Chapter 1

Classification and Spectral Property with Structure

We first show that a tile digit set in ZS must be an integer tile ($i.e. D \oplus L = Z^s$ for some discrete set L). This allows us to combine the technique of Coven and Meyerowitz on integer tiling on R_1 together with results to characterize explicitly all tile digit sets D ⊂ Z with A = $p^\alpha q$ (p, q distinct primes) as modulo product-form of some order, an advance of the previously known results for A = p^α and pq. For a regular maximal orthogonal set, we show that its completeness in L^2(μq,b) is determined by the certain growth rate of non-zero digits in the tail of the b−adic expansions of the elements. We exhibit complete orthogonal exponentials with zero Beurling dimensions. These examples show that the technical condition in Theorem 3.5 of [5] cannot be removed. For an irregular maximal orthogonal set, we show that under some condition, its completeness is equivalent to that of the corresponding regularized mapping. .We characterize the tile digit sets through some product of cyclotomic polynomials (kernel polynomials), which is a generalization of the product-form to higher order.

Section (1.1): Tile Digit Sets as Product-Forms

For A be an $s \times s$ expanding matrix (i.e. all eigenvalues have moduli > 1) with integral entries and $|detA| = b$ is a positive integer. Let $D \subset \mathbb{Z}^s$ and call it a digit set. It follows that there exists a unique compact set $T := T(A, D) \subset \mathbb{R}^s$ satisfying the set-valued relation $AT = T + D$. Alternatively, T can be expressed as a set of radix expansions with base A and digits in D:

$$T = \{\sum_{k=1}^{\infty} A^{-k} d_k : d_k \in D\}. \qquad (1)$$

It is well known that when $\#D = |detA| = b$ and T has non-empty interior, then T is a translational tile in $\mathbb{R}^s$ [2]. We call such T a self-affine tile (or self-similar tile if A is a scaling multiple of an orthonormal matrix) and D a tile digit set with respect to A. These tiles are referred as fractal tiles because their boundaries are usually fractals. See ([4], [3], [5], [6], [7], [8], [9], [10]). Note that for D to be a tile digit set, $\#D = |detA| = b$ is necessary. Hence in our consideration of tile digit sets, we will make this assumption without explicitly mentioning. For the digit sets in $\mathbb{R}^1$, we also assume , that $D \subset \mathbb{Z}^+, 0 \in D$ and $g.c.d(D) = 1$.

The interest is on the fundamental question of characterizing the tile digit sets D (i.e., $T(A, D)$ is a tile) for a given matrix A. This turns out to be a very challenging problem even in $\mathbb{R}^1$. So far the only known cases are $A = b$ with $b = p^\alpha$, a prime power [5], or $b = pq$, a product of two distinct primes [11]; there are extensions to the higher dimensional case when $|det(A)| = p$ is a prime ([13], [5])., we will advance our knowledge to the case when $b = p^\alpha q$, based on the theory that was developed in [6] and the new techniques here.

The most basic tile digit set $D \subset \mathbb{Z}$ is $D \equiv \mathcal{E}(mod\, b)$ where $b = \#D$ and $E = \{0, 1, \cdots, b-1\}$ (i.e., D is a complete residue set modulo b). According to the earlier study in [5,6], it was suggested that a tile digit set should possess certain product-form structure, i.e., D is obtained by decomposing $\mathcal{E}$ into direct sums according to the factors of $b(= \#D)$, together with certain modulation by b. [15], investigated this idea of modulo product-forms in detail, and discovered the new classes of k-th order modulo product-forms of tile digit set . The main tool they used is the cyclotomic polynomials in

elementary number theory, from which such product-forms are formulated algebraically in terms of product of cyclotomic factors of P_D, the mask polynomial of D.
The very interesting part was a tree structure on the cyclotomic polynomials (Φ-tree), and the following theorem was proved
Theorem (1.1.1)[1]: [5] $D \subset \mathbb{Z}$ is a tile digit set if and only if there is a blocking N in the Φ-tree such that

$$P_D(x) = \left(\prod_{\Phi_d \in N} \Phi_d(x) \right) Q(x).$$

By a blocking in a tree, we mean a finite subset N of vertices in the tree such that any paths from the root must pass through one and only one vertex in the set N. The above product is called a kernel polynomial. If D is a modulo product-form, then the kernel polynomial plays the role of product-form .
First we will relate the tile digit sets with another well-known class of integer sets in number theory. A finite subset A in $\mathbb{Z}^s$ is called an integer tile if there exists $\mathcal{L} \subset \mathbb{Z}^s$ such that $A \oplus \mathcal{L} = \mathbb{Z}^s$ (The set $A \oplus B = \{a + b : a \in A, b \in B\}$ with $\oplus$ means that all elements are distinct). It is obvious that A is an integer tile if and only if $A + [0,1]^s$ is a translational tile of $\mathbb{R}^s$. The study of integer tiling dates back to the 40's, when Hajos and de Bruijn studied the factorization of abelian groups, and used it to solve some conjectures on Minkowski's geometry of numbers. see [5] for some details on the development of this topic. Our first main theorem, which holds in any dimension, is
Theorem (1.1.2) [1]: Let A be an integral expanding matrix. Suppose $D \subset \mathbb{Z}^s$ is a tile digit set (i.e., $T(A, D)$ is a self-affine tile), then D is an integer tile.
The proof makes use of the self-replicating tiling sets of T [12] to construct the tiling set $\mathcal{L}$ so that $D \oplus \mathcal{L} = \mathbb{Z}^s$. For the one-dimensional case, the theorem enables us to use the classical factorization techniques of cyclic groups to study the structure of tile digit sets. In particular, the most important technique we employ is the decomposition of the integer tiles A when $\#A = p^\alpha q^\beta$ developed by Coven and Meyerowitz [16]. We will combine this decomposition method and Theorem (1.1.1) to give the explicit expression of the kernel polynomial in the Φ-tree , and to conclude the following .
Theorem (1.1.3) [1]: Let $D \subset \mathbb{Z}$ be a digit set with $\#D = p^\alpha q$ and p, q are primes. Then D is a tile digit set if and only if it is a k^{th}-order modulo product-form for some k.
The proof of Theorem (1.1.3) relies heavily on the algebraic operations on the cyclotomic polynomials, which offers a lot more flexibility in handling the mask polynomial P_D than the set D itself.
We prove Theorem (1.1.2) and recall some needed facts on integer tiles , we summarize the notions of modulo product-forms and cyclotomic tree we developed in [2]., we give a detailed study for the case $b = p^\alpha q^\beta$ and prove Theorem (1.1.3) for the case $p^2 q$, the case when $b = p^\alpha q$ is similar and is outlined , we give some further remarks on the tile digit sets and the integer tiles; the modulo product-forms and the related spectral problem of self-affine tiles are also discussed.
We use the affine pair (A, D) in $\mathbb{R}^s$ and the attractor $T := T(A, D)$ in $\mathbb{R}^s$. Let $D_{A,k} := \{\sum_{j=0}^{k-1} A^j d_{i_j} : d_{i_j} \in D\}$ and $D_{A,\infty} := \bigcup_{k=1}^{\infty} D_{A,k}$. The following are some well-known equivalent conditions for T to be a tile [5].
Theorem (1.1.4) [1]: That $T := T(A, D)$ is a self-affine tile (i.e., $T^\circ \neq \emptyset$) is equivalent to either one of the following conditions:
(i) $\mu(T) > 0$ where μ is the Lebesgue measure;

(ii) $\overline{T^\circ} = T$ and $\mu(\partial T) = 0$;
(iii) $\#D_{A,k} = b^k$ for all $k \geq 0$. (Here $b = |\det(A)|$.)
our first goal is to show that if $T(A, D)$ is a self-affine tile (i.e., D is a tile digit set with respective to A), then D must be an integer tile in $\mathbb{Z}^s$. We need a couple of lemmas to tie up some translational properties of the self-affine tiles.
Let $\|\cdot\|$ denote the Euclidean norm.
Lemma(1.1.5) [1]: let $E_k = \{\sum_{j=1}^{\infty} A^{-kj} z : z \in D_{A,k}\}$ and let $E = \bigcup_{j=1}^{\infty} E_k$, then E is dense in T.
Proof: For any $y \in T$, write $y = \sum_{j=1}^{\infty} A^{-j} d_j$ and let $y_k = \sum_{j=1}^{k} A^{-j} d_j$. Let $z_k = A^k y_k$, then $z_k \in D_{A,k}, \sum_{j=1}^{\infty} A^{-kj} z_k \in E_k$, and

$$\left\| \sum_{j=1}^{\infty} A^{-kj} z_k - y_k \right\| = \left\| \sum_{j=1}^{\infty} A^{-kj} y_k \right\| \leq \frac{\|A^{-k}\|}{1 - \|A\|^{-k}} \|y_k\|.$$

In view of the expanding property of A and $\{\|y_k\|\}_{k=1}^{\infty}$ is bounded, the above expression tends to 0 as k tends to ∞. This shows that E is dense in T.
It is easy to see from (1) that for any $\ell \geq 1, T(A, D) = T(A^\ell, D_{A,\ell})$. Moreover, if we let $\widetilde{D} := D_{A,\ell} - z^*$ be a translation of the digits, then by (1), $T(A^\ell, \widetilde{D})$ satisfies

$$T\left(A^\ell, \widetilde{D}\right) = T\left(A^\ell, D_{A,\ell}\right) - \sum_{j=1}^{\infty} A^{-j\ell} z^*. \qquad (2)$$

We can impose some properties on the tile by suitably translating the digits.
Lemma (1.1.6) [1]: Suppose $T(A, D)$ is a self-affine tile, then there exists $z^* \in D_{A,\ell}$ for some $\ell \geq 1$ such that for $\widetilde{D} := D_{A,\ell} - z^*$, the tile $\tilde{T} := T(A^\ell, \widetilde{D})$ satisfies

$$0 \in \tilde{T}^\circ \quad and \quad \partial\tilde{T} \cap \mathbb{Z}^s = \emptyset.$$

($\partial\tilde{T}$ denotes the boundary of $\tilde{T}$.)
Proof: It is known that there exists $D' := D_{A,k} - z, z \in D_{A,k}$ such that $0 \in T(A^k, D')^\circ$ ([5, Theorem 1.2]). Hence we can assume without loss of generality that $0 \in T^\circ (= T(A, D)^\circ)$. Note that $\partial T \cap \mathbb{Z}^s$ is a finite set. We let

$$\eta = \min\{dist(w, \partial T) : w \in \mathbb{Z}^s \backslash (\partial T \cap \mathbb{Z}^s)\} (> 0). \qquad (3)$$

Let $0 < \epsilon < \eta$ such that $B(0, \epsilon) \subset T$. It follows from $\mu(\partial T) = 0$ (Theorem (1.1.4)(ii)) and the density of E in T (Lemma (1.1.5)) that there exists $u \in B(0, \epsilon/2)$ with $u = \sum_{j=1}^{\infty} A^{-kj} z^*, z^* \in D_{A,k}$, and $v + u \notin \partial T$ for each $v \in \partial T \cap \mathbb{Z}^s$. This, together with (3), yields

$$(\partial T - u) \cap \mathbb{Z}^s = \emptyset. \qquad (4)$$

Let $\widetilde{D} = D_{A,k} - z^*$ and let $\tilde{T} = T(A^k, \widetilde{D})$. Then by (2),

$$\tilde{T} = T - \sum_{j=1}^{\infty} A^{-kj} z^* = T - u.$$

Now by our choice of u and ϵ, we have $0 \in B(0, \epsilon/2) - u \subset \tilde{T}$. Hence $0 \in \tilde{T}^\circ$. Note also that $\partial\tilde{T} = \partial T - u$, we have, by (4), $\partial\tilde{T} \cap \mathbb{Z}^s = \emptyset$.
For a self-affine tile T, there exists a self-replicating tiling set $J \subset \mathbb{Z}^s$ [9], i.e., there exists $k \geq 1$ such that

$$A^k J \oplus D_{A,k} = J. \qquad (5)$$

Basically if $0 \in T^\circ$, we can take $J = D_{A,\infty}$ in (5). If $0 \in \partial T$, we translate T so that 0 is in the interior of the translated tile, then choose the J accordingly. The direct sum in (5) is easy to check.
Recall a finite set $A \subset \mathbb{Z}^s$ is an integer tile if there exists $\mathcal{L}$ such that $A \oplus \mathcal{L} = \mathbb{Z}^s$.

Theorem (1.1.7) [1]: Let $T(A,D)$ be a self-affine tile. Then D tiles $\mathbb{Z}^s$, i.e., $D \oplus \mathcal{L} = \mathbb{Z}^s$ for some $\mathcal{L}$ in $\mathbb{Z}^s$.

Proof: Let $\widetilde{D} = D_{A,\ell} - z^*$ be a digit set so chosen that the conclusion of Lemma (1.1.6) holds. As $0 \in T(A^\ell, \widetilde{D})^\circ$, it follows that $J := \widetilde{D}_{A^l,\infty}$ is a self-replicating tiling set in $\mathbb{Z}^s$. Hence,

$$A^\ell J \oplus \widetilde{D} = J. \qquad (6)$$

Let $\tilde{T} = T(A^\ell, \widetilde{D})$ and let $B = \mathbb{Z}^s \cap \tilde{T}$. We claim that $J \oplus B = \mathbb{Z}^s$. First, for any $w \in \mathbb{Z}^s$, we have $w \in \tilde{T} + t$ for some $t \in J$. This means that $w - t \in B$. Hence, $J + B = \mathbb{Z}^s$. To show that the representation is unique, we let $w = t + z = t' + z'$ where $t, t' \in J$ and $z, z' \in B$. By the tiling assumption, we must have $w \in \partial\tilde{T} + t$. This shows that $w - t \in B \cap \partial\tilde{T}$. But this is impossible since $\partial\tilde{T} \cap \mathbb{Z}^s = \emptyset$ by our choice of $\widetilde{D}$.

Now, by adding B to both side of (6), we have

$$A^\ell J \oplus \widetilde{D} \oplus B = J \oplus B = \mathbb{Z}^s.$$

Let $J' = A^\ell J \oplus B$. As $\widetilde{D} = D_{A,\ell} - z^*$, we have from the above, $D_{A,\ell} \oplus (J' - z^*) = \mathbb{Z}^s$. This implies

$$D \oplus (AD \oplus \cdots \oplus A^{\ell-1}D \oplus (J' - z^*)) = \mathbb{Z}^s.$$

The theorem follows by setting $\mathcal{L} = AD \oplus \cdots \oplus A^{\ell-1}D \oplus (J' - z^*)$.

The converse of Theorem (1.1.7) is false in general, as is seen in the second part of the following example.

Example (1.1.8) [1]: Let $A = 4, D = \{0,1,8,9\} = \{0,1\} \oplus 4\{0,2\}$, then $T(4,D) = [0,1] \cup [2,3]$ is a self-similar tile (as it satisfies $4T = T + D$), and the tiling set for T is $J = \{0,1\} \oplus 4\mathbb{Z}$. By Theorem (1.1.7), D tiles $\mathbb{Z}$ also, and the tiling set for D (as an integer tile) is $\mathcal{L} = \{0,2,4,6\} \oplus 16\mathbb{Z}$.

If we let $D = \{0,1,4,5\} = \{0,1\} \oplus 4\{0,1\}$, then D tiles $\mathbb{Z}$ with the tiling set $\mathcal{L} = \{0,2\} \oplus 8\mathbb{Z}$. On the other hand, $T(4,D)$ is not a tile of $\mathbb{R}$ since $\#(D + 4D) = 12 < 4^2$ (by Theorem (1.1.4)(iii)).

We remark that the explicit expression of the prime-power integer tiles and tile digit sets will

$a|b$ means a divides b, and $a \nmid b$ means a does not divide b. The notations apply to both integers and polynomials. We will give a brief summary on the cyclotomic polynomials and integer tiles in $\mathbb{Z}^1$.

Let $\Phi_d(x)$ be the d-th cyclotomic polynomial, which is the minimal polynomial of the primitive d-th root of unity, i.e., $\Phi_d(e^{2\pi i/d}) = 0$. It is well-known that

$$x^n - 1 = \textstyle\sum_{d|n} \Phi_d(x) \qquad (7)$$

and the formula provides a constructive way to find Φ_d inductively. The class of cyclotomic polynomials plays a fundamental role . Its basic manipulation rules are recalled below, they will be used extensively.

Proposition (1.1.9) [1]: Cyclotomic polynomials satisfy the following:

(i) If p is a prime, then $\Phi_p(x) = 1 + x + \ldots + x^{p-1}$ and $\Phi_{p^{\alpha+1}}(x) = \Phi_p(x^{p^\alpha})$;

(ii) $\Phi_s(x^p) = \Phi_{sp}(x)$ if p is prime and $p|s$, and

$\Phi_s(x^p) = \Phi_s(x)\Phi_{sp}(x)$ if p is prime but $p \nmid s$;

(iii) $\Phi_s(1) = \begin{cases} 0, & if\ s = 1; \\ p, & if\ s = p^\alpha; \\ 1, & otherwise. \end{cases}$

The class of integer tiles on $\mathbb{Z}$ has been studied in depth in connection with the factorization of cyclic groups and cyclotomic polynomials ([3], [5], [7], [8], [10], [11]). Let $\mathbb{Z}^+$ be the set of non-negative integers. For $A \subset \mathbb{Z}^+$, we let

$$P_A(x) = \sum_{a \in A} x^a$$

and call it the mask polynomial of A. The next simple lemma is well-known (see for example [5]) and it connects cyclotomic polynomials with the factorization of $\mathbb{Z}_n$, the cyclic group with n elements.

Lemma (1.1.10) [1]: Let n be a positive integer and let A, B be two finite sets of non-negative integers, then the following are equivalent.

(i) $A \oplus B \equiv \mathbb{Z}_n$,

(ii) $P_A(x)P_B(x) \equiv 1 + x + \ldots + x^{n-1} (mod\ x^n - 1)$.

(iii) $n = P_A(1)P_B(1)$, and for every $d|n, \Phi_d(x)$ divides either $P_A(x)$ or $P_B(x)$.

The notion in (i) means that $A \oplus B \equiv \{0, 1, \cdots, n-1\}\ (mod\ n)$. It is clear in this case A is an integer tile with the tiling set $\mathcal{L} = B \oplus n\mathbb{Z}$. (iii) relates the zeros $\{e^{2\pi i/d} : d|n\}$ of P_A and P_B on the unit circle.

For a finite set $A \subset \mathbb{Z}^+$, we use

$$S_A = \left\{p^\alpha > 1 : p\ prime, \Phi_{p^\alpha}(x) \middle| P_A(x)\right\} \tag{8}$$

to denote the prime-power spectrum of A, and $\tilde{S}_A = \{s > 1 : \Phi_s(x) | P_A(x)\}$ the spectrum of A. In [6], Coven and Meyerowitz made use of the following two conditions to study the integer tiles:

(T1) $\#A = P_A(1) = \prod_{s \in S_A} \Phi_s(1)$,

(T2) For any distinct prime powers $s_1, \ldots, s_n \in S_A$, then $s = s_1 \cdots s_n \in \tilde{S}_A$.

Theorem (1.1.11) [1]: [5] Let $A \subset \mathbb{Z}^+$ be a finite set, and satisfies conditions $(T1)$ and $(T2)$, then A tiles $\mathbb{Z}$ with period $n = l.c.m.(S_A)$.

Conversely, if A is an integer tile, then $(T1)$ holds; if in addition $\#A = p^\alpha q^\beta, \alpha, \beta \geq 0$, then (T2) holds.

It is still an open question whether an integer tile must satisfy (T2) in general.

we will recall some basic results about modulo product-forms and cyclotomic trees developed in [5]. They will be used for the explicit characterization of the tile digit sets D. Let $b \geq 2$, the product-form digit set of b is defined as

$$D = \mathcal{E}_0 \oplus b^{l_1}\mathcal{E}_1 \oplus \ldots \oplus b^{l_k}\mathcal{E}_k \tag{9}$$

where $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b$, and $0 \leq l_1 \leq l_2 \leq \ldots \leq l_k$ [5]; if $\mathcal{E} = \{0, 1, 2, \ldots, b-1\}$, then D is called a strict product-form [5]. The tile digit set in Example (1.1.8) is such an example. However, such simple expression is far from covering all tile digit sets even when $b = 4$. In [15], we have introduced some more general classes of product-forms as tile digit sets.

First we consider the product-form (9) in terms of the mask polynomial and the product of cyclotomic polynomials. Observe that

$$P_{\mathcal{E}}(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}(x) \cdots P_{\mathcal{E}_k}(x) = \prod_{d|b, d>1} \Phi_d(x)Q(x)\,,$$

and

$$P_D(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}(x^{b^{l_1}}) \cdots P_{\mathcal{E}_k}(x^{b^{l_k}}) = \prod_{d|b, d>1} \Phi_d(x^{b_d})Q'(x)$$

where b_d is defined in the obvious way. Based on the above product, we will generate more tile digit sets by taking modulo on each component. We need more notations. Let $S_i = \{d > 1: d|b, \Phi_d(x)|P_{\mathcal{E}_i}(x)\}$ and let

$$\Psi_i(x) = \prod_{d\in S_i} \Phi_d(x)\,.$$

Then $\Psi_i(x)|P_{\mathcal{E}_i}(x)$, hence $\Psi_i(x^{b^{l_i}})|P_{\mathcal{E}_i}(x^{b^{l_i}})$. Let

$$K_1^{(i)}(x) = \Psi_0(x)\Psi_1\left(x^{b^{l_1}}\right)\ldots\Psi i\left(x^{b^{l_i}}\right), \quad 0 \le i \le k\,. \tag{10}$$

It is clear that for the product form, $K_1^{(k)}(x) = \prod_{d|b,d>1} \Phi_d(x^{b_d})$, which divides $P_D(x)$.

Definition(1.1.12) [1]: Let $\mathcal{E} = \mathcal{E}_0 \oplus\ldots\oplus \mathcal{E}_k \equiv \mathbb{Z}_b$ and $0 \le l_1 \ldots \le l_k$. For $n_i = l.c.m.\{s: \Phi_s(x)|K_1^{(i)}(x)\}$, we define $D^{(0)} \equiv \mathcal{E}_0 (mod\ n_0)$ and

$$D^{(i)} \equiv D^{(i-1)} \oplus b^{l_i}\mathcal{E}_i (mod\ n_i), \quad 1 \le i \le k. \tag{11}$$

We call $D = D^{(k)}$ the modulo product-form with respect to $\mathcal{E}$.

It is clear that product-form is a special case of modulo product-form. We also note that in the mask polynomial, (11) is equivalent to

$$P_{D^{(i)}}(x) = P_{D^{(i-1)}}(x)P_{\mathcal{E}_i}(x^{b^{l_i}}) + (x^{n_i} - 1)Q_{i+1}(x).$$

By the choice of n_i in (11), we can prove that $K^{(i)}(x)|P_D(x)$. This is used to show that the modulo product-form is a tile digit set. This extension, however, still does not cover all tile digit sets, as was shown by an example of D for $b = 12$ in [11, Example 3.6]. On the other hand, the example suggests the following higher order product-forms. In view of this, we call the above D a 1^{st}-order modulo product-form.

Definition (1.1.13) [1]: D is called a 2nd-order product-form if it is a product-form of $\mathcal{G}$ with $0 \le r_1 \cdots \le r_\ell$, where $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus\ldots\oplus \mathcal{G}_\ell$, and $\mathcal{G}$ itself is a 1st-order modulo product-form possibly in another decomposition different from the $\mathcal{G}_i$).

For the above $\mathcal{G}$ (as a 1st-modulo product-form), we let $K_{\mathcal{G}} = K_1^{(k)}$ be as in (10),

$$S_i^{(2)} = \{s: \Phi_s(x)|P_{\mathcal{G}_i}(x), \Phi_s(x)|K_{\mathcal{G}}(x)\},$$

$K_2^{(i)}(x) = \prod_{j=0}^{i}\prod_{s\in S_j^{(2)}} \Phi_s(x^{b^{r_j}})$ and $n_i = l.c.m.\{s: \Phi_s(x)|K_2^{(i)}(x)\}$ for $0 \le i \le \ell$. We use the same procedure as in Definition (1.1.12) to define the 2nd-order modulo product-form.

Inductively, we can define the k^{th}-order product-form and modulo product-forms.

we can produce new tile digit sets as follows: we start with the basic tile digit set $\mathcal{E}$ (complete residue class), we construct the 1st-order modulo product-forms. We then rearrange those digits to form a product, and use them to construct the 2nd-order modulo product-forms, and likewise for the higher orders. The interesting question is whether these higher order modulo product-forms will characterize all the tile digit sets. For this, we reformulate the question by expressing the mask polynomial into cyclotomic polynomials, and make use of the algebraic operations in Proposition (1.1.9) to study the question.

Let $b \ge 2$. We define a tree of cyclotomic polynomials (with respect to b), which we call it a Φ-tree, as follows: the set of vertices of this tree at level 1 are Φ_d, where $d|b$ and $d > 1$; the offsprings of $\Phi_{d'}$ in each level are the cyclotomic factors of $\Phi_{d'}(x^b)$, they are determined by Proposition (1.1.9)(ii). All Φ_d in the tree are different [5, Proposition 2.2], hence it has a well-defined tree structure (see e.g. Figure) . We call a finite subset of vertices N a blocking if every infinite path starting from the root intersects exactly one element of N. The following is one of the main results proved in [9].

Theorem (1.1.14) [1]: Let $D \subset \mathbb{Z}^+$ with $\#D = b$. Then D is a tile digit set (with respect to b) if and only if there is a blocking N in the Φ-tree such that

$$P_D(x) = \left(\prod_{\Phi_d \in N} \Phi_d(x) \right) Q(x).$$

We denote the above product by $K(x)$ and call it a kernel polynomial of D. The kernel polynomials play a central role. It is seen that for the 1st-order modulo product-form, $K(x) = K_1^{(k)}(x)$ of $\mathcal{E}$ in (10) is its kernel polynomial; the $K_2^{(k)}(x)$ is also a kernel polynomial. It is possible that by varying $Q(x)$, the same kernel can represent different tile digit sets D. On the other hand, there are kernel polynomials that do not generate any tile digit set (see Remark 3 of Theorem 5.6 in [5]). For the case $b = p^\alpha q$, we will determine the admissible kernel polynomials and show that all the tile digit sets are k^{th}-order modulo product-forms for some k. To do so, we need to first specify the structure of the prime-power spectrum of a tile digit set, which is to be presented in the following theorem [5, Theorem 2.4].

Theorem (1.1.15) [1]: Let $b = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the product of prime powers and let D be a tile digit set of b. Then the prime power spectrum of D is given by

$$S_D = \bigcup_{j=1}^{k} \{p_j^a : a \in E_j\}$$

where $E_j = \{a : \Phi_{p_j^a}(x) | P_D(x)\}$ and $E_j \equiv \{0, \cdots, \alpha_{j-1}\} (mod\ \alpha_j)$.

The theorem can be proved directly from the Kenyon's criterion of self-similar tiles [10, Theorem 2.4]. It can also be seen from Theorem (1.1.14) by considering the branches of the prime-power factors d of b in the first level.

For integer tiles A in $\mathbb{Z}$, there are special structural results when $\#A = p^\alpha q^\beta$, a product of two prime powers [6]. Our main purpose is to apply these results to prove, among the other results, a special factorization lemma for A of cardinality $p^\alpha q$ (Lemma (1.1.22)), which will be essential. For convenience, we assume that $A \subset \mathbb{Z}^+$, also we can assume that $g.c.d.(A) = 1$ whenever it is needed [4, Lemma 1.4(1)]. First, We start with a general lemma.

Lemma (1.1.16) [1]: Let $P(x) \in \mathbb{Z}^+[x]$ and suppose that $P(x) = (x^n - 1)g(x) + h(x)$ where $g(x)$ and $h(x)$ are respectively the quotient and remainder of $P(x)$ when it is divided $(x^n - 1)$. Then $g(x), h(x) \in \mathbb{Z}^+[x]$. Moreover, if the coefficients of $h(x)$ are 0 or 1 only, then the same is for the coefficients of $P(x)$.

Proof: Write $P(x) = \sum_{i=1}^{N} a_i x^{m_i}$, where $a_i > 0$, and let $m_i = \ell_i + nr_i$ with $0 \le \ell_i < n$. Then

$$P(x) = \sum_{i=1}^{N} a_i (x^{m_i} - x^{\ell_i} + x^{\ell_i})$$

$$= \sum_{i=1}^{N} a_i x^{\ell_i} (x^{nr_i} - 1) + \sum_{i=1}^{N} a_i x^{\ell_i}$$

$$= (x^n - 1) \sum_{i=1}^{N} a_i x^{\ell_i} (1 + x^n + \ldots + x^{n(r_i - 1)}) + \sum_{i=1}^{N} a_i x^{\ell_i}.$$

Hence, $g(x) = a_i x^{\ell_i}(1 + x^n + \ldots + x^{n(r-1)})$ and $h(x) = \sum_{i=1}^{N} a_i x^{\ell_i}$ $(deg\ h < n)$. This means that $g(x), h(x)$ have non-negative coefficients. The second part is also clear.

Indeed, the condition on $h(x)$ implies all the ℓ_i are distinct, and the above expression of $P(x)$ implies that its coefficients equal to some a_i, and hence either 0 or 1.

The major techniques we use are two well-known decomposition theorems (Theorems (1.1.17) and (1.1.19)). The first one is due to de Bruijn [5]. Let $\mathbb{Z}^+[x]$ denote the set of polynomials with non-negative integer coefficients, and $F(x)(mod\ x^n-1)$ the remainder of $F(x)$ divided by (x^n-1).

Theorem (1.1.17) [1]: (de Bruijn) Let $n=p^\lambda q^\mu$ where $\lambda,\mu\geq 0$. Suppose $f(x)\in\mathbb{Z}^+[x]$ and $\Phi_n(x)|f(x)$, then there exist polynomials $P(x),Q(x)\in\mathbb{Z}^+[x]$ such that

$$f(x)(mod\ x^n-1)=P(x)\Phi_{p^\lambda}\left(x^{q^\mu}\right)+Q(x)\Phi_{q^\mu}\left(x^{p^\lambda}\right). \quad (12)$$

The theorem allows us to give an explicitly characterization of all integer tiles of prime power as modulo product-form. (Note that the modulo product-form is originally defined for tile digit sets in (11), however the same modulo procedure also works for integer tiles.) Given an integer tile $A\subset\mathbb{Z}^+$ with $\#A=p^\alpha$, denote $S_A=\{p^{k_1},\dots,p^{k_\alpha}\}$ where $1\leq k_1<\dots<k_\alpha$ (if $g.c.d.(A)=1$, then $k_1=1$, also condition $(T1)$ ensures S_A has exactly α elements). Set $\mathcal{E}_{k_j-1}=p^{k_j-1}\{0,1,\dots,p-1\}$. For $A'=\mathcal{E}_{k_1-1}\oplus\dots\oplus\mathcal{E}_{k_\alpha-1}$,

$$P_{A'}(x)=P_{\mathcal{E}_{k_1-1}}(x)\dots P_{\mathcal{E}_{k_\alpha-1}}(x)=\prod_{i=1}^{\alpha}\Phi_{p^{k_i}}(x).$$

It is seen that A' is a product-form integer tile (by Theorem (1.1.11)).

Theorem (1.1.18) [1]: Let $A\subset\mathbb{Z}^+$ with $\#A=p^\alpha$. Then A is an integer tile if and only if A is a modulo product-form of $\mathcal{E}_{k_1-1}\oplus\dots\oplus\mathcal{E}_{k_{\alpha-1}-1}$, in the sense that, $A=A^{(\alpha)}$ with $A^{(0)}=\{0\}$ and

$$A^{(i)}\equiv A^{(i-1)}\oplus\mathcal{E}_{k_i-1}(mod\ p^{k_i}), \quad (13)$$

for some $1\leq k_1<\dots<k_\alpha, 1\leq i\leq\alpha$.

Proof: If A is a modulo product-form as in (13), then $P_A(x)=\prod_{i=1}^{\alpha}\Phi_{p^{k_i}}(x)Q(x)$. Then it satisfies $(T1)$ and $(T2)$, and Theorem (1.1.11) implies it is an integer tile.

Conversely, assume A is an integer tile, then $S_A=\{p^{k_1},\dots,p^{k_\alpha}\}$. We observe that with $n=p^{k^\alpha}$, (13) is reduced to

$$P_A(x)\left(mod\ x^{p^{k_\alpha}}-1\right)=\Phi_{p^{k_\alpha}}(x)Q_{\alpha-1}(x) \quad (14)$$

and $Q_{\alpha-1}(x)$ has nonnegative coefficients as the remainder has non-negative coefficients by Lemma (1.1.16) and $deg\ (Q_{\alpha-1})<p^{k_\alpha}-deg(\Phi_{p^{k_\alpha}})=p^{k_\alpha-1}$. Clearly, $Q_{\alpha-1}(1)=p^{\alpha-1}$. Observe that $\Phi_{p^{k_i}}(x)|Q_{\alpha-1}(x)$ for each i, we can repeat the same argument to obtain

$$Q_{\alpha-1}(x)(mod\ x^{p^{k_{\alpha-1}}}-1)=\Phi_{p^{k_{\alpha-1}}}(x)\ Q_{\alpha-1}(x),$$

where $Q_{\alpha-2}(x)$ has nonnegative coefficients and $Q_{\alpha-2}(1)=p^{\alpha-2}$. Inductively we reach $Q_1(x)$ with the following identity,

$$Q_1(x)\left(mod\ x^{p^{k_1}}-1\right)=\Phi_{p^{k_1}}(x)Q_0(x), \quad (15)$$

where $Q_0(x)$ has nonnegative coefficients and $Q_0(1)=1$. Since $Q_0(x)$ has nonnegative coefficients, we must have $Q_0\equiv 1$.

We now claim that A must be a modulo product-form. Note that $Q_1(x)$ is a polynomial with coefficients 0 or 1 (by (15) and $Q_0(x)\equiv 1$), $Q_1(x)$ determines a digit set $A^{(1)}$. By (15) again and observe that $\Phi_p(x^{p^{k_1-1}})=\Phi_{p^{k_1}}(x)$,

$$A^{(1)}\equiv\mathcal{E}_{k_1-1}(mod\ p^{k_1}).$$

Now, from $Q_2(x)(mod\ x^{p^{k_2}}-1)=\Phi_{p^{k_2}}(x)Q_1(x)$, we have $degQ_1<p^{k_2}-(p-1)p^{k_2-1}=p^{k_2-1}$. This means that $\Phi_{p^{k_2}}(x)Q_1(x)$ is a polynomial with coefficients 0 or 1, and by Lemma (1.1.16), the same is for $Q_2(x)$. Let $A^{(2)}$ be the digit set determined by Q_2, we have

$$A^{(2)}\equiv A^{(1)}\oplus\mathcal{E}_2(mod\ p^{k_2}).$$

Continuing this process, we finally reach A as in (14). Hence A is a modulo product-form in (13).

Theorem (1.1.18) provides a characterization of the structure of integer tiles of prime powers as some modulo product-forms. In particular, the techniques used in the proof of the theorem will appear again. Next, we will state another well-known decomposition theorem for the $p^\alpha q^\beta$ integer tiles, which is also derived from de Bruijn's theorem ([10]).

Theorem(1.1.19) [1]: If A is an integer tile with $\#A=p^\alpha q^\beta$, where p,q are primes and $\alpha,\beta\geq 1$, and $A\oplus B\equiv\mathbb{Z}_n$, then there is a prime factor of $\#A$, say p, such that

$$either\ A\subset p\mathbb{Z}\ or\ B\subset p\mathbb{Z}.$$

In the latter case (e.g., $g.c.d.(A)=1$), we have

$$A=\bigcup_{j=0}^{p-1}(\{a_j\}\oplus pA_j),\qquad(16)$$

where $a_j=\min\{a\in A: a\equiv j(mod\ p)\}$, and $A_j=\{n\geq 0: a_j+np\in A\}$ are integer tiles. In this case $\{a_j: 0\leq j\leq p-1\}$ forms a complete residue set $(mod\ p)$ and all $\#A_j$ are equal.

We call (16) a decomposition of A along p. Putting the above in terms of the prime-power spectrum, we have

Corollary (1.1.20) [1]: Suppose A is an integer tile with $\#A=p^\alpha q^\beta$ and $g.c.d.(A)=1$, then we have

$$p\in S_A\quad or\quad q\in S_A\ (can\ be\ both).$$

In the case $p\in S_A, A$ has a decomposition along p as in (16), and

$$S_{pA_j}=S_A\backslash\{p\}\quad\forall 0\leq j\leq p-1.\qquad(17)$$

Proof: There is a finite set B such that $A\oplus B\equiv\mathbb{Z}_n$. Note that A cannot be a subset of $p\mathbb{Z}$ since $g.c.d.(A)=1$. Theorem (1.1.19) implies that there is a prime factor, say p, such that $B\subset p\mathbb{Z}$. Writing $P_B(x)=P(x^p)$ and note that

$$P_B(e^{2\pi i/p})=P((e^{2\pi i/p})^p)=P(1)=P_B(1)=\#B\neq 0,$$

which means $\Phi_p(x)$ does not divide $P_B(x)$. But $\Phi_p(x)|P_A(x)P_B(x)$ (Lemma (1.1.10)(ii)). This implies $\Phi_p(x)|P_A(x)$, i.e., $p\in S_A$.

That A has decomposition along p follows from (16). Notice that $A\oplus B\equiv\mathbb{Z}_n$ and $A_j\oplus B/p\equiv\mathbb{Z}_{n/p}$. These imply that

$$S_A\cup S_B=S_{\mathbb{Z}}\ and\ S_{pA_j}\cup S_B=S_{\mathbb{Z}_n}\backslash\{p\},$$

and the last statement of the corollary follows.

In the following we use the above theorems to consider the cyclotomic factors of P_A with $\#A=p^\alpha q^\beta$.

Lemma (1.1.21) [1]: Let A be an integer tile with $\#A=p^\alpha q^\beta$. If A admits a decomposition along p, and $\Phi_{p^\lambda q^\mu}(x)|P_A(x)$ with $\lambda,\mu\geq 1$, then we have

(i) If $\lambda\geq 2$, then $\Phi_{p^{\lambda-1}q^\mu}(x)|P_{A_j}(x)$ for all $j=0,\dots,p-1$;

(ii) $\Phi_{p^\lambda}(x)|P_A(x)$ or $\Phi_{q^\mu}(x)|P_A(x)$.

Proof: (i) By assumption, we can write

$$P_A(x)=\Phi_{p^\lambda q^\mu}(x)\cdot Q(x)=\Phi_{p^{\lambda-1}q^\mu}(x^p)\cdot Q(x).\quad(18)$$

We use $Q_j(x), 0 \le j \le p-1$ to denote the polynomials of the terms x^{j+kp} in $Q(x)$, then $Q(x) = \sum_{j=0}^{p-1} Q_j(x)$. We define $\tilde{Q}_j(x)$ by $Q_j(x) = x^j \tilde{Q}_j(x^p)$. Together with the decomposition in (16), we have

$$x^{a_0} P_{A_0}(x^p) + \ldots + x^{a_{p-1}} P_{A_{p-1}}(x^p) = P_A(x) = \sum_{j=0}^{p-1} x^j \Phi_{p^{\lambda-1}q^\mu}(x^p) \tilde{Q}_j(x^p).$$

By comparing the terms of the two polynomials, which has sorted according to the residue class modulo p, we have

$$x^{a_j} P_{A_j}(x^p) = x^j \Phi_{p^{\lambda-1}q^\mu}(x^p) \tilde{Q}_j(x^p) \; \forall \; 0 \le j \le p-1.$$

This implies that $\Phi_{p^{\lambda-1}q^\mu}(x) | P_{A_j}(x)$.

(ii) We will prove the statement by using induction on $k = \alpha + \beta$. Let $n = p^\lambda q^\mu$. If $k = 1$, say $\#A = p$, then (12) implies that $p = pP(1) + qQ(1)$, which implies that $Q(1) = 0$, so that $Q(x) \equiv 0$. Therefore $\Phi_{p^\lambda}(x^{q^\mu}) | P_A(x)$ (by (12) again), and hence $\Phi_{p^\lambda}(x) | P_A(x)$ (by Proposition (1.1.9) (ii)).

For the induction step, by factoring out the $g.c.d.$ of A, we can assume that $g.c.d.(A) = 1$. Let us assume that $\Phi_p(x) | P_A(x)$ as in Corollary (1.1.20), hence A has a decomposition along p as in (17). If $\lambda = 1$, then we are done. If $\lambda \ge 2$, then part (i) implies that $\Phi_{p^{\lambda-1}q^\mu}(x) | P_{A_i}(x)$ for all $i = 0, \ldots, p-1$. As all S_{A_i} are identical (by Corollary (1.1.20)), we have, by the induction hypothesis, $\Phi_{p^{\lambda-1}}(x)$ or $\Phi_{q^\mu}(x)$ must divide all $P_{A_i}(x)$. It follows that $\Phi_{p^\lambda}(x)$ or $\Phi_{q^\mu}(x)$ divides $P_A(x)$

The following strengthens Lemma (1.1.21)(ii) for the case $\#A = p^\alpha q$.

Lemma (1.1.22) [1]: Let A be an integer tile such that $\#A = p^\alpha q$. If $\Phi_{p^\lambda q^\mu}(x) | P_A(x)$, then

$$\Phi_{p^\lambda}(x) \nmid P_A(x) \; \Rightarrow \Phi_{q^\mu}(x^{p^\lambda}) | P_A(x). \tag{19}$$

Proof: In view of de Bruijn's identity (Theorem (1.1.17)),

$$P_A(x) \left(mod \; x^{p^\lambda q^\mu} - 1 \right) = P(x) \Phi_{p^\lambda}\left(x^{q^\mu}\right) + Q(x) \Phi_{q^\mu}\left(x^{p^\lambda}\right), \tag{20}$$

it suffices to show that if $\Phi_{p^\lambda}(x) \nmid P_A(x)$ (i.e., $p^\lambda \notin S_A$), then $P(x) \equiv 0$. By Corollary (1.1.20), we have $p \in S_A$ or $q \in S_A$. We will divide our proof into two cases.

Case 1: $q \in S_A$. Suppose $P(x)$ in (20) is not zero, let x^t be a term with positive coefficient. By checking the terms of $x^t \Phi_{p^\lambda}(x^{q^\mu})$ in the product $P(x) \Phi_{p^\lambda}(x^{q_\mu})$, and noting that the terms in (20) are positive, we conclude that there exists $C \subset A$ and

$$C \equiv t + q^\mu \{0, p^{\lambda-1}, \ldots, (p-1)p^{\lambda-1}\} \; (mod \; p^\lambda q^\mu).$$

Since $\Phi_q(x)$ divides $P_A(x)$, A admits a decomposition along q:

$$A = \bigcup_{j=0}^{q-1} \left(\{a_j\} \oplus qA_j \right).$$

As all elements of C are in the same residue class $(mod \; q)$, we have $C \subset \{a_j\} \oplus qA_j$ for the j such that $a_j \equiv t (mod \; q)$. Hence

$$C' := q^{\mu-1} \left\{0, p^{\lambda-1}, \ldots, (p-1)p^{\lambda-1}\right\} \subset A_j \left(mod \; p^\lambda\right). \tag{21}$$

Note that $\#A = p^\alpha q$ implies $\#A_j = p^\alpha$. Let $S_{A_j} = \{p^{k_1}, \ldots, p^{k_\alpha}\}$, then by assumption, $p^\lambda \notin S_{A_j}$. Let ℓ be the largest integer such that $k_\ell < \lambda$. By Theorem (1.1.18), A_j is of the

modulo product-form defined by $A_j^{(i)}, 1 \le i \le \alpha$ as in (13). By taking modulo of p^λ, we see that

$$A_j(mod\ p^\lambda) = A_j^{(\ell)}(mod\ p^{k_\ell}).$$

This leads to a contradiction as C' is a set of p distinct elements in $A_j(mod\ p^\lambda)$, but it is a singleton set $\{0\}$ in $A_j^{(\ell)}(mod\ p^{k_\ell})$. Hence $P(x) \equiv 0$.

Case 2: $p \in S_A$. Notice that in this case, we must have $\lambda > 1$. The proof of this case is by induction on α and making use of Case 1 in the induction step.

When $\alpha = 1$, from (20), we have $pq = pP(1) + qQ(1)$. Therefore either $P(1) = 0, Q(1) = p$ or $Q(1) = 0, P(1) = q$, which implies either $P(x) \equiv 0$ or $Q(x) \equiv 0$. But $Q(x) \not\equiv 0$ (for otherwise, set $x = e^{2\pi i/p^\lambda}$ in (20), then the left-hand side is not 0 but the right-hand side is 0, a contradiction). Hence we must have $P(x) \equiv 0$, so that $\Phi_{q^\mu}(x^{p^\lambda}) | P_A(x)$.

Suppose the statement holds for $\alpha - 1$. Since $p \in S_A$, we have by (16), $P_A(x) = x^{a_0} P_{A_0}(x^p) + \ldots + x^{a_{p-1}} P_{A_{p-1}}(x^p)$. This, together with (17) and $\Phi_{p^\lambda}(x) \nmid P_A(x)$, implies that

$$\Phi_{p^{\lambda-1}}(x) \nmid P_{A_i}(x) \qquad \forall\ 0 \le i \le p-1.$$

On the other hand Lemma (1.1.21) (i) implies that $\Phi_{p^{\lambda-1}q^\mu}(x) | P_{A_i}(x)$ for all $0 \le i \le p - 1$. Now if $p \in S_{A_i}$, we apply the induction hypothesis to conclude that $\Phi_{q^\mu}(x^{p^{\lambda-1}}) | P_{A_i}(x)$ for all i; if $q \in S_{A_i}$, we can draw the same conclusion by Case 1. This implies that in either cases, $\Phi_{q^\mu}(x^{p^\lambda}) | P_A(x)$.

we will give an explicit characterization of the tile digit sets for $b = p^\alpha q$ and show that they are modulo product-form of some order. We assume that $D \subset \mathbb{Z}^+, 0 \in D$ and $g.c.d.(D) = 1$ as before. First we consider the simple case $b = p^\alpha$. Since a tile digit set D is an integer tile (Theorem (1.1.7)), they must have the form in (13). Furthermore by Theorem (1.1.15), the spectrum $S_D = \{p^{k_1}, \ldots, p^{k_\alpha}\}$ (with $k_1 = 1$) satisfies, in addition, that $\{k_i = i + \alpha t_i\}_{i=1}^{\alpha}$ is a complete residue set modulo α. In this case, $\mathcal{E}_{k_i - 1} = p^{\alpha t_i} \mathcal{E}_{i-1}$ where $\mathcal{E}_i = p^i\{0, 1, \ldots, p-1\}$. Hence

$$D' := \mathcal{E}_{k_1 - 1} \oplus \ldots \oplus \mathcal{E}_{k_\alpha - 1} = \mathcal{E}_0 \oplus \ldots \oplus p^{\alpha t_\alpha} \mathcal{E}_{\alpha - 1}.$$

This is a product-form since $\mathcal{E}_0 \oplus \ldots \oplus \mathcal{E}_{\alpha-1} = \{0, 1, \ldots, p^\alpha - 1\}$. The following theorem is immediate.

Theorem (1.1.23) [1]: Suppose $b = p^\alpha$ for some $\alpha \ge 1$ and D is a tile digit set of size b, then D is a modulo product-form of $\mathcal{E} = \mathcal{E}_0 \oplus \ldots \oplus \mathcal{E}_{\alpha-1} = \{0, 1, \ldots, p^\alpha - 1\}$.

We remark that in [10], Lagarias and Wang have a characterization of tile digit sets of prime power. Their expression is more complicate, but it is the same as the above in essence. For $b = pq$, it was proved in [11] and [13] that

Theorem (1.1.24) [1]: Let $b = pq$ and D be a tile digit set of b. Then the prime power spectrum is $S_D = \{p, q^n\}\ or\ \{p^n, q\}$. In the first case

$$D = \{0, 1, \cdots p-1\}(mod\ p) \oplus b^{n-1}\{0, p, \ldots, p(q-1)\}\ (mod\ b^n),$$

and the kernel polynomial is

$$K(x) = \Phi_p(x)\Phi_{q^n}(x^{p^n}).$$

We make use of the cyclotomic polynomial techniques developed in the previous to give a study of the tile digit sets for $b = p^\alpha q$. As the situation is rather involved in notations, we only work out the case $b = p^2 q$ in detail, and the same idea applies to $b = p^\alpha q$.

Theorem (1.1.14) will be needed when characterizing the tile digit sets by the cyclotomic Φ-tree. Figure 1 illustrates the Φ-tree for p^2q in which Φ_{p^m} has two offsprings, Φ_{q^n} has three, all the $\Phi_{p^mq^n}$ have only one. It follows that the descendants of $\Phi_{p^mq^n}$ only form one path with no branch, hence the blocking on this path is rather restricted, this becomes essential in classifying the admissible blockings (see the proof of Theorem (1.1.26)(i) and (ii)).

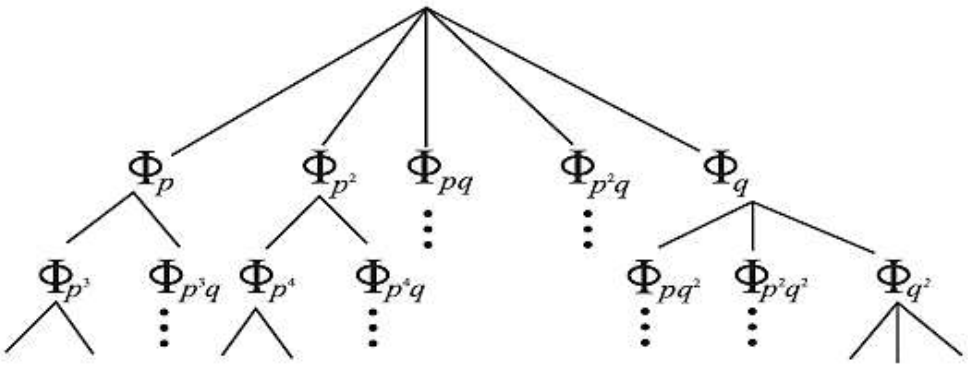

Figure 1[1]: Φ-tree for p^2q.

First we prove a simple lemma which will apply to the kernel polynomials.

Lemma (1.1.25) [1]: Let D be a tile digit set (with $g.c.d.(D) = 1$ as assumed) and let $G(x) = 1 + \sum_{j=1}^{n} a_j x^{k_j}, k_j \neq 0$, be an integer polynomial such that and $G(x)|P_D(x)$ and $G(1) = \#D$. Then the $g.c.d.$ of $\{k_1, \dots, k_n\}$ is 1.

Proof: Suppose on the contrary that $g.c.d.\{k_1, \dots, k_n\} = d > 1$. We can write $G(x) = \tilde{G}(x^d)$ and $P_D(x) = \tilde{G}(x^d)Q(x)$. For each $0 \leq j \leq d-1$, let $Q_j(x)$ be the polynomial containing the terms of the form x^{j+td} in $Q(x)$. Then $Q(x) = \sum_{j=0}^{d-1} Q_j(x)$, and we can also write $Q_j(x) = x^j \tilde{Q}_j(x^d)$. Hence

$$P_D(x) = \sum_{j=0}^{d-1} x^j \tilde{G}(x^d)\tilde{Q}_j(x^d).$$

Let $D_j = D \cap (j + d\mathbb{Z})$, the subset of D that is congruent to $j(mod\ d)$. Then $P_D(x) = \sum_{j=0}^{d-1} P_{D_j}(x)$. By comparing the power, which has been sorted according to the residue classes, we have for all $0 \leq j \leq d-1$ that

$$P_{D_j}(x) = x^j \tilde{G}(x^d)\tilde{Q}_j(x^d).$$

By taking $x = 1$ and using the assumption that $G(1)(= \tilde{G}(1)) = \#D$, we see that $\#D_j = \#D \cdot Q_j(1)$. This means that $Q_j(1)$ is non-negative. Since Q_j is a monic polynomial of integer coefficients, $Q_j(1)$ is an integer. As $\#D_j = \#D$, it follows that only one of the j satisfies $\#D_j = \#D$ and the others are 0. Since $0 \in D_0$, we must have $D = D_0$. But then this contradicts to the fact that $g.c.d.(D) = 1$ and hence the conclusion follows.

Let D be a tile digit set with $\#D = p^2q$ and $g.c.d.(D) = 1$. Then it follows from Theorem (1.1.15) that

$$S_D = \{p, p^{2m}, q^n\}\ or\ \{q, p^{2m}, p^{2n+1}\}.$$

for some $m \in \{1, 2, \cdots\}$ and $n \in \{0, 1, \cdots, \}$. Our two main theorems are

Theorem (1.1.26) [1]: Let $b = p^2q$ and let D be a tile digit set with $\#D = b$. Then the mask polynomial P_D contains the following kernel polynomials:

(i) If $S_D = \{p, p^{2m}, q^n\}$, then

(I) $K_I(x) = \Phi_p(x)\Phi_{p^{2m}}(x^{q^m})\Phi_{q^n}(x^{p^{2(n-1)+1}})$; or

(II) a factor of $K_{II}(x) = \Phi_p(x)\Phi_{p^{2m}}(x^{q^{\ell-1}})\Phi_{q^n}(x^{p^{2(n+m-\ell)}}), 1 \le \ell \le m$. (as in (28) below)
(ii) If $S_D = \{q, p^{2m}, p^{2n+1}\}$, then
(III) $K_{III}(x) = \Phi_q(x)\Phi_{p^{2m}}(x^{q^m})\Phi_{p^{2n+1}}(x^{q^{n+1}})$.
Moreover, each of the above $K_i(x)$ represents a tile digit set of p^2q.

Theorem (1.1.27) [1]: Let $b = p^2q$ and assume that $\#D = b$. Then D is a tile digit set if and only if it is a k^{th}-order modulo product-form for some $k \le m$.

Using Proposition (1.1.9)(ii), it is direct to check that $K_I(x)$ is the mask polynomial of

$$D_I = \mathcal{E}_p \oplus b^{n-1}p\mathcal{E}_q \oplus b^{m-1}pq\mathcal{E}_p, \quad (22)$$

where $\mathcal{E}_k = \{0, 1, \dots, k-1\}$; $K_{II}(x)$ is the mask polynomial of

$$D_{II} = \mathcal{E}_p \oplus b^{\ell-1}p^{2(m-\ell+1)-1}\mathcal{E}_p \oplus b^{n-1}p^{2(m-\ell+1)}\mathcal{E}_q, \quad (23)$$

and K_{III} is the mask polynomial of

$$D_{III} = \mathcal{E}_q \oplus b^n q\mathcal{E}_p \oplus b^{m-1}pq\mathcal{E}_p.$$

Lemma(1.1.28) [1]: With the above notations, D_I and D_{III} are 1st-order product-forms, and D_{II} is an $(m-\ell+1)$-order product-form.

Proof: Note that

$$\mathcal{E}_p \oplus p\mathcal{E}_q \oplus pq\mathcal{E}_p = \{0, 1, \cdots, p^2q - 1\} = \mathcal{E}.$$

It follows that D_I is a product-form of $\mathcal{E}$. The proof for D_{III} is the same.
For D_{II}, we let $t = m - \ell + 1$, and let

$$D' = \mathcal{E}_p \oplus p^{2t-1}\mathcal{E}_p \oplus p^{2t}\mathcal{E}_q. \quad (24)$$

Then D_{II} is the product-form of D'. It $t = 1$, then clearly D_{II} is a 1st-order product form. Hence we assume that $t > 1$, observe that $\mathcal{E}_p \oplus p\mathcal{E}_q = q\mathcal{E}_p \oplus \mathcal{E}_q (= \{0, 1, \cdots, pq-1\})$, we can rewrite D' as

$$D' = \mathcal{E}_p \oplus bp^{3t-1}\mathcal{E}_p \oplus p^{2t-1}\mathcal{E}_q. \quad (25)$$

Let $D'' = \mathcal{E}_p \oplus p^{2t-3}\mathcal{E}_p \oplus p^{2t-2}\mathcal{E}_q$. We claim that D' is a modulo product form of D''. Indeed, if we let $i \in \mathcal{E}_q$ and $j \in \mathcal{E}_q$, then $\{p_j\}_{j\in\mathcal{E}_q}$is a complete residue $(mod\ q)$. This implies that

$$i + p^{2t-1}j = i + p^{2t-2}(p_j) = i + p^{2t-2}(qx_j + y_j) \equiv i + p^{2t-2}y_j (mod\ p^{2t-2}q)$$

and $\{y_j\}$ is a complete residue $(mod\ q)$. Hence,

$$\mathcal{E}_p \oplus p^{2t-1}\mathcal{E}_q \equiv \mathcal{E}_p \oplus p^{2t-2}\mathcal{E}_q (mod\ p^{2t-2}q)$$

and therefore D' is a modulo product-form of D'' by ignoring the last modulo action. We continue this process for $t-1$ times by noting that

$$p^{2t-3}\mathcal{E}_p \oplus p^{2t-2}\mathcal{E}_q = p^{2t-3}(\mathcal{E}_p \oplus p\mathcal{E}_q) = p^{2t-3}(q\mathcal{E}_p \oplus \mathcal{E}_q),$$

and finally we obtain the first-order product form

$$\mathcal{E} = \mathcal{E}_p \oplus p\mathcal{E}_p \oplus p^2\mathcal{E}_q \ (\equiv \mathbb{Z}_{p^2q}).$$

This implies D_{II} must be an $(m-\ell+1)$-order product-form.
We remark that the rearrangement of (24) into (25) is the key idea of the higher order modulo product-forms in Definition (1.1.13). In term of cyclotomic polynomials, it means that $\Phi_p(x)\Phi_{p^{2t}}(x)\Phi_q(x^{p^{2t}}) = \Phi_p(x)\Phi_{p^{2t}}(x^q)\Phi_q(x^{p^{2t-1}})$ (by switching the position of $\Phi_q(x^{p^{2t}})$ from the last factor to the middle factor) and the latter product is the modulo product-form of the $\Phi_p(x)\Phi_{p^{2t-2}}(x)\Phi_q(x^{p^{2t-2}})$.
For convenience, we call a vertex in a blocking N a node. Note that if D is a tile digit set and Φ_d is a node in N_D, then $\Phi_d(x)|P_D(x)$. For the case $S_D = \{p, p^{2m}, q^n\}$ in Theorem

(1.1.26)(i), the $K_I(x)$ gives a blocking of the Φ-tree, and the nodes are determined by the following identities:

$$\Phi_{p^{2m}}\left(x^{q^m}\right) = \Phi_{p^{2m}}(x)\Phi_{p^2q}(x)\dots\Phi_{p^{2m}q^m}(x) \quad (26)$$

$$\Phi_{q^n}\left(x^{p^{2(n-1)+1}}\right) = \Phi_{q^n}(x)\Phi_{pq^n}(x)\dots\Phi_{p^{2(n-1)+1}q^n}(x). \quad (27)$$

Hence $K(x)$ in (I) is a kernel polynomial of D_I.

$K_{II}(x)$ is a variant of K_I, it is obtained by replacing the nodes $\Phi_{p^{2m}q^i}(x), \ell \le i \le m$ (factors of $\Phi_{p^{2m}}(x^{q^m})$ in (I)) with new node $\Phi_{p^{2(m+n-i)}q^n}(x)$ (See Figure 1 and Lemma (1.1.29) below). Let

$$\widetilde{K}_{II}(x) = \Phi_p(x)\left(\Phi_{p^{2m}}\left(x^{q^{\ell-1}}\right)\prod_{j=n}^{n+m-\ell}\Phi_{p^{2j}q^n}(x)\right)\Phi_{q^n}\left(x^{p^{2(n-1)+1}}\right). \quad (28)$$

It follows that $\widetilde{K}_{II}(x)$ is a kernel polynomial as its factors defines a blocking. Moreover $K_{II}(x) = \widetilde{K}_{II}(x)\tilde{Q}(x)$, with $\tilde{Q}(x) = \prod_{j=n}^{n+m-\ell-1}\Phi_{p^{2j+1}q^n}(x)$.

In order to classify the kernel polynomial of a tile digit set D in Theorem (1.1.26), we need to know more precisely about the nodes. Let γ_d denote the infinite path starts form ϑ and passes through Φ_d. The following lemma describes the possible nodes on $\gamma_{p^\lambda q^\beta}, \lambda, \beta \ge 1$.

Lemma (1.1.29) [1] :With the above notations and $S_D = \{p, p^{2m}, q^n\}$, then the nodes of N_D satisfy

(i) for $0 \le k \le m$, the node on $\gamma_{p^{2m}q^k}$ is either $\Phi_{p^{2m}q^k}$ or $\Phi_{p^{2(m+n-k)}q^n}$;

(ii) for $0 \le k \le n-1$, the node on $\gamma_{p^{2k+1}q^n}$ is either $\Phi_{p^{2k+1}q^n}$ or $\Phi_{pq^{n-k}}$;

(iii) for $1 \le k \le n-1$, the node on $\gamma_{p^{2k}q^n}$ is either $\Phi_{p^{2k}q^n}$ or $\Phi_{p^{2m}q^{n+m-k}}$.

Proof: To prove (i), note that the infinite path from $\Phi_{p^{2(m-k)}}$ pass through $\Phi_{p^{2m}q^k}$ has no other branch. If $\Phi_{p^{2m}q^k}$ is not a node, then it must be an ancestor or a descendant of $\Phi_{p^{2m}q^k}$, i.e., $\Phi_{p^{2(m-r)}q^{k-r}}$ or $\Phi_{p^{2(m+r)}q^{k+r}}$. By Lemma (1.1.22) (or Lemma (1.1.21)(ii)), we must have $\Phi_{q^{k-2}}$ or $\Phi_{q^{k+r}}$ divides P_D. They can only be q^n since by assumption $S_A = \{p, p^{2m}, q^n\}$. Hence $\Phi_{p^{2(m+n-k)}q^n}$ is the only choice. This completes the proof of (i). The proof of (ii) and (iii) are similar.

Similarly, one can also develop an analogous lemma for $S_D = \{q, p^{2m}, p^{2n+1}\}$. We can now prove our theorems.

We divide the proof into two parts.

Case 1: Assume $\Phi_{p^{2m}}(x^{q^m})|P_D(x)$, then $P_D(x) = K_I(x)Q(x)$.

Note that if $n = 1$, then $\Phi_{pq}(x)|P_D(x)$ (by Lemma (1.1.29)(ii) with $k = 0$). Together with $S_D = \{p, p^{2m}, q\}$, we have $\Phi_p(x), \Phi_q(x), \Phi_{pq}(x)$ and $\Phi_{p^{2m}}(x^{q^m})$ dividing $P_D(x)$. Hence, $P_D(x)$ must contain

$$\Phi_p(x)\Phi_{p^{2m}}(x^{q^m})\Phi_q(x^p),$$

which is of type (I).

For $n > 1$, we claim that $\Phi_{p^{2(n-1)+1}q^n}(x)|P_D(x)$, then by observing that $p^{2(n-1)+1} \notin S_D$ $(= \{p, p^{2m}, q^n\})$ and applying Lemma (1.1.22), $\Phi_{q^n}(x^{p^{2(n-1)+1}})$ will divide $P_D(x)$, and $P_D(x)$ is of type (I).

Suppose otherwise, $\Phi_{p^{2(n-1)+1}q^n}(x) \nmid P_D(x)$. Then $\Phi_{pq}(x)|P_D(x)$ (by Lemma (1.1.29)(ii)), hence $\Phi_p(x^p)|P_D(x)$. Therefore $P_D(x)$ must contain the following factor

$$G(x) = \Phi_p(x^q)\Phi_{p^{2m}}\left(x^{q^m}\right)\Phi_{q^n}(x) = \Phi_p(x^q)\Phi_{p^{2m}}\left(x^{q^m}\right)\Phi_{q^{n-1}}(x^q). \quad (29)$$

It is now direct to check that $G(1) = p^2q = \#D$, and the g.c.d. of the non-zero power k of x^k in $G(x)$ is q. This contradicts to Lemma (1.1.25) and hence Case 1 is proved.

Case 2: Assume $\Phi_{p^{2m}}(x^{q^m}) \nmid P_D(x)$, then $P_D(x) = K_{II}(x)Q(x)$.

Let ℓ be the first integer such that $1 \le \ell \le m$ and $\Phi_{p^{2m}q^\ell}(x) \nmid P_D(x)$, we claim that $\ell \neq n$. This is trivial if $m < n$. If $n \le m$, by taking $k = n$ in Lemma (1.1.29)(i), the two possibilities coincide as $\Phi_{p^{2m}q^n}(x)$. This means that $\Phi_{p^{2m}q^n}(x)$ must divide $P_D(x)$ and hence $\ell \neq n$.

It follows that $m + n - \ell \neq m$. By Lemma (1.1.29)(i), we have $\Phi_{p^{2(m+n-\ell)}q^n}(x)|P_D(x)$. Since $p^{2(m+n-\ell)} \notin S_D$, we must have $\Phi_{q^n}(x^{p^{2(m+n-\ell)}})|P_D(x)$ (by Lemma (1.1.22)). Also the choice of ℓ implies that $\Phi_{p^{2m}}(x^{q^{\ell-1}})$ divides $P_D(x)$. Hence $P_D(x)$ contains a kernel polynomial of type (II).

In this case $S_D = \{q, p^{2m}, p^{2n+1}\}$. It follows from the (T2) property of integer tiles that

$$\Phi_{p^{2m}q}(x)\big|P_D(x) and\ \Phi_{p^{2n+1}q}(x)\big|P_D(x) \quad (30)$$

Also, analogous to Lemma (1.1.29), we have

(i) for $1 \le k \le m$, either $\Phi_{p^{2m}q^k}(x)|P_D(x)$ or $\Phi_{p^{2(m-n+1)}q}(x)|P_D(x)$; and

(ii) for $1 \le l \le n+1$, either $\Phi_{p^{2n+1}q^l}(x)|P_D(x)$ or $\Phi_{p^{2(n-l+1)+1}q}(x)|P_D(x)$.

Let

$$K_{III}(x) = \Phi_q(x)\Phi_{p^{2m}}(x^{q^m})\Phi_{p^{2n+1}}(x^{q^{n+1}}),$$

then $K_{III}(x)$ is a kernel polynomial. We show that $P_D(x)$ has $K_{III}(x)$ as a factor. Suppose otherwise, let k_0 and l_0 be the first integers such that

$$\Phi_{p^{2m}q^{k_0}}(x) \nmid P_D(x)\ and\ \Phi_{p^{2n+1}q^{l_0}}(x) \nmid P_D(x) \quad (31)$$

respectively.

Note that $k_0 > 1$ (by (30)). By (i), we have $\Phi_{p^{2(m-k_0+1)}q}(x)|P_D(x)$. Hence $p^{2(m-k_0+1)} \notin S_D$ (by (T2) of Theorem (1.1.11)). It follows from Lemma (1.1.22) that $\Phi_q(x^{p^{2(m-k_0+1)}})|P_D(x)$.

By the same reasoning for the second part of (31), we have $\Phi_q(x^{p^{2(n-l_0+1)+1}})|P_D(x)$.

Let $\tau = \max\{2(m-k_0+1), 2(n-l_0+1)+1\} > 0$, we have $\Phi_q(x^{p^\tau})|P_D(x)$. Hence, $P_D(x)$ must contain the factor

$$G(x) = \Phi_q(x^{p^\tau})\Phi_{p^{2m}}(x^{q^{k_0-1}})\Phi_{p^{2n+1}}(x^{q^{l_0-1}})$$

It is clear that $G(1) = p^2q$ and the x^k of $G(x)$ has a common power p. This contradicts Lemma (1.1.25), and hence, K_{III} is a factor of P_D. This completes the proof.

The proof for the three types of $K(x)$ uses the same idea, we will only prove type (II) as it involves more variations. For this type, there are two cases: (i) $n \ge \ell$ and (ii) $n < \ell$. For simplicity, we consider only the first case.

The second case is similar by interchanging the last two factors in (32) below. Let

$$D_{II} = \mathcal{E}_p \oplus b^{\ell-1}p^{2(m-\ell+1)-1}\mathcal{E}_p \oplus b^{n-1}p^{2(m-\ell+1)}\mathcal{E}_q. \quad (32)$$

Then

$$K(x) := K_{II}(x) = \Phi_p(x)\Phi_{p^{2m}}\left(x^{q^{\ell-1}}\right)\Phi_{q^n}\left(x^{p^{2(n+m-\ell)}}\right). \quad (33)$$

and the kernel polynomial is $\widetilde{K}(x)$ in (28). We will prove that D is a modulo product-form of $D' = \mathcal{E}_p \oplus p^{2(m-\ell+1)-1}\mathcal{E}_p \oplus p^{2(m-\ell+1)}\mathcal{E}_p$. Then together with Lemma (1.1.28) (and also the proof), D is an $(m-\ell+1)$-order modulo product-form.

To this end, we write $K(x) := k_1(x)k_2(x)k_3(x)$ for the three factors in (33). We will use the similar technique as Theorem (1.1.18). Let $n_3 = l.c.m.\{s: \Phi_s(x)|\widetilde{K}(x)\}$.

As $\Phi_{p^{2(m+n-\ell)}q^n}(x)$ is in $\widetilde{K}(x)$. Then by definition $n_3 = p^{2(m+n-\ell)}q^n$, and hence $k_3(x)|x^{n_3} - 1$. Thus,

$$P_D(x)(mod\ x^{n_3} - 1) = K(x)Q(x). \quad (34)$$

Note that $K(x)Q(x)$ has non-negative coefficients (Lemma (1.1.16)), it implies that $k_1(x)k_2(x)Q(x)$ must also have non-negative coefficients. (In fact this follows from $deg(k_3) = \frac{n_3}{q}(q-1)\ and\ k_3(x) = 1 + x^{n_3/q} + \cdots$, we have

$$deg(k_1k_2Q) < n_3 - deg\ k_3 = n_3/q,$$

so that the terms of $k_1(x)k_2(x)Q(x)$ in the expansion of $K(x)Q(x)$ do not overlap.)

By considering $P_D'(x) = k_1(x)k_2(x)Q(x)$ and letting $n_2 = l.c.m.\{s : \Phi_s(x)|k_1(x)k_2(x)\}$, we have $n_2 = p^{2m}q^{\ell}$ and

$$P_D'(x)(mod\ x^{n_2} - 1) = k_2(x)\big(k_1(x)Q'(x)\big). \quad (35)$$

By the same argument as the above, $P_D''(x) = k_1(x)Q'(x)$ must have non-negative coefficients. Finally, let $n_1 = p$. We have

$$P_D''(x)(mod\ x^p - 1) = \Phi_p(x)Q''(x). \quad (36)$$

As $deg(\Phi_pQ'') < p$ and $deg(\Phi_p) = p - 1$, we must have $Q''(x) \equiv 1$. By combining (34), (35) and (36), we see that $D = D^{(2)}$ where

$$\begin{cases} D^{(0)} \equiv \mathcal{E}_p(mod\ n_1), \\ D^{(1)} \equiv D^{(0)} \oplus b^{l-1}p^{2(m-l+1)-1}\mathcal{E}_p(mod\ n_2), \\ D^{(2)} \equiv D^{(1)} \oplus b^{n-1}p^{2(m-l+1)-1}\mathcal{E}_q(mod\ n_3) \end{cases}$$

This proves the theorem.

For the case $p^{\alpha}q$, and $g.c.d.(D) = 1$, the prime power spectrum S_D is either

(i) $S_D = \{p\} \cup \{p^{m_j\alpha+j}\}_{j=2}^{\alpha} \cup \{q^n\}$, or

(ii) $S_D = \{p^{m_j\alpha+j}\}_{j=1}^{\alpha} \cup \{q\}$

where $m_j \in \mathbb{N}^+$. (We modify the notation for m_{α} slightly in comparison with the p^2q case, it is easy to check that $m_2 + 1 = m$ for the m in Theorem (1.1.26). The modification simplifies some expressions below.) We have

Theorem (1.1.30) [1]: Let $b = p^{\alpha}q$ and let D be a tile digit set with $\#D = b$. Then the mask polynomial P_D contains the following kernel polynomials

Case (i): either

(I) $K_I(x) = \Phi_p(x)\Phi_{q^n}(x^{p^{\alpha(n-1)+1}})\prod_{j=2}^{\alpha}\Phi_{p^{m_j\alpha+j}}(x^{q^{m_j+1}})$; or

(II) a factor of $K_{II}(x) = \Phi_p(x)\Phi_{q^n}(x^{p^{\alpha(n+M)+k}})\prod_{j=2}^{\alpha}\Phi_{p^{m_j\alpha+j}}(x^{q^{\ell_j-1}})$, where $1 \le \ell_j \le m_j + 2\ \forall\ j \ge 2$ and at least one $\ell_j \le m_j + 1$ with $M = \max\{m_i - \ell_i : 2 \le i \le \alpha\}$ and $k = \max\{i : m_i - \ell_i = M\}$.

Case (ii):

(III) $K_{III}(x) = \Phi_q(x)\prod_{j=1}^{\alpha}\Phi_{p^{m_j\alpha+j}}(x^{q^{m_j+1}})$.

Moreover, each of the above $K_i(x)$ represents a tile digit set of $b = p^{\alpha}q$.

Theorem (1.1.31) [1]: Let $b = p^{\alpha}q$ and let D be a tile digit set $\#D = b$. Then D must be a k^{th}-order modulo product-form.

It follows from the same idea of proof as Theorem (1.1.27), The first thing is to establish the analogue of Lemma (1.1.28) for D_I, D_{II} and D_{III} given above. It is easy to see that D_I and D_{III} are 1-st order product-form. For D_{II}, pick j so that $m_j - \ell_j = M$ and $k = j$, then we note that

$$p^{\alpha(M+1)+k}\mathcal{E}_q \oplus p^{(m_j-\ell_j+1)\alpha+j-1}\mathcal{E}_p = p^{(m_j-\ell_j+1)\alpha+j-1}(\mathcal{E}_p \oplus p\mathcal{E}_q)$$
$$= p^{(m_j-\ell_j+1)\alpha+j-1}(q\mathcal{E}_p \oplus \mathcal{E}_q).$$

From this, we can use the same argument in Lemma (1.1.28) to conclude that D_{II} is a product-form of some orders t.

Next, we need to show that any tile digit sets must be given by the modulo product-forms of D_I, D_{II} and D_{III}. This is done by arranging the powers of b of those digit sets in non-decreasing order and applying the same argument in the proof of Theorem (1.1.27). We can eventually show that all are of modulo product-form of order $t+1$.

We study the role of the modulo product-forms (and the higher order ones) in the tile digit sets, in particular, to characterize the tile digit sets to be such forms. So far we can only describe the modulo product-forms on $\mathbb{R}^1$. It will be interesting to define the analog in the higher dimensional spaces. Note that the definition of the product-form is easy to be generalized (see. e.g. [11]), however, there is no direct generalization for the modulo product-form. One of the main difficulties is to find some replacement of the cyclotomic polynomials. In another direction, it will also be useful to develop an algorithm to check for a given digit set $D \subset \mathbb{Z}^+$ to be a tile digit set.

The main techniques we use in the explicit characterization for the tile digit sets of $\#D = p^\alpha q$ is the classical results of de Bruijn about $\Phi_{p^\alpha q^\beta}(x)|P_D(x)$, and the decomposition of integer tiles A when $\#A = p^\alpha q^\beta$ (Theorem (1.1.19)). It is likely that our approach can further be improved to obtain a complete characterization of tile digit sets of $\#D = p^\alpha q^\beta$ (as well as integer tiles A of the same cardinality) as certain kind of modulo product-forms as in Theorems (1.1.30), (1.1.31). Finally, if $\#A$ or $\#D$ has more than two prime factors, some new factorization theorems may need to be developed.

The structure of the tile digit sets is closely related to the spectral set problems. Recall that a closed subset $\Omega \subset \mathbb{R}^s$ is called a spectral set if $L^2(\Omega)$ admits an exponential orthonormal basis $\{e^{2\pi i\langle\lambda,\cdot\rangle}\}_{\lambda\in\Lambda}$($\Lambda$ is called a spectrum). The well-known Fuglede conjecture asserted that Ω is a spectral set if and only if it is a translational tile. The conjecture was eventually proved to be false on $\mathbb{R}^s$ for $s \geq 3$ ([9]). The conjecture is still widely open for self-affine tiles. Our consideration on cyclotomic polynomials factors for the tile digit sets is closely linked to the spectral problem, because it also deals with zeros on the unit circle. It would be instructive to first study the spectral problem for simpler product-form as a testing case. On the other hand, there are studies of the spectral problem for integer tiles on $\mathbb{R}^1$[2]. In view of Theorem (1.1.7), the results can be applied to the tile digit sets, and it may offer some insight to investigate the spectral problem of the self-affine tiles.

Section (1.2): Cantor Measures with Consecutive Digits

For μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. We say that μ is a spectral measure if there exists a countable set $\Lambda \subset \mathbb{R}^d$ so that $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a spectrum of μ. If $\chi_\Omega dx$ is a spectral measure, then we say that Ω is a spectral set. The study of spectral measures was first initiated from B. Fuglede in 1974 [27], when he considered a functional analytic problem of extending some commuting partial differential operators to some dense subspace of L^2 functions. In his first attempt, Fuglede proved that any fundamental domains given by a discrete lattice are spectral sets with its dual lattice as its spectrum. On the other hand, he also proved that triangles and circles on $\mathbb{R}^2$ are not spectral sets, while some examples (e.g. [0, 1] ∪ [2, 3]) that are not fundamental domains can still be spectral.

From the examples and the relation between Fourier series and translation operators, he proposed a reasonable conjecture on spectral sets: $\Omega \subset \mathbb{R}^s$ is a spectral set if and only if Ω is a translational tile. This conjecture baffled experts for 30 years until 2004, Tao [28] gave the first counterexamples on $\mathbb{R}^d, d \geq 5$. The examples were modified later so that the conjecture are false in both directions on $\mathbb{R}^d, d \geq 3$ [28], [29]. It remains open in dimension 1 and 2. Despite the counterexamples, the exact relationship between spectral measures and tiling is still mysterious.

The problem of spectral measures is as exciting when we consider fractal measures. Jorgensen and Pedersen [30] showed that the standard Cantor measures are spectral measures if the contraction is $\frac{1}{2n}$, while there are at most two orthogonal exponentials when the contraction is $\frac{1}{2n+1}$. Following this discovery, more spectral self-similar/self-affine measures were also found ([31], [32] et. al.). The construction of these spectral self-similar measures is based on the existence of the compatible pairs (known also as Hadamard triples). It is still unknown whether all such spectral measures are obtained from compatible pairs. Having an exponential basis, the series convergence problem was also studied by Strichartz. It is surprising that the ordinary Fourier series of continuous functions converge uniformly for standard Cantor measures [31]. By now there are considerable amount of literatures studying spectral measures and other generalized types of Fourier expansions like the Fourier frames and Riesz bases ([36], [29], [36], [33], [34], [35] [36], [35], [30], [28], [29.

In [30], Hu and Lau made a start in studying the spectral properties of Bernoulli convolutions, the simplest class of self-similar measures. They classified the contraction ratios with infinitely many orthogonal exponentials. It was recently shown by Dai that the only spectral Bernoulli convolutions are of contraction ratio $\frac{1}{2n}$ [32]. , we study another general class of Cantor measures on $\mathbb{R}^1$. Let $b > 2$ be an integer and $q < b$ be another positive integer. We consider the iterated function system(IFS) with maps

$$f_i(x) = b^{-1}(x + i), \qquad i = 0, 1, \dots, q - 1.$$

The IFS arises a natural self-similar measure $\mu = \mu_{q,b}$ satisfying

$$\mu(E) = \sum_{i=0}^{q-1} \frac{1}{q} \mu\left(f_j^{-1}(E)\right) \tag{37}$$

for all Borel sets E. Note that we only need to consider equal weight since nonequally weighted self-similar measures here cannot have any spectrum by Theorem 1.5 in [29]. It is also clear that if $q = 2, \mu$ becomes the standard Cantor measure of b^{-1} contraction. For this class of self-similar measures, we find surprisingly that the spectral properties depend heavily on the number theoretic relationship between q and b. the first result is to show that $\mu = \mu_{q,b}$ has infinitely many orthogonal exponentials if and only if q and b is not relatively prime. If moreover, q divides b, the resulting measure will be a spectral measure (Theorem (1.2.1)). However, when q does not divide b and they are not relatively prime (e.g. $q = 4, b = 6$), variety of cases may occur and we are not sure whether there are spectral measures in these classes.

We then focus on the case when $b = qr$ in which we aim at giving a detailed classification of its spectra. The classification of spectra, for a given spectral measure, was first studied by Lagarias, Reeds and Wang [28]. They considered the spectra of $L^2([0,1)^d)$ (more generally fundamental domains of some lattices) and they showed that the spectra of $L^2([0,1)^d)$ are exactly all the tiling sets of $[0,1)^d$.

If $d = 1$, the way of tiling [0, 1) is rather rigid, and it is easy to see that the only spectrum (respectively the tiling set) is the translates of the integer lattice $\mathbb{Z}$.

Such kind of rigidity breaks down even on $\mathbb{R}^1$ if we turn to fractal measures. The first attempt of the classification of its spectra was due to [28], Dutkay, Han and Sun decomposed the maximal orthogonal sets of one-fourth Cantor measure using 4-adic expansion with digits $\{0, 1, 2, 3\}$ and put them into a labeling of the binary tree. The maximal orthogonal sets will then be obtained by reading all the infinite paths with digits ending eventually in 0 (for positive elements) or 3 (for negative elements). They also gave some sufficient conditions on the digits for a maximal orthogonal set to be a spectrum. The condition is not easy to verify. Turning to our self-similar measures with consecutive digits where the one-fourth Cantor measure is a special case, we will classify all the maximal orthogonal sets using mappings on the standard $q-$adic tree called maximal mappings. This construction improves the tree labeling method in [30] in two ways. (1) We will choose the digit system to be $\{-1, 0, 1, \cdots, b-2\}$ instead of $\{0, 1, \cdots, b-1\}$. By doing so, all integers (both positive and negative) can be expanded into $b-$adic expansions terminating .

(2) We impose restrictions on our labeling position on the tree so that together with (1), all the elements in a maximal orthogonal set can be extracted by reading some specific paths in the tree. These paths are collected in a countable set Γ_q defined in (41).

Having such a new tree structure of a maximal orthogonal set, we discover there are two possibilities for the maximal sets depending on whether all the paths in Γ_q are corresponding to some elements in the maximal orthogonal sets (i.e. the values assigned are eventually 0). If it happens that all the paths in Γ_q behave nicely as said, we call such maximal orthogonal sets regular. It turns out that regular sets cover most of the interesting cases and we can give regular sets a natural ordering $\{\lambda_n : n = 0, 1, 2, \cdots\}$. If the standard $q-$adic expansion of n has length k, we define N_n^* to be the number of non-zero digits in the $b-$adic expansion using $\{-1, 0, \cdots, b-2\}$ of λ_n after k. N_n^* is our crucial factor in determining whether the set is a spectrum. We show that if N_n^* grows slowly enough or even uniformly bounded, the set will be a spectrum, while if N_n^* grows too fast, say it is of polynomial rates, then the maximal orthogonal sets will not be a spectrum.

In [32], Dutkay $et\ al$ tried to generalize the classical results of Landau [36] about the Beurling density on Fourier frames to fractal settings. They defined the concept of Beurling dimensions for a discrete set and showed that all Bessel sequences for an IFS of similitudes with no overlap condition must have Beurling dimension not greater than its Hausdorff dimension of the attractor. Under technical assumption on the frame spectra, they showed the above two dimensions coincide. They conjectured that the assumption can be removed. However, as we see that N_n^* counts the number of non-zero digits only, we can freely add qb^m for any $m > 0$ on the tree of the canonical spectrum. These additional terms push the λ_n's as far away from each other as wanted and we therefore show that there exists spectrum of zero Beurling dimension.

We present our set-up and main results, we discuss the maximal orthogonal sets of $\mu_{q,b}$ and classify all maximal orthogonal sets via the maximal mapping on the $q-$adic when q divides b., we discuss the regular spectra and prove the growth rate criteria. Moreover, the examples of the spectra with zero Beurling dimensions will be given., we give a study on the irregular spectra.

Let Λ be a countable set in $\mathbb{R}$ and denote $E(\Lambda) = \{e_\lambda : \lambda \in \Lambda\}$ where $e_\lambda(x) = e^{2\pi i \lambda x}$. We say that Λ is a maximal orthogonal set (spectrum) if $E(\Lambda)$ is a maximal orthogonal set (an

orthonormal basis) for $L^2(\mu)$. Here $E(\Lambda)$ is a maximal orthogonal set of exponentials means that it is a mutually orthogonal set in $L^2(\mu)$ such that if $\alpha \notin \Lambda, e_\alpha$ is not orthogonal to some $e_\lambda, \lambda \in \Lambda$. If $L^2(\mu)$ admits a spectrum, then μ is called a spectral measure. Given a measure μ, the Fourier transform is defined to be

$$\hat{\mu}(\xi) = \int e^{2\pi i \xi x} d\mu(x).$$

It is easy to see that $E(\Lambda)$ is an orthogonal set if and only if

$$(\Lambda - \Lambda)\backslash\{0\} \subset Z(\hat{\mu}) := \{\xi \in \mathbb{R}: \hat{\mu}(\xi) = 0\}.$$

We call such Λ a bi-zero set of μ. For $\mu = \mu_{q,b}$, we can calculate its Fourier transform.

$$\hat{\mu}(\xi) = \prod_{j=1}^{\infty} \left[\frac{1}{q}\left(1 + e^{2\pi i b^{-j}\xi} + \ldots + e^{2\pi i b^{-j}(q-1)\xi}\right)\right]. \tag{38}$$

Denote

$$m(\xi) = \frac{1}{q}(1 + e^{2\pi i \xi} + \cdots + e^{2\pi i (q-1)\xi} \tag{39}$$

and thus $|m(\xi)| = |\frac{\sin q\pi\xi}{q \sin \pi\xi}|$. The zero set of m is

$$Z(m) = \left\{\frac{a}{q}: q \nmid a, a \in \mathbb{Z}\right\},$$

where $q \nmid a$ means q does not divide a. We can then write $\hat{\mu}(\xi) = \prod_{j=1}^{\infty} m(b^{-j}\xi)$, so that the zero set of $\hat{\mu}$ is given by

$$Z(\hat{\mu}) = \left\{\frac{b^n}{q} a: n \geq 1, q \nmid a\right\} = r\{b^n a: n \geq 0, q \nmid a\}, \tag{40}$$

where $r = b/q$.

We have the following theorem classifying which $\mu_{q,b}$ possess infinitely many orthogonal exponentials.

Theorem (1.2.1)[26]: $\mu = \mu_{q,b}$ has infinitely many orthogonal exponentials if and only if the greatest common divisor between q and b is greater than 1. If q divides b, then $\mu_{q,b}$ is a spectral measure.

We wish to give a classification on the spectra and the maximal orthogonal sets whenever they exist. To do this, it is convenient to introduce some multiindex notations: Denote $\Sigma_q = \{0, \cdots, q-1\}, \Sigma_q^0 = \{\vartheta\}$ and $\Sigma_q^n = \underbrace{\Sigma_q \times \cdots \times \Sigma_q}_{n}$. Let $\Sigma_q^* = \cup_{n=0}^{\infty} \Sigma_q^n$ be the set of all finite words and let $\Sigma_q^\infty = \Sigma_q \times \Sigma_q \times \cdots$ be the set of all infinite words. Given $\sigma = \sigma_1\sigma_2 \cdots \in \Sigma^\infty \cup \Sigma^*$, we define $\vartheta\sigma = \sigma, \sigma|_k = \sigma_1 \cdots \sigma_k$ for $k \geq 0$ where $\sigma|_0 = \vartheta$ for any σ and adopt the notation $0^\infty = 000\cdots, 0^k = \underbrace{0\cdots 0}_{k}$ and $\sigma\sigma'$ is the concatenation of σ and σ'. We start with a definition.

Definition (1.2.2) [26]: Let Σ_q^* be all the finite words defined as above. We say it is a $q-$adic tree if we set naturally the root is ϑ, all the k-th level nodes are Σ_q^k for $k \geq 1$ and all the offsprings of $\sigma \in \Sigma_q^*$ are σi for $i = 0, 1, \ldots, q-1$.

Let τ be a map from Σ_q^* to real numbers. Then the image of τ defines a q-adic tree labeling. Define Γ_q

$$\Gamma_q := \{\sigma 0^\infty: \sigma = \sigma_1 \cdots \sigma_k \in \Sigma_q^*, \sigma_k \neq 0\}. \tag{41}$$

Γ_q will play a special role in our construction.

Suppose that for some word $\sigma = \sigma' 0^\infty \in \Gamma_q, \tau(\sigma|_k) = 0$ for all k sufficiently large, we say that τ is regular on σ, otherwise irregular. Let b be another integer, if τ is regular on some $\sigma \in \Gamma_q$, we define the projection Π_b^τ from Γ_q to $\mathbb{R}$ as

$$\Pi_b^\tau(\sigma) = \sum_{k=1}^{\infty} \tau(\sigma|_k) b^{k-1}. \tag{42}$$

The above sum is finite since $\tau(\sigma|_k) = 0$ for sufficiently large k. If τ is regular on any σ in Γ_q, we say that τ is a regular mapping.

Example (1.2.3) [1]: Suppose $b = q$, let $C = \{c_0 = 0, c_1, \dots, c_{b-1}\}$ be a residue system $mod\ b$ where $c_i \equiv i\ (mod\ b)$. Define $\tau(\vartheta) = 0$ and $\tau(\sigma) = c_{\sigma_k}$ if $\sigma = \sigma_1 \cdots \sigma_k \in \Sigma_q^k \subset \Sigma_q^*$. Then it is easy to see that τ is regular on any $\sigma \in \Gamma_q$ and hence it is regular. Moreover,

$$\Pi_b^\tau(\Gamma_b) \subseteq \mathbb{Z}. \tag{43}$$

When $C = \{0, 1, \dots, b-1\}$, then the mapping Π_b^τ is a bijection from Γ_b onto $\mathbb{N} \cup \{0\}$.

In [5], putting their setup in our language, they classified maximal orthogonal sets of standard one-fourth Cantor measure via the mapping τ from Σ_2^* to $\{0, 1, 2, 3\}$.

However, some maximal orthogonal sets may have negative elements in which those elements cannot be expressed finitely in 4-adic expansions using digits $\{0, 1, 2, 3\}$.

In our classification, we will choose the digit system to be $C = \{-1, 0, 1\ \cdots, b-2\}$ in which we can expand any integers uniquely by finite $b-$adic expansion. We have the following simple but important lemma.

Lemma (1.2.4) [26]: Let $C = \{-1, 0, 1, \dots, b-2\}$ with integer $b \geq 3$ and let τ be the map defined in Example (1.2.3). Then Π_b^τ is a bijection between Γ_b and $\mathbb{Z}$.

Proof: For any $n \in \mathbb{Z}$ and $|n| < b$, it is easy to see that there exists unique $\sigma \in \Gamma_b$ such that $n = \Pi_b^\tau(\sigma)$. For example, $n = b-1$, then $n = \Pi_b^\tau(\sigma_1\sigma_2)$ where $\sigma_1 = b-1$ and $\sigma_2 = 1$. When $|n| \geq b$, then n can be decomposed uniquely as $n = \ell b + c$ where $c \in C$. We note that $|\ell| = |\frac{n-c}{b}| \leq \frac{|n|+b-2}{b} < |n|$. If $|\ell| < b$, we are done. Otherwise, we further decompose ℓ in a similar way and after finite number of steps, $|\ell| < b$.

The expansion is unique since each decomposition is unique.

We now define a $q-$adic tree labeling which corresponds to a maximal orthogonal set for $\mu_{q,b}$ when $b = qr$. We observe that for $b = qr$, we can decompose $C = \{-1, 0\ \cdots, b-2\}$ in q disjoint classes according to the remainders after being divided by q: $C = \cup_{i=0}^{q-1} C_i$ where

$$C_i = (i + q\mathbb{Z}) \cap C.$$

Definition (1.2.5) [26]: Let Σ_q^* be a $q-$adic tree and $b = qr$, we say that τ is a maximal mapping if it is a map $\tau = \tau_{q,b}: \Sigma_q^* \to \{-1, 0, \dots, b-2\}$ that satisfies

(i) $\tau(\vartheta) = \tau\ (0^n) = 0$ for all $n \geq 1$.

(ii) For all $k \geq 1, \tau\ (\sigma_1 \cdots\ \sigma_k) \in C_{\sigma_k}$.

(iii) For any word $\sigma \in \Sigma_q^*$, there exists σ' such that τ is regular on $\sigma\sigma' 0^\infty \in \Gamma_q$.

We call a tree mapping a regular mapping if it satisfies (i) and (ii) in above and is regular on any word in Γ_q. Clearly, regular mappings are maximal.

Given a maximal mapping τ, the following sets will be of our main study .

$$\Lambda(\tau) := \left\{\Pi_b^\tau(\sigma): \sigma \in \Gamma_q,\ \ \tau\ is\ regular\ on\ \sigma\right\}. \tag{44}$$

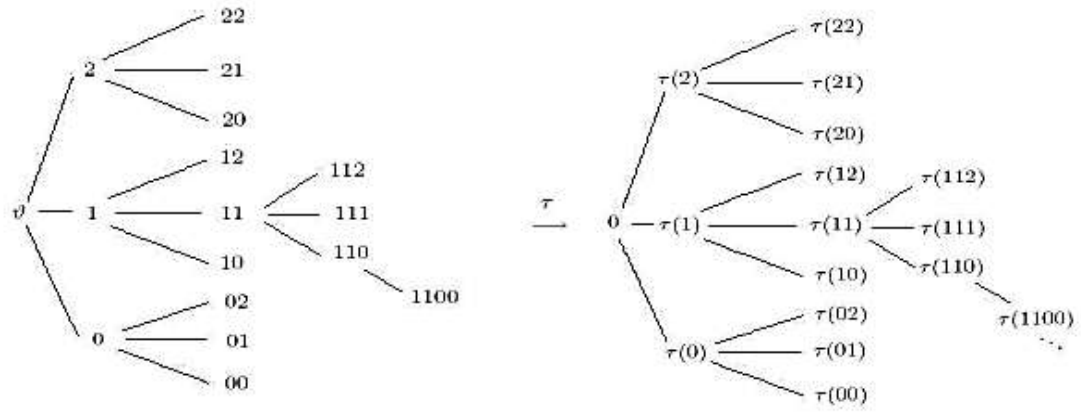

Figure 1[1]: An illustration of the 3−adic tree and the associated mapping τ.

From now on, we will assume that $b = qr, C = \{-1, 0, 1, \ldots, b-2\}$ and $0 \in \Lambda$. The main results are as follows and this is also the reason why τ is called a maximal mapping.

Theorem (1.2.6) [26]: Λ is a maximal orthogonal set of $L^2(\mu_{q,b})$ if and only if there exists a maximal mapping τ such that $\Lambda = r\Lambda(\tau)$, where $b = qr$.

For the proof, (i) in Definition (1.2.5) is to ensure $0 \in \Lambda$. (ii) is to make sure the mutually orthogonality and (iii) is for the maximal orthogonality.

If Λ is a spectrum of $L^2(\mu)$, we call the associated maximal mapping τ a spectral mapping. We will restrict our attention to regular mappings (i.e. for all $\sigma \in \Gamma_q, \tau$ is regular on σ). In this case, $\Lambda(\tau) = \{\Pi_b^\tau(\sigma): \sigma \in \Gamma_q\}$. The advantage of considering regular mappings is that we can give a natural ordering of the maximal orthogonal set $\Lambda(\tau)$. The ordering goes as follows: Given any $n \in \mathbb{N}$, we can expand it into the unique finite q−adic expansion,

$$n = \sum_{j=1}^{k} \sigma_j q^{j-1}, \quad \sigma_j \in \{0, \cdots, q-1\}, \quad \sigma_k \neq 0. \quad (45)$$

In this way n is uniquely corresponding to one word $\sigma = \sigma_1 \cdots \sigma_k$, which is called the q-adic expansion of n. For a regular mapping τ, there is a natural ordering of the maximal orthogonal set $\Lambda(\tau)$: $\lambda_0 = 0$ and

$$\lambda_n = \Pi_b^\tau(\sigma 0^\infty) = \sum_{j=1}^{k} \tau\big(\sigma|_j\big) b^{j-1} + \sum_{i=k+1}^{N_n} \tau\left(\sigma 0^{i-k}\right) b^{i-1} \quad (46)$$

where $\sigma = \sigma_1 \cdots \sigma_k$ is the q−adic expansion of n in (45), $\tau(\sigma 0^{N_n - k}) \neq 0$ and $\tau(\sigma 0^n) = 0$ for all $n > N_n - k$. Under this ordering, we have $\Lambda(\tau) = \{\lambda_n\}_{n=0}^{\infty}$. Let

$$N_n^* = \#\{\ell: \sigma = \sigma_1 \cdots \sigma_k, \tau(\sigma 0^\ell) \neq 0\},$$

where we denote $\#A$ the cardinality of the set A. The growth rate of N_n^* is crucial in determining whether $r\Lambda(\tau)$ is a spectrum of $L^2(\mu)$. To describe the growth rate, we let $N_{m,n}^* = \max\{N_k^*: q^m \leq k < q^n\}, \mathcal{L}_n^* = \min\{N_k^*: q^n \leq k < q^{n+1}\}$ and $M_n = \max\{N_k: 1 \leq k < q^n\}$. We have the following two criteria depending on the growth rate of N_n^*.

Theorem (1.2.7) [26]: Let $\Lambda = r\Lambda(\tau)$ for a regular mapping τ. Then we can find $0 < c_1 < c_2 < 1$ so that the following holds.

(i) If there exists a strictly increasing sequence α_n satisfying

$$\alpha_{n+1} - M_{\alpha_n} \to \infty, \text{ and } \sum_{n=1}^{\infty} c_1^{N^*_{\alpha_n, \alpha_{n+1}}} = \infty, \quad (47)$$

then Λ is a spectrum of $L^2(\mu)$.

(ii). If $\sum_{n=1}^{\infty} c_2^{\mathcal{L}_n^*} < \infty$, then Λ is not a spectrum of $L^2(\mu)$.

The following is the most important example of Theorem (1.2.7).

Example (1.2.8) [26]: For a regular mapping τ, if $M := \sup_n\{N_n^*\}$ is finite, then Λ must be a spectrum.

Proof: Note that $N_{m,n}^* \leq M$ and therefore any strictly increasing sequences α_n will satisfy the second condition in (47). Let $\alpha_1 = 1$ and $\alpha_{n+1} = n + M_{\alpha_n}$ for $n \geq 1$. Then the first condition holds and hence Λ must be a spectrum by Theorem (1.2.7).

We will also see that when there is some slow growth in N_n^*, Λ can still be a spectrum (see Example (1.2.16)). The exact growth rate for Λ to be a spectrum is however hard to obtain from the techniques we used.

Now, we can construct some spectra which can have zero Beurling dimension from regular orthogonal sets using Theorem (1.2.7). In fact, they can even be arbitrarily sparse.

Theorem (1.2.9) [26]: Let $\mu = \mu_{q,b}$ be a measure defined in (37) with $b > q$ and $gcd(q, b) = q$. Then given any increasing non-negative function g on $[0, \infty)$, there exists a spectrum Λ of $L^2(\mu)$ such that

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{\#(\Lambda\cap(x-R,x+R))}{g(R)} = 0. \tag{48}$$

We also make a study on the irregular spectra, although most interesting cases are from the regular one. Let τ be a maximal mapping such that it is irregular on $\{I_1 0^\infty, \dots, I_N 0^\infty\}$, where $I_i \in \Sigma^*$ and the last word in I_i is non-zero, and is regular on the others in Γ_q. We define the corresponding regularized mapping τ_R:

$$\tau_R(\sigma) = \begin{cases} 0, if\ \sigma = I_i 0^k & for\ k \geq 1; \\ \tau\ (\sigma), & otherwise. \end{cases}$$

The result is as follows:

Theorem (1.2.10) [26]: Let τ be an irregular maximal mapping of μ. Suppose τ is irregular only on finitely many σ in Γ_q. Then τ is a spectral mapping if and only if a corresponding regularized mapping τ_R is a spectral mapping.

We will prove this theorem more generally in Theorem (1.2.17) by showing the spectral property is not affected if we alter only finitely many elements in Γ_q. However, we don't know whether the same holds if the finiteness assumption on irregular elements is removed.

we discuss the existence of orthogonal sets for $\mu_{q,b}$, in particular, Theorem (1.2.1) and Theorem (1.2.6) are proved.

Let $gcd(q, b) = d$. Suppose q and b are relatively prime i.e. $d = 1$. Let

$$Z_n := \left\{\frac{b^n}{q} a : q \nmid a\right\}.$$

It is easy to see that $Z(\hat{\mu}_{q,b}) = \bigcup_{n=1}^{\infty} Z_n$. Note that for any a with $q \nmid a$, we have $q \nmid ba$ since $gcd(q, b) = 1$. Hence, if $n > 1$,

$$\frac{b^n}{q} a = \frac{b^{n-1}}{q} (ba) \in Z_{n-1}.$$

This implies that $Z_1 \supset Z_2 \supset \cdots$ and $Z(\hat{\mu}_{q,b}) = Z_1$. Let

$$Y_i = \left\{\frac{b}{q} a : q \nmid a, a \equiv i (mod\ q)\right\},$$

then $Z(\hat{\mu}_{q,b}) = \bigcup_{i=1}^{q-1} Y_i$. If there exists a mutually orthogonal set Λ for $\mu_{q,b}$ with $\#\Lambda \geq q$, we may assume $0 \in \Lambda$ so that $\Lambda\backslash\{0\} \subset Z(\hat{\mu}_{q,b})$. Hence there exists $1 \leq i \leq q-1$ such that $Y_i \cap \Lambda$ contains more than 1 elements, say λ_1, λ_2. But then $\lambda_1 - \lambda_2 = \frac{b}{q} r$ where $q | r$. This contradicts the orthogonal property of Λ.

Suppose now $d > 1$, we know $d \leq q$. We first consider $d = q$ and prove that the measure is a spectral measure. This shows also the second statement. Write now $b = qr$ and define $D = \{0, 1, \cdots, q-1\}$ and $S = \{0, r, \cdots (q-1)r\}$. Then it is easy to see that the matrix

$$H := \left[e^{2\pi i \frac{ijr}{b}}\right]_{0\leq i,j\leq q-1} = \left[e^{2\pi i \frac{ij}{q}}\right]_{0\leq i,j\leq q-1}$$

is a Hadamard matrix (i.e. $HH^* = qI$). This shows $\frac{1}{b}D$ and S forms a compatible pair as in [32]. Therefore it is a spectral measure by Theorem 1.2 in [32].

Suppose now $1 < d < q$. We have shown that $\mu_{d,b}$ is a spectral measure and hence $Z(\hat{\mu}_{d,b})$ contains an infinite bi-zero sets Λ (i.e. $\Lambda - \Lambda \subset Z(\hat{\mu}_{d,b}) \cup \{0\}$). We claim that $Z(\hat{\mu}_{d,b}) \subset Z(\hat{\mu}_{q,b})$ and hence $Z(\hat{\mu}_{q,b})$ has infinitely many orthogonal exponentials.

To justify the claim, we write $q = dt$. Note that for $d \nmid a$,

$$\frac{b^n}{d}a = \frac{b^n}{q}(ta).$$

As q cannot divide ta. Hence, $\frac{b^n}{q}(ta) \in Z(\hat{\mu}_{q,b})$. This also completes the proof of Theorem (1.2.1).

Remark (1.2.11) [26]: In view of Theorem (1.2.1), we cannot decide whether there are spectral measures when $1 < gcd(q, b) < q$. In general, $\mu_{q,b}$ is the convolutions of several self-similar measures with some are spectral and some are not spectral. If $q = 4, b = 6$, we know that $\{0, 1, 2, 3\} = \{0, 1\} \bigoplus \{0, 2\}$ and hence

$$\hat{\mu}_{4,6}(\xi) = \prod_{j=1}^{\infty}\left(\frac{1 + e^{2\pi i 6^{-j}\xi}}{2}\right) \cdot \prod_{j=1}^{\infty}\left(\frac{1 + e^{2\pi i 2 6^{-j}\xi}}{2}\right) = \hat{\nu}_1(\xi)\hat{\nu}_2(\xi)$$

where $\nu_1 = \mu_{2,6}$ and ν_2 is the equal weight self-similar measure defined by the IFS with maps $\frac{1}{6}x$ and $\frac{1}{6}(x + 2)$. Hence, $\mu_{4,6} = \nu_1 * \nu_2$. It is known that both ν_1 and ν_2 are spectral measures, but we don't know whether $\mu_{4,6}$ is a spectral measure. If $q = 6$ and $b = 10$, then $\{0, 1, \cdots, 5\} = \{0, 1\} \bigoplus \{0, 2, 4\}$ and hence $\mu_{6,10}$ is the convolution of $\mu_{2,10}$ with a non-spectral measure with 3 digits and contraction ratio $1/10$. Because of its convolutional structure, it may be a good testing ground for studying the Laba-Wang conjecture [33] and also for finding non-spectral measures with Fourier frame [35].

Suppose $\Lambda = r\Lambda(\tau)$ for some maximal mapping τ. We show that it is an maximal orthogonal set for $L^2(\mu)$. To see this, we first show Λ is a bi-zero set. Pick $\lambda, \lambda' \in \Lambda$, by the definition of $\Lambda(\tau)$, we can find two distinct σ, σ' in Γ_q such that

$$\lambda = \frac{b}{q}\Pi_b^{\tau}(\sigma), \qquad \lambda' = \frac{b}{q}\Pi_b^{\tau}(\sigma').$$

Let k be the first index such that $\sigma|_k \neq \sigma'|_k$. Then for some integer M, we can write

$$q\lambda - q\lambda' = b\sum_{i=k}^{\infty}(\tau(\sigma|_i) - \tau(\sigma'|_i))b^{i-1} = b^k((\tau(\sigma|_k) - \tau(\sigma'|_k)) + bM).$$

By (ii) in Definition (1.2.5), $\tau(\sigma|_k)$ and $\tau(\sigma'|_k)$ are in distinct residue class of q. This means q does not divide $\tau(\sigma|_k) - \tau(\sigma'|_k)$. On the other hand, q divides b. Hence, q does not divide $(\tau(\sigma|_k) - \tau(\sigma'|_k)) + bM$. By (40), $\lambda - \lambda'$ lies in $Z(\hat{\mu})$.

To establish the maximality of the orthogonal set Λ, we show by contradiction. Let $\theta \notin \Lambda$ but θ is orthogonal to all elements in Λ. Since $0 \in \Lambda, \theta \neq 0$ and $\theta = \theta - 0 \in Z(\hat{\mu})$. Hence, by (40) we may write

$$\theta = r(b^{k-1}a),$$

where q does not divide a. Expand $b^{k-1}a$ in $b-$adic expansion using digits $\{-1, 0, \cdots, b - 2\}$

$$b^{k-1}a = \epsilon_{k-1}b^{k-1} + \epsilon_k b^k + \cdots + \epsilon_{k+\ell}b^{k+\ell},$$

q does not divide ϵ_{k-1}. Note that there exists unique $\sigma_s, 0 \leq \sigma_s \leq q - 1$, such that $\epsilon_s \equiv \sigma_s (mod\ q)$ for $k - 1 \leq s \leq k + \ell$. Denote $\sigma_s = \epsilon_s = 0$ for $s > k + \ell$. Since $\theta \notin \Lambda$,

we can find the smallest integer α such that $\tau(0^{k-2}\sigma_{k-1}\sigma_k\cdots\sigma_\alpha)\neq\epsilon_\alpha$. By (iii) in the definition of τ, we can find $\sigma\in\Gamma_q$ so that $\sigma=0^{k-2}\sigma_{k-1}\sigma_k\cdots\sigma_\alpha\sigma'0^\infty$ and τ is regular on σ, then there exists M' such that

$$\theta-r\Pi_b^\tau(\sigma)=rb^\alpha(\epsilon_\alpha-\tau(0^{k-1}\sigma_{k-1}\cdots\sigma_\alpha)+M'b).$$

By (ii) in the definition of $\tau,\tau(0^{k-1}\sigma_{k-1}\cdots\sigma_\alpha)\equiv\sigma_\alpha(mod\ q)$, which is also congruent to ϵ_α by our construction. This implies $\theta-r\Pi_b^\tau(\sigma)$ is not in the zero set of $\hat{\mu}$ since q divides $\epsilon_\alpha-\tau(0^{k-1}i_{k-1}\cdots\ i_\alpha)$ and b does not divide it either. It contradicts to θ being orthogonal to all Λ.

Conversely, suppose we are given a maximal orthogonal set Λ of $L^2(\mu)$ with $0\in\Lambda$. Then $\Lambda\subset Z(\hat{\mu})$. Hence, we can write

$$\Lambda=\{ra_\lambda:\lambda\in\Lambda,a_\lambda=b^{k-1}m\ for\ some\ k\geq1\ and\ m\in\mathbb{Z}\ with\ q\nmid m\},$$

where $a_0=0$. Now, expand a_λ in $b-$adic expansion with digits chosen from $C=\{-1,0,\cdots,b-2\}$.

$$a_\lambda=\sum_{j=1}^{\infty}\epsilon_\lambda^{(j)}b^{j-1}. \tag{49}$$

Let $D(\vartheta)=\{\epsilon_\lambda^{(1)}:\lambda\in\Lambda\}$ be all the first coefficients of b-adic expressions (49) of elements in Λ, and let $D(c_1,\cdots,c_n)=\{\epsilon_\lambda^{(n+1)}:\epsilon_\lambda^{(1)}=c_1,\cdots\ \epsilon_\lambda^{(n)}=c_n,\lambda\in\Lambda\}$ be all the $n+1$-st coefficients of elements in Λ whose first n coefficients are fixed, where $c_1,c_2,\ldots,c_n\in C$. We need the following lemma.

Lemma (1.2.12) [26]: With the notations above, then $D(\vartheta)$ contains exactly q elements which are in distinct residue class $(mod\ q)$ and $0\in D(\vartheta)$. Moreover, if $D(c_1,\cdots,c_n)$ with all $c_i\in C$ is non-empty, then it contains exactly q elements which are in distinct residue class $(mod\ q)$ also. In particular, $0\in D(c_1,\cdots,c_n)$ if $c_1=\cdots=c_n=0$ for $n\geq1$.

Proof: Clearly, by (49), $0\in D(\vartheta)$ and $0\in D(c_1,\cdots,c_n)$ if $c_1=\cdots=c_n=0$ for $n\geq1$. Suppose the number of elements in $D(\vartheta)$ is strictly less than q. We let $\alpha\in C\backslash D(\vartheta)$ such that α is not congruent to any elements in $D(\vartheta)$. Then, for any $\lambda\in\Lambda$, by (49) we have

$$r\alpha-\lambda=r\left(\alpha-\sum_{j=1}^{\infty}\epsilon_\lambda^{(j)}b^{j-1}\right)=\frac{b}{q}\left(\alpha-\epsilon_\lambda^{(1)}+\sum_{j=2}^{\infty}\epsilon_\lambda^{(j)}b^{j-1}\right).$$

Note that $q\nmid(\alpha-\epsilon_\lambda^{(1)})$ for all $\lambda\in\Lambda$ by the assumption, this implies $r\alpha$ is mutually orthogonal to Λ but is not in Λ, which contradicts to maximal orthogonality. Hence $D(\vartheta)$ contains at least q elements. If $D(\vartheta)$ contains more than q elements, then there exists $a_{\lambda_1}=\sum_{j=1}^{\infty}\epsilon_{\lambda_1}^{(j)}b^{j-1}$, $a_{\lambda_2}=\sum_{j=1}^{\infty}\epsilon_{\lambda_2}^{(j)}b^{j-1}$ such that $\epsilon_{\lambda_1}^{(1)}\equiv\epsilon_{\lambda_2}^{(1)}(mod\ q)$ and $\epsilon_{\lambda_1}^{(1)}\neq\epsilon_{\lambda_2}^{(1)}$. Then $r(a_{\lambda_1}-a_{\lambda_2})=\frac{b}{q}(\epsilon_{\lambda_1}^{(1)}-\epsilon_{\lambda_2}^{(1)}+\ bM)$ for some integer M. This means $r(a_{\lambda_1}-a_{\lambda_2})$ is not a zero of $\hat{\mu}$. This contradicts to the mutually orthogonality. Hence, $D(\vartheta)$ contains exactly q elements which are in distinct residue class $(mod\ q)$.

In general, we proceed by induction. Suppose the statement holds up to $n-1$. If now $D(c_1,\cdots,c_n)$ is non-empty, then $D(c_1,\cdots,c_k)$ is also non-empty for all $k\leq n$. we now show that $D(c_1,\cdots,c_n)$ must contain at least q elements. Otherwise, we consider $\theta=r(c_1+\cdots+c_nb^{n-1}+\alpha b^n)$ where α is in $C\backslash D(c_1,\cdots,c_n)$ and α is not congruent to any elements in $D(c_1,\cdots,c_n)\ (mod\ q)$. If $\lambda\in\Lambda$ and $\lambda=r(c_1+\cdots+c_kb^{k-1}+c'_{k+1}b^k+\cdots)$ where $c_{k+1}\neq c'_{k+1}$ and $k\leq n$, then $c_{k+1},c'_{k+1}\in D(c_1,\cdots\ c_k)$ and hence θ and λ are mutually orthogonal by the induction hypothesis. If $\lambda\in\Lambda$ is such that the first

n digit expansion are equal to θ, the same argument as in the proof of $D(\vartheta)$ shows θ will be orthogonal to this λ. Therefore, θ will be orthogonal to all elements in Λ, a contradiction. Also in a similar way as the above, $D(c_1,\cdots,c_n)$ contains exactly q elements can be shown. Returning to the proof, by convention, we define $\tau(\vartheta)=0$ and on the first level, we define $\tau(\sigma_1)$ to be the unique element in $D(\vartheta)$ such that it is congruent to $\sigma_1(mod\ q)$. For $\sigma=\sigma_1\cdots\sigma_n$, we define $\tau(\sigma\sigma_{n+1})$ to be the unique element in $D(\tau(\sigma|_1),\cdots,\tau(\sigma|_n))$ (it is non-empty from the induction process) that is congruent to $\sigma_{n+1}(mod\ q)$. Then $\tau(0^k)=0$ for $k\geq 1$.

We show that τ is a maximal mapping corresponding to Λ. (i) is satisfied by above. By Lemma (1.2.12), τ is well-defined with (ii) in Definition (1.2.5). Finally, given a node $\sigma\in\Sigma_q^n$, by the construction of the τ we can find λ whose first n digits in the digit expansion (49) exactly equals the value of $\tau(\sigma|_k)$ for all $1\leq k\leq n$. Since the digit expansion of λ becomes 0 eventually, we continue following the digit expansion of λ so that (iii) in the definition is satisfied.

We now show that $\Lambda=r\Lambda(\tau)$. For each a_λ given in (49), Lemma (1.2.12) with the definition of τ shows that there exists unique path σ such that $\tau(\sigma|_n)=\epsilon_\lambda^{(n)}$ for all n. As the sum is finite, this means $\Lambda\subset r\Lambda(\tau)$. Conversely, if some $\Pi_b^\tau(\sigma)\in r\Lambda(\tau)$ is not in Λ, then from the previous proof we know $\Pi_b^\tau(\sigma)$ must be orthogonal to all elements in Λ. This contradicts to the maximal orthogonality of Λ. Thus, $\Lambda=r\Lambda(\tau)$.

We study under what conditions a maximal orthogonal set is a spectrum or not a spectrum.

Lemma (1.2.13) [26]: [33] Let μ be a Borel probability measure in $\mathbb{R}$ with compact support. Then a countable set Λ is a spectrum for $L^2(\mu)$ if and only if

$$Q(\xi):=\sum_{\lambda\in\Lambda}|\hat{\mu}(\xi+\lambda)|^2\equiv 1,\qquad for\ \xi\in\mathbb{R}.$$

Moreover, if Λ is a bi-zero set, then Q is an entire function.

Note that the first part of Lemma (1.2.13) is well-known. For the entire function property, we just note that the partial sum $\sum_{|\lambda|\leq n}|\cdots|^2$ is an entire function and it is locally uniformly bounded by applying the Bessel's inequality, hence Q is entire by the Montel's theorem in complex analysis. See [10] for the details of the proof.

Let δ_a be the Dirac measure with center a. We define

$$\delta_{\mathcal{E}}=\frac{1}{\#\mathcal{E}}\sum_{e\in\mathcal{E}}\delta_e$$

for any finite set $\mathcal{E}$. Let μ be the self-similar measure in (37). Write $D=\{0,1,\ldots,q-1\}$ and $D_k=\frac{1}{b}D+\cdots+\frac{1}{b^k}D$ for $k\geq 1$. We recall that the mask function of D is

$$m(\xi)=\frac{1}{q}(1+e^{2\pi i\xi}+\cdots+e^{2\pi i(q-1)\xi})$$

and define $\mu_k=\delta_{D_k}$, then

$$\widehat{\mu_k}(\xi)=\prod_{j=1}^{k}m(b^{-k}\xi)$$

and it is well-known that μ_k converges weakly to μ when k tends to infinity and we have

$$\hat{\mu}(\xi)=\widehat{\mu_k}(\xi)\hat{\mu}\left(\frac{\xi}{b^k}\right).\qquad(50)$$

Lemma (1.2.14) [26]: Let τ be a regular mapping and let $\Lambda=r\Lambda(\tau)=\{\lambda_k\}_{k=0}^{\infty}$ be the maximal orthogonal set determined by τ. Then for all $n\geq 1$,

$$\sum_{k=0}^{q^n-1} |\widehat{\mu_n}(\xi + r\lambda_k)|^2 \equiv 1. \tag{51}$$

Proof: We claim that $\{r\lambda_k\}_{k=0}^{q^n-1} = \frac{b}{q}\{\lambda_k\}_{k=0}^{q^n-1}$ is a spectrum of $L^2(\mu_n)$. We can then use Lemma (1.2.13) to conclude our lemma. Since this set has exactly q^n elements, we just need to show the mutually orthogonality. To see this, we note that

$$\hat{\mu}_n(\xi) = m\left(\frac{\xi}{b}\right)\cdots m\left(\frac{\xi}{b^n}\right). \tag{52}$$

Given $l \neq l'$ in $\{0,\cdots q^n-1\}$, let $\sigma = \sigma_1 \cdots \sigma_n$ and $\sigma' = \sigma_1' \cdots \sigma_n'$ be the q-adic expansions of l and l' respectively as in (45), where σ_n and σ_n' may be zero. We let $s \leq n$ be the first index such that $\sigma_s \neq \sigma_s'$. Then we can write

$$\lambda_l - \lambda_{l'} = b^{s-1}(\tau(\sigma|_s) - \tau(\sigma'|_s) + bM)$$

for some integer M. We then have from the integral periodicity of $e^{2\pi ix}$ that

$$m\left(\frac{r(\lambda_l - \lambda_{l'})}{b_s}\right) = m\left(\frac{\tau(\sigma|_s) - \tau(\sigma'|_s)}{q}\right) = 0.$$

It is equal to 0 because (ii) in the definition of maximal mapping implies that q does not divide $\tau(\sigma|_s) - \tau(\sigma'|_s)$. Hence, by (4.3), $\hat{\mu}_n(r(\lambda_l - \lambda_{l'})) = 0$.

Now, we let

$$Q_n(\xi) = \sum_{k=0}^{q^n-1} |\hat{\mu}(\xi + r\lambda_k)|^2 \ , and\ Q(\xi) = \sum_{k=0}^{\infty} |\hat{\mu}(\xi + r\lambda_k)|^2.$$

For any n and p, we have the following identity:

$$Q_{n+p}(\xi) = Q_n(\xi) + \sum_{k=q^n}^{q^n+p-1} |\hat{\mu}(\xi + r\lambda_k)|^2$$

$$= Q_n(\xi) + \sum_{k=q^n}^{q^{n+p-1}} \left|\widehat{\mu_{n+p}}(\xi + r\lambda_k)\right|^2 \left|\hat{\mu}\left(\frac{\xi + r\lambda_k}{b^{n+p}}\right)\right|^2. \tag{53}$$

Our goal is see whether $Q(\xi) \equiv 1$. Then by invoking Lemma (1.2.13), we can determine whether we have a spectrum. As Q is an entire function by Lemma (1.2.13), we just need to see the value of $Q(\xi)$ for some small values of ξ. To do this, we need to make a fine estimation of the terms $\left|\hat{\mu}(\frac{\xi+r\lambda_k}{b^{n+p}})\right|^2$ in the above. Write

$$c_{\min} = \min\left\{\prod_{j=0}^{\infty} |m(b^{-j}\xi)|^2 : |\xi| \leq \frac{b-1}{qb}\right\} > 0,$$

where $|m(\xi)| = \frac{|\sin \pi q\xi|}{q|\sin \pi\xi|}$ and $\prod_{j=0}^{\infty} |m(b^{-j}\xi)|^2 = |m(\xi)\hat{\mu}(\xi)|^2$. Denote

$$c_{\max} = \max\left\{|m(\xi)|^2 : \frac{1}{b^2} \leq |\xi| \leq \frac{b-1}{qb}\right\} < 1.$$

The following proposition roughly says that the magnitude of the Fourier transform is controlled by the number of non-zero digits in the $b-$adic expansion in a uniform way. Recall that $b = qr$ with $q, r \geq 2$.

Proposition (1.2.15) [26]: Let $|\xi| \leq \frac{r(b-2)}{b-1}$ and let

$$t = \xi + \sum_{k=1}^{N} d_i b^{n_k},$$

where $d_i \in \{1, 2, \cdots r-1\}$ and $1 \le n_1 < \cdots < n_N$. Then

$$c_{mim}^{N+1} \le |\hat{\mu}(t)|^2 \le c_{\max}^{N} \quad . \quad (54)$$

Proof: First it is easy to check that, for $|\xi| \le \frac{r(b-2)}{b-1}$ and all $d_k \in \{0, 1, 2, \ldots, r-1\}$, we have

$$\begin{aligned}\left|\frac{\xi + \sum_{k=1}^{n} d_k b^k}{b^{n+1}}\right| &\le \frac{1}{b^{n+1}}\left(\frac{r(b-2)}{b-1} + (r-1)(b + b^2 + \cdots + b^n)\right) \\ &= \frac{r(b-2) + (r-1)(b^{n+1} - b)}{b^{n+1}(b-1)} \\ &\le \frac{b-1}{qb} \qquad (55)\end{aligned}$$

for $n \ge 1$. The inequality in the last line follows from a direct comparison of the difference and $q \ge 2$. To simplify notations, we let $n_0 = 0$ and $n_{N+1} = \infty$. Then $|\hat{\mu}(t)|^2$ equals

$$\prod_{j=1}^{\infty} \left|m(b^{-j}t)\right|^2 = \prod_{i=0}^{N} \prod_{j=n_i+1}^{n_i+1} \left|m(b^{-j}t)\right|^2. \quad (56)$$

We now estimate the products one by one. By (55), we have

$$\left|\frac{\xi + \sum_{k=1}^{i} d_k b^{n_k}}{b^{n_i+1}}\right| \le \frac{b-1}{qb}.$$

Hence, together with the integral periodicity of m and the definition of $c_{\min}$, we have for all $i > 0$,

$$\begin{aligned}\prod_{j=n_i+1}^{n_i+1} \left|m(b^{-j}t)\right|^2 &= \prod_{j=n_i+1}^{n_i+1} \left|m\left(b^{-j}\left(\xi + \sum_{k=1}^{i} d_k b^{n_k}\right)\right)\right|^2 \\ \ge \prod_{j=0}^{\infty} \left|m\left(b^{-j}\left(\frac{\xi + \sum_{k=1}^{i} d_k b^{n_k}}{b^{n_i+1}}\right)\right)\right|^2 &\ge c_{\min} \quad . \quad (57)\end{aligned}$$

For the case $i = 0$, it is easy to see that $\left|\frac{\xi}{b}\right| \le \frac{b-2}{q(b-1)} < \frac{b-1}{qb}$. Hence, $\prod_{j=n_0+1}^{n_1} |m(b^{-j}t)|^2 \ge \prod_{j=0}^{\infty} |m(b^{-j}(\xi/b))|^2 \ge c_{\min}$. Putting this fact and (57) into (56), we have $|\hat{\mu}(t)|^2 \ge c_{\min}^{N+1}$.

We next prove the upper bound. From $|m(\xi)| \le 1$, (56) and the integral periodicity of m,

$$|\hat{\mu}(t)|^2 \le \prod_{i=1}^{N} \left|m\left(b^{-(n_i+1)}t\right)\right|^2 = \prod_{i=1}^{N} \left|m\left(b^{-(n_i+1)}\left(\xi + \sum_{k=1}^{i} d_k b^{n_k}\right)\right)\right|^2. \quad (58)$$

By (55) we have

$$|\xi + \sum_{k=1}^{i} d_k b^{n_k}| \ge b^{n_i} - |\xi + \sum_{k=1}^{i-1} d_k b^{n_k}| \ge b^{n_i} - \frac{b^{n_i-1}(b-1)}{q} \ge b^{n_i-1}.$$

By (55), (58), the above and the definition of c_{max}, we obtain that $|\hat{\mu}(t)|^2 \le c_{max}^{N}$.

We now prove Theorem (1.2.7). Write $c_1 = c_{\min}$ and $c_2 = c_{\max}$, where $c_{\min}$ and $c_{\max}$ are in Proposition (1.2.15). Also recall the quantities defined For any $n \in N$, the q-adic expression of n is $\sum_{j=1}^{k} \sigma_j q^{j-1}$ with $\sigma_k \ne 0$. Then for the map τ we have

$$\lambda_n = \sum_{j=1}^{k} \tau(\sigma_1 \cdots \sigma_j) b^{j-1} + \sum_{j=k+1}^{N_n} \tau(\sigma_1 \cdots \sigma_k 0^{j-k}) b^{j-1}$$

where $\tau(\sigma_1 \cdots \sigma_k 0^{N_n-k}) \neq 0$ and $N_n^* = \#\{\tau(\sigma_1 \cdots \sigma_k 0^j) \neq 0 : k+1 \leq j \leq N_n\}$. Moreover, $N_{m,n}^* = \max_{q^m \leq k < q^n}\{N_k^*\}, \mathcal{L}_n^* = \min_{q^n \leq k < q^{n+1}}\{N_k^*\}$ and $M_n = \max_{1 \leq k < q^n}\{N_k\}$.

Let α_n be the increasing sequence satisfying (47) and let $|\xi| \leq \frac{b-2}{b-1}$. Recall (53),

$$Q_{\alpha_{n+1}}(q^{-1}\xi) = Q_{\alpha_n}(q^{-1}\xi) + \sum_{k=q^{\alpha_n}}^{q^{\alpha_{n+1}-1}} \left|\widehat{\mu_{\alpha_{n+1}}}(q^{-1}\xi + r\lambda_k)\right|^2 \left|\hat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}}\right)\right|^2 .$$

For $k = q^{\alpha_n}, \cdots, q^{\alpha_{n+1}} - 1$, we may write λ_k as

$$\lambda_k = \sum_{j=0}^{\alpha_{n+1}-1} c_j b^j + \sum_{j=1}^{M_k} d_j q b^{n_j},$$

where $c_j \in \{-1, \cdots, b-2\}, d_j \in \{1, \cdots, r-1\}$ and $\alpha_{n+1} \leq n_1 < n_2 < \cdots < n_{M_k}$ with $n_{M_k} = N_k$ and $M_k \leq N_k^*$, where N_k, N_k^* was defined in (46) or see the above. Note also that the second term on the right hand of the above is zero whenever $N_k < \alpha_{n+1}$. Now,

$$\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}} = \frac{q^{-1}\xi + q^{-1}\sum_{j=1}^{\alpha_{n+1}} c_j b^j}{b^{\alpha_{n+1}}} + \sum_{j=1}^{M_k} d_j b^{n_j - \alpha_{n+1}+1}.$$

Note that

$$\left|\frac{\xi}{q} + \frac{1}{q} c_j b^j\right| \leq \frac{b-2}{q(b-1)} + (b-2)\frac{b^{k+1}-b}{q(b-1)} \leq \frac{b-2}{q(b-1)} b^{k+1} = \frac{r(b-2)}{b-1} b^k$$

for all $k \geq 1$. Hence, Proposition (1.2.15) implies that

$$\left|\hat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}} b\right)\right|^2 \geq c_1^{1+M_k} \geq c_1^{1+N_k^*} \geq c_1^{1+N_{\alpha_n,\alpha_{n+1}}^*}$$

for all $q^{\alpha_n} \leq k < q^{\alpha_{n+1}}$. Therefore, together with Lemma (1.2.14),

$$Q_{\alpha_{n+1}}(q^{-1}\xi) \geq Q_{\alpha_n}(q^{-1}\xi) + c_1^{1+N_{\alpha_n,\alpha_{n+1}}^*} \sum_{k=q^{\alpha_n}}^{q^{\alpha_{n+1}-1}} \left|\widehat{\mu_{\alpha_{n+1}}}(q^{-1}\xi + r\lambda_k)\right|^2$$

$$= Q_n(q^{-1}\xi) + c_1^{1+N_{\alpha_n,\alpha_{n+1}}^*}\left(1 - \sum_{k=0}^{q^{\alpha_{n-1}}} \left|\widehat{\mu_{\alpha_{n+1}}}(q^{-1}\xi + r\lambda_k)\right|^2\right). \qquad (59)$$

From elementary analysis, there exists $\delta, 0 < \delta < 1$, such that $|\hat{\mu}(\xi)|^2$ is decreasing on $(0, \delta)$ (In fact, since $\hat{\mu}(0) = 1$ and $|\hat{\mu}(\xi)|^2$ is entire in complex plane, there exists $\eta > 0$ such that $|\hat{\mu}(\xi)| < 1$ for all $0 < \xi < \eta$. If $|\hat{\mu}(\xi)|^2$ is not decreasing on $(0, \delta)$ for any $\delta > 0$, we can find a sequence $\xi_n \to 0$ such that $(|\hat{\mu}|^2)'(\xi_n) = 0$ and thus $(|\hat{\mu}|^2)' \equiv 0$ by the entire function property of $|\hat{\mu}|^2$, this is impossible). In the proof, it is also useful to note that $|\hat{\mu}(-\xi)| = |\hat{\mu}(\xi)|$. We now argue by contradiction.

Suppose there exists Λ such that Theorem (1.2.7) (i) holds but is not a spectrum, then there exists $t_0 < \min\{\delta, \frac{b-2}{b-1}\}$ such that $Q(q^{-1}t_0) < 1$ because Q is entire. For $0 \leq k \leq q^{\alpha_n} - 1$, we have

$$\left|\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}\right| \leq \frac{1 + rb^{M_{\alpha_n}}}{b^{\alpha_{n+1}}} := \beta_n.$$

By the assumption that $\alpha_{n+1} - M_{\alpha_n} \to \infty$, we have for all n large, say $n \geq M$, $\beta_n < \delta$ so that $\left|\hat{\mu}\left(\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}\right)\right|^2 \geq |\hat{\mu}(\beta_n)|^2$ and we can find $r < 1$ such that

$$|\hat{\mu}(\beta_n)|^{-2} Q(q^{-1}t_0) \leq r < 1, \qquad for\, n \geq M$$

because β_n tends to zero when n tends to infinity and $\hat{\mu}(0) = 1$. According to $\hat{\mu}(\xi) = \hat{\mu}_{\alpha_{n+1}}(\xi)\hat{\mu}(\xi/b^{\alpha_{n+1}})$, we have

$$\begin{aligned} |\hat{\mu}(q^{-1}t_0 + r\lambda_k)|^2 &= \left|\hat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)\hat{\mu}\left(\frac{q^{-1}t_0 + r\lambda_k}{b^{\alpha_{n+1}}}\right)\right|^2 \\ &\geq |\hat{\mu}_{\alpha_{n+1}}(q^{-}t_0 + r\lambda_k)|^2 |\hat{\mu}(\beta_n)|^2 \\ &\geq \frac{Q(q^{-1}t_0)}{r} |\hat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)|^2 \end{aligned}$$

From (59) and for all $n \geq M$,

$$\begin{aligned} &Q_{\alpha_{n+1}}(q^{-1}t_0) \\ &\geq Q_{\alpha_n}(q^{-1}t_0) + \left(1 - \frac{r}{Q(q^{-1}t_0)} \sum_{k=0}^{q^{\alpha_{n-1}}} \left|\hat{\mu}_{\alpha_{n+1}}(q^{-1}t_0 + r\lambda_k)\right|^2\right) c_1^{1+N^*_{\alpha_n,\alpha_{n+1}}}. \\ &\geq Q_{\alpha_n}(q^{-1}t_0) + (1-r)c_1^{1+N^*_{\alpha_n,\alpha_{n+1}}}. \end{aligned}$$

Taking summation on n from M to $M + p$ where $p > 0$ and note that $Q_n(t) \leq 1$ for any n we have

$$1 \geq Q_{\alpha_{M+p+1}}(q^{-1}t_0) \geq Q_{\alpha_M}(q^{-1}t_0) + (1-r) \sum_{n=M}^{M+p} c_1^{1+N^*_{\alpha_n,\alpha_{n+1}}}.$$

As $\sum_{n=M}^{\infty} c_1^{1+N^*_{\alpha_n,\alpha_{n+1}}} = \infty$ by the assumption, the right hand side of the above tends to infinity. This is impossible. Hence, Λ must be a spectrum.

(ii). The proof starts again at (53) with $p = 1$, we have

$$Q_{n+1}(q^{-1}\xi) = Q_n(q^{-1}\xi) + \sum_{k=q^n}^{q^{n+1}-1} |\widehat{\mu_{n+1}}(q^{-1}\xi + r\lambda_k)|^2 \left|\hat{\mu}\left(\frac{q^{-1}\xi + r\lambda_k}{b^{\alpha_{n+1}}}\right)\right|^2.$$

Since $N_k^* \geq \mathcal{L}_n^*$ for $q^n \leq k < q^{n+1}$, so Proposition (1.2.15) implies that

$$Q_{n+1}(q^{-1}\xi) \leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}_n^*} \sum_{k=q^{\alpha_n}}^{q^{\alpha_{n+1}-1}} \left|\widehat{\mu_{\alpha_{n+1}}}(q^{-1}\xi + r\lambda_k)\right|^2.$$

Using Lemma (1.2.14) and note that $|\widehat{\mu_{n+1}}(\xi)|^2 \geq |\hat{\mu}(\xi)|^2$, we have

$$\begin{aligned} Q_{n+1}(q^{-1}\xi) &\leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}_2^*}\left(1 - \sum_{k=0}^{q^n-1} |\widehat{\mu_{n+1}}(q^{-1}\xi + r\lambda_k)|^2\right) \\ &\leq Q_n(q^{-1}\xi) + c_2^{\mathcal{L}_n^*}(1 - Q_n(q^{-1}\xi)). \end{aligned}$$

Hence,

$$1 - Q_{n+1}(q^{-1}\xi) \geq \left(1 - Q_n(q^{-1}\xi)\right)\left(1 - b^{\mathcal{L}_n^*}\right) \geq \cdots \geq \left(1 - Q_1(q^{-1}\xi)\right) \prod_{k=1}^{n} \left(1 - c_2^{\mathcal{L}_k^*}\right) \qquad (60)$$

Since $\sum_n c_2^{\mathcal{L}_n^*} < \infty, B := \prod_{k=1}^{\infty}(1 - c_2^{\mathcal{L}_n^*}) > 0$ and hence as n tends to infinity in (60), we have

$$1 - Q(q^{-1}\xi) \geq (1 - Q_1(q^{-1}\xi)) \cdot B > 0.$$

Therefore, τ is not a spectral mapping.
As known , τ is a spectral mapping if $\sup\{N_n^*\}$ is finite. Now, we give an example of spectrum with slow growth rate of N_n^*.

Example(1.2.16) [26]: Let τ be a regular mapping so that $N_n \le \log_q n + \log_{c_1^{-2}} \log_q n$ and $N_n^* \le log_{c_1^{-2}} log_q n$ for $n \ge 1$, where c_1 is given in Theorem (1.2.7). Then $r\Lambda(\tau)$ is a spectrum of $L^2(\mu)$.

Proof: Take $\alpha_n = n^n$. Recall $M_n = \max_{1\le k<q^n} N_k$, we have

$$\alpha_{n+1} - M_{\alpha_n} \ge (n+1)^2 - n^2 - \log_{c_1^{-2}} n^2 ,$$

which tends to infinity when n tends to infinity. Note that

$$N^*_{\alpha_n,\alpha_{n+1}} = \max_{q^{\alpha_n}\le k<q^{\alpha_{n+1}}} N_k^* \le log_{c_1^{-2}} log_q\, q^{(n+1)^2} = log_{c_1^{-2}}(n+1)^2 .$$

Then

$$\sum_{n=1}^{\infty} c_1^{N^*_{\alpha_n,\alpha_{n+1}}} \ge \sum_{n=1}^{\infty} c_1^{\,log_{c_1^{-2}}(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

By Theorem (1.2.7) the result follows.
On the other hand, if N_n^* is so that $\mathcal{L}_n^* \ge (1+\epsilon)\ log_{c_2^{-1}} n$, for some $\epsilon > 0$ and $n \ge 1$, then $r\Lambda(\tau)$ is not a spectrum. This is done by checking the condition of Theorem (1.2.7)(ii) using the similar method in the above. We therefore omit its detail. Finally, we prove Theorem (1.2.9).
For any $n \in \mathbb{N}, n$ can be expressed as

$$n = \textstyle\sum_{j=1}^{k} \sigma_j q^{j-1} , \qquad (61)$$

where all $\sigma_j \in \{0,1,\dots,q-1\}$ and $\sigma_k \ne 0$. Let $\{m_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers with $m_1 \ge 2$. We now define a maximal mapping in terms of this sequence by $\tau(\vartheta) = \tau(0^k) = 0$ for $k \ge 1$ and for n as in (61),

$$\tau(\sigma) = \begin{cases} \sigma_k, & if\ \sigma = \sigma_1 \cdots \sigma_k, \sigma_k \ne 0; \\ 0, & if\ \sigma = \sigma_1 \cdots \sigma_k 0^{\ell}, \ell \ne m_n; \\ q, & if\ \sigma = \sigma_1 \cdots \sigma_k 0^{\ell}\ and\ \ell = m_n. \end{cases}$$

By the definition we have $\lambda_0 = 0$ and

$$\lambda_n = \sum_{j=1}^{k} \tau(\sigma_1 \cdots \sigma_j) b^{j-1} + q b^{m_n} ,$$

consequently, $N_n^* = 1$ and by Theorem (1.2.7)(i), $\Lambda := \{\lambda_n\}_{n=0}^{\infty}$ is a spectrum for $L^2(\mu)$.
We now find Λ with density in (48) zero by choosing m_n. To do this, we first note that there exists a strictly increasing continuous function $h(t)$ from $[0,\infty)$ onto itself such that $h(t) \le g(t)$ for $t \ge 0$ and it is sufficient to replace $g(t)$ by $h(t)$ in the proof. In this way, the inverse of $h(t)$ exists, and we denote it by $h^{-1}(t)$.
Now, note that

$$\lambda_n \le q\frac{b^k - 1}{b-1} + q b^{m_n} \le (q+1) b^{m_n}.$$

Hence,

$$\lambda_{n+1} - \lambda_n \ge q b^{m_{n+1}} - (q+1) b^{m_n} \ge b^{m_n+1}. \quad (62)$$

Therefore, we choose m_n so that $b^{m_n} \ge 2h^{-1}(b^{n+1})$ for all $n \ge 1$. For any $h(R) \ge 1$, there exists unique $s \in \mathbb{N}$ such that $b^{s-1} \le h(R) < b^s$. Then

$$\frac{\sup_{x\in\mathbb{R}} \#(\Lambda \cap (x-R, x+R))}{h(R)} \le \frac{\sup_{x\in\mathbb{R}} \#(\Lambda \cap (x-h^{-1}(b^s), x+h^{-1}(b^s)))}{b^{s-1}}.$$

Note from (62) that the length of the open intervals $(x-h^{-1}(b^s), x+h^{-1}(b^s))$ is less than $\lambda_{n+1}-\lambda_n$ whenever $n \geq s$. This implies that the set $\Lambda \cap (x-h^{-1}(b^s), x+h^{-1}(b^s))$ contains at most one λ_n where $n \geq s$. We therefore have

$$\sup_{x\in\mathbb{R}} \#(\Lambda \cap (x-h^{-1}(b^s), x+h^{-1}(b^s))) \leq s+1.$$

Thus the result follows by taking limit.

Let τ be a maximal mapping (not necessarily regular) for $\mu = \mu_{q,b}$ with $b = qr$. Given any $I = \sigma_1 \cdots \sigma_k \in \Sigma_q^k = \{0,1,\dots,q-1\}^k$ with $\sigma_k \neq 0$. Define a map τ' by

$$\tau'(\sigma) = \begin{cases} 0, & \sigma = I0^\ell \ for\ \ell \geq 1; \\ \tau(\sigma), & otherwise. \end{cases}$$

Clearly τ' is a maximal mapping. The main result is as follows:

Theorem (1.2.17) [26]: With the notation above, τ is a spectral mapping if and only if τ' is a spectral mapping.

This result shows that if we arbitrarily change the value of τ along an element in Γ_q as above, the spectral property of τ is unaffected. In particular, Theorem (1.2.10) follows as a corollary because we can alter the irregular elements one by one using Theorem (1.2.17) recursively.

We now prove Theorem (1.2.17). Note that we can decompose

$$\Gamma_q = \{\sigma 0^\infty : \sigma \in \Sigma_q^*\} = \cup_{I\in\Sigma_q^n} I\Gamma_q \tag{63}$$

for all $n \geq 1$. And recall that

$$\Lambda(\tau) = \{\Pi_b^\tau(J) : J \in \Gamma_q, \tau\ is\ regular\ on\ J\},$$

where $\Pi_b^n(J) = \sum_{k=1}^\infty \tau(J|_k) b^{k-1}$. Denote naturally $\Pi_b^\tau(I) = \sum_{k=0}^n \tau(I|_k) b^{k-1}$ if $\in \Sigma_q^n$, and $\Pi_{b,I}^\tau(J) = \sum_{k=1}^\infty \tau(Ij_1\cdots j_k) b^{k-1}$ for $J = j_1 j_2 \cdots \in \Gamma_q$ where IJ is regular for τ. Define also

$$\Lambda_I(\tau) = \{\Pi_{b,I}^\tau(J) : J \in \Gamma_q, \tau\ is\ regular\ on\ J\}.$$

By (63) we have

$$\Lambda(\tau) = \bigcup_{I\in\Sigma_q^n} (\Pi_b^\tau(I) + b^n \Lambda_I(\tau)).$$

The following is a simple lemma which was also observed in [28].

Proposition (1.2.18) [26]: Let τ be a tree mapping and $n \geq 1$. Then $r\Lambda(\tau)$ is a spectrum for μ if and only if all $r\Lambda_I(\tau), I \in \Sigma_q^n$, are spectra.

Proof: Recall that μ_k satisfies $\hat{\mu}(\xi) = \hat{\mu}_k(\xi)\hat{\mu}(b^{-k}\xi)$ and $\hat{\mu}_k(\xi) = \prod_{j=1}^k m(b^{-j}\xi)$ where $m(\xi) = \frac{1}{q}\sum_{j=1}^q e^{2\pi i(j-1)\xi}$. Write $Q_I(\xi) = \sum_{\lambda\in\Lambda_I(\tau)} |\hat{\mu}(\xi+r\lambda)|^2$. Note that

$$\begin{aligned} Q(\xi) &= \sum_{\lambda\in\Lambda(\tau)} |\hat{\mu}(\xi+r\lambda)|^2 \\ &= \sum_{I\in\Sigma_q^n, \lambda\in\Lambda_I(\tau)} |\hat{\mu}_n(\xi+r\Pi_b^\tau(I)+rb^n\lambda)|^2 \left|\hat{\mu}\left(\frac{\xi+r\Pi_b^\tau(I)}{b^n}\right)+r\lambda\right|^2 \\ &= \sum_{I\in\Sigma_q^n, \lambda\in\Lambda_I(\tau)} \left|\hat{\mu}_n\left(\xi+r\Pi_b^\tau(I)\right)\right|^2 \left|\hat{\mu}\left(\frac{\xi+r\Pi_b^\tau(I)}{b^n}\right)+r\lambda\right|^2 \\ &= \sum_{I\in\Sigma_q^n} \left|\hat{\mu}_n\left(\xi+r\Pi_b^\tau(I)\right)\right|^2 \cdot Q_I\left(\frac{\xi+r\Pi_b^\tau(I)}{b^n}\right). \end{aligned}$$

In a similar proof of Lemma (1.2.14), we have

$$1 \equiv \sum_{I\in\Sigma_q^n} \left|\hat{\mu}_n\left(\xi + r\Pi_b^\tau(I)\right)\right|^2.$$

Invoking Lemma (1.2.13), the result follows.

Proposition (1.2.18) asserted that spectral property are determined by a finite number of nodes. The following two lemmas show that the spectral property of a particular node σ can be determined by infinitely many of its offsprings and is independent of the regularity of $\sigma 0^\infty$. These are the key lemmas to the proof of Theorem (1.2.17).

Lemma (1.2.19) [26]: Let $I \in \Sigma_q^*$ with $I \neq \vartheta$, the empty word. If τ is regular on $I0^\infty$, then

$$\Lambda_I(\tau) = \{\Pi_{b,I}^\tau(0^\infty)\} \cup \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{q-1}\left(\Pi_{b,I}^\tau(0^{k-1}j) + b^k\Lambda_{I0^{k-1}j}(\tau)\right),$$

where $\Pi_{b,I}^\tau(0^{k-1}j) = \tau\,(I0) + \tau(I0^2)b + \cdots + \tau\,(I0^{k-1}j)b^{k-1}$. If τ is irregular on $I0^\infty$, then

$$\Lambda_I(\tau) = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{q-1}\left(\Pi_{b,I}^\tau(0^{k-1}j) + b^k\Lambda_{I0^{k-1}j}(\tau)\right).$$

Lemma (1.2.20) [26]: Let τ be a maximal mapping and let $I \in \Sigma_q^*$. Then $\Lambda_I(\tau)$ is a spectrum of μ if and only if $\Lambda_{I0^{k-1}j}(\tau)$ are spectra of μ for all $k \geq 1$ and $j = 1, \cdots, q-1$.

Proof: The necessity is clear from Proposition (1.2.18). We now prove the sufficiency. Assume that $\Lambda_{I0^{k-1}j}(\tau)$ are spectra for all $k \geq 1$ and $j = 1, \cdots, q-1$. We need to show that $Q_I(\xi) = \sum_{\lambda\in\Lambda_I(\tau)} |\hat{\mu}(\xi + r\lambda)|^2 \equiv 1$.

By the integral periodicity of m and Lemma (1.2.13) which will be used in the second equality below, we have for all $k \geq 2$,

$$\sum_{j=1}^{q-1} |\hat{\mu}_k(\xi + r\Pi_{b,I}^\tau(0^{k-1}j)|^2$$

$$= \sum_{j=1}^{q-1} |\hat{\mu}_{k-1}(\xi + r\Pi_{b,I}^\tau(0^{k-1})|^2 \left|\hat{\mu}_1\left(\frac{\xi + r\Pi_{b,I}^\tau(0^{k-1}) + rr(0^{k-1}j)b^{k-1}}{b^k}\right)\right|^2$$

$$= |\hat{\mu}_{k-1}(\xi + r\Pi_{b,I}^\tau(0^{k-1})|^2\left(1 - \left|\hat{\mu}_1\left(\frac{\xi + r\Pi_{b,I}^\tau(0^k)}{b^k}\right)\right|^2\right)$$

$$= |\hat{\mu}_{k-1}(\xi + r\Pi_{b,I}^\tau(0^{k-1})|^2 - |\hat{\mu}_k(\xi + r\Pi_{b,I}^\tau(0^k)|^2.$$

If $k = 1$, the above becomes $\sum_{j=1}^{q-1} |\hat{\mu}_1(\xi + r\Pi_{b,I}^\tau(j)|^2 = 1 - |\hat{\mu}_1(\xi + r\Pi_{b,I}^\tau(0)|^2$. Now we simplify the following terms which is corresponding to the unions of the sets in Lemma (1.2.19),

$$\sum_{k=1}^{\infty}\sum_{j=1}^{q-1}\sum_{\lambda\in\Lambda_{\sigma 0^{k-1}j}(\tau)} \left|\hat{\mu}\left(\xi + r\Pi_{b,I}^\tau(0^{k-1}j) + rb^k\lambda\right)\right|^2$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{q-1}\sum_{\lambda\in\Lambda_{\sigma 0^{k-1}j}(\tau)} \left|\hat{\mu}_k\left(\xi + r\Pi_{b,I}^\tau(0^{k-1}j)\right)\right|^2 \left|\hat{\mu}_1\left(\frac{\xi + r\Pi_{b,I}^\tau(0^{k-1}j)}{b^k} + \lambda\right)\right|^2$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{q-1} \left|\hat{\mu}_k\left(\xi + r\Pi_{b,I}^\tau(0^{k-1}j)\right)\right|^2$$

$$= 1 - \lim_{N\to\infty} \left|\hat{\mu}_N\left(\xi + r\Pi^{\tau}_{b,I}(0^N)\right)\right|^2$$
$$= 1 - \prod_{j=1}^{\infty} \left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^j)}{b^j}\right)\right|^2. \quad (64)$$

We now divide the proof into two cases.

Case (i). If $I0^{\infty}$ is regular, then $\left|\hat{\mu}\left(\xi + r\Pi^{\tau}_{b,I}(0^{\infty})\right)\right|^2 = \prod_{j=1}^{\infty}\left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^j)}{b^j}\right)\right|^2$. Hence, by Lemma (1.2.19),

$$Q_I(\xi) = \left|\hat{\mu}\left(\xi + r\Pi^{\tau}_{b,I}(0^{\infty})\right)\right|^2 + (64) \equiv 1.$$

This shows $r\Lambda_I(\tau)$ is a spectrum.

Case (ii). If $I0^{\infty}$ is irregular, then Lemma (1.2.19) shows that

$$Q_I(\xi) = (64) = 1 - \prod_{j=1}^{\infty}\left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^j)}{b^j}\right)\right|^2.$$

Note that, by (ii) in the definition of the maximal mapping, we may write

$$r\Pi_{b,I}(0^n) = r\sum_{j=1}^{n}\tau\left(I0^j\right)b^{j-1} = r\sum_{j=1}^{n}(s_j q)b^{j-1} = \sum_{j=1}^{n} s_j b^j \quad (65)$$

for $s_j \in \{0, 1, \cdots, r-1\}$. Then

$$Q_I(\xi) = 1 - \prod_{j=1}^{\infty}\left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^{j-1})}{b^j}\right)\right|^2. \quad (66)$$

Suppose on the contrary $Q_I(\xi) < 1$ for some $\xi > 0$. Since Q_I is entire, we may assume ξ is small, say $|\xi| < \frac{r-1}{b-1}$. From (66), we must have $\prod_{j=1}^{\infty}\left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^{j-1})}{b^j}\right)\right|^2 > 0$.

For those n such that $s_n \neq 0$ in (65).

$$\frac{1}{b} \le \frac{s_n}{b} \le \left|\frac{\xi + \sum_{j=1}^{n} s_j b^j}{b^{n+1}}\right| \le \frac{r-1}{b(b-1)} < \frac{1}{q(b-1)}.$$

Hence, letting $c = \max\{|m(\xi)|^2 : \frac{1}{b} \le |\xi| < \frac{1}{q(b-1)}\} < 1$, we have

$$\left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^n)}{b^{n+1}}\right)\right|^2 = \left|m\left(\frac{\xi + \sum_{j=1}^{n} s_j b^j}{b^{n+1}}\right)\right| \le c.$$

And

$$\prod_{j=1}^{\infty}\left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^{j-1})}{b^j}\right)\right|^2 = \lim_{N\to\infty}\prod_{j=1}^{N}\left|m\left(\frac{\xi + r\Pi^{\tau}_{b,I}(0^{j-1})}{b^j}\right)\right|^2 \le \lim_{N\to\infty} c^{\#\{n : s_n \neq 0, n \le N\}}.$$

As $I0^{\infty}$ is irregular, there exists infinitely many $s_j \neq 0$. The above limit is zero. This is a contradiction and hence $\Lambda_I(\tau)$ must be a spectrum.

By the definition of τ and τ', $\Lambda_{I'}(\tau) = \Lambda_{I'}(\tau')$ for all $I' \neq I$ and $I' \in \Sigma_q^k$. Moreover, $\Lambda_{I0^{n-1}j}(\tau) = \Lambda_{I0^{k-1}j}(\tau')$ for all $k \ge 1$ and $j = 1, \cdots q-1$. Therefore, if τ is a spectrum, then $\Lambda_{I'}(\tau')$ are spectra of μ for all $\sigma' \neq \sigma$ and $\sigma' \in \Sigma_q^k$ by Proposition (1.2.18). On the other hand, $\Lambda_{I0^{k-1}j}(\tau')$ are spectra of μ also as τ is a spectrum. By Lemma (1.2.20), $\Lambda_I(\tau')$ is also a spectrum. We therefore conclude that $\Lambda(\tau')$ is a spectrum of μ by Proposition (1.2.18) again. The converse also holds by reversing the role of τ and τ'. This completes the whole proof.

Section (1.3): Digit Sets of Self-Similar Tiles on R^1

For A be an $s \times s$ expanding matrix (i.e., all eigenvalues have moduli > 1) with integral entries and $|detA| = b$. Also let $D \subset \mathbb{Z}^s$ with $\#D = b$; we call D a digit set. Consider the iterated function system (IFS) defined by $\{S_j\}_{j=0}^{m-1}$, where

$$S_j(x) = A^{-1}(x + d_j), \quad 0 \le j \le m-1. \tag{67}$$

Then these affine maps are contractive under certain norms in $\mathbb{R}^s$, and it is well known that there exists a unique compact set $T := T(A, D) \subset \mathbb{R}^s$ satisfying the set-valued relation $AT = T + D$. Alternatively, $T(A, D)$ can be expressed as a set of radix expansions with base A and digits in D:

$$T(A, D) = \left\{ \sum_{k=1}^{\infty} A^{-k} d_k : d_k \in D \right\}. \tag{68}$$

In [57], Bandt proved that if $|detA| = b = \#D$ and $T^\circ \neq \emptyset$, then $T(A, D)$ is a translational tile. This means that there exists $J \subset \mathbb{R}^s$ such that $\bigcup_{t \in J}(T + t) = \mathbb{R}^s$, and the Lebesgue measure of $(T + t) \cap (T + t')$ is zero for all $t, t' \in J, t \neq t'$. In that case we call T an (integral) self-affine tile, and a self-similar tile if the matrix A is, in addition, a scalar times an orthonormal matrix. The digit set D is called a (self-affine) tile digit set with respect to A, and J is called a tiling set for T.

The study of self-affine tiles and the tiling theory was initiated by Thurston [56] and Kenyon [51], and the foundation and basic properties were laid down in a series by Lagarias and Wang ([51], [52], [53], [54]). By now there is a wealth of literature on this subject, with topics including the tiling theory, the geometric and fractal structures, the topological properties, the classification problems and applications to wavelet theory ([52], [54]). The related topic of the Fuglede's problem on tiles and spectral sets has also received a lot of attention recently ([52], [53], [59], [60]). However, despite such intensive studies, many of these aspects are still not fully understood. Among them is the following basic question:

(Q) Given an expanding integral matrix A, can we classify all the tile digit sets D, and what is the structure of such a D?

The question turns out to be rather intriguing and challenging; it is not clear even in $\mathbb{R}^1$. A well-known sufficient condition (it is also true in any dimension) for D to be a tile digit set is: D is a complete residue set $(mod A)$ [52] (which is called a standard tile digit set [53]). This condition is also necessary in $\mathbb{R}^1$ if $b = p$ is a prime [55]. It is also true in higher dimensions when an additional hypothesis is assumed ([53], [52]).

The situation is far more complicated if b is not a prime number. For example in $\mathbb{R}^1$ if $A = [b]$ with $b = 4$ and $D = \{0, 1, 8, 9\}$, then D is not a complete residue $(mod\ 4)$, but $T(4, D) = [0, 1] \cup [2, 3]$ is a tile with a tiling set $J = \{0, 1\} \oplus 4\mathbb{Z}$. In reference to the work of Odlyzko [56], Lagarias and Wang proposed a class of digit sets in $\mathbb{R}^s$ called a product-form [63] as a generalization of the complete residue class. They also showed that for $b = p^\ell$ a prime power, then a tile digit set D of b must be a "product-form-like" set. More recently, Lau and Rao introduced a weak product-form and used this idea to classify all tile digit sets $D(\subset \mathbb{Z})$ for $b = pq$, a product of two primes [60].

, we only focus on self-similar tiles in $\mathbb{R}^1$. We assume that $b > 1, A = [b]$ and $\#D = b$. We also assume, without loss of generality, that $0 \in D \subset \mathbb{Z}^+$ and $g.c.d.(D) = 1$. We first recall that a cyclotomic polynomial $\Phi_d(x)$ is theminimal polynomial of $e^{2\pi i/d}$. It is easy to see that a product-form can be expressed into a product of cyclotomic polynomials. Our main result is to obtain a characterization of the tile digit sets as a product of cyclotomic polynomials. We call

$$P_D(x) = \sum_{d\in D} x^d = 1 + x^{d_1} + \cdots + x^{d_{b-1}}$$

a mask polynomial of D. A well-known necessary and sufficient condition for $T(b, D)$ to be a self-similar tile, due to Kenyon [58], is that the mask polynomial satisfies: for any $m > 0$, there exists k such that

$$P_D(e^{2\pi i m/b^k}) = 0.$$

The Kenyon criterion shows a close link of the zero set of the mask polynomials with the cyclotomic polynomial. Note that $P_D(e^{2\pi i/d}) = 0$ if and only if $\Phi_d(x)|P_D(x)$.

We call the set $\{d > 1: \Phi_d(x)|P_D(x)\}$ the spectrum of D, and

$$S_D = \{p^\alpha > 1: p\ prime, \Phi_{p^\alpha}(x)|P_D(x)\}$$

the prime-power spectrum of D. They play an important role in the consideration.

The spectrum was used extensively by Coven and Meyerowitz [56] in their study of integer tiles A (i.e., $\#A$ is finite, and there exists $\mathcal{L}$ such that $A \oplus \mathcal{L} = \mathbb{Z}$). They gave an inductive characterization of such an A when $\#A = p^\alpha q^\beta, p, q$ are primes. Later, Laba and Wang [56], [56] used the spectrum to investigate the spectral sets and spectral measures. In self-similar tile digit sets D, we showed that D is an integer tile (see [54]). We prove the following theorem, which characterizes the prime-power spectrum of D..

Theorem (1.3.1)[50]: Let $b = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be a product of prime powers and let D be a tile digit set of b. Then

$$S_D = \bigcup_{j=1}^{k} S_{p_j},$$

where $S_{p_j} = \{p_j^{a_{j,1}}, \ldots, p_j^{a_{j,\alpha_j}}\}$ and $\{a_{j,1}, \ldots, a_{j,\alpha_j}\}$ is a complete residue set modulo α_i.

It is well known that if $D = \{0, 1, \ldots, b-1\}$, then the mask polynomial

$$P_D(x) = 1 + x\ + \cdots + x^{b-1} = \prod_{d|b, d>1} \Phi_d(x)\,;$$

if D is a complete residue set modulo b, then $P_D(x) = \left(\prod_{d|b,d>1} \Phi_d(x)\right)Q(x)$ for some integral polynomial $Q(x)$. If D is a product-form, i.e. $D = \mathcal{E}_0 \oplus b^{\ell_1}\mathcal{E}_1 \oplus \cdots \oplus b^{\ell_k}\mathcal{E}_k$, where $\mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b$, we can see that D satisfies

(P_1): for any factor $d(> 1)$ of b, there exists $j(d) \geq 0$ such that $\Phi_d(x^{b^j})|P_D(x)$.

In fact, if D is a tile digit set that satisfies (P_1), we can write the mask polynomial as

$$P_D(x) = \left(\prod_{d|b,d>1} \Phi_d\left(x^{b^{j(d)}}\right)\right) Q(x) \tag{69}$$

for some integral polynomial $Q(x)$. It is also easy to check that if D is a weak product-form [56], then D satisfies condition (P_1) .. Actually, condition (P_1) is satisfied for the more general modulo product-form, which is defined through the product-form by allowing certain modulo action of each factor .

On the other hand, such a simple expression of P_D in (69) is not sufficient to cover all tile digit sets. For this we introduce the notion of $(P_k), k \geq 1$ and, correspondingly, the more concrete k^{th}-order modulo product-forms . The case $k = 1$ corresponds to the ordinary product-forms. We have the following theorem:

Theorem (1.3.2) [50]: If D is a k^{th}-order modulo product-forms, then D satisfies the condition (P_k) and it is a tile digit set.

A second-order product-form can be constructed as follows: we take a modulo product-form and rearrange the digits to form a product, then use this to construct a new product-form. By doing so, we are able to construct new tile digit sets that are not known.

By using a device of Protasov [56], we can represent the mask polynomial by a "tree" of cyclotomic polynomials (a Φ-tree) with respect to b, and a blocking N of the tree: a finite subset such that every path on the tree meets one and only one element of N the main theorem is:

Theorem (1.3.3) [50]: Let D be a set of non-negative integers with $\#D = b$. Then the following are equivalent:

(i) D is a tile digit set;

(ii) there is a blocking N in the Φ-tree such that

$$P_D(x) = \left(\prod_{\Phi_d \in N} \Phi_d(x) \right) Q(x);$$

(iii) $P_D(x)$ satisfies condition (P_k) for some $k \geq 1$.

We call the product in (ii) a kernel polynomial of D. It plays a central role on the structure of the tile digit sets. For $b = pq$ with the prime-power spectrum equal to $\{p, q^\ell\}$, the digit set D is a (1^{st}-order) modulo product-form, and the kernel polynomial is of the form $\Phi_p(x)\Phi_{q^\ell}(x^{p^\ell})$. This was proved in [89] (an errata in the statement of Theorem 1.5 and 5.1 in [58]: $\mathcal{E}_0$ should be $\{0, 1, \ldots, p-1\}$ $(mod p)$; it was shown in the proof there). For the more general case where $b = p^\alpha q$, we determine, in a separate [56] all the kernel polynomials that generate tile digit sets and we also show that all tile digit sets are of modulo product-forms up to some order depending on the prime-power spectrum.

we first provide some basic properties of the cyclotomic polynomials and integer tilings; we also bring in the Kenyon criterion to study the mask polynomial P_D and prove Theorem (1.3.1). we consider the condition (P_1) and the various kinds of product-forms, and we set up the (P_k) condition and the higher-order product-form and prove Theorem (1.3.2). we make use of Protasov's device to study the Φ-tree and prove Theorem (1.3.3). We conclude with some remarks.

We give a brief summary on the cyclotomic polynomials which is needed for later discussions. We use $\Phi_d(x)$ to denote the d-th cyclotomic polynomial.

It is the minimal polynomial of the primitive d-th root of unity, i.e., $\Phi_d(e^{2\pi i/d}) = 0$.

Note that the degree of $\Phi_d(x)$ is equal to the Euler-phi function $\varphi(d)$ (the number of relatively prime integers in $1, \ldots, d-1$). It is well known that

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{70}$$

The formula provides a constructive way to find Φ_d inductively. The following basic properties of cyclotomic polynomials will be used.

Proposition (1.3.4) [50]: The cyclotomic polynomials satisfy the following:

(i) If p is a prime, then $\Phi_p(x) = 1 + x + \ldots + x^{p-1}$ and $\Phi_{p^{\alpha+1}}(x) = \Phi_p(x^{p^\alpha})$;

(ii) $\Phi_s(x^p) = \Phi_{sp}(x)$ if p is prime and $p|s$, and $\Phi_s(x^p) = \Phi_s(x)\Phi_{sp}(x)$ if p is prime but $p \nmid s$;

(iii) $\Phi_s(1) = \begin{cases} 0, & if\ s = 1; \\ p, & if\ s = p^\alpha; \\ 1, & otherwise. \end{cases}$

Proposition (1.3.5) [50]: Let $b \geq 2$ be an integer. Then for any two distinct factors d_1, d_2 $(\neq 1)$ of b, and for any integers $u_1, u_2 \geq 0$, $\Phi_{d_1}(x^{b^{u_1}})$ and $\Phi_{d_2}(x^{b^{u_2}})$ have no common factor.

Proof: Note that if $f(x)$ and $g(x)$ have no common divisor, then so do $f(x^n)$ and

$g(x^n)$ (use $a(x)f(x)+b(x)g(x)=1$ for some polynomials $a(x),b(x)$). Hence, we only need to show that $\Phi_{d_1}(x)$ and $\Phi_{d_2}(x^{b^{u_2-u_1}})$ (assuming $u_1 \le u_2$) have no common divisor. This is simple as $e^{2\pi i/d_1}$ is not a root of the polynomial $\Phi_{d_2}(x^{b^k})$ for all $k \ge 0$.

Let $\mathbb{Z}^+$ be the set of non-negative integers. For $A \subset \mathbb{Z}^+$, we let

$$P_A(x) = \sum_{a \in A} x^a$$

and call it the mask polynomial of A. For a finite set $A \subset \mathbb{Z}^+$, we use

$$S_A = \{p^\alpha > 1 : p\ prime, \Phi_{p^\alpha}(x) | P_A(x)\} \tag{71}$$

to denote the prime-power spectrum of A, and $\bar{S}_A = \{s > 1 : \Phi_s(x) | P_A(x)\}$ the spectrum of A.

We call a finite set $A \subset \mathbb{Z}$ an integer tile if there exists $\mathcal{L}$ such that $A \oplus \mathcal{L} = \mathbb{Z}$.

The class of integer tiles on $\mathbb{Z}$ has been studied in depth in connection with the factorization of cyclic groups and cyclotomic polynomials ([52], [56]). In [58],

Coven and Meyerowitz introduced the following two conditions to study the integer tiles:

(T1) $\#A = P_A(1) = \sum_{s \in S_A} \Phi_s(1)$.

(T2) For any distinct prime powers $s_1, \dots, s_n \in S_A$, then $s_1 \dots s_n \in \bar{S}_A$.

They showed that if $A \subset \mathbb{Z}^+$ is a finite set, and suppose (T1) and (T2) hold, then A tiles $\mathbb{Z}$ with period $n = l.c.m.(S_A)$. Conversely, if A is an integer tile, then (T1) holds; if in addition $\#A = p^\alpha q^\beta, \alpha, \beta \ge 0$, then (T2) holds. It is still an open question whether an integer tile must satisfy (T2) in general. The class of integer tiles and the (self-affine) tile digit sets are closely related. In fact it is shown in [58] that all tile digit sets (in any dimension) are integer tiles.

In the following, we start to consider the tile digit sets of b in $\mathbb{R}$. We assume that $D \subset \mathbb{Z}^+, 0 \in D$, and $g.c.d.(D) = 1$. For $T := T(b,D)$, we let χ_T denote the characteristic function of T. The self-similar identity $bT = T + D$ can be expressed as a refinement equation

$$\chi_T(x) = \sum_{d \in D} \chi_T(bx - d), for\ a.e.\ x \in \mathbb{R},$$

and the Fourier transform is $\hat{\chi}_T(\xi) = P_D(e^{2\pi i \xi / b^k})$. By using the Riemann- Lebesgue lemma, it is not difficult to derive from the product that for χ_T to be an L^1-function (i.e., T is a self-similar tile since it has positive Lebesgue measure), then $\hat{\chi}_T(m) = 0$ for any integer $m \neq 0$, so that one of the factors is 0. The converse can be proved by considering a certain tempered distribution (see, e.g., [59]). This is the basic idea of the following criterion due to Kenyon (actually holds in $\mathbb{R}^s$).

Theorem (1.3.6) [50]: ([56]). $T(b,D)$ is a self-similar tile if and only if for each integer $m > 0$, there exists $k \ge 1$ (depending on m) such that

$$P_D(e^{2\pi i m / b^k}) = 0.$$

The factorization of $P_D(x)$ is closely connected with the cyclotomic polynomials.

By using the Kenyon criterion, we prove the following theorem, which characterizes the prime power spectrum of D.

Theorem (1.3.7) [50]: Let $b = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the product of prime powers and let D be a tile digit set of b. Then for each p_j, and $0 \le \ell \le \alpha_j - 1$, there exists a unique $a_{j,\ell} \equiv \ell\ (mod\ \alpha_j)$ such that $\Phi_{p_j^{a_{j,\ell}}}(x)$ divides $P_D(x)$, and the prime power spectrum of D is

$$S_D = \bigcup_{j=1}^{k} \{p_j^{a_{j,1}}, \ldots, p_j^{a_{j,\alpha_j}}\}.$$

Proof: We prove the theorem for p_1. For simplicity, we write $b = p_1^{\alpha_1} t$ and $e(x) = e^{2\pi i x}$. For a fixed $0 \le \ell \le \alpha_j - 1$ and for each $n \ge 1$, we set $a_n = p_1^{\alpha_1 - \ell}\, t^n$. By the Kenyon criterion, for any n, there exists $m = m(n)$ such that $e(a_n/b^m)$ is a root of $P_D(x)$.

We claim that there exists n such that $n \ge m(n)$. Suppose otherwise; for any $n \ge 1$, we have $n < m(n)$. Let $u_n = p_1^{\ell+(m-1)\alpha_1} t^{m-n}$. Then $e(a_n/b^m) = e(1/u_n)$ are roots of $P_D(x)$. Note that $u_n > p_1^{\ell+(m-1)\alpha_1}$; hence u_n tends to infinity as n tends to infinity (since $n < m(n) := m$). This means that u_n contains infinitely many distinct numbers, which is a contradiction since $P_D(x)$ is a finite degree polynomial.

Hence, we can find n such that $n \ge m(n)$ and

$$e(a_n/b^m) = e(t^{n-m}/p_1^{\ell+(m-1)\alpha_1})$$

is a root of $P_D(x)$. Letting $a_{1,\ell} = \ell + (m-1)\alpha_1$, we have $\Phi_{p_1^{a_{1,\ell}}}(x) | P_D(x)$. This shows the existence part in the theorem.

For the uniqueness, we note that we have $P_D(1) = \#D = b$. On the other hand, each $\Phi_{p_j^{a_{j,\ell}}}(1) = p_j$ (by Proposition (1.3.4)), and the product of all these $\Phi_{p_j^{a_{j,\ell}}}(1)$ is $p_1^{\alpha_1} \cdots p_k^{\alpha_k} = b$. Moreover, $\Phi_{p_j^{a_{j,\ell}}}(x)$ divides $P_D(x)$. It follows that the set of all these $p_j^{a_{j,\ell}}$ is precisely the prime power spectrum S_D and the uniqueness follows.

It follows from (70) that if D is a complete residue set modulo b (standard digit set [56]), then $P_D(x) = \left(\prod_{d|b, d>1} \Phi_d(x)\right) Q(x)$ for some integral polynomial $Q(x)$. We see that the S_D in Theorem (1.3.7) is simply the set p_j^i, with $i = 1, \ldots, \alpha_i$.

The theorem implies that if we change to the other tile digit sets of b, the prime power factors are preserved in a modulo way.

Example (1.3.8) [50]: Let $D = \{0, 1, 8, 9\}$ and $b = 4 = 2^2$. It is known that $T(b, D) = [0, 1] \cup [2, 3]$ is a tile. Its mask polynomial is

$$P_D(x) = 1 + x + x^8 + x^9 = \Phi_2(x)\Phi_{2^4}(x).$$

We see that $\{1, 4\}$ is a complete residue $(mod\ 2)$.

If $A = \{0, 1, 4, 5\}$ (it is an integer tile), we have

$$P_A(x) = 1 + x\ + x^4 + x^5 = \Phi_2(x)\Phi_{2^3}(x).$$

Note that $\{1, 3\}$ is not a complete residue $(mod\ 2)$; this shows that A is not a tile digit set by Theorem (1.3.7).

We denote $\mathbb{Z}_b$ to be the complete residue class of integers $(mod\ b)$. It is known that if b is a prime, then D is a tile digit set if and only if $D \equiv \mathbb{Z}_b$ [56]. But when b is not a prime, the problem is vastly more complicated. The following classes of D are some of the basic tile digit sets that are known.

Definition (1.3.9) [50]: D is called a product-form digit set (with respect to b) [55] if

$$D = \mathcal{E}_0 \oplus b^{l_1}\mathcal{E}_1 \oplus \ldots \oplus b^{l_k}\mathcal{E}_k, \quad (72)$$

where $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_k \equiv \mathbb{Z}_b$, and $0 \le l_1 \le l_2 \le \ldots \le l_k$; if $\mathcal{E} = \{0, 1, 2, \ldots, b-1\}$, then D is called a strict product-form [O].

Furthermore if $D' \equiv D (mod\ b^{l_k+1})$ for D in (72), then D' is called a weak product-form [58].

In terms of the mask polynomials, a product-form can be expressed as

$$P_D(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}\left(x^{b^{l_1}}\right)\dots P_{\mathcal{E}_k}\left(x^{b^{l_k}}\right), \quad (73)$$

where

$$P_{\mathcal{E}}(x) = P_{\mathcal{E}_0}(x)\dots P_{\mathcal{E}_k}(x) \equiv 1 + x + \dots + x^{b-1}(mod\ x^b - 1). \quad (74)$$

For the weak product-form,

$$P_D(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}\left(x^{b^{l_1}}\right)\dots P_{\mathcal{E}_k}\left(x^{b^{l_k}}\right) + \left(x^{b^{l_k+1}} - 1\right)Q(x) \quad (75)$$

for some integral polynomial $Q(x)$.

In view of the expressions in (73)-(75), we introduce a new condition on D:

(**P**1) for any $d > 1$ and $d|b$, there exists $j \geq 0$ (depends on d) such that $\Phi_d(x^{b^j})|P_D(x)$.

We also say that a polynomial $P(x)$ satisfies (P_1) if the above holds with $P_D(x)$ replaced by $P(x)$.

Theorem (1.3.10) [50]: Suppose D satisfies condition (P_1). Then D is a tile digit set, and

$$P_D(x) = \prod_{d|b,d>1}\Phi_d\left(x^{b^{j(d)}}\right)Q(x), \quad (76)$$

where $j(d)$ is an integer depending on d.

Proof: Letting $m > 0$, we claim that there exists $d|b, d > 1$ such that $\Phi_d(e^{2\pi im/b^k}) = 0$. Consider the standard tile digit set $\mathcal{E} = \{0, 1, \dots, b-1\}$. By Theorem (1.3.6), there exists $k \geq 1$ such that

$$P_{\mathcal{E}}(e^{2\pi im/b^k}) = 0.$$

Let $a = g.c.d.(m, b^k)$ and let $s = m/a, \ell = b^k/a$; then $s, \ell(> 1)$ are relatively prime and $P_{\mathcal{E}}(e^{2\pi is/\ell}) = 0$. It follows that

$$\Phi_\ell(x)|P_{\mathcal{E}}(x). \quad (77)$$

As $P_{\mathcal{E}}(x) = \prod_{d|b,d>1}\Phi_d(x)$, we have $\ell = d$ for some $d > 1$ and the claim follows.

By condition (P_1), there exists $j := j(d) \geq 0$ such that

$$P_D(x) = \Phi_d\left(x^{b^j}\right)Q_d(x) \quad (78)$$

for some polynomial $Q_d(x)$. Hence we have

$$P_D(e^{2\pi im/b^{k+j}}) = \Phi_d(e^{2\pi im/b^k})Q_d(e^{2\pi im/b^{k+j}}) = 0.$$

This implies that D satisfies the Kenyon criterion and hence it is a tile digit set. For the expression in (76), we only need to use (78) and observe that those $\Phi_d(x^{b^{j(d)}})$'s are distinct factors of $P_D(x)$ (by Proposition (1.3.5)).

Corollary (1.3.11) [50]: A product-form (or a weak product-form) digit set is a tile digit set.

Proof: It follows from (74) that

$$P_{\mathcal{E}_0}(x)\dots P_{\mathcal{E}_k}(x) = P_{\mathcal{E}}(x) = 1 + x + \dots + x^{b-1} + (x^b - 1)Q(x)$$

for some polynomial $Q(x)$. This implies that for any $d|b$ and $d > 1, \Phi_d(x)|P_{\mathcal{E}_i}(x)$ for some $i = 0, \dots, k$. Hence $\Phi_d(x^{b^{l_i}})|P_{\mathcal{E}_i}(x^{b^{l_i}})$. On the other hand, by (73),

$$P_D(x) = P_{\mathcal{E}_0}(x)P_{\mathcal{E}_1}(x^{b^{l_1}})\dots P_{\mathcal{E}_k}(x^{b^{l_k}}),$$

we obtain $\Phi_d(x^{b^{l_i}})|P_D(x)$, and hence D is a tile digit .

For the weak product-form, we need to observe that the last factor in (75) is divisible by $\Phi_d(x^{b^{l_i}})$.

We now introduce a more general kind of digit set that satisfies (P_1). Consider the product-form D in Definition (1.3.9). We define a decomposition of the spectrum: let $S_i = \{d > 1: d|b, \Phi_d(x)|P_{\mathcal{E}_i}(x)\}$ and let

$$\Psi_i(x) = \prod_{d\in S_i}\Phi_d(x). \quad (79)$$

Then $\Psi_i(x)|P_{\mathcal{E}_i}(x)$; hence $\Psi_i(x^{b^{l_i}})|P_{\mathcal{E}_i}(x^{b^{l_i}})$. Let

$$K_D^{(i)}(x) = \Psi_0(x)\Psi_1\left(x^{b^{l_1}}\right)\ldots\Psi_i\left(x^{b^{l_i}}\right), \quad 0 \le i \le k\,, \tag{80}$$

and denote $K_D^{(k)}(x)$ by $K_D(x)$. Then $K_D(x)|P_D(x)$. Moreover, $K_D(x)$ satisfies (P_1). This is because for any $d > 1$ a factor of b, we have $d \in S_i$ for some i; hence $\Phi_d(x^{b^{l_i}})|\Psi_i(x^{b^{l_i}})$ so that $\Phi_d(x^{b^{l_i}})|K_D(x)$ also.

Definition (1.3.12[50]): We say that D is a modulo product-form if $D := D^{(k)}$ is defined through a product-form $D' = \mathcal{E}_0 \oplus b^{l_1}\mathcal{E}_1 \oplus \ldots \oplus b^{l_k}\mathcal{E}_k, 0 \le l_1 \le \ldots \le l_k$ as follows: let

$$n_i = l.c.m.\{s\colon \Phi_s(x)|\, K_{D'}^{(i)}(x)\}$$

and

$$\begin{cases} D^{(0)} \equiv \mathcal{E}_0 (mod\ n0), \\ \vdots \\ D^{(i)} \equiv D^{(i-1)} \oplus b^{l_i}\mathcal{E}_i (mod\ n_i), \\ \vdots \\ D^{(k)} \equiv D^{(k-1)} \oplus b^{l_k}\mathcal{E}_k (mod\ n_k). \end{cases} \tag{81}$$

Theorem (1.3.13) [50]: Let D be a modulo product-form derived from the product form D'. Then the polynomial $K_{D'}(x)$ in (80) divides $P_D(x)$; hence D satisfies condition (P_1) and is a tile digit set.

Proof: For each $i = 0, \ldots, k$, by the definition of $n_i, x^{n_i} - 1 = \prod_{d|n_i}\Phi_d(x)$ and $\Psi_i(x^{b^{l_i}})$ is a product of cyclotomic polynomials, we have $\Psi_i(x^{b^{l_i}})|(x^{n_i} - 1)$. Moreover, we know $\Psi_i(x^{b^{l_i}})|P_{\mathcal{E}_i}(x^{b^{l_i}})$. Hence in view of

$$P_{D^{(i+1)}}(x) = P_{D^{(i)}}(x)P_{\mathcal{E}_i}(x^{b^{l_i}}) + (x^{n_i} - 1)Q_{i+1}(x),$$

we conclude that $\Psi_d(x^{b^{l_i}})|P_{D^{(i+1)}}(x)$.

Note that the definition of n_{i+1} implies that $\Psi_i(x^{b^{l_i}})|(x^{n_i+1} - 1)$; this together with the expression of $P_{D^{(i+2)}}(x)$ implies that $\Psi_i(x^{b^{l_i}})|P_{D^{(i+2)}}(x)$. Continuing the process, we obtain $\Psi_i(x^{b^{l_i}})|P_D^{(k)}(x)$. This shows that $K_{D'} = K_{D'}^{(k)}$ in (80) divides P_D. As $K_{D'}$ satisfies (P_1), we have that $D = D^{(k)}$ satisfies (P_1) also, and D is a tile digit set following from Theorem (1.3.10).

We end by giving an example to illustrate the construction of a modulo product-form digit set.

Example (1.3.14) [50]: Let $b = 12$ and write $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2 = \{0,1\} \oplus \{0,4,8\} \oplus \{0,2\}$. Let

$$D' = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus 12\mathcal{E}_2.$$

Then D' is a (strict) product-form digit set. Let $D^{(0)} = \{0,1\}$; then $n_0 = 2$. To construct $D^{(1)}$, we first evaluate n_1. Since $1 + x^4 + x^8 = \Phi_3(x^4)$, we have

$$K_{D'}^{(1)}(x) = \Psi_0(x)\Psi_1(x) = \Phi_2(x)\Phi_3(x^4) = \Phi_2(x)\Phi_3(x)\Phi_6(x)\Phi_{12}(x).$$

Hence $n_1 = 12$. We can choose

$$D^{(1)} = \{0,1,4,8,9,17\} \equiv D^{(0)} \oplus \mathcal{E}_1\ (mod\ 12).$$

Next we observe that $K_{D'}^{(2)}(x) = \Psi_0(x)\Psi_1(x)\Psi_2(x^{12})$ is given by

$$\Phi_2(x)\Phi_3(x^4)\Phi_{2^2}(x^{12}) = \Phi_2(x)\Phi_3(x)\Phi_6(x)\Phi_{12}(x)\Phi_{16}(x)\Phi_{48}(x).$$

This implies that $n_2 = 48$. We let

$$D = \{0,1,4,8,9,17,25,33,41,72,76,80\} \equiv D^{(1)} \oplus 12\mathcal{E}_2 (mod\ 48).$$

Then D is a modulo product-form.

On the other hand, D is not a product-form. For if so, then D can only be of the form $\mathcal{E}_0 \oplus 12\mathcal{E}_1$ with $\mathcal{E}_0 \oplus \mathcal{E}_1 \equiv \mathbb{Z}_{12}$. It is direct to check that $\mathcal{E}_0 = \{0,1,4,8,9,17\}$ necessarily. But then it cannot be written in the product-form as needed.

It is clear that D is not a weak product-form either, since taking modulo 12^n with $n \geq 2$ reduces back to the original set which is not a product-form, while taking modulo 12 contains only 6 digits.

The condition (P_1) and the modulo product-form in the do not cover all tile digit sets (see Example (1.3.18)). In this regard, we set up a higher-order analog, which will be studied in detail in the following.

Let $\Phi_d(x^{b^j})$ be as in the definition of (P_1). Then by Proposition (1.3.4)(ii),

$$\Phi_d(x^{b^j}) = \Phi_{t_1}(x) \dots \Phi_{t_n}(x),$$

and each $\Phi_{t_j}(x)$ is a factor of $P_D(x)$. We observe that a more relaxed condition that $\Phi_{t_j}(x^{b^{i_j}})$ is a factor of $P_D(x)$ also suffices for the Kenyon criterion to hold. We formulate this in the following. For $d|b$, let

$$j_1 = j_1(d) := \min\{j: \exists\ a\ factor\ \Phi_t(x) of\ \Phi_d(x^{b^j}) s.t. \Phi_t(x) | P_D(x)\} \quad (82)$$

(and $j_1 = \infty$ if no factor $\Phi_t(x)$ exists). Define

($\boldsymbol{P_2}$) For each $d|b, d > 1, j_1(d) < \infty$ and for any factor $\Phi_{t_1}(x)$ of $\Phi_d(x^{b^{j_1}})$, there exists $j_2 \geq 0$ (depends on t_1) with $\Phi_{t_1}(x^{b^{j_2}}) | P_D(x)$.

Likewise we can repeat the same procedure of defining $j_2 = j_2(t_1)$ and find j_3 for the factors of $\Phi_{t_1}(x^{b^{j_2}})$ to define (P_3), and inductively for $(P_k), k \geq 3$.

It is clear that $(P_1) \Rightarrow (P_2)$ by putting $j_1 = j$ and $j_2 = 0$. Also $(P_{k-1}) \Rightarrow (P_k)$.

Proposition(1.3.15) [50]: Suppose D satisfies (P_k). Then D is a tile digit set.

Proof: We will check for the case (P_2) that the Kenyon criterion is fulfilled; the general case follows from the same idea. As in Theorem (1.3.10), for $m > 0$, there exists $\ell_1 > 1$ with $\ell_1 | b$ and $\Phi_{\ell_1}(e^{2\pi i m/b^k}) = 0$. Taking j_1 as in the assumption of (P_2), note that

$$\Phi_{\ell_1}((e^{2\pi i m/b^{j_1+k}})^{b^{j_1}}) = \Phi_{\ell_1}(e^{2\pi i m/b^k}) = 0.$$

Let $a = g.c.d.(m, b^{j_1+k})$ and let $\ell_2 = b^{j_1+k}/a, c = m/a$; they are relatively prime.

From the above, we have $\Phi_{\ell_1}((e^{2\pi i c/l_2})^{b^{j_1}}) = 0$, which implies that $\Phi_{\ell_2}(x) | \Phi_{l_1}(x^{b^{j_1}})$.

The (P_2) assumption implies there exists $j_2 \geq 0$ such that $P_D(x) = \Phi_{\ell_2}(x^{b^{j_2}})Q(x)$ for some polynomial $Q(x)$. Let $j_1 + j_2 + k \geq 1$. Then we have

$$P_D(e^{2\pi i m/b^{j_1+j_2+k}}) = \Phi_{\ell_2}(e^{2\pi i m/b^{j_1+k}}) Q(e^{2\pi i m/b^{j_1+j_2+k}}) = 0.$$

This verifies the Kenyon criterion for D, and hence D is a tile digit set.

Next we give a concrete class of digit sets that satisfies the (P_k) condition. We regard the product-form and modulo product-form in Definitions (1.3.9) and (1.3.12) as 1st-order. We define

Definition (1.3.16) [50]: D is called a 2nd-order product-form (with respect to b) if

$$D = \mathcal{G}_0 \oplus b^{l_1}\mathcal{G}_1 \oplus \dots \oplus b^{l_k}\mathcal{G}_k, \quad (83)$$

where $0 \leq l_1 \leq l_2 \leq \dots \leq l_k, \mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_k$, and $\mathcal{G}$ itself is a modulo product-form as in Definition (1.3.12) (possibly in another decomposition different from the $\mathcal{G}_i$).

For the above G, let $S_i^{(2)} = \{s : \Phi_s(x) | P_{\mathcal{G}_i}(x), \Phi_s(x) | K_{\mathcal{G}}(x)\}$, where $K_{\mathcal{G}}$ is defined in (80). We use the same procedure as in Definition (1.3.12) to define the 2nd-order modulo product-form and the corresponding K_D. Inductively we can define the k^{th}-order modulo product-form.

Theorem (1.3.17) [50]: Let D be a k^{th}-order product-form (or a k^{th}-order modulo productform).
Then it satisfies the (P_k) condition and hence a tile digit set.
Proof: We only prove the case when D is a 2nd-order product-form; the other cases are similar. Let $d|b$ and $d > 1$. Since $\mathcal{G}$ satisfies (P), by Theorem (1.3.10), there exists j such that $\Phi_d(x^{b^j})|P_{\mathcal{G}}(x)$. As $P_{\mathcal{G}}(x) = P_{\mathcal{G}_0}(x) \dots P_{\mathcal{G}_k}(x)$, we let i be the first index such that there exists a factor $\Phi_t(x)$ of $\Phi_d(x^{b^j})$ that divides $P_{\mathcal{G}_i}(x)$. Letting j_1 be as in (82) corresponding to D, it follows from (83) that $j_1 = j_1(d) = j + l_i$.
Now for each factor $\Phi_e(x)$ of $\Phi_d(x^{b^{j_1}})$, by

$$\Phi_d(x^{b^{j_1}}) = \prod_{\Phi_{e'}(x)|\Phi_d(x)} \Phi_{e'}(x^{b^{l_i}}),$$

we have $\Phi_e(x)|\Phi_{e'}(x^{b^{l_i}})$ for some e' in the above product. In view of $\Phi_d(x^{b^j})|P_{\mathcal{G}}(x)$, there exists m such that $\Phi_{e'}(x)|P_{\mathcal{G}_m}(x)$. By the choice of i, we have $m \geq i$. Let $j_2 = j_2(e) = l_m - l_i$. Then

$$\Phi_e(x^{b^{j_2}})|\Phi_{e'}(x^{b^{l_m}}) \quad and \quad \Phi_{e'}(x^{b^{l_m}})|P_{\mathcal{G}_m}(x^{b^{l_m}}).$$

As $P_{\mathcal{G}_m}(x^{b^{l_m}})|P_D(x)$, this shows that $\Phi_e(x^{b^{j_2}})|P_D(x)$.
The following diagram indicates the implications of the new classes of tile digit sets for a given b:
1st-order mpf ⇒ 2nd-order mpf ⇒ ..

$$\begin{array}{ccccc} \Downarrow & & \Downarrow & & \\ (P_1) & \Rightarrow & (P_2) & \Rightarrow \dots \Rightarrow & D\text{is a tile digit set} \end{array}$$

(where mpf means modulo product-form). The (P_k) can be understood better through the tree structure of cyclotomic polynomials to be developed. We show that the converse of the last implication also holds; i.e., every tile digit set of b must satisfy condition (P_k) for some k. On the other hand, it is still unclear that if a digit set satisfies (P_k) for some k, then the digit set must be a k^{th}-order product-form.
We conclude by constructing a non-trivial 2nd-order product-form digit set.
Example (1.3.18) [50]: Let $b = 12$ and

$$D = \{0,1\} \oplus 2^4\{0,6\} \oplus 2^7 \cdot 3^2\{0,2,4\}.$$

Then D is a 2nd-order product-form and it satisfies (P_2), but not (P_1).
Indeed by rearranging the terms, we can write D as

$$D = \{0,1\} \oplus 12\{0,8\} \oplus (12)^2\{0,16,32\} := \mathcal{G}_0 \oplus 12\mathcal{G}_1 \oplus (12)^2\mathcal{G}_2.$$

It is clear that $\mathcal{G} := \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 = \{0,1,8,9,16,17\} \oplus 12\{0,2\}$ is a 1st-order product-form. Therefore D is a 2nd-order product-form.
On the other hand, D does not satisfy $(P1)$. Suppose otherwise, consider

$$\begin{aligned} P_D(x) &= \Phi_2(x)\Phi_2\left(x^{6\cdot 2^4}\right)\Phi_3\left(x^{2^8\cdot 3^2}\right) \\ &= \Phi_2(x) \cdot \Phi_{2^6}(x)\Phi_{2^6\cdot 3}(x) \cdot \Phi_{3^3}(x)\Phi_{2\cdot 3^3}(x) \dots \Phi_{2^8\cdot 3^3}(x) \end{aligned} \tag{84}$$

and for the factor $d = 4$, there exists $j \geq 0$ such that $\Phi_4(x^{12^j})|P_D(x)$. Since $\Phi_{2^{2j+2}}(x)|\Phi_4(x^{12^j})$, hence $\Phi_{2^{2j+2}}(x)|P_D(x)$. From (84), we must have $2j + 2 = 6$, so that $j = 2$. Thus $\Phi_4(x^{12^2})|P_D(x)$. But this is impossible because

$$\Phi_4(x^{12^2}) = \Phi_{2^6}(x)\Phi_{2^6\cdot 3}(x)\Phi_{2^6\cdot 3^2}(x),$$

and $\Phi_{2^6\cdot 3^2}(x)$ does not divide $P_D(x)$.
It is also interesting to observe the following:

Example (1.3.19) [50]: If we multiply one more factor $\Phi_{2^6\cdot 3^2}(x)$ to (84), then it is the mask polynomial for another tile digit set $D = \{0,1\} \oplus 12^2\{0,2\} \oplus 12^2\{0,16,32\}$, which is a 1st-order product-form of $b = 12$.

The expression of D follows from

$$P_D(x) = \Phi_2(x) \cdot \Phi_{2^6}(x)\Phi_{2^6\cdot 3}(x)\Phi_{2^6\cdot 3^2}(x) \cdot \Phi_{3^3}(x)\Phi_{2\cdot 3^3}(x) \dots \Phi_{2^8\cdot 3^3}(x)$$
$$= \Phi_2(x)\Phi_2(x^{12^2\cdot 2})\Phi_3(x^{12^2\cdot 2^4}) .$$

It is a 1st-order product form as $\{0,1\} \oplus \{0,2\} \oplus \{0,16,32\} \equiv \mathbb{Z}_{12}$.

We will come back to these two examples

we will study the mask polynomial by using a graph-theoretic consideration, and then use it to consider the condition (P_k). We make use of a setup by Protasov on the refinement equations [56]. Let $V_0 = \{\vartheta\}$ be the root,

$$V_k = \{\boldsymbol{j} = j_k \dots j_1 : j_\ell \in \{0,1,\dots,b-1\}, j_1 \neq 0\}, k \geq 1,$$

and $V = \bigcup_{k\geq 0} V_k$. (We reverse the usual ordering on the index as we are dealing with the integers in the b-adic expansion instead of the decimals.) For any $\boldsymbol{j} \in V_k, k \neq 0$, it has b offsprings $j_{k+1}\boldsymbol{j}$ (note that by assumption, ϑ has only $b-1$ offsprings in V, which are the elements of V_1). Let E be the set of edges connecting those $\boldsymbol{j}$ and $j_{k+1}\boldsymbol{j}$. Then (V,E) is a tree with ϑ as the root, and we call it a Protasov tree (associated with b).

For each $\boldsymbol{j} \in V_k$, we let $m_j = j_k b^{k-1} + \dots + j_2 b + j_1$, which is the b-adic expansion determined by $\boldsymbol{j}$. Note that $j_1 \neq 0$ by the assumption on the Protasov tree. It follows that there is a one-to-one correspondence between V_k and the set of integers in $\{1,\dots,b^k - 1\}$ which are not divisible by b.

We call $B \subset V\backslash\{\vartheta\}$ a blocking if it is a finite set and every infinite path starting from ϑ must intersect exactly one element of B. The following criterion is due to Protasov [58] on the refinement equation adjusted to the present situation. For each $\boldsymbol{j} \in V_k$, we use $e(m_j)$ to denote $e^{2\pi i m_j/b^k}$ for simplicity.

Theorem (1.3.20) [50]: D is a tile digit set if and only if there is a blocking B such that for any $\boldsymbol{j} \in B$,

$$P_D\left(e(m_j)\right) = 0. \quad (85)$$

(We call such a B a P_D-blocking.)

Proposition (1.3.21) [50]: With the above notation, we have

(i) if $\tau(\boldsymbol{i}) = \tau(\boldsymbol{j})$, then $\boldsymbol{i}$ and $\boldsymbol{j}$ lie in the same V_k, and $\tau(\boldsymbol{i}^-) = \tau(\boldsymbol{j}^-)$;

(ii) $C_d \subset V_k$ for some k; $\#C_d = deg\Phi_d$ and

$$\Phi_d(x) = \prod_{\boldsymbol{j}\in C_d}\left(x - e(m_j)\right); \quad (86)$$

(iii) if $\tau(\boldsymbol{j}) = \Phi_d$, then $\tau(L_j) = \{\Phi_e : \Phi_e(x) | \Phi_d(x^b)\}$ and

$$\Phi_d(x^b) = \prod_{\boldsymbol{j}'\in C^*}\left(x - e(m_{j'})\right). \quad (87)$$

Proof: (i) Suppose that $\boldsymbol{i} \in V_{k_1}, \boldsymbol{j} \in V_{k_2}$ with $k_1 < k_2$ and $\tau(\boldsymbol{i}) = \tau(\boldsymbol{j}) = \Phi_d$. Let $a_i = g.c.d.(b^{k_1}, m_i), a_j = g.c.d.(b^{k_2}, m_j)$. Then

$$d = b^{k_1}/a_i = b^{k_2}/a_j.$$

Hence $a_j = b^{k_2-k_1}a_i$, so that b divides m_j, which implies $j_j = 0$. This contradicts the assumption that $j_j \neq 0$ for $\boldsymbol{j} \in V$.

To prove the second part, we let $\boldsymbol{i} = i_k \cdots i_1$ and $\boldsymbol{j} = j_k \cdots j_1$. Since $\tau(\boldsymbol{i}) = \tau(\boldsymbol{j})$, we have $g.c.d.(b^k, m_i) = g.c.d.(b^k, m_j)$. This is equivalent to

$$m_j = c_1 b^k + c_2 m_i \quad and \quad m_i = c_1' b^k + c_2' m_j$$

for some integers c_1, c_2, c_1', c_2'. Hence

$$m_j = (c_1 b)b^{k-1} + c_2 m_i \quad and \quad m_i = (c_1' b)b^{k-1} + c_2' m_j,$$

so that $g.c.d.(b^{k-1}, m_i) = g.c.d.(b^{k-1}, m_j)$. So $g.c.d.(b^{k-1}, m_i) = g.c.d.(b^{k-1}, m_{j'})$.

(ii) $C_d \subset V_k$ for some k follows from (i). From this, we have $C_d = \{\boldsymbol{j} \in V_k : \Phi_d(e(m_j)) = 0\}$, so that $\#C_d \leq deg\Phi_d$. On the other hand, let $a_j = g.c.d.(b^k, m_j)$.

Then $d = b^k/a_j$ for some $\boldsymbol{j} \in C$. Note that $deg\Phi_d$ equals the Euler-phi function $\varphi(d)$, i.e.,

$$deg\Phi_d = \#\{r: 1 \leq r < d, g.c.d.(r, d) = 1\}. \quad (88)$$

Hence for each such r, we have $r < d = b^k/a_j$. This implies that $ra_j < b^k$ and we can write $ra_j = m_{j'}$ for a unique $\boldsymbol{j}' \in V_k$. In this case, $e^{2\pi ir/d} = e(m_{j'})$. We have shown that each r in (88) corresponds to exactly one $\boldsymbol{j}' \in V_k$ with $\tau(\boldsymbol{j}') = \Phi_d$. Hence $deg\Phi_d \leq \#C_d$.

For $\boldsymbol{j} \in C_d \subset V_k, e(m_j) := e^{2\pi i m_j/b^k}$ is a root of $\Phi_d(x)$. This implies that the product in (86) is a factor of $\Phi_d(x)$. Since $deg\Phi_d = \#C_d$ and the polynomials involved are monic, identity (86) must hold.

(iii) Given $\boldsymbol{j}$ such that $\tau(\boldsymbol{j}) = \Phi_d$, by noting that $e^{2\pi i(\ell b^k + m_j)/b^{k+1}}$ with $0 \leq \ell \leq b-1$ and $\boldsymbol{j} \in C_d$ are roots of $\Phi_d(x^b)$, we have $\tau(L_j) \subset \{\Phi_e: \Phi_e(x) | \Phi_d(x^b)\}$.

Conversely, letting $a_j = g.c.d.(b^k, m_j)$, we have $m_j = ra_j, b^k = da_j$ and r, d are relatively prime. We then write $b = sd'$, where d' contains all prime factors appearing in d. We note that

$$\frac{\ell b^k + m_j}{b^{k+1}} = \frac{\ell da_j + ra_j}{bda_j} = \frac{\ell d + r}{bd} = \frac{\ell d + r}{sd'd} \quad (89)$$

and $g.c.d.(\ell d + r, d'd) = 1$. Now, given any $\Phi_e(x) | \Phi_d(x^b), e = s'dd'$, where s' is a factor of s (by Proposition (1.3.4)(ii)). Since $g.c.d.(d, s) = 1, \{\ell d + r: 0 \leq \ell \leq b-1\}$ contains a complete residue set $(mod\ s)$. This implies that there exists ℓ_0 such that $\ell_0 d + r \equiv s/s'(mod\ s)$ and so that $\frac{\ell_0 d + r}{s} = \frac{c}{s'}$ with $g.c.d.(c, s') = 1$. By (89),

$$\frac{\ell_0 b^k + m_j}{b^{k+1}} = \frac{c}{s'd'd}$$

and $g.c.d.(c, s'd'd) = 1$. This implies that $\tau(\ell_0 m_j) = \Phi_e$, and $\{\Phi_e: \Phi_e(x) | \Phi_d(x^b)\} \subset \tau(L_j)$ follows.

The identity (87) follows from $e^{2\pi i(\ell b^k + m_j)/b^{k+1}}$ with $0 \leq \ell \leq b-1$ and $\boldsymbol{j} \in C_d$ are roots of $\Phi_d(x^b)$, and the degree of the two polynomials in the identity is $b\ deg(\Phi_d)$.

By this proposition, we can associate a Protasov tree with a tree of cyclotomic polynomials (with respect to b), which we call a Φ-tree. The set of vertices of this tree at level 1 are $\tau(V_1)$. The offsprings of Φ_d are the cyclotomic factors of $\Phi_d(x^b)$; they are determined by Proposition (1.3.4)(ii). By (iii), an edge joining $\boldsymbol{j}$ to its offspring $\ell\boldsymbol{j}$ corresponds to an edge joining Φ_d and a cyclotomic factor of $\Phi_d(x^b)$. We can see easily from (iii) that the map τ is surjective on the Φ-tree. Moreover, all Φ_d in the tree are different by Proposition (1.3.5).

A blocking B in V is called a symmetric blocking if $\tau^{-1}(\tau(\boldsymbol{j})) \subset B$ for every $\boldsymbol{j} \in B$.

Lemma (1.3.22) [50]: Let B be a symmetric blocking in the Protasov tree; then $N_B := \tau(B)$ is a blocking in the Φ-tree. Conversely, if N is a blocking in the Φ-tree, then $B_N = \tau^{-1}(N)$ is a symmetric blocking in the Protasov tree.

Proof: It is clear that N_B is a finite set. Let $\Gamma = \{\vartheta, \Phi_{d_1}, \Phi_{d_2}, \cdots\}$ be an infinite path in the Φ-tree (i.e. $\Phi_{d_{k-1}}(x)$ is a factor of $\Phi_{d_{k-1}}(x^b)$). Since B is symmetric, by Proposition (1.3.21)(iii), there exists an infinite path $\gamma = \{\vartheta, \boldsymbol{j_1}, \boldsymbol{j_2}, \cdots\}$ in the Protasov tree such that $\tau(\boldsymbol{j_k}) = \Phi_{d_k}$ for all k. Since B is a blocking, we can find a vertex $\boldsymbol{j_r}$

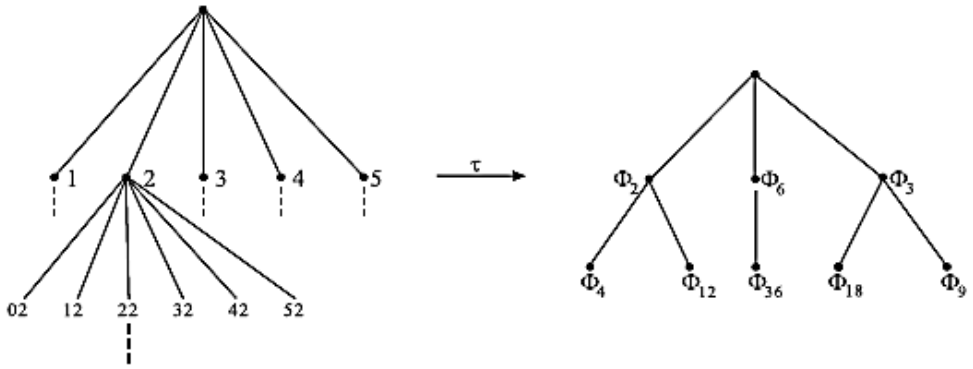

Figure 2[50]: An illustration of the Protasov tree and the associated Φ-tree for the case $b = 6$. On the first level, $\tau(1) = \tau(5) = \Phi_6; \tau(2) = \tau(4) = \Phi_3; \tau(3) = \Phi_2$. On the second level, $\Phi_3(x^6) = \Phi_9(x)\Phi_{18}(x)$; hence Φ_3 has two descendants Φ_9 and Φ_{18}.
The map τ acts on the descendant of 2 as: $\tau(02) = \tau(22) = \tau(32) = \Phi_9; \tau(12) = \tau(32) = \tau(52) = \Phi_{18}$.

on γ such that $\boldsymbol{j_r} \in B$. Hence $\Phi_{d_r} \in N_B$. That Γ hits N_B exactly once is an easy consequence from the blocking B in V.
Conversely, let $\gamma = \{\vartheta, \boldsymbol{j_1}, \boldsymbol{j_2}, \dots\}$ be an infinite path in V. Then Proposition (1.3.21)(iii) implies that $\tau(\gamma) = \{\vartheta, \tau(\boldsymbol{j_1}), \dots\}$ is a path in the Φ-tree. Hence, there is a unique $\tau(\boldsymbol{j_k}) = \Phi_d \in N \cap \tau(\gamma)$. It follows that $\boldsymbol{j_k} \in B_N$, i.e., γ meets B_N at $\boldsymbol{j_k}$ and it is the unique point by Proposition (1.3.21)(i). This shows that B_N is a blocking, and it is clearly symmetric.

Definition (1.3.23) [50]: Letting N be a blocking of the Φ-tree, we call

$$K(x) = \prod_{\Phi_d \in N} \Phi_d(x)$$

a kernel polynomial (with respect to b).
Note that the most basic kernel polynomial is $K(x) = \prod_{d|b,d>1} \Phi_d(x)$. We can generate all the kernel polynomials by adopting the following simple procedure step by step, starting from $K_{\tau(\mathcal{E})}(x)$: let K_N be as in the definition, and let $\Phi_d \in N$.
The descendants of Φ_d, denoted by N_d, are the factors of $\Phi_d(x^b)$. It follows that $N' = (N \backslash \{\Phi_d\}) \cup N_d$ is again a blocking, and the corresponding kernel polynomial is

$$K_{N'}(x) = \frac{\Phi_d(x^b) K_N(x)}{\Phi_d(x)}. \qquad (90)$$

Lemma (1.3.24) [50]: A minimal P_D-blocking (i.e., smallest cardinality) is symmetric.
Proof: Let B be a minimal P_D-blocking and let $\boldsymbol{j}$ be an element of B. Suppose there exists $\boldsymbol{i}$ such that $\tau(\boldsymbol{i}) = \tau(\boldsymbol{j})$, but $\boldsymbol{i} \notin B$. If there is no ancestor of $\boldsymbol{i}$ belonging to B, then we can form a B' by including $\boldsymbol{i}$ into B and dropping all its offsprings in B. This B' is a P_D-blocking since $\tau(\boldsymbol{i}) = \tau(\boldsymbol{j})$, and has smaller cardinality than B, so it is impossible by the minimality assumption. Hence there is an ancestor $\boldsymbol{i'}$ of $\boldsymbol{i}$ such that $\boldsymbol{i'} \in B$. Let $\boldsymbol{j'}$ be the ancestor of $\boldsymbol{j}$ such that $|\boldsymbol{j'}| = |\boldsymbol{i'}|$. Then by Proposition (1.3.21)(i), $\tau(\boldsymbol{i'}) = \tau(\boldsymbol{j'})$. By dropping all the offsprings of $\boldsymbol{j'}$ and adding $\boldsymbol{j}$ to B, we obtain a smaller P_D-blocking. We see that it is again impossible, and the lemma follows.

the main conclusion is the following characterization of the tile digit sets.

Theorem (1.3.25) [50]: Let $b > 1$ be an integer and let D be a digit set with $\#D = b$. Then the following are equivalent:

(i) D is a tile digit set of b;

(ii) there is a symmetric P_D-blocking in the Protasov tree of b;

(iii) there is a blocking N in the Φ-tree of b such that

$$K(x) := \prod_{\Phi_d \in N} \Phi_d(x)$$

is a kernel polynomial and $K(x)|P_D(x)$;

(iv) $P_D(x)$ satisfies condition (P_k) for some $k \geq 1$.

Proof: $(i) \Rightarrow (ii)$. Since we can always find a minimal blocking, the assertion follows from the Protasov criterion (Theorem (1.3.20)) and Lemma (1.3.24).

$(ii) \Rightarrow (iii)$. For a symmetric blocking B in (ii), Lemma (1.3.22) implies that N_B is a blocking in the Φ-tree. For any $\Phi_d \in N_B, \Phi_d = \tau(\boldsymbol{j})$ for some $\boldsymbol{j} \in B$. By (ii), $P_D\left(e\left(m_j\right)\right) = 0$; this shows that $\Phi_d(x)|P_D(x)$. Since N_B is a blocking, $K(x)$ is a kernel polynomial by definition, and clearly it divides $P_D(x)$.

$(iii) \Rightarrow (iv)$. We can localize the (P_k) condition by checking (P_{k_d}) for each $d|b$, then take $k = \max\{k_d : d|n\}$. Note that by assumption, $P_D(x) = K(x)Q(x)$; hence it suffices to prove that for each $d|b, K(x)$ satisfies some (P_{k_d}) condition.

We trace through the steps of producing $K(x) := K_N(x)$ as in (90). For a fixed $d|b$, there exists a smallest j_1 such that some factors of $\Phi_d(x^{b^{j_1}})$ divide $K(x)$ (as N is a blocking). If all factors of $\Phi_d(x^{b^{j_1}})$ divide $K(x)$, then $K(x)$ satisfies (P_1) for d.

Otherwise, there exists at least one factor in $\Phi_d(x^{b^{j_1}})$ that does not divide $K(x)$, say e_1. We repeat the same procedure on Φ_{e_1} as for Φ_d and find j_2 such that some factors in $\Phi_{e_1}(x^{b^{j_2}})$ will divide $K(x)$. We check whether all factors of $\Phi_{e_1}(x^{b^{j_2}})$ will divide $K(x)$, i.e, to satisfy condition (P_2). If not, we continue on with the same procedure. Since the blocking N is a finite set, the process must stop finally. This factor d must satisfy some (P_{k_d}) condition.

$(iv) \Rightarrow (i)$. This has been proved in Proposition (1.3.15).

Remarks(1.3.26) [50]: (i) For the 1st-order product-form D, we have seen that K_D defined in (80) is a polynomial satisfying (P_1) and it divides $P_D(x)$; it is a kernel polynomial for D. Similarly, for the D in (83), the polynomial K_D is also a kernel polynomial of D; it satisfies (P_2).

(ii) For a tile digit set D, P_D may contain more than one kernel polynomial. For example, let $P_D(x)$ be the mask polynomial of D in Example(1.3.19). It contains the factors $\Phi_{2^7 3^3}(x), \Phi_{2^8 3^3}(x)$ and $\Phi_{2^6 \cdot 3^2}(x)$. We write

$$P_D(x) = K_1(x)\Phi_{2^7 3^3}(x)\Phi_{2^6 \cdot 3^2}(x) = K_2(x)\Phi_{2^7 3^3}(x)\Phi_{2^8 3^3}(x)$$

with $K_1(x), K_2(x)$ defined in an obvious way. It is seen that $K_1(x)\Phi_{2^7 3^3}(x)$ is the mask polynomial of the tile digit set in Example (1.3.18). This $K_1(x)$ forms a blocking in the Φ-tree of $b = 12$ and is thus a kernel polynomial of D (the blocking property can be checked easily using the diagram of Figure 2). Moreover, note that both factors $\Phi_{2^6 \cdot 3^2}(x)$ and $\Phi_{2^8 3^3}(x)$ are on the same path in the Φ-tree, and the path has no branches. We can interchange the two factors to form $K_2(x)$, which is still a blocking, and therefore is another kernel polynomial of D.

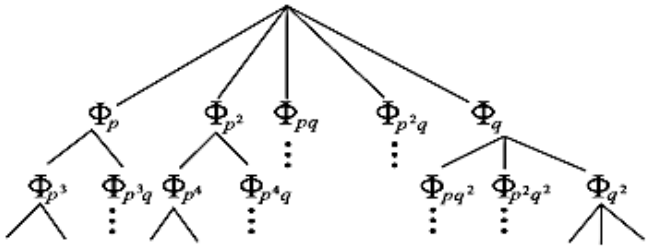

Figure 3[50]:. The Φ-tree for p^2q.

(iii) It is possible that a kernel polynomial $K(x)$ does not generate any tile digit set at all. For example, let $b = pq$ and consider

$$K(x) = \Phi_p(x)\Phi_q(x)\Phi_{p^kq^k}(x), k \geq 1. \quad (91)$$

It is clear that $K(x)$ is a blocking. For $k = 1, K(x)$ corresponds to $\{0, \dots, b-1\}$. However for $k > 1, K(x)$ cannot generate a tile digit set (i.e., $K(x)$ does not divide $P_D(x)$ for any tile digit set D. To see this, we note that from [54] the weak product-form characterization implies that the kernel polynomial can only be the form $\Phi_p(x)\Phi_{q^\ell}(x^{p^\ell})$. Now from (91) the prime power spectrum is $\{p, q\}$.

Hence, $\ell = 1$ and k in (91) can only be 1 in order to obtain a tile digit set for$b = pq$.

It is not clear which kernel polynomials can generate a tile digit set. A complete answer is known only for $b = p^\alpha$ (every kernel polynomial can generate tile digit sets) and q . For the more general case that $b = p^\alpha q$, we can use Theorem (1.3.25) and the technique of integer tiling in [56] to determine all the kernel polynomials that generate tile digit sets. Moreover, we will give a partial answer to $b = p^\alpha q^\beta$. These details will be given in [58].

we have shown that a k^{th}-order modulo product-form satisfies the (P_k) condition. Actually for $b = p^\alpha q$, the converse is also true. This is contained in our classification of the kernel polynomials for $b = p^\alpha q$ in [57]. In here we provide a simple sufficient condition to the kernel polynomial that generates tile digit sets.

Proposition (1.3.27) [50]: Let $K(x)$ be a kernel polynomial associated with b and let $Q(x)$ be a polynomial in $\mathbb{Z}[x]$ with $Q(1) = 1$. If $P(x) = K(x)Q(x)$ has non-negative coefficients, then $P(x) = P_D(x)$ for some tile digit set D.

Proof: Write $P(x)$ explicitly as

$$P(x) = a_0 + a_1x^{d_1} + \cdots + a_mx^{d_m},$$

where $a_j \geq 1$ are integers and $\sum_{\ell=1}^{m} a_\ell = P(1) = K(1)Q(1) = b$. Hence $m \leq b$.

Let $D = \{0 = d_0, d_1, \dots, d_m\}$. We claim that $m = b$. Supposing $m < b$, we note that (b, D) define an IFS (as in (67)) and a self-similar set $T(b, D)$. Let

$$p_j = \frac{a_j}{\sum_{\ell=1}^{m} a_\ell}$$

be a set of probability on the IFS. It induces a self-similar measure μ with support on $T(b, D)$ [58]. Note that $P(x)$ is the mask polynomial of μ, and $K(x)$ is a kernel polynomial. They imply that $P(x)$ satisfies the Protasov criterion for a refinement equation [56]. It follows that μ is absolutely continuous, and its support $T(b, D)$ has positive Lebesgue measure. However, $\#D = m < b$ implies that $T(b, D)$ has zero Lebesgue measure. This is a contradiction. Hence $m = b$. This means that a_j must all be 1 and D must be a tile digit set with $P(x) = P_D(x)$.

Is in terms of tile digit sets; in fact, many of the theorems also hold for the absolutely continuous self-similar measures. Let $D \subset \mathbb{Z}^+$ such that $D = m$ (may not equal b), and let $\phi_d(x) = b^{-1}(x + d)$ be the corresponding affine map. It is well known that there exists a unique probability measure of equal weight $\mu = \mu(b, m, D)$ which satisfies

$$\mu(E) = \sum_{d\in D} \frac{1}{m}\mu\left(\phi_d^{-1}(E)\right), \tag{92}$$

for any Borel set E.

A natural question is to determine when μ is absolutely continuous with respect to the Lebesgue measure. It is known that the general criterion of Kenyon and Protasov for the mask polynomial of D is necessary and sufficient for μ to be absolutely continuous ([58], [52]). We can modify Theorem (1.3.7) and Theorem (1.3.25) as follows:

Let $\mu = \mu(b, D)$ be the self-similar measure defined in (92). Then:

(i) If μ is absolutely continuous with respect to the Lebesgue measure, then Theorem (1.3.7) holds except for the uniqueness of the prime-power spectrum.

(ii) If μ is absolutely continuous, then the equivalence of Theorem (1.3.25) holds.

It is easy to show that if the above μ is absolutely continuous, then $m \geq b$. As an application, we prove the following interesting proposition.

Proposition (1.3.28) [50]: If $\mu = \mu(b, m, D)$ is absolutely continuous, then b divides m.

Proof: Let $b = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be its unique prime factorization. If $\mu(b, m, D)$ is absolutely continuous, then Theorem (1.3.7) shows that for each p_j there exists $\{a_{j,\ell}\}_{\ell=0}^{\alpha_j-1} = 0$ with $a_{j,\ell} \equiv \ell (mod\ \alpha_j)$ and $\Phi_{p_j^{a_{j,\ell}}}(x)$ divides $P_D(x)$. Let

$$S(x) = \prod_{j=1}^{k} \prod_{\ell=0}^{\alpha_j-1} \Phi_{p_j^{a_{j,\ell}}}(x).$$

We have $P_D(x) = S(x)Q(x)$ for some integral polynomial $Q(x)$. Note that the cyclotomic factors are all distinct. By Proposition (1.3.4)(iii) we have $S(1) = p_1^{\alpha_1} \dots p_k^{\alpha_k} = b$. Hence,

$$m = P_D(1) = S(1)Q(1)$$

and $Q(1)$ is an integer since $Q(x)$ is integral polynomial. This completes the proof.

Chapter 2

Non-Spectral Problem and Spectral Property

We establish a large class of such measures, and give a necessary and sufficient condition on the spectrum of a spectral Cantor measure. These results extend the studies by Jorgensen and Pedersen . We study one of the basic problems in Fourier analysis associated with. We give some conditions under which the measure is a spectral measure, i.e., there exists a discrete subset $\Lambda \subseteq \mathrm{R}$ such that $E(\Lambda)$ is an orthonormal basis .

Section (2.1): Spectral Cantor Measures

It is known that certain Cantor measures in $\mathbb{R}^d$ have an orthonormal basis consisting of complex exponentials. This was first observed in Jorgensen and Pedersen [9] and studied further in Strichartz [90]. Let μ be a probability measure in $\mathbb{R}^d$. We call μ a spectral measure if there exists a $\Lambda \subset \mathbb{R}^d$ such that the set of complex exponentials $\{e(\lambda t): \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$ (we use $e(t)$ to denote $e^{2\pi it}$ throughout). The set Λ is called a spectrum for μ; we also say that (μ, Λ) is a spectral pair. It should be pointed out that a spectral measure often has more than one spectrum.

We study spectral Cantor measures in $\mathbb{R}$. Our Cantor measures are self-similar measures associated with iterated function systems (IFS). Consider the iterated functions system (IFS) $\{\phi_j\}_{j=1}^{q}$ given by

$$\phi_j(x) = \rho(x + a_j), \tag{1}$$

where $a_j \in \mathbb{R}$ and $|\rho| < 1$. It is well known (see e.g. Falconer [87]) that for any given probability weights $p_1, \dots, p_q > 0$ with $\sum_{j=1}^{q} p_j = 1$ there exists a unique probability measure μ satisfying

$$\mu = \sum_{j=1}^{q} p_j \mu \circ \phi_j^{-1}. \tag{2}$$

We ask the following question: Under what conditions is μ a spectral measure?

The familiar middle 3rd Cantor measure given by $\rho = 1/3$ and $a_1 = 0, a_1 = 2$ with $p_1 = p_2 = 1/2$ is not a spectral measure, see Jorgensen and Pedersen [89]. The first known example of a spectral measure whose support has non-integer dimension was given, who showed that the measure μ corresponding to $\rho = 1/4$, $a_1 = 0, a_2 = 1$ and $p_1 = p_2 = 1/2$ is spectral. A spectrum of μ is

$$\Lambda = \left\{ \sum_{k=0}^{m} d_k 4^k : m \geq 0, d_k = 0 \text{ or } 2 \right\}.$$

Strichartz [99] gave an alternative proof of this result, and found other examples of spectral Cantor measures. Later Strichartz [100] applied his method to a more general setting to show that a class of measures having a self-similar type of structure (but not necessarily self-similar in the more traditional sense) are spectral measures, provided that an implicit condition on the zero set of certain trigonometric polynomials is satisfied. This condition, however, is not necessary and is somewhat difficult to check.

Spectral measures are a natural generalization of spectral sets. A measurable set Ω in $\mathbb{R}^d$ with positive and finite measure is called spectral if $L^2(\Omega)$ has an orthogonal basis consisting of complex exponentials. Spectral sets have been studied rather extensively, particularly in recent years. The major unsolved problem concerning spectral sets is the following conjecture of Fuglede [82]:

Fuglede's Spectral Set Conjecture: Let Ω be a set in $\mathbb{R}^d$ with positive and finite Lebesgue measure. Then Ω is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation.

The conjecture remains open in either direction, even in dimension one and for sets that are unions of unit intervals. As we shall see, the spectral measures–tiling connection seems to be equally compelling.

We study self-similar measures satisfying

$$\mu = \sum_{j=1}^{q} \frac{1}{q} \mu \circ \phi_j^{-1} \tag{3}$$

where $\phi_j(x) = \frac{1}{N}(x + d_j), N \in \mathbb{Z}$ and $|N| > 1$, and $D = \{d_j\} \subset \mathbb{Z}$. We use $\mu_{N,D}$ to denote the unique probability measure satisfying (3). In addition, for each finite subset A of $\mathbb{R}$ we use $T(N, A)$ to denote the set

$$T(N, A) := \left\{ \sum_{j=1}^{\infty} a_j N^{-j} : a_j \in A \right\},$$

which is in fact the attractor of the IFS $\{\phi_a(x) := \frac{1}{N}(x + a) : a \in A\}$, see [87]. Finally, denote

$$\Lambda(N, A) := \left\{ \sum_{j=0}^{k} a_j N^{-j} : k \geq 1 \text{ and } a_j \in A \right\}.$$

Two finite sets $A = \{a_j\}$ and $S = \{s_j\}$ of cardinality q in $\mathbb{R}$ form a compatible pair, following the terminology of [85], if the matrix $M = \left[\frac{1}{\sqrt{q}}\, e(a_j s_k)\right]$ is a unitary matrix. In other words (δ_A, S) is a spectral pair, where

$$\delta_A := \sum_{a\in A} \frac{1}{q}\delta(x-a).$$

For each finite set A in $\mathbb{R}$ define its symbol by

$$m_A(\xi) := \frac{1}{|A|}\sum_{a\in A} e(-a\xi)\,.$$

Strichartz [90] proves the following theorem:

Theorem (2.1.1)[80]: (Strichartz). Let $N \in \mathbb{Z}$ with $|N| > 1$ and D be a finite set of integers. Let $S \subset \mathbb{Z}$ such that $0 \in S$ and $(\frac{1}{N}D, S)$ is a compatible pair. Suppose that $m_{D/N}(\xi)$ does not vanish on $T(N,S)$. Then the self-similar measure $\mu_{N,D}$ is a spectral measure with spectrum $\Lambda(N,A)$.

Unfortunately the condition that $m_{D/N}$ does not vanish on $T(N,S)$ is not a necessary condition, and can be very difficult to check, even when both D and S are simple. In general we know very little about the zeros $m_{D/N}$. Our objective here is to remove the above condition. We prove that a compatible pair automatically yields a spectral measure. We also give a necessary and sufficient condition for $\Lambda(N,A)$ to be a spectrum.

Theorem (2.1.2) [80]: Let $N \in \mathbb{Z}$ with $|N| > 1$ and let D be a finite set of integers. Let $S \subset \mathbb{Z}$ such that $0 \in S$ and $(\frac{1}{N}D, S)$ is a compatible pair. Then the self-similar measure $\mu_{N,D}$ is a spectral measure. If moreover $gcd(D-D) = 1, 0 \in S$ and $S \subseteq [2-|N|, |N|-2]$, then $\Lambda(N,S)$ is a spectrum for $\mu_{N,D}$.

In Lemma (2.1.6) we prove that if $(\frac{1}{N}D, S)$ is a compatible pair for some $S \subset \mathbb{Z}$, then there is also a set $\hat{S} \subset \mathbb{Z}$ satisfying the additional conditions of Theorem (2.1.2) (i.e., $0 \in \hat{S}$ and $\hat{S} \subseteq [2-|N|, |N|-2]$) such that $(\frac{1}{N}D, \hat{S})$ is a compatible pair.

The following theorem gives a necessary and sufficient condition for $\Lambda(N,A)$ to be a spectrum. It also leads to a simple algorithm,.

Theorem (2.1.3) [80]: Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $gcd(D) = 1$. Let $S \subset \mathbb{Z}$ with $0 \in S$ such that $(\frac{1}{N}D, S)$ is a compatible pair. Then $(\mu_{N,D}, \Lambda(N,S))$ is NOT a spectral pair if and only if there exist $s_j^* \in S$ and nonzero integers $\eta_j, 0 \le j \le m-1$, such that $\eta_{j+1} = N^{-1}(\eta_j + s_j^*)$ for all $0 \le j \le m-1$ (with $\eta_m := \eta_0$ and $s_m^* := s_0^*$).

The proof of Theorem (2.1.3) depends on the analysis of the extreme values of the eigen-functions of the Ruelle transfer operator. The Ruelle transfer operator was studied in [89].

There appears to be a strong link between compatible pairs, tiling of integers and Fuglede's Conjecture. All examples suggest that if D is a finite set of integers and is part of a compatible pair then D tiles $\mathbb{Z}$. A finite set $D \subset \mathbb{Z}$ is called a complementing set $(mod\ N)$ if there exists a $\mathcal{E} \subset \mathbb{Z}$ such that $D \oplus \mathcal{E}$ is a complete residue system $(mod\ N)$. It is known that D tiles $\mathbb{Z}$ if and only if it is a complementing set $(mod\ N)$ for some N. We prove:

Theorem (2.1.4) [80]: Let $D \subset \mathbb{Z}$ be a complementing set $(mod\ N)$ with $|N| > 1$. Suppose that $|D|$ has no more than two distinct prime factors. Then $\mu_{N,D}$ is a spectral measure.

we shall prove the results just stated. We give some examples and state some open problems.

We first state several lemmas, many of which have been proved in Jorgensen and Pedersen [89] or Strichartz [90].

Lemma (2.1.5) [80]: Let $A, B \subset \mathbb{R}$ be finite sets of the same cardinality. Then the following are equivalent:
(a) (A, B) is a compatible pair.
(b) $m_A(b_1 - b_2) = 0$ for any distinct $b_1, b_2 \in B$.
(c) $\sum_{b \in B} |m_A(\xi + b)|^2 \equiv 1$.
Proof: Note that the condition (b) says precisely that the rows of the matrix $M = [\frac{1}{\sqrt{|A|}} e(a_j b_k)]$ are orthonormal. So (a) and (b) are clearly equivalent. To see (a) and (c) are equivalent, let $\delta_A := \frac{1}{|A|} \sum_{a \in A} \delta(x - a)$. Then δ_A is a probability measure with $\hat{\delta}_A(\xi) = m_A(\xi)$. Furthermore, (A, B) is a compatible pair if and only if (δ_A, B) is a spectral pair, see [20]. The equivalence of (a) and (c) follows immediately from Lemma 2.3 of [90].

Lemma (2.1.6) [80]: Let $D, S \subset \mathbb{Z}$ and $N \in \mathbb{Z}, |N| > 1$ such that $(\frac{1}{N} D, S)$ is a compatible pair. Then
(a) $(\frac{1}{N} D + a, S + b)$ is a compatible pair for any $a, b \in \mathbb{R}$.
(b) Suppose that $\hat{S} \subset \mathbb{Z}$ such that $\hat{S} \equiv S \ (mod\ N)$. Then $(\frac{1}{N} D, \hat{S})$ is a compatible pair.
(c) The elements in both D and S are distinct modulo N.
(d) Suppose that $|N| > 2$. Then there exists an $\hat{S}$ with $0 \in \hat{S}$ and $\hat{S} \subseteq [2 - |N|, |N| - 2]$ such that $(\frac{1}{N} D, \hat{S})$ is a compatible pair.
(e) Denote $D_k = D + ND + \cdots + N^{k-1} D$ and $S_k = S + NS + \cdots + N^{k-1} S$. Then $(\frac{1}{N^k} D_k, S_k)$ is a compatible pair.

Proof: (a) is essentially trivial from Lemma (2.1.5). It is also well known from the fact that any translate of a spectrum is also a spectrum, and any translate of a spectral measure is also a spectral measure with the same spectra.
For (b), observe that if $s \equiv \hat{s} \ (mod\ N)$ then $m_{D/N}(\xi + s) = m_{D/N}(\xi + \hat{s})$. Therefore

$$\sum_{\hat{s} \in \hat{S}} |m_{D/N}(\xi + \hat{s})|^2 = \sum_{s \in S} |m_{D/N}(\xi + s)|^2 = 1.$$

This proves (b).
For (c), assume that $S = \{s_j\}$ has $s_1 \equiv s_2 \ (mod\ N)$. Then we may replace s_2 by s_1 in S and still have a compatible pair by (b). This means the matrix M used for defining compatible pairs has two identical columns, so it cannot be unitary, a contradiction. So elements in S are distinct modulo N. Similarly M will have two identical rows if elements in D are not distinct modulo N, again a contradiction.
To prove (d), we first translate S so that $0 \in S_1 := S + a$ for some $a \in \mathbb{Z}$. $(\frac{1}{N} D, S_1)$ is still a compatible pair. Now $[2 - |N|, |N| - 2]$ contains a complete set of residues $(mod\ N)$ because it contains at least $|N|$ consecutive integers. Choose $\hat{S} \subseteq [2 - |N|, |N| - 2]$ so that $0 \in \hat{S}$ and $\hat{S} \equiv S_1 (mod\ N)$. Then $(\frac{1}{N} D, \hat{S})$ is a compatible pair.
Finally (e) is a special case of Lemma 2.5 in [90].

Lemma (2.1.7) [80]: Under the assumptions of Theorem (2.1.2), let $Q(\xi) := \sum_{\lambda \in \Lambda(N,S)} |\hat{\mu}(\xi + \lambda)|^2$.
Then
(a) The set of exponentials $\{e(\lambda \xi) : \lambda \in \Lambda(N, S)\}$ is orthonormal in $L^2(\mu)$.
(b) $Q(\xi) \leq 1$ for all $\xi \in \mathbb{R}$ and $\{e(\lambda \xi) : \lambda \in \Lambda(N, S)\}$ is an orthonormal basis for $L^2(\mu)$ if and only if $Q(\xi) \equiv 1$.

(c) The function $Q(\xi)$ is the restriction of an entire function of exponential type to the real line. Furthermore it satisfies

$$Q(\xi) = \sum_{s\in S} \left|m_D\left(N^{-1}(\xi+s)\right)\right|^2 Q\left(N^{-1}(\xi+s)\right). \quad (4)$$

Proof: See Jorgensen and Pedersen [89]. The right hand side of (4) is known as the Ruelle transfer operator (operated on Q).

Denote $\Lambda := \Lambda(N,S)$ and $\mu := \mu_{N,D}$.

($\Leftarrow$) We prove that (μ,Λ) is not a spectral pair by proving that $Q(\xi) \not\equiv 1$, where $Q(\xi)$ is defined in Lemma (2.1.7). In fact, we prove that $Q(\eta_0) = 0$.

Observe that $m_D(\eta_j) = 1$ because $\eta_j \in \mathbb{Z}$. Now $\sum_{s\in S} |m_D(N^{-1}(\eta_j+s))|^2 = 1$ by (c) of Lemma (2.1.5). Since $m_D(N^{-1}(\eta_j + s_j^*)) = m_D(\eta_j + 1) = 1$, it follows that $m_D(N^{-1}(\eta_j+s)) = 0$ for all $s \neq s_j^*$ in S.

Take any $\lambda \in \Lambda$ and write $\lambda = \sum_{k=0}^{\infty} s_k N^k$ where $s_k \in S$ and of course only finitely many $s_k \neq 0$. We have

$$\hat{\mu}(\eta_0+\lambda) = \prod_{j=1}^{\infty} m_D(N^{-j}(\eta_0+\lambda)).$$

Note that

$$m_D(N^{-1}(\eta_0+\lambda)) = m_D\left(N^{-1}(\eta_0+s_0) + \sum_{k=0}^{\infty} s_{k+1}N^k\right) = m_D(N^{-1}(\eta_0+s_0)).$$

Hence $m_D\left(N^{-1}(\eta_0+\lambda)\right) = 1$ for $s_0 = s_0^*$ and $m_D(N^{-1}(\eta_0+\lambda)) = 0$ otherwise. Suppose $s_0 = s_0^*$. Then the same argument together with the fact $\eta_1 = N^{-1}(\eta_0 + s_0^*)$ yield $m_D(N^{-2}(\eta_0+\lambda)) = 1$ for $s_1 = s_1^*$ and $m_D(N^{-2}(\eta_0+\lambda)) = 0$ otherwise. By induction we easily obtain $m_D(N^{-j}(\eta_0+\lambda)) \neq 0$ if and only if $s_j = s_j^*(mod\ m)$. Therefore $\hat{\mu}(\eta_0+\lambda) \neq 0$ only if $s_j = s^*_{j(mod\ m)}$ for all $j \geq 0$. But this is impossible since $s_j = 0$ for all sufficiently large j. Thus $\hat{\mu}(\eta_0+\lambda) = 0$ and $Q(\eta_0) = 0$.

($\Rightarrow$) Assume that (μ,Λ) is NOT a spectral pair. Then $Q(\xi) \not\equiv 1$. Note that this can not happen if $|D| = |S| = 1$, in which case $\mu = \delta_0$ and $\Lambda = \{0\}$. So we may assume that $q = |D| = |S| > 1$. It is well known that in this case $T := T(N,S)$ is a compact set with infinite cardinality.

Since $Q(\xi) \not\equiv 1, Q(\xi) \not\equiv 1$ for $\xi \in T$, because Q is extendable to an entire function on the complex plane and T is an infinite compact set. Denote $X^- := \{\xi \in T : Q(\xi) = \min_{\eta\in T} Q(\eta)\}$. It follows from $Q(0) = 1$ that $0 \notin X^-$.We apply the Ruelle transfer operator to derive a contradiction.

For any $s \in \mathbb{R}$ denote $\phi_s(\xi) = N^{-1}(\xi+s)$. Then $T = \bigcup_{s\in S} \phi_s(T)$. Hence $\phi_s(\xi) \in T$ for all $\xi \in T$. Now choose any $\xi_0 \in X^-$ and set $Y_0 = \{\xi_0\}$. Define recursively

$$Y_{n+1} = \{\phi_s(\xi): s \in S, \xi \in Y_n, \phi_s(\xi) \in X^-\} \quad (counting\ multiplicity).$$

Claim 1: We have $|Y_{n+1}| \geq |Y_n|$ (counting multiplicity).

Proof of Claim 1. Let $\xi^* \in X^-$. By (c) of Lemma (2.1.7)

$$\min_{\eta\in T} Q(\eta) = Q(\xi^*) = \sum_{s\in S} |m_D(\phi_s(\xi^*))|^2 Q(\phi_s(\xi^*)).$$

But $\sum_{s\in S} |m_D(\phi_s(\xi^*))|^2 = 1$ by Lemma (2.1.7) and $Q(\phi_s(\xi^*)) \geq min_{\eta\in T} Q(\eta)$. Thus $Q(\phi_s(\xi^*)) = \min_{\eta\in T} Q(\eta)$ whenever $m_D(\phi_s(\xi^*)) \neq 0$. In other words,

$$\phi_s(\xi^*) \in X^- whenever\ \ m_D\left(\phi_s(\xi^*)\right) \neq 0. \quad (5)$$

Hence for each $\xi \in Y_n$ there exists at least one $s \in S$ such that $\phi_s(\xi) \in Y_{n+1}$, proving Claim 1.

Claim 2: The elements of Y_n in fact all have multiplicity one.

Proof of Claim 2. It is easy to see that elements in Y_n have the form $\phi_{s_n} \circ \cdots \circ \phi_{s_1}(\xi_0)$. If some element in Y_n has multiplicity more than one, then there are two distinct sequences $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ in S such that

$$\phi_{s_n} \circ \cdots \circ \phi_{s_1}(\xi_0) = \phi_{t_n} \circ \cdots \circ \phi_{t_1}(\xi_0).$$

Expanding the two expressions yields

$$\frac{1}{N^n}(\xi_0 + s_1 + Ns_2 + \cdots + N^{n-1}s_n) = \frac{1}{N^n}(\xi_0 + t_1 + Nt_2 + \cdots + N^{n-1}t_n).$$

But this is clearly not possible, since all elements of S are in different residue classes $(mod\ N)$.

Now X^- is finite. It follows that all Y_n have the same cardinality for sufficiently large n, say $n \geq n_0$. Therefore for $n \geq n_0$ any $\xi^* \in Y_n$ has a unique offspring $\phi_s(\xi^*) \in Y_{n+1}$. Furthermore, for any $t \in S$ and $t \neq s$ we must have $m_D(\phi_t(\xi^*)) = 0$, because otherwise by (5) we would have $\phi_t(\xi^*) \in Y_{n+1}$ as another offspring, a contradiction. Thus starting with a $\xi_{n_0} \in Y_{n_0}$ we obtain a sequence $\{\xi_n\}_{n \geq n_0}$ in which $\xi_n \in Y_n$ is the unique offspring of $\xi_{n-1} \in Y_{n-1}, \xi_n = \phi_{s_{n-1}}(\xi_n - 1)$ for some $s_{n-1} \in S$. It follows from the finiteness of X^- that there exist $k > n_0$ and $m > 0$ such that

$$\xi_k = \xi_{k+m} = \frac{1}{N^m}(\xi_k + s_k + s_{k+1}N + \cdots + s_{k+m}N^{m-1}).$$

In particular $\xi_k \in \mathbb{Q}$. Set $\eta_j = \xi_{j+k}$ and $s_j^* = s_{j+k}$ for $0 \leq j \leq m-1$ ($\eta_m := \eta_0$ and $s_m^* := s_0^*$. Only the case $j = m-1$ needs to be check). . Then $\eta_j \in \mathbb{Q}$ for all j. Furthermore $\eta_{j+1} = \phi_{s_j^*}(\eta_j) = N^{-1}(\eta_j + s_j^*)$ for all $0 \leq j \leq m-1$

Note that $m_D(\phi_s(\eta_j)) = 0$ for all $0 \leq j \leq m-1$ and $s \neq s_j^*$, as shown above. This yields

$$|m_D(\eta_j)|^2 = \sum_{s \in S} |m_D(\phi_{s_{j-1}^*}(\eta_{j-1}))|^2 = 1, \qquad 1 \leq j \leq m.$$

But $m_D(\eta_j) = \frac{1}{|D|}\sum_{d \in D} e(d\eta_j)$ and $0 \in D$. The only way $|m_D(\eta_j)| = 1$ can hold is $e(d \cdot \eta_j) = 1$ for all $d \in D$. Hence $\eta_j \cdot d \in \mathbb{Z}$ for all $d \in D$. Since all $\eta_j \neq 0$ and $gcd(D) = 1$, it follows that all $\eta_j \in \mathbb{Z}$. The theorem is proved.

Suppose that $\mu_{N,D}$ is a spectral measure with spectrum Λ then $\mu_{N,aD}$ is a spectral measure with spectrum $a^{-1}\Lambda$. Therefore to prove that $\mu_{N,D}$ is spectral we may without loss of generality assume that $0 \in D$ and $gcd(D) = 1$.

If $|N| = 2$ then $D = \{0, 1\}$ or $D = \{0, -1\}$. The corresponding self-similar measure $\mu_{N,D}$ is the Lebesgue measure supported on a unit interval. Clearly in this case $\mu_{N,D}$ is a spectral measure with spectrum $\mathbb{Z}$.

For $|N| \geq 3$, by Lemma (2.1.6) we may replace S by $\hat{S} \subseteq [2 - |N|, |N| - 2]$ such that $0 \in \hat{S}$ and $(\frac{1}{N}D, \hat{S})$ is a compatible pair. Now for any such $\hat{S}$, we have $T(N, \hat{S}) \subseteq [-\frac{N-2}{N-1}, \frac{N-2}{N-1}]$ for $N > 0$ or $T(N, \hat{S}) \subseteq [-\frac{N^2+N-2}{N^2-1}, \frac{N^2+N-2}{N^2-1}]$ for $N < 0$. In either case $T(N, \hat{S})$ contains no integer other than 0. Now suppose that $(\mu_{N,D}, \Lambda(N, \hat{S}))$ is not a spectral pair. Then there exist s_j^* in S and nonzero integers η_j satisfying the condition of Theorem (2.1.3). Starting with $T_0 = \{\eta_j\}$ we see that $T_0 \subseteq \bigcup_{s \in S} \phi_s(T_0)$. This yields

$T_0 \subseteq T(N,S)$. But this is a contradiction since $T(N,\hat{S})$ contains no integer other than 0. Therefore $(\mu_{N,D}, \Lambda(N,\hat{S}))$ is a spectral pair by Theorem (2.1.3).

It is known that if D is a complementing set $(mod\ N)$ then there exists an $L|N$ whose prime factors are precisely those of $|D|$ such that D is a complementing set $(mod\ L)$. We prove that there exists an $S \in \mathbb{Z}$ such that $(\frac{1}{N}D, S)$ is a compatible pair, using a theorem of Coven and Meyerowitz [83]. The argument below is essentially a repetition of the proof of Theorem 1.5(i) in [93] specialized to the case of two prime factors.

Let $\Phi_n(z)$ denote the n-th cyclotomic polynomial. Let also $D(z) = \sum_{d\in D} z^d$ so that $m_D(\xi) = D(e^{2\pi i\xi})$. Assume that $|D| = p^\alpha q^\beta$ and $L = p^{\alpha'} q^{\beta'}$, where p, q are distinct primes.

(If $|D|$ is a prime power, the proof below works and is simpler.) Let

$$P := \left\{p^k : \Phi_{p^k}(z) \middle| D(z), k \le \alpha'\right\}, \qquad Q := \{q^k : \Phi_{q^k}(z) | D(z), k \le \beta'\}.$$

Coven and Meyerowitz [3] prove that $|P| = \alpha, |Q| = \beta$, and that $\Phi_{p^k q^l}(z)\ |m_D(z)$ for all k, l such that $p^k \in P, q^l \in Q$.

We construct the set S. Write $P = \{p^{k_j}\}$ and $Q = \{q^{l_j}\}$, where $k_1 < k_2 < \cdots < k_\alpha$ and $l_1 < l_2 < \cdots l_\beta$. Define

$$\mathcal{E} = \left\{ \sum_{j=1}^{\alpha} a_j p^{-k_j} + \sum_{j=1}^{\beta} b_j q^{-l_j} : 0 \le a_j < p, 0 \le b_j < q \right\},$$

and let $S = N\mathcal{E}$. Clearly $S \subset \mathbb{Z}$. To prove that $(\frac{1}{N}D, S)$ is a compatible pair it suffices to prove that $m_{D/N}(s_1 - s_2) = 0$ for any distinct $s_1, s_2 \in S$. Equivalently, we only need to show that $m_D(\lambda_1 - \lambda_2) = 0$ for any distinct $\lambda_1, \lambda_2 \in \mathcal{E}$. Note that

$$\lambda_1 - \lambda_2 = \sum_{j=1}^{\alpha} c_j p^{-k_j} + \sum_{j=1}^{\beta} d_j q^{-l_j}, -p < c_j < p, -q\ < d_j < q.$$

If all $c_j = 0$ then not all $d_j = 0$. So $\lambda_1 - \lambda_2 = r/q^{l_{j^*}}$ for some r with $gcd(r,q) = 1$, where j^* is the largest j such that $d_j \neq 0$. Therefore $\Phi_{q^{l_{j^*}}}(e^{2\pi i(\lambda_1-\lambda_2)}) = 0$ and hence $m_D(\lambda_1 - \lambda_2) = 0$. Similarly $m_D(\lambda_1 - \lambda_2) = 0$ if all $d_j = 0$. Finally, assume none of the above is true. Let j_1 be the largest j such that $c_j \neq 0$ and j_2 be the largest j such that $d_j \neq 0$. Then $\lambda_1 - \lambda_2 = r_1/p^{k_{j1}} + r_2/q^{l_{j2}}$ with $gcd(r_1, p) = gcd(r_2, q) = 1$. This yields

$$\lambda_1 - \lambda_2 = \frac{r_1 q^{l_{j2}} + r_2 q^{k_{j1}}}{p^{k_{j1}} q^{l_{j2}}}.$$

The numerator is clearly coprime with the denominator. Hence $\Phi_{p^{k_{j1}} q^{l_{j2}}}(e^{2\pi i(\lambda_1-\lambda_2)}) = 0$ and therefore $m_D((\lambda_1 - \lambda_2) = 0$.

So $(D/N, S)$ is a compatible pair, and hence $\mu_{N,D}$ is a spectral measure.

Theorem (2.1.3) leads to an algorithm for determining whether $\Lambda(N,S)$ is a spectrum for $\mu_{N,D}$. To find whether the integer sequence $\{\eta_j\}$ exists we only need to check a finite number of integers. This is because we have shown in the proof of Theorem (2.1.3) that if the sequence $\{\eta_j\}$ exists, it must be contained in $T(N,S)$. However, $T(N,S)$ is compact. In fact $T(N,S) \subseteq [\frac{a}{N-1}, \frac{b}{N-1}]$ for $N > 0$ and $T(N,S) \subseteq [\frac{a+Nb}{N^2-1}, \frac{b+Na}{N^2-1}]$ for $N < 0$, where a, b are the smallest and the largest elements in S, respectively.

Example (2.1.8) [80]: Our first example addresses the condition in Theorem (2.1.1) by Strichartz.

Let $N = 5$ and $D = \{0, \pm 2, \pm 11\}$. Since D is a residue system $(mod\ N)$, the set $T(N, D)$ is a fundamental domain of $\mathbb{Z}$ and $\mu_{N,D}$ is simply the restriction of the Lebesgue measure to $T(N, D)$, see Grochenig and Haas [5]. Let $S = \{0, \pm 1, \pm 2\}$. Then $\Lambda(N, S) = \mathbb{Z}$ is a spectrum for $\mu_{N,D}$. However, $m_D(\xi)$ has a zero in $[0, \frac{1}{4}]$, which is contained in $T(N, S) = [-\frac{1}{2}, \frac{1}{2}]$.

The condition can also be hard to check. Consider the same D as above, let $N = 10$ and $S = \{0, 2, 4, 6, 8\}$. Then $(D/N, S)$ is a compatible pair. By Theorem (2.1.2) $\Lambda(N, S)$ is a spectrum for $\mu_{N,D}$. It is difficult to check whether $T(N, S)$ contains a zero of $m_D(\xi)$.

Example (2.1.9) [80]: Our next example illustrates that a spectral measure can have many spectra. Take $N = 6$ and $D = \{0, 1, 2\}$. Then $(D/N, S)$ are compatible pairs for both $S = \{0, 2, 4\}$ or $S = \{0, -2, 2\}$. By Theorem (2.1.2) $\Lambda(N, S)$ are spectra of $\mu_{N,D}$ for both S. A far more striking example is to take $S' = \{0, 4, 8\}$. Then $(D/N, S')$ is a compatible pair because $S' \equiv S = \{0, 2, 4\}$ $(mod\ N)$. One can check using the algorithm described earlier that $\Lambda(N, S')$ is indeed a spectrum of $\mu_{N,D}$, as is $\Lambda(N, S)$. But $\Lambda(N, S') = 2\Lambda(N, S)$! This is rather striking because $\Lambda(N, S')$ is intuitively "twice as sparse as" $\Lambda(N, S)$.

The study also leaves several questions unanswered. For example, do Theorem (2.1.2) and Theorem (2.1.3), or something similar, hold in higher dimensions? The difficulty is that an analytic function of two or more variables may attain its infimum at infinitely many points, even on a compact set. The technique of the Ruelle transfer operator employed has its origin in the study of wavelets and self-affine tiles, see [85], in dimension 1. It was extended to higher dimensions in Lagarias and Wang [87]. Could the techniques there be applied to higher dimensions to yield results on spectral measures?

We have studied spectral Cantor measures in which the probability weights are equally distributed. Is this a general rule?

Conjecture (2.1.10) [80]: Let μ be the self-similar measure associated with the IFS $\phi_j(x) = \rho(x + a_j), 1 \le j \le q$, with probability weights $p_1, \dots, p_q > 0$, where $|\rho| < 1$. Suppose that μ is a spectral measure. Then

(i) $\rho = \frac{1}{N}$ for some $N \in \mathbb{Z}$.

(ii) $p_1 = \cdots = p_q = \frac{1}{q}$.

(iii) Suppose that $0 \in A = \{a_j\}$. Then $A = \alpha D$ for some $\alpha \in \mathbb{R}$ and $D \subset \mathbb{Z}$. Furthermore, D must be a complementing set $(mod\ N)$.

Section (2.2): A Class of Planar Self-Affine Measures

Invariant measures, such as self-similar measures, have recently found wide use in the theory of fractals, in dynamics, in harmonic analysis and in quasicrystals (cf. [96]). A measure μ is self-similar if it is a convex combination of a given set S of transformations applied to the measure itself. One usually restricts attention to the case where the set S is finite. Then, an iterated function system (IFS) results, and varying S yields a rich family of measures μ. To get a manageable problem, further restrictions are placed on the transformations from S. E.g., that they are contractive, and that they fall in a definite class, such as conformal maps (giving equilibrium measures on Julia sets), or affine mappings. Here the affine case is considered. Our IFS $\{\phi_d(x)\}_{d \in D}$ consists of the following affine maps on $\mathbb{R}^n$,

$$\phi_d(x) = M^{-1}(x + d) \quad (x \in \mathbb{R}^n),$$

where $M \in M_n(\mathbb{R})$ is an $n \times n$ expanding real matrix (that is, all the eigenvalues of the real matrix M have moduli > 1), and $D \subset \mathbb{R}^n$ is a finite subset of the cardinality $|D|$. We denote the corresponding measure by $\mu_{M,D}$, which is a unique probability measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}. \tag{6}$$

Such a measure $\mu_{M,D}$ is supported on the attractor (or invariant set) $T(M, D)$ of the affine IFS $\{\phi_d(x)\}_{d \in D}$ (cf. [107,108]), and is called a self-affine measure.

Since this affine case includes restrictions of n-dimensional Lebesgue measure, Cantor measures, and IFS fractal measures, say on Sierpinski gaskets, it is natural to ask for Fourier duality.

Can one get some kind of Fourier representation for $\mu_{M,D}$? We know from prior research on $L^2(\mu_{M,D})$ that a naive notion of orthogonal Fourier series is not feasible in general for affine IFSs. For example, the familiar middle 3rd Cantor set $T(M, D)$ corresponding to $M = 3$ and $D = \{0, 2\}$, Jorgensen and Pedersen [108, Theorem 6.1] proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 2 elements. In the case when $M = p, p > 1$, is odd and $D = \{0, 1\}$, Dutkay and Jorgensen [114, Theorem 5.1(i)] proved that there are no 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. we will explore planar affine IFS-examples when the obstruction to getting a Fourier basis is extreme.

Recall that for a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that the exponential function system $E_\Lambda := \{e^{2\pi i(\lambda, x)} : \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$. The set Λ is then called a spectrum for μ; we also say that (μ, Λ) is a spectral pair (cf. [109]).

Spectral measure is a natural generalization of spectral set introduced by Fuglede [100] whose famous spectrum-tiling conjecture and its related problems have received much attention (cf. [115]). The spectral self-affine measure problem at the present day consists in determining conditions under which $\mu_{M,D}$ is a spectral measure, and has been studied [111] (see also [115]). In the opposite direction, the non-spectral Lebesgue measure problem has been studied see [113] and [115,116] where the conjecture that the disk has no more than 3 orthogonal exponentials is still unsolved. Correspondingly, the non-spectral problem on the self-affine measure consists of the following two classes:

(I) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$ orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (cf. [99]).

(II) There are natural infinite families of orthogonal exponentials, but non of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The main question is whether some of theses families can be combined to form larger collections of orthogonal exponentials. The other questions concerning this class can be found in [111].

Except the case that there might be no more than two orthogonal exponentials, the problem on a non-spectral measure $\mu_{M,D}$ in fact falls into one of the above two classes. The first problem we meet is how to determine a measure $\mu_{M,D}$ is non-spectral. There are some results in this direction, such as [114, Theorem 3.1], but we are still far from settling this problem.

Relating to the questions of the class (I), we first recall the following results.

(i) The plane Sierpinski gasket $T(M, D)$ corresponding to

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad and \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{7}$$

Li [115, p. 65], [28, Example 1] proved that $\mu_{M,D}$-orthogonal exponentials contain at most 3 elements and found such 3 elements-orthogonal exponentials. Dutkay and Jorgensen [114, Theorem 5.1(ii)] proved that if $p \in \mathbb{Z}$ is not a multiple of 3 and

$$M = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} (p \geq 2) and\ D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (8)$$

then there are no 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

(ii) The generalized plane Sierpinski gasket $T(M, D)$ corresponding to

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad and \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (9)$$

see [114, Fig. 3 and Example 3.1], by applying [114, Theorem 3.1], Dutkay and Jorgensen [114] obtained that $\mu_{M,D}$-orthogonal exponentials contain atmost 7 elements. More recently, Li [129] proved that for the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} 2 & b \\ 0 & 2 \end{bmatrix} (b \in \mathbb{Z}) and\ D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (10)$$

there are at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best.

In the plane, the above set D (usually called the digit set) which consists of the canonical vectors in $\mathbb{R}^2$ is fundamental, many digit sets can be obtained from this set. From (7)–(9) and (10), we see that the condition $3 \nmid det(M)$ is always satisfied or assumed. In fact, the condition $|D| \in W(m)$ is necessary for the discussion of $\mu_{M,D}$-orthogonality in the integer case, where $|det(M)| = m = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r} (p_1 < p_2 < \cdots < p_r$ are prime numbers, $b_j > 0)$ is the standard prime factorization and $W(m)$ denotes the non-negative integer combination of $p_1, p_2, \ldots, p_r$ (cf. [115, Section 4.2], [118, Section 3]). The known results provide some supportive evidence that the following Conjecture (2.2.1) should be true, although we cannot prove it.

Conjecture (2.2.1)[102]: For an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, if $|D| \notin W(m)$, then $\mu_{M,D}$ is a non-spectral measure and the non-spectral problem on this $\mu_{M,D}$ falls in the class (I).

Motivated by the previous research, especially the above Conjecture (2.2.1), in the present we will consider the non-spectral problem for a class of more general planar self-affine measures. We have the following.

Theorem(2.2.2) [102]: Let $a, b, c \in \mathbb{Z}, |a| > 1, |c| > 1$ and $ac \in \mathbb{Z} \backslash (3\mathbb{Z})$. For the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad and\ D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (11)$$

there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best.

This generalizes the above-mentioned results on the non-spectral self-affine measure problem. The proof of Theorem depends mainly on the characterization of the zero set $Z(\hat{\mu}_{M,D})$ of the Fourier transform $\hat{\mu}_{M,D}$. It is different from the previous research that we find more inclusion relations inside the zero set $Z(\hat{\mu}_{M,D})$. Some facts concerning this zero set are given. Based on these established facts, we prove Theorem. Finally we give a concluding remark on a related question.

For a general expanding matrix $M \in M_n(\mathbb{R})$ and a finite subset $D \subset \mathbb{R}^n$, the Fourier transform of the self-affine measure $\mu_{M,D}$ is

$$\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i(\xi,t)} d\mu_{M,D}(t) \qquad (\xi \in \mathbb{R}^n).$$

From (6), we have

$$\hat{\mu}_{M,D}(\xi) = m_D(M^{*-1}\xi)\hat{\mu}_{M,D}(M^{*-1}\xi) \quad (\xi \in \mathbb{R}^n), (12)$$

which yields

$$\hat{\mu}_{M,D}(\xi) = \prod_{j=1}^{\infty} m_D\left(M^{*-j}\xi\right), (13)$$

by iteration, where

$$m_D(t) := \frac{1}{|D|}\sum_{d\in D} e^{2\pi i(d,t)}, \ (14)$$

and M^* denotes the conjugate transpose of M, in fact $M^* = M^t$. The infinite product (13) converges absolutely for all $\xi \in \mathbb{R}^n$. It also converges uniformly on compact subsets of $\mathbb{R}^n$.

The self-affine measure $\mu_{M,D}$ and its Fourier transform $\hat{\mu}_{M,D}$ given by (13) play an important role in analysis and geometry. Previous research on such measure and its Fourier transform revealed some surprising connections with a number of areas in mathematics, such as harmonic analysis, dynamical systems, number theory, and others, see [111]. Here we are interested in the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$ which is highly important to the spectral and non-spectral problems on the self-affine measures.

For any $\lambda_1, \lambda_2 \in \mathbb{R}^n, \lambda_1 \neq \lambda_2$, the orthogonality condition

$$\langle e^{2\pi i(\lambda_1,x)}, e^{2\pi i(\lambda_2,x)}\rangle_{L^2(\mu_{M,D})} = \int e^{2\pi i(\lambda_1-\lambda_2,x)} d\mu_{M,D}$$
$$= \hat{\mu}_{M,D}(\lambda_1 - \lambda_2) = 0 \quad (15)$$

directly relates to the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$. From (13), we have

$$Z\left(\hat{\mu}_{M,D}\right) = \left\{\xi \in \mathbb{R}^n : \exists j \in \mathbb{N} \ such \ that \ m_D\left(M^{*-j}\,\xi\right) = 0\right\}. (16)$$

Furthermore, if we let $\Theta_j = \{\xi \in \mathbb{R}^n : m_D(M^{*-j}\xi) = 0\}$ $(j = 1, 2, \dots)$, we have the following properties:

(i') $Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} \Theta_j$;

(ii') $\xi_0 \in Z(\hat{\mu}_{M,D}) \Leftrightarrow -\xi_0 \in Z(\hat{\mu}_{M,D})$ or $\xi_0 \in \Theta_j \Leftrightarrow -\xi_0 \in \Theta_j$ for $= 1, 2, \dots$;

(iii') $\Theta_{j+1} = M^*\left(\Theta_j\right) for \ j = 1, 2, \dots$.

In the following, we will restrict our discussion on the special M and D given by (11), and find out some characteristic properties on the set $Z(\mu_{M,D})$.

For the given M and D in (11), we first have

$$M^j = \begin{bmatrix} a^i & bq(j) \\ 0 & c^j \end{bmatrix} \qquad (j = 1, 2, \dots)$$

and

$$m_D\left(M^{*-j}\xi\right) = \frac{1}{3}\left\{1 + e^{2\pi i\frac{\xi_1}{a^j}} + e^{2\pi i\frac{a^j\xi_2 - bq(j)\xi_1}{a^j c^j}}\right\}, \qquad (17)$$

where $\xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2$ and

$$q(j) = \sum_{k=0}^{j-1} a^k c^{j-k-1}.$$

Then, we get from (16) and (17) that

$$Z\left(\hat{\mu}_{M,D}\right) = \bigcup_{j=1}^{\infty}\left(Z_j \cup \tilde{Z}_j\right), \ (18)$$

where

$$Z_j = \left\{\begin{pmatrix} a^j/3 \\ (bq(j) + 2c^j)/3 \end{pmatrix} + \begin{pmatrix} a^j k_1 \\ bq(j)k_1 + c^j k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{Z}\right\} \subset \mathbb{R}^2 \qquad (19)$$

and

$$\tilde{Z}_j = \left\{ \begin{pmatrix} a^j/3 \\ (bq(j)+2c^j)/3 \end{pmatrix} + \begin{pmatrix} a^j\tilde{k}_1 \\ bq(j)\tilde{k}_1 + c^j\tilde{k}_2 \end{pmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2 \qquad (20).$$

From (19) and (20), we first have the following.

Proposition (2.2.3) [102]: The sets Z_j and $\tilde{Z}_j$ given by (19) and (20) satisfy the following properties:

(i) $(x,y)^t \in Z_j \Leftrightarrow (-x,-y)^t \in \tilde{Z}_j, that\ is, Z_j = -\tilde{Z}_j\ or\ \tilde{Z}_j = -Z_j (j = 1,2,\dots)$;

(ii) $Z_j - Z_j \subseteq \mathbb{Z}^2\ and\ \tilde{Z}_j - \tilde{Z}_j \subseteq \mathbb{Z}^2 (j = 1,2,\dots)$;

(iii) $Z_j + Z_j \subseteq \tilde{Z}_j\ and\ \tilde{Z}_j + \tilde{Z}_j \subseteq Z_j (j = 1,2,\dots)$.

In order to find more relations inside the zero set $Z(\hat{\mu}_{M,D})$, we will reduce the fractional expressions in (19) and (20) to their lowest terms. The denominator of all such fractional expressions is the number 3. So we consider the integers a, b and c according to the residue class modulo-3 where these integers belong. The condition $b \in \mathbb{Z}$ can be divided into the following three cases:

$$b = 3g\ (g \in \mathbb{Z});\ b = 3g+1\ (g \in \mathbb{Z});\ b = 3g+2\ (g \in \mathbb{Z}). (21)$$

The assumption that $a, c \in \mathbb{Z}, |a| > 1, |c| > 1$ and $ac \in \mathbb{Z}\backslash(3\mathbb{Z})$ implies that a and c satisfy one of the following four cases:

(A) $a = 3l+1\ (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l'+1 (l' \in \mathbb{Z}\backslash\{0\})$;

(B) $a = 3l+1 (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l'+2 (l' \in \mathbb{Z}\backslash\{-1\})$;

(C) $a = 3l+2\ (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l'+1 (l' \in \mathbb{Z}\backslash\{0\})$;

(D) $a = 3l+2 (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l'+2 (l' \in \mathbb{Z}\backslash\{-1\})$.

We therefore divide our discussion into the following according to (2.10). deals with the case $b = 3g\ (g \in \mathbb{Z})$, is the case $b = 3g+1 (g \in \mathbb{Z})$ and is the case $b = 3g+2\ (g \in \mathbb{Z})$. In each we will discuss Z_j and $\tilde{Z}_j$ according to the above four cases (A), (B), (C) and (D). The main goal of each is to simplify the expression of the zero set $Z(\hat{\mu}_{M,D})$ in (18). The detailed process is given in the first they are presented briefly.

In the case when $b = 3g\ (g \in \mathbb{Z})$, we can rewrite Z_j and $\tilde{Z}_j$ in (19) and (20) as

$$Z_j = \left\{ \begin{pmatrix} a^j/3 \\ 2c^j/3 \end{pmatrix} + \begin{pmatrix} a^j k_1 \\ gq(j) + 3gq(j)k_1 + c^j k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2 \quad (22)$$

and

$$\tilde{Z}_j = \left\{ \begin{pmatrix} 2a^j/3 \\ c^j/3 \end{pmatrix} + \begin{pmatrix} a^j\tilde{k}_1 \\ 2gq(j) + 3gq(j)\tilde{k}_1 + c^j\tilde{k}_2 \end{pmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2 \ (23)$$

respectively. Furthermore, if a and c satisfy one of the four conditions (A), (B), (C) and (D), we will find some interesting inclusion relations between Z_j and $\tilde{Z}_j$. Therefore, we further divide our discussion into the following four according to (A), (B), (C) and (D).

Under the conditions $b = 3g\ (g \in \mathbb{Z})$ and (A), that is

$b = 3g\ (g \in \mathbb{Z}), a = 3l+1\ \ (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l'+1\ \ (l \in \mathbb{Z}\backslash\{0\}), (24)$

we can rewrite Z_j in (22) as

$$Z_j = \left\{ \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} x(l,l',g,j;\ k_1,k_2) \\ y(l,l',g,j;k_1,k_2) \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \qquad (25)$$

where

$$x(l,l',g,j;k_1,k_2) = \frac{(3l+1)^j - 1}{3} + (3l+1)^j\ k_1 \in \mathbb{Z}, \qquad (26)$$

$$y(l,l',g,j;k_1,k_2) = \frac{2\left((3l'+1)^j - 1\right)}{3} + gq(j) + 3gq(j)k_1 + (3l'+1)^j\ k_2 \in \mathbb{Z}, \qquad (27)$$

and $q(j)=\sum_{k=0}^{j-1}(3l+1)^k(3l'+1)^{j-k-1}$.
The case $j=1$ plays an important role in (26) and (27). In fact, we find, from (26) and (27), that there exist $k_1'\in\mathbb{Z}, k_2'\in\mathbb{Z}$ such that

$$x(l,l',g,j;k_1,k_2)=x(l,l',g,1;k_1',k_2'),$$
$$y(l,l',g,j;k_1,k_2)=y(l,l',g,1;k_1',k_2'). \quad (28)$$

This shows that

$$Z_j\subseteq Z_1 \; for \; j\geq 1. \quad (29)$$

In the same way, we can rewrite $\tilde{Z}_j$ in (23) as

$$\tilde{Z}_j=\left\{\begin{pmatrix}2/3\\1/3\end{pmatrix}+\begin{pmatrix}\tilde{x}(l,l',g,j;\,\tilde{k}_1,\tilde{k}_2)\\ \tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2)\end{pmatrix}:\tilde{k}_1,\tilde{k}_2\in\mathbb{Z}\right\}\subset\mathbb{R}^2, \quad (30)$$

where

$$\tilde{x}\big(l,l',g,j;\tilde{k}_1,\tilde{k}_2\big)=\frac{2\big((3l+1)^j-1\big)}{3}+(3l+1)^j\tilde{k}_1\in\mathbb{Z}, \quad (31)$$

$$\tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2)=\frac{(3l'+1)^j-1}{3}+2gq(j)$$
$$+3gq(j)\tilde{k}_1+(3l'+1)^j\tilde{k}_2\in\mathbb{Z}, \quad (32)$$

and $q(j)=(3l+1)^k(3l'+1)^{j-k-1}$. Then, one can verify that there exist $\tilde{k}_1'\in\mathbb{Z},\tilde{k}_2'\in\mathbb{Z}$ such that

$$\tilde{x}(l,l',g,j;\tilde{k}_1,\tilde{k}_2)=\tilde{x}(l,l',g,1;\tilde{k}_1',\tilde{k}_2'),$$
$$\tilde{y}\big(l,l',g,j;\tilde{k}_1,\tilde{k}_2\big)=\tilde{y}\big(l,l',g,1;\tilde{k}_1',\tilde{k}_2'\big). \quad (33)$$

This also shows that

$$\tilde{Z}_j\subseteq\tilde{Z}_1 \quad for\; j\geq 1. \quad (34)$$

Hence, from (18), (29) and (34), we have the following.

Proposition (2.2.4) [102]: Let $b=3g\;(g\in\mathbb{Z}), a=3l+1\;(l\in\mathbb{Z}\backslash\{0\})$ and $c=3l'+1\;(l'\in\mathbb{Z}\backslash\{0\})$. For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ is given by

$$Z\big(\hat{\mu}_{M,D}\big)=Z_1\cup\tilde{Z}_1 \quad (35)$$

with

$$Z_1\cap\tilde{Z}_1=\big(Z_1\cup\tilde{Z}_1\big)\cap\mathbb{Z}^2=\emptyset, \quad (36)$$

where Z_1 and $\tilde{Z}_1$ are given by (25) and (30) respectively.

The cases $b=3g\;(g\in\mathbb{Z})$ and (B)

Under the conditions $b=3g\;(g\in\mathbb{Z})$ and (B), that is

$$b=3g\;(g\in\mathbb{Z}),\;\; a=3l+1\quad(l\in\mathbb{Z}\backslash\{0\})\quad \text{and } c=3l'+2(l\in\mathbb{Z}\backslash\{-1\}), \quad (37)$$

we can rewrite Z_j in (22) as

$$Z_j=\left\{\begin{pmatrix}1/3\\2^{j+1}/3\end{pmatrix}+\begin{pmatrix}x(l,l',g,j;\,k_1,k_2)\\ y(l,l',g,j;k_1,k_2)\end{pmatrix}:k_1,k_2\in\mathbb{Z}\right\}\subset\mathbb{R}^2, \quad (38)$$

where

$$x(l,l',g,j;k_1,k_2)=\frac{(3l+1)^j-1}{3}+(3l+1)^jk_1\in\mathbb{Z}, \quad (39)$$

$$y(l,l',g,j;k_1,k_2)$$
$$=\frac{2\big((3l'+2)^j-2^j\big)}{3}+gq(j)+3gq(j)k_1+(3l'+2)^jk_2\in\mathbb{Z}, \quad (40)$$

and $q(j)=(3l+1)^k(3l'+2)^{j-k-1}$.

A little difference from the above case, we find, from (39) and (40), that there exist $k_1'\in\mathbb{Z},k_2'\in\mathbb{Z}$ such that

$$x(l,l',g,j+2;k_1,k_2) = x(l,l',g,j;k_1',k_2'),$$
$$2^{j+1} + y(l,l',g,j+2;k_1,k_2) = y(l,l',g,j;k_1',k_2'). \quad (41)$$

This shows that

$$Z_{j+2} \subseteq Z_j \quad for \quad j \geq 1. \quad (42)$$

In the same way, we can rewrite $\tilde{Z}_j$ in (23) as

$$\tilde{Z}_j = \left\{ \begin{pmatrix} 2/3 \\ 2^j/3 \end{pmatrix} + \begin{pmatrix} \tilde{x}(l,l',g,j;\ \tilde{k}_1,\tilde{k}_2) \\ \tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2) \end{pmatrix} : \tilde{k}_1,\tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (43)$$

where

$$\tilde{x}\big(l,l',g,j;\tilde{k}_1,\tilde{k}_2\big) = \frac{2\big((3l+1)^j - 1\big)}{3} + (3l+1)^j \tilde{k}_1 \in \mathbb{Z}, \quad (44)$$

$$\tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2)$$
$$= \frac{(3l'+2)^j - 2^j}{3} + 2gq(j) + 3gq(j)\tilde{k}_1 + (3l'+2)^j \tilde{k}_2 \in \mathbb{Z}, \quad (45)$$

and $q(j) = \sum_{k=0}^{j-1} (3l+1)^k (3l'+2)^{j-k-1}$. Then, one can verify that there exist $\tilde{k}_1' \in \mathbb{Z}, \tilde{k}_2' \in \mathbb{Z}$ such that

$$\tilde{x}(l,l',g,j+2;\tilde{k}_1,\tilde{k}_2) = \tilde{x}(l,l',g,j;\tilde{k}_1',\tilde{k}_2'),$$
$$2^j + \tilde{y}\big(l,l',g,j+2;\tilde{k}_1,\tilde{k}_2\big) = \tilde{y}\big(l,l',g,j;\tilde{k}_1',\tilde{k}_2'\big). \quad (46)$$

This also shows that

$$\tilde{Z}_{j+2} \subseteq \tilde{Z}_j \quad for \quad j \geq 1. \quad (47)$$

Hence, from (18), (42) and (47), we have the following.

Proposition (2.2.5) [102]: Let $b = 3g$ $(g \in \mathbb{Z}), a = 3l+1$ $(l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l'+2$ $(l' \in \mathbb{Z}\backslash\{-1\})$. For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ is given by

$$Z\big(\hat{\mu}_{M,D}\big) = Z_1 \cup Z_2 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \quad (48)$$

with

$Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$ are mutually disjoint and $\bigcup_{j=1}^{2}(Z_j \cup \tilde{Z}_j) \cap Z^2 = \emptyset$ (49),

where Z_1 and Z_2 are given by (38), $\tilde{Z}_1$ and $\tilde{Z}_2$ are given by (43).

2.1.3. The cases $b = 3g$ $(g \in \mathbb{Z})$ and (C)

Under the conditions $b = 3g$ $(g \in \mathbb{Z})$ and (C), that is

$b = 3g$ $(g \in \mathbb{Z}), a = 3l+2$ $(l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l'+1$ $(l' \in \mathbb{Z}\backslash\{0\})$, (50)

as in the above we first rewrite Z_j in (22) and $\tilde{Z}_j$ in (23) as

$$Z_j = \left\{ \begin{pmatrix} 2^j/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} x(l,l',g,j;\ k_1,k_2) \\ y(l,l',g,j;k_1,k_2) \end{pmatrix} : k_1,k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (51)$$

$$\tilde{Z}_j = \left\{ \begin{pmatrix} 2^{j+1}/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} \tilde{x}(l,l',g,j;\ \tilde{k}_1,\tilde{k}_2) \\ \tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2) \end{pmatrix} : \tilde{k}_1,\tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (52)$$

where

$$x(l,l',g,j;k_1,k_2) = \frac{(3l+2)^j - 2^j}{3} + (3l+2)^j k_1 \in \mathbb{Z}, \quad (53)$$

$$y(l,l',g,j;k_1,k_2)$$
$$= \frac{2\big((3l'+1)^j - 1\big)}{3} + gq(j) + 3gq(j)k_1 + (3l'+1)^j k_2 \in \mathbb{Z}, \quad (54)$$

$$\tilde{x}\big(l,l',g,j;\tilde{k}_1,\tilde{k}_2\big) = \frac{2\big((3l+2)^j - 2^j\big)}{3} + (3l+2)^j \tilde{k}_1 \in \mathbb{Z}, \quad (55)$$

$$\tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2)$$

$$= \frac{(3l'+1)^j - 1}{3} + 2gq(j) + 3gq(j)\tilde{k}_1 + (3l'+1)^j \tilde{k}_2 \in \mathbb{Z}, \quad (56)$$

and $q(j) = (3l+2)^k (3l'+1)^{j-k-1}$.

Then, one can verify that

$$Z_{j+2} \subseteq Z_j \quad and \; \tilde{Z}_{j+2} \subseteq \tilde{Z}_j \quad for \; j \geq 1. \quad (57)$$

Hence, from (18) and (57), we have the following.

Proposition (2.2.6) [102]: Let $b = 3g \; (g \in \mathbb{Z}), a = 3l + 2 \; (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l' + 1 (l' \in \mathbb{Z}\backslash\{0\})$. For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (48) and (49), where Z_1 and Z_2 are given by (51), $\tilde{Z}_1$ and $\tilde{Z}_2$ are given by (52).

The cases $b = 3g \; (g \in \mathbb{Z})$ and (D)

Under the conditions $b = 3g \; (g \in \mathbb{Z})$ and (D), that is

$$b = 3g \; (g \in \mathbb{Z}), a = 3l + 2 \quad (l \in \mathbb{Z}\backslash\{-1\}) \text{ and } c = 3l' + 2 \; (l' \in \mathbb{Z}\backslash\{-1\}), \quad (58)$$

as in the above we first rewrite Z_j in (22) and $\tilde{Z}_j$ in (23) as

$$Z_j = \left\{ \begin{pmatrix} 2^j/3 \\ 2^{j+1}/3 \end{pmatrix} + \begin{pmatrix} x(l,l',g,j;\; k_1,k_2) \\ y(l,l',g,j;k_1,k_2) \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (59)$$

$$\tilde{Z}_j = \left\{ \begin{pmatrix} 2^{j+1}/3 \\ 2^j/3 \end{pmatrix} + \begin{pmatrix} \tilde{x}(l,l',g,j;\; \tilde{k}_1,\tilde{k}_2) \\ \tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2) \end{pmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subset \mathbb{R}^2, \quad (60)$$

where

$$x(l,l',g,j;\; k_1,k_2) = \frac{(3l+2)^j - 2^j}{3} + (3l+2)^j \; k_1 \in \mathbb{Z}, \quad (61)$$

$$y(l,l',g,j;\; k_1,k_2) = \frac{2\left((3l'+2)^j - 2^j\right)}{3} + gq(j) + 3gq(j)k_1 + (3l'+2)^j \; k_2 \in \mathbb{Z}, \quad (62)$$

$$\tilde{x}\left(l,l',g,j;\tilde{k}_1,\tilde{k}_2\right) = \frac{2\left((3l+2)^j - 2^j\right)}{3} + (3l+2)^j \tilde{k}_1 \in \mathbb{Z}, \quad (63)$$

$$\tilde{y}(l,l',g,j;\tilde{k}_1,\tilde{k}_2) = \frac{(3l'+2)^j - 2^j}{3} + 2gq(j) + 3gq(j)\tilde{k}_1 + (3l'+2)^j \tilde{k}_2 \in \mathbb{Z}, \quad (64)$$

and $q(j) = \sum_{k=0}^{j-1} (3l+2)^k (3l'+2)^{j-k-1}$.

Then, we find, with a little difference from the above cases, that the following inclusion relations

$$Z_{j+1} \subseteq \tilde{Z}_j \quad and \; \tilde{Z}_{j+1} \subseteq Z_j \; for \; j \geq 1 \quad (65)$$

hold. Hence, from (18) and (65), we have the following.

Proposition (2.2.7) [102]: Let $b = 3g \; (g \in \mathbb{Z}), a = 3l + 2 \; (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l' + 2 (l' \in \mathbb{Z}\backslash\{-1\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (35) and (36), where Z_1 and $\tilde{Z}_1$ are given by (59) and (60) respectively.

The case $b = 3g + 1 (g \in \mathbb{Z})$

In the case when $b = 3g + 1 (g \in \mathbb{Z})$, we can rewrite the planar sets Z_j and $\tilde{Z}_j$ in (19) and (20) as

$$Z_j = \left\{ \begin{pmatrix} \frac{a^j}{3} \\ \frac{q(j)+2c^j}{3} \end{pmatrix} + \begin{pmatrix} a^j k_1 \\ gq(j) + (3g+1)q(j)k_1 + c^j k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \quad (66)$$

and

$$\tilde{Z}_j = \left\{ \begin{pmatrix} \frac{2a^j}{3} \\ \frac{2q(j)+c^j}{3} \end{pmatrix} + \begin{pmatrix} a^j\tilde{k}_1 \\ gq(j) + (3g+1)q(j)\tilde{k}_1 + c^j\tilde{k}_2 \end{pmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} (67)$$

respectively, where $q(j) = \sum_{k=0}^{j-1} a^k c^{j-k-1}$. Furthermore, if a and c satisfy one of the four conditions (A), (B), (C) and (D), we will find certain inclusion relations between Z_j and $\tilde{Z}_j$) by applying the same technique.

The cases $b = 3g + 1$ $(g \in \mathbb{Z})$ and (A)

Under the conditions $b = 3g + 1$ $(g \in \mathbb{Z})$ and (A), that is

$b = 3g + 1$ $(g \in \mathbb{Z})$, $a = 3l + 1$ $(l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 1$ $(l \in \mathbb{Z}\backslash\{0\}), (68)$

we find that the following inclusion relations

$$Z_{j+3} \subseteq Z_j \; and \; \tilde{Z}_{j+3} \subseteq \tilde{Z}_j (j = 1, 2, \dots) \; (69)$$

hold. Hence, from (18) and (69), we have the following.

Proposition (2.2.8) [102]: Let $b = 3g + 1$ $(g \in \mathbb{Z}), a = 3l + 1$ $(l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 1$ $(l' \in \mathbb{Z}\backslash\{0\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ is given by

$$Z(\hat{\mu}_{M,D}) = Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3 \; (70)$$

with

$Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ are mutually disjoint and $\bigcup_{j=1}^{3}(Z_j \cup \tilde{Z}_j) \cap \mathbb{Z}^2 = \emptyset, (71)$

where Z_j and $\tilde{Z}_j$ $(j = 1, 2, 3)$ are given by (66) and (67) respectively with a, b, c given by (68).

The cases $b = 3g + 1$ $(g \in \mathbb{Z})$ and (B)

Under the conditions $b = 3g + 1$ $(g \in \mathbb{Z})$ and (B), that is

$b = 3g + 1$ $(g \in \mathbb{Z})$, $a = 3l + 1 (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 2 (l' \in \mathbb{Z}\backslash\{-1\}), (72)$

we find that the following inclusion relations

$$Z_{j+1} \subseteq Z_j \; and \; \tilde{Z}_{j+1} \subseteq \tilde{Z}_j \; (j = 1, 2, \dots) \quad (73)$$

hold. Hence, from (18) and (73), we have the following.

Proposition (2.2.9) [102]: Let $b = 3g + 1$ $(g \in \mathbb{Z}), a = 3l + 1$ $(l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 2 (l' \in \mathbb{Z}\backslash\{-1\})$. For the self-affine measure $\mu_{M,D}$ corresponding to (1.6), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (35) and (36), where Z_1 and $\tilde{Z}_1$ are given by (66) and (67) respectively with a, b, c given by (72).

The cases $b = 3g + 1 (g \in \mathbb{Z})$ and (C)

Under the conditions $b = 3g + 1$ $(g \in \mathbb{Z})$ and (C), that is

$b = 3g + 1$ $(g \in \mathbb{Z}), a = 3l + 2 (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l' + 1 (l' \in \mathbb{Z}\backslash\{0\}), (74)$

we find that the following inclusion relations

$$Z_{j+2} \subseteq Z_j \quad and \; \tilde{Z}_{j+2} \subseteq \tilde{Z}_j \; (j = 1, 2, \dots) \; (75)$$

hold. Hence, from (18) and (75), we have the following.

Proposition (2.2.10) [102]: Let $b = 3g + 1$ $(g \in \mathbb{Z}), a = 3l + 2$ $(l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l' + 1$ $(l' \in \mathbb{Z}\backslash\{0\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (48) and (49), where Z_j and $\tilde{Z}_j$ $(j = 1, 2)$ are given by (66) and (67) respectively with a, b, c given by (74).

2.2.4. The cases $b = 3g + 1$ $(g \in \mathbb{Z})$ and (D)

Under the conditions $b = 3g + 1$ $(g \in \mathbb{Z})$ and (D), that is

$b = 3g + 1$ $(g \in \mathbb{Z}), a = 3l + 2$ $(l \in \mathbb{Z}\backslash\{-1\})$and $c = 3l' + 2 (l' \in \mathbb{Z}\backslash\{-1\}), (76)$

we find that the following inclusion relations

$$Z_{j+3} \subseteq \tilde{Z}_j \; and \; \tilde{Z}_{j+3} \subseteq Z_j \quad (j = 1, 2, \dots) \; (77)$$

hold. Hence, from (18) and (77), we have the following.

Proposition (2.2.11) [102]: Let $b = 3g + 1 \; (g \in \mathbb{Z}), a = 3l + 2 (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l' + 2 (l' \in \mathbb{Z}\backslash\{-1\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (70) and (71), where Z_j and $\tilde{Z}_j \; (j = 1, 2, 3)$ are given by (66) and (67) respectively with a, b, c given by (76).

The case $b = 3g + 2 (g \in \mathbb{Z})$

In the case when $b = 3g + 2 (g \in \mathbb{Z})$, we can rewrite the planar sets Z_j and $\tilde{Z}_j$ in (19) and (20) as

$$Z_j = \left\{ \begin{pmatrix} \frac{a^j}{3} \\ \frac{2q(j)+2c^j}{3} \end{pmatrix} + \begin{pmatrix} a^j k_1 \\ gq(j) + (3g+2)q(j)k_1 + c^j k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \; (78)$$

and

$$\tilde{Z}_j = \left\{ \begin{pmatrix} 2a^j/3 \\ (q(j) + c^j)/3 \end{pmatrix} + \begin{pmatrix} a^j \tilde{k}_1 \\ (2g+1)q(j) + (3g+2)q(j)\tilde{k}_1 + c^j \tilde{k}_2 \end{pmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \; (79)$$

respectively, where $q(j) = \sum_{k=0}^{j-1} a^k c^{j-k-1}$. Furthermore, if a and c satisfy one of the four conditions (A), (B), (C) and (D), we will find certain inclusion relations between Z_j and $\tilde{Z}_j$ (a little difference from those) by applying the same method as above.

2.3.1. The cases $b = 3g + 2 \; (g \in \mathbb{Z})$ and (A)

Under the conditions $b = 3g + 2 \; (g \in \mathbb{Z})$ and (A), that is

$b = 3g + 2 \; (g \in \mathbb{Z}), a = 3l + 1 (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 1 \; (l \in \mathbb{Z}\backslash\{0\}), (80)$

we find that the following inclusion relations

$$Z_{j+3} \subseteq Z_j \; and \; \tilde{Z}_{j+3} \subseteq \tilde{Z}_j \; (j = 1, 2, \dots) \; (81)$$

hold. Hence, from (18) and (81), we have the following.

Proposition (2.2.12) [102]: Let $b = 3g + 2 \; (g \in \mathbb{Z}), a = 3l + 1 \; (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 1 \; (l' \in \mathbb{Z}\backslash\{0\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (70) and (71), where Z_j and $\tilde{Z}_j \; (j = 1, 2, 3)$ are given by (78) and (79) respectively with a, b, c given by (80).

The cases $b = 3g + 2 \; (g \in \mathbb{Z})$ and (B)

Under the conditions $b = 3g + 2 (g \in \mathbb{Z})$ and (B), that is

$b = 3g + 2 (g \in \mathbb{Z}), a = 3l + 1 (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 2 (l \in \mathbb{Z}\backslash\{-1\}), (82)$ we find that the following inclusion relations

$$Z_{j+2} \subseteq Z_j \; and \; \tilde{Z}_{j+2} \subseteq \tilde{Z}_j \; (j = 1, 2, \dots) \quad (83)$$

hold. Hence, from (18) and (83), we have the following.

Proposition (2.2.13) [102]: Let $b = 3g + 2 (g \in \mathbb{Z}), a = 3l + 1 \; (l \in \mathbb{Z}\backslash\{0\})$ and $c = 3l' + 2 (l' \in \mathbb{Z}\backslash\{-1\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (48) and (49), where Z_j and $\tilde{Z}_j \; (j = 1, 2)$ are given by (78) and (79) respectively with a, b, c given by (82).

The cases $b = 3g + 2 \; (g \in \mathbb{Z})$ and (C)

Under the conditions $b = 3g + 2 \; (g \in \mathbb{Z})$ and (C), that is

$b = 3g + 2(g \in \mathbb{Z}), a = 3l + 2(l \in \mathbb{Z}\backslash\{-1\})$and $c = 3l' + 1(l' \in \mathbb{Z}\backslash\{0\}),$ (84)
we find that the following inclusion relations

$$Z_{j+1} \subseteq \tilde{Z}_j \; and \; \tilde{Z}_{j+1} \subseteq Z_j (j = 1, 2, \dots) \quad (85)$$

hold. Hence, from (18) and (85), we have the following.

Proposition (2.2.14) [102]: Let $b = 3g + 2(g \in \mathbb{Z}), a = 3l + 2\ (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l' + 1(l' \in \mathbb{Z}\backslash\{0\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (35) and (36), where Z_1 and $\tilde{Z}_1$ are given by (78) and (79) respectively with a, b, c given by (84).

The cases $b = 3g + 2(g \in \mathbb{Z})$ and (D)

Under the conditions $b = 3g + 2(g \in \mathbb{Z})$ and (D), that is
$b = 3g + 2\ (g \in \mathbb{Z}), a = 3l + 2(l \in \mathbb{Z}\backslash\{-1\})$and $c = 3l' + 2(l' \in \mathbb{Z}\backslash\{-1\}),$ (86)
we find that the following inclusion relations

$$Z_{j+3} \subseteq \tilde{Z}_j \; and \; \tilde{Z}_{j+3} \subseteq Z_j \ (j = 1, 2, \dots) \quad (87)$$

hold. Hence, from (18) and (87), we have the following.

Proposition (2.2.15) [102]: Let $b = 3g + 2(g \in \mathbb{Z}), a = 3l + 2\ (l \in \mathbb{Z}\backslash\{-1\})$ and $c = 3l' + 2(l' \in \mathbb{Z}\backslash\{-1\})$.

For the self-affine measure $\mu_{M,D}$ corresponding to (11), the zero set $Z(\hat{\mu}_{M,D})$ satisfies (70) and (71), where Z_j and $\tilde{Z}_j\ (j = 1, 2, 3)$ are given by (78) and (79) respectively with a, b, c given by (86).

Summary of the above three cases (twelve subcases)

The above discussion involves the three cases: $b = 3g\ (g \in \mathbb{Z})$ $b = 3g + 1(g \in \mathbb{Z})$ and $b = 3g + 2(g \in \mathbb{Z})$ Each contains four subsections. Propositions (2.2.4)–(2.2.15) correspond to the twelve subsections. These established propositions characterize the zero set $Z(\hat{\mu}_{M,D})$. They can be divided into three types:

Type 1. Propositions (2.2.4), (2.2.7), (2.2.9) and (2.2.14) illustrate that $Z(\hat{\mu}_{M,D})$ satisfies (35) and (36).

Type 2. Propositions (2.2.5), (2.2.6), (2.2.10) and (2.2.13) illustrate that $Z(\hat{\mu}_{M,D})$ satisfies (48) and (49).

Type 3. Propositions (2.2.8), (2.2.11), (2.2.12) and (2.2.15) illustrate that $Z(\hat{\mu}_{M,D})$ satisfies (70) and (71).

The above three types correspond to three kinds of representations for $Z(\hat{\mu}_{M,D})$ which will help us to prove Theorem(2.2.2).

If $\lambda_j (j = 1, 2, 3, 4) \in \mathbb{R}^2$ are such that the exponential functions

$$e^{2\pi i(\lambda_1, x)}, e^{2\pi i(\lambda_2, x)}, e^{2\pi i(\lambda_3, x)}, e^{2\pi i(\lambda_4, x)},$$

are mutually orthogonal in $L^2(\mu_{M,D})$, then the differences $\lambda_j - \lambda_k (1 \le j \ne k \le 4)$ are in the zero set $Z(\hat{\mu}_{M,D})$. That is, we have

$$\lambda_j - \lambda_k \in Z\big(\hat{\mu}_{M,D}\big)(1 \le j \ne k \le 4). \quad (88)$$

We will use the above established facts on the zero set $Z(\hat{\mu}_{M,D})$ to deduce a contradiction. The proof will divide into three according.

In the case of Type 1, we obtain from (35) and (88) that

$$\lambda_1 - \lambda_k \in Z_1 \cup \tilde{Z}_1\ (1 \le j \ne k \le 4) \quad (89)$$

and (36) hold. In particular, the following three differences

$$\lambda_1 - \lambda_2, \ \ \lambda_1 - \lambda_3, \ \ \lambda_1 - \lambda_4 \quad (90)$$

are in $Z_1 \cup \tilde{Z}_1$. The well-known pigeon hole principle, combined with (36), (89) and Proposition (2.2.3), immediately deduces a contradiction, since any two of three

differences in (90) cannot belong to the same set Z_1 or $\tilde{Z}_1$. For example, if $\lambda_1 - \lambda_2 \in \tilde{Z}_1$ and $\lambda_1 - \lambda_4 \in \tilde{Z}_1$, then, by Proposition (2.2.3)(2),

$$\lambda_4 - \lambda_2 = (\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_4) \in \tilde{Z}_1 - \tilde{Z}_1 \subseteq \mathbb{Z}^2$$

which contradicts (36) and (89). Hence any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements. One can obtain many such orthogonal systems which contain three elements.

For instance, the exponential function system E_Λ with Λ given by

$$\Lambda = \{0, s_1, s_2\} \subset \mathbb{R}^2 \text{ for each } s_1 \in Z_1 \text{ and } s_2 \in \tilde{Z}_1 \quad (91)$$

is a three-elements orthogonal system in $L^2(\mu_{M,D})$. This shows that the number 3 is the best.

In the case of Type 2, we obtain from (48) and (88) that

$$\lambda_j - \lambda_k \in Z_1 \cup Z_2 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \ (1 \le j \ne k \le 4) \quad (92)$$

and (49) hold. We will use Proposition (2.2.3), (49) and (92) to deduce a contradiction.

Observe that the following six differences:

$$\begin{gathered} \lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4, \\ \lambda_2 - \lambda_3, \quad \lambda_2 - \lambda_4, \\ \lambda_3 - \lambda_4, \quad (93) \end{gathered}$$

belong to the four sets $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$. By Proposition (2.2.3) and (49), the elements (or differences) in each row of (93) (except the final row where there is only one element $\lambda_3 - \lambda_4$) and the elements (or differences) in each column of (93) (except the first column where there is only one element $\lambda_1 - \lambda_2$) cannot belong to the same set. In particular, the following three elements

$$\lambda_1 - \lambda_2, \quad \lambda_1 - \lambda_3, \quad \lambda_1 - \lambda_4$$

in the first row will be in the three different sets of the four sets $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$. There are 24 distribution methods. We only consider the following typical case.

Typical case. $\lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_1$.

The other cases (by applying Proposition (2.2.3)) can be proved in the same manner.

By Propositions (2.2.3), (2.2.3), we see that in this Typical case, each set contains elements (or differences) in the following Box 1:

Z_1	Z_2	$\tilde{Z}_1$	$\tilde{Z}_2$
$\lambda_1 - \lambda_2$ $\lambda_4 - \lambda_1$ $\lambda_2 - \lambda_4$	$\lambda_1 - \lambda_3$	$\lambda_1 - \lambda_4$ $\lambda_2 - \lambda_1$ $\lambda_4 - \lambda_2$	$\lambda_3 - \lambda_1$

Box 1[102]:

The other elements in (93), that is, the elements $\lambda_2 - \lambda_3$ and $\lambda_3 - \lambda_4$, are also in certain small boxes of Box 1. Firstly, we have the following fact that

$\lambda_2 - \lambda_3$ cannot belong to the sets (or small boxes) $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$ (94).

The reason is as follows.

(i) If $\lambda_2 - \lambda_3 \in Z_1$, then, by Proposition (2.2.3),

$$\lambda_4 - \lambda_3 = (\lambda_2 - \lambda_3) - (\lambda_2 - \lambda_4) \in Z_1 - Z_1 \subseteq \mathbb{Z}^2 \ (95)$$

which contradicts (49) and (92). Also, if $\lambda_2 - \lambda_3 \in Z_1$, then, by Proposition (2.2.3)(3),

$$\lambda_1 - \lambda_3 = (\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) \in Z_1 + Z_1 \subseteq \tilde{Z}_1, \ (96)$$

which contradicts (49) and $\lambda_1 - \lambda_3 \in Z_2$. The same reason shows that $\lambda_2 - \lambda_3 \notin \tilde{Z}_1$.

(ii) If $\lambda_2 - \lambda_3 \in Z_2$, then, by Proposition (2.2.3),

$$\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) - (\lambda_1 - \lambda_3) \in Z_2 - Z_2 \subseteq \mathbb{Z}^2, (97)$$

which contradicts (49) and (92) or contradicts (49) and $\lambda_2 - \lambda_1 \in \tilde{Z}_1$.

(iii) if $\lambda_2 - \lambda_3 \in \tilde{Z}_2$, then, by Proposition (2.2.3),

$$\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) + (\lambda_3 - \lambda_1) \in \tilde{Z}_2 + \tilde{Z}_2 \subseteq Z_2, \quad (98)$$

which contradicts (49) and $\lambda_2 - \lambda_1 \in \tilde{Z}_1$. Hence (94) holds.

Similarly, we have the following facts that $\lambda_3 - \lambda_4$ cannot belong to the sets (or small boxes) $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$. (99)

The established facts (94) and (99) clearly contradict (92). This proves the Typical case. The other cases can be proved in the same manner.

Therefore, any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements. One can obtain many such orthogonal systems which contain 3 elements. For example, the exponential function systems E_Λ with Λ given by (91) or with Λ given by

$$\Lambda = \{0, s_1, s_2\} \subset \mathbb{R}^2 \text{ for each } s_1 \in Z_2 \text{ and } s_2 \in \tilde{Z}_2 \quad (100)$$

are also the three-elements orthogonal system in $L^2(\mu_{M,D})$. This shows that the number 3 is the best.

In the case of Type 3, we obtain from (70) and (88) that

$$\lambda_j - \lambda_k \in Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3 \ (1 \le j \ne k \le 4) \ (101)$$

and (71) hold. We will use Proposition (2.2.3), (71) and (104) to deduce a contradiction.

In this case, the six differences in (93) belong to the six sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$. By Proposition (2.2.3) and (71), the elements (or differences) in each row of (93) (except the final row where there is only one element $\lambda_3 - \lambda_4$) and the elements (or differences) in each column of (93) (except the first column where there is only one element $\lambda_1 - \lambda_2$) cannot belong to the same set. In particular, the following three elements in the first row of (93)

$$\lambda_1 - \lambda_2, \qquad \lambda_1 - \lambda_3, \qquad \lambda_1 - \lambda_4$$

will be in the three different sets of the six sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$. There are 120 distribution methods. One can use the method presented in [119] to deal with each case. For completeness, we use this method to deal with the following three typical cases:

Case 1. $\lambda_1 - \lambda_2 \in \tilde{Z}_1, \lambda_1 - \lambda_3 \in \tilde{Z}_2, \lambda_1 - \lambda_4 \in \tilde{Z}_3$.

Case 2. $\lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_3$.

Case 3. $\lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_1$.

The other cases (by applying Proposition (2.2.3)) can be proved in the same manner.

Case 1. By Proposition (2.2.3), we see that in this case, each set contains elements (or differences) in the following Box 2:

Z_1	Z_2	Z_3	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$
$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_4 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_1 - \lambda_3$	$\lambda_1 - \lambda_4$

Box 2[102]:

The other elements in (93) are also in certain small boxes of Box 2. Firstly, we have the following fact that

$\lambda_2 - \lambda_3$ cannot belong to the sets (or small boxes) $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$ (102).

The reason is as follows.

(i) If $\lambda_2 - \lambda_3 \in Z_2$, then, by Proposition (2.2.3),

$$\lambda_3 - \lambda_1 = (\lambda_2 - \lambda_1) - (\lambda_2 - \lambda_3) \in Z_1 - Z_1 \subseteq \mathbb{Z}^2, \quad (103)$$

which contradicts (71) and $\lambda_3 - \lambda_1 \in Z_2$. The same reason shows that $\lambda_2 - \lambda_3 \notin \tilde{Z}_2$.

(ii) If $\lambda_2 - \lambda_3 \in Z_2$, then, by Proposition (2.2.3),

$$\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) + (\lambda_3 - \lambda_1) \in Z_2 + Z_2 \subseteq \tilde{Z}_2, \quad (104)$$

which contradicts (71) and $\lambda_2 - \lambda_1 \in Z_1$. The same reason shows that $\lambda_2 - \lambda_3 \notin \tilde{Z}_1$.

Similarly, we have the following facts that:

$\lambda_2 - \lambda_4$ cannot belong to the sets (or small boxes) $Z_1, Z_3, \tilde{Z}_1, \tilde{Z}_3$ (105);

$\lambda_3 - \lambda_4$ cannot belong to the sets (or small boxes) $Z_2, Z_3, \tilde{Z}_2, \tilde{Z}_3$ (106).

Hence, from (102), (105) and (106), we have

$$\lambda_2 - \lambda_3 \in Z_3 \text{ or } \tilde{Z}_3;\ \lambda_2 - \lambda_4 \in Z_2 \text{ or } \tilde{Z}_2;\ \lambda_3 - \lambda_4 \in Z_1 \text{ or } \tilde{Z}_1 \quad (107)$$

which is impossible. To see this, we only consider the following two typical cases:

(i') If

$$\lambda_2 - \lambda_3 \in Z_3, \qquad \lambda_2 - \lambda_4 \in Z_2, \qquad \lambda_3 - \lambda_4 \in Z_1,$$

then, by Proposition (2.2.3), the above Box 2 becomes the following Box 3:

Z_1	Z_2	Z_3	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$
			$\lambda_1 - \lambda_2$	$\lambda_1 - \lambda_3$	$\lambda_1 - \lambda_4$
$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_4 - \lambda_1$			
$\lambda_3 - \lambda_4$	$\lambda_2 - \lambda_4$	$\lambda_2 - \lambda_3$			
			$\lambda_4 - \lambda_3$	$\lambda_4 - \lambda_2$	$\lambda_3 - \lambda_2$

Box 3[102]:

We apply Propositions (2.2.3), (2.2.3) to the elements of sets Z_1 and Z_3 (or $\tilde{Z}_1$ and $\tilde{Z}_3$) respectively. Since

$$(\lambda_2 - \lambda_1) - (\lambda_3 - \lambda_4) = (\lambda_4 - \lambda_1) + (\lambda_2 - \lambda_3),$$

the left-hand side is in $Z_1 - Z_1 \subseteq \mathbb{Z}^2$ and the right-hand side is in $Z_3 + Z_3 \subseteq \tilde{Z}_3$, which leads to a contradiction by (71). Also, by Proposition (2.2.3), the elements in Z_1 and Z_2 (or in $\tilde{Z}_1$ and $\tilde{Z}_2$) have the character that

$$(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_4) = (\lambda_3 - \lambda_1) + (\lambda_2 - \lambda_4) \in \tilde{Z}_1 \cap \tilde{Z}_2,$$

which contradicts (71).

(ii') If

$$\lambda_2 - \lambda_3 \in \tilde{Z}_3, \qquad \lambda_2 - \lambda_4 \in \tilde{Z}_2, \qquad \lambda_3 - \lambda_4 \in Z_1,$$

then, by Proposition (2.2.3), the above Box 2 becomes the following Box 4:

Z_1	Z_2	Z_3	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$
			$\lambda_1 - \lambda_2$	$\lambda_1 - \lambda_3$	$\lambda_1 - \lambda_4$
$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	$\lambda_4 - \lambda_1$			
$\lambda_3 - \lambda_4$				$\lambda_2 - \lambda_4$	$\lambda_2 - \lambda_3$
	$\lambda_4 - \lambda_2$	$\lambda_3 - \lambda_2$	$\lambda_4 - \lambda_3$		

Box 4[102]:

By Proposition (2.2.3), the elements in Z_2 and Z_3 (or in $\tilde{Z}_2$ and $\tilde{Z}_3$) have the character that

$$(\lambda_3 - \lambda_1) + (\lambda_4 - \lambda_2) = (\lambda_4 - \lambda_1) + (\lambda_3 - \lambda_2) \in \tilde{Z}_2 \cap \tilde{Z}_3,$$

which contradicts (71). Another way to deduce a contradiction is to apply Propositions (2.2.3), (2.2.3) on the elements of sets Z_1 and Z_2 (or $\tilde{Z}_1$ and $\tilde{Z}_2$) respectively. Since

$$(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_4) = (\lambda_3 - \lambda_1) - (\lambda_4 - \lambda_2),$$

the left-hand side is in $Z_1 + Z_1 \subseteq \tilde{Z}_1$ and the right-hand side is in $Z_2 - Z_2 \subseteq \mathbb{Z}^2$, which also leads to a contradiction by (71). This completes the proof of Case 1.

Case 2. By Proposition (2.2.3), we see that in this case, each set contains elements (or differences) in the following Box 5:

Z_1	Z_2	Z_3	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$
$\lambda_1 - \lambda_2$	$\lambda_1 - \lambda_3$				$\lambda_1 - \lambda_4$
		$\lambda_4 - \lambda_1$	$\lambda_2 - \lambda_1$	$\lambda_3 - \lambda_1$	

Box 5[102]:

The other elements in (93) are also in certain small boxes of Box 5. As in Case 1, we have the following facts that:

$\lambda_2-\lambda_3$ cannot belong to the sets (or small boxes) $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$ (108);

$\lambda_2-\lambda_4$ cannot belong to the sets (or small boxes) $Z_1, Z_3, \tilde{Z}_1, \tilde{Z}_3$ (109);

$\lambda_3-\lambda_4$ cannot belong to the sets (or small boxes) $Z_2, Z_3, \tilde{Z}_2, \tilde{Z}_3$ (110).

Hence, from (108)–(110), we have

$$\lambda_2-\lambda_3 \in Z_3 \text{ or } \tilde{Z}_3; \lambda_2-\lambda_4 \in Z_2 \text{ or } \tilde{Z}_2;\ \lambda_3-\lambda_4 \in Z_1 \text{ or } \tilde{Z}_1 \quad (111)$$

which is impossible. The reason is the same as in Case 1. This completes the proof of Case 2.

Case 3. By Proposition (2.2.3), (2.2.3), we see that in this case, each set contains elements (or differences) in the following Box 6:

Z_1	Z_2	Z_3	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$
$\lambda_1-\lambda_2$ $\lambda_2-\lambda_1$ $\lambda_3-\lambda_4$	$\lambda_1-\lambda_3$		$\lambda_1-\lambda_4$ $\lambda_2-\lambda_1$ $\lambda_4-\lambda_2$	$\lambda_3-\lambda_1$	

Box 6[102]:

The other elements in (93) are also in certain small boxes of Box 6. As in Case 1, we have the following facts that:

$\lambda_2-\lambda_3$ cannot belong to the sets (or small boxes) $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$ (112)

$\lambda_3-\lambda_4$ cannot belong to the sets (or small boxes) $Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2$ (113)

Hence, from (112) and (37), we have

$$\lambda_2-\lambda_3 \in Z_3 \text{ or } \tilde{Z}_3;\ \lambda_3-\lambda_4 \in Z_3 \text{ or } \tilde{Z}_3 \quad (114)$$

which is impossible. The reason is the same as in Case 1. This completes the proof of Case 3.

Hence any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements. One can obtain many such orthogonal systems which contain 3 elements. For example, the exponential function systems E_Λ with Λ given by (91) or with Λ given by

$$\Lambda=\{0,s_1,s_2\}\subset\mathbb{R}^2 \text{ for each } s_1\in Z_2 \text{ and } s_2\in\tilde{Z}_2 \quad (115)$$

or with Λ given by

$$\Lambda=\{0,s_1,s_2\}\subset\mathbb{R}^2 \text{ for each } s_1\in Z_3 \text{ and } s_2\in\tilde{Z}_3 \quad (116)$$

are also the three elements orthogonal systems in $L^2(\mu_{M,D})$ (note that in each type, Z_j and $\tilde{Z}_j$ have different representations according to the corresponding Proposition (2.2.4)–(2.2.15) or twelve subcases). This shows that the number 3 is the best. The proof of Theorem is complete.

The non-spectral self-affine measure problem mentioned depends fundamentally on the characterization of the zero set $Z(\hat{\mu}_{M,D})$. For any finite set $D\subset\mathbb{R}^2$ of the cardinality $|D|=3$ or 4, one can obtain the certain expression for the set $Z(\hat{\mu}_{M,D})$ similar to (18). But it is more difficult to obtain some characteristic properties on this set. On the other hand, the condition $ac\in\mathbb{Z}\backslash(3\mathbb{Z})$ in Theorem is just $|D|\notin W(m)$. Following Conjecture (2.2.1), it is reasonable to extend Theorem further as shown in Conjecture (2.2.16) below.

Conjecture (2.2.16) [102]: For an expanding integer matrix M and the three elements digit set D given by

$$M=\begin{bmatrix}a & b\\ d & c\end{bmatrix}\in M_2(\mathbb{Z}) \quad and \quad D=\left\{\binom{0}{0},\binom{1}{0},\binom{0}{1}\right\} \quad (117)$$

if $ac-bd\notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 3 is the best.

We have proved Conjecture (2.2.16) in the case when $d=0$ or $b=0$. It should be pointed out that

Conjecture (2.2.16) also holds if $a + c = 0$. In fact, in this interesting case, we can write $\hat{\mu}_{M,D}(\xi)$ as

$$\hat{\mu}_{M,D}(\xi) = \hat{\mu}_{\widetilde{M},D}(\xi) \cdot m_D(M^{*-1}\xi) \cdot \hat{\mu}_{\widetilde{M},D}(M^{*-1}\xi), \quad (118)$$

where $\widetilde{M} = (a^2 + bd)I_2$ and I_2 is the 2×2 identity matrix. Since the zero set $Z(\hat{\mu}_{\widetilde{M},D}) = Z_1 \cup \tilde{Z}_1$ with $Z_1 \cap \tilde{Z}_1 = (Z_1 \cup \tilde{Z}_1) \cap \mathbb{Z}^2 = \emptyset$ for two concrete sets Z_1 and $\tilde{Z}_1$

$$\left\{\xi \in \mathbb{R}^2 : \hat{\mu}_{\widetilde{M},D}(M^{*-1}\xi) = 0\right\} = M^*\left(Z_1 \cup \tilde{Z}_1\right) \subseteq \{\xi \in \mathbb{R}^2 : m_D(M^{*-1}\xi) = 0\}. \quad (119)$$

The set $\{\xi \in \mathbb{R}^2 : m_D(M^{*-1}\xi) = 0\}$ can be expressed as two disjoint sets, say Q_1 and $\tilde{Q}_1$. Then $Z(\hat{\mu}_{M,D}) = Z_1 \cup \tilde{Z}_1 \cup Q_1 \cup \tilde{Q}_1$. Such expression on the zero set $Z(\hat{\mu}_{M,D})$ can be further simplified as. Hence, the same method shows that Conjecture (2.2.16) holds in the case when $a + c = 0$. Here the number 3 matches the cardinality of $|D|$, and we need not divide $|det(M)|$ or $|D|$ into the two cases: $|D| < |det(M)|$ and $|D| > |det(M)|$. The all known results on the non-spectral self-affine measure problem are in the case $|D| < |det(M)|$. In the IFS $\{\phi_d\}_{d\in D}$, the condition $|D| \geq |det(M)|$ is necessary for $T(M, D)$ to have positive Lebesgue measure. For the integral self-affine tile $T(M, D)$, there are infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$ (cf. [28, p. 636]). However this conclusion does not hold in the case when $|D| > |det(M)|$, even if $T(M, D)$ has positive Lebesgue measure. For example, consider the pair (M, D) given by

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{Z}) \quad and \quad D = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}; \quad (120)$$

we see that $T(M, D)$ has positive Lebesgue measure and $|D| > |det(M)|$, but there are at most 3 mutually orthogonal exponentials in $L^2(\mu_{M,D})$, and the number 3 is the best.

Finally we would like to point out that for any 2×2 expanding matrix $M_1 \in M_2(\mathbb{R})$ and any digit sets $D_1 = \{0, d_1, d_2\} \subset \mathbb{R}^2$ (not necessarily an integer matrix and an integer set), if $P = [d_1, d_2]$ is an invertible 2×2 matrix (whose column vectors are d_1 and d_2) such that

$$P^{-1}M_1P = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \text{ with } a, b, c \in \mathbb{Z} \text{ and } ac \in \mathbb{Z}\backslash(3\mathbb{Z}), \quad (121)$$

then μ_{M_1,D_1} -orthogonal exponentials contain at most 3 elements and the number 3 is the best. This follows from the fact that

$$Z\left(\hat{\mu}_{M,D}\right) = P^*\left(Z\left(\hat{\mu}_{M_1,D_1}\right)\right) \quad (122)$$

(see [117, p. 208]) and the above Theorem.

Section (2.3): A Class of Moran Measures on R

For μ be a compactly supported Borel probability measure on $\mathbb{R}$. We call μ a spectral measure if there exists a discrete set $\Lambda \subseteq \mathbb{R}$ such that $E(\Lambda) = \{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ forms an orthonormal basis (Fourier basis) for $L^2(\mu)$. The set Λ is called a spectrum for μ; we also say that (μ, Λ) is a spectral pair. We will study the spectral property of a class of Moran measures $\mu_{b,D,\{n_k\}}$ on $\mathbb{R}$ generated by the Moran iterated function system (Moran IFS)

$$\{f_{k,d}(x) = b^{n_{k-1}-n_k}(x + d) : d \in D, k \in \mathbb{N}\},$$

where $b \geq 2$ is an integer, D is a finite subset of $\mathbb{Z}$, $n_0 = 0$ and $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ is a strictly increasing sequence. We use $\mathbb{N}$ to denote the set of positive integers, and $\#D$ to denote the cardinality of the finite set D.

More explicitly, the Moran measure $\mu_{b,D,\{n_k\}}$ considered here can be expressed as the infinite convolution product measure

$$\mu_{b,D,\{n_k\}} = \delta_{b^{-n_1}D} * \delta_{b^{-n_2}D} * \cdots * \delta_{b^{-n_k}D} * \cdots \quad (123)$$

in the weak convergence and is supported on a uniquely nonempty compact set $T_{b,D,\{n_k\}}$, which is called a Moran set of the Moran IFS $\{f_{k,d}: d \in D, k \in \mathbb{N}\}$ (see [138]). Here, the symbol $\delta_D$denotes the atomic measure

$$\delta_D = \frac{1}{\#D} \sum_{d\in D} \delta_d \ , \quad (124)$$

where δ_d is the Dirac point mass measure at the point d and $rE = \{rx : x \in E\}$. The compact set $T_{b,D,\{n_k\}}$ has an explicit expression by the radix expansion

$$T_{b,D,\{n_k\}} := \sum_{k=1}^{\infty} b^{-n_k} D = \left\{ \sum_{k=1}^{\infty} b^{-n_k} d_k : d_k \in D \right\}. \quad (125)$$

In general, the Moran sets $T(b, D, \{n_k\}) := T_{b,D,\{n_k\}}$behave like a fractal, highly non-linear, and they include complicated geometries, and the Moran measures $\mu_{b,D,\{n_k\}}$include the classical self-similar measure $\mu_{b,D} := \mu_{b,D,\mathbb{N}}$ and the restriction of 1-dimensional Lebesgue measure $\mathcal{L}$. These Moran sets and Moran measures have connections with a number of areas in mathematics, such as harmonic analysis, wavelet theory, multifractal analysis, algebraic number theory, dynamical system, and others (see [139]).

It is well known that the study of spectral measure has a long history. It was attracted more attention from Fuglede'sspectral set conjecture [143]in 1974, which said that the d-dimensional Lebesgue measure $\mathcal{L}^d$ restricted on a Borel set $\Omega \subseteq \mathbb{R}^d$is a spectral measure if and only if Ω is a translation tile. Although the conjecture has been proved to be false by Tao and others in both directions in dimension three or higher [140], it is still suggestive in the study of spectral measure.

There has been a wide range of interests in expanding the classical Fourier analysis to fractal or more general probability measures after the pioneer work of Jorgensen and Pedersen [136]in 1998 [138]. One of the central themes of this area of research involves constructing Fourier bases in $L^2(\mu)$, where μ is a measure which is generated by the iterated function systems. If so, one can develop L^2-Fourierrepresentation of periodic functions on the real line (or $\mathbb{R}^d$) for general classes of fractal measures including $\mu_{b,D,\{n_k\}}$. Indeed, there is relatively few classes of fractal spectral measures that is known. We give some new singular spectral measures.

Recall that a pair (b, D) is called admissible if there exists a finite subset $C \subseteq \mathbb{Z}$ such that $\#D = \#C = q$ and the matrix

$$q^{-1/2}[e^{2\pi i b^{-1} dc}]_{d\in D, c\in C}$$

is unitary (usually the pair $(b^{-1}D, C)$ is called a compatible pair, see Definition(2.3.3)). In $\mathbb{R}$, the well known result proved by Łaba and Wang [149]is that if (b, D) is admissible, then $\mu_{b,D}$ is a spectral measure. An and He [140] proved that $\mu_{b,D,\{n_k\}}$ is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ and a consecutive digit set $D = \{0, 1, \cdots, r-1\}$with $r|b$. In the present , we follow their research to consider the spectral property of $\mu_{b,D,\{n_k\}}$ and formulate the following conjecture.

Conjecture (2.3.1)[137]: Let $b \geq 2$ be an integer. Let D be a finite subset of $\mathbb{Z}$ such that (b, D) is admissible. Then the Moran measure $\mu_{b,D,\{n_k\}}$ is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty}$in $\mathbb{N}$.

We will prove that Conjecture(2.3.1)is true in several cases. We first show that Conjecture(2.3.1)is true under a condition of Strichartz [16] 3; as a consequence, we obtain Conjecture(2.3.1) when the cardinality of the finite digit set D is 2 or 3. Also, we obtain

that $\mu_{b,D,\{n_k\}}$ is a spectral measure for certain 'lacunary' sequence $\{n_k\}_{k=1}^{\infty}$in. We then get a sufficient condition such that Conjecture(2.3.1)holds, in which the condition is inspired by [145]. we get that the measure $\mu_{b,D,\{n_k\}}$ is a spectral measure when the sequence $\{n_k\}_{k=1}^{\infty}$is ultimately periodic.

We introduce a basic definition and some facts; see especially Proposition(2.3.2)and Lemma(2.3.5). The data of compatible pair (Definition(2.3.3)) allows us to create orthogonal set and then to check the conditions in Proposition(2.3.2)in the following.

The Fourier transform plays an important role in the study of spectral measure. From(123), the Fourier transform $\hat{\mu}_{b,D,\{n_k\}}$ of the Moran measure $\mu_{b,D,\{n_k\}}$ is

$$\hat{\mu}_{b,D,\{n_k\}}(\xi) = \int e^{2\pi i\xi x} d\mu_{b,D,\{n_k\}}(x) = \prod_{k=1}^{\infty} \hat{\delta}_D(b^{-n_k}\xi) \ (\xi \in \mathbb{R}), \quad (126)$$

where

$$\hat{\delta}_D(\xi) = \int e^{2\pi i\xi x} d\delta_D(x) = \frac{1}{\#D} \sum_{d\in D} e^{2\pi id\xi} \ (\xi \in \mathbb{R}). \quad (127)$$

Then, it is clear that a discrete set $\Lambda \subseteq \mathbb{R}$ such that the family $E(\Lambda) = \{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ is an orthogonal system in $L^2(\mu_{b,D,\{n_k\}})$if and only if

$$\sum_{\lambda\in\Lambda} \left|\hat{\mu}_{b,D,\{n_k\}}(\xi+\lambda)\right|^2 \le 1 \ (\xi \in \mathbb{R}). \quad (128)$$

In this case, we also say that Λ is an orthogonal set for $\mu_{b,D,\{n_k\}}$.

Let

$$Q_\Lambda(\xi) := \sum_{\lambda\in\Lambda} \left|\hat{\mu}_{b,D,\{n_k\}}(\xi+\lambda)\right|^2 \ (\xi \in \mathbb{R}).$$

The well known result of Jorgensen and Pedersen [146] shows that Q_Λ has an entire analytic extension to C if Λ is an orthogonal set for $\mu_{b,D,\{n_k\}}$. The following is a universal test which allows us to decide whether an orthogonal set Λ is a spectrum for the measure $\mu_{b,D,\{n_k\}}$.

Proposition (2.3.2) [137]: (See [146].)If Λ is an orthogonal set for $\mu_{b,D,\{n_k\}}$, then the following statements are equiv-alent:

(i) The orthogonal set Λ is a spectrum for $\mu_{b,D,\{n_k\}}$;

(ii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} \left|\hat{\mu}_{b,D,\{n_k\}}(\xi+\lambda)\right|^2 = 1, \forall\, \xi \in \mathbb{R}$;

(iii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} \left|\hat{\mu}_{b,D,\{n_k\}}(\xi+\lambda)\right|^2 = 1, \forall\, \xi \in (-a,a), a > 0.$

The equivalence of (i) and (ii) is a direct consequence of Stone–Weierstrass theorem and Parseval identity, while the equivalence of (ii) and (iii) is due to the analytic property of the function Q_Λ.

Definition (2.3.3) [137]: Let $b \ge 2$ be a positive integer, and let D and C be two finite subsets of $\mathbb{Z}$. We say that $(b^{-1}D, C)$ forms a compatible pair if $\#D = \#C = q$ and the matrix

$$H_{b^{-1}D,C} := q^{-1/2}[e^{2\pi ib^{-1}dc}]_{d\in D, c\in C} \quad (129)$$

is unitary, i.e. $H_{b^{-1}D,C}, H^*_{H_{b^{-1}D,C}} = I_q$.

we will assume that $(b^{-1}D, C)$ is a compatible pair and $0 \in D \cap C$. Given any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, we will always assume that

$$\mu_k = \delta_{b^{-n_1}D} * \cdots * \delta_{b^{-n_k}D} \quad (k \in \mathbb{N}),$$

$$\Lambda_k(b, C, \{n_k\}) := b^{n_1-1}C + \cdots + b^{n_k-1}C \quad (k \in \mathbb{N}),$$

$$\Lambda(b, C, \{n_k\}) := \{ \sum_{finite} b^{n_j-1}c_j : c_j \in C, j \in \mathbb{N}\}.$$

It is easy to check that the following proposition holds.

Proposition (2.3.4) [137]: (See [16].)Let $b \geq 2$ be a positive integer, D and C be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}D, C)$ is a compatible pair and $0 \in D \cap C$. Then

(i) $\sum_{\lambda \in \Lambda_k(b,C,\{n_k\})} |\hat{\mu}_k(\xi + \lambda)|^2 = 1 (\xi \in \mathbb{R})$;

(ii) The set $\Lambda(b, C, \{n_k\})$ is an infinitely orthogonal set for $\mu_{b,D,\{n_k\}}$.

We denote the zero set of the continuous function g by $Z(g)$:

$$Z(g) = \{\xi : g(\xi) = 0\}.$$

The following lemma gives a characteristic of the zero set $Z(\hat{\delta}_D)$, provided that $(b^{-1}D, C)$is a compatible pair.

Lemma (2.3.5) [137]: Let $b \geq 2$ be a positive integer, D and C be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}D, C)$is a compatible pair. Then for each $\beta \in Z(\hat{\delta}_D)$ there exists $c \in C$ depending on β such that $\beta + \frac{c}{b} \notin Z(\hat{\delta}_D)$.

Proof: Let $D = \{d_1, \cdots, d_q\} \subseteq \mathbb{Z} (q \leq b)$. On the contrary, we suppose that there exists $\beta \in Z(\hat{\delta}_D)$ such that $\beta + \frac{c}{b} \in Z(\hat{\delta}_D)$ for all $c \in C$. Then by (127), we have

$$e^{2\pi i d_1 \beta} e^{2\pi i d_1 \frac{c}{b}} + \cdots + e^{2\pi i d_q \beta} e^{2\pi i d_q \frac{c}{b}} = 0 \; for all \; c \in C,$$

which means that the non-zero column vector $(e^{2\pi i d_1 \beta}, \cdots, e^{2\pi i d_q \beta})^T$ is the solution of the $q \times q$ linear system $Mx = 0$, where $M = [e^{2\pi i d_j \frac{c}{b}}]_{1 \leq j \leq q, c \in C}$ is a $q \times q$ non-singular matrix. This leads to a contradiction.

We prove, under a condition of Strichartz [146], that $\mu_{b,D,\{n_k\}}$ is a spectral measure for any finite subset D of $\mathbb{Z}$ and any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{Z}$. Furthermore, we show that $\mu_{b,D,\{n_k\}}$ is a spectral measure for any finite digit set D when the strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ keeps a certain gap.

Theorem (2.3.6) [137]: Let $b \geq 2$ be a positive integer, D and C be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}D, C)$ is a compatible pair and $0 \in D \cap C$. If the zero set $Z(\hat{\delta}_{b^{-1}D})$ is disjoint from the set $T(b, C)$, then $\left(\mu_{b,D,\{n_k\}}, \Lambda(b, C, \{n_k\})\right)$ is a spectral pair.

Proof: The proof is identical to Theorem 2.8 of [138]. We know by Theorem 3.1 in [24], that the following two statements are equivalent:

(i) $Z(\hat{\delta}_{b^{-1}D}) \cap T(b, C) = \emptyset$;

(ii) $Z(\hat{\mu}_{b,D}) \cap T(b, C) = \emptyset$.

Note that $Z(\hat{\mu}_{b,D})$is closed and $T(b, C)$ is compact. It follows from (ii) that there is a $\delta > 0$ such that $d(Z(\hat{\mu}_{b,D}), T(b, C)) \geq \delta$. Here, d denotes the usual metric on the real line $\mathbb{R}$. Since the function $\hat{\mu}_{b,D}$ is continuous on $\mathbb{R}$, there is an $\epsilon > 0$ such that

$$|\hat{\mu}_{b,D}(t)|^2 > \epsilon, \quad (130)$$

if $t \in \{x \in \mathbb{R} : d(x, T(b, C)) \leq \frac{1}{2}\delta\}$.

Fix $\xi \in (-1, 1)$ and set

$$f_k(\lambda) = \begin{cases} |\hat{\mu}_k(\xi + \lambda)|^2, \lambda \in \Lambda_k(b, C, \{n_k\}); \\ 0, \qquad\qquad otherwise, \end{cases}$$

and

$$f(\lambda) = \begin{cases} |\hat{\mu}_{b,D,\{n_k\}}(\xi + \lambda)|^2, \lambda \in \Lambda(b, C, \{n_k\}); \\ 0, \qquad\qquad otherwise. \end{cases}$$

Then, $\lim_{k \to \infty} f_k(\lambda) = f(\lambda)$ for all $\lambda \in \Lambda(b, C, \{n_k\})$. By Proposition(2.3.4), we have

$$\sum_{\lambda\in\Lambda_k(b,C,\{n_k\}} f_k(\lambda) = 1.$$

Obviously, $|\hat{\delta}_{b^{-n_k}D}| \le 1$ for $k \ge 1$. From (126), we have

$$|\hat{\mu}_{b,D,\{n_k\}}(\xi+\lambda)|^2 = |\hat{\mu}_k(\xi+\lambda)|^2 \prod_{j=1}^{\infty} |\hat{\delta}_{b^{-n_{k+j}}D}(\xi+\lambda)|^2$$

$$\ge |\hat{\mu}_k(\xi+\lambda)|^2 \prod_{j=n_k+1}^{\infty} |\hat{\delta}_{b^{-j}D}(\xi+\lambda)|^2$$

$$= |\hat{\mu}_k(\xi+\lambda)|^2 |\hat{\mu}_{b,D}(b^{-(n_{k+1}-1)}(\xi+\lambda))|^2, \lambda \in \Lambda_k(b,C,\{n_k\}).$$

Note that $b^{-(n_{k+1}-1)}\lambda \in T(b,C)$ if $\lambda \in \Lambda_k(b,C,\{n_k\})$. Hence we can choose k_0 large enough such that $k \ge k_0$ implies that $\left|b^{-(n_{k+1}-1)}\right| \le \frac{1}{2}\delta$. Whence from (130), we have

$$f_k(\lambda) \le \epsilon^{-1} f(\lambda), \qquad k \ge k_0.$$

By (128), we have

$$\sum_{\lambda\in\Lambda(b,C,\{n_k\})} \epsilon^{-1} f(\lambda) \le \epsilon^{-1} < \infty.$$

Applying Lebesgue's dominated convergence theorem to $\{f_k\}_{k=1}^{\infty}$, we obtain that

$$\sum_{\lambda\in\Lambda(b,C,\{n_k\})} |\hat{\mu}_{b,D,\{n_k\}}(\xi+\lambda)|^2 = 1, \qquad \xi \in (-1,1).$$

So by Proposition(2.3.2), $\Lambda(b,C,\{n_k\})$ is a spectrum for $\mu_{b,D,\{n_k\}}$. This completes the proof of Theorem(2.3.6).

Inspired by the proof of Theorem(2.3.6), we can determine the following class of sequences $\{n_k\}_{k=1}^{\infty}$ such that the measures $\mu_{b,D,\{n_k\}}$ are spectral measures.

Theorem (2.3.7) [137]: Let $b \ge 2$ be a positive integer, and let D be a finite subset of $\mathbb{Z}$ such that (b,D) is admissible. If there is a $K_0 \in \mathbb{N}$ such that $k \ge K_0$ implies that $n_{k+1} \ge n_k + N$ holds for some N (only depending on b,D), then $(\mu_{b,D,\{n_k\}}, \Lambda(b,C,\{n_k\}))$ is a spectral pair.

Proof: We first choose $C \subseteq [-b+2, b-2]$ such that $(b^{-1}D, C)$ forms a compatible pair. Next let f_k, f be as in the proof of Theorem(2.3.6). By doing the same procedure for f_k as in the proof of Theorem(2.3.6), we obtain that

$$f(\lambda) \ge f_k(\lambda)\left|\hat{\mu}_{b,D}(b^{-(n_{k+1}-1)}(\xi+\lambda))\right|^2, \lambda \in \Lambda_k(b,C,\{n_k\}).$$

In order to use Lebesgue's dominated convergence theorem, it is sufficient to show that there is an $\epsilon > 0$ such that

$$\left|\hat{\mu}_{b,D}(b^{-(n_{k+1}-1)}(\xi+\lambda))\right|^2 > \epsilon, \quad \forall\, \lambda \in \Lambda_k(b,C,\{n_k\}), \quad (131)$$

for k large enough and ξ small.

Since $\hat{\mu}_{b,D}$ is continuous and $\hat{\mu}_{b,D}(0) = 1$, there exist positive constants ϵ and δ such that

$$|\hat{\mu}_{b,D}(x)|^2 > \epsilon, \quad \forall\, x \in (-\delta,\delta). \quad (132)$$

Let N be the integer satisfied that $b^{-(N-2)} < \delta/2$. Note that

$$b^{-(n_{k+1}-1)}\lambda \in b^{-(n_{k+1}-1)}(b^{n_1-1}C + b^{n_2-1}C + \cdots + b^{n_k-1}C)$$

for any $\lambda \in \Lambda_k(b,C,\{n_k\})$. Hence, it follows from the assumption of the sequence $\{n_k\}$ that

$$\left|b^{-(n_{k+1}-1)}\lambda\right| \le (b-1)^{-1}b^{-(n_{k+1}-n_k-2)} \le (b-1)^{-1}b^{-(N-2)} < \frac{\delta}{2} \quad (133)$$

holds for $k \ge K_0$. Whence, from (132) and (133), the desired result (131) holds for all $\xi \in [-\delta/4, \delta/4]$. By Proposition(2.3.2), we finished the proof of Theorem(2.3.7).

The following two examples serve to illustrate Theorem(2.3.7).

Example(2.3.8) [137]: If the sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ is Hadamard lacunary, i.e., there exists a constant $q > 1$ such that $n_{k+1} > qn_k$ for all $k \in \mathbb{N}$, then the measure $\mu_{b,D,\{n_k\}}$ is a spectral measure with a spectrum $\Lambda(b, C, \{n_k\})$.

Indeed, we know, from the proof of Theorem(2.3.7), that the condition about the sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ in Theorem(2.3.7) can be replaced by $n_{k+1} \geq n_k + f(k)$ for $k \in \mathbb{N}$, where $f(k)$ is an increasing sequence. The lacunary sequence is contained in the latter case. Also, there are many other sequences, such as classical Fibonacci sequence

Example (2.3.9) [137]: The purpose of this example is to show how to determine N in concrete case. Let $b = 10$, and let $D = \{0, 1, 2, 3, 4\}$. Let $C \subseteq \mathbb{Z}$ such that $(b^{-1}D, C)$ is a compatible pair. So we can check that if the sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ satisfies that $n_{n+1} \geq n_k + 3$ eventually, then the measure $\mu_{b,D,\{n_k\}}$ admits a spectrum $\Lambda(b, C, \{n_k\})$.

Proof: The proof is similar to that of Theorem(2.3.7). From (126) and (127), one has

$$Z(\hat{\mu}_{10,D}) = \bigcup_{k=1}^{\infty} 10^k Z(\hat{\delta}_D),$$

where $Z(\hat{\delta}_D) = \{\frac{a}{5} : a \in \mathbb{Z}\backslash 5\mathbb{Z}\}$. Hence

$$\begin{aligned} Z(\hat{\mu}_{10,D}) &= \{10^k \frac{a}{5} : k \geq 1 \text{ and } a \in \mathbb{Z}\backslash 5\mathbb{Z}\} \\ &= 2\{10^k a : k \geq 0 \text{ and } a \in \mathbb{Z}\backslash 5\mathbb{Z}\}. \end{aligned}$$

This means that 2 is the positive and smallest zero of $\hat{\mu}_{10,D}$. Throughout the proof of Theorem(2.3.7), any number in the interval $(0,2)$ can be chosen as the δ in (132). Without loss of generality, we choose $\delta = 1$. It follows from (133) that the inequality $10^{-(N-2)} < 1/2$ implies that $N \geq 3$. Hence if the sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ satisfies that $n_{k+1} \geq n_k + 3$ eventually, then (131) holds for all $\xi \in [-1/4, 1/4]$. By Proposition(2.3.2), the corresponding measure $\mu_{10,D,\{n_k\}}$ is a spectral measure.

At the end, we will deal with the spectral property of the measure $\mu_{b,D,\{n_k\}}$ when the cardinality of the finite set is 2 or 3.

Theorem (2.3.10) [137]: Let $b \geq 2$ be a positive integer, and let D be a finite subset of $\mathbb{Z}$ with the cardinality 2 such that (b, D) is admissible. Then for any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, the measure $\mu_{b,D,\{n_k\}}$ is a spectral measure.

Proof: Without loss of generality, we assume that $\{n_k\}_{k=1}^{\infty}$ is a proper subset of $\mathbb{N}$, and $D = \{0, 1\}$. Since (b, D) is admissible, there exists $C = \{0, c\} \subseteq \mathbb{Z}$ such that $(b^{-1}D, C)$ is a compatible pair, which is equivalent to say that 2×2 matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & e^{2\pi i \frac{c}{b}} \end{pmatrix}$$

is unitary. This implies that $c \in \frac{b}{2}(2\mathbb{Z} + 1)$. Hence $b \in 2\mathbb{N}$.

Let $b = 2b_0, b_0 \in \mathbb{N}$. By taking $c = b_0 p$ with $p \in 2\mathbb{Z} + 1$, we get a compatible pair $(b^{-1}D, C)$. Furthermore, by (127) and (125), we obtain that

$$\hat{\delta}_{b^{-1}D}(\xi) = \frac{1}{2}\left(1 + e^{2\pi i b^{-1}\xi}\right) = 0 \Leftrightarrow \xi \in b_0(2\mathbb{Z} + 1), \quad (134)$$

and

$$T(b, C) = \left\{\sum_{j=1}^{\infty} b^{-j} c_j : c_j \in C\right\} = b_0 p \left\{\sum_{j=1}^{\infty} b^{-j} c_j : c_j \in \{0, 1\}\right\} \subseteq b_0 p[0, \tfrac{1}{b-1}). \quad (135)$$

Therefore if $p = 1$, we obtain, from (134)and (135), that $T(b, C) \cap Z(\hat{\delta}_{b^{-1}D}) = \emptyset$. By Theorem(2.3.6), $\mu_{b,D,\{n_k\}}$ is a spectral measure. The proof of Theorem(2.3.10)is complete.

Theorem (2.3.11) [137]: Let $b \geq 2$ be a positive integer, and let D be a finite subset of $\mathbb{Z}$ with the cardinality 3 such that (b, D) is admissible. Then for any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, the measure $\mu_{b,D,\{n_k\}}$is a spectral measure.

Proof: We may assume that $D = \{0, d_1, d_2\} \subseteq \mathbb{Z}$. Since (b, D) is admissible, there exists $C = \{0, c_1, c_2\} \in \mathbb{Z}$ such that $(b^{-1}D, C)$ is a compatible pair, which is equivalent to say that 3×3 matrix

$$H_{b^{-1}D,C} := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i \frac{d_1}{b} c_1} & e^{2\pi i \frac{d_1}{b} c_2} \\ 1 & e^{2\pi i \frac{d_2}{b} c_1} & e^{2\pi i \frac{d_2}{b} c_2} \end{pmatrix} \quad (136)$$

is unitary. Because the distinct rows of the matrix $H_{b^{-1}D,C}$ are mutually orthogonal, we obtain that $d_j c_k \in b(\pm\frac{1}{3} + \mathbb{Z})$ for $j, k = 1, 2$. Hence $b \in 3\mathbb{N}$.

Let $b = 3b_0, b_0 \in \mathbb{N}$. With an easy computation, from (127),we have

$$\hat{\delta}_D(\xi) = \frac{1}{3}(1 + e^{2\pi i d_1 \xi} + e^{2\pi i d_2 \xi}) = 0 \Leftrightarrow \xi \in \pm\frac{1}{3} + \mathbb{Z} \; and \; \{d_1, d_2\} \equiv \{1, 2\} \, (mod 3).$$

Hence,

$$\hat{\delta}_{b^{-1}D}(\xi) = 0 \Leftrightarrow \xi \in b_0(\pm 1 + 3\mathbb{Z}) and \, \{d_1, d_2\} \equiv \{1, 2\} (mod 3). \, (137)$$

Taking $C = \{0, (3n+1)b_0, (3n-1)b_0\}$, where $n \in \mathbb{N} \cup \{0\}$, we obtain that $(b^{-1}D, C)$ is a compatible pair. Furthermore,

$$T(b, C) = \left\{\sum_{j=1}^{\infty} b^{-j} c_j : c_j \in C\right\} \subseteq (3n+1)\left[0, \frac{b_0}{b-1}\right]. \, (138)$$

It follows from (137) and (138) that $Z(\hat{\delta}_{b^{-1}D}) \cap T(b, C) = \emptyset$ if $C = \{0, b_0, -b_0\}$. Then by Theorem(2.3.6), the measure $\mu_{b,D,\{n_k\}}$ is a spectral measure. The proof of Theorem(2.3.11) is complete.

We will prove that $\mu_{b,D,\{n_k\}}$ is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ under the assumption that $0 \in C \subseteq \{0, 1, \cdots, b-1\}$ such that each element in the attractor $T(b, C \cup (-C))$ (see formula (125)) has a unique radix expansion in base b. Before stating our main result, we first introduce and not include the proof of the following technique lemma, since it follows readily from Lemma 3.1 and Lemma 4.3 in [145].

Lemma (2.3.12) [137]: Let $b \geq 2$ be a positive integer and D be a finite set of $\mathbb{Z}$. Suppose that $C + C \subseteq \{0, 1, \cdots, b-1\}$ such that $(b^{-1}D, C)$forms a compatible pair. Then there is a positive integer L and an integer set $\tilde{C}$ such that $(b^{-1}D + b^{-2}D + \cdots + b^{-L}D, \tilde{C})$ is a compatible pair and

$$Z(\hat{\mu}_{b,D}) \cap T(b^L, \tilde{C}) = \emptyset.$$

Corollary (2.3.13) [137]: Let $b \geq 2$ be a positive integer, and D be a finite set of $\mathbb{Z}$. Suppose that $0 \in C \subseteq \{0, 1, \cdots, b-1\}$ such that $(b^{-1}D, C)$ is a compatible pair and each element in $T(b, C \cup (-C))$ has a unique radix expansion in base b. Then there is a positive integer L and an integer set $\tilde{C}$ such that $(b^{-1}D + b^{-2}D + \cdots + b^{-L}D, \tilde{C})$ is a compatible pair and

$$Z(\hat{\mu}_{b,D}) \cap T(b^L, \tilde{C}) = \emptyset. \quad (139)$$

Proof: We will omit the proof, one can see Lemma 3.1 and Lemma 4.3 in [146]. We would like to point out that the proof of Lemma 4.3 in [145] is easier with the help of Lemma(2.3.5)in our setting. The reader may refer to [148] for more detail.

As a consequence of Corollary(2.3.13), we obtain the main result.

Theorem (2.3.14) [137]: Let $b \geq 2$ be a positive integer, D and C be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}D, C)$ is a compatible pair and $0 \in C \subseteq \{0, 1, \cdots$

$, b-1\}$. Suppose that each element in $T(b, C \cup (-C))$ has a unique radix expansion in base b. Then for any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}, \mu_{b,D,\{n_k\}}$ is a spectral measure.

Proof: Given a sequence $\{n_k\}_{k=1}^{\infty}$. We choose Las in Corollary(2.3.13) and set

$$A_j := (\{n_k\}_{k=1}^{\infty} - jL) \cap (0, L], \qquad j \in \mathbb{N} \cup \{0\}.$$

Then, from properties of compatible pairs (see Lemma 2.2 in [19]) and Corollary(2.3.13), there exist $\tilde{C}_j \subseteq \tilde{C}$ such that $(\sum_{i\in A_j} b^{-i} D, \tilde{C}_j)$ form compatible pairs, where $\tilde{C}$ as in Corollary(2.3.13). Denote by

$$\widetilde{D}_j = \sum_{i\in A_j} b^{L-i} D \,, j \in \mathbb{N} \cup \{0\}.$$

Then $\widetilde{D}_j \subseteq \mathbb{Z}$ and $(b^{-L}\widetilde{D}_j, \tilde{C}_j)$ are compatible pairs. It is easy to check that

$$\mu_{b,D,\{n_k\}} = \delta_{b^{-L}\overline{D}_0} * \delta_{b^{-2L}\overline{D}_1} * \cdots .$$

The proof to be left only need us to use the same standard approximating method as in but we write it down for the sake of completeness.

Fix $k \in \mathbb{N}$. Let $\Gamma_k = \tilde{C}_0 + b^L \tilde{C}_1 + \cdots + b^{(k-1)L} \tilde{C}_{k+1}$ and $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$. We will show that Γ is a spectrum for $\mu_{b,D,\{n_k\}}$. For the purpose of this, we assume that

$$v_k = \delta_{b^{-L}\overline{D}_0} * \delta_{b^{-2L}\overline{D}_1} * \cdots \delta_{b^{-kL}\overline{D}_k}.$$

Then, (v_k, Γ_k) are spectral pairs, and Γ is an orthogonal set for $\mu_{b,D,\{n_k\}}$ as a direct result of compatible pairs. According to Proposition(2.3.4), we have

$$\sum_{\gamma\in\Gamma_k} |\hat{v}_k(\xi+\gamma)|^2 = 1 \ \ and \ \ \sum_{\gamma\in\Gamma} \left|\hat{\mu}_{b,D,\{n_k\}}(\xi+\gamma)\right|^2 \le 1 \quad (\xi \in \mathbb{R}).$$

Fix $\xi \in (-1, 1)$and set

$$f_k(\gamma) = \begin{cases} |\hat{v}_k(\xi+\gamma)|^2, if\ \gamma \in \Gamma_k; \\ 0, \qquad\quad otherwise, \end{cases} \ and \ f(\gamma) = \begin{cases} \left|\hat{\mu}_{b,D,\{n_k\}}(\xi+\gamma)\right|^2, if\ \gamma \in \Gamma; \\ 0, \qquad\quad otherwise. \end{cases}$$

Then $\lim_{k\to\infty} f_k(\gamma) = f(\gamma)$ for $\gamma \in \Gamma$. For any $\gamma \in \Gamma_k$, we have $b^{-(k+1)L}\gamma \in T(b^L, \tilde{C})$and

$$\begin{aligned} \left|\hat{\mu}_{b,D,\{n_k\}}(\xi+\gamma)\right|^2 &= |\hat{v}_k(\xi+\gamma)|^2 \prod_{j>k} \left|\hat{\delta}_{b^{-jL}\overline{D}_j}(\xi+\gamma)\right|^2 \\ &\ge |\hat{v}_k(\xi+\gamma)|^2 \prod_{j\ge 1} \left|\hat{\delta}_{b^{-kL-j}D}(\xi+\gamma)\right|^2 \\ &= |\hat{v}_k(\xi+\gamma)|^2 \left|\hat{\mu}_{b,D}(b^{-kL}(\xi+\gamma))\right|^2. \end{aligned}$$

From Corollary(2.3.13), we choose δ, ϵ to satisfy the following condition

$$|\hat{\mu}_{b,D}(\xi)|^2 \ge \epsilon, \quad for\ all\ \xi \in \left(T\left(b^L, \tilde{C}\right)\right)_{\delta}, \quad (140)$$

where $(T(b^L, \tilde{C}))_{\delta}$ is the δ-neighborhood of $T(b^L, \tilde{C})$. Whence $f(\gamma) \ge \epsilon f_k(\gamma)$ if $b^{-kL} < \delta/2$. Hence by Lebesgue's dominated convergence theorem and Proposition(2.3.2), $\mu_{b,D,\{n_k\}}$ is a spectral measure. The proof of Theorem(2.3.14) is complete.

We give a proposition and an example to illustrate Theorem(2.3.14). The following proposition shows that the cardinality of the digit set D is no more than $\left\lfloor \frac{1}{2}(b+1) \right\rfloor$ under the assumption of Theorem(2.3.14). Here, the symbol $\lfloor x \rfloor$ stands for the lower integer part of the real number x.

Proposition (2.3.15) [137]: Let $b \ge 2$ be a positive integer, $0 \in C \subseteq \{0, 1, \cdots, b-1\}$, and every point in $T(b, C \cup (-C))$ has a unique radix expansion in base b, then $2\#C \le b+1$.

Proof: We first have $\#A = 2\#C - 1$. Suppose that $\#A > b$ and fix $n \ge 1$. Then the set

$$E_n := \left\{ \sum_{j=1}^{n} a_j b^{-j} : a_j \in A, a_n \neq 0 \right\}$$

contains at least $(b+1)^{n-1}b$ different expansions, since each element of $E_n (\subseteq T(b,A))$ has a unique radix expansion in base b. However, for any $w = \sum_{j=1}^{n} a_j b^{-j} \in E_n$, we have

$$|b^n w| \leq (b-1)(b^{n-1} + \cdots + b + 1) \leq b^n - 1.$$

Hence there is no more than $2(b^n - 1) + 1 = 2b^n - 1$ possible different values in E_n. Note that $(b+1)^{n-1}b > b^n - 1$, hence some number in E_n must have at least two different expansions in base b. It is a contradiction. Which completes the proof.

Example (2.3.16) [137]: Let $b = 8$, and let $D = \{0, d_1, d_2, d_3\}$, where $d_j \in D_j = j + 4\mathbb{Z}$ for $j = 1, 2, 3$. Then the measure $\mu_{b,D,\{n_k\}}$is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$.

Proof: Let $C = \{0, 2, 4, 6\}$. Then the matrix in (129) is

$$H_{8^{-1}D,C} := \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

is unitary so $(8, D)$ is admissible. By Theorem(2.3.14), it is enough to show that each element in $T(b, C \cup (-C))$ has a unique radix expansion in base b.

Indeed, it follows from (125)that the self-similar set $K := T(b, C \cup (-C))$ of IFS $\{\varphi_a(x) = b^{-1}(x + a) : a \in C \cup (-C))\}$ is contained in the interval $I := [-6/7, 6/7]$. It is easy to check that $\varphi_a(I) \cap \varphi_{a'}(I) = \emptyset$ for distinct $a, a' \in C \cup (-C)$. Hence $\varphi_a(K) \cap \varphi_{a'}(K) (\subseteq \varphi_a(I) \cap \varphi_{a'}(I)) = \emptyset$ for distinct $a, a' \in C \cup (-C)$. That is, the IFS $\{\varphi_n\}$ satisfies the strong separation condition (SSC). As we all know, SSC implies that each element in $T(b, C \cup (-C))$ has a unique radix expansion in base b. The proof is complete.

We will deal with a class of sequences called ultimately periodic sequence, which is a generalization of arithmetic procession.

Definition (2.3.17) [137]: A sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ is ultimately periodic (respectively periodic) if there exist constants $K, m, T \geq 1$ such that

$$n_{k+T} = n_k + m, \qquad k \geq K \ (respectively\ n_{k+T} = n_k + m, k \geq 1).$$

If $K = T = 1$, then the sequence $\{n_k\}_{k=1}^{\infty}$ is an arithmetic procession.

For example, the sequence $\{1, 4, 5, 7, 8, 10, 11, \cdots\}$ is ultimately periodic, where $K = 2, T = 2, m = 3$. Remove the first term, the sequence $\{4, 5, 7, 8, 10, 11, \cdots\}$ is periodic.

Theorem (2.3.18) [137]: Let $b \geq 2$ be a positive integer, and let D be a finite subset of $\mathbb{Z}$ such that$(b, D)$is admissible. Suppose that the sequence $\{n_k\}_{k=1}^{\infty}$is ultimately periodic. Then the measure $\mu_{b,D,\{n_k\}}$ is a spectral measure.

In order to prove Theorem(2.3.18), we need the following fundamental theorem.

Theorem (2.3.19) [137]: (See [149].)Let $b \geq 2$ be a positive integer, D and C be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}D, C)$ is a compatible pair. Then the self-similar measure $\mu_{b,D}$is a spectral measure. Moreover, if $0 \in C \subseteq [-b + 2, b - 2]$, then $\mu_{b,D}$ is a spectral measure with a spectrum

$$\Lambda(b, C) = \left\{ \sum_{j=1}^{n} b^{j-1} c_j : c_j \in C, for\ all\ n \in \mathbb{N} \right\}.$$

According to the definition of ultimately periodic sequence, there exist positive integers K, T, m with $T \leq m$ such that

$$\left(n_{K+jT},\cdots,n_{K+(j+1)T-1}\right)=(n_K,\cdots,n_{K+T-1})+\ jm,\qquad j\in\mathbb{N}.$$

Then the infinite convolution product measure

$$\mu=\delta_{b^{-n_{K+T}D}}*\ \delta_{b^{-n_{K+T+1}D}}*\cdots$$

exists in the weak convergence. Hence $\mu_{b,D,\{n_k\}}=\mu_{K+T-1}*\mu$. It is sufficient to show that μ is a spectral measure with a spectrum

$$\Lambda'=b^{n_{K+T}-1}C+b^{n_{K+T+1}-1}C+\cdots,\qquad when\ C\subseteq[-b\ +\ 2,b\ -\ 2].$$

In fact, by Proposition(2.3.4), we first have

$$\sum_{\dot\lambda\in\Lambda_{K+T-1}(b,C,\{n_k\})}\left|\hat\mu_{K+T-1}(\xi+\dot\lambda)\right|^2=1\ (\xi\in\mathbb{R}).\ (141)$$

In view of that fact that (μ,Λ') is a spectral pair if and only if

$$\sum_{\ddot\lambda\in\Lambda'}\left|\hat\mu(\xi+\ddot\lambda)\right|^2=\ 1\quad(\xi\in\mathbb{R}),\ (142)$$

it follows from (141), (142)and the identity $\mu_{b,D,\{n_k\}}(\xi)=\hat\mu_{K+T-1}(\xi)\hat\mu(\xi)$ that

$$\begin{aligned}&\sum_{\lambda\in\Lambda(b,C,\{n_k\})}\left|\hat\mu_{b,D,\{n_n\}}(\xi+\lambda)\right|^2\\&\sum_{\dot\lambda\in\Lambda_{K+T-1}(b,C,\{n_k\})}\sum_{\ddot\lambda\in\Lambda'}\left|\hat\mu_{b,D,\{n_n\}}(\xi+\dot\lambda+\ddot\lambda)\right|^2\\&=\sum_{\dot\lambda\in\Lambda_{K+T-1}(b,C,\{n_k\})}\sum_{\ddot\lambda\in\Lambda'}\left|\hat\mu_{K+T-1}(\xi+\dot\lambda+\ddot\lambda)\right|^2\left|\hat\mu(\xi+\dot\lambda+\ddot\lambda)\right|^2\\&=\sum_{\dot\lambda\in\Lambda_{K+T-1}(b,C,\{n_k\})}\left|\hat\mu_{K+T-1}(\xi+\dot\lambda)\right|^2\sum_{\ddot\lambda\in\Lambda'}\left|\hat\mu(\xi+\dot\lambda+\ddot\lambda)\right|^2\\&=\sum_{\dot\lambda\in\Lambda_{K+T-1}(b,C,\{n_k\})}\left|\hat\mu_{K+T-1}(\xi+\dot\lambda)\right|^2\\&=1\quad(\xi\in\mathbb{R}),\end{aligned}$$

which shows that $(\mu_{b,D,\{n_k\}},\Lambda(b,C,\{n_k\}))$is a spectral pair.

Next we show that (μ,Λ') is a spectral pair. Set

$$\widetilde{D}=b^{n_{K+T-1}}\sum_{i=K}^{K+T-1}b^{-n_i}D\ \ and\ \ \tilde{C}=b^{m-n_{K+T-1}}\sum_{i=K}^{K+T-1}b^{n_i-1}C\,.$$

Then both $\widetilde{D}$ and $\tilde{C}$ are subsets of $\mathbb{Z}$, and $(b^{-m}\widetilde{D},\tilde{C})$ forms a compatible pair. Since the sequence $\{n_k\}_{K=K+T}^{\infty}$ is periodic, we have

$$\mu=\delta_{b^{n_{K+T-1}}b^{-m}\bar{D}}*\delta_{b^{n_{K+T-1}}b^{-2m}\bar{D}}*\cdots$$

in the weak convergence and

$$\Lambda'=b^{n_{K+T-1}}\left(\tilde{C}+b^m\tilde{C}+\cdots\right)=:\ b^{n_{K+T-1}}\Lambda\left(b^m,\tilde{C}\right).\ (143)$$

With an easy calculation, $\tilde{C}\subseteq[2\ -b^m,b^m-2]$. So by Theorem (2.3.19), $\Lambda(b^m,\tilde{C})$ is a spectrum for the self-similar measure

$$\mu_{b^m,\bar{D}}=\delta_{b^{-m}\bar{D}}*\delta_{b^{-2m}\bar{D}}*\cdots.$$

That is,

$$\sum_{\lambda\in\Lambda(b^m,\tilde{C})}\left|\hat\mu_{b^m,\bar{D}}(\xi+\lambda)\right|^2=\ 1\qquad(\xi\in\mathbb{R}).\quad(144)$$

It follows from (143), (144) and the identity $\hat\mu(\xi)=\hat\mu_{b^m,\bar{D}}(b^{-n_{K+T-1}}\xi)$ that

$$\sum_{\lambda'\in\Lambda'}|\hat\mu(\xi+\lambda')|^2=\sum_{\lambda\in\Lambda(b^m,\tilde{C})}\left|\hat\mu_{b^m,\bar{D}}(b^{-n_{K+T-1}}\xi+\lambda)\right|^2=\ 1\qquad(\xi\in\mathbb{R}),$$

which shows that (μ,Λ') is a spectral pair. The proof of Theorem(2.3.18) is complete.

Chapter 3

Mmock Fouier Series and Spectrality

We obtain a necessary and sufficient condition on the spectrality . The study generalizes the one dimensional results proved. We shall determine the spectrality and non-spectrality of a class of self-affine measures with decomposable digit sets. We present a method to deal with such case, and clarify the spectrality and non-spectrality of a class of self-affine measures by applying this method

Section (3.1): Transforms Associated with Certain Cantor Measures

For certain Cantor measures μ in $\mathbb{R}^n$, there exists an orthonormal basis for $L^2(\mu)$ of complex exponentials $\{e(x \cdot \lambda)\}_{\lambda \in \Lambda}$ for a discrete set Λ (we use the abbreviation $e(t) = e^{2\pi i t}$. We call Λ a spectrum for μ. The existence of such spectral pairs $\{\mu, \Lambda)$ was established by Jorgensen and Pedersen [170], and in [171] we gave a simplification of part of the proof. The spectrum is by no means unique, as, for example, any translate of a spectrum is again a spectrum; but more radically different choices are also available. We propose to call expansions of functions in $L^2(\mu)$ in terms of such orthonormal bases mock Fourier series because of their resemblance to the genuine article. The original motivation for studying spectral pairs came from a problem of Irving Segal and Fuglede [172], in which the measure μ is taken to be the restriction of Lebesgue measure to an open set Ω in $\mathbb{R}^n$. In that case, the orthonormal basis of exponentials can be used to construct a commuting family of skew-adjoint operators on $L^2(\Omega)$ which extend the derivatives $\{\partial/\partial x_j\}$ on the domain $C_0^\infty(\Omega)$, however, there does not appear to be any natural class of operators which are diagonalized by the basis of exponentials.

A simple example of a spectral pair from [173] is the "middle half" Cantor measure, obtained by successively deleting the middle half of the interval [-1/3, 1/3] and assigning equal measure to the remaining quarters. This is the self-similar probability measure characterized by the identity

$$\mu(A) = \frac{1}{2}\mu(4A - 1) + \frac{1}{2}\mu(4A + 1).$$

A spectrum for this measure is $\Lambda = \{0, 1, 4, 5, 16, 17, 20, 21, \dots\}$, the set of nonnegative integers expressible in base 4 using only the digits 0 and 1.

We extend the construction to a larger class of measures and spectra. We shall see that self-similarity is not an essential feature, but rather we require infinite product structures for both μ and Λ that are linked. We give a complete, self-contained exposition of the construction , using ideas from [178] and [176]. In particular, it will become apparent how much freedom we have in choosing the spectrum Λ. We also give the analogue of the fast Fourier transform for computing mock Fourier series.

The second theme is to develop the L^1 theory of mock Fourier series expansions. Since the exponentials are uniformly bounded, it is natural to consider mock Fourier series expansions for functions in $L^1(\mu)$. Do we have uniqueness? Does the series converge to f in some sense? We are able to answer these questions affirmatively under suitable hypotheses, but the estimates involved are sometimes quite delicate. For the basic example above, it turns out that we do have uniqueness and a very weak form of convergence, owing to the fact that $1 + 2\pi/3 < 4$. The method we use is to find a dense

class of continuous functions whose mock Fourier series converge uniformly. In fact, these functions will be characteristic functions of the basic intervals that define the Cantor set K supporting μ. This argument actually gives uniqueness for mock Fourier series of all finite measures supported by K. Note that this argument has no analogue for genuine Fourier series, where no nontrivial characteristic functions are continuous. On the other hand, we have not found any analogue of the summability methods that work for ordinary Fourier series.

The third theme is to pass from mock Fourier series to mock Fourier transforms, in analogy with the passage from ordinary Fourier series to Fourier transforms. As pointed out in [173], the spectrum Λ has a "fractal in the large" structure ([173]) dual to the more standard fractal structure of μ. But we can also conjoin the two structures into one and consider infinite Cantor measures $\tilde{\mu}$ which are sums of translates of the small Cantor measuresμ by discrete sets with structure similar to the spectrum Λ. The dual object will be another infinite measure $\tilde{\mu}'$ with both small and large scale product structure. The mock Fourier transform will then be an isometry between $L^2(\tilde{\mu})$ and $L^2(\tilde{\mu}')$, set up by the ordinary Fourier transform of $f d\tilde{\mu}$ for f in a suitable dense subspace of $L^2(\tilde{\mu})$. The idea of extending the notion of spectral pairs to such a correspondence between two L^2 spaces occurs already in [179].

For the example $\{\mu, \Lambda\}$ given above, we can take $\tilde{\mu} = \mu + 2\Lambda$ and $\tilde{\mu}' = \mu' + \Lambda$, where $\mu'(A) = \mu(2A)$. To establish the properties of the mock Fourier transform, we need to use the estimates concerning the uniform convergence of mock Fourier series. As a consequence, we obtain a new kind of uncertainty principle for the usual Fourier transform. Let $\widetilde{K}$ and $\widetilde{K}'$ denote the unbounded Cantor sets supporting the measures $\tilde{\mu}$ and $\tilde{\mu}'$. If v is any nonzero finite measure supported on $\widetilde{K}$ (or any translate of $\widetilde{K}$), then $\hat{v}$ cannot vanish identically on $\widetilde{K}'$ (or any translate of $\widetilde{K}'$). This is an uncertainty principle in the spirit of [178] or [179], since the assumption on the support of v means it is highly localized, while the conclusion about $\hat{v}$ means it is highly diffuse. It should be kept in mind that this result is conditional on the precise quantitative relationship between the sets $\widetilde{K}$ and $\widetilde{K}'$, and it is unlikely that there is any qualitative generic statement that implies it.

For example, it is easy to give examples of measures on the line supported on the integers (a set of dimension 0) with Fourier transform vanishing on an open set (a set of dimension 1). So one should not look to the dimensions of $\widetilde{K}$ and $\widetilde{K}'$ for an explanation here. A relationship between the dimension of K and Λ is given in [179].

We expect that there is an analogue of the Poisson summation formula connecting mock Fourier transforms and series via an analogue of the periodization map. Although we are able to write down the formula, for technical reasons we are unable to establish it for an interesting class of functions.

The fourth theme is the analogue of the Shannon Sampling Theorem. Here we study the Fourier transforms of finite measures supported on K; we call such functions K-bandlimited. We show, under the hypotheses, that such functions are determined by their values on Λ, and we give an explicit formula for recovering the function analogous to the Shannon formula, with $\hat{\mu}$ replacing the sine function. A numerical implementation of our sampling formula for some examples has been carried out by Nina Tillman. There are many interesting questions that we cannot answer. We do not know if the hypothesis in Theorem (3.1.8) is necessary. The conditions in Theorems (3.1.13) and (3.1.14) are certainly not sharp; perhaps they are not needed at all for the uniqueness of L^1 mock Fourier series, And there are many aspects of classical Fourier analysis that have not yet been explored. For example, we might weaken the orthonormal basis condition to the

frame (or exact frame) condition. For the examples where a spectrum Λ exists, how much can we perturb Λ and still have a frame of exponentials? For measures for which there is no spectrum, can we find a frame of exponentials? For other work pertaining to Cantor measures and Fourier analysis, see [179] and [172].

Most of the material can be found, explicitly or implicitly, in [174], but we present full details.

Definition (3.1.1)[169]: Let B and L be finite subsets of $\mathbb{R}^n$ of the same cardinality N. We say $\{B, L\}$ is a compatible pair if the $N \times N$ matrix

$$\{N^{-1/2} e(b \cdot \ell)\}_{b \in B, \ell \in L} \tag{1}$$

is unitary, where $e(x) = e^{2\pi i x}$.

Note that if $\{B, L\}$ is a compatible pair, then we can translate either B or L and obtain another compatible pair. Thus, we assume without essential loss of generality that 0 belongs to both B and L. Another trivial transformation among compatible pairs is to replace $\{B, L\}$ by $\{R^{-1}B, R^{*}L\}$ for any $R \in GL(n, \mathbb{R})$.

Lemma (3.1.2) [169]: If $\{B, L\}$ is a compatible pair, then L is a spectrum for the measure

$$\mu = N^{-1} \sum_{b \in B} \delta_b, \tag{2}$$

meaning that $\{e(x \,.\, \ell)\}_{\ell \in L}$ is an orthonormal basis for $L^2(\mu)$.

Lemma (3.1.3)[169]: For any probability measure μ of compact support on $\mathbb{R}^n$, the functions $\{e(x.\lambda)\}_{\lambda \in \Lambda}$ are orthonormal in $L^2(\mu)$ if and only if

$$\hat{\mu}(\lambda_1 - \lambda_2) = 0 \text{ for all } \lambda_1, \lambda_2 \in \Lambda \text{ with } \lambda_1 \neq \lambda_2. \tag{3}$$

Furthermore, they form a complete orthonormal system if and only if also

$$\textstyle\sum_{\lambda \in \Lambda} |\hat{\mu}(t - \lambda)|^2 = 1 \text{ for } t \in \mathbb{R}^n \tag{4}.$$

Proof: Condition (3) is exactly the orthogonality of $e(x.\lambda_1)$ and $e(x \cdot \lambda_2)$ for $\lambda_1 \neq \lambda_2$. Condition (4) is exactly the Parseval equality for the functions $e(x \cdot t)$, which is necessary for completeness, and also sufficient under the hypotheses that μ has compact support.

From (2), we have

$$\hat{\mu}(t) = N^{-1} \sum_{b \in B} e(-b \cdot t). \tag{5}$$

The condition (3) (for $\Lambda = L$) is exactly the orthogonality of the rows of the matrix (1) (functions of b for fixed g), proving the orthonormality of the family $\{e(x \cdot \ell)\}_{\ell \in L}$. (Note that the fact that μ is a probability measure automatically normalizes these functions.) We could deduce completeness by a dimension argument, but it is preferable to observe that we can obtain it directly by verifying (4). Indeed,

$$\begin{aligned} \sum_{\ell \in L} |\hat{\mu}(t - \ell)|^2 &= N^{-2} \sum_{\ell \in L} \sum_{b_1 \in B} \sum_{b_2 \in B} e(-b_1 \cdot (t - \ell)) e(b_2 \cdot (t - \ell)) \\ &= N^{-2} \sum_{\ell \in L} \sum_{b \in B} |e(-b \cdot (t - \ell))|^2 = 1 \end{aligned}$$

by the orthogonality of columns of the matrix (1) (functions of ℓ for fixed b).

Example(3.1.4) [169]: (a) Choose an integer $N \geq 2$, and let $B \subseteq N^{-1}\mathbb{Z}$ be chosen so that $\{Nb\}_{b \in B}$ is a complete set of residues rood N. Similarly, let L be a complete set of residues mod N. Then $\{B, L\}$ is a compatible pair by a theorem known to Gauss (perhaps even before him, to Lagrange). We shall refer to this as a Gaussian compatible pair. Although the choice of residue sets has no influence on the matrix (1), the different compatible pairs that arise are not isometric.

(b) In $\mathbb{R}^2$ let $B = \{(0,0), (\frac{1}{3}, 0), (0, \frac{1}{3}) \}$ and $L = \{(0,0), (2,1), (1,2)\}$. In this case, the matrix (1) is the same as for a Guassian compatible pair with $N = 3$, but this example is not reducible to the Gaussian case.

(c) In $\mathbb{R}^3$ let

$$\boldsymbol{B} = \left\{(0,0\ ,0), \left(\frac{1}{2}, 0,0\right), \left(0, \frac{1}{2}, 0\right), \left(0,0, \frac{1}{2}\right)\right\}$$

and

$$L = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}.$$

In this case, the matrix (1) is not the same as in the Gaussian case. Also, this example is not attainable via the product construction described below.

Lemma (3.1.5) [169]: Let $\{B_0, L_0\}$ and $\{B_1, L_1\}$ be compatible pairs of cardinality N_0 and N_1. Suppose $R_1 \in GL(n, \mathbb{R})$ has the property that

$$(R_1 b_0) \cdot \ell \in \mathbb{Z} \text{ for all } b_0 \in B_0, \ell \in L_1 \quad (6).$$

Then $\{B_0 + R_1^{-1} B_1, L_0 + R_1^* L_1\}$ is a compatible pair of eardinality $N_0 N_1$.

Proof: We need to show

$$\sum_{b_0 \in B_0} \sum_{b_1 \in B_1} e((b_0 + R_1^{-1} b_1) \cdot (\ell_0 - \ell_0' + R_1^*(\ell_1 - \ell_1')) = 0 \quad (7)$$

for all $\ell_0, \ell_0' \in L_0$ and $\ell_1, \ell_1' \in L_1$ such that $(\ell_0, \ell_1) \neq (\ell_0', \ell_1')$. Consider first the case $\ell_0 \neq \ell_0'$. By factoring the exponential and taking the sum over B_0 in (7), we get 0 because $\{B_0, L_0\}$ is a compatible pair, and $e(b_0 \cdot R_1^*(\ell_1 - \ell_1')) = 1$ by (6). In the other case, $\ell_0 = \ell_0'$; so $\ell_1 \neq \ell_1'$, and the sum over B_1 in (7) produces 0 because $\{B_1, L_1\}$ is a compatible pair. Note that (7) also shows that the $N_0 N_1$ elements $\ell_0 + R_1^* \ell_1$ of $L_0 + R_1^* L_1$ are distinct, and this in turn implies that $B_0 + R_1^{-1} B_1$ also has cardinality $N_0 N_1$.

The measure μ associated to $B = B_0 + R_1^{-1} B_1$ via (2) is just the convolution product

$$\mu = \mu_0 * (\mu_1 \circ R_1), \quad (8)$$

where μ_0 and μ_1 are the measures associated with B_0 and B_1 and $\mu_1 \circ R_1(A) = \mu_1(R_1 A)$ for any measurable set A. The Fourier transform is just the ordinary product

$$\hat{\mu}(t) = \hat{\mu}_0(t) \hat{\mu}_1(R_1^*)^{-1} t). \quad (9)$$

It was not strictly speaking necessary to introduce the matrix R_1 into the description, since we could always eliminate it by modifying to a second compatible pair. However, it turns out to be very convenient, since we can then make some further normalizations concerning the compatible pairs we allow. From now on, we always assume $L \subseteq \mathbb{Z}^n$ and $B \subseteq M^{-1} \mathbb{Z}^n$ for some positive integer M. Condition (6) will then be automatically satisfied if we take $R_1 \in GL(n, M_0 \mathbb{Z})$. There is perhaps some loss of generality in making these assumptions, but we are not aware of any examples which cannot be accommodated (allowing trivial modifications). At any rate, it makes it much easier to describe the conditions that guarantee (6) when we iterate the product construction.

Definition (3.1.6) [169]: A compatible tower (one sided) is a sequence (finite or infinite) of compatible pairs $\{B_0, L_0\}, \{B_1, L_1\}, \{B_2, L_2\}, \ldots$ with $B_j \subseteq M_j^{-1} \mathbb{Z}^n$ and $L_j \subseteq \mathbb{Z}^n$, and matrices $R_j \in GL(n, M_{j-1} \mathbb{Z}^n)$ for $j \geq 1$.

For a finite compatible tower, we can form the associated measure

$$\mu = \mu_0 * (\mu_1 \circ R_1) * \left(\mu_2 \circ (R_2 R_1)\right) * \cdots * \left(\mu_k \circ (R_k R_{k-1} \cdots R_1)\right) \quad (10)$$

whose Fourier transform is

$$\hat{\mu}(t) = \hat{\mu}_0(t) \hat{\mu}_1((R_1^*)^{-1} t) \hat{\mu}_2((R_2^*)^{-1} (R_1^*)^{-1} t) \cdots \hat{\mu}_k((R_k^*)^{-1} (R_{k-1}^*)^{-1} \cdots (R_1^*)^{-1} t). \quad (11)$$

Of course, this is just the measure associated with

$$\{B_0 + R_1^{-1} B_1 + R_1^{-1} R_2^{-1} B_2 + \cdots + R_1^{-1} R_2^{-1} \cdots R_k^{-1} B_k, \quad (12)$$
$$L_0 + R_1^* L_1 + R_1^* R_2^* L_2 + \cdots + R_1^* R_2^* \cdots R_k^* L_k\},$$

which is a compatible pair by Lemma (3.1.5) iterated. But we are really interested in infinite compatible towers, where we do not have the description (12) but may still hope to have (10) and (11) hold as infinite products. For this to happen, we need some additional assumptions, the chief one being that the matrices R_j arc expanding. We also need some quantitative control of the expansion rate. The following theorem does not attempt to make the weakest possible hypotheses; in fact, it is easy to construct examples not covered by the theorem for which the conclusions still hold. However, the hypotheses arc easy to check; and, in their absence, it may require very delicate estimates to obtain the same conclusions.

Theorem (3.1.7) [169]: Given an infinite compatible tower whose compatible pairs $\{S_j, L_j)$ and expanding matrices $\{R_j\}$ are all chosen from a finite set of compatible pairs and expanding matrices, the infinite convolution product measure

$$\mu = \mu_0 * (\mu_1 \circ R_1) * \cdots * \left(\mu_k \circ (R_k \cdots R_1)\right) * \cdots \tag{13}$$

exists as a weak limit and is a compactly supported probability measure, whose Fourier transform is given by the infinite product

$$\hat{\mu}(t) = \prod_{k=0}^{\infty} \hat{\mu}_k((R_k^*)^{-1} \cdots (R_1^*)^{-1} t). \tag{14}$$

Moreover, the set of functions $\{e(x \cdot \lambda)\}_{\lambda \in \Lambda}$ for

$$\Lambda = L_0 + R_1^* L_1 \ + \cdots + R_1^* \cdots R_k^* L_k + \cdots \tag{15}$$

is an orthonormal set (each element of (15) is a finite sum, since we are assuming that 0 belongs to each L_k).

Proof: Because the matrices are expanding and finite in number, there exists $r > 1$ such that $\left\|R_j x\right\| \geq r\|x\|$ for all j($\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$). Again because the measures μ_k are finite in number, the infinite product (14) converges uniformly; and this suffices to show that the measures (10) converge weakly to the measure (13). The orthonormality of the functions $\{e(x.\lambda)\}$ for λ in $\Lambda_j = L_0 + R_1^* L_1 + \cdots + R_1^* \cdots R_j^* L_j$ with respect to the measure (10) for $k \geq j$ follows from Lemma (3.1.5) iterated and Lemma (3.1.2). (This also implies that each element of Λ_j has a unique representation.) It is then easy to pass to the limit as $k \to \infty$, keeping j fixed, to conclude that the functions $\{e(x.\lambda)\}_{\lambda \in \Lambda_j}$ are orthonormal with respect to the measure (13). Finally, since $\Lambda = \bigcup_j \Lambda_j$, it follows that $\{e(x \cdot \lambda)\}_{\lambda \in \Lambda}$ is orthonormal.

We are interested in conditions under which Λ is a spectrum for μ; that is, the functions $\{e(x \cdot \lambda)\}_{\lambda \in \Lambda}$ are complete in $L^2(\mu)$. This is not universally the case, since it is easy to obtain μ equal to Lebesgue measure on [0,1] with $\Lambda =$ the nonnegative integers simply by taking all $B_j = \{0, 1/2\}, L_j = \{0, 1\}$ and $R_j = 2$. We would like to verify condition (4) of Lemma (3.1.3). We know this holds for the finite compatible tower (truncating at k), so

$$\sum_{\lambda \in \Lambda_k} \left| \prod_{j=0}^{k} \hat{\mu}_j((R_j^*)^{-1} \cdots (R_1^*)^{-1}(t - \lambda)) \right|^2 = 1. \tag{16}$$

The issue is whether or not we can take the limit in (16) as $k \to \infty$. By Fatou's Lemma the limit is always less than or equal to 1, but in the example above it does go below 1 for t not equal to a positive integer. It is natural to try to use the dominated convergence theorem to prevent this from happening. We follow the method in [174].

Theorem (3.1.8) [169]: Under the hypotheses of Theorem (3.1.7), suppose that the zero set Z_k of the trigonometric polynomial $\hat{\mu}_k$ is separated from the set

$$(R_k^*)^{-1} \cdots (R_1^*)^{-1} L_0 + (R_k^*)^{-1} \cdots (R_2^*)^{-1} L_1 + \cdots + (R_k^*)^{-1} L_{k-1} \tag{17}$$

by a distance $\delta > 0$, uniformly in k, for all large k. Then Λ is a spectrum for μ.

Proof: Fix t, and define

$$f_k(\lambda) = \begin{cases} \prod_{j=0}^{k} \left|\hat{\mu}_j\left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}(t-\lambda)\right)\right|^2 & if \lambda \in \Lambda_k, \\ 0 & otherwise. \end{cases} \tag{18}$$

Let $f(A) = \lim_{k\to\infty} f_k(\lambda)$ and $g(\lambda) = \sup_{k\geq k_0} f_k(\lambda)$, where k_0 will be chosen (depending on t, but not λ). The condition (4) is

$$\sum_{\lambda\in\Lambda} f(\lambda) = I, \tag{19}$$

and we know the sum is ≤ 1 by Fatou's Lemma. Then $g(\lambda)$ will serve as a dominator to establish (19) if we can verify $\sum_{\lambda\in\Lambda} g(\lambda) < \infty$, and this will follow from a lower bound $f(\lambda) \geq \varepsilon g(\lambda)$ uniform in λ. Now let $k_1(\lambda)$ denote $\min\{k: k \geq k_0 \; and \; \lambda \in \Lambda_k\}$. Then, in fact, $g(\lambda) = f_{k_1(\lambda)}(\lambda)$, since the sequence $f_k(\lambda)$ is decreasing in k once $\lambda \in \Lambda_k$, Of course,

$$f(\lambda) = f_{k_1(\lambda)}(\lambda) \prod_{k=k_1(\lambda)+1}^{\infty} \left|\hat{\mu}_k\left((R_k^*)^{-1} \cdots (R_1^*)^{-1}(t-\lambda)\right)\right|^2, \tag{20}$$

so we need a uniform lower bound for the infinite product in (20). The term involving t will be controlled by choosing k_0 large enough that

$$\left|(R_{k_0}^*)^{-1} \cdots (R_1^*)^{-1} t\right| \leq \delta/2.$$

The minus sign is harmless; and since $\lambda \in \Lambda_{k_1(\lambda)}$, we have

$$(R_k^*)^{-1} \cdots (R_1^*)^{-\mathbf{1}}\lambda = (R_k^*)^{-1} \cdots (R_1^*)^{-1}\ell_0 + (R_k^*)^{-1} \cdots (R_2^*)^{-\mathbf{1}}\ell_1 + \cdots + (R_k^*)^{-1} \cdots (R_{k_1(\lambda)+1}^*)^{-1}\boldsymbol{\ell_{k_1}}$$

for some $\ell_j \in L_j$. Note that this is an element in the set (17), since we can always choose $\ell_j = 0$ for $j > k_1$. By the hypothesis, we are always evaluating $\hat{\mu}_k$ in (20) at a point of distance at least $\delta/2$ from the zero set. Since there are only a finite number of functions $\hat{\mu}_k$, this gives a uniform lower bound for all the factors in the infinite product (20).

The rest of the argument is routine, since the sequence of points $(R_k^*)^{-1} \cdots (R_1^*)^{-1}(t-\lambda)$ converges to 0 at an exponential rate, and $\hat{\mu}_k(0) = 1$.

So, in fact, there is an integer k_2, independent of λ, such that the product

$$\prod_{k=k_1(\lambda)+k_2}^{\infty} \left|\hat{\mu}_k\left((R_k^*)^{-1} \cdots (R_1^*)^{-1}(t-\lambda)\right)\right|^2$$

can be bounded below, independent of λ. We are using the finiteness assumptions here to obtain uniform control of

$$\left\|\left(R_{k_1(\lambda)}^*\right)^{-1} \cdots (R_1^*)^{-1}(t-\lambda)\right\| \quad for \; \lambda \in \Lambda_{k_1}.$$

Thus we only need to control the product in (20) with $k_1(\lambda) + 1 \leq k \leq k_1(\lambda) + k_2$, and ε^{k_2} provides the bound if ε is the lower bound for each factor.

Example (3.1.9) [169]: (a) It is not difficult to verify the hypotheses of the theorem in specific examples, simply by estimating the size of elements of (17). For example, suppose we choose $B_j = \{0, \pm 1/2\}$ and $L_j = \{0, \pm 1\}$, with all $R_j = 4$, allowing arbitrary and independent choices of sign. The zero sets Z_j are just the odd integers, while the set (17) lies in the interval $(-1/3, 1/3)$, so $\delta = 2/3$ will do.

This already gives an uncountable number of different measures (not translates of each other), and for each measure an uncountable number of different spectra (not translates of each other). These measures are not self-similar, unless the sequence of signs is periodic.

(b) If we choose the self-similar measure with all $B_j = \{0, 1/2\}$ and $R_j = 4$, we could also choose all $L_j = \{0,5\}$. In this case, the set (17) consists of all numbers of the form $5x$ where $x = 4^{-k_1} + \cdots + 4^{-k_n}$ for some sequence $1 \le k_1 < k_2 < \cdots < k_n$ (in other words, x is represented base 4 by a "decimal" with digits 0 and 1). Since $x < 1/3$, the only zero of $\hat{\mu}_j$ we have to avoid is 1. If $k_1 = 1$, then $5x \ge 5/4$, while if $k_1 \ge 2$, then $5x < 5/12$, so we have a separation of at least $1/4$. On the other hand, if we try to replace 5 by 3 in the choice of L_j, then the theorem does not apply, since 1 is contained in the closure of the set (1 7). It is not known whether or not we have a spectrum in this example.

(c) Here is a kind of "minimal" example. Let $N_j \ge 2$ be chosen arbitrarily from a finite set of integers, let $B_j = \{0, 1/N_j, \ldots, (N_j - 1)/N_j\}, L_j = \{0, 1, \ldots, N_j - 1\}$ and let $R_j = N_{j-1}$ or $2N_{j-1}$, chosen arbitrarily. If we choose $R_j = N_{j-1}$ always, then μ is Lebesgue measure, Λ is the nonnegative integers, and so Λ is not a spectrum. If we make the choice $R_j = 2N_{j-1}$ for infinitely many j, then it is easy to see that μ is supported on a Cantor set. However, the theorem does not always apply. The condition we need is that the set $\{j: R_j = 2N_{j-1}\}$ not have arbitrarily large gaps; i.e., there exists J such that for every k, there exists j satisfying $k \le j \le k + J$ for which $R_j = 2N_{j-1}$. Under this assumption, it is easy to see that every element in the set (17) lies between 0 and $1 - N^{-J}/2$, while the first positive zero of any $\hat{\mu}_j$ is at 1, so we may take $\delta = N^{-J}/2$.

(d) If we choose all (B_j, L_j) equal to the compatible pair in Example (3.1.4) (b), then we can take $R_j = 3I$ (or any multiple of 3). In this case,

$$\hat{\mu}_j(x, y) = \frac{1}{3}(1 + e^{2\pi i x/3} + e^{2\pi i y/3})$$

and the zero set is just $(1, -1) \cup (-1, 1)$ mod $3\mathbb{Z}^2$. So the smallest vector in the zero set has length $\sqrt{2}$, while the vectors of the form (17) have length bounded above by $\sqrt{5}/2$.

(e) In [S4], we discussed some examples in which $\{B_j, L_j\}$ is taken to be the 3-dimensional compatible pair in Example (3.1.4) (c). It was noted there that in this case

$$\hat{\mu}_j(x, y, z) = \frac{1}{4}(1 + e^{i\pi x} + e^{i\pi y} + e^{i\pi z})$$

with zero set generated by points of the form $(1, y, y + 1)$ under permutations of the variables, mod $2\mathbb{Z}^3$, which excludes the region $|x| + |y| + |z| < 2$. By the same reasoning as in (c) above, we may take $R_j = 2I$ or $4I$ arbitrarily as long as the set $\{j: R_j = 4I\}$ does not have arbitrarily large gaps. It was observed in [178] that we can also take all $R_j = -2I$. This is interesting, because it produces a measure whose support K is a connected fractal. Note that the choice of all $R_j = 2I$ leads to a 3-dimensional Sierpinski gasket. Using $R_j = -2I$ introduces an inversion that complicates the geometry, but the situation can be clarified by iterating one more time to produce proper dilations. Instead of thinking of K as a self-similar set generated by the 4-element iterated function system $\{-\frac{1}{2}x + b\}_{b \in B}$, we think of it as generated by the 16-element iterated function system $\{\frac{1}{4}x - \frac{1}{2}b + b'\}_{b,b' \in B}$. Since this system contains the mappings $\frac{1}{4}x + (\frac{k}{4}, 0, 0)$ for $k = -1, 0, 1, 2$, whose attractor is the interval $(x, 0, 0)$ for $-\frac{1}{3} \le x \le \frac{2}{3}$ along the axis, it follows that this interval is contained in K. Similarly, K contains intervals along the other axes; and from this it follows that K is connected.

We now describe a fast algorithm for the computation of approximations to mock Fourier series that is the exact analog of the FFT. We take a finite compatible tower, which may

be thought of as the truncation of an infinite one, which yields the compatible pair (12), which we denote $\{B, L\}$. For a function f defined on B, we want to compute the coefficients $\hat{f}(\lambda)$ of its expansion

$$f(x) = \sum_{\lambda \in L} \hat{f}(\lambda) e(x \cdot \lambda) \quad (21)$$

in a way that minimizes the number of complex multiplications. The coefficients are given by

$$\hat{f}(\lambda) = N^{-1} \sum_{b \in B} f(b) e(-b \cdot \lambda) \quad (22)$$

for $N = N_0 \cdot N_1 \cdots N_k$. If we simply perform all the multiplications called for in (22), then the number is N^2 (we ignore the multiplication by N^{-1} in (22) in counting multiplications), since there are N choices of λ and N distinct multiplications for each λ. By exploiting the product structure of the sets and the triangular form of the dot products $\lambda \cdot b \bmod \mathbb{Z}$, we can reduce the number considerably, down to $O(N \log N)$ when all N_j are of comparable size.

Algorithm (3.1.10) [169]: Define operators $T_1, \ldots, T_{k+1}$ inductively by

$$T_1 f(\ell_0, b_1, \ldots, b_k) = \sum_{b_0 \in B_0} f(b_0 + R_1^{-1} b_1 + \cdots + R_1^{-1} R_2^{-1} \cdots R_k^{-1} b_k) e(-b_0 \cdot \ell_0) \quad (23)$$

and

$$T_{j+1} f(\ell_0, \ldots, \ell_j, b_{j+1}, \ldots, b_k) \quad (24)$$
$$= \sum_{b_j \in B_j} T_j f(\ell_0, \ldots, \ell_{j-1}, b_j, \ldots, b_k)$$
$$e(-b_j \cdot ((R_j^*)^{-1} \ldots (R_1^*)^{-1} \ell_0 + (R_j^*)^{-1} \ldots (R_2^*)^{-1} \ell_1 + \cdots + \ell_j))$$

for $1 \le j \le k$. Then

$$\hat{f}(\ell_0 + R_1^* \ell_1 + \cdots + R_1^* \cdots R_k^* \ell_k) = N^{-1} T_{k+1} f(\ell_0, \ell_1, \ldots, \ell_k), \quad (25)$$

and the number of complex multiplications needed to compute $\hat{f}$ on L is

$$N(N_0 + N_1 + \cdots + N_k). \quad (26)$$

Proof: Note that $b.\lambda$ in (22) can be written as

$$(b_0 + R_1^{-1} b_1 + \cdots + R_1^{-1} R_2^{-1} \cdots R_k^{-1} b_k) \quad (27)$$
$$\cdot (\ell_0 + R_1^* \ell_1 + R_1^* R_2^* \ell_2 + \cdots + R_1^* \cdots R_k^* \ell_k)$$
$$\equiv b_0 \cdot \ell_0 + b_1((R_1^*)^{-1} \ell_0 + \ell_1) + \cdots$$
$$+ b_k \cdot ((R_k^*)^{-1} \cdots (R_1^*)^{-1} \ell_0 + (R_k^*)^{-1} \cdots (R_2^*)^{-1} \ell_1 + \cdots + \ell_k) mod\, \mathbb{Z}$$

since the other terms produce integers by the compatibility condition of the tower. We substitute (27) into (22), factor the exponentials, and write the sum over B as a successive sum over $b_0, b_1, \ldots, b_k$, to obtain

$$\hat{f}(\ell_0 + R^* \ell_1 + \cdots + R_1^* \cdots R_k^* \ell_k) \quad (28)$$
$$= N^{-1} \sum_{b_0 \in B_0} \cdots \sum_{b_k \in B_k} f(b_0 + R_1^{-1} b_1 + \cdots + R_1^{-1} R_2^{-1} \cdots R_k^{-1} b_k)$$
$$e(-b_0 \cdot \ell_0) e(-b_1 \cdot ((R_1^*)^{-1} \ell_0 + \ell_1)) \cdots$$
$$e(-b_k \cdot ((R_k^*)^{-1} \cdots (R_1^*) \ell_0 + \cdots + \ell_k).$$

Doing the sums in that order successively computes $T_1 f, T_2 f, \ldots, T_{k+1} f$, and we obtain (25). The computation of $T_{j+1} f$ for all values of the N variables costs $N N_j$ complex multiplications, which proves (26).

There is a similar fast algorithm to compute the inverse transform (21) for the same cost in complex multiplications.

Let μ and Λ be as in Theorem (3.1.8). We then have L^2 convergence of mock Fourier series based on Λ for functions in $L^2(\mu)$. It is natural to hope for uniform convergence for

functions in some nice subspace of continuous functions and some sort of weak convergence, and hence uniqueness, for functions in $L^1(\mu)$.
Not surprisingly, the two questions are linked. We are interested in the case when the support of μ is a Cantor set (totally disconnected but not discrete). This is typically the case, although there are examples in higher dimensions of measures μ satisfying the hypotheses of Theorem (3.1.8) which are supported on connected fractal sets (see Example (3.1.9) (e)). The support of μ being totally disconnected implies that there are characteristic functions of sets that are continuous, and we are able to show, under suitable hypotheses, that these functions have uniformly convergent mock Fourier series. By duality, this will imply uniqueness, and a kind of weak convergence, for the mock Fourier series of functions in $L^1(\mu)$ or, more generally, measures supported on the Cantor set. It would be desirable also to find a summability method, like the Cesaro or Abel method, which produces uniform limits for all continuous functions and L^1 limits for all L^1 functions
The method is based on the following simple observation.
Lemma (3.1.11) [169]: Let f be a continuous function with mock Fourier series coefficients

$$\hat{f}(\lambda) = \int e(-y \cdot \lambda) f(y) d\mu(y) \quad for\ \lambda \in \Lambda. \tag{29}$$

Suppose that

$$\sum_{\lambda \in \Lambda} |\hat{f}(\lambda)| < \infty. \tag{30}$$

Then the mock Fourier series

$$\sum_{\lambda \in \Lambda} \hat{f}(\lambda) e(x \cdot \lambda) \tag{31}$$

converges to f uniformly and absolutely on the support of μ.
Proof: By Theorem (3.1.8), the series (31) converges to f in $L^2(\mu)$. But (30) implies that (31) converges uniformly and absolutely; in particular, the uniform and L^2 limits must be equal μ-almost everywhere. Since f is continous, the limit of (31) must equal f everywhere on the support of μ.
Let K denote the support of μ. We introduce some notation. Write

$$\tilde{B}_k = B_0 + R_1^{-1} B_1 + \cdots + R_1^{-1} R_2^{-1} \cdots R_k^{-1} B_k. \tag{32}$$

Then we have

$$K = cl\left(\bigcup_{k=0}^{\infty} \tilde{B}_k \right). \tag{33}$$

Now let

$$K_m = cl\left(\bigcup_{k=m+1}^{\infty} \tilde{B}_k \right). \tag{34}$$

Then for any fixed m, K decomposes as

$$K = \bigcup_{b \in \tilde{B}_m} (K_m + b). \tag{35}$$

Under suitable hypotheses, K will be a Cantor set and the decomposition (35) will be disjoint and, in fact, separated. We want to show that (30) holds for f equal to the characteristic function of one of the sets $K_m + b$ in (35). Of course, the translation by b only affects the phase of $\hat{f}(\lambda)$, so we can restrict attention to $f = \chi_{K_m}$.

Lemma (3.1.12) [169]: Suppose (30) holds for $f = \chi_{K_m}$ for every m. Then for any finite measure v supported on K,

$$v(K_m + b) = \sum_{\lambda \in \Lambda} \hat{v}(\lambda) \int_{k_m + b} e(x \cdot \lambda) d\mu(x) \tag{36}$$

(absolute convergence), where

$$\hat{v}(\lambda) = \int e(-y \cdot \lambda) dv(y). \tag{37}$$

In particular, v is uniquely determined by $\{\hat{v}(\lambda)\}_{\lambda \in \Lambda}$. We also have the analogue of the Riemann−Lebesgue Lemma: if v is absolutely continuous with respect to μ, then $\hat{v}(\lambda) \to 0 \; as \; \lambda \to \infty$ in Λ.

Proof: Take $f = \chi_{K_m + b}$. Since the convergence of (31) to f is uniform, we can integrate with respect to v to obtain

$$v(K_m + b) = \sum_{\lambda \in \Lambda} \int \hat{f}(\lambda) e(y \cdot \lambda) dv(y).$$

We can replace A by $-A$ with no essential change, and then substitute the definition of $\hat{f}$ to obtain

$$v(K_m + b) = \sum_{\lambda \in \Lambda} \iint e\big((x - y) \cdot \lambda\big) d\mu(x) dv(y) \; ;$$

this yields (36). The uniqueness follows, since the field of sets generated by $K_m + b$ for all m generates the σ-field of Borel subsets of K.

For the Riemann-Lebesgue Lemma, we note that it holds for $v = f d\mu$ and $f = XK_m + b$. It also holds for all linear combinations of these functions, and these form a dense subspace of $L^1(\mu)$. Since we have the obvious $L^1 \to L^\infty$ bound for mock Fourier series, the Riemann-Lebesgue Lemma holds for all $L^1(\mu)$.

Before discussing the general case, we show how to verify the hypothesis of

Lemma (3.1.12) in the basic example where $B_j = \{0, 1/2\}, L_j = \{0, 1\}$ and $R_j = 4$ for all j. In this case,

$$\left|\hat{\mu}_j(t)\right| = \left|\cos \frac{\pi}{2} t\right|$$

and

$$\left|\hat{X} K_m(\lambda)\right| = 2^{-m} \prod_{k=m}^{\infty} \left| cos \frac{\pi}{2} \frac{\lambda}{4^k} \right|. \tag{38}$$

Each nonzero $\lambda \in \Lambda$ can be represented uniquely as

$$\lambda = \textstyle\sum_{j=1}^{p} 4^{k_j} \quad \text{for} \quad k_1 < k_2 < \ldots < k_p, \text{ for some} p \tag{39}.$$

In estimating $\left|\hat{X} K_m(\lambda)\right|$, we consider only the factors in (38) with $k = k_j$ (the other factors are ≤ 1). Note that

$$4^{-k_j} \lambda \equiv 4^{k_1 - k_j} + \cdots + 4^{k_{j-1} - k_j} + 1 \;\; mod \; 4,$$

SO

$$\left|\cos \frac{\pi}{2} \frac{\lambda}{4^{k_j}}\right| = \left|\sin \frac{\pi}{2} \left(4^{k_1 - k_j} + \cdots + 4^{k_{j-1} - k_j}\right)\right| \leq \frac{2\pi}{3} 4^{k_{j-1} - k_j}. \tag{40}$$

Using (40) in (38) yields

$$\left|\hat{X} K_m(\lambda)\right| \leq 2^{-m} \left(\frac{2\pi}{3}\right)^p 4^{m+1-k_p}. \tag{41}$$

Write $\Lambda_{(k)}$ for the set of $\lambda \in \Lambda$ with $k_p = k$ in (39). For each $p \le k$, there are $\binom{k}{p}$ elements in $\Lambda_{(k)}$ corresponding to that value of p. From (41), we obtain

$$\sum_{\lambda\in\Lambda_{(k)}} \left|\hat{X}K_m(\lambda)\right| \le 2^{-m} \sum_{p=1}^{k} \binom{k}{p} \left(\frac{2\pi}{3}\right)^p 4^{m+1-k} \le 2^{-m} \left(1+\frac{2\pi}{3}\right)^k 4^{m+1-k} \quad (42).$$

When we sum over k, we obtain the desired convergence because

$$1 + \frac{2\pi}{3} < 4 \quad . \quad (43)$$

Next, we present two different methods that can be applied in general. The first, which is probably the best approach for a generic example, uses a different method of estimating $\hat{X}_{K_m}(\lambda)$. But in some cases the method outlined above seems to give better results. Both methods begin with the observation that $\hat{\mu}_j(\ell_j) = 0$ for all $\ell_j \in L_j$ with $\ell_j \neq 0$. The first method uses an estimate of the form

$$\sum_{\ell_j\in L_j} \left|\ell_j\right| \left|\hat{\mu}_j(\ell_j + t)\right| \le A_j|t| \quad \text{for } |t| \le 1, \quad (44)$$

where we simply choose the constant A_j to be the smallest for which (44) holds (or close to the smallest if it is difficult to compute the best constant exactly). In the above example, all $A_j = \pi/2$. By our finiteness assumption, there are only a finite number of different values for A_j. We write

$$r_j = \left\|(R_j^*)^{-1}\right\|. \quad (45)$$

Note that we can use any norm on $\mathbb{R}^n$ for $\left|\ell_j\right|$ in (44), but we must use the same norm to define $\left\|(R_j^*)^{-1}\right\|$ in (45).

Theorem (3.1.13) [169]: Under the hypotheses of Theorem (3.1.8), assume also that

$$\sum_{k=1}^{\infty} \prod_{j=1}^{k} (A_j + N_j) r_j < \infty. \quad (46)$$

Then the mock Fourier series of f converges uniformly to f for $f = \chi_{K_m+b}$ for all m and $b \in \tilde{B}_m$; and for anyfinite measure v supported in K, we have convergence of its mock Fourier series in the sense of (36).

Proof: We have

$$\hat{\chi}_{K_m}(\lambda) = (N_0 \cdots N_{m-1})^{-1} \prod_{j=m}^{\infty} \hat{\mu}_j\left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}\lambda\right). \quad (47)$$

Now each nonzero $\lambda \in \Lambda$ can be written uniquely as

$$\lambda = \sum_{j=0}^{k} R_1^* \cdots R_j^* \ell_j \quad \text{for} \quad \ell_j \in L_j \text{ and } \ell_k \neq 0. \quad (48)$$

Denote by $\Lambda_{(k)}$the set of λ's with the representation (48) for fixed k. We substitute (48) into (47) and estimate only $\hat{\mu}_j$ for $j \le k$ (since all $\left|\hat{\mu}_j(t)\right| \le 1$ for all t). In view of (46), it suffices to show

$$\sum_{\lambda\in\Lambda_{(k)}} \prod_{j=m}^{k} \left|\hat{\mu}_j\left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}\lambda\right)\right| \le c \prod_{j=m}^{k} (A_j + N_j) r_j \quad (49)$$

We begin by noting that

$$\sum_{\substack{\ell_k\in L_k \\ \ell_k\neq 0}} \left|\hat{\mu}_j\left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}\lambda\right)\right| = \sum_{\substack{\ell_k\in L_k \\ \ell_k\neq 0}} \left|\hat{\mu}_j\left(\ell_k + \sum_{j=0}^{k-1} (R_k^*)^{-1} \cdots \left(R_{j+1}^*\right)^{-1} \ell_j\right)\right| \quad (50)$$

$$\leq A_k \sum_{j=0}^{k-1} r_k \cdots r_{j+1} |\ell_j|$$

(this is a sloppy estimate since we do not have the factors $|\ell_k|$ which appear in (44), but we shall be sloppy only this one time). Next, we use different estimates for each term on the right side of (50), namely we use

$$\sum_{\substack{\ell_k \in L_k \\ \ell_k \neq 0}} \left|\hat{\chi}_{K_m}(\lambda)\right| \leq A_k \sum_{j=0}^{k-1} r_k \cdots r_{j+1} |\ell_j| \prod_{i=m}^{k} |\hat{\mu}_i((R_i^*)^{-1} \cdots (R_1^*)^{-1}\lambda)| .$$

We do the sum over $\ell_{k-1} \in L_{k-1}$ next. Note that the term corresponding to $j = k-1$ contains the factor

$$\sum_{\ell_{k-1} \in L_{k-1}} |\ell_{k-1}| \left| \hat{\mu}_{k-1} \left(\ell_{k-1} + \sum_{p=0}^{k-2} (R_{k-1}^*)^{-1} \cdots \left(R_{p+1}^*\right)^{-1} \ell_p \right) \right|,$$

which can be estimated by

$$A_{k-1} \sum_{p=0}^{k-2} r_{k-1} \cdots r_{p+1} |\ell_p|$$

using (44). The remaining terms are independent of ℓ_{k-1}, and so pick up a factor of N_{k-1} under the sum. Thus we have

$$\sum_{\substack{\ell_k \in L_k \\ \ell_k \neq 0}} \sum_{\ell_{k-1} \in L_{k-1}} \left|\hat{\chi}_{K_m}(\lambda)\right| \leq A_k (A_{k-1} + N_{k-1}) r_k r_{k-1}$$

$$\cdot \sum_{j=0}^{k-2} r_{k-2} \cdots r_{j+1} |\ell_j| \prod_{i=m}^{j} |\hat{\mu}_i((R_i^*)^{-1} \cdots (R_1^*)^{-1}\lambda)|.$$

The argument just given can be iterated until we reach the cutoff value $i = m$. The result is

$$\sum_{j=m}^{k} \sum_{\ell_j \in L_j} \left|\hat{\chi}_{K_m}(\lambda)\right| \leq \left(\prod_{j=m}^{k} (A_j + N_j) r_j \right) |\ell_{m-1}|.$$

The sum over ℓ_j for $j < m$ just produces a constant that depends on m. But m is fixed, so we have the desired estimate (49).

We have sacrified some sharpness in the formulation of the theorem, in return for the simplicity of condition (46). In the example discussed earlier, condition (46) still holds because

$$\frac{\pi}{2} + 2 < 4, \tag{51}$$

but there is less wiggle room here than in (43).

The second method uses the estimate

$$\left|\hat{\mu}_j(\ell_j + t)\right| \leq a|t| \text{ for all } \ell_j \neq 0, \tag{52}$$

$\ell_j \in L_j$, in place of(44). We assume that all $\{B_j, L_j\}$ are the same, but not that all R_j are the same. We also need the constants

$$L_{max} = \sup\{|\ell_j| : \ell_j \in L_j\} \tag{53}$$

and

$$c(r) = \sup\{1 + r_j + r_j r_{j-1} + \cdots + r_j r_{j-1} \cdots r_1\}. \tag{54}$$

This method appears to work best when $|\ell_j|$ is the same for all nonzero $\ell_j \in L_j$ and the best constant in (52) is similarly independent of ℓ_j.

Theorem (3.1.14) [169]: Under the hypotheses of Theorem (3.1.8), assume also that all $\{B_j, L_j\}$ are the same and

$$\sum_{k=1}^{\infty} (1 + (N-1)ac(r)L_{max})^k \prod_{j=1}^{k} r_j < \infty. \tag{55}$$

Then the conclusions of Theorem (3.1.13) hold.

Proof: In place of (48), we use the analogue of (39), which we express as

$$\lambda = \textstyle\sum_{j\in J} R_1^* \cdots R_j^* \ell_j \text{ where } \ell_j \neq 0, \ell_j \in L_j \tag{56}$$

for all j; here J denotes any finite set of nonnegative integers. If k is the largest element of J, then $\lambda \in \Lambda_{(k)}$. To estimate (47), we use only factors corresponding to $j \in J$, and then

$$\left|\hat{\mu}_j\left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}\lambda\right)\right| \leq a \sum_{\substack{p\in J \\ p<j}} \left|\left(R_j^*\right)^{-1} \cdots \left(R_{p+1}^*\right)^{-1} \ell_p\right| \leq ac(r)L_{max} \prod_{p=i+1}^{j} r_p\,,$$

where i is the next element in J below j. Multiplying these estimates (for $j \geq m$), we obtain

$$\left|\hat{\chi}_{K_m}(\lambda)\right| \leq c_m (ac(r)L_{max})^{\#J} \prod_{j=1}^{k} r_j$$

(the constant c_m takes care of terms for $j < m$). If we fix the number of elements of J to be p, there will be $\binom{k}{p}(N-1)^p$ distinct choices of λ, since there are $N-1$ choices for each ℓ_j. Thus we have

$$\sum_{\lambda\in\Lambda_{(k)}} \left|\hat{\chi}_{K_m}(\lambda)\right| \leq c_m \sum_{p=1}^{k} \binom{k}{p} ((N-1)ac(r)L_{max})^p \prod_{j=1}^{k} r_j$$

$$\leq c_m (1 + (N-1)ac(r)L_{max})^k \prod_{j=1}^{k} r_j\,,$$

and (55) gives the desired convergence.

We next study some special cases in which the condition (46) holds. We begin by looking at some of the minimal examples described in Example (3.1.9) (c).

Corollary(3.1.15) [169]: Assume $B_j = \{0, 1/N, 2/N, \ldots, (N-1)/N\}$ and $L_j = \{0, 1, 2, \ldots, N\}$ for all j, and $R_j = n_j N$, where n_j are positive integers and $\{j: n_j \neq 1\}$ has no arbitrarily long gaps. There exists a constant γ_N, with

$$\gamma_N = O(log\ N) \quad as\ N \to \infty, \tag{57}$$

such that the conclusions of Theorem (3.1.13) hold provided that

$$\liminf_{k\to\infty} \quad \left(\prod_{j=1}^{k} n_j\right) > \gamma_N. \tag{58}$$

Proof: In this case, we have

$$\left|\hat{\mu}_j(t)\right| = \left|\frac{\sin \pi t}{N sin \frac{\pi}{N} t}\right| \quad for\ all\ \ j.$$

It is difficult to compute the value of A_j exactly, but we can easily get the order of magnitude estimate $O(NlogN)$. Consider the family of functions

$$\frac{\sin a \sin \pi t}{t \sin(a + at/k)} \quad for\ 0 \le a \le \frac{\pi}{2}, \qquad |t| \le 1$$

and $k = 1, 2, \ldots$. These functions are continuous, and their behavior improves as k increases; so there is a uniform bound c (in the vicinity of 2). This enables us to estimate

$$\left|\hat{\mu}_j(N - k + t)\right| = \left|\hat{\mu}_j(k + t)\right| \le \frac{c\pi|t|}{N \sin \pi k/N} \quad for\ 1 \le k \le N/2.$$

Thus we have the estimate

$$A_j \le c\pi \sum_{k=1}^{[N/2]} \frac{(k + N - k)}{N \sin \pi k/N} = O(NlogN),$$

as claimed.

Now we verify the condition (46). Let $\gamma_N = N^{-1}A_j + 1$, so (57) holds. Then

$$\prod_{j=1}^{k}(A_j + N_j)r_j = \gamma_N^k \prod_{j=1}^{k} n_j^{-1},$$

and (58) is exactly what is needed to obtain the convergence in (46) by comparison with a geometric series.

Note that the gap condition, though of a more qualitative nature, is not a consequence of the density condition (58). Moreover, we cannot verify the hypotheses of Theorem (3.1.8) using a density condition alone, so the gap condition cannot be eliminated.

When $N = 2$, we know $A_j = \pi/2$, so the best value for $\gamma_2$is $1 + \pi/4 \approx 1.7853981$.

Thus we can satisfy the condition by taking $R_j = 2$ or 4, provided we choose 4 most of the time. Using Theorem (3.1.14) does not yield better results, because the minimum value for $c(r)$ is $5/3$ if we allow $R_j = 2$.

Corollary (3.1.16) [169]: Let $\{B_j, L_j\}$ be the compatible pair in Example (3.1.4) (b), and take $R_j = 3n_j$. Then the conclusions of Theorem (3.1.13) hold provided that

$$\lim_{k\to\infty} \inf \Big(\prod_{j=1}^{k} n_j\Big)^{1/k} > (1 + \pi)/3 \approx 1.3805309.$$

Proof: We verify condition (55) of Theorem (3.1.14), since it gives a better result than (46). As observed in Example (3.1.9) (d), the hypotheses of Theorem (3.1.8) hold without any condition on the choice of n_j. Now

$$\hat{\mu}_j(2 + s, 1 + t) = \frac{1}{3}\left(1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)e^{2\pi is/3} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)e^{2\pi it/3}\right)$$

$$= \frac{1}{3}(e^{2\pi is/3} - 1)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + \frac{1}{3}(e^{2\pi it/3} - 1)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),$$

so

$$\left|\hat{\mu}_j(2 + s, 1 + t)\right| \le \frac{1}{3}\left(\left|\sin\frac{\pi s}{3}\right| + \left|\sin\frac{\pi t}{3}\right|\right) \le \frac{\pi}{9}(|s| + |t|);$$

and the same estimate holds if we permute 2 and 1. Thus we can take $a = \pi/9$ in (52) using the ℓ^1 norm, so $L_{max} = 3$; and we always have $c(r) \le 3/2$, as this is the value we obtain in the worst case when all $R_j = 3I$. Thus

$$(1+(N-1)ac(r)L_{max})^k \prod_{j=1}^{k} r_j = \left(\frac{1+\pi}{3}\right)^k \prod_{j=1}^{k} n_j\,,$$

so (59) suffices to verify (55).

Corollary (3.1.17) [169]: Let $\{B_j, L_j\}$ be the compatible pair in Example (3.1.4) (c), and take $R_j = 2n_j$ with all $n_j \geq 2$. Then the conclusions of Theorem (3.1.13) hold provided that

$$\lim_{k\to\infty} \inf \Big(\prod_{j=1}^{k} n_j\Big)^{1/k} > \frac{1+\pi}{2} \approx 2.0707963. \qquad (59)$$

Proof: As observed in Example (3.1.9) (e), the condition $n_j \geq 2$ suffices to verify the hypotheses of Theorem (3.1.8). In this case, we again use Theorem (3.1.14) and do the estimates in the ℓ^1 norm. We have

$$\hat{\mu}_j(1+x,1+y,z) = \frac{1}{4}\left(1 - e^{\pi i x} - e^{\pi i y} + e^{\pi i z}\right)$$
$$= \frac{1}{4}\left(1 - e^{\pi i x}\right) + \frac{1}{4}\left(1 - e^{\pi i y}\right) - \frac{1}{4}\left(1 - e^{\pi i z}\right),$$

so that

$$\left|\hat{\mu}_j(1+x,1+y,z)\right| \leq \frac{1}{4}\left(\left|\sin\frac{\pi}{2}x\right| + \left|\sin\frac{\pi}{2}y\right| + \left|\sin\frac{\pi}{2}z\right|\right) \leq \frac{\pi}{8}(|x|+|y|+|z|).$$

Thus we may take $a = \pi/8$ and $L_{max} = 2$. In this case, $N = 4$ and we may take $c(r) = 4/3$ (corresponding to all $R_j = 4I$). Thus

$$(1+(N-1)ac(r)L_{max})^k \prod_{j=1}^{k} r_j = \left(\frac{1+\pi}{2}\right)^k \prod_{j=1}^{k} n_j\,,$$

we consider structures in which the product construction extends infinitely in both small and large scales, combining the features of the Cantor set K with the spectrum Λ. We use a tilde to denote objects with such two-sided product structure.

Definition (3.1.18) [169]: A two-sided compatible tower is a sequence (indexed by $\mathbb{Z}$) of compatible pairs $\{B_j, L_j\}$ with $B_j \subseteq M_j^{-1}\mathbb{Z}^n$ and $L_j \subseteq \mathbb{Z}^n$, and matrices $R_j \in GL(n, M_{j-1}\mathbb{Z}^n)$ for all $j \in \mathbb{Z}$. We associate with such a tower an infinite measure $\tilde{\mu}$ defined by

$$\tilde{\mu} = \lim_{j\to\infty} N_{-j}\cdots N_{-1}(\mu_{-1} \circ (R_0 R_{-1}\cdots R_{-j+1})^{-1}) \qquad (60)$$
$$* \left(\mu_{-j+1} \circ \left(R_0 R_1 \cdots R_{-j+2}\right)^{-1}\right) * \cdots * (\mu_{-1} \circ R_0^{-1}) * \mu,$$

where μ is the measure given by (10) associated with the one-sided compatible tower obtained by restricting $j \geq 0$. In fact, $\tilde{\mu}$ is just the sum of translates of μ by vectors in

$$R_0 B_{-1} + R_0 R_{-1} B_{-2} + \cdots + R_0 \cdots R_{-j+1} B_{-j} + \cdots, \qquad (61)$$

so $\tilde{\mu}$ is supported on the unbounded Cantor set $\tilde{K}$ that is the sum of K and the discrete set (61) (note that (61) is a subset of $\mathbb{Z}^n$ and each element has a unique representation). The dual two-sided tower (denoted by a prime) is defined by

$$(B_j', L_j') = ((R_{-j}^*)^{-1} L_{-j-1}, R_{-j} B_{-j-1})$$

with $M_j' = M_{-j-1}$, and $R_j' = R_{-j+1}^*$. We denote the associated measure by $\tilde{\mu}'$.

Note that the set (61) for the dual two-sided tower is just A.

Our goal is to establish an isometry between $L^2(\tilde{\mu})$ and $L^2(\tilde{\mu}')$ given by the mock Fourier transform formulas

$$\hat{f}(\lambda) = \int e(-y \cdot \lambda) f(y) d\tilde{\mu}(y) \tag{62}$$

and

$$f(x) = \int e(x \cdot \lambda) \hat{f}(\lambda) d\tilde{\mu}'(\lambda), \tag{63}$$

with

$$\int |f(x)|^2 d\tilde{\mu}(x) = \int \left|\hat{f}(\lambda)\right|^2 d\tilde{\mu}'(\lambda). \tag{64}$$

Of course, (62) and (63) will only be valid in a pointwise sense for a dense space of functions, while for a general $f \in L^2(\tilde{\mu})$, the mock Fourier transform $\hat{f}$ is defined as an $L^2(\tilde{\mu}')$ limit. Note that $f d\tilde{\mu}$ is well-defined as a tempered distribution for any $f \in L^2(\tilde{\mu})$, so the ordinary Fourier transform .$\mathcal{F}(f d\tilde{\mu})$ is defined as a tempered distribution; and $\hat{f}$ is the restriction to $\widetilde{K}'$ of . $\mathcal{F}(f d\tilde{\mu})$ for a dense subspace of functions (for which the distribution $\mathcal{F}(f d\tilde{\mu})$ coincides with a continuous function).

It is by no means clear, however, how to define the restriction to $\widetilde{K}'$ of $\mathcal{F}(f d\tilde{\mu})$ for the general f in $L^2(\tilde{\mu})$.

We establish the basic formulas (62−64) for the mock Fourier transform in the same way that the ordinary Fourier transform formulas are derived from Fourier series and a limiting process. In this case, we consider, for each $k \geq 0$, the one-sided tower that begins with $\{B_{-k}, L_{-k}\}$ instead of $\{B_0, L_0\}$. So we write $\{B_j^{(k)}, L_j^{(k)}\} = \{B_{j-k}, L_{j-k}\}$ for $j \geq 0$ and $R_j^{(k)} = R_{j-k}$. Also, let $\mu^{(k)}$ be the corresponding measure and $\Lambda^{(k)}$ the corresponding spectrum. Assume the hypotheses of Theorem (3.1.8). It is easy to see that if it holds for $k = 0$, then it holds for all k. We then take the conclusion of Theorem (3.1.8) and transform it by replacing the measure $\mu^{(k)}$ by

$$\mu_{(k)} = N_{-k} \cdots N_{-1} \mu^{(k)} \circ (R_0 R_{-1} \cdots R_{-k+1})^{-1}, \tag{65}$$

simultaneously replacing $\Lambda^{(k)}$ by

$$\begin{aligned} \Lambda_{(k)} &= (R_{-k+1}^* \cdots R_0^*)^{-1} \Lambda^{(k)} \\ &= (R_0^*)^{-1} \cdots (R_{-k+1}^*)^{-1} L_{-k} + (R_0^*)^{-1} \cdots (R_{-k+2}^*)^{-1} L_{-k+1} \\ &\quad + \cdots + (R_0^*)^{-1} L_{-1} + L_0 + R_1^* L_1 + R_1^* R_2^* L_2 + \cdots. \end{aligned} \tag{66}$$

(Note that this is not the same as the notation.) We then have

$$f(x) = \frac{1}{N_{-k} \cdots N_{-1}} \sum_{\lambda \in \Lambda_{(k)}} \hat{f}(\lambda) e(x \cdot \lambda) \tag{67}$$

on the support of $\mu_{(k)}$ for

$$\hat{f}(\lambda) = \int e(-y \cdot \lambda) f(y) d\mu_{(k)}(y) \tag{68}$$

and

$$\int |f(y)|^2 d\mu_{(k)}(y) = \frac{1}{N_{-k} \cdots N_{-1}} \sum_{\lambda \in \Lambda_{(k)}} \left|\hat{f}(\lambda)\right|^2 . \tag{69}$$

Note that $\mu_{(k)}$ is the sum of translates of μ by vectors in

$$R_0 B_{-1} + R_0 R_{-1} B_{-2} + \cdots + R_0 \cdots R_{-k+1} B_{-k}\,, \tag{70}$$

so $\mu_{(k)} \to \tilde{\mu}$ and $\mu_{(k)} = \tilde{\mu}$ when restricted to any fixed compact set if k is large enough.

Now suppose f has compact support. Then for k large enough, we can replace $\mu_{(k)}$ by $\tilde{\mu}$ in (68) and (69). Thus (68) becomes (62). We would like to pass to the limit as $k \to \infty$ to obtain (63) from (67) and (64) from (69). It is clear that $(N_{-k} \cdots N_{-1})^{-1}$ times counting

measure on $\Lambda_{(k)}$ converges weakly to the measure $\tilde{\mu}'$. We use the dominated convergence theorem to obtain the desired limits, and this requires estimates of the type used.

Theorem (3.1.19) [169]: Given a two-sided compatible tower, suppose that the hypotheses of Theorems (3.1.8) and (3.1.13) (or (3.1.14)) hold for the one-sided tower obtained by restricting $j \geq 0$. Then there is a dense subspace of $L^2(\tilde{\mu})$ on which the identities (62), (63) and (64) hold as absolutely convergent integrals. The mock Fourier transform (62) then extends to all of $L^2(\tilde{\mu})$ as an isometry into $L^2(\tilde{\mu}')$; so (64) holds on all of $L^2(\tilde{\mu})$, and (63) is valid in a limiting sense. Finally, if the same assumptions hold for the dual tower, then the mock Fourier transform is an isometry onto $L^2(\tilde{\mu}')$.

Proof: For the dense subspace we take the finite linear combinations of characteristic sets of $K_m + b$, where b ranges over the sum of the set $\bar{B}_m$ defined by (32) and the set (61). We give the argument just for $f = \chi_{K_m}$, as the extension to the linear combinations of translates is routine.

With $\hat{f}(\lambda)$ defined by (68) for k large enough, we rewrite (67) and (69) so that the sum is replaced by an integral with respect to $\tilde{\mu}'$. Let K'_m denote the closure of

$$(R_0^*)^- \cdots (R_{-m}^*)^{-1} L_{-m-1} + (R_0^*)^{-1} \cdots (R_{-m-1}^*)^{-1} L_{-m-2} + \cdots$$

(the dual tower analogue of K_m), so that $\widetilde{K}' = K'_m + \Lambda_{(m)}$. For fixed x, we set

$$g_m(t) = N_{-m} \cdots N_{-1} \int_{K'_m + \lambda} \hat{f}(y) e(x \cdot y) d\tilde{\mu}'(y) \quad if\ t \in K'_m + \lambda \quad (71)$$

for $\lambda \in \Lambda_{(m)}$. Then (67) is the same as

$$f(x) = \int g_m(t) d\tilde{\mu}'(t). \quad (72)$$

It is clear that $g_m(t) \to \hat{f}(t) e(x \cdot t)$ pointwise as $m \to \infty$. To obtain (63), it suffices to dominate $|g_m(t)|$ by an integrable function g. We take

$$g(t) = \sup_{s \in K'} \left|\hat{f}(\lambda + s)\right| \quad if \quad t \in K' + \lambda \quad (73)$$

for $\lambda \in \Lambda$. It is clear that $|g_m(t)| \leq g(t)$ because $g_m(t)$ is an average over a subset of $K' + \lambda$ while $g(t)$ is the sup. The integrability of g is almost the same as the convergence of $\sum_{\lambda \in \Lambda} \left|\hat{f}(\lambda)\right|$ proved in Theorem (3.1.13) or (3.1.14); one has only to check that the replacement of λ by $\lambda + s$ in (73) does not significantly affect the estimates.

The argument for (64) is similar. The fact that we have an L^2 norm means we have to show $g \in L^2$, or that $\sum_{\lambda \in \Lambda} \left|\hat{f}(\lambda)\right|^2$ converges; but this follows from the L^1 estimate. The extension to all of $L^2(\mu)$ is routine.

Finally, assume that the dual tower satisfies the same hypotheses. Given any $F \in L^2(\tilde{\mu}')$, we want to find $f \in L^2(\tilde{\mu})$ such that $\hat{f} = F$. To do this, we set $f(x) = \int e(x \cdot \lambda) F(\lambda) d\tilde{\mu}'(\lambda)$, interpreted in the limiting sense as (62) for the dual system (aside from the harmless minus sign). Then (63) for the dual system (with the minus sign) gives $\hat{f} = F$, as desired.

Of course, we can polarize the mock Plancherel formula (64) to obtain the mock Parseval identity

$$\int f_1(x) \overline{f_2(x)} d\tilde{\mu}(x) = \int \hat{f}_1(\lambda) \overline{\hat{f}_2(\lambda)} d\tilde{\mu}'(\lambda) \quad (74)$$

for any $f_1, f_2 \in L^2(d\tilde{\mu})$.

We can also extend the mock Fourier transform to L^1 functions or, more generally, finite measures, supported on $\widetilde{K}$, by

$$\hat{v}(\lambda) = \int e(-y \cdot \lambda) dv(y) \ for\ \lambda \in \widetilde{K}'. \quad (75)$$

Note that this is just the restriction to $\widetilde{K}'$ of the usual Fourier-Stieltjes transform.

Corollary (3.1.20) [169]: Under the assumptions of Theorem (3.1.19), the finite measure v is determined by its mock Fourier transform via

$$v(K_m + b) = \int_{\overline{K'}} \left(\int_{K_m + b} \hat{v}(\lambda) e(x \cdot \lambda) d\tilde{\mu}'(x) \right) d\tilde{\mu}'(\lambda) \qquad (76)$$

(iterated integral only), for every b in the sum set of $\tilde{B}_m$ and the set (61). In particular, if $\hat{v}$ vanishes on $\widehat{\widetilde{K}'}$, then v, is zero. Moreover, if v is absolutely continuous with respect to $\hat{\mu}$, then $\hat{v}$ vanishes at infinity in $\widehat{\widetilde{K}'}$ (Riemann-Lebesgue Lemma).

Corollary (3.1.21) [169]: (Uncertainty Principle). Let v be any non-zero finite measure supported on a translate of $\widetilde{K}$,. Then the Fourier-Stieltjes transform $\hat{v}$ cannot vanish identically on any translate of $\widetilde{K}'$.

Proof: Translation of v or $\hat{v}$ does not influence the argument.

We expect that there will be a Poisson summation formula in this context, of the form

$$\sum_{b \in \Lambda'} f(b) = \sum_{\lambda \in \Lambda} \hat{f}(\lambda), \qquad (77)$$

where Λ' denotes the set (61) (this is, indeed, the analogue of Λ for the dual tower), valid for a suitable class of well-behaved functions f. On a formal level, we may obtain (77) by performing the analogue of periodization, namely taking

$$F(x) = \sum_{b \in \Lambda'} f(x + b) \ for \quad x \in K. \qquad (78)$$

Then the mock Fourier series coefficients $\hat{F}(\lambda)$ for $\lambda \in \Lambda$ agree with the restriction to Λ of the mock Fourier transform of f, since

$$\hat{f}(\lambda) = \sum_{b \in \Lambda'} \int_K f(y + b) e(-(y + b) \cdot \lambda) d\mu(y) = \int_K F(y) e(-y \cdot \lambda) d\mu(y) \quad (79)$$

for $\lambda \in \Lambda$ since $b \cdot \lambda \in \mathbb{Z}$ for $b \in \Lambda'$. This step requires only that $f \in L^1(\tilde{\mu})$, which implies $F \in L^1(\mu)$. If F has a convergent mock Fourier series, then $F(0) = \sum_{\lambda \in \Lambda} \hat{F}(\lambda)$; and this yields (77). To make this into a theorem, we need a reasonable class of functions with convergent mock Fourier series and then to find a class of functions f on $\widetilde{K}$ such that the function F in (78) ends up in this class.

Since we know the convergence only for step functions, we do not at present have an interesting class of functions for which (77) holds.

The Shannon Sampling Theorem tells us how to recover a function F on $\mathbb{R}^n$ from its values on a lattice, under the assumption that F is in L^2 and its Fourier transform has compact support. If μ and Λ are as in Theorem (3.1.8), and

$$F(x) = \int e(-x \cdot y) f(y) d\mu(y) \quad (80)$$

for some $f \in L^2(\mu)$, we can rephrase the conclusion of Theorem (3.1.8) to say that the values of F on Λ form an arbitrary sequence in $\ell^2(\Lambda)$ and completely determine F via

$$F(x) = \sum_{\lambda \in \Lambda} F(\lambda) \hat{\mu}(x - \lambda), \quad (81)$$

with uniform convergence. We may regard (81) as a sampling formula in this context. We show that the sampling formula holds for a wider class of functions, under the assumptions.

Definition (3.1.22) [169]: Let K denote a fixed compact set in $\mathbb{R}^n$. A function F on $\mathbb{R}^n$ is said to be K-bandlimited if

$$F(x) = \int e(-x \cdot y) dv(y) \qquad (82)$$

for a finite (complex-valued) measure v supported in K.

Theorem (3.1.23) [169]: Let μ, Λ, K be as in Theorems (3.1.13) or (3.1.14). Then (81) holds for any K-bandlimited function, with uniform convergence on compact sets. Moreover, the series (81) may be differentiated term-by-term any number of times.

Proof: We begin by showing that the right side of (81) converges, just using the boundedness of the coefficients $F(\lambda)$. We claim the estimate

$$\sum_{\lambda \in \Lambda} |\hat{\mu}(x-\lambda)| \leq c(x), \tag{83}$$

where $c(x)$ has at most polynomial growth in x. Suppose

$$c(r_1 \cdots r_m)^{-1} \leq |x| \leq c'(r_1 \cdots r_m)^{-1} \tag{84}$$

for convenient constants c and c', Then we use the estimate

$$|\hat{\mu}(x-\lambda)| \leq \prod_{j=m}^{\infty} \left| \hat{\mu}_j \left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}(x-\lambda) \right) \right| \tag{85}$$

as in the proof of Theorem (3.1.13) or (3.1.14). It is not hard to see that the x term does not affect the estimate. Also, the constant depending on m that arises in both proofs is easily bounded by an exponential in m, which in turn is bounded by a power of $|x|$.

Since the right side of (81) converges, in order to identify the limit with F it suffices to show

$$\sum_{\lambda \in \Lambda} F(\lambda) \int \hat{\mu}(x-\lambda)\psi(x)dx = \int F(x)\psi(x)dx \tag{86}$$

for every $\psi \in S(\mathbb{R}^n)$. In fact, due to the estimate (83), this follows if we can verify (86) for ψ in a dense subspace of S. We shall use the term step function on K for any finite linear combination of the functions χ_{K_m+b} considered.

Suppose $\hat{\psi}|_K$ is a step function on K. Then

$$\begin{aligned} \sum_{\lambda \in \Lambda} F(\lambda) \int \hat{\mu}(x-\lambda)\psi(x)dx &= \sum_{\lambda \in \Lambda} \iint F(\lambda) e(\lambda \cdot y) e(-x \cdot y)\psi(x)dxd\mu(y) \\ &= \sum_{\lambda \in \Lambda} F(\lambda) \int F(\lambda) e(\lambda \cdot y)\hat{\psi}(y)d\mu(y) \\ &= \int \hat{\psi}(y)dv(y) \end{aligned}$$

by (36) (here $F(\lambda) = \hat{v}(\lambda)$). Since

$$\int \hat{\psi}(y)dv(y) = \int F(x)\psi(x)dx\,,$$

we have established (86) for such functions ψ. But it is routine to show that this is a dense subspace of S.

To verify that (81) can be differentiated term-by-term we first obtain the analogue of (83) for derivatives of $\hat{\mu}$. This is relatively straightforward, so we only sketch the argument. For example, we have

$$\left(\frac{\partial}{\partial x_1}\right)\hat{\mu}(x) = \sum_{j=0}^{\infty} \frac{\partial}{\partial x_1}\left(\hat{\mu}_j \left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}x \right) \right) \prod_{k \neq j} \hat{\mu}_k((R_k^*)^{-1} \cdots (R_1^*)^{-1}x)\,. \tag{87}$$

We can estimate

$$\left| \frac{\partial}{\partial x_1}\left(\hat{\mu}_j \left(\left(R_j^*\right)^{-1} \cdots (R_1^*)^{-1}x \right) \right) \right| \leq c r_1 \cdots r_j,$$

and the infinite series $\sum_{j=1}^{\infty} r_1 \cdots r_j$ is convergent. The estimate for the finite product in (87) is almost the same as before, except for the absence of the $k = j$ term; but this

requires only a minor modification of the argument. Other derivatives are handled similarly.

To show that

$$\sum_{\lambda\in\Lambda} F(\lambda)\left(\frac{\partial}{\partial x}\right)^{\alpha}\hat{\mu}(x-\lambda) \tag{88}$$

actually converges to $\left(\frac{\partial}{\partial x}\right)^{\alpha} F(x)$, we verify

$$\sum_{\lambda\in\Lambda} F(\lambda)\int \psi(x)\left(\frac{\partial}{\partial x}\right)^{\alpha}\hat{\mu}(x-\lambda)dx = \int \psi(x)\left(\frac{\partial}{\partial x}\right)^{\alpha} F(x)dx \tag{89}$$

for all $\psi \in S$. We obtain this by replacing ψ by $(-1)^{|\alpha|}\left(\frac{\partial}{\partial x}\right)^{\alpha}\psi(x)$ in (86) and integrating by parts on both sides of the equation.

Note that although $\left(\frac{\partial}{\partial x}\right)^{\alpha} F(x)$ is also K-bandlimited, the series (88) is a different representation from what is obtained using (81) with F replaced by $\left(\frac{\partial}{\partial x}\right)^{\alpha} F(x)$.

The representation (81), under the hypotheses of Theorems (3.1.13) or (3.1.14), has much better convergence properties than the Shannon formula

$$F(x) = \sum_{\lambda\in\mathbb{Z}} F(\lambda)\frac{\sin\pi(x-\lambda)}{\pi(x-\lambda)} \quad (90)$$

for $K = [-\frac{1}{2}, \frac{1}{2}]$. In fact, it is necessary to assume more than just the K-bandlimited hypothesis to ensure the convergence of (90), since $F(x) = e(x/2)$ gives a counterexample. Moreover, if we choose a spectrum Λ supported in $[0, \infty)$ (for $n = 1$), the representation (81) will supply the values of F at negative x as well, so it extrapolates as well as interpolates from the sampled data.

We do not know any physical systems which can be expected to produce only K-bandlimited signals for K a Cantor set. Thus we cannot claim any practical applications of (81). Also missing from the theory is an intrinsic characterization of K-bandlimited functions in terms of their analytic continuation to $\mathbb{C}^n$, analogous to the Paley-Wiener type theorems.

Section (3.2): Self-Affine Measures on R^n

For μ be a Borel probability measure with compact support on $\mathbb{R}^n$.We call μ a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that the set of complex exponentials $E(\Lambda) := \{e^{2\pi i<\lambda,x>} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. The set Λ is called a spectrum for μ. The interest of spectral measures was initialedby Fuglede [189] and his famous conjecture: $\chi_\Omega dx$ is a spectral measure on $\mathbb{R}^n$ if and only if Ω is a translational tile. Later it was proved that the conjecture is false in both directions on $\mathbb{R}^n$ for $n \geq 3$ [186]; but it is still open for $n = 1, 2$. The problem of spectral measures is also very attractive when we replace the Lebesgue measure μ by fractal measures such as self-similar/affine measures.

For $M \in M_n(\mathbb{Z})$ be an expanding matrix (i.e., all the moduli of eigenvalues of M are strictly larger than one) and $D \subset \mathbb{Z}^n$ be a finite digit set. Then the maps

$$\phi_d(x) = M^{-1}(x+d), \qquad d \in D$$

are contractive with respect to a suitable norm in $\mathbb{R}^n$[16]and it is well known that there is a unique nonempty compact set $T := T(M, D)$ satisfying the set-valued equation [178]:

$$T = \bigcup_{d\in D} \phi_d(T).$$

Such T is called the self-affine set (or attractor) of the iterated function system $(IFS)\{\phi_d\}_{d\in D}$. A self-affine set can be equipped with a unique invariant probability measure $\mu := \mu_{M,D}$ defined by

$$\mu = \frac{1}{|D|}\sum_{d\in D} \mu \circ \phi_d^{-1},$$

and $\mu_{M,D}$is supported on T. We call $\mu_{M,D}$ a self-affine measure [191]. In particular, if M is a multiple of an orthonormal matrix, T and $\mu_{M,D}$ are often called self-similar set and self-similar measure, respectively.

It is natural to ask whether $\mu_{M,D}$ is a spectral measure. Jorgenson and Pederson [192] first studied the spectral property of certain Cantor measures. It was found that the 1/4-Cantor measure $\mu_{1/4}$ on $\mathbb{R}$ is a spectral measure while the $1/3$-Cantor measure $\mu_{1/3}$ is not. Hu and Lau [190] further investigated the spectrality of Bernoulli convolutions μ_ρ, and observed that $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if the contraction ratio ρ is the n-th root of a fraction p/q where p is odd and q is even. Recently, Dai [193] completely settled the problem that the only spectral Bernoulli convolutions are of the contraction ratio $1/2k$. As a generalization of the Bernoulli convolution, Dai, He and Lai [194]studied the self-similar measure generated by a positive integer band consecutive digits $D = \{0, 1, \dots, q-1\}$, it was proved that $L^2(\mu_{b,D})$ has infinitely many orthogonal exponentials if and only if $gcd(q, b) > 1$, and $\mu_{b,D}$ is a spectral measure provided $q|b$. After that, replacing the integer b by any real number $b' > 1$, Dai, He and Lau [185]showed that $\mu_{b',D}$ is a spectral measure if and only if $b' \in \mathbb{N}$ and $q|b'$. For more general cases such as Moran measures, see [189].

There are few results on the spectrality of self-affine measures [189], among which Li [185]mainly considered the spectrality or non-spectrality of some self-affine measures in low dimensions. For example, Li and Wen [22] studied the measure $\mu_{M,D}$ generated by $M \in M_2(\mathbb{Z})$ and $D = \{0, 1\}v$, where $v \in \mathbb{Z}^2\backslash\{0\}$. They obtained that $\mu_{M,D}$ admits an infinite orthonormal set if $det(M)$ is even; if v is the eigenvector of M with eigenvalue ℓ, then $\mu_{M,D}$ is a spectral measure if and only if ℓ is even.

We study the spectrality of self-affine measures on $\mathbb{R}^n$ with consecutive collinear digits. First we need a decomposition of integer matrices, which may be known, so we provide a proof (see Lemma(3.2.13)).

Let $M \in M_n(\mathbb{Z}), v \in \mathbb{Z}^n\backslash\{0\}$, and let r be the rank of the set of vectors $\{v, Mv, \dots, M^{n-1}v\}$. Then there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that $Bv = (x_1, \dots x_r, 0, \dots, 0)^t$ and

$$BMB^{-1} = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix} \quad (91)$$

where $M_1 \in M_r(\mathbb{Z}), M_2 \in M_{n-r}(\mathbb{Z})$ and $C \in M_{r,n-r}(\mathbb{Z})$. By making use of this matrix decomposition, we have the main theorem. .

Theorem (3.2.1)[182]: Let $M \in M_n(\mathbb{Z})$ be an expanding matrix, $D = \{0, 1, \dots, q-1\}v$ be a digit set where $q \geq 2$ is an integer and $v \in \mathbb{Z}^n\backslash\{0\}$. If the set of vectors $\{v, Mv, \dots, M^{n-1}v\}$ has rank r and $M, M_1$are expressed as in (91), then

(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if $gcd(q, det(M_1)) > 1$;

(ii) $\mu_{M,D}$ is a spectral measure if $q|det(M_1)$.

In particular, if the characteristic polynomial of M_1 is of the simple form $f(x) = x^r + c$, then we have the following necessary and sufficient condition on the spectrality of $\mu_{M,D}$.

Theorem (3.2.2) [182]: Under the same assumption of Theorem(3.2.1), if the characteristic polynomial of M_1 is $f(x) = x^r + c$, then

(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if and only if $gcd(q, det(M_1)) > 1$;

(ii) $\mu_{M,D}$ is a spectral measure if and only if $q|det(M_1)$.

According to the dependence of the set of vectors $\{v, Mv, \dots, M^{n-1}v\}$, we will prove the above theorems in two steps: $r = n$ (Theorems(3.2.9), (3.2.11)) and $r < n$ (Theorem(3.2.15)). The main idea is to use the techniques of linear algebra and matrix analysis.

If the rank $r = 1$, then v becomes an eigenvector of M and $M_1 = \ell$ with characteristic polynomial $f(x) = x - l$, by Theorem(3.2.2), the following corollary is immediate.

Corollary (3.2.3) [182]: Under the same assumption of Theorem(3.2.1), if $r = 1$, in this case v is an eigenvector of M and $M_1 = \ell$ is the corresponding eigenvalue, then

(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if and only if $gcd(q, \ell) > 1$;

(ii) $\mu_{M,D}$is a spectral measure if and only if $q|\ell$.

Definition (3.2.4) [182]: Let $M \in M_n(\mathbb{Z})$ be an expanding matrix. Let D and S be finite subsets of $\mathbb{Z}^n$ with the same cardinality q. We say $(M^{-1}D, S)$is an integral compatible pair if the matrix

$$H = \frac{1}{\sqrt{q}}\left[e^{2\pi i<M^{-1}d,s>}\right]_{d\in D, s\in S}$$

is unitary, i.e., $H^*H = I$, where H^* denotes the transposed conjugate of H. At this time, (M, D, S) is called Hadamard triple.

It is fairly easy to construct an infinite mutually orthogonal set of exponential functions using the Hadamard triple assumption. However, to check these exponentials form a Fourier basis for $L^2(\mu)$ is a much more difficult task. Jorgensen and Pedersen conjectured that all Hadamard triples will generate self-affine spectral measures. The conjecture was solved in dimension one by Laba and Wang [185]. Under various additional conditions, it was also valid in the high dimensions (see [178]). Until recently, Dutkay, Haussermann and Lai [187] have completely proved the conjecture.

Lemma (3.2.5) [182]: ([187]) Let (M, D, S) be a Hadamard triple. Then the self-affine measure $\mu_{M,D}$ is a spectral measure.

The Fourier transform of $\mu_{M,D}$plays a key role in studying the spectralityof the measure. We recall the definition by

$$\hat{\mu}_{M,D}(\xi) := \int e^{2\pi i<x,\xi>} d\mu_{M,D}(x) = \prod_{j=0}^{\infty} m_D\left(M^{*-j}\xi\right), \xi \in \mathbb{R}^n \quad (92)$$

where $m_D(\cdot) = \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,\cdot\rangle}$ is the mask polynomial of the digit set D. It is easy to see that m_D is a $\mathbb{Z}^n$-periodic function for $D \subset \mathbb{Z}^n$. Let $Z_D^n := \{x \in [0,1)^n : m_D(x) = 0\}$ be the zero set of $m_D$in $[0,1)^n$. Then we have the following useful criterion for the existence of infinitely many orthogonal systems.

Lemma (3.2.6) [182]: ([199]) Let $M \in M_n(\mathbb{Z})$ be an expanding matrix and $D \subset \mathbb{Z}^n$ be a finite digit set, if there exist $\alpha \in Z_D^n$ and $\ell \in \mathbb{N}$ such that $M^{*\ell}\alpha \in \mathbb{Z}^n$. Then there are infinitely many orthogonal exponentials $E(\Lambda)$ in $L^2(\mu_{M,D})$ with $\Lambda \subseteq \mathbb{Z}^n$.

Let $M, \widetilde{M}$ be $n \times n$ integer matrices, and the finite sets $D, S, \widetilde{D}, \tilde{S}$ be in $\mathbb{Z}^n$. We say that two triples (M, D, S) and $(\widetilde{M}, \widetilde{D}, \tilde{S})$ are conjugate(through the matrix B) if there exists an integer matrix B such that $\widetilde{M} = B^{-1}MB, \widetilde{D} = B^{-1}D$ and $\tilde{S} = B^*S$. The following lemma is trivial.

Lemma (3.2.7) [182]: Suppose that (M, D, S) and $(\widetilde{M}, \widetilde{D}, \tilde{S})$ are two conjugate triples, through the matrix B. Then
(i) if (M, D, S) is a Hadamard triple then so is $(\widetilde{M}, \widetilde{D}, \tilde{S})$;
(ii) $\mu_{M,D}$ is a spectral measure with spectrum Λ if and only if $\mu_{\widetilde{M},\widetilde{D}}$ is a spectral measure with spectrum $B^*\Lambda$;
(iii) $\mu_{M,D}$ has infinitely many orthogonal exponentials $E(\Lambda)$ in $L^2(\mu_{M,D})$if and only if $\mu_{\widetilde{M},\widetilde{D}}$ has infinitely many orthogonal exponentials $E(B^*\Lambda)$ in $L^2(\mu_{\widetilde{M},\widetilde{D}})$.
Proof: The proofs of (i) and (ii) can be found in [198](or [197]). In fact, it is easy to see that $E(\Lambda)$ is an orthogonal set in $L^2(\mu_{M,D})$ if and only if $(\Lambda - \Lambda)\backslash\{0\} \subset Z(\hat{\mu}_{M,D}) := \{\xi \in \mathbb{R}^n : \hat{\mu}_{M,D}(\xi) = 0\}$. Let $\lambda_1, \lambda_2 \in \Lambda$, then

$$m_{\widetilde{D}}(\widetilde{M}^{*-j}B^*(\lambda_1 - \lambda_2)) = m_{\widetilde{D}}(B^*M^{*-j}(\lambda_1 - \lambda_2))$$
$$= \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle B^{-1}d, B^*M^{*-j}(\lambda_1-\lambda_2)\rangle} = m_D(M^{*-j}(\lambda_1 - \lambda_2)).$$

Hence by (92), $\lambda_1 - \lambda_2 \in Z(\hat{\mu}_{M,D})$if and only if $B^*\lambda_1 - B^*\lambda_2 \in Z(\hat{\mu}_{\widetilde{M},\widetilde{D}})$, proving (iii).
we first consider the case that $\{v, Mv, \dots, M^{n-1}v\}$ are linearly independent for expanding matrix $M \in M_n(\mathbb{Z})$ and $v \in \mathbb{Z}^n\backslash\{0\}$. We will get the sufficient conditions for $\mu_{M,D}$ to have infinitely many orthogonal exponentials or to be a spectral measure that only depend on the value of $det(M)$. The following simple lemma is an important tool in constructing a suitable conjugate Hadamard triple.
Lemma (3.2.8) [182]: Let $M \in M_n(\mathbb{Z})$ be an integer matrix with characteristic polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ and $v = (x_1, \dots, x_n)^t \in \mathbb{Z}^n\backslash\{0\}$. If the set of vectors $\{v, Mv, \dots, M^{n-1}v\}$is linearly independent, then there exists an integer matrix B, such that

$$\widetilde{M} := \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} = B^{-1}MB \tag{93}$$

and $B^{-1}v = (0, \dots, 0, 1)^t$.
Proof: Let $B = [M^{n-1}v, M^{n-2}v, \dots, Mv, v]$, then B is invertible and $B(0, \dots, 0, 1)^t = v$. Hence $B^{-1}v = (0, \dots, 0, 1)^t$. To prove (93), we only need to show that $B\widetilde{M} = MB = [M^nv, M^{n-1}v, \dots, M^2v, Mv]$. Indeed,

$$B\widetilde{M} = [M^{n-1}v, M^{n-2}v, \dots, Mv, v]\begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}$$
$$= \left[\sum_{i=1}^{n} -a_iM^{n-i}v, M^{n-1}v, \dots, M^2v, Mv\right]$$
$$= \left[\left(-\sum_{i=1}^{n} a_iM^{n-i}\right)v, M^{n-1}v, \dots, M^2v, Mv\right]. \tag{94}$$

Since $f(x)$ is the characteristic polynomial of M, it follows that $f(M) = M^n + a_1M^{n-1} + \cdots + a_{n-1}M + a_nI = 0$ and $M^n = -\sum_{i=1}^{n} a_iM^{n-i}$. By (94), we have $B\widetilde{M} = [M^nv, M^{n-1}v, \dots, M^2v, Mv] = MB$.
Theorem (3.2.9) [182]: Let $D = \{0, 1, \dots, q-1\}v$ with $q \geq 2, v \in \mathbb{Z}^n\backslash\{0\}$, and let $M \in M_n(\mathbb{Z})$ be an expanding matrix. If $\{v, Mv, \dots, M^{n-1}v\}$ is linearly independent, then

(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if $gcd(q, det(M)) > 1$;
(ii) $\mu_{M,D}$ is a spectral measure if $q|det(M)$.

Proof: Suppose $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ is the characteristic polynomial of M. Let $\widetilde{M}$ be as in (93) and $\tilde{v} = (0, \ldots, 0, 1)^t$. By Lemmas(3.2.7) and (3.2.8), it suffices to prove the theorem for the measure $\mu_{\widetilde{M},\widetilde{D}}$ which is generated by $\widetilde{M}$ and $\widetilde{D} = \{0, 1, \ldots, q-1\}\tilde{v}$.

(i) For $\xi = (\xi_1, \ldots, \xi_n)^t \in \mathbb{R}^n$, note that

$$m_{\widetilde{D}}(\xi) = \frac{1}{q}\sum_{\tilde{d}\in\widetilde{D}} e^{2\pi i\langle \tilde{d},\xi\rangle} = \frac{1}{q}(1 + e^{2\pi i\xi_n} + e^{2\pi i2\xi_n} + \cdots + e^{2\pi i(q-1)\xi_n}).$$

Then

$$Z_{\widetilde{D}}^n := \{\xi \in [0,1)^n : m_{\widetilde{D}}(\xi) = 0\}$$
$$= \{(\xi_1, \ldots, \xi_{n-1}, j/q)^t : \xi_1, \ldots, \xi_{n-1} \in [0,1), j = 1, \ldots, q-1\}.$$

Suppose $gcd(q, det(M)) = s > 1$, let $\alpha := (0, \ldots, 0, 1/s)^t \in \mathbb{R}^n$, then $\alpha \in Z_{\widetilde{D}}^n$ and

$$\widetilde{M}^*\alpha = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{-a_n}{s} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

As $a_n = (-1)^n det(M) = (-1)^n det(\widetilde{M})$, then $\frac{-a_n}{s} \in \mathbb{Z}$ and $\widetilde{M}^*\alpha \in \mathbb{Z}^n$. Lemma(3.2.6) implies that $\mu_{\widetilde{M},\widetilde{D}}$ has infinitely many orthogonal exponentials.

(ii) If $q|det(M)$, let $u = (\frac{-a_n}{q}, 0, \ldots, 0)^t, \tilde{S} = \{0, 1, \ldots, q-1\}u$, then $|\widetilde{D}| = |\tilde{S}| = q$ and $\tilde{S} \subset \mathbb{Z}^n$. Moreover,

$$\widetilde{M}^{-1}\tilde{v} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{-1}{a_n} \\ 1 & 0 & \cdots & 0 & \frac{-a_1}{a_n} \\ & & & & \vdots \\ \vdots & \vdots & \ddots & 0 & \frac{-a_{n-2}}{a_n} \\ 0 & 0 & \cdots & 0 & \frac{-a_{n-1}}{a_n} \\ 0 & 0 & \cdots & 1 & \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{a_n} \\ \frac{-a_1}{a_n} \\ \vdots \\ \frac{-a_{n-2}}{a_n} \\ \frac{-a_{n-1}}{a_n} \end{bmatrix}.$$

It yields that

$$H = \frac{1}{\sqrt{q}}\left[e^{2\pi i\langle \widetilde{M}^{-1}\tilde{d},\tilde{s}\rangle}\right]_{\tilde{d}\in\widetilde{D},\tilde{s}\in\tilde{S}} = \frac{1}{\sqrt{q}}\left[e^{2\pi i\frac{k\ell}{q}}\right]_{k,\ell\in\{0,1,\ldots,q-1\}}$$

is unitary. Hence $(\widetilde{M}, \widetilde{D}, \tilde{S})$ is a Hadamard triple, and $\mu_{\widetilde{M},\widetilde{D}}$is a spectral measure by Lemma(3.2.5).

In particular, if the characteristic polynomial of M is of the simple form: $f(x) = x^n + c$, we obtain a sufficient and necessary condition for $\mu_{M,D}$ to have infinitely many orthogonal exponentials or to be a spectral measure. The following lemma was due to Dai, He and Lau [185].

Lemma (3.2.10) [182]: Let $\mu = \mu_1 * \mu_2$ be the convolution of two probability measures $\mu_i, i = 1, 2$, and they are not Dirac measures. Suppose that $E(\Lambda)$ is an orthogonal set of $\mu_1$with $0 \in \Lambda$, then $E(\Lambda)$ is also an orthogonal set of μ, but it cannotbe a spectrum of μ.

Theorem (3.2.11) [182]: Under the same assumption of Theorem(3.2.9) and the matrix M with characteristic polynomial $f(x) = x^n + c$, then
(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if and only if $gcd(q, det(M)) > 1$;
(ii) $\mu_{M,D}$ is a spectral measure if and only if $q|det(M)$.
Proof: Likewise Theorem(3.2.9), we consider the conjugate situation where the matrix

$$\widetilde{M} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c & 0 & 0 & \cdots & 0 \end{bmatrix} = B^{-1}MB$$

and digit set $\widetilde{D} = \{0, 1, \dots, q-1\}\tilde{v}$ where $\tilde{v} = (0, \dots, 0, 1)^t$ as in (93). Then the sufficiencies of (i) and (ii) come from Theorem(3.2.9), we now prove the necessities.
(i) Let $Z_0 := \{\xi = (\xi_1, \dots, \xi_n)^t \in \mathbb{R}^n : m_{\widetilde{D}}(\xi) = 0\}$ be the zero set of the mask function $m_{\widetilde{D}}$. Then

$$Z_0 = \left\{ \left(\xi_1, \dots, \xi_{n-1}, k + \frac{i}{q} \right)^t \in \mathbb{R}^n : \xi_1, \dots, \xi_{n-1} \in \mathbb{R}, k \in \mathbb{Z}, 1 \le i \le q-1 \right\}.$$

For $1 \le j \le n$, by letting $\xi_n = k + \frac{i}{q}$ as above, we observe that

$$Z_j := \widetilde{M}^{*j}(Z_0) = \left\{ \begin{bmatrix} -c\xi_{n-j+1} \\ \vdots \\ -c\xi_{n-1} \\ -c\xi_n \\ \xi_1 \\ \vdots \\ \xi_{n-j} \end{bmatrix} : \xi_1, \dots, \xi_{n-1} \in \mathbb{R}, \xi_n = k + \frac{i}{q}, k \in \mathbb{Z}, 1 \le i \le q-1 \right\}$$

$$\subset \left\{ \begin{bmatrix} \xi'_1 \\ \vdots \\ \xi'_{j-1} \\ k' - \frac{ci}{q} \\ \xi'_{j+1} \\ \vdots \\ \xi'_n \end{bmatrix} : \xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_n \in \mathbb{R}, k' \in \mathbb{Z}, 1 \le i \le q-1 \right\}.$$

By (92), the zero set $Z(\hat{\mu}_{\widetilde{M},\widetilde{D}})$ of the Fourier transform $\hat{\mu}_{\widetilde{M},D}$ can be written as

$$Z(\hat{\mu}_{\widetilde{M},\widetilde{D}}) = \bigcup_{j=1}^{\infty} Z_j .$$

If $gcd(q, det(M)) = 1$, it follows from $c = (-1)^n det(M)$ that

$$\{k' - \frac{ci}{q} : k' \in \mathbb{Z}, 1 \le i \le q-1\} \subset \{k + \frac{i}{q} : k \in \mathbb{Z}, 1 \le i \le q-1\}.$$

Hence $Z_n \subset Z_0$, which shows that

$$Z(\hat{\mu}_{\widetilde{M},\widetilde{D}}) = \bigcup_{j=1}^{n} Z_j . \tag{95}$$

We now prove that there are only finite mutually orthogonal exponentials in $L^2(\mu_{\widetilde{M},\widetilde{D}})$. If otherwise, there exists a mutually orthogonal set $E(\Lambda)$ with $\Lambda = \{\tau_\ell\}_{\ell=1}^{\infty}$ for $\mu_{\widetilde{M},\widetilde{D}}$, we may assume $\tau_1 = 0$ so that $\Lambda \backslash \{0\} \subset Z(\hat{\mu}_{\widetilde{M},\widetilde{D}})$. By (95), there exists $1 \le j_0 \le n$ such that Z_{j_0}

contains infinitely many elements of Λ. Without loss of generality, we let Z_1 contain infinite elements of Λ, say $\{\tau_{\ell_i}\}_{j=1}^{\infty}$. According to the form of Z_1 above, there exist an $i_0 \in \{1,\dots,q-1\}$ and a subsequence $\{\tau_{\ell_{i_j}}\}_{j=1}^{\infty}$satisfying

$$\left\{\tau_{\ell_{i_j}}\right\}_{j=1}^{\infty} \subset \left\{\left(k-\frac{ci_0}{q},\xi_2,\dots,\xi_n\right)^t : \xi_2,\dots,\xi_n \in \mathbb{R}, k \in \mathbb{Z}\right\}.$$

Then for $j \geq 2$, the difference

$$\tau_{\ell_{i_j}} - \tau_{\ell_{i_1}} \in \{(k_1,\xi_2',\dots,\xi_n')^t : \xi_2',\dots,\xi_n' \in \mathbb{R}, k_1 \in \mathbb{Z}\}.$$

Hence $\tau_{\ell_{i_j}} - \tau_{\ell_{i_1}} \notin Z_1$ for all $j \geq 2$. Since $(\Lambda-\Lambda)\backslash\{0\} \subset Z(\hat{\mu}_{\widetilde{M},\widetilde{D}})$, it follows that

$$\left\{\tau_{\ell_{i_j}} - \tau_{\ell_{i_1}}\right\}_{j=2}^{\infty} \subset \bigcup_{j=2}^{n} Z_j\,.$$

By continuing the above process n times, finally we obtain an infinite sequence of differences $\{\tau_{\ell_m} - \tau_{\ell^*}\}_{m=1}^{\infty}$such that

$$\tau_{\ell_m} - \tau_{\ell^*} \in \{(k_1,k_2,\dots,k_n)^t : k_i \in \mathbb{Z}, i=1,\dots,n\}.$$

That means $\tau_{\ell_m} - \tau_{\ell^*} \notin Z(\hat{\mu}_{\widetilde{M},\widetilde{D}})$, which is impossible. Therefore, there are only finite mutually orthogonal exponentials, and $\mu_{\widetilde{M},\widetilde{D}}$is not a spectral measure.

(ii) As $c=(-1)^n det(M)$, we only need to prove the $\mu_{\widetilde{M},\widetilde{D}}$ is a non-spectral measure for $gcd(q,c)=d$ where $1<d<q$. Let $q=q'd, c=c'd$, then $gcd(c',q')=1$. Denote by

$$\widetilde{D} = \{0,1,\dots,d-1\}\tilde{v} \oplus d\{0,1,\dots,q'-1\}\tilde{v} := \widetilde{D}_1 \oplus \widetilde{D}_2.$$

Let

$$\mu_1 = \delta_{\widetilde{M}^{-1}\widetilde{D}} * \cdots * \delta_{\widetilde{M}^{-n}\widetilde{D}} * \delta_{\widetilde{M}^{-(n+1)}\widetilde{D}_1} * \delta_{\widetilde{M}^{-(n+2)}\widetilde{D}} * \cdots \text{ and } \mu_2 = \delta_{\widetilde{M}^{-(n+1)}\widetilde{D}_2}$$

where $\delta_E = \frac{1}{\#E}\Sigma_{e\in E}\delta_e$for a finite set E and Dirac mass measure $\delta_e$at the point e, and the infinite convolutions converge in the weak sense. Then we have $\mu_{\widetilde{M},\widetilde{D}} = \mu_1 * \mu_2$.

It can be seen that

$$Z(\hat{\delta}_{\overline{M}^{-1}\widetilde{D}}) = \{(-c\frac{m}{q},\eta_1,\dots,\eta_{n-1})^t : m \in \mathbb{Z}, q \nmid m, \eta_1,\dots,\eta_{n-1} \in \mathbb{R}\}$$

and

$$Z(\hat{\mu}_2) = \left\{\left(-c^2\frac{m'}{dq'},\eta_1',\dots,\eta_{n-1}'\right)^t : m' \in \mathbb{Z}, q' \nmid m', \eta_1',\dots,\eta_{n-1}' \in \mathbb{R}\right\}.$$

Write $c^2\frac{m'}{db'} = -c\frac{-cm'}{q}$. Since $\frac{-cm'}{q} = \frac{-cm'}{dq'} = \frac{-c'm'}{q'}, gcd(c',q')=1$ and $q' \nmid m'$, it follows that $q \nmid -cm'$, and $Z(\hat{\mu}_2) \subset Z(\hat{\delta}_{\overline{M}^{-1},\widetilde{D}})$. Hence $Z(\hat{\mu}_{\overline{M},\widetilde{D}}) = Z(\hat{\mu}_1)$. This shows that any orthogonal set $E(\Lambda)$ of $\mu_{\overline{M},\widetilde{D}}$ is an orthogonal set of μ_1, proving that $\mu_{\overline{M},\widetilde{D}}$ is a non-spectral measure by Lemma(3.2.10).

Example (3.2.12) [182]: Let

$$M = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 2 \\ -1 & -1 & -4 \end{bmatrix}$$

and $D=\{0,1,\dots,q-1\}v$ where $q \geq 2, v=(0,0,1)^t$. Then M is an expanding matrix and $\{v,Mv,M^2v\}$is linearly independent. Moreover, the characteristic polynomial of M is $f(x)=x^3+36$. Hence by Theorem(3.2.11), we have

(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if and only if $gcd(36,q)>1$;

(ii) $\mu_{M,D}$ is a spectral measure if and only if $q|36$.

On the other hand, if the set $\{v, Mv, \dots, M^{n-1}v\}$ is linearly dependent, by using some techniques of linear algebra, we can reduce the dimension and get the similar results as Theorems(3.2.9) and (3.2.11).

Lemma (3.2.13) [182]: Let $v \in \mathbb{Z}^n \backslash \{0\}$ and $M \in M_n(\mathbb{Z})$. If $\{v, Mv, \dots, M^{n-1}v\}$ is linearly dependent with $rank\, r < n$, then there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that $Bv = (x_1, \dots, x_r, 0, \dots, 0)^t$and

$$\widetilde{M} = BMB^{-1} = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix} \tag{96}$$

where $M_1 \in M_r(\mathbb{Z}), M_2 \in M_{n-r}(\mathbb{Z})$ and $C \in M_{r,n-r}(\mathbb{Z})$.

Proof: Let $A = [M^{r-1}v, \dots, Mv, v] = [a_{ij}]_{1 \le i \le n, 1 \le j \le r}$. It is well known that by a series of elementary (unimodular) row operations on A which is equivalent to multiple a unimodular B from the left side of A, we can get $BA = \begin{bmatrix} A'_{r,r} \\ 0_{n-r,r} \end{bmatrix} \in M_{n,r}(\mathbb{Z})$, i.e.,

$$BA = B\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} \end{bmatrix} = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1r} \\ 0 & a'_{22} & \cdots & a'_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a'_{rr} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{97}$$

Now we show that the matrix B is the desired one. As $v = (a_{1r}, a_{2r}, \dots, a_{nr})^t$, it follows from (97) that $Bv = (a'_{1r}, \dots a'_{rr}, 0, \dots, 0)^t := (x_1, \dots, x_r, 0, \dots, 0)^t$.

Suppose

$$BMB^{-1} = \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix}.$$

Then

$$BM = \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix} B.$$

Multiplying $A = [M^{r-1}v, \dots, Mv, v]$ from both right sides of the above identity, we have

$$BMA = \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix} BA$$

$$= \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix} \begin{bmatrix} A'_{r,r} \\ 0_{n-r,r} \end{bmatrix} = \begin{bmatrix} M_{r,r}A'_{r,r} \\ D_{n-r,r}A'_{r,r} \end{bmatrix}. \tag{98}$$

It is known that $\{M^{r-1}v, M^{r-2}v, \dots, Mv, v\}$ is a maximal linear independent group as the rank of $\{M^{n-1}v, M^{n-2}v, \dots, Mv, v\}$ is $r < n$. Then there exist $b_0, b_1, \dots, b_{r-1} \in \mathbb{R}$ such that

$$M^r v = b_0 v + b_1 Mv + \cdots + b_{r-1}M^{r-1}v$$

and by (97),

$$BM^r v = (c_1, \dots, c_r, 0, \dots, 0)^t$$

where $c_r = b_0 a'_{rr} \neq 0$. Moreover,

$BMA = BM[M^{r-1}v, \dots, Mv, v] = [BM^r v, \dots, BM^2 v, BMv]$

$$= \begin{bmatrix} c_1 & a'_{11} & \cdots & a'_{1r-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r-1} & 0 & \cdots & a'_{r-1r-1} \\ c_r & 0 & \cdots & a'_{rr} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} := \begin{bmatrix} Q_{r,r} \\ 0_{n-r,r} \end{bmatrix}. \quad (99)$$

From (98) and (99), it follows that $D_{n-r,r}A'_{r,r} = 0$, and $D_{n-r,r} = 0$ as $A'_{r,r}$is invertible. Thus we have

$$BMB^{-1} = \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ 0 & N_{n-r,n-r} \end{bmatrix}.$$

Since B is a unimodular matrix, B^{-1} and BMB^{-1} are also integer matrices. Therefore $M_{r,r}, N_{n-r,n-r}$ and $C_{r,n-r}$ are integer matrices as desired.

Example (3.2.14) [182]: Let

$$M = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

and $v = (1,1,2)^t$, then $Mv = 4v, Bv = (1,0,0)^t$ and

$$BMB^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Theorem (3.2.15) [182]: Let $M \in M_n(\mathbb{Z})$ be an expanding matrix and $D = \{0,1,\dots,q-1\}v$ where $q \geq 2$ and $v \in \mathbb{Z}^n \backslash \{0\}$. If $\{v, Mv, \dots, M^{n-1}v\}$ is linearly dependent with $rank\ r < n$, and M, M_1 are expressed as in (96), then

(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if $gcd(q, det(M_1)) > 1$;

(ii) $\mu_{M,D}$is a spectral measure if $q|det(M_1)$, where M_1 is as in Lemma3.5;

In particular, if the characteristic polynomial of $M_1$is $f(x) = x^r + c$, then

(iii) $\mu_{M,D}$ has infinitely many orthogonal exponentials if and only if $gcd(q, det(M_1)) > 1$;

(iv) $\mu_{M,D}$ is a spectral measure if and only if $q|\ det(M_1)$.

Proof: By Lemma(3.2.13), there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that $Bv = (x_1,\dots,x_r,0,\dots,0)^t := \tilde{v}$ and

$$\widetilde{M} = BMB^{-1} = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix}.$$

Let $\widetilde{D} = BD = \{0,1,\dots,q-1\}\tilde{v}$. Hence it suffices to consider the conjugate case for $\widetilde{M}$ and $\widetilde{D}$. For $j = 1, 2, ...$, we have

$$\widetilde{M}^{*-j} = \begin{bmatrix} M_1^{*-j} & 0 \\ \times & M_2^{*-j} \end{bmatrix}.$$

Let $v' = (x_1,\dots,x_r)^t$ and $D' = \{0,1,\dots,q-1\}v'$. For $\xi = (\xi_1,\dots,\xi_n)^t \in \mathbb{R}^n$, we denote$\xi' = (\xi_1,\dots,\xi_r)^t$. It follows from (92) that

$$\hat{\mu}_{\overline{M},\widetilde{D}}(\xi) = \prod_{j=1}^{\infty}\frac{1}{|\widetilde{D}|}\sum_{d\in\widetilde{D}} e^{2\pi i\langle d,\overline{M^*}^{-j}\xi\rangle} = \prod_{j=1}^{\infty}\frac{1}{q}\sum_{k=0}^{q-1} e^{2\pi i\langle k\tilde{v},\overline{M^*}^{-j}\xi\rangle}$$

$$= \prod_{j=1}^{\infty}\frac{1}{q}\sum_{k=0}^{q-1} e^{2\pi i\langle kv', M_1^{*-j}\xi\rangle} = \hat{\mu}_{M_1,D'}(\xi').$$

If $E(\Lambda)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{\overline{M},\widetilde{D}})$, we let $\Lambda_r = \{\lambda' = (\lambda_1,\dots,\lambda_r)^t : \lambda = (\lambda_1,\dots,\lambda_n)^t \in \Lambda\}$, then $E(\Lambda_r)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{M_1,D'})$. Conversely, if $E(\Lambda_r)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{M_1,D'})$, we let

$$\Lambda := \left\{ \begin{pmatrix} \lambda' \\ \varphi(\lambda') \end{pmatrix} : \lambda' \in \Lambda_r \right\}$$

where $\varphi : \Lambda_r \to \mathbb{R}^{n-r}$ is an arbitrary single-valued function, then $E(\Lambda)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{\overline{M},\widetilde{D}})$. Therefore, $\mu_{M_1,D'}$ and $\mu_{\overline{M},\widetilde{D}}$ have the same spectrality.

By the assumption, $\{v', M_1 v', \dots, M_1^{r-1} v'\}$ is linearly independent. It follows from Theorems(3.2.9)and (3.2.11)that (i)–(iv) hold for $\mu_{M_1,D}$, hence hold for $\mu_{M,D}$.

If the rank of $\{v, Mv, \dots, M^{n-1}v\}$ is $r = 1$, then v becomes an eigenvector of M and $M_1 = \ell$ is the corresponding eigenvalue. Indeed, from Lemma(3.2.13), it can be seen that $Mv = B^{-1}BMB^{-1}Bv = B^{-1}\widetilde{M}Bv = B^{-1}(\ell x_1, 0, \cdots, 0)^t = \ell B^{-1}Bv = \ell v$. By Theorem3.6, the following corollary is immediate.

Corollary (3.2.16) [182]: Under the same assumption as Theorem(3.2.15) with the $rank\, r = 1$, in this case v is an eigenvector of M and $M_1 = \ell$ (as in (96)) is the corresponding eigenvalue, then

(i) $\mu_{M,D}$ has infinitely many orthogonal exponentials if and only if $gcd(q, \ell) > 1$;

(ii) $\mu_{M,D}$ is a spectral measure if and only if $q|\ell$.

Example (3.2.17) [182]: Let M, B and v be as in Example(3.2.14). Let $D_1 = \{0,1,2,3,4,5\}v$ and $D_2 = \{0,1\}v$, then μ_{M,D_1}has infinitely many orthogonal exponentials but not a spectral measure and $\mu_{M,D_2}$is a spectral measure by Corollary(3.2.16).

Section (3.3): A Class of Self-Affine Measures with Decomposable Digit Sets

For a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e^{2\pi i\langle \lambda, x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. The set Λ is then called a spectrum for μ; we also say that (μ, Λ) is a spectral pair. We shall determine the spectrality and non-spectrality of a class of self-affine measures with decomposable digit sets. The self-affine measure considered here is a unique probability measure $\mu := \mu_{M,D}$ satisfying the self-affine identity

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}, \tag{100}$$

where $\phi_d(x) = M^{-1}(x + d)$ is an affine mapping, $M \in M_n(\mathbb{R})$ is an expanding real matrix (that is, all the eigenvalues of M have modulus greater than 1), and $D \subset \mathbb{R}^n$ (usually called the digit set) is a finite subset of cardinality $|D|$. The measure $\mu_{M,D}$ is supported on the attractor or invariant set $T(M,D)$ of the affine iterated function system (IFS) $\{\phi_d(x)\}_{d\in D}$. This is a unique nonempty compact subset $T := T(M,D) \subset \mathbb{R}^n$ such that $T = \bigcup_{d\in D} \phi_d(T)$ (cf. [192]).

Let $S \subset \mathbb{R}^n$ be a finite subset of the cardinality $|S| = |D|$ and $0 \in S$. Corresponding to the dual IFS $\{\psi_s(x) = M^* x + s\}_{s\in S}$, we use $\Lambda(M,S)$ to denote the expansive orbit of 0 under $\{\psi_s(x)\}_{s\in S}$, that is, $\Lambda(M,S) := \{M^{*j} s_j : k \geq 0 \text{ and all } s_j \in S\}$, where M^* denotes the transposed conjugate of M, in fact $M^* = M^t$. Since $M^*\Lambda(M,S) + S = \Lambda(M,S)$, we also call $\Lambda(M,S)$ the invariant set of the dual IFS $\{\psi_s(x)\}_{s\in S}$.

It is known that for certain M, D and S, the corresponding $\mu_{M,D}$ is a spectral measure with spectrum $\Lambda(M,S)$. This was first observed by Jorgensen and Pedersen [210], and studied further by Strichartz [205–207], Laba and Wang [213], Dutkay et al. [205], Hu and Lau [211], Li and Wen [214–224], and others. All these researches focus mainly on the conditions under which $\mu_{M,D}$ is a spectral measure or a non-spectral measure. Among the obtained results, the condition that $(M^{-1}D, S)$ is a compatible pair (or (M, D, S) is a Hadamard triple) plays an important role (see [224, Conjecture 2.5], [216, Problem 1] and [218, Conjecture 1.1]). Recall that for two finite subsets B and P of $\mathbb{R}^n$ of the same cardinality q, we say (B, P) is a compatible pair if the $q \times q$ matrix $H_{B,P} := [q^{-1/2} e^{2\pi i\langle b,p\rangle}]_{b\in B, p\in P}$ is unitary, i.e., $H_{B,P}H_{B,P}^* = I_q$. The well-known result of Jorgensen and Pedersen [210] shows that if $(M^{-1}D, S)$ is a compatible pair with the expanding matrix $M \in M_n(\mathbb{Z})$ and $D, S \subset \mathbb{Z}^n$, then $E(\Lambda(M,S))$ is an infinite orthogonal system in $L^2(\mu_{M,D})$. Except for the compatible pair condition, there are several additional conditions to guarantee that $(\mu_{M,D}, \Lambda(M,S))$ is a spectral pair. Also there are two conjectures related to the spectrality and non-spectrality of self-affine measures (see [224, Conjecture 2.5], [228, Conjecture 1.1], [226, Problem 1] and [217, Conjecture 1]). It should be pointed out that the measure considered here is an invariant measure in dynamics, it has very close connections with the harmonic analysis, the number theory, and the theory of wavelets. And many problems on the spectrality or non-spectrality of this measure remain open. The researches on these problems have led to the spectral theory for fractal measures and have been received much attention in recent years.

We shall determine the spectrality and non-spectrality of a class of self-affine measures when the digit set D can be decomposed in the form $D = D_1 + D_2 = D_1 \oplus D_2$ (direct sum).

We present a method to deal with such case, and clarify the spectrality and non-spectrality of a class of self-affine measures by applying this method.

For a given pair (M, D), the spectrality or non-spectrality of $\mu_{M,D}$ is directly connected with the Fourier transform $\hat{\mu}_{M,D}(\xi)$. From (100), we have

$$\hat{\mu}_{M,D}(\xi) := \int e^{2\pi i\langle x,\xi\rangle} d\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D\left(M^{*-j}\xi\right),\ \xi \in \mathbb{R}^n, \quad (101)$$

where

$$m_D(x) = \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,x\rangle}, x \in \mathbb{R}^n.$$

If $D = D_1 + D_2 = D_1 \oplus D_2$ (direct sum), then

$$m_D(x) = m_{D_1}(x) m_{D_2}(x), \quad x \in \mathbb{R}^n. \quad (102)$$

It follows from (101) and (102) that

$$\hat{\mu}_{M,D}(\xi) = \hat{\mu}_{M,D_1}(\xi)\hat{\mu}_{M,D_2}(\xi), \quad \xi \in \mathbb{R}^n, \quad (103)$$

$$Z\left(\hat{\mu}_{M,D}\right) = Z\left(\hat{\mu}_{M,D_1}\right) \cup Z\left(\hat{\mu}_{M,D_2}\right), \quad (104)$$

where $Z(\hat{\mu}_{M,D}) = \{\xi \in \mathbb{R}^n : \hat{\mu}_{M,D}(\xi) = 0\}$.

Let $\Theta_1 := \{x \in \mathbb{R}^n : m_{D_1}(x) = 0\}$ and $\Theta_2 := \{x \in \mathbb{R}^n : m_{D_2}(x) = 0\}$. Then,

$$Z\left(\hat{\mu}_{M,D_1}\right) = \cup_{j=1}^{\infty} M^{*j}\Theta_1 = M^*\Theta_1 \cup M^{*2}\Theta_1 \cup M^{*3}\Theta_1 \cup \cdots; \quad (105)$$

$$Z\left(\hat{\mu}_{M,D_2}\right) = \cup_{j=1}^{\infty} M^{*j}\Theta_2 = M^*\Theta_2 \cup M^{*2}\Theta_2 \cup M^{*3}\Theta_2 \cup \cdots. \quad (106)$$

In the case when $D_2 = M^k D_1$ for some integer $k > 0$, we have $m_{D_2}(x) = m_{D_1}(M^{*k}x)$, which yields $\Theta_2 = M^{*-k}\Theta_1$ or $\Theta_1 = M^{*k}\Theta_2$. It follows from (105) and (106) that $Z(\hat{\mu}_{M,D_1}) \subseteq Z(\hat{\mu}_{M,D_2})$. Combined with (104), we have

$$Z\left(\hat{\mu}_{M,D}\right) = Z\left(\hat{\mu}_{M,D_2}\right). \quad (107)$$

Lemma (3.3.1)[208]: Let $M \in M_n(\mathbb{R})$ be an expanding matrix and let $\Lambda \subset \mathbb{R}^n$ be a discrete set. Suppose that $D \subset \mathbb{R}^n$ has the decomposable form $D = D_1 \bigoplus D_2$ with $D_2 = M^k D_1$ for some integer $k > 0$. Then the following three statements are equivalent:
(i) $E(\Lambda)$ is an orthogonal system in $L^2(\mu_{M,D})$;
(ii) $E(\Lambda)$ is an orthogonal system in $L^2(\mu_{M,D_2})$;
(iii) $E(M^{*k}\Lambda)$ is an orthogonal system in $L^2(\mu_{M,D_1})$.
Proof: In view of the fact that $E(\Lambda)$ is an orthogonal system in $L^2(\mu_{M,D})$ if and only if $(\Lambda - \Lambda) \subseteq Z(\hat{\mu}_{M,D}) \cup \{0\}$, the equivalence of (i) and (ii) follows from (107). For the digit sets with $D_2 = M^k D_1$, we have $Z(\hat{\mu}_{M,D_1}) = M^{*k} Z(\hat{\mu}_{M,D_2})$ or $Z(\hat{\mu}_{M,D_2}) = M^{*-k} Z(\hat{\mu}_{M,D_1})$, which yields the equivalence of (ii) and (iii).
Lemma (3.3.2) [208]: Let $M \in M_n(\mathbb{R})$ be an expanding matrix and let $\Lambda \subset \mathbb{R}^n$ be a discrete set. Suppose that $D \subset \mathbb{R}^n$ has the decomposable form $D = D_1 \bigoplus D_2$ with $D_2 = M^k D_1$ for some integer $k > 0$. If $(\mu_{M,D}, \Lambda)$ is a spectral pair, then both (μ_{M,D_2}, Λ) and $(\mu_{M,D_1}, M^{*k}\Lambda)$ are spectral pairs.
Proof: It follows from Lemma (3.3.1)(i), (ii) that $E(\Lambda)$ is an orthogonal system in $L^2(\mu_{M,D_2})$, equivalently,

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_{M,D_2}(\xi + \lambda)|^2 \leq 1, \qquad \forall\, \xi \in \mathbb{R}^n.$$

Since $|\hat{\mu}_{M,D_1}(\xi)| \leq 1, \forall\, \xi \in \mathbb{R}^n$, we know, from (103) and the spectral pair $(\mu_{M,D}, \Lambda)$, that for all $\xi \in \mathbb{R}^n$,
$1 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D}(\xi + \lambda)|^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D_1}(\xi + \lambda)|^2 \, |\hat{\mu}_{M,D_2}(\xi + \lambda)|^2 \leq \sum_{\lambda\in\Lambda} \left|\hat{\mu}_{M,D_2}(\xi + \lambda)\right|^2 \leq 1$ (108),
which yields

$$\sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D_2}(\xi + \lambda)|^2 = 1, \quad \forall\, \xi \in \mathbb{R}^n, \quad (109)$$

equivalently, (μ_{M,D_2}, Λ) is a spectral pair. From $\hat{\mu}_{M,D_2}(\xi) = \hat{\mu}_{M,D_1}(M^{*k}\xi)$ and the above (109), we also know that $(\mu_{M,D_1}, M^{*k}\Lambda)$ is a spectral pair.
For the self-affine measure $\mu_{M,D}$ with decomposable digit set, we first give a non-spectral condition, and then we obtain a spectral condition. Some applications of these conditions are also presented.
We follow to deal with the non-spectrality of self-affine measures.
Proposition (3.3.3) [208]: Let $M \in M_n(\mathbb{R})$ be an expanding matrix. Suppose that $D \subset \mathbb{R}^n$ has the decomposable form $D = D_1 \bigoplus D_2$ with $D_2 = M^k D_1$ for some integer $k > 0$. If there exists a point $\xi_0 \in \mathbb{R}^n$ such that

$$0 < |\hat{\mu}_{M,D_1}(\xi_0)| < 1 \text{ and } \prod_{j=0}^{k-1} m_{D_1}(M^{*j}\xi_0) \neq 0, \quad (110)$$

then $\mu_{M,D}$ is a non-spectral measure.
Proof: From $\mu_{M,D_2}(\xi) = \hat{\mu}_{M,D_1}(M^{*k}\xi)$, we have

$$\hat{\mu}_{M,D_2}(\xi) = \hat{\mu}_{M,D_1}(\xi) \prod_{j=0}^{k-1} m_{D_1}(M^{*j}\xi), \qquad \forall\, \xi \in \mathbb{R}^n.$$

Hence, the condition (110) implies that

$$0 < \left|\hat{\mu}_{M,D_1}(\xi_0)\right| < 1 \quad and \quad \hat{\mu}_{M,D_2}(\xi_0) \neq 0. \quad (111)$$

In fact, the conditions (110) and (111) are equivalent.
Suppose that $\mu_{M,D}$ is a spectral measure with spectrum $\Lambda \subset \mathbb{R}^n$. Then, by Lemma (3.3.2), (μ_{M,D_2}, Λ) is a spectral pair. Equivalently, we have

$$1 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D}(\xi+\lambda)|^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D_1}(\xi+\lambda)|^2\, |\hat{\mu}_{M,D_2}(\xi+\lambda)|^2, \forall\, \xi \in \mathbb{R}^n,$$

and

$$\sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D_2}(\xi+\lambda)|^2 = 1, \qquad \forall\, \xi \in \mathbb{R}^n.$$

Now, choosing a point $\tilde{\xi} \in \mathbb{R}^n$ and $\tilde{\lambda} \in \Lambda$ such that $\tilde{\xi} + \tilde{\lambda} = \xi_0$, we obtain that

$$1 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D}(\tilde{\xi}+\lambda)|^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D_1}(\tilde{\xi}+\lambda)|^2\, |\hat{\mu}_{M,D_2}(\tilde{\xi}+\lambda)|^2$$

$$= |\hat{\mu}_{M,D_1}(\xi_0)|^2 |\hat{\mu}_{M,D_2}(\xi_0)|^2 + \sum_{\lambda\in\Lambda\setminus\{\tilde{\lambda}\}} |\hat{\mu}_{M,D_1}(\tilde{\xi}+\lambda)|^2\, |\hat{\mu}_{M,D_2}(\tilde{\xi}+\lambda)|^2$$

$$< |\hat{\mu}_{M,D_2}(\tilde{\xi}+\tilde{\lambda})|^2 + \sum_{\lambda\in\Lambda\setminus\{\tilde{\lambda}\}} |\hat{\mu}_{M,D_2}(\tilde{\xi}+\lambda)|^2$$

$$= \textstyle\sum_{\lambda\in\Lambda} |\hat{\mu}_{M,D_2}(\tilde{\xi}+\lambda)|^2 = 1. \quad (112)$$

This contradiction implies that $\mu_{M,D}$ cannot be a spectral measure.

Proposition (3.3.4) [208]: Let $M \in M_n(\mathbb{R})$ be an expanding matrix. Suppose that $D \subset \mathbb{R}^n$ has the decomposable form $D = D_1 \oplus D_2$ with $\Theta_1 \subseteq \Theta_2$ or with $Z(\hat{\mu}_{M,D_1}) \subseteq Z(\hat{\mu}_{M,D_2})$. If there exists a point $\xi_0 \in \mathbb{R}^n$ such that $0 < |\hat{\mu}_{M,D_1}(\xi_0)| < 1$ and $\hat{\mu}_{M,D_2}(\xi_0) \neq 0$, then $\mu_{M,D}$ is a non-spectral measure.

Example(3.3.5) [208]: (I). Let $p, p_1 \in \mathbb{Z}$ and $|p_1| > 1$. Consider the pair (M, D) given by

$$M = \begin{bmatrix} 2 & 0 \\ p & p_1 \end{bmatrix} \quad and \quad D = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\p\end{pmatrix}, \begin{pmatrix}3\\p\end{pmatrix}\right\}, \quad (113)$$

then the following statements hold:

(i) $\mu_{M,D}$ is a non-spectral measure;

(ii) There are infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$;

(iii) There is no subset $S \subset \mathbb{Z}^2$ such that $(M^{-1}D, S)$ is a compatible pair;

(iv) There exists a subset $S \subset \mathbb{R}^2$ such that $(M^{-1}D, S)$ is a compatible pair.

In fact, we first have $D = D_1 \oplus D_2$, where

$$D_1 := \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\} \quad and \quad D_2 := \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2\\p\end{pmatrix}\right\} = MD_1.$$

Now, $m_{D_1}(x) = e^{\pi i x_1} cos(\pi x_1)$, where $x = (x_1, x_2)^t \in \mathbb{R}^2$, we have

$$\Theta_1 := \{x \in \mathbb{R}^2 : m_{D_1}(x) = 0\} = \left\{\begin{pmatrix} k + \frac{1}{2} \\ b \end{pmatrix} \in \mathbb{R}^2 : k \in \mathbb{Z}, b \in \mathbb{R}\right\},$$

and for $\xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2$,

$$\hat{\mu}_{M,D_1}(\xi) = \prod_{j=1}^{\infty} m_{D_1}\left(M^{*-j}\xi\right) = \prod_{j=1}^{\infty} e^{\pi i \frac{\xi_1 - p\xi_2 \sum_{l=}^{j-1} 2^l p_1^{-l-1}}{2j}} \cos\left(\pi \frac{\xi_1 - p\xi_2 \sum_{l=}^{j-1} 2^l p_1^{-l-1}}{2j}\right).$$

Taking $\xi_0 = (a, 0)^t \in \mathbb{R}^2$ with $0 < |a| < \frac{1}{2}$, we obtain

$$0 < |\hat{\mu}_{M,D_1}(\xi_0)| = \left|e^{\pi i a} \textstyle\prod_{j=0}^{\infty} \cos\left(\frac{\pi a}{2^j}\right)\right| = \left|\frac{sin(\pi a)}{\pi a}\right| < 1 \; and \; m_{D_1}(\xi_0) \neq 0. \quad (114)$$

Hence, the conclusion (i) follows from Proposition (3.3.3).

Next, by taking $S_1 = D_1$, we obtain that $(M^{-1}D_1, S_1)$ is a compatible pair. So, $E(\Lambda(M, S_1))$ is an infinite orthogonal system in $L^2(\mu_{M,D_1})$. By Lemma (3.3.1), $E(M^{*-1}\Lambda(M, S_1))$ is an infinite orthogonal system in $L^2(\mu_{M,D})$. That is, (ii) holds.

Finally, for the conclusions (iii) and (iv), we would like to give a general result. In the integer case: $M \in M_n(\mathbb{Z})$ and $D \subset \mathbb{Z}^n$, if $D = D_1 \oplus D_2 = D_1 \oplus M^k D_1$ with $0 \in D_1$ and $k > 0$, then for any $d_1 \in D_1$ with $d_1 \neq 0$, we have $0, d_1, M^k d_1 \in D$, which shows $M^k d_1 \equiv 0 (mod\ M)$. Also, if there is a subset $S \subset \mathbb{Z}^n$ such that $(M^{-1}D, S)$ is a compatible pair, then the elements in D are distinct modulo M [214, Proposition 2.2]. Hence (iii) holds. On the other hand, if there is a subset $S_1 \subset \mathbb{Z}^n$ such that $(M^{-1}D_1, S_1)$ is a compatible pair, then

$$(M^{-2}(D_1 \oplus MD_1), S_1 \oplus M^* S_1)\ and\ (M^{-1}(D_1 \oplus MD_1), M^{*-1}(S_1 \oplus M^* S_1))$$

are two compatible pairs. Hence (iv) holds with $S := M^{*-1}(S_1 \oplus M^* S_1) \subset \mathbb{R}^n$. Note that in this case, the set $S = M^{*-1}(S_1 \oplus M^* S_1) \not\subset \mathbb{Z}^n$ and $\mu_{M^*,S}$ is a non-spectral meature.

Example (3.3.5) [208] (II): Let $p, p_1 \in \mathbb{Z}$ and $|p_1| > 1$. Consider the pair (M, D) given by

$$M = \begin{bmatrix} 2 & 0 \\ p & p_1 \end{bmatrix} and\ D = \left\{ \binom{0}{0}, \binom{1}{1}, \binom{2}{p+p_1}, \binom{3}{p+p_1+1} \right\}, (115)$$

then the following statements hold:

(i) $\mu_{M,D}$ is a non-spectral measure;

(ii) There are infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$;

(iii) There is no subset $S \subset \mathbb{Z}^2$ such that $(M^{-1}D, S)$ is a compatible pair;

(iv) There exists a subset $S \subset \mathbb{R}^2$ such that $(M^{-1}D, S)$ is a compatible pair.

In fact, we first have $D = D_1 \oplus D_2$, where

$$D_1 := \left\{ \binom{0}{0}, \binom{1}{1} \right\} \quad and \quad D_2 := \left\{ \binom{0}{0}, \binom{2}{p+p_1} \right\} = MD_1.$$

Now, $m_{D_1}(x) = e^{\pi i(x_1+x_2)}\ cos\ \pi(x_1 + x_2)$, where $x = (x_1, x_2)^t \in \mathbb{R}^2$, we have

$$\Theta_1 := \{x \in \mathbb{R}^2 : m_{D_1}(x) = 0\} = \left\{ \binom{x_1}{x_2} \in \mathbb{R}^2 : x_1 + x_2 = k + \frac{1}{2}, k \in \mathbb{Z} \right\}.$$

As the above Example (3.3.5)(I), taking $\xi_0 = (a, 0)^t \in \mathbb{R}^2$ with $0 < |a| < \frac{1}{2}$, we obtain

$$0 < |\hat{\mu}_{M,D_1}(\xi_0)| = \left| \frac{sin(\pi a)}{\pi a} \right| < 1\ and\ m_{D_1}(\xi_0) \neq 0.$$

Hence, we have (i) by Proposition (3.3.3). The other conclusions can be proved in the same way as Example (3.3.5)(I).

In the same manner, one can show that if $p, p_1 \in \mathbb{Z}$ and $|p_1| > 1$, then the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} p_1 & p \\ 0 & 2 \end{bmatrix} \quad and \quad D = \left\{ \binom{0}{0}, \binom{0}{1}, \binom{p}{2}, \binom{p}{3} \right\} \quad (116)$$

or

$$M = \begin{bmatrix} p_1 & p \\ 0 & 2 \end{bmatrix} \quad and \quad D = \left\{ \binom{0}{0}, \binom{1}{1}, \binom{p+p_1}{2}, \binom{p+p_1+1}{3} \right\} \quad (117)$$

satisfies the above properties (i)–(iv) in Examples (3.3.5)(I) and (3.3.5)(II).

It should be pointed out that the above Examples (3.3.5)(I) and (3.3.5)(II) give us many non-spectral measures by letting p and p_1 be two proper integers. For example, the self-affine measure $\mu_{M,D}$ corresponding to

$$M = \begin{bmatrix} 2 & p \\ 0 & p_1 \end{bmatrix}, p \in \mathbb{Z}, \quad p_1 \in \mathbb{Z}\backslash\{0, \pm 1\}\ and\ D = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0} \right\};$$

$$M = \begin{bmatrix} 2 & 0 \\ 1 & p_1 \end{bmatrix}, \quad p_1 \in \mathbb{Z}\backslash\{0, \pm 1\}\ and\ D = \left\{ \binom{0}{0}, \binom{1}{0}, \binom{2}{1}, \binom{3}{1} \right\};$$

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad and \quad D = \left\{ \binom{0}{0}, \binom{1}{1}, \binom{2}{2}, \binom{3}{3} \right\};$$

$$M = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\ and\ D = \left\{ \binom{0}{0}, \binom{1}{1}, \binom{2}{3}, \binom{3}{4} \right\}$$

are all non-spectral measures, even if we have an infinite orthogonal family of exponential functions in each of the space $L^2(\mu_{M,D})$. Note that in the first example and in the third example, D is a collinear digit set. Since $D = D_1 \oplus MD_1$ in these examples, we know that the spectrality or non-spectrality of $\mu_{M^2,D}$ is the same as μ_{M,D_1}. So, from the spectrality of μ_{M,D_1} (see [224]), we also have a spectral measure $\mu_{M,D}$ corresponding to a nontrivial collinear digit set D. For example, if the pair (M, D) is given by

$$M = \begin{bmatrix} 4 & p(p_1+2) \\ 0 & p_1^2 \end{bmatrix} \quad and \quad D = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}\right\},$$

where $p \in \mathbb{Z}, p_1 \in \mathbb{Z}\setminus\{0, \pm 1\}$, or if the pair (M, D) is given by

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad and \quad D = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}\right\},$$

then $\mu_{M,D}$ is a spectral measure. This also illustrates that there are M and D such that $\mu_{M^2,D}$ is a spectral measure but $\mu_{M,D}$ is a non-spectral measure. On the other hand, by applying the similarity over $\mathbb{R}$, it follows from (113) or (115) that for the pair (M, D) given by

$$M = \begin{bmatrix} 2+p_1 & 1 \\ -2p_1 & 0 \end{bmatrix}, \qquad p_1 \in \mathbb{Z}\setminus\{0, \pm 1\} \; and \; D = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\},$$

the properties (i)–(iv) listed in Example (3.3.5)(I) or Example (3.3.5)(II) still hold.

Finally, from the above discussion and applications, we state and prove the following main result:

Theorem (3.3.6) [208]: Let $M \in M_n(\mathbb{R})$ be an expanding matrix. If $D \subset \mathbb{R}^n$ has the decomposable form $D = D_1 \oplus D_2$ with $D_2 = M^k D_1$ for some integer $k > 0$, then $\mu_{M,D}$ is a non-spectral measure.

Proof: We only need to prove that there always exists a point $\xi_0 \in \mathbb{R}^n$ such that the condition (110) in Proposition (3.3.3) holds. This is based on the following observation:

Since the function $\hat{\mu}_{M,D_1}(\xi) = \int e^{2\pi i\langle x,\xi\rangle} d\mu_{M,D_1}(x)$ is analytic in each variable $\xi_1, \xi_2, \ldots, \xi_n$, the set of zeroes of this function has empty interior. Indeed, otherwise there is a ball $B(\xi_0, \delta)$ with center ξ_0 and radius $\delta > 0$, such that $\hat{\mu}_{M,D_1}(\xi) = 0, \forall\, \xi \in B(\xi_0, \delta)$. Then for any $\xi \in \mathbb{R}^n$, the function $t \to \hat{\mu}_{M,D_1}((1-t)\xi_0 + t\xi)$ is analytic and is zero on some open interval, therefore it is zero everywhere so $\hat{\mu}_{M,D_1}(\xi) = 0$, a contradiction.

Also, the set where $|\hat{\mu}_{M,D_1}(\xi)| = 1$ has empty interior. This can be seen from the triangle inequality

$$|\hat{\mu}_{M,D_1}(\xi)| = \left|\int e^{2\pi i\langle x,\xi\rangle} d\mu_{M,D_1}(x)\right| \le \int \left|e^{2\pi i\langle x,\xi\rangle}\right| d\mu_{M,D_1}(x) = 1.$$

Since we want to have equality, we must have that there is some constant $e^{2\pi i\alpha}$ such that $e^{2\pi i\langle x,\xi\rangle} = e^{2\pi i\alpha}$ for μ_{M,D_1}-a.e.x. Then $\langle x, \xi\rangle = \alpha + k$ for some $k \in \mathbb{Z}$ and for μ_{M,D_1}-a.e.x. But even for a fixed non-zero x, the set of such points ξ, is just a union of hyperplanes. Therefore it has empty interior.

Finally, since the function $\prod_{j=0}^{k-1} m_{D_1}(M^{*j}\xi)$ is analytic in each variable, the set of its zeroes has empty interior.

In conclusion, the set of points where the condition (110) is not satisfied is a union of three closed sets with empty interior, therefore it cannot cover $\mathbb{R}^n$ (Baire's theorem). Hence there always exists a point $\xi_0 \in \mathbb{R}^n$ such that the condition (110) holds.

Similarly, corresponding to Proposition (3.3.4), we have the following:

Theorem (3.3.7) [208]: Let $M \in M_n(\mathbb{R})$ be an expanding matrix. If $D \subset \mathbb{R}^n$ has the decomposable form $D = D_1 \oplus D_2$ with $\Theta_1 \subseteq \Theta_2$ or with $Z(\hat{\mu}_{M,D_1}) \subseteq Z(\hat{\mu}_{M,D_2})$, then $\mu_{M,D}$ is a non-spectral measure.

Let $M \in M_n(\mathbb{R})$ be an expanding matrix. Suppose that $D \subset \mathbb{R}^n$ has the decomposable form $D = D_1 \oplus D_2$ with $0 \in D_1 \cap D_2$. If μ_{M,D_1} is a spectral measure with spectrum $\Lambda_1 \subset \mathbb{R}^n$ and μ_{M,D_2} is a spectral measure with spectrum $\Lambda_2 \subset \mathbb{R}^n$ such that

$$\left|\hat{\mu}_{M,D_1}(\xi + \lambda_2)\right| = \left|\hat{\mu}_{M,D_1}(\xi)\right|, \quad \forall \xi \in \mathbb{R}^n, \lambda_2 \in \Lambda_2 \quad (118)$$

or

$$\left|\hat{\mu}_{M,D_2}(\xi + \lambda_1)\right| = \left|\hat{\mu}_{M,D_2}(\xi)\right|, \quad \forall \xi \in \mathbb{R}^n, \lambda_1 \in \Lambda_1, \quad (119)$$

then $\mu_{M,D}$ is a spectral measure with spectrum $\Lambda_1 + \Lambda_2 = \Lambda_1 \oplus \Lambda_2 \subset \mathbb{R}^n$. In fact, from (103), we have

$$\sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} |\hat{\mu}_{M,D}(\xi + \lambda_1 + \lambda_2)|^2$$

$$= \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} |\hat{\mu}_{M,D_1}(\xi + \lambda_1 + \lambda_2)|^2 \, |\hat{\mu}_{M,D_2}(\xi + \lambda_1 + \lambda_2)|^2 \; (apply\ (118))$$

$$= \sum_{\lambda_1 \in \Lambda_1} |\hat{\mu}_{M,D_1}(\xi + \lambda_1)|^2 \sum_{\lambda_2 \in \Lambda_2} |\hat{\mu}_{M,D_2}(\xi + \lambda_1 + \lambda_2)|^2$$

$$= \textstyle\sum_{\lambda_1 \in \Lambda_1} |\hat{\mu}_{M,D_1}(\xi + \lambda_1)|^2 = 1, \quad \forall\, \xi \in \mathbb{R}^n. \quad (120)$$

Since $\hat{\mu}_{M,D}(0) = 1$, the above (120) first shows that $\Lambda_1 + \Lambda_2 = \Lambda_1 \oplus \Lambda_2$. This is due to the fact that if $\lambda_1 + \lambda_2 = \tilde{\lambda}_1 + \tilde{\lambda}_2 := \lambda$ with $\lambda_1, \tilde{\lambda}_1 \in \Lambda_1$ and $\lambda_2, \tilde{\lambda}_2 \in \Lambda_2$, then, by taking $\xi = -\lambda$ in (120), the left-hand side of (120) will be greater than 2. Hence, an element $\lambda \in \Lambda_1 + \Lambda_2$ can be written uniquely as $\lambda = \lambda_1 + \lambda_2$ with $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$. Secondly, the above (120) shows that $\mu_{M,D}$ is a spectral measure with spectrum $\Lambda_1 \oplus \Lambda_2$.

As an application of the above observation, we shall prove the following:

Theorem (3.3.8) [208]: For the digit set D given by

$$D = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}d_1\\0\end{pmatrix}, \begin{pmatrix}0\\d_2\end{pmatrix}, \begin{pmatrix}d_1\\d_2\end{pmatrix}\right\}, \quad d_1, d_2 \in \mathbb{R}\backslash\{0\}, (121)$$

and for the expanding matrices given by

(i) $M_1 = \begin{bmatrix}p_1 & 0\\0 & p_2\end{bmatrix}, p_1, p_2 \in 2\mathbb{Z}\backslash\{0\}; \quad (122)$

(ii) $M_2 = \begin{bmatrix}p_1 & p\\0 & p_2\end{bmatrix}, \quad p_1, p_2 \in 2\mathbb{Z}\backslash\{0\},\ p \in \mathbb{Z}; \quad (123)$

(iii) $M_3 = \begin{bmatrix}p_1 & 0\\p & p_2\end{bmatrix}, \quad p_1, p_2 \in 2\mathbb{Z}\backslash\{0\},\ p \in \mathbb{Z}, \quad (124)$

respectively, the corresponding self-affine measures $\mu_{M_1,D}, \mu_{M_2,D}$ and $\mu_{M_3,D}$ are all spectral measures.

Proof: For the digit set D given by (121), let

$$D_1 := \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}d_1\\0\end{pmatrix}\right\} = d_1 \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\} := d_1 \widetilde{D}_1;$$

$$D_2 := \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\d_2\end{pmatrix}\right\} = d_2 \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} := d_2 \widetilde{D}_2.$$

We first know that $D = D_1 + D_2 = D_1 \oplus D_2$ holds. Then, we have, for the pair (M_1, D),

$$\begin{cases} \hat{\mu}_{M_1,\widetilde{D}_1}(\xi) = e^{\pi i \xi_1/(p_1-1)} \prod_{j=1}^{\infty} \cos\left(\frac{\pi \xi_1}{p_1^j}\right), \\ \hat{\mu}_{M_1,\widetilde{D}_2}(\xi) = e^{\pi i \xi_2/(p_2-1)} \prod_{j=1}^{\infty} cos\left(\frac{\pi \xi_2}{p_2^j}\right); \end{cases} \quad (125)$$

for the pair (M_2, D),

$$\begin{cases} \hat{\mu}_{M_2,\widetilde{D}_1}(\xi) = e^{\pi i \xi_1/(p_1-1)} \prod_{j=1}^{\infty} cos\left(\frac{\pi \xi_1}{p_1^j}\right) = \hat{\mu}_{M_1,\widetilde{D}_1}(\xi), \\ \hat{\mu}_{M_2,\widetilde{D}_2}(\xi) = \prod_{j=1}^{\infty} e^{\pi i\left(p_2^{-j}\xi_2 - p\xi_1 \sum_{l=0}^{j-1} p_1^{l-j} p_2^{-l-1}\right)} cos\,\pi(p_1^{-j}\xi_2 - p\xi_1 \sum_{l=0}^{j-1} p_1^{l-j} p_2^{-l-1}); \end{cases} \quad (126)$$

and for the pair (M_3, D),

$$\begin{cases} \hat{\mu}_{M_3,\widetilde{D}_1}(\xi) = \prod_{j=1}^{\infty} e^{\pi i\left(p_1^{-j}\xi_1 - p\xi_2 \sum_{l=0}^{j-1} p_1^{l-j} p_2^{-l-1}\right)} cos\,\pi(p_1^{-j}\xi_1 - p\xi_2 \sum_{l=0}^{j-1} p_1^{l-j} p_2^{-l-1}); \\ \hat{\mu}_{M_3,\widetilde{D}_2}(\xi) = e^{\pi i \xi_2/(p_2-1)} \prod_{j=1}^{\infty} cos\left(\frac{\pi \xi_2}{p_2^j}\right) = \hat{\mu}_{M_1,\widetilde{D}_2}(\xi), \end{cases} \quad (127)$$

where $\xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2$.

Let

$$\Lambda_{11} := \left\{\sum_{j=0}^{k} p_1^j s_j : k \geq 0 \; and \; all \; s_j \in \left\{0, \frac{p_1}{2}\right\}\right\} \subseteq \mathbb{R}, \quad (128)$$

and

$$\Lambda_1 := \left\{\begin{pmatrix} \lambda_{11} \\ 0 \end{pmatrix} \in \mathbb{R}^2 : \lambda_{11} \in \Lambda_{11}\right\}. \quad (129)$$

Then, $\mu_{M_1,\widetilde{D}_1}$ and $\mu_{M_2,\widetilde{D}_1}$ are spectral measures with the same spectrum Λ_1 (see [24]), hence μ_{M_1,D_1} and μ_{M_2,D_1} are spectral measures with the same spectrum $d_1^{-1}\Lambda_1$.

Similarly, let

$$\Lambda_{22} := \left\{\sum_{j=0}^{k} p_2^j s_j : k \geq 0 \; and \; all \; s_j \in \{0, p_2/2\}\right\} \subseteq \mathbb{R}, \quad (130)$$

and

$$\Lambda_2 := \left\{\begin{pmatrix} 0 \\ \lambda_{22} \end{pmatrix} \in \mathbb{R}^2 : \lambda_{22} \in \Lambda_{22}\right\}. \quad (131)$$

Then, $\mu_{M_1,\widetilde{D}_2}$ and $\mu_{M_3,\widetilde{D}_2}$ are spectral measures with the same spectrum Λ_2, hence μ_{M_1,D_2} and μ_{M_3,D_2} are spectral measures with the same spectrum $d_2^{-1}\Lambda_2$.

Now, we prove Theorem (3.3.8) from the simple case.

(i) For the given pair (M_1, D), in view of the fact that $(\mu_{M_1,D_1}, d_1^{-1}\Lambda_1)$ and $(\mu_{M_1,D_2}, d_2^{-1}\Lambda_2)$ are two spectral pairs, one can check, from (125), that the following two equalities:

$$|\hat{\mu}_{M_1,D_1}(\xi + \lambda_2)| = |\hat{\mu}_{M_1,D_1}(\xi)|, \forall \xi \in \mathbb{R}^2, \qquad \lambda_2 \in d_2^{-1}\Lambda_2$$
$$|\hat{\mu}_{M_1,D_1}(\xi + \lambda_1)| = |\hat{\mu}_{M_1,D_2}(\xi)|, \forall \xi \in \mathbb{R}^2, \qquad \lambda_1 \in d_1^{-1}\Lambda_1$$

are all satisfied. Hence $\mu_{M_1,D}$ is a spectral measure with spectrum $d_1^{-1}\Lambda_1 \oplus d_2^{-1}\Lambda_2$. This proves Theorem (3.3.8)(i).

(ii) For the pair (M_2, D), we have known that $(\mu_{M_2,D_1}, d_1^{-1}\Lambda_1)$ is a spectral pair. As for the measure μ_{M_2,D_2}, we shall prove the following conclusions:

(iia) If $p_2 \neq 2$, then $(\mu_{M_2,\widetilde{D}_2}, \tilde{\Lambda}_2)$ is a spectral pair, equivalently, $(\mu_{M_2,D_2}, d_2^{-1}\Lambda_2)$ is a spectral pair;

(iib) If $p_2 = 2$, then $(\mu_{M_2,\widetilde{D}_2}, \tilde{\Lambda}_2)$ is a spectral pair, equivalently, $(\mu_{M_2,D_2}, d_2^{-1}\tilde{\Lambda}_2)$ is a spectral pair, where $\tilde{\Lambda}_2 = \{(0, k)^t \in \mathbb{R}^2 : k \in \mathbb{Z}\}$.

In fact, by letting

$$S_2 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p_2/2 \end{pmatrix} \subset \mathbb{Z}^2,$$

we first obtain that $(M_2^{-1}\widetilde{D}_2, S_2)$ is a compatible pair. Then,

$$T(M_2^*, S_2) = \left\{\sum_{j=1}^{\infty} M_2^{*-j} s_j : all\ s_j \in S_2\right\} = \left\{\begin{pmatrix} 0 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \in \tilde{T}\right\}, \quad (132)$$

where

$$\tilde{T} = \frac{p_2}{2}\left\{\sum_{j=1}^{\infty} p_2^{-j} s_{2,j} : all\ s_{2,j} \in \{0,1\}\right\}.$$

Proof of (iia). If $p_2 \neq 2$, then $p_2 \in 2\mathbb{Z}\backslash\{0,2\}$, which yields

$$T(M_2^*, S_2) \subseteq \left\{\begin{pmatrix} 0 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -1 < x_2 < 1\right\}. \quad (133)$$

Since the zero set $Z(m_{M_2^{-1}\widetilde{D}_2})$ of the function $m_{M_2^{-1}\widetilde{D}_2}(x)$ is given by

$$Z(m_{M_2^{-1}\widetilde{D}_2}) = \{x \in \mathbb{R}^2 : m_{M_2^{-1}\widetilde{D}_2}(x) = 0\}$$

$$= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \frac{-px_1}{p_1 p_2} + \frac{x_2}{p_2} = k + \frac{1}{2}, k \in \mathbb{Z}\right\}, \quad (134)$$

we obtain, under the given condition, that

$$T(M_2^*, S_2) \cap Z\left(m_{M_2^{-1}\widetilde{D}_2}\right) = \emptyset. \quad (135)$$

By appling a criterion of Strichartz [225], we get that $(\mu_{M_2\widetilde{D}_2}, \Lambda(M_2, S_2))$ is a spectral pair. Since $\Lambda(M_2, S_2) = \{\sum_{j=0}^{k} M_2^{*j} s_j : k \geq 0$ and all $s_j \in S_2\} = \Lambda_2$, we know that μ_{M_2,D_2} is a spectral measure with spectrum $d_2^{-1}\Lambda_2$. Hence the conclusion (iia) holds.

Proof of (iib). If $p_2 = 2$, then we have, from (132) that

$$T(M_2^*, S_2) = \left\{\begin{pmatrix} 0 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : 0 \leq x_2 \leq 1\right\}. \quad (136)$$

In this case, we have

$$T(M_2^*, S_2) \cap Z\left(m_{M_2^{-1}\widetilde{D}_2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2, \quad (137)$$

so the criterion of Strichartz [225] cannot be applied. We shall apply [6, Theorem 2.38] to prove (iib).

Note that $(M_2, \widetilde{D}_2, S_2)$ is a Hadamard triple and (136) holds. One can see that the function $m_{\widetilde{D}_2}(x)$ has only one zero $(0, 1/2)^t$ on the attractor $T(M_2^*, S_2)$. Also, the $m_{\widetilde{D}_2}$-cycles consist of two one-cycles C_1 and C_2, where $C_1 = \{(0,0)^t\}$ and $C_2 = \{(0,1)^t\}$. For the cycle C_1, let $s_0 = (0,0)^t$, we have

$$S(C_1) := \left\{\sum_{j=0}^{k} M_2^{*j}\omega_j - M_2^{*(k+1)}(M_2^* - I)^{-1}s_0 : all\ \omega_j \in S_2, k \geq 0\right\}$$

$$= \Lambda(M_2, S_2) = \left\{(0,k)^t \in \mathbb{R}^2 : k \in \mathbb{N} \cup \{0\}\right\}. \quad (138)$$

For the cycle C_2, let $s_1 = (0,1)^t$. We have

$S(C_2) := \left\{\sum_{j=0}^{k} M_2^{*j}\omega_j - M_2^{*(k+1)}(M_2^* - I)^{-1}s_1 : all\ \omega_j \in S_2, k \geq 0\right\} = \{(0,-k)^t \in \mathbb{R}^2 : k \in \mathbb{N}\}.$ (139)

So, $\tilde{\Lambda}_2 := S(C_1) \cup S(C_2) = \{(0,k)^t \in \mathbb{R}^2 : k \in \mathbb{Z}\}$ is the smallest set that contains $-C_1, -C_2$ and satisfies

$$M_2^*\tilde{\Lambda}_2 + S_2 \subset \tilde{\Lambda}_2. \quad (140)$$

One can directly verify that $\tilde{\Lambda}_2 := \{(0,k)^t \in \mathbb{R}^2 : k \in \mathbb{Z}\}$ is the smallest set that contains $-C_1, -C_2$ and satisfies (140). Hence, it follows from Theorem 2.38 in [226] that $\tilde{\Lambda}_2$ is a spectrum for the measure $\mu_{M_2,\widetilde{D}_2}$, and this spectrum does not depend on p_1 and p. The conclusion (iib) holds.

Finally, one can check, from (126), that the following equality:

$$|\hat{\mu}_{M_2,D_1}(\xi + \lambda_2)| = |\hat{\mu}_{M_2,D_1}(\xi)|, \forall\ \xi \in \mathbb{R}^2, \lambda_2 \in d_2^{-1}\ \Lambda_2\ or\ \lambda_2 \in d_2^{-1}\tilde{\Lambda}_2$$

holds. Hence $\mu_{M_2,D}$ is a spectral measure with spectrum $d_1^{-1}\Lambda_1 \oplus d_2^{-1}\Lambda_2$ in the case when $p_2 \neq 2$ or with spectrum $d_1^{-1}\Lambda_1 \oplus d_2^{-1}\Lambda_2$ in the case when $p_2 = 2$. This completes the proof of Theorem (3.3.8)(ii).

Note that, from the above discussion, we know that the proof of Theorem (3.3.8)(ii) needs more technology than that of Theorem (3.3.8)(i), although the former can be reduced to the latter if we let $p = 0$. Also, for the pair (M_2, D_2), the conclusions (iia) and (iib) are independent of the parity of p_1. In these two conclusions, we only need $p_1 \in \mathbb{Z}\setminus\{0, \pm 1\}$, i.e., $p_1 \in \mathbb{Z}$ and $|p_1| > 1$. However, for the pair (M_2, D_1), we require that $p_1 \in 2\mathbb{Z}\setminus\{0\}$. In order to understand the method better, we present an analogue proof on Theorem (3.3.8)(iii) below.

(iii) For the pair (M_3, D), we have known that $(\mu_{M_3,D_1}, d_1^{-1}\Lambda_1)$ is a spectral pair. With the same method as the above case (ii), we can prove the following conclusions:

(iiia) If $p_1 \neq 2$, then $(\mu_{M_3,\widetilde{D}_1}, \Lambda_1)$ is a spectral pair, equivalently, $(\mu_{M_3,D_1}, d_1^{-1}\Lambda_1)$ is a spectral pair;

(iiib) If $p_1 = 2$, then $(\mu_{M_3,\widetilde{D}_1}, \tilde{\Lambda}_1)$ is a spectral pair, equivalently, $(\mu_{M_3,D_1}, d_1^{-1}\Lambda_1)$ is a spectral pair, where $\tilde{\Lambda}_1 = \{(k,0)^t \in \mathbb{R}^2 : k \in \mathbb{Z}\}$.

In fact, a little difference with the above case (ii) is by letting

$$S_1 := \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}p_1/2\\2\end{pmatrix}\right\} \subset \mathbb{Z}^2,$$

to obtain that $(M_3^{-1}\widetilde{D}_1, S_1)$ is a compatible pair. Then,

$$T(M_3^*, S_1) = \left\{\sum_{j=1}^{\infty} M_3^{*-j} s_j : all\ s_j \in S_1\right\} = \left\{\begin{pmatrix}x_1\\0\end{pmatrix} \in \mathbb{R}^2 : x_1 \in \tilde{T}_1\right\}, (141)$$

where

$$\tilde{T}_1 = \frac{p_1}{2}\left\{\sum_{j=1}^{\infty} p_1^{-j} s_{1,j} : all\ s_{1,j} \in \{0,1\}\right\}.$$

Proof of (iiia). If $p_1 \neq 2$, then $p_1 \in 2\mathbb{Z}\setminus\{0,2\}$, which yields

$$T(M_3^*, S_1) \subseteq \left\{\begin{pmatrix}x_1\\0\end{pmatrix} \in \mathbb{R}^2 : -1 < x_1 < 1\right\}. \quad (142)$$

Since the zero set $Z(m_{M_2^{-1}\widetilde{D}_1})$ of the function $m_{M_3^{-1}\widetilde{D}_1}(x)$ is given by

$$Z\left(m_{M_3^{-1}\widetilde{D}_1}\right) = \left\{x \in \mathbb{R}^2 : m_{M_3^{-1}\widetilde{D}_1}(x) = 0\right\} = \left\{\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2 : \frac{x_1}{p_1} - \frac{px_2}{p_1 p_2} = k + \frac{1}{2}, k \in \mathbb{Z}\right\},$$

(143)

we obtain, under the given condition, that

$$T(M_3^*, S_1) \cap Z\left(m_{M_3^{-1}\widetilde{D}_1}\right) = \emptyset. \quad (144)$$

Applying a criterion of Strichartz [225], we get that $(\mu_{M_3\widetilde{D}_1}, \Lambda(M_3, S_1))$ is a spectral pair. Since $\Lambda(M_3, S_1) = \{\sum_{j=0}^{k} M_3^{*j} s_j : k \geq 0 \text{ and all } s_j \in S_1\} = \Lambda_1$, we know that $\mu_{M_3 D_1}$ is a spectral measure with spectrum $d_1^{-1}\Lambda_1$. Hence the conclusion (iiia) holds.

Proof of (iiib). If $p_1 = 2$, then we have, from (141) that

$$T(M_3^*, S_1) \subseteq \left\{\begin{pmatrix}x_1\\0\end{pmatrix} \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\right\}. \quad (145)$$

In this case, we have

$$T(M_3^*, S_1) \cap Z\left(m_{M_3^{-1}\widetilde{D}_1}\right) = \begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2, \quad (146)$$

so the criterion of Strichartz [225] cannot be applied as well. We shall apply [226, Theorem 2.38] to prove (iiib).

Note that the function $m_{\widetilde{D}_1}(x)$ has only one zero $(1/2,0)^t$ on the attractor $T(M_3^*,S_1)$. Also, the $m_{\widetilde{D}_1}$-cycles consist of two one-cycles $\tilde{C}_1$ and $\tilde{C}_2$, where $\tilde{C}_1=\{(0,0)^t\}$ and $\tilde{C}_2=\{(1,0)^t\}$. For the cycle $\tilde{C}_1$, let $\tilde{s}_0=(0,0)^t$, we have

$$S(\tilde{C}_1):=\left\{\sum_{j=0}^{k} M_3^{*j}\omega_j - M_3^{*(k+1)}(M_3^*-I)^{-1}\bar{s}_0 : all\ \omega_j\in S_1, k\geq 0\right\}$$
$$=\Lambda(M_3,S_1)=\{(0,k)^t\in\mathbb{R}^2: k\in\mathbb{N}\cup\{0\}\}. \quad (147)$$

For the cycle $\tilde{C}_2$, let $\tilde{s}_1=(1,0)^t$, we have

$S(\tilde{C}_2):=\left\{\sum_{j=0}^{k} M_3^{*j}\omega_j - M_3^{*(k+1)}(M_3^*-I)^{-1}\bar{s}_1 : all\ \omega_j\in S_1, k\geq 0\right\}=\{(-k,0)^t\in\mathbb{R}^2: k\in\mathbb{N}\}.$ (148)

So, $\tilde{\Lambda}_1:=S(\tilde{C}_1)\cup S(\tilde{C}_2)=\{(k,0)^t\in\mathbb{R}^2: k\in\mathbb{Z}\}$ is the smallest set that contains $-\tilde{C}_1,-\tilde{C}_2$ and satisfies

$$M_3^*\tilde{\Lambda}_1+S_1\subset\tilde{\Lambda}_1. \quad (149)$$

One can directly verify that $\tilde{\Lambda}_1:=\{(k,0)^t\in\mathbb{R}^2: k\in\mathbb{Z}\}$ is the smallest set that contains $-\tilde{C}_1,-\tilde{C}_2$ and satisfies (149). Hence, it follows from Theorem 2.38 in [6] that $\tilde{\Lambda}_1$ is a spectrum for the measure $\mu_{M_3,\widetilde{D}_1}$.

This proves (iiib).

It should be pointed out that for the pair (M_3,D_1), the conclusions (iiia) and (iiib) are independent of the parity of p_2. In these two conclusions, the entry p_2 of M_3 in (124) is not necessarily an even integer, we only need $p_2\in\mathbb{Z}\backslash\{0,\pm1\}$. However, for the pair (M_3,D_2), we require that $p_2\in 2\mathbb{Z}\backslash\{0\}$.

Finally, one can check, from (127), that the following equality:

$$|\hat{\mu}_{M_3,D_2}(\xi+\lambda_1)|=|\hat{\mu}_{M_3,D_2}(\xi)|, \forall\ \xi\in\mathbb{R}^2, \lambda_1\in d_1^{-1}\Lambda_1\ or\ \lambda_1\in d_1^{-1}\tilde{\Lambda}_1$$

holds. Hence $\mu_{M_3,D}$ is a spectral measure with spectrum $d_1^{-1}\Lambda_1\oplus d_2^{-1}\Lambda_2$ in the case when $p_1\neq 2$ or with spectrum $d_1^{-1}\Lambda_1\oplus d_2^{-1}\Lambda_2$ in the case when $p_1=2$. This completes the proof of Theorem (3.3.8)(iii).

Chapter 4

Spectrality and Some Moran Measures

We show under certain conditions in terms of (b_k,D_k), that the associated Moran measure μ is a spectral measure, i.e., there exists a countable set $\Lambda\subset N$ such that is an orthonormal basis for $L^2(\mu)$ We show that the class of infinite convolution (Moran measure) is a spectral measure provided that there is a common $C\subset$ Z+ compatible to all the (b,D_k) and $C+C\subseteq\{0,1,\ldots,b-1\}$. We also give some examples to illustrate the hypotheses and results, in particular, the last condition on C is essential

Section (4.1): Spectral Moran measures

For μ be a Borel probability measure with compact support in $\mathbb{R}^d$. One of basic problems in harmonic analysis associated μ is whether there exists a countable set Λ such that $E(\Lambda)=\{e^{2\pi i\langle x,\lambda\rangle}: \lambda\in\Lambda\}$ is an orthonormal basis for $L^2(\mu)$. If there exists such a set Λ, we say μ is a spectral measure and Λ is a spectrum of μ. The first general conjecture on this issue, was due to Fuglede [240].

Conjecture(4.1.1)[236]: Let Ω be a Borel subset of $\mathbb{R}^d$ with finite Lebesgue measure. Then the Lebesgue measure restricted on Ω is a spectral measure if and only if Ω is a translation tile ,i.e., there exists a set Γ in $\mathbb{R}^d$ such that $\{\Omega+t: t\in\Gamma\}$ is a partition of $\mathbb{R}^d$ (up to Lebesgue measure zero).

Although the conjecture has been proved to be false in both directions in dimension 3 and in higher dimensions [245], it is still suggestive in the research of spectral measure theory.

In many cases, the existence of spectral measures is equivalent to the existence of tiles. In dimensions 1 and 2, the conjecture is still open in both directions.

The second general result was given by He, Lai and Lau[256]; they showed that if μ is a frame spectral measure, then μ is absolutely continuous or singularly continuous with respect to the Lebesgue measure, or a finite counting measure (spectral measures are frame spectral ones). It is known that the problem with respect to a finite counting measure is either very easy or very difficult [241]. Dutkay and Lai recently studied the frame spectral measures when μ is absolutely continuous [258].

There exists a big difference in the spectral theory between absolutely continuous measures and singular measures (see [245]). The first example of singularly spectral measure was given by Jorgensen and Pedersen [254]. They showed that the 4th Cantor measure is a spectral measure. Later on, many researchers concentrated their work on investigating self-similar/self-affine measures which are generated by iterated function systems (IFS)[257] and the construction of their spectrums [245]. An IFS is a family of contraction functions $\{f_i(x)\}_{i=1}^n$, which determines a unique nonempty compact set T, called an attractor, and a Borel probability measure μ supported on T satisfying

$$T = \bigcup_{i=1}^{n} f_i(T), \qquad \mu(\cdot) = \sum_{i=1}^{n} p_i \mu \circ f_i^{-1}(\cdot),$$

where $\{p_i\}_{i=1}^n$ are called probability weights, that is, $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. The 4th Cantor measure mentioned before is generated by the IFS $f_1(x) = \frac{x}{4}, f_2(x) = \frac{x+2}{4}$ with equal probability weight.

In $\mathbb{R}$, a large class of self-similar measures have been proved to be spectral measures by Łaba and Wang [247]. Let$\{f_d(x) = \frac{1}{b}(x+d)\}_{d\in D}$be an IFS, where $b \geq 2$ is an integer and $D \subset \mathbb{Z}$ is a digit set with $0 \in D$. In this case the corresponding self-similar measure with equal weight probability is denoted by $\mu_{b,D}$. Łaba and Wang showed that, if D is an integer tile,i.e., there exists $\Gamma \subset \mathbb{Z}$ such that $\{\Gamma + d\}_{d\in D}$ is a partition of $\mathbb{Z}$, and the cardinality of D (denoted by $\#D$) has no more than two distinct prime factors, then there exists N such that $\#D$ is a factor of N and $\mu_{N,D}$ is a spectral measure.

A pair (b, D) is called admissible if there exists a finite set $C \subset \mathbb{Z}$ with $\#C = \#D = q$ such that the matrix $[q^{-1/2}e^{2\pi idc/b}]_{d\in D, c\in C}$ is unitary (usually $(b^{-1}D, C)$ is called a compatible pair). Łaba and Wang [247] proved that $\mu_{b,D}$ is a spectral measure if (b, D) is admissible.

We consider the following more general function system: let $\{b_n\}_{n=1}^{\infty}$be a sequence of integer numbers with all $b_n \geq 2$ and let $\{D_n\}_{n=1}^{\infty}$be a sequence of digit sets with $0 \in D_n \subset \mathbb{N}$ for each $n \geq 1$. We call the function system $\{f_{k,d}(x = b_k^{-1}(x+d): d \in D_k\}_{k=1}^{\infty}$ a Moran IFS, which is a generalization of an IFS.

We have the following simple proposition:

Proposition(4.1.2) [236]: Let $D = \{0,1,\ldots,q-1\}$ be a digit set with $q > 1$. Then (b, D) is admissible if and only if q is a factor of b.

Let $\delta_a$be the Dirac measure and denote

$$\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$$

for a finite set E. For the completeness, we introduce the following known theorem (see [244]).

Theorem(4.1.3) [236]: Let$\{f_{k,d}(x) = b_k^{-1}(x+d): d \in D_k\}_{k=1}^{\infty}$ be a Moran IFS. Suppose that $\sup\{x: x \in b_k^{-1}D_k, k \geq 1\} < \infty$, then the sequence of measures

$$\mu_k = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_k)^{-1}D_k}$$

converges to a Borel probability measure μ_T with compact support T in a weak sense, where $*$ is the convolution sign and

$$T = \sum_{k=1}^{\infty}(b_1b_2\cdots\ b_k)^{-1}D_k\ = \left\{\sum_{k=1}^{\infty}(b_1b_2\cdots\ b_k)^{-1}d_k: d_k \in D_k, k \geq 1\right\}.$$

The measure μ_T in Theorem(4.1.3) is called a Moran measure. In order to construct a spectral measure from a Moran IFS $\{f_{k,d}(x) = b_k^{-1}(x+d): d \in D_k\}_{k=1}^{\infty}$, we assume that all (b_k, D_k) are admissible according to Proposition(4.1.2). In this case, it was called an infinite compatible tower by Strichartz [284] (note that the choices of (b_k, D_k) for $k \geq 1$ are finite in Strichartz's case). Under some other conditions, he proved that the measure μ_T in Theorem(4.1.3) is a spectral measure for an infinite compatible tower. Many such examples can be found in [248]. However, all these examples are indeed the convolution of a counting measure supported on a finite set and a self-similar spectral measure. Here we show that

Theorem(4.1.4) [236]: For any $k \geq 1$, let $D_k = \{0,1,\ldots,q_k - 1\}$ with $q_k > 1$ and $q_k|b_k$. Let $\{f_{k,d}(x) = b_k^{-1}(x+d): d \in D_k\}_{k=1}^{\infty}$ be a Moran IFS. Then the measure μ_T given in Theorem(4.1.3) is a spectral measure.

If $sup\{x: x \in b_k^{-1}D_k, k \geq 1\} < \infty$, we believe that the measure μ_T given in Theorem(4.1.3) is a spectral measure if and only if all (b_k, D_k) are admissible.

To summarize, there are three methods to show a measure to be a spectral one. The first one is due to Jorgensen and Pedersen [254]. They proved it by showing that the Ruelle transfer operator has a unique solution. The second is due to Strichartz [243], who proved it by approximation in terms of the weak convergence of measures. The third is due to Dutkay etal.[243] and Daietal.[245]. They proved it by constructing a concrete spectral set. The proof of Theorem(4.1.4) is inspired by the ideas from [246].

To prove Proposition(4.1.2), we define the mask function of a finite set A in $\mathbb{R}$ by

$$m_A(\xi) = \frac{1}{\#A}\sum_{a\in A} e^{-2\pi i a\xi}\ .$$

As usually, the Fourier transform of a probability measure μ in $\mathbb{R}$ is defined by

$$\hat{\mu}(\xi) = \int e^{-2\pi i\xi x}d\mu(x)\,.$$

Then $m_A(\xi) = \hat{\delta}_A(\xi)$.

To show the sufficiency, we write $b = qr$ and choose $C = \{0, r, 2r, \ldots, (q-1)r\}$. It is easy to see that $(b^{-1}D, C)$ is a compatible pair. Hence (b, D) is admissible.

Conversely, let $gcd(q, b) = d$ and $q = dq_1, b = db_1$. If (b, D) is admissible, then there exists $C \subset \mathbb{Z}$ with $\#C = \#D$ and $0 \in C$ such that $(b^{-1}D, C)$ is a compatible pair. Moreover $m_D(b^{-1}(c_j - c_k)) = 0$ for all $0 \leq j < k \leq q-1$. It is known that, if $C \equiv C'(mod b)$, then $(b^{-1}D, C')$ is a compatible pair too. So we can assume that all c_k satisfy $0 = c_0 < \cdots < c_{q-1} < b$.For $1 \leq k < q$, we have

$$|m_D(b^{-1}c_k)| = \left|\frac{\sin b^{-1}qc_k\pi}{q\sin b^{-1}c_k\pi}\right| = 0.$$

This yields $c_k = \alpha_k b_1$for $1 \le k < q$, where all α_k are integers and $0 < \alpha_1 <\cdots< \alpha_{q-1}$. Hence $\alpha_{q-1} \ge q-1$. By the assumption that all $c_k < b$, we have $b > \alpha_{q-1} b_1 \ge (q-1)b_1$, and then $d > q-1$. On the other hand, $d = gcd(q, b)$, thus $d = q$.

For the sake of convenience, we introduce some notations from symbolic dynamical system. Let $\{D_n\}_{n=0}^{\infty}$ be a sequence of digit sets in $\mathbb{N}$.Denote $D^0 = \{\vartheta\}$ and

$$D^n = \{\sigma_1\sigma_2 \cdots \sigma_n : \sigma_j \in D_j, 1 \le j \le n\}$$

for $n \ge 1$.Then the collection of all finite words is $D^* = \bigcup_{n=0}^{\infty} D^n$ and the set of all infinite words is denoted by

$$D^{\infty} = \{\sigma_1\sigma_2 \cdots : \sigma_j \in D_j, j \ge 1\}.$$

Let $B(0,r)$ be the open interval $(-r,r)$ in $\mathbb{R}$. Then

$$f_{k,d}\big(B(0,r)\big) \subseteq B(0,r), \qquad for\ d \in D_k\ and\ k \ge 1,$$

Where r satisfies that $r \ge 2\sup\{x : x \in b_k^{-1}D_k, k \ge 1\}$.

For any $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in D^n$, we write $f_\sigma(x) = f_{1,\sigma_1} \circ f_{2,\sigma_2} \circ\cdots\circ f_{n,\sigma_n}(x)$, i.e., $f_\sigma(x)$ is the composition of $\{f_{i,\sigma_i}(x)\}_{i=1}^{n}$. Then it is easy to check that

$$T = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in D^n} f_\sigma\big(B(0,r)\big)$$

and thus it is a compact set.

Recall that $\mu_n = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} *\cdots* \delta_{(b_1b_2\cdots b_n)^{-1}D_n}$.Then

$$\hat{\mu}_n(\xi) = \hat{\delta}_{D_1}(b_1^{-1}\xi)\hat{\delta}_{D_2}((b_1b_2)^{-1}\xi) \cdots \hat{\delta}_{D_n}((b_1b_2\cdots b_n)^{-1}\xi).$$

By a standard check, the product converges uniformly on each compact set in complex space to an entire function $f(z) = \prod_{n=1}^{\infty} \hat{\delta}_{D_n}((b_1b_2\cdots b_n)^{-1}z)$. By Levy's continuity theorem [253,p.167], there exists a probability measure μ such that $\hat{\mu}(x) = f(x)$ and μ_n converges weakly to μ. Moreover, the support of μ is the compact set T.

Let Λ be a countable set. The following well-known lemma, which was proved in [254], gives a criterion for Λ to be a spectrum of a probability measure ν.

Lemm(4.1.5) [236]: Let ν be a Borel probability measure in $\mathbb{R}$ with compact support. Then a countable set Γ is a spectrum for $L^2(\nu)$ if and only if

$$Q(\xi) := \sum_{\gamma \in \Gamma} |\hat{\nu}(\xi+\gamma)|^2 \equiv 1, \qquad for\ \xi \in \mathbb{R}.$$

Moreover, if Γ is a bi-zero set of ν,i.e., $(\Gamma-\Gamma)\backslash\{0\} \subseteq Z(\hat{\nu}) := \{\xi : \hat{\nu}(\xi) = 0\}$, then $Q(z)$ is an entire function.

We denote $D_n = \{0,1,\ldots,q_n-1\}$ with $q_n > 1$ and $q_n | b_n$.

Note that, if $q_2 = b_2$, then we have

$$\delta_{b_1^{-1}\{0,1,\ldots,q_1-1\}} * \delta_{(b_1b_2)^{-1}\{0,1,\ldots,q_2-1\}} = \delta_{b_1^{-1}\{0,1,\ldots,q_1-1\}+(b_1b_2)^{-1}\{0,1,\ldots,q_2-1\}}$$
$$= \delta_{(b_1b_2)^{-1}\{0,1,\ldots,q_1q_2-1\}}. \quad (1)$$

Hence, we prove Theorem(4.1.4) by considering the following two cases:

Case1. There are only finitely many integers k such that $b_k > q_k$. In particular, if $b_k = q_k$ for $k \ge 1$, then

$$\mu_n = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots* \delta_{(b_1b_2\cdots b_n)^{-1}D_n} = \delta_{(b_1b_2\cdots b_n)^{-1}\{0,1,\ldots,b_1b_2-1\}}$$

tends weakly to the Lebesgue measure restricted on [0,1], which is denoted by $\mathcal{L}|_{[0,1]}$, when n goes to infinity. This shows that the limit measure $\mu = \mathcal{L}|_{[0,1]}$is a spectral measure. By the assumption that there are only finitely many integers k such that $b_k > q_k$, there exists $\ell \ge 1$ such that $b_k = q_k$ for $k > \ell$. In this case, we have $\mu = \mu_\ell * \nu$, where

$$\nu(E) = \delta_{(b_1b_2\cdots b_{\ell+1})^{-1}D_{\ell+1}} * \delta_{(b_1b_2\cdots b_{\ell+2})^{-1}D_{\ell+2}} * \cdots\ (E)\ = \mathcal{L}|_{[0,1]}\left((b_1b_2\ \cdots\ b_\ell)E\right)$$

For a Borel set E. Next, we will show that

$$\Lambda := D_1 + b_1D_2 + \cdots + b_1b_2\ \cdots\ b_{\ell-1}D_\ell + b_1b_2\cdots\ b_\ell\mathbb{Z}$$

is a spectrum of μ. Note that $\hat{\mu}_\ell(\xi) = \prod_{k=1}^{\ell}\hat{\delta}_{D_k}((b_1b_2\cdots b_k)^{-1}\xi)$ and

$$\hat{\nu}(\xi) = \hat{\chi}_{[0,1]}\left((b_1b_2\cdots\ b_\ell)^{-1}\xi\right),$$

where $\hat{\chi}_{[0,1]}(\xi) = \int_0^1 e^{-2\pi i\xi x}dx$ and $\chi_{[0,1]}$ is the characteristic function of [0,1]. We have

$$\begin{aligned}
&\sum_{\lambda\in\Lambda}|\hat{\mu}(\xi+\lambda)|^2\\
&= \sum_{\lambda\in D_1+b_1D_2+\cdots+b_1b_2\cdots b_{\ell-1}D_\ell}\ \sum_{n\in\mathbb{Z}}|\hat{\mu}_\ell(\xi+\lambda+\ b_1b_2\cdots b_\ell n)|^2\\
&\qquad\times\left|\hat{\chi}_{[0,1]}(\ (b_1b_2\ \cdots\ b_\ell)^{-1}(\xi+\lambda)+n)\right|^2\\
&= \sum_{\lambda\in D_1+b_1D_2+\cdots+b_1b_2\cdots b_{\ell-1}D_\ell}\ \sum_{n\in\mathbb{Z}}|\hat{\mu}_\ell(\xi+\lambda)|^2\left|\hat{\chi}_{[0,1]}(\ (b_1b_2\ \cdots\ b_\ell)^{-1}(\xi+\lambda)+n)\right|^2\\
&= \sum_{\lambda\in D_1+b_1D_2+\cdots+b_1b_2\cdots b_{\ell-1}D_\ell}|\hat{\mu}_\ell(\xi+\lambda)|^2\\
&\equiv 1.
\end{aligned}$$

Hence, the assertion follows.

Case2. There exist infinitely many integers $k \geq 1$ such that $b_k > q_k$. We divide this case into two subcases:

Subcase I. In this part, we always assume that $b_n > q_n$ for $n \geq 1$.Then $b_n/q_n = r_n \geq 2$. Note that the measure we focus on is $\mu = \mu_T = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} *\cdots$, and its Fourier transform is

$$\hat{\mu}(\xi\ ) = \prod_{n=1}^{\infty} m_{D_n}\left(\frac{\xi}{b_1b_2\cdots b_n}\right). \quad (2)$$

Observe that the zero set of $m_{D_n}$is

$$Z\left(m_{D_n}\right) = \left\{\xi\in\mathbb{R}: m_{D_n}(\xi\ ) =\ 0\right\} = \left\{\frac{a}{q_n} : a\in\mathbb{Z}\backslash q_n\mathbb{Z}\right\}.$$

Hence, the zero set of $\hat{\mu}$ can be expressed explicitly by

$$\begin{aligned}
Z(\hat{\mu}) &= \left\{\frac{b_1b_2\cdots b_n}{q_n}\ a: n\geq 1, a\in\mathbb{Z}\backslash q_n\mathbb{Z}\right\}\\
&= \{b_1b_2\cdots\ b_{n-1}r_na: n\geq 1, a\in\mathbb{Z}\backslash q_n\mathbb{Z}\}. \quad (3)
\end{aligned}$$

Now we construct a countable set $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ in terms of $(\{q_n\},\{b_n\})$. For any $n\in\mathbb{N}, n\neq 0$, there exists a unique $\sigma = \sigma_1\cdots\sigma_k \in D^k, \sigma_k \neq 0$, such that

$$n = \sigma_1 + \sigma_2q_1 + \cdots + \sigma_kq_1\cdots\ q_{k-1} = \sum_{j=1}^{k}\sigma_jq_1\cdots\ q_{j-1}. \quad (4)$$

We define $\lambda_0 = 0$ and

$$\lambda_n = \sigma_1r_1 + \sigma_2r_2b_1 + \cdots + \sigma_kr_kb_1\cdots\ b_{k-1} = \sum_{j=1}^{k}\sigma_jr_jb_1b_2\cdots\ b_{j-1}, (5)$$

Where $\sigma = \sigma_1\cdots\sigma_k$ is the word given in (4).We call $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ the set determined by $(\{q_n\},\{b_n\})$.

Claim. The countable set $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ determined by $(\{q_n\},\{b_n\})$ is an orthonormal basis for $L^2(\mu)$.

If the claim is true, then Theorem(4.1.4) holds for Subcase I. We will prove the claim later.

Subcase II. According to (1) and Subcase I, we can assume without loss of generality that $b_1 = q_1$ and $b_k > q_k$ for $k\geq 2$. Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of $\nu = \delta_{b_2^{-1}D_2} *$

$\delta_{(b_2b_3)^{-1}D_3} *\cdots$ given in Subcase I. Now we prove that $D_1 + b_1\Lambda$ is a spectrum of μ. Note that $\hat{\mu}(\xi) = m_{D_1}(b_1^{-1}\xi)\hat{\nu}(b_1^{-1}\xi)$ and

$$Q(\xi) = \sum_{\lambda \in D_1 + b_1\Lambda} |\hat{\mu}(\xi + \lambda)|^2$$

$$= \sum_{i=0}^{q_1-1} \sum_{n=0}^{\infty} \left| m_{D_1}\left(b_1^{-1}(\xi + i + b_1\lambda_n)\right)\right|^2 |\hat{\nu}(b_1^{-1}(\xi + i + b_1\lambda_n))|^2$$

$$= \sum_{i=0}^{q_1-1} \left| m_{D_1}(b_1^{-1}(\xi + i))\right|^2 \equiv 1.$$

By Lemma(4.1.5), Theorem(4.1.4) for Subcase II follows.

In the following, we will concentrate on proving the claim. The following theorem says that $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$is an orthogonal set.

Theorem(4.1.6) [236]: The countable set $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ determined by $(\{q_n\}, \{b_n\})$ is an orthogonal set for $L^2(\mu)$.

Proof: It is equivalent to show that Λ is a bi-zero set. For any $n > n' \geq 0$, there exist $\sigma = \sigma_1 \cdots \sigma_k$ and$\sigma' = \sigma_1' \cdots \sigma_{k'}' \in D^*$such that

$$n = \sum_{j=1}^{k} \sigma_j q_1 \cdots q_{j-1}, \qquad n' = \sum_{j=1}^{k'} \sigma_j' q_1 \cdots q_{j-1}.$$

If $k > k'$, let $\sigma_j' = 0$ for $k' + 1 \leq j \leq k$. Then, by the definition of Λ, we have

$$\lambda_n = \sum_{j=1}^{k} \sigma_j r_j b_1 b_2 \cdots b_{j-1}, \qquad \lambda_{n'} = \sigma_j' r_j b_1 b_2 \cdots b_{j-1}.$$

Let s be the first index such that $\sigma_s \neq \sigma_s'$. Then

$$\lambda_n - \lambda_{n'} = b_1 b_2 \cdots b_{s-1}(\sigma_s r_s - \sigma_s' r_s + M b_s) = \frac{b_1 \cdots b_s(\sigma_s - \sigma_s' + M q_s)}{q_s}$$

for some integer $M. \sigma_s, \sigma_s' \in D_s$ implies that $q_s \nmid (\sigma_s - \sigma_s')$. Hence, $\lambda_n - \lambda_{n'} \in Z(\hat{\mu})$ by (3).

Next we show the completeness of $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$. Denote $\mu_N = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} *\cdots* \delta_{(b_1b_2\cdots b_N)^{-1}D_N}, \nu_N = \delta_{b_{N+1}^{-1}D_{N+1}} * \delta_{(b_{N+1}b_{N+2})^{-1}D_{N+2}} *\cdots$, then

$$\hat{\mu}_N(\xi) = \prod_{j=1}^{N} m_{D_j}\left(\frac{\xi}{b_1 b_2 \cdots b_j}\right), \hat{\nu}_N(\xi) = \prod_{j=1}^{\infty} m_{D_{N+j}}\left(\frac{\xi}{b_{N+1} b_{N+2} \cdots b_{N+j}}\right).$$

It is known that μ_N converges weakly to μ when N tends to infinity (Theorem(4.1.3)) and

$$\hat{\mu}(\xi) = \hat{\mu}_N(\xi)\hat{\nu}_N\left(\frac{\xi}{b_1 b_2 \cdots b_N}\right). \quad (6)$$

For the measure μ_N, we have

Theorem(4.1.7) [236]: Let $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ be the orthogonal set determined by $(\{q_n\}, \{b_n\})$. Then for all $N \geq 1$,

$$\sum_{n=0}^{q_1 \cdots q_N - 1} |\hat{\mu}_N(\xi + \lambda_N)|^2 \equiv 1. \quad (7)$$

Proof: First, we show the mutual orthogonality of $\{\lambda_n\}_{n=0}^{q_1 \cdots q_N - 1}$ for $L^2(\mu_N)$. Note that

$$\hat{\mu}_N(\xi) = \prod_{j=1}^{N} m_{D_j}\left(\frac{\xi}{b_1 b_2 \cdots b_j}\right).$$

For any $n, n' \in \{0,1,\ldots, q_1 \cdots q_N - 1\}$ with $n \neq n'$, by (4), we have

$$n = \sum_{j=1}^{N} \sigma_j q_1 \cdots q_{j-1} \text{ and } n' = \sum_{j=1}^{N} \sigma_j' q_1 \cdots q_{j-1},$$

Where $\sigma = \sigma_1 \cdots \sigma_N, \sigma' = \sigma_1' \cdots \sigma_N' \in D^N, \sigma_j$ and σ_j' may be zero for $1 \le j \le N$. Let $s \le N$ be the first index such that $\sigma_s \ne \sigma_s'$. Then, we can write

$$\lambda_n - \lambda_{n'} = b_1 b_2 \cdots b_{s-1}(\sigma_s r_s - \sigma_s' r_s + M b_s)$$

for some integer M.It follows from the periodicity of the exponential function $e^{2\pi i x}$ that

$$m_{D_s}\left(\frac{\lambda_n - \lambda_{n'}}{b_1 b_2 \cdots b_s}\right) = m_{D_s}\left(\frac{\sigma_s - \sigma_s'}{q_s}\right) = 0.$$

Hence, $\hat{\mu}_N(\lambda_n - \lambda_{n'}) = 0$. Since $\{\lambda_n\}_{n=0}^{q_1 \cdots q_N - 1}$ has exactly $q_1 \cdots q_N$ elements, so it is a spectrum for $L^2(\mu_N)$. The result follows by Lemma(4.1.5).

The next lemma says that $\hat{v}_N(\xi)$ is uniformly bounded in modulus from below for $N \ge 1$ and $0 < \xi \le 1$.

Lemma(4.1.8) [236]: For any $0 < \xi \le 1$ and $N \ge 1, |\hat{v}_N(\xi)|^2 \ge \alpha$ for some positive constant α.

Proof: Recall that

$$\hat{v}_N(\xi) = \prod_{j=1}^{\infty} \left| m_{D_{N+j}}\left(\frac{\xi}{b_{N+1} \cdots b_{N+j}}\right)\right|.$$

By simple calculation, we have $\sin x / x \ge 1 - x^2/6$ for $x > 0$. Since $b_k/q_k = r_k \ge 2$ for $k \ge 1$, then

$$\left|m_{D_n}(b_n^{-1}\xi)\right| = \frac{\sin b_n^{-1} q_n \pi \xi}{q_n \sin \pi b_n^{-1} \xi} \ge \frac{\sin b_n^{-1} q_n \pi \xi}{b_n^{-1} q_n \pi \xi} \ge 1 - \frac{1}{6}(b_n^{-1} q_n \pi \xi)^2 \quad (8)$$

and for $j \ge 1$

$$\frac{q_{N+j} \pi \xi}{b_{N+1} \cdots b_{N+j}} \le \frac{\pi}{2^j}.$$

Consequently,

$$|\hat{v}_N(\xi)| \ge \prod_{j=1}^{\infty}\left(1 - \frac{1}{6}\left(\frac{\pi}{2^j}\right)^2\right) > 0,$$

which follows from the convergence of the sum $\sum_{j=1}^{\infty} \frac{1}{6}(\frac{\pi}{2^j})^2$.

From the proof of Lemma(4.1.8), we have the following inequality which will be used in the proof of the claim:

$$|\hat{v}_N(\xi)| \ge \prod_{j=1}^{\infty}\left(1 - \frac{1}{6}\left(\frac{q_{N+j}\pi\xi}{b_{N+1} \cdots b_{N+j}}\right)^2\right), \forall 0 < \xi \le 1. \quad (9)$$

Now we can prove the claim. The measure $\mu := \mu_T$ is given in Theorem(4.1.3). $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ is the orthogonal set determined by $(\{q_n\}, \{b_n\})$,i.e.,

$$\lambda_n = \sum_{j=1}^{k} \sigma_j r_j b_1 \cdots b_{j-1},$$

if $n = \sum_{j=1}^{k} \sigma_j q_1 \cdots q_{j-1}$, where $\sigma = \sigma_1 \cdots \sigma_k \in D^k$.

Let

$$Q_N(\xi) = \sum_{n=0}^{q_1 \cdots q_N - 1} |\hat{\mu}(\xi + \lambda_n)|^2 \text{ and } Q(\xi) = \sum_{n=0}^{\infty} |\hat{\mu}(\xi + \lambda_n)|^2.$$

For any $N \ge 1$, using (6), we have the following identity:

$$Q_{2N}(\xi) = Q_N(\xi) + \sum_{n=q_1\cdots q_N}^{q_1\cdots q_{2N}-1} |\hat{\mu}(\xi+\lambda_n)|^2$$

$$= Q_N(\xi) + \sum_{n=q_1\cdots q_N}^{q_1\cdots q_{2N}-1} |\hat{\mu}_{2N}(\xi+\lambda_n)|^2 \left|\hat{v}_{2N}\left(\frac{\xi+\lambda_n}{b_1 b_2\cdots b_{2N}}\right)\right|^2. \quad (10)$$

Our goal is to prove that $Q(\xi) = \sum_{n=0}^{\infty} |\hat{\mu}(\xi+\lambda_n)|^2 \equiv 1$. Since Q is an entire function by Theorem(4.1.6) and Lemma(4.1.5), we only need to determine the value of $Q(\xi)$ for some small values of ξ.Let $0 < \xi \leq 1$. For $q_1 \cdots q_N \leq n \leq q_1 \cdots q_{2N} - 1$, we may write λ_n as

$$\lambda_n = \sum_{j=1}^{2N} \sigma_j r_j b_1 b_2 \cdots b_{j-1},$$

where $\sigma_j \in \{0,1,\ldots,q_j - 1\}$. Recall that $b_k = q_k r_k$ and $r_k \geq 1$for $k \geq 1$. Then we have

$$\frac{\xi+\lambda_n}{b_1 b_2\cdots b_{2N}} \leq \frac{r_1 + (q_1-1)r_1 + (q_2-1)r_2 b_1 + \cdots + (q_{2N}-1)r_{2N} b_1\cdots b_{2N-1}}{b_1 b_2 \cdots b_{2N}}$$

$$= \frac{b_1 + (q_2-1)r_2 b_1 + \cdots + (q_{2N}-1)r_{2N} b_1\cdots b_{2N-1}}{b_1 b_2 \cdots b_{2N}}$$

$$\leq \frac{r_2 + (q_2-1)r_2 + \cdots + (q_{2N}-1)r_{2N} b_2\cdots b_{2N-1}}{b_2 \cdots b_{2N}}$$

$$\leq \frac{r_{2N}+(q_{2N}-1)r_{2N}}{b_{2N}} = 1. \quad (11)$$

Therefore, using Lemma(4.1.8), Theorem(4.1.7) and the inequality (11), we have

$$Q_{2N}(\xi) \geq Q_N(\xi) + \alpha \sum_{n=q_1\cdots q_N}^{q_1\cdots q_{2N}-1} |\hat{\mu}_{2N}(\xi+\lambda_n)|^2$$

$$= Q_N(\xi) + \alpha\left(1 - \sum_{n=0}^{q_1\cdots q_N-1} |\hat{\mu}_{2N}(\xi+\lambda_n)|^2\right). \quad (12)$$

We now argue by contradiction. Assume that $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ is not a spectrum of μ. Then there exists a $\xi_0, 0 < \xi_0 < 1$, such that $Q(\xi_0) < 1$ and $\hat{\mu}(\xi_0) \neq 0$ since Q is entire and $\hat{\mu}(0) = 1$. Let β be so that $Q(\xi_0) < \beta < 1$. For $0 \leq n < q_1 \cdots q_N$, we have, by (11),

$$\frac{\xi_0+\lambda_n}{b_1 b_2\cdots b_{2N}} = \frac{1}{b_{N+1}\cdots b_{2N}} \frac{\xi_0+\lambda_n}{b_1 b_2\cdots b_{2N}} \leq \frac{1}{b_{N+1}\cdots b_{2N}} \leq \frac{1}{4^N}. \quad (13)$$

It follows from (9) that

$$\left|\hat{v}_{2N}\left(\frac{\xi_0+\lambda_n}{b_1 b_2\cdots b_{2N}}\right)\right|^2 \geq \prod_{j=1}^{\infty}\left(1 - \frac{1}{6}\left(\frac{q_{N+j}\pi}{b_{2N+1}\cdots b_{2N+j}} \frac{\xi_0+\lambda_n}{b_1 b_2\cdots b_{2N}}\right)^2\right). \quad (14)$$

By (13) and the assumption that $b_n \geq 4$,

$$\sum_{j=1}^{\infty} \frac{1}{b_{2N+1}\cdots b_{2N+j-1}} \frac{q_{N+j}\pi}{b_{2N+j}} \frac{\xi_0+\lambda_n}{b_1 b_2\cdots b_{2N}} \leq \frac{\pi}{2}\frac{1}{4^N}\sum_{j=1}^{\infty}\frac{1}{4^{j-1}} < \frac{\pi}{4^N}.$$

Hence, by the fact that $\ln(1-x) \geq -2x$ for $0 \leq x \leq 1/2$, (14) and the above inequality, we have

$$\ln\left|\hat{v}_{2N}\left(\frac{\xi_0+\lambda_n}{b_1 b_2\cdots b_{2N}}\right)\right| \geq \frac{\pi}{3\cdot 4^N}$$

uniformly for $0 \leq n < q_1 \cdots q_N$. This implies that, if N is large enough, say $N \geq M > 0$,

$$\left|\hat{v}_{2N}\left(\frac{\xi_0+\lambda_n}{b_1 b_2\cdots b_{2N}}\right)\right|^2 \geq \frac{Q(\xi_0)}{\beta}$$

For $0 \leq n < q_1 \cdots q_N$. Consequently, for $N \geq M$and $0 \leq n < q_1 \cdots q_N$,

$$|\hat{\mu}(\xi_0+\lambda_n)|^2=\left|\hat{\mu}_{2N}(\xi_0+\lambda_n)\hat{v}_{2N}\left(\frac{\xi_0+\lambda_n}{b_1b_2\cdots b_{2N}}\right)\right|^2\geq\frac{Q(\xi_0)}{\beta}\,|\hat{\mu}_{2N}(\xi_0+\lambda_n)|^2.$$

Combining (12) we have

$$Q_{2N}(\xi_0)\geq Q_N(\xi_0)+\alpha\left(1-\frac{Q(\xi_0)}{\beta}\sum_{n=0}^{q_1\cdots q_N-1}|\hat{\mu}(\xi_0+\lambda_n)|^2\right)$$
$$\geq Q_N(\xi_0)+\alpha(1-\beta).\quad(15)$$

Therefore

$$1\geq Q_{2^\ell N}(\xi_0)\geq Q_N(\xi_0)+\ell\alpha(1-\beta)\geq\ell\alpha(1-\beta)$$

for$\ell\geq 1$, which is impossible. Hence, Λ must be a spectrum.

Section (4.2): A Class of Infinite Convolutions

For μ be a Borel probability measure on $\mathbb{R}^d$with compact support. We call a countable subset $\Lambda\subset\mathbb{R}^d$ a spectrum of μ if the set of complex exponentials $E_\Lambda=\{e_\lambda:\lambda\in\Lambda\}$ forms an orthonormal basis for $L^2(\mu)$, where $e_\lambda(x)=e^{-2\pi i\langle x,\lambda\rangle}$. If $L^2(\mu)$ admits a spectrum, then μ is called a spectral measure; in this case, μ is of pure type, i.e., it is either discrete and finite, or absolutely continuous or singular continuous with respect to the Lebesgue measure [264]. In particular, if μ is the normalized Lebesgue measure supported on a Borel set Ω, then Ω is called a spectral set. The interest of the spectral set was first raised by Fuglede [273] in 1974, who conjectured that Ω is a spectral set if and only if it is a translational tile.

Fuglede's conjecture was disproved in 2004 by Tao [270] who showed that a spectral set is not necessary a tile for $\mathbb{R}^d, d\geq 5$. Tao's work was subsequently improved by Kolountza-kis and Matolcsi [277,18]who proved that the conjecture is false in both directions for $d\geq 3$. There are positive results with additional conditions on the under-lying sets, and the conjecture is still valid for $\mathbb{R}^d, d\leq 2$. In another direction, Jorgensen and Pedersen [276] made a head start to study the spectral property of the self-similar measures. See [270]. Among those, one of the best known resultsis that if $\rho=1/q$ for some integer $q>1$, then $\mu_{\rho,\{0,1\}}$is a spectral measure if and only if q is an even integer [286]. In [272], Dai characterized that these $\mu_{1/q,\{0,1\}}$ are the only spectral measures among the Bernoulli convolutions.

Until now, there are only a few classes of singular spectral measures that are known. , we will consider a new class of singular spectral measures defined by some simple conditions. We will study the Moran measures, which is a non-self-similar extension of the Cantor measure (as well as the Bernoulli convolution) through the infinite convolution. For any finite set, denote $\delta_E=\frac{1}{\#E}\sum_{a\in E}\delta_a$, where $\#E$ is the cardinality of the set E and δ_a is the Dirac point mass measure at a. Let $\{b_k\}_{k=1}^{\infty}$be a sequence of integers >1, and let $\{D_k\}_{k=1}^{\infty}$be a sequence of finite digit sets in $\mathbb{Z}$. If $\sup_{k\geq 1}|D_k|:=\sup\{|d|:d\in D_k,k\geq 1\}<\infty$, then associated to these two sequences, there exists a Borel probability measure with compact support defined by the convolution

$$\mu_{\{b_k\},\{D_k\}}=\delta_{b_1^{-1}D_1}*\delta_{(b_1b_2)^{-1}D_2}*\cdots$$

(where $rE=\{rx:x\in E\}$) and the convergence is in the weak sense. The measure $\mu_{\{b_k\},\{D_k\}}$ is called a Moran measure, as its support is the Moran set

$$T(\{b_k\},\{D_k\}) = \left\{\sum_{k=1}^{\infty}(b_1\cdots b_k)^{-1}d_k : d_k \in D_K, k \geq 1\right\} := \sum_{k=1}^{\infty}(b_1 \cdots b_k)^{-1}D_k. \quad (16)$$

Moran sets and Moran measures appear frequently in dynamic systems, multifractal analysis and geometry number theory (see [271]). To simplify notations, we write $\mu_{b,\{D_k\}} = \mu_{\{b_k\},\{D_k\}}$ if all b_k are equal to b, and $\mu_{b,D} = \mu_{\{b_k\},\{D_k\}}$if all (b_k, D_k) are equal to (b, D). In this case $\mu_{b,D}$ is a self-similar measure.

Observe that $\delta_{b^{-1}D}$ is a spectral measure if and only if there exists a set C such that

$$q^{-1/2}[e^{-2\pi idc/b}]_{d\in D, c\in C}$$

is a unitary matrix, where q is the cardinality of D and also C. In this case we call $(b^{-1}D, C)$ a compatible pair, or just say (b, D) is admissible for short. The most natural idea to prove the spectrality of the self-similar measure $\mu_{b,D}$ is to approximate $\mu_{b,D}$ by the finite convolutions $\nu_n = \delta_{b^{-1}D} * \delta_{b^{-2}D} *\cdots* \delta_{b^{-n}D}$. This idea is feasible if $C \subset \mathbb{Z}$ (see, e.g. [268]). Wecall such Cintegrally compatible with respect to (b, D), and we will adopt this in our consideration. Note that there are examples such that $\mu_{b,D}$is a spectral measure with $C \not\subset \mathbb{Z}$[268].

To study the spectrality of the Moran measure $\mu_{\{b_k\},\{D_k\}}$, the natural assumption is that all $\{b_k, D_k\}$ are admissible. The follow example (Example(4.2.13)) says that this assumption is not sufficient for the Moran measure being a spectral measure.

Example (4.2.1)[262]: Let $b = 2, D_1 = \{0,1\}$ and $D_k = \{0,3\}$ for $k \geq 2$. Then all $(b^{-1}D_k, C)$ are compatible pairs with common $C = \{0,1\}$, but the measure $\mu_{b,\{D_k\}}$ is not a spectral measure.

If all (b_k, D_k) are admissible, there are various methods to construct an infinite set Λ such that $\mathbb{E}_\Lambda$ is an orthonormal set of $L^2(\mu_{\{b_k\},\{D_k\}})$[279]. In general the difficult part is to verify the completeness of $\mathbb{E}_\Lambda$. Strichartz in [279] showed that (actually his theorem is for $\mathbb{R}^d$).

Theorem (4.2.2) [262]: (See [279].) For the above Moran measure $\mu_{\{b_k\},\{D_k\}}$, if it satisfies

(i) there exists $\{C_k\}_{k=0}^{\infty}$ with $0 \in C_k$ for all k such that all $(b_k^{-1}D_k, C_k)$ are choosing from some finite family of integrally compatible pairs;

(ii)the set of zeros of the Fourier transform $\hat{\mu}_{\{b_k\},\{D_k\}}$is uniformly disjoint from the sets $(b_1\cdots b_n)^{-1}C_1 + (b_2\cdots b_n)^{-1}C_2 \cdots + b_n^{-1}C_n, n > 0$.

Then $\mu_{\{b_k\},\{D_k\}}$is a spectral measure and has a spectrum $\Lambda = \bigcup_{n=1}^{\infty}(C_1 + b_1C_2 \cdots + b_1 \cdots b_{n-1}C_n)$.

From condition (i), it is easy to see that the Λ in the theorem is an orthonormal set. To show the completeness of $\mathbb{E}_\Lambda$ by approximating with the finite convolutions, Strichartz needed condition (ii), which is rather difficult to check. Moreover, $\mu_{\{b_k\},\{D_k\}}$ may still be a spectral measure even Λ is not its spectrum. (For example, the self-similar spectral measure $\mu_{2,\{0,1\}}$ is the Lebesgue measure restricting on $[0,1]$, but the Λ as in Theorem(4.2.2) is $\mathbb{Z}^+$, which is not a spectrum of $\mu_{2,\{0,1\}}$.)For the special case of a self-similar measure $\mu_{b,D}$ on $\mathbb{R}$, Łaba and Wang [270]proved that it is a spectral measure if (b, D) is admissible and the condition of the zeros are not needed.

Theorem (4.2.3) [262]: Let $b \geq 2$ be an integer, and let $\{D_k\}_{k=1}^{\infty}$be a sequence of digit sets that is uniformly bounded. Suppose there exists $C \subset \mathbb{Z}^+$ such that $C + C \subseteq \{0,1,\ldots,b-1\}$ and C is compatible with respect to (b, D_k) for each $k \geq 1$. Then the Moran measure $\mu_{b,\{D_k\}}$ is a spectral measure.

Note that $\{(b, D_k)\}_k$ is called universally admissible if the family is compatible with a common C, as in the theorem. Such notion has been used in [278] to study Fuglede's

conjecture. The techniques to prove the theorem are the construction of a suitable Λ, which is different from all spectrums appeared. The condition $C + C \subseteq \{0, 1, \ldots, b - 1\}$ is to guarantee that the Moran set $T(b, \pm C)$ satisfies the strong separation condition, i.e., $(T(b, \pm C) + c_1) \cap (T(b, \pm C) + c_2) = \emptyset$ for any pair $c_1 \neq c_2$ in $\pm C$, which is equivalent to that each element in the $T(b, \pm C)$ has a unique expansion, as is needed in the proof of Theorem(4.2.3). We will also illustrate the hypotheses and the result with some examples, in particular, we show that the condition $C + C \subseteq \{0, 1, \ldots, b - 1\}$ (Example(4.2.13)) cannot be omitted.

Let μ be a Borel probability measure with compact support on $\mathbb{R}$. The Fourier trans-form of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x).$$

It is clear that a set Λ such that $\mathbb{E}_\Lambda$ is an orthonormal family for $L^2(\mu)$ if and only if

$$(\Lambda - \Lambda)\backslash\{0\} \subset Z(\hat{\mu}),$$

where $Z(f) := \{\xi : f(\xi) = 0\}$ is the set of the roots of f. In this case, we just call the set Λ an orthonormal set of μ for convenience. We will assume that 0 is in the orthonormal set Λ, and thus $\Lambda \subset \Lambda - \Lambda$. Firstly, we recall the fundamental criterion for spectral measures [16], which is a directed application of Parseval's identity.

Proposition (4.2.4) [262]: Let μ be a Borel probability measure with compact support and let $Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2$ for $\Lambda \subset \mathbb{R}$. Then

(i) Λ is an orthonormal set of μ if and only if $Q_\Lambda(\xi) \leq 1$for $\xi \in \mathbb{R}$;

(ii) Λ is a spectrum of μ if and only if $Q_\Lambda(\xi) \equiv 1$for $\xi \in \mathbb{R}$.

Moreover, if Λ is an orthonormal set, then $Q_\Lambda(z)$ is an entire function.

The following properties of compatible pairs can be found in [270] or to be checked directly.

Lemma (4.2.5) [262]: Let $D, C \subset \mathbb{Z}$ and let $b \geq 2$ be an integer such that $(b^{-1}D, C)$is a compatible pair. Then

(i) $b^{-1}C$ is a spectrum of the measure δ_D;

(ii) the elements in D are in different cosets of $\mathbb{Z}/b\mathbb{Z}$;

(iii) $(b^{-1}D, C + a)$ for $a \in \mathbb{R}$ and $(b^{-1}D, -C)$ are compatible pairs;

(iv) suppose that $C \subset \mathbb{Z}$ such that $\tilde{C} = C(mod\ b)$. Then $(b^{-1}D,\ C)$ is a compatible pair;

(v) if all $(b^{-1}D_k, C_k)$ are compatible pairs, then $(b^{-1}D_1 + \cdots + b^{-k}D_k, C_1 + \cdots + b^{k-1}C_k)$ is a compatible pair for each $k > 1$.

We assume that $\{(b, D_k)\}_{k=1}^{\infty}$is compatible with a common C, where $C = \{c_0 = 0, c_1, \ldots, c_{q-1}\} \subseteq \{0, 1, \ldots, b - 1\}$. This implies that $\#D_k = \#C = q$ for all k. According to the hypothesis $\sup_{k \geq 1} |D_k| < \infty$, there can only be finitely many distinct D_k's with q elements, we re-arrange these distinct D_k's as $\mathcal{E}_1, \cdots, \mathcal{E}_N$. For any $1 \leq n \leq N$, let $\{f_{n,e}(x) = b^{-1}(x + e) : e \in \mathcal{E}_n\}$ be an iterated function system (IFS), and define the self-similar measure $\mu_{b,\mathcal{E}_n}$to be

$$\mu_{b,\mathcal{E}_n}(\cdot) = \frac{1}{q} \sum_{e \in \mathcal{E}_n} \mu_{b,\mathcal{E}_n} \circ f_{n,e}^{-1}(\cdot).$$

Clearly,

$$\hat{\mu}_{b,D_k}(\xi) = \prod_{k=1}^{\infty} M_{D_k}(b^{-k}\xi),\ \hat{\mu}_{b,\mathcal{E}_n}(\xi) = \prod_{k=1}^{\infty} M_{\mathcal{E}_n}(b^{-k}\xi),\ (17)$$

where $M_A(\xi) = \frac{1}{\#A} \sum_{a \in A} e^{-2\pi i a \xi}$ is the mask polynomial of A. So we have $Z(M_{D_k}) \subseteq \bigcup_{n=1}^{N} Z(M_{\mathcal{E}_n})$ and $Z(\hat{\mu}_{b,D_k}) \subseteq \bigcup_{n=1}^{N} Z(\hat{\mu}_{b,\mathcal{E}_n})$. To simplify notations, we write

$$Z_{\mathcal{E}} := \bigcup_{n=1}^{N} Z(M_{\mathcal{E}_n})\ and\ Z_{b,\mathcal{E}} := \bigcup_{n=1}^{N} Z(\hat{\mu}_{b,\varepsilon_n}).\quad (18)$$

It follows from (17) that

$$Z_{b,\mathcal{E}} = \cup_{k=1}^{\infty} b^k Z_{\mathcal{E}} . \quad (19)$$

For the given C, we let, for $k > 1$,

$$C_k = C + bC + \cdots + b^{k-1}C, \qquad C_k' = b^{-k}C_k = b^{-1}C + \cdots + b^{-k}C.$$

Note that for each $k, b^{-1}C\backslash\{0\} \subset Z_{\mathcal{E}_k} \subset Z_{\mathcal{E}}$and $C_k\backslash\{0\} \subset Z_{b,\mathcal{E}}$. Moreover, the closure of $\cup_{k=1}^{\infty} C_k'$ is the self-similar set $T(b,C) = \sum_{k=1}^{\infty} b^{-k}C$ (as in (16)). In this setup and notations, condition (ii) in Theorem(4.2.2)is Zb,E∩T(b, C) =∅, and the spectrum is $\cup_{k=1}^{\infty} C_k = \sum_{k=1}^{\infty} b^{k-1}C$(finite sum)., we will remove condition (ii) by constructing another $\tilde{C}$ that has a similar property, and show that $\cup_{k=1}^{\infty}(\tilde{C})_k$ is the spectrum of μ_{b,D_k}.

As $Z_{\mathcal{E}}$ has period 1, and has at most finitely many elements in (0,1). The set

$$Z_{\mathcal{E}} \cap \bigcup_{k=1}^{\infty} C_k'$$

is a finite set. Observe that $C_k' \subset C_{k+1}' \subset [0,1)$ for $k \geq 1$ and all elements in C_k' are distinct (Lemma(4.2.5)(ii)), hence there exists $\ell \geq 1$ such that

$$Z_{\mathcal{E}} \cap (C_k'\backslash C_\ell') = \emptyset, \quad \forall\, k > \ell. \quad (20)$$

When $Z_{\mathcal{E}} \cap C_\ell' \neq \emptyset$, then all its elements are of the form $\{b^{-m_j}\lambda_j\}_{j=1}^{s}$, where $m_j \geq 0$ and

$\lambda_j = b^{-1}c_{j1} + b^{-2}c_{j2} + \cdots + b^{-k_j}c_{jk_j}$ with $c_{ji} \in C$, and $c_{j1}, c_{jk_j} \neq 0$. (21)

In view of (19), the above finiteness conclusion (20) also applies directly to $Z_{b,\mathcal{E}}$. Note that for any $\beta \in Z_{b,\mathcal{E}} \cap \cup_{k=1}^{\infty} C_k', \beta = b^{-n}\lambda_j$for some j and $n \geq 0$ (by (19) and (21)). Let n_j be the largest of such n. Then it is easy to see that $b^{-i}\lambda_j \in Z_{b,\mathcal{E}} \cap \cup_{k=1}^{\infty} C_k'$ for all $0 \leq i \leq n_j$ also. Summarizing the above discussion, we have

Lemma (4.2.6) [262]: If $Z_{b,\mathcal{E}} \cap \cup_{k=1}^{\infty} C_k' \neq \emptyset$, then $Z_{b,\mathcal{E}} \cap \cup_{k=1}^{\infty} C_k'$ is a finite set, there exists $\ell \geq 1$ such that

$$\left(Z_{\mathcal{E}} \cup Z_{b,\mathcal{E}}\right) \cap (C_k'\backslash C_\ell') = \emptyset \quad \forall k > \ell, \quad (22)$$

and

$$Z_{b,\mathcal{E}} \cap \cup_{k=1}^{\infty} C_k' = \cup_{j=1}^{\infty}\{b^{-i}\lambda_j : 0 \leq i \leq n_j\}, \quad (23)$$

with λ_j defined as in (21).

We always assume $C \subseteq \{0,\cdots, b-1\}$by Lemma(4.2.5) (iv).

Lemma (4.2.7) [262]: Suppose $C + C \subseteq \{0,1,\ldots,b-1\}$. If $Z_{b,\mathcal{E}} \cap \cup_{k=1}^{\infty} C_k' = \emptyset$, then there exist integer $L \geq 1$ and integral digit set $\tilde{C}$ with cardinality q^L such that $(b^{-L}\mathcal{E}_{i_1} + \cdots + b^{-1}\mathcal{E}_{i_L}, \tilde{C})$ is a compatible pair for any $i_j \in \{1,2,\ldots,N\}$ and

$$Z_{b,\mathcal{E}} \cap T\left(b^L, \tilde{C}\right) = \emptyset. (24)$$

Consequently, there exists $\eta, \alpha > 0$ such that

$$\prod_{n=1}^{N} |\hat{\mu}_{b,\mathcal{E}_n}(\xi)| \geq \alpha > 0, \qquad \forall \xi \in (T(b^L, \tilde{C}))_\eta.$$

(Here E_η denotes $\{x: dist\{x,E\} \leq \eta\}$.)

Proof: Let

$$T(b, \pm C) = \pm b^{-1}C \pm b^{-2}C + \cdots$$

be the self-similar set generated by $(b, C \cup (-C))$. Clearly $T(b,\pm C) \subseteq [-1/2, 1/2]$(as $C \subset \mathbb{Z}^+$ and $C + C \subseteq \{0,1,\ldots,b-1\}$). Since $\prod_{n=1}^{N} \hat{\mu}_{b,\mathcal{E}_n}(z)$ is an entire functions, we have $Z_{b,\mathcal{E}} \cap T(b,\pm C)$ is a finite set. is empty, then the first assertion follows by letting $L = 1$ and $\tilde{C} = C$; is nonempty, we write it as $\{x_1, x_2, \ldots, x_m\}$. As $x_i \in T(b,\pm C)$, we can express it as

$$x_i = \frac{c_{i1}}{b} + \frac{c_{i2}}{b^2} + \frac{c_{i3}}{b^3} + \cdots \; with\; c_{ij} \in \pm C. \quad (25)$$

We divide $\{x_1, x_2, \ldots, x_m\}$ into three parts. By rearrangement, we assume that for $1 \le i \le p_1, x_i$ has a finite series expression with the highest term k_i; for $p_1 + 1 \le i \le p_2, x_i$ has an infinite series expression with at least one term $c_{ik_i} < 0$ (we fix this k_i); and for $p_2 + 1 \le i \le m, x_i$ has an infinite series expression with all $c_{ij} \ge 0$. Denote $K = \max_{1 \le i \le p^2} k_i$. Let $L > K$ be an integer such that for each $p_2 + 1 \le i \le m, c_{ik_i} > 0$ for some $K < k_i \le L$; we fix these k_i's also. Now we define $\tilde{C}$ by

$$\tilde{C} = (b^{L-1}C + \cdots + b^{L-K}C) - (b^{L-(K+1)}C + \cdots + C).$$

Then $\#\tilde{C} = q^L$. We will show that $T(b^L, \tilde{C}) = \sum_{n=1}^{\infty} b^{-nL}\tilde{C}$ fulfills the lemma.

Indeed, it follows from Lemma(4.2.5) (iii) and (v) that $(b^{-L}\mathcal{E}_{i_1} + \cdots + b^{-1}\mathcal{E}_{i_L}, \tilde{C})$ is a compatible pair for any $i_j \in \{1, 2, \ldots, N\}$. It remains to show that $x_i \notin T(b^L, C)$. Suppose otherwise, there is $x_i \in T(b^L, \tilde{C})$. As $T(b^L, C) \subset T(b, \pm C)$, we can write

$$x_i = \sum_{j=1}^{\infty} \frac{a_{ij}}{b^j},$$

where $a_{ij} \in C$ if $j \in \{1, \cdots, K\} + L\mathbb{Z}^+$, and $a_{ij} \in -C$ otherwise. In comparing with the expression in (25), we let t be the smallest integer such that $c_{it} \ne a_{it}$. Then

$$c_{it} - a_{it} = \sum_{j=t+1}^{\infty} \frac{a_{ij}}{b^{j-t}} - \sum_{j=t+1}^{\infty} \frac{c_{ij}}{b^{j-t}}. \quad (26)$$

We claim that $t \le L$:

(i) if $1 \le i \le p_1$, then $t \le K (< L)$. For otherwise $t > K \ge k_i$, we have $x_i = \sum_{j=1}^{k_i} \frac{c_{ij}}{b^j} = \sum_{j=1}^{k_i} \frac{a_{ij}}{b^j} \in C_k'$, which contradicts to the hypothesis $Z_{b,\mathcal{E}} \cap \bigcup_{k=1}^{\infty} C_k' = \emptyset$;

(ii) if $p_1 < i \le p_2$, then $t \le K (< L)$, following from the choice of k_i and $\tilde{C}$;

(iii) if $p_2 < i \le m$, then $t \le L$, also following from the choice of k_i and $\tilde{C}$.

Hence the claim follows. As $a_{ij} \le 0$ for $L + K < j \le 2L$and $C + C \subset \{0, \cdots, b-1\}$, the claim implies

$$\sum_{j=t+1}^{\infty} \frac{a_{ij}}{b^{j-t}} = \sum_{j=t+1}^{L+K} \frac{b-1}{2b^{j-t}} + \sum_{j=2L+1}^{\infty} \frac{b-1}{2b^{j-t}} < \frac{1}{2}.$$

Similarly, we have $\sum_{j=t+1}^{\infty} \frac{a_{ij}}{b^{j-t}} > -1/2$ by estimating with $-(b-1)$ instead.

Finally observe that $\left|\sum_{j=t+1}^{\infty} \frac{c_{ij}}{b^{j-t}}\right| \le 1/2$, it follow from (26)and the above estimate that $-1 < c_{it} - a_{it} < 1$, which is impossible as they are distinct integers. Hence $x_i \notin T(b^L, C)$ for all i. In view of $T(b^L, \tilde{C}) \subset T(b, \pm C)$, we have $Z_{b,\mathcal{E}} \cap T(b^L, \tilde{C}) = \emptyset$.

The last inequality is immediate by compactness of $T(b^L, \tilde{C})$.

The following is Theorem(4.2.3)for the case $Z_{b,\mathcal{E}} \cap \bigcup_{k=1}^{\infty} C_k' = \emptyset$.

Theorem (4.2.8) [262]: Suppose that all $(b^{-1}\mathcal{E}_n, C), 1 \le n \le N$, are compatible pair and $C + C \subseteq \{0, 1, \ldots, b-1\}$. If $Z_{b,\mathcal{E}} \cap \bigcup_{k=1}^{\infty} C_k' = \emptyset$. Then the Moran measure $\mu_{b,\{D_k\}}$ is a spectral measure with a spectrum $\Lambda = \tilde{C} + b^L\tilde{C} + b^{2L}\tilde{C} + \cdots$ (finite sum), where $\tilde{C}$ is given in Lemma(4.2.7).

Proof: We define $\widetilde{D}_k = \sum_{i=1}^{L} b^{L-i} D_{i+(k-1)L}$ for $k \ge 1$. Then by Lemma(4.2.7), $(b^{-L}\widetilde{D}_k, \tilde{C}), k \ge 1$, are compatible pairs. We use the approximating method as in [28] to show that $\mu_{b^L,\{\widetilde{D}_k\}}$ is a spectral measure with a spectrum as stated. For completeness, we will provide a proof.

For any $m \ge 1$, let $\Lambda_m = \tilde{C} + b^L\tilde{C} + \cdots + b^{(m-1)L}\tilde{C}$ and $\Lambda = \bigcup_{m=1}^{\infty} \Lambda_m$. Denote

$\mu_m = \delta_{b^{-L}D_1} * \delta_{b^{-L}D_2} * \cdots * \delta_{b^{-mL}D_m}$.
Then, by Lemma(4.2.5) (i) and (v), Λ_m is a spectrum of μ_m, and Λ is an orthogonal set of $\mu_{b,\{D_k\}}$ by direct check. According to Proposition(4.2.4), we have

$$Q_m(\xi) = \sum_{\lambda\in\Lambda_m}|\hat{\mu}_m(\xi+\lambda)|^2 \equiv 1 \; and \; Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda}\left|\hat{\mu}_{b,\{D_k\}}(\xi+\lambda)\right|^2 \le 1. \quad (27)$$

Since Λ_m increases to Λ, and μ_m converges to $\mu_{b,\{D_k\}}$weakly, we will use the dominated convergence theorem to justify $Q_\Lambda(\xi) \equiv 1$. Define

$$f_m(\lambda) = \begin{cases} |\hat{\mu}_m(\xi+\lambda)|^2, if\ \lambda\in\Lambda_m; \\ 0, \qquad\quad otherwise, \end{cases} \quad f(\lambda) = \begin{cases} |\hat{\mu}_{b,\{D_k\}}(\xi+\lambda)|^2, if\ \lambda\in\Lambda; \\ 0, \qquad\quad otherwise. \end{cases}$$

Then $\lim_{m\to\infty} f_m(\lambda) = f(\lambda)$ for $\lambda\in\Lambda$. By (27) we have $\sum_{\lambda\in\Lambda}|\hat{\mu}_{b,\{D_k\}}(\xi+\lambda)|^2 = \sum_{\lambda\in\Lambda} f(\lambda) \le 1$for any $\xi\in\mathbb{R}$. Now we construct a dominated function by means of $f(\lambda)$. For any $\lambda\in\Lambda_m$, we have $b^{-mL}\lambda \in T(b^L, \tilde{C})$ and

$$\begin{aligned} f(\lambda) = |\hat{\mu}_{b,\{D_k\}}(\xi+\lambda)|^2 &= \prod_{k=1}^{\infty}|M_{D_k}(b^{-k}(\xi+\lambda))|^2 = \prod_{k=1}^{\infty}|M_{\tilde{D}_k}(b^{-kL}(\xi+\lambda))|^2 \\ &= |\hat{\mu}_m(\xi+\lambda)|^2\prod_{k=1}^{\infty}|M_{\tilde{D}_{k+m}}(b^{-kL}(b^{-mL}\xi+b^{-mL}\lambda))|^2 \\ &= f_m(\lambda)\prod_{k=1}^{\infty}|M_{D_{k+mL}}(b^{-k}(b^{-mL}\xi+b^{-mL}\lambda))|^2 \\ &\ge f_m(\lambda)\prod_{n=1}^{N}|\hat{\mu}_{b,\varepsilon_n}(b^{-mL}\xi+b^{-mL}\lambda)|^2. \end{aligned}$$

When $\xi\in(0,1)$ and $b^{-mL} < \eta$, where η is given in Lemma(4.2.7), we have $f(\lambda) \ge \alpha f_m(\lambda)$for $\lambda\in\Lambda_m$. Hence, $\alpha^{-1}f$ is a dominated function and so $Q_\Lambda(\xi)\equiv 1$for $\xi\in(0,1)$. The assertion follows by Proposition(4.2.4).

we always assume that $Z_{b,\mathcal{E}}\cap\bigcup_{k=1}^{\infty}C_k' \ne \emptyset$. For $C_k' = b^{-1}C + \cdots b^{-k}C$, we use

$$[b^{-1}c]_k = b^{-1}c + (b^{-2}C + \cdots\ b^{-k}C), \qquad c\in C,$$

to denote a "cell" (or a "cylinder set") in C_k'. In the following lemmas, we will modify Lemma(4.2.7)under a weaker assumption.

Lemma (4.2.9) [262]: Let ℓ be as in Lemma(4.2.6). Assume that $Z_{\mathcal{E}}\cap([b^{-1}c]_\ell\backslash\{b^{-1}c\}) = \emptyset$ for some $c\in C$. Then there exists $C_\ell^\dagger \subset \sum_{k=1}^{\infty}b^{k-1}C$satisfying $C_\ell^\dagger = C_\ell(mod b^\ell)$ and

$$Z_{b,\mathcal{E}}\cap\left(b^{-\ell}C_\ell^\dagger\right) = \emptyset. \quad (28)$$

Proof. Let $\lambda_j, j = 1,\cdots,s$, as in (21)and (23), and let ℓ be such that $Z_{b,\mathcal{E}}\cap(C_k'\backslash C_\ell') = \emptyset$ for all $k > \ell$ as in Lemma(4.2.6). We construct $C_\ell^\dagger$ by replacing elements in C_ℓ as follows.

Firstly, we include all elements λ of C_ℓ such that $b^{-\ell}\lambda\notin Z_{b,\mathcal{E}}$into $C_\ell^\dagger$. Our main task is to consider those elements in $Z_{b,\mathcal{E}}\cap b^{-\ell}C_\ell$. Note that every element $\beta\in Z_{b,\mathcal{E}}\cap b^{-\ell}C_\ell$ has the expression $\beta = b^{-i}\lambda_j$ (by (23)). For a give β, we let $m_1\ge 1$ be the smallest integer such that $\beta\in b^{m_1}Z_{\mathcal{E}}$(see (19)). By assumption and (22),

$$b^{-m_1}(b^{m_1-1}c+\beta) = b^{-1}c + b^{-m_1}\beta \notin Z_{\mathcal{E}}. \quad (29)$$

We let $\beta_1 = \beta$, and $\beta_2 = b^{m_1-1}c + \beta_1$. If $\beta_2\notin Z_{b,\mathcal{E}}$, we replace $b^\ell\beta_1\in C_\ell$ by the integer $b^\ell\beta_2$ as an element in $C_\ell^\dagger$. If $\beta_2\in Z_{b,\mathcal{E}}$, then there exists a smallest $m_2\ge 1$ such that

$$\beta_2\in b^{m_2}Z_{\mathcal{E}}. \quad (30)$$

We claim that $m_2 > m_1$. In fact, if $m_2 < m_1$, then $b^{-m_2}\beta_2 = b^{-m_2}(b^{m_1-1}c + \beta_1) \in Z_{\mathcal{E}}$, which implies $b^{-m_2}\beta_1 \in Z_{\mathcal{E}}$ by the periodic property of $Z_{\mathcal{E}}$, this contradicts the choice of m_1; if $m_2 = m_1$, then $b^{-m_2}\beta_2 = b^{-1}c + b^{-m_1}\beta_1 \in Z_{\mathcal{E}}$, which is impossible by (29). Sothe claim follows, and (30) reduces to

$$b^{-m_2}\beta_2 \in Z_{\mathcal{E}} \cap (0,1). \quad (31)$$

By assumption again, we have

$$b^{-1}c + b^{-m_2}\beta_2 \notin Z_{\mathcal{E}}.$$

We then repeat the above procedure to $\beta_3 = b^{m_2-1}c + \beta_2$ to have either $b^{\ell}\beta_3 \in \mathbb{Z}$ as an element in C_{ℓ}^{+}, or there exists a smallest $m_3 > m_2$ such that

$$b^{-m_3}\beta_3 \in Z_{\mathcal{E}} \cap (0,1). \quad (32)$$

We continue this inductively. Note that $Z_{\mathcal{E}} \cap (0,1)$ is a finite set and all the $b^{-m_r}\beta_r$'s constructed are distinct by Lemma(4.2.5)(iii). Let $K > 1$ be the first one such that

$$\sum_{r=1}^{K} b^{m_r-1}c + \beta = \beta_K \notin Z_{b,\mathcal{E}}.$$

We let $b^{\ell}\beta_K$ to be in $C_{\ell}^{\dagger}$. As we go through all $\beta \in Z_{b,\mathcal{E}} \cap b^{-\ell}C_{\ell}$, together with the part we choose in the beginning of the proof, we obtained the desirable $C_{\ell}^{\dagger}$.

Corollary (4.2.10) [262]: With the same notation and assumption as in Lemma(4.2.9), we have

$$Z_{b,\mathcal{E}} \cap \bigcup_{p=0}^{\infty} b^{-\ell}\left(C_{\ell}^{\dagger} + b^{-1}C + \cdots + b^{-p}C\right) = \emptyset. \quad (33)$$

Proof: Suppose $Z_{b,\mathcal{E}} \cap b^{-\ell}(C_{\ell}^{\dagger} + b^{-1}C + \cdots + b^{-p}C) = \emptyset$, and let x be in the intersection. From the proof of Lemma(4.2.9), there exists K such that

$$x = b^{m_r-1}c + \beta + b^{-(\ell+1)}c_{i_1} + \cdots + b^{-(\ell+m)}c_{i_m},$$

for some $1 \le m \le p, \beta \in b^{-\ell}C_{\ell}, c_{i_j} \in C$ and $c_{i_m} \neq 0$. From $x \in Z_{b,\mathcal{E}}$, we have $x = b^n x'$ for some $x' \in Z_{\mathcal{E}}$ and thus $x' = M + x''$ for some integer M and $x'' \in C'_{\ell+m+n} \cap Z_{\mathcal{E}}$, which contradicts the choice of ℓ in Lemma(4.2.6).

The expression in (33) is an adjustment of the assumption $Z_{b,\mathcal{E}} \cap \bigcup_{k=1}^{\infty} C'_k = \emptyset$ in Lemma(4.2.7). It is used to prove the following lemma, which has the conclusion as Lemma(4.2.7)under a weaker assumption.

Lemma (4.2.11) [262]: Suppose $C + C \subseteq \{0, 1, \ldots, b-1\}$ and adopt the same notation and as-sumption as in Lemma(4.2.9). Then there exists $\tilde{C}$ such that $\#\tilde{C} = q^{\ell M}$ for some $M > 0, (b^{-\ell M}(\mathcal{E}_{i_1} + b\mathcal{E}_{i_2} + \cdots + b^{\ell M-1}\mathcal{E}_{i_{\ell M}}), \tilde{C})$ is a compatible pair for any $i_j \in \{1, 2, \ldots, \mathbb{N}\}$, and

$$Z_{b,\mathcal{E}} \cap T\left(b^{\ell M}, \tilde{C}\right) = \emptyset. \quad (34)$$

Consequently, there exists $\eta, \alpha > 0$ such that

$$\prod_{n=1}^{N} |\hat{\mu}_{b,\mathcal{E}_n}(\xi)| \ge \alpha > 0, \qquad \forall\, \xi \in (T(b^{\ell M}, \tilde{C}))_{\eta}.$$

Proof: As a self-similar set, $T(b^{\ell}, \pm C_{\ell}) = b^{-\ell}\left((\pm C_{\ell}) + T\left(b^{\ell}, \pm C_{\ell}\right)\right)$. We let

$$T = b^{-\ell}\left(C_{\ell}^{\dagger} + T\left(b^{\ell}, \pm C_{\ell}\right)\right),$$

where $C_{\ell}^{\dagger}$ is constructed in Lemma(4.2.9). In view of Corollary (4.2.10) which gives the analogous condition in Lemma(4.2.7), we will proceed with a proof similar to Lemma(4.2.7)by replacing $T(b, \pm C)$ there with T. For the same reason, we can write $Z_{b,\mathcal{E}} \cap T = \{x_1, x_2, \cdots, x_m\}$ (if non-empty), and each x_i is expressed by

$$x_i = \frac{c_{i1}}{b^{\ell}} + \frac{c_{i2}}{b^{2\ell}} + \cdots, \quad (35)$$

where $c_{i1} \in C_\ell^\dagger$, and all the other coefficients are in $\pm C_\ell$. We decompose $\{x_1, \cdots, x_m\}$ into three groups: $\{x_1, \cdots, x_{p1}\}$ are those with finite series expression, let k_i be the last term; $\{x_{p_1+1}, \cdots, x_{p_2}\}$ are those associated with an infinite series, and there is a coefficient $c_{ik_i} < 0$; and $\{x_{p_{2+1}}, \cdots, x_m\}$ are those such that all the coefficients $c_{ik_i} \geq 0$.

Let $K = \max_{1 \leq i \leq p_2} k_i$, and let $L > K$ be such that for $p_2 < i \leq m$, there exists $c_{ik_i} > 0$ for some $K < k_i \leq L$. Let

$$F_\ell = b^{(L-1)\ell} C_\ell + \cdots + b^{(L-K)\ell} C_\ell - (b^{(L-K-1)\ell} C_\ell + \cdots + C_\ell)$$

be a subset in $\mathbb{Z}$, and let $F_\ell^\dagger$ denote the set that are obtained by replacing the first term of F_ℓ with $b^{(L-1)\ell} C_\ell^\dagger$. Let

$$T_1 = b^{-\ell L}(F_\ell^\dagger + T(b^{\ell L}, F_\ell)).$$

Then $T_1$is a compact set, and by using the same proof as in Lemma(4.2.7), we obtain $Z_{b,\mathcal{E}} \cap T_1 = \emptyset$. Hence there exists $\delta > 0$ such that the closed neighborhood $(T_1)_\delta$ of T_1 is disjoint with $Z_{b,\mathcal{E}}$.

Finally, we will modify the above T_1 to be a self-similar set as in (34). Let r be an integer such that $(\max |F_\ell^\dagger - F_\ell|) \sum_{j=1}^{\infty} b^{-(jrL+1)\ell} \leq \delta$. Note that

$$T_1 = b^{-\ell L} F_\ell^\dagger + b^{-2\ell L} F_\ell + \cdots + b^{-r\ell L} F_\ell + \cdots.$$

Replacing the $(kr+1)$-st term $b^{-(kr+1)\ell L} F_\ell$ of the expression of $T_1$by $b^{-(kr+1)\ell L} F_\ell^\dagger$for $k \geq 1$, we obtain a self-similar set $T(b^{r\ell L}, \tilde{C})$, where

$$\tilde{C} = F_\ell + b^{\ell L} F_\ell + \cdots + b^{(r-2)\ell L} F_\ell + b^{(r-1)\ell L} F_\ell^\dagger.$$

Then $\#\tilde{C}_\ell = q^{r\ell L}$by Lemma(4.2.5) (ii) and

$$dist\left(T_1, T\left(b^{r\ell L}, \tilde{C}\right)\right) \leq (\max |F_\ell^\dagger - F_\ell|) \sum_{j=1}^{\infty} b^{-(jrL+1)\ell} \leq \delta.$$

Thus $Z_{b,\mathcal{E}} \cap T(b^{r\ell L}, \tilde{C}) = \emptyset$. (Note that if $Z_{b,\mathcal{E}} \cap T = \emptyset$, we choose the same $\tilde{C}$ and still have $Z_{b,\mathcal{E}} \cap T(b^{r\ell L}, \tilde{C}) = \emptyset$.)

Write $M = rL$. According to Lemma(4.2.5) (iii), (iv) and (v), we see that $(b^{-\ell M} E_{i_1} + b^{-(\ell M-1)} \mathcal{E}_{i_2} + \cdots + b^{-1} \mathcal{E}_{i_{\ell M}}, \tilde{C})$ is a compatible pair for any $i_j \in \{1, 2, \ldots, N\}$, and the first assertion follows. Thesecond assertion is a simple consequence of the first one.

In view of Theorem(4.2.8), we only need to verify the case $Z_{b,\mathcal{E}} \cap \bigcup_{k=1}^{\infty} C_k' \neq \emptyset$. There are two cases:

(i) There exists $c \in C$such that $Z_{\mathcal{E}} \cap ([b^{-1}c]_\ell \backslash \{b^{-1}c\}) = \emptyset$, then by applying Lemma(4.2.11) we can conclude the theorem by using the same proof of Theorem(4.2.8).

(ii) For any $c \in C, Z_{\mathcal{E}} \cap ([b^{-1}c]_\ell \backslash \{b^{-1}c\}) \neq \emptyset$. Define

$$\mathcal{G}_k = b^{\ell-1} D_{1+(k-1)\ell} + b^{\ell-2} D_{2+(k-1)\ell} + \cdots + D_{k\ell}.$$

Then

$$\mu_{b,\{D_k\}} = \delta_{b^{-\ell}\mathcal{G}_1} * \delta_{b^{-2\ell}\mathcal{G}_2} * \cdots = \mu_{b^\ell,\{\mathcal{G}_k\}}.$$

Now we consider $\mu_{b^\ell,\{\mathcal{G}_k\}}$instead of $\mu_{b,\{D_k\}}$. In this case C is replaced by $C_\ell, Z_{\mathcal{E}}$is replaced by $Z_{\mathcal{E}}^{(\ell)} := \bigcup_{i=1}^{\ell} b^i Z_{\mathcal{E}}$. It suffices to prove the assumption in Lemma(4.2.11)holds for $\mu_{b^\ell,\{\mathcal{G}_k\}}$. Indeed, for any $k \geq 2$, there exists a nonzero $a \in C_\ell$ such that

$$Z_{\mathcal{E}}^{(\ell)} \cap \left(\left[b^{-\ell}a\right]_k \backslash \left\{b^{-\ell}a\right\}\right) = \emptyset. \quad (36)$$

(Here $[b^{-\ell}a]_k = b^{-\ell}a + b^{-2\ell}C_\ell + \cdots + b^{-\ell k}C_\ell$.)If otherwise, it is direct to show that there exist $m \geq 2$ such that

$$Z_{\mathcal{E}}^{(\ell)} \cap \left(b^{-\ell m}C_{\ell m} \backslash b^{-\ell}C_\ell\right) \neq \emptyset. \quad (37)$$

Note that $Z_{\mathcal{E}}^{(\ell)} \subseteq Z_{\mathcal{E}} \cup Z_{b,\mathcal{E}}$, we see that (37)contradicts (22). Hence, $\mu_{b,\{D_k\}} = \mu_{b^\ell,\{\mathcal{G}_k\}}$ is a spectral measure, and Theorem(4.2.3) follows.

5. Some examples

The first example is a simple application of Theorem(4.2.3).

Example (4.2.12) [262]: Let p be a positive integer. For $k \geq 1$, let $b_k = 4p$, and $D_k = \{a_k, b_k\}$ so that $b_k - a_k$ is an odd integer, and $\sup_{k\geq 1}|b_k - a_k| < \infty$. Then the measure

$$\mu_{4p,\{D_k\}} = \delta_{(4p)^{-1}D_1} * \delta_{(4p)^{-2}D_2} * \delta_{(4p)^{-3}D_3} * \cdots$$

is a spectral measure.

Proof: Note that $\mu_{4p,\{D_k\}}$ is a spectral measure if and only if $\mu_{4p,\{2D_k\}}$is a spectral measure. Let $C = \{0, p\}$, then $C + C \subset \{0,\cdots, 4p - 1\}$, and $((4p)^{-1}2D_k, C), k \geq 1$ are compatible pairs with orthogonal matrix

$$\begin{bmatrix} 1 & e^{-\pi i a_k} \\ 1 & e^{-\pi i b_k} \end{bmatrix}.$$

Hence $\mu_{4p,\{D_k\}}$ is a spectral measure by Theorem(4.2.3).

The second example explains that the assumption that all (b_k, D_k) are universal admissible cannot guarantee that $\mu_{\{b_k\},\{D_k\}}$is a spectral measure.

Example (4.2.13) [262]: Let $b_k = 2, D_1 = \{0, 1\}$ and $D_k = \{0, 3\}$for $k \geq 2$. Then all $(b_k^{-1}D_k, \{0, 1\})$ are compatible pairs, but the measure

$\mu_{2,\{D_k\}} = \delta_{2^{-1}D_1} * \delta_{2^{-2}D_2} * \delta_{2^{-3}D_3} * \cdots$

is not a spectral measure. In this case, the condition $C + C \subseteq \{0, 1, \ldots, b - 1\}$ in Theorem(4.2.3) is not satisfied.

Proof: Note that

$$\mathcal{L}_{[0,1]} = \delta_{2^{-1}\{0,1\}} * \delta_{2^{-2}\{0,1\}} * \cdots,$$

where $\mathcal{L}_{[a,b]}$ is the Lebesgue measure restricting on the interval $[a, b]$. Then it is easy to check that

$$\mu_{2,\{D_k\}} = \delta_{2^{-1}D_1} * \frac{2}{3}\mathcal{L}_{[0,3/2]}$$
$$= \frac{1}{3}\mathcal{L}_{[0,1/2]} + \frac{2}{3}\mathcal{L}_{[1/2,3/2]} + \frac{1}{3}\mathcal{L}_{[3/2,2]}.$$

According to Theorem1.3 in [9], $\mu_{b,\{D_k\}}$ is not a spectral measure.

The third example says that the universal admissibility is not a necessity in the theorem.

Example (4.2.14) [262]: Let $b = 4, D_{2k-1} = \{0, 1\}$ and $D_{2k} = \{0, 2\}$ for $k \geq 1$. Then the measure

$$\mu_{b,\{D_k\}} = \delta_{4^{-1}D_1} * \delta_{(4\cdot 2)^{-1}D_2} * \delta_{(4\cdot 2^2)^{-1}D_3} * \cdots$$

is a spectral measure. Note that the sequence $\{(b, D_k)\}_k$ does not satisfy the universal admissible property in Theorem(4.2.3).

Proof: Note that both $(4^{-1}D_{2k-1}, \{0, 2\})$ and $(4^{-1}D_{2k}, \{0, 1\})$ are compatible pairs, but not universal admissible. On the other hand,

$$\mu_{b,\{D_k\}} = \left(\delta_{4^{-1}D_1} * \delta_{4^{-2}D_2}\right) * \left(\delta_{4^{-3}D_3} * \delta_{4^{-4}D_4}\right) * \cdots$$
$$= \delta_{4^{-2}\{0,2,4,6\}} * \delta_{4^{-4}\{0,2,4,6\}} * \cdots$$
$$= \mu_{4^{-2},\{0,2,4,6\}}.$$

It is a spectral measure according to Theorem1.1 in [273].

Section (4.3): The Existence of Fourier Basis

The exponential functions $\{e^{2\pi i<\lambda,x>}: \lambda \in \mathbb{Z}^d\}$form a Fourier basis for $L^2([0,1]^d)$ and it is now one of the fundamental pillars in modern mathematics. It is natural to ask what other measures have this property, that there is a family of exponential functions which form an orthonormal basis for their L^2-space?

In 1998, Jorgensen and Pedersen [316] made a surprising discovery: they constructed a fractal measure μ on a Cantor set which admits an orthonormal basis for $L^2(\mu)$. This opened up a new field in researching the orthogonal harmonic analysis of fractal measures including self-similar/self-affine measures and general Moran measures. In [304], Strichartz proved the surprising result that the Fourier series for the Jorgensen–Pedersen example have much better convergence properties than their classical counterparts on the unit interval.

Definition (4.3.1)[293]: Let μ be a compactly supported Borel probability measure on $\mathbb{R}$. We say that μ is a spectral measure if there exists a countable set Λ of $\mathbb{R}$ such that $E(\Lambda) := \{e^{2\pi i\lambda x}: \lambda \in \Lambda\}$forms an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a spectrum of μ and (μ, Λ) is called a spectral pair.

Later on, more fractal spectral measures were constructed, even in higher dimensions [303]. However, most of these fractal measures were generated by self-affine iterated function systems (IFSs) [301], we will study measures generated by Moran IFSs (see Definition(4.3.2)), which are generalizationsof self-affineIFSs.

Definition (4.3.2) [293]: Let $\{b_n\}_{n=1}^{\infty}$be a sequence of integers with all $b_n \geq 2$ and let $\{D_n\}_{n=1}^{\infty}$be a sequence of digit sets with $0 \in D_n \subset \mathbb{Z}$ for each $n \geq 1$. We call the function system $\{f_{n,d}(x) = b_n^{-1}(x+d): d \in D_n\}_{n=1}^{\infty}$ a Moran *IFS*.

If $\sup\{x: x \in b_n^{-1}D_n, n \geq 1\} < \infty$, then there exists a Borel probability measure with compact support defined by the convolution

$$\mu_{\{b_n\},\{D_n\}} = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots,$$

where $\delta_{rE} = \frac{1}{\#E}\sum_{a\in E}\delta_{ra}$ ($\#E$is the cardinality of E) and δ_{ra} is the Dirac measure at ra, the sign $*$ means the convolution and the convergence is in weak sense. In this case, $\mu_{\{b_n\},\{D_n\}}$is called a Moran measure, and its support is the Moran set

$$T = \sum_{n=1}^{\infty}(b_1b_2\cdots b_n)^{-1}D_n = \left\{\sum_{n=1}^{\infty}(b_1b_2\cdots b_n)^{-1}d_n : d_n \in D_n, n \geq 1\right\}.$$

Moran sets and Moran measures appear frequently in dynamic systems, multifractal analysis and geometry number theory (see[312]), etc. Until now, there are only a few results on the spectrality of Moran measures [304].

The main question addressed in spectral measure theory is the following:

Question. When are the above Moran measures $\mu_{\{b_n\},\{D_n\}}$ spectral?

The following is our partial answer to this question.

Theorem (4.3.3) [293]: Let $\{b_n\}_{n=1}^{\infty}$be a sequence of integers bigger than1, and let $\{D_n\}_{n=1}^{\infty}$be a sequence of digit sets with $D_n = \{0, r_n, 2r_n, \cdots, (q_n-1)r_n\}$ in $\mathbb{Z}$. Then the associated Moran measure

$$\mu_{\{b_n\},\{D_n\}} = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}D_n} * \cdots$$

is a spectral measure if $r_nq_n | b_n$.

Corollary (4.3.4) [293]: Let $\{b_n\}_{n=1}^{\infty}$be a sequence of integers bigger than1, and let $\{D_n\}_{n=1}^{\infty}$be a sequence of digit sets with $D_n = r\{0,1,2,\cdots,(q_n-1)\}$ in $\mathbb{Z}$. Then the associated Moran measure

$$\mu_{\{b_n\},\{D_n\}} = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}D_n} * \cdots$$

is a spectral measure if $q_n|b_n$.

If $r = 1$, the Corollary(4.3.4) is the main result in [301, Theorem1.4].

To be general, there are three main methods to show a measure to be a spectral one. The first one is due to Jorgensen and Pedersen [316]. They proved it by showing that the Ruelle transfer operator has a unique solution. The second one is due to Strichartz [302], who proved it by approximation in terms of the weak convergence of measures. The third one is due to Dutkay etal. [307], Dai etal. [305]and An etal. [301]. They proved it by constructing a concrete spectral set. We hope to use the methods in the literatures to complete the proof of Theorem(4.3.3)directly, but failed. In order to complete the proof of Theorem(4.3.3), we have to find some new techniques. In fact, we are not only concerned about what measure is a spectral measure, but also interested in the structure of spectra of the measure. Since spectra are related to the convergence of Fourier series [305], we always try our best to find more spectra. The result is stated as follows.

Theorem (4.3.5) [293]: Let $\{b_n\}_{n=1}^{\infty}$be a sequence of integers bigger than1, and let $\{D_n\}_{n=1}^{\infty}$ be a sequence of digit sets with $D_n = \{0, r_n, 2r_n, \cdots, (q_n-1)r_n\}$ in $\mathbb{Z}$. If $r_nq_n|b_n$ and there exists an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{Z}$ such that $\frac{r_{n_k}q_{n_k}}{b_{n_k}} < 1$, then

$$\bigcup_{n=1}^{\infty}(c_1\sigma_1D_1' + b_1c_2\sigma_2D_2' + \cdots + b_1b_2\ldots b_{n-1}c_n\sigma_nD_n')$$

are spectra of $\mu_{\{b_n\},\{D_n\}}$for each $\sigma_i \in \{-1,1\}, 1 \le i \le n$, where $c_n = \frac{b_n}{r_nq_n}, D_n' = \{0,1,2,\cdots,q_n-1\}$.

We introduce some preliminary results. The main difficulty in the proof of Theorem(4.3.3)is to construct candidate spectrum, and check it is a spectrum of $\mu_{\{b_n\},\{D_n\}}$, they are displayed. In the final we prove Theorem(4.3.5), and give some examples to illustrate our conclusions.

For μ be a probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i\xi x}d\mu(x).$$

Then we have

$$\hat{\mu}_{\{b_n\},\{D_n\}}(\xi) = \prod_{n=1}^{\infty} m_{D_n}\left(\frac{\xi}{b_1b_2\cdots b_n}\right),$$

where the mask polynomial

$$m_{D_n}(\xi) = \frac{1}{q_n}(1 + e^{-2\pi i\xi r_n} + e^{-2\pi i\xi 2r_n} + \cdots + e^{-2\pi i\xi(q_n-1)r_n})$$

$$= \begin{cases} e^{-2\pi i\xi(q_n-1)r_n}\ \dfrac{sin\ q_nr_n\pi\xi}{q_n\ sin\ \pi r_n\xi}, \xi r_n \notin \mathbb{Z}; \\ 1, \qquad\qquad otherwise. \end{cases}$$

Let $Z_f := \{\xi : f(\xi) = 0\}$ be the set of the roots of the function $f(\xi)$. Then

$$Z_{m_{D_n}} = \left\{\frac{a}{r_nq_n} : q_n \nmid a, a \in \mathbb{Z}\right\},$$

where $q_n \nmid a$ means q_n does not divide a, and

$$Z_{\widehat{\mu}_{\{b_n\},\{D_n\}}} = \left\{\frac{b_1 b_2 \cdots b_n}{r_n q_n}\, a : n \geq 1, q_n \nmid a\right\}.$$

It is clear that $E(\Lambda)$ is an orthonormal family for $L^2(\mu)$ if and only if

$$(\Lambda - \Lambda)\backslash\{0\} \subset Z_{\widehat{\mu}}. \quad (38)$$

In this case, we just call the set Λ an orthonormal set of μ for convenience. Without loss of generality, we will assume that $0 \in \Lambda$, and thus $\Lambda \subset \Lambda - \Lambda$. The following proposition is a basic criterion for the spectrality of a measure μ [306], which is a direct application of Stone–Weierstrass theorem and Parseval's identity.

Proposition (4.3.6) [293]: Let μ be a Borel probability measure with compact support and let $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2$ for $\Lambda \subset \mathbb{R}$. Then

(i) Λ is an orthonormal set of μ if and only if $Q_\Lambda(\xi) \leq 1$ for $\xi \in \mathbb{R}$;

(ii) Λ is a spectrum of μ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}$.

Moreover, if Λ is an orthonormal set, then $Q_\Lambda(z)$ is an entire function.

The following lemma will be used in Example(4.3.13)and was proved by [306].

Lemma (4.3.7) [293]: Let $\mu = \mu_0 * \mu_1$be the convolution of two probability measure $\mu_i, i = 0, 1$, and they are not Dirac measures. Suppose that Λ is an orthogonal set of μ_0 with $0 \in \Lambda$, then Λ is also an orthogonal set of μ, but cannot be a spectrum of μ.

, we focus on the measure

$$\mu_{\{b_n\},\{D_n\}} = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}D_n} * \cdots$$

where $D_n = \{0, r_n, 2r_n, \cdots, (q_n - 1)r_n\}$. We denote

$$\mu_n = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}D_n}$$

and

$$\mu_{>n} = \delta_{b_{n+1}^{-1}D_{n+1}} * \delta_{(b_{n+1}b_{n+2})^{-1}D_{n+2}} * \cdots.$$

Then $\mu_{\{b_n\},\{D_n\}}(\cdot) = \mu_n * \mu_{>n}\left(\frac{1}{b_1b_2\cdots b_n}\cdot\right)$.

In order to complete Theorem(4.3.3), we need to consider three cases. The first one is that there exist infinitely many integers $n \geq 1$such that $\frac{r_n q_n}{b_n} < 1$. The second one is that $\frac{r_n q_n}{b_n} = 1$ for all $n \geq 1$. The third one is that there are only finitely many integers n such that $\frac{r_n q_n}{b_n} < 1$.

Case 1. There exist infinitely many integers $n \geq 1$ such that $\frac{r_n q_n}{b_n} < 1$. Then there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$of $\mathbb{N}$ such that $\frac{r_{n_k+1} q_{n_k+1}}{b_{n_k+1}} < 1$.

For simplicity, we denote $c_n = \frac{b_n}{r_n q_n}$ and define

$$\Lambda_n = c_1 D_1' + b_1 c_2 D_2' + \cdots + b_1 b_2 \cdots b_{n-1} c_n D_n'$$

for $n \geq 1$, where $D_n' = \{0, 1, 2, \cdots, q_n - 1\}$.

Lemma (4.3.8) [293]: With the above notations, Λ_n is a spectrum of μ_n and the set $\Lambda := \sum_{n=1}^{\infty} \Lambda_n$is an orthogonal set of the measure $\mu_{\{b_n\},\{D_n\}}$.

Proof: For any $\lambda_1 \neq \lambda_2 \in \Lambda$, without loss of generality, we assume that

$$\lambda_1 = c_1 d_1 + b_1 c_2 d_2 + \cdots + b_1 \cdots b_{k-1} c_k d_k$$

and

$$\lambda_2 = c_1 d_1' + b_1 c_2 d_2' + \cdots + b_1 \cdots b_{m-1} c_m d_m',$$

where $k \geq m$ and $d_i, d_i' \in D_i$. We denote that s is the minimal index such that $ds \neq d_s'$. Then

$$\lambda_1 - \lambda_2 = b_1 b_2 \cdots b_{s-1}(c_s(d_s - d_s') + M b_s)$$

$$= \frac{b_1 b_2 \cdots b_{s-1} b_s (c_s(d_s - d_s') + M b_s)}{c_s r_s q_s}$$
$$= \frac{b_1 b_2 \cdots b_{s-1} b_s ((d_s - d_s') + M r_s q_s)}{r_s q_s}$$

for some M. Since $d_s, d_s' \in D_s$, we have $q_s \nmid (d_s - d_s')$. By (38), we obtain that Λ is an orthogonal set of the measure $\mu_{\{b_n\},\{D_n\}}$.

Similarly, it is easy to show that Λ_n is orthogonal set for μ_n. Λ_n is a spectrum of μ_n because the dimension of $L^2(\mu_n)$ is the same with the cardinality of the set Λ_n.

Lemma (4.3.9) [293]: Let $\{n_k\}_{k=1}^{\infty}$be an increasing sequence of $\mathbb{N}$ such that $\frac{r_{n_k+1} q_{n_k+1}}{b_{n_k+1}} < 1$. Then there exists a constant α such that

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1 \cdots b_{n_k}}\right)\right| > \alpha > 0$$

for $0 < \xi < 1$ and $\lambda \in \Lambda_{n_k}$.

Proof: Given $0 < \xi < 1, \lambda \in \Lambda_{n_k}$. Recall that

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1 \cdots b_{n_k}}\right)\right| = \prod_{j=1}^{\infty}\left|m_{D_{n_k+j}}\left(\frac{1}{b_1 b_2 \cdots b_{n_k}} \cdot \frac{\xi+\lambda}{b_{n_k+1} \cdots b_{n_k+j}}\right)\right|,$$

and

$$\left|m_{D_{n_k+j}}\left(\frac{1}{b_1 b_2 \cdots b_{n_k}} \cdot \frac{\xi+\lambda}{b_{n_k+1} \cdots b_{n_k+j}}\right)\right|$$
$$= \frac{\sin((b_1 \cdots b_{n_k+j})^{-1} r_{n_k+j} q_{n_k+j} \pi(\xi+\lambda))}{r_{n_k+j} q_{n_k+j} \sin(\pi(b_1 \cdots b_{n_k+j})^{-1}(\xi+\lambda))}$$
$$\geq 1 - \frac{1}{6}\left(\frac{\xi+\lambda}{b_1 b_2 \cdots b_{n_k}} \cdot \frac{r_{n_k+j} q_{n_k+j}}{b_{n_k+j}} \cdot \frac{\pi}{b_{n_k+1} \cdots b_{n_k+j-1}}\right)^2. \quad (39)$$

Now, we focus on estimating the values of $\left|\frac{\xi+\lambda}{b_1 b_2 \cdots b_{n_k}}\right|$for $k \geq 1$. Firstly,

$$\left|\frac{\xi+\lambda}{b_1 b_2 \cdots b_{n_k}}\right| \leq \frac{c_1 + c_1(q_1 - 1) + b_1 c_2(q_2 - 1) + \cdots + b_1 b_2 \cdots b_{n_k-1} c_{n_k}(q_{n_k} - 1)}{b_1 b_2 \cdots b_{n_k}}$$
$$\leq \frac{c_1 + c_1(r_1 q_1 - 1) + b_1 c_2(r_2 q_2 - 1) + \cdots + b_1 b_2 \cdots b_{n_k-1} c_{n_k}(r_{n_k} q_{n_k} - 1)}{b_1 b_2 \cdots b_{n_k}}$$
$$\leq \frac{c_2 + c_2(r_2 q_2 - 1) + \cdots + b_2 \cdots b_{n_k-1} c_{n_k}(r_{n_k} q_{n_k} - 1)}{b_2 \cdots b_{n_k}}$$
$$\leq \frac{c_{n_k} + c_{n_k}(r_{n_k} q_{n_k} - 1)}{b_{n_k}}$$
$$= 1.$$

For $j = 1$, we have the following estimate

$$\frac{\xi+\lambda}{b_1 b_2 \cdots b_{n_k}} \cdot \frac{r_{n_k+1} q_{n_k+1} \pi}{b_{n_k+1}} \leq \frac{\pi}{2}.$$

Consequently,

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1\cdots b_{n_k}}\right)\right| \geq \left(1-\frac{1}{6}\left(\frac{\pi}{2}\right)^2\right)$$
$$= \prod_{j=2}^{\infty}\left(1-\frac{1}{6}\left(\frac{\xi+\lambda}{b_1 b_2\cdots b_{n_k}}\cdot\frac{r_{n_k+j}q_{n_k+j}}{b_{n_k+j}}\cdot\frac{\pi}{b_{n_k+1}\cdots b_{n_k+j-1}}\right)^2\right)$$
$$\geq \left(1-\frac{1}{6}\left(\frac{\pi}{2}\right)^2\right)\prod_{j=2}^{\infty}\left(1-\frac{1}{6}\left(\frac{\pi}{2^{j-1}}\right)^2\right).$$

Let $\alpha = \left(1-\frac{1}{6}\left(\frac{\pi}{2}\right)^2\right)\prod_{j=2}^{\infty}\left(1-\frac{1}{6}\left(\frac{\pi}{2^{j-1}}\right)^2\right)$, we complete the proof.

Define

$$Q_n(\xi) = \sum_{\lambda\in\Lambda_n}|\hat{\mu}_{\{b_n\},\{D_n\}}(\xi+\lambda)|^2.$$

Proof of Case 1. Without loss of generality, we assume that the sequence $\{n_n\}_{k=1}^{\infty}$satisfies $n_k - n_{k-1} \geq l$ for some $l \in \mathbb{N}$. Let ξ be so that $0 < \xi < 1$. Then

$$Q_{n_k}(\xi) = Q_{n_{k-1}}(\xi) + \sum_{\lambda\in\Lambda_{n_k}\setminus\Lambda_{n_{k-1}}}|\hat{\mu}_{\{b_n\},\{D_n\}}(\xi+\lambda)|^2$$
$$= Q_{n_{k-1}}(\xi) + \sum_{\lambda\in\Lambda_{n_k}\setminus\Lambda_{n_{k-1}}}|\hat{\mu}_{n_k}(\xi+\lambda)|^2\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1\cdots b_{n_k}}\right)\right|^2$$
$$\geq Q_{n_{k-1}}(\xi) + \alpha^2\left(1-\sum_{\lambda\in\Lambda_{n_{k-1}}}\left|\hat{\mu}_{n_k}(\xi+\lambda)\right|^2\right).$$

If $Q_\Lambda(\xi) \not\equiv 1$, then there exists $\xi_1, 0 < \xi_1 < 1$, such that $Q_\Lambda(\xi_1) < 1$. Let η_1 so that $\max\{Q_\Lambda(\xi_1), e^{\frac{-\pi}{3}}\} < \eta_1 < 1$. Without loss of generality, we can assume that $n_k - n_{k-1} \geq l \geq 1 + \log_2(-\frac{\pi}{3}\frac{1}{\ln\sqrt{\eta_1}})$ for $k \geq 2$. Now we prove that

$$\sum_{\lambda\in\Lambda_{n_{k-1}}}\left|\hat{\mu}_{n_k}(\xi+\lambda)\right|^2 \leq \frac{Q_\Lambda(\xi_1)}{\eta_1} < 1 \quad forall\ k > K_1.$$

It is easy to see

$$\frac{\xi_1+\lambda}{b_1 b_2\cdots b_{n_{k-1}}}\cdot\frac{r_{n_k+j}q_{n_k+j}}{b_{n_k+j}}\cdot\frac{\pi}{b_{n_k+1}\cdots b_{n_k+j-1}} \leq \frac{1}{2^{n_k-n_{k-1}}}\cdot\frac{\pi}{2^{j-1}}.$$

Note that, for $\lambda \in \Lambda_{n_{k-1}}$ and $k > K_1$,

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi_1+\lambda}{b_1\cdots b_{n_k}}\right)\right| \geq \prod_{j=1}^{\infty}\left(1-\frac{1}{6}\left(\frac{\pi}{2^{n_k-n_{k-1}+j-1}}\right)^2\right)$$

Hence, by the fact that $\ln(1-x) \geq -2x$ for $0 \leq x \leq 1/2$ and the above inequality, we have

$$\ln\left|\hat{\mu}_{>n_k}\left(\frac{\xi_1+\lambda}{b_1\cdots b_{n_k}}\right)\right| \geq \frac{\pi}{3\cdot 2^{l-1}}.$$

Then

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi_1+\lambda}{b_1\cdots b_{n_k}}\right)\right| \geq \exp\left(-\frac{\pi}{3\cdot 2^{l-1}}\right) \geq \sqrt{\eta_1}.$$

Recall that $\mu_{\{b_n\},\{D_n\}} = \mu_n * \mu_{>n}\left(\frac{1}{b_1 b_2\cdots b_n}\cdot\right)$, we have

$$|\hat{\mu}_{\{b_n\},\{D_n\}}(\xi_1+\lambda)|^2 = |\hat{\mu}_{n_k}(\xi_1+\lambda)|^2 \quad \left|\hat{\mu}_{>n_k}\left(\frac{\xi_1+\lambda}{b_1\cdots b_{n_k}}\right)\right|^2 \geq \eta_1\left|\hat{\mu}_{n_k}(\xi_1+\lambda)\right|^2.$$

Consequently,

$$\sum_{\lambda\in\Lambda_{n_{k-1}}} |\hat{\mu}_{n_k}(\xi_1+\lambda)|^2 \leq \frac{1}{\eta_1}\sum_{\lambda\in\Lambda_{n_{k-1}}} \left|\hat{\mu}_{\{b_n\},\{D_n\}}(\xi_1+\lambda)\right|^2 \leq \frac{Q_\Lambda(\xi_1)}{\eta_1}.$$

Hence,

$$\begin{aligned} Q_{n_k}(\xi_1) &\geq Q_{n_{k-1}}(\xi_1) + \alpha^2\left(1 - \sum_{\lambda\in\Lambda_{n_{k-1}}} \left|\hat{\mu}_{n_k}(\xi_1+\lambda)\right|^2\right) \\ &\geq Q_{n_{k-1}}(\xi_1) + \alpha^2\left(1-\eta_1^{-1}Q_\Lambda(\xi_1)\right). \end{aligned}$$

Therefore,

$$1 \geq Q_\Lambda(\xi_1) \geq Q_{n_k}(\xi_1) \geq Q_{n_{k_1}}(\xi_1) + (k-K_1)\alpha^2\left(1-\eta_1^{-1}Q_\Lambda(\xi_1)\right)$$

for $k \leq K_1$, which is impossible for klarge enough. Hence, $Q_\Lambda(\xi) \equiv 1$ and the proof is complete by Proposition (4.3.6). Thus, we complete the proof of Theorem(4.3.3) in the Case1.

Case 2. For all $n \in \mathbb{N}$, $\frac{r_n q_n}{b_n}$ are equal to 1. We need to divide this case into three subcases. The first one is that all r_n are equal to 1 for $n \geq 1$. The second one is that there exist infinitely many integers n such that $r_n > 1$. The third one is that there are only finitely many integers n such that $r_n > 1$.

Subcase 1. For all $n \in \mathbb{N}$, we have $r_n = 1$ i.e. $b_n = q_n$. Therefore,

$\mu_n = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}D_n} = \delta_{(b_1b_2\cdots b_n)^{-1}\{0,1,\ldots b_1b_2\cdots b_n-1\}}$

tends weakly to the Lebesgue measure restricted on [0,1], which is denoted by $\mathcal{L}|_{[0,1]}$, when n goes to infinity. This shows that the limit measure $\mu_{\{b_n\},\{D_n\}} = \mathcal{L}|_{[0,1]}$ is a spectral measure.

Subcase 2. There exist infinitely many integers n such that $r_n > 1$. Then there is an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that $r_{n_k} > 1$.

We define

$$\Lambda_n = D_1' - b_1D_2' + \cdots + (-1)^{n+1}b_1b_2\cdots b_{n-1}D_n'$$

for $n \geq 1$, where $D_n' = \{0,1,2,\cdots,q_n-1\}$, and

$$\Lambda = \bigcup_{n=1}^{\infty}\Lambda_n.$$

Next, we focus on estimating $\frac{\xi+\lambda}{b_1\cdots b_{n_k}}$ for $\lambda \in \Lambda_{n_k}, 0 < \xi < 1$ and $k \geq 1$.

If n_k is odd, we denote $n_k = 2m+1$ and $\lambda_{2m+1} \in \Lambda_{2m+1}$. Consequently, we have

$$\begin{aligned} \frac{\xi+\lambda_{2m+1}}{b_1\cdots b_{2m+1}} &\leq \frac{\xi + (q_1-1) + b_1b_2(q_3-1) + \cdots + b_1b_2\cdots b_{2m}(q_{2m+1}-1)}{b_1b_2\cdots b_{2m+1}} \\ &\leq \frac{b_1b_2 + b_1b_2(q_3-1) + \cdots + b_1b_2\cdots b_{2m}(q_{2m+1}-1)}{b_1b_2\cdots b_{2m+1}} \\ &\leq \frac{q_3 + b_3b_4(q_5-1) + \cdots + b_3b_4\cdots b_{2m}(q_{2m+1}-1)}{b_3b_4\cdots b_{2m+1}} \\ &\leq \frac{1}{r_{2m+1}}, \end{aligned}$$

and

$$\frac{\xi+\lambda_{2m+1}}{b_1\cdots b_{2m+1}} \le -\frac{b_1(q_2-1)+b_1b_2b_3(q_4-1)+\cdots+b_1\cdots b_{2m-1}(q_{2m}-1)}{b_1b_2\cdots b_{2m+1}}$$
$$\le -\frac{b_2b_3+b_2b_3(q_4-1)+\cdots+b_2\cdots b_{2m-1}(q_{2m}-1)}{b_2b_3\cdots b_{2m+1}}$$
$$\le -\frac{q_{2m}}{b_{2m}b_{2m+1}}$$

If n_k is even, we denote $n_k=2m$ and $\lambda_{2m}\in\Lambda_{2m}$. It is not difficult to estimate that

$$\frac{\xi+\lambda_{2m}}{b_1\cdots b_{2m}} \le \frac{\xi+(q_1-1)+b_1b_2(q_3-1)+\cdots+b_1b_2\cdots b_{2m-2}(q_{2m-1}-1)}{b_1b_2\cdots b_{2m}}$$
$$\le \frac{q_{2m-1}}{b_{2m-1}b_{2m}},$$

and

$$\frac{\xi+\lambda_{2m}}{b_1\cdots b_{2m}} \ge \frac{b_1(q_2-1)+b_1b_2b_3(q_4-1)+\cdots+b_1\cdots b_{2m-1}(q_{2m}-1)}{b_1b_2\cdots b_{2m}}$$
$$\ge -\frac{q_{2m}}{b_{2m}}.$$

In general, for $\lambda\in\Lambda_{n_k}, r_{n_k}>1$ and $k\ge 1$, we have

$$\left|\frac{\xi+\lambda}{b_1\cdots b_{n_k}}\right| \le \frac{1}{2}, \qquad for\ 0<\xi<1.$$

Lemma(4.3.10) [293]: Suppose there exists an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ such that $r_{n_k}>1$. Then there exists a constant α such that

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1\cdots b_{n_k}}\right)\right| > \alpha > 0$$

for $0<\xi<1$ and $\lambda\in\Lambda_{n_k}$.

Proof: By (39), we have

$$\left|m_{D_{n_k+j}}\left(\frac{1}{b_1b_2\cdots b_{n_k}}\cdot\frac{\xi+\lambda}{b_{n_k+1}\cdots b_{n_k+j}}\right)\right|$$
$$\ge 1-\frac{1}{6}\left(\frac{\xi+\lambda}{b_1b_2\cdots b_{n_k}}\cdot\frac{r_{n_k+j}q_{n_k+j}}{b_{n_k+j}}\cdot\frac{\pi}{b_{n_k+1}\cdots b_{n_k+j-1}}\right)^2$$

Hence,

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1\cdots b_{n_k}}\right)\right| \ge \prod_{j=1}^{\infty}\left(1-\frac{1}{6}\left(\frac{\xi+\lambda}{b_1b_2\cdots b_{n_k}}\cdot\frac{r_{n_k+j}q_{n_k+j}}{b_{n_k+j}}\cdot\frac{\pi}{b_{n_k+1}\cdots b_{n_k+j-1}}\right)^2\right)$$
$$\ge \left(1-\frac{1}{6}\left(\frac{\pi}{2}\right)^2\right)\prod_{j=2}^{\infty}\left(1-\frac{1}{6}\left(\frac{\pi}{2^{j-1}}\right)^2\right).$$

Let $\alpha=\left(1-\frac{1}{6}\left(\frac{\pi}{2}\right)^2\right)\prod_{j=2}^{\infty}\left(1-\frac{1}{6}\left(\frac{\pi}{2^{j-1}}\right)^2\right)$, we complete the proof.

The remainder of the Subcase2 is analogous to that in Case1. Furthermore, the difficulty of proving Subcase2 is the construction of spectrum Λ_n. It is a new method which is different from the classic methods in the literature[307].

Subcase 3. There are only finitely many integers n such that $r_n>1$. We assume that N is the maximal index n such that $r_n>1$. Then

$$\mu_{\{b_n\},\{D_n\}}(\cdot)=\mu_N * \mu_{>N}\left(\frac{1}{b_1\cdots b_N}\cdot\right).$$

Next, we only need to prove that $\mu_{\{b_n\},\{D_n\}} = \mu_N * \mu_{>N}\left(\frac{1}{b_1\cdots b_N}\cdot\right)$ is spectral measure. The following lemma shows us that, under certain conditions, the convolution of the two spectral measures is still spectral measure.

Lemma (4.3.11) [293]: Let $\mu_n = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}D_n}$ with its spectrum Λ_1. Let ν be a spectral measure with spectrum Λ_2. Suppose $Z_{\hat{\nu}} \subset b_1b_2\cdots b_n\mathbb{Z}$. Then $\mu = \mu_n * \nu$ is a spectral measure with its spectrum $\Lambda_1 + \Lambda_2$.

Proof: Since $Z_{\hat{\nu}} \subset b_1b_2\cdots b_n\mathbb{Z}$, it is easy to know that $\Lambda_2 \subset b_1b_2\cdots b_n\mathbb{Z}$ Thus

$$\begin{aligned}\hat{\mu}_n(\xi+\lambda) &= \hat{\mu}_n(\xi+\lambda_1+\lambda_2)\\ &= \prod_{j=1}^{n} m_{D_n}\left(\frac{\xi+\lambda_1+\lambda_2}{b_1b_2\cdots b_j}\right)\\ &= \prod_{j=1}^{n} m_{D_n}\left(\frac{\xi+\lambda_1}{b_1b_2\cdots b_j}\right)\\ &= \hat{\mu}_n(\xi+\lambda_1),\end{aligned}$$

where $\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2$. Then we have

$$\begin{aligned}Q_{\Lambda_1+\Lambda_2}(\xi) &= \sum_{\lambda\in\Lambda_1+\Lambda_2} |\hat{\mu}(\xi+\lambda)|^2\\ &= \sum_{\lambda_1\in\Lambda_1}\sum_{\lambda_2\in\Lambda_2} |\hat{\mu}(\xi+\lambda_1+\lambda_2)|^2\\ &= \sum_{\lambda_1\in\Lambda_1}\sum_{\lambda_2\in\Lambda_2} |\hat{\mu}_n(\xi+\lambda_1+\lambda_2)|^2|\hat{\nu}(\xi+\lambda_1+\lambda_2)|^2\\ &= \sum_{\lambda_1\in\Lambda_1} |\hat{\mu}_n(\xi+\lambda_1)|^2 \sum_{\lambda_2\in\Lambda_2} |\hat{\nu}(\xi+\lambda_1+\lambda_2)|^2\\ &= \sum_{\lambda_1\in\Lambda_1} |\hat{\mu}_n(\xi+\lambda_1)|^2 \equiv 1.\end{aligned}$$

By Proposition(4.3.6), we complete the proof of Lemma(4.3.11).

Clearly, μ_N is spectral measure. By Subcase 1, we know that the $\mu_{>N}$ is a spectral measure, then $\mu_{>N}\left(\frac{1}{b_1\cdots b_N}\cdot\right)$is a spectral measure and its spectrum is a subset of $b_1\cdots b_N\mathbb{Z}$. By Lemma(4.3.11), we complete the proof of Subcase3. By these three subcases, we complete the proof of Theorem(4.3.3)in the Case2.

Now, we consider the Case 3.

Case 3. There are only finitely many integers n such that $\frac{r_nq_n}{b_n} < 1$. We assume that M is the maximal index n such that $\frac{r_nq_n}{b_n} < 1$. Then

$$\mu_{\{b_n\},\{D_n\}}(\cdot) = \mu_M * \mu_{>M}\left(\frac{1}{b_1\cdots b_M}\cdot\right).$$

By Lemma(4.3.8), μ_M is spectral measure. By Case 2, we know that the $\mu_{>M}$ is a spectral measure, then $\mu_{>M}(\frac{1}{b_1\cdots b_M}\cdot)$is a spectral measure and its spectrum is a subset of $b_1\cdots b_M\mathbb{Z}$. By Lemma(4.3.11), we complete the proof of Theorem(4.3.3)in the Case3.

By these three cases, we complete the proof of Theorem(4.3.3).

Let $\{b_n\}_{n=1}^{\infty}$be a sequence of integers bigger than1, and let $\{D_n\}_{n=1}^{\infty}$be a sequence of digit sets in $\mathbb{Z}$. Recall that

$$\mu_{\{b_n\},\{D_n\}} = \delta_{b_1^{-1}D_1} * \delta_{(b_1b_2)^{-1}D_2} * \cdots * \delta_{(b_1b_2\cdots b_n)^{-1}D_n} * \cdots,$$

where $D_n = \{0, r_n, 2r_n, \cdots, (q_n - 1)r_n\}$.

We first prove a claim: $\mu_{\{b_n\},\{C_n\}}$is a spectral measure if and only if $\mu_{\{b_n\},r\{C_n\}}$ is a spectral measure, where C_n is a digit set in $\mathbb{Z}$.

Suppose $\mu_{\{b_n\},\{C_n\}}$ is a spectral measure with its spectrum Λ, then

$$Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2 \equiv 1$$

for $\xi \in \mathbb{R}$. Let $\Lambda' = \frac{1}{r}\Lambda$, we have

$$\begin{aligned}
&\sum_{\lambda\in\Lambda'} |\hat{\mu}_{\{b_n\},r\{C_n\}}(\xi + \lambda)|^2 \\
&= \sum_{\lambda\in\frac{1}{2}\Lambda} |\hat{\mu}_{\{b_n\},\{C_n\}}(r\xi + r\lambda)|^2 \\
&= \sum_{\lambda'\in\Lambda} |\hat{\mu}_{\{b_n\},\{C_n\}}(r\xi + \lambda')|^2 \equiv 1.
\end{aligned}$$

Hence, $\mu_{\{b_n\},r\{C_n\}}$ is a spectral measure. Similarly, the converse of the conclusion is true. By Theorem(4.3.3), $\mu_{\{b_n\},\{C_n\}}$ is a spectral measure if $r_nq_n|b_n$. In particular, all r_n are equal to1. Then $D_n = \{0, 1, 2, \cdots, (q_n - 1)\}$ and $\mu_{\{b_n\},\{C_n\}}$ is a spectral measure if $q_n|b_n$. By the above claim, we complete the proof of Corollary(4.3.4). _

For $\sigma \in \{-1, 1\}^N$, i.e., $\sigma = \sigma_1\sigma_2 \cdots$ with all $\sigma_k \in \{-1, 1\}$, we denote $\sigma|_n = \sigma_1\sigma_2 \cdots \sigma_n$. Now we construct a countable set Λ_σ. Let

$$\Lambda_{n,\sigma|_n} = c_1\sigma_1D_1' + b_1c_2\sigma_2D_2' + \cdots + b_1 \ldots b_{n-1}c_n\sigma_nD_n'$$

for $n \geq 1$ and

$$\Lambda_\sigma = \bigcup_{n=1}^{\infty} \Lambda_{n,\sigma|_n},$$

where $c_n = \frac{b_n}{r_nq_n}$, $D_n' = \{0, 1, 2, \cdots, q_n - 1\}$.

Given $0 < \xi < 1, \lambda \in \Lambda_{n,\sigma|_n}, n \geq 1$. We have

$$|\lambda| \leq c_1(q_1 - 1) + b_1c_2(q_2 - 1) + \cdots + b_1b_2 \cdots b_{n-1}c_n(q_n - 1).$$

Hence,

$$\begin{aligned}
\left|\frac{\xi + \lambda}{b_1b_2\cdots b_n}\right| &\leq \frac{1 + c_1(q_1 - 1) + b_1c_2(q_2 - 1) + \cdots + b_1b_2 \cdots b_{n-1}c_n(q_n - 1)}{b_1b_2 \cdots b_n} \\
&\leq \frac{1 + b_1 - c_1 + b_1b_2 - b_1c_2 + \cdots + b_1b_2 \cdots b_{n-1}b_n - b_1b_2 \cdots b_{n-1}c_n}{b_1b_2 \cdots b_n} \\
&\leq \frac{1 + b_n - c_n}{b_n} \\
&\leq 1. \quad (40)
\end{aligned}$$

Suppose $\{n_k\}_{k=1}^{\infty}$isan increasing subsequence of $\mathbb{N}$ such that $\frac{r_{n_k}q_{n_k}}{b_{n_k}} < 1$. By (40), we have

$$\left|\frac{\xi + \lambda}{b_1b_2 \cdots b_{n_k}}\right| \leq 1,$$

for $0 < \xi < 1, \lambda \in \Lambda_{n_k}$. Recall that

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1\cdots b_{n_k}}\right)\right| = \prod_{j=1}^{\infty}\left|m_{D_{n_k+j}}\left(\frac{1}{b_1 b_2\cdots b_{n_k}}\cdot\frac{\xi+\lambda}{b_{n_k+1}\cdots b_{n_k+j}}\right)\right|.$$

By Lemma(4.3.9), there exists $\beta > 0$ such that

$$\left|\hat{\mu}_{>n_k}\left(\frac{\xi+\lambda}{b_1\cdots b_{n_k}}\right)\right| \geq \beta.$$

The proof left is similar to that of Case 1, the proof of Theorem(4.3.5) is completed.

Example(4.3.12) [293]: Let $b_1 = 4, b2 = 6, D_1 = \{0,1\}, D_2 = \{0,3\}$, and $b_n = 8, D_n = \{0,2,4,6\}$ for $n \geq 3$. Then the measure $\mu_{\{b_n\},\{D_n\}}$ is a spectral measure.

Proof: Note that $r_1 = 1, r_2 = 3, q_1 = q_2 = 2$, and $r_n = 2, q_n = 4$ for $n \geq 3$, then $r_n q_n | b_n$ for $n \geq 1$. According to Theorem(4.3.3), the measure $\mu_{\{b_n\},\{D_n\}}$is a spectral measure.

Motivated by Fu et al. [313] and He et al. [314], we give an example which does not satisfy $r_n p_n | b_n$ for all $n \geq 1$ but is spectral measure.

Example (4.3.13) [293]: Let $b_1 = 2, b_2 = 7, D_1 = \{0,1\}, D_2 = \{0,3\}$, and $b_n = 2, D_n = \{0,1\}$ for $n \geq 3$. Then the measure $\mu_{\{b_n\},\{D_n\}}$ is not a spectral measure.

Proof: In this case

$$\mu_{\{b_n\},\{D_n\}} = \delta_{2^{-1}\{0,1\}} * \delta_{(7\cdot 2)^{-1}\{0,3\}} * \delta_{(7\cdot 2^2)^{-1}\{0,1\}} * \delta_{(7\cdot 2^3)^{-1}\{0,1\}} * \cdots.$$

It is easy to calculate that

$$\hat{\mu}_{\{b_n\},\{D_n\}}(\xi) = m_{\{0,1\}}\left(\frac{\xi}{2}\right) m_{\{0,3\}}\left(\frac{\xi}{14}\right)\prod_{k=2}^{\infty} m_{\{0,1\}}\left(\frac{\xi}{7\cdot 2^k}\right)$$

$$= m_{\{0,1\}}\left(\frac{\xi}{2}\right) m_{\{0,3\}}\left(\frac{\xi}{14}\right)\hat{\mu}_{2,\{0,1\}}\left(\frac{\xi}{14}\right),$$

where $m_{\{0,a\}}(\xi) = \frac{1}{2} + \frac{1}{2}e^{-2\pi i a\xi}, a \in \{1,3\}$ and

$$\mu_{2,\{0,1\}} = \delta_{2^{-1}\{0,1\}} * \delta_{2^{-2}\{0,1\}} * \cdots.$$

Write $\Lambda_1 = 2\mathbb{Z} + 1 = 2Z_{m_{\{0,1\}}}, \Lambda_2 = \frac{7}{3}(2\mathbb{Z}+1) = 14Z_{m_{\{0,3\}}}$and

$$\Lambda_3 = 7\bigcup_{j=1}^{\infty} 2^j(2\mathbb{Z}+1) = 14Z_{\hat{\mu}_{2,\{0,1\}}}.$$

Then the roots of $\hat{\mu}_{\{b_n\},\{D_n\}}$are

$$Z_{\hat{\mu}_{\{b_n\},\{D_n\}}} = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.$$

Suppose that $\mu_{\{b_n\},\{D_n\}}$ is a spectral measure with a spectrum $\Lambda, 0 \in \Lambda$. Then by (38)

$$(\Lambda - \Lambda)\backslash\{0\} \subset \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.$$

First, we claim that

$$\Lambda\backslash\{0\} \subset \Lambda_i \cup \Lambda_3 \; for \; i = 1 \; or 2.$$

Indeed, by the orthogonal property (38), it sufficesto prove the assertion by showing that for $\lambda_1 \in \Lambda \cap (\Lambda_1\backslash\Lambda_2)$ and $\lambda_2 \in \Lambda \cap (\Lambda_2\backslash\Lambda_1), \lambda_1 - \lambda_2 \notin Z_{\hat{\mu}_{\{b_n\},\{D_n\}}}$.

Otherwise, by the orthogonal property, we have $\lambda_1 - \lambda_2 \in \Lambda_1$ or $\lambda_1 - \lambda_2 \in \Lambda_2$ or $\lambda_1 - \lambda_2 \in \Lambda_3$. Let $\lambda_1 = 2b_1 + 1, 7 \nmid (2b_1 + 1)$, and $\lambda_2 = \frac{7}{3}(2b_2+1), 3 \nmid (2b_2 + 1)$ for $b_1, b_2 \in \mathbb{Z}$.

When $\lambda_1 - \lambda_2 \in \Lambda_1$, we obtain that

$$\lambda_1 - \lambda_2 = 2b_1 + 1 - \frac{7}{3}(2b_2+1) = 2b_3 + 1 \; forsome \; b_3 \in \mathbb{Z}.$$

It is easy to see that $3|(2b_2 + 1)$, which is a contradiction.

When $\lambda_1 - \lambda_2 \in \Lambda_3$, we have

$$2b_1 + 1 - \frac{7}{3}(2b_2 + 1) = \frac{7}{3}(2b_3 + 1)\ for some\ b_3 \in \mathbb{Z}.$$

It is easy to see that $2b_1 + 1 = \frac{14}{3}(b_3 + b_2 + 1)$, this yields a contradiction.

When $\lambda_1 - \lambda_2 \in \Lambda_3$, we have

$$2b_1 + 1 - \frac{7}{3}(2b_2 + 1) = 7 \cdot 2^j(2b_3 + 1) for\ some\ \ j\ \ and\ b_3 \in \mathbb{Z}.$$

This is a contradiction to $3 \nmid (2b_2 + 1)$. In one word, the claim follows.

Next, we will show that there exist contradictions to the above hypothesis by two cases.

Case 1. Suppose $\Lambda\backslash\{0\} \subset \Lambda_1 \cup \Lambda_3$, we have $(\Lambda - \Lambda)\backslash\{0\} \subset \Lambda_1 \cup \Lambda_3$.

We need to show that

$$\lambda_1 - \lambda_2 \in \Lambda_1 \cup \Lambda_3,$$

where $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in (\Lambda \cap \Lambda_1) \cup (\Lambda \cap \Lambda_3)$. That is, we need to check the following two subcases:

Subcase A: $\lambda_1 - \lambda_2 \in \Lambda_1 \cup \Lambda_3$, where $\lambda_1, \lambda_2 \in \Lambda \cap \Lambda_k$ for $k \in \{1, 3\}$ and $\lambda_1 \neq \lambda_2$.

Let $\lambda_1 = 2b_1 + 1 \in \Lambda_1, \lambda_2 = 2b_2 + 1 \in \Lambda_1$. Then

$$\lambda_1 - \lambda_2 = (2b_1 + 1) - (2b_2 + 1) = 2(b_1 - b_2) \notin \Lambda_2,$$

that is $\lambda_1 - \lambda_2 \in \Lambda_1 \cup \Lambda_3$; Let $\lambda_1 = 7 \cdot 2^{j_1}(2b_1 + 1) \in \Lambda_3, \lambda_1 = 7 \cdot 2^{j_2}(2b_2 + 1) \in \Lambda_3$ for some $j_1, j_2 \geq 1$. Then

$$\lambda_1 - \lambda_2 = 7 \cdot 2^{j_1}(2b_1 + 1) - 7 \cdot 2^{j_2}(2b_2 + 1) \in \Lambda_3.$$

Subcase B: $\lambda_1 - \lambda_2 \in \Lambda_1 \cup \Lambda_3$, where $\lambda_1 \in \Lambda \cap \Lambda_1, \lambda_2 \in \Lambda \cap \Lambda_3$.

Let $\lambda_1 = 2b_1 + 1 \in \Lambda_1, \lambda_2 = 7 \cdot 2^j(2b_2 + 1) \in \Lambda_3$ for some $j \geq 1$. Then

$$\lambda_1 - \lambda_2 = (2b_1 + 1) - 7 \cdot 2^j(2b_2 + 1) \in \Lambda_1$$

In one word, the Case 1 follows. It guarantees that Λ is an orthogonal set of the measure ν, which satisfies that $\mu_{\{b_n\},\{D_n\}} = \delta_{14^{-1}\{0,3\}} * \nu$. This yields a contradiction by Lemma(4.3.7).

Case 2. Suppose $\Lambda\backslash\{0\} \subset \Lambda_2 \cup \Lambda_3$, we have $(\Lambda - \Lambda)\backslash\{0\} \subset \Lambda_2 \cup \Lambda_3$.

Otherwise, there exists $\lambda \in \Lambda_1\backslash\Lambda_2$ such that $\lambda \in \Lambda$, then $\lambda = 2b + 1, 7 \nmid (2b + 1)$. Let $\lambda_1 = 0 \in \Lambda$. It is easy to see that

$$\lambda - \lambda_1 = 2b + 1 \notin \Lambda_2 \cup \Lambda_3,$$

which is a contradiction to assumption. So Λ is an orthogonal set of the measure ν, which satisfies that $\mu_{\{b_n\},\{D_n\}} = \delta_{2^{-1}\{0,1\}} * \nu$. This yields a contradiction again by Lemma(4.3.7). Hence, the measure $\mu_{\{b_n\},\{D_n\}}$ is not a spectral measure.

The following example tells us that the condition in Theorem(4.3.3)is not sufficient.

Example (4.3.14) [293]: Let $b_1 = 4, b_2 = 7, D_1 = \{0, 1\}, D_2 = \{0, 2\}$, and $b_n = 6, D_n = \{0, 2, 4\}$ for $n \geq 3$. Then the measure $\mu_{\{b_n\},\{D_n\}}$ is a spectral measure.

Proof: In this case,

$$\mu_{\{b_n\},\{D_n\}} = \delta_{4^{-1}\{0,1\}} * \delta_{(4\times7)^{-1}\{0,2\}} * \delta_{(4\times7\times6)^{-1}\{0,2,4\}} * \delta_{(4\times7\times6^2)^{-1}\{0,2,4\}} * \cdots.$$

Let $\nu = \delta_{(4\times7\times6)^{-1}\{0,2,4\}} * \delta_{(4\times7\times6^2)^{-1}\{0,2,4\}} * \cdots$. By Theorem(4.3.3), ν is a spectral measure with $Z_{\hat{\nu}} \subset 28\mathbb{Z}$. Hence, the measure $\mu_{\{b_n\},\{D_n\}}$ is a spectral measure by Lemma(4.3.11).

Chapter 5

Spectrality and Cantor–Moran Measures

We study the existence of Fourier basis for find a discrete set Λ such that EΛ is an orthonormal basis . We give some sufficient conditions for this aim and some examples to explain the theory. Let denote the uniformly discrete probability measure on a finite set E. We show that the infinite convolution (Moran measure) admits an orthonormal basis of exponential provided that is a uniformly bounded sequence of 4-digit spectral

sets, with q > 1 an odd integer, and l sufficiently large. We also give some examples to illustrate the result.

Section (5.1): Infinite Convolutions with Three-Element Digit Sets

For μ be a compactly supported Borel probability measure on $\mathbb{R}^n$. The measure μ is called a spectral measure if there exists a discrete set $\Lambda \subseteq \mathbb{R}^n$ such that the collection of exponential functions $E(\Lambda) := \{e^{2\pi i(\lambda,x)} : \lambda \in \Lambda\}$ forms an orthonormal basis (Fourier basis) for $L^2(\mu)$. In this case, the set Λ is called a spectrum for μ and (μ, Λ) is called a spectral pair. In particular, if μ is the Lebesgue measure restricted on a Borel set Ω, then the set Ω is called a spectral set. The existence of the spectrum provides a natural L^2-Fourier representation (Fourier series) of functions in $L^2(\mu)$.

The study of spectral measures dates back to the work of Fuglede [321] in 1974, which conjectured that Ω is a spectral set if and only if the set Ω tiles $\mathbb{R}^n$ by translation. Although this conjecture was proved to be false by Tao [339] and others [327,328,335] in dimension three and higher, it has attracted a lot of attention, see [334]. In 1998, Jorgensen and Pedersen [326] found that some singular, non-atomic fractal measures can be spectral ones. Since then, the study of spectral property of singular measures becomes an active research area which involves constructing new class of spectral measures [337], and classifying their possible spectra [330].

It is surprising to find many distinctive phenomenon that singular measures do have but the Lebesgue measure does not. The convergence of the associated Fourier series is very different for distinct spectra in the space of continuous functions [333]. However, there are only a few classes of singular spectral measures to be found. We find new singular spectral measures on $\mathbb{R}$ generated by infinite convolution product. An et al. [333] has investigated the spectral property of infinite Bernoulli convolutions. If $0 < \rho < 1$ and $\{b_n - a_n\}_{n=1}^{\infty}$ is a sequence of positive integers with upper bound, there is a Borel probability measure $\mu_{\rho,\{a_n,b_n\}}$, which is called an infinite Bernoulli convolution, defined by

$$\mu_{\rho,\{a_n,b_n\}} = \delta_{\rho\{a_1,b_1\}} * \delta_{\rho^2\{a_2,b_2\}} * \delta_{\rho^3\{a_3,b_3\}} * \cdots,$$

in the weak convergence. The following is their main result [319, Theorem 1.1].

Theorem (5.1.1)[318]: If the infinite Bernoulli convolution $\mu_{\rho,\{a_n,b_n\}}$ is a spectral measure, then $\rho = \frac{1}{2k}$ for some positive integer $k \in \mathbb{N}$.

We consider the spectral property of infinite convolutions with three-element digit sets. Let $0 < \rho < 1$ and let $\{a_n, b_n\}_{n=1}^{\infty}$ be
a sequence of integers with bounded from upper and lower. The probability measure $\mu_{\rho,\{0,a_n,b_n\}}$ considered here is the following infinite convolution product measure

$$\mu_{\rho,\{0,a_n,b_n\}} = \delta_{\rho\{0,a_1,b_1\}} * \delta_{\rho^2\{0,a_2,b_2\}} * \delta_{\rho^3\{0,a_3,b_3\}} * \cdots$$

in the weak convergence and is supported on the following nonempty compact set:

$$T(\rho,\{0,a_n,b_n\}) := \sum_{n=1}^{\infty} \rho^n\{0,a_n,b_n\}.$$

Here, $rE = \{rx : x \in E\}$ and the sign $*$ denotes the convolution of two measures and the discrete measure δ_A supported on a finite set A is

$$\delta_A = \frac{1}{\#A}\sum_{a\in A} \delta_a,$$

where δ_a is the Dirac measure at a and by $\#A$ we denote the number of A.

These measures $\mu_{\rho,\{0,a_n,b_n\}}$ are a class of infinite convolutions generated by discrete measures. These types of measures were first studied by Strichartz in [337], which were generated by compatible tower there. see [337] for more spectral infinite convolutions.

We generalize and refine the arguments from [10] to investigate under what conditions there exists a discrete set Λ such that the set $E(\Lambda)$ forms an orthonormal basis for $L^2(\mu_{\rho,\{0,a_n,b_n\}})$. The following is the main result.

Theorem(5.1.2) [318]: Let $0 < \rho < 1$ and let $\{a_n, b_n\}_{n=1}^{\infty}$ be a sequence of integers such that $gcd(a_n, b_n) = 1$ for all $n \in \mathbb{N}$ and $\sup_n\{|a_n|, |b_n|\} < \infty$. Then $\mu_{\rho,\{0,a_n,b_n\}}$ is a spectral measure if and only if $\rho^{-1} \in 3\mathbb{N}$ and $\{a_n, b_n\} \equiv \{1, 2\}$ $(mod\ 3)$ for all $n \in \mathbb{N}$.

We remark here that the sufficiency of Theorem (5.1.2) follows from constructing a family of compatible pairs and then constructing a canonical spectrum.

We show the necessity by proving that $\mu_{\rho,\{0,a_n,b_n\}}$ is not a spectral measure in the following cases:

• $\rho = \left(\frac{p}{q}\right)^{1/r}$ be an irrational number, where $r > 1$ (Proposition (5.1.6));

• $\rho \neq \left(\frac{p}{q}\right)^{1/r}$ be an irrational number, where $r > 1$ (Proposition (5.1.7));

• $\rho = \frac{p}{q}$ be a rational number and $gcd(3, q) = 1$ (Proposition (5.1.11));

• $\rho = \frac{p}{q}$ be a rational number, $p > 1$ and $q \in 3\mathbb{N}$ (Proposition (5.1.13)).

We shall denote the number $\frac{p}{q}$ by the reduced representation, that is, if $\frac{p}{q} = \frac{2}{3} = \frac{4}{6}$, we take $\frac{p}{q} = \frac{2}{3}$.

Repeating the arguments in the proof of Theorem (5.1.2), we also have the following result.

Theorem (5.1.3) [318]: Let $0 < \rho < 1$ and let $\{a_n, b_n, c_n\}_{n=1}^{\infty}$ be a sequence of integers such that $gcd(b_n - a_n, c_n - a_n) = 1$ for all $n \in \mathbb{N}$ and $\sup_n\{|a_n|, |b_n|, |c_n|\} < \infty$. Then $\mu_{\rho,\{a_n,b_n,c_n\}}$ is a spectral measure if and only if $\rho^{-1} \in 3\mathbb{N}$ and $\{b_n - a_n, c_n - a_n\} \equiv \{1, 2\}$ $(mod\ 3)$ for all $n \in \mathbb{N}$.

We include the basic criteria of spectral property, the necessary property of the measure $\mu_{\rho,\{0,a_n,b_n\}}$, and the proof of the sufficiency of Theorem (5.1.2)., we focus on the non-spectrality of $\mu_{\rho,\{0,a_n,b_n\}}$ when ρ is an irrational number , we show that $\mu_{\rho,\{0,a_n,b_n\}}$ is a non-spectralmeasure when ρ is a rational number but $\rho^{-1} \notin 3\mathbb{N}$.We end by giving an example to show the condition in Theorem (5.1.2) is essential (see Example (5.1.19)).

The purpose is to collect necessary facts that we need in the following, and to give the proof of the sufficiency of Theorem (5.1.2).

For a compactly supported Borel probability measure μ on $\mathbb{R}$, the Fourier transform of μ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} d\mu(x) \ \ (\xi \in \mathbb{R}).$$

It is easy to check that the set family $E(\Lambda)$ forms an orthogonal set for $L^2(\mu)$ is equivalent to the condition

$$\sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2 \leq 1 \quad (\xi \in \mathbb{R}), \ (1)$$

as well as to the condition

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq Z(\hat{\mu}), \ (2)$$

where $Z(f)$ denotes the zero set of the function f , i.e., $Z(f) = \{x \in \mathbb{R}: f(x) = 0\}$.

We also call Λ an orthogonal set (resp. spectrum) for μ if $E(\Lambda)$ forms an orthogonal set (resp. Fourier basis) for $L^2(\mu)$. Since the properties of spectrum are invariant under translation, it will be convenient to assume that $0 \in \Lambda$, and hence $\Lambda \subseteq \Lambda - \Lambda$.

Theorem (5.1.4) [318]: (see [326]) Let μ be a compactly supported Borel probability measure on $\mathbb{R}$ and let Λ be an orthogonal set for μ. Then the following statements are equivalent.

(i) The orthogonal set Λ is a spectrum for μ;

(ii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda}|\hat{\mu}(\xi+\lambda)|^2 = 1, \quad \forall\, \xi \in \mathbb{R}$;

(iii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda}|\hat{\mu}(\xi+\lambda)|^2 = 1, \quad \forall\, \xi \in (-r, r\,)$, where $r > 0$.

Lemma (5.1.5) [318]: (see [320]) Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures $\mu_i, i = 0, 1$, and they are not Dirac measures. Suppose that Λ is an orthogonal set for μ_0, then Λ is also an orthogonal set for μ, but cannot be a spectrum for μ.

For any integer digit set $D := \{0, a, b\}$ with $a, b \in \mathbb{Z}\backslash\{0\}$, the Fourier transform of δ_D is $\hat{\delta}_D(\xi) = \frac{1}{3}(1 + e^{-2\pi i a\xi} + e^{-2\pi i b\xi})$. It is easy to check that the zero set of the Fourier transform $\hat{\delta}_D$ is non-empty if and only if $\{a, b\} \equiv \{1, 2\}(mod\ 3)$.

Consequently, if there is an (a_n, b_n) such that $\{a_n, b_n\} \equiv \{1, 2\}(mod\ 3)$ fails to hold, then any orthogonal set Λ for $\mu_{\rho,\{0,a_n,b_n\}}$ is actually the orthogonal set for the following probability measure

$$\delta_{\rho\{0,a_1,b_1\}} * \cdots * \delta_{\rho^{n-1}\{0,a_{n-1},b_{n-1}\}} * \delta_{\rho^{n+1}\{0,a_{n+1},b_{n+1}\}} * \cdots.$$

Now Lemma (5.1.5) implies that the set Λ is not a spectrum for $\mu_{\rho,\{0,a_n,b_n\}}$. Thus, from now on, we always assume that $\{0, a_n, b_n\} \equiv \{0, 1, 2\}(mod\ 3)$ for all $n \in \mathbb{N}$.

For simplicity, we let $D_n = \{0, a_n, b_n\}$ and denote $\mu_{\rho,\{0,a_n,b_n\}}$ by $\mu_{\rho,\{D_n\}}$. Observe that

$$Z\big(\hat{\delta}_{D_n}\big) = \frac{1}{3}(3\mathbb{Z} + \{1, 2\}), \qquad \forall\, n \in \mathbb{N}. \quad (3)$$

Then

$$Z(\hat{\mu}_{\rho,\{D_n\}}) = Z(\hat{\delta}_{\rho^n\{0,a_n,b_n\}}) = \bigcup_{n=1}^{\infty} \rho^{-n}\frac{1}{3}(3\mathbb{Z} + \{1, 2\})\,.$$

Recall that for two finite subsets D and C of $\mathbb{R}$ with the same cardinality q, we say that (D, C) forms a compatible pair if the $q \times q$ matrix

$$H_{D,C} := \left[\frac{1}{\sqrt{q}} e^{2\pi i dc}\right]_{d\in D, c\in C}$$

is unitary, i.e., $H_{D,C}\, H^*_{D,C} = I_q$.

Fix $n \in \mathbb{N}$ and set $C = \{0, -1, 1\}$. It is easy to check that the matrix $H_{3^{-1}D_n,C}$ is a unitary matrix, which is equivalent to say that $H_{(3k)^{-1}D_n,kC}$ is a unitary matrix for all $k \in \mathbb{N}$, and hence $((3k)^{-1}D_n, kC)$ forms a compatible pair. Define

$$\Lambda(3k, kC) := \left\{\sum_{n=0}^{m}(3k)^n c_n : c_n \in kC\ for\ 0 \le n \le m, m \in \mathbb{N}_0\right\}.$$

Here and below, the symbol $\mathbb{N}_0$ denotes the set of nonnegative integers. From the property of compatible pairs (e.g. see [2, Lemma 2.2]), we know that the set $\Lambda(3k, kC)$ forms an orthogonal set for the measure $\mu_{(3k)^{-1},\{D_n\}}$.

Proof of the sufficiency of Theorem (5.1.2) Suppose $\{a_n, b_n\} \equiv \{1, 2\}\,(mod\ 3)$ and $\rho^{-1} = 3k$ for some $k \in \mathbb{N}$. Since $\sup_n\{|a_n|, |b_n|\} < \infty$, there are only finitely many distinct digit sets in all $\{D_n\}_{n=1}^{\infty}$, say that $D(1), D(2), \ldots, D(N)$ for some $N \in \mathbb{N}$.

Set $C = \{0, -1, 1\}$. With an easy calculation, we get that

$$Z\left(\hat{\delta}_{(3k)^{-1}D(i)}\right) = k(\pm 1 + 3\mathbb{Z}), \forall\, i = 1, 2, \ldots, N, (4)$$

and

$$T(3k, kC) = \{\sum_{n=1}^{\infty}(3k)^{-n}c_n : c_n \in kC\} \subseteq \left[-\frac{k}{3k-1}, \frac{k}{3k-1}\right] \subseteq (-1, 1). (5)$$

Therefore, from (4) and (5), we get that $Z(\hat{\delta}_{(3k)^{-1}D(i)}) \cap T(3k, kC) = \emptyset$ for all $i = 1, 2, \ldots, N$. By [320, Theorem 3.3], the measure $\mu_{(3k)^{-1},\{D_n\}}$ is a spectral measure with a spectrum $\Lambda(3k, kC)$.

we prove the necessity of Theorem (5.1.2) when ρ is an irrational number.

For any positive integer $r \in \mathbb{N}$, we let

$$\mathbb{Q}^r = \{u^{1/r} : u \in \mathbb{Q} \cap (0, 1)\}.$$

Proposition(5.1.6) [318]: Let $\rho \in \mathbb{Q}^r, r > 1$, be an irrational number. Then $\mu_{\rho,\{D_n\}}$ is not a spectral measure.

Proof : Let $= \left(\frac{p}{q}\right)^{1/r}$, where $p, q, r \in \mathbb{N}$ with $gcd(p, q) = 1$ and $p < q$. Then the minimal polynomial of ρ in the ring $\mathbb{Z}[x]$ is $qx^r - p$. For $i = 1, 2, \ldots, r$, set

$$\nu_i = \delta_{\rho^i D_i} * \delta_{\rho^{i+r} D_{i+r}} * \delta_{\rho^{i+2r} D_{i+2r}} * \cdots.$$

Then $\mu_{\rho,\{D_n\}} = \nu_1 * \nu_2 \cdots * \nu_r$ and the Fourier transform of $\mu_{\rho,\{D_n\}}$ is

$$\hat{\mu}_{\rho,\{D_n\}}(\xi) = \prod_{i=1}^{r} \widehat{\nu_i}(\xi) = \prod_{i=1}^{r}\prod_{j=0}^{\infty} \hat{\delta}_{D_{i+jr}}(\rho^{i+jr}\xi)\,.$$

Suppose $\mu_{\rho,\{D_n\}}$ is a spectral measure with a spectrum $0 \in \Lambda$. Then for any $\lambda_s \in \Lambda, s = 1, 2$, we get that $\hat{\mu}_{\rho,\{D_n\}}(\lambda_s) = 0$, therefore $\hat{\delta}_{D_{is+jsr}}(\rho^{is+jsr}\lambda_s) = 0$ for some $1 \le i_s \le r$ and $j_s \in \mathbb{N}_0$. By (3), each λ_s has the form

$$\lambda_s = \rho^{-(i_s + j_s r)}\frac{a_s}{3}, \qquad where\ a_s \in 3\mathbb{Z} + \{1, 2\}.$$

Since $\Lambda - \Lambda \subseteq \hat{\mu}_{\rho,\{D_n\}} \cup \{0\}$, then

$$\lambda_1 - \lambda_2 = \rho^{-(i_1 + j_1 r)}\frac{a_1}{3} - \rho^{-(i_2 + j_2 r)}\frac{a_2}{3} = \rho^{-(i + jr)}\frac{a}{3} \quad (6)$$

for some $1 \le i \le r, j \in \mathbb{N}_0$ and $a \in 3\mathbb{Z} + \{1, 2\}$. We may as well assume that $j_1, j_2 \le j$. Thus, (6) means that

$$\rho^{i-i_1}\left(\frac{p}{q}\right)^{j-j_1} a_1 - \rho^{i-i_2}\left(\frac{p}{q}\right)^{j-j_2} a_2 = a,$$

which yields that $i = i_1 = i_2$, since the minimal polynomial of ρ is $qx^r - p$. Hence Λ is an orthogonal set for some ν_i , so by Lemma (5.1.5) it cannot be the spectrum for $\mu_{\rho,\{D_n\}}$.

We will show that:

Proposition (5.1.7): If $\rho \notin \mathbb{Q}^r, r \in \mathbb{N}$ and $r > 1$, is an irrational number, then $\mu_{\rho,\{D_n\}}$ is not a spectral measure.

We need some lemmas.

Lemma (5.1.8) [318]: Let Λ be a spectrum for $\mu_{\rho,\{D_n\}}$ and $0 \in \Lambda$, and let

$$\Lambda_k = \Lambda \cap \left\{\rho^{-k}\frac{a}{3} : a \in 3\mathbb{Z} + \{1, 2\}\right\}.$$

Then $\#\Lambda_k \le 2$.

Proof: If $\#\Lambda_k \ge 3$, there exist $\lambda_1, \lambda_2 \in \Lambda_k$ such that $\lambda_1, \lambda_2 \in \rho^{-k}3^{-1}(3\mathbb{Z} + i)$ for some $i = 1$ or 2. In either case, we have

$$\lambda_1 - \lambda_2 \in \rho^{-k}\mathbb{Z}.$$

That is, $\lambda_1 - \lambda_2 \notin Z(\hat{\mu}_{\rho,\{D_n\}})$. It is a contradiction.

Lemma(5.1.9) [318]: Let Λ be a spectrum for $\mu_{\rho,\{D_n\}}$ and $0 \in \Lambda$. Then ρ is an algebraic integer.

Proof*:* Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ with $\lambda_0 = 0 \in \Lambda$ be a spectrum for $\mu_{\rho,\{D_n\}}$. Since $\Lambda \subseteq Z(\hat{\mu}_{\rho,\{D_n\}}) \cup \{0\}$, then each λ_k has the following form

$$\lambda_k = \rho^{-n_k}\frac{a_k}{3}, \text{ where } a_k \in 3\mathbb{Z} + \{1,2\}.$$

We can assume that $n_k \leq n_{k+1}$ for all $k \in \mathbb{N}$. Fix $\ell \in \mathbb{N}$. One can conclude from $k \geq \ell$ that $n_k \geq n_\ell$, and hence there is $n_{k,\ell} > n_\ell$ and $a_{k,\ell} \in 3\mathbb{Z} + \{1,2\}$ such that

$$\rho^{-n_k}\frac{a_k}{3} - \rho^{-n_\ell}\frac{a_\ell}{3} = \rho^{-n_{k,\ell}}\frac{a_{k,\ell}}{3}.$$

It follows that there is a polynomial $P(x) = a_\ell x^s + a_k x^t + c$ with $s > t \geq 0$ satisfying that $P(\rho) = 0$. Let $\varphi(x) = c_0 x^m + c_1 x^{m-1} + \cdots + c_m \in \mathbb{Z}[x]$ be the minimal polynomial of ρ. Then $\varphi(x)|P(x)$ and $c_0|a_\ell$. Since $a_\ell/c_0 \in 3\mathbb{Z} + \{1,2\}$, then $\frac{1}{c_0}\lambda_\ell = \rho^{-n_\ell 3^{-1}}\frac{a_\ell}{c_0} \in Z(\hat{\mu}_{\rho,\{D_n\}})$. Letting ℓ runs through all $\lambda_\ell \in \Lambda$, we get that

$$\frac{1}{c_0}\Lambda\backslash\{0\} \subseteq Z(\hat{\mu}_{\rho,\{D_n\}}).$$

We claim that $\frac{1}{c_0}\Lambda$ is an orthogonal set for $\mu_{\rho,\{D_n\}}$, i.e.

$$\frac{1}{c_0}(\Lambda - \Lambda)\backslash\{0\} \subseteq Z(\hat{\mu}_{\rho,\{D_n\}}).$$

Indeed, for any $\lambda_{k_1}, \lambda_{k_2} \in \Lambda$, there exists $n_{k_1,k_2} \in \mathbb{N}$ and $a_{k_1,k_2} \in 3\mathbb{Z} + \{1,2\}$ such that

$$\lambda_{k_1} - \lambda_{k_2} = \rho^{-n_{k_1,k_2}}\frac{a_{k_1,k_2}}{3}.$$

By Lemma (5.1.8), there exists $k \in \mathbb{N}$ such that $\min\{n_{k,k_1}, n_{k,k_2}\} > n_{k_1,k_2}$ and

$$\rho^{-n_{k_1,k_2}}\frac{a_{k_1,k_2}}{3} = (\lambda_{k_1} - \lambda_k) - (\lambda_{k_2} - \lambda_k) = \rho^{-n_{k,k_2}}\frac{a_{k,k_1}}{3} - \rho^{-n_{k,k_2}}\frac{a_{k,k_2}}{3}.$$

As before, one can show that $c_0|\, a_{k_1,k_2}$. Notice that $a_{k_1,k_2}/c_0 \in 3\mathbb{Z} + \{1,2\}$.We get that

$$\frac{1}{c_0}(\lambda_{k_1} - \lambda_{k_2}) \in Z(\hat{\mu}_{\rho,\{D_n\}}).$$

The claim is proved.

Repeating the same argument, we see that $\frac{1}{c_0^k}(\Lambda - \Lambda)\backslash\{0\} \subseteq Z(\hat{\mu}_{\rho,\{D_n\}})$ for all $k \in \mathbb{N}$. This forces that $c_0 = 1$. This finishes the proof.

For any $x \in \mathbb{R}$, we shall use the symbol $\langle\cdot\rangle$ to denote the signed distance to the nearest integer, i.e.,

$$\langle x\rangle = \{x\} \;\; if \;\; \{x\} < \frac{1}{2} \;\; and \; \langle x\rangle = \{x\} - 1 \; if \; \frac{1}{2} \leq \{x\} < 1,$$

where $\{x\}$ is the fractional part of a real number x. Also, we shall use $\|x\|$ to denote the absolute value of the difference between x and the nearest integer, i.e., $\|x\| = |\langle x\rangle|$.

Clearly, $\langle x\rangle \in [-\frac{1}{2}, \frac{1}{2})$ and $x - \langle x\rangle$ is the nearest integer to x.

Lemma (5.1.10) [318]: Let ρ be a root of $x^m + c_1 x^{m-1} + c_2 x^{m-2} + \cdots + c_m \in \mathbb{Z}[x]$. Then for any $a \in 3\mathbb{Z} + \{1,2\}$, we have

$$\max_{1\leq n\leq m}\left\|\rho^{-n}\frac{a}{3}\right\| \geq \left(3\sum_{n=1}^{m}|c_n|\right)^{-1} =: \alpha > 0.$$

Proof: For each $1 \leq n \leq m$, let $a_n := \rho^{-n}\frac{a}{3} - \langle\rho^{-n}\frac{a}{3}\rangle$ be the nearest integer to $\rho^{-n}\frac{a}{3}$. Since

$$1+\sum_{n=1}^{m} c_n \rho^{-n}=0,$$

then

$$\frac{a}{3}+\sum_{n=1}^{m} c_n \langle \rho^{-n}\frac{a}{3}\rangle+\sum_{n=1}^{m} c_n a_n=0. \quad (7)$$

If $\left\|\rho^{-n}\frac{a}{3}\right\| = |\langle \rho^{-n}\frac{a}{3}\rangle| < \alpha$ for all $1 \le n \le m$, then

$$\left|\sum_{n=1}^{m} c_n \langle \rho^{-n}\frac{a}{3}\rangle\right| < \left(\sum_{n=1}^{m}|c_n|\right)\alpha = \frac{1}{3}.$$

It is a contradiction to (7) since $a \in 3\mathbb{Z}+\{1,2\}$. The proof is complete.

For each digit set $D=\{0,a,b\}$, where $\{a,b\} \equiv \{1,2\}\ (mod\ 3)$, there is a polynomial

$$M_D(\xi) := \sum_{j=1,2}\left|\hat{\delta}_D\left(\xi+\frac{j}{3}\right)\right|^2 \quad (\xi \in \mathbb{R}). \quad (8)$$

Then, from the property of the compatible pair $(3^{-1}D,\{0,1,2\})$, we get that

$$|\hat{\delta}_D(\xi)|^2 + M_D(\xi) = 1 \quad (\xi \in \mathbb{R}), \quad (9)$$

and hence $M_D(0)=0$.

On the contrary, we suppose $\mu_{\rho,\{D_n\}}$ is a spectral measure with a spectrum Λ and $0 \in \Lambda$. By Theorem (5.1.4), it suffices to show that there is a point $\xi \in \mathbb{R}$ such that

$$Q(\xi)=\sum_{\lambda\in\Lambda}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2 < 1.$$

The proof will be divided into the following three steps.

Step I. Since $\sup_n\{|a_n|,|b_n|\} < \infty$, there are only finitely many digit sets in $\{D_n\}_{n=1}^{\infty}$, say that $D(1),D(2),\dots,D(N)$ for some $N \in \mathbb{N}$. For simplicity, we define $M_i := M_{D(i)}$ and $A_i=\{k\in\mathbb{N}: D_k = D(i)\}$ for $i=1,2,\dots,N$. Then, for all $\xi \in (0,1)$, we obtain, from (9), that

$$|\hat{\mu}_{\rho,\{D_n\}}(\xi)|^2 = \prod_{i=1}^{N}\prod_{k\in A_i}\left|\hat{\delta}_{\rho^k D_k}(\xi)\right|^2 = \prod_{i=1}^{N}\prod_{k\in A_i}\left|\hat{\delta}_{D(i)}(\rho^k\xi)\right|^2$$

$$= \prod_{i=1}^{N}\prod_{k\in A_i}(1-M_i(\rho^k\xi)\ \ (since\ 1-\xi<e^{-\xi})$$

$$\le \prod_{i=1}^{N}\prod_{k\in A_i} e^{-M_i(\rho^k\xi)} = e^{-\sum_{i=1}^{N}\sum_{k\in A_i}M_i(\rho^k\xi)}$$

$$= 1-\sum_{i=1}^{N}\sum_{k\in A_i}M_i(\rho^k\xi)+o\left(\sum_{i=1}^{N}\sum_{k\in A_i}M_i(\rho^k\xi)\right), \quad (10)$$

where $o(\xi)$ satisfies that $\lim_{\xi\to 0}\ o(\xi)/\xi = 0$.

Step II. Since $\mu_{\rho,\{D_n\}}$ is a spectral measure, Lemma (5.1.9) implies that ρ is an algebraic integer. Assume that $\varphi(x)=x^m+c_1x^{m-1}+\cdots+c_m \in \mathbb{Z}[x]$ is the minimal polynomial of ρ in the ring $\mathbb{Z}[x]$.

Fix this m and define

$$\Lambda_k = \Lambda \cap \left\{\rho^{-k}\frac{a}{3}: a \in 3\mathbb{Z}+\{1,2\}\right\}\ for\ all\ k\ \in \mathbb{N}.$$

Then, by Lemma (5.1.8), $\#\Lambda_k \le 2$ for all $k \in \mathbb{N}$. Notice that for any $k \in \mathbb{N}, \lambda \in \Lambda_k$ implies that $\rho^k\lambda = \frac{a}{3}$ for some $a \in 3\mathbb{Z}+\{1,2\}$. Then, for any $k \in \mathbb{N}$, we have

$$\sum_{\lambda\in\Lambda_k}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2 \le \sum_{\lambda\in\Lambda_k}\left|\hat{\delta}_{\rho^k D_k}(\xi+\lambda)\right|^2 = \sum_{\lambda\in\Lambda_k}\left|\hat{\delta}_{D_k}(\rho^k(\xi+\lambda))\right|^2$$
$$\le M_{D_k}(\rho^k\xi). \quad (11)$$

Here, (8) and integer-periodicity of the function $\hat{\delta}_{D_k}$ are used in the last inequality. Furthermore, let α be as in Lemma (5.1.10) and let

$$\beta := \min\left\{1-\left|\hat{\delta}_{D(i)}(\xi)\right|^2 : \frac{\alpha}{2} \le |\xi| \le 1-\frac{\alpha}{2}, 1\le i\le N\right\} > 0. \quad (12)$$

Thus, if $\lambda\in\Lambda_k$ for some $k>m$, by Lemma (5.1.10), there exists $k-m\le \ell_\lambda \le k-1$ such that $\left\|\rho^{\ell_\lambda}\lambda\right\| \ge \alpha$. Hence for all $\xi\in[0,\frac{\alpha}{2}]$, we have $\left\|\rho^{\ell_\lambda}(\xi+\lambda)\right\| \ge \alpha/2$. Therefore, if $k\in A_i$, we have that

$$\sum_{\lambda\in\Lambda_k}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2 \le \sum_{\lambda\in\Lambda_k}\left|\hat{\delta}_{D_{\ell_\lambda}}(\rho^{\ell_\lambda}(\xi+\lambda))\right|^2\left|\hat{\delta}_{D_k}(\rho^k(\xi+\lambda))\right|^2$$
$$\le (1-\beta)\sum_{\lambda\in\Lambda_k}\left|\hat{\delta}_{D_k}\left(\rho^k(\xi+\lambda)\right)\right|^2 \quad (by\ (12))$$
$$\le (1-\beta)M_i(\rho^k\xi) \quad \left(by\ (11)\right). \quad (13)$$

Step III. Since $\rho\notin\mathbb{Q}^{\frac{1}{r}}, r\in\mathbb{N}$ and $r>1$, is an irrational number, it is easy to check that $\Lambda_k\cap\Lambda_\ell=\emptyset$ if $k\neq\ell$. Therefore,

$$Q(\xi) = \sum_{\lambda\in\Lambda_k}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2$$
$$= \left|\hat{\mu}_{\rho,\{D_n\}}(\xi)\right|^2 = \sum_{k=1}^{m}\sum_{\lambda\in\Lambda_k}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2 + \sum_{k=m+1}^{\infty}\sum_{\lambda\in\Lambda_k}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2$$
$$= \left|\hat{\mu}_{\rho,\{D_n\}}(\xi)\right|^2 + \sum_{k=1}^{m}\sum_{\lambda\in\Lambda_k}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2$$
$$+\sum_{i=1}^{N}\sum_{\{k\in A_i,k>m\}}\sum_{\lambda\in\Lambda_k}\left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2.$$

By (11) and (13), we get that

$$Q(\xi) \le \left|\hat{\mu}_{\rho,\{D_n\}}(\xi+\lambda)\right|^2 + \sum_{k=1}^{m}M_{D_k}(\rho^k\xi) + \sum_{i=1}^{N}\sum_{\{k\in A_i,k>m\}}(1-\beta)M_i(\rho^k\xi).$$

Combining this with (10), we obtain that

$$Q(\xi) \le 1-\beta\sum_{i=1}^{N}\sum_{\{k\in A_i,k>m\}}M_i(\rho^k\xi) + o\left(\sum_{i=1}^{N}\sum_{k\in A_i}M_i(\rho^k\xi)\right). \quad (14)$$

Since M_i is an entire function, there exists an entire function H_i and an integer $t_i>0$ such that $M_i(z)=z^{t_i}H_i(z)$ and $H_i(0)\neq 0$. Then there exist common constants $C_2>C_1>0$ such that

$$C_1 \le H_i(z) \le C_2$$

holds for all $|z|\le\eta\le\alpha/2$ and all $i=1,2,\dots,N$.

Therefore, by summing over all $k\in A_i$, we obtain, for $0<|\xi|\le\eta$ and $i=1,2,\dots,N$, that

$\xi^{t_i} \lesssim \sum_{\{k\in A_i\}}M_i(\rho^k\xi) = \sum_{\{k\in A_i\}}(\rho^k\xi)^{t_i}H_i(\rho^k\xi) = \xi^{t_i}\sum_{\{k\in A_i\}}(\rho^{t_i})^kH_i(\rho^k\xi) \lesssim \xi^{t_i}$ (15).

Here the symbol $f(x)\lesssim g(x)$ means that there is a constant C such that $f(x)\le Cg(x)$. Consequently,

$$\lim_{\xi\to 0}\sum_{i=1}^{N}\sum_{k\in A_i} M_i(\rho^k\xi)=0. \quad (16)$$

As a special case of (15), for all $k\in A_i$ with $k\geq m$, one can obtain, for $0<|\xi|\leq\eta$ and $i=1,2,\dots,N$, that

$$\xi^{t_i}\lesssim\sum_{\{k\in A_i,k>m\}} M_i(\rho^k\xi)\lesssim\xi^{t_i}. \quad (17)$$

It follows from (14), (16) and (17) that $Q(\xi)<1$ for ξ small enough, as desired.

The proof of Proposition (5.1.7) is complete.

we prove the necessity of Theorem (5.1.2) when ρ is a rational number.

We will show the following proposition.

Proposition (5.1.11) [318]: If $\rho=\frac{p}{q}$ with $p<q$ and $gcd(3,q)=1$, then the measure $\mu_{\rho,\{D_n\}}$ is not a spectral measure.

Now we state and prove a lemma that we need to prove Proposition (5.1.11).

Lemma (5.1.12) [318]: Let $\rho=\frac{p}{q}\in\mathbb{Q}$ with $p<q$ and $gcd(p,q)=1$. If Λ is a spectrum for $\mu_{\rho,\{D_n\}}$, then there exists a $m_0\in\mathbb{N}$ such that

$$\Lambda\backslash\{0\}\subseteq p^{-m_0}\left\{q^m\frac{a}{3}: m\in\mathbb{N}\ and\ m\geq m_0, a\in q\mathbb{Z}+\{1,2,\dots,q-1\}\right\}. (18)$$

Proof Recall that each element $\lambda\in\Lambda\backslash\{0\}$ corresponds to a $n_\lambda\in\mathbb{N}$ and $a_\lambda\in 3\mathbb{Z}+\{1,2\}$ such that

$$\lambda=\left(\frac{q}{p}\right)^{n_\lambda}\frac{a_\lambda}{3}.$$

Let $a_\lambda=q^{\ell_\lambda}a'_\lambda$, where $\ell_\lambda\in\mathbb{N}$ and $q\nmid a'_\lambda$. Since $q\nmid p^{\ell_\lambda}a'_\lambda$, each λ can be rewritten as

$$\lambda=\left(\frac{q}{p}\right)^{n_\lambda+\ell_\lambda}\frac{p^{\ell_\lambda}a'_\lambda}{3}=:\left(\frac{q}{p}\right)^{m_\lambda}\frac{b_\lambda}{3}, q\nmid b_\lambda, b_\lambda\in\mathbb{Z}\backslash\{0\}, m_\lambda\in\mathbb{N}. (19)$$

Let m_0 be the smallest integer among all $m'_\lambda s,\lambda\in\Lambda$. Then, by the orthogonality condition, for all $m_\lambda\geq m_0$, there exist $n\in\mathbb{N}$ and $b\in 3\mathbb{Z}+\{1,2\}$ such that

$$\left(\frac{q}{p}\right)^{m_0}\frac{b_0}{3}-\left(\frac{q}{p}\right)^{m_\lambda}\frac{b_\lambda}{3}=\left(\frac{q}{p}\right)^{n}\frac{b}{3}.$$

Notice that the condition $q\nmid b_0$ implies that $m_0=n$. This yields that b_λ is divisible by $p^{m_\lambda-m_0}$, i.e., there exists a $c_\lambda\in\mathbb{Z}\backslash\{0\}$ such that $b_\lambda=p^{m_\lambda-m_0}c_\lambda$. Hence, from (19), we get that

$$\lambda=p^{-m_0}q^{m_\lambda}\frac{c_\lambda}{3}, where\ q\nmid c_\lambda.$$

The desired result (18) holds. This finishes the proof.

Suppose on the contrary that $\mu_{\rho,\{D_n\}}$ is a spectral measure and Λ is a spectrum for $\mu_{\rho,\{D_n\}}$ with $0\in\Lambda$. Then $\Lambda\subseteq\Lambda-\Lambda\subseteq Z(\hat{\mu}_{\rho,\{D_n\}})\cup\{0\}$.

Let

$$\mu=\delta_{\rho D_1}*\cdots*\delta_{\rho^{m_0}D_{m_0}}*\delta_{\rho^{m_0+2}D_{m_0+2}}*\cdots.$$

Then $\mu_{\rho,\{D_n\}}=\delta_{\rho^{m_0+1}D_{m_0+1}}*\mu$. By Lemma (5.1.5), it suffices to show that Λ is an orthogonal set for μ.

In fact, it follows from (18) and (3) that for any $\lambda_1,\lambda_2\in\Lambda$ with the property $\lambda_1-\lambda_2\in Z(\delta_{\rho^{m_0+1}D_{m_0+1}})$, there exist $m_k,m_\ell\in\mathbb{N}$ with $m_k,m_\ell\geq m_0$ and $a_k,a_\ell\in q\mathbb{Z}+\{1,2,\dots,q-1\}$ and some $a\in 3\mathbb{Z}+\{1,2\}$ such that

$$\lambda_1-\lambda_2:=p^{-m_0}q^{m_k}\frac{a_k}{3}-p^{-m_0}q^{m_\ell}\frac{a_\ell}{3}=\rho^{-(m_0+1)}\frac{a}{3},$$

which yields that

$$q^{m_k}a_k-q^{m_\ell}a_\ell=p^{-1}q^{m_0+1}a.$$

Combining with $gcd(p,q)=1$, we get that $p|a$. Now the conditions $gcd(3,q)=1$ and $a\in 3\mathbb{Z}+\{1,2\}$ imply that $aq/p\in 3\mathbb{Z}+\{1,2\}$. Hence

$$\lambda_1-\lambda_2=\rho^{-(m_0+1)}\frac{a}{3}=\rho^{-m_0}\,\frac{aq/p}{3}\in Z(\hat{\delta}_{\rho^{m_0}D_{m_0}}),$$

that is, $\lambda_1-\lambda_2\in Z(\hat{\mu})$. The above argument shows that $\Lambda-\Lambda\subseteq Z(\hat{\mu})\cup\{0\}$, as desired. The proof is complete.

We will prove $\mu_{\rho,\{D_n\}}$ is not a spectral measure when ρ is a rational number and $\rho^{-1}\notin 3\mathbb{N}$.

Proposition (5.1.13) [318]: If $\rho=\frac{p}{q}$ with $1<p<q, gcd(p,q)=1$ and $3|q$, then $\mu_{\rho,\{D_n\}}$ is not a spectral measure.

We need some lemmas.

Lemma (5.1.14) [318]: Let $\rho=\frac{p}{q}$ with $1<p<q$ and $gcd(p,q)=1$. If $3|q$, then

$$\Lambda_n\cap\Lambda_m=\emptyset,\qquad if\ n\neq m,$$

where $\Lambda_n=\{\rho^{-n}\frac{a}{3}: a\in 3\mathbb{Z}+\{1,2\}$.

Proof: If $\lambda\in\Lambda_n\cap\Lambda_m$, where $n>m$, then

$$\left(\frac{q}{p}\right)^n\frac{a_n}{3}=\left(\frac{q}{p}\right)^m\frac{a_m}{3}.$$

where $a_n,a_m\in 3\mathbb{Z}+\{1,2\}$. That is,

$$q^{n-m}a_n=p^{n-m}a_m.$$

Since $3|q$, we have $3|p$ or $3|a_m$. It is a contradiction to the assumptions of p and a_m.

The following lemma characterizes the structure of spectrum for the measure $\mu_{\rho,\{D_n\}}$ when $\rho=p/q$ is a rational number and satisfies that $1<p<q, gcd(p,q)=1$ and $3|q$.

Lemma (5.1.15) [318]: Let $\rho=\frac{p}{q}\in\mathbb{Q}$ with $1<p<q$ and $gcd(p,q)=1$. If $3|q$ and Λ is a spectrum for $\mu_{\rho,\{D_n\}}$, then

$$\Lambda\backslash\{0\}\subseteq p^{-1}\left\{q^m\frac{a}{3}: m\in\mathbb{N}, a\in 3\mathbb{Z}+\{1,2\}\right\}, (20)$$

and

$$(\Lambda-\Lambda)\backslash\{0\}\subseteq p^{-1}\left\{q^m\frac{a}{3}: m\in\mathbb{N}, a\in 3\mathbb{Z}+\{1,2\}\right\}. (21)$$

Proof Recall that each element $\lambda\in\Lambda\backslash\{0\}$ corresponds to a $m_\lambda\in\mathbb{N}$ and $a_\lambda\in 3\mathbb{Z}+\{1,2\}$ such that

$$\lambda=\left(\frac{q}{p}\right)^{m_\lambda}\frac{a_\lambda}{3}.$$

Let m_0 be the smallest integer among all $m_\lambda's, \lambda\in\Lambda$, and denote the corresponding a_λ by a_0. We claim that $m_0=1$. In fact, by Lemmas (5.1.14) and (5.1.5), we know that $(\Lambda-\Lambda)\cap\Lambda_n\neq\emptyset$ for all $n\in\mathbb{N}$, where Λ_n is defined as in Lemma (5.1.14). Furthermore, since $\Lambda\subseteq\bigcup_{n=1}^{\infty}\Lambda_n$, we can conclude that $\Lambda\cap\Lambda_1\neq\emptyset$ if $3|q$. Otherwise, the condition $\Lambda\subseteq\bigcup_{n=2}^{\infty}\Lambda_n$ means that there exists $\lambda_1\in\Lambda_n\cap\Lambda, n>1$ and $\lambda_2\in\Lambda_m\cap\Lambda, m>1$ such that $\lambda_1-\lambda_2\in\Lambda_1$.Let $\lambda_1=\rho^{-n}\frac{a_n}{3}$ and $\lambda_2=\rho^{-m}\frac{a_n}{3}$, where $a_n,a_m\in 3\mathbb{Z}+\{1,2\}$. We can assume that $n\geq m$. Then $\lambda_1-\lambda_2=\rho^{-1}\frac{a}{3}$ for some $a\in 3\mathbb{Z}+\{1,2\}$, which yields that

$$q^{n-1}a_n-q^{m-1}p^{n-m}a_m=p^{n-1}a.$$

This is impossible since $3|q$ and $3\nmid a, 3\nmid p$. This finishes the proof of the claim.

Whence, by the orthogonality condition, for all $m_\lambda\geq 1$, there exist $n\in\mathbb{N}$ and $a\in 3\mathbb{Z}+\{1,2\}$ such that

$$\left(\frac{q}{p}\right)\frac{a_0}{3}-\left(\frac{q}{p}\right)^{m_\lambda}\frac{a_\lambda}{3}=\left(\frac{q}{p}\right)^{n}\frac{a}{3}.$$

It is easy to see that $n=1$ and a_λ is divisible by $p^{m_\lambda-1}$. Thus there exists a $c_\lambda\in\mathbb{Z}\backslash\{0\}$ such that $a_\lambda=p^{m_\lambda-1}c_\lambda$, and hence

$$\lambda=p^{-1}\,q^{m_\lambda}\frac{c_\lambda}{3},\qquad where\ 3\nmid c_\lambda.$$

This shows that (20) holds.

To prove (21). For any distinct elements $\lambda_k,\lambda_\ell\in\Lambda$, there exist $m_k,m_\ell\in\mathbb{N}$ and $a_k,a_\ell\in 3\mathbb{Z}+\{1,2\}$ such that $\lambda_k=p^{-1}q^{m_k}\frac{a_k}{3}$ and $\lambda_\ell=p^{-1}q^{m_\ell}\frac{a_\ell}{3}$. We may as well assume that $m_k\geq m_\ell$, then

$$\lambda_k-\lambda_\ell=p^{-1}q^{m_\ell}\frac{q^{m_k-m_\ell}a_k-a_\ell}{3}.$$

If $m_k>m_\ell$, then $3\nmid(q^{m_k-m_\ell}a_k-a_\ell)$ since $3|q$ and $3\nmid a_\ell$.

If $m_k=m_\ell$, there exist $\alpha\in\mathbb{N}_0$ and $a\in\mathbb{Z}\backslash\{0\}$ such that $a_k-a_\ell=q^\alpha a$, where $q\nmid a$, and hence

$$\lambda_k-\lambda_\ell=p^{-1}q^{m_\ell+\alpha}\frac{a}{3}.\quad(22)$$

On the other hand, the orthogonality of Λ implies that

$$\lambda_k-\lambda_\ell=\left(\frac{q}{p}\right)^{n}\frac{c}{3}\ \text{for some } n\in\mathbb{N}\text{ and } c\in 3\mathbb{Z}+\{1,2\}.\ (23)$$

It follows from (22) and (23) that

$$p^{n-1}q^{m_\ell+\alpha-n}a=c,$$

which yields that $3\nmid a$ and $m_\ell+\alpha=n$. If not, we will get that $3|c$, i.e., $c\in 3\mathbb{Z}$. It is a contradiction. The above argument shows that (21) holds.

Let Λ be a spectrum for $\mu_{\rho,\{D_n\}}$, where $\rho=\frac{p}{q},p<q,gcd(p,q)=1$ and $3|q$. By Lemma (5.1.14), the set Λ can be written as a countable disjoint union, i.e.,

$$\Lambda=\bigcup_{m=1}^{\infty}(\Lambda\cap\Lambda_m).$$

From (20), for any $\lambda\in\Lambda\backslash\{0\}$, there is a unique $m\in\mathbb{N}$ and $k\in\mathbb{N}$ with $k\geq m$ such that

$$\lambda=p^{-1}q^{m}\frac{a}{3}=p^{-1}q^{m}\frac{a_0+a_1q+\cdots+a_{k-m}q^{k-m}}{3},\quad(24)$$

where $a_0\in\{-1,0,\dots,q-2\}\cap(3\mathbb{Z}+\{1,2\}),a_{k-m}\neq 0$, and $a_j\in\{-1,0,\dots,q-2\}$ for $1\leq j\leq k-m$. Now we call k the degree of λ and let Γ_k be the set of elements of Λ with degree k for all $k\in\mathbb{N}$. Define $\Gamma_k=\{0\}$ and

$$\Omega_n=\bigcup_{k=0}^{n}\Gamma_k.\quad(25)$$

Lemma (5.1.16) [318]: For any $n\in\mathbb{N}$, the set Ω_n as in (25) forms an orthogonal set for $\nu_{\rho,n}$, where

$$\nu_{\rho,n}=\delta_{\rho D_1}*\delta_{\rho^2D_2}*\cdots*\delta_{\rho^nD_n}.$$

Consequently,

$$\sum_{\lambda\in\Omega_n}|\hat{\nu}_{\rho,n}(\xi+\lambda)|^2\leq 1,\qquad\forall\,\xi\in\mathbb{R}.$$

Proof: By (24) and (25), each element $\lambda\in\Omega_n$ can be expressed as

$$\lambda=p^{-1}q^{m}\frac{a_0+a_1q+\cdots+a_{n-m}q^{n-m}}{3},\quad(26)$$

where $a_0\in\{-1,0,\dots,q-2\}\cap(3\mathbb{Z}+\{1,2\})$ and $a_j\in\{-1,0,\dots,q-2\}$ for all $1\leq j\leq n-m$ (a_{n-m} may be 0). Then the integer-periodicity of $\hat{\delta}_{D_m}$ and $3|q$ imply that

$$\hat{\delta}_{\rho^m D_m}(\lambda) = \hat{\delta}_{D_m}(\rho^m \lambda) = \hat{\delta}_{D_m}(p^{m-1}a_0/3).$$

Notice that $p^{m-1}a_0 \in 3\mathbb{Z} + \{1,2\}$ and $(3^{-1}D_m, \{0, c_1, c_2\})$ forms an compatible pair, where $c_i \in 3\mathbb{Z} + i$ for $i = 1, 2$. Then $\hat{\delta}_{D_m}(p^{m-1}a_0/3) = 0$, and hence $\hat{v}_{\rho,n}(\lambda) = 0$.

On the other hand, for any $\lambda' \in \Omega_n \backslash \{0\}$, from (21), the element $\lambda - \lambda'$ must be of the form (26). By doing the same procedure as the above, we also get that $\hat{v}_{\rho,n}(\lambda - \lambda') = 0$. Thus Ω_n is an orthogonal set for $v_{\rho,n}$.

Recall that

$$\mu_{\rho,\{D_n\}} = \delta_{\rho D_1} * \delta_{\rho^2 D_@} * \delta_{\rho^3 D_3} * \cdots.$$

For any $n \in \mathbb{N}$, we define

$$\mu_{\rho,\geq n} = \delta_{\rho D_n} * \delta_{\rho^2 D_{n+1}} * \delta_{\rho^3 D_{n+2}} * \cdots.$$

The following lemma is a simple generalization of Lemma 5.4 in [10].

Lemma (5.1.17) [318]: Let $a = log\ p/\ log\ q$ and $c = \max\{|\hat{\delta}_{D(i)}(\xi)|: \frac{1}{2q} \leq \xi \leq \frac{1}{2}, i = 1, 2, \ldots, N\} < 1$. Then for any $\xi > 1$ there exist ξ' and $n \in \mathbb{N}$ such that $\rho^2 \xi^a \leq \xi' \leq \rho\xi$ and

$$|\hat{\mu}_{\rho,\{D_n\}}(\xi)|^2 \leq c\left|\hat{\mu}_{\rho,\geq n}(\xi')\right|^2. \quad (27)$$

Proof Fix $\xi > 1$ and note that

$$\hat{\mu}_{\rho,\{D_n\}}(\xi) = \hat{\delta}_{D_1}(\rho\xi)\hat{\mu}_{\rho,\geq 2}(\rho\xi), \qquad \forall\, \xi \in \mathbb{R}.$$

Recall that $\rho\xi - \langle\rho\xi\rangle \in \mathbb{Z}$. Then if $\langle\rho\xi\rangle \notin (-\frac{1}{2q}, \frac{1}{2q})$, the integer-periodicity of the function $\hat{\delta}_{D_1}$ and the property $|\hat{\delta}_{D_1}(t)| = |\hat{\delta}_{D_1}(-t)|$ imply that

$$|\hat{\mu}_{\rho,\{D_n\}}(\xi)| = |\hat{\delta}_{D_1}(\langle\rho\xi\rangle)| \cdot |\hat{\mu}_{\rho,\geq 2}(\rho\xi)| \leq c|\hat{\mu}_{\rho,\geq 2}(\rho\xi)|.$$

Then the desired inequality (27) holds if we let $\xi' = \rho\xi$ and $n = 2$.

If $\langle\rho\xi\rangle \in (-\frac{1}{2q}, \frac{1}{2q})$, the condition $\rho\xi > p/q > 1/q$ implies that the nearest integer to $\rho\xi$ is bigger or equal to 1, i.e., $\rho\xi - \langle\rho\xi\rangle \in \mathbb{N}$. So there exist $a_s \in \{1, 2, \ldots, q-1\}, s \geq 0$ and $a_j \in \{0, 1, \ldots, q-1\}$ for $j = s+1, \ldots, t$ with $a_t \neq 0$ such that

$$\rho\xi - \langle\rho\xi\rangle = a_s q^s + \cdots + a_t q^t. \quad (28)$$

Thus,

$$\langle\rho^{s+2}\xi\rangle = \langle\rho^{s+1}(\langle\rho\xi\rangle + (a_s q^s + \cdots + a_t q^t))\rangle = \langle\rho^{s+1}\langle\rho\xi\rangle + p^{s+1}a_s/q\rangle.$$

Since $|\rho^{s+1}\langle\rho\xi\rangle| < \frac{1}{2q}$ and $\frac{1}{q} \leq |\langle p^{s+1}a_s/q\rangle| \leq \frac{1}{2}$, then $\langle\rho^{s+2}\xi\rangle \notin (-\frac{1}{2q}, \frac{1}{2q})$. By the integer-periodicity of $\hat{\delta}_{D_{s+2}}$ and the property $|\hat{\delta}_{D_{s+2}}(t)| = |\hat{\delta}_{D_{s+2}}(-t)|$, we also have that

$$|\hat{\mu}_{\rho,\{D_n\}}(\xi)| = |\hat{\delta}_{D_1}(\rho\xi)| \cdots |\hat{\delta}_{D_{s+2}}(\rho^{s+2}\xi)| \cdot |\,\hat{\mu}_{\rho,\geq s+3}(\rho^{s+2}\xi)|$$
$$\leq |\hat{\delta}_{D_{s+2}}(\langle\rho^{s+2}\xi\rangle)| \cdot |\hat{\mu}_{\rho,\geq s+3}(\rho^{s+2}\xi)| \leq c|\hat{\mu}_{\rho,\geq s+3}(\rho^{s+2}\xi)|.$$

The desired inequality (27) holds if we let $\xi' = \rho^{s+2}\xi$ and $n = s + 3$.

Furthermore, from (28), we have $\xi' \geq \rho\xi \geq q^s$ .With $a = log\ p/log\ q$, we obtain that

$$\xi' = \rho^{s+2}\xi = \frac{p^s}{q^s}\rho^2\xi = \frac{q^{a_s}}{q^s}\rho^2\xi = q^{s(a-1)}\rho^2\xi \geq \xi^{a-1}\rho^2\xi = \rho^2\xi^a.$$

This finishes the proof of Lemma (5.1.17).

The following lemma is a simple generalization of Lemma 5.5 in [320].

Lemma (5.1.18) [318]: Let $r > \frac{\log a}{\log c} + 1$ be an integer, where a, c are as in Lemma (5.1.17). Then there exist constants $\alpha > 1, C > 1$ and n_0 large enough such that $n \geq n_0$ implies that

$$\left|\hat{\mu}_{\rho,\geq(n+1)^r+1}\left(\rho^{(n+1)^r}(\xi + \lambda)\right)\right| \leq \frac{C}{n^\alpha}, \qquad \forall\, \xi \in \left(\frac{0,2}{3p}\right)$$

holds for all $\lambda \in \Omega_{(n+1)^r} \backslash \Omega_{n^r}$.

Proof : From (24) and (25), each element $\lambda \in \Omega_{(n+1)^r} \backslash \Omega_{n^r}$ has the expression

$$\lambda = \frac{1}{3p}(a_0 q^m + a_1 q^{m+1} + \cdots + a_l q^l),$$

for some $m \in \mathbb{N}$, where $a_0 \in \{-1, 0, \ldots, q-2\} \cap (3\mathbb{Z} + \{1, 2\}), a_j \in \{-1, 0, \ldots, q-2\}$ for $1 \leq j \leq l, a_l \neq 0$ and $(n+1)^r \geq l > n^r$.

An easy calculation shows that

$$|\lambda| \geq \frac{1}{3p}\left[q^l - (q-2)\sum_{j=0}^{l-1} q^j\right] \geq \frac{1}{3p} q^{l-1} + \frac{2}{3p} \geq q^{l-3} + \frac{2}{3p}.$$

Let $\eta := \rho^{(n+1)^r}(\xi + \lambda)$. Combining with the assumption of $\xi \in \left(0, \frac{2}{3p}\right)$, we get that

$$|\eta| \geq \rho^{(n+1)^r}(|\lambda| - |\xi|) \geq \frac{p^{(n+1)^r}}{q^{(n+1)^r - n^r + 3}}.$$

It is easy to see that $(n+1)^r - n^r + 3 \leq n^{r-1} 2^r$ when $n \geq 3$, which means that

$$|\eta| \geq \frac{p^{(n+1)^r}}{q^{n^{r-1}2^r}} = \left(\frac{p^{(1+1/n)^r n}}{q^{2^r}}\right)^{n^{r-1}} \geq \left(\frac{p^n}{q^{2^r}}\right)^{n^{r-1}}.$$

Then there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that $p^n > q^{2^r+1}$, and hence

$$|\eta| \geq q^{n^r - 1} (> 1). \quad (29)$$

Let $\xi_0 := \eta$. Applying Lemma (5.1.17) several times, there is an increasing sequence of positive integers $n_1 < n_2 < \cdots$ and $\rho^2 \xi_i^a \leq \xi_{i+1} \leq \rho \xi_i$ for $i \geq 0$ satisfying that

$$|\hat{\mu}_{\rho,\{D_n\}}(\eta)| \leq c|\hat{\mu}_{\rho,\geq n_1}(\xi_1)| \leq c^2|\hat{\mu}_{\rho,\geq n_2}(\xi_2)| \leq \cdots \leq c^l|\hat{\mu}_{\rho,\geq n_l}(\xi_l)| \leq \cdots,$$

whenever $|\xi_i| \geq 1$ for all $i \geq 1$.

In particular, by taking $l = \left\lfloor \log_a \frac{2a^{1-r}}{1-a} \right\rfloor$, where the symbol $\lfloor x \rfloor$ denotes the biggest integer smaller or equal to x, we obtain, from (29), that

$$|\xi_l| \geq \rho^2 |\xi_{l-1}|^a \geq \cdots \geq \rho^{2+2a+\cdots+2a^{l-1}} |\eta|^{a^l} \geq \rho^{\frac{2}{1-a}} |\eta|^{a^l}$$
$$\geq p^{\frac{2}{1-a}} q^{-\frac{2}{1-a}} q^{n^{r-1} a^l} \geq p^{\frac{2}{1-a}} > 1,$$

and

$$\left|\hat{\mu}_{\rho,\{D_n\}}(\eta)\right| \leq c^l \leq c^{\log_a \frac{2a^{1-r}}{1-a}} = c^{\log_a \frac{2}{1-a}} n^{-(r-1)} \log_a c.$$

We only need to assume $C := c^{\log_a \frac{2}{1-a}}$ and $\alpha := (r-1)\log_a c$. This finishes the proof of Lemma (5.1.18).

Let Λ be a spectrum for $\mu_{\rho,\{D_n\}}$ with $0 \in \Lambda$ and let

$$Q_n(\xi) = \sum_{\lambda \in \Omega_n} |\hat{\mu}_{\rho,\{D_n\}}(\xi + \lambda)|^2 \text{ and } Q(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}_{\rho,\{D_n\}}(\xi + \lambda)|^2.$$

Let α, C, n_0 be as in Lemma (5.1.18). If $\Omega_{(n+1)^r} = \Omega_{n^r}$ for $n \geq n_0$, we have $Q_{(n+1)^r}(\xi) = Q_{n^r}(\xi)$. If $\Omega_{(n+1)^r} \neq \Omega_{n^r}$ for some $n \geq n_0$, then for any $\xi \in \left(0, \frac{2}{3p}\right)$, we get that

$$Q_{(n+1)^r}(\xi) = Q_{n^r}(\xi) + \sum_{\lambda \in \Omega_{(n+1)^r} \backslash \Omega_{n^r}} |\hat{\mu}_{\rho,\{D_n\}}(\xi + \lambda)|^2$$
$$= Q_{n^r}(\xi) + \sum_{\lambda \in \Omega_{(n+1)^r} \backslash \Omega_{n^r}} |\hat{v}_{\rho,(n+1)^r}(\xi + \lambda)|^2$$

$$\cdot |\hat{\mu}_{\rho,\geq(n+1)^r+1}(\rho^{(n+1)^r}(\xi+\lambda))|^2$$

$$\leq Q_{n^r}(\xi) + \frac{C^2}{n^{2\alpha}} \sum_{\lambda\in\Omega_{(n+1)^r}\setminus\Omega_{n^r}} \left|\hat{v}_{\rho,(n+1)^r}(\xi+\lambda)\right|^2 \ (by\ Lemma\ (5.1.18))$$

$$\leq Q_{n^r}(\xi) + \frac{C^2}{n^{2\alpha}}\left(1 - \sum_{\lambda\in\Omega_{n^r}} \left|\hat{v}_{\rho,(n+1)^r}(\xi+\lambda)\right|^2\right) \ (by\ Lemma\ (5.1.16))$$

$$\leq Q_{n^r}(\xi) + \frac{C^2}{n^{2\alpha}}(1 - Q_{n^r}(\xi)).$$

Thus,

$$1 - Q_{(n+1)^r}(\xi) \geq \left(1 - Q_{n^r}(\xi)\right)\left(1 - \frac{C^2}{n^{2\alpha}}\right) \geq \cdots \geq \left(1 - Q_{n_0^r}(\xi)\right)\prod_{k=n_0}^{n}\left(1 - \frac{C^2}{k^{2\alpha}}\right).$$

Letting $n \to \infty$, we get that

$$1 - Q(\xi) \geq \left(1 - Q_{n_0^r}(\xi)\right)\prod_{k=n_0}^{\infty}\left(1 - \frac{C^2}{k^{2\alpha}}\right).$$

Note that $\prod_{k=n_0}^{\infty}\left(1 - \frac{C^2}{k^{2\alpha}}\right) \neq 0$. From Lemma (5.1.16), $Q_{n_0^r}(\xi) \leq 1$ and $Q_{n_0^r}(\xi) \not\equiv 1$. Hence, $Q(\xi) < 1$ for some $\xi \in \left(0, \frac{2}{3p}\right)$. The proof is complete.

We end by giving an example which devotes to illustrate that the condition $gcd(a_n, b_n) = 1$ for all $n \in \mathbb{N}$ in Theorem (5.1.2) can not be removed.

Example (5.1.19) [318]: Let $D_1 = \{0, 1, 2\}$, and $D_n = \{0, 2, 4\}$ for all $n \in \mathbb{N}$ with $n \geq 2$. Then $D_n \equiv \{0, 1, 2\}\ (mod\ 3)$ holds for all $n \in \mathbb{N}$, but the measure

$$\mu_{3^{-1},\{D_n\}} = \delta_{3^{-1}D_1} * \delta_{3^{-2}D_2} * \delta_{3^{-3}D_3} * \cdots$$

is not a spectral measure.

Proof *:* Notice that

$$\delta_{3^{-2}D_2} * \delta_{3^{-3}D_3} * \cdots = \frac{3}{2}\mathcal{L}|_{[0,2/3]}.$$

Then

$$\mu_{3^{-1},\{D_n\}} = \delta_{3^{-1}D_1} * \frac{3}{2}\mathcal{L}|_{[0,2/3]} = \frac{1}{2}\mathcal{L}|_{[0,1/3]\cup[1,4/3]} + \mathcal{L}|_{[1/3,1]}.$$

By [337, Theorem 1.3], $\mu_{3,\{D_n\}}$ is not a spectral measure.

Corollary (5.1.20) [486]: Let $(1-\epsilon) \in \mathbb{Q}^{1+\epsilon}, \epsilon > 0$, be an irrational number. Then $\mu_{(1-\epsilon),\{D_n\}}$ is not a spectral measure.

Proof. Let $1 - \epsilon = \left(\frac{1+\epsilon}{1+2\epsilon}\right)^{\frac{1}{1+\epsilon}}$, where $1+\epsilon, 1+2\epsilon, 1+\epsilon \in \mathbb{N}$ with $\gcd(1+\epsilon, 1+2\epsilon) = 1$ and $\epsilon > 0$. Then the minimal polynomial of $(1-\epsilon)$ in the ring $\mathbb{Z}[x]$ is $(1+2\epsilon)x^{1+\epsilon} - (1+\epsilon)$. For $i = 1, 2, \ldots, 1+\epsilon$, set

$$\nu_i = \delta_{(1-\epsilon)^i D_i} * \delta_{(1-\epsilon)^{i+1+\epsilon}D_{i+1+\epsilon}} * \delta_{(1-\epsilon)^{i+2(1+\epsilon)}D_{i+2(1+\epsilon)}} * \cdots.$$

Then $\mu_{(1-\epsilon),\{D_n\}} = \nu_1 * \nu_2 \cdots * \nu_{1+\epsilon}$ and the Fourier transform of $\mu_{(1-\epsilon),\{D_n\}}$ is

$$\sum_s \hat{\mu}_{(1-\epsilon),\{D_n\}}(\xi_s) = \sum_s \prod_{i=1}^{1+\epsilon} \widehat{\nu_i}(\xi_s) = \sum_s \prod_{i=1}^{1+\epsilon}\prod_{j=0}^{\infty} \hat{\delta}_{D_{i+j(1+\epsilon)}}((1-\epsilon)^{i+j(1+\epsilon)}\xi_s).$$

Suppose $\mu_{(1-\epsilon),\{D_n\}}$ is a spectral measure with a spectrum $0 \in \Lambda$. Then for any $\lambda_{1+2\epsilon} \in \Lambda, 1+2\epsilon = 1, 2$, we get that $\hat{\mu}_{(1-\epsilon),\{D_n\}}(\lambda_{1+2\epsilon}) = 0$, therefore $\hat{\delta}_{D_{i(1+2\epsilon)+j(1+2\epsilon)(1+\epsilon)}}((1-$

$\epsilon)^{i(1+2\epsilon)+j(1+2\epsilon)(1+\epsilon)}\lambda_{1+2\epsilon}) = 0$ for some $1 \le i_{1+2\epsilon} \le 1+\epsilon$ and $j_{1+2\epsilon} \in \mathbb{N}_0$. each $\lambda_{1+2\epsilon}$ has the form

$$\lambda_{1+2\epsilon} = (1-\epsilon)^{-(i_{1+2\epsilon}+j_{1+2\epsilon}(1+\epsilon))}\frac{a_{1+2\epsilon}}{3}, \qquad \text{where } a_{1+2\epsilon} \in 3\mathbb{Z}+\{1,2\}.$$

Since $\Lambda - \Lambda \subseteq \hat{\mu}_{(1-\epsilon),\{D_n\}} \cup \{0\}$, then

$$\lambda_1 - \lambda_2 = (1-\epsilon)^{-(i_1+j_1(1+\epsilon))}\frac{a_1}{3} - (1-\epsilon)^{-(i_2+j_2(1+\epsilon))}\frac{a_2}{3}$$

$$= (1-\epsilon)^{-(i+j(1+\epsilon))}\frac{a}{3}$$

for some $1 \le i \le 1+\epsilon, j \in \mathbb{N}_0$ and $a \in 3\mathbb{Z}+\{1,2\}$. We may as well assume that $j_1, j_2 \le j$. Thus, means that

$$(1-\epsilon)^{i-i_1}\left(\frac{1+\epsilon}{1+2\epsilon}\right)^{j-j_1}a_1 - (1-\epsilon)^{i-i_2}\left(\frac{1+\epsilon}{1+2\epsilon}\right)^{j-j_2}a_2 = a,$$

which yields that $i = i_1 = i_2$, since the minimal polynomial of $(1-\epsilon)$ is $(1+2\epsilon)x^{1+\epsilon} - (1+\epsilon)$. Hence Λ is an orthogonal set for some ν_i , it cannot be the spectrum for $\mu_{(1-\epsilon),\{D_n\}}$.

Corollary (5.1.21) [486]: Let Λ be a spectrum for $\mu_{(1-\epsilon),\{D_n\}}$ and $0 \in \Lambda$, and let

$$\Lambda_{\frac{1}{3(1-\epsilon)}} = \Lambda \cap \left\{(1-\epsilon)^{-\frac{1}{3(1-\epsilon)}}\frac{a}{3} : a \in 3\mathbb{Z}+\{1,2\}\right\}.$$

Then $\#\Lambda_{\frac{1}{3(1-\epsilon)}} \le 2$.

Proof. If $\#\Lambda_{\frac{1}{3(1-\epsilon)}} \ge 3$, there exist $\lambda_1, \lambda_2 \in \Lambda_{\frac{1}{3(1-\epsilon)}}$ such that $\lambda_1, \lambda_2 \in (1-\epsilon)^{-\frac{1}{3(1-\epsilon)}}3^{-1}(3\mathbb{Z}+i)$ for some $i = 1$ or 2. In either case, we have

$$\lambda_1 - \lambda_2 \in (1-\epsilon)^{-\frac{1}{3(1-\epsilon)}}\mathbb{Z}.$$

That is, $\lambda_1 - \lambda_2 \notin Z(\hat{\mu}_{(1-\epsilon),\{D_n\}})$. It is a contradiction.

Corollary (5.1.22) [486]: Let Λ be a spectrum for $\mu_{(1-\epsilon),\{D_n\}}$ and $0 \in \Lambda$. Then $(1-\epsilon)$ is an algebraic integer.

Proof. Let $\Lambda = \{\lambda_{\frac{1}{3(1-\epsilon)}}\}_{\frac{2}{3}=\epsilon}^{\infty}$ with $\lambda_0 = 0 \in \Lambda$ be a spectrum for $\mu_{(1-\epsilon),\{D_n\}}$. Since $\Lambda \subseteq Z(\hat{\mu}_{(1-\epsilon),\{D_n\}}) \cup \{0\}$, then each $\lambda_{\frac{1}{3(1-\epsilon)}}$ has the following form

$$\lambda_{\frac{1}{3(1-\epsilon)}} = (1-\epsilon)^{-n_{\frac{1}{3(1-\epsilon)}}}\frac{a_{\frac{1}{3(1-\epsilon)}}}{3}, \text{ where } a_{\frac{1}{3(1-\epsilon)}} \in 3\mathbb{Z}+\{1,2\}.$$

We can assume that $n_{\frac{1}{3(1-\epsilon)}} \le n_{\frac{4-\epsilon}{3(1-\epsilon)}}$ for all $\frac{1}{3(1-\epsilon)} \in \mathbb{N}$. Fix $\ell \in \mathbb{N}$. One can conclude from $\epsilon \ge 0$ that $n_{\ell+\epsilon} \ge n_\ell$, and hence there is $n_{\ell+\epsilon,\ell} > n_\ell$ and $a_{\ell+\epsilon,\ell} \in 3\mathbb{Z}+\{1,2\}$ such that

$$(1-\epsilon)^{-n_{\ell+\epsilon}}\frac{a_{\ell+\epsilon}}{3} - (1-\epsilon)^{-n_\ell}\frac{a_\ell}{3} = (1-\epsilon)^{-n_{\ell+\epsilon,\ell}}\frac{a_{\ell+\epsilon,\ell}}{3}.$$

It follows that there is a polynomial $P(x) = a_\ell x^{1+2\epsilon} + a_{\ell+\epsilon}x^{1+\epsilon} + a + 2\epsilon$ with $\epsilon \ge -1$ satisfying that $P(1-\epsilon) = 0$. Let $\varphi(x) = (a_0+2\epsilon)x^m + (a_1+2\epsilon)x^{m-1} + \cdots + (a_m+2\epsilon) \in \mathbb{Z}[x]$ be the minimal polynomial of $(1-\epsilon)$. Then $\varphi(x)|P(x)$ and $(a_0+2\epsilon)|a_\ell$. Since $a_\ell/(a_0+2\epsilon) \in 3\mathbb{Z}+\{1,2\}$, then $\frac{1}{a_0+2\epsilon}\lambda_\ell = (1-\epsilon)^{-n_\ell 3^{-1}}\frac{a_\ell}{a_0+2\epsilon} \in Z(\hat{\mu}_{(1-\epsilon),\{D_n\}})$. Letting ℓ runs through all $\lambda_\ell \in \Lambda$, we get that

$$\frac{1}{a_0+2\epsilon}\Lambda\backslash\{0\} \subseteq Z(\hat{\mu}_{(1-\epsilon),\{D_n\}}).$$

We claim that $\frac{1}{a_0+2\epsilon}\Lambda$ is an orthogonal set for $\mu_{(1-\epsilon),\{D_n\}}$, i.e.

$$\frac{1}{a_0+2\epsilon}(\Lambda-\Lambda)\backslash\{0\}\subseteq Z(\hat{\mu}_{(1-\epsilon),\{D_n\}}).$$

Indeed, for any $\lambda_{k_1},\lambda_{k_2}\in\Lambda$, there exists $n_{k_1,k_2}\in\mathbb{N}$ and $a_{k_1,k_2}\in 3\mathbb{Z}+\{1,2\}$ such that

$$\lambda_{k_1}-\lambda_{k_2}=(1-\epsilon)^{-n_{k_1,k_2}}\frac{a_{k_1,k_2}}{3}.$$

There exists $\ell+\epsilon\in\mathbb{N}$ such that $\min\{n_{\ell+\epsilon,k_1},n_{\ell+\epsilon,k_2}\}>n_{k_1,k_2}$ and

$$\begin{aligned}(1-\epsilon)^{-n_{k_1,k_2}}\frac{a_{k_1,k_2}}{3}&=(\lambda_{k_1}-\lambda_{\ell+\epsilon})-(\lambda_{k_2}-\lambda_{\ell+\epsilon})\\&=(1-\epsilon)^{-n_{\ell+\epsilon,k_2}}\frac{a_{\ell+\epsilon,k_1}}{3}-(1-\epsilon)^{-n_{\ell+\epsilon,k_2}}\frac{a_{\ell+\epsilon,k_2}}{3}.\end{aligned}$$

As before, one can show that $(a_0+2\epsilon)|\,a_{k_1,k_2}$. Notice that $a_{k_1,k_2}/(a_0+2\epsilon)\in 3\mathbb{Z}+\{1,2\}$.We get that

$$\frac{1}{a_0+2\epsilon}(\lambda_{k_1}-\lambda_{k_2})\in Z(\hat{\mu}_{(1-\epsilon),\{D_n\}}).$$

The claim is proved.

Repeating the same argument, we see that $\frac{1}{(a_0+2\epsilon)^{\ell+\epsilon}}(\Lambda-\Lambda)\backslash\{0\}\subseteq Z(\hat{\mu}_{(1-\epsilon),\{D_n\}})$ for all $\ell+\epsilon\in\mathbb{N}$. This forces that $a_0+2\epsilon=1$. This finishes the proof.

For any $x\in\mathbb{R}$, we shall use the symbol $\langle\cdot\rangle$ to denote the signed distance to the nearest integer, i.e.,

$$\langle x\rangle=\{x\}\ \ if\ \ \{x\}<\frac{1}{2}\ \ and\ \langle x\rangle=\{x\}-1\ if\ \frac{1}{2}\leq\{x\}<1,$$

where $\{x\}$ is the fractional part of a real number x. Also, we shall use $\|x\|$ to denote the absolute value of the difference between x and the nearest integer, i.e., $\|x\|=|\langle x\rangle|$.

Clearly, $\langle x\rangle\in[-\frac{1}{2},\frac{1}{2})$ and $x-\langle x\rangle$ is the nearest integer to x.

Corollary (5.1.23) [486]: Let $(1-\epsilon)$ be a root of $x^m+(a_1+2\epsilon)x^{m-1}+(a_2+2\epsilon)x^{m-2}+\cdots+(a_m+2\epsilon)\in\mathbb{Z}[x]$. Then for any $a\in 3\mathbb{Z}+\{1,2\}$, we have

$$\max_{1\leq n\leq m}\left\|(1-\epsilon)^{-n}\frac{a}{3}\right\|\geq\left(3\sum_{n=1}^{m}|a_n+2\epsilon|\right)^{-1}=:\alpha>0.$$

Proof. For each $1\leq n\leq m$, let $a_n:=(1-\epsilon)^{-n}\frac{a}{3}-\langle(1-\epsilon)^{-n}\frac{a}{3}\rangle$ be the nearest integer to $(1-\epsilon)^{-n}\frac{a}{3}$. Since

$$1+\sum_{n=1}^{m}(a_n+2\epsilon)(1-\epsilon)^{-n}=0,$$

then

$$\frac{a}{3}+\sum_{n=1}^{m}(a_n+2\epsilon)\langle(1-\epsilon)^{-n}\frac{a}{3}\rangle+\sum_{n=1}^{m}(a_n+2\epsilon)a_n=0.$$

If $\left\|(1-\epsilon)^{-n}\frac{a}{3}\right\|=|\langle(1-\epsilon)^{-n}\frac{a}{3}\rangle|<\alpha$ for all $1\leq n\leq m$, then

$$\left|\sum_{n=1}^{m}(a_n+2\epsilon)\langle(1-\epsilon)^{-n}\frac{a}{3}\rangle\right|<\left(\sum_{n=1}^{m}(a_n+2\epsilon|\right)\alpha=\frac{1}{3}.$$

It is a contradiction since $a\in 3\mathbb{Z}+\{1,2\}$. The proof is complete.

For each digit set $D = \{0, a, a+\epsilon\}$, where $\{a, a+\epsilon\} \equiv \{1,2\}\,(mod\ 3)$, there is a polynomial

$$\sum_s M_D(\xi_s) := \sum_{j=1,2}\sum_s \left|\hat{\delta}_D\left(\xi_s + \frac{j}{3}\right)\right|^2 \quad (\xi_s \in \mathbb{R}).$$

Then, from the property of the compatible pair $(3^{-1}D, \{0,1,2\})$, we get that

$$\sum_s |\hat{\delta}_D(\xi_s)|^2 + \sum_s M_D(\xi_s) = 1 \quad (\xi_s \in \mathbb{R}),$$

and hence $M_D(0) = 0$.

Corollary (5.1.24) [486]: Let $1-\epsilon = \frac{1+\epsilon}{1+2\epsilon} \in \mathbb{Q}$ with $\epsilon > 0$ and $\gcd(1+\epsilon, 1+2\epsilon) = 1$. If Λ is a spectrum for $\mu_{(1-\epsilon),\{D_n\}}$, then there exists a $m_0 \in \mathbb{N}$ such that

$$\Lambda\backslash\{0\} \subseteq (1+\epsilon)^{-m_0}\left\{(1+2\epsilon)^m \frac{a}{3} : m \in \mathbb{N}\ and\ m \geq m_0, a \in (1+2\epsilon)\mathbb{Z} + \{1,2,\dots,2\epsilon\}\right\}.$$

Proof. Recall that each element $\lambda \in \Lambda\backslash\{0\}$ corresponds to a $n_\lambda \in \mathbb{N}$ and $a_\lambda \in 3\mathbb{Z} + \{1,2\}$ such that

$$\lambda = \left(\frac{1+2\epsilon}{1+\epsilon}\right)^{n_\lambda} \frac{a_\lambda}{3}.$$

Let $a_\lambda = (1+2\epsilon)^{\ell_\lambda} a'_\lambda$, where $\ell_\lambda \in \mathbb{N}$ and $(1+2\epsilon) \nmid a'_\lambda$. Since $(1+2\epsilon) \nmid (1+\epsilon)^{\ell_\lambda} a'_\lambda$, each λ can be rewritten as

$$\lambda = \left(\frac{1+2\epsilon}{1+\epsilon}\right)^{n_\lambda+\ell_\lambda} \frac{(1+\epsilon)^{\ell_\lambda} a'_\lambda}{3} =: \left(\frac{1+2\epsilon}{1+\epsilon}\right)^{m_\lambda} \frac{a_\lambda+\epsilon}{3}, (1+2\epsilon) \nmid a_\lambda + \epsilon,$$

$a_\lambda + \epsilon \in \mathbb{Z}\backslash\{0\}, m_\lambda \in \mathbb{N}.$

Let m_0 be the smallest integer among all $m'_\lambda(1+2\epsilon), \lambda \in \Lambda$. Then, by the orthogonality condition, for all $m_\lambda \geq m_0$, there exist $n \in \mathbb{N}$ and $(a+\epsilon) \in 3\mathbb{Z} + \{1,2\}$ such that

$$\left(\frac{1+2\epsilon}{1+\epsilon}\right)^{m_0} \frac{a_0+\epsilon}{3} - \left(\frac{1+2\epsilon}{1+\epsilon}\right)^{m_\lambda} \frac{a_\lambda+\epsilon}{3} = \left(\frac{1+2\epsilon}{1+\epsilon}\right)^{n} \frac{a+\epsilon}{3}.$$

Notice that the condition $(1+2\epsilon) \nmid a_0 + \epsilon$ implies that $m_0 = n$. This yields that $a_\lambda + \epsilon$ is divisible by $(1+\epsilon)^{m_\lambda - m_0}$, i.e., there exists a $a_\lambda + 2\epsilon \in \mathbb{Z}\backslash\{0\}$ such that $a_\lambda + \epsilon = (1+\epsilon)^{m_\lambda - m_0}(a_\lambda + 2\epsilon)$. Hence, from (4.2), we get that

$$\lambda = (1+\epsilon)^{-m_0}(1+2\epsilon)^{m_\lambda} \frac{a_\lambda + 2\epsilon}{3}, \qquad \text{where} \qquad (1+2\epsilon) \nmid (a_\lambda + 2\epsilon).$$

The desired result holds. This finishes the proof.

Corollary (5.1.25) [486]: Let $1-\epsilon = \frac{1+\epsilon}{1+2\epsilon}$ with $\epsilon > 0$ and $\gcd(1+\epsilon, 1+2\epsilon) = 1$. If $3|(1+2\epsilon)$, then

$$\Lambda_n \cap \Lambda_m = \emptyset, \qquad if\ n \neq m,$$

where $\Lambda_n = \{(1-\epsilon)^{-n} \frac{a}{3} : a \in 3\mathbb{Z} + \{1,2\}$.

Proof. If $\lambda \in \Lambda_n \cap \Lambda_m$, where $n > m$, then

$$\left(\frac{1+2\epsilon}{1+\epsilon}\right)^{n} \frac{a_n}{3} = \left(\frac{1+2\epsilon}{1+\epsilon}\right)^{m} \frac{a_m}{3}.$$

where $a_n, a_m \in 3\mathbb{Z} + \{1,2\}$. That is,

$$(1+2\epsilon)^{n-m} a_n = (1+\epsilon)^{n-m} a_m.$$

Since $3|(1+2\epsilon)$, we have $3|(1+\epsilon)$ or $3|a_m$. It is a contradiction to the assumptions of $(1+\epsilon)$ and a_m.

The following lemma characterizes the structure of spectrum for the measure $\mu_{(1-\epsilon),\{D_n\}}$ when $1-\epsilon=(1+\epsilon)/(1+2\epsilon)$ is a rational number and satisfies that $\epsilon>0, \gcd(1+\epsilon, 1+2\epsilon)=1$ and $3|(1+2\epsilon)$.

Section (5.2): On the Fourier Orthonormal Bases

For μ be a Borel probability measure on $\mathbb{R}$. It is difficult to approximate functions in $L^2(\mu)$ by some simple functions such as exponential functions. When μ has no compact support, it is classical to study the existence of a positive number a such that $\overline{span}\,\mathbb{E}_{[0,a]}=L^2(\mu)$, where $\mathbb{E}_{[0,a]}=\{e^{-2\pi i\lambda x}:\lambda\in[0,a]\}$(see, [363]); When μ has compact support, it has been extensively studied the existence of $\mathbb{E}_\Lambda=\{e^{-2\pi i\lambda x}:\lambda\in\Lambda\}$ for a countable set Λ such that it is an orthonormal basis (Fourier basis)/a Riesz basis /a Fourier frame for $L^2(\mu)$ (see, e.g. [362]). Corresponding to each case, we call μ a (Fourier) spectral/ Riesz spectral/ frame spectral measure and correspondingly Λ is called a spectrum/ Riesz spectrum/ frame spectrum respectively.

He et al.[364]proved that, if μ is a frame spectral measure, then μ is pure type, i.e., it is either discrete with finite support, or absolutely continuous or singularly continuous with respect to the Lebesgue measure. For a discrete measure μ, the spectrality of μ is closely related to the integer tiles [364]. For an absolutely continuous measure μ, the spectrality of μ is equivalent in some sense to the one of its support Ω [364]. We say Ω is a spectral set if the normalized Lebesgue measure restricted on it is a spectral measure. A famous conjecture on the spectral set was raised by Fuglede [362]: Ω is a spectral set if and only if it is a translational tile. see [368]. The spectral problem becomes more intriguing when μ is a singular measure, because it probably hassome singular properties and some useful theories for Lebesgue measure are not suitable for singular measures. Jorgensen and Pedersen [365] made a head start to study the spectral property of the self-similar measures, which are singularly continuous in general. One basic but the most important problem to find more spectral measures. Some measures were proved to be spectral [369], but there are only a few classes.

Let μ be a singular measure with compact support which admits $\mathbb{E}_\Lambda$ a Fourier basis for $L^2(\mu)$. Some surprising facts on singular spectral measures have been discovered: (1). There exists a singular spectral measure μ such that the Fourier expansion of any continuous function in $L^2(\mu)$ with respect to a spectrum Λ is uniformly convergent, but it is not convergent for some other spectrum Λ' and some continuous functions [378];. There exists a singular spectral measure μ with two spectra Λ and $\kappa\Lambda$ for some integer $\kappa>1$ [369]. There exists a singular spectral measure μ with two spectra $\Lambda_1$and Λ_2 such that the Beurling dimensions of them are equal to 0 and the Hausdorff dimension of the support of μ, respectively [369];. There exists a singular spectral measure μ and a countable set Λ such that $\mathbb{E}_\Lambda$ is the maximal orthonormal family of exponentials for $L^2(\mu)$ (not a basis) but $\kappa\Lambda$ is a spectrum of μ for some integer $\kappa>1$ [385];. There exist singular spectral measures which have been used in the convolution decomposition of Lebesgue measure restricted on [0, 1][363]and in wavelets [382]. construct a class of singular spectral measures. Let $\{p_n\}_{n=1}^\infty$be a sequence of integers bigger than 1and let $\{D_n\}_{n=1}^\infty$be a sequence of finite digit sets in $\mathbb{Z}$. Denote $P_n=p_1p_2\cdots p_n$. Associated to the sequence $\{p_n,D_n\}_{n=1}^\infty$, there exists a Borel probability measure $\mu_{\{p_n\},\{D_n\}}$, which is defined by the following infinite convolutions of finite measures:

$$\mu_{\{p_n\},\{D_n\}}=\delta_{P_1^{-1}D_1}*\delta_{P_2^{-1}D_2}*\cdots,$$

where $\delta_E=1/\#E\ \sum_{e\in E}\delta_e$, $\#E$ is the cardinality of a set E, δ_e is the Dirac measure at the point $e\in\mathbb{R}$, and the convergence is in a weak sense. We call the measure $\mu_{\{p_n\},\{D_n\}}$ a

Cantor–Moran measure, which is singular with respect to Lebesgue measure in general[389]. It is known that the support of $\mu_{\{p_n\},\{D_n\}}$is the following Cantor–Moran set:

$$T(\{p_n\},\{D_n\}) := \sum_{n=1}^{\infty} P_n^{-1} D_n ,$$

where $aE = \{ax : x \in E\}$ and $E + F = \{x + y : x \in E, y \in F\}$ for any sets E, F and a number a. The first result on the spectrality of $\mu_{\{p_n\},\{D_n\}}$was given by Strichartz [385], then further studied by An, He and Lau[2]and Dutkay and Lai [386]. The special case that $p_n = p$ and $D_n = \{0, d_n\}$ for $n \geq 1$ has been investigated by An et al.[383]under the assumption $\sup_{n\geq 1} d_n < \infty$.

we will study the spectrality of the Cantor–Moran measures $\mu_{\{p_n\},\{D_n\}}$ with all $D_n = \{0, d_n\}$, we denote it as $\mu_{\{p_n\},\{d_n\}}$. It should be mentioned that there is no assumption $\sup_{n\geq 1}\{d_n\} < \infty$ in our setting. The main result of this is the following:

Theorem (5.2.1)[358]: Let $\{d_n, p_n\}_{n=1}^{\infty}$be a sequence of integers with $0 < d_n < p_n$for all $n \geq 1$. If $2|\frac{p_n}{gcd(d_n,p_n)}$ for all $n \geq 1$, then the Cantor–Moran measure $\mu_{\{p_n\},\{d_n\}}$is a spectral measure.

According to [380], the Hausdorff dimension of the compact support $T(\{p_n\},\{d_n\})$ of $\mu_{\{p_n\},\{d_n\}}$is $\lim_{n\to\infty} \inf \frac{n \ln 2}{\ln P_n}$, which can be taken any values in [0,1] by choosing suitable ones of p_n for each $n \geq 1$. Hence, there exists a spectral measure such that its support has arbitrary given dimension including 0 and 1. We notice that the spectrality of $\mu_{\{p_n\},\{D_n\}}$ does not depend on the value of $\{d_1, p_1\}$, in this case its spectra may be not in $\mathbb{Z}$(see Corollary(5.2.15)). And we can weaken the condition of $0 < d_n < p_n$ for all $n \geq 1$as follows.

Corollary (5.2.2) [358]: Let $\{d_n, p_n\}_{n=1}^{\infty}$be a sequence of integers with $0 < d_n < p_n$ for n large enough. If $2|\frac{p_n}{gcd(d_n,p_n)}$ for all $n \geq 1$, then the Cantor–Moran measure $\mu_{\{p_n\},\{d_n\}}$ is a spectral measure.

We will give several examples to explain the above conditions in Theorem(5.2.1)and Corollary(5.2.2) at the end. Example(5.2.16) and (5.2.8) indicate that the hypotheses $2|\frac{p_n}{gcd(d_n,p_n)}$ for $n \geq 2$ and $0 < d_n < p_n$for n large enough are essential in some sense respectively. However, Example(5.2.17) says that the hypothesis $2|\frac{p_n}{gcd(d_n,p_n)}$ for $n \geq 2$ is not necessary for the Cantor–Moran measure $\mu_{\{p_n\},\{d_n\}}$ being a spectral measure.

We decompose Theorem(5.2.1)into the following Theorem(5.2.3) and Theorem(5.2.5). We let $d_n = 2^{l_n} d'_n, p_n = 2^{l_n} p'_n$ and $P_n = p_1 p_2 \cdots p_n$for $n \geq 1$, where all $l_n \geq 0$ and d'_n are odd, but the integer p'_nmay be even. For $\sigma \in \{-1, 1\}^N$, i.e., $\sigma = \sigma_1 \sigma_2 \cdots$ with all $\sigma_k \in \{-1, 1\}$, we denote $\sigma|_n = \sigma_1 \cdots \sigma_n$. Now we construct a countable set Λ^{σ}in terms of $(\{p_n\},\{l_n\},\{\sigma_n\})$. Let

$$\Lambda_n^{\sigma} = P_1\left\{0, \frac{\sigma_1}{2^{1+l_1}}\right\} + P_2\left\{0, \frac{\sigma_2}{2^{1+l_2}}\right\} + \cdots + P_n\left\{0, \frac{\sigma_n}{2^{1+l_n}}\right\} \quad (30)$$

for $n \geq 1$ and

$$\Lambda^{\sigma} = \cup_{n=1}^{\infty} \Lambda_n^{\sigma} = \sum_{n=1}^{\infty} P_n\left\{0, \frac{\sigma_n}{2^{1+l_n}}\right\} \qquad (31)$$

(all finite sums).

Theorem (5.2.3) [358]: Let $\{d_n, p_n\}_{n=1}^{\infty}$be a sequence of integers with $0 < d_n < p_n$ for all $n \geq 1$. If $2|\frac{p_n}{gcd(d_n,p_n)}$ for all $n \geq 1$ and there exists an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that $\frac{d_{n_k+1}}{p_{n_k+1}} \leq r$ for some $r < 1$, then the following statements hold:

(i) If $r < \frac{1}{2}$, then $\mu_{\{p_n\},\{d_n\}}$ is a spectral measure with spectra Λ^{σ} for each $\sigma \in \{-1, 1\}^{\mathbb{N}}$;

(ii) If $r < 1$, then $\mu_{\{p_n\},\{d_n\}}$ is a spectral measure with a spectrum $\Lambda^{\sigma}$for some $\sigma \in \{-1, 1\}^{\mathbb{N}}$.

Corollary (5.2.4) [358]: Under the conditions of Theorem(5.2.3), then both Λ^{σ} and $\kappa\Lambda^{\sigma}$are spectra of $\mu_{\{p_n\},\{d_n\}}$ for each $\sigma \in \{-1, 1\}^{\mathbb{N}}$ if $\kappa r < \frac{1}{2}$ and for some $\sigma \in \{-1, 1\}^{\mathbb{N}}$if $\kappa r < 1$, where $\kappa > 1$ is an odd integer.

Proof: Check the proof of Theorem(5.2.3). Using $\frac{1}{p_{n_k+1}} \leq r$, then one can give the desired proofs by minor adjustion.

Theorem (5.2.5) [358]: Let $\{d_n, p_n\}_{n=1}^{\infty}$be a sequence of integers with $0 < d_n < p_n$ for all $n \geq 1$. If $2 \mid \frac{p_n}{gcd(d_n,p_n)}$ for all $n \geq 1$ and d_n/p_n tends to 1 as n tends to infinity, then $\mu_{\{p_n\},\{d_n\}}$ is a spectral measure.

Different from Theorem(5.2.3), the following proposition illustrates that all the candidate spectra Λ^{σ} in (31) are not spectra for some Cantor–Moran measures.

Proposition (5.2.6) [358]: Let $\{d_n, p_n\}_{n=1}^{\infty}$be a sequence of integers with $0 < d_n < p_n$for all $n \geq 1$. If all d_n and $p_n - 1$ are odd and $\sum_{n=1}^{\infty} \frac{p_n - d_n}{p_n} < \infty$, then $(\mu_{\{p_n\},\{d_n\}}, \Lambda^{\sigma})$ is not a spectral pair for any $\sigma \in \{-1, 1\}^{\mathbb{N}}$, where $\Lambda^{\sigma} = \sum_{n=1}^{\infty} P_n\{0, \frac{\sigma_n}{2}\}$ (all finite sums).

The is organized as follows., we introduce some basic definitions and properties of spectral measures. The techniques to prove Theorem(5.2.1)are the construction of suitable spectra Λ, they are displayed., we prove Corollary(5.2.2), Proposition(5.2.6), and give some examples.

we give some necessary definitions and lemmas that we need in the proof of the main theorem. Let μ be a probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x).$$

Then we have

$$\hat{\mu}_{\{p_n\},\{d_n\}}(\xi) = \prod_{n=1}^{\infty} M_{d_n}(P_n^{-1}\xi),$$

where the mask polynomial

$$M_{d_n}(\xi) = \frac{1}{2} + \frac{1}{2} e^{-2\pi i d_n \xi} = e^{-\pi i d_n \xi} \cos \pi d_n \xi .$$

Let $Z_h := \{\xi : h(\xi) = 0\}$ be the set of the roots of the function $h(\xi)$. Then

$$Z_{M_{d_n}} = \frac{2\mathbb{Z}+1}{2d_n} \quad \text{and} \quad Z_{\overline{\mu}_{\{p_n\},\{d_n\}}} = \cup_{n=1}^{\infty} P_n \frac{2\mathbb{Z}+1}{2d_n} \quad (32).$$

We say that Λ is an *orthogonal* set of μ if $\mathbb{E}_{\Lambda}$is an orthonormal family for $L^2(\mu)$. It is easy to show that Λ is an orthogonal set of μ if and only if $\hat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i \neq \lambda_j \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq Z_{\overline{\mu}}. \quad (33)$$

Let

$$Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi+\lambda)|^2 .$$

The following theorem is a basic criterion for the spectrality of a measure μ[385].

Theorem (5.2.7) [358]: Let μ be a Borel probability measure with compact support in $\mathbb{R}$, and let $\Lambda \subset \mathbb{R}$ be a countable subset. Then

(i) $\mathbb{E}_\Lambda$ is an orthonormal family of $L^2(\mu)$ if and only if $Q_\Lambda(\xi) \le 1$ for $\xi \in \mathbb{R}$;

(ii) It is an orthonormal basis if and only if $Q_\Lambda(\xi) \equiv 1$for $\xi \in \mathbb{R}$;

(iii) $Q_\Lambda(z)$ is an entire function in the complex plane if Λ is an orthogonal set of μ.

The following lemma will be used in Example(5.2.16) and was proved by [387]. Since the proof is simple, we give it here.

Lemma (5.2.8) [358]: Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures $\mu_i, i = 0, 1$, and they are not Dirac measures. Suppose that Λ is an orthogonal set of μ_0 with $0 \in \Lambda$, then Λ is also an orthogonal set of μ, but cannot be a spectrum of μ.

Proof: Note that μ_i is not a Dirac measure is equivalent to $|\hat{\mu}_i(\xi)| \not\equiv 1$. Since $\hat{\mu}_0(0) = 1$, there exists ξ_0 such that $|\hat{\mu}_0(\xi_0)| \neq 0$ and $|\hat{\mu}_1(\xi_0)| < 1$. Hence by Theorem(5.2.7)(i),

$$Q_\Lambda(\xi_0) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi_0+\lambda)|^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_0(\xi_0+\lambda)|^2 |\hat{\mu}_1(\xi_0+\lambda)|^2 < \sum_{\lambda\in\Lambda} |\hat{\mu}_0(\xi_0+\lambda)|^2 \le 1.$$

The result follows by Theorem(5.2.7)(i) and (ii).

we denote

$$\mu_n = \delta_{P_1^{-1}\{0,d_1\}} * \cdots * \delta_{P_n^{-1}\{0,d_n\}} \quad (34)$$

and

$$\mu > n = \delta_{P_{n+1}^{-1}\{0,d_{n+1}\}} * \delta_{P_{n+2}^{-1}\{0,d_{n+2}\}} \cdots . \quad (35)$$

Then $\mu_{\{p_n\},\{d_n\}} = \mu_n * \mu > n$. More generally, for any $q = q_1 q_2 \cdots q_n \cdots \in (2\mathbb{Z}+1)^{\mathbb{N}}$, i.e., all q_n are odd, we define

$$\Lambda_n^q = P_1\left\{0, \frac{q_1}{2^{1+l_1}}\right\} + P_2\left\{0, \frac{q_2}{2^{1+l_2}}\right\} + \cdots + P_n\left\{0, \frac{q_n}{2^{1+l_n}}\right\}$$

for $n \ge 1$, and

$$\Lambda^q = \bigcup_{n=1}^{\infty} \Lambda_n^q := \sum_{n=1}^{\infty} P_n\left\{0, \frac{q_n}{2^1 + l_n}\right\} \quad (all\ finite sums).$$

Lemma (5.2.9) [358]: Let $\{d_n, p_n\}_{n=1}^{\infty}$ be a sequence of integers with $0 < d_n < p_n$ for all $n \ge 1$. With the same notations as above, the set Λ_n^q is a spectrum of the measure μ_n and Λ^q is an orthogonal set of the measure $\mu_{\{p_n\},\{d_n\}}$ for each $q \in (2\mathbb{Z}+1)^{\mathbb{N}}$.

Proof: It is easy to show that both Λ_n^q and Λ^q are orthogonal set by (33)and (32) for μ_n and $\mu_{\{p_n\},\{d_n\}}$ respectively. Λ_n^q is a spectrum of μ_n because the dimension of $L^2(\mu_n)$ is 2^n, which is the cardinality of the set Λ_n^q.

we consider the case that there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that $\frac{d_{n_k+1}}{p_{n_k+1}} \le r$ for some $r < 1$. We begin with two lemmas for the cases $r < \frac{1}{2}$ and $r < 1$ respectively. Notice that $p_n \ge 2$ for $n \ge 1$, then $P_n = p_1 p_2 \cdots p_n \ge 2^n$, which will be used many times.

Lemma (5.2.10) [358]: Let $r \in (0, \frac{1}{2})$ and let ℓ be an integer so that $1 + 2^{-\ell} < \min\left\{\frac{1}{2r}, \frac{4}{\pi}\right\}$. Suppose that $\{n_k\}_{k=1}^{\infty}$is an increasing subsequence of $\mathbb{N}$ such that $n_k \ge \ell$ and $\frac{d_{n_k+1}}{p_{n_k+1}} \le r < 1/2$ for each $k \ge 1$. Then there exists $\alpha_{r,\ell} > 0$ such that

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \alpha_{r,\ell}$$

for $k > 1, 0 < \xi < 1, \sigma \in \{-1,1\}^{\mathbb{N}}$ and $\lambda \in \Lambda_{n_k}^{\sigma}$.

Proof: By the choice of ℓ we have $\ell > 1$. Given $0 < \xi < 1, \sigma \in \{-1,1\}^{\mathbb{N}}, \lambda \in \Lambda_{n_k}^{\sigma}$and $k > 1$. Recall that

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| = \prod_{i=n_k+1}^{\infty} \left|\cos \pi P_i^{-1} d_i(\xi+\lambda)\right|$$

and

$$\frac{\Lambda_{n_k}^{\sigma}}{P_{n_k}} = \left\{0, \frac{\sigma_{n_k}}{2^{1+l_{n_k}}}\right\} + \frac{1}{p_{n_k}}\left\{0, \frac{\sigma_{n_k-1}}{2^{1+l_{n_k-1}}}\right\} + \cdots + \frac{1}{p_{n_k}p_{n_k-1}\cdots p_2}\left\{0, \frac{\sigma_1}{2^{1+l_1}}\right\}.$$

Now, we focus on estimating the values of $\left|cos\pi P_i^{-1} d_i(\xi+\lambda)\right|$ for $i \geq n_k + 1$. Notice that for $\lambda \in \Lambda_{n_k}^{\sigma}$

$$\frac{|\lambda|}{P_{n_k}} \leq \frac{1}{2} + \frac{1}{2^2} + \cdots = 1 \quad (36)$$

and thus

$$\left|\frac{\xi+\lambda}{P_{n_k}}\right| \leq \frac{1}{P_{n_k}} + 1 \leq \frac{1}{2^{n_k}} + 1 \leq \frac{1}{2^{\ell}} + 1. \quad (37)$$

For $i = n_i + 1$ and $i = n_k + 2$, by the choice of ℓ and (37), we have the following estimations:

$$\left|\frac{d_{n_k+1}(\xi+\lambda)}{P_{n_k+1}}\right| = \left|\frac{d_{n_k+1}}{p_{n_k+1}}\frac{\xi+\lambda}{P_{n_k}}\right| \leq r\left(\frac{1}{2^{\ell}} + 1\right) < \frac{1}{2},$$

and

$$\left|\frac{d_{n_k+2}(\xi+\lambda)}{P_{n_k+2}}\right| = \left|\frac{d_{n_k+2}}{p_{n_k+2}}\frac{\xi+\lambda}{P_{n_k}}\frac{1}{P_{n_k+1}}\right| < \left(\frac{1}{2^{\ell}} + 1\right)\left|\frac{d_{n_k+1}}{P_{n_k+1}}\right| + 1 \leq \left(\frac{1}{2^{\ell}} + 1\right)r < \frac{1}{2}.$$

Then

$$\left|cos\, \pi P_i^{-1} d_i(\xi+\lambda)\right| \geq \cos\left(\frac{1}{2^{\ell}} + 1\right)r\pi > 0.$$

For $i \geq n_k + 3$, denote $P_{n,m} = \prod_{j=n}^{m} p_j$, using the choice of ℓagain we have

$$\pi\left|\frac{d_i(\xi+\lambda)}{P_i}\right| = \pi\left|\frac{d_i}{p_i}\frac{\xi+\lambda}{P_{n_k}}\,\frac{1}{P_{n_k+1,i-1}}\right| \leq \pi\left(\frac{1}{2^{\ell}} + 1\right)\frac{1}{2^{i-n_k-1}} \leq \frac{1}{2^{i-n_k-3}}.$$

Since $\cos x \geq 1 - \frac{1}{2}x^2$, we obtain that

$$\prod_{i=n_k+3}^{\infty}\left|cos\, \pi\, \frac{d_i(\xi+\lambda)}{P_i}\right| \geq \prod_{i=n_k+3}^{\infty} \cos\frac{1}{2^{i-n_k-3}} \geq \prod_{i=n_k+3}^{\infty}\left(1 - \frac{1}{2}\frac{1}{4^{i-n_k-3}}\right)$$
$$= \prod_{i=0}^{\infty}\left(1 - \frac{1}{2}\frac{1}{4^i}\right). \quad (38)$$

Hence,

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| \geq \left(cos\left(1 + \frac{1}{2^{\ell}}\right)r\pi\right)^2 \prod_{i=0}^{\infty}\left(1 - \frac{1}{2}\frac{1}{4^i}\right) := \alpha_{r,\ell} > 0.$$

Lemma (5.2.11) [358]: Let $r \in (0,1)$ and let $\ell > 0$ be an integer so that $\frac{2^{\ell-1}+1}{2^{\ell-1}} < \min\left\{\frac{1}{2r}, \frac{2}{\pi}\right\}$. Suppose that $\{n_k\}_{k=1}^{\infty}$is an increasing subsequence of $\mathbb{N}$ such that $\frac{d_{n_k+1}}{p_{n_k+1}} \leq r < 1$ and $n_{k+1} - n_k \geq \ell$ for each $k \geq 1$. Then there exist $\sigma \in \{-1,1\}^{\mathbb{N}}$ and $\beta_{r,\ell} > 0$ such that

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| \geq \beta_{r,\ell}$$

for $k > 1, 0 < \xi < 1$ and $\lambda \in \Lambda_{n_k}^{\sigma}$.

Proof: By the choice of ℓ we have $\ell > 1$. Choose σ so that $\sigma_i = 1$if $i \in \{n_k : k \geq 1\}$ and $\sigma_i = -1$for otherwise. Given $0 < \xi < 1, \lambda \in \Lambda_{n_k}^{\sigma}$, and $k > 1$. Recall that

$$|\hat{\mu}_{>n_k}(\xi + \lambda)| = \prod_{i=n_k+1}^{\infty} \left| cos\, \pi\, \frac{d_i(\xi + \lambda)}{P_i} \right|$$

$$= \left| \cos \pi \frac{d_{n_k+1}}{p_{n_k+1}} \frac{\xi + \lambda}{P_{n_k}} \right| \prod_{i=n_k+2}^{\infty} \left| cos\, \pi\, \frac{d_i(\xi + \lambda)}{P_i} \right|,$$

and

$$\frac{\Lambda_{n_k}^{\sigma}}{P_{n_k}} = \left\{0, \frac{\sigma_{n_k}}{2^{1+\ell_{n_k}}}\right\} + \frac{1}{p_{n_k}} \left\{0, \frac{\sigma_{n_k-1}}{2^{1+l_{n_k-1}}}\right\} + \frac{1}{p_{n_k} p_{n_k-1}} \left\{0, \frac{\sigma_{n_k-2}}{2^{1+l_{n_k-2}}}\right\} + \cdot\cdot$$

$$\cdot + \frac{1}{p_{n_k} p_{n_k-1} \cdots p_2} \left\{0, \frac{\sigma_1}{2^{1+l_1}}\right\}.$$

Also, we estimate the values of $|cos\pi P_i^{-1} d_i(\xi + \lambda)|$ for $i \geq n_k + 1$. First, we claim that

$\frac{|\lambda|}{P_{n_k}} \leq \frac{2^{\ell-1}}{2^{\ell}-1}$ for any $\lambda \in \Lambda_{n_k}^{\sigma}$ and $k > 1$ (39).

In fact, for $\lambda \in \Lambda_{n_k-1}^{\sigma}$, by the definition of Λ_n^{σ} in (30) we have $\lambda \in P_1 \left\{0, \frac{\sigma_1}{2^{1+l_1}}\right\} + \cdots + P_{n_{k-1}} \left\{0, \frac{\sigma_{n_k-1}}{2^{1+l_{n_k-1}}}\right\}$ and

$$\frac{|\lambda|}{P_{n_k}} \leq \frac{1}{2^{1+l_{n_k-1}} p_{n_k}} + \frac{1}{2^{1+l_{n_k-2}} p_{n_k} p_{n_k-1}} + \cdots + \frac{1}{2^{1+l_1} p_{n_k} p_{n_k-1} \cdots p_2}$$

$$= \frac{1}{p_{n_k}} \left(\frac{1}{2^{1+l_{n_k-1}}} + \frac{1}{2^{1+l_{n_k-2}} p_{n_k-1}} + \cdots + \frac{1}{2^{1+l_1} p_{n_k-1} \cdots p_2} \right)$$

$$\leq \frac{1}{2^{1+\ell_{n_k}}} \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right) = \frac{1}{2^{1+l_{n_k}}} < \frac{2^{\ell-1}}{2^{\ell}-1}.$$

For $\lambda \in \Lambda_{n_k}^{\sigma} \backslash \Lambda_{n_k-1}^{\sigma}$, we have $\lambda \in P_1 \left\{0, \frac{\sigma_1}{2^{1+l_1}}\right\} + \cdots + P_{n_k-1} \left\{0, \frac{\sigma_{n_k-1}}{2^{1+l_{n_k-1}}}\right\} + P_{n_k} \frac{\sigma_{n_k}}{2^{1+l_{n_k}}}$ and

$$0 \leq \lambda / P_{n_k} \leq \frac{1}{2^{1+l_{n_k}}} + \frac{1}{p_{n_k} p_{n_k-1} \cdots p_{n_{k-1}+1} 2^{1+l_{n_k-1}}} + \cdots + \frac{1}{p_{n_k} p_{n_k-1} \cdots p_{n_1+1} 2^{1+l_{n_1}}}$$

$$\leq \frac{1}{2} + \frac{1}{2 p_{n_k} p_{n_k-1} \cdots p_{n_{k-1}+1}} + \cdots + \frac{1}{2 p_{n_k} p_{n_k-1} \cdots p_{n_1+1}}$$

$$\leq \frac{1}{2} \left(1 + \frac{1}{2^{\ell}} + \frac{1}{2^{2\ell}} + \cdots \right) = \frac{2^{\ell-1}}{2^{\ell}-1}.$$

Then the claim follows. Therefore, by the choice of ℓ, we have

$$\left| cos\, \pi \frac{d_{n_k+1}}{p_{n_k+1}} \frac{\xi + \lambda}{P_{n_k}} \right| \geq cos\, \pi r \left(\frac{1}{P_{n_k}} + \frac{2^{\ell-1}}{2^{\ell}-1} \right) \geq cos\, \frac{2^{\ell-1}+1}{2^{\ell}-1} r\pi > 0.$$

For $i \geq n_k + 2$, by the choice of ℓ again, we have

$$\frac{\pi d_i |\xi + \lambda|}{P_i} = \frac{|\xi + \lambda|}{P_{n_k}} \frac{\pi}{p_{n_k+1}} \frac{d_i}{p_i} \frac{1}{P_{n_k+2,i-1}} \leq \frac{2^{\ell-1}+1}{2^{\ell}-1} \frac{\pi}{2} \frac{1}{P_{n_k+2,i-1}} \leq \frac{1}{2^{i-n_k-2}}.$$

Similar to (38), we get

$$\prod_{i=n_k+2}^{\infty} \left| cos\, \pi P_i^{-1} d_i(\xi + \lambda) \right| \geq \prod_{i=n_k+2}^{\infty} cos\, \frac{1}{2^{i-n_k-2}} \geq \prod_{i=0}^{\infty} \left(1 - \frac{1}{2} \frac{1}{4^i} \right) > 0.$$

Hence,

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| \geq \left(cos\ \frac{2^{\ell-1}+1}{2^{\ell}-1} r\pi\right) \prod_{i=0}^{\infty} \left(1-\frac{1}{2}\frac{1}{4^i}\right) := \beta_{r,\ell} > 0.$$

Define

$Q_n(\xi) = \sum_{\lambda\in\Lambda_n^\sigma} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi+\lambda)\right|^2$ and $Q_{\Lambda^\sigma}(\xi) = \sum_{\lambda\in\Lambda^\sigma} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi+\lambda)\right|^2$. (40)

Without loss of generality we assume that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies the conditions given in Lemma(5.2.10)and (5.2.11), that is, $d_{n_k+1}/p_{n_k+1} \leq r$ and $n_{k+1}-n_k \geq \ell$ for $k \geq 1$, where ℓ is given in Lemma(5.2.10)and (5.2.11), respectively. Let $\alpha_r = \min\{\alpha_{r,\ell}, \beta_{r,\ell}\}$ and let Λ^σ be given in Lemma(5.2.10)and (5.2.11) for the corresponding case. For $0 < \xi < 1$and $k > 1$, we have

$$Q_{n_k}(\xi) = Q_{n_{k-1}}(\xi) + \sum_{\lambda\in\Lambda_{n_k}^\sigma\setminus\Lambda_{n_{k-1}}^\sigma} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi+\lambda)\right|^2$$

$$= Q_{n_{k-1}}(\xi) + \sum_{\lambda\in\Lambda_{n_k}^\sigma\setminus\Lambda_{n_{k-1}}^\sigma} \left|\hat{\mu}_{n_k}(\xi+\lambda)\right|^2 \left|\hat{\mu}_{>n_k}(\xi+\lambda)\right|^2$$

$$\geq Q_{n_{k-1}}(\xi) + \alpha_r^2 \sum_{\lambda\in\Lambda_{n_k}^\sigma\setminus\Lambda_{n_{k-1}}^\sigma} \left|\hat{\mu}_{n_k}(\xi+\lambda)\right|^2 (by\ Lemma\ (5.2.10)\ and\ (5.2.11))$$

$$= Q_{n_{k-1}}(\xi) + \alpha_r^2 \left(1 - \sum_{\lambda\in\Lambda_{n_{k-1}}^\sigma} \left|\hat{\mu}_{n_k}(\xi+\lambda)\right|^2\right) (by\ Lemma\ (5.2.9)).$$

If $Q_{\Lambda^\sigma}(\xi) \not\equiv 1$, then there exists $\xi_0, 0 < \xi_0 < 1$, such that $Q_{\Lambda^\sigma}(\xi_0) < 1$. Let η_0 so that $\max\{Q_{\Lambda^\sigma}(\xi_0), e^{-2}\} < \eta_0 < 1$. Without loss of generality we can assume that $n_k - n_{k-1} \geq \max\{\ell, 5 - log_4 \ln \eta_0^{-1/2}\}$ for $k \geq 2$ (otherwise we choose a subsequence of $\{n_k\}_{k=1}^{\infty}$to replace it). Now we prove that

$\sum_{\lambda\in\Lambda_{n_{k-1}}^\sigma} \left|\hat{\mu}_{n_k}(\xi_0+\lambda)\right|^2 \leq \frac{Q_{\Lambda^\sigma}(\xi_0)}{\eta_0} < 1\ forall\ k\ > 1.$ (41)

Note that, for $\lambda \in \Lambda_{n_{k-1}}^\sigma$and $k > 1$,

$$|\hat{\mu}_{>n_k}(\xi_0+\lambda)| = \prod_{j=n_k+1}^{\infty} \left|cos\ \pi P_j^{-1} d_j(\lambda+\xi_0)\right|.$$

By (36) and (39), for $j > n_k$ we have

$$\pi\ \frac{d_j|\lambda+\xi_0|}{P_j} = \pi\ \frac{|\lambda+\xi_0|}{P_{n_{k-1}}} \frac{d_j}{p_j} \frac{1}{P_{n_{k-1}+1,j-1}} \leq 2\pi\ \frac{1}{2^{j-1(n_{k-1}+1)+1}} < \frac{1}{2^{j-n_{k-1}-4}}.$$

Hence, according to $\cos x \geq 1 - x^2$ and $ln(1-x) \geq -5x$ for $0 \leq x \leq 4/5$, we have that

$$|\hat{\mu}_{>n_k}(\xi_0+\lambda)| \geq \prod_{j=n_k+1}^{\infty} \left|\cos\frac{1}{2^{j-n_{k-1}-4}}\right| \geq \prod_{j=n_k+1}^{\infty} \left(1-\frac{1}{4^{j-n_{k-1}-4}}\right)$$

$$\geq \exp\left(-5 \sum_{j=n_k+1}^{\infty} \frac{1}{4^{j-n_{k-1}-4}}\right).$$

Then

$$|\hat{\mu}_{>n_k}(\xi_0+\lambda)| \geq \exp\left(-\frac{5}{3}\frac{1}{4^{n_k-n_{k-1}-4}}\right) \geq \exp\left(-\frac{1}{4^{n_k-n_{k-1}-5}}\right) \geq \sqrt{\eta_0}.$$

Recall that $\mu_{\{p_n\},\{d_n\}} = \mu_n * \mu_{>n}$, we have

$$|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi_0+\lambda)|^2 = |\hat{\mu}_{n_k}(\xi_0+\lambda)|^2 |\hat{\mu}_{>n_k}(\xi_0+\lambda)|^2 \geq \eta_0 |\hat{\mu}_{n_k}(\xi_0+\lambda)|^2.$$

Consequently,

$$\sum_{\lambda\in\Lambda^{\sigma}_{n_{k-1}}}|\hat{\mu}_{n_k}(\xi_0+\lambda)|^2\le\frac{1}{\eta_0}\sum_{\lambda\in\Lambda^{\sigma}_{n_{k-1}}}|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi_0+\lambda)|^2\ \le Q_{\Lambda^{\sigma}}(\xi_0)/\eta_0.$$

Hence,

$$\begin{aligned}Q_{n_k}(\xi_0)&\ge Q_{n_{k-1}}(\xi_0)+\alpha_r^2\left(1-\sum_{\lambda\in\Lambda^{\sigma}_{n_{k-1}}}\left|\hat{\mu}_{n_k}(\xi_0+\lambda)\right|^2\right)\\&\ge Q_{n_{k-1}}(\xi_0)+\alpha_r^2\left(1-\eta_0^{-1}Q_{\Lambda^{\sigma}}(\xi_0)\right)\quad(by\ (41)).\end{aligned}$$

Therefore

$$1\ge Q_{\Lambda^{\sigma}}(\xi_0)\ge Q_{n_k}(\xi_0)\ge Q_{n_1}(\xi_0)+(k-1)\alpha_r^2\left(1-\eta_0^{-1}Q_{\Lambda^{\sigma}}(\xi_0)\right)$$

for $k>1$, which is impossible for k large enough. Hence, $Q_{\Lambda^{\sigma}}(\xi)\equiv 1$ and the proof is complete by Theorem(5.2.7).

we consider the case that $0<d_n<p_n$ for $n\ge 1$ and $\frac{d_n}{p_n}$ tends to 1 as n tends to infinity. Then p_n tends to infinity as n to infinity. To prove Theorem(5.2.5), we decompose it into two cases. For each case, we need to construct different orthogonal sets for them being spectra.

Case I. There are finitely many integers n such that $l_n\ge 1$. Then there exists $N>1$ such that $l_n=0$ for $n\ge N$. We choose a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ with $n_{k+1}-n_k\ge 2$ for $k\ge 1$ and $n_1\ge N$. For $n>N$, denote

$$\begin{aligned}\Lambda_n=P_1\left\{0,\frac{c_1}{2^{1+l_1}}\right\}+P_2\left\{0,\frac{c_2}{2^{1+l_2}}\right\}+\cdots+P_{N-1}\left\{0,\frac{c_{N-1}}{2^{1+l_{N-1}}}\right\}\\+P_N\left\{0,\frac{c_N}{2}\right\}+\cdots+P_n\left\{0,\frac{c_n}{2}\right\},\end{aligned}$$

where $c_i=1$ if $i\notin\{n_k:k\ge 1\}$ and

$$c_{n_k}=\begin{cases}\dfrac{p_{n_k+1}}{2}, & if\ 4\nmid p_{n_k+1};\\ \dfrac{p_{n_k+1}}{2}-1, & if\ 4|p_{n_k+1}\ and\ d_{n_k+1}\equiv 1\ mod 4;\\ \dfrac{p_{n_k+1}}{2}+1, & if\ 4|p_{n_k+1}\ and\ d_{n_k+1}\equiv -1\ mod 4.\end{cases}$$

Now we show that

$$\Lambda=\bigcup_{n=N+1}^{\infty}\Lambda_n$$

is a spectrum of $\mu_{\{p_n\},\{d_n\}}$. As $n_{k+1}-n_k\ge 2$ and $c_{n_k}\le p_{n_k+1}$ for $k\ge 1$, we have

$$\begin{aligned}\max\frac{\Lambda_{n_k-1}}{P_{n_k}}&\le\frac{c_{n_k-1}}{2p_{n_k}}+\frac{c_{n_k-2}}{2p_{n_k}p_{n_k-1}}+\frac{c_{n_k-3}}{2p_{n_k}p_{n_k-1}p_{n_k-2}}+\cdots\\&=\frac{1}{2p_{n_k}}+\frac{c_{n_k-2}}{2p_{n_k}p_{n_k-1}}+\frac{c_{n_k-3}}{2p_{n_k}p_{n_k-1}p_{n_k-2}}+\cdots\\&\le\frac{1}{2p_{n_k}}+\frac{1}{2p_{n_k}}+\frac{1}{2p_{n_k}p_{n_k-1}}+\frac{1}{2p_{n_k}p_{n_k-1}p_{n_k-2}}+\cdots\\&\le\frac{2}{p_{n_k}}\qquad(42)\end{aligned}$$

and

$$\max\frac{\Lambda_{n_k}}{P_{n_k}}\le\frac{c_{n_k}}{2p_{n_k+1}}+\frac{c_{n_k-1}}{2p_{n_k+1}p_{n_k}}+\frac{c_{n_k-2}}{2p_{n_k+1}p_{n_k}p_{n_k-1}}+\cdots+\frac{c_1}{2p_{n_k+1}p_{n_k}p_{n_k-1}\cdots p_2}$$

$$\leq \frac{\frac{p_{n_k+1}}{2}}{2p_{n_k+1}} + \frac{1}{2p_{n_k+1}} + \frac{1}{2p_{n_k+1}p_{n_k}} + \cdots$$
$$\leq \frac{1}{4} + \frac{1}{2p_{n_k+1}} + \frac{1}{p_{n_k+1}} < \frac{1}{4} + \frac{2}{p_{n_k+1}}. \quad (43)$$

Lemma (5.2.12) [358]: Under the conditions of Theorem(5.2.5), if Case I holds, then there exist two positive numbers $\alpha > 0$ and $M \geq N$ (N is given in Case I) such that, for $k \geq M$ and $0 < \xi < 1$, we have

$$\left|\hat{\mu}_{>n_k}(\xi + \lambda)\right| \geq \alpha \; for \; \lambda \in \Lambda_{n_k}. \quad (44)$$

Proof: Let N be the number given in Case I. By the assumptions of Theorem(5.2.5)we can assume $p_n \geq 32$ for $n \geq N$ for the following estimation. For $\lambda \in \Lambda_{n_k}, k \geq N$, and $0 < \xi < 1$, by (43)we have

$$\frac{|\lambda + \xi|}{P_{n_k+1}} \leq \frac{1}{4} + \frac{2}{p_{n_k+1}} + \frac{1}{p_{n_k+1}p_{n_k}} < \frac{3}{8}.$$

Then, when $j = n_k + 2$,

$$\frac{d_{n_k+2}|\lambda + \xi|}{P_{n_k+2}} = \frac{|\lambda + \xi|}{P_{n_k+1}} \frac{d_{n_k+2}}{p_{n_k+2}} \leq \frac{3}{8}$$

and thus

$$\left| cos \; \frac{d_{n_k+2}(\lambda + \xi)}{P_{n_k+2}} \pi \right| \geq \cos \frac{3}{8}\pi.$$

When $j > n_k + 2$, we have

$$\pi \; \frac{d_j|\lambda + \xi|}{P_j} = \pi \; \frac{|\lambda + \xi|}{P_{n_k+1}} \frac{d_j}{p_j} \frac{1}{P_{n_k+2,j-1}} \leq \frac{\pi}{32^{j-1-(n_k+2)+1}} < \frac{1}{2^{j-n_k-3}}.$$

Similar to (38) we obtain

$$\prod_{j=n_k+3}^{\infty} \left| cos \; \pi \; \frac{d_j(\lambda + \xi)}{P_j} \right| \geq \prod_{j=n_k+3}^{\infty} cos \; \frac{1}{2^{j-n_k-3}} \geq \prod_{j=0}^{\infty} \left(1 - \frac{1}{2}\frac{1}{4^j}\right) > 0.$$

The difficult case is that $j = n_k + 1$, we discuss it by two subcases. Firstly, when $j = n_k + 1$ and $\lambda \in \Lambda_{n_k} \backslash \Lambda_{n_k-1}$, we have

$$\frac{d_{n_k+1}(\lambda+\xi)}{P_{n_k+1}} = \frac{d_{n_k+1}}{p_{n_k+1}} \left(\frac{c_{n_k}}{2} + \frac{\lambda'}{P_{n_k}}\right) + \frac{d_{n_k+1}\xi}{P_{n_k+1}} \quad (45)$$

for some $\lambda' \in \Lambda_{n_k-1}$. By the choices of c_{n_k} we have that

$$\frac{d_{n_k+1}}{2p_{n_k+1}} c_{n_k} = \begin{cases} \frac{d_{n_k+1}}{4}, if \; 4 \nmid p_{n_k+1} \; and \; c_{n_k} = \frac{p_{n_k+1}}{2}; \\ \frac{d_{n_k+1}}{4} - \frac{d_{n_k+1}}{2p_{n_k+1}}, if \; 4 | p_{n_k+1} and \; d_{n_k+1} \equiv 1 \; mod 4 \\ \qquad and \; c_{n_k} = \frac{p_{n_k+1}}{2} - 1; \\ \frac{d_{n_k+1}}{4} + \frac{d_{n_k+1}}{2p_{n_k+1}}, if \; 4 | p_{n_k+1} \; and \; d_{n_k+1} \equiv -1 \; mod 4 \\ \qquad and \; c_{n_k} = \frac{p_{n_k+1}}{2} + 1. \end{cases}$$

Choose M, so that $M \geq N$ and $\frac{3}{4} < \frac{d_{n_k+1}}{p_{n_k+1}} < 1$ for $k \geq M$. Thus, if $k \geq M$, it is easy to see that

$$\frac{d_{n_k+1}}{2p_{n_k+1}}c_{n_k} \in \begin{cases} \mathbb{Z}+\left\{\pm\frac{1}{4}\right\}, & if\ 4 \nmid p_{n_k+1}\ and\ c_{n_k}=\frac{p_{n_k+1}}{2}; \\ \mathbb{Z}+\frac{1}{4}-\frac{d_{n_k+1}}{2p_{n_k+1}} \subseteq \mathbb{Z}+\left(-\frac{1}{4},-\frac{1}{8}\right), if\ 4|p_{n_k+1} and\ d_{n_k+1}\equiv 1\ mod 4 \\ \qquad and\ c_{n_k}=\frac{p_{n_k+1}}{2}-1; \\ \mathbb{Z}-\frac{1}{4}+\frac{d_{n_k+1}}{2p_{n_k+1}} \subseteq \mathbb{Z}+\left(\frac{1}{8},\frac{1}{4}\right), if\ 4|p_{n_k+1}\ and\ d_{n_k+1}\equiv -1\ mod 4 \\ \qquad and\ c_{n_k}=\frac{p_{n_k+1}}{2}+1. \end{cases}$$

For convenience and no misunderstanding, we rewrite the above relationships in one formula as follows, that is, there exists an integer m_k such that

$$\frac{d_{n_k+1}}{2p_{n_k+1}}c_{n_k}+m_k \in \left[-\tfrac{1}{4},-\tfrac{1}{8}\right)\cup\left(\tfrac{1}{8},\tfrac{1}{4}\right]. \quad (46)$$

Using (42), we have

$$0<\frac{d_{n_k+1}}{p_{n_k+1}}\frac{\lambda'}{P_{n_k}}+\frac{d_{n_k+1}\xi}{P_{n_k+1}} \le \frac{2}{p_{n_k}}+\frac{1}{p_{n_k}} \le \frac{3}{p_{n_k}}<\frac{1}{8}. \quad (47)$$

Hence, by (45), (46) and (47) we have

$$\frac{d_{n_k+1}(\lambda+\xi)}{P_{n_k+1}}+m_k \in (-\frac{1}{4},0)\cup(\frac{1}{8},\frac{3}{8})$$

and consequently, by the periodicity,

$$\left|cos\ \frac{d_{n_k+1}(\lambda+\xi)}{P_{n_k+1}}\pi\right| \ge cos\frac{3}{8}\pi;$$

Secondly, when $j=n_k+1$ and $\lambda \in \Lambda_{n_k-1}$,

$$\frac{d_{n_k+1}(\lambda+\xi)}{P_{n_k+1}}=\frac{d_{n_k+1}}{p_{n_k+1}}\frac{\lambda}{P_{n_k}}+\frac{d_{n_k+1}\xi}{P_{n_k+1}}.$$

By (47), we have

$$\left|cos\ \frac{d_{n_k+1}(\lambda+\xi)}{P_{n_k+1}}\pi\right| \ge cos\frac{1}{8}\pi.$$

In one word, we have proved that

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \left(cos\ \frac{3}{8}\pi\right)^2\prod_{j=0}^{\infty}\left(1-\frac{1}{2}\frac{1}{4^j}\right):=\alpha>0.$$

Case II. There exist infinitely many integers $n\ge 1$ such that $l_n \ge 1$. We choose an increasing subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that all $l_{n_k}\ge 1$. Denote

$$\Lambda_n=P_1\left\{0,\frac{1}{2^{1+l_1}}\right\}+P_2\left\{0,\frac{1}{2^{1+l_2}}\right\}+\cdots+P_n\left\{0,\frac{1}{2^{1+l_n}}\right\}\ and\ \Lambda=\bigcup_{n=1}^{\infty}\Lambda_n\,.$$

Clearly,

$$\begin{aligned}\max\frac{\Lambda_{n-1}}{P_n}&=\max\left\{\frac{1}{p_n}\left\{0,\frac{1}{2^{1+l_{n-1}}}\right\}+\frac{1}{p_np_{n-1}}\left\{0,\frac{1}{2^{1+l_{n-2}}}\right\}+\cdots+\frac{1}{p_np_{n-1}\cdots p_2}\left\{0,\frac{1}{2^{1+l_1}}\right\}\right\}\\ &\le\frac{1}{2p_n}+\frac{1}{2p_np_{n-1}}+\frac{1}{2p_np_{n-1}p_{n-2}}+\cdots\\ &\le\frac{1}{p_n}. \quad (48)\end{aligned}$$

Lemma (5.2.13) [358]: Under the conditions of Theorem(5.2.5), if Case II holds, then there exist two positive numbers $\alpha > 0$ and $M \geq 1$ such that, for $k \geq M$ and $0 < \xi < 1$, we have

$$|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq \alpha, for\ \lambda \in \Lambda_{n_k}.$$

Proof: By the assumptions of Theorem(5.2.5) again we can choose M so that $p_n \geq 32$ for $n \geq M$. Let $0 < \xi < 1$. For $k \geq M, j \geq n_k + 2$ and $\lambda \in \Lambda_{n_k}$, by (48) we have

$$\pi \frac{d_j|\lambda + \xi|}{P_j} = \pi \frac{|\lambda + \xi|}{P_{n_k+1}} \frac{d_j}{p_j} \frac{1}{P_{n_k+2,j-1}} \leq \frac{2\pi}{p_{n_k+1}} \frac{1}{2^{j-1-(n_k+2)+1}} < \frac{1}{2^{j-2-n_k}}.$$

Similar to (38) we obtain

$$\prod_{j=n_k+2}^{\infty} \left|cos\ \pi P_j^{-1} d_j(\xi + \lambda)\right| \geq \prod_{j=n_k+2}^{\infty} cos\ \frac{1}{2^{j-n_k-2}} \geq \prod_{j=0}^{\infty} \left(1 - \frac{1}{2}\frac{1}{4^j}\right) > 0.$$

When $j = n_k + 1$ and $\lambda \in \Lambda_{n_k} \backslash \Lambda_{n_k-1}$, we have $\lambda = P_{n_k} \frac{1}{2^{1+l_{n_k}}} + \lambda'$ for some $\lambda' \in \Lambda_{n_k-1}$ and by (48)

$$0 < \frac{d_{n_k+1}(\lambda + \xi)}{P_{n_k+1}} = \frac{d_{n_k+1}}{p_{n_k+1}} \frac{1}{2^{1+l_{n_k}}} + \frac{d_{n_k+1}}{p_{n_k+1}} \frac{\lambda'}{P_{n_k}} + \frac{d_{n_k+1}\xi}{P_{n_k+1}} \leq \frac{1}{4} + \frac{2}{p_{n_k}} < \frac{3}{8};$$

When $j = n_k + 1$ and $\lambda \in \Lambda_{n_{k-1}}$, we obtain

$$0 \leq \frac{d_{n_k+1}(\lambda + \xi)}{P_{n_k+1}} \leq 2/p_{n_k} < \frac{3}{8}.$$

Hence,

$$\cos \frac{d_{n_k+1}(\lambda + \xi)}{P_{n_k+1}} \geq \cos \frac{3}{8}\pi.$$

Consequently

$$|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq \left(cos \frac{3}{8}\pi\right) \prod_{j=0}^{\infty} \left(1 - \frac{1}{2}\frac{1}{4^j}\right) := \alpha > 0.$$

Recall

$$Q_n(\xi) = \sum_{\lambda \in \Lambda_n} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi + \lambda)\right|^2 \ and\ Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi + \lambda)\right|^2.$$

Lemma (5.2.14) [358]: Suppose that $Q_\Lambda(\xi_0) < 1$ for some $0 < \xi_0 < 1$, where Λ is given in CaseI and II respectively. Let η_0 so that $\max\{Q_\Lambda(\xi_0), e^{-2}\} < \eta_0 < 1$. For any increasing subsequence $\{n_k\}_{k=1}^{\infty}$of $\mathbb{N}$ satisfies the conditions of Lemma(5.2.12) (Lemma(5.2.13), resp.) and $n_{k+1} - n_k \geq 4 - \log_4 \ln \eta_0^{-1/2} \geq 4$ for $k \geq 1$, we have

$$|\hat{\mu}_{>n_k}(\xi_0 + \lambda)| \geq \sqrt{\eta_0}\ \ for\ \lambda \in \Lambda_{n_{k-1}}\ and\ k > M,$$

where M is given in Lemma(5.2.12)(Lemma(5.2.13), resp.).

Proof: Recall that

$$|\hat{\mu}_{>n_k}(\xi_0 + \lambda)| = \prod_{j=n_k+1}^{\infty} \left|cos\ \pi P_j^{-1} d_j(\lambda + \xi_0)\right|.$$

For $\lambda \in \Lambda_{n_{k-1}}$ and $k > M$, by (43) for Case I and (48)for Case II we have

$$\frac{|\lambda + \xi_0|}{P_{n_{k-1}+1}} \leq \frac{1}{2}$$

and thus for $j > n_k$

$$\pi\frac{d_j|\lambda+\xi_0|}{P_j}=\pi\frac{|\lambda+\xi_0|}{P_{n_{k-1}+1}}\frac{d_j}{p_j}\frac{1}{P_{n_{k-1}+2,j-1}}\le\frac{\pi}{2}\frac{1}{2^{j-1-(n_{k-1}+2)+1}}<\frac{1}{2^{j-n_{k-1}-3}}.$$

Hence, according to $\cos x\ge 1-x^2$ and $ln(1-x)\ge -5x$ for $0\le x\le 4/5$, we have

$$|\hat{\mu}_{>n_k}(\xi_0+\lambda)|\ge\prod_{j=n_k+1}^{\infty}\left|\cos\frac{1}{2^{j-n_{k-1}-3}}\right|\ge\prod_{j=n_k+1}^{\infty}\left(1-\frac{1}{4^{j-n_{k-1}-3}}\right)$$

$$\ge\exp\left(-5\sum_{j=n_k+1}^{\infty}\frac{1}{4^{j-3-n_{k-1}}}\right)\ge\exp\left(\frac{-1}{4^{n_k-n_{k-1}-4}}\right)\ge\sqrt{\eta_0}.$$

Let Λ, M and α be given in Case I or II and the corresponding lemmas. Then, for $k>M$, we get

$$Q_{n_k}(\xi)=Q_{n_{k-1}}(\xi)+\sum_{\lambda\in\Lambda_{n_k}\setminus\Lambda_{n_{k-1}}}\left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi+\lambda)\right|^2$$

$$=Q_{n_{k-1}}(\xi)+\sum_{\lambda\in\Lambda_{n_k}\setminus\Lambda_{n_{k-1}}}|\hat{\mu}_{n_k}(\xi+\lambda)|^2|\hat{\mu}_{>n_k}(\xi+\lambda)|^2$$

$$\ge Q_{n_{k-1}}(\xi)+\alpha^2\sum_{\lambda\in\Lambda_{n_k}\setminus\Lambda_{n_{k-1}}}|\hat{\mu}_{n_k}(\xi+\lambda)|^2\ \text{(by Lemma (5.2.12) or (5.2.13))}$$

$$=Q_{n_{k-1}}(\xi)+\alpha^2\left(1-\sum_{\lambda\in\Lambda_{n_{k-1}}}|\hat{\mu}_{n_k}(\xi+\lambda)|^2\right)\ \text{(by Lemma (5.2.9))}$$

If $Q_\Lambda(\xi)\not\equiv 1$, there exists $\xi_0, 0<\xi_0<1$, such that $Q_\Lambda(\xi_0)<1$ and thus there exists η_0 such that $\max\{Q_\Lambda(\xi_0), e^{-2}\}<\eta_0<1$. Without loss of generality we assume that the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies the conditions of Lemma (5.2.14)(otherwise we choose a subsequence of $\{n_k\}_{k=1}^{\infty}$ to replace it). Hence, for $\lambda\in\Lambda_{n_{k-1}}$and $k>M$, we have

$$\left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi_0+\lambda)\right|=|\hat{\mu}_{n_k}(\xi_0+\lambda)||\hat{\mu}_{>n_k}(\xi_0+\lambda)|\ge\eta_0^{\frac{1}{2}}|\hat{\mu}_{n_k}(\xi_0+\lambda)|.$$

Therefore

$$Q_{n_k}(\xi_0)\ge Q_{n_{k-1}}(\xi_0)+\alpha^2\left(1-\sum_{\lambda\in\Lambda_{n_{k-1}}}\eta_0^{-1}\left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi_0+\lambda)\right|^2\right)$$

$$\ge Q_{n_{k-1}}(\xi_0)+\alpha^2\left(1-\eta_0^{-1}Q_\Lambda(\xi_0)\right).$$

Using the same argument at the end of proof Theorem(5.2.3), we complete the proof.

For a Borel measure μ and a constant number a, we denote the measure $\nu(E)=\mu(aE)$for any Borel set E in $\mathbb{R}$ as $\nu=\mu\circ a$.

Suppose that $0<d_n<p_n$ for $n\ge N>1$. Recall that $P_{n,n+m}=p_np_{n+1}\cdots p_{n+m}$, and define that.

$$\mu_N=\delta_{P_1^{-1}\{0,d_1\}}*\cdots*\delta_{P_N^{-1}\{0,d_N\}}\ and$$

$$\mu_{N+}=\delta_{P_{N+1,N+1}^{-1}\{0,d_{N+1}\}}*\delta_{P_{N+1,N+2}^{-1}\{0,d_{N+2}\}}\cdots.$$

Then $\mu_{\{p_n\},\{d_n\}}=\mu_N*(\mu_{N+}\circ P_N)$ and

$$\hat{\mu}_{\{p_n\},\{d_n\}}(\xi)=\hat{\mu}_N(\xi)\hat{\mu}_{N+}\left(\frac{\xi}{P_N}\right).$$

Since $2|\frac{p_n}{gcd(d_n,p_n)}$ for $n\ge 1$, it is easy to check directly that (similar to Lemma(5.2.9))

$$\Lambda_N=P_1\left\{0,\frac{1}{2^{1+l_1}}\right\}+P_2\left\{0,\frac{1}{2^{1+l_2}}\right\}+\cdots+P_N\left\{0,\frac{1}{2^{1+l_N}}\right\}$$

is a spectrum of the measure μ_N. By Theorem(5.2.1), there exists a spectrum Λ_{N+} of μ_{N+} in $\mathbb{Z}$. Now we show that $\Lambda_N+P_N\Lambda_{N+}$is a spectrum of $\mu_{\{p_n\},\{d_n\}}$ by Theorem(5.2.7).

Since

$$\hat{\mu}_N(\xi) = \prod_{k=1}^{N} M_{d_k}\left(\frac{\xi}{P_k}\right)$$

and P_N is a period of $\hat{\mu}_N(\xi)$, we have that

$$\sum_{\lambda\in\Lambda_N+P_N\Lambda_{N+}} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi+\lambda)\right|^2 = \sum_{\lambda_1\in\Lambda_N,\lambda_2\in\Lambda_{N+}} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi+\lambda_1+P_N\lambda_2)\right|^2$$

$$= \sum_{\lambda_1\in\Lambda_N,\lambda_2\in\Lambda_{N+}} |\hat{\mu}_N(\xi+\lambda_1+P_N\lambda_2)|^2 \left|\hat{\mu}_{N+}\left(\frac{\xi+\lambda_1+P_N\lambda_2}{P_N}\right)\right|^2$$

$$= \sum_{\lambda_1\in\Lambda_N,\lambda_2\in\Lambda_{N+}} |\hat{\mu}_N(\xi+\lambda_1)|^2 \left|\hat{\mu}_{N+}\left(\frac{\xi+\lambda_1+P_N\lambda_2}{P_N}\right)\right|^2$$

$$= \sum_{\lambda_1\in\Lambda_N} |\hat{\mu}_N(\xi+\lambda_1)|^2 \sum_{\lambda_2\in\Lambda_{N+}} \left|\hat{\mu}_{N+}\left(\frac{\xi+\lambda_1}{P_N}+\lambda_2\right)\right|^2$$

$$= \sum_{\lambda_1\in\Lambda_N} |\hat{\mu}_N(\xi+\lambda_1)|^2 \equiv 1.$$

Hence, we complete the proof.

Corollary (5.2.15) [358]: Let $\{d_n, p_n\}_{n=1}^{\infty}$be a sequence of integers with $0 < d_n < p_n$for all $n \geq 2$. If $2|\frac{p_n}{gcd(d_n,p_n)}$ for all $n \geq 2$, then the Cantor–Moran measure $\mu_{\{p_n\},\{d_n\}}$ is a spectral measure.

Proof: We define that

$\mu_1 = \delta_{p_1^{-1}\{0,d_1\}}\ and\ \mu_2 = \delta_{p_2^{-1}\{0,d_2\}} * \delta_{p_2^{-1}p_3^{-1}\{0,d_3\}} * \cdots$.

Then $\mu_{\{p_n\},\{d_n\}} = \mu_1 * (\mu_2 \circ p_1)$. By Theorem(5.2.1), there exists a spectrum Λ of μ_2 in $\mathbb{Z}$. Define $\Lambda_1 = \left\{0, \frac{p_1}{2d_1}\right\}$ and $\tilde{\Lambda} = \Lambda_1 + p_1\Lambda$. Since $\Lambda_1$is a spectrum of the measure $\delta_{p_1}^{-1}\{0, d_1\}$, similar to the proof of Corollary(5.2.2)we have that $\tilde{\Lambda}$ is a spectrum of $\mu_{\mu_{\{p_n\},\{d_n\}}}$.

By the hypotheses, all d_n and $p_n - 1$ are odd. Hence, in this case

$$\Lambda_n^{\sigma} = P_1\left\{0, \frac{\sigma_1}{2}\right\} + P_2\left\{0, \frac{\sigma_2}{2}\right\} + \cdots + P_n\left\{0, \frac{\sigma_n}{2}\right\}\ and\ \Lambda^{\sigma} = \bigcup_{n=1}^{\infty} \Lambda_n^{\sigma}.$$

With the same notation in (34) and (35) we show that, for $\xi \in (0,1)$ and $\lambda \in \Lambda_n^{\sigma}\backslash\Lambda_{n-1}^{\sigma}, n > 1$,

$$|\hat{\mu}_{>n}(\xi+\lambda)| \leq \pi\left(\frac{p_{n+1}-d_{n+1}}{p_{n+1}} + \frac{2}{p_n}\right) := a_n.$$

Indeed,

$$|\hat{\mu}_{>n}(\xi+\lambda)| = \prod_{i=n+1}^{\infty} \left|cos\ \pi d_i P_i^{-1}(\xi+\lambda)\right| \leq |cos\ \pi d_{n+1}P_{n+1}^{-1}(\xi+\lambda)|$$

$$= \sin\pi\left(d_{n+1}P_{n+1}^{-1}(\xi+\lambda) - \frac{\sigma_n}{2}\right) \leq \pi\left|d_{n+1}P_{n+1}^{-1}(\xi+\lambda) - \frac{\sigma_n}{2}\right|$$

$$\leq \pi\left(\left|\frac{d_{n+1}}{P_{n+1}}\xi\right| + \left|\left(\frac{d_{n+1}}{p_{n+1}} - 1\right)\frac{\lambda}{P_n} + \frac{\lambda}{P_n} - \frac{\sigma_n}{2}\right|\right)$$

$$\leq \pi\left(P_n^{-1} + \frac{p_{n+1}-d_{n+1}}{p_{n+1}}\left|\frac{\lambda}{P_n}\right| + \left|\frac{\lambda}{P_n} - \frac{\sigma_n}{2}\right|\right).$$

For $\lambda \in \Lambda_n^\sigma \backslash \Lambda_{n-1}^\sigma$, we have

$$\frac{\lambda}{P_n} \in \frac{\sigma_n}{2} + \frac{1}{p_n}\left\{0, \frac{\sigma_{n-1}}{2}\right\} + \cdots + \frac{1}{p_n p_{n-1} \cdots p_2}\left\{0, \frac{\sigma_1}{2}\right\}.$$

Then

$$\left|\frac{\lambda}{P_n}\right| \le 1 \; and \; \left|\frac{\lambda}{P_n} - \frac{\sigma_n}{2}\right| \le \frac{1}{p_n}.$$

Consequently, we have

$$|\hat{\mu}_{>n}(\xi + \lambda)| \le \pi\left(\frac{p_{n+1} - d_{n+1}}{p_{n+1}} + \frac{2}{p_n}\right) = a_n.$$

With the same notation in(40), we have

$$\begin{aligned} Q_n(\xi) &= Q_{n-1}(\xi) + \sum_{\lambda \in \Lambda_n^\sigma \backslash \Lambda_{n-1}^\sigma} \left|\hat{\mu}_{\{p_n\},\{d_n\}}(\xi + \lambda)\right|^2 \\ &= Q_{n-1}(\xi) + \sum_{\lambda \in \Lambda_n^\sigma \backslash \Lambda_{n-1}^\sigma} |\hat{\mu}_n(\xi + \lambda)|^2 |\hat{\mu}_{>n}(\xi + \lambda)|^2 \\ &\le Q_{n-1}(\xi) + a_n^2 \sum_{\lambda \in \Lambda_n^\sigma \backslash \Lambda_{n-1}^\sigma} |\hat{\mu}_n(\xi + \lambda)|^2 \\ &= Q_{n-1}(\xi) + a_n^2\left(1 - \textstyle\sum_{\lambda \in \Lambda_{n-1}^\sigma} |\hat{\mu}_n(\xi + \lambda)|^2\right) \quad \text{(by Lemma (5.2.9))} \\ &\le Q_{n-1}(\xi) + a_n^2\left(1 - Q_{n-1}(\xi)\right). \end{aligned}$$

Since $\sum \frac{p_n - d_n}{p_n} < \infty$ and $a_n = \pi\left(\frac{p_{n+1} - d_{n+1}}{p_{n+1}} + \frac{2}{p_n}\right) \le \pi\left(\frac{p_{n+1} - d_{n+1}}{p_{n+1}} + \frac{2(p_n - d_n)}{p_n}\right)$, there exists $N > 1$ such that $a_n < 1$for $n > N$. Then

$$1 - Q_n(\xi) \ge \left(1 - Q_{n-1}(\xi)\right)(1 - a_n^2) \ge (1 - Q_N(\xi)) \prod_{k=N+1}^{n} (1 - a_k^2).$$

Let n tend to infinity we obtain that

$$1 - Q_{\Lambda^\sigma}(\xi) \ge (1 - Q_N(\xi)) \prod_{k=N+1}^{\infty} (1 - a_k^2).$$

By hypothesis and Theorem(5.2.7)we have that $Q_N(\xi) \le 1, Q_N(\xi) \not\equiv 1$ and $0 < \sum_{k=N+1}^{\infty} a_k^2 \le \sum_{k=N+1}^{\infty} a_k < \infty$, which implies that $Q_{\Lambda^\sigma}(\xi) < 1$ for some ξ and the assertion follows by Theorem(5.2.7).

The following example indicates that the hypothesis $2|\frac{p_n}{gcd(d_n, p_n)}$ for $n \ge 2$ is essential.

Example (5.2.16) [358]: Let $(d_1, p_1) = (1, 4), (d_2, p_2) = (a, 7)$ and $(d_n, p_n) = (1, 4)$ for $n \ge 3$. Then the measure $\mu_{\{p_n\},\{d_n\}}$ is not a spectral measure if $a \in \{1, 3, 5\}$.

Proof: Recall in this case

$\mu_{\{p_n\},\{d_n\}} = \delta_{4^{-1}\{0,1\}} * \delta_{28^{-1}\{0,a\}} * \delta_{4^{-2}7^{-1}\{0,1\}} * \delta_{4^{-3}7^{-1}\{0,1\}} * \cdots$.

It is easy to calculate that

$$\begin{aligned} \hat{\mu}_{\{p_n\},\{d_n\}}(\xi) &= M_{\{0,1\}}\left(\frac{\xi}{4}\right) M_{\{0,a\}}\left(\frac{\xi}{28}\right) \prod_{k=2}^{\infty} M_{\{0,1\}}\left(\frac{\xi}{7 \cdot 4^k}\right) \\ &= M_{\{0,1\}}\left(\frac{\xi}{4}\right) M_{\{0,a\}}\left(\frac{\xi}{28}\right) \hat{\mu}_{4,\{0,1\}}\left(\frac{\xi}{28}\right), \quad (49) \end{aligned}$$

where $M_{\{0,a\}}(\xi) = \frac{1}{2} + \frac{1}{2} e^{-2\pi i a \xi}$ and

$$\mu_{4,\{0,1\}} = \delta_{4^{-1}\{0,1\}} * \delta_{4^{-2}\{0,1\}} * \cdots.$$

Write $\Lambda_1 = 2(2\mathbb{Z}+1) = 4Z_{M_{\{0,1\}}}, \Lambda_2 = \frac{14}{a}(2\mathbb{Z}+1) = 28Z_{M_{\{0,a\}}}$ and $\Lambda_3 = 14 \cup_{j=1}^{\infty} 4^j(2\mathbb{Z}+1) = 28Z_{\overline{\mu}_{4,\{0,1\}}}$. Then the roots of $\hat{\mu}_{\{p_n\},\{d_n\}}$is

$$Z_{\overline{\mu}_{\{p_n\},\{d_n\}}} = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3. \quad (50)$$

Suppose that $\mu_{\{p_n\},\{d_n\}}$ is a spectral measure with a spectrum $\Lambda, 0 \in \Lambda$. Then by (33)

$$(\Lambda - \Lambda)\backslash\{0\} \subset \Lambda_1 \bigcup \Lambda_2 \bigcup \Lambda_3 .$$

We will show that there exist contradictions by two cases.
Note that

$$\Lambda = \bigcup_{k=1}^{3} (\Lambda \cap \Lambda_k) .$$

Then

$$(\Lambda - \Lambda)\backslash\{0\} = \cup_{i,j=1}^{3} \big(\Lambda \cap \Lambda_i - \Lambda \cap \Lambda_j\big)\backslash\{0\} \subset \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 . (51)$$

It is easy to show that

$$(\Lambda \cap \Lambda_3 - \Lambda \cap \Lambda_3)\backslash\{0\} \subset \Lambda_3 . (52)$$

Case I. Suppose that $\Lambda \cap \Lambda_2 = \emptyset$. Clearly

$$(\Lambda - \Lambda)\backslash\{0\} \subset \Lambda_1 \bigcup \Lambda_3 .$$

So Λ is an orthogonal set of the measure ν, which satisfies that $\mu_{\{p_n\},\{d_n\}} = \delta_{28^{-1}\{0,a\}} * \nu$. This yields a contradiction by Lemma(5.2.8);
Case II. Suppose that $\Lambda \cap \Lambda_2 \neq \emptyset$. We claim that

$$(\Lambda - \Lambda)\backslash\{0\} \subset \Lambda_2 \cup \Lambda_3 . \quad (53)$$

According to (51), (52) and $\Lambda_k = -\Lambda_k$ for $k \geq 1$, to prove the claim we need to show that all the following sets: $(\Lambda \cap \Lambda_k - \Lambda \cap \Lambda_2)\backslash\{0\}$ for $k \in \{2,3\}, (\Lambda \cap \Lambda_k - \Lambda \cap \Lambda_1)\backslash\{0\}$ for $k \in \{1,2,3\}$ and $\Lambda \cap \Lambda_1 - 0$ (because $0 \in \Lambda$) belong to $\Lambda_2 \cup \Lambda_3$. That is, we need to check the following three subcases:
Subcase A: $\lambda_1 - \lambda_2 \in \Lambda_2 \cup \Lambda_3$ where $\lambda_1 \in \Lambda \cap \Lambda_k$ for $k = 2,3$ and $\lambda_2 = \frac{14}{a}(2b_2+1) \in \Lambda \cap \Lambda_2$. Let $\lambda_1 = 14 \cdot 4^j(2b_1+1) \in \Lambda_3$ for some $j \geq 1$. Then

$$\lambda_1 - \lambda_2 = 14 \cdot 4^j(2b_1+1) - \frac{14}{a}(2b_2+1) = \frac{14}{a}(2b_3+1) \in \Lambda_2$$

for some $b_3 \in \mathbb{Z}$; Let $\lambda_1 = \frac{14}{a}(2b_1+1) \in \Lambda_2$ and $\lambda_1 \neq \lambda_2$. Then $\lambda_1 - \lambda_2 = \frac{28(b_1-b_2)}{a} \notin \Lambda_1$, that is $\lambda_1 - \lambda_2 \in \Lambda_2 \cup \Lambda_3$ and thus the claim follows in this case;
Subcase B: $\lambda_1 - \lambda_2 \in \Lambda_2 \cup \Lambda_3$ where $\lambda_1 \in \Lambda \cap \Lambda_k$ for $k \in \{1,2,3\}$ and $\lambda_2 = 2(2b_2+1) \in \Lambda \cap \Lambda_1$. By the assumption there exists $\lambda_0 = \frac{14}{a}(2b_0+1)$in $\Lambda \cap \Lambda_2$, since $\lambda_0 - \lambda_2 \in \cup_{k=1}^{3} \Lambda_k$ by (51), and

$$\lambda_0 - \lambda_2 = \frac{14}{a}(2b_0+1) - 2(2b_2+1) = \frac{2}{a}\big(7(2b_0+1) - a(2b_2+1)\big) = \frac{4c}{a} \notin \Lambda_1 (54)$$

for some $c \in \mathbb{Z}$. So there exists $b_3 \in \mathbb{Z}$ such that

$$\frac{14}{a}(2b_0+1) - 2(2b_2+1) = \begin{cases} \frac{14}{a}(2b_3+1), & if\ \lambda_0 - \lambda_2 \in \Lambda_2; \\ 14 \cdot 4^j(2b_3+1), & if\ \lambda_0 - \lambda_2 \in \Lambda_3\ for\ some\ j \geq 1. \end{cases}$$

This implies that $14|\frac{14}{a}(2b_0+1) - 2(2b_2+1)$ and then $7|(2b_2+1)$, so that $\lambda_2 = 2(2b_2+1) \in \Lambda_2$. By Subcase A we have proved that $\lambda_1 - \lambda_2 \in \Lambda_2 \cup \Lambda_3$when $\lambda_1 \in \Lambda_k$ for $k = 2,3$ and $\lambda_2 \in \Lambda_2$. So the remained situation in Subcase B is that $\lambda_1 = 2(2b_1+1) \in \Lambda_1$. In this situation, we have

$$\lambda_1 - \lambda_2 = 2(2b_1 + 1) - 2(2b_2 + 1) = 4(b_1 - b_2) \in \Lambda_2 \cup \Lambda_3$$

and the claim follows;

Subcase $C: \lambda_1 - \lambda_2 \in \Lambda_2 \cup \Lambda_3$ where $\lambda_1 = 2(2b_1 + 1) \in \Lambda \cap \Lambda_1$ and $\lambda_2 = 0$. If we replace λ_2 in (54) by λ_1, then we have $\lambda_1 \in \Lambda_2$.

In one word, the claim follows. The claim guarantees that Λ is an orthogonal set of the measure ν which satisfies that $\mu_{\{p_n\},\{d_n\}} = \delta_{4^{-1}\{0,1\}} * \nu$. This yields a contradiction by Lemma(5.2.8). Hence, the measure $\mu_{\{p_n\},\{d_n\}}$ is not a spectral measure for $a \in \{1, 3, 5\}$.

Surprisedly we have the following example.

Example (5.2.17) [358]: Let $(d_1, p_1) = (1, 4), (d_2, p_2) = (a, 7)$ and $(d_n, p_n) = (1, 4)$ for $n \geq 3$. Then the measure $\mu_{\{p_n\},\{d_n\}}$ is a spectral one for $a \in \{2, 4, 6\}$.

Proof: Denote that

$$\nu_a = \delta_{4^{-1}\{0,1\}} * \delta_{28^{-1}\{0,a\}}.$$

Then

$$\mu_{\{p_n\},\{d_n\}} = \nu_a * (\mu_{4,\{0,1\}} \circ 28).$$

Let

$$\Gamma_1(a) = \begin{cases} \{0, 14(2b_1 + 1), 7(4b_2 + 1), 7(4b_3 + 3)\}, if\ a = 2 or\ 6; \\ \left\{0, 14(2b_1 + 1), \frac{7}{2}(8b_2 + 1), \frac{7}{2}(8b_3 + 5)\right\}, \quad if\ a = 4. \end{cases}$$

And

$$\Gamma_2 = \{0, 2\} + 4\{0, 2\} + 4^2\{0, 2\} \cdots (all\ finite sums).$$

It is easy to check that $\Gamma_1(a)$ is a spectrum of ν_a (similar to Lemma(5.2.9)). And it is well known that $\Gamma_2$is a spectrum of $\mu_{4,\{0,1\}}$(e.g. see [21]). Then similar to the proof of Corollary(5.2.2), we have that $\Gamma_1(a) + 28\Gamma_2$ is a spectrum of $\mu_{\{p_n\},\{d_n\}}$.

The following example indicates that the hypothesis $0 < d_n < p_n$ for n large enough is essential in some sense, which was proved by An et al.[2]. We give a proof here for completion.

Example (5.2.18) [358]: Let $(d_1, p_1) = (a, 2), 0 < a < 7$, and let $(d_n, p_n) = (7, 2)$ for $n \geq 2$. Then the measure $\mu_{\{p_n\},\{d_n\}}$ is not a spectral measure.

Proof: . Observe that

$$\mu_{2,\{0,1\}} = \delta_{\frac{1}{2}\{0,1\}} * \delta_{\frac{1}{2^2}\{0,1\}} * \cdots = L_{[0,1]},$$

where $L_{[0,1]}$ is the Lebesgue measure restricted on $[0, 1]$. Then

$$\mu_{\{p_n\},\{d_n\}} = \delta_{\frac{1}{2}\{0,a\}} * \frac{2}{7} L_{[0,7/2]}$$

$$= \frac{1}{7} L_{[0,a/2]} + \frac{2}{7} L_{[a/2,7/2]} + \frac{1}{7} L_{[7/2,7/2+a/2]}.$$

Then $\mu_{\{p_n\},\{d_n\}}$ is not a spectral measure by Theorem 1.6 in [364].

Section (5.3): Moran Measures with Four-Element Digit Sets

For μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. μ is called a spectral measure if there exists a countable set $\Lambda \subset \mathbb{R}^d$ such that $E(\Lambda) := \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a spectrum of μ and (μ, Λ) is called a spectral pair. If the normalized Lebesgue measure restricting on a Borel set Ω is a spectral measure, then Ω is called a spectral set. The study of spectral measures was first initiated by B.Fuglede in 1974 [408], who conjectured that $\Omega \subset \mathbb{R}^d$ is a spectral set if and only if Ω is a translational tile. The conjecture has been studied by many, e.g., Iosevich,

Jorgensen, Kolountzakis, Laba, Lagarias, Matolcsi, Pedersen, Tao, Wang and many others ([394]), and it had baffled experts for 30 years until Tao [404] constructed the first counterexample, a spectral set which is not a tile on $\mathbb{R}^d, d \geq 5$. The example and technique were refined later to disprove the conjecture in both directions on $\mathbb{R}^d$ for $d \geq 3$. It is still open in dimensions $d = 1$ and $d = 2$. Despite the counterexamples, the exact relationship between spectral measures and tiling is still mysterious.

For non-atomic singular measures, a class of spectral measures was first found by Jorgensen and Pedersen (i.e., the $1/q$-Cantor measure where q is even) [409], and Strichartz supplemented their result with a simplified proof [401]. The result was further extended to other singular measures, and brought in a lot of interesting developments in the topic.

Let $\{f_i(x)\}_{i=1}^n$ be an iterated function system (IFS) [409], that is, all $f_i(x)$ are contractive in $\mathbb{R}^d$. Then it determines a unique nonempty compact set T, called an attractor, and a Borel probability measure μ supported on T satisfying

$$T = \bigcup_{i=1}^{n} f_i(T), \qquad \mu(\cdot) = \sum_{i=1}^{n} p_i \mu \circ f_i^{-1}(\cdot),$$

where $\{p_i\}_{i=1}^n$ is a probability weight, that is, $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. Moreover, if the IFS $\{f_i(x)\}_{i=1}^n$ are similarity transformations, then the Borel probability measure μ is called a self-similar measure and T a self-similar set. In the same way the measure μ generated by the IFS $\{f_i(x)\}_{i=1}^n$ of affine transformations is called a self-affine measure.

There is a considerable number of articles on the spectrality of self-similar /selfaffine measures and the construction of their spectrums [4–7, 14, 16, 24, 29, 33]. In particular, Fu, He and Lau [400] studied the spectrality of a self-similar measure $\mu_{4,D}$ with equal weight probability generated by $\{f_d(x) = 4^{-1}(x + d)\}_{d\in D}$ and obtained the following theorem.

Theorem (5.3.1)[396]: Let $D \subset \mathbb{Z}^+$ be a digit set with $\#D = 4$($\#D$ is the cardinality of D) and $gcd(D) = 1$. The following are equivalent

(i) $\mu_{4,D}$ is a spectral measure;

(ii) T is a tile;

(iii) $= \{0, a, 2^t l, a + 2^t l'\}$,where a, t, l, l' are odd integers. In this case $\mu_{4,D}$ is the normalized Lebesgue measure on T.

The digit set D can be expressed as $D = (\{0,1\}\ (mod\ 2) + 2^t\{0,2\})\ (mod\ 2^{2(t+1)})$, a modulo product-form studied in detail in [400] in connection with the tilings.

Definition (5.3.2) [396]: The set $D = \{0, a, 2^t l, a + 2^t l'\}$ with $gcd(D) = 1$ called a 4-digit spectral set, where a, t, l, l' are odd integers.

Let $\{b_k\}_{k=1}^\infty$ be a sequence of integer numbers with all $b_k \geq 2$ and let $\{D_k\}_{k=1}^\infty$ be a sequence of digit sets with $0 \in D_k \subset \mathbb{N}$ for each $k \geq 1$. We call the function system $\{f_{k,d}(x) = b_k^{-1}(x + d): d \in D_k\}_{k=1}^\infty$ a Moran IFS, which is a generalization of an IFS.

If $\sup\{x: x \in b_k^{-1}D_k, k \geq 1\} < \infty$, then associated to the Moran IFS, there exists a Borel probability measure with compact support defined by the convolution

$$\mu_{\{b_k\},\{D_k\}} = \delta_{b_1^{-1}D_1} * \delta_{(b_1 b_2)D_2} * \cdots,$$

where $\delta_{rE} = \frac{1}{\#E}\sum_{a\in E} \delta_{ra}$ and δ_{ra} is the Dirac measure at ra, the sign $*$ means the convolution and the convergence is in the weak sense. In this case $\mu_{\{b_k\},\{D_k\}}$ is called a Moran measure, and its support is the Moran set

$$T(\{b_k\},\{D_k\}) = \left\{\sum_{k=1}^{\infty}(b_1 b_2 \cdots b_k)^{-1} d_k : d_k \in D_k, k \geq 1\right\} := \sum_{k=1}^{\infty}(b_1 b_2 \cdots b_k)^{-1} D_k .$$

To simplify notations, we write $\mu_{b,\{D_k\}} = \mu_{\{b_k\},\{D_k\}}$ if all b_k are equal to b.

In fact, the Moran measure is a non-self-similar extension of the Cantor measure through the infinite convolution. Moran sets and Moran measures appear frequently in dynamic systems, multifractal analysis and geometry number theory (see [402]) etc. Until now, there are only a few results on the spectrality of Moran measures [405].

Obverse that $\delta_{b^{-1}D}$ is a spectral measure if and only if there exists a set C such that

$$N^{-\frac{1}{2}}[e^{-2\pi i b^{-1} dc}]_{d\in D, c\in C}$$

is a unitary matrix, where N is the cardinality of D and also C. In the case that $D, C \subset \mathbb{Z}$, we call $(b^{-1}D, C)$ a compatible pair, or just say (b, D) is admissible for short.

To study the spectrality of the Moran measure $\mu_{\{b_k\},\{D_k\}}$, the natural assumption is that all (b_k, D_k) are admissible. However, this assumption is not sufficient for the Moran measure being a spectral measure (see Example 5.2 [413]). In [403], An, He and Lau proved that the Moran measure $\mu_{b,\{D_k\}} = \delta_{b^{-1}D_1} * \delta_{b^{-2}D_2} * \cdots$ is a spectral measure provided that there is a common $C \subset \mathbb{Z}^+$ such that all the $(b^{-1}D_k, C)$ are compatible pairs and $C + C \subset \{0, 1, 2, \cdots, b-1\}$. Without the condition of common compatible pairs, An, He and Li [402] proved that $\mu_{b,\{D_k\}} = \delta_{b^{-1}D_1} * \delta_{b^{-2}D_2} * \cdots$ is a spectral measure provided that $b = 2^{l+1}q$ is an integer so that $l > L$ if $= 1$, and $l \geq L$ if $q > 1$ is odd where $D_k = \{a_k, b_k\}$ and L is the maximal integer number such that $2^L | (a_k - b_k)$ for some $k \geq 1$.

Motivated by their ideas and results, , we focus on the Moran measure $\mu_{b,\{D_k\}} = \delta_{b^{-1}D_1} * \delta_{b^{-2}D_2} * \cdots$ with $\#D_k = 4$. We always assume that $\{D_k\}_{k=1}^{\infty}$ are uniformly bounded and D_k are 4-digit spectral sets of the form $\{0, a_k, 2^{t_k} l_k, a_k + 2^{t_k} l_k'\}$ and denote $t_{\max} = \max_{k\geq 1} t_k$.

The main result is the following theorem.

Theorem (5.3.3) [396]: Let $b = 2^{l+1}q$ with $q > 1$ an odd integer, and let $\{D_k\}_{k=1}^{\infty}$ be a uniformly bounded sequence of 4-digit spectral sets. Then the Moran measure $\mu_{b,\{D_k\}}$ is a spectral measure if $l \geq t_{max}$.

The main idea of proof is to use Strichartz's technique of finite approximation [32], and is similar to the one used in the proof of Theorem 1.3 in [403]. The crucial part in the finite approximation is to analyze the set of zeros of the Fourier transform $\mu_{b,\{D_k\}}$, and to construct a candidate spectrum (see (64)) for D_k.

The case digit sets with three elements has been studied in [411], and for the prime case, some difficulty on determining the zeros, and one side of the Fuglede problem "self-similar spectral set implies tile" is still not known.

is organized as follows. we introduce preliminary results. In we prove Theorem (5.3.3) and give some examples to illustrate our conclusion.

Let μ be a probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x) .$$

It is clear that a set Λ such that $E(\Lambda)$ is an orthonormal family for $L^2(\mu)$ if and only if

$$(\Lambda - \Lambda)\backslash\{0\} \subset Z(\hat{\mu}),$$

where $Z(f) := \{\xi : f(\xi) = 0\}$ is the set of the roots of f. In this case, we just call the set Λ an orthonormal set of μ for convenience, we will assume that 0 is in the orthonormal set

Λ, and thus $\Lambda \subset \Lambda - \Lambda$. Firstly, we recall the fundamental criterion for spectral measures [409], which is a directed application of Parseval's identity.

Proposition (5.3.4) [396]: Let μ be a Borel probability measure with compact support and let $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi+\lambda)|^2$ for $\Lambda \subset \mathbb{R}$. Then
(i) Λ is an orthonormal set of μ if and only if $Q_\Lambda(\xi) \le 1$ for $\xi \in \mathbb{R}$;
(ii) Λ is a spectrum of μ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}$.
Moreover, if Λ is an orthonormal set, then $Q_\Lambda(z)$ is an entire function.

we give some results that are needed to prove Theorem(5.3.3). The following properties of compatible pairs can be found in [419] or to be checked directly.

Proposition (5.3.5) [396]: Let $D, C \subset \mathbb{Z}$ and let $b \ge 2$ be an integer such that $(b^{-1}D, C)$ is a compatible pair. Then
(i) $b^{-1}C$ is a spectrum of the measure δ_D;
(ii) the elements in D are in different cosets of $\mathbb{Z}/b\mathbb{Z}$;
(iii) $(b^{-1}D + c, C + a)$ for $a, c \in \mathbb{R}$ and $(b^{-1}D, -C)$ are compatible pairs;
(iv) suppose that $\tilde{C} \subset \mathbb{Z}$ such that $C \equiv C(mod\ b)$, then $(b^{-1}D, C)$ is a compatible pair;
(v) if all $(b^{-1}D_k, C_k)$ are compatible pairs, then $(b^{-1}D_1 + \cdots + b^{-k}D_k, C_1 + \cdots + b^{k-1}C_k)$ is a compatible pair for each $k > 1$.

Lemma (5.3.6) [396]: $Z(\hat{\delta}_D)$ is a nonempty set if and only if two of a, b, c are odd and one is even. Without loss of generality, we assume $a < c$ and a, c are odd , b is even. The elements of $Z(\hat{\delta}_D)$ are

$$\frac{2\mathbb{Z}+1}{2gcd(a,c-b)} \bigcup \frac{2\mathbb{Z}+1}{2gcd(c,b-a)} \bigcup \frac{2\mathbb{Z}+1}{2gcd(b,c-a)}.$$

Lemma (5.3.7) [396]: δ_D is a spectral measure if $D = \{0, a, 2^t l, a + 2^t l'\}$ where a, t, l, l' are odd integers. In such case, it admits a spectrum $C = \frac{1}{2^{t+1}}\{0, 1, 2^t, 1 + 2^t\}$.

For any integer $b > 1$ and finite digit set $C \subset \mathbb{Z}$, we define

$$T(b, C) := \{\sum_{k=1}^{\infty} c_k b^{-k} : c_k \in C\} := \sum_{k=1}^{\infty} b^{-k} C. \quad (55)$$

It is a compact set generated by the iterated function system $\{f_c = b^{-1}(x + c)\}_{c\in C}$ [9]. For any $x \in T(b, C)$, x can be expressed (not uniquely in general) as

$$x = \sum_{k=1}^{\infty} c_k b^{-k}, \forall c_k \in C.$$

We say that x has an infinite expansion in $T(b, C)$ if there are infinitely many nonzero c_k, s, and eventually periodic expansion if the sequence $\{c_k\}_{k=1}^{\infty}$ is eventually periodic. In this case, we say a period of $\{c_k\}_{k=l+1}$ is also a period of x. The following lemma can be found in [401].

Lemma (5.3.8) [396]: Let C be a digit set in $\mathbb{Z}$ and let $b > 1$ be an integer. Then $x \in T(b, C)$ is rational if and only if x has an eventually periodic expansion in $T(b, C)$.

Recall that $\{D_k\}_{k=1}^{\infty}$ is a uniformly bounded sequence of 4-digit spectral sets. Then $t_{max} := maxk \ge 1\ t_k < \infty$ and there are only finitely many distinct D_k's. We rearrange these distinct D_k's as $D_{\varepsilon_1}, D_{\varepsilon_2}, \cdots, D_{\varepsilon_N}$, and let $\mathcal{E} = \{D_{\varepsilon_1}, D_{\varepsilon_2}, \cdots, D_{\varepsilon_N}\}$. Clearly,

$$\hat{\mu}_{b,\{D_k\}}(\xi) = \prod_{k=1}^{\infty} M_{D_k}(b^{-k}\xi), \hat{\mu}_{b,D_{\varepsilon_n}}(\xi) = \prod_{k=1}^{\infty} M_{D_{\varepsilon_n}}(b^{-k}\xi), \quad (56)$$

where $M_{D_k}(\xi) = \hat{\delta}_{D_k}(\xi) = \frac{1}{\#D_k}\sum_{a\in D} e^{-2\pi i a\xi}$. Therefore we have $Z(M_{D_k}) \subseteq \bigcup_{n=1}^{\infty} Z(M_{D_{\varepsilon_n}})$ and $Z(\hat{\mu}_{b,D_k}) \subseteq \bigcup_{n=1}^{N} Z(\hat{\mu}_{b,D_{\varepsilon_n}})$. To simplify notations, we write

$$Z_{q\mathcal{E}} := n = 1NZ(M_{qD_{\varepsilon_n}}), \qquad Z_{b,q\mathcal{E}} := n = 1NZ(\hat{\mu}_{b,qD_{\varepsilon_n}}) \quad (57)$$

where $qD_{\varepsilon_n} = \{qx : x \in D_{\varepsilon_n}\}$, it follows from (56) that

$$Z_{b,q\mathcal{E}} = \bigcup_{k=1}^{\infty} b^k Z_{q\mathcal{E}}. \quad (58)$$

Let $l \geq t_{max}$ be fixed, and let $b = 2^{l+1}q$ where $q > 1$ is an odd integer, Lemma (5.3.7) implies that $qC_k = q\{0, 2^{l-t_k}, 2^l, 2^l + 2^{l-t_k}\}$ is an integral spectrum of $\delta_{b^{-1}D_k}$. According to Proposition (5.3.4), Λ is a spectrum of $\mu_{b,\{D_k\}}$ if and only if $\frac{1}{q}\Lambda$ is a spectrum of $\mu_{b,q\{D_k\}}$. In we consider $\mu_{b,q\{D_k\}}$ instead of $\mu_{b,\{D_k\}}$. Since all $1 \leq t_k \leq t_{max} \leq l$, we have

$$\cup_{k=1}^{\infty} C_k \subset \mathcal{G} := \{0, 1, 2, \cdots, 2^l + 2^{l-1}\}. \quad (59)$$

Lemma (5.3.9) [396]: With the above notations, each element in $T(b, \pm\mathcal{G})$ has a unique expansion.

Proof*:* Suppose that x in $T(b, \pm\mathcal{G})$ has two different expansions, then

$$x = \sum_{k=1}^{\infty} c_k b^{-k} = \sum_{k=1}^{\infty} c'_k b^{-k}, c_k, c'_k \in \pm\mathcal{G}.$$

Without loss of generality, we assume that $c_1 \neq c'_1$. Since $q > 1$ and

$$|c_k - c'_k| \leq 2(2^l + 2^{l-1}) \leq b - 2, \forall k \geq 1,$$

then we have

$$|c_1 - c'_1| = \sum_{k=2}^{\infty} |c_k - c'_k| \, b^{-k+1} \leq \sum_{k=1}^{\infty} (b-2)b^{-k} < 1,$$

which contradicts the assumption. Hence, the result follows.

Lemma (5.3.10) [396]: Suppose $x \in Z_{q\mathcal{E}} \cap b^{-1}T(b, \pm\mathcal{G})$, then x has an infinite expansion.

Proof: Suppose that x_0 in $Z_{q\mathcal{E}} \cap b^{-1}T(b, \pm\mathcal{G})$ has finite expansion. Then there is a $D_k = \{0, a_k, 2^{t_k}l_k, a + 2^{t_k}l'_k\}$ such that x_0 in $Z(M_{qD_k}) \cap b^{-1}T(b, \pm\mathcal{G})$ has finite expansion.

To simplify notations, we write $D_k = \{0, a, 2^t l, a + 2^t l'\}$. In this case, let

$$p_1 = gcd(a, a + 2t(l' - l)), p_2 = gcd(a + 2^t l', 2^t l - a) \; and \; p_3 = gcd(l, l').$$

It is easy to see that $p_i \in 2\mathbb{Z} + 1$ and $gcd(p_i, p_j) = 1$ for $i \neq j$. By Lemma (5.3.6), we know that

$$Z(M_{qD_k}) = \frac{2\mathbb{Z}+1}{2p_1 q} \bigcup \frac{2\mathbb{Z}+1}{2p_2 q} \bigcup \frac{2\mathbb{Z}+1}{2^{t+1}p_3 q}.$$

We prove that this assumption is impossible by considering the following three cases:

Case 1. There is an integer u such that

$$\frac{2u+1}{2p_1 q} = \frac{1}{b^i} \sum_{j=1}^{n} \frac{c_j}{b^j} = \frac{1}{b^{i+n}} (c_n + \cdots + b^{n-1}c_1),$$

where $i \geq 1, c_j \in \pm\mathcal{G}$ and $c_1, c_n \neq 0$. The above identity can be rewritten as

$$(2u+1)b^{i+n} = 2p_1 q(c_n + bc_{n-1} + b^2 c_{n-2} + \cdots + b^{n-1}c_1). \quad (60)$$

When c_n is odd , then $1 \leq c_n < 2^l + 2^{l-1}$. The above identity is equivalent to

$$(2u+1)2^{(l+1)(i+n)} q^{i+n-1} = 2p_1(\pm c_n + 2M)$$

for some integer M. It is impossible because the power of 2 on the two sides can not be the same.

When c_n is even, then $1 < c_n \leq 2^l + 2^{l-1}$. (60) is equivalent to

$$(2u+1)2^{(l+1)(i+n)} q^{i+n-1} = 2p_1 2^{i_n}(\pm 1 + 2M)$$

for some integer M, where $1 \leq i_n \leq l$. Similarly, the identity is impossible.

Case 2. There is an integer u such that

$$\frac{2u+1}{2p_2q}=\frac{1}{b_i}\sum_{j=1}^{n}\frac{c_j}{b^j}=\frac{1}{b^{i+n}}\ (c_n+\cdots+b^{n-1}c_1).$$

Similar to **Case 1**, it is impossible.

Case 3. There is an integer u such that

$$\frac{2u+1}{2^{t+1}p_3q}=\frac{1}{b^i}\sum_{j=1}^{n}\frac{c_j}{b^j}=\frac{1}{b^{i+n}}(c_n+\cdots+b^{n-1}c_1),$$

where $i\geq 1, c_j\in\pm\mathcal{G}$ and $c_1,c_n\neq 0$. The above identity can be rewritten as

$$(2u+1)b^{i+n}=2^{t+1}p_3q(c_n+bc_{n-1}+b^2c_{n-2}+\cdots+b^{n-1}c_1). \quad (61)$$

When c_n is odd, then $1\leq c_n<2^l+2^{l-1}$. (61) is equivalent to

$$(2u+1)2^{(l+1)(i+n)}q^{i+n}=p_3q2^{t+1}(\pm c_n+2M)$$

for some integer M. It is impossible because the power of 2 on the two sides can not be the same.

When c_n is even, then $1<c_n\leq 2^l+2^{l-1}$. (61) is equivalent to

$$(2u+1)2^{(l+1)(i+n)}q^{i+n-1}=p_3 2^{t+1}2^{i_n}(\pm 1+2M)$$

for some integer M, where $1\leq i_n\leq l, 1\leq t\leq l$. Similarly, the identity is impossible.

We define

$$\mathcal{G}_{p,b}=\mathcal{G}+b\mathcal{G}+\cdots+b^{p-1}\mathcal{G}-b^p\mathcal{G}-\cdots-b^{2p-1}\mathcal{G}.$$

By (55), we know that

$$T_{p,b}:=T(b^{2p},\mathcal{G}_{p,b})=\sum_{k=1}^{\infty}b^{-2pk}\mathcal{G}_{p,b}\,.$$

It is easy to see that

$$T_{p,b}\subset T(b,\pm\mathcal{G}).$$

Lemma (5.3.11) [396]: There exists p_0 so that $\hat{\mu}_{b,qD_{\varepsilon_n}}(\xi)$ has no roots on $T_{p,b}$ for any $1\leq n\leq N$ and $p\geq p_0$. Consequently, there exists $c,\eta>0$ depending on p such that

$$\prod_{n=1}^{N}\left|\hat{\mu}_{b,qD_{\varepsilon_n}}(\xi)\right|^2\geq c>0,\qquad \xi\in(T_{p,b})_\eta.$$

Where $(E)_\eta=\{y:d(y,E)\leq\eta\}$.

Proof: Note that

$$T_{p,b}=\frac{1}{b^{2pk}}\left(-\frac{\mathcal{G}}{b}-\cdots-\frac{\mathcal{G}}{b^p}+\frac{\mathcal{G}}{b^{p+1}}+\cdots+\frac{\mathcal{G}}{b^{2p}}\right). \quad (62)$$

By Lemma (5.3.9) and Lemma (5.3.10), there exists a unique infinite expansion for each $x\in\Omega:=Z_{q\mathcal{E}}\cap b^{-1}T(b,\pm G)$. Since $Z_{q\mathcal{E}}$ is a subset of rational number (by Lemma(5.3.6)), the expansion of x is eventually periodic by Lemma (5.3.8) and its minimal period P_x is divided by $2p$ according to the equation (62). Since $\prod_{n=1}^{N}\hat{\mu}_{b,qD_{\varepsilon_n}}(\xi)$ is an entire function, there are at most finitely many elements in Ω. Choose p_0 so that $2p_0>\max_{x\in\Omega}P_x$. Next we claim that

$$\Phi:=Z_{q\mathcal{E}}\cap\bigcup_{i=1}^{2p}b^{-i}T_{p,b}=\phi, for\ p\geq p_0. \quad (63)$$

As $b^{-i}T_{p,b}\subseteq b^{-1}T(b,\pm\mathcal{G})$ for $1\leq i\leq 2p$, then $\Phi\subseteq\Omega$. If there exists $x\in\Phi$, we have $P_x\geq 2p\geq 2p_0$. But all P_x are less than $2p_0$ for $x\in\Omega$, which yields a contradiction and thus the claim follows.

Suppose that the first assertion of the lemma is false. Recall that

$$Z_{b,q\mathcal{E}}:=\bigcup_{n=1}^{N} Z(\hat{\mu}_{b,qD_{\varepsilon_n}}) \text{ and } Z_{b,q\mathcal{E}}=\bigcup_{k=1}^{\infty} b^k Z_{q\mathcal{E}}.$$

Then there exist $k \geq 1$ so that $b^k Z_{q\mathcal{E}} \cap T_{p,b} \neq \phi$. Observing that $b^{-2pn}T_{p,b} \subseteq T_{p,b}$ for $n \geq 1$, we have $Z_{q\mathcal{E}} \cap b^{-j}T_{p,b} \neq \phi$ for some $1 \leq j \leq 2p$, which contradicts (63). Hence the first assertion follows. The last inequality is immediate by compactness of $T_{p,b}$.

We define $\widetilde{D}_k = \sum_{i=1}^{2p} b^{2p-i} q D_{i+2p(k-1)}$ for $k \geq 1$ and denote

$$\mu_m = \delta_{b^{-2p}\overline{D}_1} * \delta_{b^{-4p}\overline{D}_2} * \cdots * \delta_{b^{-2pm}\overline{D}_m}.$$

Now we construct a set

$$\Lambda_m(p,b) = \sum_{k=1}^{2pm} (-1)^{\tau(k)} b^{k-1} C_k,$$

where $C_k = \{0, 2^{l-t_k}, 2^l, 2^l + 2^{l-t_k}\}$ is a spectrum of $\delta_{b^{-1}qD_k}$, τ is a periodic function on $\mathbb{Z}$ with period $2p$ and takes values 0 for $1 \leq k \leq p$, and 1 for $p+1 \leq k \leq 2p$. It is not difficult to check that

$$b^{-2pm}\Lambda_m(p,b) \subseteq T_{p,b}.$$

Let

$$\Lambda(p,b) = \cup_{m=1}^{\infty} \Lambda_m(p,b). \quad (64)$$

Lemma (5.3.12) [396]: With the above notations, $\Lambda_m(p,b)$ is a spectrum of the measure μ_m and $\Lambda(p,b)$ is an orthogonal set of $\mu_{b,\{qD_k\}}$.

Proof: Since all $(b^{-1}qD_k, C_k)$ are compatible pairs, so is $(\sum_{k=1}^{2pm} b^{-k} qD_k, \Lambda_m(p,b))$ by Proposition (5.3.5). Hence $\Lambda_m(p,b)$ is a spectrum of the measure μ_m. Since

$$\Lambda_1(p,b) \subseteq \Lambda_2(p,b) \subseteq \cdots \subseteq \Lambda_m(p,b) \subseteq \Lambda_{m+1}(p,b) \subseteq \cdots,$$

and

$$Z(\hat{\mu}_{b,\{qD_k\}}) = \bigcup_{m=1}^{\infty} Z(\hat{\mu}_m).$$

the second assertion follows.

Now, Theorem(5.3.3) is rewritten as the following theorem.

Theorem (5.3.13) [396]: Let $b = 2^{l+1}q$ be an integer with $q > 1$ an odd number, and let $\{D_k\}_{k=1}^{\infty}$ be a uniformly bounded sequence of 4-digit spectral sets. Suppose $l \geq t_{\max}$, then the Moran measure $\mu_{b,\{qD_k\}}$ is a spectral measure with a spectrum $\Lambda(p,b)$ for $p \geq p_0$ where p_0 is given in Lemma (5.3.11).

Proof: By Lemma (5.3.12), $\Lambda_m(p,b)$ is a spectrum of the measure μ_m and $\Lambda(p,b)$ is an orthogonal set of $\mu_{b,\{qD_k\}}$. According to Proposition (5.3.4), we have $Q_m(\xi) = \sum_{\lambda\in\Lambda_m} |\hat{\mu}_m(\xi+\lambda)|^2 \equiv 1 \text{ and } Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} \left|\hat{\mu}_{b,\{qD_k\}}(\xi+\lambda)\right|^2 \leq 1. (65)$

Since Λ_m increases to Λ, and μ_m converges to $\mu_{b,\{qD_k\}}$ weakly, we will use the dominated convergence theorem to justify $Q_\Lambda(\xi) \equiv 1$. Define

$$f_m(\lambda) = \begin{cases} |\hat{\mu}_m(\xi+\lambda)|^2, if\ \lambda \in \Lambda_m; \\ 0 \qquad\qquad , otherwise, \end{cases}$$

and

$$f(\lambda) = \begin{cases} \left|\hat{\mu}_{b,\{qD_k\}}(\xi+\lambda)\right|^2, if\ \lambda \in \Lambda; \\ 0, \qquad\qquad otherwise. \end{cases}$$

Then $\lim_{n\to\infty} f_m(\lambda) = f(\lambda)$ for $\lambda \in \Lambda$. By (65) we have $\sum_{\lambda\in\Lambda}\left|\hat{\mu}_{b,\{qD_k\}}(\xi+\lambda)\right|^2 = \sum_{\lambda\in\Lambda} f(\lambda) \le 1$ for any $\xi \in \mathbb{R}$. Now we construct a dominated function by means of $f(\lambda)$. For any $\lambda \in \Lambda_m$, we have $b^{-2pm}\lambda \in T_{p,b}$ and

$$f(\lambda) = \left|\hat{\mu}_{b,\{qD_k\}}(\xi+\lambda)\right|^2 = \prod_{k=1}^{\infty}\left|M_{qD_k}\left(b^{-k}(\xi+\lambda)\right)\right|^2 = \prod_{k=1}^{\infty}\left|M_{\bar{D}_k}(b^{-2pk}(\xi+\lambda))\right|^2$$

$$= |\hat{\mu}_m(\xi+\lambda)|^2 \prod_{k=1}^{\infty}\left|M_{\bar{D}_{k+m}}(b^{-2pk}(b^{-2pm}\xi + b^{-2pm}\lambda))\right|^2$$

$$= f_m(\lambda)\prod_{k=1}^{\infty}\left|M_{qD_{k+2pm}}(b^{-k}(b^{-2pm}\xi + b^{-2pm}\lambda))\right|^2$$

$$\ge f_m(\lambda)\prod_{n=1}^{N}\left|\hat{\mu}_{b,qD_{\varepsilon_n}}(b^{-2pm}\xi + b^{-2pm}\lambda)\right|^2.$$

When $\xi \in (0,1)$ and $b^{-2pm} < \eta$, where η is given in Lemma (5.3.11), we have $f(\lambda) \ge c f_m(\lambda)$ for $\lambda \in \Lambda_m$. Hence, $c^{-1}f$ is a dominated function and so $Q_\Lambda(\xi) \equiv 1$ for $\xi \in (0,1)$.

The assertion follows by Proposition (5.3.4).

As an illustration of Theorem(5.3.3), we give the following example.

Example (5.3.14) [396]: Let $\{0, a_k, b_k, c_k\}$ be a sequence of digit sets that is uniformly bounded, $a_k = 1, b_k = 2+4m_k, c_k = 3+4m_k$ for $k \ge 1$, and $b = 4q$ for some odd number $q > 1$,m_k be positive integers with bounded M. Then the Moran measure $\mu_{4q,\{0,a_k,b_k,c_k\}}$ is a spectral measure.

Proof*:* In this example, $l = 1$ and $t_{max} = 1$, By Theorem (5.3.3), the assertion follows. Clearly, the condition $q > 1$ in Theorem (5.3.3) is not necessary. The following example will illustrate this fact.

Example (5.3.15) [396]: Let $D_1 = \{0,1,6,7\}$ and $D_k = \{0,1,2,3\}$ for $k \ge 2$. Then the Moran measure $\mu_{4,\{D_k\}}$ is a spectral measure.

Proof: It is easy to check that $(\frac{1}{4}D_1, C)$ is compatible pair, where $C = \{0,1,2,3\}$. Observe that

$$\mu_{4,\{D_k\}}(\cdot) = \delta_{4^{-1}D_1}(\cdot) * \delta_{4^{-2}D_2}(\cdot) * \cdots$$
$$= \delta_{4^{-1}D_1}(\cdot) * \mathcal{L}|_{[0,1]}(4\,\cdot)$$

where $\mathcal{L}|_{[0,1]}$ is the Lebesgue measure restricted on the interval $[0,1]$, then $\mu_{4,\{D_k\}}$ is a spectral measure (see Theorem 5.3 [406]).

Chapter 6

Spectrality of Planar Self-Affine Measures with Certain Moran Measures

We consider the spectrality and non-spectrality of planar self-affine measures with two-element digit set. We give a method to deal with the two-dimensional case, and clarify the spectrality and non-spectrality of a class of planar self-affine measures. The result here provides some supportive evidence to the two related conjectures.

Section (6.1): Two-Element Digit Set

For a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e^{2\pi i\langle \lambda, x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. The set Λ is then called a spectrum for μ. Spectral measure is a generalization of the spectral set introduced by Fuglede [436]. We will study the spectrality and non-spectrality of a class of self-affine measures in the plane $\mathbb{R}^2$. It is known that a measure μ is self-affine if it is a convex combination of a given set of affine transformations applied to the measure itself. We will restrict the discussion on the measure μ which is a unique probability measure $\mu := \mu M, D$ satisfying

$$\mu = \frac{1}{|D|}\sum_{d\in D} \mu \circ \phi_d^{-1} \ , \qquad (1)$$

where the family of transformations $\{\phi_d(x) = M^{-1}(x + d)\}_{d\in D}$ is an affine iterated function system (IFS), $M \in M_n(\mathbb{Z})$ is an expanding integer matrix (that is, all the eigenvalues of the integer matrix M have moduli larger than 1) and $D \subset \mathbb{Z}^n$ is a finite subset of the cardinality $|D|$. The support of $\mu_{M,D}$ is the attractor (or invariant set) $T(M, D)$ of the IFS $\{\phi_d(x)\}_{d\in D}$. After Jorgensen and Pedersen [429], the research on the spectrality and non-spectrality of self-affine measures has been received much attention in recent years (see e.g., [425]). The spectral and nonspectral problems at present focus mainly on the conditions under which $\mu_{M,D}$ is a spectral measure or a non-spectral measure. There are two conjectures (see Conjectures (6.1.1) and (6.1.2) below) related to the problem. One is on the compatible pair condition introduced by Jorgensen and Pedersen [439] (see also [436]). Let G and P be two finite subsets of $\mathbb{R}^n$ of the same cardinality q. We say (G, P) is a compatible pair if the $q \times q$ matrix $H_{G,P} := [q^{-1/2}e^{2\pi i\langle g,p\rangle}]_{g\in G,p\in P}$ is unitary, i.e., $H_{G,P}H_{G,P}^* = I_q$. Here, we use $*$ to denote the transposed conjugate. Dutkay and Jorgensen (see [423, Conjecture 2.5], [425, Conjecture 1.1]) formulated the following.

Conjecture (6.1.1)[432]: Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, and $D \subset \mathbb{Z}^n$ a finite digit set with $0 \in D$. If there exists a subset $S \subset \mathbb{Z}^n, 0 \in S$ such that $(M^{-1}D, S)$ is a compatible pair (or (M, D, S) is a Hadamard triple), then $\mu_{M,D}$ is a spectral measure.

Note that we may always assume that $0 \in D \cap S$. Conjecture (6.1.1) holds in the dimension $n = 1$ [440], in the case when $|D| = |S| = |\det(M)|$ [438] and in the case when $|\det(M)|$ is a prime number [442], and it also holds in higher dimensions with additional conditions [436].

The other conjecture related to the above problem is on the cardinality of the digit set D. We know that the non-spectral problem on the self-affine measure consists of the following two classes:

(I) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them.
(II) There are natural infinite families of orthogonal exponentials, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The main question is whether some of these families can be combined to form larger collections of orthogonal exponentials.
See [424, Section 3]. Let $|det(M)| = m = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ be the standard prime factorization, where $p_1 < p_2 < \cdots < p_r$ are prime numbers and $b_j > 0$. We denote by $W(m)$ the non-negative integer combination of $p_1, p_2, \ldots, p_r$. The known results on the above non-spectral problem provide some supportive evidence that the following conjecture should be true.
Conjecture (6.1.2) [432]: For an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, if $|D| \notin W(m)$, then $\mu_{M,D}$ is a non-spectral measure and the non-spectral problem on this $\mu_{M,D}$ falls in the class (I).
We proved Conjecture (6.1.2) for a class of planar self-affine measures with three-element digit set. For the four-element digit set, we also have the following.
Theorem (6.1.3) [432]: For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and the four-element digit set D given by

$$M = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \text{ and } D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad (2)$$

if $det(M) = ac - bd \notin 2\mathbb{Z}$ and $a = c = 0$ $(or\ b = d = 0)$, then there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 4 is the best.
In the case when $b = d = 0$, Theorem (6.1.3) is contained in [426, Theorem 2]. The proof in [426] is also appropriate to the case $a = c = 0$ (in fact, one only needs to replace M by M^* and change (35) in [426] to $M^{*j}(A_1) \subseteq M^*(A_2)$ and $M^{*j}(A_2) \subseteq M^*(A_1)$).
One of the main purposes of this is to extend Theorem (6.1.3) in the following form.
Theorem (6.1.4) [432]: For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a four-element digit set D given by

$$M = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \text{ and } D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} d_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d_2 \end{pmatrix}, \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \right\}, \quad (3)$$

where $d_1, d_2 \in \mathbb{Z}$ satisfy $d_1 d_2 \neq 0$, the following conclusions hold :
(i) If $det(M) = ac - bd \notin 2\mathbb{Z}$ and $a + c = 0$, then $\mu_{M,D}$ is a non-spectral measure, and there are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$;
(ii) If $det(M) = ac - bd \in 2\mathbb{Z}$, then there are infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$ we mainly consider the two-element digit set. That is, the pair (M, D) given by

$$M = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \text{ and } D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\} (\alpha^2 + \beta^2 \neq 0). \quad (4)$$

For this (M, D), we obtain several useful results relating to the spectrality or non-spectrality of $\mu_{M,D}$. We derive some results on the four-element digit set from that results on the two-element digit set, and vice versa. The above Theorem (6.1.4) is one of such corollaries. It should be pointed out that the digit set with two elements is the simplest case and the most important case in the study of self-affine measures. The one-dimensional case corresponds to the Bernoulli convolution whose spectral property is understandable [436]. The two-dimensional case is not clear. The discussion here is to clarify it although we cannot solve Conjectures (6.1.1) and (6.1.2) in this simple case. The

result here provides some supportive evidence that Conjectures (6.1.1) and (6.1.2) should be true.

We give a method to deal with the two-dimensional case.

The method enables us to consider the simplified M and D at the same time. we determine the spectrality and non-spectrality of self-affine measures in two different cases. One case is completely determined, and the other case is determined under some additional conditions. we clarify certain relations between the two-element digit set and the four-element digit set, from which, Theorem (6.1.4) is proved and a general result is obtained. Finally, we give a concluding remark on a related question.

Since the spectrality and non-spectrality of self-affine measures are invariant under the similarity (see [421, p. 208]), the general method to deal with (4) is to simplify the matrix M and the digit set D simultaneously by using similarity over $\mathbb{R}$. For the expanding integer matrix $M \in M_2(\mathbb{Z})$ and the two element digit set $D \subset \mathbb{Z}^2$ given by (4), we define an integer γ by

$$\gamma := d\alpha^2 - b\beta^2 + (c-a)\alpha\beta, \quad (5)$$

and divide our discussion into the two cases: $\gamma \neq 0$ and $\gamma = 0$.

In the case when $\gamma \neq 0$, we know that the integer matrix B defined by

$$B := \begin{bmatrix} a\alpha + b\beta & \alpha \\ d\alpha + c\beta & \beta \end{bmatrix} \quad (6)$$

is invertible and

$$B^{-1} = -\frac{1}{\gamma}\begin{bmatrix} \beta & -\alpha \\ -(d\alpha + c\beta) & a\alpha + b\beta \end{bmatrix}.$$

Now, consider the pair $(\widetilde{M}, \widetilde{D})$ given by

$$\widetilde{M} := B^{-1}MB \text{ and } \widetilde{D} := B^{-1}D, \quad (7)$$

we have

$$\widetilde{M} = \begin{bmatrix} a+c & 1 \\ bd - ac & 0 \end{bmatrix} \text{ and } \widetilde{D} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}. \quad (8)$$

Note that, if we take the integer matrix B_1 to be of the following form

$$B_1 := \begin{bmatrix} \alpha & a\alpha + b\beta \\ \beta & d\alpha + c\beta \end{bmatrix}, \quad (9)$$

then, the pair $(\widetilde{M}_1, \widetilde{D}_1)$ given by

$$\widetilde{M}_1 := B_1^{-1}MB_1 \text{ and } \widetilde{D}_1 := B_1^{-1}D \quad (10)$$

will be of the following form

$$\widetilde{M}_1 = \begin{bmatrix} 0 & bd - ac \\ 1 & a+c \end{bmatrix} \text{ and } \widetilde{D}_1 = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}. \quad (11)$$

Hence, the transformations

$$B: (M, D) \to (\widetilde{M}, \widetilde{D}) \text{ and } B_1: (M, D) \to (\widetilde{M}_1, \widetilde{D}_1)$$

reduce the expanding integer matrix M and the digit set D to their simple forms. The spectrality and non-spectrality of self-affine measures $\mu_{\widetilde{M},\widetilde{D}}$ and $\mu_{\widetilde{M}_1,\widetilde{D}_1}$ are the same as $\mu_{M,D}$. Here, the matrices $M, \widetilde{M}$ and $\widetilde{M}_1$ are integers and expanding. If we do not require that $\widetilde{M}$ and $\widetilde{M}_1$ are integer matrices, we can further simplify them. For example, let λ_1 and λ_2 be the two eigenvalues of the matrix M, then $\lambda_1 + \lambda_2 = a + c$ and $\lambda_1\lambda_2 = ac - bd$. Define $\tilde{B}$ and $\tilde{B}_1$ by

$$\tilde{B} := \begin{bmatrix} 0 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \text{and } \tilde{B}_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (12)$$

we see that $\widetilde{M}_2 := \tilde{B}^{-1}\widetilde{M}\tilde{B}, \widetilde{D}_2 := \tilde{B}^{-1}\widetilde{D}$ and $\widetilde{M}_3 := \tilde{B}_1^{-1}\widetilde{M}_2\tilde{B}_1, \widetilde{D}_3 := \tilde{B}_1^{-1}\widetilde{D}_2$ are given by

$$\widetilde{M}_2 = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \text{ and } \widetilde{D}_2 = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad (13)$$

$$\widetilde{M}_3 = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_1 \end{bmatrix} \text{ and } \widetilde{D}_3 = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \quad (14)$$

respectively. These simplified pairs $(\widetilde{M}_2, \widetilde{D}_2)$ and $(\widetilde{M}_3, \widetilde{D}_3)$ are easy to handle at least in form. From (8), (11), (13) and (14), we see that under the condition $\gamma \neq 0$, the spectrality or non-spectrality of $\mu_{M,D}$ depends only on $a + c$ and $bd - ac$ $(= -det(M))$ or depends only on $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$. Here λ_1 and λ_2 may be two complex numbers. Even so, in (13) and (14) we must distinguish the integer case and the non-integer case because the compatible pair condition behaves totally different in these two cases (see [422, Example 1]). If $\lambda_1 = \lambda_2$, then the pairs $(\widetilde{M}_2, \widetilde{D}_2)$ and $(\widetilde{M}_3, \widetilde{D}_3)$ are in the integer case, and the spectrality and non-spectrality of their corresponding measures are the same as that of the pair (M, D).

In the case when $\gamma = 0$, we know that the integer matrices B and B_1 defined by (6) and (9) are not invertible, and $det(B) = det(B_1) = 0$. However, one can see that in this case, there exists a constant $l \in \mathbb{Q}$ such that

$$\begin{cases} a\alpha + b\beta = l\alpha \\ d\alpha + c\beta = l\beta \end{cases} \quad or\ M\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = l\begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \quad (15)$$

So, l is an eigenvalue of the matrix M or l is a zero of the characteristic polynomial $det(\lambda I - M)$. Since M is expanding and $det(\lambda I - M)$ is a monic polynomial with integer coefficients, we see that

$$l \in \mathbb{Z}\backslash\{0, \pm 1\}. \quad (16)$$

Since the spectrality and non-spectrality of the two measures $\mu_{M,D}$ and $\mu_{M,\omega D}$ are the same for any non-zero number $\omega \in \mathbb{R}$ (see [11, Proposition 4.2]), without loss of generality, we may assume that $gcd(\alpha, \beta) = 1$. Then, there exist $p, q \in \mathbb{Z}$ such that

$$p\alpha - q\beta = 1 \text{ and } gcd(p, q) = 1. \quad (17)$$

Let

$$Q := \begin{bmatrix} \alpha & q \\ \beta & p \end{bmatrix} \text{ and } Q_1 := \begin{bmatrix} q & \alpha \\ p & \beta \end{bmatrix}. \quad (18)$$

Then

$$det(Q) = 1 \text{ and } Q^{-1} = \begin{bmatrix} p & -q \\ -\beta & \alpha \end{bmatrix};$$

$$det(Q_1) = -1 \text{ and } Q_1^{-1} = \begin{bmatrix} -\beta & \alpha \\ p & -q \end{bmatrix}.$$

Now, consider the pairs $(\widetilde{M}, \widetilde{D})$ and $(\widetilde{M}_1, \widetilde{D}_1)$ given by

$$\widetilde{M} := Q^{-1}MQ \text{ and } \widetilde{D} := Q^{-1}D, \quad (19)$$

$$\widetilde{M}_1 := Q_1^{-1}MQ_1 \text{ and } \widetilde{D}_1 := Q_1^{-1}D, \quad (20)$$

respectively, it follows from (15) and (17) that

$$\widetilde{M} = \begin{bmatrix} l & bp^2 - dq^2 + (a - c)pq \\ 0 & a + c - l \end{bmatrix} \text{ and } \widetilde{D} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad (21)$$

$$\widetilde{M}_1 = \begin{bmatrix} a + c - l & 0 \\ bp^2 - dq^2 + (a - c)pq & l \end{bmatrix} \text{ and } \widetilde{D} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}. \quad (22)$$

Hence, the transformations

$$Q: (M, D) \to (\widetilde{M}, \widetilde{D}) \text{ and } Q_1: (M, D) \to (\widetilde{M}_1, \widetilde{D}_1)$$

reduce the expanding integer matrix M and the digit set D to their simple forms. The spectrality and non-spectrality of self-affine measures $\mu_{\widetilde{M},\widetilde{D}}$ and $\mu_{\widetilde{M}_1,\widetilde{D}_1}$ are the same as $\mu_{M,D}$.

Note that the matrices Q and Q_1 defined by (18) are unimodular, and one can see that the pair $(\widetilde{M}, \widetilde{D})$ and $(\widetilde{M}_1, \widetilde{D}_1)$ defined by (19) and (20) are also in the integer case. Hence,

without the condition $\gamma = 0$ (or without considering the cases $\gamma = 0$ and $\gamma \neq 0$), we may always assume that the digit set D in (4) is of the following form

$$D = \left\{\binom{0}{0}, \binom{0}{1}\right\} \text{ or } D = \left\{\binom{0}{0}, \binom{1}{0}\right\}. \quad (23)$$

Also, we see that if $M \in M_2(\mathbb{Z})$ has two rational eigenvalues, then the eigenvalues of M are integers, and there exists a unimodular matrix $P \in M_2(\mathbb{Z})$ such that $P^{-1}MP$ is a upper(lower)-triangle matrix.

On the other hand, the following two results illustrate that, in the study of spectrality or non-spectrality of self-affine measure $\mu_{M,D}$, we can always reduce the discussion to the case when $D \not\subset M(\mathbb{Z}^n)$ and $0 \in D$.

Proposition (6.1.5) [432]: Let $M \in M_n(\mathbb{Z})$ be an expanding matrix, and let $D \subset \mathbb{Z}^n$ be a finite digit set. Then there exist a non-negative integer $r \in \mathbb{N}_0$ and a finite digit set $D_0 \subset \mathbb{Z}^n$ such that

$$D = M^r D_0 \text{ and } D_0 \not\subset M(\mathbb{Z}^n). \quad (24)$$

Proof: In the case when $D \not\subset M(\mathbb{Z}^n)$, we know that the desired conclusion holds if we let $r = 0$ and $D_0 = D$. On the contrary, in view of the fact that

$$M^{\sigma}(\mathbb{Z}^n) \subseteq M^{\sigma-1}(\mathbb{Z}^n)(\sigma = 1, 2, \dots) \text{and } \bigcap_{\sigma=1}^{\infty} M^{\sigma}(\mathbb{Z}^n) = \{0\}, (25)$$

see [420, Lemma 1], we obtain that if $D \subset M(\mathbb{Z}^n)$, then there exists a positive integer $r \in \mathbb{N}$ such that

$$D \subset M^r(\mathbb{Z}^n) \text{and } D \not\subset M^{r+1}(\mathbb{Z}^n). \quad (26)$$

In this case, by letting $D_0 = M^{-r}D$, we get the desired conclusion.

Proposition (6.1.6) [432]: With the same notation as Proposition (6.1.5), the spectrality or non-spectrality of $\mu_{M,D}$ is the same as μ_{M,D_0}.

Proof: For a given pair (M, D), the spectrality or non-spectrality of $\mu_{M,D}$ is directly connected with the Fourier transform $\hat{\mu}_{M,D}(\xi)$. From (1), we have

$$\hat{\mu}_{M,D}(\xi) := \int e^{2\pi i\langle x,\xi\rangle} d\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D\left(M^{*-j}\xi\right), \xi \in \mathbb{R}^n, (27)$$

where

$$m_D(x) = \frac{1}{|D|} \sum_{d\in D} e^{2\pi i\langle d,x\rangle}, x \in \mathbb{R}^n. (28)$$

Combined with (24), we further have

$$\hat{\mu}_{M,D}(\xi) = \hat{\mu}_{M,D_0}(M^{*r}\xi), \quad \xi \in \mathbb{R}^n, \quad (29)$$

which shows that Λ is a spectrum for $\mu_{M,D} \Leftrightarrow M^{*r}\Lambda$ is a spectrum for μ_{M,D_0}. Hence, the spectrality or non-spectrality of $\mu_{M,D}$ is the same as μ_{M,D_0}.

As a direct application, we obtain that for any expanding matrix $M \in M_n(\mathbb{Z})$ with $|det(M)| = 2$, and for any two-element digit set $D \subset \mathbb{Z}^n, \mu_{M,D}$ is always a spectral measure.

We follow to discuss the spectrality and non-spectrality of self-affine measures. In the case when $\gamma = 0$, the spectrality and non-spectrality depend upon the parity of the number l which can be stated as follows.

Theorem (6.1.7) [432]: For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a two-element digit set $D \subset \mathbb{Z}^n$ given by (4), define an integer γ by (5). The case $\gamma = 0$ is equivalent to the fact that $(\alpha, \beta)^t$ is an eigenvector of M with certain eigenvalue $l \in \mathbb{Z}\backslash\{0, \pm 1\}$. If l is odd, then $\mu_{M,D}$ is a non-spectral measure, and there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$; If l is even, then $det(M) = ac - bd \in 2\mathbb{Z}\backslash\{0\}$, and $\mu_{M,D}$ is a spectral measure.

Proof: We consider the pair $(\widetilde{M}, \widetilde{D})$ given by (21), and let $\tilde{\Theta}_0 := \{\xi \in \mathbb{R}^2 : m_{\tilde{D}}(\xi) = 0\}$. Then

$$\tilde{\Theta}_0 = \left\{ \begin{pmatrix} k + 1/2 \\ \rho \end{pmatrix} : k \in \mathbb{Z}, \rho \in \mathbb{R} \right\} \subset \mathbb{R}^2. \quad (30)$$

In the case when l is odd, we see that

$$\widetilde{M}^*\left(\tilde{\Theta}_0\right) \subseteq \tilde{\Theta}_0, \quad (31)$$

which shows that the zero set $Z(\hat{\mu}_{\widetilde{M},\widetilde{D}})$ of the Fourier transform $\hat{\mu}_{\widetilde{M},\widetilde{D}}(\xi)$ satisfies

$$Z\left(\hat{\mu}_{\widetilde{M},\widetilde{D}}\right) = \cup_{j=1}^{\infty} \widetilde{M}^{*j}\left(\tilde{\Theta}_0\right) = \widetilde{M}^*\left(\tilde{\Theta}_0\right) \subseteq \tilde{\Theta}_0. \quad (32)$$

If $\tau_j (j = 1, 2, 3) \in \mathbb{R}^2$ are such that the exponential functions

$$e^{2\pi i\langle \tau_1, x\rangle}, \qquad e^{2\pi i\langle \tau_2, x\rangle}, \qquad e^{2\pi i\langle \tau_3, x\rangle}$$

are mutually orthogonal in $L^2(\mu_{\widetilde{M},\widetilde{D}})$, then the differences

$$\tau_2 - \tau_1, \qquad \tau_3 - \tau_1, \qquad \tau_3 - \tau_2$$

are in the zero set $Z(\hat{\mu}_{\widetilde{M},\widetilde{D}})$ (see [424, p. 3140]). It follows from (30) and (32) that there are $k_1, k_2, k_3 \in \mathbb{Z}$ and $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$ such that

$$\tau_2 - \tau_1 = \begin{pmatrix} k_1 + \frac{1}{2} \\ \rho_1 \end{pmatrix}, \tau_3 - \tau_1 = \begin{pmatrix} k_2 + \frac{1}{2} \\ \rho_2 \end{pmatrix}, \tau_3 - \tau_2 = \begin{pmatrix} k_3 + \frac{1}{2} \\ \rho_3 \end{pmatrix}. \quad (33)$$

This shows that

$$(\tau_3 - \tau_1) - (\tau_2 - \tau_1) = \begin{pmatrix} k_2 - k_1 \\ \rho_2 - \rho_1 \end{pmatrix} \neq \begin{pmatrix} k_3 + 1/2 \\ \rho_3 \end{pmatrix} = \tau_3 - \tau_2,$$

a contradiction. Hence, there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{\widetilde{M},\widetilde{D}})$, and $\mu_{\widetilde{M},\widetilde{D}}$ is a non-spectral measure. The same conclusion also applies to the pair (M, D). This proves the first part of Theorem (6.1.7).

In the case when l is even, we still consider the pair $(\widetilde{M}, \widetilde{D})$ given by (21). First, for the upper-triangle matrix $\widetilde{M}$, we have

$$\widetilde{M}^{*-j} = \begin{bmatrix} l^{-j} & 0 \\ \times & (a + c - l)^{-j} \end{bmatrix} \quad for\ j = 1, 2, \ldots \quad (34)$$

Then, for $\xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2$, it follows from (27) that

$$\hat{\mu}_{\widetilde{M},\widetilde{D}}(\xi) = e^{\frac{\pi i \xi_1}{l-1}} \prod_{j=1}^{\infty} \cos\left(\frac{\pi \xi_1}{lj}\right). \quad (35)$$

In the case when $l = \pm 2$, the spectrality of $\mu_{\widetilde{M},\widetilde{D}}$ has been proved (see [421, p. 212]). In the following, we may assume that l is even and $|l| \geq 4$.

Now, consider the IFS:

$$\left\{\sigma_0(x) = \frac{x}{l}, \sigma_1(x) = \frac{(x+1)}{l}\right\} \quad (36)$$

in one dimension $\mathbb{R}$, where the integer expanding matrix $\bar{M} = [l]$ and digit set $\bar{D} = \{0, 1\}$, we have

$$\mu_{\bar{M},\bar{D}}(\xi_1) = e^{\frac{\pi i \xi_1}{l-1}} \prod_{j=1}^{\infty} \cos\left(\frac{\pi \xi_1}{lj}\right), \quad \forall\, \xi_1 \in \mathbb{R}, \quad (37)$$

which is connected with (35).

Let

$$s_1 := \frac{l}{2} \in \mathbb{Z} \quad and \quad \bar{S} := \{0, s_1\} \subset \mathbb{Z}. \quad (38)$$

The expansive orbit of 0 under the expansive IFS $\{\varphi_0(x) = lx, \varphi_1(x) = lx + s_1\}$ is

$$\Lambda(\bar{M}, \bar{S}) := \left\{\sum_{j=0}^{k-1} l^j s_j : k \geq 1 \text{ and } s_j \in \bar{S}\right\}. \quad (39)$$

One can check that $(\bar{M}^{-1}\bar{D}, \bar{S})$ is a compatible pair, and the conditions of Theorem 1.2 in [10] are satisfied. From this theorem, we know that $\Lambda(\bar{M}, \bar{S})$ is a spectrum for $\mu_{\bar{M},\bar{D}}$. Equivalently, we have

$$\sum_{\lambda \in \Lambda(\bar{M},\bar{S})} |\hat{\mu}_{\bar{M},\bar{D}}\,(\xi_1 - \lambda)|^2 = 1, \quad \forall\, \xi_1 \in \mathbb{R}. \quad (40)$$

Finally, we define $\tilde{\Lambda}$ by

$$\tilde{\Lambda} := \left\{ \begin{pmatrix} \lambda \\ \eta(\lambda) \end{pmatrix} \in \mathbb{R}^2 : \lambda \in \Lambda(\bar{M}, \bar{S}) \right\}, \quad (41)$$

where $\eta: \Lambda(\bar{M}, \bar{S}) \to \mathbb{R}$ is an arbitrary single-valued function. It follows from (35), (37) and (40) that

$$\sum_{\tilde{\lambda} \in \tilde{\Lambda}} |\hat{\mu}_{\widetilde{M},\widetilde{D}}(\xi - \tilde{\lambda})|^2 = \sum_{\lambda \in \Lambda(\bar{M},\bar{S})} |\hat{\mu}_{\bar{M},\bar{D}}(\xi_1 - \lambda)|^2 = 1, \forall\, \xi = (\xi_1, \xi_2)^t \in \mathbb{R}^2, \quad (42)$$

which shows that $\tilde{\Lambda}$ is a spectrum for $\mu_{\widetilde{M},\widetilde{D}}$. Since $\hat{\mu}_{M,D}(\xi) = \hat{\mu}_{\widetilde{M},\widetilde{D}}(Q^*\xi)$, we see that $Q^{*-1}\tilde{\Lambda}$ is a spectrum for $\mu_{M,D}$. This completes the proof of Theorem (6.1.7).

Note that, in the above proof, one can also consider the pair $(\widetilde{M}_1, \widetilde{D}_1)$ given by (22) instead of the pair $(\widetilde{M}, \widetilde{D})$ given by (21). Under the condition $\gamma = 0$, we have $det(M) = (a + c - l)l$. If $det(M) \notin 2\mathbb{Z}$, then l is odd, so $\mu_{M,D}$ is a non-spectral measure and there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. Furthermore, if $det(M) \in 2\mathbb{Z}$ and l is even, then $\mu_{M,D}$ is a spectral measure. If $det(M) \in 2\mathbb{Z}$ and l is odd, then $\mu_{M,D}$ is a non-spectral measure and there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. For example, let M and D be given by

$$M = \begin{bmatrix} 3 & b \\ 0 & 2 \end{bmatrix} \ (b \in \mathbb{Z}) and\ D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \quad (43)$$

Then $\gamma = 0, det(M) \in 2\mathbb{Z}$ and $l = 3$, so $\mu_{M,D}$ is a non-spectral measure and there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. However, if we let M and D be given by

$$M = \begin{bmatrix} 2 & b \\ 0 & 3 \end{bmatrix} \ (b \in \mathbb{Z}) and\ D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad (44)$$

then $\gamma = 0, det(M) \in 2\mathbb{Z}, l = 2$, and $\mu_{M,D}$ is a spectral measure.

We follow to deal with the spectrality and non-spectrality of self-affine measures. According to Conjecture (6.1.2), the discussion here are divided into two cases: $det(M) \in 2\mathbb{Z}$ and $det(M) \notin 2\mathbb{Z}$.

Theorem (6.1.8) [432]: For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a two-element digit set $D \subset \mathbb{Z}^2$ given by (4), define an integer γ by (5). If $\gamma \neq 0$ and $det(M) \in 2\mathbb{Z}$, then there are infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$. If, in addition, $a + c = 0$, then $\mu_{M,D}$ is a spectral measure.

Proof: Under the condition $\gamma \neq 0$, we consider the pair $(\widetilde{M}, \widetilde{D})$ given by (8) (the same method applies to the pair $(\widetilde{M}_1, \widetilde{D}_1)$ given by (11)). In the case when $det(M) = det(\widetilde{M}) = \pm 2$, one can verify that $\widetilde{D}$ is a complete residue system $(mod\ \widetilde{M})$. So, $\mu_{\widetilde{M},\widetilde{D}}$ is a spectral measure [439, Theorem 3.7]. In the following, we may assume that $det(\widetilde{M}) = ac - bd \in 2\mathbb{Z}\backslash\{0, \pm 2\}$.

First, we can construct a set $\tilde{S}$,

$$\tilde{S} := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right\} \subset \mathbb{Z}^2 \quad (45)$$

such that $(\widetilde{M}^{-1}\widetilde{D}, \tilde{S})$ is a compatible pair. In fact, we only need to choose the integer numbers $s_1, s_2 \in \mathbb{Z}$ such that

$$s_1 - (a + c)s_2 = (ac - bd)\left(k + \frac{1}{2}\right) \quad (46)$$

holds for some $k \in \mathbb{Z}$. There are many such integers s_1 and s_2 satisfying (46). What we really need is $s_2 = 0$ and $s_1 = (bd - ac)/2$. Once we have a compatible pair $(\widetilde{M}^{-1}\widetilde{D}, \tilde{S})$, we immediately obtain that $E(\Lambda(\widetilde{M}, \tilde{S}))$ is an infinite orthogonal exponentials in $L^2(\mu_{\widetilde{M},\widetilde{D}})$. This proves the first part of Theorem (6.1.8).

Note that there are many compatible pairs $(\widetilde{M}^{-1}\widetilde{D}, \tilde{S})$ such that the corresponding family $E(\Lambda(\widetilde{M}, \tilde{S}))$ is an infinite orthogonal exponentials in $L^2(\mu_{\widetilde{M},\widetilde{D}})$. According to Conjecture

(6.1.1), the measure $\mu_{\widetilde{M},\widetilde{D}}$ should be a spectral measure. Usually, in the two compatible pairs $(\widetilde{M}^{-1}\widetilde{D},\tilde{S}_1)$ and $(\widetilde{M}^{-1}\widetilde{D},\tilde{S}_2)$, one set $\Lambda(\widetilde{M},\tilde{S}_1)$ may be a spectrum for $\mu_{\widetilde{M},\widetilde{D}}$, and the other set $\Lambda(\widetilde{M},\tilde{S}_2)$ may not be a spectrum (see [421, Example 4.6]), or both $\Lambda(\widetilde{M},\tilde{S}_1)$ and $\Lambda(\widetilde{M},\tilde{S}_2)$ are not spectra for $\mu_{\widetilde{M},\widetilde{D}}$ even if Conjecture (6.1.1) is true.

Since Conjecture (6.1.1) is still open, to finish the proof of Theorem (6.1.8), we next show that if $a + c = 0$, then the selected set $\tilde{S}$,

$$\tilde{S} := \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}(bd-ac)/2\\0\end{pmatrix}\right\} \subset \mathbb{Z}^2 \quad (47)$$

leads to a spectrum $\Lambda(\widetilde{M},\tilde{S})$ for $\mu_{\widetilde{M},\widetilde{D}}$.

We need the following lemma due to Strichartz [437].

Lemma (6.1.9) [432]: Let $M \in M_n(\mathbb{Z})$ be expanding, D and S be finite subsets of $\mathbb{Z}^n$ such that $(M^{-1}D,S)$ is a compatible pair and $0 \in D \cap S$. Suppose that the zero set $Z(m_{M^{-1}D}(x))$ is disjoint from the set $T\ (M^*,S)$.

Then $\Lambda(M,S)$ is a spectrum for $\mu_{M,D}$.

Now, for the compatible pair $(\widetilde{M}^{-1}\widetilde{D},\tilde{S})$ with $(\widetilde{M},\widetilde{D})$ given by (8) and $\tilde{S}$ given by (47), we define M_1, D_1 and S_1 by

$$M_1 := \widetilde{M}^2 = \begin{bmatrix} bd+a^2 & 0 \\ 0 & bd+a^2 \end{bmatrix} := \nu I_2 (\nu = bd + a^2); \quad (48)$$

$$D_1 := \widetilde{D} + \widetilde{M}\widetilde{D} = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}; \quad (49)$$

$$S_1 := \tilde{S} + \widetilde{M}^*\tilde{S} = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\nu/2\\0\end{pmatrix}, \begin{pmatrix}0\\\nu/2\end{pmatrix}, \begin{pmatrix}\nu/2\\\nu/2\end{pmatrix}\right\}. \quad (50)$$

Then, $(M_1^{-1}D_1, S_1)$ is a compatible pair, $S_1 = \frac{\nu}{2}D_1$ and

$$T(M_1^*, S_1) = \left\{\sum_{j=1}^{\infty} M_1^{*-j}\, s_j : s_j \in S_1\right\} = \frac{\nu}{2}\left\{\sum_{j=1}^{\infty} \nu^{-j} d_j : d_j \in D_1\right\} = \frac{\nu}{2}T(M_1, D_1); \quad (51)$$

$$Z\left(m_{M_1^{-1}D_1}(x)\right) = Z\left(m_{D_1}\left(M_1^{*-1}(x)\right)\right) = Z\left(m_{D_1}\left(\nu^{-1}(x)\right)\right) = \nu Z\left(m_{D_1}(x)\right). \quad (52)$$

From (51) and (52), we have

$$T(M_1^*, S_1) \cap Z\left(m_{M_1^{-1}D_1}(x)\right) = \emptyset \Leftrightarrow T\ (M_1, D_1) \cap 2Z\left(m_{D_1}(x)\right) = \emptyset. \quad (53)$$

In view of the fact that $|\nu| \geq 4$, we see that $T(M_1, D_1)$ is contained in the open square $(-1,1)^2$, while $(-1,1)^2 \cap 2Z(m_{D_1}(x)) = \emptyset$. Therefore, the conditions of Lemma (6.1.9) are satisfied, and $\Lambda(M_1, S_1)$ is a spectrum for μ_{M_1,D_1}. Since $\Lambda(M_1, S_1) = \Lambda(\widetilde{M},\tilde{S})$ and $\hat{\mu}_{M_1,D_1} = \hat{\mu}_{\widetilde{M},\tilde{S}}$, we obtain that $\Lambda(\widetilde{M},\tilde{S})$ is a spectrum for $\mu_{\widetilde{M},\widetilde{D}}$. This completes the proof of Theorem (6.1.8).

In the case when the two eigenvalues λ_1 and λ_2 of the expanding matrix $M \in M_2(\mathbb{Z})$ are equal, i.e., $\lambda_1 = \lambda_2 := \lambda$, we obtain that $det(M) = \lambda_2 \in 2\mathbb{Z} \Leftrightarrow \lambda \in 2\mathbb{Z}\backslash\{0\}$. In this case, we can further prove the following.

Theorem (6.1.10) [432]: For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a two-element digit set $D \subset \mathbb{Z}^2$ given by (4), define an integer γ by (5). Suppose that M has two equal eigenvalues, i.e., $\lambda_1 = \lambda_2 := \lambda$. If $\gamma \neq 0$ and $\lambda \in 2\mathbb{Z}\backslash\{0\}$, then $\mu_{M,D}$ is a spectral measure.

Proof*:* Consider the pair $(\widetilde{M}_2, \widetilde{D}_2)$ given by (13) with $\lambda_1 = \lambda_2 := \lambda$, by letting

$$\tilde{S}_2 := \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\lambda/2\\0\end{pmatrix}\right\} \subset \mathbb{Z}^2,$$

we first obtain a compatible pair $(\widetilde{M}_2^1\widetilde{D}_2, \tilde{S}_2)$.

Then, we have

$$T(\widetilde{M}_2^*, \tilde{S}_2) = \left\{ \sum_{j=1}^{\infty} \begin{bmatrix} \lambda^{-j} & -j\lambda^{-j-1} \\ 0 & \lambda^{-j} \end{bmatrix} \begin{pmatrix} s_{1,j} \\ s_{2,j} \end{pmatrix} : \begin{pmatrix} s_{1,j} \\ s_{2,j} \end{pmatrix} \in \bar{S}_2 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \in \tilde{T}, x_2 = 0 \right\}, \quad (54)$$

Where

$$\tilde{T} = \frac{\lambda}{2} \left\{ \sum_{j=1}^{\infty} \lambda^{-j} s_{1,j} : s_{1,j} \in \{0,1\} \right\}.$$

This shows that the planar compact set $T(\widetilde{M}_2^*, \tilde{S}_2)$ is located on the x_1-axis. In particular, if $\lambda = 2$, then

$$T\left(\widetilde{M}_2^*, \tilde{S}_2\right) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : 0 \le x_1 \le 1, x_2 = 0 \right\}; (55)$$

if $\lambda = -2$, then

$$T\left(\widetilde{M}_2^*, \tilde{S}_2\right) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -\frac{1}{3} \le x_1 \le \frac{2}{3}, x_2 = 0 \right\}. (56)$$

For any $\lambda \in 2\mathbb{Z}\backslash\{-2, 0, 2\}$, we have

$$T\left(\widetilde{M}_2^*, \tilde{S}_2\right) \subseteq \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -1 \le x_1 \le 1, x_2 = 0 \right\}. (57)$$

Now, the zero set $Z(m_{\widetilde{M}_2^{-1}\widetilde{D}_2}(x))$ is given by

$$Z(m_{\widetilde{M}_2^{-1}\widetilde{D}_2}(x)) = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : m_{\widetilde{M}_2^{-1}\widetilde{D}_2}(x) = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \lambda x_1 - x_2 = \lambda^2 \left(k + \frac{1}{2}\right), k \in \mathbb{Z} \right\}. \quad (58)$$

For $\lambda \in 2\mathbb{Z}\backslash\{0\}$ and $k \in \mathbb{Z}$, the line $l_{\lambda,k}$ defined by

$$l_{\lambda,k} : \lambda x_1 - x_2 = \lambda^2 \left(k + \frac{1}{2}\right)$$

intercepts the x_1-axis at the point $a(\lambda, k) := \lambda(k + \frac{1}{2})$, where $a(\pm 2, 0) = \pm 1$ and $a(\pm 2, -1) = \mp 1$. Since $a(\lambda, k) \in \mathbb{Z}\backslash\{0\}$ and $|a(\lambda, k)| > 1$ for $\lambda \in 2\mathbb{Z}\backslash\{-2, 0, 2\}$ and $k \in \mathbb{Z}$, it follows from (56)–(58) that

$$T\left(\widetilde{M}_2^*, \tilde{S}_2\right) \cap Z\left(m_{\widetilde{M}_2^{-1}\widetilde{D}_2}(x)\right) = \emptyset \quad if\ \lambda \in 2\mathbb{Z}\backslash\{0, 2\}. (59)$$

From Lemma (6.1.9), we obtain that if $\lambda \in 2\mathbb{Z}\backslash\{0, 2\}$, then $\Lambda(\widetilde{M}_2, \tilde{S}_2)$ is a spectrum for $\mu_{\widetilde{M}_2, \widetilde{D}_2}$.

Finally, in the case when $\lambda = 2$, it follows from (55) and (58) that

$$T\left(\widetilde{M}_2^*, \tilde{S}_2\right) \cap Z\left(m_{\widetilde{M}_2^{-1}\widetilde{D}_2}(x)\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2. \quad (60)$$

So, Lemma (6.1.9) cannot be applied. And if $\lambda = 2$, then the matrix $(\widetilde{M}_2^* - I)$ is unimodular. This shows that the orthogonal system $E(\Lambda(\widetilde{M}_2, \tilde{S}_2))$ is not complete in $L^2(\mu_{\widetilde{M}_2, \widetilde{D}_2})$ and $\Lambda(\widetilde{M}_2, \tilde{S}_2)$ is not a spectrum for $\mu_{\widetilde{M}_2, \widetilde{D}_2}$ (see [11, Theorem 3.3]). However, one can use the method presented in [3–5] to show that $\mu_{\widetilde{M}_2, \widetilde{D}_2}$ is a spectral measure. In fact, by considering

$$\widetilde{M}_2^2 = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix} \quad and\ \widetilde{D}_2 + \widetilde{M}_2 \widetilde{D}_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\},$$

we can obtain the spectrality of the measure $\mu_{\widetilde{M}_2, \widetilde{D}_2}$ in the same way as [443, Example 5.2] and [4, 5]. This completes the proof of Theorem (6.1.10).

Note that the proof of Theorem (6.1.10) also yields the following more general result:

For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and the digit set D given by

$$M = \begin{bmatrix} a & b \\ d & c \end{bmatrix} \quad and \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad (61)$$

if $a \in 2\mathbb{Z}\backslash\{0\}$ and $b = 0$, then $\mu_{M,D}$ is a spectral measure.

So, the conclusion of Theorem (6.1.10) still holds if λ_1 and λ_2 are two integers, $det(M) = \lambda_1\lambda_2 \in \mathbb{Z}$ and $\gamma \neq 0$. For the pair (M, D) in (61), we have $\gamma = d$. If $d = 0$, then the spectrality or non-spectrality of $\mu_{M,D}$ is contained in Theorem (6.1.7).

Theorem (6.1.11) [432]: For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a two-element digit set $D \subset \mathbb{Z}^2$ given by (4), define an integer γ by (5). If $\gamma \neq 0$, $det(M) \notin 2\mathbb{Z}$, and $a + c = 0$, then $\mu_{M,D}$ is a non-spectral measure. Furthermore, there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 4 is the best.

Proof: With the same notation as, we see that $det(M) = -\nu \in (2\mathbb{Z} + 1)\backslash\{\pm 1\}$ and there exists no $\tilde{S} \subset \mathbb{Z}^2$ such that $(\widetilde{M}^{-1}\,\widetilde{D}, \tilde{S})$ is a compatible pair. However, we can still consider the simplified pair (M_1, D_1) given by (48) and (49). It follows from Theorem (6.1.3) that there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M_1,D_1})$, and the number 4 is the best. Since

$$\hat{\mu}_{M,D}(\xi) = \hat{\mu}_{\widetilde{M},\tilde{S}}(B^*\xi) = \hat{\mu}_{M_1,D_1}(B^*\xi), \quad (62)$$

we have $Z(\hat{\mu}_{M,D}) = B^{*-1}Z(\hat{\mu}_{M_1,D_1})$. So, $L^2(\mu_{M,D})$-space also have at most 4 mutually orthogonal exponential functions. This completes the proof of Theorem (6.1.11).

In the proof of Theorems (6.1.8) and (6.1.11), we have derived some results on the two-element digit set from that results on the four-element digit set., we will clarify certain relations between the two-element digit set and the four-element digit set, from which, Theorem (6.1.4) is proved and a general result is obtained.

For the digit set D given by (3), if we let

$$D_1 := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} d_1 \\ 0 \end{pmatrix} \right\} = d_1 \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} := d_1\widetilde{D}_1,$$

$$D_2 := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d_2 \end{pmatrix} \right\} = d_2 \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} := d_2\widetilde{D}_2,$$

we see that $D = D_1 + D_2 = D_1 \bigoplus D_2$ holds. In general, for such digit set D, we have

$$m_D(x) = m_{D_1}(x)m_{D_2}(x) = m_{\widetilde{D}_1}(d_1x)m_{\widetilde{D}_2}(d_2x)(x \in \mathbb{R}^2), (63)$$

which shows that

$$\hat{\mu}_{M,D}(\xi) = \hat{\mu}_{M,D_1}(\xi)\hat{\mu}_{M,D_2}(\xi) = \hat{\mu}_{M,\widetilde{D}_1}(d_1\xi)\hat{\mu}_{M,\widetilde{D}_2}(d_2\xi)(\xi \in \mathbb{R}^2)(64)$$

and

$$Z\left(\hat{\mu}_{M,D}\right) = Z\left(\hat{\mu}_{M,D_1}\right) \cup Z\left(\hat{\mu}_{M,D_2}\right) = d_1^{-1}Z\left(\hat{\mu}_{M,\widetilde{D}_1}\right) \cup d_2^{-1}Z\left(\hat{\mu}_{M,\widetilde{D}_2}\right). (65)$$

Hence, we have the following conclusions:

(i) The space $L^2(\mu_{M,D})$ has infinite families of orthogonal exponentials if and only if one of the two spaces $L^2(\mu_{M,D_1})$ (or $L^2(\mu_{M,\widetilde{D}_1})$) and $L^2(\mu_{M,D_2})$ (or $L^2(\mu_{M,\widetilde{D}_2})$) has infinite families of orthogonal exponentials;

(ii) The space $L^2(\mu_{M,D})$ has at most a finite number of orthogonal exponentials if and only if both the spaces $L^2(\mu_{M,D_1})$ (or $L^2(\mu_{M,\widetilde{D}_1})$) and $L^2(\mu_{M,D_2})$ (or $L^2(\mu_{M,\widetilde{D}_2})$) have at most a finite number of orthogonal exponentials.

Note that, we may let d_1 and d_2 be two nonzero real numbers (not necessarily integer numbers). Also, for such digit set D, we have $|D| = |D_1| \cdot |D_2|$. If $|D| \notin W(m)$ then $|D_1| \notin W(m)$ and $|D_2| \notin W(m)$. The above conclusion (ii) provides a supportive evidence on Conjecture (6.1.2).

Applying Theorems (6.1.7) and (6.1.11) to the pairs (M, D_1) and (M, D_2), we get Theorem (6.1.4)(i) from conclusion (ii). Applying Theorems (6.1.7) and (6.1.8) to the pairs (M, D_1)

and (M, D_2), we get Theorem (6.1.4)(ii) from conclusion (i). After obtaining the Theorem (6.1.4), we can further prove the following.

Theorem (6.1.12) [432]: For an expanding integer matrix $M \in M_2(\mathbb{Z})$ and a two-element digit set $D \subset \mathbb{Z}^2$ given by (4), define an integer γ by (5). If $\gamma \neq 0, det(M) \notin 2\mathbb{Z}$, and $a^2 + c^2 + 2bd = 0$, then $\mu_{M,D}$ is a non-spectral measure, and there are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$.

In fact, by considering the pair $(\widetilde{M}, \widetilde{D})$ given by (8), and applying Theorem (6.1.4)(i) to the pair $(\widetilde{M}^2, \widetilde{D} + \widetilde{M}\widetilde{D})$, we get Theorem (6.1.12).

Since $|det(M)| > 1$ for an expanding matrix M, the condition $a + c = 0$ in Theorem (6.1.11) and the condition $a^2 + c^2 + 2bd = (a + c)^2 - 2(ac - bd) = 0$ in Theorem (6.1.12) do not occur at the same time.

Some of the above ideas can be used to deal with the higher dimensional case without any difficulties.

For example, let $M \in M_n(\mathbb{Z})$ be a upper-triangle expanding matrix with diagonal entries $p_1, p_2, \ldots, p_n$, and let $D = \{(0, 0, \ldots, 0)^t, (d_1, 0, \ldots, 0)^t\} \subset \mathbb{R}^n$, where $|p_j| > 1$ $(j = 1, 2, \ldots, n)$ and $d_1 \neq 0$. Then, the following conclusions hold:

(i) If $|p_1|$ is even, then $\mu_{M,D}$ is a spectral measure;

(ii) If $|p_1|$ is odd, then $\mu_{M,D}$ is a non-spectral measure and there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

Finally, we would like to point out that, the spectrality and non-spectrality of self-affine measures are completely determined in the case when $\gamma = 0$. The case $\gamma \neq 0$ is determined under some additional conditions. According to Conjecture (6.1.1), the condition $a + c = 0$ in Theorem (6.1.8) is superfluous, while according to Conjecture (6.1.2), the condition $a + c = 0$ in Theorem (6.1.11) and the condition $a^2 + c^2 + 2bd = 0$ in Theorem (6.1.12) are also superfluous. This leaves an open question of how to delete these additional conditions in the corresponding theorems.

Section (6.2): One Dimensional Self-Similar Measures with Consecutive Digits

For μ be a Borel probability measure on $\mathbb{R}$. We say that μ is a spectral measure if there exists a discrete set Λ such that $E_\Lambda := \{e^{2\pi\lambda xi} : \lambda \in \Lambda\}$ forms an orthonormal basis of $L^2(\mu)$. In this case, we call Λ a spectrum of μ and (μ, Λ) a spectral pair, respectively.

Jorgenson and Pederson [464]studied the spectral property of general Cantor measures. They proved that the $1/k$-Cantor measure$\mu_{1/k}$ on $\mathbb{R}$ is a spectral measure if k is even (Strichartz provided a simplified proof in [469]). This result was investigated by Laba and Wang in more details in [465] and for the general Borel measures in [466].

Hu and Lau [463] further studied the spectral property of Bernoulli convolutions. They proved that the necessary and sufficient condition that the Bernoulli convolution has an infinite orthonormal set E_Λ of exponential functions is that the contraction ratio ρ is the n-th root of a fraction p/q, where p is odd and q is even. Dai [472] proved that the Bernoulli convolution has an orthonormal basis E_Λ of exponential functions if and only if the contraction ratio ρ is the reciprocal of an even integer.

We study the spectral property of one dimensional self-similar measures with consecutive digits.

For ρ be a real number such that $0 < |\rho| < 1$, it is well known that for any positive integer $m \geq 2$, there exists a unique probability measure, denoted by μ_ρ, such that

$$\mu_\rho(A) = \frac{1}{m}\sum_{j=0}^{m-1} \mu_\rho(\rho^{-1}(A) - j) \quad (66)$$

for all Borel set $A \in B$. μ_ρ is called a self-similar measure.

For the self-similar measure μ_ρ defined in (66), the main theorem is as follows.

Theorem (6.2.1)[453]: If m is a prime, then $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions only if $\rho = \pm\frac{1}{mk}$ for some $k \in \mathbb{N}$.

This is an extension of the result of [452].

On the other hand, Dai etc. proved that $L^2(\mu_{1/mk})$ contains an orthonormal basis of exponential functions for any $k \in \mathbb{N}$ in [452]. It is easy to see that $L^2(\mu_{-1/mk})$ also contains an orthonormal basis of exponential functions for any $k \in \mathbb{N}$. Hence we have the following.

Theorem (6.2.2) [453]: If m is a prime, then $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions if and only if $\rho = \pm\frac{1}{mk}$ for some $k \in \mathbb{N}$.

Theorem (6.2.3) [453]: Assume m is a prime, then $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if $\rho = \pm(q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with the properties: p, q are co-prime and $m|p$.

Since E_Λ forms an orthonormal set in $L^2(\mu_\rho)$ if and only if $E_{t+\Lambda}$ forms an orthonormal set in $L^2(\mu_\rho)$ for any fixed $t \in \mathbb{R}^d$.

For simplicity we assume that $0 \in \Lambda$ throughout this .

Let $\mathbb{Z}$ be the set of all integers, let $\mathbb{N}$ be the set of all positive integers. For any $x, y \in \mathbb{Z}$ and $r \in \mathbb{N}$, we use $x \equiv y(mod\ r)$ to denote $x - y \in r\mathbb{Z}$.

For the iterated function system $\{S_j\}_{j=0}^{m-1}$ with $S_j(x) = \rho(x + j)$ and the associated μ_ρ defined in (66), let $\hat{\mu}_\rho(t) = \int e^{2\pi xti} d\mu_\rho(x)$ be the Fourier transform of μ_ρ. Define

$$Z_\rho = \{t \in \mathbb{R}: \hat{\mu}_\rho(t) = 0\}$$

to be the set of zeros of $\hat{\mu}_\rho(t)$. Let

$$\mathbb{O} = \{\pm(q/p)^{1/r}: p, q, r \in \mathbb{N}\}.$$

It is clear that $\beta \in \mathbb{O}$ if and only if $|\beta|$ is an algebraic rational with a minimal polynomial $px^r - q$ for some $p, q, r \in \mathbb{N}$.

Throughout this , we always use

$$E_\Lambda = \{e^{2\pi\lambda xi}: \lambda \in \Lambda\},$$

to denote an orthonormal set of exponential functions in $L^2(\mu_\rho)$, whereΛ is a subset of $\mathbb{R}$ containing 0. For this Λ, we define

$$Q_\Lambda(t) = \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(\lambda - t)|^2 .$$

Some preliminary lemmas are given. is devoted to prove Theorem (6.2.3). While Theorem (6.2.1) is proven

We first give some preliminary results associated with the self-similar measure μ_ρ. Then we will use them to prove Theorems (6.2.1) and (6.2.3), respectively.

It is easy to prove the following.

Lemma (6.2.4) [453]: Let $\hat{\mu}_\rho(t)$ be the Fourier transform of the self-similar measure μ_ρ defined in (66), then

$$\hat{\mu}_\rho(t) = \prod_{k=1}^{n} \left[\frac{1}{m}\sum_{j=0}^{m-1} \left(e^{2\pi\rho^k tj}\right)^j\right] \hat{\mu}_\rho(\rho^n t) \quad (67)$$

for all positive integers $n > 0$.

Lemma (6.2.5) [453]: $Z_\rho = \{\frac{\ell}{m\rho^k}: k \in \mathbb{N}, \ell \in \mathbb{Z}\backslash m_{\mathbb{Z}}\}$.

Proof: Use (67), since $\hat{\mu}_\rho(\rho^n t) \to 1$ as $n \to +\infty, \hat{\mu}_\rho(\lambda) = 0$ if and only if there exists a positive integer $k > 0$ so that

$$\sum_{j=0}^{m-1}(e^{2\pi\rho^k\lambda i})^j = 0.$$

Hence $e^{2\pi\rho^k\lambda i} \neq 1$, multiplying both sides by $1 - e^{2\pi\rho^k\lambda i}$, we see that the above equation is equivalent to

$$1 - (e^{2\pi\rho^k\lambda i})^m = 0, e^{2\pi\rho^k\lambda i} \neq 1.$$

Hence $\hat{\mu}_\rho(\lambda) = 0$ if and only if there exist integers $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $2m\pi\rho^k\lambda = 2\ell\pi$ and $\rho^k\lambda$ is not an integer, i.e. $\lambda = \frac{\ell}{m\rho^k}$ with $\ell \in \mathbb{Z}\backslash m_{\mathbb{Z}}$ and $k \in \mathbb{N}$, the conclusion follows.

Lemma (6.2.6) [453]: Let Λ be a subset of $\mathbb{R}$ containing 0, then E_Λ is an orthonormal set of $L^2(\mu_\rho)$ if and only if $(\Lambda - \Lambda)\backslash\{0\} \subset Z_\rho$. Equivalently, the following two conditions are satisfied:

(i) $\Lambda = \{0\} \cup \{\frac{\ell_j}{m\rho^{k_j}} : 1 \leq j < N\}$ with $k_j \in \mathbb{N}, \ell_j \in \mathbb{Z}\backslash\, m\mathbb{Z}, 0 < j < N$, where N is a finite positive integer or the infinity.

(ii) There exist $\frac{v_{s,t}}{m\rho^{r_{s,t}}} \in Z_\rho$ with $v_{s,t} \in \mathbb{Z}\backslash m\mathbb{Z}$ such that

$$\frac{\ell_s}{m\rho^{k_s}} - \frac{\ell_t}{m\rho^{k_t}} = \frac{v_{s,t}}{m\rho^{r_{s,t}}}, 1 \leq s \neq t < N. \quad (68)$$

Proof: It is clear that E_Λ is an orthonormal set in $L^2(\mu_\rho)$ if and only if $\hat{\mu}_\rho(\lambda_1 - \lambda_2) = 0$ for any distinct $\lambda_1, \lambda_2 \in \Lambda$. Hence

(ii) follows from Lemma (6.2.5), and (i) follows from the assumption $0 \in \Lambda$.

Lemma (6.2.7) [453]: If $L^2(\mu_\rho)$ has an orthonormal set of exponential functions with at least $m + 1$ elements, then ρ^{-1} is a zero of an integral polynomial.

Proof: Let E_Λ be an orthonormal set with at least $m + 1$ elements. Then the N defined in Lemma (6.2.6) is at least $m + 1$. Hence (68) holds for all $1 \leq s < t \leq m$.

If $k_s = k_t = r_{s,t}$ for all $1 \leq s < t \leq m$, then $\ell_s - \ell_t = v_{s,t}$ for all $1 \leq s < t \leq m$. It is clear that there exist s and t such that $0 \leq s < t \leq m$ and $\ell_s - \ell_t \in m\mathbb{Z}$. Hence $v_{s,t} \in m\mathbb{Z}$, it contradicts Lemma (6.2.6)(ii). Therefore, there is a pair (s, t) so that at least two of $k_s, k_t, r_{s,t}$ are distinct, so ρ^{-1} is a zero of an integral polynomial by using (68).

Lemma (6.2.8) [453]: Assume $\beta \in \mathbb{O}$ admits a minimal polynomial $p\beta^r - (\pm 1)^r q = 0$ and satisfies $a_1\beta^k + a_2\beta^j = a_3\beta^u$, where $k, j, u \geq 0$ are nonnegative integers and $a_1, a_2, a_3 \in \mathbb{Z}\backslash\{0\}$. Then $k \equiv j \equiv u (mod\ r)$.

Proof: Let $k = k_1 r + s, j = j_1 r + t, u = u_1 r + v$ with $0 \leq s, t, v < r$. Since $p\beta^r = (\pm 1)^r q$, so β satisfies

$$b_1\beta^s + b_2\beta^t = b_3\beta^v$$

for some integers $b_1, b_2, b_3 \neq 0$. Since $px^r - (\pm 1)^r q$ is the minimal polynomial of β with order r, in view of $0 \leq s, t, v < r$, we see that $s = t = v$, the conclusion follows.

We first prove Theorem (6.2.3) for the case $\rho \in \mathbb{O}$.

Proposition (6.2.9) [453]: Let $px^r - q$ be the minimal polynomial of $|\rho|$ for some $p, q, r \in \mathbb{N}$, where p, q are co-prime. Then $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if p, m have a common divisor larger than one.

Proof: Consider the necessity. Let E_Λ be an infinite orthonormal set with $0 \in \Lambda$, then $\Lambda\backslash\{0\} \subseteq Z_\rho$.

For any $x \in Z_\rho$, there exist $u \in \mathbb{N}$ and $v \in \mathbb{Z}\backslash m\mathbb{Z}$ such that $x = \frac{v}{m|\rho|^u}$ by Lemma (6.2.5). Furthermore, there exist integers $\ell \geq 0$ and $v_0 \in \mathbb{Z}$ so that $v = p^l v_0$ with $p \nmid v_0$. Hence

$\frac{v}{m|\rho|^u} = \frac{q^l v_0}{m|\rho|^{u+rl}}, p \nmid v_0$ and $m \nmid p^l v_0$. Since $\Lambda - \Lambda \subseteq \{0\} \cup Z_\rho$. Let k_1 be the smallest positive integer such that $\frac{\ell_1}{m|\rho|^{k_1}} \in \Lambda - \Lambda \subseteq \{0\} \cup Z_\rho$ for some $\ell_1 \in (\mathbb{Z} \setminus p\mathbb{Z})$.

Let $\lambda_1, \lambda_0 \in \Lambda$ be such that $\lambda_1 - \lambda_0 = \frac{\ell_1}{m|\rho|^{k_1}}$.

For any $\lambda \in \Lambda$, Lemma (6.2.6) implies that there exists a $\lambda' \in Z_\rho$ such that $\frac{\ell_1}{m\rho^{k_1}} - (\lambda - \lambda_0) = \lambda_1 - \lambda = \lambda'$. Write $\lambda = \lambda_0 + \frac{q^s \ell}{m|\rho|^{k+rs}}$ and $\lambda' = \frac{q^t v}{m|\rho|^{u+rt}}$ with the properties: $\ell, v \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$; $k, u \in \mathbb{N}$; $s, t \geq 0$. Then we have

$$\frac{\ell_1}{m|\rho|^{k_1}} - \frac{q^s \ell}{m|\rho|^{k+rs}} = \frac{q^t v}{m|\rho|^{u+rt}}.$$

Therefore, Lemma (6.2.8) implies that $k - k_1 \equiv u - k_1 \equiv 0 \ (mod\ r)$. Let $k + rs = nr + k_1, u + rt = u'r + k_1$, then the definition of k_1 implies $n, u' \geq 0$. The above equality becomes

$$\ell_1 - q^{s-n} p^n \ell = q^{t-u'} v p^{u'}.$$

If $n = 0$, then $\lambda = \lambda_0 + \frac{q^s \ell}{m|\rho|^{k_1}}$. If $n > 0$, note that p, q are co-prime, if $u' > 0$, then the above equality implies $p|\ell_1$, a contradiction. Hence $u' = 0$, so the above equality implies $q^n | q^s \ell$. Hence λ can be written as the form $\lambda = \lambda_0 + \frac{q^n \ell'}{m|\rho|^{k_1+rn}} = \frac{p^n \ell'}{m|\rho|^{k_1}}$ with $\ell' \in \mathbb{Z}$. Therefore, there exist non-zero integers z_j such that

$$\Lambda = \{\lambda_0\} \cup \{\lambda_j\}_{j=0}^{+\infty} \text{ with } \lambda_j = \lambda_0 + \frac{z_j}{m|\rho|^{k_1}} (j > 0). \quad (69)$$

Lemma (6.2.6)(ii) implies that $\frac{z_s - z_t}{m|\rho|^{k_1}} \in Z_\rho$ for all distinct $s, t > 0$. Choose $s > t > 0$ so that $m|(z_s - z_t)$, let $z_s - z_t = mz$.

Then Lemma (6.2.5) implies that there exist $u_{s,t} \in \mathbb{Z} \setminus m\mathbb{Z}, v_{s,t} \in \mathbb{N}$ such that

$$\frac{z}{|\rho|^{k_1}} = \lambda_s - \lambda_t = \frac{u_{s,t}}{m|\rho|^{v_{s,t}}}.$$

Lemma (6.2.8) implies that $v_{s,t} - k_1 = r\xi_{s,t}$ for some integer $\xi_{s,t}$. Hence

$$\left(\frac{q}{p}\right)^{\xi_{s,t}} = \frac{u_{s,t}}{mz}. \quad (70)$$

The definition of k_1 implies $\xi_{s,t} \geq 0$. Note that $\nmid u_{s,t}$, we see that $\xi_{s,t} > 0$ and p, m have a common divisor larger than one. The necessity follows.

We now prove the sufficiency. Suppose that p, m have a common divisor larger than one. Let $m_0 > 1$ be the greatest common divisor of m and p. Since p, q are co-prime, so m_0, q are co-prime.

Let

$$\hat{\Lambda} = \{0\} \cup \left\{\frac{q^n}{m|\rho|^{nr}} : n \in \mathbb{N}\right\}.$$

Since m_0, q are co-prime and $m_0 > 1$ is the greatest common divisor of m and p, so neither $\frac{q^n}{m}$ nor $\frac{q^n(p^k-1)}{m}$ is an integer for all $n, k \in \mathbb{N}$. Therefore, both $\frac{q^n}{m|\rho|^{nr}}$ and

$$\frac{q^{n+k}}{m|\rho|^{nr+kr}} - \frac{q^n}{m|\rho|^{nr}} = \frac{q^n(p^k - 1)}{m|\rho|^{nr}}$$

are zeros of $\hat{\mu}_\rho(t)$ for all $n, k \in \mathbb{N}$ by using Lemma (6.2.5). Hence $(\hat{\Lambda} - \hat{\Lambda}) \setminus \{0\} \subset Z_\rho$, so $E_{\hat{\Lambda}}$ is an infinite orthonormal set in $L^2(\mu_\rho)$ by Lemma (6.2.6). The sufficiency follows.

To prove Theorem (6.2.3) for the case $\rho \notin \mathbb{Q}$. We will suppose on the contrary that $L^2(\mu_\rho)$ has an infinite orthonormal set of exponential functions for some $\rho \notin \mathbb{Q}$, then obtain a contradiction.

Let E_Λ be an infinite orthonormal set in $L^2(\mu_\rho)$, then Lemma (6.2.7) implies that ρ^{-1} is a zero of an integral polynomial. Let

$$g(x) = a_0 + a_1 x + \cdots + a_n x^n \quad (71)$$

be the minimal integral polynomial of ρ^{-1}, where $a_0 > 0, a_1, \ldots, a_n$ are relative prime and $a_n \neq 0$.

It is easy to see that the following lemma holds.

Lemma (6.2.10) [453]: If ρ^{-1} is a zero of an integral polynomial $f(x) = d_0 + d_1 x + \cdots + d_k x^k$, then $a_0 | d_0$ and $a_n | d_k$.

Lemma (6.2.11) [453]: Let E_Λ be an orthonormal set in $L^2(\mu_\rho)$ and $\rho \notin \mathbb{Q}$, let $\Lambda_k = \frac{Z \setminus mZ}{m\rho^k} \cap \Lambda$, then $\{\Lambda_k\}_{k>0}$ are disjoint and each one has cardinality at most $m-1$.

Proof: For the disjointness of $\{\Lambda_k\}_{k>0}$, suppose on the contrary, $\Lambda_k \cap \Lambda_r \neq \emptyset$ for some $0 < k < r$. Then there exist $\frac{s}{m\rho^k} \in \Lambda_k$ and $\frac{t}{m\rho^r} \in \Lambda_r$ so that $\frac{s}{m\rho^k} = \frac{t}{m\rho^r}$. Hence

$$\rho = \pm(|t/s|)^{\frac{1}{r-k}} \in \mathbb{Q},$$

a contradiction.

For the cardinality of Λ_k. If Λ_k has at least m elements, let $\frac{\ell_j}{m\rho^k} \in \Lambda_k, j = 1, 2, \ldots, m$. Then there exist $1 \leq j_1 < j_2 \leq m$ so that $\ell_{j_1} - \ell_{j_2} = rm$ for a non-zero integer r. Hence Lemma (6.2.6) implies that

$$\frac{r}{\rho^k} = \frac{u}{m\rho^v}$$

for some $u \in \mathbb{Z} \setminus m\mathbb{Z}$ and $v \in \mathbb{N}$. This means that $v \neq k$ and

$$\rho = \left(\frac{u}{mr}\right)^{\frac{1}{v-k}} (v > k) \text{ or } \rho = \left(\frac{mr}{u}\right)^{\frac{1}{k-v}} (v < k),$$

a contradiction to the assumption $\rho \notin O$.

Lemma (6.2.12) [453]: Assume $\rho \notin \mathbb{Q}$. If $L^2(\mu_\rho)$ has an infinite orthonormal set of exponential functions, then ρ^{-1} has a minimal polynomial $g(x)$ as (71) with $a_0 = 1$.

Proof: Lemma (6.2.7) implies that ρ^{-1} is a zero of an integral polynomial. Hence ρ^{-1} has a minimal polynomial $g(x)$ as given by (71). Since $\rho \notin \mathbb{Q}$, it is easy to see that $z\rho^{-k} \notin Z_\rho$ for all $z \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Since $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions E_Λ, Lemma (6.2.11) implies that there exist infinitely many $\lambda_j = \frac{\ell_j}{m\rho^{k_j}} \in \Lambda$ with $0 < k_j < k_{j+1}$. Without loss of generality, we assume

$$\Lambda = \{0\} \cup \left\{\frac{\ell_j}{m\rho^{k_j}} : 0 < k_1 < k_2 < \cdots\right\}.$$

By Lemma (6.2.5), we can assume that $\frac{\ell_j}{m} \notin \mathbb{Z}$. Lemma (6.2.6)(ii) then implies that there exist $v_{s,t} \in \mathbb{Z} \setminus m\mathbb{Z}, r_{s,t} \in \mathbb{N}$ such that

$$\frac{\ell_s}{m\rho^{k_s}} - \frac{\ell_t}{m\rho^{k_t}} = \frac{v_{s,t}}{m\rho^{r_{s,t}}}, \qquad \forall s \neq t \in \mathbb{N}. \quad (72)$$

Since $\rho \notin \mathbb{Q}$ and $k_s \neq k_t$ when $s \neq t$, it is easy to see that $r_{s,t} \neq k_s, k_t$ when $s \neq t$.

Claim. $r_{s,j} \to +\infty$ as $j \to +\infty$ for all $s \in \mathbb{N}$.

Proof of the Claim. For any given $s, a \in \mathbb{N}$, if there are infinitely many j such that $r_{s,j} \leq a$, then there are infinitely many identical $r_{s,j}$'s. Hence there exist $j_2 > j_1 > s$ so that $m|(v_{s,j_1} - v_{s,j_2})$ and $r_{s,j_1} = r_{s,j_2}$. This means that $\frac{\ell_{j_1}}{m\rho^{k_{j_1}}} - \frac{\ell_{j_2}}{m\rho^{k_{j_2}}} = \frac{v_{s,j_2} - v_{s,j_1}}{m\rho^{rs,j_1}}$ and $\frac{v_{s,j_2} - v_{s,j_1}}{m\rho^{rs,j_1}}$ is not a zero of $\hat{\mu}_\rho(t)$ by $\rho \notin \mathbb{O}$, a contradiction. Therefore, there are only finite j such that $r_{s,j} \leq a$, the claim follows.

For any $s \geq 1$, the claim implies that, we can choose a sufficiently large t so that $k_t, r_{s,t} > k_s$. (72) implies that ρ^{-1} is a zero of the integral polynomial

$$\ell_s - \ell_t x^{k_t - k_s} - v_{s,t} x^{r_{s,t} - k_s}.$$

Hence Lemma (6.2.10) implies

$$a_0 | \ell_s, \qquad s = 1, 2, \ldots . \quad (73)$$

For any given $s \neq t \in \mathbb{N}$, since $\lambda_s - \lambda_t = (\lambda_s - \lambda_u) - (\lambda_t - \lambda_u)$, so (72) implies

$$\frac{v_{s,t}}{m\rho^{r_{s,t}}} = \frac{v_{s,u}}{m\rho^{r_{s,u}}} - \frac{v_{t,u}}{m\rho^{r_t,u}}$$

for all $u \in \mathbb{N}$. By the above claim, we can choose a sufficiently large u so that $r_{s,u}, r_{t,u} > r_{s,t}$. Hence ρ^{-1} is a zero of the integral polynomial

$$v_{s,t} - v_{s,u} x^{r_{s,u} - r_{s,t}} + v_{t,u} x^{r_{t,u} - r_{s,t}}.$$

Hence Lemma (6.2.10) implies

$$a_0 | v_{s,t}, \qquad s \neq t = 1, 2, \ldots . \quad (74)$$

Therefore, (72)–(74) imply that $E_{a_0^{-1}\Lambda}$ is also an infinite orthonormal set in $L^2(\mu_\rho)$. Replace Λ by $a_0^{-1}\Lambda$, the above proof implies that $E_{a_0^{-2}\Lambda}$ is an infinite orthonormal set in $L^2(\mu_\rho)$. Continuing in this way, it can be proven that $E_{a_0^{-n}\Lambda}$ is an infinite orthonormal set in $L^2(\mu_\rho)$ for all $n > 0$. It must implies that $a_0 = 1$ necessary by using $a_0 \in \mathbb{N}$. The lemma is proven.

For the minimal polynomial of ρ^{-1} defined in (71) with $a_0 = 1$, we define an $(n+1) \times (n+1)$ invertible matrix

$$E = \begin{pmatrix} 1 & 0 \\ -a & I_n \end{pmatrix}, \quad (75)$$

where I_n is the $n \times n$ identical matrix, $a = (a_1, \ldots, a_n)^t$. For this matrix E, we define a sequence of $(n+1) \times 1$ vectors as following:

$$\alpha_0 \doteq (0, 1, 0, \ldots, 0)^t$$

and

$$\alpha_{k+1} \doteq (\alpha_{k+1,0}, \alpha_{k+1,1}, \ldots, \alpha_{k+1,n})^t = E(\alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,n}, 0)^t \ (76)$$

for $k \geq 0$.

Lemma (6.2.13) [453]: Assume m is a prime, then

(i) $\alpha_{s,n} = -a_n \alpha_{s,0}$ for any integer $s \geq 0$;

(ii) There is an integer $\ell > 0$ so that $\alpha_{i+\ell} \equiv \alpha_i (mod\ m)$ for all $i \geq 0$;

(iii) $\alpha_{k-n} \equiv (\alpha_{k-n,0}, 0, \ldots, 0, 1)^t \ (mod\ m)$ when $k = \ell j + 1 > n$ for some integer j, where $\alpha_{k-n,0}$ satisfies $a_n \alpha_{k-n,0} \equiv 1 (mod\ m)$.

Proof: (i) It follows directly from the definition.

(ii) Consider the residue class of $\mathbb{Z}^{n+1}$ modulo m, we see that there exist integers i and $\ell > 0$ so that $\alpha_{i+\ell} \equiv \alpha_i (mod\ m)$.

If $i > 0$, then

$$E(\alpha_{i+\ell-1,1}, \alpha_{i+\ell-1,2}, \ldots, \alpha_{i+\ell-1,n}, 0)^t \equiv E(\alpha_{i-1,1}, \alpha_{i-1,2}, \ldots, \alpha_{i-1,n}, 0)^t (mod\ m).$$

It is easy to see that $(\alpha_{i+\ell-1,1}, \alpha_{i+\ell-1,2}, \ldots, \alpha_{i+\ell-1,n}, 0)^t \equiv (\alpha_{i-1,1}, \alpha_{i-1,2}, \ldots, \alpha_{i-1,n}, 0)^t (mod\ m)$. By the

conclusion (i) and the assumption that m is a prime, we see that $\alpha_{i+\ell-1,n} \equiv \alpha_{i-1,n}\ (mod\ m)$ implies $\alpha_{i+\ell-1,0} \equiv \alpha_{i-1,0}\ (mod\ m)$. Therefore, $\alpha_{i+\ell-1} \equiv \alpha_{i-1}(mod\ m)$. Continuing in this way implies the conclusion (ii).

(iii) The conclusion (ii) implies $\alpha_{k-1} \equiv \alpha_0 = (0,1,0,\dots,0)^t(mod\ m)$. Note

$$E^{-1} = \begin{pmatrix} 1 & 0 \\ a & I_n \end{pmatrix},$$

(76) implies $(\alpha_{k-2,1}, \alpha_{k-2,2}, \dots, \alpha_{k-2,n}, 0)^t \equiv E^{-1}(0,1,0,\dots,0)^t \equiv (0,1,0,\dots,0)^t(mod\ m)$. Since m, a_n are co-prime, $\alpha_{k-2,n} = a_n\alpha_{k-2,0}$, so $\alpha_{k-2} \equiv (0,0,1,0,\dots,0)^t(mod\ m)$ if $n > 2$. Continuing in this way proves $(\alpha_{k-n,1}, \alpha_{k-n,2}, \dots, \alpha_{k-n,n}, 0)^t \equiv (0,0,\dots,0,1)^t\ (mod\ m)$. Note that m, an are co-prime, we see that the conclusion $a_n\alpha_{k-n,0} \equiv 1(mod\ m)$ follows and so (iii) is proven.

Proposition (6.2.14) [453]: If $\rho \notin \mathbb{O}, m$ is a prime and E_Λ is an orthonormal set in $L^2(\mu_\rho)$ with $0 \in \Lambda$, then Λ is a finite set.

Proof: Assume on the contrary that Λ is an infinite set. Lemma (6.2.12) implies that ρ^{-1} has a minimal polynomial $g(x)$ as (71) with $a_0 = 1$.

Since m is a prime, the assumption of Lemma (6.2.13) holds, m, c_j are co-prime.

Claim. There are infinitely many families of integers $\{c_1, c_2, c_3, s, k\}$ such that:ρ^{-1} is a zero of the integral polynomial $f(x) = c_1 - c_2x^s - c_3x^{k+n}, k + n > s > 0, c_1, c_2, c_3 \in \mathbb{Z}\backslash m\mathbb{Z}$ and, either $c_1 \equiv c_2(mod\ m)$ and $\ell|s$ or $c_1 \equiv c_3(mod\ m)$ and $\ell|(k+n)$, where ℓ is defined in Lemma (6.2.13).

By Lemma (6.2.11), there exist infinitely many $\frac{\ell_s}{m\rho^{k_s}} \in \Lambda$ such that$\ell_s \in \mathbb{Z}\backslash m\mathbb{Z}$ and $0 < k_j < k_{j+1}$. By (68) in Lemma (6.2.6)(ii), there exist $v_{s,t} \in \mathbb{Z}\backslash m\mathbb{Z}$ and $r_{s,t} \in \mathbb{N}$ such that

$$\frac{\ell_s}{m\rho^{k_s}} - \frac{\ell_t}{m\rho^{k_t}} = \frac{v_{s,t}}{m\rho^{r_{s,t}}},\ \ s \neq t \geq 1. \quad (77)$$

(a) It is easy to see that there is an i such that $\ell_i \equiv \ell_s(mod\ m)$ for infinitely many s. By choosing a subclass of Λ, without loss of generality, we can assume that $\ell_1 \equiv \ell_s(mod\ m)$ for all $s > 1$.

(b) For the ℓ defined in Lemma (6.2.13), there is an i such that $\ell|(k_t - k_i)$ for infinitely many t. By choosing a subclass of Λ, without loss of generality, we can assume that $\ell|(k_t - k_1)$ for all $t > 1$.

(c) By the claim in the proof of Lemma (6.2.12), we can choose a t_0 such that $r_{1.t} > k_1$ for all $t \geq t_0$. Hence (77) implies that ρ^{-1} is a zero of the integral polynomial $\ell_1 - \ell_t x^{k_t - k_1} - v_{s,t}x^{r_{1,t}-k_1}$ with $\ell|(k_t - k_1)$.

Since $\rho \notin \mathbb{O}$ implies that either $r_{1.t} - k_1 > k_t - k_1 > 0$ or $0 < r_{1.t} - k_1 < k_t - k_1$, the claim follows.

By Lemma (6.2.12), let $g(x) = 1 + a_1x + \cdots + a_nx^n$ be the minimal polynomial of ρ^{-1}, then $f(x) = g(x)h(x)$ for an integral polynomial $h(x)$ with order k.

For the above $g(x), f(x)$ and $h(x)$, we use the following notations. Let $h(x) = b_0 + b_1x + \cdots + b_kx^k, a = (1, a_1, \dots, a_n)^t, b = (b_0, b_1, \dots, b_k)^t$. Let e_j be the j-th column of the $(k+n+1) \times (k+n+1)$ identical matrix I_{k+n+1}.

Define a $(k+n+1) \times (k+1)$ matrix $A = (u_{t,j})_{0\leq t\leq k+n, 0\leq j\leq k}$ by $u_{j,j} = 1$ for $0 \leq j \leq k, u_{j,ij} = a_i$ for $0 \leq j \leq k$ and $1 \leq i \leq n$, and $u_{j,t} = 0$ for other (j,t).

Then $f(x) = g(x)h(x)$ if and only if $Ab = c_1e_1 - c_2e_{s+1} - c_3e_{k+n+1}$. Define $(k+n+1) \times (k+n+1)$ matrices:

$$E_1 = \begin{pmatrix} E & 0 \\ 0 & I_k \end{pmatrix}, \ldots, E_j = \begin{pmatrix} I_{j-1} & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I_{k-j+1} \end{pmatrix}, \ldots, E_{k+1} = \begin{pmatrix} I_k & 0 \\ 0 & E \end{pmatrix},$$

where I_j is the $j \times j$ identical matrix, E is defined in (75).

It is clear that $E(1, a_1, \ldots, a_n)^t = (1, 0, \ldots, 0)^t$. It follows that $E_1 A$ reduces the first column of A to $(1, 0, \ldots, 0)^t$ and the other columns remain unchanged; $E_2 E_1 A$ reduces the second column of $E_1 A$ to $(0, 1, 0, \ldots, 0)^t$ and the other columns remain unchanged. Finally $E_{k+1} \cdots E_1 A$ is the matrix with 1 on the diagonal and 0 elsewhere.

Furthermore, the last $n+1$ entries of $E_{k+1} \cdots E_1 e_1$ generate the vector α_{k+1} defined in Lemma (6.2.13). Hence

$E_{k+1} E_k \cdots E_1 e_1 \equiv (v_0, v_1, \ldots, v_{k-1}, \alpha_{k+1,0}, \alpha_{k+1,1}, \ldots, \alpha_{k+1,n})^t \ (mod\ m)$ for some $v_0, v_1, \ldots, v_{k-1}$.

For e_{s+1}, we have

$$E_{k+1} \cdots E_1 e_{s+1} = \begin{cases} E_{k+1} \cdots E_{s+1} e_{s+1}, & s \le k, \\ e_{s+1}, & s > k. \end{cases} \quad (78)$$

Hence the last $n+1$ entries of $E_{k+1} \cdots E_1 e_{s+1}$ generate the vector α_{k+1-s} when $s \le k$ and the vector $(0, \ldots, 0, 1, 0, \ldots, 0)^t$ with the 1 at the $s-k+1$-th entry when $s > k$.

Since $f(x) = g(x)h(x)$, so the last n entries of $E_{k+1} \cdots E_1 (c_1 e_1 - c_2 e_{s+1} - c_3 e_{k+n+1})$ are all zero. In the following, we will consider the $(n+1) \times 1$ vector generated by the last $n+1$ entries of $E_{k+1} \cdots E_1 (c_1 e_1 - c_2 e_{s+1} - c_3 e_{k+n+1})$ by using the above claim. Note that the last $n+1$ entries of $E_{k+1} \cdots E_1 Ab$ generates the vector $(b_k, 0, \ldots, 0)^t$. Hence

$$\begin{cases} c_1 \alpha_{k+1} - c_2 \alpha_{k+1-s} - c_3 (0, \ldots, 0, 1) = (b_k, 0, \ldots, 0)^t, & if\ s \le k, \\ c_1 \alpha_{k+1} - c_2 (0, \ldots, 0, 1, 0, \ldots, 0)^t - c_3 (0, \ldots, 0, 1) = (b_k, 0, \ldots, 0)^t, & if\ s > k. \end{cases} \quad (79)$$

Case 1. If $\ell | s$ and $k + n > s > k$. Then, use (78), the vector generated by the last $n+1$ entries of $E_{k+1} \cdots E_1 (c_1 e_1 - c_2 e_{s+1} - c_3 e_{k+n+1})$ is $c_1 \alpha_{k+1} - (0, \ldots, 0, c_2, 0, \ldots, 0, c_3)^t$ with the c_2 at the $s-k+1$-th entry. Hence (79) implies $c_1 \alpha_{k+1} = (b_k, 0, \ldots, 0, c_2, 0, \ldots, 0, c_3)^t$, so c_1 is a common divisor of b_k, c_2 and c_3, so $\alpha_{k+1} = (b_k / c_1, 0, \ldots, 0, c_2 / c_1, 0, \ldots, 0, c_3 / c_1)^t$.

By the definition of α_j, we have $\alpha_{s+1} = E(c_2 / c_1, 0, \ldots, 0, c_3 / c_1, \ldots, 0,)^t$. This contradicts $\alpha_{s+1} \equiv E(1, 0, \ldots, 0)^t (mod\ m)$, since $\ell | s$ and $c_2, c_3 \not\equiv 0 (mod\ m)$.

Case 2. If $\ell | s, s \le k$ and $c_1 \equiv c_2 (mod\ m)$. Then the last $n+1$ entries of $E_{k+1} \cdots E_1 e_{s+1}$ generate the vector α_{k+1-s} by using the definition of E_j and (78). Hence the last $n+1$ entries of $E_{k+1} \cdots E_1 (c_1 e_1 - c_2 e_{s+1})$ generate the vector $c_1 \alpha_{k+1} - c_2 \alpha_{k+1-s} \equiv 0 (mod\ m)$ by using Lemma (6.2.13)(ii) and $c_1 \equiv c_2 (mod\ m)$.

Therefore, we have

$E_{k+1} E_k \cdots E_1 (c_1 e_1 - c_2 e_{s+1} - c_3 e_{k+n+1}) \equiv (c_1 u_0, \ldots, c_1 u_{s-1}, 0, \ldots, 0, -c_3)^t (mod\ m)$ for some $u_0, u_1, \ldots, u_{s-1}$. Since $c_3 \not\equiv 0 (mod\ m)$, it contradicts to the conclusion that the last n entries of $E_{k+1} \cdots E_1 (c_1 e_1 - c_2 e_{s+1} - c_3 e_{k+n+1})$ are all zero.

Case 3. If $\ell | (k+n)$ and $c_1 \equiv c_3 (mod\ m)$. Then $E_{k+1} E_k \cdots E_1 (c_1 e_1) \equiv (v_0, v_1, \ldots, v_{k-1} c_1 \alpha_{k+1,0}, 0, \ldots, 0, c_1)^t (mod\ m)$ for some $v_0, v_1, \ldots, v_{k-1}$ by using Lemma (6.2.13)(ii). Note $E_{k+1} E_k \cdots E_1 e_{k+n+1} = e_{k+n+1}$, we see

$E_{k+1} E_k \cdots E_1 (c_1 e_1 - c_3 e_{k+n+1}) \equiv (v_0, v_1, \ldots, v_{k-1} c_1 \alpha_{k+1,0}, 0, \ldots, 0, c_1)^t \ (mod\ m)$.

Since the last n entries of $E_{k+1} \cdots E_1 (c_1 e_1 - c_2 e_{s+1} - c_3 e_{k+n+1})$ are all zero, the above equality implies

$$E_{k+1} E_k \cdots E_1 e_{s+1} \equiv (w_0, w_1, \ldots, w_k, 0, 0, \ldots, 0)^t \ (mod\ m)$$

for some $w_0, w_1, \ldots, w_k$. If $s \le k$, (78) implies $\alpha_{k-s+1} \equiv (w_k, 0, 0, \ldots, 0)^t (mod\ m)$, so Lemma (6.2.13)(i) implies $a_n w_k = a_n \alpha_{k-s+1,0} = \alpha_{k-s+1,n} \equiv 0 (mod\ m)$. Since $f(x) = g(x)h(x)$, so $a_n | c_3$. Since $c_3 \in \mathbb{Z} \backslash m\mathbb{Z}$ and m is a prime, so $\alpha_{k-s+1} \equiv 0 (mod\ m)$. Hence the definition of α_j implies $\alpha_j \equiv 0 (mod\ m)$ for all sufficiently large j, a contradiction to Lemma (6.2.13)(ii). If $s > k$, (78) implies $E_{k+1} E_k \cdots E_1 e_{s+1} = e_{s+1}$, so the above equality implies that $(w_k, 0, 0, \ldots, 0)^t \equiv (0, \ldots, 0, 1, 0, \ldots, 0)^t\ (mod\ m)$ with the 1 at the $s - k + 1$-th entry, This is obviously impossible.

Theorem (6.2.15) [453]: If $|\rho| = q/p$ with $1 < q < p$ being co-prime, then there exist constants $a > 0, C_0 > 0$ so that

$$\sup_{t \in \mathbb{R}} \{|\hat{\mu}_\rho(t)| (ln(2 + |t|))^a\} \le C_0 < +\infty.$$

Proof: Claim. There is a positive constant $b < 1$ such that: assume $|\xi| > 1$, then we can find an η such that $|\rho\xi| \ge |\eta| \ge \rho^2 |\xi|^{ln\, q / ln\, p}$ and

$$|\hat{\mu}_\rho(\xi)| \le b |\hat{\mu}_\rho(\eta)|.$$

Proof of the Claim. For any real number x, let $\lceil x \rceil$ be the number in $\left(-\frac{1}{2}, \frac{1}{2}\right]$ such that $x - \lceil x \rceil \in \mathbb{Z}$.

(a) If $\lceil \rho\xi \rceil \notin (-\frac{1}{2p}, \frac{1}{2p})$, then $|1 + \exp\{2\pi \lceil \rho\xi \rceil i\}| \le |1 + \exp\{\frac{\pi}{p} i\}| < 2$. Use (67) and $|\exp\{2\pi j \lceil \rho\xi \rceil i\}| \le 1$, we have

$$|\hat{\mu}_\rho(\xi)| = |\hat{\mu}_\rho(\rho\xi)| \cdot \left| \frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi\rho\xi i\})^j \right| = |\hat{\mu}_\rho(\rho\xi)| \cdot \left| \frac{1}{m} \sum_{j=0}^{m-1} exp\{2\pi j \lceil \rho\xi \rceil i\} \right|$$

$$\le |\hat{\mu}_\rho(\rho\xi)| \cdot \frac{m - 2 + |1 + \exp\{2\pi \lceil \rho\xi \rceil i\}|}{m} \le |\hat{\mu}_\rho(\rho\xi)| \cdot \frac{m - 2 + \left|1 + \exp\{\frac{\pi}{m} i\}\right|}{m}.$$

Let $b = \frac{m - 2 + |1 - \exp\{\frac{\pi}{p} i\}|}{m}, \eta = \rho\xi$, the claim holds.

(b) If $\lceil \rho\xi \rceil \in (-\frac{1}{2p}, \frac{1}{2p})$. Since $|\xi| > 1$, so $|\rho\xi| > |\rho| \ge \frac{1}{p}$ by $|\rho| = q/p$. Hence, we can write $|\rho\xi| = \lceil \rho\xi \rceil + \sum_{k=s}^{t} e_k p^k$ with $e_j \in \{0, 1, \ldots, p-1\}, e_s > 0, e_t > 0$ and $s \ge 0$. Hence

$$\lceil \rho^{s+2} \xi \rceil = \left\lceil (q/p)^{s+1} \lceil \rho\xi \rceil + q^{s+1}/p \sum_{k=s}^{t} e_k p^{k-s} \right\rceil = \left\lceil (q/p)^{s+1} \lceil \rho\xi \rceil + \frac{e_s q^{s+1}}{p} \right\rceil.$$

Since $\lceil \rho\xi \rceil \in (-\frac{1}{2p}, \frac{1}{2p})$ and $0 < (q/p)^{s+1} < 1$, so $(q/p)^{s+1} \lceil \rho\xi \rceil \in (-\frac{1}{2p}, \frac{1}{2p})$. Hence $\lceil \rho^{s+2} \xi \rceil \notin (-\frac{1}{2p}, \frac{1}{2p})$ by using the above equality. Therefore, similar to the above conclusion (a), we have

$$|\hat{\mu}_\rho(\xi)| = \left| \prod_{k=1}^{s+2} \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi\rho^k \xi i\})^j \right] \hat{\mu}_\rho(\rho^{s+2}\xi) \right|$$

$$\le \left| \frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi\rho^{s+2}\xi i\})^j | \cdot |\hat{\mu}_\rho(\rho^{s+2}\xi) \right|$$

$$\le \frac{m - 2 + \left|1 - \exp\{\frac{\pi}{p} i\}\right|}{m} |\hat{\mu}_\rho(\rho^{s+2}\xi)|.$$

Since $|\rho\xi| = \lceil|\rho\xi|\rceil + \sum_{k=s}^{t} e_k p^k \geq e_s p^s - \frac{1}{p} \geq |\rho| p^s$, so $s \leq \log_p |\xi|$. Hence $|\rho^{s+2}\xi| \geq |\xi|\rho^2|\rho|^{log_p |\xi|} = |\xi|\rho^2|\xi|^{log_p |\rho|} = \rho^2|\xi|^{ln\, q/ln\, p}$. Let $\eta = \rho^{s+2}\xi$ and $b = \frac{m-2+|1-\exp\{\frac{\pi}{p}i\}|}{m}$, then we have

$$\left|\hat{\mu}_\rho(\xi)\right| \leq b\left|\hat{\mu}_\rho(\eta)\right|, \qquad |\rho\xi| \geq |\eta| \geq \rho^2|\xi|^{ln\, q/\, ln\, p}$$

the claim also holds. The proof of the claim is completed.

For any real number t with $|t| > 1$, the claim shows that, there exist an integer n and real numbers ξ_j with $\xi_0 = t, |\xi_{n-1}| > 1 \geq |\xi_n|$ such that, $|\rho\xi_j| \geq |\xi_{j+1}| \geq \rho^2|\xi_j|^{ln\, q/\, ln\, p}$ and $|\hat{\mu}_\rho(\xi_j)| \leq b|\hat{\mu}_\rho(\xi_{j+1})|$.

The second inequality implies $|\hat{\mu}_\rho(t)| \leq b^n|\hat{\mu}_\rho(\xi_n)|$.While, the first inequality implies

$$\begin{aligned} 1 \geq |\xi_n| &\geq \rho^2|\xi_{n-1}|^{ln\, q/\, ln\, p} \geq \rho^{2+2(ln\, q/\, ln\, p)}|\xi_{n-2}|^{(ln\, q/\, ln\, p)^2} \\ &\geq \cdots \geq \rho^{2+2(ln\, q/\, ln\, p)+\cdots+2(ln\, q/\, ln\, p)^{n-1}}|\xi_0|^{(ln\, q/\, ln\, p)^n} \\ &\geq \rho^{2\, ln\, p/(ln\, p - ln\, q)}|t|^{(ln\, q/\, ln\, p)^n} \\ &= p^{-2}|t|^{(ln\, q/\, ln\, p)^n} \end{aligned}$$

by using the equality $x^{\frac{1}{-\ln x}} = e^{-1}$ for $0 < x < 1$. Hence

$$\ln p^2 \geq \ln|t| \cdot (ln\, q/\, ln\, p)^n = \ln|t| \cdot b^{n\, ln(ln\, q/\, ln\, p)/\, ln\, b},$$

so

$$(\ln p^2)^{ln\, b/\, ln(ln\, q/\, ln\, p)} \geq (\ln|t|)^{ln\, b/\, ln(ln\, q/\, ln\, p)} \cdot b^n, \forall |t| > 1.$$

Let $a = ln\, b/\, ln(ln\, q/\, ln\, p)$, then $a > 0$ and

$$\begin{aligned} |\hat{\mu}_\rho(t)|(\ln(2+|t|))^a &\leq b^n|\hat{\mu}_\rho(\xi_n)|(\ln(2+|t|))^a \\ &= (\ln|t|)^{ln\, b/\, ln(ln\, q/\, ln\, p)} \cdot b^n|\hat{\mu}_\rho(\xi_n)|\left(\frac{\ln(2+|t|)}{\ln|t|}\right)^a \\ &\leq (\ln p^2)^{ln\, b/\, ln(ln\, q/\, ln\, p)}|\hat{\mu}_\rho(\xi_n)|\left(\frac{\ln(2+|t|)}{\ln|t|}\right)^a, \forall |t| > 1. \end{aligned}$$

Since

$$\sup_{|t|\geq 2}\left\{\left(\frac{ln(2+|t|)}{ln|t|}\right)^a\right\} < +\infty, \qquad \sup_{|t|\leq 2}\{|\hat{\mu}_\rho(t)|(ln(2+|t|))^a\} < +\infty,$$

so

$$\sup_{t\in\mathbb{R}}\{|\hat{\mu}_\rho(t)|(ln(2+|t|))^a\} < +\infty$$

by using the fact $|\hat{\mu}_\rho(\xi)| \leq 1$. The theorem is proven.

Lemma (6.2.16) [453]: Assume $|\rho|$ admits the minimal polynomial $px^r - q$, where p, q are co-prime and $r \in \mathbb{N}$. If E_Λ is an orthonormal basis of $L^2(\mu_\rho)$, then

$$\Lambda - \Lambda \subseteq \{0\} \cup \left[\bigcup_{k\geq 0}\frac{\mathbb{Z}\backslash m\mathbb{Z}}{m|\rho|^{1+kr}}\right],$$

$$(\Lambda - \Lambda) \cap \left[\frac{\mathbb{Z}\backslash m\mathbb{Z}}{m|\rho|}\right] \neq \emptyset.$$

Proof: Since E_Λ is an orthonormal basis of $L^2(\mu_\rho)$, so $\Lambda - \Lambda \subseteq Z_p$. By Lemmas (6.2.5) and (6.2.6), we can choose the smallest positive integer k_0 so that $(\Lambda - \Lambda) \cap \frac{\mathbb{Z}\backslash m\mathbb{Z}}{m|\rho|^{k_0}} = \emptyset$. Hence there exist $\lambda_1, \lambda_0 \in \Lambda$ such that

$$\lambda_1 - \lambda_0 = \frac{mz+s}{m|\rho|^{k_0}}, \quad 1 \leq s < m. \quad (80)$$

For any $\lambda \in \Lambda \backslash \{\lambda_0, \lambda_1\}$, Lemma (6.2.6) shows that $\lambda - \lambda_j = \frac{mz_j + s_j}{m|\rho|^{n_j}}$ with $1 \le s_j < m$. The definition of k_0 shows that $n_j \ge k_0$. Hence

$$\frac{mz_0 + s_0}{m|\rho|^{n_0}} - \frac{mz_1 + s_1}{m|\rho|^{n_1}} = \lambda_1 - \lambda_0 = \frac{mz + s}{m|\rho|^{k_0}}$$

for some integers $z_0, z_1, z, s_0, s_1, s, n_0, n_1$ with $n_j \ge k_0$ and $1 \le s_j < m$. Hence Lemma (6.2.7) shows $r|(n_j - k_0), j = 0, 1$.

Therefore, $\lambda = \lambda_0 + \frac{mz_0 + s_0}{m|\rho|^{k_0 + r_k}}$ for some $k \ge 0$.

Now, for any $\gamma \in \Lambda$ different from λ, the above results show that γ has the form: $\gamma = \lambda_0 + \frac{mz + s}{m|\rho|^{k_0 + m}}$ and satisfies

$$\gamma - \lambda = \frac{mz + s}{m|\rho|^{k_0 + m}} - \frac{mz_0 + s_0}{m|\rho|^{k_0 + rk}} = \frac{mz' + s'}{m|\rho|^N}$$

for some integers z', s' and N with $1 \le s' < m$ (since $\gamma - \lambda \in Z_\rho$). Hence Lemma (6.2.7) shows $r|(N - k_0)$. Therefore, $\lambda - \gamma \in [\bigcup_{k \ge 0} \frac{\mathbb{Z} \backslash m\mathbb{Z}}{m|\rho|^{k_0 + rk}}]$. Hence

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq \bigcup_{k \ge 0} \frac{\mathbb{Z} \backslash m\mathbb{Z}}{m|\rho|^{k_0 + rk}}.$$

Noting (80), we see that, in order to prove the lemma, we need only prove $k_0 = 1$. Assume $k_0 > 1$ on the contrary.

Let $\tilde{\Lambda} = \rho^{k_0 - 1}(\Lambda - \lambda_0)$. The definition of k_0 and Λ shows that $E_{\tilde{\Lambda}}$ is also an orthonormal set of $L^2(\mu_\rho)$. For any $\lambda = \lambda_0 + \frac{mz + s}{m|\rho|^{k_0 + rk}} \in \Lambda$, let $\tilde{\lambda} = \rho^{k_0 - 1}(\lambda - \lambda_0)$ and $\tilde{t} = (t - \lambda_0)\rho^{k_0 - 1}$, then (67) implies

$$\hat{\mu}_\rho(t - \lambda) = \prod_{k=1}^{k_0 - 1} \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi\rho^k (t - \lambda) i\})^j \right] \hat{\mu}_\rho(\tilde{t} - \tilde{\lambda}).$$

Hence

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_\rho(t - \lambda)|^2 = \sum_{\tilde{\lambda} \in \tilde{\Lambda}} |\hat{\mu}_\rho(\tilde{t} - \tilde{\lambda})|^2 \cdot \left| \prod_{k=1}^{k_0 - 1} \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^k (t - \lambda) i\})^j \right] \right|^2.$$

It is easy to see that there exist a $t \in \mathbb{R}$ and a $\lambda \in \Lambda$ so that $\left| \prod_{k=1}^{k_0 - 1} \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^k (t - \lambda) i\})^j \right] \hat{\mu}_\rho(\tilde{t} - \tilde{\lambda}) \right| < 1$ and $\hat{\mu}_\rho(\tilde{t} - \tilde{\lambda}) \ne 0$.

Hence

$$\sum_{\lambda \in \Lambda} |\hat{\mu}_\rho(t - \lambda)|^2 = \sum_{\tilde{\lambda} \in \tilde{\Lambda}} |\hat{\mu}_\rho(\tilde{t} - \tilde{\lambda})|^2 \le 1,$$

where the last inequality follows from Parseval's identity and the fact that $E_{\tilde{\Lambda}}$ is also an orthonormal set of $L^2(\mu_\rho)$.

Therefore, E_Λ is not an orthonormal basis of $L^2(\mu_\rho)$ by using Parseval's identity. A contradiction, hence $k_0 = 1$. The proof completes.

Now, we prove Theorem (6.2.1) by using the conclusions proven in the above part. Assume Λ is a spectrum of μ_ρ. It is easy to see that Λ is infinite, so Theorem (6.2.3) shows that $|\rho| = \left(\frac{q}{p}\right)^{\frac{1}{r}}$, admit a minimal polynomial $px^r - q$ such that $p > q > 0$ are co-prime and $m|p$. We first prove $r = 1$, then prove $q = 1$.

As an extension of [478, Theorem 4.4], we have the following.

Proposition (6.2.17) [453]: If μ_ρ is a spectral measure, then ρ is a rational.

Proof: By Theorem (6.2.3), $|\rho|$ has a minimal polynomial as $px^r - q$. Assume $r > 1$, on the contrary. Define μ_{ρ^k} with $k \in \mathbb{N}$ to be the self-similar measure satisfying

$$\mu_{\rho^k}(A) = \frac{1}{m}\sum_{j=0}^{m-1} \mu_{\rho^k}(\rho^{-1}(A) - j)\,.$$

Let Λ be a spectrum of μ_ρ, use Lemma (6.2.16), we see that

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq \bigcup_{k\geq 0} \frac{\mathbb{Z}\backslash m\mathbb{Z}}{m|\rho|^{1+kr}}.$$

Hence

$$\rho^{-r+1}(\Lambda - \Lambda)\backslash\{0\} \subseteq \cup_{k\geq 0} \frac{\mathbb{Z}\backslash m\mathbb{Z}}{m|\rho|^{kr}} = Z_{\rho^r}. \quad (81)$$

Since $|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi\rho^k ti\})^j| \leq 1, \mathbb{N} + r - 1 \supset \{r+1\} \cup (r\mathbb{N})$ by using $r > 1$, so

$$|\hat{\mu}_\rho(t)| = \prod_{k=1}^{+\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi\rho^k ti\})^j\right|$$

$$= \prod_{k=1}^{+\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi\rho^{k+r-1}\rho^{1-r} ti\})^j\right|$$

$$\leq \left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi\rho^2 ti\})^j\right| \left|\prod_{n=1}^{+\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi\rho^{nr}\rho^{1-r} ti\})^j\right|\right|$$

$$= \left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi\rho^2 ti\})^j\right| \cdot |\hat{\mu}_{\rho^r}(\rho^{1-r}t)|.$$

Choose a $\lambda_0 \in \Lambda$ and a $t_0 \in \mathbb{R}$ so that $\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi\rho^2(t_0 - \lambda_0)i\})^j\right| < 1$ and $|\hat{\mu}_{\rho^r}(\rho^{1-r}(t_0 - \lambda_0))| > 0$. Since E_Λ is an orthonormal basis of $L^2(\mu_\rho)$, so the above inequality and Parseval's identity imply

$$1 = \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(t_0 - \lambda)|^2 < \sum_{\lambda\in\Lambda} |\hat{\mu}_{\rho^r}(\rho^{1-r}t_0 - \rho^{1-r}\lambda)|^2 = \sum_{\gamma\in\rho^{1-r}\Lambda} |\hat{\mu}_{\rho^r}(\rho^{1-r}t_0 - \gamma)|^2$$
$$\leq 1,$$

a contradiction, where the last inequality ''≤'' follows from (81) which implies that $E_{\rho^{1-r}\Lambda}$ is an orthonormal set of $L^2(\mu_{\rho^r})$.

Therefore, $r = 1$, the proposition follows.

Lemma (6.2.18) [453]: If Λ is a spectrum of μ_ρ and $|\rho| = q/p$ with $gcd(q, m) = 1$. Then

$$(\Lambda - \Lambda)\backslash\{0\} \subset \bigcup_{k\geq 0} \frac{p^k(\mathbb{Z}\backslash m\mathbb{Z})}{m|\rho|}.$$

Proof: We first prove a weaker conclusion:

$$(\Lambda - \lambda_0)\backslash\{0\} \subset \cup_{k\geq 0} \frac{p^k(\mathbb{Z}\backslash m\mathbb{Z})}{m|\rho|^\iota} \quad (82)$$

for some positive integer ι and $\lambda_0 \in \Lambda$.

For any $a \in Z_p$, it can be written as $a = \frac{\ell}{m|\rho|^k}$ for some $k \in \mathbb{N}, \ell \in \mathbb{Z}\backslash m\mathbb{Z}$. We can find a nonnegative integer $n \geq 0$ so that $\ell = p^n \ell_1$ with $\ell_1 \in (\mathbb{Z}\backslash p\mathbb{Z})\backslash m\mathbb{Z}$, then $a = \frac{q^n \ell_1}{m|\rho|^{k+n}}$ with $q^n \ell_1 \in (\mathbb{Z}\backslash p\mathbb{Z})\backslash m\mathbb{Z}$ by using the assumption $gcd(p,q) = 1$ and $gcd(q,m) = 1$. For any $\lambda \in \Lambda\backslash\{\lambda_0\}$, we have $\lambda - \lambda_0 \in Z_p$. Hence

$$\Lambda\backslash\{\lambda_0\} \subset \lambda_0 + Z_\rho \subset \lambda_0 + \left\{\frac{\ell}{m|\rho|^k} : k \in \mathbb{N}, \ell \in (\mathbb{Z}\backslash p\mathbb{Z})\backslash m\mathbb{Z}\right\}. \quad (83)$$

Since $(\Lambda - \Lambda)\backslash\{0\} \subset Z_p$, let ι be the smallest positive integer such that there is a $\lambda_1 \in \Lambda$ so that $\lambda_1 - \lambda_0 = \frac{\ell_1}{m|\rho|^\iota} \in Z_\rho$ for some $\ell_1 \in (\mathbb{Z}\backslash p\mathbb{Z})\backslash m\mathbb{Z}$.

For any $\lambda \in \Lambda\backslash\{\lambda_0, \lambda_1\}$, by (83), we have $\lambda - \lambda_0 = \frac{\ell_2}{m|\rho|^n} \in \Lambda$ with $n \in \mathbb{N}, \ell_2 \in (\mathbb{Z}\backslash p\mathbb{Z})\backslash m\mathbb{Z}$. Furthermore, Lemma (6.2.6) and the above (83) imply that there exists $\frac{v}{m|\rho|^u} \in Z_\rho$ with $v \in (\mathbb{Z}\backslash p\mathbb{Z})\backslash m\mathbb{Z}$ such that

$$\frac{\ell_1}{m|\rho|^\iota} - \frac{\ell_2}{m|\rho|^n} = \lambda_1 - \lambda = \frac{v}{m|\rho|^u}. \quad (84)$$

Let $n = t + \iota, u = s + \iota$. The definition of ι implies that either $t = 0$ or $t > 0$. If $t = 0$, i.e. $n = \iota$, then $\lambda - \lambda_0$ belongs to the right hand side of (82). Assume $t > 0$, i.e. $n > \iota$. (84) implies

$$\ell_1 - \frac{\ell_2 p^t}{q^t} = \frac{v p^s}{q^s}.$$

If $s > 0$, since p, q are co-prime, so $p|\ell_1$, a contradiction, hence $s \leq 0$. If $s < 0$, then the above equality shows $p^{|s|}|q^{|s|+t}v$, also a contradiction. Hence $s = 0$, the above equality implies that $q^t|\ell_2$, so $\lambda - \lambda_0 = \frac{p^t \ell_3}{m|\rho|^\iota}$ for some $\ell_3 \in (\mathbb{Z}\backslash p\mathbb{Z})\backslash m\mathbb{Z}$.

Therefore, (82) holds for some $\iota \in \mathbb{N}$.

Assume $\iota > 1$. Note that we have assumed $gcd(q,m) = 1$. For any distinct $\lambda, \lambda' \in \Lambda$, the above equality shows $\lambda - \lambda' = \frac{p^s x - p^t y}{m|\rho|^\iota}$ for some integers $x, y \in \mathbb{Z}\backslash p\mathbb{Z}$ and $t \geq s \geq 0$. Since $\lambda - \lambda' \in Z_\rho$, so

$$\lambda - \lambda' = \frac{p^s(x - p^{t-s}y)}{m|\rho|^\iota} = \frac{v}{m|\rho|^{\iota+n}} \quad (85)$$

for some $v \in \mathbb{Z}\backslash m\mathbb{Z}$ and $n > -\iota$. We can find $k \geq 0$ so that $v = p^k(mu + w')$ with $p - (mu + w')$ and $1 \leq w' < m$.

Since $|\rho| = q/p$ and p, q are co-prime, the second equality of (85) shows $k + n \geq s \geq 0$. Noting $gcd(q,m) = 1$, we see that $q^k(mu + w')$ belongs to $\mathbb{Z}\backslash m\mathbb{Z}$. Hence

$$|\rho|(\lambda - \lambda') = \frac{p^k(mu + w')}{m|\rho|^{\iota+n-1}} = \frac{q^k(mu + w')}{m|\rho|^{\iota+n+k-1}} \in Z_\rho$$

by using $gcd(q,m) = 1, k + n \geq s \geq 0$ and $\iota > 1$.

Therefore, $\rho(\Lambda - \Lambda) \subseteq Z_\rho \cup \{0\}$, i.e. $E_{\rho\Lambda}$ is an orthonormal set of $L^2(\mu_\rho)$ when $\iota > 1$. Since Λ is a spectrum, Parseval's identity implies

$$1 = \sum_{\lambda \in \Lambda} |\hat{\mu}_\rho(t - \lambda)|^2 = \sum_{\gamma \in \rho\Lambda} \left| \frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi i(\rho t - \gamma)\}) \right| \cdot \left|\hat{\mu}_\rho(\rho t - \gamma)\right|^2.$$

It is easy to see that there exist $t \in \mathbb{R}$ and $\gamma \in \rho\Lambda$ so that $\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi i(\rho t - \gamma)\})\right| \cdot \left|\hat{\mu}_\rho(\rho t - \gamma)\right|^2 < \left|\hat{\mu}_\rho(\rho t - \gamma)\right|^2$.

Hence, by $\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi i(\rho t-\gamma)\})\right| \le 1$, we have

$$1=\sum_{\lambda\in\Lambda}\left|\hat{\mu}_\rho(t-\lambda)\right|^2<\sum_{\gamma\in\rho\Lambda}\left|\hat{\mu}_\rho(\rho t-\gamma)\right|^2\le 1,$$

a contradiction, where the last inequality follows from the proven fact that $E_{\rho\Lambda}$ is an orthonormal set of $L^2(\mu\rho)$. Therefore, $\iota=1$.

By (82) and (85) and $\iota=1$, we see that, for any two different $\lambda,\lambda'\in\Lambda$, the v and n in (85) satisfy $n\ge 0$ and $q^n|v$. Hence $\lambda-\lambda'$ belongs to the set $\frac{p^n(\mathbb{Z}\setminus m\mathbb{Z})}{m|\rho|}$ with $n\ge 0$. The lemma follows.

The following theorem describes the structure of a spectrum.

Theorem (6.2.19) [453]: If Λ is a spectrum of μ_ρ and $|\rho|=q/p$ with $m|p$ and $gcd(p,q)=1$, then there exist spectrums $\Gamma_k\subset\mathbb{Z}$ and integers z_k such that $\Lambda=(\lambda_0+\frac{k+mz_k}{m|\rho|}+|\rho|^{-1}\Gamma_k)$ with $\lambda_0\in\Lambda$ and $0\in\Gamma_k$. Furthermore, the union is a disjoint union.

Proof: Use the notations in Lemma (6.2.18). Let

$$B=\left\{k\in\{0,1,\dots,m-1\}:\left(\lambda_0+\frac{k+m\mathbb{Z}}{m|\rho|}\right)\cap\Lambda\neq\emptyset\right\}.$$

Using Lemmas (6.2.16) and (6.2.18), there are integers z_k, for all $k\in B$, such that

$$\left|\frac{mz_k+k}{m|\rho|}\right|=\min\left\{\left|\frac{mz+k}{m|\rho|}\right|:z\in\mathbb{Z},\lambda_0+\frac{k}{m|\rho|}+\frac{mz}{m|\rho|}\in\Lambda\right\}\quad(86)$$

and

$$\Lambda=\bigcup_{k\in B}\left[\left(\lambda_0+\frac{mz_k+k}{m|\rho|}+\frac{m\mathbb{Z}}{m|\rho|}\right)\cap\Lambda\right]$$

by noting that Lemma (6.2.18) implies $\Lambda-\lambda_0\subset\frac{Z}{m|\rho|}$. Furthermore, the above union is disjoint. Let

$$\Gamma_k=\mathbb{Z}\cap\left(|\rho|\Lambda-|\rho|\lambda_0-\frac{k+mz_k}{m}\right),\qquad k\in B,$$

then

$$\Lambda=\bigcup_{k\in B}\left(\lambda_0+\frac{k+mz_k}{m|\rho|}+|\rho|^{-1}\Gamma_k\right),\qquad \Gamma_k\subset\mathbb{Z}.$$

We will prove that E_{Γ_k} is an orthonormal basis of $L^2(\mu_\rho)$ for each $k\in B$ and $B=\{0,1,\dots,m-1\}$.

For any $k\in B$, let $z,\ell\in\Gamma_k$ be distinct. Let

$$a=\lambda_0+\frac{mz_k+k}{m|\rho|}+\frac{mz}{m|\rho|},\qquad b=\lambda_0+\frac{mz_k+k}{m|\rho|}+\frac{m\ell}{m|\rho|}.$$

Then a,b belong to Λ, so $a-b=\frac{mt+s}{m|\rho|^n}$ for some integers t,s,n such that $0<s<m$ and $n>0$, hence $z-\ell=|\rho|(a-b)=\frac{mt+s}{m|\rho|^{n-1}}$. Noting $|\rho|=q/p$, we see that $m|p^{n-1}(mt+s)$, so $n>1$ by using $0<s<m$. Hence $z-\ell\in Z_\rho$ by Lemma (6.2.5). This means that E_{Γ_k} is an orthonormal set of $L^2(\mu_\rho)$ for all $k\in B$ by Lemma (6.2.6).

Since

$$\sum_{k=0}^{m-1}\exp\left\{2\pi\frac{(j_1-j_2)k}{m}i\right\}=\begin{cases}0, & j_1\neq j_2\\ m, & j_1=j_2\end{cases}\quad(87)$$

for any $j_1,j_2\in\{0,1,\dots,m-1\}$, so

$$\sum_{k=0}^{m-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi\left(\rho t-\rho\lambda_0-\frac{k}{m}\right)i\right\}\right)^j\right|^2$$

$$=\frac{1}{m^2}\sum_{k=0}^{m-1}\sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1}exp\left\{2\pi(j_1-j_2)\left(\rho t-\rho\lambda_0-\frac{k}{m}\right)i\right\}$$

$$=\frac{1}{m^2}\sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1}exp\{2\pi(j_1-j_2)(\rho t-\rho\lambda_0)i\}\sum_{k=0}^{m-1}exp\left\{2\pi\frac{(j_1-j_2)k}{m}i\right\}$$

$$=\frac{1}{m^2}\sum_{j_1}^{m-1}m=1$$

for all t. Hence, by $\Gamma_k\subseteq\mathbb{Z}$, we have

$$1=\sum_{\lambda\in\Lambda}|\hat{\mu}(t-\lambda)||^2=\sum_{\lambda\in\Lambda}|\hat{\mu}_\rho(\rho t-\rho\lambda)||^2\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi(\rho t-\rho\lambda)i\})^j\right|^2$$

$$=\sum_{k\in B}\sum_{\gamma\in\Gamma_k}|\hat{\mu}_\rho(\rho t-\rho\lambda_0-\frac{k+mz_k}{m}$$

$$-\gamma)||^2\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi(\rho t-\rho\lambda_0-\frac{k+mz_k}{m}-\gamma)i\})^j\right|^2$$

$$=\sum_{k\in B}\sum_{\gamma\in\Gamma_k}|\hat{\mu}_\rho(\rho t-\rho\lambda_0-\frac{k+mz_k}{m}-\gamma)||^2\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi(\rho t-\rho\lambda_0-\frac{k}{m})i\})^j\right|^2$$

$$\leq\sum_{k\in B}\left|\frac{1}{m}\sum_{j=0}^{m-1}(exp\{2\pi(\rho t-\rho\lambda_0-\frac{k}{m})i\})^j\right|^2$$

$$\leq\sum_{k=0}^{m-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi\left(\rho t-\rho\lambda_0-\frac{k}{m}\right)i\right\}\right)^j\right|^2=1$$

for all t. Therefore, the above two inequalities are equalities. The first one means that Γ_j is a spectrum of $L^2(\mu_\rho)$ for each $j\in B$. The second one means that $B=\{0,1,\dots,m-1\}$. While the conclusion $0\in\Gamma_k$ follows from the assumption $\lambda_0+\frac{mz_k+k}{m|\rho|}\in\Lambda$. The theorem is proven.

Repeated application, there exist spectrums $\Gamma_{k_1,\dots,k_n}$ containing zero such that

$\Lambda=\cup_{k_1=0}^{m-1}\cup_{k_2=0}^{m-1}\cdots\cup_{k_n=0}^{m-1}\left(\lambda_0+\frac{k_1+mz_{k_1}}{m|\rho|}+\frac{k_2+mz_{k_1,k_2}}{m|\rho|^2}+\cdots+\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}+\right.$

$\left.|\rho|^{-n}\Gamma_{k_1,\dots,k_n}\right)$ (88)

and the union is a disjoint union, where integers $z_{k_1,\dots,k_j}$ $(j=1,2,\dots,n)$ are chosen to satisfy $\lambda_0+\frac{k_1+mz_{k_1}}{m|\rho|}+\frac{k_2+mz_{k_1,k_2}}{m|\rho|^2}+\cdots+\frac{k_j+mz_{k_1,\dots,k_j}}{m|\rho|^j}\in\Lambda$ for $j=1,2,\dots,n$.

First, by (88) and Lemma (6.2.18), we have $|\rho|^{-n}\Gamma_{k_1,\dots,k_n} \subseteq \Lambda - \Lambda \subseteq \frac{\mathbb{Z}}{m|\rho|}$. Since $|\rho| = q/p$ and p, q are co-prime, we see $\Gamma_{k_1,\dots,k_n} \subseteq q^n \frac{\mathbb{Z}}{m|\rho|}$ and

$$\frac{k_n + mz_{k_1,\dots,k_n}}{m|\rho|^n} \in |\rho|^{1-n}\Gamma_{k_1,\dots,k_{n-1}} \subseteq \frac{p^{n-1}\mathbb{Z}}{m|\rho|}. \quad (89)$$

Second, as we have assumed (86) for the case $n = 1$, repeated application of Theorem 4.4 shows that we can choose $z_{k_1,\dots,k_j}$ so that $\frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}$ has the minimal absolute value in the set $\frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} + |\rho|^{-n}\Gamma_{k_1,\dots,k_n}$. This implies that $z_{k_1,\dots,k_j} = 0$ whenever $k_j = 0$.

For any given sequence $\{k_j\}_j$ of $\{0, 1, \dots, m-1\}$, denote $a_n = \lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}$, then $|a_n| \geq |a_{n-1}|$.

Claim. There exists a constant $c \in \mathbb{N}$ so that $|a_n| \geq p^{n-c}$ whenever $k_n \neq 0$.

Proof of the Claim. If a_n, a_{n-1} have the same sign, then $|a_n| = |a_{n-1}| + \left|\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}\right|$ by using $|a_n| \geq |a_{n-1}|$. Using (89) implies $a_n - a_{n-1} = \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} \in \frac{p^{n-1}\mathbb{Z}}{m|\rho|}$ is non-zero, so $|a_n| \geq \frac{p^{n-1}}{m|\rho|}$. Otherwise, a_n, a_{n-1} have different signs. Since $|a_n| \geq |a_{n-1}|$, so $|a_n| \geq \frac{1}{2}|\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}|$. Using (89) implies $\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} \in \frac{p^{n-1}\mathbb{Z}}{m|\rho|}$ is non-zero, so $|a_n| \geq \frac{p^{n-1}}{2m|\rho|}$. The claim is proven.

For any $\lambda \in \Lambda\backslash\{\lambda_0\}$, we can find an integer n_0 so that $|\lambda| < p^{n_0-c}$, hence the above claim shows that λ does not belong to the set $\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} + |\rho|^{-n}\Gamma_{k_1,\dots,k_n}$ for any $n \geq n_0$ with $k_n \neq 0$. On the other hand, however, (88) shows that $\lambda \in \lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} + |\rho|^{-n}\Gamma_{k_1,\dots,k_n}$ for some n. Hence λ can be written as the form $\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}$ for some n.

Therefore, we have proven the following theorem.

Theorem(6.2.20) [453]: If μ_ρ is a spectral measure with spectrum Λ and $|\rho| = q/p$ with $gcd(q, m) = 1$, then $m|p$ and Λ has the form

$$\Lambda = \left\{\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} : k_j = 0, 1, \dots, m-1; n > 0\right\} \quad (90)$$

such that $|\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}| \geq p^{n-c}$ if $k_n \neq 0$ and this sequence of absolute values increases as n increases for any given sequence $\{k_j\}_j$ of $\{0, 1, \dots, m-1\}$.

Assume μ_ρ is a spectral measure with spectrum Λ. Since m is a prime, Theorem (6.2.3) implies that $|\rho| = (q/p)^{\frac{1}{r}}$ with $gcd(p, q) = 1$ and $m|p$. Proposition (6.2.17) shows that $r = 1. m|p$ and $gcd(p, q) = 1$ imply $gcd(m, q) = 1$. Hence Theorems (6.2.19) and (6.2.20) applicable. Hence Λ has the form as (90).

Therefore, we need only prove $q = 1$. Assume, on the contrary, that $q > 1$.

Use Theorem (6.2.20), we define

$$\Lambda_n = \left\{\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} : k_j = 0, 1, \dots, m-1\right\} \quad (91)$$

for $n = 1, 2, \dots$. Then Theorem (6.2.20) implies

$$\lambda \in \Lambda \backslash \Lambda_n \Rightarrow |\lambda| \geq p^{n+1-c}. \quad (92)$$

Using Theorems (6.2.19) and (6.2.20), we see that $\frac{k_n + mz_{k_1,\ldots,k_n}}{m|\rho|^n} \in |\rho|^{-1}\mathbb{Z}$ for $n > 1$. Hence $q^{n-1}|(k_n + mz_{k_1,\ldots,k_n})$. Since we have proven $|\rho| = q/p$ and $m|p$, so

$$\rho^{\ell} \frac{k_n + mz_{k_1,\ldots,k_n}}{m|\rho|^n} \in \mathbb{Z}, \quad \forall_1 \leq \ell < n. \quad (93)$$

Using (87), we have

$$\sum_{k_n=0}^{m-1} \left\| \left[\frac{1}{m} \sum_{j=0}^{m-1} \left(\exp\left\{ 2\pi j \left(t - \frac{k_n}{m} \right) i \right\} \right) \right] \right\|^2 = \frac{1}{m^2} \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} \sum_{k_n=0}^{m-1} exp\left\{ 2\pi (j_1 - j_2) \left(t - \frac{k_n}{m} \right) i \right\}$$

$$= \frac{1}{m^2} \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} exp\{2\pi (j_1 - j_2) t i\} \sum_{k_n=0}^{m-1} exp\left\{ 2\pi \frac{(j_1 - j_2) k_n}{m} i \right\}$$

$$= \frac{1}{m^2} \sum_{j_1=0}^{m-1} m = 1$$

for all t. Hence, by (67), Theorem (6.2.20) and (93), we have

$$\sum_{\lambda \in \Lambda_n} \prod_{\ell=1}^{n} \left\| \left[\frac{1}{m} \sum_{j=0}^{m-1} \left(exp\{2\pi \rho^{\ell} j (t - \lambda) i\} \right) \right] \right\|^2$$

$$= \sum_{\gamma \in \Lambda_{n-1}} \sum_{k_n=0}^{m-1} \prod_{\ell=1}^{n-1} \left| \frac{1}{m} \sum_{j=0}^{m-1} exp\left\{ 2\pi \rho^{\ell} j \left(t - \gamma - \frac{k_n + mz_{k_1,\ldots,k_n}}{m|\rho|^n} \right) i \right\} \right|^2$$

$$= \sum_{\gamma \in \Lambda_{n-1}} \sum_{k_n=0}^{m-1} \prod_{\ell=1}^{n-1} \left| \frac{1}{m} \sum_{j=0}^{m-1} exp\{2\pi \rho^{\ell} j (t - \gamma)\} \right|^2$$

$$\cdot \left| \frac{1}{m} \sum_{j=0}^{m-1} exp\left\{ 2\pi j \left(\rho^n (t - \gamma) - \frac{k_n + mz_{k_1,\ldots,k_n}}{m} \right) i \right\} \right|^2$$

$$= \sum_{\gamma \in \Lambda_{n-1}} \prod_{\ell=1}^{n-1} \left| \frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi \rho^{\ell} j (t - \gamma) i\}) \right|^2 \cdot \left| \frac{1}{m} \sum_{j=0}^{m-1} exp\left\{ 2\pi j \left(\rho^n (t - \gamma) - \frac{k_n}{m} \right) i \right\} \right|^2$$

$$= \sum_{\gamma \in \Lambda_{n-1}} \prod_{\ell=1}^{n-1} \left| \frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi \rho^{\ell} j (t - \gamma) i\}) \right|^2$$

for all t and $n > 1$.

Repeatedly using the above equality gives

$$\sum_{\lambda \in \Lambda_n} \prod_{\ell=1}^{n} \left| \frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi \rho^{\ell} j (t - \gamma) i\}) \right|^2 = \sum_{k_1=0}^{m-1} \left| \frac{1}{m} \sum_{j=0}^{m-1} exp\left\{ 2\pi \rho j \left(t - \lambda_0 - \frac{k_1 + mz_{k_1}}{m|\rho|} \right) i \right\} \right|^2 = 1. \quad (94)$$

Let N be an integer with $N > a^{-1}$, where a is defined in Theorem (6.2.15). Let

$$Q_{\ell}(t) = \sum_{\lambda \in \Lambda_{\ell N}} |\hat{\mu}_{\rho}(t - \lambda)|^2, \ell \in \mathbb{N}.$$

Then

$$Q_{\ell+1}(t) - Q_\ell(t) = \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell N}} |\hat{\mu}_\rho(t-\lambda)|^2$$

$$= \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^s j(t-\lambda)i\}) \right] \right|^2 \cdot |\hat{\mu}_\rho(\rho^{(\ell+1)^N}(t-\lambda))|^2$$

$$\leq C_0^2 \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^s j(t-\lambda)i\}) \right] \right|^2 \cdot (\ln(2 + |\rho^{(\ell+1)^N}(t-\lambda)|))^{-2a}$$

$$\leq C_0^2 \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^s j(t-\lambda)i\}) \right] \right|^2 \cdot (ln(2 + |\rho^{(\ell+1)^N} \rho^{\ell^N - c - 1}|))^{-2a}$$

$$= C_0^2 \left[1 - \sum_{\lambda \in \Lambda_{\ell N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^s j(t-\lambda)i\}) \right] \right|^2 \right] (ln(2 + |\rho^{(\ell+1)^N} \rho^{\ell^N - c - 1}|))^{-2a}$$

for any $t \in (-p^{-c-1}, p^{-c-1})$, where the first equality follows from (67), the first inequality follows from Theorem (6.2.15), the second inequality follows from (92), the last equality follows from (94).

By (67), we have

$$Q_\ell(t) = \sum_{\lambda \in \Lambda_{\ell N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^s j(t-\lambda)i\}) \right] \right|^2 \cdot |\hat{\mu}_\rho(\rho^{(\ell+1)^N}(t-\lambda))|^2$$

$$\leq \sum_{\lambda \in \Lambda_{\ell N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (exp\{2\pi\rho^s j(t-\lambda)i\}) \right] \right|^2 .$$

Therefore,

$$1 - Q_{\ell+1}(t) \geq [1 - Q_\ell(t)][1 - C_0^2(\ln(2 + |\rho^{(\ell+1)^N} p^{\ell^N - c - 1}|))^{-2a}],$$

and so

$$1 - Q_{\ell+1}(t) \geq [1 - Q_\ell(t)] \prod_{k=n}^{\ell} \left[1 - C_0^2 \left(\ln\left(2 + \left|\rho^{(\ell+1)^N} p^{\ell^N - c - 1}\right|\right)\right)^{-2a} \right], \forall \ell > n. \quad (95)$$

The assumption $N > \frac{1}{a}$ shows that $\sum_{k=1}^{\infty} k^{-2Na} < +\infty$. Hence $\prod_{k=1}^{\ell} [1 - k^{-2Na}]$ converges to a positive number. Since

$$\lim_{\ell \to +\infty} \frac{(ln(2 + |\rho^{(\ell+1)^N} p^{\ell^N - c - 1}|))^{-2a}]}{\ell^{-2Na}} = \lim_{\ell \to +\infty} \left(\frac{(\ell+1)^N \ln|\rho| + \ell^N \ln p}{\ell^N} \right)^{-2a}$$

$$= (\ln q)^{-2a} > 0.$$

Hence, we can find $\ell_0 > 0$ so that

$$[1 - C_0^2(\ln(2 + |\rho^{(k+1)^N} p^{k^N - c - 1}|))^{-2a}] = a_0 \in (0, 1).$$

Therefore, by noting that Λ is a spectrum of μ_ρ, (95) shows

$$0 = 1 - \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(t-\lambda)|^2 = \lim_{\ell\to+\infty}[1 - Q_{\ell+1}(t)] \geq a_0[1 - Q_{\ell_0}(t)] \geq 0$$

for any $t \in (-p^{-c-1}, p^{-c-1})$. Hence $Q_{\ell_0}(t) = 1$ for all $t \in (-p^{-c-1}, p^{-c-1})$. Since $Q_{\ell_0}(t)$ can be extended to an analytic function on the complex plane, so $Q_{\ell_0}(t) = 1$ for all $t \in \mathbb{R}$. Hence $\Lambda_{\ell_0^N}$ is a spectrum. It is obviously impossible, as $\Lambda_{\ell_0^N}$ is a finite set.

Therefore, $q = 1$, so Theorem (6.2.1) is proven.

Corollary (6.2.21) [486]: $Z_{\sum_i \rho_i} = \left\{\frac{\ell}{\sum_i m\rho_i^k} : k \in \mathbb{N}, \ell \in \mathbb{Z}\backslash m_{\mathbb{Z}}\right\}$.

Proof. Use, since $\hat{\mu}_{\sum_i \rho_i}(\rho_i^n t) \to 1$ as $n \to +\infty$, $\hat{\mu}_{\sum_i \rho_i}(\lambda) = 0$ if and only if there exists a positive integer $k > 0$ so that

$$\sum_{j=0}^{m-1} \sum_i (e^{2\pi\rho_i^k \lambda i})^j = 0.$$

Hence $\sum_i e^{2\pi\rho_i^k \lambda i} \neq 1$, multiplying both sides by $1 - \sum_i e^{2\pi\rho_i^k \lambda i}$, we see that the above equation is equivalent to

$$1 - \sum_i \left(e^{2\pi\rho_i^k \lambda i}\right)^m = 0, \qquad \sum_i e^{2\pi\rho_i^k \lambda i} \neq 1.$$

Hence $\hat{\mu}_{\sum_i \rho_i}(\lambda) = 0$ if and only if there exist integers $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $2m\pi\rho_i^k \lambda = 2\ell\pi$ and $\rho_i^k \lambda$ is not an integer, i.e. $\lambda = \frac{\ell}{\sum_i m\rho_i^k}$ with $\ell \in \mathbb{Z}\backslash m_{\mathbb{Z}}$ and $k \in \mathbb{N}$, the conclusion follows.

Corollary (6.2.22) [486]: Let Λ be a subset of $\mathbb{R}$ containing 0, then E_Λ is an orthonormal set of $L^2(\mu_{\sum_i \rho_i})$ if and only if $(\Lambda - \Lambda)\backslash\{0\} \subset Z_{\sum_i \rho_i}$. Equivalently, the following two conditions are satisfied:

(i) $\Lambda = \{0\} \cup \left\{\frac{\ell_j}{\sum_i m\rho_i^{k_j}} : 1 \leq j < N\right\}$ with $k_j \in \mathbb{N}, \ell_j \in \mathbb{Z}\backslash m\mathbb{Z}, 0 < j < N$, where N is a finite positive integer or the infinity.

(ii) There exist $\frac{v_{1+\epsilon,1+2\epsilon}}{\sum_i m\rho_i^{r_{1+\epsilon,1+2\epsilon}}} \in Z_{\sum_i \rho_i}$ with $v_{1+\epsilon,1+2\epsilon} \in \mathbb{Z}\backslash m\mathbb{Z}$ such that

$$\sum_i \frac{\ell_{1+\epsilon}}{m\rho_i^{k_{1+\epsilon}}} - \sum_i \frac{\ell_{1+2\epsilon}}{m\rho_i^{k_{1+2\epsilon}}} = \sum_i \frac{v_{1+\epsilon,1+2\epsilon}}{m\rho_i^{r_{1+\epsilon,1+2\epsilon}}}, \qquad 0 < \epsilon < N.$$

Proof. It is clear that E_Λ is an orthonormal set in $L^2(\mu_{\sum_i \rho_i})$ if and only if $\hat{\mu}_{\sum_i \rho_i}(\lambda_1 - \lambda_2) = 0$ for any distinct $\lambda_1, \lambda_2 \in \Lambda$. Hence

(ii) follows from, and (i) follows from the assumption $0 \in \Lambda$.

Corollary (6.2.23) [486]: If $L^2(\mu_{\sum_i \rho_i})$ has an orthonormal set of exponential functions with at least $m + 1$ elements, then ρ_i^{-1} is a zero of an integral polynomial.

Proof. Let E_Λ be an orthonormal set with at least $m + 1$ elements. Then the N defined in is at least $m + 1$. Hence holds for all $0 \leq 1 + \epsilon < 1 + 2\epsilon \leq m$.

If $k_{1+\epsilon} = k_{1+2\epsilon} = r_{1+\epsilon,1+2\epsilon}$ for all $1 \leq 1 + \epsilon < 1 + 2\epsilon \leq m$, then $\ell_{1+\epsilon} - \ell_{1+2\epsilon} = v_{1+\epsilon,1+2\epsilon}$ for all $1 \leq 1 + \epsilon < 1 + 2\epsilon \leq m$. It is clear that there exist $(1 + \epsilon)$ and $1 + 2\epsilon$ such that $0 \leq 1 + \epsilon < 1 + 2\epsilon \leq m$ and $\ell_{1+\epsilon} - \ell_{1+2\epsilon} \in m\mathbb{Z}$. Hence $v_{1+\epsilon,1+2\epsilon} \in m\mathbb{Z}$, it contradict Therefore, there is a pair $(1 + \epsilon, 1 + 2\epsilon)$ so that at least two of $k_{1+\epsilon}, k_{1+2\epsilon}, r_{1+\epsilon,1+2\epsilon}$ are distinct, so ρ_i^{-1} is a zero of an integral polynomial.

Corollary (6.2.24) [486]: Assume $\beta \in \mathbb{O}$ admits a minimal polynomial $(1+2\epsilon)\beta^r - (\pm 1)^r(1+\epsilon) = 0$ and satisfies $a_1\beta^k + a_2\beta^j = a_3\beta^u$, where $k, j, u \geq 0$ are nonnegative integers and $a_1, a_2, a_3 \in \mathbb{Z}\backslash\{0\}$. Then $k \equiv j \equiv u(mod\ r)$.

Proof. Let $k = k_1 r + 1 + \epsilon, j = j_1 r + 1 + 2\epsilon, u = u_1 r + v$ with $0 \leq 1+\epsilon, 1+2\epsilon, v < r$. Since $(1+2\epsilon)\beta^r = (\pm 1)^r(1+\epsilon)$, so β satisfies

$$b_1\beta^{1+\epsilon} + b_2\beta^{1+2\epsilon} = b_3\beta^v$$

for some integers $b_1, b_2, b_3 \neq 0$. Since $(1+2\epsilon)x^r - (\pm 1)^r(1+\epsilon)$ is the minimal polynomial of β with order r, in view of $0 \leq 1+\epsilon, 1+2\epsilon, v < r$, we see that $1+\epsilon = v$, the conclusion follows.

Corollary (6.2.25) [486]: Let $(1+2\epsilon)x^r - (1+\epsilon)$ be the minimal polynomial of $|\rho_i|$ for some $(1+2\epsilon), (1+\epsilon), r \in \mathbb{N}$, where $(1+2\epsilon), (1+\epsilon)$ are co-prime. Then $L^2(\mu_{\sum_i \rho_i})$ contains an infinite orthonormal set of exponential functions if and only if $(1+2\epsilon), m$ have a common divisor larger than one.

Proof. Consider the necessity. Let E_Λ be an infinite orthonormal set with $0 \in \Lambda$, then $\Lambda\backslash\{0\} \subseteq Z_{\sum_i \rho_i}$.

For any $x \in Z_{\sum_i \rho_i}$, there exist $u \in \mathbb{N}$ and $v \in \mathbb{Z}\backslash m\mathbb{Z}$ such that $x = \sum_i \frac{v}{m|\rho_i|^u}$ by Furthermore, there exist integers $\ell \geq 0$ and $v_0 \in \mathbb{Z}$ so that $v = (1+2\epsilon)^l v_0$ with $(1+2\epsilon) \nmid v_0$. Hence $\sum_i \frac{v}{m|\rho_i|^u} = \sum_i \frac{(1+\epsilon)^l v_0}{m|\rho_i|^{u+rl}}, (1+2\epsilon) \nmid v_0$ and $m \nmid (1+2\epsilon)^l v_0$. Since $\Lambda - \Lambda \subseteq \{0\} \cup Z_{\sum_i \rho_i}$. Let k_1 be the smallest positive integer such that $\sum_i \frac{\ell_1}{m|\rho_i|^{k_1}} \in \Lambda - \Lambda \subseteq \{0\} \cup Z_{\sum_i \rho_i}$ for some $\ell_1 \in (\mathbb{Z}\backslash\ (1+2\epsilon)\mathbb{Z})$.

Let $\lambda_1, \lambda_0 \in \Lambda$ be such that $\lambda_1 - \lambda_0 = \sum_i \frac{\ell_1}{m|\rho_i|^{k_1}}$.

For any $\lambda \in \Lambda$, implies that there exists a $\lambda' \in Z_{\sum_i \rho_i}$ such that $\sum_i \frac{\ell_1}{m\rho_i^{k_1}} - (\lambda - \lambda_0) = \lambda_1 - \lambda = \lambda'$. Write $\lambda = \lambda_0 + \sum_i \frac{(1+\epsilon)^{1+\epsilon}\ell}{m|\rho_i|^{k+r(1+\epsilon)}}$ and $\lambda' = \sum_i \frac{(1+\epsilon)^{1+2\epsilon}v}{m|\rho_i|^{u+r1+2\epsilon}}$ with the properties: $\ell, v \in (\mathbb{Z}\backslash\ (1+2\epsilon)\mathbb{Z})\backslash m\mathbb{Z};\ k, u \in \mathbb{N};\ \epsilon \geq 0$. Then we have

$$\sum_i \frac{\ell_1}{m|\rho_i|^{k_1}} - \sum_i \frac{(1+\epsilon)^{1+\epsilon}\ell}{m|\rho_i|^{k+r1+\epsilon}} = \sum_i \frac{(1+\epsilon)^{1+2\epsilon}v}{m|\rho_i|^{u+r1+2\epsilon}}.$$

Therefore, implies that $k - k_1 \equiv u - k_1 \equiv 0\ (mod\ r)$. Let $k + r(1+\epsilon) = nr + k_1, u + r(1+2\epsilon) = u'r + k_1$, then the definition of k_1 implies $n, u' \geq 0$. The above equality becomes

$$\ell_1 - (1+\epsilon)^{1+\epsilon-n}(1+2\epsilon)^n\ell = (1+\epsilon)^{1+2\epsilon-u'}v(1+2\epsilon)^{u'}.$$

If $n = 0$, then $\lambda = \lambda_0 + \sum_i \frac{(1+\epsilon)^{1+\epsilon}\ell}{m|\rho_i|^{k_1}}$. If $n > 0$, note that $(1+2\epsilon), (1+\epsilon)$ are co-prime, if $u' > 0$, then the above equality implies $(1+2\epsilon)|\ell_1$, a contradiction. Hence $u' = 0$, so the above equality implies $(1+\epsilon)^n|(1+\epsilon)^{1+\epsilon}\ell$. Hence λ can be written as the form $\lambda = \lambda_0 + \sum_i \frac{(1+\epsilon)^n\ell'}{m|\rho_i|^{k_1+rn}} = \sum_i \frac{(1+2\epsilon)^n\ell'}{m|\rho_i|^{k_1}}$ with $\ell' \in \mathbb{Z}$. Therefore, there exist non-zero integers z_j such that

$$\Lambda = \{\lambda_0\} \cup \{\lambda_j\}_{j=0}^{+\infty} \text{ with } \lambda_j = \lambda_0 + \sum_i \frac{z_j}{m|\rho_i|^{k_1}} (j > 0).$$

(ii) implies that $\sum_i \frac{z_{1+\epsilon} - z_{1+2\epsilon}}{m|\rho_i|^{k_1}} \in Z_{\sum_i \rho_i}$ for all distinct $\epsilon \geq 0$. Choose $\epsilon \geq 0$ so that $m|(z_{1+2\epsilon} - z_{1+\epsilon})$, let $z_{1+2\epsilon} - z_{1+\epsilon} = mz$.

Then implies that there exist $u_{1+2\epsilon,1+\epsilon} \in \mathbb{Z}\backslash m\mathbb{Z}, v_{1+2\epsilon,1+\epsilon} \in \mathbb{N}$ such that

$$\sum_i \frac{z}{|\rho_i|^{k_1}} = \lambda_{1+2\epsilon} - \lambda_{1+\epsilon} = \sum_i \frac{u_{1+2\epsilon,1+\epsilon}}{m|\rho_i|^{v_{1+2\epsilon,1+\epsilon}}}.$$

implies that $v_{1+2\epsilon,1+\epsilon} - k_1 = r\xi_{1+2\epsilon,1+\epsilon}$ for some integer $\xi_{1+2\epsilon,1+\epsilon}$. Hence

$$\left(\frac{1+\epsilon}{1+2\epsilon}\right)^{\xi_{1+2\epsilon,1+\epsilon}} = \frac{u_{1+2\epsilon,1+\epsilon}}{mz}.$$

The definition of k_1 implies $\xi_{1+2\epsilon,1+\epsilon} \geq 0$. Note that $\nmid u_{1+2\epsilon,1+\epsilon}$, we see that $\xi_{1+2\epsilon,1+\epsilon} > 0$ and $(1+2\epsilon), m$ have a common divisor larger than one. The necessity follows.

We now prove the sufficiency. Suppose that $(1+2\epsilon), m$ have a common divisor larger than one. Let $m_0 > 1$ be the greatest common divisor of m and $(1+2\epsilon)$. Since $(1+2\epsilon), (1+\epsilon)$ are co-prime, so $m_0, (1+\epsilon)$ are co-prime.

Let

$$\hat{\Lambda} = \{0\} \cup \left\{\sum_i \frac{(1+\epsilon)^n}{m|\rho_i|^{nr}} : n \in \mathbb{N}\right\}.$$

Since $m_0, (1+\epsilon)$ are co-prime and $m_0 > 1$ is the greatest common divisor of m and $(1+2\epsilon)$, so neither $\frac{(1+\epsilon)^n}{m}$ nor $\frac{(1+\epsilon)^n((1+2\epsilon)^k-1)}{m}$ is an integer for all $n, k \in \mathbb{N}$. Therefore, both $\frac{(1+\epsilon)^n}{m|\rho_i|^{nr}}$ and

$$\sum_i \frac{(1+\epsilon)^{n+k}}{m|\rho_i|^{nr+kr}} - \sum_i \frac{(1+\epsilon)^n}{m|\rho_i|^{nr}} = \sum_i \frac{(1+\epsilon)^n((1+2\epsilon)^k - 1)}{m|\rho_i|^{nr}}$$

are zeros of $\hat{\mu}_{\sum_i \rho_i}(1+\epsilon)$ for all $n, k \in \mathbb{N}$ by using. Hence $(\hat{\Lambda} - \hat{\Lambda})\backslash\{0\} \subset Z_{\sum_i \rho_i}$, so $E_{\hat{\Lambda}}$ is an infinite orthonormal set in $L^2(\mu_{\sum_i \rho_i})$. The sufficiency follows.

To prove for the case $\rho_i \notin \mathbb{O}$. We will suppose on the contrary that $L^2(\mu_{\sum_i \rho_i})$ has an infinite orthonormal set of exponential functions for some $\rho_i \notin \mathbb{O}$, then obtain a contradiction.

Let E_Λ be an infinite orthonormal set in $L^2(\mu_{\sum_i \rho_i})$, implies that ρ_i^{-1} is a zero of an integral polynomial. Let

$$g(x) = a_0 + a_1 x + \cdots + a_n x^n$$

be the minimal integral polynomial of ρ_i^{-1}, where $a_0 > 0, a_1, \ldots, a_n$ are relative prime and $a_n \neq 0$.

It is easy to see that the following lemma holds

Corollary (6.2.26) [486]: Let E_Λ be an orthonormal set in $L^2(\mu_{\sum_i \rho_i})$ and $\rho_i \notin \mathbb{O}$, let $\Lambda_k = \sum_i \frac{Z\backslash mZ}{m\rho_i^k} \cap \Lambda$, then $\{\Lambda_k\}_{k>0}$ are disjoint and each one has cardinality at most $m-1$.

Proof. For the disjointness of $\{\Lambda_k\}_{k>0}$, suppose on the contrary,$\Lambda_k \cap \Lambda_r \neq \emptyset$ for some $0 < k < r$. Then there exist $\sum_i \frac{1+2\epsilon}{m\rho_i^k} \in \Lambda_k$ and $\sum_i \frac{1+\epsilon}{m\rho_i^r} \in \Lambda_r$ so that $\sum_i \frac{1+2\epsilon}{m\rho_i^k} = \sum_i \frac{1+\epsilon}{m\rho_i^r}$. Hence

$$\rho_i = \pm\left(\left|\frac{1+\epsilon}{1+2\epsilon}\right|\right)^{\frac{1}{r-k}} \in \mathbb{O},$$

a contradiction.

For the cardinality of Λ_k. If Λ_k has at least m elements, let $\sum_i \frac{\ell_j}{m\rho_i^k} \in \Lambda_k, j = 1, 2, \ldots, m$. Then there exist $1 \leq j_1 < j_2 \leq m$ so that $\ell_{j_1} - \ell_{j_2} = rm$ for a non-zero integer r. Hence implies that

$$\frac{r}{\rho_i^k} = \sum_i \frac{u}{m\rho_i^v}$$

for some $u \in \mathbb{Z}\backslash m\mathbb{Z}$ and $v \in \mathbb{N}$. This means that $v \neq k$ and

$$\rho_i = \left(\frac{u}{mr}\right)^{\frac{1}{v-k}} (v > k) \; or \; \rho_i = \left(\frac{mr}{u}\right)^{\frac{1}{k-v}} (v < k),$$

a contradiction to the assumption $\rho_i \notin O$. We have (see [490])

Corollary (6.2.27) [486]: Assume $\rho_i \notin \mathbb{Q}$. If $L^2(\mu_{\sum_i \rho_i})$ has an infinite orthonormal set of exponential functions, then ρ_i^{-1} has a minimal polynomial $g(x)$ with $a_0 = 1$.

Proof. implies that ρ_i^{-1} is a zero of an integral polynomial. Hence ρ_i^{-1} has a minimal polynomial $g(x)$ as given. Since $\rho_i \notin \mathbb{Q}$, it is easy to see that $z\rho_i^{-k} \notin Z_{\sum_i \rho_i}$ for all $z \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Since $L^2(\mu_{\sum_i \rho_i})$ contains an infinite orthonormal set of exponential functions E_Λ, implies that there exist infinitely many $\lambda_j = \sum_i \frac{\ell_j}{m\rho_i^{k_j}} \in \Lambda$ with $0 < k_j < k_{j+1}$. Without loss of generality, we assume

$$\Lambda = \{0\} \cup \left\{\sum_i \frac{\ell_j}{m\rho_i^{k_j}} : 0 < k_1 < k_2 < \cdots\right\}.$$

we can assume that $\frac{\ell_j}{m} \notin \mathbb{Z}$. then implies that there exist $v_{1+2\epsilon,1+\epsilon} \in \mathbb{Z}\backslash m\mathbb{Z}, r_{1+2\epsilon,1+\epsilon} \in \mathbb{N}$ such that

$$\sum_i \frac{\ell_{1+2\epsilon}}{m\rho_i^{k_{1+2\epsilon}}} - \sum_i \frac{\ell_{1+\epsilon}}{m\rho_i^{k_{1+\epsilon}}} = \sum_i \frac{v_{1+2\epsilon,1+\epsilon}}{m\rho_i^{r_{1+2\epsilon,1+\epsilon}}}, \qquad \forall \epsilon \neq 0 \in \mathbb{N}.$$

Since $\rho_i \notin \mathbb{Q}$ and $k_{1+2\epsilon} \neq k_{1+\epsilon}$ when $\epsilon \neq 0$, it is easy to see that $r_{1+2\epsilon,1+\epsilon} \neq k_{1+2\epsilon}, k_{1+\epsilon}$ when $\epsilon \neq 0$.

Claim. $r_{1+2\epsilon,j} \to +\infty$ as $j \to +\infty$ for all $1 + 2\epsilon \in \mathbb{N}$.

Section (6.3): Three-Element Digit Sets

For μ be a probability measure with compact support on $\mathbb{R}^n$. We call it a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda := \{e^{2\pi i\langle \xi, \lambda\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. The set Λ is then called a spectrum for μ. The existence of spectrum for μ was initiated by Fuglede in his seminal [468]. The first example of a singular, non-atomic, spectral measure was constructed by Jorgensen and Pedersen in [464]. This surprising discovery received a lot of attention. The spectral property of fractal measures becomes an active research area, and more spectral fractal measures were found in [469] and [470]. The spectral property, Fourier transform [469] and Cauchy transform [469] of fractal measure form the main topics in the analysis on fractals.

[469] and [471], constructed some classes of Moran spectral measures.

we focus on certain Moran measures with three-element digit sets. Before the statement of the main results.

Let

$$D_n = \{0, a_n, b_n\} = \{0, 1, 2\}(mod\ 3) and\ p_n \in 3\mathbb{Z}^+ \quad (96)$$

be a digit set in $\mathbb{Z}$ with $|a_n| < |b_n|$ and an integer for all $n \geq 1$, respectively, we always assume that

$$c := \lim_{n\to\infty} \sup \frac{|a_n|}{P_n} < \infty, \quad d := lim_{n\to\infty}\, sup\, \frac{|a_n|}{P_n} < \infty. \text{ (97)}$$

Write $P_n = p_1 p_2 \cdots p_n$. Under the assumptions of (96) and (97), then associated to the sequence $\{p_n, D_n\}$, there exists a Borel probability measure $\mu_{\{p_n\},\{D_n\}}$, which is defined by the following infinite convolutions of finite measures:

$$\mu_{\{p_n\},\{D_n\}} = \delta_{P_1^{-1}D_1} * \delta_{P_2^{-1}D_2} * \cdots \quad (98)$$

in the weak convergence, where $*$ is the convolution sign, $\delta_E = \frac{1}{\#E}\sum_{e\in E} \delta_e$, $\#E$ is the cardinality of a set E and δ_e is the Dirac measure at the point $e \in \mathbb{R}$. It is known that the support of $\mu_{\{p_n\},\{D_n\}}$ is the Moran set:

$$T(\{p_n\},\{D_n\}) = \sum_{n=1}^{\infty} P_n^{-1} D_n .$$

Strichartz [19] first studied the spectrality of $\mu_{\{R_n\},\{B_j\}}$ where expanding matrices $\{R_j\}_{j=1}^{\infty}$ and digit sets $\{B_j\}_{j=1}^{\infty}$ satisfy $\#\{R_1, R_2, \cdots\}, \#\{B_1, B_2, \cdots\} < \infty$. In [465], Ding considered the case p_n and D_n defined by (1.1) such that $D_n \subset \mathbb{N}, \#\{a_n, b_n : n \geq 1\} < \infty$ and $gcd(a_n, b_n) = 1$ for $n \geq 1$. , we will study the spectrality of $\mu_{\{p_n\},\{D_n\}}$ defined by (98) without the boundedness assumptions : $\#\{a_n, b_n : n \geq 1\} < \infty$.

Theorem(6.3.1)[464]: Let $\mu_{\{p_n\},\{D_n\}}$ be defined by (98) where p_n and D_n satisfy (96)-(97). Suppose that one of the following conditions holds:

(a) $\lim_{n\to\infty} sup\, \frac{|a_n|}{P_n} < \min\{\frac{2}{3}, \frac{2}{3c}\}$;

(b) $1 \leq a_n < b_n$ for all $n \geq 1$, and $\lim_{n\to\infty} inf\, \frac{b_n}{P_n} < \min\{1, \frac{1}{d}\}$.

Then $\mu_{\{p_n\},\{D_n\}}$ is a spectral measure with spectrum $\Lambda = (\sum_{i=1}^{n} P_i\{0, u_i/3, v_i/3\})$ for some choice $\{u_i\},\{v_i\}$ satisfying $u_i, v_i \in \mathbb{Z}\backslash 3\mathbb{Z}$ and $u_i \neq v_i (mod\ 3)$ for all i, where $P_i = \prod_{k=1}^{i} p_k$.

From Theorem (6.3.1), we have the following corollary.

Corollary (6.3.2) [464]: Let $\mu_{\{p_n\},\{D_n\}}$ be defined by (98) where p_n and D_n satisfy (96)-(97). If $c < 2/3$ (or $d < 1$ and $1 \leq a_n < b_n$), then $\mu_{\{p_n\},\{D_n\}}$ is a spectral measure.

Let μ be a probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined by $\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x)$. Then for our case, we have

$$\hat{\mu}_{\{p_n\},\{D_n\}}(\xi) = \prod_{n=1}^{\infty} M_{a_n,b_n}(P_n^{-1}\xi), \quad (99)$$

where

$$M_{a_n,b_n}(\xi) = \frac{1}{3}\left(1 + e^{-2\pi i a_n \xi} + e^{-2\pi i b_n \xi}\right) \quad (100)$$

is known as the mask polynomial of D_n. We use $Z_h := \{\xi : h(\xi) = 0\}$ to denote the zero set of the function h. It is easy to see from $\{a_n, b_n\} = \{1, 2\}(mod\ 3)$ and (99)-(100) that

$$Z_{M_{a_i,b_i}} = \frac{1}{3\ gcd(a_i, b_i)}(\mathbb{Z}\backslash 3\mathbb{Z}) \ \ and\ Z_{\widehat{\mu}_{\{p_n\},\{D_n\}}} = \bigcup_{i=1}^{\infty} \frac{1}{3\ gcd(a_i, b_i)}(\mathbb{Z}\backslash 3\mathbb{Z}) . \text{ (101)}$$

We say that Λ is an orthogonal set for μ if E_Λ is an orthonormal family for $L^2(\mu)$. It is easy to show that Λ is an orthogonal set for μ if and only if $\hat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i \neq \lambda_j \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda)\backslash\{0\} \subseteq Z_{\hat{\mu}}. \quad (102)$$

Let $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2$. Using the Parseval identity, we have the following basic criterion for orthogonality of E_Λ in $L^2(\mu)$.

Proposition (6.3.3) [464]: [454] Let μ be a probability measure in $\mathbb{R}^2$ with compact support, and let $\Lambda \subset \mathbb{R}^n$ be a countable subset. Then
(i) Λ is an orthonormal set for μ if and only if $Q_\Lambda(\xi) \le 1$ for $\xi \in \mathbb{R}^n$; and
(ii) Λ is a spectrum for μ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}^n$.
, we write

$$\mu_n = \delta_{P_1^{-1}D_1} * \cdots * \delta_{P_n^{-1}D_n} \; and \; \mu_{>n} = \delta_{P_{n+1}^{-1}D_{n+1}} * \delta_{P_{n+2}^{-1}D_{n+2}} * \cdots . \quad (103)$$

Then $\hat{\mu}_{\{p_n\},\{D_n\}} = \mu_n * \mu_{>n}$. We say $u \in (\mathbb{Z}\backslash 3\mathbb{Z})^{\mathbb{N}}$ if $u = \{u_i\}_{i=1}^{\infty}$ with all $u_i \in \mathbb{Z}\backslash 3\mathbb{Z}$. For any $u, v \in (\mathbb{Z}\backslash 3\mathbb{Z})^{\mathbb{N}}$ with all $u_i \ne v_i (mod\ 3)$, we write

$$\Lambda_n^{u,v} := \sum_{i=1}^{n} P_i\{0, u_i/3, v_i/3\} \; and \; \Lambda^{u,v} := \cup_{n=1}^{\infty} \Lambda_n^{u,v} \; . \quad (104)$$

The following lemma is used to illustrate that $\Lambda^{u,v}$ is an orthogonal set for $\mu_{\{p_n\},\{D_n\}}$.
Lemma (6.3.4) [464]: Let p_n and D_n be defined by (96). Then the set $\Lambda_k^{u,v}$ is a spectrum for μ_k, and $\Lambda^{u,v}$ is an orthogonal set for $\mu_{\{p_n\},\{D_n\}}$ for any $u, v \in (\mathbb{Z}\backslash 3\mathbb{Z})^{\mathbb{N}}$ with $u_i \ne v_i (mod\ 3)$ for all i.
Proof From (101) and (102), it is easy to check that $\Lambda_k^{u,v}$ and $\Lambda^{u,v}$ are orthogonal sets for μ_k and $\mu_{\{p_n\},\{D_n\}}$, respectively. Moreover, $\Lambda_k^{u,v}$ is a spectrum for μ_k because the dimension of the space $L^2(\mu_k)$ is 3^k, which is the cardinality of the set $\Lambda_k^{u,v}$.
We first state a proposition, which is weaker than Theorem (6.3.1). Indeed, we only use " sup " instead of " lim sup " in the hypotheses of Theorem (6.3.1).
Proposition (6.3.5) [464]: Let $\mu_{\{p_n\},\{D_n\}}$ be defined by (98) where p_n and D_n satisfy (96) and $\tilde{c} := \sup_{n\ge1} \frac{|a_n|}{p_n} \le \tilde{d} := \sup_{n\ge1} \frac{|b_n|}{p_n} < \infty$. In addition, suppose that one of the following conditions holds: (i) $\lim_{n\to\infty} inf \frac{|a_n|}{P_n} < min\{\frac{2}{3}, \frac{2}{3\tilde{c}}\}$; (ii) $0 < a_n < b_n$ for all $n \ge 1$ and $lim_{n\to\infty} inf \frac{b_n}{P_n} < min\{1, \frac{1}{\tilde{d}}\}$.
Then $\mu_{\{p_n\},\{D_n\}}$ is a spectral measure.
To prove Proposition (6.3.5), we need the following lemmas. Noting that $p_n \in 3\mathbb{Z}^+$, we have $P_n = p_1 p_n \cdots p_n \ge 3^n$, which will be used many times. We use the notation $\lceil x \rceil$ to denote the minimum integer greater than or equal to $x \in \mathbb{R}$, i.e., $\lceil x \rceil := \min\{n \in \mathbb{Z}: n \ge x\}$, and define a positive integer

$$N_0 := \lceil 3 + log(\max\{1, \tilde{d}\})/log\ 3 \rceil \quad (105)$$

Lemma (6.3.6) [464]: With the hypotheses of Proposition (6.3.5)(i), let $\mu_{>n}$ be defined by (103) for $n \ge 1$ and $u = -v = \{1\}_{i=1}^{\infty}$. Then there exist an increasing positive integer sequence $\{n_k\}_{k=1}^{\infty}$ with $n_k - n_{k-1} \ge N_0$, and a constant $\alpha > 0$ such that
(i) $|\hat{\mu}_{>n_k}(\xi + \lambda)| \ge \alpha$ for all $\lambda \in \Lambda_{n_k}^{u,v}, k \ge 1$ and $|\xi| \le 1$;
(ii) $|\hat{\mu}_{>n_k}(\xi + \lambda)| \ge \exp(-9^{N_0+1-n_k+n_{k-1}})$ for all $\lambda \in \Lambda_{n_{k-1}}^{u,v}, k > 1$ and $|\xi| \le 1$.
Proof: For fixed $r \in (\lim_{n\to\infty} inf \frac{|a_n|}{P_n}, min\{\frac{2}{3}, \frac{2}{3\tilde{c}}\})$, we choose a positive integer l such that

$$\frac{3^l+2}{2\cdot 3^l} < \min\left\{\frac{1}{3r}, \frac{1}{3\tilde{c}r}\right\} \quad (106)$$

We can find an increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$n_k \ge l, n_{k+1} - n_k \ge N_0, \frac{|a_{n_k+1}|}{p_{n_k+1}} \le r, \quad \forall k \ge 1. \quad (107)$$

For convenience, we write

$$J_j(\xi + \lambda) := |M_{a_j,b_j}(P_j^{-1}(\xi + \lambda))| = \frac{1}{3}1 + e^{-2\pi i P_j^{-1} a_j(\xi+\lambda)} + e^{-2\pi i P_j^{-1} b_j(\xi+\lambda)}$$

and

$$\eta_{j,1} = \eta_{j,1}(\xi+\lambda) = -\frac{a_j(\xi+\lambda)}{P_j}, \eta_{j,2} = \eta_{j,2}(\xi+\lambda) = -\frac{b_j(\xi+\lambda)}{P_j}. \quad (108)$$

It follows from (103) that $|\hat{\mu}_{>n_k}(\xi+\lambda)| = \prod_{j=n_k+1}^{\infty} J_j(\xi+\lambda)$. Now, we estimate $J_j(\xi+\lambda)$ for $j \geq n_k+1$. For

$$\lambda \in \Lambda_n^{u,v} = \sum_{i=1}^{n} P_i\{0, 1/3, -1/3\} \text{ (see (104) with } u = -v = \{1\}_{i=1}^{\infty} \quad (109)$$

we can find $\{c_i\}_{i=1}^{n} \in \{0,1,-1\}^n$ such that $\lambda = 3^{-1}\sum_{i=1}^{n} c_i P_i$, and hence $|\lambda| \leq 3^{-1}\sum_{i=1}^{n} P_i$.

Noting that $p_n \in 3\mathbb{Z}^+$ and $\frac{P_i}{P_j} = (\prod_{\ell=i+1}^{j} p_\ell)^{-1} \leq 3^{-j+1}$ for $i < j$, we have

$$\frac{|\xi+\lambda|}{P_{n_j}} \leq \frac{|\xi|}{3^{n_j}} + \frac{\sum_{i=1}^{n_j} P_i}{3P_{n_j}} \leq \frac{1}{3^{n_j}} + \sum_{i=1}^{n_j}\frac{1}{3^i} \leq \frac{1}{3^l} + \frac{1}{2}, \lambda \in \Lambda_{n_j}^{u,v}, |\xi| \leq 1. \quad (110)$$

Combining (107) and (110),

$$\begin{cases} |\eta_{n_k+1,1}(\xi+\lambda)| = \frac{|a_{n_{k+1}}|}{p_{n_{k+1}}}\frac{|\xi+\lambda|}{P_{n_k}} \leq r\left(\frac{1}{3^l}+\frac{1}{2}\right), \\ |\eta_{j,1}(\xi+\lambda)| = \frac{|a_j|}{p_j}\frac{|a_{n_{k+1}}(\xi+\lambda)|}{P_{n_k+1}}\frac{1}{|a_{n_{k+1}}|}\frac{P_{n_{k+1}}}{P_{j-1}} \leq \tilde{c}r\left(\frac{1}{3^l}+\frac{1}{2}\right)\frac{1}{3^{j-n_k-2}}, j \geq n_k+2 \end{cases}, (111)$$

where $\sup_{n\geq 1}\frac{|a_n|}{P_n} = \tilde{c}$. Put $\varepsilon = \frac{1}{3} - (\frac{1}{3^l}+\frac{1}{2})\max\{\tilde{c}r, r\}$, (106) implies $\varepsilon \in (0, \frac{1}{3})$. By (111),

$$(\eta_{j,1}(\xi+\lambda), \eta_{j,2}(\xi+\lambda)) \in [-\tfrac{1}{3}+\epsilon, \tfrac{1}{3}-\epsilon\,] \times \mathbb{R} = \cup_{k\in\mathbb{Z}}\big(\Omega + (0,k)\big) (112)$$

for $|\xi| \leq 1, \lambda \in \Lambda_{n_k}^{u,v}$ and $j \geq n_k+1$, where $\Omega := [-\frac{1}{3}+\varepsilon, \frac{1}{3}-\varepsilon] \times [0,1]$. We consider the continuous function

$$f(x,y) := \frac{1}{3}\left|1 + e^{2\pi i x} + e^{2\pi i y}\right|, \qquad (x,y)^t \in \mathbb{R}^2.$$

It is easy to see that $f(\eta_{j,1}, \eta_{j,2}) = J_j(\xi+\lambda)$ and $Z_f = \{(m, 2m+k)/3 : m \in \mathbb{Z}\backslash 3\mathbb{Z}, k \in 3\mathbb{Z}\}$, where Z_f is the zero set of f. Since $dist(\Omega, Z_f) > 0$ (see Figure 1), it follows from (112) and the integer-periodicity of f that

$$\min_{|\xi|\leq 1, \lambda\in\Lambda_{n_k}^{u,v}} J_j(\xi+\lambda) \geq \min_{(x,y)\in\Omega} f(x,y) := \beta > 0, j \geq n_k+1.$$

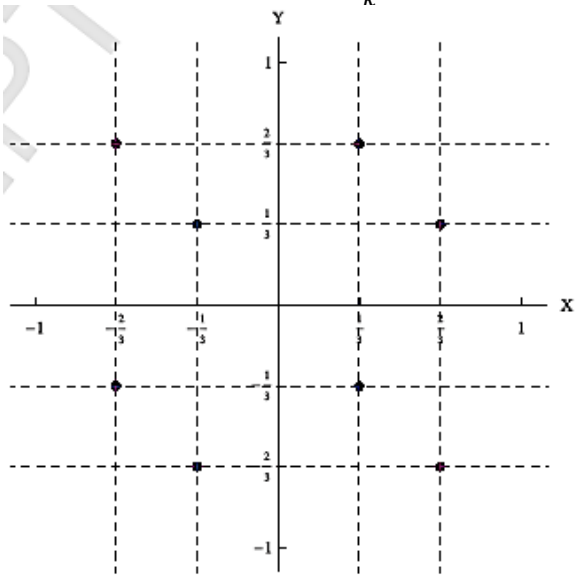

Figure 1[464]: The zero set Z_f of f.

Thus for integer $p \geq 1$,

$$\prod_{j=n_k+1}^{n_k+p} J_j(\xi+\lambda) \geq \beta^p > 0, |\xi| \leq 1, \lambda \in \Lambda_{n_k}^{u,v}. \quad (113)$$

In the following, we estimate the lower bound of $\prod_{j=n_k+N_0+1}^{\infty} J_j(\xi+\lambda)$ where N_0 is given in (105). It is easy to check that $3^{2-N_0}\tilde{d} \le 1$ where $\tilde{d} = \sup_{n\ge 1}\frac{|b_n|}{p_n}$. Similar to (111), we have

$$\left|\eta_{j,2}(\xi+\lambda)\right| = \frac{|b_j|}{p_j}\frac{|\xi+\lambda|}{P_{n_k}}\frac{P_{n_k}}{P_{j-1}} \le \frac{3^l+2}{2\cdot 3^l}\frac{\tilde{d}}{3^{N_0-2}}\frac{1}{3^{j+1-n_k-N_0}} \le \frac{3^l+2}{2\cdot 3^l}\frac{1}{3^{j+1-n_k-N_0}}$$

for $j \ge n_k+N_0+1, |\xi| \le 1$ and $\lambda \in \Lambda_{n_k}^{u,v}$. Together with (106) and (111), it yields

$$2\pi\left|\eta_{j,1}(\xi+\lambda)\right|, 2\pi\left|\eta_{j,1}(\xi+\lambda)\right| \le 3^{-j+n_k+N_0+1}, j \ge n_k+N_0+1, |\xi| \le 1, \lambda \in \Lambda_{n_k}^{u,v}. \quad (114)$$

Hence $J_j(\xi+\lambda) \ge \frac{1}{3}|1+cos\,\eta_{j,1}+cos\,\eta_{j,2}| \ge \frac{1}{3}(1+2\,cos\,3^{-j+n_k+N_0+1})$ for $j \ge n_k+N_0+1$.

Applying $cos\,x \ge 1-\frac{1}{2}x^2$, we obtain that

$$\prod_{j=n_k+N_0+1}^{\infty} J_j(\xi+\lambda) \ge \prod_{j=n_k+N_0+1}^{\infty}\frac{1}{3}\left(1+2\cos\frac{1}{3^{j-n_k-N_0-1}}\right) \ge \prod_{j=0}^{\infty}\left(1-\frac{1}{3}\frac{1}{9^j}\right) > \frac{1}{2} \quad (115)$$

for $|\xi| \le 1$ and $\lambda \in \Lambda_{n_k}^{u,v}$. This together with (113) gives $|\hat{\mu}_{>n_k}(\xi+\lambda)| = \prod_{j=n_k+1}^{\infty} J_j(\xi+\lambda) > \frac{1}{2}\beta^{N_0} := \alpha$ for $|\xi| \le 1, \lambda \in \Lambda_{n_k}^{u,v}$ and $k \ge 1$, and (i) follows.

In the following, we prove (ii). Let $\lambda \in \Lambda_{n_{k-1}}^{u,v}, k > 1$ and $|\xi| \le 1$. By (107), $j \ge n_k+1$ implies $j \ge n_{k-1}+N_0+1$; hence (3.10) becomes

$$2\pi\left|\eta_{j,1}(\xi+\lambda)\right|, 2\pi\left|\eta_{j,2}(\xi+\lambda)\right| < 3^{-j+n_{k-1}+N_0+1}, j \ge n_k+1. \quad (116)$$

Similar to (115), we have

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \prod_{j=0}^{\infty}\left(1-\frac{1}{3}\frac{1}{9^{j+n_k-n_{k-1}-N_0}}\right) = \exp\Big(\sum_{j=0}^{\infty}\log\left(1-\frac{1}{3}\frac{1}{9^{j+n_k-n_{k-1}-N_0}}\right)\Big)$$

$$\ge \exp\Big(\sum_{j=0}^{\infty}-\frac{5}{3}\frac{1}{9^{j+n_k-n_{k-1}-N_0}}\Big) \ge \exp\Big(\frac{-1}{9^{n_k-n_{k-1}-N_0-1}}\Big),$$

where the second inequality follows from $\log(1-x) \ge -5x$ $(0 \le x \le 4/5)$. The proof of (ii) is complete.

Lemma (6.3.7) [464]: Under the conditions of Proposition (6.3.5)(ii), then there exist an increasing positive integer sequence $\{n_k\}_{k=1}^{\infty}$ satisfying $n_k - n_{k-1} \ge N_0$ where N_0 is given in (105), two sequences $u, v \in (\mathbb{Z}\backslash 3\mathbb{Z})^{\mathbb{N}}$, and a constant $\beta > 0$ such that

(i) $|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \beta$ for all $\lambda \in \Lambda_{n_k}^{u,v}, k > 1$ and $\xi \in [0,1]$;

(ii) $|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \exp(-9^{N_0+1-n_k+n_{k-1}})$ for all $\lambda \in \Lambda_{n_{k-1}}^{u,v}, k > 1\ and\ \xi \in [0,1]$.

Proof: Note that $\lim_{n\to\infty} inf\,\frac{b_n}{P_n} < min\{1, \frac{1}{\tilde{d}}\}$ and $0 < a_n < b_n$ $(n \ge 1)$. For fixed $r \in \left(\lim_{n\to\infty} inf\,\frac{b_n}{P_n}, \min\{1, \frac{1}{\tilde{d}}\}\right)$, we can find a integer $l \ge N_0$ such that

$$\frac{3^{l-1}+1}{3^l-1} < \min\left\{\frac{1}{3r}, \frac{1}{3\tilde{d}r}\right\}. \quad (117)$$

It is obvious that there exists an increasing positive integer sequence $\{n_k\}_{k=1}^{\infty}$ with $n_{k+1}-n_k > l$ and $n_1 \ge l$ such that

$$\frac{a_{n_k+1}}{p_{n_k+1}} < \frac{b_{n_k+1}}{p_{n_k+1}} \le r < \min\left\{1, \frac{1}{\tilde{d}}\right\} \le \min\left\{1, \frac{1}{\tilde{c}}\right\}, k \ge 1. \quad (118)$$

Define $\delta = \{\delta_i\}_{i=1}^{\infty} \in \{-1,1\}^{\mathbb{N}}$ such that $\delta_i = 1\ for\ i \in \{n_k : k \ge 1\}$ and $\delta_i = -1$ otherwise.

Let $u=2v=\{2\delta_i\}_{i=1}^{\infty}$. For

$$\lambda\in\Lambda_n^{u,v}=\sum_{i=1}^n\delta_iP_i\,\{0,1/3,2/3\}\quad\text{(see (104))},$$

we can find $\{c_i\}_{i=1}^n\in\{0,1,2\}^n$ such that $\lambda=3^{-1}\sum_{i=1}^n c_i\delta_iP_i$. Hence

$$\frac{|\lambda|}{P_{n_k}}\le\frac{2\sum_{i=1}^{n_k-1}P_i}{3P_{n_k}}\le\frac{2}{3}\sum_{i=1}^{n_k-1}\frac{1}{3^i}<\frac{1}{3},\lambda\in\Lambda_{n_k-1}^{u,v},k\ge1.\quad(119)$$

We further claim that

$$0<\frac{\lambda}{P_{n_k}}\le\frac{2\cdot3^{l-1}}{3^l-1},\quad\lambda\in\Lambda_{n_k}^{u,v}\backslash\Lambda_{n_k-1}^{u,v},\ k\ge1.\quad(120)$$

In fact, $\lambda\in\Lambda_{n_k}^{u,v}\backslash\Lambda_{n_k-1}^{u,v}$ implies $c_{n_k}\in\{1,2\}$ and $\delta_{n_k}=1$, and hence

$$\frac{\lambda}{P_{n_k}}=\sum_{i=1}^{n_k}\frac{c_i\delta_iP_i}{3P_{n_k}}=\frac{c_{n_k}}{3}+\sum_{i=1}^{n_k-1}\frac{c_i\delta_i}{3}(\prod_{j=i+1}^{n_k}p_j)^{-1}\ge\frac{1}{3}-2\sum_{i=1}^{n_k-1}\frac{1}{3^{n_k-i+1}}>0.$$

On the other hand,

$$\frac{\lambda}{P_{n_k}}=\sum_{i=1}^{k}\frac{c_{n_j}P_{n_j}}{3P_{n_k}}-\sum_{1\le i\le n_k-1,i\ne n_j}\frac{c_iP_i}{3P_{n_k}}\le\sum_{i=1}^{k}\frac{2}{3}\frac{P_{n_i}}{3P_{n_k}}\le\frac{2}{3}\sum_{i=1}^{k}\frac{1}{3^{n_k-n_i}}<\frac{2\cdot3^{l-1}}{3^l-1},$$

where the last inequality follows from $n_k-n_i=\sum_{j=1}^{k-1}(n_{j+1}-n_j)\ge(k-i)l$. Hence, (120) holds.

We now prove Lemma (6.3.7) (i). For $\lambda\in\Lambda_{n_k}^{u,v}=\Lambda_{n_{k-1}}^{u,v}\cup(\Lambda_{n_k}^{u,v}\backslash\Lambda_{n_{k-1}}^{u,v})$, we divide it into two situations for discussion: $\lambda\in\Lambda_{n_{k-1}}^{u,v}$ and $\lambda\in\Lambda_{n_k}^{u,v}\backslash\Lambda_{n_{k-1}}^{u,v}$.

Case I: $\lambda\in\Lambda_{n_{k-1}}^{u,v}$ and $\xi\in[0,1]$. Then (119) implies $\frac{|\xi+\lambda|}{P_{n_k}}<\frac{1}{3^l}+\frac{1}{3}<\frac{3^{l-1}+1}{3^l-1}$. Using this instead of (110), and applying a similar argument as in Lemma (6.3.6)(i), we can prove that there exists $c_0>0$ such that $|\hat{\mu}_{>n_k}(\xi+\lambda)|\ge c_0$ for $\lambda\in\Lambda_{n_{k-1}}^{u,v},\xi\in[0,1]$.

Case II: $\lambda\in\Lambda_{n_k}^{u,v}\backslash\Lambda_{n_k-1}^{u,v}$ and $\xi\in[0,1]$. By (118) and (120), we have

$$\begin{cases}0\le\frac{\max\{a_{n_k+1},b_{n_k+1}\}}{p_{n_k+1}}\frac{\xi+\lambda}{P_{n_k}}\le r(\frac{1}{3^{(k-1)l+n_1}}+\frac{2\cdot3^{l-1}}{3^l-1})<2r\frac{3^{l-1}+1}{3^l-1},\\ 0\le\frac{max\{a_j,b_j\}}{p_j}\frac{\xi+\lambda}{P_{n_k+1}}\frac{P_{n_k+1}}{P_{j-1}}\le2\tilde{d}r\ \frac{3^{l-1}+1}{3^l-1}\frac{1}{3^{j-n_k-2}},j\ge n_k+2,k\ge1.\end{cases}\quad(121)$$

Let $\epsilon:=\frac{2}{3}-2r\max\{1,\tilde{d}\}\ \frac{3^{l-1}+1}{3^l-1}$. Then $\epsilon\in(0,\frac{2}{3})$ by (117). This together with (121) gives $(\eta_{j,1},\eta_{j,2})\in[-\frac{2}{3}+\epsilon,0]^2:=D$ for $j>n_k$, where $\eta_{j,1},\eta_{j,2}$ are in (108). Similar to (113), we have

$$\prod_{j=n_k+1}^{n_k+3}J_j(\xi+\lambda)\ge\left(\min_{(x,y)\in D}f(x,y)\right)^3:=\kappa^3>0,\lambda\in\Lambda_{n_k}^{u,v}\backslash\Lambda_{n_k-1}^{u,v},\xi\in[0,1].\quad(122)$$

It is easy to see, by (117) and (121), that $2\pi|\eta_{j,1}|,2\pi|\eta_{j,2}|\le4\pi\frac{3^{l-1}+1}{3^l-1}\frac{\tilde{d}r}{3^{j-n_k-2}}<\frac{1}{3^{j-n_k-4}}$ for $j\ge n_k+4$. Hence $\prod_{j=n_k+4}^{\infty}J_j(\xi+\lambda)>\frac{1}{2}$ (a similar argument as in (115)). This and (122) give $|\hat{\mu}_{>n_k}(\xi+\lambda)|>\frac{1}{2}\kappa^3$ for $\lambda\in\Lambda_{n_k}^{u,v}\backslash\Lambda_{n_k-1}^{u,v},\xi\in[0,1]$.

The above two cases tell us that (i) holds for $\beta:=\min\{c_0,\frac{1}{2}\kappa^3\}$.

The proof of (ii) is similar to that of Lemma (6.3.6)(ii); we only need to use $2\pi|\eta_{j,1}|$, $2\pi|\eta_{j,2}|\le4\pi\frac{3^{l-1}+1}{3^l-1}\frac{\tilde{d}r}{3^{j-n_{k-1}-2}}<\frac{1}{3^{j-n_{k-1}-4}}$ for $j\ge n_k+1$ and $\lambda\in\Lambda_{n_{k-1}}^{u,v}$ instead of (116). Therefore,

$$|\hat{\mu}_{>n_k}(\xi+\lambda)|\ge\exp(-9^{4-n_k+n_{k-1}})\ge\exp(-9^{N_0+1-n_k+n_{k-1}}),$$

where the last inequality follows from $N_0 \geq 3$ (see (105)).

(i) Let $\Lambda^{u,v} = \cup_{n=1}^{\infty} \Lambda_n^{u,v}$, where $\Lambda_n^{u,v}$ is given in (109). Define

$$Q_m(\xi) = \sum_{\lambda \in \Lambda_m^{u,v}} |\hat{\mu}_{\{p_n\},\{D_n\}}(\xi+\lambda)|^2 \ and \sum_{\lambda \in \Lambda^{u,v}} |\hat{\mu}_{\{p_n\},\{D_n\}}(\xi+\lambda)|^2 .$$

Let $\{n_k\}_{k=1}^{\infty}$ be given in Lemma (6.3.6). It follows from $\hat{\mu}_{\{p_n\},\{D_n\}} = \hat{\mu}_{n_k}\hat{\mu}_{>n_k}$ and Lemma (6.3.6) (i) that $|\hat{\mu}_{\{p_n\},\{D_n\}}(\xi+\lambda)|^2 \geq \alpha^2 |\hat{\mu}_{n_k}(\xi+\lambda)|^2$ for $\lambda \in \Lambda_{n_k}^{u,v}$ and $|\xi| \leq 1$. By $\sum_{\lambda \in \Lambda_{n_k}^{u,v}} |\hat{\mu}_{n_k}(\xi+\lambda)|^2 = 1$ (see Lemma (6.3.4)), we have

$$Q_{n_k}(\xi) = Q_{n_{k-1}}(\xi) + \sum_{\lambda \in \Lambda_{n_k}^{u,v} \setminus \Lambda_{n_{k-1}}^{u,v}} |\hat{\mu}_{\{p_n\},\{D_n\}}(\xi+\lambda)|^2$$

$$\geq Q_{n_{k-1}}(\xi) + \alpha^2 \left(1 - \textstyle\sum_{\lambda \in \Lambda_{n_{k-1}}^{u,v}} \left|\hat{\mu}_{n_k}(\xi+\lambda)\right|^2\right), \quad |\xi| \leq 1. \ (123)$$

If $Q_{\Lambda^{u,v}}(\xi) \not\equiv 1 (\xi \in \mathbb{R})$, then by Proposition (6.3.3) and the uniqueness theorem of analytic function, we can find $|\xi_0| < 1$ such that $Q_{\Lambda^{u,v}}(\xi_0) < 1$. Let η_0 satisfy $\max\{Q_{\Lambda^{u,v}}(\xi_0), e^{-2}\} < \eta_0 < 1$. Further, by selecting subsequences, we can assume that $n_{k+1} - n_k \geq N_0 + 1 - \log_9 \ln \eta_0^{-1/2}$ for $k \geq 1$, where N_0 is given in (105). It follows from Lemma (6.3.6)(ii) that

$$|\hat{\mu}_{>n_k}(\xi_0+\lambda)| \geq \exp(-9^{N_0+1-n_k+n_{k-1}}) \geq \sqrt{\eta_0} > 0, \lambda \in \Lambda_{n_{k-1}}^{u,v} .$$

Since $|\hat{\mu}_{\{p_n\},\{D_n\}}(\xi_0+\lambda)|^2 = |\hat{\mu}_{n_k}(\xi_0+\lambda)|^2 |\hat{\mu}_{>n_k}(\xi_0+\lambda)|^2 \geq \eta_0 |\hat{\mu}_{n_k}(\xi_0+\lambda)|^2$, summing with respect to $\lambda \in \Lambda_{n_{k-1}}^{u,v}$, we have

$$\sum_{\lambda \in \Lambda_{n_k-1}^{u,v}} \left|\hat{\mu}_{n_k}(\xi_0+\lambda)\right|^2 \leq \frac{1}{\eta_0} \sum_{\lambda \in \Lambda_{n_k-1}^{u,v}} \left|\hat{\mu}_{\{p_n\},\{D_n\}}(\xi_0+\lambda)\right|^2 \leq \frac{1}{\eta_0} Q_{\Lambda^{u,v}}(\xi_0) < 1.$$

This and (123) give $Q_{n_k}(\xi_0) \geq Q_{n_{k-1}}(\xi_0) + \alpha^2(1 - \eta_0^{-1} Q_{\Lambda^{u,v}}(\xi_0))$. By recursion,

$$1 \geq Q_{\Lambda^{u,v}}(\xi_0) \geq Q_{n_k}(\xi_0) \geq Q_{n_1} + (k-1)\alpha^2(1 - \eta_0^{-1} Q_{\Lambda^{u,v}}(\xi_0) \to +\infty \ as \ k \to +\infty.$$

This contradiction shows $Q_{\Lambda^{u,v}}(\xi) \equiv 1 (\xi \in \mathbb{R})$, and the proof of Proposition (6.3.5)(i) is complete by Proposition (6.3.3).

The proof of Proposition (6.3.5)(ii) is similar to that of Proposition (6.3.5)(i), we only need to use Lemma (6.3.7) instead of Lemma (6.3.6).

For any positive integer N, we let $\tilde{c} = c_N = \sup_{n \geq N} \frac{|a_n|}{p_n}$ and $\tilde{d} = d_N = \sup_{n \geq N} \frac{|b_n|}{p_n}$. It is easy to see $\tilde{c} \geq c, \tilde{d} \geq d$ (see (97)). By the assumptions of Theorem (6.3.1)(a) and (b), we can find a sufficiently large N such that (i) $\lim_{n\to\infty} \inf \frac{|a_n|}{p_n} < \min\{\frac{2}{3}, \frac{2}{3cN}\} \leq \min\{\frac{2}{3}, \frac{2}{3c}\}$, (ii) $\lim_{n\to\infty} inf \frac{b_n}{p_n} < \min\{1, \frac{1}{d_N}\} \leq \min\{1, \frac{1}{d}\}$ under the assumption of $1 \leq a_n \leq b_n$.

Let $\mu_N = \delta_{P_1^{-1}D_1} * \cdots * \delta_{P_N^{-1}D_N}$ and $\mu_{N+} = \delta_{P_N P_{N+1}^{-1} D_{N+1}} * \delta_{P_N P_{N+2}^{-1} D_{N+2}} * \cdots$. Then $\mu_{\{p_n\},\{D_n\}} = \mu_N * (\mu_{N+} \circ P_N)$ and $\hat{\mu}_{\{p_n\},\{D_n\}}(\xi) = \hat{\mu}_N(\xi)\, \hat{\mu}_{N+}(P_N^{-1}\xi)$. Let $\Lambda_N = \sum_{i=1}^{N} P_i\{0, 1/3, -1/3\}$. It follows from Lemma (6.3.4) that Λ_N is spectrum for μ_N. On the other hand, by the proof of Proposition (6.3.5), it is clear that there exists a spectrum $\Lambda_{N+} \subset \mathbb{Z}$ for μ_{N+}. Now we show that $\Lambda_N + P_N \Lambda_{N+}$ is a spectrum for $\mu_{\{p_n\},\{D_n\}}$ by Proposition (6.3.3).

Note that $\hat{\mu}_N(\xi) = \prod_{i=1}^{N} M_{a_i,b_i}(P_i^{-1}\xi)$, where $M_{a_i,b_i}(\xi) = \frac{1}{3}(1 + e^{-2\pi i a_i \xi} + e^{-2\pi i b_i \xi})$. Since $P_N P_i^{-1} (i \leq N)$ is an integer and $M_{a_i,b_i}(\xi)$ has integer-periodicity, we can conclude that $\hat{\mu}_N(\xi + P_N\eta) = \hat{\mu}_N(\xi)$ for any $\eta \in \mathbb{Z}$, so

$$\sum_{\lambda\in\Lambda_N+P_N\Lambda_{N+}} |\hat{\mu}_{\{p_n\},\{D_n\}}(\xi+\lambda)|^2$$

$$= \sum_{\lambda_1\in\Lambda_N,\lambda_2\in\Lambda_{N+}} |\,\hat{\mu}_N(\xi+\lambda_1+P_N\lambda_2)|^2\,|\,\hat{\mu}_{N+}(P_N^{-1}(\xi+\lambda_1+P_N\lambda_2))|^2$$

$$= \sum_{\lambda_1\in\Lambda_N} |\,\hat{\mu}_N(\xi+\lambda_1)|^2 \sum_{\lambda_2\in\Lambda_{N+}} |\,\hat{\mu}_{N+}(P_N^{-1}(\xi+\lambda_1)+\lambda_2)|^2 \equiv 1.$$

Hence, we complete the proof.

List of Symbol

Symbol		Page
$\oplus$	Orthogonal sum	1
L^2	Hilbert space	1
det	Determinant	1
g.c.d	Greatest common divisor	1
mod	Modulo	1
min	Minimum	3
Dist	Distance	3
L.c.m	Least common multiple	5
deg	Degree	7
ISF	Iterated function system	17
max	Maximum	21
a.e	Almest every where	36
L^1	Lebesgue integral an the redline	36
mpf	Modulo predud -form	41
SSc	Strong separation condition	76
Sup	Supermom	83
L^∞	Essential lebesgue space	85
inf	Infimum	88
L^2	Hilbert space of sequences	92

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