Dedication

To my parents and family

To my beloved ones

Baby

and Mobarak

I dedicate this venture.

Acknowledgements

First, my gratitude to Almighty ALLAH for helping me to complete this work.

I wish to express my sincere thanks to Prof. Shawgy Hussein AbdAlla for providing me with all the necessary facilities.

My sincere gratitude to the head and staff of mathematics department for their constant encouragement.

I extend my sincere thanks to my friends Haytham Mohammed Ali and Allaa Ahmed for their unceasing supports.

I also place on record, my sense of gratitude to everyone who directly or indirectly have lent their helping hand in this venture.

Abstract

We study the spectral properties and complemented invariant subspaces in the Bergman spaces having the codimension two property and also with the spectra of some translation and fiber dimension for invariant subspaces. We determine the index of invariant subspaces in Hilbert spaces of vector-valued analytic functions of several complex variables. We classify similarity, reducing manifolds, unitary equivalence of Volterra operators and Volterra invariant subspaces of Hardy spaces. We show nearly invariant subspaces of the backward shift and shift plus complex Volterra operator. We characterize the analytic continuability of Bergman inner functions and linear graph transformations on spaces of analytic functions. We give the algebraic properties of the index of invariant subspaces of operators and of parabolic self-maps on Banach and Hardy spaces.

الخلاصة

درسنا الخصائص الطيفية والفضاء الجزئي اللامتغير المتمم في فضاءات بيرجمان الذي له البعد المصاحب ذو الخاصيتين وأيضا مع بعض الانسحاب والبعد الليفي لأجل الفضاءات الجزئية اللامتغيرة. حددنا الدليل للفضاءات الجزئية اللامتغيرة في فضاءات هلبرت للدوال التحليلية قيمة -المتجه للمتغيرات المركبة المتعددة. صنفنا التماثلية ومتعددات الطيات المختزلة والتكافؤ الأحادي لمؤثرات فولتيرا والفضاءات الجزئية اللامتغيرة لفولتيرا لفضاءات هاردي. أوضحنا الفضاءات الجزئية اللامتغيرة القريبة للإزاحة الخلفية والإزاحة زائداً مؤثر فولتيرا المركب. شخصنا قابلية الاستمرارية التحليلية للدوال الداخلة لبيرجمان وتحويلات البيان الخطي على الفضاءات للدوال التحليلية. أعطينا الخصائص الجبرية للدليل للفضاءات الجزئية اللامتغيرة للمؤثرات وللرواسم –الذاتية المكافئية على فضاءات باناخ وهاردي.

Introduction

We study the relationship between two types of spectra associated with invariant subspaces of the Bergman space $L_a^2(\mathbb{D})$ and the function theoretic properties of the invariant subspaces themselves. We shall present a simple concrete example of an invariant subspace having the codimension 2 property.

We show that given any left translation invariant space how we can characterize the spectrum and the point spectrum. Let $\mathbb{B}_d$ be the open unit ball in $\mathbb{C}^d, d \geq 1$, and H_d^2 be the space of analytic functions on $\mathbb{B}_d$ determined by the reproducing kernel $(1 - \langle z, \lambda \rangle)^{-1}$. This reproducing kernel Hilbert space serves a universal role in the model theory for d-contractions, i.e. tuples $T = (T_1, .., T_d)$ of commuting operators on a Hilbert space $\mathcal{K}$ such that $\|T_1x_1 + .. + T_dx_d\|^2 \leq \|x_1\|^2 + .. + \|x_d\|^2$ for all $x_1, .., x_d \in \mathcal{K}$. If $\mathcal{D}$ is a separable Hilbert space then we write $H_d^2(\mathcal{D}) \cong H_d^2 \otimes \mathcal{D}$ for the space of $\mathcal{D}$ -valued H_d^2 functions and we use $M_z = \left(M_{z_1}, .., M_{z_d}\right)$ to denote the tuple of multiplication by the coordinate functions.

We deal with similarity properties. We lean heavily on results by Volterra. We show examples illustrating the basic conditions.

A formula yielding the analytic continuation of extremal functions associated to finite zero sequences is derived. As a consequence we obtain estimates on these analytic continuations. The estimates are used to show the main result. The formula derived is used to give a different proof of result of Mac Gregor and Stessin. Let $\mathcal{H}$ be a Hilbert space of analytic functions with multiplier algebra $\mathcal{M}(\mathcal{H})$, and let $\mathcal{M} = \{(f, T_1f, \ldots, T_{n-1}f): f \in \mathcal{D}\}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})^{(n)}$. Here $n \geq 2, \mathcal{D} \subseteq \mathcal{H}$ is a vector-subspace, $T_i: \mathcal{D} \to \mathcal{H}$ are linear transformations that commute with each multiplication operator $M_\varphi \in \mathcal{M}(\mathcal{H})$, and $\mathcal{M}$ is closed in $(\mathcal{H})^{(n)}$.

For an operator S on a Banach space X, let Lat $(S; X)$ be the collection of all its invariant subspaces. We consider the index function on Lat $(S; X)$ and establish various algebraic properties of it. Let $\mathcal{H}$ be a Hilbert space of analytic functions on the unit disc $\mathbb{D}$ with $\|M_z\| \leq 1$, where M_z denotes the operator of multiplication by the identity function on $\mathbb{D}$. Under certain conditions on $\mathcal{H}$ it has been shown by Aleman, Richter and Sundberg that all invariant subspaces have index 1 if and only if $\lim_{k\to\infty} \left\|M_z^k f\right\| \neq 0$ for all $f \in \mathcal{H}, f \not\equiv 0$. We study the fiber dimension of invariant subspaces for a large class of operators. We define a class of invariant subspaces called CF subspaces, which are related to the codimension-one property. We obtain several characterizations of CF subspaces, including one in terms of Samuel multiplicity.

A complete description is obtained for the subspaces of the Hardy space $H^p (p \geq 1)$ that are invariant under the Volterra integral operator. We provide a precise description of the lattice of invariant subspaces of composition operators acting on the classical Hardy space, whose inducing symbol is a parabolic non-automorphism. This is achieved with an explicit isomorphism between the Hardy space and the Sobolev Banach algebra $W^{1,2}[0,\infty)$ that induces a bijection between the lattice of the composition operator and the closed ideals of $W^{1,2}[0,\infty)$. The lattice of closed invariant subspaces of the Volterra operator acting on $L^2(0,1)$ was completely described by Sarason. On the other hand, he explicitly found the lattice of closed invariant subspaces of the shift plus Volterra operator on $L^2(0,1)$.

Table of Contents

Subject		Page
Dedication		I
Acknowledgements		II
Abstract		III
Abstract(Arabic)		IV
Introduction		V
Table of contents		VII
Chapter 1 **Spectral Properties and Invariant Subspaces**		
Section(1.1):	Invariant Subspaces in the Bergman Space	1
Section(1.2):	Bergman Space Having the Codimension two Property	6
Chapter 2 **Spectra and Index of Invariant Subspaces**		
Section(2.1):	Translation Invariant Spaces	13
Section(2.2):	Spaces of Analytic Functions of Several Complex Variables	29
Chapter 3 **Similarity and Reducing Manifolds with Nearly Invariant Subspace**		
Section(3.1):	Unitary Equivalence of Volterra Operators	50
Section(3.2):	Invariant Subspace of the Backward Shift	63
Chapter 4 **Analytic Continuability and Complemented Invariant Subspaces with Linear Graph Transformations**		
Section(4.1):	Bergman Inner Functions	71
Section(4.2):	Complemented Invariant Subspaces in Bergman Spaces	78
Section(4.3):	Spaces of Analytic Functions	90
Chapter 5 **Algebraic Properties and Index of Invariant Subspaces with Fiber Dimension**		
Section(5.1):	The Index of Invariant Subspaces of Operators on Banach Spaces	112
Section(5.2):	Hilbert Spaces of Vector-Valued Analytic Functions	114
Section(5.3):	Invariant Subspaces and Fiber Dimension	135
Chapter 6 **Volterra Invariant Subspaces**		
Section(6.1):	Volterra Invariant Subspaces of H^p	153
Section(6.2):	Parabolic Self-maps in the Hardy Space	184
Section(6.3):	Shift Plus Complex Volterra Operator	191
List of simples		204
References		205