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Solutions of Some Differential Algebraic Equations by Laplace-Padé Resummation Method

حلول بعض المعادلات التفاضلية الجبرية بإستخدام طريقة لابلاس - بادي

A project Submitted in fulfillment for the degree of M.Sc in Mathematics

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2021

الايـــــــــــــة

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ (1) الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ (2)الرَّحْمَنِ الرَّحِيمِ (3)مَالِكِ يَوْمِ الدِّينِ (4)إِيَّاكَ نَعْبُدُ وَإِيَّاكَ نَسْتَعِينُ (5)اهْدِنَا الصِّرَاطَ الْمُسْتَقِيمَ (6)صِرَاطَ الَّذِينَ أَنْعَمْتَ عَلَيْهِمْ غَيْرِ الْمَغْضُوبِ عَلَيْهِمْ وَلَا الضَّالِّينَ (7)

صدق الله العظيم

Dedication

Every challenging work needs self efforts as well as guidance of olders especially those who were very close to our heart.

My humble effort I dedicate to my sweet and loving

Mother & My family

whose affection, love, encouragement and prays of day and right make me able to get such success and honor, along with all hard working and respected

Teachers.

Acknowledgments

First we would like to thank without end to our greater AllAH and prophet MOHAMED, then we would like to express about appreciation and thanks to our supervisor Dr. Belgiss Abdelaziz. And I would like to thank for everyone who help us,

To our teachers,

To our colleagues.

To our burn candles that illuminate for others

He taught me to teach the characters.

Abstract

Differential Algebraic Equation arise a variety of application. There for their analysis and numerical treatment plays an important role. We give examples of DAEs are considered showing their importance for practical problem and known index concept. In the context of the tractability index existence and uniqueness of solution for low index linear DAEs. The main tool is a procedure to doucaple the DAE into it's dynamical and algebraic part and use result to study numerical method when applied to linear DAEs.

We present two cases application of series method to find some solution of DAEs system by processing of the series solutions with Laplace-Pade (LP). Finally found analytical solution of (PDAEs) in two system index-1 and index-3 by (LPPSM).

Abstract in Arabic

المعادلات التفاضلية الجبرية تظهر في انواع مختلفة من التطبيقات والتي لها دور مهم في المعالجات النظرية والعددية ولقد أعطينا أمثلة من المعادلات التفاضلية الجبرية (DAEs) توضح أهميتها في المشاكل العملية وقمنا بتعريف مؤشر لكل DAE وعن طريق مؤشر التتبع أثبتنا وجود ووحدانية الحل للمعادلات التفاضلية الجبرية الخطية والاداة الرئيسية هي إجراء الفصل لـDAEs إلى جزئيها الديناميكي والجبري ويتم إستخدام النتائج لدراسة سلوك الطرق العددية عند تطبيقها على المعادلات التفاضلية الجبرية الخطية .

قدمنا تطبيقين لحالتين لطريقة متسلسلة القوى (PSM) لإيجاد بعض الحلول لنظام (DAEs) من خلال معالجة الحل بإستخدام لابلاس-بادي (LP) وأخيرا أوجدنا الحل التحليلي للمعادلات التفاضلية الجبرية الجزئية لنظامين بإستخدام (LPPSM).

The Contents

Chapter 1
Introduction to Differential Algebraic Equations

We consider implicit differential equations

$$f(x'(t), x(t), t) = 0 \qquad (*)$$

On an interval $\mathcal{J} \subset \mathbb{R}$. If $\frac{\partial f}{\partial x'}$is nonsingular, then it is possible to formally solve (*) for x'in order to obtain an ordinary differential equation. However, if $\frac{\partial f}{\partial x'}$is singular, this is no longer possible and the solution $x$has to satisfy certain algebraic constraints. Thus equations (*) where $\frac{\partial f}{\partial x'}$is singular are referred to as differential-algebraic equationsor DAEs.

These notes aim at giving an introduction to differential-algebraic equations and are based on four lectures given by the author during his stay at the University of
Auckland in 2003.
We deal with examples of DAEs. Here problems from different kinds of applications are considered in order to stress the importance of DAEs when modeling practical problems.

The each DAE is assigned a number, the index, to measure its complexity concerning both theoretical and numerical treatment. Several index notions are introduced, each of them stressing different aspects of the DAE considered.
Special emphasis is given to the tractability index for linear DAEs.

The definition of the tractability index in the second section gives rise to a detailed analysis concerning existence and uniqueness of solutions. The main tool is a procedure to decouple the DAE into it's dynamical and algebraic part. In section three this analysis is carried out for linear DAEs with low index as it was established by März.

The results obtained, especially the decoupling procedure, are used in the fourth section to study the behavior of numerical methods when applied to linear DAEs.

1.1 Examples of differential-algebraic equations.

Modeling with differential-algebraic equations plays a vital role, among others, for constrained mechanical systems, electrical circuits and chemical reaction kinetics. We will give examples of how DAEs are obtained in these fields.

We will point out important characteristics of differential-algebraic equations that distinguish them from ordinary differential equations.

More information about differential-algebraic equations can be found.

1.1.1 Constrained Mechanical Systems:

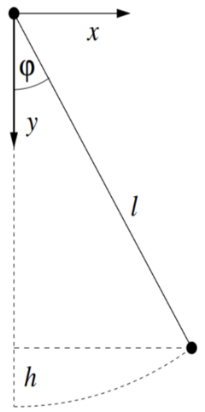


Figure 1.1: The mathematical pendulum

Consider the mathematical pendulum in figure 1.1. Let $m$be the pendulum's mass which is attached to a rod of length l. In order to describe the pendulum in Cartesian coordinates we write down the potential energy

$$U(x, y) = mgh = mgl - mgy \tag{1.1}$$

Where $(x(t), y(t))$ is the position of the moving mass at time t. The earth's acceleration of gravity is given by g, the pendulum's height is h. If we denote derivatives of x and $y$by $\dot{x}$ and $\dot{y}$ respectively, the kinetic energy is given by

$$T(\dot{x}, \dot{y}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2). \tag{1.2}$$

The term $\dot{x}^2 + \dot{y}^2$ describes the pendulum's velocity. The constraint is found to be

$$0 = g(x, y) = x^2 + y^2 - l^2. \tag{1.3}$$

(1.1)-(1.3) are used to form the Lagrange function

$$L(q, \dot{q}) = T(\dot{x}, \dot{y}) - U(x, y) - \lambda g(x, y).$$

Here $q$denotes the vector $q = (x, y, \lambda)$. Note that λ¸ serves as a Lagrange multiplier. The equations of motion are now given by Euler's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \quad k = 1,2,3.$$

We arrive at the system

$$\begin{aligned} m\ddot{x} + 2\lambda x &= 0, \\ m\ddot{y} - mg + 2\lambda y &= 0, \\ g(x, y) &= 0. \end{aligned} \tag{1.4}$$

By introducing additional variables $u = \dot{x}$ and $v = \dot{y}$ we see that (1.4) is indeed of the form (*).

When solving (1.3) as an initial value problem, we observe that each initial value
$x(t_0), y(t_0) = (x_0, y_0)$ has to satisfy the constraint (1.3) (consistent initialization).
No initial condition can be posed for λ, as λis determined implicitly by (1.4).

Of course the pendulum can be modeled by the second order ordinary differential equation

$$\ddot{\varphi} = -\frac{g}{l}\sin\varphi$$

When the angle $\varphi$is used as the dependent variable. However for practical problems a formulation in terms of a system of ordinary differential equations is often not that obvious, if not impossible.

1.1.2 Electrical circuits:

Modern simulation of electrical networks is based on modeling techniques that allow an automatic generation of the model equations. One of the techniques most widely used is the modified nodal analysis (MNA).

Example (1.1.1):

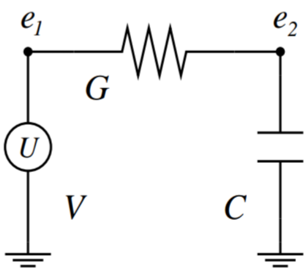


Figure 1.2: A simple circuit

To see how the modified nodal analysis works, consider the simple circuit in figure 1.2 .It consists of a voltage source$v_V = v(t)$, a resistor with conductance $G$and a capacitor with capacitance$C > 0$. The layout of the circuit can be described by

$$A_a = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

where the columns of $A_a$correspond to the voltage, resistive and capacitive branches respectively. The rows represent the network's nodes, so that -1 and 1 indicate the nodes that are connected by each branch under consideration. Thus $A_a$assigns a polarity to each branch.

By construction the rows of $A_a$are linearly dependent. However, after deleting one row the remaining rows describe a set of linearly independent equations; the node corresponding to the deleted row will be denoted as the ground node. The matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

is called the incidence matrix. It is now possible to formulate basic physical laws in terms of the incidence matrix A. Denote with $i$and $v$the vector of branch currents and voltage drops respectively and introduce the vector e of node potentials.

For each node the node potential is it's voltage with respect to the ground node.

- Kirchhoff's Current Law (KCL):

$$\text{For each node the sum of all currents is zero}\} \Longrightarrow A_i = 0.$$

- Kirchhoff's Voltage Law (KVL):

$$For\ each\ loop\ the\ sum\ of\ all\ voltages\ is\ zero.\} \Longrightarrow v = A^T e$$

For the circuit in figure 1.2 KCL and KVL read

$$-i_V + i_G = 0; \; -i_G + i_C = \; 0 \qquad (1.5.a)$$

and

$$v_V = -e_1, \qquad v_G = e_1 - e_2, \qquad v_C = e_2 \qquad (1.5.b)$$

respectively. If we assume ideal linear devices the equations modelling the resistor

and the capacitor are

$$i_G = Gv_G, \qquad i_C = C\,\frac{dv_c}{dt}\,. \qquad (1.5.c)$$

Finally we have

$$v_V = v(t) \qquad (1.5.d)$$

For the independent source which is thought of as the input signal driving the system.

The system (1.5) is called the *sparse tableau*. The equations of the modified nodal analysis are obtained from the sparse tableau by expressing voltages in terms of node potential via (1.5.b) and currents, where possible, by device equations (1.5.c):

$$\left.\begin{aligned} -i_V + G(e_1 - e_2) = 0 \\ -G(e_1 - e_2) + C\frac{de_2}{dt} = 0 \\ -e_1 = v \end{aligned}\right\}$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ C \\ 0 \end{pmatrix}\left(\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix}\right)' + \begin{pmatrix} G & -G & -1 \\ -G & G & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ i_V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \qquad (1.5)$$

The MNA equations reveal typical properties of DAEs:

(i) Only certain parts of $x = (e_1, e_2, i_V)^T$ need to be differentiable. It is sufficient if e_1 and i_V are continuous.
(ii) Any initial condition $x(t_0) = x_0$ needs to be consistent, i.e. there is a solution
passing through x_0. Here this means that we can pose an initial condition for e_2 or i_V only.
For (1.6) it is sufficient to solve the ordinary differential equation

$$e_2'(t) = -C^{-1}G\big(v(t) + e_2(t)\big).$$

$e_2(t)$ can be thought of as the output signal. The remaining components of the solution are uniquely determined as

$$e_1(t) = -v(t), i_V(t) = G\big(e_1(t) + e_2(t)\big).$$

Another important feature that distinguishes DAEs from ordinary differential equations is that the solution process often involves differentiation rather than integration.
This is illustrated in the next example.

Example (1.1.2):

If we replace the independent voltage in figure 1.2 source by a current source $i_I = i(t)$ and the capacitor by an inductor with inductance L, we arrive at the circuit in figure 1.3. The sparse tableau now reads

$$-i_I + i_G = 0, \qquad -i_G + i_L = 0, \tag{1.6.a}$$

$$v_I = -e_1, v_G = e_1 - e_2, \qquad v_L = e_2, \tag{1.6.b}$$

$$i_G = Gv_G, \qquad v_L = L\frac{di_L}{dt}, \tag{1.6.c}$$

$$i_I = i(t). \tag{1.6.d}$$

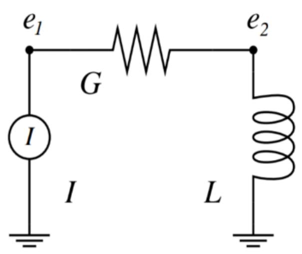


Figure 1.3: Another simple circuit

Thus the modified nodal analysis leads to

$$G(e_1 - e_2) = i(t)$$

$$-G(e_1 - e_2) + i_L = 0$$

$$L\frac{di_L}{dt} - e_2 = 0 \tag{1.7}$$

The solution is given by

$$i_L = i(t),$$
$$e_2 = L\frac{di_L}{dt} = L\frac{di(t)}{dt},$$
$$e_1 = e_2 + G^{-1}i(t) = L\frac{di(t)}{dt} + G^{-1}i(t),$$

under the assumption that the current $i(t)$ is differentiable. Notice that all component values are fixed. To solve for e_2 we need to differentiate the current i.

1.1.3 A transistor amplifier:

We will now present a more substantial example adapted . Consider the transistor amplifier circuit in figure 1.4.

The circuit consists of eight nodes, $U_e(t) = 0.1\ sin(200\pi t)$ is an arbitrary 100 Hz input signal and e_8, the node potential of the 8th node, is the amplified output. The circuit contains two transistors. We model the behavior of these semiconductor devices by voltage controlled current sources

$$I_{gate} = (1-\alpha)g\big(e_{gate} - e_{source}\big),$$
$$I_{drain} = \alpha g(e_{gate} - e_{source});$$
$$I_{source} = g(e_{gate} - e_{source})$$

with a constant $\alpha = 0.99, g$is the nonlinear function

$$g : \mathbb{R} \longrightarrow \mathbb{R};\ v \longmapsto g(v) = \beta\left(exp\left(\frac{v}{U_F}\right)\right);\ \beta = 10^{-6}, U_F = 0.026.$$

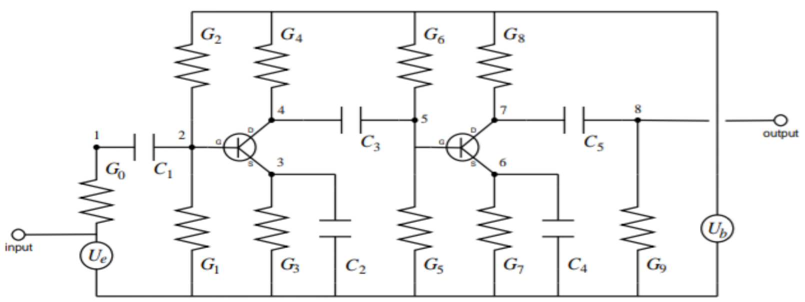


Figure 1.4: Circuit diagram for the transistor amplifier

It is also possible to use PDE models (partial differential equations) to model semiconductor devices. This approach leads to abstract differential-algebraic systems.

The modified nodal analysis can now be carried out as in the previous examples.

Consider for instance the second node. KCL implies that

$$\begin{aligned} 0 &= -i_{C_1} - i_{R_1} - i_{R_1} - i_{gate,2} \\ &= -C_1 v_{C_1}^{'} - v_{G_1} G_1 - v_{G_2} G_2 - (1-\alpha) g(e_2 - e_3) \\ &= -C_1 - (e_2 - e_1)^{'} - e_2 G_1 - (e_2 - U_b) G_2 + (\alpha - 1) g(e_2 - e_3) \\ &= C_1 - (e_1 - e_2)^{'} - e_2 (G_1 + G_2) + U_b G_2 + (\alpha - 1) g(e_2 - e_3) \end{aligned}$$

$U_b = 6$is the working voltage of the circuit and the remaining constant parameters
of the model are chosen to be

$$G_0 = 10^{-3}, G_k = \frac{1}{9} \cdot 10^{-3}, k = 1, \dots, 9, \qquad C_k = 10^{-6}, k = 1, \dots, 5.$$

A similar derivation for the other nodes leads to the quasi-linear system

$$A(Dx(t))^{'} = b(x(t)) \tag{1.8}$$

with

$$A = \begin{pmatrix} C_1 & 0 & 0 & 0 & 0 \\ -C_1 & 0 & 0 & 0 & 0 \\ 0 & -C_2 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 \\ 0 & 0 & -C_3 & 0 & 0 \\ 0 & 0 & 0 & -C_4 & 0 \\ 0 & 0 & 0 & 0 & C_5 \\ 0 & 0 & 0 & 0 & -C_5 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$b(x) = \begin{pmatrix} -U_e G_0 + e_1 G_0 \\ -U_b G_2 + e_2 (G_1 + G_2) - (\alpha - 1) g(e_2 - e_3) \\ -g(e_2 - e_3) + e_3 G_3 \\ -U_b G_4 + e_4 G_4 + \alpha g(e_2 - e_3) \\ -U_b G_6 + e_5 (G_5 + G_6) - (\alpha - 1) g(e_5 - e_6) \\ -g(e_5 - e_6) + e_6 G_7 \\ -U_b G_8 + e_7 G_8 + \alpha g(e_5 - e_6) \\ e_8 G_9 \end{pmatrix}$$

A numerical solution of (1.8) can be calculated using Dassl or Radau5.
A mathematically more general version of (1.9) is

$$A(x(t), t)(D(t)x(t))^{'} = b(x(t), t) \tag{1.9}$$

with a solution dependent matrix A. We identified $x_i$with the node potential e_i. Let us assume that $N_0(t) = kerA(x(t), t)D(t)$ does not dependent on x. We will follow and investigate (10) in more detail. With

$$f(y,x,t) = A(x(t),t)y - b(x(t),t),$$
(1.9) can be written as
$$f\left(\left(D(t)x(t)\right)', x(t), t\right) = 0. \tag{1.10}$$
Denote $B(y,x,t) = f_x'(y,x,t)$ and let $Q(t)$ be a continuous projector function onto $N_0(t)$. Calculate
$$G_1(y,x,t) = A(x,t)D(t) + B(y,x,t)Q(t).$$
For the transistor amplifier (1.10) in figure 1.4 this matrix is always nonsingular. We want to use this matrix in conjunction with the Implicit Function Theorem to derive an ordinary differential equation that determines the dynamical flow of (1.9). Let $D(t)^-$be defined by
$$DD^-D = D, \qquad DD^- = I_5,$$
$$D^-DD^- = D^-, \qquad D^-D = P := I_8 - Q.$$
$I_k$denotes the identity in $\mathbb{R}^k$and $D(t)^-$is a generalized reflexive inverse of $D(t)$. For more information on generalized matrix inverses.
For a solution x of (1.10) define
$$u(t) = D(t)x(t), w(t) = D(t)^-u'(t) + Q(t)x(t).$$
Observe that $A(Dx)' = ADw$and $x = Px + Qx = D^-Dx + Qx \ = D^-u + Qw$. Thus it holds that
$$(1.10) \iff ADw + b(x,t) \iff F(w,u,t) := f(Dw, D^-u + Qw, t) = 0.$$
Note that
$$u' = R'u + Dw,$$
since $Dw = DD^-u + DQx = (Ru)' = u' - R'u$. The mapping Fcan be studied without requiring x to be a solution. Let $(y_0, x_0, t_0) \in \mathbb{R}^{5+8+1}$, such that
$$f(y_0, x_0, t_0) = 0.$$
For $w_0 = D(t_0)^-y_0 + Q(t_0)x_0, u_0 = D(t_0)x_0$ it follows that

- $F(w_0, u_0, t_0) = f(y_0, x_0, t_0) = 0,$
- $F_w'(w_0, u_0, t_0) = G_1(y_0, x_0, t_0)$

is nonsingular.
Due to the Implicit Function Theorem there is a $\varrho > 0$ and a smooth mapping
$$\omega : B_\varrho(u_0, t_0) \times \mathcal{I} \to \mathbb{R}^m$$
satisfying
$$\omega(u_0, t_0) = w_0, \qquad F(\omega(u,t), u, t) = 0 \quad \forall (u,t) \in B_\varrho(u_0, t_0).$$
We use $\omega$to define
$$x(t) = \ D(t)^-u(t) + Q(t)\omega(u(t),t), t \in \mathcal{I}.$$
where$u$is the solution of the ordinary differential equation
$$u^-(t) = R'(t)u(t) + D(t)\omega(u(t),t), u(t_0) = D(t_0)x_0. \ (1.11)$$

x is indeed a solution of (1.9), since

$$f\left(\left(D(t)x(t)\right)', x(t), t\right) = f(u', D^- u + Q\omega(u,t), t) = F(\omega, u, t) = 0.$$

This example shows that there is a formulation of the problem in terms of an ordinary differential equation (1.11) as was the case for the mathematical pendulum in the first example. However, (1.11) is available only theoretically as it was obtained using the Implicit Function Theorem. Thus we have to deal directly with the DAE formulation (1.9) when solving the problem. Nevertheless, (1.11) will play a vital part in analyzing (10) and in analyzing numerical methods applied to (1.9).

It will be shown how (1.11) can be obtained explicitly for linear DAEs. devoted to showing that there are numerical methods that, when applied directly to (1.9), behave as if they were integrating (1.11), given that (1.11) satisfies some additional properties. In this case results concerning convergence and order of numerical methods can be transferred directly from ODE theory to DAEs.

1.1.4 The Akzo Nobel Problem:

The last example originates from the Akzo Nobel Central Research in Arnhem, the

Netherlands, and is again taken. It describes a chemical process in which two species, FLB and ZLU, are mixed while carbon dioxide is continuously added. The resulting species of importance is ZLA. The reaction equations are given.

$$2\ FLB + \frac{1}{2} CO_2 \xrightarrow{k_1} FLBT + H_2O$$

$$ZLA + FLBK \overset{\boldsymbol{k_2/K}}{\rightleftarrows} FLBT + ZHU$$

$$FLB + 2ZHU + CO_2 \xrightarrow{k_3} LB + nitrate$$

$$FLB.ZHU + \frac{1}{2} CO_2 \xrightarrow{k_1} ZLA + H_2O$$

$$ZLB + ZHU \rightleftarrows FLB.ZHU$$

The last equation describes an equilibrium where the constant

$$K_s = \frac{[FLB.ZHU]}{[FLB] \cdot [ZHU]}$$

plays a role in parameter estimation. Square brackets denote concentrations.

The chemical process is appropriately described by the reaction velocities

$$r_1 = k_1 \cdot [FLB]^4 \cdot [CO_2]^{\frac{1}{2}},$$

$$r_2 = k_2 \cdot [FLBT] \cdot [ZHU],$$

$$r_3 = \frac{k_2}{K} \cdot [FLB] \cdot [ZLA],$$

$$r_4 = k_3 \cdot [FLB] \cdot [ZHU]^2,$$

$$r5 \ = k_4 \cdot [FLB.ZHU]^2 \cdot [CO_2]^{\frac{1}{2}},$$

The inflow of carbon dioxide per volume unit is denoted by $F_{in}$and satisfies

$$F_{in} = \ klA \cdot \left(\frac{p(CO_2)}{H} - [CO_2] \right).$$

$klA$is the mass transfer coefficient, $H$the Henry constant and $p(CO_2)$ is the partial carbon dioxide pressure. It is assumed that $p(CO_2)$ is independent of $[CO_2]$. The various constants are given by

$$k_1 = 18.7, \qquad k_4 = 0.42, \quad K_s = 115.83,$$
$$k_2 = 0.58, \qquad K = 34.4, \qquad p(CO_2) = 0.9,$$
$$k_3 = 0.09, \qquad klA = 3.3, \qquad H = 737.$$

If we identify the concentrations

$$[FLB], [CO_2], [FLBT], [ZHU], [ZLA], [FLB.ZHU]$$

with $x_1, \dots, x_6$ respectively, we obtain the differential-algebraic equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} x'(t) = \begin{pmatrix} -2r_1 + r_2 - r_3 - r_4 \\ -\frac{1}{2}r_1 \quad - r_4 - \frac{1}{2}r_5 + F_{in} \\ r_1 - r_2 + r_3 \\ -r_2 + r_3 - 2r_4 \\ r_2 - r_3 \qquad\quad + r_5 \\ K_s x_1 x_4 - x_6 \end{pmatrix} \tag{1.12}$$

This DAE can be analyzed in a similar way as the previous example. The matrix

$$G_1 = AD + BQ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0.42x_6\sqrt{x_2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.84x_6\sqrt{x_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is always nonsingular. Here, $A = D = diag(1,1,1,1,1,0)$ was chosen.

1.2 Index Concepts for DAEs.

We saw that DAEs differ in many ways from ordinary differential equations. For instance the circuit in figure 1.3 lead to a DAE where a differentiation process is involved when is solving the equations. This differentiation needs to be carried out numerically, which is an unstable operation. Thus there are some problems to be expected when solving these systems. We try to measure the difficulties arising in the theoretical and numerical treatment of a given DAE.

1.2.1 The Kronecker Index:

Let's take linear differential-algebraic equations with constant coefficients as a starting point. These equations are given as

$$Ex'(t) + Fx(t) = q(t), \qquad t \in \mathcal{J} \tag{1.13}$$

with $E, F \in L(\mathbb{R}^m)$. Even for (2.1) existence and uniqueness of solutions is not apriori clear.

Example (1.2.1): For the DAE

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x'(t) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) = 0$$

a solution $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is given by$x_2(t) = g(t)$ and $x_1(t) = -\int_{t_0}^{t} g(s)\, ds$, where thefunction $g \in C(\mathcal{J}, \mathbb{R})$ can be chosen arbitrarily.

In order to exclude examples like (1.13) we consider the matrix pencil ¸$\lambda E + F$. The pair (E, F) is said to form a regular matrix pencil, if there is a λ ¸ such that $det(\lambda E + F) \neq 0$.

A simultaneous transformation of $E$and Finto Kronecker normal form makes a solution of (1.13) possible.

Theorem (1.2.2): (Kronecker) Let (E, F) form a regular matrix pencil. Then there exist nonsingular matrices U *and* V *such that*

$$UEV = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \qquad UFV = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix};$$

where $N = diag(N_1, \dots, N_k)$is a block-diagonal matrix of Jordan-blocks N_i to the eigenvalue 0.

Notice that due to the special structure of Nthere is $\mu \in N$ such that $N^{\mu-1} \neq 0$but $N^{\mu} = 0$. μ is known as N's index of nilpotency. It does not depend on the special choice of $U$and.

We solve (1.13) by introducing the transformation

$$x = V\begin{pmatrix} u \\ v \end{pmatrix}, \ \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = Uq(t).$$

Thus (1.13) is equivalent to

$$UEV(V^{-1}x(t))' + UFVV^{-1}x(t) = Uq(t)$$

$$\Leftrightarrow \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}' + \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}. \tag{1.14}$$

The first equation is an ordinary differential equation

$$u'(t) + Cu(t) = a(t)$$

for the $u$component. The second equation reads

$$v(t) = b(t) - Nv'(t) = b(t) - N\big(b'(t) - Nv''(t)\big)$$
$$= b(t) - Nb'(t) + N^2 v''(t) = \cdots = \sum_{i=0}^{\mu-1} (-N)^i b^{(i)}(t) \quad (1.15)$$

determining the $v$component completely by repeated differentiation of the right hand side b. Since numerical differentiation is an unstable process, the index $\mu$is a measure of numerical difficulty when solving (1.13).

Definition (1.2.3): Let (E, F) form a regular matrix pencil. The (Kronecker) index of (14) *is* 0 if E is nonsingular and μ, *i.e. N's* index of nilpotency, otherwise.

1.2.2 The differentiation index:

How can Definition (1.2.3) be generalized to the case of time dependent coefficients or even to nonlinear DAEs? If we consider (1.15) again, it turns out that

$$v'(t) = \sum_{i=0}^{\mu-1} (-N)^i b^{(i+1)}(t),$$

meaning that exactly μdifferentiations transform (1.14) into a system of explicit ordinary differential equations. This idea was generalized by Gear, Petzold. The following definition.

Definition (1.2.4): The nonlinear *DAE*

$$f(x'(t), x(t), t) = 0 \qquad (1.16)$$

Has (differentiation) index μ if μ is the minimal number of differentiations

$$f(x'(t), x(t), t) = 0\,, \frac{f(x'(t), x(t), t)}{dt} = 0, \ldots, \frac{d^{\mu} f(x'(t), x(t), t)}{dt^{\mu}} = 0 \quad (1.17)$$

such that the equations (1.16) allow to extract an explicit ordinary differential system $\mathrm{x}'(\mathrm{t}) = \varphi(\mathrm{x}(\mathrm{t}), \mathrm{t})$ using only algebraic manipulations.

We now want to look at four examples to get a feeling of how to calculate the differentiation index.

Example (1.2.5): For linear DAEs with constant coefficients forming a regular matrix pencil we have differentiation index μif and only if the Kronecker index is μ.

Example (1.2.6): Consider the system

$$x' = f(x, y) \qquad (1.18.a)$$
$$0 = g(x, y)\,. \qquad (1.18.b)$$

The second equation yields

$$0 = \frac{dg(x, y)}{dt} = g_x(x, y)x' + g_y(x, y)y'.$$

If $g_y(x,y)$ is nonsingular in a neighbourhood of the solution, (1.18) is transformed to

$$x' = f(x,y) \qquad (1.18.a')$$

$$y' = -g_y(x,y)^{-1}g_x(x,y)x' = -g_y(x,y)^{-1}g_x(x,y)f(x,y) \qquad (1.18.b')$$

and the differentiation index is $\mu = 1$.

The DAE (1.5) modeling the circuit in figure 1.2 is of the form (1.18) with

$$x = e_2, y = \binom{e_1}{i_v}, f(x,y) = \frac{G}{C}(e_1 - e_2) \; and \; g(x,y) = \begin{pmatrix} G(e_1 - e_2) - i_v \\ e_1 + v_V \end{pmatrix}$$

Note that $g_y(x,y) = \begin{pmatrix} G & -1 \\ 1 & 0 \end{pmatrix}$ is nonsingular so that (1.5) is an index 1 equation.

Example (1.2.7): The system

$$x' = f(x,y) \qquad (1.19.a)$$

$$0 = g(x) \qquad (1.19.b)$$

can be studied in a similar way. (1.19.b) gives

$$0 = \frac{dg(x)}{dt} = g_x(x)x' = g_x(x)f(x,y) = h(x,y). \quad (1.19.b')$$

Comparing with example (1.2.6) we know that (1.19.a), (1.19.b') is an index 1 system if $hy(x;\, y)$ remains nonsingular in a neighborhood of the solution. If this condition holds, (1.19) is of index 2, as two differentiations produce

$$x' = f(x,y) \qquad (1.19.a)$$

$$y' = -h_y(x,y)^{-1}h_x(x,y)f(x,y)$$

$$= -\left(g_x(x)f_y(x,y)\right)^{-1}\Big(g_{xx}(x)\big(f(x,y), f(x,y)\big)$$

$$+ g_x(x)f_x(x,y)f(x,y)\Big). \qquad (1.19.b'')$$

(1.19.b') defines the "hidden constraint" of the index 2 equation (1.19).

The DAE (1.7) modeling the circuit in figure 1.3 can be written as

$$i_L' = \frac{1}{L}e_2 = f(i_L, e_2) \qquad (1.20.a)$$

$$0 = i_L - i_I = g(i_L). \qquad (1.20.b)$$

The remaining variable e_1 is determined by $e_1 = e_2 + G^{-1}i_I$, where $i_I$is the input current. (1.20) is of the form (1.18) with $x = i_L$and $y = e_2. h_y(x,y) = g_x f_y = 1 \cdot \frac{1}{L}$ is nonsingular and the index is 2.

Example (1.2.8): Finally take a look at the system

$$x' = f(x,y) \qquad (1.21.a)$$

$$y' = g(x,y,z) \qquad (1.21.b)$$

$$0 = h(x). \qquad (1.21.c)$$

Differentiation of (1.2.4c) yields

$$0 = \frac{dh(x)}{dt} = h_x(x)x' = hx(x)f(x,y) = \hat{h}(x,y) \ (1.21.c')$$

and

$$x' = \binom{x}{y}' = \begin{pmatrix} f(x,y) \\ g(x,y,z) \end{pmatrix} = f(x,\mathfrak{y})(1.21.a) \text{ and } (1.21.\text{b}) \ \ (1.21.\text{a})$$

$$0 = \hat{h}(x,y) = g(x) \ (1.2.4c') \qquad (1.21.b)$$

is of the form (1.19) with $\mathfrak{x} = \binom{x}{y}$ and $\mathfrak{y} = z$. Define

$$h(\mathfrak{x},\mathfrak{y}) = g_x(x)f(\mathfrak{x},\mathfrak{y})$$

and compare with (1.19.b') to find that (1.21) is of the index 2 if

$$\mathfrak{h}_{\mathfrak{y}}(\mathfrak{x},\mathfrak{y}) = g_x(x)f_{\mathfrak{y}}(\mathfrak{x},\mathfrak{y}) = \left(\hat{h}_x\hat{h}_y\right)\binom{f_z}{g_z}$$

$$= \left(h_{xx}(f,\cdot) + h_xf_xh_{xy}(f,\cdot) + h_xf_y\right)\binom{0}{g_z} = h_xf_yg_z$$

remains nonsingular. This shows that (1.20) is an index 3 system if the matrix $h_x(x)f_y(x,y)g_z(x,y,z)$is invertible in a neighbourhood of the solution (x,y,z).

Hidden constraints are given by (1.20.c') but also by

$$\mathfrak{h}(\mathfrak{x},\mathfrak{y}) = g_x(x)f(\mathfrak{x},\mathfrak{y}) = h_{xx}(f,f) + h_xf_xf + h_xf_yg = 0,$$

which is condition (1.19.b') in terms of the index 2 system (1.21).

Consider again the mathematical pendulum from section 1.1 in the formulation

$$x' = u = f_1(x,y,u,v)$$
$$y' = v = f_2(x,y,u,v)$$
$$u' = -\frac{2}{m}\lambda_x = g_1(x,y,u,v,\lambda)$$
$$v' = +g - \frac{2}{m}\lambda_y = g_2(x,y,u,v,\lambda)$$
$$0 = x^2 + y^2 - l^2 = h(x,y). \qquad (1.22)$$

For $l > 0$ the value $h(x,y)f(u,v)g_\lambda = -\frac{4}{m}(x^2+y^2)$ is always nonsingular so that (1.12) is an index 3 problem.

1.3 The Tractability Index.

In definition (1.2.4) the function $f$is assumed to be smooth enough to calculate the derivatives (1.18). In applications this smoothness is often not given. For instance in circuit simulation input signals are continuous but often not differentiable.

We want to study the tractability index introduced by Griepentrog, M¨arz . In fact we consider the generalization of the tractability index proposed by M¨arz .

The idea is to replace the smoothness requirements for the coefficients by the requirement on certain subspaces to be smooth, to define the tractability index we introduce linear DAEs with properly stated leading terms. A second matrix $D(t)$ is used when formulating the DAE as

$$A(t)\big(D(t)x(t)\big)' + B(t)x(t) = q(t). \qquad (1.23)$$

In contrast to the standard formulation

$$E(t)x(t)' + F(t)x(t) = q(t) \qquad (1.24)$$

the leading term in (1.13) precisely figures out which derivatives are actually involved.

The formulation (1.23) was first used to study linear DAEs and their adjoint equations. For (1.24) the adjoin equation

$$(E^{*}y)' - F^{*}y = p$$

is of a different type. For the more general formulation (1.23) the adjoint equation fits nicely into this general form:

$$D^{*}(A^{*}y)' - B^{*}y = p.$$

We consider linear DAEs (1.13) with matrix coefficients

$$A \in C\big(\mathfrak{I}, L(\mathbb{R}^{n}, \mathbb{R}^{m})\big), \ \ D \in C\big(\mathfrak{I}, L(\mathbb{R}^{m}, \mathbb{R}^{n})\big), \qquad B \in C\big(\mathfrak{I}, L(\mathbb{R}^{m})\big).$$

Neither Anor Dneeds to be a projector function. Note that $A(t)$ and $D(t)$ are rectangular matrices in general. However, $A$and $D$are assumed to be well matched in the following sense.

Definition (1.3.1): The leading term of (24) is properly stated if

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^{n}, t \in \mathfrak{I},$$

and there is a continuously differentiable projector function $R \in C^{1}\big(\mathfrak{I}, L(\mathbb{R}^{n})\big) with$

$$imR(t) = imD(t), \qquad kerR(t) = kerA(t) t \in \mathfrak{I}.$$

By definition $A(t)$ and $D(t)$ have a common constant rank if the leading term is properly stated .

Definition (1.3.2): A function $x\colon \mathcal{J} \longrightarrow \mathbb{R}^{m}$ is said to be a solution of (1.23) if

$$x \in C_{D}^{1}(\mathcal{J}, \mathbb{R}^{m}) = \{x \in C(\mathcal{J}, \mathbb{R}^{m}) | D_{x} \in C^{1}(\mathcal{J}, \mathbb{R}^{n})\}$$

Satisfies (1.23) point wise.

Let us point out that a solution $x$is a continuous function, but the part $D_{x} : \mathcal{J} \longrightarrow \mathbb{R}^{n}$is differentiable.

We now define a sequence of matrix functions and possibly time-varying subspaces.

All relations are meant point wise for $t \in \mathfrak{I}$. Let $G_{0} = AD$, $B_{0} = B$and for $i \geq 0$

$$\left.\begin{array}{c} N_i = \ker G_i \\ S_i = \{z \in \mathbb{R}^m | B_i z \in imG_i\} = \{z \in \mathbb{R}^m | Bz \in imG_i\} \\ Q_i = Q_i^2, imQ_i = N_i, P_i = I - Q_i \\ G_{i+1} = G_i + B_i Q_i, \\ B_{i+1} = B_i P_i - G_{i+1} D^- C'_{i+1} D P_0 \dots P_i, \\ C_{i+1} = D P_0 \dots P_{i+1} D^- \end{array}\right\} \quad (1.25)$$

Here, $D^- : \mathcal{I} \longrightarrow (\mathbb{R}^n, \mathbb{R}^m)$denotes the reflexive generalized inverse of D such that

$$DD^-D = D, D^-DD^- = D^-, DD^- = R, D^-D = P_0. \quad (1.26)$$

Note that $D^-$is uniquely determined by (1.16) and depends only on the choice of Q_0.

We now define a sequence of matrix functions and possibly time-varying subspaces All relations are meant point wise for $t \in \mathcal{I}$. Let $G_0 = AD$; $B_0 =$ Band for $i \geq 0$

$$\left.\begin{array}{c} N_i = ker\, G_i \\ S_i = \{z \in \mathbb{R}^m | B_{iz} \in im\, G_i\} = \{z \in \mathbb{R}^m | B_z \in im\, G_i\} \\ Q_i = Q_i^2, im\, G_i = N_i, p_i = 1 - Q_i \\ G_{i+1} = G_i + B_i Q_i \\ B_{i+1} = B_i p_i - G_{1+i} D^- G'_{i+1} D p_0 \dots p_i \\ G_{1+i} = D p_0 \dots p_{i+1} D^- \end{array}\right\} \quad (1.27)$$

Here, $D^-: \mathcal{I} \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$denotes the reflexive generalized inverse of Dsuch that

$$DD^-D = D;\ D^-DD^- = D^-;\ DD^- = R;\ D^-D = P_0. \quad (1.28)$$

Note that $D^-$is uniquely determined by (1.26) and depends only on the choice of Q_0 contains more details about generalized matrix inverses.

Definition (1.3.3): The DAE (1.23) with properly stated leading term is said to be a regular DAE with tractability index μ on the interval $\mathcal{I}$ if there is a sequence (1.25) such that

- G_i*has constant rank* r_i *on* $\mathcal{I}$,
- $Q_i \in C(\mathcal{I}, L(\mathbb{R}^m)), Dp_0 \dots p_i D^- \in C^1(\mathcal{I}, L(\mathbb{R}^n)), i \geq 0$
- $Q_{i+1} Q_j = 0, j = 0, \dots i, i \geq 0$
- $0 \leq r_0 \leq \cdots \leq r_{\mu-1} < m\ and\ r_\mu = m$

(1.23) is said to be a regular DAE if it is regular with some index μ.

This index criterion does not depend on the special choice of the projector functions Q_i[28]. As proposed in [24] the sequence (1.25) can be calculated automatically. Thus the index can be calculated without the use of derivative arrays [27].

Example (1.3.4): Consider the DAE

$$\begin{pmatrix} t \\ 1 \end{pmatrix}\left((-1 \ \ t)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\right)' + \begin{pmatrix} 1 & -t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0$$

taken from [25]. With $\ker A(t) = \{0\}, \operatorname{im} D(t) = \mathbb{R}$ the leading term is properly stated. Calculate

$G_0(t) = A(t)D(t) = \begin{pmatrix} -t & t^2 \\ -1 & t \end{pmatrix}$ and $N_0(t) = \left\{z \in \mathbb{R}^2 | \exists \alpha \in \mathbb{R}, z = \alpha\binom{t}{1}\right\}$

to find that $N_0(t) \subset kerB(t)$. Independently of the choice of Q_0 in (26) we have

$$G_1(t) = G_0(t) + B(t)Q(t) = G_0(t):$$

Similarly it follows that $G_i(t) = G_0(t)$ for every $i \geq 0$. Note that for every $\gamma \in C(\mathcal{J}; \mathbb{R})$ a solution is given by $x(t) = \gamma(t)\binom{t}{1}$. Solutions are therefore not uniquely determined. This is the case in spite of the fact that for every *t* the local matrix pencil $\lambda AD + (B + AD^-)$ of the reformulated DAE.

Similarly it follows that $G_i(t) = G_0(t)$ for every $i \geq 0$. Note that for every $\gamma \in C(\mathcal{J}; \mathbb{R})$ a solution is given by $x(t) = \gamma(t)(1^t)$ Solutions are therefore not uniquely determined. This is the casein spite of the fact that for every *t* the local matrix pencil $\lambda AD + (B + AD')$ of the reformulated DAE $0 = A(t)D(t)x'(t) + (B(t) + A(t)D'(t))x(t) = \begin{pmatrix} -t & t^2 \\ -1 & t \end{pmatrix} x'(t) + x(t)$ is regular.

Is indeed a generalization of the Kronecker index, i.e. in the case of constant coefficients, the Kronecker index and the tractability index for regular $DAEs$ coincide. To show this, define the subspaces

$$S_{EF} = \left\{\mathcal{Z} \in \mathbb{R}^{m \lceil F_z \in imE}\right\}, N_E = \ker E$$

for given matrices $E, F \in L(\mathbb{R}^m)$. Obviously for fixed $t \in \mathcal{J}$ we have $N_i(t) = N_{G_i(t)}$ and $S_i(t) = S_{G_i(t)B_i(t)}$ in sequence (1.25).

Lemma (1.3.5): For matrices $E, F \in L(\mathbb{R}^m)$ the following statements are equivalent*:*

1. $N_E \cap S_{EF} = \{0\}$
2. For every projector Q_E onto N_E the matrix $E + F Q_E$ is nonsingular
3. $N_E \oplus S_{EF} = \mathbb{R}^m$
4. (E, F) form a regular matrix pencil with Kronecker index 1.

Proof.
$(1.\Rightarrow 2.)$ $(\mathrm{E} + \mathrm{F}\,\mathrm{Q_E})\mathrm{z} = 0$implies $Q_E z \in S_{EF}$.Since $Q_E z \in \mathrm{N_E}$. too, we have $Q_E z \in \mathrm{N_E} \cap S_{EF} = \{0\}$ and $Q_E z = 0$ .Thus $0 = Ez + FQ_{EZ} = Ez$and $z \in \mathrm{N_E} = \mathrm{im}\,\mathrm{Q_E}$.
Therefore $z = Q_E z = 0$ $(2.\Rightarrow 3.)$$G_{EF} = E + FQ_E$is nonsingular. Show that $Q_* = Q_E G_{EF}^{-1} F$ is the projector onto N_E along S_{EF}.
$(3.\Rightarrow 4.)$There is exactly one projector Q_*onto along S_{EF}. Since $3.\Rightarrow 1.\Rightarrow 2.$we find $Q_* = Q_* G_{EF}^{-1} F$with $G_{EF} = E + FQ_*$. Let $p_* = 1 - Q_*$.Show that $\lambda E + F$is nonsingular for $\lambda \notin spec(p_* G_{EF}^{-1} F)$so that $(E;\, F)$ form a regular matrix pencil. We are nonsingular matrices $U,\ V \in GL_{\mathbb{R}}(m)$such that

$$V\ E\ U = \begin{pmatrix} 1 & \\ & N \end{pmatrix} = \bar{E}, \qquad V\ F\ U = \begin{pmatrix} C & \\ & 1 \end{pmatrix} = \bar{F}$$

It follows that $\mathrm{N_{\bar{E}}} = \ker \bar{\mathrm{E}} = U^{-1}\mathrm{N_E}$ and $S_{\bar{E}\bar{F}} = \{\, z \in \mathbb{R}^m | \bar{F}z \in im\bar{E}\} = U^{-1} S_{EF}$so that

$$\mathrm{N_{\bar{E}}} \cap S_{\bar{E}\bar{F}} = U^{-1}(\mathrm{N_E} \cap S_{EF}) = \{0\} \qquad (1.27)$$

On the other hand

$$\mathrm{N_{\bar{E}}} = \left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^m | \mathrm{z_1} = 0, \mathrm{z_2} \in \ker \mathrm{N}\right\} and$$

$$S_{\bar{E}\bar{F}} = \left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^m | \begin{pmatrix} C\ z_1 \\ z_2 \end{pmatrix} \in \mathrm{im}\bar{\mathrm{E}}\right\} = \left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^m | z_2 \in \mathrm{imN}\right\}$$

meaning that imN $\cap$ kerN $= \{0\}$ and$N = 0$. Thus the Kronecker index is 1.
$(4.\Rightarrow 1.)$Kronecker index 1 gives $N = 0$and $S_{\bar{E}\bar{F}} = \{0\}, \mathrm{N_{\bar{E}}} \cap S_{\bar{E}\bar{F}} = \{0\}$.
Use (1.27) to see $(\mathrm{N_E} \cap S_{EF}) = U(\mathrm{N_{\bar{E}}} \cap S_{\bar{E}\bar{F}}) = \{0\}$.

As in the previous section we now want to calculate the index of the DAEs modeling the electrical circuits in figure 1.2 and 1.3.

Example (1.3.6): For (1.6) we calculate $G_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $N_0 = \mathbb{R} \times \{0\} \times \mathbb{R}$ .Choose $Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$to find that $G_1 = \begin{pmatrix} G & 0 & -1 \\ -G & C & 0 \\ -1 & 0 & 0 \end{pmatrix}$ is nonsingular.
For the circuit in figure 1.2 we therefore have index 1.
Example (1.3.7): Equation (1.8) can be written as

$$\begin{pmatrix} 0 \\ 0 \\ \mathrm{L} \end{pmatrix}\left((0 \quad 0 \quad 1)\begin{pmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \\ \boldsymbol{i_L} \end{pmatrix}\right)' + \begin{pmatrix} G & -G & 0 \\ -G & G & 0 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ i_L \end{pmatrix} = \begin{pmatrix} i(t) \\ 0 \\ 0 \end{pmatrix} \qquad (1.28)$$

leading to $G_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{pmatrix} N_0 = \mathbb{R} \times \{0\} \times \mathbb{R}$. With $Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

It turns out that $G_1 = \begin{pmatrix} G & 0 & 0 \\ -G & G & 0 \\ 0 & -1 & L \end{pmatrix}$ is singular and $N_1 = \{z \in \mathbb{R}^3 | \exists \alpha \in \mathbb{R}, z_1 = z_2 = \alpha L\,, z_3 = \alpha\}$.

$Q_1 = \begin{pmatrix} 0 & 0 & L \\ 0 & 0 & L \\ 0 & 0 & 1 \end{pmatrix}$ is a projector onto N_1 satisfying $Q_1 Q_0 = 0$Finally $G_2 = \begin{pmatrix} G & -G & 0 \\ -G & G & 1 \\ 0 & -1 & L \end{pmatrix}$is nonsingular. Thus the index is 2. Note that the terms G'_{i+1}dissappear in (1.25) as Q_0 does not depend on t.

Nevertheless, in general the derivatives of G'_{i+1}appearing in the definition of B_{i+1} in sequence (1.25) are necessary in order to determine the index correctly. We will illustrate this in the next example which can be found in [25] as well.

Example (1.3.8): The DAE

$$x'_2 = q_1 - x_1 = f(x_1) \qquad (1.29.a)$$

$$x'_3 = q_2 - (1-\eta)x_2 - \eta^t(q_1 - x_1) \qquad = g(x_1, x_2\ , x_3) \quad (1.29.b)$$

$$0 = q_3 - \eta t x_2 - x_3 \qquad = h(x_1, x_2\ , x_3) \quad (1.29.c)$$

is easily checked to have (differentiation) index 3 as repeated differentiation of (1.29.c) yields

$$0 = q'_3 - q_2 + x_2$$

$$0 = q''_3 - q'_2 + q_1 - x_1$$

$$x'_1 = q'''_3 - q''_2 + q'_1$$

The index does not depend on the value of η We now write (1.18) as

$$\begin{pmatrix} 1 & 0 \\ \eta t & 1 \\ 0 & 0 \end{pmatrix}\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)' + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\eta & 0 \\ 0 & \eta t & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad (1.29)$$

with a properly stated leading term and calculate the sequence (1.29)

$$G_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \eta t & 1 \\ 0 & 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, G_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \eta t & 1 \\ 0 & 0 & 0 \end{pmatrix}, Q_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & \eta t & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \eta t+1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & -\eta t & 1 \\ 0 & \eta t & -1 \\ 0 & \eta t(\eta t+1) & \eta t+1 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \eta t+1 & 1 \\ 0 & \eta t & 1 \end{pmatrix}$$

Since $det G_3 = 1$, (1.18) is a regular DAE with index 3 independently of η However, if we dropped the terms G_{i+1}in (1.25) and defined $\mathcal{G}_{i+1} = \mathcal{G}_i + \mathcal{B}_i \mathcal{Q}_i$, $\mathcal{B}_{i+1} = \mathcal{B}_i \mathcal{P}_i$with $\mathcal{G}_0 = AD$ and $\mathcal{B} = B$we would obtain

$$\mathcal{G}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \eta t+1+\eta & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{Q}_2 = \begin{pmatrix} 0 & \eta t & 1 \\ 0 & -\eta t & -1 \\ 0 & (\eta t+1+\eta)\eta t & \eta t+1+\eta \end{pmatrix},$$

$$\mathcal{G}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \eta t+1+\eta & 1 \\ 0 & \eta * t & 1 \end{pmatrix}$$

$det\, \mathcal{G}_3 = 1 + \eta$ shows that $\mathcal{G}_3$ is singular for $\eta = -1$. Thus the use of the simpler version of $B_i$would lead to an index criterion not recognizing the index properly.

The previous example gives rise to investigating the relationship between $G_i$and $\mathcal{G}_i$further. Due to $G_i P_i = G_i$the matrix G_{i+1}may be written as

$$G_{i+1} = (G_i + B_{i-1} P_{i-1} Q_i)(I - P_i D^- C_i' D P_0 \ldots P_{i-1} Q_i).$$

For low indices we thus find $G_0 = \mathcal{G}_0\, G_1 = \mathcal{G}_1$, $G_2 = \mathcal{G}_2\,(I - P_i D^- C_i' D P_0 Q_1)$ with the nonsingular factor $I - P_i D^- C_i' D P_0 Q_1$The matrices G_2 and $\mathcal{G}_2$ have therefore common rank and we had to choose an index 3 to show the necessity of the second term in the definition of B_{i+1}. We don't have to restrict ourselves to linear DAEs (1.23). Nonlinear DAEs

$$A(x(t),t)(d(x,,t),t)' + b(x(t),t) = 0 \qquad (1.30)$$

can also be considered. For (31) the index is defined in such a way that all linearization along solutions have the same index μ in the sense of definition (1.3.5).

1.3.1 Some technical details:

In order to define the sequence (1.25) we introduced the generalized reflexive inverse $D^-$of D. Here we want to provide a short summary of the properties of generalized matrix inverses [41]. For a rectangular matrix $M \in L(\mathbb{R}^m, \mathbb{R}^n)$a matrix $\widetilde{M} \in L(\mathbb{R}^m, \mathbb{R}^n)$is called a generalized inverse of Mif $\widetilde{M} M \widetilde{M} = \widetilde{M}$If the condition $M \widetilde{M} M = M$ holds as well, then $\widetilde{M}$ is called a reflexive generalized inverse of M. Observe that for any reflexive generalized inverse $\widetilde{M}$ of M the matrices

$$\left(M\widetilde{M}\right)^2 = M\widetilde{M}M\widetilde{M} = M\widetilde{M}\left(\widetilde{M}M\right)^2 = \widetilde{M}M\,\widetilde{M}M = \widetilde{M}M$$

are projectors. Reflexive generalized inverses are not uniquely determined. Uniqueness is obtained if we require $M\widetilde{M}$ and $\widetilde{M}M$to be special projectors. We could, for instance, require them to be ortho-projectors

$$\left(M\widetilde{M}\right)^T = M\widetilde{M}\left(\widetilde{M}M\ \right)^T = \widetilde{M}M$$

In this case $\widetilde{M}$is called the Moore-Penrose inverse of M, often denoted by M^+ In the case of DAEs with properly stated leading terms we appropriated the projectors $P_0(t) \in L(\mathbb{R}^n)$and $R(t) \in L(\mathbb{R}^n)$to determine $D^-(t) \in L(\mathbb{R}^n, \mathbb{R}^m)$uniquely. $D^-(t)$is the reflexive generalized inverse of $D(t)$ defined by

$$DD^-D = DD^-DD^- = D^- \ , DD^- = R\ D^-D = P_0 \qquad (1.31)$$

If there was another generalized inverse $\widetilde{D}^-$satisfying (2.21), then

$$\widetilde{D}^- = \widetilde{D}^-D\widetilde{D}^- = \widetilde{D}^-R \qquad \widetilde{D}^-DD^- = P_0DD^- = D^-DD^- = D^-$$

In definition 2.11 the condition

$$Q_{i+1}Q_j = 0 \quad j = 0, \dots i, \ \ i \geq 0 \qquad (1.32)$$

is required. We will show briefly that the projectors $Q_j$in sequence (1.15) can always be chosen to satisfy (1.31). If for a given DAE (1.23) there was an index i_*such that $N_{i_*+1} \cap N_{i_*} \neq \{0\}$ then (1.23) would not be a regular DAE as all $G_j$would be singular. Thus $N_0 \cap N_1 \neq \{0\}$is a necessary condition for a regular DAE and the projector Q_1 onto N_1 can be chosen such that $N_0 \subset \ker Q_1$.

For an index $i \geq 1$let the projectors $Q_j$for $j = 1, \dots i$ satisfy $Q_jQ_k = 0 \quad k = 0, \dots j - 1$.Then $N_{i+1} \cap N_i = \{0\}$implies $N_{i+1} \cap N_j = \{0\}$ for $j = 1, \dots i$and Q_{i+1} can be chosen such that $N_0 \oplus N_1 \oplus \dots \oplus N_i \subset \ker Q_{i+1}$.

1.3.2 Other index concepts:

As seen in the previous sections a DAE can be assigned an index in several ways. In the case of linear equations with constant coefficients all index notions coincide with the Kronecker index. Apart from that, each index definition stresses different aspects of the DAE under consideration. While the differentiation index aims at finding possible reformulations in terms of ordinary differential equations, the tractability index is used to study DAEs without the use of derivative arrays. There are several other index concepts available. Here we want to introduce some of them briefly.

1.3.3 The Perturbation Index:

The perturbation index was introduced for nonlinear DAEs

$$f\left(x'(t), x(t)\right) = 0 \qquad (1.33)$$

(1.32) has perturbation index μalong a solution x on $\mathcal{J} = [0,T]$ if $\mu$is the smallest integer such that, for all functions $\hat{x}$having a defect

$$f\left(\hat{x}'(t), \hat{x}(t)\right) = \delta(\mathrm{t})$$

there exists on $\mathcal{J}$ an estimate

$$\|\hat{x}(t) - x(t)\| \leq C\left(\|\hat{x}(0) - x(0)\| + \underset{0 \leq \xi \leq t}{Max}\|\delta(\xi)\| + \cdots \underset{0 \leq \xi \leq t}{Max}\|\delta\mu^{-1}(\xi)\|\right)$$

Whenever the expression on the right-hand side is sufficiently small. Here C denotes a constant which depends only on $f$and the length of $\mathcal{J}$.

1.3.4 The geometric index:

Consider the autonomous DAE

$$f(x', x) = 0 \qquad (1.34)$$

and assume that $M_0 = f^{-1}(0)$s a smooth submanifold of $\mathbb{R}^m \times \mathbb{R}^n$Then the DAE (1.23) can be written as

$$(x', x) \in M_0$$

Each solution has to satisfy $x \in W_0 = \pi(M_0)$where $\pi: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the canonical projection onto the second component. If W_0 is a submanifold of $\mathbb{R}^m$ then (x', x) belongs to the tangent bundle $TW_0$of W_0. In other words

$$(x', x) \in M_1 = M_0 \cap TW_0$$

$M_1$is called the first reduction of M_0. Iterate this process to obtain sequence $M_0, M_1, M_2, ...$of manifolds where $M_{i+1}$is the first reduction of $M_i$and

$$(x', x) \in \bigcap_{i \geq 0} M_i$$

The geometric index is defined as the smallest integer µsuch that $M_\mu = M_{\mu+1}$.

1.3.5 The Strangeness Index:

This index notion is a generalization of the Kronecker index to DAEs

$$E(t)x'(t) + F(t)x(t) = q(t), \qquad t \in \mathcal{J} \subset R, \quad (1.35)$$

with time-dependent coefficients. The matrices U and V in theorem 2.2 now depend on t, i.e. (1.32) is transformed to

$$UEVy' + (UFV - UEV')y = U_q \Leftrightarrow \hat{E}y' + \hat{F}y = \hat{q}$$

The pairs of matrix functions (E, F) and $(\hat{E}, \hat{F})$ are said to be globally equivalent. The pairs of matrix functions (E, F) and $(\hat{E}, \hat{F})$ are said to be globally equivalent. For fixed $t \in \mathcal{J}$ define matrices $T(t), \hat{T}(t)$, $Z(t)$ and $V(t)$ such that the column vectors of $T(t), \hat{T}(t)$, $Z(t)$ and $V(t)$ span the subspaces $kerE(t), imE^{T}, kerE^{T}$ and $im(Z(t)^T N(t)T(t))^{\perp}$ respectively. Use these matrices to define

$$r(t) = \text{rank}E(t); \qquad d(t) = r(t) ¡ s(t);$$
$$a(t) = \text{rank}\big(Z(t)^T N(t)T(t)\big) \quad u(t) = m - r(t) - a(t) - s(t),$$
$$s(t) = \text{rank}\left(V(t)^T Z(t)^T N(t)\hat{T}(t)\right)$$

We assume that the functions r, s and a are constant on $\mathcal{J}$. Then (E, F) is globally equivalent to the pair

$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & F_{12} & 0 & F_{14} & F_{15} \\ 0 & 0 & 0 & F_{24} & F_{25} \\ 0 & 0 & I_\alpha & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right) = (E_1, F_1)$$

The proof can be found in [21]. The value s is called the strangeness of the pair (E, F). Denote (E, F) by $(\mathrm{E}_0, \mathrm{F}_0)$ and $\mathrm{s}_0 = \mathrm{s}$. Similarly we define the strangeness s_1 of the pair $(\mathrm{E}_1, \mathrm{F}_1)$. If we repeat the procedure described above, we arrive at a sequence of globally equivalent pairs $(\mathrm{E}_1, \mathrm{F}_1)$. $i \geq 0$, each having strangeness s_i. The strangeness index or s-index is then defined by

$$\mu = \min f\,\{i = 0, 1, 2, \ldots\ j\ s_i = 0\}.$$

Chapter 2

Solvability of Linear DAE with Properly Stated Leading Term

We consider linear differential-algebraic equations

$$A(t)(D(t)x(t))' \quad + \; B(t)x(t) = \; q(t), \quad t \in \mathcal{J} \qquad (***)$$

with properly stated leading terms A, B and D are continuous matrix functions with

$$D(t) \; \in L(\mathbb{R}^m, \mathbb{R}^n) \; A(t) \; \in L(\mathbb{R}^n, \mathbb{R}^m)$$
$$B(t) \; \in L(\mathbb{R}^m, \mathbb{R}^m) q(t) \; \in L\mathbb{R}^m$$

A function $x: \mathcal{J} \to \mathbb{R}^m$is said to be a solution of (**) if s

$$x \in C_D^1(\mathcal{J}, \mathbb{R}^m) = \{x \in C(\mathcal{J}, \mathbb{R}^m) | D_x \in C^1(\mathcal{J}, \mathbb{R}^m)\}$$

satisfies (**) point wise.
As in the previous section we define for t ∈ $\mathcal{J}$ pointwise $G_0 \; = \; AD, B_0 \; = B$and for $i \; \geq 0$

$$\left.\begin{array}{c} N_i = ker\, G_i \\ S_i = \{z \in \mathbb{R}^m | B_{iz} \in im\, G_i\} = \{z \in \mathbb{R}^m | B_z \in im\, G_i\} \\ Q_i = Q_i^2, im\, G_i = N_i, p_i = 1 - Q_i \\ G_{i+1} = G_i + B_i Q_i \\ B_{i+1} = B_i p_i - G_{1+i} D^- G'_{i+1} D p_0 \dots \, p_i \\ G_{1+i} = D p_0 \dots p_{i+1} D^- \end{array}\right\} \qquad (***)$$

$D^-$is again the reflexive generalized inverse of D.
For completeness we repeat the definition of index $\mu$from the previous section

Definition (2.1): The DAE (**) with properly stated leading term is said to be *a* regular DAE with tractability index μ on the interval $\mathcal{J}$ if there is a sequence (***) such that

- G_i has constant rank r_i on $\mathcal{J}$,
- $Q_i \in C\big(\mathcal{J}, L(\mathbb{R}^m)\big), D p_0 \dots p_i D^- \in C^1\big(\mathcal{J}, L(\mathbb{R}^n)\big), i \geq 0$
- $Q_{i+1} Q_j = 0, j = 0\,, \dots i, i \geq 0$
- $0 \leq r_0 \leq \cdots \leq r_{\mu-1} < m \; and \; \; r_\mu = m$ (**) is said to be a regular DAE if it is regular with some index μ.

2.1 Decoupling of Linear Lndex-1 DAEs

Let (**) be a regular index 1 DAE with properly stated leading term. Due to definition (2.1) the Matrix G_1 is nonsingular.

Lemma (2.1.1): The matrices of sequence (***) satisfy

(a)$P_0 = D^- D = P_0 D^- = D^- \,, D P_0 = D \;\;, D P_0 D^- = D D^- = R$

(b) $RD = DD^{-}D = D,$
(c) $A = AR = ADD^{-},$
(d) $Q_0 = C_1^{-1}BQ_0$
(e) $P_0 = C_1^{-1}AD$
(f) $P_0x = P_0y \Leftrightarrow DP_0x = DP_0y \underset{(a)}{\Leftrightarrow} Dx = Dy$

Proof.

(a) and (b) are just the properties of the generalized reflexive inverse D^{-}. Remember that $R \in C^1(\mathcal{J}, \mathrm{L}(\mathbb{R}^n))$is the smooth projector function realizing the decomposition $kerA(t) \oplus imD(t) = \mathbb{R}^n$provided by the properly stated leading term. $kerA = kerR$ implies (c). $G_1Q_0 = ADQ_0 + BQ_0^2 = BQ_0$ proves (d). Similarly $G_1P_0 = ADP_0 + BQ_0P_0 = AD$shows ($e$). For ($f$) we only have to show $'' \Leftarrow ''$. If $DP_0z = 0$ then $P_0z \in kerD = kerAD = ker\, P_0$ and thus $P_0z = 0$.

$kerD = kerAD$ Holds due to the properly stated leading term.

Let's assume that x is a solution of the DAE (1). Scaling with G_1^{-1} yields

$$A(Dx)' + Bx = q \Leftrightarrow G_1^{-1}A(Dx)' + G_1^{-1}Bx + G_1^{-1}q. \qquad (2.1)$$

Note that

- $G_1^{-1}A(Dx)' \underset{c}{=} G_1^{-1}AR(Dx)' = G_1^{-1}\, DD^{-}A(Dx)' \underset{e}{=} P_0D^{-}(Dx)'$
- $G_1^{-1}Bx = G_1^{-1}BP_0x = G_1^{-1}BQ_0x \underset{d}{=} G_1^{-1}BP_0x + Q_0x$

Thus multiplication of (3) by P_0 and Q_0 from the left shows that

$$A(Dx)' + Bx = q \Leftrightarrow G_1^{-1}A(Dx)' + G_1^{-1}Bx + G_1^{-1}q$$

$$\Leftrightarrow \begin{Bmatrix} P_0D^{-}(Dx)' + P_0G_1^{-1}BP_0x & = P_0G_1^{-1}q \\ Q_0G_1^{-1}BP_0x + Q_0x & = Q_0G_1^{-1}q \end{Bmatrix}$$

$$\underset{f(x)}{\Leftrightarrow} \begin{Bmatrix} DP_0D^{-}(Dx)' + DP_0G_1^{-1}BP_0x & = DP_0G_1^{-1}q \\ Q_0G_1^{-1}BP_0x + Q_0x & = Q_0G_1^{-1}q \end{Bmatrix}$$

$$\underset{(a)}{\Leftrightarrow} \begin{Bmatrix} R(Dx)' + DG_1^{-1}BP_0x & = DG_1^{-1}q \\ Q_0G_1^{-1}BP_0x + Q_0x & = Q_0G_1^{-1}q \end{Bmatrix}$$

$$\underset{(b)}{\Leftrightarrow} \begin{Bmatrix} (Dx)' - R'Dx + DG_1^{-1}BP_0x & = DG_1^{-1}q \\ Q_0G_1^{-1}BP_0x + Q_0x & = Q_0G_1^{-1}q \end{Bmatrix}$$

$$\underset{(a)}{\Leftrightarrow} \begin{Bmatrix} (Dx)' = R'(Dx) - DG_1^{-1}\, BD^{-}(Dx) + DG_1^{-1}q \\ Q_0x = -\, Q_0G_1^{-1}\, BD^{-}(Dx)BP_0x + Q_0G_1^{-1}q \end{Bmatrix}$$

Every solution x can therefore be written as

$$x = P_0x + Q_0x = D^-(Dx) + Q_0x = D^-(Dx) - \mathrm{Q_0G_1^{-1}}BD^-(Dx) + \mathrm{Q_0G_1^{-1}q}$$
$$= (I - Q_0G_1^{-1}B)D^-u + Q_0G_1^{-1}q \quad (2.2)$$

where $u = D_x$ is a solution of the ODE

$$u' = R'u - DG_1^{-1}BD^-u + DG_1^{-1}q \quad (2.3)$$

Definition (2.1.2): The explicit ordinary differential equation (2.3) is called the inherent regular ODE of the index-1 equation (**).

Lemma (2.1.3): (i) imD is a (time varying) invariant subspace of (2.3).

(ii) (2.3) is independent of the choice of Q_0.

Proof.

(i) Because of $imD = imR = ker(I - R)$ multiplication of (5) by $I - R$gives

$$(I - R)u' = (I - R)R'u = -(1 - R)'Ru$$

And $v = (I - R)u$ satisfies the ODE $v' = (1 - R)'v$

If there is $t_* \in \mathcal{J}$such that $u(t_*) = R(t_*)u(t_*) \in im(t_*)$ then $v(t_*) = 0$ This Means $v(t) = 0$ and thus $u(t) = R(t)u(t)$ for every t.

(ii) Let $\hat{Q}_0$be another projector with $im\hat{Q}_0 = N_0$and let $\hat{\mathrm{P}}_0$,$\hat{D}^-$be defined as in (2.16). Then $\hat{G}_0 = G_1(I + Q_0\hat{Q}_0P_0)$ implies $\hat{C}_1^{-1} = (I + Q_0\hat{Q}_0P_0)C_1^{-1}$and $D\hat{C}_1^{-1} = DC_1^{-1}$. Finally note that

$$D\hat{C}_1^{-1}B\hat{D}^- \underset{(a)}{=} D\hat{C}_1^{-1}B\hat{P}_0D^- = D\hat{C}_1^{-1}B\,D^- = D\hat{C}_1^{-1}B\hat{Q}_0D^-$$
$$\underset{(d)}{=} DC_1^{-1}B\,D^- - D\hat{Q}_0D^- = DC_1^{-1}B\,D^-.$$

Theorem (2.1.4): Let (**) be a regular index 1 DAE. For each d 2 $imD(t_0)$, $t_0 \in \mathcal{J}$, the initial value problem

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad D(t_0)x(t_0) = d \quad (2.4)$$

is uniquely solvable in$C\frac{1}{D}(\mathcal{J}, \mathbb{R}^m)$.

Proof.

There is exactly one solution $u \in C\frac{1}{D}(\mathcal{J}, \mathbb{R}^m)$of the inherent ODE

$$u' = R'u - DC_1^{-1}B\,D^- + DG_1^{-1}q$$

satisfying the initial condition $u(t_0) = d$. Lemma (2.1.3) shows that $u(t) = R(t)u(t)$ for every t. Therefore

$$(I - Q_0\mathrm{G}_1^{-1}\,\mathrm{B})D^-u + \mathrm{Q_0G_1^{-1}q} \in C\frac{1}{D}(\mathcal{J}, \mathbb{R}^m)$$

is a solution of (2.4) satisfying $Dx = u$. The decoupling process shows the uniqueness.

Note that the initial condition $D(t_0)x(t_0) = \ d$ for $d \in imD(t_0)$ can be replaced by $D(t_0)x(t_0) = D(t_0)x^0 \ , x^0 \in \mathbb{R}^m$.

2.2 Decoupling of Linear Index-2 DAEs

We now want to repeat the same argument for linear index 2 differential-algebraic equations. We assume that (**) is an index 2 DAE with properly stated leading term. Due to definition (2.1) and lemma (2.1.3) we have $N_1(t) \oplus S_1(t) = \mathbb{R}^m$. We choose Q_1 to be the canonical projector onto N_1 along S_1. Lemma (2.1.3) also implies $Q_1 \ Q_0 = \ Q_1 C_2^{-1} \ B_1 \ Q_0 = 0$ as required in definition (2.1). For the sequence (***) to make sense we have to assume $DP_0D^- \in G^1(\mathcal{J}, \mathrm{L}(\mathbb{R}^m))$ Then $DQ_1D^- = -DP_1D^- + DD^- = -DP_1D^- + R$is also smooth. Note that

$DQ_1D^-$and $DP_1D^-$are projector functions. In addition to (*a*), (*b*), (*c*) and (*f*) from lemma (2.1.3) we now have

Lemma (2.2.1):

(g)$Q_1 = Q_1 G_2^{-1} B_1$,

(h)$G_2^{-1} AD = P_1 P_0$,

(i) $G_2^{-1} B = G_2^{-1} B P_0 P_1 + P_1 D^- (DP_1 D^-)' D Q_1 + Q_1 + Q_0$,

(j) $Q_1 x + Q_1 y \Leftrightarrow DQ_1 x + DQ_1 y$,

(k)$\Omega\Omega'\Omega = 0$ for every projector function $\Omega \in G^1(\mathcal{J}, L(\mathbb{R}^n))$

Proof.

(g) follows from lemma (2.1.3), (h) can be proved similar to (e) in lemma (a) (2.1.3), but (i) is a consequence of $B = BP_0 + BQ_0 + BP_0P_1 + BP_0Q_1 + BQ_0$ and $G_2Q_0 = BQ_0$, $G_2Q_1 = B_1Q_1, BP_0Q_1 = B_1Q_1G_2P_1D^-(DP_1D^-)'DQ_1$.

To show (j) assume that $DQ_1z = 0$Then $Q_1z \in kerD = ker\, P_0$ and $Q_1z = Q_1^2z = Q_1P_0Q_1z = 0$ Final $y_0 = (I - \Omega)\Omega$ implies $0 = (\mathrm{I} - \Omega)'\Omega + (\mathrm{I} - \Omega)\Omega' = -\Omega'\Omega + (\mathrm{I} - \Omega)\Omega'$.

In order to decouple (**) in the index 2 case we again assume that x is a solution of the DAE. Since G_2 is nonsingular, we find

$$A(Dx)' + \ Bx \ = \ q \Leftrightarrow G_2^{-1}A(Dx)' + G_2^{-1}Bx = G_2^{-1}q \qquad (2.5)$$

$$\Leftrightarrow P_1D^-(Dx)' + G_2^{-1}BP_0P_1x + P_1D^-(DP_1D^-)'DQ_1x + Q_1x + Q_0x = G_2^{-1}q$$

Using $(a), (c), (h)$ and (i). Due to $I = P_1 + Q_1 = P_0P_1 + Q_0P_1 + Q_1$.

we can decouple (2.5) by multiplying with P_0P_1 , Q_0P_1 and Q_1 respectively. (**) is therefore equivalent to the system

$$P_0P_1D^-(Dx)' + P_0P_1G_2^{-1}BP_0P_1x + P_0P_1D^-(DP_1D^-)'DQ_1x$$
$$= P_0P_1G_2^{-1}q \quad (2.6.a)$$

$$Q_0P_1D^-(Dx)' + Q_0P_1G_2^{-1}BP_0P_1x$$
$$+Q_0P_1D^-(DP_1D^-)'DQ_1x + Q_1 = Q_0P_1G_2^{-1}q \qquad (2.6.b)$$

$$Q_1G_2^{-1}BP_0P_1x + Q_1x = Q_1G_2^{-1}q \qquad (2.6.c)$$

With (a) and (f) equation $(2.6.a)$ takes the form

$$DP_1D^-(Dx)' + DG_2^{-1}BP_0P_1x + DP_1D^-(DP_1D^-)'DQ_1x = DG_2^{-1}q$$

Use the product rule of differentiation to find

$$DP_1D^-(DP_1x)' - (DP_1D^-)'(Dx)$$

On the other hand

$$DP_1D^-(DP_1D^-)'DQ_1 = (DP_1D^-)'DQ_1$$

As $(DP_1D^-)'DP_0P_1Q_1 = 0$ so that (2.6.a) is equivalent to

$$(DP_1x)' - (DP_1D^-)'(DP_1x) + DP_1G_2^{-1}BD^-(DP_1x) = DP_1G_2^{-1}q \qquad (2.6.a')$$

A similar analysis involving (g), (j) and (k) from lemma 3.6 yields

$$-Q_0Q_1D^-(DQ_1x)' + Q_0Q_1D^-(DQ_1D^-)'(DP_1x)$$

$$Q_0P_1G_2^{-1}BD^-(DP_1x) + Q_0x = Q_0P_1G_2^{-1}q \qquad (2.6.b')$$

$$DQ_1x \qquad = DQ_1G_2^{-1}q \qquad (2.6.c')$$

Each solution x of (**) can thus be written as

$$x = P_0x + Q_0x = D^-Dx + Q_0x = D^-(DP_1x + DQ_1x) + Q_0x \qquad (2.7)$$

$$KD^-u - Q_0Q_1D^-(DQ_1D^-)'u + (Q_0P_1 + P_1Q_0)G_2^{-1}q + (DQ_1G_2^{-1}q)'$$

where

$$K = 1 - Q_1P_1G_2^{-1}B$$

And $u = DP_1x$ satisfies the ordinary differential equation

$$u' - (DP_1D^-)'u + DP_1G_2^{-1}BD^-u = DP_1G_2^{-1}q.$$

As in the index 1 case this ODE will be referred to as the inherent regular ODE.

Definition (2.2.2): The explicit ordinary differential equation

$$u' = (DP_1D^-)'u - DP_1G_2^{-1}BD^-u + DP_1G_2^{-1}q \qquad (2.8)$$

is called the inherent regular ODE of the index 2 equation (**).

Lemma (2.2.3):

(i) ImDP1 is a (time varying) invariant subspace of (2.8).

(ii) (2.8) is independent of the choice of Q_0 and thus uniquely determined by the problem data

Proof.

To prove (i), carry out a similar analysis as in the proof of lemma (2.2.5) but with R replaced by DP_1D^-. To see (ii) consider another projector $\hat{Q}_0$ with $im\,\hat{Q}_0 = N_0$ and the relation $\hat{G}_1 = G_1\big(I + Q_0\hat{Q}_0P_0\big)$. The subspaces $\hat{N}_1 = \big(I + Q_0\hat{Q}_0P_0\big)N_1$ and $\hat{S}_1 = S_1$are given in terms of $N_1 = S_1$so that $\hat{Q}_1 = \big(I + Q_0\hat{Q}_0P_0\big)Q_1$ is the canonical projector onto $\hat{N}_1$along $\hat{S}_1$. This implies $D\hat{P}_1\hat{D}^- = DP_1D^-$. Use the representation

$$\hat{G}_2^{-1}q = \big(I + Q_0\hat{P}_0P_1P_0\big)G_2^{-1}q$$

to see that $DP_1G_2^{-1}q$ and $DP_1G_2^{-1}q\ BD^{-}$are independent of the choice of Q_0.As in the previous section we are now able to prove existence and uniqueness ofsolutions for regular index 2 DAEs with properly stated leading terms. We makeuse of the function space

$$C^1_{DQ_1G_2^{-1}}(\mathcal{J},\ \mathbb{R}^m) = \{z \in C(\mathcal{J},\ \mathbb{R}^m) | DQ_1G_2^{-1}z \in C^1(\mathcal{J},\ \mathbb{R}^n)\}$$

Theorem (2.2.84): Let (1) be a regular index 2 DAE with $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{J},\ \mathbb{R}^m)$. For each $d \in imD(t_0)P_0(t_0)t_0 \in \mathcal{J}$ the initial value problem

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t),\quad D(t_0)P_0(t_0)x(t_0) = d \quad (2.9)$$

is uniquely solvable in $C^1_D(\mathcal{J},\ \mathbb{R}^m)$

Proof.

Solve the inherent regular ODE (2.8) with initial value $u(t_0) = d$ Lemma (2.2.3) yields $u(t) \in \mathrm{im}D(t)P_1(t)$for every $t$and

$$x = KD^{-}u - Q_0Q_1D^{-}(DQ_1D^{-})'u + (Q_0P_1 + P_1Q_0)G_2^{-1}q$$
$$+ Q_0Q_1D^{-}(DQ_1G_2^{-1}q)'$$

is the desired solution of (2.9).

The initial condition $D(t_0)P_1(t_0)x(t_0) = d$ can be replaced by $D(t_0)P_1(t_0)x(t_0) = D(t_0)P_1(t_0)x^0$ for $x^0\ \mathbb{R}^m$.

Remarks (2.2.5):

We presented examples of nonlinear differential-algebraic equations $f((Dx)', x, t) = 0$ where the solution could be expressed as

$$x(t) = D(t)^{-}u(t) + Q(t)\omega(u(t), t),\quad t \in \mathcal{J}$$

u was the solution of

$$u'(t) = R'(t)u(t) + Q(t)\omega(u(t), t),\quad u(t_0)\ D(t_0)x_0 \quad (2.10)$$

And ω was implicitly defined by

$$F(\omega, u, t) = F(D\omega, D^{-}u + Q\omega, t) = 0$$

The ordinary differential equation (2.10) is thus only available theoretically.

We made use of the sequence (***) established the tractability index in order to perform a refined analysis of linear DAEs with properly stated leading terms. We were able to find explicit expressions of (2.10) for these equations with index 1 and 2. This detailed analysis lead us to results about existence and uniqueness of solutions for DAEs with low index. We were able to figure out precisely what initial conditions are to be posed, namely $D(t_0)P_1(t_0) = D(t_0)x^0$ and $D(t_0)P_1(t_0)x(t_0) = D(t_0)P_1(t_0)x^0$ in the index 1 and index 2 case respectively. These initial conditions guarantee that solutions u of the inherent regular ODE (2.3) and (2.8) lie in the corresponding invariant subspace. Let us stress that only those solutions of the regular inherent ODE

that lie in the invariant subspace are relevant for the DAE. Even if this subspace varies with t we know the dynamical degree of freedom to be $rank G_0$ and rank $G_0 + rank G_1 - m$for index 1 and 2 respectively.
The results presented can be generalized for arbitrary index μ The inherent regular ODE for an index μ DAE with properly stated leading term.
There it is also proved that the index μ is invariant under linear transformations and refactorizations of the original DAE and the inherent regular ODE remains unchanged. Finally let us point out that we assumed A, D and $B$to be continuous only. The required smoothness of the coefficients in the standard formulation

$$Ex' + Fx = q \tag{2.11}$$

Was replaced by the requirement on certain subspaces to be spanned by smooth functions. Namely, the projectors $R, DP_1D^-$and $DQ_1D^-$are differentiable if DN_1 and $DS_1$are spanned by continuously differentiable functions [1]. However, if the DAE

$$A(Dx)' + Bx = q \tag{2.12}$$

Is given with smooth coefficients and we orient on G^1-solutions, then comparisons with concepts for (2.11) can be made via

$$ADx' + (B - AD')x = q$$

On the other hand, if Ehas constant $rank$ on $\mathcal{J}$ and $P_E \in C^1(\mathcal{J}, L(\mathbb{R}^m))$is a projector function on to $kerE$, we can reformulate (2.11) as

$$E(P_E x)' + (F - EP_E')x = q$$

with a properly stated leading term.

2.3 Numerical Methods for Linear DAEs with Properly Stated Leading Term.

The last part is devoted to studying the application of numerical methods to linear DAEs of index $\mu = 1$ and $\mu = 2$. From the previous section we know that (2.2) and (2.7) are representations of the exact the solution, respectively. In fact, it turns out that (2.2) is just a special cases of (9). To see this, observe that for $\mu = 1$the matrix G_1 is nonsingular so tha $Q_1 = 0, P_1 = I$ and $G_2 = G_1$.We therefore treat index 1 and index 2 equations simultaneously in this section. We will show how to apply Runge-Kutta methods to DAEs

$$A(t)\big(D(t)x(t)\big)' + B(t)x(t) = q(t) \tag{2.13}$$

with properly stated leading terms.
When using the s-stage Runge-Kutta method

$$\begin{array}{c|c} c & \mathcal{A} \\ \hline & \end{array} , \mathcal{A} = (\alpha_{ij}) \in L(\mathbb{R}^s) \;\; c = \mathcal{A}e \,, \mathcal{B} \in \mathbb{R}^s, e = (1, \dots I)^T \in \mathbb{R}^s$$

$\mathcal{B}^T$

to solve an ordinary differential equation

$$x'(t) = F(x(t), t) \qquad (2.14)$$

Numerically with step size h, an approximation $x_{l-1}$to the exact solution $x(t_{l-1})$ is used to calculate the approximation $x_l$to $x(t_l) = x(t_{l-1} + h)$ via

$$x_l = x_{l-1} + h \sum_{i=1}^{s} \beta_i X'_{li} \qquad (2.14.a)$$

Where X'_{l_i} is defined by

$$X'_{li} = F(X'_{li}, t'_{li}) i = 1, \dots s \qquad (2.14.b)$$

And $t'_{l_i} = t_{l-1} + c_i h$ are intermediate time steps. The internal stages X_i are given by

$$X'_{li} = x_{l-1} + h \sum_{j=1}^{s} \alpha_{ij} X'_{lj} \qquad (2.14.c)$$

Observe that (2.14.a) and (2.14.c) depend on the method and only (2.14.b) depends on the equation (2.13). If the ODE (2.13) is replaced by the DAE

$$f(x'(t), x(t)t) = 0$$

we also replace (2.14.b) by

$$F(X'_{li}, X_{li}, t'_{l_i}) i = 1, \dots s \qquad (2.14.b')$$

in the Runge-Kutta scheme.

The matrix $\frac{\partial y}{\partial x'}$ is singular. Therefore some components of the increments X'_{li} need to be calculated from (2.14.c) as seen in the following trivial example.

Example (2.3.1): If $f(x', x, t) = x - q(t)$, then $x(t) = q(t)$. The numerical method (2.14.a), (2.14.b'), (2.14.c) now reads

$$x_l = x_{l-1} + h \sum_{i=1}^{s} \beta_i X'_{li} \quad q(t_{li}) = X'_{li} = x_{l-1} + h \sum_{j=1}^{s} \alpha_{ij} X'_{lj}$$

This system can be solved if and only if $\mathcal{A}$ is nonsingular.

We always assume $\mathcal{A}$ to be nonsingular. This leads to an expression of X'_{li} in terms of X_{lj}.

Lemma (2.3.2): Let $\mathcal{A} = (\alpha_{ij})$ be nonsingular and $\mathcal{A}^{-1} = (\tilde{\alpha}_{ij})$ Then

$$X_{lj} = x_{l-1} + h\sum_{j=1}^{s} \alpha_{ij} X'_{lj}\; i \;=\; 1, \dots\; s \Leftrightarrow X'_{lj} = \frac{1}{h}\sum_{j=1}^{s} \tilde{\alpha}_{ij}(X_{lj} - x_{l-1})\; i \;=\; 1 \dots\; s$$

$$\begin{pmatrix} X_{l1} \\ \cdots \\ X_{ls} \end{pmatrix} = e_m \otimes x_{l-1} + h(\mathcal{A} \otimes I_m) \begin{pmatrix} X'_{l1} \\ \cdots \\ X'_{ls} \end{pmatrix} \Leftrightarrow \begin{pmatrix} X'_{l1} \\ \cdots \\ X'_{ls} \end{pmatrix} = \frac{1}{h}(\mathcal{A}^{-1} \otimes I_m)\left[\begin{pmatrix} X_{l1} \\ \cdots \\ X_{ls} \end{pmatrix} - e_m \otimes x_{l-1}\right]$$

Now consider the linear DAE (2.13) with continuous matrix functions

$$A(t) \in L(\mathbb{R}^n, \mathbb{R}^m)\;, D(t) \in L(\mathbb{R}^m, \mathbb{R}^n)$$
$$\boldsymbol{B}(t) \in L(\mathbb{R}^m, \mathbb{R}^m)$$

and a properly stated leading term.

When applying the numerical scheme (2.14.a),(2.14.b’),(2.14.c) we don’t want to lose the additional information provided by the properly stated leading term. According to lemma (2.3.2) we therefore replace (2.14.c) by

$$[DX]'_{lj} = \frac{1}{h}\sum_{j=1}^{s} \tilde{\alpha}_{ij}(D_{lj}X_{lj} - D_{l-1}x_{l-1}) \qquad (2.14.c')$$

and solve the system

$$D_{li}[\mathrm{DX}]'_{\mathrm{lj}} + B_{li}X_{li} = q_{li} \quad ,\mathrm{i} \;=\; 1 \dots\; \mathrm{s} \quad (2.14.b'')$$

For X_{li}Here we write $D_{l-1} = D(t_{l-1})$, $D_{li} = D(t_{li})$, $A_{li} = D(t_{li})$ and so on. Using this an satz the output value

$$x_l = x_{l-1} + h\sum_{j=1}^{s} \beta_i \frac{1}{h}\sum_{j=1}^{s} \tilde{\alpha}_{ij}\big(X_{lj} - x_{l-1}\big)$$
$$= (1 - \beta^T \mathcal{A}^{-1} e)x_{l-1} \sum_{i=1}^{s}\sum_{j=1}^{s} \beta_i \tilde{\alpha}_{ij} X_{lj}$$

is computed. For RadauIIA methods this expression simplifies considerably.

Definition (2.3.3): The s-stage RadauIIA method is uniquely determined by requiring $C(s), D(s)$, $c_s = 1$and choosing $c_1, \dots, c_{s-1}$ to be the zeros of the Gauss-Legendre polynomial P_s. For the conditions $C(s), D(s)$. The Gauss-Legendre polynomialP_s. is orthogonal to every polynomial of degree less than s. RadauIIA methods are A- and L-stable and have order $p = 2s - 1$. The last row of $\mathcal{A}$ coincides with β^T.

Lemma (2.3.4): For the s-stage RadauIIA method $1-\beta^T A^{-1} e = 0$ holds and the outputvalue computed by (2.14.a), (2.14.b''), (2.14.c), (2.14.c') is given by the last stage X_{ls}.

Proof.

$1 - \beta^T A^{-1} e = 1 - Z_s(A) A^{-1} e = 1 - (0, \dots, 0, 1) e = 0$ and

$$x_l = (1-\beta^T A^{-1} e) x_{l-1} + \sum_{i=1}^{s} \sum_{j=1}^{s} \beta_i \bar{\alpha}_{ij} X_{lj} = \big((0, \dots ,0,1) \otimes I_m\big) \begin{pmatrix} X_{l1} \\ \dots \\ X_{ls} \end{pmatrix} = X_{ls}.$$

To summarize these results we present the following algorithm for solving the DAE (2.13) using RadauIIA methods.

Algorithm (2.3.5): Given an approximation x_{l-1} to the exact solution $x(t_{l-1})$ and astepsize$h$, solve

$$A_{li}[DX]'_{li} + B_{li} X_{li} = q_{li}, \quad i = 1, \dots, s \tag{2.14.b''}$$

for X_{li} where $[DX]'_{li}$ is given by

$$[DX]'_{li} = \frac{1}{h} \sum_{j=1}^{s} \bar{\alpha}_{ij} \big(D_{lj} X_{lj} - D_{l-1} x_{l-1}\big). \tag{2.14.c}$$

Return the output value $x_l = X_{ls}$ as an approximation to $x(t_l) = x(t_{l-1} + h)$. The exact solution x of (2.13) satisfies

$$x(t) \in M_0(t) = \{z \in \mathbb{R}^m \,|\, B(t)z - q(t) \in \operatorname{im}\big(A(t)D(t)\big)\} \ \forall\, t.$$

Since $X_{li} \in M_0(t_{li})$ for every i and $cs = 1$ we have

$$x_l = X_{ls} \in M_0(t_{ls}) = M_0(t_l)$$

for every RadauIIA method.

2.3.1 Decoupling of the Discretized Equation.

Algorithm (2.3.5) replaces the DAE

$$A(Dx)' + Bx = q \tag{2.15}$$

by the discretized problem

$$A_{li}[DX]'_{li} + B_{li} X_{li} = q_{li}, \qquad i = 1, \dots, s. \tag{2.16}$$

The analytic solution x of index 1 and index 2 equations (2.13) can be represented as

$$x = KD^{-}u - Q_0Q_1D^{-}(DQ_1D^{-})'u + (Q_0P_1 + P_0Q_1)G_2^{-1}q + Q_0Q_1D^{-}(DQ_1G_2^{-1}q)' \tag{2.17}$$

where $K = I - Q_0P_1G_2^{-1}B$and the component $u = DP_1x$ satisfies the inherent regular ordinary differential equation

$$u' - (DP_1D^{-})'u + DP_1G_2^{-1}\,BD - u = DP_1G_2^{-1}\,q. \tag{2.18}$$

If we applied the Runge-Kutta method directly to the inherent regular ODE, due to lemma (2.3.1) we would obtain

$$\frac{1}{h}\sum_{j=1}^{s}\bar{\alpha}_{ij}\left(U_{lj} - u_{l-1}\right) - (DP_1D^{-})'_{li}U_{li} + (DP_1G_2^{-1}BD^{-})U_{li} = (DP_1G_2^{-1}q)_{li} \tag{2.19}$$

for $i = 1, \ldots, s$. Our aim is to show that the Runge-Kutta method, when applied to (2.13), behaves as if it was integrating the inherent regular ODE (2.18). Doing so (2.18) is found to be equivalent to the system

$$\left.\begin{array}{c} (DP_1D^{-})_{li}[DX]'_{li} + (Q_0P_1G_2^{-1}BP_0P_1)_{li}X_{li} + (DP_1D^{-})'_{li}D_{li}Q_{1,li}X_{li} \\ = (DP_1G_2^{-1}q)_{li} \\ -(Q_0Q_1D^{-})_{li}[DX]'_{li} + (Q_0P_1G_2^{-1}BP - 0\;P_1)_{li}X_{li} + (Q_0P_1D^{-1})_{li}(DP_1D^{-1})'_{li}D_{li}Q_{1,li}X_{li} \\ +Q_{0,li}X_{li} = (Q - 0\;P_1G_2^{-1}q)li \\ D_{li}Q_{1,li}X_{li} = (DQ_1G_2^{-1}q)_{li} \end{array}\right\} \tag{2.19}$$

for $i = 1, \ldots, s$. The decoupled system (2.19) immediately implies the convergence of RadauII A methods applied to (2.13) on compact intervals I if the stepsize $h$tends to zero.

Theorem (2.3.6): Let (2.13) be an index μ equation, μ ∈ $\{1, 2\}$. Let the subspaces $D(\cdot)S_1(\cdot)$ and $D(\cdot)N_1(\cdot)$ be constant. Then the difference between the exact solution and the solution obtained by using a RadauIIA method can be written as

$$x(t_1) - x_l = K_lD_l^{-}(u(t_1) - u_l) + (Q_0Q_1D^{-})_l\{(DQ_1G_2^{-1}q)'_l - \frac{1}{h}\sum_{j=0}^{k}\bar{\alpha}_{sj}\left((DQ_1G_2^{-1}q)_{lj} - (DQ_1G_2^{-1}q)_{l-1}\right\}.$$

Here u_1 is exactly the RadauIIA approximation to the solution $u(t_l)$ of the inherent regular ODE(21).

Note that $\frac{1}{h}\sum_{j=0}^{k}\bar{\alpha}_{sj}\left((DQ_1G_2^{-1}q)_{lj}-(DQ_1G_2^{-1}q)_{l-1}\right)$ is exactly the Runge-Kutta approximation to $(DQ_1G_2^{-1}q)'_{lj}$. The proof of theorem (2.3. 6) will use the following lemma.

Lemma (2.3.7): DP_1D^- and DQ_1D^- are projector functions satisfying (i) $DS_1=\mathrm{im}DP_1=\mathrm{im}DP_1D^-, DN_1=\mathrm{im}DQ_1=\mathrm{im}DQ_1D^-$.

If the subspaces DS_1 and DN_1 are constant, so that there are constant projectors V,W onto DS_1 and DN_1 respectively, then the following relations hold:

$(ii) DP_1D^-V=V, DP_1D^-W=0, DQ_1D^-WW, DQ_1D^-V=0,$
$(iii)(DP_1D^-)'V=0,(DP_1D^-)'W=0,(DQ_1D^-)'W=0,(DQ_1D^-)'V=0.$

Proof.

DP_1D^- and DQ_1D^- are projector functions.

The same lemmas imply (i), so that $DP_1D^-V=V$ and $DQ_1D^-W=W$ hold as well. These relations together with (i) show (ii). Finally use (ii) to prove (iii) by noting that V and W are constant projectors and therefore do not depend on t.

Proof of theorem (2.3. 6): The proof will be divided into four parts. We analyze $(DP_1D^-)_{li}[DX]'_{li}$ and $(Q_0Q_1D^-)_{li}[DX]'_{li}$, so that we can find a representation of the numerical solution in part 2. This representation will depend on $U_{ls}=D_{ls}P_{1,ls}X_{ls}$. We show that $u_l=U_{ls}$ is exactly the RadauIIA solution of the inherent regular ODE. Analyze $(DP_1D^-)_{li}[DX]'_{li}$ and $(Q_0Q_1D^-)_{li}[DX]'_{li}$ write $U_{li}=D_{li}P_{1,li}X_{li}$ and $u_{l-1}=D_{l-1}P_{1,l-1}x_{l-1}$. Then

$$(DP_1D^-)_{li}[DX]'_{li}=\frac{1}{h}(DP_1D^-)_{li}\sum_{j=1}^{s}\bar{\alpha}_{ij}\left(D_{lj}X_{lj}-D_{l-1}x_{l-1}\right)$$
$$=\frac{1}{h}(DP_1D^-)_{li}\sum_{j=1}^{s}\bar{\alpha}_{ij}\left(U_{lj}+D_{lj}Q_{1,li}X_{lj}-u_{l-1}-(DQ_1x)_{l-1}\right).$$

Use lemma (2.3.7) to see that

$$(DP_1D^-)_{li}\left(U_{lj}-u_{l-1}\right)=(DP_1D^-)_{li}V\left(U_{lj}-u_{l-1}\right)=V\left(U_{lj}-u_{l-1}\right)$$
$$=U_{lj}-u_{l-1}$$

and

$$(DP_1D^-)_{li}\left(D_{lj}Q_{1,li}X_{lj}-(DQ_1x)_{l-1}\right)=(DP_1D^-)_{li}W\left(D_{lj}Q_{1,li}X_{lj}-(DQ_1x)_{l-1}\right)=0.$$

We arrive at

$$(DP_1D^-)_{li}[Dx]'_{li} = \frac{1}{h}\sum_{j=1}^{s} \bar{\alpha}_{ij}\left(U_{lj} - u_{l-1}\right).$$

Similarly, Lemma (2.3.7) implies

$$(DP_1D^-)_{li}[Dx]'_{li} = \frac{1}{h}\sum_{j=1}^{s} \bar{\alpha}_{ij}\left(D_{lj}Q_{1,li}X_{lj} - (DQ_1x)_{l-1}\right) \quad .$$

Because of

$$(Q_0Q_1D^-)_{li} = (Q_0(Q_1P_0Q_1)D^-)_{li} = \left((Q - 0\ Q_1D^-)(DQ_1D^-)\right)_{li},$$

it follows that

$$(Q_0Q_1D^-)_{li}[DX]'_{li} = \frac{1}{h}(Q_0Q_1D^-)_{li}\sum_{j=1}^{s} \bar{\alpha}_{ij}\left(D_{lj}Q_{1,li}X_{lj} - (DQ_1x)_{l-1}\right).$$

The discretized system (2.19) now reads

$$\left.\begin{array}{c}\frac{1}{h}\sum_{j=1}^{s} \bar{\alpha}_{ij}\left(U_{lj} - u_{l-1}\right) + (DP_1G_2^{-1}BD^-)_{li}U_{li} + (DP_1D^-)'_{li}D_{li}Q_{1,li}X_{li} = (DP_1G_2^{-1})_{li} \\ -\frac{1}{h}(Q_0Q_1D^-)_{li}\sum_{j=1}^{s} \bar{\alpha}_{ij}\left(D_{lj}Q_{1,li}X_{lj} - (DQ_1x)_{l-1}\right) + (Q_0P_1G_2^{-1}BD^-)_{li}U_{li} \\ +(Q_0P_1D^-)_{li}(DP_1D^-)'_{li}D_{li}Q_{1,li}X_{li} + Q_{0.li} = (Q_0P_1G_2^{-1}q)_{li}D_{li}Q_{1,li}X_{li} = (DQ_1G_2^{-1}q)_{li}\end{array}\right\}$$

but due to Lemma (2.3.7) this reduces to

$$\left.\begin{array}{c}\frac{1}{h}\sum_{j=1}^{s} \bar{\alpha}_{ij}\left(U_{lj} - u_{l-1}\right) + (DP_1G_2^{-1}BD^-)_{li}U_{li} = (DP_1G_2^{-1})_{li} \\ -\frac{1}{h}(Q_0Q_1D^-)_{li}\sum_{j=1}^{s} \bar{\alpha}_{ij}\left(D_{lj}Q_{1,li}X_{lj} - (DQ_1x)_{l-1}\right) + (Q_0P_1G_2^{-1}BD^-)_{li}U_{li} \\ +Q_{0.li}X_{li} = (Q_0P_1G_2^{-1}q)_{li}\end{array}\right\}$$

The numerical solution can thus be written as

$$\begin{aligned} x_l = X_{ls} &= P_{0,ls}X_{ls} + Q_{0,ls}X_{ls} = D_{ls}^-\left(D_{ls}P_{1,ls}X_{ls} + D_{ls}Q_{1,ls}X_{ls}\right) + Q_{0,ls}X_{ls} \\ &= (I - (Q_0P_1G_2^{-1}B)_{ls}D_l^-U_{ls} + (P_0Q_1 + Q_0P_1)_l(G_2^{-1}q)_l \end{aligned}$$

$$+\frac{1}{h}(Q_0Q_1D^-)_l\sum_{j=1}^{s}\bar{\alpha}_{sj}\left((DQ_1G_2^{-1}q)_{lj}-(DQ_1G_2^{-1}q)_{l-1}\right). \quad (2.20)$$

The stage approximations U_{lj} satisfy the recursion

$$\frac{1}{h}\sum_{j=1}^{s}\bar{\alpha}_{ij}\left(U_{lj}-u_{l-1}\right)+(DP_1G_2^{-1}BD^-)_{li}U_{li}=(DP_1G_2^{-1}q)_{li}. \quad (2.21)$$

Again, Lemma (2.3.7) implies

$$(DP_1D^-)'_{li}U_{li}=(DP_1D^-)'_{li}VU_{li}=0$$

in (2.19). This shows that (2.21) and (2.19) coincide. Therefore, and due to $c_s =$ 1,
$u_l = U_{ls}$is exactly the Runge-Kutta solution of the inherent regular ODE (2.20). Use Lemma (2.3.7)

$$(DQ_1D^-)'_l u(t_l)=(DQ_1D^-)'_l Vu(t_l)=0.$$

Now the assertion follows by comparing (2.18) and (2.20).

Theorem 4.6 is the central tool in analyzing the behaviour of RadauIIA methods when applied to DAEs (2.13). In the case of index μ = 1 theorem (2.3.6) shows that discretization and the decoupling procedure commute.

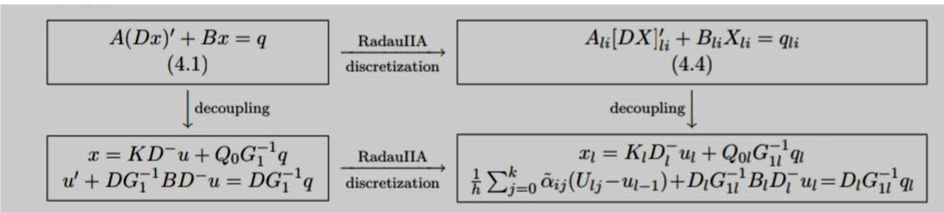
Corollary (2.3. 8): Let the DAE (2.13) be of index 1. Assume that im$D(t)$ is constant. Then we have for any RadauIIA method

$$x(t_l)-x_l=K_lD_l^-(u(t_l)-u_l), K=I-Q_0G_1^{-1}B.$$

Proof.

If the index is 1, we have $Q_1 = 0$and $P_1 = I$. Thus $N_1 = \{0\}$and $S_1 = \mathbb{R}^n$. Since im$D(t)$ is constant, the subspaces DS_1 and DN_1 are constant as well. We can therefore apply theorem (2.3.6).

Due to corollary (2.3.8) the following diagram commutes for index 1 equations with constant im *D*.



If the index is 2, we cannot expect the corresponding diagram to commute. However, the term

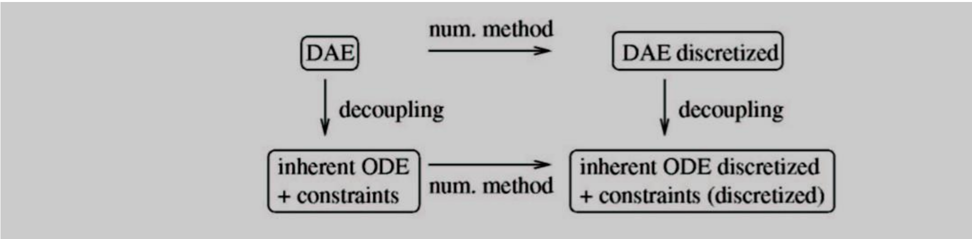
$$\frac{1}{h}\sum_{j=1}^{s} \bar{\alpha}_{sj} \, ((DQ_1G_2^{-1}q)_{lj} - (DQ_1G_2^{-1}q)_{l-1} = [DQ_1G_2^{-1}q]'_{tl}$$

appearing in theorem (2.3.6) is exactly the RadauIIA approximation to $(DQ_1G_2^{-1}q)'_l$(lemma (2.3.7)) so that

$$x(t_l) - x_l = K_l D_l^-(u(t_l) - u_l) + Q_{0l}Q_{1l}D_l^-\{(DQ_1G_2^{-1}q)'_l - [DQ_1G_2^{-1}q]'_{tl}\}.$$

We have the following statement:

When applying a RadauIIA method to problems of index $\mu \in \{1,2\}$with constant
subspaces DS_1 and DN_1, then discretization and decoupling commute.



Definition (2.3.9): The DAE (2.13) of index $\mu \in \{1,2\}$ is said to be numerically qualified, if

- $\mu = 1$ and im$D$ is constant,
- $\mu = 2$ and DS_1, DN_1 are constant.

The commutativity of discretization and the decoupling process is the desired property for DAEs since it guarantees a good behavior of the numerical method. Even though the numerical method is applied to the DAE directly, it behaves as if it was integrating the regular inherent ODE (2.18). In this case results concerning convergence on compact intervals I hold automatically. The RadauIIA method applied to a numerically qualified DAEs is convergent with the same order as for ODEs.

Chapter 3

Solutions for Differential-Algebraic Equations

Solving differential equations is an important issue in sciences because many physical phenomena are modeled using such equations. Modern methods likehomotopy perturbation method (HPM), homotopy analysis method (HAM), variation iteration method (VIM), among others, are powerful tools to approximate nonlinear dynamic problems. Nevertheless, series method is a well known classic procedure from literature that can be applied successfully to solve differential equations. This method establishes that the solution of a differential equation can be expressed as a power series of the independent variable. Therefore, in this work, we apply series method to solve two differential-algebraic equations. Additionally, we present the use of Laplace-Pade(LP) resumptions method as an useful strategy to obtain exact solutions or approximations possessing a large domain of convergence. We introduce the basic concept of the series method. The concept about Laplace-Padere summation method is explained. The solution of two differential algebraic equations is presented. Numerical simulations and a discussion about the results are provided.

3.1 Basic Concept of Series Method.

It can be considered that a nonlinear differential equation can be expressed as

$$A(u) - f(t) = 0 \qquad\qquad t \in \Omega \tag{3.1}$$

having as boundary condition

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \qquad t \in \Gamma, \tag{3.2}$$

Where A is a general operator, $f(t)$is a known analytic function, B is a boundary operator, and $\Gamma$is the boundary of domain Ω.

The series method establishes that the solution of a differential equation can be written as

$$u = \sum_{i=0}^{\infty} u_i\, t^i, \tag{3.3}$$

Where $u_0, u_1, \ldots$ are unknowns to be determined by series method. The basic process of series method can be described as:

(1) Equation (3.3) is substituted into (3.1), then we regroup equation in terms of t-powers.
(2) We equate each coefficient of the resulting polynomial to zero.
(3) The boundary conditions of (3.1) are substituted into (3.3) to generate an equation for each boundary condition.

(4) Aforementioned steps generate a nonlinear algebraic equation system (NAEs) in terms of the unknowns of (3.3).

(5) Finally, we solve the NAEs to obtain $u_0, u_1, \ldots,$ coefficients.

3.2 Laplace-Pade Resummation Method.

Several approximate methods provide power series solutions (polynomial). Nevertheless, sometimes, this type of solutions lacks of large domains of convergence. Therefore, Laplace-Pade resummation method is used in literature to enlarge the domain of convergence of solutions or inclusive to find exact solutions.
The Laplace-Pade method can be explained as follows:

(1) First, Laplace transformation is applied to power series (3.3).

(2) Next, s is substituted by $1/t$ in the resulting equation.

(3) After that, we convert the transformed series into a meromorphic function by forming its Pade approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller value than the order of the power series. In this step, the Pade approximant extends the domain of the truncated seriessolution to obtain better accuracy and convergence.

(4) Then t is substituted by $1/s$.

(5) Finally, by using the inverse Laplace s transformation, we obtain the exact or approximate solution.

This process is known as Laplace-Pade series method (LPSM).

3.3 Case Studies.

We will solve two DAE problems in order to depict the LPSM method.
3.3.1. Hessenberg Index-3 DAE.

Consider the following DAE

$$x_1' + x_1 - tx_3 + x_4 = 0,$$

$$x_2' - x_1 + x_2 - t^2 x_3 + t x_4 = 0,$$
$$x_3' - t^3 x_1 + t^2 x_2 - x_3 = 0,$$
$$t x_1 - x_2 + t x_3 - x_4 = 0,$$
$$x_1(0) = x_3(0) = 1,$$
$$x_2(0) = x_4(0) = 0, \quad (3.4)$$

where prime denotes derivative with respect to t.

We suppose that solution for (3.1) has the following fourth order expression

$$x_1(t) = \sum_{i=0}^{4} x_{1i} t^i, \qquad x_2(t) = \sum_{i=0}^{4} x_{2i} t^i$$
$$x_3(t) = \sum_{i=0}^{4} x_{3i} t^i \qquad x_4(t) = \sum_{i=0}^{4} x_{4i} t^i$$

Substituting (3.2) into (3.1), rearranging and equating terms having the same t-powers,we obtain

$$x_{11} + x_{40} + x_{10} + (x_{41} + x_{11} + 2x_{12} - x_{30})t + \cdots = 0.$$
$$x_{21} + x_{20} - x_{10} + (x_{40} + 2x_{22} + x_{21} - x_{11})t + \cdots = 0.$$
$$x_{31} - x_{30} + (-x_{31} + 2x_{32})t + \cdots = 0.$$
$$-x_{20} - x_{40} + (x_{10} - x_{41} - x_{21} + x_{30})t + \cdots = 0. \quad (3.5)$$

Next, equating the coefficients of (3.3) to zero, we obtain the following system of algebraic equations

$$t^0 : x_{11} + x_{40} + x_{10} = 0,$$
$$t^1 : x_{41} + x_{11} + 2x_{12} - x_{30} = 0,$$
$$\vdots$$
$$t^3 : .... \quad (3.6)$$

$$t^0 : x_{21} + x_{20} - x_{10} = 0,$$
$$t^1 : x_{40} + 2x_{22} + x_{21} - x_{11} = 0,$$
$$\vdots$$
$$t^3 : ... \quad (3.7)$$

$$t^0 : x_{31} - x_{30} = 0,$$

$$t^1 : -x_{31} + 2x_{32} = 0,$$

$$\vdots$$

$$t^3 : \dots \tag{3.8}$$

$$t^0 : -x_{20} - x_{40} = 0,$$

$$t^1 : x_{10} - x_{41} - x_{21} + x_{30} = 0,$$

$$\vdots$$

$$t^4 : -x_{24} + x_{33} + x_{13} - x_{44} = 0 \tag{3.9}$$

Now, in order to consider the initial condition of (3.3), we substitute them into (3.4) to obtain

$$x_{10} = 1, \qquad x_{20} = 0$$

$$x_{30} = 1 \qquad x_{40} = 0 \tag{3.10}$$

It is important to notice that, from (3.2)-(3.3), we use only the powers $t^i (i = 0,1,2,3)$,because the rest of the information needed is taken from (3.7). From (3.1), we can observe that there is not an explicit equation for variable x_4. Therefore, from (3.9), we use coefficients of powers $t^i (i = 1,2,3,4)$ to collect enough information to compensate the presence of x_4 in those equations. Furthermore, order zero term of (3.7) possesses redundant information that can be ignored. Hence, for (3.8), we can use powers $t^i (i = 1,2,3,4)$. Then, solving the NAEs composed by (3.3), (3.4), (3.5),(3.6), and (3.7) results the following approximate solution

$$x_1(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4,$$

$$x_2(t) = t - t^2 + \frac{1}{2}t^3 - \frac{1}{6}t^4$$

$$x_3(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4,$$

$$x_4(t) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 \tag{3.11}$$

Then, Laplace transformation is applied to (3.7) and then $\frac{1}{t}$ is written in place of s.

Afterwards, Pade approximant of order $[2/2]$ is applied and $\frac{1}{s}$ is written in place of t for each variable. Finally, by using the inverse Laplace s transformation, we obtain the exact solution for (3.1)

$$x_1(t) = \exp(-t),$$

$$x_2(t) = t\exp(-t),$$

$$x_3(t) = \exp(t),$$

$$x_4(t) = t\exp(t). \tag{3.12}$$

3.3.2. Index-Three Nonlinear Differential-Algebraic Equation System

Consider the following nonlinear DAE

$$y_1' = 2y_1y_2z_1z_2,$$

$$y_2' = -y_1y_2z_2^2,$$

$$z_1' = (y_1y_2 + z_1z_2)u,$$

$$z_2' = -y_1y_2^2z_2^2u,$$

$$y_1y_2^2 = 1$$

$$y_1(0) = y_2(0) = 1,$$

$$z_1(0) = z_2(0) = 1,$$

$$u(0) = 1. \tag{3.13}$$

where prime denotes derivative with respect to t.

We suppose that solution for (3.12) has the following fourth order expression

$$y_1(t) = \sum_{(i=0)}^{4} y_{1i}t^i, y_2(t) = \sum_{(i=0)}^{4} y_{2i}t^i,$$

$$z_1(t) = \sum_{i=0}^{4} z_{1i}t^i, \qquad z_2(t) = \sum_{i=0}^{4} z_{2i}t^i,$$

$$u(t) = \sum_{i=0}^{4} u_it^i. \tag{3.14}$$

Substituting (3.13) into (3.14), rearranging and equating terms having the same powers, we obtain

$$y_{11} - 2y_{10}y_{20}z_{10}z_{20} + (-2y_{10}y_{20}z_{11}z_{20} + 2y_{12} - 2y_{10}y_{21}z_{10}z_{20} - 2y_{11}y_{20}z_{10}z_{10} - 2y_{10}y_{20}z_{10}z_{21})t + \cdots = 0.$$

$$y_{21} + y_{10}y_{20}z_{20}^2 + (y_{10}y_{21}z_{20}^2 + 2y_2 2 + 2y_{10}y_{20}z_{20}z_{21} + y_{11}y_{20}z_{20}^2)t + \cdots = 0.$$

$$z_{11} - u_0y_{10}y_{20} - u_0z_{10}z_{20} + (-u_0y_{10}y_{21} - u_1z_{10}z_{20} - u_0y_{11}y_{20} - u_0z_{10}z_{21} - u_0z_{11}z_{20} - u_1y_{10}y_{20} + 2z_{12})t + \cdots = 0.$$

$$y_{10}y_{20}^2z_{20}^2u_0 + z_{21} + (2z_{22} + 2y_{10}y_{20}^2z_{20}z_{21}u_0 + y_{11}y_{20}^2z_{20}^2u_0 + y_{10}y_{20}^2z_{20}^2u_1 + 2y_{10}y_{20}y_{21}z_{20}^2u_0)t + \cdots = 0.$$

$$-1 + y_{10}y_{20}^2 + (y_{11}y_{20}62 + 2y_{10}y_{20}y_{21})t + \cdots = 0. \tag{3.15}$$

Next, equating the coefficients of (3.15) to zero, we obtain the following system of nonlinear algebraic equations

$$\begin{aligned} &t^0: y_{11} - 2y_{10}y_{20}z_{10}z_{20} = 0, \\ &t^1 \pm 2y_{10}y_{20}z_{11}z_{20} + 2y_{12} - 2y_{10}y_{21}z_{10}z_{20} - 2y_{11}y_{20}z_{10}z_{20} \\ &\quad -2y_{10}y_{20}z_{10}z_{21} = 0, \\ &\vdots \\ &t^3: \ldots \end{aligned} \tag{3.16}$$

$$\begin{aligned} &t^0: y_{21} + y_{10}y_{20}z_{20}^2 = 0, \\ &t^1: y_{10}y_{21}z_{20}^2 + 2y_2 2 + 2y_{10}y_{20}z_{20}z_{21} + y_{11}y_{20}z_{20}^2 = 0, \\ &\vdots \\ &t^3: \ldots \end{aligned} \tag{3.17}$$

$$\begin{aligned} &t^0: z_{11} - u_0y_{10}y_{20} - u_0z_{10}z_{20} = 0, \\ &t^1: -u_0y_{10}y_{21} - u_1z_{10}z_{20} - u_0y_{11}y_{20} - u_0z_{10}z_{21} - u_0z_{11}z_{20} - u_1y_{10}y_{20} \\ &\quad + 2z_{12} = 0, \\ &\vdots \\ &t^4: \ldots \end{aligned} \tag{3.18}$$

$$t^0: y_{10}y_{20}^2z_{20}^2u_0 + z_{21} = 0,$$

$$t^1: 2z_{22} + 2y_{10}y_{20}^2z_{20}z_{21}u_0 + y_{11}y_{20}^2z_{20}^2u_0 + y_{10}y_{20}^2z_{20}^2u_1 + 2y_{10}y_{20}y_{21}z_{20}^2u_0 = 0.$$

$$\vdots$$

$$t^4: .... \tag{3.19}$$

$$t^0: -1 + y_{10}y_{20}^2 = 0,$$

$$t^1: y_{11}y_{20}^2 + 2y_{10}y_{20}y_{21} = 0,$$

$$t^2: 2y_{10}y_{20}y_{22} + y_{10}y_{21}^2 + 2y_{11}y_{20}y_{21} + y_{12}y_{20}^2 = 0,$$

$$t^3: y_{11}y_{21}^2 + 2y_{11}y_{20}y_{22} + 2y_{10}y_{20}y_{23} + 2y_{10}y_{21}y_{22} + 2y_{12}y_{20}y_{21} + y_{13}y_{20}^2 = 0,$$

$$t^4: y_14y_{20}^2 + 2y_{11}y_{20_{y_{23}}} + 2y_{13}y_{20}y_{21} + y_{12}y_{21}^2 + 2y_{12}y_{20}y_{22} + 2y_{10}y_{21}y_{23} + y_{10}y_{22}^2 + 2y_{10}y_{20}y_{24} + 2y_{11}y_{21}y_{22} = 0. \tag{3.20}$$

Now, in order to consider the initial condition of (3.11), we substitute them into (3.12) to obtain

$$y_{10} = 1, \qquad y_{20} = 1,$$

$$z_{10} = 1, \qquad z_{20} = 1,$$

$$u_0 = 1 \tag{3.21}$$

It is important to notice that, from (3.14) and (3.15), we use only the powers $t^i (i = 0,1,2,3)$, because the rest of the information needed is taken from (3.21).

From (3.12), we can observe that there is not an explicit equation for variable u. Instead, $u$is implicitly included in equations for $z_1'$and z_2'. Therefore, from (3.16) and (3.17), we use coefficients of powers $t^i (i = 1,2,3,4)$ to collect enough information to compensate the presence of u in those equations. Furthermore, powers $t^i (i = 0,1,2,)$of (3.20) possesses redundant information that can be ignored. Hence, for (3.20), we can use only powers $t^3$and t^4. Then, solving the NAEs composed by (3.16), (3.17), (3.18), (3.19), (3.20), and (3.21) results the following approximate solution

$$y_1(t) = 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4,$$

$$y_2(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4,$$

$$z_1(t) = 1 + 2t + 2t^2 - \frac{4}{3}t^3 + \frac{101}{174}t^4,$$

$$z_2(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{59}{696}t^4,$$

$$u(t) = 1 + t + \frac{1}{2}t^2 - \frac{1}{174}t^3 - \frac{25}{58}t^4. \tag{3.22}$$

Then, Laplace transformation is applied to (3.22) and then $1/t$is written in place of s. Afterwards, Pade approximant of order $[2/2]$ is applied and $1/s$ is written in placeof t for each variable. Finally, by using the inverse Laplace s transformation, we obtain the exact solution for (3.3)

$$y_1(t) = \exp(2t)$$

$$y_2(t) = \exp(-t)$$

$$z_1(t) = \exp(2t)$$

$$z_2(t) = \exp(-t)$$

$$u(t) = \exp(t) \tag{3.23}$$

Chapter 4

Analytical Solutions for Systems of Partial Differential–Algebraic Equations

As widely known, the importance of research on partial differential-algebraic equations (PDAEs) is that many phenomena, practical or theoretical, can be easily modeled by such equations. Those kinds of equations arise in fields like: nanoelectronics, electrical networks and mechanical systems, among others.

In recent years, PDAEs have received much attention, nevertheless the theory in this field is still young. For linear PDAEs the convergence of Runge-Kutta method is investigated. The numerical solution of linear PDAEs with constant coefficients and the study.

Linear and nonlinear PDAEs are characterized by means of indices which play an important role in the treatment of these equations. The differentiation index is defined as the minimum number of times that all or part of the PDAE must be differentiated with respect to time, in order to obtain the time derivative of the solution, as a continuous function of the solution and its space derivatives.

Higher-index PDAEs (differentiation index greater than one) are known to be difficult to treat even numerically.

Often such problems are first transformed to index-one systems before applying numerical integration methods.

This procedure called index-reduction, can be very expensive and may change the properties of the solution. Since applications problems in science and engineering often lead to higher-index PDAEs, new techniques are required to solve these problems efficiently. Modern methods like homotopy perturbation method (HPM), homotopy analysis method (HAM), variational iteration method (VIM), generalized homotopy method, among others, are powerful tools to approximate nonlinear and linear problems. The HPM has been successfully applied to solve various kinds of nonlinear problems in science and engineering, including Volterra's integro-differential equation, nonlinear differential equations, nonlinear oscillators, partial differential equations (PDEs), bifurcation of nonlinear problems (He 2005b) and boundary-value problems. Recently, the modifications of the HPM have been used to solve. The power series method (PSM) is a well-known classic straightforward procedure from literature that can be applied successfully to solve differential equations of different kind: linear ordinary differential equations (ODEs), nonlinear ODEs, among others. This method establishes that the solution of a differential equation can be expressed as a power series of the independent variable. In this paper we present the application of a hybrid technique combining PSM, Laplace Transform (LT) and Padé Approximant (PA) to find analytical solutions for PDAEs.

4.1 Basic Concept of Power Series Method.

It can be considered that a nonlinear differential equation can be expressed as

$$A(u) - f\ (t)\ 0, \qquad t\ \in \Omega\,, \tag{4.1}$$

having as boundary condition

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \qquad t \in \Gamma\,, \tag{4.2}$$

where $A$is a general differential operator, $f\ (t)$is a known analytic function, $B$is a boundary operator, and is the boundary of domain .

PSM establishes that the solution of a differential equation can be written as

$$u(t) = \sum_{n=0}^{\infty} u_n t^n \tag{4.3}$$

Where $u_0, u_1, \ldots$are unknowns to be determined by series method.

The basic process of series method can be described as:

1. Equation (4.3) is substituted into (4.1), then we regroup the equation in terms of powers of t.

2. We equate each coefficient of the resulting polynomial to zero.
3. The boundary conditions of (4.1) are substituted into (4.3) to generate an algebraic equation for each boundary condition.

4. Aforementioned steps generate an algebraic linear system for the unknowns of (4.3).

5. Finally, we solve the algebraic linear system to obtain the coefficients $u_0, u_1, \ldots$

4.1.1 Padé Approximant.

Given an analytical function $u(t)$ with Maclaurin's expansion

$$u(t) = \sum_{n=0}^{\infty} u_n t^n , \qquad 0 \leq t \leq T. \tag{4.4}$$

The Padé approximant to $u(t)$of order $[L, M]$ which we denote by $[L/M]_u(t)$is defined by

$$[L/M]_u(t) = \frac{p_0 + p_1 t + \ldots + p_L t^L}{1 + q_1 t + \ldots + q_M t^M} , \tag{4.5}$$

where we considered $q_0 = 1$, and the numerator and denominator have no common factors.

The numerator and the denominator in (4.5) are constructed so that $u(t)$and $[L/M]_u(t)$and their derivatives agree at $t = 0$ up to $L + M$. That is

$$u(t) - [L/M]_u(t) = O(t^{L+M+1}) . \tag{4.6}$$

From (4.6), we have

$$u(t) \sum_{n=0}^{M} q_n t^n - \sum_{n=0}^{L} p_n t^n = O(t^{L+M+1}) . \tag{4.7}$$

From (4.7), we get the following algebraic linear systems

$$\begin{cases} u_L q_1 + \cdots + u_{L-M+1 q_M} = -u_{L+1} \\ u_{L+1 q_1} + \cdots + u_{L-M+2 q_M} = -u_{L+2} \\ \vdots \\ p_L = u_L + u_{L-1 q_1} + \cdots + u_0 q_L \end{cases} \tag{4.8}$$

and

$$\begin{cases} p_0 = u_0 \\ p_1 = u_1 + u_0 q_1 \\ \vdots \\ p_L = u_L + u_{L-1 q_1} + \cdots + u_0 q_L \end{cases} \tag{4.9}$$

From (4.6), we calculate first all the coefficients $q_n, 1 \leq n \leq M$. Then, we determine the coefficients $p_n, 0 \leq n \leq L$from (4.7).

Note that for a fixed value of $L + M + 1$, the error (4.8) is smallest when the numerator and denominator of (4.5) have the same degree or when the numerator has degree one higher than the denominator.

4.2 Laplace-Padéresummation Method.

Several approximate methods provide power series solutions (polynomial).

Nevertheless, sometimes, this type of solutions lacks of large domains of convergence. Therefore, Laplace-Padé resummation method is used in literature to enlarge the domain of convergence of solutions or inclusive to find exact solutions.

The Laplace-Padé method can be explained as follows:

1. First, Laplace transformation is applied to power series (4.5).

2. Next, s is substituted by $1/t$ in the resulting equation.

3. After that, we convert the transformed series into a meromorphic function by forming its Padéapproximant of order $\left[\frac{L}{M}\right]$. L and $M$are arbitrarily chosen, but they should be of smaller value than the order of the power series. In this step, the Padéapproximant extends the domain of the truncated series solution to obtain better accuracy and convergence.

4. Then, $t$is substituted by $\frac{1}{s}$.

5. Finally, by using the inverse Laplace s transformation, we obtain the exact or approximate solution.

4.3 Application of PSM to solve PDAE systems.

Since many application problems in science and engineering are often modelled by semi-explicit PDAEs, we consider therefore the following class of PDAEs

$$u_{1t} = \phi(u, u_x, u_{xx}) \,, \tag{4.10}$$

$$0 = \psi(u, u_x, u_{xx}) \,, (t, x) \in (0, T) \times (a, b) \,, \tag{4.11}$$

where u_k: $[0, T] \times [a, b] \to \mathrm{R}^{mk}$, $k = 1, 2$ and $b > a$.

System (4.10)-(4.11) is subject to the initial condition

$$u_1(0, x) = g(x) \,, a \leq x \leq b, \tag{4.12}$$

and some suitable boundary conditions

$$B\big(u(t, a), u(t, b)\,, u_x(t, a), u_x(t, b)\big) = 0, \quad 0 \leq t \leq T, \tag{4.13}$$

where $g(x)$is a given function.

We assume that the solution to initial boundary value problem (4.10)-(4.13) exists, is unique and sufficiently smooth.

To simplify the exposition of the PSM, we integrate first equation (4.10) with respect to t and use the initial condition (4.12) to obtain

$$u_1(t,x) - g\,(x) - \int_0^t \phi(u, u_x, u_{xx})\,dt = 0. \tag{4.14}$$

It is important to note that the time integration of equation (4.10) is not relevant to the solution procedure presented here, so one can apply the PSM directly to (4.10).

In view of PSM, we assume the solution components $u_k(t,x)\,, k = 1,2$ to have the form

$$u_k(t,x) = \alpha_{k,0}(x) + \alpha_{k,1}(x)t + \alpha_{k,2}(x)t^2 \; + \; \dots\,, \tag{4.15}$$

where $\alpha_{k,n}(x)$, $k = 1,2;\ n = 0,1,2,$...are unknown functions to be determined later on by the PSM. Then substitute (4.15) into system (4.11)-(4.14) and equate the coefficients of powers of $t$in the resulting polynomial equations to zero to get an algebraic linear system for these coefficients. Finally, we use equation (4.15) to obtain the exact solution components $u_k, k = 1,2$ as series. The solutions series obtained from PSM may have limited regions of convergence, even if we take a large number of terms. Therefore, we apply the Laplace-Padéresummation method to PSM truncated series to enlarge the convergence region as depicted in the next section.

4.3.1 Test Problems:

We will demonstrate the effectiveness and accuracy of the LPPSM presented in the previous section through two PDAE systems of index-one and index-three.

4.3.2 Nonlinear Index-One System:

Consider the following nonlinear index-one PDAE which arises as a similarity reduction of Navier-Stokes equations (Budd et al. 1994)

$$u_{1t} = u_{1xx} - u_2 u_{1x} + u_1^2 - 2\int_0^1 u_1^2\,dx, \tag{4.16}$$

$$0 = u_{2x} - u_1, \tag{4.17}$$

where $0 < x < 1$ and $t > 0$.

System (4.16)-(4.17) is subject to the following initial condition

$$u_1(0,x) = \cos\pi x\,, \qquad 0 \le x \le 1, \tag{4.18}$$
and boundary conditions

$$u_{1x}(t,0) = u_{1x}(t,1) = u_2(t,0) = u_2(t,1) = 0, t \ \ge\ 0. \tag{4.19}$$
The exact solution of problem (4.16)-(4.19) is

$$u_1(t,x) = e^{-\pi^2 t}\cos\pi x\,,$$

$$u_2(t,x) = (1/\pi)e^{-\pi^2 t}\sin\pi x\,, \quad 0 \le x \le 1, t \ge 0. \tag{4.20}$$
Since one time differentiation of equation (4.17) determines u_{2t}in terms of $u$and its space derivatives, then PDAE (4.16)-(4.17) is index-one. Note that no initial condition is prescribed for the variable u_2 as this is determined by the PDAE. In order to simplify the exposition of the PSM presented in section “Application of PSM to solve PDAE systems” to solve (4.16)-(4.17), we first integrate equation (4.16) with respect to $t$and use the initial condition (4.20) to get

$$u_1(t,x) - \cos\pi x - \int_0^t \left(u_{1xx} - u_2 u_{1x} + u_1^2 - 2\int_0^1 u_1^2 dx\right) dt = 0 \tag{4.21}$$

In view of the PSM, we assume the solution components $u_k, k = 1, 2$ to have the form

$$u_k(t,x) = \alpha_{k,0}(x) + \alpha_{k,1}(x)t + \alpha_{k,2}(x)t^2 + \ldots, \tag{4.22}$$
where $\alpha_{k,n}(x)$, $k = 1, 2$; $n = 0, 1, 2, \ldots$are unknown functions to be determined later on by the PSM. Then, we substitute (4.21) into equations (4.16) and (4.22) to get

$$\begin{aligned}\sum_{n=0}^{\infty} \alpha_{1,n}(x)t^n - \cos\pi x - \int_0^t \sum_{n=0}^{\infty} \alpha_{1,n}''(x)t^n dt \\ + \int_0^t \left(\sum_{n=0}^{\infty} \alpha_{2,n}(x)t^n\right)\left(\sum_{n=0}^{\infty} \alpha_{1,n}'(x)t^n\right) dt \\ - \int_0^t \left(\sum_{n=0}^{\infty} \alpha_{1,n}(x)t^n\right)^2 dt + \int_0^t \int_0^1 \left(\sum_{n=0}^{\infty} \alpha_{1,n}(x)t^n\right)^2 dx dt \\ = 0.\end{aligned} \tag{4.23}$$

$$\sum_{n=0}^{\infty} \left(\alpha_{2,n}'(x) - \alpha_{1,n}(x)\right) t^n = 0, \tag{4.24}$$

where denotes the ordinary derivative with respect to x.

Equating the coefficients of powers of t to zero in (4.24) then solving the resulting equation for $\alpha_{2,n}(x)$ and using the boundary conditions (4.19), we have

$$\alpha_{2n}(x) = \int_0^x \alpha_{1,n}\,(x)\,dx, n = 0, 1, 2, \ldots \qquad (4.25)$$

Now equation (1.21) can be written as a series

$$\left(\alpha_{1,0}(x) - \cos\pi x\right) + \sum_{n=1}^{\infty}\left(\alpha_{1,n} - (1/n)\alpha''_{1,n-1}(x) - (1/n)\sum_{k=0}^{n-1}\beta_{k,n}\,(x)\right)t^n = 0, \qquad (4.26)$$

where

$$\beta_{k,n}(x) = \alpha_{1,k}(x)\alpha_{1,n-1-k}(x) - \alpha'_{1,n-1-k}(x)\int_0^x \alpha_{1,k}(x)dx - 2\int_0^1 \alpha_{1,k}(x)\alpha_{1,n-1-k}(x)dx.$$

Equating all coefficients of powers of $t$to zero in (4.26), yields $\alpha_{1,0}(x) = \cos\pi x$and the recursive formula for

$$\alpha_{1,n}(x) = (1/n)\alpha''_{1,n-1}(x) + (1/n)\sum_{k=0}^{n-1}\beta_{k,n}\,(x), n = 1, 2, 3, \ldots \qquad (4.27)$$

From recursion (4.27), we get $\alpha_{1,1}(x) = -\pi^2\cos\pi x$and $\alpha_{1,2}(x) = (\pi/4)\cos\pi x$.

From equation (4.25), we get $\alpha_{2,0}(x) = (1/\pi)\sin\pi x$, $\alpha_{2,1}(x) = -\pi\sin\pi x$and $\alpha_{2,2}(x) = (\pi^3/2)\sin\pi x$. Using (68) and the coefficients recently obtained, we have

$$u_1(t,x) = \left(1 - \pi^2 t + \frac{1}{2}(-\pi^2 t^2)^2\right)\cos\pi x, \qquad (4.28)$$

and

$$u_2(t,x) = \left(1 - \pi^2 t + \frac{1}{2}(-\pi^2 t^2)^2\right)(1/\pi)\sin\pi x. \qquad (4.29)$$

Similarly, the coefficients $\alpha_{1,n}(x)$and $\alpha_{2,n}(x)$for $n \geq 3$can be found from (4.25) and (4.25) respectively.

The solutions series obtained from the PSM may have limited regions of convergence, even if we take a large number of terms. Accuracy can be increased by applying the Laplace-Padé post-treatment. First, we apply t-

Laplace transform to (4.28) and (4.29). Then, we substitute *s* by 1/*t*and apply *t*-Padé approximant to the transformed series.

Finally, we substitute *t* by 1/*s* and apply the inverse Laplace s-transform to the resulting expressions to get the approximate or exact solutions. Applying Laplace transforms to $u_1(t,x)$and $u_2(t,x)$yields

$$\mathcal{L}[u_1(t,x)] = \left(\frac{1}{s} - \frac{\pi^2}{s^2} + \frac{\pi^4}{s^3}\right)\cos \pi x \;, \tag{4.30}$$

and

$$\mathcal{L}[u_2(t,x)] = \left(\frac{1}{s} - \frac{\pi^2}{s^2} + \frac{\pi^4}{s^3}\right)(1/\pi)\sin \pi x \;, \tag{4.31}$$

For the sake of simplicity let $s = \frac{1}{t}$, then

$$L\,[u_1(t,x)] = (t - \pi^2 t^2 + \pi^4 t^3)\cos \pi x \;, \tag{4.32}$$

and

$$L[u_2(t,x)] = (t - \pi^2 t^2 + \pi^4 t^3)(1/\pi)\sin \pi x. \tag{4.33}$$

All of the $[L/M]t$-Padé approximants of (4.32) and (4.33) with $L \geq 1$ and $M \geq 1$ and $L + M \leq 3$ yield

$$[L/M]_{u_1}(t,x) = \left(\frac{t}{1 + \pi^2 t}\right)\cos \pi x, \tag{4.34}$$

and

$$[L/M]_{u_2}(t,x) = \left(\frac{t}{1 + \pi^2 t}\right)(1/\pi)\sin \pi x. \tag{4.35}$$

Now since $t = \frac{1}{\pi}s$, we obtain $[L/M]_{u_1}$ and $[L/M]_{u_2}$in terms of *s* as follows

$$[L/M]_{u_1}(t,x) = (\pi^2 + s)^{-1}\cos \pi x, \tag{4.36}$$

$$[L/M]_{u_2}(t,x) = (\pi^2 + s)^{-1}(1/\pi)\sin \pi x. \tag{4.37}$$

Finally, applying the inverse LT to the Padé approximants (4.36) and (4.37), we obtain the approximate solution which is in this case the exact solution (4.21) in closed form.

4.3.3 Linear Index-Three System:

Consider the following index-three PDAE system

$$u_{1tt} = u_{1xx} + u_3 \sin \pi x, \tag{4.38}$$

$$u_{2tt} = u_{2xx} + u_3 \cos \pi x, \tag{4.39}$$

$$0 = u_1 \sin \pi x + u_2 \cos \pi x - e^{-t}, \tag{4.40}$$

where $0 < x < 1$ and $t > 0$.

System (4.37)-(4.38) is subject to the following initial conditions

$$u_1(0,x) = \sin \pi x \, u_{1t}(0,x) = -\sin \pi x, \tag{4.41}$$

$$u_2(0,x) = \cos \pi x \, u_{2t}(0,x) = -\cos \pi x \; 0 \le x \le 1, \tag{4.42}$$

and the boundary conditions

$$u_1(t,0) = u_1(t,1) = 0,$$

$$u_2(t,0) = -u_2(t,1) = e^{-t} \; t \ge 0. \tag{4.43}$$

The exact solution of problem (4.37)-(4.38) is

$$u_1(t,x) = e^{-t} \sin \pi x \, u_2(t,x) = e^{-t} \cos \pi x \,,$$

$$u_3(t,x) = (1+\pi^2)e^{-t}, \;\; 0 \le x \le 1, t \ge 0. \tag{4.44}$$

Since three time differentiations of equation (3.17) determine $u3t$ in terms of the solution u and its space derivatives, then PDAE (4.39)-(4.37) is index-three. Therefore, this PDAE is difficult to solve numerically. Moreover no initial condition is prescribed for the variable $u3$ as this is determined by the PDAE. In order to simplify the exposition of the LPPSM presented in section "Application of PSM to solve PDAE systems" to solve (4.39)-(4.40), we first integrate equations (4.39) and (4.40) twice with respect to t and use the initial conditions (4.40)-(4.41) to get

$$u_1(t,x) - \sin \pi x + t \sin \pi x - \int_0^t \int_0^t u_{1xx} + u_3 \sin \pi x \, dtdt = 0, \tag{4.45}$$

$$u_2(t,x) - \cos \pi x + t \cos \pi x - \int_0^t \int_0^t u_{2xx} + u_3 \cos \pi x \, dtdt = 0, \tag{4.46}$$

In view of the PSM, we assume the solution components $u_k(t,x), k = 1,2,3$ to have the form

$$u_k(t,x) = \alpha_{k,0}(x) + \alpha_{k,1}(x)t + \alpha_{k,2}(x)t^2 + \ldots, \tag{4.47}$$

where $\alpha_{k,n}(x), k = 1,2,3;\ n = 0,1,2,\ldots$are unknown functions to be determined later on by the PSM.

Substituting (93) into equations (4.39), (45) and (4.47) we get the system

$$\sum_{n=0}^{\infty} \alpha_{1,n}(x)t^n - \sin \pi x + t \sin \pi x - \int_0^t \int_0^t \sum_{n=0}^{\infty} \alpha''_{1,n}(x)\, t^n dtdt$$

$$- \sin \pi x \int_0^t \int_0^t \sum_{(n=0)}^{\infty} \alpha_{3,n}(x)t^n dtdt = 0, \tag{4.48}$$

$$\sum_{n=0}^{\infty} \alpha_{2,n}(x)\, t^n - \cos \pi x + t \cos \pi x - \int_0^t \int_0^t \sum_{(n=0)}^{\infty} \alpha''_{2,n}\,(x) t^n dt dt$$

$$- \cos \pi x \int_0^t \int_0^t \sum_{(n=0)}^{\infty} \alpha_{3,n}\,(x) t^n dt dt = 0, \qquad (4.49)$$

and

$$\sum_{(n=0)}^{\infty} \big(\alpha_{1,n}(x) \sin \pi x + \alpha_{2,n}(x) \cos \pi x\big)\, t^n - e^{-t} = 0, \qquad (4.50)$$

where denotes the ordinary derivative with respect to x.

System (4.46)-(4.48) can be rewritten as series

$$\big(\alpha_{1,0}(x) - \sin \pi x\big) + \big(\alpha_{1,1}(x) + \sin \pi x\big)t - \sum_{n=2}^{\infty} \left(\frac{\alpha''_{1,n-2}(x) + \alpha_{3,n-2}(x) \sin \pi x}{(n-1)n}\right) t^n = 0$$

$$\big(\alpha_{2,0}(x) - \cos \pi x\big) + \big(\alpha_{2,1}(x) + \cos \pi x\big)t - \sum_{n=2}^{\infty} \left(\frac{\alpha''_{2,n-2}(x) + \alpha_{3,n-2}(x) \cos \pi x}{(n-1)n}\right) t^n = 0 \quad (4.51)$$

$$\sum_{n=0}^{\infty} \left(\alpha_{1,n}(x) \sin \pi x + \alpha_{2,n}(x) \cos \pi x - \frac{(-1)^n}{n!}\right) t^n = 0,$$

Equating the coefficient of powers of t to zero in (4.46) then solving the resulting system we find the coefficients $\alpha_{k,n}(x)$, for$k = 1,2,3$ and$n = 0,1,2, \ldots$

$$\alpha_{1,0}(x) = \sin \pi x\,, \alpha_{1,1}(x) = -\sin \pi x\,,$$

$$\alpha_{2,0}(x) = \cos \pi x\,, \alpha_{2,1}(x) = -\cos \pi x\,,$$

and the nonsingular algebraic linear system for the unknown functions $\alpha_{1,n}, \alpha_{2,n}$and $\alpha_{3,n-2}$

$$\alpha_{1,n}(x) = \frac{\alpha_{3,n-2}(x) \sin \pi x}{(n-1)n} = \frac{\alpha''_{1,n-2}(x)}{(n-1)n},$$

$$\alpha_{2,n}(x) = \frac{\alpha_{3,n-2}(x) \cos \pi x}{(n-1)n} = \frac{\alpha''_{2,n-2}(x)}{(n-1)n} \qquad (4.52)$$

$$\alpha_{1,n}(x)\sin\pi x+\alpha_{2,n}(x)\cos\pi x-\frac{(-1)^n}{n!}\text{for } n=2,3,...$$

Solving system (4.6) exactly, we obtain the recursions

$$\alpha_{1,n}(x)=\frac{(-1)^n}{n!}\sin\pi x+\frac{\delta_n(x)\cos\pi x}{(n-1)n},$$

$$\alpha_{2,n}(x)=\frac{(-1)^n}{n!}\cos\pi x+\frac{\delta_n(x)\sin\pi x}{(n-1)n}, \quad (4.53)$$

$$\alpha_{3,n-2}(x)=\frac{(-1)^n}{(n-2)!}-\alpha''_{1,n-2}(x)\sin\pi x-\alpha''_{2,n-2}(x)\cos\pi x,$$

where $\delta_n(x)=\alpha''_{1,n-2}(x)\cos\pi x-\alpha''_{2,n-2}(x)\sin\pi x$.

For $n=2,3,4$, we have $\delta_n(x)=0$ and hence

$$\alpha_{1,2}(x)=\frac{1}{2}\sin\pi x\,,\alpha_{2,2}(x)=\frac{1}{2}\cos\pi x\,,\alpha_{3,0}(x)=1+\pi^2,$$
$$\alpha_{1,3}(x)=-\frac{1}{6}\sin\pi x\,,\alpha_{2,3}(x)=-\frac{1}{6}\cos\pi x\,,\alpha_{3,1}(x)=(-1+\pi^2)\,,$$

and

$$\alpha_{1,4}(x)=\frac{1}{24}\sin\pi x\,,\alpha_{2,4}(x)=\frac{1}{24}\cos\pi x\,,\alpha_{3,2}(x)=\frac{1}{2}(1+\pi^2)\,.$$

Using (4.46) and the coefficients recently obtained, we get

$$u_1(t,x)=\left(1-t+\frac{1}{2}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4\right)\sin\pi x\,, \quad (4.54)$$

$$u_2(t,x)=\left(1-t+\frac{1}{2}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4\right)\cos\pi x, \quad (4.55)$$

and

$$u_3(t,x)=(1+\pi^2)\left(1-t+\frac{1}{2}t^2\right). \quad (4.56)$$

Similarly, the coefficients $\alpha_{1,n}(x),\alpha_{2,n}(x)$and $\alpha_{3,n-2}(x)$for $n\geq 5$ can be found from (4.53). The solutions series obtained from the PSM may have limited regions of convergence, even if we take a large number of terms. Therefore, we apply the t-Padé approximation technique to these series to increase the convergence region. First t-Laplace transform is applied to (4.54), (4.55) and (4.56).

Then, s is substituted by $1/t$and the t-Padé approximant is applied to the transformed series. Finally, $t$is substituted by $1/s$and the inverse Laplace s-

transform is applied to the resulting expressions to get the approximate or exact solutions.

Applying Laplace transforms to $u_1(t,x), u_2(t,x)$ and $u_3(t,x)$ yields

$$\mathcal{L}\left[u_1(t,x)\right] = \left(\frac{1}{s} - \frac{\pi^2}{s^2} + \frac{\pi^4}{s^3}\right)\sin \pi x, \tag{4.57}$$

$$\mathcal{L}[u_2(t,x)] = \left(\frac{1}{s} - \frac{\pi^2}{s^2} + \frac{\pi^4}{s^3}\right)\cos \pi x, \tag{4.58}$$

and

$$\mathcal{L}[u_3(t,x)] = (1+\pi^2)\left(\frac{1}{s} - \frac{\pi^2}{s^2} + \frac{\pi^4}{s^3}\right). \tag{4.59}$$

For the sake of simplicity let $s = 1/t$, then

$$\mathcal{L}[u_1(t,x)] = (t - \pi^2 t^2 + \pi^4 t^3)\sin \pi x, \tag{4.60}$$

$$\mathcal{L}[u_2(t,x)] = (t - \pi^2 t^2 + \pi^4 t^3)\cos \pi x, \tag{4.61}$$

and

$$\mathcal{L}[u_3(t,x)] = (1+\pi^2)(t - \pi^2 t^2 + \pi^4 t^3). \tag{4.62}$$

All of the $[L/M]t$-Padé approximants of (4.60), (4.61) and (4.62) with $L \geq 1$ and $M \geq 1$ and $L + M \leq 3$ yield

$$[L/M]_{u_1}(t,x) = \left(\frac{t}{1+t}\right)\sin \pi x, \tag{4.63}$$

$$[L/M]_{u_2}(t,x) = \left(\frac{t}{1+t}\right)\cos \pi x, \tag{4.64}$$

and

$$[L/M]_{u_3}(t,x) = (1+\pi^2)\left(\frac{t}{1+t}\right), \tag{4.65}$$

Now since $t = 1/s$, we obtain $[L/M]_{u_1}$, $[L/M]_{u_2}$ and $[L/M]_{u_3}$ in terms of s as follows

$$[L/M]_{u_1}(t,x) = (\pi^2 + s)^{-1}\sin \pi x, \tag{4.66}$$

$$[L/M]_{u_2}(t,x) = (\pi^2 + s)^{-1}\cos \pi x, \tag{4.67}$$

and

$$[L/M]_{u_3}(t,x) = (1+\pi^2)(\pi^2 + s)^{-1}. \tag{4.68}$$

Finally, applying the inverse Laplace transform to the Padé approximants (4.66), (4.67) and (4.68), we obtain the approximate solution which is in this case the exact solution (4.32) in closed form.

Discussion.

We presented the power series method (PSM) as a useful analytical tool to solve partial differential algebraic equations (PDAEs). Two PDAE problems of index-one and index-three were solved by this method leading to the some solutions. The method has successfully handled the index-three PDAE without the need for a preprocessing step of index-reduction. For each of the two problems solved here, the PSM transformed the PDAE into an easily solvable linear algebraic system for the coefficient functions of the power series solution. To improve the PSM solution, a Laplace-Padé (LP) post-treatments applied to the PSM's truncated series leading to the some solution. Additionally, the solution procedure does not involve unnecessary computation like that related to noise terms. This greatly reduces the volume of computation and improves the efficiency of the method. It should be noticed that the high complexity of these problems was effectively handled by LPPSM method due to the malleability of PSM and resummation capability of Laplace-Padé. What is more, there is not any standard analytical or numerical methods to solve higher-index PDAEs, converting the LPPSM method into an attractive tool to solve such problems. On one hand, semi-analytic methods like HPM, HAM, VIM among others, require an initial approximation for the sought solutions and the computation of one or several adjustment parameters. If the initial approximation is properly chosen the results can be highly accurate, nonetheless, no general methods are available to choose such initial approximation. This issue motivates the use of adjustment parameters obtained by minimizing the least-squares error with respect to the numerical solution. On the other hand, PSM or LPPSM methods do not require any trial equation as requisite for the starting the method. What is more, PSM obtains its coefficients using an easy computable straightforward procedure that can be implemented into programs like Maple or Mathematica. Finally, if the solution of the PDAE is not expressible in terms of known functions then the LP post treatement will provide a larger domain of convergence.

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