Sudan University of Science and Technology
College of Graduate Studies

Solving Fractional Telegraphy Equations Using Double Laplace Transform Method

حل معادلات التلغراف الكسرية باستخدام طريقة تحويل لابلاس الثنائي

Submitted in Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

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Dedication

This thesis is dedicated to my parents, wife and children for their love, endless support and encouragement.

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My praise and thanks be to Almighty Allah. I would like to express my sincere gratitude to my supervisor Prof. Hassan Eltayeb and Dr. Mohamed H. Khabir for the their continuous support of my Ph.D project and related research. I thank them for their patience, motivation and immense knowledge. Their guidance has gratly helped me throughout my research and the writing up of this thesis. I could not have imagined having a better supervisor and mentor during my Ph.D study.

Abstract

This research project deals with a new two methods called Fractional Double Laplace Decomposition Method and Fractional Natural Decomposition Method. The first method was developed by combining the fractional double Laplace transform and adomain decomposition method. The second method was developed by combining the fractional natural transform and adomain decomposition method. We applied the above methods to find an exact and approximate solution of different types of fractional telegraph equation. adomian polynomials were used to decompose the nonlinear terms of the differential equations. The two techniques are described and illustrated with some examples using the Matlab to plot the solution in order to compare the exact outcome. Then the two techniques lead us to say that these methods have highly accurate and very efficient solutions and can be applied to other nonlinear terms.

الخلاصة

يتعامل هذا البحث مع طريقتين جديدتين هما طريقة تحويل لابلاس الكسرية الثنائية وطريقة التحويل الطبيعي الكسرية. تم التوصل للطريقة الأولى من خلال الدمج ين طريقة تحويل لابلاس الكسرية ااالثنائية وطريقة أدومين، اما الطريقة الثانية فقد تم تطويرها من خلال الدمج بين التحويل الطبيعي الكسري وطريقة تحلل أدومين. لقد قمنا بتطبيق الطرق المذكورة أعلاه لإيجاد حل دقيق وتقريبي لأنواع مختلفة من معادلة التلغراف الكسرية، وتم استخدام كثير الحدود لأدومين للتعامل مع الحدود غير الخطية للمعادلات التفاضلية. كما تم وصف وتوضيح الطرق مع بعض الأمثلة باستخدام ماتلاب لرسم الحل من أجل مقارنة النتيجة الدقيقة. قادنا الأسلوبان إلى القول بأن هذه الطرق لها حلول عالية الدقة وفعالة للغاية ويمكن تطبيقها على معادلات أخرى غير خطية.

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Chapter 1

Introduction

1.1 Introduction

Fractional Calculus is not a modern mathematical subject “why”? Fractional calculus is more general than calculus which studies (non-integer) order implies that the calculus is sub-subject from fractional calculus. Perhaps, this branch of mathematics changes many concept of different field in nature. Thereby, can use fractional calculus as instrument to describe various natural phenomenon in a better away. For this reason most scientists’ orientation is to research about the fractional calculus and the applications such as engineering, finance, science and applied mathematics to discover more details about these areas and how to apply fractional calculus in this field. I believe, next decade we will see the fractional calculus exploitation to the humanity is going to make a revolution in the world. Fractional partial differential equation is an equation that contains an unknown function of several variables. Telegraph equation is one of the most important problem in physics engineering. This equation describes the voltage and current on an electrical transmission line with distance and time, it represents transmission line model. The equation came from Oliver Heaviside in the 1880.

1.2 Literature Review

Many authors solved the fractional telegraph equation by different methods to obtain the exact and approximate solutions. In the literature, authors used powerful methods to solve fractional telegraph equation, for example in [1] the author applied Laplace transform method, in [2] the author suggested Laplace variational iteration method to obtain the approximate solution. In [3] the author obtained the approximate solution of the telegraph equation by using double Laplace transform method, in [4] Homotopy perturbation technique was used, in [5] radial basis functions and in [6] the author introduced the mixture of a new integral transform and homotopy perturbation method (HPM).

In this thesis, we used a new two methods, namely fractional double Laplace adomain decomposition method (FDLDM) and fractional natural adomain decomposition method (FNDM) for solving fractional telegraph equation.

This first method is a combination of the double Laplace transform method and adomain decomposition method and the second method is a combination of the natural transform method and adomain decomposition method. The nonlinear terms can be easily handled by the use of Adomian polynomials. The technique is described and illustrated with some examples and uses Matlabe to plot the solution.

This thesis is organized as follows:

In chapter one we took a glance at the literature review and how this research is organized. In chapter two we discussed and reviewed some concepts concerning the basics of single and double Laplace transformation, natural transformation and the introduction of decomposition method and all the tools that are used and are useful in this thesis. In chapter three we applied the fractional double Laplace adomain decomposition method to solve linear, non linear, singular and coupled systems of the fractional telegraph equation. Some examples are given to illustrate our method. In chapter four we used fractional natural adomain decomposition method to solve linear, non-linear, and a singular fractional telegraph equation. Some examples are given to support the present method.

1.3 Publications

- Eltayeb, Hassan, Yahya T Abdalla, Imed Bachar, and Mohamed H Khabir. "Fractional Telegraph Equation and its Solution by Natural Transform Decomposition Method." Symmetry 11, 3 (2019): 334 , DOI: 10.3390/ sym11030334.
- Yahya T. Abdallah, "Applications of Double Laplace Transform Method to Coupled Systems of Fractional Telegraph Equations", International Journal of Science and Research (IJSR), https://www.ijsr.net/archive/v8i4/ ART20196834.pdf, Volume 8 Issue 4, April 2019, 534 - 537

Chapter 2

Mathematical Tools

In this chapter, some basic definition, theorems and special functions are also mentioned, and reviews are utilized in the other chapters. Some information is given here on the Gamma and beta functions, Mittag-Leffler function, Wright Function, Grünwald-Letnikov fractional derivative, Caputo fractional, single Laplace transform, double Laplace transform, natural transform and Adomian decomposition method. Several theorem are stated without proof and they refer to the reference for more details.

2.1 Laplace Transform of Fractional Order Derivative

Laplace transform is a very powerful technique for solving fractional telegraph equation. In the present section, we compile the foundations of the theory and the basic properties of the Laplace transform that are useful in further thesis.

Single Laplace Transform

Let us recall some basic facts about the Laplace transform.

Definition (2.1.1) : [8] Let f is a real or complex-valued function of the variable $t > 0$ and s is a real or complex parameter. we define the Laplace transform of f as follows:

$$F(s) = \mathscr{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t) dt. \tag{2.1}$$

Whenever the limit exists, and when it does, the integral (2.1) is said to converge. If the limit does not exist, the integral is said to diverge and there is no Laplace transform defined for f.

Definition (2.1.2) : [8] The original $f(t)$ can be restored from the Laplace transform $F(s)$ which defined the inverse Laplace transform as:

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\,ds, \qquad c = \Re(s) > c_0. \tag{2.2}$$

where c_0 lies in the half plane of the absolute convergence of the Laplace transform (2.1) .

2.2 Double Laplace Transform

Definition (2.2.1) : [9] Is defined double Laplace transform as:

$$\mathcal{L}_x\mathcal{L}_t[f(x,t)] = F(p,s) = \int_0^\infty e^{-px}\int_0^\infty e^{-st}f(x,t)dt\,dx, \tag{2.3}$$

where $x, t > 0$ and p, s are complex values, and further double Laplace transform of the first order partial derivatives is given by

$$\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial u(x,t)}{\partial x}\right] = pU(p,s) - U(0,s). \tag{2.4}$$

Similarly, the double Laplace transform for second partial derivative with respect to x and t are defined as follows

$$\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = p^2U(p,s) - pU(0,s) - \frac{\partial u(0,s)}{\partial x}\,, \tag{2.5}$$

$$\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2U(p,s) - sU(p,0) - \frac{\partial u(p,0)}{\partial t}\,, \tag{2.6}$$

Definition (2.2.2) : [3] The double Laplace transform formulas for the partial derivatives of an arbitrary integer order are

$$\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}\right] = p^\alpha U(p,s) - \sum_{j=0}^{n-1} p^{\alpha-j-1}\mathscr{L}_t\left[\frac{\partial^j u(0,t)}{\partial x^j}\right]\,, \tag{2.7}$$

$$\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}\right]=s^{\alpha}U(p,s)-\sum_{i=0}^{m-1}s^{\alpha-i-1}\mathscr{L}_x\left[\frac{\partial^{i}u(x,0)}{\partial t^{i}}\right], \tag{2.8}$$

Definition (2.2.3) : [10] The inverse double Laplace transform $\mathscr{L}_p^{-1}\mathscr{L}_s^{-1}\left[F\left(p,s\right)\right]=f(x,t)$ is defined by the complex double integral formula as:

$$\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[F\left(p,s\right)\right]=f(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{px}dp\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}e^{st}F\left(p,s\right)ds, \tag{2.9}$$

where $F\left(p,s\right)$ must be an analytic function for all p and s in the region defined by the inequalities $\Re e\geq c$ and $\Re s\geq s$, where c and d are real constants to be chosen suitably.

Example (2.2.1) : If $f(t)=1$ for $t>0$, then

$$\begin{aligned}\mathscr{L}\left[1\right] &= \int_0^{\infty}e^{-st}(1)dt\\ &= \lim_{\tau\to\infty}\left(\left.\frac{-e^{-st}}{s}\right|_0^{\tau}\right)\\ &= \lim_{\tau\to\infty}\left(\frac{-e^{-st}}{s}+\frac{1}{s}\right)\\ &= \frac{1}{s}.\end{aligned}$$

Example (2.2.2) : If $f(t)=t$ for $t\geq 0$, then

$$\begin{aligned}\mathscr{L}\left[t\right] &= \int_0^{\infty}e^{-st}(t)dt\\ &= \left.\frac{-te^{-st}}{s}\right|_0^{\infty}+\frac{1}{s}\int_0^{\infty}e^{-st}dt\\ &= \frac{1}{s}\mathscr{L}\left[1\right]\\ &= \frac{1}{s^2}.\end{aligned}$$

(provided $Re(s)>0$).

Now, by induction, one can show that in general

$$\mathscr{L}\left[t^n\right] = \frac{n!}{s^{n+1}} \qquad Re(s) > a, \ \ for\ n = 1, 2, 3..... \tag{2.10}$$

Theorem (2.2.4) : [8] (First Translation Theorem). If $F(s) = \mathscr{L}\left[f(t)\right]$ for $\Re(s) > 0$,then

$$F(s-a) = \mathscr{L}\left[e^{at}f(t)\right] \quad (a\ real, \Re(s) > a).$$

Proof. For $\Re(s) > a$,

$$\begin{aligned} F(s-a) &= \int_0^\infty e^{-(s-a)}f(t)dt \\ &= \int_0^\infty e^{-(s-a)}f(t)dt \\ &= \mathscr{L}\left[e^{at}f(t)\right] \end{aligned}$$

Definition (2.2.5) : [8] Now from equation (2.10) we want to generalize the $\mathscr{L}\left[t^\upsilon\right]$ from non-integer value of υ, consider

$$\mathscr{L}\left[t^\upsilon\right] = \int_0^\infty e^{-s}t^\upsilon dt, \quad (\upsilon > -1)\,.$$

By a change of variables, $x = st\ (s > 0)$,

$$\begin{aligned} \mathscr{L}\left[t^\upsilon\right] &= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^\upsilon\left(\frac{1}{s}\right)dx \\ &= \frac{1}{s^{\upsilon+1}}\int_0^\infty x^\upsilon e^{-x}dx. \end{aligned}$$

The quantity $\int_0^\infty e^{-x}t^{z-1}dx$ is gamma function, Although the improper integral exists and is a continuous function of $\Re(z) > 0$. Then

$$\mathscr{L}\left[t^\upsilon\right] = \tfrac{\Gamma(\upsilon+1)}{s^{\upsilon+1}}, \qquad (\upsilon > -1, s > 0)$$

Yields , $\Gamma(\upsilon + 1) = \upsilon!$

Example (2.2.3) : For $\upsilon = -\frac{1}{2}$

$$\mathscr{L}\left[t^{-\frac{1}{2}}\right] = \tfrac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \sqrt{\tfrac{\pi}{s}}, \qquad (s > 0)\,.$$

2.3 Special Functions of the Fractional Calculus

Gamma Function:

Euler's gamma function $\Gamma(z)$ is a pillar of fractional calculus which plays an important role in the theory of this branch.

We will give the the main concept of gamma function and properties of it. All the theory and definition in this section by [7].

Definition of the Gamma Function:

The $(Euler)$ gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \qquad \Re e(z) > 0 \tag{2.11}$$

Limit Representation of the Gamma Function

The gamma function $\Gamma(z)$ can be also represented by the limit due to Euler such as:

$$\Gamma(z) = \lim_{n \to \infty} \frac{n!\,(n+1)^z}{z(z+1)...(z+n)}, \tag{2.12}$$

where we initially suppose $\Re e(z) > 0$.

Some Properties of the Gamma Function:

There are basic prominent properties of the gamma function one of these properties is:

$$\Gamma(z+1) = z\Gamma(z), \tag{2.13}$$

can be easily proved (2.13) by using (2.11) and integrating by parts:

$$\Gamma(z+1) = \int_0^{\infty} e^{-t} t^{z} dt = \left[-e^{-t} t^{z}\right]_{t=0}^{t=\infty} + z \int_0^{\infty} e^{-t} t^{z-1} dt = z\Gamma(z).$$

In the case $z = 1$ then $\Gamma(1) = 1$ and using (2.11) we obtained for $z = 2, 3, ...$:

$$\Gamma(2) = 1.\Gamma(1) = 1 = 1!,$$

$$\Gamma(3) = 2.\Gamma(2) = 2.1! = 2!,$$

$$\Gamma(4) = 3.\Gamma(3) = 3.2! = 3!,$$

$$\Gamma(z+1) = z.\Gamma(z) = z.(z-1)! = z!.$$

Beta Function:

In many cases it is more comfortable to use the beta function instead of a certain combination of values of the gamma function.

The beta function $\mathrm{B}(z)$ is defined by:

$$\mathrm{B}(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}\,d\tau, \qquad (\Re e(z) > 0, \qquad \Re e(w) > 0) \tag{2.14}$$

The relationship between the gamma function and the beta function

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

Example (2.3.1) :Evaluate the value of $\Gamma(\frac{1}{2})$

Solution.

by definition of Gamma function

$$\Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$$

We know that

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Let $p = 0$ and $q = 0$ then,

$$I = \frac{1}{2}\mathrm{B}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$I = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2$$

and

$$I=\int_0^{\frac{\pi}{2}} d\theta=\frac{\pi}{2}$$

$$\therefore \frac{1}{2}\Gamma\left(\frac{1}{2}\right)^2=\frac{\pi}{2}$$

$$\therefore \Gamma(\frac{1}{2})=\sqrt{\pi}$$

Mittag-Leffler Function:

The exponential function, e^z, can be defined as series by using Taylor series which is represent as summation series in one-parameter generalization, and denoted by

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \tag{2.15}$$

was introduced by G.M. Mittag-Leffler.

A two-parameter function of Mittag-Leffler type plays a very important role in the fractional calculus, is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (\alpha>0,\beta>0) \tag{2.16}$$

Mittag-Leffler and Relation to some other functions:

From the definition (2.16) we get some relation with other function, for example:

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z, \tag{2.17}$$

$$E_{1,2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{e^z-1}{z}, \tag{2.18}$$

$$E_{1,3}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+2)!}=\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+2)!}=\frac{e^z-1-z}{z^2}, \tag{2.19}$$

and in general

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right\}. \tag{2.20}$$

The hyperbolic sine and cosine are also defined by Mittag-Leffler function,

$$E_{2,1}\left(z^2\right) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{(2k)!} = \cosh(z), \tag{2.21}$$

$$E_{2,2}\left(z^2\right) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}. \tag{2.22}$$

The hyperbolic functions of order n can be generalizations of the hyperbolic sine and cosine and represented in the terms of Mittag-Leffler function:

$$h_r(z,n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1} E_{n,r}(z^n), \qquad (r = 1, 2, 3, ...), \tag{2.23}$$

as well as the trigonometric functions of order n can be generalizations of the sine and cosine functions:

$$E_r(z,n) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{nj+r-1}}{(nj+r-1)!} = z^{r-1} E_{n,r}(-z^n), \qquad (r = 1, 2, 3, ...) \tag{2.24}$$

The function $\varepsilon_t(\nu, a)$, introduced for solving differential equations of rational order, is a particular case of Mittag-Leffler function (2.16):

$$\varepsilon_t(\nu, a) = t^{\nu} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu+k+1)} = t^{\nu} E_{1,\nu+1}, (at). \tag{2.25}$$

Yu. N. Rabotnov's function $\ni_\alpha(\beta, t)$ is a particular case of the Mittag-Leffler function (2.16) too:

$$\ni_\alpha(\beta, t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma\left((k+1)(1+\alpha)\right)} = t^{\alpha} E_{\alpha+1,\alpha+1}, (\beta t^{\alpha+1}). \tag{2.26}$$

Laplace Transform of the Mittag-Leffler in two-parameter:

Let us consider the relationship (2.16) be led to find the Laplace transform of the Mittag-Leffler in two-parameter $E_{\alpha,\beta}(z)$. For this purpose, let the Laplace transform of the function $t^k e^{\pm zt}$ in an untraditional way.

First, let us prove that

$$\int_0^\infty e^{-t} e^{\pm zt} dt = \frac{1}{1 \mp z}, \qquad |z| < 1. \tag{2.27}$$

Indeed, using the series expansion for e^z, we obtain

$$\int_0^\infty e^{-t} e^{zt} dt = \frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{(\pm z)^k}{k!} \int_0^\infty e^{-t} e^k dt = \sum_{k=0}^{\infty} (\pm z)^k = \frac{1}{1 \mp z}. \tag{2.28}$$

Second, we differentiate both side of equation (2.27) with respect to z. We get

$$\int_0^\infty e^{-t} e^{\pm zt} dt = \frac{1}{1-z} = \frac{k!}{(1-z)^{k+1}}, \qquad |z| < 1, \tag{2.29}$$

from equation (2.29) a pair Laplace transform of the function $t^k e^{\pm at}$:

$$\int_0^\infty e^{-pt} t^k e^{\pm at} dt = \frac{k!}{(p \mp a)^{k+1}}, \qquad (\Re e(p) > |a|)\,. \tag{2.30}$$

Now, the Mittag-Leffler function (2.16) can be substituted in the integral below leads us to

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(zt^\alpha) dt = \frac{1}{1-z}, \qquad (|z| < 1)\,. \tag{2.31}$$

and we obtain from (2.31) apair of Laplace transform of the function $t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm zt^\alpha)$, $\left(E^{(k)}_{\alpha,\beta}(y) \equiv \frac{d^k}{dy^k} E_{\alpha,\beta}(y)\right)$

$$\int_0^\infty e^{-pt} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm at^\alpha) dt = \frac{k! p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}, \qquad \left(\Re e(p) > |a|^{\frac{1}{\alpha}}\right). \tag{2.32}$$

The particular case of (2.32) for $\alpha = \beta = \frac{1}{2}$

$$\int_0^\infty e^{-pt} t^{\frac{k-1}{2}} E^{(k)}_{\frac{1}{2},\frac{1}{2}}\left(\pm a\sqrt{t}\right) dt = \frac{k!}{\left(\sqrt{p} \mp a\right)^{k+1}}, \qquad \left(\Re(p) > |a|^2\right). \tag{2.33}$$

Derivative of the Mittag-Leffler Function:

By the Riemann-Lioville fractional-order differentiation ${}_0D_t^\gamma$ (γ is an arbitrary real number) of series representation (2.16) we obtain

$${}_0D_t^\gamma \left(t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\lambda t^\alpha)\right) = t^{\alpha k+\beta-\gamma-1} E^{(k)}_{\alpha,\beta-\gamma}(\lambda t^\alpha). \tag{2.34}$$

The particular case of relationship (2.34) for $k = 0, \lambda = 1$ and integer γ equation (2.34) has the form

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1} E_{\alpha,\beta}(t)^\alpha\right) = t^{\beta-m-1} E_{\alpha,\beta-m}(t^\alpha), \qquad (m = 1, 2, 3, ...). \tag{2.35}$$

If we take $\alpha = \frac{m}{n}$, where m and n are natural numbers, we obtain

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1} E_{\frac{m}{n},\beta}\left(t^{\frac{m}{n}}\right)\right) = t^{\beta-1} E_{\frac{m}{n},\beta}\left(t^{\frac{m}{n}}\right) + t^{\beta-1} \sum_{k=1}^n \frac{t^{-\frac{m}{n}k}}{\Gamma(\beta - \frac{m}{n}k)}, (m, n = 1, 2, 3, ...) \tag{2.36}$$

We obtain from (2.36) that

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1} E_{m,\beta}(t^m)\right) = t^{\beta-1} E_{m,\beta}(t^m), (m = 1, 2, 3, ...; \quad \beta = 0, 1, 2, ...). \tag{2.37}$$

Performing the substitution $t = z^{\frac{n}{m}}$ in (2.35) we obtain

$$\left(\frac{m}{n} z^{1-\frac{n}{m}} \frac{d}{dz}\right)^m \left(z^{(\beta-1)} E_{\frac{m}{n},\beta}(z)\right) = z^{(\beta-1)\frac{n}{m}} E_{\frac{m}{n},\beta}(z) + t^{(\beta-1)\frac{n}{m}} \sum_{k=1}^n \frac{z^k}{\Gamma(\beta - \frac{m}{n}k)}, (m, n = 1, 2, 3, ...) \tag{2.38}$$

Taking $m = 1$ in (2.28), we obtain the following expression:

$$\frac{1}{n}\frac{d}{dz} \left(z^{(\beta-1)n} E_{\frac{1}{n},\beta}(z)\right) = z^{(\beta n-1)} E_{\frac{1}{n},\beta}(z) + z^{\beta n-1} \sum_{k=1}^n \frac{z^{-k}}{\Gamma(\beta - \frac{k}{n})}, (n = 1, 2, 3, ...) \tag{2.39}$$

Differential Equation for the Mittag-Leffler Function:

It is worthwhile noting that relationships (2.26)-(2.30) can also be interpreted as differential equations for the Mittag-Leffler Function; namely, if we denote

$$y_1(t) = t^{\beta-1} E_{\frac{m}{n},\beta}\left(t^{\frac{m}{n}}\right),$$

$$y_2(t) = t^{\beta-1} E_{m,\beta}(t^m),$$

$$y_3(t) = t^{(\beta-1)\frac{n}{m}} E_{\frac{m}{n},\beta}(t),$$

$$y_4(t) = t^{(\beta-1)n} E_{\frac{1}{n},\beta}(t),$$

then these functions satisfy the following differential equations respectively:

$$\frac{d^m y_1}{dt^m} - y_1(t) = t^{\beta n-1} \sum_{k=1}^{n} \frac{t^{-\frac{m}{n}k}}{\Gamma(\beta - \frac{m}{n}k)}, \qquad (m, n = 1, 2, 3, ...)\,, \tag{2.40}$$

$$\frac{d^m y_2}{dt^m} - y_2(t) = 0, \qquad (m = 1, 2, 3, ... \quad \beta = 0, 1, 2, ..., m)\,, \tag{2.41}$$

$$\left(\frac{m}{n} t^{1-\frac{n}{m}} \frac{d}{dt}\right)^m y_3(t) - y_3(t) = t^{(\beta-1)\frac{n}{m}} \sum_{k=1}^{n} \frac{t^{-k}}{\Gamma(\beta - \frac{m}{n}k)}, \qquad (m, n = 1, 2, 3, ...)\,, \tag{2.42}$$

$$\frac{1}{n} \frac{dy_4 t}{dt} y_4(t) - t^{n-1} y_4(t) = t^{\beta n-1} \sum_{k=1}^{n} \frac{t^{-k}}{\Gamma(\beta - \frac{k}{n})}, \qquad (n = 1, 2, 3, ...)\,, \tag{2.43}$$

Integration Equation for the Mittag-Leffler Function:

Integrating (2.4) term-by-term, we obtain

$$\int_0^z E_{\alpha,\beta}\left(\lambda t^\alpha\right) t^{\beta-1} dt = z^\beta E_{\alpha,\beta+1}\left(\lambda z^\alpha\right), \qquad (\beta > 1)\,. \tag{2.44}$$

Wright Function:

The Wright function plays an important role in the solution of linear partial differential equations, e.g. the fractional telegraph equation.

For convenience, we adopt here Madinardi's notation for the Wright function $W(z; \alpha, \beta)$.

The Wright function defined as:

$$W(z;\alpha,\beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k+\beta)}. \tag{2.45}$$

2.4 Fractional Derivatives and Integrals:

In this section, several approaches to the generalization of the notion of differentiation and integration are considered. The choice has been reduced to those definitions which are related to applications.

Grünwald-Letnikov Derivative:

Unification of integer-order Derivatives and Integrals:

We derive the derivative of integer order n and n-fold integrals. As will be shown below, these notions are closer to each than one usually assumes.

Let us consider a continuous function $y = f(x)$. According to the well-definition, the first-order derivative of the function $f(x)$ is defined by

$$Df(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}. \tag{2.46}$$

Applying this definition again gives the second and third-order derivative.

$$D^2 f(x) = \frac{d^2 f}{dx^2} \lim_{h\to 0} \frac{f(x)-2f(x-h)+f(x-2h)}{h^2} \tag{2.47}$$

$$D^3 f(x) = \frac{d^3 f}{dx^3} = \lim_{h\to 0} \frac{f(x)-3f(x-h)+3f(x-2h)-f(x-3h)}{h^3} \tag{2.48}$$

As can be easily seen and proved by induction for any natural number n,

$$D^n f(x) = \frac{d^n f}{dx^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f\left(x+(n-m)h\right). \tag{2.49}$$

Where

$$\binom{n}{m} = C_m^n = \frac{n!}{m!(n-m)!}$$

Or equivalently,

$$D^n f(x) = \frac{d^n f}{dx^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh) \tag{2.50}$$

The case of $n = 0$ can be included as well.

In view of this expression one asks immediately if it can be generalized to any non-integer, real or complex number n. There are some reasons that can make us think so:

(i) The fact that for any natural number n the calculation of the n-st derivative is given by an explicit formula (2.47) or (2.49).

(ii) That the generalization of the factorial by the gamma function allows

$$\binom{n}{m} = C_m^n = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} \tag{2.51}$$

which is also valid for non-integer values.

(iii) He likens of (2.47) to the binomial formula

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m \tag{2.52}$$

which can be generalized to any complex number α by

$$(a+b)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} a^{\alpha-n} b^n \tag{2.53}$$

which is convergent if

$$|b| < a \tag{2.54}$$

There are some desirable properties that could be required to the fractional derivative,

(i) Existance and continuity for m times derivable functions, for any n which modulus is equal or less than m.

(ii) For $n = 0$ the result should be the function itself; for $n > 0$ integer values it should be equal to the ordinary derivative and for $n < 0$ integer values it should be equal to ordinary integration -regardless the integration constant.

(iii) Iterating should not give problems,

$$D^{\alpha+\beta} f(x) = D^{\alpha} D^{\beta} f(x) \tag{2.55}$$

(iv) Linearity,

$$D^{\alpha} \left(af(x) + bg(x) \right) = aD^{\alpha} f(x) + bD^{\alpha} g(x) \tag{2.56}$$

(v) Allowing Taylor's expansion in some other way.

(vi) Its characteristic property should be preserved for the exponential function,

$$D^{\alpha} e^{x} = e^{x} \tag{2.57}$$

Example (2.4.1) :**Exponential Function:**

$$\begin{aligned} D^{\alpha} e^{ax} &= \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{n=0}^{\alpha} (-1)^{n} \binom{\alpha}{n} e^{a\left(x+(\alpha-n)h\right)} \\ &= e^{ax} \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{n=0}^{\alpha} (-1)^{n} \binom{\alpha}{n} \left(e^{ah}\right)^{\alpha-n} \\ &= e^{ax} \lim_{h \to 0} \frac{1}{h^{\alpha}} \left(e^{ah} - 1\right)^{\alpha} = a^{\alpha} e^{ax} \end{aligned} \tag{2.58}$$

Example (2.4.2) :**Powers Function:**

The case of powers of x also has some simplicity that allows its generalization. The case of integer order derivatives

$$D^{n} x^{a} = x^{a-n} \prod_{m=a}^{n-1} (a - m) = \frac{a!}{(a-n)} x^{a-n}$$

one can be generalized to non-integer order derivatives

$$D^{\alpha} x^{a} = \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} x^{a-\alpha}$$

Binomial Formula:

In the expression (2.47) the exponential function allows the substitution of the binomial formula as done in (2.58), but this is not possible for any given function. For applying this substitution we require the following displacement operator,

$$d_h f(x) = f(x+h) \tag{2.59}$$

whose iteration yields

$$d_h^\alpha f(x) = f(x+\alpha h) \tag{2.60}$$

what allows the application of the binomial formula (2.44) for natural numbers and the generalized binomial formula (2.45) for complex numbers,

$$D^n f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f\left(x+(n-m)h\right)$$

$$= \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} d_h^{n-m} f(x)$$

$$= \lim_{h\to 0} \left(\frac{d_h - 1}{h}\right)^n f(x) \tag{2.61}$$

so that it can be generalized for any complex number α

$$D^\alpha f(x) = \lim_{h\to 0} \left(\frac{d_h - 1}{h}\right)^\alpha f(x) \tag{2.62}$$

This sheds more light on the derivative and its generalization using the expression of the generalized binomial formula (2.45) for non-integer numbers,

$$D^\alpha f(x) = \lim_{h\to 0} \left(\frac{d_h - 1}{h}\right)^\alpha f(x)$$

$$= \lim_{h\to 0} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} (-1)^n d_h^{\alpha-n} f(x)$$

$$= \lim_{h\to 0} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} f x + (\alpha - n)h \tag{2.63}$$

In the case of integer values the summation only extends α terms and it is equal to the ordinary derivative. Finally, it is obvious that as h goes to 0 and the last equation is equivalent to the following

$$D^{\alpha} f(x) = \lim_{h \to 0} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} f(x-nh) \tag{2.64}$$

2.5 Natural Transform Fractional Derivatives:

Definition (2.5.1) : [12] The Natural transform of a function $f(t) \geq 0$ is the function $R(p,s)$ and defined by

$$N[f(t)] = R(p,s) = \int_0^{\infty} e^{-st} f(pt) dt; \;\; s > 0, p > 0 \tag{2.65}$$

Where s and p are the transform variables.

Natural transform of elementary function:

Exponential function

Let$f(t) = e^{at}$ when $t \geq 0$, where a is constant, the natural transform of this function can be written as:

$$N[e^{at}] = \int_0^{\infty} e^{apt} e^{-st} dt$$

$$= \lim_{b \to \infty} \int_0^{b} e^{-(s-ap)t} dt$$

$$= \lim_{b \to \infty} \left[\frac{e^{-(s-ap)t}}{s-ap} \right]_0^{b} = \frac{1}{s-ap}.$$

when $p = 1$, Eq.(2.65) converges to Laplace transform Eq.(2.66) and when $s = 1$, Eq.(2.65) converge to Sumudu transform Eq.(2.67) respectively defined by

$$\mathscr{L}[f(t)] = F(s) = \int_0^{\alpha} e^{-st} f(t)\; dt \tag{2.66}$$

$$S[f(t)] = G(s) = \int_0^{\alpha} e^{-t} f(pt)\; dt, \tag{2.67}$$

Definition (2.5.2) : [13] The Natural Transform of Mittag-Leffler function $E_{\alpha,\beta}$ is defined as follows:

$$N^{+}[f(x,t)] = \int_0^{\infty} e^{-st} f(x,pt) dt = \sum_{k=0}^{\infty} \frac{s^{k+1}\Gamma(k+\beta)}{p^{k+1}\Gamma(\alpha k+\beta)} \tag{2.68}$$

Definition (2.5.3) : [18] The Natural transform of $\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}$ w.r.t (t) can be calculated as

$$N^+[\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}] = \frac{s^\alpha}{p^\alpha}R(x,s,p) - \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{p^{\alpha-k}} \left[\lim_{x\to 0} \frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right] \tag{2.69}$$

Definition (2.5.4) [13, 14]: The inverse natural transform of a function is defined by

$$\mathbb{N}^{-1}[R(s,p)] = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{st}{p}} R(s,p)ds \tag{2.70}$$

Properties of natural transform:

Linearity property:

Theorem (2.5.5) : If a and b are any constants and $f(t)$ and $g(t)$ are functions, then

$$N\left[af(t) + bg(t)\right] = aN\left[f(t)\right] + bN\left[g(t)\right]. \tag{2.71}$$

First translation or shifting property:

Theorem (2.5.6) : Let f(t) be a continuous functions and $t \geq 0$ then

$$N\left[e^{at}f(t)\right] = \frac{s}{s-ap}\left[\frac{ps}{s-ap}\right]. \tag{2.72}$$

Change of scale property:

Theorem (2.5.7) : If $N\left[f(t)\right] = R\left(s,p\right)$ then

$$N\left[f(at)\right] = \frac{1}{a}R\left(s,p\right). \tag{2.73}$$

2.6 Adomain Decomposition Method

Adomian Decomposition Method (ADM) is a technique for solving ordinary, linear and nonlinear partial differential equations. Using this method, it is possible to express analytic solutions in terms of a rapidly converging series. In a nutshell, the method identifies and separates the linear and nonlinear parts of a differential equation. By inverting and applying the highest order differential operator that is contained in the linear part of the equation, it is possible to express the solution in terms of the the rest of the equation affected by this inverse operator. This method was introduced and developed by George Adomian in [11] and is well addressed in the literature. The Adomian decomposition method consists of decomposing the unknown function $u(x,t)$ of any equation into a sum of an infinite number of components defined by the decomposition series.

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{2.74}$$

differential equation written in an operator form by

$$Lu + Ru = g, \tag{2.75}$$

where L is a lower order derivative which is assumed to be invertible, R is another linear differential operator, and g is a source term. Now apply the inverse operator L^{-1} to both sides of Eq.(2.75) and using the given condition to obtain:

$$u = f - L^{-1}\left[Ru\right], \tag{2.76}$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. By using Eq.(2.74) then Eq.(2.76) becomes

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left[R\sum_{n=0}^{\infty} u_n\right]. \tag{2.77}$$

To construct the recursive relation needed for the determination the components u_0, u_1, u_2,... it is important to note the Adomian method suggests that the zeroth component u_0 is usually defined by the function f described above, i.e. According, the formal recursive relation is defined by

$$u_0 = f, \tag{2.78}$$

$$u_{n+1} = -L^{-1}\left[Ru_n\right], n \geq 0 \tag{2.79}$$

and consider the nonlinear differential equation

$$Lu + Ru + F(u) = g, \tag{2.80}$$

the nonlinear term $F(u)$ such as $u^2, u^3, u^4, \sin u, e^u, uu_x, u_x^2$,ect. can be expressed by an infinite series of the so-called Adomain polynomials A_n given in the form,

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, u_3, ...). \tag{2.81}$$

The Adomain polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression;

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F(\sum_{i=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}, n = 0, 1, 2, 3, ... \tag{2.82}$$

Assuming that the nonlinear function $F(u)$ therefore by using Eq.(2.82), Adomain polynomials are given by

$$A_0 = F(u_0), \tag{2.83}$$

$$A_1 = u_1 F^{'}(u_0), \tag{2.84}$$

$$A_2 = u_2 F^{'}(u_0) + \frac{1}{2!}u_1^2 F^{''}(u_0), \tag{2.85}$$

$$A_3 = u_3 F^{'}(u_0) + u_1 u_2 F^{''}(u_0) + \frac{1}{3!}u_1^3 F^{'''}(u_0). \tag{2.86}$$

Other polynomials can be generated in a similar manner. By substituting Eq.(2.86) into Eq.(2.81) we have

$$F(u) = A_0 + A_1 + A_2 + A_3 + ... \tag{2.87}$$

$$F(u) = F(u_0) + u_1 F^{'}(u_0) + u_2 F^{'}(u_0) \tag{2.88}$$

$$+\frac{1}{2!}u_1^2 F^{''}(u_0) + u_3 F^{'}(u_0) \tag{2.89}$$

$$+u_1 u_2 F^{''}(u_0) + \frac{1}{3!}u_1^3 F^{'''}(u_0) + ..., \tag{2.90}$$

$$
\begin{aligned}
F(u) &= F(u_0) + (u_1 + u_2 + u_3 + ...)F^{'}(u_0) & (2.91)\\
&+\frac{1}{2!}(u_1^2 + 2u_1u_2 + u_2^2 + ...)F^{''}(u_0) & (2.92)\\
&+\frac{1}{3!}(u_1^3 + 3u_1^2u_2 + 3u_1^2u_3 + ...)F^{'''}(u_0) + ... & (2.93)
\end{aligned}
$$

The last expansion confirms the fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used. In the following, we will calculate Adomain polynomials for several forms of nonlinearity that may arise in nonlinear problem.

Case 1: We consider the nonlinear polynomial in the following form

$$F(u) = u_x^2. \tag{2.94}$$

where $A_n = A_n(u_0, u_1, u_2, ..., u_n)$ are the so-called Adomain polynomials. The first few polynomials are given by

$$A_0 = u_{0x}^2,$$

$$A_1 = 2u_{0x}u_{1x},$$

$$A_2 = 2u_{0x}u_{2x} + (u_{1x})^2,$$

$$A_3 = 2u_{0x}u_{3x} + 2u_{1x}u_{2x},$$

similar, we get $(u_x)^3$, $(u_x)^4$, (u_x^5)...ect

Case 2:

$$F(u) = u_x. \tag{2.95}$$

The A_n polynomials in this case given by:

$$A_0 = F(u_0) = u_0u_{0x},$$

$$A_1 = u_{0x}u_1 + u_0u_{1x},$$

$$A_2 = u_{0x}u_2 + u_0u_{2x} + u_1u_{1x},$$

$$A_3 = u_{0x}u_3 + u_0u_{3x} + u_{1x}u_2 + u_1u_{2x}.$$

In a parallel manner, Adomian polynomials can be calculated for nonlinear polynomials of higher degrees.

Chapter 3

Application of Double Laplace Decomposition Method for Solving Fractional Telegraph Equation

Introduction

The fractional telegraph equation is used in signal analysis for the transmission and propagation of electrical signals and also used modeling reaction diffusion [15]. Recently, many authors studied fractional telegraph equation by several methods for more specifics we refer to [2, 3, 4, 5, 6, 15].

In this chapter, we present a new technique which is used for solving different types of fractional telegraph equation which is called fractional double Laplace decomposition method (FDLDM). It is worth mentioning that the proposed method is an elegant combination of fractional double Laplace transform method and Adomain decomposition method.

3.1 Linear Fractional Telegraph Equations

In this section, we are going to apply fractional double Laplace decomposition methods (FDLDM) to solve linear fractional telegraph equations. In a general case, we consider linear fractional telegraph equations in the form:

$$\begin{aligned} &\tfrac{\partial^\alpha u}{\partial x^\alpha} = \tfrac{\partial^2 u}{\partial t^2} + \tfrac{\partial u}{\partial t} + u, \\ &t \geq 0, 0 < \alpha \leq 2, \end{aligned} \tag{3.1}$$

subject to:

$$u(0,t) = f_1(t) \quad \text{and} \quad u_x(0,t) = f_2(t). \tag{3.2}$$

In order to solve the solution of Eq.(3.1), we apply the following steps.

Step 1: Applying fractional double Laplace transform to Eq.(3.1), we get

$$p^\alpha [U(p,s) - p^{-1}U(0,s) - p^{-2}U_x(0,s)] = \mathscr{L}_x \mathscr{L}_t \left[u_{tt}(x,t) + u_t(x,t) + u \right]. \tag{3.3}$$

Step 2: Using the single Laplace transform for Eq.(3.2), we have

$$\begin{aligned} \mathscr{L}_t \left[u(0,s) \right] &= \mathscr{L}_t \left[f(x) \right] = F_1 \left(s \right), \\ \mathscr{L}_t \left[u_x(0,s) \right] &= \mathscr{L}_t \left[g(x) \right] = F_2 \left(s \right). \end{aligned} \tag{3.4}$$

where $F_1 \ (s)$ and $F_2 \ (s)$ are single Laplace transform for boundary condition.

Step 3: By substituting Eq.(3.4) into Eq.(3.3), we obtain

$$U(p,s) = p^{-1}F_1 \left(s \right) + p^{-2}F_2 \left(s \right) + \frac{1}{p^\alpha} \mathscr{L}_x \mathscr{L}_t \left[u_{tt}(x,t) + u_t(x,t) + u(x,t) \right]. \tag{3.5}$$

Step 4: Operating the inverse double Laplace to Eq.(3.5), we have

$$u(x,t) = H(x,t) + \mathscr{L}_x^{-1} \mathscr{L}_t^{-1} \left[\frac{1}{p^\alpha} \mathscr{L}_x \mathscr{L}_t \left[u_{tt}(x,t) + u_t(x,t) + u(x,t) \right] \right]. \tag{3.6}$$

where $H(x,t)=\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[p^{-1}F_1(s)+p^{-2}F_2(s)\right]$.

The solution of Eq.(3.6) is given by infinite series as follows

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{3.7}$$

Step 5: The general solution of Eq.(3.1), is denoted by

$$\sum_{n=0}^{\infty}u_n(x,t)=H(x,t)+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}u_{ntt}(x,t)+\sum_{n=0}^{\infty}u_{nt}(x,t)+\sum_{n=0}^{\infty}u_n(x,t)\right]\right]. \tag{3.8}$$

We assume that the inverse double Laplace transform exists for each terms in the right side of Eq.(3.8).

The following examples are selected to demonstrate the above method.

Example (3.1.1) : In [1] consider the following linear fractional telegraph equation:

$$\begin{gathered}\frac{\partial^{\alpha}u}{\partial x^{\alpha}}=\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+u,\\ t\geq 0,\ 0<\alpha\leq 2.\end{gathered} \tag{3.9}$$

subject to:

$$\begin{gathered}u(0,t)=e^{-t},\quad u_x(0,t)=e^{-t},\\ t\geq 0,\ 0<x\leq 1.\end{gathered} \tag{3.10}$$

Step 1: By taking the double Laplace transform to Eq.(3.9), we obtain

$$p^{\alpha}\left[U(p,s)-p^{-1}U(0,s)-p^{-2}U_x(0,s)\right]=\mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t)+u_t(x,t)+u(x,t)\right], \tag{3.11}$$

Step 2: Applying single Laplace transform with respect to t for Eq.(3.10), we have

$$\mathscr{L}_t\left[u(0,t)\right]=\mathscr{L}_t\left[u_x(0,t)\right]=\mathscr{L}_t\left[e^{-t}\right]=\frac{1}{s+1}. \tag{3.12}$$

Step 3: Substituting Eq.(3.12) into Eq.(3.11), we get

$$p^{\alpha}U(p,s)-\frac{p^{\alpha-1}}{s+1}-\frac{p^{\alpha-2}}{s+1}=\mathscr{L}_x\mathscr{L}_t[u_{tt}(x,t)+u_t(x,t)+u(x,t)]. \tag{3.13}$$

By simplify Eq.(3.13), we get

$$U(p,s) = \frac{1}{p(s+1)} + \frac{1}{p^2(s+1)} + \frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t[u_{tt} + u_t + u]. \tag{3.14}$$

Step 4: On using inverse of double Laplace transform to Eq.(3.14), we get

$$u(x,t) = e^{-t} + xe^{-t} + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\frac{1}{p^\alpha}\left[\mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t) + u_t(x,t) + u(x,t)\right]\right], \tag{3.15}$$

the solution of Eq.(3.8) is given by infinite series as

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t), \\ u_{tt}(x,t) + u_t(x,t) &= A_0. \end{aligned} \tag{3.16}$$

Step 5: Inserting Eq.(3.16) into Eq.(3.15), to gives

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{-t} + xe^{-t} + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\frac{1}{p^\alpha}\left[\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} u_n(x,t)\right]\right] \tag{3.17}$$

we identify the zeroth component $u_0(x,t)$ by

$$u_0(x,t) = e^{-t} + xe^{-t}, \tag{3.18}$$

we obtain the recursive relations

$$u_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[u_n(x,t) + A_n\right]\right], n \geq 0. \tag{3.19}$$

From Eq.(3.19), we obtain

$$u_1(x,t) = \frac{x^{\alpha}}{\Gamma(\alpha+1)}e^{-t} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^{-t} \quad ,$$

$$u_2(x,t) = \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}e^{-t} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^{-t} \, ,$$

$$u_3(x,t) = \frac{x^{3\alpha}}{\Gamma(3\alpha+1)}e^{-t} + \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^{-t} \, , \tag{3.20}$$

$$\vdots$$

therefore,

$$u_n(x,t) = e^{-t}\left[\frac{x^{n\alpha}}{\Gamma(n\alpha+1)} + \frac{x^{n\alpha+1}}{\Gamma(n\alpha+2)}\right], \qquad \forall \quad n \in N, 0 < \alpha \leq 2. \tag{3.21}$$

The solution can be written as

$$u(x,t) = e^{-t}\left[1 + x + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)} + ...\right]. \tag{3.22}$$

In order to obtain the exact solution of Eq.(3.9), we set $\alpha = 2$, thus

$$u(x,t) = e^{x-t} \tag{3.23}$$

Example (3.1.2) :In [1] consider the following linear fractional telegraph equation:

$$\begin{gathered} \frac{\partial^{\alpha} u}{\partial x^{\alpha}} = \frac{\partial^2 u}{\partial t^2} + 4\frac{\partial u}{\partial t} + 4u, \\ t \geq 0, \ 0 < \alpha \leq 2, \end{gathered} \tag{3.24}$$

subject to:

$$\begin{gathered} u(0,t) = 1 + e^{-2t} \quad \text{and} \quad u_x(0,t) = 2 \\ t \geq 0, \ 0 < x \leq 1. \end{gathered} \tag{3.25}$$

By using our method (FDLDM) for Eq.(3.24) we have

$$\sum_{n=0}^{\infty} u_n(x,t) = 1+2x+e^{-2t}+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty} u_{ntt}(x,t) + \sum_{n=0}^{\infty} 4u_{nt}(x,t) + \sum_{n=0}^{\infty} A_n\right]\right], \tag{3.26}$$

zeroth component u_0 is given by

$$u_0(x,t) = 1 + 2x + e^{-2t}, \tag{3.27}$$

and we obtain the recursive relations

$$u_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[u_{ntt}(x,t) + 4u_{nt}(x,t) + A_n\right]\right], n \geq 0. \tag{3.28}$$

According to (FDLDM) we obtain

$$\begin{aligned} u_1(x,t) &= \tfrac{4x^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{8x^{\alpha+1}}{\Gamma(\alpha+2)}, \\ u_2(x,t) &= \tfrac{16x^{2\alpha}}{\Gamma(2\alpha+1)} + \tfrac{32x^{2\alpha+1}}{\Gamma(2\alpha+2)}, \\ u_3(x,t) &= \tfrac{64x^{3\alpha}}{\Gamma(3\alpha+1)} + \tfrac{128x^{3\alpha+1}}{\Gamma(3\alpha+2)}, \\ &. \\ &. \\ &. \end{aligned} \tag{3.29}$$

therefore

$$u_n(x,t) = 4^n\left[\frac{x^{n\alpha}}{\Gamma(n\alpha+1)} + \frac{2x^{n\alpha+1}}{\Gamma(n\alpha+2)}\right], \qquad \forall \quad n \in N, 0 < \alpha \leq 2. \tag{3.30}$$

Hence, the solution of problem Eq.(3.24) is denoted by

$$u(x,t) = e^{-2t} + 1 + 2x + \tfrac{4x^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{8x^{\alpha+1}}{\Gamma(\alpha+2)} + \tfrac{16x^{2\alpha}}{\Gamma(2\alpha+1)} + \tfrac{32x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \tfrac{64x^{3\alpha}}{\Gamma(3\alpha+1)} + \tfrac{128x^{3\alpha+1}}{\Gamma(3\alpha+2)} + \dots . \tag{3.31}$$

Or

$$u(x,t) = e^{-2t}+1+2x+4^n\left[\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}+\frac{2x^{n\alpha+1}}{\Gamma(n\alpha+2)}\right], \quad \forall \quad n\in N, 0<\alpha\leq 2. \quad (3.32)$$

When $\alpha = 2$, then the exact solution Eq.(3.24) is given by $u(x,t) = e^{2x} + e^{-2t}$.

3.2 Nonlinear Fractional Telegraph Equations

In this section, we apply fractional double Laplace decomposition method (FDLDM) to obtain the approximate solution of nonlinear fractional telegraph equations.

To illustrate the basic idea of the (FDLDM) for the nonlinear fractional telegraph equation, we consider the following nonlinear fractional telegraph equation as:

$$\begin{gathered}\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + Nu(x,t) + h(x,t)\\ 0<\alpha\leq 2, x, t\geq 0,\end{gathered} \quad (3.33)$$

with boundary conditions:

$$u(0,t) = g_1(t) \text{ and } u_x(0,t) = g_2(t). \quad (3.34)$$

Where N is a nonlinear term and $h(x,t)$ is given function of x and t.

Analysis of the method:

In order to solve the Eq.(3.33), we apply the following steps:

Step 1: By taking double Laplace transform for both sides of Eq.(3.33) and single Laplace transform for Eq.(3.34), we obtain

$$U(p,s) = \frac{1}{p}G_1(s)+\frac{1}{p^2}G_2(s)+\frac{1}{p^\alpha}H(p,s)+\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial u(x,t)}{\partial t}+Nu(x,t)\right], \quad (3.35)$$

where $H(p,s)$ is double Laplace transform of $h(x,t)$ and $G_1(s), G_2(s)$ are single Laplace transform of functions $g_1(t), g_2(t)$ respectively.

Step 2: On using double inverse Laplace transform for Eq.(3.35), we have

$$u(x,t) = \Psi + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2 u(x,t)}{\partial t^2} + \tfrac{\partial u(x,t)}{\partial t} + Nu(x,t)\right]\right], \tag{3.36}$$

where $\Psi = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p}G_1(s) + \frac{1}{p^2}G_2(s) + \frac{1}{p^\alpha}H(p,s)\right]$.

Step 3: Applying the decomposition method, then we consider the solution as an infinite series given as follows

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.37}$$

the nonlinear terms $N(u)$ is decomposed as follow:s

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, ..., u_n), \tag{3.38}$$

where $A_n = A_n(u_0, u_1, u_2, ..., u_n)$, $n \geqslant 0$ are the Adomian polynomials that represent the nonlinear term $u^2(x,t)$.

Step 4: By substituting Eq.(3.37) and Eq.(3.38) into Eq.(3.36) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = \Psi + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\tfrac{\partial^2 u_n(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\tfrac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty}A_n(x,t)\right]\right], \tag{3.39}$$

We assume that the inverse double Laplace transform of each terms in the right side of Eq.(3.39) exists.

In the next example, we demonstrate the applicability of previous method as follows:

Example (3.2.1) : In [3] consider the following fractional nonlinear telegraph equation:

$$\begin{gathered}\tfrac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \tfrac{\partial^2 u(x,t)}{\partial t^2} + \tfrac{\partial u(x,t)}{\partial t} + u^2(x,t) - e^{-2t}(x-x^2)^2 - 2e^{-t}\left(\tfrac{x^{(2-\alpha)}}{\Gamma(3-\alpha)}\right), \\ x,t \geq 0 \quad \text{and} \quad 1 < \alpha \leq 2,\end{gathered} \tag{3.40}$$

subject to:

$$u(0,t) = 0, \quad u_x(0,t) = e^{-t} \tag{3.41}$$

We follow the above method as:

Step 1: By taking double Laplace transform for both sides of Eq.(3.40) and single Laplace transform for Eq.(3.41), we have:

$$U(p,s)= \frac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial u(x,t)}{\partial t}+u^2(x,t)\right] - \left[\frac{2}{p^{\alpha+3}(s+2)}-\frac{12}{p^{\alpha+4}(s+2)}+\frac{24}{p^{\alpha+5}(s+2)}+\frac{2}{p^3(s+1)}-\frac{1}{p^2(s+1)}\right]. \tag{3.42}$$

Step 2: On using inverse double Laplace to Eq.(3.42), we obtain:

$$u(x,t)=\left(x-x^2\right)e^{-t}-\left(\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{12x^{\alpha+3}}{\Gamma(\alpha+4)}+\frac{24x^{\alpha+4}}{\Gamma(\alpha+5)}\right)e^{-2t} +\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial u(x,t)}{\partial t}+u^2(x,t)\right]\right]. \tag{3.43}$$

Where the nonlinear terms $u^2(x,t)$ is defined by:

$$u^2(x,t)=\sum_{n=0}^{\infty}A_n(x,t), \tag{3.44}$$

Step 3: Inserting Eq.(3.44) into Eq.(3.43) yields:

$$\sum_{n=0}^{\infty}u_n(x,t)=\left(x-x^2\right)e^{-t}-\left(\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{12x^{\alpha+3}}{\Gamma(\alpha+4)}+\frac{24x^{\alpha+4}}{\Gamma(\alpha+5)}\right)e^{-2t} +\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 u(x,t)}{\partial t^2}+\sum_{n=0}^{\infty}\frac{\partial u(x,t)}{\partial t}+\sum_{n=0}^{\infty}A_n(x,t)\right]\right]. \tag{3.45}$$

Step 4: The components of $u(x,t)$ can be craftily determined by using the recursive relations

$$u_0(x,t) = \left(x - x^2\right) e^{-t},$$

$$u_1(x,t) = -\left(\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{12x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{24x^{\alpha+4}}{\Gamma(\alpha+5)}\right) e^{-2t} + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u_0(x,t)}{\partial t^2} + \frac{\partial u_0(x,t)}{\partial t} + A_0(x,t)\right]\right],$$

and

$$u_{n+2}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_{n+1}(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\frac{\partial u_{n+1}(x,t)}{\partial t} + \sum_{n=0}^{\infty} A_{n+1}(x,t)\right]\right]$$

$$n \geq 0, \tag{3.46}$$

then the first term is given by:

$$u_1(x,t) = -e^{-2t}\left[\frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{6x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{24x^{\alpha+4}}{\Gamma(\alpha+5)}\right] + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\left(\left(x - x^2\right) e^{-t}\right)^2\right]\right]$$
$$= 0. \tag{3.47}$$

The rest terms are given by:

$$u_{k+2}(x,t) = 0, \tag{3.48}$$

therefore,

$$u(x,t) = \left(x - x^2\right) e^{-t}. \tag{3.49}$$

Example (3.2.2) :Consider the following fractional nonlinear telegraph equation:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial u(x,t)}{\partial t} + u^2(x,t) - xu(x,t)u_x(x,t),$$
$$x,t \geq 0 \quad \text{and} \quad 0 < \alpha \leq 2, \tag{3.50}$$

with boundary conditions

$$u(0,t) = 0 \text{ and } u_x(0,t) = e^t. \tag{3.51}$$

By applying the above steps which is given in the solution of Eq.(3.33), we obtain:

$$u_0(x,t) = xe^t. \tag{3.52}$$

and

$$u_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u_n(x,t)}{\partial t^2} - \frac{\partial u_n(x,t)}{\partial t} + A_n(x,t) - xB_n(x,t)\right]\right], n \geq 0. \tag{3.53}$$

By using Eq.(3.52) and Eq.(3.53), we obtain u_1 as follows:

$$\begin{aligned} u_1(x,t) &= \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2 u_0(x,t)}{\partial t^2} - \tfrac{\partial u_0(x,t)}{\partial t} + A_n(x,t) - xB_0(x,t)\right]\right] \\ &= 0. \end{aligned} \tag{3.54}$$

Consequently, $u_n = 0, n \geq 1$. Having determined the components of $u(x,t)$.
Therefore, the exact solution of Eq.(3.50) is given by:

$$u(x,t) = xe^t. \tag{3.55}$$

3.3 Singular Fractional Telegraph Equations:

In this section we derive the main idea of fractional double Laplace decomposition method to solve singular fractional telegraph equation.

Analysis of the method:

We consider singular fractional telegraph equation with boundary condition as follows:

$$D_x^{2\alpha}u(x,t) = \tfrac{1}{t}\left(tu_t(x,t)\right)_t + u(x,t) + h(x,t), \text{ and } 0 < \alpha \leq 1 \text{ and } x,t \geq 0, \tag{3.56}$$

subject to:

$$u(0,t) = f_1(t) \text{ and } u_x(0,t) = f_2(t), \tag{3.57}$$

where $\frac{1}{t}\left(tu_t(x,t)\right)_t$ is the Bessel operator and $h(x,t)$ is a continuous function.

Proof. In order to obtain the solution of Eq.(3.56), we use the following procedure:

First: Applying the fractional double Laplace transform for Eq.(3.56), we get:

$$p^{2\alpha}\left[U(p,s)-p^{-1}U(0,s)-p^{-2}U_x(0,s)\right]=\mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t)+\tfrac{1}{t}u_t(x,t)+u(x,t)+h(x,t)\right]. \tag{3.58}$$

Second: Using the single Laplace transform for Eq.(3.57) and substituting into Eq.(3.58), we obtain:

$$U(p,s)=\tfrac{1}{p^{2\alpha}}\,\mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t)+\tfrac{1}{t}u_t(x,t)+u\right]+\tfrac{1}{p^{2\alpha}}H(p,s)+\tfrac{1}{p}F_1(s)+\tfrac{1}{p^2}F_2(s), \tag{3.59}$$

where $H(p,s)$ is double Laplace transform of the function $h(x,t)$ and $F_1(s), F_2(s)$ are single Laplace transform of functions $f_1(t), f_2(t)$ respectively.

Third: By applying inverse double Laplace transform to Eq.(3.59), we get:

$$u(x,t)=G(x,t)+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t)+\frac{1}{t}u_t(x,t)+u\right]\right], \tag{3.60}$$

where $G(x,t)=\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}H(p,s)+\frac{1}{p}F_1(s)+\frac{1}{p^2}F_2(s)\right]$.

Fourth: By using Eq.(3.7) then Eq.(3.60) becomes:

$$\sum_{n=0}^{\infty}u_n(x,t)=G(x,t)+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}u_{ntt}(x,t)+\tfrac{1}{t}\sum_{n=0}^{\infty}u_{nt}(x,t)+\sum_{n=0}^{\infty}u_n\right]\right], \tag{3.61}$$

Eq.(3.61) can be written as:

$$\begin{aligned}u_0(x,t)=&G(x,t),\\ u_{n+1}(x,t)=&\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[u_{ntt}+\tfrac{1}{t}u_{nt}+u_n\right]\right], \qquad n\geq 0.\end{aligned} \tag{3.62}$$

Provided that the inverse double Laplace transform of each terms in the right side of Eq.(3.62) exists.

Now, we demonstrate the applicability of our method by applying numerical problems.

Example (3.3.1) : We consider the following singular fractional telegraph equation:

$$D_x^{2\alpha}u(x,t) = \frac{1}{t}\left(tu_t(x,t)\right)_t + u + 2t^2 - 4x^2 - x^2t^2, \tag{3.63}$$

$$x, t \geq 0 \text{ and } 0 < \alpha \leq 1.$$

Subject to:

$$u(0,t) = 0, \quad u_x(0,t) = 0. \tag{3.64}$$

Appling the double Laplace transform on both sides of Eq.(3.63), we have:

$$p^{2\alpha}U(p,s) - p^{2\alpha-1}U(0,s) - p^{2\alpha-2}U_x(0,s) = 2\frac{2!}{ps^3} - 4\frac{2!}{p^3s} - \frac{2!2!}{p^3s^3} + \mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t) + \frac{1}{t}u_t(x,t) + u\right], \tag{3.65}$$

by using single Laplace transform to Eq.(3.64) and substitute in Eq.(3.65), gives

$$U(p,s) = 2\frac{2!}{p^{2\alpha+1}s^3} - 4\frac{2!}{p^{2\alpha+3}s} - \frac{2!2!}{p^{2\alpha+3}s^3} + \frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t) + \frac{1}{t}u_t(x,t) + u\right], \tag{3.66}$$

using the inverse double Laplace transform for Eq.(3.66) we have:

$$u(x,t) = 2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}t^2 - 8\frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} - 2\frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)}t^2 + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[u_{tt}(x,t) + \frac{1}{t}u_t(x,t) + u\right]\right], \tag{3.67}$$

inserting Eq.(3.7) into Eq.(3.67), gives:

$$\sum_{n=0}^{\infty} u_n(x,t) = 2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}t^2 - 8\frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} - 2\frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)}t^2 + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty} u_{ntt}(x,t) + \frac{1}{t}\sum_{n=0}^{\infty} u_{nt}(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \tag{3.68}$$

by using the Eq.(3.68) we find a few terms series as:

$$u_0(x,t) = 2\tfrac{x^{2\alpha}}{\Gamma(2\alpha+1)}t^2 - 8\tfrac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} - 2\tfrac{x^{2\alpha+2}}{\Gamma(2\alpha+3)}t^2 \,,$$
$$u_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[u_{ntt}(x,t) + \tfrac{1}{t}u_{nt}(x,t) + u_n(x,t)\right]\right], \quad n \geq 0, \tag{3.69}$$

then

$$u_1(x,t) = 8\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} - 16\frac{x^{4\alpha+2}}{\Gamma(4\alpha+3)} + 2\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}t^2 - 2\frac{x^{4\alpha+2}}{\Gamma(4\alpha+3)}t^2, \tag{3.70}$$

$$u_2(x,t) = 16\frac{x^{6\alpha}}{\Gamma(6\alpha+1)} - 24\frac{x^{6\alpha+2}}{\Gamma(6\alpha+3)} + 2\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}t^2 - 2\frac{x^{6\alpha+2}}{\Gamma(6\alpha+3)}t^2, \tag{3.71}$$

therefore, the (FDLDM) gives us the series solution of Eq.(3.63) as follows:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

$$u(x,t) = 2\tfrac{x^{2\alpha}}{\Gamma(2\alpha+1)}t^2 - 8\tfrac{x^{2\alpha+2}}{\Gamma(2\alpha+3)} - 2\tfrac{x^{2\alpha+2}}{\Gamma(2\alpha+3)}t^2 + 8\tfrac{x^{4\alpha}}{\Gamma(4\alpha+1)} - 16\tfrac{x^{4\alpha+2}}{\Gamma(4\alpha+3)} + 2\tfrac{x^{4\alpha}}{\Gamma(4\alpha+1)}t^2$$
$$-2\tfrac{x^{4\alpha+2}}{\Gamma(4\alpha+3)}t^2 + 16\tfrac{x^{6\alpha}}{\Gamma(6\alpha+1)} - 24\tfrac{x^{6\alpha+2}}{\Gamma(6\alpha+3)} + 2\tfrac{x^{6\alpha}}{\Gamma(6\alpha+1)}t^2 - 2\tfrac{x^{6\alpha+2}}{\Gamma(6\alpha+3)}t^2 + ... \;. \tag{3.72}$$

If we take $\alpha = 1$, we get an exact solution of Eq.(3.63) as $u(x,t) = x^2t^2$.

Example (3.3.2) : Consider the following singular fractional telegraph equation:

$$D_x^{2\alpha}u(x,t) = \tfrac{1}{t}\left(tu_t(x,t)\right)_t + u - x\ln(t),$$
$$x, t \geq 0 \;\text{ and }\; 0 < \alpha \leq 1, \tag{3.73}$$

subject to:

$$u(0,t) = 0, \quad u_x(0,t) = \ln(t). \tag{3.74}$$

According to the above analysis method, we have:

$$u(x,t) = x\ln(t) + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\ln(t) + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[D_{tt}u(x,t) + \frac{1}{t}D_t u(x,t) + u\right]\right], \tag{3.75}$$

now we define the function $u(x,t)$ by the decomposition series as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.76}$$

inserting Eq.(3.76) into Eq.(3.75), gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) &= x\ln(t) - \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\ln(t) + \\ &\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty} u_{ntt}(x,t) + \frac{1}{t}\sum_{n=0}^{\infty} u_{nt}(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \end{aligned} \tag{3.77}$$

by using Eq.(3.77) we find a few terms of the series of $u(x,t)$

$$\begin{aligned} u_0 &= x\ln(t) - \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\ln(t), \\ u_{n+1}(x,t) &= \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[u_{ntt} + \frac{1}{t}u_{nt} + u_n\right]\right], \quad n \geq 0, \end{aligned} \tag{3.78}$$

the components $u_1(x,t), u_2(x,t), u_3(x,t), \ldots$ are thus determined as follows:

$$\begin{aligned} u_1(x,t) &= \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\ln(t) - \frac{x^{2\alpha+3}}{\Gamma(2\alpha+4)}\ln(t), \\ u_2(x,t) &= \frac{x^{2\alpha+3}}{\Gamma(2\alpha+3)}\ln(t) - \frac{x^{2\alpha+5}}{\Gamma(2\alpha+6)}\ln(t), \\ u_3(x,t) &= \frac{x^{2\alpha+5}}{\Gamma(2\alpha+5)}\ln(t) - \frac{x^{2\alpha+7}}{\Gamma(2\alpha+8)}\ln(t), \\ &\vdots \end{aligned} \tag{3.79}$$

therefore, the (FDLDM) gives us the series solution of Eq.(3.73) is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

$$u(x,t) = x\ln(t) - \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\ln(t) + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\ln(t) - \frac{x^{2\alpha+3}}{\Gamma(2\alpha+4)}\ln(t) + \frac{x^{2\alpha+3}}{\Gamma(2\alpha+3)}\ln(t)$$

$$-\frac{x^{2\alpha+5}}{\Gamma(2\alpha+6)}\ln(t) + \frac{x^{2\alpha+5}}{\Gamma(2\alpha+5)}\ln(t) - \frac{x^{2\alpha+7}}{\Gamma(2\alpha+8)}\ln(t) + ... \ . \tag{3.80}$$

At $\alpha = 1$ the exact solution of Eq.(4.6) is given by $u(x,t) = x\ln(t)$.

3.4 Fractional Telegraph Equation Coupled System:

In this section, the fractional double Laplace decomposition method is effectively implemented for solving coupled systems of fractional telegraph equations.We discuss and derive the analytical solution of the coupled systems of fractional telegraph equations with boundary conditions as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) + v(x,t) + f(x,t),$$

$$\frac{\partial^\alpha v(x,t)}{\partial x^\alpha} = \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial v(x,t)}{\partial t} + v(x,t) + u(x,t) + g(x,t), \tag{3.81}$$

$$0 < \alpha \leq 2 \ \text{ and } \ x, t \geq 0,$$

subject to:

$$u(0,t) = f_1(t) \text{ and } u_x(0,t) = f_2(t),$$

$$v(0,t) = g_1(t) \text{ and } v_x(0,t) = g_2(t). \tag{3.82}$$

In order to solve the above system we apply the folowing steps.

First: By applying the fractional double Laplace transform for Eq.(3.81), we get:

$$p^{\alpha}U(p,s) - p^{\alpha-1}U(0,s) - p^{\alpha-2}U_x(0,s) =$$

$$\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) + v(x,t) + f(x,t)\right],$$

(3.83)

$$p^{\alpha}V(p,s) - p^{\alpha-1}V(0,s) - p^{\alpha-2}V_x(0,s) =$$

$$\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial v(x,t)}{\partial t} + v(x,t) + u(x,t) + g(x,t)\right],$$

Second: On using the single Laplace transform for Eq.(3.82), we have:

$$\mathscr{L}_t u(0,s) = \mathscr{L}_t\left[f_1(t)\right] = \hat{f}_1(s),$$

$$\mathscr{L}_t u_x(0,s) = \mathscr{L}_t\left[f_2(t)\right] = \hat{f}_2(s),$$

(3.84)

$$\mathscr{L}_t v(0,s) = \mathscr{L}_t\left[g_1(t)\right] = \hat{g}_1(s),$$

$$\mathscr{L}_t v_x(0,s) = \mathscr{L}_t\left[g_2(t)\right] = \hat{g}_2(s),$$

Third: By substituting Eq.(3.84) in Eq.(3.83), we obtain:

$$U(p,s) = \frac{1}{p^{2\alpha}}\,\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) + v(x,t)\right] + \frac{1}{p^{\alpha}}\hat{f}(x,t) + \frac{1}{p}\hat{f}_1(s) + \frac{1}{p^2}\hat{f}_2(s),$$

$$V(p,s) = \frac{1}{p^{2\alpha}}\,\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial v(x,t)}{\partial t} + v(x,t) + u(x,t)\right] + \frac{1}{p^{\alpha}}\hat{g}(x,t) + \frac{1}{p}\hat{g}_1(s) + \frac{1}{p^2}\hat{g}_2(s),$$

(3.85)

where $\hat{f}(x,t)$, $\hat{g}(x,t)$ are double Laplace transform of $f(x,t)$ and $g(x,t)$ respectively.

Fourth: Taking inverse double Laplace transform to Eq.(3.85), we get:

$$u(x,t) = \psi(x,t) + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2 u(x,t)}{\partial t^2} + \tfrac{\partial u(x,t)}{\partial t} + u(x,t) + v(x,t)\right]\right],$$

$$v(x,t) = \zeta(x,t) + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2 v(x,t)}{\partial t^2} + \tfrac{\partial v(x,t)}{\partial t} + v(x,t) + u(x,t)\right]\right], \tag{3.86}$$

where

$$\psi(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{\alpha}}\hat{f}(x,t) + \tfrac{1}{p}\hat{f}_1(s) + \tfrac{1}{p^2}\hat{f}_2(s)\right],$$

$$\zeta(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{\alpha}}\hat{f}(x,t) + \tfrac{1}{p}\hat{g}_1(s) + \tfrac{1}{p^2}\hat{g}_2(s)\right], \tag{3.87}$$

The solution of Eq.(3.81) can be written as infinite series terms such as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{3.88}$$

By substituting Eq.(3.88) into Eq.(3.86), we have

$$\sum_{n=0}^{\infty} u_n(x,t) = \psi(x,t) + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\tfrac{\partial^2 u_n(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\tfrac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} v_n(x,t)\right]\right],$$

$$\sum_{n=0}^{\infty} v_n(x,t) = \zeta(x,t) + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\tfrac{\partial^2 v_n(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\tfrac{\partial v_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} v_n(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \tag{3.89}$$

we define the following recursively relations

$$u_0(x,t) = \psi(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{\alpha}}\hat{f}(x,t) + \tfrac{1}{p}\hat{f}_1(s) + \tfrac{1}{p^2}\hat{f}_2(s)\right],$$

$$v_0(x,t) = \zeta(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{\alpha}}\hat{f}(x,t) + \tfrac{1}{p}\hat{g}_1(s) + \tfrac{1}{p^2}\hat{g}_2(s)\right], \tag{3.90}$$

and

$$u_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u_n(x,t)}{\partial t^2} + \frac{\partial u_n(x,t)}{\partial t} + u_n(x,t) + v_n(x,t)\right]\right],$$

$$v_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^{2\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 v_n(x,t)}{\partial t^2} + \frac{\partial v_n(x,t)}{\partial t} + v_n(x,t) + u_n(x,t)\right]\right], n \geq 0. \tag{3.91}$$

Assume that the inverse double Laplace transform of each term on the right side of Eq.(3.91) exists.

Numerical Examples

We demonstrate the applicability of our method by applying the following examples:

Example (3.4.1) : We consider the following system of fractional telegraph equation:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) + v(x,t) - 3xe^t - xt,$$

$$\frac{\partial^\alpha v(x,t)}{\partial x^\alpha} = \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial v(x,t)}{\partial t} + v(x,t) + u(x,t) - x - xt - xe^t, \tag{3.92}$$

$$0 < \alpha \leq 2 \text{ and } x, t \geq 0,$$

subject to:

$$u(0,t) = 0 \text{ and } u_x(0,t) = e^t,$$

$$v(0,t) = 0 \text{ and } v_x(0,t) = t. \tag{3.93}$$

Solution (3.4.1) : Applying the fractional double Laplace transform to Eqs.(3.92), we get:

$$p^{\alpha}U(p,s)-p^{\alpha-1}U(0,s)-p^{\alpha-2}U_x(0,s)=$$

$$-\tfrac{3}{p^2(s-1)}-\tfrac{1}{p^2s^2}+\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2u(x,t)}{\partial t^2}+\tfrac{\partial u(x,t)}{\partial t}+u(x,t)+v(x,t)\right],$$

$$p^{\alpha}V(p,s)-p^{\alpha-1}V(0,s)-p^{\alpha-2}V_x(0,s)=$$

$$-\tfrac{1}{p^2s}-\tfrac{1}{p^2s^2}-\tfrac{1}{p^2(s-1)}+\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2v(x,t)}{\partial t^2}+\tfrac{\partial v(x,t)}{\partial t}+v(x,t)+u(x,t)\right], \tag{3.94}$$

Taking single Laplace transform to Eq.(3.93) and substitute into Eqs.(3.94), gives:

$$U(p,s)=\tfrac{1}{p^2(s-1)}-\tfrac{3}{p^{\alpha+2}(s-1)}-\tfrac{1}{p^{\alpha+2}s^2}+\tfrac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2u(x,t)}{\partial t^2}+\tfrac{\partial u(x,t)}{\partial t}+u(x,t)+v(x,t)\right],$$

$$V(p,s)=\tfrac{1}{p^2s^2}-\tfrac{1}{p^{\alpha+2}s}-\tfrac{1}{p^{\alpha+2}s^2}-\tfrac{1}{p^{\alpha+2}(s-1)}+\tfrac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2v(x,t)}{\partial t^2}+\tfrac{\partial v(x,t)}{\partial t}+v(x,t)+u(x,t)\right], \tag{3.95}$$

By using the inverse double Laplace transform transform for Eqs.(3.95), we have:

$$u(x,t)=xe^t-3\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t-\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}t$$

$$+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2u(x,t)}{\partial t^2}+\tfrac{\partial u(x,t)}{\partial t}+u(x,t)+v(x,t)\right]\right],$$

$$\tag{3.96}$$

$$v(x,t)=xt-\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}-\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}t-\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t$$

$$+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\tfrac{1}{p^{\alpha}}\mathscr{L}_x\mathscr{L}_t\left[\tfrac{\partial^2v(x,t)}{\partial t^2}+\tfrac{\partial v(x,t)}{\partial t}+v(x,t)+u(x,t)\right]\right],$$

insert Eqs.(3.88) into Eqs.(3.96), gives:

$$\sum_{n=0}^{\infty} u_{n+1}(x,t) = xe^t - 3\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t$$

$$+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} v_n(x,t)\right]\right],$$

$$\sum_{n=0}^{\infty} v_{n+1}(x,t) = xt - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t$$

$$+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 v_n(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\frac{\partial v_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} v_n(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \tag{3.97}$$

by applying Eqs.(3.90) and Eqs.(3.91) into Eqs.(3.97) , we have

$$u_0(x,t) = xe^t - 3\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t$$

$$u_{n+1}(x,t) =\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u_n(x,t)}{\partial t^2} + \frac{\partial u_n(x,t)}{\partial t} + u_n(x,t) + v_n(x,t)\right]\right], n \geq 0, m$$

$$v_0(x,t) = xt - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t$$

$$u_{n+1}(x,t) =\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 v_n(x,t)}{\partial t^2} + \frac{\partial v_n(x,t)}{\partial t} + v_n(x,t) + u_n(x,t)\right]\right], n \geq 0, \tag{3.98}$$

then,

$$\begin{aligned} u_1(x,t) =& 3\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - 10\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t - 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} - 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t, \\ v_1(x,t) =& \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t - 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - 6\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t, \end{aligned} \tag{3.99}$$

$$\begin{aligned} u_2(x,t) =& -36\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t + 10\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t - 4\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}t - 6\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)} + 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t, \\ v_2(x,t) =& 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} - 6\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)} - 4\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}t - 28\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t + 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t + 6\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t, \end{aligned} \tag{3.100}$$

$$u(x,t) = xe^t - \frac{3x^{\alpha+1}}{\Gamma(\alpha+2)}e^t + 3\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - 10\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t - \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t$$

$$+3\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t - 10\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t - \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}t + \ldots.$$

$$v(x,t) = xt - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t - \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t$$

$$-6\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t - \frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}t - 6\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t + \ldots. \tag{3.101}$$

If we take $\alpha = 2$ then we get exact solution of Eqs.(3.92) as:

$$u(x,t) = xe^t.$$
$$v(x,t) = xt. \tag{3.102}$$

Example (3.4.2) : We consider the following system of fractional telegraph equation:

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) + v(x,t) - 3xe^t - x\sin t,$$
$$\frac{\partial^\alpha v(x,t)}{\partial x^\alpha} = \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial v(x,t)}{\partial t} + v(x,t) + u(x,t) - xe^t - x\cos t, \tag{3.103}$$
$$0 < \alpha \leq 2 \text{ and } x,t \geq 0,$$

subject to:

$$u(0,t) = 0 \text{ and } u_x(0,t) = e^t,$$
$$v(0,t) = 0 \text{ and } v_x(0,t) = \sin t, \tag{3.104}$$

Solution (3.4.2) : By using our method (FDLDM) for Eqs.(3.103), we have:

$$\sum_{n=0}^{\infty} u(x,t) = xe^t - 3\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\sin t$$

$$+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} v_n(x,t)\right]\right],$$

$$\sum_{n=0}^{\infty} v(x,t) = x\sin t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\cos t$$

$$+\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\sum_{n=0}^{\infty}\frac{\partial^2 v(x,t)}{\partial t^2} + \sum_{n=0}^{\infty}\frac{\partial v_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} v_n(x,t) + \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \tag{3.105}$$

by applying Eqs.(3.90) and Eqs.(3.91) into Eqs.(3.105) , we have

$$u_0(x,t) = xe^t - 3\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\sin t$$

$$u_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 u_n(x,t)}{\partial t^2} + \frac{\partial u_n(x,t)}{\partial t} + u_n(x,t) + v_n(x,t)\right]\right], n \geq 0$$

$$v_0(x,t) = x\sin t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\cos t$$

$$v_{n+1}(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left[\frac{1}{p^\alpha}\mathscr{L}_x\mathscr{L}_t\left[\frac{\partial^2 v_n(x,t)}{\partial t^2} + \frac{\partial v_n(x,t)}{\partial t} + v_n(x,t) + u_n(x,t)\right]\right], n \geq 0. \tag{3.106}$$

Then,

$$u_1(x,t) = 3\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - 10\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\sin t - 2\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\cos t,$$
$$\tag{3.107}$$
$$v_1(x,t) = \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - 6\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\cos t,$$

$$u_2(x,t) = 10\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t - 36\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t + 2\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\sin t + 2\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos t,$$
$$\tag{3.108}$$
$$v_2(x,t) = 6\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t - 28\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t - 2\frac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos t,$$

therefore,

$$u(x,t) = xe^t - 3\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}\sin t + 3\tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - 10\tfrac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t + \tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}\sin t$$

$$-2\tfrac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\cos t + 10\tfrac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t - 36\tfrac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t + 2\tfrac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\sin t + 2\tfrac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos t + ...$$

$$v(x,t) = x\sin t - \tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - \tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}\cos t + \tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}e^t - 6\tfrac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t + \tfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}\cos t$$

$$+6\tfrac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}e^t - 28\tfrac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}e^t - 2\tfrac{x^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos t + ... \tag{3.109}$$

If we take $\alpha = 2$ then we get exact solution of Eq.(3.103) as

$$u(x,t) = xe^t,$$

$$v(x,t) = x\sin t. \tag{3.110}$$

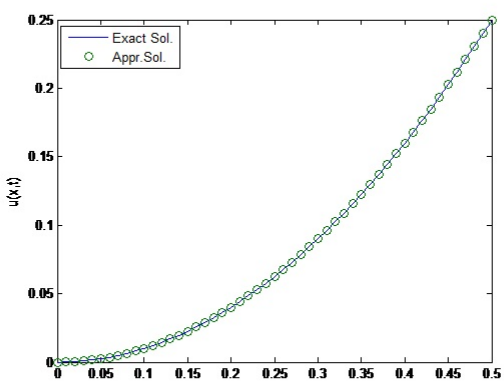

Figure 3.1: The graph of exact and approximate solutions of $u(x,t)$ for Example (3.3.1).

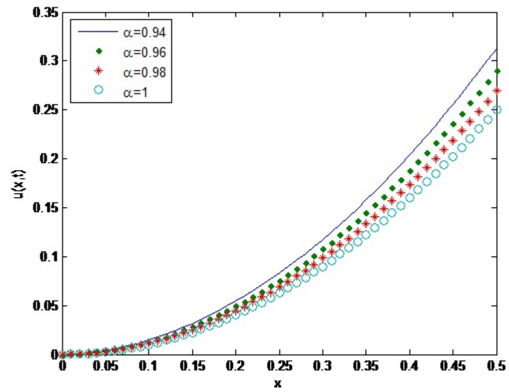

Figure 3.2: The approximate solutions of $u(x,t)$ for Example (3.3.1). for α=0.94,0.96,0.98, and 1.

t	Exact Solution	Approximate Solution $\alpha = 1$	$\text{E}_8(u) = \lvert u - u_8 \rvert$ for $\alpha = 1$
0.2	0.010000000000000	0.010000000000000	1.734723475976807e-18
0.4	0.040000000000000	0.040000000000000	0
0.6	0.090000000000000	0.090000000000000	1.387778780781446e-17
0.8	0.160000000000000	0.160000000000000	2.775557561562891e-17
1.0	0.250000000000000	0.250000000000000	2.775557561562891e-17

Table 3.1: Exact solution and approximate solution of $u(x,t)$ for Example (3.3.1). with $n = 51$ at $\alpha = 1$.

3.5 Numerical results:

We shall illustrate the accuracy and efficiency of the fractional double Laplace decomposition method (FDLDM).

Figure (3.1) By discuss the exact solution and approximate solution of example $(3.3.1)$, we get infinitesimal error equal $(2.775557561562891e^{-17})$ that means the present method is a forceful and accurate method.

Figure (3.2) approximate solution of example $(3.3.1)$ the behaviour of the function with

various values of fractional $\alpha = 0.94, 0.96, 0.98$ and 1, we see that the function $u(x,t)$ increasing when α is decreasing with increasing the x at the value of $t = 1$.
Table (3.1) tells us the absolute error for example $(3.3.2)$ by comparing the exact solution and approximate solution u_8 obtained by the (FDLDM) at $\alpha = 1$ and different values of t.
In example (3.3.2), the exact solution and approximate solution are equal $\ln(t^x)$ by cancelling the noise terms, notice that the solution is verified in Eq.(4.6) when $\alpha = 1$.

Conclusion:

We have successfully applied fractional double Laplace transform and Adomian decomposition method to obtain the approximate solutions of the fractional telegraph equation. The (FDLDM) gives us small error and high convergence. As seen in Table 1, these techniques lead us to say the method has highly accurate and efficient solutions.

Chapter 4

Application of Natural Transform Decomposition Method for Solving Fractional Telegraph Equation

Introduction

In this chapter, we apply the fractional natural transform decomposition method (FNTDM) to solve the linear, nonlinear and singular fractional telegraph equations. This method is a combination of the natural transform and Adomian decomposition methods. The natural transform method was first proposed by [12] (2008) and was successfully applied to partial differential equations. In addition, we prove the convergence of our method. Finally, their examples have been employed to illustrate the preciseness and effectiveness of the proposed method.

4.1 Linear Fractional Telegraph Equations:

In the current investigation, the approach is different as we use the natural transform decomposition method to solve linear fractional of telegraph equation. Furthermore, some example are given to demonstrating the efficience of the proposed method.In general we consider the linear fractional telegraph equation with indicated initial conditions as:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{2}u(x,t)}{\partial x^{2}} - \frac{\partial u(x,t)}{\partial t} - u(x,t) + h(x,t), \tag{4.1}$$

$$0 < \alpha \leq 2 \text{ and } x, t \geq 0.$$

With initial conditions:

$$u(x,0) = f_1(x) \text{ and } u_t(x,0) = f_2(x), \tag{4.2}$$

where $h(x,t)$ is source function.

In order to apply natural transform decomposition technique for Eq.(4.1), we use the followinf steps.

Step 1: Using definition of the fractional double Laplace transform of partial derivatives for Eq.(4.1) and single Laplace transform for initial condition, we get:

$$R(x,s,p) = \frac{1}{s}f_1(x) + \frac{p}{s^2}f_2(x) + \frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\frac{\partial^{2}u(x,t)}{\partial x^{2}} - \frac{\partial u(x,t)}{\partial t} - u(x,t) + h(x,t)\right]. \tag{4.3}$$

Step 2: Now implementing the inverse natural transform for Eq.(4.3), we obtain:

$$u(x,t) = \Phi(x,t) + \mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\frac{\partial^{2}u(x,t)}{\partial x^{2}} - \frac{\partial u(x,t)}{\partial t} - u(x,t)\right]\right], \tag{4.4}$$

where

$$\Phi(x,t) = \mathbb{N}^{-1}\left[f_1(x) + tf_2(x) + \frac{p^{\alpha}}{s^{\alpha}}N^{+}\left[h(x,t)\right]\right], \tag{4.5}$$

Step 3: inserting Eq.(3.7) into Eq.(4.4), we have:

$$\sum_{n=0}^{\infty} u_n(x,t) = \Phi(x,t) + \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} - \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} - \sum_{n=0}^{\infty} u_n(x,t)\right]\right]. \tag{4.6}$$

We assume that the inverse natural transform of each term on the right side of Eq.(4.6) exists. The first term

$$u_0(x,t) = \Phi(x,t), \tag{4.7}$$

consequently, the first few components can be written as:

$$\begin{aligned}
u_1(x,t) &= \mathbb{N}^{-1}\left[\tfrac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\tfrac{\partial^2 u_0(x,t)}{\partial x^2} - \tfrac{\partial u_0(x,t)}{\partial t} - u_0(x,t)\right]\right],\\
u_2(x,t) &= \mathbb{N}^{-1}\left[\tfrac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\tfrac{\partial^2 u_1(x,t)}{\partial x^2} - \tfrac{\partial u_1(x,t)}{\partial t} - u_1(x,t)\right]\right],\\
u_3(x,t) &= \mathbb{N}^{-1}\left[\tfrac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\tfrac{\partial^2 u_2(x,t)}{\partial x^2} - \tfrac{\partial u_2(x,t)}{\partial t} - u_2(x,t)\right]\right],\\
&.\\
&.\\
&.
\end{aligned} \tag{4.8}$$

then we have:

$$u_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\frac{\partial^2 u_n(x,t)}{\partial x^2} - \frac{\partial u_n(x,t)}{\partial t} - u_n(x,t)\right]\right],\ n \geq 0. \tag{4.9}$$

4.2 Nonlinear Fractional Telegraph Equations

The main aims of this section to generalise the fractional natural transform decomposition method to solve the nonlinear fractional telegraph equation, we also study the convergence and the error behavior to the nonlinear fractional telegraph equation. In addition, we demonstrate the applicability of the method by some examples and also the solution has been plotted for different values of α and comparing with exact solution. In order to elucidate the solution procedure of the fractional natural transform decomposition algorithm, we consider the following nonlinear fractional telegraph equation as:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} - Nu(x,t) + h(x,t), \tag{4.10}$$

$$0 < \alpha \leq 2 \ \text{ and } \ x, t \geq 0,$$

with the initial conditions:

$$u(x,0) = g_1(x) \text{ and } u_t(x,0) = g_2(x). \tag{4.11}$$

Where N is nonlinear and $h(x,t)$ is a source term.

In order to obtain the solution of Eq.(4.10) we use the fractional natural transform decomposition method as follows .

Step 1: By using Eq.(2.69) for Eq.(4.10) and single Laplace transform for Eq.(4.11), we have:

$$R(x,s,p) = \frac{1}{s} g_1(x) + \frac{p}{s^2} g_2(x) + \frac{p^\alpha}{s^\alpha} \mathbb{N}^+ \left[\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} - Nu(x,t) + h(x,t) \right], \tag{4.12}$$

Step 2: On using the inverse natural transform for Eq.(4.12), we obtain:

$$u(x,t) = \Phi(x,t) + \mathbb{N}^{-1} \left[\frac{p^\alpha}{s^\alpha} \mathbb{N}^+ \left[\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial u(x,t)}{\partial t} - Nu(x,t) \right] \right], \tag{4.13}$$

where

$$\Phi(x,t) = \mathbb{N}^{-1}\left[g_1(x) + tg_2(x) + \frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[h(x,t)\right]\right], \tag{4.14}$$

Provided that the inverse natural transform of each term on the right side of Eq.(4.13) exists. When we use Eq.(3.37) and Eq.(3.38) into Eq.(4.14), we obtain:

$$\sum_{n=0}^{\infty} u_n(x,t) = \Phi(x,t) + \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} - \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} - \sum_{n=0}^{\infty} A_n\right]\right], \tag{4.15}$$

by using the recursive relations

$$u_0(x,t) = \Phi(x,t), \tag{4.16}$$

consequently,

$$\begin{aligned}
u_1(x,t) &= \mathbb{N}^{-1}\left[\tfrac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\tfrac{\partial^2 u_0(x,t)}{\partial x^2} - \tfrac{\partial u_0(x,t)}{\partial t} - A_0\right]\right],\\
u_2(x,t) &= \mathbb{N}^{-1}\left[\tfrac{p^\alpha}{s^\alpha}\mathbb{N}^{+1}\left[\tfrac{\partial^2 u_1(x,t)}{\partial x^2} - \tfrac{\partial u_1(x,t)}{\partial t} - A_1\right]\right],\\
u_3(x,t) &= \mathbb{N}^{-1}\left[\tfrac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\tfrac{\partial^2 u_2(x,t)}{\partial x^2} - \tfrac{\partial u_2(x,t)}{\partial t} - A_2\right]\right],\\
&.\\
&.\\
&.
\end{aligned} \tag{4.17}$$

then we have:

$$u_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\frac{\partial^2 u_n(x,t)}{\partial t^2} + \frac{\partial u_n(x,t)}{\partial t} + A_n\right]\right],\ n \geq 0. \tag{4.18}$$

the solution $u_n(x,t)$ can be written as convergent series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{4.19}$$

4.2.1 Convergence Analysis

In this section, the sufficient condition that guarantees the existence of a unique solution is introduced and we discuss the convergence of the solution.
In the next theorem we follow [19]

Theorem (4.2.1) : **(Uniqueness theorem):** Eq.(4.18) has a unique solution whenever $0 < \varepsilon < 1$ where $\varepsilon = \frac{(L_1+L_2+L_3)t^{\alpha+1}}{(\alpha-1)!}$

proof (4.2.1). Let $E = (C[I], \|.\|)$ be the Banach space of all continuous functions on $I = [0,T]$ with the norm $\|.\|$, we define a mapping $F : E \to E$ where

$$u_{n+1}(x,t) = \Phi(x,t) + \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[L\left[u_n(x,t)\right] + M\left[u_n(x,t)\right] + N\left[u_n(x,t)\right]\right]\right],\ n \geq 0$$

where $L[u(x,t)] \equiv \frac{\partial^2 u(x,t)}{\partial x^2}$ and $M[u(x,t)] \equiv \frac{\partial u(x,t)}{\partial t}$. Now suppose $M[u(x,t)]$ and $L[u(x,t)]$ is also Lipschitzian with $\left|Mu - M\widehat{u}\right| < L_1\left|u - \widehat{u}\right|$and $\left|Lu - L\widehat{u}\right| < L_2\left|u - \widehat{u}\right|$ where L_1 and L_2 is Lipschitz constant respectively and $u,\widehat{u}$ is different values of the function.

$$\left\|Fu - F\widehat{u}\right\| = \max_{t\in I}\left|\begin{array}{c} \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[L\left[u(x,t)\right] + M\left[u(x,t)\right] + N\left[u(x,t)\right]\right]\right] \\ \\ -\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[L\left[\widehat{u}(x,t)\right] + M\left[\widehat{u}(x,t)\right] + N\left[\widehat{u}(x,t)\right]\right]\right] \end{array}\right|,$$

$$\leq \max_{t\in I}\left|\begin{array}{c} \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[L\left[u(x,t)\right] - L\left[\widehat{u}(x,t)\right]\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[M\left[u(x,t)\right] - M\left[\widehat{u}(x,t)\right]\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[N\left[u(x,t)\right] - N\left[\widehat{u}(x,t)\right]\right]\right] \end{array}\right|,$$

$$\leq \max_{t\in I}\left[\begin{array}{c} L_1\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left|u(x,t) - \widehat{u}(x,t)\right|\right] \\ +L_2\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left|u(x,t) - \widehat{u}(x,t)\right|\right] \\ +L_3\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left|u(x,t) - \widehat{u}(x,t)\right|\right] \end{array}\right],$$

$$\leq \max_{t\in I}(L_1+L_2+L_3)\left[\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^{+}\left|u(x,t)-\widehat{u}(x,t)\right|\right]\right],$$

$$\leq (L_1+L_2+L_3)\left[\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^{+}\left\|u(x,t)-\widehat{u}(x,t)\right\|\right]\right],$$

$$=\frac{(L_1+L_2+L_3)t^{(\alpha-1)}}{(\alpha-1)!}\left\|u(x,t)-\widehat{u}(x,t)\right\|.$$

Under the condition $0<\varepsilon<1$, the mapping is contraction. Therefore, by Banach fixed point theorem for contraction, there exists a unique solution to Eq.(4.19).
This ends the proof of theorem (4.2.1).

Theorem (4.2.2) :**(Convergence Theorem):** The solution of Eq.(4.1) and Eq.(4.10) in general forum will be convergence.

proof (4.2.2). Let S_n be the n^{th} partial sum, i.e., $S_n=\sum_{i=0}^{n}u_i(x,t)$. We shall prove that $\{S_n\}$ is a Cauchy sequence in Banach space E. By using a new formulation of Adomian polynomials we get:

$$R(S_n)=\widehat{A}_n+\sum_{r=0}^{n-1}\widehat{A}_r$$

$$N(S_n)=\widehat{A}_n+\sum_{c=0}^{n-1}\widehat{A}_c$$

$$\|S_n-S_m\|=\max_{t\in I}|S_n-S_m|=\max_{t\in I}\left|\sum_{i=m+1}^{n}\widehat{u}_i(x,t)\right|,$$

$$\leq \max_{t\in I}\left|\begin{array}{c}\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^{+}\left[\sum_{i=m+1}^{n}L\left[u_{n-1}(x,t)\right]\right]\right]\\ +\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^{+}\left[\sum_{i=m+1}^{n}M\left[u_{n-1}(x,t)\right]\right]\right]\\ +\mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^{+}\left[\sum_{i=m+1}^{n}A_{n-1}(x,t)\right]\right]\end{array}\right|,$$

$$= \max_{t\in I}\left|\begin{array}{c} \mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum\limits_{i=m}^{n-1}L\left[u_n(x,t)\right]\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum\limits_{i=m}^{n-1}M\left[u_n(x,t)\right]\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum\limits_{i=m+1}^{n}A_n(x,t)\right]\right] \end{array}\right|,$$

$$\leq \max_{t\in I}\left|\begin{array}{c} \mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum\limits_{i=m}^{n-1}L(S_{n-1})-L(S_{m-1})\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum\limits_{i=m}^{n-1}M(S_{n-1})-M(S_{m-1})\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum\limits_{i=m+1}^{n}N(S_{n-1})-N(S_{m-1})\right]\right] \end{array}\right|,$$

$$\leq \max_{t\in I}\left|\begin{array}{c} \mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[L(S_{n-1})-L(S_{m-1})\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[M(S_{n-1})-R(S_{m-1})\right]\right] \\ +\mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[N(S_{n-1})-N(S_{m-1})\right]\right] \end{array}\right|,$$

$$\begin{array}{l} \leq L_1\max\limits_{t\in I}\mathbb{N}^{-1}\left|\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[(S_{n-1})-(S_{m-1})\right]\right]\right|, \\ +L_2\max\limits_{t\in I}\mathbb{N}^{-1}\left|\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[(S_{n-1})-(S_{m-1})\right]\right]\right|, \\ +L_3\max\limits_{t\in I}\mathbb{N}^{-1}\left|\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[(S_{n-1})-(S_{m-1})\right]\right]\right|. \end{array}$$

$$= \frac{(L_1+L_2+L_3)t^{(\alpha-1)}}{(\alpha-1)!}\left\|S_{n-1}+S_{m-1}\right\|$$

Let $n = m + 1$; then

$$\left\|S_{m+1}-S_m\right\| \leq \varepsilon\left\|S_m-S_{m-1}\right\| \leq \varepsilon^2\left\|S_{m-1}-S_{m-2}\right\| \leq ... \leq \varepsilon^m\left\|S_1-S_0\right\|.$$

where $\varepsilon = \frac{(L_1+L_2+L_3)t^{(\alpha-1)}}{(\alpha-1)!}$ similarly, we have, from the triangle inequality we have

$$\left\|S_n-S_m\right\| \leq \left\|S_{m+1}-S_m\right\| + \left\|S_{m+2}-S_{m+1}\right\| + ... + \left\|S_n-S_{n-1}\right\|,$$

$$\leq \left[\varepsilon^m+\varepsilon^{m+1}+...+\varepsilon^{n-1}\right] \leq \left\|S_1+S_0\right\|,$$

$$\leq \varepsilon^m\left(\frac{1-\varepsilon^{n-m}}{\varepsilon}\right)\left\|u_1\right\|,$$

since $0 < \varepsilon < 1$ we have $\left(1 - \varepsilon^{n-m}\right) < 1$: then,

$$\|S_n - S_m\| \leq \frac{\varepsilon^m}{1-\varepsilon} \max_{t \in I} \|u_1\| .$$

However, $|u_1| < \infty$ (since $u(x,t)$ is bounded) so, as $m \to \infty$ then $\|S_n - S_m\| \to 0$, hence $\{S_n\}$ is a Cauchy sequence in E so, the series $\sum_{n=0}^{\infty} u_n$ converges and the proof is complete.

Theorem (4.2.3) **(Error estimate:)**: The maximum absolute truncation error of the series solution Eq.(4.18) to Eq.(4.19) is estimated to be:

$$\max_{t \in I} \left| u(x,t) - \sum_{n=1}^{m} u_n(x,t) \right| \leq \frac{\varepsilon^m}{1-\varepsilon} \max_{t \in I} \|u_1\| ,$$

proof (4.2.3). From Eq.(3.18) and theorem (4.2.2) we have:

$$|S_n - S_m| \leq \frac{\varepsilon^m}{1-\varepsilon} \max_{t \in I} \|u_1\| ,$$

as $n \to \infty$ then $S_n \to u(x,t)$ so we have

$$\|u(x,t) - S_m\| \leq \frac{\varepsilon^m}{1-\varepsilon} \max_{t \in I} \|u_1(x,t)\| .$$

Finally, the maximum absolute truncation error in the interval I is

$$\max_{t \in I} \left| u(x,t) - \sum_{n=1}^{m} u_n(x,t) \right| \leq \max_{t \in I} \frac{\varepsilon^m}{1-\varepsilon} |u_1(x,t)| = \frac{\varepsilon^m}{1-\varepsilon} \|u_1(x,t)\| .$$

thus, completing the proof of Theorem (4.2.3).

To incorporate our discussion, three examples of of linear and nonlinear fractional telegraph equations with specific initial conditions, will be investigated the reliability of the proposed scheme:

Example (4.2.1) : Consider the following space-fractional homogeneous telegraph equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} - u(x,t),$$
$$x, t \geq 0 \quad \text{and} \quad 0 < \alpha \leq 2, \tag{4.20}$$

with the initial conditions

$$u(x,0) = e^{-x} \text{ and } u_t(x,0) = -e^{-x}. \tag{4.21}$$

Solution (4.2.1) :: By applying natural transform for Eq.4.20) w.r.t (t) and substituting the initial conditions Eq.(4.21), we get:

$$R(x,s,p) = \frac{1}{s}e^x - \frac{p}{s^2}e^x + \frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} - u(x,t)\right], \tag{4.22}$$

On using the inverse natural transform for Equation (4.22), we have:

$$u(x,t) = e^x - te^x + \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} - u(x,t)\right]\right], \tag{4.23}$$

the correction function for Equation (4.23), is given by

$$\sum_{n=0}^{\infty} u_{n+1}(x,t) = e^{-x} - te^{-x} + \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} - \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} - \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \tag{4.24}$$

the initial term

$$u_0(x,t) = e^x - te^x, \tag{4.25}$$

Then we have

$$u_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{p^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} - \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} - \sum_{n=0}^{\infty} u_n(x,t)\right]\right], n \geq 0, \tag{4.26}$$

the first 3^{rd} terms is given by

$$u_1(x,t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}e^x,$$

$$u_2(x,t) = -\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}e^x,$$

$$u_3(x,t) = \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}e^x, \tag{4.27}$$

.

.

.

then general form is successive approximation is given by

$$u(x,t) = e^x\left(1 - t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - ...\right), \tag{4.28}$$

$$u(x,t) = e^x\left[1 + \sum_{k=0}^{\infty}(-1)^{k+1}\left[\frac{t^{k\alpha-k+1}}{\Gamma(k\alpha-k+2)}\right]\right], \tag{4.29}$$

In order to prove the efficiency of our method, we replace the fractional order $\alpha = 2$, we get

$$u(x,t) = e^{x-t}. \tag{4.30}$$

Example (4.2.2) :Consider the following space-fractional non-homogeneous telegraph equation:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} - u(x,t) + x^2 + t - 1, \tag{4.31}$$

$$x,t \geq 0 \text{ and } 0 < \alpha \leq 2,$$

with the initial conditions

$$u(x,0) = x^2 \text{ and } u_t(x,0) = 1. \tag{4.32}$$

Solution (4.2.2) :By using the technique of natural transform decomposition (Eq.(4.5) then

Eq.(4.31), becomes:

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)}x^2 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$+\mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} - \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} - \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \tag{4.33}$$

the initial term

$$u_0(x,t) = x^2 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)}x^2 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{4.34}$$

then we have:

$$\sum_{n=0}^{\infty} u_{n+1}(x,t) = \mathbb{N}^{-1}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} - \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} - \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \quad n \geq 0 \tag{4.35}$$

Now the components of the series solution are given by:

$$\begin{aligned}
u_1(x,t) &= \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} - \tfrac{t^{\alpha}}{\Gamma(\alpha+1)}x^2 - \tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
&\quad - \tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}x^2 - \tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)}x^2, \\
u_2(x,t) &= -2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}x^2 \\
&\quad + \tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)}x^2 - 5\tfrac{t^{3\alpha-1}}{\Gamma(3\alpha+2)} - 3\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 2\tfrac{t^{3\alpha-1}}{\Gamma(3\alpha)}x^2 \\
&\quad - \tfrac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + \tfrac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}x^2 + \tfrac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}x^2, \\
u_3(x,t) &= 5\tfrac{t^{3\alpha-1}}{\Gamma(3\alpha+2)} + 3\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 2\tfrac{t^{3\alpha-1}}{\Gamma(3\alpha)}x^2 + \tfrac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \tfrac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \\
&\quad - \tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}x^2 - \tfrac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}x^2 + \tfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \tfrac{t^{4\alpha}}{\Gamma(4\alpha+1)}x^2 + 7\tfrac{t^{4\alpha-1}}{\Gamma(4\alpha)} - 3\tfrac{t^{4\alpha-1}}{\Gamma(4\alpha)}x^2 \\
&\quad - 2\tfrac{t^{4\alpha+1}}{\Gamma(4\alpha)} + 5\tfrac{t^{4\alpha-1}}{\Gamma(4\alpha)} + 8\tfrac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} - 3\tfrac{t^{4\alpha-2}}{\Gamma(4\alpha-1)}x^2 + \tfrac{t^{4\alpha-3}}{\Gamma(4\alpha-2)} - \tfrac{t^{4\alpha-3}}{\Gamma(4\alpha-2)}x^2,
\end{aligned} \tag{4.36}$$

Then the solution is given by

$$
\begin{aligned}
\sum_{n=0}^{\infty} u_n(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+...\\
u(x,t) &= x^2+t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}x^2+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&-\frac{t^{\alpha}}{\Gamma(\alpha+1)}x^2-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\\
&-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}x^2-\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}x^2-2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\\
&+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}x^2+\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}x^2-5\frac{t^{3\alpha-1}}{\Gamma(3\alpha+2)}-3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+2\frac{t^{3\alpha-1}}{\Gamma(3\alpha)}x^2\\
&-\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}+\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}x^2+\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}x^2+5\frac{t^{3\alpha-1}}{\Gamma(3\alpha+2)}\\
&+3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-2\frac{t^{3\alpha-1}}{\Gamma(3\alpha)}x^2+\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}x^2\\
&-\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}x^2+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}-\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}x^2+7\frac{t^{4\alpha-1}}{\Gamma(4\alpha)}-3\frac{t^{4\alpha-1}}{\Gamma(4\alpha)}x^2\\
&-2\frac{t^{4\alpha+1}}{\Gamma(4\alpha)}+5\frac{t^{4\alpha-1}}{\Gamma(4\alpha)}+8\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)}-3\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)}x^2+\frac{t^{4\alpha-3}}{\Gamma(4\alpha-2)}\\
&-\frac{t^{4\alpha-3}}{\Gamma(4\alpha-2)}x^2+...
\end{aligned}
$$

at $\alpha = 2$, we obtain the exact solution of standard telegraph equation.

$$u(x,t) = t + x^2 \tag{4.37}$$

Example (4.2.3) : Consider the following space-fractional nonlinear telegraph equation:

$$
\begin{gathered}
\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial t} - u^2(x,t) + xu(x,t)u_x(x,t),\\
x,t \geq 0 \quad \text{and} \quad 0 < \alpha \leq 2,
\end{gathered} \tag{4.38}
$$

with the initial conditions

$$u(x,0) = x \text{ and } u_t(x,0) = x. \tag{4.39}$$

Solution 4.2.3

By taking natural transform for Eq.(4.38), we have:

$$\frac{s^{\alpha}}{p^{\alpha}}R(x,s,p)-\frac{s^{\alpha-1}}{p^{\alpha}}u(x,0)-\frac{s^{\alpha-2}}{p^{\alpha-1}}u_t(x,0)=\ \mathbb{N}^{+}\left[\frac{\partial u^2(x,t)}{\partial x^2}+\frac{\partial u(x,t)}{\partial t}-u^2(x,t)+xu(x,t)u_x(x,t)\right],\tag{4.40}$$

and by the arrangement and substitution the initial condition (4.39) in (4.40), we get

$$R(x,s,p)=\frac{1}{s}x+\frac{p}{s^2}x+\ \left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\frac{\partial u^2(x,t)}{\partial x^2}+\frac{\partial u(x,t)}{\partial t}-u^2(x,t)+xu(x,t)u_x(x,t)\right]\right],\tag{4.41}$$

and by applying the inverse natural transform for Eq.(4.41), we have

$$u(x,t)=x+tx+\ \mathbb{N}^{-}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\frac{\partial u^2(x,t)}{\partial x^2}+\frac{\partial u(x,t)}{\partial t}-u^2(x,t)+xu(x,t)u_x(x,t)\right]\right],\tag{4.42}$$

hence,

$$\begin{aligned}&\sum_{n=0}^{\infty}u_{n+1}(x,t)=(x+tx)\\&+\mathbb{N}^{-}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^2u_n(x,t)}{\partial x^2}+\sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t}-\sum_{n=0}^{\infty}A_n(x,t)+x\sum_{n=0}^{\infty}B_n(x,t)\right]\right],\end{aligned}\tag{4.43}$$

the initial term

$$u_0(x,t)=(x+tx)\,.\tag{4.44}$$

Now the components of the series solution are given by:

$$u_{n+1}(x,t)=\mathbb{N}^{-}\left[\frac{p^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\frac{\partial^2u_n(x,t)}{\partial x^2}+\frac{\partial u_n(x,t)}{\partial t}-A_n(x,t)+xB_n(x,t)\right]\right],n\geq 0\tag{4.45}$$

$$\begin{aligned}u_1(x,t)&=\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}x\right),\\u_2(x,t)&=\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x\right),\\u_3(x,t)&=\left(\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}x\right).\end{aligned}\tag{4.46}$$

Since

$$u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ... \tag{4.47}$$

$$u(x,t) = x + tx + \frac{t^{\alpha}}{\Gamma(\alpha+1)}x + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x + \frac{t^{\alpha+2}}{\Gamma(\alpha+3)}x + ... \tag{4.48}$$

by substituting $\alpha = 2$ in Equation (4.48), we obtain the exact solution of the standard telegraph equation in the following form:

$$u(x,t) = xe^{t} \tag{4.49}$$

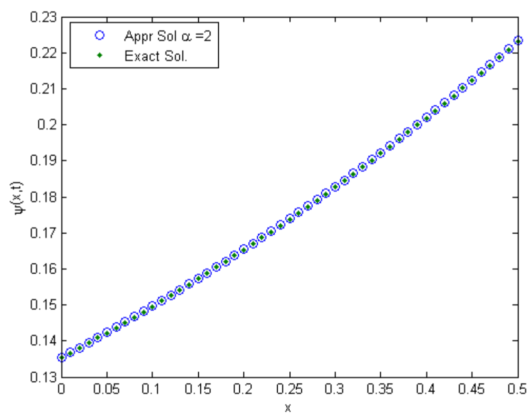

Figure 4.1: The exact and approximate solutions of $u(x,t)$ for Example 4.2.1 for $\alpha = 2$.

Table 4.1: Exact and approximate solution of $u(x,t)$ for Example 4.2.1.

t	Exact Solution u	Approximate Sol u_{10}	$u = \|u - u_{10}\|$
0.0	1.648721270700128	1.648721270700128	0.0
0.5	1.0	1.000000000040401	4.040101586 $e{-11}$
1.0	0.606530659712633	0.606530742852590	8.313995659 $e{-8}$
1.5	0.367879441171442	0.367886690723836	7.249552393 $e{-6}$
2.0	0.223130160148429	0.223303762933655	1.569783692 $e{-4}$

Table 4.2: Approximate solution of $u(x,t)$ for Example 4.2.1.

t	$\alpha = 1.99$	$\alpha = 1.98$	$\alpha = 1.97$
0.0	1.648721270700128	1.648721270700128	1.648721270700128
0.5	1.002243362235993	1.004498274095028	1.006764389796932
1.0	0.609569757949665	0.612585971492061	0.615579284214978
1.5	0.369222108754806	0.371788163947318	0.370522335669580
2.0	0.221291547575669	0.219269844467107	0.217240584657229

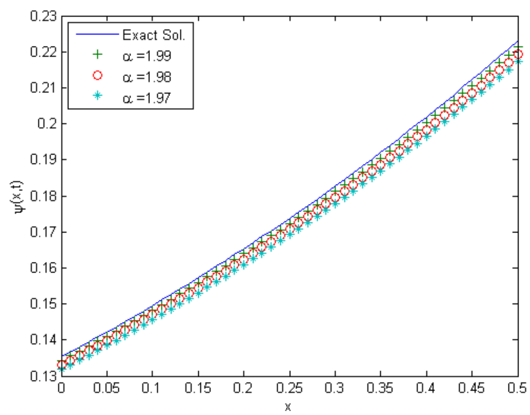

Figure 4.2: The Exact solutions and approximate solutions of $u(x,t)$ for Example 4.2.1 for different value for α.

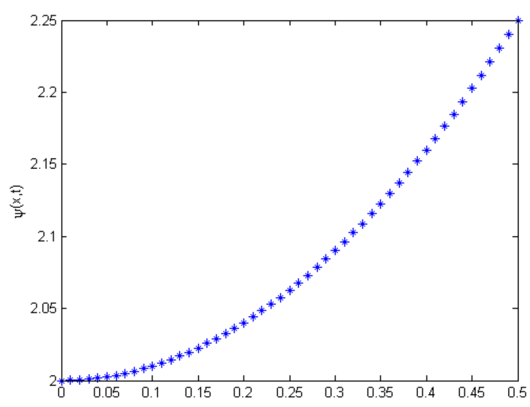

Figure 4.3: The exact solution of $u(x,t)$ for Example 4.2.2.

Table 4.3: Approximate solution of $u(x,t)$ for Example 4.2.3.

t	Exact Solution	$\alpha = 1.95$	$\alpha = 1.90$	$\alpha = 1.85$
0.0	0.5	0.5	0.5	1.5
5.0	0.824360635350064	0.830817752645242	0.837755999175080	0.845202109327201
1.0	1.359140914229523	1.378259288907402	1.398076433466764	1.418592017094073
1.5	2.240844535169032	2.276244404149126	2.312171661003479	2.348587393416824
2.0	3.694528049465325	3.748855797997422	3.803171995755493	3.857406067787722

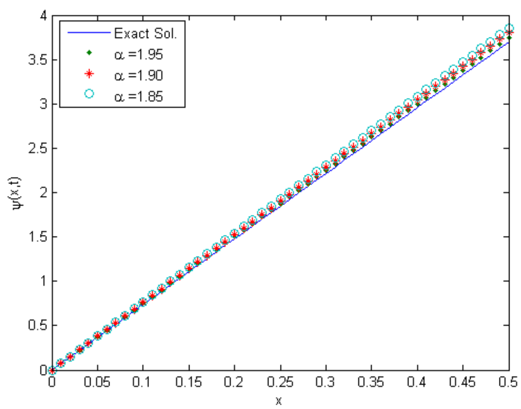

Figure 4.4: The approximate solutions of $u(x,t)$ for Example 4.2.3 for $\alpha = 1.95$, $\alpha = 1.90$, $\alpha = 1.85$ and exact solution.

4.3 Numerical Results

In this section, we shall illustrate the accuracy and efficiency of the (NTDM) by comparing the approximate and the exact solution.

Figure (4.1) confirms the accuracy and efficiency of the natural transform and Adomian decomposition method and discuss the behaviour of exact solution and approximate solutions Eq.(4.20) obtained by (NTDM) for the special case $\alpha = 2$ for Example (4.2.1). We see that Table 4.1 illustrated the absolute error by computing $u = |u - u_{10}|$ where u is the exact solution and u_{10} is approximate solution of Eq.(4.20) obtained by truncating the respective solution series Eq.(4.29) at u_{10}. Approximate solutions converge very swiftly to the exact solutions in only the 10th order approximations i.e, approximate solutions are nearly identical to the exact solutions. The accuracy of the result can be ameliorated by generating more terms of the approximate solutions.

Figure (4.2) shows the exact solution and the approximate solution Equation (4.20) obtained by natural transform and Adomian decomposition method when α decreasing then the u decreasing.

Table (4.2) discuss the solution of Example (4.2.1) by choosing different values of $t = \{0, 0.5, 1, 1.5, 2\}$ and the values of $u(x,t)$ decreasing when t increasing for different values of $\alpha = 1.99, 1.98$ and 1.97.

Figure (4.3) Shows when setting $\alpha = 2$ in the n^{th} approximations and cancelling noise terms yields the exact solution $u = |u - u_{10}|$ as $n \to \infty$. The analytical solution for the exact solution and the approximate solution Equation (4.31) obtained by natural transform and Adomian decomposition method. In addition, the exact solution is presented graphically in Figure (4.3).

The exact and approximate solutions of Equation (4.38) are presented graphically in Figure (4.4), the approximate solution is given at $\alpha = 1.99, 1.98$ and 1.97. The value of the solution satisfies Equation (4.38) see in Table (4.3) for the values $\alpha = 1.99, 1.98$ and 1.97.

4.4 Singular Fractional Telegraph Equations:

In this section, we derive the main idea of fractional natural transform decomposition method to solve a singular fractional telegraph equation.

Theorem (4.4.1) : We consider singular fractional telegraph equation with initial condition as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{x}\left(x\frac{\partial u(x,t)}{\partial x}\right)_x + \frac{\partial u(x,t)}{\partial t} + u(x,t) + h(x,t),\ 0 < \alpha \leq 2 \text{ and } x,t \geq 0, \tag{4.50}$$

subject to:

$$u(x,0) = f_1(x) \text{ and } u_t(x,0) = f_2(x), \tag{4.51}$$

where $\frac{1}{x}\left(x\frac{\partial u(x,t)}{\partial x}\right)_x$ is the Bessel operator and $h(x,t)$ is a continuous function.

Then the solution of Eq.(4.50) is given by

$$\begin{aligned} u_0(x,t) =& \tfrac{1}{s}H(x,t) + \tfrac{1}{s^2}F_1(s) - \tfrac{u}{s^2}F_2(s) \\ u_{n+1}(x,t) =&\ N^{-1}\left[\tfrac{u^\alpha}{s^\alpha}N^+\left[\left[\sum_{n=0}^{\infty}\tfrac{\partial^2 u_n(x,t)}{\partial x^2} + \tfrac{1}{x}\sum_{n=0}^{\infty}\tfrac{\partial u_n(x,t)}{\partial x} + \sum_{n=0}^{\infty}\tfrac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty}u_n(x,t)\right]\right]\right] \\ & , n \geq 0 \end{aligned} \tag{4.52}$$

Proof. We apply the natural transform of partial derivatives for equation Eq.(4.50), we get:

$$\tfrac{s^\alpha}{u^\alpha}U(p,s) - \tfrac{s^{\alpha-1}}{u^\alpha}U(p,0) - \tfrac{s^{\alpha-2}}{u^{\alpha-1}}U_t(p,0) = N^+\left[\tfrac{\partial^2 u(x,t)}{\partial x^2} + \tfrac{1}{x}\tfrac{\partial u(x,t)}{\partial x} + \tfrac{\partial u(x,t)}{\partial t} + u(x,t) + h(x,t)\right], \tag{4.53}$$

By substituting Eq.(4.51) into Eq.(4.53) and by using use the property of natural transform and simplifying, we obtain:

$$U(p,s) = \tfrac{u^\alpha}{s^\alpha}N^+\left[\tfrac{\partial^2 u(x,t)}{\partial x^2} + \tfrac{1}{x}\tfrac{\partial u(x,t)}{\partial x} + \tfrac{\partial u(x,t)}{\partial t} + u(x,t)\right] + \tfrac{1}{s}H(x,t) + \tfrac{1}{s^2}F_1(s) - \tfrac{u}{s^2}F_2(s), \tag{4.54}$$

where $H(p,s)$ is natural transform of $h(x,t)$.

Taking the inverse natural transform to Eq.(4.54), we get:

$$u(x,t) = G(x,t) + N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{x}\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} + u(x,t)\right]\right], \quad (4.55)$$

where $G(x,t)$ is the function comes from continuous function and initial condition.
The solution of Eq.(4.50) can be written as infinite series terms (Adomian decomposition method) such as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (4.56)$$

Then Eq.(4.55) becomes:

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) + N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} + \frac{1}{x}\sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial x} + \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty}u_n(x,t)\right]\right]\right], \quad (4.57)$$

The method suggests that the zero component $u_0(x,t)$ is identified by terms that are not included under N^{-1} in Eq.(4.57).

$$\begin{aligned} u_0(x,t) &= G(x,t) \\ u_{n+1}(x,t) &= N^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}N^{+}\left[\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} + \frac{1}{x}\sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial x} + \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty}u_n(x,t)\right]\right]\right], n \geq 0 \end{aligned} \quad (4.58)$$

Note that the inverse natural transform of each terms on the right side of Eq.(4.58) exists.

Numerical Examples:

In this section, we demonstrate the applicability and stability of our method by applying numerical examples.

Example (4.4.1) :Consider the following singular fractional telegraph equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{x}\left(x\frac{\partial u(x,t)}{\partial x}\right)_x + \frac{\partial u(x,t)}{\partial t} + u(x,t) + 2x^2 - 2x^2 t - x^2 t^2 - 4t^2, \quad x,t \geq 0 \text{ and } 0 < \alpha \leq 2, \quad (4.59)$$

subject to the initial condations:

$$u(x,0)=0, \quad u_t(x,0)=0. \tag{4.60}$$

By applying the natural transform on both sides of Eq.(4.59) to get:

$$\begin{aligned} \tfrac{s^{\alpha}}{u^{\alpha}}U(p,s)-\tfrac{s^{\alpha-}}{u^{\alpha}}U(p,0)-\tfrac{s^{\alpha-2}}{u^{\alpha-1}}U_x(p,0)= \\ 2\tfrac{2!}{p^3 s}-2\tfrac{2!}{p^3 s^2}-\tfrac{2!2!}{p^3 s^3}-4\tfrac{2!}{ps^3}+N^{+}\left[\tfrac{\partial^2 u(x,t)}{\partial x^2}+\tfrac{1}{x}\tfrac{\partial u(x,t)}{\partial x}+\tfrac{\partial u(x,t)}{\partial t}+u(x,t)\right], \end{aligned} \tag{4.61}$$

substituting the initial conditions Eq.(4.60) in Eq.(4.61), gives:

$$U(p,s)=2\tfrac{2!}{p^3 s^{\alpha+1}}-2\tfrac{2!}{p^3 s^{\alpha+2}}-\tfrac{2!2!}{p^3 s^{\alpha+3}}-4\tfrac{2!}{ps^{\alpha+3}}+\tfrac{1}{s^{\alpha}}N^{+}\left[\tfrac{\partial^2 u(x,t)}{\partial x^2}+\tfrac{1}{x}\tfrac{\partial u(x,t)}{\partial x}+\tfrac{\partial u(x,t)}{\partial t}+u(x,t)\right], \tag{4.62}$$

and by using the inverse natural transform for Eq.(4.62) we have:

$$\begin{aligned} u(x,t)=2x^2\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}-2x^2\tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}-2x^2\tfrac{t^{\alpha+2}}{\Gamma(\alpha+3)}-8\tfrac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ +N^{-}\left[\tfrac{1}{s^{\alpha}}N^{+}\left[\tfrac{\partial^2 u(x,t)}{\partial x^2}+\tfrac{1}{x}\tfrac{\partial u(x,t)}{\partial x}+\tfrac{\partial u(x,t)}{\partial t}+u(x,t)\right]\right], \end{aligned} \tag{4.63}$$

Now we define the function $u(x,t)$ by the decomposition series

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{4.64}$$

Insert Eq.(4.64) into both sides Eq.(4.63), gives:

$$\begin{aligned} \sum_{n=0}^{\infty}u_n(x,t)=2x^2\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}-2x^2\tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}-2x^2\tfrac{t^{\alpha+2}}{\Gamma(\alpha+3)}-8\tfrac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ +N^{-}\left[\tfrac{1}{s^{\alpha}}N^{+}\left[\sum_{n=0}^{\infty}\tfrac{\partial^2 u_n(x,t)}{\partial x^2}+\tfrac{1}{x}\sum_{n=0}^{\infty}\tfrac{\partial u_n(x,t)}{\partial x}+\sum_{n=0}^{\infty}\tfrac{\partial u_n(x,t)}{\partial t}+\sum_{n=0}^{\infty}u_n(x,t)\right]\right], \end{aligned} \tag{4.65}$$

By using the Eq.(4.65) we find a few terms of the series of $u(x,t)$.

$$u_0(x,t) = 2x^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2x^2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 2x^2\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} - 8\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}$$

$$u_{n+1}(x,t) = N^{-}\left[\frac{1}{s^{\alpha}}N^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} + \frac{1}{x}\sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial x} + \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty}u_n(x,t)\right]\right], n \geq 0 \quad (4.66)$$

then,

$$u_1(x,t) = 2x^2\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - 2x^2\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 4x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 16\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 16\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 8\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (4.67)$$

$$u_2(x,t) = 2x^2\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + 2x^2\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - 4x^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 2x^2\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} - 6x^2\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}$$
$$-24\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} - 48\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 16\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} \quad (4.68)$$

$$u_3(x,t) = 2x^2\frac{t^{4\alpha-3}}{\Gamma(4\alpha-2)} + 4x^2\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} - 2x^2\frac{t^{4\alpha-1}}{\Gamma(4\alpha)} - 10x^2\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - 8x^2\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}$$
$$-2x^2\frac{t^{4\alpha+2}}{\Gamma(4\alpha+3)} - 32\frac{t^{4\alpha+2}}{\Gamma(4\alpha+3)} - 96\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} - 72\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + 24\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} \quad (4.69)$$

Therefore, the (NTDM) series solution is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

$$u(x,t) = 2x^2\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - 2x^2\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 4x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 16\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 16\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 8\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+2x^2\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - 2x^2\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 4x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 16\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} - 16\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 8\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2x^2\frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}$$
$$+2x^2\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} - 4x^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 2x^2\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} - 6x^2\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 24\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)} - 48\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - 8\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$+16\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + 2x^2\frac{t^{4\alpha-3}}{\Gamma(4\alpha-2)} + 4x^2\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} - 2x^2\frac{t^{4\alpha-1}}{\Gamma(4\alpha)} - 10x^2\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - 8x^2\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} - 2x^2\frac{t^{4\alpha+2}}{\Gamma(4\alpha+3)}$$
$$-32\frac{t^{4\alpha+2}}{\Gamma(4\alpha+3)} - 96\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)} - 72\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + 24\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} + ... \quad (4.70)$$

If we take $(\alpha = 2)$ then we get exact solution Eq.(4.59) of standard telegraph equation as.

$$u(x,t) = x^2 t^2 \tag{4.71}$$

Example (4.4.2) : Consider the following singular fractional telegraph equation:

$$\begin{gathered} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{x}\left(x\frac{\partial u(x,t)}{\partial x}\right)_x + \frac{\partial u(x,t)}{\partial t} + u(x,t) - \ln x - t\ln x, \\ x,t \geq 0 \text{ and } 0 < \alpha \leq 2, \end{gathered} \tag{4.72}$$

subject to:

$$u(x,0) = 0, \quad u_t(x,0) = \ln x. \tag{4.73}$$

By applying the natural transform on both sides of Eq.(4.72), we get

$$\begin{aligned} s^\alpha U(p,s) - s^{\alpha-1}U(p,0) - s^{\alpha-2}U_t(p,0) = \frac{\ln(p)+\gamma}{ps} + \frac{\ln(p)+\gamma}{ps^2} \\ +N^+\left[\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{x}\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} + u(x,t)\right], \end{aligned} \tag{4.74}$$

substituting initial conditions Eq.(4.73) in Eq.(4.72), gives:

$$U(p,s) = \frac{\ln(p)+\gamma}{ps^{\alpha+1}} + \frac{\ln(p)+\gamma}{ps^{\alpha+2}} - \frac{\ln(p)+\gamma}{ps^2} + \frac{1}{s^\alpha}N^+\left[\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1}{x}\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} + u(x,t)\right], \tag{4.75}$$

and by using the inverse natural transform transform for Eq.(4.75), we have:

$$u(x,t) = \ln(x)\frac{t^\alpha}{\Gamma(\alpha+1)} + \ln(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + t\ln(x) \tag{4.76}$$

Now we define the function $u(x,t)$ by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{4.77}$$

Inserting Eq.(4.77) into both sides of Eq.(4.76) gives:

$$\sum_{n=0}^{\infty} u_n(x,t) = \ln(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \ln(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + t\ln(x)$$
$$+N^{-}\left[\frac{1}{s^{\alpha}}N^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} + \frac{1}{x}\sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial x} + \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \tag{4.78}$$

By using Eq.(4.78) we find a few terms of the series of $u(x,t)$:

$$u_0 = \ln(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \ln(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + t\ln(x)$$

$$u_{n+1}(x,t) = N^{-1}\left[\frac{1}{s^{\alpha}}N^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2} + \frac{1}{x}\sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial x} + \sum_{n=0}^{\infty}\frac{\partial u_n(x,t)}{\partial t} + \sum_{n=0}^{\infty} u_n(x,t)\right]\right], \quad n \geq 0 \tag{4.79}$$

Then the next terms are

$$u_1(x,t) = \ln(x)\Big(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big) \tag{4.80}$$

$$u_2(x,t) = \ln(x)\Big(\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \frac{t^{2\alpha-1}}{\Gamma(2\alpha)}\Big)$$

$$u_3(x,t) = \ln(x)\left(6\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - 3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 4\frac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} - 3\frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + \frac{t^{4\alpha-3}}{\Gamma(4\alpha-2)} - \frac{t^{3\alpha-2}}{\Gamma(3\alpha-1)}\right) \tag{4.81}$$

Therefore ,the (NTDM) series solution is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

$$u(x,t) = \ln(x)\Big(\tfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)} + t + \tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \tfrac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \tfrac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$\tfrac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \tfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 3\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3\tfrac{t^{3\alpha-1}}{\Gamma(3\alpha)} + \tfrac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} - \tfrac{t^{2\alpha-1}}{\Gamma(2\alpha)}$$

$$+6\tfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} - 3\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 4\tfrac{t^{4\alpha-2}}{\Gamma(4\alpha-1)} - 3\tfrac{t^{3\alpha-1}}{\Gamma(3\alpha)} + \tfrac{t^{4\alpha-3}}{\Gamma(4\alpha-2)} - \tfrac{t^{3\alpha-2}}{\Gamma(3\alpha-1)} + ...\Big) \tag{4.82}$$

The solution of Eq.(4.82) is equal to the exact solution of the standard telegraph equation (when $\alpha = 2$):

$$u(x,t) = t\ln(x). \tag{4.83}$$

Conclusion

We have successfully applied double Laplace transform and Adomian decomposition method to obtain the approximate solutions of the fractional telegraph equation.The (DLADM) gives us high convergence and leads us to say this method has highly accurate and efficient solutions.

We have successfully applied the natural transform and Adomian decomposition method to obtain the approximate solutions of the fractional telegraph equation. The (NTDM) gives us small error and high convergence. As seen in Tables 4.1–4.3, errors are very small, and they sometimes deflate as shown in Table 4.3. These techniques lead us to say that the method is accurate and efficient according to the theoretical analysis and examples 3 and 4 in the exact solution and approximate solution of $u(x,t)$ are equal at $\alpha = 2$ the absolute error equal zero.

we see no difference in the solution in all examples and the two methods double Laplace transform decomposition method and natural transform decomposition method lead us to the same solution.

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