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Reduction Theory with Nuclear and Quasi-States on C^* -Algebras

نظرية الإختزال مع النووية وشبه-الحالات على جبريات- C^*

**A Thesis Submitted in Partial Fulfillment for the Degree of
M.Sc. in Mathematics**

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Dedication

To my Family

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First I would like to thank without limits to our greatest Allah, then I would like to express my appreciation and thanks to my

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Abstract

The theory of direct integral decompositions of both bounded and unbounded operators is further developed. We study Banach and C^* -algebras generated by Toeplitz operators acting on weighted Bergman Spaces over the complex unit ball. We provide examples of ambient nuclear C^* -algebras of non-nuclear C^* -algebras with no proper intermediate C^* -algebras. We characterize the continuous quasi – states on C^* -algebras.

الخلاصة

تم النموالأوسع لنظرية التفكيكات التكاملية المباشرة للمؤثرين معاً المحدود وغير المحدود. درسنا باناخ وجبريات- C^* المولدة بواسطة مؤثرات تبوليتز الفاعلة على فضاءات بارجمان المرجحة فوق كرة الوحدة المركبة. أشرطنا أمثلة لجبريات- C^* النووية المحيطه لجبريات- C^* غير النووية مع جبريات- C^* المتوسطة التامة. شخصنا شبه الحالات المستمرة على جبريات- C^* .

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Chapter 1

Brown Measure for Closed Unbounded Operators

Results about spectral projections, functional calculus and affiliation to von Neumann algebras are shown. For operators belonging to or affiliated to a tracial von Neumann algebra that is a direct integral von Neumann algebra, the Brown measure is shown to be given by the corresponding integral of Brown measure.

Section (1.1): Spectral Projections and Functional Calculus for Bounded Operators

Reduction theory is a way of decomposing von Neumann algebras as direct integrals (a generalization of direct sums) of other von Neumann algebras. It is commonly employed, when the direct integral decomposition is done over the center of the von Neumann algebra, to see that an arbitrary von Neumann algebra is a direct integral of factors. However, the direct integral decomposition can be done over any von Neumann subalgebra of the center.

We show that, tracial von Neumann algebras and certain unbounded operators affiliated to such von Neumann algebras, the Brown spectral distribution measure behaves well with respect to direct integral decompositions. This result finds immediate application that extends results about Schur upper-triangular forms to certain unbounded operators affiliated to finite von Neumann algebras.

We will now describe some of the theory of Brown measure and the Fuglede-Kadison determinant, on which it depends. Given a tracial von Neumann algebra (\mathcal{M}, τ) , by which we mean a von Neumann algebra \mathcal{M} and a normal, faithful, tracial state τ , the Fuglede-Kadison determinant is the map $\Delta = \Delta_\tau: \mathcal{M} \rightarrow [0, \infty)$ defined by

$$\Delta(T) = \exp(\tau(\log |T|)) := \lim_{\epsilon \rightarrow 0^+} \exp(\tau(\log |T| + \epsilon)).$$

Fuglede and Kadison proved that it is multiplicative: $\Delta(AB) = \Delta(A)\Delta(B)$. The Brown measure ν_T was introduced by L.G. Brown. It is a sort of spectral distribution measure for elements $T \in \mathcal{M}$ (and for certain unbounded operators affiliated to \mathcal{M}). It is defined to be the Laplacian (in the sense of distributions in \mathbb{C}) of the function $f(\lambda) = \frac{1}{2\pi} \log \Delta(T - \lambda)$; Brown proved, among other properties, that it is a probability measure whose support is contained in the spectrum of T .

Later, Haagerup and Schultz proved that the Fuglede-Kadison determinant and Brown measure are defined and have nice properties for all closed, densely defined, possibly unbounded operators T affiliated to \mathcal{M} such that $\tau(\log^+ |T|) < \infty$, where $\log^+(x) = \max(\log(x), 0)$. We will use $\exp(\mathcal{L}_1)(\mathcal{M}, \tau)$ for this set. It is easy to see that $\exp(\mathcal{L}_1)(\mathcal{M}, \tau)$ is an \mathcal{M} -bimodule; it is, in fact, a $*$ -algebra containing \mathcal{M} as a $*$ -subalgebra. A characterization of the Brown measure ν_T of $T \in \exp(\mathcal{L}_1)(\mathcal{M}, \tau)$ is as the unique probability measure on \mathbb{C} satisfying

$$\int_{\mathbb{C}} \log^+ |z| d\nu_T(z) < \infty \quad (1)$$

and

$$\int_{\mathbb{C}} \log |z - \lambda| d\nu_T(z) = \log \Delta(T - \lambda) (\lambda \in \mathbb{C}). \quad (2)$$

Brown measure is naturally defined on elements of $\exp(\mathcal{L}^1)$; we will need reduction theory also for unbounded operators in Hilbert space. Nussbaum introduced this theory and developed several aspects of it. We will show and make use of some further results about direct integral decompositions of unbounded operators, for example, about (a) functional

calculus for decomposable unbounded self-adjoint operators, (b) polar decompositions and (c) affiliated operators.

We will recall elements of the reduction theory for von Neumann algebras as expounded by Dixmier and some definitions and results from Nussbaum on reduction theory for unbounded operators. We let ω be a fixed σ -finite positive measure on a standard Borel space Z , namely a Polish space endowed with the Borel σ -algebra.

(A) Direct integrals of Hilbert spaces: A measurable field of Hilbert spaces is a function $\zeta \mapsto \mathcal{H}(\zeta)$, ($\zeta \in Z$), where each $\mathcal{H}(\zeta)$ is a Hilbert space, together with a set S of vector fields (namely, functions $\zeta \mapsto x(\zeta) \in \mathcal{H}(\zeta)$) that are said to be measurable and that satisfy (i) that the function $\zeta \mapsto \langle x(\zeta), y(\zeta) \rangle$ is measurable for all $x, y \in S$ and (ii) if v is a vector field and the function $\zeta \mapsto \langle x(\zeta), v(\zeta) \rangle$ is measurable for each $x \in S$, then $v \in S$.

The direct integral Hilbert space

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(\zeta) d\omega(\zeta)$$

consists of all measurable vector fields $x \in S$ for which the function $\zeta \mapsto \|x(\zeta)\|^2$ is integrable with respect to ω . The inner product on \mathcal{H} is given by

$$\langle x, y \rangle = \int_Z \langle x(\zeta), y(\zeta) \rangle d\omega(\zeta).$$

(B) Fields of Bounded Operators: A field $\zeta \mapsto T(\zeta) \in B(\mathcal{H}(\zeta))$ ($\zeta \in Z$) of bounded operators is said to be measurable if for every measurable vector field $x \in S$ (as in (A)) the field $\zeta \mapsto T(\zeta)x(\zeta)$ is measurable. In this case, the map $\zeta \mapsto \|T(\zeta)\|$ is measurable.

(C) Decomposable and Diagonal Bounded Operators: If T is a measurable field of bounded operators as in (B) and if the map

$$\zeta \mapsto \|T(\zeta)\| \tag{3}$$

is essentially bounded, where $\|\cdot\|$ is the operator norm, then T describes a bounded linear operator, also denoted by T , on the direct integral Hilbert space \mathcal{H} , by $(Tx)(\zeta) = T(\zeta)x(\zeta)$, and we write

$$T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta). \tag{4}$$

The norm of T equals the essential supremum of the map (3). Such operators T on \mathcal{H} are said to be decomposable. The set of decomposable operators, which we will denote \mathcal{E} , is a subalgebra of $B(\mathcal{H})$ and the $*$ -algebra operations have the obvious almost-everywhere-pointwise interpretation. In particular, T is self-adjoint if and only if $T(\zeta)$ is self-adjoint for almost every ζ and $T \geq 0$ if and only if $T(\zeta) \geq 0$ for almost every ζ . The diagonal operators are the decomposable operators T for which each $T(\zeta)$ is a scalar multiple of the identity operator on $\mathcal{H}(\zeta)$.

The algebra of all diagonal operators, which we shall denote \mathcal{D} , is a von Neumann algebra isomorphic to $\mathcal{L}^1(Z, \omega)$, and its commutant is the von Neumann algebra \mathcal{E} of decomposable operators.

(D) Fields of von Neumann algebras: All of the von Neumann algebras considered will be assumed to be countably generated. If \mathcal{A} is a von Neumann algebra in $B(\mathcal{H})$ that is generated by the algebra \mathcal{D} of diagonalizable operators together with a countable set $\{T_i \mid i \geq 1\}$ of decomposable operators, then \mathcal{A} is said to be decomposable. Letting $\mathcal{A}(\zeta)$ be the von Neumann algebra in $B(\mathcal{H}(\zeta))$ generated by $\{T_i(\zeta) \mid i \geq 1\}$, we have that

whenever T is a decomposable operator, then $T \in \mathcal{A}$ if and only if $T(\zeta) \in \mathcal{A}(\zeta)$ for almost every ζ . We write

$$\mathcal{A} = \int_Z^{\oplus} \mathcal{A}(\zeta) d\omega(\zeta).$$

Note that the von Neumann algebra \mathcal{D} of diagonal operators is contained in the center of \mathcal{A} .

(E) Measurable fields of traces: Suppose $\mathcal{A} = \int_Z^{\oplus} \mathcal{A}(\zeta) d\omega(\zeta)$ is a decomposable von Neumann algebra and $\zeta \mapsto \tau_\zeta$ is a field of traces, each τ_ζ being a trace on $\mathcal{A}(\zeta)^+$ taking values in $[0, +\infty]$. The field of traces is said to be measurable if for every $T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta) \in \mathcal{A}$, the function $\zeta \mapsto \tau_\zeta(T(\zeta))$ is measurable. In this case

$$\tau = \int_Z^{\oplus} \tau_\zeta d\omega(\zeta)$$

denotes the trace on \mathcal{A}^+ defined as follows. When $T \in \mathcal{A}^+$, writing T as in (4), we have

$$\tau(T) = \int_Z \tau_\zeta(T(\zeta)) d\omega(\zeta).$$

(F) Direct integral decomposition of a finite von Neumann algebra and trace. If $\mathcal{A} = \int_Z^{\oplus} \mathcal{A}(\zeta) d\omega(\zeta)$ is a decomposable von Neumann algebra and τ is a normal, faithful, tracial state on \mathcal{A} , then there is a measurable field $\zeta \mapsto \tau_\zeta$ of normal, faithful, finite traces τ_ζ on $\mathcal{A}(\zeta)$, so that

$$\tau = \int_Z^{\oplus} \tau_\zeta d\omega(\zeta).$$

After redefining ω , if necessary, we may without loss of generality assume each τ_ζ is a tracial state.

(G) Measurable fields of unbounded operators: We will denote the domain of a closed (possibly unbounded) operator T on a Hilbert space by $\text{dom}(T)$. Let $\zeta \mapsto T(\zeta)$ be a field of closed operators on $\mathcal{H}(\zeta)$. Let $P(\zeta) = \left(P_{ij}(\zeta) \right)_{1 \leq i, j \leq 2} \in M_2(B(\mathcal{H}(\zeta)))$ be the projection onto the graph of $T(\zeta)$. Nussbaum introduced the following notion of measurability: the field of operators is measurable if for all i and j , the field $P_{ij}(\zeta)$ of bounded operators is measurable, in the sense of (B). It shows that in the case of an essentially bounded field of bounded operators, measurability in the above sense is equivalent to measurability as found in (B). The field $\zeta \mapsto T(\zeta)$ is said to be weakly measurable if for every measurable vector field $\zeta \mapsto x(\zeta)$ of vectors such that for all ζ , $x(\zeta) \in \text{dom}(T(\zeta))$, the vector field $\zeta \mapsto T(\zeta)x(\zeta)$ is measurable.

Nussbaum proves that every measurable field $\zeta \mapsto T(\zeta)$ of closed operators is weakly measurable, while the converse statement was shown to be false.

(H) Decomposable unbounded operators: Given a measurable field $\zeta \mapsto T(\zeta)$ of closed operators as in (G) and letting $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(\zeta) d\omega(\zeta)$ be the direct integral Hilbert space, define the operator T to have domain equal to the set of vectors $x \in \mathcal{H}$ defined by square integrable vector fields $\zeta \mapsto x(\zeta)$ such that $x(\zeta) \in \text{dom}(T(\zeta))$ for all ζ and such that the vector field $\zeta \mapsto T(\zeta)x(\zeta)$ is square integrable, and for such an x to have value Tx equal to the vector field

$$(Tx)(\zeta) = T(\zeta)x(\zeta).$$

T is a closed operator. A closed operator that arises in this way from a measurable field of closed operators is said to be decomposable. A closed operator in \mathcal{H} is decomposable if and only if it permutes with all the bounded diagonalizable operators, as described in (C). A closed operator in \mathcal{H} is decomposable if and only if it is affiliated with the von Neumann algebra \mathcal{E} of all bounded decomposable operators.

Suppose $T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta)$ is a decomposable closed operator. Hence

- (a) $T(\zeta)$ is densely defined in $\mathcal{H}(\zeta)$ for almost every ζ if and only if T is densely defined in \mathcal{H} ;
- (b) $T(\zeta)$ is self-adjoint for almost every ζ if and only if T is self-adjoint;

We treat spectral projections and functional calculus of bounded decomposable operators, with respect to a fixed direct integral decomposition of Hilbert space

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(\zeta) d\omega(\zeta).$$

We let $\sigma(\cdot)$ denote the spectrum of an operator.

Lemma (1.1.1)[1]. Suppose

$$X = \int_Z^{\oplus} X(\zeta) d\omega(\zeta)$$

is a bounded, decomposable operator. Then for almost every ζ , we have $\sigma(X(\zeta)) \subseteq \sigma(X)$.

By appeal to the standard $*$ -algebra operations, we have:

Lemma (1.1.2)[1]. Let X be a bounded, decomposable operator. Then X is a normal operator if and only if $X(\zeta)$ is normal for almost all ζ .

We consider now the continuous functional calculus, which is quite straightforward to prove, and must be well known.

Lemma (1.1.3)[1]. Let X be a bounded, normal, decomposable operator. Using Lemmas (1.1.1) and (1.1.2), by redefining $X(\zeta)$ for ζ in a null set, if necessary, we may suppose $X(\zeta)$ is normal and has spectrum contained in $\sigma(X)$ for all ζ . Suppose $f : \sigma(X) \rightarrow \mathbb{C}$ is a continuous function. Then in the continuous functional calculus, we have

$$f(X) = \int_Z^{\oplus} f(X(\zeta)) d\omega(\zeta).$$

Proof. Take a sequence $(g_k)_{k=1}^{\infty}$ of polynomials in z and \bar{z} such that $g_k(z, \bar{z})$ converges uniformly to $f(z)$ for all $z \in \sigma(X)$. Letting $\epsilon_k = \max_{z \in \sigma(X)} |f(z) - g_k(z, \bar{z})|$, we have

$\lim_{k \rightarrow \infty} \epsilon_k = 0$. But $\|f(X) - g_k(X, X^*)\| = \epsilon_k$ and for each ζ , since $\sigma(X(\zeta)) \subseteq \sigma(X)$, we have $\|f(X(\zeta)) - g_k(X(\zeta), X(\zeta)^*)\| \leq \epsilon_k$ and from this we get (see (C)),

$$\left\| \int_Z^{\oplus} f(X(\zeta)) d\omega(\zeta) - \int_Z^{\oplus} g_k(X(\zeta), X(\zeta)^*) d\omega(\zeta) \right\| \leq \epsilon_k.$$

Since the C^* -algebra operations thread through decompositions, we have

$$g_k(X, X^*) = \int_Z^{\oplus} g_k(X(\zeta), X(\zeta)^*) d\omega(\zeta).$$

Taking $k \rightarrow \infty$ finishes the proof.

We next consider spectral projections. For a normal operator X and a Borel subset B of \mathbb{C} , we will denote by $E_X(B)$ the corresponding spectral projection. The following result is a special case.

Proposition (1.1.4)[1]. Suppose $X = \int_Z^{\oplus} X(\zeta) d\omega(\zeta)$ is a bounded, normal, decomposable operator and, as above, assume without loss of generality $X(\zeta)$ is normal and has spectrum contained in $\sigma(X)$ for all ζ . Let B be a Borel subset of \mathbb{C} . Then

$$E_X(B) = \int_Z^{\oplus} E_{X(\zeta)}(B) d\omega(\zeta). \quad (5)$$

Proof. First suppose that B is a nonempty open, bounded rectangle in \mathbb{C} . Let $(f_n)_{n=1}^{\infty}$ be an increasing sequence of continuous functions on \mathbb{C} , each taking values in $[0, 1]$ and vanishing outside of B and such that f_n converges pointwise to 1_B (the characteristic function of B) as $n \rightarrow \infty$. By Lemma (1.1.3), we have

$$f_n(X) = \int_Z^{\oplus} f_n(X(\zeta)) d\omega(\zeta).$$

Since f_n is increasing to 1_B , by the spectral theorem, $f_n(X)$ converges in strong operator topology to $E_X(B)$. Similarly, for every ζ , $f_n(X(\zeta))$ converges in strong operator topology to $E_{X(\zeta)}(B)$, for all ζ . Thus, $f_n(X)$ converges strongly to $\int_Z^{\oplus} E_{X(\zeta)}(B) d\omega(\zeta)$. This yields the equality (5) when B is an open rectangle.

We now show that the set β of Borel sets B with the property (5) is a σ -algebra.

First, if $B \in \beta$, then

$$\begin{aligned} E_X(B^c) &= 1 - E_X(B) = 1 - \int_Z^{\oplus} E_{X(\zeta)}(B) d\omega(\zeta) = \int_Z^{\oplus} (1 - E_{X(\zeta)}(B)) d\omega(\zeta) \\ &= \int_Z^{\oplus} E_{X(\zeta)}(B^c) d\omega(\zeta), \end{aligned}$$

so $B^c \in \beta$. Now let $(B_n)_{n=1}^{\infty}$ be a sequence of sets from β . For any $i, j \in \mathbb{N}$ we have

$$\begin{aligned} E_X(B_i \cup B_j) &= E_X(B_i) + E_X(B_j) - E_X(B_i)E_X(B_j) \\ &= \int_Z^{\oplus} (E_{X(\zeta)}(B_i) + E_{X(\zeta)}(B_j) - E_{X(\zeta)}(B_i)E_{X(\zeta)}(B_j)) d\omega(\zeta) \\ &= \int_Z^{\oplus} E_{X(\zeta)}(B_i \cup B_j) d\omega(\zeta), \end{aligned}$$

So $B_i \cup B_j \in \beta$. Hence β is closed under finite unions. Thus, for every n , we have

$$E_X\left(\bigcup_{i=1}^n B_i\right) = \int_Z^{\oplus} E_{X(\zeta)}\left(\bigcup_{i=1}^n B_i\right) d\omega(\zeta).$$

But $E_X(\bigcup_{i=1}^n B_i)$ converges in strong operator topology to $E_X(\bigcup_{i=1}^{\infty} B_i)$, and for each ζ , $E_{X(\zeta)}(\bigcup_{i=1}^n B_i)$ converges in strong operator topology to $E_{X(\zeta)}(\bigcup_{i=1}^{\infty} B_i)$. We get

$$E_X\left(\bigcup_{i=1}^{\infty} B_i\right) = \int_Z^{\oplus} E_{X(\zeta)}\left(\bigcup_{i=1}^{\infty} B_i\right) d\omega(\zeta).$$

Thus β is a σ -algebra.

Since β contains all of the bounded open rectangles, it is the whole Borel σ -algebra of \mathbb{C} .

From the above result, it is easy to show that an analogue of Lemma (1.1.3) holds for the Borel functional calculus.

Proposition (1.1.5)[1]. Let $X = \int_Z^{\oplus} (X(\zeta)) d\omega(\zeta)$ be a bounded, normal, decomposable operator. Using Lemmas (1.1.1) and (1.1.2), by redefining $X(\zeta)$ for ζ in a null set, if necessary, we may suppose $X(\zeta)$ is normal and has spectrum contained in $\sigma(X)$ for all ζ . Suppose $f : \sigma(X) \rightarrow \mathbb{C}$ is a bounded Borel function. Then taking the Borel functional calculus, we have

$$f(X) = \int_Z^{\oplus} f(X(\zeta)) d\omega(\zeta).$$

Proof. Let $\epsilon > 0$ and let $g = \sum_{j=1}^n a_j 1_{B_j}$ be a Borel measurable simple function such that $\sup_{z \in \sigma(X)} |f(z) - g(z)| < \epsilon$. By Proposition (1.1.4), we have

$$g(X) = \int_Z^{\oplus} g(X(\zeta)) d\omega(\zeta).$$

But $\|g(X) - f(X)\| < \epsilon$. Moreover, for all ζ we have $\|g(X(\zeta)) - f(X(\zeta))\| < \epsilon$, so we get

$$\left\| \int_Z^{\oplus} g(X(\zeta)) d\omega(\zeta) - \int_Z^{\oplus} f(X(\zeta)) d\omega(\zeta) \right\| \leq \epsilon.$$

This yields

$$\left\| f(X) - \int_Z^{\oplus} f(X(\zeta)) d\omega(\zeta) \right\| < 2\epsilon.$$

Letting $\epsilon \rightarrow 0$ finishes the proof.

Section (1.2): Affiliation for Unbounded Operators and Tracial von Neumann Algebras with Brown Measure

We show a result about functional calculus for decomposable self-adjoint, possibly unbounded operators, as well as a result about the polar decomposition of decomposable unbounded operators and one about affiliation to decomposable von Neumann algebras.

Lemma (1.2.1)[1]. Let $T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta)$ be a closed, (possibly unbounded), self-adjoint, decomposable operator. Then the Cayley transform $(T + i)(T - i)^{-1}$ of T is equal to the direct integral

$$\int_Z^{\oplus} (T(\zeta) + i)(T(\zeta) - i)^{-1} d\omega(\zeta) \quad (6)$$

of Cayley transforms.

Proof. Note that the operator (6) is unitary. By evaluating at measurable vector fields $\zeta \mapsto x(t)$ belonging to $\text{dom}(T)$, we have

$$(T - i)x = \int_Z^{\oplus} (T(\zeta) - i)x(\zeta) d\omega(\zeta)$$

and

$$\begin{aligned} & \left(\int_Z^{\oplus} (T(\zeta) + i)(T(\zeta) - i)^{-1} d\omega(\zeta) \right) (T - i)x \\ &= \int_Z^{\oplus} (T(\zeta) + i)x(\zeta) d\omega(\zeta) = (T + i)x. \end{aligned}$$

Thus, the two unitary operators $(T + i)(T - i)^{-1}$. and

$$\int_Z^{\oplus} (T(\zeta) + i)(T(\zeta) - i)^{-1} d\omega(\zeta),$$

agree on a dense subset of \mathcal{H} , so they must be equal, as required.

Now using the Cayley transform to go from unbounded self-adjoint operators to unitary operators, we easily get the following analogues of Propositions (1.1.4) and (1.1.5). Here, for a Borel set B , we denote the corresponding spectral projection of also an unbounded self-adjoint operator T by $E_T(B)$.

Proposition (1.2.2)[1]. Let $T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta)$ be a closed, (possibly unbounded), self-adjoint, decomposable operator. For every Borel subset $B \subset \mathbb{R}$, we have

$$E_T(B) = \int_Z^{\oplus} E_{T(\zeta)}(B) d\omega(\zeta). \quad (7)$$

Moreover, for every (possibly unbounded) Borel measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$f(T) = \int_Z^{\oplus} f(T(\zeta)) d\omega(\zeta). \quad (8)$$

Proof. Consider the map $h : \mathbb{R} \rightarrow \mathbb{T}$ given by $h(t) = \frac{t+i}{t-i}$. Let $U = (T + i)(T - i)^{-1}$ be the Cayley transform of T and let $U(\zeta) = (T(\zeta) + i)(T(\zeta) - i)^{-1}$. Then for all ζ we have

$$E_T(B) = E_U(h(B)) \quad \text{and} \quad E_{T(\zeta)}(B) = E_{U(\zeta)}(h(B)).$$

Thus, applying Proposition (1.1.4) to U and $h(B)$ yields (7). Now, by approximating f in norm with simple Borel measurable functions, as was done for bounded operators in the proof of Proposition (1.1.5), we obtain (8).

Nussbaum proved that given a densely defined, decomposable, (possibly unbounded) closed operator

$$T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta),$$

its absolute value is the direct integral of absolute values:

$$|T| = \int_Z^{\oplus} |T(\zeta)| d\omega(\zeta). \quad (9)$$

Proposition (1.2.3)[1]. With T as above, let $T = V|T|$ be the polar decomposition of T . Then the polar part V is decomposable and we have

$$V = \int_Z^{\oplus} V(\zeta) d\omega(\zeta), \quad (10)$$

where $V(\zeta)$ is the polar part in the polar decomposition

$$T(\zeta) = V(\zeta)|T(\zeta)| \text{ of } T(\zeta).$$

Proof. Let W be the bounded, decomposable operator defined by the right-hand-side of (10). Then W is a partial isometry. By evaluating on vector fields $x \in \mathcal{H}$ in $\text{dom}(T) = \text{dom}(|T|)$, and using (9), we find

$$|T|x = \int_Z^{\oplus} |T(\zeta)|x(\zeta) d\omega(\zeta)$$

and

$$W|T|x = \int_Z^{\oplus} V(\zeta)|T(\zeta)|x(\zeta) d\omega(\zeta) = \int_Z^{\oplus} T(\zeta)x(\zeta) d\omega(\zeta) = Tx.$$

Thus we have

$$W|T| = T. \quad (11)$$

Moreover, $V(\zeta)^*V(\zeta)$ is the range projection $E_{|T(\zeta)|}((0, \infty))$ of $|T(\zeta)|$. Thus,

$$W^*W = \int_Z^{\oplus} V(\zeta)^*V(\zeta) d\omega(\zeta) = \int_Z^{\oplus} E_{|T(\zeta)|}((0, \infty)) d\omega(\zeta) = E_{|T|}((0, \infty)),$$

where the last equality is provided by Proposition (1.2.2). This, together with (11), implies that $T = W|T|$ is the polar decomposition of T .

Recall that for a closed, densely defined operator T in \mathcal{H} and a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$, we say that T is affiliated to \mathcal{M} if, letting $T = V|T|$ denote the polar decomposition of T , we have $V \in \mathcal{M}$ and $E_{|T|}(B) \in \mathcal{M}$ for every Borel subset B of \mathbb{R} .

The following is the analogue for unbounded operators of the fundamental fact about decompositions of von Neumann algebras stated in (D).

Proposition (1.2.4)[1]. Suppose

$$\mathcal{M} = \int_Z^{\oplus} \mathcal{M}(\zeta) d\omega(\zeta)$$

is decomposable von Neumann algebra (see (D)). Let T be a closed (possibly unbounded) operator in \mathcal{H} . Then T is affiliated to \mathcal{M} if and only if (a) T is decomposable and (b) writing out the decomposition as

$$T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta), \quad (12)$$

we have that $T(\zeta)$ is affiliated to $\mathcal{M}(\zeta)$ for almost every ζ .

Proof. First we show \Leftarrow . Suppose T is decomposable and is written as in (12). Let $T = V|T|$ and $T(\zeta) = V(\zeta)|T(\zeta)|$ be the polar decompositions. For almost every ζ we have $V(\zeta) \in \mathcal{M}(\zeta)$; using Proposition (1.2.3), we have $V \in \mathcal{M}$. Similarly, for every Borel subset $B \subseteq \mathbb{R}$, we have $E_{|T(\zeta)|}(B) \in \mathcal{M}(\zeta)$ for almost every ζ , so using Proposition (1.2.2), we find $E_{|T|}(B) \in \mathcal{M}$. Thus, T is affiliated to \mathcal{M} .

To show \Rightarrow , we suppose T is affiliated to \mathcal{M} . Let $T = V|T|$ be the polar decomposition of T . Since $V \in \mathcal{M}$ and all spectral projections $E_{|T|}(B)$ are in \mathcal{M} , they all commute with all the diagonalizable operators; from this, we easily see that T permutes with all diagonalizable operators. By Nussbaum, T is decomposable; we write it as in (12). Let $T(\zeta) = V(\zeta)|T(\zeta)|$ be the polar decomposition. Since $V \in \mathcal{M}$, using Proposition (1.2.3) we get $V(\zeta) \in \mathcal{M}(\zeta)$ for almost every ζ .

Similarly, but using Proposition (1.2.2), for every Borel set B , since $E_{|T|}(B) \in \mathcal{M}$, there is a null set N_B such that for all $\zeta \notin N_B$, we have $E_{|T(\zeta)|}(B) \in \mathcal{M}(\zeta)$. Let N be the union of the sets N_B as B ranges over the open intervals with rational endpoints in \mathbb{R} .

Then N is a null set and for all $\zeta \notin N$ we have $E_{|T(\zeta)|}((a, b)) \in \mathcal{M}(\zeta)$ for all rational numbers $a < b$. From this, we deduce $E_{|T(\zeta)|}(B) \in \mathcal{M}(\zeta)$ for all Borel subsets $B \subseteq \mathbb{R}$.

Thus, we have that $T(\zeta)$ is affiliated to $\mathcal{M}(\zeta)$ for almost every ζ .

We will specialize to the case of operators in or affiliated to tracial von Neumann algebras, by which we mean, pairs (\mathcal{M}, τ) consisting of a von Neumann algebra \mathcal{M} and a fixed normal, faithful, tracial state τ on it. Recall that, given such a pair, we let $\exp(\mathcal{L}^1)(\mathcal{M}, \tau)$ denote the bimodule of closed operators T affiliated to \mathcal{M} such that $\tau(\log^+(|T|)) < \infty$.

Here is a technical lemma that we will need later; it is convenient to prove it here.

Lemma (1.2.5)[1]. Let $T \in \exp(\mathcal{L}_1)(\mathcal{M}, \tau)$. Then the mapping $\lambda \mapsto \Delta_\tau(|T - \lambda|^2 + 1)(\lambda \in \mathbb{C})$ is continuous.

Proof. By shifting T , it suffices to prove that our mapping is continuous at 0. To see this, note that

$$\begin{aligned} \Delta(|T - \lambda|^2 + 1) &= \Delta(|T|^2 + 1) \Delta\left((1 + |T|^2)^{-\frac{1}{2}}(|T - \lambda|^2 + 1)(1 + |T|^2)^{-\frac{1}{2}}\right) \\ &= \Delta(|T|^2 + 1) \Delta\left(1 + (1 + |T|^2)^{-\frac{1}{2}}(|T - \lambda|^2 - |T|^2)(1 + |T|^2)^{-\frac{1}{2}}\right). \end{aligned}$$

It will, thus, suffice to show

$$\lim_{\lambda \rightarrow 0} \Delta\left(1 + (1 + |T|^2)^{-\frac{1}{2}}(|T - \lambda|^2 - |T|^2)(1 + |T|^2)^{-\frac{1}{2}}\right) = 1. \quad (13)$$

It is immediate that

$$|T - \lambda|^2 - |T|^2 = |\lambda|^2 - \lambda T^* - \bar{\lambda} T = |\lambda|^2 - \lambda |T| U^* - \bar{\lambda} U |T|,$$

where $T = U|T|$ is the polar decomposition. Thus,

$$\begin{aligned} (1 + |T|^2)^{-\frac{1}{2}}(|T - \lambda|^2 - |T|^2)(1 + |T|^2)^{-\frac{1}{2}} &= \\ &= |\lambda|^2 (1 + |T|^2)^{-1} - \lambda \left(\frac{|T|}{(1 + |T|^2)^{\frac{1}{2}}} \right) U^* \left(\frac{1}{(1 + |T|^2)^{\frac{1}{2}}} \right) \end{aligned}$$

$$-\bar{\lambda} \left(\frac{1}{(1 + |T|^2)^{\frac{1}{2}}} \right) U \left(\frac{|T|}{(1 + |T|^2)^{\frac{1}{2}}} \right).$$

Thus, we have the estimate of operator norm

$$\left\| (1 + |T|^2)^{-\frac{1}{2}} (|T - \lambda|^2 - |T|^2) (1 + |T|^2)^{-\frac{1}{2}} \right\| \leq 2|\lambda| + |\lambda|^2.$$

So when $|\lambda| \leq \frac{1}{3}$, we have

$$\begin{aligned} \log(1 - 2|\lambda| - |\lambda|^2) &\leq \log \Delta \left(1 + (1 + |T|^2)^{-\frac{1}{2}} (|T - \lambda|^2 - |T|^2) (1 + |T|^2)^{-\frac{1}{2}} \right) \\ &\leq \log(1 + 2|\lambda| + |\lambda|^2), \end{aligned}$$

which proves (13). This concludes the proof.

We suppose $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra equipped with a normal, faithful tracial state τ and that $\mathcal{M} \subseteq \mathcal{E}$ consists of decomposable operators. Using Dixmier's reduction theory (described in (F)), and by modifying the measure ω to be a probability measure, we may write

$$\mathcal{M} = \int_Z^{\oplus} \mathcal{M}(\zeta) d\omega(\zeta), \quad \text{and} \quad \tau = \int_Z^{\oplus} \tau_{\zeta} d\omega(\zeta),$$

for tracial von Neumann algebras $(\mathcal{M}(\zeta), \tau_{\zeta})$, with $\mathcal{M}(\zeta) \subseteq B(\mathcal{H}(\zeta))$. By Proposition (1.2.4) if T is affiliated to \mathcal{M} , then T is decomposable and may be written

$$T = \int_Z^{\oplus} T(\zeta) d\omega(\zeta), \tag{14}$$

with $T(\zeta)$ affiliated to $\mathcal{M}(\zeta)$ for almost every ζ .

For an element $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$, we let ν_T denote the Brown measure of T . For any self-adjoint, closed operator T affiliated to \mathcal{M} , we let μ_T denote the distribution of T , namely, τ composed with spectral measure of T . In fact, when $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$ is self-adjoint, we have $\nu_T = \mu_T$ (this follows immediately from the characterization provided by Equations (1) and (2)) so there would be no conflict in using the same notation for both; but for clarity of meaning, we will distinguish them.

Proposition (1.2.2) yields the following formula for spectral distributions of self-adjoint (possibly unbounded) operators.

Proposition (1.2.6)[1]. Let T be self-adjoint and affiliated to \mathcal{M} . Then for every Borel subset B of \mathbb{R} , the function $\zeta \mapsto \mu_{T(\zeta)}(B)$ is measurable and

$$\mu_T(B) = \int_Z^{\oplus} \mu_{T(\zeta)}(B) d\omega(\zeta).$$

We let $\mathcal{L}^1(\mathcal{M}, \tau)$ denote the set of all closed operators affiliated to \mathcal{M} such that $\tau(|T|) < \infty$.

Lemma (1.2.7)[1]. Suppose $T \in \mathcal{L}^1(\mathcal{M}, \tau)$ and $T \geq 0$; use the decomposition (14). Then $T(\zeta) \in \mathcal{L}^1(\mathcal{M}(\zeta), \tau_{\zeta})$ for almost every ζ and

$$\tau(T) = \int_Z^{\oplus} \tau_{\zeta}(T(\zeta)) d\omega(\zeta). \tag{15}$$

Proof. We have $T(\zeta) \geq 0$ for almost every ζ . Since the decompositions of T and τ are measurable, the function $\zeta \mapsto \tau_{\zeta}(T(\zeta))$ is measurable. Let $(f_n)_{n=1}^{\infty}$ be an increasing

sequence of simple functions, each having finitely many values, that converges point-wise to the identity function $t \mapsto t$ on $[0, \infty)$. Then $\tau(f_n(T))$ is increasing in n and converges to $\tau(T)$ while for every ζ such that $T(\zeta) \geq 0$, the sequence $\tau_\zeta(f_n(T(\zeta)))$ is increasing in n and converges to $\tau_\zeta(T(\zeta))$. Now fixing n and writing $f_n = \sum_{k=1}^m a_k 1_{B_k}$ for some $a_k \geq 0$ and some Borel sets B_k , using Proposition (1.2.6), we find

$$\tau(f_n(T)) = \sum_k a_k \mu_T(B_k) = \sum_k a_k \int_Z \mu_{T(\zeta)}(B_k) d\omega(\zeta) = \int_Z \tau_\zeta(f_n(T(\zeta))) d\omega(\zeta).$$

Letting $n \rightarrow \infty$, the Monotone Convergence Theorem implies the equality (15). This, in turn, implies $\tau_\zeta(T(\zeta)) < \infty$ for almost every ζ .

Now we turn to the $\exp(\mathcal{L}^1)$ class and the Fuglede–Kadison determinant.

Lemma (1.2.8)[1]. Let $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$ and use the decomposition (14). Then $T(\zeta) \in \exp(\mathcal{L}^1)(\mathcal{M}(\zeta), \tau_\zeta)$ for almost every ζ . Moreover, we have

$$\tau(\log^+(|T|)) = \int_Z^\oplus \tau_\zeta(\log^+(|T(\zeta)|)) d\omega(\zeta) \quad (16)$$

$$\log \Delta_\tau(T) = \int_Z \log \Delta_{\tau_\zeta}(T(\zeta)) d\omega(\zeta). \quad (17)$$

Proof. Equation (9) — We may without loss of generality assume $T \geq 0$, which entails $T(\zeta) \geq 0$ for almost every ζ . Now using Proposition (1.2.2), we get

$$\log^+(T) = \int_Z^\oplus \log^+(T(\zeta)) d\omega(\zeta).$$

Since $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$, we have $\log^+(T) \in \mathcal{L}^1(\mathcal{M}, \tau)$. Now Lemma (1.2.7) yields (16) and we deduce $\log^+ T(\zeta) \in \mathcal{L}^1(\mathcal{M}(\zeta), \tau_\zeta)$, namely, $T(\zeta) \in \exp(\mathcal{L}^1)(\mathcal{M}(\zeta), \tau_\zeta)$, for almost every ζ .

Now we show (17). Let $\epsilon > 0$. Using the function $f_\epsilon(t) = \log(t + \epsilon)$ ($t \geq 0$) and using Proposition (1.2.2) to apply the functional calculus to T , we get

$$\log(T + \epsilon) = \int_Z^\oplus \log(T(\zeta) + \epsilon) d\omega(\zeta). \quad (18)$$

Now Lemma (1.2.7) applies (if we first add $-\log \epsilon$ to both sides of (18) to make the operators positive) and we have

$$\tau(\log(T + \epsilon)) = \int_Z \tau_\zeta(\log(T(\zeta) + \epsilon)) d\omega(\zeta).$$

Letting $\epsilon \rightarrow 0$ and using the Monotone Convergence Theorem, we get

$$\log \Delta_\tau(T) = \tau(\log(T)) = \int_Z \tau_\zeta(\log(T(\zeta))) d\omega(\zeta) = \int_Z \log \Delta_{\tau_\zeta}(T(\zeta)) d\omega(\zeta),$$

as required.

Recall that, for $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$, we let v_T denote the Brown measure of T .

Lemma (1.2.9)[1]. Let $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$ and use the decomposition (14). Then for every Borel subset $B \subseteq \mathbb{C}$ the mapping $\zeta \mapsto v_{T(\zeta)}(B)$ is measurable.

Proof. By Lemma (1.2.8), $T(\zeta) \in \exp(\mathcal{L}^1)(\mathcal{M}(\zeta), \tau_\zeta)$ for almost all ζ , and we will confine ourselves to such ζ . It will suffice to prove measurability when B is an open, bounded rectangle in \mathbb{C} , for the collection of such sets generates the Borel σ -algebra. Fix a sequence $\{f_n\}_{n \geq 0}$ of Schwartz functions having support in B and increasing pointwise to the characteristic function of B . Then by the Monotone Convergence Theorem, we have

$$v_{T(\zeta)}(B) = \lim_{n \rightarrow \infty} \int_{\mathbb{C}} f_n(\lambda) dv_{T(\zeta)}(\lambda).$$

By definition of the Brown measure, we have

$$\int_{\mathbb{C}} f_n(\lambda) dv_{T(\zeta)}(\lambda) = \frac{1}{2\pi} \int_{\mathbb{C}} \tau_\zeta(\log(|T(\zeta) - \lambda|)) \nabla^2 f_n(\lambda) d\lambda,$$

where $d\lambda$ means Lebesgue measure on \mathbb{C} . Note that $\tau_\zeta(\log(|T(\zeta) - \lambda|))$ is bounded above for λ in compact subsets of \mathbb{C} . Fixing n for the moment and writing $\nabla^2 f_n(\lambda) = h_1 - h_2$, where h_1 and h_2 are positive Schwartz functions, it follows that both of the integrals

$$\int_{\mathbb{C}} \tau_\zeta(\log(|T(\zeta) - \lambda|)) h_1(\lambda) d\lambda \quad \text{and} \quad \int_{\mathbb{C}} \tau_\zeta(\log(|T(\zeta) - \lambda|)) h_2(\lambda) d\lambda$$

are finite. It follows from the Monotone Convergence Theorem that

$$\begin{aligned} \int_{\mathbb{C}} \tau_\zeta(\log(|T(\zeta) - \lambda|)) h_1(\lambda) d\lambda &= \frac{1}{2} \lim_{m \rightarrow \infty} \int_{\mathbb{C}} \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) h_1(\lambda) d\lambda, \\ \int_{\mathbb{C}} \tau_\zeta(\log(|T(\zeta) - \lambda|)) h_2(\lambda) d\lambda &= \frac{1}{2} \lim_{m \rightarrow \infty} \int_{\mathbb{C}} \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) h_2(\lambda) d\lambda. \end{aligned}$$

Thus, we have

$$\int_{\mathbb{C}} \tau_\zeta(\log(|T(\zeta) - \lambda|)) \nabla^2 f_n(\lambda) d\lambda = \frac{1}{2} \lim_{m \rightarrow \infty} \int_{\mathbb{C}} \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) \nabla^2 f_n(\lambda) d\lambda$$

and, since each $\nabla^2 f_n$ vanishes outside of the rectangle B ,

$$v_{T(\zeta)}(B) = \frac{1}{4\pi} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_B \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) \nabla^2 f_n(\lambda) d\lambda.$$

By Lemma (1.2.5), the mapping

$$\lambda \mapsto \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) \nabla^2 f_n(\lambda)$$

is continuous and, therefore, is Riemann integrable over B . Thus,

$$\begin{aligned} \int_B \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) \nabla^2 f_n(\lambda) d\lambda \\ = \lim_{k \rightarrow \infty} \frac{1}{k^2} \sum_{\lambda \in \frac{1}{k}(\mathbb{Z} + i\mathbb{Z})} \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) \nabla^2 f_n(\lambda), \end{aligned}$$

where the sum is actually finite. Thus,

$$v_{T(\zeta)}(B) = \frac{1}{4\pi} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k^2} \sum_{\lambda \in \frac{1}{k}(\mathbb{Z} + i\mathbb{Z})} \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right) \nabla^2 f_n(\lambda).$$

Because the decompositions of T and of τ are measurable, for each fixed λ the mapping

$$\zeta \rightarrow \tau_\zeta \left(\log \left(|T(\zeta) - \lambda|^2 + \frac{1}{m^2} \right) \right)$$

is measurable. Since the pointwise limit of the sequence of measurable functions is again a measurable function, the lemma is proved.

Here is the main theorem about decomposition of Brown measure.

Theorem (1.2.10)[1]. Let $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$ and write

$$T = \int_{\mathbb{Z}}^{\oplus} T(\zeta) d\omega(\zeta).$$

Then the Brown measure ν_T of T is given by

$$\nu_T(B) = \int_{\mathbb{Z}} \nu_{T(\zeta)}(B) d\omega(\zeta) \quad (19)$$

for every Borel subset $B \subseteq \mathbb{C}$.

Proof. By Lemma (1.2.9), the right-hand-side of (19) defines a probability measure on \mathbb{C} , which we will denote by the symbol ρ . We will show that ρ satisfies

$$\int_{\mathbb{C}} \log^+ |z| d\rho(z) < \infty \quad (20)$$

$$\int_{\mathbb{C}} \log |z - \lambda| d\rho(z) = \log \Delta_{\tau}(T - \lambda) \quad (\lambda \in \mathbb{C}). \quad (21)$$

From the uniqueness property of Brown measure expressed with Equations (1) and (2), this will imply $\rho = \nu_T$.

To prove (20), let f_n be an increasing sequence of simple functions on \mathbb{C} , each taking only finitely many values, that converges pointwise to the function $w \mapsto \log^+(w)$. For each n , we have

$$\int_{\mathbb{C}} f_n(w) d\rho(w) = \int_{\mathbb{Z}} \int_{\mathbb{C}} f_n(w) d\nu_{T(\zeta)}(w) d\omega(\zeta). \quad (22)$$

Applying the Monotone Convergence Theorem, we get

$$\int_{\mathbb{C}} \log^+ |w| d\rho(w) = \int_{\mathbb{Z}} \int_{\mathbb{C}} \log^+(w) d\nu_{T(\zeta)}(w) d\omega(\zeta).$$

For each ζ , we have

$$\int_{\mathbb{C}} \log^+(|w|) d\nu_{T(\zeta)}(w) \leq \tau_{\zeta}(\log^+(|T(\zeta)|)).$$

Since $T \in \exp(\mathcal{L}^1)(\mathcal{M}, \tau)$, using Lemma (1.2.8), we have

$$\int_{\mathbb{Z}} \tau_{\zeta}(\log^+(|T(\zeta)|)) d\omega(\zeta) < \infty.$$

This implies (20).

Now fix $\lambda \in \mathbb{C}$ and $\epsilon > 0$ and let $(f_n)_{n=1}^{\infty}$ be an increasing sequence of simple Borel measurable functions on \mathbb{C} , each taking only finitely many values, that converges pointwise to the function $w \mapsto \log(|w - \lambda| + \epsilon)$. Again we have (22). Using the Monotone Convergence Theorem and taking $n \rightarrow \infty$ we get

$$\int_{\mathbb{C}} \log(|w - \lambda| + \epsilon) d\rho(w) = \int_{\mathbb{Z}} \int_{\mathbb{C}} \log(|w - \lambda| + \epsilon) d\nu_{T(\zeta)}(w) d\omega(\zeta).$$

Using (20), we see that the left-hand-side above is not $+\infty$. Thus, letting $\epsilon > 0$ and using the Monotone Convergence Theorem, we get

$$\begin{aligned}
\int_{\mathbb{C}} \log(|w - \lambda| + \epsilon) \, d\rho(w) &= \int_Z \int_{\mathbb{C}} \log(|w - \lambda| + \epsilon) \, dv_{T(\zeta)}(w) \, d\omega(\zeta) \\
&= \int_Z \log \Delta_{\tau_{\zeta}}(T(\zeta) - \lambda) \, d\omega(\zeta).
\end{aligned}$$

From (17) of Lemma (1.2.8), we get (21).

Definition (1.2.11)[5]. [The Fuglede-kadison determinant]

We denote by \mathcal{M}^{Δ} the set of operators $T \in \bar{\mathcal{M}}$ fulfilling the condition:

$$\tau(\log^+ |T|) = \int_0^{\infty} \log^+(t) d\mathcal{M}_{|T|}(t) < \infty$$

For $T \in \mathcal{M}^{\Delta}$, the integral

$$\int_0^{\infty} \log t \, d\mathcal{M}_{|T|}(t) \in \mathbb{R} \cup \{-\infty\}.$$

is well-defined, and we define the Fuglede-kadison determinant of T ,

$$\Delta(T) \in [0, \infty), \text{ by:}$$

$$\Delta(T) = \exp \left(\int_0^{\infty} \log t \, d\mathcal{M}_{|T|}(t) \right).$$

Chapter 2

Toeplitz Operators and their Representations

We discuss various examples. In the case of $S = C(\overline{\mathbb{D}})$ and $S = C(\overline{\mathbb{D}}) \otimes L_\infty(0, 1)$ we characterize all irreducible representations of the resulting Toeplitz operator C^* -algebras. Their Calkin algebras are described and index formulas are provided.

Section (2.1): Bergman Space Representation and Action of Toeplitz Operators with Commutative Algebras Generated by Toeplitz Operators

In the study of Toeplitz operators T_a consists in selecting symbol subclasses S of L_∞ so that the properties of T_a with $a \in S$ and of the algebra generated by them admit a reasonable description. To study an algebra generated by Toeplitz operators (rather than just Toeplitz operators themselves) lies, first, in a possibility to apply more tools, in particular those coming from the algebraic toolbox. Secondly, the results obtained are applicable not only for generating Toeplitz operators but for all elements of the algebra.

A fundamental result due to Coburn, describes the structure of the C^* -algebra generated by Toeplitz operators with $C(\overline{\mathbb{B}^n})$ -symbols. This work initiated an extensive study of algebras generated by Toeplitz operators with symbols from certain predefined classes. The majority of the results obtained deal with Toeplitz operators that act on the Bergman space on the unit disk. The multidimensional setting, even the case of the unit ball, is more difficult as, beyond the class of continuous symbols, the symbol-functions may have more sophisticated behavior than for the one-dimensional case of the unit disk.

We study algebras generated by Toeplitz operators which act on weighted Bergman spaces over the complex two-dimensional unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. Here the dimension $n = 2$ of the underlying domain is minimal such that the proposed approach is meaningful.

Discussing this lowest dimensional case permits us to present the main ideas in a more simple and transparent form. However, a similar approach can be applied in the higher dimensional framework in which some new features are present.

It has been observed that Toeplitz operators, with symbols invariant under the action of the (maximal Abelian) subgroup \mathbb{T}^2 of all biholomorphisms of \mathbb{B}^2 , generate a commutative C^* -algebra on any weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^2)$. In this case there exists a unitary operator R_λ that maps $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ onto the one-sided sequence space $\ell_2 = \ell_2(\mathbb{Z}_+)$ (the direct sum of one-dimensional Hilbert spaces \mathbb{C}). For Toeplitz operators T_a^λ with bounded and group invariant symbols $a = a(|z_1|, |z_2|)$, these one-dimensional spaces \mathbb{C} are invariant for the operator $R_\lambda T_a^\lambda R_\lambda^*$. In particular, $R_\lambda T_a^\lambda R_\lambda^*$ acts on each of these spaces as multiplication by a constant operator, and the commutativity result trivially follows.

We consider symbols that are invariant under the action of the subgroup $\{1\} \times \mathbb{T} \cong \mathbb{T}$ of \mathbb{T}^2 . On the one hand replacing \mathbb{T}^2 by a strict subgroup enlarges the class of admissible symbols but on the other hand it destroys the commutativity property of Toeplitz operators. At the same time, there still exists the unitary operator U , of the form (7) mapping $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ onto the direct sum of the weighted Bergman spaces $\mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D})$, $\alpha_2 \in \mathbb{Z}_+$ of holomorphic L_2 -functions on the unit disk \mathbb{D} .

We consider various subclasses of symbols of the form

$$a(z_1), \quad b\left(\frac{|z_2|}{\sqrt{1-|z_1|^2}}\right), \quad \text{and} \quad a(z_1)b\left(\frac{|z_2|}{\sqrt{1-|z_1|^2}}\right), \quad (1)$$

where $a \in L_\infty(\mathbb{D})$ and $b \in L_\infty(0, 1)$. Note that these functions are invariant under the action of the group $\{1\} \times \mathbb{T}$. Again each space $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$ is invariant for the operators $UT_c^\lambda U^*$ where T_c^λ is the Toeplitz operator on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ with a symbol c of one of the three types in (1). Moreover precisely, the restrictions act as follows:

$$UT_a^\lambda U^*|_{\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})} = T_a^{\alpha_2+\lambda+1} \text{ and } UT_b^\lambda U^*|_{\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})} = \gamma_b^\lambda I, \text{ where } \gamma_b^\lambda \in \mathbb{C}.$$

Here $T_a^{\alpha_2+\lambda+1}$ denotes the Toeplitz operator with symbol a acting on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$.

In summary, the invariance of the symbols under a certain subgroup of biholomorphisms of the unit ball \mathbb{B}^2 permits us to diminish the dimension of the problem: the study of the algebra generated by Toeplitz operators on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ with such invariant symbols reduces to the study of the algebras generated by Toeplitz operators on a countable number of differently weighted Bergman spaces $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$, $\alpha_2 \in \mathbb{Z}_+$ over the unit disk \mathbb{D} . Known results on C^* -algebras generated by Toeplitz operators acting on Bergman spaces over the unit disk \mathbb{D} can be successfully applied to describe algebras generated by Toeplitz operators on the two-dimensional ball \mathbb{B}^2 .

A unitary operator U between $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ and a countable sum of differently weighted Bergman spaces over \mathbb{D} is defined. Given a $\{1\} \times \mathbb{T}$ -invariant symbol $c \in L_\infty(\mathbb{B}^2)$, each space in this orthogonal decomposition is invariant under the action of $UT_c^\lambda U^*$. Moreover, this action is described in Corollaries (2.1.3) and (2.1.5).

We devoted to the description of the commutative algebras, both C^* and Banach, that are generated by various subclasses of the above invariant symbols. Among other cases we show that the commutative C^* -algebras in $\mathcal{L}(A_\mu^2(\mathbb{D}))$ generated by Toeplitz operators induce commutative subalgebras in $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^2))$ of the corresponding type (quasi-elliptic, quasi-parabolic, quasi-hyperbolic).

We study non-commutative Toeplitz C^* -algebras in $\mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^2))$ that originate from the construction. The first algebra corresponds to the classical Toeplitz C^* -algebra and is generated by operators $\{T_c^\lambda : c(z_1, z_2) = a(z_1) \text{ where } a \in C(\overline{\mathbb{D}})\}$. The second C^* -algebra under consideration is larger: it is generated by elements from the first algebra and Toeplitz operators with the componentwise radial symbols b described in (1).

We give a complete list of irreducible representations of both algebras. Different from the case of the classical Toeplitz algebra over the unit disk or ball, an additional series of irreducible representations arise via a quantization effect. This effect is based on the appearance of weighted Bergman spaces with weight parameter tending to infinity in the

orthogonal sum decomposition. An explicit expression of these representations involves limits of the Berezin transforms for operators on each Bergman space $\mathcal{A}_\mu^2(\mathbb{D})$ (as a weight parameter tends to infinity) of the above direct sum decomposition of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$.

Finally, for both C^* -algebras we give explicit direct sum expressions for their elements and characterize their Fredholmness. It is shown that Fredholm operators in the first algebra always have index zero; in case of the second algebra we provide an index formula.

We simultaneously use differently weighted Bergman spaces both on the unit ball \mathbb{B}^2 and the unit disk \mathbb{D} , as well as various objects (functions, operators, etc) that correspond to these two different settings (unit ball \mathbb{B}^2 and unit disk \mathbb{D}). To distinguish them, we will write in **bold** the objects that correspond to the unit ball setting.

Recall that, given a weight parameter $\lambda \in (-1, \infty)$, the weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ is the closed subspace of $L_2(\mathbb{B}^2, dv_\lambda)$ which consists of functions that are complex analytic in \mathbb{B}^2 . Here the standard weighted measure dv_λ is given by

$$dv_\lambda(z) = \frac{\Gamma(\lambda + 3)}{\pi^2 \Gamma(\lambda + 1)} (1 - |z|^2)^\lambda dv(z),$$

where $z = (z_1, z_2) \in \mathbb{B}^2$ and dv denotes the Lebesgue volume form on $\mathbb{C}^2 \cong \mathbb{R}^4$. Put $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ and recall that the normalized monomials

$$e_\lambda(\alpha_1, \alpha_2) := \frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\lambda + 3)} z_1^{\alpha_1} z_2^{\alpha_2}, (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$$

form an orthonormal basis in $\mathcal{A}_\lambda^2(\mathbb{B}^2)$.

We denote by B_λ the orthogonal (Bergman) projection from $L_2(\mathbb{B}^2, dv_\lambda)$ onto the Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^2)$. The Toeplitz operator T_a^λ with a symbol $a \in L_\infty(\mathbb{B}^2)$ acts on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ by

$$T_a^\lambda f = B_\lambda(af).$$

In what follows we will consider as well the weighted Bergman spaces $\mathcal{A}_\mu^2(\mathbb{D})$ on the unit disk and Toeplitz operators acting on them. Recall that, given a weight parameter $\mu \in (-1, \infty)$, $\mathcal{A}_\mu^2(\mathbb{D})$ is the closed subspace of $L_2(\mathbb{D}, d\eta_\mu)$ consisting of complex analytic functions in \mathbb{D} .

Here the standard weighted measure $d\eta_\mu$ is given by

$$d\eta_\mu(w) = \frac{\mu + 1}{\pi} (1 - |w|^2)^\mu dx dy, \quad w = x + iy \in \mathbb{D}.$$

Recall that the normalized monomials

$$e_\mu(n) := \sqrt{\frac{\Gamma(n + \mu + 2)}{\Gamma(n + 1)\Gamma(\mu + 2)}} w^n, \quad n \in \mathbb{Z}_+ \quad (2)$$

form an orthonormal basis in $\mathcal{A}_\mu^2(\mathbb{D})$.

We will denote by $\langle \cdot, \cdot \rangle_{\lambda, \mathbb{B}^2}$ and $\langle \cdot, \cdot \rangle_{\mu, \mathbb{D}}$ the inner products in $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ and $\mathcal{A}_\mu^2(\mathbb{D})$, respectively. The corresponding norms will be written as $\| \cdot \|_{\lambda, \mathbb{B}^2}$ and $\| \cdot \|_{\mu, \mathbb{D}}$.

We denote by B_μ the orthogonal (Bergman) projection from $L_2(\mathbb{D}, d\eta_\mu)$ onto the Bergman space $\mathcal{A}_\mu^2(\mathbb{D})$. Again, given $a \in L_\infty(\mathbb{D})$, the Toeplitz operator T_a^μ with symbol a acts on $\mathcal{A}_\mu^2(\mathbb{D})$ by the formula

$$T_a^\mu \varphi = B_\mu(a\varphi).$$

For each $\alpha_2 \in \mathbb{Z}_+$ we denote by H_{α_2} the following (closed) subspace of $\mathcal{A}_\lambda^2(\mathbb{B}^2)$

$$H_{\alpha_2} = \overline{\text{span}} \{ e_\lambda(\alpha_1, \alpha_2) : \alpha_1 \in \mathbb{Z}_+, \alpha_2 \in \mathbb{Z}_+ \text{ is fixed} \}.$$

Then we can represent $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ as a countable orthogonal sum:

$$\mathcal{A}_\lambda^2(\mathbb{B}^2) = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} H_{\alpha_2}. \quad (3)$$

Observe that

$$\begin{aligned} e_\lambda(\alpha_1, \alpha_2) &= \sqrt{\frac{(\Gamma(|\alpha| + \lambda + 3))}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\lambda + 3)}} z_1^{\alpha_1} z_2^{\alpha_2} \\ &= \sqrt{\frac{(\Gamma(|\alpha| + \lambda + 3))}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + \lambda + 3)}} z_1^{\alpha_1} \cdot \sqrt{\frac{\Gamma(\alpha_2 + \lambda + 3)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 3)}} z_2^{\alpha_2} \\ &= e_{\alpha_2 + \lambda + 1}(\alpha_1) \cdot e_{\lambda + 1}(\alpha_2). \end{aligned}$$

That is, the orthonormal basis in H_{α_2} has the form

$$\{ e_{\alpha_2 + \lambda + 1}(\alpha_1) \cdot e_{\lambda + 1}(\alpha_2) \}_{\alpha_1 \in \mathbb{Z}_+},$$

and thus we have

$$H_{\alpha_2} = \{ f(z_1) \cdot [e_{\lambda + 1}(\alpha_2)](z_2) : f \in \mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D}) \}. \quad (4)$$

According to the direct sum decomposition (3) each function $f \in \mathcal{A}_\lambda^2(\mathbb{B}^2)$ admits the unique representation:

$$\begin{aligned} f(z_1, z_2) &= \sum_{\alpha_2 \in \mathbb{Z}_+} f_{\alpha_2}(z_1) \cdot [e_{\lambda + 1}(\alpha_2)](z_2), \\ &\text{where } f_{\alpha_2} \in \mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D}) \text{ for all } \alpha_2 \in \mathbb{Z}_+, \end{aligned}$$

and

$$\|f\|_{\mathcal{A}_\lambda^2(\mathbb{B}^2)}^2 = \sum_{\alpha_2 \in \mathbb{Z}_+} \|f_{\alpha_2}\|_{\mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D})}^2. \quad (5)$$

We introduce the mapping $\omega : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{B}^2$ by $\omega(\zeta_1, \zeta_2) := (\zeta_1, \sqrt{1 - |\zeta_1|^2} \zeta_2)$. Let the unitary operator

$$u_{\alpha_2} : H_{\alpha_2} \rightarrow \mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D})$$

be defined as

$$(u_{\alpha_2} \phi)(\zeta_1) := \int_{\mathbb{D}} (\phi \circ \omega)(\zeta_1, \zeta_2) (1 - |\zeta_1|^2)^{-\frac{\alpha_2}{2}} \sqrt{\frac{(\lambda + 2)}{\alpha_2 + \lambda + 2}} \overline{[e_\lambda(\alpha_2)](\zeta_2)} d\eta_\lambda(\zeta_2). \quad (6)$$

By (4) each element $\phi \in H_{\alpha_2}$ has the form

$$\begin{aligned} \phi(z_1, z_2) &= f(z_1) \cdot [e_{\lambda+1}(\alpha_2)](z_2) = f(z_1) \cdot \sqrt{\frac{\alpha_2 + \lambda + 2}{\lambda + 2}} [e_\lambda(\alpha_2)](z_2), \\ f &\in \mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D}), \end{aligned}$$

thus

$$\begin{aligned} (u_{\alpha_2} \phi)(\zeta_1) &= \int_{\mathbb{D}} f(\zeta_1) (1 - |\zeta_1|^2)^{\frac{\alpha_2}{2}} \sqrt{\frac{\alpha_2 + \lambda + 2}{\lambda + 2}} [e_\lambda(\alpha_2)](\zeta_2) \\ &\quad \times (1 - |\zeta_1|^2)^{-\frac{\alpha_2}{2}} \sqrt{\frac{\lambda + 2}{\alpha_2 + \lambda + 2}} \overline{[e_\lambda(\alpha_2)](\zeta_2)} d\eta_\lambda(\zeta_2) \\ &= f(\zeta_1) \int_{\mathbb{D}} |[e_\lambda(\alpha_2)](\zeta_2)|^2 d\eta_\lambda(\zeta_2) = f(\zeta_1). \end{aligned}$$

Introduce now the Hilbert space

$$\mathcal{H} = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D})$$

and the unitary operator

$$U = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} u_{\alpha_2} : \mathcal{A}_\lambda^2(\mathbb{B}^2) = \bigoplus_{(\alpha_2 \in \mathbb{Z}_+)} H_{\alpha_2} \rightarrow \mathcal{H} = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D}) \quad (7)$$

acting componentwise according to the direct sum decomposition. We summarize the above observations in the following proposition.

Proposition (2.1.1)[2]. The unitary operator U , where each u_{α_2} is given by (6), gives an isometric isomorphism between the spaces in (7).

Our next aim is to characterize Toeplitz operators which (after conjugation with U) leave all spaces H_{α_2} in the decomposition (3) invariant. Hence such operators can be expressed

as a direct sum of operators acting componentwise on \mathcal{H} . We describe the componentwise action.

In what follows we will express points $z = (z_1, z_2) \in \mathbb{B}^2$ in polar coordinates: $z_k = r_k e^{i\theta_k}$, $k = 1, 2$. Given a function $c \in L_\infty(\mathbb{D} \times (0, 1))$, we consider the symbol

$$c\left(z_1, \frac{r_2}{\sqrt{1-r_1^2}}\right) \in L_\infty(\mathbb{B}^2) \quad (8)$$

and the corresponding Toeplitz operator T_c^λ acting on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$. For any pair of multi-indices $\alpha, \beta \in \mathbb{Z}_+^2$ we calculate the corresponding matrix element of

$$\begin{aligned} T_c^\lambda : \langle T_c^\lambda \mathbf{e}_\lambda(\alpha), \mathbf{e}_\lambda(\beta) \rangle_{\lambda, \mathbb{B}^2} \\ = \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\lambda + 3)} \frac{\Gamma(|\beta| + \lambda + 3)}{\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)\Gamma(\lambda + 3)}} \\ \times \langle c z_1^{\alpha_1} z_2^{\alpha_2}, z_1^{\beta_1} z_2^{\beta_2} \rangle_{\lambda, \mathbb{B}^2} \\ = \frac{1}{\Gamma(\lambda + 3)} \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \frac{\Gamma(|\beta| + \lambda + 3)}{\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)}} \\ \times \frac{\Gamma(\lambda + 3)}{\pi^2 \Gamma(\lambda + 1)} \int_{\mathbb{B}^2} c r_1^{\alpha_1 + \beta_1} r_2^{(\alpha_2 + \beta_2)} e^{i(\alpha_1 - \beta_1)\theta_1} e^{i(\alpha_2 - \beta_2)\theta_2} \\ \times (1 - r^2)^\lambda dv(z) \\ = \frac{1}{\pi^2 \Gamma(\lambda + 1)} \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \frac{\Gamma(|\beta| + \lambda + 3)}{\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)}} \\ \times \int_{\tau(\mathbb{B}^2)} \int_0^{2\pi} \int_0^{2\pi} c r_1^{\alpha_1 + \beta_1 + 1} r_2^{\alpha_2 + \beta_2 + 1} \\ \times (1 - r^2)^\lambda e^{i(\alpha_1 - \beta_1)\theta} e^{i(\alpha_2 - \beta_2)\theta_2} dr_1 dr_2 d\theta_1 d\theta_2, \end{aligned}$$

where the domain of the first integration is given by

$$\tau(\mathbb{B}^2) := \{r = (r_1, r_2) \in \mathbb{R}_+^2 : r_1^2 + r_2^2 < 1\}.$$

The integral over θ_2 vanishes whenever $\alpha_2 \neq \beta_2$ and is equals to 2π if $\alpha_2 = \beta_2$. That is, each subspace H_{α_2} is invariant for the Toeplitz operator T_c^λ . In the case of $\beta_2 = \alpha_2$ we have

$$\begin{aligned} \langle T_c^\lambda \mathbf{e}_\lambda(\alpha), \mathbf{e}_\lambda(\beta) \rangle_{\lambda, \mathbb{B}^2} \\ = \frac{2}{\pi \Gamma(\lambda + 1)\Gamma(\alpha_2 + 1)} \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3)\Gamma(|\beta| + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\beta_1 + 1)}} \\ \times \int_{\tau(\mathbb{B}^2)} \int_0^{2\pi} c r_1^{\alpha_1 + \beta_1 + 1} r_2^{2\alpha_2 + 1} (1 - r^2)^\lambda e^{i(\alpha_1 - \beta_1)\theta_1} dr_1 dr_2 d\theta_1. \end{aligned}$$

Changing variables to $r_1 = s_1, r_2 = \sqrt{1 - s_1^2} s_2$ gives

$$\langle T_c^\lambda e_\lambda(\alpha), e_\lambda(\beta) \rangle_{\lambda, \mathbb{B}^2}$$

$$\begin{aligned}
&= \frac{2}{\pi \Gamma(\lambda + 1) \Gamma(\alpha_2 + 1)} \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3) \Gamma(|\beta| + \lambda + 3)}{\Gamma(\alpha_1 + 1) \Gamma(\beta_1 + 1)}} \\
&\times \int_0^1 \int_0^{2\pi} \left[\int_0^1 c(s_1 e^{i\theta_1}, s_2) s_2^{2\alpha_2+1} (1 - s_2^2)^\lambda ds_2 \right] s_1^{\alpha_1+\beta_1+1} (1 - s_1^2)^{\alpha_2+\lambda+1} e^{i(\alpha_1-\beta_1)\theta_1} d\theta_1 ds_1 \\
&= \frac{2}{\pi \Gamma(\lambda + 1) \Gamma(\alpha_2 + 1)} \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3) \Gamma(|\beta| + \lambda + 3)}{\Gamma(\alpha_1 + 1) \Gamma(\beta_1 + 1)}} \\
&\times \int_{\mathbb{D}} \left[\int_0^1 c(z_1, s_2) s_2^{2\alpha_2+1} (1 - s_2^2)^\lambda ds_2 \right] z_1^{\alpha_1} \bar{z}_1^{\beta_1} (1 - |z_1|^2)^{\alpha_2+\lambda+1} dv(z_1) \\
&= \frac{1}{\Gamma(\lambda + 1) \Gamma(\alpha_2 + 1)} \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3) \Gamma(|\beta| + \lambda + 3)}{\Gamma(\alpha_1 + 1) \Gamma(\beta_1 + 1)}} \frac{1}{\alpha_2 + \lambda + 2} \\
&\times \int_{\mathbb{D}} \left[\int_0^1 c(z_1, \sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2 \right] z_1^{\alpha_1} \bar{z}_1^{\beta_1} \frac{\alpha_2 + \lambda + 2}{\pi} (1 - |z_1|^2)^{\alpha_2+\lambda+1} dv(z_1) \\
&= \frac{1}{\Gamma(\lambda + 1) \Gamma(\alpha_2 + 1)} \sqrt{\frac{\Gamma(|\alpha| + \lambda + 3) \Gamma(|\beta| + \lambda + 3)}{\Gamma(\alpha_1 + 1) \Gamma(\beta_1 + 1)}} \frac{1}{\alpha_2 + \lambda + 2} \\
&\times \sqrt{\frac{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + \lambda + 3)}{\Gamma(|\alpha| + \lambda + 3)} \frac{\Gamma(\beta_1 + 1) \Gamma(\alpha_2 + \lambda + 3)}{\Gamma(|\beta| + \lambda + 3)}} \\
&\times \left\langle \int_0^1 c(z_1, \sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2 \right\rangle \\
&\times e_{\alpha_2+\lambda+1}(\alpha_1), e_{\alpha_2+\lambda+1}(\beta_1) \rangle_{\alpha_2+\lambda+1, \mathbb{D}} \\
&= \langle \tilde{c}_{\alpha_2} e_{\alpha_2+\lambda+1}(\alpha_1), e_{\alpha_2+\lambda+1}(\beta_1) \rangle_{\alpha_2+\lambda+1, \mathbb{D}} \\
&= \langle T_{c\alpha_2}^{(\alpha_2+\lambda+1)} e_{\alpha_2+\lambda+1}(\alpha_1), e_{\alpha_2+\lambda+1}(\beta_1) \rangle_{\alpha_2+\lambda+1, \mathbb{D}} '
\end{aligned}$$

where

$$\begin{aligned}
\tilde{c}_{\alpha_2}(z_1) &:= \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1) \Gamma(\lambda + 1)} \\
&\int_0^1 c(z_1, \sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2.
\end{aligned} \tag{9}$$

We summarize the above calculation in the following lemma.

Lemma (2.1.2)[2]. Let $c \in L_\infty(\mathbb{D} \times (0, 1))$. Consider the Toeplitz operator T_c^λ acting on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ and having the symbol

$$c\left(z_1, \frac{r_2}{\sqrt{1-r_1^2}}\right) \in L_\infty(\mathbb{B}^2).$$

With $\alpha, \beta \in \mathbb{Z}_+^2$ we have:

$$\langle T_c^\lambda e_\lambda(\alpha), e_\lambda(\beta) \rangle_{\lambda, \mathbb{B}^2} =$$

$$\begin{cases} 0, & \text{if } \alpha_2 \neq \beta_2 \\ \langle T_{\tilde{c}_{\alpha_2}}^{\alpha_2+\lambda+1} e_{\alpha_2+\lambda+1}(\alpha_1), e_{\alpha_2+\lambda+1}(\beta_1) \rangle_{\alpha_2+\lambda+1, \mathbb{D}}, & \text{if } \alpha_2 = \beta_2 \end{cases}$$

where the function $\tilde{c}_{\alpha_2}(z_1)$ is given in (9).

Corollary (2.1.3)[2]. Under the assumptions of the previous lemma, each subspace $H_{\alpha_2}, \alpha_2 \in \mathbb{Z}_+$, is invariant for the operator T_c^λ , and its action on H_{α_2} is as follows

$$[T_c^\lambda f e_{\lambda+1}](z_1, z_2) = (T_{\tilde{c}_{\alpha_2}}^{\alpha_2+\lambda+1} f)(z_1)[e_{\lambda+1}(\alpha_2)](z_2) \text{ where } f \in \mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D}).$$

Moreover, with the operator U given in (7) one has:

$$UT_c^\lambda U^* = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_{\tilde{c}_{\alpha_2}}^{\alpha_2+\lambda+1}.$$

Remark (2.1.4)[2]. Let T^λ be a bounded operator on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ which leaves all subspaces $H_{\alpha_2}, \alpha_2 \in \mathbb{Z}_+$ invariant. Then there exists a sequence of bounded operators $\{T^{\alpha_2+\lambda+1}\}_{\alpha_2 \in \mathbb{Z}_+}$, where each $T^{\alpha_2+\lambda+1}$ acts on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$, and

$$[T^\lambda f e_{\lambda+1}](z_1, z_2) = (T^{\alpha_2+\lambda+1} f)(z_1)[e_{\lambda+1}(\alpha_2)](z_2), \text{ where } f \in \mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D}).$$

Via the unitary operator U in (7) one has

$$UT^\lambda U^* = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T^{\alpha_2+\lambda+1}. \quad (10)$$

In what follows we will abbreviate (10) by

$$T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T^{\alpha_2+\lambda+1}.$$

identifying thus the operator T^λ with its direct sum representation.

In the next corollary we collect some symbols classes that induce Toeplitz operators on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ leaving the spaces $H_{\alpha_2}, \alpha_2 \in \mathbb{Z}_+$ invariant.

Corollary (2.1.5)[2]. Let $a \in L_\infty(\mathbb{D})$ and $b \in L_\infty(0,1)$ and introduce the symbols

$$a(z_1), \quad b\left(\frac{r_2}{\sqrt{1-r_1^2}}\right), \quad \text{and} \quad b\left(\frac{r_2}{\sqrt{1-r_1^2}}\right). \quad (11)$$

The corresponding Toeplitz operators T_a^λ, T_b^λ and T_{ab}^λ acting on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ leave each subspace $H_{\alpha_2}, \alpha_2 \in \mathbb{Z}_+$ invariant and their action on H_{α_2} is as follows

$$\begin{aligned} [T_a^\lambda f e_{\lambda+1}](z_1, z_2) &= (T_a^{\alpha_2+\lambda+1} f)(z_1)[e_{\lambda+1}(\alpha_2)](z_2), \\ T_b^\lambda|_{H_{\alpha_2}} &= \gamma_b^\lambda(\alpha_2)I, \\ [T_{ab}^\lambda f e_{\lambda+1}](z_1, z_2) &= \gamma_b^\lambda(\alpha_2)(T_a^{\alpha_2+\lambda+1} f)(z_1)[e_{\lambda+1}(\alpha_2)](z_2), \end{aligned}$$

where $f \in \mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$ and

$$\gamma_b^\lambda(\alpha_2) = \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2.$$

Corollary (2.1.6)[2]. With the notation of Corollary (2.1.5) the operators T_a^λ and T_b^λ commute,

$$T_{ab}^\lambda = T_a^\lambda T_b^\lambda = T_b^\lambda T_a^\lambda,$$

and one has the identification:

$$T_{ab}^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \gamma_b^\lambda(\alpha_2) T_a^{\alpha_2 + \lambda + 1}.$$

In the following we restrict our attention to generating symbols of the form (11). The results suggest the following recipe.

We select a subclass S of $L_\infty(\mathbb{D})$ such that the algebra (or C^* -algebra) generated by Toeplitz operators T_a^λ , with $a \in S$, acting on each weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{D})$, $\lambda \in (-1, \infty)$, admits a reasonable description, and denote by $\mathcal{T}_\lambda(S, L_\infty)$ the unital Banach algebra generated by all Toeplitz operators T_a^λ and T_b^λ acting on $\mathcal{A}_\lambda^2(\mathbb{D}^2)$ with symbols

$$a = a(z_1) \in S \quad \text{and} \quad b\left(\frac{r_2}{\sqrt{1-r_1^2}}\right), \quad \text{where } b \in L_\infty(0,1).$$

Then the algebra $\mathcal{T}_\lambda(S, L_\infty)$ is generated by two of its subalgebras sharing the same identity: the C^* -algebra $\mathcal{T}_\lambda(L_\infty)$ generated by all Toeplitz operators T_b^λ and the unital Banach algebra $\mathcal{T}_\lambda(S)$ generated by all Toeplitz operators T_a^λ where $a \in S$. Note that $\mathcal{T}_\lambda(S)$ is a C^* -algebra if S is closed under complex conjugation. The C^* -algebra $\mathcal{T}_\lambda(L_\infty)$ is isomorphic to an algebra of sequences. This isomorphism is given by the following assignment of generators of $\mathcal{T}_\lambda(L_\infty)$

$$T_b^\lambda \mapsto \gamma_b^\lambda = \{\gamma_b^\lambda(\alpha_2)\}_{\alpha_2 \in \mathbb{Z}_+}. \quad (12)$$

The corresponding sequence algebra is known to coincide with the algebra $SO(\mathbb{Z}_+)$, introduced by R. Schmidt and consisting of all ℓ_∞ -sequences γ that satisfy the condition

$$\lim_{\substack{j+1 \\ k+1} \rightarrow 1} |\gamma(j) - \gamma(k)| = 0.$$

We may also interpret $SO(\mathbb{Z}_+)$ as the C^* -algebra of bounded functions $\gamma : \mathbb{Z}_+ \rightarrow \mathbb{C}$ that are uniformly continuous with respect to the logarithmic metric

$$\rho(j, k) = |\log(j+1) - \log(k+1)|, \quad j, k \in \mathbb{Z}_+.$$

More details on the isomorphism (12) in both the weighted and unweighted situation can be found.

The algebra $\mathcal{T}_\lambda(S)$ splits, according to the decomposition (3), into the direct sum of the algebras $\mathcal{T}_{\alpha_2 + \lambda + 1}(S)$ generated by all Toeplitz operators $T_a^{\alpha_2 + \lambda + 1}$, $a \in S$, acting on the weighted Bergman spaces $\mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D})$, $\alpha_2 \in \mathbb{Z}_+$. This is the place where the already known description of the algebras $\mathcal{T}_{\alpha_2 + \lambda + 1}(S)$ enters to the study.

Clearly, $\mathcal{T}_\lambda(S, L_\infty)$ will be a commutative C^* -algebra if and only if $S \subset L_\infty(\mathbb{D})$ is chosen such that all algebras $\mathcal{T}_{\alpha_2 + \lambda + 1}(S)$ of operators acting on $\mathcal{A}_{\alpha_2 + \lambda + 1}^2(\mathbb{D})$ are C^* and

commutative.

The C^* -algebras generated by Toeplitz operators which are commutative on each weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{D})$ and whose generating symbols contain the so-called 3-rich- symbols are completely classified. Up to a unitary equivalence via Möbius transformations there are only three model classes of such algebras: elliptic case – Toeplitz operators on the disk \mathbb{D} with radial symbols; parabolic case - Toeplitz operators on the upper-half plane Π with symbols depending only on the imaginary part y of $z = x + iy \in \Pi$; and hyperbolic case - Toeplitz operators on the upper half-plane Π with symbols depending only on the polar angle θ of $z = |z|e^{i\theta} \in \Pi$.

We explore now these three one-dimensional cases and show that they generate subalgebras of the known commutative C^* -algebras for the two-dimensional quasi-elliptic, quasiparabolic, and quasi-hyperbolic cases, respectively.

Example (2.1.7)[2]. Elliptic case put $S := \{a = a(r_1): a \in L_\infty(0, 1)\}$. Then the C^* -algebra $\mathcal{T}_{\alpha_2+\lambda+1}(S)$ is generated by Toeplitz operators $T_a^{\alpha_2+\lambda+1}$ that are diagonal with respect to the standard monomial basis (2) in $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$. More precisely, $T_a^{\alpha_2+\lambda+1}$ acts as follows:

$$\begin{aligned} T_a^{\alpha_2+\lambda+1} e_{\alpha_2+\lambda+1}(\alpha_1) &= \frac{\Gamma(\alpha_1 + \alpha_2 + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + \lambda + 2)} \\ &\times \int_0^1 a(\sqrt{s_1}) s_1^{\alpha_1} (1 - s_1)^{\alpha_2+\lambda+1} ds_1 e_{\alpha_2+\lambda+1}(\alpha_1). \end{aligned}$$

Consider now a separately radial symbol

$$c(r_1, r_2) = a(r_1) b\left(\frac{r_2}{\sqrt{1 - r_1^2}}\right)$$

of the form (11). Then for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and with $r = (r_1, r_2)$ we have

$$\begin{aligned} T_c^\lambda e_\lambda(\alpha) &= T_{ab}^\lambda e_{\alpha_2+\lambda+1}(\alpha_1) \cdot e_{\lambda+1}(\alpha_2) = T_a^{(\alpha_2+\lambda+1)} e_{\alpha_2+\lambda+1}(\alpha_1) \cdot T_b^\lambda e_{\lambda+1}(\alpha_2) \\ &= \frac{\Gamma(\alpha_1 + \alpha_2 + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + \lambda + 2)} \\ &\times \int_0^1 a(\sqrt{s_1}) s_1^{\alpha_1} (1 - s_1)^{\alpha_2+\lambda+1} ds_1 e_{\alpha_2+\lambda+1}(\alpha_1) \\ &\times \frac{\Gamma(\alpha_2 + \lambda + 2)\Gamma}{(\alpha_2 + 1)\Gamma(\lambda + 1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2 e_{\lambda+1}(\alpha_2) \\ &\times \frac{\Gamma(|\alpha| + \lambda + 3)}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \\ &\times \int_{0 \leq |r|^2 < 1} c(\sqrt{r_1}, \sqrt{r_2}) r^\alpha (1 - |r|^2)^\lambda dr_1 dr_2 \cdot e_\lambda(\alpha). \end{aligned}$$

The last equality follows by a change of variables: $s_1 = r_1, s_2 = (1 - r_1)^{-1}r_2$. Note that this result recovers the formula for $\gamma_{c,\lambda}(\alpha)$ in the quasi-elliptic case.

Example (2.1.8)[2]. Parabolic case

This case corresponds to the algebra generated by Toeplitz operators on the weighted Bergman space $\mathcal{A}_\lambda^2(\Pi)$ over the upper half-plane Π whose symbols depend only on $\text{Im } w, w \in \Pi$. That is, in this case $S_\Pi = \{a = a(\text{Im}): a \in L_\infty(\mathbb{R}_+)\}$. The C^* -algebra $\mathcal{T}_\lambda(S_\Pi)$ is isomorphic to a certain subalgebra of $C_b(\mathbb{R}_+)$ and this isomorphism is generated by the following mapping:

$$T_a^\lambda \mapsto \gamma_a^\lambda(\xi) = \frac{1}{\Gamma(\lambda + 1)} \int_{\mathbb{R}_+} a\left(\frac{t}{2\xi}\right) t^\lambda e^{-t} dt, \quad \xi \in \mathbb{R}_+.$$

The standard unitary operator defined by the Möbius transformation

$$w = i \frac{1 - \zeta}{1 + \zeta} \quad (13)$$

of the unit disk \mathbb{D} onto the upper half-plane Π maps $\mathcal{A}_\lambda^2(\Pi)$ onto $\mathcal{A}_\lambda^2(\mathbb{D})$ and provides the unitary equivalence of the algebras $\mathcal{T}_\lambda(S_\Pi)$ and $\mathcal{T}_\lambda(S_\mathbb{D})$, where

$$S_\mathbb{D} = \left\{ a = a\left(\frac{1 - |\zeta|^2}{1 + \zeta + \bar{\zeta} + |\zeta|^2}\right) : a \in L_\infty(\mathbb{R}_+), \zeta \in \mathbb{D} \right\}.$$

Then, by Corollary (2.1.6), the algebra $\mathcal{T}_\lambda(S_\mathbb{D}, L_\infty)$ is isomorphic to a subalgebra of $C_b(\mathbb{R}_+ \times \mathbb{Z}_+)$ with the following assignment of its generators

$$\begin{aligned} T_{ab}^\lambda &= T_a^\lambda T_b^\lambda \mapsto \gamma_a^{\alpha_2 + \lambda + 1}(\xi) \cdot \gamma_b^\lambda(\alpha_2) \\ &= \frac{1}{\Gamma(\alpha_2 + \lambda + 2)} \int_{\mathbb{R}_+} a\left(\frac{t}{2\xi}\right) t^{\alpha_2 + \lambda + 1} e^{-t} dt \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \\ &\quad \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2, \end{aligned} \quad (14)$$

where $(\xi, \alpha_2) \in \mathbb{R}_+ \times \mathbb{Z}_+$.

Introduce the two-dimensional Siegel domain $D_2 = \{(w_1, w_2) \in \mathbb{C}^2; \text{Im } w_1 - |w_2|^2 > 0\}$. The Cayley transform $w = w(z)$, where

$$w(z) = (w_1(z), w_2(z)) = \left(i \frac{1 - z_1}{1 + z_1}, i \frac{z_2}{1 + z_1}\right) : \mathbb{B}^2 \rightarrow D_2 \quad (15)$$

biholomorphically maps the unit ball \mathbb{B}^2 onto the Siegel domain D_2 . The unitary operator defined by the inverse to the Cayley transform establishes the unitary equivalence between the algebra $\mathcal{T}_\lambda(S_\mathbb{D}, L_\infty)$ and the corresponding Toeplitz operator algebra on D_2 . Under this unitary equivalence each generator $T_{ab}^\lambda \in \mathcal{T}_\lambda(S_\mathbb{D}, L_\infty)$ is mapped to the Toeplitz operator on D_2 with symbol

$$c(w_1, w_2) = a(\text{Im } w_1) b\left(\frac{|w_2|}{\sqrt{\text{Im } w_1}}\right), \quad (w_1, w_2) \in D_2, \quad (16)$$

which in turn is mapped to its “spectral function” γ_c^λ under the isomorphic description of the C^* -algebra generated by Toeplitz operators as a function subalgebra of $C_b(\mathbb{R}_+ \times \mathbb{Z}_+)$.

In our notation the function γ_c^λ has the form

$$\begin{aligned}\gamma_c^\lambda(\xi, \alpha_2) &= \frac{(2\xi)^{\alpha_2+\lambda+2}}{\Gamma(\alpha_2+1)\Gamma(\lambda+1)} \\ &\times \int_{\mathbb{R}_+^2} c\left(\operatorname{Im} w_1 + |w_2|, \sqrt{|w_2|}\right) |w_2|^{\alpha_2} (\operatorname{Im} w_1)^\lambda \\ &\times e^{-2\xi(\operatorname{Im} w_1 + |w_2|)} d\operatorname{Im} w_1 d|w_2| \\ &= \frac{1}{\Gamma(\alpha_2+1)\Gamma(\lambda+1)} \int_{\mathbb{R}_+^2} c\left(\frac{v+r}{2\xi}\right) \sqrt{\frac{r}{2\xi}} r^{\alpha_2} v^\lambda e^{-(v+r)} dv dr,\end{aligned}$$

where $v = \operatorname{Im} w_1$ and $r = |w_2|$. Replacing now c by its expression (16) and making a change of variables $t = v + r, s_2 = \frac{r}{v+r}$, gives

$$\begin{aligned}\gamma_{ab}^\lambda(\xi, \alpha_2) &= \frac{1}{\Gamma(\alpha_2+1)\Gamma(\lambda+1)} \int_{\mathbb{R}_+^2} a\left(\frac{(v+r)}{2\xi}\right) b\left(\sqrt{\frac{r}{v+r}}\right) r^{\alpha_2} v^\lambda e^{-(v+r)} dv dr \\ &= \frac{1}{\Gamma(\alpha_2+1)\Gamma(\lambda+1)} \int_{\mathbb{R}_+} \int_0^1 a\left(\frac{t}{2\xi}\right) b(\sqrt{s_2}) t^{\alpha_2} s_2^{\alpha_2} (1-s_2)^\lambda t^\lambda e^{-t} dt ds_2 \\ &= \frac{1}{\Gamma(\alpha_2+\lambda+2)} \int_{\mathbb{R}_+} a\left(\frac{t}{2\xi}\right) t^{\alpha_2+\lambda+1} e^{-t} dt \frac{\Gamma(\alpha_2+\lambda+2)}{\Gamma(\alpha_2+1)\Gamma(\lambda+1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1-s_2)^\lambda ds_2 \\ &= \gamma_a^{\alpha_2+\lambda+1}(\xi) \cdot \gamma_b^\lambda(\alpha_2), \quad (\xi, \alpha_2) \in \mathbb{R}_+ \times \mathbb{Z}_+.\end{aligned}$$

A comparison with (14) shows that the algebra $\mathcal{T}_\lambda(S_{\mathbb{D}}, L_\infty)$ is just unitary equivalent to a subalgebra of the C^* -algebra of the two-dimensional quasi-parabolic case.

Example (2.1.9)[2]. Hyperbolic case

This case corresponds to the algebra generated by Toeplitz operators on the weighted Bergman space $\mathcal{A}_\lambda^2(\Pi)$ over the upper half-plane Π whose symbols depend only on the angular variable $\theta = \arg w, w \in \Pi$, i.e.

$$S_\Pi = \{a = a(\theta): a \in L_\infty(0, \pi)\}.$$

The C^* -algebra $\mathcal{T}_\lambda(S_\Pi)$ is isomorphic to a certain subalgebra of $C_b(\mathbb{R})$ and this isomorphism is generated by the following mapping (see [20], for details):

$$T_a^\lambda \mapsto \gamma_a^\lambda(\xi) = \left(\int_0^\pi e^{-2\xi\theta} \sin^\lambda \theta d\theta \right)^{-1} \int_0^\pi a(\theta) e^{-2\xi\theta} \sin^\lambda \theta d\theta, \quad \xi \in \mathbb{R}$$

We repeat now all the steps of the previous parabolic case. The Möbius transformation (13) and the corresponding unitary operator from $\mathcal{A}_\lambda^2(\Pi)$ onto $\mathcal{A}_\lambda^2(\mathbb{D})$ provide the description of the corresponding class $S_{\mathbb{D}}$ of symbols in \mathbb{D} and the unitary equivalence of the algebras $\mathcal{T}_\lambda(S_\Pi)$ and $\mathcal{T}_\lambda(S_{\mathbb{D}})$. As a consequence of Corollary (2.1.6), the algebra $\mathcal{T}_\lambda(S_{\mathbb{D}}, L_\infty)$ is isomorphic to a subalgebra of $C_b(\mathbb{R} \times \mathbb{Z}_+)$ with the following assignment of its generators

$$\begin{aligned}
T_{ab}^\lambda &= T_a^\lambda T_b^\lambda \mapsto \gamma_a^{\alpha_2 + \lambda + 1}(\xi) \cdot \gamma_b^\lambda(\alpha_2) \\
&= \left(\int_0^\pi e^{-2\xi\theta} (\sin \theta)^{\alpha_2 + \lambda + 1} d\theta \right)^{-1} \int_0^\pi a(\theta) e^{-2\xi\theta} (\sin \theta)^{\alpha_2 + \lambda + 1} d\theta \\
&\quad \times \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2
\end{aligned}$$

where $(\xi, \alpha_2) \in \mathbb{R} \times \mathbb{Z}_+$. Then the unitary operator, defined by the inverse to the Cayley transform (15), establishes the unitary equivalence between the algebra $\mathcal{T}_\lambda(S_{\mathbb{D}}, L_\infty)$ and the corresponding Toeplitz operator algebra on D_2 . Under this unitary equivalence each generator $T_{ab}^\lambda \in \mathcal{T}_\lambda(S_{\mathbb{D}}, L_\infty)$ is mapped to the Toeplitz operator on D_2 with symbol

$$a(\arg w_1) b \left(\frac{|w_2|}{\sqrt{\operatorname{Im} w_1}} \right), \quad (w_1, w_2) \in D_2. \quad (17)$$

Such symbols, for all $a \in L_\infty(0, \pi)$ and $b \in L_\infty(0, 1)$, are invariant under the action

$$\mathbb{R}_+ \times \mathbb{T} \ni (r, t): (w_1, w_2) \in D_2 \mapsto (rw_1, r^{\frac{1}{2}}tw_2) \in D_2$$

of the *quasi-hyperbolic* group $\mathbb{R}_+ \times \mathbb{T}$ of biholomorphisms of D_2 . Thus the resulting C^* -algebra generated by Toeplitz operators with symbols of the form (17) is a subalgebra of the C^* -algebra of the two-dimensional quasi-hyperbolic case.

Example (2.1.10)[2]. More commutative C^* -algebras

All others so far known algebras \mathcal{T}_λ that are commutative for each weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^2)$, $\lambda \in (-1, \infty)$, were Banach (not C^*). Each of them was generated by two of its subalgebras: the C^* -algebra generated by Toeplitz operators with bounded radial symbols (which is isomorphic to $SO(\mathbb{Z}_+)$) and a unital Banach algebra generated by a single Toeplitz operator with a so-called (generalized) quasi-homogeneous symbol.

Now we can unhide many others, previously unexpected, similarly constructed algebras that are both C^* and commutative for each weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^2)$, $\lambda \in (-1, \infty)$. All of them are generated by the C^* -algebra $\mathcal{T}_\lambda(L_\infty)$, being isomorphic to $SO(\mathbb{Z}_+)$, and the unital C^* -algebra $\mathcal{T}_\lambda(a)$ generated by a single Toeplitz operator T_a^λ , where $a(z_1)$ can be *any* real valued L_∞ -function on \mathbb{D} .

Each such algebra admits the following description. We start with a fixed real valued L_∞ -function $a = a(z_1)$ on \mathbb{D} , and denote by $\operatorname{sp} T_a^\mu$ the spectrum of the Toeplitz operator T_a^μ acting on $\mathcal{A}_\mu^2(\mathbb{D})$. Note that $\operatorname{sp} T_a^\mu$ may depend on the weight parameter μ . Put

$$M_{\mathbb{Z}_+} = \coprod_{\alpha_2 \in \mathbb{Z}_+} \operatorname{sp} T_a^{\alpha_2 + \lambda + 1},$$

where \coprod denotes the *disjoint* union. Then the C^* -algebra $\mathcal{T}_\lambda(a, L_\infty)$ is isomorphic to a subalgebra of $C_b(M_{\mathbb{Z}_+})$ under the following assignment of the generators of $\mathcal{T}_\lambda(a, L_\infty)$:

$$\begin{aligned}
T_{ab}^\lambda &\mapsto x_{\alpha_2 + \lambda + 1} \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2, \quad x_{\alpha_2 + \lambda + 1} \\
&\in \operatorname{sp} T_a^{\alpha_2 + \lambda + 1},
\end{aligned}$$

where $b \in L_\infty(0, 1)$.

We note that, for *any* finite subset N of \mathbb{Z}_+ , the restriction of the subalgebra of $C_b(M_{\mathbb{Z}_+})$, which is isomorphic to the C^* -algebra $\mathcal{T}_\lambda(a, L_\infty)$, onto $M_N = \coprod_{\alpha_2 \in N} sp T_a^{\alpha_2 + \lambda + 1}$ coincides with $\mathcal{C}(M_N) = \bigoplus_{\alpha_2 \in N} \mathcal{C}(sp T_a^{\alpha_2 + \lambda + 1})$.

If a fixed symbol $a = a(z_1) \in L_\infty(\mathbb{D})$ is not real-valued, then the unital algebra $\mathcal{T}_\lambda(a)$ generated by the Toeplitz operator T_a^λ is Banach, and the description of the algebra $\mathcal{T}_\lambda(a, L_\infty)$ of Example (2.1.10) needs a certain adjustment.

The spectrum $sp_{\mathcal{T}} T_a^\mu$ of the Toeplitz operator T_a^μ , as an element of the unital algebra $\mathcal{T}_\mu(a)$ (generated by T_a^μ), is a polynomially convex compact subset of \mathbb{C} the set $\mathbb{C} \cap sp_{\mathcal{T}} T_a^\mu$ is connected. Recall that under these conditions Mergelyan's theorem states that any continuous complex function on $K = sp_{\mathcal{T}} T_a^\mu$ which is holomorphic in the interior of K can be uniformly approximated by holomorphic polynomials restricted to K .

Hence, for each weight parameter μ , the algebra $\mathcal{T}_\mu(a)$ is isomorphic to $\mathcal{A}(sp_{\mathcal{T}} T_a^\mu)$, the algebra of all functions that are continuous on $sp_{\mathcal{T}} T_a^\mu$ and analytic in the interior of $sp_{\mathcal{T}} T_a^\mu$. Moreover, this isomorphism is generated by the assignment

$$T_a^\mu \mapsto [w \mapsto w] \in \mathcal{A}(sp_{\mathcal{T}} T_a^\mu).$$

Consider again the set

$$M_{\mathbb{Z}_+} = \coprod_{\alpha_2 \in \mathbb{Z}_+} sp_{\mathcal{T}} T_a^{\alpha_2 + \lambda + 1}.$$

Then the Banach algebra $\mathcal{T}_\lambda(a, L_\infty)$ is isomorphic to a subalgebra of $C_b(M_{\mathbb{Z}_+})$ under the following assignment of the generators of $\mathcal{T}_\lambda(a, L_\infty)$:

$$\begin{aligned} T_{ab}^\lambda &\mapsto w_{\alpha_2 + \lambda + 1} \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2, \\ w_{\alpha_2 + \lambda + 1} &\in sp_{\mathcal{T}} T_a^{\alpha_2 + \lambda + 1}, \end{aligned} \quad (18)$$

where $b \in L_\infty(0, 1)$.

For any finite subset N of \mathbb{Z}_+ , the restriction of the subalgebra of $C_b(M_{\mathbb{Z}_+})$, which is isomorphic to the algebra $\mathcal{T}_\lambda(a, L_\infty)$, onto $M_N = \coprod_{\alpha_2 \in N} sp_{\mathcal{T}} T_a^{\alpha_2 + \lambda + 1}$ coincides with the algebra $\prod_{\alpha_2 \in N} \mathcal{A}(sp_{\mathcal{T}} T_a^{\alpha_2 + \lambda + 1})$.

Example (2.1.11)[2]. Case of $a(z_1) = z_1$

In this case T_a^μ is just the multiplication operator M_{z_1} , the so-called *Bergman shift*, and $sp_{\mathcal{T}} T_a^\mu = \mathbb{D}$ independently on the weight parameter $\mu \in (-1, \infty)$. This implies that $M_{\mathbb{Z}_+} = \mathbb{D} \times \mathbb{Z}_+$, and formula (18) takes the form

$$\begin{aligned} T_{z_1 \cdot b}^\lambda &\mapsto \gamma_{z_1 \cdot b}^\lambda = \gamma_{z_1 \cdot b}^\lambda(w, \alpha_2) \\ &= w \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2, (w, \alpha_2) \in \mathbb{D} \times \mathbb{Z}_+. \end{aligned}$$

Note that each function $\gamma_{z_1}^\lambda \cdot b, b \in L_\infty(0,1)$, as well as each element γ^λ of the function algebra generated by them obey the following properties

- for each fixed $\alpha_2 \in \mathbb{Z}_+$: $\gamma^\lambda(\cdot, \alpha_2) \in \mathcal{A}(\mathbb{D})$,
- for each fixed $w \in \mathbb{D}$: $\gamma^\lambda(w, \cdot) \in SO(\mathbb{Z}_+)$.

We note that, apart from $T_{z_1}^\lambda$, the Banach algebra $\mathcal{T}_\lambda(z_1)$ contains more Toeplitz operators. In fact, for each $a(z_1) \in \mathcal{A}(\mathbb{D})$, the Toeplitz operator $T_{a(z_1)}^\lambda = M_{a(z_1)}$ belongs to $\mathcal{T}_\lambda(z_1)$. That is, $\mathcal{T}_\lambda(z_1, L_\infty) = \mathcal{T}_\lambda(S, L_\infty)$, with $S = \mathcal{A}(\mathbb{D})$.

Example (2.1.12)[2]. Case of $S = H^\infty(\mathbb{D})$

A maximal extension of the algebra constructed in the previous example is achieved by the replacement of $S = \mathcal{A}(\mathbb{D})$ by the maximally ample class $H^\infty(\mathbb{D})$ of all bounded analytic functions in \mathbb{D} .

There are many other admissible sets S related to bounded analytic functions. Actually each (closed) subalgebra of $H^\infty(\mathbb{D})$ can serve as a class S . We give here just one example of such an algebra having important connections with other function classes that often appear in the operator theory in function spaces.

Example (2.1.13)[2]. Case of $S = COP$. We recall first the notion of Gleason parts. Let $M(H^\infty)$ denote the compact set of maximal ideals of the algebra $H^\infty = H^\infty(\mathbb{D})$. On $M(H^\infty)$ consider the pseudohyperbolic distance $\rho(m_1, m_2) = \sup\{|f(m_2)| : f \in H^\infty, \|f\|_\infty \leq 1, f(m_1) = 0\}$, $m_1, m_2 \in M(H^\infty)$. Clearly, $\rho(m_1, m_2) \leq 1$ for all m_1, m_2 and one obtains an equivalence relation on $M(H^\infty)$ by

$$m_1 \sim m_2 \Leftrightarrow \rho(m_1, m_2) < 1.$$

The equivalence classes $P(m)$ for $m \in M(H^\infty)$ are called Gleason parts and form a partition of $M(H^\infty)$. Each Gleason part $P(m), m \in M(H^\infty)$ is either an analytic disc (i.e. the range of a certain analytic map $L_m : \mathbb{D} \rightarrow M(H^\infty)$) or a single point set $\{m\}$. Recall then that $COP = COP(\mathbb{D})$ is the algebra of all bounded functions analytic in \mathbb{D} which are constant on Gleason parts $P(m)$ for all $m \in M(H^\infty) \setminus \mathbb{D}$. It is known that $COP = B_0 \cap H^\infty$, where B_0 is the little Bloch space, which consists of all functions f analytic in \mathbb{D} such that $|f'(w)|(1 - |w|^2) \rightarrow 0$ as $|w| \rightarrow 1^-$. It is known as well that COP is an “analytic” part of the class Q , i.e., $COP = Q \cap H^\infty$. Here Q is the maximal C^* -subalgebra of $L_\infty(\mathbb{D})$ such that the semicommutators of Toeplitz operators with symbols from this algebra are compact.

For each subset $S \subset H^\infty(\mathbb{D})$, the algebra $\mathcal{T}_\lambda(S, L_\infty)$ is isomorphic to a subalgebra of $C_b(\mathbb{D} \times \mathbb{Z}_+)$. The isomorphism is generated by the following mapping: given $f \in S$ and $b \in L_\infty(0,1)$,

$$\begin{aligned} T_{f \cdot b}^\lambda &\mapsto \gamma_{f \cdot b}^\lambda = \gamma_{f \cdot b}^\lambda(w, \alpha_2) \\ &= f(w) \frac{\Gamma(\alpha_2 + \lambda + 2)}{\Gamma(\alpha_2 + 1)\Gamma(\lambda + 1)} \int_0^1 b(\sqrt{s_2}) s_2^{\alpha_2} (1 - s_2)^\lambda ds_2, (w, \alpha_2) \in \mathbb{D} \times \mathbb{Z}_+. \end{aligned}$$

In the case of $S = COP$ each such function $\gamma_{f.b}^\lambda$, as well as each element γ^λ of the function algebra generated by them, obeys the following properties

- for each fixed $\alpha_2 \in \mathbb{Z}_+$: $\gamma^\lambda(\cdot, \alpha_2) \in COP$,
- for each fixed $w \in \mathbb{D}$: $\gamma^\lambda(w, \cdot) \in SO(\mathbb{Z}_+)$.

In particular, $sp T_{f.b}^\lambda = clos\{Range \gamma_{f.b}^\lambda\}$.

Section (2.2): Non-Commutative C^* -Algebras

The characterization of non-commutative C^* -algebras consists in the description of its irreducible representations (up to unitary equivalence). If a unital operator C^* -algebra \mathcal{R} contains the ideal \mathcal{K} of compact operators, then its identical representation is irreducible and all other irreducible representations are obtained by composing the natural projection onto the Calkin algebra $\hat{\mathcal{R}} = \mathcal{R}/\mathcal{K}$ with an irreducible representation of $\hat{\mathcal{R}}$. If the Calkin algebra is commutative, then the non-identical irreducible representations of \mathcal{R} are one-dimensional and parametrized by the points of the compact set of maximal ideals of \mathcal{R} .

Example (2.2.1)[2]. Case of $S = C(\bar{\mathbb{D}})$. This is the C^* -extension of the commutative Banach algebra in Example (2.1.11). The C^* -algebra $\mathcal{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$, of operators acting on the weighted Bergman space $A_{\alpha_2+\lambda+1}^2(\bar{\mathbb{D}})$, is generated by all Toeplitz operators $T_a^{\alpha_2+\lambda+1}$ with symbols $a \in C(\bar{\mathbb{D}})$. It is known, that each operator T from $\mathcal{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$ can be represented as a compact perturbation of an initial generator, i.e., $T = T_a + K$ for some $a \in C(\bar{\mathbb{D}})$ and $K \in \mathcal{K}$.

We give a more detailed description of the algebra $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$ generated by all Toeplitz operators T_a^λ , with $a = a(z_1) \in C(\bar{\mathbb{D}})$, acting on the Bergman space $A_\lambda^2(\mathbb{B}^2)$. By Corollary (2.1.6) operators $T^\lambda \in \mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$ admit a decomposition into a countable direct sum

$$T^\lambda \simeq \bigotimes_{\alpha_2 \in \mathbb{Z}_+} T^{\alpha_2+\lambda+1}, \quad (19)$$

where, for each $\alpha_2 \in \mathbb{Z}_+$, the operator $T^{\alpha_2+\lambda+1}$ belongs to the algebra $\mathcal{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$.

Then (5), together with standard arguments, implies the following lemma.

Lemma (2.2.2)[2]. For each operator $T^\lambda \in \mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$,

$$\|T^\lambda\| = \sup_{\alpha_2 \in \mathbb{Z}_+} \|T^{\alpha_2+\lambda+1}\|.$$

Corollary (2.2.3)[2]. For initial generators T_a^λ , with $a = a(z_1) \in C(\bar{\mathbb{D}})$, of the algebra $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$ we have that $\|T_a^\lambda\| = \sup_{z_1 \in \bar{\mathbb{D}}} |a(z_1)| = \|a\|_\infty$. Denote by $C_\infty(\bar{\mathbb{D}})$ the dense subset

in $C(\bar{\mathbb{D}})$ which consists of all smooth functions whose derivatives admit a continuous extension to the boundary $S^1 = \partial\mathbb{D}$. Let $\mathcal{D}_\lambda = \mathcal{D}_\lambda(C^\infty(\bar{\mathbb{D}}))$ be the dense subalgebra of $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$ that consists of finite sums of finite products of Toeplitz operators with symbols from $C^\infty(\bar{\mathbb{D}})$, i.e. elements $T^\lambda \in \mathcal{D}_\lambda$ are of the form

$$T^\lambda = \sum_{k=1}^n \prod_{j_k=1}^{m_k} T_{a_{k,j_k}}^\lambda, \quad a_{k,j_k} \in C^\infty(\overline{\mathbb{D}}).$$

Lemma (2.2.4)[2]. Each element $T^\lambda \in \mathcal{D}_\lambda$, in the direct sum representation (19), is of the form

$$T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}),$$

where $a \in C^\infty(\overline{\mathbb{D}})$, each $K^{\alpha_2} \in \mathcal{L}(A_{\alpha_2+\lambda+1}^2(\mathbb{D}))$ is compact, and $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

Proof. It is sufficient to prove the lemma for finite products of m Toeplitz operators and by induction we can assume that $m = 2$. Thus, given $a, b \in C^\infty(\overline{\mathbb{D}})$, we consider

$$T_a^\lambda T_b^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_a^{\alpha_2+\lambda+1} T_b^{\alpha_2+\lambda+1}.$$

For each $\alpha_2 \in \mathbb{Z}_+$, we write $T_a^{\alpha_2+\lambda+1} T_b^{\alpha_2+\lambda+1} = T_{ab}^{\alpha_2+\lambda+1} + K^{\alpha_2}$, where K^{α_2} is the semicommutator of $T_a^{\alpha_2+\lambda+1}$ and $T_b^{\alpha_2+\lambda+1}$, i.e., $K^{\alpha_2} = T_a^{\alpha_2+\lambda+1} T_b^{\alpha_2+\lambda+1} - T_{ab}^{\alpha_2+\lambda+1}$. This operator is known to be compact. Finally, implies that $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

Now we are ready to prove:

Theorem (2.2.5)[2]. Each operator $T^\lambda \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$, in the direct sum decomposition (19), admits the following representation

$$T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}), \quad (20)$$

where $a \in C(\overline{\mathbb{D}})$, each K^{α_2} is compact, and $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

Proof. Given $T^\lambda \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$, there exists a fundamental sequence $\{T^{\lambda,k}\}_{k \in \mathbb{N}}$ of elements from \mathcal{D}_λ that converges in norm to T^λ . By the previous lemma, each $T^{\lambda,k}$ has the form

$$T^{\lambda,k} \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_{a_k}^{\alpha_2+\lambda+1,k}, \text{ with } T_{a_k}^{\alpha_2+\lambda+1,k} = T_{a_k}^{\alpha_2+\lambda+1} + K_k^{\alpha_2},$$

where $a_k \in C^\infty(\overline{\mathbb{D}})$ for each $k \in \mathbb{N}$ and $K_k^{\alpha_2}$ is compact with $K_k^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$, for fixed k . By Lemma (2.2.2), each sequence $\{T_{a_k}^{\alpha_2+\lambda+1,k}\}_{k \in \mathbb{N}}$ is also fundamental in $\mathcal{L}(\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D}))$.

In particular, for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $n, m > N_0$ we have the following estimate (uniform in α_2):

$$\|T_{a_k}^{\alpha_2+\lambda+1,n} - T_{a_k}^{\alpha_2+\lambda+1,m}\| \leq \|T^{\lambda,n} - T^{\lambda,m}\| < \frac{\varepsilon}{2}.$$

Observe now that $T_{a_k}^{\alpha_2+\lambda+1,n} - T_{a_k}^{\alpha_2+\lambda+1,m} = T_{(a_n-a_m)}^{\alpha_2+\lambda+1} + (K_n^{\alpha_2} - K_m^{\alpha_2})$. For any fixed $n, m > N_0$ we pass to the limit as $\alpha_2 \rightarrow \infty$. Then, taking into account Corollary (2.2.3) together with the observation that $K_n^{\alpha_2}$ and $K_m^{\alpha_2}$ both tend to 0 as $\alpha_2 \rightarrow \infty$, we have

$$\lim_{\alpha_2 \rightarrow \infty} \|T_{a_k}^{\alpha_2+\lambda+1,n} - T_{a_k}^{\alpha_2+\lambda+1,m}\| = \lim_{\alpha_2 \rightarrow \infty} \|T_{(a_n-a_m)}^{\alpha_2+\lambda+1}\| = \|a_n - a_m\|_\infty \leq \frac{\varepsilon}{2}.$$

Hence the function sequence $\{a_k\}_{k \in \mathbb{N}}$ is fundamental, and thus converges to some $a \in C(\overline{\mathbb{D}})$. Then, $\|T_{a_k}^{\alpha_2+\lambda+1} - T_a^{\alpha_2+\lambda+1}\| \leq \|a_k - a\|_\infty$ implies that, for each α_2 , the sequence

$\{T_{a_k}^{(\alpha_2+\lambda+1)}\}_{k \in \mathbb{N}}$ converges in norm to the operator $T_a^{\alpha_2+\lambda+1}$. Thus, for each α_2 , the sequence of compact operators $\{K_k^{\alpha_2+\lambda+1}\}_{k \in \mathbb{N}}$, being the difference of two convergent sequences $\{T^{\alpha_2+\lambda+1,k}\}_{k \in \mathbb{N}}$ and $\{T_{a_k}^{\alpha_2+\lambda+1}\}_{k \in \mathbb{N}}$, converges in norm to a compact operator K^{α_2} .

This implies the desired representation

$$T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T^{\alpha_2+\lambda+1} = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}.$$

It remains to prove that $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$. To do this we use the standard $\frac{\varepsilon}{3}$ trick. Using the representation

$$T^{\alpha_2+\lambda+1} - T^{\alpha_2+\lambda+1,k} = T_{(a-a_k)}^{\alpha_2+\lambda+1} + K^{\alpha_2} - K_k^{\alpha_2},$$

we obtain:

$$\|K^{\alpha_2}\| \leq \|T^{\alpha_2+\lambda+1} - T^{\alpha_2+\lambda+1,k}\| + \|T_{(a-a_k)}^{\alpha_2+\lambda+1}\| + \|K_k^{\alpha_2}\|.$$

Now, given any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$\|T^{\alpha_2+\lambda+1} - T^{\alpha_2+\lambda+1,k}\| \leq \|T^\lambda - T^{\lambda,k}\| < \frac{\varepsilon}{3}$ and $\|T_{(a-a_k)}^{\alpha_2+\lambda+1}\| \leq \|a - a_k\|_\infty < \frac{\varepsilon}{3}$, both uniformly in α_2 . With this fixed k and $\frac{\varepsilon}{3}$, there exists $N_0 \in \mathbb{N}$ such that for all $\alpha_2 > N_0$ we have that $\|K_k^{\alpha_2}\| < \frac{\varepsilon}{3}$. The above implies that for all $\alpha_2 > N_0$

$$\|K^{\alpha_2}\| < 3\frac{\varepsilon}{3} = \varepsilon.$$

Further information on the structure of the algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$ is given by the following two lemmas.

Lemma (2.2.6)[2]. The representation of $T^\lambda \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$ in the form (20) is unique.

Proof. Assume that

$$T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_{a_1}^{\alpha_2+\lambda+1} + K_1^{\alpha_2}) = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_{a_2}^{\alpha_2+\lambda+1} + K_2^{\alpha_2}),$$

where $a_j \in C(\overline{\mathbb{D}})$, each $K_j^{\alpha_2}$ is compact, and $K_j^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$, for $j = 1, 2$. Then

$$T_{a_1-a_2}^{\alpha_2+\lambda+1} + (K_1^{\alpha_2} - K_2^{\alpha_2}) = 0, \quad \text{for each } \alpha_2 \in \mathbb{Z}_+, \quad (21)$$

and thus

$$\begin{aligned} 0 &= \lim_{\alpha_2 \in \mathbb{Z}_+} \|T_{a_1-a_2}^{\alpha_2+\lambda+1} + (K_1^{\alpha_2} - K_2^{\alpha_2})\| = \lim_{\alpha_2 \in \mathbb{Z}_+} \|T_{a_1-a_2}^{\alpha_2+\lambda+1}\| \\ &= \|a_1 - a_2\|_\infty. \end{aligned}$$

Thus $a_1 = a_2$, and (21) implies that $K_1^{\alpha_2} = K_2^{\alpha_2}$ for all $\alpha_2 \in \mathbb{Z}_+$.

For fixed $\alpha_2 \in \mathbb{Z}_+$, we denote by $\mathfrak{T}_{\alpha_2+\lambda+1}(C(\overline{\mathbb{D}}))$ the set (algebra) of all operators $T_a^{\alpha_2+\lambda+1} = T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}$ that appear on the “ α_2 level” in the representation (20) of the operators $T^\lambda \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$.

Lemma (2.2.7)[2]. For $\alpha_2 \in \mathbb{Z}_+$, the algebra $\mathfrak{T}_{\alpha_2+\lambda+1}(C(\overline{\mathbb{D}}))$ coincides with algebra $\mathcal{T}_{\alpha_2+\lambda+1}(C(\overline{\mathbb{D}}))$, which is generated by all Toeplitz operators $T_a^{\alpha_2+\lambda+1}$, with $a \in C(\overline{\mathbb{D}})$.

Proof. From the decomposition $T_a^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_a^{\alpha_2+\lambda+1}$ we observe that $\mathfrak{T}_{\alpha_2+\lambda+1}(C(\overline{\mathbb{D}}))$ contains all Toeplitz operators $T_a^{\alpha_2+\lambda+1}$ with $a \in C(\overline{\mathbb{D}})$. Then

$$\begin{aligned}
\langle T_w^{\alpha_2+\lambda+1} w^n, w^m \rangle_{\alpha_2+\lambda+1, \mathbb{D}} &= \langle w^{n+1}, w^m \rangle_{\alpha_2+\lambda+1, \mathbb{D}} \\
&= \begin{cases} \frac{\Gamma(n+2)\Gamma(\alpha_2+\lambda+3)}{\Gamma(n+\alpha_2+\lambda+4)}, & \text{if } m = n+1, \\ 0, & \text{otherwise,} \end{cases} \\
\langle T_{\bar{w}}^{\alpha_2+\lambda+1} w^n, w^m \rangle_{\alpha_2+\lambda+1, \mathbb{D}} &= \langle w^n, w^{m+1} \rangle_{\alpha_2+\lambda+1, \mathbb{D}} \\
&= \begin{cases} \frac{\Gamma(n+1)\Gamma(\alpha_2+\lambda+3)}{\Gamma(n+\alpha_2+\lambda+3)}, & \text{if } m = n-1, \\ 0, & \text{otherwise,} \end{cases} \\
\langle T_{\bar{w}} T_w^{\alpha_2+\lambda+1} w^n, w^m \rangle_{\alpha_2+\lambda+1, \mathbb{D}} &= \langle w^{n+1}, w^{m+1} \rangle_{\alpha_2+\lambda+1, \mathbb{D}} \\
&= \begin{cases} \frac{\Gamma(n+2)\Gamma(\alpha_2+\lambda+3)}{\Gamma(n+\alpha_2+\lambda+4)}, & \text{if } m = n, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

which implies that the operators from $\mathfrak{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$ do not have common non-trivial invariant subspaces. Therefore the identical representation of the C^* -algebra $\mathfrak{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$ is irreducible.

The algebra $\mathfrak{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$ obviously contains non-trivial compact operators, and thus by the above it contains the full ideal of compact operators. That is, $\mathfrak{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$ contains all operators of the form $T_a^{\alpha_2+\lambda+1} + K$, where $a \in C(\bar{\mathbb{D}})$ and K is compact, and thus coincides with $\mathcal{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$.

Next, we classify the irreducible representations of $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$. Since each subspace in the direct sum decomposition (3) is invariant under the action of operators in $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$, each irreducible representation of the algebra $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$ is formed by the restriction of elements in $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$ onto an invariant subspace followed (according to the result of Lemma (2.2.7)) by an irreducible representation of the corresponding algebra $\mathcal{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}}))$. All irreducible representations of the latter algebra are well known. They consist of the infinite dimensional identical representation ι and the one-dimensional representations π_t , parameterized by points $t \in S^1 = \partial\mathbb{D}$. More precisely, π_t has the form

$$\pi_t : \mathcal{T}_{\alpha_2+\lambda+1}(C(\bar{\mathbb{D}})) \ni T_a^{\alpha_2+\lambda+1} + K \mapsto a(t) \in \mathbb{C}.$$

We obtain the following (not yet complete) list of irreducible representations of the C^* -algebra $\mathcal{T}_\lambda(C(\bar{\mathbb{D}}))$:

- a countable family of infinite dimensional representations ι_{α_2} induced by the identical representations on the spaces $A_{\alpha_2+\lambda+1}^2(\mathbb{D})$, $\alpha_2 \in \mathbb{Z}_+$:

$$\iota_{\alpha_2} : T^\lambda = \bigoplus_{\beta_2 \in \mathbb{Z}_+} (T_a^{\beta_2+\lambda+1} + K^{\beta_2}) \mapsto T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}. \quad (22)$$

- an uncountable family of one-dimensional representations $\pi_{t,\alpha_2}, (t, \alpha_2) \in S^1 \times \mathbb{Z}_+$, defined by

$$\pi_{t,\alpha_2} : T^\lambda \simeq \bigoplus_{\beta_2 \in \mathbb{Z}_+} (T_a^{\beta_2 + \lambda + 1} + K^{\beta_2}) \mapsto a(t).$$

Of course, for a fixed t , all one-dimensional representations $\pi_{t,\alpha_2}, (t, \alpha_2) \in S^1 \times \mathbb{Z}_+$, are unitary equivalent. Thus for each $t \in S^1$ we have in fact just one representation

$$\pi_t : T^\lambda \simeq \bigoplus_{\beta_2 \in \mathbb{Z}_+} (T_a^{\beta_2 + \lambda + 1} + K^{\beta_2}) \mapsto a(t). \quad (23)$$

which is of infinite multiplicity.

At the same time, the infinite dimensional representations $\iota_{\alpha_2}, \alpha_2 \in \mathbb{Z}_+$, are not pairwise unitary equivalent. To see this, let us consider

$$\iota_{\alpha_2} \left(T_{1-|z_1|^2}^\lambda \right) = T_{1-|z_1|^2}^{\alpha_2 + \lambda + 1}$$

The Toeplitz operator on the right has a radial symbol and is unitary equivalent to a diagonal operator with eigenvalue sequence

$$\gamma_{1-|z_1|^2}^{\alpha_2 + \lambda + 1} = \left\{ \gamma_{1-|z_1|^2}^{\alpha_2 + \lambda + 1}(k) \right\}_{k \in \mathbb{Z}_+},$$

where

$$\begin{aligned} \gamma_{1-|z_1|^2}^{\alpha_2 + \lambda + 1}(k) &= 1 - \frac{\Gamma(k + \alpha_2 + \lambda + 3)}{\Gamma(k + 1)\Gamma(\alpha_2 + \lambda + 2)} - \int_0^1 r^{k+1}(1-r)^{\alpha_2 + \lambda + 1} dr \\ &= \frac{\alpha_2 + \lambda + 2}{k + \alpha_2 + \lambda + 3}. \end{aligned}$$

Now,

$$\left\| T_{1-|z_1|^2}^{\alpha_2 + \lambda + 1} \right\| = \sup_{k \in \mathbb{Z}_+} \gamma_{1-|z_1|^2}^{\alpha_2 + \lambda + 1}(k) = \frac{\alpha_2 + \lambda + 2}{\alpha_2 + \lambda + 3},$$

and, as the norms $\left\| T_{1-|z_1|^2}^{\alpha_2 + \lambda + 1} \right\|$ are different for different α_2 , the representations ι_{α_2} , for different α_2 , cannot be unitary equivalent.

A rather unexpected additional series of one-dimensional irreducible representations of the algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$ is induced by the Berezin quantization on the hyperbolic unit disk.

Recall that the Berezin transform of a bounded linear operator A acting on the weighted Bergman space $\mathcal{A}_\mu^2(\mathbb{D})$ is defined as follows

$$B_\mu(A)(\zeta) = \langle A k_\zeta, k_\zeta \rangle_{\mu, \mathbb{D}},$$

where $k_\zeta(w) = \frac{(1-|\zeta|^2)^{1+\mu/2}}{(1-\overline{\zeta}w)^{2+\mu}}$ is the normalized Bergman kernel in $\mathcal{A}_\mu^2(\mathbb{D})$. In particular, if $A = T_a$, with $a \in L_\infty(\mathbb{D})$, then

$$B_\mu(T_a)(\zeta) = \langle T_a k_\zeta, k_\zeta \rangle_{\mu, \mathbb{D}} = \langle a k_\zeta, k_\zeta \rangle_{\mu, \mathbb{D}} =: B_\mu(a)(\zeta).$$

Lemma (2.2.8)[2]. The mapping $\rho : \mathcal{T}_\lambda(C(\overline{D})) \longrightarrow C(\overline{D})$, defined by

$$\rho : T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T^{\alpha_2 + \lambda + 1} \mapsto \lim_{\alpha_2 \rightarrow \infty} B_{\alpha_2 + \lambda + 1}(T^{\alpha_2 + \lambda + 1}) \quad (24)$$

is a continuous $*$ -homomorphism of the C^* -algebra $\mathcal{T}_\lambda(C(\overline{D}))$ onto $C(\overline{D})$.

Proof. Recall that all operators $T^{\alpha_2 + \lambda + 1}$ are of the form $T_a^{\alpha_2 + \lambda + 1} + K^{\alpha_2}$ with a common function $a \in C(\overline{D})$ and compact operators K^{α_2} obeying the property $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

From the standard estimate

$$|B_{\alpha_2 + \lambda + 1}(K^{\alpha_2 + \lambda + 1})| \leq \|K^{\alpha_2 + \lambda + 1}\|$$

it follows that $\lim_{\alpha_2 \rightarrow \infty} B_{\alpha_2 + \lambda + 1}(K^{\alpha_2 + \lambda + 1}) = 0$. Furthermore, by (a variant of the correspondence principle for Berezin quantization), we have that

$$\lim_{\alpha_2 \rightarrow \infty} B_{\alpha_2 + \lambda + 1}(T_a^{\alpha_2 + \lambda + 1}) = a$$

uniformly on \overline{D} . Thus,

$$\begin{aligned} \lim_{\alpha_2 \rightarrow \infty} B_{\alpha_2 + \lambda + 1}(T^{\alpha_2 + \lambda + 1}) &= \lim_{\alpha_2 \rightarrow \infty} B_{\alpha_2 + \lambda + 1}(T_a^{\alpha_2 + \lambda + 1} + K^{\alpha_2}) \\ &= \lim_{\alpha_2 \rightarrow \infty} B_{\alpha_2 + \lambda + 1}(T_a^{\alpha_2 + \lambda + 1}) = a \in C(\overline{D}), \end{aligned}$$

and (24) is well-defined. The mapping ρ is onto as for each $a \in C(\overline{D})$ we have that

$\rho(T_a^\lambda) = a$. For $T_1^{\alpha_2 + \lambda + 1} = T_{a_1}^{\alpha_2 + \lambda + 1} + K_1^{\alpha_2}$ and $T_2^{\alpha_2 + \lambda + 1} = T_{a_2}^{\alpha_2 + \lambda + 1} + K_2^{\alpha_2}$ we have

$$\begin{aligned} T_1^{\alpha_2 + \lambda + 1} + T_2^{\alpha_2 + \lambda + 1} &= T_{a_1 + a_2}^{\alpha_2 + \lambda + 1} + K_+^{\alpha_2}, \\ T_1^{\alpha_2 + \lambda + 1} \times T_2^{\alpha_2 + \lambda + 1} &= T_{a_1 \cdot a_2}^{\alpha_2 + \lambda + 1} + K_\times^{\alpha_2}, \\ (T_1^{\alpha_2 + \lambda + 1})^* &= T_{a_1}^{\alpha_2 + \lambda + 1} + (K_1^{\alpha_2})^*, \end{aligned}$$

which implies that ρ is a $*$ -homomorphism.

The continuity follows from the inequality

$$\|a\|_\infty = \sup_{z_1 \in \overline{D}} |a(z_1)| = \lim_{\alpha_2 \rightarrow \infty} \|T_a^{\alpha_2 + \lambda + 1}\| \leq \sup_{\alpha_2 \in \mathbb{Z}_+} \|T_a^{\alpha_2 + \lambda + 1} + K^{\alpha_2}\| = \|T^\lambda\|.$$

Corollary (2.2.9)[2]. For each $z_1 \in \overline{D}$, the mapping $\rho_{z_1} : \mathcal{T}_\lambda(C(\overline{D})) \rightarrow \mathbb{C}$, defined by

$$\rho_{z_1} : T^\lambda \mapsto \rho(T^\lambda) = a \mapsto a(z_1) \in \mathbb{C},$$

is a one-dimensional representation of the C^* -algebra $\mathcal{T}_\lambda(C(\overline{D}))$.

Remark (2.2.10)[2]. Given $T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T^{\alpha_2 + \lambda + 1} \in \mathcal{T}_\lambda(C(\overline{D}))$, the result of Lemma (2.2.8) permits us to recover its unique (by Lemma (2.2.6)) representation (20). Indeed, all necessary data for the representation (20) are given by

$$a = \rho(T^\lambda) = \lim_{\alpha_2 \rightarrow \infty} B_{\alpha_2 + \lambda + 1}(T^{\alpha_2 + \lambda + 1}) \in C(\overline{D})$$

and

$$K^{\alpha_2} = T^{\alpha_2 + \lambda + 1} - T_a^{\alpha_2 + \lambda + 1}.$$

Our next aim is to show that the above described irreducible representations of $\mathcal{T}_\lambda(C(\overline{D}))$ exhaust all its irreducible representations. Observe first that for each

$$T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_a^{\alpha_2 + \lambda + 1} + K^{\alpha_2}) \in \mathcal{T}_\lambda(C(\overline{D})), \quad (25)$$

the operator $K = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_a^{\alpha_2} + K^{\alpha_2}$ is compact on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$, being the norm-limit of the compact operators $K_n = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_a^{\alpha_2} + K^{\alpha_2}$ as $n \rightarrow \infty$.

We denote by $\mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$ the closed two-sided ideal in $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$ generated by all such compact operators K . It is easy to see that $\mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$ is just the set of all operators of the form $K \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$. Let $\mathcal{K}_{\mathcal{A}_\lambda^2(\mathbb{B}^2)}$ stand for the closed two-sided ideal of all compact operators on $\mathcal{A}_\lambda^2(B_2)$.

Lemma (2.2.11)[2]. We have that $\mathcal{T}_\lambda(C(\overline{\mathbb{D}})) \cap \mathcal{K}_{\mathcal{A}_\lambda^2(B_2)} = \mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$.

Proof. Note that the operator T^λ in (25) is compact on $\mathcal{A}_\lambda^2(\mathbb{B}^2)$ if and only if each operator $T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}$ is compact on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$ for each $\alpha_2 \in \mathbb{Z}_+$, and

$$\lim_{\alpha_2 \rightarrow \infty} T_a^{\alpha_2+\lambda+1} + K^{\alpha_2} = 0,$$

which is equivalent to $T_a^{\alpha_2+\lambda+1}$ being compact on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$ for each $\alpha_2 \in \mathbb{Z}_+$, and

$$\lim_{\alpha_2 \rightarrow \infty} T_a^{\alpha_2+\lambda+1} = \|a\|_\infty = 0.$$

The last conditions are equivalent to $T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_a^{\alpha_2+\lambda+1} + K^{\alpha_2} \in \mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$.

Observe now that the kernel of the mapping (24) coincides with the ideal $\mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$. This implies that the Calkin algebra $\hat{\mathcal{T}}_\lambda = \mathcal{T}_\lambda(C(\overline{\mathbb{D}})) / \mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$ is isomorphic and isometric to $C(\overline{\mathbb{D}})$, and that the one-dimensional representations ρ_z of Corollary (2.2.9) as well as the representations π_t of (23) come from the one-dimensional representations of $\hat{\mathcal{T}}_\lambda$. Recall, that if J is a closed two sided ideal of a C^* -algebra \mathcal{A} then each irreducible representation of \mathcal{A} is either induced by an irreducible representation of the quotient algebra \mathcal{A}/J , or is an extension to \mathcal{A} of an irreducible representation of J .

That is, what is left, is a description of the representations being extensions to $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$ of the irreducible representation of $\mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$. Recall that each summand in (3) is an invariant subspace for $\mathcal{K}_\lambda(C(\overline{\mathbb{D}}))$, whose restriction on “the level α_2 ” coincides with the ideal of all compact operators on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$. Thus its identical irreducible representation induces the infinite dimensional irreducible representation ι_{α_2} of the form (22).

That is, we listed above all (up to unitary equivalence) irreducible representations of the C^* -algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$.

As a byproduct of the description of the Calkin algebra $\hat{\mathcal{T}}_\lambda$, we have the following proposition.

Proposition (2.2.12)[2]. An operator $T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}) \in \hat{\mathcal{T}}_\lambda(C(\overline{\mathbb{D}}))$ is Fredholm if and only if $a(z_1) \neq 0$ for all $z_1 \in \overline{\mathbb{D}}$. In the case of Fredholmness, $\text{Ind } T^\lambda = 0$. The essential spectrum of T^λ is given by $\text{ess-sp } T^\lambda = \text{Range}(a)$.

The following clarifying observation seems to be useful here. In the case of Fredholmness of the operator $T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_a^{\alpha_2+\lambda+1} + K^{\alpha_2})$, its two sided regularizer can be taken in the form $R^\lambda T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_{1/a}^{\alpha_2+\lambda+1})$. So that

$$T^\lambda R^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (I + K_1^{\alpha_2}) \text{ and } R^\lambda T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (I + K_2^{\alpha_2}),$$

where both $K_1^{\alpha_2}$ and $K_2^{\alpha_2}$ tend to 0 as α_2 tends to infinity. Thus the norms $K_1^{\alpha_2}$ and $K_2^{\alpha_2}$ become less than 1 for all α_2 greater than some α_2^0 , which implies that the operators $T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}$, with $\alpha_2 > \alpha_2^0$, are all invertible. That is, in the direct sum decomposition $T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_a^{\alpha_2+\lambda+1} + K^{\alpha_2})$ of zero index Fredholm operators, only a finite number of operators $T_a^{\alpha_2+\lambda+1} + K^{\alpha_2}$ may not be invertible. This implies that $\dim \ker T^\lambda = \dim \operatorname{coker} T^\lambda$ is finite, as it should be.

Lemma (2.2.13)[2]. The intersection $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty) \cap \mathcal{K}_{\mathcal{A}_\lambda^2}(\mathbb{B}^2)$ consists of all operators of the form $\mathbf{K}^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} K^{\alpha_2}$, where each K^{α_2} is compact on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$, and $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

Proof. As each subspace of the direct sum decomposition (3) is invariant for operators from $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty) \cap \mathcal{K}_{\mathcal{A}_\lambda^2}(\mathbb{B}^2)$, each compact operator \mathbf{K}^λ in $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$ is of the form $\bigoplus_{\alpha_2 \in \mathbb{Z}_+} K^{\alpha_2}$, where K^{α_2} is compact on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$, for all $\alpha_2 \in \mathbb{Z}_+$, and $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

Take now any sequence $\{K^{\alpha_2}\}_{\alpha_2 \in \mathbb{Z}_+}$, where K^{α_2} is compact on $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$ and $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$. It remains to show that $\mathbf{K}^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} K^{\alpha_2}$ belongs to $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$. Observe that, for each fixed $\alpha_2^0 \in \mathbb{Z}_+$, the sequence $\gamma_{\alpha_2^0}(\alpha_2) = \delta_{\alpha_2^0, \alpha_2}$ belongs to $c_0 \subset SO(\mathbb{Z}_+)$, and thus the projection $P_{\alpha_2^0}$ onto the α_2^0 -level in the decomposition (3) belongs to the algebra $\mathcal{T}_\lambda(L_\infty) \subset \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$. Then, by Lemma (2.2.7), the algebra $P_{\alpha_2^0} \mathcal{T}_\lambda(C(\overline{\mathbb{D}})) P_{\alpha_2^0} \subset \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$ contains the compact operator $\mathbf{K}_{\alpha_2^0} = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \delta_{\alpha_2^0, \alpha_2} K^{\alpha_2}$, and thus the operator

$$\mathbf{K}^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} K^{\alpha_2} = \lim_{n \rightarrow \infty} \bigoplus_{\alpha_2^0=0}^n K^{\alpha_2^0} = \lim_{n \rightarrow \infty} \sum_{\alpha_2^0=0}^n \mathbf{K}_{\alpha_2^0}$$

belongs to $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$.

Below we frequently consider the tensor product $\mathcal{A} \otimes \mathcal{B}$ of two commutative C^* -algebras \mathcal{A} and \mathcal{B} . Recall that as \mathcal{A} and \mathcal{B} are commutative, and thus nuclear, the C^* -norm on $\mathcal{A} \otimes \mathcal{B}$ is uniquely defined. In particular, if $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ are the (locally) compact sets of maximal ideals of \mathcal{A} and \mathcal{B} , respectively, then

$$\mathcal{A} \otimes \mathcal{B} \cong C(M_{\mathcal{A}}) \otimes C(M_{\mathcal{B}}) = C(M_{\mathcal{A}} \times M_{\mathcal{B}}).$$

The algebraic tensor product of \mathcal{A} and \mathcal{B} , which consists of all finite sums of the form $\sum a_k \otimes b_k$, $a_k \in \mathcal{A}$ and $b_k \in \mathcal{B}$, we will denote by $\mathcal{A} \otimes_a \mathcal{B}$.

Corollary (2.2.14)[2]. All compact Toeplitz operators from the algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$ are of the form

$$\bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_{d(z_1, \alpha_2)}^{\alpha_2+\lambda+1}, \quad \text{where } d = d(z_1, \alpha_2) \in C_0(\overline{\mathbb{D}}) \otimes c_0.$$

Here $C_0(\overline{\mathbb{D}})$ denotes the set of functions from $C(\overline{\mathbb{D}})$ that vanish on the boundary $\partial \mathbb{D} = S^1$.

The next theorem gives the description of elements from $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$.

Theorem (2.2.15)[2]. Each element $T^\lambda \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$ admits the following representation

$$T^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \left(T_{c(z_1, \alpha_2)}^{\alpha_2 + \lambda + 1} + K^{\alpha_2} \right), \quad (26)$$

where $c = c(z_1, \alpha_2) \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$, K^{α_2} is compact for all $\alpha_2 \in \mathbb{Z}_+$, and $K^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

Proof. Observe first that the theorem is valid for elements of the dense subalgebra D formed by all operators of the form

$$T^\lambda = \sum_{k=1}^n \prod_{j_k=1}^{m_k} T_{a_{k,j_k}}^\lambda b_{k,j_k},$$

where $a_{k,j_k} \in C(\overline{\mathbb{D}})$ and $b_{k,j_k} \in L_\infty(0, 1)$.

To see this it is sufficient to consider, as in Lemma (2.2.4), just a product of two operators:

$$\begin{aligned} T_{a_1 b_1}^\lambda T_{a_2 b_2}^\lambda &= \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \gamma_{b_1}^\lambda(\alpha_2) \gamma_{b_2}^\lambda(\alpha_2) \left(T_{a_1}^{\alpha_2 + \lambda + 1} + K_1^{\alpha_2} \right) \left(T_{a_2}^{\alpha_2 + \lambda + 1} + K_2^{\alpha_2} \right) \\ &= \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \gamma_{b_1}^\lambda(\alpha_2) \gamma_{b_2}^\lambda(\alpha_2) \left(T_{a_1 a_2}^{\alpha_2 + \lambda + 1} + K_{12}^{\alpha_2} \right) = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \left(T_c^{\alpha_2 + \lambda + 1} + K^{\alpha_2} \right), \end{aligned}$$

where $c = c(z_1, \alpha_2) = a_1 a_2(z_1) \gamma_{b_1}^\lambda \gamma_{b_2}^\lambda(\alpha_2) \in C(\overline{\mathbb{D}}) \otimes_a SO(\mathbb{Z}_+)$, and $K^{\alpha_2} = \gamma_{b_1}^\lambda \gamma_{b_2}^\lambda(\alpha_2) K_{12}^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

We show now that each operator of the form (26) belongs to the algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$. Indeed, given any $c = c(z_1, \alpha_2) \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$, there exists a sequence of functions $c_n \in C(\overline{\mathbb{D}}) \otimes_a SO(\mathbb{Z}_+)$ that converges uniformly to c . Thus the operator $T_c^\lambda = \lim_{n \rightarrow \infty} T_{c_n}^\lambda$, belongs to $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$, and Lemma (2.2.13) implies the conclusion.

Given $T^\lambda \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$, there is a sequence of operators

$$T_n^\lambda \bigoplus_{\alpha_2 \in \mathbb{Z}_+} T_n^{\alpha_2 + \lambda + 1} = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_{c_n}^{\alpha_2 + \lambda + 1} + K_n^{\alpha_2}) \in D$$

that converges in norm to the operator T^λ . As the sequence $\{T_n^\lambda\}_{n \in \mathbb{N}}$ is fundamental, for each $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$ such that for all $n, m > N_0$ and all $\alpha_2 \in \mathbb{N}$ we have that

$$\left\| T_n^{\alpha_2 + \lambda + 1} - T_m^{\alpha_2 + \lambda + 1} \right\| < \frac{\varepsilon}{2}.$$

Recall, that the compact set of maximal ideals (multiplicative functionals) of the algebra $SO(\mathbb{Z}_+)$ has the form $M(SO(\mathbb{Z}_+)) = \mathbb{Z}_+ \cup M_\infty$, where each $\alpha_2 \in \mathbb{Z}_+$ is identified with the evaluation functional $\gamma \in SO(\mathbb{Z}_+) \mapsto \gamma(\alpha_2)$, and the fiber M_∞ is closed, connected, and consists of all functionals μ obeying the property $\mu(\gamma) = 0$ for all $\gamma \in c_0$. The points of M_∞ are responsible for the partial limit values of sequences in $SO(\mathbb{Z}_+)$, none of the points $\mu \in M_\infty$ can be reached by subsequences of \mathbb{Z}_+ ; its topological nature requires to use nets (subnets of \mathbb{Z}_+). That is, for each point $\mu \in M_\infty$ there is a net $\{\alpha_2^\eta\}_{\eta \in E}$, valued in \mathbb{Z}_+ , converging to μ in the Gelfand topology of $M(SO(\mathbb{Z}_+))$.

Fix $\mu \in M_\infty$, and let $\{\alpha_2^\eta\}_{\eta \in E}$ be a net converging to μ . Then,

$$\begin{aligned}
\lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} \left(T_n^{\alpha_2^\eta + \lambda + 1} \right) &= \lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} \left(T_{c_n}^{\alpha_2^\eta + \lambda + 1} + K_n^{\alpha_2^\eta} \right) \\
&= \lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} \left(T_{c_n}^{\alpha_2^\eta + \lambda + 1} \right) \\
&= \lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} \left(c_n(z_1, \alpha_2^\eta) - c_n(z_1, \mu) \right) + \lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} (c_n(z_1, \mu)) \\
&= 0 + c_n(z_1, \mu) = c_n(z_1, \mu).
\end{aligned} \tag{27}$$

Thus, for the above ε, N_0 , and $n, m > N_0$, we have

$$\begin{aligned}
\|c_n(z_1, \mu) - c_m(z_1, \mu)\|_\infty &= \lim_{\alpha_2^\eta} \left\| B_{\alpha_2^\eta + \lambda + 1} \left(T_n^{\alpha_2^\eta + \lambda + 1} - T_m^{\alpha_2^\eta + \lambda + 1} \right) \right\|_\infty \\
&\leq \sup_{\alpha_2 \in \mathbb{Z}_+} \left\| T_n^{\alpha_2 + \lambda + 1} - T_m^{\alpha_2 + \lambda + 1} \right\| = \|T_n^\lambda - T_m^\lambda\| < \frac{\varepsilon}{2}.
\end{aligned}$$

The above estimate is uniform in $\mu \in M_\infty$, thus the sequence $\{c_n|_{\overline{\mathbb{D}} \times M_\infty}\}$ converges uniformly to a certain function $c_\infty = c_\infty(z_1, \mu) \in C(\overline{\mathbb{D}} \times M_\infty)$.

Fix now any $\alpha_2 \in \mathbb{Z}_+$. Since each $c_n(\cdot, \alpha_2)$ is continuous on $\overline{\mathbb{D}}$ and $K_n^{\alpha_2}$ is compact, for the above $\varepsilon, N_0, n, m > N_0$, and each $t \in S^1 = \partial \overline{\mathbb{D}}$, we have:

$$\begin{aligned}
\lim_{z_1 \rightarrow t} \left| B_{\alpha_2 + \lambda + 1} \left(T_n^{\alpha_2 + \lambda + 1} - T_m^{\alpha_2 + \lambda + 1} \right) (z_1) \right| &= \\
&= \lim_{z_1 \rightarrow t} \left| B_{\alpha_2 + \lambda + 1} \left(T_{c_n - c_m}^{\alpha_2 + \lambda + 1} \right) (z_1) + B_{\alpha_2 + \lambda + 1} \left(K_n^{\alpha_2 + \lambda + 1} - K_m^{\alpha_2 + \lambda + 1} \right) (z_1) \right| \\
&= |(c_n - c_m)(t) + 0| \leq \sup_{t \in S^1} |(c_n - c_m)(t)| \leq \|T_n^\lambda - T_m^\lambda\| < \frac{\varepsilon}{2}.
\end{aligned}$$

That is, for each $\alpha_2 \in \mathbb{Z}_+$, the sequence of restrictions $\{c_n|_{C(S^1)}\}$ converges uniformly to a certain function $c_{\alpha_2} = c_{\alpha_2}(z_1) \in C(S^1)$. Moreover, the function

$$\hat{c} = \begin{cases} c_\infty(z_1, \mu), & \text{for } (z_1, \mu) \in \overline{\mathbb{D}} \times M_\infty \\ c_{\alpha_2}(z_1), & \text{for } (z_1, \alpha_2) \in S^1 \times \mathbb{Z}_+ \end{cases}$$

is continuous on the closed subset $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$ of $\overline{\mathbb{D}} \times M(SO(\mathbb{Z}_+))$. Thus, by Tietze's theorem, \hat{c} admits a continuous extension (which we will still denote by \hat{c}) from $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$ to $\overline{\mathbb{D}} \times M(SO(\mathbb{Z}_+))$, with

$$\|\hat{c}\|_{C(\overline{\mathbb{D}} \times M(SO(\mathbb{Z}_+)))} = \|\hat{c}\|_{C((\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+))}.$$

We denote now by $c = c(z_1, \alpha_2)$ the function from $C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$ whose Gelfand transform coincides with \hat{c} .

For each $n \in \mathbb{N}$, the function

$$\hat{d}_n = \begin{cases} \hat{c}(z_1, \mu) - \hat{c}_n(z_1, \mu), & \text{for } (z_1, \mu) \in \overline{\mathbb{D}} \times M_\infty \\ c(z_1, \alpha_2) - c_n(z_1, \alpha_2), & \text{for } (z_1, \alpha_2) \in S^1 \times \mathbb{Z}_+ \end{cases}$$

where \hat{c}_n is the Gelfand transform of the function $c_n \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$, is continuous on $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$. Thus, by Tietze's theorem, \hat{d}_n admits a continuous extension preserving the norm, which we will keep denoting by \hat{d}_n , from $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$ onto $\overline{\mathbb{D}} \times M(SO(\mathbb{Z}_+))$. We denote now by $dn = dn(z_1, \alpha_2)$ the function from $C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$,

whose Gelfand transform coincides with \hat{d}_n .

Observe that for the above ε, N_0 , and all $n > N_0$ we have both

$$\|d_n\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\hat{d}_n\| < \frac{\varepsilon}{2}.$$

With these data we introduce $c'_n = c - d_n \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$, and observe that on $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$

$$\hat{c}'_n = \hat{c} - \hat{d}_n = \hat{c} - \hat{c} + \hat{c}_n = \hat{c}_n,$$

that is $c'_n - c_n \in C_0(\overline{\mathbb{D}}) \otimes c_0$, and thus (by Corollary (2.2.14)), the Toeplitz operators $T_{c'_n - c_n}^\lambda$ are compact for all $n \in \mathbb{N}$. Moreover the sequence of functions c'_n converges uniformly to $c \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$. For each $n \in \mathbb{N}$, we have

$$T_n^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \left(T_{c_n}^{\alpha_2 + \lambda + 1} + K_n^{\alpha_2} \right) = \bigoplus_{\alpha_2 \in \mathbb{Z}_+} \left(T_{c'_n}^{\alpha_2 + \lambda + 1} + K_n^{\alpha_2} + T_{c_n - c'_n}^{\alpha_2 + \lambda + 1} \right),$$

where $K_n^{\alpha_2} = K_n^{\alpha_2} + T_{c_n - c'_n}^{\alpha_2 + \lambda + 1}$ is compact for each $\alpha_2 \in \mathbb{Z}_+$, and $K_n^{\alpha_2} \rightarrow 0$ as $\alpha_2 \rightarrow \infty$.

The end of the proof now repeats literally the end of the proof of Theorem (2.2.5).

Remark (2.2.16)[2]. Theorem (2.2.15) ensures that any operator $T^\lambda \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$ admits a representation of the form (26). Given an operator T^λ , there is a simple procedure to recover this representation. First of all the operator T^λ uniquely defines a function continuous on $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$. Indeed, let $\mu \in M_\infty$, and let $\{\alpha_2^\eta\}_{\eta \in E}$ be a net converging to μ . Then, as in (27), we have

$$\begin{aligned} \lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} \left(T^{\alpha_2^\eta + \lambda + 1} \right) &= \lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} \left(T_c^{\alpha_2^\eta + \lambda + 1} + K_2^{\alpha_2^\eta} \right) \\ &= \lim_{\alpha_2^\eta} B_{\alpha_2^\eta + \lambda + 1} \left(T_c^{\alpha_2^\eta + \lambda + 1} \right) = c(z_1, \mu). \end{aligned} \quad (28)$$

Fix now any $\alpha_2 \in \mathbb{Z}_+$, then

$$\lim_{|z_1| \rightarrow 1^-} B_{\alpha_2 + \lambda + 1} \left(T_c^{\alpha_2 + \lambda + 1} + K^{\alpha_2} \right) = c|_{S^1 \times \{\alpha_2\}}.$$

Having the function c continuous on $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$ we extend it continuously to $\overline{\mathbb{D}} \times (\mathbb{Z}_+ \cup M_\infty)$ (e.g. by Tietze's theorem) and denote by the same letter c the function from $C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$ whose Gelfand transform coincides with this extension.

We note that any other extension of $c \in C((\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+))$ defines a function

$$c' \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+) \text{ with } c - c' \in C_0(\overline{\mathbb{D}}) \otimes c_0.$$

By Corollary (2.2.14), the difference $T_c^\lambda - T_{c'}^\lambda = T_{c - c'}^\lambda$ only affects the compact part of the representation (26). Finally, with such $c \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$, put $K^{\alpha_2} = T^{\alpha_2 + \lambda + 1} - T_c^{\alpha_2 + \lambda + 1}$.

Next, we describe the irreducible representations of the C^* -algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$. By Corollary (2.1.6) its generators T_{ab}^λ , where $a \in C(\overline{\mathbb{D}})$ and $b \in L_\infty(0, 1)$ split into the following direct sum of operators

$$T_{ab}^\lambda = \bigoplus_{\alpha_2 \in \mathbb{Z}} + \gamma_b^\lambda(\alpha_2) T_a^{\alpha_2 + \lambda + 1},$$

according to the direct sum decomposition (3). Such a splitting implies the following list of irreducible representations of $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$: according to the direct sum decomposition (3). Such a splitting implies the following list of irreducible representations of $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$:

- infinite dimensional (identical) representations ι_{α_2} on the spaces $\mathcal{A}_{\alpha_2+\lambda+1}^2(\mathbb{D})$, $\alpha_2 \in \mathbb{Z}_+$, defined by

$$\iota_{\alpha_2} : T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_c^{\alpha_2+\lambda+1} + K^{\alpha_2}) \mapsto T_c^{\alpha_2+\lambda+1} + K^{\alpha_2};$$

- one-dimensional representations π_{t,α_2} , $(t, \alpha_2) \in S^1 \times \mathbb{Z}_+$, defined by

$$\pi_{t,\alpha_2} : T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_c^{\alpha_2+\lambda+1} + K^{\alpha_2}) \mapsto c(t, \alpha_2).$$

It is easy to see that these representations are not pairwise unitary equivalent, and that the ambiguity of the form (26), does not effect the action of the above representations.

As in the case of the algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$, there is a series of one-dimensional representations of the C^* -algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$, induced by the Berezin quantization procedure on the hyperbolic disk \mathbb{D} .

For a fixed point $\mu \in M_\infty$, let $\{\alpha_2^\eta\}$ be a \mathbb{Z}_+ -valued net that converges to μ . Then, by Theorem (2.2.15) and (28), we have the well-defined map

$$\rho_\mu : T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_c^{\alpha_2+\lambda+1} + K^{\alpha_2}) \mapsto c(\cdot, \mu),$$

which is easily seen to be a $*$ -homomorphism of $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$ onto $C(\overline{\mathbb{D}})$. The map ρ_μ induces a family of one-dimensional representations of $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$, defined for each $(z_1, \mu) \in \mathbb{D} \times M_\infty$ as follows

$$\rho(z_1, \mu) : T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_c^{\alpha_2+\lambda+1} + K^{\alpha_2}) \xrightarrow{\rho_\mu} c(\cdot, \mu) \in C(\overline{\mathbb{D}}) \mapsto c(z_1, \mu) \in \mathbb{C}.$$

Observe now that the difference of $T_1^\lambda = T_{c_1}^\lambda + K_1^\lambda$ and $T_2^\lambda = T_{c_2}^\lambda + K_2^\lambda$ is compact if and only if $T_{c_1}^\lambda - T_{c_2}^\lambda = T_{c_1-c_2}^\lambda$ is compact, or if and only if $c_1 - c_2 \in C_0(\mathbb{D}) \otimes c_0$. This implies that the Calkin algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty) = \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty) / \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty) \cap (\mathcal{K}_{\mathcal{A}_\lambda^2(\mathbb{B}^2)})$ is isomorphic and isometric to

$$C(\mathbb{D} \times (\mathbb{Z}_+ \cup M_\infty)) / C_0(\overline{\mathbb{D}}) \otimes c_0 = C((\mathbb{D} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)).$$

The same arguments, as given for the case of the C^* -algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}))$, show that we listed above all (up to unitary equivalence) irreducible representations of the C^* -algebra $\mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$. Again, as a byproduct of the description of the Calkin algebra $\hat{\mathcal{T}}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$, we have the following proposition.

Proposition (2.2.17)[2]. An operator $T^\lambda \simeq \bigoplus_{\alpha_2 \in \mathbb{Z}_+} (T_{c(z_1, \alpha_2)}^{\alpha_2+\lambda+1} + K^{\alpha_2}) \in \mathcal{T}_\lambda(C(\overline{\mathbb{D}}), L_\infty)$ is Fredholm if and only if \hat{c} does not vanish at any point of $(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)$, where $\hat{c} \in C(\overline{\mathbb{D}} \times (\mathbb{Z}_+ \cup M_\infty))$ is the Gelfand transform of the uncton $c = c(z_1, \alpha_2) \in C(\overline{\mathbb{D}}) \otimes SO(\mathbb{Z}_+)$.

In the case of Fredholmness,

$$\text{Ind } T^\lambda = \sum_{\alpha_2 \in \mathbb{Z}_+} \text{Ind} (T_{c(z_1, \alpha_2)}^{\alpha_2+\lambda+1} + K^{\alpha_2}) = -\frac{1}{2\pi} \sum_{\alpha_2 \in \mathbb{Z}_+} \{\arg c(\cdot, \alpha_2)\}_{\partial \mathbb{D}}. \quad (29)$$

The essential spectrum of T^λ is given by $\text{ess-sp } T^\lambda = \text{Range}(\hat{c}|_{(\overline{\mathbb{D}} \times M_\infty) \cup (S^1 \times \mathbb{Z}_+)}).$

We note that the right-hand side of (29) contained only a finite number of non-zero summands. Indeed, as \hat{c} has no zeros on $\bar{\mathbb{D}} \times M_\infty$, the functions $c_{\alpha_2} = c(\cdot, \alpha_2) \in C(\bar{\mathbb{D}})$ do not vanish on $\bar{\mathbb{D}}$ for all α_2 , starting from some α_2^0 . Furthermore, all operators $T_{c(z_1, \alpha_2)}^{\alpha_2 + \lambda + 1} + K^{\alpha_2}$ are invertible for all α_2 , starting from some α_2^1 , possibly greater than α_2^0 . That is, only a finite number of the Fredholm operators $T_{c(z_1, \alpha_2)}^{\alpha_2 + \lambda + 1} + K^{\alpha_2}$ are not invertible, making thus their generically non-zero contribution to the index formula. This implies, in particular, that both $\ker T^\lambda$ and $\text{coker } T^\lambda$ are finite dimensional, as it should be.

Theorem (2.2.18)[6]. [Mergelyan's theorem].

Let $k \subseteq C$ be compact and assume that $\hat{c} \setminus k$ has only finitely many connected components if $f \in c \setminus k$ be analytic on the interior of k , then for any $\varepsilon > 0$, there exists arotional function $r(z)$ such that:

$$\sup_{z \in k} |f(z) - r(z)| < \varepsilon$$

Theorem (2.2.19)[7]. Extension of Tietze' theorem:

Let X, Y be arbitrary space, and $A \subset X$, let $f: A \rightarrow Y$ be continuous $F: X \rightarrow Y$ is called the extension of f if $F(a) = f(a)$ for every $a \in A$.

Definition (2.2.20)[8]. Cayley Transform:

Cayley's transformation parameterizes a proper orthogonal matrix C as a function of a skew-symmetric matrix Q .

It is therefore, a map $\psi = so(n) \rightarrow SO(n)$. The classical Tayley transform is given by:

$$C = \psi(Q) = (1 - Q)(1 + Q)^{-1} = (1 + Q)^{-1}(1 - Q).$$

Chapter 3

Minimal Nuclear C^* -Algebras

We give the first examples of minimal ambient nuclear C^* -algebras of non-nuclear C^* -algebras. For this purpose, we study generic Cantor systems of infinite free product groups.

Section (3.1): Some Generic Properties of Cantor Systems

Choi constructed the first example of an ambient nuclear C^* -algebra of a non-nuclear C^* -algebra. Kirchberg–Phillips show that any separable exact C^* -algebra in fact has an ambient nuclear C^* -algebra. (In fact, one can choose it to be isomorphic to the Cuntz algebra \mathcal{O}_2 .) When we consider reduced group C^* -algebras, thanks to Ozawa’s result, we have more natural ambient nuclear C^* -algebras, namely, the reduced crossed products of amenable dynamical systems. Ambient nuclear C^* -algebras play important roles in theory of both C^* - and von Neumann algebras.

We investigate how an ambient nuclear C^* -algebra of a non-nuclear C^* -algebra can be tight. Based on (new) results on topological dynamical systems, we give the first example of a minimal ambient nuclear C^* -algebra of a non-nuclear C^* -algebra.

In fact, we have a stronger result: our examples of minimal ambient nuclear C^* -algebras have no proper intermediate C^* -algebras.

Note that as shown in contrast to injectivity of von Neumann algebras, nuclearity of C^* -algebras is not preserved under taking the decreasing intersection. We also note that the increasing union of non-nuclear C^* -algebras can be nuclear. Thus there is no obvious way to provide a minimal ambient nuclear C^* -algebra. We also remark that in the von Neumann algebra case, thanks to the bicommutant theorem, for any von Neumann algebra, finding a minimal ambient injective von Neumann algebra is equivalent to finding a maximal injective von Neumann subalgebra. Popa provided the first concrete examples of maximal injective von Neumann subalgebras.

Powers invented a celebrated method to study structures of the reduced group C^* -algebras. His idea has been applied to more general situations, particularly for reduced crossed products, and to more general groups, by many hands. We combine his technique with certain properties of dynamical systems to obtain the following main theorem of the paper.

We say that a group is an infinite free product group if it is a free product of infinitely many nontrivial groups. Groups are supposed to be countable.

Let Γ be an infinite free product group with the *AP* (or equivalently, each free product component has the *AP*). Then there is an amenable action of Γ on the Cantor set X with the following property. There is no proper intermediate C^* -algebra of the inclusion $C_r^*(\Gamma) \subset C(X) \rtimes_r \Gamma$.

In particular $C(X) \rtimes_r \Gamma$ is a minimal ambient nuclear C^* -algebra of the non-nuclear C^* -algebra $C_r^*(\Gamma)$.

We remark that it is not known if there is an ambient injective von Neumann algebra (or equivalently, injective von Neumann subalgebra) of a non-injective von Neumann algebra with no proper intermediate von Neumann algebra. Here we remark that the *AP* implies exactness, while the converse is not true. We need the *AP* to determine when a given element of the reduced crossed product sits in the reduced group C^* -algebra.

In theory of both measurable and topological dynamical systems, the Baire category theorem is a powerful tool to produce an example with a nice property. For further information on this topic. We follow this strategy to construct dynamical systems as in Main Theorem. To apply the Baire category theorem, we need a nice topology on the set of dynamical systems. We deal with the following (well-known) space of topological dynamical systems.

Let X be a compact metric space with a metric d_X . Then, on the homeomorphism group $\text{Homeo}(X)$ of X , define a metric d as follows.

$$d(\varphi, \psi) := \max_{x \in X} d_X(\varphi(x), \psi(x)) + \max_{x \in X} d_X(\varphi^{-1}(x), \psi^{-1}(x)).$$

Then d defines a complete metric on $\text{Homeo}(X)$. The topology defined by d coincides with the uniform convergence topology. In particular it does not depend on the choice of d_X .

Next let Γ be a (countable) group and consider the set $S(\Gamma, X) = \text{Hom}(\Gamma, \text{Homeo}(X))$ of all dynamical systems of Γ on X . The set $S(\Gamma, X)$ is naturally identified with a closed subset of the product space $\prod_{\Gamma} \text{Homeo}(X)$, where the latter space is equipped with the product topology. Since Γ is countable, this makes $S(\Gamma, X)$ a complete metric space.

Finally, we recall some definitions from theory of topological dynamical systems. Let $\alpha: \Gamma \curvearrowright X$ and $\beta: \Gamma \curvearrowright Y$ be actions of a group on compact metrizable spaces. The α is said to be an extension of β if there is a Γ -equivariant quotient map $\pi: X \rightarrow Y$. In this case β is said to be a factor of α . The action $\alpha: \Gamma \curvearrowright X$ is said to be

- (i) Free if any $s \in \Gamma \setminus \{e\}$ has no fixed points,
- (ii) Minimal if every Γ -orbit is dense in X ,
- (iii) Prime if there is no nontrivial factor of α ,
- (iv) Amenable if for any $\epsilon > 0$ and any finite subset S of Γ , there is a continuous map $\mu: X \rightarrow \text{Prob}(\Gamma)$ satisfying $\|s \cdot \mu^x - \mu^{s \cdot x}\|_1 < \epsilon$ for all $s \in S$ and $x \in X$.

Here $\text{Prob}(\Gamma)$ denotes the space of probability measures on Γ equipped with the pointwise convergence topology (which coincides with the ℓ^1 -norm topology), and Γ acts on $\text{Prob}(\Gamma)$ by the left translation. Obviously freeness and amenability pass to extensions and minimality passes to factors. Anantharaman-Delaroche has characterized amenability of topological dynamical systems by the nuclearity of the reduced crossed product.

We say that a property of topological dynamical systems is open, G_δ , dense, G_δ -dense, respectively when the subset of $S(\Gamma, X)$ consisting of actions with this property has the corresponding property. We say that a property is generic when the corresponding set contains a G_δ -dense subset of $S(\Gamma, X)$. Note that thanks to the Baire category theorem, the intersection of countably many G_δ -dense properties is again G_δ -dense, and similarly for genericity. Although some results (e.g., genericity of amenability, minimality, primeness, for infinite free product groups) can be extended to more general spaces by minor

modifications, we concentrate on the Cantor set. This is enough for Main Theorem. For short, we call an action on the Cantor set a Cantor system.

- (i) For an action $\alpha: \Gamma \curvearrowright X$, let $C(X) \rtimes_{\text{alg}} \Gamma$ denote its algebraic crossed product, i.e., the $*$ -subalgebra of the reduced crossed product generated by $C(X)$ and Γ .
- (ii) For the simplicity of notation, in the reduced crossed product $A = C(X) \rtimes_r \Gamma$, we denote the unitary of A corresponding to $s \in \Gamma$ by the same symbol s .
- (iii) Denote by e the unit element of a group.
- (iv) Let $E: C(X) \rtimes_r \Gamma \rightarrow C(X)$ denote the canonical conditional expectation on the reduced crossed product. That is, the unital completely positive map defined by the formula $E(fs) := \delta_{e,s}f$ for $f \in C(X)$ and $s \in \Gamma$.
- (v) For a unital C^* -algebra, we denote by \mathbb{C} the C^* -subalgebra generated by the unit.
- (vi) Denote by \otimes the minimal tensor product of C^* -algebras. We use the same notation for the minimal tensor product of completely positive maps.
- (vii) For a subset S of a set, denote by χ_S the characteristic function of S .
- (viii) For a subset S of a group, denote by $\langle S \rangle$ the subgroup generated by S .

When the action $\alpha: \Gamma \curvearrowright X$ is clear from the context, we denote $\alpha_s(x)$ by $s.x$ for short. Similarly for $s \in \Gamma$ and $U \subset X$, we denote $\alpha_s(U)$ by sU when no confusion arises.

We summarize generic properties of Cantor systems. From now on we denote by X the Cantor set. We recall that the Cantor set is the topological space characterized (up to homeomorphism) by the following four properties: compactness, total disconnectedness, metrizability, and perfectness (i.e., no isolated points).

Lemma (3.1.1)[3]. For any group Γ , the following properties are G_δ in $\mathcal{S}(\Gamma, X)$.

- (i) Freeness.
- (ii) Amenability.

Proof. The first claim is well-known. For completeness, we include a proof.

(i): For $s \in \Gamma$, set $V_s := \{\alpha \in \mathcal{S}(\Gamma, X) : \alpha_s(x) \neq x \text{ for all } x \in X\}$. By the compactness of X , each V_s is open. The G_δ -set $\bigcap_{s \in \Gamma \setminus \{e\}} V_s$ consists of all free Cantor systems.

(ii): For each finite subset S of Γ , we say that an action $\alpha: \Gamma \curvearrowright X$ has property as if it admits a continuous map $\mu: X \rightarrow \text{Prob}(\Gamma)$ satisfying

$$\|s.\mu^x - \mu^{s.x}\|_1 < \frac{1}{|S|}$$

for all $s \in S$ and $x \in X$. Let $\alpha \in \mathcal{S}(\Gamma, X)$ be given and suppose we have a continuous map μ that witnesses \mathcal{A}_S of α . Then, by the continuity of μ , it guarantees \mathcal{A}_S for any β sufficiently close to α . This shows that \mathcal{A}_S is open. Now obviously, the intersection $\bigwedge_S \mathcal{A}_S$ is equivalent to amenability, where S runs over finite subsets of Γ .

The following simple lemma is crucial to show the genericity of some properties.

Lemma (3.1.2)[3]. Let $\alpha: \Gamma \curvearrowright X$ be a given Cantor system. Then the set of extensions of α is dense in $\mathcal{S}(\Gamma, X)$.

Proof. Let us regard the Cantor set X as the direct product of infinitely many copies Y of the Cantor set: $X = Y^{\mathbb{N}}$. We regard α as a dynamical system on Y via a homeomorphism $\cong Y$. For each $N \in \mathbb{N}$, define a map $\sigma_N: \mathbb{N} \rightarrow \mathbb{N}$ by

$$\sigma_N(n) := \begin{cases} n & \text{when } n < N, \\ n + 1 & \text{when } n \geq N. \end{cases}$$

Now let $\beta \in S(\Gamma, X)$ be given. Let $\gamma: \Gamma \curvearrowright Y \times X$ be the diagonal action of α and β . For each $N \in \mathbb{N}$, define a homeomorphism $\varphi_N: X \rightarrow Y \times X$ by $\varphi_N(x) := (x_N, (x_{\delta_N(n)})_{n \in \mathbb{N}})$. Put $\beta^{(N)} := \varphi_N^{-1} \circ \gamma \circ \varphi_N \in S(\Gamma, X)$. Then for each $N \in \mathbb{N}$, the projection from X onto the N th coordinate gives a factor map of $\beta^{(N)}$ onto α . Moreover the sequence $(\beta^{(n)})_{n=1}^{\infty}$ converges to β . Since β is arbitrary, this proves the claim.

The next lemma is well-known.

Lemma (3.1.3)[3]. Every group admits a free Cantor system. Also, every exact group admits an amenable Cantor system.

Proof. Let Γ be a group. We first show that the left translation action of Γ on its Stone–Čech compactification $\beta\Gamma$ is free. Let $s \in \Gamma \setminus \{e\}$ be given. Put $\Lambda := \langle s \rangle$. Take a Λ -equivariant map $\Gamma \rightarrow \Lambda$ where Λ acts on both groups by the left multiplication. This extends to the Λ -equivariant quotient map $\beta\Gamma \rightarrow \beta\Lambda$. By universality, $\beta\Lambda$ factors onto every minimal dynamical system of Λ (on a compact space). Since any cyclic group admits a minimal free action on a compact space, this shows that s has no fixed points in $\beta\Gamma$.

Let $(A_\mu)_{\mu \in M}$ be the increasing net of Γ -invariant unital C^* -subalgebras of $\ell^\infty(\Gamma) = C(\beta\Gamma)$ generated by countably many projections. Note that $\bigcup_{\mu \in M} A_\mu = \ell^\infty(\Gamma)$. Let X_μ denote the spectrum of A_μ . Obviously, each X_μ is totally disconnected and metrizable. Let $\alpha_\mu: \Gamma \curvearrowright X_\mu$ be the action induced from the action $\Gamma \curvearrowright A_\mu$. By the freeness of $\Gamma \curvearrowright \beta\Gamma$, for sufficiently large μ , the α_μ must be free. When Γ is exact, for sufficiently large μ , the α_μ must be amenable. Hence for sufficiently large μ , the diagonal action of α_μ and the trivial Cantor system gives the desired action.

We now summarize the results.

Corollary (3.1.4)[3]. For any group Γ , freeness is a G_δ -dense property in $S(\Gamma, X)$. Moreover, when Γ is exact, amenability is also a G_δ -dense property in $S(\Gamma, X)$.

Proof. Since both freeness and amenability are inherited to extensions, it follows from Lemmas (3.1.1) Theorem (3.1.3).

Section (3.2): Construction of Dynamical Systems and Further Examples

We prove Main Theorem. Let $(\Gamma_i)_{i=1}^{\infty}$ be a sequence of nontrivial groups and let $\Gamma := \ast_{i=1}^{\infty} \Gamma_i$ be their free product. By replacing Γ_i by $\Gamma^{2i-1} \ast \Gamma_{2i}$ for all i if necessary, in the rest of the paper, we assume that each free product component Γ_i contains a torsionfree element. We start with the following elementary lemmas. We remark that in the case that Γ is the free group \mathbb{F}_∞ , we do not need these lemmas.

Lemma (3.2.1)[3]. Let Λ be a group and Y be its subgroup. Then for any minimal dynamical system α of Y on a compact metrizable space, there is a Cantor system of Λ whose restriction on Y is an extension of α .

Proof. Let $\alpha: Y \curvearrowright Y$ be an action as in the statement. Fix an element $e \in Y$. Then the map $Y \rightarrow Y$ defined by $s \mapsto s \cdot e$ extends to a factor map $Y \rightarrow Y$. This induces an Y -equivariant unital embedding of $C(Y)$ into $\ell^\infty(Y)$. By the right coset decomposition of Λ with respect to Y , we have an Y -equivariant unital embedding of $\ell^\infty(Y)$ into $\ell^\infty(\Lambda)$.

We identify $C(Y)$ with a unital Y -invariant C^* -subalgebra of $\ell^\infty(Y)$ via the composite of these two embeddings. Take a Y -invariant C^* -subalgebra A of $\ell^\infty(\Lambda)$ which contains $C(Y)$ and is generated by countably many projections. Let Z be the spectrum of A . Note that Z is metrizable and totally disconnected. Let $\beta: \Lambda \curvearrowright Z$ be the action induced from the action $\Lambda \curvearrowright A$. Since A contains $C(Y)$ as a unital C^* -subalgebra, the restriction of β on Y is an extension of α . Now the diagonal action of β with the trivial Cantor system gives the desired Cantor system.

Lemma (3.2.2)[3]. Let Λ be a group. Let s be a torsion-free element of Λ . Then for any finite family $U = \{U_1, \dots, U_n\}$ of pairwise disjoint proper clopen subsets of X , there is a Cantor system $\alpha: \Lambda \curvearrowright X$ with $sU_i = U_{i+1}$ for all i . (mod n)

Proof. By Lemma (3.2.1), there is a Cantor system $\alpha: \Lambda \curvearrowright X$ whose restriction on $\langle s \rangle$ factors a transitive action on the set $\{1, \dots, n\}$. For such α , there is a partition $\{V_1, \dots, V_n\}$ of X by clopen subsets satisfying $sV_i = V_{i+1}$ for all i . Set $I := \{0, 1\}$ if $\bigcup_{i=1}^n U_i \neq X$. Otherwise we set $I := \{0\}$. Then define a new action $\beta: \Lambda \curvearrowright X \times I$ by

$$\beta_t(x, j) := \begin{cases} (\alpha_t(x), 0) & \text{when } j = 0, \\ (x, 1) & \text{otherwise} \end{cases}$$

Since nonempty clopen subsets of the Cantor set are mutually homeomorphic, there is a homeomorphism $\varphi: X \times I \rightarrow X$ which maps $V_i \times \{0\}$ onto U_i for each i . For such φ , the conjugate $\varphi \circ \beta \circ \varphi^{-1}$ gives the desired Cantor system.

We next introduce a property of Cantor systems which is one of the key of the proof of Main Theorem and show that this property is G_δ -dense for infinite free product groups.

Proposition (3.2.3)[3]. Let $\Gamma = \bigast_{i=1}^\infty \Gamma_i$ be an infinite free product group. Then the following property \mathcal{R} of Cantor systems is G_δ -dense in $\mathcal{S}(\Gamma, X)$.

(\mathcal{R}): For any finite family $u = \{U_1, \dots, U_n\}$ of mutually disjoint proper clopen subsets of X , there are infinitely many $i \in \mathbb{N}$ satisfying the following condition. The group Γ_i contains a torsion-free element s satisfying $sU_j = U_{j+1}$ for all j .

Here we put $U_{n+1} := U_1$ as before.

Proof. For any $i \in \mathbb{N}$ and a family u as stated, we say that an element $\alpha \in \mathcal{S}(\Gamma, X)$ has property $\mathcal{R}(i, u)$ if it satisfies the following condition. There are $k \geq i$ and a torsion-free element $s \in \Gamma_k$ satisfying $sU_j = U_{j+1}$ for all j . Then observe that for any two clopen subsets U and V of X , the set

$$\{\varphi \in \text{Homeo}(X): \varphi(U) = V\}$$

is clopen in $\text{Homeo}(X)$. This shows that property $\mathcal{R}(i, \mathcal{U})$ is open in $\mathcal{S}(\Gamma, X)$.

To show the density of $\mathcal{R}(i, \mathcal{U})$, for each $m \in \mathbb{N}$, take a Cantor system $\varphi_m: \Gamma_m \curvearrowright X$ as in Lemma (3.2.2). Let $\alpha \in \mathcal{S}(\Gamma, X)$ be given. Then, for each $m \in \mathbb{N}$, we define $\alpha^{(m)} \in \mathcal{S}(\Gamma, X)$ as follows.

$$\alpha^{(m)}|_{\Gamma_k} := \begin{cases} \alpha|_{\Gamma_k} & \text{for } k < m, \\ \varphi_k & \text{for } k \geq m. \end{cases}$$

Then each $\alpha^{(m)}$ satisfies property $\mathcal{R}(i, \mathcal{U})$ and the sequence $(\alpha^{(m)})_{m=1}^{\infty}$ converges to α . This proves the density of $\mathcal{R}(i, \mathcal{U})$.

Now observe that property \mathcal{R} is equivalent to the intersection $\bigwedge_{i, \mathcal{U}} \mathcal{R}(i, \mathcal{U})$. Since there are only countably many clopen subsets in X , the intersection is taken over a countable family. Now the Baire category theorem completes the proof.

Proposition (3.2.4)[3]. Assume $\alpha \in \mathcal{S}(\Gamma, X)$ satisfies \mathcal{R} . Then there is no Γ -invariant closed subspace of $\mathcal{C}(X)$ other than 0, \mathbb{C} , or $\mathcal{C}(X)$. In particular \mathcal{R} implies primeness.

Proof. Let V be a closed Γ -invariant subspace of $\mathcal{C}(X)$ other than 0 or \mathbb{C} . We first show that V contains \mathbb{C} . Take a nonzero function. Then for any $\epsilon > 0$, there is a partition $\mathcal{U} := \{U_1, \dots, U_n\}$ of X by proper clopen sets and complex numbers c_1, \dots, c_n with $|c_1| = \|f\|$ such that with $g := \sum_{i=1}^n c_i \chi_{U_i}$, we have $\|f - g\| < \epsilon$. Put $c := \frac{1}{n} \sum_{i=1}^n c_i$.

By replacing \mathcal{U} by dividing U_1 into sufficiently many clopen subsets and replacing the sequence $(c_i)_i$ suitably, we may assume $|c| \geq \|f\|/2$. By property \mathcal{R} , we can take $s \in \Gamma$ with $sU_i = U_{i+1}$ for all i . We then have $\sum_{i=1}^n s^i g s^{-i} = \sum_{i=1}^n c_i$. This yields the inequality

$$\left\| \frac{1}{n} \sum_{i=1}^n s^i f s^{-i} - c \right\| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary and $|c| \geq \|f\|/2$, we obtain $\mathbb{C} \subset V$.

From this, we can choose a nonzero function $f \in V$ with $0 \notin f(X)$. For any $\epsilon > 0$, take a partition $\mathcal{U} = \{U_0, U_1, \dots, U_n\}$ of X by proper clopen sets and complex numbers c_1, \dots, c_n such that with $g := \sum_{i=1}^n c_i \chi_{U_i}$, we have $\|f - g\| < \epsilon$. Put $c := \frac{1}{n} \sum_{i=1}^n c_i$. As before, we may assume $|c| \geq \|f\|/2$. By using property \mathcal{R} to the family $\{U_1, \dots, U_n\}$, we can take $s \in \Gamma$ satisfying $sU_0 = U_0$ and $sU_i = U_{i+1}$ for $1 \leq i < n$. Then we have, $\frac{1}{n} \sum_{i=1}^n s^i g s^{-i} = c \chi_{X \setminus U_0}$. Now let U be any proper clopen subset of X . Take $t \in \Gamma$ with $t(X \setminus U_0) = U$. (To find such t , use property \mathcal{R} twice.) We then have

$$t \left(\frac{1}{n} \sum_{i=1}^n s^i g s^{-i} \right) t^{-1} = c t(\chi_{X \setminus U_0}) t^{-1} = c \chi_U.$$

This shows the inequality

$$\left\| \left(\frac{1}{n} \sum_{i=1}^n t s^i f s^{-i} t^{-1} \right) - c_{\chi_U} \right\| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves $\chi_U \in V$. Since U is arbitrary, we obtain $V = C(X)$.

We need the following restricted version of the Powers property for free product groups.

Although the proof is essentially contained, for completeness, we include a proof.

Lemma (3.2.5)[3]. Let Λ_1, Λ_2 be groups and set $\Lambda := \Lambda_1 * \Lambda_2$. Let $s \in \Lambda_1, t \in \Lambda_2$ be torsion-free elements. Then for any finite subset F of $\Lambda \setminus \{e\}$, there are a partition $\Lambda = D \sqcup E$ of Λ and elements $u_1, u_2, u_3 \in \langle s, t \rangle$ with the following properties.

(i) $fD \cap D = \emptyset$ for all $f \in F$.

(ii) $u_j E \cap u_k E = \emptyset$ for any two distinct $j, k \in \{1, 2, 3\}$.

Proof. Let $F \subset \Lambda \setminus \{e\}$ be given. Then for sufficiently large $n \in \mathbb{N}$, with $z := ts^n$, any element of zFz^{-1} is started with t and ended with t^{-1} . Here for $u \in \Lambda_i \setminus \{e\}$, we say an element w of Λ is started with u if $w = uw_1 \dots w_n$ for some (possibly empty) sequence w_1, \dots, w_n with $w_j \in \Lambda_{k_j} \setminus \{e\}$ and $i \neq k_1 \neq k_2 \neq \dots \neq k_n$. The word “ended with u ” is similarly defined. (Thus, in our terminology, the element u_2 is not started with u .)

Let E' be the subset of Λ consisting of all elements started with t . Put $E := z^{-1}E', D := \Lambda \setminus E$, and $D' := \Lambda \setminus E'$. Then note that $fD \cap D = \emptyset$ for all $f \in F$ if and only if $f'D' \cap D' = \emptyset$ for all $f' \in zFz^{-1}$. Since elements $f' \in zFz^{-1}$ are started with t and ended with t^{-1} but D' consists of elements not started with t , we have $f'D' \cap D' = \emptyset$. Now for $j \in \{1, 2, 3\}$, put $u_j := s^j z$. Obviously each u_j is contained in $\langle s, t \rangle$. By definition, we have $u_j E = s^j E'$. This shows that $u_j E$ consists of only elements started with s^j . Therefore $u_1 E, u_2 E$, and $u_3 E$ are pairwise disjoint.

Now we prove Main Theorem. Before the proof, we remark that the AP is preserved under taking free products. Hence Γ has the AP if and only if each free product component Γ_i has it.

Theorem (3.2.6)[3]. Let Γ be an infinite free product group with the AP. Then, for $\alpha \in S(\Gamma, X)$ with property \mathcal{R} , there is no proper intermediate C^* -algebra of the inclusion $C_r^*(\Gamma) \subset C(X) \rtimes_r \Gamma$. In particular, when additionally α is amenable, then $C(X) \rtimes_r \Gamma$ is a minimal ambient nuclear C^* -algebra of the non-nuclear C^* -algebra $C_r^*(\Gamma)$.

Proof. Let A be an intermediate C^* -algebra of the inclusion $C_r^*(\Gamma) \subset C(X) \rtimes_r \Gamma$. We first consider the case $E(A) = \mathbb{C}$. In this case, we have the equality $A = C_r^*(\Gamma)$.

We next consider the case $E(A) \neq \mathbb{C}$. In this case, by Proposition (3.2.4), $E(A)$ is dense in $C(X)$. Let U be a proper clopen subset of X . Let $\epsilon > 0$ be given. Then take a self-adjoint element $x \in A$ with $E(x) - \chi_U < \epsilon$. By property \mathcal{R} , there are torsion-free elements $s_1 \in \Gamma_i$ and $s_2 \in \Gamma_j$ with $i \neq j$ which fix χ_U . Put $\Lambda := \langle s_1, s_2 \rangle$. Take $y \in C(X) \rtimes_{\text{alg}} \Gamma$ satisfying $E(y) = \chi_U$ and $\|y - x\| < \epsilon$. By Lemma (3.2.5), we can apply the Powers argument, by

elements of Λ . Iterating the Powers argument sufficiently many times, we obtain a sequence $t_1, \dots, t_n \in \Lambda$ satisfying the inequality

$$\left\| \frac{1}{n} \sum_{i=1}^n t_i (y - \chi_U) t_i^{-1} \right\| < \epsilon.$$

Since χ_U is Λ -invariant, we have

$$\left\| \frac{1}{n} \sum_{i=1}^n t_i x t_i^{-1} - \chi_U \right\| < 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this shows $\chi_U \in A$. Therefore $A = C(X) \rtimes_r \Gamma$.

Proposition (3.2.7)[3]. Let A be a simple C^* -algebra. Let Γ be an infinite free product group with the AP. Let $\alpha: \Gamma \curvearrowright X$ be a Cantor system with property \mathcal{R} . Then the inclusion $A \otimes C_r^*(\Gamma) \subset A \otimes (C(X) \rtimes_r \Gamma)$ has no proper intermediate C^* -algebra.

Proof. Let B be an intermediate C^* -algebra of the inclusion $A \otimes C_r^*(\Gamma) \subset A \otimes (C(X) \rtimes_r \Gamma)$. Put $\Phi := \text{id}_A \otimes E$. Throughout the proof, we identify A with a C^* -subalgebra of $A \otimes C(X)$ in the canonical way. Note that the image $\Phi(B)$ contains A . When the equality $\Phi(B) = A$ holds, we have $B = A \otimes C_r^*(\Gamma)$.

Suppose $\Phi(B) \neq A$. We observe first that for an element $x \in A \otimes C(X)$ satisfying $(\varphi \otimes \text{id}_{C(X)})(x) \in C$ for all pure states φ on A , we have $x = (\text{id}_A \otimes \psi)(x) \in A$ for any state ψ on $C(X)$. Hence we can choose a pure state φ on A and an element $b \in B$ satisfying $f := (\varphi \otimes \text{id}_{C(X)})(\Phi(b)) \in C(X) \setminus C$ and $\|f\| = 1$. Now let $\epsilon > 0$ be given.

By the Akemann–anderson–Pedersen excision theorem, there is a positive element $a \in A$ of norm one satisfying $\|\Phi(aba) - a^2 \otimes f\| < \epsilon$. By the simplicity of A , for any $\epsilon > 0$ and any positive contractive element $c \in A$, there is a finite sequence $x_1, \dots, x_n \in A$ satisfying the following conditions.

$$(i) \quad \left\| \sum_{i=1}^n x_i a^2 x_i^* - c \right\| \leq \epsilon.$$

$$(ii) \quad \left\| \sum_{i=1}^n x_i x_i^* \right\| \leq 2.$$

For such a sequence, we have

$$\left\| \Phi \left(\sum_{i=1}^n x_i a b a x_i^* \right) - c \otimes f \right\| < 3\epsilon.$$

This shows that the closure of $\Phi(B)$ contains $c \otimes f$. Proposition (3.2.4) then shows that the closure of $\Phi(B)$ contains the subspace $c \otimes C(X)$. Now the proof of Theorem (3.2.6) shows the equality $B = A \otimes (C(X) \rtimes_r \Gamma)$.

Chapter 4

Quasi-States on C^* -Algebras

We show that an answer to a question that positive quasi-linear functionals in C^* -algebras are linear under certain given conditions.

Section (4.1): Decomposition of Quasi-States and C^* -Algebras Generated by two Projections

Let \mathcal{A} be a C^* -algebra. A positive quasi-linear functional on \mathcal{A} is a function $\rho: \mathcal{A} \rightarrow \mathbb{C}$ such that:

- (i) ρ is a positive linear functional on each abelian C^* -subalgebra of \mathcal{A}
- (ii) $\sup\{\rho(A) | A \in \mathcal{A}, A \geq 0, \|A\| \leq 1\} < \infty$,
- (iii) $\rho(A + iB) = \rho(A) + i\rho(B)$ if A and B are self-adjoint elements of \mathcal{A}

A quasi-state on \mathcal{A} is a positive quasi-linear functional ρ on \mathcal{A} that satisfies the normalization condition $\sup\{\rho(A) | A \in \mathcal{A}, A \geq 0, \|A\| \leq 1\} = 1$. If \mathcal{A} is unital with unit I , this is equivalent to the condition $\rho(I) = 1$.

We are concerned with the question of when positive quasilinear functionals on C^* -algebras are linear. In view of the condition (iii) and since positive quasi-linear functionals are scalar multiples of quasistates, this reduces to the question of additivity of quasi-states on the self-adjoint elements of a C^* -algebra. This question and its various forms, depending both on the nature of quasi-states and on the structure of the underlying algebra, substantial progress had been made in a number of cases.

One of the most remarkable advances was the pioneering work of

which the question was settled in the affirmative for quasistates on $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators acting on a Hilbert space \mathcal{H} with the property of being completely additive on orthogonal projections, provided the dimension of \mathcal{H} is different from 2. In the two dimensional case the answer is, in general, negative-one can construct discontinuous quasi-states by simple geometrical arguments; so that one cannot expect all quasi-states to be linear on C^* -algebras that admit two-dimensional irreducible representations. Following Gleason's result, contributions to the problem were made, and in recent years the general problem. Combined together their results provide an affirmative answer for quasi-states on von Neumann algebras without central summands of type I_2 . One of the crucial points underlying this solution is the norm-continuity of quasi-states on the set of projections. For more general C^* -algebras continuity of quasi-states still remains an open problem. More can be said about additivity of continuous quasi-states. One finds, for example, that continuous quasi-states are linear on AF C^* -algebras, by applying Gleason's result to a norm-dense union of finite-dimensional subalgebras.

We study various classes of continuous quasi-states and the objective is to obtain further information on the problem. Besides continuous and uniformly continuous quasi-states we consider the so-called weakly subadditive quasi-states that satisfy $\rho(A + B) \geq \rho(A) + \rho(B)$ for all positive A and B in \mathcal{A} and approximately additive quasi-states with

the property that $|\rho(A_a + B_a) - \rho(A_a) - \rho(B_a)| \rightarrow_a 0$, whenever $\{A_a\}_{a \in A}$ and $\{B_a\}_{a \in A}$ are bounded nets of self-adjoint elements in \mathcal{A} and $\|[A_a, B_a]\| \rightarrow_a 0$. Develops some of the elementary properties of such quasi-states. We obtain a decomposition of uniformly continuous quasi-states into atomic and diffuse parts on separable C^* -algebras whose irreducible representations are of dimension different from 2. This is used together with Christensen's result to show additivity of weakly subadditive quasi-states on certain extensions of locally trivial fields of elementary algebras by finite-dimensional C^* -algebras, and, in particular, on C^* -subalgebras generated by two projections under suitable multiplicity condition. We use the information and the techniques developed to obtain new results about quasi-states on C^* -algebras containing a dense set of elements with finite spectrum. In particular, additivity is shown for arbitrary quasi-states on the Calkin algebra, and for weakly subadditive and continuous quasistates on certain stable algebras.

The letters \mathcal{A}, \mathcal{B} and \mathcal{C} will denote C^* -algebras with elements $A, B, C, D, \dots, \mathcal{A}_{s.a.}, \mathcal{A}^+$, and \mathcal{A}_1^+ are the symbols for the self-adjoint part of \mathcal{A} , positive part of \mathcal{A} and positive part of the unit ball of \mathcal{A} respectively. If \mathcal{A} is unital, I will always denote the identity of \mathcal{A} by 1 . The C^* -algebra of $n \times n$ matrices over \mathcal{A} is denoted by $M_n(\mathcal{A})$. We shall occasionally consider C^* -algebra \mathcal{A} in its universal representation. In this case \mathcal{D}^- will denote the weak closure of a subset $\mathcal{D} \subseteq \mathcal{A}$ and C_{at} stands for the minimal central projection in \mathcal{A}^- majorizing each minimal projection of \mathcal{A}^- (the atomic projection of \mathcal{A}^-). For a linear functional ρ on \mathcal{A} we shall sometimes use the same letter ρ to denote its ultraweakly continuous extension to \mathcal{A}^- . The symbols $\mathcal{B}(\mathcal{H})$ and \mathcal{K} are reserved for the algebra of all bounded operators on a Hilbert space \mathcal{H} and the algebra of compact operators on a separable Hilbert space, respectively.

If ρ is a positive quasi-linear functional on \mathcal{A} we use the notation $\|\rho\| = \sup\{\rho(A) | A \in \mathcal{A}_1^+\}$ in analogy with positive linear functionals.

Similarly, for positive quasi-linear functionals ρ and ω on \mathcal{A} the expression $\omega \leq \rho$ will mean that $\omega(A) \leq \rho(A)$ for all $A \in \mathcal{A}^+$ (equivalently, $\rho - \omega$ is a positive quasi-linear functional on \mathcal{A}).

Definition (4.1.1)[4]. A positive quasi-linear functional ρ on a C^* -algebra \mathcal{A} is said to be weakly subadditive if $\rho(A + B) = \rho(A) + \rho(B)$ for all positive

A, B in \mathcal{A} . ρ is said to be approximately additive if $|\rho(A_a + B_a) - \rho(A_a) - \rho(B_a)| \rightarrow_a 0$ for each pair of bounded nets $\{A_a\}_{a \in A}$ and $\{B_a\}_{a \in A}$ in $\mathcal{A}_{s.a.}$, such that $\|[A_a, B_a]\| \rightarrow_a 0$.

Proposition (4.1.2)[4]. Let ρ be a positive quasi-linear functional on a C^* -algebra \mathcal{A}

- (i) If ρ is weakly subadditive, then ρ is monotone on \mathcal{A}^+ . If in addition, \mathcal{A} is unital, then ρ is monotone on $\mathcal{A}_{s.a.}$, and uniformly continuous on \mathcal{A} .
- (ii) If ρ is approximately additive and $\{A_a\}_{a \in A}, \{B_a\}_{a \in A}$ are bounded nets in $\mathcal{A}_{s.a.}$, such that $\|A_a - B_a\| \rightarrow_a 0$, then $|\rho(A_a) - \rho(B_a)| \rightarrow_a 0$.

Proof. (i) Monotonicity of ρ on \mathcal{A}^+ follows immediately from the definition, since $\rho(A) - \rho(B) \geq \rho(A - B) \geq 0$ when $0 \leq B \leq A$. If \mathcal{A} is unital and $C \leq D$ for $C, D \in \mathcal{A}_{s.a.}$, then

$0 \leq C + \lambda I \leq D + \lambda I$ where $I = \max\{\|C\|, \|D\|\}$. Consequently, $\rho(C) + \lambda\rho(I) \leq \rho(D) + \lambda\rho(I)$, and $\rho(C) \leq \rho(D)$. Finally, for A and B in $\mathcal{A}_{s,a}$, we have, so that

$$B - \|A - B\|I \leq A \leq B + \|A - B\|I,$$

So that

$$\rho(B) - \|A - B\|\rho(I) \leq \rho(A) \leq \rho(B) + \|A - B\|\rho(I),$$

from the remark above. Thus $|\rho(A) - \rho(B)| \leq \rho(I)\|A - B\|$. For arbitrary A and B we obtain $|\rho(A) - \rho(B)| \leq 2\rho(I)\|A - B\|$, by applying the preceding inequality to the real and imaginary parts.

(ii) Since the nets $\{A_a\}_{a \in A}$ and $\{B_a\}_{a \in A}$ are bounded, we have

$\|[A_a, B_a]\| \rightarrow_a 0$ and hence $|\rho(A_a) - \rho(B_a) - \rho(A_a - B_a)| \rightarrow_a 0$ from the definition of ρ . On the other hand, $|\rho(A_a - B_a)| \leq \|\rho\| \cdot \|A_a - B_a\|$, since for each $a \in A$ the restriction of ρ to the abelian C^* -subalgebra generated by

$A_a - B_a$ is a positive linear functional. Therefore $\rho(A_a - B_a) \rightarrow_a 0$, and the assertion follows.

Let $Q(\mathcal{A})$ denote the set of all positive quasi-linear functionals on \mathcal{A} of norm less or equal to 1, and let $SQ(\mathcal{A})$ denote the subset of $Q(\mathcal{A})$, consisting of weakly subadditive positive quasi-linear functionals. It was shown that $Q(\mathcal{A})$ is weak*-compact and convex. The same property holds for the weak*-closed convex subset $SQ(\mathcal{A})$. From the Krein-Milman theorem the sets $Q(\mathcal{A})$ and $SQ(\mathcal{A})$ are weak*-closed convex hulls of their extreme points. It is not hard to see that the zero functional is an extreme point of each of these sets, while the nonzero extreme points are of norm 1, that is, are quasi-states of \mathcal{A} . Following this observation we shall call the nonzero extreme points of $Q(\mathcal{A})$ (resp. $Se(\mathcal{A})$) pure quasi-states (resp. pure weakly subadditive quasi-states).

Where the following proposition.

Proposition (4.1.3)[4]. Let \mathfrak{d} be a unital C^* -algebra.

(i) ρ is a pure quasi-state of \mathcal{A} if and only if each positive quasi-linear functional σ on \mathcal{A} such that $\sigma \leq \rho$ is a scalar multiple of ρ .

(ii) If ρ is a pure weakly subadditive quasi-state of \mathcal{A} and φ_1 , is a nonzero positive linear functional on \mathcal{A} such that $\varphi_1 \leq \rho$, then $\rho = (1/\|\varphi_1\|) \cdot \varphi_1$.

Proof: We shall show (ii). The proof of (i) is analogous. Let $\varphi_2 = \rho - \varphi_1$. Then φ_2 is a positive quasi-linear functional on \mathcal{A} and φ_2 is weakly subadditive, because

$$\begin{aligned} \varphi_2(A + B) &= \rho(A + B) - \varphi_1(A + B) = \rho(A + B) - (\varphi_1(A) + \varphi_1(B)) \\ &\geq \rho(A) + \rho(B) - (\varphi_1(A) + \varphi_1(B)) \\ &= \varphi_2(A) + \varphi_2(B), \quad \text{when } A, B \in \mathcal{A}^+ \end{aligned}$$

With $\lambda_1 = \|\varphi_1\|$ and $\lambda_2 = \|\varphi_2\| = (\varphi_2(I))$, we have

$$\lambda_1 + \lambda_2 = \varphi_1(I) + (\rho(I) - \varphi_1(I)) = \rho(I) = 1.$$

If $\lambda_1 = 0$, then $\varphi_2 = 0$, and $\rho = \varphi_1 = \left(\frac{1}{\|\varphi_1\|}\right) \varphi_1$. If $\lambda_2 \neq 0$, then $\omega_1 = \left(\frac{1}{\lambda_1}\right) \varphi_1$ and $\omega_2 = \left(\frac{1}{\lambda_2}\right) \varphi_2$ belong to $SQ(\mathcal{A})$, and $\rho = \lambda_1 \omega_1 + \lambda_2 \omega_2$. Since ρ is pure, this implies $\omega_1 = \omega_2 = \rho$. Consequently, $\rho = \left(\frac{1}{\|\varphi_1\|}\right) \varphi_1$.

We obtain the natural decomposition of a uniformly continuous quasi-state on a separable C^* -algebra, whose irreducible representations are of dimension different from two, into atomic and diffuse parts. Applications of this technique appear in the second part, where we consider weakly subadditive quasi-states on certain separable type I C^* -algebras.

Lemma (4.1.4)[4]. (i) The function $\rho|_{\bigcup_{\mathcal{B} \in \Lambda} \mathcal{B}}$ extends to the function $\bar{\rho}$ on $\bigcup_{\mathcal{B} \in \Lambda} \mathcal{B}^-$, given by $\bar{\rho}(A) = \omega_x|_{\mathcal{B}}$ if $A \in \mathcal{B}$ for some $\mathcal{B} \in \Lambda$ $\rho|_{\mathcal{B}} = \omega_x|_{\mathcal{B}}$. (ii) if P_1, \dots, P_n , is a finite family of mutually orthogonal minimal projections in \mathcal{A}^- , then P_1, \dots, P_n , and $\sum_{i=1}^n P_i$ belong to $\bigcup_{\mathcal{B} \in \Lambda} \mathcal{B}^-$, and $\sum_{i=1}^n \bar{\rho}(P_i) = \bar{\rho}(\sum_{i=1}^n P_i) \leq 1$.

Proof. (i) We have to show that $\omega_x(A) = \omega_y(A)$, if $A \in \mathcal{B}^- \cap \mathcal{C}^-$ for some \mathcal{B} and \mathcal{C} in Λ and $\rho|_{\mathcal{B}} = \omega_x|_{\mathcal{B}}$, $\rho|_{\mathcal{C}} = \omega_y|_{\mathcal{C}}$. Choose nets $\{B_a\}_{a \in \Lambda}$ and $\{C_a\}_{a \in \Lambda}$ contained in \mathcal{B} and \mathcal{C} , respectively, and convergent to A in the weak-operator topology. Since 0 is the weak-operator limit of the net $\{B_a - C_a\}_{a \in \Lambda}$, and bounded linear functionals on \mathcal{A} are weakoperator continuous, the Hahn-Banach separation theorem implies that 0 belongs to the norm-closed convex hull of any cofinal subnet of $\{B_a - C_a\}_{a \in \Lambda}$. Consequently, for any $\varepsilon > 0$ we can choose $a_0 \in A$ such that $|\omega_x(A) - \omega_x(B_a)| < \varepsilon$, $|\omega_y(A) - \omega_y(C_a)| < \varepsilon$ for all $a \geq a_0$ and a convex combination $\sum_j \lambda_j (B_{a_j} - C_{a_j})$ such that $a_j \geq a_0$, for all j and $|\rho(\sum_j \lambda_j B_{a_j}) - \rho(\sum_j \lambda_j C_{a_j})| < \varepsilon$, from uniform continuity of ρ . Therefore $|\omega_x(A) - \omega_y(A)| \leq \sum_j \lambda_j |\omega_x(A) - \omega_x(B_{a_j})| + |\rho(\sum_j \lambda_j B_{a_j}) - \rho(\sum_j \lambda_j C_{a_j})| + \sum_j \lambda_j |\omega_y(C_{a_j}) - \omega_y(A)| < 3\varepsilon$ and $\omega_x(A) = \omega_y(A)$.

(ii) If P is a minimal projection in \mathcal{A}^- , then P is the support projection of some pure state ω of \mathcal{A} , and $I - P$ is the open right support projection for the left kernel \mathcal{L}_ω of ω . Since $\mathcal{L}_\omega \cap \mathcal{L}_\omega^*$, is separable, it admits a strictly positive element A_ρ , whose range projection is $I - P$. Thus, P belongs to a weak-operator closure of a maximal abelian subalgebra of \mathcal{A} containing A_ρ . If P_1, \dots, P_n , is a finite family of mutually orthogonal minimal projections in \mathcal{A}^- , the corresponding pure states $\omega_1, \dots, \omega_n$, satisfy $\|\omega_j - \omega_k\| = 2$ ($j, k \in \{1, \dots, n\}$, $j \neq k$). From there is a maximal abelian C^* -subalgebra $\mathcal{B} \subseteq \mathcal{A}$ such that for each $i \in \{1, \dots, n\}$, $\omega_i|_{\mathcal{B}}$ is a pure state of \mathcal{B} , and ω_i is the unique extension of $\omega_i|_{\mathcal{B}}$ to a state of \mathcal{A} . Noting that ω_i also uniquely extends to a normal state of \mathcal{A}^- , we see that $P_i \in \mathcal{B}^-$ for each $i = 1, \dots, n$. Therefore $\sum_{i=1}^n P_i \in \mathcal{B}^-$, and from the definition of $\bar{\rho}$ in (i), $\sum_{i=1}^n \bar{\rho}(P_i) = \bar{\rho}(\sum_{i=1}^n P_i) \leq |\rho| = 1$.

Now consider an arbitrary but fixed t in $\widehat{\mathcal{A}}$ the spectrum of \mathcal{A} and fix (π, \mathcal{H}_π) in t . Let C_t be the central cover of (π, \mathcal{H}_π) in \mathcal{A} , and let Φ denote a fixed isomorphism of $\mathcal{A}^- C_t$, onto $B(\mathcal{H}_\pi)$. With each unit vector e in \mathcal{H}_π , we shall associate the minimal projection P_e in

\mathcal{A}^-C_t , such that $\Phi(P_e)$ is the one-dimensional projection onto the subspace spanned by e . This notation is implicit in the following lemma.

Lemma (4.1.5)[4]. If $\dim(\mathcal{H}_\pi) \neq 2$, then there is a positive normal linear functional σ_t , on \mathcal{A}^-C_t , such that $\sigma_t(P) = \bar{\rho}(P)$ for each minimal projection

$$P \in \mathcal{A}^-C_t.$$

Proof: Let ψ be the nonnegative function on the unit sphere of \mathcal{H}_π , given by $\psi(e) = \bar{\rho}(P_e)$ ($e \in \mathcal{H}_\pi, \|e\| = 1$). If \mathcal{S} is a finite-dimensional subspace

of \mathcal{H}_π and $\{e_i\}_{i=1}^n, \{g_i\}_{i=1}^n$, are any two orthonormal bases of \mathcal{S} , then $\sum_{i=1}^n P_{e_i} = \sum_{i=1}^n P_{g_i}$, so that

$$\sum_i \psi(e_i) = \sum_i \bar{\rho}(P_{e_i}) = \bar{\rho}\left(\sum_i P_{e_i}\right) = \bar{\rho}\left(\sum_i P_{g_i}\right) = \sum_i \bar{\rho}(P_{g_i}) = \sum_i \psi(g_i),$$

from Lemma (4.1.4) (ii). Therefore, the restriction of ψ to a unit sphere of any finite-dimensional subspace of \mathcal{H}_π is a frame function. There is a positive operator $T \in B(\mathcal{H}_\pi)$ such that $\psi(e) = \langle Te, e \rangle$ for each unit vector $e \in \mathcal{H}_\pi$, and T is of the trace class, because $\sup_m \left(\sum_{i=1}^m \bar{\rho}(P_{h_i})\right) \leq 1$ for any orthonormal basis $\{h_i\}$ of \mathcal{H}_π (Lemma (4.1.4) (ii)).

Let σ_t be the positive normal linear functional on \mathcal{A}^-C_t , given by $\sigma_t(A) = \text{Tr}(\Phi(A)T)$ ($A \in \mathcal{A}^-C_t$, where Tr denotes the usual trace on $B(\mathcal{H}_\pi)$). If P is a minimal projection of \mathcal{A}^-C_t , then $P = P_u$ for some unit vector $u \in \mathcal{H}_\pi$, and $\sigma_t(P_u) = \text{Tr}(\Phi(P_u)T) = \langle Tu, u \rangle = \psi(u) = \bar{\rho}(P_u)$.

Proposition (4.1.6)[4]. Let \mathcal{A} be a separable C^* -algebra given in its universal representation on a Hilbert space \mathcal{H} , and Λ be the set of all maximal abelian subalgebras of \mathcal{A} . Suppose ρ is a uniformly continuous quasi-state on \mathcal{A} . Then:

- (i) The set $\bigcup_{\mathcal{B} \in \Lambda} \mathcal{B}^-$ contains all finite orthogonal sums of minimal projections in \mathcal{A}^- .
- (ii) The function $\rho|_{\bigcup_{\mathcal{B} \in \Lambda} \mathcal{B}}$ extends to a function ρ on $\bigcup_{\mathcal{B} \in \Lambda} \mathcal{B}^-$, such that $\bar{\rho}(\sum_{i=1}^n P_i) = \sum_{i=1}^n \bar{\rho}(P_i)$ or any finite orthogonal family P_1, \dots, P_n of minimal projections of \mathcal{A} .
- (iii) If \mathcal{A} does not admit irreducible representations of dimension 2, then there exists the atomic positive linear functional ρ_{at} on \mathcal{A} such that $\rho_{at} \leq \rho$ and $\rho_{at}(P) = \bar{\rho}(P)$ for each minimal projection $P \in \mathcal{A}^-$.

The proof of the proposition will be broken into several steps. Note first that if $\mathcal{B} \in \Lambda$, then the restriction $\rho|_{\mathcal{B}}$ of ρ to \mathcal{B} extends to a positive linear functional on \mathcal{A} , so that $\rho|_{\mathcal{B}} = \omega_x|_{\mathcal{B}}$ for some vector $x \in \mathcal{H}$.

Proof. Lemma (4.1.4) establishes parts (i) and (ii) of the proposition. It remains to show part (iii).

Let $\{P_{(i,t)} | (i,t) \in \mathbf{I} \times \hat{\mathcal{A}}\}$ be any orthogonal family of minimal projections in \mathcal{A}^- , such that $\sum_{i \in \mathbf{I}} P_{(i,t)} = C_t$, for each $t \in \hat{\mathcal{A}}$. For any finite subset $F \subseteq \mathbf{I} \times \hat{\mathcal{A}}$ we

have $\sum_{(i,t) \in \mathbf{F}} \bar{\rho}(P_{(i,t)}) \leq 1$, from Lemma (4.1.4)(ii). Consequently, the series $\sum_{t \in \widehat{\mathcal{A}}} \sum_{i \in \mathbf{I}} \bar{\rho}(P_{(i,t)})$, is (unconditionally) convergent, and

$$\sum_{t \in \widehat{\mathcal{A}}} \sum_{i \in \mathbf{I}} \bar{\rho}(P_{(i,t)}) = \sum_{t \in \widehat{\mathcal{A}}} \sigma_t(C_t) \quad (1)$$

from Lemma (4.1.5). For each $t \in \widehat{\mathcal{A}}$ let ψ_t be the positive normal linear functional on \mathcal{A}^- , given by $\psi_t(A) = \sigma_t(AC_t)$ ($A \in \mathcal{A}^-$). From (1) there are at most countably many t for which $\psi_t \neq 0$. Relabeling these as t_1, t_2, \dots , we see that the series $\sum_{i=1}^{\infty} \psi_{t_i}$, is absolutely convergent to a positive normal linear functional ρ_{at} . We have

$$\rho_{at}(A) = \sum_{i=1}^{\infty} \sigma_{t_i}(AC_{t_i}) \quad (A \in \mathcal{A}^-), \quad (2)$$

so that $\|\rho_{at}\| = \rho_{at}(C_{at})$. Hence ρ_{at} is an atomic positive linear functional on \mathcal{A} . From (2) and Lemma (4.1.5), $\rho_{at}(P) = \bar{\rho}(P)$ for each minimal projection

$P \in \mathcal{A}^-$.

Given $A \in \mathcal{A}^+$, we shall show that $\rho_{at}(A) \leq \rho(A)$. For this it suffices to establish that $\rho_{at}(E) \leq \bar{\rho}(E)$ for each projection E of the form $E = \mathcal{X}_{(\lambda, \lambda']} (A)$, where $\mathcal{X}_{(\lambda, \lambda']}$ is the characteristic function of an interval $(\lambda, \lambda'] \subseteq \mathbb{R}$. Furthermore, from inner regularity of the measure induced on \mathbb{R} by the restrictions of ρ_{at} and ρ to the abelian C^* -subalgebra generated by A , it suffices to consider $E = \mathcal{K}(A)$, where \mathcal{K} is a compact subset of \mathbb{R} . By outer regularity, for any $\varepsilon > 0$ we can choose an open subset \mathcal{O} containing \mathcal{K} such that $\bar{\rho}(\mathcal{K}(A)) - \varepsilon < \bar{\rho}(E)$, and continuous nonnegative functions f_ε and g_ε that are identically 1 on \mathcal{K} , vanish outside \mathcal{O} , and satisfy $f_\varepsilon g_\varepsilon = g_\varepsilon$. We then have

$$\rho(f_\varepsilon(A)) - \varepsilon < \bar{\rho}(E) \leq \rho(f_\varepsilon(A)). \quad (3)$$

On the other hand, $\sum_{i=1}^{\infty} EC_{t_i} = \sum_{a \in A} P_a$, for some orthogonal family $\{P_a\}_{a \in A}$ of minimal projections of \mathcal{A}^- , so that from (2)

$$\rho_{at}(E) = \sum_{i=1}^{\infty} \sigma_{t_i}(EC_{t_i}) = \sum_{a \in A} \bar{\rho}(P_a). \quad (4)$$

From (3) and (4) it now follows that the inequality $\rho_{at}(E) \leq \bar{\rho}(E)$ will be established, once we show that

$$\sum_{j=1}^n \bar{\rho}(P_j) \leq \rho(f_\varepsilon(A)) \quad (5)$$

for any finite subfamily $\{P_j\}_{j=1}^n$ of $\{P_a\}_{a \in A}$. Let $\omega_1, \dots, \omega_n$ be the pure states of \mathcal{A} with support projections P_1, \dots, P_n , respectively. Since $\sum_{j=1}^n P_j \leq g_\varepsilon(A)$, the restriction of each ω_j ($j = 1, \dots, n$) to the hereditary C^* -subalgebra $\mathcal{J} = g_\varepsilon(A)\mathcal{A}g_\varepsilon(A)$ is a pure state of \mathcal{J} . There is a maximal abelian C^* -subalgebra \mathcal{B}_0 of \mathcal{J} such that $\omega_j|_{\mathcal{B}_0}$ is multiplicative and

$\omega_j|_{\mathcal{T}}$ is the unique extension of $\omega_j|_{\mathcal{B}_0}$ each j . Noting that $f_\varepsilon(A)$ is a unit for \mathcal{T} , it follows that P_1, \dots, P_n and $f_\varepsilon(A)$ belong to the weak closure of some abelian C^* -subalgebra $\mathcal{B} \supseteq \mathcal{B}_0$. Thus (5) follows from the definition of $\bar{\rho}$. The proof is complete.

Following Proposition (4.1.6) we see that $\rho_d = \rho - \rho_{at}$ is a positive quasilinear functional on \mathcal{A} that satisfies $\bar{\rho}_d(P) = 0$ for each minimal projection $P \in \mathcal{A}^-$. Thus, it is natural to call ρ_{at} and ρ_d the atomic and diffuse parts of ρ .

Christensen established an affirmative solution to the general quasi-state problem for C^* -algebras with Hausdorff spectrum that are representable by locally trivial continuous fields of elementary C^* -algebras nonisomorphic to $M_2(C)$. This result is used in the following proposition, where we consider weakly subadditive quasi-states on unital extensions of such algebras by finite-dimensional C^* -algebras.

Proposition (4.1.7)[4]. Let \mathcal{A} be a separable unital C^* -algebra containing a C^* -algebra \mathcal{B} with Hausdorff spectrum as a closed ideal, such that \mathcal{B} is representable by a locally trivial continuous field of elementary C^* -algebras, \mathcal{A}/\mathcal{B} is finite dimensional, and \mathcal{A} does not admit two-dimensional irreducible representations. Then weakly subadditive quasi-states on \mathcal{A} are linear.

Proof. It suffices to consider the case of pure weakly subadditive quasistate ρ on \mathcal{A} in its universal representation. From Proposition (4.1.2)(i), ρ is uniformly continuous on \mathcal{A} . Following the notation of Proposition (4.1.6), $\rho_{at} \neq 0$ if $\bar{\rho}(P) \neq 0$ for some minimal projection $P \in \mathcal{A}^-$. Hence $\rho = (1/\|\rho_{at}\|) \cdot \rho_{at}$ from Proposition (4.1.3)(ii), and ρ is linear in this case.

Otherwise, let $C_{\mathcal{B}}$ denote the open central support of \mathcal{B} in \mathcal{A}^- . Since $\mathcal{A}(I - C_{\mathcal{B}})$ is isomorphic to the finite-dimensional C^* -algebra \mathcal{A}/\mathcal{B} , we have $I - C_{\mathcal{B}} = \sum_{i=1}^m P_i$ for some finite orthogonal family of minimal projections P_1, \dots, P_m in \mathcal{A} . From Lemma (4.1.4)(ii), P_1, \dots, P_m belong to the weak closure of some maximal abelian C^* -subalgebra \mathcal{E} . By spectral theory there is an increasing sequence $\{A_n\}$ in \mathcal{E} , such that $A_n \rightarrow_n I - \sum_{i=1}^m P_i = C_{\mathcal{B}}$ in the strong-operator topology. Since $\bar{\rho}(P_i) = \dots = \bar{\rho}(P_m) = 0$, we have $\rho(A_n) \rightarrow_n 1$. In particular, $\|\rho|_{\mathcal{B}}\| = 1$. $\rho|_{\mathcal{B}}$ is a state of \mathcal{B} . Let $\{H_a\}_{a \in A} \subseteq \mathcal{B}_1^+$ be an increasing approximate unit of \mathcal{B} , which is quasi-central for \mathcal{A} , and let ω be the state of \mathcal{A} that extends $\rho|_{\mathcal{B}}$ via $\omega(A) = \lim_a \rho(H_a A H_a)$ ($A \in \mathcal{A}$). If $A \in \mathcal{A}^+$, we have $\rho\left(A^{\frac{1}{2}} H_a^2 A^{\frac{1}{2}}\right) \leq \rho(A)$ for each $a \in A$ (Proposition (4.1.2) (i)). Therefore

$$\omega(A) = \lim_a \rho(H_a A H_a) = \lim_a \rho\left(A^{\frac{1}{2}} H_a^2 A^{\frac{1}{2}}\right) \leq \rho(A),$$

from uniform continuity of ρ . Consequently, $\omega \leq \rho$, and from Proposition

(4.1.3)(ii), $\rho = \omega$ is linear. This completes the proof.

Corollary (4.1.8)[4]. Let \mathcal{A} be a unital C^* -algebra containing projections P and Q , $C^*(P, Q)$ be the C^* -subalgebra generated by P, Q , and I , and ρ be a weakly subadditive quasi-state on \mathcal{A} . If the relative commutant of $C^*(P, Q)$ in \mathcal{A} contains a unital copy of $M_n(C)$ for some $n \geq 3$, then ρ is linear on $C^*(P, Q)$.

Proof. It is well known that $C^*(P, Q)$ has at most two-dimensional irreducible representations. The two-dimensional representations are characterized up to unitary equivalence by $Sp(PQP)\setminus\{0,1\}$, so that to each $\lambda \in Sp(PQP)\setminus\{0,1\}$ corresponds the representation π_λ given by

$$\pi_\lambda(P) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \pi_\lambda(Q) = \begin{bmatrix} \lambda & (\lambda - \lambda^2)^{1/2} \\ (\lambda - \lambda^2)^{1/2} & 1 - \lambda \end{bmatrix}$$

Let \mathcal{L} denote the intersection of kernels of one-dimensional representations.

Then $\mathcal{L} \cong M_2(\mathcal{E}_0(Sp(PQP)\setminus\{0,1\}))$ and $C^*(P, Q)/\mathcal{L} \cong \sum_i \oplus \mathbb{C}$ for some integer l such that $0 \leq l \leq 3$, depending on the relative position of P and Q .

Consequently, if \mathcal{A}_0 denotes the separable C^* -subalgebra generated by $C^*(P, Q)$ and some commuting unital copy of $M_n(C)$ ($n \geq 3$), then $\mathcal{A}_0 \cong M_n(C^*(P, Q))$, and \mathcal{A}_0 contains a closed ideal $\mathcal{B}_0 \cong M_n(\mathcal{L}) \cong M_{2n}(\mathcal{E}_0(Sp(PQP)\setminus\{0,1\}))$, such that $\mathcal{A}_0/\mathcal{B}_0 \cong \sum_t \oplus M_n(C)$.

From Proposition (4.1.7), ρ is linear on \mathcal{A}_0 and, in particular, on $C^*(P, Q)$.

Section (4.2): C^* -Algebras Containing a Dense Set of Elements with Finite Spectrum

The following theorem is the crucial tool in our investigation of quasi-states on C^* -algebras containing a dense set of elements with finite spectrum. It is a slight generalization established by Christensen for quasi-states on properly infinite von Neumann algebras.

Theorem (4.2.1)[4]. Let \mathcal{A} be a unital C^* -algebra, and ρ be a quasi-state on \mathcal{A} which is linear when restricted to each C^* -subalgebra generated by two projections. Suppose that there exists a sequence of projections $\{P_n\}$ in \mathcal{A} such that

(i) $\rho(I - P_n) \rightarrow_n 0$, and

(ii) for each n there are partial isometries U_n, V_n , and W_n in \mathcal{A} such that $U_n^* U_n = V_n^* V_n = W_n^* W_n = P_n$, and $P_n, U_n U_n^*, V_n V_n^*, W_n W_n^*$ are mutually orthogonal.

Then there is a state on \mathcal{A} that coincides with ρ on the set of projections of \mathcal{A} . In particular, if \mathcal{A} contains a dense set of elements with finite spectrum and ρ is continuous, then ρ is linear.

Proof. The key point in the proof is to show that ρ is “almost linear” on the algebras $P_n \mathcal{A} P_n$ for all sufficiently large n . For this, the conditions (6) and (7) are used to construct, for each n and any two positive elements A and B satisfying $A, B < \frac{1}{2} P_n$, mutually orthogonal projections Q and R in \mathcal{A} such that

$$P_n Q P_n = A, \quad P_n R P_n = B$$

and

$$\rho(Q) \approx \rho(P_n Q P_n), \quad \rho(R) \approx \rho(P_n R P_n), \quad \rho(Q + R) \approx \rho(P_n (Q + R) P_n)$$

when n is sufficiently large. Once this is achieved,

$$\rho(A + B) \approx \rho(Q + R) = \rho(Q) + \rho(R) \approx \rho(A) + \rho(B).$$

From this it will follow that the sequence of functions $\{\varphi_n\}$, given by $\varphi_n(C) = \rho(P_n C P_n)$ ($C \in \mathcal{A}$), is pointwise convergent on the linear span of projections to the positive linear functional that coincides with ρ on each projection.

For a fixed n consider any two positive elements $A, B \in P_n \mathcal{A} P_n$, such that

$A, B \leq \frac{1}{2} P_n$. From (ii) the elements

$$X_n = A^{\frac{1}{2}} + U_n A^{\frac{1}{2}} + V_n (P_n - 2A)^{\frac{1}{2}}$$

and

$$Y_n = B^{\frac{1}{2}} - U_n B^{\frac{1}{2}} + W_n (P_n - 2B)^{\frac{1}{2}}$$

are partial isometries that satisfy

$$X_n^* X_n = Y_n^* Y_n = P_n \quad \text{and} \quad X_n^* Y_n = 0.$$

Thus, the projections $Q = X_n X_n^*$ and $R = Y_n Y_n^*$ are mutually orthogonal, so that $\rho(Q + R) = \rho(Q) + \rho(R)$. In addition, $P_n Q P_n = A$ and $P_n R P_n = B$.

Since ρ is linear on the C^* -subalgebra $C^*(Q, P_n)$, generated by Q, P_n and I , we have $|\rho(A) - \rho(Q)| = |\rho(P_n Q P_n) - \rho(Q)| \leq |\rho((P_n - I)Q P_n)| + |\rho(Q(P_n - I))| \leq 2/n$, from the Cauchy-Schwarz inequality (where it is assumed that $\rho(I - P_n) \leq 1/n^2$). Similarly,

$$|\rho(B) - \rho(R)| \leq \frac{2}{n} \quad \text{and} \quad |\rho(A + B) - \rho(Q + R)| \leq \frac{2}{n}.$$

Therefore

$$|\rho(A + B) - \rho(A) - \rho(B)| \leq \frac{6}{n} \quad \left(0 \leq A, B \leq \frac{1}{2} P_n\right). \quad (6)$$

If C and D are arbitrary self-adjoint elements of \mathcal{A} and $a = \max\{\|C\|, \|D\|\}$, $b = \max\{\|P_n C P_n + a P_n\|, \|P_n D P_n + a P_n\|\}$, then

$0 \leq \left(\frac{1}{2b}\right) (P_n C P_n + a P_n), \left(\frac{1}{2b}\right) (P_n D P_n + a P_n) \leq \frac{1}{2} P_n$; so that from (6),

$$|\rho(P_n(C + D)P_n) - \rho(P_n C P_n) - \rho(P_n D P_n)| \frac{12b}{n} \leq \frac{24a}{n}$$

This, in its turn, implies that

$$|\rho(P_n(A + B)P_n) - \rho(P_n A P_n) - \rho(P_n B P_n)| \xrightarrow[n]{} 0 \quad \text{for } A, B \in \mathcal{A} \quad (7)$$

Let $\{\varphi_n\}$ be the sequence of functions of \mathcal{A} given by $\varphi_n(A) = \rho(P_n A P_n)$ ($A \in \mathcal{A}$). For each projection $E \in \mathcal{A}$ we have $|\varphi_n(E) - \rho(E)| = |\rho(P_n E P_n) - \rho(E)| \leq 2/n$, as before. Thus, $\lim_n \varphi_n(E) = \rho(E)$, and we see from (7) that the sequence $\{\varphi_n\}_{n=1}^\infty$, converges on the linear span \mathcal{S} of projections of \mathcal{A} to the linear functional φ . If $A \in \mathcal{S}^+$, then $\varphi(A) =$

$\lim_{n \rightarrow \infty} \rho(P_n A P_n) \geq 0$. Consequently, φ is a state on \mathcal{S} and φ extends to a state of \mathcal{A} which satisfies the first assertion of the theorem.

Note that if \mathcal{F} denotes the subset of \mathcal{S} consisting of the elements with finite spectrum, then $\varphi|_{\mathcal{F}} = \rho|_{\mathcal{F}}$. Consequently, if \mathcal{F} is norm-dense in \mathcal{A} and ρ is continuous, then ρ is the unique continuous extension of $\varphi|_{\mathcal{F}}$.

Therefore ρ is the unique extension of φ to a positive linear functional on \mathcal{A} .

Corollary (4.2.2)[4]. Let \mathcal{A} be a unital C^* -algebra containing a dense set of elements with finite spectrum, and \mathcal{M} a properly infinite von Neumann algebra. If ρ is a weakly subadditive quasi-state on $\mathcal{A} \otimes_{\min} \mathcal{M}$, then ρ is linear on \mathcal{A} .

Proof. Let \mathcal{B} denote a subalgebra of \mathcal{M} , which is the relative commutant of some unital type I_3 subfactor of \mathcal{M} . Since \mathcal{B} is a properly infinite von Neumann algebra, we can choose an increasing sequence of projections $\{P_n\}$ in \mathcal{B} , such that $\rho(P_n) \geq 1 - 2^{-n}$ and $I - P_n$ contains three mutually orthogonal subprojections each equivalent to P_n . From Corollary (4.1.8), ρ is linear when restricted to each C^* -subalgebra of $\mathcal{A} \otimes_{\min} \mathcal{B} (\subset \mathcal{A} \otimes_{\min} \mathcal{M})$ generated by two projections. Furthermore, ρ is continuous on $\mathcal{A} \otimes_{\min} \mathcal{M}$ (Proposition (4.1.2) (i)). Thus, Theorem (4.2.1) applies, and ρ is linear $\mathcal{A} \otimes_{\min} \mathcal{B}$. In particular, ρ is linear on \mathcal{A} .

Corollary (4.2.3)[4]. If ρ is a quasi-state on the Calkin algebra $\mathcal{C} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, then ρ is linear.

Proof. Since each self-adjoint element of \mathcal{C} belongs to some abelian C^* -subalgebra generated by projections, any bounded linear functional on \mathcal{C} that agrees with ρ on the set of projections must coincide with ρ . Hence it suffices to show that ρ satisfies the conditions of Theorem (4.2.1). It is easy to see that ρ satisfies conditions (6) and (7). We shall show that ρ is linear on each C^* -subalgebra generated by two projections of \mathcal{C} .

Consider projections E and F in \mathcal{C} . There are projections E_1 and F_1 in $\mathcal{B}(\mathcal{H})$ such that $\phi(E_1) = E$ and $\phi(F_1) = F$, where ϕ denotes the quotient map. Indeed, if $E = \phi(A)$ for some $A \in \mathcal{B}(\mathcal{H})^+$, it suffices to choose E_1 equal to the spectral projection of A corresponding to the interval $(\frac{1}{2}, \infty)$.

The W^* -subalgebra of $\mathcal{B}(\mathcal{H})$, generated by E_1 and F_1 is the direct sum of a type I_2 W^* -algebra \mathcal{B} and an abelian W^* -algebra \mathcal{D} . Thus, with $C^*(E, F)$ -the C^* -algebra generated by E and F -we have

$$C^*(E, F) \subseteq \phi(\mathcal{B}) \oplus \phi(\mathcal{D})$$

If $\phi(\mathcal{B}) = 0$, then $C^*(E, F)$ is abelian, and ρ is linear on $C^*(E, F)$. Since \mathcal{B} is generated by the projections $E_2 = E_1 - E_1 \wedge F_1 - E_1 \wedge (E_1 \vee F_1 - F_1)$

and $F_2 = F_1 - E_1 \wedge F_1 - (E_1 \vee F_1 - E_1) \wedge F_1$, we may assume that E_2 and F_2 are infinite projections of $\mathcal{B}(\mathcal{H})$.

The algebra \mathcal{B} may be identified with $M_2(\mathcal{A}(X))$, where X is a compact hyperstonean space-the spectrum of the abelian W^* -algebra generated by E_2, F_2, E_2 . Let \mathcal{B}_0 , be the norm-

dense $*$ -subalgebra of \mathcal{B} consisting of 2×2 matrices over continuous functions that take finitely many values on X .

Then $\phi(\mathcal{B}_0)$ is norm-dense in $\phi(\mathcal{B})$. If A_0 and B_0 are two elements in $\phi(\mathcal{B}_0)$, they generate a common finite partition of the identity of $\phi(\mathcal{B})$ into nonzero projections H_1, \dots, H_k in the center of $\phi(\mathcal{B})$, and we can find projections G_1, \dots, G_k in the center of \mathcal{B} such that $\phi(G_1) = H_1, \dots, \phi(G_k) = H_k$.

Since G_1, \dots, G_k are infinite projections of $\mathcal{A}(\mathcal{H})$, each can be halved into two equivalent subprojections; and we see that A_0 and B_0 are contained in the direct sum of k copies of $M_4(\mathcal{C})$ inside \mathcal{E} . By Gleason's theorem, $\rho(A_0 + B_0) = \rho(A_0) + \rho(B_0)$. Therefore $\rho|_{\phi(\mathcal{B}_0)}$ is a positive linear functional on $\phi(\mathcal{B}_0)$, and $\rho|_{\phi(\mathcal{B}_0)}$ extends by continuity to a positive linear functional φ on $\phi(\mathcal{B})$. ρ is continuous on the set of projections in $\phi(\mathcal{B})$. But the set of projections in $\phi(\mathcal{B}_0)$ is norm-dense in the set of projections in $\phi(\mathcal{B})$. Thus ρ agrees with φ on the set of projections in $\phi(\mathcal{B})$, so that $\varphi = \rho|_{\phi(\mathcal{B})}$. Consequently, ρ is linear on $\phi(\mathcal{B}) \oplus \phi(\mathcal{D})$, and, in particular, on $C^*(E, F)$. This completes the proof.

Proposition (4.2.4)[4]. Let \mathcal{A} be a unital C^* -algebra containing a dense set of elements with finite spectrum. If ρ is a weakly subadditive and continuous quasi-state on $\mathcal{A} \otimes \mathcal{K}$ then ρ is linear.

Proof. $\mathcal{A} \otimes \mathcal{K}$ contains a dense $*$ -subalgebra that may be identified with the increasing union $\bigcup_{m \in \mathbb{N}} M_m(\mathcal{A})$, where $M_m(\mathcal{A})$ is viewed as embedded in $M_{m+1}(\mathcal{A})$ via the map i_m given by $i_m(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$. We shall adopt the proof of Theorem (4.2.1) to show that ρ is linear on $\bigcup_{m \in \mathbb{N}} M_m(\mathcal{A})$.

Since \mathcal{A} is unital and ρ is continuous, $\rho|_{CI \otimes \mathcal{K}} \neq 0$ (in fact, $\rho|_{CI \otimes \mathcal{K}}$ is a state). Therefore we can choose an increasing sequence of projections $\{P_n\}$ in $\bigcup_{m \in \mathbb{N}} M_m(\mathcal{C})$ ($\subseteq \bigcup_{m \in \mathbb{N}} M_m(\mathcal{A})$) such that $\rho(P_n) \geq \|\rho|_{CI \otimes \mathcal{K}}\| - 1/n^2$. In particular, $\rho(G - P_n) \leq 1/n^2$ if G is a projection in $\bigcup_{m \in \mathbb{N}} M_m(\mathcal{C})$ and $G \geq P_n$. It is easy to see also that for a given n there is a sufficiently large integer ρ , and partial isometries U_n, V_n, W_n in $M_\rho(\mathcal{C})$, such that $U_n^* U_n = V_n^* V_n = W_n^* W_n = P_n$ and $P_n, U_n U_n^* V_n V_n^* W_n W_n^*$ are mutually orthogonal.

For each $m \in \mathbb{N}$ let I_m , denote the identity projection of $M_m(\mathcal{A})$. If P and Q are two projections in $\bigcup_{m \in \mathbb{N}} M_m(\mathcal{A})$, then $P, Q \in M_m(\mathcal{A})$ for some k .

Let $\{E_{ij} | i, j \in \{1, 2, 3\}\}$ be the system of matrix units coming from the equivalence of projections $I_\kappa I_{2\kappa} - I_\kappa$, and $I_{3\kappa - 2\kappa}$ in $M_{3\kappa}(\mathcal{C})$. The C^* -algebra $C^*(P, Q)$ generated by P, Q , and I_κ is a hereditary subalgebra of the C^* -algebra \mathcal{B} generated by $C^*(P, Q)$ and $\{E_{ij} | i, j \in \{1, 2, 3\}\}$. From the proof of Corollary (4.1.8) it follows that \mathcal{B} contains an essential ideal $\mathcal{L}_0 \cong M_6(\mathcal{E}_0(Sp(PQP) \setminus \{0, 1\}))$ such that $\mathcal{B}/\mathcal{L}_0 \cong \sum_l \oplus M_3(\mathcal{C})$ for some integer l .

From Proposition (4.1.7), ρ is linear on \mathcal{B} and, in particular, on $C^*(P, Q)$.

The proof of Theorem (4.2.1) now applies almost verbatim (indeed, the only change is to replace the estimate $\rho(I - P_n) \leq 1/n^2$ by $\rho(G - P_n) \leq 1/n^2$, if $G \in \bigcup_{m \in \mathbb{N}} M_m(\mathcal{C})$ and

$G \geq P_n$) and shows that ρ is linear on the normdense $*$ -subalgebra $\bigcup_{m \in \mathbb{N}} M_m(\mathcal{A})$. Since ρ is continuous, it is linear on $\mathcal{A} \otimes \mathcal{K}$.

Guided by the example of pure states, it is natural to ask whether the conditions (6) and (7) of Theorem (4.2.1) (or somewhat similar properties reflecting the size of a nullspace) hold for pure quasi-states. We were not able to answer this question even for continuous pure quasi-states on C^* -algebras containing a dense set of elements with finite spectrum.

However, some additional evidence for a certain class of C^* -algebras with finite trace may be deduced by “approximate centralizer” techniques developed for finite von Neumann algebras. In the following proposition we shall say that a C^* -algebra \mathcal{A} containing a dense set of elements with finite spectrum, is strictly finite if there is a tracial state τ on \mathcal{A} such that for each pair of projections P and Q in \mathcal{A} the condition $\tau(P) \leq \tau(Q)$ implies $P \preceq Q$ (“ \preceq ” means “Murray-von Neumann subequivalent”).

Since the positive part of the unit ball of each hereditary C^* -subalgebra of \mathcal{A} is a norm-closed convex span of projections, it follows that \mathcal{A} is simple. In particular, any tracial state of \mathcal{A} is faithful.

Obvious examples of strictly finite C^* -algebras are finite von Neumann algebra factors and UHF-algebras.

Proposition (4.2.5)[4]. If \mathcal{A} is a separable unital strictly finite C^* -algebra and ρ is a continuous pure quasi-state on \mathcal{A} which is linear when restricted to each C^* -subalgebra generated by two projections, then ρ is linear.

Proof. Let τ denote a tracial state of \mathcal{A} and \mathcal{P} denote the set of projections of \mathcal{A} . We may assume that there is at least one projection G in \mathcal{P} such that $\tau(G) = \alpha \leq \frac{1}{4}$ (otherwise \mathcal{A} is isomorphic to $M_n(C)$ for some $n \leq 3$).

If $M(\alpha) = \sup\{\rho(F) \mid F \in \mathcal{P}, \tau(F) = \alpha\}$, choose a sequence of projections $\{P_n\}$ such that $\tau(P_n) = \alpha$ for each n , and $\rho(P_n) > M(\alpha) - 1/2n^2$ ($n = 1, 2, \dots$).

Which apply with only slight changes in the proof to the present case, for each projection $E \in \mathcal{P}$ we have

$$|\rho(E) - \rho(P_n Q P_n) - \rho((I - P_n)E(I - P_n))| \leq \frac{1}{n}, \quad (8)$$

$$\rho(P_n E P_n) \geq \lambda_n \tau(P_n E P_n) - \frac{1}{2n^2}, \quad (9)$$

$$\rho((I - P_n)E(I - P_n)) \leq \lambda_n \tau((I - P_n)E(I - P_n)) + \frac{1}{n^2} \quad (10)$$

for some sequence of nonnegative real numbers $\{\lambda_n\}_{n=1}^\infty$. From (8) and (9)

$$\rho(E) \geq \lambda_n \tau(P_n E P_n) - \frac{1}{2n^2} - \frac{1}{n} \quad (E \in \mathcal{P}, n = 1, 2, \dots). \quad (11)$$

We claim that $\rho(I - E_\kappa) \rightarrow_\kappa 0$ for some sequence of projections $\{E_\kappa\}$ such that $\tau(E_\kappa) = \alpha$ for each κ . Indeed, if $\inf\{\lambda_n | n \in \mathbb{N}\} = 0$, then from (10), applied for $E = I$, $\rho(I - P_n) \leq \lambda_n \tau(I - P_n) + \frac{1}{2n^2} \leq \lambda_n \frac{1}{2n^2}$, so that a suitable subsequence of $\{P_n\}_{n=1}^\infty$ can be taken for $\{E_\kappa\}_{\kappa=1}^\infty$.

On the other hand, if $\lambda_n \geq \lambda > 0$ for all n , consider the sequence of positive linear functionals $\{\varphi_n\}$ on \mathcal{A} given by $\varphi_n(A) = \lambda_n \tau(P_n A P_n)$ ($A \in \mathcal{A}$). From (9), $\|\varphi_n\| \leq \frac{3}{2}$ for each n . By weak compactness, the sequence $\{\varphi_n\}$ has a limit point φ , and $\varphi \neq 0$, because $\varphi_n(I) = \lambda_n \tau(P_n) \geq \lambda \alpha$ for each n . Furthermore, from (11), $\rho(E) \geq \varphi(E)$ for each $E \in \mathcal{P}$. Since ρ is continuous, and the elements with finite spectrum are dense in \mathcal{A} , this implies $\rho(E) \geq \varphi(E)$ for each \mathcal{A} in \mathcal{A}^+ . Therefore $\rho = (1/\|\varphi\|) \cdot \varphi$ (Proposition (4.1.3) (i)), so that ρ is a pure state. But then the maximal hereditary C^* -subalgebra $\mathcal{L}_\rho \cap \mathcal{L}_\rho^*$ has an increasing approximate identity consisting of projections. Consequently, since \mathcal{A} is strictly finite, $\mathcal{L}_\rho \cap \mathcal{L}_\rho^*$ must contain projection P such that $\tau(P) = 1 - \alpha$, and we may set $E_\kappa = I - P$ for each κ . Thus, the claim follows, and ρ is linear by Theorem (4.2.1).

Proposition (4.2.6)[4]. Let ρ be an approximately additive quasi-state on a C^* -algebra \mathcal{A} in addition, ρ is either monotone on \mathcal{A}^+ , monotone on $\mathcal{A}_{s,a}$ or weakly subadditive, then ρ extends to a quasi-state with the same properties on the multiplier algebra $\mathcal{M}(\mathcal{A})$.

Proof. Suppose first that ρ is approximately additive and monotone on \mathcal{A}^+ . Let $\{H_a\}_{a \in A} \subseteq \mathcal{A}_1^+$ be an increasing approximate identity for \mathcal{A} which is quasi-central for $\mathcal{M}(\mathcal{A})$. We note that the net $\left\{H_a^{\frac{1}{2}}\right\}_{a \in A}$ is also quasi-central for $\mathcal{M}(\mathcal{A})$. For this consider the C^* -algebra $\mathcal{B} = l^\infty(\mathcal{M}(\mathcal{A}), A) / \mathcal{I}$, where $l^\infty(\mathcal{M}(\mathcal{A}), A)$ denotes the C^* -algebra of all bounded nets $\{M_a\}_{a \in A}$ along A in $\mathcal{M}(\mathcal{A})$ under the pointwise operations and the norm $\|\{M_a\}\| = \sup_a \|M_a\|$, and \mathcal{I} is the closed ideal in $l^\infty(\mathcal{M}(\mathcal{A}), A)$, consisting of all the nets converging to 0. If $[\{M_a\}]$ denotes the image of $\{M_a\}_{a \in A}$ in \mathcal{B} under the quotient map, and $\tilde{\mathcal{M}}(\mathcal{A})$ denotes the image of $\mathcal{M}(\mathcal{A})$ in \mathcal{B} under the canonical embedding of $\mathcal{M}(\mathcal{A})$ into \mathcal{B} , then $[\{H_a\}]$ belongs to the relative commutant of $\tilde{\mathcal{M}}(\mathcal{A})$ in \mathcal{B} . Hence $[\{H_a^{1/2}\}] (= [\{H_a\}]^{1/2})$ commutes with $\tilde{\mathcal{M}}(\mathcal{A})$. This means that $\|H_a^{1/2} M - M H_a^{1/2}\| \rightarrow 0$ for each $M \in \mathcal{M}(\mathcal{A})$.

Given $A \in \mathcal{M}(\mathcal{A})$, the net $\{\rho(A^{1/2} H_a A^{1/2})\}_{a \in A}$ is increasing, since ρ is monotone on \mathcal{A}^+ , and bounded above by $\|A\|$. Therefore $\lim_a \rho(A^{1/2} H_a A^{1/2})$ exists. Since $\|(A^{1/2} H_a A^{1/2} - H_a^{1/2} A H_a^{1/2})\| \rightarrow 0$, we have $|\rho(A^{1/2} H_a A^{1/2}) - \rho(H_a^{1/2} A H_a^{1/2})| \rightarrow_a 0$, (Proposition (4.1.2)(ii)).

Consequently, $\lim_a \rho(H_a^{1/2} A H_a^{1/2})$ exists, $\lim_a \rho(H_a^{\frac{1}{2}} A H_a^{\frac{1}{2}}) = \lim_a \rho(A^{1/2} H_a A^{1/2})$, and we may define $\sigma(A) = \lim_a \rho(H_a^{1/2} A H_a^{1/2})$ for each $A \in \mathcal{M}(\mathcal{A})^+$. If A and B are commuting elements in $\mathcal{M}(\mathcal{A})^+$ then $\|H_a^{1/2} A H_a^{1/2} - H_a^{1/2} B H_a^{1/2}\| \rightarrow 0$, so that from approximate additivity of ρ ,

$$\begin{aligned}\sigma(A + B) &= \lim_a \rho(H_a^{1/2}(A + B)H_a^{1/2}) = \lim_a \rho(H_a^{1/2}AH_a^{1/2}) + \lim_a \rho(H_a^{1/2}BH_a^{1/2}) \\ &= \sigma(A) + \sigma(B).\end{aligned}$$

From this it follows that σ extends to a positive linear functional on each maximal abelian subalgebra of $\mathcal{M}(\mathcal{A})$, and to a positive quasi-linear functional (again denoted by σ) on $\mathcal{M}(\mathcal{A})$. It is clear that σ is a quasi-state.

Furthermore, if $C \in \mathcal{A}^+$, then $\sigma(C) = \lim_a \rho(C^{1/2}H_a C^{1/2}) = \rho(C)$, from Proposition (4.1.2)(ii).

If ρ is either monotone on $\mathcal{A}_{s.a.}$ or weakly subadditive, then, in particular, ρ is monotone on \mathcal{A}^+ . From the arguments above, ρ extends to a quasi-state σ on $\mathcal{M}(\mathcal{A})$. In either case, monotonicity, subadditivity, and approximate additivity of σ are easily deduced from the definition of σ and the corresponding properties of ρ .

List of Symbols

Symbol	Page
max:	maximum
\oplus :	Direct sum
dom:	domain
\otimes :	tensor product
L_∞ :	Essential Lebesgue space
\mathcal{A}_λ^2 :	weighted Bergman space
ℓ_2 :	Hilbert sequences space
L_2 :	Hilbert space
Im:	Imaginary
arg:	argument
Sp:	Spectrum
Sup:	Supremum
Ind:	Index
Hom:	homeomorphism
Prob:	Probability
ℓ^∞ :	Essential Hilbert sequences space
dim:	Dimension
min:	Minimum

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