varepsilon}\mu(D)$$

Where $c_8 > 0$ is an absolute constant.

Comparing with the estimate $\beta_{2+\varepsilon}^{(1+\varepsilon)}(\mu) \le \sqrt{3+\varepsilon}$ of Koldobsky and Zvavitch, note that the estimate in [241] is true for an arbitrary measure μ, i.e. the log-concavity of μ is not required; on the other hand, is valid for any codimension $\varepsilon \ge 0$ and the convexity of the second body D is not required.

We prove. Our main tools are the generalized Blaschke–Petkantschin formula and the Busemann–Straus–Grinberg inequality for the dual affine quermassintegrals of a convex body. For the proof of Theorem(4.3.1) we also use a functional version of the latter inequality, recently obtained by Dann , Paouris . We collect some facts for the case of volume; we obtain the following bounds for the constants $\alpha_{3+\varepsilon,1+\varepsilon}$ and $\beta_{3+\varepsilon,1+\varepsilon}$. (see [341]).

Corollary (4.3.21) [388]:. For every $0 \le \varepsilon \le \infty$ we have

$$\alpha_{3+\varepsilon,1+\varepsilon} \le \beta_{3+\varepsilon,1+\varepsilon}$$

Moreover,

$$\beta_{3+\varepsilon,1+\varepsilon} \le \bar{c_1} L_{3+\varepsilon}$$

Where $c_1 > 0$ is an absolute constant. Finally, for codimensions $1+\varepsilon$ which are proportional to $3+\varepsilon$ we have the stronger bound

$$\beta_{3+\varepsilon,1+\varepsilon} \le \bar{c_2}\sqrt{(3+\varepsilon)/(1+\varepsilon)}\,(\log(e\,(3+\varepsilon)/(1+\varepsilon)))^{3/2}$$

where $\bar{c_2} > 0$ is an absolute constant

The estimates are probably known; we just point out alternative ways to justify them. In particular, Koldobsky has proved in [228] that

$$\beta_{3+\varepsilon,1+\varepsilon}^{(1+\varepsilon)} \le \bar{c_4}\sqrt{3+\varepsilon/1+\varepsilon}\,(\log(e(3+\varepsilon)/(1+\varepsilon)))^{3/2}$$

for all $0 \le \varepsilon \le \infty$, where $c_4 > 0$ is an absolute constant; this is the symmetric analogue We finish with a general stability estimate in the spirit of Koldobsky's stability theorem. (see [241]).

Corollary (4.3.22) [388]: Let $0 \le \varepsilon \le \infty$ and let K be a compact set in $\mathbb{R}^n$. If g is a locally integrable non-negative function on $\mathbb{R}^n$ such that

$$\int_{G_{3+\varepsilon,3}} \left(\int_{K\cap F} g(x)dx \right)^{3+\varepsilon} d\nu_{3+\varepsilon,2}(F) \le \varepsilon^{3+\varepsilon}$$

for some $\varepsilon > 0$, then

$$\int_K g(x)dx \le \left(c_0\sqrt{4\varepsilon+1}\right)^{1+\varepsilon} |K|^{\frac{1+\varepsilon}{3+\varepsilon}}\varepsilon$$

Where $c_0 > 0$ is an absolute constant.

Corollary (4.3.23) [388]: There exists an absolute constant $\delta > 0$ such that, for every log-concave probability measure ν on $\mathbb{R}^m$ and every $p > 1$.

$$S_p(v) \leq \delta p^m S_1(v)$$

In particular, for every convex body D in $\mathbb{R}^m$ and every $p > 1$,

$$S_p(D) \leq \delta p^m S_1(D)$$

Proof. We use the fact that there exists an absolute constant $\delta > 0$ with the following property:
if $\nu \in p_m$ is a log-concave probability measure then, for any seminorm $u : \mathbb{R}^m \to \mathbb{R}$ and any $q > p \geq 1$,

$$\left(\int_{\mathbb{R}^m} |u(x)|^q dv(x) \right)^{1/q} \leq \frac{\delta q}{p} \left(\int_{\mathbb{R}^m} |u(x)|^{q^p} dv(x) \right)^{1/p}$$

This is a consequence of Borel'slemma (see e.g. [243, Theorem 2.4.6]). Next, recall that

$$|\mathrm{Conv}(0, x_1, \dots, x_m)| = \frac{1}{m!} |\det(x_1, \dots, x_m)|$$

The function $u_i: \mathbb{R}^m \to \mathbb{R}$ defined by $x_i \longmapsto |\det(x_1, \dots, x_n)|$ for fixed x_j in $\mathbb{R}^m$, $j \neq i$, is a seminorm, as is the function $v_i: \mathbb{R}^m \to \mathbb{R}$ defined by

$$x_i \longmapsto \int_{\mathbb{R}^m} \dots \int_{\mathbb{R}^m} |\det(x_1, \dots, x_m)| dx_{i+1} \dots dx_m$$

for fixed $x_j (1 \leq j < i)$ in R^m. By consecutive applications of Fubini's theorem and ,we obtain

The next lemma gives upper bounds for the constants $p(3+\varepsilon, 2)$ and $\gamma_{3+\varepsilon, 1+\varepsilon} = |B_2^n|^{\frac{2}{3+\varepsilon}} / |B_2^{3+\varepsilon}|$; both constants appear frequently in the next sections (see [241]).

Chapter 5
Monotonicity Properties with Rigidity and Stability

As an application we obtain the Fortuin, Kasteleyn, Ginibre correlation inequalities as well as some generalizations of the Brascamp–Lieb momentum inequalities. We show that if a 1-log-concave measure has almost the same Poincar´e constant as the Gaussian measure, then it almost splits off a Gaussian factor.

Section (5.1): Optimal Transportation and the FKG and Related Inequalities

We give some background on optimal transportation and the FKG inequalities. We are given two probability densities $f(X)$,$g(Y)$,and we want to transport the(variable X with)density f onto the(variable Y with)density g in a way that minimizes transportation costs, say for simplicity, $C(Y-X)$. Let us first say what we mean by transporting f to g.
A smooth map $Y(X)$ transports f to g if

$$g(Y(X))det\, D_X Y \;=\; f(X).$$

That is, a small differential of volume

$$g(Y)dy$$

is pulled back to

$$f(X)dx$$

by the map $Y(X)$.
A weak formulation is the following:
Definition (5.1.1)[266]: *A (weak) transport is a measurable map $Y(X)$, such that for any C_0 function $h(Y)$ the following (“change of variable”) formula is valid:*

$$\int h(Y)g(Y)dY = \int h(Y(X))f(X)dX.$$

Now, given the cost function $C(X)$, we define
The (weak) transportation $Y(X)$ is optimal if it minimizes

$$J(Y) = \int C(Y(X)) - X)f(X)dx$$

among all weak transportation.
Existence and regularity of such an optimal transportation has been studied in detail. (See [267] and [268].) We will discuss (and use) the particular case where

$$C(X-Y) = \tfrac{1}{2}|X-Y|^2.$$

The correlation inequalities holds true for more general cost functions, still convex and with the appropriate symmetries, but the proofs are technically involved and we will present it elsewhere.
The second derivative estimates for the Monge-Ampere like equations corresponding to non-quadratic cost functions, is a completely open matter. In the quadratic case, there is a rather complete existence and regularity theory ([268]). We will be interested in the following results.

Theorem (5.1.2) [266]: *Let Ω_1, Ω_2 be two open domains in $\mathbb{R}^n$, $f(X)$, $g(Y)$ two strictly positive bounded, measurable functions in $\overline{\boldsymbol{\Omega_i}}$, with*

$$\int_{\boldsymbol{\Omega_1}} f(X)dX = \int_{\boldsymbol{\Omega_2}} g(Y)dY = 1.$$

Then,

(i) *There exists a unique optimal transportation map $Y(X)$.*

(ii) *The optimal transportation $Y(X)$ (and its inverse $X(Y)$) are obtained from the following minimization process:b_1) Among all pairs of continuous functions $\varphi(X)$, $\psi(Y)$ satisfying the constraint*

$$\varphi(X) + \psi(Y) \geq \langle X, Y \rangle$$

minimize

$$J(\varphi, \psi) = \int_{\boldsymbol{\Omega_1}} \varphi(X) f(X)dX + \int_{\boldsymbol{\Omega_2}} \psi(Y)g(Y)dY.$$

(b_2) φ and ψ are unique and convex and $Y(X)$ is defined as the (possibly multiple valued) map $Y \in Y(X)$ if

$$\varphi(X) + \psi(Y) = \langle Y, X \rangle.$$

Theorem (5.1.3) [266]: *Hypothesis as before, assume further that $\boldsymbol{\Omega_1}, \boldsymbol{\Omega_2}$ are convex. Then*

(i) *If $0 < \lambda \leq f, g \leq \Lambda$, the map $Y(X)$ and its inverse $X(Y)$ are single valued, of class C^α in $\boldsymbol{\Omega_i}$ for some α.*

(ii) *If f, g are Hölder continuous, with exponent β for some β then $Y(X), X(Y)$ are of class $C^{1,\beta}$.*

(iii) *In both cases, (a) and b)), there exists a pair of convex potentials $\varphi(X), \psi(Y)$ such that*

$$Y(X) = \nabla\varphi(X), X(Y) = \nabla\psi(Y).$$

(iv) *φ satisfies the Monge–Ampére equation*

$$\det D^2\varphi(X) = \frac{f(X)}{g(\nabla\varphi(X))}$$

in case a) in the Alexandrov weak sense, in case b) in the classical sense.

(Note that $\varphi \in C^{2,\beta}$.) By approximation, we will develop all our discussion for f, g of class C^α, so we will always talk of "classical" solutions.

From the variational construction of Y, we also have a stability theorem.

Theorem (5.1.4) [266]: *Let f_j, g_j be uniformly bounded, measurable and supported in a bounded domain B_R. Assume that $f_j \to f$ in L^1, $g_j \to g$ in L^1. Then $\varphi_j \to \varphi$, $\psi_j \to \psi$ uniformly in B_R. In particular if φ_j, ψ_j are uniformly $C^{1,\alpha}$, then $\nabla\varphi_j, \nabla\psi_j$ also converge uniformly to $\nabla\varphi, \nabla\psi$.*

We complete the discussion with the following interpretation (see [270]).

If we think of $f(X), g(Y)$ as probability densities, we may think of the map $Y(X)$ as a joint probability distribution: $v_0(X, Y)$in$\Omega_1 \times \Omega_2$, sitting on the graph $X, Y(X)$ with the property that the marginals $\mu_1(X), \mu_2(Y)$ of v_0 are exactly $f(X)dx$ and $g(Y)$dy. In fact v_0 has the following minimizing property:

Among all probability measures $\nu(X,Y)$ with marginals $f(X)dX$ and $g(Y)dY, Y(X)$ minimizes

$$E(\nu) = \int |X-Y|^2 d\nu(X,Y).$$

We are interested in a theorem of Holley [269] from which the inequalities follow. Holley's Theorem establishes a monotonicity condition for probability measures μ_1, μ_2 defined on a finite lattice, Λ .

We discuss briefly his two main theorems. We consider a finite lattice Λ(that wewill think of as embedded in the set P of vertices of the unit cube of $\mathbb{R}^N$ for some N (i.e., the set of all N −tuples, $X = (x_1, \dots, x_N)$ with $x_i = 0$ *or* $1. On\ \Lambda$, we have two non-vanishing probability measures $\mu_1(X), \mu_2(X)$ with the "monotonicity property":

Given $X, Y\ in\ \Lambda$,

$$\mu_2(X \vee Y)\mu_1(X \wedge Y) \geq \mu_2(X)\mu_1(Y).$$

(As usual $\vee$ denotes taking max in each entry, $\wedge$ min.) Then

Theorem (5.1.5) [266]: ([270]**).** *There exists a joint measure*

$$\nu(X,Y)$$

with marginals $\mu_1(X), \mu_2(Y)$ *such that*

$$\nu(X,Y) \neq 0 \Rightarrow X \leq Y.$$

As a corollary, he obtains

Corollary (5.1.6) [266]: *If h is an increasing function of X, then*

$$\int_\Lambda h(X)d\mu_1(X) \leq \int h(X)d\mu_2(X)$$

(that is μ_2 is "concentrated more to the right" than μ_1).

We study the relation between optimal transportation and the FKG inequalities, in particular to show:

(i) In the continuous case, the optimal transportation from the unit cube of $\mathbb{R}^n$into itself ($\mu_1 = f(X), \mu_2 = g(Y)$) has the proper monotonicity properties ($Y(X) \geq X$) of Holley's joint probability density provided that f, g do).

(ii) If we "spread" the measures μ_i from the vertices of the unit cube to half cubes, the densities f, g so obtained satisfy these properties, recuperating from this approach Holley's theorem, for the lattice formed by all vertices of the cube.

(iii) For a general sublattice, one can extend the "spread" measure to all of the half cubes recuperating in full the theorem of Holley.

(iv) In fact the discrete optimal transportation satisfies $Y(X) \geq X$.

The proof is based on the fact that first derivatives of solutions of the Monge–Ampére equation satisfy an equation themselves. But it is also known that second derivatives are subsolutions of an elliptic equation.

we explore what the implications of that fact are in terms of correlation inequalities.

We want to stress that in the continuous case the optimal transport map $Y(X)$ interpreted as a joint probability measure

$$\nu(X,Y) = \delta_{X,Y(X)}(X,Y)f(X)dX = \delta_{X,Y(X)}(X,Y)g(Y)dY$$
is not just a joint distribution but a "change of variables", i.e., a one to one map that carries one density to the other, and it is further the gradient of a convex potential, giving the map (or the measure $\nu(X,Y)$) a lot of stability.

We start this with a reflection property of optimal transportation maps. Given $X \in \mathbb{R}^n$we denote by $\overline{X}$ its reflection with respect to $x_1, i.e.$, if $X = (x_1, x_2, \ldots, x_n)$ then $\overline{X} = (-x_1, x_2, \ldots, x_n)$.

Lemma (5.1.7) [266]: *Assume that*

(i) Ω_1, Ω_2 *are symmetric with respect to*$x_1, i.e. X \in \Omega_i \Leftrightarrow \overline{X} \in \Omega_i$,

(ii) f, g *are also symmetric, i.e.,*
$$f(X) = f(\overline{X}), \qquad g(X) = g(\overline{X}).$$
Then the optimal transportation is also symmetric, i.e.,

(i) $\varphi(X) = \varphi(\overline{X}), \quad \psi(Y) = \psi(\overline{Y})$,

(ii) $Y(\overline{X}) = Y(\overline{X})$.

Proof: By Brenier [270] φ, ψ are the unique minimizing pair of
$$\int \varphi(X)f(X)dX + \int \psi(Y)f(Y)dY$$
under the constraint
$$\varphi(X) + \psi(Y) \geq \langle X, Y \rangle.$$
By uniqueness, then,
$$\varphi(X) = \varphi(\overline{X}), \qquad \psi(Y) = \psi(\overline{Y})$$
since $\varphi(\overline{X})$,$\psi(\overline{Y})$ are a competing pair with the same energy.

Corollary (5.1.8) [266]: *Under the hypothesis and with the notation of the lemma, if Y^+ is the optimal transportation from Ω_1^+ to Ω_2^+ then $Y^+ = Y|_{\Omega_1^+}$, where Y is again $\varphi(X), \psi(Y)$ restricted to X, Y in $(\mathbb{R}^n)^+ = \{X : x_1 > 0\}$ must be the minimizing pair.*

We apply the previous lemma and corollary to densities $f(X)$ and $g(Y)$ in the unit cube of $\mathbb{R}^n$.Let f, g be densities in the unit cube of $\mathbb{R}^n, Q_1 = \{X: 0 \leq x_i \leq 1\}$ and Y be the optimal transportation.

Let us write $Y = X + V$ and respectively
$$\varphi(X) = \frac{1}{2}|X|^2 + u(X)$$
(that is $V = \nabla u$). Then

Theorem (5.1.9) [266]: *If we extend f, g to f^*, g^* on a larger cube Q by even reflections, then $u(X)$ also extends periodically to u^*, to the same cube Q^* by even reflection and $Y(X)$ to the optimal transportation map*
$$Y^* = X + \nabla u^*(X)$$
from Q^* to Q^*.

Corollary (5.1.10) [266]: *If f, g are strictly positive and C^α in the unit cube Q_1 , then $Y(X)$ maps each face of the cube to itself and both $Y(X), X(Y)$ have a $C^{1,\alpha}$ extension across ∂Q.*

Proof: It follows from the interior regularity theory (the above theorem) since each face of Q becomes interior after a reflection.

We start with a heuristic discussion. Recall that the Holley condition on μ_2, μ_1 was that

$$\mu_2(A \vee B)\mu_1(A \wedge B) \geq \mu_2(A)\mu_1(B).$$

Logarithmically

$$\log\mu_2(A \vee B) - \log\mu_2(A) \geq \log\mu_1(B) - \log\mu_1(A \wedge B).$$

Let us now think on smooth densities $f(X), g(Y)$ on the unit cube, and assume we are trying to prove, by a continuity argument that $Y(X)$ is monotone, that is $Y(X) \geq X$. So we are looking at a continuous family of densities f^t, g^t for which $Y(X) > X$ and we find a first time t_0 and a point X_0, for which $Y(X_0) \not> X_0$, that is some coordinate, say $y_1(X_0) = x_1(X_0)$. That means that $y_1(X) - x_1(X)$ has a local minimum, zero, at X_0.

But it is well known that $y_1 = D_1\varphi$, satisfies an elliptic equation, obtained by differentiating the equation for φ.From

$$logdet\, D^2\varphi = logf(X) - logg(\nabla\varphi)$$

we get

$$M_{ij}D_{ij}(D_1\varphi) = (logf(X))_1 - (logg(\nabla\varphi))_i D_{i1}\varphi.$$

. Since $\varphi_1 - x_1$ has a minimum, zero, at X_0,

$$D_{i1}\varphi = \delta_{i1},$$

and we get at $X_0, Y(X_0)$,

$$M_{ij}D_{ij}[y_1 - x_1] = (logf)_1(X) - (logg)_1(Y).$$

Since M_{ij} is a strictly positive matrix for φ strictly convex and $y_1 - x_1$ has a minimum, the left-hand side must be non-negative.

If we impose the right-hand to be non-positive we have a contradiction. About the right-hand side, we know that $Y > X$ and that

$$\langle Y - X, e_1\rangle = 0,$$

so the natural hypothesis we want to impose on f, g is that

If $Y \geq X$ and $\langle Y - X, e_i\rangle = 0$, then

$$D_i(logg)(Y) \geq D_i(logf)(X).$$

Note. If we think of $A = Y$ and $B = X + te_i$ we can argue that heuristically $B \vee A = Y + te_i$ and $B \wedge A = X$, so

$logg(Y + te_i) - logg(Y) \geq logf(X + te_i) - logf(X)$

becomes Holley's condition. We will show in fact later how to associate to a discrete "Holley" pair a continuous one satisfying our hypothesis.

But first we prove the main comparison theorem.

Theorem (5.1.11) [266]: *Let f, g be $C^{1,\alpha}$, strictly positive probability densities in the unit cube Q of $\mathbb{R}^n$. Assume that given any X, Y, e_j with $X \leq Y$, and $\langle X - Y, e_j\rangle = 0$ ($i.e.$, $y_j - x_j = 0$)*

$$(D_j\, logf)(X) \leq (D_j\, logg)(Y),$$

and let $Y(X)$ be the optimal transportation map. Then for any X in Q,

$$Y(X) \geq X.$$

Proof: As we pointed out before, we know that the potentials $\varphi(X), \psi(Y)$ are of class $C^{2,\alpha}$ across ∂Q_j and the $C^{1,\alpha}$optimal transportations $Y(X), X(Y)$ map each face of the cube into itself in a $C^{1,\alpha}$ fashion.

In particular, classical regularity theory for fully non linear equations applies to φ in the interior of the cube. More precisely, φ satisfies

$$\det D_{ij}\varphi = \frac{f(X)}{g(\nabla\varphi)}$$

(see [271]) and f, g being $C^{1,\alpha}$ (this is not kept by reflection along the faces), we have that:

$$\varphi \text{ is of class } C^{3,\alpha}(\ Q\ ).$$

We now study directional derivatives along the boundary of Q_j.

Consider $D_1\varphi$ outside the faces $x_1 = 0, x_1 = 1$.Then, across the remaining boundary of $Q_1, y_1(X) = D_1\varphi$ satisfies

$$M_{ij}D_{ij}(D_1\varphi) = D_1\ logf(X) - D_\ell(logg)D_{\ell 1}\varphi.$$

Both M_{ij} and the right-hand side are of class C^α (since D_1 log$f$ is tangential to the face). Hence $y_1(X)$ is of class $C^{2,\alpha}$ across that part of the boundary and the equation is satisfied in the classical sense.

In order to make the f, g relation strict we change g to g_ε by defining

$$1ogg_\varepsilon(Y) = 1ogg + \sum \varepsilon y_i + C_\varepsilon,$$

where the constant C_ε is chosen so that

$$\int g_\varepsilon(Y) = 1.$$

Then from the condition

$$D_j\ logf(X) \le D_j\ logg(Y)$$

for $y_j - x_j = 0$, we now have for $0 < \gamma < \delta(\varepsilon)$ small enough:

$$D_j\ logf(X) \le D_j\ logg\varepsilon(Y) - \delta$$

if $|y_{j_0} - x_{j_0}| < \gamma$ for some j_0 and $y_j - x_j > -\gamma$ for the remaining j.

We now look at the continuous family of densities f_t, g_t defined by

$$logf_t = t\ logf + C(t),$$
$$logg_t = t\ logg_\varepsilon + D(t),$$

where $C(t), D(t)$ are chosen to keep$\int f_t = \int g_t = 1$and we show

Lemma (5.1.12) [266]: *For any* $0 < t < 1$ *the corresponding (continuous in* t*) family of optimal transports* $Y_t(X)$*, satisfies*

$$y_j^t \ge x_j^t - \tfrac{1}{2}\gamma.$$

Proof: For $t = 0, Y(X)$ is the identity map, and thus the inequality is satisfied for t small. As usual, suppose there exists a first value $t_0 > 0$, for which the inequality is not satisfied. Thus, there exists X_0 and a j (say $j = 1$) such that

$$y_1(X_0) = x_1(X_0) - \tfrac{1}{2}\gamma$$

and still$y_1(X) = x_1(X) - \tfrac{1}{2}\gamma$ everywhere else.

We first note that $x_1(X_0) \ne 0,1$ because, if not

$$y_1(X_0) = x_1(X_0).$$

But everywhere else we have

$$0 \le M_{ij}D_{ij}y_1(X_0)$$

(since $y_1 - x_1$has a minimum at X_0)and

$$D_1\, logf(X_0) \leq D_1\, logg(Y(X_0)) - t\delta$$

(since $|y_1 - x_1| = \gamma/2$ and $y_j \geq x_j - \gamma/2$ for the remaining j).

This is a contradiction that completes the proof of the lemma and the theorem.

Corollary (5.1.13) [266]: *Let $0 < \lambda \leq f, g \leq \Lambda$ be measurable. Suppose that $logf$, $logg$ satisfies the hypothesis of the theorem in the sense of distributions. Then, the theorem still holds, i.e.,*

$$Y(X) \geq X.$$

Proof: Mollify $logf, logg$ to $logf_\varepsilon, logg_\varepsilon$ with a standard (radially symmetric, nonnegative, compactly supported) mollifier φ_ε.Then the hypothesis of Theorem (5.1.11) is satisfied as long as X, Y stay at distance ε from ∂Q_1.

Take as center of coordinates the center of the cube:$X = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and make a 2ε-dilation. The new $f_\varepsilon, g_\varepsilon$ satisfy the hypothesis of Theorem (5.1.11) when restricted to the unit cube. Thus Theorem (5.1.2) holds for them. By passing to the limit on the maps, the theorem holds for f, g.

Given a vertex $X \in P$, we will denote by Q_X the subcube of Q_j, of side $1/2$ that has X as a vertex

$$Q_X = \{Z : |Z - X|_{L^\infty} \leq 1/2\}.$$

We prove the following theorem.

Theorem (5.1.14) [266]: *Let f, g be step functions*

$$f = \sum_{X\in P} \mu_1(X)\chi_{Q_X},$$

$$g = \sum_{X\in P} \mu_2(X)\chi_{Q_X}.$$

Assume that given vertices $X, Y, X + e_j, Y + e_j$ with $Y \geq X$ and $\langle Y, e_j\rangle = \langle X, e_j\rangle = 0$ we have

$$1og\, \mu_2(Y + e_j) - 1og\, \mu_2(Y) \geq 1og\, \mu_1(X + e_j) - 1og\, \mu_1(X).$$

Then $Y(X) \geq X$.

Proof: As a distribution $D_i\, logf$ (resp. $D_i\, logg$) is the jump function

$$log\mu_i(X + e_j) - log\mu_1(X)$$

supported on the face of Q_X laying in the plane $x_i = 1/2$.

Corollary (5.1.15) [266]: *Let $Z_1, Z_2 \in P$. Define*

$$\nu(Z_1, Z_2) = \mu_1(Z_1)/|Q_{1/2}||\{X \in Q_{Z_1}/Y(X) \in Q_{Z_2}\}|$$
$$= \mu_2(Z_2)/|Q_{1/2}||\{Y \in Q_{Z_2}/X(Y) \in Q_{Z_1}\}|.$$

Then

a) *ν is a probability measure with marginals* $\mu_1(Z_1), \mu_2(Z_2)$,

b) $\nu(Z_1, Z_2) \neq 0 \Rightarrow Z_2 \geq Z_1$.

Given a lattice $\Lambda \subset P$, and two measures μ_1, μ_2 satisfying the Holley condition we want to extend μ_1, μ_2 to small perturbationsμ_1^*, μ_2^* in all of P keeping the inequalities.

Usually, μ_1, μ_2 are extended by zero. We state the following presentation of Λ.

Lemma (5.1.16) [266]: *There is a partition of*

$$\mathbb{R}^N = \mathbb{R}^{k_1} \otimes \mathbb{R}^{k_2} \otimes \cdots \otimes \mathbb{R}^{k_\ell}$$

and a family of elements $w_i^j (1 \le j \le \ell, 1 \le i \le k_j)$ *such that any non zero element* $X \in \Lambda$ *is the max of* w_i^j ,

$$x = \bigvee_{i,j \in I_X} w_i^j$$

and

$$w_i^j = e_i^j + v$$

with the coordinates$v_i^s = 0 \; \forall \; s \ge j$. *(More precisely* $w_i^1 = e_i^1, w_i^2 = e_i^2 + v$, *with* $v \in \mathbb{R}^{k_1}, w_i^3 = e_i^3 + v$ *with* $v \in \mathbb{R}^{k_1+k_2}$ *and so on.*

Proof: The decomposition is by first choosing the minimal elements $\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_{k_1}$ and contracting the ones in them to only one position. Next we choose minimal elements among those not in $\mathbb{R}^{k_1}$ and so on.

We now extend the lattice and the measure. Let $\overline{\Lambda}$ be the following extension of Λ:

$$\overline{\Lambda} = \Lambda \cup \Lambda_0, \text{where } w \in \Lambda_0 \Leftrightarrow max(w, e_1) \in \Lambda$$

(that is, we add to all those elements with a 1 as first coordinates, those with a zero).

Given w in $\overline{\Lambda}$ define

$$w^+ = w \vee e_1,$$

$w^- = w^+ - e_1$ (i.e., w with a zero in the position e_1).

Define

$$\mu^*(w) = \begin{cases} \mu(w) & if \; w \in \Lambda \\ \mu(w^+)/M & \text{otherwise (M large)} \end{cases}.$$

Theorem (5.1.17) [266]: $\overline{\Lambda}$ *is a lattice and* μ_1^*, μ_2^* *still satisfy*

$log\, \mu_2^*(v_1 \vee v_2) - log\, \mu_2^*(v_2) \ge log\, \mu_1^*(v_1) - log\, \mu_1^*(v_1 \wedge v_2)$.

Proof: Elements in $\overline{\Lambda}$ are $w^+ and$ w^- of elements in Λ(w^+ is always in Λ since $e_1 \in \lambda$).

Then

$$v_1 \wedge v_2 = w_1^{\pm} \wedge w_2^{\pm}$$

for $w \in \Lambda$.

If one of the signs is a $-$,

$$v_1 \wedge v_2 = (w_1 \wedge w_2)^-.$$

If not

$$v_1 \wedge v_2 = w_1 \wedge w_2 .$$

Also

$$v_1 \vee v_2 = w_1^{\pm} \vee w_2^{\pm}.$$

If one of the signs is a $+$ (since $w^+ \in \Lambda$),

$$v_1 \vee v_2 = w_1 \vee w_2 .$$

If not

$$v_1 \vee v_2 = (w_1 \vee w_2)^-.$$

About the measures μ_1^*, μ_2^*,, let us verify the proper inequalities. For that purpose we choose $M \gg \mu_i(X)$ for any X. There are several cases to consider

a) $w_1, w_2 \in \Lambda$, then $w_1 \wedge w_2, w_1 \vee w_2 \in \Lambda$ and everything is as before.

(b) $w_1 \in \Lambda, w_2 \notin \Lambda(thus\ w_2 = w_2^-)$.

b1) If $w_1 = w_1^-$, we have that $w_1 \wedge w_2 \in \Lambda$ and $w_1 \vee w_2 \notin \Lambda$ and the factor $logM$ cancels in μ_2^* the expression.

b2) If $w_1 = w_1^+$, $w_1 \vee w_2 \in \Lambda$. IF $w_1 \wedge w_2 \in \Lambda$ the eXtra factor log M in the μ_2^* expression controls everything e1se (we choose $logM \gg$ sup $|log\ \mu_i|. If\ w_1 \wedge w_2 \notin \Lambda, \mu_1^*(w_1 \wedge w_2) = \mu_1(w_1 \wedge w_2^+)/M$,and $\mu^*(w_2) = \mu(w_2^+)/M$,
thus each term has an extra $logM$ factor that cancels.

c) $w_2 \in \Lambda, w_1 \notin \Lambda$.

c1) If $w_2 = w_2^+$, then $w_1 \vee w_2 \in \Lambda$.IF $w_1 \wedge w_2 \in \Lambda$. the extra term $-logM$ in the μ_1 expression controls everything. If $w_1 \wedge w_2 \notin \Lambda$. then

$$\mu_1^*(w_1 \wedge w_2) = \mu(w_1^+ \wedge w_2)/M,$$
$$\mu_1^*(w_1) = \mu(w_1^+)/M,$$

and we have $logM$ cancellation.

c2) If $w_2 = w_2^-$, then $w_1 \wedge w_2 \in \Lambda$. IF $w_1 \vee w_2 \notin \Lambda$, and we have

$$\mu_2^*(w_1 \wedge w_2) = \mu_2(w_1^+ \vee w_2)/M,$$
$$\mu_1^*(w_1) * = \mu_1(w_1^+)/M,$$

and there is a $logM$ factor cancellation.

d) If $w_1 \notin \Lambda, w_2 \notin \Lambda$,then $w_1 \vee w_2 \notin \Lambda$. IF $w_1 \wedge w_2 \notin \Lambda$,, the factors $logM$ cancel.

If not, the extra factor $logM$ in the μ_1^* expression controls everything else.

The proof of the theorem is complete.

Theorem (5.1.18) [266]: *We are given $\Lambda \subset P$ and μ_1, μ_2. As before, let* f,g *be the step functions*

$$f = \sum_{w_i \in \Lambda} \mu_1(w_i) \chi_{Q_{w_i}},$$
$$g = \sum_{w_i \in \Lambda} \mu_2(w_i) \chi_{Q_{w_i}}.$$

Then, the optimal transportation map $Y(X)$ is monotone.

Proof: If we start with $M = M_0$ and we repeat the extension process ($M_1 \gg M_0, M_2 \geq M_1$ and so on) we exhaust P. Note that once we have extended through $e_1^1, \ldots, e_{k_1}^1$, the elements $e_1^2, \ldots, e_{k_2}^2$ belong now to the lattice and are minimal, so we can keep extending. As M_0 goes to infinity the measures μ_i^* converge to μ_i.

We complete this work by showing that, actually, the discrete optimal transportation map is monotone. In this case the map is in general multi-valued. That is the mass $\mu_1(w)$ may have to be spread through several points v. Still, for all those v's, $v(w) \geq w$.

Theorem (5.1.19) [266]: ***Let* Λ** *be a sublattice of P, the set of vertices of the unit cube on $\mathbb{R}^n$,and let μ_1, μ_2 be positive measures in satisfying the usual monotonicity condition. Let $\nu(X,Y)$ be the (discrete) optimal transportation. Then $\nu(X,Y) \neq 0 \Rightarrow Y \geq X$.*

Proof: From the previous theorem we may assume that μ_i is defined and positive in all of P. We will approximate it by bounded densities f, g that satisfy the hypothesis of Theorem (5.1.11). We define them as follows.

Let 1 be the vector $1 = (1,1,\ldots,1)$. In the strip$S_w^\varepsilon = \{\varepsilon w < X \leq w + \varepsilon 1\}$,let $N(X,\omega)$ be the number of coordinates, j, for which $w_j - x_j > \varepsilon$ and we define there, for $\delta \ll \varepsilon$,

$$f(X) = \mu_1(\omega)\delta^N.$$

Note thatS_w^εcover Q_1 disjointly (given X we determine w by those coordinates $x_j > \varepsilon$). Same definition for g.

Of course, we have to multiply as usual by a normalization constant to make $\int f = \int g = 1$, but this does not affect the logarithmic inequality. Also if δ goes to zero much faster than ε, (say like ε^{2N}) f and g converge to μ_1 and μ_2, since most of the mass concentrates in the cube $Q_\varepsilon(\omega) = \{|x_i - \omega_i| < \varepsilon\}$.

About $D_i \, logf, D_i \, logg$, they are jump functions concentrated on the planes $x_j = \varepsilon$ or $1 - \varepsilon$ so we have to check that the jump inequalities are satisfied. We also may disregard plane intersections since they will not affect $D_i f$ in the distributional sense.

So we check that

a) For $X \leq Y$ and $x_i = y_i = \varepsilon$ we have Jump$(logg) \geq$ Jump$(logf)$. Indeed when x_i, y_i go through ε we change from evaluating the measures at w_1, (resp. w_2) to $w_1 + e_i, w_2 + e_i$, and both $N(X), N(Y)$ increase by one, so the jump relation holds (they are the lattice relations plus a factor $log\delta$.

b) When x_i, y_i go through $(1-\varepsilon), w_1$ and w_2 remain unchanged and $N(X), N(Y)$ both decrease by one.

Also here the jump relation holds (both jumps are just $log\delta$).

This completes the proof.

we explore what the implications are of the fact that second derivatives of solutions to Monge–Ampére equations are subsolutions of an elliptic equation.

First an heuristic discussion: Let us take a second pure derivative of the equation

$$logdet\, D_{ij}\varphi = logf(x) - logg(\nabla\varphi).$$

We get

$$M_{ij}D_{ij}\varphi_{\alpha\alpha} + M_{ij,k\ell}D_{ij\alpha}\varphi D_{ij\beta}\varphi = D_{\alpha\alpha}\, logf - (1ogg)_{ij}\varphi_{i\alpha}\varphi_{j\alpha} - (logg)_{\,i\varphi_{\alpha\alpha i}}.$$

From the concavity of log det the second term on the left is negative. If $\varphi_{\alpha\alpha}$ reaches at X_0 the maximum value among all pure second derivatives, then the right-hand side must be negative. Let us look at the explicit case in which up to a constant, $f = e^{-Q(X)}$ and $g = e^{-(Q(Y)+F(Y))}$, where Q is a nonnegative quadratic polynomial, $a_{ij}x_ix_j$ (for instance ,near neighborhood or other "Dirichlet Integral "like interactions in field theory).

We may assume that $\alpha = e_1$. Then, we must compute

$$D_{11}(-Q(X) + Q(\nabla\varphi) + F(\nabla\varphi)\,,$$

we have

$$D_{11}(-Q)(X) = -a_{11},$$
$$D_{11}Q(\nabla\varphi) = a_{ij}\varphi_{i1}\varphi_{j1} + a_{ij}\varphi_{i11}\varphi_j.$$
But since $\varphi_{11}(X_0)$ is the maximum among all pure second derivatives, $\varphi_{11i} = 0$ for all i, and $\varphi_{1i} = 0$ for $i \neq 1$. So $D_{11}Q(\nabla\varphi(X_0)) = a_{11}(\varphi_{11})^2$. Finally, if F is convex
$$D_{11}F(\nabla\varphi) = F_{ij}\varphi_{i1}\varphi_{j1} + F_i\varphi_{i11}$$
is non-negative.

Therefore $D_{11}(R.H.S.) \geq a_{11}((\varphi_{11})^2 - 1)$. We get a contradiction if $\varphi_{11} > 1$. That is

Theorem (5.1.20) [266]: *Let, up to a multiplicative constant,*
$$f(X) = e^{-Q(X)},$$
$$g(Y) = e^{-(Q(Y)+F(Y))}$$
with F convex. Then the potential φ of the optimal transportation satisfies
$$0 \leq \varphi_{\alpha\alpha} \leq 1.$$
In particular
$$Y = X + \nabla u(X),$$
where
$$u = \varphi - \tfrac{1}{2}|X|^2$$
is concave and
$$-1 \leq u_{\alpha\alpha} \leq 0$$
(independently of dimension).

Proof: To make the previous theorem valid we have to take care of what happens when X goes to infinity.

Again by approximation we may assume that the convex function $F(X)$ is $+\infty$ outside the ball B_R (that is g is supported in the ball of radius R, and smooth bounded away from zero and infinity inside it.

We will replace the second derivative by an incremental quotient, and show that it still satisfies a maximum principle and goes to zero at infinity. Let
$$(\delta\varphi_e)(X) = \varphi(X + he) + \varphi(X - he) - 2\varphi(X).$$
We fix h, and study what happens if $\delta\varphi = \delta\varphi_{e_1}$ attains a local maximum at X_0, for all possible e. From the concavity of log det, we still have that, for the linearization coefficients M_{ij}, of log det at X_0,
$$M_{ij}\delta\varphi(X_0) \leq \delta(\log f - \log g) = \delta(-Q(X)) + Q(\nabla\varphi) + F(\nabla\varphi).$$
From the fact that $\delta\varphi_{e_1}$ realizes a maximum among X and e, we obtain

a) $\nabla\delta\varphi = \nabla\varphi(X_0 + \mathrm{h}e_1) + \nabla\varphi(X_0 - \mathrm{h}e_1) - 2\nabla\varphi(X_0) = 0$

and

b) for any $\tau \perp e_1$,
$$D_\tau\delta\varphi = \tau \cdot (\nabla\varphi(X_0 + he_1) - \nabla\varphi(X_0 - he_1) = 0.$$
Therefore
$$\nabla\varphi(X \pm he_1) = \nabla\varphi(X) \pm \lambda e_1$$
and $\delta\varphi = 2\lambda$ (λ positive). Then, from the convexity of F,

$$\delta F(\nabla\varphi(X_0)) \geq 0.$$

If we write $Q(X)$ as a bilinear form $Q(X) = B(X, X)$,

$$\delta Q(\nabla\varphi) = B(\nabla\varphi(X_0) + \lambda e_1, \nabla\varphi(X_0) + \lambda e_1)$$
$$+ B(\nabla\varphi(X_0) - \lambda e_1, \nabla\varphi(X_0) - \lambda e_1)$$
$$- 2B(\nabla\varphi(X_0), \nabla\varphi(X_0))$$
$$= \lambda^2 B(e_1, e_1).$$

Similarly $\delta Q(X) = h^2 B(e_1, e_1)$ so we get: If $\delta\varphi$ has an interior maximum at X_0, then it must hold:

$$\nabla\varphi(X_0 \pm he_1) = \nabla\varphi(X_0) \pm \lambda e_1$$

with $\lambda < h$.

But, since φ is convex

$$\varphi(X_0 \pm he_1) - \varphi(X_0) \leq \langle\nabla\varphi(X_0 \pm he_1) - \nabla\varphi(X_0) \pm he_1\rangle = \lambda h \leq h^2..$$

Thus,

$$\delta\varphi \leq 2h^2,$$

the desired inequality.

To complete the proof of the theorem it would be enough to show that $\delta\varphi$ goes to zero (for fixed δ) when X goes to infinity. We show that:

Lemma (5.1.21) [266]: *As X goes to infinity Y converges uniformly to* $R\frac{X}{|X|}$.

Proof: Let $X_0 = \lambda e_1$ for λ large and Y_0 its image. Let ν be a unit vector with

$$angle\ (\nu, e_1) \leq \frac{\pi}{2} - \varepsilon.$$

From the monotonicity of the map, any point on B_R of the form

$$Y' = Y_0 + t\nu$$

must come from a vector

$$X' = X_0 + s\mu,$$

with $\langle\mu, \nu\rangle \geq 0$.

In particular, we must have

$$angle(\mu, e_1) \leq (\pi - \varepsilon).$$

In other words if in Y space we consider the cone,

$$\Gamma = \{Y' = X_0 + t\nu, \text{with } t > 0, \text{angle}(\nu, e_1) \geq \frac{\pi}{2} - \varepsilon,$$

its intersection with B_R must be covered by the image of the (concave) cone

$$\overline{\Gamma} = \{X' = X_0 + s\mu, \text{with } s > 0 \text{ and } angle(\mu, e_1) \leq \pi - \varepsilon\}.$$

But $\overline{\Gamma}$ has very small f measure

$$\mu_f\left(\overline{\Gamma}\right) \leq (\varepsilon\lambda)^n e^{-(\varepsilon\lambda)^2}, \qquad \varepsilon\lambda > \lambda^{1/2},$$

since the ball of radius $\varepsilon\lambda$ is not contained in $\overline{\Gamma}$.

On the other hand, g is strictly positive in B_R, so

$$\mu_g(\Gamma \cap B_R) \sim |\Gamma \cap B_R| \leq \mu_f(\overline{\Gamma}).$$

This forces the exponential convergence of Y to Re_1.

This completes the proof of the lemma and the theorem, since the uniform convergence of $\nabla\varphi$ to $\frac{X}{|X|}$makes $\delta\varphi$ go to zero (for any fixed, positive h).

We state three corollaries of this last inequality. The first two are a generalization of the classic Brascamp–Lieb moment inequality and the third an eigenvalue inequality.

Corollary (5.1.22) [266]: *Let $f(X) = e^{-Q(X)}, g(H) = e^{-[Q(Y)+F(Y)]}$ with Q quadratic and F convex, and let Γ be a convex function of one variable ($|x_1|^\alpha$ in $[B-L]$). Then*

$$E_g(\,\Gamma(y_1 - E_g(y_1)) \le E_f\,(\,\Gamma(x_1)).$$

Proof: It follows from $[B-L]$ that it is enough to prove it in the one dimensional case (see Theorem 5.1 of $[B-L]$). We can also assume by a translation that $E_g(y_1) = 0$.
By the change of variable formula that means

$$\int y(x)f(x)dx = 0.$$

Also

$$E_g\big(\Gamma(y_1)\big) = \int \Gamma(y_1(x)f(x)dx.$$

But $y(x) = x + u(x)$, where$y = \varphi'(x), \varphi$ convex and $u = \psi'(x), \psi$ concave. Thus y is increasing, and u is decreasing and changes sign, since

$$\int u(x)f(x)dx = \int y(x)f(x)dx = 0.$$

Say $u(x_0) = 0$. Then, we write

$$\int \Gamma(y(x))f(x) \le \int [\Gamma(x) + \Gamma'\big(y(x)\big)(y-x)]f(x).$$

Since Γ is convex,

$$\le E_f\big(\Gamma(x)\big) + \int [\Gamma'(\mathrm{y}(x)) - \Gamma'(x_0)](y-x)f(x).$$

But at $x_0, = \Gamma'(y(x_0)) = \Gamma'(x_0)$ and $y(x_0) = x_0$, and further $\Gamma'$is increasing, while $y - x = u$ is decreasing, thus the last integrand is negative, and this completes the proof.
If we want to repeat the argument above for functions Γ that depend on more than one variable, and we want to prove that

$$E_g(\Gamma(Y - E_g(Y)) \le E_f\,(\Gamma(X))\,,$$

we may as before assume that $E_g(Y) = 0$.
That means, with $Y = X + U$,that $U(X_0) = 0$ for some X_0 (i.e., the concave function $-\psi$ has a maximum). The same computation then gives us

$$E_g\big(\Gamma(Y)\big) \le E_f\big(\Gamma(X)\big) + \int (\nabla\Gamma(Y) - \nabla(\Gamma(X_0))(-\nabla\psi(Y)f(X)dx,$$

where$\psi$and $\Gamma - \langle\nabla\Gamma(X_0), X - X_0\rangle$ are both convex with a minimum at X_0, so there is some hope that the integrand be negative.

For instance, if we are looking at statistics of k −variables we have the following corollary.

Corollary (5.1.23) [266]: *Assume that $Q(X), F(X)$ in the definition of $f(X), g(Y)$ are symmetric with respect to $(x_1, \dots, x_k)$ and that $\Gamma(x_1, \dots, x_k)$ is convex and symmetric. Then*

$$E_g(\Gamma(Y)) \leq E_f(\Gamma(X)).$$

Proof: As before we may assume the problem is k −dimensional ([275], Theorem 4.3). Since Q and F are symmetric, the potentials $\varphi(X), \psi(X)$ are symmetric. Therefore $\nabla\varphi, \nabla\psi, \nabla\Gamma = 0 \ for\ X = 0$ and further,

$sign\ \varphi_i(X) = sign\ \psi_i(X) = sign\ \Gamma_i(X) = sign\ x_i = sign\ y_i$.

From the computation above it suffices to show that for all Y,

$$\nabla\Gamma \cdot \nabla\psi \geq 0.$$

That follows since $\Gamma_i \cdot \psi_i \geq 0$ for all i.

A final consequence of the estimate $\varphi_{\alpha\alpha} \leq 1$ for log concave perturbations of the Gaussian is that any Raleigh-like quotient (log Sobolev inequality, isoperimetric inequality, Poincaré inequality) that involves a quotient between first derivatives and the function themselves is smaller for the perturbation than for the Gaussian.

For instance, let $F(t), G(t), H(t), K(t)$ be non-negative, non-decreasing functions of $t \in [0, \infty)$, then we have the

Corollary (5.1.24) [266]: *Let f, g be densities as inTheorem (5.1.20) (i.e., a Gaussian and its log concave perturbation) then consider the "Raleigh" quotient*

$$\lambda_f = \inf \frac{F(\int G(|\nabla u|)f(X)dX)}{H(\int K(|u|)f(X)dX)}.$$

Then $\lambda_g \geq \lambda_f$.

Proof: If we apply the change of variable formula to any function $u(Y)$, we get

$$\int K(|u(Y)|)g(Y)dY = \int K(|u(X)|)f(X)dX,$$

while $\nabla_X u(Y(X)) = D_X(Y)\nabla_Y u(X)$. But $D_X Y$ is a symmetric matrix with all eigenvalues less than one, so $|\nabla_X u(Y(X))| \leq |D_Y u(Y)|$ which proves the corollary.

Section (5.2): Caffarelli Log-Concave Perturbation Theorem

Let γ_n denote the centered Gaussian measure in $\mathbb{R}^n$, i.e., $\gamma_n = (2\pi)^{-n/2} e^{-|x|^2/2} dx$, and let μ be a probability measure on $\mathbb{R}^n$. By a classical theorem of Brenier [280], there exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $T = \nabla_\varphi : \mathbb{R}^n \to \mathbb{R}^n$ transports γ_n onto μ, i.e., $T_\sharp \gamma_n = $ μ, or equivalently

$$\int h \circ T\, d_{\gamma_n} = \int h d\mu \quad \text{for all continuous and bounded functions } h \in C_b(\mathbb{R}^n).$$

In the sequel we will refer to T as the *Brenier map* from γ_n to μ.

In [284,285] Caffarelli proved that if μ is "more log-concave" than γ_n, then T is 1-Lipschitz, that is, all the eigenvalues of $D^2\varphi$ are bounded from above by 1. Here is the exact statement:

Theorem (5.2.1)[279]: (*Caffarelli*)**.** *Let γ_n be the Gaussian measure in $\mathbb{R}^n$, and let* $\mu = e^{-V}\,dx$ *be a probability measure satisfying $D^2V \geq Id_n$. Consider the Brenier map $T = \nabla_\varphi$ from γ_n to* μ*. Then T is* 1-Lipschitz. *Equivalently,* $0 \leq D^2\varphi(x) \leq Id_n$ *for a.e. x.*

This theorem allows one to show that optimal constants in several functional inequalities are extremized by the Gaussian measure. More precisely, let F, G, H, L, J be continuous functions on $\mathbb{R}$ and assume that F, G, H, J are nonnegative, and that H and J are increasing. For $\ell \in \mathbb{R}_+$ let

$$\lambda(\mu,\ell) := \inf\left\{\frac{H(\int J(|\nabla_u|)d\mu)}{F(\int G(u)d\mu)} : u \in Lip(\mathbb{R}^n), \int L(u)\,d\mu = \ell\right\} \qquad (1).$$

Then

$$\lambda(\gamma_n,\ell) \leq \lambda(\mu,\ell). \qquad (2)$$

Indeed, given a function u admissible in the variational formulation for μ, we set $v := u \circ T$ and note that, since $T_\sharp\,\gamma_n = \mu$,

$$\int K(u)\,d\gamma_n = \int K(u \circ T)\,d\gamma_n = \int K(u)\,d\mu \ \text{ for } \ K = G, L\,.$$

In particular, this implies that v is admissible in the variational formulation for γ_n. Also, thanks to Caffarelli's Theorem,

$$|\nabla v| \leq |\nabla u| \circ T\,|\nabla T| \leq |\nabla u| \circ T,$$

therefore

$$H\left(\int J(|\nabla v|)d\,\gamma_n\right) \leq H\left(\int J(|\nabla v|) \circ T d\,\gamma_n\right) = H\left(\int J(|\nabla v|)d\mu\right).$$

Thanks to these formulas, (2) follows easily.

Note that the classical Poincar´e and Log-Sobolev inequalities fall in the above general framework. For instance, choosing $H(t) = F(t) = L(t) = t, \ell = 0$, and $J(t) = F(t) = |t|^p$ with $p \geq 1$, we deduce that

$$\inf\left\{\frac{\int|\nabla_u|^p d\mu}{\int|u|^p d\mu} : u \in Lip(\mathbb{R}^n), \int u\,d\mu = 0\right\} \geq \inf\left\{\frac{\int|\nabla_u|^p\,d\gamma_n}{\int|u|^p\,d\gamma_n} : u \in Lip(\mathbb{R}^n), \int u\,d\,\gamma_n = 0\right\} \qquad (3)\,.$$

Two questions that naturally arise from the above considerations are:

– *Rigidity*: What can be said about μ when $\lambda(\mu,\ell) = \lambda(\gamma_n,\ell)$?

– *Stability*: What can be said about μ when $\lambda(\mu,\ell) \approx \lambda(\gamma_n,\ell)$?

Looking at the above proof, these two questions can usually be reduced to the study of the corresponding ones concerning the optimal map T in Theorem (5.2.1) (here $|A|$ denotes the operator norm of a matrix A):

– *Rigidity*: What can be said about μ when $|\nabla T(x)| = 1$ for a.e. x?

– *Stability*: What can be said about μ when $|\nabla T(x)| \approx 1$ (in suitable sense)?

Our first main result states that if $|\nabla T(x)| = 1$ for a.e. x then μ "splits off" a Gaussian factor. More precisely, it splits off as many Gaussian factors as the number of eigenvalues of $\nabla T = D^2\varphi$ that are equal to 1. In the following statement and in the sequel, given $p \in \mathbb{R}^k$ we denote by $\gamma_{p,k}$ the Gaussian measure in $\mathbb{R}^k$ with barycenter p, that is, $\gamma_{p,k} = (2\pi)^{-k/2}e^{-|x-p|^2/2}dx$.

Theorem (5.2.2) [279]: (*Rigidity*)**.** *Let* γ_n *be the Gaussian measure in* $\mathbb{R}^n$, *and let* $\mu = e^{-V}dx$ *be a probability measure with* $D^2V \geq Id_n$. *Consider the Brenier map* $T = \nabla\varphi$ *from* γ_n *to* μ, *and let*

$$0 \leq \lambda_1(D^2\varphi(x)) \leq \cdots \leq \lambda_n(D^2\varphi(x)) \leq 1$$

be the eigenvalues of the matrix $D^2\varphi(x)$. *If* $\lambda_{n-k+1}(D^2\varphi(x)) = 1$ *for a.e. x then* $\mu = \gamma_{p,k} \otimes e^{-W(x\prime)}dx'$, *where* $W : \mathbb{R}^{n-k} \to \mathbb{R}$ *satisfies* $D^2W \geq Id_{n-k}$.

Our second main result is a quantitative version of the above theorem. Before stating it let us recall that, given two probability measures $\mu, \nu \in P(\mathbb{R}^n)$, the 1-Wasserstein distance between them is defined as

$$W_1(\mu,\nu) := \inf\{\int |x - y| d\sigma(x,y) : \sigma \in P(\mathbb{R}^n \times \mathbb{R}^n) such\ that\ (pr_1)_{\#}\sigma = \mu, (pr_2)_{\#}\sigma = \nu\},$$

where pr_1 (resp. pr_2) is the projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto the first (resp. second) factor. Our stability result is formulated in terms of the W_1 −distance between probability measures as this distance naturally comes out from our strategy of proof. Our result could also be proved with other notions of distances meterizing the weak topology (for instance, any Wasserstein distance W_p), as well as stronger notion of distances (such as the total variation), but we shall not investigate this here.

Theorem (5.2.3) [279]: (*Stability*)**.** *Let* γ_n *be the Gaussian measure in* $\mathbb{R}^n$ *and let* $\mu = e^{-V}dx$ *be a probability measure with* $D^2V \geq Id_n$. *Consider the Brenier map* $T = \nabla\varphi$ *from* γ_n *to* μ, *and let*

$$0 \leq \lambda_1(D^2\varphi(x)) \leq \cdots \leq \lambda_n(D^2\varphi(x)) \leq 1$$

be the eigenvalues of $D^2\varphi(x)$. *Let* $\varepsilon \in (0,1)$ *and assume that*

$$1 - \varepsilon \leq \int \lambda_{n-k+1}\big(D^2\varphi(x)\big) d\,\gamma_n(x) \leq 1 \qquad (4).$$

Then there exists a probability measure $\nu = \gamma_{p,k} \otimes e^{-W(x\prime)}dx'$, *with* $W : \mathbb{R}^{n-k} \to \mathbb{R}$ *satisfying* $D^2W \geq Id_{n-k}$, *such that*

$$W_1(\mu,\nu) \lesssim \frac{1}{|\log\varepsilon|^{1/4}} \qquad (5).$$

In the above statement, we are employing the following notation:

$X \lesssim Y^{\beta}$- if $X \leq C(n,\alpha)Y^{\alpha}$ for all $\alpha < \beta$.

Analogously,

$X \gtrsim Y^{\beta}$- if $C(n,\alpha)X \geq Y^{\alpha}$ for all $\alpha < \beta$.

Remark (5.2.4) [279]: We do not expect the stability estimate in the previous theorem to be sharp. In particular, in dimension 1 an elementary argument (but completely specific to the one dimensional case) gives a linear control in ε. Indeed, assuming (up to translating μ) that

$$\int x d\mu = 0 \quad (6),$$

set $\psi(x) := x^2/2 - \varphi(x)$. Then, since $\psi'' = (x - T)' > 0$, our assumption can be rewritten as

$$\int |(x - T)'| d\gamma_1 = \psi'' \, d\gamma_1 \leq \varepsilon.$$

Also, since $T_{\#}\gamma_1 = \mu$, (6) yields

$$\int T(x) d\gamma_1 = 0 = \int x d\gamma_1.$$

Hence, by the L^1 −Poincar´e inequality for the Gaussian measure we obtain

$$W_1(\mu, \gamma_1) \leq \int |x - y| d\sigma_T(x, y) = \int |x - T(x)| d\gamma_1 \leq C \int |(x - T)'| d\gamma_1 \leq C\varepsilon,$$

where $\sigma_T := (Id \times T)_{\#}\gamma_1$.

As explained above, Theorems (5.2.2) and (5.2.3) can be applied to study the structure of 1-log-concave measures (i.e., measures of the form $e^{-V} dx$ with $D^2 V \geq Id_n$) that almost achieve equality in (2). To simplify the presentation and emphasize the main ideas, we limit ourselves to a particular instance of (1), namely the optimal constant in the L^2 −Poincar´e inequality for μ:

$$\lambda_\mu := \inf \left\{ \frac{\int |\nabla_u|^2 d\mu}{\int u^2 d\mu} : u \in Lip(\mathbb{R}^n), \int u \, d\mu = 0 \right\}.$$

It is well-known that $\lambda_{\gamma_n} = 1$ and that $\{u_i(x) = x_i\}_{1 \leq i \leq n}$ are the corresponding minimizers. In particular it follows by (3) that, for every 1-log-concave measure μ,

$$\int u^2 d\mu \leq \int |\nabla u|^2 d\mu \text{ for all } u \in Lip(\mathbb{R}^n) \text{ with } \int u \, d\mu = 0 (7).$$

As a consequence of Theorems (5.2.2) and (5.2.3) we have:

Theorem (5.2.5) [279]: *Let $\mu = e^{-V} dx$ be a probability measure with $D^2 V \geq Id_n$, and assume there exist k functions $\{u_i\}_{1 \leq i \leq k} \subset W^{1,2}(\mathbb{R}^n, \mu)$, $k \leq n$, such that*

$$\int u_i \, d\mu = 0, \int u_i^2 \, d\mu = 1, \qquad \cdot \nabla u_j \, d\mu = 0 \; \forall i \neq j,$$

and

$$\int |\nabla u_i|^2 \, d\mu \leq 1 + \varepsilon$$

for some $\varepsilon > 0$. Then there exists a probability measure $\nu = \gamma_{p,k} \otimes e^{-W(x\prime)} dx'$, with $W : \mathbb{R}^{n-k} \to \mathbb{R}$ satisfying $D^2 W \geq Id_{n-k}$, such that

$$W_1(\mu, \nu) \lesssim \frac{1}{|\log \varepsilon|^{1/4}}.$$

In particular, if there exist n orthogonal functions $\{u_i\}_{1 \leq i \leq n}$ that attain the equality in (7) then $\mu = \gamma_{n,p}$.

We conclude recalling that the rigidity version of the above theorem (i.e., the case $\varepsilon = 0$) has already been proved by Cheng and Zho in [286, Theorem 2] with completely different techniques.

To prove Theorem (5.2.2), we first recall the following classical estimate due to Alexandrov (see for instance [288, Theorem 2.2.4 and Example 2.1.2(1)] for a proof):

Lemma (5.2.6) [279]: *Let Ω be an open bounded convex set, and let $u : \Omega \to \mathbb{R}$ be a$C^{1,1}$ convex function such that $u = 0$ on ∂Ω. Then there exists a dimensional constant $C_n > 0$ such that*

$$|u(x)|^n \leq C_n \, diam(\Omega) n^{-1} \, dist(x, \partial\Omega) \int_\Omega det D^2 u \forall x \in.$$

Set $\psi(x) := |x|^2/2 - \varphi(x)$ and note that, as a consequence of Theorem (5.2.1), $\psi : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,1}$convex function with $0 \le D^2\psi \le Id$. Also, our assumption implies that

$$\lambda_1\big(D^2\psi(x)\big) =\cdots= \lambda_k\big(D^2\psi(x)\big) = 0 \text{ for a.e. } x \in \mathbb{R}^d \qquad (8).$$

We are going to show that ψ depends only on $n-k$ variables. As we shall show later, this will immediately imply the desired conclusion. In order to prove the above claim, we note it is enough to prove it for $= 1$, since then one can argue recursively on $\mathbb{R}^{n-1}$and so on.

Note that (8) implies that

$$det D^2\psi \equiv 0 \ (9).$$

Up to translate μ we can subtract a linear function to ψ and assume without loss of generality that $\psi(x) \ge \psi(0) = 0$.

Consider the convex set $\Sigma := \{\psi = 0\}$. We claim that Σ contains a line. Indeed, if not, this set would contain an exposed point $\bar{x}$. Up to a rotation, we can assume that $\bar{x} = ae_1$ with $a \ge 0$. Also, since $\bar{x}$ is an exposed point,

$$\Sigma \subset \{x_1 \le a\} \, and \, \Sigma \cap \{x_1 = a\} = \{\bar{x}\}.$$

Hence, by convexity of Σ, the set $\Sigma \cap \{x_1 \ge -1\}$ is compact.

Consider the affine function

$$\ell_\eta(x) := \eta(x_1 + 1), \eta > 0 \text{ small},$$

and define $\Sigma_\eta := \{\psi \le \ell_\eta\}$. Note that, as $\eta \to 0$, the sets Σ_η converge in the Hausdorff distance to the compact set $\Sigma \cap \{x_1 \ge -1\}$. In particular, this implies that Σ_η is bounded for η sufficiently small.

We now apply Lemma (5.2.6) to the convex function $\psi - \ell_\eta$ inside Σ_η, and it follows by (9) that (note that $D^2\ell_\eta \equiv 0$)

$$\left|\psi(x) - \ell_\eta(x)\right|^n \le C_n \operatorname{diam}\,(\Sigma_\eta))^n \int_{\Sigma_\eta} det\, D^{\,2}\psi \in 0 \ \forall x \in \Sigma_\eta \,.$$

In particular this implies that $\psi(0) = \ell_\eta(0) = \eta$, a contradiction to the fact that $\psi(0) = 0$.

Hence, we proved that $\{\psi = 0\}$ contains a line, say $\mathbb{R}_{e1}$. Consider now a point $x \in \mathbb{R}^n$. Then, by convexity of ψ,

$$\psi(x) + \nabla\psi(x) \cdot (se_1 - x) \le \psi(se_1) = 0 \ \forall s \in \mathbb{R},$$

and by letting $s \to \pm\infty$ we deduce that $\partial_1\psi(x) = \nabla\psi(x) \cdot e_{\,1} = 0$. Since x was arbitrary, this means that $\partial_1\psi \equiv 0$, hence $\psi(x) = \psi(0, x'), x' \in \mathbb{R}^{n-1}$.

Going back to φ, this proves that

$$T(x) = (x_1, x' - \nabla\psi(x')),$$

and because $\mu = T\#\gamma_n$ we immediately deduce that $\mu = \gamma_{\,1} \otimes \mu_1$ where $\mu_1 := (Id_{n-1} - \nabla\psi)\#\gamma_{n-1}$.

Finally, to deduce that $\mu_1 = e^{\,-W}dx'$ with $D^2\,W \ge Id_{n-1}$we observe that $\mu_1 = (\pi')\#\mu$ where $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$is the projection given by $\pi'(x_1, x') := x'$. Hence, the result is a consequence of the fact that

1-log-concavity is preserved when taking marginals, see [281, Theorem 4.3] or [289, Theorem 3.8].

we first recall a basic properties of convex sets (see for instance [283, Lemma 2] for a proof).

Lemma (5.2.7) [279]: *Given S an open bounded convex set in $\mathbb{R}^n$ with barycenter at 0, let ε denote an ellipsoid of minimal volume with center 0 and containing S. Then there exists a dimensional constant $\kappa_n > 0$ such that $\kappa_n \varepsilon \subset S$.*

We can prove the following simple geometric lemma:

Lemma (5.2.8) [279]: *Let κ_n be as in Lemma (5.2.7), set $c_n := \kappa_n/2$, and consider $S \subset \mathbb{R}^n$ an open convex set with barycenter at 0. Assume that $S \subset B_R$ and $\partial S \cap \partial B_R / \neq \emptyset$. Then there exists a unit vector $v \in \mathbb{S}^{n-1}$ such that $\pm c_n R v \in S$.*

Proof: By scaling we can assume that $R = 1$.

Let $v \in \partial S \cap \partial B_1$, and consider the ellipsoid E provided by Lemma (5.2.7). Since $v \in \bar{\mathcal{E}}$ and $\mathcal{E}$ is symmetric

with respect to the origin, also $-v \in \bar{\mathcal{E}}$. Hence

$$\pm c_n v \in c_n \bar{\mathcal{E}} \subset \kappa_n \mathcal{E} \subset S,$$

as desired.

In order to complete the proof of Theorem (5.2.3) we recall the following geometric result, see [283, Lemma 1].

Lemma (5.2.9) [279]: *Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a nonnegative convex function with $(0) = 0$. Assume that ψ is finite in a neighborhood of 0 and that the graph of ψ does not contain lines. Then there exists $p \in \mathbb{R}^n$ such that the open convex set*

$$S_1 := \{x : \psi(x) \le p \cdot x + 1\}$$

is nonempty, bounded, and with barycenter at 0.

As in the proof of Theorem (5.2.2) we set $\psi := |x|^2/2 - \varphi$. Then, inequality (4) gives

$$\int \lambda_k(D^2\psi) d\gamma_n \le \varepsilon. \tag{10}$$

Up to subtract a linear function (i.e., substituting μ with one of its translation, which does not affect the conclusion of the theorem) we can assume that $\psi(x) \ge \psi(0) = 0$, therefore $\nabla\psi(0) = \nabla\phi(0) = 0$. Since $(\nabla\varphi)\#\gamma_n = \mu$ and $\| D^2\varphi \|_\infty \le 1$, these conditions imply that

$$\int |x| d\mu(x) = \int |\nabla\varphi(x)| d\gamma_n(x) = \int |\nabla\varphi(x) - \nabla\varphi(0)| d\gamma_n(x) \le \int |x| d\gamma_n(x) \le C_n.$$

In particular

$$W_1(\mu, \gamma) \le W_1(\mu, \delta_0) + W_1(\delta_0, \gamma) \le C_n.$$

This proves that (5) holds true with $\nu = \gamma_n$ and with a constant $C \approx |log\varepsilon_0|^{1/4}$ whenever $\varepsilon \ge \varepsilon_0$. Hence, when showing the validity of (5), we can safely assume that $\varepsilon \le \varepsilon_0(n) \ll 1$. Furthermore, we can assume that the graph of ψ does not contain lines (otherwise, by the proof of Theorem (5.2.2), we would deduce that μ splits a Gaussian factor, and we could simply repeat the argument in $\mathbb{R}^{n-1}$).

Thus we can apply Lemma (5.2.9) to deduce the existence of a slope $p \in \mathbb{R}^n$ such that

$$S_1 = \{x \in \mathbb{R}^n : \psi(x) < p \cdot x + 1\}$$

is nonempty, bounded, and with barycenter at 0. Applying Lemma (5.2.6) to the convex function $\tilde{\psi}(x) := \psi(x) - p \cdot x - 1$ inside the set S_1, we get (note that $D^2\tilde{\psi} = D^2\psi$)

$$1 \le \left(\underset{S_1}{-min}\tilde{\psi}\right)^n \le C_n\,(diam(S_1))^n \int_{S_1} \det\, D^2\psi. \qquad (11)$$

Consider now the smallest radius $R > 0$ such that $S_1 \subset B_R$ (note that $R < +\infty$ since S_1 is bounded). Since $\gamma_N \ge c_N e - R^2/2$ in B_R and $\lambda_n(D_n\psi) \le 1$ for all $i = 1,\dots,n$, (10) implies that

$$\int_{B_R} detD^2\psi \le C_n e^{R^2/2}\,\varepsilon.$$

Hence, using (11), since $\text{diam}(S_1) \le 2R$ we get

$$1 \le C_n R^n e^{R^2/2}\,\varepsilon$$

which yields

$$R \gtrsim |log\varepsilon|^{\frac{1}{2}+}. \qquad (12)$$

Now, up to a rotation and by Lemma (5.2.8), we can assume that

$$\pm c_n R e_1 \in S_1.$$

Consider $1 \ll \rho \ll R^{1/2}$ to be chosen. Since $S_1 \subset B_R$ and $\psi \ge 0$ we get that $|p| \le 1/R$, therefore $\psi \le 2$ on $S_1 \subset B_R$. Hence

$$2 \ge \psi(z) \ge \psi(x) + \langle \nabla\psi(x), z - x\rangle \ge \langle \nabla\psi(x), z - x\rangle\ \forall z \in S_1, x \in B_\rho.$$

Thus, since $|\nabla\psi| \le \rho\ in\ B\rho\ (by\ \| D^2\psi \| L^\infty(\mathbb{R}^n) \le 1\ and\ |\nabla\psi(0)| = 0)$, choosing $z = \pm c_n R e_1$ we get

$$|\partial_1\psi| \le \frac{C_n\,\rho^2}{\mathcal{R}} \qquad \text{inside } B_\rho \quad (13).$$

Consider now $\bar{x}_1 \in [-1,1]$ (to be fixed later) and define $\psi_1(x') := \psi(\bar{x}_1, x')$ with $x' \in \mathbb{R}^{n-1}$. Integrating (13) with respect to x_1 inside $B_{\rho/2}$, we get

$$|\psi - \psi_1| \le C_n \frac{\rho^3}{R}\, inside\ B_{\rho/2}.$$

Thus, using the interpolation inequality

$$\|\nabla\psi - \nabla\psi_1\|^2_{L^\infty(B_{\rho/})} \le C_n \|\nabla\psi - \nabla\psi_1\|_{L^\infty(B_{\rho/2})} \|D^2\psi - D^2\psi_1\|_{L^\infty(B_{\rho/2})}$$

and recalling that $\| D^2\psi \|_{L^\infty(\mathbb{R}^n)} \le 1$ (hence $\| D^2\psi_1 \|_{L^\infty(\mathbb{R}^{n-1})} \le 1$), we get

$$|\nabla\psi - \nabla\psi_1| \le C_n \frac{R^{3/2}}{R^{1/2}} \text{ inside } B_{\rho/4}.$$

If $k = 1$ we stop here, otherwise we notice that (10) implies that

$$\int_{\mathbb{R}} d\gamma_1(x_1) \int_{\mathbb{R}^{n-1}} det\, D^2_{x',x'}\psi(x_1, x')d\gamma_{n-1}(x')$$
$$\leq \int_{\mathbb{R}} d\gamma_1(x_1) \int_{\mathbb{R}^{n-1}} \lambda_2(D^2\psi)(x_1, x')d\gamma_{n-1}(x') \leq \varepsilon,$$

where we used that[1] $\lambda_1(D^2\psi|_{\{0\}\times\mathbb{R}^{n-1}}) \leq (D^2\psi)$ and that (since $0 \leq D^2\psi \leq Id_n$)

$$drt\, D^2_{x',x'}\psi(x_1, x') \leq \lambda_1(D^2\psi|_{\{0\}\times\mathbb{R}^{n-1}}).$$

Hence, by Fubini's Theorem, there exists $\bar{x}_1 \in [-1,1]$ such that $\psi_1(x') = \psi(\bar{x}_1, x')$ satisfies

$$\int_{\mathbb{R}^{n-1}} det\ D^2\psi_1 d\gamma_{n-1}(x) \leq C_n\varepsilon.$$

This allows us to repeat the argument above in $\mathbb{R}^{n-1}$ with

$$\bar{\psi}_1(x') := \psi_1(x') - \nabla x'\psi_1(0) \cdot x' - \psi_1(0)$$

[1] This inequality follows from the general fact that, given $A \in \mathbb{R}^{n\times n} \times n$ symmetric matrix and $W \subset \mathbb{R}^n$ a k-dimensional vector space,

$$\lambda_1(A|_W) = \min_{v\in W} \frac{|Av.v}{|v|^2} \leq \max_{\substack{v\in w' \\ w'}} \ \min_{\substack{\subset\mathbb{R}^n \\ k-dim}} \ \min_{w'} \frac{|Av.v}{|v|^2} = \lambda_{n-k+1}(A).$$

in place of ψ, and up to a rotation we deduce that

$$\left|\nabla\widetilde{\psi_1} - \nabla\psi_2\right| \leq C_n \frac{R^{3/2}}{R^{1/2}} R \text{ inside } B_{\rho/4}$$

where $\psi_2(x'') := \psi 1(\bar{x}_2, x'')$, where $\bar{x}_2 \in [-1,1]$ is arbitrary. By triangle inequality, this yields

$$|\nabla\psi + p' - \nabla\psi_2| \leq C_n \frac{R^{3/2}}{R^{1/2}} R \text{ inside } B_{\rho/4},$$

where $p' = -(0, \nabla_{x'}\psi(\overline{x_1}, 0))$. Note that, since $|\overline{x_1}| \leq 1, \nabla\psi(0) = 0$, and $\| D^2\psi\|\infty \leq 1$, we have $|p| \leq 1$.

Iterating this argument k times, we conclude that

$$|\nabla\psi + \bar{p} - \nabla\psi_k| \leq C_n \frac{R^{3/2}}{R^{1/2}} \quad \text{inside } B_{\rho/4},$$

Where $\bar{p} = (p, p'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$ with $|\bar{p}| \leq C_n$,

$$\psi_k(y) := \psi(\overline{x_1}, \dots, \overline{x_k}, y), y \in \mathbb{R}^{n-k},$$

and $\overline{x_\iota} \in [-1,1]$. Recalling that $\nabla\varphi = x - \nabla\psi$, we have proved that

$$T(x) = \nabla\varphi(x) = (x_1 + p_1, \dots, x_k + p_k, S(y) + p'') + Q(x),$$

where $Q := -(\nabla\psi - \nabla\psi_k + \overline{p})$ satisfies

$$\|Q\|_{L^\infty(B_\rho)} \leq C_n \frac{\rho^{3/2}}{R^{1/2}} \text{ and } |Q(x)| \leq C_n(1 + |x|)$$

(in the second bound we used that $T(0) = \nabla\varphi(0) = 0, |p| \leq C_n$, and T is 1-Lipschitz). Hence, if we set $\nu := (S + p'')_{\#}\gamma_{n-k}$, we have

$W_1(\mu, \gamma_{p,k} \otimes \nu) \leq \int |Q| d\gamma_n \leq C_n \frac{\rho^{\frac{3}{2}}}{R^{\frac{1}{2}}} + \int_{\mathbb{R}^n \setminus B_\rho} |x| d\gamma_n = C_n \frac{\rho^{\frac{3}{2}}}{R^{\frac{1}{2}}} + C_n \rho^n e^{-\rho^2/2}$,

so, by choosing $\rho := (logR)^{1/2}$, we get

$$W_1(\mu, \gamma_{p,k} \otimes \nu) \lesssim \frac{1}{R^{1/2_-}} .$$

Consider now $\pi_k : \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{\pi}_{n-k} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ the orthogonal projection onto the first k and the last $n-k$ coordinates, respectively. Define $\mu_1 := (\pi_k)_{\#}(e^{-V}dx), \mu_2 := (\bar{\pi}_{n-k})_{\#}(e^{-V}dx)$, and note that these are 1-log-concave measures in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively (see [281, Theorem 4.3] or [289, Theorem 3.8]). In particular $\mu_2 = e^{-W}$ with $D^2W \geq Id_{n-k}$. Moreover, since W_1 decreases under orthogonal projection,

$$W_1(\mu_2, \nu) = W_1((\bar{\pi}_{n-k})_{\#}\mu, (\bar{\pi}_{n-k})_{\#}(\gamma_{p,k} \otimes \nu) \leq$$
$$W_1(\mu, \gamma_{p,k} \otimes \nu) \lesssim \frac{1}{R^{1/2_-}},$$

thus

$$W_1(\mu, \gamma_{p,k} \otimes \mu_2) \leq W_1(\mu, \gamma_{p,k} \otimes \nu) + W_1(\mu, \gamma_{p,k} \otimes \nu, \gamma_{p,k} \otimes \mu_2)$$
$$\leq W_1(\mu, \gamma_{p,k} \otimes \nu) + W_1(\nu, \mu_2) \lesssim \frac{1}{R^{1/2_-}}$$

where we used the elementary fact that $W_1(\mu, \gamma_{p,k} \otimes \nu, \gamma_{p,k} \otimes \mu_2) \leq W_1(\nu, \mu_2)$. Recalling (12), this proves that

$$W_1(\mu, \gamma_{p,k} \otimes \mu_2) \lesssim \frac{1}{|\log \varepsilon|^{1/4_-}},$$

concluding the proof.

As in the proof of Theorem (5.2.3), it is enough to prove the result when $\varepsilon \leq \varepsilon_0 \ll 1$.

Let $\{u_i\}_{1 \leq i \leq k}$ be as in the statement, and set $u_i := u_i \circ T$, where $T = \nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is the Brenier map from γ_n to μ. Note that since $T_{\#}\gamma_n = \mu$,

$$\int u_i d\gamma_n = \int u_i \circ T d\gamma_n = \int u_i d\mu = 0.$$

Also, since $|\nabla T| \leq 1$ and by our assumption on u_i,

$$\int |\nabla u_i|^2 d\gamma_n \leq \int |\nabla u_i|^2 \circ T d\gamma_n = \int |\nabla u_i|^2 d\mu$$

$\leq (1 + \varepsilon) \int u_i^2 \, d\mu = (1 + \varepsilon) \int u_i^2 \, d\gamma_n \leq (1 + \varepsilon) \int |\nabla u_i|^2 d\gamma_n$,

where the last inequality follows from the Poincar´e inequality for γ_n applied to u_i. Since

$$\int |\nabla u_i|^2 d\mu \leq 1 + \varepsilon,$$

this proves that

$$0 \leq \int (|\nabla u_i|^2 \circ T - |\nabla u_i|^2) d\gamma_n \leq \varepsilon \int |\nabla u_i|^2 d\mu \leq \varepsilon(1 + \varepsilon) \qquad (14) \quad .$$

Moreover, by Theorem (5.2.1), $\nabla T = D^2\varphi$ is a symmetric matrix $0 \leq \nabla T \leq Id_n$ therefore $(Id - \nabla T)^2 \leq Id - (\nabla T)^2$. Hence, since $\nabla u_i =$ Satisfying , $\nabla T \cdot \nabla u_i \circ T$, it follows by (14) that

$$\int |\nabla u_i \circ T - \nabla u_i|^2 d\gamma_n = \int |\nabla u_i \circ T - \nabla u_i|^2 d\gamma_n =$$

$$= \int (Id_n - (\nabla T))^2 [\nabla u_i \circ T, \nabla u_i \circ T] d\gamma_n$$

$$\leq \int (Id_n - (\nabla T)^2) [\nabla u_i \circ T, \nabla u_i \circ T] d\gamma_n$$

$$= \int (|\nabla u_i|^2 \circ T - |\nabla u_i|^2) d\gamma_n \leq 2\varepsilon \qquad (15)$$

where, given a matrix A and a vector v, we have used the notation $A[v, v]$ for $Av \cdot v$. In particular, recalling the orthogonality constraint $\int \nabla u_i . \nabla u_j \, d\mu = 0$, we deduce that

$$\int \nabla u_i . \nabla u_j \, d\gamma_n = O(\sqrt{\varepsilon}) \qquad (16).$$

In addition, if we set

$$f_i(x) := \frac{\nabla u_i \circ T(x)}{|\nabla u_i \circ T(x)|}$$

then, using again that $|\nabla T| \leq 1$,

$$\int |\nabla(u_i \circ T)|^2 \left(1 - |\nabla T \cdot f_i|^2 \, d\gamma \leq \int (|\nabla u_i|^2 \circ T - (1 - |\nabla T . f_i|^2\right) d\gamma_n$$

$$\leq 2\varepsilon \cdot \qquad (17)$$

Now, for $j \in \mathbb{N}$, let $H_J : \mathbb{R} \to \mathbb{R}$ be the one dimensional Hermite polynomial of degree j:

$$H_J(t) = \frac{(-1)^j}{\sqrt{j!}} e^{t^2/2} (\frac{d}{dt})^j e^{-t^2/2}$$

see [287, Section 9.2]. It is well known (see for instance [287, Theorem 9.7]) that for $J = (j_1, \dots, j_n) \in \mathbb{N}^n$ the functions

$$H_J(x_1, \dots, x_n) := H_{j_1}(x_1) H_{j_2}(x_2) \dots . H_{j_n}(x_n)$$

form a Hilbert basis of $L^2(\mathbb{R}^n, \gamma_n)$. Hence, since $\alpha_0^i = \int u_i \, d\gamma_n = 0$, we can write

$$u_i = \sum_{J \in \mathbb{N}^n \setminus \{0\}} \alpha_j^i \, H_J.$$

By elementary computations (see for instance [287, Proposition 9.3]), we get

$$1 = \int u_i^2 \, d\gamma_n = \sum_{J \in \mathbb{N}^n \setminus \{0\}} (\alpha_j^i)^2, \int (|\nabla u_i|^2 \, d\gamma_n = \sum_{J \in \mathbb{N}^n \setminus \{0\}} |J| (\alpha_j^i)^2,$$

where $|J| = \sum_{m=1}^n j_m$. Hence, combining the above equations with the bound $\int (|\nabla u_i|^2 \, d\gamma_n \leq (1 + \varepsilon)$,

$$\varepsilon \geq \int |\nabla u_i|^2 \, d\gamma_n - \int u_i^2 \, d\gamma_n = \sum_{J \in \mathbb{N}^n, |J| \geq 2} (|J| - 1)(\alpha_j^i)^2 \geq \frac{1}{2} \sum_{J \in \mathbb{N}^n, |J| \geq 2} |J| (\alpha_j^i)^2 \, .$$

Recalling that the first Hermite polynomials are just linear functions (since $H_1(t) = t$), using the notation

$$\alpha_j^i := \alpha_J^i \qquad \text{with } J = e_j \in \mathbb{N}^n$$

we deduce that

$$u_i(x) = \sum_{j=1}^n \alpha_j^i x_j + z(x), \qquad \text{with} \|z\|^2_{W^{1,2}(\mathbb{R}^n,\gamma_n)} = O(\varepsilon).$$

In particular, if we define the vector

$$V_i := \sum_{j=1}^n \alpha_j^i e_j \in \mathbb{R}^n,$$

and we recall that $\int |\nabla u_i|^2 \, d\gamma_n = 1 + O(\varepsilon)$ and the almost orthogonality relation (16), we infer that $|V_i| = 1 + O(\varepsilon)$ and $|V_i \cdot V_l| = O(\sqrt{\varepsilon})$ for all $i \neq l \in \{1, \dots, k\}$.

Hence, up to a rotation, we can assume that $|V_i - e_i| = O(\sqrt{\varepsilon})$ for all $i = 1, \dots, k$, and (15) yields

$$\int |\nabla(u_i \circ T) - e_i|^2 d\gamma_n \leq C\,\varepsilon \qquad (18).$$

Since $0 \leq 1 - |\nabla T \cdot f_i|^2 \leq 1$, it follows by (17) and (18) that

$$\int (1 - |\nabla T \cdot f_i|^2)\, d\gamma_n \leq 2 \int (|\nabla(u_i \circ T)|^2 + |\nabla(u_i \circ T) - e_i|^2)(1 - |\nabla T \cdot f_i|^2) d\gamma_n \leq C\,\varepsilon \quad (19)$$

Set $w_i := \nabla u_i \circ T$ so that $f_i = \frac{w_i}{|w_i|}$. We note that, since all the eigenvalues of $\nabla T = D^2\varphi$ are bounded by 1 , given $\delta \ll 1$ the following holds: whenever

$$|\nabla T \cdot w_i - e_i| \leq \delta \qquad \text{and } |\nabla T \cdot f_i| \geq 1 - \delta$$

then $|w_i| = 1 + O(\delta)$. In particular,

$$|\nabla T \cdot f_i - e_i| \leq C\delta.$$

Hence, if $\delta \leq \delta_0$ where δ_0 is a small geometric constant, this implies that the vectors f_i are a basis of $\mathbb{R}^k$, and

$$\nabla T|_{span(f_1,\dots,f_k)} \geq (1 - C_0\delta)\, Id_k$$

for some dimensional constant C_0. Defining $\psi(x) := |x|^2/2 - \varphi(x)$, this proves that

$$\left\{x: \sum_{i=1}^k [|\nabla T(x). w_i(x) - e_i| + (1 - |\nabla T(x). f_i(x)|)]\right\} \leq \delta \subset \left\{x: \lambda_{n-k+1}\big(D^2\psi(x)\big) \leq C_0\delta\right\} (20)$$

for all $0 < \delta \leq \delta_0$. Hence, by the layer-cake formula, (18), and (19),

$$\int_{\{\lambda_{n-k+1(D^2\psi)\leq C_0\delta_0}\}} \lambda_{n-k+1(D^2\psi)} d\gamma_n = C_0 \int_0^{\delta_0} \gamma_n(\{\lambda_{n-k+1(D^2\psi)>C_0 s}\})\, ds$$

$$\leq C_0 \int_0^{\delta_0} \gamma_n\left(\left\{\sum_{i=1}^k [|\nabla T(x). w_i(x) - e_i| + (1 - |\nabla T(x). f_i(x)|)] > s\right\}\right) ds$$

$$\leq C_0 \sum_{i=1}^k \int \big(|\nabla T. w_i - e_i| + (1 - |\nabla T. f_i|)\big) d\gamma_n \leq C\sqrt{\varepsilon} \quad (21).$$

On the other hand, it follows by (20) that

$$\{x: \lambda_{n-k+1}(D^2\psi(x)) > C_0\delta\} \subset \bigcup_{i=1}^k \left[\left\{x: |\nabla T(x). w_i(x) - e_i| > \frac{\delta}{2k}\right\} \cup \left\{x: (1 - |\nabla T(x). f_i(x)|) > \frac{\delta}{2k}\right\}\right].$$

Thus, (18), (19), and Chebyshev's inequality yield

$$\gamma_n(\{\lambda_{n-k+1}(D^2\psi(x) > C_0\delta_0) \leq \sum_{i=1}^{k}\gamma_n\left(\left\{|\nabla T.w_i - e_i| > \frac{\delta_0}{2k}\right\}\right) + \sum_{i=1}^{k}\gamma_n\left(\left\{1 - |\nabla T.f_i| > \frac{\delta_0}{2k}\right\}\right) \leq C\frac{\varepsilon}{\delta_0^2} \quad (22) \quad .$$

Hence, since δ_0 is a small but fixed geometric constant, combining (21) and (22), and recalling that $\lambda_{n-k+1}(D^2\psi) \leq 1$, we obtain

$$\int \lambda_{n-k+1}(D^2\psi)d\gamma_n \leq C\sqrt{\varepsilon}.$$

This implies that (4) holds with $C\sqrt{\varepsilon}$ in place of ε, and the result follows by Theorem (5.2.3).

Chapter 6
Brunn-Minkowski Inequalities

Throughout the study attention is paid to precise equality conditions and conditions on the coefficients of dilatation. Interesting links are found to the S-inequality and the (B) conjecture. An example is given to show that convexity is needed in the (B) conjecture. It is shown that these two families of inequalities are "equivalent" in that once either of these inequalities is established, the other must follow as a consequence. All of the conjectured inequalities are established for plane convex bodies. We establish the stability near a Euclidean ball of two conjectured inequalities: the dimensional Brunn–Minkowski inequality for radially symmetric log-concave measures in R^n, and of the log-Brunn–Minkowski inequality.

Section (6.1): Gaussian Brunn-Minkowski Inequalities

This focuses on two fundamental ingredients of mathematics: Gauss measure, the most important probability measure in $\mathbb{R}^n$, and the Brunn-Minkowski inequality, one of the most powerful inequalities in analysis and geometry.

The Brunn-Minkowski inequality for convex bodies K and L in $\mathbb{R}^n$ states that

$$V_n(K+L)^{1/n} \geq V_n(K)^{1/n} + V_n(L)^{1/n}, \qquad (1)$$

where $K+L$ is the Minkowski or vector sum of K and L, V_n denotes n-dimensional Lebesgue measure, and equality holds if and only if K is homothetic to L. By the homogeneity of V_n, this is equivalent to

$$V_n(sK+tL)^{1/n} \geq \ sV_n(K)^{1/n} + tV_n(L)^{1/n}, \qquad (2)$$

where $s, t \geq 0$.

It is known that (1) and (2) still hold when the sets concerned are Lebesgue measurable, and indeed the Brunn-Minkowski inequality reaches far beyond geometry. No less than three recent surveys cover its extensive generalizations, variations, connections, and applications in probability and statistics, information theory, Banach space theory, algebraic geometry, geometric tomography, interacting gases, and crystallography; see [290], [294], and [298].

The Brunn-Minkowski inequality (1) is a cornerstone of the vast Brunn-Minkowski theory, expounded in [299]. This harbors the tools, such as Minkowksi sum, for metrical problems on convex bodies and their projections onto subspaces. Around 1975, Lutwak [297] observed that when the Minkowski sum of two sets is replaced by an operation he called radial sum, in which only sums of parallel vectors are taken into account, a theory arises that is ideal for treating metrical problems about sets star-shaped with respect to the origin, and their intersections with subspaces. This newer theory, now called the dual Brunn-Minkowski theory, has attracted much attention and counts among its successes the solution of the 1956 Busemann-Petty problem on volumes of central of o-symmetric convex bodies; see [295].

Corresponding in the dual theory to the Brunn-Minkowski inequality (1) is the dual Brunn-Minkowski inequality for bounded Borel star sets C and D in $\mathbb{R}^n$, which states that

$$V_n(C \mp D)^{1/n} \leq V_n(C)^{1/n} + V_n(D)^{1/n}, \qquad (3)$$

where $\tilde{+}$ denotes radial sum, with equality if and only if C is a dilatate of D. See, for example, [293, (B.30)] and [298, Section 3]. This is equivalent to

$$V_n(sC \tilde{+} tD)^{1/n} \leq sV_n(C)^{1/n} + tV_n(D)^{1/n}, \quad (4)$$

where $s, t \geq 0$. The reversal of the inequality sign in the passage from (1) to (3) is a standard, but not yet fully understood, feature of the duality at play. Here we are interested in inequalities of the Brunn-Minkowski type for Gauss measure γ_n in $\mathbb{R}^n$. Despite the fact that γ_n is not translation invariant, such inequalities have been found. The most powerful, due to Ehrhard [391], [392], states that for $0 < t < 1$ and closed convex sets K and L in $\mathbb{R}^n$, we have

$$\Phi^{-1}\left(\gamma_n\big((1-t)K + tL\big)\right) \geq (1-t)\Phi^{-1}\left(\gamma_n(K)\right) + t\Phi^{-1}\big(\gamma_n(L)\big), \quad (5)$$

where $\Phi(x) = \gamma_1((-\infty, x))$. By [292, p. 154], equality holds when $\gamma_n(K)\gamma_n(L) > 0$ if and only if $K = \mathbb{R}^n, L = \mathbb{R}^n, K = L$, or both K and L are half-spaces, one contained in the other. Since the function Φ is (strictly) log concave (i.e., $log\Phi$ is (strictly) concave), Ehrhard's inequality and its equality condition imply that for $0 < t < 1$ and closed convex sets K and L in $\mathbb{R}^n$,

$$\gamma_n\big((1-t)K + tL\big) \geq \gamma_n(K)^{1-t}\gamma_n(L)^t, \quad (6)$$

with equality when $\gamma_n(K)\gamma_n(L) > 0$ if and only if $K = L$. Inequality (6), proved independently by Borell [293], [294] and Brascamp and Lieb [299], is also an easy consequence of the Prekopa-Leindler inequality and the fact that the density function of γ_n is log concave, and moreover (6) holds when the sets concerned are Borel sets (see, for example, [294, p. 378]). On the other hand it was only recently that Borell [296] proved that (5) also holds for Borel sets. (Note that what Borell in [295] calls the Brunn-Minkowski inequality for Gauss measure is none of the above inequalities but is rather an isoperimetric inequality that follows from (5); see [292].)

One of the main results, is the following new inequality for Borel star sets C and D in $\mathbb{R}^n$ and $s, t \geq 1$:

$$\gamma_n(sC \tilde{+} tD)^{1/n} \leq s\gamma_n(C)^{1/n} + t\gamma_n(D)^{1/n}.$$

See Theorem (6.1.2), which also gives precise equality conditions. What is remarkable about this Gaussian dual Brunn-Minkowski inequality (compare (4)) is not its proof, which does not require innovative techniques, but that it exists. The discussion after Theorem (6.1.2) shows that the inequality is the best possible from several points of view. In particular, the restriction $s, t \geq 1$ on the coefficients of dilatation is necessary. This may seem strange at first, since (4) has no such restriction. However, γ_n is not homogeneous, and the restriction $s, t \geq 1$ becomes natural when we see that it also applies to (4) when the exponent $1/n$ is replaced by $0 < p < 1/n$.

where we examine the role of the coefficients of dilatation in several inequalities, including, for the first time as far as we know, those for (6).

Also we find that when the exponent $1/n$ in (4) is replaced by $p > 1/n$, the appropriate condition on the coefficients is $s + t \leq 1$, which includes the important special case of the convex combination where $s = 1 - t$. This raises the question (see Question (6.1.6)) as to whether there is a Gaussian dual Brunn -Minkowski inequality that holds when $s + t \leq 1$.

Our investigation turns up an interesting connection with the so-called S-inequality of Latala and Oleszkiewicz [294], but our results suggest that there may be no satisfactory answer to this question.
In the course of our detailed investigation into Gaussian dual Brunn-Minkowski inequalities, we were led to the following intriguing question (see Question (6.1.7)): If $0 < t < 1$ and K and L are closed convex sets containing the origin in $\mathbb{R}^n$, is it true that

$$\gamma_n((1-t)K + tL)^{1/n} \geq (1-t)\gamma_n(K)^{1/n} + t\gamma_n(L)^{1/n}?$$

we note that the restriction on the position of K and L is necessary, but in view of the direct analogy with (2), it is amazing that the inequality seems to have been overlooked. It does not follow from Ehrhard's inequality (5), and if true it would be stronger than (6) when K and L contain the origin. We provide evidence in its favor by showing that it is true when K and L are coordinate boxes, when either K or L is a slab, and when K and L are both dilatates of the same o-symmetric closed convex set. Even the latter special case is not at all easy. We establish it by means of a fascinating link (see Theorem (6.1.12)) with Banaszczyk's conjecture—the (B) conjecture—that $\gamma_n(e^t K_0)$ is log concave in t when K_0 is an o-symmetric convex body, recently proved by Cordero-Erasquin, Fradelizi, and Maurey [290]. It is not known if the symmetry is necessary for the truth of the (B) conjecture, but we give an example to show that the convexity is necessary. In Theorem (6.1.14) we prove a Gaussian Prekopa-Leindler inequality that follows from earlier results.
We are very grateful to Franck Barthe for his helpful suggestions and comments, in particular the contribution given in detail at the end.
As usual, S^{n-1} denotes the unit sphere, B the unit ball, o the origin, and $\|\cdot\|$ the norm in Euclidean n-space $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, then $x \cdot y$ is the inner product of x and y and $[x, y]$ denotes the line segment with endpoints x and y.
If X is a set, $dimX$ is its dimension, that is, the dimension of its affine hull, and ∂X is its boundary. A set is o-symmetric if it is centrally symmetric, with center at the origin. If $r \in \mathbb{R}$, the set $rX = \{rx : x \in X\}$ is called a dilatate of X. If X and Y are sets in $\mathbb{R}^n$, then

$$X + Y = \{x + y : x \in X, y \in Y\}$$

is the Minkowski or vector sum of X and Y.
A body is a compact set equal to the closure of its interior.
We write V_k for k-dimensional Lebesgue measure in $\mathbb{R}^n$, where $k = 1, \dots, n$ and where we identify V_k with k-dimensional Hausdorff measure. If K is a k–dimensional body in $\mathbb{R}^n$, then we refer to $V_k(K)$ as its volume. Define $\kappa_n = V_n(B)$. The notation dz will always mean $dV_k(z)$ for the appropriate $k = 1, \dots, n$.
A set in $\mathbb{R}^n$is called a convex body if it is convex and compact with nonempty interior. The treatise of Schneider [299] is an excellent general reference for convex sets.
A (possibly unbounded) set C is star shaped at the origin if every line through the origin that meets C does so in a (possibly degenerate) closed line segment, a closed half-infinite ray, or in the line itself. If C is a set that is star shaped at the origin, its radial function ρ_C is defined, for all $u \in S^{n-1}$ such that the line through the origin parallel to u intersects C, by

$$\rho_C(u) = sup\{c \in \mathbb{R} : cu \in C\}.$$

Note that C may not contain the origin and that ρ_C may take negative or infinite values. A Borel star set is a Borel set that contains the origin and is star shaped at the origin.
By a star body in $\mathbb{R}^n$ we mean a body L star shaped at the origin such that ρ_L, restricted to its support, is continuous. This definition, introduced in [299] (see also [293, Section 0.7]), allows bodies not containing the origin, unlike previous definitions; in particular, every convex body is a star body in this sense.
If $x, y \in \mathbb{R}^n$, then the radial sum $x \mp y$ of x and y is defined to be the usual vector $sumx + y$ if x and y are contained in a line through o, and o otherwise. If C and D are Borel star sets in $\mathbb{R}^n$ and $s, t \in \mathbb{R}$, then

$$sC \mp tD = \{sx \mp ty: x \in C, y \in D\}$$

and

$$\rho_{sC \mp tD} = s\rho_C + t\rho_D. \tag{7}$$

The standard Gauss measure γ_n is defined for measurable subsets E of $\mathbb{R}^n$ by

$$\gamma_n(E) = \ c_n e^{-\|x\|^2/2} dx, \tag{8}$$

where dx denotes integration with respect to V_n and

$$c_n = (2\pi)^{-n/2}. \tag{9}$$

For $n \in \mathbb{N}$ and $r \in \mathbb{R}$, define

$$\Psi_n(r) = \gamma_n(rB). \tag{10}$$

From (8) it follows by substitution that if E is a measurable subset of $\mathbb{R}^n$, then

$$\gamma_n(sE)^{1/n} \geq s_{\gamma_n}(E)^{1/n} if\ 0 \leq s \leq 1\ and\ \gamma_n(sE)^{1/n} \leq s_{\gamma_n}(E)^{1/n}\ if\ s \geq 1. \tag{11}$$

Equality holds in each inequality if and only if $s = 1$ or $\gamma_n(E) = 0$.
Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\ dt, \tag{12}$$

and note that $\Phi(x) = \gamma_1((-\infty, x))$.
It will be convenient to define, for $a \geq 0$,

$$\phi_n(a) = \left(\int_0^a e^{-\frac{t^2}{2}} t^{n-1}\ dt \right)^{1/n}. \tag{13}$$

Then if C is a Borel star set in $\mathbb{R}^n, n \geq 2$, a change to polar coordinates yields

$$\gamma_n(C) = c_n \int_{S^{n-1}} \int_0^{\rho_C(u)} e^{-\frac{r^2}{2}} r^{n-1} dr\, du = c_n \int_{S^{n-1}} \phi_n\big(\rho_C(u)\big)^n du, \tag{14}$$

where c_n is given by (9), an analog of the familiar polar coordinate expression for the V_n-measure of a Borel star set.
If C is a Borel set contained in the ball εB for $\varepsilon > 0$, it follows from (8) that

$$cne^{-\frac{\varepsilon^2}{2}} V_n(C) \leq \gamma_n(C) \leq c_n V_n(C). \tag{15}$$

Since γ_n is not homogeneous, it makes sense to carefully examine the precise conditions on the coefficients of dilatation in inequalities involving Gauss measure.
In [297] (see also [292]), Borell resolved this issue for Ehrhard's inequality (5) by showing that

$$\Phi^{-1}(\gamma_n(sK + tL)) \geq s\Phi^{-1}(\gamma_n(K))\ t\Phi^{-1}(\gamma_n(L))\ ,$$

where Φ is defined by (12), holds for $s, t \geq 0$, even for Borel sets, when $s + t \geq 1$ and $|s - t| \leq 1$, and not generally unless these conditions are satisfied. In [298], Borell shows that, remarkably, the corresponding condition for convex K and L is different; here only $s + t \geq 1$ is required.

The corresponding analysis for the weaker inequality (6) does not appear. We claim that the inequality

$$\gamma_n(sK + tL) \geq \gamma_n(K)^s \gamma_n(L)^t \qquad (16)$$

holds generally for Borel star sets K and L and $s, t \geq 0$ if and only if $s + t \geq 1$. To see this, note first that if $s + t < 1$ and $K = L$, (16) implies that

$$\gamma_n(K) > \gamma_n((s + t)K) \geq \gamma_n(K)^{s+t},$$

a contradiction since $\gamma_n(K) \leq 1$. Suppose, then, that $s + t \geq 1$. Let

$$f(s, t) = log(\gamma_n(sK + tL)) - s\, log(\gamma_n(K)) - t\, log(\gamma_n(L)) .$$

Clearly $\gamma_n(sK + tL)$ increases with s and $t, log(\gamma_n(K)) \leq 0$, and $log(\gamma_n(L)) \leq 0$, so $\partial f/\partial s \geq 0$ and $\partial f/\partial t \geq 0$. If $s, t \geq 1$, this yields $f(s, t) \geq f(0, 1) = 0$, as required. On the other hand if $t < 1$, say, then $f(s, t) \geq f(1 - t, t) \geq 0$ by (6), completing the proof of the claim.

Note, however, that for convex K and L, (16) holds generally for $s, t \geq 0$ if and only if $s = 1 - t$. In view of the previous paragraph, we need only consider the case when $s + t > 1$. Let $n = 1$, let $K = L = [x, x + 1], x > 0$, and let $s + t = a > 1$.

Then (16) and crude estimates give

$$\frac{a}{\sqrt{2\pi}} e^{-(ax)^2/2} > \gamma_1([ax, ax + a]) \geq \gamma_1([x, x + 1])^a > \left(\frac{1}{\sqrt{2\pi}} e^{-(x+1)^2/2}\right)^a$$

or

$$\frac{a}{\sqrt{2\pi}} e^{-a^2x^2/2} > \frac{1}{(2\pi)^{a/2}}\ e^{-a(x+1)^2/2}.$$

Since $a^2 > a$, this is clearly false for sufficiently large x.

In view of the connection (15) between Gauss and Lebesgue measure, we revisit the classical and dual Brunn-Minkowski inequality for exponents $p > 0$.

To deal with this, first note that if $p > 0$ and $a, b, s, t \geq 0$, the weighted pth means $(sa^p + tb^p)^{1/p}$ increase with p for all $a, b \geq 0$ if and only if $s + t \leq 1$ and decrease with p for all $a, b \geq 0$ if and only if $s, t \geq 1$. (The cases $s = 1 - t$ and $s = t = 1$ are usually called the pth mean and pth sum of a and b, respectively.) See [290, (2.10.4) and (2.10.5), p. 29]. In particular, the inequality

$$(sa + tb)^p \geq sa^p + tb^p \qquad (17)$$

is true for all $a, b \geq 0$ when $p = 1$, when $p < 1$ and $s + t \leq 1$, and when $p > 1$ and $s, t \geq 1$, and it is false for all $a, b > 0$ when $p > 1$ and $s + t \leq 1$, and when $p < 1$ and $s, t \geq 1$. Moreover, it does not generally hold otherwise. To see this, it suffices to check that when $s < 1$ and $t > 1$, (17) is false for $p < 1$ and sufficiently small a and for $p > 1$ and sufficiently small b.

The above monotonicity properties of the weighted means and (2) imply that

$$V_n(sK + tL)^p \geq sV_n(K)^p + tV_n(L)^p \qquad (18)$$

holds for $s, t \geq 0$ and all convex bodies K and L in $\mathbb{R}^n$ when $p = 1/n$, when $0 < p < 1/n$ and $s + t \leq 1$, and when $p > 1/n$ and $s, t \geq 1$. By using the homogeneity of volume and the remarks above concerning the inequality (17), we see that (18) is otherwise generally false for $K = aB, L = bB$ and small $a, b \geq 0$, and it is always false for $K = aB, L = bB, a, b > 0$, when $p > 1/n$ and $s + t \leq 1$, and when $0 < p < 1/n$ and $s, t \geq 1$.
In a similar fashion, it can be seen that

$$V_n(sC \mp tD)^p \leq sV_n(C)^p + tV_n(D)^p \quad (19)$$

holds for $s, t \geq 0$ and all bounded Borel star sets C and D in $\mathbb{R}^n$ when $p = 1/n$, when $p > 1/n$ and $s + t \leq 1$, and when $0 < p < 1/n$ and $s, t \geq 1$. It is otherwise generally false for $C = aB, D = bB$ and small $a, b \geq 0$, and it is always false for $C = aB, D = bB, a, b > 0$, when $0 < p < 1/n$ and $s + t \leq 1$, and when $p > 1/n$ and $s, t \geq 1$.

Lemma (6.1.1)[289]: The function ϕ_n defined by (13) is sublinear, i.e.,

$$\phi_n(a + b) \leq \phi_n(a) + \phi_n(b),$$

for $a, b \geq 0$, with equality if and only if $a = 0$ or $b = 0$.

Proof: For fixed $b > 0$ and all $a \geq 0$, define

$$f(a) = \phi_n(a + b) - \phi_n(a) - \phi_n(b).$$

Then $f(0) = 0$, and it suffices to show that $f'(a) < 0$ for all $a \geq 0$. In view of (13), we have

$$nf'(a) = (a + b)^{n-1} e^{-(a+b)^2/2} \phi_n(a + b)^{1-n} - a^{n-1} e^{-a^2/2} \phi_n(a)^{1-n}.$$

If $n = 1$, it is clear from this that $f'(a) < 0$ for $a \geq 0$. Suppose that $n \geq 2$. Using (13) again, we see that $f'(a) < 0$ is equivalent to

$$(a + b)^{-n} e^{n(a+b)^2/(2(n-1))} \int_0^{a+b} e^{-t^2/2} t^{n-1} dt > a^{-n} e^{na^2/(2(n-1))} \int_0^a e^{-t^2/2} t^{n-1} dt$$

or

$$e^{n(a+b)^2/(2(n-1))} \int_0^1 e^{-(s(a+b))^2/2} s^{n-1} ds > e^{na^2/(2(n-1))} \int_0^1 e^{-(sa)^2/2} s^{n-1} ds.$$

Rearranging, we obtain

$$\int_0^1 e^{(n/(n-1)-s^2)(a+b)^2/2} s^{n-1} ds > \int_0^1 e^{(n/(n-1)-s^2)a^2/2} s^{n-1} ds\,.$$

The previous inequality holds since $s^2 \leq 1 < n/(n-1)$, and this proves the lemma.

Theorem (6.1.2) [289]: Let C and D be Borel star sets in $\mathbb{R}^n$, and let $s, t \geq 1$. Then

$$\gamma_n(sC \mp tD)^{1/n} \leq s\gamma_n(C)^{1/n} + t\gamma_n(D)^{1/n}. \quad (20)$$

Suppose that C and D are properly contained in $\mathbb{R}^n$. Equality holds when $s = t = 1$ if and only if $\gamma_n(C) = 0, \gamma_n(D) = 0$, or $n = 1$ and both C and D are (possibly degenerate or infinite) intervals with one endpoint at the origin, each on opposite sides of the origin. Equality holds when $s > 1$ and $t = 1$ (or $s = 1$ and $t > 1$, or $s > 1$ and $t > 1$) if and only if $\gamma_n(C) = 0$ (or if and only if $\gamma_n(D) = 0$, or if and only if $\gamma_n(C) = 0$ and $\gamma_n(D) = 0$, respectively).

Proof: Suppose first that $s = t = 1$.

If $n = 1$ and C and D are bounded, then $C = [-a_1, b_1]$ and $D = [-a_2, b_2]$ for nonnegative a_1, a_2, b_1, and b_2, and (20) is equivalent to

$$\phi_1(a_1 + a_2) + \phi_1(b_1 + b_2) \leq (\phi_1(a_1) + \phi_1(b_1)) + (\phi_1(a_2) + \phi_1(b_2)) .$$

This follows immediately from Lemma (6.1.1), and its equality condition shows that either $a_1 = 0$ *or* $a_2 = 0$ and either $b_1 = 0$ or $b_2 = 0$. The same conclusion is reached if C or D is unbounded. This yields the required equality condition when $n = 1$.

Suppose that $n \geq 2$. By (14), (7), Lemma (6.1.1), and Minkowski's inequality for integrals, we have

$$\gamma_n(C \mp D)^{1/n} = \left(c_n \int_{S^{n-1}} \phi_n(\rho_C \mp D(u))^n du\right)^{1/n}$$

$$= \left(c_n \int_{S^{n-1}} \phi_n\left(\rho_C(u) + \rho_D(u)\right)^n du\right)^{1/n}$$

$$\leq \left(c_n \int_{S^{n-1}} (\phi_n(\rho_C(u)) + \phi_n(\rho_D(u)))^n du\right)^{1/n}$$

$$\leq \left(c_n \int_{S^{n-1}} \phi_n\left(\rho_C(u)\right)^n du\right)^{1/n} + \left(c_n \int_{S^{n-1}} \phi_n\left(\rho_D(u)\right)^n du\right)^{1/n}$$

$$= \gamma_n(C)^{1/n} + \gamma_n(D)^{1/n}.$$

Suppose, in addition to our assumption that $s = t = 1$, that equality holds in (20). Then for almost all $u \in S^{n-1}$, equality holds in Lemma (6.1.1) when $a = \rho_C(u)$ and $b = \rho_D(u)$, and hence for almost all $u \in S^{n-1}$ we have either $\rho_C(u) = 0$ or $\rho_D(u) = 0$. But equality also holds in Minkowski's inequality for integrals, so there is a constant c such that $\phi_n(\rho_C(u)) = c\phi_n(\rho_D(u))$ for almost all $u \in S^{n-1}$. It follows that either $\rho_C(u) = 0$ for almost all $u \in S^{n-1}$ or $\rho_D(u) = 0$ for almost all $u \in S^{n-1}$, and therefore either $\gamma_n(C) = 0$ or $\gamma_n(D) = 0$.

We have proved (20) and its equality conditions when $s = t = 1$. Using this and (11), for general $s, t \geq 1$ we obtain

$$\gamma_n(sC \mp tD)^{1/n} \leq \gamma_n(sC)^{1/n} + \gamma_n(tD)^{1/n} \leq s\gamma_n(C)^{1/n} + t\gamma_n(D)^{1/n},$$

as required. The equality conditions for $s > 1$ or $t > 1$ follow from those of (11).

Inequality (20) does not hold generally when either $s < 1$ or $t < 1$. Indeed, if $s < 1$, (20) is false when $D = \varepsilon B$ and $\varepsilon > 0$ is sufficiently small, in view of (11). Inequality (20) is false for arbitrary Borel sets star shaped at the origin. To see this, let $s = t = 1$, and for each $m \in \mathbb{N}$, let $C_m = \{(r, \theta) \in \mathbb{R}^n : m \leq r \leq m + 1, 0 \leq \theta \leq \pi/2\}$ and $D_m = -C_m$. Then $C_m \mp D_m = C_0 \cup (-C_0)$, so $\gamma_2(C_m \mp D_m)$ is positive and independent of m while $\gamma_2(C_m) = \gamma_2(D_m) \to 0$ as $m \to \infty$. Note that C_m and D_m are actually star bodies.

The monotonicity properties of the weighted pth means $(sa^p + tb^p)^{1/p}$ summarized at the end imply that Theorem (6.1.2) holds for $s, t \geq 1$ and $0 < p \leq 1/n$.

However, the exponent $1/n$ in (20) is the best possible; it does not hold when $1/n$ is replaced by $p > 1/n$, as can be seen by taking $C = aB$ and $D = bB$ for sufficiently small positive a and b, and using (15) and the remarks concerning (19). Similarly, using the remarks concerning (18) instead, we see that it is also not true that (20) holds when $1/n$ is replaced by $p > 1/n$ and the inequality is reversed.

When C and D are convex bodies containing the origin, we have $sC \mp tD \subset sC + tD,$ so in this case the inequality $\gamma_n(sC + tD)^{1/n} \leq s\gamma_n(C)^{1/n} + t\gamma_n(D)^{1/n}$ would be stronger than (20). However, by (2), its equality condition, and (15), this is false in general when C and D are sufficiently small nonhomothetic convex bodies containing the origin.

We consider the possibility that

$$\ominus_n^{-1}\,\gamma_n sC\widetilde{+}tD \leq\; s\,\ominus_n^{-1}\left(\gamma_n(C)\right) + t\,\ominus_n^{-1}\left(\gamma_n(D)\right) \tag{21}$$

holds for Borel star sets C and D in $\mathbb{R}^n$ and $s,t \geq 1$, where $\ominus_n$ is some standard function related to Gauss measure. Certainly (21) is not generally true when $s = t = 1$ and $\ominus_n = \Psi_n$, the function defined by (10). To see this, let C and D be half-spaces in $\mathbb{R}^n$ bounded by a common hyperplane through the origin, so that $C\widetilde{+}D = \mathbb{R}^n$ and $\gamma_n(C) = \gamma_n(D) = 1/2$. Then the left-hand side of (21) with $s = t = 1$ and $\ominus_n = \Psi_n$ is infinite, while the right-hand side is bounded. Of course the same argument shows that (21) is not generally true when $\ominus_n = \Psi_1\ or\ \ \ominus_n = \Phi$ (defined by (12)).

In view of Theorem (6.1.2) and the dual Brunn-Minkowski inequality in the form (19), it is natural to ask whether there is a $p > 0$ such that

$$\gamma_n(sC\widetilde{+}tD)^p \leq s\gamma_n(C)^p + t\gamma_n(D)^p \tag{22}$$

holds for $s,t \geq 0, s + t \leq 1$, and Borel star sets C and D in $\mathbb{R}^n$. We shall see that the answer is negative for $s,t > 0$, even for o-symmetric balls. To this end, the following lemma will be useful.

Lemma (6.1.3) [289]: The function

$$F_n(r) = \left(\int_0^r e^{-\frac{t^2}{2}} t^{n-1}\,dt\right)^p \tag{23}$$

is strictly concave when (i) $0 < p < 1$ and $r \geq \sqrt{n-1}$, (ii) $p \geq 1$ and $r > \sqrt{np-\ 1}$, and (iii) $0 < p \leq 1/n$ and $r > 0$.

Proof: Let

$$I_n(r) = e^{-\frac{t^2}{2}} t^{n-1}\,dt, \tag{24}$$

so that $F_n(r) = I_n(r)^p$. A straightforward calculation yields

$$F_n''(r) = pI_n(r)^{p-2} e^{-\frac{r^2}{2}} r^{n-2}\left((p-1)e^{-\frac{r^2}{2}} r^n + I_n(r)(n-1-r^2)\right). \tag{25}$$

Note that a trivial estimate gives $I_n(r) > e^{-r^2/2} r^n/n$ for $r > 0$, so if $r \geq \sqrt{n-1}$, we obtain $F_n''(r) = pI_n(r)^{p-2} e^{-r^2/2} r^{2n-2}(np-1-r^2)/n.$ Fromthiswe see that $F_n''(r) < 0$ when, in addition, $p < 1$, establishing (i), and (ii) also follows immediately.

In proving (iii) we may suppose that $p = 1/n$, since pth means increase with p. Substituting $p = 1/n$ into (25), we see that it suffices to show that

$$G_n(r) = -(n-1)e^{-r^2/2} r^n + nI_n(r)(n-1-r^2) < 0$$

for $r > 0$. Now $G_n(0) = 0$, and

$$G_n'(r) = e^{-r^2/2} r^{n+1} - 2nrI(r) < 0$$

for $r > 0$. It follows that $G_n(r) < 0$ for $r > 0$, as required.
No attempt was made to obtain best possible estimates in cases (i) and (ii) of the previous lemma, since those found are sufficient for our purposes. Case (iii) of the previous lemma is equivalent to the concavity of $\phi_n(r)$ for $r > 0$, and this is also implied by a result of Koenig and Tomczak-Jaegermann [291, p. 1218].

Corollary (6.1.4) [289]: Let $s, t \geq 0, s + t \leq 1$, and let C and D be o-symmetric balls in $\mathbb{R}^n$. Then
$$\gamma_n(sC \tilde{+} tD)^p \geq s\gamma_n(C)^p + t\gamma_n(D)^p \qquad (26)$$
holds for $0 < p \leq 1/n$. Equality holds for $s, t > 0$ if and only if $C = D$.

Proof: Note that when $n = 1, \gamma_1(rB) = \gamma_1([-r, r]) = 2c_1 I_1(r)$, where $I_n(r)$ is given by (24). If $n \geq 2$, by (14), we have
$$\gamma_n(rB) = c_n \int_{S^{n-1}} \phi_n(r)^n du \ = n\kappa_n c_n I_n(r)$$
for $r > 0$. Thus if the function $F_n(r)$ given by (23) is concave for $0 < a < r < b$, then
$$\gamma_n((1-t)C \tilde{+} tD)^p \geq (1-t)\gamma_n(C)^p + t\gamma_n(D)^p \qquad (27)$$
holds when $C = r_0 B, D = r_1 B$, and $0 < a < r_0, r_1 < b$. By Lemma (6.1.3)(iii), $F_n(r)$ is actually strictly concave for $0 < p \leq 1/n$, and this yields the corollary together with the equality condition when $s = 1 - t$.
For general $s, t \geq 0$ with $s + t \leq 1$, let $\alpha = s/(1-t) \leq 1$ and note that by (27) and (11), for $0 < p \leq 1/n$, we have
$$\begin{aligned}\gamma_n(sC \tilde{+} tD)^p = \gamma_n((1-t)(\alpha C) \tilde{+} tD)^p &\geq (1-t)\gamma_n(\alpha C)^p + t\gamma_n(D)^p \\ &\geq (1-t)\alpha^{pn}\gamma_n(C)^p + t\gamma_n(D)^p \\ &\geq (1-t)\alpha\gamma_n(C)^p + t\gamma_n(D)^p \\ &= s\gamma_n(C)^p + t\gamma_n(D)^p,\end{aligned}$$
as required. If equality holds, then equality holds in (11), implying that $\alpha = 1$, and then $C = D$ from the equality condition for (27).

Corollary (6.1.5) [289]: For given $s, t > 0, s + t \leq 1$, and $p > 0$, inequality (22) is false in general, even for o-symmetric balls.

Proof. Corollary (6.1.4) and its equality condition yield the result for $0 < p \leq 1/n$.
Suppose that $p > 1/n$. By Lemma (6.1.3)(i) and (ii) we can choose the radii of o-symmetric balls C and D in $\mathbb{R}^n$ so that with $s = 1 - t$,
$$\gamma_n(sC \tilde{+} tD)^p > s\gamma_n(C)^p + t\gamma_n(D)^p, \qquad (28)$$
and therefore so that (22) is false. It remains to consider the case when $s + t < 1$.
Let $C = aB$ and $D = aB$ for $a > 0$. Then (28) is equivalent to
$$\gamma_n((s+t)aB)^p > (s+t)\gamma_n(aB)^p.$$
As $a \to \infty$, the left-hand side approaches 1, while the right-hand side approaches $s + t < 1$. It follows that (28) holds for sufficiently large a.
Note that Corollary (6.1.4) holds even for $p < 0$, at least when $s = 1 - t$. This is because pth means increase with real p; see [290, Section 2.9]. Consequently Corollary (6.1.5) also holds when $s = 1 - t$ and $p < 0$.

Corollary (6.1.4) does not hold in general, even when both C and D are dilatates of a fixed o-symmetric Borel star set E. To see this, let $E_1 = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$, $E_2(a) = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq r \leq a, \pi/2 \leq \theta \leq \pi\}$, and let $E(a) = E_1 \cup (-E_1) \cup E_2(a) \cup (-E_2(a))$. Letting

$$f(t) = \gamma_2\big(tE(a)\big)^{1/2} = \left(\frac{1}{2} + \frac{1}{2}\big(1 - e^{-t^2a^2/2}\big)\right)^{1/2} = I(t)^{1/2},$$

say, we obtain

$$f''(t) = \frac{a^2 e^{-t^2a^2/2}}{16 I(t)^{3/2}}\big(-t^2a^2e^{-t^2a^2/2} + 4I(t)(1 - t^2a^2)\big).$$

Using the inequalities $1 - x \leq e^{-x} \leq 1 - x + x^2/2$ for $x \geq 0$, we have

$$-t^2a^2e^{-t^2a^2/2} + 4I(t)(1 - t^2a^2) = 4 - 4t^2a^2 - 2e^{-t^2a^2/2} + t^2a^2e^{-t^2a^2/2}$$
$$\geq \frac{1}{4}(8 - 8t^2a^2 - 3t^4a^4).$$

The latter quantity is positive for $0 \leq t \leq 1$, and hence $f(t)$ is convex there, when $a \leq a_0 = ((2\sqrt{10} - 4)/3)^{1/2} = 0.8802\ldots$. It follows that if $0 < a_1 < a_3 < a_0$, $C = E(a_1)$, and $D = E(a_2)$, then (26) is false for $0 < s = 1 - t < 1$ when $n = 2$ and $p = 1/2$. By replacing E_1 with $E_1' = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq r \leq b, 0 \leq \theta \leq \pi/2\}$ for sufficiently large b and then approximating, we can clearly also find sets C and D in $\mathbb{R}^n$, each dilatates of a fixed o-symmetric star body, such that (26) is false for $0 < s = 1 - t < 1$ when $n = 2$ and $p = 1/2$.
The results of the previous and the existence of Ehrhard's inequality (5) raise the following question.

Question (6.1.6) [289]: Let $n \in \mathbb{N}$. Is there a natural nonconstant function Θ_n such that for $0 < t < 1$ and Borel star sets C and D in $\mathbb{R}^n$,

$$\Theta_n^{-1}\big(\gamma_n((1-t)C \widetilde{+} tD)\big) \leq (1-t)\Theta_n^{-1}\big(\gamma_n(C)\big) + t\Theta_n^{-1}\big(\gamma_n(D)\big)? \qquad (29)$$

For $n = 1$, we can take $\Theta_1 = 1 - \Phi$, for then, noting that $1 - \Phi(x) = \Phi(-x)$, we have $\Theta^{-1} = -\Phi^{-1}$, and since the radial sum equals the Minkowski sum when $n = 1$, (29) becomes Ehrhard's inequality (5)! However, we cannot take $\Theta_n = 1 - \Phi$ when $n \geq 2$. To see this, note that this would imply that Ehrhard's inequality (5) is true when $n \geq 2$, K and L are Borel star sets, and the Minkowski sum is replaced by the radial sum. But this is false. Indeed, recall that since Φ is log concave, this would imply that (6) also holds when $n \geq 2$, K and L are Borel star sets, and the Minkowski sum is replaced by the radial sum. Moreover, from the equality conditions for (5) we can conclude that the radial sum version of (6) would hold with strict inequality when K and L are dilatates with $K \neq L$. By (15), we would then have

$$V_n((1-t)K \widetilde{+} tL) > V_n(K)^{1-t}V_n(L)^t$$

for sufficiently small nonequal dilatates K and L. By a standard argument (see, for example, [294, p. 362]), this would contradict (4).
Any Θ_n for which (29) holds for o-symmetric Borel star sets must be decreasing. To see this, let C and D be o-symmetric infinite double cones such that $C \cap D = \{o\}$.

Then $(1-t)C = C, tD = D,$ and $(1-t)C\tilde{+}tD = C \cup D$. If $\gamma_n(C) = a$ and $\gamma_n(D) = b,$ then (29) yields

$$\Theta_n^{-1}\,(a+b) \le (1-t)\Theta_n^{-1}(a) + t\Theta_n^{-1}\,(b).$$

As $t \to 0$, we obtain $\Theta_n^{-1}(a+b) \le \Theta_n^{-1}(a)$. Therefore Θ_n^{-1} is decreasing on $[0,1]$ and hence Θ_n is also decreasing. In particular, we cannot take $\Theta_n = \Phi, \Psi_1$, or Ψ_n (see (10)).

Despite all this, we claim that for all $n \in \mathbb{N}$, (29) is true when $\Theta_n = \Psi_1, C = \{o\}$, and D is o-symmetric and convex. To see this, let $0 < t < 1$ and consider an o-symmetric slab (the closed region between two parallel hyperplanes) P of half-width a, and note that $\gamma_n(P) = \gamma_1([-a,a]) = \Psi_1(a), or\ a = \Psi_1^{-1}\,(\gamma_n(P))$. Suppose that P is chosen so that $\gamma_n(P) = \gamma_n(D)$. Then P has half-width $\Psi_1^{-1}(\gamma_n(D))$, so tp has half-width $t\Psi_1^{-1}(\gamma_n(D))\ and\ \gamma_n(tP) = \Psi_1(t\Psi_1^{-1}(\gamma_n(D)))$. By the so-called S-inequality (see [292] and [294]), we have

$$\gamma_n(tD) \le \gamma_n(tP) = \Psi_1(t\Psi_1^{-1}(\gamma_n(D))),$$

which is (29) for the special case under consideration.

The previous observation suggests that Question (6.1.6) should be revisited under the restriction that the sets C and D are o-symmetric closed convex sets. In fact, it turns out that we still cannot take $\Theta_n = \Phi, \Psi_1, or\ \Psi_2$, but different arguments are required.

To see that it is still not possible to take $\Theta_n = \Phi$, let C and D be different parallel o-symmetric slabs. Then $(1-t)C\tilde{+}tD = (1-t)C + tD$, so (29) with $\Theta_n = \Phi$ would contradict (5) and its equality conditions.

Next, note that if Question (6.1.6) has a positive answer for o-symmetric closed convex sets, then $\Theta_n^{-1}(\Psi_1(x))$ must be convex. Indeed, let C and D be parallel o-symmetric slabs of half-widths x and y, respectively, so that $(1-t)C\tilde{+}tD$ is an o-symmetric slab of half-width $(1-t)x + ty$. Then $\gamma_n(C) = \Psi_1(x), \gamma_n(D) = \Psi_1(y)$, and $\gamma_n((1-t)C\tilde{+}tD) = \Psi_1((1-t)x + ty)$, so (29) becomes

$$\Theta_n^{-1}(\Psi_1((1-t)x+ty)) \le (1-t)\Theta_n^{-1}(\Psi_1(x)) + t\Theta_n^{-1}(\Psi_1(y)),$$

which holds for all $x, y \ge 0$ if and only if $\Theta_n^{-1}(\Psi_1(x))$ is convex.

Let $f(x) = \Psi_n^{-1}(\Psi_1(x)), n \ge 2$. Using (14) and differentiating $\Psi_n(f(x)) = \Psi_1(x)$ with respect to x, we obtain

$$c_n n \kappa_n e^{-f(x)^2/2} f(x)^{n-1} f'(x) = \sqrt{\frac{2}{\pi}}\, e^{-x^2/2}$$

or

$$f'(x) = d_n \frac{e^{(f^2(x)-x^2)/2}}{f(x)^{n-1}},$$

for some constant d_n. It follows that

$$f''(x) = -d_n \frac{e^{(f^2(x)-x^2)/2}}{f(x)^n}(xf(x) + (n-1-f(x)^2)f'(x).$$

As $x \to 0_+$, we have $f(x) \to 0$ and $f'(x) \to \infty$. Therefore $f''(x)$ must be negative for small x, so $f(x)$ is not convex. By the previous paragraph, we still cannot take $\Theta_n = \Psi_n$ for $n \ge 2$.

The previous argument does not eliminate the possibility $\Theta_n = \Psi_1$. To deal with this we first observe by taking C and D to be o-symmetric balls of radius x and y, respectively, that if Question (6.1.6) has a positive answer for o-symmetric closed convex sets, then $\Theta_n^{-1}(\Psi_n(x))$ must be convex. We shall show that $g(x) = \Psi_1^{-1}\,(\Psi_n(x))$ is not convex for $n = 2$.

To this end, note first that $\Psi_2(x) = 1 - e^{-x^2/2}, \Psi_2'(x) = xe^{-x^2/2}$, and $\Psi_1'(x) = \sqrt{2/\pi}\, e^{-x^2/2}$. By differentiating $\Psi_1(g(x)) = \Psi_2(x)$, we obtain

$$g'(x) = \sqrt{\frac{\pi}{2}}xe^{(g^2-x^2)/2},$$

and hence

$$g''(x) = \sqrt{\frac{\pi}{2}}e^{(g^2-x^2)/2}(1 + x(g'g - x))\,.$$

So it suffices to study the sign of

$$h(x) = 1 + x(g'g - x) = 1 + x\left(\sqrt{\frac{\pi}{2}}xe^{\frac{g^2-x^2}{2}} - x\right). \tag{30}$$

From $\Psi_1(g(x)) = \Psi_2(x)$ we also obtain

$$\sqrt{\frac{2}{\pi}}\int_0^g e^{-t^2/2}dt = 1 - e^{-x^2/2},$$

which yields

$$\sqrt{\frac{\pi}{2}}e^{-x^2/2} = \int_g^\infty (1/t)te^{-t^2/2}dt$$

$$= \frac{1}{g}e^{-g^2/2} - \int_g^\infty \frac{1}{t^2}\, e^{-\frac{t^2}{2}}dt \tag{31}$$

$$= \frac{1}{g}e^{-g^2/2} - \frac{1}{g^3}e^{-\frac{g^2}{2}} + \,3\int_g^\infty \frac{1}{t^4}e^{-\frac{t^2}{2}}dt \tag{32}$$

$$< \frac{1}{g}e^{-g^2/2} - \frac{1}{g^3}e^{-g^2/2} + \,3\int_g^\infty \frac{t}{g^5}e^{-t^2/2}dt$$

$$= \,e^{-g^2/2}\left(\frac{1}{g} - \frac{1}{g^3} + \frac{3}{g^5}\right). \tag{33}$$

From (30) and (33), we have

$$h(x) < \frac{1}{g^2}\left(g^2 - x^2 + \frac{3x^2}{g^2}\right). \tag{34}$$

Now (31) gives

$$\sqrt{\frac{\pi}{2}}e^{(g^2-x^2)/2} < \frac{1}{g},$$

and hence

$$(g^2 - x^2) < \,-ln\left(\frac{\pi g^2}{2}\right). \tag{35}$$

Similarly (32) yields

$$(g^2 - x^2) > \,-ln\left(\frac{\pi g^6}{2(g^2-1)^2}\right). \tag{36}$$

By (34), (35), and (36), we conclude that

$$h(x) < \frac{1}{g^2}\left(-ln(\pi g^2/2) + 3 + \frac{3}{g^2} ln\left(\frac{\pi g^6}{2(g^2-1)^2}\right)\right),$$

which is negative for sufficiently large x, since $g(x) \to \infty$ as $x \to \infty$.

Question (6.1.7) [289]: Let $0 < t < 1$ and let K and L be closed convex sets containing the origin in $\mathbb{R}^n$. Is it true that

$$\gamma_n\left((1-t)K + tL\right)^{1/n} \geq (1-t)\gamma_n(K)^{1/n} + t\gamma_n(L)^{1/n}? \qquad (37)$$

The exponent $1/n$ is the best possible. Indeed, by the relation (15) and the remarks after (18) concerning the classical Brunn-Minkowski inequality with exponent p, we see that (37) does not hold in general when $1/n$ is replaced by $p > 1/n$. On the other hand, if (37) is true, then the remarks about the weighted pth means ensure that (37) remains true when $1/n$ is replaced by $0 < p \leq 1/n$.

A positive answer to Question(6.1.7) would imply that (37) remains true when $(1-t)$ is replaced by $s > 0$, under the condition $s + t \leq 1$, as can be verified by the same argument used at the end of the proof of Corollary (6.1.2).

We gave an example after Corollary (6.1.5) showing that the stronger inequality

$$\gamma_n\left((1-t)K \tilde{+} tL\right)^{1/n} \geq (1-t)\gamma_n(K)^{1/n} + t\gamma_n(L)^{1/n}$$

is false in general for K and L which are both dilatates of the same o-symmetric star body. It is also false for sufficiently small star bodies K and L containing the origin that are not dilatates, by (4), its equality condition, and (15).

Some restriction on the position of the sets is necessary. To see this, let $0 < t < 1, K = B$, and $L = B + x_1 e_1$, where $x_1 > 0$ and e_1 is a unit vector in the direction of the positive first coordinate axis. Then $(1-t)K + tL = B + tx_1e_1$, so the left-hand side of (37) approaches zero as $x_1 \to \infty$, while the right-hand side remains bounded away from zero.

If it is true, (37) would be stronger than (6) for closed convex sets containing the origin, and it does not follow from Ehrhard's inequality (5). Indeed, we claim that this is even the case when K and L are o-symmetric balls. To prove this, for fixed $0 < t < 1$ and $r_0 > 0$, consider the function

$$f(r) = \Phi(1-t)\Phi^{-1}\left(\gamma_n(r_0B)\right) + t\Phi^{-1}\left(\gamma_n(rB)\right)$$
$$-\left((1-t)\gamma_n(r_0B)^{1/n} + t\gamma_n(rB)^{1/n}\right).$$

If r_0 is chosen so that $\gamma_n(r_0B) = 1/2$, then $\Phi^{-1}(\gamma_n(r_0B)) = 0$ and we have $f(r) < 0$ if and only if

$$\Phi(t\Phi^{-1}(\gamma_n(rB))) < \left((1-t)2^{-1/n} + t\gamma_n(rB)^{1/n}\right)^n$$

Or

$$t\Phi^{-1}\left(\gamma_n(rB)\right) < \Phi^{-1}\left((1-t)2^{-1/n} + t\gamma_n(rB)^{1/n}\right)^n.$$

Now as $r \to 0_+$, the left-hand side of the previous inequality approaches $-\infty$, while the right-hand approaches $\Phi^{-1}((1-t)^n/2)$. Therefore $f(r) < 0$ for sufficiently small $r > 0$, proving the claim.

Corollary (6.1.4) shows that the answer to Question (6.1.4) is positive if K and L are o-symmetric balls, since in this case the radial sum and Minkowski sum coincide.

Theorem (6.1.8) [289]: Question (6.1.7) has a positive answer when $n = 1$.

Proof*:* Let $0 < t < 1$ and let $K = [-a, b]$ and $L = [-c, d]$ for nonnegative reals a, b, c, and d. Note that since $n = 1$, radial and Minkowski addition coincide. Then, by the first statement of Corollary (6.1.4) with $n = 1$, we have

$$\begin{aligned}
\gamma_1((1-t)K + tL) &= \gamma_1((1-t)[-a,b] + t[-c,d]) \\
&= \gamma_1([-(1-t)a - tc, 0]) + \gamma_1([0, (1-t)b + td]) \\
&= \frac{1}{2}\gamma_1((1-t)[-a,a] + t[-c,c]) \\
&\quad + \frac{1}{2}\gamma_1((1-t)[-b,b] + t[-d,d]) \\
&\geq \frac{1}{2}((1-t)\gamma_1([-a,a]) + t\gamma_1([-c,c])) \\
&\quad + \frac{1}{2}((1-t)\gamma_1([-b,b]) + t\gamma_1([-d,d])) \\
&= (1-t)\gamma_1([-a,0]) + t\gamma_1([-c,0]) \\
&\quad + (1-t)\gamma_1([0,b]) + t\gamma_1([0,d]) \\
&= (1-t)\gamma_1([-a,b]) + t\gamma_1([-c,d]) \\
&= (1-t)\gamma_1(K) + t\gamma_1(L),
\end{aligned}$$

as required. The argument still applies if one or both of K and L is an infinite interval.

The following theorem generalizes the previous result. A different generalization is given in Theorem (6.1.15).

Theorem (6.1.9) [289]: Question (6.1.7) has a positive answer when K and L are coordinate boxes containing the origin in $\mathbb{R}^n$.

Proof: Let $0 < t < 1$, and let $K = \prod_{i=1}^n I_i$ and $L = \prod_{i=1}^n J_i$ for closed (possibly unbounded) intervals I_i and J_i in $\mathbb{R}$ containing the origin, $1 \leq i \leq n$. Then

$$(1-t)K + tL = \prod_{i=1}^{n} ((1-t)I_i + tJ_i).$$

An inequality of Minkowski (see [290, (2.13.8), p. 35]) states that for nonnegative reals x_i and y_i, $1 \leq i \leq n$,

$$\left(\prod_{i=1}^n (x_i + y_i)\right)^{1/n} \geq \left(\prod_{i=1}^n x_i\right)^{1/n} + \left(\prod_{i=1}^n y_i\right)^{1/n}. \qquad (38)$$

Using the fact that Gauss measure is a product measure, Theorem (6.1.8), and (38), we obtain

$$\begin{aligned}
\gamma_n((1-t)K + tL)^{1/n} &= \left(\prod_{i=1}^{n} \gamma_1((1-t)I_i + tJ_i)\right)^{1/n} \\
&\geq \left(\prod_{i=1}^{n} ((1-t)\gamma_1(I_i) + t\gamma_1(J_i))\right)^{1/n} \\
&\geq \left(\prod_{i=1}^{n} ((1-t)\gamma_1(I_i))\right)^{1/n} + \left(\prod_{i=1}^{n} (t\gamma_1(J_i))\right)^{1/n} \\
&= (1-t)\gamma_n(K)^{1/n} + t\gamma_n(L)^{1/n}.
\end{aligned}$$

Corollary (6.1.10) [289]: Question (6.1.7) has a positive answer when one set is a slab containing the origin in $\mathbb{R}^n$.

Proof: Without loss of generality, let $L = [-a, b] \times \mathbb{R}^{n-1}, a, b \geq 0$, be a slab, and let $K_S = [-c, d] \times \mathbb{R}^{n-1}, c, d \geq 0$, be a parallel slab such that the hyperplanes $x_1 = -c$ and $x_1 = d$ support K. Then $K \subset K_S$ and $(1-t)K + tL = (1-t)K_S + tL$. Therefore, by Theorem (6.1.9),

$$\begin{aligned}\gamma_n((1-t)K + tL)^{1/n} &= \gamma_n((1-t)K_S + tL)^{1/n} \\ &\geq (1-t)\gamma_n(K_S)^{1/n} + t\gamma_n(L)^{1/n} \\ &\geq (1-t)\gamma_n(K)^{1/n} + t\gamma_n(L)^{1/n}.\end{aligned}$$

Our next result is related to the so-called (B) conjecture proposed by W. Banaszczyk, which asks whether the function $\gamma_n(e^t K)$ is log concave in t when K is an o-symmetric closed convex set in $\mathbb{R}^n$. This was proved by Cordero-Erasquin, Fradelizi, and Maurey [290]. The following lemma merely rephrases the log concavity and is essentially part of the proof in [290] (see inequality (4) in that paper).

Lemma (6.1.11) [289]: Let K be a closed convex set in $\mathbb{R}^n$ such that $\gamma_n(K) > 0$. Then $\gamma_n(e^t K)$ is log concave in t if and only if

$$\frac{\int_K \|x\|^4 e^{-\|x\|^2/2}\, dx}{\gamma_n(K)} - \left(\frac{\int_K \|x\|^2 e^{-\|x\|^2/2}\, dx}{\gamma_n(K)}\right)^2 - 2\frac{\int_K \|x\|^2 e^{-\|x\|^2/2}\, dx}{\gamma_n(K)} \leq 0. \qquad (39)$$

Theorem (6.1.12) [289]: Let K_0 be a closed convex set containing the origin in $\mathbb{R}^n$ such that $\gamma_n(K_0) > 0$, and suppose that $\gamma_n(e^t K_0)$ is log concave in t. Then Question (6.1.7) has a positive answer when K and L are both dilatates of K_0.

Proof: Let K_0 satisfy the hypotheses of the theorem and define

$$f(t) = c_n^{-1/n}\, \gamma_n(tK_0)^{1/n}.$$

For $m = 0, 1, 2, \ldots$, let

$$I_{K_0,m}(t) = \int_{K_0} \|x\|^m e^{-t^2\|x\|^2/2}\, dx = t^{-(m+n)} \int_{tK_0} \|x\|^m e^{-\|x\|^2/2} dx = t^{-(m+n)} I_{L,m}(1),$$

where $L = tK_0$. Then

$$f(t) = \left(\int_{tK_0} e^{-\|x\|^2/2} dx\right)^{1/n} = tI_{K_0,0}(t)^{1/n}.$$

Note that

$$I'_{K_0,m}(t) = -tI_{K_0,m+2}(t). \qquad (40)$$

To prove the theorem, it suffices to show that $f(t)$ is concave for $0 < t < 1$. By direct calculation, using (40), we find

$$f'(t) = \frac{I_{K_0,0}(t)^{1/n}}{n}\left(n - t^2\, \frac{I_{K_0,2}(t)}{I_{K_0,0}(t)}\right)$$

and

$$f''(t) = \frac{tI_{K_0,0}(t)^{1/n}}{n^2}\left(t^2\left(n\frac{I_{K_0,4}(t)}{I_{K_0,0}(t)} - (n-1)\left(\frac{I_{K_0,2}(t)}{I_{K_0,0}(t)}\right)^2\right) - 3n\frac{I_{K_0,2}(t)}{I_{K_0,0}(t)}\right)$$

$$= \frac{I_{L,0}(1)^{1/n}}{n^2t^2}\left(n\frac{I_{L,4}(1)}{I_{L,0}(1)} - (n-1)\left(\frac{I_{L,2}(1)}{I_{L,0}(1)}\right)^2 - \ 3n\frac{I_{L,2}(1)}{I_{L,0}(1)}\right)$$

$$= \frac{I_{L,0}(1)^{1/n}}{n^2t^2}\left(nJ_L + \left(\frac{I_{L,2}(1)}{I_{L,0}(1)^2}\right)(I_{L,2}(1) - nI_{L,0}(1))\right), \qquad (41)$$

where

$$J_L = \frac{I_{L,4}(1)}{I_{L,0}(1)} - \left(\frac{I_{L,2}(1)}{I_{L,0}(1)}\right)^2 - 2\frac{I_{L,2}(1)}{I_{L,0}(1)}.$$

Now

$$I_{L,2}(1) = \int_L \|x\|^2 e^{-\|x\|^2/2}dx$$

$$= -\int_{S^{n-1}}\int_0^{\rho_L(u)} e^{-r^2/2}r^{n+1}\,dr\,du$$

$$= -\rho_L(u)^n e^{-\rho_L(u)^2/2}du + n\int_{S^{n-1}}\int_0^{\rho_L(u)} e^{-r^2/2}r^{n-1}dr\,du$$

$$\le n\int_{S^{n-1}}\int_0^{\rho_L(u)} e^{-\frac{r^2}{2}}r^{n-1}dr\,du = nI_{L,0}(1). \qquad (42)$$

By (41) and (42), it suffices to show that $J_L \le 0$. But this is precisely (39) with K replaced by $L = tK_0$. Our assumption that $g(t) = log\,\gamma_n(e^tK_0)$ concave in t implies that for any $s > 0$, $g(t + logs) = log\,\gamma_n(e^t(sK_0))$ is concave, so (39) also holds when K is replaced by any dilatate of K_0. This completes the proof.

As was mentioned above, the (B) conjecture was proved by Cordero-Erasquin, Fradelizi, and Maurey [290]. The same authors state that they do not know if the o-symmetry is needed, and they show that in some cases it is not. Specifically, they define $G(K)$ to be the group of isometries ϕ of $\mathbb{R}^n$ such that $\phi K = K$, and they define

$$Fix(K) = \{x \in \mathbb{R}^n : \phi x = x\ for\ all\ \phi \in G(K)\}.$$

Then, in [290, Section 3], it is proved that $\gamma_n(e^tK)$ is log concave in t when $Fix(K) = \{o\}$; for example, when K is a regular simplex with centroid at the origin.

Corollary (6.1.13) [289]: Question (6.1.7) has a positive answer when K and L are both dilatates of the same o-symmetric closed convex set, or more generally, of the same closed convex set K_0 with $Fix(K_0) = \{o\}$.

We remark that calculations very similar to those in the example given just before Question (6.1.6) show that the function $\gamma_n(e^tK)$ is not log concave in general when K is an o-symmetric star body. Indeed, let $E_1 = \{(x,y) \in \mathbb{R}^2 : x, y \ge 0\}$, $E_2(a) = \{(r,\theta) \in \mathbb{R}^2 : 0 \le r \le a, \pi/2 \le \theta \le \pi\}$, and $E(a) = E_1 \cup (-E_1) \cup E_2(a) \cup (E_2(a))$. Define

$$f(t) = log\left(\gamma_2\left(e^tE(a)\right)\right) = log\left(\frac{1}{2} + \frac{1}{2}\left(1 - e^{-e^{2t}a^2/2}\right)\right) = logI(t),$$

say. Then

$$f''(t) = \frac{e^{2t}a^2e^{-e^{2t}a^2/2}}{2I(t)^2}\left(2 - e^{2t}a^2 - e^{-e^{2t}a^2/2}\right).$$

Using the inequality $e^{-x} \le 1 - x + x^2/2 \; for \; x = e^{2t}a^2/2 \ge 0$, we have

$$2 - e^{2t}a^2 - e^{-e^{2t}a^2/2} \ge \frac{1}{8}\,(8 - 4e^{2t}a^2 - e^{4t}a^4).$$

The latter quantity is positive for $0 \le t \le 1$, and hence $f(t)$ is convex there, when

$$a \le a_0 = (2\sqrt{3} - 2)^{1/2}e^{-1} = 0.4451 \ldots.$$

If we replace E_1 with $E_1' = \{(r, \theta) \in \mathbb{R}^2 : 0 \le r \le b, 0 \le \theta \le \pi/2\}$ for sufficiently large b and approximate, we can find an o-symmetric star body E such that $\gamma_n(e^tE)$ is not log concave.

For some time we considered the possibility that if $0 < t < 1$ and K and L are o-symmetric closed convex sets in $\mathbb{R}^n$, then

$$\Psi_n^{-1}\left(\gamma_n\big((1-t)K + tL\big)\right) \ge (1-t)\Psi_n^{-1}\big(\gamma_n(K)\big) + t\Psi_n^{-1}\big(\gamma_n(L)\big), \qquad (43)$$

where Ψ_n is defined by (10), with equality if and only if K and L are o-symmetric balls. The motivation was the fact that for arbitrary convex sets K and L, (43) implies (37). Indeed, using (43) and the fact that by the first statement of Corollary (6.1.4) the function $\Psi_n(r)^{1/n}$ is concave for $r > 0$, we obtain

$$\begin{aligned}\gamma_n\big((1-t)K + tL\big)^{1/n} &\ge \Psi_n(1-t)\Psi_n^{-1}(\gamma_n(K)) + t\Psi_n^{-1}(\gamma_n(L))^{1/n} \\ &\ge (1-t)\Psi_n(\Psi_n^{-1}(\gamma_n(K)))^{1/n} + t\Psi_n(\Psi_n^{-1}(\gamma_n(L)))^{1/n} \\ &= (1-t)\gamma_n(K)^{1/n} + t\gamma_n(L)^{1/n},\end{aligned}$$

which is (37).

However, inequality (43) is false in general for arbitrary o-symmetric convex sets. We are grateful to Franck Barthe for the following proof of this fact. (A similar argument is used by Latala [292, p. 816].)

Let K and L be o-symmetric convex sets in $\mathbb{R}^n$, let $0 < t < 1$, and let $h > 0$.

In (43), replace K by $(1-t)^{-1}K$ and let $L = (h/t)B$. Then, on letting $t \to 0$, we obtain from (43) the inequality

$$\Psi_n^{-1}\left(\gamma_n(K + hB)\right) \ge \Psi_n^{-1}\big(\gamma_n(K)\big) + h. \qquad (44)$$

Choose $r > 0$ so that $\gamma_n(rB) = \gamma_n(K)$. Then (44) yields

$$\gamma_n(K + hB) \ge \Psi_n(\Psi_n^{-1}(\gamma_n(rB)) + h) = \Psi_n(r + h) = \gamma_n(rB + hB).$$

Therefore

$$\lim_{h \to 0_+} \frac{\gamma_n(K + hB) - \gamma_n(K)}{h} \ge \lim_{h \to 0_+} \frac{\gamma_n(rB + hB) - \gamma_n(rB)}{h}.$$

However, by [295, Lemma 3], the previous inequality is false when $n = 2, K = \{(x, y) \in \mathbb{R}^2 : y \in [-a, a]\}$ is a slab, and $a > 0$ is sufficiently large.

Indeed, it can be seen by direct calculation that (43) is false when $K = \{(x, y) \in \mathbb{R}^2 : y \in [-1/(1-t), 1/(1-t)]\}, L = (1/t)B$, and $0 < t < 0.04$. It is interesting to note that by Corollary (6.1.10), sets of this form cannot supply a negative answer to Question (6.1.7).

If f is a nonnegative measurable function on $\mathbb{R}^n$ and $s \ge 0$, the superlevel set $L(f, s)$ is defined by

$$L(f, s) = \{x : f(x) \ge s\}.$$

Note that

$$c_n \int_{\mathbb{R}^n} f(x) e^{-\|x\|^2/2}\, dx = c_n \int_{\mathbb{R}^n} \int_0^{f(x)} e^{-\|x\|^2/2} ds\, dx$$
$$= c_n \int_0^\infty \int_{L(f,s)} e^{-\|x\|^2/2} dx\, ds = \int_0^\infty \gamma_n(L(f,s))\, ds. \qquad (45)$$

The standard Prekopa-Leindler inequality (see, for example, [294, Theorem 7.1]) holds when Lebesgue measure is replaced by any log concave measure, in particular, by γ_n. Theorem (6.1.8) yields the following stronger inequality when $n = 1$, for a restricted class of functions.

Theorem (6.1.14) [289]: Let $0 < t < 1$ and let $f, g,$ and h be nonnegative integrable functions on $\mathbb{R}$ such that superlevel sets of f and g are either empty or intervals containing the origin. If

$$h((1-t)x + ty) \geq (1-t)f(x) + tg(y),$$

for all $x, y \in \mathbb{R}$, then

$$\int_{\mathbb{R}} h(x) e^{-\|x\|^2/2} dx \geq (1-t) \int_{\mathbb{R}} f(x) e^{-\|x\|^2/2} dx + t \int_{\mathbb{R}} g(x) e^{-\|x\|^2/2} dx.$$

Proof. If $s \geq 0, f(x) \geq s$, and $g(y) \geq s$, then $h((1-t)x + ty) \geq s$. Therefore,

$$L(h,s) \supseteq (1-t)L(f,s) + tL(g,s).$$

Then, by (45), the fact that $L(f,s)$ and $L(g,s)$ are intervals containing the origin, and Theorem (6.1.8), we obtain

$$\int_{\mathbb{R}} h(x) e^{-\|x\|^2/2} dx = \frac{1}{c_1} \int_0^\infty \gamma_1(L(h,s))\, ds$$
$$\geq \frac{1}{c_1} \int_0^\infty \gamma_1((1-t)L(f,s) + tL(g,s))\, ds$$
$$\geq \frac{1-t}{c_1} \int_0^\infty \gamma_1(L(f,s))\, ds + \frac{t}{c_1} \int_0^\infty \gamma_1(L(g,s))\, ds$$
$$= (1-t) \int_{\mathbb{R}} f(x) e^{-\|x\|^2/2} dx + tg(x) e^{-\|x\|^2/2}\, dx.$$

We do not know whether the assumption on the superlevel sets of f and g is necessary. It could be removed if Theorem (6.1.8) holds when K and L are arbitrary Borel sets containing the origin. We have the following generalization of Theorem (6.1.8), inspired by work of Latala [293].

Theorem (6.1.15) [289]: Question (6.1.7) has a positive answer when $n = 1, K$ is an interval containing the origin, and L is any Borel set containing the origin.

Proof: Let $K = [a,b]$ and $L = \cup_{i=-m}^{n} [x_i, y_i]$, where

$$x_{-m} \leq y_{-m} < x_{-m-1} \leq y_{-m-1} < \cdots < x_n \leq y_n,$$

$o \in [a,b]$, and $o \in [x_0, y_0]$. Then

$$(1-t)K + tL = \bigcup_{i=-m}^{n} [(1-t)a + tx_i, (1-t)b + ty_i].$$

We claim that we may assume that the intervals in this union are disjoint. Otherwise, for some $-m \leq i \leq n$, since $x_i < x_{i+1}$ and $y_i < y_{i+1}$, we have

$$\emptyset \neq [(1-t)a + tx_i, (1-t)b + ty_i] \cap [(1-t)a + tx_{i+1}, (1-t)b + ty_{i+1}]$$
$$= [(1-t)a + tx_i, (1-t)b + ty_{i+1}] = (1-t)[a,b] + t[x_i, y_{i+1}].$$
Let
$$L' = \bigcup_{k=-m,k\neq i,i+1}^{n} [x_k, y_k] \cup [x_i, y_{i+1}].$$
Then $(1-t)K + tL = (1-t)K + tL'$ and $\gamma_1(L') \geq \gamma_1(L)$, the set L' consists of fewer intervals than L, and $o \in L'$. So we may replace L by L'. We can repeat the argument, if necessary, until all the intervals in the union are disjoint.
Since $o \in [a,b]$, we have
$$\bigcup_{i=-m}^{n} [(1-t)a + tx_i, (1-t)b + ty_i]$$
$$\supseteq [(1-t)a + tx_0, (1-t)b + ty_0] \cup \bigcup_{i=-m,i\neq 0}^{n} [tx_i, ty_i].$$
Now we can use Theorem (6.1.8) and (11) to obtain
$$\gamma_1((1-t)K + tL) \geq \gamma_1\left([(1-t)a + tx_0, (1-t)b + ty_0] \cup \bigcup_{i=-m,i\neq 0}^{n} [tx_i, ty_i] \right)$$
$$= \gamma_1([(1-t)a + tx_0, (1-t)b + ty_0]) + \sum_{i=-m,i\neq 0}^{n} \gamma_1(t[x_i, y_i])$$
$$\geq (1-t)\gamma_1(K) + t\gamma_1([x_0, y_0]) + t \sum_{i=-m,i\neq 0}^{n} \gamma_1([x_i, y_i])$$
$$= (1-t)\gamma_1(K) + t\gamma_1(L).$$
Therefore the result holds when L is a finite union of intervals, and the theorem is then proved by a standard approximation argument.
The previous result allows the assumptions in Theorem (6.1.14) to be weakened.

Corollary (6.1.16) [388]: The function ϕ_n defined by (13) is sublinear, i.e.,
$$\phi_n(2a + \epsilon) \leq \phi_n(a) + \phi_n(a + \epsilon),$$
for $\epsilon \geq 0$, with equality if and only if $a = 0$ or $\epsilon = a$.
Proof. For fixed $\epsilon > 0$ and all $a \geq 0$, define
$$f(a) = \phi_n(2a + \epsilon) - \phi_n(a) - \phi_n(a + \epsilon).$$
Then $f(0) = 0$, and it suffices to show that $f'(a) < 0$ for all $a \geq 0$. In view of (13), we have
$$nf'(a) = (2a + \epsilon)^{n-1} e^{-(a+a+\epsilon)^2/2} \phi_n(2a + \epsilon)^{1-n} - a^{n-1} e^{-a^2/2} \phi_n(a)^{1-n}.$$
If $n = 1$, it is clear from this that $f'(a) < 0$ for $a \geq 0$. Suppose that $n \geq 2$. Using (13) again, we see that $f'(a) < 0$ is equivalent to

$$(2a\ +\epsilon)^{-n}e^{n(2a+\epsilon)^2/(2(n-1))}\int_0^{2a+\epsilon} e^{-(1+\epsilon)^2/2}(1+\epsilon)^{n-1}d\epsilon$$
$$> a^{-n}e^{na^2/(2(n-1))}\int_0^{a} e^{-(1+\epsilon)^2/2}(1+\epsilon)^{n-1}d\epsilon$$

or

$$e^{n(2a+\epsilon)^2/(2(n-1))}\int_0^1 e^{-((1+2\epsilon)(2a+\epsilon))^2/2}(1+2\epsilon)^{n-1}d\epsilon$$
$$> e^{na^2/(2(n-1))}\int_0^1 e^{-((1+2\epsilon)a)^2/2}(1+2\epsilon)^{n-1}d\epsilon.$$

Rearranging, we obtain

$$\int_0^1 e^{(n/(n-1)-(1+2\epsilon)^2)(2a+\epsilon)^2/2}(1+2\epsilon)^{n-1}d\epsilon > \int_0^1 e^{(n/(n-1)-(1+2\epsilon)^2)a^2/2}(1+2\epsilon)^{n-1}d\epsilon\,.$$

The previous inequality holds since $(1+2\epsilon)^2 \le 1 < n/(n-1)$, and this proves the lemma.

Corollary (6.1.17) [388]: C^m and D^m be Borel star sets in $\mathbb{R}^n$, and let $\epsilon \ge 0$. Then

$$\gamma_n\big((1+2\epsilon)C^m \mp (1+\epsilon)D^m\big)^{1/n} \le (1+2\epsilon)\gamma_n(C^m)^{1/n} + (1+\epsilon)\gamma_n(D^m)^{1/n}.$$

Suppose that C^m and D^m are properly contained in $\mathbb{R}^n$. Equality holds when $\epsilon = 1$ if and only if $\gamma_n(C^m) = 0, \gamma_n(D^m) = 0$, or $n = 1$ and both C^m and D^m are (possibly degenerate or infinite) intervals with one endpoint at the origin, each on opposite sides of the origin. Equality holds when $\epsilon \ge 0$ and $\epsilon = 1$ (or $\epsilon = 1$ and $\epsilon \ge 0$, or $\epsilon > 1$ and $\epsilon > 1$) if and only if $\gamma_n(C^m) = 0$ (or if and only if $\gamma_n(D^m) = 0$, or if and only if $\gamma_n(C^m) = 0$ and $\gamma_n(D^m) = 0$, respectively).

Proof. Suppose first that $\epsilon = 1$.

If $n = 1$ and C^m and D^m are bounded, then $C^m = [-a_1, (a+\epsilon)_1]$ and $D^m = [-a_2, (a+\epsilon)_2]$ for nonnegative $a_1, a_2, (a+\epsilon)_1$, and $(a+\epsilon)_2$, and (20) is equivalent to

$$\phi_1(a_1+a_2) + \phi_1(b_1+b_2) \le (\phi_1(a_1) + \phi_1(b_1)) + (\phi_1(a_2) + \phi_1(b_2))\,.$$

This follows immediately from Lemma 4.1, and its equality condition shows that either $a_1 = 0\ or\ a_2 = 0$ and either $(a+\epsilon) = 0$ or $(a+\epsilon)_2 = 0$. The same conclusion is reached if C^m or D^m is unbounded. This yields the required equality condition when $n = 1$.

Suppose that $n \ge 2$. By (14), (7), , and Minkowski's inequality for integrals, we have

$$\gamma_n(C^m \mp D^m)^{1/n} = \left(c_n \int_{S^{n-1}} \phi_n(\rho_{C^m} \mp D^m(u_m))^n du_m\right)^{1/n}$$
$$= \left(c_n \int_{S^{n-1}} \phi_n\ (\rho_{C^m}(u_m) + \rho_{D^m}(u_m))^n\ du_m\right)^{1/n}$$
$$\le \left(c_n \int_{S^{n-1}} (\phi_n(\rho_{C^m}(u_m)) + \phi_n(\rho_{D^m}(u_m)))^n du_m\right)^{1/n}$$
$$\le \left(c_n \int_{S^{n-1}} \phi_n\ (\rho_{C^m}(u_m))^n\ du_m\right)^{1/n} + \left(c_n \int_{S^{n-1}} \phi_n\ (\rho_{D^m}(u_m))^n\ du_m\right)^{1/n}$$

$$= \gamma_n(C^m)^{1/n} + \gamma_n(D^m)^{1/n}.$$

Suppose, in addition to our assumption that $\epsilon \geq 0$, that equality holds in (20). Then for almost all $u_m \in S^{n-1}$, equality holds when $a = \rho_{C^m}(u_m)$ and $a + \epsilon = \rho_{D^m}(u_m)$, and hence for almost all $u_m \in S^{n-1}$ we have either $\rho_{C^m}(u_m) = 0$ or $\rho_{D^m}(u_m) = 0$. But equality also holds in Minkowski's inequality for integrals, so there is a constant c such that $\phi_n(\rho_{C^m}(u_m)) = c\phi_n(\rho_{D^m}(u_m))$ for almost all $u_m \in S^{n-1}$. It follows that either $\rho_{C^m}(u_m) = 0$ for almost all $u_m \in S^{n-1}$ or $\rho_{D^m}(u_m) = 0$ for almost all $u_m \in S^{n-1}$, and therefore either $\gamma_n(C^m) = 0$ or $\gamma_n(D^m) = 0$.

We have proved (20) and its equality conditions when $\epsilon \geq 0$. Using this and (11), for general $\epsilon \geq 0$ we obtain

$$\gamma_n\big((1+2\epsilon)C^m \mp (1+\epsilon)D^m\big)^{1/n} \leq \gamma_n\big((1+2\epsilon)C^m\big)^{1/n} + \gamma_n\big((1+\epsilon)D^m\big)^{1/n}$$
$$\leq (1+2\epsilon)\gamma_n(C^m)^{1/n} + (1+\epsilon)\gamma_n(D^m)^{1/n},$$

as required. The equality conditions for $\epsilon \geq 0$ follow from those of (11).

Inequality (20) does not hold generally when either $\epsilon \geq 0$. Indeed, if $\epsilon < 1$, (20) is false when $D^m = \varepsilon B$ and $\varepsilon > 0$ is sufficiently small, in view of (11). Inequality (20) is false for arbitrary Borel sets star shaped at the origin. To see this, let $\epsilon \geq 0$, and for each $m \in \mathbb{N}$, let $C_m^m = \{(r,\theta) \in \mathbb{R}^n : m \leq r \leq m+1, 0 \leq \theta \leq \pi/2\}$ and $D_m^m = -C_m^m$. Then $C_m^m \mp D_m^m = C_0^m \cup (-C_0^m)$, so $\gamma_2(C_m^m \mp D_m^m)$ is positive and independent of m while $\gamma_2(C_m^m) = \gamma_2(D_m^m) \to 0$ as $m \to \infty$. Note that C_m^m and D_m^m are actually star bodies.

The monotonicity properties of the weighted $(1+\epsilon)$th means $((1+2\epsilon)a^{1+\epsilon} + (1+\epsilon)(a+\epsilon)^{1+\epsilon})^{1/1+\epsilon}$ summarized at the end of Section 2 imply that Theorem 4.2 holds for $\epsilon \geq 0$ and $1+\epsilon \leq 1/n$.

However, the exponent $1/n$ in (20) is the best possible; it does not hold when $1/n$ is replaced by $\epsilon \geq 0$, as can be seen by taking $C^m = aB$ and $D^m = (a+\epsilon)B$ for sufficiently small positive a and $a+\epsilon$, and using (15) and the remarks concerning (19). Similarly, using the remarks concerning (18) instead, we see that it is also not true that (20) holds when $1/n$ is replaced by $\epsilon \geq 0$ and the inequality is reversed.

When C^m and D^m are convex bodies containing the origin, we have $(1+2\epsilon)C^m \mp (1+\epsilon)D^m \subset (1+2\epsilon)C^m + (1+\epsilon)D^m$, so in this case the inequality $\gamma_n((1+2\epsilon)C^m + (1+\epsilon)D^m)^{1/n} \leq (1+2\epsilon)\gamma_n(C^m)^{1/n} + (1+\epsilon)\gamma_n(D^m)^{1/n}$ would be stronger than (20). However, by (2), its equality condition, and (15), this is false in general when C^m and D^m are sufficiently small nonhomothetic convex bodies containing the origin.

As a final remark, we consider the possibility that

$$\ominus_n^{-1}\,\gamma_n(1+2\epsilon)C^m \widetilde{\mp} (1+\epsilon)D^m \leq (1+2\epsilon)\ \ominus_n^{-1}\,(\gamma_n(C^m)) + (1+\epsilon)\ \ominus_n^{-1}\,(\gamma_n(D^m))$$

holds for sequences of Borel star sets C^m and D^m in $\mathbb{R}^n$ and $\epsilon \geq 0$, where $\ominus_n$ is some standard function related to Gauss measure. Certainly (21) is not generally true when $\epsilon \geq 0$ and $\ominus_n = \Psi_n$, the function defined by (10). To see this, let C^m and D^m be half-spaces in $\mathbb{R}^n$ bounded by a common hyperplane through the origin, so that $C^m \widetilde{\mp} D^m = \mathbb{R}^n$ and $\gamma_n(C^m) = \gamma_n(D^m) = 1/2$. Then the left-hand side of (21) with $\epsilon \geq 0$ and $\ominus_n = \Psi_n$ is infinite, while

the right-hand side is bounded. Of course the same argument shows that (21) is not generally true when $\ominus_n = \Psi_1$ *or* $\ominus_n = \Phi$ (defined by (12)).
In view and the dual Brunn-Minkowski inequality in the form (19), it is natural to ask whether there is a $\epsilon \geq 0$ such that

$$\gamma_n\left((1+2\epsilon)C^m \tilde{+} (1+\epsilon)D^m\right)^{1+\epsilon} \leq (1+2\epsilon)\gamma_n(C^m)^{1+\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1+\epsilon}$$

holds for $\epsilon \geq 0, \epsilon \leq 1$, and sequences of Borel star sets C^m and D^m in $\mathbb{R}^n$. We shall see that the answer is negative for $\epsilon \geq 0$, even for o-symmetric balls. To this end, the following will be useful(see[325]).

Corollary (6.1.18) [388]: The function

$$F_n(r) = \left(\int_0^r e^{-(1+\epsilon)^2/2}(1+\epsilon)^{n-1}\, d\epsilon\right)^{1+\epsilon}$$

is strictly concave when (i) $0 < \epsilon < 1$ and $r \geq \sqrt{n-1}$, (ii) $\epsilon \geq 0$ and $r > \sqrt{n(1+\epsilon) - 1}$, and (iii) $1 + \epsilon \leq 1/n$ and $r > 0$.

Proof. Let

$$I_n(r) = e^{-(1+\epsilon)^2/2}(1+\epsilon)^{n-1}\, d\epsilon,$$

so that $F_n(r) = I_n(r)^{1-\epsilon}$. A straightforward calculation yields

$$F_n''(r) = pI_n(r)^{-\epsilon-1}e^{-r^2/2}r^{n-2}\left((\epsilon)e^{-r^2/2}r^n + I_n(r)(n-1-r^2)\right).$$

Note that a trivial estimate gives $I_n(r) > e^{-r^2/2}r^n/n$ for $r > 0$, so if $r \geq \sqrt{n-1}$, we obtain $F_n''(r) = pI_n(r)^{-\epsilon-1}e^{-\frac{r^2}{2}}r^{2n-2}(n(1-\epsilon) - 1 - r^2)/n$. From this we see that $F_n''(r) < 0$ when, in addition, $\epsilon \leq 0$, establishing (i), and (ii) also follows immediately.
In proving (iii) we may suppose that $1 - \epsilon = 1/n$, since $(1+\varepsilon)$th means increase with $1 - \epsilon$. Substituting $1 - \epsilon = 1/n$ into (25), we see that it suffices to show that

$$G_n(r) = -(n-1)e^{-r^2/2}r^n + nI_n(r)(n-1-r^2) < 0$$

for $r > 0$. Now $G_n(0) = 0$, and

$$G_n'(r) = e^{-r^2/2}r^{n+1} - 2nrI(r) < 0$$

for $r > 0$. It follows that $G_n(r) < 0$ for $r > 0$, as required.
No attempt was made to obtain best possible estimates in cases (i) and (ii) of the previous lemma(6.1.3), since those found are sufficient for our purposes. Case (iii) of the previous lemma (6.1.3)is equivalent to the concavity of $\phi_n(r)$ for $r > 0$, and this is also implied by a result of Koenig and Tomczak-Jaegermann [326, p. 1218].

Corollary (6.1.19) [388]: Let $\epsilon \geq 0, \epsilon \leq 1$, and let C^m and D^m be o-symmetric balls in $\mathbb{R}^n$. Then

$$\gamma_n\left((1+2\epsilon)C^m \tilde{+} (1+\epsilon)D^m\right)^{1-\epsilon} \geq (1+2\epsilon)\gamma_n(C^m)^{1-\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1-\epsilon}$$

holds for $1 - \epsilon \leq 1/n$. Equality holds for $\epsilon \geq 0$ if and only if $C^m = D^m$.

Proof. Note that when $n = 1, \gamma_1(rB) = \gamma_1([-r,r]) = 2c_1I_1(r)$, where $I_n(r)$ is given by (24). If $n \geq 2$, by (14), we have

$$\gamma_n(rB) = c_n\int_{S^{n-1}} \phi_n(r)^n du_m = n\kappa_n c_n I_n(r)$$

for $r > 0$. Thus if the function $F_n(r)$ given by (23) is concave for $0 < a < r < a + \epsilon$, then

$$\gamma_n(-\epsilon C^m \tilde{+} (1+\epsilon)D^m)^{1-\epsilon} \geq -\epsilon\gamma_n(C^m)^{1-\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1-\epsilon}$$

holds when $C^m = r_0 B, D^m = r_1 B,$ and $0 < a < r_0, r_1 < a + \epsilon$. By Lemma (6.1.3)(iii), $F_n(r)$ is actually strictly concave for $1-\epsilon \leq 1/n$, and this yields the corollary together with the equality condition when $\epsilon \geq 0$.

For general $\epsilon \geq 0$ with $\epsilon \leq 1$, let $\alpha = \frac{1+2\epsilon}{-\epsilon} \leq 1$ and note that by (27) and (11), for $1-\epsilon \leq 1/n$, we have

$$\begin{aligned}\gamma_n\left((1+2\epsilon)C^m \tilde{+} (1+\epsilon)D^m\right)^{1-\epsilon} &= \gamma_n(-\epsilon(\alpha C^m) \tilde{+} (1+\epsilon)D^m)^{1-\epsilon} \\ &\geq -\epsilon\gamma_n(\alpha C^m)^{1-\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1-\epsilon} \\ &\geq -\epsilon\alpha^n\gamma_n(C^m)^{1-\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1-\epsilon} \\ &\geq -\epsilon\alpha\gamma_n(C^m)^{1-\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1-\epsilon} \\ &= (1+2\epsilon)\gamma_n(C^m)^{1-\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1-\epsilon},\end{aligned}$$

as required. If equality holds, then equality holds in (11), implying that $\alpha = 1$, and then $C^m = D^m$ from the equality condition for (27).

Corollary (6.1.20) [388]: For given $\epsilon > 0, \epsilon \leq 1$, and $\epsilon \geq 0$, inequality (22) is false in general, even for o-symmetric balls.

Proof. and its equality condition yield the result for $1-\epsilon \leq 1/n$.

Suppose that $1-\epsilon > 1/n$. By Lemma 5.1(i) and (ii) we can choose the radii of o-symmetric balls C^m and D^m in $\mathbb{R}^n$ so that with $s = 1-t$,

$$\gamma_n\left((1+2\epsilon)C^m \tilde{+} (1+\epsilon)D^m\right)^{1-\epsilon} > (1+2\epsilon)\gamma_n(C^m)^{1-\epsilon} + (1+\epsilon)\gamma_n(D^m)^{1-\epsilon},$$

and therefore so that (22) is false. It remains to consider the case when $\epsilon \leq 1$.

Let $C^m = aB$ and $D^m = aB$ for $a > 0$. Then (28) is equivalent to

$$\gamma_n((2+3\epsilon)aB)^{1-\epsilon} > (2+3\epsilon)\gamma_n(aB)^{1-\epsilon}.$$

As $a \to \infty$, the left-hand side approaches 1, while the right-hand side approaches $\epsilon \leq 1$. It follows that holds for sufficiently large a.

Note that holds even for $\epsilon \leq 0$, at least when $\epsilon \geq 0$. This is because $(1+\epsilon)$th means increase with real $1-\epsilon$; see [320, Section 2.9]. Consequently Corollary (6.1.13) also holds when $\epsilon \geq 0$ and $\epsilon \leq 0$.

Section (6.2): Log-Brunn-Minkowski Inequality

The fundamental Brunn-Minkowski inequality states that for convex bodies K, L in Euclidean nspace, $\mathbb{R}^n$, the volume of the bodies and of their Minkowski sum $K + L = \{x + y: x \in K\ and\ y \in L\}$, are related by

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

with equality if and only if K and L are homothetic. As the first milestone of the Brunn-Minkowski theory, the Brunn-Minkowski inequality is a far-reaching generalization of the isoperimetric inequality. The Brunn-Minkowski inequality exposes the crucial log-concavity property of the volume functional because the Brunn-Minkowski inequality has an equivalent formulation as: for all real $\lambda \in [0,1]$,

$$V\big((1-\lambda)K+\lambda L\big) \geq V(K)^{1-\lambda}V(L)^{\lambda}, \qquad (46)$$

and for $\lambda \in (0,1)$, there is equality if and only if K and L are translates. A big part of the classical Brunn-Minkowski theory is concerned with establishing generalizations and analogues of the Brunn-Minkowski inequality for other geometric invariants. The excellent survey article of Gardner [326] gives a comprehensive account of various aspects and consequences of the Brunn- Minkowski inequality.

If h_K and h_L are the support functions (see (60) for the definition) of K and L, the Minkowski combination $(1-\lambda)K+\lambda L$ is given by an intersection of half-spaces,

$$(1-\lambda)K+\lambda L = \bigcap_{u\in S^{n-1}} \{x \in \mathbb{R}^n : x\cdot u \leq (1-\lambda)h_K(u)+\lambda h_L(u)\},$$

where $x\cdot u$ denotes the standard inner product of x and u in $\mathbb{R}^n$. Assume that K and L are convex bodies that contain the origin in their interiors, then the geometric Minkowski combination, $(1-\lambda)\cdot K +_{\circ} \lambda\cdot L$, is defined by

$$(1-\lambda)\cdot K +_o \lambda\cdot L = \cap_{u\in S^{n-1}}\big\{x \in \mathbb{R}^n : x\cdot u \leq h_K(u)^{1-\lambda}+h_L(u)^{\lambda}\big\}. \qquad (47)$$

The arithmetic-geometric-mean inequality shows that for convex bodies K, L and $\lambda \in [0,1]$,

$$(1-\lambda)\cdot K +_o \lambda\cdot L \subseteq (1-\lambda)K+\lambda L. \qquad (48)$$

What makes the geometric Minkowski combinations difficult to work with is that while the convex body $(1-\lambda)K+\lambda L$ has $(1-\lambda)h_K+\lambda h_L$ as its support function, the convex body $(1-\lambda)\cdot K +_o \lambda\cdot L$ is the Wulff shape of the function $h_K^{1-\lambda}h_L^{\lambda}$.

The authors conjecture that for origin-symmetric bodies (i.e., unit balls of finite dimensional Banach spaces), there is a stronger inequality than the Brunn-Minkowski inequality (46), the log-Brunn-Minkowski inequality:

Problem (6.2.1)[321]: Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then for all $\lambda \in [0,1]$,

$$V\big((1-\lambda)\cdot K +_o \lambda\cdot L\big) \geq V(K)^{1-\lambda}V(L)^{\lambda}. \qquad (49)$$

That the log-Brunn-Minkowski inequality (49) is stronger than its classical counterpart (46) can be seen from the arithmetic-geometric mean inequality (48). Simple examples (e.g. an origincentered cube and one of its translates) shows that (49) cannot hold for all convex bodies.

As is well known, the classical Brunn-Minkowski inequality (46) has as a consequence an inequality of fundamental importance: the Minkowski mixed-volume inequality. One of the aims is to show that the log-Brunn-Minkowski inequality (49) also has an important consequence, the log-Minkowski inequality:

Problem (6.2.2) [321]: Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then

$$\int_{S^{n-1}} \log\frac{h_L}{h_K}\, d\bar{V}_K \;\geq \frac{1}{n}\log\frac{V(L)}{V(K)}. \qquad (50)$$

Here $\bar{V}_K$ is the cone-volume probability measure of K (see definitions (64), (65), (67)).

Just as the log-Brunn-Minkowski inequality (49) is stronger than its classical counterpart (46), the log-Minkowski inequality (50) turns out to be stronger than its classical counterpart.
The classical Minkowski mixed-volume inequality and the classical Brunn-Minkowski inequality are "equivalent" in that once either of these inequalities has been established, then the other can be obtained as a simple consequence. One of the aims is to demonstrate that the log-Brunn-Minkowski inequality (49) and the log-Minkowski inequality (50) are "equivalent" in that once either of these inequalities has been established, then the other can be obtained as a simple consequence.
Even in the plane the above problems are non-trivial and unsolved. Establish the plane log-Brunn-Minkowski inequality along with its equality conditions:
Theorem (6.2.3) [321]: If K and L are origin-symmetric convex bodies in the plane, then for all $\lambda \in [0,1]$,

$$V\big((1-\lambda)\cdot K +_o \lambda \cdot L\big) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{51}$$

When $\lambda \in (0,1)$, equality in the inequality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.
In addition, in the plane, we will establish the log-Minkowski inequality along with its equality conditions:
Theorem (6.2.4) [321]: If K and L are origin-symmetric convex bodies in the plane, then,

$$\int_{S^1} \log \frac{h_L}{h_K} d\bar{V}_K \geq \frac{1}{2} \log \frac{V(L)}{V(K)}, \tag{52}$$

with equality if and only if, either K and L are dilates or K and L are parallelograms with parallel sides.
The above Minkowski combinations and problems are merely two (important) frames of a long film. In the early 1960's, Firey (see e.g. Schneider [324, p. 383]) defined for each $p \geq 1$, what have become known as Minkowski-Firey L_p-combinations (or simply L_p-combinations) of convex bodies.
If K and L are convex bodies that contain the origin in their interiors and $0 \leq \lambda \leq 1$ then the Minkowski-Firey L_p-combination, $(1-\lambda)\cdot K +_p \lambda \cdot L$, is defined by

$$(1-\lambda)\cdot K +_p \lambda \cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq ((1-\lambda) h_K(u)^p + \lambda h_L(u)^p)^{1/p}\}. \tag{53}$$

Firey also established the L_p-Brunn-Minkowski inequality (also known as the Brunn-Minkowski-Firey inequality): If $p > 1$, then

$$V\left((1-\lambda)\cdot K +_p \lambda \cdot L\right) \geq V(K)^{1-\lambda} V(L)^{\lambda}, \tag{54}$$

with equality for $\lambda \in (0,1)$ if and only if $K = L$. In the mid 1990's, it was shown in [325, 36], that a study of the volume of Minkowski-Firey L_p-combinations leads to an embryonic L_p-Brunn-Minkowski theory. This theory has expanded rapidly. (See e.g. [324].)
Note that definition (53) makes sense for all $p > 0$. The case where $p = 0$ is the limiting case given by (47). The crucial difference between the cases where $0 < p < 1$ and the cases where $p \geq 1$ is that the function $((1-\lambda)h_K^p + \lambda h_L^p)^{1/p}$ is the support function of

$(1-\lambda)\cdot K +_p \lambda \cdot L$ when $p \geq 1$, but it is not whenever $0 < p < 1$. When $0 < p < 1$, the convex body $(1-\lambda)\cdot K +_p \lambda \cdot L$ is the Wulff shape of $((1-\lambda)h_K^p + \lambda h_L^p)^{1/p}$.
Unfortunately, progress in the L_p-Brunn Minkowski theory for $p < 1$ has been slow. The present work is a step in that direction.
It is easily seen from definition (53) that for fixed convex bodies K, L and fixed $\lambda \in [0,1]$, the L_p-Minkowski-Firey combination $(1-\lambda)\cdot K +_p \lambda \cdot L$ is increasing with respect to set inclusion, as p increases; i.e., if $0 \leq p \leq q$,

$$(1-\lambda)\cdot K +_p \lambda \cdot L \subseteq (1-\lambda)\cdot K +_q \lambda \cdot L. \qquad (55)$$

From (55) one sees that the classical Brunn-Minkowski inequality (46) (i.e. the case $p = 1$ of (54)) immediately yields Firey's L_p-Brunn-Minkowski inequality (54) for each $p > 1$. The difficult situation arises when $p \in [0,1)$ because now we are seeking inequalities that are stronger than the classical Brunn-Minkowski inequality.
The L_p-Brunn-Minkowski inequality (54) cannot be established for all convex bodies that contain the origins in their interiors, for any fixed $p < 1$. Even an origin-centered cube and one of its translates show that. However, the following problem is of fundamental importance in the L_p-Brunn-Minkowski theory:
Problem (6.2.5) [321]: Suppose $0 < p < 1$. Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then for all $\lambda \in [0,1]$,

$$V\left((1-\lambda)\cdot K +_p \lambda \cdot L\right) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \qquad (56)$$

From the monotonicity of the L_p-Minkowski combination (55), it is clear that the log-Brunn-Minkowski inequality implies the L_p-Brunn-Minkowski inequalities for each $p > 0$. We note that there are easy examples that show that the L_p-Brunn-Minkowski inequality (56) fails to hold for any $p < 0$ — even if attention were restricted to simple origin symmetric bodies.
We show that the L_p-Brunn-Minkowski inequality (50) can be formulated equivalently as the L_p-Minkowski inequality:
Problem (6.2.6) [321]: Suppose $0 < p < 1$. Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then

$$\left(\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p d\bar{V}_K\right)^{\frac{1}{p}} \geq \left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}. \qquad (57)$$

For each $p \geq 1$, the inequalities (56) and (57) are well known to hold for all convex bodies (that contain the origin in their interior) and are also well known to be equivalent, in that given one, the other is an easy consequence.
From Jensen's inequality it can be seen that the L_p-Minkowski inequality (57) for the case $p = 0$, the log-Minkowski inequality (50), is the strongest of the L_p-Minkowski inequalities (57). The L_p-Minkowski inequality for the case $p = 1$, the classical Minkowski mixed-volume inequality, is weaker than all the cases of (57) where $p \in (0,1)$.

Even in the plane the above problems are non-trivial and unsolved. One of the aims of this is to solve the problems in the plane. Solutions in higher dimensions would be highly desirable.

We will prove the following theorems.

Theorem (6.2.7) [321]: Suppose $0 < p < 1$. If K and L are origin-symmetric convex bodies in the plane, then for all $\lambda \in [0, 1]$,

$$V\left((1-\lambda) \cdot K +_p \lambda \cdot L\right) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{58}$$

When $\lambda \in (0, 1)$, equality in the inequality holds if and only if $K = L$.

Observe that the equality conditions here are different than those of Theorem (6**.2.3).**

Theorem (6.2.8) [321]: Suppose $0 < p < 1$. If K and L are origin-symmetric convex bodies in the plane, then,

$$\left(\int_{S^1} \left(\frac{\mathrm{h}_L}{\mathrm{h}_K}\right)^p d\bar{V}_K\right)^{\frac{1}{p}} \geq \left(\frac{V(L)}{V(K)}\right)^{\frac{1}{2}}, \tag{59}$$

with equality if and only if K and L are dilates.

Observe that the equality conditions here are different than those of Theorem (6.2.4).

The approach used to establish the geometric inequalities of these theorems is new.

Good general references for the theory of convex bodies are provided by the books of Gardner [325], Gruber [327], Schneider [324], and Thompson [328].

The support function $h_K: \mathbb{R}^n \to \mathbb{R}$, of a compact, convex set $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \tag{60}$$

and uniquely determines the convex set. Obviously, for a pair $K, L \subset \mathbb{R}^n$ of compact, convex sets, we have

$$h_k \leq h_L, if\ and\ only\ if, K \subseteq L. \tag{61}$$

Note that support functions are positively homogeneous of degree one and subadditive.

A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. A boundary point $x \in \partial K$ of the convex body K is said to have $u \in S^{n-1}$ as one of its outer unit normals provided $x \cdot u = h_K(u)$. A boundary point is said to be singular if it has more than one unit normal vector. It is well known (see, e.g., [324]) that the set of singular boundary points of a convex body has $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ equal to 0.

Let K be a convex body in $\mathbb{R}^n$ and $v_K: \partial K \to S^{n-1}$ the generalized Gauss map. For arbitrary convex bodies, the generalized Gauss map is properly defined as a map into subsets of S^{n-1}.

However, $\mathcal{H}^{n-1}$-almost everywhere on ∂K it can be defined as a map into S^{n-1}. For each Borel set$\omega \subset S^{n-1}$, the inverse spherical image $v_K^{-1}(\omega)$ of ω is the set of all boundary points of K which have an outer unit normal belonging to the set ω. Associated with each convex body K in $\mathbb{R}^n$ is a Borel measure S_K on S^{n-1} called the Aleksandrov-Fenchel-Jessen surface area measure of K, defined by

$$S_K(\omega) = \mathcal{H}^{n-1}\left(v_K^{-1}(\omega)\right), \tag{62}$$

for each Borel set $\omega \subseteq S^{n-1}$; i.e., $S_K(\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all points on ∂K that have a unit normal that lies in ω.

The set of convex bodies will be viewed as equipped with the Hausdorff metric and thus a sequence of convex bodies, K_i, is said to converge to a body K, i.e.,

$$\lim_{i\to\infty} K_i = K,$$

provided that their support functions converge in $C(S^{n-1})$, with respect to the max-norm, i.e.,

$$\left\|h_{K_i} - h_K\right\|_\infty \to 0.$$

We shall make use of the weak continuity of surface area measures; i.e., if K is a convex body and K_i is a sequence of convex bodies then

$$\lim_{i\to\infty} K_i = K \Longrightarrow \lim_{i\to\infty} S_{K_i} = S_K, \tag{63}$$

weakly.

Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior. The cone-volume measure V_K of K is a Borel measure on the unit sphere S^{n-1} defined for a Borel $\omega \subseteq S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n}\int_{x\in v_K^{-1}(\omega)} x \cdot v_K(x) d\mathcal{H}^{n-1}(x), \tag{64}$$

and thus

$$dV_K = \frac{1}{n} h_K\, dS_K. \tag{65}$$

Since,

$$V(K) = \frac{1}{n}\int_{u\in S^{n-1}} h_K(u) dS_K(u), \tag{66}$$

we can turn the cone-volume measure into a probability measure on the unit sphere by normalizing it by the volume of the body. The cone-volume probability measure $\bar{V}_K$ of K is defined

$$\bar{V}_K = \frac{1}{V(K)} V_K. \tag{67}$$

Suppose K, L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. For $p \neq 0$, the L_p-mixed volume $V_p(K,L)$ can be defined as

$$V_p(K,L) = \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p dV_K. \tag{68}$$

We need the normalized L_p-mixed volume $\bar{V}_p(K,L)$, which was first defined in [43],

$$\bar{V}_p(K,L) = \left(\frac{V_p(K,L)}{V(K)}\right)^{\frac{1}{p}} = \left(\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p d\bar{V}_K\right)^{\frac{1}{p}}.$$

Letting $p \to 0$ gives

$$\bar{V}_0(K,L) = exp\left(\int_{S^{n-1}} log\frac{h_L}{h_K} d\bar{V}_K\right),$$

which is the normalized log-mixed volume of K and L. Obviously, from Jensen's inequality we know that $p \longmapsto \bar{V}_p(K,L)$ is strictly monotone increasing, unless h_L/h_K is constant on $\sup p\, S_K$.

Suppose that the function $k_t(u) = k(t,u)$: $I \times S^{n-1} \to (0,1)$ is continuous, where $I \subset \mathbb{R}$ is an interval. For fixed $t \in I$, let

$$K_t = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \le k(t,u)\}$$

be the Wulff shape (or Aleksandrov body) associated with the function k_t. We shall make use of the well-known fact that

$$h_{K_t} \le k_t \quad and \quad h_{K_t} = k_t, a.e.w.r.t.S_{K_t}, \tag{69}$$

for each $t \in I$. If kt happens to be the support function of a convex body then $h_{K_t} = k_t$, everywhere.

The following lemma (proved in e.g. [323]) will be needed.

Lemma (6.2.9) [321]: Suppose $k(t,u): I \times S^{n-1} \to (0,1)$ is continuous, where $I \subset \mathbb{R}$ is an open interval. Suppose also that the convergence in

$$\frac{\partial k(t,u)}{\partial t} = \lim_{s \to 0} \frac{k(t+s,u) - k(t,u)}{s}$$

is uniform on S^{n-1}. If $\{K_t\}_{t \in I}$ is the family of Wulff shapes associated with k_t, then

$$\frac{dV(K_t)}{dt} = \int_{S^{n-1}} \frac{\partial k(t,u)}{\partial t} dS_{K_t}(u).$$

Suppose K, L are convex bodies in $\mathbb{R}^n$. The inradius $r(K,L)$ and outradius $R(K,L)$ of K with respect to L are defined by

$$r(K,L) = \sup\{t > 0 : x + tL \subset K \ and \ x \in \mathbb{R}^n\},$$
$$R(K,L) = \inf\{t > 0 : x + tL \,.\, K \ and \ x \in \mathbb{R}^n\}.$$

If L is the unit ball, then $r(K,L)$ and $R(K,L)$ are the radii of maximal inscribable and minimal circumscribable balls of K, respectively. Obviously from the definition, it follows that

$$r(K,L) = \frac{1}{R(L,K)}. \tag{70}$$

If K, L happen to be origin-symmetric convex bodies, then obviously

$$r(K,L) = \lim_{u \in S^{n-1}} \frac{h_K(u)}{h_L(u)} \ and \ R(K,L) = \max_{u \in S^{n-1}} \frac{h_K(u)}{h_L(u)}. \tag{71}$$

It will be convenient to always translate K so that for $0 \le t < r = r(K,L)$, the function $k_t = h_K - th_L$ is strictly positive. Let K_t denote the Wulff shape associated with the function k_t; i.e., let K_t be the convex body given by

$$K_t = \{x \in \mathbb{R}^n : x \cdot u \le h_K(u) - th_L(u) for \ all \ u \in S^{n-1}\}. \tag{72}$$

Note that $K_0 = K$, and that obviously

$$\lim_{t \to 0} K_t = K_0 = K.$$

From definition (72) and (61) we immediately have

$$K_t = \{x \in \mathbb{R}^n : x + tL \subseteq K\}. \tag{73}$$

Using (73) we can extend the definition of K_t for the case where $t = r = r(K,L)$:

$$K_r = \{x \in \mathbb{R}^n : x + rL \subseteq K\}.$$

It is not hard to show (see e.g. the proof of (6.5.11) in [334]) that K_r is a degenerate convex set (i.e. has empty interior) and that

$$\lim_{t \to r} V(K_t) = V(K_r) = 0. \tag{74}$$

From Lemma (6.2.9) and (68), we obtain the well-known fact that for $0 < t < r = r(K, L)$,

$$\frac{d}{dt} V(K_t) = -nV_1(K_t, L). \qquad (75)$$

Integrating both sides of (75), and using (74), gives

Lemma (6.2.10) [321]: Suppose K and L are convex bodies, and for $0 \le t < r = r(K, L)$, the body K_t is the Wulff shape associated with the positive function $k_t = h_K - th_L$. Then, for $0 \le t \le r = r(K, L)$,

$$V(K) - V(K_t) = n \int_0^t V_1(K_s, L) ds, \qquad (76)$$

where $K_r = \{x \in \mathbb{R}^n : x + rL \subseteq K\}$.

we show that for each fixed $p \ge 0$ the L_p-Brunn-Minkowski inequality and the L_p-Minkowski inequality are equivalent in that one is an easy consequence of the other. In particular, the log-Brunn-Minkowski inequality and the log-Minkowski inequality are equivalent.

Suppose $p > 0$. If K and L are convex bodies that contain the origin and $s, t \ge 0$ (not both zero) the L_p-Minkowski combination $s \cdot K +_p t \cdot L$, is defined by

$$s \cdot K +_p t \cdot L = \{x \in \mathbb{R}^n : x \cdot u \le (sh_K(u)^p + th_L(u)^p)^{1/p} \; for \; all \; u \in S^{n-1}\}.$$

We see that for a convex body K and real $s \ge 0$ the relationship between the L_p-scalar multiplication, $s \cdot K$, and Minkowski scalar multiplication sK is given by:

$$s \cdot K = s^{\frac{1}{p}} K.$$

Suppose $p > 0$ is fixed and suppose the following "weak" L_p-BrunnMinkowski inequality holds for all origin-symmetric convex bodies K and L in $\mathbb{R}^n$ such that $V(K) = 1 = V(L)$:

$$V\left((1-\lambda) \cdot K +_p \lambda \cdot \bar{L}\right) \ge 1, \qquad (77)$$

for all $\lambda \in (0, 1)$. We claim that from this it follows that the following seemingly "stronger"

L_p-Brunn-Minkowski inequality holds: If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then

$$V(s \cdot K +_p t \cdot L)^{\frac{p}{n}} \ge sV(K)^{\frac{p}{n}} + tV(L)^{\frac{p}{n}}, \qquad (78)$$

for all $s, t \ge 0$. To see this assume that the "weak" L_p-Brunn-Minkowski inequality (77) holds and that K and L are arbitrary origin-symmetric convex bodies. Let $\bar{K} = V(K)^{-\frac{1}{n}} K$ and $\bar{L} = V(L)^{-\frac{1}{n}} L$. Then (77) gives

$$V\left((1-\lambda) \cdot \bar{K} +_p \lambda \cdot \bar{L}\right) \ge 1. \qquad (79)$$

Let $\lambda = \frac{V(L)^{\frac{p}{n}}}{V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}}$. Then

$$(1-\lambda) \cdot \bar{K} +_p \lambda \cdot \bar{L} = \frac{1}{\left(V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}\right)^{\frac{1}{p}}} (K +_p L).$$

Therefore, from (79), we get

$$V(K+_p L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}.$$

If we now replace K by $s \cdot K$ and L by $t \cdot L$ and note that $V(s \cdot K)^{\frac{p}{n}} = sV(K)^{\frac{p}{n}}$, we obtain the desired "stronger" L_p-Brunn-Minkowski inequality (78).

Lemma (6.2.11) [321]: Suppose $p > 0$. For origin symmetric convex bodies in $\mathbb{R}^n$, the L_p-Brunn-Minkowski inequality (56) and the L_p-Minkowski inequality (57) are equivalent.

Proof: Suppose K and L are fixed origin-symmetric convex bodies in $\mathbb{R}^n$. For $0 \leq \lambda \leq 1$, let

$$Q_\lambda = (1-\lambda) \cdot K +_p \lambda \cdot L;$$

i.e., Q_λ is the Wulff shape associated with the function $q_\lambda = ((1-\lambda)h_K^p + \lambda h_L^p)^{\frac{1}{p}}$. It will be convenient to consider q_λ as being defined for λ in the open interval $(-\epsilon_o, 1+\epsilon_o)$, where $\epsilon_o > 0$ is chosen so that for $\lambda \in (-\epsilon_o, 1+\epsilon_o)$, the function q_λ is strictly positive. We first assume that the L_p-Minkowski inequality (57) holds. From (66), the fact that $h_{Q_\lambda} = ((1-\lambda)h_K^p + \lambda h_L^p)^{\frac{1}{p}}$ $a.e.$ with respect to the surface area measure S_{Q_λ}, (65) and (68), and finally the L_p-Minkowski inequality (57), we have

$$\begin{aligned} V(Q_\lambda) &= \frac{1}{n}\int_{S^{n-1}} h_{Q_\lambda} dS_{Q_\lambda} \\ &= \frac{1}{n}\int_{S^{n-1}} ((1-\lambda)h_K^p + \lambda h_L^p) h_{Q_\lambda}^{1-p} dS_{Q_\lambda} \\ &= (1-\lambda)V_p(Q_\lambda, K) + \lambda V_p(Q_\lambda, L) \\ &\geq (1-\lambda)V(Q_\lambda)^{\frac{n-p}{n}} V(K)^{\frac{p}{n}} + \lambda V(Q_\lambda)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}. \end{aligned} \tag{80}$$

This gives,

$$V(Q_\lambda) \geq \left((1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}\right)^{n/p} \geq V(K)^{1-\lambda} V(L)^{\lambda}, \tag{81}$$

which is the L_p-Brunn-Minkowski inequality (56).

Now assume that the L_p-Brunn-Minkowski inequality (56) holds. As was seen at the beginning, this inequality (in fact a seemingly weaker one) implies the seemingly stronger L_p-Brunn-Minkowski inequality (78). But this inequality tells us that the function $f: [0,1] \to (0,\infty)$, given by $f(\lambda) = V(Q_\lambda)^{\frac{p}{n}}$ for $\lambda \in [0,1]$, is concave.

The convex body Q_λ is the Wulff shape of the function $q_\lambda = ((1-\lambda)h_K^p + \lambda h_L^p)^{1/p}$. Now, the convergence as $\lambda \to 0$ in

$$\frac{q_\lambda - q_0}{\lambda} \longrightarrow \frac{h_K^{1-p}}{p}(h_L^p - h_K^p) = \frac{h_K^{1-p} h_L^p - h_K}{p},$$

is uniform on S^{n-1}. By Lemma (6.2.9), (65) and (68), and (66),

$$\left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0} = \int_{S^{n-1}} \frac{h_K^{1-p} h_L^p - h_K}{p} dS_K = \frac{n}{p}[V_p(K,L) - V(K)].$$

Therefore, the concavity of f yields

$$V(K)^{\frac{p-n}{n}}(V_p(K,L)-V(K))=f'(0)\geq f(1)-f(0)=V(L)^{\frac{p}{n}}-\ V(K)^{\frac{p}{n}},$$
which gives the L_p-Minkowski inequality (57).

Lemma (6.2.12) [321]: For origin symmetric convex bodies in $\mathbb{R}^n$, the log-Brunn-Minkowski inequality (49) and the log-Minkowski inequality (50) are equivalent.

Proof: Suppose K and L are fixed origin-symmetric convex bodies in $\mathbb{R}^n$. For $0\leq\lambda\ \leq 1$, let

$$Q_\lambda=(1-\lambda)\cdot K+_o\lambda\cdot L;$$

i.e., Q_λ is the Wulff shape associated with the function $q_\lambda=h_K^{1-\lambda}h_L^\lambda$. It will be convenient to consider q_λ as being defined for all λ in the open interval $(-\epsilon_o,1+\ \epsilon_o)$, for some sufficiently small $\epsilon_o>0$ and let Q_λ be the Wulff shape associated with the function q_λ. Observe that since q_0 and q_1 are the support functions of convex bodies, $Q_0=K$ and $Q_1=L$.

First suppose that we have the log-Minkowski inequality (50) for K and L. Now $h_{Q_\lambda}=h_K^{1-\lambda}h_L^\lambda$ a.e. with respect to S_{Q_λ}, and thus,

$$0=\frac{1}{nV(Q_\lambda)}\int_{S^{n-1}}h_{Q_\lambda}log\frac{h_K^{1-\lambda}h_L^\lambda}{h_{Q_\lambda}}dS_{Q_\lambda}$$
$$=(1-\lambda)\frac{1}{nV(Q_\lambda)}\int_{S^{n-1}}h_{Q_\lambda}log\frac{h_K}{h_{Q_\lambda}}dS_{Q_\lambda}+\lambda\frac{1}{nV(Q_\lambda)}\int_{S^{n-1}}h_{Q_\lambda}log\frac{h_L}{h_{Q_\lambda}}dS_{Q_\lambda}$$
$$\geq(1-\lambda)\frac{1}{n}log\frac{V(K)}{V(Q_\lambda)}+\lambda\frac{1}{n}log\frac{V(L)}{V(Q_\lambda)}\qquad(82)$$
$$=\frac{1}{n}log\frac{V(K)^{1-\lambda}V(L)^\lambda}{V(Q_\lambda)}.$$

This gives the log-Brunn-Minkowski inequality (49) .

Suppose now that we have the log-Brunn-Minkowski inequality (49) for K and L. The body Q_λ is the Wulff shape associated wit the function $q_\lambda=h_K^{1-\lambda}h_L^\lambda$, and the convergence as $\lambda\to 0$ in

$$\frac{q_\lambda-q_0}{\lambda}\longrightarrow h_K\ log\frac{h_L}{h_K},$$

is uniform on S^{n-1}. By Lemma (6.2.9),

$$\left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0}=\int_{S^{n-1}}h_K\ log\frac{h_L}{h_K}dS_K\qquad(83)$$

But the log-Brunn-Minkowski inequality (49) tells us that $\lambda\longmapsto logV(Q_\lambda)$ is a concave function, and thus

$$\frac{1}{V(Q_0)}\left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0}\geq V(Q_1)-V(Q_0)=log\ V(L)-logV(K).\qquad(84)$$

When (83) and (84) are combined the result is the log-Minkowski inequality (50).

We shall work exclusively in the Euclidean plane. We will make use of the properties of mixed-volumes of compact convex sets, some of which might possibly be degenerate (i.e. lower-dimensional).

Suppose K, L are plane compact convex sets. Of fundamental importance is the fact that for real $s, t \geq 0$, the area, $V(sK + tL)$, of the Minkowski linear combination $sK + tL = \{sx + ty : x \in K \; and \; y \in L\}$ is a homogeneous polynomial of degree 2 in s and t:
$$V(sK + tL) = s^2V(K) + 2stV(K, L) + t^2V(L). \qquad (85)$$
The coefficient $V(K, L)$, the mixed area of K and L, is uniquely defined by (85) if we require (as we always will) it to be symmetric in its arguments; i.e.
$$V(K, L) = V(L, K). \qquad (86)$$
From its definition, we see that the mixed area functional $V(\cdot,\cdot)$ is obviously invariant under independent translations of its arguments. Obviously, for each K,
$$V(K, K) = V(K). \qquad (87)$$
The mixed area of K, L is just the mixed volume $V_1(K, L)$ in the plane and thus (from (68) we see) it has the integral representation
$$V(K, L) = \frac{1}{2}\int_{S^1} h_L(u)dS_K(u). \qquad (88)$$
If K is degenerate with $K = \{su: -c \leq s \leq c\}$, where $u \in S^1$ and $c > 0$, then S_K is an even measure concentrated on the two point set $\{\pm u^\perp\}$ with total mass $4c$.
From (85), or from (88), we see that for plane compact convex K, L, L' and real $s, s' \geq 0$,
$$V(K, sL + s'L') = sV(K, L) + s'V(K, L'). \qquad (89)$$
But this, together with (86), shows that the mixed area functional $V(\cdot,\cdot)$ is linear with respect to Minkowski linear combinations in both arguments.
From (88) we see that for plane compact convex K, L, L', we have
$$L \subseteq L' \Longrightarrow V(K, L) \leq V(K, L'),$$
with equality if and only if $h_L = h_{L'}$ a.e. w.r.t. S_K (90)

The basic inequality, inequality (91), is Blaschke's extension of the Bonnesen inequality. It was proved using integral geometric techniques. It has been a valuable tool used to establish various isoperimetric inequalities, see e.g., [325], [326], Since the equality conditions of inequality (91) are one of the critical ingredients in the proof of the log-Brunn-Minkowski inequality, we present a complete proof of inequality (91), with its equality conditions.

Theorem (6.2.13) [321]: If K, L are plane convex bodies, then for $r(K, L) \leq t \leq R(K, L)$,
$$V(K) - 2tV(K, L) + t^2V(L) \leq 0. \qquad (91)$$
The inequality is strict whenever $r(K, L) < t < R(K, L)$. When $t = r(K, L)$, equality will occur in (91) if and only if K is the Minkowski sum of a dilation of L and a line segment. When $t = R(K, L)$, equality will occur in (91) if and only if L is the Minkowski sum of a dilation of K and a line segment.

Proof: Let $r = r(K, L)$ and suppose $t \in [0, r]$. Recall from (72) that
$$K_t = \{x \in \mathbb{R}^n: x \cdot u \leq h_K(u) - th_L(u) \; for \; all \; u \in S^{n-1}\},$$
and that from (73), we have
$$K_t + tL \subseteq K. \qquad (92)$$
But (92), together with the monotonicity (90), linearity (89), and symmetry (86) of mixed volumes, together with (87) gives
$$V(K, L) \geq V(K_t + tL, L) = V(K_t, L) + tV(L). \qquad (93)$$

Now Lemma (6.2.10) and (93) gives,

$$V(K) - V(K_t) = 2\int_0^t V(K_s, L)ds$$
$$\leq 2\int_0^t \big(V(K,L) - sV\,(L)\big)ds \qquad (94)$$
$$= 2tV(K,L) - t^2V(L).$$

Thus,

$$V(K) - 2tV(K,L) + t^2V(L) \leq V(K_t). \qquad (95)$$

From (93) and (94) we see that equality holds in (95) if and only if,

$$V(K,L) = V(K_s + sL, L), \quad \text{for all } s \in [0,t] \qquad (96),$$

which, from (92) and (90), gives

$$h_K = h_{K_s} + sh_L, \quad a.e.w.r.t.S_L$$

for all $s \in [0,t]$.

By (74) we know $V(K_r) = 0$ and thus K_r is a line segment, possibly a single point. Therefore, from (95) we have

$$V(K) - 2rV(K,L) + r^2V(L) \leq 0. \qquad (97)$$

We will now establish the equality conditions in (97). To that end, suppose:

$$V(K) - 2rV(K,L) + r^2V(L) = 0. \qquad (98)$$

Then by (96) we have,

$$V(K,L) = V(K_r + rL, L).$$

But this in (98) gives:

$$V(K) - 2rV(K_r + rL, L) + r^2V(L) = 0,$$

which, using (89), can be rewritten as

$$V(K) - 2rV\,(K_r, L) - \,r^2V(L) = 0,$$

and since $V(K_r) = 0$ can be written, using (89), as

$$V(K) - V(K_r + rL) = 0.$$

Since $K_r + rL \subseteq K$, the equality of their volumes forces us to conclude that in fact $K_r + rL = K$.

Therefore, K is the Minkowski sum of a dilation of L and the line segment K_r (which may be a point).

Since $1/R(K,L) = r(L,K)$ from (71), from inequality (97), and its established equality conditions, we get

$$V(L) - 2r'V\,(L,K) + r'^2V\,(K) \leq 0, \quad \text{where } r' = r(L,K) = 1/R(K,L),$$

with equality if and only if L is the Minkowski sum of a dilation of K and a line segment. But, using the symmetry of mixd volumes (86), this means that

$$V(K) - 2RV(K,L) + R^2V(L) \leq 0, \text{ where } R = R(K,L), \qquad (99)$$

with equality if and only if L is the Minkowski sum of a dilation of K and a line segment. Finally, inequalities (97) and (99) together with the well-known properties of quadratic functions show that

$$V(K) - 2tV(K,L) + t^2V(L) < 0, \text{ whenever } r(K,L) < t < R(K,L).$$

Given a finite Borel measure on the unit sphere, under what necessary and sufficient conditions is the measure the cone-volume measure of a convex body? This is the unsolved

log-Minkowski problem. It requires solving a Monge-Ampere equation and is connected with some important curvature flows (see e.g. [233], [334], [336]). Uniqueness for the log-Minkowski problem is more difficult than existence. Even in the plane, the uniqueness of cone volume measure has not been settled. If the cone-volume measure is that of a smooth origin-symmetric convex body that has positive curvature, uniqueness for plane convex bodies was established by Gage [334] and in the case of even, discrete, measures in the plane is treated by Stancu [336].
We shall establish the uniqueness of cone-volume measure for arbitrary symmetric plane convex bodies. For non-symmetric plane convex bodies the problem remains both open and important.
The uniqueness of cone-volume measure is related to Firey's worn stone problem. In determining the ultimate shape of a worn stone, Firey [331] showed that if the cone-volume measure of a smooth origin-symmetric convex body in $\mathbb{R}^n$ is a constant multiple of the Lebesgue measure (on S^{n-1}), then the convex body must be a ball. This established uniqueness for the worn stone problem for the symmetric case. In $\mathbb{R}^3$, Andrews [332] established the uniqueness of solutions to the worn stone problem by showing that a smooth (not necessarily symmetric) convex body in $\mathbb{R}^3$ must be a ball if its cone volume measure is a constant multiple of Lebesgue measure on S^2.
The following inequality (100) was established by Gage [14] when the convex bodies are smooth and of positive curvature. A limit process gives the general case, but the equality conditions do not follow. As will be seen, the equality conditions are critical for establishing the uniqueness of cone-volume measures in the plane.
Lemma (6.2.14) [321]: If K, L are origin-symmetric plane convex bodies, then

$$\int_{S^1} \frac{h_K^2}{h_L} dS_K \leq \frac{V(K)}{V(L)} \int_{S^1} h_L \, dS_K, \tag{100}$$

with equality if and only if K and L are dilates, or K and L are parallelograms with parallel sides.
Proof: Since K and L are origin symmetric, from (71) we have

$$r(K,L) \leq \frac{h_K(u)}{h_L(u)} \leq R(K,L),$$

for all $u \in S^1$. Thus, from Theorem (6.2.13) we get

$$V(K) - 2\frac{h_K(u)}{h_L(u)} V(K,L) + \left(\frac{h_K(u)}{h_L(u)}\right)^2 V(L) \leq 0.$$

Integrating both sides of this, with respect to the measure $h_L dS_K$, and using (88) and (66), gives

$$0 \geq \int_{S^1} \left(V(K) - 2\frac{h_K(u)}{h_L(u)} V(K,L) + \left(\frac{h_K(u)}{h_L(u)}\right)^2 V(L) \right) h_L(u) dS_K(u)$$

$$= -2V(K)V(K,L) + V(L) \int_{S^1} \frac{h_K(u)^2}{h_L(u)} dS_K(u).$$

This yields the desired inequality (100).

Suppose there is equality in (100). Thus,

$$V(K)-2\frac{h_K(u)}{h_L(u)}V(K,L)+\left(\frac{h_K(u)}{h_L(u)}\right)^2V(L)=0, for\ all\ u\in supp\ S_K. \quad (101)$$

If K and L are dilates, we're done. So assume that K and L are not dilates. But $K\neq L$ implies that $r(K,L)<\ R(K,L)$. From Theorem (6.2.13), we know that when

$$r(K,L)<\frac{h_K(u)}{h_L(u)}<R(K,L),$$

it follows that

$$V(K)-2\frac{h_K(u)}{h_L(u)}V(K,L)+\left(\frac{h_K(u)}{h_L(u)}\right)^2V(L)<0,$$

and thus we conclude that

$$h_K(u)/h_L(u)\in\{r(K,L),R(K,L)\} \text{ for all } u\in supp\ S_K \quad (102).$$

Note that since K is origin symmetric $supp\ S_K$ is origin symmetric as well. Either there exists $u'\in supp\ S_K$ so that $h_K(u_0)/h_L(u_0)=r(K,L)$ or $h_K(u_0)/h_L(u_0)\ =R(K,L)$. Suppose that $h_K(u_0)/h_L(u_0)=r(K,L)$. Then from (101) and the equality conditions of Theorem (6.2.13) we know that K must be a dilation of the Minkowski sum of L and a line segment. But K and L are not dilaltes, so there exists an $x_0\neq 0$ so that

$$h_K(u)=|x_0\cdot u|+r(K,L)h_L(u),$$

for all unit vectors u.This together with $h_K(u_0)/h_L(u_0)=r(K,L)$ shows that x_0 is orthogonal to u_0 and that the only unit vectors at which $h_K/h_L=r(K,L)$ are u_0 and $-u_0$. But $supp\ S_K$ must contain at least one unit vector $u_1\in supp\ S_K$ other than $\pm u_0$. From (102), and the fact that the only unit vectors at which $h_K/h_L=r(K,L)$ are u_0 and $-u_0$, we conclude $h_K(u_1)/h_L(u_1)=R(K,L)$ and by the same argument we conclude that the only unit vectors at which $h_K/h_L=R(K,L)$ are u_1 and $-u_1$. Now (102) allows us to conclude that

$$supp\ S_K=\{\pm u_0,\pm u_1\}.$$

This implies that K is a parallelogram. Since K is the Minkowski sum of a dilate of L and a line segment, L must be a parallelogram with sides parallel to those of K. If we had assumed that $h_K(u_0)/h_L(u_0)=R(K,L)$, rather than $r(K,L)$, the same argument would lead to the same conclusion.

It is easily seen that the equality holds in (100) if K and L are dilates. A trivial calculation shows that equality holds in (100) if K and L are parallelograms with parallel sides.

The following theorem was established by Gage [325] when the convex bodies are smooth and have positive curvature. When the convex bodies are polytopes it is due to Stancu [326].

Theorem (6.2.15) [321]: If K and L are plane origin-symmetric convex bodies that have the same conevolume measure, then either $K=L$ or else K and L are parallelograms with parallel sides.

Proof: Assume that $K\neq L$. Since

$$V_K=V_L,$$

it follows that $V(K) = V(L)$. Thus, since $K \neq L$, the bodies cannot be dilates. Thus inequality (100) becomes

$$\int_{S^1} \frac{h_L}{h_K} dV_K \geq \int_{S^1} \frac{h_K}{h_L} dV_K \ \ and \ \ \int_{S^1} \frac{h_K}{h_L} dV_L \geq \int_{S^1} \frac{h_L}{h_K} dV_L, \qquad (103)$$

with equality, in either inequality, if and only if K and L are parallelograms with parallel sides. Using (103) and the fact that $V_K = V_L$, both twice, we get

$$\int_{S^1} \frac{h_L(u)}{h_K(u)} dV_K(u) \geq \int_{S^1} \frac{h_K(u)}{h_L(u)} dV_K(u)$$
$$= \int_{S^1} \frac{h_K(u)}{h_L(u)} dV_L(u)$$
$$\geq \int_{S^1} \frac{h_L(u)}{h_K(u)} dV_l(u)$$
$$= \int_{S^1} \frac{h_L(u)}{h_K(u)} dV_K(u).$$

Thus, we have equality in both inequalities of (103) and from the equality conditions of (103) we conclude that K and L are parallelograms with parallel sides.

Lemma (6.2.16) [321]: Suppose K is a plane origin-symmetric convex body, with $V(K) = 1$, that is not a parallelogram. Suppose also that P_k is an unbounded sequence of origin-symmetric parallelograms all of which have orthogonal diagonals, and such that $V(P_k) \geq 2$. Then, the sequence

$$log\ h_{P_k}(u)\ dV_K(u)$$

is not bounded from above.

Proof: Let $u_{1,k}, u_{2,k}$ be orthogonal unit vectors along the diagonals of P_k. Denote the vertices of P_k by $\pm h_{1,k}u_{1,k}, \pm h_{2,k}u_{2,k}$. Without loss of generality, assume that $0 < h_{1,k} \leq h_{2,k}$. The condition $V(P_k) \geq 2$ is equivalent to $h_{1,k}h_{2,k} \geq 1$. The support function of P_k is given by

$$h_{P_k}(u) = \max\{h_{1,k}|u \cdot u_{1,k}|, h_{2,k}|u \cdot u_{2,k}|\}, \qquad (104)$$

for $u \in S^1$. Since S^1 is compact, the sequences u_1, k and u_2, k have convergent subsequences. Again, without loss of generality, we may assume that the sequences u_1, k and u_1, k are themselves convergent with

$$\lim_{k\to\infty} u_{1,k} = u_1 \text{ and } \qquad \lim_{k\to\infty} u_{2,k} = u_2,$$

where u_1 and u_2 are orthogonal.

It is easy to see that if the cone-volume measure, $V_K(\{\pm u_1\})$, of the two-point set $\{\pm u_1\}$ is positive, then K contains a parallelogram whose area is $2V_K(\{\pm u_1\})$. Since K itself is not a parallelogram and $V(K) = 1$, it must be the case that

$$V_K(\{\pm u_1\}) < \tfrac{1}{2}. \qquad (105)$$

For $\delta \in (0, \frac{1}{3})$, consider the neighborhood, U_δ, of $\{\pm u_1\}$, on S^1,

$$U_\delta = \{u \in S^1 : |u \cdot u_1| > 1 - \delta\}.$$

Since $V_K(S^1) = V(K) = 1$, we see that for all or $\delta \in (0, \frac{1}{3})$

$$V_K(U_\delta) + V_K(U_\delta^c) = 1, \qquad (106)$$

where U_δ^c is the complement of U_δ.

Since the U_δ are decreasing (with respect to set inclusion) in δ and have a limit of $\{\pm u_1\}$,

$$\lim_{\delta \to 0^+} V_K(U_\delta) = V_K(\{\pm u_1\}).$$

This together with (105), shows the existence of a $\delta_\circ > 0$ such that

$$V_K(U_{\delta_o}) < \frac{1}{2}.$$

But this implies that there is a small $\epsilon_o \in (0, \frac{1}{2})$ so that

$$\tau_o = V_K\big(U_{\delta_o}\big) - \tfrac{1}{2} + \epsilon_o < 0. \qquad (107)$$

This together with (106) gives

$$V_K(U_{\delta_o}) = \tfrac{1}{2} - \epsilon_o + \tau_o \qquad \text{and} \qquad V_K\big(U_{\delta_o}^c\big) = \tfrac{1}{2} + \epsilon_o - \tau_o. \qquad (108)$$

Since u_{ik} converge to u_i, we have, $|u_{ik} - u_i| < \delta_o$ whenever k is sufficiently large (for both $k = 1$ and $k = 2$). Then for $u \in U_{\delta_o}$ and k sufficiently large, we have

$$\begin{aligned} |u \cdot u_{1,k}| &\geq |u \cdot u_1| - |u \cdot (u_{1,k} - u_1)| \\ &\geq |u \cdot u_1| - |u_{1,k} - u_1| \\ &\geq 1 - \delta_o - \delta_o \\ &\geq \delta_o, \end{aligned}$$

where the last inequality follows from the fact that $\delta_o < \frac{1}{3}$. For all $u \in S^1$, we know that $|u \cdot u_1|^2 + |u \cdot u_2|^2 = 1$. Thus, for $\in U_{\delta_o}^c$, we have $|u \cdot u_2| > (1 - (1 - \delta_o)^2)^{\frac{1}{2}} > 2\delta_o$, which shows that when k is sufficiently large,

$$\begin{aligned} |u \cdot u_{2,k}| &\geq |u \cdot u_2| - |u \cdot (u_{2,k} - u_2)| \\ &\geq |u \cdot u_2| - |u_{2,k} - u_2| \\ &\geq 2\delta_o - \delta_o \\ &= \delta_o. \end{aligned}$$

From the last paragraph and (104) it follows that when k is sufficiently large,

$$h_{P_k}(u) \geq \begin{cases} \delta_o h_{1,k} & if\ u \in U_{\delta_o}, \\ \delta_o h_{2,k} & if\ \in U_{\delta_o}^c. \end{cases} \qquad (109)$$

By (109) and (106), (108), the fact that $0 < h_{1,k} \leq h_{2,k}$ together with (107), and finally the fact that $h_{1,k}h_{2,k} \geq 1$ together with $\epsilon_o \in (0, \frac{1}{3})$, we see that for sufficiently large k,

$$\begin{aligned} \int_{S^1} log\ h_{P_k} dV_K &= \int_{U_{\delta_o}} log\ h_{P_k}\, dV_K + \int_{U_{\delta_o}^c} log\ h_{P_k} dV_K \\ &\geq log\ \delta_o + V_K(U_{\delta_o})\ log\ h_{1,k} + V_K(U_{\delta_o}^c)\ log\ h_{2,k} \\ &= log\delta_o + (\frac{1}{2} + \tau_o - \epsilon_o)\ log\ h_{1,k} + (\frac{1}{2} - \tau_o + \epsilon_o)\ log\ h_{2,k} \\ &= log\delta_o + 2\epsilon_o log\ h_{2,k} + (\frac{1}{2} - \epsilon_o)\ log(h_{1,k}h_{2,k}) + \tau_o(log\ h_{1,k} - log\ h_{2,k}) \\ &\geq log\delta_o + 2\epsilon_o\ log\ h_{2,k}. \end{aligned}$$

Since P_k is not bounded, the sequence $h_{2,k}$ is not bounded from above. Thus, the sequence

$$\int_{S^1} log\ h_{P_k} dV_K$$

is not bounded from above.

Lemma (6.2.17) [321]: If K is a plane origin-symmetric convex body that is not a parallelogram, then there exists a plane origin-symmetric convex body K_0 so that $V(K_0) = 1$ and

$$log\ h_Q\ dV_K \geq log\ h_{K_0}\ dV_K$$

for every plane origin-symmetric convex body Q with $V(Q) = 1$.

Proof: Obviously, we may assume that $V(K) = 1$. Consider the minimization problem,

$$inf \int_{S^1} log\ h_Q\ dV_K$$

where the infimum is taken over all plane origin-symmetric convex bodies Q with $V(Q) = 1$. Suppose that Q_k is a minimizing sequence; i.e., Q_k is a sequence of origin-symmetric convex bodies with $V(Q_k) = 1$ and such that $\int_{S^1} log\ h_{Q_k}\ dV_k$ tends to the infimum (which may be $-\infty$).

We shall show that the sequence Q_k is bounded and the infimum is finite.

By John's Theorem, there exist ellipses E_k centered at the origin so that

$$E_k \subset Q_k \subset \sqrt{2} E_k. \qquad (110)$$

Let $u_{1,k}, u_{2,k}$, be the principal directions of E_k so that

$$h_{1,k} \leq h_{2,k}, \quad where \quad h_{1,k} = h_{E_k}(u_1, k) \quad and\ h_{2,k} = h_{E_k}(u_2, k).$$

Let P_k be the origin-centered parallelogram that has vertices $\{\pm h_{1,k} u_{1,k}, \pm h_{2,k} u_{2,k}\}$. (Observe that by the Principal Axis Theorem the diagonals of P_k are perpendicular.)

Because of $E_k \subset \sqrt{2} P_k$, it follows from (110) that

$$P_k \subset Q_k \subset 2P_k. \qquad (111)$$

From this and $V(Q_k) = 1$, we see that $V(P_k) \geq \frac{1}{4}$.

Assume that Q_k is not bounded. Then P_k is not bounded. Applying Lemma (6.2.16) to $\sqrt{8} P_k$ shows that the sequence $\int_{S^1} log\ h_{P_k} dV_k$ is not bounded from above. Therefore, from (111) we see that the sequence $\int_{S^1} log\ h_{Q_k} dV_k$ cannot be bounded from above. But this is impossible because Q_k was chosen to be a minimizing sequence.

We conclude that Q_k is bounded. By the Blaschke Selection Theorem, Q_k has a convergent subsequence that converges to an origin-symmetric convex body K_0, with $V(K_0) = 1$. It follows that $\int_{S^1} log\ h_{K_0}\ dV_K$ is the desired infimum.

We repeat the statement of Theorem (6.2.4):

Theorem (6.2.18) [321]: If K and L are plane origin-symmetric convex bodies, then

$$\int_{S^1} log \frac{h_L}{h_K} d\ \bar{V}_K \geq \frac{1}{2} log \frac{V(L)}{V(K)},$$

with equality if and only if either K and L are dilates or when K and L are parallelograms with parallel sides.

Proof: Without loss of generality, we can assume that $V(K) = V(L) = 1$. We shall establish the theorem by proving

$$log\ h_L\ dV_K \ \geq log\ h_K\ dV_K,$$

with equality if and only if either K and L are dilates or if they are parallelograms with parallel sides.

First, assume that K is not a parallelogram. Consider the minimization problem

$$min \int_{S^1} log\ h_Q dV_K\ ,$$

taken over all plane origin-symmetric convex bodies Q with $V(Q) = 1$. Let K_0 denote a solution, whose existence is guaranteed by Lemma (6.2.17). (Our aim is to prove that $K_0 = K$ and thereby demonstrate that K itself can be the only solution to this minimization problem.)

Suppose f is an arbitrary but fixed even continuous function. For some sufficiently small $\delta_o > 0$, consider the deformation of h_{K_0} , defined on $(-\delta_o, \delta_o) \times S^1$, by

$$q_t(u) = q(t, u) = h_{K_0}(u)e^{tf(u)}.$$

Let Q_t be the Wulff shape associated with q_t. Observe that Q_t is an origin symmetric convex body and that since q_0 is the support function of the convex body K_0, we have $Q_0 = K_0$.

Since K_0 is an assumed solution of the minimization problem, the function defined on $(-\delta_o, \delta_o)$ by

$$t \longmapsto V(Q_t)^{-\frac{1}{2}} \exp\left\{\int_{S^1} log\ h_{Q_t} dV_K\right\},$$

attains a minimal value at $t = 0$. Since $h_{Q_t} \leq q_t$ this function is dominated by the differentiable function defined on $(-\delta_o, \delta_o)$ by

$$t \longmapsto V(Q_t)^{-\frac{1}{2}}\ exp\left\{\int_{S^1} log\ q_t dV_K\right\}.$$

But clearly both functions have the same value at 0 and thus the latter function attains a local minimum at 0. Thus, differentiating the latter function at $t = 0$, by using Lemma (6.2.9), and recalling that $V(Q_0) = V(K_0) = 1$, shows that

$$-\frac{1}{2}\int_{S^1} h_{K_0}(u) f(u)\ dS_{K_0}(u)\ + \int_{S^1} f(u)\ dV_K(u) = 0.$$

Thus, since f was an arbitrary even function, we conclude that

$$\int_{S^1} f(u)\ dV_{K_0}(u) = f(u)\ dV_K(u)$$

for every even continuous f, and therefore,

$$V_K = V_{K_0}.$$

By Theorem (6.2.15), and the assumption that K is not a parallelogram, we conclude that $K_0 = K$.

Thus, for each L such that $V(L) = 1$,

$$\int_{S^1} log\, h_L\, dV_K \geq \int_{S^1} log\, h_K\, dV_K\,,$$

with equality if and only if $K = L$. This is the desired result when K is not a parallelogram. If K is a parallelogram the proof is trivial, but for the sake of completeness we shall include it.

Assume that K is the parallelogram whose support function, for $u \in S^1$, is given by

$$h_K(u) = a_1|v_1 \cdot u| + a_2|v_2 \cdot u|,$$

where $v_1, v_2 \in S^1$ and $a_1, a_2 > 0$. Then $suppS_K = \{\pm v_1^{\perp}, \pm v_2^{\perp}\}$, while $V_K(\{\pm v_i^{\perp}\}) = 2a_1a_2|v_1 \cdot v_2^{\perp}|$, and $|v_1 \cdot v_2^{\perp}| = |v_2 \cdot v_1^{\perp}|$. It is easily seen that $V(K) = 4a_1a_2|v_1 \cdot v_2| = 1$, and that

$$exp \int_{S^1} log\, h_L\, dV_K = \sqrt{h_L(v_1^{\perp})h_L(v_2^{\perp})}. \qquad (112)$$

Recall that $V(L) = 1$. The parallelogram circumscribed about L with sides parallel to those of K has volume

$$4h_L(v_1^{\perp})h_L(v_2^{\perp})|v_1 \cdot v_2^{\perp}|^{-1} = 16a_1a_2h_L(v_1^{\perp})h_L(v_2^{\perp}),$$

and thus, $16a_1a_2h_L(v_1^{\perp})h_L(v_2^{\perp}) \geq V(L) = 1$, or equivalently

$$h_L(v_1^{\perp})h_L(v_2^{\perp}) \geq \frac{1}{16a_1a_2},$$

with equality if and only if L itself is a parallelogram with sides parallel to those of K. Thus, by (112), the functional $\int_{S^1} log\, h_L dV_K$ attains its minimal value if and only if

$$h_L(v_1^{\perp})h_L(v_2^{\perp}) = \frac{1}{16a_1a_2};$$

i.e., if and only if L is a parallelogram with sides parallel to those of K.

Lemma (6.2.12) shows that the log-Minkowski inequality of Theorem (6.2.18) yields the log-Brunn-Minkowski inequality (51) of Theorem (6.2.3). To obtain the equality conditions of the log-Brunn-Minkowski inequality (51), we need to analyze the equality conditions of the inequality (82) in the proof of Lemma (6.2.12). The equality conditions for the log-Minkowski inequality of Theorem (6.2.18) show that equality in inequality (82) would imply that either K, L and Q_λ are dilates or that K, L and Q_λ are parallelograms with parallel sides. This establishes the equality conditions of Theorem (6.2.3).

Jensen's inequality (along with its equality conditions), shows that the L_p-Minkowski inequality, for $p > 0$, of Theorem (6.2.8) follows from the L_0-Minkowski inequality of Theorem (6.2.18).

Lemma (6.2.11) shows that the L_p-Minkowski inequality of Theorem (6.2.8) yields the L_p-Brunn-Minkowski inequality of Theorem (6.2.7).

To obtain the equality conditions of the L_p-Brunn-Minkowski inequality (58) of Theorem (6.2.7) we need to analyze the equality conditions of inequalities (80) and (81) of Lemma (6.2.11) which were used to derive the L_p-Brunn-Minkowski inequality of Theorem (6.2.7) from the L_p-Minkowski inequality of Theorem (6.2.8).

From the equality conditions of Theorem (6.2.8), we know that equality in inequality (80) implies that K and L are dilates. But inequality (81) is a direct consequence of the concavity of the log function and this concavity is strict. Hence, equality in inequality (81) implies that $V(K) = V(L)$.
Thus we conclude that equality in the L_p-Brunn-Minkowski inequality (58) of Theorem (6.2.7) implies that $K = L$.

Section (6.3): Stability of Brunn–Minkowski Type Inequalities

The classical Brunn–Minkowski inequality states that for $\lambda \in [0,1]$ and for Borel measurable sets A and B in $\mathbb{R}^n$, such that $(1-\lambda)A + \lambda B$ is measurable as well,

$$|\lambda A + (1-\lambda)B|^{\frac{1}{n}} \geq \lambda|A|^{\frac{1}{n}} + (1-\lambda)|B|^{\frac{1}{n}}. \qquad (113)$$

Here $|\cdot|$ denotes the Lebesgue measure, the addition between sets is the standard vector addition, and multiplication of sets by non-negative reals is the usual dilation.
This inequality has found many important applications in Geometry and Analysis (see $e.g.$Gardner [372]for an exhaustive survey on this subject). For example, the classical isoperimetric inequality can be deduced in a few lines from (113). Also, Maurey [373]deduced from this inequality the Poincaré inequality for the Gaussian measure and Gaussian concentration properties. Based on Maurey's results, Bobkov and Ledoux proved that the Brunn–Minkowski inequality implies Brascamp–Lieb and log-Sobolev inequalities [372]; they also deduced sharp Sobolev and Gagliardo–Nirenberg inequalities [373]. A different argument was developed by [374]to deduce Poincaré type inequalities on the boundary of convex bodies from the Brunn–Minkowski inequality.
Recall that a convex body is a convex compact set with non-empty interior. The family of convex bodies of $\mathbb{R}^n$will be denoted by $\mathcal{K}^n$. For the theory of convex bodies see Ball [371], Bonnesen, Fenchel [374], Koldobsky [374], Milman and Schechtman [379], Schneider [375] and others. A measure γ on $\mathbb{R}^n$ is called log-concave if for any pair of sets A and B and for any scalar $\lambda \in [0,1]$,

$$\gamma(\lambda A + (1-\lambda)B) \geq \gamma(A)^{\lambda}\gamma(B)^{1-\lambda}. \qquad (114)$$

Borell showed [375] that a measure is log-concave if it has a density (with respect to the Lebesgue measure) which is log-concave (see also Prékopa [376], Leindler [377]). In particular, the Lebesgue measure on $\mathbb{R}^n$ is log-concave:

$$|\lambda A + (1-\lambda)B| \geq |A|^{\lambda}|B|^{1-\lambda}. \qquad (115)$$

Inequality (113)implies (115)by the arithmetic–geometric mean inequality. Conversely, a simple argument based on the homogeneity of Lebesgue measure shows that (115) implies (113) (see [372]). In general, a property analogous to (113) may not hold for log-concave measures which are not homogeneous. The transposition of (113) to a measure γ,

$$\gamma(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \geq \lambda\gamma(A)^{\frac{1}{n}} + (1-\lambda)\gamma(B)^{\frac{1}{n}}, \quad \forall\, \lambda \in [0,1], \qquad (116)$$

as A and B vary in some class of sets, will be called. If γ is the Gaussian measure, $A = \{p\}, p \in \mathbb{R}^n$, and B is measurable set with positive measure, then the set $A + B$ is the translate of B by p. Hence, letting $|p| \to \infty$, and keeping B fixed, (116) fails. Moreover, Nayar and Tkocz [380] constructed an example in which (116) fails for the Gaussian measure while

both A and B contain the origin. Gardner and Zvavitch [373] proved that, for the Gaussian measure, (116) holds if the sets A and $B$are convex symmetric dilates of each other. They also proposed a conjecture for the Gaussian measure, that we state it in a more general form.
Conjecture (6.3.1)[362]: (Gardner, Zvavitch – generalized). Let $n \geq 2$. Let γ be a log-concave symmetric measure (i.e. $\gamma(A) = \gamma(-A)$for every measurable set A) on $\mathbb{R}^n$. Let $K$and L be symmetric convex bodies in $\mathbb{R}^n$. Then

$$\gamma(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda\gamma(K)^{\frac{1}{n}} + (1-\lambda)\gamma(L)^{\frac{1}{n}} \qquad (117).$$

Next, we pass to describe the log-Brunn–Minkowski inequality. For a scalar $\lambda \in [0,1]$ and for convex bodies $K$and L containing the origin in their interior, with support functions h_K and h_L, respectively (for the definition), define their geometric average as follows:

$$K^\lambda L^{1-\lambda} := \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K^\lambda(u) h_L^{1-\lambda}(u) \forall u \in \mathbb{S}^{n-1}\right\}, \qquad (118)$$

where $\langle \cdot,\cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. This set is again a convex body, whose support function is, in general, smaller than the geometric mean of the support functions of K and L. The following is widely known as log-Brunn–Minkowski conjecture (see [6]).
Conjecture (6.3.2) [362]: (Böröczky, Lutwak, Yang, Zhang). Let $n \geq 2$ be an integer. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then

$$\left|K^\lambda L^{1-\lambda}\right| \geq |K|^\lambda |L|^{1-\lambda}. \qquad (119)$$

Important applications and motivations for Conjecture(6.3.2)can be found in [378].
It is not difficult to see that the condition of symmetry is necessary , Böröczky, Lutwak, Yang and Zhang showed that this conjecture holds for $n = 2$. Saroglou [373] and Cordero, Fradelizi, Maurey [371]proved that (119) is true when the sets $K$and L are unconditional (i.e. they are symmetric with respect to every coordinate hyperplane). Rotem [372]showed that log-Brunn–Minkowski conjecture holds for complex convex bodies. Saroglou showed [374]that the validity of Conjecture(6.3.2)would imply the same statement for every log-concave symmetric measure γ on $\mathbb{R}^n$: for every symmetric $K, L \in \mathcal{K}^n$ and for every $\lambda \in [0,1]$,

$$\gamma\left(K^\lambda L^{1-\lambda}\right) \geq \gamma(K)^\lambda \gamma(L)^{1-\lambda}. \qquad (120)$$

Note that the straightforward inclusion

$$K^\lambda L^{1-\lambda} \subset \lambda K + (1-\lambda)L$$

tells us that (120)is stronger than (114), for every measure.
In [376] Nayar and Zvavitch showed that (120) implies (117) for every ray-decreasing measure γ on $\mathbb{R}^n$and for every pair of convex sets K and L. Therefore, Conjecture(6.3.1)holds on the plane and for unconditional sets.
The main results are the two theorems below.
Theorem (6.3.3) [362]: (The dimensional Brunn–Minkowski inequality near a ball). Let γ be a rotation invariant log-concave measure on $\mathbb{R}^n$. Let $R \in (0,\infty)$. Let $\psi \in C^2(S^{n-1})$. Then there exists a sufficiently small $a > 0$ such that for every $\epsilon_1, \epsilon_2 \in (0, a)$ and for every $\lambda \in [0,1]$, one has

$$\gamma(\lambda K_1 + (1-\lambda)K_2)^{\frac{1}{n}} \geq \lambda\gamma(K_1)^{\frac{1}{n}} + (1-\lambda)\gamma(K_2)^{\frac{1}{n}},$$

where K_1 is the convex set with the support function $h_1 = R + \epsilon_1\psi$ and K_2 is the convex set with the support function $h_2 = R + \epsilon_2\psi$.

Theorem (6.3.4) [362]: (The log-Brunn–Minkowski inequality near a ball). Let γ be a rotation invariant log-concave measure on $\mathbb{R}^n$. Let $R \in (0,\infty)$. Let $\varphi \in C^2(\mathbb{S}^{n-1})$ be **even** and strictly positive. Then there exists a sufficiently small $a > 0$ such that for every $\epsilon_1, \epsilon_2 \in (0,a)$ and for every $\lambda \in [0,1]$, one has

$$\gamma(K_1^\lambda K_2^{1-\lambda}) \geq \gamma(K_1)^\lambda \gamma(K_2)^{1-\lambda},$$

where K_1 is the convex set with the support function $h_1 = R\varphi^{\epsilon_1}$ and K_2 is the convex set with the support function $h_2 = R\varphi^{\epsilon_2}$.

Theorem(6.3.4) can be used to obtain a local uniqueness result for log-Minkowski problem (see Böröczky, Lutwak, Yang, Zhang [367]), and the corresponding investigation shall be carried out in a separate manuscript.

We work in the n-dimensional Euclidean space $\mathbb{R}^n$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. We set $B_2^n := \{x \in \mathbb{R}^n : |x| \leq 1\} and\ \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, to denote the unit ball and the unit sphere, respectively. We shall denote the Lebesgue measure (the volume) in $\mathbb{R}^n$ by $|\cdot|$.

We say that a set $A \subset \mathbb{R}^n$ is symmetric if for every $x \in A$one has $-x \in A$. All measures under consideration will be tacitly assumed to be Radon measures, and all sets will be assumed to be measurable. A measure γ on $\mathbb{R}^n$is called symmetric if for every set $S \subset \mathbb{R}^n, \gamma(S) = \gamma(-S)$. If the measure has a density then it is symmetric whenever the density is an even function.

A measure γ on $\mathbb{R}^n$ is said to be rotation invariant if for every set $A \subset \mathbb{R}^n$, and for every rotation $T, \gamma(A) = \gamma(TA)$. If a rotation invariant measure γ has a density F, we may write F in the form:

$$F(x) = f(|x|),$$

for a suitable $f: [0,\infty) \to [0,\infty)$.

For $K \in \mathcal{K}^n$, the support function of $K, h_K: \mathbb{S}^{n-1} \to \mathbb{R}$, is defined as

$$h_K(u) = sup_{x\in K}\langle x, u\rangle.$$

By the geometric viewpoint, $h_K(u)$ represents the (signed) distance from the origin of the supporting hyperplane to $K$with outer unit normal u. We shall use the notation $H_K(x)$ for the 1-homogenous extension of h_K, that is,

$$H_K(x) = \begin{cases} |x| h_K \left(\frac{x}{|x|}\right) & if\ x \neq 0, \\ 0 & if\ x = 0. \end{cases}$$

The function H_K is convex in $\mathbb{R}^n$, for every $K \in \mathcal{K}^n$. Vice versa, for every continuous 1-homogeneous convex function Hon $\mathbb{R}^n$, there exists a unique convex body Ksuch that $H = H_K$.

Note that $K \in \mathcal{K}^n$contains the origin (resp., in its interior) if and only if $h_K \geq 0 (resp.\ h_K > 0)$ on $\mathbb{S}^{n-1}$. For convex bodies K and L, and for $\alpha, \beta \geq 0$, we have:

$$h_{\alpha K+\beta L}(u) = \alpha h_K(u) + \beta h_L(u). \qquad (120)$$

We say that a convex body K is $C^{2,+}$ if ∂K is of class C^2 and the Gauss curvature is strictly positive at every $x \in \partial K$. In particular, if K is $C^{2,+}$then it admits outer unit normal $\nu_K(x)$ at every boundary point x. Recall that the Gauss map $\nu_K: \partial K \to \mathbb{S}^{n-1}$is the map assigning the unit normal to each point of ∂K.

$C^{2,+}$ convex bodies can be characterized through their support function. We recall that an orthonormal frame on the sphere is a map which associates a collection of $n-1$orthonormal vectors to every point of $\mathbb{S}^{n-1}$. Let $\psi \in C^2(\mathbb{S}^{n-1})$. We denote by $\psi_i(u)$ and $\psi_{ij}(u), i, j \in \{1, \ldots, n-1\}$, the first and second covariant derivatives of ψ at $u \in \mathbb{S}^{n-1}$, with respect to a fixed local orthonormal frame on an open subset of $\mathbb{S}^{n-1}$. We define the matrix

$$Q(\psi; u) = (q_{ij})_{i,j=1,\ldots,n-1} = (\psi_{ij}(u) + \psi(u)\delta_{ij})_{i,j=1,\ldots,n-1}, \qquad (122)$$

where the δ_{ij}'s are the usual Kronecker symbols. On an occasion, instead of $Q(\psi; u)$ we write $Q(\psi)$. Note that $Q(\psi; u)$ is symmetric by standard properties of covariant derivatives. The meaning of this matrix becomes particularly important when ψ is the support function of a convex body K. In this case we shall call it curvature matrixof K. The proof of the following proposition can be deduced from Schneider [375, Section 2.5].

Proposition (6.3.5) [362]: Let $K \in \mathcal{K}^n$ and let h be its support function. Then $K$is of class $C^{2,+}$ if and only if $h \in C^2(\mathbb{S}^{n-1})$ and

$$Q(h; u) > 0 \quad \forall\, u \in \mathbb{S}^{n-1}.$$

In view of the previous results it is convenient to introduce the following set of functions

$$C^{2,+}(\mathbb{S}^{n-1}) = \{h \in C^2(\mathbb{S}^{n-1}): Q(h; u) > 0\ \forall\, u \in \mathbb{S}^{n-1}\}.$$

Hence $C^{2,+}(\mathbb{S}^{n-1})$is the set of support functions of convex bodies of class $C^{2,+}$.

Remark (6.3.6) [362]: Let $\psi \in C^1(\mathbb{S}^{n-1})$. The notation $\nabla_{\sigma\psi}$ stands for the spherical gradient of ψ, i.e. the vector $(\psi_1, \ldots, \psi_{n-1})$, where $\psi_i$are the covariant derivatives of ψ with respect to the i-th element of a fixed orthonormal system on $\mathbb{S}^{n-1}$. Let Φ be the 1-homogeneous extension of ψ to $\mathbb{R}^n$. Then we have

$$|\nabla\Phi(u)|^2 = \psi^2(u) + |\nabla_{\sigma\psi}(u)|^2 \qquad (123)$$

for every $u \in \mathbb{S}^{n-1}$.

We denote the family of centrally symmetric convex bodies by K_s^n. The notation $C_e^{2,+}(\mathbb{S}^{n-1})$will stand for the set of support functions of centrally symmetric $C^{2,+}$ convex bodies, i.e. functions from $C^{2,+}(\mathbb{S}^{n-1})$ which are additionally even.

Let h be the support function of a $C^{2,+}$ convex body K, and let $\psi \in C^{2,+}(\mathbb{S}^{n-1})$; then, by Proposition(6.3.5),

$$h_s := h + s\psi \in C^{2,+}(\mathbb{S}^{n-1}) \qquad (124)$$

if s is sufficiently small, say $|s| \le a$ for some appropriate $a > 0$. Hence for every s in this range there exists a unique $C^{2,+}$ convex body K_s with the support function h_s. For an interval I, we define the one-parameter family of convex bodies:

$$\boldsymbol{K}(h, \psi, I) := \{K_s : h_{K_s} = h + s\psi, s \in I\}.$$

Lemma (6.3.7) [362]: Assume that γ is a symmetric log-concave measure with continuously differentiable density. Conjecture(6.3.1)holds for γ if and only if for every one-parameter family $\boldsymbol{K}(h, \psi, I)$, with even h and ψ,

$$\frac{d^2}{ds^2}[\gamma(K_s)]|_{s=0} \cdot \gamma(K_0) \le \frac{n-1}{n}\left(\frac{d}{ds}[\gamma(K_s)]\Big|_{s=0}\right)^2 \qquad (125).$$

In particular, if (125) holds for $K_s$in a fixed family $\boldsymbol{K}(h,\psi,I)$, then Conjecture(6.3.1)holds for all sets K_s in that family.

Proof: Assume first that γ satisfies (117)on the system $\boldsymbol{K}(h,\psi,I)$. Then the equality $h_{K_s} = h + s\psi, \ \ s \in I$, and the linearity of support function with respect to Minkowski addition, imply that for every $s,t \in I$ and for every $\lambda \in [0,1]$

$$K_{\lambda s+(1-\lambda)t} = \lambda K_s + (1-\lambda)K_t.$$

By (117),

$$\gamma(K_{\lambda s+(1-\lambda)t})^{\frac{1}{n}} = \gamma(\lambda K_s + (1-\lambda)K_t)^{\frac{1}{n}} \ge \lambda\gamma(K_s)^{\frac{1}{n}} + (1-\lambda)\gamma(K_t)^{\frac{1}{n}},$$

which means that the function $\gamma(K_s)^{\frac{1}{n}}$is concave on I. Inequality (125) follows.

Conversely, suppose that for every system $\boldsymbol{K}(h,\psi,I)$ the function $\gamma(K_s)^{\frac{1}{n}}$ has non-positive second derivative at 0, i.e. (125) holds. We observe that this implies concavity of $\gamma(K_s)^{\frac{1}{n}}$ on the entire interval I. Indeed, given s0in the interior of I, consider $\tilde{h} = h + s_0\psi$, and define a new system $\widetilde{\boldsymbol{K}}(\tilde{h},\psi,J)$, where J is a new interval such that $\tilde{h} + s\psi = h + (s+s_0)\psi \in C^{2,+}$for every $s \in J$. Then the second derivative of $\gamma(K_s)^{\frac{1}{n}}$ at $s = s_0$ is negative, as it is equal to the second derivative of $\gamma(\widetilde{K}_s)^{\frac{1}{n}}$ at $s = 0$. On the other hand, the concavity $\gamma(K_s)^{\frac{1}{n}}$ on the family $\boldsymbol{K}(h,\psi,I)$ is equivalent to the validity of (117)on this family.

A similar approach can be used for the log-Brunn–Minkowski inequality. In order to do this we introduce a corresponding type of one-parameter families of convex bodies. In this case, additive perturbations are replaced by multiplicative perturbations.

Let $h \in C^{2,+}(\mathbb{S}^{n-1})$ and $\varphi \in C^2(\mathbb{S}^{n-1})$, with $\varphi > 0$ on $\mathbb{S}^{n-1}$. Then there exists $a > 0$ such that

$$h_s := h\varphi^s \in C^{2,+}(\mathbb{S}^{n-1}) \qquad \forall s \in [-a,a].$$

In particular, by Proposition(6.3.5), for every $s \in [-a,a]$there exists a $C^{2,+}$convex body $Q_s$whose support function is h_s.

We introduce the corresponding 1-dimensional systems.

$$\boldsymbol{Q}(h,\varphi,I) := \ Q_s \in K^n : h_{Q_s} = h_{\varphi^s}, s \in I\}.$$

Lemma (6.3.8) [362]: Let γ be a symmetric log-concave measure with continuously differentiable density. Assume that Conjecture(6.3.2) holds for a measure γ, i.e. (120)is valid for every pair of symmetric convex sets $K$and L and for every $\lambda \in [0,1]$. Then for every one-parameter family $Q_s \in \boldsymbol{Q}(h,\varphi,I)$, with h and φ even,

$$\frac{d^2}{ds^2}\ log(\gamma(Q_s))\Big|_{s=0} \le 0. \qquad (126)$$

The converse is true locally: if (126) holds for all $Q_s$in a fixed family $\boldsymbol{Q}(h,\varphi,I)$, then Conjecture(6.3.2) holds for all sets Q_s in $\boldsymbol{Q}(h,\varphi,[0,\epsilon])$for a small enough interval $[0,\epsilon] \subset I$.

Proof: Let $h \in C^{2,+}(\mathbb{S}^{n-1})$ and $\varphi \in C^2(\mathbb{S}^{n-1})$ be strictly positive even functions on $\mathbb{S}^{n-1}$; there exists $a > 0$ such that $h_s := h_{\varphi^s}$ is the support function of a convex body Q_s for all $s \in [-a, a]$. Note that for $s, t \in [-a, a]$ we get

$$h_{\lambda s+(1-\lambda)t} = h_s^{\lambda} h_t^{1-\lambda},$$

and thus

$$Q_{\lambda s+(1-\lambda)t} = Q_s^{\lambda} Q_t^{1-\lambda}.$$

If the Conjecture(6.3.2)is true, then

$$\gamma(Q_{\lambda s+(1-\lambda)t}) = \gamma(Q_s^{\lambda} Q_t^{1-\lambda}) \geq \gamma(Q_s)^{\lambda}\gamma(Q_t)^{1-\lambda},$$

which means that $\gamma(Q_s)$ is log-concave in $[-a, a]$.

The following Lemma is the key step in proving Theorem(6.3.3). To prove it, we express a measure of a convex set in terms of its support function and run a long and technical computation, involving integration by parts; the complete proof is outlined.

Lemma (6.3.9) [362]: Let $R > 0$. Let γ be a rotation invariant measure with density $f(|x|)$, and let $A = \int_0^1 t^{n-1} f(Rt)dt$. In the case $h_K = R$, (125)is equivalent to the validity of the following inequality for every $\psi \in C^2(\mathbb{S}^{n-1})$:

$$\frac{Af(R)}{|\mathbb{S}^{n-1}|}\left((n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du\right) + \frac{ARf'(R)}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \psi^2 du \leq \frac{n-1}{n} f(R)^2 \left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi du\right)^2. \quad (127)$$

By Lemma(6.3.7), to prove the Theorem, it suffices to show the validity of (127). Let us denote the quadratic operators appearing in the left-hand side and in the right-hand side of the inequality (127) by $B_1(\psi)$ and $B_2(\psi)$, correspondingly. That is,

$$B_1(\psi) = \frac{Af(R)}{|\mathbb{S}^{n-1}|}\left((n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du\right) + \frac{ARf'(R)}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \psi^2 du,$$

and

$$B_2(\psi) = \frac{n-1}{n} f(R)^2 \left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi du\right)^2.$$

The next step is to decompose ψ as the sum of a constant function and a function which is orthogonal to constant functions. Let us write

$$\psi = \psi_0 + \psi_1$$

where

$$\psi_0 = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \psi du \text{ and } \int_{\mathbb{S}^{n-1}} \psi_1 du = 0.$$

Note that

$$\int_{\mathbb{S}^{n-1}} \psi^2 d\sigma = \int_{\mathbb{S}^{n-1}} \psi_0^2 d\sigma + \int_{\mathbb{S}^{n-1}} \psi_1^2 d\sigma.$$

Therefore,

$$B_1(\psi) = B_1(\psi_0) + B_1(\psi_1),$$

as well as

$$B_2(\psi) = B_2(\psi_0) + B_2(\psi_1).$$

Since γ is radially symmetric, one has $f' \leq 0$. Moreover, by the standard Poincaré inequality on the unit sphere,

$$(n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma \psi|^2 du \leq 0, \quad (128)$$

for every ψ such that

$$\int_{\mathbb{S}^{n-1}} \psi du = 0. \quad (129)$$

Thus

$$B_1(\psi_1) \leq 0 = B_2(\psi_1).$$

To prove (127) it remains to show that

$$B_1(\psi_0) \leq B_2(\psi_0). \quad (130)$$

This condition is equivalent to

$$\gamma(\lambda_{r_1} B_2^n + (1-\lambda) r_2 B_2^n)^{\frac{1}{n}} \geq \lambda\gamma(r_1 B_2^n)^{\frac{1}{n}} + (1-\lambda)\gamma(r_2 B_2^n)^{\frac{1}{n}}, \quad (131)$$

for some $r_1, r_2 \in [R, R+\epsilon]$. As was shown in [366] (see also [377]), this statement follows from log-Brunn–Minkowski conjecture in the case of log-concave spherically invariant measures and when $K$and L are Euclidean balls. The latter is indeed true: it follows from the results of [371] and [363].

As before, we start with a Lemma, which shall be rigorously proved.

Lemma (6.3.10) [362]: Let $R > 0$. Let γ be a rotation invariant measure with density $f(|x|)$, and let $A = \int_0^1 t^{n-1} f(Rt) dt$. In the case $h_K = R$, (126)is equivalent to the following inequality:

$$A[nf(R) + Rf'(R)] \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \psi^2 du - Af(R) \frac{1}{|\mathbb{S}^{n-1}|} |\nabla_\sigma \psi|^2 du \leq$$

$$f(R)^2 \left(\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \psi d\sigma\right)^2, \quad (132)$$

for every even $\psi \in C^2(\mathbb{S}^2)$.

We follow the argument of the previous and split the proof into two cases.

Case 1. Consider an even $\psi \in C^2(\mathbb{S}^{n-1})$ such that $\int \psi = 0$. Here we use some basic facts from the theory of spherical harmonics, which can be found, for instance in [375, Appendix], where will find hints to the corresponding literature. We denote by $\Delta_\sigma$the spherical Laplace operator (or Laplace–Beltrami operator), on $\mathbb{S}^{n-1}$. The first eigenvalue of Δ_σ is 0, and the corresponding eigenspace if formed by constant functions. Hence the zero-mean condition on ψ implies that ψ is orthogonal to such eigenspace. The second eigenvalue of Δ_σ is $n-1$, and the corresponding eigenspace is formed by the restrictions of linear functions of $\mathbb{R}^n$to $\mathbb{S}^{n-1}$. As each of them is odd and ψ is even, ψ is orthogonal to this eigenspace as well. Finally, the third eigenvalue is $2n$. Then the inequality (132) amounts to

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \psi^2 du \leq \frac{f(R)}{nf(R)+Rf'(R)} \frac{1}{|\mathbb{S}^{n-1}|} |\nabla_\sigma \psi|^2 du. \quad (133)$$

Hence

$$\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \psi^2 du \leq \frac{1}{2n} \frac{1}{|\mathbb{S}^{n-1}|} |\nabla_\sigma \psi|^2 du. \quad (134)$$

Since f is decreasing, we have $f'(R) \leq 0$, and hence

$$\frac{f(R)}{nf(R)+Rf\prime(R)} \geq \frac{1}{n} > \frac{1}{2n}. \quad (135)$$

The inequalities (134)and (135)imply (133).

Case 2. Let ψ be a constant function. The inequality (132) holds for constant functions because, once again, the log-Brunn–Minkowski inequality holds in the case of spherically invariant measures and Euclidean balls.

To summarize, we established (132) separately for constant functions and centered functions. A polarization argument analogous to the one presented in the proof of Theorem(6.3.3)finishes the proof.

A formula expressing a measure of a convex set in terms of its support function

Let γ be a probability measure on $\mathbb{R}^n$; we assume that γ has a density F with respect to the Lebesgue measure, and that F is sufficiently regular ($e.g.$continuous).

Lemma (6.3.11) [362]: Let K be a $C^{2,+}$convex body; let hand H be the support function of K and its homogenous extension, respectively. Assume that the origin is in the interior of K. Then

$$\gamma(K) = \int_{\mathbb{S}^{n-1}} h(y) det Q(h;\ y) \int_0^1 t^{n-1} F\left(t\nabla H(y)\right) dt dy. \qquad (136)$$

The cofactor matrix and related notions

Let $M = (m_{ij})$ be an $N \times N$ symmetric matrix, $N \in \mathbb{N}$. We define $C[M]$, the cofactor matrix of M, as follows

$$C[M] = \left(c_{ij}\ [M]\right)_{i,j=1,\dots,N}\ where\ c_{ij}[M] = \frac{\partial det}{\partial m_{ij}}(M)\ i,j\ =\ 1,\dots,N.$$

$C[M]$ is an $N \times N$ symmetric matrix. Using the homogeneity of the determinant we get

$$\sum_{i,j=1}^{N} c_{ij}\ [M] m_{ij} = N\ det(M). \qquad (137)$$

We shall also consider the second derivatives of the determinant of a matrix with respect to its entries:

$$c_{ij,kl}[M] = \frac{\partial^2 det}{\partial m_{ij} \partial m_{kl}}(M).$$

By homogeneity we have that, for every $i, j = 1, \dots, N$

$$\sum_{i,j=1}^{N} c_{ij,kl}\ [M] m_{kl} =\ (N-1) c_{ij}\ [M]. \qquad (138)$$

Let $h \in C^{2,+}(\mathbb{S}^{n-1})$, and assume additionally that $h \in C^3(\mathbb{S}^{n-1})$. Consider the cofactor matrix $y \to C[Q(h;\ y)]$. This is a matrix of functions on $\mathbb{S}^{n-1}$. The lemma of Cheng and Yau asserts that each column of this matrix is divergence-free.

Lemma (6.3.12) [362]: (Cheng–Yau). Let $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$. Then, for every index $j \in \{1, \dots, n\ -1\}$ and for every $y \in \mathbb{S}^{n-1}$,

$$\sum_{i=1}^{n-1} (c_{ij}\ [Q(h;\ y)])_i = 0,$$

where the sub-script idenotes the derivative with respect to the i-th element of an orthonormal frame on $\mathbb{S}^{n-1}$.

We shall often write $C(h), c_{ij}(h)$ and $c_{ij,kl}(h)$ in place of $C[Q(h)], c_{ij}[Q(h)]$ and $c_{ij,kl}[Q(h)]$ respectively.

As a corollary of the previous result we have the following integration by parts formula. If $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$ and $\psi, \phi \in C^2(\mathbb{S}^{n-1})$, then

$$\int_{\mathbb{S}^{n-1}} \phi c_{ij}(h)(\psi_{ij} + \psi\, \delta_{ij})dy \ = \int_{\mathbb{S}^{n-1}} \psi\, c_{ij}(h)(\varphi_{ij} + \varphi\delta_{ij})dy. \quad (139)$$

The Lemma of Cheng and Yau admits the following extension (see by the first-named author, Hug and Saorin-Gomez [370]).

Lemma (6.3.13) [362]: Let $\psi \in C^2(\mathbb{S}^{n-1})$ and $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1})$. Then, for every $k \in \{1, \dots, n-1\}$ and for every $y \in \mathbb{S}^{n-1}$

$$\sum_{i=1}^{n-1} (c_{ij,kl}[Q(h;\ y)](\psi_{ij} + \psi\delta_{ij}))_l = 0.$$

Correspondingly we have, for every $h \in C^{2,+}(\mathbb{S}^{n-1}) \cap C^3(\mathbb{S}^{n-1}), \psi, \phi, \varphi \in C^2(\mathbb{S}^{n-1})$ and $i, j \in \{1, \dots, n-1\}$

$$\int_{\mathbb{S}^{n-1}} \psi\, c_{ij,kl}(h)(\varphi_{ij} + \varphi\delta_{ij})((\phi)_{kl} + \phi\delta_{kl})dy$$
$$= \int_{\mathbb{S}^{n-1}} \phi c_{ij,kl}(h)(\varphi_{ij} + \varphi\delta_{ij})((\psi)_{kl} + \psi\delta_{kl})dy. \quad (140)$$

As usual, γ is a radially symmetric log-concave measure on $\mathbb{R}^n$, with density F with respect to Lebesgue measure; in particular, we write Fin the form:

$$F(x) = f(|x|).$$

We will assume that f is smooth, more precisely $f \in C^2([0, \infty))$. Let us fix $h \in C^{2,+}(\mathbb{S}^{n-1})$ and let K be a convex body with support function h. Let $\psi \in C^2(\mathbb{S}^{n-1})$ and consider the one-parameter system of convex bodies $\boldsymbol{K}(h, \psi, [-a, a])$ for a suitably small $a > 0$. In particular for every $s \in [-a, a]$ there exists a convex body K_s such that $h_{K_s} = h_s$. Hence we may consider the function

$$g: [-a, a] \to \mathbb{R}, \qquad g(s) = \gamma(K_s).$$

The aim of this subsection is to derive formulas for the first and second derivative of $g(s)$ at $s = 0$. We start from the expression:

$$g(s) = \int_{\mathbb{S}^{n-1}} h_s(u)\, det(Q(h_s; u)) \int_0^1 t^{n-1} f(t\sqrt{h_s^2(u) + |\nabla_\sigma h_s(u)|^2})dtdu\,,$$

where we used Lemma(6.3.11), the rotation invariance of γ, and Remark(6.3.6). To simplify notations we set

$$Q_s = Q(h_s;\ u), \qquad Q = Q_0;\ \ D_s = [h_s^2(u) + |\nabla_\sigma h_s(u)|^2]^{1/2}, \qquad D = D_0;$$
$$A_s = \int_0^1 t^{n-1} f(tD_s)dt\,,\ A = A_0;\ \ \ B_s = \int_0^1 t^n f'(tD_s)dt\,,\ B = B_0;$$
$$C_s = \int_0^1 t^{n+1} f''(tD_s)dt\ \ , C = C_0.$$

Then

$$g'(s) = \int_{\mathbb{S}^{n-1}} \psi \det(Q_s)\, A_s du\ + \int_{\mathbb{S}^{n-1}} h_s c_{ij}(h_s)(\psi_{ij} + \psi\delta_{ij})A_s du$$
$$+ \int_{\mathbb{S}^{n-1}} h_s\, det(Q_s)B_s\ \frac{h_s\psi + \langle\nabla_\sigma h_s, \nabla_\sigma\psi\ \rangle}{D_s} du. \quad (141)$$

Passing to the second derivative (for $s = 0$) we get

$$g''(0) = 2\int_{\mathbb{S}^{n-1}} \psi c_{ij}(h)(\psi_{ij} + \psi\delta_{ij})Adu$$

$$+ 2\int_{\mathbb{S}^{n-1}} \psi\, det(Q) B\, \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\, du$$
$$+ 2\int_{\mathbb{S}^{n-1}} hc_{ij}(h)(\psi_{ij} + \psi\delta_{ij}) B\, \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\, du$$
$$+\int_{\mathbb{S}^{n-1}} Ahc_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})(\psi_{kl} + \psi\delta_{kl})du$$
$$+ \int_{\mathbb{S}^{n-1}} h\, det(Q) C \left[\frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\right]^2 du$$
$$+\int_{\mathbb{S}^{n-1}} h\, det(Q) B \left[D(h^2 + |\nabla_\sigma \psi|^2) - \frac{[h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle]^2}{D}\right] \frac{1}{D^2} du. \qquad (142)$$

We now focus on the fourth summand of the last expression. Applying formulas (140)and (138)we get

$$\int_{\mathbb{S}^{n-1}} Ahc_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})(\psi_{kl} + \psi\delta_{kl})du$$
$$= \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})((Ah)_{kl} + Ah\delta_{kl})du$$
$$= \int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})(A(h_{kl} + h\delta_{kl}) + 2A_k h_l + hA_{kl})du$$
$$= \int_{\mathbb{S}^{n-1}} A\psi c_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})(h_{kl} + h\delta_{kl})du$$
$$+\int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})(2A_k h_l + hA_{kl})du$$
$$= (n-2) \int_{\mathbb{S}^{n-1}} A\psi c_{ij}(h)(\psi_{ij} + \psi\delta_{ij})du$$
$$+\int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})(2A_k h_l + hA_{kl})du$$

Hence

$$g''(0) = n\int_{\mathbb{S}^{n-1}} \psi c_{ij}(h)\big(\psi_{ij} + \psi\delta_{ij}\big)A du + 2\int_{\mathbb{S}^{n-1}} \psi\, det(Q) B\, \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\, du$$
$$+ 2\int_{\mathbb{S}^{n-1}} hc_{ij}(h)(\psi_{ij} + \psi\delta_{ij}) B\, \frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\, du$$
$$+\int_{\mathbb{S}^{n-1}} \psi c_{ij,kl}(h)(\psi_{ij} + \psi\delta_{ij})(2A_k h_l + hA_{kl})du$$
$$+ \int_{\mathbb{S}^{n-1}} h\, det(Q) C \left[\frac{h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle}{D}\right]^2 du$$
$$+\int_{\mathbb{S}^{n-1}} h\, det(Q) B \left[D(\psi^2 + |\nabla_\sigma \psi|^2) - \frac{[h\psi + \langle \nabla_\sigma h, \nabla_\sigma \psi\rangle]^2}{D}\right] \frac{1}{D^2} du. \quad (143)$$

Let $h \equiv R, R > 0$. This choice considerably simplifies the situation as:

$$Q = RI_{n-1}; \quad \nabla_\sigma \equiv R; \quad D \equiv R; \quad c_{ij}(h) \equiv \mathbb{R}^{n-1}\delta_{ij};$$
$$A = \int_0^1 t^{n-1} f(Rt)dt, \qquad B = \int_0^1 t^n f'(Rt)dt, \qquad C = \int_0^1 t^{n+1} f''(Rt)dt.$$

Here I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix. In particular A does not depend on the point u on $\mathbb{S}^{n-1}$, so that

$$A_i \equiv A_{ij} \equiv 0 \qquad on\mathbb{S}^{n-1}.$$

Hence $g(0) = |\mathbb{S}^{n-1}|R^n A$, and

$$g'(0) = R^{n-1}A\int_{\mathbb{S}^{n-1}} \psi du + R^{n-1}A\int_{\mathbb{S}^{n-1}} (\Delta_\sigma\psi + (n-1)\psi)du + R^n B\int_{\mathbb{S}^{n-1}} \psi du$$
$$= R^{n-1}(nA + RB)\int_{\mathbb{S}^{n-1}} \psi du. \qquad (144)$$

Here we used the fact that, by the divergence theorem on $\mathbb{S}^{n-1}$,

$$\int_{\mathbb{S}^{n-1}} \Delta_\sigma\psi du = 0.$$

As for the second derivative, we haveg

$$g''(0) = nR^{n-2}A\int_{\mathbb{S}^{n-1}} \psi(\Delta_\sigma\psi + (n-1)\psi)du + 2R^{n-1}B\int_{\mathbb{S}^{n-1}} \psi^2 du$$
$$+2R^{n-1}B\int_{\mathbb{S}^{n-1}} \psi(\Delta_\sigma\psi + (n-1)\psi))du + R^n C\int_{\mathbb{S}^{n-1}} \psi^2 du$$
$$+ R^{n-1}B\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du.$$

By the divergence theorem,

$$\int_{\mathbb{S}^{n-1}} \psi\Delta_\sigma\psi du = -\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du,$$

and thus

$$g''(0) = R^{n-2}(A_n(n-1) + 2nRB + R^2C)\int_{\mathbb{S}^{n-1}} \psi^2 du - R^{n-2}(nA + RB)\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du. \qquad (145)$$

Integrating by parts in t, we get

$$f(R) = nA + RB,$$

and

$$f'(R) = (n+1)B + RC.$$

Thus we obtain

$$g'(0) = R^{n-1}f(R)\int_{\mathbb{S}^{n-1}} \psi du, \qquad (146)$$

and

$$g''(0) = R^{n-2}[(n-1)f(R) + Rf'(R)]\int_{\mathbb{S}^{n-1}} \psi^2 du - R^{n-2}f(R)\int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du$$
$$= R^{n-2}f(R)\left((n-1)\int_{\mathbb{S}^{n-1}} \psi^2 du - \int_{\mathbb{S}^{n-1}} |\nabla_\sigma\psi|^2 du\right) +$$
$$R^{n-1}f'(R)\int_{\mathbb{S}^{n-1}} \psi^2 du. \qquad (147)$$

This concludes the proof of Lemma(6.3.9).

Proof of the Lemma(6.3.10)

Firstly, we state the following.

Lemma (6.3.14) [362]: Let $n \geq 2$. Let γ be a measure on $\mathbb{R}^n$. Fix $h \in C^{2,+}(\mathbb{S}^{n-1}), \varphi \in C^2(\mathbb{S}^{n-1}), \varphi > 0$ and set $\psi = h\, log\varphi$. Let $\boldsymbol{K}(h, \psi, I)$, with $I = [-a, a]$ and $a > 0$, be the corresponding one-parameter family. Consider the function $f(s) = \gamma(K_s)$. Introduce the additional notation for the operator $F(h, \psi) := f'(0)$. Set

$$A(h, \psi) := \left.\frac{dF\left(h, \frac{h+s\psi}{h}\psi\right)}{ds}\right|_{s=0}. \qquad (148)$$

Consider the one-parameter family $\boldsymbol{Q}(h, \phi, [-a, a])$, i.e. the collection of sets with support functions $h_s = h\varphi^s, s \in [-a, a]$. Let $g(s) = \gamma(Q_s)$. Then
•$g(0) = f(0)$;
•$g'(0) = f'(0)$;
•$g''(0) = f''(0) + A(h, \psi)$.
The proof of the Lemma immediately follows from the fact that
$$h\varphi^s = h + sh \log \varphi + o(s), \qquad as\ s \to 0,$$
with the selection $\psi = h\ log\varphi$. When $h \equiv R > 0$, the additional term introduced in Lemma(6.3.14)can be written as follows:
$$A(h, \psi) = f(R) \int_{\mathbb{S}^{n-1}} \psi^2 du\,.$$
That, together with Lemma(6.3.9), implies Lemma(6.3.10).
Finally, we note that Lemma(6.3.14)implies the following result.
Theorem (6.3.15) [362]: (Infinitesimal form of Log-Brunn–Minkowski conjecture). Let $n \geq 2$ be an integer. If Conjecture(6.3.2)is true, then for every $h \in C_e^{2,+}(\mathbb{S}^{n-1}), \psi \in C^2(\mathbb{S}^{n-1}), \psi$ even and strictly positive,
$$\int_{\mathbb{S}^n} \psi^2 \frac{1+tr\left(Q^{-1}(h)\right)h}{h^2} d\bar{V}_{h-n} \left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2 \leq \int_{\mathbb{S}^{n-1}} \frac{1}{h} \langle Q^{-1}(h)\nabla\psi, \nabla\psi\rangle d\bar{V}_h. \quad (149)$$
Here h is the support function of $K, Q(h)$ is the curvature matrix of K and
$$d\bar{V}_h = \frac{1}{|K|}\frac{1}{n} h_K(u)\ detQ(h_K(u))du$$
is the normalized cone measure of the convex body K.
A corresponding infinitesimal Brunn–Minkowski inequality for Lebesgue measure was obtained by [379]and reads as:
$$\int_{\mathbb{S}^{n-1}} \psi^2 \frac{tr(Q^{-1}(h))}{h} d\bar{V}_h - (n-1)\left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2 \leq$$
$$\int_{\mathbb{S}^{n-1}} \frac{1}{h} \langle Q^{-1}(h)\nabla\psi, \nabla\psi\rangle d\bar{V}_h. \quad (150)$$
Note that by the Cauchy–Schwarz inequality,
$$\int_{\mathbb{S}^{n-1}} \frac{\psi^2}{h^2} d\bar{V}_h \geq \left(\int_{\mathbb{S}^{n-1}} \frac{\psi}{h} d\bar{V}_h\right)^2.$$
Hence, (149) is indeed a strengthening of (150).
In particular, letting $\varphi \equiv 1$we arrive to the following corollary of Theorem(6.3.15).
Corollary (6.3.16) [362]: (A strengthening of Minkowski's second inequality.). Let K be a convex symmetric set in the plane, or a convex unconditional set in $\mathbb{R}^n$. Then,
$$V_n(K)\left(V_{n-2}(K) + \int_{\partial K} \frac{1}{\langle y, \nu_K(y)\rangle} d\sigma(y)\right) \leq V_{n-1}(K)^2, \quad (151)$$
where V_{n-i} are the intrinsic volumes of $K, \nu_K(y)$stands for the unit normal at $y \in \partial K$and $d\sigma(y)$ is the surface area measure on ∂K.
Minkowski's second inequality, which states that for every convex set $K \subset \mathbb{R}^n$ one has
$$V_n(K)V_{n-2}(K) \leq \frac{n-1}{n} V_{n-1}(K)^2,$$

is deduced from (151)by using the Cauchy–Schwarz inequality. For a more general version of this inequality see, for example, Schneider [375, Chapter 4] .

List of Symbols

Symbol	page
Inf : infimum	4
Max: maxaimum	4
L_q:Dual of Lebesgue Space	5
Conv: convex	5
Min: minimum	13
GBP: Buse mann-problem	13
$vol_i(.)$:i-dimensiomal lebesgue measure	13
Gr: Grassmannian	14
L^1:Banach space	15
L^p:lebesgue space	23
$\otimes$: tensor product	26
Esssup:essential supremum	28
$\oplus$: $Divect\ sum$	29
a.e: almost everywhere	32
Const: constant	37
Det: determinant	43
Diam : diameter	46
Dist : distance	46
Bv: Bounded variation	63
Prab: probability	72
Vr:volume patio	79
Cap: capacity	88
$\dot{+}$:Blaschke addition	89
$\dagger_n$:harmonic addition	94
Ent: Entropy	97
$vol_n(.)$:Euchidean volume	98
L^2:Hilbert space	98
Int: interiov	111
Cov: covariant	111
V.Rad: volume-raduis	112
Proj: projection	114
Hess: Hessian	114
o.v.r: outer volume ratio	133
FKG: fortuin ,Kasteley	145
Lip: lipschitz	180
$w^{1,2}$:Sobolev space	180
L^∞:Essential Lebesgue space	183
Dim:dimension	191
Supp:support	224

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