CHAPTER 1

Proximal-Type Methods inVector Variational Inequality Problems

We employ the obtained results to propose a class of proximal-type method to solve the vector variational inequality problems, carry out convergent analysis on the method and prove convergence of the generated sequence to a solution of the vector variational inequality problems under some mild conditions.

Sec (1.1) :Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$and let $T : H \rightrightarrows H$ be a maximal monotone operator. Consider the following problem: finding an $x \in H$ such that

$$0 \in T(x).$$

This problem is very important in both theory and methodology of mathematical programming and some related fields. One of the efficient algorithms for the above problem is the proximal point algorithm (PPA, in short). This algorithm was first introduced by Martinet and its celebrated progress was attained in the work of Rocks fellar . The classical proximal point algorithm generated a sequence $\{z^{\mathrm{k}}\} \subset H$ with an initial point z^0 through the following iteration.

$$z^{k+1} = (I + c_k T)^{-1} z^k \tag{1}$$

where $\{c_k\}$ is a sequence of positive real numbers bounded away from zero. Rockafellar proved that for a maximal monotone operator T, the sequence $\{z^k\}$ weakly converges to a zero of T under some mild conditions. From then on, many works have been devoted to investigate the proximal point algorithm, its applications and generalizations and the references therein for scalar-valued problems for vector-valued optimization problems[vector optimization is asubarea of mathematical optimization where optimization problems with avector-valued opjective functions are optimized with respect to agiven partial ordering and supject to certain constraints.Amulti-opjective optimization problem:The objective space is the finite dimensional Euclidean space partially ordered by the component-wise”less than or equal to”ordered] .

On the other hand, the concept of vector variational inequality was firstly introduced by Giannessi in finite dimensional spaces. The vector variational inequality problems have found a lot of important applications in multiobjective decision making problems, network equilibrium problems, traffic equilibrium problems and so on. Because of these significant applications, the study of vector variational inequalities has attracted wide attention. Chen and Yang investigated general vector variational inequality problems and vector complementary problems in infinite dimensional spaces. Chen considered the vector variational inequality problems with a variable ordering structure. Yang studied the inverse vector variational inequality problems and their relations with some vector optimization problems.

Recently, Huang, Fang and Yang obtained some necessary and sufficient conditions for the nonemptiness and compactness of the solution set of a pseudomonotone vector variational inequality defined in a finite-dimensional space. Through the last twenty years of development, existence results of solutions, duality theorems and topological properties of solution sets of several kinds of vector variational inequalities have been derived.

However there is no numerical method has be designed for solving vector variational inequality problems, even no conceptual one. Motivated by the classical results of Rockafellar’s, in this section we firstly try to construct a class of vector-valued proximal-type method for solving a weak vector variational inequality problemand prove the sequence generated by our method converges to a solution of the weak vector variational inequality problem under some mild conditions.

we present some basic concepts, assumptions and preliminary results, we introduce the proximal-type method and carry out convergence analysis on the method, we draw a conclusion and make some remarks.

In this section, we present some basic definitions and propositions for the proof of our main results.

Let $C = R^m_+ \subset R^m \; and \; C_1 = \{x \epsilon R^m_+ \|x\| = 1\}$. We define, for any $y_1, y_2 \in R^m$,

$$y_1 \leq_C y_2 \text{ if and only if } y_2 - y_1 \in C;$$

$$y_1 \nleq_{int} y_2 \text{ if and only if } y_2 - y_1 \notin intC.$$

The extended space of R^m is $R^m = R^m \cup \{-\infty C, +\infty C\}$, where $-\infty C$ is an imaginary point, each of the coordinates is $-\infty$ and the imaginary point $+\infty C$ is analogously understood

(with the conventions $\infty C + \infty C = \infty C, \mu(+\infty C) = +\infty C$ for each positive number μ). The point $y \in R^m$ is a column vector and its transpose is denote by $y > T$. The inner product in R^m is denoted by $\langle .,. \rangle$

Let X_0 be a nonempty subset of R^n and let $Ti : X_0 \to R^n, i \in [1, \ldots, m]$ be vector-valued functions. Let $T := (T_1, \ldots, T_1)$ be a a $n \times m$ matrix which columns are $T_i(x)$, and let

$$T(x) = (T_1(x), \ldots, T_m(x)), T(x)^T(v) = (\langle T_1(x), v \rangle, \ldots, \langle T_m(x), v \rangle)^T$$

for every $x \in X_0$ and $v \in R^n$. For any $\lambda \in C_1$, a mapping $\lambda(T): X_0 \to R^n$ is defined by

$$\lambda(T)(x) = \sum_{i=1}^{m} \lambda_i T_i(x), x \in X_0. \quad (2)$$

Definition (1.1.1)[1]: Avector variational inequality (VVI in short) is a problem of finding $x^* \in X_0$ such that

$$(VVI) T\ (x^*)^T(x - x^*) \nleq_{C\setminus\{0\}} 0, \forall x \in X_0$$

where x^* is called a solution of problem (VVI).

Definition (1.1.2)[1]: Aweak variational inequality (WVVI in short) is a problem of finding

$x^* \in X_0$ such that

$$(WVVI) \qquad T(x^*)^T(x - x^*) \forall_{intC} 0, \forall x \in X_0,$$

where x^* is called a solution of problem (WVVI). Denote by $X^*$the solution set of problem (WVVI).

Let $\lambda \in C_1$, consider the corresponding scalar-valued variational inequality problem of finding

$x^* \in X_0$ such that:

$$(VIP)_\lambda \langle \lambda(T)(x^*), x - x^* \rangle \geq 0 \forall x \in X_0.$$

Denote by x^* be the solution set of(VIP_λ).

It is worth noticing that the partial order $\leq intc$ is closed in the sense that if $x_k \longrightarrow x^*$

as $k \longrightarrow \infty$, $x_k \leq_{intC} 0$, then we have $x^* \leq_{intC} 0$. This is because of the closeness of the set

$$S =: R^m \setminus (-intC).$$

Definition (1.1.3)[1]: Let $X_0 \subset R^n$ be nonempty, closed and convex, and $F : X_0 \longrightarrow R^n$ be a single-valued mapping.

(i) F is said to be monotone on X_0 if, for any $x_1, x_2 \epsilon X_0$, there holds

$$\langle F(x_1) - \langle F(x_2), x_1 - x_2 \rangle \geq 0.$$

(ii) F is said to be pseudomonotone X_0 if, for any $x_1, x_2 \in X_0$, there holds

$$< \langle F(x_2), x_1 - x_2 \rangle \geq 0 \Longrightarrow \langle F(x_1), x_1 - x_2 \rangle \geq 0 .$$

Clearly, a monotone map is pseudomonotone.

Now we give the definitions of C-monotonicity of a matrix-valued map.

Definition (1.1.4)[1]: $Let\ X_0 \longrightarrow R^n$ be nonempty, closed and convex. $T : X_0 \longrightarrow R^{n\times m}$ is amapping, which is said to be C-monotone on X_0 if, for any $x_1, x_2 \in X_0$, there holds

$$(T(x_1) - T(x_2))^T(x_1 - x_1) \geq_C 0.$$

Proposition (1.1.5)[1]: $Let\ X_0$ and T be defined as we have the following statements:

(i) T is C-monotone if and only if, for any $\lambda \in C_1$, the mapping $\lambda(T) : X_0 \longrightarrow R^n$ is monotone.

(ii) if T is $C - monotone$, then for any $\lambda \in C_1, \lambda(T) : X_0 \longrightarrow R^n is\ pseudomonotone$.

Definition (1.1.6)[1]: Let $L \subset R^{n\times m}$ be a nonempty set. The weak and strong C-polar cones of L are defined, respectively, by

$$L_C^{w0} := \{x \epsilon R^n : l(x) \ngeq_c 0, \quad \forall l \in L\}; \tag{3}$$

And

$$L_C^{s0} := \{x \epsilon R^n : l(x) \leq_c 0, \quad \forall l \in L\}; \tag{4}$$

Definition (1.1.7)[1]: Let $K \subset R^n$ be nonempty, closed and convex, $F : K \subset R^n \rightarrow R^m \cup \{+\infty C\}$ be a vector-valued mapping. $A\ n \times m$ matrix V is said to be a strong subgradient of $F\ at\ \bar{x} \in K\ if$

$$F(x) - F(\overline{x}) - V^T(x - \overline{x}) \geq_C 0 \quad \forall_x \in K.$$

A $n \times m$ matrix V is said to be a weak subgradient of F at ¯x ∈ K if

$$F(x) - F(\bar{x}) - V^{\top}(x - \bar{x}) \nleq_{intC} 0 \quad \forall\ x \in K$$

Denote by $\partial_c^w F(\bar{x})$ the set of weak subgradients of F on K at $\overline{x}$.

Let $K \subset R^n$ be nonempty, closed and convex. A vector-valued indicator function $\partial(x \mid K)$ of K at x is defined by

$$\delta(x \mid K) = \begin{cases} 0 \in R^n, \text{ if } x \in K; \\ +\infty_C, \quad \text{if } x \notin K. \end{cases}$$

An important and special case in the theory of weak subgradient is that when $F(x) = \partial(x \mid K)$ becomes a vector-valued indicator function of K, we obtain $\vee \epsilon \partial_c^w \delta(x^* | K)$ if and only if

$$V^\top(x - x^*) \not\geq_{intC} 0 \quad \forall x \in K. \tag{5}$$

Definition (1.1.8)[1]: A set $V\, N_K^w\,(x^*) \subset R^{n\times m}$ is said to be a weak normality operator set to K at x^*, if for every $V \epsilon V\, N_K^w\,(x^*)$ the inequality holds.

Clearly, $V\, N_K^w\,(x^*) = \partial_c^w(x^* \mid K)$. As for the scalar-valued case, we know that $v^* \in \partial\delta_K\,(x^*) = N_K(x^*)$ if and only if

$$\langle v^*, x - x^* \rangle \leq 0 \quad \forall x \in K \tag{6}$$

where $\partial K(x)$ is the scalar-valued indicator function of K. The inequality (1.1.6) means that $v*$ is normal to K at x^*.

Definition (1.1.9)[1]: Let $V\, N_K^w\,(.) : R^n \Rightarrow R^{n\times m}$ be a set-valued mapping, which is said to be a weak normal mapping for K, if for any $y \in K, V \in V\, N_K^w$ (y) such that

$$V^\top(x - y) \not\geq_{intC} 0, \quad \forall x \in K. \tag{7}$$

$V\, N_K^s\,(.)$is said to be strong normal mapping for K, if for any $y \in K, V \in V\, N_K^s$

$$V^\top(x - y) \leq_C 0, \quad \forall x \in K. \tag{8}$$

As in , the normal mapping for K is a set-valued mapping, which is defined as follows: if for any $y \in K, V \in V N_K^w$ such that

$$\|AX\| \leq \|A\|M\|X\|$$

Let $\|A\|_{\mathrm{M}}$ be a matrix norm of the matrix $A \in R^{n\times m}$. In this section, we always assume that the matrix norm $\|A\|_{\mathrm{M}}$ is compatible with $\|.\|, i.e.,$

$$\langle v, x - y\rangle \leq 0, \quad \forall x \in K$$

for all $A \in R^{m\times n}$ and x $\in R^n$. We now introduce a new notion.

Definition (1.1.10)[1]: Let $T : X_0 \to R^{n\times m}$ be a mapping, which is said to be norm sequentially bounded if for any bounded sequence $\{x_k\} \subset X_0$, it holds that the sequence $\{\|T(xk)\|_M\}$ is bounded.

Next we will introduce the definition and some basic results about the maximal monotone mapping.

Definition (1.2.11)[1]: Let a set-valued map $G : X_0 \subset R^n \rightrightarrows R^n$ be given, it is said to be monotone if

$$\langle z - \bar{z}, w - \bar{w}\rangle \geq 0$$

for all z and z in X_0, all w in G(z) and $\bar{}w \; in \; G(\bar{}z)$. It is said to be maximal monotone if, in addition, the graph

$$gph(G) = \{(z, w) \in R^n \times R^n \mid w \in G(z)\}$$

is not properly contained in the graph of any other monotone operator from $R^n$to R^n.

Lemma (1.2.12)[1]: Let K be a nonempty closed and convex subset of R^n. Let $T_1 : R^n \rightrightarrows R^n$ be the normal mapping to K and $T_2 : R^n \to R^n$ be any single-

valued monotone operator such that K $\cap\, dom(T_2) \neq \emptyset$; and T_2 is continuous on K. Then, we have $T_1 + T_2$ is a maximal
monotone operator.

Lemma (1.1.13)[1]: (Minty's theorem) Let $\lambda > 0$ and T : $R^n \rightrightarrows R^n$ be monotone. Then $(I + \lambda T)^{-1}$ is monotone and nonexpansive. Moreover, T is maximal monotone if and only if rge$(I + \lambda T\,)$= Rn. In that case $(I + \lambda T)^{-1}$ is maximal monotone too, and it is a single-valued mapping from all of R^n into itself.

Next we will introduce some fundamental definitions of the asymptotic analysis.

Definition (1.1.14)[1]: Let K be a nonempty set in R^n. Then the asymptotic cone of the set K, denoted by K_1, is the set of all vectors $d \in R^n$ that are limits in the direction of the sequence $\{X_k\} \subset$ K, namely

$$K^\infty = \{d \in R^n |\ \exists t_k \to +\infty,\ and\ x_k \in K,\ \lim_{k\to+\infty} \frac{x_k}{t_k} = d\} \tag{9}$$

In the case that K is convex and closed, then, for any $x_0 \in K$,

$$K^\infty = \{d \in R^n |\ x_0 + td \in K, \forall\ t > 0\}. \tag{10}$$

Definition (1.1.15)[1]: A set-valued mapping $S : R^n \rightrightarrows R^m$ is said to be outer semicontinuous $(osc\ in\ short)\ at\ \bar{x}$ if

$$\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x}),$$

where

$$\limsup_{x\to\bar{x}} S(x) := \bigcup_{x_k\to\bar{x}} \limsup_{k\to+\infty} S(x_k)$$

$= \{u | \exists X_k \to \overline{X}, \exists N_k \to u, with\ , N_k\ \in S(x_k))$

Sec (1.2) :Main Results

Proposition (1.2.1)[1]: Let $X_0 \subset R^n$ be nonempty, closed and convex, and $V\, N_{X_0}^{w}\,(.)$ be a weak normal mapping for X_0. For any $x^* \epsilon X_0$, and $\varphi \in V\, N_{X_0}^{w}\,(x^*)$, there exists $a\, \lambda \,\epsilon\, C_1$ such that $\varphi(\lambda)\, \epsilon\, N_{X_0}(x^*)$.

Proof: By the definition of the weak normal mapping, we know that

$$\varphi^{\top}(x - x^*) \not\geq_{intC} 0, \quad \forall x \in X_0.$$

It follows that

$$\varphi^{\top}(x - x^*) \in R^m \backslash intC, \quad \forall x \in X_0.$$

and

$$\varphi^{\top}(X_0 - x^*) \subset R^m \backslash intC.$$

That is

$$\varphi^{\top}(X_0 - x^*) \cap intC = \emptyset.$$

By the convexity of X_0, one has there exists $a\, \lambda^* \in C \backslash \{0\}$ such that

$$\langle \varphi \lambda^*, x - x^* \rangle \leq 0, \quad \forall x \in X_0.$$

Since $\|\lambda^*\| > 0$, one obtains that

$$\langle \varphi \frac{\lambda^*}{\|\lambda^*\|}, x - x^* \rangle \leq 0, \quad \forall x \in X_0.$$

Clearly, we have $\frac{\lambda^*}{\|\lambda^*\|} \in C_1$ Without loss of generality, let $\lambda = \frac{\lambda^*}{\|\lambda^*\|}$, one has

$$\langle \varphi \lambda, x - x^* \rangle \leq 0 \quad \forall x \in X_0.$$

That is $\varphi\lambda \in N_{X_0}(x^*)$ The proof is complete.

We propose the following exact proximal-type method (PTM, in short) for solving the problem (WVVI):

Step (1) : Taken $X_0 \in X_0$;

Step (2) : Given any $X_k \in X_0$, if $X_k \in x^*$. Then, the algorithm stops; otherwise goes to step (3);

Step (3) : If $X_k \notin X^*$. We define X_{k+1} by the following conclusion:

$$0 \in T(x_{k+1})\lambda_k + VN^w_{X_0}(x_{k+1})\lambda_k + \varepsilon_k(x_{k+1} - x_k) \quad (11)$$

where the sequence $\lambda_K \in C_1, \mathcal{E}_k \in (0, \mathcal{E}], \mathcal{E} > 0$ and V $N^{w}_{X_0}(.)$ is the weak normal mapping to X_0. Go to step (2).

Remark (1.2.2)[1]: The algorithm PTM is actually a kind of exact proximal point algorithm, where

the sequence $\lambda_K \in C_1$ is called as scalarization parameter, a bounded exogenous sequence of positive real numbers $\{\mathcal{E}_k\}$ is called as regularization parameter. For every $x_k \notin X^*$, we try to find a x_{k+1} such that $0 \in R^n$ belongs to the inclusion (11).

Next we will show the following results.

Theorem (1.2.3)[1]: Let $X_0 \subset R^n$ be nonempty, closed and convex, $T : X_0 \longrightarrow R^{n\times m}$ be continuous and C-monotone on X_0, if dom $T \cap intX_0 \neq \theta$ The sequence $\{x_K\}$ generated by the method (PTM) is well-defined.

Proof: Let $x_0 \in X_0$ be an initial point and suppose that the method (PTM) reaches step k. We then show that the next iterate x_{k+1} does exist. By the assumptions, T(.) is continuous and C-monotone on X_0, we have $\lambda(T)$ is monotone and continuous on X_0 for any $\lambda \in C_1$. From the Proposition (1.2.1), there exists $a\,\bar{\lambda} \in C_1$ such that the mapping V $N^{w}_{X_0}$ $(.)\bar{\lambda}$ is a normal mapping on X_0. Thus, by the assumption dom $T \cap intX_0 \neq \phi$ and Lemma (1.1.11), one has that for any $x \in X_0$, the mapping $(V\, N_{X0}(x) + T(x))\bar{\lambda}$ is maximal monotone. Without loss of generality, let $\lambda_k = \bar{\lambda}$. By Lemma (1.1.12), one obtains that

$$rge\{(VN^w_{X_0}(.)+T(.))\lambda_k+\varepsilon_k I(.)\}=R^n$$

Hence, for any given $\mathcal{E}_k x_k R^n$, there exists a$x_{k+1} \in X_0$ such that

$$\varepsilon_k x_k \in (T+VN^w_{X_0})(x_{k+1})\lambda_k+\varepsilon_k x_{k+1} \tag{12}$$

and

$$0 \in (T+VN^w_{X_0})(x_{k+1})\lambda_k+\varepsilon_k(x_{k+1}-x_k)$$

That is the inclusion (11) holds. The proof is complete.

Theorem (1.2.4)[1]: Let the same assumptions as in Theorem(1.2.3) hold. Further suppose that $X_0^\infty \cap [\mathrm{T}(X_0)]_C^{w0} = \{0\}$ and X^* is nonempty and compact . Then, the sequence $\{x_k\}$ generated by the method (PTM) is bounded.

Proof: From the method (PTM), we know that if the algorithm stops at some iteration, the point x_k will be a constant thereafter. Now we assume that the sequence $\{x_k\}$ will not stop after a finite number of iteratives. From the Proposition (1.2.1), we know that there exists $\lambda_k \epsilon C_1$ and $\varphi_{k+1} \in V\,N^w_{X_0}(X_{k+1})$ such that$\varphi_{k+1}\lambda_k \in V\,N^w_{X_0}(X_{k+1})$From the inclusion (11), one has that

$$0=T(x_{k+1})\lambda_k+\varphi_{k+1}\lambda_k+\varepsilon_k(x_{k+1}-x_k).$$

By the fact of $\varphi_{k+1}\lambda_k \in V\,N^w_{X_0}(X_{k+1})$, we obtain that

$$\langle \varphi_{k+1}\lambda_k, x-x_{k+1}\rangle \le 0, \quad \forall x \in X_0. \tag{13}$$

It follows that

$$\langle T(x_{k+1})\lambda_k+\varepsilon_k(x_{k+1}-x_k), x-x_{k+1}\rangle \ge 0, \quad \forall x \in X_0. \tag{14}$$

On the other hand, we know that for any given $\lambda_k \in C_1$, the following scalar-valued variational inequality problem $(V\,IP_{\lambda k})$ has a nonempty solution set, where

$$(VIP_{\lambda_k}) \quad \langle T(x^*)\lambda_k, x - x^* \rangle \geq 0, \quad \forall\, x \in X_0$$

Without loss of generality, let $x^* \in X^*$ and x^* is also a solution of problem $(V\,IP_{\lambda k})$. Hence, we have

$$\langle T(x^*)\lambda_k, x^* - x_{k+1} \rangle \leq 0.$$

By the $C - monotonicity$ of T, one has that

$$\langle T(x_{k+1})\lambda_k, x^* - x_{k+1} \rangle \leq 0.$$

(15)

Combining (14) with (15), we obtain that

$$\langle \varepsilon_k(x_{k+1} - x_k), x^* - x_{k+1} \rangle \geq 0.$$

From the method (PTM), we know that $\mathcal{E}_k > 0$. It follows that

$$\langle x_{k+1} - x_k, x^* - x_{k+1} \rangle \geq 0$$

$$2\langle x_{k+1} - x_k, x^* \rangle + 2\langle x_k - x_{k+1}, x_{k+1} \rangle \geq 0$$

$$\|x_k\|^2 - 2\langle x_k, x^* \rangle + \|x^*\|^2 - \|x_k\|^2 + 2\langle x_k, x_{k+1} \rangle - \|x_{k+1}\|^2 - \|x_{k+1}\|^2 + 2\langle x_{k+1}, x^* \rangle - \|x^*\|^2 \geq 0.$$

That is

$$\| x_{k+1} - x^* \|^2 \leq \| x_k - x^* \|^2 - \| x_k - x_{k+1} \|^2.$$

(16)

Clearly, the sequence $\{\|x_k - x^*\|^2\}$ is nonnegative and nonincreasing. Furthermore $\{\|x_k - x^*\|^2\}$ is also bounded below, as denoted by l^* the lower bound of the sequence. By the fact (15), we have

$$\sum_{k=0}^{\infty} \| x_k - x_{k+1} \|^2 \leq \| x_0 - x^* \|^2 - l^* \leq \| x_0 - x^* \|^2 < \infty$$

and

$$\lim_{k \to +\infty} \| x_k - x_{k+1} \| = 0. \qquad (17)$$

From the inequality (15), one has that

$$\|x_k - x^*\| \leq \|x_0 - x^*\|$$

for all $x^* \in X^*$. By the nonemptiness and compactness of X^*, we conclude that $\{x_k\}$ is bounded. The proof is complete.

Theorem (1.2.5)[1]: Let the same assumptions as in Theorem (1.2.3) hold. We also assume that T is norm sequentially bounded. Then any accumulation point of $\{x_k\}$ is a solution of problem (WV V I).

Proof: If there exists $k_0 \geq 1$ such that $x_{k_0+p} = x_{k_0}$, $\forall p \geq 1$. Then, it is clear that x_{k0} is the unique cluster point of $\{x_k\}$ and it is also a solution of problem (WV V I). Suppose that the algorithm does not terminate finitely. Then, by Theorem (1.2.3), we have that $\{x_k\}$ is bounded and it has some cluster points. Next we show that all of cluster points are solutions of problem (WV VI). Let $\hat{x}$ be a cluster points of $\{x_k\}$ and $\{x_{kj}\}$ be a subsequence of $\{x_k\}$, which converges to $\hat{x}$. From the limit (16), we know that $lim_{j \to \infty} \|x_{kj+1} - x_{kj}\| = 0$. That is $x_{kj+1} - xas\ j \to \infty$By the inclusion (11), one has that there exist $\lambda_{kj} \in C_1$ such that

$$\varphi_{kj+1} \in \vee N_{x_0}^{w}$$

$$T(x_{k_j+1})\lambda_{k_j} + (\varphi_{k_j+1})\lambda_{k_j} + \varepsilon_{k_j}(x_{k_j+1} - x_{k_j}) = 0$$

and

$$\|T(x_{k_j+1})\lambda_{k_j} + (\varphi_{k_j+1})\lambda_{k_j} + \varepsilon_{k_j}(x_{k_j+1} - x_{k_j})\| = 0.$$

It follows that

$$0 \geq \| T(x_{k_j+1})\lambda_{k_j} + (\varphi_{k_j+1})\lambda_{k_j} \| - \varepsilon_{k_j} \| (x_{k_j} - x_{k_j+1}) \| \tag{18}$$

From (17), we know that $lim_{j\to\infty}\|x_{kj} - x_{kj+1}\| = 0$. Since $\lambda_{kj} \in C_1$, by the compactnessof C_1, we know that the sequence $\{\lambda_{kj}\}$ has a convergent subsequence. Without loss ofgenerality, we assume that $\lambda_{kj} \longrightarrow \bar{\lambda}$. Furthermore we have $\bar{\lambda} \in C_1$ and $\bar{\lambda} \neq 0$. Thus, taking the limit in (17), we deduce the following:

$$\lim_{j\to+\infty} \|T(x_{k_j+1})\lambda_{k_j} + (\varphi_{k_j+1})\lambda_{k_j}\| = 0. \tag{19}$$

We claim that the sequence $\{\varphi_{kj+1}\lambda_{kj}\}$ is bounded. Suppose that, in contrast, without loss of generality, we assume that $\|\varphi_{kj+1}\lambda_{kj}\| \to \infty$ and $\frac{\varphi_{kj+1}\lambda_{kj}}{\|\varphi_{kj+1}\lambda_{kj}\|} \longrightarrow \overline{w} \in R^n \overline{w} \neq 0$From (18), we know that

$$0 = \lim_{j\to+\infty} \frac{\|T(x_{k_j+1})\lambda_{k_j} + \varphi_{k_j+1}\lambda_{k_j}\|}{\|\varphi_{k_j+1}\lambda_{k_j}\|} = \lim_{j\to+\infty} \| \frac{T(x_{k_j+1})\lambda_{k_j}}{\|\varphi_{k_j+1}\lambda_{k_j}\|} + \frac{\varphi_{k_j+1}\lambda_{k_j}}{\|\varphi_{k_j+1}\lambda_{k_j}\|} \| = \|0 + \bar{\omega}\|, \quad (3$$

since T is norm sequentially bounded, which yields that

$$\|T(x_{k_j+1})\lambda_{k_j}\| \leq \|T(x_{k_j+1})\|_M \|\lambda_{k_j}\| = \|T(x_{k_j+1})\|_M \leq \mu < +\infty \tag{20}$$

for some $\mu > 0$. Obviously, the equality (20) contradicts with the assumption $\overline{w} \neq 0$.

Thus, the sequence $\{\varphi_{kj+1}\lambda_{kj}\}$ is bounded. Without loss of generality, we assume that.

$\varphi_{kj+1}\lambda_{kj} \longrightarrow \widehat{w} \in R^n$ Furthermore, from (19) and the continuity of T, we derive that

$$\|T(\hat{x})\bar{\lambda} + \hat{\omega}\| = 0.$$

Hence, we have

$$T(\hat{x})\bar{\lambda} + \hat{\omega} = 0.$$

Meanwhile, from the definition of weak normal mapping and Proposition ??, we have $\overline{w} \in N_{x0}(\hat{x})$. By the definition of $N_{x0}(\hat{x})$, we know that

$$\langle \hat{\omega}, x - \hat{x} \rangle \leq 0 \quad \forall x \in X_0.$$

That is

$$\langle T(\hat{x})\bar{\lambda}, x - \hat{x} \rangle \geq 0 \quad \forall x \in X_0. \tag{21}$$

Thus

$$T(\hat{x})^\top (x - \hat{x}) \notin -intC \quad \forall x \in X_0. \tag{22}$$

We conclude that ˆx is a solution of problem (WV V I). The proof is complete.

Theorem (1.2.6)[1]: Let the same assumptions as those in Theorem (1.2.5) hold. Then the whole sequence $\{x_k\}$ converges to a solution of problem (WV V I).

Proof: Suppose that, in contrast, both $\hat{x}$ and $\tilde{x}$ are two distinct cluster points of $\{x_k\}$ and

$$\lim_{j \to +\infty} x_{k_j} = \hat{x}, \quad \lim_{i \to +\infty} x_{k_i} = \tilde{x}.$$

By Theorem (1.2.5), we know that $\hat{x}$ and $\tilde{x}$ are solutions of problem (WV V I). By virtue of Theorem (1.2.3) and the proof of Theorem (1.2.4), we know that there exist $\hat{\lambda}$ and $\tilde{\lambda} \in C_1$ such that

$$\langle T(\hat{x})\hat{\lambda}, \hat{x} - x_{k+1} \rangle \leq 0, \quad \langle T(\tilde{x})\tilde{\lambda}, \tilde{x} - x_{k+1} \rangle \leq 0 \tag{23}$$

By the C-monotonicity of T, one obtains

$$\langle T(x_{k+1})\hat{\lambda}, \hat{x} - x_{k+1} \rangle \leq 0, \quad \langle T(x_{k+1})\tilde{\lambda}, \tilde{x} - x_{k+1} \rangle \leq 0. \tag{24}$$

From (19), one has

$$\langle x_{k+1} - x_k, \hat{x} - x_{k+1} \rangle \geq 0, \quad \langle x_{k+1} - x_k, \tilde{x} - x_{k+1} \rangle \geq 0. \tag{25}$$

Similarly with (20), we know

$$\| x_{k+1} - \hat{x} \|^2 \leq \| x_k - \hat{x} \|^2 - \| x_k - x_{k+1} \|^2, \tag{26}$$

and

$$\| x_{k+1} - \tilde{x} \|^2 \leq \| x_k - \tilde{x} \|^2 - \| x_k - x_{k+1} \|^2 . \tag{27}$$

Combining (25)with(26), we obtain that both sequences $\{\|x_k - \hat{x}\|^2\} and \; \{\|x_k - \hat{x}\|^2\}$ are nonnegative and nonincreasing, hence they are convergent. So there exist $\hat{B}, \; \tilde{B} \in \; R$ such that

$$\lim_{k\to+\infty} \| x_k - \tilde{x} \| = \tilde{\beta}, \quad \lim_{k\to+\infty} \| x_k - \hat{x} \| = \hat{\beta}. \tag{28}$$

Clearly, we have

$$\| x_k - \hat{x} \|^2 = \| x_k - \tilde{x} \|^2 + 2\langle x_k - \tilde{x}, \tilde{x} - \hat{x} \rangle + \| \tilde{x} - \hat{x} \|^2 \tag{29}$$

Combining (28) with (29), we deduce the following

$$\lim_{k\to+\infty} \langle x_k - \tilde{x}, \tilde{x} - \hat{x} \rangle = \frac{1}{2}(\hat{\beta}^2 - \tilde{\beta}^2 - \| \tilde{x} - \hat{x} \|^2). \tag{30}$$

Taking $k = k_i$ in (30), we obtain that

$$\hat{\beta}^2 - \tilde{\beta}^2 = \| \tilde{x} - \hat{x} \|^2$$

Changing the places of $\hat{x} and \; \tilde{x}$ in (28) and repeating $k = k_i$in (30), we have that

$$\| \tilde{x} - \hat{x} \|^2 = \tilde{\beta}^2 - \hat{\beta}^2.$$

Thus, we conclude that

$$\| \tilde{x} - \hat{x} \| = 0,$$

which establishes the uniqueness of the cluster points of $\{x_k\}$. The proof is complete.

CHAPTER 2

Variational Inequalities in Finite Dimensional Spaces

Some existence theorems of Carath´eodory weak solutions for the differential inverse variational inequality are also established under suitable conditions. An application to the time-dependent spatial price equilibrium control problem is also given.

Sec (2.1) :Main Result

Let K $\subset R^{n}$ be a nonempty, closed, and convex set and $g : R^{n} \to R^{n}$ be a function. An inverse variational inequality (denoted by IVI(K, g)) is formulated as follows: find $x^* \in R^{n}$, such that

$$g(x^*) \in K, < \tilde{g} - g(x^*), x^* > i \geq 0, \forall \tilde{g} \in K. \quad (1)$$

Let SOLIVI(K, g) denote the solution set of this problem. We write $\dot{x} := \frac{dx}{dt}$for the time derivative of a function x(t). In this article, we introduce and study the following differential Received July 15, 2013; revised March 17, 2014. The work was inverse variational inequality (denoted by DIVI):

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t), \\ u(t) \in \text{SOLIVI}(K, G(t, x(t)) + F(\cdot)), \\ x(0) = x_0, \end{cases} \quad (2)$$

where $\Omega := [0, T] \times R^{m}, (f, B, G) : \to R^{m} \times R^{m\times n} \times R^{n}$ are given functions and $F : R^{n} \to R^{n}$ is a single-valued linear function. A point (x, u) is called a Carath´eodory weak solution of DIVI (1) if and only if x is an absolutely continuous function on [0, T] and u is an integrable function on [0, T] such that the differential equation satisfied for almost all t ∈[0, T] and u(t)

$\in$SOLIVI(K,$G(t, x(t)) + F(\cdot)$) for almost all t $\in$[0, T]. The set of all Carath´eodory weak solutions (x, u) of the initial-value DIVI (1) is denoted by SOLDIVI(K,G + F).

It is well known that the variational inequality theory has wide applications in optimization, engineering, economics, and transportation.

And ordinary differential equation with smooth input functions are a classical paradigm in applied mathematics that have existed for centuries. Yet, as evidenced by the growing literature that has surfaced in recent years on multi-rigid-body dynamics with frictional contacts and on hybrid engineering systems, ordinary differential equations are inadequate to deal with many naturally occurring engineering problems that contain inequalities and disjunctive conditions. For solving these problems, and studied differential variational inequality (DVI) in finite-dimensional Euclidean spaces which significantly extends these differential equations and open up a broad paradigm for the enhanced modeling of complex engineering system. Recently, introduced and investigated a class of differential mixed variational inequalities in finite dimensional spaces. Very recently, and studied differential vector variational inequalities in finite-dimensional spaces.

On the other hand, first introduced and studied the inverse variational inequalities in finite dimensional Euclidean spaces. They pointed out that there are many control problems appearing in economics, transportation, and management science and energy networks can be modeled as the inverse variational inequalities, but they are difficult to be formulated as the classical variational inequalities. Furthermore, developed a proximal point based algorithm for solving the inverse variational inequality. proposed two projection-based methods for solving the inverse variational inequality. considered the dynamic power price problem and characterized the optimal price as a solution of an inverse variational inequality.

studied the time-dependent spatial price equilibrium control problem and modeled it as an evolutionary inverse variational inequality. Some related work concerned with the inverse variational inequalities; and the references therein. Obviously, if the function f is single-valued, setting u = f(x) and $g(u) = f^{-1}$ (u), then the inverse variational inequality is transformed into the classical variational inequality. However, this transformation fails when f is set-valued. Moreover, in many real applications, explicit forms of function cannot be obtained which also causes failure of this transformation .Therefore, it is important and interesting to consider an ordinary differential equation whose right-hand function is parameterized by an algebraic variable that is required to be a solution of an inverse variational inequality containing the state variable of the system.

We give the linear growth of the solution set for the differential inverse variational inequality (1) under various conditions. Moreover, we show the existence theorems concerned with the Carath´eodory weak solutions for the differential inverse variational inequality (1) in finite-dimensional spaces. We also give an application to the time-dependent spatial price equilibrium control problem under some suitable conditions.

we will introduce some basic notations and preliminary results.

Definition(2.1.1)[2]: A map $f : R^n \to R^n$ is said to be (i) para-monotone on a convex set $K \subset R^n$ if f is monotone on K, that is

$$\langle f(v) - f(u), v - u \rangle \geq 0, \quad \forall v, u \in K,$$

and the following property holds: for any v, u ∈K, we have

$$\langle f(v) - f(u), v - u \rangle = 0 \Rightarrow f(v) = f(u).$$

(ii) strongly monotone on K if there exists a constant $\alpha > 0$ such that, for any v, u ∈K, we have

$$\langle f(v) - f(u), v - u\rangle \geq \alpha \|v - u\|^2$$

Definition(2.1.2)[2]: A map $F: \Omega \to R^n$ (respectively, $B: \Omega \to R^{n\times m}$) is said to be Lipschitz continuous if there exists a constant $L_F > 0$ (respectively, $L_B > 0$) such that, for any

$(t_1, x), (t_2, y) \in \Omega$, we have

$$\|F(t_1, x) - F(t_2, y)\| \leq L_F(|t_1 - t_2| + \|x - y\|),$$

$$\text{(respectively, } \|B(t_1, x) - B(t_2, y)\| \leq L_B(|t_1 - t_2| + \|x - y\|)).$$

In the rest of this article, we assume that the following conditions (A) and (B) hold:

(A) f, B, and G are Lipschitz continuous functions on Ω with Lipschitz constants $L_f > 0$, $L_B > 0$, and $L_G > 0$, respectively;

(B) B is bounded on Ω with $\sigma B := sup_{(t,x)\in\Omega}\|B(t,x)\| < \infty$.

Let

$$\mathbb{F}(t, x) := \{f(t, x) + B(t, x)u : u \in SOLIVI(K, G(t, x) + F)\}. \quad (3)$$

Lemma (2.1.3)[2]: Let $F : \Omega \rightrightarrows R^m$ be an upper semicontinuous set-valued map with nonempty closed convex values. Suppose that there exists a scalar $\rho F > 0$ satisfying

$$\sup\{\|y\| : y \in \mathbb{F}(t, x)\} \leq \rho_{\mathbb{F}}(1 + \|x\|), \quad \forall (t, x) \in \Omega \quad (4)$$

Then, for every $x^0 \in R^n$, $DI: x^{\cdot} \in F(t, x), x(0) = x^0$ has a weak solution in the sense of Carath´eodory,

Lemma (2.1.4)[2]: Let $h: \Omega \times R^m \to R^n$ be a continuous function and $U: \Omega \rightrightarrows R^m$ be a closed set-valued map such that for some constant ηU > 0,

$$\sup_{u\in U(t,x)} \|u\| \leq \eta_U(1+\|x\|), \quad \forall(t,x)\in\Omega.$$

let $v : [0,T] \to R^n$ be a measurable function and $x : [0,T] \to R^n$ be a continuous function satisfying $v(t) \in h(t,x(t),U(t,x(t)))$ for almost all $t \in [0,T]$. Then, there exists a measurable function $u : [0,T] \to R^m$ such that $u(t) \in U(t,x(t))$ and $v(t) = h(t,x(t),u(t))$ for almost all $t \in [0,T]$.

Lemma (2.1.5)[2]: Let (f,G,B) satisfy conditions (A) and (B), and $F : R^n \to R^n$ be a continuous map. Suppose that there exists a constant $\rho > 0$ such that, for all $q \in G(\Omega)$,

$$\sup\{\|u\| : u\in SOLIVI(K,q+F)\} \leq \rho(1+\|q\|). \tag{5}$$

Then, there exists a constant $\rho F > 0$ such that (2) holds for the map $F > 0$ defined by (1).

Hence, F is an upper semicontinuous closed-valued map on ρ .

Proof: Because f and G are Lipschitz continuous on Ω, we know that f,G have linear growth on Ω in x, that is, for some positive constants ρF and pG and for any $(t,x) \in \Omega$,

$$\|f(t,x)\| \leq \rho_f(1+\|x\|) \tag{6}$$

and

$$\|G(t,x)\| \leq \rho_G(1+\|x\|). \tag{7}$$

from (3), (4), and (5), we can obtain the fact that there exists $\rho F > 0$ such that (2) holds. Thus, F has linear growth.

Next, we prove that F is upper semicontinuous on Ω. We need only to prove that F is closed. Let sequence $\{(t_n, x_n)\} \subset \Omega$ be a sequence converging to some vector $(t_0, x_0) \in \Omega$

and $\{f(t_n, x_n) + B(t_n, x_n)u_n\}$converges to some vector $z_0 \in R^m$ as $n \to \infty, where\ u_n \in$

$SOLIV\ I(K, G(t_n n, x_n) + F(\cdot))$ for every n. It follows that the sequence $\{u_n\}$ is bounded, and has a convergent subsequence, denoted again by $\{u_n\}$, with a limit $u_0 \in R_n$. As F is continuous and K is nonempty, closed, and convex, it is easy to obtain

$$f(t_n, x_n) + B(t_n, x_n)u_n \to z_0 = f(t_0, x_0) + B(t_0, x_0)u_0 \in \mathbb{F}(t_0, x_0$$

and so F is closed.

Lemma (2.1.5)[2]: Let (f,G,B) satisfy conditions (A) and (B), and $F : R^n \to R^n$ be a continuous and para-monotone map on R^n. Suppose that $SOLIV\ I\big(K, q + F(\cdot)\big) \neq \emptyset\ for\ any\ q \in G(\Omega)$.

Then, SOLIV I(K, q + F(·)) is closed and convex for all $q \in G(\Omega)$.

Proof: Let$\{u_n\} \subset SOLIV\ I(K, q + F(\cdot))\ with\ u_n \to u_0$. Applying the closedness and convexity of K and the continuity of F, we deduce that $u_0 \in SOLIV\ I(K, q + F(\cdot))$ and so $SOLIV\ I(K, q + F(\cdot))$ is closed for all $q \in G(\Omega)$. Next, we prove that $SOLIV\ I(K, q + F(\cdot))$is convex for all $q \in G(\Omega). Let\ u_1, u_2 \in SOLIV\ I(K, q + F(\cdot))$. Then,

$$q + F(u_1) \in K, q + F(u_2) \in K. \quad (8)$$

Moreover, for any $\tilde{F} \in K$, we have

$$\langle \tilde{F} - q - F(u_1), u_1 \rangle \geq 0 \quad (9)$$

and

$$\langle \tilde{F} - q - F(u_2), u_2 \rangle \geq 0 \quad (10)$$

It follows from (6) that, for every $\lambda \in [0,1]$, we have

$$\lambda(q + F(u_1)) + (1 - \lambda)(q + F(u_2)) = q + \lambda F(u_1) + (1 - \lambda)F(u_2)$$

$$= q + F(\tilde{u}) \in K, (11)$$

where

$$\tilde{u} = \lambda u_1 + (1 - \lambda)u_2.$$

Letting $\tilde{F} = q + F(u_2)$ *in* (8) and $\tilde{} F = q + F(u_1)$ i, respectively, one has

$\langle F(u_2) - F(u_1), u_1 - u_2\rangle \geq 0$ (12)

Because F is para-monotone, we know that $F(u_2) = F(u_1)$. It follows from(8)and(9)that

$$\langle \tilde{F} - q - F(u_1), \lambda u_1 + (1 - \lambda)u_2\rangle \geq 0,$$

which means that

$$\langle \tilde{F} - q - F(\tilde{u}), \tilde{u}\rangle \geq 0.$$

This shows that $\tilde{u} \in SOLIV\, I(K, q + F(\cdot))$ and so $SOLIV\, I(K, q + F(\cdot))$ is convex for any $q \in G(\Omega)$.

Lemma(2.1.6)[2]: Let (f,G,B) satisfy conditions (A) and (B), and $F : R^n \to R^n$ *be a continuous* and para-monotone map. Suppose that there exists a constant $p > 0$ such that (3)holds for any $q \in G(\Omega)$, and SOL$(K, q + F) \neq \emptyset$for any $q \in G(\Omega)$. Then, DIVI(2) has a weak solution in the sense of Carath´eodory.

Proof: Similar to the proof of Proposition 6.1 in [19], by Lemmas (2.1.2), we can deduce that DIVI(1) has a weak solution in the sense of Carath eodory.

Theorem (2.1.7)[2]: Let $K \subset R^n$ be a nonempty compact convex subset and F : $R^n \to R^n$ be a continuous and para-monotone map. Suppose that q + F is invertible and $(q + F)^{-1}$ is continuous on R^n. Then, $SOLIV\, I(K, q + F(\cdot))$ is a nonempty compact convex set in K for any $q \in R^n$, and there exists $p > 0$ such that (3) holds for any $q \in R^n$.

Proof: For any $u \in R^n$, let

$$g(u) = (q + F)^{-1}(u) = y.$$

Then,

$$\langle g(u_1) - g(u_2), u_1 - u_2\rangle = \langle y_1 - y_2, q + F(y_1) - q - F(y_2)\rangle$$
$$= \langle y_1 - y_2, F(y_1) - F(y_2)\rangle.$$

Now, the monotonicity of F implies that g is monotone on R_n. For any $q \in R^n$, we know that $SOL(K, g)$ is nonempty and so there exists $u \in K$ such that

$$\langle \tilde{u} - u, g(u\rangle) \geq 0, \forall \tilde{u} \in K \quad (13)$$

It follows from (3) that there exists y $\in R_n$ such that $q + F(y) \in K$ and$\langle \tilde{u} - q - F(y), y\rangle \geq 0, \forall \tilde{u} \in K$, which means that $SOLIV\ I(K, q + F)$ is nonempty for any q $\in R_n$. Thus, Lemma (2.1.5) yields that SOLIV $I(K, q + F(\cdot))$ is a nonempty, closed and convex set for every $q \in R^n$. Because K is compact, it follows that SOLIV $I(K, q + F(\cdot))$ is a nonempty compact convex set for any $q \in R^n$. This shows that there exists a constant $p > 0$ such that (3) holds for any q $\in R^n$

Theorem (2.1.8)[2]: Let K $\subset R^n$ be nonempty compact convex set. Assume that $F : R^n \to R^n$be a continuous and strictly monotone map such that $q + F$ is surjective for any q $\in R^n$. Then,SOLIV $I(K, q + F(\cdot))$ is a singleton for any q $\in R^n$ and there exists a constant $p > 0$ such that(3) holds for any $q \in R^n$.

Proof: Because F is continuous and strictly monotone on R^n, it is easy to see that $q + F$ is continuous and strictly monotone on R^n. This implies that $(q + F)^{-1}$ is strictly monotone and continuous on R^n. we know that $SOL(K, (q + F)^{-1})$ is nonempty. From Theorem (2.1.7), it yields that SOLIV $I(K, (q + F))$ is nonempty. For any $u_1, u_2 \in SOLIV\ I(K, (q + F))$, we have

$$q + F(u_1) \in K, \langle \tilde{F} - q - F(u_1), u_1\rangle \geq 0, \forall \tilde{F} \in K$$

and

$$q + F(u_2) \in K, \langle \tilde{F} - q - F(u_2), u_2\rangle \geq 0, \forall \tilde{F} \in K.$$

It follows that

$$\langle F(u_1) - F(u_2), u_1 - u_2 \rangle \leq 0.$$

Now, the strictly monotonicity of F shows that $u_1 = u_2$ and so there exists a constant $\rho > 0$

such that (3) holds for any $q \in R^n$.

Theorem (2.1.9)[2]: Let $F : R^n \to R^n$ be a continuous and para-monotone map. Suppose that there exist $u_0, y_0 \in R^n$ such that, for any $u, y \in R^n$,

$$\frac{\langle q + F(u), u - u_0 \rangle - \langle u, y_0 \rangle + \langle u_0, y \rangle}{(\|u\|^2 + \|y\|^2)^{\frac{1}{2}}} \to +\infty \quad \text{as} \quad \|u\|^2 + \|y\|^2 \to +\infty. \tag{14}$$

Moreover, assume that there exists $F^0 \in R^n$ such that

$$\liminf_{\|u\| \to \infty} \frac{\langle F(u) - F^0, u \rangle}{\|u\|^2} > 0. \tag{15}$$

Then, SOLIV $I(R^n, q + F(\cdot))$ is a nonempty, closed, and convex set for all $q \in R^n$ and there exists a constant $\rho > 0$ such that (3) holds for any $q \in R^n$. **Proof**: The problem $IVI(R^n, q + F)$: $find\, u \in R^n$such that $q + F(u) \in R^n$ and $\langle \tilde{F} - q - F(u), u \rangle \geq 0, \forall \tilde{F} \in R^n$,is equivalent to the problem $V\,I(R^{2n}, P)$: find $v \in R^{2n}$ such that

$$\langle \tilde{v} - v, P(v) \rangle \geq 0, \forall \tilde{v} \in R^{2n},$$

where

$$v = \begin{pmatrix} u \\ y \end{pmatrix}, \quad P(v) = \begin{pmatrix} q + F(u) - y \\ u \end{pmatrix}.$$

By the monotonicity of F, one has

$$\langle P(v_1) - P(v_2), v_1 - v_2 \rangle = \begin{pmatrix} F(u_1) - y_1 - F(u_2) + y_2 \\ u_1 - u_2 \end{pmatrix}^T \begin{pmatrix} u_1 - u_2 \\ y_1 - y_2 \end{pmatrix}$$

$$\begin{aligned} &= \langle F(u_1) - F(u_2) - y_1 + y_2, u_1 - u_2 \rangle + \langle u_1 - u_2, y_1 - y_2 \rangle \\ &= \langle F(u_1) - F(u_2), u_1 - u_2 \rangle \\ &\geq 0, \end{aligned}$$

which implies that P is monotone on R^{2n}. Thus, there exists $v_0 = \binom{u_0}{y_0}$ such that

$$\frac{\langle P(v), v - v_0 \rangle}{\|v\|} = \frac{\langle q + F(u) - y, u - u_0 \rangle + \langle u, y - y_0 \rangle}{(\|u\|^2 + \|y\|^2)^{\frac{1}{2}}} \to +\infty \quad \text{as} \quad \|u\|^2 + \|y\|^2 \to +\infty,$$

which means that

$$\frac{\langle P(v), v - v_0 \rangle}{\|v\|} \to +\infty \quad \text{as} \quad \|v\| \to +\infty.$$

By Theorem (2.1.8), we know that SOL(R^{2n}, P) is a nonempty set and so SOLIV $I(Rn, q + F(\cdot))$ is nonempty. It follows from Lemma (2.1.6) that SOLIV $I(R^n, q + F(\cdot))$ is a nonempty closed convex set for every $q \in R^n$.

Next, we prove the second assertion. Suppose to the contrary, there exist $\{qk\} \subset R^n$ and $\{u^k\} \subset R^n$ such that, for any $\tilde{F} \in R^n$,

$$\langle \tilde{F} - q^k - F(u^k), u^k \rangle \geq 0, \tag{16}$$

And

$$\|u^k\| > k(1 + \|q^k\|).$$

Obviously, $\{u^k\}$ is unbounded. It follows from (16) that

$$\langle F^0 - q^k - F(u^k), u^k \rangle \geq 0$$

and so

$$\langle F(u^k) - F^0, u^k \rangle \leq \langle -q^k, u^k \rangle.$$

Dividing by $\|u^k\|^2$, we have

$$\liminf_{k\to\infty} \frac{\langle F(u^k) - F^0, u^k \rangle}{\|u^k\|^2} \leq 0,$$

Which contradicts (15).This shows that there exists a constant $p > 0$ such that (3) holds for any $q \in R^n$.

Theorem (2.1.10)[2]: Let $F : R^n \to R^n$ be a continuous and para-monotone map. Suppose that

SOLIV $I(R^n, q + F(\cdot)) \neq \emptyset$for any q $\in R^n$ and there exists $F^0 \in R^n$ such that

$$\frac{\langle F(u) - F^0, u \rangle}{\|u\|} \to +\infty \quad \text{as} \quad \|u\| \to +\infty. \tag{17}$$

Then, SOLIV $I(R^n, q + F(\cdot))$ is a nonempty closed convex set for all q $\in R^n$ and there exists aconstant $p > 0$ such that (3) holds for all q ∈S, where S is bounded set.

Proof : Similar to the proof of Theorem (2.1.7), we know SOLIV $I(R^n, q + F(\cdot))$ is a nonempty

closed convex set for all $q \in R^n$.

Now, we prove the second assertion. If the assertion is not true, then there exist $\{q^k\} \subset S$ and $\{u^k\} \subset R^n$ such that for any $\tilde{F} \in R^n$,

$$\langle \tilde{F} - q^k - F(u^k), u^k \rangle \geq 0, \tag{18}$$

and

$$\|u^k\| > k(1 + \|q^k\|).$$

It is clear that $\{u^k\}$ is unbounded. From (18), one has

$$\langle F^0 - q^k - F(u^k), u^k \rangle \geq 0,$$

which means

$$\langle F(u^k) - F^0, u^k \rangle \leq \langle -q^k, u^k \rangle.$$

Dividing by $\|u^k\|$, we have

$$\frac{\langle F(u^k) - F^0, u^k \rangle}{\|u^k\|} \leq \frac{\langle -q^k, u^k \rangle}{\|u^k\|}.$$

Because $\{q^k\}$ is bounded, there exists a constant C such that

$$\frac{\langle F(u^k) - F^0, u^k \rangle}{\|u^k\|} \leq C,$$

which contradicts (17).

In the rest of this article, let

$$S := \{v \in R^n : \langle Fv, v \rangle = 0\}$$

Obviously, S is a linear subspace of R^n and $S^{\perp}$is also a linear subspace of R^n.

Theorem (2.1.11)[2]: Let $F_{n\times n}$ be a positive semi-defined matrix. Suppose that for any $n \in N$,

we have

$$SOLIVI\left(R^n, q + (1 - \frac{1}{n})F + \frac{1}{n}I\right) \neq \emptyset,$$

where I is the identity map on R^n. Then,

(i) SOLIV I$(R^n, q + F(\cdot))$ is a nonempty closed convex set for all q ∈S⊥.

(ii) there exists a constant $p > 0$ such that

$$\sup\{\|u\| : u \in SOLIVI(R^n, q + F(\cdot))\} \leq \rho(1 + \|q\|).$$

Proof: We denote SOLIV $I(R^n.,q + (1 - \frac{1}{n})F + \frac{1}{n}I)$ by SOLIV$I_n(F_n)$. Assume for the sake of contrary that the contrary holds. Suppose that $\cup_{n \in N}$ $SOLIV\ In(F_n)$ is unbounded. Then, there exists a sequence $\{u_n\} \subset R^n$ such that, for any $\tilde{F} \in R^n$,

$$\left\langle \tilde{F} - q - (1 - \frac{1}{n})F(u_n) - \frac{1}{n}I(u_n), u_n \right\rangle \geq 0, \tag{19}$$

where $\|u_n\| \to \infty$. Let

$$\lim_{n \to \infty} \frac{u_n}{\|u_n\|} = u_\infty.$$

Dividing by $\| u_n \|^2$ and taking n → ∞ in (19), we have

$$\langle F(u_\infty), u_\infty \rangle \leq 0.$$

As F is positive semi-defined, one has $\langle F(u_\infty), u_\infty \rangle = 0$ and so u∞ ∈S. Because

$$\left\langle (1 - \frac{1}{n})F(u_n) + \frac{1}{n}I(u_n), u_n \right\rangle \geq 0,$$

it follows from (19) that

$$\langle \tilde{F} - q, u_n \rangle \geq 0$$

and so

$$\left\langle \tilde{F} - q, \frac{u_n}{\|u_n\|} \right\rangle \geq 0.$$

Letting n $\to \infty$ in the above inequality, we have

$$\langle \tilde{F} - q, u_\infty \rangle \geq 0.$$

It follows from $u_\infty \in S$ and $q \in S^\perp$ that

$$\langle \tilde{F}, u_\infty \rangle \geq 0.$$

Taking $\widetilde{F} = -u_\infty$, we obtain a contradiction. Therefore, $U_{n\in N}$ SOLIV $I_n(F_n)$ is bounded and so there exists a convergent subsequence with a limit u0. It follows from (19) that for any $\widetilde{F} \in R^n$,
$\langle \widetilde{F} - q - F(u_0), u_0 \rangle \geq 0$,which implies that $u_0 \in$SOLIV $I(R^n, q + F(\cdot))$ and so SOLIV I$(R^n, q + F(\cdot))$ is nonempty for all $q \in S^\perp$ Similar to the proof of Lemma (2.1.8), we can prove that SOLIV I$(R^n, q + F(\cdot))$ is nonempty, closed and convex set.

Next, we prove the second assertion. If not, then there exist $\{q^k\} \subset S\frac{1}{B}$ *and* $\{u^k\}$such that, for any given $\widetilde{F} \in R^n$,

$$\langle \tilde{F} - q^k - F(u^k), u^k \rangle \geq 0 \qquad (20)$$

and

$$\|u^k\| > k(1 + \|q^k\|).$$

It follows that

$$\lim_{k\to\infty} \|u^k\| = \infty, \quad \lim_{k\to\infty} \frac{\|q^k\|}{\|u^k\|} = 0.$$

Because $\{q^k\} \subset S\frac{1}{B}$ is bounded, without loss of generality, we can assume that

$$\lim_{n \to \infty} q^k = q^\infty \in S_B^\perp$$

and

$$\lim_{k \to \infty} \frac{u_k}{\|u_k\|} = u_\infty.$$

From (20), we have

$$\left\langle \frac{\tilde{F} - q^k - F(u^k)}{\|u^k\|}, \frac{u^k}{\|u^k\|} \right\rangle \geq 0.$$

Letting $k \to \infty$in the above inequality, one has

$$\langle F(u_\infty), u_\infty \rangle \leq 0.$$

As F is semi-defined, we obtain

$$\langle F(u_\infty), u_\infty \rangle = 0$$

and so u∞ ∈S. Moreover, it follows from (20) that for any $\tilde{F} \in R^n$,

$$\langle \tilde{F} - q^k, u^k \rangle \geq 0.$$

This means that

$$\langle \tilde{F} - q^k, \frac{u^k}{\|u^k\|} \rangle \geq 0$$

and so

$$\langle \tilde{F} - q^\infty, u_\infty \rangle \geq 0.$$

As $u_\infty \in S$ and $q_\infty \in S^\perp$, we have

$$\langle \tilde{F}, u_\infty \rangle \geq 0,$$

which is a contradiction.

Lemma (2.1.12)[2]: Let K be a nonempty closed convex set and $F : R^n \to R^n$ be a paramonotone and continuous map. Assume that SOLIV $I(K, q + F(\cdot)) \neq \emptyset$for any $q \in R^n$ and the linear growth (3) holds. Then, $A : R^n \to R^n$ is continuous, where A is defined by $A(q) = F(u)$for any $q \in R^n$ and u ∈SOLIV $I(K, q + F(\cdot))$.

Proof: Let $q_n \to q$ and u_n∈SOLIV $I(K, q^n + F(\cdot))$. Then, $q_n + F(u_n) \in K$ and for any $\tilde{F} \in K$,

$$\langle \tilde{F} - q_n - F(u_n), u_n \rangle \geq 0.$$

It follows that $\{u_n\}$ is bounded and so there exists a convergent subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, with a limit u_0. Because K is closed and F is continuous, we have $q + F(u_0) \in K$

and

$$\langle \tilde{F} - q_0 - F(u_0), u_0 \rangle \geq 0, \forall \tilde{F} \in K.$$

This means that u_0 ∈SOLIV I(K, q + F(·)). Suppose that there exists another convergent subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, with a limit u_1. Then, u_1∈SOLIV $I(K, q + F(\cdot))$.

From the proof of Lemma (2.1.8), it is easy to see that F(u) is a constant for all $u \in SOLIV\, I(K, q + F(\cdot))$ and so $F(u_0) = F(u_1)$. It follows that

$$A(q_n) = F(u_n) \to F(u_1) = A(q)$$

and so $A : R^n \to R^n$is continuous.

Theorem (2.1.13)[2]: Let $F \in R^{n\times n}$ be a psd-plus matrix [positive-definite matrix:In linear algebra,asymmetric n×n real matrix M is said to be positive definite if the scalar z^1 Mz is positive for every non-zero column vector z of n real numbers.Here z^T denotes the transpose of z.

More generally,an n×n Hermition matrix M is said to be positive definite if the scalar$z^* Mz\ is\ real\ and\ positive\ for\ all\ nonzero\ column\ vector\ z\ of$ n complex number.Here z^* denotes the conjugate transpose of z]suppose that SOLIV $I(R^n, q + F(\cdot)) \neq \emptyset$for all q $\in R^n$ and there exists a constant $p > 0$ such that (3) holds. Let $D : R^n \to R^n$ be a continuous map such that

$$\|D(u)\| \leq L_D \|u\|, \quad \forall u \in R^n \tag{21}$$

for some constant $L_D \in (0, \frac{1}{\rho})$. Then, for any $q \in R^n$, SOLIV $I(R^n, q + H)$ is a nonempty closed set, where $H = F + D$, and

$$\sup\{\|u\| : u \in SOLIVI(R^n, q + H)\} \leq \frac{\rho(1 + \|q\|)}{1 - \rho L_D}. \tag{22}$$

Assume further that there exist constants $L_A > 0$ and $L \in (0, \frac{1}{L_A})$ such that

$$\begin{cases} \|A(q_1) - A(q_2)\| \leq L_A \|q_1 - q_2\|, & \forall q_1, q_2 \in R^n, \\ \|D(u_1) - D(u_2)\| \leq L\|F(u_1) - F(u_2)\|, & \forall u_1, u_2 \in R^n, \end{cases} \tag{23}$$

where A is defined as that in Lemma (2..1.18) Then, for any $q^i \in R^n$ and u^i ∈SOLIV $I(R^n, q + H)$ with i = 1, 2,

$$\|Fu_1 - Fu_2\| \leq \frac{L_A \|q_1 - q_2\|}{1 - L_A L}, \tag{24}$$

and for every $q \in R^n$,

$$SOLIVI(R^n, q + H) = F^{-1}v(q) \bigcap \{v : \langle F' - w(q), v \rangle \geq 0, \forall F' \in R^n\}$$

where $v(q) = F\,\hat{u}, w(q) = q + H(\hat{u})$ for any $\hat{u} \in SOLIV\ I(R^n, q + H)$, and $F^{-1}v(q)$ is the inverse image of v(q). Consequently, SOLIV $I(R^n, q + H)$ is a convex set.

Proof: Similar to the proof of Theorem (2.1.11), we can obtain all the results except for the last one. Now, we prove the last result. For any $u_1, u_2 \in SOLIV\ I(Rn, q + H)$, by the inequality (24), we know that $\|Fu_1 - Fu_2\| = 0$. This means that Fu is a constant vector for all $u \in SOLIV\ I(R^n, q + H)$. Furthermore, it follows from (23) that $\|Du_1 - Du_2\| = 0$. Thus, Du is a constant vector and so is H(u) for all $u \in SOLIV\ I(R^n, q + H)$.

For any $u \in SOLIV\ I(R^n, q + H)$ and $\tilde{F} \in R^n$, one has $Fu = F\hat{u} = v(q)$ and so $u \in F^{-1}v(q)$. As

$$w(q) = q + H(\hat{u}), \hat{u} \in SOLIV\ I(R^n, q + H),$$

we know that v(q) and w(q) are constants. Moreover, for any u ∈SOLIV $I(R^n, q + H)$, we have

$$\langle \tilde{F} - q - H(u), u \rangle \geq 0,$$

which implies that

$$\langle \tilde{F} - q - H(\hat{u}), u \rangle \geq 0$$

and so

$$\langle \tilde{F} - w(q), u \rangle \geq 0.$$

It follows that Conversely, for any $u \in F^{-1}v(q) \cap \{v : \langle \tilde{F} - w(q), vi \geq 0, \forall \tilde{F} \in R^n\}$, we have

$$Fu = v(q) = F\hat{u},$$

where $\hat{u}$∈SOLIV I(R^n, q + H). It follows from (23) that $Du = D\hat{u}$and so $H(u) = H(\hat{u})$.

Consequently, we have

$$0 \leq \langle \tilde{F} - w(q), u \rangle = \langle \tilde{F} - q - H\hat{u}, u \rangle = \langle \tilde{F} - q - Hu, u \rangle$$

and so u ∈SOLIV I$(R^n, q + H)$. This shows that

$$SOLIVI(R^n, q + H) = F^{-1}v(q) \bigcap \{v : \langle \tilde{F} - w(q), v\rangle \geq 0,\ \forall F' \in R^n\}$$

Next, we show that SOLIV I$(R^n, q + H)$ is a convex set. In fact, for any $u_1, u_2 \in$ SOLIV I$(R^n, q + H)$, we only need to show that $\hat{u} = \lambda u_1 + (1 - \lambda)u_2 \in$ SOLIV I$(R^n, q + H)$for all $\lambda \in [0, 1]$.Because $F(u_1) = F(u_1) = v(q)$, one has

$$F(\lambda u_1 + (1 - \lambda)u_2) = \lambda F(u_1) + (1 - \lambda)F(u_2) = v(q),$$

which means that $\hat{u} \in F^{-1}$v(q). Moreover, for any $\tilde{F} \in R^n$, we have

$$\langle \tilde{F} - w(q), u_1\rangle \geq 0, \quad \langle \tilde{F} - w(q), u_2\rangle \geq 0.$$

It follows that

$$\langle \tilde{F} - w(q), \hat{u}\rangle \geq 0$$

and so

$$\hat{u} \in F^{-1}v(q) \bigcap \{v : \langle \tilde{F} - w(q), v\rangle \geq 0,\ \forall \tilde{F} \in R^n\},$$

which shows that $\hat{u} \in SOLIV\ I(R^n, q + H)$.

Theorem (2.1.14)[2]: *Let* $F : R^n \to R^n$ be a given linear map and (f,G,B) satisfy conditions (A) and (B). Then, DIVI(1) has a weak solution in the sense of Carath´eodory under any one of the following conditions:

(a) $K \subset R^n$is a nonempty compact convex set, and $F : R^n \to R^n$ is continuous and para-monotone such that q + F is invertible and $(q + F)^{-1}$ is continuous on Rn for all $q \in R^n$;

(b) $K \subset R^n$ is a nonempty, compact and convex set, and $F : R^n \to R^n$ is surjective, continuous, and strictly monotone;

(c) $K = R^n, F : R^n \to R^n$is continuous and para-monotone, and there exist $u_0, y_0, F^0 \in R^n$ such that (14) and (15) hold;

(d) $K = R^n, F : R^n \to R^n$ is continuous and para-monotone, and there exist $u_0, y_0, F^0 \in R^n$ such that (15) and (18) hold;

(e) F is a positive semi-define matrix such that, for any n ∈N

$$SOLIVI\left(R^n, q + (1 - \frac{1}{n})F + \frac{1}{n}I\right) \neq \emptyset,$$

where I is the identity map on R^n;

(f) $F = \hat{F} + D$, where $\hat{F} \in R^{n \times n}$ is a psd-plus matrix such that (15) and (16) hold and D is a continuous map such that (22) and (24) hold.

Proof: It follows from Theorems (2.1.7)–(2.1.13) that SOLIV I(K, q + F) is a nonempty, closed and convex set and satisfies condition (3). By Lemma(2.1.5), we know that DIVI (1) has a weak solution in the sense of Carath´eodory.

Sec (2.2): An Application

In this section, we will give an application of the DIVI to the time-dependent spatial price equilibrium control problem.

we consider the time-dependent spatial price equilibrium control problem. Assume that a single commodity is produced at m supply markets, with typical supply market denoted by i and is consumed at n demand markets, with typical demand market denoted by j, during the time interval [0, T] with T > 0. (i, j) denotes the typical pair of producers and consumers for i = 1, · · · ,m and j = 1, · · · , n. Let Si(t) be the supply of the commodity produced at supply market i at time t ∈[0, T] and group the supplies into a column

vector

$$S(t) = (S_1(t), S_2(t), \cdots, S_m(t)) \in R^m.$$

Let D_j(t) be the demand of the commodity associated with demand market j at time t ∈[0, T] and group the demands into a column vector

$$D(t) = (D_1(t), D_2(t), \cdots, D_n(t)) \in R^n.$$

Let x_{ij}(t) be the commodity shipment from supply market i to demand market j at time $t \in [0, T]$ and group the commodity shipments into a column vector $x(t) \in R^{mn}$. Suppose that for all $t \in [0, T]$,

$$S_i(t) = \sum_{j=1}^{n} x_{ij}(t), \qquad D_j(t) = \sum_{i=1}^{m} x_{ij}(t).$$

Now, we consider the problem from the policy-maker's point of view and present the time dependent optimal control equilibrium problem. Under this perspective, by adjusting taxes u(t), it is possible to control the resource exploitations S(x(t), u(t)) at supply markets and the consumption D(x(t), u(t)) at demands markets. It is known that the tax adjustment is an efficient means of regulating production and consumption. Specifically, if the policy-maker is concerned with restricting production or consumption of a certain commodity, then higher taxes will be imposed; whereas if the government aims to encourage production or consumption of some commodities, subsidies will be imposed . we introduce the function of commodity shipments x(t) and regulatory taxes u(t) as follows:

$$W(t, x(t), u(t)) = (S(x(t), u(t)), D(x(t), u(t)))T, \forall t \in [0, T].$$

Obviously, the map W is defined as $W : [0, T] \times R^{mn} \times R^{m+n} \rightarrow R^{m+n}$. We assume that then map W(t, x, u) can be written as

$$W(t, x(t), u(t)) = G(t, x(t)) + F(u(t)), \forall t \in [0, T]$$

such that G(t, x) is a Carath´eodory function (that is, it is measurable in t for all $x \in R^{mn}$and continuous with respective to x) and F(u) is Lipschitz continuous. Moreover, assume that there exists γ(t) ∈L^2(0, T) such that

$$\|G(t, x)\| \leq \gamma(t) + \|x\|.$$

Thus, it is easy to know that

$$W : [0, T] \times L^2([0, T], R^{mn}) \times L^2([0, T], R^{m+n}) \rightarrow L^2([0, T], R^{m+n}).$$

Finally, we suppose that the following lower and upper capacity constrains are satisfied:

$$\underline{w}(t) = (\underline{S}(t), \underline{D}(t)), \overline{w}(t) = (\overline{S}(t), \overline{D}(t)),$$

where $\underline{S}(t), \overline{S}(t) \in L^2([0, T], R^m), \underline{D}(t), \overline{D}(t) \in L^2([0, T], R^n), 0 \leq \underline{S}(t) < \overline{S}(t)$ for almost all $t \in [0, T]$ *and* $0 \leq \underline{D}(t) < \overline{D}(t)$ for almost all $t \in [0, T]$.

We note that the capacity constrains are assumed to be independent of x and u.

We introduce the set of feasible states as follows:

$\{K = w \in L2([0, T], R^{m+n}): \underline{w}(t) \leq w(t) \leq \overline{w}(t)$ *for almost all* $t \in [0, T]\}$.

we say that u^*(t) is an optimal regulatory tax if it makes the corresponding state W(t, x(t), u^*(t)) satisfying the constraint W(t, x(t), u^*(t)) ∈ K

and for almost all t ∈ [0, T], the following three conditions hold:

$W_r(t, x(t), u^*(t)) = \overline{w}r(t) \Rightarrow u_r^*(t) \geq 0, r = 1, 2, \cdots, m + n,$

$W_r(t, x(t), u^*(t)) = \underline{w}r(t) \Rightarrow u_r^*(t) \leq 0, r = 1, 2, \cdots, m + n,$

$\underline{w}_r(t) < Wr(t, x(t), u^*(t)) < \overline{w}r(t) \Rightarrow u_r^*(t) = 0, r = 1, 2, \cdots, m + n.$

It is easy to see that a regulatory tax vector $u^*(t) \in L^2([0, T], R^{m+n})$ is optimal if and only if it solves the following inverse variational inequality:

$$W((x(t), u^*(t))\epsilon K, \int_0^T \langle w(t) - W(t, x(t), u^*(t), u^*(t)dt \leq 0, \forall w(t) \in K\rangle \quad (24)$$

On the other hand, we know that there is a relationship between the change rate of commodity shipments x(t) and regulatory taxes u(t) with the commodity shipments x(t). We require that

$$\dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t), \qquad \text{for almost all } t \in [0, T], \quad (25)$$

where $f : [0, T] \times R^{mn} \to R^{mn}$ and $B : [0, T] \times R^{mn} \to R^{mn\times(m+n)}$ are two maps satisfying some suitable conditions.

Combining (24) and (25), we know that (x(t), u(t)) is a Carath´eodory weak solution of the following DIVI problem:

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t), \\ u(t) \in \text{SOLIVI}(-\mathcal{K}, -G(t, x(t)) - F(\cdot)), \\ x(0) = x_0. \end{cases} \quad (26)$$

Specially, suppose that $\underline{w}_r(t)$ and $\overline{w}_r$ (t) are constants for $r = 1, 2, \cdots, m + 1$ and

$$f(t,x) = \begin{pmatrix} \alpha_1 t \\ \vdots \\ \alpha_m t \end{pmatrix} + \beta x,$$

$$B(t,x) = \begin{pmatrix} t\sin(x_1) & 0 & \cdots & 0 \\ t\sin(x_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ t\sin(x_m) & 0 & \cdots & 0 \end{pmatrix},$$

$$G(t,x) = \begin{pmatrix} \lambda_1 e^t + x_1 \\ \vdots \\ \lambda_m e^t + x_m \\ 0 \end{pmatrix},$$

$$F(u) = u,$$

where $x = (x_1, \cdots, x_m)^T$. Then, all the conditions of (b) in Theorem (2.1.14) are satisfied and so it shows that DIVI (26) has a Carath´eodory weak solution (x(t), u(t)).

CHAPTER 3

Theory in Reflexive Smooth Banach Spaces and Applications to P-Laplacian Elliptic Inequalities

Variational inequality theorems are proved and applied to study existence of nonzero positive weak solutions for p-Laplacian elliptic inequalities and a population model of one species arising in mathematical biology.

Sec(3.1) : A variational inequality theory in reflexive smooth Banach spaces

We develop a theory for variational inequalities of the form

$$(Jx - Ax, x - v) \leq 0 \; for \; v \in K \tag{1}$$

in a reflexive smooth Banach space X, where $J : X \rightarrow X^*$is a duality map with a gauge function and $A : D \subset X \rightarrow X^*$is a demi continuous S-contractive map.

A theory for variational inequalities (1) with J = I, the identity map, and A being a demicontinuous S-contractive map in Hilbert spaces was established in , and an index theory for such variational inequalities with condensing maps in Hilbert spaces was developed in . However, these theories cannot be applied to treat p-Laplacian elliptic inequalities with $p \neq 2$. An index theory for (1) with J being strictly monotone and coercive and A compact was established in.

The key requirements are that A is compact and the map rA must be continuous, where r is the unique solution map of (1) with J. However, it is known that rA may not be continuous if A is demicontinuous. We refer to for the related study on a class of maps of S-type and to for the study of the fixed point equation $x = rAx$.

To develop the theory for variational inequalities (1) in reflexive smooth Banach spaces, we employ the method used in , where the variational inequality theory for demicontinuous S-contractive map in Hilbert spaces is established. The main ideas originate from the Granas topological transversality which was developed in order to study existence of fixed points for nonlinear maps .

Following , we introduce the essential maps for the variational inequality (1) in the class of demicontinuous S-contractive maps in reflexive smooth Banach spaces and prove three standard properties of variational inequalities:

existence property, normalization and homotopy property. These properties are generalizations of those in , where the spaces involved are Hilbert spaces. Sufficient conditions for maps to be essential or non-essential are provided. These conditions are similar to those used in the fixed point index theories or variational inequality theory , namely, the Leray–Schauder type conditions and the conditions implying that the fixed point index is zero. Some variational theoremsare proved, where the generalized projections introduced by Alber play important roles. The proofs of these results are more difficult than those in Hilbert spaces .

As applications of the variational inequality theory, we study existence of nonzero positive weak solutions for the following p-Laplacian elliptic inequalities

$$\begin{cases} -\Delta_p u(x) \geq f(x, u(x)) & \text{for almost every (a.e.) } x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Δ_p is the p-Laplacian operator and Ω is a bounded and connected open set in R^n.

Existence of positive or nonzero positive weak solutions of the Laplacian elliptic inequalities (2) when $= 2$, where $2 \leq p < n$ and the critical point theory was applied.

To the best of our knowledge, when $2 \leq n < p$, there is little study on existence of nonzero positive weak solutions of the p-Laplacian elliptic inequality (2).

Our theory is suited to treating (2) with $2 \leq n < p$. One of our conditions imposed on f is

$|f\ (x, u)|\ \leq\ gr\ (x)\ for\ a.e. x\ \in \overline{\Omega}\ and\ all\ u\ \in\ [0, r]$ (3)

This condition (3) is more general than those used in , where suitable upper bound conditions related to uα are imposed on $|f\ (x, u)|$. We refer for the study of p-Laplacian equations with $p\ > 2$, where a condition imposed on f is stronger than (3).

we establish the variational inequality theory in reflexive smooth Banach spaces., we prove some variational inequality principles. we apply this variational inequality theory to study (2). we obtain results on the existence of nonzero positive weak solutions for (2) with the specific nonlinearity arising in mathematical biology.

Let X be a Banach space and X^*its dual space. Recall that X is strictly convex if $\| x\ +\ y \| \leq\ 2$ for x, $y\ \in \partial B1 := \{x\ \in X : \| x \|\ =\ 1\}$with $x \neq y$; is smooth if the limit $lim_{t\rightarrow 0} t^{-1}(\| x\ +\ ty \| - \| x \|)$exists for x, $y\ \in \partial B_1$. It is known that if X is reflexive, then the following assertions hold:

(i)X is strictly convex if and only if $X^*$is smooth;

(ii) X is smooth if and only if $X^*$is strictly convex. Recall that X has property (H) if $y_n \rightharpoonup y$ and $\|y_n\| \rightarrow \|y\|$ together imply $y_n \rightarrow\ y$. Every locally uniformly convex Banach space is reflexive, strictly convex and has the property (H).

Recall that a continuous function Φ : $R_+\ \rightarrow R_+$is said to be a gauge function if $\Phi$is a strictly increasing function with Φ(0) = 0 and $limt_{t\rightarrow\infty}$ Φ(t) = ∞. Assume that $X^*$is strictly convex. A map $J : X\ \rightarrow\ X^*$is said to be a duality map with gauge function Φ if, for each x ∈X, (J(x), x) = Φ(‖x‖)‖x‖and ‖Jx‖ = Φ(‖x‖). When Φ(t) = t, J is called a normalized duality map. J is a bounded single-valued map and is demicontinuous, that is, if $\{x_n\ \}$ ⊂X and x_n → x ∈X together imply Jx_n ⇀Jx, where the symbols→and ⇀to indicate strong and weak convergence, respectively.

Moreover, J is monotone and if we assume further that X is strictly convex, then J is strictly monotone, that is,

$$(Jy - Jx, y - x) > 0 \; for \; x, y \in X \; with \; x \neq y \tag{4}$$

Note that the smoothness of X or the strict convexity of $X^*$is not sufficient for a duality map to be strictly monotone .

A map $T : D \subset X \rightarrow X^*$is of S^+-type if $\{y_n\}$⊂D with $y_n \rightharpoonup y \in X$ and $lim \; sup(Ty_n, y_n - y) \leq 0$ together imply $y_n \rightarrow y$. It is easy to verify that J is of S^+-type if either X has the property (H) or there exist $\sigma > 0$ and $\alpha > 0$ such that

$$(Ju - Jv, u - v) \geq \sigma \parallel u - v \parallel^\alpha \; for \; u, v \in X \tag{5}$$

A map $A : D \subset X \rightarrow X^*$is said to be compact if A is continuous and A(Ω) is relatively compact for each bounded subset Ω of D. If T: $D \subset X \rightarrow X^*$is of S^+-type and $A : D \subset X \rightarrow X^*$is compact, then $T + A$ is of S^+-type.

A map A : $D \subset X \rightarrow X^*$is said to be S-contractive (on D) if J − A is of S^+-type. It is obvious that if A is S-contractive on D, then A is S-contractive on Ω for every subset Ω of D. Moreover, the sum of an S-contractive map and a compact map is S-contractive. Now, we establish a theory for variational inequality of the form

$$(Jx - Ax, x - v) \leq 0 \; for \; v \in K \tag{6}$$

where $J : X \rightarrow X^*$is a duality map with gauge function Φ and $A : D \subset X \rightarrow X^*$is an S-contractive map on D.

In the rest of this section, we always assume that X is a reflexive smooth Banach space. Hence, its dual space $X^*$is strictly convex.

Variational inequalities for maps of monotone types arise in physics, mechanics, engineering, control, optimization, nonlinear potential theory and elliptic inequalities and have been widely studied, The theories of variational inequalities (1) in Hilbert spaces were established where $J = I$ and A is a demicontinuous S-contractivemapor a condensing map. However, these theories cannot be applied to tackle the p-Laplacian elliptic inequalities with $p \neq 2$. The related studies on the fixed point equations and on variational inequalities for maps of S-type

The variational inequality (3) is said to have a solution in D if there exists x ∈D such that (3) holds. The complementarity problem of A:

$$(Jx - Ax, x) = 0 \; and \; (Jx - Ax, v) \geq 0 \; for \; v \in K \tag{7}$$

is said to have a solution in D if there exists $x \in D$ such that(4) holds.

A closed convex set K in X is called a wedge if $\lambda x \in K$ for x ∈K and λ ≥ 0. If a wedge K also satisfies $K \cap (-K) = \{0\}$, then K is called a cone. A wedge which is neither a cone nor a subspace of X is called a proper wedge. It is well known that if K is a wedge in X, then x ∈D is a solution of the variational inequality (6) if and only if x ∈D is a solution of the complementary problem (7). If K is a subspace of X^*, then $x \in D$ is a solution of the variational inequality (6) if and only if $(Jx - Ax, v) = 0$for all v ∈K, that is, $Jx - Ax$ is orthogonal to K .

Let K be a closed convex set in X and let D be a bounded open set in X such that D_k = D ∩ K≠∅. We denote by $\overline{D_k}$and ∂D_K the closure and the boundary, respectively, of D_k relative to K. for some properties among these sets. We denote by $V(D_k, X^*)$the set of all demicontinuous S-contractive maps A : $\overline{D_k} \to X^*$such that (6) has no solutions on ∂D_K .we generalize the definition of essential maps related to variational inequalities from Hilbert spaces to reflexive smooth Banach spaces.

Definition (3.1.1)[3]: A map $A \in V(D_K, X^*)$is said to be essential on D_K if for each map φ ∈V(D_K, X^*) with φ(x) = Ax for x ∈∂DK , the variational inequality of φ

$$(Jx - \varphi(x), x - v) \leq 0 \; for \; v \in K \tag{8}$$

has a solution in D_K .

The following important properties of essential maps are generalizations from Hilbert spaces to reflexive smooth Banach spaces.

Theorem(3.1.2)[3]: Let K be a closed convex set in a reflexive smooth Banach space X and let D be a bounded open set in X such that $D_K \neq \emptyset$. Then the following assertions hold.

(P1) (Existence property) If $A \in V(D_K, X^*)$is essential on , then (6) has a solution in D_k .

(P2) (Normalization) Assume that J is of S^+-type and strictly monotone. If u ∈ D_K, then $J\hat{u}$ is essential on DK , where J$\hat{u}$(x) ≡ Ju for x ∈$\overline{D_k}$.

(P3) (Homotopy property) Let $\overline{D_k} \neq K$ and let $A, B : \overline{D_K} \to X^*$be demicontinuous S-contractive maps. Assume that the variational inequality of $h(t, \cdot\ )$has no solutions on ∂D_K for each t ∈ $[0,1]$, where $h : [0,1] \times \overline{D_K} \to X^*$is defined by $h(t, x) = tAx + (1 - t)$Bx.

Then A is essential onD_K if and only if B is essential onD_K .

Proof: (P_1)The result follows from Definition (3.1.1) with φ = A.

$(P_2)\ If\ \overline{D_K} = K$, then K is bounded since D is bounded. Since J is of S+-type, J$\hat{u}$: K → $X^*$is a demicontinuous S-contractive map. $Let\ \varphi \in V(D_K, X^*)$with $\varphi(x) = J\hat{u}$ for x ∈ ∂D_K . Since $D_K = K$ the variational inequality of φ has a solution in K and $J\hat{}u$ is essential on D_K . If $D_K \neq K$, then the variational inequality of $J\hat{}u$ has a unique solution in $\overline{D_K}$ and has no solutions on $\partial\overline{D_K}$. Hence, $J\hat{u} \in V(D_K, X^*). Let\ \varphi \in V(D_K, X^*)$with $\varphi(x) = J\hat{u}(x) = Ju$ for x∈ ∂D_K . Define a map $T : K \to X^*$by

$$Tx = \begin{cases} \varphi(x) \text{ if } x \in \overline{D}_K \\ J(u) \text{ if } x \in K \setminus D_K \end{cases},$$

Then T is a demicontinuous S-contractive map. If K is bounded, the variational inequality of T has a solution in K. If K is unbounded, noting that D_K is bounded, we have for every $x_0 \in K$,

$$\lim_{x\in K, \|x\|\to\infty} \sup \frac{(Tx, x - x_0)}{(Jx, x)} = \lim_{x\in K, \|x\|\to\infty} \sup \frac{(J(u), x - x_0)}{(Jx, x)} = 0 < 1.$$

By a method similar to the first part of the proof of Theorem(3.1.1) we can show that the variational inequality of T has a solution $x \in K$. We prove x ∈ $\overline{D_K}$. In fact, if x ∈K\D_K , then $x \neq u$ and $Tx = Ju$. By (4) and the strict monotonicity of J, we have

$$0 < (Jx - Ju, x - u) = (Jx - Tx, x - u) \leq 0,$$

a contradiction. Hence, $\varphi(x) = Tx$ and x is a solution of the variational inequality of φ. By Definition (3.1.1), $J\hat{u}$ is essential on D_K .

(P3) Assume that B is essential on DK . $Let\ \varphi \in V(D_K, X^*)$with φ(x) = A(x) for x ∈$\partial D_K$ Define $h^* : [0,1] \times D_K \to X^*$by $h^*(t,x) = t\varphi(x) + (1 - t)B(x)$.Let F be the set of all the solutions in DK of variational inequality ofh^*(t, ·) for t ∈ [0, 1]. Then F≠∅since B is essential on D_K . We prove that F is closed in X. In fact, let $\{u_n\}$ ⊂F with un → u and $\{t_n\}$ ⊂ [0, 1] with $t_n \to t_0$. Then

$$(Ju_n - h^*(t_n, u_n), un - v) \leq 0\ for\ v \in K \tag{9}$$

Since J, B and φ are demicontinuous, $Ju_n - h^*(t_n, u_n) \rightharpoonup Ju - h^*(t_0, u)$and $\{Ju_n - h*(t_n, u_n)\}$is bounded. This implies

$$(Ju_n - h^*(t_n, u_n), u - v) \to (Ju - h^*(t_0, u), u - v), for\ v \in K \tag{10}$$

Noting that $u_n \to u$ and

$$|(Ju_n - h^*(t_n, u_n), u_n - u)| \leq \| Jun - h^*(t_n, u_n) \| \| u_n - u \|,$$

we have

$$\lim_{n\to\infty} (Ju_n - h^*(t_n, u_n), u_n - u) = 0 \tag{11}$$

Let v ∈K. Then

$$(Ju_n - h^*(t_n, u_n), u_n - u) + (Ju_n - h^*(t_n, u_n), u - v)$$
$$= (Ju_n - h^*(t_n, u_n), u_n - v).$$

This, together with (6)–(8), implies

$$(Ju - h^*(t0, u), u - v) \le 0 \; for\; v \in K$$

and $u \in F$.

By Urysohn's lemma there exists a continuous function$\lambda : D_K \to [0,1]$such that $\lambda(x) = 0$ for $x \in \partial D_K$ and $\lambda(x) = 1\; for\; x \in F$. Define a map $T : D_K \to X^*$by

$$Tx = \lambda(x)\varphi(x) + (1 - \lambda(x))B(x).$$

Then T is a demicontinuous S-contractive map. Since

$$Tx = B(x) = h^*(0, x)\; for\; x \in \partial D_k$$

and the variational inequality of $h^*(0, \cdot\ )$has no solutions on ∂D_K , $T \in V(D_K, X^*)$. Since B is essential on D_K, by Definition (3.1.1), the variational inequality of T has a solution $x_0 \in D_k$. $Let\; t_0 = \lambda(x_0)$. Then $Tx_0 = h^*(t_0, x_0)$and x0 is a solution of variational inequality of $h^*(t_0, \cdot\ )$. Hence, $x_0 \in F$, $\lambda(x_0) = 1$ and $Tx_0 = \varphi(x_0)$. It follows that x_0 is a solution of variational inequality ofφ. By Definition (3.1.1), A is essential on D_K . For the converse, the proof is exactly same.

Sec(3.2): Variational Inequality Theorems

In this section we prove some results on existence of solutions of (3). We first prove the following result under the Leray–Schauder type condition.

Theorem (3.2.1)[3]: Let K be a closed convex set in a reflexive smooth Banach space X and D a bounded open set in X such that $D_k = \emptyset$and $\overline{D_K} \neq K$. Assume that J is of S^+-type and strictly monotone. Assume that $A : \overline{D_K} \to X^*$is a demicontinuous S-contractive map satisfying the following condition.

(L_S) There exists $x_0 \in D_K$ such that the variational inequality of tA + (1 − t)J$\hat{x}_0$has no solutions on ∂D_K for each t ∈(0, 1).

Then (3) has a solution in D_k . Moreover, if (3) has no solutions on ∂D_K , then A is essential on D_k .

Proof: Assume that (3) has no solutions on ∂D_K . Define $h : [0,1] \times \overline{D}_k \to X^* by$

$$h(t,x) = tAx + (1 - t)Jx_0.$$

By Theorem(3.1.2) (P2), $J\hat{x}_0$ is essential on D_k. Note that the variational inequality of J$\hat{x}_0$ has no solutions on ∂D_K . It follows from

(P3) with $B = J\hat{x}_0$ that A is essential on D_K .

The following result provides general conditions which ensure that (3) has nonzero positive solutions from Hilbert spaces to reflexive smooth Banach spaces.

Theorem (3.2.2)[3]: Let K be a closed convex set in a reflexive smooth Banach space X and let D^1, D be bounded open sets in X such that $D_{\mathrm{K}}^1 \neq \emptyset$, D_{K}^1 ≠K and $\overline{D}_{\mathrm{K}}^1 \subset D_{\mathrm{K}}$. Assume that $A : \overline{D}_{\mathrm{K}} \to X^*$satisfies the following conditions.

(H1) $A \in V(D_{\mathrm{K}}, X^*)$is essential on D_{K} .

(H2) $A \in V(\overline{D}_{\mathrm{K}}^1, X^*)$is not essential on D_{K}^1

Then (3) has a solution in $D_{\mathrm{K}} \backslash \overline{D_{\mathrm{K}}^1}$.

Proof: Since A is not essential on D_{K}^1 , there exists $\varphi \in V(D_{\mathrm{K}}^1, X^*)$with $\varphi(x) = Ax\ for\ x \in \partial D_{\mathrm{K}}^1$ such that the variational inequality of φ has no solutions in D_{K}^1 . Define a map $T: \overline{D_{\mathrm{K}}} \to X^* by$

$$Tx = \begin{cases} \varphi(x)\ if\ x \in \overline{D_{\mathrm{K}}^1} \\ Ax\ if\ x \in \overline{D_{\mathrm{K}}} \backslash D_{\mathrm{K}}^1 \end{cases}, .$$

Then T is a demicontinuous S-contractive map on $\overline{D_{\mathrm{K}}}$. Moreover, $T \in V(D_{\mathrm{K}}, X^*)\ and\ Tx = Ax\ for\ x \in \partial D_{\mathrm{K}}$. Since A is essential on D_{K} , it follows from (P1) that the variational inequality of T has a solution x_0 in D_{K} . Since the variational inequality of φ has no solutions in D_{K}^1 , we have $x_0 \in D_{\mathrm{K}} \backslash \overline{D}_{\mathrm{K}}^1$ and thus x_0 is a solution of (3).

The following result gives conditions under which the maps are not essential.

Lemma (3.2.3)[3]: Let K be a wedge in a reflexive smooth Banach space X and D a bounded open set in X such that $D_{\mathrm{K}} \neq \emptyset$. Assume that $A \in V(D_{\mathrm{K}}, X^*)$is bounded and satisfies the following condition.

(E_1) There exists $e \in K$ with $\| e \| = 1$ such that the variational inequality of $A + \beta J\hat{e}$ has no solutions on ∂D_K for each $\beta > 0$.

Then A is not essential on D_{K} .

Proof: Since J, A and D_{K} are bounded, $\tau := sup\{\| Jx - Ax \|: x \in \overline{D_{\mathrm{K}}}\} < \infty. Let\ \beta 0 > \tau / \| Je \|. Define\ a\ map\ S: \overline{D_{\mathrm{K}}} \to X^* by$

$Sx = Ax + \beta_0 Je$. Then S is a demicontinuous S-contractive map and it follows from (E_1) that $S \in V(D_{\mathrm{K}}, X^*)$. We prove that the variational inequality of S has no solutions on D_K . In fact, if not, there exists x $\in D_{\mathrm{K}}$ such that $(Jx - Sx, x - v) \leq 0$ for each $v \in K$. Taking $v = x + e$ implies $(Jx - Sx, e) \geq 0$. Hence, $(Jx - Ax, e) \geq (\beta_0 Je, e)$and

$\beta_0 \| Je \| = (\beta_0 Je, e) \leq (Jx - Ax, e) \leq \| Jx - Ax \| \| e \| \leq \tau$, which contradicts the choice of β_0. (P1), S is not essential on D_K . Define $h :$ $[0,1] \times D_K \to X^*$by $h(t,x) = tAx + (1-t)Sx$. Then

$$h(t,x) = Ax + \beta_0(1-t)Je \; for \; (t,x) \in [0,1] \times \overline{D}_K.$$

By (E1) and $A \in V(D_K, X^*)$, the variational inequality of h(t, ・) has no solutions on ∂DK for each t ∈ $[0,1]$.

(P3), A is not essential on D_K.

Combining Theorem(3.2.1)and Lemma(3.2.3),and using Theorem(3.2.2) we obtain the following result. Its proof is similar to that of Theorem(3.2.4) and we omit it.

Theorem (3.2.4)[3]: Let K be a wedge in a reflexive smooth Banach space X and let D^1, D be bounded open sets in X such that $0 \in D_1$ and $\overline{D_K^1} \subset D_K$. Assume that J is of S+-type and strictly monotone. Assume that $A : \overline{D_K} \to X^*$is a bounded demicontinuous S-contractive map satisfying the following conditions:

(i) (LS) of Theorem(3.2.1) holds on ∂D_K .

(ii) (E_1) of Lemma(3.2.3) holds on ∂D_K^1

Then (4) has a solution on $\overline{D_k} \backslash D_K^1$.

In the following, we generalize Theorem(3.2.4) and study existence of eigenvalues for variational inequalities. a function $d^* : X^* \times X \to R$is defined by

$$d^*(u,x) = \| x \|^2 - 2(u,x) + \| u \|^2 \qquad (9)$$

Definition (3.2.5)[3]: Let K be a nonempty closed convex set in a Banach space X. A map $r : X^* \to K$ is said to be the (generalized) projection from $X^*$to K if it satisfies

$$d^*(u, r(u)) = d^*(u,K) := inf\{d^*(u,x) : x \in K\} \; for \; u \in X^*.$$

Lemma (3.2.6)[3]: Let K be a nonempty closed convex set in a reflexive and strictly convex Banach space X. Then there exists a unique

projection $r : X^* \to K$.

Proof: Since X is reflexive, it follows from Theorem (3.1.2) that r(u) exists for $u \in X*$. Since X is strictly convex, by Theorem (3.1.2) is unique.

From now on, we always assume that X is a reflexive, strictly convex and smooth Banach space and $J : X \to X^*$is the normalized dual map. Since X is strictly convex, J is strictly monotone .

Lemma (3.2.7)[3]: Let K be a nonempty closed convex subset of X. Let $u \in X^*$and $x \in K$. Then the following assertions are equivalent:

(i) $(Jx - u, x - v) \leq 0 \, for\, all\, v \in K$.

(ii) $x = r(u)$.

By Lemma (3.2.7), it is easy to prove the following result.

Lemma (3.2.8)[3]: Assume that $0 \in K$. Then $r(u) = 0$ if and only if $(u, v) \leq 0$ for $v \in K$ if and only if $u \in -K^*$if and only if $\| u \|^2 = d^*(u, K)$, where

$$K^* = \{u \in X^* : (u, v) \geq 0 \, for\, v \in K\}$$

is the dual cone of K.

The following result gives relations between J and d^*

Lemma (3.2.9)[3]: (i) $d^*(u, x) = 0$ if and only if $u = Jx$.

(ii) $u \in J(K)$if and only if $d^*(u, K) = 0$.

Proof: (i) Assume that $d^*(u, x) = 0$. Since $(u, x) \leq \| u \| \| x \|$, we have

$0 = \| x \|^2 - 2(u, x) + \| u \|^2 \geq \| u \|^2 - 2 \| u \| \| x \| + \| u \|^2 = (\| x \| - \| u \|)^2$.

This implies that $\| u \| = \| x \|$and $(u, x) = \frac{1}{2}[\| u \|^2 + \| u \|^2] = \| u \| \| x \|$. Since J is a single-valued map, u = Jx. Conversely, assume that $u = Jx$. Since J is a normalized duality map from $X \, to \, X^*$, $(J(x), x) = \| Jx \| \| x \| \, and \, \| Jx \| = \| x \|. By$ (8),

$d^*(u, x) = d^*(Jx, x) = \| x \|^2 - 2(Jx, x) + \| Jx \|^2 = \| x \|^2 - 2 \| x \|^2 + \| x \|^2 = 0$.

(ii) Let u ∈J(K) and let x ∈K be such that $u = Jx$. Then

$$d^*(u, r(u)) = d^*(u, K) = d^*(Jx, K) \leq d^*(Jx, x) = 0$$

and $d^*(u, r(u)) = d^*(u, K) = 0$. Conversely, if $d^*(u, r(u)) = 0$, then by (i), we have $u = J(r(u)) \in K$.

(i)Where X is assumed to be auniformly convex and uniformly smooth Banach space. Lemma (3.2.9)(ii)We give two examples of generalized projections in $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$.

Let $W_0^{1,p} := W_0^{1,p}(\Omega)$ is the Sobolev space with the standard norm

$$\|u\|W_0^{1,p} = \left(\int_\Omega |\nabla u(x)|p \, dx\right)^{1/p} \qquad (12)$$

where $\nabla u(x) = \frac{\partial u(x)}{\partial x1}, \ldots, \frac{\partial u(x)}{\partial xm}$, $|\nabla u(x)| = \left[\sum_{k=1}^{m} \left(\frac{\partial u}{\partial xk}^2\right)\right]^{\frac{1}{2}}$ and Ω is a bounded and connected open set in $R^n (n \geq 1)$. Itis known that $W_0^{1,p}$

is a uniformly convex and smooth Banach space. Hence, $W_0^{1,p}$ is a reflexive, strictly convex and smooth Banach space with property (H). The dual space of $W_0^{1,p}$ is denoted by $W^{-1,p}(\Omega)$,where $1/p + 1/p' = 1$.

We denote by P the standard positive cone of $W_0^{1,p}$, that is,

$$P = \{u \in W_0^{1,p} : u(x) \geq 0 \; a.e. \, on \, \Omega\} \qquad (13)$$

We need the following weak comparison principle.

Lemma(3.2.10)[3]:. Assume that w, u $\in W_0^{1,p}$ satisfy

$(Jw(x), v(x)) \leq (Ju(x), v(x)) \; for \; v \in P \; and \; a.e. x \in \Omega$.Then $w(x) \leq u(x) \; for \; a.e. x \in \Omega$.

Remark (3.2.11)[3]: By Lemma (3.2.10) we see that if $Ju \in p^*$, then u ∈P. Moreover, if $Ju(x) \geq 0$ for a.e. x ∈Ω, then $u \in P$.

Example (3.2.12)[3]: The map r : $W^{-1,p'} \rightarrow$ P defined by

$$r(u)(x) = max\{Jp'u(x), 0\}$$

is the generalized projection from $W^{-1,p'}$ to P, where

$J_p u(x) = \| u \|^{2-p}_{W_0^{1,p}} (-\Delta P u(x)$ for x $\in \Omega$ (14)

is the normalized dual map from $W_0^{1,p}$ to $W^{-1,p'}$ and

$\Delta_p u(x) = \sum_{i=1}^{n} \frac{\partial}{\partial xi} (|\nabla u(x)|^{p-2} \frac{\partial u(x)}{\partial xi})$.

Proof: Let u $\in W_0^{1,p}$, $\Omega_+ = \{x \in \Omega : J_{p'} u(x) \geq 0\}$ and $\Omega - = \{x \in \Omega : J_{p'} u(x) < 0\}$. $Let\ w(x) = max\{J_{p'} u(x), 0\}$ $for\ x \in \Omega$.

Then $w(x) = J_p w(x) = 0$ $for\ x \in \Omega -$and

$w(x) = J_{p'} u(x)$ and $J_p w(x) = u(x)$ for x $\in \Omega_+$

since $J_p J_{p'}$u(x) = u(x) for x $\in \Omega$. Since $J_{p'} u(x) < 0$ for $x \in \Omega -$, it follows from Remark(3.2.10) that $u(x) \leq 0$ $for\ x \in \Omega -$. Let

$v \in P$ and $\xi = (J_p w - u, w - v)$. Then

$$\xi = \int_{\Omega} [J_p w(x) - u(x)][w(x) - v(x)]\, dx$$

$= \int_{\Omega+} [J_p w(x) - u(x)][w(x) - v(x)]\, dx + \int_{\Omega-} [J_p w(x) - u(x)][w(x) - v(x)]\, dx$

$= \int_{\Omega-} u(x) v(x)\, dx \leq 0$

The result follows from Lemmas (3.2.7) and (3.2.6).

By a proof similar to that of Example(3..2.12), we obtain the following result.

Example (3.2.13)[3]: The map $r : L^q(\Omega) \rightarrow K_p$ defined by

$$r(u)(x) = max\{J_q u(x), 0\}$$

is the generalized projection from $L^q(\Omega)$ to K_p, where

$$J_p u(x) = \| u \|^{2-p}{}_{Lp(\Omega)} |u(x)|^{p-2} u(x)\ for\ x \in \Omega$$

and $K_p := \{u \in L^p(\Omega) : u(x) \geq 0 \, a.e. \, on \, \Omega\}$.

Theorem (3.2.14)[3]: Let K be a wedge in X with $J(K) \cap K^* \neq \{0\}. Let \, D^1, D$ be bounded open sets in X such that $0 \in D^1$ and $\overline{D_K^1} \subset D_K$ .Suppose J is of S_+type, $A : \overline{D_K} \to X^*$is a bounded demicontinuous S-contractive map and $B : \overline{D_K^1} \to X^*$is a compact map.

Assume that the following conditions hold.

(h_1) A satisfies (L_S) of Theorem (3..2.1) on .

(h_2) Either $\overline{B(\partial D_K^1)} \cap (-K^*) = \emptyset$or the following conditions hold.

(i) inf$\{\| Bx \|: x \in \partial D_K^1\} > 0$.

(ii) $\overline{B(\partial D_K^1)} \cap ((-K^*)\backslash J(K)) = \emptyset$.

(h_3) The variational inequality of $A + \lambda B$ has no solutions on $\partial D_K^1 \, for \, \lambda > 0$.

Then (6)) has a solution on $\overline{D_K}\backslash D_K^1$.

Proof: Assume that (6) has no solutions on $D_K \cup \partial D_K^1$. By (L_S) and Theorem (3.2.1), A is essential on D_K . Since$J(K) \cap K^* \neq \{0\}$,there exists $e \in K$ with $\| e \| = 1$ such that $Je \in K^*$. We prove the following assertion:

(E_2) There exists $\lambda_0 > 1$ such that the variational inequality of $A + \lambda_0 B + \beta \hat{J} e$ has no solutions on ∂D_K^1 for each$\beta \geq 0$.

In fact, if not, there exist $\{x_n\} \subset \partial D_K^1, \{\lambda_n\} \subset (1, \infty)$ with $\lambda_n \to \infty, \{\beta_n\} \subset [0, \infty)$such that

$$(Jx_n - (Ax_n + \lambda_n Bx_n + \beta nJe), x_n - v) \leq 0 \, for \, all \, v \in K \tag{15}$$

Taking $v = x_n + e \, in$ (13) implies

$$\beta_n \| Je \| = \beta_n(Je, e) \leq (Jx_n - Ax_n, e) - \lambda n(Bx_n, e)$$

and

$$\frac{\beta n}{\lambda_n} \leq \frac{\| Jxn - Ax_n \|}{\| Je \| \lambda_n} + \frac{\| Bx_n \|}{\| Je \|}$$

Since $\overline{D_K^1}, J, A$ and B are bounded and $\lambda_n \to \infty, \{\beta_n/\lambda_n\}$is bounded. We may assume that $\beta_n/\lambda_n \to \beta_0, Bx_n \to w$ and $x_n \rightharpoonup x \in K. Let\ v \in K$. By(15),

$$\left(\frac{J\lambda_n - A\lambda_n}{\lambda_n}\right) - (Bx_n - w) - (\frac{\beta_n}{\lambda_n} - \beta_0)Je, x_n - v) - (w + \beta_0 Je, x_n - v)$$

$$\leq 0.$$

Taking limit implies $-(w + \beta_0 Je, x - v) \leq 0$ for $v \in K$ and

$(w + \beta_0 Je, u) \leq 0\ for\ u \in K$ (16)

This, together with $Je \in K^*$, implies

$(w, u) \leq -(\beta_0 Je, u) \leq 0\ for\ u \in K$ (17)

By Lemma(3.2.8), we have $r(w) = 0$.

(i) If the first condition of (h_2) holds, then $r(w) = 0$ implies that

$inf\{\| r(Bx) \|: x \in \partial D_K^1\} = 0,$

where r is the same as in Lemma (3.2.6). Hence, there exists $\{u_n\} \subset \partial D_K^1$such that $r(Bu_n) \to 0$. Since B is compact, we may assume that $Bu_n \to w \in B(\partial \overline{D_K^1})$. Since r is continuous, $r(w) = 0$. By Lemma (3.2.8), $w \in -K^*$. Hence, $B(\partial \overline{D_K^1}) \cap (-K^*) \neq \emptyset$,

which contradicts the hypothesis $B(\partial \overline{D_K^1}) \cap (-K^*) = \emptyset$.

(ii) Under the second condition of (h2), if $w \in J(K)$, then by Lemmas (3.2.9)(ii) and (3.2.8),

$$\| w \|^2 = d^*(w, r(w)) = d^*(w, K) = 0.$$

Hence, we have $w = 0$, which contradicts inf$\{\| Bx \|: x \in \partial D_K^1\} > 0. If\ w/\in J(K)$, then noting that r(w) = 0, we have by Lemma (3.2.8), $w \in -K^*$. Hence, $w \in (-K^*)\backslash J(K)$and $B(\partial D_K^1) \cap (-K^*)\backslash J(K) \neq \emptyset$, a contradiction.

Define a map $T : \overline{D_K^1} \to X^*$by $Tx = Ax + \lambda_0 Bx$. Then T is a demi-continuous S-contractive map and the variational inequality of T has no solutions on D_K^1 . Hence, T$\in V(D_K^1, X^*)$and T is bounded since A and B are bounded. It is shown above that the variational inequality of $A + \lambda_0 B + \hat{\beta} e$ has no solutions on ∂D_K^1 for each $\beta \geq 0$. By Lemma(3.2..3), T is not essential on D_K^1 . By (h_3), the variational inequality of $tA + (1 - t)T = A + t\beta_0 B$ has no solutions on ∂D_K^1 for $t \in [0, 1]$.

It follows from Theorem (3.1.2)(P3) that A is not essential on D_K^1 . By Theorem (3.2.1), (6) has a solution in $\overline{D_{\mathrm{K}}} \backslash D_K^1$. The result follows.

Remark (3.2.15)(3]: It is easy to show that if B$(\overline{D_K^1}) \subset J(K)$, then Theorem (3.2.14)(ii) is satisfied, and if $J(K) \cap K^* \neq \{0\}, then\ K \neq -K$.

By Lemma (3.2.8) one can prove that (h2) is equivalent to inf{‖r(Bx)‖ :$x \in \partial D_K^1\} > 0$.

By the proof of Theorem(3.2.14), we obtain the following result on the existence of eigenvalues of variational inequalities.

Theorem (3.2.16)[3]: Let K be a wedge in X with J(K)∩$K^* \neq$ {0} and D a bounded open set in X such that $\partial D_K \neq \emptyset$. Suppose J is of S^+type, A : $\overline{D_k} \rightarrow X^*$is a bounded demicontinuous S-contractive map and $B : D_K \rightarrow X^*$is a compact map. Assume that (h_1)–$(h_2)$of Theorem 3.4 hold on ∂D_K . Then there exists $\lambda \geq 0$ such that the variational inequality of $A + \lambda B$ has a solution on ∂D_K.

Proof: The proof is by contradiction. We may assume that (6) has no solutions on ∂D_K . If the result were false, then (h3) of Theorem (3.2.7) holds on . By the proof of Theorem (3.2.14) we see that under (h2), A is not essential on D_K . On the other hand,by (h_1) and Theorem (3.2.1), A is essential on D_{K} .

In Theorem(3.2.14), K is required to satisfy$J(K) \cap K^* \neq \{0\}$. From the following result, we see that the last condition can be dropped if K is a proper wedge.

Theorem(3.2.17)[3]: Let K be a proper wedge in $X. Let\ D^1, D, A, B$ be the same as in Theorem (3.2.14). Assume that (h_1) of Theorem (3.2.14) holds on ∂D_K and (h_3) of Theorem(3.2.14). holds on D_{K}^1 . Assume that the following conditions hold.

(h_2') $\mathrm{B}(\partial D_{\mathrm{K}}^1) \cap \mathrm{J}((\mathrm{K} \cap (-\mathrm{K})) = \emptyset$.

(h_2'') $d^*(w, K) < d^*(w, K \cap (-K))$for $w \in B(\partial D_{\mathrm{K}}^1)$ with $d^*(w, K) > 0$.

Then (6) has a solution on $\overline{D_K} \backslash D_K^1$.

Proof: The proof is similar to that of Theorem (3.2.14) and we sketch the proof. We can choose $e \in K \cap (-K)$with $\| e \| = 1$. We prove that (E_2) holds. In fact, if not, there exists $\{x_n\} \subset \partial D_K^1$such that $Bx_n \to w$and (3.2.9) holds. Let $v \in K$ and $u = \beta_0 e + v$.

Then $u \in K$. Note that J is homogeneous and odd operator BY(17), we have

$$(J(-\beta_0 e) - w, (-\beta_0 e) - v) = (w + \beta_0 Je, \beta_0 e + v) = (w + \beta_0 Je, u) \leq 0. (17)$$

Since $e \in K \cap (-K)$, we have $-\beta_0 e \in K$. By (18) and Lemma (3.2.7), $r(w) = -\beta_0 e$. This implies that

$$d^*(w, K \cap (-K)) \leq d^*(w, -\beta_0 e) = d^*(w, r(w)) = d^*(w, K) \leq d^*(w, K \cap (-K))$$

and $d^*(w, K \cap (-K)) = d^*(w, K)$. Since $w \in B\overline{(\partial D_K^1)}$, it follows from (h_2'') that $d^*(w, K) = 0$. Hence, $d^*(w, K \cap (-K)) = 0$.

By Lemma (3.2.9)(ii), $w \in J(K \cap (-K))$and $\overline{B(\partial D_K^1)} \cap J(K \cap (-K)) \neq \emptyset$, which contradicts (h_2').

In Theorems (3.2.8)–(3.2.10), K is not a subspace of X. To obtain results when K is an infinite dimensional subspace in X, we first prove the following lemma.

Lemma (3.2.18)[3]: Let K be a wedge in X such that $\partial K_1 = \{x \in K : \| x \| = 1\}$is not compact. Assume that D^1 is a bounded open set in X such that $D_K^1 \neq \emptyset$.

Assume that B : $\overline{D_K^1} \to X^*$is a compact map such that the first condition of (h2) in Theorem (3.1.4) holds.Then there exists e $\in \partial K_1$such that

$$-\overline{r(B(\partial D_K^1))} \cap \{\beta e : \beta \geq 0\} = \emptyset \quad (18)$$

Proof: The proof is by contradiction. If (18) were false, then for each $x \in \partial K_1$, there exists β_xx ≥ 0 such that $\beta_x x \in -\overline{r(B(\partial D_K^1))}$.

Let $= inf\{\| r(Bx) \|: x \in \partial D_K^1\}$. Then by the first condition of (h_2) and Remark (3.2.15) $\beta_x \geq \alpha > 0$ for each x $\in \partial D_K^1$. Let $Q = \{\beta_x x : x \in \partial K_1\}$. Then

$$\partial K_\alpha := \{x \in K : \| x \| = \alpha\} \subset \overline{co}(Q \cup \{0\}) \subset \overline{co}(-\overline{r(B(\partial D_K^1))} \cup \{0\}).$$

Since B is compact and r is continuous, $r(B(\partial D_K^1))$is relatively compact and $\partial K\alpha$is compact, which contradicts noncompactness of ∂K_1.

Theorem (3.2.19)[3]: Let K be an infinite dimensional subspace in $X. Let\ D^1, D$ be bounded open sets in X such that $0 \in D^1$ and $\overline{D_k^1} \subset DK$.

Suppose J is of S_+type, $A : \overline{D_K} \to X^*$is a bounded demicontinuous S-contractive map and $B : \overline{D_k^1} \to X^*$is a compact map.

Assume that (h_1), the first condition of (h_2), and (h_3) of Theorem (3.2.14) hold. Then there exists $x \in D_K \backslash D_K^1$ such that x − Ax is orthogonal to K.

Proof: Assume that (6) has no solutions on $\cup\, \partial D_K^1$. By (LS) and Theorem(3.2.1), A is essential on D_K . By Lemma (3.1.18), there exists e ∈K with ‖e‖ = 1 such that (18) holds. We prove that (E_2) holds. In fact, if not, a similar proof to that of Theorem (18)

shows that (15) holds. Let v ∈K and $u = \beta_0 e + v$. Then $u \in K$. By (15), we have

$$(J(-\beta_0 e) - w, (-\beta_0 e) - v) = (w + \beta_0 Je, \beta_0 e + v) = (w + \beta_0 Je, u)$$
$$\leq 0.$$

By Lemma (3.2.7)$r(w) = -\beta_0 e$. Hence, we have $\beta_0 e \in -r(\overline{B(\partial D_K^1)})$, which contradicts (18). An argument similar to that ofTheorem (3.2.14) shows that A is not essential on D_K^1 . The result follows from Theorem (3.2.14).

By a similar proof to that of Theorem (3.2.16), we obtain the following result on the existence of eigenvalues.

Theorem (3.2.20)[3]: Let K be an infinite dimensional subspace in X and D a bounded open set in X such that $\partial D_K \neq \emptyset$. Assume that J is of $S+$ type, $A : \overline{D_K} \to X^*$is a bounded demicontinuous S-contractive map and B : $\overline{D_K} \to X^*$is a compact map. Assume that (h_1) and the first condition of (h_2) of Theorem (3.2.14) hold on D_K . Then there exists $\lambda \geq 0$ such that the variational inequality of $A + \lambda B$ has a solution on ∂D_K .

In this section, we apply the results obtained to study the existence of nonzero positive weak solutions for p-Laplacian elliptic inequalities

$$\begin{cases} -\Delta pu(x) \geq f\,(x, u(x)) for\ a.e. x \in \Omega \\ \qquad\qquad , u(x) = 0\ on\ \partial\Omega, \end{cases} \tag{19}$$

where Ω is a bounded and connected open set in R^n with meas$(\Omega) > 0$.

The p-Laplacian elliptic inequalities (19) and equations arise in the study of Newtonian fluids (p = 2) and non-Newtonian fluids $(p \neq 2)$such as dilatant fluids $(p > 2)$and pseudoplastic fluids $(1 < p < 2)$.

In the following, we study the case when 2 ≤ n <p. We always assume that the following conditions hold.

(C_0) $n \in N$, the set of natural numbers, and $2 \leq n < p < \infty$.

(C_1) $f : \Omega \times R+ \to R$ satisfies the Carathéodory conditions, that is, $f\,(\cdot\ , u)$ is measurable for each fixed $u \in R_+$and f (x, ·)

is continuous for a.e. $x \in \Omega$.

(C_2) For each $r > 0$ there exists $gr \in L_+^1$ (Ω) such that

$$|f\ (x,u)| \leq\ g_r(x) for\ a.e.x\ \in \Omega\ and\ all\ u\ \in\ [0,r] \tag{20}$$

We note that the condition (C_2) do not require the upper bound of $|f\ (x,u)|$to depend on u, so it is more general than those used, where f satisfies suitable lower and upper bound conditions depending on u.

We define a map $J:\ W_0^{1,p} \rightarrow\ W^{-1,p\prime}$by

$$Ju(x) =\ -\Delta_p u(x) \tag{21}$$

Then J is a duality map from $W_0^{1,p}$ to $W^{-1,p}$ with the gauge function $\Phi(t) = t^{p-1} for\ t\ \in R_+$, and J is (p−1)-homogeneous,that is, $J(cu) = c^{p-1}J(u)\ for\ c\ \in R_+ and\ u\ \in W_0^{1,p}$. Moreover,

$$(Ju,v) = \sum_{i=1}^{n}\int_{\Omega}\ \left(|\nabla u(x)|^{p-2}\frac{du}{\partial xi}\right)\frac{\partial\sqrt{u}}{\partial xi}dx\ for\ u,v\ \in W_0^{1,p} \tag{22}$$

and

$$(Ju,u) = \| u \|^p_{W_0^{1,p}} \quad for\ u\ \in W_0^{1,p} \tag{23}$$

Since $W_0^{1,p}$has the property (H) and is strictly convex, J is of S_+type and is strictly monotone.

We denote by P the standard positive cone of $W_0^{1,p}0$ given in (12). We define a map $A:\ P\ \rightarrow\ W^{-1,p\prime} by$

$$(Au,v)\ = \int_{\Omega}\ f\ \big(x,u(x)\big)v(x)dx \tag{24}$$

Since P is a cone in $W_0^{1,p}$, we see that u ∈P is a solution of the variational inequality

$$(Ju\ -\ Au,u\ -\ v)\ \leq\ 0\ for\ v\ \in P \tag{25}$$

if and only if u ∈P is a solution of the complementary problem

$$(Ju,u)\ =\ (Au,u) \tag{26}$$

and

$$(Ju, v) \geq (Au, v) \, for \; v \in P \tag{27}$$

Definition (3.2.21)[3]: A function $u \in W_0^{1,p}$ is called a positive weak solution of the p-Laplacian elliptic inequality (3.2.1) if $u \in P$ and u satisfies the following inequality:

$$\sum_{i=1}^{n} \int_{\Omega} \left(|\nabla u(x)|^{p-2} \frac{\partial u}{\partial xi} \right) dx \geq \int_{\Omega} f\left(x, u(x)\right) v(x) dx \; for \; v \in P \tag{28}$$

By (22), (24) and Definition (3.2.21), we see that u $\in W_0^{1,p}$ is a positive weak solution of (19) if and only if $u \in P$ and usatisfies (27). Hence, if $u \in P$ is a solution of the variational inequality (19), then u is a positive weak solution of (19). This allows one to apply the theory developed to the variational inequality (27) to study existence of positive weak solution of the p-Laplacian elliptic inequality(19).

Lemma (3.2.22)[3]: Under the hypothesis (C_0), the following assertions hold.

(i) $W_0^{1,p} \subset C(\overline{\Omega})$.

(ii)

$\|u\|_{C(\overline{\Omega})} \leq c_0 \|u\|_{W_0^{1,p}} \, for \; u \in W_0^{1,p}, \, where \; c_0 = [\mathrm{meas}(\Omega)]^{\left(\frac{1}{n}-\frac{1}{p}\right)} \frac{\xi^{\nu(\xi)}}{\sqrt{n}}, \; \xi = \frac{n(p-1)}{(n-1)p} \; and \; \nu(\xi) = \sum_{k=1}^{\infty} \frac{k}{\xi^k}$

(iii) $If \; \{u_k\} \subset W_0^{1,p} \; with \; u_k \rightharpoonup u \in W_0^{1,p}, then \; u_k \to u \; in \; C\overline{(\Omega)}$.

$Let \; r > 0 \; and \; let \; P_r = \{u \in P : \| u \|_{W_0^{1,p}} < r\} \; and \; \partial P_r = \{u \in P : \| u \|_{W_0^{1,p}} = r\}$.

Now, we prove the following result which shows that the map A defined in (24) maps P into $W^{-1,p'}$ and is compact.

Lemma (3.2.23)[3]: Under the hypotheses (C_0)–(C_2), the map A defined in (24) maps P into $W^{-1,p'}$ and is compact.

Proof: Let $r > 0$ and let $u \in C_+(\overline{\Omega})$ $with$ $\| u \|_{c(\overline{\Omega})} \leq r. By$ (C_2), there exists $g_r \in L^1_+(\Omega)$such that (18) holds. Hence,

$$\left|f\left(x, u(x)\right)\right| \leq g_r(x) for\ a.e.x \in \overline{\Omega} \tag{29}$$

We prove that the Nemytskii operator f defined by

$$fu(x) = f(x, u(x))$$

maps $C_+(\Omega)\ to\ L_1$ and is continuous. In fact, let $u \in C_+(\Omega)$and r = $\| u \|_{C(\Omega)}. By$ $(C_1), f(\cdot, u(\cdot))$is measurable and by (29),

we have

$$\int_{\Omega} |f(x, u(x))|\, dx \leq \int_{\Omega} g_r(x)\, dx < \infty \tag{30}$$

and $fu \in L^1$ $for\ u \in C_+\overline{(\Omega)}. Let\ \{uk\} \subset C_+\overline{(\Omega)} with\ u_k \to u \in C_+(\Omega)\ in\ C\overline{(\Omega)}, that\ is, \| u_k - u \|_{\overline{C(\Omega)}} \to 0. Then\ u_k(x) \to u(x)$

$$for\ x \in \Omega\ and\ by\ (C_1),$$

$$f\left(x, u_k(x)\right) \to f\left(x, u(x)\right) for\ a.e.x \in \overline{\Omega} \tag{31}$$

Let $r = sup\{\| uk \|_{C(\Omega)}, \| u \|_{C(\Omega)}\}. Then\ r < \infty. By$ (29), we have

$|f(x, u_k(x)) - f(x, u(x))| \leq |f(x, u_k(x))| + |f(x, u(x))| \leq$
$2g_r(x)\ for\ a.e.x \in \overline{\Omega}$.

This, together with (31) and the Lebesgue dominated convergence theorem, implies that

$$\lim_{K\to\infty} \| fu_k - fu \|_{L^1} = \lim_{k\to\infty} \int_{\Omega} |f(x, u_k(x)) - f(x, u(x))|\, dx$$

$$= \int_{\Omega} \lim_{k\to\infty} |f(x, u_k(x)) - f(x, u(x))|\, dx$$

$$\int_{\Omega} \lim_{k\to\infty} |f(x, u_k(x)) - f(x, u(x))|\, dx = 0.$$

Hence, $f : c_+(\bar{\Omega}) \to L_1$ is continuous.

Now, we prove that A maps P into $W^{-1,p'}$and is compact. In fact, let u∈ P and $v \in W_0^{1,p}$. By Lemma (3.2.22)(i) and (ii), we see that

$$v(x) \le \| v \|_{\overline{C(\Omega)}} \le c_0 \| v \|_{W_0^{1,p}} \; for\; x \in \overline{\Omega} \tag{32}$$

where c_0 is the same as in Lemma (3.2.22)(ii), and

$$|(Au, v)| \le \int_\Omega |f(x, u(x))|\,|v(x)|\,dx \le c_0 \| v \|_{W_0^{1,p}} \int_\Omega |f(x, u(x))|\,dx < \infty.$$

This shows that Au is well defined. Let $v_n, v \in W_0^{1,p}$ $with$ $v_n \to v$ in $W_0^{1,p}$. By Lemma (3.2.22)(ii), $\| vn - v \|_{C(\Omega)} \to 0$. Since

$$|(A_u, v_n) - (Au, v)| \le \int_\Omega \left|f\left(x, u(x)\right)\right| |v_n(x) - v(x)| dx \le$$

$$\| v_n - v \|_{C(\Omega)} \int_\Omega |f(x, u(x))|\,dx$$

we obtain $(A_u, v_n) \to (A_u, v)$ and $A_u, \in W^{-1,p'}$. Hence, A maps P into $W^{-1,p'}$. By Lemma (3.2.22)(iii), $A : P \to W^{-1,p'}$is completely continuous and is compact.

Let $g \in L_+^\infty(\Omega)\backslash\{0\}$ and let

$$\mu_g = \inf\left\{ \int_{|\Omega} |\nabla u(x)|^p\,dx \Big/ \int_\Omega g(x)|u(x)|^p\,dx : u \in (W_0^{1,p})_+(\overline{\Omega})\backslash\{0\} \right\}. \tag{33}$$

for each $g \in L_+^\infty(\Omega)\backslash\{0\}$, there exists $Q_g \in W_0^{1,p} \cap (C_+(\overline{\Omega})\backslash\{0\})$such that the followingp-Laplacian equation holds:

$$\begin{cases} -\Delta_p \varphi_g(x) = \mu_g g(x) \varphi_g^{p-1}(x) & \text{for a.e. } x \in \Omega, \\ \varphi_g(x) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{34}$$

Now, we prove our main result on the existence of nonzero positive weak solutions of (19).

Theorem (3.2.24)[3]: Assume that $(C_0)-(C_2)$ and the following conditions hold:

(i) There exist $r_0 > 0, \varepsilon > 0$ and $\varphi_{r_0} \in L_+^\infty(\Omega)\setminus\{0\}$ such that

$$f(x,u) \le (\mu_{\varphi r0} - \varepsilon)\varphi_{r0}(x)u^{p-1} \text{ for a.e. } x \in \Omega \text{ and all } u \in [r_0,\infty) \qquad (35)$$

(ii) There exist $\rho_0 > 0, \varepsilon > 0$ and $\psi\rho_0 \in L_+^\infty(\Omega)\setminus\{0\}$ such that

$$f(x,u) \ge \left(\mu_{\varphi_{r_0}} + \varepsilon\right)\psi_{\rho 0}(x)u^{p-1} \text{ for a.e. } x \in \Omega \text{ and all } u \in [0,\rho_0] \qquad (36)$$

Then (19) has a nonzero positive weak solution in P.

Proof: By Lemma (3.2.23), $A : P \to W^{-1,p'}$ is compact. By (C_2), for this r_0 given in the condition (i), there exists $g_{r_0} \in L_+^1(\Omega)$

such that

$$|f(x,u)| \le gr_0(x) \text{ for a.e. } x \in \bar{\Omega} \text{ and all } u \in [0,r_0],$$

$$|f(x,u)| \le gr_0(x) + (\mu\varphi r0 - \varepsilon)\varphi r_0(x)u^{p-1} \text{ for a.e. } x \in \bar{\Omega} \text{ and all } u \in R_+ \qquad (37)$$

Let

$$r > \max\left\{\rho_0, \left(\varepsilon^{-1}c_0\mu_{\phi_{r_0}}\|g_{r_0}\|_{L^1}\right)^{\frac{1}{p-1}}\right\}. \qquad (38)$$

We prove that the variational inequality of tA has no solutions on ∂P_r for $t \in [0,1]$. In fact, if not, there exist $u \in \partial P_r$ and $t \in [0,1]$ such that

$$(Ju - tAu, u - v) \le 0 \text{ for } v \in P.$$

By (26), we have

$$(Ju,u) = (tAu,u) = t\int_\Omega f(x,u(x))u(x)dx \qquad (39)$$

By (11) and (33), we have

$\mu\varphi_{r_0}\int_\Omega \varphi_{r_0}(x)u^p(x)\,dx \le \| u \|^p_{W_0^{1,p}}$ (40)

By (22), (37),(39),(40) and Lemma (3.2.22)(ii), we have

$\| u \|^p_{W_0^{1,p}}=(Ju,u)=t\int_\Omega f(x,u(x))u(x)dx \le \int_\Omega |f(x,u(x))| u(x)dx$

$$\begin{aligned}
&\le \int_\Omega g_{r_0}(x)u(x)\,dx + (\mu_{\phi_{r_0}} - \varepsilon)\int_\Omega \phi_{r_0}(x)u^p(x)\,dx \\
&\le \|u\|_{C(\overline{\Omega})}\|g_{r_0}(x)\|_{L^1} + (\mu_{\phi_{r_0}} - \varepsilon)\mu^{-1}_{\phi_{r_0}}\|u\|^p_{W_0^{1,p}} \\
&\le c_0\|u\|_{W_0^{1,p}}\|g_{r_0}(x)\|_{L^1} + \|u\|^p_{W_0^{1,p}} - \varepsilon\mu^{-1}_{\phi_{r_0}}\|u\|^p_{W_0^{1,p}}.
\end{aligned}$$

This implies that

$$\varepsilon \| u \|^p_{W_0^{1,p}} \le c_0\mu\varphi_{r_0} \| u \|_{W_0^{1,p}} \| g_{r_0}(x) \|_{L^1} \text{ and we have}$$

$$r = \| u \|_{W_0^{1,p}} \le \left(\varepsilon^{-1}c_0\mu_{\varphi_{r_0}} \| g_{r_0}(x) \|_{L_1}\right)^{\frac{1}{p-1}} < r,$$

a contradiction. Hence, A satisfies Theorem(3.2.4)(LS) on $:= \partial P_r$.

Let $0 < \rho < \min\{r, c_0^{-1}\rho_0\}$, where r is the same in (38). By Lemma (3.2.22)(ii),

$$u(x) \le \|u\|_{C(\overline{\Omega})} \le c_0\|u\|_{W_0^{1,p}} = c_0\rho < \rho_0 \quad \text{for } x \in \overline{\Omega} \text{ and all } u \in \partial P_\rho$$

and by (36), we obtain

$$f(x, u(x)) \ge (\mu_{\psi_{\rho 0}} + \varepsilon)\psi_{\rho 0}(x)u^{p-1}(x) \quad \text{for a.e. } x \in \overline{\Omega} \text{ and all } u \in \partial P_\rho. \tag{41}$$

Let

$$e(x) = \varphi_{\psi_{\rho 0}}(x) \quad \text{for } x \in \overline{\Omega},$$

where $\phi_{\psi_{\rho 0}}$ satisfies (34) with g = $\psi_{\rho 0}$. Hence, we have

$$(Je, v) = \mu_{\psi_{\rho 0}}\int_\Omega \psi_{\rho 0}(x)e^{p-1}(x)v(x)\,dx \quad \text{for } v \in P. \tag{42}$$

We prove that the variational inequality of $A + \beta\hat{J}e$ has no solutions on ∂P_{ρ} for $\beta > 0$. In fact, if not, there exist $u \in \partial P\rho$ and $\beta > 0$ such that

$$(Ju - Au - \beta Je, v) \geq 0 \quad \text{for } v \in P. \tag{43}$$

BY(41)we see that $f\ (x, u(x)) \geq 0\ for\ a.e.x \in \overline{\Omega}$ and $u \in \partial P\rho$. Hence,

$$(Au, v) = \int_{\Omega} f(x, u(x))v(x)\,dx \geq 0 \quad \text{for } u \in \partial P_{\rho} \text{ and } v \in P.$$

This, together with (43), implies

$$(Ju, v) \geq (Au, v) + \beta(Je, v) \geq \beta(Je, v) = \left(J\left(\beta^{\frac{1}{p-1}}e\right), v\right) \quad \text{for } v \in P.$$

for $v \in P$.

By Lemma (3.2.10), we have

u(x) ≥ β

$$u(x) \geq \beta^{\frac{1}{p-1}}e(x) \quad \text{for a.e. } x \in \Omega. \tag{44}$$

Let

$$\tau = \sup\left\{\zeta > 0 : u(x) \geq \zeta^{\frac{1}{p-1}}e(x) \text{ for a.e. } x \in \Omega\right\}. \tag{45}$$

Then by (44) we see that $0 < \beta \leq \tau < \infty$and

$$u(x) \geq \tau^{\frac{1}{p-1}}e(x) \quad \text{for a.e. } x \in \Omega. \tag{46}$$

By (41), (46) and (42), we have for $v \in P$,

$$\begin{aligned}(Ju, v) &\geq (Au, v) + \beta(Je, v) \geq (Au, v) = \int_{\Omega} f(x, u(x))v(x)\,dx \\ &\geq (\mu_{\psi_{\rho_0}} + \varepsilon)\int_{\Omega} \psi_{\rho_0}(x)u^{p-1}(x)v(x)\,dx \\ &\geq (\mu_{\psi_{\rho_0}} + \varepsilon)\tau\int_{\Omega} \psi_{\rho_0}(x)e^{p-1}(x)v(x)\,dx = \sigma(Je, v) = \left(J\left(\sigma^{\frac{1}{p-1}}e, v\right)\right.\end{aligned}$$

where $\sigma = \mu_{\psi_{\rho_0}}^{-1}(\mu_{\psi_{\rho_0}} + \varepsilon)\tau$. By Lemma (3.2.10), we have

$$u(x) \geq \sigma^{\frac{1}{p-1}} e(x) \quad \text{for a.e. } x \in \Omega.$$

By (45), we have $\geq \sigma > \tau$, a contradiction. Hence, A satisfies Theorem (3.2.4) (E1) on $D_K^1 := \partial P_\rho$. By Theorem (3.2.4), (19) has a nonzero positive weak solution in P.

As a special case of Theorem (3.2.24), we consider existence of nonzero positive weak solutions for the p-Laplacian elliptic inequalities

$$\begin{cases} -\Delta_p u(x) \geq f(u(x)), & x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{47}$$

By Theorem (3.2.24), we obtain the following result which is easily verified in applications when the nonlinearity is independent of the variable x.

Corollary (3.2.25)[3]: Assume that (C_0) holds and $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition:

$$\limsup_{u\to\infty} \frac{f(u)}{u^{p-1}} < \mu_1 < \liminf_{u\to 0+} \frac{f(u)}{u^{p-1}}, \tag{48}$$

where $\mu_1 = \mu_g$ with $g \equiv 1$ is given by (33).

Then (47) has a nonzero positive weak solution in P.

As illustrations, we study existence of nonzero positive weak solutions of the p-Laplacian elliptic inequality

$$\begin{cases} -\Delta_p u(x) \geq r u^{p-1}(x) \left(1 - \frac{u(x)}{K}\right) - \frac{a u^\alpha(x)}{b + u^\gamma(x)} & \text{for a.e. } x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{49}$$

arising in mathematical biology, where u(x) denotes the population density of one species at location x, r is the intrinsic growth rate of the species, $K > 0$ is the carrying capacity of the species, the term $u^{p-1}(x)(1 - u(x))$ represents the logistic growth rate of order p, and the term $\frac{au^{\alpha}(x)}{b+u^{\gamma}(x)}$ contains the functional response of Holling type III, where α = γ , and the parameters $a \geq 0, b, \alpha, \gamma > 0$.

To make the population persist on every location x ∈Ω, one needs to find nonzero positive solutions or weak solutions u satisfying $u(x) > 0 \; for \; x \in \Omega$.

It is well known that the Laplacian elliptic equation with logistic growth rates

$$\begin{cases} -\Delta u(x) = ru(x)\big(1 - u(x)\big) & \text{for a.e. } x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{50}$$

has a unique nonzero positive solution in $C(\overline{\Omega})$ if $r \in (\mu_1, \infty)$, and has no nonzero positive solutions in $C(\overline{\Omega})$) if r ∈ $(0, \mu 1]$, where n = 1, where n ≥ 1. for the study of the Laplace equations related to (50). Hence, it is interesting to know whether (49) has nonzero positive solutions in $W_0^{1,p}$ even when $a = 0$.

In the following, using Corollary (3.2.25), we prove a result on existence of nonzero positive weak solutions in $W_0^{1,p}$ of (49)under the assumption (C_0), where $n \in N$and $2 \leq n < p < \infty$, and we allow $a > 0$ and $\alpha \neq \gamma$.

Theorem (3.2.26)[3]: Assume that (C_0) holds, $a \geq 0$ and $b > 0. Let \, p \in (0, \infty)$, $\alpha \in (p, \infty)$ and $\in (0, \infty)$. Then (49) has a nonzero positive weak solution in P for $r \in (\mu_1, \infty)$.

Proof: We define a function $\mathbb{R}_+ \to \mathbb{R}$ by

$$f(u) = ru^{p-1}\left(1 - \frac{u}{K}\right) - \frac{au^{\alpha}}{b + u^{\gamma}}.$$

Then

$$\lim_{u\to 0^+}\frac{f(u)}{u^{p-1}} = r\lim_{u\to 0^+}\left(1-\frac{u}{K}\right) - \lim_{u\to 0^+}\frac{au^{\alpha-p}}{b+u^{\gamma}} = r > \mu_1$$

and

$$\lim_{u\to \infty}\frac{f(u)}{u^{p-1}} = r\lim_{u\to \infty}\left(1-\frac{u}{K}\right) - \lim_{u\to \infty}\frac{au^{\alpha-p}}{b+u^{\gamma}} = -\infty < \mu_1.$$

The result follows from Corollary (3.2.25).

We end this section by considering the following eigenvalue problems on variational inequalities:

$$\begin{cases} -\|u\|^{2-p}_{W_0^{1,p}}\Delta_p u(x) \geq f(x,u(x)) + \lambda g(x,u(x)) & \text{for a.e. } x \in \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{51}$$

We first prove the following result.

Lemma (3.2.27)[3]: Let J_{p} and J be the same as in (11) and (19), respectively. Then

$$J(P)\cap P^* \neq \{0\} \quad \textit{and} \quad J_p(P)\cap P^* \neq \{0\}.$$

Proof: for each $g \in L^{\infty}_{+}$
$(\Omega)\backslash\{0\}$, there exists $\phi g \in W_0^{1,p} \cap (C + (\Omega)\backslash\{0\})$ such that

$$\begin{cases} J\varphi_g(x) = -\Delta_p\varphi_g(x) = \mu_g g(x)\varphi_g^{p-1}(x) \neq 0 & \text{for a.e. } x \in \Omega, \\ \varphi_g(x) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{52}$$

It follows that $J_{\phi g} \neq 0\ and J_{\phi g} \in J(P)$. Moreover,

$$(J\varphi_g, v) = \int_{\Omega} \mu_g g(x)\varphi_g^{p-1}(x)v(x)\,dx \geq 0 \quad \text{for } v \in P$$

and $J_{\phi g} \in P^*$. Hence, $J(P) \cap P^* \neq \{0\}$. It is obvious that the second result follows from the first one and (11).

By applying Theorem (3.2.16), we prove the following eigenvalue result on the variational inequalities (51).

Theorem (3.2.28)[3]: Assume that (C_0) holds and f and g satisfy (C_1) and (C_2). Assume further that the following conditions hold.

(i) There exists $u_0 > 0$ and $m \in (0[\quad c_0^2\, meas(\Omega) \quad]^{-1})$

such that

$$|f\ (x, u)| \leq mu\ for\ a.e. x\ \in \Omega\ and\ u\ \in\ [0, u_0] \tag{53}$$

whereC_0 is the same as in Lemma (3.2.22).

(ii) There exists $\varsigma > 0$ such that

$$g(x, u)\ \geq\ \varsigma\ for\ a.e. x\ \in \Omega\ and\ all\ u\ \in \mathbb{R}_+ \tag{54}$$

Then for each $\in (0.\frac{u_0}{c_0}]$, there exists $\lambda > 0$ such that (51) has a positive weak solution in ∂P_r .

Proof: Let Jp be the normalized duality map defined in (11). By Lemma (3.2.27), we have $J_p(P)\ \cap\ P^* \neq \{0\}. Let\ r\ \in (0, \frac{u_0}{c_0}]$.

We prove that the variational inequality of tA has no solutions on ∂P_r for $t\ \in\ [0, 1]$. In fact, if not, there exist u $\in \partial P_r$ and $t\ \in\ [0, 1]$such that

$$(J_p u - tAu, u - v) \leq 0 \quad \text{for } v \in P$$

By Lemma (3.2.22)(ii), we have

$$u(x) \leq \|u\|_{C(\overline{\Omega})} \leq c_0 \|u\|_{W_0^{1,p}} = c_0 r \leq u_0 \quad \text{for } x \in \overline{\Omega}.$$

By (26), (53) and Lemma (3.2.22)(ii), we have

$$\|u\|^2_{W_0^{1,p}} = \|u\|^2_{W_0^{1,p}} = (J_p u, u) = (tAu, u) = t \int_\Omega f(x, u(x))u(x)\, dx$$
$$\leq \int_\Omega f(x, u(x))u(x)\, dx \leq m \int_\Omega u^2(x)\, dx$$

$$\leq m\|n\|^2_{c(\overline{\Omega})} meas\ (\Omega) \leq m\ C_0^2 meas\ (\Omega) \|u\|^2_{W_0^{1,p}} \leq \|u\|^2_{W_0^{1,p}}$$

a contradiction. Hence, A satisfies Theorem (3.2.15) (h1) on $D_K := \partial Pr$.

We define a map $B : P \to W^{-1,p'}$by

$$(Bu, v) = \int_\Omega g(x, u(x)) v(x)\, dx. \tag{55}$$

Since (C_0) holds and g satisfies (C_1) and (C_2), by Lemma (3.2.23), the map B defined in (55) maps P intoW−1,p′ and is compact.

For each $v \in P\backslash\{0\}$, by (55) and (54), we have

$$\|Bu\|\ \|v\|_{W_0^{1,p}} \geq (Bu, v) \geq \varsigma \int_\Omega v(x)\, dx > 0 \quad \text{for}\ u, v \in P$$

This implies that $Bu \in P^*$and

$$\|Bu\| \geq \sigma \sup\left\{\int_\Omega w(x)\, dx :\ w \in \partial P_1\right\} > 0 \quad \text{for}\ u \in P.$$

Hence, $\overline{B(\partial P_r\ )} \cap (-P^*) = \emptyset$and the first condition of (h_2) in Theorem 3.4 holds on $D_K := \partial P_r$. The result follows from Theorem (3.2.13).

Chapter 4

AGlobal Error Bounds for Generalized Mixed Quasi Variational Inequalities

By using these gap functions we obtain global error bounds for the solution of generalized mixed quasi variationalin equality problem s in Hilbert spaes.The results given in this chapter generalize and improve some corresponding knownresults.

Sec (4.1): Preliminaries and basic facts

In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities related to set-valued operators, non convex optimization and non monotone operators. A useful and important generalization of variational inequalities is a mixed variational inequality containing the nonlinear term. For the applications of the mixed variational inequalities, see for example and the references therein. Due to the presence of the nonlinear term, one cannot develop the projection-type algorithms for solving the mixed quasi-variational inequalities, which motivated authors to develop another technique. This technique is related to the resolvent of the maximal monotone operator. The main idea of this technique was introduced by Brezis Further by using the concept of the resolvent operator technique, many authors introduced and studied the various resolvent equations to develop the sensitivity analysis for mixed variational inequalities.

One of the classical approach in the analysis of variational inequality problem is to transform it into an equivalent optimization problem via the notion of gap function, see for example and the references therein. This enables us to develop descent-like algorithms to solve variational inequality problem. Besides these, gap functions also turned out to be very useful in designing new globally convergent algorithms, in analyzing the rate of convergence of some iterative methods and in obtaining the error bounds. Gap functions have turned out to be very useful in deriving the error bounds, which provide a measure of the distance between solution set and an arbitrary point. Recently, many error bounds for various kinds of variational inequalities have been established, see for example and the references therein.

Throughout this section, let H be a real Hilbert space, whose inner product and norm are denoted by ·, · and · , respectively. Let C(H) be a family of nonempty compact subsets of H. Let S, T, : H → C(H) be the set-valued operators and g : H → H be asingle-valued operator. $Let \varphi(\cdot,\cdot) : H \times H \to R \cup \{+\infty\}$ be a continuous bifunction with respect to both arguments. Let F : H × H → R be a bifunction satisfying F(x, x) = 0, for all x ∈ H. For given nonlinear operator N(·, ·) : H × H → H, we consider the following generalized mixed quasi variational inequality problem, denoted by GMQVIP, which consists in finding x ∈ H, u ∈ S(x), v ∈ T(x) such that

$$F(g(x), g(y)) + N(u, v), g(y) - g(x) + \varphi(g(x), g(y)) - \varphi(g(x), g(x)) \geq 0, \forall y \in \qquad (1)$$

The quasi variational inequality problems are definitely most notable one among the several variants of variational inequality problems. An important reason for this is that a number of problems involving the non convex, and nonsmooth operators arising in optimization, mechanics and structural engineering theory can be studied via the generalized mixed quasi variational inequalities, see for example and the references therein.

If g ≡ I, the identity operator and F ≡ 0, then GMQVIPis equivalent to generalized mixed set-valued variational inequality problem, denoted by GMSVVIP, which consists in finding x ∈ H, u ∈ S(x), v ∈ T(x) such that

$$N(u, v), y - x + \varphi(x, y) - \varphi(x, x) \geq 0, \forall y \in H \qquad (2)$$

a problem studied by using the auxiliary principle techniques.

If φ(x, y) = φ(y), S ≡ 0 and T : H → C(H) are set-valued operator, N(u, v) = T(x), then problem GMSVVIP (2) collapses to set-valued mixed variational inequality problem, denoted by SVMVIP, which consists in finding x ∈ H such that

$$\exists u \in T(x) : u, y - x + \varphi(y) - \varphi(x) \geq 0, \forall y \in H, \quad (3)$$

which was considered by Tang They introduced two regularized gap functions for above SVMVIPand studied there differentiable properties.

If T is single valued, then problem SVMVIPreduces to mixed variational inequality problem, denoted by MVIP, which consists in finding x ∈ H such that,

$$\langle T(x), y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in H, \quad (4)$$

We introduced three gap functions for MVIP and by using these We obtained error bounds.

If the function φ(·) is an indicator function of a closed set K in H, then problem MVIP (4) reduces to set-valued variational inequality problem, denoted by SVVIP, which consists in finding x ∈ K such that:

$$\exists u T(x): \langle u , y - x\rangle + \phi(y) - \phi(x) \geq 0 , \forall \in H \quad (5)$$

They obtained some existence results for global error bounds for gap function under strong monotonicity. Later,defined gap functions and by using it they obtained finiteness and error bounds properties for above set-valued variational inequalities.

If T is single valued and K : H → C(H) be a set-valued mapping, such that K(x) is a closed convex set in H, for each x ∈ H, then above problem SVVIP(5)is equivalent to quasi variational inequality problem, denoted by QVIP, which consists in finding x ∈ K(x) such that:

$$\langle T(x), y - x\rangle \geq 0, \forall y \in K(x) \quad (6)$$

They derived local and global error bounds for above quasi variational inequality problems in terms of the regularized gap function and the D-gap function.

Inspired and motivated by the recent research work above, we introduce gap functions and error bounds for generalized mixed quasi variational inequality problems. Since this class is the most general and includes the previously studied some classes of variational inequalities as special cases, therefore our results cover and extend the previously known results under weaker conditions.

Further we define normal residual vector R(x, θ) to derive the global error bounds for the solution of GMQVIP(1)we introduce a regularized gap function for GMQVIP(1)and derived error bounds without using Lipschitz continuity assumption, we introduce D-gap function and derive error bounds for the solution of the GMQVIP(1)under some weaker conditions.

In order to establish resolvent equations for the GMQVIP(1)we needed the following definitions and results.

Dentition (4.1.1)[4]: Let $F : H \times H \to \mathbb{R}$ and $\phi : H \times H \to \mathbb{R}$ be two bifunctions. Then

(a) F is said to be monotone if, $F(x,y) + F(y,x) \leq 0, \forall x,y \in H$;

(b) ϕ is said to be skew-symmetric if, $\phi(x,x) - \phi(x,y) - \phi(y,x) + \phi(y,y) \geq 0, \forall x,y \in H$.

Remark (4.1.2)[4]: Clearly if the skew-symmetric bifunction φ(·, ·) is bilinear, then φ(x, x) ≥ 0, ∀x ∈ H. In fact,

$$\phi(x,x) - \phi(x,y) - \phi(y,x) + \phi(y,y) = \phi(x-y, x-y) \geq 0, \forall x,y \in H.$$

The skew-symmetric bifunctions have the properties which can be considered an analog of monotonicity of gradient and non negativity of second derivative for the convex function.

Definition (4.1.3)[4]: $Let S, T, : H \to C(H)$ be the set-valued operators, $N(\cdot,\cdot) : H \times H \to H$ be the nonlinear operator and $g : H \to H$ be a single-valued operator, then

(a) N is said to be strongly mixed g-monotone, if there exists a constant $\alpha > 0$ such that

$N(u, v) - N(u0, v0), g(x) - g(x0) \geq \alpha \| g(x) - g(x_0), \|^2$

for all x, $x_0 \in H$, $u \in S(x)$, $u_0 \in S(x_0)$, $v \in T(x)$, $v_0 \in T(x_0)$;

(b) N is said to be mixed Lipschitz continuous, if there exist constants $\beta, \delta > 0$ such that

$$\|N(u, v) - N(u0, v0)| \leq \beta \|u0 - u\| + \delta \|v0 - v\|^2,$$

for all $x, x_0 \in H, u \in S(x), u_0 \in S(x_0), v \in T(x), v_0 \in T(x_0)$;

(c)T is said to be M-Lipschitz continuous, if there exists a constant $\mu > 0$ such that

$M(T(x), T(x_0)) \leq \mu \|x - x_0\|, \forall x, x_0 \in H$

where $M(\cdot, \cdot)$ is the Hausdorff metric on C(H).

(d)g is said to be Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|g(x) - g(x_0))\| \leq L\|x - x_0\|, \forall x, x_0 \in H;$$

(e)g is said to be strongly nonexpanding, if there exists a constant $\tau > 0$ such that

$$\|g(x) - g(x_0)\| \geq \tau \|x - x_0\|, \forall x, x_0 \in H$$

Remark (4.1.4)[4]: From (d) and (e)

$$\tau\|x - x_0\| \leq \|g(x) - g(x_0))\| \leq L\|x - x_0\|,$$

implies that $\tau \leq L$. A mapping $g : R \to R$ defined as $g(x) = \frac{1}{x^2}, \forall x \in [1, 2]$ is Lipschitz continuous and strongly nonexpanding with $L = 4$ and $\tau = \frac{1}{8}$, respectively, while g(x) is not affine.

The following theorem is a special case of results given by Chang

Theorem (4.1.5)[4]: Let X be a closed convex subset of a Hausdorff topological vector space E and $G : X \times X \to R$ be a bifunction. Assume that the following conditions hold:

(i) $G(x, x) \geq 0, \forall x \in X$;

(ii) G is monotone;

(iii) For each y ∈ X fixed, the function x → G(x, y) is upper-hemicontinuous, i.e.,

$$\lim_{t\to 0} sup\, G(tw + (1 - t)x, y) \le G(x, y)\ \forall x, y, w \in X, t \in [0,1]$$

(iv) For each $x \in X$ fixed, the function $y \to G(x, y)$ is convex and lower semicontinuous;

(v) there exists a compact subset K of E and there exists $y_0 \in K \cap X$ such that $G(x, y_0) < 0\ for\ each\ x \in X\backslash K$.

Then the set {x^*∈ X : G$(x^*, y) \ge 0, \forall y \in X$} is nonempty convex and compact.

$Let\ \theta > 0$ be a number. For a given bifunction F, the associated Yosida approximation,F_θ , over K ⊂ H and the corresponding regularized operator, A_θ^F , are defined as follows:

$$F_\theta(x, y) = \langle \frac{1}{\theta}(x - J_\theta^F(x), y - x\rangle\ and\ A_\theta^F(x) = \frac{1}{\theta}\ (x - J_\theta^F(x))$$

Where $J_\theta^F : H \to H\ defined\ as\ J_\theta^F$(x)=$(1 + \theta J_\theta^F)^{-1}(x)\ is\ resolvent\ operater.$

Remark (4.1.6)[4]: (i) If F_θ(x, y) = $sup_{u\in Mx}\langle u, y - x\rangle$ and K = H, M being a maximal monotone operator, it directly yields

$$J_\theta^F(x) = (1 + \theta M)^{-1}(x). A_\theta^F(x) = M_\theta(x)$$

where $M_\theta := \frac{1}{\theta}$ (I − (I + θ M)−1) is the Yosida approximation of M and I is the identity operator;

(ii) Resolvent operator J_θ^F is nonexpansive, i.e.

$$\left\|J_\theta^F(x) - J_\theta^F(y)\right\| \le \|x - y\|, \forall x, y \in H.$$

(iii) From above, we get

$$J_{\theta,\phi}^F(x) = (1 + 0\, \partial\phi(x, .)^{-1} \equiv (1 + \partial\phi(x))^{-1}$$

where $\phi: H \times H \to R \cup \{+\infty\}$ is a convex, proper and lower-semicontinuous function in second argument. The subdiffer- ential ∂φ of φ is maximal monotone with respect to the second argument, where $\partial\phi(x) \equiv \partial\phi(x,\cdot)$.

Now we prove following important result for the characterization of resolvent operator $J^{F}_{\theta,\phi}$,(x).

Lemma (4.1.7)[4]: Let H be a real Hilbert space $H. Let\, F : H \times H \to R\; and\; \phi : H \times H$ → R be nonlinear bifunctions and let $\theta > 0$. Suppose that the following conditions are satisfied:

(i)F satisfies condition (i)–(iv) in Theorem(4.1.5).

(ii) ϕ is skew-symmetric, convex in second argument and continuous;

(iii) For each fixed $z \in H$, there exists a compact subset K of E and $y_0 \in K \cap H$ such that $\theta F(x, y_0) + \langle x - z, y_0\rangle + \theta\phi(x, y_0) - \theta\phi(x.x) < 0 foreach\; x \in H\backslash K$

Then for each fixed z ∈ H, find x ∈ H such that

$$\theta F(x,y) + x - z, y - x + \theta\,\phi(x,y) - \theta\,\phi(x,x) \geq 0, \forall y \in H \qquad (7)$$

has a unique solution if and only if x = $J^{F}_{\theta\phi(x),}$[z].

Proof: For each fixed z ∈ H, define G : H × H → R by

$$G(x,y) = \theta\, F(x,y) + x - z, y - x + \theta\,\phi(x,y) - \theta\,\phi(x,x) \geq 0, \qquad \forall x,y \in H.$$

Evidently $G(x,x) = 0, \forall x \in H$ and condition (i) of Theoorem(4.1.5)is satisfied. Further since F is monotone and ϕ is skew-symmetric, then we have

$$G(x,y) + G(y,= \theta\,[F(x,y) + F(y,x)] + x - z, y - x + y - z, x - y + \theta\,[\phi(x,y) - \theta\,\phi(x,x) + \phi(y,x) - \phi(y,y)]$$

$$\leq -x - y2$$

$$\leq 0,$$

i.e., G is monotone and thus condition (ii) of Theorem(4.1.5) is satisfied. Since F is upper hemicontinuous and φ are continuous, we have that for each $x, y, w \in H, t \in$ [0, 1],

$$\lim_{t\to 0} sup\, G(tw + (1 - t)x, y)$$

$$\leq \lim_{t\to 0}\sup \theta F(tw + (1-t)x, y) + \lim_{t\to 0} sup\langle tw + (1-t)x - z, y - tw(1-t)x\rangle$$

$$+ \theta \lim_{t\to 0} sup\, [\phi(tw + (1-t)x, y) - \phi(tw + (1-t)x, tw + (1-t)x)]$$

$$\leq \theta F(x,y) + \lim_{t\to 0} sup\, \langle t(w-z) + (1-t)(x-z), t(y-w) + (1-t)(y-x) + \theta\phi(x,y) - \theta\phi(x,x)\rangle$$

$$\leq \theta F(x,y) + \lim_{t\to 0} sup\ [\langle t^2\langle w-z, y-w\rangle + t\,(1-t)\langle x-z, y-w\rangle + (1-t)^2(x-z, y-x)\rangle]$$

$$+\theta\phi(x,y) - \theta\phi(x.x)$$

$$\leq \theta F(x,y) + \langle x-z, y-x\rangle + \theta\phi(x,x)$$

$=C(x,y).$

Thus condition (iii) of Theorem(4.1.5) of is satisfied. Since for each x ∈ H, F(x, ·) is convex and lower semicontinuous and ϕ is convex in the second argument and continuous, it is easily observe that for each $x \in H, G(x,\cdot)$ is convex and lower semicontinuous and thus condition (iv) of is satisfied. Evidently condition (iii) implies that G satisfies condition (v) of Theorem(4.1.5) Hence it follows from Theorem(4.1.5) that there exists a point x ∈ H such that $G(x,y) = 0, \forall y$ ∈ H, that is, for each fixed $z \in H$, there exist x ∈ H such that

$$\theta\, F(x,y) + x - z, y - x + \theta\,\phi(x,y) - \theta\,\phi(x,x) \geq 0, \forall y \in H.$$

In order to show that x ∈ H is unique solution of(7), for each fixed $z \in H$, let $x1, x2 \in H$ be any two solutions of(7) Then,

we have

$$\theta\, F(x_1,y) + x_1 - z, y - x_1 + \theta\,\phi(x_1,y) - \theta\,\phi(x_1,x1) \geq 0, \forall y \in H. \qquad (8)$$

$$\theta\, F(x_2,y) + x_2 - z, y - x_2 + \theta\,\phi(x_2,y) - \theta\,\phi(x2,x2) \geq 0, \forall y \in H. \qquad (9\,)$$

Taking y = x_2 in(8) and y = x_1 in(9) and then adding these two inequalities, we get

$$\theta\,(F(x_1,x_2) + F(x_2,x_1)) - \theta\,[\phi(x_1,x_1) - \phi(x_1,x_2) - \varphi(x_2,x_1)$$

$$+ \phi(x_2,x_2)] \geq \langle x_1 - x_2, x_1 - x_2\rangle.$$

Since F is monotone, ϕ is skew-symmetric and θ > 0, the preceding inequality reduces to

$$\|x_1 - x_2\|^2 \leq 0$$

which implies that x_1=x_2. Hence x ∈ H is unique solution of (7)

Therefore, it follows that for each z ∈ H, write the unique solution of (7) as

x = $J^F_{0,\phi(x)}[z]$∈ H. Then for all y ∈ H, we have

$$\theta F\left(J^F_{\theta,\phi(x)}[z], y\right) + \langle J^F_{\theta,\phi(x)}[z] - z, y - J^F_{\theta,\phi(x)}[z]\rangle + \theta\phi\left(J^F_{\theta,\phi(x)}[z], J^F_{\theta,\phi(x)}[z]\right) \geq 0 \quad (10)$$

Hence$J^F_{\theta,\phi(x)}$: $H \rightarrow H$ is well defined and single-valued mapping. Further, we observe from Remark(4.1.6) that x = $J^F_{\theta,\phi(x)}[z]$ if and only if x is a solution of This completes the proof.

Lemma (4.1.8)[4]: $Any\, x \in H, u \in S(x), v \in T(x)$ is a solution of GMQVIP(1) if and only if $x \in H, u \in S(x), v \in T(x)$ satisfies the relation:

$$g(x) = J^F_{0,\phi(x)}\,[g(x) - \theta\, N(u,v)],$$

where $\theta > 0$ is a constant and $J^F_{\theta,\phi(x)}$ is resolvent operator.

Proof: Let $x \in H, u \in S(x), v \in T(x)$ be solution of GMQVIP (1)then

$F(g(x), g(y)) + N(u,v), g(y) - g(x) + \phi(g(x), g(y)) - \phi(g(x), g(x)) \geq 0, \forall y \in H,$

which can be written as

$\theta\, F(g(x), g(y)) + g(x) - [g(x) - \theta\, N(u,v)], g(y) - g(x) + \theta\, \phi(g(x), g(y)) - \theta\, \phi(g(x), g(x)) \geq 0, \forall y \in H.$

Thus, by invoking Lemma(4.1.7) we have

$$g(x) = J^F_{0,\phi(x)}[g(x) - \theta\, N(u,v)],$$

the required result.

Definition (4.1.9)[4]: Let K be the domain of the GMQVIP(1) A function p : K → R is said to be a gap function for the GMQVIP(1)if it satisfies the following properties:

$(i) p(x) \geq 0, \forall x \in K;$

(ii)p(x*) = 0, if and only if x^* solves the GMQVIP(1).

We now define the residual vector R(x, θ) by relation

$$R(x,\theta) = g(x) - J^F_{0,\phi(x)}[g(x) - \theta N(u,v)]. \quad (11)$$

Invoking Lemma(4.1.8) one can observe that $x \in H, u \in S(x), v \in T(x)$ is a solution of GMQVIP(1)if and only if, x ∈ H is a root of the equation

$$R(x,\theta) = 0. \quad (12)$$

The residual vector $R(x,\theta)$ is a gap function for GMQVIP(1)

Now by using residual vector $R(x,\theta)$ i.e. gap function, we derive the global error bounds for the solution of GMQVIP(1)

Theorem (4.1.10)[4]: Assume that all conditions of Lemma(4.1.7)hold. Let x0 ∈ H be a solution of GMQVIP(1) let N(·, ·) be strongly mixed g-monotone with constant $\alpha > 0$ and mixed Lipschitz continuous with constants $\beta, \delta > 0$, respectively. Let g : $H \to H$ be Lipschitz continuous with constant $L > 0$ and strongly nonexpanding with constant $\tau > 0$. Suppose $S,T : H \to C(H)$ be a M-Lipschitz continuous with constants $\eta, \mu > 0$, respectively. If for any ρ > 0,

$$\left\|J^F_{\theta.\phi(x)}(W) - J^F_{\theta.\phi(x)}(W)\right\| \le p\|x-y\|, \forall x, yw \in H \quad (13)$$

then

$$\frac{1}{c_1}\|R(x,\theta)\| \le \|x - x_0\| \le c_2\|R(x,\theta)\|, \quad \forall x \in H$$

where $R(x,\theta)$ is residual vector defined by(11) and c_1, c_2 are generic constants .

proof: let $x_0 \in H$, $u_0 \in S(x), v_0 \in T(x) be\ a\ solution\ of\ G\ M\ Q\ VIP\ (1) then.$

$F(g(x0), g(y)) + N(u0, v0), g(y) - g(x0) + \varphi(g(x0), g(y)) - \varphi(g(x0), g(x0)) \ge 0, \forall y \in H.$

Substituting $g(y) = J^F_{\theta.\phi(x)}[g(x) - \theta N(u,v)])$ in above inequality , we have.

$$F\left(g(x_0), J^F_{\theta,\phi(x)}[g(x)\right) - \theta N(u,v)] + \langle N(u_0,v_0), J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)] - g(x_0)\rangle$$

$$+ \phi\left(g(x_0), J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]\right) - \phi\left(g(x_0)\right) \ge 0 \quad (14)$$

Taking $z = g(x) - \theta N(u,v)$ and $y = g(x_0) in (10)$ we get

$$\theta\, F\left(J^F_{\theta,\phi(x)}[g(x)\,-\,\theta\, N(u,v))\right)], g(x_0$$

$$+\ \langle J^F_{\theta,\phi(x)}[g(x)\,-\,\theta\, N(u,v)]\,-\,g(x)\,+\,\theta\, N(u,v), g(x_0)\,-\,J^F_{\theta,\phi(x)}[g(x)$$

$$-\ \theta\, N(u,v)]\rangle$$

$$+\ \theta\ \varphi\ J^F_{\theta,\phi(x)}[g(x)-\ \theta\, N(u,v)], g(x_0)-\ \theta\ \varphi J^F_{\theta,\phi(x)}$$

$$[g(x)\,-\,\theta\, N(u,v), J^F_{\theta,\phi(x)}, [g(x)\,-\,\theta\, N(u,v)]\ \geq\ 0.$$

Which implies that

$$F\left(J^F_{\theta,\phi(x)}[g(x)-\theta N(u.v)]g(x_0)\right)$$

$$-\ \langle N(v,v)+\frac{1}{\theta}\left(J^F_{\theta,\phi(x)}[g(x)-\theta N(u.v)]-g(x)J^F_{\theta,\phi(x)}\right]g(x)-\theta N(u,v)$$

$$-\,g(x_0))\rangle$$

$$+\left(J^F_{\theta,\phi(x)}[g(x)-\theta N(u.v)]-g(x)J^F_{\theta,\phi(x)}\right]J^F_{\theta,\phi(x)}[g(x)$$

$$-\,\theta N(u.v)]J^F_{\theta,\phi(x)}[g(x)$$

$$-\,\theta N(u.v)] \tag{15}$$

Adding (14) and (15) , we get

$$F\left(J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)], g(x_0)\right)+F\left(g(x_\theta)J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)]\right)$$

$$+\langle N(u_0,v_0)-N(u,v)+\frac{1}{\theta}\left(g(x)-\ J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)]\right)J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)]$$

$$-\,g(x_\theta)\rangle$$

$$+\phi(J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)], g(x_0)-\phi(J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)], (J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)])$$

$$+\phi(g(x_0), J^F_{\theta\phi(x),}[g(x)-\theta N(u,v)], -\phi\left(g(x_0), g(x_0)\right)\geq 0\ \forall y\in H.$$

Since F(0,0) is monotone and ϕ(0,0) is skew – symmetric, therefore

$$\langle N(u_0,v_0)-\ N(u,v)+\frac{1}{\theta}\left(g(x)\ -J^F_{0,\phi(x)}[g(x)\,-\,\theta\, N(u,v))\right]\ , J^F_{0,\phi(x)}[g(x)$$

$$-\ \theta\, N(u,v)]\,-\,g(x)\rangle\geq 0$$

Which implies that

$$\left\langle N(u_0, v_0) - N(u, v), g(x_0) - J^F_{\theta,\phi(x)}[g(x) - \theta N(u, v)] \right\rangle$$
$$\leq \frac{1}{\theta} \left\langle g(x) - J^F_{\theta,\phi(x)}[g(x) - \theta N(u, v)], J^F_{\theta,\phi(x)}[g(x) - \theta N(u, v)] - g(x_0) \right\rangle . \quad (16)$$

Since. N(·, ·) is strongly mixed g-monotone and g is strongly nonexpanding, therefore for α > 0, we have

$$\alpha t^2 \parallel x_0 - x \parallel^2 \leq \alpha \parallel g(x_0) - g(x) \parallel^2$$
$$\leq \langle \quad N(u_0, v_0) - N(u, v), g(x_0) - g(x) \quad \rangle$$
$$= \langle N(u_0, v_0) - N(u, v), g(x_0) -, J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] + , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] - g(x)\rangle$$
$$= \langle N(u_0, v_0) - N(u, v), g(x_0) -, J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)]\rangle$$
$$+\langle N(u_0, v_0) - N(u, v), , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] - g(x)\rangle$$
$$\leq \frac{1}{\theta}\langle g(x)- , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)], , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] - g(x_0)\rangle$$
$$+\langle N(u_0, v_0) - N(u, v), , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] - g(x)\rangle$$
$$\leq -\frac{1}{\theta}\langle g(x)- , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)], , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] - g(x_0)\rangle$$
$$+ N(u0, v0) - N(u, v), J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] - g(x)$$
$$\leq -\frac{1}{\theta}\langle g(x)- , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)], g(x) - \theta\, N(u, v)]\rangle$$
$$+ \frac{1}{\theta}\langle g(x)- , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)], g(x) - g(x_0)\rangle$$
$$+ \frac{1}{\theta}\, g(x) - , J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)], g(x) - g(x_0)$$

$$+\langle N(u_0, v_0) - N(u, v), J^F_{\theta,\phi(x)}[g(x) - \theta\, N(u, v)] - g(x)\rangle$$
$$\leq -\frac{1}{\theta}\|R(x,\theta)\|^2 + \frac{1}{\theta}\|R(x,\theta)\|\|g(x) - g(x_0)\| + \|N(u_0, v_0) - N(u, v)\|\|R(x,\theta)\|$$

Now using the mixed Lipschitz continuity of the operator N(·, ·) and the M-Lipschitz continuity of S and T, we have

$$\leq -\frac{1}{\theta}\|(x,\theta)\|^2 + \frac{L}{\theta}\|R(x,\theta)\|\|x-x_0\| + (\beta\|u_0-u\| + \|v_0-v\|)\|R(x,\theta)\|$$

$$\leq -\frac{1}{\theta}\|(x,\theta)\|^2 + \frac{L}{\theta}\|R(x,\theta)\|\|x-x_0\| + (\beta\|x_0-x\| + \delta\mu\|u_0-u\|)\|R(x,\theta)\|$$

$$\leq \frac{L}{\theta} \leq \left(\frac{L}{\theta} + \beta\eta + \delta\mu\right)\|x_0-x\|\|R(x,\theta)\|.\|R(x,\theta)\|^2$$

$$+ \left(\frac{L}{\theta} + \beta\eta + \delta\mu\right)\|x_0-x\|\|R(x,\theta)\|$$

Which implies that $\|x - x_0\| \leq \frac{1}{\alpha\tau^2}\left(\frac{L}{\theta} + \beta\eta + \delta\mu\right) \| R(x,\theta) \|$, from which we have$\| x - x_0 \| \leq c_2 \| R(x,\theta) \|, where\ c_2 = \frac{1}{\alpha\tau^2}(\theta L + \beta\eta + \delta\mu)$.

From the definition of residual vector(11) we have

$$\|R(x,\theta)\| = \left\|g(x) - J^F_{\theta,\phi(x)}(x)[g(x) - \theta N(u,v)]\right\|$$

$$= \left\|g(x) - g(x_0) + J^F_{\theta,\phi(x)}(x_0)[g(x_0) - \theta N(u_0,v_0)] - J^F_{\theta,\phi(x)}(x)[g(x) - \theta N(u,v)]\right\|.$$

$$\leq \left\| g(x) - g(x_0) + J^F_{\theta,\phi(x)}(x_0)[g(x_0) - \theta N(u_0,v_0)] - J^F_{\theta,\phi(x)}(x_0)[g(x) - \theta N(u,v)]\right\|$$

$$+ \left\|J^F_{\theta,\phi(x)}(x_0)[g(x) - \theta N(u,v)] - J^F_{\theta,\phi(x)}(x)[g(x) - \theta N(u,v)]\right\|$$

By using Lipschitz continuity of g, nonexpansiveness of $J^F_{\theta,\emptyset}$,φ and assumption(13) we have

$$R(x,\theta) \leq L\|x - x_0\| + \|g(x_0) - g(x)\| + \theta\|(N(u,v) - N(u_0,v_0)\|$$

$$+ \rho\|x_0 - x\|.$$

$$\leq (2L + \rho)\|x - x_0\| + \theta\|(N(u,v) - N(u_0,v_0)\|.$$

Now from the mixed Lipschitz continuity of the operator N(·, ·) and M-Lipschitz continuity of S and T, we have

$$\|R(x,\theta)\| \leq (2L + \rho + \theta(\beta\eta + \delta\mu))\|x - x_0\| = k_1\|x - x_0\|,$$

which implies that

$$\|x - x_0\| \geq \frac{1}{k_1}\|R(x,\theta)\|$$

where $k_1 = 2L + \rho + \theta(\eta\beta + \delta\mu)$. This completes the proof.

In this section our main motivation is to overcome the non differentiability of normal residual vector R(x, θ) i.e. the gap function, defined bywhich is a serious drawback of the normal residual gap function. Now, by using an approach due to Fukushima, we construct another gap function associated with problem GMQVIP(1.1), which can be viewed as a regularized gap function. For θ > 0, the functions G_θ is defined by

$$G_\theta(x) = \max_{y\in H.g(y)\in H}\left\{-F\big(g(x),g(y)\big) + \langle N(u,v), g(x)-g(y)\rangle - \phi\big(g(x),g(x)\big) - \frac{1}{2\theta}\|g(x) - g(y)\|^2\right\} \quad (17)$$

which is finite valued everywhere and is differentiable whenever all operators involved in $G_\theta(x)$, are differentiable. We note that the function $G_\theta(x)$ can be written as

$$G_\theta(x) = -F(g(x), J^F_{\theta,\phi(x)}(x)[g(x) - \theta N(u,v)])$$
$$+ N(u,v), g(x) - J^F_{\theta,\phi(x)}(x)[g(x) - \theta N(u,v)] - \phi(g(x), J^F_{\theta,\phi(x)}(x)[g(x) - \theta N(u,v)])$$
$$g(x) - J^F_{0,\phi(x)}(x)\frac{1}{2_\theta}\|[g(x) - \theta N(u,v)]2\|. \quad (18)$$

Theorem (4.1.11)[4]: Assume that all conditions of Lemma hold and R(x, θ) is residual vector defined by(17) then the function G_θ (x) for θ > 0 defined, is a gap function for GMQVIP(1)

Proof: Taking z = g(x) − θ N(u, v)and y = g(x)in(10), we get

$$F(J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)], g(x)) + J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)] - g(x)$$
$$+ \theta N(u,v), g(x) - J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]$$

$$+\ \phi(J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], g(x))\ -\ \theta\, \phi(J^F_{\theta,\phi(x)}[g(x)$$
$$-\ \theta N(u,v)], J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)])\ \geq\ 0.$$

Now we have

$$F(J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], g(x))\ +\ \langle\ \ N(u,v)\ -\ \tfrac{1}{\theta}(g(x)\ - J^F_{\theta,\phi(x)}[g(x)$$
$$-\ \theta N(u,v)]), g(x)\ -\ J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)]\ \ \rangle$$
$$+\ \phi(J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], g(x))\ -\ \phi(J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], J^F_{\theta,\phi(x)} g(x)$$
$$-\ \theta N(u,v)])\ \geq\ 0,$$

which can be written as,

$$F(J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], g(x))\ +\ N(u,v), g(x)\ -\ J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)]$$
$$+\ \phi(J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], g(x))\ - \phi(J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], J^F_{\theta,\phi(x)}[g(x)$$
$$-\ \theta N(u,v)])$$
$$\geq \frac{1}{\theta} g(x)\ -\ J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)], g(x)\ -\ J^F_{\theta,\phi(x)}[g(x)\ -\ \theta N(u,v)]. \qquad (19)$$

Adding(18)and(19) also by using monotonicity of F(·, ·) and skew-symmetry of $\phi(\cdot,\cdot)$, we get

$$G_\theta\ (x)\ \geq\ \frac{1}{\theta}\ g(x)\ -\ J^F_{\theta,\phi(x)}[g(x)\ -\ \theta\ N(u,v)], g(x)\ -\ J^F_{\theta,\phi(x)}[g(x)$$
$$-\ \theta\ N(u,v)]\ -\ \frac{1}{2\theta} g(x)\ -\ J^F_{\theta,\phi(x)} \|[g(x)\ -\ \theta\ N(u,v)]\|^2$$
$$\geq\ \frac{1}{\theta}\|R(x,\theta\ )\|^2\ -\ \frac{1}{2\theta}\|R(x,\theta\ )\|^2\ =\ \frac{1}{2\theta}\|R(x,\theta\ )\|^2.$$

Clearly, we have $G_\theta(x)\ \geq\ 0$, for all $x\ \in\ H$.

Now from the above conclusion, if G_θ (x) = 0, then $R(x,\theta\ )\ =\ 0$. Hence by, we see that $x\ \in\ H$ is a solution of GMQVIPConversely, if x ∈ H is a solution of GMQVIP(1), then g(x) = $J^F_{\theta,\phi(x)}[g(x)\ -\ \theta\ N(u,v)]$, consequently, from and with condition $F(x,x) = 0$, for all $x \in H$, we have that $G_\theta(x) = 0$. This completes the proof.

Now, we derive the error bounds without using the Lipschitz continuity of the $N(\cdot,\cdot)$.

Theorem (4.1.12)[4]: Let x_0 is a solution of GMQVIP(1) Suppose that $N(\cdot,\cdot)$ is strongly mixed g-monotone with constant $\alpha > 0, F(\cdot,\cdot)$ is monotone, $\phi(\cdot,\cdot)$ is skew-symmetric and g is strongly nonexpanding with constant τ > 0, then

$$\|x - x_0\| \leq \frac{1}{\tau\sqrt{(\alpha - \frac{1}{2\theta})}}\sqrt{G_\theta} \quad \forall x \in H, \qquad \theta > \frac{1}{2\alpha}$$

Proof: From(17), it can be written as,

$$G_\theta\,(x) \geq -F(g(x), g(x_0)) + N(u,v), g(x) - g(x_0) - \phi(g(x), g(x_0)) + \phi(g(x), g(x)) - \frac{1}{2\theta}\|g(x) - g(x_0)\|^2.$$

By using strongly mixed g-monotonicity of N(·, ·), we have

$$G_\theta(x) \geq -F\big(g(x), g(x_0)\big) + N(u,v), g(x) - g(x_0) - \phi\big(g(x), g(x_0)\big) - \phi\big(g(x), g(x_0)\big) + g(x), g(x)) - \frac{1}{2\theta}\|g(x) - g(x_0)\|^2 \qquad (20)$$

Since $x_0 \in H, u_0 \in S(x), v_0 \in$ T(x) be a solution of GMQVIP(1), then

$$F\big(g(x), g(x_0)\big) + N(u,v), g(x) - g(x_0) - \phi\big(g(x), g(x_0)\big) - \phi\big(g(x), g(x_0)\big) + g(x), g(x))$$

Taking y = x in above inequality

$$F(g(x_0), g(x)) + N(u_0, v_0), g(x) - g(x_0) + \phi(g(x_0), g(x)) - \phi(g(x_0), g(x_0)) \geq 0. \qquad (21)$$

Combining(20)(21)then using monotonicity of F and skew-symmetry of ϕ, respectively, we get

$$G_\theta(x) \geq \alpha\|g(x) - g(x_0)\|^2 - \frac{1}{2\theta}\|g(x) - g(x_0)\|^2$$

Further, using the strongly nonexpandicity of g, we have

$G_\theta(x) \geq \left(\alpha - \frac{1}{2\theta}\right)\tau^2\|x - x_0\|^2,$

Whichimplies

$\|x - x_0\| \leq \frac{1}{\tau\sqrt{(\alpha - \frac{1}{2\theta})}}\sqrt{G_\theta(x)}.$ This completes the proof.

Sec (4.2):Global Error Bounds for GMQVIP(1)

In this section, we consider another gap function associated with GMQVIP(1), which can be viewed as a difference of two regularized gap functions with distinct parameters, known as D-gap function. The D-gap function for GMQVIP(1) with parameters $\theta > \psi > 0$ is defined as

$$G_{\theta\psi}\,,(x) = G_\theta(x) - G_\psi(x), \forall x \in H,$$

Now, D-gap function associated with the GMQVIP(1) is given by

$$G_{\theta\psi}\,,(x) = \max_{y \in H.g(y) \in H}\{-F(g(x), g(y)) + \langle N(u,v), g(x) - g(y)\rangle - \phi(g(x), g(y)) + \phi(g(x), g(x))$$

$$+ \frac{1}{2\psi}\|g(x) - g(y)\|^2 - \frac{1}{2\theta}\|g(x) - g(y)\|^2\}, x \in H, \tilde{\theta} > \psi > 0 \qquad (22)$$

The D-gap function defined by(22)can be written as

$$G_{\theta\psi}\,,(x) = -\mathrm{F}(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v), J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)] + N(u,v), J^F_{\theta,\phi(x)} g(x) - \psi N(u,v)] - J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]$$

Further, it can be written as,

$$G_{\theta\psi}(x) = -F(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]) + N(u,v), R(x,\theta) - R(x,\psi) - \varphi(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)],, J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)])$$

$$+ \phi(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)]) + \frac{1}{2\psi}\|R(x,\psi)\|^2 - \frac{1}{2\theta}\|R(x,\theta)\|^2.$$

(23)

Next, we derive global error bounds for GMQVIP(1).

Theorem (4.2.1)[4]: Assume that all conditions of Lemma(4.1.8) hold and $R(x,\theta)$ is residual vector defined by(10) then for all $x \in H$, $\tilde{\theta} > \psi > 0$, we have

$$\frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\psi)\|^2 \le \|D_{\theta,\psi}(x)\| \le \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\theta)\|^2.$$

In particular $D_{\theta,\psi}(x) = 0$, if and only if, $x \in H$ solves GMQVIP(1)

Proof: Taking $z = g(x) - \theta N(u,v)$ and $y = J^{F}_{\theta,\varphi(x)}[g(x) - \psi N(u,v)]$ in(19), we get

$$\begin{aligned}
&\theta F(J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)], J^F_{\theta,\phi(x)}[g(x)-\psi N(u,v)]) \\
&\quad + \langle J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)] - g(x) + \theta N(u,v), J^F_{\theta,\phi(x)}[g(x)-\psi N(u,v)] - J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)]\rangle \\
&\quad + \theta\phi(J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)], J^F_{\theta,\phi(x)}[g(x)-\psi N(u,v)]) \\
&\quad - \theta\phi(J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v), J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)]) \ge 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&F(J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)], J^F_{\theta,\phi(x)}[g(x)-\psi N(u,v)]) + \langle N(u,v), R(x,\theta)-R(x,\psi)\rangle \\
&\quad \ge \frac{1}{\theta}\langle R(x,\theta), R(x,\theta)-R(x,\psi)\rangle - \phi(J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)], J^F_{\theta,\phi(x)}[g(x)-\psi N(u,v)]) \\
&\qquad + \phi(J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v), J^F_{\theta,\phi(x)}[g(x)-\theta N(u,v)]).
\end{aligned} \tag{23}$$

Combining(22)and(23) also by using monotonicity of $F(\cdot,\cdot)$ and skew-symmetry of $\varphi(\cdot,\cdot)$, we get

$$\begin{aligned}
D_{\theta,\psi}(x)\| &\ge \frac{1}{\theta}\langle R(x,\theta), R(x,\theta)-R(x,\psi)\rangle + \frac{1}{2\psi}\|R(x,\psi)\|^2 - \frac{1}{2\theta}\|R(x,\theta)\|^2 \\
&= \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\psi)\|^2 + \frac{1}{\theta}\langle R(x,\theta), R(x,\theta)-R(x,\psi)\rangle \\
&\quad - \frac{1}{2\theta}\|R(x,\theta)-R(x,\psi)\|^2 - \frac{1}{\theta}\langle R(x,\psi), R(x,\theta)-R(x,\psi)\rangle \\
&= \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\psi)\|^2 + \frac{1}{2\theta}\|R(x,\theta)-R(x,\psi)\|^2 \\
&\ge \frac{1}{2}\left(\frac{1}{\psi}-\frac{1}{\theta}\right)\|R(x,\psi)\|^2,
\end{aligned} \tag{24}$$

which implies the left-most inequality in the assertion.

In a similar way, by taking $x = J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], \tilde{z} = g(x) - \psi N(u,v)$ and $y = J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]$ in(9) we get

$$\begin{aligned}
&\psi F(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]) \\
&+ \langle J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)] - g(x) + \psi N(u,v), J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)] - J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)]\rangle \\
&+ \psi\phi(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]) \\
&- \psi\phi(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)]) \geq 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&F(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]) - \langle N(u,v), R(x,\theta) - R(x,\psi)\rangle \\
&\geq -\frac{1}{\psi}\langle R(x,\psi), R(x,\theta) - R(x,\psi)\rangle - \phi(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \theta N(u,v)]) \\
&\quad + \phi(J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)], J^F_{\theta,\phi(x)}[g(x) - \psi N(u,v)]).
\end{aligned} \tag{25}$$

Combining, we get

$$\begin{aligned}
D_{\theta,\psi}(x) &\leq \frac{1}{\psi}\langle R(x,\psi), R(x,\theta) - R(x,\psi)\rangle + \frac{1}{2\psi}\|R(x,\psi)\|^2 - \frac{1}{2\theta}\|R(x,\theta)\|^2 \\
&= \frac{1}{2}\left(\frac{1}{\psi} - \frac{1}{\theta}\right)\|R(x,\theta)\|^2 - \frac{1}{2\psi}\|R(x,\theta) - R(x,\psi)\|^2 \\
&\leq \frac{1}{2}\left(\frac{1}{\psi} - \frac{1}{\theta}\right)\|R(x,\theta)\|^2,
\end{aligned} \tag{26}$$

which implies the right-most inequality in the assertion. Combining(24)and(26) we obtain the required result.

Finally, we derive a global error bound for GMQVIP(1)

Theorem(4.2.2)[4]: Let x0 is a solution of GMQVIP(1) Suppose that N is strongly mixed g-monotone with constant $\alpha > 0$, F(·, ·) is monotone, $\varphi(\cdot,\cdot)$ is skew-symmetric and g is strongly nonexpanding with constant $\tau > 0$, then

$$\|x - x_0\| \leq \frac{1}{\tau\sqrt{[\alpha + \frac{1}{2}\left(\frac{1}{\psi} - \frac{1}{\theta}\right)]}}\sqrt{D_{\theta,\psi}(x)} \qquad \forall x \in H, \quad \alpha > \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{\psi}\right).$$

Proof: From(21), it can be written as,

$$D_{\theta,\psi}(x) \geq -F(g(x), g(x_0)) + \langle N(u, v), g(x) - g(x_0)\rangle - \phi(g(x), g(x_0)) + \phi(g(x), g(x))$$
$$+ \frac{1}{2\psi}\|g(x) - g(x_0)\|^2 - \frac{1}{2\theta}\|g(x) - g(x_0)\|^2.$$

By using strongly mixed g-monotonicity of N(·, ·), we have

$$D_{\theta,\psi}(x) \geq -F(g(x), g(x_0)) + \langle N(u_0, v_0), g(x) - g(x_0)\rangle + \alpha\|g(x) - g(x_0)\|^2$$
$$-\phi(g(x), g(x_0)) + \phi(g(x), g(x)) + \frac{1}{2\psi}\|g(x) - g(x_0)\|^2 - \frac{1}{2\theta}\|g(x) - g(x_0)\|^2. \quad (27)$$

Since x_0 ∈ H, u_0 ∈ S(x), v_0 ∈ T(x) be a solution of GMQVIP(1), then

F(g(x_0), g(y)) + N(u_0, v_0), g(y) − g(x_0) + φ(g(x_0), g(y)) − φ(g(x_0), g(x_0)) ≥ 0.

Taking y = x in above inequality

$$F(g(x_0), g(y)) + \langle N(u_0, v_0), g(y) - g(x_0)\rangle + \phi(g(x_0), g(y)) - \phi(g(x_0), g(x_0)) \geq 0. \quad (28)$$

Combining(27)and(28) then using monotonicity of F and skew-symmetry of φ, respectively, we get

$$D_{\theta,\psi}(x) \geq \alpha\|g(x) - g(x_0)\|^2 + \frac{1}{2\psi}\|g(x) - g(x_0)\|^2 - \frac{1}{2\theta}\|g(x) - g(x_0)\|^2.$$

Further, using the strongly nonexpandicity of g, we have

$$D_{\theta,\psi}(x) \geq \left(\alpha + \frac{1}{2\psi} - \frac{1}{2\theta}\right)\tau^2\|x - x_0\|^2,$$

which implies

$$\|x - x_0\| \leq \frac{1}{\tau\sqrt{[\alpha + \frac{1}{2}\left(\frac{1}{\psi} - \frac{1}{\theta}\right)]}}\sqrt{D_{\theta,\psi}(x)}.$$

Which completes the proof.