كلية الدراسات العليا

Sudan University of Science and Technol

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Berezin Transforms on Noncommutative Varieties with Harmonic Bergman Spaces and of two Arguments

تحويلات بيرزين علي التشكيلات غير التبديلية مع فضاءات بيرجمان التوافقيه ولحجتين

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by:

Muzdalifa Mohammed Ahmed Mohammed

Supervisor :

Prof.Dr. Shawgy Hussein Abd Alla

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Dedication

To my Family

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Abstract

This study based on operator theory and joint similarity to operators on noncommutative varieties with strong deformation estimates deification and an excursion for the Berezin-Toeplitz quantization . The basic asymptotic expansion of the harmonic Berezin transform on the half-space and the determine Gauss hypergeometric function for a specific large constant are dealt with. The Berezin transform and perfect weighted reproducing Kernels with Toeplitz operators on harmonic Bergman spaces on the real ball are definitely considered and also in addition the Berezin transforms on noncommutative varieties in Polydomains and of two arguments are construded and weighted explained .

الخلاصة

هذه الدراسة أسست علي نظرية المؤثر والتشابه المشترك الي المؤثرات علي التشكيلات غير التبديلية مع تقديرات التنبؤات القوية والتطابق والانحراف لاجل تكميم بيرزين-تبوليتز. قمنا بالتعامل مع مفكوك المقاربة الاساسية لتحويل بيرزين التوافقي علي نصف الفضاء وتجديد دالة جاوس الهندسية الفائقة لاجل الثابت الكبير المحدد . قمنا بالاعتبار القطعي لتحويل بيرزين ونويات اعادة الانتاج المرجحة التامة مع مؤثرات تبوليتز علي فضاءات بيرجمان التوافقية علي الكرة الحقيقة وايضا اضافة تحويلات بيرزين علي التشكيلات غير التبديلية في مجالات متعددة وله حجتين وتم بناءه وله شرح علي نطاق واسع

Introduction

We develop a dilation theory for row contractions $T := [T_1, \ldots, T_n]$ subject to constraints such as $p(T_1, \ldots, T_n) = 0, p \in P$, where P is a set of noncommutative polynomials. The model n-tuple is the universal row contraction $[B_1, \ldots, B_n]$ satisfying the same constraints as T, which turns out to be, in a certain sense, the maximal constrained piece of the n-tuple $[S_1, \ldots, S_n]$ of left creation operators on the full Fock space on n generators. The theory is based on a class of constrained Poisson kernels associated with T and representations of the C^*-algebra generated by $B_1, \ldots, B_n$ and the identity. Under natural conditions on the constraints we have uniqueness for the minimal dilation. A characteristic function Θ_r is associated with any (constrained) row contraction T and it is proved that $I - \Theta_T \Theta_T^* = K_T K_T^*$
where K_T is the (constrained) Poisson kernel of T. Consequently, for pure constrained row contractions, we show that the characteristic function is a complete unitary invariant and provide a model.

Deformation estimates for the Berezin-Toeplitz quantization of C^n are obtained. We give a complete identification of the deformation quantization which was obtained from the Berezin-Toeplitz quantization on an arbitrary compact Kahler manifold. The deformation quantization with the opposite star-product proves to be a differential deformation quantization with separation of variables whose classifying form is explicitly calculated .

We present an introduction to the Berezin and Berezin-Toeplitz quantizations, starting from their historical origins and relationships with other quantization methods. We give a complete asymptotic expansion of the Berezin transform associated to the Bergman space of harmonic functions on the half-spacethat are square-integrable with respect to the measure $y^{\alpha}\, dx\, dy\ (\alpha \geq 0)\ as\ \alpha \to \infty$

We an attempt to unify the multivariable operator model theory for ball-like domains and commutative polydiscs, and extend it to a more general class of noncommutative polydomains. An important role in our study is played by noncommutative Berezin transforms associated with the elements of the polydomain. These transforms are used to prove that each such polydomain has a universal model consisting of weighted shifts acting on a tensor product of full Fock spaces. We introduce the noncommutative Hardy algebra as the weakly closed algebra generated by and the identity, and use it to provide a WOT-continuous functional calculus for completely non-coisometric tuples, which are identified. It is shown that the Berezin transform is a completely isometric isomorphism between and the algebra of bounded free holomorphic functions on the radial part . A characterization of the Beurling type joint invariant subspaces under is also provided.

We consider the Gauss hypergeometric function $F(a, b + 1; c + 2; z)$. We derive a convergent expansion of $F(a, b + 1; c + 2; z)$ in terms of rational functions of a, b, c and z valid for $|b||z| < |c - bz|$ and $|c - b||z| < |c - bz|$.

For the standard weighted Bergman spaces on the complex unit ball, the Berezin transform of a bounded continuous function tends to this function pointwise as the weight parameter tends to infinity. We show that this remains valid also of harmonic Bergman spaces on the real unit ball of any dimension. We describe the asymptotic behavior of the Berezin transform of two arguments which generalizes the standard notion of the Berezin transform

The Contents

Chapter 1
Operator Noncommutative Varieties

We show that the curvature invariant and Euler characteristic asssociated with a Hilbert module generated by an arbitrary (resp. commuting) row contraction T can be expressed only in terms of the (resp. constrained) characteristic function of T. We provide a commutant lifting theorem for pure constrained row contractions and obtain a Nevanlinna-Pick interpolation result. We use some of the results to provide Wold type decompositions and triangulations for n-tuples of operators in noncommutative varieties $V_{f,p}^{1}(H)$, which parallel the classical Sz.-Nagy–Foias triangulations for contractions but also provide new proofs. As consequences, we show the existence of joint invariant subspaces for certain classes of operators in $V_{f,p}^{1}(H)$.

Section (1.1): Operator Theory

There has been exciting progress in multivariable dilation theory , in the attempt to extend the classical Nagy-Foias theory of contractions [38].

In the noncommutative case ,significant results were obtained in [14], [9],[20],[21],[22],[24], and recently in [10]. Some of these results were further extended by Muhly and solel [17]to representations of tensor algebras over C^* −correspondences. We develops a dilation theory for row contractions subject to constrains determined by sets of noncomutative polynomials .The theory includes ,in particular , the commutative (see[13],[30],and [4]) and q-commuative (see [2],[7]) cases ,while the standard noncommuative dilation theory for row contractions serves as a ''universal model ''. An n-tuple $T:=[T_1, \dots, T_n]$ of bounded linear operators acting on common Hilbert space **H** is called row contraction if $TT^* = T_1 T^*_1 + \cdots + T_n T^*_n \le I$.

A distinguished role among row contractions is played by n-tuple $S:=[S_1, \dots, S_n]$of left creation operators on the full Fock space $F^2(H_n)$,which satisfies the noncommutative von Neumann inequality [25]

$$\| p(T_1, \dots, T_n)\| \le \| p(S_1, \dots, S_n)\|$$

For any polynomial $P(X_1, \dots, X_n)$ in n noncommuting indeterminates . for the classical von Neumann inequality [39] (case n=1) and a nice survey , see [19].

Based on the left creation operators and their representations , a noncommutative dilation theory and model theory for row contractions was developed in [14], [9], [20] ,[21], [22],[24] , ect .

Assume now that T is subject to the constraints

$$T_i T_j = T_j T_i, \qquad i, j = 1, \dots, n,$$

In this commutative case , the noncomutative dilation theory can be applied but , in many respects , it is not satisfactory due to the fact that the models shift $S:=[S_1, \dots, S_n]$ does not satisfy the same contraints as T . however , the universal commutative row contraction is a piece of S , namely

$$B:=[B_1, \dots, B_n], \ B_i := P_{F_s^2} S_i \big|_{F_s^2}, \ i=1, \dots, n,$$

Where$F_s^2 \subset F^2(H_n)$ is the symmetric Fock space . in this setting , the natural von Neumann inequality (see [13] ,[30] and [4])is

$$\| p(T_1, \dots, T_n)\| \le \| p(B_1, \dots, B_n)\|$$

For any polynomial $p(z_1, \dots, z_n)$ in n commuting indeterminates . A dilation theory for commuting row contractions based on the model shift $B:=[B_1, \dots, B_n]$ and its representations was considered by drury [13] and [30] to a certain extent , and by Arveson [4] in greater details this circle of ideas was extended to row contractions satisfying the constraints

$$T_j T_i = q_{ij} T_i T_j, \ 1 \le i < j \le n,$$

Where $q_{ij} \in \mathbb{C}$. In this setting , a von Neumann inequality was obtained by Arias and [2] . This was used further by B.V.R . Bhat and T . Bhattacharyya [7] to obtain a model theory for q-commuting row contractions .

we develop a dilation theory for row contractions $T:=[T_1, \dots, T_n]$ subject to more general constrains such as

$$p(T_1, \dots, T_n) = 0,\ p \in P,$$

Where p is a set of noncommuative polynomials . If T is a pruerow contraction , then p can be any WOT-sided ideal of noncomutative analytic . Toeplitz algebra F_n^∞ . The model n-tuple is the universal row contraction $[B_1, \dots, B_n]$ satisfying the same constraints as T , which turns out to be , in a certain sense , the maximal constrained piece of the n-tuple of left creation operators on the full Fock space with n generators .

we provide basic results concerning the constrained shift $[B_1, \dots, B_n]$ and the w^*-closed algebra generated by $B_1, \dots, B_n$ and the identity we obtain a Beurling type theorem characterizing the invariant subspace under each operator $B_1 \otimes I_{\mathcal{H}}, \dots, B_n \otimes I_{\mathcal{H}}$ and a characterizing of cyclic co- invariant subspaces under the same operators . We also provide Wold type decompositions for non degenerate *representations of C^*-algebra $C^*(B_1, \dots, B_n)$ and prove that two constrained shifts $[B_1 \otimes I_{\mathcal{H}}, \dots, B_n \otimes I_{\mathcal{H}}]$ and $[B_1 \otimes I_k, \dots, B_n \otimes I_k]$ are similar if and only if dim $\mathcal{H}$= dim k.

We develop a dilation theory for constrained row contractions . The theory is based on a class of constrained poission kernels (see[30] ,[2],[35], and [6] for n=1) associated with $T:=[T_1, \dots, T_n]$ and representations of the C^*- algebra generated by $B_1, \dots, B_n$ and the identity . In particular , if the set p consists of homogenous polynomials , then we show that there exists a Hilbert space k_π such that H can be identified with a subspace of $\tilde{K} :=(N_J \otimes \overline{\Delta_T H}) \oplus K_\pi$ and

$$T_i^* = V_i^* \big|_H ,\ i = 1, \dots n,$$

Where $\Delta_T := (I - T_1T_1^* - \cdots - T_nT_n^*)^{1/2}$,

$$V_i := \begin{bmatrix} B_i \otimes I_{\overline{\Delta_T H}} & 0 \\ 0 & \pi(B_i) \end{bmatrix},\ i=1, \dots, n,$$

And $\pi : C^*(B_1, \dots, B_n)$ is a *-representations which annihilates the compact operators and

$$\pi(B_1)\pi(B_1)^* + \dots + \pi(B_n)\pi(B_n)^* = I_{K_\pi}$$

Under certain natural conditions on the constrains , we have uniqueness for the minimal dilation of T . We introduce and evaluate the dilation index , a numerical invariant for row contractions , and show that it does not depend on the constraints .

we provide new properties for the standard characteristic function Θ_T associated with an arbitrary row contraction T (see [22]), and introduce a new characteristic function associated with constrained row contractions . the constrained characteristic function is a multi-analytic operator (with respect to the constrained shifs $B_1, \dots, B_n$)

$$\Theta_{J,T} : N_J \otimes D_{T^*} \to N_J \otimes D_T$$

Uniquely defined by the formel Fourier representation

$$-I_{N_J} \otimes T + \left(I_{N_J} \otimes \Delta_T\right)\left(I_{N_J \otimes H} - \sum_{i=1}^n W_i \otimes T_i^*\right)^{-1} [W_1 \otimes I_H, \dots, W_n \otimes I_H](I_{N_J} \otimes \Delta_{T^*})$$

. we prove a factorization result for the constrained characteristic function , namely

$$I - \Theta_{J,T}{\Theta^*}_{J,T} = K_{J,T}{K^*}_{J,T},$$

Where $K_{J,T}$ is the constrained Poisson kernel associated with T . Consequently , for the class of prue constrained row contractions , we show that the characteristic function is a complete unitary invariant and provide a model . All the results apply , in particular , to commutative row contractions We obtain a commutant lifting theorem for pure contractions and a Nevanlinna –pick interpolation result in our setting . These results are based on the more general non commutative commutant lifting theorem (see [21],[24]) and some results from previous . The above –mentioned factorization result for the characteristic function has important consequences in multivariable operator theory . We point out some of them which are considered, Arveson introduced a notion of curvature and Euler characteristic for finite rank contractive Hilbert modules over $[z_1, \dots, z_n]$ the complex unital algebra of all polynomials in n commuting variables .

The canonical operators $T_1, \dots, T_n$ is associated with the $\mathbb{C}[z_1, \dots, z_n]$–module structure are commuting and $T:=[T_1, \dots, T_n]$ is a row contraction with rank $\Delta_T < \infty$. Non commutative analogues of these notions were introduced and studied [32] and , independendently , by D.Kribs

[15] . We show that the curvature and the Euler characteristic (in both the commutative and non commutative case) depend only on the properties of the characteristic function of T .
For example , in the commutative case , if $T:=[T_1,\dots,T_n]$ is a commuatative row contraction with rank $\Delta_T < \infty$, *and* K(T) and $\mathcal{X}$(T)denote Arveson's curvature and Euler characteristic, respectively, then we prove that

$$K(T) = \int_{\partial\beta_n} \lim_{r\to 1} \text{trace}\,[\,I - \Theta_{J_c,T}(r\,\boldsymbol{\xi})\,\Theta_{J_c,T}(r\,\xi)^*]d\sigma(\boldsymbol{\xi})$$

$$= \text{rank}\,\Delta_T - (n\text{-}1)!\,\lim_{m\to\infty} \frac{trace\,[\Theta_{J_c,T}\ \Theta^*{}_{J_c,T}(Q_m \otimes I_{D_T})]}{n^m},$$

where Q_m is the projection of Arveson's space H^2 onto the subspace of homogeneous polynomials of degree m, and

$$(T)=n!\,\lim_{m\to\infty} \frac{rank[(I - \theta_{J_c,T}\ \theta^*{}_{J_c,T}(Q_{\le m} \otimes I_{D_T})}{m^n},$$

where $Q_{\le m}$ is the projection of H^2onto the subspace of all polynomials of degree ≤ m. Here, the operator $\theta_{.J_c,T} : H^2 \otimes D_{T^*} \to H^2 \otimes D_T$ stands for the constrained characteristic function associated with T, which, in this particular case, is a multiplier (multiplication operator) defined by its symbol (for which we use the same notation)

$$\theta_{J_c,T}(z) := -T + \Delta_T(I - z_1 T_1^* - \dots - z_n T_n^*)^{-1}(z_1 I_{\mathcal{H}},\dots,z_n I_{\mathcal{H}}]\Delta_{T^*}\ ,\ z \in \beta_n$$

which is a bounded operator-valued analytic function on the open unit ball of $\mathbb{C}^n$.
Constrained shifts: invariant subspaces and Wold decompositions
we provide basic results concerning the constrained shift $[B_1,\dots,B_n]$ associated with every WOT-closed two-sided ideal J of the noncommutative analytic Toeplitz algebra F^∞ . We obtain a Beurling type theorem characterizing the invariant subspaces under each operator $B_n \otimes I_{\mathcal{H}},\dots,B_n \otimes I_{\mathcal{H}}$, and a characterization of cyclic co-invariant subspaces under the same operators. We also provide Wold type decompositions for the non degenerate ∗-representations of the Toeplitz C^*-algebra $C^*(B_1,\dots,B_n)$ generated by $B_1,\dots,B_n$ and the identity. The dilation theory developed in will be based on the constrained shift $[B_1,\dots,B_n]$.
Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1,\dots,e_n$, where n∈{1,2, ….} .we consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k\ge 0} H_n^{\otimes k},$$

Where $H_n^{\otimes 0} := \mathbb{C}_1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n . Define the left creation operators $S_i : F^2(H_n) \to F^2(H_n)$, i= 1, . . . , n, by

$S_{i\varphi} := e_i \oplus \varphi\ ,\ \varphi \in F^2(H_n)$.
where n ∈ {1, 2, . . . }. We consider the full Fock space of Hn defined by
The noncommutative analytic Toeplitz algebra F_n^∞ and its norm closed version, the noncommutative disc algebra A_n , were introduced by [25], [26], [28] in connection with a multivariable noncommutative von Neumann inequality. F_n^∞ is the algebra of left multipliers of the Fock space $F^2(H_n)$ and can be identified with the weakly closed (or w^*-closed) algebra generated by the left creation operators $S_1,\dots,S_n$acting on $F^2(H_n)$ and the identity. The noncommutative disc algebra A_n is the norm closed algebra generated by$S_1,\dots,S_n$, and the identity. When n = 1, F_1^∞ can be identified with $H^\infty(\mathbb{D})$, the algebra of bounded analytic functions on the open unit disc. The noncommutative analytic Toeplitz algebra F_n^∞ can be viewed as a multivariable noncommutative analogue of $H^\infty(\mathbb{D})$. There are many analogies with the invariant subspaces of the unilateral shift on$H^2(\mathbb{D})$, inner-outer factorizations, analytic operators, Toeplitz operators, $H^\infty(\mathbb{D})$–functional calculus, bounded (resp. spectral) interpolation, etc .
Let F_n^+ be the unital free semigroup on n generators $g_1,\dots,g_n$, and the identity g_0. The length of $\propto \in F_n^+$ is defined by $|\propto| := k$ if $\propto = g_{i_1} g_{i_2} \dots g_{i_k}$, and $|\propto| := 0$ if $\propto = g_0$ .If $T_1,\dots,T_n \in B(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$, define $T_\propto := T_{i_1} T_{i_2} \dots T_{i_k}$ if $\propto = g_{i_1} g_{i_2} \dots g_{i_k}$, and $T_{g0} := I_{\mathcal{H}}$. Similarly, we denote $e_\propto := e_{i_1} \otimes \cdots \otimes e_{i_k}$

and $e_{g0} := 1$. We need to recall from [22], [23], [25], [26], [27], and [33] a few facts concerning multianalytic operators on Fock spaces. We say that a bounded linear operator M acting from $F^2(H_n) \otimes K$ to $F^2(H_n) \otimes K'$ is multi analytic if $M(S_i \otimes I_k) = (S_i \otimes I_{K'})M$ for any i= 1, . . . , n .
Notice that M is uniquely determined by the "coefficients" $\theta_{(\alpha)} \in B(K, K')$ given by
$\langle \theta_{(\tilde{\alpha})} K, K' \rangle := \langle M(1 \otimes K), e_\alpha \otimes K' \rangle$, k∈ $\mathcal{K}$, k'∈ $\mathcal{K}'$,$\alpha \in \mathbb{F}_n^+$
where $\tilde{\alpha}$ is the reverse of α, i.e., $\tilde{\alpha} = g_{i_k} \ldots g_{i_1}$ if $\alpha = g_{i_1} \ldots g_{i_k}$. We can associate with M a unique formal Fourier expansion

$$M(R_1, \ldots, R_n) := \sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes \theta_{(\alpha)} ,$$

where $R_i := U^* S_i U$, i = 1, . . . , n, are the right creation operators on $F^2(H_n)$
and U is the (flipping) unitary operator on $F^2(H_n)$ mapping $e_{i_1} \otimes \cdots \otimes e_{i_k}$ into $e_{i_k} \otimes \cdots \otimes e_{i_1}$.
Since the operator M acts like its Fourier representation on "polynomials", we will identify them for simplicity. Based on the noncommutative von Neumann inequality, we proved that
$M(R_1, \ldots, R_n) = \text{SOT} - \lim_{r \to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} R_\alpha \otimes \theta_{(\alpha)}$
where, for each r ∈ [0, 1), the series converges in the uniform norm. Moreover, the set of all multi-analytic operators in $B(F^2(H_n) \otimes K, F^2(H_n) \otimes K')$ coincides with $R_n^\infty \otimes B(K,K')$, the WOT-closed operator space generated by the spatial tensor product, where $R_n^\infty = U^* F_n^\infty U$. A multi-analytic operator is called inner if it is an isometry.
Let J be a WOT-closed two-sided ideal of F_n^∞ such that $J \neq F_n^\infty$, and define the subspaces of the full Fock space $(F^2(H_n))$ by setting
$\mathcal{M}_J := \overline{J(F^2(H_n))}$ and $\mathcal{N}_J := (F^2(H_n)) \ominus \mathcal{M}_J$.

Notice that

$$\mathcal{M}_J = \overline{\{\varphi(1) : \varphi \in J\}} \text{ and } \mathcal{N}_J = \cap_{\varphi \in J} Ker\, \varphi^* .$$

Based on a Beurling type theorem [22] for the left creation operators $S_1, \ldots, S_n$, a characterization of all WOT-closed two-sided ideal of $F_n^\infty$was obtained by Davidson and Pitts in [11]. One can easily obtain the following result.
Lemma (1.1.1)[1]: Let J be a WOT-closed two-sided ideal of F_n^∞ .
(i) If f(0) = 0 for any f ∈ J, then $1 \in \mathcal{N}_J$.
(ii) If $\mathcal{N}_J \neq 0$ and only if $J \neq F_n^\infty$,
(iii) The subspaces $\mathcal{N}_J$ and U $\mathcal{N}_J$are invariant under $S_1^*, \ldots, S_n^*$,and $R_1^*, \ldots, R_n^*$.
Proof : The first part is obvious. The second part is a consequence of the fact (see [10], [12]) that, for any $\varphi \in F_n^\infty$ $d(\varphi, J) = \left\| P_{\mathcal{N}_J} \varphi(S_1, \ldots, S_n) \mid \mathcal{N}_J \right\|$.
Part (iii) is straightforward.
Define the constrained left (resp. right) creation operators by setting

$$B_i := P_{\mathcal{N}_J} S_i \mid \mathcal{N}_J , \text{ and } W_i := P_{\mathcal{N}_J} S_i \mid \mathcal{N}_J , \ i = 1, \ldots, n.$$

Let $W(B_1, \ldots, B_n)$ be the w∗-closed algebra generated by $B_1, \ldots, B_n$and the identity. We proved in [10] that $W(B_1, \ldots, B_n)$ has the $\mathbb{A}_1(1)$ property and therefore the $w^*$and WOT topologies coincide on this algebra. Moreover, we showed that
$W(B_1, \ldots, B_n) = P_{\mathcal{N}_J} F_n^\infty \mid \mathcal{N}_J = \{f(B_1, \ldots, B_n) : f(S_1, \ldots, S_n) \in F_n^\infty\}$,

where, according to the F_n^∞ -functional calculus for row contractions [26],

$$f(B_1, \ldots, B_n) = \text{SOT-} \lim_{r \to 1} f(rB_1, \ldots, rB_n) .$$

Note that if $\varphi \in J$, then $\varphi(B_1, \ldots, B_n) = 0$. Similar results hold for $W(W_1, \ldots, W_n)$ thew∗-closed algebra generated by $W_1, \ldots, W_n$ and the identity. Moreover, we proved in [12] that

$W(B_1, \ldots, B_n)' = W(W_1, \ldots, W_n)$ and $W(W_1, \ldots, W_n)' = W(B_1, \ldots, B_n)$,

where $'$ stands for the commutant . An operator $M \in B(\mathcal{N}_J \otimes K, \mathcal{N}_J \otimes K')$ is called multianalytic with respect to $B_1, \ldots, B_n$ the constrained shifts if

$$M(B_i \otimes I_{\mathcal{K}}) = (B_i \otimes I_{\mathcal{K}'})M, \ i = 1, \ldots, n.$$

If in addition M is partially isometric, then we call it inner. We recall from [33] that the set of all multi-analytic operators with respect to $B_1, \ldots, B_n$ coincides with

$W(W_1, \ldots, W_n) \overline{\otimes} B(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{K}}') = P_{\mathcal{N}_J \otimes \mathcal{K}'}[R_n^\infty \overline{\otimes} B(\boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{K}}')] \, |\mathcal{N}_J \otimes K$,
and a similar result holds for the algebra $W(B_1, \ldots, B_n)$.
The next result provides a Beurling type theorem characterizing the invariant subspaces under the constrained shifts $B_1, \ldots, B_n$.

Theorem (1.1.2) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of F_n^∞ and let be $B_1, \ldots, B_n$ the corresponding constrained left creation operators on $\mathcal{N}_J$. A subspace $\mathcal{M} \subseteq \mathcal{N}_J \otimes K$ is invariant under each operator $B_i \otimes I_{\mathcal{K}}$, $i = 1, \ldots, n$, if and only if there exists a Hilbert space G and an inner operator

$$\theta(W_1, \ldots, W_n) \in . W(W_1, \ldots, W_n) \overline{\otimes} B(G, \boldsymbol{\mathcal{K}})$$

Such that

$$\boldsymbol{\mathcal{M}} = \theta(W_1, \ldots, W_n)(\mathcal{N}_J \otimes G)$$

Proof : According to Lemma (1.1.1), the subspace $\mathcal{N}_J \otimes K$ is invariant under each operator $S_i^* \otimes I_{\mathcal{K}}$, $i = 1, \ldots, n$, and

$(S_i^* \otimes I_{\mathcal{K}})|\mathcal{N}_J \otimes K = B_i^* \otimes I_{\mathcal{K}}$ $i = 1, \ldots, n$.
Since the subspace $[\mathcal{N}_J \otimes K] \ominus M$ is invariant under, $B_i^* \otimes I_{\mathcal{K}}$ $i = 1, \ldots, n$, we deduce that it is also invariant under each operator, $S_i^* \otimes I_{\mathcal{K}}$ $i = 1, \ldots, n$. Therefore, the subspace

$$E := [F^2(H_n) \otimes K] \ominus \{[\mathcal{N}_J \otimes \boldsymbol{\mathcal{K}}] \ominus \boldsymbol{\mathcal{M}}\} = [\mathcal{M}_J \otimes K] \oplus \boldsymbol{\mathcal{M}} \qquad (1)$$

is invariant under $S_i \otimes I_{\mathcal{K}}$, $i = 1, \ldots, n$, where $\mathcal{M}_J := F^2(H_n) \ominus \mathcal{N}_J$. Using the Beurling type characterization of the invariant subspaces under the left creation operators (see Theorem 2.2 from [22]) and the characterization of multi-analytic operators from [27] (see also [33]),we find a Hilbert space G and an inner multi-analytic operator
$\theta(R_1, \ldots, R_n) \in R_n^\infty \overline{\otimes} B(G,K)$

such that

$$\varepsilon = \theta(R_1, \ldots, R_n)[F^2(H_n) \otimes G],$$

where $\theta(R_1, \ldots, R_n)$ is essentially unique up to a unitary diagonal multi-analytic operator. Since $\theta(R_1, \ldots, R_n)$ is an isometry, we have

$$P_\varepsilon = \theta(R_1, \ldots, R_n)\theta(R_1, \ldots, R_n)^*, \qquad (2)$$

where P_ε is the orthogonal projection of $F^2(H_n)\otimes K$ onto ε . According to Lemma(1.1.1), the subspace $\mathcal{N}_J \otimes K$ is invariant under the operators $R_i^* \otimes I_{\mathcal{K}}$,
i = 1, . . . , n. Moreover, using the remarks preceding the theorem we have
$\theta(W_1, \ldots, W_n) = P_{\mathcal{N}_J \otimes K}\theta(R_1, \ldots, R_n)|\mathcal{N}_J \otimes K$.
Hence, and compressing equation (2) to the subspace $\mathcal{N}_J \otimes K$, we obtain
$P_{\mathcal{N}_J \otimes K} P_E \mid \mathcal{N}_J \otimes K = \theta(W_1, \ldots, W_n)\theta(W_1, \ldots, W_n)^*$.
Notice that, due to (1), the left hand side of this equality is equal to $P_{\mathcal{M}}$, the orthogonal projection of $\mathcal{N}_J \otimes K$ onto M. Hence,

$$P_{\mathcal{M}} = \theta(W_1, \ldots, W_n)\theta(W_1, \ldots, W_n)^*$$

and $\theta(W_1, \ldots, W_n)$ is a partial isometry. Therefore,

$$\boldsymbol{\mathcal{M}} = \theta(W_1, \ldots, W_n)[\mathcal{N}_J \otimes G]$$

and the proof is complete.
We remark that in the particular case when the ideal J is generated by the polynomials $S_iS_j - S_jS_i$, i, j = 1, . . . , n, then $\mathcal{N}_J = F_s^2$, the symmetric Fock space, and B_i, i = 1, . . . , n ,are the creation operators on the symmetric Fock space. In this case Theorem (1.1.2) provides a Beurling type theorem for Arveson's space H^2 which was also obtained in [16] using different
methods. From now on, throughout, we assume that J is a WOT-closed two-sided ideal of F_n^∞ such that $1 \in \mathcal{N}_J$.
Theorem(1.1.3) [1]: Let J be a WOT-closed two-sided ideal of $\mathcal{N}_J$such that $1 \in \mathcal{N}_J$. Then all the compact operators in $B(\mathcal{N}_J)$are contained in the operator space
$\overline{span}\{B_\propto {B^*}_\beta : \propto, \beta \in \mathbb{f}_n^+\}$.
Moreover, the C^*-algebra $C^*(B_1, \ldots, B_n)$ is irreducible.
Proof : Since $\mathcal{N}_J$ is an invariant subspace under each operator S_i^*, i = 1, . . . , n, and contains the constants, we have

$$I_{\mathcal{N}_J} - B_1B_1^* - \cdots - B_nB_n^* = P_{\mathcal{N}_J}(I - S_1S_1^* - \cdots - S_nS_n^*)|\mathcal{N}_J$$

$$= P_{\mathcal{N}_J} P_{\mathbb{C}} |\mathcal{N}_J$$

$$= P_{\mathbb{C}}^{\mathcal{N}_J},$$

where $P_{\mathbb{C}}^{\mathcal{N}_J}$ is the orthogonal projection of $\mathcal{N}_J$ onto $\mathbb{C}$. Let
$g(S_1, \ldots, S_n) := \sum_{|\propto| \le m} a_\propto S_\propto$
and

$$\varepsilon := \sum_{\beta \in \mathbb{f}_n^+} b_\beta e_\beta \in \mathcal{N}_J \subset F^2(H_n) .$$

Notice that
$P_{\mathbb{C}}^{\mathcal{N}_J} g(B_1, \ldots, B_n)^* \varepsilon = \sum_{|\propto| \le m} P_\propto \overline{a_\propto} S_\propto^* \varepsilon = \sum_{|\propto| \le m} \overline{a_\propto}\ b_\propto$

$$\langle \varepsilon, \textstyle\sum_{|\propto| \le m} a_\propto e_\propto \rangle = \langle \varepsilon, g(B_1, \ldots, B_n)(1) \rangle .$$

Therfore,

$$f(B_1, \ldots, B_n)\, P_{\mathbb{C}}^{\mathcal{N}_J} g(B_1, \ldots, B_n)^* \varepsilon = \langle \varepsilon\, g(B_1, \ldots, B_n)(1) \rangle\, f(B_1, \ldots, B_n)(1) \quad (3)$$

for any polynomial $f(B_1, \ldots, B_n) := \sum_{|\gamma| \le p} c_\gamma B_\gamma$
Hence, $f(B_1, \ldots, B_n)\, P_{\mathbb{C}}^{\mathcal{N}_J} g(B_1, \ldots, B_n)^*$is a rank one operator in $B(\mathcal{N}_J)$. Moreover, since $P_{\mathbb{C}}^{\mathcal{N}_J} = I_{\mathcal{N}_J} - B_1B_1^* - \cdots - B_nB_n^*$, we deduce that the above operator is also in the operator space $\overline{\text{span}}\{B_\alpha B_\beta^* : \alpha, \beta \in \mathbb{f}_n^+\}$.

Since the polynomials$\sum_{|\alpha| \leq m} a_\alpha e_\alpha$,m ∈N, $a_\alpha \in \mathbb{C}$, are dense in $F^2(H_n)$
it is clear thatthe set

$$L := \{(\sum_{|\alpha| \leq m} a_\alpha B_\alpha)(1) : m \in N, a_\alpha \in \mathbb{C}\}$$

dense in $\mathcal{N}_J$. Using this density and relation (3), we deduce that all compact operators in $B(\mathcal{N}_J)$ are included in the operator space $\overline{\text{span}}\{B_\alpha B_\beta^* : \alpha, \beta \in \mathbb{F}_n^+\}$. To prove the last part of this lemma , let $\mathcal{M}$ be a nonzero subspace of $\mathcal{N}_J$ which is jointly reducing for. $B_1, \dots, B_n$Take f ∈ , f ≠0 and assume that f = $a_0 + \sum_{|\alpha| \geq 1} a_\alpha e_\alpha$. If a_β is a nonzero coefficient of f, then

$$B_\beta^* f = P_{\mathcal{N}_J} S_\beta^* f = P_{\mathcal{N}_J}\left(a_\beta + \sum_{|\gamma| \geq 1} a_{\beta_\gamma} e_\alpha\right) \quad (4)$$

Is in M. Since 1 ∈ $\mathcal{N}_J$ we have $\langle P_{\mathcal{N}_J} e_\gamma, 1\rangle = \langle e_\gamma, 1\rangle = 0$for any $\gamma \in$ with $|\gamma| \geq 1$. Hence, we deduce that

$P_{\mathcal{N}_J} 1 = 1$ and $P_{\mathbb{C}} P_{\mathcal{N}_J}\left(\sum_{|\gamma| \geq 1} a_{\beta_\gamma} e_\alpha\right) = 0$

Therefore, relation (4) implies $P_{\mathbb{C}} P_{\mathcal{N}_J}$ f = a_β.

On the other hand, since $P_{\mathbb{C}}^{\mathcal{N}_J} = I_{\mathcal{N}_J} - B_1 B_1^* - \cdots - B_n B_n^*$ and $\mathcal{M}$is reducing for, $B_1, \dots, B_n$ we infer that $a_\beta \in \mathcal{M}$, so $1 \in \mathcal{M}$. Using again that $\mathcal{M}$ is invariant under, $B_1, \dots, B_n$we have $\mathcal{L} \subseteq$M. Since $\mathcal{L}$ is dense in $\mathcal{N}_J$, we deduce that $\mathcal{N}_J \subset \mathcal{M}$ and therefore $\mathcal{N}_J = \mathcal{M}$. This completes the proof.

We say that two row contractions $[T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, and $[T_1', \dots, T_n']$, $T_i' \in B(\mathcal{H}')$, are unitarily equivalent if there exists a unitary operator U : $\mathcal{H} \to \mathcal{H}'$ such that $T_i = U^* T_i' U$ for any i = 1, . . . , n. If $[B_1, \dots, B_n]$ is a constrained shift as above, then $[B_1 \otimes I_{\mathcal{H}}, \dots, B_n I_{\mathcal{H}} \otimes]$ is called constrained shift with multiplicity dim $\mathcal{H}$.

Proposition (1.1.4) [1]: Two constrained shifts are unitarily equivalent if and only if their multiplicities are equal.

Proof : Let $[B_1 \otimes I_{\mathcal{H}}, \dots, B_n I_{\mathcal{H}} \otimes]$ and $[B_1 \otimes I_{\mathcal{H}'}, \dots, B_n \otimes I_{\mathcal{H}'}]$ be two constrained shifts and let U : $\mathcal{N}_J \otimes \mathcal{H} \to \mathcal{N}_J \otimes \mathcal{H}'$ be a unitary operator such that

$U(B_i \otimes I_{\mathcal{H}}) = (B_1 \otimes I_{\mathcal{H}'})U$, i = 1, . . . , n.

Since U is unitary , we deduce that

$U(B_i^* \otimes I_{\mathcal{H}})\ ) = (B_i^* \otimes I_{\mathcal{H}'})U$, i = 1, . . . , n.

Since, according to Theorem (1.1.3), the C^*algebra $C^*(B_1, \dots, B_n)$ is irreducible, we infer that U = $I_{\mathcal{N}_J} \otimes$ W for some unitary operator W ∈ $B(\mathcal{H}, \mathcal{H}')$. Therefore, dim $\mathcal{H}$ = dim $\mathcal{H}'$. The converse is obvious.

We need a few more definitions. Let $\mathcal{S} \subseteq B(\mathcal{K})$ be a set of operators acting on the Hilbert space $\mathcal{K}$. Denote by A(S) the non self a djoint algebra generated by $\mathcal{S}$ and the identity, and let $C^*(\mathcal{S})$ be the C^*-algebra generated by and the identity. A subspace $\mathcal{H} \subseteq \mathcal{K}$ is called
∗-cyclic for $\mathcal{S}$ if

$$\mathcal{K} = \bigvee\{X h : X \in C^*(\mathcal{S}), h \in \mathcal{H}\},$$

i.e., $\mathcal{K}$ is the smallest reducing sub pace for $\mathcal{S}$ which contains $\mathcal{H}$. We call $\mathcal{H}$ cyclic for $\mathcal{S}$ if

$$\mathcal{K} = \bigvee\{X h : X \in A(\mathcal{S}), h \in \mathcal{H}\},$$

i.e., $\mathcal{K}$ is the smallest invariant sub pace under $\mathcal{S}$ which contains $\mathcal{H}$. Finally, a subspace $\mathcal{H} \subseteq \mathcal{K}$ is called co-invariant under $\mathcal{S}$ if $X^*\mathcal{H} \subseteq \mathcal{H}$ for any X ∈ $\mathcal{S}$.

Theorem (1.1.5) [1]: Let J be a WOT-closed two-sided ideal of F_n^∞such that 1 ∈ $\mathcal{N}_J$ and let be a Hilbert space. If $\mathcal{M} \subseteq \mathcal{N}_J \otimes \mathcal{D}$ is a co-invariant subspace under $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n, then there exists a subspace $\mathcal{E} \subseteq \mathcal{D}$ such that

$$\overline{Span}\{(B_\alpha \otimes I_{\mathcal{D}})\mathcal{M} : \alpha \in \mathbb{F}_n^+\} = \mathcal{N}_J \otimes \mathcal{E}. \quad (5)$$

Proof : Denote by $P_0 := P_{\mathbb{C}} \otimes I_{\mathcal{D}}$ the orthogonal projection of $\mathcal{N}_J \otimes \boldsymbol{\mathcal{D}}$ onto the subspace $1 \otimes \boldsymbol{\mathcal{D}}$, which is identified with $\boldsymbol{\mathcal{D}}$. Define the subspace $\boldsymbol{\mathcal{E}} \subset \boldsymbol{\mathcal{D}}$ by setting $\boldsymbol{\mathcal{E}} := P_0 \boldsymbol{\mathcal{M}}$ and let f be nonzero element of $\boldsymbol{\mathcal{M}}$ having the Fourier representation

$f = \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes h_\alpha$, $h_\alpha \in \boldsymbol{\mathcal{D}}$

Let $\beta \in \mathbb{F}_n^+$ be such that $h_\beta \neq 0$ and note that

$$(B_\beta^* \otimes I_{\mathcal{D}})f = (P_{\mathcal{N}_J} \otimes I_{\mathcal{D}})(S_\beta^* \otimes I_{\mathcal{D}}))f = (P_{\mathcal{N}_J} \otimes I_{\mathcal{D}})\left(1 \otimes h_\beta + \sum_{|\gamma| \geq 1} e_\gamma \otimes h_{\beta_\gamma}\right) \quad (6)$$

As in the proof of Theorem (1.1.3), since $1 \in \mathcal{N}_J$, we have $P_{\mathcal{N}_J} = 1$ and $P_{\mathbb{C}} P_{\mathcal{N}_J} e_\gamma = 0$ for $|\gamma| \geq 1$. Hence and using (6), we obtain $P_0(B_\beta^* \otimes I_{\mathcal{D}})f = h_\beta$. Since is a co-invariant subspace under $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n, it is clear that h_ ∈ E. Using this and taking into account that 1 ∈ NJ, we deduce that

$$(B_\beta \otimes I_{\mathcal{D}})(1 \otimes h_\beta) = P_{\mathcal{N}_J} e_\beta \otimes h_\beta \in \otimes \mathcal{N}_J \otimes \boldsymbol{\mathcal{E}}.$$

Now, since $f \in \boldsymbol{\mathcal{M}} \subseteq \mathcal{N}_J \otimes \boldsymbol{\mathcal{D}}$, we infer that

$$f = (P_{\mathcal{N}_J} \otimes I_{\mathcal{D}})f = \lim_{K \to \infty} P_{\mathcal{N}_J} e_\beta \otimes h_\beta$$

is in $\mathcal{N}_J \otimes \boldsymbol{\mathcal{E}}$. This shows that $\boldsymbol{\mathcal{M}} \subset \mathcal{N}_J \otimes \boldsymbol{\mathcal{E}}$ and therefore

$\mathcal{Y} := \overline{\text{span}}\, \{(B_\alpha \otimes I_{\mathcal{D}})\boldsymbol{\mathcal{M}}: \boldsymbol{\alpha} \in \mathbb{F}_n^+ \} \subset \mathcal{N}_J \otimes \boldsymbol{\mathcal{E}}$

For the other inclusion, we prove first that $\boldsymbol{\mathcal{E}} \subset$. If $h_0 \in \boldsymbol{\mathcal{E}} \in E$, $h_0 \neq 0$, then there exists $g \in \boldsymbol{\mathcal{M}}$ such that $g = 1 \otimes h_0 + \sum_{|\alpha| \geq 1} e_\alpha \otimes h_\alpha$. Due to the first part of the proof, we have

$$h_0 = P_0 g = (I - \sum_{i=1}^{n}((B_i \otimes I_{\mathcal{D}})((B_i \otimes I_{\mathcal{D}})^*)\, g.$$

Acoording to the proof of Theorem (1.1.3), we have $P_{\mathbb{C}}^{\mathcal{N}_J} = I_{\mathcal{N}_J} - \sum_{i=1} B_i B_i^* =$. Using this and the fact that $\boldsymbol{\mathcal{M}}$ is co-invariant under $B_i \otimes I_{\mathcal{D}}$ i = 1, . . . , n, we deduce that $h_0 \in \mathcal{Y}$ for any $h_0 \in E$, i.e., $\boldsymbol{\mathcal{E}} \subset \mathcal{Y}$. The latter inclusion shows that $(B_{\boldsymbol{\alpha}} \otimes I_{\mathcal{D}}\ (1 \otimes \boldsymbol{\mathcal{E}}) \subset \mathcal{Y}$ for any $\boldsymbol{\alpha} \in \mathbb{F}_n^+$,which implies

$$P_{\mathcal{N}_J} e_\alpha \otimes \boldsymbol{\mathcal{E}} \subset \mathcal{Y}, \boldsymbol{\alpha} \in \mathbb{F}_n^+ . \quad (7)$$

Let $\varphi \in \mathcal{N}_J \otimes \boldsymbol{\mathcal{E}} \subset F^2(H_n) \otimes \boldsymbol{\mathcal{E}}$ be with Fourier representation $\varphi = \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha k_\alpha$, $k_\alpha \in \boldsymbol{\mathcal{E}}$. Using relation (7), we have

$$\varphi = (P_{\mathcal{N}_J} \otimes I_{\mathcal{E}}) = \lim_{k \to \infty} \sum_{|\alpha| = k} P_{\mathcal{N}_J} e_\alpha \otimes k_\alpha \in \mathcal{Y}.$$

Therefore, $\mathcal{N}_J \otimes \boldsymbol{\mathcal{E}} \subseteq \mathcal{Y}$, which completes the proof. _

Corollary (1.1.6) [1] : Let J be a WOT-closed two-sided ideal of $\mathbb{F}_n^\infty$ such that $1 \in \mathcal{N}_J$ and let $\boldsymbol{\mathcal{D}}$ be a Hilbert space. Let $\boldsymbol{\mathcal{M}} \subseteq \mathcal{N}_J \otimes \boldsymbol{\mathcal{D}}$ be a co-invariant subspace under $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n.

Then the following statements are equivalent:

(i) M is a cyclic subspace for $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n;

(ii) $P_0 \boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{D}}$;

(iii) $\mathcal{M}^\perp \cap \boldsymbol{\mathcal{D}} = \{0\}$.

Proof: The equivalence (i) ↔ (ii) is clear from Theorem (1.1.5) and the definition of cyclic subspace. To prove that (ii) ↔ (iii), notice first that if there exists $h \in \mathcal{M}^\perp \cap \boldsymbol{\mathcal{D}}$, $h \neq 0$, then taking into account that i$\mathcal{M}^\perp$s invariant under $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n, we deduce that $\mathcal{N}_J \otimes h \subset .\mathcal{M}^\perp$ .This shows that h $P_0 \boldsymbol{\mathcal{M}}$ which means that is $P_0 \boldsymbol{\mathcal{M}}$ not equal to $\boldsymbol{\mathcal{D}}$. Now, assume that there exists $k \in D$, $k \neq 0$, such that $k \perp P_0 \boldsymbol{\mathcal{M}}$. Since $1 \in \mathcal{N}_J$,we have $k \perp M$ which shows that $k \in \boldsymbol{\mathcal{D}} \cap \mathcal{M}^\perp$. The proof is complete. _

Corollary (1.1.7) [1]: Let J be a WOT-closed two-sided ideal of F_n^∞ such that $1 \in \mathcal{N}_J$ and let $\boldsymbol{\mathcal{D}}$ be a Hilbert space. A subspace $\boldsymbol{\mathcal{M}} \subseteq \mathcal{N}_J \otimes \boldsymbol{\mathcal{D}}$ is reducing under each operator $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n, if and only if there exists a subspace $\boldsymbol{\mathcal{E}} \subseteq \boldsymbol{\mathcal{D}}$ such that

$$\boldsymbol{\mathcal{M}} = \mathcal{N}_J \otimes \boldsymbol{\mathcal{E}}.$$

The next result, is a Wold type decomposition for nondegenerate $*$-representations of the C^*-algebra $C^*(B_1,\dots,B_n)$

Theorem(1.1.8) [1]: Let J be a WOT-closed two-sided ideal of F_n^∞ such that $1 \in \mathcal{N}_J$, and let $\pi: C^*(B_1,\dots,B_n) \to B(K)$ be a non degenerate $*$-representation of $C^*(B_1,\dots,B_n)$ on a separable Hilbert space K. Then π decomposes into a direct sum

$$\pi = \pi_0 \oplus \pi_1 \text{ on } K = K_0 \oplus K_1,$$

where π_0, π_1 are disjoint representations of $C^*(B_1,\dots,B_n)$ on the Hilbert spaces $K_0 := \overline{\operatorname{span}}\{\pi(B_\alpha)(I - \sum_{i=1}^n \pi(B_i)\pi(B_i)^*)K : \alpha \in \mathbb{F}_n^+\}$

and $K_1 := K_0^\perp$, respectively, such that, up to an isomorphism,

$$K_0 \simeq \mathcal{N}_J \otimes G(X) = X \otimes IG,\ X \in C^*(B_1,\dots,B_n), \tag{8}$$

for some Hilbert space G with $\dim G = \dim[range\ (I - \sum_{i=1}^n \pi(B_i)\pi(B_i)^*)]$,
and π_1 is a $*$-representation which annihilates the compact operators and
$\pi_1(B_1)\pi_1(B_1)^* + \cdots + \pi_1(B_n)\pi_1(B_n)^* = I_{K_1}$.
Moreover, if π' is another non degenerate $*$-representation of $C^*(B_1,\dots,B_n)$ on a separable Hilbert space K', then π is unitarily equivalent to π' if and only if $\dim G = \dim G'$ and π_1 is unitarily equivalent to π_1'.

Proof: Since the subspace $\mathcal{N}_J$ contains the constants, Theorem (1.1.3) implies that all the compact operators $LC(\mathcal{N}_J)$ in $B(\mathcal{N}_J)$ are contained in $C^*(B_1,\dots,B_n)$ according to the standard theory of representations of the $C*$-algebras, the representation π decomposes into a direct sum $\pi = \pi_0 \oplus \pi_1$ on $K = K_0 \oplus K_1$, where $K_0 := \overline{\operatorname{span}}\{\pi(X)K : X \in LC(\mathcal{N}_J)\}$ and $K_1 := K_0^\perp$, and the representations $\pi_1: C^*(B_1,\dots,B_n) \to K_j$ are defined by $\pi_j(X) := \pi(X)|K_j$, j = 0, 1. Now, it is clear that π_0, π_1 are disjoint representations of $C^*(B_1,\dots,B_n)$ such that π_1 annihilates the compact operators in $B(\mathcal{N}_J)$, and π_0, is uniquely determined by the action of π on the ideal $LC(\mathcal{N}_J)$). Since every representation of $LC(\mathcal{N}_J)$) is equivalent to a multiple of the identity representation (see [3]), we deduce (8). Now, we show that the space K_0
coincides with the one defined in the theorem. Using Theorem (1.1.3) and its proof, we deduce that
$K_0 := \overline{\operatorname{span}}\{\pi(X)K : X \in LC(\mathcal{N}_J)\}$
$= \overline{\operatorname{span}}\{\pi(B_\alpha P_{\mathbb{C}}^{\mathcal{N}_J} B_\beta^*)K : \alpha, \beta \in \mathbb{F}_n^+\}$
$= \overline{\operatorname{span}}\{\pi(B_\alpha)(I - \sum_{i=1}^n \pi(B_i)\pi(B_i)^*)K : \alpha \in \mathbb{F}_n^+\}$.

On the other hand, since $P_{\mathbb{C}}^{\mathcal{N}_J} = (I - \sum_{i=1}^n \pi(B_i)\pi(B_i)^*)$ is a rank one projection in $C^*(B_1,\dots,B_n)$ (see Theorem (1.1.3)), we deduce that

$$\sum_{i=1}^n \pi_1(B_i)\pi_1(B_i)^* = I_{K_1}, \text{ and}$$

$\dim G = \dim[\text{range}\ \pi(P_{\mathbb{C}}^{\mathcal{N}_J})] = \dim[\text{range}(I - \sum_{i=1}^n \pi(B_i)\pi(B_i)^*)]$.

To prove the uniqueness, note that according to the standard theory of representations of C^*-algebras, π and π' are unitarily equivalent if and only if π_0 and π_0'(resp. π_1 and π_1') are unitarily equivalent. Using Proposition (1.1.4),
we deduce that $\dim G = \dim G'$ and complete the proof. _

Corollary(1.1.9) [1]: Under the hypotheses and notations of Theorem (1.1.8) and setting $V_i := \pi(B_i)$, i = 1, . . . , n, we have:
(i) Q := $I_K - \sum_{i=1}^{n} V_i V_i^*$ is an orthogonal projection and $Q_K = \bigcap_{i=1}^{n} ker\ V_i^*$;

(ii) K_0 ={ k ∈ K: $\lim_{k\to\infty} \sum_{|\alpha| = k} \| V_\alpha^* k \|^2 = 0$} ;

(iii) K_1 ={ k ∈ K: $\sum_{|\alpha| = k} \| V_\alpha^* k \|^2 = \| k \|^2$ for any $k = 1, 2, \ldots$} ;

(iv) SOT- $\lim_{k\to\infty} \sum_{|\alpha| = k} V_\alpha V_\alpha^* = P_{K_1}$;

(v) $\sum_{k=0}^{\infty} \sum_{|\alpha| = k} V_\alpha V_\alpha^* = P_{k_0}$;

Proof: Since $I_{\mathcal{N}_J} - \sum_{i=1}^{n} B_i B_i^* = P_{\mathbb{C}}^{\mathcal{N}_J}$ is an orthogonal projection (see the proof of Theorem (1.1.3), so is Q = π ($P_{\mathbb{C}}^{\mathcal{N}_J}$). Therefore,

QK ={k ∈ K :(I-$\sum_{i=1}^{n} V_i V_i^*$)k = k }

={k ∈ K : $\sum_{i=1}^{n} V_i V_i^*$ k = 0}

= $\bigcap_{i=1}^{n} \ker V_i^*$
which proves (i). Using Theorem (1.1.8), we have

$$\sum_{|\alpha| = k} V_\alpha V_\alpha^* = \begin{bmatrix} \sum_{|\alpha| = k} B_\alpha B_\alpha^* \otimes I_G & 0 \\ 0 & I_{k_1} \end{bmatrix}. \qquad (9)$$

Since $\mathcal{N}_J$ is co-invariant under $S_1, \ldots, S_n$, and $B_i^* = S_i^* | \mathcal{N}_J$, i = 1, . . . , n, we have

SOT-$\lim_{k\to\infty} \sum_{|\alpha| = k} B_\alpha B_\alpha^* \otimes I_G$ = SOT- $\lim_{k\to\infty} [P_{\mathcal{N}_J} (\sum_{|\alpha| = k} S_\alpha S_\alpha^*) | \mathcal{N}_J] \otimes I_G = 0.$
The latter equality holds due to the fact that [S1, . . . , Sn] is a pure contraction. Therefore, (iv) holds. Hence, and taking into account that
$\sum_{i=1}^{m} \sum_{|\alpha| = k} V_\alpha Q V_\alpha^* = I - \sum_{|\alpha| = m+1} V_\alpha V_\alpha^* *$,
we deduce (v). Now, let k ∈ K = $K_0 \oplus K_1$, k = $k_0 + k_1$, with $k_0 \in K_0$ and $k_1 \in K_1$. By (9),we have

$$\sum_{|\alpha| = m} \| V_\alpha^* k \|^2 = \langle (\sum_{|\alpha| = m} B_\alpha B_\alpha^* \otimes I_G) k_0, k_0 \rangle + \| k_1 \|^2, m = 1,2, \ldots \qquad (10)$$

Hence, $\lim_{m\to\infty} \sum_{|\alpha| = m} \| V_\alpha^* k \|^2 = 0$ if and only if k_1 = 0, i.e., k = $k_0 \in K_0$, which proves (ii).
On the other hand, (10) shows that $\sum_{|\alpha| = m} \| V_\alpha^* k \|^2 = \| k \|^2$ for any m = 1, 2, . . ., if and only if $\langle (\sum_{|\alpha| = m} B_\alpha B_\alpha^* \otimes I_G) k_0, k_0 \rangle = \| k_0 \|^2$ for any m = 1, 2, ... Since [$B_1, \ldots, B_n$] is a pure row contraction, SOT- $\lim_{m\to\infty} \sum_{|\alpha| = m} B_\alpha B_\alpha^* = 0$. Therefore , the above equality holds for any
m = 1, 2, . . ., if and only if k_0 = 0, which is equivalent to k = $k_1 \in K_1$. This completes the proof. _
Corollary (1.1.10) [1]: Let _ be a non degenerate ∗-representation of $C^*(B_1, \ldots, B_n)$ on a separable Hilbert space K, and let $V_i := \pi(B_i)$, i = 1, . . . , n. Then the following statements are equivalent:
(i) V := [$V_1, \ldots, V_n$] is a constrained shift ;
(ii) K = $\overline{\text{span}}$ { $V_\alpha (I - \sum_{i=1}^{n} V_i V_i^*)$K : $\alpha \in \mathbb{F}_n^+$};
(iii) SOT- $\lim_{k\to\infty} \sum_{|\alpha| = k} V_\alpha V_\alpha^* = 0$.
In this case, the multiplicity of V (denoted by mult (V)) satisfies the equality

$$\text{mult}(V) = \dim(I - V_1 V_1^* - \cdots - V_n V_n^*)K, \qquad (11)$$

and it is also equal to the minimum dimension of a cyclic subspace for $V_1, \ldots, V_n$.

Proof : The above equivalences are consequences of Corollary (1.1.9), and relation (11) follows from Theorem (1.1.8.) We prove the last part of the corollary. According to (ii) and Corollary(1.1.9), L $:=\bigcap_{i=1}^{n} V_i^* = (I - V_1V_1^* - \cdots - V_nV_n^*)$K

is a cyclic subspace for $V_1, \ldots, V_n$. Now, let $\boldsymbol{\mathcal{E}}$ be any cyclic subspace for$V_1, \ldots, V_n$., i.e., K $=\bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{E}$, and denote A := $P_L|\boldsymbol{\mathcal{E}} \in B(\boldsymbol{\mathcal{E}},L)$, where P_L is the orthogonal projection of K onto L. Assume that k ∈ L ⊖ T$\boldsymbol{\mathcal{E}}$ and let h ∈ $\boldsymbol{\mathcal{E}}$. Notice that

$\langle h, k\rangle = \langle h, P_L k\rangle = \langle Ah, k\rangle = 0$.

On the other hand, $V_\alpha^* k = 0$ for any k ∈ L, we have $\langle V_\alpha h, k\rangle = 0$ for any $\alpha \in \mathbb{F}_n^+$ with$|\alpha| \geq 1$. Therefore, k $\perp V_\alpha \mathcal{E}$ for any $\alpha \in \mathbb{F}_n^+$. Since $\boldsymbol{\mathcal{E}}$ is a cyclic sub space for $V_1, \ldots, V_n$, we deduce that k = 0, and therefore A$\boldsymbol{\mathcal{E}}$ = L. This shows that $A^* \in B(L, \boldsymbol{\mathcal{E}})$ is one-to-one and, consequently, dim L ≤ dim $\boldsymbol{\mathcal{E}}$. This completes the proof. _

An easy consequence of Corollary (1.1.10) and Proposition (1.1.4)is the following.

Proposition (1.1.11) [1]: Two constrained shifts are similar if and only if they are unitarily equivalent.

Proof: One implication is obvious. Let V := [$V_1, \ldots, V_n$], $V_i \in B(K)$, and V ′ := [$V_1', \ldots, V_n'$], $V_i' \in$ B(K′), be two constrained shifts and let X : K → K′ be an invertible operator such that

$$XV_i = V_i' X, \quad i = 1, \ldots, n.$$

If M is a cyclic subspaces for $V_1, \ldots, V_n$, then

$K' = XK = X(\bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \boldsymbol{\mathcal{M}})$

$$\subseteq \bigvee_{\alpha \in \mathbb{F}_n^+} XV_\alpha \boldsymbol{\mathcal{M}} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha' X \boldsymbol{\mathcal{M}} \subseteq K'.$$

Therefore , $K' = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha' X \boldsymbol{\mathcal{M}}$, which shows that X $\boldsymbol{\mathcal{M}}$ is cyclic for V ′. Since X is invertible, dim $\boldsymbol{\mathcal{M}}$ = dim X $\boldsymbol{\mathcal{M}}$. Hence an using Corollary (1.1.10), we conclude that the two constrained shifts have the same multiplicity. By Proposition (1.1.4), the result follows.

we develop a dilation theory for row contractions T := [$T_1, \ldots, T_n$] subject to constraints such as

$$p(T_1, \ldots, T_n) = 0, \quad p \in P,$$

where P is a set of non commutative polynomials. The model n-tuple is the universal row contraction [$B_1, \ldots, B_n$] satisfying the same constraints as T. The theory is based on a class of constrained Poisson kernels associated with T and representations of the C^*-algebra generated by $B_1, \ldots, B_n$ and the identity. Under natural conditions on the constraints we have uniqueness for the minimal dilation. We introduce and evaluate the dilation index, a numerical invariant for row contractions, and show that it does not depend on the constraints. These results are used in connection with characteristic functions and models for constrained row contractions.

We need to recall from [30] a few facts about non commutative Poisson transforms associated with row contractions T := [$T_1, \ldots, T_n$)], $T_i \in B(\mathcal{H})$. For each 0 < r ≤ 1, define the defect operator $\Delta_{T,r} := (I - r^2T_1T_1^* - \cdots - r^2T_nT_n^*)^{1/2}$(The Poisson kernel associated with T is the family of operators

$$K_{T,r}: \mathcal{H} \to F^2(H_n) \otimes \overline{\Delta_{T,r}\mathcal{H}}, \quad 0 < r \leq 1,$$

defined by

$$K_{T,r}h := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} e_\alpha \otimes r^{|\alpha|} \Delta_{T,r} T_\alpha^* h, \quad h \in \mathcal{H}. \tag{12}$$

When r = 1, we denote $\Delta_T := \Delta_{T,1}$ and $K_T := K_{T,1}$The operators $K_{T,r}$are isometries if 0 < r < 1, and

$$K_T^* K_T = I - \text{SOT-}\lim_{k\to\infty} \sum_{|\alpha|=k} T_\alpha T_\alpha^*.$$

This shows that K_T is an isometry if and only if T is a pure row contraction ([21]), i.e.,

$$\text{SOT-}\lim_{k\to\infty} \sum_{|\alpha|=k} T_\alpha T_\alpha^* = 0.$$

A key property of the Poisson kernel is that

$$K_{T,r}(r^{|\alpha|}\ T_\alpha^*\ ) = (\ S_\alpha^* \otimes I)K_{T,r} \text{for any } 0 < r \leq 1,\ \alpha \in \mathbb{F}_n^+ . \qquad (13)$$

In [30], we introduced the Poisson transform associated with $T := [\,T_1, \ldots, T_n]$ as the unital completely contractive linear map $\varphi_T : C*(\,S_1, \ldots, S_n) \to B(\mathcal{H})$ defined by

$$\varphi_T\,(f) := \lim_{r\to 1} K_{T^*,r}\,(f \otimes I)K_{T,r}$$

where the limit exists in the norm topology of $B(\mathcal{H})$. Moreover, we have
$(S_\alpha, S_\beta^*\,) = T_\alpha, T_\beta^*\ \ \alpha, \beta \in \mathbb{F}_n^+$.

When T is a completely non-coisometric (c.n.c.) row-contraction, i.e., there is no $h \in \mathcal{H}$, $h \neq 0$, such that

$$\sum_{|\alpha|=k} \| T_\alpha^* h \|^2 = \| h \|^2 \ \text{ for any } k = 1, 2, \ldots,$$

an F_n^∞-functional calculus was developed in [26]. We showed that if

$$f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha S_\alpha, \text{ is in } F_n^\infty,$$

then
$T_T\,(f) = f(T_1, \ldots, T_n) := \text{SOT-}\lim_{r\to 1} \sum_{k=0}^\infty \sum_{|\alpha|=k} r^{|\alpha|}\ a_\alpha T_\alpha$

exists and$T_T\ : F_n^\infty \dashrightarrow B(\mathcal{H})$ is a WOT-continuous completely contractive homomorphism.
More about noncommutative Poisson transforms on C^*-algebras generated by isometries can be found in [30], [2], [31], [32], and [34].
Let $J \neq F_n^\infty$-n be a WOT closed two-sided ideal of F_n^∞- generated by a family of polynomials $P_J \subset F_n^\infty$ and let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction such that $p(T_1, \ldots, T_n) = 0$ for any $p \in PJ$
Let D and K be Hilbert spaces and let $Z_i \in B(K)$ be bounded operators such that $[T_1, \ldots, T_n]$ is a Cuntz row contraction, i.e., $Z_1Z_1^* + \cdots + Z_nZ_n^* = I_k$.
An n-tuple $V := [V_1, \ldots, V_n]$ of operators with

$$V_i := \begin{bmatrix} B_i \otimes I_{\mathcal{D}} & 0 \\ 0 & Z_i \end{bmatrix},\ i = 1, \ldots, n, \qquad (14)$$

where the n-tuple $[B_1, \ldots, B_n]$ is the constrained shift associated with J, is called constrained (or J-constrained) dilation of T if:
(i) $p(V_1, \ldots, V_n) = 0$ for any $p \in P_J$;
(ii) $\mathcal{H}$ can be identified with a co-invariant subspace under $V_1, \ldots, V_n$such that

$$T_i = P_{\mathcal{H}} V_i|\mathcal{H},\ i = 1, \ldots, n.$$

The dilation is minimal if $\mathcal{H}$ is cyclic for $V_1, \ldots, V_n$, i.e.,
$(\mathcal{N}_J \otimes \boldsymbol{\mathcal{D}}) \oplus K = \bigvee_{\alpha\in\mathbb{F}_n^+} V_\alpha \mathcal{H}$.
We introduce the dilation index of T, denoted by dil-ind (T), to be the minimum dimension of $\boldsymbol{\mathcal{D}}$ such that V is a constrained dilation of T.

Our first dilation result for constrained row contractions is the following.

Theorem (1.1.12) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of generated by a family of polynomials $P_J \subset F_n^\infty$ and let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction such that

$$p(T_1, \ldots, T_n) = 0 \text{ for any } p \in P_J . \tag{15}$$

Then there exists a Hilbert space K and some operators $Z_i \in B(K)$ with the properties

$Z_1Z_1^* + \cdots + Z_nZ_n^* = I_k$ and

$$p(Z_1 \ldots, Z_n) = 0 \text{ for any } p \in P_J ,$$

such that:

(i) $\mathcal{H}$ can be identified with a co-invariant subspace of $\check{k} := (\mathcal{N}_J \otimes \overline{\Delta_T\mathcal{H}}) \oplus K$ under the operators

$$V_i := \begin{bmatrix} B_i \otimes I_{\overline{\Delta_T\mathcal{H}}} & 0 \\ 0 & Z_i \end{bmatrix}, \ i = 1, \ldots, n;$$

(ii)$T_i^* = V_i^*|\mathcal{H}$, i = 1, . . . , n.

Moreover, K = {0} if and only if $[T_1, \ldots, T_n]$ is a pure row contraction.

Proof: Consider the subspace

$\boldsymbol{\mathcal{M}} := \overline{\text{span}}\ \{S_\alpha p (S_1, \ldots, S_n)S_\beta(1) : p \in P_J , \alpha,\beta \in \mathbb{F}_n^+ \}$.

It is clear that $\boldsymbol{\mathcal{M}} \subseteq \mathcal{M}_J$. To prove that $\mathcal{M}_J \subseteq \boldsymbol{\mathcal{M}}$, it is enough to show that $\mathcal{M}^\perp \subseteq \mathcal{M}_J^\perp$.

Let $g \in F^2(H_n)$ be such that

$\langle g, \varphi(S_1, \ldots, S_n)p (S_1, \ldots, S_n)S_\beta(1)\rangle = 0$ for any $p \in P_J$, $\alpha,\beta \in \mathbb{F}_n^+$.

It is known (see [4], [12]) that for any $\varphi(S_1, \ldots, S_n) \in F_n^\infty$, there is a sequence of polynomials $\{q_m(S_1, \ldots, S_n)\}_{m=1}^\infty$ which is SOT-convergent to $\varphi(S_1, \ldots, S_n)$ as $m \to \infty$. Consequently

$\langle g, \varphi(S_1, \ldots, S_n)p (S_1, \ldots, S_n)S_\beta(1)\rangle = 0$

for any $\varphi(S_1, \ldots, S_n) \in \mathbb{F}_n^+$, $p \in P_J$,and $\alpha,\beta \in \mathbb{F}_n^+$, Hence, $g \in \mathcal{M}_J^\perp$. Therefore, $\mathcal{M}_J = \boldsymbol{\mathcal{M}}$. Now, using the properties of the Poisson kernel K_T (see (13)) and that $p(T_1, \ldots, T_n) = 0$ for any $p \in P_J$, we obtain

$\langle K_T k S_\alpha p (S_1, \ldots, S_n)S_\beta(1) \otimes h\rangle = \langle k, T_\alpha p (T_1, \ldots, T_n)S_\beta(1)T_{\beta\Delta_T} h\rangle = 0$

for any $k \in \mathcal{H}$, $h \in \overline{\Delta_T\mathcal{H}}$, and $p \in P_J$. Since $\mathcal{M}_J = \boldsymbol{\mathcal{M}}$, we infer that

$$\text{range } K_T \subseteq (\mathcal{M}_J \otimes \overline{\Delta_T\mathcal{H}})^\perp = \mathcal{N}_J \otimes \overline{\Delta_T\mathcal{H}}. \tag{16}$$

Consider the constrained Poisson kernel $K_{J,T} : \mathcal{H} \to \mathcal{N}_J \otimes \overline{\Delta_T\mathcal{H}}$ defined by

$$K_{J,T} := (P_{\mathcal{N}_J} \otimes I_{\overline{\Delta_T\mathcal{H}}})K_T ,$$

where K_T is the Poisson kernel defined by (12). Using relations (13) and (16), we obtain

$$K_{J,T}\, T_\alpha^* = (B_\alpha^* \otimes I_H)K_{J,T}\ \alpha \in \mathbb{F}_n^+ \tag{17}$$

Define the contraction $Q := \text{SOT-}\lim_{k\to\infty} \sum_{|\alpha| = k} T_\alpha T_\alpha^*$

and the operator

$Y : \mathcal{H} \to K\ \ \overline{Q^{1/2}\mathcal{H}}$:= Q1/2H by $Y h := Q^{1/2}\mathcal{H}$, $h \in \mathcal{H}$.

For each i = 1, . . . , n, define $A_i : Q^{1/2}\mathcal{H} \to K$ by setting

$$A_i\, Y\, h := Y\, T_i^* h,\ h \in \mathcal{H}. \tag{18}$$

The operators A_i, i = 1, . . . , n, are well-defined since

$$\sum_{i=1}^{n} \| A_i\, Y\, h\ \|^2 = \langle \sum_{i=1}^{n} T_i Q T_i^* h, h\rangle = \| Q^{\frac{1}{2}} h \|^2$$

Therefore the operator A_i can be extended to a bounded operator on K, which will also be denoted by A_i. Now, setting $Z_i := A_i^*$, i = 1, . . . , n, relation (18)implies

$$Y^* Z_i = T_i\, Y^*,\ i = 1, \dots, n. \tag{19}$$

Notice that

$$Y^* \left(\sum_{i=1}^{n} Z_i Z_i^* \right) Y = \sum_{i=1}^{n} T_i Y^* Y\, T_i^*$$

$$= \sum_{i=1}^{n} T_i Q\, T_i^* = Q = Y\, Y^*$$

Hence,

$$\langle \sum_{i=1}^{n} (Z_i\, Z_i^*) Yh, Yh \rangle = (Yh, Yh),$$

which implies

$\sum_{i=1}^{n} Z_i Z_i^* = I_k$. Using relations (19) and (15), we get

$Y^* p(Z_1, \dots, Z_n) = p(T_1, \dots, T_n)\, Y^* = 0$, $p \in P_J$.

Since $Y^*$is injective on K = Y H, we deduce that $p(Z_1, \dots, Z_n) = 0$ for any $p \in P_J$.

Define the operator $V : \mathcal{H} \rightarrow [\mathcal{N}_J \otimes \mathcal{H}] \oplus K$ by setting

$$V := \begin{bmatrix} K_{T,J} \\ Y \end{bmatrix}.$$

Note that

$$\| Vh \|^2 = \| K_{T,Y} h \|^2 + \| Yh \|^2$$

$$= \| h \|^2 \text{ -SOT-} \lim_{k\to\infty} \langle \ \sum_{|\alpha| = k} T_\alpha T_\alpha^* \ \rangle + \| Yh \|^2$$

$$= \| h \|^2 - \langle Qh, h \rangle + \langle Qh, h \rangle$$

$$= \| h \|^2$$

for any $h \in \mathcal{H}$. Therefore, V is an isometry. On the other hand, using relations (17) and(18), we deduce that

$$V\, T_i^* = \begin{bmatrix} K_{T,J} \\ Y \end{bmatrix} T_i^* = K_{T,J} T_i^*\ h \oplus Y\, T_i^*\ h$$

$$= (B_i^* \otimes I_{\mathcal{H}}) K_{T,J}\ h \oplus Z_i^*\ Y\ h$$

$$= \begin{bmatrix} B_i \otimes I_{\overline{\Delta_T \mathcal{H}}} & 0 \\ 0 & Z_i \end{bmatrix} Vh,$$

Since V is an isometry we can identify $\mathcal{H}$ with V $\mathcal{H}$ and complete the proof of (i) and (ii).The last part of the theorem is obvious. _

Corollary(1.1.13) [1]: In the particular case when n = 1 and $P_J = 0$, we obtain the classical iso-metric dilation theorem for contractions obtained by S_Z.-Nagy (see [38]) by different methods.

Now we can evaluate the dilation index of a constrained row contraction and show that it does not depend on the constraints. We show that the dilation index coincides with rank Δ_T , [10].

Corollary(1.1.14) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of generated by a family of polynomials $P_J \subset F_n^\infty$ and let $T := [T_1, . . . ,T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction such that

$$p(T_1, . . . ,T_n) = 0 \text{ for any } p \in P_J$$

Then the dilation index of T satisfies the equation

dil-ind (T) = rank Δ_T .

Proof : Let $\mathcal{D}$ and K be Hilbert spaces and let $Z_i \in B(K)$ be bounded operators such that $Z_1Z_1^* + \cdot \cdot \cdot + Z_nZ_n^* = I_k$

Assume that the n-tuple $V := [V_1, . . . , V_n]$ given by

$$V_i := \begin{bmatrix} B_i \otimes I_{\mathcal{D}} & 0 \\ 0 & Z_i \end{bmatrix}, \; i = 1, . . . , n \;; \tag{20}$$

is a constrained dilation of T. Since $\mathcal{H}$ is co-invariant under $V_1, . . . , V_n$, and $\mathcal{N}_J$ is co-invariant under the left creation operators $S_1, . . . , S_n$, we have

$$I_{\mathcal{H}} - \sum_{i=1}^n T_iT_i^* = P_{\mathcal{H}} \begin{bmatrix} \left[P_{\mathcal{N}_J}(I - \sum_{i=1}^n S_iS_i^*)| \; \mathcal{N}_J\right] \otimes I_{\mathcal{D}} & 0 \\ 0 & 0 \end{bmatrix} | \; \mathcal{H}$$

Hence, and taking into account that $(I - \sum_{i=1}^n S_iS_i^*)$ is a rank one operator, we deduce that rank $\Delta_T \leq$ rank $[P_{\mathcal{N}_J}(I - \sum_{i=1}^n S_iS_i^*)| \; \mathcal{N}_J \otimes I_{\mathcal{D}}] \leq$ dim .

Now, using Theorem (1.1.12), we conclude that dil ind (T) = rank Δ_T. _

Theorem(1.1.15) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of generated by a family of polynomials $P_J \subset F_n^\infty$ and let $T := [T_1, . . . ,T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction such that

$$p(T_1, . . . ,T_n) = 0 \text{ for any } p \in P_J. \tag{21}$$

Then there exists a separable Hilbert space k_π and a $*$-representation π: of $C^* (B_1 , ... , B_n) \to B(k_\pi)$ which annihilates the compact operators and

$$\pi(B_1) \pi (B_1)^* + \cdot \cdot \cdot + \pi(B_n) \pi (B_n)^* = I_{k_\pi} .$$

such that:

(i) $\mathcal{H}$ can be identified with a co-invariant subspace of $\check{k} := (\mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}}) \oplus K$ under the operators

$$V_i := \begin{bmatrix} B_i \otimes I_{\overline{\Delta_T \mathcal{H}}} & 0 \\ 0 & Z_i \end{bmatrix}, \; i = 1, . . . , n;$$

(ii) $T_i^* = V_i^*|\mathcal{H}$, i = 1, . . . , n.

Proof: According to [35], if P_J consists of homogeneous polynomials, then Range $K_{T,r} \subseteq \mathcal{N}_J \otimes \mathcal{H}$ for any r ∈ (0, 1), the constrained Poisson kernel $K_{J,T,r} := (P_{\mathcal{N}_J} \otimes I_{\mathcal{H}}) K_{T,r}$ is an isometry, and there is a unique unital completely contractive linear map

$$\Phi_{J,T}: \overline{\text{span}}\{B_\alpha B_\alpha^* : \alpha, \beta \in \mathbb{F}_n^+\}$$

such that $\Phi_{J,T}(B_\alpha B_\beta^*) = T_\alpha T_\beta^* : \alpha,\beta \in \mathbb{F}_n^+$. Applying Arveson extension theorem [2] to the map $\Phi_{J,T}$, we obtain a unital completely positive linear map
$\Psi_{J,T}: C^*(B_1, . . . ,B_n) \to B(\mathcal{H})$. Let $\breve{\pi} : C^*(B_1, . . . ,B_n) \to B(\breve{K})$ be a minimal Stinespring dilation of $\Psi_{J,T}$: i.e.,

$$\Psi_{J,T}(X) = P_{\mathcal{H}}\breve{\pi}(X)|\mathcal{H},\ X \in C^*(B_1, . . . ,B_n)$$

and $\breve{K} = \overline{\text{span}}\{\breve{\pi}(X)h : h \in \mathcal{H}\}$.

Notice that, for each i = 1, . . . , n,

$$\Psi_{J,T}(B_iB_i^*) = T_iT_i^* = P_{\mathcal{H}}\breve{\pi}(B_i)\breve{\pi}(B_i^*)|\ \mathcal{H}$$

$$=P_{\mathcal{H}}\breve{\pi}(B_i)(P_{\mathcal{H}} + P_{\mathcal{H}^\perp})\breve{\pi}(B_i^*)|\ \mathcal{H}$$

$$=\Psi_{J,T}(B_iB_i^*)+P_{\mathcal{H}}\breve{\pi}(B_i)|_{\mathcal{H}^\perp})(P_{\mathcal{H}^\perp}\breve{\pi}(B_i^*)|\ \mathcal{H})$$

Hence, we infer that $P_{\mathcal{H}}\breve{\pi}(B_i)|_{\mathcal{H}^\perp} = 0$ and

$$\Psi_{J,T}(B_\alpha X)= P_{\mathcal{H}}(\breve{\pi}(B_\alpha)\breve{\pi}(X))|\ \mathcal{H}$$

$$=(P_{\mathcal{H}}\breve{\pi}(B_\alpha)|\ \mathcal{H})(P_{\mathcal{H}}\breve{\pi}(X))|\ \mathcal{H}) \tag{22}$$

$$=\Psi_{J,T}(B_\alpha)\ \Psi_{J,T}(X)$$

for any $X \in C^*(B_1, . . . ,B_n)$and $\alpha \in \mathbb{F}_n^+$. Note that the Hilbert space $\breve{K}$ is separable and $\mathcal{H}$ is an invariant subspace under each $\breve{\pi}(B_i)^*$,i = 1, . . . , n, due to the fact that $P_{\mathcal{H}}\breve{\pi}(B_i)|_{\mathcal{H}^\perp} = 0$.
This means that

$$\breve{\pi}(B_i)^*|\ \mathcal{H} =\Psi_{J,T}(B_i^*)=T_i^* ,\ i = 1, . . . , n. \tag{23}$$

Now, since the subspace $\mathcal{N}_J$ contains the constants, we can apply Theorem (1.1.3) and deduce that all the compact operators $LC(\mathcal{N}_J))$ in $B(\mathcal{N}_J)$ are contained in $C^*(B_1, . . . ,B_n)$. According to Theorem (1.1.8), the representation $\breve{\pi}$ decomposes into a direct sum $\breve{\pi} = \pi_0 \oplus \pi$ on $\breve{K} = K_0 \oplus K_\pi$,
where π_0, π are disjoint representations of $C^*(B_1, . . . ,B_n)$on the Hilbert spaces $K_0$and K_π , respectively, such that

$$K_0 \simeq \mathcal{N}_J \otimes G,\ \pi_0(X) = X \otimes I_G,\ X \in C^*(B_1, . . . ,B_n) \tag{24}$$

for some Hilbert space G, and is a representation such that $\pi(LC(\mathcal{N}_J)) = 0$. Since $P_{\mathbb{C}}^{\mathcal{N}_J} = I - \sum_{i=1}^n B_iB_i^*$
is a rank one projection in $C^*(B_1, . . . ,B_n)$we deduce that

$$\sum_{i=1}^n \pi(B_i)\pi(B_i)^* = I_{k_\pi}$$

and
dim G = dim(range $\breve{\pi}(P_{\mathbb{C}}^{\mathcal{N}_J})$).

Using the minimality of the Stinespring representation $\breve{\pi}$ and the proof of Theorem (1.1.3), we have

$$\text{range}\ \breve{\pi}(P_{\mathbb{C}}^{\mathcal{N}_J}) = \overline{span}\{\breve{\pi}(P_{\mathbb{C}}^{\mathcal{N}_J})\breve{\pi}(X)h : X \in C^*(B_1,\dots,B_n),\ h \in \mathcal{H}\}$$

$= \overline{span}\ \{\breve{\pi}\ (P_{\mathbb{C}}^{\mathcal{N}_J})\breve{\pi}\ (Y)h\colon Y \in C_0, h\ \in\ \mathcal{H}\}$

$=\overline{span}\ \{\breve{\pi}\ (P_{\mathbb{C}}^{\mathcal{N}_J})\breve{\pi}\ (B_\alpha P_{\mathbb{C}}^{\mathcal{N}_J} B_\beta^*)h\ \colon \alpha, \beta \epsilon\ \mathbb{F}_n^+, h\ \in\ \mathcal{H}\}$
$=\overline{span}\ \{\breve{\pi}\ (P_{\mathbb{C}}^{\mathcal{N}_J})\breve{\pi}\ (B_\beta^*)h\colon \beta \epsilon\ \mathbb{F}_n^+, h\ \in\ \mathcal{H}\}$

On the other hand, using relation (22), we have

$$\langle \breve{\pi}\ \left(P_{\mathbb{C}}^{\mathcal{N}_J}\right) \breve{\pi}\ (B_\alpha^*)h, \breve{\pi}\ \left(P_{\mathbb{C}}^{\mathcal{N}_J}\right) \breve{\pi}\ \left(B_\beta^*\right)k\rangle = \langle h, \pi(B_\alpha)\pi\ \left(P_{\mathbb{C}}^{\mathcal{N}_J}\right) \pi\ \left(B_\beta^*\right)h\rangle$$

$$=\langle h, T_\alpha(I_k - \textstyle\sum_{i=1}^{n} T_i T_i^*)T_\beta^* h\rangle$$

$$=\langle \Delta_T T_\alpha^* h, \Delta_T T_\beta^* k\rangle$$

for any h, k $\in \mathcal{H}$. This shows that one can define a unitary operator A : range $\breve{\pi}\ \left(P_{\mathbb{C}}^{\mathcal{N}_J}\right) \to \overline{\Delta_T\mathcal{H}}$ by setting

$$A(\breve{\pi}\ \left(P_{\mathbb{C}}^{\mathcal{N}_J}\right), \breve{\pi}\ (B_\alpha^*)h) := \Delta_T T_\alpha^*\ h \in H\ ,$$

and extending it by linearity and continuity. Therefore, we have dim[range $\pi\ \left(P_{\mathbb{C}}^{\mathcal{N}_J}\right)$] $=\overline{\Delta_T\mathcal{H}}$ = dim G.

Hence, making the appropriate identification of G with $\overline{\Delta_T\mathcal{H}}$and using relations (23) and(24), we obtain the required dilation. This completes the proof. _

Corollary(1.1.16) [1]: Let V := [V_1, . . . , V_n] be the dilation of Theorem (1.1.15). Then,
(i) V is a constrained shift if and only if T is a pure constrained row contraction;
(ii) V is a Cuntz type representation if and only if T is a constrained row contraction such that

$$T_1 T_1^* + \cdot \cdot \cdot + T_n T_n^* = I.$$

Proof: Notice that

$$\textstyle\sum_{|\alpha| = k} T_\alpha T_\alpha^* = P_{\mathcal{H}} \begin{bmatrix} \sum_{|\alpha| = k} B_\alpha B_\alpha^* \otimes I_{\overline{\Delta_T\mathcal{H}}} & 0 \\ 0 & I_{K_\pi} \end{bmatrix} |\ \mathcal{H}$$

and therefore,

$$\text{SOT-}\textstyle\sum_{|\alpha| = k} T_\alpha T_\alpha^* = P_{\mathcal{H}} \begin{bmatrix} 0 & 0 \\ 0 & I_{K_\pi} \end{bmatrix} |\ \mathcal{H}.$$

This shows that T is a pure row contraction if and only if $P_{\mathcal{H}} P_{K_\pi} | \mathcal{H} = 0$. The latter condition is equivalent to $\mathcal{H} \perp (0 \oplus K_\pi)$, which implies $\mathcal{H} \subset \mathcal{N}_J \otimes \overline{\Delta_T\mathcal{H}}$. Now, since $\mathcal{N}_J \otimes \overline{\Delta_T\mathcal{H}}$ is reducing for each operator V_i, i = 1, . . . , n, and $\breve{K}$ is the smallest reducing subspace for the same operators, which contains $\mathcal{H}$, we conclude that $\breve{K} = \mathcal{N}_J \otimes \overline{\Delta_T\mathcal{H}}$., which proves (i).

Now assume that the dilation V is a Cuntz type representation, i.e.,
$\sum_{|\alpha| = k} V_i V_i^* = I_{\breve{K}}$. Since

$$\textstyle\sum_{|\alpha| = k} V_\alpha V_\alpha^* = \begin{bmatrix} \sum_{|\alpha| = k} B_\alpha B_\alpha^* \otimes I_{\overline{\Delta_T\mathcal{H}}} & 0 \\ 0 & I_{K_\pi} \end{bmatrix}$$

we deduce that

$$\sum_{|\alpha| = k} B_\alpha B_\alpha^* \otimes I_{\overline{\Delta_T\mathcal{H}}} = I_{K_0}$$

for any k = 1, 2, ... Due to the fact that
SOT-$\lim\limits_{k\to\infty} \sum_{|\alpha| = k} B_\alpha B_\alpha^*$ = 0, we must have K_0= {0}. Using the proof of Theorem (1.1.15), we get G = {0}, which means $\Delta_T = 0$. The proof is complete. Under additional hypotheses, one can obtain the following remarkable particular case of Theorem(1.1.15) where the dilation is unique up to a unitary equivalence.

Corollary(1.1.17) [1]: If in addition to the hypotheses of Theorem (1.1.15)

$$\overline{\text{span}}\{B_\alpha B_\alpha^* : \alpha, \beta \in \mathbb{F}_n^+\} = C^*(B_1, \ldots, B_n), \quad (25)$$

then the dilation of Theorem(1.1.15) is minimal, i.e., $\breve{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$, and it is unique up to a unitary equivalence.

Let $T' := [T_1', \ldots, T_n']$, $T_i' \in B(\mathcal{H}')$, be another row contraction subject to the same constraints as T and let $V' := [V_1', \ldots, V_n']$ be the corresponding dilation. Then T and T′ are unitarily equivalent if and only if

Dim $\overline{\Delta_T \mathcal{H}}$ = dim $\overline{\Delta_{T'} \mathcal{H}'}$ and there are unitary operators $A : \mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}} \rightarrow \mathcal{N}_J \otimes \overline{\Delta_{T'} \mathcal{H}'}$ and $\Gamma : K_\pi \rightarrow K_{\pi'}$ such that

$A (B_i \otimes \overline{\Delta_T \mathcal{H}}) = (B_i \otimes \overline{\Delta_{T'} \mathcal{H}'})A$ and $\Gamma_\pi(B_i) = \pi'(B_i)\Gamma$ for $i = 1, \ldots, n$,

and

$$\begin{bmatrix} A & 0 \\ 0 & \Gamma \end{bmatrix} \mathcal{H} = \mathcal{H}'.$$

Proof: A closer look at the proof of Theorem (1.1.15) reveals that, under condition (25), the map $\Psi_{J,T}$ is unique. Using the uniqueness of the minimal Stinespring representation (see [37], [4]), one can prove the uniqueness of the minimal dilation of Theorem(1.1.12) .The last part of this corollary follows using standard arguments concerning representation theory of C^*- algebras [3] and the uniqueness of minimal completely positive dilations of completely positive maps of C^*-algebras. In what follows we present several examples when the condition (25) is satisfied.

Example (1.1.18) [1]: Let $P_J \subset F_n^\infty$ by a set of polynomials and let P_J be the WOT-closed two-sided ideal of F_n^∞ generated by P_J . The condition (25) is satisfied in the following particular cases.

(i) If $P_J := 0$, then $\mathcal{N}_J = F^2(H_n)$, $B_i = S_i$, and therefore $S_j^* S_i = \delta_{i,j} I$. In this case, Theorem (1.1.15) and Corollary(1.1.17) imply the standard non commutative isometric dilation theorem for row contraction [21].

(ii) If $P_J := \{S_i S_j - S_j S_i : I, j = 1, \ldots, n\}$, then $\mathcal{N}_J = F_s^2$, the symmetric Fock space,
and B_i, $i = 1, \ldots, n$, are the creation operators on the symmetric Fock space. We obtain in this case the dilation theorem for commuting g row contractions (see [13], [4], and [30]).

(iii) If $B_1, \ldots, B_n$ are essentially normal.

(iv) Let P_J be a set of homogenous polynomials in F_n^∞ . According to Lemma (1.1.1), $U\mathcal{N}_J$ is a subspace invariant under S_i^* , $i = 1, \ldots, n$. Using the characterization of the invariant subspace for the left creation operators [22], there exists an essentially unique sequence
$\{\varphi_p(S_1, \ldots, S_n)\}_{p=1}^N$, $N = 1, 2, \ldots, \infty$, of isometries with orthogonal ranges
such that

$$P_{\mathcal{N}_J} = I - \sum_{p=1}^N \varphi_p(S_1, \ldots, S_n)\varphi_p(S_1, \ldots, S_n)^*$$

where the series is SOT-convergent if $N = \infty$. If the above sequence is finite ($N < \infty$)and $\varphi_p(S_1, \ldots, S_n)$ $p = 1, \ldots, N$, are in the non commutative disc algebra A_n, then condition (25) holds. Indeed, in this case we have

$$B_i^* B_j = P_{\mathcal{N}_J} S_i^* \left(I - \sum_{p=1}^N \varphi_p(S_1, \ldots, S_n)\varphi_p(S_1, \ldots, S_n)^* \right) S_j | \mathcal{N}_J$$

Since $S_i^* S_j = \delta_{i,j} I$ and $\mathcal{N}_J$ is invariant under S_i^* , $i = 1, \ldots, n$, we deduce that $B_i^* B_j$ is in span $\{B_\alpha B_\beta^* : \alpha, \beta \in \mathbb{F}_n^+\}$

(v) If $P_J := \{S_\alpha : |\alpha| = m\}$, then $P_{\mathcal{N}_J} = I - \sum_{|\alpha|=m} S_\alpha S_\alpha^*$ .In this case, we have

$B_i^* B_j = P_{\mathcal{N}_J} S_i^* \left(I - \sum_{|\alpha|=m} S_\alpha S_\alpha^* \right) S_j | \mathcal{N}_J$

and $B_i^* B_j$ is in span $\{B_\alpha B_\beta^* : \alpha, \beta \epsilon \mathbb{f}_n^+\}$.

(vi) If $P_J := \{S_\alpha : |\alpha| = m\} \cup \{S_i S_j - S_j S_i : i, j = 1, \ldots, n\}$.
(vii) If $P_J := S_j S_i - q_{ji} S_i S_j : - i < j, i, j = 1, \ldots, n\}$ for some $q_{ji} \in \mathbb{C}$, then $B_j B_j^* = 0$ if $i \neq j$ and $B_i^* B_i$ can be written as a linear combination of the identity and $B_j B_j^*$,$j = 1, \ldots, n$. In this case we obtain the dilation result from [7].
Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction and let $C \subset F_n^\infty$. If T is a c.n.c. row contraction and $\varphi \in C$, then $\varphi(T_1, \ldots, T_n)$ is defined by the F_n^∞ -functional calculus for row contractions [26]. When T is an arbitrary row contraction, then we assume that C consists of polynomials.
Denote by $\mathcal{M}_c$ the closed span of all co-invariant spaces $M \subseteq \mathcal{H}$ under $T_1, \ldots, T_n$ such that $\varphi(P_{\mathcal{M}} T_1 | \mathcal{M}, \ldots, P_{\mathcal{M}} T_n | \mathcal{M}) = 0$ for any $\varphi \in C$.
We call the row contraction

$$[P_{\mathcal{M}_C} T_1 | \mathcal{M}_C, \ldots, P_{\mathcal{M}_C} T_n | \mathcal{M}_C]$$

the maximal C-constrained piece of $[T_1, \ldots, T_n]$.
Lemma(1.1.19) [1]: If $V := [V_1, \ldots, V_n]$, $V_i \in B(\mathcal{H})$, is a row contraction then

$$\mathcal{M}_C = \overline{span}\{V_\alpha \varphi(V_1, \ldots, V_n)\mathcal{H} : \varphi \in C, \alpha \in \mathbb{f}_n^+\}^\perp = \bigcap_{\varphi \in C, \alpha \in \mathbb{f}_n^+} \varphi(V_1, \ldots, V_n)^* V_\alpha^*$$

Proof : Denote := $\operatorname{span}\{V_\alpha \varphi(V_1, \ldots, V_n)\mathcal{H} : \varphi \in C, \alpha \in \mathbb{f}_n^+\}$

and note that $\mathcal{E}^\perp$ is co-invariant under $V_1, \ldots, V_n$. If $h \in \mathcal{E}^\perp$ and $k \in \mathcal{H}$, then

$0 = \langle \varphi(V_1, \ldots, V_n) k, h \rangle = \langle k, \varphi(V_1, \ldots, V_n)^* h \rangle$.

Hence, we get $\varphi(P_{\mathcal{E}^\perp} V_1 | \mathcal{E}^\perp, \ldots, P_{\mathcal{E}^\perp} V_n | \mathcal{E}^\perp) = 0$. Let M be a co-invariant subspace under $V_1, \ldots, V_n$ such that $\varphi(P_{\mathcal{M}} V_1 | \mathcal{M}, \ldots, P_{\mathcal{M}} V_n | \mathcal{M}) = 0$. For any $h \in \mathcal{M}$ and $\in \mathbb{f}_n^+$, we have $V_\alpha^* h \in \mathcal{M}$, therefore $\varphi(V_1, \ldots, V_n)^* V_\alpha^* h = 0$ This implies $\langle h, \varphi(V_1, \ldots, V_n) k \rangle = 0$ for any $k \in \mathcal{H}$, which shows that $M \subseteq \mathcal{E}^\perp$ and completes the proof.
Using this lemma and the definition of the subspace $\mathcal{N}_J$, one can easily prove the following.
Proposition(1.1.20) [1]: Let: Let $J \neq F_n^\infty$ be an arbitrary WOT-closed two-sided ideal of F_n^∞ , $S_1, \ldots, S_n$ be the left creation operators on the full Fock space $F^2(H_n)$, and $B_i := P_{\mathcal{N}_J} S_i | \mathcal{N}_J$, $i = 1, \ldots, n$. Then $[B_1, \ldots, B_n]$ is the maximal J-constrained piece of $[S_1, \ldots, S_n]$.We consider now the particular case when the row contraction $T := [T_1, \ldots, T_n]$ is pure, i.e. SOT-$\lim_{|k| \to \infty} \sum_{|\alpha|} T_\alpha T_\alpha^* = 0$. In this case, the resultstion can be extended to a larger class of constrained row contractions.
Theorem(1.1.21) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of generated by a family of polynomials $P_J \subset F_n^\infty$ and let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction such that $f(T_1, \ldots, T_n) = 0$ for any $f \in J$.
Then the following statements hold:
(i) The constrained Poisson kernel $K_{J,T} : \mathcal{H} \to \mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}}$ defined by setting

$$K_{J,T} := (P_{\mathcal{N}_J} \otimes I) K_T$$

is an isometry , $K_{J,T}\mathcal{H}$ is co-invariant under $B_i \otimes I_{\overline{\Delta_T\mathcal{H}}}$, i = 1, . . . , n, and

$$T_i = K^*_{T,J}(B_i \otimes I_{\overline{\Delta_T\mathcal{H}}})\, K_{J,T},\ i = 1, \ldots, n. \tag{26}$$

(ii) dil-ind (T) = rank Δ_T .
(iii) If $1 \in \mathcal{N}_J$, then the dilation provided by (26) is minimal.
If, in addition, $1 \in \mathcal{N}_J$ and

$$\overline{span}\{B_\alpha B^*_\beta : \alpha, \beta \in \mathbb{F}^+_n\} = C^*(B_1, \ldots, B_n), \tag{27}$$

then we have:
(iv) The minimal dilation provided by (26) is unique up to an isomorphism.
(v) The minimal dilation is the maximal J-constrained piece of the standard non commutative isometric dilation of T.
(vi) A pure constrained row contraction has rank Δ_T = m, m = 1, 2, . . . ,∞, if and only if it is unitarily equivalent to one obtained by compressing $[B_1 \otimes I_{\mathbb{C}^m}, \ldots, B_n \otimes I_{\mathbb{C}^m}]$ to a co-invariant subspace $\mathcal{M} \subset \mathcal{N}_J \otimes \mathbb{C}^m$ under $B_1 \otimes I_{\mathbb{C}^m}, \ldots, B_n \otimes I_{\mathbb{C}^m}$, with the property that dim $P_0\mathcal{M}$ = m, where P_0 is the orthogonal projection of $\mathcal{N}_J \otimes \mathbb{C}^m$ onto the subspace $1 \otimes \mathbb{C}^m$.

Proof: Part (i) follows from [12]. To prove (ii), let D be a Hilbert space such that $\mathcal{H}$ can be identified with a co-invariant subspace of $\mathcal{N}_J \otimes \mathcal{D}$ under $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n, and such that $T_i = P_{\mathcal{H}}(B_i \otimes I_{\mathcal{D}})|\mathcal{H}$ for i = 1, . . . , n. Then

$$I_{\mathcal{H}} - \sum_{i=1}^{n} T_i T_i^* = P_{\mathcal{H}}^{\mathcal{N}_J \otimes \mathcal{D}}\left[\left(I_{\mathcal{N}_J} - \sum_{i=1}^{n} B_i B_i^*\right) \otimes I_{\mathcal{D}}\right] |\, \mathcal{H}$$

$$= P_{\mathcal{H}}^{\mathcal{N}_J \otimes \mathcal{D}}\left[P_{\mathcal{N}_J}\left(I - \textstyle\sum_{i=1}^{n} S_i S_i^*\right)\right)|\, \mathcal{N}_J \otimes I_{\mathcal{D}}\Big]|\, \mathcal{H}$$

$$= P_{\mathcal{H}}^{\mathcal{N}_J \otimes \mathcal{D}}\left[P_{\mathcal{N}_J} P_{\mathbb{C}}|\, \mathcal{N}_J \otimes I_{\mathcal{D}}\right]|\, \mathcal{H}$$

Hence, rank $\Delta_T \le \dim\mathcal{D}$. Using (i), we deduce that the dilation index of T is equal to rank Δ_T .
Assume now that $1 \in \mathcal{N}_J$. As in the proof of Theorem (1.1.5), we obtain $P_{\mathbb{C}}^{\mathcal{N}_J} P_{\mathcal{N}_J} e_\alpha = 0$ for any $\alpha \in \mathbb{F}^+_n$, $|\alpha| \ge 1$. On the other hand, the definition of the constrained Poisson kernel $K_{J,T}$ implies

$$P_0 K_{J,T}\, h = \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{|\alpha| = k} P_{\mathbb{C}}^{\mathcal{N}_J} P_{\mathcal{N}_J} e_\alpha \otimes \Delta_T T^*_\alpha h\ ,\ h \in \mathcal{H}$$

where $P_0 := P_{\mathbb{C}}^{\mathcal{N}_J} \otimes I_{\overline{\Delta_T\mathcal{H}}}$. Therefore, $P_0 K_{J,T}\,\mathcal{H} = \overline{\Delta_T\mathcal{H}}$. Using Corollary (1.1.6) in the particular case when := $K_{J,T}\mathcal{H}$ and D := $\overline{\Delta_T\mathcal{H}}$, we deduce that $K_{J,T}\mathcal{H}$ is cyclic for $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n, which proves the minimality of the dilation (26), i.e.,

$$\mathcal{N}_J \otimes \overline{\Delta_T\mathcal{H}} = \bigvee_{\alpha \in \mathbb{F}^+_n} (B_\alpha \otimes I_{\overline{\Delta_T\mathcal{H}}})\, K_{J,T}\mathcal{H} \tag{28}$$

Now we assume that $1 \in \mathcal{N}_J$ and that relation (27) holds. Consider another minimal dilation of T, i.e.,

$$T_i = V^*(B_i \otimes I_{\mathcal{D}})V, \tag{29}$$

where $V : \mathcal{H} \to \mathcal{N}_J \otimes \mathcal{D}$ is an isometry , $V\mathcal{H}$ is co-invariant under $B_i \otimes I_{\mathcal{D}}$, i = 1, . . . , n, and

$$\mathcal{N}_J \otimes \mathcal{D} = \bigvee_{\alpha \in \mathbb{F}^+_n} (B_\alpha \otimes I_{\mathcal{D}})\, V\mathcal{H}. \tag{30}$$

We know (see [1]) that there exists a unital completely positive linear map

$$\Phi: \text{span}\{B_\alpha B^*_\alpha : \alpha, \beta \in \mathbb{F}^+_n\} \to B(\mathcal{H})$$

such that $\Phi(B_\alpha B_\alpha^*) = T_\alpha T_\alpha^*$, $\alpha, \beta \in \mathbb{F}_n^+$. Due to (27), Φ has a unique extension to $C^*(B_1, \ldots, B_n)$. Consider the ∗-representations
$\pi_1 : C^*(B_1, \ldots, B_n) \to B(\mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}})$, $\pi_1(X) := X \otimes I_{\overline{\Delta_T \mathcal{H}}}$
$\pi_2 : C^*(B_1, \ldots, B_n) \to B(\mathcal{N}_J \otimes \boldsymbol{\mathcal{D}})$, $\pi_2(X) := X \otimes I_{\mathcal{D}}$.

It is easy to see that due to relations (26), (29), (27), and the co-invariance of the subspaces $K_{J,T}\mathcal{H}$ and $V\mathcal{H}$, we have
$\Phi(X) = K_{J,T}^* \pi_1(X) K_{J,T} = V^* \pi_2((X)V$, $X \in C^*(B_1, \ldots, B_n)$.

Now, due to the minimality conditions (28) and (30), and relation (27), we deduce that π_1 and π_2 are minimal Stinespring dilations . Since they are unique, there exists a unitary operator $U : \mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}} \to \mathcal{N}_J \otimes \boldsymbol{\mathcal{D}}$ such that

$$U(B_i \otimes I_{\overline{\Delta_T \mathcal{H}}}) = (B_i \otimes I_{\mathcal{D}})U, \ i = 1, \ldots, n. \qquad (31)$$

and $UK_{J,T} = V$. Hence, we also have

$U(B_i^* \otimes I_{\overline{\Delta_T \mathcal{H}}})) = (B_i^* \otimes I_{\mathcal{D}})U$, $i = 1, \ldots, n$.

By Theorem (1.1.3), $C^*(B_1, \ldots, B_n)$. is irreducible, so we must have $U = I_{\mathcal{N}_J} \otimes W$, where $W \in B(\overline{\Delta_T \mathcal{H}}, \boldsymbol{\mathcal{D}})$ is a unitary operator. Therefore, dim $\overline{\Delta_T \mathcal{H}}$ = dim $\boldsymbol{\mathcal{D}}$ and $UK_{J,T}\mathcal{H} = V\mathcal{H}$, which proves that the two dilations are unitarily equivalent.
In the particular case when $J = \{0\}$, part (iv) shows that

$$S := [S_1 \otimes I_{\overline{\Delta_T \mathcal{H}}}, \ldots, S_n \otimes I_{\overline{\Delta_T \mathcal{H}}}]$$

is a realization of the standard minimal isometric dilation of $[T_1, \ldots, T_n]$. Using Lemma(1.1.19)and Proposition (1.1.20), one can easily see that the maximal J-constrained piece of S coincides with $[B_1 \otimes I_{\overline{\Delta_T \mathcal{H}}}, \ldots, B_n \otimes I_{\overline{\Delta_T \mathcal{H}}}]$, which proves (v). Now, we prove (vi). The implication " $\Longrightarrow$ " follows from part (i). Conversely, assume that

$$T_i = P_{\mathcal{H}}(B_i \otimes I_{\mathbb{C}^m}) |\mathcal{H}, \ i = 1, \ldots, n,$$

where $\mathcal{H} \subset \mathcal{N}_J \otimes \mathbb{C}^m$ is a co-invariant subspace under $B_i \otimes I_{\mathbb{C}^m}$, $i = 1, \ldots, n$, with dim $P_0\mathcal{H} = m$(recall $P_0 := P_{\mathbb{C}}^{\mathcal{N}_J} \otimes I_{\mathbb{C}^m}$). It is clear that $T := [T_1, \ldots, T_n]$ is a pure J-constrained row contraction. Consider first the case when $m < \infty$. Since $P_0\mathcal{H} \subseteq \mathbb{C}^m$ and dim$P_0\mathcal{H} = m$, we deduce that $P_0\mathcal{H} = \mathbb{C}^m$. By Corollary (1.1.6), we have

$$\mathcal{H}^\perp \cap \mathbb{C}^m = \{0\}. \qquad (32)$$

On the other hand, since $I_{\mathcal{N}_J} - \sum_{i=1}^n B_i B_i^* = P_{\mathbb{C}}^{\mathcal{N}_J}$
, we obtain

$$\text{rank } \Delta_T = \text{rank } P_{\mathcal{H}}\left[(I_{\mathcal{N}_J} - \textstyle\sum_{i=1}^n B_i B_i^*) \otimes I_{\mathbb{C}^m}\right| \mathcal{H}$$
$$= \text{rank } P_{\mathcal{H}} P_0 |H = \dim P_{\mathcal{H}} P_0 \mathcal{H}$$
$= \dim P_{\mathcal{H}}$Cm.
If rank $\Delta_T < m$, then there exists a nonzero vector $h \in \mathbb{C}^m$ with $P_{\mathcal{H}} h = 0$, which contradicts relation (32). Therefore, we must have rank $\Delta_T = m$.
Now, we consider the case $m = \infty$. According to Theorem (1.1.5), setting $:= P_0\mathcal{H}$, we have

$\bigvee_{\alpha\in\mathbb{F}_n^+}(B_i \otimes I_{\mathbb{C}^m})\mathcal{H} = \mathcal{N}_J \otimes \boldsymbol{\mathcal{E}}$,

which is reducing for $B_i \otimes I_{\mathbb{C}^m}$, i = 1, . . . , n. Therefore,
$T_i = P_{\mathcal{H}}((B_i \otimes I_{\mathcal{E}})|\mathcal{H}$, i = 1, . . . , n.
Using the uniqueness of the minimal dilation of T (see (iv)), we deduce that $\dim\overline{\Delta_T\mathcal{H}} = \dim\boldsymbol{\mathcal{E}} = \infty$.
The proof is complete. _
Let J be a WOT-closed two-sided ideal of F_n^∞ generated by a family P_J of homogeneous polynomials such that $1 \in \mathcal{N}_J$ and condition(5) holds. If $[T_1, \ldots, T_n]$ is a arbitrary J-constrained row contraction, is it true that the minimal dilation provided by Theorem (1.1.15) is the maximal J-constrained piece of the standard non commutative isometric dilation of T ?We should mention that the answer to this question is positive in the commutative case, i.e., when $P_J := \{S_iS_j - S_jS_i : i, j = 1, \ldots, n\}$ (see [8]).Under certain natural conditions on the ideal J, we can characterize the pure J-constrained row contractions of rank one.

Corollary (1.1.22) [1]: Let J be a WOT-closed two-sided ideal of F_n^∞ such that $1 \in \mathcal{N}_J$ and condition (27) is satisfied. If $M \subset \mathcal{N}_J$ is a co-invariant subspace under $B_1, \ldots, B_n$ then the n-tuple $T := [T_1, \ldots, T_n]$ $T_i := B_{\mathcal{M}}B_i|\boldsymbol{\mathcal{M}}$, i = 1, . . . , n, is a pure J-constrained row contraction such that rank $\Delta_T = 1$.
If $\boldsymbol{\mathcal{M}}'$ is another co-invariant subspace for $B_1, \ldots, B_n$, which gives rise to a row contraction T′, then T and T′ are unitarily equivalent if and only if $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}'$.
Every pure constrained row contraction with rank $\Delta_T = 1$ is unitarily equivalent to one obtained by compressing $[B_1, \ldots, B_n]$ to a co-invariant subspace for $B_1, \ldots, B_n$.

Proof: Since $\boldsymbol{\mathcal{M}} \subset \mathcal{N}_J$ is a co-invariant subspace under $B_1, \ldots, B_n$, we have
$f(T_1, \ldots, T_n) = P_{\mathcal{M}} f(B_1, \ldots, B_n)|\boldsymbol{\mathcal{M}} = 0$, $f \in J$,
and $I_{\mathcal{M}} - \sum_{i=1}^n T_iT_i^* = P_{\mathcal{M}}(I_{\mathcal{N}_J} - \sum_{i=1}^n B_iB_i^*)|\ \mathcal{M}$
$= P_{\mathcal{M}} P_{\mathbb{C}}^{\mathcal{N}_J}|\ \boldsymbol{\mathcal{M}}$.
Hence, $[T_1, \ldots, T_n]$ is a constrained row contraction with rank $\Delta_T \le 1$. On the other hand, since

$$\sum_{|\alpha| = k} T_\alpha T_\alpha^* = P_{\mathcal{M}}(\sum_{|\alpha| = k} B_\alpha B_\alpha^*,\ k = 1, 2, \ldots,$$

and $[B_1, \ldots, B_n]$ is a pure row contraction, we deduce that $[T_1, \ldots, T_n]$ is pure. This also implies that $\Delta_T \neq 0$, so rank $\Delta_T \ge 1$. Consequently, we have rank $\Delta_T = 1$.
To prove the second part of this corollary, notice that, as in the proof of Theorem(1.1.21) part (iv), one can show that T and T′ are unitarily equivalent if and only if there exists a unitary operator A : $\mathcal{N}_J \to \mathcal{N}_J$ such that $AB_i = B_iA$, for i = 1, . . . , n, and $A\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}'$.
This implies that A commutes with $C^*(B_1, \ldots, B_n)$ which, due to Theorem(1.1.3), is irreducible. Therefore, A must be a scalar multiple of the identity. Hence, we have $\boldsymbol{\mathcal{M}} = A\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}'$.
Finally, the last part of this corollary follows from Theorem(1.1.15) and Corollary (1.1.17). The purpose is to provide new properties for the standard characteristic function associated with an arbitrary row contraction and show that $I - \Theta_T\Theta_T^* = K_TK_T^*$, where K_T is the Poisson kernel of T. Consequently, we will show that the curvature invariant and Euler characteristic asssociated with a Hilbert module over $\mathbb{F}_n^+$ generated by an arbitrary row contraction T can be expressed only in terms of the characteristic function of T.
The characteristic function associated with an arbitrary row contraction $T := [T_1, \ldots, T_n]$,$T_i \in B(\mathcal{H})$, was introduce in [22] (see [38] for the classical case n = 1) and it was proved to be a complete unitary invariant for completely non-coisometric (c.n.c.) row contractions.
Using the characterization of multi-analytic operators on Fock spaces (see [27], [31]), one can easily see that the characteristic function of T is a multi-analytic operator

$$\Theta_T(R_1, \ldots, R_n) : F^2(H_n) \otimes D_{T^*} \to F^2(H_n) \otimes D_T$$

with the formal Fourier representation

$$-I_{F^2(H_n)} \otimes T + (I_{F^2(H_n)} \otimes \Delta_T)\left(I_{F^2(H_n)\otimes\mathcal{H}} - \sum_{i=1}^n R_iT_i^*\right)^{-1}$$

$$[R_1 \otimes I_{\mathcal{H}}, \ldots, R_n \otimes I_{\mathcal{H}}](I_{F^2(H_n)} \otimes \Delta_{T^*}),$$

where $R_1, \ldots, R_n$ are the right creation operators on the full Fock space $F^2(H_n)$ Here, we need to clarify some notations since some of them are different from those considered in [22]. The defect operators associated with a row contraction $T := [T_1, \ldots, T_n]$ are

$$\Delta_T := (I_{\mathcal{H}} - \sum_{i=1}^{n} T_i T_i^*)^{1/2} \in B(\mathcal{H}) \text{ and } \Delta_{T^*} = (I - T^*T)^{1/2} \in B(\mathcal{H}^{(n)}),$$

while the defect spaces are $\mathcal{D}_T := \overline{\Delta_T \mathcal{H}}$ and $\mathcal{D}_{T^*} := \overline{\Delta_{T^*} \mathcal{H}^{(n)}}$, where $\mathcal{H}$(n) denotes the direct sum of n copies of H. In what follows we need the following result.

Lemma(1.1.23) [1]: If $\Theta(R_1, \ldots, R_n) \in R_n^\infty \overline{\otimes} B(\mathcal{H}, K)$, then SOT- $\lim_{r\to 1} \Theta(rR_1, \ldots, rR_n)^* = \Theta(R_1, \ldots, R_n)^*$.

Proof: We know that any multi-analytic operator $(R_1, \ldots, R_n)$ with formal Fourier representation

$$(R_1, \ldots, R_n) \sim \sum_{k=0}^{\infty} \sum_{|\alpha|=k} R_\alpha \otimes \theta_{(\alpha)},\ \theta_{(\alpha)} \in B(\mathcal{H}, K),$$

has the property that

$$(R_1, \ldots, R_n) = \text{SOT-}\lim_{r\to 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} R_\alpha \otimes \theta_{(\alpha)}$$

where the series converges in the uniform norm for each r ∈ (0, 1). Now, note that for every $\beta \in \mathbb{F}_n^+$, h ∈ $\mathcal{H}$, and g ∈ $F^2(H_n) \otimes K)$, we have

$$\langle \Theta(R_1, \ldots, R_n)^* (e_\beta \otimes h), g \rangle = \langle e_\beta \otimes h, \Theta(R_1, \ldots, R_n) g \rangle$$

$$= \langle e_\beta \otimes h, (\sum_{\alpha \in \mathbb{F}_n^+ |\alpha| \le |\beta|} R_\alpha \otimes \theta_{(\alpha)}) g \rangle$$

$$= \langle \sum_{\alpha \in \mathbb{F}_n^+ |\alpha| \le |\beta|} R_\alpha^* \otimes \theta_{(\alpha)}^*)(e_\beta \otimes h), g \rangle$$

Therefore,

$$\Theta(R_1, \ldots, R_n)^* (e_\beta \otimes h) = (\sum_{\alpha \in \mathbb{F}_n^+ |\alpha| \le |\beta|} R_\alpha^* \otimes \theta_{(\alpha)}^*)(e_\beta \otimes h)$$

Similarly, we have

$$\Theta(rR_1, \ldots, rR_n)^* (e_\beta \otimes h) = (\sum_{\alpha \in \mathbb{F}_n^+ |\alpha| \le |\beta|} r^{|\alpha|} R_\alpha^* \otimes \theta_{(\alpha)}^*)(e_\beta \otimes h)$$

Using the last two equalities, we obtain

$$\lim_{r\to 1} \Theta(rR_1, \ldots, rR_n)^* (e_\beta \otimes h) = \Theta(R_1, \ldots, R_n)^* (e_\beta \otimes h)$$

for any $\beta \in \mathbb{F}_n^+$,and h ∈ $\mathcal{H}$. On the other hand, according to the noncommutative von Neumann inequality,

$\|\Theta(rR_1, \ldots, rR_n)^*\| \le \|\Theta(R_1, \ldots, R_n)^*\|$, for any r ∈ (0, 1).

Hence, and due to the fact that the closed span of all vectors $e_\alpha \otimes h$ with $\beta \in \mathbb{F}_n^+$, h ∈ H, coincides with $F^2(H_n) \otimes \mathcal{H}$, we deduce (using standard arguments) that

SOT- $\lim_{r\to 1} \Theta(rR_1, \ldots, rR_n)^* = \Theta(R_1, \ldots, R_n)^*$.

The proof is complete.

The following factorization result will play an important role in our investigation.

Theorem (1.1.24) [1]: Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction. Then

$$I - \Theta_T \Theta_T^* = K_T K_T^* \qquad (33)$$

where Θ_T is the characteristic function of T and K_T is the corresponding Poisson kernel.

Proof: Denoting $\breve{T} := [I_{F^2(H_n)} \otimes T_1, \ldots, I_{F^2(H_n)} \otimes T_n]$and $\breve{R} := [R_1 \otimes I_{\mathcal{H}}, \ldots, R_n \otimes I_{\mathcal{H}}]$, the characteristic function of T has the representation

$\Theta_T(R_1, \ldots, R_n) = \text{SOT-}\lim_{r\to 1}\left[-\breve{T} + \Delta_{\breve{T}}\left(I_{F^2(H_n)\otimes\mathcal{H}} - r\breve{R}\breve{T}^*\right)^{-1} r\breve{R}\Delta_{\breve{T}^*}\right]$ (34)

Define the operators

$A := \breve{T}^* B := \Delta_{\breve{T}^*}$, $C := \Delta_{\breve{T}^*}$, $D := -\breve{T}$, and $Z := r\breve{R}$, $0 < r < 1$, .

and notice that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \breve{T}^* & \Delta_{\breve{T}^*} \\ \Delta_{\breve{T}} & -\breve{T} \end{pmatrix}$$

is a unitary operator. Therefore,

$$AA^* + BB^* = I,\ CC^* + DD^* = I, \text{ and } AC^* + BD^* = 0. \qquad (35)$$

Define

$$(Z) := D + C(I - ZA)^{-1}ZB$$

and notice that using relation (35), we have

$I - (Z)(Z)^* = I - DD^* - C(I - ZA)^{-1}ZBD^* - DB^*Z^*(I - A^*Z^*)^{-1}C^*$
$- C(I - ZA)^{-1}ZB\, B^*Z^*(I - A^*Z^*)^{-1}C^*$

$= CC^* + C(I - ZA)^{-1}ZAC^* + CA^*Z^*(I - A^*Z^*)^{-1}C^*$
$- (I - ZA)^{-1}ZZ^*(I - A^*Z^*)^{-1}C^*$
$+ C(I - ZA)^{-1}ZAA^*Z^*(I - A^*Z^*)^{-1}C^*$

$= C(I - ZA)^{-1}\,[(I - ZA)(I - A^*Z^*) + ZA(I - A^*Z^*)$
$+(I - ZA)A^*Z^* - ZZ^* + ZAA^*Z^*](I - A^*Z^*)^{-1}C^*$

$= C(I - ZA)^{-1}\,(I - ZZ^*)(I - A^*Z^*)^{-1}C^*.$

Therefore,

$$I - (Z)(Z)^* = C(I - ZA)^{-1}\,(I - ZZ^*)(I - A^*Z^*)^{-1}C^{**}. \qquad (36)$$

Therefore, according to our notations, for any $r \in (0, 1)$, the defect operator

$$I - \Theta_T(rR_1, \ldots, rR_n)\,\Theta_T(rR_1, \ldots, rR_n)^*$$

is equal to the product

$$\Delta_{\breve{T}}\,(I - r\breve{R}\breve{T}^*)^{-1}(I - r^2\breve{R}\breve{R}^*)(I - r\breve{T}\breve{R}^*)^{-1}\Delta_{\breve{T}}$$

$= (I \otimes \Delta_T)(I - r\sum_{i=1}^{n} R_i \otimes T_i^*)^{-1}[(I - r^2\sum_{i=1}^{*} R_iR_i^*) \otimes I](I - r\sum_{i=1}^{n} R_i^* \otimes T_i)^{-1}(I \otimes \Delta_T)$

$=(\sum_{k=0}^{\infty}\sum_{|\gamma|=k} r^{|\gamma|} R_\gamma \Delta_T T_{\breve{\gamma}}^*)\left[[(I - r^2\sum_{i=1}^{n} R_iR_i^*) \otimes I]\right]\left(\sum_{p=0}^{\infty}\sum_{|\beta|=p} r^{|\beta|} R_\beta^* \otimes T_{\breve{B}}\Delta_T\right)$

$=\sum_{k,p=0}^{\infty}{}_{|\gamma|=k,}\sum_{|\beta|=p} r^{|\gamma|+|\beta|} R_\gamma\,(I - r^2\sum_{i=1}^{n} R_iR_i^*)R_\beta^* \otimes \Delta_T T_{\breve{\gamma}}^* T_{\tilde{B}}\Delta_T$

Now, for every $\alpha, \in \mathbb{F}_n^+$, $h \in \mathcal{D}_T$, $k \in \mathcal{D}_T$, we have

$([I - \Theta_T(rR_1, \ldots, rR_n)\,\Theta_T(rR_1, \ldots, rR_n)^*](e_\alpha \otimes h), e_\omega \otimes k)$

$=\sum_{\gamma\in\mathbb{F}_n^+,|\gamma|\le|\omega|}\sum_{|\beta|\le|\alpha|}\langle r^{|\gamma|+|\beta|}R_\gamma(I-r^2\sum_{i=1}^n R_iR_i^*)R_\beta^*e_\alpha,e_\omega\rangle\langle\Delta_T T_{\tilde\gamma}^*T_{\tilde B}\Delta_T h,k\rangle$

Using Lemma (1.1.23), we have

$$\text{SOT-}\lim_{r\to 1}\Theta_T(rR_1,\dots,rR_n)^* = \Theta_T(rR_1,\dots,rR_n)^*$$

Therefore, the above computations imply that

$\langle I-\Theta_T(R_1,\dots,R_n)\Theta_T(R_1,\dots,R_n)^*(e_\alpha\otimes h),e_\omega\otimes k\rangle$

$$=\sum_{\gamma\in\mathbb{F}_n^+,|\gamma|\le|\omega|}\sum_{|\beta|\le\alpha}\langle R_\gamma R_{\mathbb{C}}R_\beta^*e_\alpha,e_\omega\rangle\langle\Delta_T T_{\tilde\gamma}^*T_{\tilde B}\Delta_T h,k\rangle$$

$$=\sum_{\gamma\in\mathbb{F}_n^+,|\gamma|\le\omega}\langle R_\gamma(1),e_\omega\rangle\langle\Delta_T T_{\tilde\gamma}^*T_{\tilde B}\Delta_T h,k\rangle$$

$$=\langle\Delta_T T_\omega^*T_\alpha\Delta_T h,k\rangle$$

Here, we used the fact that $R_{\mathbb{C}}R_\beta^*e_\alpha\neq 0$ if and only if $\beta=\tilde\alpha$ (recall that $\tilde\alpha$ is the reverse of), and that R(1) = $\tilde\gamma$. On the other hand, using the definition of the Poisson kernel associated with a row contraction, we deduce that $K_T^*(e_\alpha\otimes h)=T_\alpha\Delta_T$
and

$$\langle K_TK_T^*(e_\alpha\otimes h),e_\omega\otimes k\rangle=\langle K_TT_\alpha\Delta_T h,e_\omega\otimes k\rangle$$

$$=\left(\sum_{\gamma\in\mathbb{F}_n^+}e_\gamma\otimes\Delta_T T_{\tilde\gamma}^*T_\gamma\Delta_T h,e_\omega\otimes k\right)$$

$$=(\Delta_T T_\omega^*T_\alpha\Delta_T h,e_\omega\otimes k)$$

for any h, k $\in\mathcal{D}_T$ and $\alpha,\omega\in\mathbb{F}_n^+$. Summing up the above computations, we deduce that
$I-\Theta_T(R_1,\dots,R_n)\Theta_T(R_1,\dots,R_n)^*=K_TK_T^*$,
which completes the proof.
We recall that the spectral radius of an n-tuple of operators X := $[X_1,\dots,X_n]$ is definedby
$r(X):=\lim_{k\to\infty}\|\sum_{|\alpha|}X_\alpha X_\alpha^*\|^{1/2k}$
A closer look at the proof of Theorem (1.1.24) reveals the following factorization result. We should add that the operator $I-\breve{X}\widetilde{T^*}$ is invertible because r(X) < 1.

Corollary (1.1.25) [1]: Let T := $[T_1,\dots,T_n]$, $T_i\in B(\mathcal{H})$, be a row contraction and let Θ_T be its characteristic function. If X := $[X_1,\dots,X_n]$, $X_i\in B(K)$, is a row contraction with spectral radius r(X) < 1, then $I_{k\otimes\mathcal{D}_T}-\Theta_T(X_1,\dots,X_n)\Theta_T(X_1,\dots,X_n)^*=\Delta_{\tilde T}\left(I-\hat X\widehat{T^*}\right)^{-1}\left(I-\hat X\widehat{X^*}\right)(I-\hat T\widehat{X^*})^{-1}\Delta_{\tilde T}$

where $\hat X:=[X_1\otimes I_{\mathcal{H}},\dots,X_n\otimes I_{\mathcal{H}}]$ and the other notations are from the proof of Theorem (1.1.24).
Let be the complex free semi group algebra generated by the free semi group $\mathbb{F}_n^+$with generators $g_1,\dots,g_n$ and neutral element g_0. Any n-tuple $T_1,\dots,T_n$of bounded operators on a Hilbert space $\mathcal{H}$ gives rise to a Hilbert (left) module over $\mathbb{C}\mathbb{F}_n^+$in the natural way

$$f\cdot h:=f(T_1,\dots,T_n)h,\ f\in\mathbb{F}_n^+,\ h\in\mathcal{H}.$$

We say that $\mathcal{H}$ is a contractive $\mathbb{C}\mathbb{F}_n^+$-module if T := $[T_1,\dots,T_n]$ is a row contraction, which is equivalent to

$$\|g_1h_1+\cdots+g_nh_n\|^2\le\|h_1\|^2+\cdots+\|h_n\|^2,\ h_1,\dots,h_n\in\mathcal{H}.$$

We say that $\mathcal{H}$ is of finite rank if rank ($\mathcal{H}$) := rank $\Delta_T<\infty$. The curvature invariant and Euler characteristic associated with an arbitrary row contraction T (or the Hilbert module $\mathcal{H}$ associated with T) were introduced and studied in [32] and [15]. We recall that

$$\text{curv}(T) = \lim_{m\to\infty} \frac{trace[I-\Phi_T^m(I)]}{1+n+\cdots+n^{m-1}}$$

and

$$(T)= \lim_{m\to\infty} \frac{rank[I-\Phi_T^m(I)]}{1+n+\cdots+n^{m-1}},$$

where Φ_T is the completely positive map associated with T, i.e.,

$$\Phi_T(X) := \sum_{i=1}^{n} T_i X T_i^*$$

Using Theorem (1.1.24) and some results from [32], we can show that the curvature and the Euler characteristic of a row contraction T can be expressed only in terms of the standard characteristic function Θ_T.

Theorem (1.1.26) [1]: Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a row contraction with rank $\Delta_T < \infty$, and let curv (T) and χ(T) denote its curvature and Euler characteristic, respectively. Then

$$\text{curv}(T) = \text{rank}\Delta_T - \lim_{m\to\infty} \frac{trace[\Theta_T\Theta_T^*(P_m \otimes I)]}{n^m}$$

and

$$(T) = \lim_{m\to\infty} \frac{rank[I-\Theta_T\Theta_T^*(P_{\le m} \otimes I)]}{1+n+\cdots+n^{m-1}}$$

where P_m (resp. $P_{\le m}$) is the orthogonal projection of the full Fock space $F^2(H_n)$onto the subspace of all homogeneous polynomials of degree m (resp. polynomials of degree ≤ m).

Proof : According to Theorem (2.3) and Corollary 2.7 from [32], we have

$$\text{curv}(T) = \lim_{m\to\infty} \frac{trace[K_T K_T^*(P_m \otimes I)]}{n^m}.$$

Using the factorization result of Theorem (1.1.24), the first result follows.

Now, according to Theorem (4.1)of [32], we have

$$(T) = \lim_{m\to\infty} \frac{rank[K_T^*(P_{\le m} \otimes I)K_T]}{1+n+\cdots+n^{m-1}} \qquad (37)$$

Since $K_T^*(P_{\le m} \otimes I)$has finite rank, we have rank $[(K_T^*(P_{\le m} \otimes I)K_T]$ = rank $[K_T^*(P_{\le m} \otimes I)]$.

On the other hand, since K_T is one-to-one on the range of $K_T^*(P_{\le m} \otimes I)$,we also have rank $[K_T^*(P_{\le m} \otimes I)]$ = rank $[K_T K_T^*(P_{\le m} \otimes I)]$

Hence, using relation (37) and Theorem (1.1.24), we complete the proof. a constrained characteristic function is associated with any constrained row contraction. For pure constrained row contractions, we show that this characteristic function is a complete unitary invariant and provide a model in terms of it. We also show that Arveson's curvature invariant and Euler characteristic asssociated with a Hilbert module over $C[z_1, \ldots, z_n]$ generated by a commuting row contraction T can be expressed only in terms of the constrained characteristic function of T.

Let J be a WOT-closed two-sided ideal of the non commutative analytic Toeplitz algebra F_n^∞ generated by a family of polynomials P_J . We define the constrained characteristic function associated with a J-constrained row contraction $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, to be the multi-analytic operator (with respect to the constrained shifts $B_1, \ldots, B_n$)

$$\Theta_{J,T}(W_1, \ldots, W_n) : \mathcal{N}_J \otimes \mathcal{D}_{T^*} \to \mathcal{N}_J \otimes \mathcal{D}_T$$

defined by the formal Fourier representation

$$-I_{\mathcal{N}_J} \otimes T + (I_{\mathcal{N}_J} \otimes \Delta_T)(I_{\mathcal{N}_J \otimes \mathcal{H}} - \textstyle\sum_{i=1}^{n} W_i T_i^*)^{-1} [W_1 \otimes I_{\mathcal{H}}, \ldots, W_n \otimes I_{\mathcal{H}}]\,(I_{\mathcal{N}_J} \otimes \Delta_{T^*})$$

Taking into account that $\mathcal{N}_J$ is a co-invariant subspace under $R_1, \ldots, R_n$, we can see that $\Theta_{J,T}$ is the maximal J-constrained piece of the standard characteristic function Θ_T of the row contraction T. More precisely, we have

$\Theta_T(R_1,\ldots,R_n)^*(\mathcal{N}_J \otimes \mathcal{D}_T) \subseteq \mathcal{N}_J \otimes \mathcal{D}_{T^*}$ and

$$P_{\mathcal{N}_J \otimes \mathcal{D}_T} \Theta_T (R_1, \ldots, R_n)| \mathcal{N}_J \otimes \mathcal{D}_{T^*} = \Theta_{J,T} (W_1, \ldots, W_n) \quad (38)$$

We remark that the above definition of the constrained characteristic function makes sense (and has the same properties) when J is an arbitrary WOT-closed two-sided ideal o f F_n^∞ and $T := [T_1, \ldots, T_n]$ is an arbitrary c.n.c. J-constrained row contraction.

Theorem(1.1.27[1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of F_n^∞ generated by a family of polynomials P_J. Let $T := [T_1, \ldots, T_n]]$, $T_i \in B(H)$, be a J-constrained row contraction.

Then

$$I_{\mathcal{N}_J \otimes \mathcal{D}_T} - \Theta_{J,T}\Theta_{J,T}^* = K_{J,T}K_{J,T}^* \quad (39)$$

where $\Theta_{J,T}$ is the constrained characteristic function of T and $K_{J,T}$ is the corresponding constrained Poisson kernel.

Proof: The constrained Poisson kernel associated with T is $K_{J,T}: \mathcal{H} \to \mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}}$ defined by

$$K_{J,T} := (P_{\mathcal{N}_J} \otimes I_{\overline{\Delta_T \mathcal{H}}})\, K_T \ , \quad (40)$$

where K_T is the standard Poisson kernel of T. According to the proof of Theorem (1.1.12) range $K_T \subseteq \mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}}$. Using Theorem (1.1.24) and taking the compression of relation (33) to the subspace $\mathcal{N}_J \otimes \mathcal{D}_T \subset F^2(H_n) \otimes \mathcal{D}_T$, we obtain

$I_{\mathcal{N}_J \otimes \mathcal{D}_T} - P_{\mathcal{N}_J \otimes \mathcal{D}_T}\ \Theta_T (R_1,\ldots,R_n)\, \Theta_T(R_1,\ldots,R_n)^*(\mathcal{N}_J \otimes \mathcal{D}_T) \mid \mathcal{N}_J \otimes \mathcal{D}_T = P_{\mathcal{N}_J \otimes \mathcal{D}_T}\ K_T K_T^* \mid \mathcal{N}_J \otimes \mathcal{D}_T$.

Taking into account relations (38), (40), and that $W_i^* = R_i^* |\mathcal{N}_J$, i = 1, . . . , n, we infer that

$$I_{\mathcal{N}_J \otimes \mathcal{D}_T} - \Theta_{J,T} (W_1, \ldots, W_n)\ \Theta_{J,T}(W_1,\ldots,W_n)^* = K_{J,T}K_{J,T}^* .$$

As in the proof of Theorem(1.1.27), one can use Corollary (1.1.25) to obtain the following constrained version of it.

Corollary (1.1.28) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of F_n^∞ generated by a family of polynomials P_J. Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a J-constrained row contraction. If $X := [X_1, \ldots, X_n]$, $X_i \in B(K)$, is a J-constrained row contraction with spectral radius $r(X) < 1$, then

$$I_{k \otimes \mathcal{D}_T} - \Theta_{J,T} (X_1, \ldots, X_n)\ \Theta_{J,T}(X_1,\ldots,X_n)^* = \Delta_{\tilde{T}}\left(I - \hat{X}\,\widehat{T^*}\right)^{-1}\left(I - \hat{X}\,\widehat{X^*}\right)\left(I - \hat{T}\,\widehat{X^*}\right)^{-1}\Delta_{\tilde{T}}$$

where $\hat{X} := [X_1 \otimes I_{\mathcal{H}},\ldots,X_n \otimes I_{\mathcal{H}}]$ and the other notations are from the proof of Theorem (1.1.24).

Now we present a model for pure constrained row contractions in terms of characteristic functions.

Theorem (1.1.29) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of F_n^∞ and $T := [T_1, \ldots, T_n]$ be a pure J-constrained row contraction. Then the constrained characteristic function $\Theta_{J,T} \in W(W_1, \ldots, W_n) \overline{\otimes} B(\mathcal{D}_{T^*}, \mathcal{D}_T)$ is a partial isometry and T is unitarily equivalent to the row contraction

$$P_{\mathbb{H}_{J,T}}(B_1 \otimes I_{\mathcal{D}_T})| \mathbb{H}_{J,T} , \ldots , P_{\mathbb{H}_{J,T}}(B_n \otimes I_{\mathcal{D}_T})| \mathbb{H}_{J,T} \quad (41),$$

where $P_{\mathbb{H}_{J,T}}$ is the orthogonal projection of $\mathcal{N}_J \otimes \mathcal{D}_T$ on the Hilbert space

$\mathbb{H}_{J,T} := (\mathcal{N}_J \otimes \mathcal{D}_T) \ominus \Theta_{J,T}(\mathcal{N}_J \otimes \mathcal{D}_{T^*})$.

Proof : According to Theorem (1.1.21), the constrained Poisson kernel $K_{J,T}: \mathcal{H} \to \mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}}$ is an isometry, $K_{J,T}\mathcal{H}$ is a co-invariant subspace under $B_i \otimes I_{\overline{\Delta_T \mathcal{H}}}$, i = 1, . . . , n, and

$$T_i = K_{J,T}^*(B_i \otimes I_{\overline{\Delta_T \mathcal{H}}}) K_{J,T},\ i = 1, \dots, n. \tag{42}$$

Consequently, $K_{J,T}K_{J,T}^*$ is the orthogonal projection of $\mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}}$ onto $K_{J,T}\mathcal{H}$. According to Theorem(1.1.27), relation (39) shows that $K_{J,T}K_{J,T}^*$ and $\Theta_{J,T}\Theta_{J,T}^*$, are mutually orthogonal projections such that

$$K_{J,T}K_{J,T}^* + \Theta_{J,T}\Theta_{J,T}^* = I_{\mathcal{N}_J \otimes \overline{\Delta_T \mathcal{H}}} .$$

Therefore,

$$K_{J,T}\mathcal{H} = (\mathcal{N}_J \otimes \mathcal{D}_T) \ominus \Theta_{J,T}(\mathcal{N}_J \otimes \mathcal{D}_{T^*}) ,$$

Now, since is $K_{J,T}$ an isometry, we identify $\mathcal{H}$ with $\mathbb{H}_{J,T} := K_{J,T}\mathcal{H}$ and, using (42), we deduce that T is unitarily equivalent to the row contraction given by (41). This completes the proof. _

Let $\Phi \in W(W_1, \dots, W_n)\overline{\otimes}B(\mathcal{K}_1, \mathcal{K}_2)$ and $\Phi' \in W(W_1, \dots, W_n)\overline{\otimes}B(\mathcal{K}_1', \mathcal{K}_2')$ be two multianalytic operators with respect to $B_1, \dots, B_n$. We say that Φ and Φ'oincide if there are two unitary multi-analytic operators $U_j : \mathcal{N}_J \otimes \mathcal{K}_j \to \mathcal{N}_J \otimes \mathcal{K}_j'$ such that the diagram

$$\mathcal{N}_J \otimes \mathcal{K}_j \to \mathcal{N}_J \otimes \mathcal{K}_j$$

$$\mathcal{N}_J \otimes \mathcal{K}_j' \to \mathcal{N}_J \otimes \mathcal{K}_j'$$

is commutative, i.e., $\Phi' U_1 = \Phi U_2$. Since

$$U_j(B_i \otimes I_{\mathcal{K}_1}) = (B_i \otimes I_{\mathcal{K}_1'})U_j,\ i = 1, \dots, n,$$

And U_j are unitary operators, we also deduce that

$$U_j(B_i^* \otimes I_{\mathcal{K}_1}) = (B_i^* \otimes I_{\mathcal{K}_1'})U_j,\ i = 1, \dots, n.$$

Taking into account that $C^*(B_1, \dots, B_n)$ is irreducible (see Theorem (1.1.3)), we conclude that

$$U_j = I_{\mathcal{N}_J} \otimes \mathcal{T}_j ,\ j = 1, 2,$$

for some unitary operators $\mathcal{T}_j \in B(\mathcal{K}_j , \mathcal{K}_j')$.

The next result shows that the constrained characteristic function is a complete unitary invariant for pure constrained row contractions.

Theorem (1.1.30) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of F_n^∞ And Let $T := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, and $T' := [T_1', \dots, T_n']$, $T_i' \in B(\mathcal{H}')$, be two J-constrained pure row contractions .Then T and T′ are unitarily equivalent if and only if their constrained characteristic functions $\Theta_{J,T}$ and $\Theta_{J,T'}$ coincide.

Proof : Assume that T and T′ are unitarily equivalent and let U : H → H′ be a unitary operator such that $T_i = U^* T_i' U$ for any i = 1, . . . , n. Simple computations reveal that

$$U\Delta_T = \Delta_{T'} U \text{ and } (\oplus_{i=1}^n U)\Delta_{T^*} = \Delta_{T'^*} (\oplus_{i=1}^n U)$$

Define the unitary operators $\mathcal{T}$ and $\mathcal{T}'$ by setting

$$:= U|\mathcal{D}_T : \mathcal{D}_T \to \mathcal{D}_{T'} \text{ and } \mathcal{T}' := (\oplus_{i=1}^n U)\mathcal{D}_{T^*} : \mathcal{D}_{T^*} \to \mathcal{D}_{T'^*} .$$

Taking into account the definition of the constrained characteristic function, it is easy to see that

$$(I_{\mathcal{N}_J} \otimes \mathcal{T})\, \Theta_{J,T} = \Theta_{J,T'}\, (I_{\mathcal{N}_J} \otimes \mathcal{T}').$$

Conversely, assume that the characteristic functions of T and T′ coincide. According to the remarks preceding the theorem, there exist unitary operators $\mathcal{T}: \mathcal{D}_T \to \mathcal{D}_{T'}$ and $\mathcal{T}_*: \mathcal{D}_{T^*} \to \mathcal{D}_{T'^*}$ such that the following diagram
′
is commutative, i.e.,

$$(I_{\mathcal{N}_J} \otimes \mathcal{T})\, \Theta_{J,T} = \Theta_{J,T'}\, (I_{\mathcal{N}_J} \otimes \mathcal{T}_*) \qquad (43)$$

Hence, we deduce that

$$\mathcal{T} := (I_{\mathcal{N}_J} \otimes \mathcal{T})|\, \mathbb{H}_{J,T}: \mathbb{H}_{J,T} \to \mathbb{H}_{J,T'}$$

is a unitary operator, where $\mathbb{H}_{J,T}$and $\mathbb{H}_{J,T'}$ are the model spaces for T and T′, respectively(see Theorem (1.1.29)). Since

$$(B_i^* \otimes I_{\mathcal{D}_T})(I_{\mathcal{N}_J} \otimes \mathcal{T}^*) = (I_{\mathcal{N}_J} \otimes \mathcal{T}^*)(B_i^* \otimes I_{\mathcal{D}_{T'}}),\ i = 1, \ldots, n,$$

and $\mathbb{H}_{J,T}$ (resp. $\mathbb{H}_{J,T'}$ is a co-invariant subspace under $B_i \otimes I_{\mathcal{D}_T}$ (resp. $B_i \otimes I_{\mathcal{D}_{T'}}$), i = 1, . . . , n, we deduce that

$$[(B_i^* \otimes I_{\mathcal{D}_T})|\, \mathbb{H}_{J,T}]\, \Gamma^* = \Gamma^* (B_i^* \otimes I_{\mathcal{D}_{T'}})|\mathbb{H}_{J,T'},\ i = 1, \ldots, n.$$

Hence, we obtain
$[P_{\mathbb{H}_{J,T}} (B_i \otimes I_{\mathcal{D}_T})\, |\mathbb{H}_{J,T} = [P_{\mathbb{H}_{J,T'}} (B_i \otimes I_{\mathcal{D}_{T'}})|\mathbb{H}_{J,T'}]$, i = 1, . . . , n.
Now, using Theorem (1.1.29), we conclude that T and T′ are unitarily equivalent. The proof is complete. _

Theorem (1.1.31) [1]: Let J be a WOT-closed two-sided ideal of F_n^∞ such that $1 \in \mathcal{N}_J$ and condition(25) is satisfied. If $\boldsymbol{\mathcal{M}} \subseteq \mathcal{N}_J$ is an invariant subspace under $B_1, \ldots, B_n$, and $T := [T_1, \ldots, T_n]$, $T_i := P_{\mathcal{M}^\perp} B_i|\mathcal{M}^\perp$, i = 1, . . . , n,
then

$$\boldsymbol{\mathcal{M}} = \Theta_{J,T}(\mathcal{N}_J \otimes \mathcal{D}_{T^*}),$$

where $\Theta_{J,T}$is the constrained characteristic function of T.

Proof: According to Corollary (1.1.22), T is a pure J-constrained row contraction with rank Δ_T =1. Therefore, we can identify the subspace $\mathcal{D}_T$ with $\mathbb{C}$. Hence, and due to Theorem (1.1.29), we have
$\mathbb{H}_{J,T} = \mathcal{N}_J \ominus \Theta_{J,T}(\mathcal{N}_J \otimes \mathcal{D}_{T^*})$
and T is unitarily equivalent to

$$[P_{\mathbb{H}_{J,T}} B_1|\mathbb{H}_{J,T}, \ldots, P_{\mathbb{H}_{J,T}} B_n|\mathbb{H}_{J,T}].$$

Using again Corollary (1.1.22), we deduce that $\mathbb{H}_{J,T} = \mathcal{M}^\perp$ and therefore $\boldsymbol{\mathcal{M}} = \Theta_{J,T}(\mathcal{N}_J \otimes \mathcal{D}_{T^*})$,
This completes the proof. _

Theorem (1.1.32) [1]: Let $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a commutative row contraction with rank $\Delta_T < \infty$, and let K(T) and χ(T) denote Arveson's curvature and Euler characteristic, respectively. Then

$$K(T) = \int_{\partial \mathbb{B}_n} \lim_{r \to 1} trace[I - \Theta_{J_c,T}(r\xi)\Theta_{J_c,T}(r\xi)^*]\, d\sigma(\xi)$$

$$= \text{rank}\, \Delta_T - (n-1)!\, \lim_{m \to \infty} \frac{trace[\Theta_{J_c,T}\Theta_{J_c,T}^*(Q_m \otimes I_{\mathcal{D}_T})}{n^m}$$

where Q_m is the projection of H^2 onto the subspace of homogeneous polynomials of degree m, and

$$(T) = n!\, \lim_{m \to \infty} \frac{rank[(I - \Theta_{J_c,T}\Theta_{J_c,T}^*(Q_{\le m} \otimes I_{\mathcal{D}_T})}{m^n},$$

where $Q_{\leq m}$is the projection of H^2 onto the subspace of all polynomials of degree ≤ m.
Proof: Using the factorization result of Corollary (1.1.28) in our particular case, we obtain $I - \Theta_{J_c,T}(z)\Theta_{J_c,T}(z)^* = (1-|z|^2)\Delta_T(I - z_1T_1^* - \ldots - z_nT_n^*)^{-1}(I - \bar{z}_1T_1 - \cdots - \bar{z}_nT_n)^{-1}\Delta_T$ for any z $\in \mathbb{B}_n$.
The first formula follows from the definition of the curvature [5] and the above-mentioned factorization for the constrained characteristic function of T. and Corollary (2.8): from [32], we have

$$K(T) = (n-1)! \lim_{m\to\infty} \frac{trace[(P_m \otimes I)K_T K_T^*(P_m \otimes I)]}{m^{n-1}}, m\to\infty$$

where K_T is the Poisson kernel of T and P_m is the orthogonal projection of $F^2(H_n)$ onto the subspace of all homogeneous polynomials of degree m. Since T is a commutative row contraction, i.e., J_c-constrained, we have range $K_T \subset F_s^2 \otimes \mathcal{D}_T$ and the constrained Poisson kernel satisfies the equation $K_{J_c,T} = (P_{F_s^2} \otimes I)\, K_T$, where $F_s^2$is the symmetric Fock space.
Using the standard properties for the trace and the above relation, we deduce that

$$K(T) = (n-1)! \lim_{m\to\infty} \frac{trace[K_{J_c,T}K_{J_c,T}^*(Q_m \otimes I)]}{m^{n-1}} \qquad (44)$$

where $Q_m := P_{F_s^2}P_m|\, F_s^2$ is the projection of F_s^2onto the subspace of homogeneous polynomials of degree m. According to Theorem (1.1.27), we have

$$I - \Theta_{J_c,T}\Theta_{J_c,T}^* = K_{J_c,T}K_{J_c,T}^* \qquad (45)$$

.
Taking onto account relations (44) and (45), we deduce the second formula for the curvarure. Here, of course, we used Arveson's identification of the symmetric Fock space F_s^2 with his space H^2.
Arveson [5] showed that his Euler characteristic satisfies the equation
$(T) = n! \lim_{m\to\infty} \frac{rank[I - \Phi_T^m(1)]}{m^n}$
where Φ_T is the completely positive map associated with T. in [32] that

$$(T) = n! \lim_{m\to\infty} \frac{rank[K_T^*(P_{\leq m} \otimes I)K_T]}{m^n} \qquad (46)$$

where $P_{\leq m}$is the orthogonal projection of $F^2(H_n)$ on the subspace of all polynomials of degree ≤ m.
Using again that range $K_T \subset F_s^2 \otimes \mathcal{D}_T$ and the constrained Poisson kernel satisfies the equation $K_{J_c,T} = (P_{F_s^2} \otimes I)\, K_T$, we deduce that rank $[K_T^*(P_{\leq m} \otimes I)K_T]$ = rank $[K_T^*(P_{F_s^2} \otimes I)(P_{\leq m} \otimes I)(P_{F_s^2} \otimes I)K_T]$

$= rank[K_{J_c,T}^*(Q_{\leq m} \otimes I)K_{J_c,T}]$
$= rank[K_{J_c,T}^*(Q_{\leq m} \otimes I)]$
$=rank[K_{J_c,Y}K_{J_c,T}^*(Q_{\leq m} \otimes I)]$

where $Q_{\leq m}$ is the projection of F_s^2 onto the subspace of all polynomials of degree ≤ m. The last two equalities hold since the operator$K_{J_c,T}^*(Q_{\leq m} \otimes I)$
has finite rank and $K_{J_c,T}$is oneto-one on the range of. $K_{J_c,T}^*(Q_{\leq m} \otimes I)$Now, using relation (46), the above equalities, and the factorization (45), we obtain the last formula of the theorem. The proof is complete. _
we provide a Sarason [36] type commutant lifting theorem for pure constrained
row contractions and obtain a Nevanlinna-Pick [18] interpolation result in our setting. Let $[T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, be a pure row contraction, and let J be a WOT-closedtwo-sided ideal of F_n^∞such that

$$(T_1, \ldots, T_n) = 0 \text{ for any } (S_1, \ldots, S_n) \in J, \qquad (47)$$

where $\varphi(T_1. . . , T_n)$ is defined using the F_n^∞ -functional calculus for row contractions. any pure constrained row contraction is unitarily equivalent to the compression of $[B_1 \otimes I_k, . . . ,B_n \otimes I_k]$ to a co-invariant subspace ε under each operator$B_i \otimes I_k$, i = 1, . . . , n. Therefore, we have
$T_i = P_\varepsilon(B_i \otimes I_k)|\boldsymbol{\varepsilon}$, i = 1, . . . , n.
The following result is a commutant lifting theorem for pure constrained row contractions.
Theorem(1.1.33) [1]: Let $J \neq F_n^\infty$be a WOT-closed two-sided ideal of the noncommutative analytic Toeplitz algebra F_n^∞ and let $[B_1, . . . ,B_n]$ and $[W_1, . . . ,W_n]$ be the corresponding constrained shifts acting on $\mathcal{N}_J$. For each j = 1, 2, let K_j be a Hilbert space and $\varepsilon_j \subseteq \mathcal{N}_J \otimes K_j$ be a co-invariant subspace under each operator $B_i\otimes I_{k_j}$, i = 1, . . . , n. If $X : \varepsilon_1 \to \varepsilon_2$ is a bounded operator such that

$$X[P_{\varepsilon_1} (B_i\otimes I_{k_1})| \varepsilon_1] = [P_{\varepsilon_2} (B_i\otimes I_{k_2})]| \varepsilon_2 X, \ i = 1, . . . , n, \quad (48)$$

then there exists
$G(W_1, . . . ,W_n) \in W(W_1, . . . ,W_n)\overline{\otimes}B(k_1,k_2)$
such that $G(W_1,\dots,W_n)^*\varepsilon_2 \subseteq \varepsilon_1$,
$G(W_1,\dots,W_n)^*| \varepsilon_2 = X^*$, and $\|G(W_1, . . . ,W_n)\| = \|X\|$.
In particular, if ε_j :=G $\otimes K_j$, where G is a co-invariant subspace under each operator B_i and W_i, i = 1, . . . , n, then the above implication becomes an equivalence.
Proof: According to Lemma (1.1.1), the subspace $\mathcal{N}_J \otimes K_j$ is invariant under each operator $S_i^* \otimes I_{K_j}$, i = 1, . . . , n, and
$(S_i^* \otimes I_{K_j})| \mathcal{N}_J \otimes K_j = B_i^* \otimes I_{K_j}$, i = 1, . . . , n.*9
Since $\varepsilon_j \subseteq \mathcal{N}_J \otimes K_j$is invariant under $B_i^* \otimes I_{K_j}$it is also invariant under $S_i^* \otimes I_{K_j}$ and
$(S_i^* \otimes I_{K_j})|\varepsilon_j = (B_i^* \otimes I_{K_j})|\varepsilon_j$, i = 1, . . . , n.
Hence, relation (48) implies

$$XP_{\varepsilon_1} (S_i \otimes I_{K_1})|\varepsilon_1 = P_{\varepsilon_2} (S_i \otimes I_{K_2})|\varepsilon_2 X, \ i = 1, . . . , n. \quad (49)$$

For each j = 1, 2, the n-tuple $[S_1 \otimes I_{K_j} , . . . , S_n \otimes I_{K_j}]$is an isometric dilation of the rowcontraction $[P_{\varepsilon_j} (S_1 \otimes I_{K_j})|\varepsilon_j \ , . . . , P_{\varepsilon_j} (S_n \otimes I_{K_j})|\varepsilon_j]$
Applying the noncommutative commutant lifting theorem ([21], [24]), we find a multi-analytic operator $\Phi(R_1, . . . ,R_n) \in R_n^\infty \overline{\otimes}B(K_1,K_2)$ such that $\Phi(R_1,\dots,R_n)^*\varepsilon_2 \subseteq \varepsilon_1$,

$$\Phi(R_1,\dots,R_n)^*|\varepsilon_2 = X^* \text{and } \|(R_1,\dots,R_n)\| = \| X \|. \quad (50)$$

Let $G(W_1,\dots,W_n) := P_{\mathcal{N}_J\otimes K_2}\Phi(R_1,\dots,R_n)| \mathcal{N}_J \otimes K_1$. According to the remarks preceding Theorem (1.1.2), we have
$G(W_1,\dots,W_n) \in [P_{\mathcal{N}_J}R_n^\infty |\mathcal{N}_J] \overline{\otimes}B(K_1,K_2) = W(W_1,\dots,W_n) \overline{\otimes}B(K_1,K_2)$.
Since $\Phi(R_1,\dots,R_n)^* (\mathcal{N}_J \otimes K_2) \subseteq \mathcal{N}_J \otimes K_1$and $\varepsilon_j \subseteq \mathcal{N}_J \otimes K_2$, relation(50) implies
$G(W_1,\dots,W_n)^*\varepsilon_2 \subseteq\varepsilon_1$and $G(W_1,\dots,W_n)^*|\varepsilon_2 =X^*$.
Hence, and using again (50), we have
$\|X\| \le \|G(W_1,\dots,W_n)\| \le \|(R_1,\dots,R_n)\| = \|X\|$.
Therefore, $\|G(W_1,\dots,W_n)\| = \|X\|$.
Now, let us prove the last part of the theorem. The implication " =⇒ " is clear from the first part of the theorem. For the converse, let $X = P_{G\otimes K_2}\Psi(W_1,\dots,W_n)|G \otimes K1$, where $\Psi(W_1,\dots,W_n) \in W(W_1,\dots,W_n) \overline{\otimes}B(K_1,K_2)$ Since $B_iW_j = W_jB_i$ for i, j = 1, . . . , n, we have
$(B_i^* \otimes I_{K_1})\Psi(W_1,\dots,W_n)^* = \Psi(W_1,\dots,W_n)^*(B_i^* \otimes I_{K_2})$, i = 1, . . . , n.
Now, taking into account that G is an invariant subspace under each of the operators B_i^* and W_i^* , i = 1, . . . , n, we deduce (48). The proof is complete.
Corollary(1.1.34) [1]: Let $J \neq F_n^\infty$ be a WOT-closed two-sided ideal of the noncommutative analytic Toeplitz algebra F_n^∞ and let $B_1,\dots,B_n$and $W_1,\dots,W_n$be the corresponding constrained shifts acting on $\mathcal{N}_J$. If K is a Hilbert space and $G \subseteq \mathcal{N}_J$is an invariant subspace under each operator $B_i^*$and W_i^* , i = 1, . . . , n, then $\{[P_G W(B_1,\dots,B_n)|G] \otimes I_K\}' = [P_G W(W_1,\dots,W_n)|G]\overline{\otimes}\, B(K)$.
We remark can be extended to the following more general setting. The proof follows exactly the same lines so we shall omit it. For each j = 1, 2, let be a WOT-closed two-sided ideal of $F_n^\infty$and let $[B_1^{(j)}$, .

. . , $B_n^{(j)}$]be the corresponding constrained shift acting on $\mathcal{N}_{J_j}$. Let $\varepsilon_j \subseteq \mathcal{N}_{J_j} \otimes K_j$ be an invariant subspace under each operator $B_1^{(j)^*} \otimes I_{K_j}$, i = 1, . . . , n, where K_j is a Hilbert space. If $X : \varepsilon_1 \to \varepsilon_2$ is a bounded operator such that
$XP_{\varepsilon_1}(B_i^{(1)} \otimes I_{K_1})|\varepsilon_1 = P_{\varepsilon_2}(B_i^{(2)} \otimes I_{K_2})|\varepsilon_2 X$, i = 1, . . . , n, then there exists $G \in [P_{\mathcal{N}_{J_2}} R_n^\infty |\mathcal{N}_{J_1}] \overline{\otimes} B(K_1,K_2)$ such that
$P_{\varepsilon_2} G|\varepsilon_1 = X$ and $\|G\| = \|X\|$.
Now we can obtain the following Nevanlinna-Pick interpolation result in our setting. We only sketch the proof which is similar to that of Theorem 2.4 from [15] but uses Theorem (1.1.33), and point out what is new.

Theorem (1.1.35) [1]: Let J be a WOT-closed two-sided ideal of F_n^∞ and let $B_1, . . . ,B_n$ be the corresponding constrained shifts acting on $\mathcal{N}_J$. Let $\lambda_1, . . . , \lambda_k$ be k distinct points in the zero set
$Z_J := \{\lambda \in \mathbb{B}_n : f(\lambda) = 0 \text{ for any } f \in J\}$,
and let $A_1, . . . ,A_K \in B(K)$. Then there exists $(B_1, . . . ,B_n) \in W(B_1, . . . ,B_n)^{-}\otimes B(K)$ such that

$\|(B_1, . . . ,B_n)\| \le 1$ and $\Phi(\lambda_j) = A_j$, j = 1, . . . , k,
if and only if the operator matrix

$$\left[\frac{I_k - A_i A_j^*}{1-\langle \lambda_i, \lambda_j \rangle}\right]_{k\times k} \tag{51}$$

is positive semi definite.

Proof: Let $\lambda_j := (\lambda_{j1}, . . . , \lambda_{jn}) \in \mathbb{C}^n$, j = 1, . . . , k, and denote $\lambda_{j\alpha} := \lambda_{ji_1}\lambda_{ji_2} . . . \lambda_{ji_m}$ if $\alpha = g_{ji_1}g_{ji_2} . . . g_{ji_m} \in \mathbb{f}_n^\infty$, and $\lambda_{jg0} := 1$. Define
$z_{\lambda_j} := \sum_{\alpha \in \mathbb{f}_n^+} \overline{\lambda_{j\alpha}} e_\alpha$, j = 1, 2, . . . , k.

Notice that, for any $f \in J$, $\lambda \in Z_J$, and $\alpha, \beta \in \mathbb{f}_n^+$, we have

$\langle [S_\alpha f(S_1, ..., S_n) S_\beta](1), z_\lambda \rangle = \lambda_\alpha f(\lambda)\lambda_\beta = 0$,

which implies $z_\lambda \in \mathcal{N}_J$ for any $\lambda \in Z_J$. Note also that, since $B_i^* = S_i^* |\mathcal{N}_J$ for i = 1, . . . , n, we have
$B_i^* z_{\lambda_j} = \overline{\lambda_{ji}}\, z_{\lambda_j}$ for i = 1, . . . , n, and j = 1, . . . , k.
Define the subspace

$$M := \text{span}\{z_{\lambda_j} : j = 1, . . . , k\}$$

and the operators $X_i \in B(M \otimes K)$ by setting $X_i = B_m B_i|M \otimes I_k$, i = 1, . . . , n. Since $z_{\lambda_1}, . . . , z_{\lambda_k}$ are linearly independent, we can define an operator $T \in B(M \otimes K)$ by setting

$$T^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes A_j^* h$$

for any $h \in K$ and j = 1, . . . , k. Notice that $TX_i = X_i T$ for i = 1, . . . , n.
Since $\mathcal{M}$ is a co-invariant subspace under each operator B_i, i = 1, . . . , n, we can apply Theorem (1.1.33) and find $\Phi(W_1, . . . ,W_n) \in W(W_1, . . . ,W_n)\overline{\otimes} B(K)$ such that

$$\Phi(W_1, ..., W_n)^* \mathcal{M} \subset \mathcal{M},\ \Phi(W_1, ..., W_n)^* |\mathcal{M} = T^* \tag{52}$$

and $\|(W_1, . . . ,W_n)\| = \|T\|$. As in [15], one can prove that $(\lambda_j) = A_j$, j = 1, . . . , k, if and only if (52) holds. Moreover, $\|(W_1, . . . ,W_n)\| \le 1$ if and only if $TT^* \le I_{\mathcal{M}}$, which is equivalent to the fact that the operator matrix (51) is positive semi definite. This completes the proof. _
We should remark that in the commutative case when $J = J_c$ (see Example (1.1.18) part (ii)), we recover the result obtained in [12], [29], and [12].

Section (1.2): Joint Similarity

For $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. Two operators $A, B \in B(\mathcal{H})$ are called similar if there is an invertible operator $S \in B(\mathcal{H})$ such that $A = S^{-1}BS$. The problem of characterizing the operators similar to contractions, i.e., the operators in the unit ball

$$[B(\mathcal{H})]_1 := \{X \in B(\mathcal{H}) : XX^* \leq I\},$$

or similar to special contractions such as parts of shifts, isometries, unitaries, etc., has been considered by many and has generated deep results in operator theory and operator algebras. We shall mention some of the classical results on similarity that strongly influenced us in writing.

In 1947, Sz.-Nagy [50] found necessary and sufficient conditions for an operator to be similar to a unitary operator. In particular, an operator T is similar to an isometry if and only if there are constants $a, b > 0$ such that

$$a\|h\| \leq \|T^n h\| \leq b\|h\|, \qquad h \in \mathcal{H}, \ n \in \mathbb{N}.$$

The fact that the unilateral shift on the Hardy space $H^2(\mathbb{T})$ plays the role of universal model in $B(\mathcal{H})$ was discovered by Rota [49]. Rota's model theorem asserts that any operator with spectral radius less than one is similar to a contraction, or more precisely, to a part of a backward unilateral shift. This result was refined furthermore by Foia¸s [49] and by de Branges and Rovnyak [46], who proved that every strongly stable contraction is unitarily equivalent to a part of a backward unilateral shift.

It is well-known that if $T \in B(\mathcal{H})$ is similar to a contraction then, due to the von Neumann inequality [43], it is polynomially bounded, i.e., there is a constant $C > 0$ such that, for any polynomial p,

$$\|p(T)\| \leq C\|p\|_\infty,$$

where $\|p\|_\infty := \sup_{|z|=1} |p(z)|$. A remarkable result obtained by Paulsen [44] shows that similarity to a contraction is equivalent to complete polynomial boundedness. Halmos' famous similarity problem [51] asked whether any polynomially bounded operator is similar to a contraction. This long standing problem was answered by Pisier [56] in a remarkable where he shows that there are polynomially bounded operators which are not similar to contractions. For more information on similarity problems and completely bounded maps see Pisier [57] and Paulsen [45]. In the noncommutative multivariable setting, joint similarity problems to row contractions, i.e., n-tuples of operators in the unit ball

$$[B(\mathcal{H})_n]_1 := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : X_1 X_1^* + \ldots + X_n X_n^* \leq I\},$$

were considered by Bunce [43], (see [58], [52], [53], [56]), and recently by Douglas, Foia¸s, and Sarkar [48]. In this setting, the universal model for the unit ball $[B(\mathcal{H})^n]_1$ is the n-tuple $(S_1, \ldots, S_n)$ of left creation operators on the full Fock space with n generators.

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0. The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \ldots g_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

If $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ we denote $X_\alpha := X_{i_1} \ldots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$, the identity on $\mathcal{H}$.

In [28] (case $m = 1$) and [25] (case $m \geq 2$), we studied more general noncommutative domains

$$\mathbf{D}_p^m(\mathcal{H}) := \{X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n : \left(id - \Phi_{p,X}\right)^s(I) \geq 0 \ \text{ for } s = 1, \ldots, m\},$$

where id is the identity map on $B(\mathcal{H})$,

$$\Phi_{p,X}(Y) := \sum_{|\alpha| \geq 1} a_\alpha X_\alpha Y X_\alpha^*, \qquad Y \in B(\mathcal{H}),$$

and $p = \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ is a positive regular noncommutative polynomial, i.e., its coefficients are positive scalars and $a_\alpha > 0$ if $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = 1$. We remark that if $q = X_1 + \cdots + X_n$ and $m \geq 1$, then $\mathbf{D}_q^m(\mathcal{H})$is a starlike domain which concides with the set of all row contractions $(X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1$ satisfying the positivity condition

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} X_\alpha X_\alpha^* \geq 0.$$

The elements of the domain $\mathbf{D}_q^m(\mathcal{H})$ can be seen as multivariable noncommutative analogues of Agler's m-hypercontractions [51]. The case $n = 1$ was recently studied by Olofsson ([52], [53]).

We showed ([58], [55]) that each domain $\mathbf{D}_p^m(\mathcal{H})$ has a universal model $(W_1,\ldots,W_n)$ of weighted left creation operators acting on the full Fock space with n generators. The study of the domain $\mathbf{D}_p^m(\mathcal{H})$ and the dilation theory associated with it are close related to the study of the weighted shifts $W_1,\ldots,W_n$, their joint invariant subspaces, and the representations of the algebras they generate: the domain algebra $A_n(\mathbf{D}_p^m)$, the Hardy algebra $F_n^\infty(\mathbf{D}_p^m)$, and the C^* −algebra $C^*(W_1,\ldots,W_n)$.

We consider problems of4 e31` joint similarity to classes of n −tuples of operators in noncommutative domains $\mathbf{D}_p^m(\mathcal{H}), m \geq 1$, and noncommutative varieties

$V_{p,P}^m(\mathcal{H}) := \{(X_1,\ldots,X_n) \in \mathbf{D}_p^m(\mathcal{H}) : q(X_1,\ldots,X_n) = 0 \text{ for any } q \in P\}$,

where P is a family of noncommutative polynomials in n indeterminates.

expanding on ([55], [57], [58]) on noncommutative Berezin transforms, we introduce a new class of generalized Berezin transforms which will play an important role. Given

$A := (A_1,\ldots,A_n) \in B(\mathcal{H})^n$, our similarity problems to n −tuples of operators in the noncommutative variety $V_{p,P}^m(\mathcal{H})$ are linked to the noncommutative cone $C(p,A)^+$ of all positive operators $D \in B(\mathcal{H})$ such that

$$(id - \Phi_{p,A})^s(D) \geq 0, \qquad s = 1,\ldots,m.$$

For example, $(A_1,\ldots,A_n)$ is jointly similar to an n −tuple of operators in $V_{p,P}^m(\mathcal{H})$ if and only if there is an invertible operator in $C(p,A)^+$. Under natural conditions, we show that there is a one-to-one correspondence between the elements of the noncommutative cone $C(p,A)^+$ and a class of generalized Berezin transforms, to be introduced.

a pure version of the above-mentioned result is established, even in a more general setting. In particular, when $m = 1$ and $T := (T_1,\ldots,T_n) \in V_{p,P}^1(\mathcal{H})$ is pure, i.e., $\Phi_{p,T}^k(I) \to 0$ strongly, as $k \to \infty$, we determine the noncommutative cone $C(p,T)^+$ by showing that all its elements have the form $P_{\mathcal{H}}\Psi\Psi^*|_{\mathcal{H}}$, where Ψ is a multi-analytic operator with respect to the universal n −tuple $(B_1,\ldots,B_n)$ associated with the variety $V_{p,P}^1(\mathcal{H})$. More precisely, $\Psi \in R_n^\infty(V_{p,P}^1) \overline{\otimes} B(K,K')$ for some Hilbert spaces K and K', where $R_n^\infty(V_{p,P}^1)$ is the commutant of the noncommutative Hardy algebra $F_n^\infty(V_{p,P}^1)$. We remark that in the particular case when $n = m = 1, p = X, P = \{0\}$, and $\Phi_{p,T}(X) := TXT^*$ with $\|T\| \leq 1$, the corresponding cone $C(p,T)^+$ was studied by Douglas in [47] and by Sz.-Nagy and Foias [42] in connection with T −Toeplitz operators (see also [44] and [45]).

we provide necessary and sufficient conditions for an n −tuple $A := (A_1,\ldots,A_n) \in B(\mathcal{H})^n$ to be jointly similar to an n-tuple of operators $T := (T_1,\ldots,T_n)$ in the noncommutative variety $V_{p,P}^m(\mathcal{H})$ or the distinguished sets

$$\{X \in V_{p,P}^m(\mathcal{H}) : (id - \Phi_{p,X})^m(I) = 0\} \text{ and } \{X \in V_{p,P}^m(\mathcal{H}) : (id - \Phi_{p,X})^m(I) > 0\},$$

where P is a set of noncommutative polynomials. Given $(A_1,\ldots,A_n) \in B(\mathcal{H})^n$, we find necessary and sufficient conditions for the existence of an invertible operator $Y : \mathcal{H} \to \mathcal{G}$ such that

$$A_i^* = Y^{-1}[(B_i^* \otimes I_{\mathcal{H}})|_{\mathcal{G}}]Y, \qquad i = 1,\ldots,n$$

where $\mathcal{G} \subseteq \mathcal{N}_p \otimes \mathcal{H}$ is an invariant subspace under each operator $B_i^* \otimes I_{\mathcal{H}}$ and $(B_1,\ldots,B_n)$ is the universal model associated with the noncommutative variety $V_{f,P}^m(\mathcal{H})$. In particular, we obtain an analogue of Foia¸s [49] and de Branges–Rovnyak [46] model theorem, for pure n −tuples of operators in $V_{f,P}^m(\mathcal{H})$. We also obtain the following Rota type [29] model theorem for the noncommutative variety$V_{f,P}^m(\mathcal{H})$. If $A := (A_1,\ldots,A_n) \in B(\mathcal{H})^n$ is such that $q(A_1,\ldots,A_n) = 0$ for $q \in P$ and

$$\sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{p,A}^k(I) \leq bI$$

for some constant $b > 0$, then the above-mentioned joint similarity holds. Moreover, we prove that the joint spectral radius $r_p(A_1,\ldots,A_n) < 1$ if and only if $(A_1,\ldots,A_n)$is jointly similar to an n −tuple $T := (T_1,\ldots,T_n) \in V_{p,P}^m(\mathcal{H})$ with $(id - \Phi_{p,T})^m(I) > 0$, i.e., positive invertible operator.

We also provide necessary and sufficient conditions for an n −tuple

$A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ to be jointly similar to an n −tuple of operators $T := (T_1, \ldots, T_n) \in V^m_{p,P}(\mathcal{H})$ with $(\mathrm{id} - \Phi_{p,T})^m(I) = 0$. Our noncommutative analugue of Sz.-Nagy's similarity result [30] asserts that there is an invertible operator $Y \in B(\mathcal{H})$ such that $A_i = Y^{-1}T_iY$, $i = 1, \ldots, n$, if and only if there exist positive constants $0 < c \leq d$ such that

$$cI \leq \Phi^k_{p,A}(I) \leq dI, \qquad k \in \mathbb{N}.$$

In particular, we obtain a multivariable analogue of Douglas' similarity result [47].

If $(A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is jointly similar to an n −tuple of operators in a radial noncommutative variety $V^m_{p,P}(\mathcal{H})$, where P is a set of homogeneous noncommutative polynomials, then the polynomial calculus $g(B_1, \ldots, B_n) \longmapsto g(A_1, \ldots, A_n)$ can be extended to a completely bounded map on the noncommutative variety algebra $A_n(V^m_{p,P})$, the norm closed algebra generated by $B_1, \ldots, B_n$ and the identity. Using Paulsen's similarity result [44], we can prove that the converse is true if $m = 1$, but remains an open problem if $m \geq 2$.

We obtain Wold type decompositions and prove the existence of triangulations of type

$$\begin{pmatrix} C_0 & 0 \\ * & C_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_c & 0 \\ * & C_{cnc} \end{pmatrix}$$

for any n −tuple of operators in the noncommutative variety $V^1_{p,P}(\mathcal{H})$, which parallel the Sz.-Nagy–Foia¸s[31] triangulations for contractions. The proofs seem to be new even in the classical case $n = 1$, since they don't involve, at least explicitly, the dilation space for contractions. As consequences, we prove the existence of joint invariant subspaces for certain classes of operators in $V^1_{p,P}(\mathcal{H})$.

We should mention that the results are presented in a more general setting when the polynomials p in the definition of $V^m_{p,P}(\mathcal{H})$ is replaced by positive regular free holomorphic functions.

we introduce a class of generalized Berezin transforms which will play an important We use them to study the noncommutative cone $C(f, A)^+$ of all positive solutions of the operator inequalities

$$(id - \Phi_{f,A})^s(X) \geq 0, \qquad s = 1, \ldots, m.$$

First, we recall ([45], [48]) the construction of the universal model associated with the noncommutative domain $\mathbf{D}^m_f(\mathcal{H}), m \geq 1$. Throughout, we assume that $f := \sum_{\alpha \in \mathbb{F}^+_n} a_\alpha X_\alpha$, $a_\alpha \in \mathbb{C}$, is a positive regular free holomorphic function in n variables $X_1, \ldots, X_n$. This means

(i) $\limsup_{k \to \infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2\right)^{1/2k} < \infty$,

(ii) $a_\alpha > 0$ for any $\alpha \in \mathbb{F}^+_n$, $a_{g_0} = 0$, and $a_{g_i} > 0$ for $i = 1, \ldots, n$.

Given $m \in \mathbb{N} := \{1, 2, \ldots\}$and a positive regular free holomorphic function f as above, we define the noncommutative domain $\mathbf{D}^m_f$ whose representation on a Hilbert space $\mathcal{H}$ is

$$\mathbf{D}^m_f(\mathcal{H}) := \{X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n : (id - \Phi_{f,X})^s(I) \geq 0 \text{ for } s = 1, \ldots, m\},$$

where $\Phi_{f,X}: B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\Phi_{f,X}(Y) := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha Y X^*_\alpha, \qquad Y \in B(\mathcal{H}),$$

and the convergence is in the weak operator topology. $\mathbf{D}^m_f(\mathcal{H})$ can be seen as a noncommutative Reinhardt domain, i.e., $(e^{i\theta_1} X_1, \ldots, e^{i\theta_n} X_n) \in \mathbf{D}^m_f(\mathcal{H})$ for any $(X_1, \ldots, X_n) \in \mathbf{D}^m_f(\mathcal{H})$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

Let H_n be an n −dimensional complex Hilbert space with orthonormal basis $e_1, \ldots, e_n$, where $n \in \mathbb{N}$ or $n = \infty$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k \geq 0} H_n^{\oplus k}$$

where $H_n^{\oplus 0} := \mathbb{C}1$ and $H_n^{\oplus k}$ is the (Hilbert) tensor product of k copies of H_n. Set $e_\alpha := e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}$ if $\alpha = g_{i_1} g_{i_2} \dots g_{i_k} \in \mathbb{F}_n^+$ and $e_{g_0} := 1$. It is clear that $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis of $F^2(H_n)$.

Define the left creation operators $S_i : F^2(H_n) \to F^2(H_n)$, $i = 1, \dots, n$, by $S_i f := e_i \otimes f, f \in F^2(H_n)$.

Let $D_i : F^2(H_n) \to F^2(H_n), i = 1, \dots, n$, be the diagonal operators given by

$$D_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}}\, e_\alpha, \qquad \alpha \in \mathbb{F}_n^+$$

where

$$b_{g_0}^{(m)} := 1 \ \text{ and } b_\alpha^{(m)} := \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \dots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \dots, |\gamma_j| \geq 1}} a_{\gamma_1} \dots a_{\gamma_j} \binom{j+m-1}{m-1} \ \text{ if } \alpha \in \mathbb{F}_n^+,$$
$$|\alpha| \geq 1. \qquad (53)$$

We have

$$\|D_i\| \sup_{\alpha \in \mathbb{F}_n^+} \sqrt{\frac{b_\alpha^{(m)}}{b_{g_i\alpha}^{(m)}}} \leq \frac{1}{\sqrt{a_{g_i}}}, \qquad i = 1, \dots, n$$

Define the weighted left creation operators $W_i : F^2(H_n) \to F^2(H_n), i = 1, \dots, n$, associated with the noncommutative domain $\mathbf{D}_f^m$ by setting $W_i := S_i D_i$, where $S_1, \dots, S_n$ are the left creation operators on the full Fock space $F^2(H_n)$. Note that

$$W_i e_\alpha := \frac{\sqrt{b_\alpha^{(m)}}}{\sqrt{b_{g_i\alpha}^{(m)}}}\, e_{g_i\alpha}, \qquad \alpha \in \mathbb{F}_n^+$$

One can easily see that

$$W_\beta e_\gamma := \frac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\beta_\gamma}^{(m)}}}\, e_{\beta_\gamma} \ \text{ and } \ W_\beta^* e_\alpha := \begin{cases} \dfrac{\sqrt{b_\gamma^{(m)}}}{\sqrt{b_{\beta_\gamma}^{(m)}}}\, e_\gamma & \text{if } \alpha = \beta_\gamma \\ 0 \ \ otherwise \end{cases} \qquad (54)$$

for any $\alpha, \beta \in \mathbb{F}_n^+$. According to Theorem 1.3 from [45], the weighted left creation operators $W_1, \dots, W_n$ associated with $\mathbf{D}_f^m$ have the following properties:

(i) $\sum_{k=1}^\infty \sum_{|\beta|=k} a_\beta \, W_\beta \, W_\beta^* \leq I$, where the convergence is in the strong operator topology;

(ii) $\left(id - \Phi_{f,W}\right)^m (I) = P_{\mathbb{C}}$, where $P_{\mathbb{C}}$ is the orthogonal projection from $F^2(H_n)$ onto $\mathbb{C}_1 \subset F^2(H_n)$, and $\lim_{p\to\infty} \Phi_{f,W}^p(I) = 0$ in the strong operator topology.

The n −tuple $(W_1, \dots, W_n) \in \mathbf{D}_f^m(F^2(H_n))$ plays the role of universal model for the noncommutative domain $\mathbf{D}_f^m$. The domain algebra $A_n(\mathbf{D}_f^m)$ associated with the noncommutative domain $\mathbf{D}_f^m$ is the norm closure of all polynomials in $W_1, \dots, W_n$, and the identity, while the Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ is the SOT-(WOT-, or w^*-) version.

We remark that, one can also define the weighted right creation operators $\Lambda_i : F^2(H_n) \to F^2(H_n)$ by setting $\Lambda_i := R_i G_i$, $i = 1, \dots, n$, where $R_1, \dots, R_n$ are the right creation operators on the full Fock space $F^2(H_n)$ and each diagonal operator G_i is defined by

$$G_i e_\alpha := \sqrt{\frac{b_\alpha^{(m)}}{b_{\alpha g_i}^{(m)}}}\, e_\alpha, \quad \alpha \in \mathbb{F}_n^+$$

where the coefficients $b_\alpha^{(m)}, \alpha \in \mathbb{F}_n^+$, are given by relation (53). It turns out that $(\Lambda_1, \ldots, \Lambda_n)$ is in the noncommutative domain $\mathbf{D}_{\tilde{f}}^m(F^2(H_n))$, where $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ and $\tilde{\alpha} = g_{ik} \ldots g_{i1}$ denotes the reverse of $\alpha = g_{ik} \ldots g_{i1} \in \mathbb{F}_n^+$. Moreover, $W_i\Lambda_j = \Lambda_j W_i$ and $U^*\Lambda_i U = W_i, i = 1, \ldots, n$, where $U \in B\big(F^2(H_n)\big)$
is the unitary operator defined by equation $Ue_\alpha := e_{\tilde{\alpha}}, \alpha \in \mathbb{F}_n^+$. Consequently, we have

$$F_n^\infty\big(\mathbf{D}_f^m\big)' = R_n^\infty\big(\mathbf{D}_f^m\big) \quad \text{ and } \quad R_n^\infty\big(\mathbf{D}_f^m\big)' = F_n^\infty\big(\mathbf{D}_f^m\big),$$

where $'$ stands for the commutant and $R_n^\infty\big(\mathbf{D}_f^m\big)$is the SOT-(WOT-, or w^*-) closure of all polynomials in $\Lambda_1, \ldots, \Lambda_n$, and the identity. More on these noncommutative Hardy algebras can be found in [49], [55], and [68].

We introduce a noncommutative Berezin kernel associated with any quadruple (f, m, A, R) satisfying the following compatibility conditions:
(i) $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ is a positive regular free holomorphic function and $m \in \mathbb{N}$;
(ii) $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent;
(iii) $R \in B(\mathcal{H})$ is a positive operator such that

$$\sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,A}^k(R) \leq bI,$$

for some constant $b > 0$.
The noncommutative Berezin kernel associated with the compatible quadruple (f, m, A, R) is the operator $K_{f,A,R}^{(m)}: \mathcal{H} \to F^2(H_n) \otimes \overline{R^{1/2}(\mathcal{H})}$ given by

$$K_{f,A,R}^{(m)} h = \sum_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}} e_\alpha \otimes R^{\frac{1}{2}} A_\alpha^* h, \qquad h \in \mathcal{H}. \tag{55}$$

Lemma (1.2.1)[41]: The noncommutative Berezin kernel $K_{f,A,R}^{(m)}$ associated with a compatible quadruple (f, m, A, R) is a bounded operator and

$$K_{f,A,R}^{(m)} A_i^* = (W_i^* \otimes I_{\mathcal{R}}) K_{f,A,R}^{(m)}, \qquad i = 1, \ldots, n,$$

where $R := \overline{R^{1/2}(\mathcal{H})}$ and $(W_1, \ldots, W_n)$ is the universal model associated with the noncommutative domain $\mathbf{D}_f^m$. Moreover,

$$\big(K_{f,A,R}^{(m)}\big)^* K_{f,A,R}^{(m)} = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,A}^k(R)$$

Proof. Since (f, m, A, R) is a compatible quadruple, $R \in B(\mathcal{H})$ is a positive operator such that

$$\sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,A}^k(R) \leq bI \tag{56}$$

for some constant $b > 0$. Note that due to relations (53) and (55), we have

$$\begin{aligned}
\big\|K_{f,A,R}^{(m)} h\big\|^2 &= \sum_{\alpha \in \mathbb{F}_n^+} b_\beta^{(m)} \langle A_\beta R A_\beta^* h, h\rangle = \langle Rh, h\rangle + \sum_{m=1}^{\infty} \sum_{|\beta|=m} \langle b_\beta^{(m)} A_\beta R A_\beta^* h, h\rangle \\
&= \langle Rh, h\rangle + \sum_{m=1}^{\infty} \sum_{|\beta|=m} \langle \left(\sum_{j=1}^{|\beta|} \binom{j+m-1}{m-1} \sum_{\substack{\gamma_1 \ldots \gamma_j = \beta \\ |\gamma_1|\geq 1 \ldots |\gamma_j|\geq 1}} a_{\gamma_1} \ldots a_{\gamma_j} \right) A_{\gamma_1 \ldots \gamma_j} R A_{\gamma_1 \ldots \gamma_j}^* h, h\rangle \\
&= \langle Rh, h\rangle + \sum_{m=1}^{\infty} \langle \binom{k+m-1}{m-1} \Phi_{f,A}^k(R) h, h\rangle
\end{aligned}$$

for any $h \in \mathcal{H}$. Hence and due to relation (56), we deduce that $K_{f,A,R}^{(m)}$ is a well-defined bounded operator and

$$\left(K_{f,A,R}^{(m)}\right)^* K_{f,A,R}^{(m)} = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,A}^k(R)$$

On the other hand, due to relations (55) and (54), we have

$$\begin{aligned}(W_i^* \otimes I_{\mathcal{R}})K_{f,A,R}^{(m)}h &= \sum_{\alpha\in\mathbb{F}_n^+} \sqrt{b_\alpha^{(m)}}\, W_i^* e_\alpha \otimes R^{\frac{1}{2}}A_\alpha^* h \\ &= \sum_{\gamma\in\mathbb{F}_n^+} \sqrt{b_{g_i\gamma}^{(m)}}\, W_i^* e_{g_i\gamma} \otimes R^{\frac{1}{2}}A_{g_i\gamma}^* h \\ &= \sum_{\gamma\in\mathbb{F}_n^+} \sqrt{b_\gamma^{(m)}}\, e_\gamma \otimes R^{\frac{1}{2}}A_\gamma^* A_i^* h \\ &= K_{f,A,R}^{(m)}A_i^* h\end{aligned}$$

for any $h \in \mathcal{H}$. Hence,

$$K_{f,A,R}^{(m)}A_i^* = (W_i^* \otimes I_{\mathcal{R}})K_{f,A,R}^{(m)}, \qquad i = 1,\dots,n,$$

and the proof is complete.

Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $W_1,\dots,W_n$ and $\Lambda_1,\dots\Lambda_n$ be the weighted left and right creation operators, respectively, associated with the noncommutative domain $\mathbf{D}_f^{\mathrm{m}}$. Let P be a family of noncommutative polynomials and define the noncommutative variety $V_{f,p}^m$ whose representation on a Hilbert space $\mathcal{H}$ is

$$V_{f,p}^m(\mathcal{H}) := \left\{(X_1,\dots,X_n) \in \mathbf{D}_f^{\mathrm{m}}(\mathcal{H}) : p(X_1,\dots,X_n) = 0 \quad \text{for any } p \in P\right\}.$$

We associate with $V_{f,p}^m$ the operators $B_1,\dots,B_n$ defined as follows. Consider the subspaces

$$M_p := \overline{\operatorname{span}}\{W_\alpha p(W_1,\dots,W_n)W_\beta(1) : p \in P, \alpha,\beta \in \mathbb{F}_n^+\}$$

and $\mathcal{N}_p := F^2(H_n) \ominus M_P$. Throughout, unless otherwise specified, we assume that $\mathcal{N}_p \neq \{0\}$. It is easy to see that $\mathcal{N}_p$ is invariant under each operator $W_1^*,\dots,W_n^*$ and $\Lambda_1^*,\dots,\Lambda_n^*$. Define

$$B_i := P_{\mathcal{N}_p} W_i|_{\mathcal{N}_p} \quad \text{and} \quad C_i := P_{\mathcal{N}_p} \Lambda_i|_{\mathcal{N}_p}, \qquad i = 1,\dots,n,$$

Where $P_{\mathcal{N}_p}$ is the orthogonal projection of $F^2(H_n)$ onto $\mathcal{N}_p$

The n −tuple of operators $B := (B_1,\dots,B_n) \in V_{f,p}^m(\mathcal{N}_p)$ plays the role of universal model for the noncommutative variety $V_{f,p}^m$. The noncommutative variety algebra $A_n\left(V_{f,p}^m\right)$ is the norm-closed algebra generated by $B_1,\dots,B_n$ and the identity, while the Hardy algebra $F_n^\infty\left(V_{f,p}^m\right)$ is the w^*-version. More on these Hardy algebras associated with noncommutative varieties can be found in [58] and [55].

Let (f,m,A,R) be a compatible quadruple. Assume that the n −tuple $A := (A_1,\dots,A_n) \in B(\mathcal{H})^n$ has, in addition, the property that $p(A_1,\dots,A_n) = 0,\ p \in P$.
Under these conditions, the tuple $q := (f,m,A,R,P)$ is called compatible. We define the (constrained) noncommutative Berezin kernel associated with the tuple q to be the operator $K_q : \mathcal{H} \to \mathcal{N}_p \otimes \overline{R^{1/2}(\mathcal{H})}$ given by

$$K_q := \left(P_{\mathcal{N}_p} \otimes I_{\overline{R^{\frac{1}{2}}(\mathcal{H})}}\right)K_{f,A,R}^{(m)},$$

Where $K_{f,A,R}^{(m)}$ is the Berezin kernel associated with the quadruple (f,m,A,R) and defined by relation (55).

Lemma (1.2.2) [41]: Let K_q be the noncommutative Berezin kernel associated with a compatible tuple $q := (f,m,A,R,P)$. Then

$$K_q A_i^* = (B_i^* \otimes I_{\mathcal{R}})K_q, \qquad i = 1,\dots,n,$$

Where $\mathcal{R} := \overline{R^{1/2}(\mathcal{H})}$ and $(B_1,\dots,B_n)$ is the universal model associated with the noncommutative variety $V_{f,p}^m$. Moreover,

$$K_q^* K_q = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,A}^k(R).$$

Proof: Using Lemma (1.2.1) and the fact that $p(A_1,\dots,A_n) = 0$ for all $p \in P$, we obtain

$$\langle K_{f,A,R}^{(m)} x, [W_\alpha p(W_1,\dots,W_n) W_\beta(1)] \otimes y\rangle = \langle x, A_\alpha p(A_1,\dots,A_n) A_\beta \left(K_{f,A,R}^{(m)}\right)^* (1 \otimes y)\rangle = 0$$

for any $x \in \mathcal{H},\ y \in \overline{R^{1/2}(\mathcal{H})}$, and $p \in P$. Hence, we deduce that

$$\text{range } K_{f,A,R}^{(m)} \subseteq \mathcal{N}_p \otimes \overline{R^{\frac{1}{2}}(\mathcal{H})}. \qquad (57)$$

Taking into account the definition of the constrained Berezin kernel $K_q : \mathcal{H} \to \mathcal{N}_p \otimes \overline{R^{1/2}(\mathcal{H})}$, one can use Lemma (1.2.1) and relation (57) to complete the proof.

We introduce now the noncommutative Berezin transform $\mathbf{B}_q$ associated with the compatible tuple $q := (f, m, A, R, P)$ to be the operator $\mathbf{B}_q : B(\mathcal{N}_p) \to B(\mathcal{H})$ given by

$$\mathbf{B}_q[\chi] := K_q^*[\chi \otimes I_{\mathcal{R}}]K_q, \qquad \chi \in B(\mathcal{N}_p).$$

where $\mathcal{R} := \overline{R^{1/2}(\mathcal{H})}$. This transform will play an important role. To justify the terminology, we shall consider the particular case when the n −tuple $A := (A_1,\dots,A_n)$ has the joint spectral radius

$$r_f(A_1,\dots,A_n) := \lim_{k\to\infty} \left\|\Phi_{f,A}^k(I)\right\|^{1/2k} < 1.$$

Then, as in the particular case considered in [55], one can show that

$$\langle \mathbf{B}_q[\chi]x, y\rangle = \langle \left(I - \sum_{|\alpha|\ge 1} a_{\tilde{\alpha}} C_\alpha^* \otimes A_{\tilde{\alpha}}\right)^{-m} (\chi \otimes R) \left(I - \sum_{|\alpha|\ge 1} a_{\tilde{\alpha}} C_\alpha \otimes A_{\tilde{\alpha}}^*\right)^{-m} (1 \otimes x),\ 1 \otimes y\rangle$$

for any $x, y \in \mathcal{H}$, where $C_i := P_{\mathcal{N}_p} \Lambda_i|_{\mathcal{N}_p}$ for $i = 1,\dots,n$ and $\tilde{\alpha}$ is the reverse of $\alpha \in \mathbb{F}_n^+$. We present a sketch of the proof. First, one can show that

$$r\left(I - \sum_{|\alpha|\ge 1} a_{\tilde{\alpha}} C_\alpha \otimes A_{\tilde{\alpha}}^*\right) \le r_f(A_1,\dots,A_n) < 1,$$

where $r(Y)$ is the usual spectral radius of a bounded operator Y . Hence, the operator

$$\left(I - \sum_{|\alpha|\ge 1} a_{\tilde{\alpha}} C_\alpha \otimes A_{\tilde{\alpha}}^*\right)^{-1} = \sum_{k=0}^{\infty} \left(I - \sum_{|\alpha|\ge 1} a_{\tilde{\alpha}} C_\alpha \otimes A_{\tilde{\alpha}}^*\right)^k$$

is well-defined, where the convergence is in the operator norm topology. Consequently, using the definition of $\Lambda_1,\dots,\Lambda_n$ and relation (55), we obtain

$$K_{f,T}^{(m)} h = \left(I_{F^2(H_n)} \otimes R^{\frac{1}{2}}\right)\left(I - \sum_{|\alpha|\ge 1} a_{\tilde{\alpha}} \Lambda_\alpha \otimes T_{\tilde{\alpha}}^*\right)^{-m} (1 \otimes h), \quad h \in \mathcal{H}.$$

Combining the above-mentioned results with the fact that $K_q := \left(P_{\mathcal{N}_p} \otimes I_{\overline{R^{\frac{1}{2}}(\mathcal{H})}}\right) K_{f,A,R}^{(m)}$, one can complete the proof of our assertion.

We remark that in the particular case when: $n = m = 1$, $f = X$, $\mathcal{H} = \mathbb{C}$, $A = \lambda \in \mathbb{D}, R = I$, and $P = \{0\}$, we recover the Berezin transform [36] of a bounded operator on the Hardy space $H^2(\mathbb{D})$, i.e.,

$$\mathbf{B}_\lambda[g] = (1 - |\lambda|^2)\, \langle g k_\lambda, k_\lambda\rangle, \qquad g \in B(H^2(\mathbb{D})),$$

where $k_\lambda(z) := \left(1 - \bar{\lambda} z\right)^{-1}$ and $z, \lambda \in \mathbb{D}$.

The following technical lemma is a slight extension of Lemma 1.4 and 2.2 from [55], where the operator D was positive. In our extension, D is a self-adjoint operator and the condition (a) is new. However, since the proof is similar to those from [55], we shall omit it. A linear map $\varphi: B(\mathcal{H}) \to B(\mathcal{H})$ is called power bounded if there exists a constant $M > 0$ such that $\|\varphi^k\| \le M$ for any $k \in \mathbb{N}$.

Lemma (1.2.3) [41]: Let $\varphi: B(\mathcal{H}) \to B(\mathcal{H})$ be a positive linear map and let $D \in B(\mathcal{H})$ be a self-adjoint operator and $m \in \mathbb{N}$. Then the following statements hold:

(i) If φ is power bounded, then

$(id - \varphi)^m(D) \ge 0$ if and only if $(id - \varphi)^s(D) \ge 0, \; s = 1, 2, \ldots, m.$

(ii) Under either one of the conditions:

(a) $(id - \varphi)^s(D) \ge 0$ for any $s = 1, \ldots, m$, or

(b) φ is power bounded and $(id - \varphi)^m(D) \ge 0$, the following limit exists and

$$\lim_{k\to\infty} k^d \langle \varphi^k (id - \varphi)^d(D)h, h\rangle = \begin{cases} \lim_{k\to\infty} k^d \langle \varphi^k h, h\rangle & \text{if } d = 0 \\ 0 & \text{if d} = 1,2,\ldots,\text{m} - 1 \end{cases}$$

for any $h \in \mathcal{H}$.

In what follows we also need the following result.

For information on completely bounded (resp. positive) maps, see [55] and [56].

Lemma (1.2.4) [41]: Let $f := \sum_{|\alpha|\ge 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ be an n-tuple of operators such that $\sum_{|\alpha|\ge 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology. Then the map $\Phi_{f,A}: B(\mathcal{H}) \to B(\mathcal{H})$, defined by

$$\Phi_{f,A}(X) = \sum_{|\alpha|\ge 1} a_\alpha A_\alpha X A_\alpha^*, \qquad X \in B(\mathcal{H}),$$

where the convergence is in the weak operator topology, is a completely positive linear map which is WOT-continuous on bounded sets. Moreover, if $0 < r < 1$, then

$$\Phi_{f,A}(X) = \text{WOT} - \lim_{r\to 1} \Phi_{f,rA}(X), \qquad X \in B(\mathcal{H}).$$

Proof: Note that, for any $x, y \in \mathcal{H}$ and any finite subset $\Lambda \subset \{\alpha \in \mathbb{F}_n^+ : |\alpha| \ge 1\}$, we have

$$\sum_{\alpha\in\Lambda} |\langle a_\alpha A_\alpha X A_\alpha^* x, y\rangle| \le \|X\| \sum_{\alpha\in\Lambda} a_\alpha \|A_\alpha^* x\| \|A_\alpha^* y\| \le \|X\| \left(\sum_{\alpha\in\Lambda} a_\alpha \|A_\alpha^* x\|^2\right)^{1/2} \left(\sum_{\alpha\in\Lambda} a_\alpha \|A_\alpha^* y\|^2\right)^{1/2}.$$

Now, since $\sum_{|\alpha|\ge 1} a_\alpha A_\alpha A_\alpha^*$is convergent in the weak operator topology it is easy to see that the series $\Phi_{f,A}(X) = \sum_{|\alpha|\ge 1} a_\alpha A_\alpha X A_\alpha^*$convergence is in the weak operator topology. Moreover, the above-mentioned inequality is true for any subset Λ in $\{\alpha \in \mathbb{F}_n^+ : |\alpha| \ge 1\}$. In particular, we deduce that

$$|\langle \Phi_{f,A}(X)x, y\rangle| \le \|X\| \langle \Phi_{f,A}(I)x, x\rangle^{\frac{1}{2}} \langle \Phi_{f,A}(I)y, y\rangle^{\frac{1}{2}}, \qquad x, y \in \mathcal{H}.$$

On the other hand, since the map $\Phi_{f,A}^{(k)} := \sum_{1\le|\alpha|\le} a_\alpha A_\alpha X A_\alpha^*, X \in B(\mathcal{H})$, is completely positive for each $k \in \mathbb{N}$ and $\Phi_{f,A}(X) = \text{WOT} - \lim_{k\to\infty} \Phi_{f,A}^{(k)}(X)$, we deduce that $\Phi_{f,A}$ is a completely positive map on $B(\mathcal{H})$. Since $\sum_{|\alpha|\ge 1} a_\alpha A_\alpha X A_\alpha^*$
is convergent in the weak operator topology, for any $\epsilon > 0$ and $x, y \in \mathcal{H}$,
there is $N_0 \in \mathbb{N}$ such that

$$\sum_{|\alpha|>N_0} \langle a_\alpha A_\alpha A_\alpha^* x, x\rangle < \epsilon \quad \text{and} \quad \sum_{|\alpha|>N_0} \langle a_\alpha A_\alpha A_\alpha^* y, y\rangle < \epsilon.$$

Using the above-mentioned inequalities, we deduce that

$$\sum_{|\alpha|>N_0} |\langle a_\alpha A_\alpha X A_\alpha^* x, y\rangle| \le \epsilon \|X\|$$

Now, it is easy to see that $\Phi_{f,A}$ is WOT-continuous on bounded sets. On the other hand, we also have $\sum_{|\alpha|>N_0} |\langle a_\alpha r^{|\alpha|} A_\alpha X A_\alpha^* x, y\rangle| \le \epsilon \|X\|$ for any $r \in [0, 1]$. This can be used to show that $\Phi_{f,A}(X) = \text{WOT} - \lim_{r\to 1} \Phi_{f,rA}(X)$ for any $X \in B(\mathcal{H})$. The proof is complete.

We remark that Lemma (1.2.4) remains true if $\{a_\alpha\}_{\alpha\geq 1}$is just a sequence of positive numbers and $A := (A_1,\dots,A_n) \in B(H)^n$ is an n-tuple of operators such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology.

We denote by $C(f,A)^+$ the cone of all positive operators $D \in B(\mathcal{H})$ such that

$$\left(id - \Phi_{f,A}\right)^s(D) \geq 0 \quad \text{for} \quad s = 1,\dots,m.$$

We denote by $C_{rad}(f,A)^+$ the set of all operators $D \in C(f,A)^+$ such that there is $\delta \in (0,1)$ with the property that $D \in C(f,rA)^+$ for any $r \in (\delta,1]$.

A few examples are necessary. Note that if $m = 1$ then we always have $C(f,A)^+ = \mathrm{C}_{\mathrm{rad}}(f,A)^+$. We remark that if $m \geq 2$ and $p = a_1X_1 + \dots + a_nX_n, a_i > 0$, then we also have $C(p,A)^+ = \mathrm{C}_{\mathrm{rad}}(p,A)^+$. Indeed, it is enough to see that if $0 < r \leq 1$, then

$$\begin{aligned}\left(id - \Phi_{p,rA}\right)^k(D) &= \left[\left(id - \Phi_{p,A}\right) + (1-r)\Phi_{p,A}\right]^k(D)\\ &= \sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}\Phi_{p,A}^{k-j}\left(id - \Phi_{p,A}\right)^j(D)\end{aligned}$$

for any $k = 1,\dots,m$. Since $\left(id - \Phi_{p,A}\right)^j(D) \geq 0$ for $j = 1,\dots,m$ and using the fact that $\Phi_{p,A}^j$ is a positive linear map, we deduce that $\left(id - \Phi_{p,rA}\right)^k(D) \geq 0$ for $k = 1,\dots,m$ and $r \in (0,1]$, which proves our assertion. Note also that when $m \geq 1$ and q is any positive regular noncommutative polynomial so that, for each $s = 1,\dots,m, \left(id - \Phi_{q,A}\right)^s(D)$ is a positive invertible operator, then $D \in \mathrm{C}_{\mathrm{rad}}(q,A)^+$.

We say that $\mathbf{D}_f^m(\mathcal{H})$ is a radial domain if there exists $\delta \in (0,1)$ such that $(rW_1,\dots,rW_n) \in \mathbf{D}_f^m(F^2(H_n))$ for any $r \in (\delta,1]$, where $(W_1,\dots,W_n)$ is the universal model associated with $\mathbf{D}_f^m$. We remark that the notion of radial domain does not depend on the Hilbert space $\mathcal{H}$. Note that if $m = 1$, then $\mathbf{D}_f^1(\mathcal{H})$ is always a radial domain. This case was extensively studied in [58]. When $m \geq 2$, we point out the particular case $p := a_1X_1 + \dots + a_nX_n, a_i > 0$, when $\mathbf{D}_f^m(\mathcal{H})$ is also a radial domain.

We show that, for radial domains $\mathbf{D}_f^m(\mathcal{H})$, the elements of the noncommutative cone $\mathrm{C}_{\mathrm{rad}}(f,A)^+$ are in one-to-one correspondence with the elements of a class of noncommutative Berezin transforms.

Theorem (1.2.5) [41]: Let $\mathbf{D}_f^m(\mathcal{H})$ be a radial domain, where $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ is a positive regular free holomorphic function and $m \geq 1$. Let P be a family of noncommutative homogeneous polynomials and let $B := (B_1,\dots,B_n)$ be the universal model associated with the noncommutative variety $V_{f,p}^m$. If $A := (A_1,\dots,A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent and $p(A_1,\dots,A_n) = 0,\ p \in P$, then there is a bijection P

$$\Gamma: CP\left(A, V_{f,p}^m\right) \to \mathrm{C}_{\mathrm{rad}}(f,A)^+, \qquad \Gamma(\varphi) := \varphi(I),$$

where $CP\left(A, V_{f,p}^m\right)$ is the set of all completely positive linear maps $\varphi: S_{f,p} \to B(\mathcal{H})$ such that

$$\varphi\left(B_\alpha B_\beta^*\right) = A_\alpha \varphi(I) A_\beta^*, \qquad \alpha,\beta \in \mathbb{F}_n^+.$$

where $S_{f,p} := \overline{\mathrm{span}}\{B_\alpha B_\beta^* : \alpha,\beta \in \mathbb{F}_n^+\}$. Moreover, if $D \in \mathrm{C}_{\mathrm{rad}}(f,A)^+$, then $\Gamma^{-1}(D)$ coincides with the noncommutative Berezin transform associated with $q := (f,m,A,R,P)$ and defined by

$$\overline{\mathbf{B}}_q[\chi] := \lim_{r\to 1} K_{qr}^*(\chi \otimes I)K_{qr}, \qquad \chi \in S_{f,p},$$

where $q_r := (f,m,rA,R_r,P)$ and $R_r := \left(id - \Phi_{f,rA}\right)^m(D),\ r \in [0,1]$, and the limit exists in the operator norm topology.

Proof: We recall that the subspace $\mathcal{N}_p \neq \{0\}$ is invariant under each operator $W_1^*,\dots,W_n^*$ and $B_i := W_i|_{\mathcal{N}_p}, i = 1,\dots,n$. Setting $B := (B_1,\dots,B_n)$ and taking into account that $\Phi_{f,W}(I) \leq I$, we deduce that $\Phi_{f,B}(I) \leq I$ and, consequently, $\Phi_{f,rB}(I) = \sum_{k=1}^{\infty}\sum_{|\alpha|=k} a_\alpha r^{|\alpha|} B_\alpha B_\alpha^* \leq I$, where the convergence is in the operator norm topology. This implies $\Phi_{f,rB}(I) \in S_{f,p}$ for any $r \in [0,1)$. The fact that $\mathbf{D}_f^m$ is radial domain implies $(rW_1,\ldots,rW_n)\in\mathbf{D}_f^m(F^2(H_n))$, $r\in(\delta,1)$, for some $\delta\in(0,1)$ and, consequently, $(id-\Phi_{f,rB})^s(I)\geq 0$ for $s=1,\ldots,m$ and $r\in(\delta,1)$. Since

$$\Phi_{f,rB}^j(I)=\sum_{k=1}^{\infty}\sum_{|\alpha|=k}a_\alpha r^{|\alpha|}B_\alpha\Phi_{f,rB}^{j-1}(I)\ B_\alpha^*,\qquad j\in\mathbb{N},$$

and $\|\Phi_{f,rB}^k(I)\|\leq 1$ for any $k\in\mathbb{N}$, it is clear that $\Phi\ {}_{f,rB}^{j}(I)\in S_{f,p}$. Taking into account that

$$(id-\Phi_{f,rB})^s(I)=\sum_{j=0}^{s}(-1)^j\binom{s}{j}\ \ \Phi\ {}_{f,rB}^{j}(I),\qquad j\in\mathbb{N},$$

we deduce that $(id-\Phi_{f,rB})^s(I)\in S_{f,p}$ completely positive linear map such that for $s=1,\ldots,m$. Now, assume that $\varphi:S_{f,p}\to B(\mathcal{H})$ is a completely positive linear map such that

$$\varphi(B_\alpha B_\beta^*)=A_\alpha\varphi(I)A_\beta^*,\qquad \alpha,\beta\in\mathbb{F}_n^+.$$

Then, setting $D:=\varphi(I)$, we deduce that $D\geq 0$ and

$$(id-\Phi_{f,rA})^s(D)=\varphi\left((id-\Phi_{f,rB})^s(I)\right)\geq 0,\qquad r\in(\delta,1),$$

for any $s=1,\ldots,m$. Since the series $\sum_{|\alpha|\geq 1}a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent one can use Lemma (1.2.4) to deduce that $\Phi_{f,A}^k(D)=\text{WOT}-\lim_{r\to 1}\Phi_{f,rA}^k(D)$ for $k\in\mathbb{N}$ and, moreover,

$$(id-\Phi_{f,A})^s\ (D)=\text{WOT}-\lim_{r\to 1}(id-\Phi_{f,rA})^s\ (D)\geq 0$$

for any $s=1,\ldots,m$. This shows that $D\in \mathrm{C}_{\mathrm{rad}}(f,A)^+$. To prove that Γ is one-to-let φ_1 and φ_2 completely positive linear maps on $S_{f,p}$ such that $\varphi_j(B_\alpha B_\beta^*)=A_\alpha\varphi_i(I)A_\beta$, $\alpha,\beta\in\mathbb{F}_n^+$, and assume that $\Gamma(\varphi_1)=\Gamma(\varphi_2)$, i.e., $\varphi_1(I)=\varphi_2(I)$. Then we have $\varphi_1(B_\alpha B_\beta^*)=\varphi_2(B_\alpha B_\beta^*)$ for $\alpha,\beta\in\mathbb{F}_n^+$. Taking into account the continuity of ϕ1 and ϕ2 in the operator norm, we deduce that $\varphi_1=\varphi_2$.

To prove surjectivity, fix $D\in \mathrm{C}_{\mathrm{rad}}(f,A)^+$. Then $D\in B(\mathcal{H})$ is a positive operator with the property that there is $\delta\in(0,1)$ such that $(id-\Phi_{f,rA})^s(D)\geq 0$ for any $s=1,\ldots,m$ and $r\in(\delta,1)$. Since the set P consists of homogeneous noncommutative polynomials , we have $p(rA_1,\ldots,rA_n)=0$ for any $p\in P$ and $r\in(\delta,1)$. We show now that, for each $r\in(\delta,1)$, the tuple $q_r:=(f,m,rA,R_r,P)$, where $R_r:=(id-\Phi_{f,rA})^m(D)$, is compatible. Indeed, we can use the equality

$$\binom{i+j}{j}-\binom{i+j-1}{j}=\binom{i+j-1}{j-1},\qquad i,j\in\mathbb{N}$$

and Lemma (1.2.3) to obtain

$$\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}\Phi_{f,rA}^k(R_r)=D-WOT-\lim_{k\to\infty}\sum_{j=0}^{m-1}\binom{k+j}{j}\Phi_{f,rA}^{k+1}(id-\Phi_{f,rA})^j(D)$$
$$=D-WOT-\lim_{k\to\infty}\Phi_{f,rA}^k(D).$$

Since $\Phi_{f,rA}^k(D)\leq r^{2k}\Phi_{f,A}^k(D)\leq r^{2k}D$, we have $D-WOT-\lim_{k\to\infty}\Phi_{f,rA}^k(D)=0$. Therefore, we deduce that

$$\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}\Phi_{f,rA}^k(R_r)=D,\qquad r\in(\delta,1).\tag{58}$$

According to Lemma (1.2.2), the constrained noncommutative Berezin kernel K_{q_r} , $r\in(\delta,1)$, associated with the compatible tuple $q_r:=(f,rA,R_r,P)$, has the property that

$$K_{q_r}(rA_i^*)=(B_i^*\otimes I_{\mathcal{H}})K_{q_r},\qquad i=1,\ldots,n,\tag{59}$$

where $(B_1,\ldots,B_n)$ is the $n-$tuple of constrained weighted left creation operators associated with the noncommutative variety $V_{f,p}^m$, and

$$K_{q_r}^* K_{q_r} = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \Phi_{f,rA}^k(R_r) = D, \qquad r \in (\delta, 1).$$

where $R_r := \left(id - \Phi_{f,rA}\right)^m(D)$. Hence and using relation (58), we obtain

$$K_{q_r}^* K_{q_r} = D, \qquad r \in (\delta, 1). \qquad (60)$$

For each $r \in (\delta, 1)$, define the operator $\mathbf{B}_{q_r} : S_{f,p} \to B(\mathcal{H})$ by setting

$$\mathbf{B}_{q_r}(\chi) := K_{q_r}^*(\chi \otimes I_{\mathcal{H}})K_{q_r}, \qquad \chi \in S_{f,p}. \qquad (61)$$

Using relation (59) and (60), we have

$$K_{q_r}^*\left(B_\alpha B_\beta^* \otimes I\right)K_{q_r} = r^{|\alpha|+|\beta|} A_\alpha D A_\beta^*, \qquad \alpha, \beta \in \mathbb{F}_n^+, \quad r \in (\delta, 1). \qquad (62)$$

Hence, and using relations (60) and (61), we infer that $\mathbf{B}_{q_r}$ is a completely positive linear map with $\mathbf{B}_{q_r}(I) = D$ and $\|\mathbf{B}_{q_r}\| = \|D\|$ for $r \in (\delta, 1)$.

Now, we show that $\lim_{r\to 1} \mathbf{B}_{q_r}(\chi)$ exists in the operator norm topology for each $\chi \in S_{f,p}$. Given a polynomial $\varphi(B_1, \ldots, B_n) := \sum_{\alpha,\beta\in\mathbb{F}_n^+} a_{\alpha\beta} B_\alpha B_\beta^*$ in the operator system $S_{f,p}$, we define

$$\varphi_D(A_1, \ldots, A_n) := \sum_{\alpha,\beta\in\mathbb{F}_n^+} a_{\alpha\beta} A_\alpha D A_\beta^*.$$

The definition is correct since, according to relation (62), we have the following von Neumman type inequality

$$\|\varphi_D(A_1, \ldots, A_n)\| \leq \|D\| \|\varphi(B_1, \ldots, B_n)\|. \qquad (63)$$

Now, fix $\chi \in S_{f,p}$ and let $\varphi^{(k)}(B_1, \ldots, B_n)$ be a sequence of polynomials in $S_{f,p}$ convergent to χ, in the operator norm topology. Define the operator

$$\chi_D(A_1, \ldots, A_n) := \lim_{k\to\infty} \varphi_D^{(k)}(A_1, \ldots, A_n). \qquad (64)$$

Taking into account relation (63), it is clear that the operator $\chi_D(A_1, \ldots, A_n)$ is well-defined and

$$\|\chi_D(A_1, \ldots, A_n)\| \leq \|D\| \|\chi\|.$$

According to relation (62), we have

$$\left\|\varphi_D^{(k)}(rA_1, \ldots, rA_n)\right\| \leq \|D\| \left\|\varphi^{(k)}(B_1, \ldots, B_n)\right\|$$

for any $r \in (\delta, 1)$. Taking into account that $\mathbf{B}_{q_r}$ is a bounded linear operator and using again relation (62), we deduce that

$$\lim_{k\to\infty} \varphi_D^{(k)}(rA_1, \ldots, rA_n) = \lim_{k\to\infty} k_{qr}^*\left(\varphi^{(k)}(B_1, \ldots, B_n) \otimes I\right) K_{q_r} = \mathbf{B}_{q_r}[\chi] \qquad (65)$$

for any $r \in (\delta, 1)$. Using relations (64), (65), the fact that $\left\|\chi - \varphi^{(k)}(B_1, \ldots, B_n)\right\| \to 0$ as $k \to \infty$, and

$$\lim_{r\to\infty} \varphi_D^{(k)}(rA_1, \ldots, rA_n) = \varphi_D^{(k)}(A_1, \ldots, A_n),$$

we can deduce that

$$\lim_{r\to 1} \mathbf{B}_{q_r}[\chi] = \chi_D(A_1, \ldots, A_n)$$

in the norm topology. Indeed, note that

$$\begin{aligned}
&\left\|\chi_D(A_1, \ldots, A_n) - \mathbf{B}_{q_r}[\chi]\right\| \\
&\quad \leq \left\|\chi_D(A_1, \ldots, A_n) - \varphi_D^{(k)}(A_1, \ldots, A_n)\right\| + \left\|\varphi_D^{(k)}(A_1, \ldots, A_n) - \mathbf{B}_{q_r}\left(\varphi^{(k)}\right)\right\| \\
&\qquad + \left\|\mathbf{B}_{q_r}\left(\varphi^{(k)}\right) - \mathbf{B}_{q_r}(\chi)\right\| \\
&\quad \leq \left\|\chi - \varphi^{(k)}(B_1, \ldots, B_n)\right\| \|D\| + \left\|\varphi_D^{(k)}(A_1, \ldots, A_n) - \varphi_D^{(k)}(rA_1, \ldots, rA_n)\right\| \\
&\qquad + \left\|\chi - \varphi^{(k)}(B_1, \ldots, B_n)\right\| \|D\|.
\end{aligned}$$

For any $r \in (\delta, 1)$, $\mathbf{B}_{q_r}$ is a completely positive linear map. Hence, and using relation (62), we infer that

$$\overline{\mathbf{B}}_q[\chi] := \lim_{r\to 1} K_{q_r}^*(\chi \otimes I)K_{q_r}, \qquad \chi \in S_{f,p}$$

is a completely positive map with $\overline{\mathbf{B}}_q(I) = D$ and $\overline{\mathbf{B}}_q(B_\alpha B_\beta^*) = A_\alpha \overline{\mathbf{B}}_q(I) A_\beta$, $\alpha, \beta \in \mathbb{F}_n^+$.

The proof is complete.

The following result is an extension of the noncommutative von Neumann inequality (see [53], [59], [51], [55]).

Corollary (1.2.6): Under the hypotheses of Theorem (1.2.5), if $D \in \mathrm{C}_{\mathrm{rad}}(f, A)^+$, then we have the following von Neumann type inequality:

$$\left\| \sum_{\alpha,\beta \in \Lambda} A_\alpha D A_\beta^* \otimes C_{\alpha,\beta} \right\| \le \|D\| \left\| \sum_{\alpha,\beta \in \Lambda} B_\alpha B_\beta^* \otimes C_{\alpha,\beta} \right\|$$

for any finite set $\Lambda \subset \mathbb{F}_n^+$ and $C_{\alpha,\beta} \in B(\mathcal{E})$, where $\mathcal{E}$ is a Hilbert space. If, in addition, D is an invertible operator, then the map $u: A_n(V_{f,p}^m) \to B(\mathcal{H})$ defined by

$$u(p(B_1, \ldots, B_n)) := p(A_1, \ldots, A_n)$$

is completely bounded with $\|u\|_{cb} \le \|D^{-1/2}\| \|D^{1/2}\|$.

Proof: Due to relation (62), we have

$$(K_{q_r}^* \otimes I_\mathcal{E})(B_\alpha B_\beta^* \otimes I \otimes C_{\alpha,\beta})(K_{q_r} \otimes I_\mathcal{E}) = r^{|\alpha|+|\beta|} A_\alpha D A_\beta^* \otimes C_{\alpha,\beta}, \alpha, \beta \in \mathbb{F}_n^+, \qquad r \in (\delta, 1).$$

Since $K_{q_r}^* K_{q_r} = D$ for $r \in (\delta, 1)$, one can easily deduce the von Neumann type inequality. To prove the second part, note that, if D is invertible, then the first part of this corollary implies

$$\begin{aligned}
\|p(A_1, \ldots, A_n)\|^2 &\le \left\|D^{-\frac{1}{2}}\right\|^2 \left\|p(A_1, \ldots, A_n) D^{1/2}\right\|^2 \\
&= \left\|D^{-\frac{1}{2}}\right\|^2 \|p(A_1, \ldots, A_n) D p(A_1, \ldots, A_n)^*\| \\
&\le \left\|D^{-\frac{1}{2}}\right\|^2 \|D\| \|p(B_1, \ldots, B_n) p(B_1, \ldots, B_n)^*\| \\
&= \left\|D^{-\frac{1}{2}}\right\|^2 \left\|D^{1/2}\right\|^2 \|p(B_1, \ldots, B_n)\|^2
\end{aligned}$$

for any noncommutative polynomial p. A similar result holds if we pass to matrices. Therefore, we deduce that u is completely bounded with $\|u\|_{cb} \le \|D^{-1/2}\| \|D^{1/2}\|$. The proof is complete.

Example (1.2.7) [41]: (i) When $m = 1, f = X_1 + \cdots + X_n$, and $D = I$, we obtain the noncommutative Poisson transform introduced in [51] (case $P = \{0\}$) and [54] (case $P \neq \{0\}$).

(ii) When $m = 1, f = X_1 + \cdots + X_n, P = \{0\}$, and $D \ge 0$ such that $\sum_{i=1}^n A_i D A_i^* \le D$, we obtain the noncommutative Poisson transform from [52].

(iii) When $m \ge 1, D = I$, and f is an arbitrary positive regular free holomorphic function, we obtain the noncommutative Berezin transforms associated with noncommutative domains $\mathbf{D}_f^m$ or noncommutative varieties $\mathrm{V}_{f,p}^m$, which were studied in [55] and [58].

We study the noncommutative cone $\mathrm{C}_{\mathrm{pure}}(f, A)^+$ of all pure solutions of the operator inequalities $(id - \Phi_{f,A})^s (X) \ge 0, s = 1, \ldots, m$. When A is a pure $n-$tuple of operators in the noncommutative variety $V_{f,p}^1(\mathcal{H})$, we obtain a complete description of the noncommutative cone $C(f, A)^+$.

Let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ be such that $\sum_{|\alpha| \ge 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology and recall that

$$\Phi_{f,A}(X) := \sum_{|\alpha| \ge 1} a_\alpha A_\alpha A_\alpha^*, \qquad X \in B(\mathcal{H}),$$

where the convergence is in the weak operator topology. We assume that $\Phi_{f,A}$ is power bounded. A self-adjoint operator $C \in B(\mathcal{H})$ is called pure solution of the inequality $(id - \Phi_{f,A})^m (X) \ge 0$ if

$$(id - \Phi_{f,A})^m (C) \ge 0 \quad \text{and} \quad \mathrm{SOT} - \lim_{k \to \infty} \Phi_{f,A}^k (C) = 0$$

Note that since $\Phi_{f,A}$ is power bounded, Lemma (1.2.3) implies $\Phi_{f,A}(C) \le C$. This can be used to show that a pure self-adjoint solution is always a positive operator. In what follows we present a canonical decomposition for the self-adjoint solutions of the operator inequality $\left(id - \Phi_{f,A}\right)^m(X) \geq 0$.

Theorem (1.2.8) [41]: Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m \geq 1$. Let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ be such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is convergent in the weak operator topology and $\Phi_{f,A}$ is power bounded. If $Y = Y^* \in B(\mathcal{H})$ is such that $\left(id - \Phi_{f,A}\right)^m(Y) \geq 0$, then there exist operators $B, C \in B(\mathcal{H})$ with the following properties:

(i) $Y = B + C$;

(ii) $B = B^*$ and $\Phi_{f,A}(B) = B$;

(iii) $C \geq 0, \left(id - \Phi_{f,A}\right)^m(C) \geq 0$, and SOT-$\lim_{k\to\infty} \Phi_{f,A}^k(C) = 0$.

Moreover, the decomposition $Y = B + C$ is unique with the above-mentioned properties and

$$B = \text{SOT} - \lim_{k\to\infty} \Phi_{f,A}^k(Y) = \text{SOT} - \lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} \Phi_{f,A}^j(Y).$$

Proof: Let $Y = Y^* \in B(\mathcal{H})$ be such that $\left(id - \Phi_{f,A}\right)^m(Y) \geq 0$. Since $\Phi_{f,A}$ is power bounded, Lemma (1.2.3) implies $\Phi_{f,A}(Y) \leq Y$. Consequently, the sequence of self-adjoint operators $\left\{\Phi_{f,A}^k(Y)\right\}_{k=0}^\infty$ is bounded and decreasing. Thus it converges strongly to a selfadjoint operator $B := \text{SOT} - \lim_{k\to\infty} \Phi_{f,A}^k(Y)$. Since $\Phi_{f,A}$ is a W OT -continuous map, we have $\Phi_{f,A}(B) = B$. Note that $C := Y - B \geq 0$ satisfies the inequality $\Phi_{f,A}(C) \leq C$, and $\left(id - \Phi_{f,A}\right)^m(C) = \left(id - \Phi_{f,A}\right)^m(Y) \geq 0$. Moreover, $\Phi_{f,A}^k(C) \to 0$ strongly, as $k \to \infty$.

To prove the uniqueness of the decomposition, suppose $Y = B_1 + C_1$, where B1 and C_1 have the same properties as B and C, respectively. Then

$$B - B_1 = \Phi_{f,A}^k(B - B_1) = \Phi_{f,A}^k(C_1 - C), \qquad k \in \mathbb{N}.$$

Taking $k \to \infty$, we get $B = B_1$ and, consequently, $C = C_1$. Since $0 \leq \Phi_{f,A}^k(C) \leq C, k \in \mathbb{N}$, and $\text{SOT} - \lim_{k\to\infty} \Phi_{f,A}^k(C) = 0$, a standard argument shows that $\text{SOT} - \lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} \Phi_{f,A}^j(C) = 0$. On the other hand, since $Y = B + C$ and $\Phi_{f,A}(B) = B$, we infer that

$$\frac{1}{k}\sum_{j=0}^{k-1} \Phi_{f,A}^j(Y) = B + \frac{1}{k}\sum_{j=0}^{k-1} \Phi_{f,A}^j(C).$$

Hence, the result follows. The proof is complete.

We denote by $C_{pure}(f, A)^+$ the set of all operators $D \in B(\mathcal{H})$ such that

$$\left(id - \Phi_{f,A}\right)^s(D) \geq 0, \qquad s = 1, \ldots, m,$$

and $\Phi_{f,A}^k(D) \to 0$ strongly, as $k \to \infty$. Note that such an operator D is always positive.

Theorem (1.2.9) [41]: Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m \geq 1$. Let P be a family of noncommutative polynomials with $\mathcal{N}_p \neq \{0\}$ and let $B := (B_1, \ldots, B_n)$ be the universal model associated with the noncommutative variety $V_{f,p}^m$. If $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ is such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is is SOT-convergent and $p(A_1, \ldots, A_n) = 0,\ p \neq P$, then there is a bijection

$$\Gamma: CP^{w^*}\left(A, V_{f,p}^m\right) \to C_{\text{pure}}(f, A)^+, \qquad \Gamma(\varphi) := \varphi(1),$$

where $CP^{w^*}\left(A, V_{f,p}^m\right)$ is the set of all w^*-continuous completely positive linear maps $\varphi : S_{f,p}^{w^*} \to B(\mathcal{H})$ such that

$$\varphi\left(B_\alpha B_\beta^*\right) = A_\alpha \varphi(I) A_\beta^*, \qquad \alpha, \beta \in \mathbb{F}_n^+,$$

where $S_{f,p}^{w^*} := \overline{\text{span}}^{w^*}\{B_\alpha B_\beta^* : \alpha, \beta \in \mathbb{F}_n^+\}$. In addition, if $D \in C_{\text{pure}}(f, A)^+$, then $\Gamma^{-1}(D)$ coincides with the noncommutative Berezin transform associated with $q := (f, m, A, R, P)$ and defined by

$$\mathbf{B}_q[\chi]:= K_q^*(\chi\otimes I)K_q, \qquad \chi\in S_{f,p}^{w^*},$$

Where $R := \left(id-\Phi_{f,A}\right)^m(D)$.

Moreover, an operator $D\in B(\mathcal{H})$ is in $\mathrm{C}_{\text{pure}}\,(f,A)^+$ if and only if there is a Hilbert space $\mathcal{D}$ and an operator $K:\mathcal{H}\to\mathcal{N}_p\otimes\mathcal{D}$ such that

$$D=K^*K \quad \text{and} \quad KA_i^* = (B_i^*\otimes I_{\mathcal{D}})K, \qquad i=1,\ldots,n.$$

Proof. Assume that $\varphi : S_{f,p}^{w^*}\to B(\mathcal{H})$ is a w^*-continuous completely positive linear map such that

$$\varphi\left(B_\alpha B_\beta^*\right)=A_\alpha\varphi(I)A_\beta^*, \qquad \alpha,\beta\in\mathbb{F}_n^+.$$

Then, setting $D:=\varphi(I)$ and taking into account that $\Phi_{f,B}=\sum_{|\alpha|\geq 1}a_\alpha B_\alpha B_\alpha^*$ is SOT convergent, we deduce that

$$\left(id-\Phi_{f,A}\right)^s(D)=\varphi\left(\left(id-\Phi_{f,B}\right)^s(I)\right)\geq 0, \quad s=1,\ldots,m.$$

On the other hand, recall that $\left\{\Phi_{f,B}^k(I)\right\}_{k=1}^\infty$ is a bounded decreasing sequence of positive operators which converges strongly to 0, as $k\to\infty$. Since $\Phi_{f,A}^k(D)=\varphi\left(\Phi_{f,B}^k(I)\right)$ for all $k\in\mathbb{N}$, one can easily see that $\left\{\Phi_{f,B}^k(D)\right\}_{k=1}^\infty$ is a bounded decreasing sequence of positive operators which converges strongly, as $k\to\infty$. Taking into account that φ is continuous in the w^*-topology, which coincides with the weak operator topology on bounded sets, we deduce that $\Phi_{f,A}^k(D)\to 0$ strongly, as $k\to\infty$. Therefore, $\mathrm{D}\in\mathrm{C}_{\text{pure}}(f,A)^+$. To prove that Γ is one-to-one, let φ_1 and φ_2 be w^*-continuous completely positive linear maps on $S_{f,p}^{w^*}$ such that $\varphi_j\left(B_\alpha B_\beta^*\right)=A_\alpha\varphi_j(I)A_\beta^*, \alpha,\beta\in\mathbb{F}_n^+$,, and assume that $\Gamma(\varphi_1)=\Gamma(\varphi_2)$, i.e., $\varphi_1(I)=\varphi_2(I)$. Then we have $\varphi_1\left(B_\alpha B_\beta^*\right)=\varphi_2\left(B_\alpha B_\beta^*\right)$ for $\alpha,\beta\in\mathbb{F}_n^+$. Since ϕ_1 and ϕ_2 are w$*$-continuous, we deduce that $\varphi_1=\varphi_2$.

We prove now that is a surjective map. Let $\mathrm{D}\in\mathrm{C}_{\text{pure}}(f,A)^+$ be fixed. According to Lemma (1.2.2), the constrained noncommutative Berezin kernel K_q associated with the compatible tuple $q := (f,m,A,R,P)$, has the property that

$$K_qA_i^*=(B_i^*\otimes I_{\mathcal{H}})K_q, \qquad i=1,\ldots,n, \tag{66}$$

where $(B_1,\ldots,B_n)$ is the universal model associated with the noncommutative variety $V_{f,p}^m$, and

$$K_q^*K_q=\sum_{k=0}^\infty\binom{k+m-1}{m-1}\Phi_{f,A}^k(R),$$

where $R:=\left(id-\Phi_{f,A}\right)^m(D)$. As in the proof of Theorem (1.2.5), we can use Lemma (1.2.5) and the fact that WOT-$\lim_{k\to\infty}\Phi_{f,A}^k(D)=0$, to obtain

$$K_q^*K_q=\sum_{k=0}^\infty\binom{k+m-1}{m-1}\Phi_{f,A}^k(R)=D-\mathrm{WOT}-\lim_{k\to\infty}\Phi_{f,A}^k(D)=D$$

Define the operator $\boldsymbol{B}_q: S_{f,p}^{w^*}\to B(\mathcal{H})$by setting

$$\boldsymbol{B}_q(\chi):= K_q^*(\chi\otimes I_{\mathcal{H}})K_q, \qquad \chi\in S_{f,p}^{w^*}.$$

Now, due to relation (66) it is easy to see that

$$\boldsymbol{B}_q\left(B_\alpha B_\beta^*\right)=K_q^*\left(B_\alpha B_\beta^*\otimes I\right)K_q=A_\alpha DA_\beta^*, \qquad \alpha,\beta\in\mathbb{F}_n^+.$$

Consequently, $\boldsymbol{B}_q\in CP^{w^*}(A,V_{f,p}^m)$ has the required properties.

To prove the last part of the theorem, note that the direct implication follows if we take K to be the noncommutative Berezin kernel K_q. To prove the converse, assume that there is a Hilbert space $\mathcal{D}$ and an operator $K:\mathcal{H}\to\mathcal{N}_p\otimes\mathcal{D}$ such that

$$D=K^*K \quad \text{and} \quad KA_i^*=(B_i^*\otimes I_{\mathcal{D}})K, \qquad i=1,\ldots,n.$$

Then

$$\left(id-\Phi_{f,A}\right)^s(D)= K^*\left[\left(id-\Phi_{f,B}\right)^s(I)\otimes I_{\mathcal{D}}\right]K\geq 0, \quad s=1,\ldots,m.$$

Since $\Phi_{f,A}^k(D) = K^*[\Phi_{f,B}^k(I) \otimes I_{\mathcal{D}}]K$, $\|\Phi_{f,B}^k(I)\| \leq 1$, and $\Phi_{f,B}^k(I) \to 0$ strongly, as $k \to 0$, we deduce that $D \in C_{\text{pure}}(f, A)^+$. The proof is complete.

We remark that, in Theorem (1.2.9), the set P is of arbitrary noncommutative polynomials with $\mathcal{N}_p \neq \{0\}$, while, in Theorem (1.2.5), P consists of homogeneous polynomials.

The proof of the next result is similar to that of Corollary (1.2.6).

Corollary (1.2.10) [41]: Under the hypotheses of Theorem (1.2.9), if $D \in C_{\text{pure}}(f, A)^+$, then we have the following von Neumann type inequality:

$$\left\| \sum_{\alpha,\beta\in\Lambda} A_\alpha D A_\beta^* \otimes C_{\alpha,\beta} \right\| \leq \|D\| \left\| \sum_{\alpha,\beta\in\Lambda} B_\alpha B_\beta^* \otimes C_{\alpha,\beta} \right\|$$

for any finite set $\Lambda \subset \mathbb{F}_n^+$and $C_{\alpha,\beta} \in B(\mathcal{E})$, where $\mathcal{E}$ is a Hilbert space.

If, in addition, D is an invertible operator, then the polynomial calculus $p(B_1, \ldots, B_2) \mapsto p(A_1, \ldots, A_n)$extends to a completely bounded map
$u: F_n^\infty(V_{f,p}^m) \to B(\mathcal{H})$ by setting

$$u(\varphi) := K_q^*[\varphi \otimes I_{\mathcal{H}}]K_q D^{-1}, \qquad \varphi \in F_n^\infty(V_{f,p}^m),$$

where K_q is the noncommutative Berezin kernel associated with the compatible tuple $q := (f, m, A, R, P)$ and $R := (id - \Phi_{f,A})m(D)$. Moreover, $\|u\|_{cb} \leq \|D^{-1/2}\|\|D^{1/2}\|$.

Theorem (1.2.11) [41]: Let $f := \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and $m \geq 1$. Let P be a family of noncommutative polynomials and let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^n$ be such that $\sum_{|\alpha|\geq 1} a_\alpha A_\alpha A_\alpha^*$ is SOT-convergent and $p(A_1, \ldots, A_n) = 0$ for $p \in P$. Then a positive operator $G \in B(\mathcal{H})$ is in $C(f, A)^+$
if and only if there exists an n −tuple $T := (T_1, \ldots, T_n) \in V_{f,p}^m(\mathcal{H})$

$$A_i G^{1/2} = G^{1/2} T_i, \quad i = 1, \ldots, n.$$

In addition, $G \in C_{\text{pure}}(f, A)^+$ if and only if $I_{\mathcal{H}} \in C_{\text{pure}}(f, T)^+$.

Proof: First, assume that $T := (T_1, \ldots, T_n) \in V_{f,p}^m(\mathcal{H})$ satisfies $A_i G^{1/2} = G^{1/2} T_i$, for any $i = 1, \ldots, n$. Then we have

$$(id - \Phi_{f,A})^s(G) = G^{\frac{1}{2}}\left[(id - \Phi_{f,T})^s(I)\right] G^{\frac{1}{2}} \geq 0, \qquad s = 1, \ldots, m.$$

Taking into account that $\Phi_{f,A}^k(G) = G^{\frac{1}{2}} \Phi_{f,T}^k(I) G^{1/2}$, $k \in \mathbb{N}$, it is clear that if $\Phi_{f,T}^k(I) \to 0$ strongly, as $k \to \infty$, then $G \in C_{\text{pure}}(f, A)^+$.

To prove the converse, assume that $G \in B(\mathcal{H})$ is in $C(f, A)^+$. Since

$$\sum_{|\alpha|\geq 1} \left\|G^{1/2} \sqrt{a_\alpha} A_\alpha^* x\right\|^2 = \langle \Phi_{f,A}(G)x, x\rangle \leq \left\|G^{1/2} x\right\|^2$$

for any $x \in \mathcal{H}$, we deduce that $\left\|G^{1/2} A_i^* x\right\|^2 \leq \left\|G^{1/2} x\right\|^2$, for any $x \in \mathcal{H}$. Recall that $a_{g_i} \neq 0$, so we can define the operator $\Lambda_i: G^{\frac{1}{2}}(\mathcal{H}) \to G^{1/2}(\mathcal{H})$ by setting

$$\Lambda_i G^{1/2} x := G^{1/2} A_i^* x, \qquad x \in \mathcal{H}, \tag{67}$$

for $i = 1, \ldots, n$. It is obvious that Λ_i can be extended to a bounded operator (also denoted by Λ_i) on the subspace $M := \overline{G^{1/2}(\mathcal{H})}$. Set $M_i := \Lambda_i^*$, $i = 1, \ldots, n$, and note that

$$G^{1/2}\left[(id - \Phi_{f,M})^s(I_M)\right]G^{1/2} = (id - \Phi_{f,A})^s(G) \geq 0, \quad s = 1, \ldots, m.$$

An approximation argument shows that

$$(id - \Phi_{f,M})^s(I_M) \geq 0, \qquad s = 1, \ldots, m.$$

Define $T_i := M_i \oplus 0$, $i = 1, \ldots, n$, with respect to the decomposition $H = M \oplus M^\perp$, and note that $(id - \Phi_{f,T})^s(I) \geq 0$, $s = 1, \ldots, m$. Due to relation (67), if $p \in P$, then we have

$$p(M_1, \ldots, M_n)^* G^{1/2} = G^{\frac{1}{2}} p(A_1, \ldots, A_n)^* = 0.$$

Hence, $p(M_1,\dots,M_n) = 0$ and, consequently, $p(T_1,\dots,T_n) = 0$ for all $p \in P$. Therefore, $(T_1,\dots,T_n) \in V_{f,p}^m(\mathcal{H})$ and $A_i G^{1/2} = G^{1/2} T_i, \quad i = 1,\dots,n$.

Assume now that $G \in \mathrm{C}_{\text{pure}}(f,A)^+$, i.e., $\Phi_{f,A}^k(G) \to 0$ strongly, as $k \to \infty$. Since

$$\langle \Phi_{f,T}^k(I) G^{1/2} x, G^{1/2} x\rangle = \langle \Phi_{f,A}^k(G) x, x\rangle, \qquad x \in \mathcal{H},$$

we have $\text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I) y = 0$ for any $y \in$ range G. Taking into account that $\|\Phi_{f,T}^k(I)\| \le k \in \mathbb{N}$, an approximation argument shows that $\text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I) y = 0$ for any $y \in \overline{G^{1/2}(\mathcal{H})}$. On the other hand, we have $\Phi_{f,T}^k(I) z = 0$ for any $z \in M^\perp$. Consequently, $I_{\mathcal{H}} \in \mathrm{C}_{\text{pure}}(f,T)^+$. This completes the proof.

We consider the case when $m = 1$. Let $f := \sum_{|\alpha|\ge 1} a_\alpha X_\alpha$ be a positive regular free holomorphic function and let P be a family of noncommutative polynomials such that $\mathcal{N}_p \ne \{0\}$. We have

$$\mathbf{D}_f^1(\mathcal{H}) := \{(X_1,\dots,X_n) \in B(\mathcal{H})^n : \sum_{|\alpha|\ge 1} a_\alpha X_\alpha X_\alpha^* \le I\}.$$

Let $B := (B_1,\dots,B_n)$ be the universal model associated with the noncommutative variety $V_{f,p}^1$. We introduced in [48] the noncommutative Hardy algebra $F_n^\infty(V_{f,p}^1)$ to be the w^*-closed algebra generated by $B_1,\dots,B_n$ and the identity. We also showed that $F_n^\infty(V_{f,p}^1) = P_{\mathcal{N}_p} F_n^\infty(\mathbf{D}_f^1)|_{\mathcal{N}_p}$. Similar results hold for $R_n^\infty(V_{f,p}^1)$, the w^*-closed algebra generated by $C_1,\dots,C_n$ and the identity, where $C_i := P_{\mathcal{N}_p} \Lambda_i|_{\mathcal{N}_p}$, and $\Lambda_1,\dots,\Lambda_n$ are the weighted right creation operators associated with $\mathbf{D}_f^1$. Moreover, we proved that

$$F_n^\infty(V_{f,p}^1)' = R_n^\infty(V_{f,p}^1) \quad \text{and} \quad R_n^\infty(V_{f,p}^1)' = F_n^\infty(V_{f,p}^1),$$

where $'$ stands for the commutant. An operator $M \in B(\mathcal{N}_p \otimes K, \mathcal{N}_p \otimes K')$ is called multi-analytic with respect to the constrained weighted shifts $B_1,\dots,B_n$ if

$$M(B_i \otimes I_K) = (B_i \otimes I_{K'})M, \qquad i = 1,\dots,n.$$

According to [48], the set of all multi-analytic operators with respect to $B_1,\dots,B_n$ coincides with

$$R_n^\infty(V_{f,p}^1) \mathbin{\overline{\otimes}} B(K,K') = P_{\mathcal{N}_p\otimes K'} \left[R_n^\infty(V_{f,p}^1) \mathbin{\overline{\otimes}} B(K,K')\right]|_{\mathcal{N}_p\otimes K},$$

and a similar result holds for the Hardy algebra $F_n^\infty(V_{f,p}^1)$. For more information on multi-analytic operators, see [40] and [48].

Theorem (1.2.12) [41]: Let P be a family of noncommutative polynomials with $\mathcal{N}_p \ne \{0\}$ and let $B := (B_1,\dots,B_n)$ be the universal model associated with the noncommutative variety $V_{f,p}^1$, where $f := \sum_{|\alpha|\ge 1} a_\alpha X_\alpha$ is a positive regular free holomorphic function. If $T := (T_1,\dots,T_n)$ is a pure n-tuple of operators in the noncommutative variety $V_{f,p}^1(\mathcal{H})$, then

$$C(f,T)^+ = \mathrm{C}_{\text{pure}}(f,T)^+$$

and any operator in $C(f,T)^+$ has the form $G = P_{\mathcal{H}} \Psi\Psi^*|_{\mathcal{H}}$, where Ψ is a multi-analytic operator with respect to $B_1,\dots,B_n$.

Proof: Assume that $T := (T_1,\dots,T_n)$ is a pure n-tuple of operators in the noncommutative variety $V_{f,p}^1(\mathcal{H})$, i.e., $\Phi_{f,T}^k(I) \to 0$ strongly, as $k \to \infty$. If $G \in C(f,T)^+$, then $G \ge 0$ and $\Phi_{f,T}(G) \le G$. Since

$$0 \le \Phi_{f,T}^k(G) \le \|G\| \Phi_{f,T}^k(I), \qquad k = 1,2,\dots,$$

we infer that $G \in \mathrm{C}_{\text{pure}}(f,T)^+$. Consequently, we have $C(f,T)^+ = \mathrm{C}_{\text{pure}}(f,T)^+$. Now, fix an operator $G \in \mathrm{C}_{\text{pure}}(f,T)^+$. Due to Theorem (1.2.11), we find $D_i \in B(\mathcal{H})$ satisfying

$$T_i G^{1/2} = G^{1/2} D_i, \qquad i = 1,\dots,n,$$

where $(D_1,\dots,D_n) \in V_{f,p}^1(\mathcal{H})$ and $\Phi_{f,D}^k(I) \to 0$ strongly, as $k \to 0$. According to Theorem 3.20 from [48], there is a Hilbert space M_1 so that $(B_1 \otimes I_{M_1},\dots,B_n \otimes I_{M_1})$ is a dilation of $(T_1,\dots,T_n)$ on the Hilbert space $K_1 := \mathcal{N}_p \otimes M_1 \supseteq \mathcal{H}$, i.e., $T_i = P_{\mathcal{H}}(B_i \otimes I_{M_1})|_{\mathcal{H}}$, $i = 1,\dots,n$, and $\mathcal{H}$ is invariant under each operator $B_i^* \otimes I_{M_1}$. Similarly, let $(B_1 \otimes I_{M_2},\dots,B_n \otimes I_{M_2})$ be a dilation of $(D_1,\dots,D_n)$ on a Hilbert space $K_2 := \mathcal{N}_p \otimes M_2 \supseteq \mathcal{H}$. According to the noncommutative commutant lifting theorem from [48] (see Theorem 4.2), there exists an operator $\hat{G}: K_2 \to K_1$ such that $\hat{G}^*(\mathcal{H}) \subset \mathcal{H}, \hat{G}^*|_{\mathcal{H}} = G^{1/2}, \|\hat{G}\| = \|G^{1/2}\|$, and

$$\hat{G}^*(B_i^* \otimes I_{M_1}) = (B_i^* \otimes I_{M_2})\hat{G}^*, \quad i = 1, \dots, n.$$

It is easy to see that

$$\Phi_{f,B\otimes I_{M_1}}(\hat{G}\hat{G}^*) = \hat{G}\Phi_{f,B\otimes I_{M_2}}(I)\hat{G}^* \le \hat{G}\hat{G}^*.$$

Setting Q $:= \hat{G}\hat{G}^*$, we have $\|Q\| = \|G\|$,

$$G = P_{\mathcal{H}}\hat{G}|_{\mathcal{H}} = P_{\mathcal{H}}\hat{G}\hat{G}^*|_{\mathcal{H}} = P_{\mathcal{H}}Q|_{\mathcal{H}}$$

Note also that

$$\Phi^k_{f,B\otimes I_{M_1}}(\hat{G}\hat{G}^*) = \hat{G}\Phi^k_{f,B\otimes I_{M_2}}(I)\hat{G}^*, \quad k \in \mathbb{N}.$$

Since $\Phi^k_{f,B\otimes I_{M_2}}(I) \to 0$ strongly, as $k \to \infty$, we deduce that $\Phi^k_{f,B\otimes I_{M_1}}(\hat{G}\hat{G}^*) \to 0$ strongly. Therefore, $Q \in C_{\text{pure}}(f, B \otimes I_{M_1})^+$ and $G = P_{\mathcal{H}}Q|_{\mathcal{H}}$.

Conversely, if $Q \in C_{\text{pure}}(f, B \otimes I_{M_1})^+$, then

$$\begin{aligned}\Phi_{f,T}(P_{\mathcal{H}}Q|_{\mathcal{H}}) &= \sum_{|\alpha|\ge 1} a_\alpha T_\alpha (P_{\mathcal{H}}Q|_{\mathcal{H}}) T_\alpha^* \\ &= P_{\mathcal{H}}\left[\Phi_{f,W\otimes I_{M_1}}(P_{\mathcal{H}}Q|_{\mathcal{H}})\right]|_{\mathcal{H}} \\ &= P_{\mathcal{H}}\left[\Phi_{f,W\otimes I_{M_1}}(Q)\right]|_{\mathcal{H}} \\ &\le P_{\mathcal{H}}Q|_{\mathcal{H}}\end{aligned}$$

On the other hand, since

$$0 \le \Phi^k_{f,T}(P_{\mathcal{H}}Q|_{\mathcal{H}}) \le P_{\mathcal{H}}\Phi^k_{f,B\otimes I_{M_1}}(Q)|_{\mathcal{H}} \to 0, \quad \text{as } k \to \infty,$$

it is clear that $G := P_{\mathcal{H}}Q|_{\mathcal{H}}$ is in $C_{\text{pure}}(f, T)^+$. We have proved that

$$C_{\text{pure}}(f,T)^+ = P_{\mathcal{H}}\left[C_{\text{pure}}(f, B \otimes I_{M_1})^+\right]|_{\mathcal{H}}.$$

Now, we determine the set $C_{\text{pure}}(f, B \otimes I_{M_1})^+$. To this end, let $Q \in C_{\text{pure}}(f, B \otimes I_{M_1})^+$. According to Theorem (1.2.9), Q ∈ Cpure(f, B ⊗ IM1)+ if and only if there is a Hilbert space $\mathcal{D}$ and an operator $K: \mathcal{N}_p \otimes M_1 \to \mathcal{N}_p \otimes \mathcal{D}$ such that $Q = K^*K$ and

$$(B_i \otimes I_{M_1})K^* = K^*(B_i \otimes I_{\mathcal{D}}), \quad i = 1, \dots, n,$$

i.e., K^* is a multi-analytic operator with respect to $B_1, \dots, B_n$. The proof is complete.

Chapter 2
Deformation Estimates and Identification

We show that the estimate justify the description of CCR $+\mathcal{H}$ as a first-order quantum deformation of $AP + C_0$, where CCR is the usual C*-algebra of (Boson) canonical commutation relations, $\mathcal{H}$ is the full algebra of compact operators, AP is the algebra of almost-periodic functions and C_0 is the algebra of continuous functions which vanish at infinity. Its characteristic class (which classifies star-products up to equivalence) is obtained. The proof is based on the microlocal description of the Szego kernel of a strictly pseudoconvex domain given by Boutet de Monvel and Sjostrand.

Section (2.1): Berezin-Toeplitz Quantization

We consider the family of Gaussian probability measures

$$d\mu_r(z) = \left(\frac{r}{\pi}\right)^n e^{-r|z|^2} dv(z), r > 0$$

for $z=(z_1, \ldots z_n)$ in complex Euclidean space $\mathbb{C}^n$, $d\upsilon(z)$ ordinary Lebesgue measure, $|z|^2 = |z_1|^2 + \cdots |z_n|^n$. The space of entire $d\mu_r$square-integrable functions is denoted by $H^2(d\mu_r) \equiv H^2(\mathbb{C}^n, d\mu_r)$. For g in $L^2(d\mu_r)$, the Berezin-Toeplitz operator $T_g^{(r)}$is defined on a dense linear subspace of $H^2(d\mu_r)$ by

$$\left(T_g^r h\right)(z) = \int g(\omega)h(\omega)e^{rz.\omega} d\mu_r(\omega).$$

Here $.\omega \equiv z_1\widetilde{\omega}_1 + \cdots + z_1\widetilde{\omega}_1$ and $e^{rz.\omega}$is the Bergman reproducing kernel for $H^2(d\mu_r)$ so that, for gh in $L^2(d\mu_r)$, $T_g^r(h)$ is in $H^2(d\mu_r)$.

The map $g \to T_g^{(r)}$ has been considered by Berezin [80] and others [81] as a "quantization" in which r plays the role of the reciprocal of Planck's constant.

In this guise, with [A, B] = AB — BA, the "canonical commutation relations" are given by

$$\left[T_{\bar{z}_j}^{(r)}, T_{z_k}^{(r)}\right] = \frac{1}{r}\delta_{jk} I,$$

where

$$\boldsymbol{\delta_{jk}} = \begin{Bmatrix} \mathbf{1} & \mathbf{j = k} \\ \mathbf{0} & \mathbf{j \neq k} \end{Bmatrix}$$

While

$$\left[T_{\bar{z}_k}^{(r)}, T_{\bar{z}_k}^{(r)}\right] = 0, \left[T_{z_j}^{(r)}, T_{z_k}^{(r)}\right]=0$$

There is an isometry $B_r: L^2(\mathbb{R}^n, dv) \to H^2(\mathbb{C}^n. d\mu_r)$ due to Bargmann [82], so that for sufficiently smooth g,

$$B_r^{-1} T_g^{(r)} B_r$$

is a Weyl pseudo-differential operator.

We establish a first-order composition calculus for $T_g^{(r)}, T_g^{(r)}$ analogous to results of [84] for the Weyl calculus. To obtain such a calculus, it does not seem possible to simple apply conjugation by B_r to the results or the related results of [76]. Instead, we proceed by a combination of direct calculation and an asymptotic analysis analogous to that.

Our results are, in particular, sufficient to justify the description of CCR($\mathbb{C}^n$) + $\varkappa$ as a first-order quantum-deformation of AP($\mathbb{C}^n$) + $\mathbb{C}_0(\mathbb{C}^n)$. Here, CCR($\mathbb{C}^n$) is the standard simple $\mathbb{C}^*$-algebra generated by the canonical commutation relations

[77], $\varkappa$ is the full algebra of compact operators on a separable infinite dimensional Hubert space, $AP(\mathbb{C}^n)$ is the supremum norm closed algebra of almost periodic functions, and $\mathbb{C}_0(\mathbb{C}^n)$ is the supremum norm closure of the compactly supported continuous functions.
We write $TP(\mathbb{C}^n)$ for the algebra of trigonometric polynomials on $\mathbb{C}^n = \mathbb{R}^{2n}$. This algebra is generated by the characters $\chi_a(\omega) = \exp\{i \operatorname{Im} \omega.a\}$. We let $C_c^m(\mathbb{C}^n)$be the algebra of m times continuously differentiable functions with compact support. We have
For f,g in TP+C_c^{2n+6}, r > 0,

$$\left\| T_f^{(r)} T_g^{(r)} - T_{fg}^{(r)} + \frac{1}{r} T^{(r)}_{\Sigma_j (\partial_j f)(\bar{\partial}_j g)} \right\|_{(r)} \le C(f,g) r^{-2}$$

holds for C(f, g) independent of r.
We make use of the maps

$$t_a(z) = z - a, \gamma_a(z) = a - z$$

These maps determine unitary operators on $H^2(d\mu_r)$ and $L^2(d\mu_r)$ given by

$$(U_a^r f)(z) = k_a^{(r)}(z) f(z - a),$$
$$\left(V_a^{(r)} f\right)(z) = k_a^{(r)}(z) f(a - z)$$

Where

$$k_a^{(r)}(z) = e^{r.z.a - r|a|^2/2}$$

is the normalized reproducing kernel for $(V_a^{(r)})^2 = I$. Note that and

$$\mathbf{V_a^{(r)} T_g^{(r)} V_a^{(r)} = T_{go\gamma_a}^{(r)}}$$

We will need
Lemma (2.1.1)[75]: For g bounded and uniformly continuous on $\mathbb{C}^n$ and ε > 0 given, there is an R = R(ε), independent of w, so that

$$\int |g(\omega) - g(\omega - z)| d\mu_r(z) < \varepsilon$$

whenever r > R(ε).
Proof: Note that

$$\int_{|z|>\delta} d\mu_r(z) \le n e^{-r\delta^2/n}$$

By uniform continuity, there is a δ = δ(ε) so that $|g(z_1) - g(z_2)| < \frac{\varepsilon}{2}$whenever $|z_1 - z_2| < \varepsilon$. For this δ, write

$$\int |g(\omega) - g(\omega - z)| d\mu_r(z) = \int_{|z|<\delta} |g(\omega) - g(\omega - z)| d\mu_r(z) + \int_{|z|\ge\delta} |g(\omega) - g(\omega z)| d\mu_r(z)$$
$$< \frac{\varepsilon}{2} \int_{|z|<\delta} d\mu_r(z) + 2\|g\|_\infty \int_{|z|\ge\delta} d\mu_r(z) < \frac{\varepsilon}{2} + 2n\|g\|_\infty e^{-r\delta^2/n}$$

Thus, choosing

$$R(\varepsilon) = -\frac{n}{\delta^2} \ln\left[\frac{\varepsilon}{4n\|g\|_\infty}\right]$$

completes the proof.
We can now prove, similarly to [77] for the disc, that
Theorem (2.1.2) [75]: For g bounded and uniformly continuous on $\mathbb{C}^n$, we have

$$\lim_{r\to\infty} \left\| T_g^{(r)} \right\|_{(r)} = \|g\|_\infty$$

Proof: Write

$$g(\omega) =< T^{(r)}_{go\gamma_\omega} 1,1 >_{(r)} + \int [g(\omega) - g(w - z)] d\mu_r(z)$$

Thus,

$$|g(\omega)| \leq \|T^{(r)}_{go\gamma_\omega}\|_{(r)} + \int |g(\omega) - g(\omega - z)| d\mu_r(z)$$

Using $V^{(r)}_\omega T^{(r)}_g V^{(r)}_\omega = T^{(r)}_{go\gamma_\omega}$, we have

$$|g(\omega)| \leq \|T^{(r)}_g\|_r + \int |g(\omega) - g(\omega - z)| d\mu_r(z)$$

and, by Lemma (2.1.1),

$$\mathbf{|g(\omega)| < \|T^{(r)}_g\|_r + \varepsilon}$$

for r > R(ε). It follows that $\|g\|_\infty - \varepsilon \leq \|T^{(r)}_g\|_{(r)}$. Since $\|T^{(r)}_g\|_{(r)} \leq \|g\|_\infty$ is trivial and ε is arbitrary, the proof is complete.

We consider some differential identities which will be needed later. For f sufficiently smooth on $\mathbb{C}^n$, we write

$$\partial_1^{k_1} \dots \partial_n^{k_n} \bar{\partial}_1^{l_n} \dots \bar{\partial}_n^{l_n} f$$

where $\partial_j \equiv \frac{\partial}{\partial z_j}, \bar{\partial}_j \equiv \frac{\partial}{\partial \bar{z}_j}$; k_j, l_j are non-negative integers. For φ in $H^2(d\mu_r)$, we recall that

$$\left(U^{(r)}_{-\omega}\varphi\right)(a) = \varphi(a + \omega) k^{(r)}_{-\omega}(a)$$

We have

Lemma (2.1.3) [75]: For φ in $H^2(d\mu_r)$,

$$\partial_1^{m_1} \dots \partial_n^{m_n} \left(U^{(r)}_{-\omega}\varphi\right)(0) = e^{r|\omega|^2/2} \partial_1^{m_1} \dots \partial_n^{m_n} \left\{\varphi(\omega) e^{-r|\omega|^2}\right\}.$$

Proof: Direct calculation.

Lemma (2.1.4) [75]: For φ in $H^2(d\mu_r)$ and $m = m_1 + \cdots + m_n$

$$\partial_1^{m_1} \dots \partial_n^{m_n} \varphi(0) = r^m \int \varphi(\omega) \bar{\omega}_1^{m_1} \dots \bar{\omega}_n^{m_n} d\mu_r(\omega)$$

Proof: Write

$$\varphi(a) = \int \varphi(\omega) e^{ra.\omega} d\mu_r(\omega)$$

and check that differentiation "under the integral" is permissible.

Lemma (2.1.5) [75]: $\int |a|^{2k} d\mu_r(a) = b(k, n) r^{-k}$.

We will write BC^m for the set of functions which are bounded and continuous, with all derivatives bounded and continuous up to order m. Clearly, C^m_c is contained in BC^m. For g in $BC^{m+1}(\mathbb{C}^n)$, we will consider the Taylor series

$$g(a + \omega) = g(\omega) + (\partial_1 g)(\omega) a_1 \dots + (\partial_n g)(\omega) a_n + \left(\bar{\partial}_1 g\right)(\omega) \bar{a}_1 + \cdots + \left(\bar{\partial}_1 g\right)(\omega) \bar{a}_n + \cdots + \frac{1}{m!} \left(\bar{\partial}_n^m g\right)(\omega) \bar{a}_n^m + g_{m+1}(a, \omega)$$

Where

$$g_{m+1}(a, \omega) = \sum c(k_1, \dots, l_n) \left(\partial_1^{k_1} \dots \partial_n^{k_n} \bar{\partial}_1^{l_1} \dots \bar{\partial}_n^{l_n} g\right)(\omega^*) a_1^{k_1} \dots a_n^{k_n} \bar{a}_1^{l_1} \dots \bar{a}_n^{l_n}$$

for $k_1 + \cdots + k_n + l_1 + \cdots + l_n = m + 1$. For g in BC^{m+1} , the remainder term $g_{m+1}(a, \omega)$ can be estimated by using

Lemma (2.1.6) [75]: We have

$$|g_{m+1}(a, \omega)| \leq \sum c(k_1, \dots, l_n) \left\| \partial_1^{k_1} \dots \bar{\partial}_n^{l_n} g \right\|_{\infty} |a|^{m+1}$$

Theorem (2.1.7) [75]: Let f be in $C_c^{n+3}(\mathbb{C}^n)$ with g in $BC^{2n+6}(\mathbb{C}^n)$. Then we have a constant $C(f, g)$so that

$$\left\| T_f^{(r)} T_g^{(r)} - T_{fg}^{(r)} + \frac{1}{r} T^{(r)}_{\sum_j (\partial_j f)(\bar{\partial}_j g)} \right\|_{(r)} \leq C(f, g) r^{-2}$$

for all r > **0.**

Proof: Borrowing from [84], we write for φ, ψ in $H^2(d\mu_r)$

,

$$\begin{aligned} \langle T_f T_g \varphi, \psi \rangle &= \int f(\omega)\overline{\psi(\omega)} d\mu_r(\omega) \int e^{r\omega . z} g(z) \varphi(z) d\mu_r(z) \\ &= \int f(\omega)\overline{\psi(\omega)} d\mu_r(\omega) \int e^{r\omega.(a+\omega)} g(a+\omega)\varphi(a+\omega) d\mu_r(a+\omega) \\ &= \int f(\omega)\overline{\psi(\omega)} e^{r|\omega|^2/2} d\mu_r(\omega) \int g(a+\omega) \left(U_{-\omega}^{(r)} \varphi\right)(a) d\mu_r(a) \end{aligned}$$

Next, write

$$g(a+\omega) = \{g(a+\omega) - g_{m+1}(a, \omega) + g_{m+1}(a, \omega)$$

Using Lemmas (2.1.5) and (2.1.6), we check that for m = n + 3,

$$\left| \int f(\omega)\overline{\psi(\omega)}\, e^{r|\omega|^2/2} d\mu_r(\omega) \int g_{m+1}(a, \omega)(\, U_{-\omega}^{(r)} \varphi)(a) d\mu_r(a) \right|$$

$$\leq \|\varphi\|_r \|\psi\|_r C(g) \pi^2 b(m+1, n)^{1/2} \left\{ \int |f(\omega)|^2 dv(\omega) \right\}^{1/2} r^{-2}$$

Thus, for m = n + 3, it remains to consider the expression

$$(\dagger) \int f(\omega)\overline{\psi)(\omega)} e^{r|\omega|^2/2} d\mu_r(\omega) \int \{g(a+\omega) - g_{m+1}(a, \omega)\} \left(U_{-\omega}^{(r)} \varphi\right)(a) d\mu_r(a)$$

For $k = k_1 + \cdots + k_n, l = l_1 + \cdots + l_n$ and k + l < m the typical term in the Expansion of

$$\int \{g(a+\omega) - g_{m+1}(a, \omega)\} \left(U_{-\omega}^{(r)} \varphi\right)(a) d\mu_r(a)$$

has the form

$$(\dagger\dagger) \;\; a(k_1, \dots l_1)(\partial_1^{k_1} \dots \bar{\partial}_n^{l_n} g)(\omega) \int \bar{a}_1^{l_1} \dots \bar{a}_n^{l_n} a_1^{k_1} \dots a_n^{k_n} \left(U_{-\omega}^{(r)} \varphi\right)(a) d\mu_r(a)$$

Applying Lemmas (2.1.3) and (2.1.4), we see that $(\dagger\dagger) = 0$ unless3

$$\mathbf{l_j \geq k_j}$$

for all j . In this case, $(\dagger\dagger)$ is a sum of terms

$$r^{-l} a'(k_1, \dots, l_n) e^{r|\omega|^2/2} \left(\partial_1^{k_1} \dots \bar{\partial}_n^{l_n} g\right)(\omega)\, \partial_1^{t_1} \dots \partial_n^{t_n} \left\{\varphi(\omega) e^{-r|\omega|^2}\right\}$$

with $l_j \geq t_j$.

It follows that $(\dagger\dagger)$is a linear combination, with coefficients independent of r, of Terms

$$(\dagger\dagger\dagger)\quad r^{-l}\int f(\omega)\overline{\psi(\omega)}\left(\partial_1^{k_1}\dots\partial_n^{k_n}\bar{\partial}_1^{l_1}\dots\bar{\partial}_n^{l_n}g\right)(\omega)\times\partial_1^{t_1}\dots\partial_n^{t_n}\{\varphi(\omega)e^{-r|\omega|^2}\}\left(\frac{r}{\pi}\right)^n dn(\omega)$$

Where

$$l_j \geq k_j,\ \ l_j \geq t_j,\ \ m \geq l + k$$

Iterated application of Gauss' Theorem ("integration by parts") shows that $(\dagger\dagger\dagger)$is a linear combination, with coefficients independent of r, of terms

$$r^{-l}\int(\partial_1^{u_1}\dots\partial_n^{u_n}f)(\omega)(\partial_1^{k_1+s_1}\dots\partial_n^{k_n+s_n}\bar{\partial}_1^{l_1}\dots\bar{\partial}_n^{l_n}g)(\omega)\varphi(\omega)\overline{\psi(\omega)}d\mu_r(\omega)$$

where

$$t_j \geq u_{\jmath},\ \ t_j \geq \delta_{\jmath}.$$

Thus, for $l > 1$, we have explicit estimates.

It remains to consider the cases $l = 0, l = 1$. Going back to $(\dagger), (\dagger\dagger)$, we see that the only $l = 0$ term is

$$\int f(\omega)\overline{\psi(\omega)}e^{r|\omega|^2/2}\,d\mu_r(\omega)\int g(\omega)\left(U_{-\omega}^{(r)}\varphi\right)(a)d\mu_r(a) = \int f(\omega)g(\omega)\overline{\psi(\omega)}\varphi(\omega)d\mu_r(\omega)$$
$$= \langle T_{fg}^{(r)}\varphi,\psi\rangle_{(r)}$$

For $l = 1$, we can have, for some j with $1 \leq j \leq n$

$$\begin{Bmatrix} l_j = 1, l_{j'} = 0 & j' \neq j \\ k_j = 1, k_{j'} = 0 & j' \neq j \end{Bmatrix}$$

Or

$$\begin{Bmatrix} l_j = 1, l_{\jmath'} = 0 & j' \neq j \\ k_j = 0, k_{\jmath'} = 0 & j' \neq j \end{Bmatrix}$$

In either case, $a(k_1, \dots, l_n)$ in $(\dagger\dagger)$. Direct calculation now shows that the sum of the $l = 1$ terms is

$$-r^{-1}\int\varphi(\omega)\overline{\psi(\omega)}\left\{\sum_j(\partial_{\jmath}f)(\bar{\partial}_{\jmath}g)\right\}d\mu_r(\omega)$$

This completes the proof!

For each a in $\mathbb{C}^n$, we have the character

$$\chi_a(\omega) = \exp\{i\ \mathrm{Im}\ \ \omega.a\}$$

The algebra $TP(\mathbb{C}^n)$ consists of finite linear combinations of characters. The supremum norm closure of $TP(\mathbb{C}^n)$ is exactly $AP(\mathbb{C}^n)$. We also consider the algebra $TP + C_c^{2n+6}$. Clearly,

$$TP + C_c^{2n+6} \subset BC^{2n+6}.$$

Lemma (2.1.8) [75]: For g in $+C_c^{2n+6}$, the representation

$$g = t + u$$

with t in TP and u in C_c^{2n+6} is unique.

Proof: On $TP + C_c^{2n+6}$, we consider the functional

$$m(g) = \lim_{T\to\infty}(2T)^{-2n}\int_{-T}^{T}\dots\int_{-T}^{T}g(x_1,y_1,x_2,y_2\dots)dx_1dy_1\dots dy_n$$

where $z_{\jmath} = x_{\jmath} + iy_{\jmath}, x_{\jmath}, y_{\jmath}$real. It is easy to check that

m(u) = 0

while, for

$$t = \sum_{k-1}^{r} c_k \chi_{ak}, m\{g\bar{\chi}_{ak}\} = c_k$$

Uniqueness follows.

Theorem (2.1.9) [75]: For f, g in $TP + C_c^{2n+6}$ there is a constant $c(f, g)$so that for all r >0,

$$\left\| T_f^{(r)} T_g^{(r)} - T_{fg}^{(r)} + \tfrac{1}{r} T_{\Sigma(\partial_J f)(\bar{\partial}_J g)}^{(r)} \right\|_{(r)} \leq C(f, g) r^{-2}.$$

Proof: For $f = t_1 + u_1, g = t_2 + u_2$with t_J in TP and u_j in C_c^{2n+6}, it will suffice to check that each of the pairs $(t_1, t_2), (t_1, u_2), (u_1, t_2), (u_1, u_2)$ satisfy (*).

The pairs $(u_1, t_2), (u_1, u_2)$ are handled using Theorem(2.1.7). For (t_1, u_2), we note that $T_F^* = T_{\bar{F}}$and $\bar{\partial}_j \bar{F} = \overline{\partial_j F}$ for F in BC^{2n+6}so that

$$\left(T_{t_1}^{(r)} T_{u_2}^{(r)} - T_{t_1 u_2}^{(r)} + \tfrac{1}{r} T_{\Sigma(\partial_J t_1)(\bar{\partial}_J u_2)}^{(r)} \right)^* = T_{\bar{u}_2}^{(r)} T_{\bar{t}_1}^{(r)} - T_{\bar{u}_2 \bar{t}_1}^{(r)} + \tfrac{1}{r} T_{\Sigma(\partial_j \bar{u}_2)(\bar{\partial}_J \bar{t}_1)}^{(r)}$$

Since $\|A^*\|_{(r)} = \|A\|_r$ and $(\bar{u}_2, \bar{t}_1)$ has been handled using Theorem(2.1.7), (*) holds automatically for (t_1, u_2).

The proof is now reduced to checking (*) for (t_1, t_2). By linearity, this is, in turn, reduced to checking (*) in the case (χ_a, χ_b). Direct calculation shows that

$$T_{\chi_a}^{(r)} T_{\chi_b}^{(r)} = \exp\{b.a/4r\} T_{\chi_{a+b}}^{(r)},$$
$$\left\| T_{\chi_a}^{(r)} \right\|_{(r)} = \exp\{-|a|^2/8r\}$$

It follows that

$$\left\| T_{\chi_a}^{(r)} T_{\chi_b}^{(r)} - T_{\chi_a \chi_b}^{(r)} + \frac{1}{r} T_{\Sigma_j(\partial_J \chi_a)(\bar{\partial}_J \chi_b)}^{(r)} \right\|_{(r)} = \left| e^{b.a/4r} - 1 - \frac{b.a}{4r} \right| \exp|-|a+b|^2/8r|$$

and routine calculation now establishes (*).

Forf, g in $+C_c^{2n+6}$,

$$\left\| \left[T_f^{(r)}, T_g^{(r)} \right] - \frac{i}{r} T_{\{f,g\}}^{(r)} \right\|_{(r)} \leq 2C(f, g) r^{-2} \qquad (**)$$

Considering the above results and the correponding results of [78] for the hyperbolic disc, it is plausible that, in a very general framework involving Toeplitz operators on Bergman spaces, the appropriate generalization of (**) holds.

Section (2.2): Berezin-Toeplitz Deformation Quantization

In [91] Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer drew the attention of both physical and mathematical communities to a well posed mathematical problem of describing and classifying up to some natural equivalence the formal associative differential deformations of the algebra of smooth functions on a manifold.

The deformed associative product is traditionally denoted $\star$ and called star-product. If the manifold carries a Poisson structure, or a symplectic structure (i.e. a nondegenerate Poisson structure) or even more specific if the manifold is a K¨ahler manifold with symplectic structure coming from the K¨ahler form one naturally asks for a deformation of the algebra of smooth functions in the “direction” of the given Poisson structure. According to [91] this deformation is treated as a quantization of the corresponding Poisson manifold .Due to work of De Wilde and Lecomte [94] , Fedosove [98] , and Omori, Maeda and Yoshioka [92] it is known that every symplectic manifold admits a deformation quantizationin this sense. The deformation quantizations for a fixed symplectic structure can be classified up to equivalence by formal power series with coefficients in two-dimensional cohomology of the underlying manifold, see [95], [96], [97], [91], [90]. Kontsevich [97] showed that every Poisson manifold admits a deformation quantization and that the equivalence classes of deformation quantizations on a Poisson manifold can be parametrized by the formal deformations of the Poisson structure.

Despite the general existence and classification theorems it is of importance to study deformation quantization for manifolds with additional geometric structure and ask for deformation quantizations respecting in a certain sense this additional structure. Examples of this additional structure are the structure of a complex manifold or symmetries of the manifold.

Another natural question is how some naturally defined deformation quantizations fit into the classification of all deformation quantizations.

We will deal with Kähler manifolds. Quantization of Kähler manifolds via symbol algebras was considered by Berezin in the framework of his quantization program developed in [93],[94]. In this program Berezin considered symbol algebras with the symbol product depending on a small parameter $\hbar$ which has a prescribed semiclassical behavior as $\hbar \to 0$. To this end he introduced the covariantand contravariant symbols on Kähler manifolds. However, in order to study quantization via symbol algebras on Kähler manifolds he, as well as most of his successors, was forced to consider Kähler manifolds which satisfy very restrictive analytic conditions. These conditions were shown to be met by certain classes of homogeneous Kähler manifolds, e.g., $\mathbb{C}^n$, generalized flag manifolds, Hermitian symmetric domains etc. The deformation quantization obtained from the asymptotic expansion in $\hbar$ as $\hbar \to 0$ of the product of Berezin's covariant symbols on these classes of Kähler manifolds was studied in a number by Moreno, Ortega-Navarro ([99], [90]); Cahen, Gutt, Rawnsley ([101], [102], [103]); see also [105]. This deformation quantization is differential and respects the separation of variables into holomorphic and anti-holomorphic ones in the sense that left star-multiplication (i.e. the multiplication with respect to the deformed product) with local holomorphic functions is pointwise multiplication, and right star-multiplication with local anti-holomorphic functions is also point-wise multiplication,for the precise definition. It was shown in [102] that such deformation quantizations "with separation of variables" exist for every Kähler manifold. Moreover, a complete classification (not only up to equivalence) of all differential deformation quantizations with separation of variables was given. They are parameterized by formal closed forms of type (1, 1). The basic results are sketched below. Independently a similar existence theorem was proven by Bordemann and Waldmann [97] along the lines of Fedosov's construction. The corresponding classifying (1,1)-form was calculated in [96]. Yet another construction was given by Reshetikhin and Takhtajan in [94]. They directly derive it from Berezin's integral formulas which are treated formally, i.e., with the use of the formal method of stationary phase. The classifying form of deformation quantization from [94] can be easily obtained by the methods developed.

In [96] Engliš obtained asymptotic expansion of Berezin transform on a quite general class of complex domains which do not satisfy the conditions imposed by Berezin.

For general compact Kähler manifolds (M, ω_{-1}) which are quantizable, i.e. admit a quantum line bundle L it was shown by Bordemann, Meinrenken and Schlichenmaier [96] that the correspondence between the Berezin-Toeplitz operators and their contravariant symbols associated to L^m has the correct semi-classical behavior as m $\to \infty$. Moreover, it was shown in [95],[96], [98] that it is possible to define a deformation quantization via this correspondence. For this purpose one can not use the product of contravariant symbols since in general it can not be correctly defined.

The approach of [96] was based on the theory of generalized Toeplitz operators due to Boutet de Monvel and Guillemin [98], which was also used by Guillemin [99] in his proof of the existence of deformation quantizations on compact symplectic manifolds. The deformation quantization obtained in [95],[96], which we call the Berezin-Toeplitz deformation quantization, is defined in a natural way related to the complex structure. It fulfils the condition to be 'null on constants' (i.e. $1 \star g = g \star 1 = g$), it is self-adjoint (i.e. $\overline{f * g} = \bar{g} * \bar{f}$), and admits a trace of certain type (see [98] . We will show that the Berezin-Toeplitz deformation quantization is differential and has the property of

separation of variables, though with the roles of holomorphic and antiholomorphic variables swapped. To comply with the conventions of [82] we consider the opposite to the Berezin-Toeplitz deformation quantization (i.e., the deformation quantization with the opposite star-product) which is a deformation quantization with separation of variables in the usual sense.
We will show how the Berezin-Toeplitz deformation quantization fits into the classification scheme of [82]. Namely, we will show that the classifying formal (1,1)-form of its opposite deformation quantization is

$$\tilde{\omega} = -\frac{1}{v}\omega_{-1}\omega_{can} \qquad (1)$$

where ν is the formal parameter, ω_{-1} is the Kähler form we started with and ω_{can} is the closed curvature (1,1)-form of the canonical line bundle of M with the Hermitian fibre metric determined by the symplectic volume. Using [103] and (1) we will calculate the classifying cohomology class (classifying up to equivalence) of the Berezin-Toeplitz deformation quantization. This class was first calculated by E. Hawkins in [100] by Ktheoretic methods with the use of the index theorem for deformation quantization ([107], [101]).
In deformation quantization with separation of variables an important role is played by the formal Berezin transform $f \mapsto I(f)$ (see [104]). we associate to a deformation quantization with separation of variables also a non-associative "formal twisted product" (f, g) ↦ Q(f, g). Here the images are always in the formal power series over the space $C^{\infty}(M)$. In the compact Kähler case by considering all tensor powers L^m of the line bundle L and with the help of Berezin-Rawnsley's coherent states [103], it is possible to introduce for every level m the Berezin transform $L^{(m)}$ and also some "twisted product" $Q^{(m)}$. The key result is that the analytic asymptotic expansions of $I^{(m)}$, resp. of $Q^{(m)}$ define formal objects which coincide with I and Q for some deformation quantization with separation of variables whose classifying form ω̃ is completely determined in terms of the form $\widetilde{\omega}$ (Theorem (2.2.13)). To prove this we use the integral representation of the Szegö kernel on a strictly pseudoconvex domain obtained by Boutet de Monvel and Sjöstrand in [109] and a theorem by Zelditch [101] based on[109].
We also use the method of stationary phase and introduce its formal counterpart which we call "formal integral".
Since the analytic Berezin transform $I^{(m)}$ has the asymptotics given by the formal
Berezin transform it follows also that the former has the expansion

$$I^{(m)} = id + \frac{1}{m}\Delta + O\left(\frac{1}{m^2}\right) \qquad (2)$$

where Δ is the Laplace-Beltrami operator on M.
The above formal form ω is the formal object corresponding to the asymptotic expansion of the pullback of the Fubini-Study form via Kodaira embedding of M into the projective space related to $L^{(m)}$as m → +∞. This asymptotic expansion was obtained by Zelditch in [101] as a generalization of a theorem by Tian [109].
We recall the basic notions of deformation quantization and the construction of the deformation quantization with separation of variables given by a formal deformation of a (pseudo-)Kähler form.
formal integrals are introduced. Certain basic properties, like uniqueness are shown.
the covariant and contravariant symbols are introduced. Using Berezin-Toeplitz operators the transformation $I^{(m)}$ and the twisted product $Q^{(m)}$ are introduced. Integral formulas for them using 2-point, resp. cyclic 3-point functions defined via the scalar product of coherent states are given. contains the key result that $I^{(m)}$ and $Q^{(m)}$admit a well-defined asymptotic expansion and that the formal objects corresponding to these expansions are given by I and Q respectively. Finally the Berezin-Toeplitz star product is identified with the help of the results obtained.
Given a vector space V , we call the elements of the space of formal Laurent series

with a finite principal part V $[v_{-1}, v]]$ formal vectors. In such a way we define formal functions, differential forms, differential operators, etc. However we shall often call these formal objects just functions, operators, and so on, omitting the word formal. Now assume that V is a Hausdorff topological vector space and $v(m), m \in \mathbb{R}$ is a family of vectors in V which admits an asymptotic expansion as $m \to \infty$, $v(m) \sim \sum_{r \geq r_0}(1/m^r)v_r$, where $r_0 \in \mathbb{Z}$. In order to associate to such asymptotic families the corresponding formal vectors we use the "formalizer" $\mathbb{F}: v(m) \mapsto \sum_{r \geq r_0} v^r v_r \in V[v^{-1}, v]]$.

Let (M, ω_{-1}) be a real symplectic manifold of dimension $2n$. For any open subset U $\subset$ M denote by $\mathcal{F}(U) = C^\infty(U)[v^{-1}, v]]$ the space of formal smooth complex-valued functions on U. Set $\mathcal{F} = \boldsymbol{\mathcal{F}}(\boldsymbol{M})$. Denote by $\mathbb{K} = \mathbb{C}[v^{-1}, v]]$ the field of formal numbers.

A deformation quantization on (M, ω_{-1}) is an associative $\mathbb{K}$-algebra structure on $\boldsymbol{\mathcal{F}}$, with the product $\star$ (named star-product) given for $f = \sum v^j f_j, g = \sum v^k g_k \in \mathcal{F}$ by the following formula:

$$f * g = \sum_r v^r \sum_{i+j+k=r} C_i(f_j, g_k) \quad (3)$$

In (3) C_r, $r = 0, 1, \ldots$, is a sequence of bilinear mappings $C_r: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ where $C_0(\varphi, \psi) = \varphi\psi$and $C_1(\varphi, \psi) - C_1(\varphi, \psi) = i\{\varphi, \psi\}$for $\varphi, \psi \in C^\infty(M)$ and $\{\cdot, \cdot\}$ is the Poisson bracket corresponding to the form ω_{-1}.

Two deformation quantizations $(\mathcal{F}, *_1)$ and $(\mathcal{F}, *_2)$ on (M, ω_{-1}) are called equivalent if there exists an isomorphism of algebras $B: (\mathcal{F}, *_1) \to (\mathcal{F}, *_2)$ of the form $B = 1 + vB_1 + v^2 B_2 + \cdots$, where B_k are linear endomorphisms of $C^\infty(M)$.

We shall consider only those deformation quantizations for which the unit constant 1 is the unit in the algebra $(\mathcal{F}, \star)$.

If all $C_r, r \geq 0$, are local, i.e., bidifferential operators, then the deformation quantization is called differential. The equivalence classes of differential deformation quantizations on (M, ω_{-1}) are bijectively parametrized by the formal cohomology classes from $(1/iv)[\omega_{-1}] + H^2(M, \mathbb{C}[[v]])$. The formal cohomology class parametrizing a star-product $\star$ is called the characteristic class of this star-product and denoted $cl(*)$.

A differential deformation quantization can be localized on any open subset $U \subset M$.

The corresponding star-product on (U) will be denoted also $\star$.

For $f, g \in \boldsymbol{\mathcal{F}}$denote by L_f, R_g the operators of left and right multiplication by f, g respectively in the algebra $(\mathcal{F}, \star)$, so that $L_f g = f * g = R_g f$. The associativity of the star-product $\star$ is equivalent to the fact that L_f commutes with $R_g$for all $f, g \in \mathcal{F}$. If a deformation quantization is differential then $L_f, R_g$are formal differential operators. Now let (M, ω_{-1}) be pseudo-Kähler, i.e., a complex manifold such that the form ω_{-1} is of type (1,1) with respect to the complex structure. We say that a differential deformation quantization $(\mathcal{F}, \star)$ is a deformation quantization with separation of variables if for any open subset $U \subset M$ and any holomorphic function a and antiholomorphic function b on U the operators L_a and R_b are the operators of point-wise multiplication by a and b respectively, i.e., $L_a = a$ and $R_b = b$.

A formal form $\omega = (1/v)\omega_{-1} + \omega_0 + v\omega_1 + \cdots$ is called a formal deformation of the form $(1/v)\omega_{-1}$ if the forms $\omega_r, r \geq 0$, are closed but not necessarily nondegenerate (1,1)-forms on M.

It was shown in [102] that all deformation quantizations with separation of variables on a pseudo-Kähler manifold (M, ω_{-1}) are bijectively parametrized by the formal deformations of the form $(1/v)\omega_{-1}$.

Recall how the star-product with separation of variables $\star$ on M corresponding to the formal form $\omega = (1/v)\omega_{-1} + \omega_0 + v\omega_1 + \cdots$is constructed. For an arbitrary contractible coordinate chart $U \subset M$ with holomorphic coordinates $\{\mathcal{Z}^k\}$let$\phi = (1/v)\phi_{-1} + \phi_0 + v\phi_1 + \cdots$ be a formal potential of the form ω on U, i.e., $\omega = -i\partial\overline{\partial}\phi$ (notice that in [102] - [106] a potential ϕ of a closed (1,1)-form ω is defined via the formula $\omega = i\partial\bar{\partial}\phi$).

The star-product corresponding to the form ω is such that $L_{\partial\phi/\partial\mathcal{Z}^k} = \partial\phi/\partial\mathcal{Z}^k + \partial/\partial\mathcal{Z}^k$ and $R_{\partial\phi/\partial\mathcal{Z}^{-l}} = \partial\phi/\partial\mathcal{Z}^{-l} + \partial/\partial\mathcal{Z}^{-l}$ on U. The set $\mathcal{L}(U)$ of all left multiplication operators on U is completely described as the set of all formal differential operators commuting with the point-wise multiplication operators by antiholomorphic coordinates $R_{\mathcal{Z}^{-l}} = \mathcal{Z}^{-l}$ and the operators $R_{\partial\phi/\partial\mathcal{Z}^{-l}} = \partial\phi/\partial\mathcal{Z}^{-l} + \partial/\partial\mathcal{Z}^{-l}$. One can immediately reconstruct the star-product on $U$from the knowledge of $\mathcal{L}(U)$. The local star-products agree on the intersections of the charts and define the global star-product $\star$ on M.

One can express the characteristic class $cl(*)$of the star-product with separation of variables $\star$ parametrized by the formal form ω in terms of this form (see [103]).

Unfortunately, there were wrong signs in the formula for $cl(*)$ in [103] which should be read as follows:

$$cl(*) = (1/i)([\omega] - \varepsilon/2) \quad (4)$$

where ε is the canonical class of the complex manifold M, i.e., the first Chern class of the canonical holomorphic line bundle on M.

Given a deformation quantization with separation of variables ($\mathcal{F}$, $\star$) on the pseudo-Kähler manifold (M, ω_{-1}), one can introduce the *formal Berezin transform I* as the unique formal differential operator on M such that for any open subset U ⊂ M, holomorphic function a and antiholomorphic function b on U the relation $I(ab) = b * a$ holds (see [104]). One can check that $I = 1 + v\Delta + \cdots$, whereΔ is the Laplace-Beltrami operator corresponding to the pseudo-Kähler metric on M. The *dual* star-product $\tilde{*}$ on M defined for f, g ∈ $\mathcal{F}$ by the formula $f \,\tilde{*}\, g = I^{-1}(Ig * If)$ is a star-product with separation of variables on the pseudo-Kähler manifold $(M, -\omega_{-1})$. For this deformation quantization the formal Berezin transform equals I^{-1}, and thus the dual to $\tilde{*}$ is again *.

Denote by $\tilde{\omega} = -(1/v)\omega_{-1} + \tilde{\omega}_0 + v\tilde{\omega}_1 + \ldots$the formal form parametrizing the star-product $\tilde{*}$. The opposite to the dual star-product, $\star' = \tilde{*}^{op}$op, given by the formula $f\star'$ g $=I^{-1}(If * Ig)$, also defines a deformation quantization with separation of variables on M but with the roles of holomorphic and antiholomorphic variables swapped. Differently said, ($\mathcal{F}$,$\star$) is a deformation quantization with separation of variables on the pseudo- Kähler manifold (M, ω_{-1}) where M is the manifold M with the opposite complex structure. The formal Berezin transform I establishes an equivalence of deformation quantizations ($\mathcal{F}$, $\star$) and ($\mathcal{F}$, $\star'$).

Introduce the following non-associative operation Q(·, ·) on $\mathcal{F}$. For $f, g \in \mathcal{F}$ set $Q(f,g)If * Ig = I(f \,\acute{*}\, g) = I(g \,\acute{*}\, f)$. We shall call it formal twisted product. The importance of the formal twisted product will be revealed later.

A trace density of a deformation quantization ($\mathcal{F}$, $\star$) on a symplectic manifold M is a formal volume form μ on M for which the functional $\kappa(f) = \int_M f\mu, f \in \mathcal{F}$, has the trace property, κ(f $\star$g) = κ(g $\star$f) for all f, $g \in \mathcal{F}$ where at least one of the functions f, g has compact support. It was shown in [104] that on a local holomorphic chart $(U, \{\mathcal{Z}^k\})$ any formal trace density μ can be represented in the form $c(v)\exp(\phi\Psi)\, d\mathcal{Z}d\bar{\mathcal{Z}}$, where $c(v) \in \mathbb{K}$ is a formal constant, $d\mathcal{Z}d\bar{\mathcal{Z}} = d\mathcal{Z}^1 \ldots d\mathcal{Z}^n d\mathcal{Z}^{-1} \ldots d\mathcal{Z}^{-n}$ is the standard volume on U and $\phi = (1/v)\phi_{-1} + \cdots, \Psi = (1/v)\Psi_{-1} + \cdots$are formal potentials of the forms ω, $\tilde{\omega}$ respectively such that the relations

$$\partial\phi/\partial\mathcal{Z}^k = -I(\partial\Psi/\partial\mathcal{Z}^k), \partial\phi/\partial\mathcal{Z}^{-l} = -I(\partial\,\Psi/\partial\mathcal{Z}^{-l}), and \phi_{-1} + \Psi_{-1} = 0 \quad (5)$$

hold. Vice versa, any such form is a formal trace density.

Let $\Phi = (1/v)\Phi_{-1} + \Phi_0 + v\Phi_1 + \cdots$and $\mu = \mu_0 + v\mu_1 + \cdots$be, respectively, a smooth complex-valued formal function and a smooth formal volume form on an open set U ⊂ $\mathbb{R}^n$. Assume that $x \in$ U is a nondegenerate critical point of the function Φ_{-1} and μ_0 does not vanish at x. We call a $\mathbb{K}$-linear functional K on $\mathcal{F}(U)$ such that

(a) $K = K_0 + vK_1 + \cdots$ is a formal distribution supported at the point x;

(b) $K_0 = \delta_x$is the Dirac distribution at the point x;

(c)$K(1) = 1$ (normalization condition);

(d) for any vector field ξ on U and $f \in \mathcal{F}(U)$ $K\big(\xi f + (\xi\Phi + div_\mu\xi)f\big) = 0$, a (normalized) *formal integral at the point x associated to the pair* (∅, μ).

It is clear from the definition that a formal integral at a point x is independent of a particular choice of the neighborhood U and is actually associated to the germs of $(\emptyset, \mu)$ at x. Usually we shall consider a contractible neighborhood U such that μ_0 vanishes nowhere on U.
We shall prove that a formal integral at the point x associated to the pair $(\emptyset, \mu)$ is uniquely determined. One can also show the existence of such a formal integral, but this fact will neither be used nor proved in what follows.
We call two pairs (∅, μ) and ($\acute{\emptyset}$, μ′) equivalent if there exists a formal function $u = u_0 + \nu u_1 + \cdots$ on Usuch that, $\emptyset^{'} = \emptyset -$ u, μ′ = $e^{u}\mu$.
Since the expression $\xi\emptyset + div_{\mu}\xi$remains invariant if we replace the pair (∅, μ) by an equivalent one, a formal integral is actually associated to the equivalence class of the pair $(\emptyset, \mu)$. This means that a formal integral actually depends on the product $e^{\emptyset}\mu$ which can be thought of as a part of the integrand of a ”formal oscillatory integral”. It will be shown that one can directly produce formal integrals from the method of stationary phase.
Notice that if K is a formal integral associated to a pair (∅, μ) it is then associated to any pair $(\emptyset, c(\nu)\mu)$, where $c(\nu)$ is a nonzero formal constant.
It is easy to show that it is enough to check condition (d) for the coordinate vector fields $\partial/\partial x^k$ on U. Moreover, if U is contractible and such that μ_0 vanishes nowhere on it, one can choose an equivalent pair of the form ($\acute{\emptyset}$, dx), where $dx = dx^1 \ldots dx^n$ is the standard volume form.
Proposition (2.2.1)[83]: *A formal integral* $K = K_0 + \nu K_1 + \cdots$ *at a point* x*, associated to a pair* (Φ = (1/ν)Φ_{-1} + Φ_0 + $\nu\Phi_1 . . .$, μ) *is uniquely determined.*
Proof: We assume that K is defined on a coordinate chart $(U, \{x^k\})$, μ = dx, and take $f \in C^{\infty}(U)$. Since $div_{dx}(\partial/\partial x^k) = 0$, the last condition of the definition of a formal integral takes the form

$$K(\partial f/\partial x^k + (\partial\Phi/\partial x^k)f) = 0 \qquad (6).$$

Equating to zero the coefficient at ,ν^r r ≥ 0, of the *l.h.s.* of (6) we get $K_r(\partial f/\partial x^k) + \sum_{s=0}^{r+1} K_s((\partial\Phi_{r-s}/\partial x^k)f) = 0$, which can be rewritten as a recurrent equation

$$K_{r+1}((\partial\Phi_{-1}/\partial x^k)f) = r.h.s \qquad (7).$$

depending on K_j , j ≤ r.
Since x is a nondegenerate critical point of Φ_{-1}, the functions $\partial\Phi_{-1}/\partial x^k$ generate the ideal of functions vanishing at x. Taking into account that $K_{r+1}(1) = 0$ for r ≥0 we see from (7) that K_{r+1} is determined uniquely. Thus the proof proceeds by induction.
Let V be an open subset of a complex manifold M and Z be a relatively closed subset of V . A function $f \in C^{\infty}(V)$ is called almost analytic at Z if $\bar{\partial}$f vanishes to infinite order there. Two functions $f_1, f_2 \in C^{\infty}(V)$ are called equivalent at Z if $f_1 - f_2$ vanishes to infinite order there.
Consider open subsets U ⊂ $\mathbb{R}^n$ and $\widetilde{U} \subset \mathbb{C}^n$ such that $U = \widetilde{U} \cap \mathbb{R}^n$, and a function $f \in C^{\infty}(U)$. A function $\tilde{f} \in C^{\infty}(\widetilde{U})$ is called an almost analytic extension of $f$if it is almost analytic at $U$and $\tilde{f}/U$.
It is well known that every $f \in C^{\infty}(U)$ has an almost analytic extension uniquely determined up to equivalence.
Fix a formal deformation $\omega = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \cdots$ of the form $(1/\nu)\omega_{-1}$ on a pseudo-Kähler manifold (M, ω_{-1}). Consider the corresponding star-product with separation of variables ⋆, the formal Berezin transform I and the formal twisted product Q on M. We are going to show that for any point $x \in M$ the functional $K_x^1(f) = (If)(x)$ on $\boldsymbol{\mathcal{F}}$ and the functional K_x^Qon (M ×M) such that K_x^Q (f ⊗ g) = Q(f, g)(x) can be represented as formal integrals.
Let U ⊂ M be a contractible coordinate chart with holomorphic coordinates $\{\mathcal{Z}^k\}$Given a smooth function $f = (z, \bar{z})$ on U, where U is considered as the diagonal of $\widetilde{U} = U \times \overline{U}$, one can choose its almost analytic extension $\tilde{f}(z_1, \bar{z}_1, z_2, \bar{z}_2)$ on $\widetilde{U}$, so that $\tilde{f}(z, \bar{z}, z, \bar{z}) = f(z, \bar{z})$ It is a substitute of the *holomorphic* function $f(z_1, \bar{z}_2)$ on $\widetilde{U}$ which in general does not exist.

Let $\phi = (1/v)\phi_{-1} + \phi_0 + v\phi_1 + \cdots$ be a formal potential of the form ω on U
and $\tilde{\phi}$ its almost analytic extension on $\tilde{U}$. In particular , $\tilde{\phi}(x,x) = \phi(x)$ for$x \in U$. Introduce an analogue of the Calabi diastatic function on U × U by the formula
$D(x,y) = \tilde{\phi}(x,y) + \tilde{\phi}(y,x) - \phi(x) - \phi(y)$. We shall also use the notation
$D_k(x,y) = \tilde{\phi}_k(x,y) + \tilde{\phi}_k(y,x) - \phi_k(x) - \phi_k(y)$ so that $D = (1/v)D_{-1} + D_0 + vD_1$
Let $\tilde{\omega}$ be the formal form corresponding to the dual star-product $\tilde{\star}$ of the star-product ⋆. Choose a formal potential of the form $\tilde{\omega}$ on U, satisfying equation (5), so that $\mu_{tr} = e^{\Phi+\Psi} dz d\tilde{z}$is a formal trace density of the star-product ⋆ on U.
Theorem (2.2.2) [83]: *For any point* x ∈ U *the functional* $K_x^1(f) = (If)(x)$ *on* $\mathcal{F}(U)$*is the formal integral at* x *associated to the pair* $(\emptyset^x, \mu_{tr})$*, where* $\emptyset^x(y) = D(x,y)$*.*
Lemma (2.2.3) [83]: *For any vector field* ξ*on* U *and* $x \in U$ $I(\xi_x \emptyset^x)(x) = 0$*, where* $\emptyset^x(y) = D(x,y)$*.*
($\xi_x \emptyset^x$denotes differentiation of $\emptyset^x$ w.r.t. the parameter *x*.)
Introduce a 3-point function *T* on *U* × *U* × *U* by the formula $T(x, y, z) = \tilde{\Phi}(x, y) + \tilde{\Phi}(y, z) + \tilde{\Phi}(z, x) - \tilde{\Phi}(x) - \tilde{\Phi}(y) - \tilde{\Phi}(z)$.
Theorem (2.2.4) [83]: *For any point* $x \in U$ *the functional* K_x^Q *on* $\mathcal{F}(U \times U)$ *such that* $K_x^Q(f \otimes g) = Q(f,g)(x)$ *is the formal integral at the point* $(x,x) \in U \times U$ *associated to the pair* $(\psi^x, \mu_{tr} \otimes \mu_{tr})$*, where*$\psi^x(y,z) = T(x,y,z)$*.*
Let (M, ω_{-1}) be a compact Kähler manifold. Assume that there exists a quantum line bundle *(L, h)* on *M*, i.e., a holomorphic hermitian line bundle with fibre metric *h* such that the curvature of the canonical connection on *L* coincides with the kähler form ω_{-1}.
Let *m* be a non-negative integer. The metric *h* induces the fibre metric h^m on the tensor power $L^m = L^{\otimes m}$. Denote by $L^n(L^m)$ the Hilbert space of square-integrable of $L^m$with respect to the norm $\|s\|^2 = \int h^m(s)\Omega$, where $\Omega = (1/n!)(\omega_{-1})^2$ is the symplectic volume form on *M*. The Bergman projector B_m is the orthogonal projector in $L^2(L^m)$ onto the space $L_m = \Gamma_{hol}(L^m)$ of holomorphic L^m.
Denote by *k* the metric on the dual line bundle $\tau: L^* \to M$induced by *h*. It is a well known fact that $D = \{\alpha \in L^* | k(\alpha) < 1\}$is a strictly pseudoconvex domain in L^*.Its boundary $X = \{\alpha \in L^* | k(\alpha) = 1\}$is a s^1-principal bundle.
The of L^m are identified with the m-homogeneous functions on L^* by means of the mapping $\gamma_m: s \mapsto \psi_s$, where $\psi_s(\alpha) = \langle \alpha^{\otimes m}, s(x) \rangle$ for $\alpha \in L_x^*$. Here $\langle .,. \rangle$ denotes the bilinear pairing between $(L^*)^m$ and L^m.
There exists a unique S^1-invariant volume form $\tilde{\Omega}$ on X such that for every $f \in C^\infty(M)$ the equality$\int_X (\tau * f)\tilde{\Omega} = \int_M f\Omega$ hold
The mapping γ_m maps $L^2(L^m)$ isometrically onto the weight subspace of $L^2(X, \tilde{\Omega})$
of weight *m* with respect to the S^1-action. The Hardy space $\mathcal{H} \subset L^2(X, \tilde{\Omega})$ of square integrable traces of holomorphic functions on L^* splits up into weight spaces $\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$, where $\mathcal{H}_m = \gamma_m(H_m)$.
Denote by *S* and $\hat{B}_m$ the Szegö and Bergman orthogonal projections in $L^2(X, \tilde{\Omega})$
onto $\mathcal{H}$ and $\mathcal{H}_m$ respectively. Thus $= \sum_{m=0}^{\infty} \tilde{B}_m$. The Bergman projection $\hat{B}_m$ has a smooth integral kernel $B_m = B_m(\alpha, \beta)$ on *X* × *X*.
For each $\alpha \in L^*$ means the zero removed) one can define a coherent state $e_\alpha^{(m)}$as the unique holomorphic of L^m such that for each $s \in H_m \langle s, e_\alpha^{(m)} \rangle = \psi_s(\alpha)$ where $\langle .,. \rangle$ is the hermitian scalar product on $L^2(L^m)$ antilinear in the second argument.
Since the line bundle *L* is positive it is known that there exists a constant m_0 such
that for $m > m_0$ dim H_m> 0 and all $e_\alpha^{(m)}, \alpha \in L^* - 0$, are nonzero vectors. From now on we assume that m > m_0 unless otherwise specified.
The coherent state $e_\alpha^{(m)}$ is antiholomorphic in α and for a nonzero $c \in \mathbb{C} e_{c\alpha}^{(m)} = c^{-m} e_\alpha^{(m)}$
Notice that in [100] coherent states are parametrized by the points of $L - 0$.
For $s \in L^2(L^m) \langle s, e_\alpha^{(m)} \rangle = \langle s, B_m e_\alpha^{(m)} \rangle = \langle B_m s, e_\alpha^{(m)} \rangle = \psi B_{ms}(\alpha)$
The mapping $\gamma_m$intertwines the Bergman projectors $B_m$and $\hat{B}_m$, for$s \in L^2(L^m) \psi_{B_{ms}} = \hat{B}_m \psi_s$. Thus,

on the one hand $\langle s, e_\alpha^{(m)}\rangle = \hat{B}_m\psi_s(\alpha) = \int_X B_m(\alpha,\beta)\psi_s(\beta)\widetilde{\Omega}(\beta)$

. On the other hand, $\langle s, e_\alpha^{(m)}\rangle = \langle\psi_s, \psi_{e_\alpha}(m)\rangle = \int_X \psi_s(\beta)\overline{\psi_{e^{(m)}{}_\alpha}(\beta)}\widetilde{\Omega}(\beta)$

Taking into account that $\langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle = \psi_{e^{(m)}{}_\beta}(\alpha) = \overline{\psi_{e^{(m)}{}_\alpha}(\beta)}$

we finally get that $\langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle = \psi_{e_\beta{}^{(m)}}(\alpha) = B_m(\alpha,\beta)$. In particular, one can extend the kernel $B_m(\alpha,\beta)$ from $X \times X$ to a holomorphic function on$(L^* - 0) \times \overline{(L^* - 0)}$ such that for nonzero *c, d* $\in \mathbb{C}$

$$B_m(c\alpha, d\beta) = \left(c\bar{d}\right)^m B_m(\alpha,\beta) \tag{8}$$

For $\alpha, \beta \in L^* - 0$ the following inequality holds

$$|B_m(\alpha,\beta)| = \left|\langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle\right| \le \left\|e_\alpha^{(m)}\right\|\left\|e_\beta^{(m)}\right\| = (B_m(\alpha,\alpha)B_m(\beta,\beta))^{1/2} \quad (9)$$

The covariant symbol of an operator *A* in the space $H_m$is the function *σ(A*) on *M* such that

$$\sigma(A) = \frac{\langle Ae_\alpha^{(m)}, e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}$$

for any α ∈ $L_x^* - 0$.

Denote by Mf the multiplication operator by a function $f \in C^\infty(M)$

L^m. Define the Berezin-Toeplitz operator $T_f^{(m)} = B_m M_f B_m$ in H_m. If an operator in H_m is represented in the form $T_f^{(m)}$ for some function $f \in C^\infty(M)$ then the function *f* is called its contravariant symbol. With these symbols we associate two important operations on $C^\infty(M)$, the Berezin transform $I^{(m)}$ and a non-associative binary operation $Q^{(m)}$which we call twisted product, as follows. For $f, g \in C^\infty(M) I^m f = \sigma\left(T_f^{(m)}\right), Q^{(m)}(f,g) = \sigma(T_f^{(m)}T_g^{(m)})$ .We are going to show that both I(m) and Q(m) have asymptotic expansions in *1/m* as *m* → +∞, such that if the asymptotic parameter *1/m* in these expansions is replaced by the formal parameter ν then we get the formal Berezin transform *I* and the formal twisted product *Q* corresponding to some deformation quantization with separation of variables on *(M,* ω_{-1}*)* which can be completely identified. We shall mainly be interested in the opposite to its dual deformation quantization. We show that it coincides with the Berezin-Toeplitz deformation quantization obtained in [106],[108].

In order to obtain the asymptotic expansions of $I^{(m)}$ and $Q^{(m)}$ we need their integral representations. To calculate them it is convenient to work on *X* rather than on *M*.

We shall use the fact that for $f \in C^\infty(M), s \in \boldsymbol{\Gamma}(\boldsymbol{L^m}), \boldsymbol{\psi_{M_f s}} = (\boldsymbol{\mathcal{T}} * \boldsymbol{f}).\boldsymbol{\psi_s}$ **for** $\boldsymbol{x} \in \boldsymbol{M}$denote by X_x the fibre of the bundle *X* over *x*, $X_x = \mathcal{T}^{-1}(x) \cap X$. For *x, y, z* ∈ *M* choose $\alpha \in X_x,\ \beta \in X_y,\ \gamma \in X_z$and set

$$u_m(x) = B_m(\alpha,\alpha), v_m(x,) = B_m(\alpha,\beta)B_m(\beta,\alpha), w_m(x,) = B_m(\alpha,\beta)B_m(\beta,\gamma)B_m(\gamma,\alpha) \quad (10).$$

It follows from (8) that $u_m(x), v_m(x,y), w_m(x,y,z)$ do not depend on the choice of α, β, γ and thus relations (10) correctly define functions u_m, v_m, w_m. The function w_m is the so called cyclic 3-point function studied in [102]. Notice that

$u_m(x) = B_m(\alpha,\alpha) = \left\|e_\alpha^{(m)}\right\|^2 \ge 0, v_m(x,y) = B_m(\alpha,\beta)B_m(\beta,\alpha) = |B_m(\alpha,\beta)|^2 \ge 0$ and

$$|w_m(x,y,z)|^2 = v_m(x,y)v_m(y,z)v_m(z,x) \tag{11}$$

It follows from (9) that

$$v_m(x,y) \leq v_m(x)v_m(y) \qquad (12)$$

For $\alpha \in X_x$ we have

$$\begin{aligned}(I^{(m)}f)(x) &= \sigma(T_f^{(m)})(x) = \frac{\langle T_f^{(m)} e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m}, e_\alpha^{(m)}\rangle} = \frac{\langle B_m M_f B_m e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}{B_m(\alpha,\alpha)} = \frac{\langle M_f e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}{B_m(\alpha,\alpha)} \\ &= \frac{\langle (\tau * f)\psi_{e_\alpha{}^{(m)}}, \psi_{e_\alpha{}^{(m)}}\rangle}{B_m(\alpha,\alpha)} = \frac{1}{B_m(\alpha,\alpha)}\int_X (\tau * f)\psi_{e_\alpha{}^{(m)}}(\beta)\overline{\psi_{e_\alpha{}^{(m)}}(\beta)}\widetilde{\Omega}(\beta) \\ &= \frac{1}{B_m(\alpha,\alpha)}\int_X B_m(\alpha,\beta)B_m(\beta,\alpha)(\tau * f)(\beta)\widetilde{\Omega}(\beta) \\ &= \frac{1}{v_m(x)}\int_M v_m(x,y)f(y)\Omega(y) \qquad (13)\end{aligned}$$

Similarly we obtain that

$$\begin{aligned}Q^m(f,g)(x) &= \frac{1}{B_m(\alpha,\alpha)}\int_{X\times X} B_m(\alpha,\beta)B_m(\beta,\gamma)B_m(\gamma,\alpha)(\tau * f)(\beta)(\tau * g)(\gamma)\widetilde{\Omega}(\beta)\widetilde{\Omega}(\gamma) \\ &= \frac{1}{v_m(x)}\int_{M\times M} w_m(x,y,z)f(y)g(z)\Omega(y)\Omega(z) \ (14)\end{aligned}$$

In [109] a microlocal description of the integral kernel $\boldsymbol{S}$ of the Szegö projection S was given. The results in [109] were obtained for a strictly pseudoconvex domain with a smooth boundary in $\mathbb{C}^{n+1}$. However, according to [109], these results are still valid for the domain D in L^* (see also [106], [101]).

It was proved in [109] that the Szegö kernel $\boldsymbol{S}$ is a generalized function on $X \times X$ singular on the diagonal of $X \times X$ and smooth outside the diagonal. The Szegö kernel $\boldsymbol{S}$ can be expressed by the Bergman kernels B_m as follows, $\boldsymbol{S} = \sum_{m\geq 0} B_m$, where the sum should be understood as a sum of generalized functions.

For $(\alpha,\beta) \in X \times X$ and $\theta \in \mathbb{R}$ set $r_\theta(\alpha,\beta) = (e^{i\theta}\alpha,\beta)$. Since each $\mathcal{H}_m$ is a weight space of the S^1-action in the Hardy space $\mathcal{H}$, one can recover B_m from the Szegö kernel,

$$B_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta} r_\theta^* S d\theta \qquad (15).$$

This equality should be understood in the weak sense.

Let E_1, E_2 be closed disjoint subsets of M. Set $F_i = \tau^{-1}(E_i) \cap X, i = 1,2$. Thus F_1, F_2 are closed disjoint subsets of X or, equivalently, $F_1 \times F_2$ is a closed subset of $X \times X$ which does not intersect the diagonal. For S and B_m considered as smooth functions outside the diagonal of $X\times X$ equality (15) holds in the ordinary sense, from whence it follows immediately that

$$\sup_{F_1\times F_2} |B_m| = O\left(\frac{1}{m^N}\right) \qquad (16)$$

for any $N \in \mathbb{N}$.

Now let E be a closed subset of M and $x \in M \setminus E$. Then (16) implies that

$$\sup_{y \in E} v_m(x, y) = O\left(\frac{1}{m^N}\right) \quad (17)$$

for any $N \in \mathbb{N}$.
In [101] Zelditch proved that the function u_m on M expands in the asymptotic series $u_m \sim m^r \sum_{r \geq 0}(1/m^r) b_r$ as $m \to +\infty$, where $b_0 = 1 (n = (1/2)\, dim_{\mathbb{R}} M$. More precisely, he proved that for any $k, N \in \mathbb{N}$

$$|u_m - \sum_{r=0}^{N-1} m^{n-r} b_r|_{C^k} = O(m^{n-N}) \quad (18)$$

Therefore

$$\sup_{M} \frac{1}{u_m} O\left(\frac{1}{m^n}\right) \quad (19)$$

Using (13),(17) and (19) it is easy to prove the following proposition.
Proposition (2.2.5) [83]: *Let $f \in C^\infty(M)$ be a function vanishing in a neighborhood of a point $x \in M$. Then $\left|(I^{(m)} f)(x)\right| = O(1/m^N)$ for any $N \in \mathbb{N}$, i.e., $(I^{(m)} f)(x)$ is rapidly decreasing as $m \to +\infty$.*
Thus for arbitrary $f \in C^\infty(M)$ and $x \in M$ the asymptotics of $(I^{(m)} f)(x)$ as $m \to +\infty$ depends only on the germ of the function f at the point x.
Let E be a closed subset of M. Fix a point $x \in M \setminus E$. The function $w_m(x, \boldsymbol{y}, \boldsymbol{z})$ with $y \in$ E can be estimated using (11) and (12) as follows.

$$|w_m(x, y, z)|^2 \leq v_m(x, y) u_m(x) u_m(y) \big(u_m(z)\big)^2 \quad (20).$$

Using (17), (18) and (20) we obtain that for any $N \in \mathbb{N}$

$$\sup_{y \in E, z \in M} |w_m(x, y, z)| = O\left(\frac{1}{m^N}\right) \quad (21)$$

Similarly,

$$\sup_{y \in E, z \in E} |w_m(x, y, z)| = O\left(\frac{1}{m^N}\right) \quad (22)$$

for any $N \in \mathbb{N}$.
Using (14), (19), (21) and (22) one can readily prove the following proposition.
Proposition (2.2.6) [83]: *For $x \in M$ and arbitrary functions $f, g \in C^\infty(M)$ such that f or g vanishes in a neighborhood of* x Q(m)(f, g)(x) *is rapidly decreasing as* m → +∞.
This statement can be reformulated as follows. For arbitrary $f, g \in C^\infty(M)$ and $x \in M$ the asymptotics of $Q^{(m)}(f, g)(x)$ as $m \to +\infty$ depends only on the germs of the functions f, g at the point x.
We are going to show how formal integrals can be obtained from the method of stationary phase.
Let $\emptyset$ be a smooth function on an open subset $U \subset M$ such that (i) $Re\, \emptyset \leq 0$;
(ii) there is only one critical point $x_c \in$ U of the function $\emptyset$, which is moreover a nondegenerate critical point; (iii) $\emptyset(x_c) = 0$.
Consider a classical symbol $\rho(x, m) \in S^0(U \times \mathbb{R})$ (see [101] for definition and notation) which has an asymptotic expansion $\rho \sim \sum_{r \geq 0}(1/m^r) \rho_r(x)$ such that $\rho_0(x_c) \neq 0$, and a smooth nonvanishing volume form dx on U. Set $\mu(m) = \rho(m, x) dx$.
We can apply the method of stationary phase with a complex phase function (see

[101] and [108]) to the integral

$$S_m(f) = \int_U e^{m\emptyset} f \mu(m) \qquad (23)$$

Where $f \in C_0^\infty(M)$. Notice that the phase function in (23) is $(1/i)\emptyset$ so that the condition $((1/i)) \geq 0$is satisfied.
Taking into account that $dim_{\mathbb{R}} M = 2n$and $\emptyset(x_c) = 0$ we obtain that $S_m(f)$ expands to an asymptotic series$S_m(f) \sim \sum_{r=0}^{\infty}(1/m^{n+r})\widetilde{K}_r(f)$ as $m \to +\infty$. Here $\widetilde{K}_r, r \geq 0$ are distributions supported at $x_c$and $\widetilde{K}_0 = c_n\delta_{xc}$, where c_n is a nonzero constant. Thus$\mathbb{F}(S_m(f)) = v^n\widetilde{K}(f)$, where $\mathbb{F}$ is the "formalizer" introduced and $\widetilde{K}$is the functional defined by the formula$\widetilde{K} = \sum_{r\geq 0} v^r \widetilde{K}_r$. Consider the normalized functional $K(f) = \widetilde{K}(f)/\widetilde{K}(1)$, so that *K(1) = 1*. Then$\mathbb{F}(S_m(f)) = c(v)K(f)$, where $c(v) = v^n c_n + \cdots$is a formal constant.
Proposition (2.2.7) [83]: *For $f \in C_0^\infty(U)$ given by (23) expands in an asymptotic series in $1/m$ as $m \to +\infty$. $\mathbb{F}(S_m(f)) = c(v)K(f)$, where K is the formal integral at the point* xc *associated to the pair $((1/v)\emptyset, (\mu))$ and $c(v)$ is a nonzero formal constant.*
Proof: Conditions (a-c) of the definition of formal integral are satisfied. It remains to check condition (d). Let ξ be a vector field on U. Denote by L_ξ the corresponding Lie derivative. We have $0 = \int_U L_\xi\left(e^{m\emptyset} f\mu(m)\right) = \int_U e^{m\emptyset}(\xi f + (m\xi\emptyset + div_\mu \xi)f)\mu(m)$
Applying $\mathbb{F}$ we obtain that $0 = \mathbb{F}\left(\int_U e^{m\emptyset}(\xi f + (m\xi\emptyset + div_\mu \xi)f)\mu(m)\right) = c(v)K(\xi f + (\xi((1/v)\emptyset) + div_{\mathbb{F}(\mu)}\xi)f)$, which concludes the proof.
We get an asymptotic expansion of the Bergman kernel B_m in a neighborhood of the diagonal of $X \times X$ as $m \to +\infty$. An asymptotic expansion of B_m on the diagonal of $X \times X$ was obtained in [101]. As in [111], we use the integral representation of the Szegö kernel $\boldsymbol{S}$ given by the following theorem. We denote $n = dim_{\mathbb{C}} M$.
Theorem (2.2.8) [83]: *Let **S**(α, β) be the Szegö kernel of the boundary X of the bounded strictly pseudoconvex domain D in the complex manifold L*. There exists a classical symbol $a \in S^n(X \times X \times \mathbb{R}^+)$which has an asymptotic expansion*

$$\alpha(\alpha, \beta, t) \sim \sum_{k=0}^{\infty} t^{n-k} a_k(\alpha, \beta)$$

so that

$$S(\alpha, \beta) = \int_0^\infty e^{it\varphi(\alpha,\beta)} a(\alpha, \beta, t)dt \qquad (24),$$

where the phase $\varphi(\alpha, \beta) \in C^\infty(L^* \times L^*)$ is determined by the following properties:
•*$\varphi(\alpha, \alpha) = (1/i)(k(\alpha) - 1)$;*
• *$\bar{\partial}_\alpha\varphi$ and $\partial_\beta\varphi$ vanish to infinite order along the diagonal;*
•*$\varphi(\alpha, \beta) = \overline{-\varphi(\beta, \alpha)}$.*
The phase function φ is thus almost analytic at the diagonal of $L^* \times \overline{L^*}$. It is determined up to equivalence at the diagonal.
Fix an arbitrary point $x_0 \in M$. Let s be a local holomorphic frame of L^* over a contractible open neighborhood $U \subset M$ of the point x_0 with local holomorphic coordinates $\{z^k\}$. Then $\alpha(x) = s(x)/\sqrt{k(s(x))}$is a smooth of X over U. Set $\Phi_{-1}(x) = \log k(s(x))$, so that

$$\alpha(x) = e^{(-1/2)\Phi_{-1}(x)} s(x) \qquad (25)$$

It follows from the fact that L is a quantum line bundle (i.e., that ω_{-1} is the curvature form of the Hermitian holomorphic line bundle L) that Φ_{-1} is a potential of the form ω_{-1} on U.
Let$\widetilde{\Phi}_{-1}(x, y) \in C^{\infty}(U \times \overline{U})$ be an almost analytic extension of the potential Φ_{-1} from the diagonal of $U \times \overline{U}$. Denote $D_{-1}(x, y) := \widetilde{\Phi}_{-1}(x, y) + \widetilde{\Phi}_{-1}(y, x) - \Phi_{-1}(x) - \Phi_{-1}(y)$,
Since $\widetilde{\Phi}_{-1}(x, x) = \Phi_{-1}(x)$, we have $D_{-1}(x, x) = 0$. In local coordinates

$$D_{-1}(x, y) = -Q_{x_0}(x - y) - O(|x - y|^3) \qquad (26)$$

Where

$$Q_{x_0}(z) = \sum \frac{\partial^2 \Phi_{-1}}{\partial z^k \partial z^{-l}}(x_0) z^k z^{-l}$$

is a positive definite quadratic form (since ω_{-1} is a Kähler form).
The following statement is an immediate consequence of (26).
Lemma (2.2.9) [83]: *There exists a neighborhood $U' \subset U$ of the point x_0 such that for any two different points $x, y \in U'$ one has $ReD_{-1}\ (x, y) < 0$.*
Taking, if necessary, $(1/2)(\widetilde{\Phi}_{-1}(x, y) + \overline{\widetilde{\Phi}_{-1}(y, x)})$instead of $\widetilde{\Phi}_{-1}(x, y)$ choose $\widetilde{\Phi}_{-1}$ such that $\widetilde{\Phi}_{-1}(y, x) = \overline{\widetilde{\Phi}_{-1}(x, y)}$. Replace U by a smaller neighborhood (retaining for it the notation U) such that $ReD_{-1}(x, y) < 0$for any different x, y from this neighborhood.
For a point α in the restriction L^*/U of the line bundle L^* to U represented in the form $\alpha = vs(x)$ with $v \in \mathbb{C}, x \in U$ one has $k(\alpha) = |v|^2 k(s(x))$.
One can choose the phase function $\varphi(\alpha, \beta)$ in (24) of the form

$$\varphi(\alpha, \beta) = (1/i)\left(v \bar{w} e^{\widetilde{\Phi}_{-1}(x, y)} - 1\right) \qquad (27)$$

Where $v = vs(x), \beta = ws(y) \in {L^*}/{U}$
Denote$(\chi(x, y) := \widetilde{\Phi}_{-1}(x, y) - (1/2)\Phi_{-1}(x) - (1/2)\Phi_{-1}(y)$. Notice that $\chi(x,x) = 0$.
The following theorem is a slight generalization of Theorem 1 from [101].
Theorem (2.2.10) [83]: *There exists an asymptotic expansion of the Bergman kernel $B_m(\alpha(x), \alpha(y))$ on $U \times U$ as $m \to +\infty$, of the form*

$$B_m\big(\alpha(x), \alpha(y)\big) \sim m^n e^{m\chi(x, y)} \sum_{r \geq 0} (1/m^r) \tilde{b}_r(x, y) \qquad (28)$$

such that (i) for any compact $E \subset U \times U$ and $N \in \mathbb{N}$

$$\sup_{() \in E} \left| B_m\big(\alpha(x), \alpha(y)\big) - m^n e^{m\chi(x, y)} \sum_{r=0}^{N-1} (1/m^n) \tilde{b}_r(x, y) \right| = O(m^{n-N}) \qquad (29)$$

(ii)$\tilde{b}_r(x, y)$ is an almost analytic extension of $b_r(x)$ from the diagonal of $U \times U$, where $b_r, r > 0$, are given by (18); in particular, $\tilde{b}_0(x, x)$.
Proof: Using integral representations (15) and (24) one gets for $x, y \in U$

$$B_m\big(\alpha(x), \alpha(y)\big) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-im\theta} e^{it\varphi(r_\theta \alpha(x), \alpha(y))} a(r_\theta \alpha(x), \alpha(y), t) d\theta dt \qquad (30).$$

Changing variables $t \mapsto mt$ in (30) gives

$$B_m\big(\alpha(x), \alpha(y)\big) = \frac{m}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{im(t\varphi(r_\theta \alpha(x), \alpha(y)) - \theta)} a(r_\theta \alpha(x), \alpha(y), mt) d\theta dt \ (31)$$

In order to apply the method of stationary phase to the integral in (31) the following preparations should be made.
Using (27) and (25) express the phase function of the integral in (31) as follows:

$$Z(t,\theta;x,y) := t\varphi(r_\theta\alpha(x),\alpha(y)) - \theta = (t/i)(e^{i\theta}e^{\chi(x,y)} - 1) - \theta \qquad (32)$$

In order to find the critical points of the phase Z (with respect to the variables *(t,θ);* the variables *(x, y)* are parameters) consider first the equation

$$\partial_t Z(t,\theta;x,) = (1/i)(e^{i\theta}e^{\chi(x,y)} - 1) = 0 \qquad (33)$$

It follows from $\Phi_{-1}(y,x) = \overline{\tilde{\Phi}_{-1}(x,y)}$ that $Re\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{y}) = (\mathbf{1/2})\boldsymbol{D}_{-1}(\boldsymbol{x},\boldsymbol{y})$. Since $D_{-1}(x,y) < 0$ for $x \neq y$ one has $|e^{\chi(x,y)}| = e^{Re\chi(x,y)} < 1$for $x \neq y$ whence it follows that (33) holds only if $x = y$ and thus Z has critical points only if $x = y$. Since *χ(x, x) = 0* one gets that $\partial_t Z(t,\theta;x,x) = (1/i)(e^{i\theta} - 1)$ and$\partial_\theta Z(t,\theta;x,x) = te^{i\theta} - 1$. As in the proof of Theorem 1 from [101], one shows that for each $x \in U$ the only critical point of the phase function *Z(t, θ; x, x)* is *(t = 1, θ = 0)*. It does not depend on x and, moreover, is nondegenerate. One has $ImZ(t,\theta;x,y) = Im\left((1/i)(e^{i\theta}e^{\chi(x,y)} - 1) - \theta\right) = t(1 - Re(e^{i\theta}e^{\chi(x,y)})) \geq 0$ since $|e^{\chi(x,y)}| \leq 1$.
Finally, a simple calculation shows that the germs of the functions *Z(t, θ; x, y)* and *(1/i)χ(x, y)* at the point *(t = 1, θ = 0, x =* x_0*, y =* x_0*)* are equal modulo the ideal generated by $\partial_t Z$ and $\partial_\theta Z$.
Applying now the method of stationary phase to the integral in (31) one obtains the expansion (28) satisfying (29).
It follows from (18) and (10) that $\tilde{b}_r(x,x) = b_r$and $\tilde{b}_0(x,x) = b_0(x) = 1$. It remains to show that all $\tilde{b}_r, r \geq 0$, are almost analytic along the diagonal of U × $\overline{U}$.
One has

$$B_m(\alpha(x),\alpha(y)) = e^{(-m/2)(\Phi_{-1}(x)+\Phi_{-1}(y))}B_m(s(x),(s(y))$$

The function $B_m(s(x),s(y))$ is holomorphic on U×$\overline{U}$. Let ξ and η be arbitrary holomorphic and ant holomorphic vector fields on U, respectively. Then $\xi_y B_m(s(x),s(y))$ and $\eta_x B_m(s(x),s(y)) = 0$ (the subscripts x, y show in which variable the vector field acts). Thus

$$\left(\eta_x + \frac{m}{2}\eta_x\Phi_{-1}(x)\right)B_m(\alpha(x),\alpha(y)) = e^{(-m/2)\Phi_{-1}(x)}\eta_x e^{(m/2)\Phi_{-1}(x)}B_m(\alpha(x),(y)) = 0$$

Analogously $(\xi_y + (m/2)\xi_y\Phi_{-1}(x))B_m(\alpha(x),\alpha(y)) = 0$.Let A_N be a product of N derivations on $U \times U$. Then, using integral representation (31), expand $0 = A_N(\eta_x + (m/2)\eta_x\Phi_{-1}(x))B_m(\alpha(x),\alpha(y))$
to the asymptotic series

$$A_N(\eta_x + \frac{m}{n}\eta_x\Phi_{-1}(x))(m^n e^{m\chi(x,y)}\sum_{r\geq 0}(1/m^r)\tilde{b}_r(x,y)) = e^{m\chi(x,y)}\sum_{r\geq r_0}(1/m^r)c_r(x,y) \quad (34)$$

for some $c_r \in C^\infty(U \times U)$ and $r_0 \in \mathbb{Z}$, and with the norm estimate of the partial sums in the $r.h.s.$ term in (34) analogous to (29). Since *χ(x, x) = 0* one gets that all$c_r(x,x) = 0$. From this fact one can prove by induction over N that $\eta_x\tilde{b}_r$ vanishes to infinite order at the diagonal of U × U. Similarly, $\xi_y\tilde{b}_r$vanishes to infinite order at the diagonal. Thus $\tilde{b}_r$ is almost analytic along the diagonal.
Choose a symbol $b(x,y,m) \in S^0((U \times U) \times \mathbb{R})$ such that it has the asymptotic Expansion $b \sim \sum_{r=0}^{\infty}(1/m^r)\tilde{b}_r$. Then $B_m(\alpha(x),\alpha(y))$ is asymptotically equivalent to $m^n e^{m\chi(x,y)}b(x,y,m)$on $U \times U$. One $\chi(x,y) + \boldsymbol{\chi}(\boldsymbol{y},\boldsymbol{x}) = \tilde{\boldsymbol{\Phi}}_{-1}(\boldsymbol{x},\boldsymbol{y}) + \tilde{\boldsymbol{\Phi}}_{-1}(\boldsymbol{y},\boldsymbol{x}) - \boldsymbol{\Phi}_{-1}(\boldsymbol{x}) -$

$\boldsymbol{\Phi_{-1}(y) = D_{-1}(x,y)}$ **and** $\boldsymbol{\chi(x,y)+\chi(y,z)+\chi(z,x) = \tilde{\Phi}_{-1}(x,y)+\tilde{\Phi}_{-1}(y,z)+\tilde{\Phi}_{-1}(z,x)-\Phi_{-1}(x)-\Phi_{-1}(y)-\Phi_{-1}(z)}$
(the last equality is the definition of T_{-1}). Thus the functions
$v_m(x,y) = B_m(\alpha(x),\alpha(y))B_m(\alpha(y),\alpha(x))$and

$$w_m(x,y,z) = B_m(\alpha(x),\alpha(y))B_m(\alpha(y),\alpha(z))B_m(\alpha(z),\alpha(x))$$

are asymptotically equivalent to
$m^{2n}e^{mD_{-1}(x,y)}b(x,y,m)b(y,x,m)andm^{2n}e^{mT_{-1}(x,y,z)}b(x,y,m)b(y,z,m)b(z,x,m)$
respectively. It is easy to show that for the functions $\emptyset^x_{-1}(y) = D_{-12}(x,y)$
and $\psi^x_{-1}(x,y,z) = T_{-1}(x,y,z)$ the points y = x and (y,z) = (x, x) respectively are nondegenerate critical ones.
Since $\tilde{b}_0(x,x) = 0$ one can take a smaller contractible neighborhood $V \Subset U$ of x_0 such that$\tilde{b}_0(x,y)$ does not vanish on the closure of $V \times V$. One can choose V such that for any $x \in V$ the only critical points of the functions $\emptyset^x_{-1}(y)$ on V and $\emptyset^x_{-1}(y,z)$ on$V \times V$ are y = x and $(y,)$ = (x, x) respectively. The identity $T_{-1}(x,y,z) = (1/2)(D_{-1}(x,y)+D_{-1}(y,z)+D_{-1}(z,x))$implies that $ReT_{-1}(x,y,z) \leq 0$for x, y,$\in V$.
The symbol $b(x, y,m)$ does not vanish on $V \times V$ for sufficiently big values of m. It
follows from (18) that $1/u_m(x)$and $(m^n b(x,x,m))^{-1}$are asymptotically equivalent for $x \in V$. Denote

$$\mu_x(m) = \frac{b(x,y,m)b(y,x,m)}{b(x,x,m)}\Omega(y), \tilde{\mu}_x(m) = \frac{b(x,y,m)b(y,z,m)}{b(x,x,m)}\Omega(y)\Omega(z) \quad (35)$$

Taking into account (13) we get for $f,g \in C_0^\infty(V)$and $x \in V$ the following asymptotic equivalences,

$(I^m f)(x) \sim m^n \int_V e^{m\emptyset^x_{-1}} f\mu_x(m)$and

$$Q^{(m)}(f,g)(x) \sim m^{2n}\int_{V\times V} e^{m\psi^x_{-1}}(f\otimes g)\tilde{\mu}_x(m) \quad (36)$$

(In(36) $(f\otimes g)(y,z) = f(y)g(z)$.)
Applying Proposition (2.2.7) to the first integral in (36) we obtain that$\mathbb{F}\big((I^m f)(x)\big) = c(v,x)L^I_x(f)$, where the functional L^I_x on $\mathcal{F}(V)$ is the formal integral at the point x associated to the pair $((1/v)\emptyset^x_{-1}, \mathbb{F}(\mu_x))$and $c(v, x)$ is a formal function. It is easy to show that $c(v, x)$ is smooth.
Similarly we obtain from (36) that $\mathbb{F}\big(Q^m(f,g)(x)\big) = d(v,x)L^Q_x(f\otimes g)$ where the functional L^Q_x on $\mathcal{F}(V \times V)$ is the formal integral at the point (x, x) associated to the pair $((1/v)\psi^x_{-1}, \mathbb{F}(\tilde{\mu}_x))$and $d(v, x)$ is a smooth formal function.
Since the unit constant 1 is a contravariant symbol of the unit operator **1**, $T^{(m)}_1 = 1$, and $\sigma(1) = 1$, we have$I^{(m)}1 = 1, Q^{(m)}(1,1) = 1$, and thus $\mathbb{F}\big(I^{(m)}1\big) = 1$ and $\mathbb{F}(Q^{(m)}(1,1)) = 1$. Taking the functions f, g in (36) to be equal to 1 in a neighborhood of x and applying Proposition (2.2.5) and Proposition (2.2.6) we get that $c(v, x) = 1$ and $d(v, x) = 1$. Since $b_0(x,y)$ does not vanish on $V \times V$ we can find a formal function $\tilde{s}(x,y)$ on $V \times V$ such that$\mathbb{F}(b(x,y,m)) = e^{\tilde{s}(x,y)}$. In these notations
$\mathbb{F}(\mu_x) = \exp(\tilde{s}(x,y)+\tilde{s}(y,x)-s(x)\Omega(y)$.and $\mathbb{F}\tilde{\mu}_x = \exp(\tilde{s}(x,y)+\tilde{s}(y,z)+\tilde{s}(z,x)-s(x)\Omega(y)\Omega(z)$ (37)
It follows from Theorem (2.2.10) that $\tilde{s}$ is an almost analytic extension of the function s
from the diagonal of $V \times V$. According to (18),$\mathbb{F}(u_m) = (1/v^n)e^s$
Denote $\tilde{\Phi} = (1/v)\tilde{\Phi}_{-1}+\tilde{s}, \Phi = (1/v)\Phi_{-1}+s, D(x,y) = \tilde{\Phi}(x,y)+\tilde{\Phi}(yx)-\Phi(x)-\Phi(y) = (1/v)D_{-1}(x,y)+(\tilde{s}(x,y)+\tilde{s}(y,x)-s(x)-s(y), T(x,y,z) = \tilde{\Phi}(x,y)+\tilde{\Phi}(y,z)+\tilde{\Phi}(z,x)-\Phi(x)-\Phi(y)-\Phi(z)$
is then equivalent to the pair $(\emptyset^x, e^s\Omega)$, where $\emptyset^x(y) = D(x,y)$. Similarly, the pair $((1/v)\psi^x_{-1}, \mathbb{F}(\tilde{\mu}_x))$is equivalent to the pair $(\psi^x, e^s\Omega\otimes e^s\Omega)$, where
$\psi^x(y,z) = T(x,y,z)$.

Thus we arrive at the following proposition.
Proposition (2.2.11) [83]: *For $f, g \in C_0^\infty(V)$, $x \in V$, $(I^m f)(x)$ and $Q^{(m)}(f,g)(x)$ expand in asymptotic series in 1/m as m → +∞.* $\mathbb{F}((I^{(m)}f)(x) = L_x^I(f)$*and* $\mathbb{F}(Q^{(m)}(f,g)(x) = L_x^Q(f\otimes g)$ *where the functional L_x^I on $\mathcal{F}(V)$ is the formal integral at the point x associated to the pair $(\emptyset^x, e^s\Omega)$ and the functional L_x^Q*
on $\mathcal{F}(V \times V)$ is the formal integral at the point (x, x) associated to the pair $(\psi^x, e^s\Omega\otimes e^s\Omega)$.
Now let ⋆ denote the star-product with separation of variables on (V, ω_{-1}) corresponding to the formal deformation $\omega = -i\partial\bar{\partial}\emptyset$ of the form *(1/ν)*ω_{-1}, so that $\boldsymbol{\Phi}$ is a formal potential of ω. Let I be the corresponding formal Berezin transform, $\widetilde{\omega}$ the formal form parametrizing the dual star-product $\tilde{*}$ and Ψ the solution of (5) so that$\mu_{tr} = e^{\Phi+\Psi}dzd\bar{z}$ is a formal trace density for the star-product ⋆.
Choose a classical symbol $\rho(x,m) \in S^0(V \times \mathbb{R})$ which has an asymptotic expansion$\rho \sim \sum_{r\geq 0}(1/m^r)\rho_r$ such that

$$\mathbb{F}(\rho)e^s\Omega = \mu_{tr} \qquad (38)$$

Clearly, (38) determines *(ρ)* uniquely.
For $f \in C_0^\infty(V)$ and $x \in V$ consider the following integral

$$(P_m f)(x) = m^n \int_V e^{m\emptyset_{-1}^x} f\rho\mu_x \qquad (39),$$

where$\emptyset_{-1}^x(y) = D_{-1}(x,y)$ and μ_x is given by (35).
Proposition (2.2.12) [83]: *For $f \in C_0^\infty(V)$and $x \in V$ $(P_m f)(x)$ has an asymptotic expansion in 1/m as m → +∞.* $\mathbb{F}((P_m f)(x) = c(v)(If)(x)$, *where c(ν) is a nonzero formal constant.*
Proof: It was already shown that the phase function $(1/i)\emptyset_{-1}^x$ of integral (39) satisfies the conditions required in the method of stationary phase. Thus Proposition (2.2.7)
can be applied to (39). We get that $\mathbb{F}((P_m f)(x) = c(v,x)K_x(f)$, where K_x is a
formal integral at the point x associated to the pair $((1/v)\emptyset_{-1}^x, \mathbb{F}(\rho\mu_x)$ and c(ν, x) is a nonvanishing formal function on V . It follows from (37) and (38) that $\mathbb{F}(\rho\mu_x) = \mathbb{F}(\rho)\mathbb{F}(\mu_x) = \mathbb{F}(\rho)exp(\tilde{s}(x,y) + \tilde{s}(x,y) - s(x))\Omega(y) = \exp(\tilde{s}(x,y) + \tilde{s}(x,y) - s(x) = s(y)\mu_{tr} = \exp(D(x,y) - (1/v)D_{-1}(x,y))\,\mu_{tr} = \exp(\emptyset^x - (1/v)\emptyset_{-1}^x)$
where $\emptyset^x(y) = D(x,y)$. The pair $((1/v)\emptyset_{-1}^x, \mathbb{F}(\rho\mu_x))$ is thus equivalent to the pair $(\emptyset^x, \mu_{tr})$. Applying
Theorem (2.2.2) we get that

$$\mathbb{F}((P_m f)(x) = c(v,x)(If)(x) \qquad (40)$$

It remains to show that *c(ν, x)* is actually a formal constant. Let x_1 be an arbitrary point of V. Choose a function $\epsilon \in C_0^\infty(V)$ such that $\epsilon = 1$ in a neighborhood $W \subset V$ of x_1. Let ξ be a vector field on V. Then, using (37), we obtain

$$\frac{1}{v}\xi_x\emptyset_{-1}^x(y) + \mathbb{F}\left(\frac{\xi_x\mu_x}{\mu_x}(y)\right) = \frac{1}{v}\xi_x D_{-1}(x,y) + \xi_x(\tilde{s}(x,y)) + \tilde{s}(x,y) - s(x)) = \xi_x D(x,y) = \xi_x\emptyset^x \qquad (41).$$

On the one hand, taking into account (41) we get for $x \in W$ that

$$\mathbb{F}((\xi P_m\epsilon)(x) = \mathbb{F}\left(m^n\xi\int_V e^{m\emptyset_{-1}^x}\epsilon\rho\mu_x\right) = \mathbb{F}\left(m^n\int_V e^{m\emptyset_{-1}^x}\left(m\xi_x\emptyset_x + \frac{\xi_x\mu_x}{\mu_x}\right)\epsilon\rho\mu_x\right)$$
$$= c(v,x)I\left(\frac{1}{v}\xi_x\emptyset_{-1}^x(y) + \mathbb{F}\left(\frac{\xi_x\mu_x}{\mu_x}(y)\right)\right) = c(v,x)I(\xi_x\emptyset^x)$$
$$= 0 \qquad (42)$$
The last equality in (42) follows from Lemma (2.2.3). On the other hand, for $x \in W$ we have from (40) that $\mathbb{F}((P_m\epsilon)(x) = c(v,x)$, from whence $\mathbb{F}((P_m\epsilon)(x) = \xi\mathbb{F}((P_m\epsilon)(x) = \xi c(v,x)$. Thus we get from (42) that $\xi c(v, x) = 0$ on W for an
arbitrary vector field ξ, from which the Proposition follows.
It follows from (36) and (39) that for $f \in C_0^\infty(V)(I^{(m)}(f\rho))(x)$ is asymptotically equivalent to $(P_m f)(x)$. Passing to formal asymptotic series we get from Proposition (2.2.11)
and Proposition (2.2.12) that $c(v)(If)(x) = \mathbb{F}\big((P_m f)(x)\big) = \mathbb{F}(\big(I^{(m)}(f\rho)\big)(x) = L_x^1(f\mathbb{F}(\rho))$ where L_x^I is the formal integral at the point x associated to the pair $(\emptyset^x, e^s\Omega)$. Thus

$$c(v)(If)(x) = L_x^I\big(f\mathbb{F}(\rho)\big) \qquad (43)$$

The formal function (ρ) is invertible (see (38)). Setting $f = 1/(\rho)$ in (43) we get $c(v)(I(1/\mathbb{F}(\rho)))(x) = L_x^I(1) = 1$ for all x ∈ V . Since the formal Berezin transform is invertible and $I(1) = 1$, we finally obtain that

$$\mathbb{F}(\rho) = c(v) \qquad (44)$$

Now (38) can be rewritten as follows,

$$c(v)e^s\Omega = d\mu_{tr} = e^{\Phi+\Psi}dzd\bar{z} \qquad (45)$$

In local holomorphic coordinates the symplectic volume can be expressed as follows, $\Omega = e^\theta dzd\bar{z}$ The closed (1,1)-form $\omega_{can} = -i\partial\bar{\partial}\theta$ does not depend on the choice of local holomorphic coordinates and is defined globally on M. The form ω_{can} is the curvature form of the canonical connection of the canonical holomorphic line bundle on M equipped with the Hermitian fibre metric determined by the volume form Ω. Its de Rham class ε = $[\omega_{can}]$ is the first Chern class of the canonical holomorphic line bundle on M and thus depends only on the complex structure on M. The class ε is called the canonical class of the complex manifold M.
One can see from (45) that c(ν) = c_0 + νc_1 + . . . , where $c_0 \neq 0$. Thus there exists
a formal constant d(ν) such that $e^{d(v)} = c(v)$ and $d(v) + s + \theta = \Phi + \Psi$. Therefore the formal potential Ψ of the form $\tilde{\omega}$ is expressed explicitly, Ψ= $d(v) - (1/v)\Phi_{-1} + \theta$, from whence it follows that

$$\tilde{\omega} = -(1/v)\omega_{-1} + \omega_{can} \qquad (46).$$
Formula (46) defines $\tilde{\omega}$ globally on M. Thus the corresponding star-product $\tilde{*}$ and therefore its dual star-product ω are also globally defined.
Theorem (2.2.2), Theorem (2.2.4), Proposition (2.2.5), Proposition (2.2.6) Proposition (2.2.11), formulas (43), (44) and (45) imply the following theorem, which is the central technical result .
Theorem (2.2.13) [83]: *For any $f, g \in C^\infty(M)$ and $x \in M$ $(I^{(m)}f)(x)$ and $Q^{(m)}(f,g)(x)$ expand to asymptotic series in 1/m as m → +∞.* $\mathbb{F}\big((I^{(m)}f)(x)\big) = (If)(x)$ *and* $\mathbb{F}\big(Q^{(m)}(f,g)(x)\big) = Q(f,g)(x)$, *where* I *and* Q *are the formal Berezin transform and the formal twisted product corresponding to the star-product with separation of variables ⋆ on* (M, ω_{-1}) *whose dual star-product* $\tilde{*}$ *on* (M,−ω_{-1}) *is parametrized by the formal form* $\tilde{\omega} = -(1/v)\omega_{-1} + \omega_{can}$.

Remark(2.2.14) [83]: As shown in [107] we have the following chain of inequalities

$$\left|I^{(m)}(f)\right|_\infty = \left|\sigma(T_f^{(m)})\right|_\infty \leq \left\|T_f^{(m)}\right\| \leq |f|_\infty \qquad (47)$$

Here$\|..\||$ denotes the operator norm with respect to the norm of L^m
and $|..|_\infty$ the sup-norm on $C^\infty(M)$. Choose as $x_e \in$ M a point with $|f(x_e)| = |f|_\infty$. From Theorem (2.2.13) and the fact that the formal Berezin transform has as leading term the identity it follows that $\left|\left(I^{(m)}f\right)(x_e) - f(x_e)\right| \leq A/m$ with a suitable constant A. This implies $\left||f(x_e)| - \left|(I^{(m)}f)(x_e)\right|\right| \leq A/m$ and hence

$$|f|_\infty - \frac{A}{m} = |f(x_e)| - \frac{A}{m} \leq \left|\left(I^{(m)}f\right)(x_e)\right| \leq \left|\left(I^{(m)}f\right)\right|_\infty \qquad (48)$$

Putting (47) and (48) together we obtain

$$|f|_\infty - \tfrac{A}{m} \leq \left\|T_f^{(m)}\right\| \leq |f|_\infty \qquad (49).$$

This provides another proof of [106],. The identification of the Berezin-Toeplitz star-product will denote the star-product with separation of variables on (M, ω_{-1}) whose dual $\tilde{*}$ is the star-product with separation of variables on (M,$-\omega_{-1}$) parametrized by the formal form $\widetilde{\omega} = -(1/v)\omega_{-1} + \omega_{can}$.
Let $I = 1 + vI_1 + v^2I_2 + \cdots and\ Q = Q_0 + vQ_1 + \cdots$denote the formal Berezin transform and the formal twisted product corresponding to $\star$. Theorem (2.2.13) asserts that for given $f, g \in C^\infty(M), r \in N, x \in M$there exist constants *A*,*B* such that for sufficiently big values of m the following inequalities hold:

$$\left|\left(I^{(m)}f\right)(x) - \sum_{i=0}^{r-1}\frac{1}{m^i}I_i(f)(x)\right| \leq \frac{A}{m^r} \qquad (50)$$

$$\left|Q^{(m)}(f,g)(x) - \sum_{i=0}^{r-1}\frac{1}{m^i}Q_i(f,g)(x)\right| \leq \frac{B}{m^r} \qquad (51)$$

It was proved in [106],[108] that Berezin-Toeplitz quantization on a compact K$\ddot{a}$hler manifold *M* gives rise to a star-product on *M*. This star-product $*^{BT}$ is given by a sequence of bilinear operators $\{C_k\}$, k ≥ 0, on $C^\infty(M)$ satisfying the following conditions. For $f, g \in C^\infty(M)$ and any $r \in N$ there exists a constant *C* such that

$$\left\|T_f^{(m)}T_f^{(m)} - T_{f*_{[r]}g}^{(m)}\right\| \leq C/m^r \qquad (52)$$

Where $f_{*_{[r]}g} = \sum_{k=0}^{r-1}(1/m^k)C_k(f,g)$. The conditions (52) determine the star-product $*^{BT}$ uniquely. We call$*^{BT}$ the Berezin-Toeplitz star-product.
Recall that $f, g \in C^\infty(M)\sigma\left(T_f^{(m)}\right) = I^{(m)}(f), \sigma\left(T_f^{(m)}T_g^{(m)}\right) = Q^{(m)}(f,g)$
Passing from operators to their covariant symbols in (52) and using the inequality $|\sigma(A)| \leq \|A\|$we get that

$$\left|Q^{(m)}(f,g)(x) - I^{(m)}(f *_{[r]} g)(x)\right| \leq C/m^r \qquad (53)$$

It follows from (50) that

$$\left|\frac{1}{m^k} I^{(m)}\big(C_k(f,g)\big)(x) - \sum_{i=0}^{r-k-1} \frac{1}{m^{i+k}} I_i(C_k(f,g))(x)\right| \le \frac{A_k}{m^r} \qquad (54)$$

Summing up inequalities (51) and (54) for k = 0, 1, . . . , r − 1, we obtain that

$$\left|\big(Q^{(m)}(f,g)(x) - I^{(m)}(f *_{[r]} g)(x)\big) - \sum_{i=0}^{r-1} \frac{1}{m^i}\left(Q_i(f,g)(x) - \sum_{j+k=i} I_i(C_k(f,g)(x))\right)\right| \le \frac{D}{m^r} \qquad (55)$$

for some constant D. It follows from (53) and (55) that

$$\sum_{i=0}^{r-1} \frac{1}{m^i}\left(Q_i(f,g)(x) - \sum_{j+k=i} I_i(C_k(f,g)(x))\right) \le \frac{E}{m^r}$$

for some constant E, which infers that for i = 0, 1, . . .

$$Q_i(f,g) = \sum_{j+k=i} I_i(C_k(f,g)) \qquad (56))$$

Equalities (56) mean that $Q(f,g) = I(f *^{BT} g)$. Since I is invertible we immediately obtain that the star-products $\star'$ and $*^{BT}$ coincide. Thus the Berezin-Toeplitz deformation quantization is completely identified as the deformation quantization with separation of variables on $(\bar{M}, \omega_{-1})$ whose star-product $*^{BT}$ is opposite to . $\tilde{*}$
Using (4) we can calculate the characteristic class cl($*^{BT}$) of the Berezin-Toeplitzstar-product $*^{BT}$ It follows from (4) and (46) that the characteristic class of the star-product $\tilde{*}$ equals to$cl(\tilde{*}) = (1/i)(-[(1/v)\omega_{-1}] + \varepsilon/2)$. It is easy to show that the characteristic class of the opposite star-product $\star'$ is equal to −cl($\tilde{*}$). Since $*^{BT} = \star'$, we finally get that the characteristic class of the Berezin-Toeplitz deformation quantization is given by the formula
$cl(*^{BT}) = (1/i)(-[(1/v)\omega_{-1}] + \varepsilon/2)$.
The characteristic class of the Berezin-Toeplitz deformation quantization was first calculated by Eli Hawkins in [100] by K-theoretic methods.
As a concluding remark we would like to draw to the fact that
the classifying form ω of the star-product ⋆ is the formal object corresponding to the asymptotic expansion as $m \to +\infty$ of the pullback $\omega^{(m)}$ of the Fubini-Study form on the projective space $\mathbb{p}(H_m^*)$ via Kodaira embedding of M into $\mathbb{p}(H_m^*)$ Here (H_m^*) denotes the Hilbert space dual to $H_m = \Gamma_{hol}(L^m)$. It was proved by Zelditch [106] that $\omega^{(m)}$ admits a complete asymptotic expansion in *1/m* as $m \to +\infty$. As an easy consequence of the results obtained one can show that $(\omega^{(m)})$ = ω.

Chapter 3
An Excursion Asymptotic Expansion

We discuss various instructive examples like the Segal-Bargmann-Fock space, and culminating by highlights of proofs of the existence of these quantizations using both the Boutet de Monvel theory and the approach via Fefferman's expansion and Forelli-Rudin construction .

Section (3.1): Berezin-Toeplitz Quantization

Quantization has traditionally been understood as a recipe in physics for passing from a classical system — which, loosely speaking, is something that concerns macroscopic objects and that we are familiar with from everyday's life — to the "corresponding" quantum system, which pertains to microscopic objects where things are subject to more complicated rules. The latter should reduce to the former as the size of the objects gets large, that is, as the "Planck constant", which, heuristically, corresponds to the magnitude where the quantum phenomena become relevant, tends to zero. (This is the so-called "correspondence principle", or "classical limit".)

Over the time, it became apparent that such a concept is not totally appropriate, both mathematically and physically. From the point of view of physics, itismoreappropriatetounderstandquantizationjustasacorrespondencebetween classical and quantum systems; that is, there may be quantum systems which have no classical counterpart, as well as different quantum systems corresponding to the same classical system. From the mathematical point of view, one even encounters obstacles of a different kind — namely, various "no-go" theorems show that there can exist no mathematical recipe that would fulfill all the axioms required by the physical interpretation As a result, nowadays we face the existence of many different quantization theories, ranging from geometric quantization, deformation quantization and various related operator-theoretic quantizations to Feynman path integrals, asymptotic quantization, or stochastic quantization, to mention just a few. No one of the existing approaches solves the quantization problem completely; on the other hand, on the mathematics side all these have evolved into rich theories of their own right, and with results of great depth and beauty.

We give a flavour of two of the approaches that belong to the list above, namely the Berezin and the Berezin-Toeplitz quantizations. Compared to other similar surveys like [161] or [160], we have tried to intersperse the exposition with simple examples that illustrate the main ideas, thus keeping it — we hope — accessible even to students or newcomers to the area. organized as follows.we present in some more detail what has been mentioned in the first two paragraphs above, namely, the original aspirations of the quantization theory and the various ramifications that the subsequent developments have led to. discusses what turns out to be the simplest example of Berezin-Toeplitz quantization, namely the Toeplitz operators on the Fock space. The basic principles of the Berezin-Toeplitz and Berezin quantizations in curved (i.e. non-Euclidean) spaces and the necessary tools for them are discussed respectively, while the full account of these theories appears . contains miscellaneous additional comments, bibliographic remarks, and the like.

The original concept of quantization, going back to Weyl, von Neumann, and Dirac, consists in assigning operators to functions: $f \mapsto Q_f$.

Here the functions f are supposed to live on some manifold, called the *classical phase space*; for reasons going back to classical mechanics, the manifold is taken to be *symplectic*, meaning it is equipped with a differential form of a certain kind. (We will be more specific about this later.) The operators live on some fixed, separable infinite-dimensional Hilbert space H, and are assumed to be selfadjoint if f is real-valued. (They need not be bounded in general.) One calls the functions $classical\ observables$, while the corresponding operators Q_f are the associated $quantum\ observables$. The physical interpretation is that upon performing some experiment to measure a quantity (position, velocity, momentum, energy, ...) represented by f, the possible outcomes will have the probability distribution $\langle \prod(Q_f)\rangle$, where $\prod(Q_f)$ is the spectral measure of the operator Q_f , while $u \in H$ is a unit vector characterizing the "state" of the given quantum system. In particular, if Q_f has pure point spectrum consisting of eigenvalues $\lambda_j$with eigenvectors u_j, $\|u_j\|$= 1, then the possible outcomes of measuring f will be λ_j with probability $|\langle u, u_j\rangle|^2$; if

$u = u_j$ for some j , the measurement will be deterministic and will always return λ_j . Noncommutativity of operators corresponds to the impossibility of measuring simultaneously the corresponding observables.

The simplest example of a quantization rule as above is for $M = \mathbf{R}^{2n}$, the real $2n$-space, with elements written as $(p, q) \in \mathbf{R}^n \times \mathbf{R}^n$; one thinks of $q_1, \cdots, q_n$ as the coordinates of a particle in $\mathbf{R}^n$, and of $p_1, \cdots, p_n$ as the velocities (or, more precisely, momenta) of the particle; in other words M , is the *phase space* of a single particle moving in $\mathbf{R}^n$. We take $H = L^2(\mathbf{R}^n)$ for the Hilbert space, viewed as L^2-functions in the position variables q; and define the quantum observables , Q_f for f one of the coordinate functions on $\mathbf{R}^{2n}$, by

$$Q_{q_j}: f(q) \mapsto q_j f(q),$$
$$Q_{p_j}: f(q) \mapsto \frac{h}{2\pi i}\frac{\partial f(q)}{\partial q_j} \qquad (1)$$

(*the Schrödinger representation*). These operators satisfy the *canonical commutation relations* (or just *CCR* for short)

$$\left[Q_{q_j}, Q_{q_k}\right] = \left[Q_{p_j}, Q_{p_k}\right] = 0, \qquad \forall j, k,$$
$$\left[Q_{q_j}, Q_{p_k}\right] = 0 \quad for\ j \neq k, \qquad (2)$$
$$\left[Q_{q_j}, Q_{p_j}\right] = \frac{ih}{2\pi} I,$$

where $[A, B] := AB - BA$ denotes the commutator of two operators. The parameter h , on which this map Q also depends, is the *Planck constant* ; this should be thought of as a small positive number, and the *classical limit* $h \searrow 0$ should somehow recover the classical system from the quantum one, as already mentioned.

Note that under the physical interpretation just explained, (1) implies , in particular, that it is possible to measure simultaneously the position variables q (in fact, the joint spectral distribution of the $Q_{q_1}, \cdots, Q_{q_n}$ is just the Lebesgue measure on $\mathbf{R}^n$, so the probability of finding the particle in a state given by $u \in L^2(\mathbf{R}^n)$ to be present in some set $\Omega \subset \mathbf{R}^n$ in an experiment is equal to the integral of $|u|^2$ over Ω), or the momentum variables p , or even p_j and q_k for $j \neq k$, but notq_j and p_j; the last is a reflection of the celebrated Heisenberg uncertainty principle.As h tends to zero,even the operators $Q_{q_j}$and Q_{p_j} become commutative, and the problems with simultaneous non-measurability thus disappear.

It remains to say how to assign the operators Q_f to more general functions f than the coordinate functions. There are some requirements which such an assignment should satisfy, coming from the physical interpretation:

(A1) The map $f \mapsto Q_f$ should be linear. (A2)

(The *von Neumann rule*.) For any polynomial $\emptyset: \boldsymbol{R} \to \boldsymbol{R}$, we should have

$$Q_{\emptyset \circ f} = \emptyset(Q_f).$$

(In particular, $Q_{\mathbf{1}} = I$.)

(A3) $$\left[Q_f, Q_g\right] = -\frac{ih}{2\pi} Q_{\{f,g\}},$$ where

$$\{f, g\} = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}\right)$$

is the *Poisson bracket of* f *and* g.

Here the axiom (A2) just means that if our experiment yields λ as an outcome for measuring f with some probability, then it should yield λ^2 with the same probability when measuring f^2, or, more generally, $\phi(\lambda)$ with the same probability when measuring, $\phi(f)$. Similarly, the linearity axiom (A1) is quite natural. Finally, the last axiom (A3) has to do with the time evolution of the system, as described by the Hamiltonian formalism in classical mechanics. (The last axiom also extends in an obvious way to any other manifold M on which we have an analogue of the Poisson bracket defined — these are precisely the symplectic manifolds that we have already hinted at.) Note that for f, g the coordinate functions on $= \mathbf{R}^{2n}$, the last axiom reduces precisely to the canonical commutation relations (2).

We are thus lead to the problem of extending the rules (1) in such a way that the axioms (A1)–(A3) above are satisfied. So, what are the solutions to this extension problem? (And, more generally, what would be the solutions for some more general symplectic manifold M?)

Unfortunately, here bad news come. Namely, the above axioms are inconsistent (even in the simplest case of $M = \mathbf{R}^{2n}$).

To see that, denote for brevity $P = Q_{p_1}, Q = Q_{q_1}, p = p_1, q = q_1$; then

$$pq = \frac{(p+q)^2 - p^2 - q^2}{2}$$

implies, using (A1) and (A2), that

$$Q_{pq} = \frac{(P+Q)^2 - P^2 - Q^2}{2} = \frac{PQ + QP}{2}$$

On the other hand, by (A2) $Q_{q^2} = Q^2$ and $Q_{p^2} = P^2$, so we can apply the same argument to p^2, q^2 in the place of p , q :

$$p^2q^2 = \frac{(p^2+q^2)^2 - p^4 - q^4}{2}$$

Implies , using (A1) and (A2) ,that

$$Q_{p^2q^2} = \frac{P^2Q^2 + Q^2P^2}{2}$$

Finally, as $p^2q^2 = (pq)^2$, (A2) requires that we should have $Q_{p^2q^2} = Q_{pq}^2$. However, an easy computation,using the canonical commutation relation for $P\ and\ Q$,shows that

$$\frac{P^2Q^2 + Q^2P^2}{2} \neq \left(\frac{PQ + QP}{2}\right)^2$$

(the two sides diff er by a nonzero multiple of the identity). Thus we have arrived at a contradiction.

Note that our argument above used just (A1) and (A2), so even these two axioms alone are inconsistent. It was shown by Groenewold in 1946 (with an improvement by van Hove in 1951) that, likewise, (A1) and (A3) alone are inconsistent. Finally, noticed (much later) that also (A2) and (A3) by themselves lead to contradiction. In other words, not only the three axioms (A1)–(A3) all together — although quite innocuous and very natural from the point of view of physics — but even any two of them are already inconsistent! .The contradiction deduced above used polynomial classical observables f, i.e. very nice functions; if we allow some “wilder” functions f as observables, then it can, in fact, be shown that already the von Neumann rule (A2) *alone* and the canonical commutation relations (2) lead to a contradiction. Namely, recall that there exists a continuous function f (Peáno curve) which maps $\mathbf{R}$ continuously and surjectively onto $\mathbf{R}^{2n}$. Let g be a right inverse for f, so that $g: \mathbf{R}^{2n} \to \mathbf{R}$ and $f \circ g =$id ; such g exists owing to the surjectivity of f, and can be chosen to be measurable and locally bounded. Denote, for brevity, $T = Q_g$ and consider the functions $\emptyset = p_1 \circ f, \psi = q_1 \circ f$. Then by the axiom (A2),

$$\emptyset(T) = Q_{p_1 \circ f \circ g} = Q_{p_1,}, \qquad \psi(T) = Q_{q_1 \circ f \circ g} = Q_{q_1,}$$

And

$$0 = (\emptyset\psi - \psi\phi)(T) = \phi(T)\psi(T) - \psi(T)\phi(T) = [Q_{p_1,}, Q_{q_1,}] = -\frac{ih}{2\pi}I,$$

a contradiction.

What should we do to resolve this disappointing situation? First of all, we will work solely with continuous or, still better, smooth (infinitely diff erentiable) functions ; these are anyway the only ones that we really meet in the physical realm, and it rules out the pathologies we saw in the preceding paragraph. Next, we discard the von Neumann rule, except for $\phi = \mathbf{1}$, i.e.

$$Q_1 = I.$$

The only discrepancy left there is thus the one between the linearity axiom (A1) and the Poisson brackets axiom (A3). There are two established approaches how to deal with that.

The first approach is to actually insist on both axioms, but restrict even further the set of quantizable observables, i.e. the domain of the map $f \mapsto Q_f$ (we have already restricted it to smooth functions a few lines above). For our quantization on $M = \mathbf{R}^{2n}$, if we allow only functions f at most linear in the momentum variables p_j, then the recipe

$$Q_f : \psi \longmapsto -\frac{ih}{2\pi}\left(\sum_j \frac{\partial f}{\partial p_j}\frac{\partial \psi}{\partial q_j}\right) + \left(f - \sum_j p_i \frac{\partial f}{\partial p_j}\right)\psi,$$

where $\psi = \psi(q) \in L^2(\mathbf{R}^n)$, does the job we need: it extends the Schrödinger representation (1) and satisfies (A1) and (A3). (Note that the last makes sense, since the Poisson bracket of two functions at most linear in p is again at most linear in p .) In the case of a general symplectic manifold M in the place of$\mathbf{R}^{2n}$, one can similarly make things work by restricting, in an appropriate sense, to functions at most linear in "half of the variables". In technical terms, choosing this "half of the variables" requires the concept of the so-called $polarizations$ of the manifold; by definition, a polarization is a smooth choice of subspaces of dimension n in each fiber $T_xM,\ x \in M$, of the tangent bundle TM of M . The whole approach leads to particularly appealing results of manifolds M with nice group actions (symmetries), when methods of representation theory apply, and is known as the $geometric\ quantization$ (Kostant [164], Souriau [161]).

The second approach ,on the other hand,starts by relaxing the Poisson brackets axiom (A3) to hold only asymptotically as $h \to 0$:

$$\left[Q_f, Q_g\right] = -\tfrac{ih}{2\pi} Q_{\{f,g\}} + O(h^2) \qquad (3)$$

This is the basic idea behind the $deformation\ quantization$. Before spelling out the precise definition of the latter in detail, let us look at a simple example on $\mathbf{R}^{2n}$, which

An "arbitrary" function $f(p,q)$ on $\mathbf{R}^{2n}$ can be expanded into exponentials via the Fourier transform:

$$f(p,q) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{f}(\xi,\eta) e^{2\pi i(\xi.p+\eta.q)} d\xi d\eta \qquad (4) .$$

From the Schrödinger representation (1) and the Taylor series for the exponential, is it easy to interpret the exponentials $e^{2\pi i\xi.Q_p}$ and $e^{2\pi i\eta.Q_q}$:

$$e^{2\pi i\xi.Q_p}u(q) = u(q+h\xi), \qquad e^{2\pi i\eta Q_q \cdot}u(q) = e^{2\pi i\eta.q}u(q).$$

With a bit of effort, one can also take a good guess what $e^{2\pi i(\xi.Q_p+\eta.Q_q)}$ should be. Indeed, given an $u \in L^2(\mathbf{R}^n)$, the function

$$g(q,t) = \left[\ e^{2\pi it(\xi.Q_p+\eta.Q_q)}u\right](q), \qquad t \in \mathbf{R},$$

should be a solution to $\partial g/\partial t = 2\pi i\left(\xi.Q_p + \eta.Q_q\right)g$ subject to the initial condition $g(q,0) = u(q)$; in other words,

$$\frac{\partial g}{\partial t} - \sum_{j=1}^{n} h\xi_j \frac{\partial g}{\partial q_j} = 2\pi i\eta.qg, \qquad g(q,0) = u(q).$$

Fixing q for a moment and setting $G(t) = g(q - th\xi, t)$, this becomes

$$G'(t) = 2\pi i\eta.(q - th\xi)G(t), \qquad G(0) = u(q),$$

with the solution $G(t) = e^{2\pi it\eta.q - \pi it^2h\eta.\xi}u(q).$, or

$$g(q,t) = e^{2\pi it\eta.(q+th\xi) - \pi it^2h\eta.\xi}u(q+th\xi) = e^{2\pi it\eta.q + \pi it^2h\eta.\xi}u(q+th\xi).$$

Taking $t = 1$ we are thus lead to

$$e^{2\pi i(\xi.Q_p+\eta.Q_q)}u(q) = e^{2\pi i\eta.q - \pi ih\eta.\xi}u(q+h\xi).$$

Returning to (4), let us now postulate that

$$Q_f = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{f}(\xi,\eta) e^{2\pi i(\xi.Q_p+\eta.Q_q)} d\xi d\eta =: W_f.$$

In other words, using the previous formula,

$$\begin{aligned} W_f u(q) &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{f}(\xi,\eta) e^{2\pi i\eta.q+\pi i h\eta.\xi} u(q+h\xi) d\xi\, d\eta \\ &= h^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat{f}\left(\frac{\xi-q}{h},\eta\right) e^{\pi i\eta.(q+\xi)} u(\xi) d\xi\, d\eta \\ &= h^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f\left(p,\frac{q+y}{2}\right) e^{2\pi i(q-y).p/h} u(y) dy\, dp \end{aligned}$$

by Plancherel's theorem. This is the celebrated *Weyl calculus* of pseudodifferential operators Folland's book [167]. It can be shown that, appropriately interpreted, W_f makes sense even for any tempered distribution f on $\mathbf{R}^{2n}$, being then a continuous operator from the Schwartz space $S\,(\mathbf{R}^n)$ into the tempered distributions $S'(\mathbf{R}^n)$ on $\mathbf{R}^n$. If f is sufficiently nice — for instance, if $f \in S\,(\mathbf{R}^{2n})$ — then W_f is continuous even from $S\,(\mathbf{R}^n)$ into itself. For such f and g, the product $W_f W_g$ therefore makes sense, and it turns out that

$$W_f W_g = W_{fg} + h\; W_{C_1(f,g)} + O(h^2) \tag{5}$$

as $h \searrow 0$, where

$$C_1(f,g) = \frac{i}{4\pi}\sum_{j=1}^{n}\left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\right)$$

satisfies

$$C_1(f,g) - C_1(g,f) = -\frac{i}{2\pi}\{f,g\}.$$

Hence

$$[W_f,\; W_g] = -\frac{ih}{2\pi}\; W_{\{f,g\}} + O(h^2)$$

and so that the Weyl calculus satisfies (3).

One can even do slightly better than that. Namely, the product formula (5) can even be improved to higher order: there exist $C_2, C_3, \cdots$such that

$$W_f W_g = W_{fg} + h\; W_{C_1(f,g)} + h^2 W_{C_2(f,g)} + O(h^3),$$
$$W_f W_g = W_{fg} + h\; W_{C_1(f,g)} + h^2 W_{C_2(f,g)} + h^3 W_{C_3(f,g)} + O(h^4),$$

and so on. Symbolically,

$$W_f W_g = W_{f*g} \tag{6}$$

where

$$f * g := fg + hC_1(f,g) + h^2 C_2(f,g) + h^3 C_3(f,g) + \cdots$$

The last expression should be viewed just as a formal power series in h (no convergence is asserted!), and (6) should just be understood as above, i.e.

$$W_f W_g = \sum_{j=0}^{N-1} h^j W_{C_j(f,g)} + O(h^N),$$

for any $N = 0,1,2, \ldots$.

Ultimately, one is even led to the idea that for the quantization it not really necessary to have the operators Q_f, but it suffices to have a noncommutative product like $*$. This is the essence of the second approach to resolving the inconsistency of the axioms (A1)–(A3), called the *deformation quantization* .

Given our manifold M, consider the ring $C^\infty(M)[[h]]$ of all formal power series in h over $C^\infty(M)$. That is, the elements of $C^\infty(M)[[h]]$ are formal power series

$$f=\sum_{j=0}^{\infty} h^j f_j(x) \qquad (7)$$

with $f_j \in C^\infty(M)$, and addition and multiplication defined in the usual way.

A $star\ product$ is an associative $\mathbf{C}[[h]]$ -bilinear mapping $*$ such that

$$f*g=\sum_{j=0}^{\infty} h^j C_j(f,g), \qquad \forall f,g\in C^\infty(M), \qquad (8)$$

where the bilinear operators C_j satisfy

$$C_0(f,g)=fg, \qquad C_1(f,g)-C_1(g,f)=-\frac{i}{2\pi}\{f,g\},$$
$$C_j(f,\mathbf{1})=C_j(\mathbf{1},f)=0 \qquad \forall j\geq 1.$$

(The $\mathbf{C}[[h]]$ -bilinearity means that $f*g$ is linear in each argument and $(hf)*g=f*(hg)=h(f*g)$; consequently, for any f,g as in (7),

$$\left(\sum_{j=0}^{\infty} h^j f_j(x)\right)*\left(\sum_{j=0}^{\infty} h^k g_k(x)\right)=\sum_{j,k,m=0}^{\infty} h^{j+k+m} C_m(f_j,g_k)(x),$$

where the last sum should, be re-arranged by combining together the terms with the same powerh^{j+k+m} of h.)

We have seen at the end that the Weyl calculus, with the star product defined by (6), satisfies (8) (in fact, that is exactly how the Weyl star-product was defined). From (6) and the fact that multiplication of operators in associative, i.e. $(W_f W_g)W_k=W_f(W_g W_k)$, it is also immediate that the Weyl star-product (6) is associative. Thus the Weyl calculus from is an example of deformation quantization on $\mathbf{R}^{2n}$.

The drawback of the Weyl quantization is, however, that it does not readily extend to more general phase spaces than $\mathbf{R}^{2n}$. Indeed, its definition used heavily the Fourier transform, and the Fourier transform is something which is specific only for the Euclidean spaces and a few of other situations.

Although the definition of deformation quantization,together with its physics interpretation etc., goes back to 1977 (it was introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [134]), its existence on a general symplectic manifolds was established only years later. The first proof was given by DeWilde and Lecomte in 1983 [167], followed by different proofs by Fedosov in 1985 [135] and Omori, Maeda and Yoshioka in 1991 [138]; finally, in 1997 Kontsevich established its existence even on any Poisson (i.e. more general than symplectic) manifold [133]. These constructions also allow to describe all possible deformation quantizations of a given manifold, and it turns out that they can be bijectively classified, up to a natural "equivalence", by the elements of the formal power series ring $H^2(\Omega,\mathbf{R})[[h]]$ over the second cohomology group $H^2(\Omega,\mathbf{R})$. For wealth of further information on deformation quantization, see Gutt [129].

One disadvantage of the deformation quantizationis that it works with formal power series: no convergence is assumed, nor — it turns out — can be guaranteed in general, which makes the whole thing somewhat awkward when it comes to performing some concrete calculations. It is therefore of interest to have deformation quantizations that would be induced by some operators behind, as was the case of the Weyl quantization and the formula (6), and it would be even nicer if these operators were somehow naturally related to the geometry and analysis on the manifold in question — as was, again, the case for the Weyl transform and its relationship to the Fourier transform.

We will discuss two instances of such deformation quantizations, which exist on domains in $\mathbf{C}^n$ (or, more generally, on nice Kähler manifolds). Before plunging into the formal definitions and technicalities, let us show how things work in the simplest example when the domain in question is the entire complex space $\mathbf{C}^n$.

The $Fock$, or $Segal\_Bargmann$, space on $\mathbf{C}$ is, by definition,

$$\mathcal{F}(\mathbf{C}) = \mathcal{F} \coloneqq L^2_{hol}\left(\mathbf{C}, \pi^{-1} e^{-|z|^2}\, dz\right),$$

the subspace of all entire functions in $L^2\left(\mathbf{C}, \pi^{-1} e^{-|z|^2}\, dz\right)$. Given a function $f \in \mathcal{F}$, its Taylor series $f(z) = \sum_{j=0}^{\infty} f_j z^j$ converges on all of $\mathbf{C}$, and uniformly on any compact subset. In particular, for any $R \in (0, +\infty)$ we have

$$\begin{aligned}
\int_{|z|<R} |f(z)|^2 e^{-|z|^2} \frac{dz}{\pi} &= \int_{|z|<R} \sum_{j,k=0}^{\infty} f_j z^j \overline{f_k z^k} e^{-|z|^2} \frac{dz}{\pi} \\
&= \int_0^{2\pi} \int_0^R \sum_{j,k=0}^{\infty} f_j \bar{f}_k r^{j+k} e^{(j-k)i\theta} e^{-r^2} \frac{r dr d\theta}{\pi} \\
&= \sum_{j=0}^{\infty} \left|f_j\right|^2 \int_0^R r^{2j} e^{-r^2} 2r\, dr \\
&= \sum_{j=0}^{\infty} \left|f_j\right|^2 \int_0^{\sqrt{R}} t^j e^{-t} dt\,,
\end{aligned}$$

where we have used the polar coordinates $z = re^{i\theta}$, and the interchange of integration and summation in the third equality is justified by the uniform convergence.

Letting $R \to +\infty$ yields

$$\|f\|^2 = \sum_{j=0}^{\infty} \left|f_j\right|^2 \int_0^{\infty} t^j e^{-t} dt = \sum_{j=0}^{\infty} \left|f_j\right|^2 j! \qquad (9)$$

Thus an entire function f belongs to $\mathcal{F}$ if and only if its Taylor coefficients satisfy $\sum_j \left|f_j\right|^2 j! < \infty$.

A similar computation (using the Cauchy-Schwarz inequality, Fubini's theorem and (9) to justify some interchanges of integration and summation signs) gives a formula for the scalar product of two functions $f, g \in \mathcal{F}$ in terms of their Taylor coefficients :

$$\langle f, g \rangle = \sum_{j=0}^{\infty} f_j \bar{g}_j j!. \qquad (10)$$

In particular, the monomials $z^n, n = 0,1,2,\cdots,$ form an orthogonal basis of $\mathcal{F}$, and

$$\frac{z^n}{\sqrt{n!}}, \quad n = 0,1,2,\cdots, \qquad (11)$$

is an orthonormal basis.$\mathcal{F}$

For any $z \in \mathbf{C}$ we have, by the preceding computations,

$$\begin{aligned}
|f(z)| = \left|\sum_j f_j z^j\right| &\le \sum_j \left|f_j\right| |z|^j = \sum_j \left|f_j\right| \sqrt{j!} \frac{|z|^j}{\sqrt{j!}} \\
&\le \left(\sum_j \left|f_j\right|^2 j!\right)^{1/2} \left(\sum_j \frac{|z|^{2j}}{j!}\right)^{1/2} = \|f\| e^{|z|^2/2}.
\end{aligned}$$

Thus, first, $f \mapsto f(z)$ is a bounded linear functional on $\mathcal{F}$; and second, it is in fact uniformly bounded for z in a bounded set in $\mathbf{C}$.

The latter implies (since locally uniform limits of holomorphic functions are holomorphic) that $\mathcal{F}$ is a closed subspace in $L^2\left(\mathbf{C}, e^{-|z|^2}\, dz\right)$, hence a Hilbert space on its own right.

The former implies that there exist $K_z \in \mathcal{F}$ such that

$$f(z) = \langle f, K_z \rangle \qquad \forall f \in \mathcal{F}.$$

In fact, it is not difficult to compute what K_z is explicitly. Indeed, for any $f \in \mathcal{F}$ and $z \in \mathbf{C}$,

$$f(z) = \sum_j f_j z^j = \sum_j f_j \frac{z^j}{j!} j! = \langle f, K_z \rangle,$$

by (10), where

$$K_z(w) = \sum_j \frac{\bar{z}^j}{j!} w^j = e^{\bar{z}w}.$$

Thus $K_z(w) = e^{\bar{z}w}$. The function of two variables

$$K(w, z) := K_z(w) = e^{\bar{z}w}$$

is called the *reproducing kernel* of $\mathcal{F}$, and will play an important role throughout.

For $f \in L^\infty(\mathbf{C})$, the *Toeplitz operator* with *symbol* f is, by definition, the operator $T_f: \mathcal{F} \to \mathcal{F}$ given by

$$T_f u = P(fu)$$

where $P: L^2(\mathbf{C}, \pi^{-1} e^{-|z|^2}\, dz) \to \mathcal{F}$ is the orthogonal projection. In other words,

$$T_f = PM_f\big|_{\mathcal{F}}$$

where $M_f: u \mapsto fu$ is the operator of “multiplication by$f$ ”. There is still other way of expressing T_f , using the reproducing kernel:

$$\begin{aligned} T_f u(z) &= \langle T_f u, K_z \rangle = \langle P(fu), K_z \rangle = \langle fu, PK_z \rangle \\ &= \langle fu, K_z \rangle \qquad\qquad (since\ K_z \in \mathcal{F}, so\ PK_z = K_z) \\ &= \int_{\mathbf{C}^n} f(w)u(w)K(z,w)e^{-|w|^2} \frac{dw}{\pi}, \end{aligned}$$

showing that T_f is an integral operator with integral kernel equal to $f(w)K(z,w)$ (with respect to the weight $e^{-|z|^2}\pi^{-1}$).

Several properties of Toeplitz operators are immediate from their definition:

• The map $f \mapsto T_f$ is linear.
• $\|T_f\| \le \|M_f\| = \|f\|_\infty$; in particular, T_f is bounded for $f \in L^\infty$.
• $T_{\mathbf{1}} = I$, the identity operator on $\mathcal{F}$. •

Toeplitz operators behave nicely under taking adjoints : $T_f^* = T_{\bar{f}}$.

It is frequently convenient to consider T_f even for unbounded f, when it often makes sense as a densely defined operator. For instance, since a product of two holomorphic functions is again holomorphic,

$$T_z u = P(zu) = zu$$

if $zu \in L^2$; so T_z is just “multiplication by z” on $\mathcal{F}$ (defined on the domain $\{u \in \mathcal{F}: z_u \in \mathcal{F}\}$, which is dense in $\mathcal{F}$ since it contains the basis elements (11)). Similarly, T_{z^m} for any $m = 0,1,2,\cdots$, is just the operator of “multiplication by z^m ”, defined again on a dense domain in $\mathcal{F}$ (containing the algebraic linear span of the basis elements (11), i.e. all polynomials).

More generally, for any $f \in L^\infty$,

$$T_{zf} u = P(zfu) = P\big(fP(zu)\big) = T_f T_z u$$

if $z_u \in L^2$;thus T_{zf} again makes sense as a densely defined operator, whose domain contains that of T_z, and $T_{zf} = T_f T_z$ on dom T_z. Similarly,

$$T_{z^m f} = T_f T_{z^m} = T_f z^m \tag{12}$$

for any $m = 0,1,2,\cdots$.

Taking adjoints gives :

$$T_{\bar{z}^m f} = T_{\bar{z}^m} T_f. \tag{13}$$

(It is possible to give examples, however, that in general $T_f T_g \neq T_{fg}$.)

We compute the adjoint $T_z^* = T_{\bar{z}}$. By (10), the definition of the reproducing kernel, and (13),

$$
\begin{aligned}
(T_z^* z^m)(w) = \langle T_z^* z^m, K_w\rangle = \langle z^m, T_z K_w\rangle &= \langle z^m, zK_w\rangle \\
&= \langle z^m, z\sum_j z^j \frac{\overline{w^j}}{j!}\rangle \\
&= \langle z^m, \sum_j z^{j+1} \frac{\overline{w^j}}{j!}\rangle \\
&= \frac{w^{m-1}}{(m-1)!}\langle z^m, z^m\rangle = \frac{m!}{(m-1)!} w^{m-1} \\
&= m w^{m-1}.
\end{aligned}
$$

Thus $T_z^* z^m = m z^{m-1}$, or

$$T_z^* = \frac{\partial}{\partial z} \equiv \partial.$$

Similarly $T_{z^m}^* = \partial^m$.

From these findings, we get the commutation relation

$$[T_z, T_{\bar{z}}]u = [z, \partial]u = z\partial u - \partial(zu) = -(\partial z u) = -u,$$

or $[T_z, T_{\bar{z}}] = -I$. Setting $z = p + iq$ for the real and imaginary parts, this means

$$\left[T_p, T_q\right] = \frac{1}{2i} I,$$

which agrees with the CCR for the Schrödinger representation, except for the constant factor of $h/2$.

It is easy to make even this constant factor come out right. Let us replace the Gaussian weight $\pi^{-1} e^{-|z|^2}$, which we have been using so far, by the scaled version: $\mathcal{F}_\alpha(\mathbf{C}) = \mathcal{F}_\alpha := L_{hol}^2\left(\mathbf{C}, \frac{\alpha}{\pi} e^{-\alpha|z|^2} dz\right)$,

where $\alpha > 0$ is a positive parameter. The same calculations as above reveal that an entire function $f(z) = \sum_j f_j z^j$ belongs to $\mathcal{F}_\alpha$ if and only if

$$\sum_{j=0}^{\infty} \left|f_j\right|^2 \frac{j!}{\alpha^j} < \infty,$$

that the inner product of $f, g \in \mathcal{F}_\alpha$ is given in terms of their Taylor coefficients by

$$\langle f, g\rangle_{\mathcal{F}_\alpha} = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} f_j \bar{g}_j,$$

and that $\mathcal{F}_\alpha$ has the reproducing kernel

$$K_\alpha(z, w) = e^{\alpha \bar{w} z}.$$

We have also the Toeplitz operators on $\mathcal{F}_\alpha$,

$$T_f u = P_\alpha(fu),$$

where $P_\alpha : L^2\left(\mathbf{C}, \frac{\alpha}{\pi} e^{-\alpha|z|^2} dz\right) \to \mathcal{F}_\alpha$ is the orthogonal projection. (Thus T_f now depends also on the parameter α, although this is not reflected by the notation.) Finally, all the formulas from the end remain valid, except that a factor of α appears in T_z^*:

$$T_{zf} = T_f T_z, \qquad T_{z^m} u = T_z^m u = z^m u,$$
$$T_{\bar{z}f} = T_{\bar{z}} T_f, \qquad T_{\bar{z}^m} = T_{\bar{z}}^m = T_z^{*m},$$

and

$$T_z^* = \frac{1}{\alpha}\partial.$$

All these reduce to our previous formulas for $\mathcal{F}$ when $\alpha = 1$.

The commutation relations for $T_p, T_q, z = p + iq \in \mathbf{C} \cong \mathbf{R}^2$, now become

$$\left[T_q, T_p\right] = \frac{1}{2\alpha i} I.$$

Taking $\alpha = \pi/h$ thus exactly recovers the CCR for the Schrödinger representation (1) we have started with.

Let We explore what are the commutation relations for Toeplitz operators T_f , T_g when f, g are polynomials in $z\ and\ \overline{z}$ (or, equivalently, in $and\ p$).

Recall $T_{\bar{z}} = \frac{1}{\alpha}\partial$. By the Leibniz rule,

$$T_{\bar{z}z^m}u = T_{\bar{z}}T_{z^m}u = \frac{1}{\alpha}\partial(z^m u) = \frac{mz^{m-1}}{\alpha}u + z^m\frac{1}{\alpha}\partial u,$$

or $T_{\bar{z}z^m} = T_{z^m}T_{\bar{z}} + \frac{1}{\alpha}T_{mz^{m-1}}$. Thus

$$T_{z^m}T_{\bar{z}} = T\left[\bar{z}z^m - \frac{1}{\alpha}(z^m)'\right] = T\left[\left(\bar{z} - \frac{1}{\alpha}\partial\right)z^m\right],$$

where, for typographical reasons, we have started writing $T[f]$ instead of T_f when needed. Multiplying both sides by $T_{\bar{z}^k}$ from the left, and remembering that $T_{\bar{z}^k f} = T_{\bar{z}^k}T_f$ for any f, while ∂ commutes with $\bar{z}$, we obtain

$$T_{\bar{z}^k z^m}T_{\bar{z}} = T_{\bar{z}^k}T_{z^m}T_{\bar{z}} = T_{\bar{z}^k}T\left[\left(\bar{z} - \frac{1}{\alpha}\partial\right)z^m\right] = T\left[\bar{z}^k\left(\bar{z} - \frac{1}{\alpha}\partial\right)z^m\right] = T\left[\left(\bar{z} - \frac{1}{\alpha}\partial\right)\bar{z}^k z^m\right].$$

It follows by linearity that

$$T_f T_{\bar{z}} = T\left[\left(\bar{z} - \frac{1}{\alpha}\partial\right)f\right]$$

for any polynomial f in $z, \bar{z}$.

Iterating this m times yields

$$T_f T_{\bar{z}^m} = T\left[\left(\bar{z} - \frac{1}{\alpha}\partial\right)^m f\right],$$

which by the binomial theorem (note that $\bar{z}$ and ∂ commute!) equals

$$\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}\frac{(-1)^j}{\alpha^j}\bar{z}^{m-j}\partial^j f = \sum_j \frac{(-1)^j}{j!\,\alpha^j}\left(\bar{\partial}^j\bar{z}^m\right)\partial^j f,$$

so

$$T_f T_{\bar{z}^m} = T\left[\sum_j \frac{(-1)^j}{j!\,\alpha^j}\left(\bar{\partial}^j\bar{z}^m\right)\partial^j f\right].$$

Multiplying both sides by T_{z^k} from the right, and remembering that $T_f T_{z^k} T_{z^k f}$ for any f , while $\bar{\partial}$ commutes with z, we obtain

$$\begin{aligned}T_f T_{\bar{z}^m z^k} = T_f T_{\bar{z}^m}T_{z^k} &= T\left[\sum_j \frac{(-1)^j}{j!\,\alpha^j}\left(\bar{\partial}^j\bar{z}^m\right)\partial^j f\right]T_{z^k}\\ &= T\left[\sum_j \frac{(-1)^j}{j!\,\alpha^j}z^k\left(\bar{\partial}^j\bar{z}^m\right)\partial^j f\right]\\ &= T\left[\sum_j \frac{(-1)^j}{j!\,\alpha^j}\left(\bar{\partial}^j\bar{z}^m z^k\right)\partial^j f\right].\end{aligned}$$

By linearity again, we thus get

$$T_f T_g = T\left[\sum_j \frac{(-1)^j}{j!\,\alpha^j}\left(\bar{\partial}^j g\right)\partial^j f\right] = \sum_j \alpha^{-j}T_{(-1)^j(\bar{\partial}^j g)(\partial^j f)/j!}$$

for any polynomials f, g in $z, \bar{z}$. (Note that the sum has only finitely many nonzero terms.)

The beginning of the last expansion reads

$$T_f T_g = T_{fg} - \frac{1}{\alpha} T_{(\partial f)(\bar\partial g)} + O(\alpha^{-2}).$$

Interchanging f, g and subtracting, we thus arrive at

$$[T_f, T_g] = \frac{1}{\alpha} T_{(\partial g)(\bar\partial f)-(\partial f)(\bar\partial g)} + O(\alpha^{-2}).$$

For $\alpha = \pi/h$, this becomes

$$[T_f, T_g] = \frac{h}{\pi} T_{(\partial g)(\bar\partial f)-(\partial f)(\bar\partial g)} + O(\alpha^{-2}).$$

Upon passing from $z, \bar z$ to the real and imaginary parts $z = p + iq$ (and from the holomorphic and antiholomorphic derivatives $\partial, \bar\partial$ to the real derivatives $\partial/\partial p, \partial/\partial q$) , this turns out to exactly recover our Poisson bracket axiom (A3).

In conclusion, we see that the map

$$f \mapsto T_f \quad on\ \mathcal{F}_\alpha, \qquad \alpha = \frac{\pi}{h},$$

produces a deformation quantization on $\mathbf{C}$, with star-product given by the formula

$$f * g = \sum_j \frac{(-1)^j h^j}{j!\, \pi^j} (\bar\partial^j g) \partial^j f$$

(at least for f, g polynomials in $z, \bar z$).

Everything we have done for the Fock space on $\mathbf{C}$ extends also to the analogous spaces

$$\mathcal{F}_\alpha(\mathbf{C}^n) := L^2_{hol}\left(\mathbf{C}^n, e^{-\alpha\|z\|^2} (\alpha/\pi)^n dz\right)$$

on any $\mathbf{C}^n$, $n \geq 1$. Namely, the inner product in $\mathcal{F}_\alpha$ is still given by the formula (10), except that now $j \in \mathbf{N}^n$, $\mathbf{N} = \{0,1,2,\cdots\}$, is a multiindex. The reproducing kernel is

$$K_\alpha(z, w) = e^{\alpha\langle z, w\rangle},$$

and the Toeplitz operators satisfy

$$T_{z_j} = z_j, \qquad T^*_{z_j} = \frac{1}{\alpha}\frac{\partial}{\partial z_j} \equiv \frac{1}{\alpha}\partial_j.$$

The product of Toeplitz operators is given by the formula

$$T_f T_g = \sum_{j\ multiindex} \frac{(-1)^{|j|}}{j!\, \alpha^{|j|}} T[(\partial^j f)(\bar\partial^j g)],$$

at least for f, g polynomials in $z_j, \overline{z_j}, j = 1, \cdots, n$. Finally, setting $\alpha = \pi/h$, we again arrive at a deformation quantization on $\mathbf{C}^n$, with star-product

$$f * g = \sum_{m=0}^{\infty} h^m C_m(f, g),$$

$$C_m(f, g) = \frac{(-1)^m}{\pi^m} \sum_{j\in\mathbf{N}^n, |j|=m} \frac{1}{j!} T[(\partial^j f)(\overline{\partial^j} g)]$$

(at least for f, g polynomials in $z, \bar z$).

We remark that there is actually an isomorphism, the *Bargmann transform* , mapping $L^2(\mathbf{R}^n)$ unitarily onto $\mathcal{F}_\alpha(\mathbf{C}^n)$. Transferring the Weyl operators W_f , to $\mathcal{F}_\alpha$ via this isomorphism, W_f actually becomes precisely T_f for f a first degree polynomial in $z_j, \overline{z_j}$; but this is no longer true for more general f . Thus $f \mapsto W_f$ and $f \mapsto T_f$ are actually two *different* deformation quantizations of $\mathbf{C}^n$. We will meet yet another quantization later on.

Even though our "Toeplitz quantization" on $\mathbf{C}^n$ using Toeplitz operators on Fock spaces is simple and nice, as yet it has several shortcomings. First of all, the operators $T_z, T_{\bar z}$ above are unbounded operators; although they have a common dense domain (the polynomials in $\mathcal{F}_\alpha$), extra care would be needed to deal with all the computations above on a rigorous level. Furthermore, it is not completely apparent to what extent the formula

$$T_f T_g = \sum_{j \text{ multiindex}} \frac{(-1)^{|j|}}{j!\,\alpha^{|j|}} T\big[(\partial^j f)(\bar{\partial}^j g)\big]$$

remains valid when f, g are not polynomials. Finally, we would need to see what to do to quantize other domains than $\mathbf{C}^n$.

There are tools to handle all this, which we now introduce.

Let Ω a bounded domain in $\mathbf{C}^n$, and let us keep the notation dz for the Lebesgue measure on Ω. The subspace $L^2_{\text{hol}}(\Omega)$ of all holomorphic functions in $L^2(\Omega, \mathrm{dz})$ is known as the $Bergman\ space$. By the mean-value property of holomorphic functions, if $z \in \Omega$ and $r > 0$ is such that the polydisc $D_{z,r} := \{w \in \mathbf{C}^n : |w_j - z_j| < r \ \ \forall j = 1, \cdots, n\}$ lies wholly in Ω , then

$$f(z) = (\pi r^2)^{-n} \int_{D_{z,r}} f(w) dw,$$

so

$$|f(z)| \leq (\pi r^2)^{-n} \left(\int_{D_{z,r}} dw \right)^{1/2} \left(\int_{D_{z,r}} |f(w)|^2 dw, \right)^{1/2} \leq (\pi r^2)^{-n/2} \|f\|.$$

Consequently, the evaluation functional $f \mapsto f(z)$ is bounded on $L^2_{\text{hol}}(\Omega)$, and uniformly for z in compact subsets of Ω. From the latter it follows, first of all, that L^2_{hol} is a closed subspace of L^2, hence a Hilbert space in its own right; while the former again implies that there exists a unique $K_z \in L^2_{\text{hol}}(\Omega)$ such that

$$f(z) = \langle f, K_z \rangle \qquad \forall f \in L^2_{\text{hol}}(\Omega).$$

The function

$$K(x,y) \equiv K_y(x) = \langle K_y, K_x \rangle = \overline{K(y,x)} \tag{14}$$

is thus the $reproducing\ \ kernel$ of $L^2_{\text{hol}}(\Omega)$, called the $Bergman\ kernel$; note that from (14) it is immediate that it is holomorphic in x and anti-holomorphic in y. Furthermore, since Ω was assumed to be bounded, hence of finite Lebesgue measure, the function constant on belongs to $L^2_{\text{hol}}(\Omega)$, and, consequently,

$$1 = \mathbf{1}(x) = \langle \mathbf{1}, K_x \rangle \leq \|\mathbf{1}\| \|K_x\|, \tag{15}$$

implying that $\|K_x\| > 0$ for all $x \in \Omega$.

While quantization is a recipe for associating operators to functions, here we come across an assignment going in the other direction, i.e. mapping operators on some Hilbert space into functions on some domain. These functions are commonly called the $symbol$ of the corresponding operator, and the whole process is often called a $symbol\ calculus,$ or $dequantization.$ (Similarly, quantization is sometimes called an $operator\ calculus$ in various contexts.) Here is an instance of such process, which is characteristic for the Bergman spaces.

For an operator T on the Bergman space $L^2_{\text{hol}}(\Omega)$, the $Berezin\ symbol\ \tilde{T}$of T is the function on Ω given by

$$\tilde{T}(x) = \frac{\langle T K_x, K_x \rangle}{\langle K_x, K_x \rangle} = \langle T k_x, k_x \rangle, \qquad k_x := \frac{K_x}{\|K_x\|}.$$

Note that this definition makes sense, since the denominator is positive by (15).

There are a number of properties of the symbol map $T \mapsto \tilde{T}$ immediate from its definition:

- The mapping $T \mapsto \tilde{T}$ is linear.
- $\tilde{I} = \mathbf{1}$, i.e. the symbol of the identity operator is the function constant one.
- $\widetilde{T^*} = \bar{\tilde{T}}$.
- If T is bounded, then $\tilde{T}$ is a bounded function; in fact, $\|\tilde{T}\|_\infty \leq \|T\|$.

Moreover, the functionse $\tilde{T}$ is smooth (in fact, even real-analytic), because it is the restriction to the diagonal $x = y$ of the function of two variables

$$\tilde{T}(x,y) := \frac{\langle TK_y, K_x\rangle}{\langle K_y, K_x\rangle} = \frac{\langle TK_y, K_x\rangle}{K(x,y)}$$

holomorphic in $x, \bar{y}$ on the set where $K(x,y) \neq 0$. (Since we know that $K(x,x) = \|K_x\|^2 > 0$ by (15), by continuity $K(x,y)$ is nonzero in some neighbourhood of the diagonal.)

However, the most important property of the symbol map is that

$$T \mapsto \tilde{T} \qquad \text{is one-to-one.} \tag{16}$$

Indeed, suppose $\tilde{T}(x) = \tilde{T}(x,x) = 0\ \forall x$. Setting $x = u + iv, \bar{y} = u - iv$, it follows that $G(u,v) := \tilde{T}(u + iv, \bar{u} + i\bar{v})$ is a holomorphic function of u, v which vanishes for all u, v real. By uniqueness principle for holomorphic functions G, must vanish identically, so $\tilde{T}(x,y) = 0\ \forall x, y$, hence $\langle TK_x, K_y\rangle = TK_x(y) = 0\ \forall x, y$. However,

$$\tilde{T}^* f(x) = \langle T^* f, K_x\rangle = \langle f, TK_x\rangle = \int_\Omega f(y)\overline{TK_x(y)}dy,$$

so $T^* f(x) = 0$ for all f and x. Hence, $T^* = 0$ and $T = 0$, proving the injectivity of the map $T \mapsto \tilde{T}$.

As before, the Toeplitz operator on $L^2_{\text{hol}}(\Omega)$ with symbol $\phi \in L^\infty(\Omega)$ is defined as

$$T_\phi f = P(\phi f)$$

where $P: L^2 \to L^2_{\text{hol}}$ is the orthogonal projection (called the *Bergman projection*). All the properties familiar from the Fock space setting remain in force here:

- $f \mapsto T_f$ is linear;
- $T_{\mathbf{1}} = I$;
- $T_f^* = T_{\bar{f}}$;
- $\|T_f\| \leq \|f\|_\infty$.

Furthermore, for ϕ bounded holomorphic, T_ϕ is just the operator of "multiplication by ϕ" on the Bergman space; and for ϕ bounded holomorphic and f arbitrary,

$$T_{f\phi} = T_f T_\phi, \qquad T_{\bar{\phi}f} = T_{\bar{\phi}} T_f.$$

The difference against the Fock space is that now, since Ω is bounded, there are plenty of bounded holomorphic functions on Ω (not just the constants), e.g. all holomorphic polynomials.

We finally remark — although this is not needed, unlike the corresponding property of the Berezin symbol map, anywhere in the sequel — that the map $f \mapsto T_f$ is also one-to-one. Indeed, assume that $T_f = 0$; then $\langle T_f u, v\rangle = \langle fu, v\rangle = 0$ for any holomorphic polynomials u, v, in particular, $\langle fz^j, z^m\rangle = 0$, or

$$\int_\Omega f(z) z^j \bar{z}^m dz = 0$$

for any multiindices j, m. By the Stone-Weierstrass theorem, this implies that

$$\int f(z) g(z) dz = 0$$

for any function g continuous on the closure $\overline{\Omega}$ of Ω. By the Riesz representation theorem,this means that $f(z)dz$ is the zero measure,and,consequently,that $f = 0$ almost everywhere, as claimed.

The Toeplitz correspondence assigns the operator T_f to a function f, while the Berezin symbol map assigns the function $\tilde{T}$ to an operator T. The *Berezin trans_ form* is the composition of these two maps; that is, it assigns to a function f on Ω again a function on Ω, denoted Bf or $\tilde{f}$, and given by

$$Bf := \tilde{f} := \widetilde{T_f}.$$

Chasing through the definitions shows that B is in fact an integral operator:

$$\tilde{f}(x) = \frac{\langle fK_x, K_x\rangle}{\langle K_x, K_x\rangle} = \int_\Omega f(y) \frac{|K(x,y)|^2}{K(x,x)} dy.$$

One also checks easily that B has the following properties, which can either be derived from those

of the Toeplitz operators and the Berezin symbols, or verified directly.
• $f \mapsto B_f$ is linear;
• $B_{\mathbf{1}} = \mathbf{1}$;
• $B\bar{f} = \overline{Bf}$;
• $\|Bf\|_\infty \leq \|f\|_\infty$.. Also, Bf is always a real-analytic function on Ω , and the operator B is one-to-one.

In an obvious manner, all the objects described generalize also to the case of weighted L^2 spaces.Namely,let $w > 0$ be a positive continuous weight on Ω , integrable there with respect to the Lebesgue measure. The associated *weighted Bergman space* on Ω with respect to w is the subspace $L^2_{hol}(\Omega, w)$of all holomorphic functions in $L^2(\Omega, w)$. Using the mean-value property of harmonic functions, one again shows that the point evaluations $f \mapsto f(z)$ are continuous on $L^2_{hol}(\Omega, w)$, uniformly on compact subsets (the continuity and positivity of w is needed here); implying as before that $L^2_{hol}(\Omega, w)$ is a closed subspace of $L^2(\Omega, w)$ — hence a Hilbert space on its own — and that is possesses a reproducing kernel, the *weighted Bergman kernel* $K_w(x, y) \equiv K_{w,y}(x)$. The Berezin symbol $\tilde{T}$ of an operator T on $L^2_{hol}(\Omega, w)$ is the function on Ω

$$\tilde{T}(x) = \frac{\langle TK_{w,x}, K_{w,x}\rangle}{\langle K_{w,x}, K_{w,x}\rangle} = \langle Tk_{w,x}, k_{w,x}\rangle, \qquad k_{w,x} := \frac{K_{w,x}}{\|K_{w,x}\|}.$$

(Naturally, $\tilde{T}$ depends also on the weight w, although this is not reflected in the notation.) Here one needs that $K_w(x, x) = \|K_{w,x}\|^2 > 0$ for all $x \in \Omega$, which again follows as in (15) (and the hypothesis of the integrability of w ensures that the function constant one belongs to $L^2_{hol}(\Omega, w)$) Importantly, the Berezin symbol map $T \mapsto \tilde{T}$ is still one-to-one (with the same proof as in the unweighted case). The Toeplitz operator on $L^2_{hol}(\Omega, w)$ with symbol $\phi \in L^\infty(\Omega)$ is defined as

$$T_\phi f = P_w(\phi f)$$

where $P_w \colon L^2(\Omega, w) \to L^2_{hol}(\Omega, w)$ is the orthogonal projection (the *weighted Bergman projection*). Finally, the *weighted Berezin transform* of a function f on Ω is another function on Ω, given by

$$B_w f := \tilde{f} := \widetilde{T_f}$$

(again, the simpler notation $\tilde{f}$ does not reflect that fact that $\tilde{f}$ depends also on the weight w); and B_w is in fact an integral operator

$$B_w f(x) = \frac{\langle fK_{w,x}, K_{w,x}\rangle}{\langle K_{w,x}, K_{w,x}\rangle} = \int_\Omega f(y)\frac{|K_w(x, y)|^2}{K_w(x, x)}w(y)dy.$$

We (at last!) describe how all these concepts can be utilized for the construction of the special deformation quantizations on Ω mentioned.

For the Fock spaces $\mathcal{F}_\alpha$, $\alpha = \pi/h$, we have seen that the Toeplitz calculus assigning to a function f on $\mathbf{C}^n$ the Toeplitz operator T_f on $\mathcal{F}_\alpha$ yields a deformation quantization of $\mathbf{C}^n$. The main idea of*Berezin_Toeplitz* quantization is to use the Toeplitz operators in the same way also on a general domain Ω. Of course, what is unclear is the right substitute for the Gaussian measures $e^{-\pi|z|^2/h}$on $\mathbf{C}^n$.

The main problem in the Berezin-Toeplitz quantization is thus to find a family of weights ρ_h , $h > 0$, on the domain Ω such that the corresponding Toeplitz operators on $L^2_{\mathrm{hol}}(\Omega, \rho_h)$ satisfy

$$T_f T_g = \sum_{j=0}^{\infty} h^j T\left[C_j(f, g)\right] \qquad (17)$$

in some sense, where $C_j$are some bidifferential operators such that $C_0(f, g) = fg$ and

$$C_1(f, g) - C_1(g, f) = \frac{i}{2\pi}\{f, g\}$$

for some given Poisson bracket $\{.,.\}$ on Ω .

Recall that for $\Omega = \mathbf{C}$ and $\rho_h(z) = e^{-\pi|z|^2/h}h^{-1}dz$, this was fulfilled with $C_j(f,g) = \frac{1}{j!}(\partial^j f)(\bar{\partial}^j g)$. (And similarly for $\mathbf{C}^n$.)

The operators $C_j \equiv C_j^{BT}$ then define a star-product

$$f *_{BT} g := \sum_{j=0}^{\infty} h^j C_j^{BT}(f,g), \qquad f,g \in C^{\infty}(\Omega),$$

called $Berezin-Toeplitz\ star-product$ (and denoted $*_{BT}$ to distinguish it from the various other star-products around).

This method is not based on Toeplitz operators, but rather on the Berezin symbols.

Consider, quite generally, any weight w on Ω of the kind discussed. Since the Berezin symbol map $T \mapsto \widetilde{T}$ is one-to-one, we can introduce a noncommutative product $*_w$ on (some) functions on Ω by

$$\tilde{S} *_w \tilde{T} := \widetilde{ST}.$$

The product $f *_w g$ is thus defined only for f, g in the set

$$\mathcal{A}_w := \{\tilde{T}: T \text{ is a bounded linear operator on } L^2_{\text{hol}}(\Omega, w)\}$$

(which also depends on w). The product $f *_w g$ then also belongs to $\mathcal{A}_w$, and $*_w$ is associative (since the multiplication of operators is).

The idea is to glue these non-commutative products $*_w$, as w is let to vary with the Planck constant h, into a star product.

More precisely, the $Berezin\ quantization$ amounts to finding a family of weights ρ_h, $h > 0$, such that, first of all, the intersection

$$\mathcal{A} := \bigcap_{h>0} \mathcal{A}_{\rho_h}$$

is sufficiently large; and, second, that for $f, g \in \mathcal{A}$,

$$f *_{\rho_h} g = \sum_{j=0}^{\infty} h^j C_j(f,g)$$

asymptotically as $h \searrow 0$, where C_j are some bidifferential operators such that $C_0(f,g) = fg$ and

$$C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\}$$

for a given Poisson bracket $\{.,.\}$ on Ω.

Here "sufficiently large" means, basically, that $\mathcal{A}$ should be so large that the bilinear operators $C_j(f,g)$ are uniquely determined by their values on $f, g \in \mathcal{A}$. Since C_j are differential operators in each argument, this will be the case, for instance, whenever for any point x any finite set J of multiindices, and any set of complex numbers $c_j, j \in J$, we can find an element $f \in \mathcal{A}$ such that $\partial^j f(x) = c_j\ \forall j \in J$. In particular, it is enough if $\mathcal{A}$ contains all polynomials (in z and $\bar{z}$) on Ω.

The resulting bidifferential operators $C_j \equiv C_j^B$ then, of course, define the desired star-product

$$f *_B g := \sum_{j=0}^{\infty} h^j C_j^B(f,g), \qquad f,g \in C^{\infty}(\Omega),$$

called the $Berezin\ star-product$ (and denoted $*_B$ to distinguish it from the BerezinToeplitz star-product).

So far, we have not exhibited any example of the Berezin quantization, even on $\mathbf{C}^n$. We will do that by showing that it is in fact related to another problem, which has a very familiar answer on $\mathbf{C}^n$. In fact, the problem described can be reduced to one concerning the asymptotic behaviour of the weighted Berezin transforms B_w with the appropriate weights w. More precisely, the following holds.

Suppose we can find a family of weights ρ_h, $h > 0$ on Ω, such that as $h \to 0$, the corresponding weighted Berezin transforms $B_{\rho_h} \equiv B_h$ have an asymptotic expansion

$$B_h = Q_0 + hQ_1 + h^2 Q_2 + \cdots, \qquad (18)$$

with some differential operators Q_j, where $Q_0 = I$. Let $c_{j\alpha\beta}$ be the coefficients of Q_j, i.e.

$$Q_j f =: \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta f;$$

and
set

$$f *_{Bt} g := \sum_{j=0}^{\infty} h^j C_j(f,g),$$

where

$$C_j(f,g) \equiv C_j^{Bt}(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta} \big(\bar{\partial}^\beta f\big)(\partial^\alpha g). \tag{19}$$

If it happens that

$$C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\},$$

then $*_{Bt}$ is a star product and

$$f *_{Bt} g = f *_B g \qquad \forall f,g, \tag{20}$$

i.e. $*_{Bt}$ coincides with the Berezin star-product.

We devoted to the proof of this assertion. Once this has been done,the construction of the Berezin quantization reduces to constructing a family of weights for which the associated Berezin transforms have the nice asymptotics (18); this will be done. Furthermore, the assertion also yields immediately an easy example of a Berezin quantization on $\mathbf{C}^n$; this, as well as some other examples, will be presented below.

So let us prove (20). Suppose we have a family of weights ρ_h such that (18) holds. Denote, for brevity, by $Z_j = T_{z_j}, j = 1, \cdots, n$, the Toeplitz operator on $L^2_{\text{hol}}(\Omega, \rho_h)$ whose symbol is the coordinate function z_j; we have seen that Z_j are actually just the multiplication operators

$$Z_j: f(z) \mapsto z_j f(z).$$

Let Z_j^* be the adjoint of Z_j on $L^2_{\text{hol}}(\Omega, \rho_h)$. (Thus Z_j^* depends also on h, although it is not visible in the notation.) For $p(z,\bar{z}) = \sum_{\alpha,\beta} p_{\alpha\beta} z^\alpha \bar{z}^\beta$ a polynomial in z and $\bar{z}$, define the operators

$$V_p := \sum_{\alpha,\beta} p_{\alpha\beta} Z^\alpha Z^{*\beta}$$

on each $L^2_{\text{hol}}(\Omega, \rho_h), h > 0$ (where we are using the obvious multiindex conventions $Z^\alpha = Z_1^{\alpha_1} \cdots Z_n^{\alpha_n}$ etc.). Note that owing to the hypothesis that the domain Ω is bounded, Z_j and, hence, V_p are bounded linear operators.

Recall now our notation $K_y = K_{\rho_h}(.,y)$ for the reproducing kernels, and the notation for the "two-variable Berezin symbol" of an operator T on $L^2_{\text{hol}}(\Omega, \rho_h)$,

$$\tilde{T}(x,y) := \frac{\langle TK_y, K_x\rangle}{\langle K_y, K_x\rangle} = \frac{TK_y(x)}{K(x,y)} = \frac{\overline{T^* K_x(y)}}{K(x,y)},$$

which is defined in some neighbourhood of the diagonal in $\Omega \times \Omega$ (where $K(x,y) \neq 0$) and whose restriction to the diagonal $x = y$ coincides with the Berezin symbol $\tilde{T}(x)$ of T. Applying this in particular to the operator V_p, we get

$$\begin{aligned}
\widetilde{V_p}(x,y) &= \frac{V_p K_y(x)}{K(x,y)} = \frac{\sum_{\alpha,\beta} p_{\alpha\beta}\big(Z^\alpha Z^{*\beta} K_y\big)(x)}{K(x,y)} \\
&= \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \big(Z^{*\beta} K_y\big)(x)}{K(x,y)} = \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \langle Z^{*\beta} K_y, K_x\rangle}{K(x,y)} \\
&= \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \langle K_y, Z^\beta K_x\rangle}{K(x,y)} = \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \overline{y^\beta K_x(y)}}{K(x,y)}
\end{aligned}$$

$$= \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \bar{y}^\beta = p(x,\bar{y}) \qquad \text{for any } h.$$

In particular $\widetilde{V_p}(x) = \widetilde{V_p}(x,x) = p(x,\bar{x})$ for any h. Consequently, $p \in \mathcal{A}_{\rho_h}$ for all h, that is, $p \in \mathcal{A}$; thus $\mathcal{A}$ contains all polynomials, settling the first requirement for the Berezin quantization.

Next, for any two operators T_1, T_2 on $L^2_{\text{hol}}(\Omega, \rho_h)$,

$$\left(\widetilde{T_1 T_2}\right)(x,y) = \frac{\langle T_2 K_y, T_1^* K_x\rangle}{\langle K_y, K_x\rangle} = \frac{\int T_2 K_y(z)\overline{T_1^* K_x(z)}\, \rho_h(z)dz}{\langle K_y, K_x\rangle}$$
$$= \int \frac{\widetilde{T_2}(z,y)K_h(z,y).\widetilde{T_1}(x,z)K_h(x,z)}{\langle K_y, K_x\rangle} \rho_h(z)dz.$$

In particular,

$$\left(\widetilde{T_1 T_2}\right)(x,x) = \int \widetilde{T_1}(x,z)\widetilde{T_2}(z,x)\frac{|K_h(x,z)|^2}{K_h(x,x)}\rho_h(x)dx$$
$$= \left(B_h\left[\widetilde{T_1}(x,.)\widetilde{T_2}(.,x)\right]\right)(x).$$

Thus if (18) holds, i.e.

$$B_h = \sum_{j=0}^{\infty} h^j Q_j \qquad as\ h \to 0,$$

with some differential operators $Q_j f = \sum_{\alpha,\beta} c_{j\alpha\beta}\partial^\alpha \bar{\partial}^\beta f$, and C_j are defined by $C_j(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta}\left(\bar{\partial}^\beta f\right)(\partial^\alpha g)$, then we get for $h \to 0$

$$\left(\widetilde{T_1 T_2}\right)(x,x) = \sum_{j=0}^{\infty} h^j Q_j\left[\widetilde{T_1}(x,.)\widetilde{T_2}(.,x)\right](x)$$
$$= \sum_{j,\alpha,\beta} h^j c_{j\alpha\beta}\bar{\partial}^\beta \widetilde{T_1}(x,.)\partial^\alpha \widetilde{T_2}(.,x)\Big|_x.$$

Now since $\tilde{T}(x) = \tilde{T}(x,x)$ and $\tilde{T}(x,y)$ is holomorphic in x and anti-holomorphic in y , we have

$$\bar{\partial}^\beta \tilde{T}_1(x,.)\Big|_x = \bar{\partial}^\beta \tilde{T}_1(x)$$

(the $\tilde{T}$ on the left-hand side is the $\tilde{T}(x,y)$, and the $\tilde{T}$ on the right-hand side is the $\tilde{T}(x)$). Similarly,

$$\partial^\alpha \tilde{T}_2(.,x)\Big|_x = \partial^\alpha \tilde{T}_2(x).$$

Thus

$$\widetilde{T_1 T_2} = \sum_{j,\alpha,\beta} h^j c_{j\alpha\beta}\left(\bar{\partial}^\beta \widetilde{T_1}\right)(\partial^\alpha \widetilde{T_2})$$
$$= \sum_j h^j C_j\left(\tilde{T}_1, \tilde{T}_2\right) = \tilde{T}_1 *_{Bt} \tilde{T}_2,$$

by the definition of $*_{Bt}$. On the other hand, $\widetilde{T_1 T_2} = \tilde{T}_1 *_{\rho_h} \tilde{T}_2$, by the definition of $*_w$ (with $w = \rho_h$) so

$$\tilde{T}_1 *_{Bt} \tilde{T}_2 = \tilde{T}_1 *_{\rho_h} \tilde{T}_2.$$

Applying this to $T_1 = V_p, T_2 = V_q$ with some polynomials p, q in $z, \bar{z}$,and recalling that $\tilde{V}_p = p$, this means that

$$p *_{Bt} q = p *_{\rho_h} q$$

for any polynomials p, q in $z, \bar{z}$. Since any $f \in C^\infty(\Omega)$ can be approximated, at any given point,to any finite order by polynomials,and the $C_j(.,.)$ for both $*_{Bt}$ and $*_B$ are differential operators in each argument, necessarily $C_j^{Bt}(f,g)(x) = C_j^B(f,g)(x)$ for all $f, g \in C^\infty(\Omega)$ and $x \in \Omega$; that is, $*_{Bt} = *_B$, completing our proof.

On a slightly more heuristic level, it is possible to derive not only the Berezin, but also the Berezin-Toeplitz quantization from the asymptotics (18) of the Berezin transform; that is, to show that if (18) holds, then

$$[T_f, T_g] \approx hT_{\{f,g\}} \qquad (21)$$

as the Planck constant $h \searrow 0$. While this will not be directly needed anywhere in the sequel, we believe it is worth mentioning here.

Assume first that $f, \bar{g}$ are holomorphic. Then for any $\phi \in L^2_{\text{hol}}$

$$\langle T_f\phi, K_x\rangle = \langle f\phi, K_x\rangle = f(x)\phi(x) = f(x)\langle\phi, K_x\rangle.$$

It follows that $T_f^* K_x = \overline{f(x)} K_x$. Similarly $T_g K_x = g(x) K_x$. Hence

$$\widetilde{T_f T_g}(x) = \frac{\langle T_f T_g K_x, K_x\rangle}{\langle K_x, K_x\rangle} = \frac{\langle T_g K_x, T_f^* K_x\rangle}{\langle K_x, K_x\rangle}$$
$$= \frac{\langle g(x) K_x, \overline{f(x)} K_x\rangle}{\langle K_x, K_x\rangle} = f(x)g(x);$$

that is, $\widetilde{T_f T_g} = fg$.

On the other hand, by definition the Berezin transform and (18),

$$\tilde{T}_{fg} = B_h(fg) = fg + hQ_1(fg) + O(h^2)$$

Subtracting this from $\widetilde{T_f T_g} = fg$ gives

$$(T_f T_g - T_{fg})^{\sim} = -hQ_1(fg) + O(h^2)$$
$$= -h\widetilde{T_{Q_1(fg)}} + O(h^2).$$

"Removing the tilde" (yes, this is the heuristic part) we get, for $f, \bar{g}$ holomorphic,

$$T_f T_g - T_{fg} = -hT_{C_1^B(g,f)} + O(h^2), \qquad (22)$$

Where C_1^B is the C_1 from the Berezin quantization. Note that, as we have seen, $C_1^B(g, f)$ involves only holomorphic derivatives of f and anti-holomorphic derivatives of g (i.e. only $\partial^\alpha f$ and $\bar{\partial}^\beta g$). This also means, in particular, that for any holomorphic functions u, v,

$$C_1^B(ug, \bar{v}f) = uC_1^B(g, f)\bar{v}.$$

On the other hand, we have seen that for u, v, as above and arbitrary F and G,

$$T_G T_u = T_{uG}, \qquad T_{\bar{v}} T_F = T_{\bar{v}F}.$$

Multiplying (22) by $T_{\bar{v}}$ from the left and T_u from the right, we therefore obtain

$$T_{\bar{v}f} T_{gu} - T_{\bar{v}fgu} = T_{\bar{v}}[T_f T_g - T_{fg}]T_u$$
$$= -hT_{\bar{v}} T_{C_1^B(g,f)} T_u + O(h^2)$$
$$= -hT_{\bar{v}C_1^B(g,f)u} + O(h^2)$$
$$= -hT_{C_1^B(ug,\bar{v}f)} + O(h^2).$$

That is, (22) holds not only for $f, \bar{g}$ holomorphic, but for any f, g of the form $u\bar{v}$ with holomorphic u, v. By the same approximation argument as in the end, we conclude that actually

$$T_f T_g - T_{fg} = -hT_{C_1^B(g,f)} + O(h^2)$$

for any $f, g \in C^\infty(\Omega)$. That is, we have obtained the first two terms

$$T_f T_g = T_{C_0^{BT}(f,g)} + hT_{C_1^{BT}(f,g)} + O(h^2)$$

of the Berezin-Toeplitz star-product (17), showing, incidentally, that $C_0^{BT}(f, g) = fg$ and)

$$C_1^{BT}(f, g) = -C_1^B(g, f). \qquad (23)$$

It is clear how to continue this argument to obtain also the higher-order terms C_j^{BT} and, hence, the entire Berezin-Toeplitz star-product.

The relationship (23) between the Berezin and the Berezin-Toeplitz operator C_1 can actually be put into a rather neat form.Recall that we have our three mappings $f \mapsto T_f$ (the Toeplitz operators), $T \mapsto \tilde{T}$ (the Berezin symbol), and their composition $f \mapsto \tilde{T}_f = B_h f$ (the Berezin transform). In terms of these, the BerezinToeplitz star product was defined by

$$T_f T_g = T_{f *_{BT} g}, \qquad (24)$$

while the Berezin star product was, essentially, defined by

$$\tilde{T} *_B \tilde{S} = \widetilde{TS}.$$

Applying the last formula to $T = T_f$, $S = T_g$, and using (24), gives

$$\tilde{T}_f *_B \tilde{T}_g = \widetilde{T_f T_g} = \widetilde{T_{f *_{BT} g}},$$

or

$$Bf *_B Bg = B(f *_{BT} g).$$

In other words, the Berezin and the Berezin-Toeplitz star-products are intertwined (conjugate) by the Berezin transform. From this, one easily gets also the higherorder analogues of the relation (23), i.e. involving C_j^B and C_j^{BT} (and the operators Q_j) for $j \geq 1$.

We have already worked out the Berezin-Toeplitz quantization on $\mathbf{C}^n$ in some detail; let us see how the other approaches discussed work out in this case.

Thus, let $\Omega = \mathbf{C}^n$ and $\rho_h(z) = e^{-\alpha|z|^2}(\alpha/\pi)^n dz$, with $\alpha = \pi/h > 0$; note that the "classical limit" $h \searrow 0$ now corresponds to $\alpha \to +\infty$. Since we know the reproducing kernel to be given by $K_\alpha(x,y) = e^{\alpha\langle x,y\rangle}$, the formula for the Berezin transform becomes

$$B_\alpha f(x) = \int_{\mathbf{C}^n} f(y) \frac{|K_h(x,y)|^2}{K_h(x,x)} \rho_h(y) dy$$

$$= \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbf{C}^n} f(y) e^{-\alpha\|x-y\|^2} dy$$

This is precisely the heat solution operator at the time $t = 1/4\alpha$:

$$B_\alpha f = e^{\Delta/4\alpha} f.$$

In particular, as $\alpha \to +\infty$, we get $B_\alpha f \to f$, more precisely there is even an asymptotic expansion

$$B_\alpha f(x) = e^{\Delta/4\alpha} f(x) = f(x) + \frac{\Delta f(x)}{4\alpha} + \frac{\Delta^2 f(x)}{2!\,(4\alpha)^2} + \cdots,$$

or more briefly

$$B_\alpha = e^{\Delta/4\alpha} = \sum_{j=0}^{\infty} \alpha^{-j} \frac{\Delta^j}{j!\,4^j}.$$

We conclude that the Berezin quantization works for the above choice of weights ρ_h on $\mathbf{C}^n$, with

$$C_j(f,g) = C_j^B(f,g) := \frac{1}{j!} \sum_{|\alpha|=j} \left(\bar{\partial}^\alpha f\right)(\partial^\alpha g).$$

This can be compared with the Berezin-Toeplitz quantization formula for the same choice of weights:

$$C_j(f,g) = C_j^{BT}(f,g) := \frac{(-1)^j}{j!} \sum_{|\alpha|=j} (\partial^\alpha f)\left(\bar{\partial}^\alpha g\right).$$

Both quantize the Euclidean Poisson bracket on $\mathbf{C}^n$ (spelled out in the axiom (A3).

The second example which can be worked out explicitly to some level is the unit disc $\Omega = \mathbf{D} := \{z \in \mathbf{C}: |z| < 1\}$ in $\mathbf{C}$, with weights $\rho_h(z) = \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha, \alpha > -1$; the parameter α again plays the role of the reciprocal of h, so that $h \searrow 0$ corresponds to $\alpha \to +\infty$. A standard calculation in polar coordinates, similar to the one we did for the Fock space, shows that the reproducing kernels are

$$K_\alpha(x,y) = \frac{1}{(1-x\bar{y})^{\alpha+2}}.$$

This gives the formula for the Berezin transform:

$$B_\alpha f(x) = \frac{\alpha+1}{\pi} \int_{\mathrm{D}} f(y) \frac{(1-|x|^2)^{\alpha+2}}{(1-x\bar{y})^{2\alpha+4}} (1-|y|^2)^\alpha dy.$$

With some work, it can again be shown that as $\alpha \to +\infty$,

$$B_\alpha f = f + \frac{\tilde{\Delta} f}{4\alpha} + \cdots$$

where

$$\tilde{\Delta} f = (1-|z|^2)^2 \Delta$$

is the invariant Laplacian on **D** . (The Q_j for $j>1$ are already a bit complicated and involve Bernoulli numbers; an explicit expression for general j is not known.)
The results thus again tell us that the Berezin quantization on **D** works for the above choice of weights, with

$$C_0^B(f,g)=fg, \qquad C_1^B(f,g)=(1-|z|^2)\bar\partial f\partial g.$$

Similarly, the Berezin-Toeplitz quantization works, with

$$C_0^{BT}(f,g)=fg, \qquad C_1^{BT}(f,g)=-(1-|z|^2)\partial f\bar\partial g.$$

Explicit expressions for C_j^B and C_j^{BT} for general $j\geq 2$ are again unknown.

Both methods quantize the Poisson bracket

$$\{f,g\}=(1-|z|^2)^2(\bar\partial f\partial g-\partial g\bar\partial f)$$

associated to the invariant (= Poincaré, Lobachevsky) metric on **D**.

Our third and final example concerns the unit ball $\Omega=\mathbf{B}^n:=\{z\in\mathbf{C}^n:|z|<1\}$ in $\mathbf{C}^n$, with weights $\rho_h(z)=c_\alpha(1-|z|^2)^\alpha$, where $\alpha=1/h\to+\infty$ and c_α is a normalizing constant making ρ_h to be of total mass 1. The reproducing kernel equals

$$K_\alpha(x,y)=\frac{1}{(1-\langle x,y\rangle)^{\alpha+n+1}},$$

yielding the expression for the Berezin transform

$$B_\alpha f(x)=c_\alpha\int_{\mathbf{B}^n}f(y)\frac{(1-|x|^2)^{\alpha+n+1}}{(1-\langle x,y\rangle)^{2\alpha+2n+2}}(1-|y|^2)^\alpha dy.$$

Again,

$$B_\alpha f=f+\frac{\tilde\Delta f}{4\alpha}+\cdots$$

as $\alpha\to+\infty$, with $\tilde\Delta$ the invariant Laplacian on $\mathbf{B}^n$. Both the Berezin and the Berezin-Toeplitz quantizations work for the above choice of weights, and their coefficients C_j are given by formulas of a similar nature as for the disc.

For a later occasion, it is instructive to summarize some observations from these examples here. Looking at the weights and the corresponding reproducing kernels in the three cases, namely,

$$\rho_\alpha(z)=\left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2}, \qquad K_\alpha(x,y)=e^{\alpha\langle x,y\rangle}$$

for the Fock space on $\mathbf{C}^n$;

$$\rho_\alpha(z)=\frac{\alpha+1}{\pi}(1-|z|^2)^\alpha, \qquad K_\alpha(x,y)=(1-x\bar y)^{-\alpha-2}$$

for the disc; and

$$\rho_\alpha(z)=c_\alpha(1-|z|^2)^\alpha, \qquad K_\alpha(x,y)=(1-\langle x,y\rangle)^{-\alpha-n-1}$$

for the ball, we observe that $K_\alpha(x,x)$ is just the reciprocal of the weight $\rho_h(x)$, up to the normalization constants and possibly a shift in the exponent α.

Furthermore, we have seen in all three cases that the Berezin transform B_α is an approximate identity as $\alpha\to+\infty$, more precisely

$$B_\alpha=I+\frac{Q_1}{\alpha}+\frac{Q_2}{\alpha^2}+\cdots,$$

where Q_1 is, up to a constant factor, some kind of "invariant Laplacian" on the domain in question.

We will later that both these observation, in fact, remain in force in a much more general setting. The main problem for carrying out both the Berezin and the Berezin-Toeplitz quantization is thus to find the weights $\rho_h, h>0$, on Ω so that (17) and (18) hold. There is a way to see what should be the right choice, which we now describe.

It is time we gave a precise definition of the object we wish to quantize, the *Poisson bracket* on our domain (or manifold) Ω . Quite generally, a *symplectic manifold* is a real manifold equipped with a 2-form

$$\omega=\sum_{j,k=1}^m g_{jk}dx_j\wedge dx_k$$

which is non-degenerate (i.e.the matrix $\{g_{jk}\}_{j,k=1}^{m}$ is invertible) and closed ($d\omega = 0$). Here m is the real dimension of the manifold, which must necessarily be even. The Poisson bracket is then defined as

$$\{f,g\} = \sum_{j,k=1}^{m} g^{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}$$

where $\{g^{jk}\}_{j,k=1}^{m}$ is the inverse matrix to $\{g_{jk}\}_{j,k=1}^{m}$. For the case of complex manifolds that we have here, it is furthermore important that the symplectic form be compatible with the complex structure, and also it is more convenient to use the complex coordinates $z_j, \overline{z_j}, j = 1, \cdots, n$, rather than the real coordinates $x_k, k = 1, \cdots, m, m = 2n$. On the level of the form ω, this translates into the fact that ω is $K\ddot{a}hler$, meaning that (in local coordinates)

$$\omega = \sum_{j,k=1}^{n} g_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

with some positive-definite matrix $\{g_{j\bar{k}}\}_{j,k=1}^{n}$ satisfying

$$\partial_l g_{j\bar{k}} = \partial_j g_{l\bar{k}}, \qquad \partial_{\bar{l}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{l}}. \tag{25}$$

The Poisson bracket is then given by

$$\{f,g\} = \sum_{j,k=1}^{n} g^{\bar{j}k} \left(\bar{\partial}_j f \partial_k g - \partial_j f \bar{\partial}_k g\right), \tag{26}$$

where $\{g^{\bar{j}k}\}_{j,k=1}^{n}$ is the inverse matrix to$\{g_{j\bar{k}}\}$. Finally, the 2-form ω determines (both in the symplectic and in the Kähler case)also a nonvanishing volume element ω^non Ω .

To find the right choice of the weights ρ_h, we take guidance from group invariance.

Assume there is a group G acting on Ω by biholomorphic transformations preserving the form ω . Naturally, we would then want our quantizations to be G-invariant, i.e. to satisfy

$$(f \circ \emptyset) * (g \circ \emptyset) = (f * g) \circ \emptyset, \qquad \forall \emptyset \in G.$$

On the level of the Berezin quantization, this means that the operators Q_j in (18), and, hence, B itself, should commute with the action of G . An examination of the formula defining the Berezin transform with respect to some weight ρ shows that this happens if and only if

$$\frac{\left|K_\rho(x,y)\right|^2}{K_\rho(y,y)} \rho(x)dx = \frac{\left|K_\rho(\emptyset(x),\emptyset(y))\right|^2}{K_\rho(\emptyset(y),\emptyset(y))} \rho(\emptyset(x)) d\emptyset(x).$$

In particular, the ratio

$$\frac{\rho\big(\emptyset(x)\big) d\emptyset(x)}{\rho(x)dx} = \frac{\left|K_\rho(x,y)\right|^2}{K_\rho(y,y)} \frac{K_\rho(\emptyset(y),\emptyset(y))}{\left|K_\rho(\emptyset(x),\emptyset(y))\right|^2}$$

has to be the squared modulus of a holomorphic function. Writing

$$\rho(z)dz = w(z).\,\omega^n(z) \tag{27}$$

with the (G-invariant) volume element ω^n and some (positive) weight function w , the last condition translates into

$$w\big(\emptyset(z)\big) = w(z)|f_\emptyset(z)|^2$$

for some holomorphic functions $f_\emptyset$ .In other words ,the form $\partial\bar{\partial}\log w$ is Ginvariant.

But the simplest examples of G -invariant 2-forms (and if G is sufficiently "ample", the only ones) are clearly the constant multiples of ω. Thus we are led to

$$\partial\bar{\partial}\log w = -c\omega$$

with some constant c. It follows that

$$\omega = \partial\bar{\partial}\Phi, \qquad \Phi := -\frac{1}{c}\log w,$$

i.e. that $\Phi = -\frac{1}{c}\log w$ is a real-valued $K\ddot{a}hler\ potential$ for ω . This gives for the volume element

$$\omega^n(z) = \det\left[\partial\,\bar{\partial}\Phi(z)\right] dz,$$

and (27) gives

$$\rho(\mathrm{z}) = \mathrm{e}^{-\mathrm{c}\Phi(z)} \det\left[\partial\,\bar{\partial}\Phi(z)\right] dz.$$

Returning the Planck constant dependence into play, we therefore see that the sought weights ρ_h should be of the form

$$\rho_h = e^{-\mathrm{c}\Phi} \det\left[\partial\,\bar{\partial}\Phi\right],$$

with some $c = c(h)$ depending only on h .

Note that the condition $\omega = \partial\bar{\partial}\Phi$ means that

$$g_{j\bar{k}}(z) = \frac{\partial^2\Phi(\mathrm{z})}{\partial z_j \partial \bar{z}_k}.$$

The fact that this matrix is positive-definite, for each $z \in \Omega$, means precisely that the potential Φ is *strictly plurisubharmonic* on Ω . We will usually abbreviate "strictly plurisubharmonic" to "strictly PSH".

Finally, the condition

$$C_1(f,g) - C_1(g,f) = -\frac{i}{2\pi}\{f,g\} \tag{28}$$

in the Berezin quantization will be satisfied if the operator Q_1in (18) equals

$$Q_1 = \sum_{j,k=1}^{n} g^{\bar{j}k}\partial_k\bar{\partial}_j =: \Delta,$$

the *Laplace − Beltrami* operator associated to ω. Indeed, in that case we have by (19)

$$C_1(f,g) = \sum_{j,k=1}^{n} g^{\bar{j}k}\left(\bar{\partial}_j f\right)(\partial_k g),$$

and (28) follows by (26).

We have thus arrived at a final recipe for the Berezin and Berezin-Toeplitz quantizations on a domain $\Omega \subset \mathbf{C}^n$ equipped with a Kähler form ω and the corresponding Poisson bracket. Namely:

(i) . There must exist a Kähler potential Φ for ω, i.e. a strictly PSH function Φ such that $\omega = \partial\bar{\partial}\Phi$.

(ii) . We take the Bergman spaces $L^2_{\mathrm{hol}}\left(\Omega, e^{-\mathrm{c}\Phi}\det\left[\partial\,\bar{\partial}\Phi\right]\right)$ where $c \in \mathbf{R}$ is a parameter. Denote by $K_c(x,y)$ the reproducing kernel of this space, by B_c the associated Berezin transform, and by $T_f^{(c)}$ the Toeplitz operator on this space with symbol f.

(iii) . See if $c = c(h)$ can be chosen so that

$$B_c = I + h\Delta + h^2 Q_2 + h^3 Q_3 + \cdots \qquad as\ h \to 0$$

with some differential operators $Q_j, Q_0 = I, Q_1 = \Delta$ (for the Berezin quantization); and

$$T_f^{(c)}T_g^{(c)} = \sum_{j=0}^{\infty} h^j T^{(c)}_{C_j(f,g)} \qquad as\ h \searrow 0$$

in some sense,with $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = -\frac{i}{2\pi}\{f,g\}$ (for the Berezin-Toeplitz quantization).

It turns out that under suitable hypothesis on Ω and Φ , this recipe indeed works, with

$$c(h) = 1/h.$$

For brevity, let us denote by $d\mu_h$the corresponding measures

$$d\mu_h(z) := e^{-\Phi(\mathrm{z})/h} \det\left[g_{k\bar{j}}(z)\right] \mathrm{d}z, \qquad h > 0,$$

and by $L^2_{\mathrm{hol},h} = L^2_{\mathrm{hol}}(\Omega, d\mu_h)$ the associated weighted Bergman spaces; also K_c, B_c and $T_f^{(c)}$ will be written as K_h, B_h and T_f , respectively. We will also sometimes use our earlier notation $\alpha = 1/h$ for $\frac{1}{h}$ rather than c.

For simplicity, we have so far really discussed only the situation when Ω is a domain in $\mathbf{C}^n$. It turns out that the whole formalism works also on arbitrary Kähler manifolds, just with some minor technical adjustments. The most conspicuous of them is that instead of considering Bergman spaces of functions on Ω , one needs to consider, more generally, spaces of a holomorphic line bundle $\mathcal{L}$, equipped with a Hermitian metric (in the fibers) given locally by $e^{-\Phi}$ the curvature form of this Hermitian metric should coincide with the given Kähler form ω). For such $\mathcal{L}$ to exist, it is necessary that the cohomology class of ω be integral. The role of the weighted Bergman spaces $L^2_{\text{hol}}(\Omega, d\mu_h)$ is then played by the spaces of holomorphic L^2 of the tensor powers $\mathcal{L}^{\otimes m}, m = 1/h = 1,2,\cdots$; in particular, the Planck constant can approach 0 only through a discrete set of values. However, the whole formalism — weighted Bergman kernels, Berezin symbols, Toeplitz operators, and Berezin transforms — still makes perfect sense, and so does the above recipe for Berezin and BerezinToeplitz quantizations.

Since both B_h and T_f are defined by formulas involving the weighted Bergman kernels K_h, the key to proving the viability of our recipe is obviously an understanding of the behaviour of $K_h(x,y)$ as $h \searrow 0$. Historically, there are two approaches how to handle this problem, which both appeared independently around $1997-1998$. The first one was developed of compact manifolds by Zelditch [44], who gave, in our language, the asymptotics of the reproducing kernels $K_h(x,x)$ on the diagonal as $h \to 0$; this was subsequently extended also away from the diagonal by Catlin [133]. These did not consider B_h and T_f, but rather were inspired by certain geometric applications going back to Tian in 1990 [143] (with a follow-up by Ruan [139]). The proofs rely on a theory, due to Boutet de Monvel and Guillemin [151], of Fourier integral operators of Hermite type, which was in exactly the same way used, in fact, already in 1994 by Bordemann, Meinrenken and Schlichenmaier [9] to establish the result about T_f on compact manifolds directly without those for K_h and B_h (thus by passing the Berezin quantization).

The second approach, dealt with domains in $\mathbf{C}^n$ not manifolds, and relied on somewhat simpler methods (Feff erman's expansion and $\bar{\partial}$-techniques) to obtain the asymptotics on K_h and B_h [128] [129] [130]; naturally, some hypothesis on the behaviour of Φ at the boundary were needed. The result for T_f can, however, be established in this case only for bounded domains, and one still has to resort to the more sophisticated machinery used by Bordemann, Meinrenken and Schlichenmaier [129].

Prior to these general results, Berezin and Berezin-Toeplitz quantizations had been established only ad hoc in some special cases, such as in dimension $n = 1$ (i.e. for Riemann surfaces) with the Poincaré metric by Klimek and Lesniewski in 1991 [132] (using uniformization), for $\Omega = \mathbf{C}^n$ with the Euclidean metric by Coburn in 1993 [134], or for bounded symmetric domains with the invariant metric by Borthwick, Lesniewski and Upmeier in 1994 [130]. The basic idea, in any case, goes back — as the terminology rightly suggests — to Berezin in 1975 [136]. The equivalence of the Berezin quantization and the asymptotic expansion of the Berezin transform is due to Karabegov [131]. Some recent extensions and generalizations of the theory are discussed e.g. [137] by Ma and Marinescu, [137] by Berndtsson, Berman and Sjöstrand.

We will first handle the case of the Berezin quantization by the second of the above-mentioned approaches. Then we proceed to deal with the Berezin-Toeplitz quantization via the first approach, adapted to the context to which we have also restricted ourselves hitherto domains in $\mathbf{C}^n$ rather than compact manifolds.

Recall that a smooth function $\Phi\colon \Omega \to \mathbf{R}$ on a domain Ω in $\mathbf{C}^n$ is called *strictly plurisubharmonic* (strictly-PSH) if for any $z \in \Omega$ and $v \in \mathbf{C}^n$, the function of one complex variable

$$t \mapsto \Phi(z + tv), \qquad t \in \mathbf{C}$$

is strictly subharmonic where defined. Equivalently, Φ is strictly- PSH if the matrix of mixed second derivatives

$$\left[\frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k}\right]_{j,k=1}^{n}$$

is positive definite.

A bounded domain $\Omega \subset \mathbf{C}^n$ with smooth boundary is called strictly *strictly pseudoconvex* if there exists a smooth function r such that

$$r > 0 \quad \text{on } \Omega\,, \qquad r = 0,\ \|\nabla r\| > 0 \quad \text{on } \partial\Omega,$$
$$-r \text{ is strictly- PSH in a neighbourhood of } \overline{\Omega}.$$

One calls r a strictly- PSH defining function for Ω.

For completeness, we remark that there are also (not necessarily strictly) plurisubharmonic (PSH) functions, for which $t \mapsto \Phi(z_t v)$ is assumed to be only subharmonic (not necessarily strictly), or, equivalently, the matrix of mixed second-order derivatives is only positive semi-definite; and (not necessarily strictly) pseudoconvex domains, which can be defined as increasing unions of strictly pseudoconvex domains. (This is *not* the same thing as having a — not necessarily strictly — PSH defining function.)

Pseudoconvex domains are the natural domains in $\mathbf{C}^n$ on which holomorphic functions live: if Ω is not pseudoconvex, then there exist a larger domain Ω' such that every holomorphic function on Ω in fact extends holomorphically to Ω'.
An example of non-pseudoconvex domain is the domain $\Omega = \{z \in \mathbf{C}^n : 1 < |z| < 2\},\ n > 1$, for which $\Omega' = \{z \in \mathbf{C}^n : |z| < 2\}$ In dimension $n = 1$, as we all know from basic complex analysis, *all* domains are pseudoconvex.

Strictly pseudoconvex domains are those whose boundary is, additionally, in some sense "non-degenerate", which makes it possible to establish results which have as yet no known counterparts in the non-strictly pseudoconvex case. We will come across some of these results.

The upshot of all the above is that pseudoconvex domains are the ones on which it makes sense to study holomorphic functions; strictly pseudoconvex domains are the manageable ones.
Let $\Omega \subset \mathbf{C}^n$ *be smoothly bounded and strictly pseudoconvex , and* Φ *a strictly-PSH function on* Ω *such that* $e^{-\Phi} = r$ *is a defining function for* Ω.

Then for the weights $w = e^{-\alpha\Phi}\det\left[\partial\,\bar{\partial}\Phi\right]$, *we have as* $\alpha \to +\infty, \alpha \in \boldsymbol{Z}$,

$$K_\alpha(x,x) \approx e^{\alpha\Phi(x)}\,\frac{\alpha^n}{\pi^n}\sum_{j=0}^{\infty}\frac{b_j(x)}{\alpha^j},$$

with some functions $b_j \in C^\infty(\Omega), b_0 = \det\left[\partial\,\bar{\partial}\Phi\right]$ *; and*

$$B_\alpha f = \sum_{j=0}^{\infty}\frac{Q_j f}{\alpha^j}$$

where Q_j *are some differential operators, in particular* $Q_0 = I$ *and*

$$Q_1 = \sum_{j,k=1}^{n} g^{\bar{j}k}\,\frac{\partial^2}{\partial z_k \partial \bar{z}_j} =: \Delta,$$

the Laplace-Beltrami operator. Here $g^{\bar{j}k}$ *is the inverse matrix to* $g_{j\bar{k}} := \frac{\partial^2\Phi}{\partial z_j \partial \bar{z}_k}$.

It follows, as explained ,that denoting by $c_{j\alpha\beta}$ the coefficients of theoperators Q_j,

$$Q_j f = \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta}\,\partial^\alpha\bar{\partial}^\beta f,$$

and setting

$$f *_{Bt} g := \sum_{j=0}^{\infty} h^j C_j(f,g),$$

where

$$C_j(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta}\left(\bar{\partial}^\beta f\right)(\partial^\alpha g),$$

we obtain a Berezin quantization on the domain Ω with the Poisson bracket associated to the Kähler form $\omega = \partial\bar{\partial}\Phi$.

It is instructive to see how Theorem B applies in the examples. For the unit ball $\Omega = \mathbf{B}^n$ (which includes $\Omega = \mathbf{D}$ for $n = 1$), take

$$\Phi(z) = \log \frac{1}{1 - |z|^2},$$

which is a Kähler potential for the invariant metric on $\mathbf{B}^n$. Then Φ is strictly-PSH,

$$e^{-\Phi(z)} = 1 - |z|^2$$

is a strictly- PSH defining function for $\mathbf{B}^n$, and

$$b_0(z) = \det\left[\frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k}\right] = \frac{1}{(1 - |z|^2)^{n+1}}.$$

We thus recover the formulas (b_0 explains the “shift in the exponent α”). Also, we see that $c_\alpha \sim \alpha^n$.

For the Fock space on $\Omega = \mathbf{C}^n$, a Kähler potential for the Euclidean metric is $\Phi(z) = |z|^2$. In that case $b_0(z) = \det[\delta_{jk}] = 1$, so there is no “shift” this time, and again recovers the asymptotics of $K_\alpha$and B_α on the Fock space .

We need to review a few prerequisites before giving a proof of the theorem.

For a domain $\Omega \subset \mathbf{C}^n$ and a real-valued smooth function $\emptyset$ on it, the *Hartogs domain* with base Ω and radius-function $e^{-\emptyset}$ is

$$\widetilde{\Omega} := \{(z, t) \in \Omega \times \mathbf{C} : \ |t|^2 < e^{-\emptyset(z)}\}.$$

It can be shown that $\widetilde{\Omega}$ is pseudoconvex if and only if Ω is pseudoconvex and $\emptyset$ is PSH ; and that $\widetilde{\Omega}$ is strictly pseudoconvex and smoothly bounded if Ω is strictly-pseudoconvex, $\emptyset$ is strictly- PSH and $e^{-\emptyset} = r$ is a defining function for Ω. Furthermore,

$$\tilde{r}(z,t) := r(z) - |t|^2 = e^{-\emptyset(z)} - |t|^2 \qquad (29)$$

is a defining function for $\widetilde{\Omega}$.

Thus the hypotheses guarantee precisely that taking for $\emptyset$ the Kähler potential Φ, the corresponding Hartogs domain $\widetilde{\Omega}$ over Ω will be smoothly bounded and strictly pseudoconvex, with a defining function given by (29).

Continuing with the notations from the preceding paragraph, consider the compact manifold

$$X := \partial\widetilde{\Omega}$$

equipped with the measure

$$d\sigma := \frac{J|\tilde{r}|}{\|\partial\tilde{r}\|} dS, \qquad (30)$$

where dS stands for the surface measure on X and $J[\tilde{r}]$ for the $Monge - Ampére\ determinant$

$$J[\tilde{r}] = -\det\begin{bmatrix} \tilde{r} & \bar{\partial}\tilde{r} \\ \partial\tilde{r} & \partial\bar{\partial}\tilde{r} \end{bmatrix} > 0.$$

Let $H^2(X) = H^2$ be the subspace in $L^2(X, d\sigma)$ of functions whose Poisson extension into $\widetilde{\Omega}$ is holomorphic. (Alternatively, $H^2(X)$ is the closure in $L^2(X, d\sigma)$ of functions continuous on the closure $\overline{\widetilde{\Omega}}$ of $\widetilde{\Omega}$ and holomorphic in its interior.)

One calls $H^2(X)$ the *Hardy space* on X .

We remark that the measure (30) — which at first sight may look a bit artificial — is actually a familiar object in differential geometry. Namely, the restriction v of the differential form $\operatorname{Im} \partial\tilde{r} = \frac{1}{2i}(\partial\tilde{r} - \bar{\partial}\tilde{r})$ to X is a *contact form* on X , meaning that $v \wedge (\partial\bar{\partial}v)^n$ is a non-vanishing volume element on X. Up to a constant factor, this volume element is precisely (30).

For each $(z, t) \in \widetilde{\Omega}$, the evaluation functional $f \mapsto f(z, t)$ on H^2 turns out to be continuous, hence is given by the scalar product with a certain element $k_{(z,t)} \in H^2$.

The function

$$K_{\text{Szegö}}((x, t), (y, s)) := \langle k_{(y,s)}, k_{(x,t)} \rangle_{H^2}$$

on $\widetilde{\Omega} \times \widetilde{\Omega}$ is called the *Szegö kernel* .

In other words, $K_{\text{Szegö}}$ is the reproducing kernel of the Hardy space $H^2(X)$, viewed as a space of holomorphic functions on $\widetilde{\Omega}$ (rather than just their boundary values on X).

There is a simple relationship between the Hardy space $H^2(X)$ and the weighted Bergman spaces $L^2_{\mathrm{hol},h}$ on the base Ω, as well as between the $Szegö$ kernel $K_{\mathrm{Szegö}}$ and the weighted Bergman kernels of $L^2_{\mathrm{hol},h}$, which we now explain.
The boundary X of $\widetilde{\Omega}$ can be parameterized as

$$X = \{(z, e^{i\theta}e^{-\phi(z)/2}): z \in \Omega, \theta \in [0.2\pi]\}.$$

In these coordinates, and recalling our notations $r(z) = e^{-\phi(z)}$, $\tilde{r}(z,t) = r(z) - |t|^2$, easy computations show that

$$dS = \sqrt{r + \|\partial r\|^2}\, dz\, d\theta, \qquad \|\partial \tilde{r}\| = \sqrt{r + \|\partial r\|^2},$$
$$J[\tilde{r}] = J[r] = e^{-(n+1)\phi} \det[\partial\bar{\partial}\phi], \tag{31}$$

so

$$d\sigma(z,t) = e^{-(n+1)\phi} \det[\partial\bar{\partial}\phi]\ dz\, d\theta. \tag{32}$$

Consider now a holomorphic function f on $\widetilde{\Omega}$. Taking Taylor expansion in the fiber variable, we can write

$$f(z,t) = \sum_{j=0}^{\infty} f_j(z) t^j, \qquad (z,t) \in \widetilde{\Omega},$$

with f_j holomorphic on Ω. Expressing t in polar coordinates, one also sees immediately that

$$f(z)t^j \perp g(z)t^k \qquad \forall f, g \text{ if } k \neq j$$

(orthogonality is meant in H^2). For the norm of f in $H^2(X)$, we thus get, using (32),

$$\int_X |f(z,t)|^2 d\sigma(z,t)$$
$$= \sum_{j=0}^{\infty} \int_\Omega |f_j(z)|^2 \left(\int_0^{2\pi} \left|e^{i\theta} e^{-\phi(z)/2}\right|^{2j} d\theta \right) e^{-(n+1)\phi(z)} \det[\partial\bar{\partial}\phi(z)]\ dz$$
$$= \sum_{j=0}^{\infty} 2\pi \int_\Omega |f_j|^2 e^{-(j+n+1)\phi} \det[\partial\bar{\partial}\phi(z)]\ dz.$$

It follows that

$$H^2(X) = \bigoplus_{j=1}^{\infty} L^2_{\mathrm{hol}}(\Omega, 2\pi e^{-(j+n+1)\phi} \det[\partial\bar{\partial}\phi(z)])\, dz,$$

and

$$K_{\mathrm{Szegö}}((x,t),(y,s)) = \frac{1}{2\pi} \sum_{k=0}^{\infty} K_{e^{-(j+n+1)\phi} \det[\partial\bar{\partial}\phi(z)]}(x,y)(t\bar{s})^j.$$

In other words, the weighted Bergman kernels of our spaces $L^2_{\mathrm{hol},h}$ are just the Taylor coefficients, with respect to the fiber variable, of the Szegö kernel of $H^2(X)$. This result is due to Ligocka [135]; the basic idea goes back to Forelli and Rudin [128].
This celebrated result of Fefferman [126] and Boutet de Monvel and Sjöstrand [132] describes the boundary behaviour of the Szegö kernel of an arbitrary (nice) domain in $\mathbf{C}^n$, thus including, in particular, the kernel $K_{\mathrm{Szegö}}$ of our Hartogs domain $\widetilde{\Omega}$. Here is the result.

For $D \subset \mathbf{C}^n$ be a bounded strictly pseudoconvex domain with smooth bound-ary, and r a defining function for D. As in the special case of $D = \widetilde{\Omega}$ discussed before, one defines the Hardy space $H^2(\partial D)$ as the subspace in $L^2(\partial D, d\sigma)$ (with some non-vanishing volume element σ on ∂D) of all functions whose Poisson extensions into D are not only harmonic but holomorphic; and the Szegö kernel $K_{\mathrm{Szegö}}(z,w), z, w \in D$, as the reproducing kernel of $H^2(\partial D)$, viewed as a space of functions on D (not just of their boundary values on ∂D).

Then there are functions $a, b \in C^\infty(\mathbf{C}^n)$ such that (a) for $x \in \partial D$,

$$a(x) = \frac{n!}{\pi^n} J[r](x) > 0; \tag{33}$$

(b) the Szegö kernel on the diagonal is given by the formula

$$K_{\text{Szegö}}(x,x)=\frac{a(x)}{r(x)^n}+b(x)\log r(x).$$

This formula also extends to $K_{\text{Szegö}}(x,y)$ with $x\neq y$, namely,

$$K_{\text{Szegö}}(x,y)=\frac{a(x,y)}{r(x,y)^n}+b(x,y)\log r(x,y),$$

where $a(x,y), b(x,y)$and $r(x,y)$ are $almost-sesquiholomorphic\ extensions$ of $a(x)=a(x,x), b(x)=b(x,x)$ and $r(x)=r(x,x)$, respectively. The latter means that $\partial a(x,y)/\partial y$ and $\partial a(x,y)/\partial\bar{x}$ both vanish to infinite order on the diagonal $x=y$, and similarly for $b(x,y)$ and $r(x,y)$. Such extensions always exist, and it is a consequence of the strict pseudoconvexity that $r(x,y)$ can be chosen so that Re $r(x,y)>0$ for all $x,y\in D$, so that the logarithm can be defined as the principal branch.

(c) $K_{\text{Szegö}}(x,y)$is smooth on $\overline{\Omega\times\Omega}\setminus\mathcal{U}$,for any neighbourhood $\mathcal{U}$ of the boundary diagonal $\{(x,x): x\in\partial\Omega\}$.

Finally, there is a device for converting this description of the boundary behaviour into the description of the Taylor components from Ligocka's formula.

Recall that the power series $\sum_{k=0}^{\infty}k^jz^k$ converges on the unit disc $\mathbf{D}$, and its sum equals

$$\sum_{k=0}^{\infty}k^jz^k=\frac{j!}{(1-z)^{j+1}}+\sum_{k=1}^{j}\frac{a_{jk}}{(1-z)^k},$$

with some constants a_{jk}, if $j=0,1,2,\cdots$; and

$$\sum_{k=0}^{\infty}k^jz^k=\frac{(-1)^j}{j!}(1-z)^j\log(1-z)+F_j(z),$$

with some $F_j\in C^{-j}(\overline{\mathbf{D}})$, if $j=-1,-2,-3,\cdots$ .Also, by the familiar Cauchy estimates, if a holomorphic function $f(z)=\sum_k f_kz^k$ on the disc belongs to$C^j(\overline{\mathbf{D}})$, then its Taylor coefficients satisfy

$$f_k=O(k^{-j})\qquad \text{as } k\to+\infty.$$

Now suppose that $f(z)=\sum_k f_kz^k$ is a holomorphic function on $\mathbf{D}$ which satisfies

$$f(z)=\frac{a(z)}{(1-z)^{n+1}}+b(z)\log(1-z)$$

for some $a,b\in C^{\infty}(\mathbf{C})$. Taking the Taylor expansions of a,b around $z=1$, this implies that there exist $\alpha_1,\cdots\alpha_{n+1}$ and $\beta_0,\beta_1,\beta_2,\cdots$, with $\alpha_{n+1}=a(1)$, such that, for any $M=0,1,2,\cdots$,

$$f(z)=\sum_{j=1}^{n+1}\frac{\alpha_j}{(1-z)^j}+\sum_{j=0}^{M}\beta_j(1-z)^j\log(1-z)+F_M(z),$$

with $F_M\in C^M(\overline{\mathbf{D}})$. Combining this with the observations in the preceding paragraph, it transpires that

$$f_k\approx a_nk^n+a_{n-1}k^{n-1}+\cdots+a_0+\frac{a_{-1}}{k}+\cdots,\qquad a_n=\frac{a(1)}{n!},\tag{34}$$

for some constants $a_n,a_{n-1},\cdots$, as $k\to\infty$.

As already mentioned, the hypotheses of the theorem guarantee that the Hartogs domain

$$\widetilde{\Omega}=\{(z,t)\in\Omega\times\mathbf{C}:\ |t|^2<e^{-\Phi(z)}\}$$

is smoothly bounded, strictly pseudoconvex, and with a defining function

$$\tilde{r}(z,t):=e^{-\Phi(z)}-|t|^2.$$

Consider the Hardy space $H^2(X)$ on the boundary $X=\partial\widetilde{\Omega}$. By Ligocka's formula we have

$$H^2(X)=\bigoplus_{k=n+1}^{\infty}L^2_{\text{hol}}(\Omega,e^{-k\Phi}\det[\partial\bar{\partial}\Phi])\tag{35}$$

(where $n=\dim\Omega$, so $n+1=\dim\widetilde{\Omega}$), and

$$K_{\text{Szegö}}\big((x,s),(y,t)\big)=\frac{1}{2\pi}\sum_{k=0}^{\infty}K_{k+n+1}(x,y)(s\bar{t})^k,$$

where,for brevity, we are denoting the reproducing kernel of $L^2_{\text{hol}}(\Omega,e^{-k\Phi}\det[\partial\bar{\partial}\Phi])$ by $K_k(x,y)$.

Feff erman's theorem for the Szegö kernel tells us that

$$K_{\text{Szegö}} = \frac{a}{\tilde{r}^{n+1}} + b\log\,\tilde{r},$$

in particular,

$$\begin{aligned}\frac{1}{2\pi}\sum_{k=0}^{\infty} K_{k+n+1}(x,x)s^k &= \tilde{k}_{\text{Szegö}}((x,s),(x,1))\\ &= \frac{a(x,s)}{(e^{-\Phi(x)}-s)^{n+1}} + b(x,s)\log(e^{-\Phi(x)}-s)\\ &= \frac{a(x,s)e^{(n+1)\Phi(x)}}{(1-\underbrace{se^{\Phi(x)}}_{=:z})^{n+1}} + b(x,s)\log\big(1-se^{\Phi(x)}\big) - b(x,s)\Phi(x)\\ &= \frac{A(x,z)}{(1-z)^{n+1}} + B(x,z)\log(1-z),\end{aligned}$$

where $A(x,z) = a\big(x,ze^{-\Phi(x)}\big)e^{(n+1)\Phi(x)} - b(x,ze^{-\Phi(x)})\Phi(x)(1-z)^{n+1}$ and $B(x,z) = b\big(x,ze^{-\Phi(x)}\big)$. So for each $x \in \Omega$,

$$\sum_{k=0}^{\infty} e^{-k\Phi(x)}K_{k+n+1}(x,x)z^k = \frac{A(x,z)}{(1-z)^{n+1}} + B(x,z)\log(1-z)$$

with functions $A,B \in C^{\infty}(\overline{\Omega}\times\overline{\mathbf{D}})$. Employing the resolution of singularities implies

$$K_k(x,x) \approx \frac{k^n}{\pi^n}e^{k\Phi(x)}\sum_{j=0}^{\infty}\frac{b_j(x)}{k^j}$$

as $k \to +\infty$, proving the first part. (The formula for b_0 follows from (31), (33) and (34).)

With a bit of technicalities which we omit, the last result can be extended also to $x \neq y$:

$$K_k(x,y) \approx \frac{k^n}{\pi^n}e^{k\Phi(x,y)}\sum_{j=0}^{\infty}\frac{b_j(x,y)}{k^j} \tag{36}$$

for (x,y) near the diagonal, where $\Phi(x,y), b_j(x,y)$ are almost-sesquiholomorphic extensions of $\Phi(x) = \Phi(x,x)$ and $b_j(x) = b_j(x,x)$. (The technicalities involve an improved version of the resolution of singularities from, where $f(z)$, holomorphic in $z \in \mathbf{D}$, is replaced by $f(x,z)$, depending smoothly on x and holomorphic in z in the disc $|z| < r(x)$, where the radius $r(x)$ also depends smoothly on x ; see Lemma 7 in [130].)

The second part (concerning the asymptotics of the Berezin transform) is then proved by first showing that in the integral defining B_α,

$$B_\alpha f(x) = \int_\Omega f(y)\frac{|K_\alpha(x,y)|^2}{K_\alpha(x,x)}e^{-\alpha\Phi(y)}\det\big[\partial\bar{\partial}\Phi(y)\big]\,dy$$

the main contribution, as $\alpha \to +\infty$ comes from a small neighbourhood of x. In that neighbourhood, one then replaces $K_\alpha(x,y)$ by the asymptotic expansion (36). This reduces the problem to finding the asymptotics as $\alpha \to +\infty$ of integrals of the form

$$\int_{\text{neighbourhood of } x} F(y)e^{\alpha(\Phi(x,y)+\Phi(y,x)-\Phi(x)-\Phi(y))}dy,$$

where F is an expression involving $f, \det\big[\partial\bar{\partial}\Phi\big]$ and the coefficient functions b_j from (36). Finally, this kind of integrals is handled by the standard stationaryphase (Laplace, WJKB) method, yielding the result in the theorem.

The first two terms in the asymptotic expansion for B_α can be evaluated explicitly, giving the desired outcomes $Q_0 = I$ and $Q_1 = \Delta$, and thus finishing completely the proof.

For $f \in L^\infty(\Omega)$ let us denote, for brevity, the Toeplitz operator with symbol f on $L^2_{\text{hol}}\big(\Omega, e^{-m\Phi}\det\big[\partial\bar{\partial}\Phi\big]\big)$ by $T_f^{(m)}$. The main result on the Berezin-Toeplitz quantization then reads as follows. . *Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$, and $\Phi\colon \Omega \to \mathbf{R}$ a* *smooth strictly-PSH function such that* $e^{-\Phi} =: r$ *is a defining function for* Ω *. Then there exist bilinear differential operators* $C_j (j = 0,1,2,\cdots)$ *such that for any* $f, g \in C^\infty(\overline{\Omega})$ *and any* $M = 0,1,2,\cdots$,

$$\left\| T_f^{(m)} T_g^{(m)} - \sum_{j=0}^{M} m^{-j} T_{C_j(f,g)}^{(m)} \right\| = O(m^{-M-1}) \qquad as\ m \to \infty.$$

Furthermore,

$$C_0(f,g) = fg, \qquad C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\}.$$

Consequently, $f * g := \sum_{j=0}^{\infty} h^j C_j(f,g)$ *defines a star-product on* Ω.

Observe that the theorem establishes the expansion for the product of two Toeplitz operators (17) in the strongest possible sense, namely, in the operator norm.

As already mentioned, the proof involves a sophisticated machinery, due to Boutet de Monvel and Guillemin, of Fourier integral operators of Hermite type — more specifically, of Toeplitz operators with pseudodifferential symbols. It is not our intention to introduce all the necessary notions and technicalities here; we will, however, try to highlight at least the main ideas.

Consider again the Hartogs domain $\widetilde{\Omega}$,

$$\widetilde{\Omega} = \{(z,t) \in \Omega \times \mathbf{C} : |t|^2 < e^{-\Phi(z)}\}.$$

Again, the hypotheses guarantee that $\widetilde{\Omega}$ is smoothly bounded, strictly pseudoconvex, and admits

$$\tilde{r}(z,t) := e^{-\Phi(z)} - |t|^2$$

as a defining function.

As before, consider the Szegö kernel on the compact manifold $X = \partial\widetilde{\Omega}$ with respect to the measure

$$d\sigma := \frac{J[\tilde{r}]}{\|\partial\tilde{r}\|} dS.$$

We have already seen that (Ligocka's formula)

$$K_{\text{Szegö}}(x,s;y,t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} K_{k+n+1}(x,y)(s\bar{t})^k,$$

$$H^2(X) = \bigoplus_{k=n+1}^{\infty} L^2_{\text{hol}}(\Omega, e^{-k\Phi}\det[\partial\bar{\partial}\Phi]). \qquad (37)$$

The space $H^2(X)$ also admits its own "Hardy-space" Toeplitz operators: namely, if F is a function in, say, $C^\infty(X)$, one defines the Toeplitz operator T_F on $H^2(X)$ with symbol F as

$$T_{F\psi} := P_{\text{Szegö}}(F\psi), \qquad \psi \in H^2(X),$$

where $P_{\text{Szegö}}: L^2(X, d\sigma) \to H^2(X)$ is the orthogonal projection (the $Szegö\ projection$).

Now if f is a smooth function on $\overline{\Omega}$, we can lift it to a function $F \in C^\infty(\overline{\widetilde{\Omega}})$by composing with the projection on the first variable, i.e.

$$F(x,t) := f(x).$$

An easy verification then reveals that under the orthogonal decomposition (37), the Toeplitz operators $T_f^{(m)}$ on $L^2_{\text{hol}}(\Omega, e^{-m\Phi}\det[\partial\bar{\partial}\Phi])$. and the Toeplitz operator T_F on $H^2(X)$ are related by

$$T_F = \bigoplus_{m=n+1}^{\infty} T_f^{(m)}.$$

The main ingredient in the whole proof is that, following the ideas of Boutet de Monvel and Guillemin, we can define Toeplitz operators T_Q on $H^2(X)$ by the same recipe not only for functions, but also for *pseudodifferential operators* (ΨDO for short) Q on X as symbols. That is, for a ΨDO Q on X, we define

$$T_Q\psi := P_{\text{Szegö}}\, Q\,\psi.$$

For Q the operator of multiplication by a function $F \in C^\infty(X)$ this recovers the Toeplitz operators T_F above as a particular case. Toeplitz operators on $H^2(X)$ with ΨDO symbols are often called *generalized Toeplitz operators*.

One proceeds to define the order ord(T_Q) and the symbol $\sigma(T_Q)$ of T_Q as the order of Q and the restriction of the principal symbol $\sigma(Q)$ of Q to the symplectic submanifold

$$\sum := \{(x,\xi): \xi = t(\bar{\partial}\tilde{r} - \tilde{r})_x, t > 0\}$$

of the cotangent bundle of X, respectively. It can be shown that these two definitions are unambiguous: although it may happen that $T_Q = T_{\acute{Q}}$ for two different ΨDOs $Q, \acute{Q}$ (which is peculiar for ΨDO symbols — it is never the case that $T_F = T_{\acute{F}}\ for\ F \neq \acute{F}$), in that case either $Q, \acute{Q}$ have the same order and their symbols coincide on $\sum$, or one of them — say, Q — has greater order then the other and its symbol vanishes on $\sum$ to order ord(Q)–ord$(\acute{Q})$. Also, the order and the symbol of T_Q obey the usual rules one would expect, as well as some additional ones:

(P1) the generalized Toeplitz operators form an algebra under composition (i.e. $\forall Q_1, Q_2\ \exists Q_3$: $T_{Q_1}T_{Q_2} = T_{Q_3}$);

(P2) ord$(T_{Q_1}T_{Q_2})$ = ord(T_{Q_1})+ord(T_{Q_2}); $\sigma(T_{Q_1}T_{Q_2}) = \sigma(T_{Q_1})\sigma(T_{Q_2})$;

(P3) $\sigma([T_{Q_1}, T_{Q_2}]) = \{\sigma(T_{Q_1}), \sigma(T_{Q_2})\}_{\sum}$;

(P4) if ord$(T_Q) = 0$, then T_Q is bounded operator on H^2;

(P5) if ord(T_{Q_1}) = ord$(T_{Q_2}) = k$ and $\sigma(T_{Q_1}) = \sigma(T_{Q_2})$, then ord$(T_{Q_1} - T_{Q_2}) \leq k - 1$;

(P6) for $F \in C^\infty(X)$ and $(x,\xi) \in \sum$, $\sigma(T_F)(x,\xi) = F(x)$.

Returning to the proof of Theorem BT, let $\mathcal{T}$ be the subalgebra of all generalized Toeplitz operators on $H^2(X)$ which commute with the rotations

$$U_\theta: f(z,w) \mapsto f(z, e^{i\theta}w), \quad (z,w) \in X, \ \theta \in \mathbf{R},$$

in the fiber variable. Clearly, the operators T_F with $F(x,t) = f(x)$ for some function $f \in C^\infty(\overline{\Omega})$ (i.e. with F constant along fibers) belong to $\mathcal{T}$.

Denote by $D : H^2(X) \to H^2(X)$ the infinitesimal generator of the semi-group U_θ . Then D acts as multiplication by im on the m-th summand in (37), for each m:

$$D = \bigoplus_m imI;$$

and also

$$D = T_{\partial/\partial\theta}$$

is a generalized Toeplitz operator of order 1.

Using (P1)–(P6) it can be shown that if $T \in \mathcal{T}$ is of order 0, then

$$T = T_F + D^{-1}R$$

for some (uniquely determined) $F \in C^\infty(X)$ which is constant along the fibers (hence, descends to a function on Ω), and $R \in \mathcal{T}$ of order 0. Repeated application of this formula shows that, for each $k \geq 0$,

$$T = \sum_{j=0}^{k} D^{-j}T_{F_j} + D^{-k-1}R_k,$$

with $F_j(x,t) = f_j(x)$ for some $f_j \in C^\infty(\overline{\Omega})$ and $R_k \in \mathcal{T}$ of order 0. Invoking the fact that zero order operators are bounded, it follows that

$$D^{k+1}\left(T - \sum_{j=0}^{k} D^{-j}T_{F_j}\right) = R_k$$

is a bounded operator on H^2.

In view of the decomposition $T_F = \bigoplus_m T_f^{(m)}$, this means that

$$\left\| T|_{L^2(\Omega, e^{-m\Phi}\det[\partial\bar{\partial}\Phi])} - \sum_{j=0}^{k} m^{-j}T_{f_j}^{(m)} \right\| = O(m^{-k-1})$$

as $m \to +\infty$. Taking for T the product $T_F T_G$, with $F(x,t) = f(x), G(x,t) = g(x)$ for some $f, g \in C^\infty(\overline{\Omega})$, and setting $C_j(f,g) := f_j$, we obtain the desired asymptotic expansion for $T_f^{(m)}T_g^{(m)}$.

Finally, the assertions concerning $C_0$and C_1 follow from the above properties (P2) and (P3) of the symbol by a routine calculation by no means intended as an exhaustive survey of quantization methods, or even of the Berezin and the Berezin-Toeplitz quantizations .From the many surveys and overviews of various quantization techniques, see [131] for a somewhat more in-depth account of many (but not all) things discuss here, as well as for abundant references to other literature. Two good surveys of traditional deformation quantization (i.e. on the level of formal power series) are Gutt [129] and Sternheimer [142]; a very nice recent overview focused on the Berezin-Toeplitz quantization discussed here is Schlichenmaier [140]. Some more technical aspects of several points left out here can be found in the author's article [121]. An excellent reading about the material discussed are several books by Folland, in particular [127].

It should, finally, be mentioned that the subject of Berezin and BerezinToeplitz quantization is still far from being understood completely, and there are many things waiting still to be resolved in a satisfactory way. For instance, in both Theorem B and Theorem BT the semiclassical limit $\alpha = \frac{1}{h} \to +\infty$ is taken only for α ranging through the integers; this is of course natural if Ω is a compact manifold (as was the original context in [139]), but is only an artifact of the methods of proof for Ω a domain in $\mathbf{C}^n$. Removing this restriction, i.e. extending the asymptotics of the reproducing kernels K_α, the Berezin transforms B_α, and the Toeplitz operators $T_f^{(\alpha)}$ also to non-integer $\alpha \to +\infty$ would be most desirable.

Another highly active area concerns the generalizations of Feff erman's theorem on the Szegö kernel from (and the analogous theorem of his for the Bergman kernel, which was not mentioned here) to domains which are only weakly (i.e. not necessarily strictly) pseudoconvex; at the moment, there are only some partial results for special types of domains (see e.g. [130]). Having a result of that kind would make it possible to extend Theorems B and BT to more general domains. Similarly, having a result of that kind for domains which are not necessarily smoothly bounded — more specifically, for Hartogs domains $\widetilde{\Omega}$ whose the radiusfunction $e^{-\emptyset}$has a logarithmic singularity at the boundary of Ω — would make it possible to quantize metrics whose Kähler potential behaves like that at the boundary; the latter includes, for instance, the important Cheng-Yau metric on Ω (the Kähler -Einstein metric; see [135] for more information on this). Carrying out the Berezin-Toeplitz quantization in the last case by the method described would also require an extension of the Boutet de Monvel and Guillemin theory of generalized Toeplitz operators to noncompact manifolds, which is another open problem at present.

Closely related ideas concern also the boundary behavior of weighted Bergman kernels with respect to weights having some kind of singularity at the boundary (e.g. involving the logarithm of the defining function); some results of the present author in that direction can be found in [122]. Interestingly, the same technique can also be used to establish that the weighted Bergman kernels $K_\alpha(x, y)$ appearing in the previous can be continued to meromorphic functions of α in the entire complex plane [124]; this is somewhat reminiscent of the resonances occuring in scattering theory, and is related to zeta functions of elliptic operators. A subject of a completely diff erent flavour is the extension of the Theorems B and BT above also to the setting of harmonic, rather than holomorphic, functions; although this seems not to have any direct relevance for quantization, the results are equally interesting, and, apparently, much more intriguing, than in the holomorphic case (see e.g. [123]).

There is also a variety of problems, though again not directly related to quantization, concerning the range of the Berezin symbol map $T \mapsto \tilde{T}$ (see e.g. Coburn [135] and Bommier-Hato [128]), while notable applications of Toeplitz operators and the Berezin transform appear in operator theory and in time-frequency analysis; let us mention at least [136], [136], [132], [133] and [145].

Section (3.2): Harmonic Berezin Transform on the Half-Space

For $\mathcal{H}_\alpha$ be the Hilbert space of all functions harmonic on the half-space $H := \{(x, y) \in \mathbb{R}^{n+1}; x \in \mathbb{R}^n, y > 0\}$ and square- integrable with respect to the measure $y^\alpha dxdy$, where dx and dy are the standard $Lebesgue$ measures on$\mathbb{R}^n$ and $\mathbb{R}_+$, respectively, and α ≥ 0 is an arbitrary nonnegative real number. By the volume version of the mean-value property for harmonic functions, the (linear) functional $e_{(x,y)}: \mathcal{H}_\alpha \ni f \mapsto f(x,y) \in \mathbb{C}$ is bounded for each fixed $(x,y) \in H$ and therefore continuous.This is precisely the necessary and sufficient condition for the existence of the so called $reproducing\ kernel$, i.e.a mapping $K_\alpha: H \times H \to \mathbb{C}$ which satisfies $K_\alpha(.;h) \in \mathcal{H}_\alpha$ for every $h = (x,y) \in H$ and has the following $reproducing\ property$: for all $f \in \mathcal{H}_\alpha$ and $(x,y) \in H$,

$$f(x,y) = \langle f, K_\alpha(.;x,y)\rangle_{L^2} \equiv \int_{\mathbb{R}^n}\int_0^\infty f(z,w)\overline{K_\alpha(z,w;x,y)}\, w^\alpha dw\, dz.$$

It can be shown that K_α is in fact real-valued and symmetric,i.e $K_\alpha(z,w;x,y) = \overline{K_\alpha(z,w;x,y)} = K_\alpha(x,y;z,w)$; see[1] for more details on K_α .Recall that for $f \in L^\infty(H)$ there is the $harmonic\ Berezin\ transform\ B_\alpha f$,associated to the kernels K_α ,and defined on H by the formula

$$B_\alpha f(x,y) := \frac{1}{K_\alpha(x,y;x,y)}\int_{\mathbb{R}^n}\int_0^\infty f(z,w)|\ K_\alpha(x,y;z,w)|^2\, w^\alpha dw\, dz \qquad (40).$$

It is a well known fact that in the case of the spaces $\mathcal{H}_\alpha^{hol}$ of functions that are $holomorphic$ (rather than harmonic)on a given domain $\Omega \subset \mathbb{C}^n$ and square-integrable with respect to w^α ,where w is an appropriate (positive) weight function on Ω , there are reproducing kernels $\mathrm{K}_\alpha^{hol}(x,y), x,y \in \Omega$ (the so called weighted Bergman kernels), and that in these spaces one has the associated ‘‘holomorphic’’Berezin transform

$$B_\alpha^{hol} f(y) := \frac{1}{K_\alpha(y,y)}\int_\Omega f(x)|\ K_\alpha(x,y)|^2\, w(x)^\alpha dx,$$

first introduced by F.A.Berezin[173] for Ω a bounded symmetric domain in $\mathbb{C}^n$and w a certain natural weight on it.Berezin himself was able to show that

$$B_\alpha^{hol} f = f + \frac{Q_1 f}{\alpha} + o(\alpha^{-2}) \quad as\ \alpha \to +\infty \qquad (41),$$

where Q_1 was a kind of Laplace–Beltrami operator,and used this fact to construct a certain quantization procedure for phase space Ω (nowadays known as the $Berezin\ quantization$). Later, (41) was extended to the complete asymptotic expansion in negative powers of α,and the Berezin quantization became one of the first nontrivial examples of the so-called $deformation\ quantization$ of Bayen,Flato,Fronsdal,Lichnerowitz and Sternheimer[172]. Subsequently,(complete) asymptotic expansions of this type have been extended to a much larger class of domains (and even manifolds) than symmetric spaces by Klimek and Lesniewski[170],Engliš[4,5],Karabegov and Schlichenmaier[179], and others,and in these days they serve as an indispensable tool in applications to quantization on Kähler manifolds (see[175]). Having a complete asymptotic expansion analogous to (41) even for spaces of harmonic functions would be of great interest from many aspects (for instance, it would eliminate the need for holomorphic structure,thus extending the whole theory also to real, instead of complex, domains and manifolds). Unfortunately, the problem turns out to be much harder than in the holomorphic case. The first result of this kind is due to C. Liu [181], who proved that for essentially bounded functions on the unit disc, and also for essentially bounded functions f on the unit ball B^n of $\mathbb{R}^n$, n > 1, that are radial (i.e. $f(x)$ depends only on$\|x\|$), the Berezin transform B_α^{ball} associated to the spaces $\mathcal{H}_\alpha^{ball}$ of all harmonic functions on B^n square-integrable with respect to the weights $(1 - \|x\|^2)^\alpha$,$-1 < \alpha < \infty$, satisfies $B_\alpha^{ball} f(x) \to f(x)$ for every $x \in B^n$ as $\alpha \to +\infty$. The radiality assumption was then removed by Otáhalová [173]. Only quite recently, the case of harmonic functions on the entire space $\mathbb{C}^n \cong \mathbb{R}^{2n}$ square-integrable with respect to the Gaussian weightse $e^{-\alpha\|x\|^2}$, $\alpha > 0$ was done by Engliš[176].There seem to be no similar results for any other domains. We show that a complete asymptotic expansion of the harmonic Berezin transform in negative powers of α is available for essentially bounded smooth functions defined on H.Thus our main result can be stated in the following way:

Theorem (3.2.1)[171]: For any $f \in L^\infty(H) \cap C^\infty(H)$,

$$(B_\alpha f)(x,y) \approx \sum_{j=0}^{\infty} \frac{R_j f(x,y)}{\alpha^j}$$

$as\ \alpha \to +\infty\ for\ every\ (x,y) \in H$ where R_j are certain differential operators, with $R_0 f(x,y) = f(x,y)$ $(R_0 thus\ being\ the\ identity\ operator)$ and

$$R_1 f(x,y) = y^2 \frac{\Delta f(x,y)}{n} + (1-n) y \frac{\partial f}{\partial y}(x,y) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y),$$

$where\ \ \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$.

Here the notation means ,of course,that for every N =0,1,2,...,

$$(B_\alpha f)(x,y) - \sum_{j=0}^{N-1} \frac{R_j f(x,y)}{\alpha^j} = o(\alpha^{-N}), \qquad as\ \alpha \to +\infty.$$

The proof of Theorem (3.2.1) occupies the rest and is divided into several steps. we show first of all that it is enough to prove the theorem for $(x,y) = (0,1)$ .We then proceed to give an explicit expression for the kernel K_α at this point by means of the Fourier transform; this is done. The proof is finally completed where we employ a little trick to reduce it to the case in which the classical multidimensional Laplace method for asymptotic expansion of integrals is directly applicable.

We prove two technical lemmas owing to which our situation simplifies substantially.

Lemma (3.2.2) [171]: For every $(x,y) and\ (a,b) in\ H,$

$$K_\alpha(a,b;x,y) = b^{-n-\alpha-1} K_\alpha\left(0,1; \frac{x-a}{b}, \frac{y}{b}\right).$$

Proof: Let $f \in \mathcal{H}_\alpha$. Clearly,the function $f(x+a,y)$ is harmonic iff $f(x,y)$ is harmonic.Thus we can write

$$\int_{\mathbb{R}^n} \int_0^\infty f(x,y) K_\alpha(a,b;x,y)\, y^\alpha dy\, dx = f(a,b)$$
$$= \int_{\mathbb{R}^n} \int_0^\infty f(x+a,y)\, K_\alpha(0,b;x,y) y^\alpha dy dx$$
$$= \int_{\mathbb{R}^n} \int_0^\infty f(x,y) K_\alpha(0,b;x-a,y) y^\alpha dy\, dx$$

where in the second and the third equality we used the reproducing property of the kernel K_α and the change of variables $x \mapsto x-a$, respectively.Since $f \in \mathcal{H}_\alpha$ was arbitrary,this implies the equality $K_\alpha(a,b;x,y) = K_\alpha(0,b;x-a,y)$. Similarly we can show that for all $t > 0$

$K_\alpha(ta,tb;x,y) = t^{-n-\alpha-1} K_\alpha\left(a,b; \frac{x}{t}, \frac{y}{t}\right)$ in which case the proof runs as follows:

$$\int_{\mathbb{R}^n} \int_0^\infty f\left(\frac{x}{t}, \frac{y}{t}\right) K_\alpha\left(a,b; \frac{x}{t}, \frac{y}{t}\right) \left(\frac{y}{t}\right)^\alpha \frac{dy}{t} \frac{dx}{t^n} = \int_{\mathbb{R}^n} \int_0^\infty f(x,y)\, K_\alpha(a,b;x,y) y^\alpha dy\, dx$$
$$= f(a,b)$$
$$= \int_{\mathbb{R}^n} \int_0^\infty f\left(\frac{x}{t}, \frac{y}{t}\right) K_\alpha(ta,tb;x,y) y^\alpha dy\, dx,$$

where again the reproducing property and the change of variables $(x,y) \mapsto \left(\frac{x}{t}, \frac{y}{t}\right)$ were used.Taking both these results into account we see that the assertion of Lemma(3.2.2)is true. Next,for any function $f: H \to \mathbb{C}$ and for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}_+$ we denote $f^{a,b}(x,y) := f(bx + a, by)$.We then have the following

Lemma(3.2.3) [171]: Let $f \in L^\infty(H), a \in \mathbb{R}^n\ and\ b \in \mathbb{R}_+\ be\ given.Then$
$(B_\alpha f)(a,b) = (B_\alpha f^{a,b})(0,1)$.

Proof:

$$(B_\alpha f)(a,b) = \frac{1}{K_\alpha(a,b;a,b)} \int_{\mathbb{R}^n} \int_0^\infty f(z,w) |K_\alpha(a,b;z,w)|^2\, w^\alpha dw\, dz$$
$$= \frac{|b^{-n-\alpha-1}|^2}{b^{-n-\alpha-1} K_\alpha(0,1;0,1)} \int_{\mathbb{R}^n} \int_0^\infty f(z,w) \left|K_\alpha(0,1; \frac{z-a}{b}, \frac{w}{b})\right|^2 w^\alpha dw\, dz$$
$$= \frac{b^{-\alpha-1-n}}{K_\alpha(0,1;0,1)} \int_{\mathbb{R}^n} \int_0^\infty f(bx+a, by) |K_\alpha(0,1;x,y)|^2\, (by)^\alpha . b. |b|^n dy dx$$
$$= \frac{b^{-\alpha-1-n}}{K_\alpha(0,1;0,1)} \int_{\mathbb{R}^n} \int_0^\infty f(bx+a, by) |K_\alpha(0,1;x,y)|^2\, b^{\alpha+1+n} y^\alpha dy dx$$
$$= \frac{1}{K_\alpha(0,1;0,1)} \int_{\mathbb{R}^n} \int_0^\infty f(bx+a, by) |K_\alpha(0,1;x,y)|^2\, y^\alpha dy dx$$
$$= (B_\alpha f^{a,b})(0,1),$$

using the definition of B_α ,Lemma(3.2.2) and the change of variable $\left(\frac{z-a}{b}, \frac{w}{b}\right) = (x,y)$,respectively.

Corollary(3.2.4) [171]: In proving Theorem (3.2.1) we can confine ourselves to the case $(x,y) = (0,1)$.

Proof: Pick $(a,b) \in H$ and suppose that $(B_\alpha f^{a,b})(0,1) \approx \sum_{k=0}^{\infty} \frac{Q_k f^{a,b}(0,1)}{\alpha^k}$. Then, because $(B_\alpha f^{a,b})(0,1) = (B_\alpha f)(a,b)$,we see that

$$(B_\alpha f)(a,b) \approx \sum_{k=0}^{\infty} \frac{Q_k f(a,b)}{\alpha^k}, \qquad \alpha \to +\infty,$$

as desired,with $Q_k f(a,b) = Q_k f^{a,b}(0,1)$.

For brevity ,denote $K_\alpha(x,y;0,1) \coloneqq H_\alpha(x,y), f_y(x) \coloneqq f(x,y)$ and let $\widehat{f_y}(\xi)$ be the Fourier transform of $f_y(x)$ with respect to x:

$$\widehat{f_y}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_y(x) e^{-ix.\xi} dx.$$

Lemma(3.2.5) [171]: For any $f \in \mathcal{H}_\alpha$,

$$\widehat{f_y}(\xi) = e^{-|\xi| y} \widehat{f_0}(\xi), \tag{42}$$

where $\widehat{f_0}(\xi)$ is a function defined on $\mathbb{R}^n$ such that

$$\Gamma(\alpha+1) \int_{\mathbb{R}^n} \left|\widehat{f_0}(\xi)\right|^2 (2|\xi|)^{-\alpha-1} d\xi < \infty. \tag{43}$$

Conversely, every such function $\widehat{f_0}$corresponds to a function $f \in \mathcal{H}_\alpha$. Moreover,for any two functions $f, g \in \mathcal{H}_\alpha$

$$\int_{\mathbb{R}^n} \int_0^\infty f(x,y) \overline{g(x,y)}\, y^\alpha dy dx = \Gamma(\alpha+1) \int_{\mathbb{R}^n} \widehat{f_0}(\xi) \overline{\widehat{g_0}(\xi)}\, (2|\xi|)^{-\alpha-1} d\xi \tag{44}$$

Proof: Harmonicity means that

$$\frac{\partial^2 f(x,y)}{\partial y^2} + \sum_{j=1}^n \frac{\partial^2 f(x,y)}{\partial x_j^2} = 0$$

and we have the following chain of equivalences (here $\mathcal{F}_x f_y(\xi) = \widehat{f_y}(\xi)$):

$$\frac{\partial^2 f(x,y)}{\partial y^2} + \sum_{j=1}^n \frac{\partial^2 f(x,y)}{\partial x_j^2} = 0$$

$$\Leftrightarrow \mathcal{F}_x \left(\frac{\partial^2 f(x,y)}{\partial y^2} + \sum_{j=1}^n \frac{\partial^2 f(x,y)}{\partial x_j^2} \right) = 0$$

$$\Leftrightarrow \frac{\partial^2 \hat{f}_y(\xi)}{\partial y^2} + \sum_{j=1}^n \mathcal{F}_x \frac{\partial}{\partial x_j} \left(\frac{\partial f_y(x)}{\partial x_j} \right) = 0$$

$$\Leftrightarrow \frac{\partial^2 \hat{f}_y(\xi)}{\partial y^2} + \sum_{j=1}^n (-i\xi_j).(-i\xi_j) \hat{f}_y(\xi) = 0$$

$$\Leftrightarrow \frac{\partial^2 \hat{f}_y(\xi)}{\partial y^2} - \|\xi\|^2 \hat{f}_y(\xi) = 0. \tag{45}$$

The general solution of (45) is

$$\hat{f}_y(\xi) = A(\xi) e^{-|\xi| y} + B(\xi) e^{|\xi| y}.$$

But the Plancherel formula

$$\int_{\mathbb{R}^n} |f(x,y)|^2 dx = \int_{\mathbb{R}^n} \left|\hat{f}_y(\xi)\right|^2 d\xi \tag{46}$$

implies that

$$\int_{\mathbb{R}^n} \int_0^\infty |f(x,y)|^2 y^\alpha dy\, dx = \int_{\mathbb{R}^n} \int_0^\infty \left|\hat{f}_y(\xi)\right|^2 y^\alpha dy\, d\xi. \tag{47}$$

Since $f \in \mathcal{H}_\alpha$,it follows that f is square-integrable on H with respect to $y^\alpha\, dy\, dx$. This means that the left-hand side in (47) is convergent hence

$$\int_{\mathbb{R}^n} \int_0^\infty \left|\hat{f}_y(\xi)\right|^2 y^\alpha dy\, d\xi < \infty$$

and so

$$\int_0^\infty \left|\hat{f}_y(\xi)\right|^2 y^\alpha dy < \infty \quad \text{for a.a. } \xi$$

But this implies that $B(\xi) \equiv 0$.Putting $A(\xi) \eqqcolon \hat{f}_0(\xi)$ concludes the proof of (42).

To prove the second part, note first that

$$\int_0^\infty e^{-2|\xi| y}\, y^\alpha dy = \frac{1}{2|\xi|} \int_0^\infty e^{-t} \left(\frac{t}{2|\xi|} \right)^\alpha dt \tag{48}$$

$$= \frac{1}{(2|\xi|)^{\alpha+1}} \int_0^\infty e^{-t} t^\alpha\, dt$$

$$= (2|\xi|)^{-\alpha-1} \Gamma(\alpha+1), \tag{49}$$

where the change of variables $t := 2|\xi| y$ was used in (48). It follows that

$$\infty > \int_{\mathbb{R}^n}\int_0^\infty \left|\hat{f}_y(\xi)\right|^2 y^\alpha dy dx = \int_{\mathbb{R}^n}\int_0^\infty \left|e^{-|\xi|y}\hat{f}_0(\xi)\right|^2 y^\alpha dy\, d\xi$$
$$=\int_{\mathbb{R}^n}\left|\hat{f}_0(\xi)\right|^2 \int_0^\infty e^{-2|\xi|y}\, y^\alpha dy\, d\xi$$
$$\overset{(49)}{=}\Gamma(\alpha+1)\ \int_{\mathbb{R}^n}\left|\hat{f}_0(\xi)\right|^2 (2|\xi|)^{-\alpha-1} d\xi$$
and (43) is thus proven. Finally, (44) is a direct consequence of the Plancherel formula: for any $f, g \in L^2$,
$$\int_{\mathbb{R}^n} f(x,y)\overline{g(x,y)}\, dx = \int_{\mathbb{R}^n} \hat{f}_y(\xi)\overline{\hat{g}_y(\xi)}\, d\xi,$$
and Fubini's theorem.

Proposition (3.2.6) [171]: We have
$$\widehat{H}_{\alpha,y}(\xi) = \frac{(2|\xi|)^{\alpha+1}e^{-(1+y)|\xi|}}{(2\pi)^{\frac{n}{2}}\Gamma(\alpha+1)} \qquad (50)\ .$$
Proof: First note that,due to the reproducing property of K_α once again, for $f \in \mathcal{H}_\alpha$ we can write
$$f(0,1) = \int_{\mathbb{R}^n}\int_0^\infty f(x,y)\overline{H_\alpha(x,y)}\, y^\alpha dx\, dy$$
$$\overset{(44)}{=}\Gamma(\alpha+1)\int_{\mathbb{R}^n}\hat{f}_0(\xi)\,\overline{\widehat{H}_{\alpha,0}(\xi)}(2|\xi|)^{-\alpha-1}d\xi. \qquad (51)$$
On the other hand, by the Fourier inversion formula
$$f(0,1) = f_1(0) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\hat{f}_1(\xi)d\xi \overset{(42)}{=} (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\hat{f}_0(\xi)e^{-|\xi|}d\xi \qquad (52).$$
Since the Eqs.(51) and (52) hold for any $f \in \mathcal{H}_\alpha$ we arrive at the formula
$$\Gamma(\alpha+1)\widehat{H}_{\alpha,0}(\xi)(2|\xi|)^{-\alpha-1} = (2\pi)^{-\frac{n}{2}}\, e^{-|\xi|}$$
which is equivalent to
$$\widehat{H}_{\alpha,0}(\xi) = \frac{(2|\xi|)^{\alpha+1}e^{-|\xi|}}{(2\pi)^{\frac{n}{2}}\Gamma(\alpha+1)} \qquad (53)\ .$$
Since by (42)
$$\widehat{H}_{\alpha,y}(\xi) = e^{-|\xi|y}\widehat{H}_{\alpha,0}(\xi),$$
using (53) we obtain
$$\widehat{H}_{\alpha,y}(\xi) = \frac{(2|\xi|)^{\alpha+1}e^{-(1+y)|\xi|}}{(2\pi)^{\frac{n}{2}}\Gamma(\alpha+1)}$$
and the proof is complete.

Employing the inverse Fourier transform ,the last proposition gives
$$H_\alpha(x,y) = \frac{1}{(2\pi)^n\Gamma(\alpha+1)}\int_{\mathbb{R}^n}(2|\xi|)^{\alpha+1}e^{-(y+1)|\xi|}e^{i\xi.x}d\xi \qquad (54).$$
In the spherical coordinates $\xi = r\zeta, r > 0, \zeta \in \boldsymbol{S}^{n-1}, d\zeta = r^{n-1}dr\, d\sigma(\zeta)$, the formula (54)becomes
$$H_\alpha(x,y) = \frac{2^{\alpha+1}}{(2\pi)^n\Gamma(\alpha+1)}\int_0^\infty\int_{\boldsymbol{S}^{n-1}} r^{\alpha+n}e^{-(y+1)r}e^{ir\zeta.x}d\sigma(\zeta)dr\ .$$
We make an additional change of variables $r \mapsto \frac{r}{y+1}$
$$H_\alpha(x,y) = \frac{2^{\alpha+1}}{(2\pi)^n\Gamma(\alpha+1)(y+1)^{\alpha+n+1}}\int_0^\infty\int_{\boldsymbol{S}^{n-1}} r^{\alpha+n}e^{-r}e^{i\frac{r\zeta.x}{y+1}}d\sigma(\zeta)dr.$$
Finally we make the change of variable $r \longmapsto \alpha r$ to obtain:
$$H_\alpha(x,y) = \frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n\Gamma(\alpha+1)(y+1)^{\alpha+n+1}e^\alpha}\int_0^\infty\int_{\boldsymbol{S}^{n-1}} r^{\alpha+n}e^{(1-r)\alpha}e^{i\frac{r\alpha\zeta.x}{y+1}}d\sigma(\zeta)dr. \qquad (55)$$
Note that $r^{\alpha+n}e^{(1-r)\alpha} = r^n e^{\alpha(\ln r+1-r)}$ and $lnr + 1 - r < 0\ for\ all\ r \in (0,\infty)\setminus\{1\}$, so that the last integrand stays bounded as $\alpha \to +\infty$.

First we employ the very definition of the Berezin transform with the kernel as given by (55) to obtain this rather huge formula: $B_\alpha f(0,1) = \frac{1}{H_\alpha(0,1)}\left(\frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n\Gamma(\alpha+1)e^\alpha}\right)^2\int_0^\infty\int_{\boldsymbol{R}^n} f(z,w)\frac{w^\alpha}{(w+1)^{2\alpha+2n+2}}\times$
$$\int_0^\infty\int_0^\infty\int_{\boldsymbol{S}^{n-1}}\int_{\boldsymbol{S}^{n-1}} r^n s^n\left(\frac{r}{e^{r-1}}\frac{s}{e^{s-1}}\right)^\alpha e^{\frac{i\alpha(r\zeta-s\eta).z}{w+1}}d\sigma(\zeta)d\sigma(\eta)ds\, dr\, dz\, dw \qquad (56).$$
At this point we would like to invoke some results from harmonic analysis. Consider the orthogonal group $\mathrm{O}(n)$.It is a well known fact that $\mathrm{O}(n)$is a compact Lie group and that there is a normalized

left and right invariant Haar measure dg on $O(n)$ such that for every function F that is continuous on $\boldsymbol{S}^{n-1}$ we have

$$\int_{\boldsymbol{S}^{n-1}} F(\zeta)d\sigma(\zeta) = \omega_n \int_{O(n)} F(ge_1)dg, \qquad (57)$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the total volume of the sphere, g $\in O(n)$ and $e_1 = (1,0,\dots,0) \in \boldsymbol{S}^{n-1}$. Now we can apply (57) to some of the integrals from (56),namely:

$$\begin{aligned} \jmath_o(\alpha,r,s,w) &:= \int_{\boldsymbol{R}^n}\int_{\boldsymbol{S}^{n-1}}\int_{\boldsymbol{S}^{n-1}} f(z,w)e^{\frac{i\alpha(r\zeta-s\eta).z}{w+1}} d\sigma(\zeta)\, d\sigma(\eta)\, dz \\ &= \omega_n^2 \int_{O(n)}\int_{O(n)}\int_{\boldsymbol{R}^n} f(z,w)e^{\frac{i\alpha(rge_1-she_1).z}{w+1}} dz\, dg\, dh \\ &= \omega_n^2 \int_{O(n)}\int_{O(n)}\int_{\boldsymbol{R}^n} f(hz,w)e^{\frac{i\alpha(rhge_1-she_1).hz}{w+1}} dz\, dg\, dh \\ &= \omega_n^2 \int_{O(n)}\int_{O(n)}\int_{\boldsymbol{R}^n} f(hz,w)e^{\frac{i\alpha(rge_1-se_1).z}{w+1}} dz\, dg\, dh, \end{aligned}$$

where in the third and fourth equality we respectively used the change of variables $g \mapsto hg, z \mapsto hz$ and the fact that h preserves the inner product on $\boldsymbol{R}^n$. Denoting

$$f^*(t,w) := \int_{O(n)} f(hte_1,w)\, dh = \frac{1}{\omega_n}\int_{\boldsymbol{S}^{n-1}} f(t\zeta,w)\, d\sigma(\zeta)$$

we can turn the last integral into

$$\jmath_o(\alpha,r,s,w) = \omega_n^2 \int_{O(n)}\int_{\boldsymbol{R}^n} f^*(|z|,w)e^{\frac{i\alpha(rge_1-se_1).z}{w+1}}\, dz\, dg\, .$$

Using spherical coordinates $z = t\tau$ and (57) once more, we obtain

$$\begin{aligned} \jmath_o(\alpha,r,s,w) &= \omega_n^2 \int_0^\infty \int_{\boldsymbol{S}^{n-1}}\int_{O(n)} f^*(t,w)t^{n-1}e^{\frac{i\alpha(rge_1-se_1).t\tau}{w+1}}\, dg\, d\sigma(\tau)dt \\ &= \omega_n^2 \int_0^\infty \int_{\boldsymbol{S}^{n-1}}\int_{\boldsymbol{S}^{n-1}} f^*(t,w)t^{n-1}e^{\frac{i\alpha(r\zeta-se_1).t\tau}{w+1}}\, d\sigma(\zeta)d\sigma(\tau)dt. \end{aligned}$$

The next step is to insert this last expression for the integral $\jmath_o(\alpha,r,s,w)$ back into the formula (56) for $B_\alpha f(0,1)$:

$$B_\alpha f(0,1) = \frac{\omega_n}{H_\alpha(0,1)}\left(\frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n\Gamma(\alpha+1)e^\alpha}\right)^2 \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_{\boldsymbol{S}^{n-1}}\int_{\boldsymbol{S}^{n-1}} \frac{f^*(t,w)t^{n-1}r^n s^n}{(w+1)^{2n+2}} \times \left(\frac{w}{(w+1)^2}\frac{r}{e^{r-1}}\frac{s}{e^{s-1}}e^{\frac{i(r\zeta-se_1).t\tau}{w+1}}\right)^\alpha d\sigma(\zeta)d\sigma(\tau)\, dt\, dr\, ds\, dw.$$

Finally, let us undo the polar coordinates , say, $r\zeta = y$ and $t\tau = z$ to obtain

$$B_\alpha f(0,1) = \frac{\omega_n}{H_\alpha(0,1)}\left(\frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n\Gamma(\alpha+1)e^\alpha}\right)^2 \int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n} \frac{f^*(|z|,w)|y|s^n}{(w+1)^{2n+2}} \times \left(\frac{w}{(w+1)^2}\frac{|w|}{e^{|w|-1}}\frac{s}{e^{s-1}}e^{\frac{i(y-se_1).z}{w+1}}\right)^\alpha dy\, dz\, ds\, dw \qquad (58)\, .$$

But this is an integral of the form

$$\jmath_o(\alpha) = \int_\Omega F(x)e^{\alpha s(x)}dx\, ,$$

to which the so called *multidimensional Laplace method* applies. Namely, under the conditions that:

(1) the function $F(x)$ is smooth in Ω ,

(2) there is a unique stationary nondegenerate point $x_0 \in \Omega$ of the function $S(x)$, (3) the integral $\jmath(\alpha)$ exists for at least one α,

(4) the point x_0 is the maximum point of Re$S(x)$, and

(5) Re$S(x) \to -\infty$ as $x \to \partial\Omega$ or ∞,

(see [177] or [178] for the details), there is the following asymptotic expansion:

$$\jmath(\alpha) \approx \left(\frac{2\pi}{\alpha}\right)^{\frac{dim\Omega}{2}} \frac{e^{\alpha S(x_0)}}{|\text{Hess } S(x_0)|^{1/2}} \sum_{j=0}^\infty \left(\sum_{k=j}^{3j} \frac{1}{k!(k-j)!2^k} L_S^k (S(x,x_0)^{k-j}F(x))\Big|_{x=x_0}\right)\alpha^{-j},$$

as $\to +\infty$. Here Hess $S(x_0)$ is the determinant of the matrix

$$A = -\left(\frac{\partial^2 S(x_0)}{\partial x_j x_k}\right)_{j,k=1}^{dim\Omega} \qquad (59)$$

L_S is the constant-coefficient differential operator on Ω given by the formula

$$L_S = \sum_{j,k=1}^{dim\Omega}(A^{-1})_{j,k}\frac{\partial^2}{\partial x_j x_k}$$

where A^{-1} is the inverse of the matrix (59), and

$$S(x,x_0) := S(x) - S(x_0) + \frac{1}{2}\langle A(x-x_0), x-x_0\rangle .$$

In the present case we may take $\Omega := \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}), x = (w,s,z,y) \in \Omega$,

$$F(x) := \frac{f^*(|z|,w)|y|s^n}{(w+1)^{2n+2}}$$

and

$$S(x) := \ln\frac{w}{(w+1)^2} + (\ln s + 1 - s) + \frac{i(w-se_1).z}{w+1} + (\ln|y| + 1 - |y|).$$

It is clear that $F(x)$ is a smooth function on Ω (that is why we omitted the axis $y = 0$ from Ω, which has no effect on the integral (58) since it is a set of measure zero) and it is also clear that the integral $\jmath(\alpha)$ exists in fact even for any $\alpha \geq 0$, since (1) does. It is also obvious that $\mathrm{Re}S(x) \to -\infty$ as $x \to \partial\Omega$ or ∞. Further,the point $x_0 = (w_o,s_o,z_o,y_0) = (1,1,0,e_1)$ is the only stationary point of the function $S(x)$,i.e. $\nabla S(x_0) = 0$, and it is also a nondegenerate one since the matrix (59), which in this case reads as its inverse. Last but not least, the point x_0 can also be easily verified to be the maximum point of the function $\mathrm{Re}S(x)$. Thus in the context of the integral (58) we obtain the asymptotic expansion

$$B_\alpha f(0,1) \approx c_{\alpha,n} \sum_{j=0}^{\infty}\left(\sum_{k=j}^{3j}\frac{1}{k!(k-j)!2^k}L_S^k(S(x,x_0)^{k-j}F(x))\Big|_{x=x_0}\right)\alpha^{-j} \qquad (60)$$

where

$$c_{\alpha,n} := \frac{\omega_n}{H_\alpha(0,1)}\left(\frac{2^{\alpha+1}\alpha^{\alpha+n+1}}{(2\pi)^n\Gamma(\alpha+1)e^\alpha}\right)^2 \left(\frac{2\pi}{\alpha}\right)^{n+1}\frac{e^{\alpha S(x_0)}}{|\mathrm{Hess}\ S(x_0)|^{1/2}}$$

(here, we have in fact $e^{\alpha S(x_0)} = 4^{-\alpha}$ and $|\mathrm{Hess}\ S(x_0)|^{1/2} = 1/2^n$) ,and

$$S(x,x_0) := \ln\frac{w}{(w+1)^2} + (\ln s + 1 - s) + \frac{i(w-se_1).z}{w+1} + (\ln|y| + 1 - |y|) + 2.\ln 2 + \frac{1}{4}(w-1)^2 + \frac{1}{2}\left(s - 1 + \frac{1}{2}iz_1\right)(s-1) + \left(\frac{1}{4}i(s-1) - \frac{1}{4}i(y_1-1)\right)z_1 - \frac{1}{4}iy_2z_2 - \cdots - \frac{1}{4}iy_nz_n + \frac{1}{2}\left(y_1 - \frac{1}{2}iz_1 - 1\right)(y_1-1) - \frac{1}{4}iz_2y_2 - \cdots - \frac{1}{4}iz_ny_n.$$

The derivatives of F implicit in (60) can be expressed in terms of those of $f(z,w)$ at $(z,w) = (0,1)$. Namely,the Taylor expansion of the function $f^*(t,w)$ at the point $(t_0,w_0) = (0,1)$ is (see [176], formulas (34) , (35))

$$f^*(t,w) = \sum_{j,m=0}^{\infty}\frac{\left(\Delta_z^m\partial_w^j f\right)(0,1)}{j!m!4^m\left(\frac{n}{2}\right)_m}t^{2m}(w-1)^j \qquad (61)$$

where Δ_z denotes the Laplace operator applied to $f(z,w)$ with respect to the z variable. In this way we arrive at an asymptotic expansion of $B_\alpha f(0,1)$ in terms of certain differential operators M_j on Ω ,this time acting on f and evaluated at (0,1):

$$B_\alpha f(0,1) \approx c_{\alpha,n}\sum_{j=0}^{\infty}\frac{M_jf(0,1)}{\alpha^j}, \quad \text{as } \alpha \to +\infty \qquad (62)$$

We pause to calculate $M_0f(0,1)$ and $M_1f(0,1)$ explicitly. To that end we adopt the following temporary notation that will play no role in the sequel: $L_j = \sum_{k=j}^{3j}\frac{1}{k!(k-j)!2^k}L_S^k(S(x,x_0)^{k-j}F(x))\Big|_{x=x_0}$,

and $a_{i,j}$ and $a^{i,j}$ denote the entries in the i-th row and j-th column of the matrix A and A^{-1}, respectively. It can be readily seen that L_0 is just $F(x_0)$ whence we infer,due to (61), that

$$M_0f(0,1) = \frac{f^*(0,1)}{2^{2n+2}} = \frac{f(0,1)}{2^{2n+2}} \qquad (63) .$$

Now for something slightly more complicated, the coefficient L_1 runs :

$$L_1 = \sum_{k=1}^{3} \frac{1}{k!(k-1)!2^k} L_S^k(S(x,x_0)^{k-1}F(x))\Big|_{x=x_0}.$$
$$= \frac{1}{2} L_S(F(x))|_{x=x_0} + \frac{1}{8} L_S^2(S(x,x_0)F(x))|_{x=x_0} + \frac{1}{96} L_S^3(S^2(x,x_0)F(x))|_{x=x_0}$$
$$= \frac{1}{2}\sum_{j,k=1}^{2n+2} a^{j,k} \frac{\partial^2 F(x)}{\partial x_j \partial x_k}\Big|_{x=x_0} + \frac{1}{8}\sum_{j,k,l,m=1}^{2n+2} a^{j,k} a^{l,m} \frac{\partial^4 (S(x,x_0)F(x))}{\partial x_j \partial x_k \partial x_l \partial x_m}\Big|_{x=x_0}$$
$$+ \frac{1}{96}\sum_{j,k,l,m,p,q=1}^{2n+2} a^{j,k} a^{l,m} a^{p,q} \frac{\partial^6 (S^2(x,x_0)F(x))}{\partial x_j \partial x_k \partial x_l \partial x_m \partial x_p \partial x_q}\Big|_{x=x_0} =: L_{1,1} + L_{1,2} + L_{1,3}$$

The matrix A^{-1} is quite sparse so it is enough to compute only those derivatives that correspond to such values of the indices (j,k), (j,k,l,m) and (j,k,l,m,p,q) for which the respective entries $a^{j,k}$, $a^{l,m}$ and $a^{p,q}$of the matrix A^{-1} are nonzero . To a certain extent ,the fact that the matrix A^{-1} and the corresponding derivatives are symmetric is helpful , too , since it makes , say , the upper diagonal entries $a^{i,j}$, $j \geq i$ those that really matter and thus helps us have control over the situation by counting only their occurrences according to their ‘‘multiplicity’’: the $L_{1,1}$-part of L_1 is straightforward, the (j,k) entries with $j = k$ count once , the ones with $j \neq k$ twice.To count the multiplicities of the $L_{1,2}$ –part and the $L_{1,3}$-part of L_1 we sort the 2-tuples $(j,k), (l,m)$ with the help of the following notation: $(j,k) \cong (l,m)$ means that (j,k) is equal to (l,m) up to a permutation , while $(j,k) \ncong (l,m)$ means that (j,k) is not equal to (l,m) nor to (m,l).

A moment's reflection now shows that in $L_{1,2}$ the entries (j,k), (l,m) with $j = k = l = m$ count once, entries (j,k), (l,m) with $j = k$ and $l = m$ but $(j,k) \ncong (l,m)$ count twice , entries with $j = k$, $l \neq m$ and $(j,k) \ncong (l,m)$ count four times ,entries with $j \neq k, l \neq m$ and $(j,k) \ncong (l,m)$ eight times and entries with $j \neq k, l \neq m$ but$(j,k) \cong (l,m)$ count four times. With $L_{1,3}$ the situation gets already a little bit messy : of course , entries with $j = k = l = m = p = q$ count once. Now, entries with $j = k = l = m$ and $p = q$ but $(l,m) \ncong (p,q)$ count three times ,entries with $j = k = l = m$, $p \neq q$ and $(l,m) \ncong (p,q)$ count six times , entries with $j = k$, $l = m$, $p = q$ but $(j,k) \ncong (l,m) \ncong (p,q)$ count six times , entries with $j = k$,l $l \neq m$, $p \neq q$ and $(j,k) \ncong (l,m) \ncong (p,q)$ count twenty-four times, entries with $j \neq k, l \neq m, p \neq q$ and $(j,k) \ncong (l,m) \ncong (p,q)$ count forty-eight times, entries with $j \neq k, l \neq m, p \neq q$ and $(j,k) \cong (l,m) \cong (p,q)$ count eight times,entries with $j = k, l \neq m, p \neq q$and $(j,k) \ncong (l,m)$ but $(l,m) \cong (p,q)$ count twelve times, entries with $j = k$, $l = m, p \neq q$ and $(j,k) \ncong (l,m) \ncong (p,q)$ count twelve times and, finally, entries with $j \neq k, l \neq m, p \neq q$ and $(j,k) \cong (l,m)$ but $(l,m) \ncong (p,q)$ count twenty-four times. With these conventions , the only nonzero contributions to $L_{1,1}$ occur for (j,k) equal to (1,1), (2,2), (3,3), (2, $n+3$). Similarly the only nonzero contributions to $L_{1,2}$ appear for (j,k,l,m) equal to (1,1,1,1) , (1,1,2,3) , (1,1,α,$n+\alpha$) where α =3,..., n +2,(2,2,2,2),(2,2,2,3), (2,2,2, n +3), (2, n +3, n +3, n +3), (3, n +3, n +3, n +3) and (n +3, n +3, n+3, n+3).Finally,the only nonzero contributions to $L_{1,3}$ occur for (j,k,l,m,p,q) equal to (1,1,1,1,1,1),(1,1,1,1,2,3),(1,1,1,1, α, n + α),α =3,..., n +2, (1, 1, 2, 2, 3, 3), (1, 1, 2, 3, 2, 3), (1,1,2,3, α, n + α), α = 3,..., n + α,(1,1,2, n +3,3,3), (1,1,3,3, n +3,n+3), (1,1,3, n + 3, α, n + α), α = 3,..., n + 2, (1,1, α, n + α, α, n + α), α = 3,..., n + 2, (1,1, α, n + α, β, n + β), α = 4,..., n +2, β= α +1,..., n +2, (2,2,2,2,2,2), (2,2,2, n +3, n +3, n +3),(2, n +3,2, n +3,2, n +3) and (n +3, n +3, n +3, n +3, n +3, n +3).Taking everything into account we get after tedious but routine calculations

$$L_{1,1} = \left((2n+2)(2n+3)2^{-2n-4} + n(n-1)2^{-2n-4} + n2^{-2n-3}\right)f(0,1) - (2n+2)2^{-2n-2}\partial_w f(0,1) + 2^{-2n-2}\partial_w^2 f(0,1) + 2^{-2n-2}\frac{\Delta_z f}{n}(0,1),$$
$$L_{1,2} = 2^{-2n-3}\left((-2n^2 - 8n - 11)f(0,1) + (2n+6)\partial_w f(0,1)\right),$$
$$L_{1,3} = \frac{2^{-2n-3}}{6}(3n^2 + 21n + 50)f(0,1),$$
so that

$$L_1 = L_{1,1} + L_{1,2} + L_{1,3} = \frac{2^{-2n-3}}{3}\big((3n^2+3n+1)f(0,1) - 6(n-1)\partial_w f(0,1) + 12\partial_w^2 f(0,1) + 6n^{-1}\Delta_z f(0,1)\big).$$

Finally, we use the obvious fact that $B_\alpha \mathbf{1} = \mathbf{1}$ for all α (by the reproducing property of the kernel K_α), so

$$1 \approx c_{\alpha,n} \sum_{j=0}^{\infty} \frac{M_j \mathbf{1}(0,1)}{\alpha^j}.$$

Since $M_0 \mathbf{1}(0,1) = 2^{-2n-2} \neq 0$ by (63), the formal power series on the right can be inverted:

$$c_{\alpha,n} \approx \frac{1}{\sum_{j=0}^{\infty} \alpha^{-j} M_j \mathbf{1}(0,1)} =: \sum_{k=0}^{\infty} \frac{N_k}{\alpha^k},$$

where N_k can be obtained from the recursion formula

$$\sum_{j+k=m} M_j \mathbf{1}(0,1) N_k = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m > 0 \end{cases}$$

Dividing the two formal series corresponding to the respective asymptotic expansions of $B_\alpha f(0,1)$ and $B_\alpha \mathbf{1}(0,1)$ gets us rid of the $c_{\alpha,n}$-term in (62) and we arrive at the desired formula

$$B_\alpha f(0,1) \approx \sum_{j=0}^{\infty} \frac{R_j f(0,1)}{\alpha^j},$$

as $\alpha \to \infty$, where R_j are differential operators that are given recursively by the standard formula for the product of formal power series :

$$R_j f(0,1) = \sum_{k=0}^{j} N_{j-k} M_k f(0,1),$$

(see[132]) .Carrying out this procedure we see that

$$R_0 f(0,1) = f(0,1),$$

so that

$$R_0 f(x,y) = R_0 f^{x,y}(0,1) = f^{x,y}(0,1) = f(x,y)$$

for arbitrary $(x,y) \in H$, and that

$$R_1 f(0,1) = \frac{(\Delta_x f)(0,1)}{n} + (1-n)\frac{\partial f}{\partial y}(0,1) + \frac{\partial^2 f}{\partial y^2}(0,1),$$

so that

$$R_1 f(x,y) = R_1 f^{x,y}(0,1) = y^2 \frac{\Delta f}{n}(x,y) + (1-n) y \frac{\partial f}{\partial y}(x,y) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y)$$

for arbitrary $(x,y) \in H$ with $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ .This completes the proof of Theorem(3.2.1).

Chapter 4
Noncommutative Polydomains and Berezin Transforms

We show an open problem for quite some time to find significant classes of elements in the commutative polidisc for whicSh a theory of characteristic functions and model theory can be developed along the lines of the Sz.-Nagy–Foias theory of contractions. We give a positive answer to this question, in our more general setting, providing a characterization for the class of tuples of operators in which admit characteristic functions. The characteristic function is constructed explicitly as an artifact of the noncommutative Berezin kernel associated with the polydomain, and it is proved to be a complete unitary invariant for the class of completely non-coisometric tuples. Using noncommutative Berezin transforms and C^*-algebras techniques, we develop a dilation theory on the noncommutative polydomain .

Section (4.1): Noncommutative Polydomains

We denote by $B(\mathcal{H})$ the algebra of bounded linear operators on a Hilbert space H. A polynomial $q \in \mathbb{C}[Z_1,\dots,Z_n]$ in n noncommuting indeterminates is called positive regular if all its coefficients are positive, the constant term is zero, and the coefficients of the linear terms $Z_1,\dots,Z_n$ are different from zero. If $X = (X_1,\dots,X_n) \in B(\mathcal{H})^n$ and $q = \sum_\alpha a_\alpha Z_\alpha$, we define the map $\Phi_{q,X} : B(\mathcal{H}) \to B(\mathcal{H})$ by setting $\Phi_{q,X}(Y) := \sum_\alpha a_\alpha X_\alpha Y X_\alpha^*$.

Given two k-tuples $\mathbf{m} := (m_1,\dots,m_k)$ and $\mathbf{n} := (n_1,\dots,n_k)$ with $m_i, n_i \in \mathbb{N} := \{1,2,\dots\}$, and a k-tuple $\mathbf{q} = (q_1,\dots,q_k)$ of positive regular polynomials $q_i \in C[Z_1,\dots,Z_{n_i}]$, we associate with each element $\mathbf{X} = (X_1,\dots,X_k) \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ the defect mapping $\Delta_{\mathbf{q},\mathbf{X}}^{\mathbf{m}}: B(\mathcal{H}) \to B(\mathcal{H})$ defined by

$$\Delta_{\mathbf{q},\mathbf{X}}^{\mathbf{m}} := \left(id - \Phi_{q_1,X_1}\right)^{m_1} \circ \cdots \circ \left(id - \Phi_{q_k,X_k}\right)^{m_k}.$$

We denote by $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ the set of all tuples $= (X_1,\dots,X_k) \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$, where $X_i := (X_{i,1},\dots,X_{i,n_i}) \in B(\mathcal{H})^{n_i}, i \in \{1,\dots,k\}$, with the property that, for any $p,q \in \{1,\dots,k\}, p \neq q$, the entries of X_p are commuting with the entries of X_q. In this case we say that X_p and X_q are commuting tuples of operators. Note that the operators $X_{i,1},\dots,X_{i,n_i}$ are not necessarily commuting.

We develop an operator model theory and a theory of free holomorphic functions on the noncommutative polydomains

$$\mathrm{D}_{\mathrm{q}}^{\mathrm{m}}(\mathcal{H}) := \left\{\mathrm{X} = (X_1,\dots,X_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} : \Delta_{\mathrm{q},\mathrm{X}}^{\mathrm{p}}(I) \geq 0 \text{ for } 0 \leq \mathrm{p} \leq \mathrm{m}\right\}.$$

Our study is an attempt to unify the multivariable operator model theory for the ball-like domains and commutative polydiscs, and to extend it further to the above-mentioned polydomains. The main tool in our investigation is a Berezin [183] type transform associated with the abstract noncommutative domain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}} := \{\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H}) : \mathcal{H} \text{ is a Hilbert space}\}$.

In the last sixty years, this type of polydomains has been studied in several particular cases. Most of all, we mention the study of the closed operator unit ball

$$[B(\mathcal{H})]_1^- := \{X \in B(\mathcal{H}) : I - XX^* \geq 0\}$$

(which corresponds to the case $k = n_1 = m_1 = 1$, and $q_1 = Z$) which has generated the celebrated Sz.-Nagy–Foias [154] theory of contractions on Hilbert spaces and has had profound implications in function theory, interpolation, and linear systems theory. When $k = n_1 = 1, m_1 \geq 2$, and $q_1 = Z$, the corresponding domain coincides with the set of all m-hypercontractions studied by Agler in [189], [192], and recently by Olofsson [185], [186].

In several variables, the case when $k = 1, n_1 \geq 2, m_1 = 1$, and $q_1 = Z_1 + \cdots + Z_n$, corresponds to the closed operator ball

$$[B(\mathcal{H})^n]_1^- := \{(X_1,\dots,X_n) \in B(\mathcal{H})^n : I - X_1X_1^* - \cdots - X_nX_n^* \geq 0\}$$

and its study has generated a free analogue of Sz.-Nagy–Foias theory (see [187], [188]. The commutative case was considered by Drurry [181], extensively studied by Arveson [195], [196], and also in [188], [195], [192], and [190]. We should remark that, in recent years, many results concerning the theory of row contractions were extended by Muhly and Solel ([196], [197], [198]) to representations of tensor algebras over C^*-correspondences and Hardy algebras. We mention that in the particular case when $k = 1$ and q_1 is a positive regular polynomial, the corresponding domain was studied in [196], if $m_1 = 1$, and in [192], [198], [199], when $m_1 \geq 2$. The commutative case when $m_1 \geq 2, n_1 \geq 2$, and $q_1 = Z_1 + \cdots + Z_n$, was studied by Athavale [197], Müller [194], Müller-Vasilescu [195], Vasilescu [186], and Curto-Vasilescu [196]. Some of these results were extended by S. Pott [191] when q_1 is a positive regular polynomial in commuting indeterminates.

The commutative polydisc case, i.e, $k \geq 2, n_1 = \cdots = n_k = 1$, and $q = (Z_1, \ldots, Z_k)$, was first considered by Brehmer [195] in connection with regular dilations. Motivated by Agler's work [192] on weighted shifts as model operators, Curto and Vasilescu developed a theory of standard operator models in the polydisc in [197], [198]. Timotin [195] was able to obtain some of their results from Brehmer's theorem. The polyball case, when $k \geq 2$ and $q_i = Z_1 + \cdots + Z_{n_i}, i \in \{1, \ldots, k\}$, was considered in [198] and [191] for the noncommutative and commutative case, respectively. As far as we know, unlike the ball case, there is no theory of characteristic functions, analoguos to the Sz.-Nagy–Foias theory, for significant classes of operators in the polydisc (or polyball) case.

We work out some basic properties of the noncommutative polydomains $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$. One of the main results, which plays an important role, states that any podydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ is radial, i.e., $r\mathbf{X} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ whenever $\mathbf{X} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ and $r \in [0, 1)$. This fact has also an important consequence in the particular case when $k = 1$, namely, that all the results from [192], [198], [199], which were proved in the setting of the radial part of $\mathbf{D}_{\mathbf{q_1}}^{\mathbf{m_1}}(\mathcal{H})$, are true for any domain $\mathbf{D}_{\mathbf{q_1}}^{\mathbf{m_1}}(\mathcal{H})$.

We introduce the noncommutative Berezin transform at $\mathbf{T} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ to be the mapping $\mathbf{B}_{\mathbf{T}} : B(\otimes_{i=1}^{k} F^2(H_{n_1})) \to B(\mathcal{H})$ defined by

$$\mathbf{B}_{\mathrm{T}}[g] := \mathbf{K}_{\mathbf{q},\mathbf{T}}^{*}(g \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{q},\mathbf{T}}, \quad g \in B(\otimes_{i=1}^{k} F^2(H_{n_i})),$$

where $F^2(H_{n_i})$ is the full Fock space on ni generators and

$$\mathbf{K}_{\mathbf{q},\mathbf{T}} : \mathcal{H} \to F^2(H_{n_i}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$$

is the noncommutative Berezin kernel associated with $\mathbf{T}$, which is defined in terms of the coefficients of the positive regular polynomials $q_1, \ldots, q_k$. We remark that in the particular case when $\mathcal{H} = \mathbb{C}, q = (Z_1, \ldots, Z_k), \mathbf{T} = \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{D}^k$, and $m_i = n_i = 1$ for any $i \in \{1, \ldots, k\}$, we recover the Berezin transform of a bounded linear operator on the Hardy space $H^2(\mathbb{D}^k)$, i.e.,

$$\mathbf{B}_{\boldsymbol{\lambda}}[g] = \prod_{i=1}^{k}(1 - |\lambda_i|^2)\langle g k_\lambda, k_\lambda\rangle , g \in B(H^2(\mathbb{D}^k)),$$

where $k_\lambda(z) := \prod_{i=1}^{k}\left(1 - \bar{\lambda}_i z_i\right)^{-1}$ and $z = (z_1, \ldots, z_k) \in \mathbb{D}^k$.

The noncommutative Berezin transforms are used to prove the main result of this (Theorem (4.1.11)) which shows that each polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ has a universal model $\mathbf{W} = \{\mathbf{W}_{i,j}\}$ consisting of weighted shifts acting on a tensor product of full Fock spaces. We show that a tuple of operators $\mathbf{X}$ is in the noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ if and only if there exists a completely positive linear map $\Psi : C^*(\mathbf{W}_{i,j}) \to B(\mathcal{H})$ such that

$$\Psi(p(\mathbf{W})r(\mathbf{W})^*) = p(X)r(X)^*,$$

for any $p(\mathbf{W}), r(\mathbf{W})$ polynomials in $\{\mathbf{W}_{i,j}\}$ and the identity.

We introduce the noncommutative Hardy algebra $F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$ as the weakly closed algebra generated by $\{\mathbf{W}_{i,j}\}$ and the identity, and use it to provide a WOT-continuous functional calculus for completely non-coisometric tuples $\mathbf{T} = \{T_{i,j}\}$ in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$, which are identified. We show that

$$\Phi(\varphi) := \mathrm{SOT} - \lim_{r \to 1} \varphi\left(rT_{i,j}\right), \varphi = \varphi(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}),$$

exists in the strong operator topology and defines a map $\Phi : F^\infty\left(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}\right) \to B(\mathcal{H})$ with the property that $\Phi(\varphi) = \text{SOT-}\lim_{r\to 1}\mathbf{B}_{r\mathbf{T}}[\varphi]$, where $\mathbf{B}_{r\mathbf{T}}$ is the noncommutative Berezin transform at $r\mathbf{T} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$.

Moreover, Φ is a unital completely contractive homomorphism, which is WOT-continuous (resp. SOT-continuous) on bounded sets.
We introduce the algebra $Hol(\mathbf{D}^{\mathbf{m}}_{\mathbf{q},\mathrm{rad}})$ of all free holomorphic functions on the abstract radial polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{q},\mathrm{rad}}$. We identify the polydomain algebra $A(\mathbf{D}^{\mathbf{m}}_{\mathbf{q}})$ (the closed algebra generated by $\{\mathbf{W}_{i,j}\}$ and the identity) and the Hardy algebra $F^{\infty}(\mathbf{D}^{\mathbf{m}}_{\mathbf{q}})$ with subalgebras of $Hol(\mathbf{D}^{\mathbf{m}}_{\mathbf{q},\mathrm{rad}})$. For example, it is shown that the noncommutative Berezin transform is a completely isometric isomorphism between $F^{\infty}(\mathbf{D}^{\mathbf{m}}_{\mathbf{q}})$ and the algebra of bounded free holomorphic functions on $\mathbf{D}^{\mathbf{m}}_{\mathbf{q},\mathrm{rad}}$.
We remark that there is an important connection between the theory of free holomorphic functions on abstract radial polydomains $\mathbf{D}^{\mathbf{m}}_{\mathbf{q},\mathrm{rad}}$, and the theory of holomorphic functions on polydomains in $\mathbb{C}^d$ (see [193], [192]). Indeed, if $\mathcal{H} = \mathbb{C}^p$ and $p \in \mathbb{N}$, then $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathbb{C}^p)$ can be seen as a subset of $\mathbb{C}^{(n_1+\cdots+n_k)p^2}$ with an arbitrary norm. Given a free holomorphic function φ on the abstract radial polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{q},\mathrm{rad}}$, we prove that its representation on $\mathbb{C}^p$, i.e., the map $\hat{\varphi}$ defined by

$$\mathbb{C}^{(n_1+\cdots+n_k)p^2} \supset \mathbf{D}^{\mathbf{m}}_{\mathbf{q},\mathrm{rad}}(\mathbb{C}^p) \ni (\lambda_{i,j}) \mapsto \varphi(\lambda_{i,j}) \in \mathbb{C}^{p^2}$$

is a holomorphic function on the interior of $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathbb{C}^p)$. In addition, $\hat{\varphi}$ is bounded when $\varphi \in F^{\infty}(\mathbf{D}^{\mathbf{m}}_{\mathbf{q}})$, and it has continuous extension to $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathbb{C}^p))$ when $\varphi \in \mathcal{A}(\mathbf{D}^{\mathbf{m}}_{\mathbf{q}})$.
We obtain a characterization of the Beurling [192] type joint invariant subspaces under $\{\mathbf{W}_{i,j}\}$. We prove that a subspace $\mathcal{M} \subset \bigotimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{H}$ has the form $\mathcal{M} = \Psi\,(\bigotimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{E}$ for some inner multi-analytic operator with respect to the universal model $\mathbf{W}$, if and only if

$$\Delta^{\mathbf{p}}_{\mathbf{q},\mathbf{W}\otimes\mathbf{I}}(P_{\mathcal{M}}) \geq 0 \text{ for any } \boldsymbol{p} \in \mathbb{Z}_+^k, \mathbf{p} \leq \mathbf{m},$$

where $P_{\mathcal{M}}$ is the orthogonal projection of the Hilbert space $\bigotimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{H}$ onto $\mathcal{M}$. In the particular case when $m = (1,\ldots,1)$, the latter condition is satisfied when $\mathbf{W} \otimes I|_{\mathcal{M}}$ is a doubly commuting tuple. We also characterize the reducing subspaces under $\{\mathbf{W}_{i,j}\}$ and present several results concerning the model theory for pure elements in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$.
We provide a characterization for the class of tuples of operators in $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ which admit characteristic functions. We say that $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ has characteristic function if there is a multi-analytic operator Ψ with respect to the universal model $\mathbf{W}$ such that

$$\mathbf{K}_{\mathbf{q},\mathbf{T}}, \mathbf{K}^*_{\mathbf{q},\mathbf{T}} + \Psi\Psi^* = I,$$

where $\mathbf{K}_{\mathbf{q},\mathbf{T}}$ is the noncommutative Berezin kernel associated with $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$. In this case, Ψ is essentially unique. We prove that $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ has characteristic function if and only if

$$\Delta^{\mathbf{p}}_{\mathbf{q},\mathbf{W}\otimes\mathbf{I}}(I - \mathbf{K}_{\mathbf{q},\mathbf{T}}, \mathbf{K}^*_{\mathbf{q},\mathbf{T}}) \geq 0, \text{for any } \boldsymbol{p} \in \mathbb{Z}_+^k, \mathbf{p} \leq \mathbf{m}$$

The characteristic function is constructed explicitly and it is proved to be a complete unitary invariant for the class of completely non-coisometric tuples. We provide an operator model for this class of elements in $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ in terms of their characteristic functions.
Using several results from the previous and C^*-algebras techniques, we develop a dilation theory on the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$. The main result states that if $\mathbf{T} = \{T_{i,j}\}$ is a tuple in $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$, then there exists a $*$-representation $\pi : C^*(\mathbf{W}_{i,j}) \to B(\mathcal{K}_\pi)$ on a separable Hilbert space $\mathcal{K}_\pi$, which annihilates the compact operators and $\Delta^{\mathbf{m}}_{\mathbf{q},\boldsymbol{\pi}(\mathbf{W})}(I_{\mathcal{K}_\pi}) = 0$ such that $\mathcal{H}$ can be identified with a $*$-cyclic co-invariant subspace of

$$\tilde{\mathcal{K}} := \left[(\bigotimes_{i=1}^{k} F^2(H_{n_i})) \otimes \overline{\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}\right] \oplus \mathcal{K}_\pi$$

under each operator

$$V_{i,j} := \begin{bmatrix} \mathbf{W}_{i,j} \otimes I & 0 \\ 0 & \pi(\mathbf{W}_{i,j}) \end{bmatrix}$$

and such that $T^*_{i,j} = V^*_{i,j}|_{\mathcal{H}}$ for all i,j.Under a certain additional condition on the universal model $\mathbf{W}$, the dilation above is minimal and unique up to unitary equivalence. We also obtain Wold type decompositions for non-degenerate $*$-representations of the C^*-algebra $C^*(\mathbf{W}_{i,j})$.

We mention that the results are presented in a more general setting, when q is replaced by a k-tuple $\mathbf{f} = (f_1,\ldots,f_k)$ of positive regular free holomorphic functions in a neighborhood of the origin.

Also, the results are used in [190] to develop an operator model theory for varieties in noncommutative polydomains. This includes various commutative cases which are presented in close connection with the theory of holomorphic functions in several complex variables.
For each $i \in \{1,\ldots,k\}$, let $\mathbb{F}_{n_i}^+$ be the unital free semigroup on n_i generators $g_1^i,\ldots,g_{n_i}^i$ and the identity g_0^i . The length of $\alpha \in \mathbb{F}_{n_i}^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0^i$ and $|\alpha| := p$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i$, where $j_1,\ldots,j_p \in \{1,\ldots,n_i\}$. If $Z_1,\ldots,Z_{n_i}$ are noncommuting indeterminates, we denote $Z_\alpha := Z_{j_1}\cdots Z_{j_p}$ and $Z_{g_0^i} := 1$. Let $f_i := \sum_{\alpha\in\mathbb{F}_{n_i}^+} a_{i,\alpha} Z_\alpha$, $a_{i,\alpha} \in \mathbb{C}$, be a formal power series in n_i noncommuting indeterminates $Z_1,\ldots,Z_{n_i}$. We say that f_i is a positive regular free holomorphic function if the following conditions hold: $a_{i,\alpha} \geq 0$ for any $\alpha \in \mathbb{F}_{n_i}^+$, $a_{i,g_0^i} = 0$, $a_{i,g_j^i} > 0$ for $j = 1,\ldots,n_i$, and

$$\limsup_{k\to\infty}\left(\sum_{|\alpha|=k}|a_{i,\alpha}|^2\right)^{1/2k} < \infty.$$

Given $X_i := (X_{i,1},\ldots,X_{i,n_i}) \in B(\mathcal{H})^{n_i}$, define the map $\Phi_{f_i,X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ by setting

$$\Phi_{f_i,X_i}(Y) := \sum_{k=1}^{\infty}\sum_{\alpha\in\mathbb{F}_{n_i}^+,|\alpha|=k} \alpha_{i,\alpha}X_{i,\alpha}Y X_{i,\alpha}^*, \qquad Y \in B(\mathcal{H}),$$

where the convergence is in the week operator topology.

Let $\mathbf{n} := (n_1,\ldots,n_k)$ and $\mathbf{m} := (m_1,\ldots,m_k)$, where $n_i , m_i \in \mathbb{N} := \{1,2,\ldots\}$ and $i \in \{1,\ldots,k\}$, and let $\mathbf{f} := (f_1,\ldots,f_k)$ be a k-tuple of positive regular free holomorphic functions. We introduce the noncommutative polydomain $\mathbf{D}_{\mathrm{f}}^{\mathrm{m}}(\mathcal{H})$ to be the set of all k-tuples

$$\mathbf{X} := (X_1,\ldots,X_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$$

with the property that $\Phi_{f_i,X_i}(I) \leq I$ and

$$\left(id - \Phi_{f_1,X_1}\right)^{\epsilon_1 m_1}\cdots\left(id - \Phi_{f_k,X_k}\right)^{\epsilon_k m_k}(I) \geq 0$$

for any $i \in \{1,\ldots,k\}$ and $\epsilon_i \in \{0,1\}$. We use the convention that $\left(id - \Phi_{f_i,X_i}\right)^0 = id$. We remark that $\mathbf{D}_{\mathrm{f}}^{\mathrm{m}}(\mathcal{H})$ contains a polyball $[B(\mathcal{H})^{n_i}]_{r_1}^- \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_{r_k}^-$ for some $r_1,\ldots,r_k > 0$, where

$$[B(\mathcal{H})^{n_i}]_{r_i}^- := \{(Y_1,\ldots,Y_{n_i}) \in B(\mathcal{H})^{n_i} : Y_1Y_1^* + \cdots + Y_{n_i}Y_{n_i}^* \leq r_i^2\}.$$

We refer to $\mathbf{D}_{\mathrm{q}}^{\mathrm{m}} := \{\mathbf{D}_{\mathrm{q}}^{\mathrm{m}}(\mathcal{H}) : \mathcal{H}$ is a Hilbert space$\}$ as the abstract noncommu-tative polydomain, and $\mathbf{D}_{\mathrm{q}}^{\mathrm{m}}(\mathcal{H})$ as its representation on the Hilbert space $\mathcal{H}$.

A linear map $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ is called power bounded if there exists a constant $M > 0$ such that $\|\varphi^k\| \leq M$ for any $k \in \mathbb{N}$. For information on completely bounded (resp. positive) maps, we refer to [191] and [192]. If $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_+^k$ and $\mathbf{q} := (q_1,\ldots,q_k) \in \mathbb{Z}_+^k$, we set $\mathbf{p} \leq \mathbf{q}$ iff $p_i \leq q_i$ for all $i \in \{1,\ldots,k\}$, where $\mathbb{Z}_+ := \{0,1,\ldots\}$.

Proposition (4.1.1)[186]: Let $\varphi_i : B(\mathcal{H}) \to B(\mathcal{H}), i \in \{1,\ldots,k\}$, be power bounded positive linear maps such that

$$\varphi_i\varphi_j = \varphi_j\varphi_i \qquad i,j \in \{1,\ldots,k\}.$$

If $Y \in B(\mathcal{H})$ is a self-adjoint operator and $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_+^k$ with $p_i \geq 1$, then the following statements are equivalent.

$(i) (id-\varphi_1)^{\epsilon_1 p_1}\cdots(id-\varphi_k)^{\epsilon_k p_k}(Y) \geq 0 \; for\; all\; \epsilon_i \in \{0,1\} \; with\; \epsilon := (\epsilon_1,\ldots,\epsilon_k) \neq 0 \; and\; i \in \{1,\ldots,k\}$.

$(ii) (id-\varphi_1)^{q_1}\cdots(id-\varphi_k)^{q_k}(Y) \geq 0 \; for\; all\; \mathbf{q} := (q_1,\ldots,q_k) \in \mathbb{Z}_+^k with\; \mathbf{q} \leq \mathbf{p}\; and\; q \neq 0$.

Proof: Note that it is enough to prove that $(id-\varphi_1)^{p_1}\cdots(id-\varphi_k)^{p_k}(Y) \geq 0$ if and only if $(id-\varphi_1)^{q_1}\cdots(id-\varphi_k)^{q_k}(Y) \geq 0$ for all $\mathbf{q} := (q_1,\ldots,q_k) \in \mathbb{Z}_+^k$ with $q_i \leq p_i$ and $q_i \geq 1$. We proceed by induction over $k \in \mathbb{N}$. Let $k = 1$, and assume that $(id-\varphi_1)^{p_1}(Y) \geq 0$ and $p_1 \geq 2$. Suppose that there is $h_0 \in \mathcal{H}$ such that $\langle(id-\varphi_1)^{p_1-1}(Y)h_0,h_0\rangle < 0$. Set $y_j := \varphi_1^j\langle(id-\varphi_1)^{p_1-1}(Y)h_0,h_0\rangle$,

$j = 0,1,\ldots$, and note that $\{y_j\}_{j=0}^{\infty}$ is a decreasing sequence with $y_j \le y_0 < 0$. Consequently, We deduce that $\sum_{j=0}^{\infty} y_j = -\infty$. On the other hand, we have

$$\left|\sum_{j=0}^{\infty} y_j\right| := \left|\langle\left(id - \varphi_1^{p+1}\right)(id - \varphi_1)^{p_1-2}(Y)h_0, h_0\rangle\right|$$
$$\le \left(1 + \left\|\varphi_1^{p+1}(I)\right\|\right)\|(id - \varphi_1)^{p_1-2}(Y)\|\|h_0\|.$$

Since φ_1 is power bounded, we get a contradiction. Therefore, we must have $(id - \varphi_1)^{p_1-1}(Y) \ge 0$. Continuing this process, we show that $(id - \varphi)^{p_1}(Y) \ge 0$ if and only if $(id - \varphi)^{s}(Y) \ge 0$ for $s = 1,2,\ldots,p_1$.

Now, assume that

$$(id - \varphi_1)^{p_1} \cdots (id - \varphi_k)^{p_k}(id - \varphi_{k+1})^{p_{k+1}}(Y) \ge 0.$$

Due to the fact that $\varphi_i\varphi_j = \varphi_j\varphi_i$ for all $i,j \in \{1,\ldots,k\}$, we deduce that $(id - \varphi_{k+1})^{p_{k+1}}(Y_k) \ge 0$, where $Y_k := (id - \varphi_1)^{p_1} \cdots (id - \varphi_k)^{p_k}(Y)$. On the other hand, due to the identity

$$(id - \varphi_k)^{p_k}(Y) = \sum_{p=0}^{p_k} (-1)^p \binom{p_k}{p} \varphi_k^p(Y),$$

the operator $(id - \varphi_k)^{p_k}(Y)$ is self-adjoint whenever φ_k is a positive linear map and Y is a self-adjoint operator. Inductively, one can easily see that Y_k is a self-adjoint operator. Now, applying the case $k = 1$, we deduce that $(id - \varphi_{k+1})^{p_{k+1}}(Y_k) \ge 0$ if and only if $(id - \varphi_{k+1})^{q_{k+1}}(Y_k) \ge 0$ for all $q_{k+1} \in \{0,1,\ldots,p_{k+1}\}$. Hence,

$$(id - \varphi_1)^{p_{k+1}} \cdots (id - \varphi_k)^{p_k}(id - \varphi_{k+1})^{q_{k+1}}(Y) \ge 0.$$

Due to the induction hypothesis, we deduce that

$$(id - \varphi_1)^{q_{k+1}} \cdots (id - \varphi_k)^{q_k}(id - \varphi_{k+1})^{q_{k+1}}(Y) \ge 0.$$

for all $(q_1,\ldots,q_{k+1}) \in \mathbb{Z}_+^{k+1}$ with $q_i \le p_i$ and $q_i \ge 1$. This completes the proof.

Let $\Phi = (\varphi_1,\ldots,\varphi_k)$ be a k-tuple of power bounded, positive linear maps on $B(\mathcal{H})$ such that $\varphi_i\varphi_j = \varphi_j\varphi_i, i,j \in \{1,\ldots,k\}$. For each $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_+^k$, we define the linear map $\Delta_{\boldsymbol{\Phi}}^{\boldsymbol{p}}: B(\mathcal{H}) \to B(\mathcal{H})$ by setting

$$\Delta_{\boldsymbol{\Phi}}^{p_1,\ldots,p_k} = \Delta_{\boldsymbol{\Phi}}^{\boldsymbol{p}} := (id - \varphi_1)^{p_1} \cdots (id - \varphi_k)^{p_k}$$

Lemma (4.1.2) [186]: Let $\mathbf{m} \in \mathbb{N}^k$ and let $Y \in B(\mathcal{H})$ be a self-adjoint operator such that $\Delta_{\boldsymbol{\Phi}}^{\boldsymbol{p}}(Y) \ge 0$ for all $p \in \mathbb{Z}_+^k$ with $\mathbf{p} \le \mathbf{m}$ and $\mathbf{p} \ne 0$. If $\mathbf{q} \in \mathbb{Z}_+^k$ with $\mathbf{q} \ne 0$ and $\mathbf{q} \le \mathbf{m}$, then

$$\Delta_{\boldsymbol{\Phi}}^{\boldsymbol{m}}(Y) \le \Delta_{\boldsymbol{\Phi}}^{\boldsymbol{q}}(Y).$$

Proof: Set $\mathbf{m} := (m_1,\ldots,m_k) \in \mathbb{N}^k$ and $m' := (m_1 - 1, m_2,\ldots,m_k)$. Since $\Delta_{\boldsymbol{\Phi}}^{\boldsymbol{m}'}(Y) \ge 0$ and φ_1 is a positive map, we deduce that

$$\Delta_{\boldsymbol{\Phi}}^{\boldsymbol{m}}(Y) = \Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y) - \varphi_1(\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)) \le \Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)$$

Using the fact that $\varphi_i\varphi_j = \varphi_j\varphi_i$ for $i,j \in \{1,\ldots,k\}$, one can continue this process and complete the proof.

Proposition (4.1.3) [186]: Let $Y \in B(\mathcal{H})$ be a self-adjoint operator, $\mathbf{m} \in \mathbb{Z}_+^k$, $\mathbf{m} \ne 0$, and let $\boldsymbol{\Phi} = (\varphi_1,\ldots,\varphi_k)$ be a k-tuple of commuting, power bounded, positive linear maps on $B(\mathcal{H})$ such that

(i) $\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}}(Y) \ge 0$, and

(ii) each φ_1 is pure, i.e., $\varphi_i^p(I) \to 0$ strongly as $p \to \infty$.

Then $\Delta_{\boldsymbol{\Phi}}^{\mathbf{q}}(Y) \ge 0$ for any $q \in \mathbb{Z}_+^k$ with $\mathbf{q} \le \mathbf{m}$. In particular, $Y \ge 0$.

Proof: Set $\mathbf{m}' := (m_1 - 1, m_2,\ldots,m_k)$ and note that due to the fact that $\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}}(Y) \ge 0$ and φ_1 is a positive linear map, we have

$$0 \le \Delta_{\boldsymbol{\Phi}}^{\mathbf{m}}(Y) = \Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y) - \varphi_1(\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)).$$

Hence, we deduce that $\varphi_1^p(\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)) \le \Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)$ for any $p \in \mathbb{N}$. Since $\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)$ is a self-adjoint operator, we have

$$-\left\|\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)\right\|\varphi_1^p(I) \le \varphi_1^p(\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(y)) \le \left\|\Delta_{\boldsymbol{\Phi}}^{\mathbf{m}'}(Y)\right\|\varphi_1^p(I)$$

Now, taking into account that $\varphi_i^p(I) \to 0$ strongly as $p \to \infty$, we conclude that $\Delta_{\mathbf{\Phi}}^{\mathbf{m}'}(Y) \geq 0$. Using the commutativity of $\varphi_1, \ldots, \varphi_k$, one can continue this process and complete the proof.

For each $i \in \{1, \ldots, k\}$, let $f_i := \sum_{\alpha_i \in \mathbb{F}_{n_i}^+, |\alpha| \geq 1} a_{i,\alpha} Z_\alpha$ be a positive regular free holomorphic function in n_i variables and let $A := (A_1, \ldots, A_n) \in B(\mathcal{H})^{n_i}$ be an n_i-tuple of operators such that $\sum_{|\alpha| \geq 1} a_{i,\alpha} A_\alpha A_\alpha^*$ is convergent in the weak operator topology. One can easily prove that the map $\Phi_{f_i,A}: B(\mathcal{H}) \to B(\mathcal{H})$, defined by

$$\Phi_{f_i,A}(X) = \sum_{|\alpha| \geq 1} a A_\alpha X A_\alpha^*, \qquad X \in B(\mathcal{H}),$$

where the convergence is in the weak operator topology, is a completely positive linear map which is WOT-continuous on bounded sets. Moreover, if $0 < r < 1$, then

$$\Phi_{f_i,A}(X) = \text{WOT} - \lim_{r \to 1} \Phi_{f_i,rA}(X), \qquad X \in B(H).$$

These facts will be used in the proof of the next theorem.

Let $= (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$, where $T_i := (T_{i,1}, \ldots, T_{i,n_1}) \in B(\mathcal{H})^{n_i}$ for all $i = 1, \ldots, k$, be such that $\Phi_{f_i,T_i}(I)$ is well-defined in the weak operator topology. If $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ and $\mathbf{f} := (f_1, \ldots, f_k)$, we define the defect mapping $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}: B(\mathcal{H}) \to B(\mathcal{H})$ by setting

$$\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}} := \left(id - \Phi_{f_1,T_1}\right)^{p_1} \cdots \left(id - \Phi_{f_k,T_k}\right)^{p_k}.$$

Given $r \geq 0$, we set $r\mathbf{T} := (rT_1, \ldots, rT_k)$ and $rT_i := (rT_{i,1}, \ldots, rT_{i,n_i})$ for $i \in \{1, \ldots, k\}$. We say that the k-tuple $\mathbf{T}$ has the radial property with respect to $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ if there exists $\delta \in (0, 1)$ such that $r\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ for any $r \in (\delta, 1]$.

Theorem (4.1.4) [186]: Let $\mathbf{T} = (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ be such that $\Phi_{f_i,T_i}(I) \leq I$ for any $i \in \{1, \ldots, k\}$, and let $\mathbf{q} \in \mathbb{Z}_+^k$ be with $\mathbf{q} \neq 0$. Then the following statements are equivalent:

(i) $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$;

(ii) for any $p_i \in \{0, 1, \ldots, m_i\}$ and $i \in \{1, \ldots, k\}$,

$$\left(id - \Phi_{f_i,T_i}\right)^{p_1} \cdots \left(id - \Phi_{f_k,T_k}\right)^{p_k}(I) \geq 0;$$

(iii) $\Delta_{\mathbf{f},\mathbf{rT}}^{\mathbf{m}}(I) \geq 0$ for any $r \in [0, 1]$;

(iv) there exists $\delta \in (0, 1)$ such that $\Delta_{\mathbf{f},\mathbf{rT}}^{\mathbf{m}}(I) \geq 0$ for any $r \in (\delta, 1)$;

(v) $\mathbf{T}$ has the radial property with respect to $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

Proof: The equivalence of (i) with (ii) is due to Proposition (4.1.1), when applied to $\varphi_i = \Phi_{f_i,T_i}$.

We prove that (ii) implies (iii). First, note that if $D \in B(\mathcal{H}), D \geq 0$, then, for each $i \in \{1, \ldots, k\}$,

$$\left(id - \Phi_{f_i,T_i}\right)(D) \geq 0 \Longrightarrow \left(id - \Phi_{f_i,rT_i}\right)(D) \geq 0, \quad r \in [0, 1]. \tag{1}$$

Indeed, if $\Phi_{f_i,T_i}(D) \leq D$, then $\Phi_{f_i,rT_i}(D) \leq D$ for any $r \in [0, 1]$. Now, assume that (ii) holds. If $\mathbf{p} \in \mathbb{Z}_+^k$ with $\mathbf{p} \geq e_1 := (1, 0, \ldots, 0) \in \mathbb{Z}_+^k$, then $(id - \Phi_{f_1,T_1})\left(\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-e_1}(I)\right) \geq 0$ for any $\mathbf{p} \in \mathbb{Z}_+^k$ with $e_1 \leq \mathbf{p} \leq \mathbf{m}$.

Consequently, due to (1), we have

$$(id - \Phi_{f_1,rT_1})\left(\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-e_1}(I)\right) \geq 0 \tag{2}$$

for any $r \in [0, 1]$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $e_1 \leq \mathbf{p} \leq \mathbf{m}$. Due to the commutativity of $\Phi_{f_i,T_i}, \ldots, \Phi_{f_k,T_k}$, the latter inequality is equivalent to

$$(id - \Phi_{f_1,T_1})(\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-2e_1}(id - \Phi_{f_1,rT_1})(I)) \geq 0$$

for any $r \in [0, 1]$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $2e_1 \leq \mathbf{p} \leq \mathbf{m}$. Due to (2), we have $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-2e_1}(id - \Phi_{f_1,rT_1})(I)$ and, applying again relation (1), we deduce that

$$(id - \Phi_{f_1,T_1})(\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-3e_1}(id - \Phi_{f_1,rT_1})^2(I)) \geq 0$$

for any $r \in [0, 1]$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $3e_1 \leq \mathbf{p} \leq \mathbf{m}$. Continuing this process, we obtain the inequality

$$\left(id - \Phi_{f_2,T_2}\right)^{p_2} \cdots \left(id - \Phi_{f_k,T_k}\right)^{p_k} (id - \Phi_{f_1,rT_1})^{p_1}(I) \geq 0$$
for any $\mathbf{p} \in \mathbb{Z}_+^k$ with $e_1 \leq \mathbf{p} \leq \mathbf{m}$, and any $r \in [0,1]$. Similar arguments lead to the inequality $\Delta_{\mathbf{f},\mathbf{rT}}^{\mathbf{m}}(I) \geq 0$ for any $r \in [0,1]$. Since the implications (iii) $\Longrightarrow$ (iv) and (v) $\Longrightarrow$ (i) are clear, it remains to prove that (iv) $\Longrightarrow$ (v).

To this end, assume that there exists $\delta \in (0,1)$ such that $\Delta_{\mathbf{f},\mathbf{rT}}^{\mathbf{m}}(I) \geq 0$ for any $r \in (\delta, 1)$. Since $\Phi_{f_i,rT_i} \leq rI$, it is clear that Φ_{f_i,rT_i} is pure for each $i \in \{1,\ldots,k\}$. Applying Proposition (4.1.3), we deduce that $\Delta_{\mathbf{f},\mathbf{rT}}^{\mathbf{p}}(I) \geq 0$ for any $r \in (\delta,1)$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$. Note that $\Delta_{\mathbf{f},\mathbf{rT}}^{\mathbf{p}}(I)$ is a linear combination of products of the form $\Phi_{f_1,rT_1}^{q_1} \cdots \Phi_{f_k,rT_k}^{q_k}(I)$, where $(q_1,\ldots,q_k) \in \mathbb{Z}_+^k$. On the other hand
$$\Phi_{f_1,rT_1}^{q_1} \cdots \Phi_{f_k,T_k}^{q_k}(I) = \text{WOT} - \lim_{j\to\infty} \sum_{\substack{\alpha_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_1|+\cdots+|\alpha_k| \leq j}} c_{\alpha_1,\ldots,\alpha_k} T_{1,\alpha_1} \cdots T_{k,\alpha_k} T_{k,\alpha_k}^* \cdots T_{1,\alpha_1}^* \leq I$$
for some positive constants $c_{\alpha_1,\ldots,\alpha_k} \geq 0$. Given $x \in \mathcal{H}$ and $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that
$$\sum_{\substack{\alpha_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_1|+\cdots+|\alpha_k| \leq j}} c_{\alpha_1,\ldots,\alpha_k} r^{2(|\alpha_1|+\cdots+|\alpha_k|)} \langle T_{1,\alpha_1} \cdots T_{k,\alpha_k} T_{k,\alpha_k}^* \cdots T_{1,\alpha_1}^* x, x \rangle < \epsilon$$
for any $j \geq N_0$ and $r \in (\delta,1)$. This can be used to show that
$$\Phi_{f_1,T_1}^{q_1} \cdots \Phi_{f_k,T_k}^{q_k}(I) = \text{WOT} - \lim_{r\to 1} \Phi_{f_1,rT_1}^{q_1} \cdots \Phi_{f_k,rT_k}^{q_k}(I)$$
Hence, we deduce that $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(I) = \text{WOT} - \lim_{r\to 1} \Delta_{\mathbf{f},\mathbf{rT}}^{\mathbf{p}}(I) \geq 0$ for any $\mathbf{p} \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$. Consequently, $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and it has the radial property. This completes the proof.

As expected, the domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is called radial if any $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ has the radial property.

Corollary (4.1.5) [186]: The noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is radial.

In the particular case when $k = 1$, Theorem (4.1.4) shows that any noncommutative domain $\mathbf{D}_{f_1}^{m_1}(\mathcal{H})$, $m_1 \in \mathbb{N}$, is radial. An important consequence is the following

Corollary (4.1.6) [186]: All the results from [192], [198], [199], which were proved in the setting of the radial part of $\mathbf{D}_{f_1}^{m_1}(\mathcal{H})$, are true for any domain $\mathbf{D}_{f_1}^{m_1}(\mathcal{H})$.

Another consequence is the following

Corollary (4.1.7) [186]:

The following statements hold:

(i) If $\mathbf{f} = (f_1,f_2)$, and $\mathbf{T} = (T_1,T_2) \in \mathbf{D}_{f_1}^{m_1}(\mathcal{H}) \times \mathbf{D}_{f_2}^{m_2}(\mathcal{H})$ with $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(I) \geq 0$, then $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

(ii) If $\mathbf{T} = (T_1,\ldots,T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ and $\Phi_{f_i,T_i}(I) = I, i \in \{1,\ldots,k\}$, then $\mathbf{T}$ is in the polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

We say that a k-tuple $\mathbf{T} = (T_1,\ldots,T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is pure if
$$\lim_{\mathbf{q}=(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} (id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I) = I.$$
We remark that $\{(id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I)\}_{\mathbf{q}=(q_1,\ldots,q_k)\in\mathbb{Z}_+^k}$ is an increasing sequence of positive operators. Indeed, due to Theorem (4.1.4), $(id - \Phi_{f_k,T_k}) \cdots (id - \Phi_{f_1,T_1})(I) \geq 0$. Taking into account that

$\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}$ are commuting, we have
$$(id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I) = \sum_{s=0}^{q_k-1} \Phi_{f_k,T_k}^s \cdots \sum_{s=0}^{q_1-1} \Phi_{f_1,T_1}^s (id - \Phi_{f_k,T_k}) \cdots (id - \Phi_{f_1,T_1})(I),$$
which proves our assertion. Note also that
$$(id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I) \leq (id - \Phi_{f_{k-1},T_{k-1}}^{q_{k-1}}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I) \leq \cdots$$
$$\leq (id - \Phi_{f_1,T_1}^{q_1})(I) \leq I.$$
Hence, we can deduce the following result.

Proposition (4.1.8) [186]: A k-tuple $\mathbf{T} = (T_1, \dots, T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is pure if and only if, for each $i \in \{1, \dots, k\}$, $\Phi_{f_i,T_i}^{p}(I) \to 0$ strongly as $p \to \infty$.

A k-tuple $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is called doubly commuting if $T_{i,p}T_{j,p}^{*} = T_{j,q}^{*}T_{i,p}$ for any $i, j \in \{1, \dots, k\}$ with $i \neq j$, and $p \in \{1, \dots, n_i\}, q \in \{1, \dots, n_j\}$. The next results provides some classes of elements in $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

Proposition (4.1.9) [186]: Let $\mathbf{T} = (T_1, \dots, T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ be such that $\Phi_{f_i,T_i}(I) \le I$ and let $m = (m_1, \dots, m_k) \in \mathbb{N}^k$. Then the following statements hold.

(i) If $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) \ge 0$ and, for each $i \in \{1, \dots, k\}$, $\Phi_{f_i,T_i}^{p}(I) \to 0$ strongly as $p \to \infty$, then $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

(ii) If $\mathbf{T} \in \mathbf{D}_{\mathrm{f}_1}^{\mathrm{m}_1}(\mathcal{H}) \times_c \cdots \times_c \mathbf{D}_{\mathrm{f}_k}^{\mathrm{m}_k}(\mathcal{H})$ is doubly commuting, then $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

(iii) If $m_1\Phi_{f_1,T_1}(I) + \cdots + m_k\Phi_{f_k,T_k}(I) \le I$, then $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

(iv) If $\mathbf{T} = (T_1, \dots, T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, then $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) = 0$ if and only if

$$(id - \Phi_{f_1,T_1}) \cdots (id - \Phi_{f_k,T_k})(I) = 0.$$

Proof: Applying Proposition (4.1.1) and Proposition (4.1.3), when $\Phi = (\Phi_{f_1,T_1}, \dots, \Phi_{f_k,T_k})$, we deduce part (i).

To prove part (ii), note that since $T_i \in \mathbf{D}_{\mathrm{f}_i}^{\mathrm{m}_i}(\mathcal{H})$, we have $\left(id - \Phi_{f_i,T_i}\right)^{p_i}(I) \ge 0$ for any $p_i \in \{0, 1, \dots, m_i\}$.

Using the fact that $\mathbf{T}$ is doubly commuting, we deduce that

$$\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(I) = \left(id - \Phi_{f_1,T_1}\right)^{p_1}(I) \cdots \left(id - \Phi_{f_k,T_k}\right)^{k}(I) \ge 0$$

for any $\mathbf{p} \in \mathbb{Z}_+^k$ with $\mathbf{p} \le \mathbf{m}$, which shows that $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

Now, we prove part (iii). Let $p := m_1 + \cdots + m_k$ and set $i_j := 1$ if $1 \le j \le m_1, i_j := 2$ if $m_1 + 1 \le j \le m_1 + m_2, \dots$, and $i_j := k$ if $m_1 + \cdots + m_{k-1} + 1 \le j \le m_1 + \cdots + m_k$. Due to Theorem (4.1.4), to prove (iii) is equivalent to showing that if $\sum_{j=1}^{p} \Phi_{f_{i_j},T_{i_j}}(I) \le I$, then

$$(id - \Phi_{f_{i_1},T_{i_1}}) \cdots (id - \Phi_{f_{i_p},T_{i_p}})(I) \ge 0.$$

Set $Y_{i_0} = I$ and $Y_{i_j} := (id - \Phi_{f_{i_j},T_{i_j}})(Y_{i_{j-1}})$ if $j \in \{1, \dots, p\}$. We proceed inductively. Note that

$I = Y_{i_0} \ge Y_{i_1} = (id - \Phi_{f_{i_1},T_{i_1}})(I) \ge 0$. Let $n < p$ and assume that

$$I \ge Y_{i_n} \ge (id - \Phi_{f_{i_1},T_{i_1}} - \cdots - \Phi_{f_{i_n},T_{i_n}})(I) \ge 0.$$

Hence, we deduce that

$$\begin{aligned} I \ge Y_{i_n} \ge Y_{i_{n+1}} &= Y_{i_n} - \Phi_{f_{i_{n+1}},T_{i_{n+1}}}(Y_{i_n}) \\ &\ge (id - \Phi_{f_{i_1},T_{i_1}} - \cdots - \Phi_{f_{i_n},T_{i_n}})(I) - \Phi_{f_{i_{n+1}},T_{i_{n+1}}}(I), \end{aligned}$$

which proves our assertion.

Now, we prove part (iv). If $\mathbf{T} = (T_1, \dots, T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, Theorem (4.1.4) implies that

$$(id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_k,T_k})^{p_k}(I) \ge 0$$

for any $p_i \in \{0, 1, \dots, m_i\}$ and $i \in \{1, \dots, k\}$. Due to Lemma 6.2 from [192], if $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ is a power bounded positive linear map such that $D \in B(\mathcal{H})$ is a positive operator with $(id - \varphi)(D) \ge 0$, and $\gamma \ge 1$, then

$$(id - \varphi)^{\gamma}(D) = 0 \qquad \text{if and only if} \quad (id - \varphi)(D) = 0.$$

Applying this result in our setting when $\varphi = \Phi_{f_1,T_1}, \gamma = m_1$, and $D = \left(id - \Phi_{f_2,T_2}\right)^{m_2} \cdots \left(id - \Phi_{f_k,T_k}\right)^{m_k}(I) \ge 0$, we deduce that relation $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) = 0$ is equivalent to $(id - \Phi_{f_1,T_1})(D) = 0$. Due to the commutativity of $\Phi_{f_1,T_1} \cdots \Phi_{f_k,T_k}$, the latter equality is equivalent to $\left(id - \Phi_{f_2,T_2}\right)^{m_2}(\Lambda) = 0$, where $\Lambda := (id - \Phi_{f_3,T_3})^{m_3} \cdots (id - \Phi_{f_k,T_k})^{m_k} (id - \Phi_{f_1,T_1})(I) \ge 0$. Applying again the result mentioned above, we deduce that the latter equality is equivalent to $(id - \Phi_{f_2,T_2})(\Lambda) = 0$. Continuing this process, we can complete the proof of part (iv).

$\mathbf{D_f^m}(\mathcal{H})$ has a universal model $\mathbf{W} = \{\mathbf{W}_{i,j}\}$ consisting of weighted shifts acting on a tensor product of full Fock spaces.

Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e_1^i, \ldots, e_{n_i}^i$. We consider the full Fock space of H_{n_i} defined by

$$F^2(H_{n_i}) := \bigoplus_{p \geq 0} H_{n_i}^{\otimes p},$$

where $H_{n_i}^{\otimes p} := \mathbb{C}1$ and $H_{n_i}^{\otimes p}$ is the (Hilbert) tensor product of p copies of H_{n_i} . Set $e_\alpha^i := e_{j_1}^i \otimes \cdots \otimes e_{j_p}^i$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $e_{g_0^i}^i := 1 \in \mathbb{C}$. It is clear that $\{e_\alpha^i : \alpha \in \mathbb{F}_{n_i}^+\}$ is an orthonormal basis of $F^2(H_{n_i})$.

Let $m_i, n_i \in \mathbb{N} := \{1, 2, \ldots\}, i \in \{1, \ldots, k\}$, and $j \in \{1, \ldots, n_i\}$. We define the weighted left creation operators $W_{i,j} : F^2(H_{n_i}) \to F^2(H_{n_i})$, associated with the abstract noncommutative domain $\mathbf{D}_{f_i}^{m_i}$ by setting

$$W_{i,j} e_\alpha^i := \frac{\sqrt{b_{i,\alpha}^{(m_i)}}}{\sqrt{b_{i,g_j\alpha}^{(m_i)}}} e_{g_j\alpha}^i, \qquad \alpha \in \mathbb{F}_{n_i}^+, \tag{3}$$

where

$$b_{i,g_0}^{(m_i)} := 1 \quad and \quad b_{i,\alpha}^{(m_i)} := \sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_p \in \mathbb{F}_{n_i}^+ \\ \gamma_1,\ldots,\gamma_p = \alpha \\ |\gamma_1|,\ldots,|\gamma_p| \geq 1}} a_{i,\gamma_1} \cdots a_{i,\gamma_p} \binom{p + m_i - 1}{m_i - 1} \tag{4}$$

for all $\alpha \in \mathbb{F}_{n_i}^+$ with $|\alpha| \geq 1$.

Lemma (4.1.10) [186]: For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, we define the operator $\mathbf{W}_{i,j}$ acting on the tensor Hilbert space $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$\mathbf{W}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1\, times} \otimes W_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i\, times},$$

where the operators $W_{i,j}$ are defined by relation (3). If $\mathbf{W}_{i,j} := (\mathbf{W}_{i,1}, \ldots, \mathbf{W}_{i,n_i})$, then the following statements hold.

(i) $(id - \Phi_{f_1,\mathbf{W}_1})^{m_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = \mathbf{P}_\mathbb{C}$, where $\mathbf{P}_\mathbb{C}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_1})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^k F^2(H_{n_1})$, where $\mathbb{C}1$ is identified with $\mathbb{C}1 \otimes \cdots \otimes \mathbb{C}1$.

(ii) $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D_f^m}(\otimes_{i=1}^k F^2(H_{n_1}))$.

Proof: Note that, due to relation (3), for each $i \in \{1, \ldots, k\}$ and $\beta_i \in \mathbb{F}_{n_i}^+$, we have

$$W_{i,\beta_i} W_{i,\beta_i}^* e_{\alpha_i}^i = \begin{cases} \dfrac{b_{i,\gamma_i}^{(m_i)}}{b_{i,\alpha_i}^{(m_i)}} e_{\alpha_i}^i & \text{if } \alpha_i = B_i, \gamma_i, \quad \gamma_i \in \mathbb{F}_{n_i}^+ \\ 0 & \text{otherwise.} \end{cases}$$

As in Lemma 1.2 from [192], straightforward computations reveal that $(id - \Phi_{f_i,\mathbf{W}_i})^{m_i}(I) = I \otimes \cdots \otimes I \otimes \boldsymbol{P}_\mathbb{C} \otimes I \otimes \cdots \otimes I$, where $\boldsymbol{P}_\mathbb{C}$ is on the i^{th} position and denotes the orthogonal projection from $F^2(H_{n_1})$ onto $\mathbb{C}1 \subset F^2(H_{n_1})$. Since the k-tuple $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ is doubly commuting, we deduce that

$$(id - \Phi_{f_1,\mathbf{W}_1})^{m_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = (id - \Phi_{f_1,\mathbf{W}_1})^{m_1}(I) \cdots (id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = \mathbf{P}_\mathbb{C}.$$

which proves part (i). To prove part (ii), note first that relation (3) implies $\Phi_{f_i,W_i}^p(I) e_\alpha^i = 0$ if $p > |\alpha|, \alpha \in \mathbb{F}_{n_i}^+$. Since $\|\Phi_{f_i,\mathbf{W}_i}^p(I)\| \leq 1$ for any $p \in \mathbb{N}$, we deduce that $\lim_{p\to\infty} \Phi_{f_i,\mathbf{W}_i}^p(I) = 0$ in the strong operator topology. Taking into account that $\Delta^{\mathbf{m}}_{\mathbf{f,W}}(I) = \mathbf{P}_{\mathbb{C}}$, we can use Proposition (4.1.3) to conclude that $\mathbf{W}$ is in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\otimes_{i=1}^{k} F^2(H_{n_1}))$. Moreover, due to Proposition (4.1.8), $\mathbf{W}$ is a pure k-tuple in $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\otimes_{i=1}^{k} F^2(H_{n_1}))$.

We mention that one can define the weighted right creation operators $\Lambda_{i,j} : F^2(H_{n_1}) \to F^2(H_{n_1})$ by setting

$$\Lambda_{i,j} e^i_\alpha := \frac{\sqrt{b^{(m_i)}_{i,\alpha}}}{\sqrt{b^{(m_i)}_{i,g_j\alpha}}} e^i_{\alpha g_j}, \qquad \alpha \in \mathbb{F}^+_{n_i}$$

As in Lemma (4.1.10), it turns out that $\mathbf{\Lambda} := (\mathbf{\Lambda}_1, \ldots, \mathbf{\Lambda}_k)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\tilde{\mathrm{f}}}(\otimes_{i=1}^{k} F^2(H_{n_1}))$, where $\tilde{\mathrm{f}} = (\tilde{f}_1, \ldots, \tilde{f}_k)$ with $\tilde{f}_i := \sum_{|\alpha|\geq 1} a_{i,\tilde{\alpha}} Z_\alpha$ and $\tilde{\alpha} = g^i_{j_p} \cdots g^i_{j_1}$ denotes the reverse of $g^i_{j_1} \cdots g^i_{j_p} \in \mathbb{F}^+_{n_i}$.

Throughout, the k-tuple $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ of Lemma (4.1.10) will be called the universal model associated with the abstract noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}$. We introduce the noncommutative Berezin kernel associated with any element $\mathbf{T} = \{T_{i,j}\}$ in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ as the operator

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\Delta^{\mathbf{m}}_{\mathbf{f,T}}(I)(\mathcal{H})}$$

defined by

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} h := \sum_{\beta_i \in \mathbb{F}^+_{n_i}, i=1,\ldots,k} \sqrt{b^{(m_1)}_{1,\beta_1}} \cdots \sqrt{b^{(m_k)}_{k,\beta_k}}\, e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \Delta^{\mathbf{m}}_{\mathbf{f,T}}(I)^{1/2} T^*_{1,\beta_1} \cdots T^*_{k,\beta_k} h,$$

where the defect operator is defined by

$$\Delta^{\mathbf{m}}_{\mathbf{f,T}}(I) := (id - \Phi_{f_1,T_1})^{m_1} \cdots (id - \Phi_{f_k,T_k})^{m_k}(I),$$

and the coefficients$b^{(m_1)}_{1,\beta_1}, \ldots, b^{(m_k)}_{k,\beta_k}$ are given by relation (4). The fact that $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a well-defined bounded operator will be proved in the next theorem.

Theorem (4.1.11) [186]: The noncommutative Berezin kernel associated with a k-tuple $\mathbf{T} = (T_1, \ldots, T_k)$ in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ has the following properties.

(i) $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a contraction and

$$\mathbf{K}^*_{\mathbf{f},\mathbf{T}} \mathbf{K}_{\mathbf{f},\mathbf{T}} = \lim_{q_k \to \infty} \ldots \lim_{q_1 \to \infty} (id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I),$$

where the limits are in the weak operator topology.

(ii) If T is pure, then

$$\mathbf{K}^*_{\mathbf{f},\mathbf{T}} \mathbf{K}_{\mathbf{f},\mathbf{T}} = I_{\mathcal{H}}.$$

(iii) For any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$,

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} T^*_{i,j} = (\mathbf{W}^*_{i,j} \otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}}.$$

Proof: Let $\mathbf{T} = (T_1, \ldots, T_k)$ be in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ and let $X \in B(\mathcal{H})$ be a positive operator such that

$$\Delta^{\mathbf{p}}_{\mathbf{f,T}}(X) := (id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_k,T_k})^{p_k}(X) \geq 0$$

for any $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}^k_+$ with $p_i \in \{0, 1, \ldots, m_i\}$ and $i \in \{1, \ldots, k\}$. Fix $i \in \{1, \ldots, k\}$ and assume that $1 \leq p_i \leq m_i$. Then, due to the commutativity of $\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}$, we have

$$(id - \Phi_{f_i,T_i})\Delta^{\mathrm{p}-\mathrm{e}_\mathrm{i}}_{\mathbf{f,T}}(X) = \Delta^{\mathbf{p}}_{\mathbf{f,T}}(X) \geq 0,$$

where $\{e_i\}_{i=1}^k$ is the canonical basis in $\mathbb{C}^k$. Hence, and using Lemma (4.1.2), we have

$$0 \leq \Phi_{f_i,T_i}(\Delta^{\mathrm{p}-\mathrm{e}_\mathrm{i}}_{\mathbf{f,T}}(X)) \leq \Delta^{\mathrm{p}-\mathrm{e}_\mathrm{i}}_{\mathbf{f,T}}(X) \leq X,$$

which proves that $\{\Phi^s_{f_i,T_i}(\Delta^{\mathrm{p}-\mathrm{e}_\mathrm{i}}_{\mathbf{f,T}}(X))\}_{s=0}^\infty$ is a decreasing sequence of positive operators which is convergent in the weak operator topology. Since Φ_{f_i,T_i} is WOT-continuous on bounded sets and $\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}, T_k$ are commuting, we deduce that

$$\lim_{s\to\infty} \Phi^s_{f_i,T_i}(\Delta^{\mathrm{p}-\mathrm{e}_\mathrm{i}}_{\mathbf{f,T}}(X)) = \Delta^{\mathrm{p}-\mathrm{e}_\mathrm{i}}_{\mathbf{f,T}}(\lim_{s\to\infty} \Phi^s_{f_i,T_i}(X)). \qquad (5)$$

Then we have

$$D_i^{(1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) := \sum_{s=0}^{\infty} \Phi_{f_i,T_i}^{s}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \sum_{s=0}^{\infty} \Phi_{f_i,T_i}^{s}\left[\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p-e_i}}(X) - \Phi_{f_i,T_i}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p-e_i}}(X))\right]$$
$$= \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p-e_i}}(X) - \lim_{q_1\to\infty} \Phi_{f_i,T_i}^{q_1}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p-e_i}}(X)) \leq \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p-e_i}}(X) \leq X.$$

Due to relation (5), we deduce that

$$0 \leq D_i^{(1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p-e_i}}\left(X - \lim_{q_1\to\infty} \Phi_{f_i,T_i}^{q_1}(X)\right), \mathbf{p} \leq \mathbf{m}, 1 \leq p_i.$$

Define $D_i^{(j)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) := \sum_{s=0}^{\infty} \Phi_{f_i,T_i}^{s}(D_i^{(j-1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)))$, where $j = 2, \dots p_i$. Inductively, we can prove that

$$0 \leq D_i^{(j)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) := \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-j\mathrm{e_i}}\left(X - \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_j}(X)\right) \leq \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-j\mathrm{e_i}}(X) \leq X, \ \ j \leq p_i. \quad (6)$$

Indeed, if $j \leq p_i - 1$ and setting $Y := X - \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_j}(X)$, relation (6) implies

$$D_i^{(j+1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \lim_{q_{j+1}\to\infty} \sum_{s=0}^{q_{j+1}} \Phi_{f_i,T_i}^{s}\left[\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-j\mathrm{e_i}}(Y)\right]$$
$$= \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-(j+1)\mathrm{e_i}}\left[Y - \lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(Y)\right]$$
$$= \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-(j+1)\mathrm{e_i}}(Y) - \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-(j+1)\mathrm{e_i}}\left(\lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(Y)\right)$$

On the other hand, we have

$$\lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(Y) = \lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}\left(X - \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_j}(X)\right)$$
$$= \lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(X) - \lim_{q_{j+1}\to\infty} \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}\left(\Phi_{f_i,T_i}^{q_j}(X)\right) = 0$$

Combining these results, we obtain

$$D_i^{(j+1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-(j+1)\mathrm{e_i}}\left(X - \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_j}(X)\right) \leq \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-(j+1)\mathrm{e_i}}(X) \leq \mathrm{X}$$

for any $\mathbf{p} := (p_1, \dots, p_k) \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$ and $p_i \geq 1$, which proves our assertion. When $j = p_i$, relation (6) becomes

$$0 \leq D_i^{(p_i)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-p_i\mathrm{e_i}}\left(X - \lim_{q\to\infty} \Phi_{f_i,T_i}^{q}(X)\right) \leq X$$

On the other hand, taking into account that we can rearrange WOT-convergent series of positive operators, we deduce that, for each $d \in \mathbb{N}$,

$$\Phi_{f_i,T_i}^{d}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \sum_{\alpha_1\in\mathbb{F}_{n_1}^+, |\alpha_1|\geq 1} a_{i,\alpha_1} T_{i,\alpha_1}\left(\cdots \sum_{\alpha_d\in\mathbb{F}_{n_1}^+, |\alpha_d|\geq 1} a_{i,\alpha_d} T_{i,\alpha_d}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) T_{i,\alpha_d}^* \cdots\right) T_{i,\alpha_1}^*$$
$$= \sum_{\gamma\in\mathbb{F}_{n_1}^+, |\gamma|\geq d} \sum_{\substack{\alpha_1,\dots,\alpha_d\in\mathbb{F}_{n_1}^+ \\ \alpha_1,\dots,\alpha_d=\gamma \\ |\alpha_1|\geq 1,\dots,|\alpha_d|\geq 1}} a_{i,\alpha_1} \cdots a_{i,\alpha_d} T_{i,\gamma} \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X) T_{i,\gamma}^*$$

and

$$D_i^{(1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \sum_{s=0}^{\infty} \Phi_{f_i,T_i}^{s}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X))$$

$$= \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X) + \sum_{\gamma\in\mathbb{F}_{n_1}^+, |\gamma|\geq 1} \left(\sum_{d=1}^{|\gamma|} \sum_{\substack{\alpha_1,\ldots,\alpha_d\in\mathbb{F}_{n_i}^+ \\ \alpha_1,\ldots,\alpha_d=\gamma \\ |\alpha_1|\geq 1,\ldots,|\alpha_d|\geq 1}} a_{i,\alpha_1}\cdots a_{i,\alpha_d} \right) T_{i,\gamma}\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)T_{i,\gamma}^*$$

Since $D_i^{(j)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) := \sum_{s=0}^{\infty}\Phi_{f_i,T_i}^{s}(D_i^{(j-1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)))$ for $j = 2,\ldots,p_i$, using a combinatorial argument and rearranging WOT-convergent series of positive operators, one can prove by induction over p_i that

$$D_i^{(p_i)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) = \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X) + \sum_{\alpha\in\mathbb{F}_{n_1}^+, |\alpha|\geq 1} \left(\sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_p\in\mathbb{F}_{n_i}^+ \\ \gamma_1,\ldots,\gamma_p=\alpha \\ |\gamma_1|\geq 1,\ldots,|\gamma_p|\geq 1}} a_{i,\gamma_1}\cdots a_{i,\gamma_p} \binom{p+p_i-1}{p_i-1} \right) T_{i,\alpha}\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)T_{i,\alpha}^*$$

$$= \sum_{\alpha\in\mathbb{F}_{n_i}^+} b_{i,\alpha}^{(p_i)} T_{i,\alpha}\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)T_{i,\alpha}^*$$

For each $i \in \{1,\ldots,k\}$, let $\Omega_i \subset B(\mathcal{H})$ be the set of all $Y \in B(\mathcal{H}), Y \geq 0$, such that the series $\sum_{\beta_i\in\mathbb{F}_{n_i}^+} b_{i,\beta_i}^{(m_i)} T_{i,\beta_i} Y T_{i,\beta_i}^*$ is convergent in the weak operator topology, where

$$b_{i,g_0}^{(m_i)} := 1 \quad \text{and} \quad b_{i,\alpha}^{(m_i)} := \sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_p\in\mathbb{F}_{n_i}^+ \\ \gamma_1,\ldots,\gamma_p=\alpha \\ |\gamma_1|\geq 1,\ldots,|\gamma_p|\geq 1}} a_{i,\gamma_1}\cdots a_{i,\gamma_p} \binom{p+m_i-1}{m_i-1}$$

for all $\alpha \in \mathbb{F}_{n_1}^+$ with $|\alpha| \geq 1$. We define the map $\Psi_i : \Omega_i \to B(\mathcal{H})$ by setting

$$\Psi_i(Y) := \sum_{\beta_i\in\mathbb{F}_{n_1}^+} b_{i,\beta_i}^{(m_i)} T_{i,\beta_i} Y T_{i,\beta_i}^* .$$

Due to the results above, we have

$$\begin{aligned} 0 \leq \Psi_i(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) &= D_i^{(m_i)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(X)) \\ &= \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-m_i\mathrm{e}_i}\left(id - \lim_{q_i\to\infty}\Phi_{f_i,T_i}^{q_i}\right)(X) \qquad (7) \\ &\leq \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}-m_i\mathrm{e}_i}(X) \leq X, \end{aligned}$$

for any $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$ and $p_i \geq m_i$. Since $\mathbf{T} = (T_1,\ldots,T_k)$ is in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, Theorem (4.1.4) implies

$$\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}}(I) := \left(id - \Phi_{f_1,T_1}\right)^{p_1}\cdots\left(id - \Phi_{f_k,T_k}\right)^{p_k}(I) \geq 0$$

for any $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_+^k$ with $p_i \in \{0,1,\ldots,m_i\}$ and $i \in \{1,\ldots,k\}$. Applying relation (7) in the particular case when $i = 1, p_1 = m_1$, and $X = I$, we have

$$0 \leq \Psi_1(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}'}(I)) = D_1^{(m_1)}(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}'}(I)) = \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}'-m_1\mathrm{e}_1}(I - \lim_{q_1\to\infty}\Phi_{f_1,T_1}^{q_1}(I)) \leq \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}'-m_1\mathrm{e}_1}(I) \leq I$$

for any $\mathbf{p}' = (m_1,p_2,\ldots,p_k)$ with $\mathbf{p}' \leq \mathbf{m}$. Hence and using again relation (7), when $i = 2, p = (0,m_2,p_3\ldots,p_k)$, and $\lim_{q_1\to\infty}(id - \Phi_{f_1,T_1}^{q_1})(I) \geq 0$, we obtain

$$\begin{aligned} 0 \leq \Psi_2(\Psi_1(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}''}(I))) &= \Psi_2\left(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}''-m_1\mathrm{e}_1}\left(I - \lim_{q_1\to\infty}\Phi_{f_1,T_1}^{q_1}(I)\right)\right) \\ &= \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}''-m_1\mathrm{e}_1-m_2\mathrm{e}_2}\lim_{q_2\to\infty}\lim_{q_1\to\infty}(id - \Phi_{f_2,T_2}^{q_2})(id - \Phi_{f_1,T_1}^{q_1})(I) \\ &\leq \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{p}''-m_1\mathrm{e}_1-m_2\mathrm{e}_2}(I) \leq I \end{aligned}$$

for any $\mathbf{p}'' = (m_1, m_2, p_3, \ldots, p_k)$. Continuing this process, a repeated application of (7), leads to the relation

$$0 \leq (\Psi_k \circ \cdots \circ \Psi_1)(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{m}}(I)) = \lim_{q_k\to\infty} \lim_{q_1\to\infty} (id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I) \leq I,$$

where $\mathbf{m} = (m_1, \ldots, m_k)$. To prove item (i), note that the results above imply

$$\|\mathbf{K}_{\mathbf{f},\mathbf{T}} h\|^2 = \sum_{\beta_k \in \mathbb{F}_{n_k}} \cdots \sum_{\beta_1 \in \mathbb{F}_{n_1}} b_{1,\beta_1}^{(m_1)} \cdots b_{k,\beta_k}^{(m_k)} \langle T_{k,\beta_k} \cdots T_{1,\beta_1} \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{m}}(I) T_{1,\beta_1}^* \cdots T_{k,\beta_k}^* h, h\rangle$$
$$= \langle (\Psi_k \circ \cdots \circ \Psi_1)(\Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{m}}(I)) h, h\rangle \leq \|h\|^2$$

for any $h \in \mathcal{H}$, and

$$\mathbf{K}_{\mathbf{f},\mathbf{T}}^* \mathbf{K}_{\mathbf{f},\mathbf{T}} = \lim_{q_k\to\infty} \cdots \lim_{q_1\to\infty} \left(id - \Phi_{f_k,T_k}^{q_k}\right) \cdots \left(id - \Phi_{f_1,T_1}^{q_1}\right)(I).$$

Now, item (ii) is clear. To prove part (iii), note that

$$W_{i,j}^* e_{\beta_i}^i = \begin{cases} \dfrac{\sqrt{b_{i,\gamma_i}^{(m_i)}}}{\sqrt{b_{i,\alpha_i}^{(m_i)}}} e_{\gamma_i}^i & \text{if } \beta_i = g_j^i \gamma_i, \quad \gamma_i \in \mathbb{F}_{n_i}^+ \\ 0 & \text{otherwise.} \end{cases} \qquad (8)$$

for any $\beta_i \in \mathbb{F}_{n_i}^+$ and $j \in \{1, \ldots, n_i\}$. Hence, and using the definition of the noncommutative Berezin kernel, we have

$$\left(\mathbf{W}_{i,j}^* \otimes I\right) \mathbf{K}_{\mathbf{f},\mathbf{T}} h$$
$$= \sum_{\beta_p \in \mathbb{F}_{n_p}, p \in \{1,\ldots,k\}} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}}\, e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_{i-1}}^{i-1} \otimes W_{i,j}^* e_{\beta_i}^i \otimes e_{\beta_{i+1}}^{i+1} \otimes \cdots \otimes e_{\beta_k}^k$$
$$\otimes \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{m}}(I)^{\frac{1}{2}} T_{1,\beta_1}^* \cdots T_{k,\beta_k}^* h$$
$$= \sum_{\beta_p \in \mathbb{F}_{n_p}^+, p \in \{1,\ldots,k\}\{i\}} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{i,\gamma_i}^{(m_i)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}}\, e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_{i-1}}^{i-1} \otimes e_{\gamma_i}^i \otimes e_{\beta_{i+1}}^{i+1} \otimes \cdots \otimes e_{\beta_k}^k$$
$$\otimes \Delta_{\mathbf{f},\mathbf{T}}^{\mathrm{m}}(I)^{1/2} T_{1,\beta_1}^* \cdots T_{i-1,\beta_i}^* T_{1,g_j^i \gamma_i}^* T_{i+1,\beta_{i+1}}^* \cdots T_{k,\beta_k}^* h$$

for any $h \in \mathcal{H}$. Using the commutativity of the tuples $T_1, \ldots, T_k$, we deduce that

$$\left(\mathbf{W}_{i,j}^* \otimes I\right) \mathbf{K}_{\mathbf{f},\mathbf{T}} = \mathbf{K}_{\mathbf{f},\mathbf{T}} T_{i,j}^*$$

for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. The proof is complete.

Theorem (4.1.12) [186]: Let $\mathbf{T} = \{T_{i,j}\}$ be in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and let

$$S := \overline{\operatorname{span}}\{p(\mathbf{W}_{i,j}) q(\mathbf{W}_{i,j})^* : p(\mathbf{W}_{i,j}), q(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})\},$$

where the closure is in the operator norm. Then there is a unital completely contractive linear map $\mathbf{\Psi}_{\mathbf{f},\mathbf{T}} : S \to B(\mathcal{H})$ such that

$$\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g) = \lim_{r\to 1} \mathbf{B}_{r\mathbf{T}}[g], \qquad g \in S,$$

where the limit exists in the norm topology of $B(\mathcal{H})$, and

$$\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}\left(\sum_{\gamma=1}^{s} p_\gamma(\mathbf{W}_{i,j}) q_\gamma(\mathbf{W}_{i,j})^*\right) = \sum_{\gamma=1}^{s} p_\gamma(T_{i,j}) q_\gamma(T_{i,j})^*$$

for any $p_\gamma(\mathbf{W}_{i,j}) q_\gamma(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})$ and $s \in \mathbb{N}$. In particular, the restriction $\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}$ to the polydomain algebra $\mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ is a completely contractive homomorphism. If, in addition, $\mathbf{T}$ is a pure k-tuple, then

$$\lim_{r\to 1} \mathbf{B}_{r\mathbf{T}}[g] = \mathbf{B}_{\mathbf{T}}[g], \qquad g \in S,$$

***Proof*:** According to Theorem (4.1.4), $r\mathbf{T} = (rT_1, \ldots, rT_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ for any $r \in (0,1)$. Since we have $\Phi_{f_i, rT_i}^n(I) \leq r^n \Phi_{f_i,T_i}^n(I) \leq r^n I$ for any $n \in \mathbb{N}$, Proposition (4.1.8) shows that $r\mathbf{T}$ is a pure k-tuple in $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

Using Theorem (4.1.11), we deduce that the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},r\mathbf{T}}$ is an isometry and

$\mathbf{K}^*_{\mathbf{f},r\mathbf{T}}\left[p(\mathbf{W}_{i,j})q(\mathbf{W}_{i,j})^* \otimes I_{\mathcal{H}}\right]\mathbf{K}_{\mathbf{f},r\mathbf{T}} = p(rT_{i,j})q(rT_{i,j})^*, \quad p(\mathbf{W}_{i,j})q(\mathbf{W}_{i,j})^* \in \mathcal{P}(\mathbf{W}).$ (9)

Hence, we obtain the von Neumann [57] type inequality

$$\left\|\sum_{\gamma=1}^{s} p_\gamma(rT_{i,j})q_\gamma(rT_{i,j})^*\right\| \le \left\|\sum_{\gamma=1}^{s} p_\gamma(\mathbf{W}_{i,j})q_\gamma(\mathbf{W}_{i,j})^*\right\| \qquad (10)$$

for any $p_\gamma(\mathbf{W}_{i,j}), q_\gamma(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})$, $s \in \mathbb{N}$, and $r \in [0,1]$. Fix $g \in S$ and let $\{\chi_n(\mathbf{W}_{i,j}, \mathbf{W}^*_{i,j})\}_{n=0}^\infty$ be a sequence of operators in the span of $\mathcal{P}(\mathbf{W})\mathcal{P}(\mathbf{W})^*$ which converges to g in norm, as $n \to \infty$. Define $\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g) := \lim_{n\to\infty} \chi_n(T_{i,j}, T^*_{i,j})$. The inequality (10) shows that $\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g)$ is well-defined and $\|\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g)\| \le \|g\|$. Using the matrix version of (9), we deduce that $\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}$ is a unital completely contractive
linear map. Now we prove that $\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g) = \lim_{r\to 1} \mathbf{B}_{r\mathbf{T}}[g]$. Note that relation (9) implies

$$\chi_n(rT_i, rT_i^*) = \mathbf{K}^*_{\mathbf{f},r\mathbf{T}}(\chi_n(\mathbf{W}_{i,j}, \mathbf{W}^*_{i,j}) \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{f},r\mathbf{T}} = \mathbf{B}_{r\mathbf{T}}[\chi_n(\mathbf{W}_{i,j}, \mathbf{W}^*_{i,j})]$$

for any $n \in \mathbb{N}$ and $r \in (0,1)$. Using the fact that $\mathbf{\Psi}_{\mathbf{f},r\mathbf{T}}(g) : \lim_{n\to\infty} \chi_n(rT_i, rT_i^*)$ exists in norm, we deduce that

$$\mathbf{\Psi}_{\mathbf{f},r\mathbf{T}}(g) = \mathbf{K}^*_{\mathbf{f},r\mathbf{T}}(g \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{f},r\mathbf{T}} = \mathbf{B}_{r\mathbf{T}}[g]. \qquad (11)$$

Given $\epsilon > 0$ let $s \in \mathbb{N}$ be such that $\|\chi_s(W_i, W_i^*) - g\| < \frac{\epsilon}{3}$. Due to the first part of the theorem, we have

$$\|\mathbf{\Psi}_{\mathbf{f},r\mathbf{T}}(g) - \chi_s(rT_i, rT_i^*)\| \le \|g - \chi_s(\mathbf{W}_i, \mathbf{W}_i^*)\| < \frac{\epsilon}{3}$$

for any $r \in [0,1]$. On the other hand, since $\chi_s(\mathbf{W}_i, \mathbf{W}_i^*)$ has a finite number of terms, there exists $\delta \in (0,1)$ such that

$$\|\chi_s(rT_i, rT_i^*) - \chi_s(T_i, T_i^*)\| < \frac{\epsilon}{3}$$

for any $r \in (\delta, 1)$. Now, using these inequalities and relation (11), we deduce that

$$\begin{aligned} \|\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g) - \mathbf{B}_{r\mathbf{T}}[g]\| &= \|\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g) - \mathbf{\Psi}_{\mathbf{f},r\mathbf{T}}(g)\| \\ &\le \|\mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g) - \chi_s(T_i, T_i^*)\| + \|\chi_s(T_i, T_i^*) - \chi_s(rT_i, rT_i^*)\| \\ &\quad + \|\chi_s(rT_i, rT_i^*) - \mathbf{\Psi}_{\mathbf{f},r\mathbf{T}}(g)\| < \epsilon \end{aligned}$$

for any $r \in (\delta, 1)$, which proves our assertion. Now, we assume that $\mathbf{T} = (T_1, \ldots, T_k)$ is a pure k-tuple in $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$. Due to Theorem (4.1.11), we have

$$\mathbf{B}_{\mathbf{T}}[\chi_n(\mathbf{W}_{i,j}, \mathbf{W}^*_{i,j})] := \mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\chi_n(\mathbf{W}_{i,j}, \mathbf{W}^*_{i,j}) \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{f},\mathbf{T}} = \chi_n(T_{i,j}, T^*_{i,j})$$

Taking into account that $\{\chi_n(\mathbf{W}_{i,j}, \mathbf{W}^*_{i,j})\}_{n=0}^\infty$ is a sequence of operators in the span of $\mathcal{P}(\mathbf{W})\mathcal{P}(\mathbf{W})^*$ which converges to g in norm, we conclude that

$$\mathbf{B}_{\mathbf{T}}[g] = \mathbf{\Psi}_{\mathbf{f},\mathbf{T}}(g) = \lim_{r\to 1} \mathbf{B}_{r\mathbf{T}}[g], g \in S.$$

This completes the proof.

We remark that Theorem (4.1.12) shows that the noncommutative polydomain algebra $\mathcal{A}(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ is the universal algebra generated by the identity and a doubly commuting k-tuple in the abstract polydomain domain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}$.

We denote by $C^*(\mathbf{W}_{i,j})$ the C^*-algebra generated by the operators $\mathbf{W}_{i,j}$, where $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$, and the identity.

Corollary (4.1.13) [186]: Let $q = (q_1, \ldots, q_k)$ be a k-tuple of positive regular noncommutative polynomials and let

$$\mathbf{X} := (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}.$$

Then $\mathbf{X}$ is in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ if and only if there exists a unital completely positive linear map $\Psi : C^*(\mathbf{W}_{i,j}) \to B(\mathcal{H})$ such that

$$\Psi_{\mathbf{q},\mathbf{T}}(p(\mathbf{W}_{i,j})r(\mathbf{W}_{i,j})^*) = p(X_{i,j})r(X_{i,j})^*, \; p(\mathbf{W}_{i,j})r(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W}),$$

where $\mathbf{W} := \{\mathbf{W}_{i,j}\}$ is the universal model associated with the abstract noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}$.

Proof: The direct implication is due to Theorem (4.1.12) and Arveson's extension theorem [193]. For the converse, note that, due to Lemma (4.1.10), Proposition (4.1.8), and Proposition (4.1.3), we have

$$\left(I - \Phi_{q_1,X_1}\right)^{p_1} \cdots \left(I - \Phi_{q_k,X_k}\right)^{p_k}(I) = \Psi_{\mathbf{q},\mathbf{T}}[\left(I - \Phi_{q_1,\mathbf{W}_1}\right)^{p_1} \cdots \left(I - \Phi_{q_k,\mathbf{W}_k}\right)^{p_k}(I)] \geq 0$$

for any $p_i \in \{0,1,\dots,m_i\}$ and $i \in \{1,\dots,k\}$. Using now Theorem (4.1.4) we can complete the proof.

We remark that under the condition

$$\overline{\operatorname{span}}\{p(\mathbf{W}_{i,j})r(\mathbf{W}_{i,j})^* : p(\mathbf{W}_{i,j}), r(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})\} = C^*(\mathbf{W}_{i,j}),$$

Corollary (4.1.13) shows that $C^*(\mathbf{W}_{i,j})$ is the universal C^*-algebra generated by the identity and a doubly commuting k-tuple in the abstract polydomain domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. We remark that the condition above holds, for example, if $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is the noncommutative polyball $[B(\mathcal{H})^{n_1}]_1^- \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_1^-$.

Hardy algebra $F^\infty(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ and provide a WOT-continuous functional calculus for completely non-coisometric tuples in in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

Let $\varphi(\mathbf{W}_{i,j}) = \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\dots,\beta_k} \mathbf{W}_{1,\beta_1} \cdots \mathbf{W}_{k,\beta_k}$ be a formal sum with $c_{\beta_1,\dots,\beta_k} \in \mathbb{C}$ and such that

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\dots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)} \cdots b_{k,\beta_k}^{(m_k)}} < \infty.$$

We prove that $\varphi(\mathbf{W}_{i,j})(e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k)$ is in $\otimes_{i=1}^k F^2(H_{n_i})$, for any $\gamma_1 \in \mathbb{F}_{n_1}^+, \dots, \gamma_k \in \mathbb{F}_{n_k}^+$. Indeed, due to relation (3), we have

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\dots,\beta_k} \mathbf{W}_{1,\beta_1} \cdots \mathbf{W}_{k,\beta_k} \left(e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k\right)$$

$$= \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\dots,\beta_k} \sqrt{\frac{b_{1,\gamma_1}^{(m_1)}}{b_{1,\beta_1\gamma_1}^{(m_1)}}} \cdots \sqrt{\frac{b_{k,\gamma_k}^{(m_k)}}{b_{k,\beta_k\gamma_k}^{(m_k)}}} e_{\beta_1\gamma_1}^1 \otimes \cdots \otimes e_{\beta_k\gamma_k}^k$$

Let $i \in \{1,\dots,k\}$ and $\alpha,\beta \in \mathbb{F}_{n_i}$ be such that $|\alpha| \geq 1$ and $|\beta| \geq 1$. Note that, for any $j \in \{1,\dots,|\alpha|\}$ and $k \in \{1,\dots,|\beta|\}$,

$$\binom{j+m_i-1}{m_i-1}\binom{k+m_i-1}{m_i-1} \leq C_{i,|\beta|}^{(m_i)} \binom{j+k+m_i-1}{m_i-1},$$

where $C_{i,|\beta|}^{(m_i)} := \binom{|\beta|+m_i-1}{m_i-1}$. Using relation (4) and comparing the product $b_{i,\alpha}^{(m_i)} b_{i,\beta}^{(m_i)}$ with $b_{i,\alpha\beta}^{(m_i)}$ we deduce that

$$b_{i,\alpha}^{(m_i)} b_{i,\beta}^{(m_i)} \leq C_{i,|\beta|}^{(m_i)} b_{i,\alpha\beta}^{(m_i)} \qquad (12)$$

and

$$b_{i,\alpha}^{(m_i)} b_{i,\beta}^{(m_i)} \leq C_{i,|\alpha|}^{(m_i)} b_{i,\alpha\beta}^{(m_i)}$$

for any $\alpha,\beta \in \mathbb{F}_{n_i}^+$. Hence, we deduce that

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\dots,\beta_k}|^2 \frac{b_{1,\gamma_1}^{(m_1)}}{b_{1,\beta_1\gamma_1}^{(m_1)}} \cdots \frac{b_{k,\gamma_k}^{(m_k)}}{b_{k,\beta_k\gamma_k}^{(m_k)}} \leq C_{1,|\gamma_1|}^{(m_i)} \cdots C_{k,|\gamma_k|}^{(m_i)} \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\dots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)} \cdots b_{k,\beta_k}^{(m_k)}} < \infty,$$

which proves our assertion. Let $\mathcal{P}$ be the linear span of the vectors $e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_k}$ for $\gamma_1 \in \mathbb{F}_{n_1}^+, \dots, \gamma_k \in \mathbb{F}_{n_k}^+$. If

$$\sup_{p\in\mathcal{P},\|p\|\le 1}\left\|\sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\beta_1,\dots,\beta_k}\,\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}(p)\right\| < \infty,$$

then there is a unique bounded operator acting on $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$, which we denote by $\varphi(\mathbf{W}_{i,j})$, such that

$$\varphi(\mathbf{W}_{i,j})p = \sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\beta_1,\dots,\beta_k}\,\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}(p) \text{ for any } p\in\mathcal{P}.$$

The set of all operators $\varphi(\mathbf{W}_{i,j})\in B(\otimes_{i=1}^{k} F\ 2(Hni\ ))$ satisfying the above-mentioned properties is denoted by $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$. One can prove that $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ is a Banach algebra, which we call Hardy algebra associated with the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}$.

In a similar manner, one can define the Hardy algebra $R^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$. For each $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, we define the operator $\mathbf{\Lambda}_{i,j}$ acting on the Hilbert space $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$ by setting

$$\mathbf{\Lambda}_{i,j} := \underbrace{I\otimes\cdots\otimes I}_{i-1\text{ times}}\otimes\Lambda_{i,j}\otimes\underbrace{I\otimes\cdots\otimes I}_{k-1\text{ times}}.$$

Set $\mathbf{\Lambda}_{i,j} := (\Lambda_{i,1},\dots,\Lambda_{i,n_i})$. As in Lemma (4.1.10), one can prove that, $\mathbf{\Lambda} := (\mathbf{\Lambda}_1,\dots,\mathbf{\Lambda}_{\mathbf{k}})$ is in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\tilde{\mathbf{f}}}(\otimes_{i=1}^{k} F^2(H_{n_i}))$, where $\tilde{\mathbf{f}} = (\tilde{f}_1,\dots,\tilde{f}_k)$.

$Let\ \chi(\mathbf{\Lambda}_{i,j}) = \sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k}\,\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}$ be a formal sum with $c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k}\in\mathbb{C}$ and such that

$$\sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} |c_{\beta_1,\dots,\beta_k}|^2\frac{1}{b^{(m_1)}_{1,\beta_1}\cdots b^{(m_k)}_{k,\beta_k}} < \infty.$$

And

$$\sup_{p\in\mathcal{P},\|p\|\le 1}\left\|\sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k}\,\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}(p)\right\| < \infty,$$

Then there is a unique bounded operator acting on $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$, which we denote by $\chi(\mathbf{\Lambda}_{i,j})$ such that

$$\chi(\mathbf{\Lambda}_{i,j})p = \sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k}\,\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}(p) \quad \text{for any } p\in\mathcal{P}.$$

The set of all operators $\chi(\mathbf{\Lambda}_{i,j})\in B(\otimes_{i=1}^{k} F^2(H_{n_i}))$ satisfying the above-mentioned properties is a Banach algebra which is denoted by $R^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$.

Proposition (4.1.14) [186]: The following statements hold:

(i) $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})' = R^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$, where $'$ stands for the commutant;

(ii) $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})'' = F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$;

(iii) $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ is WOT-closed in $B(\otimes_{i=1}^{k} F^2(H_{n_i}))$.

Proof: Let $U\in B(\otimes_{i=1}^{k} F^2(H_{n_i}))$ be the unitary operator defined by equation

$$U(e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k}) := (e^1_{\tilde{\gamma}_1}\otimes\cdots\otimes e^k_{\tilde{\gamma}_k}), \gamma_1\in\mathbb{F}^+_{n_1},\dots,\gamma_k\in\mathbb{F}^+_{n_k},$$

and note that $U^*\mathbf{\Lambda}_{i,j}U = \mathbf{W}^{\tilde{f}}_{i,j}$ for any $i = 1,\dots,k$ and $j\in\{1,\dots,n_i\}$, where $\mathbf{W}^{\tilde{f}}_{i,j}$ is the universal model associated with $\mathbf{D}^{\mathbf{m}}_{\tilde{\mathbf{f}}}$. Consequently, we have $U^*F^\infty(\mathbf{D}^{\mathbf{m}}_{\tilde{\mathbf{f}}})U = R^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$. On the other hand, since $\mathbf{W}_{i_1,j_1}\mathbf{\Lambda}_{i_2,j_2} = \mathbf{\Lambda}_{i_2,j_2}\mathbf{W}_{i_1,j_1}$ for any $i_1,i_2\in\{1,\dots,k\}, j_1\in\{1,\dots,n_{i_1}\}$, and $j_2\in\{1,\dots,n_{i_2}\}$. We deduce that $R^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})\subseteq F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})'$. Now, we prove the reverse inclusion. Let $G\in F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})'$ and note that, since $G(1)\in\otimes_{i=1}^{k} F^2(H_{n_i})$, we have

$$G(1) = \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k} \frac{1}{\sqrt{b_{1,\tilde{\beta}_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\tilde{\beta}_k}^{(m_k)}}} e_{\tilde{\beta}_1}^1 \otimes\cdots\otimes e_{\tilde{\beta}_k}^k$$

for some coefficients $c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k} \in \mathbb{C}$ with

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\dots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} < \infty.$$

Taking into account that $G\mathbf{W}_{i,j} = \mathbf{W}_{i,j}G$ for $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$, relations (8) and its analogue for $\mathbf{\Lambda}_{i,j}$ imply

$$\begin{aligned} G\left(e_{\alpha_1}^1 \otimes\cdots\otimes e_{\alpha_k}^k\right) &= \sqrt{b_{1,\alpha_1}^{(m_1)}}\cdots\sqrt{b_{k,\alpha_k}^{(m_k)}}\, GW_{1,\alpha_1}\cdots W_{k,\alpha_k}(1) \\ &= \sqrt{b_{1,\alpha_1}^{(m_1)}}\cdots\sqrt{b_{k,\alpha_k}^{(m_k)}}\, W_{1,\alpha_1}\cdots W_{k,\alpha_k}G(1) \\ &= \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k} \frac{\sqrt{b_{1,\alpha_1}^{(m_1)}}}{\sqrt{b_{1,\alpha_1\tilde{\beta}_1}^{(m_1)}}}\cdots\frac{\sqrt{b_{k,\alpha_k}^{(m_k)}}}{\sqrt{b_{k,\alpha_k\tilde{\beta}_k}^{(m_k)}}} e_{\alpha_1\tilde{\beta}_1}^1 \otimes\cdots\otimes e_{\alpha_k\tilde{\beta}_k}^k \\ &= \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k}\, \mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}\left(e_{\alpha_1}^1 \otimes\cdots\otimes e_{\alpha_k}^k\right) \end{aligned}$$

for any $\alpha_1 \in \mathbb{F}_{n_1}^+, \dots, \alpha_k \in \mathbb{F}_{n_k}^+$. Therefore,

$$G(p) = \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k}\, \mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}(p)$$

for any polynomial for any $p \in \mathcal{P}$. Since G is a bounded operator,

$$g(\mathbf{\Lambda}_{i,j}) := \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde{\beta}_1,\dots,\tilde{\beta}_k}\, \mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}$$

is in $R^\infty(\mathbf{D_f^m})$ and $G = g(\mathbf{\Lambda}_{i,j})$. Therefore, $R^\infty(\mathbf{D_f^m}) = F^\infty(\mathbf{D_f^m})'$. The item (ii) follows easily applying part (i). Now, item (iii) is clear. This completes the proof.

Similarly to the proof of Proposition (4.1.14), one can prove that if $S \subset B(\mathcal{K})$ and $I_{\mathcal{K}} \in S$, then

$$(F^\infty(\mathbf{D_f^m}) \otimes S)' = R^\infty(\mathbf{D_f^m})\, \overline{\otimes}\, S' \ \ and \quad (R^\infty(\mathbf{D_f^m}) \otimes S)' = F^\infty(\mathbf{D_f^m})\, \overline{\otimes}\, S',$$

where $F^\infty(\mathbf{D_f^m})\, \overline{\otimes}\, S'$ is the WOT-closed algebra generated by the spatial tensor product of the two algebras. Moreover, for each $i \in \{1,\dots,k\}$, the commutant of the set

$$\{W_{i,j} \otimes I_{\mathcal{H}} : j \in \{1,\dots,n_i\}\} \cup \{I_{F^2(H_{n_i}} \otimes Y : Y \in S\}$$

is equal to $R^\infty(\mathbf{D}_{f_1}^{m_1})\, \overline{\otimes}\, S'$. A repeated appplication of these results shows that, if $\mathrm{f} = (f_1,\dots,f_k)$ and $m = (m_1,\dots,m_k)$, then

$$F^\infty(\mathbf{D_f^m})\, \overline{\otimes}\, B(\mathcal{H}) = F^\infty(\mathbf{D}_{f_1}^{m_1})\, \overline{\otimes}\cdots\overline{\otimes}\, F^\infty(\mathbf{D}_{f_k}^{m_k})\, \overline{\otimes}\, B(\mathcal{H})$$

In the same manner, one can prove the corresponding result for $R^\infty(\mathbf{D_f^m})\, \overline{\otimes}\, B(\mathcal{H})$. Another consequence of the results above is the following Tomita-type theorem in our non-selfadjoint setting: if $\mathcal{M}$ is a von Neumann algebra, then

$$(F^\infty(\mathbf{D_f^m})\, \overline{\otimes}\, \mathcal{M})'' = F^\infty(\mathbf{D_f^m})\, \overline{\otimes}\, \mathcal{M}.$$

Proposition (4.1.15) [186]: The noncommutative Hardy algebra $F^\infty(\mathbf{D_f^m})$ is the sequential SOT- (resp. WOT-,w^*-) closure of all polynomials in $W_{i,j}$ and the identity, where $i \in \{1,\dots,k\}, j \in \{1,\dots,n_k\}$.

Proof: Let $P_n, n \geq 0$, be the orthogonal projection of $F^2(H_{n_1}) \otimes\cdots\otimes F^2(H_{n_k})$ on the the subspace span $\{e_{\alpha_1} \otimes\cdots\otimes e_{\alpha_k}: |\alpha_1| + \cdots + |\alpha_k| = n, \alpha_1 \in \mathbb{F}_{n_1}^+, \dots, \alpha_k \in \mathbb{F}_{n_k}^+\}$. Define the completely contractive projection $\Gamma_j : B(\otimes_{i=1}^k F^2(H_{n_i})) \to B(\otimes_{i=1}^k F^2(H_{n_i})), j \in \mathbb{Z}$, by

$$\Gamma_j(A) := \sum_{n\geq max\{0,-j\}} P_n A P_{n+j} \ .$$

The Cesaro operators on $B(\otimes_{i=1}^{k} F^2(H_{n_i}))$, defined by

$$\chi_n(A) := \sum_{|j|<n} \left(1 - \frac{|j|}{n}\right) \Gamma_j(A)\,, \qquad n \ \geq\ 1,$$

are completely contractive and $\chi_n(A)$ converges to A in the strong operator topology. Let $A \ \in F^\infty(\mathbf{D_f^m})$ have the Fourier representation $\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k}\, \mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$ .Taking into account the definition of the operators $W_{i,j}$, one can easily check that

$$P_{n+j}AP_j \ = \left(\sum_{\substack{|\beta_1|+\cdots|\beta_k|=n\\ \beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}} c_{\beta_1,\ldots,\beta_k}\, \mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\right) P_j\,, \qquad n \ \geq\ 0, j \ \geq\ 0,$$

and $P_jAP_{n+j} =\ 0$ if $n \ \geq\ 1$ and $j \ \geq\ 0$. Therefore,

$$\chi_k(A) \ = \sum_{0\leq q\leq n-1} \left(1 - \frac{q}{n}\right)\left(\sum_{\substack{|\beta_1|+\cdots|\beta_k|=q\\ \beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}} c_{\beta_1,\ldots,\beta_k}\, \mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\right)$$

converges to A, as $k\ \to\ \infty$, in the strong operator topology. The proof is complete.

Lemma (4.1.16) [186]: Let $\mathbf{W} :=\ (\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D_f^m}$, where $\mathbf{W}_i :=\ (\mathbf{W}_{i,1},\ldots,\mathbf{W}_{i,n_i})$ for $i\ \in\ \{1,\ldots,k\}$. If

$$\varphi(\mathbf{W}_{i,j}) \ = \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k}\, \mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$$

is in the noncommutative Hardy algebra $F^\infty(\mathbf{D_f^m})$, then the following statements hold.

(i) The series

$$\varphi(r\mathbf{W}_{i,j}) \ = \sum_{q=0}^{\infty} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=q}} r^q c_{\beta_1,\ldots,\beta_k}\, \mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$$

converges in the operator norm topology for any $r\ \in\ [0,1)$.

(ii) The operator $\varphi(r\mathbf{W}_{i,j})$ is in the noncommutative domain algebra $\mathcal{A}(\mathbf{D_f^m})$ and

$$\|\varphi(r\mathbf{W}_{i,j})\| \leq\ \|\varphi(\mathbf{W}_{i,j})\|.$$

(iii) $\varphi(\mathbf{W}_{i,j}) =$ SOT- $\lim_{r\to1} \varphi(r\mathbf{W}_{i,j})$ and

$$\|\varphi(\mathbf{W}_{i,j})\| =\ \sup_{0\leq r<1} \|\varphi(r\mathbf{W}_{i,j})\| = \lim_{r\to1}\|\varphi(r\mathbf{W}_{i,j})\|\,.$$

Proof: First, we prove that

$$\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\cdots,|\beta_k|=p_k}} b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}\, \mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\mathbf{W}_{k,\beta_k}^*\cdots\mathbf{W}_{1,\beta_1}^*$$
$$\leq \binom{p_1 +\ m_1\ -\ 1}{m_1\ -\ 1}\cdots\binom{p_k + \ m_k\ -\ 1}{m_k\ -\ 1} I \qquad (13)$$

According to relations (3) and (12), for each $i\ \in\ \{1,\ldots,k\}$, and $p_i\ \in \mathbb{N}$, the operators $\{W_{i,\beta_i}\}_{\beta_i\in F_{n_i},|\beta_i|=p_i}$ have orthogonal ranges and

$$\|W_{i,\beta_i}x\| \leq \frac{1}{\sqrt{b_{i,\beta_i}^{(m_i)}}}\binom{|\beta_i|+m_i-1}{m_i-1}^{1/2}\|x\|, x \in F^2(H_{n_i}).$$

Consequently, we deduce that

$$\sum_{\beta_i\in\mathbb{F}_{n_i}^+,|\beta_i|=p_i} b_{i,\beta_i}^{(m_i)} W_{i,\beta_i}W_{i,\beta_i}^* \leq \binom{p_i+m_i-1}{m_i-1} I \quad \text{for any} \quad p_i \in \mathbb{N}.$$

A repeated application of this inequality proves our assertion. Since $\varphi(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$, we have

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} < \infty. \qquad (14)$$

Hence, using relation (13) and Cauchy-Schwarz inequality, we deduce that, for $0 \leq r < 1$,

$$\sum_{p=0}^\infty r^p \left\| \sum_{\substack{p_1,\ldots,p_k \in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\cdots,|\beta_k|=p_k}} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k} \right\|$$

$$\leq \sum_{p=0}^\infty r^p \sum_{\substack{p_1,\ldots,p_k \in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \left(\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\cdots,|\beta_k|=p_k}} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} \right)^{1/2}$$

$$\left\| \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\cdots,|\beta_k|=p_k}} b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\mathbf{W}_{k,\beta_k}^*\cdots\mathbf{W}_{1,\beta_1}^* \right\|^{\frac{1}{2}}$$

$$\leq \sum_{p=0}^\infty r^p \sum_{\substack{p_1,\ldots,p_k \in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \binom{p_1+m_1-1}{m_1-1}^{\frac{1}{2}}\cdots\binom{p_k+m_k-1}{m_k-1}^{\frac{1}{2}}$$

$$\left(\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\cdots,|\beta_k|=p_k}} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} \right)^{1/2}$$

$$\leq \left(\sum_{p=0}^\infty r^{2p} \sum_{\substack{p_1,\ldots,p_k \in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \binom{p_1-m_1-1}{m_1-1}\cdots\binom{p_k-m_k-1}{m_k-1} \right)^{1/2}$$

$$\left(\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} \right)$$

Now, using relation (14) we obtain

$$\sum_{p=0}^\infty r^{2p} \sum_{\substack{p_1,\ldots,p_k \in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \binom{p_1+m_1-1}{m_1-1}\cdots\binom{p_k+m_k-1}{m_k-1} \leq \sum_{p=0}^\infty r^{2p}(p+M)^{Mk-k}(p+1)^k$$

$$< \infty. \qquad (15)$$

where $M := \max\{m_1, \ldots, m_k\}$, and deduce that the series

$$\varphi(r\mathbf{W}_{i,j}) = \sum_{q=0}^{\infty} \sum_{\substack{\beta_1 \in \mathbb{F}_{n_1}^+, \ldots, \beta_k \in \mathbb{F}_{n_k}^+ \\ |\beta_1| + \cdots + |\beta_k| = q}} r^q c_{\beta_1, \ldots, \beta_k} \mathbf{W}_{1,\beta_1} \cdots \mathbf{W}_{k,\beta_k}$$

converges in the operator norm topology. Therefore $\varphi(r\mathbf{W}_{i,j})$ is in the noncommutative domain algebra $\mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$. In what follows, we show that

$$\varphi(\mathbf{W}_{i,j}) = \text{SOT} - \lim_{r \to 1} \varphi(r\mathbf{W}_{i,j}) \qquad (16)$$

for any $\varphi(\mathbf{W}_{i,j}) = \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \ldots, \beta_k \in \mathbb{F}_{n_k}^+} c_{\beta_1, \ldots, \beta_k} \mathbf{W}_{1,\beta_1} \cdots \mathbf{W}_{k,\beta_k}$ in the noncommutative Hardy algebra $F^\infty(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$.

According to the first part of this lemma,

$$\varphi(r\mathbf{W}_{i,j}) = \text{SOT} - \lim_{n \to \infty} p_n(r\mathbf{W}_{i,j}) \qquad (17)$$

where $p_n(\mathbf{W}_{i,j}) := \sum_{q=0}^{n} \sum_{\substack{\beta_1 \in \mathbb{F}_{n_1}^+, \ldots, \beta_k \in \mathbb{F}_{n_k}^+ \\ |\beta_1| + \cdots + |\beta_k| = q}} c_{\beta_1, \ldots, \beta_k} \mathbf{W}_{1,\beta_1} \cdots \mathbf{W}_{k,\beta_k}$ and the convergence is in the operator norm topology. For each $i \in \{1, \ldots, k\}$, let $\gamma_i, \sigma_i, \epsilon_i \in \mathbb{F}_{n_1}^+$ and set $n := |\gamma_1| + \cdots + |\gamma_k|$. Since $\mathbf{W}_{1,\beta_1}^* \cdots \mathbf{W}_{k,\beta_k}^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k) = 0$ for any $\beta_i \in \mathbb{F}_{n_i}^+$ with $|\beta_1| + \cdots + |\beta_k| > n$, we have

$$\varphi(r\mathbf{W}_{i,j})^* (e_{\alpha_1}^1 \otimes \cdots \otimes e_{\alpha_k}^k) = p_n(r\mathbf{W}_{i,j})^* (e_{\alpha_1}^1 \otimes \cdots \otimes e_{\alpha_k}^k)$$

for any $\alpha_i \in \mathbb{F}_{n_1}^+$ with $|\alpha_1| + \cdots + |\alpha_k| \leq n$ and any $r \in [0,1)$. Due to Lemma (4.1.10) and Theorem (4.1.4), $r\mathbf{W} := (r\mathbf{W}_1, \ldots, r\mathbf{W}_n)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\otimes_{i=1}^k F^2(H_{n_i}))$ for any $r \in [0,1)$. Applying Theorem (4.1.11), we obtain

$$\mathbf{K}_{\mathbf{f},r\mathbf{W}} p_n(r\mathbf{W}_{i,j})^* = [p_n(r\mathbf{W}_{i,j})^* \otimes I_{(\otimes_{i=1}^k F^2(H_{n_i}))}] \mathbf{K}_{\mathbf{f},r\mathbf{W}}$$

for any $r \in [0,1)$. Using all these facts and the definition of the noncommutative Berezin kernel, careful calculations reveal that

$$\begin{aligned}
&\langle \mathbf{K}_{\mathbf{f},r\mathbf{W}} \varphi(r\mathbf{W}_{i,j})^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), (e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k) \otimes (e_{\epsilon_1}^1 \otimes \cdots \otimes e_{\epsilon_k}^k) \rangle \\
&= \langle \mathbf{K}_{\mathbf{f},r\mathbf{W}} p_n(r\mathbf{W}_{i,j})^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), (e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k) \otimes (e_{\epsilon_1}^1 \otimes \cdots \otimes e_{\epsilon_k}^k) \rangle \\
&= \langle [p_n(r\mathbf{W}_{i,j})^* \otimes I_{(\otimes_{i=1}^k F^2(H_{n_i}))}] \mathbf{K}_{\mathbf{f},r\mathbf{W}} (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), (e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k) \\
&\qquad \otimes (e_{\epsilon_1}^1 \otimes \cdots \otimes e_{\epsilon_k}^k) \rangle \\
&= \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\ldots,k} r^{|\beta_1| + \cdots + |\beta_k|} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} \langle p_n(r\mathbf{W}_{i,j})^* (e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k), e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k \rangle \\
&\qquad \times \langle \mathbf{W}_{1,\beta_1}^* \cdots \mathbf{W}_{k,\beta_k}^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), \Delta_{\mathbf{f},r\mathbf{W}}^{\mathbf{m}}(\boldsymbol{I})^{1/2} (e_{\epsilon_1}^1 \otimes \cdots \otimes e_{\epsilon_k}^k) \rangle \\
&= \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\ldots,k} r^{|\beta_1| + \cdots + |\beta_k|} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} \langle \varphi(r\mathbf{W}_{i,j})^* (e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k), e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k \rangle \\
&\qquad \times \langle \mathbf{W}_{1,\beta_1}^* \cdots \mathbf{W}_{k,\beta_k}^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), \Delta_{\mathbf{f},r\mathbf{W}}^{\mathbf{m}}(\boldsymbol{I})^{1/2} (e_{\epsilon_1}^1 \otimes \cdots \otimes e_{\epsilon_k}^k) \rangle \\
&= \langle [\varphi(r\mathbf{W}_{i,j})^* \otimes I_{(\otimes_{i=1}^k F^2(H_{n_i}))}] \mathbf{K}_{\mathbf{f},r\mathbf{W}} (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), (e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k) \\
&\qquad \otimes (e_{\epsilon_1}^1 \otimes \cdots \otimes e_{\epsilon_k}^k) \rangle
\end{aligned}$$

for any $r \in [0,1)$ and $\gamma_i, \sigma_i, \epsilon_i \in \mathbb{F}_{n_1}^+, i \in \{1, \ldots, k\}$. Hence, since $\varphi(r\mathbf{W}_{i,j})$ and $\varphi(\mathbf{W}_{i,j})$ are bounded operators on $(\otimes_{i=1}^k F^2(H_{n_i}))$, we deduce that

$$\mathbf{K}_{\mathbf{f},r\mathbf{W}} \varphi(r\mathbf{W}_{i,j})^* = [\varphi(\mathbf{W}_{i,j})^* \otimes I_{(\otimes_{i=1}^k F^2(H_{n_i}))}] \mathbf{K}_{\mathbf{f},r\mathbf{W}}$$

for any $r \in [0,1)$. Since $r\mathbf{W} := (r\mathbf{W}_1, \ldots, r\mathbf{W}_n)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\otimes_{i=1}^k F^2(H_{n_i})$ for any $r \in [0,1)$, Theorem (4.1.11) shows that the Berezin kernel $\mathbf{K}_{\mathbf{f},r\mathbf{W}}$ is an isometry and, therefore, the equality above implies

$$\|\varphi(r\mathbf{W}_{i,j})\| \leq \|\varphi(\mathbf{W}_{i,j})\| \quad \text{for any } r \in [0,1). \qquad (18)$$

Hence, and due to the fact that $\varphi(\mathbf{W}_{i,j})\left(e^1_{\alpha_1}\otimes\cdots\otimes\ e^k_{\alpha_k}\right)=\lim_{r\to1}\varphi(r\mathbf{W}_{i,j})\left(e^1_{\alpha_1}\otimes\cdots\otimes\ e^k_{\alpha_k}\right)$ for any $\alpha_i\in\mathbb{F}^+_{n_i}$,
an approximation argument implies relation (16). Note that if $0\ <\ r_1<r_2<1$, then

$$\|\varphi(r_1\mathbf{W}_{i,j})\|\ \leq\|\varphi(r_2\mathbf{W}_{i,j})\|.$$

Indeed, since $\varphi(r_2\mathbf{W}_{i,j})$ is in the polydomain algebra $\mathcal{A}(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$, Theorem (4.1.12) implies $\|\varphi(rr_2\mathbf{W}_{i,j})\|\leq\|\varphi(r_2\mathbf{W}_{i,j})\|$ for any $r\ \in\ [0,1)$. Taking $:=\frac{r_1}{r_2}$, we prove our assertion. Now one can easily complete the proof of the theorem.

Lemma (4.1.17) [186]: Let $\mathbf{T}\ =\ (T_1,\ldots,T_k)$ be in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ and let $\varphi(\mathbf{W}_{i,j})$ be in the Hardy algebra $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$. Then the noncommutative Berezin kernel satisfies the relations

$$\varphi(rT_{i,j})\mathbf{K}^*_{\mathbf{f},\mathbf{T}}=\mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\varphi(r\mathbf{W}_{i,j})\otimes\ I_{\mathcal{H}})$$

and

$$\varphi(rT_{i,j})\mathbf{K}^*_{\mathbf{f},\mathbf{rT}}=\mathbf{K}^*_{\mathbf{f},\mathbf{rT}}(\varphi(\mathbf{W}_{i,j})\otimes\ I_{\mathcal{H}})$$

for any $r\ \in\ [0,1)$.

Proof: Due to Theorem (4.1.11), we have

$$T_{i,j}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}=\mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\mathbf{W}_{i,j}\otimes\ I_{\mathcal{H}})$$

for any $i\ \in\ \{1,\ldots,k\}$ and $j\ \in\ \{1,\ldots,n_i\}$. Hence, using Theorem (4.1.12) and part (i) of Lemma (4.1.16), we deduce that

$$\varphi\left(rT_{i,j}\right)=\sum_{q=0}^{\infty}\sum_{\substack{\beta_1\in\mathbb{F}^+_{n_1},\ldots,\beta_k\in\mathbb{F}^+_{n_k}\\|\beta_1|+\cdots+|\beta_k|=q}}r^q c_{\beta_1,\ldots,\beta_k}\,T_{1,\beta_1}\cdots T_{k,\beta_k}$$

converges in the operator norm topology and $\varphi(rT_{i,j})\mathbf{K}^*_{\mathbf{f},\mathbf{T}}=\mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\varphi(r\mathbf{W}_{i,j})\otimes\ I_{\mathcal{H}})$ for all $r\ \in\ [0,1)$. Now, we prove the second part of this lemma. Using again Theorem (4.1.11), we obtain

$$\mathrm{K}^*_{\mathrm{f},\mathrm{rT}}\left[p(\mathrm{W}_{i,j})\otimes\ I_{\mathcal{H}}\right]=p(rT_{i,j})\mathrm{K}^*_{\mathrm{f},\mathrm{rT}}\qquad(19)$$

for any polynomial $p(\mathbf{W}_{i,j})$ and $r\ \in\ [0,1)$. Since $r\mathbf{T}\ :=\ (rT_1,\ldots,rT_n)\in D^m_f(\mathcal{H})$ (see Theorem (4.1.4)), relation (17) and Theorem (4.1.12) imply

$$\varphi\left(rtT_{i,j}\right)=\lim_{n\to\infty}\sum_{q=0}^{n}\sum_{\substack{\beta_1\in\mathbb{F}^+_{n_1},\ldots,\beta_k\in\mathbb{F}^+_{n_k}\\|\beta_1|+\cdots+|\beta_k|=q}}(rt)^q c_{\beta_1,\ldots,\beta_k}\,T_{1,\beta_1}\cdots T_{k,\beta_k}\,,\qquad r,t\ \in\ [0,1),$$

where the convergence is in the operator norm topology. Consequently, an approximation argument shows that relation (19) implies

$$\mathbf{K}^*_{\mathbf{f},\mathbf{rT}}\left[\varphi(t\mathbf{W}_{i,j})\otimes\ I_{\mathcal{H}}\right]=\varphi(rtT_{i,j})\mathbf{K}^*_{\mathbf{f},\mathbf{rT}}\quad\text{for }r,t\ \in\ (0,1).\qquad(20)$$

On the other hand, let us prove that

$$\lim_{t\to1}\varphi(rtT_{i,j})=\varphi(rT_{i,j}),\qquad(21)$$

where the convergence is in the operator norm topology. Notice that, due to relation (15), if $\epsilon>\ 0$, there is $m_0\in\ \mathbb{N}$ such that $\sum_{p=0}^{\infty}r^{2p}\sum_{\substack{p_1,\ldots,p_k\ \in\mathbb{N}\cup\{0\}\\p_1+\cdots+p_k=p}}\binom{p_1-\ m_1\ -\ 1}{m_1\ -\ 1}\cdots\binom{p_k-\ m_k\ -\ 1}{m_k\ -\ 1}<\frac{\epsilon^2}{4k^2}$,where $K:=\|\varphi(\mathbf{W}_{i,j})(1)\|$. Since $:=\ (T_1,\ldots,T_n)\ \in\ \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$, Theorem (4.1.12) and relation (13) imply

$$\sum_{\substack{\beta_1\in\mathbb{F}^+_{n_1},\ldots,\beta_k\in\mathbb{F}^+_{n_k}\\|\beta_1|=p_1,\cdots,|\beta_k|=p_k}}b^{(m_1)}_{1,\beta_1}\cdots b^{(m_k)}_{k,\beta_k}\,T_{1,\beta_1}\cdots T_{k,\beta_k}T^*_{k,\beta_k}\cdots T^*_{1,\beta_1}$$
$$\leq\binom{p_1-\ m_1\ -\ 1}{m_1\ -\ 1}\cdots\binom{p_k-\ m_k\ -\ 1}{m_k\ -\ 1}I$$

Now, as in the proof of Lemma (4.1.16), we can deduce that

$$\sum_{p=m_0}^{\infty} r^p \left\| \sum_{\substack{p_1,\dots,p_k \in \mathbb{N}\cup\{0\} \\ p_1+\cdots+p_k=p}} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+ \\ |\beta_1|=p_1,\cdots,|\beta_k|=p_k}} c_{\beta_1,\dots,\beta_k} T_{1,\beta_1}\cdots T_{k,\beta_k} \right\|$$

$$\leq \left(\sum_{p=m_0}^{\infty} r^{2p} \sum_{\substack{p_1,\dots,p_k \in \mathbb{N}\cup\{0\} \\ p_1+\cdots+p_k=p}} \binom{p_1+m_1-1}{m_1-1}\cdots\binom{p_k+m_k-1}{m_k-1} \right)^{1/2} \|\varphi(\mathbf{W}_{i,j})(1)\|$$

$\leq \frac{\epsilon}{2}$.

Consequently, setting $T_{(\beta)} := T_{1,\beta_1}\cdots T_{k,\beta_k}$, there exists $0 < d < 1$ such that

$$\left\| \sum_{p=0}^{\infty} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=p}} (rt)^{|\beta_1|+\cdots+|\beta_k|} c_{\beta_1,\dots,\beta_k} T_{(\beta)} - \sum_{p=0}^{\infty} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=p}} r^{|\beta_1|+\cdots+|\beta_k|} c_{\beta_1,\dots,\beta_k} T_{(\beta)} \right\|$$

$$\leq \epsilon + \left\| \sum_{p=1}^{m_0-1} r^p(t^p-1) \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=p}} c_{\beta_1,\dots,\beta_k} T_{(\beta)} \right\| \|\varphi(\mathbf{W}_{i,j})(1)\|$$

$$\leq 2\epsilon$$

for any $t \in (d,1)$. Hence, we deduce relation (21). On the other hand, due to Lemma (4.1.16), we have $\varphi(\mathbf{W}_{i,j}) = \text{SOT-}\lim_{t\to 1}\varphi(t\mathbf{W}_{i,j})$. Since the map $Y \mapsto Y \otimes I_{\mathcal{H}}$ is SOT-continuous on bounded sets, we deduce that

$$\left[\text{SOT} - \lim_{t\to 1}[\varphi(t\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}}\right] = \varphi(\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}}. \qquad (22)$$

Consequently, using relation (21) and passing to limit in (20), as $t \to 1$, we complete the proof.

In what follows we show that the restriction of the noncommutative Berezin transform to the Hardy algebra $F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$ provides a functional calculus associated with each pure tuple of operators in the noncommutative domain $\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$. Moreover, we obtain a Fatou type result.

Theorem (4.1.18) [186]: Let $\mathbf{T} = (T_1,\dots,T_k)$ be a pure k-tuple in the noncommutative polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ and define the map

$$\Psi_\mathbf{T} : F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m}) \to B(\mathcal{H}) \text{ by } \Psi_\mathbf{T}(\varphi) := \mathbf{B}_\mathbf{T}[\varphi],$$

Where $\mathbf{B}_\mathbf{T}$ is the noncommutative Berezin transform at $\mathbf{T} \in \mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$. Then

(i) $\Psi_\mathbf{T}$ is WOT-continuous (resp. SOT-continuous) on bounded sets;

(ii) $\Psi_\mathbf{T}$ is a unital completely contractive homomorphism and

$$\Psi_\mathbf{T}(\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}) = T_{1,\beta_1}\cdots T_{k,\beta_k}, \qquad \beta_i \in \mathbb{F}_{n_i}^+, i \in \{1,\dots,k\}$$

(iii) for any $\varphi(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$,

$$\mathbf{B}_{r\mathbf{T}}[\varphi(\mathbf{W}_{i,j})] = \varphi(rT_{i,j}) = \mathbf{B}_\mathbf{T}[\varphi(r\mathbf{W}_{i,j})]$$

and

$$\Psi_\mathbf{T}(\varphi(\mathbf{W}_{i,j})) = \text{SOT} - \lim_{r\to 1}\varphi(rT_{i,j}).$$

Proof: Since

$$\Psi_\mathbf{T}(\varphi(\mathbf{W}_{i,j})) = \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\varphi(\mathbf{W}_{i,j})) \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{f},\mathbf{T}}, \quad \varphi(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m}), \qquad (23)$$

using standard facts in functional analysis, we deduce part (i).

Now, we prove part (ii). Since T is pure, Theorem (4.1.11) shows that $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is an isometry. Consequently, relation (23) implies

$$\left\|\left[\Psi_\mathbf{T}(\varphi_{i,j})\right]_{k\times k}\right\| \leq \left\|\left[\varphi_{i,j}\right]_{k\times k}\right\|$$

for any operator-valued matrix $[\varphi_{i,j}]_{k\times k}$ in $M_{k\times k}(F^\infty(\mathbf{D_f^m}))$, which proves that $\Psi_{\mathbf{T}}$ is a unital completely contractive linear map. Due to Theorem (4.1.12), $\Psi_{\mathbf{T}}$ is a homomorphism on the set $\mathcal{P}(\mathbf{W})$ of polynomials in $\{\mathbf{W}_{i,j}\}$. By Proposition (4.1.15), the polynomials in $\mathbf{W}_{i,j}$ and the identity are sequentially WOT-dense in $F^\infty(\mathbf{D_f^m})$. On the other hand, due to part (i), Ψ_T is WOT- continuous on bounded sets. Using the principle of uniform boundedness we deduce that Ψ_T is also a homomorphism on $F^\infty(\mathbf{D_f^m})$.

Due to Lemma (4.1.17) and taking into account that $\mathbf{K_{f,T}}$ and $\mathbf{K_{f,rT}}$ are isometries, we have

$$\begin{aligned}\mathbf{B}_{r\mathbf{T}}[\varphi(\mathbf{W}_{i,j})] &= \mathbf{K}^*_{\mathbf{f},r\mathbf{T}}(\varphi(\mathbf{W}_{i,j}))\otimes I)\,\mathbf{K}_{\mathbf{f},r\mathbf{T}}\\ &= \varphi(rT_{i,j}) = \mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\varphi(r\mathbf{W}_{i,j}))\otimes I)\,\mathbf{K}_{\mathbf{f},\mathbf{T}}\\ &= \mathbf{B}_{\mathbf{T}}[\varphi(r\mathbf{W}_{i,j})].\end{aligned}$$

Now, due to relation (22) we have

$$\text{SOT}-\lim_{t\to 1}[\varphi(t\mathbf{W}_{i,j})\otimes I_{\mathcal{H}}] = \varphi(\mathbf{W}_{i,j})\otimes I_{\mathcal{H}}$$

Hence, and using the equalities above, we deduce that

$$\begin{aligned}\mathbf{B}_{\mathbf{T}}[\varphi(\mathbf{W}_{i,j})] &:= \mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\varphi(\mathbf{W}_{i,j}))\otimes I)\,\mathbf{K}_{\mathbf{f},\mathbf{T}}\\ &= \text{SOT}-\lim_{r\to 1}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\varphi(r\mathbf{W}_{i,j}))\otimes I)\,\mathbf{K}_{\mathbf{f},\mathbf{T}}\\ &= \text{SOT}-\lim_{r\to 1}\varphi(rT_{i,j})\end{aligned}$$

This completes the proof.

We say that $\mathbf{T} = (T_1,\ldots,T_k) \in D_f^m(\mathcal{H})$ is completely non-coisometric if there is no $h\in\mathcal{H}, h\neq 0$ such that

$$\langle (id - \Phi^{q_1}_{f_1,T_1})\cdots(id - \Phi^{q_k}_{f_k,T_k})(I_{\mathcal{H}})h,h\rangle = 0$$

for any $(q_1,\ldots,q_k)\in\mathbb{N}^k$. This is equivalent to the condition

$$\lim_{q=(q_1,\ldots,q_k)\in\mathbb{N}^k}\langle (id - \Phi^{q_1}_{f_1,T_1})\cdots(id - \Phi^{q_k}_{f_k,T_k})(I_{\mathcal{H}})h,h\rangle = 0.$$

In what follows we present an $F^\infty(\mathbf{D_f^m})$-functional calculus for the completely non-coisometric part of the noncommutative polydomain $\mathbf{D_f^m}(\mathcal{H})$.

Theorem (4.1.19) [186]: Let $\mathbf{T} = (T_1,\ldots,T_k)$ be a completely non-coisometric k-tuple in the noncommutative polydomain $\mathbf{D_f^m}(\mathcal{H})$. Then

$$\Phi(\varphi) := \text{SOT}-\lim_{r\to 1}\varphi(rT_{i,j}),\ \varphi = \varphi(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D_f^m}),$$

exists in the strong operator topology and defines a map $\Phi: F^\infty(\mathbf{D_f^m}) \to B(\mathcal{H})$ with the following properties:

(i) $\Phi(\varphi) := \text{SOT}-\lim_{r\to 1}\mathbf{B}_{r\mathbf{T}}[\varphi]$, where $\mathbf{B}_{r\mathbf{T}}$ is the noncommutative Berezin transform at $r\mathbf{T}\in\mathbf{D_f^m}(\mathcal{H})$;

(ii) Φ is WOT-continuous (resp. SOT-continuous) on bounded sets;

(iii) Φ is a unital completely contractive homomorphism.

Proof: According to Theorem (4.1.4), $r\mathbf{T}\in\mathbf{D_f^m}(\mathcal{H})$ and $r\mathbf{W}\in\mathbf{D_f^m}(\otimes_{i=1}^k F^2(H_{n_i}))$ for any $r\in[0,1)$. Due to relations (18) and (22), we have $\text{SOT}-\lim_{t\to 1}[\varphi(t\mathbf{W}_{i,j})\otimes I_{\mathcal{H}}] = \varphi(\mathbf{W}_{i,j})\otimes I_{\mathcal{H}}$. Taking the limit in the first relation of Lemma (4.1.17), as $r\to 1$, we deduce that the map Λ : range $\mathbf{K}^*_{\mathbf{f},\mathbf{T}}\to\mathcal{H}$ given by $\Lambda y := \lim_{r\to 1}\varphi(tT_{i,j})\,y$, $y\in$ range $\mathbf{K}^*_{\mathbf{f},\mathbf{T}}$, is well-defined, linear, and

$$\|\Lambda\mathbf{K}^*_{\mathbf{f},\mathbf{T}}x\| \le \limsup_{r\to 1}\|\varphi(rT_{i,j})\|\|\mathbf{K}^*_{\mathbf{f},\mathbf{T}}x\| \le \|\varphi(\mathbf{W}_{i,j})\|\|\mathbf{K}^*_{\mathbf{f},\mathbf{T}}x\|$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$.

Since $T = (T_1,\ldots,T_k)$ is completely non-coisometric, Theorem (4.1.11) implies that the noncommutative Berezin kernel $\mathbf{K_{f,T}}$ is one-to-one and, therefore, the range of $\mathbf{K}^*_{\mathbf{f},\mathbf{T}}$ is dense in $\mathcal{H}$. Consequently, the map Λ has a unique extension to a bounded linear operator on $\mathcal{H}$, denoted also by Λ, with $\|\Lambda\| \le \|\varphi(\mathbf{W}_{i,j})\|$. We show that

$$\lim_{r\to 1}\varphi(rT_{i,j})\,h = \Lambda h \quad\text{for any } h\in\mathcal{H}. \tag{24}$$

Let $h \in \mathcal{H}$ and let $\{y_k\}_{k=1}^{\infty}$ be a sequence of vectors in the range of $\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$, which converges to h. According to Theorem (4.1.12) and relations (17), (18), we have

$$\|\varphi(rT_{i,j})\| \leq \|\varphi(r\mathbf{W}_{i,j})\| \leq \|\varphi(\mathbf{W}_{i,j})\|$$

for any $r \in [0,1)$. Note that

$$\|\Lambda h - \varphi(rT_{i,j}))\| \leq \|\Lambda h - \Lambda y_k\| + \|\Lambda y_k - \varphi(rT_{i,j}))y_k\| + \|\varphi(rT_{i,j}))y_k - \varphi(rT_{i,j}))h\|$$
$$\leq 2\|\varphi(\mathbf{W}_{i,j})\|\|h - y_k\| + \|\Lambda y_k - \varphi(rT_{i,j}))y_k\|.$$

Consequently, since $\lim_{r\to 1} \varphi(rT_{i,j})\, y_k = \Lambda \mathrm{y}_k$, relation (24) follows. Due to Lemma (4.1.17), we have

$$\varphi(rT_{i,j}) = \mathbf{K}_{\mathbf{f},r\mathbf{T}}^*[\varphi(\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}}]\mathbf{K}_{\mathbf{f},r\mathbf{T}}, \qquad (25)$$

which together with relation (24) imply part (i) of the theorem.

Now we prove part (ii). Since $\|\varphi(rT_{i,j})\| \leq \|\varphi(\mathbf{W}_{i,j})\|$ we deduce that $\|\Phi(\varphi)\| \leq \|\varphi\|$ for $\varphi \in F^{\infty}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$. Taking $r \to 1$ in the first relation of Lemma (4.1.14) and using the first part of this theorem, we obtain

$$\Phi(\varphi)\mathbf{K}_{\mathbf{f},\mathbf{T}}^* = \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\varphi \otimes I), \quad \varphi \in F^{\infty}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}). \qquad (26)$$

If $\{g_\iota\}$ be a bounded net in $F^{\infty}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ such that $g_\iota \to g \in F^{\infty}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ in the weak (resp. strong) operator topology, then $g_\iota \otimes I$ converges to $g \otimes I$ in the same topologies. By relation (26), we have $\Phi(g_\iota)\mathbf{K}_{\mathbf{f},\mathbf{T}}^* = \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(g_\iota \otimes I)$. Since the range of $\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$ is dense in $\mathcal{H}$ and $\{\Phi(g_\iota)\}$ is bounded, an approximation argument shows that $\Phi(g_\iota) \to \Phi(g)$ in the weak (resp. strong) operator topology.

Now, we prove (iii). Relation (25) and the fact that $\mathbf{K}_{\mathbf{f},r\mathbf{T}}$ is an isometry for $r \in [0,1)$ imply

$$\|[\varphi_{st}(rT_{i,j})]_{k\times k}\| \leq \|[\varphi_{st}]_{k\times k}\|$$

for any operator-valued matrix $[\varphi_{st}]_{k\times k} \in M_{k\times k}F^{\infty}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ and $r \in [0,1)$. Hence, and using the fact that $\Phi(\varphi_{st}) = \text{SOT} - \lim_{r\to 1} \varphi_{st}(rT_{i,j})$, we deduce that Φ is completely contractive map. Due to Theorem (4.1.12), Φ is a homomorphism on polynomials in $\mathbf{W}_{i,j}$ and the identity. Since, due to Proposition (4.1.15), these polynomials are sequentially WOT-dense in $F^{\infty}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ and Φ is WOT-continuous on bounded sets, we deduce part (iii) of the theorem. The proof is complete.

We introduce the algebra $Hol(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$ of all free holomorphic functions on the abstract radial poly-domain $\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}$. We identify the polydomain algebra $\mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ and the Hardy algebra $F^{\infty}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ with subalgebras of $Hol(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$.

For each $i \in \{1,\ldots,k\}$, let $Z_i := (Z_{i,1},\ldots,Z_{i,n_i})$ be an n_i-tuple of noncommuting indeterminates and assume that, for any $p,q \in \{1,\ldots,k\}$, $p \neq q$, the entries in Z_p are commuting with the entries in Z_q. We set $Z_{i,\alpha_i} := Z_{i,j_1}\cdots Z_{i,j_p}$ if $\alpha_i \in \mathbb{F}_{n_i}^+$ and $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i$, and $Z_{i,g_0^i} := 1$, where g_0^i is the identity in $\mathbb{F}_{n_i}^+$. We consider formal power series

$$\varphi = \sum_{\alpha_1\in\mathbb{F}_{n_i}^+,\ldots,\alpha_k\in\mathbb{F}_{n_i}^+} a_{\alpha_1,\ldots,\alpha_k} Z_{1,\alpha_1}\cdots Z_{k,\alpha_k}\,, \qquad a_{\alpha_1,\ldots,\alpha_k} \in \mathbb{C},$$

in ideterminates $Z_{i,j}$. Denoting $(\alpha) := (a_{\alpha_1,\ldots,\alpha_k}) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+, Z_{(\alpha)} := Z_{1,\alpha_1}\cdots Z_{k,\alpha_k}$, and $(\alpha) := (a_{\alpha_1,\ldots,\alpha_k})$, we can also use the abbreviation $\varphi = \sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}$.

Given a Hilbert space $\mathcal{H}$, we introduce the radial polydomain

$$\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}(\mathcal{H}) := \bigcup_{0\leq r<1} r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}) \subseteq \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}).$$

A formal power series φ, having the representation above, is called free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}} := \{\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}(\mathcal{H}) : \mathcal{H} \text{ is a Hilbert space}\}$ if the series

$$\varphi(X_{i,j}) := \sum_{q=0}^{\infty} \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} a_{(\alpha)}X_{(\alpha)}$$

is convergent in the operator norm topology for any $X = (X_{i,j}) \in \mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}(\mathcal{H})$ with $i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}$, and any Hilbert space $\mathcal{H}$. We denote by $Hol(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$ the set of all free holomorphic functions on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}$.

Lemma (4.1.20) [186]: LetQ= $\sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{(\alpha)}Z_{(\alpha)}$ be a formal power series and let $\mathbf{W} = \{\mathbf{W}_{i,j}\}$ be the universal model associated with the abstract noncommutative polydomain $\mathbf{D_f^m}$. Then the following statements are equivalent.

(i) φ is a free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}$.

(ii) For any $r \in [0,1)$, the series

$$\varphi(r\mathbf{W}_{i,j}) := \sum_{q=0}^{\infty} \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} a_{(\alpha)} r^{|\alpha_1|+\cdots+|\alpha_k|}\mathbf{W}_{(\alpha)}$$

is convergent in the operator norm topology.

(iii) The inequality

$$\limsup_{n\to\infty} \left\| \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=n}} a_{(\alpha)}\mathbf{W}_{(\alpha)} \right\|^{1/n} \leq 1.$$

Proof: The equivalence of (i) with (ii) is due to Theorem (4.1.12). Using standard arguments, one can easily prove that (ii) is equivalent to (iii).

We remark that the coefficients of a free holomorphic function are uniquely determined by its representation on an infinite dimensional Hilbert space. Indeed, under the above notations, let $0 < r < 1$ and assume that $\varphi(r\mathbf{W}_{i,j}) = 0$. Taking into account relation (8), we have

$$\langle \varphi(r\mathbf{W}_{i,j})1, \mathbf{W}_{(\alpha)}1 \rangle = r^{|\alpha_1|+\cdots+|\alpha_k|} a_{(\alpha)} \frac{1}{b_{1,\alpha_1}^{(m_1)}} \cdots \frac{1}{b_{k,\alpha_k}^{(m_k)}} = 0$$

for any $(\alpha) = (\alpha_1,\ldots,\alpha_k) \in \mathbb{F}_{n_1}^+ \otimes \cdots \otimes \mathbb{F}_{n_k}^+$. Therefore $a_{(\alpha)} = 0$, which proves our assertion.

Due to Lemma (4.1.20), if $\varphi \in Hol(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$, then $\varphi(r\mathbf{W}_{i,j})$ is in the domain algebra $\mathcal{A}(\mathbf{D_f^m})$ for any $r \in [0,1)$. Using the results from the previous, one can see that $Hol(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$ is an algebra. Let $H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$ denote the set of all elements φ in $Hol(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$ such that

$$\|\varphi\|_\infty := \sup\|\varphi(X_{i,j})\| < \infty,$$

where the supremum is taken over all $(X_{i,j}) \in \mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. One can show that $H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$. For each $p \in \mathbb{N}$, we define the norms $\|\cdot\|_p : M_{p\times p}\left(H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})\right) \to [0,\infty)$ by setting

$$\|[\varphi_{st}]_{p\times p}\| := \sup\|[\varphi_{st}(X_{i,j})]_{p\times p}\|,$$

where the supremum is taken over all $(X_{i,j}) \in \mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. It is easy to see that the norms $\|\cdot\|_p, p \in \mathbb{N}$, determine an operator space structure on $H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$, in the sense of Ruan ([31]). Let φ be a free holomorphic function on the abstract radial polydomain Dmf,rad. Note that if $0 < r_1 < r_2 < 1$, then $r_1\mathbf{D_f^m}(\mathcal{H}) \subset r_2\mathbf{D_f^m}(\mathcal{H}) \subset \mathbf{D_f^m}(\mathcal{H})$. Since $\varphi(r_2X_{i,j})$ is in the polydomain algebra

$\mathcal{A}(\mathbf{D_f^m})$, Theorem (4.1.12) implies $\|\varphi(rr_2\mathbf{W}_{i,j})\| \leq \|\varphi(r_2\mathbf{W}_{i,j})\|$ for any $r \in [0,1)$. Taking $r := \frac{r_1}{r_2}$, we deduce that

$$\|\varphi(r_1\mathbf{W}_{i,j})\| \leq \|\varphi(r_2\mathbf{W}_{i,j})\|.$$

On the other hand, if $0 < r < 1$, then we can use again Theorem (4.1.12) to show that the mapping $g: r\mathbf{D_f^m}(\mathcal{H}) \to B(\mathcal{H})$ defined by

$$g(X_{i,j}) := \varphi(X_{i,j}), \quad (X_{i,j}) \in r\mathbf{D_f^m}(\mathcal{H}),$$

is continuous and $\|g(X_{i,j})\| \leq \|g(r\mathbf{W}_{i,j})\|$. Moreover, the series defining g converges uniformly on $r\mathbf{D_f^m}(\mathcal{H})$ in the operator norm topology.

Given $\varphi \in F^\infty(\mathbf{D_f^m})$ and a Hilbert space $\mathcal{H}$, the noncommutative Berezin transform associated with the abstract noncommutative polydomain $\mathbf{D_f^m}$ generates a function whose representation on $\mathcal{H}$ is

$$\mathbf{B}[\varphi] : \mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}}(\mathcal{H}) \to B(\mathcal{H})$$

defined by

$$\mathbf{B}[\varphi](X_{i,j}) := \mathbf{B}_X[\varphi], \qquad X := (X_{i,j}) \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}}(\mathcal{H}),$$

where $\mathbf{B}_X$ is the Berezin transform at X. We call $\mathbf{B}[\varphi]$ the Berezin transform of φ. In what follows, we identify the noncommutative algebra $F^\infty(\mathbf{D_f^m})$ with the Hardy subalgebra $H^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}})$ of bounded free holomorphic functions on $\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}}$.

Theorem (4.1.21) [186]: The map $\Phi : H^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}}) \to F^\infty(\mathbf{D_f^m})$ defined by

$$\Phi\left(\sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}\right) := \sum_{(\alpha)} a_{(\alpha)} \mathbf{W}_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $g := \sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}$ is a free holomorphic function on the abstract radial polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}}$, then the following statements are equivalent:

(i) $g \in H^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}})$;

(ii) $\sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| < \infty$, where $g(r\mathbf{W}_{i,j}) := \sum_{q=0}^\infty \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)} \mathbf{W}_{(\alpha)}$;

(iii) there exists $\varphi \in F^\infty(\mathbf{D_f^m})$ with $g = \mathbf{B}[\varphi]$, where $\mathbf{B}$ is the noncommutative Berezin transform associated with the abstract polydomain $\mathbf{D_f^m}$.

In this case,

$$\Phi(g) = \mathrm{SOT}-\lim_{r\to1} g(r\mathbf{W}_{i,j}), \quad \Phi^{-1}(\varphi) = B[\varphi], \varphi \in F^\infty(\mathbf{D_f^m}),$$

and

$$\|\Phi(g)\| = \sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| = \lim_{r\to1} \|g(r\mathbf{W}_{i,j})\|$$

Proof: To show that the map Φ is well-defined, let $g := \sum_{(\beta)} a_{(\beta)} Z_{(\beta)}$ be in the Hardy algebra $H^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}})$.

Since $(r\mathbf{W}_{i,j}) \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f},\mathrm{rad}}(F^2(H_n))$, Lemma (4.1.20) shows that $g(r\mathbf{W}_{i,j})$ is well-defined for any $r \in [0,1)$ and $\sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| \le \|g\|_\infty < \infty$. We need to show that $g(\mathbf{W}_{i,j}) := \sum_{(\beta)} a_{(\beta)} \mathbf{W}_{(\beta)}$ is the Fourier representation of an element in $F^\infty(\mathbf{D_f^m})$. Taking into account relation (8), we deduce that

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} r^{|\beta_1|+\cdots+|\beta_k|} |a_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} = \|g(r\mathbf{W}_{i,j})(1)\| \le \sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| < \infty$$

for any $0 \le r < 1$. Consequently, $\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} |a_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} < \infty$ As in the latter relation implies that $g(\mathbf{W}_{i,j})p$ is in the tensor product $\otimes_{i=1}^k F^2(H_{n_i})$ for any polynomial $p \in \otimes_{i=1}^k F^2(H_{n_i})$ Now assume that $g(\mathbf{W}_{i,j}) \notin F^\infty(\mathbf{D_f^m})$. According to the definition of $F^\infty(\mathbf{D_f^m})$, for any fixed positive number M , there exists a polynomial $q \in \otimes_{i=1}^k F^2(H_{n_i})$ with $\|q\| = 1$ such that $\|g(\mathbf{W}_{i,j})q\| > M$. Since $\|g(r\mathbf{W}_{i,j})(1) - g(\mathbf{W}_{i,j})(1)\| \to 0$ as $r \to 1$, we have $\|g(\mathbf{W}_{i,j})q - g(r\mathbf{W}_{i,j})q\| \to 0$, as $r \to 1$. Consequently, there is $r_0 \in (0,1)$ such that $\|g(r_0\mathbf{W}_{i,j})q\| > M$, which implies $\|g(r_0\mathbf{W}_{i,j})\| > M$. This contradicts the fact that $\sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| < \infty$. Therefore, $g(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D_f^m})$, which proves that the map Φ is well-defined.

Moreover, due to Theorem (4.1.12) , we have $\|g(X_{i,j}))\| \leq \|g(r\mathbf{W}_{i,j})\|$ for any $(X_{i,j}) \in r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$. Using now Lemma (4.1.16), we deduce that

$$\|g(\mathbf{W}_{i,j})\| = \sup_{0\leq r<1} \|g(r\mathbf{W}_{i,j})\| = \|g\|_\infty$$

and

$$\Phi(g) = g(\mathbf{W}_{i,j}) = \text{SOT} - \lim_{r\to 1} g(r\mathbf{W}_{i,j}).$$

Therefore, Φ is a well-defined isometric linear map. We show now that Φ is a surjective map. To this end, let $\varphi(\mathbf{W}_{i,j}) := \sum_{(\beta)} a_{(\beta)}\mathbf{W}_{(\beta)}$ be in $F^\infty(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$. Using Lemma (4.1.16) and Theorem (4.1.20), we deduce that $g := \sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}$ is a free holomorphic function on the noncommutative domain $\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}$ and

$$\|g(X_{i,j})\| \leq \|g(r\mathbf{W}_{i,j})\| \leq \|g(\mathbf{W}_{i,j})\|$$

for any $(X_{i,j}) \in r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and $r \in [0,1)$. Hence, we deduce that

$$\sup_{(X_{i,j})\in\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}(\mathcal{H})} \|g(X_{i,j})\| \leq \|g(\mathbf{W}_{i,j})\| < \infty,$$

which proves that $g \in H^\infty(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}})$. This shows that the map Φ is surjective. Therefore, we have proved that Φ is an isometric isomorphism of operator algebras. Using the same techniques and passing to matrices, one can prove that Φ is a completely isometric isomorphism. Moreover, note that if $X := (X_{i,j}) \in \mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}$, then there is $r \in (0,1)$ such that $X = rY$ with $Y = (Y_{i,j}) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$. Applying Theorem (4.1.18) part (iii), we deduce that $\varphi(X) = \mathbf{B}_X[\varphi]$. Now, the equivalences mentioned in the theorem can be easily deduced from the considerations above. The proof is complete.

For the rest, we assume that $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is closed in the operator norm topology for any Hilbert space $\mathcal{H}$. Then we have $\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}(\boldsymbol{H})^- = \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$. Note that the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, which we denote by $Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}))$, is a subset of $\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}$. We remark that if $q = (q_1,\ldots,q_k)$ is a k-tuple of positive regular noncommutative polynomials, then $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$ is closed in the operator norm topology.

We denote by $A(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}})$ the set of all elements g in $Hol(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}})$ such that the mapping

$$\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}(\mathcal{H}) \ni (X_{i,j}) \mapsto g(X_{i,j}) \in B(\mathcal{H})$$

has a continuous extension to $[\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}(\mathcal{H})]^- = \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. One can show that $A(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}(\mathcal{H}))$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$, and it has an operator space structure under the norms $\|\cdot\|_p, p \in \mathbb{N}$. Moreover, we can identify the polydomain algebra $A(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ with the subalgebra $A(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}})$. Using Theorem (4.1.12), Theorem (4.1.21), and an approximation argument, one can obtain the following result.

Corollary (4.1.22) [186]: The map $\Phi : A(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}) \to \mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ defined by

$$\Phi\left(\sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}\right) := \sum_{(\alpha)} a_{(\alpha)}\mathbf{W}_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $g := \sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}$ is a free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}$, then the following statements are equivalent:

(i) $g \in A(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}})$;

(ii) $g(r\mathbf{W}_{i,j}) := \sum_{q=0}^{\infty} \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)}\mathbf{W}_{(\alpha)}$ is convergent in the norm topology as $r \to 1$;

(iii) there exists $\varphi \in A(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ with $g = \mathbf{B}[\varphi]$, where $\mathbf{B}$ is the noncommutative Berezin transform associated with the abstract polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$.

In this case,

$$\Phi(g) = \lim_{r\to 1} g(r\mathbf{W}_{i,j}) \quad and \quad \Phi^{-1}(\varphi) = \mathbf{B}[\varphi], \ \varphi \in \mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}).$$

We remark that there is an important connection between the theory of free holomorphic functions on abstract radial polydomains $\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}$, and the theory of holomorphic functions on polydomains in

$\mathbb{C}^d$. Indeed, consider the case when $\mathcal{H} = \mathbb{C}^d$ and $p = 1, 2 \ldots$. Then $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$ can be seen as a subset of $\mathbb{C}^{(n_1+\cdots+n_k)p^2}$ with an arbitrary norm. We denote by $Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p))$ the interior of the closed set $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$.

In the particular case when $p = 1$, the interior $Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p))$ is a Reinhardt domain, i.e., $(\xi_{i,j}\lambda_{i,j}) \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p))$ for any $(\lambda_{i,j}) \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}))$ and $\xi_{i,j} \in \mathbb{T}$. Let $M_{p\times p}(\mathbb{C})$ denote the set of all $p \times p$ matrices with entries in $\mathbb{C}$.

Proposition (4.1.23) [186]: If $p \in \mathbb{N}$ and φ is a free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}$, then its representation on $\mathbb{C}^p$, i.e., the map $\hat{\varphi}$ defined by

$$\mathbb{C}^{(n_1+\cdots+n_k)p^2} \supset \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p) \ni (\lambda_{i,j}) \mapsto \varphi(\lambda_{i,j}) \in M_{p\times p}(\mathbb{C}) \subset \mathbb{C}^{p^2}$$

is a holomorphic function on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$. Moreover, the following statements hold:

(i) if $\varphi \in F^\infty\left(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}}\right)$, then $\hat{\varphi}$ is bounded on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$;

(ii) if $\varphi \in A(\mathbf{D}_{\mathbf{f},\mathbf{rad}}^{\mathbf{m}})$, then $\hat{\varphi}$ is continuous on $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$ and holomorphic on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$.

Proof: If K is a compact subset in the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$, then there exists $r \in (0,1)$ such that $K \subset r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$. Indeed, if $\lambda := (\lambda_{i,j}) \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)) \subset \mathbb{C}^{(n_1+\cdots+n_k)p^2}$, then there exists $\epsilon_\lambda > 0$ and $r \in (0,1)$ such that $\frac{1}{r_\lambda}\mu \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p))$ for any $\mu \in B_{\epsilon_\lambda}(\lambda) := \{z \in \mathbb{C}^{(n_1+\cdots+n_k)p^2} : \|\lambda - z\| < \epsilon_\lambda\}$. Since K is a compact set and $K \subset \cup_{\lambda\in K} B_{\epsilon_\lambda}(\lambda)$, there exists $\lambda_1, \ldots \lambda_l \in K$ such that $K \subset \cup_{i=1}^{l} B_{\epsilon_{\lambda_i}}(\lambda_i)$.

Consequently, for any $\mu \in K$, we have $\frac{1}{r_{\lambda_i}}\mu \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)) \subset \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$ for some $i \in \{1, \ldots, n\}$.

Taking into account that $r_1\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p) \subset r_2\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$ if $r_1, r_2 \in (0,1)$ and $r_1 \le r_2$, we conclude that $K \subset r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$, where $r := \max\{r_1, \ldots, r_l\}$.

Note that if $\varphi := \sum_{q=0}^{\infty} \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} a_{(\alpha)} Z_{(\alpha)}$, then

$$\left\| \varphi(\lambda_{i,j}) - \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|\le n}} a_{(\alpha)}\lambda_{(\alpha)} \right\| \le \sum_{s=n+1}^{\infty} \left\| \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=s}} r^{|\alpha_1|+\cdots+|\alpha_k|} a_{(\alpha)}\mathbf{W}_{(\alpha)} \right\|$$

for any $(\lambda_{i,j}) \in K$. Using, we deduce that $\sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|\le n}} a_{(\alpha)}\lambda_{(\alpha)}$ converges to $\varphi(\lambda_{i,j})$ uniformly on K, as $n \to \infty$. Therefore, the map $(\lambda_{i,j}) \mapsto \varphi(\lambda_{i,j})$ is holomorphic on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$. Now, the items (i) and (ii) are consequences of Theorem (4.1.21) and Corollary (4.1.22). The proof is complete.

We remark that one can obtain versions of all the results in the setting of free holomorphic functions with operator-valued coefficients. Since the proofs are very similar we shall omit them. We also mention that, in the particular case when $k = m_1 = 1$ and $f_1 = Z_1 + \cdots + Z_n$, we recover some of the results concerning the free holomorphic functions on the unit ball of $B(\mathcal{H})^n$ (see [190], [205], [207]).

We obtain a Beurling type factorization and a characterization of the Beurling [212] type joint invariant subspaces under $\{W_{i,j}\}$. We also characterize the reducing subspaces under $\{W_{i,j}\}$ and present several results concerning the model theory for pure elements in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

We recall that a subspace $\mathcal{H} \subseteq \mathcal{K}$ is called co-invariant under $S \subset B(\mathcal{K})$ if $X^*\mathcal{H} \subseteq \mathcal{H}$ for any $X \in S$.

Theorem (4.1.24) [186]: Let $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. If $\mathcal{K}$ be a Hilbert space and $\mathcal{M} \subseteq (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{K}$ is a co-invariant subspace under each operator $W_{i,j} \otimes I_{\mathcal{K}}$ for $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$, then there exists a subspace $\mathcal{E} \subseteq \mathcal{K}$ such that

$$\overline{\text{span}}\{(\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\otimes I_{\mathcal{K}})\mathcal{M}:\ \beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\}=(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}.$$

Consequently, a subspace $\mathcal{M}\subseteq(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K}$ is reducing under each operator $W_{i,j}\otimes I_{\mathcal{K}}$ for $i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\}$, if and only if there exists a subspace $\mathcal{E}\subseteq\mathcal{K}$ such that

$$\mathcal{M}=(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}.$$

Proof: Define the subspace $\mathcal{E}\subseteq\mathcal{K}$ by $\mathcal{E}:=(\mathbf{P}_{\mathbb{C}}\otimes I_{\mathcal{K}})\mathcal{M}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1\subset\otimes_{i=1}^k F^2(H_{n_i})$. Let φ be a nonzero element of $\mathcal{M}$ with representation

$$\varphi=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} e_{\beta_1}^1\otimes\cdots\otimes e_{\beta_k}^k\otimes h_{\beta_1,\ldots,\beta_1},$$

where $h_{\beta_1,\ldots,\beta_1}\in\mathcal{K}$ and $\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}\|h_{\beta_1,\ldots,\beta_k}\|^2<\infty$. Let $\sigma_1\in\mathbb{F}_{n_1}^+,\ldots,\sigma_k\in\mathbb{F}_{n_k}^+$ be such that $h_{\sigma_1,\ldots,\sigma_k}\neq 0$ and note that

$$(\mathbf{P}_{\mathbb{C}}\otimes I_{\mathcal{K}})(\mathbf{W}_{1,\sigma_1}^*\cdots\mathbf{W}_{k,\sigma_k}^*\otimes I_{\mathcal{K}})\varphi=1\otimes\frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}}h_{\sigma_1,\ldots,\sigma_k}.$$

Consequently, since $\mathcal{M}$ is a co-invariant subspace under $\mathbf{W}_{i,j}\otimes I_{\mathcal{K}}$ for $i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\}$, we deduce that $h_{\sigma_1,\ldots,\sigma_k}\in\mathcal{E}$. This implies

$$(\mathbf{W}_{1,\sigma_1}\cdots\mathbf{W}_{k,\sigma_k}\otimes I_{\mathcal{K}})(1\otimes h_{\sigma_1,\ldots,\sigma_k})=\frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}}e_{\sigma_1}^1\otimes\cdots\otimes e_{\sigma_k}^k\otimes h_{\sigma_1,\ldots,\sigma_k}.$$

is a vector in $\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{E}$. Therefore,

$$\varphi=\lim_{n\to\infty}\sum_{q=0}^n\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\|\alpha_1|+\cdots+|\alpha_k|=q}} e_{\beta_1}^1\otimes\cdots\otimes e_{\beta_k}^k\otimes h_{\beta_1,\ldots,\beta_1}\qquad(27)$$

is in $\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{E}$. Hence, $\mathcal{M}\subset\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{E}$ and

$$\mathcal{Y}:=\overline{\text{span}}(\mathbf{W}_{1,\sigma_1}\cdots\mathbf{W}_{k,\sigma_k}\otimes I_{\mathcal{K}})\mathcal{M}:\ \sigma_1\in\mathbb{F}_{n_1}^+,\ldots,\sigma_k\in\mathbb{F}_{n_k}^+\subset(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}.$$

To prove the reverse inclusion, we show first that $\mathcal{E}\subset\mathcal{Y}$. If $h_0\in\mathcal{E}, h_0\neq 0$, then there exists $g\in\mathcal{M}\subset\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{E}$ such that

$$g=1\otimes h_0+\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\|\beta_1|+\cdots+|\beta_k|\geq 1}} e_{\beta_1}^1\otimes\cdots\otimes e_{\beta_k}^k\otimes h_{\beta_1,\ldots,\beta_1}$$

and $1\otimes h_0=(\mathbf{P}_{\mathbb{C}}\otimes I_{\mathcal{K}})g$. Consequently, due to Lemma (4.1.10), we have

$$1\otimes h_0=(\mathbf{P}_{\mathbb{C}}\otimes I_{\mathcal{K}})g=(I-\Phi_{q_1,\mathbf{W}_1\otimes I_{\mathcal{K}}})^{m_1}\cdots(I-\Phi_{q_k,\mathbf{W}_k\otimes I_{\mathcal{K}}})^{m_k}(I)g.$$

Taking into account that $\mathcal{M}$ is co-invariant under $\mathbf{W}_{i,j}\otimes I_{\mathcal{K}}$ for $i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\}$, we deduce that $h_0\in\mathcal{Y}$ for any $h_0\in\mathcal{E}$, i.e., $\mathcal{E}\subset\mathcal{Y}$. The latter inclusion shows that $(\mathbf{W}_{1,\sigma_1}\cdots\mathbf{W}_{k,\sigma_k}\otimes I_{\mathcal{K}})(1\otimes\mathcal{E})\subset\mathcal{Y}$, for any $\sigma_1\in\mathbb{F}_{n_1}^+,\ldots,\sigma_k\in\mathbb{F}_{n_k}^+$, which implies

$$\frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}}e_{\sigma_1}^1\otimes\cdots\otimes e_{\sigma_k}^k\otimes\mathcal{E}\subset\mathcal{Y}.$$

Hence, if $\varphi\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}$ has the representation (27), we deduce that $\varphi\in\mathcal{Y}$. Therefore, $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}\subseteq\mathcal{Y}$. The last part of the theorem is now obvious. The proof is complete.

Let $\mathbf{W}:=(\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. An operator $M:(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K}$ is called multi-analytic with respect to $\mathbf{W}$ if

$$M(\mathbf{W}_{i,j}\otimes I_{\mathcal{H}})=(\mathbf{W}_{i,j}\otimes I_{\mathcal{K}})M$$

for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$. In case M is a partial isometry, we call it inner multi-analytic operator.

Theorem (4.1.25) [186]: Let $\mathbf{W} := (\mathbf{W}_1,\dots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D_f^m}$ and let $W_i \otimes I_{\mathcal{H}} := (\mathbf{W}_{i,1} \otimes I_{\mathcal{H}},\dots,\mathbf{W}_{i,n_i} \otimes I_{\mathcal{H}})$ for $i \in \{1,\dots,k\}$, where $\mathcal{H}$ is a Hilbert space. If $Y \in B(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$ then the following statements are equivalent.

(i) There is a multi-analytic operator $M : \left(\otimes_{i=1}^k F^2(H_{n_i})\right) \otimes \mathcal{E} \to \left(\otimes_{i=1}^k F^2(H_{n_i})\right) \otimes \mathcal{H}$ with respect to $\mathbf{W}$, where $\mathcal{E}$ is a Hilbert space, such that

$$Y = MM^*.$$

(ii) For any $\mathbf{p} := (p_1,\dots,p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \le \mathbf{m}, \mathbf{p} \neq 0$,

$$\left(id - \Phi_{f_1,\mathbf{W}_1 \otimes I_{\mathcal{H}}}\right)^{p_1} \cdots \left(id - \Phi_{f_k,\mathbf{W}_k \otimes I_{\mathcal{H}}}\right)^{p_k}(Y) \ge 0.$$

Proof: Setting $\Delta_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}} := \left(id - \Phi_{f_1,\mathbf{W}_1 \otimes I_{\mathcal{H}}}\right)^{p_1} \cdots \left(id - \Phi_{f_k,\mathbf{W}_k \otimes I_{\mathcal{H}}}\right)^{p_k}$, it is easy to see that if item (i) holds, then

$$\Delta_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}}(Y) = M\Delta^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{E}}}(I)M^* \ge 0$$

for any $\mathbf{p} := (p_1,\dots,p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \le \mathbf{m}, \mathbf{p} \neq 0$.

To prove the converse, assume that (ii) holds. In particular, we have $\Phi_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}}(\Delta^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}}(Y)) \le \Delta^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}}(Y)$, where $m' = (m_1 - 1, m_2,\dots,m_k)$. Consequently, $\Phi^n_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}}(\Delta^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}}(Y)) \le \Delta^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}}(Y)$ for any $n \in \mathbb{N}$.

Since $\mathbf{W} := (\mathbf{W}_1,\dots,\mathbf{W}_k)$ is a pure k-tuple, we have SOT-$\lim_{n\to\infty} \Phi^n_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}}(\Delta^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}}(Y)) = 0$, which implies $\Delta^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}}(Y) \ge 0$. Continuing this process, we deduce that $Y \ge 0$.

Let $\mathcal{M} := \overline{\text{range}\, Y^{1/2}}$ and define

$$A_{i,j}(Y^{1/2}x) := Y^{1/2}\left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{H}}\right)x, \qquad x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}, \qquad (28)$$

for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$. Since $\Phi_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(Y) \le Y$, we have

$$\sum_{\alpha\in\mathbb{F}^+_{n_i},|\alpha|\ge 1} a_{i,\alpha}\|A_{i,\bar{\alpha}}Y^{1/2}x\|^2 = \langle \Phi_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(Y)x,x\rangle \le \|Y^{1/2}x\|^2 k_2$$

for any $x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$. Consequently, we deduce that $a_{i,g^i_j}\|A_{i,j}Y^{1/2}x\|^2 \le \|Y^{1/2}x\|^2$, for any $x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$. Since $a_{i,g^i_j} \neq 0$ each $A_{i,j}$ can be uniquely be extended to a bounded operator (also denoted by $A_{i,j}$) on the subspace M. Denoting $X_{i,j} := A^*_{i,j}$ for $i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$, an approximation argument shows that $\Phi_{f_i,X_i}(I_{\mathcal{M}}) \le I_{\mathcal{M}}$ and relation (28) implies

$$X^*_{i,j}\left(Y^{1/2}x\right) = Y^{1/2}\left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{H}}\right)x, \quad x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H},$$

for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$. This implies

$$Y^{1/2}\Delta^{\mathbf{p}}_{\mathbf{f},\mathbf{X}}(I_{\mathcal{M}})Y^{1/2} = \Delta^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{H}}} Y) \ge 0$$

for any $\mathbf{p} := (p_1,\dots,p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \le \mathbf{m}, \mathbf{p} \neq 0$. On the other hand, we have

$$\langle \Phi_{f_i,X_i}(I_{\mathcal{M}})Y^{1/2}x, Y^{1/2}x\rangle = \langle \Phi^n_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(Y)x,x\rangle \le \|Y\|\langle \Phi^n_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(I)x,x\rangle$$

for any $x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ and $n \in \mathbb{N}$. SOT-$\lim_{n\to\infty} \Phi^n_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(I) = 0$, we have SOT-$\lim_{m\to\infty} \Phi^n_{f_i,X_i}(I_{\mathcal{M}}) = 0$, which, due to Proposition (4.1.8) shows that $\mathbf{X} := (X_1,\dots,X_k)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D_f^m}(\mathcal{M})$. Set $\mathcal{E} := \overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{X}}(I_{\mathcal{M}})(\mathcal{M})}$. According to Theorem (4.1.11), the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{X}} : \mathcal{M} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}$ is an isometry with the property that

$$X_{i,j}\mathbf{K}^*_{\mathbf{f},\mathbf{X}} = \mathbf{K}^*_{\mathbf{f},\mathbf{X}}(\mathbf{W}_{i,j} \otimes I_{\mathcal{E}})$$

for any $i \in \{1,\dots,k\}$ and any $j \in \{1,\dots,n_i\}$. Now, define the bounded linear operator $M := Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}} : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ and note that

$$M(\mathbf{W}_{i,j} \otimes I_{\mathcal{E}}) = Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}}(\mathbf{W}_{i,j} \otimes I_{\mathcal{E}}) = Y^{1/2}X_{i,j}\mathbf{K}^*_{\mathbf{f},\mathbf{X}}$$
$$= (\mathbf{W}_{i,j} \otimes I_{\mathcal{H}})Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}} = (\mathbf{W}_{i,j} \otimes I_{\mathcal{H}})M$$
for any $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$, which proves that M is a multi-analytic operator with respect to $\mathbf{W}_{i,j}$. We also have $MM^* = Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}}\mathbf{K}_{\mathbf{f},\mathbf{X}}Y^{1/2} = Y$. This completes the proof.
We say that $\mathcal{M} \subset (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ is a Beurling type invariant subspace under the operators $\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}, i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}$, if there is an inner multi-analytic operator with respect to $\mathbf{W}$,
$$\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H},$$
such that $\mathcal{M} = \Psi\,((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E})$.
Corollary (4.1.26) [186]: Let $\mathcal{M} \subset (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ be an invariant subspace under the operators $\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}, i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}$. Then $\mathcal{M}$ is of Beurling type if and only if
$$(id - \Phi_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}})^{p_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k\otimes I_{\mathcal{H}}})^{p_k}(P_{\mathcal{M}}) \geq 0$$
for any $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \leq \mathbf{m}$, where $P_{\mathcal{M}}$ is the orthogonal projection of the Hilbert space $(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ onto $\mathcal{M}$. In the particular case when $\mathbf{m} = (1,\ldots,1)$, the condition above is satisfied when $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}} := (\mathbf{W}_1 \otimes I_{\mathcal{H}}|_{\mathcal{M}},\ldots,\mathbf{W}_k \otimes I_{\mathcal{H}}|_{\mathcal{M}})$ is doubly commuting.
Proof: If $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ is a inner multianalytic operator and $\mathcal{M} = \Psi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E})$, then $P_{\mathcal{M}} = \Psi\Psi^*$. Taking into account Lemma (4.1.10), we deduce that
$$(id - \Phi_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}})^{p_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k\otimes I_{\mathcal{H}}})^{p_k}(P_{\mathcal{M}}) = \Psi(P_C \otimes I_{\mathcal{E}})\Psi^* \geq 0$$
for any $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \leq \mathbf{m}$. The converse is a consequence of Theorem (4.1.25), when we take $Y = P_{\mathcal{M}}$.
Now, we consider the case when $\mathbf{m} = (1,\ldots,1)$. Note that if $\mathcal{M}$ is an invariant subspace under the operators $\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}$, then $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting if and only if $P_{\mathcal{M}}(\mathbf{W}_{i_1,j_1} \otimes I_{\mathcal{H}})P_{\mathcal{M}}$ commutes with $P_{\mathcal{M}}(\mathbf{W}_{i_2,j_2} \otimes I_{\mathcal{H}})P_{\mathcal{M}}$ for any $i_1, i_2 \in \{1,\ldots,k\}, i_1 \neq i_2$, and any $j_1 \in \{1,\ldots,n_{i_1}\}, j_2 \in \{1,\ldots,n_{i_2}\}$. The latter condition is equivalent to
$$P_{\mathcal{M}}(\mathbf{W}_{i_1,\alpha} \otimes I_{\mathcal{H}})P_{\mathcal{M}} \quad (29)$$
commutes with $\quad P_{\mathcal{M}}(\mathbf{W}^*_{i_2,\beta} \otimes I_{\mathcal{H}})P_{\mathcal{M}}$
for any $\alpha \in \mathbb{F}_{n_i}^+$ and $\beta \in \mathbb{F}_{n_i}^+$. Assume that $\mathcal{M}$ is invariant subspace under the operators $\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}$ and $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting. Then, due to relation (29), for any $\alpha_i \in \mathbb{F}_{n_i}^+, i \in \{1,\ldots,k\}$, we have
$$(\mathbf{W}_{1,\alpha_1} \otimes I_{\mathcal{H}}) \cdots (\mathbf{W}_{k,\alpha_k} \otimes I_{\mathcal{H}})P_{\mathcal{M}}(\mathbf{W}_{k,\alpha_k} \otimes I_{\mathcal{H}}) \cdots (\mathbf{W}_{1,\alpha_1} \otimes I_{\mathcal{H}}) \quad (30)$$
$$= (\mathbf{W}_{1,\alpha_1} \otimes I_{\mathcal{H}})P_{\mathcal{M}}(\mathbf{W}^*_{1,\alpha_1} \otimes I_{\mathcal{H}}) \cdots (\mathbf{W}_{k,\alpha_k} \otimes I_{\mathcal{H}})P_{\mathcal{M}}(\mathbf{W}^*_{k,\alpha_k} \otimes I_{\mathcal{H}}).$$
Consequently, we deduce that
$$(id - \Phi_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}})^{p_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k\otimes I_{\mathcal{H}}})^{p_k}(P_{\mathcal{M}})$$
$$= (P_{\mathcal{M}} - \Phi_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}}(P_{\mathcal{M}}))^{p_1} \cdots (P_{\mathcal{M}} - \Phi_{f_k,\mathbf{W}_k\otimes I_{\mathcal{H}}}(P_{\mathcal{M}}))^{p_k}$$
for any $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \leq (1,\ldots,1)$. Now, since $\mathbf{W}_1,\ldots,\mathbf{W}_k$ are commuting tuples, we deduce that $P_{\mathcal{M}} - \Phi_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(P_{\mathcal{M}}), i \in \{1,\ldots,k\}$, are commuting operators. On the other hand, they are also positive operators. Indeed, let $\{a_{i,\alpha_i}\}_{\alpha\in\mathbb{F}_{n_i}^+}$ be the coefficients of the positive regular free holomorphic function f_i, and let $x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ have the representation $x = x_1 + x_2$ with respect to the orthogonal decomposition $\mathcal{M} \oplus \mathcal{M}^\perp$. Note that
$$\langle \Phi_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(P_{\mathcal{M}})x, x\rangle = \langle \Phi_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(P_{\mathcal{M}})x_1, x_1\rangle = \sum_{|\alpha|\geq 1} a_{i,\alpha_i}\|P_{\mathcal{M}}(\mathbf{W}_{i,\alpha_i} \otimes I_{\mathcal{H}})x_1\|^2$$
$$\leq \langle \Phi_{f_i,\mathbf{W}_i\otimes I_{\mathcal{H}}}(I)x_1, x_1\rangle \leq \|x_1\|^2 = \langle P_{\mathcal{M}}x, x\rangle\,,$$
which proves our assertion. Therefore, we can deduce that
$$(id - \Phi_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}})^{p_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k\otimes I_{\mathcal{H}}})^{p_k}(P_{\mathcal{M}}) \geq 0$$

for any $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \leq (1, \ldots, 1)$. Due to the first part of this corollary, we conclude that $\mathcal{M}$ is a Beurling type invariant subspace under the operators $\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}$. The proof is complete.

Let $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$, and let $\Phi: (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}$ be a multi-analytic operator with respect to $\mathbf{W}$, i.e., if $\Phi(\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}) = (\mathbf{W}_{i,j} \otimes I_{\mathcal{K}})\Phi$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. We introduce the support of Φ as the smallest reducing subspace supp $(\Phi) \subset \otimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{H}$ under each operator $\mathbf{W}_{i,j}$, containing the co-invariant subspace $\mathcal{M} := \overline{\Phi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K})}$. Using Theorem (4.1.24) and its proof, we deduce that

$$\text{supp}\,(\Phi) = \bigvee_{(\alpha) \in \mathbb{F}_{n_i}^+ \times \cdots \times \mathbb{F}_{n_k}^+} \left(\mathbf{W}_{(\alpha)} \otimes I_{\mathcal{H}}\right)(\mathcal{M}) = (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L},$$

where $\mathcal{L} := \overline{(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}})\Phi^*(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K})}$.

Assume that $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ is the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. We remark that if $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ is an isometric multi-analytic operator and $\mathcal{M} = \Psi\,(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}$, then $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting. Since this is a straightforward computation, we omit it. The converse of this implication holds true for the noncommutative polyball.

Corollary(4.1.27) [186]: Let $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ be the universal model associated to the noncommutative polyball $[B(\mathcal{H})^{n_1}]_1^- \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_1^-, i.e., \mathbf{m} = (1, \ldots, 1)$ and $f_i := Z_{i,1} + \cdots + Z_{i,n_i}$ for $i \in \{1, \ldots, k\}$. If $\mathcal{M} \subset (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ is a nonzero invariant subspace under the operators $\mathbf{W} \otimes I_{\mathcal{H}}$, then $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting if and only if there is a Hilbert space $\mathcal{L}$ and an isometric multi-analytic operator $\Phi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ such that $\mathcal{M} = \Phi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L})$.

Proof: Due to the remarks preceding this corollary, it remains to prove the direct implication. Assume that $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting. Corollary (4.1.26) and Theorem (4.1.25) imply the existence of an inner multianalytic operator $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ such that $\mathcal{M} = \Psi\,(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E})$. Since $\mathbf{W}_{i,j}$ are isometries, the initial space of Ψ, i.e., $\Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}) = \{x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}) : \|\Psi x\| = \|x\|\}$ is reducing under each $\mathbf{W}_{i,j}$. On the other hand, the support of Ψ is the the smallest reducing subspace supp $(\Psi) \subset F^2(H_{n_i}) \otimes \mathcal{H}$ under each operator $\mathbf{W}_{i,j}$, containing the co-invariant subspace $\Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$. Therefore, we must have supp $(\Psi) = \Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$. Note that $\Phi := \Psi|_{\text{supp}\,(\Psi)}$ is an isometric multi-analytic operator. Since supp $(\Psi) = (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L}$, where $\mathcal{L} := (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{E}})\Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$ and $\mathcal{M} = \Phi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L})$, the proof is complete.

We remark that in the particular case when $n_1 = \cdots = n_k = 1$, Corollary (4.1.27) is a Beurling type result for the the Hardy space $H^2(\mathbb{D}^k)$ of the polydisc, which seems to be new if $k > 2$.

We recall that $\mathcal{P}(\mathbf{W})$ is the set of all polynomials $p(\mathbf{W}_{i,j})$ in the operators $\mathbf{W}_{i,j}, i \in \{1, \ldots, k\}, j \in \{1, \ldots, ni\}$, and the identity.

Lemma (4.1.28) [186]: If $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ is the universal model associated to the abstract noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$, then the C^*-algebra $C^*(\mathbf{W}_{i,j})$ is irreducible.

Proof: Let $\mathcal{M} \neq \{0\}$ be a subspace of $\otimes_{i=1}^k F^2(H_{n_i})$, which is jointly reducing for each operator $\mathbf{W}_{i,j}$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. Let $\varphi \in \mathcal{M}, \varphi \neq 0$, and assume that

$$\varphi = \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} a_{\beta_1,\ldots,\beta_1} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k .$$

If $a_{\beta_1,\ldots,\beta_1}$ is a nonzero coefficient of φ, then, using relation (8), we deduce that

$$\mathbf{P}_{\mathbb{C}}\mathbf{W}_{1,\beta_1}^* \cdots \mathbf{W}_{k,\beta_k}^*\varphi = \frac{1}{\sqrt{b_{1,\beta_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\beta_k}^{(m_k)}}} a_{\beta_1,\ldots,\beta_1}.$$

On the other hand, according to Lemma (4.1.10), $(I - \Phi_{q_1,\mathbf{W}_1})^{m_1} \cdots (I - \Phi_{q_k,\mathbf{W}_k})^{m_k}(I) = \mathbf{P}_{\mathbb{C}}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^k F^2(H_{n_i})$. Hence, and using the fact that $\mathcal{M}$ is reducing for each $\mathbf{W}_{i,j}$, we deduce that $a_{\beta_1,\ldots,\beta_1} \in \mathcal{M}$, so $1 \in \mathcal{M}$. Using again that $\mathcal{M}$ is invariant
under the operators $\mathbf{W}_{i,j}$, we have $\mathcal{M} = \otimes_{i=1}^k F^2(H_{n_i})$. This completes the proof.

Let $\mathbf{T} = (T_1,\ldots,T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and $\mathbf{T}' = (T_1',\ldots,T_k') \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}')$ be k-tuples with $T_i := (T_{i,1},\ldots,T_{i,n_i})$ and $T_i' := (T_{i,1}',\ldots,T_{i,n_i}')$. We say that $\mathbf{T}$ is unitarily equivalent to $\mathbf{T}'$ if there is a unitary operator $U:\mathcal{H} \to \mathcal{H}'$ such that $T_{ij} = U^*T_{ij}'U$ for all $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$.

Theorem (4.1.29) [186]: Let $\mathbf{T} = (T_1,\ldots,T_k)$ be a pure k-tuple in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and let

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$$

be the noncommutative Berezin kernel. Then the subspace $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ is co-invariant under each operator $\mathbf{W}_{i,j} \otimes I_{\overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}\mathcal{H}}}$ for any $i \in \{1,\ldots,k\}, j \in \{1,\ldots,n_i\}$ and the dilation provided by the relation

$$T_{(\alpha)} = \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\mathbf{W}_{(\alpha)} \otimes I_{\overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}})\mathbf{K}_{\mathbf{f},\mathbf{T}}, \quad (\alpha) \in \mathbb{F}_{n_i}^+ \times \cdots \times \mathbb{F}_{n_k}^+,$$

is minimal. If $\mathbf{f} = \mathbf{q} = (q_1,\ldots,q_k)$ is a k-tuple of positive regular noncommutative polynomials and

$$\overline{\operatorname{span}}\{\mathbf{W}_{(\alpha)}\mathbf{W}_{(\beta)} : (\alpha),(\beta) \in \mathbb{F}_{n_i}^+ \times \cdots \times \mathbb{F}_{n_k}^+\} = C^*(\mathbf{W}_{i,j}),$$

then the minimal dilation of $\mathbf{T}$ is unique up to an isomorphism.

Proof: Due to Theorem (4.1.11), we have $\mathbf{K}_{\mathbf{f},\mathbf{T}}T_{i,j}^* = (\mathbf{W}_{i,j} \otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}}$ for any $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$, where the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is an isometry. On the other hand, the definition of the Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ implies

$$(\mathbf{P}_{\mathbb{C}} \otimes I_{\overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}})\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H} = \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}.$$

Using Theorem (4.1.24) in the particular case when $\mathcal{M} := \mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ and $\mathcal{E} := \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$, we deduce that the subspace $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ is cyclic for $\mathbf{W}_{i,j} \otimes I_{\mathcal{E}}$ for $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$, which proves the minimality of the dilation, i.e.,

$$(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})} = \bigvee_{(\alpha)\in\mathbb{F}_{n_i}^+\times\cdots\times\mathbb{F}_{n_k}^+} (\mathbf{W}_{(\alpha)} \otimes I_{\overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}})\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}. \quad (31)$$

To prove the last part of the theorem, assume that $\mathbf{f} = \mathbf{q} = (q_1,\ldots,q_k)$ is a k-tuple of positive regular noncommutative polynomials and that the relation in the theorem holds. Consider another minimal dilation of $\mathbf{T}$, i.e.,

$$\mathbf{T}_{(\alpha)} = V^*(\mathbf{W}_{(\alpha)} \otimes I_{\mathcal{D}})V, \qquad (\alpha) \in \mathbb{F}_{n_i}^+ \times \cdots \times \mathbb{F}_{n_k}^+, \qquad (32)$$

where $V : \mathcal{H} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$ is an isometry, $V\mathcal{H}$ is co-invariant under each operator $\mathbf{W}_{i,j} \otimes I_{\mathcal{D}}$, and

$$\left(\otimes_{i=1}^k F^2(H_{n_i})\right) \otimes \mathcal{D} = \bigvee_{(\alpha)\in\mathbb{F}_{n_i}^+\times\cdots\times\mathbb{F}_{n_k}^+} (\mathbf{W}_{(\alpha)} \otimes I_{\mathcal{D}})V\mathcal{H}. \qquad (33)$$

Due to Theorem (4.1.12), there exists a unique unital completely positive linear map $\Psi_{\mathbf{q},\mathbf{T}}$: $C^*(\mathbf{W}_{i,j}) \to B(\mathcal{H})$ such that

$$\Psi_{\mathbf{q},\mathbf{T}}\left(\sum_{\gamma=1}^{s} p_\gamma(\mathbf{W}_{i,j}) q_\gamma(\mathbf{W}_{i,j})^*\right) = \sum_{\gamma=1}^{s} p_\gamma(T_{i,j}) q_\gamma(T_{i,j})^*$$

for any $p_\gamma(\mathbf{W}_{i,j}) q_\gamma(\mathbf{W}_{i,j})^* \in \mathcal{P}(\mathbf{W})$ and $s \in \mathbb{N}$. Consider the $*$-representations

$$\pi_1 : C^*(\mathbf{W}_{i,j}) \to B(\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}, \pi_1(X) := X \otimes I_{\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}}.$$

and

$$\pi_2 : C^*(\mathbf{W}_{i,j}) \to B(\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{D}, \qquad \pi_2(X) := X \otimes I_{\mathcal{D}}.$$

Since the subspaces $\mathbf{K}_{\mathbf{q},\mathbf{T}}\mathcal{H}$ and $V\mathcal{H}$ are co-invariant for each operator $\mathbf{W}_{i,j} \otimes I_{\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}}$, the relation (32) implies

$$\mathbf{\Psi}_{\mathbf{q},\mathbf{T}}(X) = \mathbf{K}_{\mathbf{q},\mathbf{T}}^* \pi_1(X) \mathbf{K}_{\mathbf{q},\mathbf{T}} = V^* \pi_2(X) V, \qquad X \in C^*(\mathbf{W}_{i,j}).$$

Due to relations (31) and (33), we deduce that π_1 and π_2 are minimal Stinespring dilations of the completely positive linear map $\mathbf{\Psi}_{\mathbf{q},\mathbf{T}}$. Since these representations are unique up to an isomorphism, there exists a unitary operator $U : B(\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})} \to (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{D}$ such that

$$U(\mathbf{W}_{i,j} \otimes I_{\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}}) = (\mathbf{W}_{i,j} \otimes I_{\mathcal{D}})U$$

and $\mathbf{K}_{\mathbf{q},\mathbf{T}} = V$. Taking into account that U is unitary, we deduce that

$$U(\mathbf{W}_{i,j}^* \otimes I_{\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}}) = (\mathbf{W}_{i,j}^* \otimes I_{\mathcal{D}})U$$

Since $C^*(\mathbf{W}_{i,j})$ is irreducible (see Lemma (4.1.28)), we must have $U = I \otimes Z$, where $Z \in B(\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(\mathcal{H})}, \mathcal{D})$ is a unitary operator. This implies that $\dim \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(\mathcal{H})} = \dim \mathcal{D}$ and $U\mathbf{K}_{\mathbf{q},\mathbf{T}}\mathcal{H} = V\mathcal{H}$, which proves that the two dilations are unitarily equivalent. The proof is complete.

Let $\mathcal{D}$ be a Hilbert space such that the Hilbert space $\mathcal{H}$ can be identified with a co-invariant subspace of $(\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{D}$ under each operator $\mathbf{W}_{i,j} \otimes I_{\mathcal{D}}$ for any $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$ and such that $\mathbf{T}_{(\alpha)} = V^*(\mathbf{W}_{(\alpha)} \otimes I_{\mathcal{D}})V$ for $(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. The dilation index of $\mathbf{T}$ is the minimum dimension of $\mathcal{D}$ with the above mentioned property. We remark that the dilation index of $\mathbf{T}$ coincides with rank $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)$. Indeed, since $\Delta_{\mathbf{f},\mathbf{W}}^{\mathbf{m}}(I) = \mathbf{P}_{\mathbb{C}}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^{k} F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^{k} F^2(H_{n_i})$, we deduce that $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) = P_{\mathcal{H}} [\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{D}}]|\mathcal{H}$. Hence, rank $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) \le \dim \mathcal{D}$.

Now, Theorem (4.1.29) implies that the dilation index of T is equal to rank $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)$.

Proposition (4.1.30) [186]: Let $\mathbf{q} = (q_1, \ldots, q_k)$ is a k-tuple of positive regular noncommutative polynomials such that

$$\overline{\text{span}}\{\mathbf{W}_{(\alpha)} \mathbf{W}_{(\beta)}^*) : (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\} = C^*(\mathbf{W}_{i,j}).$$

A pure k-tuple $\mathbf{T} = (T_1, \ldots, T_k) \in D_q^m(\mathcal{H})$ has rank $\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I) = n, n = 1, 2, \ldots, \infty$, if and only if it is unitarily equivalent to one obtained by compressing $(\mathbf{W}_1 \otimes I_{\mathbb{C}^n}, \ldots, \mathbf{W}_k \otimes I_{\mathbb{C}^n})$ to a co-invariant subspace $\mathcal{M} \subset \otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathbb{C}^n$ under each operator $\mathbf{W}_{i,j} \otimes I_{\mathbb{C}^n}, i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, with the property that $\dim[(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathbb{C}^n})\mathcal{M}] = n$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^{k} F^2(H_{n_i})$ onto $\mathbb{C}1$.

Proof: The direct implication is a consequence of Theorem (4.1.29). To prove the converse, assume that

$$\mathbf{T} = P_{\mathcal{H}}(\mathbf{W}_{(\alpha)} \otimes I_{\mathbb{C}^n})|_{\mathcal{H}}, \qquad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$$

where $H \subset (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathbb{C}^n$ is a co-invariant subspace under each operator $\mathbf{W}_{i,j} \otimes I_{\mathbb{C}^n}$ for any $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$, such that $\dim(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathbb{C}^n})\mathcal{H} = n$. Note that $\mathbf{T}$ is a pure element in the noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$. First, we consider the case when $n < \infty$. Since $(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathbb{C}^n})\mathcal{H} \subseteq \mathbb{C}^n$ and $\dim(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathbb{C}^n})\mathcal{H} = n$, we deduce that $\dim(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathbb{C}^n})\mathcal{H} = \mathbb{C}^n$. This condition is equivalent to the equality $\mathcal{H}^\perp \cap \mathbb{C}^n = \{0\}$. Since $\Delta_{\mathbf{q},\mathbf{W}}^{\mathbf{m}}(I) = \mathbf{P}_{\mathbb{C}}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^{k} F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^{k} F^2(H_{n_i})$, we deduce that $\Delta_{\mathbf{q},\mathbf{W}}^{\mathbf{m}}(I) =$

$P_{\mathcal{H}}[P_{\mathbb{C}} \otimes I_{\mathbb{C}^n}]|_{\mathcal{H}} = P_{\mathcal{H}}\mathbb{C}^n$. Consequently, we have rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) = \dim P_{\mathcal{H}}\mathbb{C}^n$. If we assume that rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) < n$, then there exists $h \in \mathbb{C}^n, h \neq 0$, with $P_{\mathcal{H}}h = 0$. This contradicts the fact that $\mathcal{H}^{\perp} \cap \mathbb{C}^n = \{0\}$. Therefore, we must have rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) = n$.

Now, we consider the case when n = ∞. According to Theorem (4.1.24) and its proof, we have

$$\otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{E} = \bigvee_{(\alpha)\in \mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}} (\mathbf{W}_{(\alpha)} \otimes I_{\mathbb{C}^n})\mathcal{H}$$

where $\mathcal{E} := (P_{\mathbb{C}} \otimes I_{\mathbb{C}^n})\mathcal{H}$. Since $\otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{E}$ is reducing for each operator $\mathbf{W}_{i,j} \otimes I_{\mathbb{C}^m}$, we deduce that $\mathbf{T}_{(\alpha)} = P_{\mathcal{H}}(\mathbf{W}_{(\alpha)} \otimes I_{\mathcal{E}})|_{\mathcal{H}}, (\alpha) \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$. The uniqueness of the minimal dilation of $\mathbf{T}$ (see Theorem (4.1.29)) implies $\dim \overline{\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathbf{I})\mathcal{H}} = \dim \mathcal{E} = \infty$. This completes the proof.

We can characterize now the pure n-tuples of operators in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$, having rank one, i.e., rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) = 1$.

Corollary (4.1.31) [186]: Under the hypothesis of Proposition (4.1.30), the following statements hold.

(i) If $\mathcal{M} \subset \otimes_{i=1}^{k} F^2(H_{n_i})$ is a co-invariant subspace under each operator $\mathbf{W}_{i,j}$, where $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$, then

$$\mathbf{T} := (T_1,\dots,T_k),\ \ T_i := (P_{\mathcal{M}}\mathbf{W}_{i,1}|_{\mathcal{M}},\dots,P_{\mathcal{M}}\mathbf{W}_{i,n_i}|_{\mathcal{M}},$$

is a pure k-tuple in $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{M})$ such that rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}} = 1$.

(ii) If $\mathcal{M}'$ is another co-invariant subspace under each operator $\mathbf{W}_{i,j}$, which gives rise to an k-tuple $\mathbf{T}'$, then $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent if and only if $\mathcal{M} = \mathcal{M}'$.

Proof: Since $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) = P_{\mathcal{M}}\mathbf{P}_{\mathbb{C}}|_{\mathcal{M}}$ we have rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) \leq 1$. On the other hand, it is clear that $\mathbf{T}$ is pure. This also implies that $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) \neq 0$, so rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) \geq 1$. Therefore, rank $\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I) = 1$.

To prove (ii), note that, as in the proof of Theorem (4.1.29), one can show that $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent if and only if there exists a unitary operator $\Lambda : \otimes_{i=1}^{k} F^2(H_{n_i}) \to \otimes_{i=1}^{k} F^2(H_{n_i})$ such that $\Lambda\mathbf{W}_{i,j} = \mathbf{W}_{i,j}\Lambda, i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$, and $\Lambda\mathcal{M} = \mathcal{M}'$. Hence $\Lambda\mathbf{W}^*_{i,j} = \mathbf{W}^*_{i,j}\Lambda$. Since $C^*(\mathbf{W}_{i,j})$ is irreducible, Λ must be a scalar multiple of the identity. Therefore, we have $\mathcal{M} = \Lambda\mathcal{M} = \mathcal{M}'$. We provide a characterization for the class of tuples of operators in $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ which admit characteristic functions. We prove that the characteristic function is a complete unitary invariant for the class of completely non-coisometric tuples and provide an operator model for this class of elements in terms of their characteristic functions.

Let $\mathbf{W} := (\mathbf{W}_1,\dots,\mathbf{W}_k)$ be the the universal model associated with the abstract noncommutative domain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}$. We say that two multi-analytic operator $\Phi : (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{K}_1 \to (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{K}_2$ and $\Phi' : (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{K}'_1 \to (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{K}'_2$ coincide if there are two unitary operators $\tau_j \in B(\mathcal{K}_j, \mathcal{K}'_j)$ such that

$$\Phi'(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau_1) = (I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau_2)\Phi.$$

Lemma (4.1.32) [186]: Let $\Phi_s : (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{H}_s \to (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{K}, s = 1, 2$, be multi-analytic operators with respect to $\mathbf{W} := (\mathbf{W}_1,\dots,\mathbf{W}_k)$ such that $\Phi_1\Phi_1^* = \Phi_2\Phi_2^*$. Then there is a unique partial isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\Phi_1 = \Phi_2(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes V),$$

where $(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes V)$ is an inner multi-analytic operator with initial space supp (Φ_1) and final space supp (Φ_2). In particular, the multi-analytic operators $\Phi_1|_{\text{supp}\,(\Phi_1)}$ and $\Phi_2|_{\text{supp}\,(\Phi_1)}$ coincide.

Proof: Due to Lemma (4.1.10), $(id - \Phi_{f_1,\mathbf{W}_1})^{m_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = \mathbf{P}_{\mathbb{C}}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^{k} F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^{k} F^2(H_{n_i})$. Since Φ_1, Ψ_2 are multi-analytic operators with respect to $\mathbf{W}$, we deduce that $\Phi_1(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_1})\Phi_1^* = \Phi_2(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_2})\Phi_2^*$. Consequently, we have

$$\|(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_1})\Phi_1^* x\| = \|(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_2})\Phi_2^* x\|, \quad x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}.$$

Set $\mathcal{L}_s := \overline{(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_s})\Phi_s^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K})}, s = 1, 2$, and define the unitary operator $U : \mathcal{L}_1 \to \mathcal{L}_2$ by

$$U(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_1})\Phi_1^* x := (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{H}_2})\Phi_2^* x, \qquad x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}.$$

This implies that there is a unique partial isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ with initial space $\mathcal{L}_1$ and final space $\mathcal{L}_2$, extending U . Moreover, we have $\Phi_1 V^* = \Phi_2|_{1\otimes\mathcal{H}_2}$. Since Φ_1, Ψ_2 are multi-analytic operators with respect to $\mathbf{W}$, we deduce that $\Phi_1(I_{\otimes_{i=1}^k F^2(H_{n_i})} \otimes V^*) = \Phi_2$. Hence, the result follows. Now, the last part of the lemma is clear.

We say that $\mathbf{T} = (T_1, \dots, T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ has characteristic function if there is a Hilbert space $\mathcal{E}$ and a multi-analytic operator $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)\mathcal{H}}$ with respect to $\mathbf{W}_{i,j}, i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$, such that

$$\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^* + \Psi\Psi^* = I.$$

According to Lemma (4.1.32), if there is a characteristic function for $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, then it is essentially unique.

We give now an example of a class of elements $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ which have characteristic function. Let $\Psi: (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{G}$ be an inner multi-analytic operator with $\Psi(0) = 0$ and consider the subspace $\mathcal{M} := \Psi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E})$. Note that M is invariant under each operator $\mathbf{W}_{i,j}$ and define $T_{i,j} := P_{\mathcal{M}^\perp}(\mathbf{W}_{i,j} \otimes I_{\mathcal{G}})|_{\mathcal{M}^\perp}$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Set $\mathbf{T} := (T_1, \dots, T_k)$, where $T_i = (T_{i,1}, \dots, T_{i,j})$, and note that

$$\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I_{\mathcal{M}^\perp}) = P_{\mathcal{M}^\perp}\Delta_{\mathbf{f},\mathbf{W}\otimes I_{\mathcal{G}}}^{\mathbf{m}}(I_{\mathcal{G}})|_{\mathcal{M}^\perp} = P_{\mathcal{M}^\perp}(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{G}})|_{\mathcal{M}^\perp}.$$

Since $\Psi(0) = 0$, we have $1 \otimes \mathcal{G} \subset \mathcal{M}^\perp$ and, consequently, $\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I_{\mathcal{M}^\perp})^{1/2} = (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{G}})|_{\mathcal{M}^\perp}$. Consider an arbitrary vector

$$h = \sum_{\beta \in \mathbb{F}_{n_i}^+, i=1,\dots,k} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes h_{\beta_1,\dots,\beta_k}$$

in $\mathcal{M}^\perp \subset (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{G}$. Using the definition of the noncommutative Berezin kernel and relation (8), we obtain

$$\begin{aligned}\mathbf{K}_{\mathbf{f},\mathbf{T}} h &:= \sum_{\beta \in \mathbb{F}_{n_i}^+, i=1,\dots,k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{G}})(\mathbf{W}_{1,\beta_1}^* \cdots \mathbf{W}_{1,\beta_1}^* \otimes I_{\mathcal{G}})^* h \\ &= \sum_{\beta \in \mathbb{F}_{n_i}^+, i=1,\dots,k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes \frac{1}{\sqrt{b_{1,\beta_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\beta_k}^{(m_k)}}} (1 \otimes h_{\beta_1,\dots,\beta_k}) = h\end{aligned}$$

Consequently, $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ can be identified with the injection of $\mathcal{M}^\perp$ into $(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{G}$, and $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$ can be identified with the orthogonal projection $P_{\mathcal{M}^\perp}$. Therefore, $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^* + \Psi\Psi^* = I$, which proves our assertion.

We also remark that in the particular case when $k = 1$ and $m_1 = 1$, all the elements in the noncom-mutative domain $D_{f_1}^1$ have characteristic functions.

Theorem (4.1.33) [186]: A k-tuple $\mathbf{T} = (T_1, \dots, T_k)$ in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ admits a char-acteristic function if and only if

$$\Delta_{\mathbf{f},\mathbf{W}\otimes I}^{\mathbf{p}}(I - \mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*) \geq 0$$

for any $\mathbf{p} := (p_1, \dots, p_k) \in \mathbb{Z}_k^+$ such that $\mathbf{p} \leq \mathbf{m}$, where $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is the noncommutative Berezin kernel associated with $\mathbf{T}$.

Proof: If $\mathbf{T}$ has characteristic function, then there is a multi-analytic operator Ψ with the property that $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^* + \Psi\Psi^* = I$. Using the multi-analyticity of Ψ, we have

$$\Delta^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I}\left(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}\right)=\Psi\Delta^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I}(I)\Psi^*\geq 0$$

for any $\mathbf{p}:=(p_1,\dots,p_k)\in\mathbb{Z}^+_k$ such that $\mathbf{p}\leq\mathbf{m}$. For the converse, we apply Theorem (4.1.25) to the operator $Y=I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}$ and complete the proof.

If **T** has characteristic function, the multi-analytic operator $\mathcal{M}$ provided by the proof of Theorem (4.1.25) when $Y=I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}$, which we denote by $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}$, is called the characteristic function of **T**. More precisely, $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}$ is the multi-analytic operator

$$\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}:\ (\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{M}_\mathbf{T}}(I)(\mathcal{M}_T)}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$$

defined by $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}:=(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{M}_\mathbf{T}}$, where

$$\mathbf{K}_{\mathbf{f},\mathbf{T}}:\mathcal{H}\to F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$$

is the noncommutative Berezin kernel associated with **T** and

$$\mathbf{K}_{\mathbf{f},\mathbf{M}_\mathbf{T}}:\mathcal{H}\to F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{M}_\mathbf{T}}(I)(\mathcal{M}_T)}$$

is the noncommutative Berezin kernel associated with $\mathbf{M}_\mathbf{T}\in\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{M}_\mathbf{T})$. Here, we have

$$\mathcal{M}_\mathbf{T}:=\overline{\text{range}\,(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})}$$

and $\mathbf{M}_\mathbf{T}:=(M_1,\dots,M_k)$ is the k-tuple with $M_i:=(M_{i,1},\dots,M_{i,n_i})$ and $M_{i,j}\in B(\mathcal{M}_\mathbf{T})$ given by $M_{i,j}:=A^*_{i,j}$, where $A_{i,j}\in B(\mathcal{M}_\mathbf{T})$ is uniquely defined by

$$A_{i,j}[(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})^{1/2}x]:=(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})^{1/2}(\mathbf{W}_{i,j}\otimes I)x$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$. According to Theorem (4.1.25), we have $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}+\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}\mathbf{\Theta}^*_{\mathbf{f},\mathbf{T}}=I$.

Theorem (4.1.34) [186]: Let $\mathbf{T}=(T_1,\dots,T_k)$ be a k-tuple in $\mathcal{C}^m_f(\mathcal{H})$. Then **T** is pure if and only if the char-acteristic function $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}$ is an inner multi-analytic operator. Moreover, in this case $\mathbf{T}=(T_1,\dots,T_k)$ is unitarily equivalent to $\mathbf{G}=(G_1,\dots,G_k)$, where $G_i:=(G_{i,1},\dots,G_{i,n_i})$ is defined by

$$G_{i,j}:=P_{\mathbf{H}_{\mathbf{f},\mathbf{T}}}(\mathbf{W}_{ij}\otimes I)|_{\mathbf{H}_{\mathbf{f},\mathbf{T}}}$$

and $P_{\mathbf{H}_{\mathbf{f},\mathbf{T}}}$ is the orthogonal projection of $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$ onto

$$\mathbf{H}_{\mathrm{f},\mathrm{T}}=\left\{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}\right\}\ominus\text{range}\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}.$$

Proof: Assume that **T** is a pure k-tuple in $\mathcal{C}^{\mathrm{m}}_{\mathrm{f}}(\mathcal{H})$. Theorem (4.1.11) shows that the Non-commutative Berezin kernel associated with **T**, i.e.,

$$\mathbf{K}_{\mathrm{f},\mathrm{T}}:\ \mathcal{H}\to F^2(H_{n1})\otimes\ldots\otimes F^2(H_{nk})\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$$

is an is ometry, the subspace $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ is coinvariant under the operators $\mathbf{W}_{i,j}\otimes I_{\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}}$, $i\in\{1,\dots,k\},j\in\{1,\dots,n_i\}$, and $T_{\mathrm{i,j}}=\mathbf{K}^*_{\mathrm{f},\mathrm{T}}(\mathbf{W}_{i,j}\otimes I_{\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}})\mathbf{K}_{\mathbf{f},\mathbf{T}}$. Since $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}$ is the orthogonal projection of $(\otimes_{i=1}^k F^2\left(H_{n_i}\right)\otimes\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$ onto $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ and $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}+\mathbf{\Theta}_{\mathbf{f},\mathbf{t}}\mathbf{\Theta}^*_{\mathbf{f},\mathbf{T}}=I$, we deduce that $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}$ is a partial isometry and $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}=\mathbf{H}_{\mathbf{f},\mathbf{T}}$. Since $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is an isometry, we can identify $\mathcal{H}$ with $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$. Therefore, $\mathbf{T}=(T_1,\ldots,T_k)$ is unitarily equivalent to $\mathbf{G}=(G_1,\dots,G_k)$.

Conversely, if we assume that $\mathbf{\Theta}_{\mathbf{f},\mathbf{t}}$ is inner, then it is a partial isometry. Due to the fact that $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}+\mathbf{\Theta}_{\mathbf{f},\mathbf{t}}\mathbf{\Theta}^*_{\mathbf{f},\mathbf{T}}=I$, the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a partial isometry. On the other hand, since **T** is completely non-coisometric, $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a one-to-one partial isometry and, therefore, isometry. Due to Theorem (4.1.11), we have

$$\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}=\lim_{\mathbf{q}=(q_1,\dots,qk)\in\mathbb{Z}^k_+}(id-\Phi^{q_k}_{f_k,T_k})\ldots(id-\Phi^{q_1}_{f_1,T_1})(I)=I$$

Consequently, **T** is a pure k-tuple. The proof is complete.

We provide a model theorem for class of the completely non-coisometric k-tuple of operators in $\boldsymbol{\mathcal{C}}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.

Theorem (4.1.35) [186]: Let $\mathbf{T}=(T_1,\ldots,T_{\mathrm{k}})$ be a completely non-coisometric k-tuple in $\boldsymbol{\mathcal{C}}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ and let $\mathbf{W}:=(W_1,\ldots,W_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}$. Set

$$\mathcal{D}:=\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})},\qquad\mathcal{D}_*:=\overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{M}_\mathbf{T}}(I)(\mathcal{M}_T)},$$

and $\Delta_{\Theta_{f,T}} := (I - \mathbf{\Theta}^*_{\mathbf{f},\mathbf{T}} \mathbf{\Theta}_{\mathbf{f},\mathbf{T}})^{1/2}$, where $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}$ is the characteristic function of $\mathbf{T}$. Then $\mathbf{T}$ is unitarily equivalent to $:= (\mathbb{T}_1, \ldots, \mathbb{T}_k) \in \mathcal{C}^{\mathbf{f}}_{\mathbf{m}}(\mathbb{H}_{\mathbf{f},\mathbf{T}})$, where $\mathbb{T}_i := (\mathbb{T}_{i,1}, \ldots, \mathbb{T}_{i,n_i})$ and $\mathbb{T}_{i,j}$ is a bounded operator acting on the Hilbert space

$$\mathbb{H}_{\mathbf{f},\mathbf{T}} := \left[\left(\left(\otimes_{i=1}^k F^2(H_{n_i})\right) \otimes \mathcal{D}\right) \otimes \overline{\Delta_{\Theta_{f,T}}((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}_*)}\right]$$
$$\ominus \left\{ \Theta_{f,T}\varphi \oplus \Delta_{\Theta_{f,T}}\varphi : \varphi \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}_* \right\}$$

and is uniquely defined by the

$$(\boldsymbol{P}_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \mathcal{D}|\mathbb{H}_{f,T}})\ \mathbb{T}^*_{i,j} x = \left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{D}}\right)\left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \mathcal{D}|\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right) x$$

for any $x \in \mathbb{H}_{\mathbf{f},\mathbf{T}}$ Here, $P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \mathcal{D}}$ is the orthogonal projection of the Hilbert space

$$\mathcal{K}_{\mathbf{f},\mathbf{T}} := ((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}) \oplus \overline{\Delta_{\Theta_{f,T}}((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}_*)}$$

onto the subspace $(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$.

Proof: First, we show that there is a unique unitary operator $\Gamma : \mathcal{H} \to \mathbb{H}_{f,T}$ such that

$$\Gamma(\mathbf{K}^*_{\mathbf{f},\mathbf{T}} g) = P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0), \qquad g \in \left(\otimes_{i=1}^k F^2(H_{n_i})\right) \otimes \mathcal{D} \qquad (34)$$

where $P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}$ the orthogonal projection of $\mathcal{K}_{\mathbf{f},\mathbf{T}}$ onto the subspace $\mathbb{H}_{\mathbf{f},\mathbf{T}}$. Indeed, note that the operator $\Phi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D} \to \mathcal{K}_{\mathbf{f},\mathbf{T}}$ defined by

$$\Phi\varphi := \Theta_{f,T}\varphi \oplus \Delta_{\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}}\varphi, \qquad \varphi \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}_*,$$

is an isometry and

$$\Phi^*\left(g \bigoplus 0\right) = \Theta^*_{f,T} g, \qquad g \in \left(\otimes_{i=1}^k F^2(H_{n_i})\right) \otimes \mathcal{D}_* \qquad (35)$$

This leads to

$$\|g\|^2 = \left\|P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0)\right\|^2 + \|\Phi\Phi^*(g \oplus 0)\|^2 = \left\|P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0)\right\|^2 = \left\|\mathbf{\Theta}^*_{\mathbf{f},\mathbf{T}} g\right\|^2$$

for any $g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$. Now, taking into account that

$$\left\|\mathbf{K}^*_{\mathbf{f},\mathbf{T}} g\right\|^2 + \left\|\mathbf{\Theta}^*_{\mathbf{f},\mathbf{T}} g\right\|^2 = \|g\|^2, g \epsilon \otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$$

we deduce that

$$\left\|\mathrm{K}^*_{f,Tg}\right\| = \left\|P_{\mathbb{H}_{f,T}}(g \oplus 0)\right\|, g \epsilon \otimes_{i=1}^k F^2(Hni)) \otimes \mathcal{D} \qquad (36)$$

Since the k-tuple $\mathbf{T} = (T_1, \ldots, T_k)$ is completely non-coisometric, the noncommutative Berezin kernel $\mathbf{K}_{f,T}$ is a one-to-one operator and, consequently, range $\mathbf{K}^*_{f,T}$ is dense in $\mathcal{H}$. Now, let $x \in \mathbb{H}_{\mathbf{f},\mathbf{T}}$ and assume that $\langle x, P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0)\rangle = 0$ for any $g(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$. Using the definition of $\mathbb{H}_{f,T}$ and the fact that $\mathcal{K}_{\mathbf{f},\mathbf{T}}$ coincides with the span of all vectors $g \oplus 0$ for $g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$ *and* $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}\varphi \oplus \Delta_{\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}}\boldsymbol{\varphi}$ for $\varphi \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$, we deduce that $x = 0$. This shows that

$$\mathbb{H}_{\mathbf{f},\mathbf{T}} = \left\{P_{\mathbb{H}_{f,T}}(g \oplus 0) : g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}\right\}$$

Using relation (36), we conclude that there is a unique unitary operator Γ satisfying relation (34).

For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, let $\mathbb{T}_{i,j} : \mathbb{H}_{\mathbf{f},\mathbf{T}} \to \mathbb{H}_{\mathbf{f},\mathbf{T}}$ be defined by

$$\mathbb{T}_{i,j} := \Gamma\, T_{i,j} \Gamma^*, \qquad i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\},$$

In what follows, we prove that

$$\left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \mathcal{D}|\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right) \mathbb{T}^*_{i,j} x = \left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{D}}\right)\left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \mathcal{D}|\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right) x \qquad (37)$$

for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$,, and $x \in \mathbb{H}_{\mathbf{f},\mathbf{T}}$. Using relations (34) and (35), and the fact that Φ is an isometry, we deduce that

$$P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}}\Gamma \boldsymbol{K}_{\boldsymbol{f},\boldsymbol{T}} g = \boldsymbol{P}_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}} P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0) =$$
$$g - P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}}\Phi\Phi^*(g \oplus 0) = g - \boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}}\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}}^* g = \mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^* g$$

for any $g \in (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{D}$. That Taking into account that the range of $\mathbf{K}_{\mathbf{f},\mathbf{T}}^* g$ is dense in $\mathcal{H}$, we deduce that

$$P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}}\Gamma = \mathbf{K}_{\mathbf{f},\mathbf{T}} \qquad (38)$$

Hence, and using the fact that the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is one-to-one, we can see that

$$P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}} = \mathbf{K}_{\mathbf{f},\mathbf{T}}\Gamma^*$$

is a one-to-one operator acting from $\mathbb{H}_{\mathbf{f},\mathbf{T}}$ to $(\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \mathcal{D}$. Relation (38) and Theorem (4.1.11) imply

$$\left(P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}}\right)\mathbb{T}_{i,j}^*\Gamma h = \left(P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}}\right)\Gamma T_{i,j}^* h = \mathbf{K}_{\mathbf{f},\mathbf{T}} T_{i,j}^* h$$
$$= (\mathbf{W}_{i,j}^* \otimes I_{\mathcal{D}})\mathbf{K}_{\mathbf{f},\mathbf{T}} h = (\mathbf{W}_{i,j}^* \otimes I_{\mathcal{D}})\left(P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}}\right)\Gamma h$$

for any $i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$, and $h \in \mathcal{H}$. Now, we can deduce relation (37). Note that, since the operator $P_{(\otimes_{i=1}^{k} F^2(H_{n_i}))\otimes \mathcal{D}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}}$ is one-to-one, the relation (37) uniquely determines each operator $\mathbb{T}_{i,j}^*$ $for\ all\ i \in \{1,\dots,k\} and\ j\{1,\dots,n_i\}$, this completes the proof .

In what follows, we show that the characteristic function $\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}}$ is a complete unitary invariant for the completely non-coisometric part of the noncommutative domain $\boldsymbol{C}_{\mathbf{f}}^{\mathbf{m}}$.

Theorem (4.1.36) [186]: Let $\mathbf{T} := (T_1, \dots, T_k) \in \boldsymbol{C}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and $\mathbf{T'} := (T_1' \dots, T_k') \in \boldsymbol{C}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}')$ be two completely non-coisometric k-tuples. Then $\mathbf{T}$ and $\mathbf{T'}$ are unitarily equivalent if and only if their characteristic functions

$\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}}$ and $\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T'}}$ coincide ..

Proof: Assume that the k-tuples $\mathbf{T}$ and $\mathbf{T'}$ are unitarily equivalent and let $U : \mathcal{H} \to \mathcal{H}'$ be a unitary operator such that $T_{i,j} = U^* T_{i,j}' U$ for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$. It is easy to see that $U\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) = \Delta_{\mathbf{f},\mathbf{T'}}^{\mathbf{m}}(I)U$ and, consequently $U\mathcal{D} = \mathcal{D}'$ where

$$\mathcal{D} := \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}, \qquad \mathcal{D}' := \overline{\Delta_{\mathbf{f},\mathbf{T'}}^{\mathbf{m}}(I)(\mathcal{H}')},$$

Using the definition of the noncommutative Berezin kernel associated with $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$ one can easily check that $(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U)\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T'}}U$. This implies

$$\left(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U\right)(I - \mathbf{K}_{\mathbf{f},\mathbf{T}}, \mathbf{K}_{\mathbf{f},\mathbf{T'}}^*)\left(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U\right) = I - \boldsymbol{K}_{\mathrm{f},\mathrm{T}}, \boldsymbol{K}_{\mathrm{f},\mathrm{T'}}^* \qquad \mathbf{(39)}$$

and $(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U)\mathcal{M}_T = \mathcal{M}_{\mathrm{T'}}$, , where $\mathcal{M}_{\mathrm{T}} := \overline{\text{range } (I - \mathbf{K}_{\mathbf{f},\mathbf{T}}, \mathbf{K}_{\mathbf{f},\mathbf{T}}^*)}$ and $\mathcal{M}_{T'}$ is defined similarly .Recall that $\mathbf{M_T} := (M_1, \dots, M_k)$ is the k-tuple with $M_i := (M_{i,1}, \dots, M_{1i,n_i})$ and M_i,j $\in B(\mathcal{M}_{\mathrm{T'}})$, and it is given by $M_{i,j} := A_{i,j}^*$, where $A_{i,j} \in B(\mathcal{M}_{\mathrm{T}})$ is uniquely defined by

$$A_{i,j}\left[(I - \mathbf{K}_{\mathbf{f},\mathbf{T}}, \mathbf{K}_{\mathbf{f},\mathbf{T}}^*)^{1/2}x\right] := (I - \mathbf{K}_{\mathbf{f},\mathbf{T}}, \mathbf{K}_{\mathbf{f},\mathbf{T}}^*)^{1/2}(W_{i,j} \otimes I)x$$

for any $x \in (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$. Similarly, we define the k-tuple $\mathcal{M}_{\mathrm{T'}}$,and the operators $A_{i,j}' \in B(\mathcal{M}_{\mathrm{T'}})$. Note that

$$A_{i,j}(I - \boldsymbol{K}_{\mathrm{f},\mathrm{T}}, \boldsymbol{K}_{\mathrm{f},\mathrm{T}}^*)^{1/2}x = (I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U^*)A_{i,j}'(I - \boldsymbol{K}_{\mathrm{f},\mathrm{T'}}, \boldsymbol{K}_{\mathrm{f},\mathrm{T'}}^*)^{1/2}(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U^*)^x$$
$$= (I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U^*)A_{i,j}'(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes U)(I - \mathbf{K}_{\mathbf{f},\mathbf{T}}, \mathbf{K}_{\mathbf{f},\mathbf{T}}^*)^{1/2}x$$

for any $x \in (\otimes_{i=1}^{k} F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$ Hence, we deduce that

$$A_{i,j} = \left(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes U^*\right) A'_{i,j} \left(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes U\right).$$

Now, we can see that $\left(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \mathrm{U}\right)\mathcal{D}_* = \mathcal{D}'_*$, where $\mathcal{D}_* := \overline{\Delta^{\mathbf{m}}_{\mathbf{f},\mathbf{mT}}(I)(\mathcal{M}_{\mathrm{T}})}$ *and* $\mathcal{D}'_*$ is defined similarly. We introduce the unitary operators τ and τ' by setting

$$\tau := U\,|_{D} : \mathcal{D} \to \mathcal{D}' \text{ and } \tau_* := (I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes U)|_{D_*} : \mathcal{D}_* \to \mathcal{D}'_*.$$

Using the definition of the characteristic function, it is easy to show that

$$(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \tau)\, \boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}} = \boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}'}\, (I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \tau_*).$$

To prove the converse, assume that the characteristic functions of **T** and **T′** coincide. Then there exist unitary operators $\tau : \mathcal{D} \to \mathcal{D}'$ and $\tau_* : \mathcal{D}_* \to \mathcal{D}'_*$ such that

$$(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \tau)\, \boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}} = \boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}'}\, (I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \tau_*)$$

It is clear that this relation implies

$$\Delta_{\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}}} = \left(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \tau_*\right)^* \Delta_{\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}'}} \left(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \tau_*\right)$$

and

$$(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau_*)\overline{\Delta_{\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}}} ((\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}_*)} = \overline{\Delta_{\boldsymbol{\Theta}_{\mathbf{f},\mathbf{T}'}} (\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}'_*)}.$$

Define now the unitary operator $U : \boldsymbol{\mathcal{K}}_{\mathbf{f},\mathbf{T}} \to \boldsymbol{\mathcal{K}}_{\mathbf{f},\mathbf{T}'}$ by setting

$$U := \left(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau\right) \oplus \left(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau_*\right).$$

Note that the operator $\Phi : (\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}_* \to \boldsymbol{\mathcal{K}}_{\boldsymbol{f},\boldsymbol{T}}$, defined by

$$\Phi\varphi := \Theta_{\mathrm{f,T}}\varphi,\ \varphi \oplus \Delta_{\Theta_{\mathrm{f,T}}\varphi}\, \varphi \in (\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}_*,$$

and the corresponding $\boldsymbol{\Phi'}$ satisfy the following relations:

$$U\Phi \left(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau_*\right)^* = \Phi' \qquad (40)$$

and

$$\left(I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau\right) P^{\mathcal{K}_{\mathbf{f},\mathbf{T}}}_{\otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{D}}\, U^* = P^{\mathcal{K}_{\mathbf{f},\mathbf{T}'}}_{\otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{D}'}, \qquad (41)$$

where $P^{\mathcal{K}_{f,\mathrm{T}}}_{\otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{D}}$ is the orthogonal projection of $\mathcal{K}_{f,\mathrm{T}}$ onto $(\otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{D}$ Note also that relation (40) implies

$$U\mathbb{H}_{\mathrm{f,T}} = \mathrm{U}\,\mathcal{K}_{f,\mathrm{T}} \ominus U\Phi((\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}_*)$$
$$= \mathcal{K}_{f,\mathrm{T}'} \ominus \Phi'\, (I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{T}_*})\, ((\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)) \otimes \mathcal{D}_*)$$
$$= \mathcal{K}_{f,\mathrm{T}'} \ominus \Phi'\, ((\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)) \otimes \mathcal{D}_*).$$

This shows that the operator $U|_{\mathbb{H}_{\mathrm{f,T}}} : \mathbb{H}_{\mathrm{f,T}} \to \mathbb{H}_{\mathrm{f,T}'}$ is unitary. Note also that

$$\left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{D}'}\right)\left(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{T}}\right) = \left(I_{\otimes_{i=1}^{k} F^2(H_{n_i}) \otimes \mathcal{T}}\right)\left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{D}}\right) \qquad (42).$$

Let $\mathbb{T} := (\mathbb{T}_1, \ldots \mathbb{T}_n)$ and $\mathbb{T}' := (\mathbb{T}'_1, \ldots \mathbb{T}'_n)$ be the model operators provided by Theorem (4.1.35) for **T** and **T′**, respectively. Using the relation (37) for **T′** and **T**, as well as (41) and (42), we have

$$P^{\mathcal{K}_{f,\mathrm{T}'}}_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}'} {\mathbb{T}'_{i,j}}^* U\chi = \left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{D}'}\right) P^{\mathcal{K}_{\mathrm{f,T}}}_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}}\, U\chi$$
$$= \left(\mathbf{W}^*_{i,j} \otimes I_{\mathcal{D}'}\right)(I_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)} \otimes \tau)\, P^{\mathcal{K}_{\mathrm{f,T}}}_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right) \otimes \mathcal{D}}\chi$$

$$= (I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau) \left(\mathbf{W}_{i,j}^* \otimes I_{\mathcal{D}} \right) P^{\mathcal{K}_{\mathrm{f,T}}}_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)\otimes \mathcal{D}^{\mathcal{X}}}$$
$$= (I_{\otimes_{i=1}^{k} F^2(H_{n_i})} \otimes \tau) P^{\mathcal{K}_{\mathrm{f,T}}}_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)\otimes \mathcal{D}} \mathbb{T}_i^* \mathcal{X}$$
$$= P^{\mathcal{K}_{f,\mathrm{T}'}}_{\otimes_{i=1}^{k} F^2(H_{n_i})\otimes \mathcal{D}'} U \mathbb{T}_{i,j}^* \mathcal{X}$$

for any $i = \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$, and $x \in \mathbb{H}_{\mathrm{f,T}}$ $since\ P^{\mathcal{K}_{\mathrm{f,T}'}}_{\otimes_{i=1}^{k} F^2\left(H_{n_i}\right)\otimes \mathcal{D}'} |_{\mathbb{H}_{\mathrm{f,T}'}}$ is an one-to-one operator (see Theorem (4.1.35)), we obtain $(U|_{\mathbb{H}_{\mathrm{f,T}}}) = \mathbb{T}_{i,j}^* = (\mathbb{T}_{i,j}')^* \, (U|_{\mathbb{H}_{\mathrm{f,T}}})$. Due to Theorem (4.1.35), we conclude that the k-tuples $\mathbf{T}$ and $\mathbf{T}'$are unitarily equivalent. The proof is complete .

Proposition (4.1.37) [186]: If $\mathbf{T} = (T_1, \ldots, T_k) \in \mathbf{D}_{\boldsymbol{f}}^{\boldsymbol{m}}(\mathcal{H})$, then the following statements hold.

(i) $\mathbf{T}$ is unitarily equivalent to $(\mathbf{W_1} \otimes I_{\mathcal{K}}, \ldots, \mathbf{W_k} \otimes I_{\mathcal{K}})$ for some Hilbert space $\mathcal{K}$ if and only if $\mathbf{T} \in \boldsymbol{\mathcal{C}}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is completely non-coisometric and the characteristic function $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}} = 0$.

(ii) If $\mathbf{T} \in \boldsymbol{\mathcal{C}}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, then $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}}$ has dense range if and only if there is no nonzero vector $h \in \mathcal{H}$ such that

$$\lim_{q=(q_1,\ldots,q_k)\in\mathbb{N}^k} \langle \left(id - \Phi^{q_1}_{f_1,T_1}\right) \ldots \left(id - \Phi^{q_1}_{f_1,T_1}\right)(I_{\mathcal{H}})h, h\rangle = \|h\|$$

Proof: Note that if $\mathbf{T} = (\mathbf{W_1} \otimes I_{\mathcal{K}}, \ldots, \mathbf{W_k} \otimes I_{\mathcal{K}})$ for some Hilbert space $\mathcal{K}$, then $\mathbf{K}_{\mathrm{f,T}} = \mathrm{I}$. Since $\mathbf{K}_{\mathrm{f,T}}\mathbf{K}_{\mathrm{f,T}}^* + \mathbf{\Theta}_{\mathbf{f},\mathbf{T}}, \mathbf{\Theta}_{\mathrm{f,T}}^* = I$, we deduce that $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}} = 0$. Conversely, if $\mathbf{T} \in \boldsymbol{\mathcal{C}}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is completely non-coisometric and the characteristic function $\mathbf{\Theta}_{\mathbf{f},\mathbf{T}} = 0$, then $\mathbf{K}_{\mathrm{f,T}}\mathbf{K}_{\mathrm{f,T}}^* = \mathrm{I}$. Using Theorem (4.1.35), the result follows.

Due to Theorem (4.1.11), the condition in item (ii) is equivalent to $\ker\left(I - \mathbf{K}_{\mathrm{f,T}}^* \mathbf{K}_{\mathrm{f,T}}\right) = \{0\}$, which is equivalent to $\ker\,(I - \mathbf{K}_{\mathrm{f,T}}\mathbf{K}_{\mathrm{f,T}}^*) = \{0\}$ and, therefore, to $\ker \mathbf{\Theta}_{\mathbf{f},\mathbf{T}}, \mathbf{\Theta}_{\mathrm{f,T}}^* = \{0\}$. Hence, the result follows. The proof is complete.

We develop a dilation theory on the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and obtain Wold type decompositions for non-degenerate $*$-representations of the C^*-algebra $C^*(\mathbf{W}_{i,j})$.

We recall that $\mathcal{P}(\mathbf{W})$ is the set of all polynomials $\mathcal{P}(\mathbf{W}_{\mathrm{i,j}})$ in the operators $\mathbf{W}_{i,j}$, $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$, and the identity.

Lemma (4.1.38) [186]: Let $\mathbf{q} = (q_1, \ldots, q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $\mathbf{W} = (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ be the universal model associated with the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. Then all the compact operators in $\mathrm{B}(\otimes_{i=1}^{k} F^2(H_{n_i}))$ are contained in the operator space

$$\mathcal{S} := \overline{\operatorname{span}}\, \{p(\mathbf{W}_{i,j})q(\mathbf{W}_{i,j})^* : p(\mathbf{W}_{i,j}), q\,(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})\},$$

where the closure is in the operator norm.

Proof: According to Lemma (4.1.10), we have

$$(I - \Phi_{q_1,\mathbf{W}_1})^{m_1} \ldots (I - \Phi_{q_k,\mathbf{W}_k})^{m_k}\,(I) = \mathbf{P}_{\mathbb{C}} \qquad (43)$$

where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $(\otimes_{i=1}^{k} F^2(H_{n_i})$ onto $\mathbb{C}1 \subset (\otimes_{i=1}^{k} F^2(H_{n_i})$. Fix

$$g\left(\mathbf{W}_{i,j}\right) := \sum_{\substack{\beta_1\in\mathbb{F}^+_{n_1},\ldots,\beta_k\in\mathbb{F}^+_{n_k}\\ |\beta_1|+\cdots+|\beta_k|\le n}} d_{\beta_1,\ldots\beta_k} \mathbf{W}_{1,\beta_1} \ldots \mathbf{W}_{k,\beta_k} \text{ and}$$

$$\xi : \sum_{\beta_1\in\mathbb{F}^+_{n_1},\ldots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\beta_1,\ldots\beta_k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k}$$

and note that $\mathbf{P}_{\mathbb{C}} g(\mathbf{W}_{i,j})^* \xi = \langle \xi, g(\mathbf{W}_{i,j})(1)\rangle$. Consequently, we have

$$\chi\left(\mathbf{W}_{i,j}\right)\mathbf{P}_{\mathbb{C}} g\left(\mathbf{W}_{i,j}\right)^* \xi = \langle \xi, g\left(\mathbf{W}_{i,j}\right)(1)\rangle\, \chi\left(\mathbf{W}_{i,j}\right)(1) \qquad (44)$$

for any polynomial $\chi(\mathbf{W}_{i,j})$. Using relation (43), we deduce that the operator $\chi(\mathbf{W}_{i,j})\mathbf{P}_{\mathbb{C}}g(\mathbf{W}_{i,j})^*$ has rank one and it is in the operator space S. On the other hand, due to the fact that the set of all vectors of the form

$$\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|\le n}} d_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\ldots\mathbf{W}_{k,\beta_k}(1) \text{ with } n\in\mathbb{N}, d_{\beta_1,\ldots,\beta_k}\in\mathbb{C}, \text{ is dense in}$$

$\otimes_{i=1}^k F^2(H_{n_i})$, relation (44) implies that all the compact operators in $B(\otimes_{i=1}^k F^2(H_{n_i}))$ are contained in S. This completes the proof.

Let $C^*(\Gamma)$ be the C^*- algebra generated by a set of operators $\Gamma\subset B(\mathcal{K})$ and the identity. A subspace $\mathcal{H}\subset\mathcal{K}$ is called $*$ - cyclic for Γ if $\mathcal{K} = \overline{\text{span}}\{Xh, X\in C^*(\Gamma), h\in\mathcal{H}\}$. The main result is the following dilation theorem for the elements of the noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$.

Theorem (4.1.39) [186]: Let $\mathbf{q} = (q_1,\ldots,q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $\mathbf{W}=(\mathbf{W_1},\ldots,\mathbf{W}_k)$ be the universal model associated with the abstract noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}$. If $\mathbf{T}=(\mathrm{T}_1,\ldots,\mathrm{T}_k)$ is a k-tuple in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$, then there exists a $*$-representation $\pi: C^*(\mathbf{W}_{i,j})\to B(\mathcal{K}_\pi)$ on a separable Hilbert space $\mathcal{K}_\pi$, which annihilates the compact operators and

$$\left(I-\Phi_{q_1,(\mathbf{W}_1)}\right)\ldots\left(I-\Phi_{q_k,\pi(\mathbf{W}_k)}\right)(I_{K_\pi}) = 0,$$

where $(\mathbf{W}_i) := (\pi(\mathbf{W}_{i,1}),\ldots,\pi(\mathbf{W}_{i,n_i}))$, such that $\mathcal{H}$ can be identified with a $*$-cyclic co-invariant subspace of

$$\tilde{\mathcal{K}} := \left[\left(\otimes_{i=1}^k F^2(H_{n_i})\right)\otimes\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}\right]\oplus\mathcal{K}_\pi$$

under each operator

$$V_{i,j} := \begin{bmatrix}\mathbf{W}_{i,j}\otimes I_{\overline{\Delta_{q,T}^{m}(\mathcal{H})(I)}} & 0\\ 0 & \pi(\mathbf{W}_{i,j})\end{bmatrix}, i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\}$$

where $\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I) := \left(id-\Phi_{q_1,T_1}\right)^{m_1}\ldots\left(id-\Phi_{q_k,\mathrm{T}_k}\right)^{m_k}(I)$, and such that

$T_{i,j}^* = V_{i,j}^*|_{\mathcal{H}}$ for all $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$.

Proof: Applying Arveson extension theorem [193] to the map $\Psi_{\mathbf{q},\mathbf{T}}$ of Theorem (4.1.12), we find a unital com-pletely positive linear map $\Psi_{\mathbf{q},\mathbf{T}}: C^*(\mathbf{W}_{i,j})\to B(\mathcal{H})$ such that $\Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)}\mathbf{W}_{(\beta)})^* = \mathbf{T}_{(\alpha)}\mathbf{T}_{(\beta)}^*$, where $\mathbf{T}_{(\alpha)} := T_{1,\alpha_1}\ldots T_{k,\alpha_k}$ for $(\alpha) := (\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\ldots\times\mathbb{F}_{n_k}^+$, *and* $\mathbf{W}_{(\alpha)}$ is defined similarly. Let $\tilde{\pi}: C^*(\mathbf{W}_{i,j})\to B(\tilde{\mathcal{K}})$ be the minimal Stinespring dilation [193] of $\Psi_{\mathbf{q},\mathbf{T}}$. Then we have

$$\Psi_{\mathbf{q},\mathbf{T}}(X) = P_{\mathcal{H}}\tilde{\pi}(X)|\mathcal{H}, \qquad X\in C^*(\mathbf{W}_{i,j}),$$

and $\tilde{\mathcal{K}}\ \overline{\text{span}}\{\tilde{\pi}(X)h: X\in C^*(\mathbf{W}_{i,j}), h\in\mathcal{H}\}$ Now, we prove that that $P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp} = 0$ for an $(\alpha) := (\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\ldots\times\mathbb{F}_{n_k}^+$. Indeed, we have

$$\begin{aligned}\Psi_{\mathbf{q},\mathbf{T}}\left(\mathbf{W}_{(\alpha)}\mathbf{W}_{(\alpha)}^*\right) &= \mathbf{T}_{(\alpha)}\mathbf{T}_{(\alpha)}^* = P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})\tilde{\pi}\left(\mathbf{W}_{(\alpha)}^*\right)|_{\mathcal{H}}\\ &= P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})(P_{\mathcal{H}}+P_{\mathcal{H}^\perp})\tilde{\pi}(\mathbf{W}_{(\alpha)}^*)|_{\mathcal{H}}\\ &= \Psi_{\boldsymbol{q},\boldsymbol{T}}(\mathbf{W}_{(\alpha)}\mathbf{W}_{(\alpha)}^*) + (P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp})(P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp})^*.\end{aligned}$$

Consequently, we deduce that $P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp} = 0$ and, therefore, $\mathcal{H}$ is an invariant subspace under each operator $\tilde{\pi}(\mathbf{W}_{i,j})^*$ and

$$\tilde{\pi}(\mathbf{W}_{i,j})^*|_{\mathcal{H}} = \Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{i,j}^*) = T_{\mathrm{i,j}}^* \qquad (45)$$

for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$.

All the compact operators $C(\otimes_{i=1}^k F^2(H_{n_i}))$ in $B(\otimes_{i=1}^k F^2(H_{n_i}))$ are con-tained in the C^*- algebra $C^*(\mathbf{W}_{i,j})$. Due to standard theory of representations of C^* -algebras [194], the representation $\tilde{\pi}$ decomposes into a direct sum $\tilde{\pi}=\pi_0\oplus\pi$ *on* $\tilde{\mathcal{K}} = \mathcal{K}_0\oplus\mathcal{K}_\pi$, where π_0,π are disjoint representations of $C^*(\mathbf{W}_{i,j})$ on the Hilbert spaces

$\mathcal{K}_0 := \overline{\text{span}}\{\tilde{\pi}(X)\tilde{\mathcal{K}} : X \in \mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))\}$ and $\mathcal{K}_\pi := \mathcal{K}_0^\perp$,

respectively, such that π annihilates the compact operators in $B(\otimes_{i=1}^k F^2(H_{n_i}))$, and π_0 is uniquely determined by the action of $\tilde{\pi}$ on the ideal $\mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))$ of compact operators. Since every representation of $\mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))$ is equivalent to a multiple of the identity representation, we deduce that .

$$\mathcal{K}_0 \simeq (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{G}, \ \pi_0(X) = X \otimes I_{\mathcal{G}}, \ X \in C^*(\mathbf{W}_{i,j}), \qquad (46)$$

for some Hilbert space $\mathcal{G}$. And its proof, one can easily see that

$$\mathcal{K}_0 := \overline{\text{span}}\{\tilde{\pi}(X)\tilde{\mathcal{K}} : X \in \mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))\}$$
$$= \overline{\text{span}}\{\tilde{\pi}(\mathbf{W}_{(\alpha)}\boldsymbol{P}_{\mathbb{C}}\mathbf{W}^*_{(\beta)})\tilde{\mathcal{K}} : (\alpha),(\beta) \in \mathbb{F}^+_{n_1} \times \dots \times \mathbb{F}^+_{n_k}\}$$
$$= \overline{span}\left\{\tilde{\pi}(\mathbf{W}_{(\alpha)})\left[\left(I - \Phi_{q_1,\tilde{\pi}(\mathbf{W}_1)}\right)^{m_1} \dots \left(I - \Phi_{q_k,\tilde{\pi}(\mathbf{W}_k)}\right)^{m_k}(I_{\tilde{\mathcal{K}}})\right]\tilde{\mathcal{K}} : (\alpha) \in \mathbb{F}^+_{n_1} \times \dots \times \mathbb{F}^+_{n_k}\right\}$$

Since $(I - \Phi_{q_1,W_1})^{m_1} \dots \left(\mathrm{I} - \Phi^{m_k}_{q_k,\mathbf{W}_k}\right)(I) = \mathbf{P}_{\mathbb{C}}$, is a projection of rank one in $C^*(\mathbf{W}_{i,j})$ we deduce that $(I - \Phi_{q_1,\pi(\mathbf{W}_1)})^{m_1} \dots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})^{m_k}(\mathrm{I}_{\mathcal{K}_\pi}) = 0$ and dim $\mathcal{G}$ = dim [range $\tilde{\pi}(\mathbf{P}_{\mathbb{C}})$] . On the other hand, since the Stinespring representation $\tilde{\pi}$ is minimal, we can use the proof to deduce that

$$\text{range}\ \tilde{\pi}(\boldsymbol{P}_{\mathbb{C}}) = \overline{\text{span}}\{\tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}^*_{(\beta)})h : (\beta) \in \mathbb{F}^+_{n_1} \times \dots \times \mathbb{F}^+_{n_k}, h \in \mathcal{H}\}.$$

Indeed, we have

$$\text{range}\ \tilde{\pi}(\boldsymbol{P}_{\mathbb{C}}) = \overline{\text{span}}\{\tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(X)h : X \in C^*(\mathbf{W}_{i,j}), h \in \mathcal{H}\}$$
$$= \overline{\text{span}}\{\tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(Y)h : Y \in \mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i})), h \in \mathcal{H}\}$$
$$= \overline{\text{span}}\{\tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}_{(\alpha)}\boldsymbol{P}_{\mathbb{C}}\mathbf{W}^*_{(\beta)})h : (\alpha),(\beta) \in \mathbb{F}^+_{n_1} \times \dots \times \mathbb{F}^+_{n_k}, h \in \mathcal{H}\}$$
$$= \overline{\text{span}}\{\tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}^*_{(\beta)})h : (\beta) \in \mathbb{F}^+_{n_1} \times \dots \times \mathbb{F}^+_{n_k}, h \in \mathcal{H}\}$$

Now , using the fact that

$$\Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)}X) = P_{\mathcal{H}}(\tilde{\pi}(\mathbf{W}_{(\alpha)})\tilde{\pi}(X))|\mathcal{H}$$
$$= (P_{\mathcal{H}}\tilde{\pi}\,\mathbf{W}_{(\alpha)})|\mathcal{H})(P_{\mathcal{H}}\tilde{\pi}(X)|\mathcal{H})$$
$$= \Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)})\Psi_{\mathbf{q},\mathbf{T}}(X)$$

for any $X \in C^*(\mathbf{W}_{i,j})$ *and* $(\alpha) \in \mathbb{F}^+_{n_1} \times \dots \times \mathbb{F}^+_{n_k}$, , it is easy to see that

$$\langle\tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}^*_{(\alpha)})h, \tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}^*_{(\beta)})k\rangle = \langle h, \mathbf{T}_{(\alpha)}\left[\left(id - \Phi_{q_1,T_1}\right)^{m_1} \dots \left(id - \Phi_{q_k,T_k}\right)^{m_k}(I_{\mathcal{H}})\right]\mathbf{T}^*_{(\beta)}h]\rangle$$
$$= \langle\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathrm{I})\mathbf{T}^*_{(\alpha)}h, \Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathrm{I})\mathbf{T}^*_{(\beta)}k\rangle$$

for any $h, k \in \mathcal{H}$ *and* $(\alpha),(\beta) \in \mathbb{F}^+_{n_1} \times \dots \times \mathbb{F}^+_{n_k}$. This implies the existence of a unitary operator , $\Lambda :$ range $\tilde{\pi}(P_{\mathbb{C}}) \to \overline{\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathrm{I})\mathcal{H}}$ defined by

$$\Lambda[\tilde{\pi}(\boldsymbol{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}^*_{(\alpha)})h] := \Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)\mathbf{T}^*_{(\alpha)}h, \qquad h \in \mathcal{H}, \alpha \in \mathbb{F}^+_n.$$

This shows that

$$\dim[range\ \pi(\boldsymbol{P}_{\mathbb{C}})] = \dim\overline{\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)\mathcal{H}} = \dim\mathcal{G}.$$

Using relations (45) and (46), and identifying $\mathcal{G}$ with $\overline{\Delta^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)\mathcal{H}}$, we obtain the required dilation. On the other hand, due to the fact that $(I - \Phi_{q_1,\pi(\mathbf{W}_1)})^{m_1} \dots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})^{m_k}(I_{\mathcal{K}_\pi}) = 0$, we can use Proposition(4.1.9) to deduce that $(I - \Phi_{q_1,\pi(\mathbf{W}_1)}) \cdots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})(I_{\mathcal{K}_\pi}) = 0$. The proof is complete.

We remark that if we replace $\mathbf{q} = (q_1, \dots, q_k)$, in Theorem (4.1.39), by a k-tuple $\mathbf{f} := (f_1, \dots, f)$ of positive regular free holomorphic functions we obtain a dilation theorem for any $\mathbf{T} = (T_1, \dots, T_k)$ in $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$. More precisely, one can show that there is a ∗-representation $\tilde{\pi} : C^*(\mathbf{W}_{i,j})^* \to B(\tilde{\mathcal{K}})$ such that $\mathcal{H}$ is an invariant subspace under each operator $\tilde{\pi}(\mathbf{W}_{i,j})^*$ and $T^*_{i,j} = \tilde{\pi}(\mathbf{W}_{i,j})^*|_{\mathcal{H}}$ for any $i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$. On the other hand , note that, using the proof of Theorem (4.1.39) and due to the standard theory of representations of C^*-algebras, one can deduce the following Wold type decomposition for non-degenerate $*$-representations of the C^*-algebra $C^*(\mathbf{W}_{i,j})$.

Corollary (4.1.40) [186]: Let $\mathbf{q} = (q_1, \ldots, q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $\mathbf{W} = (\mathbf{W}_{i,j})$ be the universal model associated with the abstract noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. If $\pi : C^*\left(\mathbf{W}_{i,j}\right) \to B(\mathcal{K})$ is a nondegenerate $*$-representation of $C^*\left(\mathbf{W}_{i,j}\right)$ on a separable Hilbert space $\mathcal{K}$, then π decomposes into a direct sum

$$\pi = \pi_0 \oplus \pi_1 \ on \ \mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1,$$

where π_0 and π_1 are disjoint representations of $C^*\left(\mathbf{W}_{i,j}\right)$ on the Hilbert spaces

$$\mathcal{K}_0 \coloneqq \overline{\text{span}}\{ \pi\left(\mathbf{W}_{(\alpha)}\right)\left[\left(I - \Phi_{q_1,\pi(\mathbf{W}_1)}\right)^{m_1} \ldots \left(I - \Phi_{q_k,\pi(\mathbf{W}_k)}\right)^{m_k}(I_{\mathcal{K}})\right]$$
$$\mathcal{K}: (\alpha) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}\}$$

And $\mathcal{K}_1 \coloneqq \mathcal{K}_0^{\perp}$, respectively, where $\pi(\mathbf{W}_i) := \left(\pi\left(\mathbf{W}_{i,1}\right), \ldots, \pi\left(\mathbf{W}_{i,n_i}\right)\right)$ Moreover, up to an isomorphism,

$$\mathcal{K}_0 \simeq (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{G},\ \pi_0(X) = X \otimes I_{\mathcal{G}} \text{ for any } X \in C^*\left(\mathbf{W}_{i,j}\right),$$

where $\mathcal{G}$ is a Hilbert space with

$$\dim \mathcal{G} = \dim\left\{\text{range}\left[(I - \Phi_{q_1,\pi(\mathbf{W}_1)})^{m_1} \ldots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})^{m_k}(I_{\mathcal{K}})\right]\right\},$$

and π_1 is a $*$-representation which annihilates the compact operators and

$$(I - \Phi_{q_1,\pi(\mathbf{W}_1)}) \ldots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})\left(I_{\mathcal{K}_1}\right) = 0.$$

If π' is another nondegenerate $*$-representation of $C^*\left(\mathbf{W}_{i,j}\right)$ on a separable Hilbert space $\mathcal{K}'$, then π is unitarily equivalent to π' if and only if $\dim \mathcal{G} = \dim \mathcal{G}'$ and π_1 is unitarily equivalent to π_1'.

Note that in the particular case when $\mathbf{m} = (1, \ldots, 1), q_i := Z_{i,1} + \cdots + Z_{i,n_i}$ for $i \in \{1, \ldots, k\}$, and $V_i = (V_{i,1} \ldots, V_{i,n_i})$ are row isometries such that $\mathbf{V} = (V_{i,j})$ are doubly commuting, Corollary (4.1.40) provides a Wold type decomposition for $\mathbf{V}$. We also remark that under the hypotheses and notations of Corollary (4.1.40), and setting $V_{i,j} := \pi(\mathbf{W}_{i,j})$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, the following statements are equivalent:

(i) $\mathbf{V} = (V_1 \ldots, V_k)$ is a pure element in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{K})$;

(ii) for each $i \in \{1, \ldots, k\}$, $\lim_{p\to\infty} \Phi_{q_i,V_i}^p(I) = 0$ in the strong operator topology;

(iii) $\mathcal{K} := \overline{\text{span}}\left\{ V_{(\alpha)} \left[\left(I - \Phi_{q_i,V_i}\right)^{m_1} \ldots \left(I - \Phi_{q_k,V_k}\right)^{m_k} (I_{\mathcal{K}})\right] (\mathcal{K}) : (\alpha) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}\right\}$.

We mention that, under the additional condition that $\overline{\text{span}}\{ \mathbf{W}_{(\alpha)}\mathbf{W}_{(\beta)}^*: (\alpha)(\beta) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+ \}$ is equel to $C^*\left(\mathbf{W}_{i,j}\right)$, (eg. for the polyball) the map $\Psi_{\mathbf{q},\mathbf{T}}$ in the proof of Theorem (4.1.39) is unique and the dilation of $\mathbf{T}$ is minimal, i.e., $\tilde{\mathcal{K}}$ is the closed span of all $V_{(\alpha)}\mathcal{H}$ $(\alpha) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+$ Taking into account the uniqueness of the minimal Stinespring representation and the Wold type decomposition mentioned above, one can prove the uniqueness, up to unitary equivalence, of the minimal dilation provided by Theorem (4.1.39). Moreover, let $\mathbf{T}' = (T_1', \ldots, T_k')$ be another k-tuple in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H}')$ and let $\mathbf{V}' = (V_1', \ldots, V_k')$ be the corresponding dilation. Using standard arguments concerning the representation theory of C^*-algebras , one can prove that $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent if and only if $\dim \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})} = \dim \overline{\Delta_{\mathbf{q},\mathbf{T}'}^{\mathbf{m}}(I)(\mathcal{H}')}$ and there are unitary operators

$$U : \left(\otimes_{i=1}^k F^2\left(H_{n_i}\right)\right) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})} \to \left(\otimes_{i=1}^k F^2\left(H_{n_i}\right)\right) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}'}^{\mathbf{m}}(I)(\mathcal{H}')}$$

$\Gamma: \mathcal{K}_{\pi} \to \mathcal{K}_{\pi'}$ such that

$$U(\mathbf{W}_{i,j} I_{\overline{\Delta_{q,T}^m(I)(\mathcal{H})}} = \mathbf{W}_{i,j} I_{\overline{\Delta_{\boldsymbol{q},\boldsymbol{T}'}^{\boldsymbol{m}}(I)(\mathcal{H}')}}) U, \Gamma_{\pi}\left(\mathbf{W}_{i,j}\right) = \pi'\left(\mathbf{W}_{i,j}\right)\Gamma$$

for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, and $\begin{bmatrix} U & 0 \\ 0 & T \end{bmatrix} \mathcal{H} = \mathcal{H}'$.

Corollary (4.1.41) [186]: Let $\mathbf{V} := (V_1 \ldots, V_k) \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\widetilde{\mathcal{K}})$ be the dilation of $\mathbf{T} := (T_1 \ldots, T_k) \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ given by Theorem (4.1.39). Then,

(i) $\mathbf{V}$ is a pure element in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\widetilde{\mathcal{K}})$ if and only if $\mathbf{T}$ is a pure element in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$;

(ii) $(I - \Phi_{q_1,v_1}) \cdots (I - \Phi_{q_k,V_k})(I_{\widetilde{\mathcal{K}}}) = 0$ if and only if $(I - \Phi_{q_1,T_1}) \cdots (I - \Phi_{qk,T_k})(I_{\mathcal{H}}) = 0$

Proof: According to Theorem (4.1.39) , we have

$$(id - \Phi_{q_k,T_k}^{p_k}) \cdots (id - \Phi_{q_1,T_1}^{p_1})(I_{\mathcal{H}}) = P_{\mathcal{H}}\begin{bmatrix} (id - \Phi_{q_k,\mathbf{W}_k}^{p_k}) \cdots (\text{id} - \Phi_{q_1,\mathbf{W_1}}^{p_1})(I_{\otimes_{i=1}^k F^2(H_{n_i})}) \otimes I_{\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(\mathcal{H})}} & 0 \\ 0 & 0 \end{bmatrix} |\mathcal{H}.$$

Hence, we deduce that $\lim_{\mathbf{q}=(q_1,\ldots,q_k)\in\mathbb{Z}_+^k} (id - \Phi_{q_k,T_k}^{p_k}) \ldots (\text{id} - \Phi_{q_1,T_1}^{p_1})(I_{\mathcal{H}}) =$ I if and only if $P_{\mathcal{H}}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} |\mathcal{H} = I$ consequently , $\mathbf{T}$ is pure if and only if $\mathcal{H} \perp (0 \oplus \mathcal{K}_\pi)$. According to Theorem (4.1.39) this is equivalent to $\mathcal{H} \subset (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$ On the other hand, since $(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$ is reducing for each $V_{i,j}$, and $\widetilde{\mathcal{K}}$ is the smallest reducing subspace For $V_{i,j}$ which contins $\mathcal{H}$, we must have $\widetilde{\mathcal{K}} = (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$. Therefore, item (i) holds.

To prove part (ii), note that

$$\Delta_{\mathbf{q},\mathbf{V}}^{\mathbf{m}}(I_{\widetilde{\mathcal{K}}}) = \begin{bmatrix} \Delta_{\mathbf{q},\mathbf{W}}^{\mathbf{m}}(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathrm{I}_{\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(\mathcal{H})}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we deduce that $\Delta_{\mathbf{q},\mathbf{V}}^{\mathbf{m}}(I_{\widetilde{\mathcal{K}}}) = 0$ if and only if $\Delta_{q,W}^{m}(I_{\otimes_{i=1}^k F^2(H_{n_i})}) \otimes I_{\overline{\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(\mathcal{H})}}$ On the other hand, we know that $\Delta_{\mathbf{q},\mathbf{W}}^{\mathbf{m}}(I_{\otimes_{i=1}^k F^2(H_{n_i})}) = \mathbf{P}_{\mathbb{C}}$. Consequently, the relation above holds if and only if $\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}} = 0$. Now, using Proposition (4.1.9), we obtain the equivalence in part (ii). The proof is complete.

We remark that every pure k-tuple $\mathbf{T} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}},(\mathcal{H})$ with rank $\Delta_{\mathbf{q},\mathbf{T}}^{\mathbf{m}} = 1$ unitarily equivalent to one obtained by compressing $(\mathbf{W}_1, \ldots, \mathbf{W}_n)$ to a co-invariant subspace under $\mathbf{W}_{i,j}$ where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. Indeed , this follows from Theorem (4.1.39), Corollary (4.1.41), and the remarks preceding Corollary (4.1.41).

Section (4.2): Noncommutative Varieties in Polydomains

We denote by $B(H)^{n_1} \times_c \cdots \times_c B(H)^{n_k}$ the set of all tuples $X := (X_1, \ldots, X_k)$ in $B(H)^{n_1} \times \cdots \times B(H)^{n_k}$ with the property that the entries of $X_s := (X_{s,1}, \ldots, X_{s,n_s})$ are commuting with the entries of $X_t := (X_{t,1}, \ldots, X_{t,n_t})$ for any $s, t \in \{1, \ldots, k\}, s \neq t$. In an attempt to unify the multivariable operator model theory for the ball-like domains and commutative polydiscs, we developed in [249] an operator model theory and a theory of free holomorphic functions on regular polydomains of the form

$$D_q^m(H) := \{X = (X_1, \ldots, X_k) \in B(H)^{n_1} \times_c \cdots \times_c B(H)^{n_k} : \Delta_{q,X}^p(I) \geq 0 \; for \; 0 \leq p \leq m\},$$

where $m := (m_1, \ldots, m_k)$ and $n := (n_1, \ldots, n_k)$ are in $\mathbb{N}^k$, the defect mapping $\Delta_{q,X}^m : B(H) \to B(H)$ is defined by

$$\Delta_{q,X}^m := (id - \Phi_{q_1,X_1})^{m_1} \circ \cdots \circ (id - \Phi_{q_k,X_k})^{m_k},$$

and $q = (q_1, \ldots, q_k)$ is a k-tuple of positive regular polynomials $q_i \in \mathbb{C}[Z_{i,1}, \ldots, Z_{i,n_i}]$, i.e., all the coefficients of q_i are positive, the constant term is zero, and the coefficients of the linear terms $Z_{i,1}, \ldots, Z_{i,n_i}$ are different from zero. If the polynomial q_i has the form $q_1 = \sum_\alpha a_{i,\alpha} Z_{i,\alpha}$, the completely positive linear map $\Phi_{q_i,X_i} : B(H) \to B(H)$ is defined by setting $\Phi_{q_i,X_i}(Y) := \sum_\alpha a_{i,\alpha} X_{i,\alpha} Y X_{i,\alpha}^*$ for $Y \in B(H)$.

We study noncommutative varieties in the polydomain $D_q^m(H)$, given by

$$V_Q(H) := \{X \in D_q^m(H): \quad g(X) = 0 \; for \; all \; g \in Q\},$$

where Q is a set of polynomials in noncommutative indeterminates $Z_{i,j}$, which generates a nontrivial ideal in $\mathbb{C}[Z_{i,j}]$. We understand the structure of this noncommutative variety, determine its elements and classify them up to unitary equivalence, for large classes of sets $Q \subset \mathbb{C}[Z_{i,j}]$. This study can be seen as an attempt to initiate noncommutative algebraic geometry in polydomains.
Let H_{n_i} be an n_i-dimensional complex Hilbert space.
We consider the full Fock space of H_{n_i} defined by

$$F^2(H_{n_i}) := \bigoplus_{p \geq 0} H_{n_i}^{\otimes p},$$

where $H_{n_i}^{\otimes 0} := \mathbb{C}1$ and $H_{n_1}^{\otimes p}$ is the (Hilbert) tensor product of p copies of H_{n_i}. Let $\mathbb{F}_{n_i}^+$ be the unital free semigroup on n_i generators $g_1^i, \ldots, g_{n_i}^i$ and the identity g_0^i. We use the notation $Z_{i,\alpha_i} := Z_{i,j_1} \cdots Z_{i,j_p}$ if $\alpha_i \in \mathbb{F}_{n_i}^+$ and $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i$, and $Z_{i,g_0^i} := 1$. If $(\alpha) := (\alpha_1, \ldots, \alpha_k)$ is in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, we set $Z(\alpha) := Z_{1,\alpha_1} \cdots Z_{k,\alpha_k}$. after setting up the notation and recalling some basic results from [183], we show that the abstract variety $V_Q := \{V_Q(H): H$ is a Hilbert space$\}$ has a universal model $S = \{S_{i,j}\}$ such that $g(S) = 0, g \in Q$, where each $S_{i,j}$ is acting on a subspace N_Q of a tensor product of full Fock spaces. For each element $T \in V_Q(H)$ we introduce the constrained noncommutative Berezin transform at T as the *map* $B_{T,Q}: B(N_Q) \to B(H)$ defined by setting

$$B_{T,Q}[\varphi] := K_{q,T,Q}^*(\varphi \otimes I_H)K_{q,T,Q}, \quad \varphi \in B(N_J),$$

where $K_{f,T,Q}$ is the constrained Berezin kernel. This Berezin [189] type transform will play an important role. We show that the pure elements of the noncommutative variety $V_Q(H)$ are detected by a class of completely positive linear maps. More precisely, given $T = \{T_{i,j}\} \in B(H)^{n_1} \times \cdots \times B(H)^{n_k}$, we prove that T is a pure element of $V_Q(H)$ if and only if there is a unital completely positive and w^*-continuous linear map

$$\Psi : \overline{span}^{w^*}\{S_{(\alpha)}S_{(\beta)}^*: (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\} \to B(H)$$

such that

$$\Psi\big(S_{(\alpha)}S_{(\beta)}^*\big) = T_{(\alpha)}T_{(\beta)}^*, \qquad (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+.$$

Every map Ψ with the above-mentioned properties is the constrained Berezin transform $B_{T,Q}$at a pure element $T \in V_Q(H)$. A similar result (see Theorem (4.2.4)) characterizing the noncommutative variety $V_Q(H)$ is provided under the condition that $Q \subset \mathbb{C}[Z_{i,j}]$ is a left ideal generated by homogeneous polynomials.
We use the noncommutative Berezin transforms to show that a tuple $T = \{T_{i,j}\}$ in $B(H)^{n_1} \times \cdots \times B(H)^{n_k}$ is a pure element in $V_Q(H)$ if and only if it is unitarily equivalent to the compression of a multiple of the universal model to a co-invariant subspace. In this case, we have

$$T_{(\alpha)} = B_{T,Q}\big[S_{(\alpha)} \otimes I_D\big], \qquad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+,$$

the constrained Berezin kernel $K_{q,T,Q}$ is an isometry, and the subspace $K_{q,T,Q}H$ is co-invariant under each operator $S_{i,j} \otimes I_D$, where D is the closure of the range of the defect operator $\Delta_{q,T}^m(I)$. For a certain class of noncommutative varieties $V_Q(H)$, this leads to a characterization of the pure elements $T \in V_Q(H)$ with $dimD = n \in \mathbb{N}$. In particular, we obtain the following description and classification of the pure elements $T \in V_Q(H)$ with $dimD = 1$. We show that they have the form $T = \{P_M S_{i,j}|_M\}$, where M is a co-invariant subspace under each operator $S_{i,j}$. Moreover, if M' is another co-invariant subspace under S_{ij}, which gives rise to an element $T' \in V_Q(H)$, then T and T' are unitarily equivalent if and only if $M = M'$. This extends a result of Douglas and Foias [249] for the Hardy space $H^2(\mathbb{D}^n)$ over the polydisc.
We also obtain a characterization of the Beurling [240] type joint invariant subspaces under the universal model $S = \{S_{i,j}\}$. We prove that a subspace $M \subset N_Q \otimes H$ has the form $M = M\ (N_Q \otimes$

$\mathcal{E}$) for some partially isometric multi-analytic operator $M: N_Q \otimes \mathcal{E} \to N_Q \otimes H$ with respect to the universal model S, i.e., $M(S_{i,j} \otimes I_H) = (S_{i.j} \otimes I_K)M$ for all i, j, if and only if

$$\Delta^{p}_{q,S\otimes I_H}(P_M) \geq 0, \qquad for\ any\ p \in \mathbb{Z}^k_+, p \leq m,$$

where P_M is the orthogonal projection of the Hilbert space $N_Q \otimes H$ onto M.
There is a strong connection between the noncommutative varieties in polydomains, the theory of functions in several complex variables, and the classical complex algebraic geometry. Note that the representation of the abstract variety V_Q on the complex plane $\mathbb{C}$ is the compact set

$$V_Q(\mathbb{C}) = D_q(C) \cap \{\lambda \in \mathbb{C}^n: g(\lambda) = 0\ for\ all\ g \in Q\}$$

and $D^\circ_q(\mathbb{C}) = \{\lambda \in \mathbb{C}^n: \Delta_{q,\lambda}(1) > 0\}$ is a Reinhardt domain in $\mathbb{C}^n$, where $n = n_1 + \cdots + n_k$ is the number of indeterminates in $q = (q_1, \ldots, q_k)$.
We determine all the joint invariant subspaces of co-dimension one of the universal model $S = \{S_{i,j}\}$. We show that the joint eigenvectors for $S^*_{i,j}$ are precisely the noncommutative constrained Berezin kernels $K_{q,\lambda,Q}$, where $\lambda \in V_Q(\mathbb{C})) \cap D^\circ_q(\mathbb{C})$. We introduce the variety algebra $A(V_Q)$ as the norm closed algebra generated by the $S_{i,j}$ and the identity, and the Hardy algebra $F^\infty(V_Q)$ as the WOT-closed version. We identify the w^*-continuous and multiplicative linear functionals of the Hardy algebra $F^\infty(V_Q)$ as the maps, indexed by $\lambda \in V_Q(\mathbb{C}) \cap D^\circ_q(\mathbb{C})$, defined by $\Phi_\lambda(A) := B_{\lambda,Q}[A]$ for $A \in F^\infty(V_Q)$. If $Q \subset \mathbb{C}[Z_{i,j}]$ is a left ideal generated by noncommutative homogenous polynomials, then we show that the right joint spectrum $\sigma_r(S)$ coincides with $V_Q(\mathbb{C})$. On the other hand, it turns out that the variety $V_Q(\mathbb{C})$ is homeomorphic to the space $M_{A(V_Q)}$ of all characters of the variety algebra $A(V_Q)$, via the mapping $\lambda \mapsto \Phi_\lambda$, where Φ_λ is the evaluation functional.
Special attention is given to the commutative case when $Q = Q_c$, the left ideal generated by the commutators $Z_{i,j}Z_{s,t} - Z_{s,t}Z_{i,j}$ of the indeterminates in $\mathbb{C}[Z_{i,j}]$. In this case, the universal model associated with V_{Q_c}, denoted by $L = \{L_{i,j}\}$, is acting on the Hilbert space N_{Q_c} which coincides with the closed span of all vectors K_{q,λ,Q_c} with $\lambda \in D^\circ_q(\mathbb{C})\}$, and it is identified with a Hilbert space $H^2(D^\circ_q(\mathbb{C}))$ of holomorphic functions on $D^\circ_q(\mathbb{C})$, namely, the reproducing kernel Hilbert space with kernel defined by

$$\kappa^c_q(\mu, \lambda) := \frac{1}{\prod_{i=1}^k \left(1 - q_i\left(\mu_i \bar{\lambda}_i\right)\right)^{m_i}}, \qquad \mu, \lambda \in D^\circ_q(\mathbb{C}).$$

We prove that the Hardy algebra $F^\infty(V_{Q_c})$ is reflexive and coincides with the multiplier algebra of the Hilbert space $H^2(D^\circ_q(\mathbb{C}))$. Under this identification, $L_{i,j}$ is the multiplier by the coordinate function $\lambda_{i,j}$. We remark that when $n_1 = \cdots = n_k$ and Q_{cc} is the left ideal generated by Q_c and the polynomials $Z_{i,j} - Z_{p,j}$, the universal model associated with $V_{Q_{cc}}$ is acting on the Hilbert space $N_{Q_{cc}}$ which can be identified with the reproducing kernel Hilbert space with kernel

$$\kappa^{cc}_q(\mu, w) := \frac{1}{\prod_{i=1}^k \left(1 - q_i(z\bar{w})\right)^{m_i}}, \qquad z, w \in \bigcap_{i=1}^k D^\circ_{q_i}(\mathbb{C})\,.$$

In the particular case when $f_1 = \cdots = f_k = Z_1 + \cdots + Z_n$ and $m_1 = \cdots = m_k = 1$, we obtain the reproducing kernel $(z, w) \mapsto \frac{1}{(1-\langle z,w\rangle)^k}$ on the unit ball $\mathbb{B}_n$. In this case, the reproducing kernel Hilbert spaces are the Hardy-Sobolev spaces (see [247]), which include the Drurry-Arveson space (see [250], [255], [248],[246]), the Hardy space of the ball and the Bergman space (see [255]). All the results are true in these commutative settings.
We show that the isomorphism problem for the universal polydomain algebras is closed connected to the biholomorphic equivalence of Reinhardt domains in several complex variables. Let $q = (q_1, \ldots, q_k)$ and $g = (g_1, \ldots, g_{k'})$ be tuples of positive regular polynomials with n and ℓ indeterminates, respectively, and let $m \in \mathbb{N}^k$ and $\in \mathbb{N}^{k'}$. We prove that if the polydomain algebras $A(D^m_q)$ and $A(D^d_g)$ are unital completely contractive isomorphic, then the Reinhardt domains $D^\circ_q(\mathbb{C})$ and $D^\circ_g(\mathbb{C})$ are biholomorphic equivalent and $n = \ell. A$ similar result holds for the commutative

variety algebras $A(V^m_{q,Q_c})$ and $A(V^d_{g,Q_c})$. We remark that when $q = Z_1 + \cdots + Z_n$ and $g = (Z_1, \ldots, Z_n)$, the corresponding domain algebras are the universal algebra of a commuting row contraction $A(V^1_{q,Q_c})$ and the commutative polydisc algebra $A(V^1_{g,Q_c})$, respectively. Since $\mathbb{B}_n$ and $\mathbb{D}^n$ are not biholomorphic equivalent domains in $\mathbb{C}^n$ if $n \geq 2$ (see [251]), our result implies that the two algebras are not isomorphic. The classification problem for polydomain algebras will be pursued.
We develop a dilation theory for noncommutative varieties in polydomains. For the class of noncommutative varieties $V_Q(H)$, where $Q \subset \mathbb{C}[Z_{i,j}]$ is an ideal generated by homogeneous polynomials, the dilation theory is refined. We obtain Wold type decompositions for non-degenerate $*$-representations of the C^*-algebra $C^*(V_Q)$ generated by the universal model $S_{i,j}$ and the identity, and coisometric dilations for the elements of $V_Q(H)$. Under natural conditions, the dilation is unique up to unitary equivalence. In the particular case when $k = m = 1, q = Z_1 + \cdots + Z_n$, and $Q = Q_c$, we recover Arveson's results [255] concerning the dilation theory for commuting row contractions.
We provide a characterization for the class of tuples of operators in the noncommutative variety $V_Q(H)$ which admit constrained characteristic functions. The characteristic function is a complete unitary invariant for the completely non-coisometric tuples. We also provide operator models in terms of the constrained characteristic functions. These results extend the corresponding ones from [258], [255], [257], [258], [251], [252], [251], and [253], to varieties in noncommutative polydomains.
We remark that the results are presented in a more general setting, when q is replaced by a k-tuple $f = (f_1, \ldots, f_k)$ of positive regular free holomorphic functions in a neighborhood of the origin, and Q is replaced by a WOT-closed left ideal of the Hardy algebra $F^\infty(D^m_f)$.
We mention that noncommutative varieties in ball-like domains were studied in several (see [256], [257], [258], [259], [250], [251], [252], and the references there in). The commutative case when $m_1 \geq 2, n_1 \geq 2$, and $q_1 = Z_1 + \cdots + Z_n$, was studied by Athavale [256], Muller [252], Muller-Vasilescu [253], Vasilescu [250], and Curto-Vasilescu [255]. Some of these results were extended by S. Pott [254] when q_1 is a positive regular polynomial in commuting indeterminates (see also [252]). The commutative polydisc case, i.e, $k \geq 2, n_1 = \cdots = n_k = 1$, and $q = (Z_1, \ldots, Z_n)$, was first considered by Brehmer [254] in connection with regular dilations. Motivated by Agler's work [251] on weighted shifts as model operators, Curto and Vasilescu developed a theory of standard operator models in the polydisc in [256], [257]. Timotin [259] obtained some of their results from Brehmer's theorem. The polyball case, when $k \geq 2$ and $q_i = Z_1 + \cdots + Z_{n_i}, i \in \{1, \ldots, k\}$, was considered in [256] and [258] for the noncommutative and commutative case, respectively.
We consider noncommutative varieties $V^m_{f,J}(H) \subset D^m_f(H)$ determined by left ideals J in either one of the following algebras: $\mathbb{C}[Z_{i,j}], \mathbb{C}[W_{i,j}], A(D^m_f)$, or $F^\infty(D^m_f)$. We associate with each such a variety a universal model $S = (S_1, \ldots, S_n) \in V^m_{f,J}(N_J)$, where N_J is an appropriate subspace of a tensor product of full Fock spaces. We introduce a constrained noncommutative Berezin transform and use it to characterize noncommutative varieties in polydomains.
We begin by recalling from [253] some definitions and basic properties of the universal model associated with the abstract noncommutative polydomain D^m_f and of the associated Berezin kernel. For each $i \in \{1, \ldots, k\}$, let $\mathbb{F}^+_{n_i}$ be the unital free semigroup on n_i generators $g^i_1, \ldots, g^i_{n_i}$ and the identity g^i_0. The length of $\alpha \in \mathbb{F}^+_{n_i}$ is defined by $|\alpha| := 0 \; if \; \alpha = g^i_0$ and $|\alpha| := p \; if \; \alpha = g^i_{j_1} \cdots g^i_{j_p}$, where $j_1, \ldots, j_p \in \{1, \ldots, n_i\}$. If $Z_{i,1}, \ldots, Z_{i,n_i}$ are noncommuting indeterminates, we denote $Z_{i,\alpha} := Z_{i,j_1} \cdots Z_{i,j_p}$ and $Z_{i,g^i_0} := 1$. Let $f_i := \sum_{\alpha \in \mathbb{F}^+_{n_i}} a_{i,\alpha} Z_\alpha, a_{i,\alpha} \in \mathbb{C}$, be a formal power series in ni noncommuting indeterminates $Z_{i,1}, \ldots, Z_{i,n_i}$. We say that f_i is a positive regular free holomorphic function if $a_{i,\alpha} \geq 0$ for any $\alpha \in \mathbb{F}^+_{n_i}, a_{i,g^i_0} = 0, \; a_{i,g^i_j} > 0$ for $j \in \{1, \ldots, n_i\}$, and
$\limsup_{k \to \infty} \left(\sum_{|\alpha| = k} |a_{i,\alpha}|^2 \right)^{1/2k} < \infty$. We denote by $B(H)$ the algebra of bounded linear operators on a separable Hilbert space H.

Given $X_i := (X_{i,1}, \ldots, X_{i,n_i}) \in B(H)^{n_i}$, define the map $\Phi_{f_i,X_i}: B(H) \to B(H)$ by setting

$$\Phi_{f_i,X_i}(Y) := \sum_{k=1}^{\infty} \sum_{\alpha\in\mathbb{F}_{n_i}^+, |\alpha|=k} a_{i,\alpha} X_{i,\alpha} Y X_{i,\alpha}^*, \qquad Y \in B(H),$$

where the convergence is in the week operator topology. Let $n := (n_1, \ldots, n_k)$ and $m := (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{N} := \{1, 2, \ldots\}$ and $i \in \{1, \ldots, k\}$, and let $f := (f_1, \ldots, f_k)$ be a k-tuple of positive regular free holomorphic functions. We associate with each element $X = (X_1, \ldots, X_k) \in B(H)^{n_1} \times \cdots \times B(H)^{n_k}$ and $p = (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ the defect mapping $\Delta_{f,X}^p: B(H) \to B(H)$ defined by

$$\Delta_{f,X}^p := (id - \Phi_{f_1,X_1})^{p_1} \circ \cdots \circ (id - \Phi_{f_k,X_k})^{p_k}.$$

We use the convention that $(id - \Phi_{f_i,X_i})^0 = id$. We denote by $B(H)^{n_1} \times_c \cdots \times_c B(H)^{n_k}$ the set of all tuples $X = (X_1, \ldots, X_k) \in B(H)^{n_1} \times \cdots \times B(H)^{n_k}$, where $X_i := (X_{i,1}, \ldots, X_{i,n_i}) \in B(H)^{n_i}, i \in \{1, \ldots, k\}$, with the property that, for any $p, q \in \{1, \ldots, k\}, p \neq q$, the entries of X_p are commuting with the entries of X_q. In this case we say that X_p and X_q are commuting tuples of operators. Note that, for each $i \in \{1, \ldots, k\}$, the operators $X_{i,1}, \ldots, X_{i,n_i}$ are not necessarily commuting.

In [273], we developed an operator model theory and a theory of free holomorphic functions on the noncommutative polydomain

$$D_f^m(H) := \{X = (X_1, \ldots, X_k) \in B(H)^{n_1} \times_c \cdots \times_c B(H)^{n_k}: \Delta_{f,X}^p(I) \geq 0 \; for \; 0 \leq p \leq m\}.$$

We refer to $D_f^m := \{D_f^m(H): H$ is a Hilbert space$\}$ as the abstract noncommutative polydomain, while $D_f^m(H)$ is its representation on the Hilbert space H.

Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e_1^i, \ldots, e_{n_i}^i$. We consider the full Fock space of H_{n_i} defined by

$$F^2(H_{n_i}) := \mathbb{C}1 \oplus \bigoplus_{p\geq 1} H_{n_i}^{\otimes p},$$

where $H_{n_i}^{\otimes p}$ is the (Hilbert) tensor product of p copies of H_{n_i}. Set $e_\alpha^i := e_{j_1}^i \otimes \cdots \otimes e_{j_p}^i$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $e_{g_0^i}^i := 1 \in \mathbb{C}$. Note that $\{e_\alpha^i: \alpha \in \mathbb{F}_{n_i}^+\}$ is an orthonormal basis of $F^2(H_{n_i})$. Let $m_i, n_i \in \mathbb{N} := \{1, 2, \ldots\}, i \in \{1, \ldots, k\}$, and $j \in \{1, \ldots, n_i\}$. We define the weighted left creation operators $W_{i,j}: F^2(H_{n_i}) \to F^2(H_{n_i})$, associated with the abstract noncommutative domain $D_{f_i}^{m_i}$ by setting

$$W_{i,j} e_\alpha^i := \frac{\sqrt{b_{i,\alpha}^{(m_i)}}}{\sqrt{b_{i,g_j\alpha}^{(m_i)}}} e_{g_j\alpha}^i, \qquad \alpha \in \mathbb{F}_{n_i}^+,$$

where

$$b_{i,g_0}^{(m_i)} := 1 \; and \;\; b_{i,\alpha}^{(m_i)} := \sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_p\in\mathbb{F}_{n_i}^+ \\ \gamma_1\cdots\gamma_p=\alpha \\ |\gamma_1|\geq 1,\ldots,|\gamma_p|\geq 1}} a_{i,\gamma_1} \cdots a_{i,\gamma_p} \binom{p + m_i - 1}{m_i - 1} \quad (47)$$

for all $\alpha \in \mathbb{F}_{n_i}^+$ with $|\alpha| \geq 1$. For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, we define the operator $W_{i,j}$ acting on the tensor Hilbert space $F^2(H_{n_i}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$W_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1\; times} \otimes W_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i\; times}.$$

The k-tuple $W := (W_1, \ldots, W_k)$, where $W_i := (W_{i,1}, \ldots, W_{i,n_i})$, is an element in the noncommutative polydomain $D_f^m(\otimes_{i=1}^k F^2(H_{n_i}))$ and it is called the universal model associated with the abstract noncommutative polydomain D_f^m . We say that $T = (T_1, \ldots, T_k) \in D_f^m(H)$ is completely non-coisometric if there is no $h \in H, h \neq 0$ such that

$$\langle (id - \Phi_{f_1,T_1}^{q_1}) \cdots (id - \Phi_{f_k,T_k}^{q_k})(I_H)h, h\rangle = 0$$

for any $(q_1, \ldots, q_k) \in \mathbb{N}^k$. The k-tuple T is called pure if

$$\lim_{q=(q_1,\ldots,q_k)\in Z_+^k} \left(id - \Phi_{f_k,T_k}^{q_k}\right) \ldots (id - \Phi_{f_1,T_1}^{q_1})(I) = I .$$

The noncommutative Berezin kernel associated with any element $T = \{T_{i,j}\}$ in the noncommutative polydomain $D_t^m(\mathcal{H})$ is the operator

$$K_{f,T}: \mathcal{H} \to F^2(H_{n_1}) \otimes \ldots \otimes F^2(H_{n_k}) \otimes \overline{\Delta_{f,T}^m(I)(\mathcal{H})}$$

defined by

$$K_{f,T}h \coloneqq \sum_{\beta_i \in F_{n_i}^+, i=1,\ldots,k} \sqrt{b_{1,\beta_1}^{(m_1)}} \ldots \sqrt{b_{k,\beta_k}^{(m_k)}} e_{\beta_1}^1 \otimes \ldots \otimes e_{\beta_k}^k \otimes \Delta_{f,T}^m(I)^{1\backslash 2} T_{1,\beta_1}^* \ldots T_{k,\beta_k}^* h,$$

where the defect operator is defined by

$$\Delta_{f,T}^m(I) \coloneqq \left(id - \Phi_{f_1,T_1}\right)^{m_1} \ldots \left(id - \Phi_{f_k,T_k}\right)^{m_k}(I),$$

and the coefficients $b_{1,\beta_1}^{(m_1)}, \ldots, b_{k,\beta_k}^{(m_k)}$ are given by relation (47). The noncommutative Berezin kernel $K_{f,T}$ is a contraction and

$$K_{f,T}^* K_{f,T} = \lim_{q_k\to\infty} \ldots \lim_{q_k\to\infty} \left(id - \Phi_{f_k,T_k}^{q_k}\right) \ldots \left(id - \Phi_{f_1,T_1}^{q_1}\right)(I),$$

where the limits are in the weak operator topology. Moreover, for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$,

$$K_{f,T} T_{i,j}^* = (W_{i,j}^* \otimes I) K_{f,T}.$$

The *noncommutative Berezin transform* at $T \in D_f^m(\mathcal{H})$ is the mapping $B_T: B(\otimes_{i=1}^k F^2(H_{n_i})) \to B(\mathcal{H})$ given by

$$B_T[g] \coloneqq K_{f,T}^*(g \otimes I_{\mathcal{H}}) K_{f,T}, \quad g \in B(\otimes_{i=1}^k F^2(H_{n_i})).$$

The polydomain algebra $A(D_f^m)$ is the norm closed algebra generated by $W_{i,j}$ and the identity. Let

$$S \coloneqq \overline{span}\{W_{(\alpha)} W_{(\beta)}^* : (\alpha), (\beta) \in F_{n_1}^+ \times \ldots \times F_{n_k}^+\},$$

where the closure is in the operator norm. We proved in [273] that there is a unital completely contractive linear map $\Psi_{f,T}: S \to B(\mathcal{H})$ such that

$$\Psi_{f,T}(g) = \lim_{r\to 1} B_{rT}[g], \qquad g \in S,$$

where the limit exists in the norm topology of $B(\mathcal{H})$,and

$$\Psi_{f,T}\left(W_{(\alpha)} W_{(\beta)}^*\right) = T_{(\alpha)} T_{(\beta)}^*, \quad (\alpha), (\beta) \in F_{n_1}^+ \times \ldots \times F_{n_k}^+,$$

where $W_{(\alpha)} \coloneqq W_{1,\alpha_1} \ldots W_{k,\alpha_k}$ for $(\alpha) \coloneqq (\alpha_1, \ldots, \alpha_k)$. In particular, the restriction of $\Psi_{f,T}$ to the polydomain algebra $A(D_f^m)$ is a completely contractive homomorphism. For information on completely bounded (resp. positive) maps, see [254].

The noncommutative Hardy algebra $F^\infty(D_f^m)$ is the sequential SOT-(resp. WOT-, $w^* -$) closure of all polynomials in $W_{i,j}$ and the identity, where $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_k\}$. Each elemeny $\varphi(W_{i,j})$ in $F^\infty(D_f^m)$has a unique Fourier type representation

$$\varphi(W_{i,j}) = \sum_{(\beta)\in F_{n_1}^+\times\ldots\times F_{n_k}^+} c_{(\beta)} W_{(\beta)}, \qquad c_{(\beta)} \in \mathbb{C},$$

and $\varphi(W_{i,j}) = SOT - \mathrm{Lim}_{r\to 1}\, \varphi(rW_{i,j})$, where $\varphi(rW_{i,j})$is in the polydomain algebra $A\left(D_f^m\right)$. We recall [253] the following result concerning the $F^\infty(D_f^m)$–functional calculus for the completely non-coisometric part of the noncommutative polydomain $D_f^m(\mathcal{H})$. Let $T = (T_1, \ldots, T_k)$ be a completely non-coisometric k-tuple in the noncommutative polydomain $D_f^m(\mathcal{H})$. Then

$$\Psi_T \coloneqq SOT - \lim_{r\to 1} \varphi\left(rT_{i,j}\right), \quad \Psi_T = \varphi\left(W_{i,j}\right) \in F^\infty\left(D_f^m\right),$$

exists in the strong operator topology and defines a map $\Psi_T: F^\infty\left(D_f^m\right) \to B(\mathcal{H})$ with the following properties:

(i)$\Psi_T(\varphi) = SOT - \lim_{r\to 1} B_{rT}[\varphi]$, where B_{rT} is the Berezin transform at $rT \in D_f^m(\mathcal{H})$;

(ii)Ψ_T is WOT-continuous (resp. SOT-continuous) on bounded sets;

(iii)Ψ_T is a unital completely contractive homomorphism and
$$\Psi_T(W_{(\beta)}) = T_{(\beta)}, \qquad (\beta) \in \mathbb{F}_{n_1}^+ \times \dots \times \mathbb{F}_{n_k}^+$$
If T is a pure k-tuple, then $\Psi_T(\varphi) = B_T[\varphi]$.
For each $i \in \{1, \dots, k\}$, let $Z_i := (Z_{i,1}, \dots, Z_{i,n_i})$ be an n_i-tuple of noncommuting indeterminates and assume that, for any $s, t \in \{1, \dots, k\}, s \neq t$, the entries in Z_s are commuting with the entries in Z_t. The algebra of all polynomials in indeterminates $Z_{i,j}$ is denoted by $\mathbb{C}[Z_{i,j}]$.
Let $W := \{W_{i,j}\}$ be the universal model associated with the abstract noncommutative polydomain D_f^m . If Q is a left ideal of polynomials in$\mathbb{C}[Z_{i,j}]$, we let $QW := \{q(W_{i,j}): q \in Q\}$ be the corresponding ideal in the algebra $\mathbb{C}[W_{i,j}]$of all polynomials in $W_{i,j}$and the identity. Using the $A(D_f^m)$-functional calculus, one can easily show that the norm-closed left ideal generated by QW in the polydomain algebra $A(D_f^m)$ coincides with the norm closure $\bar{Q}W$. Similarly, using the $F^\infty(D_f^m)$–functional calculus, one can prove that the WOT-closed left ideal generated by QW in the Hardy algebra $F^\infty(D_f^m)$ coincides with $\bar{Q}_W^{wot}$. If J is a left ideal in $\mathbb{C}[W_{i,j}], A(D_f^m)$, or $F^\infty(D_f^m)$, we introduce the subspace M_J to be the closed image of J in $\otimes_{i=1}^k F^2(H_{n_i})$, i.e., $M_J := \overline{J(\otimes_{i=1}^k F^2(H_{n_i})}$. We also introduce the space
$$N_J := [\otimes_{i=1}^k F^2(H_{n_i})] \ominus M_J .$$
When Q is a left ideal of polynomials in$\mathbb{C}[Z_{i,j}]$, we set $M_Q := M_{QW}$ and $N_Q := [\otimes_{i=1}^k F^2(H_{n_i})] \ominus M_Q$. We remark that in this case we have
$$N_Q = N_{\bar{Q}W} = N_{\bar{Q}_W^{wot}}.$$
We consider J to denote a left ideal in either one of the following algebras:
$\mathbb{C}[Z_{i,j}], \mathbb{C}[W_{i,j}], A(D_f^m)$, or $F^\infty(D_f^m)$. We always assume that $N_J \neq \{0\}$. It is easy to see that N_J is invariant under each operator $W_{i,j}^*$ for $i \in \{1, \dots, k\}$,
$j \in \{1, \dots, n_i\}$. Define $S_{i,j} := P_{N_J} W_{i,j|N_J}$, where P_{N_J} is the orthogonal projection of $\otimes_{i=1}^k F^2(H_{n_i})$ onto N_J . Using the properties of the universal model $W = \{W_{i,j}\}$ and the fact that N_J is invariant under each operator $W_{i,j}^*$, one can obtain the following result.
Lemma (4.2.1)[244]: Let J be *a* left ideal in either one of the following algebras*:*
$\mathbb{C}[Z_{i,j}], \mathbb{C}[W_{i,j}], A(D_f^m)$, or $F^\infty(D_f^m)$. The k-tuple $S := (S_1, \dots, S_k)$, where $S_i := (S_{i,1} \dots, S_{i,n_i})$ and $S_{i,j} := P_{N_J} W_{i,j|N_J}$ has the following properties.
(i) S *is a* pure tuple in the polydomain $D_f^m(N_J)$.
(ii) Under the $F^\infty(D_f^m)$-functional calculus,
$$g(S_1, \dots, S_k) = 0, \qquad g \in \bar{J}^{wot}.$$
(iii) If $P_{\mathbb{C}}$ denotes the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}_1$, then
$$(id - \Phi_{f_1,S_1})^{m_1} \dots (id - \Phi_{f_k,s_k})^{m_k}(I_{N_J}) = P_{N_J} P_{\mathbb{C}|N_J}.$$
Proof*:* Since N_J is invariant under each operator $W_{i,j}^*$, we have $\Phi_{f_1,S_1}^{q_1}(I) = P_{N_J} \Phi_{f_1,S_1}^{q_1}(I) |_{N_J}$. Taking into account that W is a pure element in $D_f^m(\otimes_{i=1}^k F^2(H_n))$, we deduce that SOT-$\lim_{q_i \to \infty} \Phi_{f_i,W_i}^{q_i}(I) = 0$, which implies that S is a pure tuple in the polydomain $D_f^m(N_J)$. To prove part (ii), note that if $g(W_{i,j}) \in \bar{J}^{mot}$, then the range of $g(W_{i,j})$ is in N_J .Using the $F^\infty(D_f^m)$-functional calculus, we deduce that
$$g(S_1, \dots, S_k) = SOT - \lim_{r \to 1} g(rS_{i,j}) = SOT - \lim_{r \to 1} g(rW_{i,j}) |_{N_J} = P_{N_J} g(W_{i,j})\big|_{N_J} = 0.$$
Part (iii) follows from the fact that $\Delta_{f,W}^m(I) = P_C$ and N_J is invariant under each operator $W_{i,j}^*$. Indeed, we have $\Delta_{f,W}^m(I) = P_{N_J} \Delta_{f,W}^m(I)\big|_{N_J} = P_{N_J} P_C|_{N_J}$.
We define the noncommutative variety $\mathcal{V}_{f,J}^m(\mathcal{H})$ in the polydomain $D_f^m(\mathcal{H})$ by setting
$$\mathcal{V}_{f,J}^m(\mathcal{H}) := \{X = \{X_{i,j}\} \in D_f^m(\mathcal{H}): g(X) = 0 \ for\ any\ g \in J\}.$$

We remark that this variety is well-defined if J is a left ideal in $\mathbb{C}[Z_{i,j}], \mathbb{C}[W_{i,j}]$, or $A(D_f^m)$. In the case when J is a WOT-closed left ideal in$F^\infty(D_f^m)$, we can use the $F^\infty(D_f^m)$–functional calculus to define the variety $\mathcal{V}_{f,J,cnc}^m(\mathcal{H})$of all completely non coisometric $(c.n.c.)$ tuples $X \in D_f^m(\mathcal{H})$ satisfying the equation $g(X) = 0$ for any $g \in J$.
According to Lemma (4.2.1), the k-tuple $S := (S_1, \dots, S_k)$ is in the noncommutative variety$\mathcal{V}_{f,J}^m(\mathcal{N}_J)$. We remark that S will play the role of universal model for the abstract noncommutative variety

$$\mathcal{V}_{f,J}^m := \{\mathcal{V}_{f,J}^m(\mathcal{H}): \mathcal{H}\ is\ a\ Hilbert\ space\}.$$

We introduce the constrained noncommutative Berezin kernel associated with $T \in \mathcal{V}_{f,J}^m(\mathcal{H})$ as the bounded operator $K_{f,T,J}: \mathcal{H} \to \mathcal{N}_J \otimes \overline{\Delta_{f,T}^m(I)(\mathcal{H})}$ defined by

$$K_{f,T,J} := \left(P_{\mathcal{N}_J} \otimes I_{\overline{\Delta_{f,T}^m(I)(\mathcal{H})}}\right) K_{f,T},$$

where $K_{f,T}$ is the noncommutative Berezin kernel associated with $T \in D_f^m(\mathcal{H})$. The next result shows that the main properties of the noncommutative Berezin kernel remain true for the constrained Berezinkernel associated with the elements of the noncommutative variety$\mathcal{V}_{f,J}^m(\mathcal{H})$.
Proposition (4.2.2) [244]: Let $T = (T_1, \dots, T_K)$, with $T_i(T_{i,1}, \dots, T_{i,n_i})$,be in the noncommutative variety$\mathcal{V}_{f,J}^m(\mathcal{H})$, where J is a left ideal in$\mathbb{C}[Z_{i,j}], \mathbb{C}[W_{i,j}]$, or $A(D_f^m\ )$. The constrained noncommutative Berezin kernel associated with T has the following properties.
(i) $K_{f,T,J}$ is a contraction and

$$K_{f,T,J}^* K_{f,T,J} = \lim_{q_k} \dots \lim_{q_k} \left(id - \Phi_{f_k,T_k}^{q_k}\right) \dots \left(id - \Phi_{f_1,T_1}^{q_1}\right)(I),$$

where the limits are in the weak operator topology.
(ii) For any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$,

$$K_{f,T,J} T_{f,J}^* = \left(S_{i,j}^* \otimes I\right) K_{f,T,J}.$$

(iii) If T is pure, then

$$K_{f,T,J}^* K_{f,T,J} = I_{\mathcal{H}}.$$

If J is a WOT-closed left ideal in $F^\infty(D_f^m)$ and $T \in \mathcal{V}_{f,J,cnc}^m(\mathcal{H})$, all the properties above remain true.
Proof: Since $K_{f,T} T_{f,J}^* = \ (W_{i,j}^* \otimes I) K_{f,T}$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, we deduce that

$$\langle K_{f,T}x, q(W_{i,j}) W_{(\alpha)}(1) \otimes y\rangle = \langle x, q(T_{i,j}) T_{(\alpha)} K_{f,T}^*(1 \otimes y)\rangle =$$
$$\langle x, q(T_{i,j}) T_{(\alpha)} \Delta_f^m(I)^{1/2} y\rangle \qquad (48)$$

for any $x \in \mathcal{H}, y \in \overline{\Delta_{f,T}^m(I)\mathcal{H}}, (\alpha) \in \mathbb{F}_{n_1}^+ \otimes \dots \otimes \mathbb{F}_{n_k}^+$, and any polynomial $q(W_{i,j}) \in \mathbb{C}[W_{i,j}]$.Consequently, if J is a left ideal in $\mathbb{C}[Z_{i,j}]$ or $\mathbb{C}[W_{i,j}]$, then $q(T_{i,j}) = 0$ for any $q \in J$ and therefore

$$range K_{f,T} \subseteq N_J \otimes \overline{\Delta_{f,T}^m(I)\mathcal{H}}. \qquad (49)$$

Assume that J is a norm-closed left ideal of $A(D_f^m\ )$ and let $g(W_{i,j}) \in J$. Choose a sequence of polynomials $q_n(W_{i,j})$ which converges in norm to $g(W_{i,j})$. This implies that $q_n(T_{i,j})$ converges in norm to $g(T_{i,j})$.
Using equation (48), we deduce a similar one where $q(W_{i,j})$ is replaced by $g(W_{i,j})$. As above, we deduce that relation (49) remains true in this case. Now, we consider the case when J is a WOT-closed left ideal in $F^\infty(D_f^m)$ and $T \in \mathcal{V}_{f,J,cnc}^m(\mathcal{H})$. Let $\varphi(W_{i,j})$ be in $J \subset F^\infty(D_f^m)$ with Fourier representation

$$\varphi(W_{i,j}) = \sum_{(\beta) \in \mathbb{F}_{n_1}^+ \times \dots \times \mathbb{F}_{n_k}^+} c_{(\beta)} W_{(\beta)}.$$

Then $\varphi(W_{i,j}) = \ SOT - \lim_{r \to 1} \varphi(rW_{i,j})$, and $\varphi(rW_{i,j})$ is in the polydomain algebra $A(D_f^m)$. Relation (48) implies

$$\langle K_{f,T}x, \varphi(rW_{i,j}) W_{(\alpha)}(1) \otimes y\rangle = \langle x, \varphi(rW_{i,j}) T_{(\alpha)} \Delta_f^m(I)^{1/2} y\rangle$$

for any $r \in [0,1), x \in \mathcal{H}, y \in \overline{\Delta_{f,T}^m(I)\mathcal{H}}$., and $(\alpha) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+$. Due to the $F^\infty(D_f^m)$–functional calculus, we have $0 = \varphi(T_{i,j}) = SOT - \lim_{r\to 1} \varphi(rT_{i,j})$. Consequently.$\langle K_{f,T}x, \varphi(W_{i,j})W_{(\alpha)}(1) \otimes y\rangle = 0$ for any $\varphi(W_{i,j}) \in J, y \in \overline{\Delta_{f,T}^m(I)\mathcal{H}}$, and$(\alpha) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+$. Therefore, relation (49) holds also in
this case. It is clear that due to relation (49), we have $K_{f,T,J}^* K_{f,T,J} = K_{f,T}^* K_{f,T}$. Now, one can easily complete the proof using the appropriate properties of the noncommutative Berezin kernel $K_{f,T}$ and the definition of the constrained Berezin kernel.
For each n-tuple $T \coloneqq \{T_{i,j}\} \in \mathcal{V}_{f,J}^m(\mathcal{H})$, we introduce the constrained noncommutative Berezin trans-form at T as the map $B_{T,J}: B(N_J) \to B(\mathcal{H})$ defined by setting

$$B_{T,J}[g] \coloneqq K_{f,T,J}^*(g \otimes I_{\mathcal{H}})K_{f,T,J}, \qquad g \in B(N_J),$$

where J is a left ideal in $\mathbb{C}[Z_{i,j}]$, $\mathbb{C}[W_{i,j}]$, $A(D_f^m)$, or $F^\infty(D_f^m)$. Note that $B_{T,J}$is a completely contractive, completely positive, and w^*-continuous linear map. Consequently, $B_{T,J}$ is WOT-continuous (resp.SOT-continuous) on bounded sets. Note that T is pure if and only if $B_{T,J}(I) = I$.
Theorem (4.2.3) [244]: Let $T = (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times \ldots \times B(\mathcal{H})^{n_k}$ and let J be *a* w^*-closed left ideal of$F^\infty(D_f^m)$. Then T is *a* pure element of the noncommutative variety $\mathcal{V}_{f,J}^m(\mathcal{H})$ if and only if there is *a* unital completely positive and w^*-continuous linear map

$$\Psi: \overline{span}^{w^*}\{S_{(\alpha)}S_{(\beta)}^*: (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+\} \to B(\mathcal{H})$$

such that

$$\Psi\left(S_{(\alpha)}S_{(\beta)}^*\right) = T_{(\alpha)}T_{(\beta)}^*, \qquad (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+.$$

Proof*:* Due to Proposition (4.2.2), if $T \coloneqq (T_1, \ldots, T_k)$ is a pure tuple in the noncommutative variety $\mathcal{V}_{f,J}^m(\mathcal{H})$, then $K_{f,T,J}$ is an isometry and the constrained noncommutative Berezin transform is a unital completely contractive and w^*-continuos linear map such that

$$B_{T,J}[S_{(\alpha)}S_{(\beta)}^*] = K_{f,T,J}^*[S_{(\alpha)}S_{(\beta)}^* \otimes I_{\mathcal{H}}]K_{f,T,J} = T_\alpha T_\beta^*$$

for any $(\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+$. To prove the converse, assume that Ψ has the required properties. Since $(S_1, \ldots, S_k)$ is a commuting tuple and Ψ is a homomorphism when restricted to $\mathbb{C}[S_{i,j}]$, we deduce that $(T_1, \ldots, T_k)$ is a commuting tuple. Taking into account that Φ_{f_i,S_i} is a w^*-continuous map, and $\Delta_{f,S}^P$ is a linear combination of products of the form $\Phi_{f_1,S_1}^{q_1} \ldots \Phi_{f_k,S_k}^{q_k}$, where $(q_1, \ldots, q_k) \in \mathbb{Z}_+^k$, we deduce that $\Delta_{f,S}^P$is a w^*-continuous map. Since Ψ is a completely positive w^*-continuous linear map such that $\Psi(S_{(\alpha)}S_{(\beta)}) = T_{(\alpha)}T_{(\beta)}^*$ for any $(\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \ldots \times \mathbb{F}_{n_k}^+$, we obtain

$$\Delta_{f,S}^P(I) = \Psi(\Delta_{f,S}^P(I)) \geq 0$$

for any $p = (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ with $p \leq m$. Therefore, $\in D_f^m(\mathcal{H})$. On the other hand, for each $i \in \{1, \ldots, k\}$, we have

$$\lim_{q_i\to\infty} \Phi_{f_i,T_i}^{q_i}(I) = \Psi(\lim_{q_i\to\infty} \Phi_{f_i,S_i}^{q_i}(I)) = \Psi(0) = 0,$$

which shows that T is a pure tuple in the polydomain $D_f^m(H)$. To prove that T is in the noncommutative variety $V_{f,J}^m(H)$, fix $g \in J$ and recall that $g(W_{i,j}) = SOT\text{-}\lim_{r\to1} g(rW_{i,j})$, where $g(rW_{i,j})$ is in the polydomain algebra $A(D_f^m)$, and $\|g(rW_{i,j})\| \leq \|g(W_{i,j})\|$ for any $r \in [0, 1)$. Using the the $F^\infty(D_f^m)$–functional calculus for pure elements in $D_f^m(H)$ and the fact that WOT and w^*-topology coincide on bounded sets, we deduce that

$$g(T_{i,j}) = WOT - \lim_{r\to1} g(rT_{i,j}) = WOT - \lim_{r\to1} \Psi\left(g(rS_{i,j})\right)$$
$$= \Psi(WOT - \lim_{r\to1} g(rS_{i,j}) = \Psi\left(g(S_{i,j})\right) = \Psi(0) = 0.$$

Therefore, T is in the noncommutative variety$\mathcal{V}_{f,J}^m(\mathcal{H})$. The proof is complete.
Theorem (4.2.4) [244]: Let $Q \subset \mathbb{C}[Z_{i,j}]$ be *a* left ideal generated by noncommutative homogenous polynomials and let

$$T \coloneqq (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times \ldots \times B\ (\mathcal{H})^{n_1}.$$

Then T is in the noncommutative variety $\mathcal{V}_{q,Q}^m(\mathcal{H})$, where $q = (q_1, \dots, q_k)$ is a k-tuple of positive regular noncommutative polynomials, if and only if there is a unital completely positive linear map $\Psi: \bar{S} \to B(\mathcal{H})$,
where $\bar{S} \coloneqq \overline{span}\{S_{(\alpha)}S_{(\beta)}^*: (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \dots \times \mathbb{F}_{n_k}^+\}$, such that

$$\Psi\big(S_{(\alpha)}S_{(\beta)}^*\big) = T_{(\alpha)}T_{(\beta)}, \qquad (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \dots \times \mathbb{F}_{n_k}^+,$$

where $S := \{S_{i,j}\}$ is the universal model associated with the abstract noncommutative variety $V_{1,Q}^m$.

Proof: Assume that $T \in V_{q,Q}^m(H)$. Since $D_q^m(H)$ is a radial domain [253], $rT \in D_{q,Q}^m(H)$ for any $r \in [0, 1)$.

Note that, due to the fact that $Q \subset \mathbb{C}[Z_{i,j}]$ is a left ideal generated by noncommutative homogenous polynomials, if $g \in Q$, then $g(T_{i,j}) = 0$ and $g(rT_{i,j}) = 0$. Thus $rT \in V_{q,Q}^m(H)$ and, as in the proof of Theorem (4.2.3), one can show that range $K_{q,rT} \subseteq N_Q \otimes H$ for any $r \in [0, 1)$, where $K_{q,rT}$ is the Berezin kernel associated with $rT \in D_q^m(H)$. Moreover,

$$K_{q,rT,Q}(r^{|\alpha|+|\beta|}T_{(\alpha)}T_{(\beta)}^*) = (S_{(\alpha)}S_{(\beta)}^* \otimes IH)K_{q,rT,Q}, (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+.$$

Since rT is pure, $K_{q,rT,Q}$ is an isometry. Consequently, for any $n \times n$ matrix with entries $\psi_{st}(S_{i,j})$ in the linear span S of all products $S_{(\alpha)}S_{(\beta)}^*$, where $(\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times F_{n_k}^+$, we have the von Neumann type inequality

$$\|[\psi_{st}(rT_{i,j})]_{n\times n}\| \le \big\|[\psi_{st}\big(S_{i,j}\big)]\big\|_{n\times n}, \qquad r \in [0, 1).$$

Taking $r \to 1$, we deduce that $\|[\psi_{st}(rT_{i,j})]_{n\times n}\| \le \big\|[\psi_{st}\big(S_{i,j}\big)]\big\|_{n\times n}$. We define the unital completely contractive linear map $\Psi_{f,q,Q}: S \to B(H)$ by setting $\Psi_{q,T,Q}(S_{(\alpha)}S_{(\beta)}^*) := T_{(\alpha)}T_{(\beta)}^*$, for all $(\alpha), (\beta)$ in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. Now, it is clear that Ψ has a unique extension to a unital completely contractive linear map on $\bar{S}$.

To prove the converse, assume that Ψ has the required properties and note that, due to Lemma (4.2.1) and the fact that $1 \in N_Q$, we have

$$(I - \Phi_{q_1,T_1})^{p_1} \cdots (I - \Phi_{q_k,T_k})^{p_k}(I) = \Psi[(I - \Phi_{q_1,S_1})^{p_1} \cdots (I - \Phi_{q_k,S_k})^{p_k}(I_{N_Q})] \ge 0$$

for any $p_i \in \{0, 1, \dots, m_i\}$ and $i \in \{1, \dots, k\}$. Since $(S_1, \dots, S_k)$ is a commuting tuple and Ψ is a homomorphism when restricted to $\mathbb{C}[S_{i,j}]$, we deduce that $(T_1, \dots, T_k)$ is a commuting tuple. Therefore, $T \in D_f^m(H)$. On the other hand, since $g(S_{i,j}) = 0$ for any $g \in Q$, we have $g(T_{i,j}) = \Psi(g(S_{i,j})) = 0$, which shows that $T \in V_{q,Q}^m(H)$. The proof is complete.

Proposition (4.2.5) [244]: Let $Q \subset \mathbb{C}[Z_{i,j}]$ be a left ideal generated by noncommutative homogenous polynomials, and let $T := (T_1, \dots, T_n)$ be in the noncommutative variety $V_{f,Q}^m(H)$, where $f = (f_1, \dots, f_k)$ is a k-tuple of positive regular free holomorphic functions. Then there is a unital completely contractive linear map $\Psi_{f,T,Q}: \bar{S} \to B(H)$, where $\bar{S} := \overline{span}\{S_{(\alpha)}S_{(\beta)}^*: (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\}$, such that

$$\Psi_{f,T,Q}(g) = \lim_{r\to 1} B_{rT,Q}[g], \qquad g \in \bar{S},$$

where the limit exists in the norm topology of $B(H)$, and

$$\Psi_{f,T,Q}\big(S_{(\alpha)}S_{(\beta)}^*\big) = T_{(\alpha)}T_{(\beta)}^*, \qquad (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+.$$

In particular, the restriction of $\Psi_{f,T,Q}$ to the variety algebra $A(V_{f,Q}^m)$ is a unital completely contractive homomorphism. If, in addition, T is a pure k-tuple of operators, then

$$\lim_{r\to 1} B_{rT,Q}[g] = B_{T,Q[g]}, \qquad g \in \bar{S},$$

where the limit exists in the norm topology of $B(H)$.

Proof: Following the proof of the direct implication of Theorem (4.2.4), we can show that the linear map $\Psi_{f,T,Q}: S \to B(H)$ defined by $\Psi_{f,T,Q}(S_{(\alpha)}S_{(\beta)}^*) := T_{(\alpha)}T_{(\beta)}^*$, for all $(\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, is unital and completely contractive. Given $g = g(S_{i,j}) \in S$, we define $\Psi_{f,T,Q}(g) := \lim_{n\to\infty} \Psi_{f,T,Q}(g_n)$, where $g_n \in S$ with $\|g - g_n\| \to 0$, as $n \to \infty$. Note that $\Psi_{f,T,Q}(g)$ does not depend on the choice of the sequence $\{g_n\}$ and

$$\|\Psi_{f,T,Q}(g) - B_{rT,Q}[g]\| \leq \|\Psi_{f,T,Q}(g) - \Psi_{f,T,Q}(g_n)\| + \|\Psi_{f,T,Q}(g_n) - B_{rT,Q}[g_n]\| + \|B_{rT,Q}[g_n - g]\| \leq 2\|g - g_n\| + \|\Psi_{f,T,Q}(g_n) - B_{rT,Q}[g_n]\|.$$

Hence, we deduce that $\Psi_{f,T,Q}(g) = \lim_{r\to 1} B_{rT,Q}[g]$ for any $g \in \bar{S}$. Now, we assume that T is a pure k-tuple in $V_{f,Q}^m(H)$. Since

$$B_{T,Q}[g_n] := K_{f,T,Q}^*(g_n \otimes I_H)K_{f,T,Q} = g_n(T_{i,j})$$

and taking into account that $g_n \in S$ with $\|g - g_n\| \to 0$, as $n \to \infty$, we conclude that $B_{T,Q}[g] = \Psi_{f,T,Q}(g)$ for any $g \in S$. This completes the proof.

we obtain a characterization of the Beurling [250] type joint invariant subspaces under the universal model $S = \{S_{i,j}\}$ of $V_{f,J}^m$, and a characterization of the joint reducing subspaces of $S \otimes I$. We use noncommutative Berezin transforms to characterize the pure elements in noncommutative varieties $V_{f,J}^m$ and obtain a classification result for the pure elements of rank one.

Denote by $C^*(S_{i,j})$ the C^*-algebra generated by the operators $S_{i,j}$, where $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}$, and the identity.

Theorem (4.2.6) [244]: Let $q = (q_1, \ldots, q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $S = (S_1, \ldots, S_k)$ be the universal model associated with the abstract noncommutative variety $V_{q,J}^m$, where J is a WOT-closed two sided ideal of $F^\infty(D_q^m)$ such that $1 \in N_J$. Then all the compact operators in $B(N_J)$ are contained in the operator space

$$\bar{S} := \overline{span}\{S_{(\alpha)}S_{(\beta)}^* : (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\}.$$

Moreover, the C^*-algebra $C^*(S_{i,j})$ is irreducible.

Proof: Since $1 \in N_J$, Lemma (4.2.1) implies

$$(I - \Phi_{q_1,S_1})^{m_1} \cdots \left(I - \Phi_{q_k,S_k}\right)^{m_k}\left(I_{N_J}\right) = P_{N_J}P_{\mathbb{C}}|_{N_J} = P_{\mathbb{C}}^{N_J}, \qquad (50)$$

where $P_{\mathbb{C}}^{N_J}$ is the orthogonal projection of N_J onto $\mathbb{C}$. Fix a polynomial $g(W_{i,j}) := \sum_{\substack{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|\le n}} d_{(\beta)}W_{(\beta)}$ and let $\xi := \sum_{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} c_{(\beta)}e_{(\beta)}$ be in $N_J \subset \bigotimes_{i=1}^k F^2(H_{n_i})$, where we denote $e_{(\beta)} := e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k$, if $(\beta) := (\beta_1, \ldots, \beta_k)$. It is easy to see that $P_{\mathbb{C}}^{N_J} g(S_{i,j})^*\xi = \langle \xi, g(S_{i,j})(1)\rangle$. Consequently, we have

$$\chi(S_{i,j})P_{\mathbb{C}}^{N_J} g\left(S_{i,j}\right)^*\xi = \langle \xi, g(S_{i,j})(1)\rangle\, \chi\left(S_{i,j}\right)(1) \qquad (51)$$

for any polynomial $\chi(S_{i,j})$. Employing relation (50), we deduce that the operator $\chi(S_{i,j})P_{\mathbb{C}}^{N_J} g(S_{i,j})^*$ has rank one and it is in the operator space $\bar{S}$. On the other hand, due to the fact that the set of all vectors of the form $\sum_{\substack{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|\le n}} d_{(\beta)}S_{(\beta)}(1)$ with $n \in \mathbb{N}, d_{(\beta)} \in \mathbb{C}$, is dense in N_J, relation (51) implies that all the compact operators in $B(N_J)$ are contained in S.

To prove the last part of this theorem, let $\mathcal{E} \neq \{0\}$ be a subspace of $N_J \subset \bigotimes_{i=1}^k F^2(H_{n_i})$, which is jointly reducing for the operators $S_{i,j}, i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. Let $\varphi \in \mathcal{E}, \varphi \neq 0$, and assume that $\varphi = \sum_{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{(\beta)}e_{(\beta)}$. If $a_{(\beta)}$ is a nonzero coefficient of φ, then we have

$$P_{\mathbb{C}}S_{1,\beta_1}^* \cdots S_{k,\beta_k}^*\varphi = P_{\mathbb{C}}W_{1,\beta_1}^* \cdots W_{k,\beta_k}^*\varphi = \frac{1}{\sqrt{b_{1,\beta_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\beta_k}^{(m_k)}}}a_{(\beta)}.$$

Due to relation (50) and using the fact that $\mathcal{E}$ is reducing for each $S_{i,j}$, we deduce that $a_{(\beta)} \in \mathcal{E}$, so $1 \in \mathcal{E}$. Using again that $\mathcal{E}$ is invariant under the operators $S_{i,j}$, we deduce that $\mathcal{E} = N_J$. This completes the proof.

Let $T = (T_1, \ldots, T_k) \in D_f^m(H)$ and $T' = (T_1', \ldots, T_k') \in D_f^m(H')$ be k-tuples with $T_i := (T_{i,1}, \ldots, T_{i,n_i})$ and $T_i' := (T_{i,1}', \ldots, T_{i,n_i}')$. We say that T is unitarily equivalent to T' if there is a unitary operator $U: H \to H'$ such that $T_{i,j} = U^*T_{i,j}'U$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$.

Corollary (4.2.7) [244]: Let $S = \{S_{i,j}\}$ be the universal model associated with the abstract noncommutative variety $V_{q,J}^m$, where J is a WOT-closed left ideal of $F^\infty(D_q^m)$ such that $1 \in N_J$. If H, K are Hilbert spaces, then $\{S_{i,j} \otimes I_H\}$ is unitarily equivalent to $\{S_{i,j} \otimes I_K\}$ if and only if $dimH = dimK$.

Proof: Let $U: N_J \otimes H \to N_J \otimes K$ be a unitary operator such that $U(S_{i,j} \otimes I_H) = (S_{i,j} \otimes I_K)U$ for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Then $U(S_{i,j}^* \otimes I_H) = (S_{i,j}^* \otimes I_k)U$ and, due to the fact that the C^*-algebra $C^*(S_{i,j})$ is irreducible, we must have $U = I_{N_J} \otimes A$, where $A \in B(H, K)$ is a unitary operator.

Therefore, $dimH = dimK$. The proof is complete.

We recall that a subspace $H \subseteq K$ is called co-invariant under $A \subset B(K)$ if $X^*H \subseteq H$ for any $X \in A$.

Theorem (4.2.8) [244]: Let $S = \{S_{i,j}\}$ be the universal model associated with the abstract noncommutative variety $V_{f,J}^m$, where J is a WOT-closed two sided ideal of $F^\infty(D_f^m)$ such that $1 \in N_J$. If K be a Hilbert space and $M \subseteq N_J \otimes K$ is a co-invariant subspace under each operator $S_{i,j} \otimes I_k$ for $i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$, then there exists a subspace $\mathcal{E} \subseteq K$ such that

$$\overline{span}\{(S_{(\beta)} \otimes I_k)M: (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\} = N_J \otimes \mathcal{E}.$$

Proof: Set $\mathcal{E} := (P_{\mathbb{C}} \otimes I_k)M \subseteq K$, where $P_{\mathbb{C}}$ is the orthogonal projection from N_J onto $\mathbb{C}1 \subset N_J$ and let φ be a nonzero element of M with representation

$$\varphi = \sum_{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} e_{(\beta)} \otimes h_{(\beta)} \in M \subset N_J,$$

where $h_{(\beta)} \in K$ and $\sum_{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \|h_{(\beta)}\|^2 < \infty$. Assume that $h_{(\sigma)} \neq 0$ for some $\sigma = (\sigma_1, \dots, \sigma_k)$ in $\mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+$ and note that

$$\begin{aligned}(P_{\mathbb{C}} \otimes I_K)(S_{1,\sigma_1}^* \cdots S_{k,\sigma_k}^* \otimes I_K)\varphi &= (P_{\mathbb{C}} \otimes I_K)(W_{1,\sigma_1}^* \cdots W_{k,\sigma_k}^* \otimes I_K)\varphi \\ &= 1 \otimes \frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}} h_{(\sigma)}.\end{aligned}$$

Consequently, since M is a co-invariant subspace under each operator $S_{i,j} \otimes I_K$, we must have $h_{(\sigma)} \in \mathcal{E}$.

Since $1 \in N_J$, we deduce that

$$(S_{1,\sigma_1} \cdots S_{k,\sigma_k} \otimes I_K)(1 \otimes h_{(\sigma)}) = \frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}} P_{N_J}(e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k) \otimes h_{(\sigma)}$$

is a vector in $N_J \otimes \mathcal{E}$. Therefore,

$$\varphi = \lim_{n\to\infty} \sum_{q=0}^n \sum_{\substack{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=q}} P_{N_J}(e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k) \otimes h_{(\beta)} \quad (52)$$

is in $N_J \otimes \mathcal{E}$. Hence, $M \subset N_J \otimes \mathcal{E}$ and

$$Y := \overline{span}\{(S_{(\sigma)} \otimes I_K)M: (\sigma) \in \mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+\} \subset N_J \otimes \mathcal{E}.$$

Now, we prove the reverse inclusion. If $h_0 \in \mathcal{E}, h_0 \neq 0$, then there exists $\xi \in M \subset N_J \otimes \mathcal{E}$ such that

$$\xi = 1 \otimes h_0 + \sum_{\substack{(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|\geq 1}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes h_{(\beta)}$$

and $1 \otimes h_0 = (P_{\mathbb{C}} \otimes I_k)\xi$. Consequently, due to Lemma (4.2.1), we have

$$1 \otimes h_0 = (P_{\mathbb{C}} \otimes I_K)\xi = (id - \Phi_{f_1,S_1\otimes I_k})^{m_1} \cdots (id - \Phi_{f_k,S_k\otimes I_k})^{m_k} (I_{N_J} \otimes I_K)\xi.$$

Taking into account that M is co-invariant under each operator $S_{i,j} \otimes I_K$, we deduce that $h_0 \in Y$ for any $h_0 \in \mathcal{E}$. Therefore, $\mathcal{E} \subset Y$. This inclusion shows that $(S_{(\sigma)} \otimes I_K)(1 \otimes \mathcal{E}) \subset Y$ for any $(\sigma) \in \mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+$, which implies

$$\frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}}P_{N_J}(e_{\sigma_1}^1\otimes\cdots\otimes e_{\sigma_k}^k)\otimes\mathcal{E}\subset Y.$$

Consequently, if $\varphi\in N_J\otimes\mathcal{E}$ has the representation (52), we conclude that $\varphi\in Y$. Therefore, $N_J\otimes\mathcal{E}\subseteq Y$.

The proof is complete.

Now, we can easily deduce the following result.

Corollary (4.2.9) [244]: Let $S:=(S_1,\ldots,S_k)$ be the universal model associated to the abstract noncommutative variety $V_{f,J}^m$, where J is a WOT-closed two sided ideal of $F^\infty(D_f^m)$ such that $1\in N_J$. If K is a Hilbert space, then a subspace $M\subseteq N_J\otimes K$ is reducing under each operator $S_{i,j}\otimes I_K$ for $i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\}$, if and only if there exists a subspace $\mathcal{E}\subseteq K$ such that

$$M=N_J\otimes\mathcal{E}.$$

Let $S:=\{S_{i,j}\}$ be the universal model associated to the abstract noncommutative variety $V_{f,J}^m$. An operator $M\colon N_J\otimes H\to N_J\otimes K$ is called multi-analytic with respect to S if

$$M(S_{i,j}\otimes I_H)=(S_{i,j}\otimes I_K)M$$

for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$. In case M is a partial isometry, we call it inner multi-analytic operator.

The next result is an extension of Theorem 5.2 from [253] to varieties in noncommutative polydomains.

The constructions from the proof are needed in a forthcoming to define characteristic functions associated with noncommutative varieties.

Theorem (4.2.10) [244]: Let $S:=(S_1,\ldots,S_k)$ be the universal model associated to the abstract noncommutative variety $V_{f,J}^m$ and let $S_i\otimes I_H:=(S_{i,1}\otimes I_H,\ldots,S_{i,n_i}\otimes I_H)$ for $i\in\{1,\ldots,k\}$, where H is a Hilbert space.

If $G\in B(N_J\otimes H)$ then the following statements are equivalent.

(i) There is a multi-analytic operator $\Gamma: N_J\otimes\mathcal{E}\to N_J\otimes H$ with respect to S, where $\mathcal{E}$ is a Hilbert space, such that

$$G=\Gamma\Gamma^*.$$

(ii) For any $p:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $p\le m, p\ne 0$,

$$(\Delta_{f,S\otimes I_H}^p(G)\ge 0.$$

Proof: Assume that item (i) holds. Then we have

$$\Delta_{f,S\otimes I_H}^p(G)=(id-\Phi_{f_1,S_1\otimes I_H})^{p_1}\cdots(id-\Phi_{f_k,S_k\otimes I_H})^{p_k}(G)=\Gamma\Delta_{f,S\otimes I_{\mathcal{E}}}^p(I)\Gamma^*\ge 0$$

for any $p:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $p\le m, p\ne 0$.

Now, assume that (ii) holds. In particular, we have $\Phi_{f_1,S_1\otimes I_H}(\Delta_{f,S\otimes I_H}^{m'}(G))\le\Delta_{f,S\otimes I_H}^{m'}(G)$, where $m'=(m_1-1,m_2,\ldots,m_k)$, which implies $\Phi_{f_1,S_1\otimes I_H}^n(\Delta_{f,S\otimes I_H}^{m'}(G))\le\Delta_{f,S\otimes I_H}^{m'}(G)$ for any $n\in\mathbb{N}$. Since $S:=(S_1,\ldots,S_k)$ is a pure k-tuple, we have SOT-$\lim_{n\to\infty}\Phi_{f_1,S_1\otimes I_H}^n(\Delta_{f,S\otimes I_H}^{m'}(G))=0$. Consequently, $\Delta_{f,S\otimes I_H}^{m'}(G)\ge 0$. Continuing this process, we deduce that $G\ge 0$.

Let $G:=\overline{rangeG^{1/2}}$ and define

$$A_{i,j}\left(G^{\frac{1}{2}}x\right):=G^{\frac{1}{2}}\left(S_{i,j}^*\otimes I_H\right)_x,\ x\in N_J\otimes H,\qquad(53)$$

for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$. Taking into account that $\Phi_{f_i,S_i\otimes I}(G)\le G$, we have

$$\sum_{\alpha\in\mathbb{F}_{n_i}^+,|\alpha|\ge 1}a_{i,\alpha}\left\|A_{i,\tilde{\alpha}}G^{1/2}x\right\|^2=\langle\Phi_{f_i,S_i\otimes I_H}(G)x,x\rangle\le\left\|G^{1/2}x\right\|^2$$

for any $x\in N_J\otimes H$, where $\bar{\alpha}=g_{j_p}^i\cdots g_{j_1}^i$ denotes the reverse of $\alpha=g_{j_1}^i\cdots g_{j_p}^i\in\mathbb{F}_{n_i}^+$.

Consequently, $a_{i,g_j^i}\left\|A_{i,j}G^{1/2}x\right\|^2\le\left\|G^{1/2}x\right\|^2$, for any $x\in N_J\otimes H$. Since $a_{i,g_j^i}\ne 0$ each $A_{i,j}$ can be uniquely be extended to a bounded operator (also denoted by $A_{i,j}$) on the subspace $\mathcal{G}$. Set $X_{i,j}:=$

$A_{i,j}^*$ for $i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$. An approximation argument shows that $\Phi_{f_i,X_i}(I_{\mathcal{G}}) \le I_{\mathcal{G}}$ and relation (53) implies

$$X_{i,j}^*\left(G^{\frac{1}{2}}x\right) = G^{\frac{1}{2}}\left(S_{i,j}^* \otimes I_H\right)_x, \; x \in N_J \otimes H. \qquad (54)$$

This implies $G^{1/2}\Delta_{f,X}^p(I_M)G^{1/2} = \Delta_{f,S\otimes I_H}^p(G) \ge 0$ for any $p := (p_1,\dots,p_k) \in \mathbb{Z}_+^k$ such that $p \le m, p \ne 0$. Now, note that

$$\langle \Phi_{f_i,X_i}^n(I_{\mathcal{G}})Y^{1/2}x, G^{1/2}x\rangle = \langle \Phi_{f_i,S_i\otimes I_H}^n(G)x,x\rangle \le \|G\|\langle \Phi_{f_i,S_i\otimes I_H}^n(I)x,x\rangle$$

for any $x \in N_J \otimes H$ and $n \in \mathbb{N}$. Since SOT- $\lim_{n\to\infty}\Phi_{f_i,S_i\otimes I_H}^n(I) = 0$, we have SOT-$\lim_{m\to\infty}\Phi_{f_i,X_i}^n(I_{\mathcal{G}}) = 0$.

Therefore, $X := (X_1,\dots,X_k)$ is a pure k-tuple in the noncommutative variety $D_f^m(M)$. Due to the $F^\infty(D_f^m)$–functional calculus, relation (54) implies

$$G^{1/2}g(X_{i,j}) = g(S_{i,j})G^{1/2} = 0, g \in J.$$

Consequently, $g(X_{i,j}) = 0$ for any $g \in J$. This shows that $X := (X_1,\dots,X_k)$ is a pure k-tuple in the noncommutative variety $V_{f,J}^m(\mathcal{G})$. According to Proposition (4.2.2), the noncommutative Berezin kernel $K_{f,X,J}: G \to N_J \otimes \mathcal{E}$ is an isometry with the property that $X_{i,j}K_{f,X,J}^* = K_{f,X,J}^*(S_{i,j}\otimes I_{\mathcal{E}})$. Set $E := \overline{\Delta_{f,X}^m(I_{\mathcal{G}})(\mathcal{G})}$ and define the bounded linear operator $\Gamma := G^{1/2}K_{f,X,J}^*: N_J \otimes \mathcal{E} \to N_J \otimes H$. Note that

$$\Gamma(S_{i,j}\otimes I_{\mathcal{E}}) = G^{1/2}K_{f,X,J}^*(S_{i,j}\otimes I_{\mathcal{E}}) = G^{1/2}X_{i,j}K_{f,X,J}^*$$
$$= (S_{i,j}\otimes I_H)G^{1/2}K_{f,X,J}^* = (S_{i,j}\otimes I_H)\Gamma$$

for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$, which proves that Γ is a multi-analytic operator with respect to the universal model $S = \{S_{i,j}\}$. Note that $\Gamma\Gamma^* = G^{1/2}K_{f,X,J}^*K_{f,X,J}G^{1/2} = G$. The proof is complete.

Following the classical case [260], we say that $M \subset N_J \otimes H$ is a Beurling type invariant subspace under the operators $S_{i,j}\otimes I_H$ for $i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$, if there is an inner multi-analytic operator with respect to $S = \{S_{i,j}\}$,

$$\Psi: N_J \otimes \mathcal{E} \to N_J \otimes H,$$

such that $M = \Psi(N_J \otimes \mathcal{E})$.

Corollary (4.2.11) [244]: Let $M \subset N_J \otimes H$ be an invariant subspace under the operators $S_{i,j}\otimes I_H$ for any $i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$. Then M is Beurling type invariant subspace if and only if

$$\Delta_{f,S\otimes I_H}^P(P_M) \ge 0, \qquad \textit{for any } p \in \mathbb{Z}_+^k, p \le m,$$

where P_M is the orthogonal projection of the Hilbert space $N_J \otimes H$ onto M.

Proof: If $M: N_J \otimes \mathcal{E} \to N_J \otimes H$ is a inner multi-analytic operator and $M = M(N_J \otimes \mathcal{E})$, then $P_M = MM^*$. Taking into account Lemma (4.2.1), we deduce that

$$\Delta_{f,S\otimes I_H}^P(P_M) = \Psi(P_C \otimes I_{\mathcal{E}})\Psi^* \ \ge 0$$

for any $p := (p_1,\dots,p_k) \in \mathbb{Z}_+^k$ such that $p \le m$. The converse is a consequence of Theorem (4.2.10), when we take $G = P_M$. The proof is complete.

We remark that in the particular case when $m = (1,\dots,1)$, the condition in Corollary (4.2.11) is satisfied when $S \otimes I_H|_M := \{S_{i,j}\otimes I_H|_M\}$ is doubly commuting. The proof is very similar to that of the corresponding result from [253].

Theorem (4.2.12) [244]: Let $S = \{S_{i,j}\}$ be the universal model associated with the abstract noncommutative variety $V_{f,J}^m$, where J is a WOT-closed left ideal of $F^\infty(D_f^m)$, and let $T = \{T_{i,j}\}$ be a pure element in the noncommutative variety $V_{f,J}^m(H)$. If

$$K_{f,T,J}: H \to N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$$

is the noncommutative constrained Berezin kernel, then the subspace $K_{f,T,J}H$ is co-invariant under each operator $S_{i,j}\otimes I_{\overline{\Delta_{f,T}^m H}}$,TH for any $i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$. If $1 \in N_J$, then the dilation provided by the relation

$$T_{(\alpha)} = K_{f,T,J}^*(S_{(\alpha)}\otimes I_{\overline{\Delta_{f,T}^m(I)(H)}})K_{f,T,J}, \qquad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+,$$

is minimal. If, in addition, $f = q = (q_1, \dots, q_k)$ is a k-tuple of positive regular noncommutative polynomials and

$$\overline{span}\,\{S_{(\alpha)}S_{(\beta)}^*) : (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\} = C^*(S_{i,j}),$$

then the minimal dilation of T is unique up to an isomorphism.

Proof: According to Proposition (4.2.2),

$$K_{f,T,J}T_{i,j}^* = (S_{i,j}^* \otimes I)K_{f,T,J}, \qquad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\},$$

and the noncommutative Berezin kernel $K_{f,T,J}$ is an isometry. Due to the definition of the constrained Berezin kernel $K_{f,T,J}$, we obtain $(P_C \otimes I_D)K_{f,T,J}H = D$, where $D := \overline{\Delta_{f,T}^m(I)(H)}$. Now, using Theorem (4.2.8) in the particular case when $M := K_{f,T,J}H$ and $\mathcal{E} := D$, we deduce that the subspace $K_{f,T,J}H$ is cyclic for the operators $S_{i,j} \otimes I_{\mathcal{E}}$, where $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. This implies the minimality of the dilation, i.e.,

$$N_J \otimes D = \bigvee_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}\big(S_{(\alpha)} \otimes I_D\big)K_{f,T,J}H. \qquad (55)$$

Now, assume that $f = q = (q_1, \dots, q_k)$ is a k-tuple of positive regular noncommutative polynomials and that the relation in the theorem holds. Consider another minimal dilation of T, i.e.,

$$T_{(\alpha)} = V^*\big(S_{(\alpha)} \otimes I_{D'}\big)V, \quad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+, \qquad (56)$$

where $V: H \to N_J \otimes D'$ is an isometry, VH is co-invariant under each operator $S_{i,j} \otimes I_{D'}$, and

$$N_J \otimes D' = \bigvee_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}\big(S_{(\alpha)} \otimes I_{D'}\big)\,VH. \qquad (57)$$

According to Theorem (4.2.3), there exists a unique unital completely positive linear map $\Psi: C^*(S_{i,j}) \to B(H)$ with the property that

$$\Psi\big(S_{(\alpha)}S_{(\beta)}^*\big) = T_{(\alpha)}T_{(\beta)}^*, \qquad (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+.$$

Now, we consider the $*$-representations $\pi_1: C^*(S_{i,j}) \to B(N_J \otimes D), \pi_1(X) := X \otimes I_D$, and $\pi_2: C^*(S_{i,j}) \to B(N_J) \otimes D'), \pi_2(X) := X \otimes I_{D'}$. Since the subspaces $K_{q,T,J}H$ and VH are co-invariant for each operator $S_{i,j} \otimes I_D$, relation (56) implies

$$\Psi(X) = K_{q,T,J}^*\pi_1(X)K_{q,T,J} = V^*\pi_2(X)V, \qquad X \in C^*(S_{i,j}).$$

Relations (55) and (57) show that π_1 and π_2 are minimal Stinespring dilations of the completely positive linear map Ψ. Since these representations are unique up to an isomorphism, there exists a unitary operator $U: N_J \otimes D \to N_J \otimes D'$ such that $U(S_{i,j} \otimes I_D) = (S_{i,j} \otimes I_{D'})U$ for $i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$, and $UK_{q,T,J} = V$. Taking into account that U is unitary, we deduce that $U(S_{i,j}^* \otimes I_D) = (S_{i,j}^* \otimes I_{D'})U$.

Since the C^*-algebra $C^*(S_{i,j})$ is irreducible, due to Theorem (4.2.6), we must have $U = I \otimes W$, where $W \in B(D, D')$ is a unitary operator. This implies that $dimD = dimD'$ and $UK_{q,T,J}H = VH$. Consequently, the two dilations are unitarily equivalent. The proof is complete.

Proposition (4.2.13) [244]: Let $S = \{S_{i,j}\}$ be the universal model associated with the abstract noncommutative variety $V_{q,J}^m$, where J is a WOT-closed left ideal of $F^\infty(D_q^m)$ such that $1 \in N_J$, and $q = (q_1, \dots, q_k)$ is a k-tuple of positive regular noncommutative polynomials such that

$$\overline{span}\ \{S_{(\alpha)}S_{(\beta)}^*) : (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\} = C^*(S_{i,j}).$$

A pure element $T = \{T_{i,j}\} \in V_q^m(H)$ has

$$rank\ \Delta_{q,T}^m(I) = n, \qquad n = 1, 2, \dots, \infty,$$

if and only if it is unitarily equivalent to one obtained by compressing $\{S_{i,j} \otimes I_{\mathbb{C}^n}\}$ to a co-invariant subspace $M \subset N_J \otimes \mathbb{C}^n$ under each operator $S_{i,j} \otimes I_{\mathbb{C}^n}$ with the property that $dim[(P_{\mathbb{C}} \otimes I_{\mathbb{C}^n})M] = n$, where $P_{\mathbb{C}}$ is the orthogonal projection from N_J onto $\mathbb{C}1$.

Proof: Note that the direct implication is a consequence of Theorem (4.2.12). We prove the converse. Assume that

$$T_{(\alpha)} = P_H\big(S_{(\alpha)} \otimes I_{\mathbb{C}^n}\big)|_H, \qquad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$$

where $H \subset N_J \otimes \mathbb{C}^n$ is a co-invariant subspace under each operator $S_{i,j} \otimes I_{\mathbb{C}^n}$ such that $dim(P_{\mathbb{C}} \otimes I_{\mathbb{C}^n})H = n$. It is clear that T is a pure element in the noncommutative variety $V_q^m(H)$. First, we consider the case when $n < \infty$. Since $(P_{\mathbb{C}} \otimes I_{\mathbb{C}^n})H \subseteq \mathbb{C}^n$ and $dim(P_{\mathbb{C}} \otimes I_{\mathbb{C}^n})H = n$, we must have

$(P_{\mathbb{C}} \otimes I_{\mathbb{C}^n})H = \mathbb{C}^n$. The later condition is equivalent to the equality $H^{\perp} \cap \mathbb{C}^n = \{0\}$. Since $\Delta_{q,S}^m(I) = P_{\mathbb{C}}$, we have $\Delta_{q,T}^m(I) = P_H[P_{\mathbb{C}} \otimes I_{\mathbb{C}^n}]|_H = P_H\mathbb{C}^n$. Consequently, $rank\ \Delta_{q,T}^m(I) = dim\ P_H\mathbb{C}^n$. If we assume that $rank\ \Delta_{q,T}^m(I) < n$, then there exists $h \in \mathbb{C}^n, h \neq 0$, with $P_H h = 0$, which contradicts the relation $H^{\perp} \cap \mathbb{C}^n = \{0\}$. Therefore, we must have $rank\ \Delta_{q,T}^m(I) = n$.
Now, assume that $n = \infty$. According to Theorem (4.2.8) and its proof, we have

$$N_J \otimes \mathcal{E} = \bigvee_{(\alpha)\in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+} (S_{(\alpha)} \otimes I_{\mathbb{C}^n})\, H$$

where $\mathcal{E} := (P_{\mathbb{C}} \otimes I_{\mathbb{C}^n})H$. Since $N_J \otimes \mathcal{E}$ is reducing for each operator $S_{i,j} \otimes I_{\mathbb{C}^m}$, we deduce that $T_{(\alpha)} = P_{(\alpha)}(S_{(\alpha)} \otimes I_{\mathcal{E}})|_H$, for all $(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. Due to Theorem (4.2.12), the minimal dilation of T is unique. Consequently, we have $\dim \overline{\Delta_{q,T}^m(I)H} = \dim \mathcal{E} = \infty$. This completes the proof.
We characterize the pure elements of rank one in the noncommutative variety $V_{q,J}^m(H)$ and obtain a classification result.
Corollary (4.2.14) [244]: Under the hypothesis of Proposition (4.2.13), the following statements hold.
(i) If $M \subset N_J$ is a co-invariant subspace under each operator $S_{i,j}$, then $T := \{P_M S_{i,j}|_M\}$ is a pure element in the noncommutative variety $V_{q,J}^m(M)$ and $rank\ \Delta_{q,T}^m = 1$.
(ii) If M' is another co-invariant subspace under each operator $S_{i,j}$, which gives rise to T', then T and T' are unitarily equivalent if and only if $M = M'$.
Proof: To prove (i), note that $\Delta_{q,T}^m(I) = P_M P_{\mathbb{C}}|_M$ and, consequently, $rank\ \Delta_{q,T}^m(I) \leq 1$. Since S is pure (see Lemma (4.2.1)) and $M \subset N_J$ is a co-invariant subspace under each operator $S_{i,j}$, we deduce that T is pure. Hence, $\Delta_{q,T}^m(I) \neq 0$, so $rank\ \Delta_{q,T}^m(I) \geq 1$. Therefore, $rank \Delta_{q,T}^m(I) = 1$.
To prove (ii), note that, as in the proof of Theorem (4.2.12), one can show that T and T' are unitarily equivalent if and only if there exists a unitary operator $\Lambda: N_J \to N_J$ such that $\Lambda S_{i,j} = S_{i,j}\Lambda$ for all i, j, and $\Lambda M = M'$. Since $\Lambda S_{i,j}^* = S_{i,j}^*\Lambda$ and $C^*(S_{i,j})$ is irreducible, Λ must be a scalar multiple of the identity. Therefore, we must have $M = \Lambda M = M'$. The proof is complete.
we find all the joint eigenvectors for $S_{i,j}^*$, where $S = \{S_{i,j}\}$ is the universal model associated with the noncommutative variety $V_{f,J}^m$ and J is a WOT-closed left ideal of the Hardy space $F^{\infty}(D_f^m)$. As consequences, we determine the joint right spectrum of S and identify the character space of the noncommutative variety algebra $A(V_{f,J}^m)$. When J_c is the commutator ideal of $F^{\infty}(D_f^m)$, we show that the WOT-closed algebra $F^{\infty}(V_{f,J_c}^m)$ generated by $S_{i,j}$ and the identity coincides with the multiplier algebra of a reproducing kernel Hilbert space of holomorphic functions on a certain polydomain in $\mathbb{C}^n$.
The results show that there is a strong connection between the study of noncommutative varieties in polydomains and the analytic function theory in $\mathbb{C}^n$.
Let $f := (f_1, \ldots, f_k)$ be a k-tuple of positive regular free holomorphic functions and define the polydomain

$$D_{f,>}^m(\mathbb{C}) := \{z = (z_1, \ldots z_k) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k} : \Delta_{f,z}^m(1) > 0\}.$$

Note that $D_{f,>}^m(\mathbb{C}) = D_{f_1,>}^1(\mathbb{C}) \times \cdots \times D_{f_k,>}^1(\mathbb{C})$, where $f_i := \sum_{\alpha\in\mathbb{F}_{n_i}^+} a_{i,\alpha} Z_\alpha$ and

$$D_{f_i,>}^1(\mathbb{C}) := \{z_i = (z_{i,1}, \ldots, z_{i,n_i}) \in \mathbb{C}^{n_i} : \sum_{\alpha\in\mathbb{F}_{n_i}^+} a_{i,\alpha}|z_{i,\alpha}|^2 < 1\}.$$

Let J be a WOT-closed left ideal of the Hardy space $F^{\infty}(D_f^m)$. We consider the set

$$V_{f,J,>}^m(\mathbb{C}) := \{z = (z_1, \ldots z_k) \in D_{f,>}^m(\mathbb{C}) : g(z_1, \ldots, z_k) = 0\ for\ g \in J\} \subset \mathbb{C}^n,$$

where $n = n_1 + \cdots + n_k$ is the number of indeterminates in $f := (f_1, \ldots, f_k)$.
Theorem (4.2.15) [244]: Let $S = \{S_{i,j}\}$ be the universal model associated with the noncommutative variety $V_{f,J}^m$, where J is a WOT-closed left ideal of the Hardy space $F^{\infty}(D_f^m)$. The joint eigenvectors for $S_{i,j}^*$ are precisely the noncommutative constrained Berezin kernels

$$\Gamma_\lambda := \Delta_{f,\lambda}^m(1)^{1/2} \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\dots,k} \bar{\lambda}_{1,\beta_1} \cdots \bar{\lambda}_{k,\beta_k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k$$

for $\lambda = (\lambda_1, \dots, \lambda_n) \in V_{f,J,>}^m(\mathbb{C})$, where $\Delta_{f,\lambda}^m(1) := (1 - \Phi_{f_1,\lambda_1}(1))^{m_1} \cdots (1 - \Phi_{f_k,\lambda_k}(1))^{m_k}$. They satisfy the equations

$$S_{i,j}^* \Gamma_\lambda = \bar{\lambda}_{i,j} \Gamma_\lambda \quad for\ i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\},$$

where $\lambda_i = (\lambda_{i,1}, \dots, \lambda_{i,n_i})$.

Proof: First, note that if $\lambda = (\lambda_1, \dots, \lambda_n) \in V_{f,J,>}^m(\mathbb{C})$, then λ is a pure element. The noncommutative constrained Berezin kernel at λ is $K_{f,\lambda,J}: \mathbb{C} \to N_J \otimes \mathbb{C}$ defined by

$$K_{f,\lambda,J}(w) = \Delta_{f,\lambda}^m(1)^{1/2} \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\dots,k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes \bar{\lambda}_{1,\beta_1} \cdots \bar{\lambda}_{k,\beta_k} w,$$

$$w \in \mathbb{C}.$$

According to Proposition (4.2.2), we have $(S_{i,j}^* \otimes I_\mathbb{C}) K_{f,\lambda,J} = K_{f,\lambda,J}(\bar{\lambda}_{i,j} I_\mathbb{C})$ for $i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$.

Identifying $N_J \otimes \mathbb{C}$ with N_J, we have $K_{f,\lambda,J} = \Gamma_\lambda$ and $S_{i.j}^* \Gamma_\lambda = \bar{\lambda}_{i,j} \Gamma_\lambda$.

Conversely, let $h = \beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+ \; c\beta_1, \dots, \beta_k e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k$ be a non-zero vector in $N_J \subset \otimes_{i=1}^k F^2(H_{n_i})$ and assume that there exists $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^{n_i} \times \cdots \times \mathbb{C}^{n_k}$, where $\lambda_i = (\lambda_{i,1}, \dots, \lambda_{i,n_i})$, such that $S_{i,j}^* h = \bar{\lambda}_{i,j} h$ for any $i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$. Since N_J is invariant under $W_{i,j}^*$, we also have $W_{i,j}^* h = \bar{\lambda}_{i,j} h$. Using the definition of the operators $W_{i,j}$, we deduce that

$$c_{\beta_1,\dots,\beta_k} = \langle h, e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \rangle = \langle h, \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} W_{1,\beta_1} \cdots W_{k,\beta_k}(1) \rangle$$

$$= \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} \langle W_{1,\beta_1}^* \cdots W_{k,\beta_k}^* h, 1 \rangle = \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} \bar{\lambda}_{1,\beta_1} \cdots \bar{\lambda}_{k,\beta_k} \langle h, 1 \rangle$$

$$= c_0 \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} \bar{\lambda}_{1,\beta_1} \cdots \bar{\lambda}_{k,\beta_k}$$

for any $\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+$. Hence, we obtain

$$h = c_0 \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\dots,k} \bar{\lambda}_{1,\beta_1} \cdots \bar{\lambda}_{k,\beta_k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k.$$

Since $h \in \otimes_{i=1}^k F^2(H_{n_i})$, we must have $\sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+} |\lambda_{1,\beta_1}|^2 \cdots |\lambda_{k,\beta_k}|^2 b_{1,\beta_1}^{(m_1)} \cdots b_{k,\beta_k}^{(m_1)} < \infty$.
On the other hand, relation (47) implies

$$\prod_{i=1}^k \left(\sum_{s=0}^{p_i} \left(\sum_{|\alpha_i| \geq 1} a_{i,\alpha_i} |\lambda_{i,\alpha_i}|^2 \right)^s \right)^{m_i} \leq \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+} |\lambda_{1,\beta_1}|^2 \cdots |\lambda_{k,\beta_k}|^2 b_{1,\beta_1}^{(m_1)} \cdots b_{k,\beta_k}^{(m_1)} < \infty$$

for any $p_1, \dots, p_k \in \mathbb{N}$. Letting $p_i \to \infty$ in the relation above, we must have $\sum_{|\alpha_i| \geq 1} a_{i,\alpha_i} |\lambda_{i,\alpha_i}|^2 < 1$, for each $i \in \{1, \dots, k\}$. Therefore, $\lambda = (\lambda_1, \dots, \lambda_n) \in D_{f,>}^m(\mathbb{C})$. On the other hand, if $g \in J$, then relation $S_{i,j}^* h = \bar{\lambda}_{i,j} h$ and an approximation argument in the norm topology imply

$$\langle h, g(rS_{i,j}) h \rangle = \langle g(rS_{i,j})^* h, h \rangle = \overline{g(r\lambda_{i,j})} \|h\|^2.$$

Using the $F^\infty(D_f^m)$-functional calculus for pure elements and taking the limit as $r \to 1$ in the relation above, we obtain

$$\langle h, g(S_{i,j}) h \rangle = \overline{g(\lambda_{i,j})} \|h\|^2.$$

Since, due to Lemma (4.2.1), $g(S_{i,j}) = 0$ and $h \neq 0$, we conclude that $g(\lambda_{i,j}) = 0$, which shows that $\lambda \in V_{f,J,>}^m(\mathbb{C})$. The proof is complete.

Let $S = \{S_{i,j}\}$ be the universal model associated with the noncommutative variety $V_{f,J}^m$, where J is a WOT-closed left ideal of the Hardy algebra $F^\infty(D_f^m)$. We introduce the Hardy algebra $F^\infty(V_{f,J}^m)$ as the WOT-closed algebra generated by $S_{i,j}$ and the identity.

Theorem (4.2.16) [244]: Let J be a WOT-closed left ideal of the Hardy algebra $F^\infty(D_f^m)$ such that $1 \in N_J$. Then $\Phi: F^\infty(V_{f,J}^m) \to \mathbb{C}$ is a w^*-continuous and multiplicative linear functional if and only if there exists $\lambda \in V_{f,J,>}^m(\mathbb{C})$ such that

$$\Phi(A) = \langle A(1), u_\lambda \rangle\,, \qquad A \in F^\infty(V_{f,J}^m),$$

where $u_\lambda := \frac{1}{\Delta_{f,\lambda}^m(1)^{1/2}} \Gamma_\lambda$ and Γ_λ is given by Theorem (4.2.15). Moreover, in this case, $A^* u_\lambda = \overline{\Phi(A)} u_\lambda$ and

$$\Phi(A) = \langle A\Gamma_\lambda, \Gamma_\lambda \rangle, \qquad A \in F^\infty(V_{f,J}^m).$$

Proof: For each $\lambda \in V_{f,J,>}^m(\mathbb{C})$, let $\Phi_\lambda: F^\infty(V_{f,J}^m) \to \mathbb{C}$ be given by $\Phi_\lambda(A) = \langle A(1), u_\lambda \rangle$. It is clear that Φ_λ is w^*-continuous. To prove that Φ_λ is multiplicative, let $\varphi, \psi \in F^\infty(V_{f,J}^m)$ and let $\{p_\iota(S_{i,j})\}$ and $\{q_\kappa(S_{i,j})\}$ be nets of polynomials such that $p_\iota(S_{i,j}) \to \varphi$ and $q_\kappa(S_{i,j}) \to \psi$ in the weak operator topology.

Note that, due to Theorem (4.2.15), we have $p_\iota(\lambda) = \langle p_\iota(W_{i,j})1, u_\lambda \rangle = \langle p_\iota(S_{i,j})1, u_\lambda \rangle$ and, consequently, $\lim_\iota p_\iota(\lambda) = \langle \varphi(1), u_\lambda \rangle$. Similarly, we obtain $\lim_\kappa q_\kappa(\lambda) = \langle \psi(1), u_\lambda \rangle$. Hence, it is easy to see that

$$\Phi_\lambda(\varphi\psi) = \langle \psi\psi(1), u_\lambda \rangle = \lim_\kappa \langle q_\kappa(1), \varphi^*(u_\lambda) \rangle$$
$$= \lim_\kappa \lim_\iota \langle p_\iota(S_{i,j}) q_\kappa(S_{i,j})(1), u_\lambda \rangle = \lim_\kappa \lim_\iota p_\iota(\lambda) q_\kappa(\lambda)$$
$$= \langle \varphi(1), u_\lambda \rangle \lim_\kappa q_\kappa(\lambda) = \langle \varphi(1), u_\lambda \rangle \langle \psi(1), u_\lambda \rangle = \Phi_\lambda(\varphi)\Phi_\lambda(\psi).$$

Note that, due to Theorem (4.2.15), we have

$$p_\iota(S_{i,j})^* u_\lambda = \overline{p_\iota(\lambda)} u_\lambda = \overline{\langle p_\iota(S_{\iota,J})1, u_\lambda \rangle} u_\lambda.$$

Since $p_\iota(S_{i,j}) \to \varphi$ in the weak operator topology, we deduce that $\varphi^* u_\lambda = \overline{\langle \varphi(1), u_\lambda \rangle} u_\lambda$. Hence, we deduce that

$$\langle \varphi\Gamma_\lambda, \Gamma_\lambda \rangle = \Delta_{f,\lambda}^m(1) \langle u_\lambda, \varphi^* u_\lambda \rangle = \varphi(\lambda) = \Phi_\lambda(\varphi).$$

Now, assume that $\Phi: F^\infty(V_{f,J}^m) \to \mathbb{C}$ is a w^*-continuous and multiplicative linear functional and let $\chi := \ker \Phi$. Then χ is a w^*-closed two-sided ideal of $F^\infty(V_{f,J}^m)$ of codimension one. We claim that $M_\chi := \overline{X N_J}$ is a subspace in N_J of codimension one and $M_\chi + \mathbb{C}1 = N_J$. By contradiction, assume that there is a vector $y \in N_J$ which is perpendicular to $M_X + \mathbb{C}1$ and $\|y\| = 1$. Since

$$\overline{span}\{p(W_{i,j})(1): p \in \mathbb{C}[Z_{i,j}]\} = \otimes_{i=1}^k F^2(H_{n_i})$$

and taking the projection onto N_J , we deduce that $\overline{span}\{p(S_{i,j})(1): p \in \mathbb{C}[Z_{i,j}]\} = N_J$. Consequently, we can choose a polynomial $p(S_{i,j}) \in F^\infty(V_{f,J}^m)$ such that $\|p(S_{i,j})(1) - y\| < 1$. On the other hand, since $p(S_{i,j}) - \Phi(p(S_{i,j})) I_{N_J}$ is in $X = \ker \Phi$ and $1 \in N_J$, we have $p(S_{i,j})(1) - \Phi(p(S_{i,j})) \in M_X$. Taking into account that y is perpendicular to $M_X + \mathbb{C}1$, we have

$$\|y\| = \langle y - \Phi(p(S_{i,j})), y \rangle$$
$$\le |\langle y - p(S_{i,j})(1), y \rangle| + |\langle p(S_{i,j})(1) - \Phi(p(S_{i,j})), y \rangle|$$
$$= |\langle y - p(S_{i,j})(1), y \rangle| \le \|y - p(S_{i,j})(1)\| \|y\| < 1,$$

which contradicts the fact that $\|y\| = 1$ and proves our assertion. Therefore, $M_X \subset N_J$ has codimension one and it is invariant under each operator $S_{i,j}$ for $i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}$. According to Theorem (4.2.15), there exists $\lambda \in V_{f,J,>}^m(\mathbb{C})$ such that $M_X = \{u_\lambda\}^\perp$. As shown in the first part of the proof, Φ_λ is a w^*-continuous and multiplicative linear functional. Note that, if $A \in X := \ker \Phi$, then $A(1) \in M_X = \{u_\lambda\}^\perp$, which implies $\langle A(1), u_\lambda \rangle = 0$. Hence, $A \in \ker \Phi_\lambda$ and, therefore, $\ker \Phi \subset \ker \Phi_\lambda$. Since $\ker \Phi$ and $\ker \Phi_\lambda$ are w^*-closed two sided maximal ideals of $F^\infty(V_{f,J}^m)$ of codimension one, we must have $\ker \Phi = \ker \Phi_\lambda$. Therefore, $\Phi = \Phi_\lambda$. This completes the proof.

We make a few remarks concerning the particular case when $J = \{0\}$. First, we note that if $\lambda = (\lambda_1, \dots, \lambda_n) \in D_{f,>}^m(\mathbb{C})$ and $\varphi(W_{i,j}) = \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+} c_{\beta_1,\dots,\beta_k} W_{1,\beta_1} \cdots W_{k,\beta_k}$ is in the Hardy algebra $F^\infty(D_f^m)$, then $\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+}|c_{\beta_1,\dots,\beta_k}||\lambda_{1,\beta_1}|\cdots|\lambda_{k,\beta_k}|<\infty$. Indeed, since $\varphi(W_{i,j})(1)\in\bigotimes_{i=1}^k F^2(H_{n_i})$, we have

$$K_1:=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+}|c_{\beta_1,\dots,\beta_k}|^2\frac{1}{b_{1,\beta_1}^{(m_1)}\cdots\ b_{k,\beta_k}^{(m_1)}}<\infty.$$

On the other hand, since $\lambda=(\lambda_1,\dots,\lambda_n)\in D_{f,>}^m(\mathbb{C})$, we deduce that

$$K_2:=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+}|\lambda_{1,\beta_1}|^2\cdots|\lambda_{k,\beta_k}|^2 b_{1,\beta_1}^{(m_1)}\cdots\ b_{k,\beta_k}^{(m_1)}<\infty.$$

Applying Cauchy's inequality, we obtain

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+}|c_{\beta_1,\dots,\beta_k}||\lambda_{1,\beta_1}|\cdots|\lambda_{k,\beta_k}|\le(K_1K_2)^{1/2}.$$

We note that the w^*-continuous and multiplicative map $\Phi_\lambda: F^\infty(D_f^m)\to\mathbb{C}$ satisfies the equation $\Phi_\lambda(\varphi(W_{i,j})):=\varphi(\lambda)$. Indeed, in this case we have

$$\begin{aligned}\langle\varphi(W_{i,j})1,u_\lambda\rangle&=\langle\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+}c_{\beta_1,\dots,\beta_k}\frac{1}{\sqrt{b_{1,\beta_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\beta_k}^{(m_1)}}}e_{\beta_1}^1\otimes\cdots\otimes e_{\beta_k}^k,u_\lambda\rangle\\&=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+}c_{\beta_1,\dots,\beta_k}\lambda_{1,\beta_1}\cdots\ \lambda_{k,\beta_k}=\varphi(\lambda).\end{aligned}$$

We recall that the joint right spectrum $\sigma_r(T_1,\dots,T_n)$ of an n-tuple $(T_1,\dots,T_n)$ of operators in $B(H)$ is the set of all n-tuples $(\mu_1,\dots,\mu_n)$ of complex numbers such that the right ideal of $B(H)$ generated by the operators $\mu_1 I-T_1,\dots,\mu_n I-T_n$ does not contain the identity operator. We recall [250] that $(\mu_1,\dots,\mu_n)\notin\sigma_r(T_1,\dots,T_n)$ if and only if there exists $\delta>0$ such that $\sum_{i=1}^n(\mu_i I-T_i)(\overline{\mu_i}I-T_i^*)\ge\delta I$.

Proposition (4.2.17) [244]: Let J be a WOT-closed left ideal of the Hardy space $F^\infty(D_f^m)$ and let $S=\{S_{i,j}\}$ be the universal model associated with the abstract noncommutative variety $V_{f,J}^m$. If the set $V_{f,J,>}^m(\mathbb{C})$ is dense in $V_{f,J}^m(\mathbb{C})$, then the right joint spectrum $\sigma_r(S)$ coincide with $V_{f,J}^m(\mathbb{C})$.
In particular, if $Q\subset\mathbb{C}[Z_{i,j}]$ is a left ideal generated by noncommutative homogenous polynomials, then the right joint spectrum $\sigma_r(S)=V_{f,Q}^m(\mathbb{C})$.

Proof: Let $\lambda=\{\lambda_{i,j}\}\in\sigma_r(S)$. Since the left ideal of $B(N_Q)$ generated by the operators $S_{i,j}^*-\bar\lambda_{i,j}I$ does not contain the identity, there is a pure state φ on $B(N_Q)$ such that $\varphi(X(S_{i,j}^*-\bar\lambda_{i,j}I))=0$ for any $X\in B(N_Q)$ and $i\in\{1,\dots,k\}, j\in\{1,\dots,n_i\}$. In particular, we have $\varphi(S_{i,j})=\lambda_{i,j}=\overline{\varphi(S_{i,j}^*)}$ and

$$\varphi\left(S_{(\alpha)}S_{(\alpha)}^*\right)=\bar\lambda_{(\alpha)}\varphi\left(S_{(\alpha)}\right)=\left|\lambda_{(\alpha)}\right|^2,\qquad(\alpha)=(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\ \mathbb{F}_{n_k}^+.$$

Hence, we deduce that

$$\sum_{\alpha\in\mathbb{F}_{n_i}^+,1\le|\alpha|\le m}a_{i,\alpha}|\lambda_{i,\alpha}|^2=\varphi\left(\sum_{\alpha\in\mathbb{F}_{n_i}^+,1\le|\alpha|\le m}a_{i,\alpha}S_{i,\alpha}S_{i,\alpha}^*\right)\le\left\|\sum_{\alpha\in\mathbb{F}_{n_i}^+,1\le|\alpha|\le m}a_{i,\alpha}S_{i,\alpha}S_{i,\alpha}^*\right\|\le 1$$

for any $n\in\mathbb{N}$. Therefore, $\sum_{\alpha\in\mathbb{F}_{n_i}^+}a_{i,\alpha}\left|\lambda_{i,\alpha}\right|^2\le 1$, which proves that $(\lambda_{i,1},\dots,\lambda_{i,n_i})\in D_{f_i}^1(\mathbb{C})$. Hence, we deduce that $\lambda:=\{\lambda_{i,j}\}\in D_f^m(\mathbb{C})$. On the other hand, if $g\in Q$, then $g(S_{i,j})=0$ and, consequently, we obtain $g(\lambda_{i,j})=\varphi(g(S_{i,j}))=0$. Therefore, $\lambda\in V_{f,Q}^m(\mathbb{C})$. Now, let $\mu:=\{\mu_{i,j}\}\in V_{f,Q}^m(\mathbb{C})$ and assume that there is $\delta>0$ such that

$$\sum_{i=1}^n\sum_{j=1}^{n_i}\left\|(S_{i,j}-\ \mu_{i,j}I)^*h\right\|^2\ge\delta\|h\|^2\quad for\ all\ h\in N_Q.$$

Take

$$h=\Gamma_\lambda:=\Delta_{f,\lambda}^{m}(1)^{1/2}\sum_{\beta_i\in\mathbb{F}_{n_i}^+,i=1,\dots,k}\bar{\lambda}_{1,\beta_1}\cdots\bar{\lambda}_{k,\beta_k}\sqrt{b_{1,\beta_1}^{(m_1)}}\cdots\sqrt{b_{k,\beta_k}^{(m_k)}}e_{\beta_1}^1\otimes\cdots\otimes e_{\beta_k}^k$$

for $\lambda\in V_{f,Q,>}^m(\mathbb{C})$ in the inequality above. Due to Theorem (4.2.15), we have $S_{i,j}^*\Gamma_\lambda=\bar{\lambda}_{i,j}\Gamma_\lambda$ for any $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$. Consequently, we deduce that

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}|\lambda_{i,j}-\mu_{i,j}|^2\geq\delta\quad for\ all\quad \lambda=\{\lambda_{i,j}\}\in V_{f,Q,>}^m(\mathbb{C}).$$

Since the set $V_{f,J,>}^m(\mathbb{C})$ is dense in $V_{f,J}^m(\mathbb{C})$, this leads to a contradiction.

Note that if $Q\subset\mathbb{C}[Z_{i,j}]$ is a left ideal generated by noncommutative homogenous polynomials, then $\{r\mu_{i,j}\}\in V_{f,Q,>}^m(\mathbb{C})$ for any $\{\mu_{i,j}\}\in V_{f,Q}^m(\mathbb{C})$ and $r\in[0,1)$. Consequently, $V_{f,Q,>}^m(\mathbb{C})$ is dense in $V_{f,Q,>}^m(\mathbb{C})$.

The proof is complete.

Let $Q\subset\mathbb{C}[Z_{i,j}]$ be a left ideal generated by noncommutative homogenous polynomials. We recall that the variety algebra $A(V_{f,Q}^m)$ is the norm closed algebra generated by the $S_{i,j}$ and the identity, and the Hardy algebra $F^\infty(V_{f,Q}^m)$ is the WOT-closed version. We identify the characters of the noncommutative variety algebra $A(V_{f,Q}^m)$. Due to Proposition (4.2.5), if $\lambda\in V_{f,Q}^m(\mathbb{C})$, then the evaluation functional

$$\Phi_\lambda: A(V_{f,Q}^m)\to\mathbb{C},\qquad \Phi_\lambda(p(S_{i.j}))=p(\lambda_{i,j}),$$

is a character of $A(V_{f,Q}^m)$.

Theorem (4.2.18) [244]: Let $Q\subset\mathbb{C}[Z_{i,j}]$ be a left ideal generated by noncommutative homogenous polynomials and let $M_{A(V_{f,Q}^m)}$ be the set of all characters of $A(V_{f,Q}^m)$. Then the map

$$\Psi: V_{f,Q}^m(\mathbb{C})\to M_{A\left(V_{f,Q}^m\right)},\qquad \Psi(\lambda)=\Phi_\lambda,$$

is a homeomorphism of $V_{f,Q}^m(\mathbb{C})$ onto $M_{A(V_{f,Q}^m)}$.

Proof: The injectivity of Ψ is clear. To prove that Ψ is surjective assume that $\Phi: A(V_{f,Q}^m)\to\mathbb{C}$ is a character. Setting $\lambda_{i,j}:=\ \Phi(S_{i,j})$ for $i\in\{1,\dots,k\}, j\in\{1,\dots,n_i\}$, we deduce that $\Phi(p(S_{i,j}))=p(\lambda_{i,j})$ for any polynomial $p(S_{i,j})$ in $A(V_{f,Q}^m)$. Since Φ is a character, it is completely contractive. Consequently, $(\lambda_{i,1},\dots,\lambda_{i,n_i})\in D_{f_i}^1(\mathbb{C})$ for each $i\in\{1,\dots,k\}$, which implies $(\lambda_{i,j}I_{\mathbb{C}})\in D_f^m(\mathbb{C})$. On the other hand, if $g\in Q$, then $g(S_{i,j})=0$ and, consequently, $g(\lambda_{i,j})=\Phi(g(S_{i,j}))=0$. Therefore, $\{\lambda_{i,j}\}\in V_{f,Q}^m(\mathbb{C})$. Since

$$\Phi(p(S_{i,j}))=p(\lambda_{i,j})=\Phi_\lambda(p(S_{i,j}))$$

for any polynomial $p(S_{i,j})$ in $A(V_{f,Q}^m)$, we must have $\Phi=\Phi_\lambda$. To prove that Ψ is a homeomorphism, let $\lambda^\alpha:=(\lambda_{i,j}^\alpha),\alpha\in\Lambda$, be a net in $V_{f,Q}^m(\mathbb{C})$ such that $\lim_{\alpha\in\Lambda}\lambda^\alpha=\lambda:=(\lambda_{i,j})$. It is clear that

$$\lim_{\alpha\in\Lambda}\Phi_{\lambda^\alpha}(p(S_{i,j}))=\lim_{\alpha\in\Lambda}p(\lambda^\alpha)=\ p(\lambda)=\Phi_\lambda(p(S_{i,j})).$$

Since the set of all polynomials $p(S_{i,j})$ is dense in $A(V_{f,Q}^m)$ and $\sup_{\alpha\in\Lambda}\|\Phi_{\lambda^\alpha}\|\leq 1$, it follows that Ψ is continuous. On the other hand, since both $V_{f,Q}^m(\mathbb{C})$ and $M_{A(V_{f,Q}^m)}$ are compact Hausdorff spaces and Ψ is a bijection, the result follows. The proof is complete.

Let $W=\{W_{i,j}\}$ be the universal model associated with the abstract noncommutative polydomain D_f^m and let Q_c be the left ideal generated by all polynomials of the form

$$Z_{i,j_1}Z_{i,j_2}-Z_{i,j_2}Z_{i,j_1},\qquad i\in\{1,\dots,k\}\ and\ j_1,j_2\in\{1,\dots,n_i\}.$$

The universal model associated with the abstract variety V_{f,Q_c}^m is the tuple $L=(L_1,\dots,L_k)$ with $L_i:=(L_{i,1},\dots,L_{i,n_i})$, where the operators $L_{i,j}$ are defined on N_{Q_c} by setting

$$L_{i,j}:=P_{N_{Q_c}}W_{i,j}|_{N_{Q_c}}.$$

We recall that $N_{Q_c}:=(\otimes_{i=1}^k F^2(H_{n_i}))\ominus M_Q$, where the subspace M_{Q_c} of $\otimes_{i=1}^k F^2(H_{n_i})$ is defined by setting

$$M_{Q_c} := \overline{span}\{W_{(\alpha)} q(W_{i,j}) W_{(\beta)}(1) : (\alpha), (\beta) \in \mathbb{F}_{n_i}^+ \times \cdots \times \mathbb{F}_{n_k}^+, q \in Q_c\}.$$

In what follows, we will identify the space N_{Q_c} with a reproducing kernel Hilbert space of holomorphic functions in several complex variables and the Hardy algebra $F^\infty(V_{f,Q_c}^m)$ is identified with the corresponding multiplier algebra.

Let $f := (f_1, \ldots, f_k)$ be a k-tuple of positive regular free holomorphic functions with $f_i := \sum_{\alpha \in \mathbb{F}_{n_i}^+} a_{i,\alpha} Z_\alpha$.

For each $\lambda_i = (\lambda_{i,1}, \ldots, \lambda_{i,n_i}) \in \mathbb{C}^{n_i}$ and each n_i-tuple $k_i := (k_{i,1}, \ldots, k_{i,n_i}) \in \mathbb{N}_0^{n_i}$, where $\mathbb{N}_0 := \{0, 1, \ldots\}$, let $\lambda_i^{k_i} := \lambda_{i,1}^{k_{i,1}} \cdots \lambda_{i,n}^{k_{i,n_i}}$. If $k_i \in \mathbb{N}_0^{n_i}$, we denote

$$\Lambda_{k_i} := \{\alpha_i \in \mathbb{F}_{n_i}^+ : \lambda_{i,\alpha_i} = \lambda_i^{k_i} \ for\ all\ \lambda_i \in \mathbb{C}^{n_i}\}$$

and define the vector

$$w_i^{k_i} := \frac{1}{\gamma_{k_i}^{(m_i)}} \sum_{\alpha_i \in \Lambda_{k_i}} \sqrt{b_{i,\alpha_i}^{(m_i)}}\, e_{\alpha_i}^i \in F^2(H_{n_i}), where\ \gamma_{k_i}^{(m_i)} := \sum_{\alpha_i \in \Lambda_{k_i}} b_{i,\alpha_i}^{(m_i)}$$

and the coefficients $b_{i,\alpha_i}^{(m_i)}, \alpha_i \in \mathbb{F}_{n_i}^+$, are defined by relation (47). It is easy to see that the set $\{w_1^{k_1} \otimes \cdots \otimes w_k^{k_k} : k_i \in \mathbb{N}_0^{n_i}, i \in \{1, \ldots, k\}\}$ consists of orthogonal vectors in $\otimes_{i=1}^k F^2(H_{n_i})$ and

$$\|w_1^{k_1} \otimes \cdots \otimes w_k^{k_k}\| = \frac{1}{\sqrt{\gamma_{k_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{\gamma_{k_k}^{(m_k)}}}.$$

Let $F_s^2(D_f^m)$ be the closed span of these vectors. The Hilbert space $F_s^2(D_f^m) \subset \otimes_{i=1}^k F^2(H_{n_i})$ is called the symmetric tensor product Fock space associated with the abstract noncommutative domain D_f^m.

For $z = (z_1, \ldots, z_n)$ and $w := (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we use the notation $z\bar{w} := (z_1\bar{w}_1, \ldots, z_n\bar{w}_n)$.

Theorem (4.2.19) [244]: Let $W = \{W_{i,j}\}$ be the universal model associated with the noncommutative polydomain D_f^m, and let Q_c be the left ideal generated by all polynomials of the form

$$Z_{i,j_1} Z_{1,j_2} - Z_{i,j_2} Z_{i,j_1}, \qquad i \in \{1, \ldots, k\} and\ j_1, j_2 \in \{1, \ldots, n_i\}.$$

Then the following statements hold.

(i) $F_s^2(D_f^m) = \overline{span}\{\Gamma_\lambda : \lambda \in D_{f,>}^m(\mathbb{C})\} = N_{Q_c} := (\otimes_{i=1}^k F^2(H_{n_i})) \ominus M_{Q_c}$.

(ii) The space $F_s^2(D_f^m)$ can be identified with the Hilbert space $H^2(D_{f,>}^m(\mathbb{C}))$ of all functions $\varphi : D_{f,>}^m(\mathbb{C}) \to \mathbb{C}$ which admit a power series representation

$$\varphi(\lambda_{i,j}) = \sum_{k_1 \in \mathbb{N}_0^{n_1}, \ldots, k_k \in \mathbb{N}_0^{n_k}} c_{k_1,\ldots,k_k} \lambda_1^{k_1} \cdots \lambda_k^{k_k}$$

with

$$\|\varphi\|_2^2 = \sum_{k_1 \in \mathbb{N}_0^{n_1}, \ldots, k_k \in \mathbb{N}_0^{n_k}} |c_{k_1,\ldots,k_k}|^2 \frac{1}{\gamma_{k_1}^{(m_1)}} \cdots \frac{1}{\gamma_{k_1}^{(m_1)}} < \infty.$$

More precisely, every element $\varphi = \sum_{k_1 \in \mathbb{N}_0^{n_1}, \ldots, k_k \in \mathbb{N}_0^{n_k}} c_{k_1,\ldots,k_k} w_1^{k_1} \otimes \cdots \otimes w_k^{k_k}$ in $F_s^2(D_f^m)$ has a functional representation on $D_{f,>}^m(\mathbb{C})$ given by

$$\varphi(\lambda) := \langle \varphi, u_\lambda \rangle = \sum_{k_1 \in \mathbb{N}_0^{n_1}, \ldots, k_k \in \mathbb{N}_0^{n_k}} c_{k_1,\ldots,k_k} \lambda_1^{k_1} \cdots \lambda_k^{k_k}, \lambda = (\lambda_1, \ldots, \lambda_k) \in D_{f,>}^m(\mathbb{C}),$$

and

$$|\varphi(\lambda)| \le \frac{\|\varphi\|_2}{\sqrt{\Delta_{f,\lambda}^m(1)}}, \qquad \lambda = (\lambda_1, \ldots, \lambda_k) \in D_{f,>}^m(\mathbb{C}),$$

where $\Delta_{f,\lambda}^m(1) = (1 - \Phi_{f_1,\lambda_1}(1))^{m_1} \cdots (1 - \Phi_{f_k,\lambda_k}(1))^{m_k}$ and $u_\lambda := \frac{1}{\Delta_{f,\lambda}^m(1)^{1/2}} \Gamma_\lambda$.

(iii) The mapping $\kappa_f^c: D_{f,>}^m(\mathbb{C}) \times D_{f,>}^m(\mathbb{C}) \to \mathbb{C}$ defined by

$$\kappa_f^c(\mu,\lambda) := \frac{1}{\prod_{i=1}^k \left(1 - f_i(\mu_i \bar{\lambda}_i)\right)^{m_i}},$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ are in $D_{f,>}^m(\mathbb{C})$, is positive definite and

$$\kappa_f^c(\mu,\lambda) = \langle u_\lambda, u_\mu \rangle.$$

Proof: We prove that

$$\overline{span}\{\Gamma_\lambda : \lambda \in D_{f,>}^m(\mathbb{C})\} \subseteq F_s^2(D_f^m) \subseteq N_{Q_c}.$$

Note that the first inclusion is due to the fact that

$$u_\lambda = \sum_{k_1 \in \mathbb{N}_0^{n_1}, \ldots, k_k \in \mathbb{N}_0^{n_k}} \lambda_1^{k_1} \cdots \lambda_k^{k_k} \gamma_{k_1}^{(m_1)} \cdots \gamma_{k_k}^{(m_k)} w_1^{k_1} \otimes \cdots \otimes w_k^{k_k} \qquad (58)$$

for $\lambda = (\lambda_1, \ldots, \lambda_n) \in D_{f,>}^m(\mathbb{C})$. To prove the second inclusion, note that, due to the definition of the universal model $W = \{W_{i,j}\}$, we have

$$\langle w_i^{k_i}, W_{i,\gamma_i}(W_{i,j_1}W_{i,j_2} - W_{i,j_2}W_{i,j_1})W_{i,\beta_i}(1)\rangle$$

$$= \frac{1}{\gamma_{k_i}^{(m_i)}} \langle \sum_{\alpha_i \in \Lambda_{k_i}} \sqrt{b_{i,\alpha_i}^{(m_i)}}\, e_{\alpha_i}^i, \frac{1}{\sqrt{b_{i,\gamma_i g_{j_1} g_{j_2} \beta_i}^{(m_i)}}} e_{\gamma_1 g_{j_1} g_{j_2} \beta_i}^i - \frac{1}{\sqrt{b_{i,\gamma_i g_{j_2} g_{j_1} \beta_i}^{(m_i)}}} e_{\gamma_i g_{j_2} g_{j_1} \beta_i}^i \rangle = 0$$

for any $k_i \in \mathbb{N}_0^{n_i}, \gamma_i, \beta_i \in \mathbb{F}_{n_i}^+, i \in \{1, \ldots, k\}$. This implies that $w_1^{k_1} \otimes \cdots \otimes w_k^{k_k} \in N_{Q_c}$ and proves our assertion. To complete the proof of part (i), it is enough to show that

$$N_{Q_c} \subseteq \overline{span}\{\Gamma_\lambda : \lambda \in D_{f,>}^m(\mathbb{C})\}.$$

To this end, assume that there is a vector $x := \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \ldots, \beta_k \in \mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k, \in N_{Q_c}$ and $x \perp u_\lambda$ for all $\lambda \in D_{f,>}^m(\mathbb{C})$. Then, using relation (58), we obtain

$$\langle \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \ldots, \beta_k \in \mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k, u_\lambda \rangle$$

$$= \sum_{k_1 \in \mathbb{N}_0^{n_1}, \ldots, k_k \in \mathbb{N}_0^{n_k}} \left(\sum_{\beta_i \in \Lambda_{k_i}, i \in \{1,\ldots,k\}} c_{\beta_1,\ldots,\beta_k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} \right) \lambda_1^{k_1} \cdots \lambda_k^{k_k} = 0$$

for any $\lambda \in D_{f,>}^m(\mathbb{C})$. Since $D_{f,>}^m(\mathbb{C})$ contains an open polydisc in $\mathbb{C}^{n_1+\cdots+n_k}$, we deduce that $\sum_{\beta_i \in \Lambda_{k_i}, i \in \{1,\ldots,k\}} c_{\beta_1,\ldots,\beta_k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} = 0$ for all $k_i \in \mathbb{N}_0^{n_i}, i \in \{1, \ldots, k\}$(59).

For each $\gamma_i \in \mathbb{F}_{n_i}^+$ and $i \in \{1, \ldots, k\}$, set $\Omega(\gamma_1, \ldots, \gamma_k) := \frac{c_{\gamma_1,\ldots,\gamma_k}}{\sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}}}$. Fix $\beta_i^0 \in \Lambda_{k_i}$ and let $\beta_i \in \Lambda_{k_i}$ be such that β_i is obtained from β_i^0 by transposing just two generators. We can assume that $\beta_i^0 = \gamma_i g_{j_1}^i g_{j_2}^i \omega_i$ and $\beta_i = \gamma_i g_{j_2}^i g_{j_1}^i \omega_i$ for some $\gamma_i, \omega_i \in \mathbb{F}_{n_i}^+$ and $j_1 \neq j_2, j_1, j_2 \in \{1, \ldots, n_i\}$. Since $x \in N_{Q_c} = \otimes_{i=1}^k F^2(H_{n_i}) \ominus M_{Q_c}$, we must have

$$x, \otimes_{i=1}^k [W_{i,\gamma_i}(W_{i,j_1}W_{i,j_2} - W_{i,j_2}W_{i,j_1})W_{i,\omega_i}(1)] = 0,$$

which implies $\Omega(\beta_1^0, \ldots, \beta_k^0) = \Omega(\beta_1, \ldots, \beta_k)$.

Since any element $\gamma_i \in \Lambda_{k_i}$ can be obtained from β_i^0 by successive transpositions, repeating the above argument, we deduce that $\Omega(\beta_1^0, \ldots, \beta_k^0) = \Omega(\gamma_1, \ldots, \gamma_k)$. Setting $t := (\beta_1^0, \ldots, \beta_k^0)$, we have $c_{\gamma_1,\ldots,\gamma_k} = t\sqrt{b_{1,\gamma_1}^{(m_1)}} \cdots \sqrt{b_{k,\gamma_k}^{(m_k)}}, \gamma_i \in \Lambda_{k_i}$, and relation (59) implies $t = 0$. Therefore, $c_{\gamma_1,\ldots,\gamma_k} = 0$ for any $\gamma_i \in \Lambda_{k_i}$ and $k_i \in \mathbb{N}_0^{n_i}$, which implies $x = 0$. Therefore, we have $N_{Q_c} = \overline{span}\{\Gamma_\lambda : \lambda \in D_{f,>}^m(\mathbb{C})\}$.

Now, we prove part (ii) of the theorem. Any element $\varphi \in F_s^2(D_f^m)$ has a unique representation $\varphi = \sum_{k_1 \in N_0^{n_1}, \ldots, k_k \in N_0^{n_k}} c_{k_1,\ldots,k_k} w_1^{k_1} \otimes \cdots \otimes w_k^{k_k}$ with

$$\|\varphi\|_2^2 = \sum_{k_1\in N_0^{n_1},\dots,k_k\in N_0^{n_k}} |c_{k_1,\dots,k_k}|^2 \frac{1}{\gamma_{k_1}^{(m_1)}} \dots \frac{1}{\gamma_{k_k}^{(m_k)}} < \infty.$$

It is easy to see that

$$\langle w_1^{k_1} \otimes \cdots \otimes w_k^{k_k}, u_\lambda\rangle = \lambda_1^{k_1} \dots \lambda_k^{k_k}$$

for any $\lambda \in D_{f,>}^m(\mathbb{C})$ and$k_i \in \mathbb{N}_0^{n_i}, i \in \{1, \dots, k\}$. Consequently,$\varphi$ has a functional representation on $D_{f,>}^m(\mathbb{C})$ given by

$$\varphi(\lambda) \coloneqq (\varphi, u_\lambda) = \sum_{k_1\in N_0^{n_1},\dots,k_k\in N_0^{n_k}} c_{k_1,\dots,k_k}\lambda_1^{k_1} \dots \lambda_k^{k_k}, \quad \lambda = (\lambda_1, \dots, \lambda_n) \in D_{f,>}^m(\mathbb{C}),$$

and

$$|\varphi(\lambda)| \le \frac{\|\varphi\|_2}{\sqrt{\Delta_{f,\lambda}^m(1)}}.$$

This shows that $F_s^2(D_f^m)$ can be identified with $H^2\big(D_{f,>}^m(\mathbb{C})\big)$. Now, we prove part (iii). Note that if $(\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ are in $D_{f,>}^m(\mathbb{C})$ then

$$\left|\sum_{\alpha\in F_{n_i}^+} \alpha_{i,\alpha_i}\lambda_{i,\alpha_i}\bar{\mu}_{i,\alpha_i}\right| \le \left(\sum_{\alpha\in F_{n_i}^+} \alpha_{i,\alpha_i}|\lambda_{i,\alpha_i}|^2\right)^{\frac{1}{2}} \left(\sum_{\alpha\in F_{n_i}^+} \alpha_{i,\alpha_i}|\mu_{i,\alpha_i}|^2\right)^{\frac{1}{2}} < 1.$$

Using relation (47), we deduce that

$$\begin{aligned} k_f^c(\mu,\lambda) &= \prod_{i=1}^{k}\big(1 - f_i(\mu_i\bar{\lambda}_i)\big)^{-m} = \prod_{i=1}^{k}\left(1 - \sum_{\alpha\in F_{n_i}^+} \alpha_{i,\alpha_i}\lambda_{i,\alpha_i}\bar{\mu}_{i,\alpha_i}\right)^{-m_i} \\ &= \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\dots,\beta_k\in\mathbb{F}_{n_k}^+} b_{1,\beta_1}^{(m_1)} \cdots\ b_{k,\beta_k}^{(m_1)}\lambda_{1,\beta_1} \dots \lambda_{k,\beta_k}\bar{\mu}_{1,\beta_1} \dots \bar{\mu}_{k,\beta_k} \\ &= \langle u_\lambda, u_\mu\rangle. \end{aligned}$$

The proof is complete.

Theorem (4.2.20) [244]: The Hardy algebra $F^\infty\big(V_{f,Q_c}^m\big)$ coincides with the algebra $H^\infty\big(D_{f,>}^m(\mathbb{C})\big)$ of all multipliers of the Hilbert space $H^2\big(D_{f,>}^m(\mathbb{C})\big)$.

Proof: Let $\varphi\big(W_{i,j}\big) \in F_n^\infty\big(D_f^m\big)$ and set $M_\varphi \coloneqq P_{F_s^2(D_f^m)}\varphi\big(W_{i,j}\big)\big|_{F_s^2(D_f^m)}$. According to Theorem (4.2.15),Proposition (4.2.17), and Theorem (4.2.19), we have $F_s^2\big(D_f^m\big) = N_{Q_c}$, the vector Γ_λ is in $F_s^2\big(D_f^m\big)$ for $\lambda \in D_{f,>}^m(\mathbb{C})$, and $\varphi\big(W_{i,j}\big)^*\Gamma_\lambda = \overline{\varphi(\lambda)}\Gamma_\lambda$. Consequently, we obtain

$$\begin{aligned} [M_\varphi\psi](\lambda) &= \langle M_\varphi\psi, u_\lambda\rangle = \langle \varphi\big(W_{i,j}\big)\psi, u_\lambda\rangle \\ &= \langle \psi, \varphi\big(W_{i,j}\big)^* u_\lambda\rangle = \langle \psi, \overline{\varphi(\lambda)}u_\lambda\rangle = \varphi(\lambda)\psi(\lambda) \end{aligned}$$

for any $\psi \in F_s^2\big(D_f^m\big)$ and $\lambda \in D_{f,>}^m(\mathbb{C})$. Therefore, M_φ is a multiplier of $F_s^2\big(D_f^m\big)$. In particular, the operator $L_{i,j}$ is the multiplier by the coordinate function $\lambda_{i,j}$.Now, we show that $H^\infty\big(D_{f,>}^m(\mathbb{C})\big)$is included in $F^\infty\big(V_{f,Q_c}^m\big)$, the weakly closed algebra generated by the operators $L_{i,j}$ and the identity. Suppose that $g = \sum_{k_1\in\mathbb{N}_0^{n_1},\dots,k_k\in\mathbb{N}_0^{n_k}} c_{k_1,\dots,k_k} w_1^{k_1} \otimes \cdots \otimes w_k^{k_k}$ is a bounded multiplier, i.e., $M_g \in B\left(F_s^2\big(D_f^m\big)\right)$. As in [253], using Cesaro means, one can find a sequence p_n of polynomials in $w_1^{k_1} \otimes \cdots \otimes w_k^{k_k}$ where $k_1 \in \mathbb{N}_0^{n_1}, \dots, k_k \in \mathbb{N}_0^{n_k}$
, such that M_{p_n} converges to M_g in the strong operator topology and,consequently, in the WOT -topology. Since M_{p_n} is a polynomial in $L_{i,j}$ and the identity, our assertion follows.

Conversely, assume that the operator $Y \in B\left(F_s^2\big(D_f^m\big)\right)$ is in $F^\infty\big(V_{f,Q_c}^m\big)$. Then Y leaves invariant all the invariant subspaces under each operator $L_{i,j}$. Due to Theorem (4.2.15), we have $L_{i,j}^* u_\lambda = \bar{\lambda}_{i,j}u_\lambda$

for any $\lambda \in D_{f,>}^m(\mathbb{C})$.Therefore, the vector u_λ must be an eigenvector for Y^*. Consequently, there is a function $\varphi: D_{f,>}^m(\mathbb{C}) \to \mathbb{C}$ such that $Y^* u_\lambda = \overline{\varphi(\lambda)} u_\lambda$ for any $\lambda \in D_{f,>}^m(\mathbb{C})$. Note that, if $f \in F_s^2(D_f^m)$, then, due to Theorem (4.2.19), $Y\,f$ has the functional representation

$$(Yf)(\lambda) = \langle Yf, u_\lambda \rangle = \langle f, Y^* u_\lambda \rangle = \varphi(\lambda) f(\lambda), \qquad \lambda \in D_{f,>}^m(\mathbb{C}).$$

In particular, if $f = 1$, then the the functional representation of $Y(1)$ coincide with φ. Consequently, φ admits a power series representation on $D_{f,>}^m(\mathbb{C})$ and can be identified with $Y(1) \in F_s^2(D_f^m)$. Moreover, the equality above shows that $\varphi f \in H^2(D_{f,>}^m(\mathbb{C}))$ for any $f \in F_s^2(D_f^m)$. The proof is complete.

We need to recall some definitions. The set of all invariant subspaces of $A \in B(H)$ is denoted by Lat A. Given $U \subset B(H)$, we define Lat $U = T_{A\in U}$ Lat A. If S is any collection of subspaces of H, then we define $Alg\ A$ by setting $Alg\ S \coloneqq \{A \in B(H): S \subset Lat\ A\}$. The algebra $U \subset B(H)$ is called reflexive if $U = Alg\ Lat\ U$.

A closser look at the proof of Theorem (4.2.20) reveals the following result.

Corollary (4.2.21) [244]: The Hardy algebra $F^\infty(V_{f,Q_c}^m)$ is reflexive.

Now, we make a few remarks in the particular case when $n_1 = \cdots = n_k = n$. Let Q_{cc} be the left ideal of $\mathbb{C}[Z_{i,j}]$ generated by the polynomials $Z_{i,j_1} Z_{i,j_2} - Z_{i,j_2} Z_{i,j_1}$ and $Z_{i,j} - Z_{p,j}$, where $i, p \in \{1, \dots, k\}$ and $j_1, j_2, j \in \{1, \dots, n\}$. The universal model associated with the variety $V_{f,Q_{cc}}^m$ is the n tuple $C = (C_1, \dots, C_n)$, where $C_j \coloneqq P_{N_{Q_{cc}}} W_{1,j}|_{N_{Q_{cc}}}$ for $j \in \{1, \dots, n\}$. Note that, in this case, we have $V_{f,Q_{cc},>}^m(\mathbb{C}) = \bigcap_{i=1}^k D_{f_i,>}^1(\mathbb{C})$. Similarly to Theorem (4.2.19), one can show that the space $N_{Q_{cc}}$can be identified with a reproducing kernel Hilbert space with kernel

$$k_f^{cc}(z,w) \coloneqq \frac{1}{\prod_{i=1}^k \left(1 - f_i(z\overline{w})\right)^{m_i}},$$

where $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n)$ are in the set $V_{f,Q_{cc},>}^m(\mathbb{C}) \subset \mathbb{C}^n$. We remark that in the particular case when $f_1 = \cdots = f_k = Z_1 + \cdots + Z_n$ and $m_1 = \cdots = m_k = 1$, we obtain the reproducing kernel $(z,w) \mapsto \frac{1}{(1-\langle z,w\rangle)^k}$ on the unit ball $\mathbb{B}_n$. In this case, the reproducing kernel Hilbert spaces are the Hardy-Sobolev spaces (see [267]). The case when $k = n$ corresponds to the Hardy space of the ball, and the case when $k = n+1$ corresponds to the Bergman space.

we show that the isomorphism problem for the universal polydomain algebras is closed related to to the biholomorphic equivalence of Reinhardt domains in several complex variables. Our results also show that there are many non-isomorphic polydomain algebras.

Given a Hilbert space H, the radial polydomain associated with the abstract D_f^m is the set

$$D_{f,rad}^m(H) \coloneqq \bigcup_{0 \le r < 1} r D_f^m(H) \subseteq D_f^m(H).$$

A formal power series $\varphi = \sum_{(\alpha)\in \mathbb{F}_{n_1}^+ \times \dots \times \mathbb{F}_{n_k}^+} a_{(\alpha)} Z_{(\alpha)}, a_{(\alpha)} \in \mathbb{C}$, in ideterminates $Z_{i,j}$ is called free holomorphic function on the abstract radial polydomain $D_{f,rad}^m \coloneqq \{D_{f,rad}^m(H): H\ is\ a\ Hilbert\ space\}$ if the series

$$\varphi(X_{i,j}) \coloneqq \sum_{q=0}^\infty \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+ \times \dots \times \mathbb{F}_{n_k}^+ \\ |\alpha_1| + \cdots + |\alpha_k| = q}} a_{(\alpha)} X_{(\alpha)}$$

is convergent in the operator norm topology for any $X = \{X_{i,j}\} \in D_{f,rad}^m(H)$ and any Hilbert space H. We denote by $H\,ol(D_{f,rad}^m)$ the set of all free holomorphic functions on the abstract radial polydomain $D_{f,rad}^m$ Let $H^\infty(D_{f,rad}^m)$ denote the set of all elements φ in $H\,ol(D_{f,rad}^m)$ such that

$$\|\varphi\|_\infty \coloneqq \sup \|\varphi(X_{i,j})\| < \infty,$$

where the supremum is taken over all $\{X_{i,j}\}\} \in D^m_{f,rad}(H)$ and any Hilbert space H. One can show that $H^\infty(D^m_{f,rad})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$. For each $p \in N$, we define the norms $\|\cdot\|_p : M_{p\times p}\left(H^\infty(D^m_{f,rad})\right) \to [0,\infty)$ by setting

$$\left\|[\varphi_{st}]_{p\times p}\right\|_p := \sup\left\|[\varphi_{st}(X)]_{p\times p}\right\|,$$

where the supremum is taken over all $X := \{X_{i,j}\} \in D^m_{f,rad}(H)$ and any Hilbert space H. The norms $\|\cdot\|_p, p \in \mathbb{N}$, determine an operator space structure on $H^\infty\left(D^m_{f,rad}\right)$, in the sense of Ruan ([248]). Throughout, we assume that $D^m_f(H)$ is closed in the operator norm topology for any Hilbert space H. Then we have $D^m_{f,rad}(H)^- = D^m_f(H)$. Note that the interior of $D^m_f(H)$, which we denote by $Int\left(D^m_f(H)\right)$, is a subset of $D^m_{f,rad}(H)$. We remark that if $q = (q_1, \dots, q_k)$ is a k-tuple of positive regular noncommutative polynomials, then $D^m_q(H)$ is closed in the operator norm topology.

We denote by $A\left(D^m_{f,rad}\right)$ the set of all elements g in $H\,ol\left(D^m_{f,rad}\right)$ such that the mapping

$$D^m_{f,rad}(H) \ni X \mapsto g(X) \in B(H)$$

has a continuous extension to $\left[D^m_{f,rad}(H)\right]^- = D^m_f(H)$ for any Hilbert space H. We remark that $A\left(D^m_{f,rad}\right)$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$ and it has an operator space structure under the norms $\|\cdot\|_p, p \in \mathbb{N}$. Moreover, we can identify the polydomain algebra $A\left(D^m_f\right)$ with the subalgebra

$A\left(D^m_{f,rad}\right)$, as follows. The map $\Phi: A\left(D^m_{f,rad}\right) \to A\left(D^m_f\right)$ defined by

$$\Phi\left(\sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}\right) := \sum_{(\alpha)} a_{(\alpha)} W_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras. If $g := \sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}$ is a free holomorphic function on the abstract radial polydomain $D^m_{f,rad}$, then $g \in A\left(D^m_{f,rad}\right)$ if and only if $g\left(rW_{i,j}\right) := \sum_{q=0}^\infty \sum_{\substack{(\alpha)\in\mathbb{F}^+_{n_1}\times\dots\times\mathbb{F}^+_{n_k} \\ |\alpha_1|+\dots+|\alpha_k|=q}} r^q a_{(\alpha)} W_{(\alpha)}$

is convergent in the norm topology as $r \to 1$. In this case, there exists a unique $\varphi \in A\left(D^m_f\right)$ with $g = B[\varphi]$, where B is the noncommutative Berezin transform associated with the abstract polydomain D^m_f , with the properties

$$\Phi(g) = \lim_{r\to 1} g\left(rW_{i,j}\right) \text{ and } \Phi^{-1}(\varphi) = B[\varphi],\ \ \varphi \in A\left(D^m_f\right).$$

We proved in [253](see Proposition 4.4) that if $p \in \mathbb{N}$ and φ is a free holomorphic function on the abstract radial polydomain $D^m_{f,rad}$ then its representation on $\mathbb{C}^p$, i.e., the map $\hat{\varphi}$ defined by

$$\mathbb{C}^{(n_1+\dots+n_k)p^2} \supset D^m_{f,rad}(\mathbb{C}^p) \ni A \longmapsto \varphi(A) \in M_{p\times p}(\mathbb{C}) \subset \mathbb{C}^{p^2}$$

is a holomorphic function on the interior of $D^m_f(\mathbb{C}^p)$. Moreover, if $\varphi \in A\left(D^m_{f,rad}\right)$, then its representation on $\mathbb{C}^p$ has a continuous extension to$D^m_f(\mathbb{C}^p)$ and it is holomorphic on the interior of $D^m_f(\mathbb{C}^p)$. The continuous extension is defined by $\hat{\varphi}(A) := \lim_{r\to 1} B_{rA}[\varphi]$ for $A \in D^m_f(\mathbb{C}^p)$.

Let Ω_1, Ω_2 be domains (open and connected sets) in $\mathbb{C}^d$. If there exist holomorphic maps $\zeta: \Omega_1 \to \Omega_2$ and $\psi : \Omega_2 \to \Omega_1$ such that $\zeta \circ \psi = id_{\Omega_2}$and $\circ\zeta = id_{\Omega_1}$, then $\Omega_1$and Ω_2 are called biholomorphic equivalent and φ and ψ are called biholomorphic maps.

Theorem (4.2.22) [244]: Let $f = (f_1, \dots, f_k)$ and $g = (g_1, \dots, g_{k'})$ be tuples of positive regular free holomorphic functions with n and ℓ indeterminates, respectively, and let $m := (m_1, \dots, m_k) \in \mathbb{N}^k$ and $d := (d_1, \dots, d_{k'}) \in \mathbb{N}^{k'}$. If $\overline{\Psi}: A\left(D^m_f\right) \to A\left(D^d_g\right)$ is a unital completely contractive isomorphism, then the map $\varphi: D^d_g(\mathbb{C}) \to D^m_f(\mathbb{C})$

defined by

$$\varphi(\lambda) := \left[\lim_{r\to 1} B_{g,r\lambda}\left[\overline{\Psi}\left(W^{(f)}_{i,j}\right)\right] : i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}\right], \qquad \lambda \in D^d_g(\mathbb{C}),$$

where $W^{(f)} := \left\{W_{i,j}^{(f)}\right\}$ is the universal model of the abstract polydomain D_f^m and $B_{g,r\lambda}$ is the Berezin transform at $r\lambda \in D_{g,>}^d(\mathbb{C})$, is a homeomorphism which is a biholomorphic function from $Int\left(D_g^d(\mathbb{C})\right)$ onto $Int\left(D_f^m(\mathbb{C})\right)$and $n = \ell$.

Proof: Denote

$$\bar{\varphi}_{i,j} := \bar{\Psi}\left(W_{i,j}^{(f)}\right) \in A\left(D_g^d\right), \qquad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}, \quad (60)$$

where $W^{(f)} := \left\{W_{i,j}^{(f)}\right\}$ is the universal model of the abstract polydomain D_f^m. Assume that f_i has the representation $f_i := \sum_{(\alpha)\in\mathbb{F}_{n_i}^+} a_{i,\alpha} Z_{i,\alpha}$. Taking into account that $0 \le \Phi_{f_i, W_i^{(f)}}(I) \le I$, we deduce that $0 \le \sum_{(\alpha)\in\mathbb{F}_{n_i}^+, |\alpha|\le N} a_{i,\alpha} W_{i,\alpha}^{(f)}\left(W_{i,\alpha}^{(f)}\right)^* \le I$

for any $N \in \mathbb{N}$. Using the fact that $a_{i,\alpha} \ge 0$ and $\bar{\Psi}$ is a completely contractive homomorphism, one can easily see that $0 \le \Phi_{f_i,\bar{\varphi}_i}(I) \le I$, where $\bar{\varphi}_i := \left(\bar{\varphi}_{i,1}, \dots, \bar{\varphi}_{i,n_i}\right)$ and$\bar{\varphi} := (\bar{\varphi}_1, \dots, \bar{\varphi}_k)$. Due to the remarks preceding the theorem, for each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, the map $\varphi_{i,j}: D_g^d(\mathbb{C}) \to \mathbb{C}$ given by

$$\varphi_{i,j}(\lambda) := \lim_{r\to 1} B_{g,r\lambda}\left[\bar{\varphi}_{i,j}\right]$$

is continuous on $D_g^d(\mathbb{C})$ and holomorphic on $Int\left(D_g^d(\mathbb{C})\right)$ Now, we define the function $\varphi: D_g^d(\mathbb{C}) \to \mathbb{C}^\ell$ by setting $\varphi(\lambda) := \left(\varphi_1(\lambda), \dots, \varphi_k(\lambda)\right)$ where $\varphi_i(\lambda) := \left(\varphi_{i,1}(\lambda), \dots, \varphi_{i,n_i}(\lambda)\right)$ for all $\lambda \in D_g^d(\mathbb{C})$. Since $0 \le \Phi_{f_i,\bar{\varphi}_i}(I) \le I$ we have $0 \le \sum_{(\alpha)\in\mathbb{F}_{n_i}^+, |\alpha|\le N} a_{i,\alpha} \bar{\varphi}_{i,\alpha} \bar{\varphi}_{i,\alpha}^* \le I$ for all N ∈ N. Apply the Berezin transform at $r\lambda \in D_{g,>}^d(\mathbb{C})$, $r \in [0,1)$, we obtain

$$0 \le \sum_{\alpha\in\mathbb{F}_{n_i}^+, |\alpha|\le N} a_{i,\alpha} \varphi_{i,\alpha}(r\lambda)\overline{\varphi_{i,\alpha}(r\lambda)} \le 1, \qquad N \in \mathbb{N}.$$

Taking $r \to 1$ and $N \to \infty$, we deduce that $0 \le \Phi_{f_i,\varphi_i(\lambda)}(1) \le 1$. Consequently, $\varphi(\lambda) \in D_f^m(\mathbb{C})$ for all $\lambda \in D_g^d(\mathbb{C})$. Moreover, the map $\varphi: D_g^d(\mathbb{C}) \to D_f^m(\mathbb{C})$ is continuous on $D_g^d(\mathbb{C})$ and holomorphic on $Int(D_g^d(\mathbb{C}))$. Now, we set

$$\bar{\xi}_{i,j} := \widehat{\Psi}^{-1}\left(W_{i,j}^{(g)}\right) \in A\left(D_f^m\right), \quad i \in \{1, \dots, k'\},\ j \in \{1, \dots, \ell_i\}, \qquad (61)$$

where $W^{(g)} := \{W_{i,j}^{(g)}\}$ is the universal model of the abstract polydomain D_g^d. Since $0 \le \Phi_{g_i, W_i^{(g)}}(I) \le I$ and $\widehat{\Psi}^{-1}$ is a completely contractive homomorphism, we deduce that $0 \le \Phi_{g_i,\bar{\xi}_i}(I) \le I$, where we set $\bar{\xi}_i := (\bar{\xi}_{i,1}, \dots, \bar{\xi}_{i,\ell_i})$ and $\bar{\xi} := (\bar{\xi}_1, \dots, \bar{\xi}_{k'})$. As above, for each $i \in \{1, \dots, k'\}$ and $j \in \{1, \dots, \ell_i\}$, the map $\xi_{i,j}: D_f^m(\mathbb{C}) \to \mathbb{C}$, given by

$$\xi_{i,j}(\mu) := \lim_{r\to 1} B_{f,r\mu}[\bar{\xi}_{i,j}]$$

is continuous on $D_g^m(\mathbb{C})$ and holomorphic on $Int(D_f^m(\mathbb{C}))$. Set $\xi(\mu) := (\xi_1(\mu), \dots, \xi_{k'}(\mu))$ and $\xi_i(\mu) := (\xi_{i,1}(\mu), \dots, \xi_{i,\ell_i}(\mu))$ for all $\mu \in D_f^m(\mathbb{C})$. Since $0 \le \Phi_{g_i,\bar{\xi}_i}(I) \le I$, we can show that $0 \le \Phi_{g_i,\xi_i(\mu)}(1) \le 1$.

Hence, we deduce that $\xi(\mu) \in D_g^d(\mathbb{C})$ for all $\mu \in D_f^m(\mathbb{C})$. Therefore, the map $\xi: D_f^m(\mathbb{C}) \to D_g^d(\mathbb{C})$ is continuous on $D_f^m(\mathbb{C})$ and holomorphic on $Int(D_f^m(\mathbb{C}))$.

Now, each $\bar{\xi}_{i,j} \in A(D_f^m), i \in \{1, \dots, k'\}, j \in \{1, \dots, \ell_i\}$, has a unique Fourier representation $\sum_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{(\alpha)} W_{(\alpha)}^{(f)}$ such that

$$\bar{\xi}_{i,j} = \lim_{r\to 1} \sum_{q=0}^{\infty} \sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)} W_{(\alpha)}^{(f)},$$

where the limit is in the operator norm topology. Hence, using the continuity of $\widehat{\Psi}$ in the operator norm, and relations (61) and (60), we obtain

$$W_{i,j}^{(g)} = \widehat{\Psi}(\bar{\xi}_{i,j}) = \widehat{\Psi}\left(\lim_{r\to 1}\sum_{q=0}^{\infty}\sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)} W_{(\alpha)}^{(f)}\right)$$

$$= \lim_{r\to 1}\sum_{q=0}^{\infty}\sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)}\widehat{\Psi}\left(W_{(\alpha)}^{(f)}\right) = \lim_{r\to 1}\sum_{q=0}^{\infty}\sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)}\bar{\varphi}_{(\alpha)}.$$

Consequently, using the continuity in the operator norm of the noncommutative Berezin transform at $\lambda \in D_{g,>}^d(\mathbb{C})$ on the polydomain algebra $A(D_g^d)$, and relations $\varphi_{i,j}(\lambda) := B_{g,\lambda}[\bar{\varphi}_{i,j}]$ for all $\lambda \in D_{g,>}^d(\mathbb{C})$, and $\xi_{i,j}(\mu) \coloneqq \lim_{r\to 1} B_{f,r\mu}[\bar{\xi}_{i,j}]$ for $\mu \in D_f^m(\mathbb{C})$, we have

$$\lambda_{i,j} = B_{g,\lambda}[W_{i,j}^{(g)}] = B_{g,\lambda}\left[\lim_{r\to 1}\sum_{q=0}^{\infty}\sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)}\bar{\varphi}_{(\alpha)}\right]$$

$$= \lim_{r\to 1}\sum_{q=0}^{\infty}\sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)} B_{g,\lambda}\left[\bar{\varphi}_{(\alpha)}\right] = \lim_{r\to 1}\sum_{q=0}^{\infty}\sum_{\substack{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)}\varphi_{(\alpha)}(\lambda)$$

$$= \lim_{r\to 1} B_{f,r\varphi(\lambda)}[\bar{\xi}_{i,j}] = \xi_{i,j}(\varphi(\lambda))$$

for each $i \in \{1,\ldots,k'\}, j \in \{1,\ldots,\ell_i\}$, and any $\lambda \in D_{g,>}^d(\mathbb{C})$. Hence $(\xi \circ \varphi)(\lambda) = \lambda$ for all $\lambda \in D_{g,>}^d(\mathbb{C})$.

Now, using the fact that the functions $\varphi: D_g^d(\mathbb{C}) \to D_f^m(\mathbb{C})$ and $\xi: D_f^m(\mathbb{C}) \to D_g^d(\mathbb{C})$ are continuous, and $D_{g,>}^d(\mathbb{C})$ is dense in $D_g^d(\mathbb{C})$, we conclude that $(\xi \circ \varphi)(\lambda) = \lambda$ for all $\lambda \in Dg_g^d(\mathbb{C})$. Similarly, one can prove that $(\varphi \circ \xi)(\mu) = \mu$ for $\mu \in D_f^m(\mathbb{C})$. Therefore, the map $\varphi: D_g^d(\mathbb{C}) \to D_f^m(\mathbb{C})$ is a homeomorphism such that φ and $\varphi^{-1} := \xi$ are holomorphic functions on $Int(D_g^d(\mathbb{C}))$ and $Int(D_f^m(\mathbb{C}))$, respectively. Now, a standard argument using Brouwer's invariance of domain theorem [253] shows that φ is a biholomorphic function from $Int(D_g^d(\mathbb{C}))$ onto $Int(D_f^m(\mathbb{C}))$ and $n = \ell$. The proof is complete.

Corollary (4.2.23) [244]: Let $f = (f_1,\ldots,f_k)$ and $g = (g_1,\ldots,g_{k'})$ be tuples of positive regular free holomorphic functions with n and ℓ indeterminates, respectively, and let $m \in \mathbb{N}^k$ and $d \in \mathbb{N}^{k'}$. If the domain algebras $A(D_f^m)$ and $A(D_g^d)$ are unital completely contractive isomorphic, then $n = \ell$ and there exists a permutation σ of the set $\{1,\ldots,n\}$ and scalars $t_1,\ldots,t_n > 0$ such that the map

$$Int(D_f^m(\mathbb{C})) \ni (z_1,\ldots,z_n) \mapsto (t_1 z_{\sigma(1)},\ldots,t_n z_{\sigma(n)}) \in Int(D_g^d(\mathbb{C}))$$

is a biholomorphic map.

Proof: Note that the sets $Int(D_f^m(\mathbb{C})) \subset \mathbb{C}^n$ and $Int(D_g^d(\mathbb{C})) \subset \mathbb{C}^\ell$ are Reinhardt domains which contain 0. Due to Theorem (4.2.22), there is a biholomorphic function from $Int(D_g^d(\mathbb{C}))$ onto $Int(D_f^m(\mathbb{C}))$ and $n = \ell$.

Using Sunada's result [257], we complete the proof.

Proposition (4.2.24) [244]: Let $Q \subset \mathbb{C}[Z_{i,j}]$ be a left ideal generated by noncommutative homogenous polynomials and let $A(V_{f,Q}^m)$ be the corresponding noncommutative variety algebra. If $\varphi \in A(V_{f,Q}^m)$, then the map $\check{\varphi}: V_{f,Q}^m(H) \to B(H)$ defined by

$$\check{\varphi}(Y) := \lim_{r\to 1} B_{rY,Q[\varphi]}, \qquad Y \in V_{f,Q}^m(H),$$

is continuous, where the convergence is in the operator norm topology and $B_{f,rY,Q}$ is the constrained noncommutative Berezin tranform.

Proof: First, note that the map $\breve{\varphi}$ is well-defined due to Proposition (4.2.5). Let $p_n(S_{i,j})$ be a sequence of polynomials in the variety algebra $A(V_{f,Q}^m)$ such that $p_n(S_{i,j}) \to \varphi$ in the operator norm. Given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\|\varphi - p_N(S_{i,j})\| < \frac{\epsilon}{4}$. Fix $A \in V_{f,Q}^m(H)$ and and choose $\delta > 0$ such that $\|p_N(Y) - p_N(A)\| < \frac{\epsilon}{2}$, whenever $Y \in V_{f,Q}^m(H)$ and $\|Y - A\| < \delta$. Now, using again Proposition (4.2.5), we have

$$\begin{aligned} \|\breve{\varphi}(Y) - \breve{\varphi}(A)\| &\le \lim_{r\to\infty} \sup \|B_{rY,Q[\varphi]} - B_{rA,Q[\varphi]}\| \\ &= \lim_{r\to\infty} \sup\{\|B_{rY,Q[\varphi - p_N(S_{i,j})]}\| + \|B_{rY,Q[p_N(S_{i,j})]} - B_{rA,Q[p_N(S_{i,j})]}\| \\ &\quad + \|B_{rA,Q[p_N(S_{i,j}) - \varphi]}\|\} \\ &\le 2\|\varphi - p_N(S_{i,j})\| + \lim_{r\to 1} \sup \|p_N(rY) - p_N(rA)\| \\ &\le 2\|\varphi - p_N(S_{i,j})\| + \|p_N(Y) - p_N(A)\| \le \epsilon \end{aligned}$$

for any $Y \in V_{f,Q}^m(H)$ with $\|Y - A\| < \delta$. The proof is complete

Consider the particular case when $Q = Q_c$. According to Theorem (4.2.20), the Hardy algebra $F^\infty(V_{f,Q_c}^m)$ coincides with the algebra $H^\infty(D_{f,>}^m(\mathbb{C}))$ of all multipliers of the Hilbert space $H^2(D_{f,>}^m(\mathbb{C}))$. We remark that each $\varphi \in A(V_{f,Q_c}^m)$ can be identified with a multiplier ξ of $H^2(D_{f,>}^m(\mathbb{C}))$ which admits a continuous extension to $D_f^m(\mathbb{C})$. Moreover,

$$\xi(\lambda) = \lim_{r\to 1} B_{r\lambda,Q_c[\varphi]}, \lambda \in D_{f,>}^m(\mathbb{C}).$$

Indeed, due to Theorem (4.2.20), φ can be identified with a multiplier ξ which is given by the relation $\xi(\lambda) = \langle \varphi(1), u_\lambda \rangle$ for all $\lambda \in D_{f,>}^m(\mathbb{C})$. On the other hand, due to Proposition (4.2.24), the map $\breve{\varphi}: V_{f,Q}^m(\mathbb{C}) \to \mathbb{C}$ defined by $\breve{\varphi}(\lambda) := \lim_{r\to 1} B_{r\lambda,Q[\varphi]}$ is continuous on $V_{f,Q}^m(\mathbb{C}) = D_f^m(\mathbb{C})$. According to Theorem (4.2.16) and the remarks that follow, we deduce that $\xi(\lambda) = \langle \varphi(1), u_\lambda \rangle = \breve{\varphi}(\lambda)$ for all $\lambda \in D_{f,>}^m(\mathbb{C})$, which proves our assertion.

Theorem (4.2.25) [244]: Let $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_{k'})$ be tuples of positive regular free holomorphic functions with n and ℓ indeterminates, respectively, let $m := (m_1, \ldots, m_k) \in \mathbb{N}^k$ and $d := (d_1, \ldots, d_{k'}) \in \mathbb{N}^{k'}$, and let Q be a left ideal generated by homogenous polynomials in $\mathbb{C}[Z_{i,j}]$. If $\widehat{\Psi}: A(V_{f,Q}^m) \to A(V_{g,Q}^d)$ is a unital completely contractive isomorphism, then the map $\varphi: V_{g,Q}^d(\mathbb{C}) \to V_{f,Q}^m(\mathbb{C})$ defined by

$$\varphi(\lambda) := \left[\lim_{r\to 1} B_{g,r\lambda,Q\left[\widehat{\Psi} S_{i,j}^{(f)}\right]} : i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}\right], \lambda \in V_{g,Q}^d(\mathbb{C}),$$

where $S^{(f)} := \{S_{i,j}^{(f)}\}$ is the universal model of the abstract variety $V_{f,Q}^m$ and $B_{g,r\lambda,Q}$ is the constrained Berezin transform at $r\lambda$, is a homeomorphism of $V_{g,Q}^d(\mathbb{C})$ onto $V_{f,Q}^m(\mathbb{C})$.

In the particular case when $Q = Q_c$, the map φ is, in addition, a biholomorphic function from $Int(V_{g,Q_c}^d(\mathbb{C}))$ onto $Int(V_{f,Q_c}^m(\mathbb{C}))$ and $n = \ell$.

Proof: We only sketch the proof, since it is very similar to that of Theorem (4.2.22), and point out the differences. Denote

$$\bar{\varphi}_{i,j} := \widehat{\Psi}\left(S_{i,j}^{(f)}\right) \in A\left(V_{g,Q}^d\right),\ i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}, \qquad (62)$$

where $S^{(f)} := \{S_{i.j}^{(f)}\}$ is the universal model of the abstract variety $V_{f,Q}^m$. Due to Proposition (4.2.24), the map $\varphi_{i,j}: V_{g,Q}^d(\mathbb{C}) \to \mathbb{C}$, given by

$$\varphi_{i,j}(\lambda) := \lim_{r\to 1} B_{g,r\lambda,Q[\bar{\varphi}_{i,j}]}$$

is well-defined and continuous. Consider the function $\varphi: V_{g,Q}^d(\mathbb{C}) \to \mathbb{C}^\ell$ given by $\varphi(\lambda) := (\varphi_1(\lambda), \ldots, \varphi_k(\lambda))$, where $\varphi_i(\lambda) := (\varphi_{i,1}(\lambda), \ldots, \varphi_{i,n_i}(\lambda))$ for all $\lambda \in V_{g,Q}^d(\mathbb{C})$ and note that $\varphi(\lambda) \in D_f^m(\mathbb{C})$ for all $V_{g,Q}^d(\mathbb{C})$.

On the other hand, since $q(S^{(f)}) = 0$ for any $q \in Q$, and $\widehat{\Psi}$ is a homomorphism, one can deduce that $q(\bar{\varphi}) = 0$. Applying the constrained Berezin transform $B_{g,r\lambda,Q}$ and taking the limit as $r \to 1$, we obtain that $q(\varphi(\lambda)) = 0$ for any $q \in Q$. Therefore $\varphi(\lambda) \in V_{f,Q}^m(\mathbb{C})$ and the map $\varphi: V_{g,Q}^m(\mathbb{C}) \to V_{f,Q}^m(\mathbb{C})$ is continuous. Similarly, setting

$$\bar{\xi}_{i,j} := \widehat{\Psi}^{-1}\left(S_{i,j}^{(g)}\right) \in A\left(V_{f,Q}^m\right), i \in \{1, \ldots, k'\}, j \in \{1, \ldots, \ell_i\}, \quad (63)$$

where $S^{(g)} := \{S_{i,j}^{(g)}\}$ is the universal model of the abstract variety $V_{g,Q}^d$, Proposition (4.2.24) shows that the map $\xi_{i,j}: V_{f,Q}^m(\mathbb{C}) \to \mathbb{C}$ given by $\xi_{i,j}(\mu) := \lim_{r\to 1} B_{f,r\mu,Q[\bar{\xi}_{i,j}]}$ is well-defined and continuous. Now, one can prove that the map $\xi: V_{f,Q}^m(\mathbb{C}) \to V_{g,Q}^d(\mathbb{C})$ defined by $\xi(\mu) := (\xi_1(\mu), \ldots, \xi_{k'}(\mu))$, where $\xi_i(\mu) := (\xi_{i,1}(\mu), \ldots, \xi_{i,\ell_i}(\mu))$, is continuous.

For each $\bar{\xi}_{i,j} \in A(V_{f,Q}^m), i \in \{1, \ldots, k'\}, j \in \{1, \ldots, \ell_i\}$, let $p_s(S^{(f)}) = \sum_{(\alpha)\in \mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+} a_{(\alpha)}^{(s)} S_{(\alpha)}^{(f)}, s \in \mathbb{N}$, be a sequence of polynomials such that $\bar{\xi}_{i,j} = \lim_{s\to\infty} p_s(S^{(f)})$ where the convergence is in the operator norm. Using the continuity of $\widehat{\Psi}$ in the operator norm, and relations (63) and (62), we obtain

$$S_{i,j}^{(g)} = \widehat{\Psi}(\bar{\xi}_{i,j}) = \widehat{\Psi}\left(\lim_{s\to\infty} p_s(S^{(f)})\right) = \lim_{s\to\infty} p_s(\bar{\varphi}).$$

Consequently, using the continuity in the operator norm of the constrained noncommutative Berezin transform at $\lambda \in V_{g,Q,>}^d(\mathbb{C})$ on the variety algebra $A(V_{g,Q}^d)$ and the relations above, we obtain

$$\begin{aligned}\lambda_{i,j} = B_{g,\lambda,Q}[S_{i,j}^{(g)}] &= B_{g,\lambda,Q}\left[\lim_{s\to\infty} p_s(\bar{\varphi})\right] \\ = \lim_{s\to\infty} p_s(\varphi(\lambda)) &= \lim_{s\to\infty} B_{f,\varphi(\lambda),Q}[p_s(S^{(f)})] \\ &= \xi_{i,j}(\varphi(\lambda))\end{aligned}$$

for each $i \in \{1, \ldots, k'\}, j \in \{1, \ldots, \ell_i\}$, and any $\lambda \in V_{g,Q,>}^d(\mathbb{C})$. Hence $(\xi \circ \varphi)(\lambda) = \lambda$ for all $\lambda \in V_{g,Q,>}^d(\mathbb{C})$.

Now, using the fact that the functions $\varphi: V_{g,Q}^d(\mathbb{C}) \to V_{f,Q}^m(\mathbb{C})$ and $\xi: V_{f,Q}^m(\mathbb{C}) \to V_{g,Q}^d(\mathbb{C})$ are continuous, and $V_{g,Q,>}^d(\mathbb{C})$ is dense in $V_{g,Q}^d(\mathbb{C})$, we conclude that $(\xi \circ \varphi)(\lambda) = \lambda$ for all $\lambda \in V_{g,Q}^d(\mathbb{C})$. Similarly, one can prove that $(\varphi \circ \xi)(\mu) = \mu$ for $\mu \in V_{f,Q}^m(\mathbb{C})$. Therefore, the map φ is a homeomorphism. Note that in the particular case when $Q = Q_c$, we have $V_{f,Q_c}^m(\mathbb{C}) = D_f^m(\mathbb{C})$ and $V_{g,Q_c}^d(\mathbb{C}) = D_g^d(\mathbb{C})$. Using Theorem (4.2.22), one can complete the proof.

We remark that a result similar to Corollary (4.2.23) holds in the commutative setting. Therefore, if the variety algebras $A(V_{f,Q_c}^m)$ and $A(V_{g,Q_c}^d)$ are unital completely contractive isomorphic, then $n = \ell$ and there exists a permutation σ of the set $\{1, \ldots, n\}$ and scalars $t_1, \ldots, t_n > 0$ such that the map

$$Int(V_{f,Q_c}^m(\mathbb{C})) \ni (z_1, \ldots, z_n) \mapsto (t_1 z_{\sigma(1)}, \ldots, t_n z_{\sigma(n)}) \in Int(V_{g,Q_c}^d(\mathbb{C}))$$

is a biholomorphic map.

The results show that there are many non-isomorphic polydomain algebras. We consider the following particular case. If $f = Z_1 + \cdots + Z_n$, then $A(V_{f,Q_c}^1)$ is the universal algebra of commuting row contractions, and $Int(V_{f,Q_c}^1(\mathbb{C}) = \mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$. When $g = (Z_1, \ldots, Z_n)$, then $A\left(V_{g,Q_c}^1\right)$ is the commutative polydisc algeba. In this case, we have $Int(V_{f,Q_c}^1(\mathbb{C}) = \mathbb{D}^n$. Since $\mathbb{B}_n$ and $\mathbb{D}^n$ are not biholomorphic domains in $\mathbb{C}^n$ if $n \geq 2$, Theorem (4.2.25) shows that the universal algebras $A(V_{f,Q_c}^1)$ and $A(V_{g,Q_c}^1)$ are not isomorphic.

We develop a dilation theory on abstract noncommutative varieties $V_{f,J}^m$, where J is a norm-closed two sided ideal of the noncommutative polydomain algebra $A(D_f^m)$ such that $N_J \neq \{0\}$.

The dilation theory can be refined for the class of noncommutative varieties $V_{q,Q}^m$, where $Q \subset \mathbb{C}[Z_{i,j}]$ is an ideal generated by homogeneous polynomials and $q = (q_1, \ldots, q_k)$ is a k-tuple of positive regular noncommutative polynomials. we also obtain Wold type decompositions for non-degenerate $*$-representations of the C^*-algebra $C^*(S_{i,j})$ generated by the universal model.

Lemma (4.2.26) [244]: Let $T = (T_1, \ldots, T_k)$ be in the noncommutative polydomain $D_f^m(H)$ and let $X \in B(H)$ be a positive operator such that $\Delta_{f,T}^p(X) \geq 0$ for any $p := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ with $p \leq m$. Then

$$0 \leq \lim_{q_k\to\infty} \cdots \lim_{q_1\to\infty} (id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(X) \leq X.$$

Proof: For each $i \in \{1, \ldots, k\}$, let $\Omega_i \subset B(H)$ be the set of all $Y \in B(H), Y \geq 0$, such that the series $\sum_{\beta_i\in\mathbb{F}^+_{n_i}} b^{(m_i)}_{i,\beta_i} T_{i,\beta_i} Y T^*_{i,\beta_i}$ is convergent in the weak operator topology, where

$$b^{(m_i)}_{i,g_0} := 1 \; and \; b^{(m_i)}_{i,\alpha_i} := \sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_p\in\mathbb{F}^+_{n_i} \\ \gamma_1\cdots\gamma_p=\alpha \\ |\gamma_1|\geq 1,\ldots,|\gamma_p|\geq 1}} a_{i,\gamma_1} \cdots a_{i,\gamma_p} \binom{p+m_i-1}{m_i-1}$$

for all $\alpha \in \mathbb{F}^+_{n_i}$ with $|\alpha| \geq 1$. We define the map $\Psi_i: \Omega_i \to B(H)$ by setting

$$\Psi_i(Y) := \sum_{\beta_i\in\mathbb{F}^+_{n_i}} b^{(m_i)}_{i,\beta_i} T_{i,\beta_i} Y T^*_{i,\beta_i}.$$

Fix $i \in \{1, \ldots, k\}$ and assume that $1 \leq p_i = m_i$. In [253] (see the proof of Theorem 2.2), we proved that

$$0 \leq \Psi_i(\Delta^p_{f,T}(X)) = \Delta^{p-m_ie_i}_{f,T}\left(id - \lim_{q_i\to\infty} \Phi^{q_i}_{f_i,T_i}\right)(X) \leq \Delta^{p-m_ie_i}_{f,T}(X) \leq X, \quad (64)$$

for any $p := (p_1, \ldots, p_k) \in \mathbb{Z}^k_+$ with $p \leq m$ and $p_i = m_i$. A repeated application of (64), leads to the relation

$$0 \leq (\Psi_k \circ \cdots \circ \Psi_1)(\Delta^m_{f,T}(X)) = \lim_{q_k\to\infty} \cdots \lim_{q_1\to\infty} (id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(X) \leq X.$$

The proof is complete.

Lemma (4.2.27) [244]: Let $T = (T_1, \ldots, T_k)$ be in the noncommutative polydomain $D^m_f(H)$ and let $K_{f,T}$ be the associated Berezin kernel. Then

$$\Delta^p_{f,T}(K^*_{f,T}K_{f,T}) \leq \Delta^p_{f,T}(I)$$

for any $p := (p_1, \ldots, p_k) \in \mathbb{Z}^k_+$ with $p \leq m$. The equality occurs if $p \geq (1, \ldots, 1)$.

Proof: Let $s \in \{1, \ldots, k\}$ and let $Y \geq 0$ be such that $(id - \Phi_{f_s,T_s}) \cdots (id - \Phi_{f_1T_1})(Y) \geq 0$. Note that $\{(id - \Phi^{q_s}_{f_s,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(Y)\}_{q=(q_1,\ldots,q_s)\in\mathbb{Z}^s_+}$ is an increasing sequence of positive operators. Indeed, since $\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}$ are commuting, we have

$$0 \leq (id - \Phi^{q_s}_{f_s,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(Y) = \sum_{t=0}^{q_s-1} \Phi^t_{f_s,T_s} \cdots \sum_{t=0}^{q_1-1} \Phi^t_{f_1,T_1}(id - \Phi_{f_s,T_s}) \cdots (id - \Phi_{f_1T_1})(Y),$$

which proves our assertion. If $p = 0$, the inequality in the lemma is due to the fact that $K^*_{f,T}K_{f,T} \leq I$. Assume that $p \neq 0$. Without loss of generality, we can assume that $p_j \geq 1$ for any $j \in \{1, \ldots, s\}$ for some $s \in \{1, \ldots, k\}$, and $p_j = 0$ for any $j \in \{s+1, \ldots, k\}$ if $s < k$. Since

$$K^*_{f,T}K_{f,T} = \lim_{q_k\to\infty} \cdots \lim_{q_1\to\infty} (id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I),$$

and taking into account that the maps $\Phi_{f_i T_i}$ are WOT-continuous and commuting, we deduce that

$$(id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_s,T_s})^{p_s} (K^*_{f,T}K_{f,T}) = \lim_q (id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_{s+1}}_{f_{s+1},T_{s+1}})\left[(id - \Phi_{f_s,T_s})^{p_s}(id - \Phi^{q_s}_{f_s,T_s})\right] \cdots \cdot \left[(id - \Phi_{f_1,T_1})^{p_1}(id - \Phi^{q_1}_{f_1,T_1})\right](I)$$

Now, let $j \in \{1, \ldots, s\}$ and let $Y \geq 0$ be such that $(id - \Phi_{f_j,T_j})(Y) \geq 0$. Due to the remark at the beginning of the proof, WOT-$\lim_{q_j\to\infty}(id - \Phi^{q_j}_{f_j,T_j})(Y)$ exists and, since Φ_{f_i,T_i} is WOT-continuous, we have

$$\lim_{q_j\to\infty} (id - \Phi_{f_j,T_j})^{p_j}(id - \Phi^{q_j}_{f_j,T_j})(Y) = (id - \Phi_{f_j,T_j})^{p_j} \lim_{q_j\to\infty} (id - \Phi_{f_j,T_j})(id - \Phi^{q_j}_{f_j,T_j})(Y) = (id - \Phi_{f_j,T_j})^{p_j}(Y).$$

Applying this result repeatedly, when $j = 1$ and $Y = I$, when $j = 2$ and $Y = (id - \Phi_{f_1,T_1})^{p_1}(I)$, and so on, when $j = s$ and $Y = (id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_s,T_s})^{p_s}(I)$, we obtain

$$\lim_{q_s\to\infty} \cdots \lim_{q_1\to\infty}\left[(id - \Phi_{f_s,T_s})^{p_s}(id - \Phi_{f_s,T_s}^{q_s})\right] \cdots \left[(id - \Phi_{f_1,T_1})^{p_1}(id - \Phi_{f_1,T_1}^{q_1})\right](I)$$
$$= (id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_s,T_s})^{p_s}(I)$$

Summing up the results above and using Lemma (4.2.26), we deduce that

$$(id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_s,T_s})^{p_s}(K_{f,T}^* K_{f,T})$$
$$= \lim_{(q_{s+1},\dots,q_k)} (id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_{s+1},T_{s+1}}^{q_{s+1}})(id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_s,T_s})^{p_s}(I)$$
$$\le (id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_s,T_s})^{p_s}(I).$$

The last part of this lemma is now obvious. The proof is complete.

Let $f = (f_1, \dots, f_k)$ be a k-tuple of positive regular free holomorphic functions and let $S = (S_1, \dots, S_n)$ with $S_i = (S_{i,1}, \dots, S_{i,n_i})$ be the universal model associated with the abstract noncommutative variety $V_{f,J}^m$, where J is a norm-closed two sided ideal of the noncommutative domain algebra $A(D_f^m)$ such that $N_J \neq \{0\}$. Let $U = \{U_{i,j}\} \in V_{f,J}^m(K)$ be such that

$$\left(id - \Phi_{f_k,U_k}\right) \dots \left(id - \Phi_{f_1,U_1}\right)(I) = 0,$$

where $U_i = (U_{i,1}, \dots, U_{i,n_i})$. A tuple $V := \{V_{i,j}\}$ having the matrix representation

$$V_{i,j} := \begin{bmatrix} S_{i,j} \otimes I_D & 0 \\ 0 & U_{i,j} \end{bmatrix}, \quad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}, \qquad (65)$$

is called constrained (or J-constrained) dilation of $T = \{T_{i,j}\} \in V_{f,J}^m(H)$ if H can be identified with a co-invariant subspace under each operator $V_{i,j}$ such that

$$T_{(\alpha)}^* = V_{(\alpha)}^*\big|H, \quad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+.$$

The dilation is called minimal if

$$(N_J \otimes D) \oplus K = \overline{span}\left\{V_{(\alpha)}H : (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\right\}.$$

The *dilation* index of T is the minimum dimension of D such that V is a constrained dilation of T.

Our first dilation result on the abstract noncommutative variety $V_{f,J}^m$ is the following.

Theorem (4.2.28) [244]: Let $S = \{S_{i,j}\}$ be the universal model associated with the abstract noncommutative variety $V_{f,J}^m$, where J is a norm-closed two sided ideal of the noncommutative polydomain algebra $A(D_f^m)$. If $T := \{T_{i,j}\}$ is an element in the noncommutative variety $V_{f,J}^m(H)$, then there exists a Hilbert space K and $U = \{U_{i,j}\} \in V_{f,J}^m(K)$ with

$$\left(id - \Phi_{f_k,U_k}\right) \dots \left(id - \Phi_{f_1,U_1}\right)(I) = 0$$

and such that H can be identified with a co-invariant subspace of $\tilde{K} := [N_J \otimes \overline{\Delta_{f,T}^m(I)H}] \oplus K$ under each operator

$$V_{i,j} := \begin{bmatrix} S_{i,j} \otimes I_{\overline{\Delta_{f,T}^m(I)H}} & 0 \\ 0 & U_{i,j} \end{bmatrix}, \qquad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\},$$

where $\Delta_{f,T}^m(I) := (id - \Phi_{f_1,T_1})^{m_1} \cdots (id - \Phi_{f_k,T_k})^{m_k}(I)$, and

$$T_{i,j}^* = V_{i,j}^*|_H, \qquad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}.$$

Moreover, the following statements hold.

(i) The dilation index of T coincides with rank$\Delta_{f,T}^m(I)$.

(ii) T is a pure element in $V_{f,J}^m(H)$ if and only if the dilation $V := \{V_{i,j}\}$ is pure.

Proof: We recall that the constrained noncommutative Berezin kernel associated with the $T \in V_{f,J}^m(H)$ is the bounded operator $K_{f,T,J} : H \to N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$ defined by

$$K_{f,T,J} := \left(P_{N_J} \otimes I_{\overline{\Delta_{f,T}^m(I)(H)}}\right) K_{f,T},$$

where $K_{f,T}$ is the noncommutative Berezin kernel associated with $T \in D_f^m(H)$. Taking into account the properties of the Berezin kernel and the fact that range $K_{f,T} \subseteq N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$, we have

$$K_{f,T,J} T_{(\alpha)}^* = \left(S_{(\alpha)}^* \otimes I_H\right) K_{f,T,J}, \quad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \qquad (66)$$

and $K_{f,T,J}^*K_{f,T,J} = K_{f,T}^*K_{f,T}$. We consider the Hilbert space $K := \overline{(I - K_{f,T}^*K_{f,T})H}$ and denote $Y := I - K_{f,T}^*K_{f,T}$. For each $i \in \{1,\dots,k\}, j \in \{1,\dots,n_i\}$, define the operator $L_{i,j}: K \to K$ by setting

$$L_{i,j}Y^{1/2}h := Y^{1/2}T_{i,j}^*h, \qquad h \in H.$$

Note that each $L_{i,j}$ is well-defined. Indeed, due to Lemma (4.2.27), we have $\Delta_{f,T}^{(1,\dots,1)}(K_{f,T}^*K_{f,T}) \leq \Delta_{f,T}^{(1,\dots,1)}(I)$. Hence, we deduce that $\Phi_{f_i,T_i}(Y) \leq Y$. Therefore,

$$\sum_{\alpha\in\mathbb{F}_{n_i}^+,|\alpha|\geq 1} a_{i,\alpha}\left\|L_{i,\tilde{\alpha}}Y^{1/2}h\right\|^2 = \langle\Phi_{f_i,T_i}(Y)h,h\rangle \leq \left\|Y^{1/2}h\right\|^2$$

for any $h \in H$, where $\tilde{\alpha}$ is the reverse of α. Consequently, we have $a_{i,g_j^i}\left\|L_{i,j}Y^{1/2}x\right\|^2 \leq \left\|Y^{1/2}x\right\|^2$, for any $x \in N_J \otimes H$. Since $a_{i,g_j^i} \neq 0$ each $L_{i,j}$ can be uniquely be extended to a bounded operator (also denoted by $L_{i,j}$) on the subspace K. Denoting $U_{i,j} := L_{i,j}^*$ and setting $U = (U_1,\dots,U_k)$ with $U_i = (U_{i,1},\dots,U_{i,n_i})$, an approximation argument shows that $\Phi_{f_i,U_i}(I_M) \leq I_M$ for $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$. The definition of $L_{i,j}$ implies

$$U_{i,j}^*(Y^{1/2}h) = Y^{1/2}T_{i,j}^*h, \qquad h \in H, \qquad (67)$$

for any $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$. Hence, and using again Lemma (4.2.27), we deduce that

$$Y^{1/2}\Delta_{f,U}^P(I_K)Y^{1/2} = \Delta_{f,T}^P(I - K_{f,T}^*K_{f,T}) \geq 0$$

for any $p := (p_1,\dots,p_k) \in \mathbb{Z}_+^k$ such that $p \leq m, p \neq 0$, and $Y^{1/2}(id - \Phi_{f_k,U_k}) \cdots (id - \Phi_{f_1,U_1})(I)Y^{1/2} = 0$.

Since $Y^{1/2}$ is injective on $K = \overline{YH}$, we conclude that $U = (U_1,\dots,U_k) \in V_{f,J}^m(K)$ and

$$(id - \Phi_{f_k,U_k}) \cdots (id - \Phi_{f_1,U_1})(I) = 0.$$

On the other hand, relation (67) implies

$$Y^{1/2}q(U) = q(T)Y^{1/2} = 0, q \in \mathbb{C}[Z_{i,j}].$$

Using the von Neumann type inequality for the elements in the abstract noncommutative polydomain D_f^m and the fact that the polynomials in $W_{i,j}$ and the identity are dense in the noncommutative polydomain algebra $A(D_f^m)$, an approximation argument shows that $Y^*g(U) = 0$ for any $g \in J$. Once again, since $Y^{1/2}$ is injective on $K = \overline{YH}$, we have $g(U) = 0$ for any $q \in J$. Let $V: H \to [N_J \otimes H] \oplus K$ be defined by

$$V := \begin{bmatrix} K_{f,T,J} \\ Y \end{bmatrix}.$$

Note that

$$\|Vh\|^2 = \left\|K_{f,T,J}h\right\|^2 + \left\|(I - K_{f,T,J}^*K_{f,T,J})^{1/2}h\right\|^2 = \|h\|^2$$

for any $h \in H$. Therefore, V is an isometry. Using relations (66) and (67), we obtain

$$\begin{aligned} VT_{i,j}^* &= \begin{bmatrix} K_{f,T,J} \\ Y \end{bmatrix} T_{i,j}^*h = K_{f,T,J}T_{i,j}^*h \oplus YT_{i,j}^*h \\ &= (S_{i,j}^* \otimes I_H)K_{f,T,J}h \oplus U_{i,j}^*Yh \\ &= \begin{bmatrix} S_{i,j}^* \otimes I_{\overline{\Delta_{f,T}^m H}} & 0 \\ 0 & U_{i,j}^* \end{bmatrix} Vh. \end{aligned}$$

Identifying H with VH, we deduce that $T_{i,j}^* = V_{i,j}^*|H$ for $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$.

Now, we prove the second part of the theorem. Assume that T has the dilation V given by relation (65). Since $\Delta_{f,U}^m(I) = 0$ and $\Delta_{f,S}^m(I) = P_{\mathbb{C}}$, where $P_{\mathbb{C}}$ is the orthogonal projection from N_J onto $\mathbb{C}1 \subset N_J$, we deduce that $\Delta_{f,T}^m(I) = P_H[P_{\mathbb{C}} \otimes I_D]|H$. Hence, rank $\Delta_{f,T}^m(I) \leq dim D$. The reverse inequality is due to the first part of the theorem. To prove item (ii), note that if T is pure then $K_{f,T}$ is an isometry and, consequently, $K = \{0\}$. This implies $V = S$, which is pure. Conversely, if we assume that V is pure, we must have

$$\lim_{q=(q_1,\dots,q_k)\in\mathbb{N}^k} (id - \Phi_{f_1,V_1}^{q_1}) \cdots (id - \Phi_{f_k,V_k}^{q_k})(I_{\tilde{K}}) = I_{\tilde{K}}.$$

Taking into account the matrix representation of each operator $V_{i,j}$ and the fact that

$$\lim_{q=(q_1,\dots,q_k)\in\mathbb{N}^k}(id - \Phi_{f_1,U_1}^{q_1}) \cdots (id - \Phi_{f_k,U_k}^{q_k})(I_K) = 0,$$

we deduce that $K = \{0\}$. This shows that the noncommutative Berezin kernel $K_{f,T}$ is an isometry, which is equivalent to the fact that T is pure. The proof is complete.

We provide a Wold type decomposition for non-degenerate $*$-representations of the C^*-algebra $C^*(S_{i,j})$.

Theorem (4.2.29) [244]: Let $q = (q_1, \dots, q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $S = (S_1, \dots, S_k)$ be the universal model associated with the abstract noncommutative variety $V_{q,J}^m$, where J is a WOT-closed two sided ideal of $F^\infty(D_q^m)$ such that $1 \in N_J$. If $\pi: C^*(S_{i,j}) \to B(K)$ is a nondegenerate $*$-representation of $C^*(S_{i,j})$ on a separable Hilbert space K, then π decomposes into a direct sum

$$\pi = \pi_0 \oplus \pi_1 \ on\ K = K_0 \oplus K_1,$$

where π_0 and π_1 are disjoint representations of $C^*(S_{i,j})$ on the Hilbert spaces

$$K_0 := \overline{span}\{\pi(S_{(\alpha)})\Delta_{q,\pi(S)}^m(I_K)K : (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}\}$$

and $K_1 \coloneqq K_0^\perp$, respectively, where $\pi(S) := (\pi(S_1), \dots, \pi(S_k))$ and $\pi(S_i) := (\pi(S_{i,1}), \dots, \pi(S_{i,n_i}))$. More-over, up to an isomorphism,

$$K_0 \simeq N_J \otimes \mathcal{G}, \qquad \pi_0(X) = X \otimes I_{\mathcal{G}}\ for\ any\ \ X \in C^*(S_{i,j}),$$

where $\mathcal{G}$ is a Hilbert space with

$$dim\ \mathcal{G} = \dim\{range\Delta_{q,\pi(S)}^m(I_K)\},$$

and π_1 is a $*$-representation which annihilates the compact operators and

$$(I - \Phi_{q_1,\pi_1(S_1)}) \cdots (I - \Phi_{q_1,\pi_1(S_1)})(I_{K_1}) = 0.$$

If π' is another nondegenerate $*$-representation of $C^*(S_{i,j})$ on a separable Hilbert space K', then π is unitarily equivalent to π' if and only if $dim\mathcal{G} = dim\ \mathcal{G}'$ and π_1 is unitarily equivalent to π_1'.

Proof: Note that, due to Theorem (4.2.6), the ideal $C(N_J)$ of compact operators in $B(N_J)$ is contained in the C^*-algebra $C^*(S_{i,j})$. Due to standard theory of representations of the C^*-algebras [254], the representation π decomposes into a direct sum $\pi = \pi_0 \oplus \pi_1$ on $K = \widetilde{K}_0 \oplus \widetilde{K}_1$, where

$$\widetilde{K}_0 \coloneqq \overline{span}\{\pi(X)K : X \in C(N_J)\}\ and\ \ \widetilde{K}_1 \coloneqq \widetilde{K}_0^\perp,$$

and the representations $\pi_J: C^*(S_{i,j}) \to B(\widetilde{K}_j)$ are defined by $\pi_j(X) \coloneqq \pi(X)|_{\widetilde{K}_j}$ for $j = 0, 1$. We remark that π_0, π_1 are disjoint representations of $C^*(S_{i,j})$ such that π_1 annihilates the compact operators in $B(N_J)$, and π_0 is uniquely determined by the action of π on the ideal $C(N_J)$ of compact operators. Since every representation of $C(N_J)$ is equivalent to a multiple of the identity representation, we deduce that

$K_0 \simeq N_J \otimes \mathcal{G}, \pi_0(X) = X \otimes I_{\mathcal{G}}$, for any $X \in C^*(S_{i,j})$, where $\mathcal{G}$ is a Hilbert space. Using Theorem (4.2.6) and its proof, one can show that the space $\widetilde{K}_0$ coincides with the space K_0. Taking into account that $(I - \Phi_{q_1,S_1})^{m_1} \cdots (I - \Phi_{q_k,S_k})^{m_k}(I) = P_{\mathbb{C}}$ is a projection of rank one in $C^*(S_{i,j})$, we deduce that $(I - \Phi_{q_1,\pi(S_1)})^{m_1} \cdots \left(I - \Phi_{q_k,\pi(S_k)}\right)^{m_k}(I_{K_\pi}) = 0$ and $dim\ \mathcal{G} = dim\ [range\ \pi(P_{\mathbb{C}})]$.

The uniqueness of the decomposition is due to standard theory of representations of C^*-algebras.

We remark that under the hypotheses and notations of Theorem (4.2.29), and setting $V_{i,j} := \pi(S_{i,j})$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, one can see that $V := \{V_{i,j}\}$ is a pure element in $V_{q,J}^m(K)$ if and only if $K \coloneqq \overline{span}\{V_{(\alpha)}\Delta_{q,V}^m(I_K)(K) : (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}$.

We can obtain a more refined dilation theorem for the class of noncommutative varieties $V_{q,J}^m(H)$, where $Q \subset \mathbb{C}[Z_{i,j}]$ is an ideal generated by homogeneous polynomials and $q = (q_1, \dots, q_k)$ is a k-tuple of positive regular noncommutative polynomials.

Let $C^*(\Gamma)$ be the C^*-algebra generated by a set of operators $\Gamma \subset B(K)$ and the identity. A subspace $H \subset K$ is called $*$-cyclic for Γ if $K = \overline{span}\{Xh, X \in C^*(\Gamma), h \in H\}$.

Theorem (4.2.30) [244]: Let $q = (q_1, \dots, q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $S = \{S_{i,j}\}$ be the universal model associated with the abstract noncommutative variety $V_{q,Q}^m$, where $Q \subset \mathbb{C}[Z_{i,j}]$ is an ideal generated by homogeneous polynomials. If $T = \{T_{i,j}\}$ is in $V_{q,Q}^m(H)$, then there exists a $*$-representation $\pi: C^*(S_{i,j}) \to B(K_\pi)$ on a separable Hilbert space K_π, which annihilates the compact operators and

$$(I - \Phi_{q_1,\pi(S_1)}) \cdots (I - \Phi_{q_k,\pi(S_k)})(I_{K_\pi}) = 0,$$

where $\pi(S) := (\pi(S_1), \dots, \pi(S_k))$ and $\pi(S_i) := (\pi(S_{i,1}), \dots, \pi(S_{i,n_i}))$, such that H can be identified with a $*$-cyclic co-invariant subspace of

$$\widetilde{K} := \left[N_Q \otimes \overline{\Delta_{f,T}^m(I)(H)}\right] \oplus K_\pi$$

under each operator

$$V_{i,j} := \begin{bmatrix} S_{i,j} \otimes I_{\overline{\Delta_{f,T}^m(I)(H)}} & 0 \\ 0 & \pi(S_{i,j}) \end{bmatrix}, \qquad i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\},$$

where $\Delta_{q,T}^m(I) := (id - \Phi_{q_1,T_1})^{m_1} \cdots (id - \Phi_{q_k,T_k})^{m_k}(I)$, and such that

$$T_{i,j}^* = V_{i,j}^*|_H \; for\ all\ i \in \{1, \dots, k\}, j \in \{1, \dots, n_i\}.$$

Proof: Applying Arveson extension theorem [253] to the map Ψ of Theorem (4.2.4), we find a unital completely positive linear map $\Psi : C^*(S_{i,j}) \to B(H)$ such that $\Psi(S_{(\alpha)}S_{(\beta)})^* = T_{(\alpha)}T_{(\beta)}^*$ for all $(\alpha), (\beta)\ in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}$. Let $\tilde{\pi}: C^*(S_{i,j}) \to B(\widetilde{K})$ be the minimal Stinespring dilation [36] of Ψ. Then we have

$$\Psi(X) = P_H \tilde{\pi}(X)|H, \qquad X \in C^*(S_{i,j}),$$

and $\widetilde{K} = \overline{span}\{\tilde{\pi}(X)h: X \in C^*(S_{i,j}), h \in H\}$. Now, one can show that that that $P_H \tilde{\pi}(S_{(\alpha)})|_{H^\perp} = 0$ for any $(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}$. Consequently, H is an invariant subspace under each operator $\tilde{\pi}(S_{i,j})^*$ and

$$\tilde{\pi}(S_{i,j})^*|_H = \Psi(S_{i,j}^*) = T_{i,j}^*$$

for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Applying the Wold decomposition of Theorem (4.2.29) to the Stinespring representation $\tilde{\pi}$, one can complete the proof of the theorem. We omit the details since the proof is now very similar to the corresponding result from [253].

Let V be the dilation of T given by Theorem (4.2.30). One can easily prove that V is a pure element in $V_q^m(\widetilde{K})$ if and only if T is a pure element in $V_q^m(H)$, and $(I - \Phi_{q_1,V_1}) \cdots (I - \Phi_{q_k,V_k})(I_{\widetilde{K}}) = 0$ if and only if $(I - \Phi_{q_1,T_1}) \cdots (I - \Phi_{q_k,T_k})(I_H) = 0$. We remark that under the additional condition that

$$\overline{span}\{S_{(\alpha)}S_{(\beta)}^*: (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}\} = C^*(S_{i,j}),$$

which holds, for example, for the polyballs (commutative or noncommutative), one can show that the dilation provided by Theorem (4.2.30) is minimal. Taking into account the uniqueness of the minimal Stinespring representation and the Wold type decomposition of Theorem (4.2.29), we can prove that the dilation is unique up to unitary equivalence.

We provide a characterization for the class of elements in the abstract noncommutative variety $V_{f,J}^m$ which admit constrained characteristic functions. The characteristic function is a complete unitary invariant for completely non-coisometric tuples. We obtain operator models in terms of the constrained characteristic functions.

Let $S := \{S_{i,j}\}$ be the universal model associated to the abstract noncommutative variety $V_{f,J}^m$ and let $\Phi: N_J \otimes H \to N_J \otimes K$ be a multi-analytic operator with respect to S, i.e.,

$$\Phi(S_{i,j} \otimes I_H) = (S_{i,j} \otimes I_K)\Phi$$

for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. The support of Φ is the smallest reducing subspace $supp(\Phi)$ of $N_J \otimes H$ under each operator $S_{i,j}$ containing the co-invariant subspace $M := \overline{\Phi^*(N_J \otimes K)}$. Using Theorem (4.2.8) and its proof, we deduce that if $1 \in N_J$, then

$$\sup p\,(\Phi) = \bigvee_{(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}} (S_{(\alpha)} \otimes I_K)(M) = N_J \otimes \mathcal{L},$$

where $\mathcal{L} := (P_{\mathbb{C}} \otimes I_H)\overline{\Phi^*(N_J \otimes K)}$. We say that two multi-analytic operator $\Phi: N_J \otimes K_1 \to N_J \otimes K_2$ and $\Phi: N_J \otimes K_1' \to N_J \otimes K_2'$ coincide if there are two unitary operators $\tau_j \in B(K_j, K_j')$ such that

$$\Phi'(I_{N_J} \otimes \tau_1) = (I_{N_J} \otimes \tau_2)\Phi.$$

As in [33], one can prove that if $\Phi_s: N_J \otimes H_s \to N_J \otimes K,\ s = 1, 2$, are multi-analytic operators with respect to $S := \{S_{i,j}\}$ such that $\Phi_1\Phi_1^* = \Phi_2\Phi_2^*$, then there is a unique partial isometry $V: H_1 \to H_2$ such that $\Phi_1 = \Phi_2(I_{N_J} \otimes V)$, where $(I_{N_J} \otimes V)$ is a inner multi-analytic operator with initial space $\sup p\ (\Phi_1)$ and final space $\sup p\ (\Phi_2)$. In particular, the multi-analytic operators $\Phi_1|_{sup\, p\ (\Phi_1)}$ and $\Phi_1|_{sup\, p\ (\Phi_1)}$ coincide.

Definition (4.2.31) [244]: A k-tuple $T \in V_{f,J}^m(H)$ is said to have constrained characteristic function if there is a Hilbert space $\mathcal{E}$ and a multi-analytic operator $\Psi: N_J \otimes \mathcal{E} \to N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$ with respect to $S = \{S_{i,j}\}$ such that

$$K_{f,T,J}K_{f,T,J}^* + \Psi\Psi^* = I,$$

where $K_{f,T,J}$ is the constrained noncommutative Berezin kernel associated with $T \in V_{f,J}^m(H)$.

According to the remarks above, if $1 \in N_J$ and there is a constrained characteristic function for $T \in V_{f,J}^m(H)$, then it is essentially unique. We also remark that in the particular case when $k = 1$ and $m_1 = 1$, all the elements in the noncommutative variety $V_{f_1}^1$ have constrained characteristic functions.

Using Theorem (4.2.10), one can deduce the following result.

Theorem (4.2.32) [244]: An element $T = \{T_{i,j}\}$ in the noncommutative variety $V_{f,J}^m(H)$ admits a constrained characteristic function if and only if

$$\Delta_{f,S\otimes I}^p(I - K_{f,T,J}K_{f,T,J}^*) \geq 0$$

for any $p := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ such that $p \leq m$, where $K_{f,T,J}$ is the constrained Berezin kernel associated with T and $S := \{S_{i,j}\}$ is the universal model of $V_{f,J}^m$.

If T has characteristic function, the multi-analytic operator Γ provided by the proof of Theorem (4.2.10) when $G = I - K_{f,T,J}K_{f,T,J}^*$, which we denote by $\Theta_{f,T,J}$, is called the constrained characteristic function of T. More precisely, $\Theta_{f,T,J}$ is the multi-analytic operator

$$\Theta_{f,T,J}: N_J \otimes \overline{\Delta_{f,M_T}^m(I)(M_T)} \to N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$$

defined by $\Theta_{f,T,J} := (I - K_{f,T,J}K_{f,T,J}^*)^{1/2}K_{f,T,J}^*$, where

$$K_{f,T,J}: H \to N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$$

is the constrained noncommutative Berezin kernel associated with T and

$$K_{f,M_T,J}: H \to N_J \otimes \overline{\Delta_{f,M_T}^m(I)(M_T)}$$

is the constrained noncommutative Berezin kernel associated with $M_T \in V_f^m(M_T)$. Here, we have

$$M_T := \overline{range\ (I - K_{f,T,J}K_{f,T,J}^*)}$$

and $M_T := \{M_{i,j}\}$, where $M_{i,j} \in B(M_T)$ is given by $M_{i,j} := A_{i,j}^*$ and $A_{i,j} \in B(M_T)$ is uniquely defined by

$$A_{i,j}\left[(I - K_{f,T,J}K_{f,T,J}^*)^{1/2}x\right] := (I - K_{f,T,J}K_{f,T,J}^*)^{1/2}(S_{i,j} \otimes I)x$$

for any $x \in N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$. According to Theorem (4.2.10), we have

$$K_{f,T,J}K_{f,T,J}^* + \Theta_{f,T,J}\Theta_{f,T,J}^* = I.$$

We denote by $C_{f,J}^m(H)$ the set of all $T = \{T_{i,j}\} \in V_{f,J}^m(H)$ which admit constrained characteristic functions. W provide a model theorem for class of the completely non-coisometric elements in $C_{f,J}^m(H)$. Due to the results obtained the proof is now similar to that of Theorem 6.4 from [253].

Theorem (4.2.33) [244]: Let $T = \{T_{i,j}\}$ be a completely non-coisometric element in $C_{f,J}^m(H)$ and let $S := \{S_{i,j}\}$ be the universal model associated to the abstract noncommutative variety $V_{f,J}^m$. Set

$$D := \overline{\Delta_{f,T}^m(I)(H)}, \qquad D_* := \overline{\Delta_{f,M_T}^m(I)(M_T)},$$

and $\Delta_{\Theta_{f,T,J}} := \left(I - \Theta_{f,T,J}^*\Theta_{f,T,J}\right)^{1/2}$, where $\Theta_{f,T,J}$ is the characteristic function of T. Then T is unitarily equivalent to $\mathbb{T} := \{T_{i,j}\} \in C_f^m(\mathbb{H}_{f,T,J})$, where $\mathbb{T}_{i,j}$ is a bounded operator acting on the Hilbert space

$$\mathbb{H}_{f,T,J} := \left[(N_J \otimes D) \oplus \overline{\Delta_{\Theta_{f,T,J}}(N_J \otimes D^*)}\right]$$

$$\ominus \{\Theta_{f,T,J}\varphi \oplus \Delta_{\Theta_{f,T,J}}\varphi : \varphi \in N_J \otimes D_*\}$$

and is uniquely defined by the relation

$$\left(P_{N_J\otimes D}|_{\mathbb{H}_{f,T,J}}\right)\mathbb{T}_{i,j}^* x = (S_{i,j}^* \otimes I_D)\left(P_{N_J\otimes D}|_{\mathbb{H}_{f,T,J}}\right)x$$

for any $x \in \mathbb{H}_{f,T,J}$. Here, $P_{N_J\otimes D}$ is the orthogonal projection of the Hilbert space

$$K_{f,T,J} := (N_J \otimes D) \oplus \overline{\Delta_{\Theta_{f,T,J}}(N_J \otimes D_*)}$$

onto the subspace $N_J \otimes D$.

Corollary (4.2.34) [244]: Let $T = \{T_{i,j}\}$ be an element in $C_{f,J}^m(H)$. Then T is pure if and only if the constrained characteristic function $\Theta_{f,T,J}$ is an inner multi-analytic operator with respect to $S := \{S_{i,j}\}$. Moreover, in this case $T = \{T_{i,j}\}$ is unitarily equivalent to $G = \{G_{i,j}\}$, where

$$G_{i,j} := P_{H_{f,T,J}}(S_{i,j} \otimes I)|_{H_{f,T,J}}$$

and $P_{H_{f,T,J}}$ is the orthogonal projection of $N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}$ onto

$$H_{f,T,J} := \left\{N_J \otimes \overline{\Delta_{f,T}^m(I)(H)}\right\} \ominus range\Theta_{f,T,J}.$$

As consequences of the results above, we can easily show that if $T = \{T_{i,j}\} \in V_{f,J}^m(H)$, then T is unitarily equivalent to $\{S_{i,j} \otimes I_K\}$ for some Hilbert space K if and only if $T \in C_{f,J}^m$ is completely noncoisometric and the characteristic function $\Theta_{f,T,J} = 0$. On the other hand, if $T \in C_{f,J}^m$, then $\Theta_{f,T,J}$ has dense range if and only if there is no nonzero vector $h \in H$ such that

$$\lim_{q=(q_1,\dots,q_k)\in\mathbb{N}^k} \langle \left(id - \Phi_{f_1,T_1}^{q_1}\right) \cdots \left(id - \Phi_{f_k,T_k}^{q_k}\right)(I_H)h, h\rangle = \|h\|,$$

where $T_i := (T_{i,j}, \dots, T_{i,n_i})$ for any $i \in \{1, \dots, k\}$.

An important consequence of Theorem (4.2.33) is that the constrained characteristic function $\Theta_{f,T,J}$ is a complete unitary invariant for the completely non-coisometric part of the noncommutative domain $C_{f,J}^m$.

The proof is similar to that of Theorem 6.5 from [253].

Theorem (4.2.35) [244]: Let $T = \{T_{i,j}\} \in C_{f,J}^m(H)$ and $T' = \{T'_{i,j}\} \in C_{f,J}^m(H')$ be two completely non-coisometric elements. Then T and T' are unitarily equivalent if and only if their constrained characteristic functions $\Theta_{f,T,J}$ and $\Theta_{f,T',J}$ coincide.

Chapter 5
Berezin Transform on Harmonic Bergman Spaces

We show that an expansion has the additional property of being asymptotic for large *c* with fixed *a* uniformly in *b* and *z* (with bounded *b/c*). Moreover, the asymptotic character of the expansion holds for a larger set of *b*, *c* and *z* specified below. We provide the full asymptotic expansion of the harmonic Berezin transform on the unit ball in R^n purely by means of transformations of hypergeometric functions and function's "hypergeometrization".

Section (5.1): Gauss Hypergeometric Function

The asymptotic behaviour of the Gauss hypergeometric function F(a, b; c; z) when different combinations of a, b, c and z are large is a subject of recent interest [286]. The hypergeometric function is defined by the power series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, |z| < 1, c \neq 0, -1, -2, \ldots \qquad (1)$$

This is an asymptotic expansion of $F(a, b; c; z)$ for $z \to 0$ and/or $c \to \infty$. The condition $|z| < 1$ may be relaxed still keeping the asymptotic character of the expansion for large c [291]. A translation formula for $F(a, b; c; z)$ [288, p. 113, Eq. (5.11)], can be used to obtain an asymptotic representation of $F(a, b; c; z)$ for large values of z with $|\arg(-z)| < \pi$ [288, p. 127]:

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (1-c+a)_n}{(1-b+a)_n n!} \frac{1}{z^n}$$
$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} \sum_{n=0}^{\infty} \frac{(b)_n (1-c+b)_n}{(1-a+b)_n n!} \frac{1}{z^n}$$

But when one or several of the parameters a, b, c or z are large (except when only c or z are large), the asymptotic study is more difficult. Some have obtained asymptotic expansions of $F(a, b; c; z)$ with certain restrictions on the parameters. Wagner provides in [282] an asymptotic expansion of $F(a, b; c; z)$ when $c \to \infty$with $a^2 = O(c)$ and $b^2 = O(c)$. This result is obtained from an integral representation of $F(a, b; c; z)$ followed by contour deformations and series expansions.
Several have focused their attention on the asymptotic behaviour of

$$F(a + e_1\lambda, b + e_2\lambda; c + e_3\lambda; z), e_j = 0, \pm 1, \lambda \to \infty \qquad (2)$$

In [293],Watson obtained an asymptotic expansion of $F(a + \boldsymbol{\lambda}, \mathbf{b} + \boldsymbol{\lambda}; \mathbf{c} + 2\boldsymbol{\lambda}; z), \mathbf{F}(\mathbf{a} + \boldsymbol{\lambda}, \mathbf{b} - \boldsymbol{\lambda}; \mathbf{c}; z)$ and$\mathbf{F}(\mathbf{a}, \mathbf{b}; \mathbf{c} + \boldsymbol{\lambda}; z)$in terms of inverse powers of $\boldsymbol{\lambda}$ by contour integrals and the steepest descent method, see also [287, Chapter 5,Section 9]. However, these expansions are only valid in small regions of z. In [284], Jones obtains a uniform asymptotic expansion of $F(a + \boldsymbol{\lambda}, \mathbf{b} - \boldsymbol{\lambda}; \mathbf{c}; \mathbf{1/2} - \mathbf{1/2}\, z)$when $\lambda \to \infty$ with $|\arg z| < \pi$ in terms of Bessel functions. Jones uses for his analysis Olver's method [287], which is based on the linear second order differential equation satisfied by $F(a, b; c; z)$. Olde Daalhuis has obtained an asymptotic expansion of $F(a, b - \lambda; c + \lambda; -z)$ in terms of parabolic cylinder functions and of $F(a + \lambda, b + 2\lambda; c; -z)$ in terms of Airy functions [286]. These expansions hold for fixed values of a, b and c, and are uniformly valid for z with $|\arg z| < \pi$. Olde Daalhuis uses Bleinstein's method applied on a contour
integral representation of $F(a, b; c; z)$ in which a saddle point and a branch point coalesce. In [289], Temme has shown that the set of 26 possible cases in (2) can be reduced to only four cases:

$$(A) e_1 = e_2 = 0, e_3 = 1 \quad , (B) e_1 = 1, e_2 = -1, e_3 = 0$$

$$(C) e_1 = 0, e_2 = -1, e_3 = 1, (D) e_1 = 1, e_2 = 2, e_3 = 0$$

For case (A), Temme obtains the uniform asymptotic expansion

$$F(a, b; c+\lambda; z) \sim \frac{\Gamma(c+\lambda)\xi^{b-a}}{\Gamma(c+\lambda-b)} \sum_{s=0}^{\infty} g_s(z)(b)_s \xi^s U(b+s, b-a+1+s, \xi\lambda) \quad (3)$$

where U is the confluent hypergeometric function, $\xi = \ln[(z-1)/z]$ and $g_s$are the coefficients of the Taylor expansion $g(t) = (t+\xi)^a[(e^t-1)/t]^{b-1}e^{(1-c)t}(1-z+ze^{-t})^{-a}$at $t = 0: g(t) = \sum_{s=0}^{\infty} g_s(z)t^s$. Formula (3) is an asymptotic expansion when $\lambda \to \infty$, uniformly with respect to bounded values of ξ(z bounded away from the origin). We are concerned with a generalization of cases (A) and (C). We study asymptotic expansions of $F(a, b;\ c;\ z)$ for large values of c uniformly in b with bounded b/c. In [288] we used a modification of the steepest descent method (see [283]) to derive uniform asymptotic expansions of the incomplete gamma functions $\Gamma(a, z)$ and $\gamma(a, z)$ for large values of a and z in terms of elementary functions. We apply here the same idea to derive a uniform asymptotic expansion of $F(a, b;\ c;\ z)$ for large b and c using the integral representation (4) of $F(a, b;\ c;\ z)$ given below. The approach consists of: (i) a factorization of the integrand in that integral in an exponential factor times
another factor and (ii) an expansion of this second factor at the asymptotically relevant point of the exponential factor. The main benefit of this procedure is the derivation of easy asymptotic expansions (in terms of elementary functions) with easily computable coefficients.
We derive a convergent expansion of $F(a, b+1;\ c+2;\ z)$ valid under the restrictions $|b||z| < |c - bz|$ and $|c-b||z| < |c-bz|$ which has also an asymptotic character for large c uniformly in b and z with bounded b/c. This expansion is not new, it was already obtained by NØrlund in [285, Eq. (1.21)], although with more restrictive conditions for the convergence and without mention to its asymptotic properties. we show that the expansion obtained in the previous keeps its asymptotic character for large c (uniformly in b and z with bounded b/c) even if the restrictions $|b||z| < |c - bz|$ and $|c-b||z| < |c-bz|$ do not hold. We consider $a, b, c \in \mathbb{C}, c \neq -2, -3, -4, \ldots$ and $|\arg(1-z)| < \pi$.
The Gauss hypergeometric function may be written in the form [288, p. 110, Eq. (5.4)]

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, \Re a > \Re b > 0$$

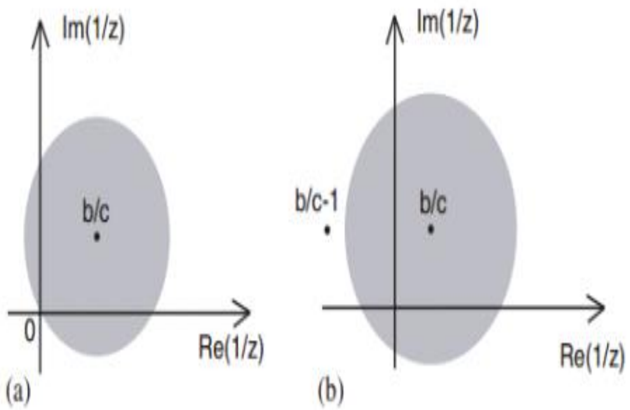

Fig. 1. Eqs. (7) are equivalent to $|z^{-1} - b/c| > |b/c|$ and $|z^{-1} - b/c| > |b/c - 1|$ (last inequality in formula (6) for $t = 0$ and $t = 1$). Then, z^{-1} must be outside of a disk of center b/c and radius $\mathrm{Max}\{|b/c|, |b/c-1|\}$. When $\Re(b/c) \geqslant \frac{1}{2}$, the origin is on the boundary of the disk. When $\Re(b/c) < \frac{1}{2}$, it is inside the disk. (a) $\Re(b/c) \geqslant 1/2$ (b) $\Re(b/c) < 1/2$.

For convenience, we consider a shift in the parameters b and c and write the hypergeometric function in the form

$$F(a, b+1; c+2; z) = \frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)} \int_0^1 e^{cf(t)}\, g(t) dt \qquad (4)$$

With

$$f(t) \equiv \frac{b}{c}\log t + \left(1 - \frac{b}{c}\right)\log(1-t), g(t) \equiv (1-tz)^{-a}, \Re c + 1 > \Re b > -1 \quad (5)$$

The unique saddle point of f (t) is located at t = b/c .We replace the function g(t) in (5) by its Taylor expansion at t = b/c with convergence radius $|1/z - b/c|$ [283]

$$g(t) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!\,(1-(b/c)z)^{k+a}} \left(t - \frac{b}{c}\right)^k, \left|t - \frac{b}{c}\right| < \left|\frac{1}{z} - \frac{b}{c}\right| \qquad (6)$$

This expansion converges uniformly with respect to t ∈ [0, 1] when the following conditions hold:

$$|b||z| < |c - bz| \text{ and } |c - b||z| < |c - bz|. \qquad (7)$$

Several possible z-regions are illustrated in Fig. 1.

For the values of z, b and c verifying (7), we can introduce (6) in (4) to obtain, after interchanging summation and integration,

$$F(a, b+1; c+2; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!\,(1-(b/c)z)^{k+a}} \Phi_k(b, c), \qquad (8)$$

where the functions $\Phi_k(b, c)$ are defined by

$$\Phi_k(b, c) = \frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)} \int_0^1 e^{cf(t)} \left(t - \frac{b}{c}\right)^k dt \qquad (9)$$

Using again the integral representation (4) we see that $\Phi_k(b, c)$ is a very simple hypergeometric function which is also a rational function of b and c:

$$\Phi_k(b, c) = \left(-\frac{b}{c}\right)^k F\left(-k, b+1; c+2; \frac{c}{b}\right) = \sum_{j=0}^{k} \binom{k}{j} \left(-\frac{b}{c}\right)^{k-j} \frac{(b+1)_j}{(c+2)_j} \quad (10)$$

The first few functions $\Phi_k(b, c)$ are detailed in Table 1.

We have derived the above expansion under the restrictions $\Re c + 1 > \Re b > -1$, $|b||z| < |c - bz|$ and $|c-b||z| < |c - bz|$. But the restriction $\Re c + 1 > \Re b > -1$ is superfluous: for large values of k we have that [290]

$$F(-k, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-b)} (kz)^{-b}[1 + o(1)] + \frac{\Gamma(c)}{\Gamma(b)} e^{\pm \pi i(b-c)} (1-z)^{c-b+k} (kz)^{b-c}[1 + o(1)]$$

Table 1[285]:

First few functions $\Phi_k(b, c)$ defined in (9) and used in (8)

k	$\Phi_k(b,c)$
0	1
1	$\frac{c-2b}{c(c+2)}$
2	$\frac{(2+b)c^2-b(6+b)c+6b^2}{c^2(c+2)(c+3)}$
3	$\frac{(6+5b)c^3-3b(8+5b)c^2+2b^2(18+5b)c-24b^3}{c^3(c+2)(c+3)(c+4)}$

when $k \to \infty$ with b and c fixed complex numbers, $c \neq 0,-1,-2,\ldots, z \neq 0$ and $|\arg(1-z)| < \pi$.
Therefore, $\Phi_k(b,c) = \vartheta(\gamma(k)\alpha^k)$ when $k\to\infty$
with $\gamma(k) \equiv \mathrm{Max}\{k^{-b-1}, k^{b-c-1}\}$ and $\alpha \equiv \mathrm{Max}\{|(b/c)|, |1-(b/c)|\}$.
Then, the terms of the series (8) verify

$$\frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}}\Phi_k(b,c) = \vartheta(k^{a-1}\gamma(k)\beta^k) \text{ when } k\to\infty,$$

With

$$\beta \equiv \mathrm{Max}\left\{\left|\frac{bz}{c-bz}\right|, \left|\frac{(c-b)z}{c-bz}\right|\right\} < 1$$

Therefore, expansion (8) has almost a power rate of convergence under the restrictions $|b||z|<|c-bz|$ and $|c-b||z|<|c-bz|$ (and the restrictions $\Re c + 1 > \Re b > -1$ are not necessary).
On the other hand, from [291, Eq. (15.2.10)] we find the recurrence

$$G_k(b,c) = \frac{1}{c+k+1}\left[k\left(1-2\frac{b}{c}\right) + (k-1)\frac{b}{c}\left(1-\frac{b}{c}\right)G_{k-1}^{-1}(b,c), k \geq 2\right] \quad (11)$$

Where

$$G_k(b,c) \equiv \frac{\Phi_k(b,c)}{\Phi_{k-1}(b,c)}, k = 1,2,3,\ldots$$

From the explicit values of Φ_0 and Φ_1 given in Table 1 we see that $G_1(b,c) = \vartheta(c^{-1})$ and $G_2(b,c) = \vartheta(b/c)$ when $|b| + |c| \to \infty$ with bounded b/c and $b \neq 0$.
From this behaviour of G_1 and G_2 and the above recurrence, it may be shown by induction that, for $b \neq 0$,
$G_k(b,c) = \vartheta(b/c)$ when $|b| + |c| \to \infty$ with bounded b/c and even k.
$G_k(b,c) = \vartheta(c^{-1})$ when $|b| + |c| \to \infty$ with bounded b/c and odd k.
Therefore, for $b \neq 0$,

$$\Phi_k(b,c) = \vartheta\left(\frac{b^{1-k \bmod 2}}{c}\Phi_{k-1}(b,c)\right) \text{when } |b| + |c| \to \infty \text{ with bounded } b/c. \quad (12)$$

The asymptotic properties of the expansion (8) improve when the saddle point t=b/c of (4) coalesces with an end point of the contour of integration $t = 0$ or $t = 1$, that is, when b = 0 or b = c. In these cases, from the recurrence (11)
we have that

$$\Phi_k(b,c) = (-1)^k\Phi_k(c,c) = \frac{k!}{(c+2)_k} \quad (13)$$

Table 2 [285]:

A numerical experiment about the relative error in the approximation of F(−i,b+1;c+2;−5−3i) for several values of b and c by using (8) with n terms

b+1	c+2	n=1	n=3	n=5	n=7	n=9
$10e^{i\pi/4}$	$20e^{i\pi/6}$	0.015019	0.001762	0.000234	0.000039	7.814(-6)

$50e^{-i\pi/3}$	$100e^{-i\pi/4}$	0.006754	0.000072	1.682(-6)	6.055(-8)	2.886(-9)
100-30i	200+2i	0.003617	0.000019	2.255(-7)	3.987(-9)	9.308(-11)
$200e^{i\pi/18}$	$400e^{-i\pi/18}$	0.001036	2.672(-6)	1.359(-8)	1.065(-10)	5.651(-12)
10-500i	40-300i	0.003759	0.000010	5.701(-8)	4.285(-10)	2.930(-12)

Then, (8) is an asymptotic expansion for large c with fixed a uniformly in b and z with bounded b/c. Table 2 contains some numerical experiments which show the accuracy achieved by expansion (8). The expansion (8)was already obtained by NØrlund in [295, Eq. (1.21)], although without any mention to the asymptotic properties of the expansion. Also, the conditions for the convergence of (8) given there are more restrictive: $(|b| + |c|)|z| < |c - bz|$.
We have shown that expansion (8) is convergent and asymptotic for large c (uniformly in b and $z$with bounded b/c) if b, c and z satisfy (7). we will show that the expansion (8) keeps that asymptotic character if 0<b/c<1 (even if conditions (7) do not hold). In the remaining we consider 0<b/c<1 and $-1<\Re b<\Re c + 1$.
Expansion (6) is not uniformly convergent for t ∈ [0, 1] if conditions (7) do not hold. We can approximate the integral (4) by replacing the function g(t) by its Taylor expansion at the point t = b/c:

$$g(t) = \sum_{k=0}^{n-1} \frac{(a)_k z^k}{k!\,(1-(b/c)z^{k+a}}\left(t-\frac{b}{c}\right)^k + g_n(t) \qquad (14)$$

with $g_n(t) = \vartheta((t-b/c)^n)$ when t → b/c. Introducing (14) in (4) and interchanging summation and integration we obtain

$$F(a, b+1; c+2; z) = \sum_{k=0}^{n-1} \frac{(a)_k z^k}{k!\,(1-(b/c)z)^{k+a}} \Phi_k(b,c) + R_n(a,b;c;z) \quad (15)$$

where the functions Φ_k(b, c) are given in (9) or (10) and

$$R_n(a,b;c;z) = \frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)} \int_0^1 e^{cf(t)} g_n(t)dt \qquad (16)$$

The key point here is to use the idea given in [293]: the critical point b/c ∈ (0, 1). Then, the Laplace method can be applied to the integrals (9) to obtain their asymptotic behaviour for large c (or large b and c with bounded b/c) [293]:
$\Phi_k(b,c) = \vartheta(c^{-(n+1)/2})$when |b| + |c| → ∞ with bounded b/c. (17)
On the other hand, we can also apply the Laplace's method to the remainder R_n(a,b; c; z) in (16) to obtain [293]:
$R_n(a,b;c;z) = \vartheta(c^{-(n+1)/2})$when |b| + |c| → ∞ with bounded b/c. (18)
Thus, from (12) or (13) and (18), we see that (15) is an asymptotic expansion of F(a, b + 1; c + 2; z) for large c (uniformly in b with bounded b/c). Moreover, from the Lagrange form for the Taylor remainder we have

$$g_n(t) = \frac{(a)_n (t-b/c)^n}{n!\,(1/z-\xi)^{n+a}}, \xi \in (t, b/c) \subset [0,1]$$

Table 3[285]:
A numerical experiment about the relative error in the approximation of F(4, b + 1; c + 2;−12) for different values of b and c by using (8) with n terms

b+1	c+2	n=1	n=3	n=5	n=7	n=9
20	50	0.1675	0.025198	0.003424	0.000256	0.000084
50	100	0.07295	0.005519	0.0000468	0.0000444	4.769(-6)
100	200	0.036612	0.001398	0.000060	2.972(-6)	1.640(-7)

250	500	0.014674	0.000225	3.947(-6)	7.886(-8)	1.780(-9)
500	1000	0.007342	0.000056	4.965(-7)	4.988(-9)	5.673(-11)

Conditions (7) do not hold for these values of b, c and z.
Then,

$$|g_n(t)| \le \frac{1}{n!}\left|\frac{(a)_n}{z^a}\right| \Lambda(z,a,n)e^{\pi|\Im a|}\left|t-\frac{b}{c}\right|^n$$

With

$$\Lambda(z,a,n) \equiv \begin{Bmatrix} |\Im(z^{-1})|^{-n-\Re a} & \text{if } 0<\Re z^{-1}<1 \text{ and } \Re a+n>0 \\ \mathrm{Max}\left\{|z|^{n+\Re a}, \left|\frac{z}{1-z}\right|^{n+\Re a}\right\} & \text{in the remainingcases} \end{Bmatrix}$$

Then, for real b and c and even n we have

$$|R_n(a,b,c,z)| \le \frac{1}{n!}\left|\frac{(a)_n}{z^a}\right| \boldsymbol{\Lambda(z,a,n)e^{\pi|\Im a|}\Phi_n(b,c)}$$

We remark that the asymptotic properties of the sequence $\{\Phi_k(b,c)\}_k$ obtained in (12) or (13) making use of (11) are slightly better than those derived from the Laplace's method in (17).
Table 3 shows a numerical experiment which illustrates the approximation supplied by (8) for large positive real values of b and c with c>b>0 when (7) does not hold.
Write t=x+iy and b/c=u+iv, with x, y, u, v ∈ ℝ. Consider a contour Γ defined as (see Fig. 2): $\Gamma \equiv \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$
With

$$\Gamma_1 = \left\{\left(-\sqrt{v^2/4-(y-v/2)^2}, y\right); 0<y<v\right\}$$
$$\Gamma_2 \equiv \{(x,v); 0<x<1\}$$
$$\Gamma_3 = \left\{\left(1+\sqrt{v^2/4-(y-v/2)^2}, y\right); 0<y<v\right\}$$

Consider the domain bounded by Γ∪ [0, 1] and defined by

$$\Omega \equiv \left\{t \in \mathbb{c}, -\sqrt{(\Im t)\left(\Im\frac{b}{c}-\Im t\right)} \le \Re t \le 1+\sqrt{(\Im t)\left(\Im\frac{b}{c}-\Im t\right)}\right\} \quad (19)$$

we extend the results of the previous to the case b ∈ C with 0<$\Re$b<c and $z^{-1} \in \mathbb{c}/\Omega$. In the remaining of we consider 0<$\Re$b<c and $z^{-1} \in \mathbb{c}/\Omega$and use the ideas of the modified saddle point method introduced in [293].
The integrand in (4) is an analytic function of t ∈ ℂ with branch cuts at (−∞, 0], [1,∞) and, if a ∉ℤ, also at [1/z,∞). Then, if $z^{-1} \notin \Omega$, the integrand in (4) is an analytic function of t in the interior of (see Fig. 2).

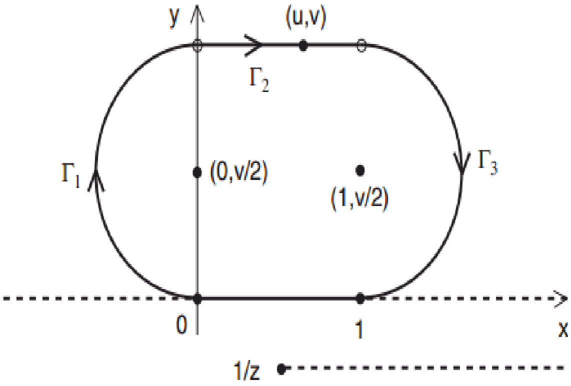

Fig. 2. The contour Γ is the union of the arc Γ_1, the segment Γ_2 and the arc Γ_3. The arc Γ_1 is a half of the circle $x^2 + (y - v/2)^2 = v^2/4$ of center $(0, v/2)$ and radius $v/2$. The segment Γ_2 is the segment $y = v, 0 < x < 1$. The arc Γ_3 is a half of the circle $(x - 1)^2 + (y - v/2)^2 = v^2/4$ of center $(1, v/2)$ and radius $v/2$. The functions f and g are analytic in Ω if $z^{-1} \notin \Omega$.

Using the Cauchy's Residue Theorem, we deform the integration contour [0, 1] in (4) to the contour Γ:

$$F(a, b+1; c+2; z) = \frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)} \int_{\Gamma} e^{cf(t)} g(t) dt \qquad (20)$$

The real part of the function f (t) in the exponent of the integrand in (20) reads

$$\Re(f(t)) = h(x,t) = u \log\sqrt{x^2+y^2} + (1-u)\log\sqrt{(1-x)^2+y^2} + v\tan^{-1}[y/(x-1)] - v\tan^{-1}(y/x) \qquad (21)$$

and verifies the following properties:

(i) For x ∈ [0, 1], the function h(x, v) has an absolute maximum at x = u. It is a strictly increasing function of x for x ∈ [0, u) and strictly decreasing for x ∈ (u, 1]. That is, it has an absolute maximum at x = u over Γ_2.

(ii) The function $h(-\sqrt{v^2/4-(y-v/2)^2}, y)$is an strictly increasing function of y for y ∈ [0, v]. That is, it is strictly increasing overΓ_1 .

(iii) The function $h(1+\sqrt{v^2/4-(y-v/2)^2}, y)$ is an strictly increasing function of y for y ∈ (0, v). That is, it is strictly decreasing overΓ_3 .

Taking into account (i)–(iii) we conclude that, over the path Γ, (f (t)) has an absolute maximum at t = b/c.

We divide the path _ in two pieces: = $\Gamma_S \cup \Gamma_T$, where is that part of $\Gamma_S$contained inside a circle of center b/c and radius r ≡ |1/z − b/c|, and $\Gamma_T = \Gamma/\Gamma_S$ (see Fig. 3).

Then,

F(a, b + 1; c + 2; z) =Γ_S (a, b; c; z) + Γ_T (a, b; c; z), (22)

With

$$F_S(a, b; c; z) = \frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)} \int_{\Gamma_S} e^{cf(t)} g(t) dt$$

And

$$F_T(a, b; c; z) = \frac{\Gamma(c+2)}{\Gamma(b+1)\Gamma(c-b+1)} \int_{\Gamma_T} e^{cf(t)} g(t) dt$$

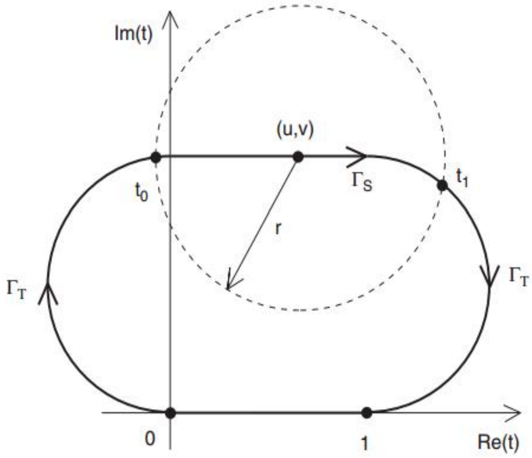

Fig. 3. Γ_S is the piece of the path Γ inside the circle of center (u, v) and radius $r \equiv |1/z - b/c|$. Expansion (6) is uniformly convergent for $t \in \Gamma_S$ (for those points t of Γ_S located between t_0 and t_1).

On the one hand, (f (t)) has an absolute maximum at t =b/c and increases from t=0 up to t =b/c and decreases from t = b/c up to t = 1 following the path Γ. On the other hand, g(t) is bounded on Γ. Then

$$\int_{\Gamma_T} e^{cf(t)} g(t)dt = \vartheta\left(e^{cf(t_0)} + e^{cf(t_1)}\right) \qquad \text{when } c \to \infty \quad (23)$$

where t_0 and t_1 are the points of the path Γ located at a distance r from b/c (see Fig. 2).
On the other hand, because of the expansion (6) is uniformly convergent for t inside the circle of radius r and center b/c, we can repeat the reasoning to conclude that

$$F_S(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}} \Phi_k^{(S)}(b, c) \qquad (24)$$

where the functions $\Phi_k^{(S)}(b, c)$ are defined by

$$\Phi_k^{(S)}(b, c) = \frac{\Gamma(c+2)}{\Gamma(c+1)\Gamma(c-b+1)} \int_{\Gamma_S} e^{cf(t)} \left(t - \frac{b}{c}\right)^k dt$$

Using again that (f (t)) has an absolute maximum over Γ at t = b/c we have that

$$\Phi_k^{(S)}(b, c) = \frac{\Gamma(c+2)}{\Gamma(c+1)\Gamma(c-b+1)} \left\{ \int_{\Gamma} e^{cf(t)} \left(t - \frac{b}{c}\right)^k dt + \vartheta(e^{cf(t_0)} + e^{cf(t_1)}) \right\} \qquad (25)$$

when c→∞. Using that $e^{cf(t)}(t - (b/c))^k$ is an analytic function of t ∈ C with branch cuts at (−∞, 0] and [1,∞)
we deform the integration contour Γ above back to [0, 1]: Γ→ [0, 1]. Then, using the results we have:

$$\Phi_k^{(S)}(b, c) = \Phi_k(b, c) + \frac{\Gamma(c+2)}{\Gamma(c+1)\Gamma(c-b+1)} \vartheta(e^{cf(t_0)} + e^{cf(t_1)} \text{when } c \to \infty, \qquad (26)$$

where Φ_k(b, c) are defined in (9), calculated in (10) and verify the recurrence (11).

Therefore, joining (22)–(24) and (26) we have that, even if the right-hand side of (8) is not convergent, it is an asymptotic expansion of F(a, b + 1; c + 2; z) for large c and fixed a uniformly in b and z (with bounded b/c,0<$\mathfrak{R}$b<c and z^{-1}∉Ω). Tables 4 and 5 show numerical experiments which illustrates the approximation supplied by (24) for large values of b and c with b complex and c>0 when (7) does not hold.

Table 4 [285]:

A numerical experiment about the relative error in the approximation of F(6−5i, b +1; c +2;−4+3i) for several values of b and c by using (24) with n terms

b+1	c+2	n=1	n=3	n=5	n=7	n=9
50+27i	160	0.138447	0.006438	0.000817	0.000281	0.000055
115+16i	290	0.074796	0.003316	0.000104	2.185(-6)	2.910(-7)
155+2i	375	0.059992	0.002348	0.000079	2.632(-6)	5.259(-8)

Conditions (7) do not hold for these values of b, c and z.

Table 5[285]:

A numerical experiment about the relative error in the approximation of F(−4 + 7i, b + 1; c + 2;−7 − 3i) for several values of b and c by using (24) with n terms

b+1	c+2	n=1	n=3	n=5	n=7	n=9
60-2i	140	0.177146	0.013372	0.000620	0.000023	9.267(-7)
130-30i	240	0.134019	0.005836	0.000034	8.676(-6)	6.385(-7)
150-7i	345	0.068925	0.001922	0.000031	4.468(-7)	7.242(-9)

Conditions (7) do not hold for these values of b, c and z.

We can resume the analysis of the previous the following theorems.

Theorem (5.1.1)[285]: For a, b, c ∈ ℂ, c ≠ −2,−3,−4, . . . , | arg(1 − z)|<π, |b||z|<|c − bz| and |c − b||z|<|c − bz|,

$$F(a,b;c+2;z)=\sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!\,(1-(b/c)z)^{k+a}}\Phi_k(b,c)$$

Where

$$\Phi_k(b,c)=\left(-\frac{b}{c}\right)^k F\left(-k,b+1;c+2;\frac{c}{b}\right) \tag{27}$$

$\Phi_k(b,c)=\vartheta((b^{1-k \bmod 2}/c)\Phi_{k-1}(b,c)$when |b| + |c| → ∞ uniformly in b(≠ 0) with bounded b/c and verify the recurrence

$$\Phi_k(b,c)=\frac{1}{c+k+1}\left[k\left(1-2\frac{b}{c}\right)\Phi_{k-1}(b,c)+(k-1)\frac{b}{c}\left(1-\frac{b}{2}\right)\Phi_{k-2}(b,c)\right],k\geq 2$$

For the particular cases b = 0 or b = c we have

$$\Phi_k(0,c)=(-1)^k\Phi_k(c,c)=\frac{k!}{(c+2)_k}$$

Theorem (5.1.2) [285]: For fixed a ∈ ℂ, | arg(1 − z)|<π, −1<$\mathfrak{R}$b<$\mathfrak{R}$c + 1 and

(i) $0<b/c<1$ or

(ii) $0<\Re b<c$, bounded b/c and $z^{-1}\in ℂ/\Omega$

$F(a,b+1;c+2;z)\sim\sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!(1-(b/c)z)^{k+a}}\Phi_k(b,c)$when $c\to\infty$

uniformly in b and z. The functions giv$\Phi_k(b,c)$are given in (27) and is defined in (19).

Section (5.2): Harmonic Bergman Spaces on the real Ball

Consider the harmonic Bergman space $L^2_{harm}\,(\mathbb{B}^n,d\mu^n_\alpha)$ on the unit ball $\mathbb{B}^n$in $\mathbb{R}^n$, consisting of all functions that are harmonic and square integrable with respect to the measure

$$d\mu^n_\alpha(y):=c_\alpha(1-|y|^2)^\alpha d^n y,\qquad \alpha>-1,$$

where $d^n y$ is the usual n-dimensional Lebesgue measure and the coefficient c_α is chosen so that $\mathbb{B}^n$ has measure 1. Specifically,

$$c_\alpha = \frac{\Gamma\left(\alpha + \frac{n}{2} + 1\right)}{\pi^{n/2}\Gamma(\alpha + 1)}.$$

It is known that such a space possesses so-called "Bergman kernel", i.e. there exists a function $R_\alpha : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{R}$ harmonic and square integrable in both arguments satisfying the reproducing property

$$\int_{\mathbb{B}^n} f(y) R_\alpha(x,y) d\mu_\alpha^n(y) = f(x) \quad \forall f \in L^2_{harm}(\mathbb{B}^n, d\mu_\alpha^n), \forall x \in \mathbb{B}^n. \qquad (28)$$

Consider now the integral transform

$$(\mathrm{B}_\alpha \mathrm{f})(\mathrm{x}) := \int_{\mathbb{B}^n} f(y) \frac{R_\alpha^2(x,y)}{R_\alpha(x,x)} d\mu_\alpha^n(y), \qquad (29)$$

which is an example of the *so*-called Berezin transform appearing in the Berezin-Toeplitz quantization with many applications in mathematical physics (quantization on Kahler manifolds, see [304], [302], [303], [306],
[309]). The parameter α represents essentially the reciprocal value of the Planck constant.
It was shown by $C.\ Liu$ in 2007 [310] that if $n = 2$ then for $f \in C(\overline{\mathbb{B}^2})$,

$$\mathrm{B}_\alpha \mathrm{f} \to \mathrm{f} \text{ uniformly as } \alpha \to \infty,$$

i.e. classical physics is recovered when the Planck constant goes to zero – or equivalently, as the observer's perspective gets larger and larger.
Subsequently, R. Otahalova in 2008 [313] generalized this result to an arbitrary dimension $n \geq 2$.
Meanwhile, M. Engliš [307] was able to prove a similar a result for the Berezin transform of functions on $\mathbb{C}^n$ providing moreover a full asymptotic expansion, with an occurrence of interesting Stokes phenomenon – asymptotic behavior changes abruptly for $x = 0$.
We provide the full asymptotic expansion of the Berezin transform (29) as $\alpha \to \infty$ generalizing thus the work of Otahalova and with the same occurrence of the Stokes phenomenon as in [317].
We have main result is the following theorem.

Theorem (5.2.1)[299]: For $x \in \mathbb{B}^n, x \neq 0, n > 1$, and $f \in C^\infty(\mathbb{B}^n)$,there exist differential operators $Q_i := Q_i(\Delta, x \cdot \nabla, |x|^2)$, involving only the Laplace operator Δ, the directional derivative $x \cdot \nabla$ and the quantity $|x|^2$, such that

$$(B_\alpha f)(x) := \int_{\mathbb{B}^n} f(y) \frac{R_\alpha^2(x,y)}{R_\alpha(x,x)} d\mu_\alpha^n(y) \approx \sum_{i=0}^{\infty} \frac{(Q_i f)(x)}{\alpha^i} \ (\alpha \to \infty),$$

where $Q_0 = 1$ and

$$Q_1 = \frac{n-2}{2} \frac{1-|x|^2}{|x|^2} x \cdot \nabla + \frac{(n-2)(1-|x|^2)^2}{4(n-1)|x|^2} (x \cdot \nabla)^2 + \frac{1}{4(n-1)} (1-|x|^2)^2 \Delta.$$

Finally, for $x = 0$ it holds

$$(B_\alpha f)(0) \approx \sum_{i=0}^{\infty} \frac{(\Delta^i f)(0)}{\Delta^i \left(\alpha + \frac{n}{2} + 1\right)_i} \qquad (\alpha \to \infty).$$

The symbol $\approx$ stands for the usual Poincaré asymptotic expansion, i. e. $f(\alpha) \approx \sum_{i=0}^{\infty} C_i \alpha^{-i}$ if and only if for all $N = 0,1, \ldots$ we have $f(\alpha) - \sum_{i=0}^{N-1} c_i \alpha^{-i} = O(\alpha^{-N})$ as $\alpha \to \infty$.
The method used to prove this theorem differs substantially from methods used, [313] and [307]. Notably, Otahalova's approach gives no hope to achieve this, on the other hand it does not look entirely impossible to exploit the tools of the [307] to obtain our result but only for even dimensions.
based on representing the Berezin transform in terms of generalized hypergeometric functions and then make use of their many known properties (notably due to [301], [311]) including asymptotic expansions for large parameters in some cases. Interestingly, the distinction between odd and even dimension, which burdens heavily [313] and [307], does not prove itself as important in this setting.

The definition of generalized hypergeometric functions as well as some of their properties will be shown. we exhibit a connection between the Berezin transform of a polynomial and a linear combination of functions $\frac{{}_5F_4}{{}_2F_1}$ with some parameters.

This is then used to prove Finally, along the way we prove Theorem (5.2.2) which bears some significance of its own, since it provides means of computing the Bergman projection (28) of more general functions than just harmonic ones:

Theorem (5.2.2) [299]: For $\forall p \in \mathbb{N}_0, \beta \geq \alpha$ and $f \in C^p(\mathbb{B}^n)\colon \Delta f = 0$ it holds:

$$\int_{\mathbb{B}^n} R_\alpha(x,y) f(y)(x\cdot y)^p \, d\mu_\alpha^n(y)$$

$$= \frac{p!}{2p} \sum_{j+2l+m=p} \frac{|x|^{2(j+l)}(\tilde{\alpha})_j (2b)_j}{j!\, m!\, l!\, (\tilde{\beta})_{j+m+l} (b)_j} {}_3((x\cdot\nabla)^m f)_3 \left(\begin{matrix} \tilde{\alpha}+j & 2b+j & b \\ \tilde{\beta}+j+l+m & b+j & 2b \end{matrix}; x\right),$$

where $b := \frac{n}{2} - 1$ and $\tilde{x} := x + \frac{n}{2} + 1$. The "hypergeometrization" ${}_m f_n$ of a function f, which is a special case of a Hadamard product and which appears naturally in this setting, will be introduced. Remember that the generalized hypergeometric function is defined by the series

$${}_F F_q \left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k x^k}{(c_1)_k \dots (c_p)_k k!},$$

where $(a)_k$ is the Pochhammer symbol $(a)_k := a(a+1) \dots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$. Obviously, the definition makes no sense when some lower parameter c_i is a negative integer or zero.

In the case $p = q + 1$, which we will be most interested in, the series converges for $|x| < 1$ and can be analytically continued to $\mathbb{C} \backslash [1, \infty)$.

The asymptotic expansions of these functions for large argument *x* are known [311], the asymptotic expansion for large parameters can be worked out in many cases. The simplest one is when more of the lower parameters are large than the upper ones. In that case the asymptotic expansion is simply the Taylor series. More precisely, for $r < s, x \notin [1, \infty)$:

$${}_p F_q \left(\begin{matrix} a_1+\alpha \dots a_r+\alpha & a_{r+1}+1 \dots a_p \\ c_1+\alpha \dots c_s+\alpha & c_{s+1}+1 \dots c_q \end{matrix}; x\right)$$

$$\approx \sum_{k=0}^{\infty} \frac{(a_1+\alpha)_k \dots (a_r+\alpha)_k (a_{r+1})_k \dots (a_p)_k}{(c_1+\alpha)_k \dots (c_s+\alpha)_k (c_{s+1})_k \dots (c_q)_k} \frac{x^k}{k!}, (\alpha \to +\infty). \quad (30)$$

(see [311] §7.3, [317]). Much less is known when some parameter is large and negative even in the case of a lower parameter. Notice that for them the problem is even somewhat ill-posed, because the lower parameter c_i cannot be a negative integer. least in case of the Gauss hypergeometric function the above result still remains in force [318], [316]:

For a, b, z fixed and $Re(z) < \frac{1}{2}, -c \notin \mathbb{N}_0$ it holds for every $m \in \mathbb{N}$

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; z\right) = \sum_{k=1}^{m-1} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} + O(c^{-m}) \qquad (|c| \to \infty), \qquad (31)$$

Some other cases can be worked out by the aid of transformations which hold for hypergeometric functions. For example there is the Pfaff transformation

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; x\right) = (1-x)^{-b} {}_2F_1\left(\begin{matrix} c-a & b \\ & c \end{matrix}; \frac{x}{x-1}\right), \qquad (32)$$

which effectively solves the asymptotic expansion of the type

$${}_2F_1\left(\begin{matrix} a+\alpha & b \\ & c+\alpha \end{matrix}; x\right) \approx (1-x)^{-b} {}_2F_1\left(\begin{matrix} c-a & b \\ & c+\alpha \end{matrix}; \frac{x}{x-1}\right) \quad (\alpha \to +\infty).$$

The transformation

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}; x\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-x)^{-a} {}_2F_1\left(\begin{matrix} a & 1+a-c \\ & 1+a-b \end{matrix}; \frac{1}{x}\right) \qquad (33)$$

$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-x)^{-b} {}_2F_1\left(\begin{matrix} b & 1+b-c \\ & 1+b-a \end{matrix}; \frac{1}{x}\right) \qquad (34)$$

which holds for $x < 0, a - b \notin \mathbb{Z}$ can be used to determine the asymptotic expansion when a is large and so on.
Our task will be to handle the case

$${}_5F_4\begin{pmatrix} \alpha & \alpha & b_1 & b_2 & b_3 \\ \alpha + a & & c_1 & c_2 & c_3 \end{pmatrix}; x\Big),$$

for large α.
Unfortunately, there are not as many transformation available for the function ${}_5F_4$ as for the ${}_2F_1$ or analogous transformations as for example (32) could be established but the final result is not expressible in terms of ${}_5F_4$ functions, but rather in terms of their certain multi-variable generalizations.
Still, the following result holds.
Lemma (5.2.3) [299]: Let $b_1, b_2, b_3 > 0$ be positive real numbers, one of them strictly less than the other two. Let $\alpha - a - \gamma \notin \mathbb{Z}, -c_i \notin \mathbb{N}_0$ and $x \in (0,1)$. Then we have

$${}_5F_4\left(\begin{matrix} \alpha & \alpha & b_1 & b_2 & b_3 \\ \alpha + a & & c_1 & c_2 & c_3 \end{matrix}; x\right) \approx \prod_{i=1}^{3} \frac{\Gamma(c_i)}{\Gamma(b_i)} \frac{(\alpha x)^{-\gamma}}{(1-x)^{\alpha-\gamma-a}} \left(1 + \sum_{k=1}^{\infty} \frac{d_k}{\alpha^k}\right) \ (\alpha \to +\infty)$$

where $\gamma = \sum_{j=1}^{3}(c_j - b_j)$ and d_k are constants independent of α.
Proof: Using the integral representation

$${}_{p+1}F_{q+1}\left(\begin{matrix} b_1 \dots b_p & b \\ c_1 \dots c_q & c \end{matrix}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\, {}_pF_q\left(\begin{matrix} b_1 \dots b_p \\ c_1 \dots c_q \end{matrix}; xt\right) dt,$$

which is valid for $c > b > 0$, in turn three times on pairs of parameters $(c_1, b_1), (c_2, b_2), (c_3, b_3)$,we get

$${}_5F_4\left(\begin{matrix} \alpha & \alpha & b_1 & b_2 & b_3 \\ \alpha + a & & c_1 & c_2 & c_3 \end{matrix}; x\right)$$

$$= \prod_{i=1}^{3} \frac{\Gamma(c_i)}{\Gamma(b_i)\Gamma(c_i - b_i)} \int_0^1\int_0^1\int_0^1 \prod_{i=1}^{3} t_i^{b_i-1}(1-t_i)^{c_i-b_i-1}\, {}_2F_1\left(\begin{matrix} \alpha & \alpha \\ \alpha + a & \end{matrix}; xt_1t_2t_3\right) dt_1dt_2dt_3.$$

Double application of the transformation (32) gives us the Euler transform

$${}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix}; x\right) = (1-x)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a & c-b \\ c & \end{matrix}; x\right),$$

which leaves us with

$$\prod_{i=1}^{3} \frac{\Gamma(c_i)}{\Gamma(b_i)\Gamma(c_i - b_i)} \int_0^1\int_0^1\int_0^1 \prod_{i=1}^{3} t_i^{b_i-1}(1-t_i)^{c_i-b_i-1}(1 - xt_1t_2t_3)^{a-\alpha}\, {}_2F_1\left(\begin{matrix} a & a \\ \alpha + a & \end{matrix}; xt_1t_2t_3\right) dt_1dt_2dt_3.$$

A triple integral of this kind can be rearranged in the following way:

$$\int_0^1\int_0^1\int_0^1 \varphi(t_1, t_2, t_3) G(t_1t_2t_3) dt_1dt_2dt_3 \qquad (35)$$

$$= \int_0^1 G(1-r_3) \int_0^1\int_0^1 \varphi\left(1 - r_1r_2r_3, \frac{1-r_2r_3}{1-r_1r_2r_3}, \frac{1-r_3}{1-r_2r_3}\right) \frac{r_2r_3^2}{(1-r_1r_2r_3)(1-r_2r_3)} dr_1dr_2dr_3.$$

(This is nothing more than a series of changes of variables. Firstly, let $s_1 = t_1, s_2 = t_1t_2, s_3 = t_1t_2t_3$.
Jacobian is $\frac{1}{s_1s_2}$ and the integral becomes:

$$\int_0^1\int_0^{s_1}\int_0^{s_2} \varphi\left(s_1, \frac{s_2}{s_1}, \frac{s_3}{s_2}\right) G(s_3) \frac{1}{s_1s_2} ds_3ds_2ds_1. \qquad (36)$$

Now we swap the order of integration:

$$\int_0^1\int_0^{s_1}\int_0^{s_2} \varphi\left(s_1, \frac{s_2}{s_1}, \frac{s_3}{s_2}\right) G(s_3) \frac{1}{s_1s_2} ds_3ds_2ds_1 = \int_0^1\int_{s_3}^1\int_{s_2}^1 \varphi\left(s_1, \frac{s_2}{s_1}, \frac{s_3}{s_2}\right) G(s_3) \frac{1}{s_1s_2} ds_1ds_2ds_3$$

and finally three changes of variable are performed: firstly $1-s_1=r_1(1-s_2)$, then $1-s_2=r_2(1-s_3)$ and lastly $1-s_3=r_3$.)
Applying this to our original triple integral we get:

$$\int_0^1 (1-x+xr_3)^{a-\alpha}\,{}_2F_1\left({a\quad a \atop \alpha+a};x(1-r_3)\right)$$

$$\int_0^1\int_0^1\prod_{i=1}^{3} r_i^{\gamma_i-1}(1-r_1)^{c_2-b_2-1}(1-r_2)^{c_3-b_3-1}(1-r_3)^{b_3-1}(1-r_2r_3)^{b_2-c_3}(1-r_1r_2r_3)^{b_1-c_2}dr_1dr_2dr_3,$$

where $\gamma_i=\sum_{k=1}^{i}(c_k-b_k)$.
After a small manipulation this gives

$$\int_0^1\int_0^1\int_0^1\prod_{i=1}^{3} t_i^{b_i-1}(1-t_i)^{c_i-b_i-1}(1-xt_1t_2t_3)^{a-\alpha}\,{}_2F_1\left({a\quad a \atop \alpha+a};xt_1t_2t_3\right)dt_1dt_2dt_3 \quad (37)$$

$$=(1-x)^{a-\alpha}\int_0^1 t^{\gamma_3-1}(1-t)^{b_3-1}F(t)\,{}_2F_1\left({a\quad a \atop \alpha+a};x(1-t)\right)\left(1-\frac{x}{x-1}t\right)^{a-\alpha}dt,$$

where

$$F(t)\int_0^1 r_2^{\gamma_2-1}(1-r_2)^{c_3-b_3-1}(1-r_2t)^{b_2-c_3}\int_0^1 r_1^{\gamma_1-1}(1-r_1)^{c_2-b_2-1}(1-r_1r_2t)^{b_1-c_2}dr_1dr_2$$

$$=\frac{\Gamma(\gamma_1)\Gamma(c_2-b_2)}{\Gamma(\gamma_2)}\int_0^1 r_2^{\gamma_2-1}(1-r_2)^{c_3-b_3-1}(1-r_2t)^{b_2-c_3}\,{}_2F_1\left({c_2-b_1\quad \gamma_1 \atop \gamma_2};r_2t\right)dr_2.$$

We expand the hypergeometric function into Taylor series to get the form

$$\mathrm{F(t)}=\frac{\Gamma(c_1-b_1)\Gamma(c_2-b_2)\Gamma(c_3-b_3)}{\Gamma(\gamma_3)}\sum_{j=0}^{\infty}\frac{(c_2-b_1)_j(\gamma_1)_j}{(\gamma_3)_j}\frac{t^j}{j!}\,{}_2F_1\left({\gamma_2+j\quad c_3-b_2 \atop \gamma_3+j};t\right).$$

We should talk about the convergence of the integral on the right hand side of the equation (37). For that it is necessary to understand the behavior of the function $F(t)$ at the end points of the interval of integration, notably in the neighborhood of $t=1$ (the behavior near $t=0$ is evident). It is well known that

$${}_2F_1\left({a\quad b \atop c};1\right)=\frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}$$

for $c>a+b$. That means that the hypergeometric function in the infinite series is left-continuous at $t=1$ if $\gamma_3+j>\gamma_2+j+c_3-b_2$ or equivalently $b_2>b_3$. In such case the infinite series for $t=1$ takes the form:

$$\frac{\Gamma(\gamma_3)\Gamma(b_2-b_3)}{\Gamma(c_2+c_1-b_1-b_3)}\sum_{j=0}^{\infty}\frac{(c_2-b_1)_j(\gamma_1)_j}{(c_2+c_1-b_1-b_3)_j j!},$$

which is a convergent series for $c_2+c_1-b_1-b_3>c_2-b_1+\gamma_1$, i.e. $b_1>b_3$. (Indeed, the series is actually equal to

$${}_2F_1\left({c_2-b_1\quad \gamma_1 \atop c_2+c_1-b_1-b_3};1\right)$$

and the formula above can be used.)
This can be summarized by saying

$$F(t)=O(1)\qquad (t\nearrow 1),$$

which holds for $b_1>b_3, b_2>b_3$ and the integral on the right hand side of the equation (37) converges under the conditions $b_3>0, b_1>b_3, b_2>b_3, \gamma_3>0$. Those are significantly less restraining conditions than in the triple integral on the left hand side of the same equation, which converges for $c_i>b_i>0\forall i$.

It is an example, therefore, of an analytic continuation. Furthermore, since hypergeometric functions are symmetric with respect to permutation of the parameters b_i, we can choose b_3 to be the smallest one.
We can summarize now that for $x < 1, \gamma_3 > 0, b_1 > b_3 > 0, b_2 > b_3$ it holds

$$
{}_5F_4\left(\begin{matrix}\alpha & \alpha & b_1 & b_2 & b_3\\ \alpha+a & c_1 & c_2 & c_3\end{matrix}; x\right)
= \frac{(1-x)^{a-\alpha}}{\Gamma(\gamma_3)} \prod_{i=1}^{3} \frac{\Gamma(c_i)}{\Gamma(b_i)} \int_0^1 t^{\gamma_3-1}(1-t)^{b_3-1} F_\alpha(t) \left(1-\frac{x}{x-1}t\right)^{a-\alpha} dt,
$$

where

$$
F_\alpha(t) = {}_2F_1\left(\begin{matrix}a & a\\ \alpha+a\end{matrix}; x(1-t)\right) \sum_{j=0}^{\infty} \frac{(c_2-b_1)_j (\gamma_1)_j}{(\gamma_3)_j} \frac{t^j}{j!}\, {}_2F_1\left(\begin{matrix}\gamma_2+j & c_3-b_2\\ & \gamma_3+j\end{matrix}; t\right),
$$

and $\gamma_i = \sum_{j=1}^{i} c_j - b_j$. As a next step we replace the function $F_\alpha(t)$ by its Taylor series expansion:

$$
F_\alpha(t) = \sum_{k=0}^{N-1} \frac{F_\alpha^{(k)}(0)}{k!} t^k + \frac{F_\alpha^{(N)}(\xi)}{N!} t^N,
$$

where $0 < \xi < t$.
Substituting this we get

$$
\begin{aligned}
&{}_5F_4\left(\begin{matrix}\alpha & \alpha & b_1 & b_2 & b_3\\ \alpha+a & c_1 & c_2 & c_3\end{matrix}; x\right) \qquad (38)\\
&= \frac{(1-x)^{a-\alpha}}{\Gamma(\gamma_3)} \prod_{i=1}^{3} \frac{\Gamma(c_i)}{\Gamma(b_i)} \sum_{k=0}^{N-1} \frac{F_\alpha^{(k)}(0)}{k!} \frac{\Gamma(\gamma_3+k)\Gamma(b_3)}{\Gamma(\gamma_3+b_3+k)}\, {}_2F_1\left(\begin{matrix}\alpha-a & \gamma_3+k\\ & \gamma_3+b_3+k\end{matrix}; \frac{x}{x-1}\right)\\
&+ \frac{(1-x)^{a-\alpha}}{\Gamma(\gamma_3)} \prod_{i=1}^{3} \frac{\Gamma(c_i)}{\Gamma(b_i)} \int_0^1 t^{\gamma_3+N-1}(1-t)^{b_3-1} \frac{F_\alpha^{(N)}(\xi)}{N!} \left(1-\frac{x}{x-1}t\right)^{a-\alpha} dt.
\end{aligned}
$$

Notice that the term $F_\alpha^{(N)}(\xi)$ in the above line still remains $O(1)$ for $t \nearrow 1$ because it can be written as

$$
F_\alpha^{(N)}(\xi) = N!\, t^{-N} \left(F_\alpha(t) - \sum_{k=0}^{N-1} \frac{F_\alpha^{(k)}(0)}{k!} t^k\right). \qquad (39)
$$

Therefore, the integral in the same line converges under conditions $\gamma_3 + N > 0$ and $b_3 > 0$. The first of these is fulfilled for sufficiently large N, hence for the right hand side of the equation to be meaningful it is only required that $b_3 > 0$. This is the largest analytic continuation as we can get. From the form of the remainder term (39) we also see that $F^{(N)}(\xi)$ is a continuous function on [0, 1].
We can therefore estimate it by its maximum on this interval, which will in general depend on α, but the asymptotic behavior of $F_\alpha^{(N)}(\xi)$ for $\alpha \to \infty$ is $F_\alpha^{(N)}(\xi) = O(1)$ uniformly for all $t \in [0,1]$. (This can be seen again from the form of the remainder term (39) – the ${}_2F_1$ in the F_α which contains the parameter α has this uniform behavior due to the (30) and terms $F_\alpha^{(k)}(0)$ are just some linear combinations of the same ${}_2F_1$ function, only possibly with parameters shifted due to the differentiations. In such case even additional negative powers of α appear.)
Hence

$$
\left| \int_0^1 t^{\gamma_3+N-1}(1-t)^{b_3-1} \frac{F^{(N)}(\xi)}{N!} \left(1-\frac{x}{x-1}t\right)^{a-\alpha} dt \right|
\le C \int_0^1 t^{\gamma_3+N-1}(1-t)^{b_3-1} \left(1-\frac{x}{x-1}t\right)^{a-\alpha} dt
$$

$$= O\left({}_2F_1\left(\begin{matrix}\alpha - a & \gamma_3 + k \\ & \gamma_3 + b_3 + k\end{matrix}; \frac{x}{x-1}\right)\right) \quad (\alpha \to \infty).$$

The problem of finding an asymptotic expansion of the function ${}_5F_4$ for large α is now effectively reduced to the problem of finding an expansion for the functions of the form:

$$(1-x)^{a-\alpha}\, {}_2F_1\left(\begin{matrix}\alpha - a & \gamma_3 + k \\ & \gamma_3 + b_3 + k\end{matrix}; \frac{x}{x-1}\right).$$

The large parameter cases for ${}_2F_1$ function has been studied by several authors (see [315, 318]). The logic goes as follows: Combining the transformations (32) and (33), we can see that

$${}_2F_1\left(\begin{matrix}a & b \\ & c\end{matrix}; x\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{b-c}(1-x)^{c-b-a}\, {}_2F_1\left(\begin{matrix}1-b & c-b \\ & 1+a-b\end{matrix}; \frac{1}{x}\right)$$
$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}\, {}_2F_1\left(\begin{matrix}b & 1+b-c \\ & 1+b-a\end{matrix}; \frac{1}{x}\right),$$

for $x < 0, a - b \notin \mathbb{Z}$. Applying this we get

$$(1-x)^{a-\alpha}\, {}_2F_1\left(\begin{matrix}\alpha - a & \gamma_3 + k \\ & \gamma_3 + b_3 + k\end{matrix}; \frac{x}{x-1}\right)$$
$$= \frac{\Gamma(\gamma_3 + b_3 + k)\Gamma(\gamma_3 + k - \alpha + a)}{\Gamma(\gamma_3 + k)\Gamma(\gamma_3 + b_3 + k - \alpha + a)} x^{-b_3}\, {}_2F_1\left(\begin{matrix}1-\gamma_3 - k & b_3 \\ & 1+\alpha - a - \gamma_3 - k\end{matrix}; 1 - \frac{1}{x}\right)$$
$$+ \frac{\Gamma(\gamma_3 + b_3 + k)\Gamma(\alpha - a - \gamma_3 - k)}{\Gamma(\alpha - a)\Gamma(b_3)} \frac{x^{-\gamma_3 - k}}{(1-x)^{\alpha - a - \gamma_3 - k}}\, {}_2F_1\left(\begin{matrix}\gamma_3 + k & 1 - b_3 \\ & 1 + \gamma_3 + k - \alpha + a\end{matrix}; 1 - \frac{1}{x}\right),$$

for $\alpha - a - \gamma_3 \notin \mathbb{Z}$ (we can always run the limit through such α so that this condition is fulfilled). But in the light of (30) and (31) we can deduce that the first term is negligible with respect to the second term, because it displays only polynomial growth and does not contain the exponentially large term $(1-x)^{-\alpha}$ (remember $x \in (0,1)$). Asymptotic behavior is, therefore, dictated by the second term, which is $O((1-x)^{-\alpha}\alpha^{-\gamma_3 - k})$ as $\alpha \to \infty$.

Substituting into the equation (38) we get:

$${}_5F_4\left(\begin{matrix}\alpha & \alpha & b_1 & b_2 & b_3 \\ \alpha + a & c_1 & c_2 & c_3\end{matrix}; x\right)$$
$$= \prod_{i=1}^{3} \frac{\Gamma(c_i)}{\Gamma(b_i)} \sum_{k=0}^{N-1} \frac{F_\alpha^{(k)}(0)}{k!} \frac{(\gamma_3)_k \Gamma(\alpha - a - \gamma_3 - k) x^{-\gamma_3 - k}}{\Gamma(\alpha - a)(1-x)^{\alpha - a - \gamma_3 - k}}\, {}_2F_1\left(\begin{matrix}\gamma_3 + k & 1 - b_3 \\ & 1 + \gamma_3 + k - \alpha + a\end{matrix}; 1 - \frac{1}{x}\right)$$
$$+ O((1-x)^{-\alpha}\alpha^{-\gamma_3 - N}).$$

It only remains to combine known asymptotic expansions of terms:

$$\frac{\Gamma(\alpha - a - \gamma_3 - k)}{\Gamma(\alpha - a)} \approx \alpha^{-\gamma_3 - k}\left(1 + \frac{c_1}{\alpha} + \cdots\right)$$
$${}_2F_1\left(\begin{matrix}\gamma_3 + k & 1 - b_3 \\ & 1 + \gamma_3 + k - \alpha + a\end{matrix}; 1 - \frac{1}{x}\right) \approx 1 + \frac{(\gamma_3 + k)(1 - b_3)}{1 + \gamma_3 + k - \alpha + a} \frac{x-1}{x} + \cdots$$
$$F_\alpha^{(k)}(0) \approx F^{(k)}(0) + \frac{d_1}{\alpha} + \cdots,$$

where

$$F(t) = \sum_{j=0}^{\infty} \frac{(c_2 - b_1)_j (\gamma_1)_j}{(\gamma_3)_j} \frac{t^j}{j!}\, {}_2F_1\left(\begin{matrix}\gamma_2 + j & c_3 - b_2 \\ & \gamma_3 + j\end{matrix}; t\right),$$

and rearrange the terms.

For $p \in \mathbb{N}_0, \beta \geq \alpha$ and $f \in C^p(\mathbb{B}^n): \Delta f = 0$ it holds:

$$\int_{\mathbb{B}^n} R_\alpha(x,y) f(y)(x \cdot y)^p d\mu_\beta^n(y)$$
$$= \frac{p!}{2^p} \sum_{j+2l+m=p} \frac{|x|^{2(j+1)}(\tilde{\alpha})_j (2b)_j}{j!\,m!\,l!\,(\tilde{\beta})_{j+m+l}(b)_j}\, {}_3((x \cdot \nabla)^m f)_3\left(\begin{matrix}\tilde{\alpha} + j & 2b + j & b \\ \tilde{\beta} + j + l + m & b + j & 2b\end{matrix}; x\right).$$

Theorem (5.2.2) deserves a bit of a clarification. From now on we set

$$b := \frac{n}{2} - 1, \quad \tilde{x} \coloneqq x + \frac{n}{2} + 1.$$

For a real (or complex) function f of a real argument we define its hypergeometrization by the series

$${}_pf_q\left({a_1 \dots a_p \atop c_1 \dots c_q}; t\right) \coloneqq \sum_{m=0}^{\infty} \frac{t^m f^{(m)}(0)}{m!} \frac{(a_1)_m \cdots (a_p)_m}{(c_1)_m \cdots (c_q)_m},$$

whenever this defines some analytic function in a neiqhbourhood of zero – i.e. the radius of convergence R is strictly greater than zero and none of the lower parameters c_i is a non-positive integer.

This can be also understand as a Hadamard product (or convolution)

$${}_pf_q\left({a_1 \dots a_p \atop c_1 \dots c_q}; t\right) = {}_{p+1}F_q\left({a_1 \dots a_p \quad 1 \atop c_1 \dots c_q}; t\right) \star f(t),$$

where the Hadamard product of the two formal power series $g(t) = \sum_{k\geq 0} g_k t^k$, $h(t) = \sum_{k\geq 0} h_k t^k$ is defined

$$g(t) \star h(t) \coloneqq \sum_{k=0}^{\infty} g_k\, h_k t^k.$$

A linear operator which brings a function to its Hadamard product with some hypergeometric function (i.e. to its hypergeometrization) appear in [305] and elsewhere. But Hadamard product is a fairly general operation. Hadamard product with hypergeometric functions – deserves its own name, mainly since it possesses many properties the general Hadamard product does not have. Some of them are listed bellow.

But first notice that although the function f is supposed to be of a real argument, once the hypergeometrization is performed and the radius of convergence is strictly positive, the result can be always treated unambiguously as a function of a complex argument and its domain can be extended, if possible, by means of analytic continuation. In that sense, this generalizes the notion of classical hypergeometric functions ${}_pf_q$, which – in our setting – should be written as ${}_pexp_q$ but we keep the historical notation, i.e. F instead of exp.

For a real function $f(x)$ of a vector argument, $x \in \mathbb{R}^n, n > 1$ we define

$${}_pf_q\left({a_1 \dots a_p \atop c_1 \dots c_q}; x\right) \coloneqq {}_pf_q\left({a_1 \dots a_p \atop c_1 \dots c_q}; tx\right)\Big|_{t=1} = {}_{p+1}F_q\left({a_1 \dots a_p \quad 1 \atop c_1 \dots c_q}; t\right) \star f(tx)\Big|_{t=1},$$

that is the hypergeometrization is performed on the real function $f(tx)$ of the real argument t and if the corresponding radius of convergence is strictly grater than 1 then the function is evaluated at the point $t = 1$.

(i) For $p = q$ hypergeometrization does not change the radius of convergence. (Indeed, lets assume that the Taylor series of the function $f(x)$ converges for $\|x\| < r$ in some norm, then the Taylor series of the function $f(tx)$ converges for $\|tx\| = |t|\|x\| < r$, in other words $|t| < \frac{r}{\|x\|}$, so theradius of convergence is $\frac{r}{\|x\|} > 1$ for any x such that $\|x\| < r$ and we can evaluate $t = 1$. The region of convergence is also unchanged by the presence of the Pochhammer symbols since there are same number of them in the numerator as in the denominator.)

(ii) If the function depends on more than one n-tuple and is symmetric with respect to them in a sense that $F(tx, y) = F(x, ty)\ \forall t$, (for example the Bergman kernel $R_\alpha(x, y)$ has this property) so it does not matter with respect to which variable the hypergeometrization is performed, we will use the simplified notation:

$${}_1F_1\left({a \atop c}; x, y\right) \coloneqq {}_1\big(F(\cdot, y)\big)_t\left({a \atop c}; x\right) = {}_1\big(F(x, \cdot)\big)_t\left({a \atop c}; x\right).$$

Specially, the conditions are fulfilled for functions of the form $F(x, y) = f(x \cdot y, |x|^2|y|^2)$.

Some properties of the hypergeometrization will be important later on.

(i) Obviously, ${}_1f_1\left({a \atop a}; x\right) = f(x)$.

(ii) For $Re c > Re a > 0$:

$$ {}_1f_1\binom{a}{c};x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}f(tx)dt. $$

(iii) When $a \neq 0$:

$$ {}_1f_1\left(\begin{matrix} a+1 \\ c \end{matrix}; x\right) = \frac{1}{a}(a + x\cdot\nabla_x)\,{}_1f_1\left(\begin{matrix} a \\ c \end{matrix}; x\right) = \frac{1}{a}(a + t\partial_t)\,{}_1f_1\left(\begin{matrix} a \\ c \end{matrix}; tx\right)\Big|_{t=1} $$

The statements can be proved easily via Taylor series expansion.

For the proof of Theorem (5.2.2) it is useful to get acquainted with some representations of the Bergman kernel. It is shown of Otahalova [313] (and elsewhere) that (although in different notation)

$$ R_\alpha(x,y) = {}_1P_1\left(\begin{matrix} \tilde{\alpha} \\ b+1 \end{matrix}; x,y\right), $$

where $P(x,y)$ is the Poisson kernel, the generating function of the zonal harmonics:

$$ P(x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{\frac{n}{2}}} = \sum_{m=0}^{\infty} Z_m\,(x,y). $$

The Bergman kernel can also be represent in terms of the Appell F_1 function

$$ F_1\left(\begin{matrix} a \\ c \end{matrix}; \begin{matrix} b_1 & b_2 \\ - & \end{matrix}; x,y\right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c)_{j+k}j!\,k!} x^j y^k, $$

as

$$ R_\alpha(x,y) = F_1\left(\begin{matrix} \tilde{\alpha} \\ b \end{matrix}; \begin{matrix} b & b \\ - & \end{matrix}; z,\tilde{z}\right), \qquad (40) $$

where $z = x\cdot y + i\sqrt{|x|^2|y|^2-(x\cdot y)^2}$

Indeed, notice that

$$ P(x,y) = \frac{1-|z|^2}{|1-z|^{2b+2}} = F_1\left(\begin{matrix} b+1 \\ b \end{matrix}; \begin{matrix} b & b \\ - & \end{matrix}; z,\tilde{z}\right). $$

The last equality is a direct consequence of the known transformation rule for the Appell F_1 function

$$ F_1\left(\begin{matrix} a \\ c \end{matrix}; \begin{matrix} b_1 & b_2 \\ - & \end{matrix}; x,y\right) = (1-x)^{-b_1}(1-y)^{-b_2}F_1\left(\begin{matrix} c-a \\ c \end{matrix}; \begin{matrix} b_1 & b_2 \\ - & \end{matrix}; \frac{x}{x-1},\frac{y}{y-1}\right) $$

and the fact that

$$ F_1\left(\begin{matrix} -1 \\ c \end{matrix}; \begin{matrix} b_1 & b_2 \\ - & \end{matrix}; x,y\right) = 1 - \frac{b_1}{c}x - \frac{b_2}{c}y. $$

So what we need to do is only the hypergeometrization

$$ R_\alpha(x,y) = {}_1P_1\left(\begin{matrix} \tilde{\alpha} \\ b+1 \end{matrix}; x,y\right)\,{}_1(F_1)_1\left(\begin{matrix} b+1 \\ b \end{matrix}\ \begin{matrix} \tilde{\alpha} \\ b+1 \end{matrix}; \begin{matrix} b_1 & b_2 \\ - & \end{matrix}; z,\bar{z}\right) = F_1\left(\begin{matrix} \tilde{\alpha} \\ b \end{matrix}; \begin{matrix} b & b \\ - & \end{matrix}; z,\bar{z}\right). $$

From this representation we can see that the Bergman kernel (as a function of y) is an analytic function with radius of convergence$|x|^{-1}$, because it is known that F_1 is a holomorphic function in its arguments with singularity at the point 1 and because of the inequality

$$ |z| = |x||y| < |x|, $$

For $y \in \mathbb{B}^n$.

From known properties of the Appell function it is also clear that when $y = x$

$$ R_\alpha(x,x) = {}_2F_1\left(\begin{matrix} \tilde{\alpha} & 2b \\ & b \end{matrix}; |x|^2\right). $$

For the proof of Theorem (5.2.2) we also need two lemmas.

Lemma (5.2.4) [299]: For$f \in C^1(\overline{\mathbb{B}^n})$:

$$ \int_{\mathbb{B}^n} z\cdot y f(y)d\mu_\gamma^n(y) = \frac{1}{2\tilde{\gamma}}\int_{\mathbb{B}^n} z\cdot\nabla f(y)d\mu_{\gamma+1}^n(y) $$

Proof: By the Stokes theorem,

$$\int_{\mathbb{B}^n} z\cdot\nabla f(y)(1-|y|^2)^{\gamma+1}c_{\gamma+1}d^ny = \int_{\partial\mathbb{B}^n} f(y)\underbrace{(1-|y|^2)^{\gamma+1}}_{=0}c_{\gamma+1}z\cdot y d\sigma(y)$$
$$+\frac{2c_{\gamma+1}(\gamma+1)}{c_\gamma}\int_{\mathbb{B}^n} z\cdot yf(y)(1-|y|^2)^{\gamma}c_\gamma d^ny$$
$$=2\tilde{\gamma}\int_{\mathbb{B}^n} z\cdot yf(y)d\mu_\gamma^n(y),$$
since $c_\gamma=\frac{\Gamma(\tilde{\gamma})}{\pi^{\frac{n}{2}}\Gamma(\gamma+1)}$.

Lemma (5.2.5) [299]:

$$x\cdot\nabla_y R_\alpha(x,y)=\frac{\tilde{\alpha}|x|^2}{b}(2b+x\cdot\nabla_x)_1(R_{\alpha+1})_1\left(\begin{matrix}b\\ b+1\end{matrix};x,y\right).$$

Proof: Recall that

$$R_\alpha(x,y)=F_1\left(\begin{matrix}\tilde{\alpha}\\ b\end{matrix};\begin{matrix}b\\ -\end{matrix}\begin{matrix}b\\ \end{matrix};z,\bar{z}\right),$$

where$z=x\cdot y+i\sqrt{|x|^2|y|^2-(x\cdot y)^2}$. From that we can see

$${}_1(R_{\alpha+1})_1\left(\begin{matrix}b\\ b+1\end{matrix};x,y\right)=F_1\left(\begin{matrix}\tilde{\alpha}+1\\ b+1\end{matrix};\begin{matrix}b\\ -\end{matrix}\begin{matrix}b\\ \end{matrix};z,\bar{z}\right).$$

Next,

$$x\cdot\nabla_y z=x\cdot\nabla_y\tilde{z}=|x|^2,\qquad x\cdot\nabla_x z^j=jz^j,\qquad x\cdot\nabla_x\bar{z}^k=k\bar{z}^k,$$

and

$$x\cdot\nabla_y F_1\left(\begin{matrix}\tilde{\alpha}\\ b\end{matrix};\begin{matrix}b\\ -\end{matrix}\begin{matrix}b\\ \end{matrix};z,\bar{z}\right)=|x|^2\tilde{\alpha}\left(F_1\left(\begin{matrix}\tilde{\alpha}+1\\ b+1\end{matrix};\begin{matrix}b+1\\ -\end{matrix}\begin{matrix}b\\ \end{matrix};z,\bar{z}\right)+F_1\left(\begin{matrix}\tilde{\alpha}+1\\ b+1\end{matrix};\begin{matrix}b\\ \end{matrix}\begin{matrix}b+1\\ -\end{matrix};z,\bar{z}\right)\right)$$
$$=\tilde{\alpha}|x|^2\sum_{j,k=0}^{\infty}\frac{(\tilde{\alpha}+1)_{j+k}}{(b+1)_{j+k}}\frac{z^j\bar{z}^k}{j!\,k!}\left((b+1)_j(b)_k+(b)_j(b+1)_k\right)$$
$$=\frac{\tilde{\alpha}|x|^2}{b}\sum_{j,k=0}^{\infty}\frac{(\tilde{\alpha}+1)_{j+k}}{(b+1)_{j+k}}\frac{z^j\bar{z}^k}{j!\,k!}(b)_j(b)_k(2b+j+k)$$
$$=\frac{\tilde{\alpha}|x|^2}{b}(2b+x\nabla_x)F_1\left(\begin{matrix}\tilde{\alpha}+1\\ b+1\end{matrix};\begin{matrix}b\\ -\end{matrix}\begin{matrix}b\\ \end{matrix};z,\bar{z}\right).$$

Proof: The proof of Theorem (5.2.2) will be done by induction on p.

$p=0$. For $\beta>\alpha$ ($\tilde{\alpha}$ is always positive from the assumption $\alpha>-1$) we get:

$$R_\alpha(x,y)={}_1(R_\beta)_1\left(\begin{matrix}\tilde{\alpha}\\ \tilde{\beta}\end{matrix};x,y\right)$$
$$=\frac{\Gamma(\tilde{\beta})}{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta}-\tilde{\alpha})}\int_0^1 t^{\tilde{\alpha}-1}(1-t)^{\tilde{\beta}-\tilde{\alpha}-1}R_\beta(tx,y)dt.$$

We substitute this into the integral and swap the order of integration:

$$\int_{\mathbb{B}^n} f(y)R_\alpha(x,y)d\mu_\beta^n(y)=\frac{\Gamma(\tilde{\beta})}{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta}-\tilde{\alpha})}\int_0^1 t^{\tilde{\alpha}-1}(1-t)^{\tilde{\beta}-\tilde{\alpha}-1}\int_{\mathbb{B}^n} f(y)R_\beta(tx,y)d\mu_\beta^n(y)dt$$
$$=\frac{\Gamma(\tilde{\beta})}{\Gamma(\tilde{\alpha})\Gamma(\tilde{\beta}-\tilde{\alpha})}\int_0^1 t^{\tilde{\alpha}-1}(1-t)^{\tilde{\beta}-\tilde{\alpha}-1}f(tx)dt={}_1f_1\left(\begin{matrix}\tilde{\alpha}\\ \tilde{\beta}\end{matrix};x\right).$$

When$\beta=\alpha$, this is just the reproducing property of the Bergman kernel.

$p=p+1$. We can see that the function $g(y):=R_\alpha(x,y)f(y)(x\cdot y)^p$ meets the condition of Lemma (5.2.4), hence:

$$\int_{\mathbb{B}^n} R_\alpha(x,y)f(y)(x\cdot y)^{p+1}d\mu_\beta^n(y)=\frac{1}{2\tilde{\beta}}\int_{\mathbb{B}^n} x\cdot\nabla_y(R_\alpha(x,y)f(y)(x\cdot y)^p)d\mu_{\beta+1}^n(y),$$

which divides the proof into three parts:

$$= \frac{1}{2\tilde{\beta}} \int_{\mathbb{B}^n} \left(x \cdot \nabla_y R_\alpha(x,y)\right) f(y) (x \cdot y)^p d\mu^n_{\beta+1}(y) \quad (41)$$

$$+ \frac{1}{2\tilde{\beta}} \int_{\mathbb{B}^n} R_\alpha(x,y) \left(x \cdot \nabla_y f(y)\right) (x \cdot y)^p d\mu^n_{\beta+1}(y) \quad (42)$$

$$+ \frac{1}{2\tilde{\beta}} \int_{\mathbb{B}^n} R_\alpha(x,y) f(y) p (x \cdot y)^{p-1} |x|^2 d\mu^n_{\beta+1}(y). \quad (43)$$

Notice that for a general series of the form

$$\frac{p!}{2^p} \sum_{j+2l+m=p} \frac{A(j,l,m)}{j!\,l!\,m!}$$

the transition $p \to p+1$ also divides this series into the three parts, namely:

$$\frac{(p+1)!}{2^{p+1}} \sum_{j+2l+m=p+1} \frac{A(j,l,m)}{j!l!m!} = \frac{p!}{2^{p+1}} \sum_{j+2l+m=p} \frac{A(j+1,l,m)}{j!l!m!} \quad (44)$$

$$+ \frac{p!}{2^{p+1}} \sum_{j+2l+m=p} \frac{A(j,l,m+1)}{j!l!m!} \quad (45)$$

$$\frac{p!}{2^p} \sum_{j+2l+m=p-1} \frac{A(j,l+1,m)}{j!l!m!}. \quad (46)$$

(The logic is as follows:

$$\sum_{j+2l+m=p+1} \frac{A(j,l,m)}{j!\,l!\,m!} = \sum_{j,l,m=0}^{\infty} \frac{\partial_t^{p+1}}{(p+1)!} t^{j+2l+m} \frac{A(j,l,m)}{j!\,l!\,m!}\Bigg|_{t=0}$$

$$= \sum_{j,l,m=0}^{\infty} \frac{\partial_t^{p}}{(p+1)!} (j+2l+m) t^{j+2l+m-1} \frac{A(j,l,m)}{j!\,l!\,m!}\Bigg|_{t=0}$$

$$= \sum_{j,l,m=0}^{\infty} \frac{\partial_t^{p}}{(p+1)!} t^{j+2l+m-1} \frac{A(j,l,m)}{(j-1)!\,l!\,m!}\Bigg|_{t=0} + \cdots$$

$$= \sum_{j,l,m=0}^{\infty} \frac{\partial_t^{p}}{(p+1)!} t^{j+2l+m} \frac{A(j+1,l,m)}{j!\,l!\,m!}\Bigg|_{t=0} + \cdots = \frac{1}{p+1} \sum_{j+2l+m=p} \frac{A(j+1,l,m)}{j!\,l!\,m!} + \cdots,$$

where the dots represent the other two terms, where the procedure is analogous.)

We will show that the corresponding parts are equal to each other, i.e. (41)=(44), (42)=(45) and (43)=(46) when

$$A(j,l,m) = \frac{|x|^{2(j+1)} (\tilde{\alpha})_j (2b)_j}{(\tilde{\beta})_{j+m+l} (b)_j} {}_3((x \cdot \nabla)^m f)_3 \left(\begin{matrix} \tilde{\alpha}+j & 2b+j & b \\ \tilde{\beta}+j+l+m & b+j & 2b \end{matrix}; x \right).$$

The equalities (42)=(45) and (43)=(46) are trivial. It remains only to prove the equality (41)=(44),

$$\frac{1}{2\tilde{\beta}} \int_{\mathbb{B}^n} \left(x \cdot \nabla_y R_\alpha(x,y)\right) f(y) (x \cdot y)^p d\mu^n_{\beta+1}(y) = \frac{p!}{2^{p+1}} \sum_{j+2l+m=p}^{\infty} \frac{A(j+1,l,m)}{j!\,l!\,m!}.$$

In the integral we use Lemma (5.2.5) to obtain:

$$\frac{1}{2\tilde{\beta}} \int_{\mathbb{B}^n} (x \cdot y)^p \frac{\tilde{\alpha}|x|^2}{b} (2b + x \cdot \nabla_x)_1 (R_{\alpha+1})_1 \left(\begin{matrix} b \\ b+1 \end{matrix}; x, y \right) f(y) d\mu^n_{\beta+1}(y),$$

which according to the Leibniz rule equals

$$\frac{\tilde{\alpha}|x|^2}{2\tilde{\beta} b} (2b + x \cdot \nabla_x - p) \int_{\mathbb{B}^n} (x \cdot y)^p {}_1(R_{\alpha+1})_1 \left(\begin{matrix} b \\ b+1 \end{matrix}; x, y \right) f(y) d\mu^n_{\beta+1}(y).$$

Using the integral form of hypergeometrization

$${}_1(R_{\alpha+1})_1 \left(\begin{matrix} b \\ b+1 \end{matrix}; x, y \right) = b \int_0^1 t^{b-1} R_{\alpha+1}(tx, y) dt$$

we get:

$$\frac{\tilde{\alpha}|x|^2}{2\tilde{\beta}}(2b + x\cdot\nabla_x - p)\int_0^1 t^{b-1-p}\int_{\mathbb{B}^n}(tx\cdot y)^p R_{\alpha+1}(tx,y)f(y)d\mu^n_{\beta+1}(y)dt.$$

By the induction hypothesis, this is equal to

$$\frac{\tilde{\alpha}|x|^2}{2\tilde{\beta}}(2b + x\cdot\nabla_x - p)\int_0^1 t^{b-1-p}$$

$$\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{t^{2j+2l+m}|x|^{2(j+1)}(\tilde{\alpha}+1)_j(2b)_j}{j!m!l!(\tilde{\beta}+1)_{j+m+l}(b)_j}\,{}_3((x\cdot\nabla)^m f)_3\left(\begin{matrix}\tilde{\alpha}+j+1 & 2b+j & b\\ \tilde{\beta}+j+1+l+m & b+j & 2b\end{matrix};tx\right)dt,$$

From the knowledge that

$$\int_0^1 t^{b+j-1}g(tx)dt = \frac{1}{b+j}\,{}_1g_1\left(\begin{matrix}b+j\\ b+j+1\end{matrix};x\right)$$

we obtain

$$|x|^2(2b + x\cdot\nabla_x - p)$$

$$\frac{p!}{2^{p+1}}\sum_{j+2l+m=p}\frac{|x|^{2(j+1)}(\tilde{\alpha})_{j+1}(2b)_j}{j!m!l!(\tilde{\beta})_{j+1+m+l}(b)_{j+1}}\,{}_3((x\cdot\nabla)^m f)_3\left(\begin{matrix}\tilde{\alpha}+j+1 & 2b+j & b\\ \tilde{\beta}+j+1+l+m & b+j+1 & 2b\end{matrix};x\right).$$

By the Leibniz rule and some manipulation we finally arrive at:

$$\frac{p!}{2^{p+1}}\sum_{j+2l+m=p}\frac{|x|^{2(j+1+l)}(\tilde{\alpha})_{j+1}(2b)_{j+1}}{j!\,m!\,l!\,(\tilde{\beta})_{j+1+m+l}(b)_{j+1}}$$

$$\frac{2b + x\cdot\nabla_x + j - m}{2b+j}\,{}_3((x\cdot\nabla)^m f)_3\left(\begin{matrix}\tilde{\alpha}+j+1 & 2b+j & b\\ \tilde{\beta}+j+1+l+m & b+j+1 & 2b\end{matrix};x\right).$$

To finish the proof it now only remains to show that the last formula is equal to

$${}_3((x\cdot\nabla)^m f)_3\left(\begin{matrix}\tilde{\alpha}+j+1 & 2b+j+1 & b\\ \tilde{\beta}+j+1+l+m & b+j+1 & 2b\end{matrix};x\right),$$

but from the property of hypergeometrization it follows generally that

$${}_1((x\cdot\nabla)^m g)_1\left(\begin{matrix}a+1\\ c\end{matrix};x\right) = \frac{1}{a}(a+t\partial_t)\,{}_1((x\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ c\end{matrix};tx\right)\Big|_{t=1}$$

$$= \frac{1}{a}(a+t\partial_t)t^{-m}\,{}_1((tx\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ c\end{matrix};tx\right)\Big|_{t=1} = \frac{1}{a}(a-m+t\partial_t)\,{}_1((tx\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ c\end{matrix};tx\right)\Big|_{t=1}$$

$$= \frac{1}{a}(a-m+x\cdot\nabla_x)\,{}_1((x\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ c\end{matrix};x\right).$$

Set $a = 2b + j$and we are done.

Corollary (5.2.6) [299]: For $p \in \mathbb{N}_0$ and $\beta \geq \alpha > -1$,

$$\int_{\mathbb{B}^n}(x\cdot y)^p R^2_\alpha(x,y)d\mu^n_\beta(y)$$

$$\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{|x|^{2(j+l+m)}(\tilde{\alpha})_j(\tilde{\alpha})_m(2b)_j(2b)_m}{(\tilde{\beta})_{j+m+l}(b)_j(b)_m j!m!l!}\,{}_5F_4\left(\begin{matrix}\tilde{\alpha}+j & \tilde{\alpha}+m & 2b+j & 2b+m & b\\ \tilde{\beta}+j+m+l & b+j & b+m & 2b\end{matrix};|x|^2\right).$$

Proof: We use Theorem (5.2.2) with $f(y) = R_\alpha(x,y)$:

$$\int_{\mathbb{B}^n}R^2_\alpha(x,y)(x\cdot y)^p d\mu^n_\beta(y)$$

$$\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{|x|^{2(j+l)}(\tilde{\alpha})_j(2b)_j}{j!\,m!\,l!\,(\tilde{\beta})_{j+m+l}(b)_j}\,{}_3\tilde{f}_3\left(\begin{matrix}\tilde{\alpha}+j & 2b+j & b\\ \tilde{\beta}+j+l+m & b+j & 2b\end{matrix};x\right),$$

where $\tilde{f}(y) := (x\cdot\nabla_y)^m R_\alpha(x,y)$. From the fact $x\cdot\nabla_y z = x\cdot\nabla_y\bar{z} = |x|^2$ we have

$$\tilde{f}(y) = \left(x\cdot\nabla_y\right)^m R_\alpha(x,y) = |x|^{2m}(\partial_z+\partial_z)^m F_1\left(\begin{matrix}\tilde{\alpha}\\ b\end{matrix};\begin{matrix}b\\ -\end{matrix}\;\begin{matrix}b\\ \end{matrix};z,\bar{z}\right)$$

$$= |x|^{2m}\frac{(\tilde{\alpha})_m}{(b)_m}\sum_{k=0}^{m}\binom{m}{k}(b)_{m-k}(b)_k F_1\left(\begin{matrix}\tilde{\alpha}+m\\ b+m\end{matrix}; \begin{matrix}b+m-k & b+k\\ - & \end{matrix}; z,\bar{z}\right).$$

Performing hypergeometrization (notice that z and.$\bar{z}$ are homogeneous of the degree 1) we get

$${}_3\tilde{f}_3\left(\begin{matrix}\tilde{\alpha}+j & 2b+j & b\\ \tilde{\beta}+j+m+l & b+j & 2b\end{matrix}; x\right)$$

$$= |x|^{2m}\frac{(\tilde{\alpha})_m}{(b)_m}$$

$$\sum_{k=0}^{m}\binom{m}{k}(b)_{m-k}(b)_k F_1\left(\begin{matrix}\tilde{\alpha}+m & \tilde{\alpha}+j & 2b+j & b\\ b+m & \tilde{\beta}+j+m+l & b+j & 2b\end{matrix}; \begin{matrix}b+m-k & b+k\\ - & \end{matrix}; |x|^2,|x|^2\right)$$

$$= |x|^{2m}\frac{(\tilde{\alpha})_m}{(b)_m}\sum_{k=0}^{m}\binom{m}{k}(b)_{m-k}(b)_k\, {}_5F_4\left(\begin{matrix}\tilde{\alpha}+m & \tilde{\alpha}+j & 2b+j & b & 2b+m\\ b+m & \tilde{\beta}+j+m+l & b+j & & 2b\end{matrix}; |x|^2\right).$$

Here, by the Appell function with more parameters we mean the Kampé de Fériet function (see [314])

$$F_1\left(\begin{matrix}a_1\dots a_4\\ c_1\dots c_4\end{matrix}; \begin{matrix}b_1 & b_2\\ - & \end{matrix}; x,y\right) := F^{4:1;1}_{4:0;0}\left(\begin{matrix}a_1\dots a_4: b_1; b_2\\ c_1\dots c_4: -.-;\end{matrix}; x,y\right)$$

$$= \sum_{j,k=0}^{\infty}\frac{(a_1)_{k+j}\dots(a_4)_{k+j}}{(c_1)_{k+j}\dots(c_4)_{k+j}}\frac{(b_1)_j(b_2)_k}{j!\,k!}x^j y^k,$$

and the last equality was obtained using the similar reduction formula like in the case of Appell F_1 function of the same argument

$$F_1\left(\begin{matrix}a_1\dots a_4\\ c_1\dots c_4\end{matrix}; \begin{matrix}b_1 & b_2\\ - & \end{matrix}; x,x\right) = {}_5F_4\left(\begin{matrix}a_1\dots a_4 & b_1+b_2\\ c_1\dots c_4 & \end{matrix}; x\right).$$

To complete the proof it is only necessary to become conscious of the fact that

$$\sum_{k=0}^{m}\binom{m}{k}(b)_{m-k}(b)_k = (2b)_m$$

and substitute everything into the series at the beginning.

Corollary (5.2.7) [299]: As $\alpha \to \infty$,

$$\frac{{}_5F_4\left(\begin{matrix}\tilde{\alpha} & \tilde{\alpha} & 2b & 2b & b\\ \tilde{\alpha}+c_1 & b+c_2 & b+c_3 & 2b+c_4 & \end{matrix}; |x|^2\right)}{{}_2F_1\left(\begin{matrix}\tilde{\alpha} & 2b\\ b & \end{matrix}; |x|^2\right)}$$

$$\approx (1-|x|^2)^{c_1}\left(\frac{1-|x|^2}{\alpha|x|^2}\right)^{c_2+c_3+c_4}(b)_{c_2}(b)_{c_3}(2b)_{c_4}\left(1+\sum_{k=1}^{\infty}\frac{d_k}{\alpha^k}\right),$$

where d_k are some constants independent of α.

Theorem (5.2.8) [299]: For $M,N \in \mathbb{N}_0$,

$$\int_{\mathbb{B}^n}(y\cdot x-|x|^2)^M(|x|^2-|y|^2)^N R^2_\alpha(x,y)\,d\mu^n_\alpha(y)$$

$$= \sum_{\substack{l,r,q\\ k_1\dots k_5}}^{\infty}\frac{|x|^{2(M-1)}M!\,(-1)^M(-N)_q(b+1)_q(b)_{k_1}(-b)_{k_2}(-b)_{k_3}(r+l+k_4)_{k_5}}{2^M(M-2l)!\,(\tilde{\alpha})_q(\tilde{\alpha})_{r+l+k_4+k_5}(2b)_{k_1}(b)_{k_2}(b)_{k_3}\,l!\,r!\,q!\,k_1!\,k_2!\,k_3!\,k_4!\,k_5!}$$

$$C(|x|^2)\,{}_5F_4\left(\begin{matrix}\tilde{\alpha} & \tilde{\alpha} & 2b & 2b & b\\ \tilde{\alpha}+l+r+k_4+k_5 & b+k_3 & b+k_2 & 2b+k_1 & \end{matrix}; |x|^2\right),$$

where

$$C(|x|^2) := \left(-\partial_{t_1}\right)^{k_1}\dots\left(-\partial_{t_5}\right)^{k_5}\partial^{k_4}_{t_6}\partial^{r}_{t_7}\left(-\partial_{t_0}\right)^{r}$$

$$t_6^{r+l+k_4-1}t_7^{l+r-1}(t_0-|x|^2 t_{1-7})^q(|x|^2(1-t_{1-7})+t_0-1)^{N-q}(2-t_{6735}-t_{724})^{M-2l}\big|_{t_0\dots t_7=1}$$

Here $t_{1-7} := t_1 t_2 \dots t_7, t_{6735} := t_6 t_7 t_3 t_5$ and so on. The summation indices are bound by the following inequalities

$$k_1 + k_2 + k_3 + 2k_4 + k_5 + 2r + q + 2l \geq M + N$$
$$2l \leq M$$
$$q, k_1, r \leq N$$
$$k_2, k_3, k_4, k_5, 1 + l \leq N + M$$

Proof: We expand the parentheses in the integral into binomial series

$$(y \cdot x - |x|^2)^M = \sum_{p=0}^{\infty} \binom{M}{P} (-|x|^2)^{M-p} (y \cdot x)^p,$$

$$(|x|^2 - |y|^2)^N = \sum_{q=0}^{\infty} \binom{N}{q} (|x|^2 - 1)^{N-q} (1 - |y|^2)^q,$$

to get

$$\sum_{p,q=0}^{\infty} \binom{M}{P} \binom{N}{q} (-|x|^2)^{M-p} (|x|^2 - 1)^{N-q} \frac{c_\alpha}{c_{\alpha+q}} \int_{\mathbb{B}^n} (x \cdot y)^p R_\alpha^2(x, y) d\mu_{\alpha+q}^n(y),$$

Where $c_\gamma = \frac{\Gamma(\tilde{\gamma})}{\pi^{\frac{n}{2}} \Gamma(\gamma+1)}$. By Corollary (5.2.6), this equals

$$\sum_{p,q=0}^{\infty} \binom{M}{P} \binom{N}{q} (-|x|^2)^{M-p} (|x|^2 - 1)^{N-q} \frac{(\alpha+1)_q}{(\tilde{\alpha})_q} \frac{p!}{2^p} \sum_{j+2l+m=p} \frac{|x|^{2(j+l+m)} (\tilde{\alpha})_j (\tilde{\alpha})_m (2b)_j (2b)_m}{(\tilde{\alpha}+q)_{j+m+l} (b)_j (b)_m j!\, l!\, m!}$$
$${}_5F_4 \left(\begin{matrix} \tilde{\alpha}+j & \tilde{\alpha}+m & 2b+j & 2b+m & b \\ \tilde{\alpha}+q+j+m+l & b+j & b+m & 2b \end{matrix} ; |x|^2 \right).$$

Now we sum over p by the procedure:

$$\sum_{p=0}^{\infty} \binom{M}{P} x^{M-p} \frac{p!}{2^p} \sum_{j+2l+m=p} A_{jlm} = \sum_{p=0}^{\infty} \binom{M}{P} x^{M-p} \frac{p!}{2^p} \sum_{j,l,m=0}^{\infty} \frac{\partial_t^p}{p!} t^{j+2l+m} A_{jlm} \Big|_{t=0}$$
$$= \sum_{j,l,m=0}^{\infty} \sum_{p=0}^{\infty} \binom{M}{P} x^{M-p} \frac{\partial_t^p}{p!} t^{j+2l+m} A_{jlm} \Big|_{t=0} = \sum_{j,l,m=0}^{\infty} \binom{M}{j+2l+m} \frac{x^{M-j-2l-m} (j+2l+m)!}{2^{j+m} 4^l} A_{jlm}.$$

This yields

$$\sum_{q=0}^{\infty} \binom{N}{q} (|x|^2 - 1)^{N-q} \frac{(\alpha+1)_q}{(\tilde{\alpha})_q} \sum_{j,l,m=0}^{\infty} \frac{|x|^{2(M-l)} M! (-1)^{M+j+m} (\tilde{\alpha})_j (\tilde{\alpha})_m (2b)_j (2b)_m}{(M-2l-j-m)! 4^l 2^{j+m} (\tilde{\alpha}+q)_{j+m+l} (b)_j (b)_m j! l! m!} \quad (47)$$
$${}_5F_4 \left(\begin{matrix} \tilde{\alpha}+j & \tilde{\alpha}+m & 2b+j & 2b+m & b \\ \tilde{\alpha}+q+j+m+l & b+j & b+m & 2b \end{matrix} ; |x|^2 \right).$$

We would like to sum over q as well but we are unable to do that since the index is present also in the hypergeometric function. To remove this difficulty we make use of the following lemma

Lemma (5.2.9) [299]: For $r \in \mathbb{N}_0$,

$$\underline{\Delta}_\beta^r \frac{1}{(\beta)_k} {}_1f_1 \left(\begin{matrix} \gamma \\ \beta+k \end{matrix} ; x \right) = (-1)^r (k)_r \sum_{j=0}^{\infty} \frac{(-1)^j (-r)_j}{j! (k)_j (\beta)_{k+r}} x^j \partial_x^j {}_1f_1 \left(\begin{matrix} \gamma \\ \beta+k+r \end{matrix} ; x \right),$$

Where $\underline{\Delta}_\beta g(\beta) := g(\beta+1) - g(\beta)$.

The proof could be easily done by induction, but our approach will be much more direct.

Proof: Firstly

$$\underline{\Delta}_\beta \frac{1}{(\beta)_k} {}_1f_1 \left(\begin{matrix} \gamma \\ \beta+k \end{matrix} ; x \right) = \sum_{j=0}^{\infty} \underline{\Delta}_\beta \frac{f^{(j)}(0) (\gamma)_j}{j! (\beta)_{k+j}} x^j = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)(\gamma)_j}{j!} x^j \left(\frac{1}{(\beta+1)_{k+j}} - \frac{1}{(\beta)_{k+j}} \right)$$
$$= \sum_{j=0}^{\infty} \frac{f^{(j)}(0)(\gamma)_j}{j!} x^j \frac{-k-j}{(\beta)_{k+j+1}} = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)(\gamma)_j}{j!} \frac{-k - x\partial_x}{(\beta)_{k+j+1}} x^j = -\frac{k + x\partial_x}{(\beta)_{k+1}} {}_1f_1 \left(\begin{matrix} \gamma \\ \beta+k+1 \end{matrix} ; x \right).$$

Hence obviously

$$\underline{\Delta}_\beta^r \frac{1}{(\beta)_k}\, {}_1f_1\left({\gamma \atop \beta+k};x\right) = (-1)^r (k+x\partial_x)_r \frac{1}{(\beta)_{k+r}}\, {}_1f_1\left({\gamma \atop \beta+k+r};x\right)$$

We claim that

$$(k+x\partial_x)_r = \sum_{j=0}^{r} \frac{(-1)^j(-r)_j(k)_r}{j!\,(k)_j} x^j\partial_x^j.$$

It is enough to check this equation on monomials x^m since the expressions on both sides are linear combinations of operators $x^l\partial_x^l$. Since $x\partial_x x^m = mx^m$this reduces the problem to verifying the equality

$$(k+m)_r = \sum_{j=0}^{r} \frac{(-1)^j(-r)_j(k)_r}{j!\,(k)_j}\frac{m!}{(m-j)!}.$$

But this can be rewritten as

$$(k)_r m! \binom{r+k-1+m}{m} = (k)_r m! \sum_{j=0}^{r} \binom{r}{j}\binom{k-1+m}{m-j},$$

which is (aside the factor $(k)_r m!$) the so-called Chu-Vandermonde identity.
We will use this lemma in the following way. Obviously

$$\frac{1}{(\beta+q)_k}\, {}_1f_1\left({\gamma \atop \beta+q+k};x\right) = \sum_{r=0}^{\infty}\binom{q}{r}\underline{\Delta}_\beta^r \frac{1}{(\beta)_k}\, {}_1f_1\left({\gamma \atop \beta+k};x\right).$$

By the lemma with

$$f(|x|^2) := {}_4F_3\left({\tilde\alpha+m \quad 2b+j \quad 2b+m \quad b \atop b+j \quad b+m \quad 2b};|x|^2\right),$$

$\gamma := \tilde\alpha + j, \beta := \tilde\alpha$ and $k := j+l+m$ we get

$$\frac{1}{(\tilde\alpha+q)_{j+l+m}}\, {}_5F_4\left({\tilde\alpha+j \quad \tilde\alpha+m \quad 2b+j \quad 2b+m \quad b \atop \tilde\alpha+q+j+m+l \quad b+j \quad b+m \quad 2b};|x|^2\right)$$

$$= \sum_{r=0}^{\infty}\binom{q}{r}(-1)^r(j+l$$

$$+m)_r \sum_{s=0}^{\infty}\frac{(-1)^s(-r)_s|x|^{2s}\partial_{|x|^2}^s}{s!\,(j+l+m)_s(\tilde\alpha)_{j+m+l+r}}\, {}_5F_4\left({\tilde\alpha+j \quad \tilde\alpha+m \quad 2b+j \quad 2b+m \quad b \atop \tilde\alpha+r+j+m+l \quad b+j \quad b+m \quad 2b};|x|^2\right)$$

$$\sum_{r=0}^{\infty}\binom{q}{r}(-1)^r(j+l+m)_r \sum_{s=0}^{\infty}\frac{(-1)^s(-r)_s|x|^{2s}}{s!\,(j+l+m)_s}\frac{(\tilde\alpha+j)_s(\tilde\alpha+m)_s(2b+j)_s(2b+m)_s(b)_s}{(\tilde\alpha)_{j+m+l+r+s}(b+j)_s(b+m)_s(2b)_s}$$

$${}_5F_4\left({\tilde\alpha+j+s \quad \tilde\alpha+m+s \quad 2b+j+s \quad 2b+m+s \quad b+s \atop \tilde\alpha+r+j+m+l+s \quad b+j+s \quad b+m+s \quad 2b+s};|x|^2\right).$$

Substituting this into (47), with some manipulations and performing a transformation of the summation index $r \to r+s$ we get

$$\sum_{q=0}^{\infty}\binom{N}{q}(|x|^2-1)^{N-q}\frac{(\alpha+1)_q}{(\tilde\alpha)_q} \qquad (48)$$

$$\sum_{j,l,m,r,s=0}^{\infty}\frac{|x|^{2(M-l+s)}M!(-1)^{M+j+m}(\tilde\alpha)_{j+s}(\tilde\alpha)_{m+s}(2b)_{j+s}(2b)_{m+s}(b)_s(-q)_{r+s}(j+l+m)_{r+s}}{(M-2l-j-m)!4^l2^{j+m}(b)_{j+s}(b)_{m+s}(2b)_s(\tilde\alpha)_{j+m+l+r+2s}j!l!m!r!s!(j+l+m)_s}$$

$${}_5F_4\left({\tilde\alpha+j+s \quad \tilde\alpha+m+s \quad 2b+j+s \quad 2b+m+s \quad b+s \atop \tilde\alpha+r+j+m+l+2s \quad b+j+s \quad b+m+s \quad 2b+s};|x|^2\right)$$

We can sum over q now. The series in question is

$$\sum_{q=0}^{\infty}\binom{N}{q}(|x|^2-1)^{N-q}\frac{(\alpha+1)_q}{(\tilde\alpha)_q}(-q)_{r+s}.$$

By the representation $(-q)_{r+s} = (-\partial_t)^{r+s} t^q|_{t=1}$(from now on every parameter that contains the letter $t-t, t_1$, and so on – will be understood to be evaluated at 1; we will not explicitly mention this) we get

$$(-\partial_t)^{r+s}(|x|^2-1)^N {}_2F_1\left(\begin{matrix}-N & \alpha+1\\ & \tilde{\alpha}\end{matrix}; \frac{t}{1-|x|^2}\right).$$

The known transformation

$${}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix}; x\right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix}a & c-b\\ & c\end{matrix}; \frac{x}{x-1}\right)$$

enables us to write this as

$$(-\partial_t)^{r+s}(|x|^2-1)^N {}_2F_1\left(\begin{matrix}-N & b+1\\ & \tilde{\alpha}\end{matrix}; \frac{t}{|x|^2-1+t}\right)$$
$$= \sum_{q=0}^{\infty} \frac{(-N)_q(b+1)_q}{(\tilde{\alpha})_q q!}(-\partial_t)^{r+s} t^q (|x|^2-1+t)^{N-q}.$$

We did not manage to sum the series explicitly but this will do. Substituting this result into (48) we get the second intermediate result:

$$\sum_{\substack{j,l,m\\ r,s,q}}^{\infty} \frac{|x|^{2(M-l+s)} M!(-1)^{M+j+m}(-N)_q(b+1)_q(\tilde{\alpha})_{j+s}(\tilde{\alpha})_{m+s}(2b)_{j+s}(2b)_{m+s}(b)_s(j+l+m)_{r+s}}{(M-2l-j-m)!4^l 2^{j+m}(b)_{j+s}(b)_{m+s}(2b)_s(\tilde{\alpha})_q(\tilde{\alpha})_{j+m+l+r+2s} j!l!m!r!s!q!(j+l+m)_s}$$
$$C_{r,s}(|x|^2)\,{}_5F_4\left(\begin{matrix}\tilde{\alpha}+j+s & \tilde{\alpha}+m+s & 2b+j+s & 2b+m+s & b+s\\ \tilde{\alpha}+r+j+m+l+2s & b+j+s & b+m+s & 2b+s & \end{matrix}; |x|^2\right), \quad (49)$$

Where $C_{r,s}(|x|^2) = (-\partial_t)^{r+s} t^q (|x|^2-1+t)^{N-q}$.

Sadly, this form is not of much use to us. As it is clear from Lemma (5.2.3) all functions ${}_5F_4$ with these parameters have the same principal asymptotic behavior as $\alpha \to \infty$. To get a more effective form we exploit the known relation between contiguous hypergeometric functions

$$F\left(\begin{matrix}a+1\\ c+1\end{matrix}\right) = \frac{c}{a}\left(F\left(\begin{matrix}a\\ c\end{matrix}\right) - \frac{c-a}{c} F\left(\begin{matrix}a\\ c+1\end{matrix}\right)\right),$$

which holds for any hypergeometric function with at least one upper and one lower parameter.
By iteration we get:

$$F\left(\begin{matrix}a+m\\ c+m\end{matrix}\right) = \frac{(c)_m}{(a)_m}\sum_{j=0}^{\infty}\frac{(-m)_j(c-a)_j}{(c)_j j!} F\left(\begin{matrix}a\\ c+j\end{matrix}\right).$$

We apply this to the function

$${}_5F_4\left(\begin{matrix}\overbrace{\tilde{\alpha}+j+s}^{k_5} & \overbrace{\tilde{\alpha}+m+s}^{k_4} & 2b+j+s & 2b+m+s & b+s\\ \underbrace{\tilde{\alpha}+r+j+m+l+2s}_{k_4 k_5} & & \underbrace{b+j+s}_{k_3} & \underbrace{b+m+s}_{k_2} & \underbrace{2b+s}_{k_1}\end{matrix}; |x|^2\right) \quad (50)$$

as indicated, five times in total. That will get us 5 new series with 5 new summation indices, which we name $k_1 \ldots k_5$. The role of m will be played in turn by the parameters $s, m+s, j+s, m+s$ and $j+s$. The lower indices c will be in this $2b, b, b, \tilde{\alpha}+r+j+s+l$ and$\tilde{\alpha}+r+l+k_4$.
This way the expression (50) will change form to:

$$\sum_{k_1 \ldots k_5} \frac{(2b)_s(b)_{m+s}(b)_{j+s}(\tilde{\alpha}+r+j+s+l)_{m+s}(\tilde{\alpha}+r+l+k_4)_{j+s}}{(b)_s(2b)_{m+s}(2b)_{j+s}(\tilde{\alpha})_{m+s}(\tilde{\alpha})_{j+s}}$$
$$\frac{(-s)_{k_1}(-m-s)_{k_2}(-j-s)_{k_3}(-m-s)_{k_4}(-j-s)_{k_5}(b)_{k_1}(-b)_{k_2}(-b)_{k_3}(r+j+s+l)_{k_4}(r+l+k_4)_{k_5}}{(2b)_{k_1}(b)_{k_2}(b)_{k_3}(\tilde{\alpha}+r+j+s+l)_{k_4}(\tilde{\alpha}+r+l+k_4)_{k_5} k_1!k_2!k_3!k_4!k_5!}$$
$${}_5F_4\left(\begin{matrix}\tilde{\alpha} & & \tilde{\alpha} & 2b & 2b & b\\ \tilde{\alpha}+l+r+k_4+k_5 & b+k_3 & b+k_2 & 2b+k_1 & & \end{matrix}; |x|^2\right).$$

Substituting this into (49) will fortunately reduce the number of terms, for many of them will cancel out each other. It can be checked by an easy calculation, for example, that the terms containing α but not q will squeeze to a single expression$1/(\tilde{\alpha})_{r+l+k_4+k_5}$. We end up with this much more tolerable expression:

$$\sum_{\substack{jlmrsq\\k_1\ldots k_5}}^{\infty}\frac{|x|^{2(M-l+s)}M!(-1)^{M+j+m}(-N)_q(b+1)_q(j+l+m+s)_r}{(M-2l-j-m)!4^l2^{j+m}(\tilde{\alpha})_q(\tilde{\alpha})_{r+l+k_4+k_5}j!l!m!r!s!q!k_1!k_2!k_3!k_4!k_5!} \qquad (51)$$

$$\frac{(-s)_{k_1}(-m-s)_{k_2}(-j-s)_{k_3}(-m-s)_{k_4}(-j-s)_{k_5}(b)_{k_1}(-b)_{k_2}(-b)_{k_3}(r+j+s+l)_{k_4}(r+l+k_4)_{k_5}}{(2b)_{k_1}(b)_{k_2}(b)_{k_3}}$$

$$C_{r,s}(|x|^2)\,{}_5F_4\left(\begin{matrix}\tilde{\alpha} & & \tilde{\alpha} & 2b & 2b & b\\ \tilde{\alpha}+l+r+k_4+k_5 & b+k_3 & b+k_2 & 2b+k_1\end{matrix};|x|^2\right).$$

We can reduce the complexity of this formula further by summing over all indices which do not appear in the hypergeometric function or depend on $\tilde{\alpha}$, i.e. over indices j, m, s. This gives

$$\sum_{\substack{l,r,q\\k_1\ldots k_5}}^{\infty}\frac{|x|^{2(M-l)}M!(-1)^M(-N)_q(b+1)_q(b)_{k_1}(-b)_{k_2}(-b)_{k_3}(r+l+k_4)_{k_5}}{4^l(\tilde{\alpha})_q(\tilde{\alpha})_{r+l+k_4+k_5}(2b)_{k_1}(b)_{k_2}(b)_{k_3}l!r!q!k_1!k_2!k_3!k_4!k_5!} \qquad (52)$$

$$C(|x|^2)\,{}_5F_4\left(\begin{matrix}\tilde{\alpha} & & \tilde{\alpha} & 2b & 2b & b\\ \tilde{\alpha}+l+r+k_4+k_5 & b+k_3 & b+k_2 & 2b+k_1\end{matrix};|x|^2\right),$$

where

$$C(|x|^2) :=$$

$$\sum_{j,m,s}^{\infty}\frac{|x|^{2s}(-1)^{j+m}(j+l+m+s)_r(-s)_{k_1}(-m-s)_{k_2}(-j-s)_{k_3}(-m-s)_{k_4}(-j-s)_{k_5}(r+j+s+l)_{k_4}}{(M-2l-j-m)!2^{j+m}j!m!s!}$$

$$(-\partial_t)^{r+s}t^q(|x|^2-1+t)^{N-q}.$$

We must deal now with the coefficient $C(|x|^2)$. For that purpose we represent each Pochhammer symbol in the series by $(-a)_k = (-\partial_t)^k t^a$ whenever the argument is negative and by $(a)_k = \partial_t^k t^{a+k-1}$in the opposite case (again, the default understanding is that every parameter t_i is to be evaluated, without
explicitly saying so, at the point 1). Thus we get

$$C(|x|^2) = (-\partial_{t_1})^{k_1}\ldots(-\partial_{t_5})^{k_5}\partial_{t_6}^{k_4}\partial_{t_7}^{r}(-\partial_t)^r$$

$$\sum_{j,m,s}^{\infty}\frac{|x|^{2s}(-1)^{j+m}t_{1-7}^{s}t_{6735}^{j}t_{724}^{m}(-\partial_t)^s}{(M-2l-j-m)!\,2^{j+m}j!\,m!\,s!}t_6^{r+l+k_4-1}t_7^{l+r-1}t^q(|x|^2-1+t)^{N-q},$$

where $t_{1-7} := t_1t_2t_3t_4t_5t_6t_7, t_{6735} = t_6t_7t_3t_5$ and so on. The sum over s is essentially the Taylor series.

As for the other two indices, it is clear that

$$\sum_{j,m}^{\infty}\frac{A^jB^m}{(M-2l-j-m)!\,j!\,m!} = \frac{1}{(M-2l)!}(1+A+B)^{M-2l}.$$

We thus finally get

$$C(|x|^2) = (-\partial_{t_1})^{k_1}\ldots(-\partial_{t_5})^{k_5}\partial_{t_6}^{k_4}\partial_{t_7}^{r}(-\partial_t)^r$$

$$\frac{1}{(M-2l)!}t_6^{r+l+k_4-1}t_7^{l+r-1}(t-|x|^2t_{1-7})^q(|x|^2(1-t_{1-7})+t-1)^{N-q}\left(1-\frac{1}{2}t_{6735}-\frac{1}{2}t_{724}\right)^{M-2l}$$

Many things can be learnt from this form. Firstly: the last two parentheses are equal to zero when all $t-s$ are evaluated at the point 1. To avoid this we must differentiate them out. For that at least $N-q+M-2l$ differentiations are needed. Available to us are $k_1+k_2+k_3+2k_4+k_5+2r$ of them. Hence:

$$k_1+k_2+k_3+2k_4+k_5+2r+q+2l \geq M+N.$$

Secondly: from the perspective of the parameter t we differentiate r-times a polynomial of degree N. In order for the factor $C_{\tilde{\alpha}}$ not to be zero it must hold $r \leq N$. Analogously, the degree of t_7 is $r-l-1+N+M$ and this tells us that $1+l \leq N+M$. The same reasoning can be applied to any parameter t_i. From those and other facts, such as that in the formula (52) there appears the term $(-N)_q$, or from the presence of the term $1/(M-2l)!$, we can easily compute upper bounds on summation indices. They are:

$$\begin{matrix} & r \leq N & \\ 2l \leq M & \wedge & 1+l \leq N+M \\ & q \leq N & \end{matrix}$$

$$k_1 \leq N$$
$$k_i \leq N + M \qquad \forall_i = 2 \dots 5.$$

Corollary (5.2.10) [299]: For $M, N \in \mathbb{N}_0$ and $x \neq 0$:

$$\int_{\mathbb{B}^n} (y \cdot x - |x|^2)^M (|x|^2 - |y|^2)^N \frac{R_\alpha^2(x,y)}{R_\alpha(x,x)} d\mu_\alpha^n(y) = O\left(\alpha^{-\left\lceil \frac{N+M}{2} \right\rceil}\right) \qquad (\alpha \to \infty).$$

Proof: From the fact

$$R_\alpha(x,x) = {}_2F_1\left(\begin{matrix}\tilde{\alpha} & 2b \\ & b\end{matrix}; |x|^2\right),$$

from Corollary (5.2.7) and from the presence of the factor $(\tilde{\alpha})_q (\tilde{\alpha})_{l+r+k_4+k_5}$in the denominator in Theorem (5.2.8) it follows that for $x \neq 0$

$$\int_{\mathbb{B}^n} (y \cdot x - |x|^2)^M (|x|^2 - |y|^2)^N \frac{R_\alpha^2(x,y)}{R_\alpha(x,x)} d\mu_\alpha^n(y) = \sum_{rlqk_1 \dots k_5} O\left(\alpha^{-(r+l+q+k_1+\cdots+k_5)}\right) \quad (\alpha \to \infty).$$

That is the speed of asymptotic decay grows with each summation index $k_1, k_2, k_3, k_4, k_5, r, q, l$ as $\alpha \to \infty$. The slowest decay (and therefore the leading term) we get for the lowest possible values of these parameters. But since $k_1 + k_2 + k_3 + 2k_4 + k_5 + 2r + q + 2l \geq M + N$, lowest values are achieved for $2k_4 + 2r + 2l = M + N$and $k_1 = \cdots = k_5 = 0$ if this is possible, i.e. if $M + N$ is an even number; if $M + N$ is odd, the decay is one negative power of alpha faster. Hence the leading order term is

$$O\left(\alpha^{-\left\lceil \frac{N+M}{2} \right\rceil}\right) \qquad (\alpha \to \infty).$$

Corollary (5.2.11) [299]: For $M \in \mathbb{N}_0$ and $x \neq 0$

$$\int_{\mathbb{B}^n} |y - x|^{2M} \frac{R_\alpha^2(x,y)}{R_\alpha(x,x)} d\mu_\alpha^n(y) = O\left(\alpha^{-\left\lceil \frac{M}{2} \right\rceil}\right) \qquad (\alpha \to \infty).$$

Proof: The statement follows directly from the representation

$$|y - x|^{2M} = (|y|^2 - 2y \cdot x + |x|^2)^M = (|y|^2 - |x|^2 - 2(y \cdot x - |x|^2))^M$$
$$= (-2)^M \sum_{N=0}^{\infty} \binom{M}{N} (y \cdot x - |x|^2)^{M-N} (|x|^2 - |y|^2)^N.$$

The integral is therefore a series of terms, whose behavior is by Corollary (5.2.10):

$$O\left(\alpha^{-\left\lceil \frac{M-N+N}{2} \right\rceil}\right) \qquad (\alpha \to \infty).$$

Lemma (5.2.12) [299]: For $m \in \mathbb{N}_0, x \neq 0$ and $n > 1$:

$$\int_{\mathbb{B}^n} (z \cdot y - z \cdot x)^m f(|y|, x \cdot y) d\mu_\alpha^n(y)$$
$$= \sum_{k=0}^{\infty} \frac{\left(-\frac{m}{2}\right)_k \left(\frac{1-m}{2}\right)_k (z \cdot x)^{m-2k} v_{z,x}^{2k}}{\left(b + \frac{1}{2}\right)_k k! \, |x|^{2m}} \int_{\mathbb{B}^n} (x \cdot y - |x|^2)^{m-2k} v_{y,x}^{2k} f(|y|, x \cdot y) d\mu_\alpha^n(y)$$

where $v_{u_1,u_2}^2 = |u_1|^2 |u_2|^2 - (u_1 \cdot u_2)^2$.

Proof: The integral

$$\int_{\mathbb{B}^n} (z \cdot y - z \cdot x)^m f(|y|, x \cdot y) d\mu_\alpha^n(y)$$

is unchanged upon replacing x, y, z by Ux, Uy, Uz for any orthogonal transformation U. Without loss of generality, we can thus assume that $x = (|x|, 0, 0, \dots)$ and $z = (z_1, z_2, 0, 0, \dots)$ with $z_2 \geq 0$. Then $x \cdot y = |x| y_1, z_1 = \frac{z \cdot x}{|x|}, z_2 = \frac{v_{x,x}}{|x|}$ and

$$z \cdot x = \frac{z \cdot x}{|x|} y_1 + v_{z,x} \frac{y_2}{|x|}.$$

We now change variables to hyper-spherical coordinates

$$y_1 = r \cos \varphi$$

$$y_2 = r\,sin\,\varphi\,cos\,\psi$$
$$\dots$$
$$y_{n-1} = r\,sin\,\varphi\,sin\,\psi\,sin\,\theta_1 \dots\, sin\,\theta_{n-4} cos\,\theta_{n-3}$$
$$y_n = r\,sin\,\varphi\,sin\,\psi\,sin\,\theta_1 \dots sin\,\theta_{n-4}\,sin\,\theta_{n-3}$$
$$d\mu_\alpha^n = \frac{(\alpha + n\backslash 2)!}{\pi^{n\backslash 2}\Gamma(\alpha+1)}(1-r^2)^\alpha r^{n-1}\,sin^{n-2}\varphi\,sin^{n-3}\psi \dots sin\,\theta_{n-4} dr d\varphi d\psi \dots d\theta_{n-3}.$$
The integration bounds are: $r \in [0,1], \varphi \in [0,\pi], \psi \in [0,\pi], \theta_1 \in [0,\pi], \dots, \theta_{n-4} \in [0,\pi], \theta_{n-3} \in [0,2\pi]$.
For the sake of brevity put $d^2\Phi \coloneqq (1-r^2)^\alpha r^{n-1}\,sin^{n-2}\varphi dr d\varphi$.
Integration over all θ_i will give us some constant C since the integrand does not depend on them.
For the rest we have
$$C\int_0^1\int_0^\pi\int_0^\pi\left(\frac{z\cdot x}{|x|} r\cos\varphi + v_{z,x}\frac{r\sin\varphi\cos\psi}{|x|} - z.x\right)^m f(r,|x|r\cos\varphi)d^2\Phi\sin^{n-3}\psi\,d\psi \quad (53)$$
$$= C\sum_{l=0}^{\infty}\binom{m}{l}\int_0^1\int_0^\pi\left(\frac{z\cdot x}{|x|} r\,cos\,\varphi - z.x\right)^{m-l}\left(v_{z,x}\frac{r\,sin\,\varphi}{|x|}\right)^l f(r,|x|r\,cos\,\varphi)d^2\Phi\int_0^\pi \cos^l\psi\,sin^{n-3}\psi\,d\psi.$$
Here and in the rest of the proof we assume $n > 3$ otherwise (in the case $n = 3$) integration over the interval $[0,2\pi]$ would rest with the parameter ψ and in the case $n = 2$ ψ would not be present at all.
These cases would however require only minor changes in the proof which continues as follows.
Let
$$A \coloneqq \frac{z\cdot x}{|x|} r\,cos\,\varphi - z.x, \qquad B \coloneqq v_{z,x}\frac{r\,sin\,\varphi}{|x|}.$$
By an easy computation we have:
$$\int_0^\pi cos^l\psi\,sin^{n-3}\psi\,d\psi = \begin{cases} \dfrac{\sqrt{\pi}\Gamma(b)}{\Gamma\left(b+\frac{1}{2}\right)}\dfrac{\left(\frac{1}{2}\right)_k}{\left(b+\frac{1}{2}\right)_k} & l = 2k, \\ 0 & l \neq 2k. \end{cases}$$
Together with fact that
$$\binom{m}{2k} = \frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!}$$
we obtain that (53) equals
$$C\sum_{k=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k}{\left(b+\frac{1}{2}\right)_k k!}\int_0^1\int_0^\pi A^{m-2k}B^{2k}f(r,|x|r\,cos\,\varphi)d^2\Phi.$$
Now we take hyper-spherical coordinates back. First we had $x.y = |x|y_1 = |x|r\cos\varphi$and $|y| = r$, so our result can be again interpreted as an integral over the unit ball in $\mathbb{R}^n$ if we replace $r\cos\varphi$ by $\frac{x.y}{|x|}$and put $r = |y|$. Therefore
$$A = \frac{z\cdot x}{|x|} r\,cos\,\varphi - z.x = \frac{z.x}{|x|}\frac{x.y}{|x|} - z.x = \frac{z.x(x.y - |x|^2)}{|x|^2},$$
$$B^2 = v_{z,x}^2\frac{r^2\,sin^2\,\varphi}{|x|^2} = v_{z,x}^2\frac{r^2(1-\cos^2\varphi)}{|x|^2} = v_{z,x}^2\frac{\left(|y|^2 - \left(\frac{x.y}{|x|}\right)^2\right)}{|x|^2} = \frac{v_{z,x}^2 v_{y.x}^2}{|x|^4}.$$
Altogether we have

$$C\sum_{k=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k}{\left(b+\frac{1}{2}\right)_k k!\,|x|^{2m}}\int_{\mathbb{B}^n}(z\cdot x(x.y-|x|^2))^{m-2k}v_{z,x}^{2k}v_{y,x}^{2k}f(|y|,x\cdot y)d\mu_\alpha^n(y)$$

The result, if necessary, can be easily checked by performing change of variables into the hyper-spherical coordinates.

Lastly, we must determine the constant C into which we have collected all unimportant constants. But comparing our original integral with the result for $m=0$ gives us the equality

$$\int_{\mathbb{B}^n} f(|y|,x\cdot y)d\mu_\alpha^n(y)=C\int_{\mathbb{B}^n} f(|y|,x\cdot y)d\mu_\alpha^n(y),$$

hence $C=1$.

Corollary (5.2.13) [299]: For $m\in\mathbb{N}_0$

$$\int_{\mathbb{B}^n}(z\cdot y-z\cdot x)^m\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)$$
$$=\sum_{k,j,p=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k(-1)^k(z\cdot x)^{m-2k}v_{z,x}^{2k}2^p}{\left(b+\frac{1}{2}\right)_k(k-j-p)!\,j!\,p!\,|x|^{2(m-p-j)}}\int_{\mathbb{B}^n}(x\cdot y$$
$$-|x|^2)^{m-2j-p}(|x|^2-|y|^2)^j\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y).$$

Proof: The Bergman kernel depends only on $|y|^2|x|^2$ and $x\cdot y-a$ fact easily seen from the representation in terms of the Appell function (40). We can therefore apply Lemma (5.2.12). Notice that the factor $v_{y,x}^{2k}$ can be written as follows

$$v_{y,x}^{2k}=(|y|^2|x|^2-(x\cdot y)^2)^k=(|y|^2|x|^2-(x\cdot y-|x|^2)^2-2|x|^2(x\cdot y-|x|^2)-|x|^4)^k,$$

so we can expand it into a finite combination of terms$(x\cdot y-|x|^2)$, $(|x|^2-|y|^2)$.Specifically,

$$v_{y,x}^{2k}=(-1)^k\sum_{j,p=0}^{\infty}\frac{k!\,2^p|x|^{2(p+j)}}{(k-j-p)!\,j!\,p!}(x\cdot y-|x|^2)^{2k-2j-p}(|x|^2-|y|^2)^j.$$

Substituting this into the expression in Lemma (5.2.12) and performing some manipulations we get the required result.

Corollary (5.2.14) [299]: For $m\in\mathbb{N}_0$ and $x\neq 0$:

$$\int_{\mathbb{B}^n}(z\cdot y-z\cdot x)^m\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)=O\left(\alpha^{-\left\lceil\frac{m}{4}\right\rceil}\right)\quad(\alpha\to\infty).$$

Proof: According to Corollary (5.2.13) and Corollary (5.2.10) we get for $x\neq 0$

$$\int_{\mathbb{B}^n}(z\cdot y-z\cdot x)^m\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)$$
$$\sum_{k,j,p=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k(-1)^k(z\cdot x)^{m-2k}v_{z,x}^{2k}2^p}{\left(b+\frac{1}{2}\right)_k(k-j-p)!\,j!\,p!\,|x|^{2(m-p-j)}}O\left(\alpha^{-\left\lceil\frac{m-j-p}{2}\right\rceil}\right).$$

Since $j+p\le k$ and $2k\le m$ the principal term is at most

$$O\left(\alpha^{-\left\lceil\frac{m-\left\lceil\frac{m}{2}\right\rceil}{2}\right\rceil}\right)=O\left(\alpha^{-\left\lceil\frac{m}{4}\right\rceil}\right).$$

We are now ready to prove the main result. We repeat the statement:

For $x\in\mathbb{B}^n, x\neq 0, n>1$, and $f\in C^\infty(\mathbb{B}^n)$, there exist differential operators $Q_i:=Q_i(\Delta,x,\nabla,|x|^2)$, involving only the Laplace operator Δ, the directional derivative $x\cdot\nabla$ and the quantity $|x|^2$, such that

$$(B_\alpha f)(x) := \int_{\mathbb{B}^n} f(y)\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y) \approx \sum_{i=0}^{\infty}\frac{Q_i f(x)}{\alpha^i} \quad (\alpha \to \infty),$$
where $Q_0 = 1$ and
$$Q_1 = \frac{n-2}{2}\frac{1-|x|^2}{|x|^2}x\cdot\nabla + \frac{(n-2)(1-|x|^2)^2}{4(n-1)|x|^2}(x\cdot\nabla)^2 + \frac{1}{4(n-1)}(1-|x|^2)^2\Delta.$$
Finally, for $x = 0$ it holds
$$(B_\alpha f)(0) \approx \sum_{i=0}^{\infty}\frac{\Delta^i f(0)}{4^i\left(\alpha+\frac{n}{2}+1\right)_i} \quad (\alpha \to \infty).$$
Proof: Let us deal with the simpler case $x = 0$ first on which the general approach will be demonstrated.
The problem is to determine the asymptotic expansion of the integral
$$I_\alpha f := \int_{\mathbb{B}^n} f(y)d\mu_\alpha^n(y).$$
Remember that
$$d\mu_\alpha^n(y) := c_\alpha(1-|y|^2)^\alpha d^n y, \qquad c_\alpha = \frac{\Gamma(\tilde{\alpha})}{\pi^{n\backslash 2}\Gamma(\alpha+1)}.$$
We expand the function $f(y)$ into its Taylor series
$$f(y) = \sum_{k=0}^{2M-1}\frac{(y\cdot\nabla)^k f(0)}{k!} + H_{2M}(y),$$
and plug in to get
$$I_\alpha f = \sum_{k=0}^{M-1}\frac{1}{(2k)!}\int_{\mathbb{B}^n}(y\cdot\nabla)^{2k}f(0)d\mu_\alpha^n(y) + \int_{\mathbb{B}^n}H_{2M}(y)d\mu_\alpha^n(y).$$
Notice that only the terms of even degree in the first integral survived. We can estimate the remainder term $H_{2M}(y)$ by the Taylor theorem as follows
$$|\, H_{2M}(y)| \le C \max_{|\gamma|-2M}\max_{y\in\mathbb{B}^n}|\partial^\gamma f(y)||y^\gamma| \le C|y|^{2M}.$$
So
$$\int_{\mathbb{B}^n} H_{2M}d\mu_\alpha^n(y) \le C\int_{\mathbb{B}^n}|y|^{2M}d\mu_\alpha^n(y) = C\int_{\mathbb{B}^n}(|y|^2-1+1)^M d\mu_\alpha^n(y)$$
$$= C\sum_{k=0}^{M}\binom{M}{k}(-1)^k\int_{\mathbb{B}^n}d\mu_{\alpha+k}^n(y)\frac{c_\alpha}{c_{\alpha+k}}$$
$$C\sum_{k=0}^{M}\binom{M}{k}(-1)^k\frac{(\alpha+1)_k}{(\tilde{\alpha})_k} = C_2F_1\left(\begin{matrix}-M & \alpha+1\\ & \tilde{\alpha}\end{matrix};1\right) = C\frac{\Gamma(\tilde{\alpha})\Gamma(b+M)}{\Gamma(\tilde{\alpha}+M)\Gamma(b)} = C\frac{(b)_M}{(\tilde{\alpha})_M} = O(\alpha^{-M}).$$
This stems again from the identity
$${}_2F_1\left(\begin{matrix}a_1 & a_2\\ & c\end{matrix};1\right) = \frac{\Gamma(c)\Gamma(c-a_1-a_2)}{\Gamma(c-a_1)\Gamma(c-a_2)},$$
which is true for $c > a_1 + a_2$.
It remains to deal with integrals
$$\int_{\mathbb{B}^n}(y\cdot\nabla)^{2k}d\mu_\alpha^n(y)f(0).$$
Now, we argue that the operator ∇can be treated as an ordinary vector, i.e. it suffices to compute the expression
$$\int_{\mathbb{B}^n}(y\cdot z)^{2k}d\mu_\alpha^n(y)$$

and then in the result replace every occurrence of z_j by ∂_j. To conclude the first part of this proof it therefore suffices to show that

$$\int_{\mathbb{B}^n} (y\cdot z)^{2k} d\mu_\alpha^n(y) = \frac{(2k)!}{k!\,4^k(\tilde{\alpha})_k}|z|^{2k}.$$

This could be, of course, computed directly but Corollary (5.2.6) can also be used in the light of the following representation

$$\int_{\mathbb{B}^n} (y\cdot z)^{2k} d\mu_\alpha^n(y) = \lim_{t\to 0} t^{-2k} \int_{\mathbb{B}^n} (y\cdot tz)^{2k} R_\alpha^2(tz,y)d\mu_\alpha^n(y),$$

i.e. apply Corollary (5.2.6) to $x = tz$ then divide by t^{2k} and let $t \to 0$.

In the case $x \neq 0$ the approach is almost identical. First we expand the function f into Taylor series around x

$$f(y) = \sum_{k=0}^{2M-1} \frac{\left((y-x)\cdot\nabla\right)^k f(x)}{k!} + H_{2M,x}(y).$$

We have

$$(B_\alpha f)(x) = \sum_{k=0}^{2M-1} \underbrace{\frac{1}{k!}\left(B_\alpha\left(\left((y-x)\cdot\nabla\right)^k f(x)\right)\right)(x)}_{=:c_k} + \left(B_\alpha H_{2M,x}\right)(x).$$

The remainder term can be estimated using the Taylor theorem:

$$\left|H_{2M,x}(y)\right| \leq C \max_{|\gamma|=2M} \max_{y\in\mathbb{B}^n} |\partial^\gamma f(y)||(y-x)^\gamma| \leq C|y-x|^{2M}$$

for some constant C, whence

$$\left(B_\alpha H_{2M,x}\right)(x) = O\left(B_\alpha(|y-x|^{2m})(x)\right) = O\left(\alpha^{-\left\lceil\frac{M}{2}\right\rceil}\right) \quad (\alpha\to\infty),$$

where the last equality holds by Corollary (5.2.11).

Thus again, we have to deal only with the terms c_k and they can further be modified by replacing ∇ by$z$:

$$c_k := \frac{1}{k!}B_\alpha\left(\left((y-x)\cdot z\right)^k\right)(x).$$

So we must only be able to determine asymptotic behavior of the Berezin transform of a polynomial.

The fact that in this case there exists an asymptotic expansion in negative powers of α follows from Corollary (5.2.6) and Lemma(5.2.12), from where it is clear that terms c_i can be written as finite combinations of functions $\frac{{}_5F_4}{{}_2F_1}$ whose asymptotic expansions are of this type. From that representation it is also possible to see the Stokes phenomenon, since for $x = 0$ the ratio equals 1 but for $0 < |x| < 1$ it decays in a way described in Corollary (5.2.7).

Dependence of differential operators Q_i on $\Delta, x\cdot\nabla$ and $|x|^2$ only (that means on $|z|^2, x\cdot z$ and $|x|^2$) is a direct consequence . That $Q_0 = 1$ stems from the fact that $c_0 = 1$ and c_1 is, according to Corollary(5.2.14) , $O(\alpha^{-1})$.

To compute Q_1 much more work is needed. We are dealing with the expression $c_1 + c_2 + c_3 + c_4$ – the term c_5 is according to Corollary (5.2.14) already $O(\alpha^{-2})$. Application of Corollary (5.2.13) to$c_1 + c_2 + c_3 + c_4$, in general, leaves us with 19 terms. Fortunately many of them are negligible according to Corollary (5.2.10) (those for which $m - j - p > 2$) and collecting expressions involving the same integral will reduce the number to 5 terms:

$$\int_{\mathbb{B}^n} \sum_{m=1}^4 \frac{1}{m!}(z\cdot y - z\cdot x)^m \frac{R_\alpha^2(x,y)}{R_\alpha(x,x)} d\mu_\alpha^n(y) = \qquad (54)$$

$$\left(\frac{x\cdot z}{|x|^2} - \frac{v_{z,x}^2}{(2b+1)|x|^2}\right) \int_{\mathbb{B}^n} (y\cdot x - |x|^2)\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)} d\mu_\alpha^n(y)$$

$$-\frac{v_{z,x}^2}{(2b+1)|x|^2}\int_{\mathbb{B}^n}(|x|^2-|y|^2)\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)$$
$$+\left(\frac{v_{z,x}^2 x\cdot z}{(2b+1)|x|^4}+\frac{v_{z,x}^4}{(2b+1)(2b+3)|x|^8}\right)\int_{\mathbb{B}^n}(y\cdot x-|x|^2)\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)$$
$$+\left(\frac{(x\cdot z)^2}{2|x|^4}-\frac{v_{z,x}^2(1+2z\cdot x)}{2(2b+1)|x|^4}+\frac{v_{z,x}^4}{2(2b+1)(2b+3)|x|^4}\right)\int_{\mathbb{B}^n}(y\cdot x-|x|^2)^2\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)$$
$$+\frac{v_{z,x}^4}{8(2b+1)(2b+3)|x|^4}\int_{\mathbb{B}^n}(|x|^2-|y|^2)^2\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)+O(\alpha^{-2}).$$

Each integral is by Theorem (5.2.8) (if we for a moment put aside the factor $1/R_\alpha(x,x)$) a sum of functions ${}_5F_4$. Numbers of terms in these sums are, in general, again very high (in the case $N=2, M=0$ even as high as 108), however since we are interested only in principal terms and the order of asymptotic decay grows with summation indices $k_1,k_2,k_3,k_4,k_5,r,q,l,$ as can be seen from the proof of Theorem (5.2.8), it is enough to consider only those summands for which

$$k_1+k_2+k_3+k_4+k_5+r+q+l=1.$$

This together with the condition

$$k_1+k_2+k_3+k_4+k_5+2r+q+2l\geq M+N$$

substantially reduces the number of terms. For the above-mentioned case $N=2, M=0$ we will be left with only 2 terms, both of which in addition contain the same hypergeometric function, so they can be combined together. Let us work this case out with more details, so we can demonstrate the approach. From Theorem (5.2.8) we see that when $M=0$ then$l=0$. We substitute $k_1+k_2+k_3+k_4+k_5+r+q=1$ into the inequality to get

$$r+k_4\geq 1,$$

but it also must be the case that $r\leq 1$ and $k_4\leq 1$. This is only possible in two cases: $r=1$ or $k_4=1$ (with all other indices equal to zero). We find

$$\int_{\mathbb{B}^n}(|x|^2-|y|^2)^2R_\alpha^2(x,y)d\mu_\alpha^n(y)=2\frac{|x|^2}{\tilde\alpha}(1-|x|^2)\,{}_3F_2\left(\begin{matrix}\tilde\alpha & \tilde\alpha & 2b\\ \tilde\alpha+1 & b & \end{matrix};|x|^2\right)+O_2.$$

where O_2 denotes a term of order $O\left(\alpha^{-2}\,{}_2F_1\left(\begin{matrix}\tilde\alpha & 2b\\ b & \end{matrix};|x|^2\right)\right)$.

Similar considerations in the other cases give us:

$$\int_{\mathbb{B}^n}(y\cdot x-|x|^2)^2R_\alpha^2(x,y)d\mu_\alpha^n(y)=\frac{1}{2}\frac{|x|^2}{\tilde\alpha}(1-|x|^2)\,{}_3F_2\left(\begin{matrix}\tilde\alpha & \tilde\alpha & 2b\\ \tilde\alpha+1 & b & \end{matrix};|x|^2\right)+O_2,$$

$$\int_{\mathbb{B}^n}(y\cdot x-|x|^2)(|x|^2-|y|^2)R_\alpha^2(x,y)d\mu_\alpha^n(y)=-\frac{|x|^2}{\tilde\alpha}(1-|x|^2)\,{}_3F_2\left(\begin{matrix}\tilde\alpha & \tilde\alpha & 2b\\ \tilde\alpha+1 & b & \end{matrix};|x|^2\right)+O_2,$$

$$\int_{\mathbb{B}^n}(y\cdot x-|x|^2)R_\alpha^2(x,y)d\mu_\alpha^n(y)=|x|^2\,{}_2F_1\left(\begin{matrix}\tilde\alpha & 2b\\ b+1 & \end{matrix};|x|^2\right)+O_2,$$

$$\int_{\mathbb{B}^n}(|x|^2-|y|^2)R_\alpha^2(x,y)d\mu_\alpha^n(y)=\frac{|x|^2}{2}\,{}_3F_2\left(\begin{matrix}\tilde\alpha & 2b & 2b\\ & b & 2b+1\end{matrix};|x|^2\right)-\frac{b+1}{\tilde\alpha}(1-|x|^2)$$
$${}_2F_1\left(\begin{matrix}\tilde\alpha & 2b\\ b & \end{matrix};|x|^2\right)-2|x|^2\,{}_2F_1\left(\begin{matrix}\tilde\alpha & 2b\\ b+1 & \end{matrix};|x|^2\right)$$
$$+\frac{|x|^2}{\tilde\alpha}\,{}_3F_2\left(\begin{matrix}\tilde\alpha & \tilde\alpha & 2b\\ \tilde\alpha+1 & b & \end{matrix};|x|^2\right)+O_2.$$

Substituting this into (54) and performing some manipulations we obtain

$$\int_{\mathbb{B}^n}\sum_{m=1}^{4}\frac{1}{m!}(y\cdot z-x\cdot z)^m\frac{R_\alpha^2(x,y)}{R_\alpha(x,x)}d\mu_\alpha^n(y)=\frac{1}{2}\frac{v_{x,x}^2(b+1)(1-|x|^2)}{(2b+1)|x|^2\tilde\alpha}+z\cdot x\frac{{}_2F_1\left(\begin{matrix}\tilde\alpha & 2b\\ b+1 & \end{matrix};|x|^2\right)}{{}_2F_1\left(\begin{matrix}\tilde\alpha & 2b\\ b & \end{matrix};|x|^2\right)}$$

$$+\frac{1}{4}\frac{2(z\cdot x)^2\big(1+b(1-|x|^2)\big)-|z|^2|x|^2(1+|x|^2)}{|x|^2(2b+1)\tilde{\alpha}}\frac{{}_2F_1\left(\begin{matrix}\tilde{\alpha} & \tilde{\alpha} & 2b \\ \tilde{\alpha}+1 & & b\end{matrix};|x|^2\right)}{{}_2F_1\left(\begin{matrix}\tilde{\alpha} & 2b \\ & b\end{matrix};|x|^2\right)}$$

$$-\frac{1}{4}\frac{v_{z,x}^2}{2b+1}\frac{{}_3F_2\left(\begin{matrix}\tilde{\alpha} & 2b & 2b \\ & b & 2b+1\end{matrix};|x|^2\right)}{{}_2F_1\left(\begin{matrix}\tilde{\alpha} & 2b \\ & b\end{matrix};|x|^2\right)}+O(\alpha^{-2}).$$

We complete the proof by using Corollary (5.2.7), replacing $v_{x,x}^2=|z|^2|x|^2-(z\cdot x)^2$ and $z_j\to\partial_j$and remembering that $b=\frac{n}{2}-1$.

Corollary (5.2.15) [349]. Let $a_1+\varepsilon, a_2+\varepsilon, a_3+\varepsilon>0$ be positive real numbers, one of them strictly less than the other two. Let $\varepsilon-1-a-\gamma\notin\mathbb{Z}, -(a+2\varepsilon)_i\notin\mathbb{N}_0$ and $x_r\in(0,1)$. Then we have

$${}_5F_4\left(\begin{matrix}\varepsilon-1 & a_1+\varepsilon & a_2+\varepsilon & a_3+\varepsilon \\ \varepsilon-1+a & a_1+2\varepsilon & a_2+2\varepsilon & a_3+2\varepsilon\end{matrix};x_r\right)$$
$$\approx\prod_{i=1}^{3}\frac{\Gamma((a+2\varepsilon)_i)}{\Gamma((a+\varepsilon)_i)}\frac{(\varepsilon-1x_r)^{-\gamma}}{(1-x_r)^{\varepsilon-1-\gamma-a}}\left(1+\sum_{k=1}^{\infty}\frac{d_k}{\varepsilon-1^k}\right)\ (\varepsilon\to+\infty)$$

where $\gamma=\sum_{j=1}^{3}((a+2\varepsilon)_j-(a+\varepsilon)_j)$ and d_k are constants independent of $\varepsilon-1$.

Proof. Using the integral representation

$${}_{p+1}F_{q+1}\left(\begin{matrix}a_1+\varepsilon\ldots a_p+\varepsilon & a+\varepsilon \\ a_1+2\varepsilon\ldots a_q+2\varepsilon & a+2\varepsilon\end{matrix};x_r\right)$$
$$=\frac{\Gamma(a+2\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(\varepsilon)}\int_0^1 t^{a+\varepsilon-1}(1-t)^{\varepsilon-1}\,{}_pF_q\left(\begin{matrix}a_1+\varepsilon\ldots a_p+\varepsilon \\ a_1+2\varepsilon\ldots a_q+2\varepsilon\end{matrix};x_rt\right)dt,$$

which is valid for $a+2\varepsilon>a+\varepsilon>0$, in turn three times on pairs of parameters $(a_1+2\varepsilon,a_1+\varepsilon),(a_2+2\varepsilon,a_2+\varepsilon),(a_3+2\varepsilon,a_3+\varepsilon)$,we get

$${}_5F_4\left(\begin{matrix}\varepsilon-1 & \varepsilon-1 & a_1+\varepsilon & a_2+\varepsilon & a_3+\varepsilon \\ & \varepsilon-1+a & a_1+2\varepsilon & a_2+2\varepsilon & a_3+2\varepsilon\end{matrix};x_r\right)$$
$$=\prod_{i=1}^{3}\frac{\Gamma((a+2\varepsilon)_i)}{\Gamma((a+\varepsilon)_i)\Gamma((a+2\varepsilon)_i-(a+\varepsilon)_i)}\int_0^1\int_0^1\int_0^1\prod_{i=1}^{3}t_i^{(a+\varepsilon)_i-1}(1$$
$$-t_i)^{(a+2\varepsilon)_i-(a+\varepsilon)_i-1}\,{}_2F_1\left(\begin{matrix}\varepsilon-1 & \varepsilon-1 \\ & \varepsilon-1+a\end{matrix};x_rt_1t_2t_3\right)dt_1dt_2dt_3.$$

Double application of the transformation (5) gives us the Euler transform

$${}_2F_1\left(\begin{matrix}a & a+\varepsilon \\ & a+2\varepsilon\end{matrix};x_r\right)=(1-x_r)^{a+3\varepsilon}\,{}_2F_1\left(\begin{matrix}2\varepsilon & 3\varepsilon \\ & a+2\varepsilon\end{matrix};x_r\right),$$

which leaves us with

$$\prod_{i=1}^{3}\frac{\Gamma((a+2\varepsilon)_i)}{\Gamma((a+\varepsilon)_i)\Gamma((a+2\varepsilon)_i-(a+\varepsilon)_i)}\int_0^1\int_0^1\int_0^1\prod_{i=1}^{3}t_i^{(a+\varepsilon)_i-1}(1-t_i)^{(a+2\varepsilon)_i-(a+\varepsilon)_i-1}(1$$
$$-x_rt_1t_2t_3)^{a-\varepsilon-1}\,{}_2F_1\left(\begin{matrix}a & a \\ \varepsilon-1+a & \end{matrix};x_rt_1t_2t_3\right)dt_1dt_2dt_3.$$

A triple integral of this kind can be rearranged in the following way:

$$\int_0^1\int_0^1\int_0^1 \varphi(t_1,t_2,t_3)G(t_1t_2t_3)dt_1dt_2dt_3$$
$$= \int_0^1 G(1-r_3)\int_0^1\int_0^1 \varphi\left(1-r_1r_2r_3, \frac{1-r_2r_3}{1-r_1r_2r_3}, \frac{1-r_3}{1-r_2r_3}\right)\frac{r_2r_3^2}{(1-r_1r_2r_3)(1-r_2r_3)}dr_1dr_2dr_3.$$

(This is nothing more than a series of changes of variables. Firstly, let $s_1 = t_1, s_2 = t_1t_2, s_3 = t_1t_2t_3$.
Jacobian is $\frac{1}{s_1s_2}$ and the integral becomes:

$$\int_0^1\int_0^{s_1}\int_0^{s_2} \varphi\left(s_1, \frac{s_2}{s_1}, \frac{s_3}{s_2}\right)G(s_3)\frac{1}{s_1s_2}ds_3ds_2ds_1.$$

Now we swap the order of integration:

$$\int_0^1\int_0^{s_1}\int_0^{s_2} \varphi\left(s_1, \frac{s_2}{s_1}, \frac{s_3}{s_2}\right)G(s_3)\frac{1}{s_1s_2}ds_3ds_2ds_1 = \int_0^1\int_{s_3}^1\int_{s_2}^1 \varphi\left(s_1, \frac{s_2}{s_1}, \frac{s_3}{s_2}\right)G(s_3)\frac{1}{s_1s_2}ds_1ds_2ds_3$$

and finally three changes of variable are performed: firstly $1-s_1 = r_1(1-s_2)$, then $1-s_2 = r_2(1-s_3)$ and lastly $1-s_3 = r_3$.)
Applying this to our original triple integral we get:

$$\int_0^1 (1-x_r+x_rr_3)^{a-\varepsilon-1}\ {}_2F_1\left(\begin{matrix} a \quad a \\ \varepsilon-1+a\end{matrix}; x_r(1-r_3)\right)$$

$$\int_0^1\int_0^1 \prod_{i=1}^{3} r_i^{\gamma_i-1}(1-r_1)^{3\varepsilon-1}(1-r_2)^{3\varepsilon-1}(1-r_3)^{a_3+\varepsilon-1}(1-r_2r_3)^{a_2+3\varepsilon-a_3}(1-r_1r_2r_3)^{a_1+3\varepsilon-a_2}dr_1dr_2dr_3$$

where $\gamma_i = \sum_{k=1}^{i}(a_k+2\varepsilon-(a_k+\varepsilon))$.
After a small manipulation this gives

$$\int_0^1\int_0^1\int_0^1 \prod_{i=1}^{3} t_i^{(a+\varepsilon)_i-1}(1-t_i)^{(a+2\varepsilon)_i-(a+\varepsilon)_i-1}(1-x_rt_1t_2t_3)^{a-\varepsilon-1}{}_2F_1\left(\begin{matrix} a \quad a \\ \varepsilon-1+a\end{matrix}; x_rt_1t_2t_3\right)dt_1dt_2dt_3$$
$$= (1-x_r)^{a-\varepsilon-1}\int_0^1 t^{\gamma_3-1}(1-t)^{a_3+\varepsilon-1}F(t)\,{}_2F_1\left(\begin{matrix} a \quad a \\ \varepsilon-1+a\end{matrix}; x_r(1-t)\right)\left(1-\frac{x_r}{x_r-1}t\right)^{a-\varepsilon-1}dt, (10)$$

where

$$F(t) = \int_0^1 r_2^{\gamma_2-1}(1-r_2)^{3\varepsilon-1}(1-r_2t)^{a_2+3\varepsilon-a_3}\int_0^1 r_1^{\gamma_1-1}(1-r_1)^{3\varepsilon-1}(1-r_1r_2t)^{a_1+3\varepsilon-a_2}dr_1dr_2$$
$$= \frac{\Gamma(\gamma_1)\Gamma(\varepsilon)}{\Gamma(\gamma_2)}\int_0^1 r_2^{\gamma_2-1}(1-r_2)^{\varepsilon-1}(1-r_2t)^{a_2+\varepsilon-a_3}\ {}_2F_1\left(\begin{matrix} a_2+\varepsilon-a_1 \quad \gamma_1 \\ \gamma_2\end{matrix}; r_2t\right)dr_2.$$

We expand the hypergeometric function into Taylor series to get the form

$$\mathrm{F(t)} = \frac{3\Gamma(\varepsilon)}{\Gamma(\gamma_3)}\sum_{j=0}^{\infty}\frac{(a_2+\varepsilon-a_1)_j(\gamma_1)_j}{(\gamma_3)_j}\frac{t^j}{j!}\,{}_2F_1\left(\begin{matrix}\gamma_2+j & a_3+\varepsilon-a_2\\ & \gamma_3+j\end{matrix};t\right).$$

We should talk about the convergence of the integral on the right hand side of the equation (10). For that it is necessary to understand the behavior of the function $F(t)$ at the end points of the interval of integration, notably in the neighborhood of $t=1$ (the behavior near $t=0$ is evident). It is well known that

$${}_2F_1\left(\begin{matrix}a & a+\varepsilon\\ & a+2\varepsilon\end{matrix};1\right) = \frac{\Gamma(a+2\varepsilon)\Gamma(a+\varepsilon)}{\Gamma(\varepsilon)\Gamma(2\varepsilon)}$$

for $\varepsilon > a$. That means that the hypergeometric function in the infinite series is left-continuous at $t=1$ if $\gamma_3+j > \gamma_2+j+a_3+3\varepsilon-a_2)$ or equivalently $a_2 > a_3$. In such case the infinite series for $t=1$ takes the form:

$$\frac{\Gamma(\gamma_3)\Gamma(a_2+2\varepsilon-a_3)}{\Gamma(a_2+3\varepsilon-a_3)}\sum_{j=0}^{\infty}\frac{(a_2+\varepsilon-a_1)_j(\gamma_1)_j}{(a_2+3\varepsilon+a_3)_j j!},$$

which is a convergent series for $\varepsilon - a_3 > +a_1+\gamma_1$, i.e. $a_1 > a_3$. (Indeed, the series is actually equal to

$${}_2F_1\left(\begin{matrix}a_2+\varepsilon-a_1 & \gamma_1\\ a_2+2\varepsilon-a_3 & \end{matrix};1\right)$$

and the formula above can be used.)
This can be summarized by saying

$$F(t) = O(1) \qquad (t \nearrow 1),$$

which holds for $a_1 > a_3, a_2 > a_3$ and the integral on the right hand side of the equation (10) converges under the conditions $a_3 > 0, a_1 > a_3, a_2 > a_3, \gamma_3 > 0$. Those are significantly less restraining conditions than in the triple integral on the left hand side of the same equation, which converges for $(a+2\varepsilon)_i > (a+\varepsilon)_i > 0\forall i$.
It is an example, therefore, of an analytic continuation. Furthermore, since hypergeometric functions are symmetric with respect to permutation of the parameters $(a+\varepsilon)_i$, we can choose $a_3+\varepsilon$ to be the smallest one.
We can summarize now that for $x_r < 1, \gamma_3 > 0, a_1+\varepsilon > a_3+\varepsilon > 0, a_2 > a_3$ it holds

$${}_5F_4\left(\begin{matrix}\varepsilon-1 & \varepsilon-1 & a_1+\varepsilon & a_2+\varepsilon & a_3+\varepsilon\\ & \varepsilon-1+a & a_1+2\varepsilon & a_2+2\varepsilon & a_3+2\varepsilon\end{matrix};x_r\right)$$
$$= \frac{(1-x_r)^{a-\varepsilon-1}}{\Gamma(\gamma_3)}\prod_{i=1}^{3}\frac{\Gamma((a+2\varepsilon)_i)}{\Gamma((a+\varepsilon)_i)}\int_0^1 t^{\gamma_3-1}(1-t)^{a_3+\varepsilon-1}F_{\varepsilon-1}(t)\left(1 - \frac{x_r}{x_r-1}t\right)^{a-\varepsilon-1}dt,$$

where

$$F_{\varepsilon-1}(t) = {}_2F_1\left(\begin{matrix}a & a\\ \varepsilon-1+a & \end{matrix};x_r(1-t)\right)\sum_{j=0}^{\infty}\frac{(a_2+\varepsilon-a_1)_j(\gamma_1)_j}{(\gamma_3)_j}\frac{t^j}{j!}\,{}_2F_1\left(\begin{matrix}\gamma_2+j & a_3+\varepsilon-a_2)\\ & \gamma_3+j\end{matrix};t\right),$$

and $\gamma_i = \sum_{j=1}^{i}(a+2\varepsilon)_j - (a+\varepsilon)_j$. As a next step we replace the function $F_{\varepsilon+1}(t)$ by its Taylor series expansion:

$$F_{\varepsilon-1}(t) = \sum_{k=0}^{N-1}\frac{F_{\varepsilon-1}^{(k)}(0)}{k!}t^k + \frac{F_{\varepsilon-1}^{(N)}(\xi)}{N!}t^N,$$

where $0 < \xi < t$.
Substituting this we get

$${}_5F_4\left(\begin{matrix}\varepsilon-1 & \varepsilon-1 & a_1+\varepsilon & a_2+\varepsilon & a_3+\varepsilon\\ & \varepsilon-1+a & a_1+2\varepsilon & a_2+2\varepsilon & a_3+2\varepsilon\end{matrix};x_r\right)$$

$$= \frac{(1-x_r)^{a-\varepsilon-1}}{\Gamma(\gamma_3)} \prod_{i=1}^{3} \frac{\Gamma((a+2\varepsilon)_i)}{\Gamma((a+\varepsilon)_i)} \sum_{k=0}^{N-1} \frac{F_{\varepsilon-1}^{(k)}(0)}{k!} \frac{\Gamma(\gamma_3+k)\Gamma(a_3+\varepsilon)}{\Gamma(\gamma_3+a_3+\varepsilon+k)} {}_2F_1\left(\begin{matrix}\varepsilon-1-a & \gamma_3+k \\ \gamma_3+a_3+\varepsilon+k & \end{matrix}; \frac{x_r}{x_r-1}\right)$$

$$+ \frac{(1-x_r)^{a-\varepsilon-1}}{\Gamma(\gamma_3)} \prod_{i=1}^{3} \frac{\Gamma((a+2\varepsilon)_i)}{\Gamma((a+\varepsilon)_i)} \int_0^1 t^{\gamma_3+N-1}(1-t)^{a_3+\varepsilon-1} \frac{F_{\varepsilon-1}^{(N)}(\xi)}{N!} \left(1-\frac{x_r}{x_r-1}t\right)^{a-\varepsilon-1} dt.$$

Notice that the term $F_{\varepsilon-1}^{(N)}(\xi)$ in the above line still remains $O(1)$ for $t \nearrow 1$ because it can be written as

$$F_{\varepsilon-1}^{(N)}(\xi) = N!\, t^{-N} \left(F_{\varepsilon-1}(t) - \sum_{k=0}^{N-1} \frac{F_{\varepsilon-1}^{(k)}(0)}{k!} t^k\right)$$

Therefore, the integral in the same line converges under conditions $\gamma_3 + N > 0$ and $a_3 + \varepsilon > 0$. The first of these is fulfilled for sufficiently large N, hence for the right hand side of the equation to be meaningful it is only required that $a_3 + \varepsilon > 0$. This is the largest analytic continuation as we can get.

From the form of the remainder term (12) we also see that $F^{(N)}(\xi)$ is a continuous function on [0, 1].

We can therefore estimate it by its maximum on this interval, which will in general depend on $\varepsilon - 1$, but the asymptotic behavior of $F_{\varepsilon-1}^{(N)}(\xi)$ for $\varepsilon \to \infty$ is $F_{\varepsilon-1}^{(N)}(\xi) = O(1)$ uniformly for all $t \in [0,1]$. (This can be seen again from the form of the remainder term (12) – the ${}_2F_1$ in the $F_{\varepsilon-1}$ which contains the parameter $\varepsilon + 1$ has this uniform behavior due to the (3) and terms $F_{\varepsilon-1}^{(k)}(0)$ are just some linear combinations of the same ${}_2F_1$ function, only possibly with parameters shifted due to the differentiations. In such case even additional negative powers of $\varepsilon - 1$ appear.)

Hence

$$\left|\int_0^1 t^{\gamma_3+N-1}(1-t)^{a_3+\varepsilon-1} \frac{F^{(N)}(\xi)}{N!} \left(1-\frac{x_r}{x_r-1}t\right)^{a-\varepsilon+1} dt\right|$$

$$\leq C \int_0^1 t^{\gamma_3+N-1}(1-t)^{a_3+\varepsilon-1} \left(1-\frac{x_r}{x_r-1}t\right)^{a-2\varepsilon-1} dt$$

$$= O\left({}_2F_1\left(\begin{matrix}\varepsilon-1-a & \gamma_3+k \\ \gamma_3+a_3+\varepsilon+k & \end{matrix}; \frac{x_r}{x_r-1}\right)\right) \quad (\varepsilon \to \infty).$$

The problem of finding an asymptotic expansion of the function ${}_5F_4$ for large$\varepsilon + 1$ is now effectively reduced to the problem of finding an expansion for the functions of the form:

$$(1-x_r)^{a-\varepsilon-1} {}_2F_1\left(\begin{matrix}\varepsilon-1-a & \gamma_3+k \\ \gamma_3+a_3+\varepsilon+k & \end{matrix}; \frac{x_r}{x_r-1}\right).$$

The large parameter cases for${}_2F_1$ function has been studied by several authors (see [15, 8]). The logic goes as follows: Combining the transformations (5) and (6), we can see that

$${}_2F_1\left(\begin{matrix}a & a+\varepsilon \\ a+2\varepsilon & \end{matrix}; x_r\right) = \frac{\Gamma(a+2\varepsilon)\Gamma(\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(2\varepsilon)} (-x_r)^{-2\varepsilon}(1-x_r)^{\varepsilon-a} {}_2F_1\left(\begin{matrix}1-(a+\varepsilon) & \varepsilon \\ 1+\varepsilon & \end{matrix}; \frac{1}{x_r}\right)$$

$$+ \frac{\Gamma(a+2\varepsilon)}{\Gamma(a)} (-x_r)^{-(a+\varepsilon)} {}_2F_1\left(\begin{matrix}a+\varepsilon & 1-\varepsilon) \\ 1+\varepsilon & \end{matrix}; \frac{1}{x_r}\right),$$

for $x_r < 0, \varepsilon \notin \mathbb{Z}$. Applying this we get

$$(1-x_r)^{a-\varepsilon-1} {}_2F_1\left(\begin{matrix}\varepsilon-1-a & \gamma_3+k \\ \gamma_3+a_3+\varepsilon+k & \end{matrix}; \frac{x_r}{x_r-1}\right)$$

$$= \frac{\Gamma(\gamma_3+a_3+\varepsilon+k)\Gamma(\gamma_3+k-\varepsilon-1+a)}{\Gamma(\gamma_3+k)\Gamma(\gamma_3+a_3+k+1+a)} {x_r}^{-(a_3+\varepsilon)} {}_2F_1\left(\begin{matrix}1-\gamma_3-k & a_3+\varepsilon \\ \varepsilon-a-\gamma_3-k & \end{matrix}; 1-\frac{1}{x_r}\right)$$

$$+ \frac{\Gamma(\gamma_3+a_3+\varepsilon+k)\Gamma(\varepsilon-1-a-\gamma_3-k)}{\Gamma(\varepsilon-1-a)\Gamma(a_3+\varepsilon)} \frac{{x_r}^{-\gamma_3-k}}{(1-x)^{\varepsilon-1-a-\gamma_3-k}} {}_2F_1\left(\begin{matrix}\gamma_3+k & 1-a_3+\varepsilon \\ \gamma_3+k-\varepsilon+a & \end{matrix}; 1-\frac{1}{x_r}\right),$$

for $\varepsilon+1-a-\gamma_3 \notin \mathbb{Z}$ (we can always run the limit through such $\varepsilon-1$ so that this condition is fulfilled). But in the light of (3) and (4) we can deduce that the first term is negligible with respect to the second term, because it displays only polynomial growth and does not contain the exponentially large term$(1-x_r)^{-\varepsilon+1}$ (remember $x_r \in (0,1)$). Asymptotic behavior is, therefore, dictated by the second term, which is$O\left((1-x_r)^{-(\varepsilon+1)}\varepsilon+1^{-\gamma_3-k}\right)$ as $\varepsilon \to \infty$.
Substituting into the equation (11) we get:

$$_5F_4\left(\begin{matrix}\varepsilon-1 & a_1+\varepsilon & a_2+\varepsilon & a_3+\varepsilon \\ \varepsilon-1+a & a_1+2\varepsilon & a_2+2\varepsilon & a_3+2\varepsilon\end{matrix};x_r\right)$$

$$=\prod_{i=1}^{3}\frac{\Gamma(a_i+2\varepsilon)}{\Gamma(a_i+\varepsilon)}\sum_{k=0}^{N-1}\frac{F_{\varepsilon+1}^{(k)}(0)}{k!}\frac{(\gamma_3)_k\Gamma(\varepsilon-1-a-\gamma_3-k)x_r{}^{-\gamma_3-k}}{\Gamma(\alpha-a)(1-x)^{\alpha-a-\gamma_3-k}}\,{}_2F_1\left(\begin{matrix}\gamma_3+k & 1-(a_3+\varepsilon) \\ 1+\gamma_3+k-\alpha+a\end{matrix};1\right.$$

$$\left.-\frac{1}{x_r}\right)+O((1-x_r)^{-\varepsilon-1}(\varepsilon-1)^{-\gamma_3-N}).$$

It only remains to combine known asymptotic expansions of terms:

$$\frac{\Gamma(\varepsilon-1-a-\gamma_3-k)}{\Gamma(\varepsilon-1-a)}\approx(\varepsilon-1)^{-\gamma_3-k}\left(1+\frac{a_1+2\varepsilon}{\varepsilon-1}+\cdots\right)$$

$$_2F_1\left(\begin{matrix}\gamma_3+k & 1-a_3+\varepsilon \\ \gamma_3+k-\varepsilon+a\end{matrix};1-\frac{1}{x_r}\right)\approx 1+\frac{(\gamma_3+k)(1-a_3+\varepsilon)}{\gamma_3+k-\varepsilon+a}\frac{x_r-1}{x_r}+\cdots$$

$$F_{\varepsilon-1}^{(k)}(0)\approx F^{(k)}(0)+\frac{d_1}{\varepsilon-1}+\cdots,$$

where

$$F(t)=\sum_{j=0}^{\infty}\frac{(a_2+\varepsilon-a_1))_j(\gamma_1)_j}{(\gamma_3)_j}\frac{t^j}{j!}\,{}_2F_1\left(\begin{matrix}\gamma_2+j & a_3-a_2 \\ & \gamma_3+j\end{matrix};t\right),$$

and rearrange the terms.

Corollary (5.2.16) [349]: For$f \in C^1\left(\overline{\mathbb{B}^{1+\varepsilon}}\right)$:

$$\int_{\mathbb{B}^{1+\varepsilon}} z_r\cdot y_r f(y)d\mu_\gamma^{1+\varepsilon}(y_r)=\frac{1}{2\tilde{\gamma}}\int_{\mathbb{B}^{1+\varepsilon}} z_r\cdot\nabla f(y_r)d\mu_{\gamma+1}^{1+\varepsilon}(y_r)$$

Proof. By the Stokes theorem,

$$\int_{\mathbb{B}^{1+\varepsilon}} z_r\cdot\nabla f(y_r)(1-|y_r|^2)^{\gamma+1}(a+2\varepsilon)_{\gamma+1}d^{1+\varepsilon}y_r$$

$$=\int_{\partial\mathbb{B}^{1+\varepsilon}} f(y_r)\underbrace{(1-|y_r|^2)^{\gamma+1}}_{=0}(a+2\varepsilon)_{\gamma+1}z_r\cdot y_r d\sigma(y_r)$$

$$+\frac{2((a+2\varepsilon)_{\gamma+1})(\gamma+1)}{(a+2\varepsilon)_\gamma}\int_{\mathbb{B}^{1+\varepsilon}} z_r\cdot y_r f(y_r)(1-|y_r|^2)^{\gamma}(a+2\varepsilon)_\gamma d^{1+\varepsilon}y_r$$

$$=2\tilde{\gamma}\int_{\mathbb{B}^{1+\varepsilon}} z_r\cdot y_r f(y_r)d\mu_\gamma^{1+\varepsilon}(y_r),$$

since $(a+2\varepsilon)_\gamma=\frac{\Gamma(\tilde{\gamma})}{\pi^{\frac{1+\varepsilon}{2}}\Gamma(\gamma+1)}$.

Corollary(5.2.17)[349]:

$$x_r\cdot\nabla_{y_r}R_{\varepsilon-1}(x_r,y_r)=\frac{\widetilde{\varepsilon-1}|x_r|^2}{a+\varepsilon}\left(2a+2\varepsilon+x_r\cdot\nabla_{x_r}\right)_1(R_\varepsilon)_1\left(\begin{matrix}a+\varepsilon \\ a+\varepsilon+1\end{matrix};x_r,y_r\right).$$

Proof. Recall that

$$R_{\varepsilon-1}(x_r,y_r)=F_1\left(\begin{matrix}(\widetilde{\varepsilon-1}); & a+\varepsilon & a+\varepsilon \\ a+\varepsilon & - & \end{matrix};z_r,\overline{z_r}\right),$$

Where $z_r=x_r\cdot y_r+i\sqrt{|x_r|^2|y_r|^2-(x_r\cdot y_r)^2}$. From that we can see

$$ {}_1(R_\varepsilon)_1\left(\begin{matrix} a+\varepsilon \\ a+\varepsilon+1 \end{matrix}; x_r, y_r\right) = F_1\left(\begin{matrix}\widetilde{\varepsilon-1}+1; a+\varepsilon \quad a+\varepsilon \\ a+\varepsilon+1 \quad\quad - \end{matrix}; z_r, \overline{z_r}\right). $$

Next,

$$ x_r \cdot \nabla_{y_r} z_r = x_r \cdot \nabla_{y_r} \tilde{z}_r = |x_r|^2, \qquad x_r \cdot \nabla_{x_r}(z_r)^j = j(z_r)^j, \qquad x_r \cdot \nabla_{x_r} \overline{z_r}^k = k\overline{z_r}^k, $$

and

$$ x_r \cdot \nabla_{y_r} F_1\left(\begin{matrix}(\widetilde{\varepsilon-1}); a+\varepsilon \;\; a+\varepsilon \\ a+\varepsilon \quad\quad - \end{matrix}; z_r, \overline{z_r}\right) $$

$$ = |x_r|^2 2\widetilde{\varepsilon-1}\left(F_1\left(\begin{matrix}(\widetilde{\varepsilon-1})+1; a+\varepsilon+1 \quad a+\varepsilon \\ a+\varepsilon+1 \quad\quad - \end{matrix}; z_r, \overline{z_r}\right)\right. $$

$$ \left. + F_1\left(\begin{matrix}(\widetilde{\varepsilon-1})+1; a+\varepsilon \quad a+\varepsilon+1 \\ a+\varepsilon+1 \quad\quad - \end{matrix}; z_r, \overline{z_r}\right)\right) $$

$$ = (\widetilde{\varepsilon-1})|x_r|^2 \sum_{j,k=0}^{\infty} \frac{\left((\widetilde{\varepsilon-1})+1\right)_{j+k}}{(a+\varepsilon+1)_{j+k}} \frac{z_r{}^j \overline{z_r}^k}{j!\,k!}\left((a+\varepsilon+1)_j(a+\varepsilon)_k + (a+\varepsilon)_j(a+\varepsilon+1)_k\right) $$

$$ = \frac{(\widetilde{\varepsilon-1})|x_r|^2}{a+\varepsilon} \sum_{j,k=0}^{\infty} \frac{(\varepsilon)_{j+k}}{(a+\varepsilon+1)_{j+k}} \frac{z_r{}^j \overline{z_r}^k}{j!\,k!}(a+\varepsilon)_j(a+\varepsilon)_k(2a+2\varepsilon+j+k) $$

$$ = \frac{(\widetilde{\varepsilon-1})|x_r|^2}{a+\varepsilon}\left(2a+2\varepsilon+x_r\nabla_{x_r}\right)F_1\left(\begin{matrix}(\widetilde{\varepsilon-1})+1; a+\varepsilon \quad a+\varepsilon \\ a+\varepsilon+1 \quad\quad - \end{matrix}; z_r, \overline{z_r}\right). $$

Bold the proof will be done by induction on p.

$p = 0$. For $\varepsilon > 0$, $((\widetilde{\varepsilon-1})$ is always positive from the assumption $\varepsilon > 0)$ we get:

$$ R_{\varepsilon-1}(x_r, y_r) = {}_1(R_{2\varepsilon-1})_1\left(\begin{matrix}(\widetilde{\varepsilon-1}) \\ \widetilde{2\varepsilon-1}\end{matrix}; x_r, y_r\right) $$

$$ = \frac{\Gamma(\widetilde{2\varepsilon-1})}{\Gamma(\widetilde{\varepsilon-1})\Gamma(\widetilde{3\varepsilon}-3)} \int_0^1 t^{\widetilde{\varepsilon-1}-1}(1-t)^{\tilde{\beta}-\widetilde{\varepsilon-1}-1} R_{2\varepsilon-1}(tx_r, y_r)dt. $$

We substitute this into the integral and swap the order of integration:

$$ \int_{\mathbb{B}^{1+\varepsilon}} f(y_r)R_{\varepsilon-1}(x_r, y_r)d\mu_{2\varepsilon-1}^{1+\varepsilon}(y_r) $$

$$ = \frac{\Gamma(\widetilde{2\varepsilon-1})}{\Gamma(\widetilde{\varepsilon-1})\Gamma(3\varepsilon-1)} \int_0^1 t^{\tilde{\varepsilon}}(1-t)^{(\widetilde{\varepsilon-1})} \int_{\mathbb{B}^{1+\varepsilon}} f(y_r)R_{2\varepsilon-1}(tx_r, y_r)d\mu_{2\varepsilon-1}^{1+\varepsilon}(y_r)dt $$

$$ = \frac{\Gamma(\widetilde{2\varepsilon-1})}{\Gamma(\widetilde{\varepsilon-1})\Gamma(\widetilde{3\varepsilon-3})} \int_0^1 t^{2\varepsilon-2}(1-t)^{\varepsilon-2} f(tx_r)dt = {}_1f_1\left(\begin{matrix}\widetilde{\varepsilon-1} \\ \widetilde{2\varepsilon-1}\end{matrix}; x_r\right). $$

When $\varepsilon = 0$, this is just the reproducing property of the Bergman kernel.

$p = p + 1$. We can see that the function $g(y_r) := R_{(\varepsilon-1)}(x_r, y_r)f(y_r)(x_r \cdot y_r)^p$ meets the condition of Lemma 2, hence:

$$ \int_{\mathbb{B}^{1+\varepsilon}} R_{(\varepsilon-1)}(x_r, y_r)f(y_r)(x_r \cdot y_r)^{p+1}\, d\mu_{2\varepsilon-1}^{1+\varepsilon}(y_r) $$

$$ = \frac{1}{2\tilde{\beta}} \int_{\mathbb{B}^{1+\varepsilon}} x_r \cdot \nabla_{y_r}\left(R_{(\varepsilon-1)}(x_r, y_r)f(y_r)(x_r \cdot y_r)^p\right)d\mu_{2\varepsilon}^{1+\varepsilon}(y_r), $$

which divides the proof into three parts:

$$ = \frac{1}{4\varepsilon-2} \int_{\mathbb{B}^{1+\varepsilon}} \left(x_r \cdot \nabla_{y_r} R_{(\varepsilon-1)}(x_r, y_r)\right) f(y_r)(x_r \cdot y_r)^p d\mu_{2\varepsilon}^{1+\varepsilon}(y_r) $$

$$ + \frac{1}{4\widetilde{\varepsilon-2}} \int_{\mathbb{B}^{1+\varepsilon}} R_{(\varepsilon-1)}(x_r, y_r)\left(x_r \cdot \nabla_{y_r} f(y_r)\right)(x_r \cdot y_r)^p d\mu_{2\varepsilon}^{1+\varepsilon}(y_r) $$

$$+\frac{1}{4\widetilde{\varepsilon-2}}\int_{\mathbb{B}^{1+\varepsilon}} R_{(\varepsilon-1)}(x_r,y_r)f(y_r)p(x_r\cdot y_r)^{p-1}|x_r|^2 d\mu_{2\varepsilon}^{1+\varepsilon}(y_r)$$

Notice that for a general series of the form

$$\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{A(j,l,m)}{j!\,l!\,m!}$$

the transition $p\to p+1$ also divides this series into the three parts, namely:

$$\frac{(p+1)!}{2^{p+1}}\sum_{j+2l+m=p+1}\frac{A(j,l,m)}{j!\,l!\,m!}=\frac{p!}{2^{p+1}}\sum_{j+2l+m=p}\frac{A(j+1,l,m)}{j!\,l!\,m!}$$
$$+\frac{p!}{2^{p+1}}\sum_{j+2l+m=p}\frac{A(j,l,m+1)}{j!\,l!\,m!}$$
$$\frac{p!}{2^p}\sum_{j+2l+m=p-1}\frac{A(j,l+1,m)}{j!\,l!\,m!}$$

(The logic is as follows:

$$\sum_{j+2l+m=p+1}\frac{A(j,l,m)}{j!\,l!\,m!}=\sum_{j,l,m=0}^{\infty}\frac{\partial_t^{p+1}}{(p+1)!}t^{j+2l+m}\frac{A(j,l,m)}{j!\,l!\,m!}\bigg|_{t=0}$$
$$=\sum_{j,l,m=0}^{\infty}\frac{\partial_t^{p}}{(p+1)!}(j+2l+m)t^{j+2l+m-1}\frac{A(j,l,m)}{j!\,l!\,m!}\bigg|_{t=0}$$
$$=\sum_{j,l,m=0}^{\infty}\frac{\partial_t^{p}}{(p+1)!}t^{j+2l+m-1}\frac{A(j,l,m)}{(j-1)!\,l!\,m!}\bigg|_{t=0}+\cdots$$
$$=\sum_{j,l,m=0}^{\infty}\frac{\partial_t^{p}}{(p+1)!}t^{j+2l+m}\frac{A(j+1,l,m)}{j!\,l!\,m!}\bigg|_{t=0}+\cdots=\frac{1}{p+1}\sum_{j+2l+m=p}^{\infty}\frac{A(j+1,l,m)}{j!\,l!\,m!}+\cdots,$$

where the dots represent the other two terms, where the procedure is analogous.)

We will show that the corresponding parts are equal to each other, i.e. (14)=(17), (15)=(18) and (16)=(19) when

$$A(j,l,m)$$
$$=\frac{|x_r|^{2(j+1)}\left(\widetilde{\varepsilon-1}\right)_j(2a+2\varepsilon)_j}{(2\varepsilon-1)_{j+m+l}(a+\varepsilon)_j}\,{}_3((x_r\cdot\nabla)^m f)_3\left(\begin{matrix}\widetilde{\varepsilon-1}+j & 2a+2\varepsilon+j & a+\varepsilon\\ \widetilde{2\varepsilon-1}+j+l+m & a+\varepsilon+j & 2a+2\varepsilon\end{matrix};x_r\right).$$

The equalities (15)=(18) and (16)=(19) are trivial. It remains only to prove the equality (14)=(17),

$$\frac{1}{4\widetilde{\varepsilon-2}}\int_{\mathbb{B}^{1+\varepsilon}}\left(x_r\cdot\nabla_{y_r}R_{\varepsilon-1}(x_r,y_r)\right)f(y_r)(x_r\cdot y_r)^p d\mu_{2\varepsilon}^{1+\varepsilon}(y_r)=\frac{p!}{2^{p+1}}\sum_{j+2l+m=p}^{\infty}\frac{A(j+1,l,m)}{j!\,l!\,m!}.$$

In the integral we use to obtain:

$$\frac{1}{4\widetilde{\varepsilon-2}}\int_{\mathbb{B}^{1+\varepsilon}}(x_r\cdot y_r)^p\frac{\widetilde{\varepsilon-1}|x_r|^2}{a+\varepsilon}\left(2a+2\varepsilon+x_r\cdot\nabla_{x_r}\right)_1(R_\varepsilon)_1\left(\begin{matrix}a+\varepsilon\\ a+\varepsilon+1\end{matrix};x_r,y_r\right)f(y_r)d\mu_{2\varepsilon}^{1+\varepsilon}(y_r),$$

which according to the Leibniz rule equals

$$\frac{?\,(\widetilde{\varepsilon-1})|x_r|^2}{5\varepsilon-2a}\left(2a+\varepsilon+x_r\cdot\nabla_{x_r}-p\right)\int_{\mathbb{B}^{1+\varepsilon}}(x_r\cdot y_r)^p{}_1(R_\varepsilon)_1\left(\begin{matrix}a+\varepsilon\\ a+\varepsilon+1\end{matrix};x_r,y_r\right)f(y_r)d\mu_{2\varepsilon}^{1+\varepsilon}(y_r).$$

Using the integral form of hypergeometrization

$${}_1(R_\varepsilon)_1\left(\begin{matrix}a+\varepsilon\\ a+\varepsilon+1\end{matrix};x_r,y_r\right)=a+\varepsilon\int_0^1 t^{a+\varepsilon-1}R_\varepsilon(tx_r,y_r)dt$$

we get:

$$\frac{(\widetilde{\varepsilon-1})|x_r|^2}{4\varepsilon-2}\left(2a+2\varepsilon+x_r\cdot\nabla_{x_r}-p\right)\int_0^1 t^{a+\varepsilon-1-p}\int_{\mathbb{B}^{1+\varepsilon}}(tx_r\cdot y_r)^p R_\varepsilon(tx_r,y_r)f(y_r)d\mu_\varepsilon^{1+\varepsilon}(y_r)dt.$$

By the induction hypothesis, this is equal to

$$\frac{(\widetilde{\varepsilon-1})|x_r|^2}{4\varepsilon-2}\left(2a+2\varepsilon+x_r\cdot\nabla_{x_r}-p\right)\int_0^1 t^{a+\varepsilon-1-p}$$

$$\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{t^{2j+2l+m}|x_r|^{2(j+1)}\left(\widetilde{\varepsilon-1}+1\right)_j(2a+2\varepsilon)_j}{j!\,m!\,l!\,(4\varepsilon-1)_{j+m+l}(a+\varepsilon)_j}\,{}_3((x_r\cdot\nabla)^m f)_3\left(\begin{matrix}\widetilde{\varepsilon-1}+j+1 & 2a+2\varepsilon+j & a+\varepsilon\\ 2\widetilde{\varepsilon-1}+j+1+l+m & a+\varepsilon+j & 2a+\varepsilon\end{matrix};tx\right)dt,$$

From the knowledge that

$$\int_0^1 t^{a+\varepsilon+j-1}g(tx_r)dt=\frac{1}{a+\varepsilon+j}\,{}_1g_1\left(\begin{matrix}a+\varepsilon+j\\ a+\varepsilon+j+1\end{matrix};x_r\right)$$

we obtain

$$|x_r|^2\left(2a+\varepsilon+x_r\cdot\nabla_{x_r}-p\right)$$

$$\frac{p!}{2^{p+1}}\sum_{j+2l+m=p}\frac{|x|^{2(j+1)}(\widetilde{\varepsilon-1})_{j+1}(2a+2\varepsilon)_j}{j!m!l!(2\varepsilon-1)_{j+1+m+l}(a+\varepsilon)_{j+1}}\,{}_3((x_r\cdot\nabla)^m f)_3\left(\begin{matrix}\varepsilon+j & 2a+2\varepsilon+j & a+\varepsilon\\ 2\widetilde{\varepsilon-1}+j+1+l+m & a+\varepsilon+j+1 & 2a+\varepsilon\end{matrix};x_r\right).$$

By the Leibniz rule and some manipulation we finally arrive at:

$$\frac{p!}{2^{p+1}}\sum_{j+2l+m=p}\frac{|x_r|^{2(j+1+l)}\left(\widetilde{\varepsilon-1}\right)_{j+1}(2a+2\varepsilon)_{j+1}}{j!\,m!\,l!\left(2\widetilde{\varepsilon-1}\right)_{j+1+m+l}(a+\varepsilon)_{j+1}}$$

$$\frac{2a+2\varepsilon+x_r\cdot\nabla_{x_r}+j-m}{2a+2\varepsilon+j}\,{}_3((x_r\cdot\nabla)^m f)_3\left(\begin{matrix}(\widetilde{\varepsilon-1})+j+1 & 2a+2\varepsilon+j & a+\varepsilon\\ 2\varepsilon+j+l+m & a+\varepsilon+j+1 & 2a+2\varepsilon\end{matrix};x_r\right).$$

To finish the proof it now only remains to show that the last formula is equal to

$${}_3((x_r\cdot\nabla)^m f)_3\left(\begin{matrix}\widetilde{\varepsilon-1}+j+1 & 2a+2\varepsilon+j+1 & a+\varepsilon\\ 2\varepsilon+j+l+m & a+\varepsilon+j+1 & 2a+2\varepsilon\end{matrix};x_r\right),$$

but from the property of hypergeometrization it follows generally that

$${}_1((x_r\cdot\nabla)^m g)_1\left(\begin{matrix}a+1\\ a+2\varepsilon\end{matrix};x_r\right)=\frac{1}{a}(a+t\partial_t)\,{}_1((x_r\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ a+2\varepsilon\end{matrix};tx_r\right)\Big|_{t=1}$$

$$=\frac{1}{a}(a+t\partial_t)t^{-m}\,{}_1((tx_r\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ a+2\varepsilon\end{matrix};tx_r\right)\Big|_{t=1}$$

$$=\frac{1}{a}(a-m+t\partial_t)\,{}_1((tx_r\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ a+2\varepsilon\end{matrix};tx_r\right)\Big|_{t=1}$$

$$=\frac{1}{a}\left(a-m+x_r\cdot\nabla_{x_r}\right)\,{}_1((x_r\cdot\nabla)^m g)_1\left(\begin{matrix}a\\ a+2\varepsilon\end{matrix};x_r\right).$$

Set $a=2\varepsilon+j$ and we are done.

Corollary (5.2.18) [349]: For $p\in\mathbb{N}_0$ and $\varepsilon\geq 0$,

$$\int_{\mathbb{B}^{1+\varepsilon}}(x_r\cdot y_r)^p R_{\varepsilon-1}^2(x_r,y_r)d\mu_{2\varepsilon-1}^{1+\varepsilon}(y_r)$$

$$=\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{|x_r|^{2(j+l+m)}(\widetilde{\varepsilon-1})_j(\widetilde{\varepsilon-1})_m\left(2(a+\varepsilon)\right)_j\left(2(a+\varepsilon)\right)_m}{(2\varepsilon-1)_{j+m+l}(a+\varepsilon)_j(a+\varepsilon)_m j!m!l!}\,{}_5F_4$$

$$\left(\begin{matrix}\widetilde{\varepsilon-1}+j & \varepsilon-1+m & 2a+\varepsilon+j & 2(a+\varepsilon)+m & a+\varepsilon\\ 2\widetilde{\varepsilon-1}+j+m+l & & a+\varepsilon+j & a+\varepsilon+m & 2a+\varepsilon\end{matrix};|x_r|^2\right).$$

Proof. We use with $f(y_r)=R_{\varepsilon-1}(x_r,y_r)$:

$$\int_{\mathbb{B}^{1+\varepsilon}}R_{(\varepsilon-1)}^2(x_r,y_r)(x_r\cdot y_r)^p d\mu_{(2\varepsilon-1)}^{1+\varepsilon}(y_r)$$

$$\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{|x_r|^{2(j+l)}\left(\widetilde{\varepsilon-1}\right)_j(2a+\varepsilon)_j}{j!\,m!\,l!\left(2\widetilde{\varepsilon-1}\right)_{j+m+l}(a+\varepsilon)_j}\,{}_3\tilde{f}_3\left(\begin{matrix}(\widetilde{\varepsilon-1})+j & 2a+2\varepsilon+j & a+\varepsilon\\ 2\widetilde{\varepsilon-1}+j+l+m & a+\varepsilon+j & 2a+2\varepsilon\end{matrix};x_r\right),$$

where $\tilde{f}(y_r):=\left(x_r\cdot\nabla_{y_r}\right)^m R_{2\varepsilon-1}(x_r,y_r)$. From the fact $x_r\cdot\nabla_{y_r}z_r=x_r\cdot\nabla_{y_r}\overline{z_r}=|x_r|^2$ we have

$$\tilde{f}(y_r)=\left(x_r\cdot\nabla_{y_r}\right)^m R_{\varepsilon-1}(x_r{-}{-}{-},y_r)=|x_r|^{2m}\left(2\partial_{z_r}\right)^m F_1\left(\begin{matrix}(\widetilde{\varepsilon-1})\\ a+\varepsilon\end{matrix};\begin{matrix}a+\varepsilon & a+\varepsilon\\ & -\end{matrix};z_r,\bar{z}_r\right)$$

$$=|x_r|^{2m}\frac{(\varepsilon-1)_m}{(a+\varepsilon)_m}\sum_{k=0}^m\binom{m}{k}(a+\varepsilon)_{m-k}(a$$
$$+\varepsilon)_k F_1\left(\begin{matrix}\widetilde{\varepsilon-1}+m\\ a+\varepsilon+m\end{matrix};\begin{matrix}a+\varepsilon+m-k & a+\varepsilon+k\\ & -\end{matrix};z_r,\overline{z_r}\right).$$

Performing hypergeometrization (notice that z and.$\bar{z}$ are homogeneous of the degree 1) we get

$${}_3\tilde{f}_3\left(\begin{matrix}\varepsilon-1+j & 2a+2\varepsilon+j & a+\varepsilon\\ 2\widetilde{\varepsilon-1}+j+m+l & a+\varepsilon+j & 2a+2\varepsilon\end{matrix};x_r\right)$$

$$=|x_r|^{2m}\frac{(\varepsilon-1)_m}{(a+\varepsilon)_m}$$

$$\sum_{k=0}^m\binom{m}{k}(a+\varepsilon)_{m-k}(a$$

$$+\varepsilon)_k F_1\left(\begin{matrix}\varepsilon-1+m & \varepsilon-1+j & 2a+2\varepsilon+j & a+\varepsilon\\ a+\varepsilon+m & \tilde{\beta}+j+m+l & a+\varepsilon+j & 2a+2\varepsilon\end{matrix};\begin{matrix}a+\varepsilon+m-k & a+\varepsilon+k\\ & -\end{matrix};|x_r|^2,|x_r|^2\right)$$

$$=|x_r|^{2m}\frac{\left(\widetilde{\varepsilon-1}\right)_m}{(a+\varepsilon)_m}\sum_{k=0}^m\binom{m}{k}(a+\varepsilon)_{m-k}(a$$
$$+\varepsilon)_k\,{}_5F_4\left(\begin{matrix}\varepsilon-1+m & \varepsilon-1+j & 2a+2\varepsilon+j & a+\varepsilon & 2a+2\varepsilon+m\\ a+\varepsilon+m & 2\varepsilon-1+j+m+l & a+\varepsilon+j & 2a+2\varepsilon\end{matrix};|x_r|^2\right).$$

Here, by the Appell function with more parameters we mean the Kampé de Fériet function (see [14])

$$F_1\left(\begin{matrix}a_1\ldots a_4\\ a_1+2\varepsilon\ldots a_4+2\varepsilon\end{matrix};\begin{matrix}a_1+\varepsilon & a_2+\varepsilon\\ & -\end{matrix};x_r,y_r\right)$$
$$:=F^{4:1;1}_{4:0;0}\left(\begin{matrix}a_1\ldots a_4\\ a_1+2\varepsilon\ldots a_4+2\varepsilon\end{matrix}\begin{matrix}:a_1+\varepsilon;a_2+\varepsilon\\ :-.-;\end{matrix};x_r,y_r\right)$$
$$=\sum_{j,k=0}^{\infty}\frac{(a_1)_{k+j}\ldots(a_4)_{k+j}}{(a_1+2\varepsilon)_{k+j}\ldots(a_4+2\varepsilon)_{k+j}}\frac{(a_1+\varepsilon)_j(a_2+\varepsilon)_k}{j!\,k!}x_r{}^j y_r{}^k,$$

and the last equality was obtained using the similar reduction formula like in the case of Appell F_1 function of the same argument

$$F_1\left(\begin{matrix}a_1\ldots a_4\\ a_1+2\varepsilon\ldots a_4+2\varepsilon\end{matrix};\begin{matrix}a_1+\varepsilon & a_2+\varepsilon\\ & -\end{matrix};x_r,x_r\right)={}_5F_4\left(\begin{matrix}a_1\ldots a_4 & a_1+\varepsilon+a_2+\varepsilon\\ & a_1+2\varepsilon\ldots a_4+2\varepsilon\end{matrix};x_r\right).$$

To complete the proof it is only necessary to become conscious of the fact that

$$\sum_{k=0}^m\binom{m}{k}(a+\varepsilon)_{m-k}(a+\varepsilon)_k=(2a+2\varepsilon)_m$$

and substitute everything into the series at the beginning. As $\varepsilon\to\infty$,

$$\frac{{}_5F_4\left(\begin{matrix}(\varepsilon-1) & & (\widetilde{\varepsilon-1}) & 2a+2\varepsilon & 2a+2\varepsilon & a+\varepsilon\\ \widetilde{\varepsilon-1}+a_1+2\varepsilon & a+a_2+3\varepsilon & a+3\varepsilon+a_3 & & 2a+4\varepsilon+a_4 & \end{matrix};|x_r|^2\right)}{{}_2F_1\left(\begin{matrix}\widetilde{\varepsilon-1} & 2a+2\varepsilon\\ & a+\varepsilon\end{matrix};|x_r|^2\right)}$$
$$\approx(1-|x_r|^2)^{a_1+2\varepsilon}\left(\frac{1-|x_r|^2}{\alpha|x_r|^2}\right)^{a_2+6\varepsilon+a_3+a_4}(a+\varepsilon)_{a_2+2\varepsilon}(a+\varepsilon)_{a_3+2\varepsilon}(2a$$
$$+2\varepsilon)_{a_4+2\varepsilon}\left(1+\sum_{k=1}^{\infty}\frac{d_k}{(\varepsilon-1)^k}\right),$$

where d_k are some constants independent of $\varepsilon-1$.

Corollary (5.2.19) [349]: . For $M,N\in\mathbb{N}_0$,

$$\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot x_r-|x_r|^2)^M(|x_r|^2-|y_r|^2)^N R^2_{\varepsilon-1}(x_r,y_r)\,d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

$$=\sum_{\substack{l,r,q\\k_1\dots k_5}}^{\infty}\frac{|x_r|^{2(M-1)}M!\,(-1)^M(-N)_q(a+\varepsilon+1)_q(a+\varepsilon)_{k_1}(-(a+\varepsilon)_{k_2}(-(a+\varepsilon)_{k_3}(r+l+k_4)_{k_5}}{2^M(M-2l)!\left(\widetilde{\varepsilon-1}\right)_q\left(\widetilde{\varepsilon-1}\right)_{r+l+k_4+k_5}(2a+2\varepsilon)_{k_1}(a+\varepsilon)_{k_2}(a+\varepsilon)_{k_3}l!\,r!\,q!\,k_1!\,k_2!\,k_3!\,k_4!\,k_5!}$$

$$C(|x_r|^2)\,{}_5F_4\left(\begin{matrix}\widetilde{\varepsilon-1} & & \widetilde{\varepsilon-1} & 2a+2\varepsilon & 2a+2\varepsilon & a+\varepsilon\\ \widetilde{\varepsilon-1}+l+r+k_4+k_5 & a+\varepsilon+k_3 & a+\varepsilon+k_2 & & 2a+2\varepsilon+k_1 & \end{matrix};|x_r|^2\right),$$

where

$$C(|x_r|^2):=\left(-\partial_{t_1}\right)^{k_1}\dots\left(-\partial_{t_5}\right)^{k_5}\partial_{t_6}^{k_4}\partial_{t_7}^{r}\left(-\partial_{t_0}\right)^{r}$$

$$t_6^{r+l+k_4-1}t_7^{l+r-1}(t_0-|x_r|^2t_{1-7})^q(|x_r|^2(1-t_{1-7})+t_0-1)^{N-q}(2-t_{6735}-t_{724})^{M-2l}\Big|_{t_0\dots t_7=1}$$

Here $t_{1-7}:=t_1t_2\dots t_7,t_{6735}:=t_6t_7t_3t_5$and so on. The summation indices are bound by the following inequalities

$$k_1+k_2+k_3+2k_4+k_5+2r+q+2l\geq M+N$$
$$2l\leq M$$
$$q,k_1,r\leq N$$
$$k_2,k_3,k_4,k_5,1+l\leq N+M$$

Proof. We expand the parentheses in the integral into binomial series

$$(y_r\cdot x_r-|x_r|^2)^M=\sum_{p=0}^{\infty}\binom{M}{P}(-|x_r|^2)^{M-p}(y_r\cdot x_r)^p,$$

$$(|x_r|^2-|y_r|^2)^N=\sum_{q=0}^{\infty}\binom{N}{q}(|x_r|^2-1)^{N-q}(1-|y_r|^2)^q,$$

to get

$$\sum_{p,q=0}^{\infty}\binom{M}{P}\binom{N}{q}(-|x_r|^2)^{M-p}(|x_r|^2-1)^{N-q}\frac{(a+2\varepsilon)_{\varepsilon-1}}{(a+2\varepsilon)_{\varepsilon-1+q}}\int_{\mathbb{B}^{1+\varepsilon}}(x_r\cdot y_r)^pR^2_{\varepsilon-1}(x_r,y_r)\,d\mu^{1+\varepsilon}_{\varepsilon-1+q}(y_r),$$

Where $(a+2\varepsilon)_\gamma=\frac{\Gamma(\tilde{\gamma})}{\pi^{\frac{1+\varepsilon}{2}}\Gamma(\gamma+1)}$. By Corollary 1, this equals

$$\sum_{p,q=0}^{\infty}\binom{M}{P}\binom{N}{q}(-|x_r|^2)^{M-p}(|x_r|^2$$

$$-1)^{N-q}\frac{(\varepsilon)_q}{(\varepsilon-1)_q}\frac{p!}{2^p}\sum_{j+2l+m=p}\frac{|x_r|^{2(j+l+m)}(\varepsilon-1)_j(\varepsilon-1)_m(2a+2\varepsilon)_j(2a+2\varepsilon)_m}{(\varepsilon-1+q)_{j+m+l}(a+\varepsilon)_j(a+\varepsilon)_m j!\,l!\,m!}$$

$${}_5F_4\left(\begin{matrix}\varepsilon-1+j & \varepsilon-1+m & 2a+2\varepsilon+j & 2a+2\varepsilon+m & a+\varepsilon\\ \tilde{\alpha}+q+j+m+l & a+\varepsilon+j & a+\varepsilon+m & 2a+2\varepsilon\end{matrix};|x_r|^2\right).$$

Now we sum over p by the procedure:

$$\sum_{p=0}^{\infty}\binom{M}{P}{x_r}^{M-p}\frac{p!}{2^p}\sum_{j+2l+m=p}A_{jlm}=\sum_{p=0}^{\infty}\binom{M}{P}x_r^{\;M-p}\frac{p!}{2^p}\sum_{j,l,m=0}^{\infty}\frac{\partial_t^p}{p!}t^{j+2l+m}A_{jlm}\Big|_{t=0}$$

$$=\sum_{j,l,m=0}^{\infty}\sum_{p=0}^{\infty}\binom{M}{P}{}^{M-p}\frac{\partial_t^p}{p!}t^{j+2l+m}A_{jlm}\Big|_{t=0}=\sum_{j,l,m=0}^{\infty}\binom{M}{j+2l+m}\frac{(x_r)^{M-j-2l-m}(j+2l+m)!}{2^{j+m}4^l}A_{jlm}.$$

This yields

$$\sum_{q=0}^{\infty}\binom{N}{q}(|x_r|^2-1)^{N-q}\frac{(\varepsilon+2)_q}{(\widetilde{\varepsilon-1})_q}\sum_{j,l,m=0}^{\infty}\frac{|x_r|^{2(M-l)}M!\,(-1)^{M+j+m}(\widetilde{\varepsilon-1})_j(\widetilde{\varepsilon-1})_m(2a+2\varepsilon)_j(2a+2\varepsilon)_m}{(M-2l-j-m)!\,4^l2^{j+m}(\varepsilon-1+q)_{j+m+l}(a+\varepsilon)_j(a+\varepsilon)_m j!\,l!\,m!} \quad (20)$$

$${}_5F_4\left(\begin{matrix}\widetilde{\varepsilon-1}+j & \widetilde{\varepsilon-1}+m & 2a+2\varepsilon+j & 2a+2\varepsilon+m & a+\varepsilon\\ \widetilde{\varepsilon-1}+q+j+m+l & a+\varepsilon+j & a+\varepsilon+m & 2a+2\varepsilon\end{matrix};|x_r|^2\right).$$

We would like to sum over q as well but we are unable to do that since the index is present also in the hypergeometric function. To remove this difficulty we make use of the following lemma(see[289])

For $r\in\mathbb{N}_0$,

$$\underline{\Delta}_{2\varepsilon-1}^{r}\frac{1}{(2\varepsilon-1)_k}\,{}_1f_1\left(\begin{matrix}\gamma\\2\varepsilon-1+k\end{matrix};x_r\right)$$
$$=(-1)^r(k)_r\sum_{j=0}^{\infty}\frac{(-1)^j(-r)_j}{j!\,(k)_j(2\varepsilon-1)_{k+r}}(x_r)^j\partial_{x_r}^j\,{}_1f_1\left(\begin{matrix}\gamma\\2\varepsilon-1+k+r\end{matrix};x_r\right),$$

Where $\underline{\Delta}_{2\varepsilon-1}g(2\varepsilon-1):=g(2\varepsilon)-g(2\varepsilon-1)$.

The proof could be easily done by induction, but our approach will be much more direct.

Proof. Firstly

$$\underline{\Delta}_{2\varepsilon-1}\frac{1}{(2\varepsilon-1)_k}\,{}_1f_1\left(\begin{matrix}\gamma\\2\varepsilon-1+k\end{matrix};x_r\right)=\sum_{j=0}^{\infty}\underline{\Delta}_{2\varepsilon-1}\frac{f^{(j)}(0)(\gamma)_j}{j!\,(2\varepsilon-1)_{k+j}}x_r{}^j$$
$$=\sum_{j=0}^{\infty}\frac{f^{(j)}(0)(\gamma)_j}{j!}\left(\frac{1}{(2\varepsilon)_{k+j}}-\frac{1}{(2\varepsilon-1)_{k+j}}\right)$$
$$=\sum_{j=0}^{\infty}\frac{f^{(j)}(0)(\gamma)_j}{j!}\frac{-k-j}{(2\varepsilon-1)_{k+j+1}}=\sum_{j=0}^{\infty}\frac{f^{(j)}(0)(\gamma)_j}{j!}\frac{-k-x_r\partial_{x_r}}{(\beta)_{k+j+1}}$$
$$=-\frac{k+x\partial_{x_r}}{(2\varepsilon-1)_{k+1}}\,{}_1f_1\left(\begin{matrix}\gamma\\\beta+k+1\end{matrix};x_r\right).$$

Hence obviously

$$\underline{\Delta}_{2\varepsilon-1}^{r}\frac{1}{(2\varepsilon-1)_k}\,{}_1f_1\left(\begin{matrix}\gamma\\2\varepsilon-1+k\end{matrix};x_r\right)=(-1)^r\left(k+x_r\partial_{x_r}\right)_r\frac{1}{(2\varepsilon-1)_{k+r}}\,{}_1f_1\left(\begin{matrix}\gamma\\2\varepsilon-1+k+r\end{matrix};x_r\right)$$

We claim that

$$\left(k+x_r\partial_{x_r}\right)_r=\sum_{j=0}^{r}\frac{(-1)^j(-r)_j(k)_r}{j!\,(k)_j}(x_r)^j\partial_{x_r}^j.$$

It is enough to check this equation on monomials $(x_r)^m$ since the expressions on both sides are linear combinations of operators $(x_r)^l\partial_{x_r}^l$. Since $x_r\partial_{x_r}(x_r)^m=m(x_r)^m$this reduces the problem to verifying the equality

$$(k+m)_r=\sum_{j=0}^{r}\frac{(-1)^j(-r)_j(k)_r}{j!\,(k)_j}\frac{m!}{(m-j)!}.$$

But this can be rewritten as

$$(k)_rm!\binom{r+k-1+m}{m}=(k)_rm!\sum_{j=0}^{r}\binom{r}{j}\binom{k-1+m}{m-j},$$

which is (aside the factor $(k)_rm!$) the so-called Chu-Vandermonde identity.

We will use this lemma in the following way. Obviously

$$\frac{1}{(2\varepsilon-1+q)_k}\,{}_1f_1\left(\begin{matrix}\gamma\\2\varepsilon-1+q+k\end{matrix};x_r\right)=\sum_{r=0}^{\infty}\binom{q}{r}\underline{\Delta}^r_{2\varepsilon-1}\frac{1}{(2\varepsilon-1)_k}\,{}_1f_1\left(\begin{matrix}\gamma\\2\varepsilon-1+k\end{matrix};x_r\right).$$

By the lemma with

$$f(|x_r|^2):={}_4F_3\left(\begin{matrix}\varepsilon-1+m & 2a+2\varepsilon+j & 2a+2\varepsilon+m & a+\varepsilon\\ & a+\varepsilon+j & a+\varepsilon+m & 2a+2\varepsilon\end{matrix};|x_r|^2\right),$$

$\gamma:=(\widetilde{\varepsilon-1})+j, 2\varepsilon-1:=(\widetilde{\varepsilon-1})$ and $k:=j+l+m$ we get

$$\frac{1}{(\varepsilon-1+q)_{j+l+m}}\,{}_5F_4\left(\begin{matrix}\varepsilon-1+j & \varepsilon-1+m & 2a+2\varepsilon+j & 2a+2\varepsilon+m & a+\varepsilon\\ \varepsilon-1+q+j+m+l & & a+\varepsilon+j & a+\varepsilon+m & 2a+2\varepsilon\end{matrix};|x_r|^2\right)$$

$$=\sum_{r=0}^{\infty}\binom{q}{r}(-1)^r(j+l$$

$$+m)_r\sum_{s=0}^{\infty}\frac{(-1)^s(-r)_s|x_r|^{2s}\partial^s_{|x|^2}}{s!\,(j+l+m)_s(\varepsilon-1)_{j+m+l+r}}\,{}_5F_4\left(\begin{matrix}\widetilde{\varepsilon-1}+j & \widetilde{\varepsilon-1}+m & 2a+2\varepsilon+j & 2a+2\varepsilon+m & a+\varepsilon\\ \widetilde{\varepsilon-1}+r+j+m+l & & a+\varepsilon+j & a+\varepsilon+m & 2a+2\varepsilon\end{matrix};|x_r|^2\right)$$

$$\sum_{r=0}^{\infty}\binom{q}{r}(-1)^r(j+l$$

$$+m)_r\sum_{s=0}^{\infty}\frac{(-1)^s(-r)_s|x_r|^{2s}}{s!\,(j+l+m)_s}\frac{\left(\widetilde{\varepsilon-1}+j\right)_s\left(\widetilde{\varepsilon-1}+m\right)_s(2a+2\varepsilon+j)_s(2a+2\varepsilon+m)_s(a+\varepsilon)_s}{\left(\widetilde{\varepsilon-1}\right)_{j+m+l+r+s}(a+\varepsilon+j)_s(a+\varepsilon+m)_s(2a+2\varepsilon)_s}$$

$${}_5F_4\left(\begin{matrix}\widetilde{\varepsilon-1}+j+s & \varepsilon-1+m+s & 2a+2\varepsilon+j+s & 2a+2\varepsilon+m+s & a+\varepsilon+s\\ \widetilde{\varepsilon-1}+r+j+m+l+s & & a+\varepsilon+j+s & a+\varepsilon+m+s & 2a+2\varepsilon+s\end{matrix};|x_r|^2\right).$$

Substituting this into (20), with some manipulations and performing a transformation of the summation index $r\to r+s$ we get

$$\sum_{q=0}^{\infty}\binom{N}{q}(|x_r|^2-1)^{N-q}\frac{(\varepsilon-1+1)_q}{(\varepsilon-1)_q}$$

$$\sum_{j,l,m,r,s=0}^{\infty}\frac{|x|^{2(M-l+s)}M!\,(-1)^{M+j+m}\left(\widetilde{\varepsilon-1}\right)_{j+s}\left(\widetilde{\varepsilon-1}\right)_{m+s}(2a+2\varepsilon)_{j+s}(2a+2\varepsilon)_{m+s}(a+\varepsilon)_s(-q)_{r+s}(j+l+m)_{r+s}}{(M-2l-j-m)!\,4^l2^{j+m}(a+\varepsilon)_{j+s}(a+\varepsilon)_{m+s}(2a+2\varepsilon)_s\left(\widetilde{\varepsilon-1}\right)_{j+m+l+r+2s}j!\,l!\,m!\,r!\,s!\,(j+l+m)_s}$$

$${}_5F_4\left(\begin{matrix}\widetilde{\varepsilon-1}+j+s & \widetilde{\varepsilon-1}+m+s & 2a+2\varepsilon+j+s & 2a+2\varepsilon+m+s & a+\varepsilon+s\\ \widetilde{\varepsilon-1}+r+j+m+l+2s & & a+\varepsilon+j+s & a+\varepsilon+m+s & 2a+2\varepsilon+s\end{matrix};|x_r|^2\right)$$

We can sum over q now. The series in question is

$$\sum_{q=0}^{\infty}\binom{N}{q}(|x_r|^2-1)^{N-q}\frac{(\varepsilon)_q}{(\varepsilon-1)_q}(-q)_{r+s}.$$

By the representation $(-q)_{r+s}=(-\partial_t)^{r+s}t^q|_{t=1}$(from now on every parameter that contains the letter t– t, t_1, and so on – will be understood to be evaluated at 1; we will not explicitly mention this) we get

$$(-\partial_t)^{r+s}(|x_r|^2-1)^N\,{}_2F_1\left(\begin{matrix}-N & \varepsilon+2\\ & \varepsilon-1\end{matrix};\frac{t}{1-|x_r|^2}\right).$$

The known transformation

$${}_2F_1\left(\begin{matrix}a & a+\varepsilon\\ & a+2\varepsilon\end{matrix};x_r\right)=(1-x_r)^{-a}\,{}_2F_1\left(\begin{matrix}a & \varepsilon\\ & a+2\varepsilon\end{matrix};\frac{x_r}{x_r-1}\right)$$

enables us to write this as

$$(-\partial_t)^{r+s}(|x_r|^2-1)^N\,{}_2F_1\left(\begin{matrix}-N & a+\varepsilon+1\\ & \varepsilon-1\end{matrix};\frac{t}{|x_r|^2-1+t}\right)$$

$$=\sum_{q=0}^{\infty}\frac{(-N)_q(a+\varepsilon+1)_q}{(\varepsilon-1)_q q!}(-\partial_t)^{r+s}t^q(|x_r|^2-1+t)^{N-q}.$$

We did not manage to sum the series explicitly but this will do. Substituting this result into (21) we get the second intermediate result:

$$\sum_{\substack{j,l,m\\ r,s,q}}^{\infty} \frac{|x_r|^{2(M-l+s)} M!\,(-1)^{M+j+m}(-N)_q(a+\varepsilon+1)_q(\varepsilon-1)_{j+s}(\varepsilon-1)_{m+s}(2a+2\varepsilon)_{j+s}(2a+2\varepsilon)_{m+s}(a+\varepsilon)_s(j+l+m)_{r+s}}{(M-2l-j-m)!\,4^l 2^{j+m}(a+\varepsilon)_{j+s}(a+\varepsilon)_{m+s}(2a+2\varepsilon)_s(\varepsilon-1)_q(\varepsilon-1)_{j+m+l+r+2s}\, j!\,l!\,m!\,r!\,s!\,q!\,(j+l+m)_s}$$

$$C_{r,s}(|x|^2)\,{}_5F_4\left(\begin{matrix}\varepsilon-1+j+s & \varepsilon-1+m+s & 2a+2\varepsilon+j+s & 2a+2\varepsilon+m+s & a+\varepsilon+s\\ \widetilde{\varepsilon-1}+r+j+m+l+2s & & a+\varepsilon+j+s & a+\varepsilon+m+s & 2a+2\varepsilon+s\end{matrix}; |x_r|^2\right)$$

Where $C_{r,s}(|x_r|^2) = (-\partial_t)^{r+s} t^q (|x_r|^2 - 1 + t)^{N-q}$.

As it is clear from Lemma 1 all functions ${}_5F_4$ with these parameters have the same principal asymptotic behavior as $\varepsilon \to \infty$. To get a more effective form we exploit the known relation between contiguous hypergeometric functions

$$F\left(\begin{matrix}a+1\\ a+2\varepsilon+1\end{matrix}\right) = \frac{a+2\varepsilon}{a}\left(F\left(\begin{matrix}a\\ a+2\varepsilon\end{matrix}\right) - \frac{2\varepsilon}{a+2\varepsilon} F\left(\begin{matrix}a\\ a+2\varepsilon+1\end{matrix}\right)\right),$$

which holds for any hypergeometric function with at least one upper and one lower parameter. By iteration we get:

$$F\left(\begin{matrix}a+m\\ a+2\varepsilon+m\end{matrix}\right) = \frac{(a+2\varepsilon)_m}{(a)_m}\sum_{j=0}^{\infty}\frac{(-m)_j(2\varepsilon)_j}{(a+2\varepsilon)_j j!} F\left(\begin{matrix}a\\ a+2\varepsilon+j\end{matrix}\right).$$

We apply this to the function

$${}_5F_4\left(\begin{matrix}\overbrace{\varepsilon-1+j+s}^{k_5} & \overbrace{\widetilde{\varepsilon-1}+m+s}^{k_4} & 2a+2\varepsilon+j+s & 2a+\varepsilon+m+s & a+\varepsilon+s\\ \underbrace{\varepsilon-1+r+j+m+l+2s}_{k_4 k_5} & & \underbrace{a+\varepsilon+j+s}_{k_3} & \underbrace{a+\varepsilon+m+s}_{k_2} & \underbrace{2a+2\varepsilon+s}_{k_1}\end{matrix}; |x_r|^2\right)$$

as indicated, five times in total. That will get us 5 new series with 5 new summation indices, which we name $k_1 \dots k_5$. The role of m will be played in turn by the parameters $s, m+s, j+s, m+s$ and $j+s$. The lower indices $a+2\varepsilon$ will be in this $2a+2\varepsilon, a+\varepsilon,, \tilde{\alpha}+r+j+s+l$ and $\widetilde{\varepsilon-1}+r+l+k_4$.

This way the expression (23) will change form to:

$$\sum_{k_1\dots k_5}\frac{(2a+2\varepsilon)_s(a+\varepsilon)_{m+s}(a+\varepsilon)_{j+s}(\varepsilon-1+r+j+s+l)_{m+s}(\varepsilon-1+r+l+k_4)_{j+s}}{(a+\varepsilon)_s(2a+2\varepsilon)_{m+s}(2a+2\varepsilon)_{j+s}(\varepsilon-1)_{m+s}(\varepsilon-1)_{j+s}}$$

$$\frac{(-s)_{k_1}(-m-s)_{k_2}(-j-s)_{k_3}(-m-s)_{k_4}(-j-s)_{k_5}(a+\varepsilon)_{k_1}(-a+\varepsilon)_{k_2}(-a+\varepsilon)_{k_3}(r+j+s+l)_{k_4}(}{(2a+2\varepsilon)_{k_1}(a+\varepsilon)_{k_2}(a+\varepsilon)_{k_3}(\varepsilon-1+r+j+s+l)_{k_4}(\varepsilon-1+r+l+k_4)_{k_5}k_1!\,k_2!\,k_3!\,k_4!}$$

$${}_5F_4\left(\begin{matrix}\varepsilon-1 & & \varepsilon-1 & 2a+2\varepsilon & 2a+2\varepsilon & a+\varepsilon\\ \widetilde{\varepsilon-1}+l+r+k_4+k_5 & a+\varepsilon+k_3 & a+\varepsilon+k_2 & & 2a+2\varepsilon+k_1 & \end{matrix}; |x_r|^2\right).$$

Substituting this into (22) will fortunately reduce the number of terms, for many of them will cancel out each other. It can be checked by an easy calculation, for example, that the terms containing $\varepsilon-1$ but not q will squeeze to a single expression$1/(\widetilde{\varepsilon-1})_{r+l+k_4+k_5}$. We end up with this much more tolerable expression:

$$\sum_{\substack{jlmrsq\\ k_1\dots k_5}}^{\infty}\frac{|x|^{2(M-l+s)}M!\,(-1)^{M+j+m}(-N)_q(a+\varepsilon+1)_q(j+l+m+s)_r}{(M-2l-j-m)!\,4^l 2^{j+m}\left(\widetilde{\varepsilon-1}\right)_q(\varepsilon-1)_{r+l+k_4+k_5}j!\,l!\,m!\,r!\,s!\,q!\,k_1!\,k_2!\,k_3!\,k_4!\,k_5!}$$

$$\frac{(-s)_{k_1}(-m-s)_{k_2}(-j-s)_{k_3}(-m-s)_{k_4}(-j-s)_{k_5}(a+\varepsilon)_{k_1}(-a-\varepsilon)_{k_2}(-a-\varepsilon)_{k_3}(r+j+s+l)_{k_4}(r+l+k_4)_{k_5}}{(2a+2\varepsilon)_{k_1}(a+\varepsilon)_{k_2}(a+\varepsilon)_{k_3}}$$

$$C_{r,s}(|x_r|^2)\,{}_5F_4\left(\begin{matrix}\varepsilon-1 & & \widetilde{\varepsilon-1} & 2(a+\varepsilon) & 2(a+\varepsilon) & a+\varepsilon\\ \widetilde{\varepsilon-1}+l+r+k_4+k_5 & a+\varepsilon+k_3 & a+\varepsilon+k_2 & & 2(a+\varepsilon)+k_1 & \end{matrix}; |x_r|^2\right).$$

We can reduce the complexity of this formula further by summing over all indices which do not appear in the hypergeometric function or depend on $\tilde{\alpha}$, i.e. over indices j, m, s. This gives

$$\sum_{\substack{l,r,q\\k_1\dots k_5}}^{\infty}\frac{|x_r|^{2(M-l)}M!\,(-1)^M(-N)_q(a+\varepsilon+1)_q(a+\varepsilon)_{k_1}(-a-\varepsilon)_{k_2}(-a-\varepsilon)_{k_3}(r+l+k_4)_{k_5}}{4^l(\varepsilon-1)_q\left(\widetilde{\varepsilon-1}\right)_{r+l+k_4+k_5}(2a+2\varepsilon)_{k_1}(a+\varepsilon)_{k_2}(a+\varepsilon)_{k_3}l!\,r!\,q!\,k_1!\,k_2!\,k_3!\,k_4!\,k_5!}$$

$$C(|x_r|^2)\,{}_5F_4\left(\begin{matrix}\varepsilon-1 & & \widetilde{\varepsilon-1} \quad 2a+2\varepsilon & 2a+2\varepsilon \quad a+\varepsilon\\ \varepsilon-1+l+r+k_4+k_5 & a+\varepsilon+k_3 & a+\varepsilon+k_2 & 2a+2\varepsilon+k_1\end{matrix};|x_r|^2\right),$$

where

$$C(|x_r|^2):=$$

$$\sum_{j,m,s}^{\infty}\frac{|x_r|^{2s}(-1)^{j+m}(j+l+m+s)_r(-s)_{k_1}(-m-s)_{k_2}(-j-s)_{k_3}(-m-s)_{k_4}(-j-s)_{k_5}(r+j+s+l)_{k_4}}{(M-2l-j-m)!\,2^{j+m}j!\,m!\,s!}$$

$$(-\partial_t)^{r+s}t^q(|x_r|^2-1+t)^{N-q}.$$

We must deal now with the coefficient $C(|x_r|^2)$. For that purpose we represent each Pochhammer symbol in the series by $(-a)_k=(-\partial_t)^k t^a$ whenever the argument is negative and by $(a)_k=\partial_t^k t^{a+k-1}$in the opposite case (again, the default understanding is that every parameter t_i is to be evaluated, without explicitly saying so, at the point 1). Thus we get

$$C(|x_r|^2)=(-\partial_{t_1})^{k_1}\dots(-\partial_{t_5})^{k_5}\partial_{t_6}^{k_4}\partial_{t_7}^{r}(-\partial_t)^r$$

$$\sum_{j,m,s}^{\infty}\frac{|x_r|^{2s}(-1)^{j+m}t_{1-7}^{s}t_{6735}^{j}t_{724}^{m}(-\partial_t)^s}{(M-2l-j-m)!\,2^{j+m}j!\,m!\,s!}t_6^{r+l+k_4-1}t_7^{l+r-1}t^q(|x_r|^2-1+t)^{N-q},$$

where $t_{1-7}:=t_1t_2t_3t_4t_5t_6t_7, t_{6735}=t_6t_7t_3t_5$ and so on. The sum over s is essentially the Taylor series.

As for the other two indices, it is clear that

$$\sum_{j,m}^{\infty}\frac{A^jB^m}{(M-2l-j-m)!\,j!\,m!}=\frac{1}{(M-2l)!}(1+A+B)^{M-2l}.$$

We thus finally get

$$C(|x_r|^2)=(-\partial_{t_1})^{k_1}\dots(-\partial_{t_5})^{k_5}\partial_{t_6}^{k_4}\partial_{t_7}^{r}(-\partial_t)^r$$

$$\frac{1}{(M-2l)!}t_6^{r+l+k_4-1}t_7^{l+r-1}(t-|x_r|^2t_{1-7})^q(|x_r|^2(1-t_{1-7})+t-1)^{N-q}\left(1-\frac{1}{2}t_{6735}-\frac{1}{2}t_{724}\right)^{M-2l}$$

Many things can be learnt from this form. Firstly: the last two parentheses are equal to zero when all $t-s$ are evaluated at the point 1. To avoid this we must differentiate them out. For that at least $N-q+M-2l$ differentiations are needed. Available to us are $k_1+k_2+k_3+2k_4+k_5+2r$ of them. Hence:

$$k_1+k_2+k_3+2k_4+k_5+2r+q+2l\geq M+N.$$

Secondly: from the perspective of the parameter t we differentiate r-times a polynomial of degree N. In order for the factor $C_{\varepsilon-1}$ not to be zero it must hold $r\leq N$. Analogously, the degree of t_7 is $r-l-1+N+M$ and this tells us that $1+l\leq N+M$. The same reasoning can be applied to any parameter t_i. From those and other facts, such as that in the formula (25) there appears the term $(-N)_q$, or from the presence of the term $1/(M-2l)!$, we can easily compute upper bounds on summation indices. (see[19]).They are:

$$\begin{gathered}r\leq N\\ 2l\leq M \quad\wedge\quad 1+l\leq N+M\\ q\leq N\\ k_1\leq N\\ k_i\leq N+M \qquad \forall_i=2\dots5.\end{gathered}$$

Corollary (5.2.20) [349]: For $M,N\in\mathbb{N}_0$ and $x_r\neq 0$:

$$\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot x_r-|x_r|^2)^M(|x_r|^2-|y_r|^2)^N\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)=O\left(\varepsilon-1^{-\left\lceil\frac{N+M}{2}\right\rceil}\right)\qquad(\varepsilon\to\infty).$$

Proof. From the fact

$$R_{\varepsilon-1}(x_r,x_r) = {}_2F_1\left(\begin{matrix}\varepsilon-1 & 2a+2\varepsilon\\ & a+\varepsilon\end{matrix}; |x_r|^2\right),$$

from and from the presence of the factor $(\widetilde{\varepsilon-1})_q(\varepsilon-1)_{l+r+k_4+k_5}$in the denominator it follows that for $x_r \neq 0$

$$\int_{\mathbb{B}^{1+\varepsilon}} (y_r\cdot x_r-|x_r|^2)^M(|x_r|^2-|y_r|^2)^N \frac{R^2_{\varepsilon-1}(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

$$= \sum_{rlqk_1\dots k_5} O\left(\varepsilon-1^{-(r+l+q+k_1+\cdots+k_5)}\right) \quad (\varepsilon\to\infty).$$

That is the speed of asymptotic decay grows with each summation index $k_1,k_2,k_3,k_4,k_5,r,q,l$ as $\varepsilon\to\infty$. The slowest decay (and therefore the leading term) we get for the lowest possible values of these parameters. But since $k_1+k_2+k_3+2k_4+k_5+2r+q+2l\geq M+N$, lowest values are achieved for $2k_4+2r+2l=M+N$and $k_1=\cdots=k_5=0$ if this is possible, i.e. if $M+N$ is an even number; if $M+N$ is odd, the decay is one negative power of alpha faster. Hence the leading order term is

$$O\left((\varepsilon-1)^{-\left\lceil\frac{N+M}{2}\right\rceil}\right) \quad (\varepsilon\to\infty).$$

Corollary (5.2.21) [349]. For $M\in\mathbb{N}_0$ and $x_r\neq 0$

$$\int_{\mathbb{B}^{1+\varepsilon}} |y_r-x_r|^{2M}\frac{R^2_{\varepsilon-1}(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r) = O\left((\varepsilon-1)^{-\left\lceil\frac{M}{2}\right\rceil}\right) \quad (\varepsilon\to\infty).$$

Proof. The statement follows directly from the representation

$$|y_r-x_r|^{2M} = (|y_r|^2-2y_r\cdot x_r+|x_r|^2)^M = (|y_r|^2-|x_r|^2-2(y_r\cdot x_r-|x_r|^2))^M$$

$$= (-2)^M\sum_{N=0}^{\infty}\binom{M}{N}(y_r\cdot x_r-|x_r|^2)^{M-N}(|x_r|^2-|y_r|^2)^N.$$

The integral is therefore a series of terms, whose behavior:

$$O\left((\varepsilon-1)^{-\left\lceil\frac{M-N+N}{2}\right\rceil}\right) \quad (\varepsilon\to\infty).$$

Corollary (5.2.22) [349]. For $m\in\mathbb{N}_0, x_r\neq 0$ and $\varepsilon>0$:

$$\int_{\mathbb{B}^{1+\varepsilon}} (z_r\cdot y_r-z_r\cdot x_r)^m f(|y_r|,x_r\cdot y_r)d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

$$= \sum_{k=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k(z\cdot x)^{m-2k}v^{2k}_{z_r,x_r}}{\left(a+\varepsilon+\frac{1}{2}\right)_k k!\,|x|^{2m}}\int_{\mathbb{B}^{1+\varepsilon}}(x_r\cdot y_r-|x_r|^2)^{m-2k}v^{2k}_{y_r,x_r}f(|y_r|,x_r\cdot y_r)d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

where $v^2_{u_1,u_2}=|u_1|^2|u_2|^2-(u_1\cdot u_2)^2$.

Proof. The integral

$$\int_{\mathbb{B}^{1+\varepsilon}} (z_r\cdot y_r-z_r\cdot x_r)^m f(|y_r|,x_r\cdot y_r)d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

is unchanged upon replacing x_r,y_r,z_r by Ux_r,Uy_r,Uz_r for any orthogonal transformation U. Without loss of generality, we can thus assume that $x_r=(|x_r|,0,0,\dots)$ and $z_r=((z_r)_1,(z_r)_2,0,0,\dots)$ with $(z_r)_2\geq 0$. Then $x_r\cdot y_r=|x_r|(y_r)_1, (z_r)_1=\frac{z_r\cdot x_r}{|x_r|}, (z_r)_2=\frac{v_{x_r,x_r}}{|x_r|}$ and

$$z_r\cdot x_r = \frac{z_r\cdot x_r}{|x_r|}y_1+v_{z_r,x_r}\frac{(y_r)_2}{|x_r|}.$$

We now change variables to hyper-spherical coordinates

$$(y_r)_1 = r\,cos\,\varphi$$
$$(y_r)_2 = r\,sin\,\varphi\,cos\,\psi$$
$$\dots$$
$$(y_r)_{\varepsilon-1} = r\,sin\,\varphi\,sin\,\psi\,sin\,\theta_1\dots sin\,\theta_{\varepsilon-3}cos\,\theta_{\varepsilon-2}$$

$$(y_r)_{\varepsilon+1} = r\, sin\, \varphi\, sin\, \psi\, sin\, \theta_1 \ldots sin\, \theta_{\varepsilon-3}\, sin\, \theta_{\varepsilon-2}$$

$$d\mu_{\varepsilon-1}^{\varepsilon+1} = \frac{(\varepsilon - 1 + (\varepsilon+1)\backslash 2)!}{\pi^{(\varepsilon+1)\backslash 2}\Gamma(\varepsilon)}(1-r^2)^{\varepsilon-1} r^{\varepsilon}\, sin^{\varepsilon-1}\varphi\, sin^{\varepsilon-2}\psi \ldots sin\, \theta_{\varepsilon-3} dr d\varphi d\psi \ldots d\theta_{\varepsilon-2}.$$

The integration bounds are: $r \in [0,1], \varphi \in [0,\pi], \psi \in [0,\pi], \theta_1 \in [0,\pi], \ldots, \theta_{\varepsilon-3} \in [0,\pi], \theta_{\varepsilon-2} \in [0,2\pi]$.

For the sake of brevity put $d^2\Phi \coloneqq (1-r^2)^{\varepsilon-1} r^{\varepsilon}\, sin^{\varepsilon-1}\varphi dr d\varphi$.

Integration over all θ_i will give us some constant C since the integrand does not depend on them. For the rest we have

$$C\int_0^1\int_0^\pi\int_0^\pi \Big(\frac{z_r \cdot x_r}{|x_r|} r\cos\varphi + v_{z_r,x_r}\frac{r\sin\varphi\cos\psi}{|x_r|} - z_r.x_r\Big)^m f(r, |x_r| r\cos\varphi) d^2\Phi \sin^{\varepsilon-2}\psi\, d\psi$$

$$= C\sum_{l=0}^{\infty}\binom{m}{l}\int_0^1\int_0^\pi \Big(\frac{z_r \cdot x_r}{|x_r|} r\, cos\,\varphi - z_r.x_r\Big)^{m-l}\Big(v_{z_r,x_r}\frac{r\, sin\,\varphi}{|x_r|}\Big)^l f(r, |x_r| r\, cos\,\varphi) d^2\Phi \int_0^\pi \cos^l\psi\, sin^{\varepsilon-2}\psi\, d\psi.$$

Here and in the rest of the proof we assume $\varepsilon > 0$ otherwise (in the case $\varepsilon = 0$) integration over the interval $[0,2\pi]$ would rest with the parameter ψ and in the case $3+\varepsilon = 2$ ψ would not be present at all.

These cases would however require only minor changes in the proof which continues as follows.

Let

$$A \coloneqq \frac{z_r \cdot x_r}{|x_r|} r\, cos\,\varphi - z_r.x_r, \qquad B \coloneqq v_{z_r,x_r}\frac{r\, sin\,\varphi}{|x_r|}.$$

By an easy computation we have:

$$\int_0^\pi cos^l\,\psi\, sin^{\varepsilon}\,\psi\, d\psi = \begin{cases} \dfrac{\sqrt{\pi}\Gamma(a+\varepsilon)}{\Gamma\left(a+\varepsilon+\frac{1}{2}\right)}\dfrac{\left(\frac{1}{2}\right)_k}{\left(a+\varepsilon+\frac{1}{2}\right)_k} & l = 2k, \\ 0 & l \neq 2k. \end{cases}$$

Together with fact that

$$\binom{m}{2k} = \frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!}$$

we obtain that (26) equals

$$C\sum_{k=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k}{\left(a+\varepsilon+\frac{1}{2}\right)_k k!}\int_0^1\int_0^\pi A^{m-2k}B^{2k} f(r, |x_r| r\, cos\,\varphi) d^2\Phi.$$

Now we take hyper-spherical coordinates back. First we had $x_r.y_r = |x_r|(y_r)_1 = |x_r| r\cos\varphi$and $|y_r| = r$, so our result can be again interpreted as an integral over the unit ball in $\mathbb{R}^{3+\varepsilon}$ if we replace $r\cos\varphi$ by $\frac{x_r.y_r}{|x_r|}$and put $r = |y_r|$. Therefore

$$A = \frac{z_r \cdot x_r}{|x_r|} r\, cos\,\varphi - z_r.x_r = \frac{z_r.x_r}{|x_r|}\frac{x_r.y_r}{|x_r|} - z_r.x_r = \frac{z_r.x_r(x_r.y_r - |x_r|^2)}{|x_r|^2},$$

$$B^2 = v_{z_r,x_r}^2\frac{r^2\, sin^2\,\varphi}{|x_r|^2} = v_{z_r,x_r}^2\frac{r^2(1-\cos^2\varphi)}{|x_r|^2} = v_{z_r,x_r}^2\frac{\left(|y_r|^2 - \left(\frac{x_r.y_r}{|x_r|}\right)^2\right)}{|x_r|^2} = \frac{v_{z_r,x_r}^2 v_{y_r.x_r}^2}{|x_r|^4}.$$

Altogether we have

$$C\sum_{k=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k}{\left(a+\varepsilon+\frac{1}{2}\right)_k k!\,|x_r|^{2m}}\int_{\mathbb{B}^{3+\varepsilon}}(z_r\cdot x_r(x_r.y_r-|x_r|^2))^{m-2k}v_{z_r,x_r}^{2k}v_{y_r,x_r}^{2k}f(|y_r|,x_r\cdot y_r)d\mu_{\varepsilon-1}^{3+\frac{\varepsilon}{2}}(y_r)$$

The result, if necessary, can be easily checked by performing change of variables into the hyper-spherical coordinates.

Lastly, we must determine the constant C into which we have collected all unimportant constants. But comparing our original integral with the result for $m=0$ gives us the equality

$$\int_{\mathbb{B}^{1+\varepsilon}} f(|y_r|,x_r\cdot y_r)d\mu_{\varepsilon-1}^{1+\frac{\varepsilon}{2}}(y_r)=C\int_{\mathbb{B}^{1+\varepsilon}} f(|y_r|,x_r\cdot y_r)d\mu_{\varepsilon-1}^{1+\frac{\varepsilon}{2}}(y_r),$$

hence $C=1$.

Corollary (5.2.23) [349]: For $m\in\mathbb{N}_0$

$$\int_{\mathbb{B}^{1+\varepsilon}}(z_r\cdot y_r-z_r\cdot x_r)^m\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$

$$=\sum_{k,j,p=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k(-1)^k(z_r\cdot x_r)^{m-2k}v_{z_r,x_r}^{2k}2^p}{\left(a+\varepsilon+\frac{1}{2}\right)_k(k-j-p)!\,j!\,p!\,|x_r|^{2(m-p-j)}}\int_{\mathbb{B}^{1+\varepsilon}}(x_r\cdot y_r-|x_r|^2)^{m-2j-p}(|x_r|^2-|y_r|^2)^j\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r).$$

Proof. The Bergman kernel depends only on $|y_r|^2|x_r|^2$ and $x_r\cdot y_r-a$ fact easily seen from the representation in terms of the Appell function (13). We can therefore apply.

Notice that the factor v_{y_r,x_r}^{2k} can be written as follows

$$v_{y_r,x_r}^{2k}=(|y_r|^2|x_r|^2-(x_r\cdot y_r)^2)^k$$
$$=(|y_r|^2|x_r|^2-(x_r\cdot y_r-|x_r|^2)^2-2|x_r|^2(x_r\cdot y_r-|x_r|^2)-|x_r|^4)^k,$$

so we can expand it into a finite combination of terms$(x_r\cdot y_r-|x_r|^2),(|x_r|^2-|y_r|^2)$.Specifically,

$$v_{y_r,x_r}^{2k}=(-1)^k\sum_{j,p=0}^{\infty}\frac{k!\,2^p|x_r|^{2(p+j)}}{(k-j-p)!\,j!\,p!}(x_r\cdot y_r-|x_r|^2)^{2k-2j-p}(|x_r|^2-|y_r|^2)^j.$$

Substituting this into the expression and performing some manipulations we get the required result.

Corollary (5.2.24) [349]: For $m\in\mathbb{N}_0$ and $x_r\neq 0$:

$$\int_{\mathbb{B}^{1+\varepsilon}}(z_r\cdot y_r-z_r\cdot x_r)^m\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)=O\left((\varepsilon-1)^{-\left\lceil\frac{m}{4}\right\rceil}\right)\quad(\varepsilon\to\infty).$$

Proof. According we get for $x_r\neq 0$

$$\int_{\mathbb{B}^{1+\varepsilon}}(z_r\cdot y_r-z_r\cdot x_r)^m\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$

$$\sum_{k,j,p=0}^{\infty}\frac{\left(-\frac{m}{2}\right)_k\left(\frac{1-m}{2}\right)_k(-1)^k(z_r\cdot x_r)^{m-2k}v_{z_r,x_r}^{2k}2^p}{\left(a+\varepsilon+\frac{1}{2}\right)_k(k-j-p)!\,j!\,p!\,|x_r|^{2(m-p-j)}}O\left((\varepsilon-1)^{-\left\lceil\frac{m-j-p}{2}\right\rceil}\right).$$

Since $j+p\le k$ and $2k\le m$ the principal term is at most

$$O\left((\varepsilon-1)^{-\left\lceil\frac{m-\left\lceil\frac{m}{2}\right\rceil}{2}\right\rceil}\right)=O\left((\varepsilon-1)^{-\left\lceil\frac{m}{4}\right\rceil}\right).$$

Corollary (5.2.25) [349]: For $x_r\in\mathbb{B}^{1+\varepsilon},x_r\neq 0,\varepsilon>0$, and $f\in C^\infty(\mathbb{B}^{1+\varepsilon})$, there exist differential operators $Q_i:=Q_i(\Delta,x_r,\nabla,|x_r|^2)$, involving only the Laplace operator Δ, the directional derivative $x_r\cdot\nabla$ and the quantity $|x_r|^2$, such that

$$(B_{\varepsilon-1}f)(x_r) := \int_{\mathbb{B}^{1+\varepsilon}} f(y_r)\frac{R^2_{\varepsilon-1}(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r) \approx \sum_{i=0}^{\infty}\frac{Q_i f(x_r)}{(\varepsilon-1)^i} \quad (\varepsilon \to \infty),$$

where $Q_0 = 1$ and

$$Q_1 = \frac{\varepsilon-1}{2}\frac{1-|x_r|^2}{|x_r|^2}x_r\cdot\nabla + \frac{(\varepsilon-1)(1-|x_r|^2)^2}{4(\varepsilon)|x_r|^2}(x_r\cdot\nabla)^2 + \frac{1}{4(\varepsilon)}(1-|x_r|^2)^2\Delta.$$

Finally, for $x_r = 0$ it holds

$$(B_{\varepsilon-1}f)(0) \approx \sum_{i=0}^{\infty}\frac{\Delta^i f(0)}{4^i\left(\varepsilon-1+\frac{(1+\varepsilon)}{2}+1\right)_i} \quad (\varepsilon\to\infty).$$

Proof. Let us deal with the simpler case $x_r = 0$ first on which the general approach will be demonstrated.

The problem is to determine the asymptotic expansion of the integral

$$I_{\varepsilon-1}f := \int_{\mathbb{B}^{1+\varepsilon}} f(y_r)d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r).$$

Remember that

$$d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r) := (a+2\varepsilon)_{\varepsilon-1}(1-|y_r|^2)^{\varepsilon-1}d^{1+\varepsilon}y_r, \qquad (a+2\varepsilon)_{\varepsilon-1} = \frac{\Gamma(\widetilde{\varepsilon-1})}{\pi^{(1+\varepsilon)\backslash 2}\Gamma(\varepsilon)}.$$

We expand the function $f(y_r)$ into its Taylor series

$$f(y_r) = \sum_{k=0}^{2M-1}\frac{(y_r\cdot\nabla)^k f(0)}{k!} + H_{2M}(y_r),$$

and plug in to get

$$I_{\varepsilon-1}f = \sum_{k=0}^{M-1}\frac{1}{(2k)!}\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot\nabla)^{2k}f(0)d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r) + \int_{\mathbb{B}^{1+\varepsilon}}H_{2M}(y_r)d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r).$$

Notice that only the terms of even degree in the first integral survived. We can estimate the remainder term $H_{2M}(y_r)$ by the Taylor theorem as follows

$$|H_{2M}(y_r)| \le C \max_{|\gamma|-2M}\max_{y\in\mathbb{B}^{1+\varepsilon}}|\partial^\gamma f(y_r)||y_r{}^\gamma| \le C|y_r|^{2M}.$$

So

$$\int_{\mathbb{B}^{1+\varepsilon}}H_{2M}d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r) \le C\int_{\mathbb{B}^{1+\varepsilon}}|y_r|^{2M}d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r) = C\int_{\mathbb{B}^{1+\varepsilon}}(|y_r|^2)^M d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

$$= C\sum_{k=0}^{M}\binom{M}{k}(-1)^k\int_{\mathbb{B}^{1+\varepsilon}}d\mu^{1+\varepsilon}_{\varepsilon-1+k}(y_r)\frac{(a+2\varepsilon)_{\varepsilon-1}}{(a+2\varepsilon)_{\varepsilon-1+k}}$$

$$C\sum_{k=0}^{M}\binom{M}{k}(-1)^k\frac{(\varepsilon)_k}{(\widetilde{\varepsilon-1})_k} = C_2F_1\left(\begin{matrix}-M & \varepsilon\\ & (\widetilde{\varepsilon-1})\end{matrix};1\right) = C\frac{\Gamma(\widetilde{\varepsilon-1})\Gamma(a+\varepsilon+M)}{\Gamma(\widetilde{\varepsilon-1}+M)\Gamma(a+\varepsilon)} = C\frac{(a+\varepsilon)_M}{(\widetilde{\varepsilon-1})_M}$$

$$= O((\varepsilon-1)^{-M}).$$

This stems again from the identity

$${}_2F_1\left(\begin{matrix}a_1 & a_2\\ a+2\varepsilon & \end{matrix};1\right) = \frac{\Gamma(a+2\varepsilon)\Gamma(a+2\varepsilon-a_1-a_2)}{\Gamma(a+2\varepsilon-a_1)\Gamma(a+2\varepsilon-a_2)},$$

which is true for $a+2\varepsilon > a_1+a_2$.

It remains to deal with integrals

$$\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot\nabla)^{2k}d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)f(0).$$

Now, we argue that the operator ∇can be treated as an ordinary vector, i.e. it suffices to compute the expression

$$\int_{\mathbb{B}^{1+\varepsilon}} (y_r \cdot z_r)^{2k} d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$

and then in the result replace every occurrence of($z_r)_j$ by ∂_j. To conclude the first part of this proof it therefore suffices to show that

$$\int_{\mathbb{B}^{1+\varepsilon}} (y_r \cdot z_r)^{2k} d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r) = \frac{(2k)!}{k!\,4^k(\varepsilon-1)_k}|z_r|^{2k}.$$

This could be, of course, computed directly can also be used in the light of the following representation

$$\int_{\mathbb{B}^{1+\varepsilon}} (y_r \cdot z_r)^{2k} d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r) = \lim_{t\to 0} t^{-2k} \int_{\mathbb{B}^{1+\varepsilon}} (y_r \cdot tz_r)^{2k} R_{\varepsilon-1}^2(tz_r, y_r) d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r),$$

i.e. apply to $x_r = tz_r$ then divide by t^{2k} and let $t \to 0$.

In the case $x_r \neq 0$ the approach is almost identical. First we expand the function f into Taylor series around x_r

$$f(y_r) = \sum_{k=0}^{2M-1} \frac{\left((y_r - x_r)\cdot \nabla\right)^k f(x_r)}{k!} + H_{2M,x_r}(y_r).$$

We have

$$(B_{\varepsilon-1}f)(x_r) = \sum_{k=0}^{2M-1} \underbrace{\frac{1}{k!}\left(B_{\varepsilon-1}\left(\left((y_r - x_r\ )\cdot \nabla\right)^k f(x_r)\right)\right)(x_r)}_{=:(a+2\varepsilon)_k} + \left(B_{\varepsilon-1}H_{2M,x_r}\right)(x_r).$$

The remainder term can be estimated using the Taylor theorem:

$$\left|H_{2M,x}(y_r)\right| \le C \max_{|\gamma|=2M} \max_{y_r\in\mathbb{B}^{1+\varepsilon}} |\partial^\gamma f(y)||(y_r - x_r)^\gamma| \le C|y_r - x_r|^{2M}$$

for some constant C, whence

$$\left(B_{\varepsilon-1}H_{2M,x_r}\right)(x_r) = O\left(B_{\varepsilon-1}(|y_r - x_r|^{2m})(x_r)\right) = O\left((\varepsilon-1)^{-\left\lceil\frac{M}{2}\right\rceil}\right) \quad (\varepsilon\to\infty),$$

where the last equality.

Thus again, we have to deal only with the terms $(a+2\varepsilon)_k$ and they can further be modified by replacing ∇ by z_r:

$$(a+2\varepsilon)_k := \frac{1}{k!} B_{\varepsilon-1}\left(\left((y_r - x_r)\cdot z_r\right)^k\right)(x_r).$$

So we must only be able to determine asymptotic behavior of the Berezin transform of a polynomial. The fact that in this case there exists an asymptotic expansion in negative powers of $\varepsilon+1$ follows from, from where it is clear that terms $(a+2\varepsilon)_i$ can be written as finite combinations of functions $\frac{{}_5F_4}{{}_2F_1}$ whose asymptotic expansions are of this type. From that representation it is also possible to see the Stokes phenomenon, since for $x_r = 0$ the ratio equals 1 but for $0 < |x_r| < 1$ it decays in a way described .

Dependence of differential operators Q_i on $\Delta, x_r\cdot\nabla$ and $|x|^2$ only (that means on $|z_r|^2, x_r\cdot z_r$ and $|x_r|^2$) is a direct consequence 5. That $Q_0 = 1$ stems from the fact that $a_0 + 2\varepsilon = 1$ and $a_1 + 2\varepsilon$ is, according to Corollary6, $O((\varepsilon-1)^{-1})$.

To compute Q_1 much more work is needed. We are dealing with the expression $a_1 + 2\varepsilon + a_2 + a_3 + a_4 + 6\varepsilon$ –the term $a_5 + 2\varepsilon$ is according already $O((\varepsilon-1)^{-2})$. Application to $a_1 + a_2 + a_3 + a_4 + 6\varepsilon$, in general, leaves us with 19 terms. Fortunately many of them are negligible according (those for which $m - j - p > 2$) and collecting expressions involving the same integral will reduce the number to 5 terms:

$$\int_{\mathbb{B}^{1+\varepsilon}} \sum_{m=1}^{4} \frac{1}{m!}(z_r\cdot y_r - z_r\cdot x_r)^m \frac{R_{\varepsilon-1}^2(x_r, y_r)}{R_{\varepsilon-1}(x_r, x_r)} d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r) =$$

$$\left(\frac{x_r\cdot z_r}{|x_r|^2}-\frac{v_{z_r,x_r}^2}{(2a+2\varepsilon+1)|x_r|^2}\right)\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot x_r-|x_r|^2)\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$
$$-\frac{v_{z_r,x_r}^2}{(2a+2\varepsilon+1)|x_r|^2}\int_{\mathbb{B}^{1+\varepsilon}}(|x_r|^2-|y_r|^2)\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$
$$+\left(\frac{v_{z_r,x_r}^2 x\cdot z_r}{(2a+2\varepsilon+1)|x_r|^4}+\frac{v_{z_r,x_r}^4}{(2a+2\varepsilon+1)(2a+2\varepsilon+3)|x_r|^8}\right)\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot x_r-|x|^2)\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$
$$+\left(\frac{(x_r\cdot z_r)^2}{2|x|^4}-\frac{v_{z_r,x_r}^2(1+2z_r\cdot x_r)}{2(2a+2\varepsilon+1)|x_r|^4}+\frac{v_{z_r,x_r}^4}{2(2a+2\varepsilon+1)(2a+2\varepsilon+3)|x_r|^4}\right)\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot x_r-|x_r|^2)^2\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$
$$+\frac{v_{z_r,x_r}^4}{8(2a+2\varepsilon+1)(2a+2\varepsilon+3)|x_r|^4}\int_{\mathbb{B}^{1+\varepsilon}}(|x_r|^2-|y_r|^2)^2\frac{R_{\varepsilon-1}^2(x_r,y_r)}{R_{\varepsilon-1}(x_r,x_r)}d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$
$$+O((\varepsilon-1)^{-2}).$$

Each integral (if we for a moment put aside the factor $1/R_{\varepsilon+1}(x_r,x_r)$) a sum of functions ${}_5F_4$. Numbers of terms in these sums are, in general, again very high (in the case $N=2, M=0$ even as high as 108), however since we are interested only in principal terms and the order of asymptotic decay grows with summation indices $k_1,k_2,k_3,k_4,k_5,r,q,l,$ as can be seen from the proof , it is enough to consider only those summands for which

$$k_1+k_2+k_3+k_4+k_5+r+q+l=1.$$

This together with the condition

$$k_1+k_2+k_3+k_4+k_5+2r+q+2l\geq M+N$$

substantially reduces the number of terms. For the above-mentioned case $N=2, M=0$ we will be left with only 2 terms, both of which in addition contain the same hypergeometric function, so they can be combined together. Let us work this case out with more details, so we can demonstrate the approach. From we see that when $M=0$ then$l=0$. We substitute $k_1+k_2+k_3+k_4+k_5+r+q=1$ into the inequality to get

$$r+k_4\geq 1,$$

but it also must be the case that $r\leq 1$ and $k_4\leq 1$. This is only possible in two cases: $r=1$ or $k_4=1$ (with all other indices equal to zero). We find

$$\int_{\mathbb{B}^{1+\varepsilon}}(|x_r|^2-|y_r|^2)^2R_{\varepsilon-1}^2(x_r,y_r)d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$
$$=2\frac{|x_r|^2}{\widetilde{\widetilde{\varepsilon-1}}}(1-|x_r|^2)\,{}_3F_2\left(\begin{matrix}\widetilde{\varepsilon-1} & \widetilde{\varepsilon-1} & 2a+2\varepsilon\\ & \widetilde{(\varepsilon-1)}+1 & a+\varepsilon\end{matrix};|x_r|^2\right)+O_2.$$

where O_2 denotes a term of order $O\left((\varepsilon-1)^{-2}\,{}_2F_1\left(\begin{matrix}\widetilde{\varepsilon-1} & 2a+2\varepsilon\\ & a+\varepsilon\end{matrix};|x_r|^2\right)\right)$.

Similar considerations in the other cases give us:

$$\int_{\mathbb{B}^{1+\varepsilon}}(y_r\cdot x_r-|x_r|^2)^2R_{\varepsilon-1}^2(x_r,y_r)d\mu_{\varepsilon-1}^{1+\varepsilon}(y_r)$$
$$=\frac{1}{2}\frac{|x_r|^2}{\widetilde{\widetilde{\varepsilon-1}}}(1-|x_r|^2)\,{}_3F_2\left(\begin{matrix}\widetilde{\varepsilon-1} & \widetilde{\varepsilon-1} & 2a+2\varepsilon\\ & \varepsilon-1 & a+\varepsilon\end{matrix};|x_r|^2\right)+O_2,$$

$$\int_{\mathbb{B}^{1+\varepsilon}} (y_r \cdot x_r - |x_r|^2)(|x_r|^2 - |y_r|^2) R^2_{\varepsilon-1}(x_r, y_r) d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

$$= -\frac{|x_r|^2}{\varepsilon - 1}(1 - |x_r|^2)\, {}_3F_2\left(\begin{matrix}\varepsilon - 1 & \widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & \tilde{\alpha} + 1 & a + \varepsilon\end{matrix}; |x_r|^2\right) + O_2,$$

$$\int_{\mathbb{B}^{1+\varepsilon}} (y_r \cdot x_r - |x_r|^2) R^2_{\varepsilon-1}(x_r, y_r) d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r) = |x_r|^2\, {}_2F_1\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & a + \varepsilon + 1\end{matrix}; |x|^2\right) + O_2,$$

$$\int_{\mathbb{B}^{1+\varepsilon}} (|x_r|^2 - |y_r|^2) R^2_{\varepsilon-1}(x_r, y_r) d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

$$= \frac{|x_r|^2}{2}\, {}_3F_2\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) & 2(a+\varepsilon) \\ & a + \varepsilon & 2(a+\varepsilon)+1\end{matrix}; |x_r|^2\right) - \frac{a + \varepsilon + 1}{\varepsilon - 1}(1 - |x_r|^2)$$

$${}_2F_1\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & a + \varepsilon\end{matrix}; |x_r|^2\right) - 2|x|^2\, {}_2F_1\left(\begin{matrix}\varepsilon - 1 & 2(a+\varepsilon) \\ & a + \varepsilon + 1\end{matrix}; |x_r|^2\right)$$

$$+ \frac{|x_r|^2}{\varepsilon - 1}\, {}_3F_2\left(\begin{matrix}\widetilde{\varepsilon - 1} & \widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & \varepsilon - 1 & a + \varepsilon\end{matrix}; |x_r|^2\right) + O_2.$$

Substituting this into (27) and performing some manipulations we obtain

$$\int_{\mathbb{B}^{1+\varepsilon}} \sum_{m=1}^{4} \frac{1}{m!}(y_r \cdot z_r - x_r \cdot z_r)^m \frac{R^2_{\varepsilon-1}(x_r, y_r)}{R_{\varepsilon-1}(x_r, x_r)} d\mu^{1+\varepsilon}_{\varepsilon-1}(y_r)$$

$$= \frac{1}{2}\frac{v^2_{x_r, x_r}(a + \varepsilon + 1)(1 - |x_r|^2)}{(2(a+\varepsilon)+1)|x_r|^2 \varepsilon - 1} + z_r \cdot x_r \frac{{}_2F_1\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & a + \varepsilon + 1\end{matrix}; |x_r|^2\right)}{{}_2F_1\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & a + \varepsilon\end{matrix}; |x_r|^2\right)}$$

$$+ \frac{1}{4}\frac{2(z_r \cdot x_r)^2\big(1 + a + \varepsilon)(1 - |x_r|^2)\big) - |z_r|^2|x_r|^2(1 + |x_r|^2)}{|x_r|^2(2(a+\varepsilon)+1)\varepsilon - 1} \frac{{}_2F_1\left(\begin{matrix}\widetilde{\varepsilon - 1} & \widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & \widetilde{\varepsilon - 1} + 1 & a + \varepsilon\end{matrix}; |x_r|^2\right)}{{}_2F_1\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & a + \varepsilon\end{matrix}; |x_r|^2\right)}$$

$$- \frac{1}{4}\frac{v^2_{z_r, x_r}}{2b + 1} \frac{{}_3F_2\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) & 2(a+\varepsilon) \\ & a + \varepsilon & 2(a+\varepsilon)+1\end{matrix}; |x_r|^2\right)}{{}_2F_1\left(\begin{matrix}\widetilde{\varepsilon - 1} & 2(a+\varepsilon) \\ & a + \varepsilon\end{matrix}; |x_r|^2\right)} + O((\varepsilon - 1)^{-2}).$$

We complete the proof by using, replacing $v{x_r}^2_{,x_r} = |z_r|^2|x_r|^2 - (z_r \cdot x_r)^2$ and $(z_r)_j \to \partial_j$and remembering that $a = -(\frac{1+\varepsilon}{2})$.

Chapter 6

Weighted Reproducing and Berezin Transform

We generalize the recent result of C. Liu for the unit disc, as well as the original assertion concerning the holomorphic case. We also obtain a formula for the corresponding weighted harmonic Bergman kernels. We give is in the setting of the holomorphic and the harmonic Fock spaces, respectively.

Section (6.1): Weighted Reproducing Kernels and Toeplitz Operators on Harmonic Bergman Spaces on the Real Ball

For $\mathbb{B}^n$ be the ball in $\mathbb{R}^n, n \geq 2$, and dz the Lebesgue measure on $\mathbb{B}^n$. For $\alpha > -1$, consider the measure

$$dA_\alpha(z) := c_\alpha(1-|z|^2)^\alpha dz,$$

where

$$c_\alpha = \frac{\Gamma\left(\alpha+\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}}\Gamma(\alpha+1)}$$

is chosen so as to make dA_α a probability measure. For simplicity, we will usually assume that α is an integer.

The harmonic Bergman space $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ consists, by definition, of all harmonic functions in $L^2(\mathbb{B}^n, dA_\alpha)$. It is known that point evaluation functionals are continuous on the harmonic Bergman space, so it possesses a reproducing kernel; i.e., there exists a function $R_\alpha(x,y)$ on $\mathbb{B}^n \times \mathbb{B}^n$, harmonic in each variable, such that

$$f(x) = \int_{\mathbb{B}^n} f(y)R_\alpha(x,y)dA_\alpha(y)$$

for each $f \in L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ and for each $x \in \mathbb{B}^n$.

The Berezin transform of a bounded linear operator T on $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ is the

function $\tilde{T}^\alpha(z)$ on $\mathbb{B}^n$defined by

$$\tilde{T}^\alpha(z) = \frac{\langle T^{(\alpha)}R_{\alpha z}, R_{\alpha z}\rangle}{\langle R_{\alpha z}, R_{\alpha z}\rangle} = \frac{T^{(\alpha)}R_{\alpha z}(z)}{R_\alpha(z,z)},$$

where, for the sake of brevity, we have denoted $R_{\alpha z}(w) := R_\alpha(z,w)$.

Finally, for $f \in L^\infty(\mathbb{B}^n)$, the Toeplitz operator T_f with symbol f is the operator on $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ defined by

$$T_f g = Q_\alpha(fg),$$

Where $Q_\alpha: L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha) \to L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ is the orthogonal projection. That

is,

$$T_f g(z) = \int_{\mathbb{B}^n} g(x)f(x)R_\alpha(z,x)dA_\alpha(x).$$

It was shown by C. Liu [327] that if $n = 2$ (so that $\mathbb{B}^2$ is just the unit disc in the complex plane $\mathbb{C}$), then for $f \in C(\overline{\mathbb{B}^n})$,

$$\tilde{T}_f^{(\alpha)} \to \ f \tag{1}$$

uniformly, and

$$\left\|T_f^{(\alpha)}\right\| \to \|f\|_\infty \tag{2}$$

as $\alpha \ \to \ \infty$.
This extends the same result known previously for Toeplitz operators on Bergman spaces of holomorphic functions, which finds important applications in mathematical physics (quantization on K¨ahler manifolds; see e.g. [324]). generalize Liu's result also to $n \geq 3$. We first establish a (reasonably) explicit formula for the kernels $R_\alpha(x, y)$; this is done. Our main result (the generalization of (1) and (2)) is proved. We remark that we actually obtain a somewhat stronger result than (1); namely, we show that for any $f \in BC(\mathbb{B}^n) := C(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n)$ we also have

$$\tilde{T}_f^{(\alpha)}(z) \ \to \ f\ (z)$$

as $\alpha \ \to \ \infty$ for all $z \in \mathbb{B}^n$. This gives a new piece of information even for the original case $n = 2$.

In this part we will find an explicit formula for the reproducing kernel of the space $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$ of all harmonic functions on the unit ball in $\mathbb{R}^n$ square-integrable with respect to the measure $dA_\alpha(y)$, when α is an integer. Let $\mathcal{H}_m(\mathbb{R}^n)$ denote the space of the harmonic polynomials on $\mathbb{R}^n$ that are homogeneous of degree m.

For each $x \in \mathbb{R}$, there exists a unique function $Ƶ_m(\cdot, x) \in \mathcal{H}_m(S)$ such that

$$p(x) = \int\limits_S p(\xi)\overline{Ƶ_m(\xi, x)}\, d\sigma(\xi), \tag{3}$$

for all $p \in \mathcal{H}_m(S)$. The polynomial $Ƶ_m(\cdot, x)$ is called the zonal harmonic of degree m and pole x. See e.g. [323, p. 94]. It extends to a function on $\mathbb{R}^n \times \mathbb{R}^n$ by setting

$$Ƶ_m(x, y) = |x|^m |y|^m Ƶ_m(x/|x|, y/|y|)$$

for $m > 0$, and for $m = 0, Ƶ_0 = \ 1$.

Passing to polar coordinates $z \ = \ r\xi\ (r \ > \ 0, \xi \ \in \ S)$, the Lebesgue measure becomes

$$dz = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \ r^{n-1}\, dr\, d\sigma(\xi).$$

From (3), we have therefore for any $p \in \mathcal{H}_m$,

$$\int\limits_{\mathbb{B}^n} p(y)Ƶ_m(x, y) dA_\alpha(y) - SZ$$

$$= \frac{2\Gamma\left(\frac{n}{2}+\alpha+1\right)}{\alpha!\,\Gamma\left(\frac{n}{2}\right)}\int_0^1 r^{n-1}(1-r^2)^{\alpha}\int_S p(r\xi)\mathbb{Z}_m(x,r\xi)d\sigma(\xi)\,dr$$

$$= \frac{2\Gamma\left(\frac{n}{2}+\alpha+1\right)}{\alpha!\,\Gamma\left(\frac{n}{2}\right)}\int_0^1 r^{n+2m-1}(1-r^2)^{\alpha}\left(\int_S p(\xi)\mathbb{Z}_m(x,\xi)d\sigma(\xi)\right)dr$$

$$= \frac{2\Gamma\left(\frac{n}{2}+\alpha+1\right)}{\alpha!\,\Gamma\left(\frac{n}{2}\right)}p(x)\int_0^1 r^{n+2m-1}(1-r^2)^{\alpha}\,dr$$

$$= \frac{2\Gamma\left(\frac{n}{2}+\alpha+1\right)\alpha!\left(m+\frac{n}{2}-1\right)!}{\alpha!\,\Gamma\left(\frac{n}{2}\right)\left(\alpha+m+\frac{n}{2}\right)!}p(x) \qquad (4)$$

for each $x \in \mathbb{R}^n$.

Now recall that for any orthonormal basis $\{\varphi_j\}$ of $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$, the reproducing kernel R_α is given by

$$R_\alpha(x,\overline{y}) = \sum_{j=1}^{\infty}\varphi_j(x)\overline{\varphi_j(y)}\,.$$

(See e.g. [332].) Now each $\mathcal{H}_m$ is a closed subspace of $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$, the spaces $\mathcal{H}_m$ and $\mathcal{H}_k$ are orthogonal if $m \neq k$, and the span of all $\mathcal{H}_m, m \geq 0$, is the whole $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$; in other words,

$$\bigoplus_{m=1}^{\infty}\mathcal{H}_m = L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha).$$

Thus if we choose a basis $\{\varphi_{mj}\}_{j=1}^{\dim\mathcal{H}_m}$ in each $\mathcal{H}_m$ so that

$$\sum_{j=1}^{\dim\mathcal{H}_m}\varphi_j(x)\overline{\varphi_j(y)} \coloneqq K_m(x,y)$$

is the reproducing kernel of $\mathcal{H}_m$, then $\bigcup_{m=0}^{\infty}\{\varphi_{mj}\}_{j=1}^{\dim\mathcal{H}_m}$ is a basis for the whole $L^2_{\text{harm}}(\mathbb{B}^n, dA_\alpha)$; consequently,

$$K(x,y) = \sum_m^{\infty} K_m(x,y). \qquad (5)$$

On the other hand, from (4) we get

$$K_m(x,y) = \frac{\Gamma\left(\frac{n}{2}\right)\left(\alpha+m+\frac{n}{2}\right)!}{\Gamma\left(\frac{n}{2}+\alpha+1\right)\left(m+\frac{n}{2}-1\right)!}\mathbb{Z}_m(x,y).$$

Thus we arrive at the following result.

Proposition (6.1.1)[318]:

$$1R_\alpha(x,y) = \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{n}{2}\right)\left(\alpha+m+\frac{n}{2}\right)!}{\Gamma\left(\frac{n}{2}+\alpha+1\right)\left(m+\frac{n}{2}-1\right)!} Ƶ_m(x,y).$$

It should be noted that this result is definitely not new, see e.g. [325, p. 32 (1)], or [326, Section 2], or [329, Proposition 3], but it is convenient to recall it here.

For α an integer, the last sum can be summed explicitly. Recall [323, p. 178] that the usual Poisson kernel

$$P\ (x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{\frac{n}{2}}}$$

is equal to

$$P(x,y) = \sum_{m=0}^{\infty} Ƶ_m(x,y)$$

for $x, y \in B$. Hence, for α an integer ($\alpha = 0, 1, 2, \ldots$) we have

$$\left(t\frac{d}{dt}+\alpha\right)P(tx,y) = \sum_{m=0}^{\infty}\left(t\frac{d}{dt}+\alpha\right)Ƶ_m(x,y)$$

$$= \sum_{m=0}^{\infty}\left(t\frac{d}{dt}+\alpha\right)t^mƵ_m(x,y)$$

$$= \sum_{m=0}^{\infty}(m+\alpha)t^mƵ_m(x,y)$$

By iteration it follows that

$$\left(t\frac{d}{dt}+\frac{n}{2}\right)\cdots\left(t\frac{d}{dt}+\frac{n}{2}+\alpha\right)P(tx,y) = \sum_{m=0}^{\infty}\left(m+\frac{n}{2}\right)\cdots\left(m+\frac{n}{2}+\alpha\right)t^mƵ_m(x,y)$$

Consequently,

$$\left(t\frac{d}{dt}+\frac{n}{2}\right)\cdots\left(t\frac{d}{dt}+\frac{n}{2}+\alpha\right)P(tx,y)|_{t=1} = \sum_{m=0}^{\infty}\left(m+\frac{n}{2}\right)\cdots\left(m+\frac{n}{2}+\alpha\right)t^mƵ_m(x,y)$$

$$= \sum_{m=0}^{\infty}\frac{\left(m+\frac{n}{2}+\alpha\right)!}{\left(m+\frac{n}{2}-1\right)!}Ƶ_m(x,y)$$

$$= \frac{2\Gamma\left(\frac{n}{2}+\alpha+1\right)!}{\alpha!\,\Gamma\left(\frac{n}{2}\right)!}\alpha!\,R_\alpha(x,y)$$

so we get the following formula.

Proposition (6.1.2) [318]: Let $x, y \in B$. Then

$$R_\alpha(x,y)=\frac{\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n}{2}+\alpha+1\right)}\mathcal{D}_{\alpha+1}P(tx,y)|_{t=1} \qquad (6)$$

where

$$\mathcal{D}_{\alpha+1}=\left(t\frac{d}{dt}+\frac{n}{2}\right)\left(t\frac{d}{dt}+\frac{n}{2}+1\right)\cdots\left(t\frac{d}{dt}+\frac{n}{2}+\alpha\right)$$

Recall that the Mobius transformation φ_z is the smooth map of $\mathbb{B}^n$ onto itself defined for each $z \in \mathbb{B}^n$ by

$$\varphi_z(w)=\frac{|w-z|^2z-(1-|z|^2)(w-z)}{1-2\langle w,z\rangle+|w|^2|z|^2}$$

Where $\langle w,z\rangle := w_1z_1+\cdots+w_nz_n$ denotes the usual scalar product in $\mathbb{R}^n$. In the next lemma, we summarize the properties of the mapping φ_z. The proofs can be found e.g. in [331].

Lemma (6.1.3) [318]: For every $z \in \mathbb{B}^n$, φ_z has the following properties:

(i) $\varphi_z(0) = z$ and $\varphi_z(z) = 0$,

(ii) φ_z is an involution, i.e. $\varphi_z \circ \varphi_z = id$, the identity mapping,

(iii) the identity

$$1-|\varphi_z(w)|^2=\frac{(1-|z|^2)(1-|w|^2)}{1-2\langle w,z\rangle+|w|^2|z|^2}$$

holds for every $z, w \in \mathbb{B}^n$,

(iv) the identity

$$|\varphi_z'(w)|=\frac{1-|z|^2}{1-2\langle w,z\rangle+|w|^2|z|^2}$$

holds for every $z, w \in \mathbb{B}^n$.

Theorem (6.1.4) [318]: If $f \in BC(\mathbb{B}^n)$, the space of all bounded continuous functions on $\mathbb{B}^n$, then for each $z \in \mathbb{B}^n$,

$$\tilde{T}_f^{(\alpha)}(z) \to f(z)$$

as $\alpha \to \infty$ through the integers.

For $f \in C(\overline{\mathbb{B}^n})$, the convergence is even uniform on $\mathbb{B}^n$.

Proof: By the definitions,

$$\tilde{T}_f^{(\alpha)}(z)=\frac{\langle T_f^\alpha R_{\alpha z},R_{\alpha z}\rangle}{R_\alpha(z,z)}=\frac{\langle Q_\alpha fR_{\alpha z},R_{\alpha z}\rangle}{R_\alpha(z,z)}=\frac{\langle fR_{\alpha z},R_{\alpha z}\rangle}{R_\alpha(z,z)}=\int_{\mathbb{B}^n}\frac{f(w)R_{\alpha z}(w)\overline{R_{\alpha z}(w)}}{R_\alpha(z,z)}dA_\alpha(w)$$

$$= \int_{\mathbb{B}^n} f(w) \frac{|R_\alpha(z,w)|^2}{R_\alpha(z,z)} dA_\alpha(w)$$

Also, by reproducing property,

$$\int_{\mathbb{B}^n} f(z) \frac{|R_\alpha(z,w)|^2}{R_\alpha(z,z)} dA_\alpha(w) = f(z) \frac{\langle R_{\alpha z}, R_{\alpha z}\rangle}{R_\alpha(z,z)} = f(z)$$

Hence

$$\tilde{T}_f^{(\alpha)}(z) - f(z) = \int_{\mathbb{B}^n} \big(f(w) - f(z)\big) \frac{|R_\alpha(z,w)|^2}{R_\alpha(z,z)} dA_\alpha(w) \int_{|\varphi_z(w)|\le\delta} + \int_{|\varphi_z(w)|>\delta},$$

for any $0 < \delta < 1$.

By the continuity of f, we may, for each fixed z and $\varepsilon > 0$, choose $\delta > 0$ so small that

$$|f(w) - f(z)| < \varepsilon \qquad (7)$$

Whenever $|\varphi_z(w)| < \delta$. Then

$$\int_{|\varphi_z(w)|\le\delta} \le \varepsilon \underbrace{\int \frac{|R_\alpha(z,w)|^2}{R_\alpha(z,z)} dA_\alpha(w)}_{1} = \varepsilon$$

To estimate the second integral we use the following lemma:

Lemma (6.1.5) [318]: There exist constants c and C, depending only on α and n, such that for all $z, w \in \mathbb{B}^n$,

$$|R_\alpha(z,w)| \le \frac{C}{[z,w]^{\frac{n+\alpha}{2}}} \qquad (8)$$

and

$$\frac{c}{[z,z]^{\frac{n+\alpha}{2}}} \le R_\alpha(z,z) \le \frac{C}{[z,z]^{\frac{n+\alpha}{2}}} \qquad (9)$$

Here, for the sake of brevity, we have introduced the notation

$$[z,w] = 1 - 2\,\langle z,w\rangle + |z|^2|w|^2.$$

Postponing the proof of the lemma for a moment, using (8) and (9) we can estimate the integral over $|\varphi_z(w)| \ge \delta$ by

$$\int_{|\varphi_z(w)|>\delta} \le 2\|f\|_\infty \int_{|\varphi_z(w)|>\delta} \frac{|R_\alpha(z,w)|^2}{R_\alpha(z,z)} dA_\alpha(w) \le 2\|f\|_\infty \frac{C^2}{c} \int_{|\varphi_z(w)|>\delta} \frac{(1-|z|^2)^{n+\alpha}}{[z,w]^{n+\alpha}} dA_\alpha(w)$$

$$\le 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int_{|\varphi_z(w)|>\delta} \frac{(1-|z|^2)^{n+\alpha}(1-|w|^2)^{n+\alpha}}{[z,w]^{n+\alpha}} \frac{dw}{(1-|w|^2)^n}$$

$$\le 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int_{|\varphi_z(w)|>\delta} (1-|\varphi_z(w)|^2)^{n+\alpha} \frac{dw}{(1-|w|^2)^n}$$

Note that the measure

$$d\lambda(x) = \frac{dx}{(1-|x|^2)^n}$$

is invariant on $\mathbb{B}^n$, in the sense that

$$d\lambda(\varphi_a(x)) = \frac{|\varphi_a(x)|^n dx}{(1-|\varphi_a(w)|^2)^{n+\alpha}} = d\lambda(x),$$

by Lemma (6.1.3), parts (3) and (4).

Hence, making the change of variable $\varphi_z(w) = x$, we can continue with

$$\int\limits_{|\varphi_z(w)|>\delta} \leq 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int\limits_{|x|>\delta} (1-|x|^2)^{n+\alpha} d\lambda(x) = 2\|f\|_\infty \frac{C^2}{c} c_\alpha \int\limits_{|x|>\delta} (1-|x|^2)^\alpha dx$$

$$\leq 2\|f\|_\infty \frac{C^2}{c} c_\alpha (1-\delta^2)^\alpha$$

Since $c_\alpha \sim \alpha^{n/2}$ as $\alpha \to \infty$, the right-hand side tends to zero as $\alpha \to \infty$.

Hence

$$\lim_{\alpha\to\infty}\left(\int\limits_{|\varphi_z(w)|\leq\delta} + \int\limits_{|\varphi_z(w)|>\delta}\right) \leq \varepsilon.$$

If f is not only bounded and continuous on the ball but even continuous on its closure, then for each $\varepsilon > 0$ we can choose $\delta > 0$ so that (7) holds for all $z \in \mathbb{B}^n$ simultaneously, by uniform continuity. This completes the proof of Theorem (6.1.4).

The proof of (8) actually occurs in [328, Lemma 3.1] taking $t = 0, s = \alpha$ in the operator $Q_{s,t}$ there. For $|y| = 1$, see also [326, Lemma 2.7]. A simple proof of (9) can be found in [329, Proposition 4.1]. For α an integer, it is possible to give another proof using the formula (2). Since this might be useful for other applications we include it for completeness.

Consider the sets of functions

$$A_{\beta,r} := \left\{\frac{p(t,z,w)}{[tz,w]^{\frac{\beta}{2}}}\left(\frac{1-t^2|z|^2|w|^2}{[tz,w]^{\frac{1}{2}}}\right)^r : p \text{ a polynomial}\right\},$$

$$A_\beta := \left\{\frac{p(t,z,w)}{[tz,w]^{\frac{\beta}{2}}} q\left(\frac{1-t^2|z|^2|w|^2}{[tz,w]^{\frac{1}{2}}}\right)^r : p, q \text{ a polynomial}\right\},$$

Differentiation yields

$$t\frac{d}{dt}A_{\beta,r} = t\frac{d}{dt}\left(\frac{p(t,z,w)(1-t^2|z|^2|w|^2)^r}{[tz,w]^{\frac{\beta+r}{2}}}\right)$$

$$= \frac{t\frac{\partial p}{\partial t}(t,z,w)\cdot(1-t^2|z|^2|w|^2)^r}{[tz,w]^{\frac{\beta+r}{2}}}\cdot(1-t^2|z|^2|w|^2)^{r-1}$$

$$+\frac{t\cdot r\cdot p(t,z,w)\cdot(-2t^2|z|^2|w|^2)}{[tz,w]^{\frac{\beta+r}{2}}}\cdot(1-t^2|z|^2|w|^2)^{r-1}$$

$$-\frac{\frac{\beta+r}{2}\cdot p(t,z,w)(1-t^2|z|^2|w|^2)^r\cdot t\cdot(-2\langle z,w\rangle+2t|z|^2|w|^2)}{[tz,w]^{\frac{\beta+r}{2}+1}} \quad (10)$$

Since

$$-2\langle z,w\rangle+2t^2|z|^2|w|^2=[tz,w]-1-t^2|z|^2|w|^2+2t^2|z|^2|w|^2=[tz,w]-1+t^2|z|^2|w|^2$$
$$=-(1-t^2|z|^2|w|^2)+[tz,w]$$

The last term in (10) can be rewritten as

$$-\frac{\frac{\beta+r}{2}\cdot p(t,z,w)\cdot(1-t^2|z|^2|w|^2)}{[tz,w]\frac{\beta+r}{2}}\cdot(1-t^2|z|^2|w|^2)^{r-1}$$

$$+\frac{\frac{\beta+r}{2}\cdot p(t,z,w)\cdot(1-t^2|z|^2|w|^2)^{r+1}}{[tz,w]\frac{\beta+r+2}{2}}$$

$$[tz,w]\beta+r+2$$

Thus we get

$$t\frac{d}{dt}A_{\beta,r}\in A_{\beta+1,r-1}+A_{\beta+1,r+1}, \qquad r>0,$$

$$t\frac{d}{dt}A_{\beta,0}\in A_{\beta+1,0}+A_{\beta+1,1}, \qquad r=0.$$

For arbitrary $c\in\mathbb{R}$ we thus get

$$\left(t\frac{d}{dt}+c\right)A_{\beta,r}\in A_\beta+A_{\beta+1}$$

and

$$\mathcal{D}_{\alpha+1}A_{\beta,r}\in A_\beta+\cdots+A_{\beta+\alpha+1}.$$

Applying this to $\frac{1-t^2|z|^2|w|^2}{[tz,w]^{n/2}}\in A_{n-1,1}$ we thus obtain by (22)

$$R_\alpha(z,w)\in A_{n-1}+\cdots+A_{n+\alpha}|_{t=1}. \qquad (11)$$

Now

$$[z,w]=1-2\langle z,w\rangle+|z|^2|w|^2\geq 1-2|z||w|+|z|^2|w|^2$$

$$=\quad(1-|z||w|)^2>0$$

And

$$[z,w] \leq 1 + 2|z||w| + |z|^2|w|^2 = (1 + |z||w|)^2 \leq 2^2,$$

so

$$0 \leq \frac{1-|z|^2|w|^2}{[z,w]^{\frac{1}{2}}} \leq \frac{1-|z|^2|w|^2}{1-|z||w|} = 1+|z||w| \leq 2,$$

and thus every function from A_β has for $t = 1$ the form

$$\frac{\text{something bounded on } \mathbb{B}^n}{[z,w]^{\beta/2}}.$$

From (11) we thus get

$$R_\alpha(z,z) = \frac{\text{something bounded on } \mathbb{B}^n}{[z,w]^{(n+\alpha)/2}}$$

which proves (8).

To prove the second half of the lemma, note that for $z = w$,

$$\begin{aligned} R_\alpha(z,z) &= \left(t\frac{d}{dt}+n\right)\dots\left(t\frac{d}{dt}+n+\alpha\right)\frac{1-t^2|z|^4}{(1-t|z|^2)^n} \\ &= \left(t\frac{d}{dt}+n\right)\dots\left(t\frac{d}{dt}+n+\alpha\right)\frac{1-t^2|z|^4}{(1-t|z|^2)^{n-1}} \end{aligned}$$

Now again, for any polynomial p,

$$\begin{aligned} \left(t\ \frac{d}{dt}+c\right)\frac{p(t,|z|^2)}{(1-t|z|^2)^\beta} &= c\,\frac{p(t,|z|^2)}{(1-t|z|^2)^\beta} + \frac{t\frac{\partial}{\partial t}p(t,|z|^2)}{(1-t|z|^2)^\beta} + \frac{tp(t,|z|^2)\beta|z|^2}{(1-t|z|^2)^{\beta+1}} \\ &= \frac{\left(cp(t,|z|^2)+t\frac{\partial}{\partial t}p(t,|z|^2)\right)(1-t|z|^2)}{(1-t|z|^2)^{\beta+1}} + \frac{tp(t,|z|^2)\beta|z|^2}{(1-t|z|^2)^{\beta+1}} = \frac{p^*(t,|z|^2)}{(1-t|z|^2)^{\beta+1}}, \end{aligned}$$

where p^* is also a polynomial, which further satisfies $p^*(1,1) = \beta p(1,1)$. By iteration and taking $|z| = 1, t = 1$, we get

$$R_\alpha(z,z) = \frac{p(|z|^2)}{(1-|z|^2)^{n+\alpha}},$$

where P is a polynomial satisfying

$$P(1) = 2\frac{(n+\alpha-1)!}{(n-2)!}. \qquad (12)$$

Thus $R_\alpha(z,z)(1-|z|^2)^{n+\alpha}$ is a positive function inside $\mathbb{B}^n$, which has a nonzero finite limit (12) at the boundary. Consequently, it has positive and finite lower and upper bounds c and C, respectively. This concludes the proof of Lemma (6.1.5).

Corollary (6.1.6) [318]: For any $f \in C(\overline{\mathbb{B}^n})$,

$$\left\|T_f^{(\alpha)}\right\| \to \|f\|_\infty \qquad as\ \alpha \to \infty.$$

Proof: From the Schwarz inequality we get

$$\left|\tilde{T}_f^{(\alpha)}(z)\right| = \frac{\left|\langle T_f^{(\alpha)} R_{\alpha z}, R_{\alpha z}\rangle\right|}{R_\alpha(z,z)} = \frac{|\langle f R_{\alpha z}, R_{\alpha z}\rangle|}{R_\alpha(z,z)} = \frac{\|f R_{\alpha z}\|_{L^2}\|R_{\alpha z}\|_{L^2}}{\|R_{\alpha z}\|_{L^2}^2} \le \frac{\|f\|_\infty \|R_{\alpha z}\|_{L^2}^2}{\|R_{\alpha z}\|_{L^2}^2} = \|f\|_\infty$$

Taking the supremum over all z gives

$$\left\|\tilde{T}_f^{(\alpha)}\right\|_\infty \le \|f\|_\infty. \qquad (13)$$

On the other hand, by Theorem (6.1.4),

$$\tilde{T}_f^{(\alpha)}(z) \to f(z),$$

for all $z \in \mathbb{B}^n$. As $\left\|\tilde{T}_f^{(\alpha)}\right\|_\infty \ge \left|\tilde{T}_f^{(\alpha)}(z)\right|$and $\left|\tilde{T}_f^{(\alpha)}(z)\right| \to |f(z)|$, thus

$$\lim_{\alpha\to\infty} \inf\left\|\tilde{T}_f^{(\alpha)}\right\| \ge |f(z)|$$

Taking again the supremum over all $z \in \mathbb{B}^n$ yields

$$\lim_{\alpha\to\infty} \inf\left\|\tilde{T}_f^{(\alpha)}\right\|_\infty \ge \sup_{z\in\mathbb{B}^n} |f(z)| = \|f\|_\infty. \qquad (14)$$

From (13) and (14) we therefore have

$$\lim_{\alpha\to\infty} \left\|\tilde{T}_f^{(\alpha)}\right\| = \|f\|_\infty,$$

Which proves the "deformation estimate" (2).

Corollary (6.1.7) [349]: $R^j{}_{(\varepsilon-1)}(x_n, y_n) = \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{2+\varepsilon}{2}\right)\left(m+\frac{3\varepsilon}{2}\right)!}{\Gamma\left(\frac{3\varepsilon}{2}+1\right)\left(m+\frac{\varepsilon}{2}\right)!} \mathrm{Z}_m^j(x_n, y_n)$.

It should be noted that this result is definitely not new, see e.g. [355,p. 32 (1)], or [356, Section 2], or [359, Proposition 3], but it is convenient to recall it here. For($\varepsilon - 1$) an integer, the last sum can be summed explicitly. Recall [353,p. 178] that the usual Poisson kernel

$$P(x_n, y_n) = \frac{1 - |x_n|^2|y_n|^2}{(1 - 2x_n \cdot y_n + |x_n|^2|y_n|^2)^{\frac{\varepsilon+2}{2}}}$$

is equal to

$$P(x_n, y_n) = \sum_{m=0}^{\infty} \mathrm{Z}_m^j(x_n, y_n)$$

for $x_n, y_n \in B$. Hence, for (ε-1) an integer ($\varepsilon - 1 = 0, 1, 2, \ldots$) we have

$$\left(t\frac{d}{dt} + \varepsilon - 1\right) P(t x_n, y_n) = \sum_{m=0}^{\infty} \left(t\frac{d}{dt} + \varepsilon - 1\right) \mathrm{Z}_m^j(x_n, y_n)$$

$$= \sum_{m=0}^{\infty}\left(t\frac{d}{dt}+\varepsilon-1\right)t^m Z_m^j(x_n,y_n)$$

$$= \sum_{m=0}^{\infty}(m+\varepsilon-1)t^m Z_m^j(x_n,y_n)$$

By iteration it follows that

$$\left(t\frac{d}{dt}+\frac{2+\varepsilon}{2}\right)\cdots\left(t\frac{d}{dt}+\frac{\varepsilon}{2}+\varepsilon\right)P(tx_n,y_n)=\sum_{m=0}^{\infty}\left(m+\frac{2+\varepsilon}{2}\right)\cdots\left(m+\frac{\varepsilon}{2}+\varepsilon\right)t^m Z_m^j(x_n,y_n)$$

Consequently,

$$\left(t\frac{d}{dt}+\frac{2+\varepsilon}{2}\right)\cdots\left(t\frac{d}{dt}+\frac{3\varepsilon}{2}\right)P(tx_n,y_n)|_{t=1}=\sum_{m=0}^{\infty}\left(m+\frac{2+\varepsilon}{2}\right)\cdots\left(m+\frac{3\varepsilon}{2}\right)t^m Z_m^j(x_n,y_n)$$

$$= \sum_{m=0}^{\infty}\frac{\left(m+\frac{3\varepsilon}{2}\right)!}{\left(m+\frac{\varepsilon}{2}\right)!}Z_m^j(x_n,y_n)$$

$$= \frac{2\Gamma\left(\frac{3\varepsilon}{2}+1\right)!}{\Gamma\left(\frac{2+\varepsilon}{2}\right)!}R^j{}_{(\varepsilon-1)}(x_n,y_n)$$

Corollary (6.1.8) [349]: Let $x_n, y_n \in B$. Then

$$R^j{}_{(\varepsilon-1)}(x_n,y_n)=\frac{\Gamma\left(\frac{2+\varepsilon}{2}\right)}{2\Gamma\left(1+\frac{3\varepsilon}{2}\right)}\mathcal{D}_\varepsilon P(tx_n,y_n)|_{t=1}$$

where

$$\mathcal{D}_\varepsilon=\left(t\frac{d}{dt}+\frac{2+\varepsilon}{2}\right)\left(t\frac{d}{dt}+\frac{\varepsilon}{2}+2\right)\cdots\left(t\frac{d}{dt}+\frac{3\varepsilon}{2}\right)$$

Remark. The formula (6) has apparently also appeared in [355,p. 32 (1)], though in the somewhat different form

$$R^j{}_{(\varepsilon-1)}(x_n,y_n)=\left[\rho^{-(1+\varepsilon)}\left(\frac{\partial}{\partial\rho}\right)^{\varepsilon}\rho^{1+2\varepsilon}P(x_n,\rho^2 y_n{}')\right]_{\rho=\sqrt{|y_n|}}.$$

Recall that the Mobius transformation φ_{z_n} is the smooth map of $\mathbb{B}^{2+\varepsilon}$ onto itself defined for each $z_n\in\mathbb{B}^{2+\varepsilon}$

$$\varphi_{z_n}(w_n)=\frac{|w_n-z_n|^2 z_n-(1-|z_n|^2)(w_n-z_n)}{1-2\langle w_n,z_n\rangle+|w_n|^2|z_n|^2}$$

Where $\langle w_n, z_n \rangle := (w_n)_1(z_n)_1 + \cdots + (w_n)_{(2+\varepsilon)}(z_n)_{(2+\varepsilon)}$ denotes the usual scalar product in $\mathbb{R}^{2+\varepsilon}$.

For every $z_n \in \mathbb{B}^{2+\varepsilon}, \varphi_{z_n}$ has the following properties:

(i) $\varphi_{z_n}(0) = z_n$ and $\varphi_{z_n}(z_n) = 0$,

(ii) φ_{z_n} is an involution, i.e. $\varphi_{z_n} \circ \varphi_{z_n} = id$, the identity mapping,

(iii) the identity

$$1 - \left|\varphi_{z_n}(w_n)\right|^2 = \frac{(1 - |z_n|^2)(1 - |w_n|^2)}{1 - 2\langle w_n, z_n \rangle + |w_n|^2|z_n|^2}$$

holds for every $z_n, w_n \in \mathbb{B}^{2+\varepsilon}$,

(iv) the identity

$$\left|\varphi'_{z_n}(w_n)\right| = \frac{1 - |z_n|^2}{1 - 2\langle w_n, z_n \rangle + |w_n|^2|z_n|^2}$$

holds for every $z_n, w_n \in \mathbb{B}^{2+\varepsilon}. Now\ we\ have (see [350])$.

Corollary (6.1.9) [349]:. If $\sum f_j \in BC(\mathbb{B}^{2+\varepsilon})$, the space of all bounded continuous functions on $\mathbb{B}^{2+\varepsilon}$, then for each $z_n \in \mathbb{B}^{2+\varepsilon}$,

$$\sum \tilde{T}_{f_j}^{(\varepsilon-1)}(z_n) \to \sum f_j(z_n)$$

as $\varepsilon \to \infty$ through the integers.

For $\sum f_j \in C(\overline{\mathbb{B}^{2+\varepsilon}})$, the convergence is even uniform on $\mathbb{B}^{2+\varepsilon}$.

Bold. By the definitions,

$$\sum \tilde{T}_{f_j}^{(\varepsilon-1)}(z_n) = \sum \frac{\langle T_{f_j}^{(\varepsilon-1)} R^j{}_{(\varepsilon-1)z_n}, R^j{}_{(\varepsilon-1)z_n} \rangle}{R^j{}_{(\varepsilon-1)}(z_n, z_n)} = \sum \frac{\langle Q_{(\varepsilon-1)} f_j R^j{}_{(\varepsilon-1)z_n}, R^j{}_{(\varepsilon-1)z_n} \rangle}{R^j{}_{(\varepsilon-1)}(z_n, z_n)} =$$

$$= \int_{\mathbb{B}^{2+\varepsilon}} \sum \frac{f_j(w_n) R^j{}_{(\varepsilon-1)z_n}(w_n) \overline{R^j{}_{(\varepsilon-1)z_n}(w_n)}}{R^j{}_{(\varepsilon-1)}(z_n, z_n)} \, dA_{(\varepsilon-1)}(z_n)$$

$$= \int_{\mathbb{B}^{2+\varepsilon}} \sum f_j(w_n) \frac{\left|R^j{}_{(\varepsilon-1)}(z_n, w_n)\right|^2}{R^j{}_{(\varepsilon-1)}(z_n, z_n)} dA_{(\varepsilon-1)}(w_n)$$

Also, by reproducing property,

$$\int_{\mathbb{B}^{2+\varepsilon}} \sum f_j(z_n) \frac{\left|R^j{}_{(\varepsilon-1)}(z_n, w_n)\right|^2}{R^j{}_{(\varepsilon-1)}(z_n, z_n)} dA_{(\varepsilon-1)}(w_n) = \sum f_j(z_n) \frac{\langle R^j{}_{(\varepsilon-1)z_n}, R^j{}_{(\varepsilon-1)z_n} \rangle}{R^j{}_{(\varepsilon-1)}(z_n, z_n)} = \sum f_j(z_n)$$

Hence

$$\tilde{T}^{(\varepsilon-1)}_{\sum f_j}(z_n) - \sum f_j\,(z_n)$$
$$= \int\limits_{\mathbb{B}^{2+\varepsilon}} \left(\sum f_j\,(w_n) - \sum f_j\,(z_n)\right) \frac{\left|R^j{}_{(\varepsilon-1)}(z_n, w_n)\right|^2}{R^j{}_{(\varepsilon-1)}(z_n, z_n)}\, dA_{(\varepsilon-1)}(w_n) \int\limits_{|\varphi_{z_n}(w_n)|\le\delta} + \int\limits_{|\varphi_{z_n}(w_n)|>\delta},$$

for any $0 < \delta < 1$.

By the continuity of $\sum f_j$, we may, for each fixed z_n and $\varepsilon > 0$, choose $\delta > 0$ so small that

$$\left|\sum f_j\,(w_n) - \sum f_j\,(z_n)\right| < \varepsilon$$

Whenever $|\varphi_{z_n}(w_n)| < \delta$. Then

$$\int\limits_{|\varphi_{z_n}(w_n)|\le\delta} \le \varepsilon \underbrace{\int \frac{\left|R^j{}_{(\varepsilon-1)}(z_n, w_n)\right|^2}{R^j{}_{(\varepsilon-1)}(z_n, z_n)}\, dA_{(\varepsilon-1)}(w_n)}_{1} = \varepsilon$$

Corollary (6.1.10) [349]:. There exist constants c and C, depending only on($\varepsilon - 1$) and $(2+\varepsilon)$, such that for all $z_n, w_n \in \mathbb{B}^{2+\varepsilon}$

$$\left|R^j{}_{(\varepsilon-1)}(z_n, w_n)\right| \le \frac{C}{[z_n, w_n]^{\frac{2\varepsilon+1}{2}}}$$

And

$$\frac{c}{[z_n, z_n]^{\frac{2\varepsilon+1}{2}}} \le R^j{}_{(\varepsilon-1)}(z_n, z_n) \le \frac{C}{[z_n, z_n]^{\frac{2\varepsilon+1}{2}}}$$

Here, for the sake of brevity, we have introduced the notation

$$[z_n, w_n] = 1 - 2\,\langle z_n, w_n\rangle + |z_n|^2|w_n|^2.$$

Postponing the proof of the lemma for a moment, using (9) and (10) we can estimate the integral over $|\varphi_{z_n}(w_n)| \ge \delta$ by

$$\int\limits_{|\varphi_{z_n}(z_n)|>\delta} \le 2\left\|\sum f_j\right\|_\infty \int\limits_{|\varphi_{z_n}(w_n)|>\delta} \frac{\left|R^j{}_{(\varepsilon-1)}(z_n, w_n)\right|^2}{R^j{}_{(\varepsilon-1)}(z_n, z_n)}\, dA_{(\varepsilon-1)}(w_n)$$
$$\le 2\left\|\sum f_j\right\|_\infty \frac{C^2}{c} \int\limits_{|\varphi_{z_n}(w_n)|>\delta} \frac{(1-|z_n|^2)^{2\varepsilon+1}}{[z_n, w_n]^{2\varepsilon+1}}\, dA_{(\varepsilon-1)}(w_n)$$
$$\le 2\left\|\sum f_j\right\|_\infty \frac{C^2}{c} c_{(\varepsilon-1)} \int\limits_{|\varphi_{z_n}(w_n)|>\delta} \frac{(1-|z_n|^2)^{2\varepsilon+1}(1-|w_n|^2)^{2\varepsilon+1}}{[z_n, w_n]^{2\varepsilon+1}} \frac{dw_n}{(1-|w_n|^2)^{2+\varepsilon}}$$
$$\le 2\left\|\sum f_j\right\|_\infty \frac{C^2}{c} c_{(\varepsilon-1)} \int\limits_{|\varphi_{z_n}(w_n)|>\delta} \left(1 - \left|\varphi_{z_n}(w_n)\right|^2\right)^{2\varepsilon+1} \frac{dw_n}{(1-|w_n|^2)^{2+\varepsilon}}$$

Note that the measure

$$d\lambda(x_n) = \frac{dx_n}{(1-|x_n|^2)^{2+\varepsilon}}$$

is invariant on $\mathbb{B}^{2+\varepsilon}$, in the sense that

$$d\lambda(\varphi_a(x_n)) = \frac{|\varphi_a'(x_n)|^{2+\varepsilon} dx_n}{(1-|\varphi_a(x_n)|^2)^{2\varepsilon+1}} = d\lambda(x_n),$$

Hence, making the change of variable $\varphi_{z_n}(w_n) = x_n$, we can continue with

$$\int_{|\varphi_{z_n}(w_n)|>\delta} \leq 2\sum \|f_j\|_\infty \frac{C^2}{c} c_{(\varepsilon-1)} \int_{|x_n|>\delta} (1-|x_n|^2)^{2\varepsilon+1} d\lambda(x_n)$$

$$= 2\sum \|f_j\|_\infty \frac{C^2}{c} c_{(\varepsilon-1)} \int_{|x|>\delta} (1-|x_n|^2)^{(\varepsilon-1)} dx_n$$

$$\leq 2\sum \|f_j\|_\infty \frac{C^2}{c} c_{(\varepsilon-1)} (1-\delta^2)^{(\varepsilon-1)}$$

Since $c_{(\varepsilon-1)} \sim (\varepsilon-1)^{2+\varepsilon/2}$ as $\varepsilon \to \infty$, the right-hand side tends to zero as $\varepsilon \to \infty$.

Hence

$$\lim_{\varepsilon\to\infty}\left(\int_{|\varphi_{z_n}(w_n)|\leq\delta} + \int_{|\varphi_{z_n}(w_n)|>\delta}\right) \leq \varepsilon$$

If f_j is not only bounded and continuous on the ball but even continuous on its closure, then for each $\varepsilon > 0$ we can choose $\delta > 0$ so that (8) holds for all $z_n \in \mathbb{B}^{2+\varepsilon}$ simultaneously, by uniform continuity. This completes the proof. The proof of (9) actually occurs in [328, Lemma 3.1] taking $t = 0, s = (\varepsilon$-1) in the operator $Q_{s,t}$ there. For $|y_n| = 1$, see also [356, Lemma 2.7]. A simple proof of (10) can be found in [359, Proposition 4.1]. For (ε-1) an integer, it is possible to give another proof using the formula. Since this might be useful for other applications we include it for completeness.

$$A_{\beta,r} := \left\{\frac{p_j(t,z_n,w_n)}{[tz_n,w_n]^{\frac{\beta}{2}}}\left(\frac{1-t^2|z_n|^2|w_n|^2}{[tz_n,w_n]^{\frac{1}{2}}}\right)^r : p_j \ a \text{ polynomial}\right\},$$

$$A_\beta := \left\{\frac{p_j(t,z_n,w_n)}{[tz_n,w_n]^{\frac{\beta}{2}}} q_j \left(\frac{1-t^2|z_n|^2|w_n|^2}{[tz_n,w_n]^{\frac{1}{2}}}\right)^r : p_j, q_j \ a \text{ polynomial}\right\},$$

Differentiation yields

$$t\frac{d}{dt}A_{\beta,r} = t\frac{d}{dt}\left(\frac{p_j(t,z_n,w_n)\left(1-t^2|z_n|^2|w_n|^2\right)^r}{[tz_n,w_n]^{\frac{\beta+r}{2}}}\right)$$

$$= \frac{t\frac{\partial p_j}{\partial t}(t, z_n, w_n)\cdot(1 - t^2|z_n|^2|w_n|^2)^r}{[tz_n, w_n]^{\frac{\beta+r}{2}}}\cdot(1 - t^2|z_n|^2|w_n|^2)^{r-1}$$
$$+ \frac{t\cdot r\cdot p_j(t, z_n, w_n)\cdot(-2t^2|z_n|^2|w_n|^2)}{[tz_n, w_n]^{\frac{\beta+r}{2}}}\cdot(1 - t^2|z_n|^2|w_n|^2)^{r-1}$$
$$- \frac{\frac{\beta + r}{2}\cdot p_j(t, z_n, w_n)(1 - t^2|z_n|^2|w_n|^2)^r\cdot t\cdot(-2\langle z_n, w_n\rangle + 2t|z_n|^2|w_n|^2)}{[tz_n, w_n]^{\frac{\beta+r}{2}+1}}$$

Since

$$-2\langle z_n, w_n\rangle + 2t^2|z_n|^2|w_n|^2 = [tz_n, w_n] - 1 - t^2|z_n|^2|w_n|^2 + 2t^2|z_n|^2|w_n|^2$$
$$= [tz_n, w_n] - 1 + t^2|z_n|^2|w_n|^2 = -(1 - t^2|z_n|^2|w_n|^2) + [tz_n, w_n]$$

The last term in (12) can be rewritten as

$$-\frac{\frac{\beta + r}{2}\cdot p_j(t, z_n, w_n)\cdot(1 - t^2|z_n|^2|w_n|^2)}{[tz_n, w_n]\,\frac{\beta + r}{2}}\cdot(1 - t^2|z_n|^2|w_n|^2)^{r-1}$$
$$+ \frac{\frac{\beta + r}{2}\cdot p_j(t, z_n, w_n)\cdot(1 - t^2|z_n|^2|w_n|^2)^{r+1}}{[tz_n, w_n]\,\frac{\beta + r + 2}{2}}$$
$$[tz, w]\beta + r + 2$$

Thus we get

$$t\frac{d}{dt}A_{\beta,r} \in A_{\beta+1,r-1} + A_{\beta+1,r+1}, \qquad r > 0,$$
$$t\frac{d}{dt}A_{\beta,0} \in A_{\beta+1,0} + A_{\beta+1,1}, \qquad r = 0.$$

For arbitrary $c \in \mathbb{R}$ we thus get

$$\left(t\frac{d}{dt} + c\right)A_{\beta,r} \in A_\beta + A_{\beta+1}$$

and

$$\mathcal{D}_\varepsilon A_{\beta,r} \in A_\beta + \cdots + A_{\beta+\varepsilon}.$$

Applying this to $\frac{1-t^2|z_n|^2|w_n|^2}{[tz_n,w_n]^{(2+\varepsilon)/2}} \in A_{(1+\varepsilon),1}$ we thus obtain by (2.2)

$$R^j{}_{(\varepsilon-1)}(z_n, w_n) \in A_{(1+\varepsilon)} + \cdots + A_{(2\varepsilon+1)}\Big|_{t=1}.$$

Now

$$[z_n, w_n] = 1 - 2\langle z_n, w_n\rangle + |z_n|^2|w_n|^2 \geq 1 - 2|z_n||w_n| + |z_n|^2|w_n|^2$$
$$= (1 - |z_n||w_n|)^2 > 0$$

And

$$[z_n, w_n] \le 1 + 2|z_n||w_n| + |z_n|^2|w_n|^2 = (1 + |z_n||w_n|)^2 \le 2^2,$$

so

$$0 \le \frac{1-|z_n|^2|w_n|^2}{[z_n, w_n]^{\frac{1}{2}}} \le \frac{1-|z_n|^2|w_n|^2}{1-|z_n||w_n|} = 1 + |z_n||w_n| \le 2,$$

and thus every function from A_β has for $t = 1$ the form

$$\frac{\text{something bounded on } \mathbb{B}^{2+\varepsilon}}{[z_n, w_n]^{\beta/2}}.$$

From (13) we thus get

$$R^j{}_{(\varepsilon-1)}(z_n, z_n) = \frac{\text{something bounded on } \mathbb{B}^{2+\varepsilon}}{[z_n, w_n]^{(2\varepsilon+1)/2}}$$

which proves (9).

To prove the second half of the lemma, note that for $\varepsilon = 0$,

$$\begin{aligned} R^j{}_{(\varepsilon-1)}(z_n, z_n) &= \left(t\frac{d}{dt} + 2 + \varepsilon\right) \ldots \left(t\frac{d}{dt} + 2\varepsilon + 1\right)\frac{1 - t^2|z_n|^4}{(1-t|z_n|^2)^{2+\varepsilon}} \\ &= \left(t\frac{d}{dt} + 2 + \varepsilon\right) \ldots \left(t\frac{d}{dt} + 2\varepsilon + 1\right)\frac{1 - t^2|z_n|^4}{(1-t|z_n|^2)^{1+\varepsilon}} \end{aligned}$$

Now again, for any polynomial p_j,

$$\begin{aligned} \left(t\,\frac{d}{dt} + c\right)\frac{p_j(t,|z_n|^2)}{(1-t|z_n|^2)^\beta} &= c\,\frac{p_j(t,|z_n|^2)}{(1-t|z_n|^2)^\beta} + \frac{t\frac{\partial}{\partial t}p_j(t,|z_n|^2)}{(1-t|z_n|^2)^\beta} + \frac{tp_j(t,|z_n|^2)\beta|z_n|^2}{(1-t|z_n|^2)^{\beta+1}} \\ &= \frac{\left(cp_j(t,|z_n|^2) + t\frac{\partial}{\partial t}p_j(t,|z_n|^2)\right)(1-t|z_n|^2)}{(1-t|z_n|^2)^{\beta+1}} + \frac{tp_j(t,|z_n|^2)\beta|z_n|^2}{(1-t|z_n|^2)^{\beta+1}} \\ &= \frac{(p_j)^*(t,|z_n|^2)}{(1-t|z_n|^2)^{\beta+1}}, \end{aligned}$$

where $(p_j)^*$ is also a polynomial, which further satisfies $(p_j)^*(1,1) = \beta p_j(1,1)$. By iteration and taking $|z_n| = 1, t = 1$, we get

$$R^j{}_{(\varepsilon-1)}(z_n, z_n) = \frac{p_j(|z_n|^2)}{(1-|z_n|^2)^{2\varepsilon+1}},$$

where P is a polynomial satisfying

$$P(1) = 4.$$

Thus $R^{j}_{(\varepsilon-1)}\,(z_n, z_n)(1-|z_n|^2)^{2\varepsilon+1}$ is a positive function inside $\mathbb{B}^{2+\varepsilon}$, which has a nonzero finite limit (14) at the boundary. Consequently, it has positive and finite lower and upper bounds c and C, respectively. This concludes the proof.

Corollary (6.1.11) [349]:. For any $\sum f_j \in C(\overline{\mathbb{B}^{2+\varepsilon}})$,

$$\sum \left\|T_{f_j}^{(\varepsilon-1)}\right\| \to \sum \|f_j\|_\infty \qquad as\ \varepsilon \to \infty.$$

Proof. From the Schwarz inequality we get

$$\sum \left|\tilde{T}_{f_j}^{(\varepsilon-1)}(z_n)\right| = \sum \frac{\left|\langle T_{f_j}^{(\varepsilon-1)} R^j{}_{(\varepsilon-1)z_n}, R^j{}_{(\varepsilon-1)z_n}\rangle\right|}{R^j{}_{\varepsilon-1}(z_n, z_n)} = \sum \frac{\left|\langle f_j R^j{}_{(\varepsilon-1)z_n}, R^j{}_{(\varepsilon-1)z_n}\rangle\right|}{R^j{}_{(\varepsilon-1)}(z_n, z_n)}$$

$$= \sum \frac{\left\|f_j R^j{}_{(\varepsilon-1)z_n}\right\|_{L^2}\left\|R^j{}_{(\varepsilon-1)z_n}\right\|_{L^2}}{\left\|R^j{}_{(\varepsilon-1)z_n}\right\|_{L^2}^2} \le \sum \frac{\|f_j\|_\infty \left\|R^j{}_{(\varepsilon-1)z_n}\right\|_{L^2}^2}{\left\|R^j{}_{(\varepsilon-1)z_n}\right\|_{L^2}^2} = \sum \|f_j\|_\infty$$

Taking the supremum over all z_n gives

$$\sum \left\|\tilde{T}_{f_j}^{(\varepsilon-1)}\right\|_\infty \le \sum \|f_j\|_\infty$$

On the other hand,

$$\sum \tilde{T}_{f_j}^{(\varepsilon-1)}(z_n) \to \sum f_j\,(z_n),$$

for all $z_n \in \mathbb{B}^{2+\varepsilon}$. As $\sum \left\|\tilde{T}_{f_j}^{(\varepsilon-1)}\right\|_\infty \ge \sum \left|\tilde{T}_{f_j}^{(\varepsilon-1)}(z_n)\right|$ and $\sum \left|\tilde{T}_{f_j}^{(\varepsilon-1)}(z_n)\right| \to \sum \left|f_j(z_n)\right|$, thus

$$\lim_{\varepsilon\to\infty} \inf \sum \left\|\tilde{T}_{f_j}^{(\varepsilon-1)}\right\| \ge \sum \left|f_j(z_n)\right|$$

Taking again the supremum over all $z_n \in \mathbb{B}^{2+\varepsilon}$ yields

$$\lim_{\varepsilon\to\infty} \inf \sum \left\|\tilde{T}_{f_j}^{(\varepsilon-1)}\right\|_\infty \ge \sup_{z\in\mathbb{B}^{2+\varepsilon}} \sum \left|f_j(z_n)\right| = \sum \|f_j\|_\infty. \quad (16)$$

From (15) and (16) we therefore have

$$\lim_{\varepsilon\to\infty} \sum \left\|\tilde{T}_{f_j}^{(\varepsilon-1)}\right\| = \left\|\sum f_j\right\|_\infty,$$

Which proves the "deformation estimate" (2).

Section (6.2); Berezin Transform of Two Arguments

For the weighted Bergman space $L^2_{hol}(\Omega, w^\alpha)$ of holomorphic and square-integrable functions on a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, the Berezin transform

$$B_\alpha f(x) = \frac{\langle f K_{\alpha,x}, K_{\alpha,x}\rangle}{\langle K_{\alpha,x}, K_{\alpha,x}\rangle} = \int_\Omega f(y) \frac{|K_\alpha(x,y)|^2}{K_\alpha(x,x)} w(y)^\alpha dy, \qquad (15)$$

where w is a weight satisfying some mild technical hypotheses, $K_{\alpha,y} := K_\alpha(\cdot, y)$ is the reproducing kernel of $L^2_{hol}(\Omega, w^\alpha)$, and f is a bounded smooth function on Ω, has the following asymptotic behavior:

$$B_\alpha f \sim \sum_{j=0}^\infty \frac{Q_j f}{\alpha^j}, \quad as\ \alpha \to \infty, \alpha \in \mathbb{Z}, \qquad (16)$$

where Q_j are certain differential operators with $Q_0 = I$, the identity operator and $Q_1 = \Delta$, the Laplace–Beltrami operator associated to w (see e.g.[326]). This result can be accordingly exploited to define certain star-product that coincides with the so-called Berezin star-product going back to Berezin [335]yielding what is known as the Berezin deformation quantization on Ω. With some little expenditure of time and effort, this can be further extended, asthe titles of some of the references already correctly suggest, even to general Kähler manifolds. For detailed information on this theme, [331] or [331]. Also, the "intertwined" sibling of the Berezin quantization, the so-called Berezin–Toeplitz deformation quantization , represents a quite well established area of interest in the realm of Kähler manifolds as witnessed for example in [333]. Yet another field where Berezin transform makes appearance is representation theory of semisimple Lie groups, notably in decompositions of tensor product of representations.

Interestingly enough, results similar to the expansion (16), albeit without any apparent direct applications to quantization procedures or to representation theory, have been recently shown to be true even in the case of harmonic Bergman spaces (though only for the unit ball [335], the half-space[342], and the whole of $\mathbb{R}^n$ with n even[330]).

Note that (15) actually represents the Berezin transform $B_\alpha f(x)$ as the restriction to the diagonal $x = z$ of a function of two variables

$$B_\alpha^2 f(x,z) = \frac{\langle f K_{\alpha,z}, K_{\alpha,x}\rangle}{\langle K_{\alpha,z}, K_{\alpha,x}\rangle} = \int_\Omega f(y)\frac{K_\alpha(x,y)K_\alpha(y,z)}{K_\alpha(x,z)} w(y)^\alpha dy\,,$$

holomorphic in x and conjugate-holomorphic in z; ofcourse, theright-hand side is only defined when $K_\alpha(x,z) \neq 0$. Infact, byaclassical result from complex function theory, $B_\alpha^2 f$ is uniquely determined by $B_\alpha f$. Now in view of (16), it is very natural to ask – albeit this has no direct relevance whatsoever either for the quantization or for the applications in representation theory mentioned above – what is the asymptotic behavior of $B_\alpha^2 f$ as $\alpha \to \infty$. Plainly, from $B_\alpha f \to Q_0 f = f$one would (heuristically) expect $B_\alpha^2 f$ to tend to some "sesqui-holomorphic extension" $f(x,z)$ of f holomorphic in $x, \bar{z}$ and satisfying $f(x,x) = f(x)$; clearly this fails to exist for general f(e.g.when f is not real-analytic), and accordingly we will thus deal for the most part only with $f(z)$ a polynomial in z and $\bar{z}$.

Furthermore, all the above makes perfect sense also of harmonic (rather than holomorphic) Bergman spaces, where actually the situation becomes even more complicated since $B_\alpha^2 f$ is then no longer uniquely determined by its restriction $B_\alpha f$ to the diagonal.

We is to investigate the asymptotic behavior of B_α^2 in the case of the Fock spaces of holomorphic functions on $\mathbb{C}^n$ (the "easy part"), and of the Bergman spaces of harmonic functions on $\mathbb{R}^n$(the "hard part"), although only for polynomial symbols f.

Consider the Segal–Bargmann or Fock space $\mathcal{F}_\alpha$of all entire functions in $\mathbb{C}^n$ that are square-integrable with respect to the measure

$$d\mu_\alpha^{2n}(y,\bar{y}) := \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|y|^2} d\lambda(y),$$

where $d\lambda(y)$ is the usual $2n$-dimensional Lebesgue volume measure with the factor $(\alpha/\pi)^n$chosen so that the whole space is of measure one.

The function $K_\alpha(x,y) = e^{\alpha x\cdot\bar{y}} = e^{\alpha(x_1\bar{y}_1+\cdots+x_n\bar{y}_n)}$ is the corresponding Bergman kernel, i.e. a mapping $K_\alpha: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ which is holomorphic in the first argument and anti-holomorphic in the second, skew-symmetric: $K_\alpha(x,y) = \overline{K_\alpha(y,x)}$, and satisfies the reproducing property

$$(P_\alpha f)(x) := \int_{\mathbb{C}^n} f(y) K_\alpha(x,y) d\mu_\alpha^{2n}(y,\bar{y}) = f(x), \qquad \forall f \in \mathcal{F}_\alpha.$$

The associated Berezin transform

$$(B_\alpha f)(x) := \int_{\mathbb{C}^n} f(y)\frac{K_\alpha(x,y)\overline{K_\alpha(x,y)}}{K_\alpha(x,x)} d\mu_\alpha^{2n}(y,\bar{y})\,,$$

then behaves asymptotically as $\alpha \to \infty$ like the identity. it follows from the standard asymptotic analysis of the Gaussian integral (the result appears also in [330]) that, as $\alpha \to \infty$,

$$(B_\alpha f)(x) \to f(x) \qquad \forall f \in L^\infty(C^n),$$

and, whenever f is smooth in the neighborhood of x, there is in addition the full asymp-totic expansion of the type(16), namely

$$(B_\alpha f)(x) \sim f(x) + \frac{\partial_x \bar{\partial}_x f(x)}{\alpha} + \frac{(\partial_x \bar{\partial}_x)^2 f(x)}{2!\alpha^2} + \cdots \tag{17}$$

We establish the asymptotic behavior of a modified Berezin transform – the Berezin transform of two arguments, with one of the x's replaced by a new vector z:

$$(B_\alpha^2 f)(x,z) := \int_{\mathbb{C}^n} f(y) \frac{K_\alpha(x,y)K_\alpha(y,z)}{K_\alpha(x,z)} d\mu_\alpha^{2n}(y,\bar{y}).$$

Note that the transformed function is of two variable x and z even though the original f is of one (vector) variable. For the purpose of iteration it is convenient to define the function $(B_\alpha^2 f)(x,z)$for f depending on two variables as follows

$$(B_\alpha^2 f)(x,z) := \int_{\mathbb{C}^n} f(y,\bar{y}) \frac{K_\alpha(x,y)K_\alpha(y,z)}{K_\alpha(x,z)} d\mu_\alpha^{2n}(y,\bar{y}).$$

We will show that the principal term of the asymptotic expansion behaves still like an identity of a kind and that there is also an asymptotic expansion strikingly similar to that in (17). More explicitly, we will prove the following:

Theorem (6.2.1)[329]: Let f be a polynomial on $\mathbb{C}^n \times \mathbb{C}^n$. Then, as $\alpha \to \infty$,

$$(B_\alpha^2 f)(x,z) \sim f(x,\bar{z}) + \frac{\partial_x \bar{\partial}_z f(x,\bar{z})}{\alpha} + \frac{(\partial_x \bar{\partial}_z)^2 f(x,\bar{z})}{\alpha^2 2!} + \cdots$$

Remark (6.2.2) [329]: Note that when $z = x$ the result is exactly the same as in(17).

It will also be clear from the proof that in this case the Berezin transform of two arguments is nothing more than a special case of the Bergman projection, namely:

$$(B_\alpha^2 f)(x,z) = (P_\alpha f_z)(x-z), \qquad f_z(y,\bar{y}) := f(y+z, \bar{y}+\bar{z}),$$

hence it does not bring anything new to the subject. This example shows what a nice behavior the Berezin transform of two arguments can act . We now move our attention , the case of harmonic Fock space, where much more interesting behavior occurs.

Consider the harmonic Fock space F_α^{harm} of all harmonic functions in $\mathbb{R}^n$ that are square-integrable with respect to the measure

$$d\mu_\alpha^n(y) := c_\alpha e^{-\alpha|y|^2} d^n y,$$

where the factor $c_\alpha := \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}}$is again chosen so that the whole space is of measure one.

The Bergman kernel for this space, $R_\alpha(x,y)$, i.e. a function with the following properties:

$$R_\alpha(x,y) = R_\alpha(y,x) \qquad \forall x,y \in \mathbb{R}^n, \tag{18}$$

$$\Delta_x R_\alpha(x,y) = \Delta_y R_\alpha(x,y) = 0 \quad \forall x,y \in \mathbb{R}^n, \tag{19}$$

$$(P_\alpha f)(x) := \int_{\mathbb{R}^n} f(y) R_\alpha(x,y) d\mu_\alpha^n(y) = f(x) \forall f \in \mathcal{F}_\alpha^{harm}, \forall x \in \mathbb{R}^n, \tag{20}$$

is not of such a pleasing form as in the complex case; however, it is known that [10]:

$$R_\alpha(x,y) = \Phi_2\left(\begin{matrix} - \\ b \end{matrix}; \begin{matrix} b & b \\ & - \end{matrix}; \alpha u_{x,y}, \alpha \bar{u}_{x,y}\right), \tag{21}$$

where $b := \frac{n}{2} - 1, u_{x,y} = x \cdot y + i\sqrt{|x|^2|y|^2 - (x \cdot y)^2}$ and Φ_2 is a hypergeometric function of two variables from Horn's list [332], defined1by means of the series:

$$\Phi_2\left(\begin{matrix} - \\ c \end{matrix}; \begin{matrix} b_1 & b_2 \\ & - \end{matrix}; x,y\right) = \sum_{j,k=0}^{\infty} \frac{(b_1)_j (b_2)_k}{(c)_{j+k}} \frac{x^j y^k}{j!k!}, \qquad \forall x,y \in \mathbb{C}. \tag{22}$$

In the same [340], it has also been shown that the Berezin transform

$$(B_\alpha f)(x) := \int_{\mathbb{R}^n} f(y) \frac{R_\alpha^2(x,y)}{R_\alpha(x,y)} d\mu_\alpha^n(y)$$

again displays an asymptotic behavior similar to that in the complex case(17). Inparticular, for every $f \in L^\infty(\mathbb{R}^{2n})$, smooth in a neighborhoodof $x \neq 0$, wehave, as $\alpha \to \infty$,

$$(B_\alpha f)(x) \sim f(x) + \frac{1}{\alpha}\Big(\frac{n-2}{2}\frac{1}{|x|^2} x \cdot \nabla$$

$$+ \frac{(n-2)}{4(n-1)|x|^2}(x \cdot \nabla)^2 + \frac{1}{4(n-1)}\Delta\Big) f(x) + \ldots, \tag{23}$$

with the additional feature that for $x = 0$, in which case the terms in the asymptotic series(23)are singular, the behavior changes abruptly[2]:

$$(B_\alpha f)(0) \sim f(0) + \frac{1}{4\alpha} \Delta f(0) + \ldots, \qquad \alpha \to \infty.$$

This is an interesting manifestation of a kind of the so-called Stokes phenomenon. Wewill show that in case of the Berezin transform of two arguments

$$(B_\alpha^2 f)(x,z) := \int_{\mathbb{R}^n} f(y) \frac{R_\alpha(x,y) R_\alpha(z,y)}{R_\alpha(x,z)} d\mu_\alpha^n(y) = (B_\alpha^2 f)(z,x), \qquad (24)$$

the Stokes phenomenon appears multiple times – the asymptotic behavior is different for x, z not collinear, for $x = \xi z$, for $x = -\xi z$ where $\xi > 0$ and for $x = 0$.

In the latter case the Berezin transform of two arguments actually collapses into the Bergman projection

$$(B_\alpha^2 f)(0,z) = (P_\alpha f)(z),$$

which is (contrary to the complex case) the only relationship between the two.

The main question is addressing is what is the limit behavior of B_α^2 and, most importantly, whether there exists a point v such that, by analogy with the single argument Berezin transform,

$$(B_\alpha^2 f)(x,z) \to f(v), \quad \alpha \to \infty. \qquad (25)$$

The answer we give is affirmative, on condition that the limit is taken through complex values of α and that x, z are not collinear. For the real values of α (and x, z real as well), there is in general no such v and the limit even does not exist for any polynomial except the constant ones. For x, z collinear the limit does exist (mostly) but it does not behave like the identity – except , when $z = x -$ the case of the usual Berezin transform.

The results can be summarized in the following two theorems:

Theorem (6.2.3) [329]: Let p be a polynomial on $\mathbb{R}^n, x, z \in \mathbb{R}^n$ not collinear. Then for $\alpha \in \mathbb{C}$ such that $(\alpha u_{x,z}) > Re(\alpha \bar{u}_{x,z}), Re(\alpha u_{x,z}) > 0, Re(\alpha) > 0$, [3] we have

$$(B_\alpha^2 p)(x,z) \to p(v), \qquad |\alpha| \to \infty,$$

where

$$v = v_{x,z} := x\, \frac{u_{x,z} - |z|^2}{u_{x,z} - \bar{u}_{x,z}} + z \frac{u_{x,z} - |x|^2}{u_{x,z} - \bar{u}_{x,z}}, u_{x,z} := x \cdot z + i\sqrt{|x|^2|z|^2 - (x \cdot z)^2},$$

and the point $v \in \mathbb{C}^n$ moreover satisfies the following relations:

$$Re\, v = \frac{x+z}{2}, v \cdot \bar{v} = |x+z|^2 + |x-z|^2, v \cdot v = u_{x,z}, \qquad x \cdot v = \frac{|x|^2 + u_{x,z}}{2},$$

$$(x \cdot v)(\overline{x \cdot v}) = \frac{|x|^2|x+z|^2}{4}, \qquad \bar{u}_{x,z}(x \cdot v) = |x|^2(\overline{z \cdot v}),$$

$$(x \cdot v)(z \cdot v) = \frac{u_{x,z}|x+z|^2}{4}.$$

For $\alpha \in \mathbb{R}$ and $\nabla p \neq 0$, the limit does not exist.

In the collinear case, the behavior is the following:

Theorem (6.2.4) [329]: Let p be a polynomial on $\mathbb{R}^n, \xi, \alpha > 0, u_{x,t}$ as above. Then

(i)For $z = \xi x$,

$$(B_\alpha^2 p)(x, \xi x) \to p(\nabla_t) e^{x \cdot t}\, \Phi_2 \left(n - 2; \frac{n}{2} - 1 \quad \frac{n}{2} - 1\, ; \frac{\xi - 1}{2} u_{x,t}, \frac{\xi - 1}{2} \bar{u}_{x,t} \right) \Bigg|_{t=0},$$

$$\alpha \to \infty.$$

(ii)For $z = 0$,

$$(B_\alpha^2 p)(x, 0) \to p(\nabla_t)\, \Phi_2 \left(\frac{n}{2} - 1; \frac{n}{2} - 1 \quad \frac{n}{2} - 1\, ; \frac{1}{2} u_{x,t}, \frac{1}{2} \bar{u}_{x,t} \right) \Bigg|_{t=0},$$

$$\alpha \to \infty.$$

(iii)For $z = -\xi x$ and $p(y) = p_1(y, |y|^2)$, where p_1 is a linear function in the first argument and a polynomial in the other, wehave

$$(B_\alpha^2 p)(x,-\xi x) \to p_1\left(x\frac{1-\xi}{2}, -\xi\,|x|^2\right), \qquad n>3 \ even,$$
$$(B_\alpha^2 p)(x,-\xi x) \to p(0), \qquad n>1 \ odd,$$
$$|(B_\alpha^2(y_1y_2))(x,-\xi x)| \to \infty, \qquad all\ n>2.$$
The polynomial y_1y_2 above serves only as an example of a divergent behavior of B_α^2 in the case $z = -\xi x$. It is not clear to the authors, what is the general behavior in this case, but we strongly suspect that there are no other polynomials for which B_α^2 would converge (for $n>2$) other than p_1.
Collecting the exponentials from the Bergman kernels and from the measure $d\mu_\alpha^{2n}$ we obtain
$$(B_\alpha^2 f)(x,z) = \int_{\mathbb{C}^n} f(y,\bar{y}) e^{-\alpha(y-x)\cdot(\bar{y}-\bar{z})} \left(\frac{\alpha}{\pi}\right)^n d\lambda(y)\,.$$
After the change of variables $y \longmapsto y+z$ we get
$$\int_{\mathbb{C}^n} f(y+z,\bar{y}+\bar{z}) e^{\alpha(x-z)\cdot\bar{y}} e^{-\alpha|y|^2} \left(\frac{\alpha}{\pi}\right) d\lambda(y) = (P_\alpha f_z)(x-z),$$
$$(f_z(y,\bar{y}) := f(y+z,\bar{y}+\bar{z})).$$
Thus the Berezin transform of two arguments can be reduced to a computation of the Bergman projection. Since any Bergman projection is a function from the Fock space, that is entire and square-integrable, we must have
$$(P_\alpha f_z)(x-z) = \sum_{k=0}^{\infty} \frac{\left((x-z)\cdot\partial_t\right)^k}{k!} (P_\alpha f_z)(t)|_{t=0},$$
where $\partial_t := \left(\frac{1}{2}\left(\frac{\partial}{\partial t_1^1} - i\frac{\partial}{\partial t_2^1}\right), \dots, \frac{1}{2}\left(\frac{\partial}{\partial t_1^n} - i\frac{\partial}{\partial t_2^n}\right)\right)$ and the series converges absolutely for any $x-z$.
But
$$\begin{aligned}(x-z)\cdot\partial_t(P_\alpha f_z)(t) &= \int_{\mathbb{C}^n} f_z(y,\bar{y})(x-z)\cdot\partial_t e^{\alpha t\cdot\bar{y}} d\mu_\alpha^{2n}(y,\bar{y}) \\ &= \int_{\mathbb{C}^n} f_z(y,\bar{y})\alpha(x-z)\cdot\bar{y} e^{\alpha t\cdot\bar{y}} e^{-\alpha y\cdot\bar{y}} \left(\frac{\alpha}{\pi}\right)^n d\lambda(y) \\ &= \int_{\mathbb{C}^n} f_z(y,\bar{y})\left(-(x-z)\cdot\partial_y\right) e^{\alpha t\cdot\bar{y}} e^{-\alpha y\cdot\bar{y}} \left(\frac{\alpha}{\pi}\right)^n d\lambda(y) \\ &= \int_{\mathbb{C}^n} \left((x-z)\cdot\partial_y f_z\right)(y,\bar{y}) e^{\alpha t\cdot\bar{y}} e^{-\alpha y\cdot\bar{y}} \left(\frac{\alpha}{\pi}\right)^n d\lambda(y) \\ &= \left(P_\alpha (x-z)\cdot\partial_1 f_z\right)(t),\end{aligned}$$
the penultimate equality follows from Stokes' theorem (the boundary term being obviously zero) and ∂_1 stands for the (holomorphic) derivative with respect to the first argument. By repeated application of this procedure we get
$$(P_\alpha f_z)(x-z) = \sum_{k=0}^{\infty}\left(P_\alpha \frac{\left((x-z)\cdot\partial_1\right)^k}{k!} f_z\right)(0) = \left(P_\alpha \sum_{k=0}^{\infty}\frac{\left((x-z)\cdot\partial_1\right)^k}{k!} f_z\right)(0).$$
We can switch the order of both operations since the sums are in fact finite. Wethus obtain
$$\begin{aligned}&\int_{\mathbb{C}^n} \sum_{k=0}^{\infty} \frac{\left((x-z)\cdot\partial_1\right)^k}{k!} f_z\,(y,\bar{y}) e^{-\alpha y\cdot\bar{y}} \left(\frac{\alpha}{\pi}\right)^n d\lambda(y) \\ &\quad= \int_{\mathbb{C}^n} f(y+x,\bar{y}+\bar{z}) e^{-\alpha y\cdot\bar{y}} \left(\frac{\alpha}{\pi}\right)^n d\lambda(y)\,.\end{aligned}$$
Finally, the last integral is nothing else than $(B_\alpha f_{x,z})(0)$, i.e. the (usual) Berezin trans-form of the function $f_{x,z}(y,\bar{y}) := f(x+y,\bar{z}+\bar{y})$ at the point 0, whose asymptotic expansion is known to be of the form(17).
It is quite difficult to guess in advance, what the point v that should be the limit of B_α^2 as $\alpha\to\infty$ looks like, even though it is fairly straightforward to compute it, since if the property (25) is to hold, wemust have

$$t \cdot v = \lim_{\alpha\to\infty} (B_\alpha^2(t \cdot y))(x,z) = \lim_{\alpha\to\infty} \int_{\mathbb{R}^n} t \cdot y \frac{R_\alpha(x,y)R_\alpha(z,y)}{R_\alpha(x,z)} d\mu_\alpha^n(y).$$

Applying Stokes' theorem gives

$$\int_{\mathbb{R}^n} (t \cdot y) g(y) d\mu_\alpha^n(y) = \frac{1}{2\alpha} \int_{\mathbb{R}^n} t \cdot \nabla g(y)\, d\mu_\alpha^n(y),$$

so that

$$\int_{\mathbb{R}^n} t \cdot y \frac{R_\alpha(x,y)R_\alpha(z,y)}{R_\alpha(x,z)} d\mu_\alpha^n(y) = \frac{1}{2\alpha} \int_{\mathbb{R}^n} t \cdot \nabla \frac{R_\alpha(x,y)R_\alpha(z,y)}{R_\alpha(x,z)} d\mu_\alpha^n(y)$$

$$= \frac{1}{2\alpha} \frac{1}{R_\alpha(x,z)} \int_{\mathbb{R}^n} \left(R_\alpha(z,y) t \cdot \nabla_y R_\alpha(x,y) + R_\alpha(x,y) t \cdot \nabla_y R_\alpha(z,y)\right) d\mu_\alpha^n(y).$$

Since $\Delta R_\alpha = 0$, the last expression is, due to the reproducing property, equal to

$$\frac{1}{2\alpha} \frac{t \cdot \nabla_z R_\alpha(x,z) + t \cdot \nabla_x R_\alpha(z,x)}{R_\alpha(x,z)}.$$

Thus, representing the Bergman kernel by the hypergeometric function (21), we have

$$(B_\alpha^2(t \cdot y))(x,z) = \frac{\Phi_2\left({- \atop b+1}; {b+1 \quad b \atop -}; \alpha u, \alpha\bar{u}\right)}{\Phi_2\left({- \atop b}; {b \quad b \atop -}; \alpha u, \alpha\bar{u}\right)} \frac{t \cdot \nabla_x + t \cdot \nabla_z}{2} u$$

$$+ \frac{\Phi_2\left({- \atop b+1}; {b \quad b+1 \atop -}; \alpha u, \alpha\bar{u}\right)}{\Phi_2\left({- \atop b}; {b \quad b \atop -}; \alpha u, \alpha\bar{u}\right)} \frac{t \cdot \nabla_x + t \cdot \nabla_z}{2} \bar{u}. \qquad (26)$$

Therefore it is clear that, to get any further, we need to understand the limiting behavior of the Φ_2 function.

In even dimensions (that is when $b \in \mathbb{N}_0$) this is not difficult, since applying the famous transformation formula for the Φ_2 function

$$\Phi_2\left({- \atop c}; {b_1 \quad b_2 \atop -}; x, y\right) = e^x \Phi_2\left({- \atop c}; {c - b_1 - b_2 \quad b_2 \atop -}; -x, y - x\right),$$

we can represent Φ_2 as a linear combination of ${}_1F_1$ as follows

$$\Phi_2\left({- \atop b}; {b \quad b \atop -}; \alpha u, \alpha\bar{u}\right) = e^{\alpha u} \Phi_2\left({- \atop b}; {-b \quad b \atop -}; -\alpha u, \alpha(\bar{u} - u)\right)$$

$$= e^{\alpha u} \sum_{j,k=0}^{\infty} \frac{(-b)_j (b)_k}{(b)_{j+k}} \frac{(-\alpha u)^j (\alpha(\bar{u} - u))^k}{j!\, k!}$$

$$= e^{\alpha u} \sum_{j=0}^{b} \frac{(-b)_j}{(b)_j} \frac{(-\alpha u)^j}{j!} {}_1F_1\left({b \atop b+j}; \alpha(\bar{u} - u)\right),$$

where

$${}_1F_1\left({a \atop c}; x\right) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}$$

is the confluent hypergeometric function whose asymptotic behavior for large argument is known to be[335]:

$${}_1F_1\left({a \atop c}; x\right) \sim e^x \frac{\Gamma(c)}{\Gamma(a)} x^{a-b} + (-x)^{-a} \frac{\Gamma(c)}{\Gamma(c-a)} \quad (|x| \to \infty). \qquad (27)$$

Hence, as$\alpha \to \infty$ for $u \notin \mathbb{R}$,

$$\Phi_2\left({- \atop b}; {b \quad b \atop -}; \alpha u, \alpha\bar{u}\right)$$

$$\sim e^{\alpha u} \sum_{j=0}^{b} \frac{(-b)_j}{(b)_j} \frac{(-\alpha u)^j}{j!} \left(e^{\alpha(\bar{u}-u)} \frac{\Gamma(b+j)}{\Gamma(b)} \left(\alpha(\bar{u} - u)\right)^{-j} + \left(\alpha(u - \bar{u})\right)^{-b} \frac{\Gamma(b+j)}{\Gamma(j)}\right)$$

$$\sim e^{\alpha\bar{u}} \sum_{j=0}^{b} \frac{(-b)_j}{j!} \left(\frac{u}{u - \bar{u}}\right)^j + e^{\alpha u} \frac{(-b)_b}{(b)_b} \frac{(-\alpha u)^b}{b!} (\alpha(u - \bar{u}))^{-b} \frac{\Gamma(2b)}{\Gamma(b)}$$

$$= e^{\alpha\bar{u}}\bar{u}^b(\bar{u}-u)^{-b} + e^{\alpha u}u^b(u-\bar{u})^{-b}.$$

With the same technique it is easy to see that

$$\Phi_2\left(\begin{matrix} - \\ b+1 \end{matrix}; \begin{matrix} b+1 & b \\ - & \end{matrix}; \alpha u, \alpha\bar{u}\right) \sim e^{\alpha\bar{u}} b\alpha^{-1}\bar{u}^b(\bar{u}-u)^{-b-1} + e^{\alpha u}u^b(u-\bar{u})^{-b},$$

$(\alpha \to \infty)$,

and therefore

$$\frac{\Phi_2\left(\begin{matrix} - \\ b+1 \end{matrix}; \begin{matrix} b+1 & b \\ - & \end{matrix}; \alpha u, \alpha\bar{u}\right)}{\Phi_2\left(\begin{matrix} - \\ b \end{matrix}; \begin{matrix} b & b \\ - & \end{matrix}; \alpha u, \alpha\bar{u}\right)} \sim \frac{e^{\alpha\bar{u}} b\alpha^{-1}\bar{u}^b(\bar{u}-u)^{-b-1} + e^{\alpha u}u^b(u-\bar{u})^{-b}}{e^{\alpha\bar{u}}\bar{u}^b(\bar{u}-u)^{-b} + e^{\alpha u}u^b(u-\bar{u})^{-b}},$$

$$(\alpha \to \infty). \qquad (28)$$

Clearly, the limit as $\alpha \to \infty$ does not exist since the fraction oscillates. This means that there is no such point v at least in even dimensions and one cannot hold high hopes for the odd-dimensional case either, since, usually, it is the more complicated one.

Fortunately, this analysis is valid only for $\alpha \in \mathbb{R}$. Nothing, however, prevents us from considering the complex values of α. The Berezin transform of two arguments (24) of a polynomial p, that is $(B_\alpha^2 p)(x,z)$, is always defined in the half-plane $Re\ \alpha > 0$ except for zeros of $R_\alpha(x,z)$, where it has poles – the Bergman kernel (as a function of α) is an entire function (as seen from the representation by the Φ_2 function which is entire in both arguments).

Consider therefore $\alpha \in \mathbb{C}$ satisfying $Re(\alpha u) > Re(\alpha\bar{u}), Re(\alpha u) > 0$, thus forcing the factor $e^{\alpha u}$ to dominate. We claim that this is possible for any fixed non-collinear x, z. Indeed, let

$$\alpha = |\alpha|e^{i\theta}, cos(\theta) > 0, \qquad u = |x|\,|z|\,e^{i\varphi}, sin\varphi > 0.$$

The condition $cos\theta > 0$ is necessary for the integral (10) to converge. The assumption $sin\varphi > 0$ holds by the definition of $u = x \cdot z + i\sqrt{|x|^2|z|^2 - (x\cdot z)^2}$, where the positive sign of $\sqrt{\cdot}$ is taken. Thus, we get

$$Re(\alpha u) > Re(\alpha\bar{u}) \qquad Re(\alpha u) > 0$$
$$cos(\theta+\varphi) > cos(\theta-\varphi) \qquad cos(\theta+\varphi) > 0$$
$$cos\,\theta\, cos\varphi - sin\,\theta\, sin\varphi > cos\,\theta\, cos\varphi + sin\theta\, sin\,\varphi \qquad cos\theta\, cos\varphi > sin\,\theta\, sin\varphi$$
$$2\, sin\theta\, sin\varphi < 0$$
$$\text{for } sin\varphi > 0 \qquad sin\theta < 0 \qquad tan\theta < cotg\varphi.$$

The inequality $sin\theta < 0$ can always be satisfied and implies that we can make the factor $tan\theta$ arbitrary large and negative letting $cos\theta$ closer and closer to 0. So for any finite value of the quantity $cotg\varphi$ (that is for $sin\varphi \neq 0$, i.e. for x, z non-collinear), the inequality $tan\theta < cotg\varphi$ can be satisfied.

Using(28)with $Re(\alpha u) > Re(\alpha\bar{u}), Re(\alpha u) > 0$, wetherefore have (in even dimen-sions):

$$\frac{\Phi_2\left(\begin{matrix} - \\ b+1 \end{matrix}; \begin{matrix} b+1 & b \\ - & \end{matrix}; \alpha u, \alpha\bar{u}\right)}{\Phi_2\left(\begin{matrix} - \\ b \end{matrix}; \begin{matrix} b & b \\ - & \end{matrix}; \alpha u, \alpha\bar{u}\right)} \to 1, \qquad |\alpha| \to \infty,$$

and

$$\frac{\Phi_2\left(\begin{matrix} - \\ b+1 \end{matrix}; \begin{matrix} b & b+1 \\ & - \end{matrix}; \alpha u, \alpha\bar{u}\right)}{\Phi_2\left(\begin{matrix} - \\ b \end{matrix}; \begin{matrix} b & b \\ - & \end{matrix}; \alpha u, \alpha\bar{u}\right)} \to 0, \qquad |\alpha| \to \infty.$$

Substituting this into (26) we finally obtain

$$t \cdot v = \lim_{|\alpha|\to\infty} (B_\alpha^2 t \cdot y)(x,z) = \frac{t\cdot\nabla_x + t\cdot\nabla_z}{2} u = t\cdot x\frac{u-|z|^2}{u-\bar{u}} + t\cdot z\ \frac{u-|x|^2}{u-\bar{u}}.$$

In other words,

$$v = x\frac{u-|z|^2}{u-\bar{u}} + z\ \frac{u-|x|^2}{u-\bar{u}}. \qquad (29)$$

Straightforward computations also reveal a few interesting properties of the point v:

$$\begin{cases} Re\, v = \frac{x+z}{2}, & v\cdot\bar{v} = |x+z|^2 + |x-z|^2, \qquad v\cdot v = u_{x,z}, \\ x\cdot v = \frac{|x|^2+u_{x,z}}{2}, & (x\cdot v)(\overline{x\cdot v}) = \frac{|x|^2|x+z|^2}{4}, \quad \bar{u}_{x,z}(x\cdot v) = |x|^2(\overline{z\cdot v}), \\ (x\cdot v)(z\cdot v) = \frac{u_{x,z}|x+z|^2}{4}, & \bar{u}_{x,v} = |x|^2, \qquad u_{x,v} = u_{x,z} = u_{v,v}. \end{cases} \tag{30}$$

Obviously, the vector v (like the number $u_{x,z}$) is symmetric with respect to inter-changing x and z, so that to every property in (30)there is its corresponding mirror counterpart with x and z replaced. We prove that the point v in(29)is the desired point – that is the property (25) is valid for a reasonably general set of admissible functions f in all dimension $n > 1$. However, since the point v is a complex vector, a necessary condition for the property (25) to hold is that the expression $f(v)$ makes sense, i.e.we must be able to evaluate the function f on complex numbers which in turn means that the function f must be analytic. Since the modulus of the point v can assume arbitrarily large values (due to(30)), the associated radius of convergence must be $+\infty$ and the function f must be entire. We should therefore consider entire functions. For this reason, aswe have already pointed out above, we restrict ourselves to polynomials.

Aside from the function (22), we will actually make use of its slightly more general variant that will be needed:

$$\Phi_2\left(\begin{matrix} a \\ c_1 \;\; c_2\end{matrix}; \begin{matrix} b_1 \;\; b_2 \\ - \end{matrix}; x, y\right) := \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c_1)_{k+1}(c_2)_{j+k}} \frac{x^j y^k}{j!\,k!},$$

defined for every $x, y \in \mathbb{C}$ (here we will use the symbol Φ_2 for both functions – this can result in no serious confusion). Both functions are in fact special instances of the generalized Kampé de Fériet function: $F^{0:1;1}_{1:0;0}(x,y)$ and $F^{1:1;1}_{2:0;0}(x,y)$, respectively (see[336]).

The following integral representation, then, is close to standard:

Lemma (6.2.5) [329]: For any $b_1 > 0, b_2 > 0, a > 0$ and $\gamma := c_1 + c_2 - b_1 - b_2 - a > 0$,

$$\Phi_2\left(\begin{matrix} a \\ c_1 \;\; c_2\end{matrix}; \begin{matrix} b_1 \;\; b_2 \\ - \end{matrix}; x, y\right) = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a)\Gamma(\gamma)}$$

$$\iint\limits_{\substack{t,s\geq 0 \\ t+s\leq 1}} t^{b_1-1}s^{b_2-1}(1-t-s)^{\gamma-1}\,{}_2F_1\left(\begin{matrix} c_1-a \;\; c_2-a \\ \gamma \end{matrix}; 1-t-s\right) e^{tx+sy}\,dt\,ds\,.$$

Proof: Expanding the exponential in the integrand into the Taylor series we get

$$\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a)\Gamma(\gamma)} \sum_{j,k=0}^{\infty} \frac{x^j y^k}{j!\,k!} \int_0^1\int_0^{1-s} t^{b_1+j-1}s^{b_2+k-1}(1-t-s)^{\gamma-1}\,{}_2F_1\left(\begin{matrix} c_1-a \;\; c_2-a \\ \gamma \end{matrix}; 1-t-s\right) dt\,ds\,.$$

Performing a series of changes of variables $t \mapsto (1-s)t, s \mapsto 1-s$ and $t \mapsto 1-t$ brings us to

$$\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a)\Gamma(\gamma)} \sum_{j,k=0}^{\infty} \frac{x^j y^k}{j!\,k!} \int_0^1\int_0^1 t^{\gamma-1}(1-t)^{b_1+j-1}s^{b_1+j+\gamma+1}(1-s)^{b_2+k-1}\,{}_2F_1\left(\begin{matrix} c_1-a \;\; c_2-a \\ \gamma \end{matrix}; ts\right) dt\,ds.$$

Finally, by repeated application of the known integral representation for the ${}_3F_2$ function that holds for any B_1, B_2, C_1, and for $C > A > 0$:

$$\int_0^1 t^{A-1}(1-t)^{C-A-1}\,{}_2F_1\left(\begin{matrix} B_1 \;\; B_2 \\ C_1 \end{matrix}; tx\right) dt = \frac{\Gamma(C-A)\Gamma(A)}{\Gamma(C)}\,{}_3F_2\left(\begin{matrix} A \;\; B_1 \;\; B_2 \\ C \;\; C_1 \end{matrix}; x\right),$$

first with respect to t and then with respect to s, we obtain after some cancellations

$$\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a)} \sum_{j,k=0}^{\infty} \frac{x^j y^k}{j!\,k!} \frac{\Gamma(b_1+j)\Gamma(b_2+k)}{\Gamma(b_2+k+b_1+j+\gamma)}\,{}_2F_1\left(\begin{matrix} c_1-a \quad c_2-a \\ b_2+k+b_1+j+\gamma \end{matrix}; 1\right).$$

This can be simplified, due to the famous summation formula for Gauss's hypergeometric function

$$ {}_2F_1\left(\begin{matrix}A & B\\ & C\end{matrix};1\right)=\frac{\Gamma(C)\Gamma(C-B-A)}{\Gamma(C-B)\Gamma(C-A)}, $$

valid for $C > A + B, C \neq 0, -1, -2, \dots$, to the form (remember that $\gamma = c_1 + c_2 - a - b_1 - b_2$)

$$ \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a)}\sum_{j,k=0}^{\infty}\frac{x^j y^k}{j!\,k!}\,\frac{\Gamma(b_1+j)\Gamma(b_2+k)\Gamma(a+j+k)}{\Gamma(c_2+j+k)\Gamma(c_2+j+k)}=\sum_{j,k=0}^{\infty}\frac{x^j y^k}{j!\,k!}\,\frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c_1)_{j+k}(c_2)_{j+k}}, $$

which concludes the proof.

The integral representation as it stands would be, however, of little use in our case, since in Eq.(26)we have $\gamma = -b$ which is never positive. Fortunately, wecan exploit the known contiguous relations between hypergeometric functions to fix that problem by raising one of the lower indices as follows

$$ \Phi_2\left(\begin{matrix}a\\ c_1\ \ c_2\end{matrix};\begin{matrix}b_1\ \ b_2\\ -\end{matrix};x,y\right)=\frac{c_1+t\partial_t}{c_1}\Phi_2\left(\begin{matrix}a\\ c_1+1\ \ \ c_2\end{matrix};\begin{matrix}b_1\ \ b_2\\ -\end{matrix};tx,ty\right)\Big|_{t=1} $$

or, generally for $k \in \mathbb{N}$,

$$ \Phi_2\left(\begin{matrix}a\\ c_1\ \ c_2\end{matrix};\begin{matrix}b_1\ \ b_2\\ -\end{matrix};x,y\right)=\frac{(c_1+t\partial_t)_k}{(c_1)_k}\Phi_2\left(\begin{matrix}a\\ c_1+k\ \ \ c_2\end{matrix};\begin{matrix}b_1\ \ b_2\\ -\end{matrix};tx,ty\right)\Big|_{t=1}. \qquad (31) $$

The validity of this claim can be checked easily by expanding the Φ_2 function into its Taylor series and noticing that the operator $(c_1 + t\partial_t)_k$ acts on powers of t, its eigenfunctions actually, inthe following
way:

$$ \frac{(c_1+t\partial_t)_k}{(c_1)_k}\frac{t^j}{(c_1+k)_j}=\frac{(c_1+j)_k}{(c_1)_{k+j}}t^j=\frac{t^j}{(c_1)_j},\forall j\in\mathbb{Z}, $$

which is what is needed. (Here we use the fact that to the Pochhammer symbol, the relation $(c)_{j+k} = (c)_j(c + j)_k$ applies.) Using the Newton series expansion of polynomials we can bring the operator into the form

$$ \frac{(c_1+t\partial_t)_k}{(c_1)_k}=\sum_{l=0}^{k}\underline{\Delta_x^l}\frac{(c_1+x)_k}{(c_1)_k}\Big|_{x=0}\binom{t\partial_t}{l}=\sum_{l=0}^{k}\frac{\underline{\Delta_x^l}}{l!}\frac{(c_1+x)_k}{(c_1)_k}\Big|_{x=0}t^l\partial_t^l, $$

where $\underline{\Delta_x}$ is the forward difference operator $\underline{\Delta_x}f(x) := f(x + 1) - f(x)$ (not to be confused with the Laplace operator Δ). Substituting this into (31) and writing the Φ_2 function in the integral form we arrive at

Lemma (6.2.6) [329]: For $k \in \mathbb{N}, \gamma := c_1 + c_2 - b_1 - b_2 - a$ with $k > -\gamma$, and $b_1 > 0, b_2 > 0, a > 0$,

$$ \Phi_2\left(\begin{matrix}a\\ c_1\ \ c_2\end{matrix};\begin{matrix}b_1\ \ b_2\\ -\end{matrix};x,y\right) $$
$$ =\sum_{l=0}^{k}\frac{\underline{\Delta_\xi^l}}{l!}\frac{(c_1+\xi)_k}{(c_1)_k}\Big|_{\xi=0}\frac{\Gamma(c_1+k)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a)\Gamma(\gamma+k)}\iint_{\substack{t,s\geq 0\\ t+s\leq 1}}t^{b_1-1}s^{b_2-1}(1-t $$
$$ -s)^{\gamma+k-1}{}_2F_1\left(\begin{matrix}c_1+k-a\ c_2-\ a\\ \gamma+k\end{matrix};1-t-s\right)(tx+sy)^l e^{tx+sy}dtds. $$

Wedeals with the asymptotic behavior of the previously defined Φ_2 functions which will ultimately prove helpful in due course as suggested by the fore-going discussion. First of all, we shall prove one auxiliary result treating the asymptotic behavior of certain integrals of the Laplace-type:

Lemma (6.2.7) [329]: Let $\Omega \subset \mathbb{R}^2$ be the set $\{(u, v) \in \mathbb{R}^2: u, v \geq 0;\ u + v \leq 1\}$ and f be C^∞ near $(u, v) = \mathbf{0}$. Suppose that $\alpha > 0, x, y \in \mathbb{C}$ with $Rex > 0$ and $Rey > 0$. Then for every $n \in \mathbb{Z}, n \geq 0$, and every $b_1, b_2 \in \mathbb{C}$ with $Reb_1 > 0$ and $Reb_2 > 0$ such that $u^{b_1-1}v^{b_2-1}f(u, v) \in L^1(\Omega)$ we have

$$ \iint_{\substack{u,v\geq 0\\ u+v\leq 1}}u^{b_1-1}v^{b_2-1}f(u,v)e^{-\alpha(xu+yv)}dudv=\frac{D^{(j,k)}f(\mathbf{0})}{j!k!}\frac{\Gamma(b_1+j)}{(\alpha x)^{b_1+j}}\frac{\Gamma(b_2+k)}{(\alpha y)^{b_2+k}}+ $$
$$ O\left(\frac{1}{\alpha^{b_1+b_2+n+1}}\right), \qquad (32) $$

as $\alpha \to \infty$.

Proof: Fix an arbitrary $\delta > 0$ and put $c := \min(Rex, Rey)$. Then

$$\left|\iint_{\substack{u,v\geq 0\\ \delta\leq u+v\leq 1}} u^{b_1-1}v^{b_2-1}f(u,v)e^{-\alpha(xu+yv)}dv\,du\right| \leq Me^{-\alpha c\delta} = o(\alpha^{-n}), \forall n \in \mathbb{N}, \quad (33)$$

as $\alpha \to \infty$. Expand the function $f(u,v)$ into its Taylor series:

$$f(u,v) = \sum_{j+k=0}^{n} \frac{D^{(j,k)}f(\mathbf{0})}{j!k!} u^j v^k + r_n(u,v), \quad (34)$$

where $r_n(u,v) = O(\|(u,v)\|^{n+1}), (u,v) \to \mathbf{0}$ (here we take $\|(u,v)\| := |u| + |v|$). Then

$$\left|\iint_{\substack{u,v\geq 0\\ u+v\leq \delta}} u^{b_1-1}v^{b_2-1}r_n(u,v)e^{-\alpha(xu+yv)}dv\,du\right|$$

$$\leq M \cdot \iint_{\substack{u,v\geq 0\\ u+v\leq \delta}} u^{Re\, b_1-1}v^{Re\, b_2-1}(u,v)^{n+1}e^{-\alpha c(u+v)}dv\,du$$

$$\leq M \cdot \iint_{u,v\geq 0} u^{Re\, b_1-1}v^{Re\, b_2-1}(u,v)^{n+1}e^{-\alpha c(u+v)}dv\,du. \quad (35)$$

Using the binomial theorem shows that the last integral in (35) is $O(\alpha^{-b_1-b_2-n-1})$. Finally, upon splitting the domain of integration in the integral

$$\iint_{\substack{u,v\geq 0\\ u+v\geq \delta}} u^{b_1-1+j}v^{b_2-1+k}e^{-\alpha(xu+yv)}dv\,du \quad (36)$$

into three parts Ω_1, Ω_2 and Ω_3 defined respectively by the conditions $\Omega_1 = \{(u,v) \in \mathbb{R}^2 : u \geq 0;\ v \geq \delta\}, \Omega_2 = \{(u,v) \in \mathbb{R}^2 : u \geq \delta;\ 0 \leq v \leq \delta\}, \Omega_3 = \{(u,v) \in \mathbb{R}^2 : 0 \leq u \leq \delta;\ \delta - u \leq v \leq \delta\}$ and estimating the three resulting integrals separately in an obvious manner, itis readily seen that(36)is $o(\alpha^{-n})$ for every $n \in \mathbb{N}$ as $\alpha \to \infty$. Since

$$\iint_{u,v\geq 0} u^{p-1}v^{q-1}e^{-\alpha(xu+yv)}dv\,du = \frac{\Gamma(p)\Gamma(q)}{(\alpha x)^p(\alpha y)^q},$$

the proof is complete.

We are now ready to prove the following result treating the asymptotic behavior of the generalized Φ_2 function:

Lemma (6.2.8) [329]: For $x, y \in \mathbb{C}, x \neq y$, and for αsuch that $Re(\alpha x) > Re(\alpha y), Re(\alpha x) > 0$, the following asymptotic expansion holds uniformly for $a, b_1, b_2, c_1, c_2 \in [\varepsilon_1, \varepsilon_2]$, for every $0 < \varepsilon_1 < \varepsilon_2$:

$$\Phi_2\left({a \atop c_1\ c_2}; {b_1\ b_2 \atop -}; \alpha x, \alpha y\right)$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)e^{\alpha x}}{\Gamma(b_1)\Gamma(a)} x^{a+b_1+b_2-c_1-c_2}(x-y)^{-b_2}\alpha^{a+b_1+b_2-c_1-c_2}(1 + O(\alpha - 1)),$$

as $|\alpha| \to \infty$.

Proof: Making the change of variables $((1-s-t), s) =: (u,v)$ in the integral representation from Lemma(6.2.6) we have

$$\Phi_2\left({a \atop c_1\ c_2}; {b_1\ b_2 \atop -}; \alpha x, \alpha y\right) = \sum_{l=0}^{k} \frac{\Delta_\xi^l}{l!} \frac{(c_1+\xi)_k}{(c_1)_k}\Bigg|_{\xi=0} \frac{\Gamma(c_1+k)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(a)\Gamma(\gamma+k)}\ \alpha^l e^{\alpha x}$$

$$\iint_{\substack{u,v\geq 0\\ u+v\leq 1}} (1-u-v)^{b_1-1}v^{b_2-1}u^{\gamma+k-1}{}_2F_1\left({c_1+k-a\ c_2-\ a \atop \gamma+k}; u\right)$$

$$((1-u-v)x + vy)^l e^{-v\alpha(x-y)-u\alpha x}du\,dv,$$

where $\gamma := c_1 + c_2 - a - b_1 - b_2$ and k is an integer such that $\gamma + k > 0$. We can use with $f(u,v) := (1-u-v)^{b_1-1}{}_2F_1\left({c_1+k-a\ c_2-\ a \atop \gamma+k}; u\right)((1\ -u-v)x + vy)^l$. Since α is a complex number, the limit in the lemma is taken through $|\alpha|$. The $e^{i\theta}$part of α is added to the numbers x and $x - y$. The condition of Lemma(6.2.7): $Rex > 0$ and $Rey > 0$ reads exactly $Re(\alpha x) > Re(\alpha y), Re(\alpha x) > 0$. Noting that the f_{00} coefficient in the Taylor expansion $f(u,v) =$

$\sum_{j,k} f_{jk}u^j v^k$is equal to x^l, and that the principal behavior is obtained when $l = k$, the result follows. The uniformity of the expansion on the compact interval $[\varepsilon_1, \varepsilon_2]$ is obvious.

Finally, it is an easy exercise to prove that in case the arguments of the second Φ_2 function are the same, the function itself collapses into the ${}_2F_2$function,

$$\Phi_2\begin{pmatrix} a & b_1 \;\; b_2 \\ c_1 \;\; c_2 & \_ \end{pmatrix}; x, x\Big) = {}_2F_2\begin{pmatrix} a & b_1 + b_2 \\ c_1 & c_2 \end{pmatrix}; x\Big), \tag{37}$$

whose asymptotic behavior for large values of the argument $x \in \mathbb{R}$ is well-known, (see[17,14]):

Lemma (6.2.9) [329]: For $x > 0$, we have

$${}_2F_2\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}; x\Big) = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)} x^{b_1+b_2-c_1-c_2} e^x (1 + O(x^{-1})),$$

as $x \to \infty$.

Direct calculations of the Bergman projection or, even worse, the Berezin transform even of a polynomial produce formulas of such length that it would be difficult not only to write them down but it would also overwhelm the reader with details that are unessential to the main argument. We will, therefore, represent our result by an operator calculus devised to simplify matters substantially while keeping the full rigor at the same time.

We define a function of an operator standardly by means of Taylor series representations. From now on, to avoid any convergence related questions, we will have to limit ourselves to the case of polynomials, i.e. we shall consider only expressions of the form

$$p(\nabla_x)f(x) := \sum_{k=0}^{m} \frac{(\nabla_x \nabla_t)^k}{k!} p(t)f(x)\Big|_{t=0},$$

where p is a polynomial of degree m and f is a smooth function.

Or, dually,

$$f(\nabla_x)p(x) := \sum_{k=0}^{m} \frac{(\nabla_x \nabla_t)^k}{k!} p(x)f(t)\Big|_{t=0},$$

where p, f are as above. That is, a smooth function acting on a polynomial produces only finitely many terms.

The sort of duality alluded to above is spelled out in the following obvious relationship:

$$p(\nabla_x)f(x)|_{x=0} = f(\nabla_x)p(x)|_{x=0}. \tag{38}$$

An important case is when $f(y) = e^{t\cdot y}$. The corresponding operator is obviously acting like a translation

$$e^{t\cdot\nabla_x}p(x) = p(x + t). \tag{39}$$

On the other hand,

$$p(\nabla_x)e^{t\cdot x}f(x) = e^{t\cdot x}p(t + \nabla_x)f(x), \tag{40}$$

which is a direct consequence of the Leibniz rule. It can be proved as follows:

$$p(\nabla_x)e^{t\cdot x}f(x) = e^{t\cdot x}e^{-t\cdot x}p(\nabla_x)e^{t\cdot x}f(x) = e^{t\cdot x}p(e^{-t\cdot x}\nabla_x e^{t\cdot x})f(x)$$
$$= e^{t\cdot x}p(t + \nabla_x)f(x).$$

Notice also that in order to compute the expression

$$e^{t\cdot\nabla_x}p(x),$$

we do not need the full Taylor series expansion of the exponential, only finitely many terms suffice. More specifically, if the polynomial is of degree m, then only the first $m + 1$terms are needed. To stress this fact we define

$$e_m^x := \sum_{k=0}^{m} \frac{x^k}{k!}. \tag{41}$$

Obviously, it holds

$$e_m^{t\cdot\nabla_x}p(x) = p(x + t) \qquad deg p \le m.$$

Nor in the dual view is the full expansion of the exponential needed. Thus in the equality

$$p(\nabla_t)e^{t\cdot x}|_{t=0} = p(x), \tag{42}$$

only the truncation e_m^x will in fact do:

$$p(\nabla_t)e_m^{t\cdot x}|_{t=0} = p(x) \qquad deg p \le m.$$

In all these cases we are thus dealing with polynomial operators acting on polynomials and no convergence related questions arise.

With the aid of the polynomial e_m^x we can also represent Taylor series truncations of other functions than just exponentials by means of the following equality

$$e_m^{\partial_s} f(sx)\big|_{s=0} = \sum_{j=0}^{m} \frac{(x\cdot\nabla)^j}{j!} f(0)\,.$$

To proceed any further, wehave to formulate the following important lemma:

Lemma (6.2.10) [329]: Let $q_M(s)$ be a polynomial of degree $M, p(s)$ an analytic function with the radius of convergence strictly greater than 1and let $p_m(s)$ be its Taylor series truncation

$$p_m(s) := \sum_{k=0}^{m} \frac{p^{(k)}(0)}{k!} s^k.$$

Then, as $m \to \infty$,

$$e_m^{\partial_s} q_M(s) p_m(s)|_{s=0} \to q_M(1)p(1).$$

Proof: The claim stems from the fact that for $m > k$ it holds

$$e_m^{\partial_s} s^k p_m(s)|_{s=0} = p_{m-k}(1).$$

Thus

$$e_m^{\partial_s} q_M(s)p_m(s)|_{s=0} = \sum_{k=0}^{M} q_k e_m^{\partial_s} s^k p_m(s)|_{s=0} = \sum_{k=0}^{M} q_k p_{m-k}(1) \to \sum_{k=0}^{M} q_k p(1)$$
$$= q_M(1)p(1).$$

It is important to understand that the polynomial e_m^x behaves much like the ordinary exponential function e^x if we dispense with any concern about terms of order higher than m. For example it is an easy exercise to prove that

$$e_m^{x+y} = e_m^x e_m^y - \sum_{\substack{j+k>m \\ j,k\le m}} \frac{x^j y^k}{j!k!}\,. \tag{43}$$

The operator case is exactly the one when no concerns about higher order terms are raised and we have

$$e_m^{x\cdot\nabla_y + t\cdot\nabla_y} p(y) = e_m^{x\cdot\nabla_y} e_m^{t\cdot\nabla_y} p(y) \qquad deg p \le m,$$

since

$$\sum_{\substack{j+k>m \\ j,k\le m}} \frac{(x\cdot\nabla y)^j (t\cdot\nabla_y)^k}{j!\,k!} p(y) = 0.$$

Also dually,

$$p(\nabla_t) e_m^{x\cdot t + y\cdot t}\big|_{t=0} = p(\nabla_t) e_m^{x\cdot t} e_m^{y\cdot t}\big|_{t=0}.$$

Another property that closely resembles one of the properties of the exponential function is the formula for the derivative

$$(e_m^{ax})' = a e_{m-1}^{ax}.$$

(It is worth mentioning that the notation is slightly abused here, because e_m^x is no power of anything. But it is convenient for our purposes and not quite an unprecedented usage since the usual exponential function in complex domain is also written as a power although it is defined by infinite series.)

The second operator we will use enables us to deal with the so-called Pochhammer symbols

$$(a)_k := a(a+1)(a+2)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The following representation obviously holds:

$$(a)_k = (a\tau_a)^k 1, \qquad \frac{1}{(a)_k} = \left(\frac{1}{a}\tau_a\right)^k 1 \quad \forall k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where τ_a is the translation operator of a by 1:

$$\tau_a f(a) := f(a+1).$$

This gives us a useful tool for recovering some of the properties of the Pochhammer symbols. For example:

$$(a)_{k+l} = (a\tau_a)^{k+l}1 = (a\tau_a)^k(a\tau_a)^l 1 = (a)_k\tau_a^k(a)_l\tau_a^l 1 = (a)_k(a+k)_l,$$

or

$$(a)_k = (a\tau_a)^k 1 = \left(\frac{\Gamma(a+1)}{\Gamma(a)}\tau_a\right)^k 1 = \left(\frac{1}{\Gamma(a)}\tau_a\Gamma(a)\right)^k 1 = \frac{1}{\Gamma(a)}\tau_a^k\Gamma(a) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Again, we will consider mainly polynomials of such operators, i.e. expressions of the form

$$p(a\tau_a x)1 = \sum_{k=0}^{m}\frac{(a)_k}{k!}(x\cdot\nabla)^k p(0)\,, p\left(\frac{1}{a}\tau_a x\right)1 = \sum_{k=0}^{m}\frac{1}{(a)_k k!}(x\cdot\nabla)^k p(0)\,,$$

that can be written, inlight of the preceding equality, as

$$p(a\tau_a x)1 = \frac{1}{\Gamma(a)}p(x\tau_a)\Gamma(a), \qquad p\left(\frac{1}{a}\tau_a x\right)1 = \Gamma(a)p(x\tau_a)\,\frac{1}{\Gamma(a)},$$

a fact that will be important later on.

Notice that the two operator calculi can be combined. It is not hard to see, for example, that the translation operator combined with the operator for the Pochhammer symbol still pretty much behaves like the translation operator, only turning the polynomial on which it acts into an operator itself. The following formula is true:

$$e_m^{a\tau_a t\cdot\nabla_x}p(x) = p(x+a\tau_a t)1, \qquad deg\, p \le m. \tag{44}$$

To prove this equality amounts only to writing the corresponding Taylor series around 0 and x for the left hand side and the right hand side, respectively, and observe that they are indeed the same.

We start by computing the Bergman projection of a polynomial. The equality (42) tells us that any polynomial can be obtained by differentiation of the generating function $e^{t\cdot y}$. Hence it is sufficient to compute the Bergman projection of the exponential function.

Lemma (6.2.11) [329]: The generating function of the harmonic Bergman projection is given by the formula

$$(P_a e^{t\cdot y})(x) := R_\alpha(x,y)e^{t\cdot y}d\mu_\alpha^n(y) = e^{\frac{|t|^2}{4\alpha}}R_{\frac{1}{2}}(x,t). \tag{45}$$

Proof: Recall that

$$d\mu_\alpha^n(y) = e^{-\alpha|y|^2}c_\alpha d^n y.$$

Completing the square in the integral we get

$$(P_\alpha e^{t\cdot y})(x) = e^{\frac{|t|^2}{4\alpha}}\int_{\mathbb{R}^n}R_\alpha(x,y)e^{-\alpha\left|y-\frac{t}{2\alpha}\right|^2}c_\alpha d^n y\,,$$

which upon the change of variables $y \to y + \frac{t}{2\alpha}$ gets the form

$$e^{\frac{|t|^2}{4\alpha}}\int_{\mathbb{R}^n}R_\alpha\left(x, y+\frac{t}{2\alpha}\right)d\mu_\alpha^n(y)\,.$$

By the mean value property of harmonic functions this can be further simplified to

$$e^{\frac{|t|^2}{4\alpha}}R_\alpha\left(x,\frac{t}{2\alpha}\right),$$

which is what we want, since

$$R_\alpha\left(x,\frac{t}{2\alpha}\right) = R_{\frac{1}{2}}(x,t).$$

The Bergman projection of any polynomial can now be obtained by differentiation under the integral sign as follows:

$$\begin{aligned}(P_\alpha p)(x) &= \int_{\mathbb{R}^n}R_\alpha(x,y)p(y)d\mu_\alpha^n(y) = \int_{\mathbb{R}^n}R_\alpha(x,y)p(\nabla_t)e^{t\cdot y}|_{t=0}d\mu_\alpha^n(y)\\ &= p(\nabla_t)\int_{\mathbb{R}^n}R_\alpha(x,y)e^{t\cdot y}d\mu_\alpha^n(y)|_{t=0} = p(\nabla_t)e^{\frac{|t|^2}{4\alpha}}R_{\frac{1}{2}}(x,t)|_{t=0}.\end{aligned}$$

Or equivalently:

$$e^{\frac{\Delta}{4\alpha}}R_{\frac{1}{2}}(x,\nabla)p(t)|_{t=0},$$

as it is clear from the duality property(38). Asalready before, neither in this case do we need the full Taylor series expansion to compute the last expression and we can write

$$(P_\alpha p)(x) = e^{\frac{\Delta}{4\alpha}}R^m_{\frac{1}{2}}(x,\nabla)p(t)|_{t=0}, \qquad m \geq deg\ p,$$

where

$$R^m_\alpha(x,y) := \sum_{j+k\leq m}\frac{(b)_j(b)_k}{(b)_{j+k}}\frac{(\alpha u_{x,y})^j(\alpha\bar{u}_{x,y})^k}{j!k!}, \tag{46}$$

and e^x_m is the truncated exponential (41). This formula indeed covers the terms of the Bergman kernel up to order m since

$$R_\alpha(x,y) = \Phi_2\left({}^{-}_{b};\ {}^{b}\ {}^{b}_{-};\alpha u_{xy},\alpha\bar{u}_{x,y}\right) := \sum_{j+k=0}^{\infty}\frac{(b)_j(b)_k}{(b)_{j+k}}\frac{(\alpha u_{x,y})^j(\alpha\bar{u}_{x,y})^k}{j!\,k!},$$

and $u,\bar{u}$ are homogeneous of degree 1.

To understand the action of the Bergman projection more clearly we have to find a more useful representation of the Bergman kernel. We achieve this by means of the following general lemma:

Lemma (6.2.12) [329]: Let β,γ be non-negative integers, then

$$e^{\partial_s}_m e_m^{2s\alpha x\cdot yBC'-s^2\alpha^2|x|^2|y|^2BC'^2}1|_{s=0} = \Phi^m_2\left({}^{-}_{b+\gamma};\ {}^{b+\beta}\ {}^{b+\beta}_{-};\alpha u_{xy},\alpha\bar{u}_{x,y}\right),$$

where

$$\Phi^m_2\left({}^{-}_{b+\gamma};\ {}^{b+\beta}\ {}^{b+\beta}_{-};\alpha u_{xy},\alpha\bar{u}_{x,y}\right) := \sum_{j+k\leq m}\frac{(b+\beta)_j(b+\beta)_k}{(b+\gamma)_{j+k}}\frac{(\alpha u_{x,y})^j(\alpha\bar{u}_{x,y})^k}{j!\,k!}$$

and B and C' are operators defined by the relations

$$B := (b+\beta)\tau_\beta, \qquad C' := \frac{1}{b+\gamma}\tau_\gamma.$$

Proof: By definition

$$e_m^{2s\alpha x\cdot yBC'-s^2\alpha^2|x|^2|y|^2BC'^2}1 = \sum_{j=0}^{m}\frac{1}{j!}(2s\alpha x\cdot yBC' - s^2\alpha^2|x|^2|y|^2BC'^2)^j 1$$

$$\sum_{j=0}^{m}\frac{(b+\beta)_j}{j!}(2s\alpha x\cdot yC' - s^2\alpha^2|x|^2|y|^2C'^2)^j 1,$$

expanding the parenthesis in the sum we get

$$\sum_{k\leq j\leq m}\frac{(b+\beta)_j}{(j-k)!\,k!}(2x\cdot y)^{j-k}(-|x|^2|y|^2)^k s^{j+k}\alpha^{j+k}C'^{j+k}1$$

$$= \sum_{k\leq j\leq m}\frac{(b+\beta)_j}{(b+\gamma)_{j+k}(j-k)!\,k!}(2x\cdot y)^{j-k}(-|x|^2|y|^2)^k s^{j+k}\alpha^{j+k}.$$

This is the same as

$$\sum_{j+k\leq m}\frac{(b+\beta)_{j+k}}{(b+\gamma)_{j+2k}j!\,k!}(2x\cdot y)^j(-|x|^2|y|^2)^k s^{j+2k}\alpha^{j+2k}.$$

Applying the operator $e^{\partial_s}_m|_{s=0}$will reduce the number of terms from $j+k\leq m$ to $j+2k\leq m$:

$$\sum_{j+2k\leq m}\frac{(b+\beta)_{j+k}}{(b+\gamma)_{j+2k}j!\,k!}(2x\cdot y)^j(-|x|^2|y|^2)^k\alpha^{j+2k}.$$

Using now the fact that $2x\cdot y = u+\bar{u}$ and $|x|^2|y|^2 = u\bar{u}$ (here we use uinstead of $u_{x,y}$ for the sake of brevity), we get

$$\sum_{j+2k\leq m}\frac{(b+\beta)_{j+k}}{(b+\gamma)_{j+2k}j!\,k!}(u+\bar{u})^j(-u\bar{u})^k\alpha^{j+2k}.$$

Expanding the term $(u+\bar{u})^j$ leaves us with

$$\sum_{j+2k\leq m}\sum_{l=0}^{j}\frac{(b+\beta)_{j+k}}{(b+\gamma)_{j+2k}(j-l)!k!l!}(\alpha u)^{j-l+k}(\alpha\bar{u})^{l+k}(-1)^k \tag{47}$$

$$=\sum_{j,k,l=0}^{\infty}\frac{B(m-j-2k)(b+\beta)_{j+k}}{(b+\gamma)_{j+2k}(j-l)!k!l!}(\alpha u)^{j-l+k}(\alpha\bar{u})^{l+k}(-1)^k. \tag{48}$$

The last series (48) is the same as the next to the last one (47), even if written in slightly different notation: the bounds on the summation indices, written explicitly in the latter, are only implicit in the former, corresponding to those values of j,k,l for which the summand function is non-zero. The fact that this is the case for only finitely many indices is ensured by the presence of factors such as $\frac{1}{(j-l)!}$ which is equal to 0 for $j<l$, since the reciprocal value of the factorial is defined to be 0 on negative integers, and the factor $H(m-j-2k)$which is just the Heaviside step function defined as $H(t)=1$for $t\geq 0$ and $H(t)=0$ otherwise.

In this form it is easier to handle changes of variables, since we do not have to worry about explicit bounds on the indices (formally they are still the same) while focusing on the change of the summands only. We let $j\rightarrowtail j+l-k$ in(34)and then $l\rightarrowtail l-k$ to get

$$\sum_{j,k,l=0}^{\infty}\frac{H(m-j-l)(b+\beta)_{j+l-k}}{(b+\gamma)_{j+l}(j-k)!\,k!\,(l-k)!}(\alpha u)^j(\alpha\bar{u})^l(-1)^k$$

$$=\sum_{j+l\leq m}\frac{(b+\beta)_{j+l}}{(b+\gamma)_{j+l}j!l!}(\alpha u)^j(\alpha\bar{u})^l\sum_{k=0}^{\infty}\frac{(b+\beta+j+l)_{-k}j!l!(-1)^k}{(j-k)!k!(l-k)!}. \tag{49}$$

Since

$$(b+\beta+j+l)_{-k}=\frac{(-1)^k}{(1-b-\beta-j-l)_k},\qquad \frac{j!\,l!}{(j-k)!\,(l-k)!}=(-j)_k(-l)_k$$

the second series in (49) can be rewritten as

$$\sum_{k=0}^{\infty}\frac{(-j)_k(-l)_k}{(1-b-\beta-j-l)_k k!}={}_2F_1\left({-j\quad -l \atop 1-b-\beta-j-l};1\right)=\frac{(1-b-\beta-j)_j}{(1-b-\beta-j-l)_j}$$

$$=\frac{(b+\beta)_l}{(b+\beta+j)_l}.$$

In the last line above we have used the so-called Chu–Vandermonde identity for the Gauss hypergeometric function

$${}_2F_1\left({-n\quad b \atop c};1\right)=\frac{(c-b)_n}{(c)_n},$$

valid for all $n=0,1,2,\ldots$[348]. Substituting this result into (49) and taking into account that $(b+\beta)_{j+l}=(b+\beta)_j(b+\beta+j)_l$ completes the proof.

The preceding lemma applies to the Bergman kernel as a special case with $\beta,\gamma=0$:

$$R_\alpha^m(x,y)=e_m^{\partial_s}e_m^{2s\alpha x\cdot yBC'-s^2\alpha^2|x|^2|y|^2BC'^2}1|_{s,\gamma,\beta=0}, \tag{50}$$

so that the operator

$$R_{\frac{1}{2}}^m(x,\nabla_y)p(y),\quad \deg p=m,$$

can be understood as follows

$$R_{\frac{1}{2}}^m(x,\nabla_y)p(y)=R_{\frac{1}{2}}^{2m}(x,\nabla_y)p(y)=e_{2m}^{\partial_s}e_{2m}^{sx\cdot\nabla_yBC'-s^2\frac{1}{4}|x|^2\Delta_yBC'^2}p(y)|_{s,\gamma,\beta=0}. \tag{51}$$

Now the second e_{2m}-term in (51) may be replaced by e_m, since, due to the presence of the operator ∇_y, no more than m derivatives is needed:

$$e_{2m}^{\partial_s}e_m^{sx\cdot\nabla_yBC'-s^2\frac{1}{4}|x|^2\Delta_yBC'^2}p(y)|_{s,\gamma,\beta=0}.$$

But then the first e_{2m}-term is acting like a translation since the polynomial on the right (in s) is of order $2m$, hence we get just

$$R^m_{\frac{1}{2}}(x,\nabla_y)p(y) = e_m^{x\cdot\nabla_y BC' - \frac{1}{4}|x|^2\Delta_y BC'^2} p(y)|_{\gamma,\beta=0}.$$

We shall summarize the obtained results in the following lemma:

Lemma (6.2.13) [329]: *(*Harmonic Bergman projection formula). The Bergman projection of a polynomial takes the form

$$(P_\alpha p)(x) = e_m^{\frac{\Delta_y}{4\alpha}} e_m^{x\cdot\nabla_y BC' - \frac{1}{4}|x|^2\Delta_y BC'^2} p(y)|_{y,\gamma,\beta=0}, \qquad m \geq \deg p,$$

where

$$B := (b+\beta)_{\tau\beta} \qquad C' := \frac{1}{b+\gamma}\tau_\gamma.$$

For example, when the polynomial $p(y)$ happens to be harmonic, all Δ_y's naturally vanish and we are left with

$$(P_\alpha p)(x) = e_m^{x\cdot\nabla_y BC'} p(y)|_{y,\gamma,\beta=0} = e^{x\cdot\nabla_y} p(y)|_{y=0} = p(y+x)|_{y=0} = p(x),$$
$$(\nabla_y p = 0),$$

thus recovering the reproducing property. The second equality is due to the fact that $(BC')^k|_{\beta,\gamma=0} = \frac{(b)_k}{(b)_k}\tau_\beta^k\tau_\gamma^k\big|_{\beta,\gamma=0} = \tau_\beta^k\tau_\gamma^k\big|_{\beta,\gamma=0}$ for every k and since nothing on the right of these operators depends on β or γ, their action produces no difference and BC' can thus be viewed to be equal to 1, i.e. $BC' = 1$. If so, no truncation of the exponential function is needed and the resulting operator is acting like a translation by (39).

One corollary of Lemma (6.2.12) that will be important in the collinear case is the following

Corollary (6.2.14) [329]: For every $m \in \mathbb{N}$ it holds

$$B^m_\alpha(x, y+\sigma x) = e_m^{\partial_s} e_m^{2s\alpha x\cdot yBC' - s^2\alpha^2|x|^2|y|^2BC'^2} {}_1F_1^m\left({2b+\gamma \atop b+\gamma}; \alpha s\sigma|x|^2\right)\Big|_{s,\beta,\gamma=0},$$

where B and C' are operators defined as

$$B := (b+\beta)_{\tau_\beta}, \qquad C' := \frac{1}{b+\gamma}\tau_\gamma,$$

and

$${}_1F_1^m\left({a \atop c}; x\right) := \sum_{k=0}^m \frac{(a)_k}{(c)_k}\frac{x^k}{k!}.$$

Proof: From the definition of the Bergman kernel we have

$$B^m_\alpha(x, y+\sigma x) = \Phi_2^m\left({- \atop b}; {b \atop -}{b \atop }; \alpha u_{x,y+\sigma x}, \alpha\bar{u}_{x,y+\sigma x}\right),$$

where as usual $u_{x,y} := x\cdot y + i\sqrt{|x|^2|y|^2 - (x\cdot y)^2}$. It is an easy computation to show that

$$u_{x,y+\sigma x} = u_{x,y} + \sigma|x|^2,$$

so that we obtain

$$\Phi_2^m\left({- \atop b}; {b \atop -}{b \atop }; \alpha u_{x,y} + \alpha\sigma|x|^2, \alpha\bar{u}_{x,y} + \alpha\sigma|x|^2\right)$$
$$e_m^{\partial_s}\Phi_2\left({- \atop b}; {b \atop -}{b \atop }; s\alpha u_{x,y} + s\alpha\sigma|x|^2, s\alpha\bar{u}_{x,y} + s\alpha\sigma|x|^2\right)\Big|_{s=0}.$$

Now we expand the $\Phi_2$function into its Taylor series around the point $(s\alpha\sigma|x|^2, s\alpha\sigma|x|^2)$:

$$e_m^{\partial_s}\Phi_2\left({- \atop b}; {b \atop -}{b \atop }; s\alpha u_{x,y} + s\alpha\sigma|x|^2, s\alpha\bar{u}_{x,y} + s\alpha\sigma|x|^2\right)\Big|_{s=0}$$
$$e_m^{\partial_s}\sum_{j,k}\frac{(b)_j(b)_k}{(b)_{j+k}}\frac{(\alpha us)^j(\alpha\bar{u}s)^k}{j!\,k!}\Phi_2\left({- \atop b+j+k}; {b+j \atop -}{b+k \atop }; s\alpha\sigma|x|^2, s\alpha\sigma|x|^2\right)\Big|_{s=0}$$
$$= e_m^{\partial_s}\sum_{j,k}\frac{(b)_j(b)_k}{(b)_{j+k}}\frac{(\alpha us)^j(\alpha\bar{u}s)^k}{j!\,k!}\,{}_1F_1\left({2b+j+k \atop b+j+k}; s\alpha\sigma|x|^2\right)\Big|_{s=0}$$

$$= e_m^{\partial_s} \sum_{j+k\le m} \frac{(b)_j(b)_k}{(b)_{j+k}} \frac{(\alpha us)^j(\alpha\bar{u}s)^k}{j!\,k!}\; {}_1F_1^m\left(\begin{matrix}2b+j+k\\ b+j+k\end{matrix}; s\alpha\sigma|x|^2\right)\Big|_{s=0}$$

$$= e_m^{\partial_s}\Phi_2\left(\begin{matrix}-\\ b\end{matrix}; \begin{matrix}b & b\\ - & \end{matrix}; s\alpha u\tau_\varepsilon, s\alpha\bar{u}\tau_\varepsilon\right) {}_1F_1^m\left(\begin{matrix}2b+\varepsilon\\ b+\varepsilon\end{matrix}; s\alpha\sigma|x|^2\right)\Big|_{s,\varepsilon=0}.$$

We represent Φ_2^m as in Lemma (6.2.12) and get

$$e_m^{\partial_s} e_m^{\partial_{s_2}} e_m^{2ss_2 x\cdot yBC'\tau_\varepsilon - s^2 s_2^2\alpha^2|x|^2|y|^2BC'^2\tau_\varepsilon^2}\; {}_1F_1^m\left(\begin{matrix}2b+\varepsilon\\ b+\varepsilon\end{matrix}; s\alpha\sigma|x|^2\right)\Big|_{s,s_2,\varepsilon,\beta,\gamma=0}.$$

This almost looks like what we need except for the additional variables ε and s_2. However, both of them can be discarded due to the following analysis: from the perspective of the variables s, s_2, weare dealing with the expression

$$e_m^{\partial_s} e_m^{\partial_{s_2}} f(ss_2, s)\,|_{s,s_2=0},$$

where $f(x,y)$ is a polynomial in both arguments. It is clear that the expression

$$e_m^{\partial_s} f(ss_2, s)\,|_{s=0}$$

alone is a polynomial in s_2 of order not exceeding m(since the order of that expression as a polynomial in sis at most m and s_2 appears only as a multiple of s), so that the operator $e_m^{\partial_{s_2}}$ is acting like a translation and the variable s_2 can indeed be cast away.

Along similar lines we can get rid of the variable ε, since in the operator C' there is the operator τ_γ which appears in the same order as τ_ε and can thus replace the action of τ_ε. Fom the perspective of variables γ, ε, weare dealing with

$$f\left(\tfrac{1}{b+\gamma}\tau_\gamma\tau_\varepsilon\right) {}_1F_1^m\left(\begin{matrix}2b+\varepsilon\\ b+\varepsilon\end{matrix}; s\alpha\sigma|x|^2\right)\Big|_{s,\gamma=0}, \qquad (52)$$

where, again, f is a polynomial whose exact form is unimportant. Clearly the expression in (52) is the same as

$$f\left(\frac{1}{b+\gamma}\tau_\gamma\right) {}_1F_1^m\left(\begin{matrix}2b+\gamma\\ b+\gamma\end{matrix}; s\alpha\sigma|x|^2\right)\Big|_{\gamma=0}.$$

The main objective is the proof of the following theorem that will become central to the proof of the two main theorems (Theorem(6.2.3)and Theorem(6.2.4)).

Theorem (6.2.15) [329]: Let p_M be a polynomial of degree M. If x and z are non-collinear, then the integral

$$\int_{\mathbb{R}^n} p_M(y)R_\alpha(x,y)R_\alpha(z,y)\, d\mu_\alpha^n(y) \qquad (53)$$

admits the following representation:

$$\int_{\mathbb{R}^n} p_M(y)R_\alpha(x,y)R_\alpha(z,y)\, d\mu_\alpha^n(y)$$
$$= e^{\frac{\Delta t}{4\alpha}} e^{x\cdot\nabla_t BC' - \frac{1}{4}|x|^2\Delta_t BC'^2}$$
$$e^{\alpha z\cdot\nabla_t\left(\frac{1}{\alpha}-|x|^2BC'^2\right)B_2C_2' - \alpha^2|z|^2\left(\frac{\Delta t}{4}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2 + \nabla_{t\cdot x}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)BC'\right)B_2C_2'^2} p_M(t)$$
$$\Phi_2\left(\begin{matrix}b+\beta\\ b+\gamma \quad b+\gamma_2\end{matrix}; \begin{matrix}b+\beta_2 & b+\beta_2\\ - & \end{matrix}; \alpha u_{z,x}, \alpha\bar{u}_{z,x}\right)\Big|_{t,\beta,\beta_2,\gamma,\gamma_2=0}. \qquad (54)$$

In case $z = \xi x$, there is the representation

$$\int_{\mathbb{R}^n} p_M(y)R_\alpha(x,y)R_\alpha(z,y)\, d\mu_\alpha^n(y)$$
$$= e^{\frac{\Delta t}{4\alpha}} e^{x\cdot\nabla_t BC' - \frac{1}{4}|x|^2\Delta_t BC'^2}$$
$$e^{\alpha\xi x\cdot\nabla_t\left(\frac{1}{\alpha}-|x|^2BC'^2\right)B_2C_2' - \frac{1}{4}\alpha^2\xi^2|x|^2\nabla_t\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2 B_2C_2'^2} p_M(t)$$
$${}_2F_2\left(\begin{matrix}2b+\gamma_2 & b+\beta\\ b+\gamma_2 & b+\gamma\end{matrix}; \alpha\xi|x|^2\right)\Big|_{t,\beta,\beta_2,\gamma,\gamma_2=0},$$

where $u_{z,x} = z\cdot x + i\sqrt{|z|^2|x|^2-(z\cdot x)^2}$and the operators B, B_2, C', C_2' are defined as

$$B = (b+\beta)_{\tau\beta}, \qquad B_2 = (b+\beta_2)_{\tau\beta_2}, \qquad C' = \frac{1}{b+\gamma}\tau_\gamma, \qquad C_2' = \frac{1}{b+\gamma_2}\tau_{\gamma_2}.$$

Proof: At first we will compute a slightly different integral with one of the Bergman kernels, say $R_\alpha(z,y)$, replaced by its truncated Taylor series:

$$\int_{\mathbb{R}^n} p_M(y) R_\alpha(x,y) R_\alpha^m(z,y)\, d\mu_\alpha^n(y) \qquad (55)$$

At the end of the day we let $m \to \infty$. This will be the only limiting process involved. Everything else is done by purely algebraic means.

Since the expression (55) can be understood as the Bergman projection of a polynomial of degree $m+M$, we can apply Lemma (6.2.13) to obtain

$$e_{m+M}^{\frac{\Delta y}{4\alpha}}\, e_{m+M}^{x\cdot\nabla_y BC' - \frac{1}{4}|x|^2 \Delta_y BC'^2} R_\alpha^m(z,y) p_M(y)|_{y,\beta,\gamma=0}.$$

Now we replace the polynomial by the standard representation

$$p_M(y) = p_M(\nabla_t) e^{t\cdot y}|_{t=0},$$

from which we get

$$p_M(\nabla_t) e_{m+M}^{\frac{\Delta y}{4\alpha}}\, e_{m+M}^{x\cdot\nabla_y BC' - \frac{1}{4}|x|^2 \Delta_y BC'^2} e^{t\cdot y} R_\alpha^m(z,y)|_{y,t,\beta,\gamma=0}.$$

The property of differential operators (40) enables us to rewrite this last expression into the following form (remember that $\Delta_y = |\nabla_y|^2$):

$$p_M(\nabla_t) e_{m+M}^{\frac{|t+\nabla_y|^2}{4\alpha}}\, e_{m+M}^{x\cdot(\nabla_y+t)BC' - \frac{1}{4}|x|^2 |t+\Delta_y|^2 BC'^2} R_\alpha^m(z,y)|_{y,t,\beta,\gamma=0}.$$

Now, the truncated Bergman kernel is a harmonic polynomial – from the definition (46) we have $R_\alpha^m(z,y) = e_m^{\partial_s} R_\alpha(sz,y)|_{s=0}$and $R_\alpha(sz,y)$is a harmonic function. Therefore all the Δ's again vanish, leaving us with

$$p_M(\nabla_t) e_{m+M}^{\frac{|t|^2+2t\cdot\nabla_y}{4\alpha}}\, e_{m+M}^{x\cdot(\nabla_y+t)BC' - \frac{1}{4}|x|^2(|t|^2+2t\cdot\nabla_y)BC'^2} R_\alpha^m(z,y)|_{y,t,\beta,\gamma=0}. \qquad (56)$$

Now we split the operators in (56) into two parts – the one that contains ∇_y and the other one that does not – as follows:

$$p_M(\nabla_t) e_{m+M}^{\frac{|t|^2}{4\alpha}} e_{m+M}^{x\cdot tBC' - \frac{1}{4}|x|^2|t|^2 BC'^2}\, e_{m+M}^{\frac{t\cdot\nabla_y}{4\alpha}} e_{m+M}^{x\cdot\nabla_y BC' - \frac{1}{2}|x|^2 t\cdot\nabla_y BC'^2} R_\alpha^m(z,y)|_{y,t,\beta,\gamma=0}. \qquad (57)$$

The last step is justified by the fact that terms (in t) of order higher than M (let alone $M+m$) are killed by the factor $p_M(\nabla_t)|_{t=0}$ and terms (in ∇_y) of order higher than mdisappear when acting on $R_\alpha^m(z,y)$.

Observing that the exponential terms in (57) containing ∇_y act like translation operators (see(44)), weget the expression(57)transformed into

$$p_M(\nabla_t) e_{m+M}^{\frac{|t|^2}{4\alpha}} e_{m+M}^{x\cdot tBC' - \frac{1}{4}|x|^2|t|^2 BC'^2} R_\alpha^m\left(z, \frac{t}{2\alpha} + xBC' - \frac{1}{2}|x|^2 tBC'^2\right) 1\Big|_{t,\beta,\gamma=0}. \qquad (58)$$

We also note that having the exponential terms truncated up at order $m+M$ is in fact superfluous and that their order being M is completely sufficient since the factor $p_M(\nabla_t)|_{t=0}$ makes all terms of order higher than M disappear, so that(58) is in fact equal to:

$$p_M(\nabla_t) e_M^{\frac{|t|^2}{4\alpha}} e_M^{x\cdot tBC' - \frac{1}{4}|x|^2|t|^2 BC'^2} R_\alpha^m\left(z, \frac{t}{2\alpha} + xBC' - \frac{1}{2}|x|^2 tBC'^2\right) 1\Big|_{t,\beta,\gamma=0}. \qquad (59)$$

The expression in (58)no longer depends on y which suggests that we are almost done. To finish the proof we have to distinguish two cases in this place, depending on whether we are dealing with the collinear or the non-collinear case.

Case 1. In the non-collinear case we use the representation(50):

$$R_\alpha^m\left(z, \frac{t}{2}\left(\frac{1}{\alpha} - |x|^2 BC'^2\right) + xBC'\right) 1$$

$$= e_m^{\partial_s} e_m^{2s\alpha z\cdot(\frac{t}{2}(\frac{1}{\alpha}-|x|^2BC'^2)+xBC')B_2C_2' - s^2\alpha^2|z|^2\left|\frac{t}{2}(\frac{1}{\alpha}-|x|^2BC'^2)+xBC'\right|^2 B_2C_2'^2} 1\Bigg|_{s,\beta_2,\gamma_2=0}, \qquad (60)$$

where B_2, C_2' are operators defined as usual:

$$B_2 := (b+\beta_2)\tau_{\beta_2}, \qquad C_2' := \frac{1}{b+\gamma_2}\tau_{\gamma_2}.$$

Splitting again the terms in (60) into those that contain tand those that do not results in turning (60) into the form

$$e_m^{\partial_s} e_m^{\alpha s z\cdot t(\frac{1}{\alpha}-|x|^2BC'^2)B_2C_2'-s^2\alpha^2|z|^2\left(\frac{|t|^2}{4}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2+t\cdot x\left(\frac{1}{\alpha}-|x|^2BC'^2\right)BC'\right)B_2C_2'^{\,2}}$$
$$e_m^{2s\alpha z\cdot xBC'B_2C_2'-s^2\alpha^2|z|^2|x|^2(BC')^2B_2C_2'^2}1\Big|_{s,\beta_2,\gamma_2=0}. \qquad (61)$$

Once again, all terms in s of order higher than m are killed due to the presence of $e_m^{\partial_s}|_{s=0}$. In addition, the factor $p_M(\nabla_t)|_{t=0}$ kills all terms in t of order higher than M, so that we can truncate the exponential terms containing t at order M instead of m. Finally, although the e_m-term in (61) is a polynomial in s of degree at most $2m$ (since it contains s^2), noterms of higher power than m are necessary. Introducing a new auxiliary variable s_2, we can therefore further restrict the expression in (61) to be equal to

$$e_m^{\partial_{s_2}} e_m^{2ss_2\alpha z\cdot xBC'B_2C_2'-s_2^2s^2\alpha^2|z|^2|x|^2(BC')^2B_2C_2'^2}\Big|_{s_2=0} \qquad (mod\ s^k, k>m). \qquad (62)$$

To (62), Lemma (6.2.12) can now be applied with $sxBC'$instead of x, yielding

$$e_m^{\partial_{s_2}} e_m^{2ss_2\alpha z\cdot xBC'B_2C_2'-s_2^2s^2\alpha^2|z|^2|x|^2(BC')^2B_2C_2'^2}\Big|_{s_2=0} \qquad (mod\ s^k, k>m)$$
$$= \Phi_2^m\left(\begin{matrix} - \\ b+\gamma_2\end{matrix}; \begin{matrix} b+\beta_2 & b+\beta_2 \\ & - \end{matrix}; s\alpha BC'u_{x,y}, s\alpha BC'\bar{u}_{x,y}\right)1$$
$$= \Phi_2^m\left(\begin{matrix} b+\beta & \\ b+\gamma & b+\ \gamma_2\end{matrix}; \begin{matrix} b+\beta_2 & b+\beta_2 \\ & - \end{matrix}; s\alpha u_{z,x}, s\alpha\bar{u}_{z,x}\right),$$

where

$$\Phi_2^m\left(\begin{matrix} a \\ c_1\ c_2\end{matrix}; \begin{matrix} b_1 & b_2 \\ & - \end{matrix}; x,y\right) = \sum_{j+k\le m}\frac{(a)_{j+k}}{(c_1)_{j+k}(c_2)_{j+k}}\frac{(b_1)_j(b_2)_k}{j!\,k!}\ x^jy^k,$$

the second Φ_2 function defined truncated at order m.

So, finally, we have

$$R_\alpha^m\left(z, \frac{t}{2}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)+xBC'\right)1$$
$$= e_m^{\partial_s} e_M^{\alpha s z\cdot t(\frac{1}{\alpha}-|x|^2BC'^2)B_2C_2'-s^2\alpha^2|z|^2\left(\frac{|t|^2}{4}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2+t\cdot x\left(\frac{1}{\alpha}-|x|^2BC'^2\right)BC'\right)B_2C_2'^{\,2}}$$
$$\Phi_2^m\left(\begin{matrix} b+\beta & \\ b+\gamma & b+\gamma_2\end{matrix}; \begin{matrix} b+\beta_2 & b+\beta_2 \\ & - \end{matrix}; s\alpha u_{z,x}, s\alpha\bar{u}_{z,x}\right)\Big|_{s,\beta_2,\gamma_2=0}. \qquad (63)$$

Substituting(63)into(59)we obtain

$$p_M(\nabla_t)e_M^{\frac{|t|^2}{4\alpha}} e_M^{x\cdot tBC'-\frac{1}{4}|x|^2|t|^2BC'^2} R_\alpha^m\left(z, \frac{t}{2\alpha}+xBC'-\frac{1}{2}|x|^2tBC'^2\right)1\Big|_{t,\beta,\gamma=0}$$
$$= p_M(\nabla_t)e_M^{\frac{|t|^2}{4\alpha}} e_M^{x\cdot tBC'-\frac{1}{4}|x|^2|t|^2BC'^2}$$
$$e_m^{\partial_s} e_M^{s\alpha z\cdot t(\frac{1}{\alpha}-|x|^2BC'^2)B_2C_2'-s^2\alpha^2|z|^2\left(\frac{|t|^2}{4}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2+t\cdot x\left(\frac{1}{\alpha}-|x|^2BC'^2\right)BC'\right)B_2C_2'^{\,2}}$$
$$\Phi_2^m\left(\begin{matrix} b+\beta & \\ b+\gamma & b+\gamma_2\end{matrix}; \begin{matrix} b+\beta_2 & b+\beta_2 \\ & - \end{matrix}; s\alpha u_{z,x}, s\alpha\bar{u}_{z,x}\right)\Big|_{s,t,\beta,\gamma,\beta_2,\gamma_2=0}. \qquad (64)$$

Now we can apply Lemma (6.2.10) (the radius of convergence being $+\infty$ in this case) to compute the limit of (64) as $m\to\infty$. It is equal to

$$= p_M(\nabla_t)e_M^{\frac{|t|^2}{4\alpha}} e_M^{x\cdot tBC'-\frac{1}{4}|x|^2|t|^2BC'^2}$$
$$e_M^{\alpha z\cdot t(\frac{1}{\alpha}-|x|^2BC'^2)B_2C_2'-\alpha^2|z|^2\left(\frac{|t|^2}{4}\left(\frac{1}{\alpha}-|x|^2BC'^2\right)^2+t\cdot x\left(\frac{1}{\alpha}-|x|^2BC'^2\right)BC'\right)B_2C_2'^{\,2}}$$

$$\Phi_2\left(\begin{matrix} b+\beta \\ b+\gamma \quad b+\gamma_2 \end{matrix}; \begin{matrix} b+\beta_2\ b+\beta_2 \\ - \end{matrix}; \alpha u_{z,x}, \alpha \bar{u}_{z,x}\right)\Big|_{t,\beta,\gamma,\beta_2,\gamma_2=0}.$$

Using the duality property (38) and dispensing with the constraints on the exponential terms containing t concludes the proof of Case1.

Case 2. We start again with the expression (59) with $z = \xi x$, this time expressing the Bergman kernel by means of Corollary(6.2.14), which gives

$$p_M(\nabla_t) e_M^{\frac{|t|^2}{4\alpha}} e_M^{x\cdot tBC' - \frac{1}{4}|x|^2|t|^2BC'^2} R_\alpha^m\left(\xi, x, \frac{t}{2\alpha} + xBC' - \frac{1}{2}|x|^2 tBC'^2\right) 1\Big|_{t,\beta,\gamma=0}$$

$$= p_M(\nabla_t) e_M^{\frac{|t|^2}{4\alpha}} e_M^{x\cdot tBC' - \frac{1}{4}|x|^2|t|^2BC'^2}$$

$$e_m^{\partial_s} e_m^{s\alpha\xi x\cdot t(\frac{1}{\alpha} - |x|^2BC'^2)B_2C_2' - \frac{1}{4}s^2\alpha^2\xi^2|x|^2|t|^2\left(\frac{1}{\alpha} - |x|^2BC'^2\right)^2 B_2C_2'^{\,2}}$$

$${}_1\mathrm{F}_1^m\left(\begin{matrix} 2b+\gamma_2 \\ b+\gamma_2 \end{matrix}; \alpha s\xi BC'|x|^2\right) 1\Big|_{s,t,\beta,\beta_2,\gamma,\gamma_2=0}, \qquad (65)$$

where the operators B_2, C_2' are defined as above. Notice that the e_m-term to the right of the operator $e_m^{\partial_s}$ in (65) can be truncated to e_M since it contains vector t and all terms of order higher than M are killed by $p_M(\nabla_t)|_{t=0}$. Also the action of the operator BC' in the ${}_1F_1^m$ function can be computed in the following manner

$${}_1F_1^m\left(\begin{matrix} a \\ c \end{matrix}; BC'x\right) 1 = 2F_2^m\left(\begin{matrix} a\ b+\ \beta \\ c\ b+\ \gamma \end{matrix}; x\right).$$

So that eventually we see that(65)is equal to

$$= p_M(\nabla_t) e_M^{\frac{|t|^2}{4\alpha}} e_M^{x\cdot tBC' - \frac{1}{4}|x|^2|t|^2BC'^2} e_m^{\partial_s}$$

$$e_M^{s\alpha\xi x\cdot t(\frac{1}{\alpha} - |x|^2BC'^2)B_2C_2' - \frac{1}{4}s^2\alpha^2\xi^2|x|^2|t|^2\left(\frac{1}{\alpha} - |x|^2BC'^2\right)^2 B_2C_2'^{\,2}}$$

$${}_2\mathrm{F}_2^m\left(\begin{matrix} 2b+\gamma_2 & b+\beta \\ b+\gamma_2 & b+\gamma \end{matrix}; s\alpha\xi|x|^2\right)\Big|_{s,t,\beta,\beta_2,\gamma,\gamma_2=0}.$$

Applying finally Lemma(6.2.10), using the duality property (38)and disregarding any constraints we get the needed result.

We are now ready to prove Theorem(6.2.3):

According to Theorem(6.2.15), the Berezin transform of a polynomial can be written as

$$R_\alpha(x,z)(B_\alpha^2 p)(x,z)$$

$$= q\left(\frac{1}{4\alpha}, \frac{1}{\alpha}B, \frac{1}{\alpha}B_2, \alpha C', \alpha C_2'\right)$$

$$\Phi_2\left(\begin{matrix} b+\beta \\ b+\gamma \quad b+\gamma_2 \end{matrix}; \begin{matrix} b+\beta_2\ b+\beta_2 \\ - \end{matrix}; \alpha u, \alpha\bar{u}\right)\Big|_{\beta,\beta_2,\gamma,\gamma_2=0}, \qquad (66)$$

where q is a polynomial in all arguments, more explicitly

$$q(l, b_1, b_2, c_1, c_2)$$

$$= e_M^{\Delta_t l + x\cdot\nabla_t b_1 c_1 - |x|^2\Delta_t l b_1 c_1^2}$$

$$e_M^{z\cdot\nabla_t(1-|x|^2b_1c_1^2)b_2c_2 - |z|^2(\Delta_t l(1-|x|^2b_1c_1^2) + \nabla_t\cdot x(1-|x|^2b_1c_1^2)b_1c_1)b_2c_2^2} p_M(t)\,|_{t=0}, \qquad (67)$$

$u = x\cdot z + i\sqrt{|x|^2|z|^2 - (x\cdot z)^2}$ and

$$B = (b+\beta)\tau_\beta, \qquad B_2 = (b+\beta_2)\tau_{\beta_2}, \qquad C' = \frac{1}{b+\gamma}\tau_\gamma, \qquad C_2' = \frac{1}{b+\gamma_2}\tau_{\gamma_2}.$$

Due to Lemma(6.2.8), we know that for $\alpha \in \mathbb{C}$ such that $Re(\alpha u) > Re(\alpha\bar{u})$ and $Re(\alpha u) > 0$, we have

$$\Phi_2\left(\begin{matrix} b+\beta \\ b+\gamma \quad b+\gamma_2 \end{matrix}; \begin{matrix} b+\beta_2\ b+\beta_2 \\ - \end{matrix}; \alpha u, \alpha\bar{u}\right)$$

$$= \frac{\Gamma(b+\gamma)\Gamma(b+\gamma_2)}{\Gamma(b+\beta)\Gamma(b+\beta_2)}\alpha^{\beta+\beta_2-\gamma-\gamma_2}u^{b+2\beta_2+\beta-\gamma-\gamma_2}(u-\bar{u})^{-b-\beta_2}e^{\alpha u}\left(1+O(\alpha^{-1})\right),$$

uniformly in $\beta, \beta_2, \gamma, \gamma_2$ as $|\alpha| \to \infty$. Denote

$$\Phi := \frac{\Gamma(b+\gamma)\Gamma(b+\gamma_2)}{\Gamma(b+\beta)\Gamma(b+\beta_2)} \alpha^{\beta+\beta_2-\gamma-\gamma_2} u^{b+2\beta_2+\beta-\gamma-\gamma_2} (u-\bar{u})^{-b-\beta_2} e^{\alpha u} \qquad (68)$$

Substituting (68) into (66) we get

$$R_\alpha(x,z)(B_\alpha^2 p)(x,z)$$
$$= q\left(\frac{1}{4\alpha}, \frac{1}{\alpha}B, \frac{1}{\alpha}B_2, \alpha C', \alpha C_2'\right)\Phi(1+O(\alpha^{-1}))\Big|_{\beta,\beta_2\gamma,\gamma_2=0}$$
$$= \Phi q\left(\frac{1}{4\alpha}, \frac{1}{\alpha}\frac{1}{\Phi}B\Phi, \frac{1}{\alpha}\frac{1}{\Phi}B_2\Phi, \alpha\frac{1}{\Phi}C'\Phi, \alpha\frac{1}{\Phi}C_2'\Phi\right)(1+O(\alpha^{-1}))\Big|_{\beta,\beta_2\gamma,\gamma_2=0}. \qquad (69)$$

It is an easy exercise to establish the relations

$$\frac{1}{\Phi}B\Phi = \alpha u \tau_\beta,$$
$$\frac{1}{\Phi}B_2\Phi = \alpha\frac{u^2}{u-\bar{u}}\tau_{\beta_2},$$
$$\frac{1}{\Phi}C'\Phi = \frac{1}{\alpha u}\tau_\gamma,$$
$$\frac{1}{\Phi}C_2'\Phi = \frac{1}{\alpha u}\tau_{\gamma_2}.$$

Applying the latter to (69) results in the equation

$$R_\alpha(x,z)(B_\alpha^2 p)(x,z)$$
$$= \left(\frac{u}{u-\bar{u}}\right)^b e^{\alpha u} q\left(\frac{1}{4\alpha}, u\tau_\beta, \frac{u^2}{u-\bar{u}}\tau_{\beta_2}, \frac{1}{u}\tau_\gamma, \frac{1}{u}\tau_{\gamma_2}\right)(1+O(\alpha^{-1}))\Big|_{\beta,\beta_2,\gamma,\gamma_2=0},$$

hence

$$R_\alpha(x,z)(B_\alpha^2 p)(x,z) \sim \left(\frac{u}{u-\bar{u}}\right)^b e^{\alpha u} q\left(0, u, \frac{u^2}{u-\bar{u}}, \frac{1}{u}, \frac{1}{u}\right).$$

Since $p \equiv 1$ implies $q \equiv 1$, we have

$$R_\alpha(x,z) = R_\alpha(x,z)(B_\alpha^2 1)(x,z) \sim \left(\frac{u}{u-\bar{u}}\right) e^{\alpha u},$$

whence the relation

$$(B_\alpha^2 p)(x,z) \longrightarrow q\left(0, u, \frac{u^2}{u-\bar{u}}, \frac{1}{u}, \frac{1}{u}\right)$$
$$= e^{x\cdot\nabla_t} e^{z\cdot\nabla_t(1-|x|^2\frac{1}{u})\frac{u}{u-\bar{u}} - |z|^2\left(\nabla_t\cdot x\left(1-|x|^2\frac{1}{u}\right)\right)\frac{1}{u-\bar{u}}} p_M(t)\Big|_{t=0} \qquad (70)$$

follows. It is now a matter of a minor manipulation to bring the last expression in (70) into the form

$$e^{\left(x\frac{u-|z|^2}{u-\bar{u}} + z\frac{u-|x|^2}{u-\bar{u}}\right)\cdot\nabla_t} p_M(t)\Big|_{t=0} = p_M\left(x\frac{u-|z|^2}{u-\bar{u}} + z\frac{u-|x|^2}{u-\bar{u}}\right).$$

This concludes the proof. The properties of the point $v = x\frac{u-|z|^2}{u-\bar{u}} + z\frac{u-|x|^2}{u-\bar{u}}$ are easy to establish and they are listed in (30).

We having to be divided into two cases:

Case 1. $z = \xi x$, where $\xi > 0$. Weapply the second part of Theorem(6.2.15)to get

$$R_\alpha(x,\xi x)(B_\alpha^2 p)(x,\xi x)$$
$$= q_2\left(\frac{1}{4\alpha}, BC', \alpha BC'^2, B_2C_2', B_2C_2'^2\right) {}_2F_2\left(\begin{matrix} 2b+\gamma_2 & b+\beta \\ b+\gamma_2 & b+\gamma \end{matrix}; \alpha\xi\,|x|^2\right)\Big|_{t,\beta,\beta_2,\gamma,\gamma_2=0}, \qquad (71)$$

where q_2 is a polynomial in all arguments, more specifically

$$q_2(l, q_1, q_2, q_3, q_4)$$
$$= e_M^{l\Delta_t} e_M^{x\cdot\nabla_t q_1 - |x|^2 l\Delta_t q_2} e_M^{\xi x\cdot\nabla_t(1-|x|^2q_2)q_3 - \frac{1}{4}\xi^2|x|^2\Delta_t(1-|x|^2q_2)^2q_4} p_M(t)|_{t=0}. \qquad (72)$$

Using Lemma(6.2.9)yields

$$ {}_2F_2\left(\begin{matrix}2b+\gamma_2\ b+\beta\\ b+\ \gamma_2\ b+\gamma\end{matrix};\ \alpha\xi\ |x|^2\right) $$
$$ =\frac{\Gamma(b+\gamma)\Gamma(b+\gamma_2)}{\Gamma(b+\beta)\Gamma(2b+\gamma_2)}(\alpha\xi|x|^2)^{b+\beta-\gamma}e^{\alpha\xi|x|^2}\left(1+O(\alpha^{-1})\right), $$

for $\alpha>0$, so that

$$ R_\alpha(x,\xi x)(B_\alpha^2 p_M)(x,\xi x) $$
$$ =q_2\left(\frac{1}{4\alpha},\frac{1}{\alpha}B,\frac{1}{\alpha}B_2,\alpha C',\alpha C_2'\right)\frac{\Gamma(b+\gamma)\Gamma(b+\gamma_2)}{\Gamma(b+\beta)\Gamma(2b+\gamma_2)} $$
$$ (\alpha\xi|x|^2)^{b+\beta-\gamma}e^{\alpha\xi|x|^2}\left(1+O(\alpha^{-1})\right)\Big|_{\beta,\beta_2,\gamma,\gamma_2=0}. $$

Put once again

$$ \Phi:=\frac{\Gamma(b+\gamma)\Gamma(b+\gamma_2)}{\Gamma(b+\beta)\Gamma(2b+\gamma_2)}(\alpha\xi|x|^2)^{b+\beta-\gamma}e^{\alpha\xi|x|^2}, $$

and upon performing manipulations similar to those used in the course of the proof of Theorem(6.2.3) we obtain

$$ R_\alpha(x,\xi x)(B_\alpha^2 p_M)(x,\xi x) $$
$$ =\Phi q_2\left(\frac{1}{4\alpha},\frac{1}{\Phi}BC'\Phi,\alpha\frac{1}{\Phi}B\Phi,\frac{1}{\Phi}B_2C_2'\Phi,\frac{1}{\Phi}B_2C_2'^2\Phi\right)(1+O(\alpha^{-1})), $$

where

$$ \frac{1}{\Phi}B\Phi=\alpha\xi|x|^2\tau_\beta, $$
$$ \frac{1}{\Phi}B_2\Phi=B_2, $$
$$ \frac{1}{\Phi}C'\Phi=\frac{1}{\alpha\xi|x|^2}\tau_\gamma, $$
$$ \frac{1}{\Phi}C_2'\Phi=\frac{1}{2b+\gamma_2}\tau_{\gamma_2}=:\ C_3', $$

so that

$$ \frac{1}{\Phi}BC'\Phi=\tau_\beta\tau_\gamma, $$
$$ \frac{1}{\Phi}BC'^2\Phi=\frac{1}{\alpha\xi|x|^2}\tau_\beta\tau_\gamma^2, $$
$$ \frac{1}{\Phi}B_2C_2'\Phi=B_2C_3', $$
$$ \frac{1}{\Phi}B_2C_2'^2\Phi=B_2C_3'^2. $$

Therefore we have

$$ R_\alpha(x,\xi x)(B_\alpha^2 p_M)(x,\xi x) $$
$$ \sim\frac{\Gamma(b)}{\Gamma(2b)}(\alpha\xi|x|^2)^b e^{\alpha\xi|x|^2}q_2\left(\frac{1}{4\alpha},\tau_\beta\tau_\gamma,\frac{1}{\xi|x|^2}\tau_\beta\tau_\gamma^2,B_2C_3',B_2C_3'^2\right)\left(1+O(\alpha^{-1})\right)\Big|_{\gamma,\gamma_2,\beta,\beta_2=0}, $$

as $\alpha\to\infty$, whence we infer that

$$ R_\alpha(x,\xi x)(B_\alpha^2 p_M)(x,\xi x)\sim\frac{\Gamma(b)}{\Gamma(2b)}(\alpha\xi|x|^2)^b e^{\alpha\xi|x|^2}q_2\left(0,1,\frac{1}{\xi|x|^2},B_2C_3',B_2C_3'^2\right)1|_{\gamma_2,\beta_2=0}. $$

Consequently,

$$ R_\alpha(x,\xi x)\sim\frac{\Gamma(b)}{\Gamma(2b)}(\alpha\xi|x|^2)^b e^{\alpha\xi|x|^2},\qquad \alpha\to\infty, $$

and, as$\alpha\to\infty$,

$$ (B_\alpha^2 p_M)(x,\xi x)\to q_2\left(0,1,\frac{1}{\xi|x|^2},B_2C_3',B_2C_3'^2\right)1|_{\gamma_2,\beta_2=0}. $$

Using (72) we get

$$q_2\left(0,1,\frac{1}{\xi|x|^2},B_2C_3',B_2C_3'^2\right)1|_{\gamma_2,\beta_2=0}$$
$$=e_M^{x\cdot\nabla_t}\, e_M^{x\cdot\nabla_t(\xi-1)B_2C_2'-\frac{1}{4}|x|^2\Delta_t(\xi-1)^2B_2C_2'^2}\; p_M(t)|_{\gamma_2,\beta_2,t=0},$$
or, equivalently,
$$q_2\left(0,1,\frac{1}{\xi|x|^2},B_2C_3',B_2C_3'^2\right)1|_{\gamma_2,\beta_2=0}$$
$$=p_M(\nabla_t)e^{x\cdot t}\, e_M^{\partial_s}e_M^{sx\cdot t(\xi-1)B_2C_2'-s^2\frac{1}{4}|x|^2|t|^2(\xi-1)^2B_2C_2'^2}\;1|_{t,s,\beta_2,\gamma_2=0}.$$
Obviously, no harm can be done by introducing a new variable s since the only effect it has is that it cancels all terms in t of order higher than M (they would be canceled anyway by the factor $p_M(\nabla_t)|_{t=0}$). Applying Lemma (6.2.12) and casting aside all the constraints on the exponential terms we get the final result
$$(B_\alpha^2p_M)(x,\xi x)\to p_M(\nabla_t)e^{x\cdot t}\,\Phi_2\left({- \atop 2b};{b \atop -}{b \atop };\frac{\xi-1}{2}u_{x,t},\frac{\xi-1}{2}\,\bar{u}_{x,t}\right)\Big|_{t=0},$$
$$\alpha\to\infty. \tag{73}$$
Notice that for the usual Berezin transform ($\xi=1$) we have only reproduced the already known result
$$(B_\alpha^2p_M)(x,x)\to p_M(\nabla_t)e^{x\cdot t}|_{t=0}=p_M(x).$$
Case 2. $z=0$. In this case we have
$$(B_\alpha^2p_M)(x,0)=(P_\alpha p_M)(x),$$
which is, according to Lemma (6.2.13), equal to
$$e^{\frac{\Delta t}{4\alpha}}R_{\frac{1}{2}}(\nabla_t,x)p_M(t)|_{t=0}.$$
Passing to the limit $\alpha\to\infty$ we therefore obtain
$$R_{\frac{1}{2}}(\nabla_t,x)p_M(t)|_{t=0},$$
or equivalently
$$p_M(\nabla_t)R_{\frac{1}{2}}(x,t)|_{t=0}=\;p_M(\nabla_t)\,\Phi_2\left({- \atop b};{b \atop -}{b \atop };\frac{1}{2}u_{x,t},\frac{1}{2}\bar{u}_{x,t}\right)\Big|_{t=0},$$
completing the proof of this case. Note that the last expression can also be written as
$$e_M^{x\cdot\nabla_tBC'-\frac{1}{4}|x|^2\Delta_tBC'^2}\;p_M(t)|_{t=0}.$$
Case 3. $z=-\xi x$, where $\xi>0$. Westart by calculating the Berezin transform of two arguments for the function f of the form $f_{s,\omega}:=e^{s|y|^2}(\omega\cdot y+\omega_0)$. Surely, any polynomial of the form $p_1(y,|y|^2)$, where $degp_1\le 1$ in the first argument, can be obtained by appropriate differentiation of the function $f_{s,\omega}$ with respect to the variables s and ω. Now, the following equality holds:
$$\int_{\mathbb{R}^n}e^{s|y|^2}(\omega\cdot y+\omega_0)R_\alpha(x,y)R_\alpha(-\xi x,y)d\mu_\alpha^n(y)$$
$$=\int_{\mathbb{R}^n}(\omega\cdot y+\omega_0)R_{\alpha-s}\left(\frac{\alpha}{\alpha-s}x,y\right)R_{\alpha-s}\left(-\xi\frac{\alpha}{\alpha-s}x,y\right)d\mu_{\alpha-s}^n(y)\left(\frac{\alpha}{\alpha-s}\right)^{\frac{n}{2}},$$
since
$$d\mu_\alpha^n(y)=e^{-\alpha|y|^2}\left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}}d^ny,\qquad and\quad R_\alpha(x,y)=R_{\alpha-s}\left(\frac{\alpha}{\alpha-s}x,y\right).$$
Application of the second part of Theorem(6.2.15) is easy because many terms cancel out since the polynomial p_M is in this case linear and we end up with the expression
$$I:=\left(\frac{\alpha}{\alpha-s}\right)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{s|y|^2}(\omega\cdot y+\omega_0)R_\alpha(x,y)R_\alpha(-\xi x,y)d\mu_\alpha^n(y)$$
$$={}_1F_1\left({2b \atop b};-\frac{\alpha^2}{\alpha-s}\xi\,|x|^2\right)\left(\frac{\alpha}{\alpha-s}x\cdot\omega+\omega_0\right)$$

$$-{}_1F_1\left(\begin{matrix}2b+1\\ b+1\end{matrix}; -\frac{\alpha^2}{\alpha-s}\xi|x|^2\right)\frac{\alpha\xi}{\alpha-s}\omega\cdot x$$
$$+{}_1F_1\left(\begin{matrix}2b+1\\ b+2\end{matrix}; -\frac{\alpha^2}{\alpha-s}\xi|x|^2\right)\frac{\alpha^3\xi|x|^2}{(b+1)(\alpha-s)^2}\omega\cdot x.$$

The asymptotic behavior of the confluent hypergeometric function ${}_1F_1$in this case depends on whether the dimension is odd or even. For even dimensions, we can use the transformation formula

$${}_1F_1\left(\begin{matrix}a\\ c\end{matrix}; x\right) = e^x\,{}_1F_1\left(\begin{matrix}c-a\\ c\end{matrix}; -x\right),$$

to get

$$I = e^{-\frac{\alpha^2}{\alpha-s}\xi|x|^2}\,{}_1F_1\left(\begin{matrix}-b\\ b\end{matrix}; \frac{\alpha^2}{\alpha-s}\xi|x|^2\right)\left(\frac{\alpha}{\alpha-s}x\cdot\omega+\omega_0\right)$$
$$-e^{-\frac{\alpha^2}{\alpha-s}\xi|x|^2}\,{}_1F_1\left(\begin{matrix}-b\\ b+1\end{matrix}; \frac{\alpha^2}{\alpha-s}\xi|x|^2\right)\frac{\alpha\xi}{\alpha-s}\omega\cdot x$$
$$+e^{-\frac{\alpha^2}{\alpha-s}\xi|x|^2}\,{}_1F_1\left(\begin{matrix}-b+1\\ b+2\end{matrix}; \frac{\alpha^2}{\alpha-s}\xi|x|^2\right)\frac{\alpha^3\xi|x|^2}{(b+1)(\alpha-s)^2}\,\omega\cdot x,$$

and all the ${}_1F_1$ functions are now polynomials (except when $b=0$, i.e. $n=2$) since $b=\frac{n}{2}-1$ is a positive integer in even dimensions. Assuch, their asymptotic behavior is governed by the highest order term. Thus we get as $\alpha\to\infty$,

$$\int_{\mathbb{R}^n} e^{s|y|^2}(\omega\cdot y+\omega_0)R_\alpha(x,y)R_\alpha(-\xi x,y)d\mu_\alpha^n(y)$$
$$\sim e^{-\frac{\alpha^2}{\alpha-s}\xi|x|^2}\frac{\Gamma(b)}{\Gamma(2b)}\,(-\alpha\xi|x|^2)^b\left(x\cdot\omega\,\frac{1-\xi}{2}+\omega_0\right).$$

Consequently,

$$R_\alpha(x,-\xi x)\sim e^{-\alpha\xi|x|^2}\frac{\Gamma(b)}{\Gamma(2b)}\,(-\alpha\xi|x|^2)^b,\qquad(\alpha\to\infty).$$

and therefore for even dimensions (except $n=2$) we have

$$B_\alpha^2(e^{s|y|^2}(y\cdot\omega+\omega_0))(x,-\xi x)\to e^{-s\xi|x|^2}\left(x\cdot\omega\frac{1-\xi}{2}+\omega_0\right),\qquad(\alpha\to\infty).$$

(Incidentally, this result is exactly the same that we would get had we used the first part of Theorem (6.2.4) for negative ξ. This is because the asymptotic behavior of ${}_1F_1$is governed by the same (exponential) term.)

For odd dimensions we apply the formula (27) to I to get

$$\int_{\mathbb{R}^n} e^{s|y|^2}(\omega\cdot y+\omega_0)R_\alpha(x,y)R_\alpha(-\xi x,y)d\mu_\alpha^n(y)\sim\frac{\Gamma(b)}{\Gamma(-b)}\,(\alpha\xi|x|^2)^b\omega_0,\qquad(\alpha\to\infty),$$

hence we arrive at

$$B_\alpha^2(e^{s|y|^2}(y\cdot\omega+\omega_0))(x,-\xi x)\to\omega_0,\qquad(\alpha\to\infty).$$

It remains to show that $(B_\alpha^2(y_1y_2))(x,-\xi x)$ diverges as $\alpha\to\infty$. We can again use the second part of Theorem(6.2.15)directly since the polynomial $y_1y_2$is quite nice (even harmonic) to get the expression

$$\int_{\mathbb{R}^n} y_1y_2R_\alpha(x,y)R_\alpha(-\xi x,y)d\mu_\alpha^n(y)$$
$$= x_1x_2\,{}_1F_1\left(\begin{matrix}2b\\ b\end{matrix}; -\alpha\xi|x|^2\right)-2\xi x_1x_2\,{}_1F_1\left(\begin{matrix}2b+1\\ b+1\end{matrix}; -\alpha\xi|x|^2\right)$$
$$+x_1x_2\frac{2\alpha\xi}{b+2}\,{}_2F_2\left(\begin{matrix}2b+1 & b+2\\ b+3 & b+1\end{matrix}; -\alpha\xi|x|^2\right)+x_1x_2\xi^2\,{}_1F_1\left(\begin{matrix}2b+2\\ b+2\end{matrix}; -\alpha\xi|x|^2\right)$$
$$-x_1x_2\frac{\alpha\xi^2|x|^2}{b+1}\,{}_2F_2\left(\begin{matrix}2b+2 & b+1\\ b+2 & b+2\end{matrix}; -\alpha\xi|x|^2\right)$$
$$+x_1x_2\frac{\alpha^2\xi^2|x|^4}{(b+2)(b+3)}\,{}_1F_1\left(\begin{matrix}2b+2\\ b+4\end{matrix}; -\alpha\xi|x|^2\right).\qquad(74)$$

It is easy to understand from (27) that any of these ${}_1F_1$functions cannot produce a divergent term when divided by $R_\alpha(x,-\xi x)$ which, being itself an ${}_1F_1$function, has exactly the same asymptotic behavior. Wecan rewrite the first ${}_2F_2$ term in (74) as a linear combination of ${}_1F_1$'s:

$$ {}_2F_2\left({2b+1\ \ b+2 \atop b+3\ \ b+1}; -\alpha\xi|x|^2\right) + {}_1F_1\left({2b+1 \atop b+3}; -\alpha\xi|x|^2\right) $$
$$ -\frac{\alpha\xi|x|^2(2b+1)}{(b+1)(b+3)}\,{}_1F_1\left({2b+2 \atop b+4}; -\alpha\xi|x|^2\right), $$

by rising its second lower parameter. The logic is the same as in (31). So the only interesting term here, capable of producing a divergent term, is

$$ {}_2F_2\left({2b+2\ \ b+1 \atop b+2\ \ b+2}; -\alpha\xi|x|^2\right), $$

which is a genuine ${}_2F_2$function (when $b \neq 0$, i.e. $n > 2$).
Applying the asymptotic formula from [14,17] we get

$$ {}_2F_2\left({2b+2\ \ b+1 \atop b+2\ \ b+2}; -\alpha\xi|x|^2\right) \sim \frac{\Gamma(b+1)\Gamma(b+2)^2}{\Gamma(2b+2)}(-\alpha\xi|x|^2)^{-b-1}, (\alpha \to \infty). $$

But the function $R_\alpha(x,-\xi x) = {}_1F_1\left({2b \atop b}; -\alpha\xi|x|^2\right)$ behaves at best (that is in odd dimensions) as $\alpha^{-2b}$and it is therefore clear that the fraction of these two will grow without bound (for $n > 2$). It is easy to see why this is happening. There are only two ways in which the ${}_1F_1$ can behave in α(exponentially or polynomially). On the other hand, there are three ways in which the ${}_2F_2$ function can behave. And it is just this third mode of behavior of the ${}_2F_2$– that one which the ${}_1F_1$ function cannot mimic – that is dominant in the region where the argument is large and negative.

List of Symbols

Symbol	The Symbols	Page
F^2	Fock space	1
WOT	Weak operator Topology	2
$\otimes$	Tensor product	2
dim	Dimension	2
H^2	Arveson space	3
F^∞	Toeplitz algebra	3
$\oplus$	Orthogonal sum	3
H^∞	Algebra of bounded Analytic functions	3
SOT	Strong operator topology	4
$\ominus$	Direct difference	4
Ker	Kernel	4
Mult	Multiplicity	11
C.n.c	Completely non-coisometric	12
Dil.ind	Dilation index	12
F_n^∞	Toeplitz algebra	27
H^2	Hardy space	28
Rad	Radial	34
K_q	Berezin kernel	35
Ap	Almost –periodic	42
L^2	Hilbert space	48
Tp	Trigonometric polynomial	52
Im	Imaginary	53
can	Canonical	53
cl	class	59
tr	trace	61
hol	Holomorphic	63
re	Real	63
L^∞	Essential lebesgue space	67
Bt	Berezin lebesgue space	83
det	Determinant	91
PSH	Plurisub harmonic	97
ord	Order	105
Hess	hessian	105
max	Maximum	113
int	Interior	135
supp	Support	142
arg	Argument	147
giv	Give	184
Harm	Harmonic	197
F_α^{harm}	Fock space of all harmonic function	205
min	Minimum	206
deg	Degree	268
mod	Modulo	275

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