Chapter 6

Incomparable and with Non Isomorphic:

General structure results for analytic equivalence relations are applied in the context of Banach spaces to show that if E_0 does not reduce to isomorphism of the subspaces of a space, in particular, if the subspaces of the space admit a classification up to isomorphism by real numbers, then any subspace with an unconditional basis is isomorphic to its square and hyperplanes and has an isomorphically homogeneous subsequence. A Banach space $\mathcal{W}$ with a Schauder basis is said to be α -minimal for some $\alpha < \omega_1$ if, for any two block subspaces $z, y \subseteq \mathcal{W}$, the Bourgain embeddability index of z into y is at least α. We show a dichotomy that characterises when a Banach space has an α -minimal subspace

Section (6.1):) Minimal Banach Spaces

The general problem of our study is a generalisation of the homogeneous space problem [238].Namely, what can be said about a Banach space with "few" non isomorphic subspaces? In particular, will such a space necessarily satisfy more regularity properties than a general space? Will it necessarily have subspaces of a given type?

Theorem (6.1.1)[234]: Let X be an infinite dimensional Banach space. Then X contains either a minimal subspace or a continuum of pairwise incomparable subspaces.

Recall that two spaces are said to be incomparable if neither of them embed into the other, and a space is minimal if it embeds into all of its infinite dimensional subspaces.

The homogeneous space problem, which was solved in the positive by the combined efforts of Gowers [235], Komorowski and Tomczak-Jaegermann [236],is the problem of whether any infinite dimensional space, isomorphic to all its infinite dimensional subspaces, must necessarily be isomorphic to ℓ_2. As a continuation of this one can ask how many isomorphism classes of subspaces a non Hilbertian space has to contain. Infinitely many? A continuum? Even for some of the classical spaces this question is still open, though recent progress has been made by Ferenczi and Galego [237].

The theorem and proof turn out to have something to say about the following two problems of Gowers. ([235], Problems 7.9 and 7.10):

(i) Determine which partial orders that can be realised as the set of sub- spaces of an infinite dimensional Banach space under the relation of embeddability. Or at least find strong conditions such a partial order must necessarily satisfy.

(ii) Find further applications of the main determinacy result in [235]. In particular, are there any applications that need its full strength, i.e., that need it to hold for analytic and not just open sets?

Theorem (6.1.1) says that any such partial order must either have a minimal element or an antichain of continuum size. And, as will be evident, the proof does in fact very much need the full strength of the determinacy result.

We mention that the proof relies heavily on methods of logic and we have therefore included a short review of the most basic notions of set theory indispensable to understand the proof. Also for the benefit of the non analyst we recall some standard notions from Banach space theory.

Before presenting the results of the second part we will first need this brief review.

A Polish space is a separable completely metrisable space. A measurable space, whose algebra of measurable sets are the Borel sets of some Polish topology, is said to be standard Borel. These spaces turn out to be completely classified up to Borel isomorphism by their cardinality, that can either be countable or equal to that of the continuum. A subset of a standard Borel space is analytic if it is the image by a Borel function of some standard Borel space and coanalytic if its complement is so. It is C-measurable if it belongs to the smallest σ-algebra containing the Borel sets and closed under the Souslin operation. In particular, analytic sets are C-measurable as they can be obtained by the Souslin operation applied to a sequence of Borel sets. C-measurable sets in Polish spaces satisfy most of the classical regularity properties, such as universal measurability and the Baire property. We denote by $\sum_1^1$, $\prod_1^1$ and $\sum_2^1$ the classes of analytic, coanalytic and Borel images of coanalytic sets respectively. A classic result of Sierpinski states that any cset is the union of $\mathcal{N}_1$ Borel sets.

Let X be a Polish space and $\mathcal{F}(X)$ denote the set of closed subsets of X. We endow $\mathcal{F}(X)$ with the following σ-algebra that renders it a standard Borel space. The generators are the following sets, where U varies over the open subsets of X.

$$\{F \in \mathcal{F}(X) \,|F \cap U \neq \emptyset\}$$

The resulting measurable space is called the Effros Borel space of X.

Fix some basis $\{U_n\}$ for the space $C(2^{\mathbb{N}})$ and define the Borel set $\mathcal{B}$ by:

$$\mathcal{B} = \{F \in \mathcal{F}(C(2^{\mathbb{N}}))|\forall n\,(0 \in U_n \rightarrow F \cap U_n \neq \emptyset) \wedge \forall n, m, l\, \forall r, t \in \mathbb{Q}$$

$$(rU_n + tU_m \subseteq U_l \wedge F \cap U_n \neq \emptyset \wedge F \cap U_m \neq \emptyset \rightarrow F \cap U_l \neq \emptyset)\}$$

This evidently consists of all the closed linear subspaces of $C(2^{\mathbb{N}})$ and, as $C(2^{\mathbb{N}})$ is isometrically universal for separable Banach spaces, any separable Banach space has an isometric copy in $\mathcal{B}$. We can therefore view $\mathcal{B}$ as the standard Borel space of all separable Banach spaces. When one wants to restrict the attention to the subspaces of some particular space X one only needs to consider the Borel subset $\{Y \in \mathcal{B}\, |Y \subseteq X\}$. Moreover, it is not hard to see that most reasonably definable properties and relations are $\sum_2^1$ in $\mathcal{B}$ or $\mathcal{B}^n$, for example, the relations of isometry and isomorphism are both analytic in $\mathcal{B}^2$ exactly as expected.

A theme of descriptive set theory, that has been extensively developed the last fifteen years or so, is the Borel reducibility ordering of analytic equivalence relations on standard Borel spaces.

This ordering is defined as follows: Suppose $E \subset X^2$ and $E \subset Y^2$ are analytic equivalence relations on standard Borel spaces X and . We say that E is Borel reducible to F, in symbols $E \leq_B F$, if there is a Borel measurable function $f : X \to Y$ such that for all $x, y \in X$:

$$xEy \Leftrightarrow f(x)Ff(y)$$

Moreover, when X and Y are Polish and f can be taken to be continuous, we write $E \leq_c F$.

Hence, X represents a class of mathematical objects (e.g., sep-arable Banach spaces) that we wish to classify up to E-equivalence (e.g., isomorphism) by complete invariants belonging to some other category of mathematical objects. A reduction $f : X \to Y$ of E to F corresponds then to a classification of X-objects up to E-equivalence by Y -objects up to F-equivalence.

Another way of viewing the Borel reducibility ordering is as a refinement of the concept of cardinality. It provides a concept of relative cardinality for quotient spaces in the absence of the axiom of choice. For a reduction of E to F is essentially an injection of X/E into Y/F admitting a Borel lifting from X to Y.

A few words on the power of the continuum: We say that an analytic equivalence relation E on a standard Borel space X has a continuum of classes if there is an uncountable Borel set $B \subset X$ consisting of pairwise E-inequivalent points. This is known to be stronger than just demanding that there should be some bijection between the set of classes and $\mathbb{R}$. The are for example analytic equivalence relations having exactly $\mathcal{N}_1$ many classes, but not having a continuum of classes (in the above sense) in any model of set theory. But an uncountable Borel set is always Borel isomorphic to $\mathbb{R}$, independently of the size of the continuum.

If A is some infinite subset of $\mathbb{N}$, we denote by $[A]^{\mathbb{N}}$ the space of all infinite subsets of A equipped with the topology induced by the product topology on 2^A. Furthermore, for two sets A and B we write $A \subset^* B$ iff $A \setminus B$ is finite. Then $A \subsetneq^* B$ iff $A \subset^* B$ but $B \not\subset^* A$. Also, when $A \subset N$ and $k \in N$ we let $A/k = \{n \in A | n > k\}$. We will occasionally also consider natural numbers as ordinals, so that $n = \{0, 1, \ldots, n-1\}$.

We will repeatedly use the following result of Ellentuck extending results of Galvin-Prikry for Borel sets and Silver for analytic sets: if $\mathcal{A} \subset [\mathbb{N}]^{\mathbb{N}}$ is a C-measurable set, then there is some $A \in [\mathbb{N}]^{\mathbb{N}}$ with either $[A]^{\mathbb{N}} \subset \mathcal{A}$ or $[A]^{\mathbb{N}} \cap \mathcal{A} = \emptyset$.

This has the consequence that if $f : [\mathbb{N}]^{\mathbb{N}} \to X$ is some C-measurable function with values in some Polish space X, then there is some $A \in [\mathbb{N}]^{\mathbb{N}}$ such that f's restriction to $[A]^{\mathbb{N}}$ is continuous.

Among the simpler analytic equivalence relations are those that admit a classification by real numbers, i.e., those that are Borel reducible to the identity relation on $\mathbb{R}$. These are said to be smooth. It turns out that among Borel equivalence relations there is a minimum, with respect to $\leq_B$, non smooth one, which we denote by E_0 (see [239]). It is defined on $[\mathbb{N}]^{\mathbb{N}}$ as the relation of eventual agreement, i.e.:

$$AE_0B \Leftrightarrow \exists_n A/n = B/n$$

To see that E_0 is non smooth, suppose towards a contradiction that $f : [\mathbb{N}]^{\mathbb{N}} \to \mathbb{R}$ is a Borel function such that $AE_0B \Leftrightarrow f(A) = f(B)$. Then there is some infinite $C \subset \mathbb{N}$ such that the restriction of f to $[C]^{\mathbb{N}}$ is continuous. But, as the equivalence class of C is dense in $[C]^{\mathbb{N}}$, this means that f is constant on $[C]^{\mathbb{N}}$, contradicting that $[C]^{\mathbb{N}}$intersects more than one equivalence class.

On the other hand any uncountable Borel set $B \subset [\mathbb{N}]^{\mathbb{N}}$ of pairwise almost disjoint sets will witness that E_0 has a continuum of classes.

From this it follows that any analytic equivalence relation to which E_0 reduces has a continuum of classes, but does not admit a classification by real numbers.

Theorem (6.1.2)[234]: Let X be a Banach space with an unconditional basis (e_n). If E_0 does not Borel reduce to isomorphism between subspaces generated by subsequences of the basis (and in particular if these admit a classification by real numbers), then any space spanned by a subsequence is isomorphic to its square and hyperplanes. Furthermore, there is a subsequence of the basis such that all of its subsequences span isomorphic spaces.

For example, as the usual basis of Tsirelson's space does not have a sub- sequence all of whose subsequences span isomorphic spaces, this shows that there is no isomorphic classification of the subspaces of Tsirelson's space by real numbers.

This result can be coupled with Gowers' dichotomy [235] proving:

Theorem (6.1.3)[234]: Let X be a separable Banach space. Either E_0 Borel reduces to isomorphism between its subspaces or X contains a reflexive subspace with an unconditional basis all of whose subsequences span isomorphic spaces.

For the above we will need some Ramsey type results for product spaces and some constructions for reducing E_0. These results seem to have an independent interest apart from their applications to Banach space theory in that they classify minimal counter examples to Ramsey properties in product spaces. Let us just state one of these:

Theorem (6.1.4)[234]: Let E be an analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ invariant under finite changes. Either E_0 Borel reduces to E or E admits a homogeneous set.

Let X be some separable Banach space and (e_i) a non zero sequence in X. We say that (e_i) is a basis for X if any vector x in X can be uniquely written as a norm convergent series $x = \sum a_i e_i$. In that case, the biorthogonal functionals $e_k^*(\sum a_i e_i) = a_k$ and the projections $P_k(\sum a_i e_i) = \sum_{i=0}^{n} a_i e_i$ are in fact continuous and moreover their norms are uniformly bounded.

If (e_i) is some non zero sequence that is a basis for its closed linear span, written $[e_i]$, we say that it is a basic sequence in X. The property of (e_i) being a basic sequence can also equivalently be stated as the existence of a constant $K \geq 1$ such that for any $n \leq m$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$:

$$\left\|\sum_{i=0}^{n} a_i e_i\right\| \leq K \left\|\sum_{i=0}^{n} a_i e_i\right\|$$

Suppose furthermore that for any $x = \sum a_i e_i$ the series actually con-verges unconditionally, i.e., for any permutation σ of $\mathbb{N}$ the series $\sum a_{\sigma(i)} e_{\sigma(i)}$ converges to x. Then the basic sequence is said to be unconditional.

Again, being an unconditional basis for some closed subspace (which will be denoted by 'unconditional basic sequence') is equivalent to there being a constant $K \geq 1$, such that for all $n, A \subset \{0, \ldots, n\}$ and $a_0, a_1, \ldots, a_m \in \mathbb{R}$

$$\left\|\sum_{i \in A}^{n} a_i e_i\right\| \leq K \left\|\sum_{i=0}^{n} a_i e_i\right\|$$

We will in general only work with normalised basic sequences, i.e., $\|e_i\| \equiv 1$, which always can be obtained by taking $e_i' = \frac{e_i}{\|e_i\|}$.

Given some vector $x \in \mathrm{span}(e_i)$ let its support, $\mathrm{supp}(x)$, be the set of indices i with $e_i^*(x) \neq 0$. For $k \in \mathbb{N}$ and $x, y \in \mathrm{span}(e_i)$ we write $k < x$ if $k < \min \mathrm{supp}(x)$ and $x < y$ if $\max \mathrm{supp}(x) < \min \mathrm{supp}(y)$. A block basis, (x_i), over a basis (e_i) is a finite or infinite sequence of vectors in span(e_i) with $x_0 < x_1 < x_2 < \ldots$. This sequence will also be basic and in fact unconditional in case (e_i) is so.

Two basic sequences $(e_i)$and (t_i) are called equivalent, in symbols $(e_i) \approx (t_i)$, provided a series Paiei converges if and only if $\sum a_i t_i$ converges. This can also be stated as saying that $T : e_i \mapsto t_i$ extends to an invertible linear operator between $[e_i]$ and $[t_i]$. The quantity $\|T\| \cdot \|T^{-1}\|$ is then the constant of equivalence between the two bases.

A basis that is equivalent to all of its subsequences is said to be sub-symmetric. A simple diagonalisation argument then shows that it must be uniformly equivalent to all of its subsequences.

Two basic sequences (e_i) and (t_i) are said to be permutatively equivalent if there is some permutation σ of $\mathbb{T}$ such that (e_i) and $(t_{\sigma(i)})$ are equivalent.

Two Banach spaces X and Y are called incomparable in case neither of them embed isomorphically into the other. X is said to be minimal if it embeds into all of its infinitely dimensional subspaces and X itself is infinite dimensional.

The proof of the first theorem will proceed by a reduction to an analysis of Borel partial orders due to L. Harrington, D. Marker and S. Shelah (see [240]). Instrumental in our reduction will be the determinacy result of Gowers on certain games in Banach spaces (see [235]), which will guarantee that some choices can be done uniformly, a fact that is needed for definability purposes. Moreover, we will use some ideas of J. Lopez-Abad on coding reals with inevitable subsets of the unit sphere of a Banach space (see [241).

We mention that it was shown by a simpler argument in [242] by V. Ferenczi and the author that any Banach spaces either contains a minimal subspace or a continuum of non isomorphic subspaces.

For facility of notation we write $X \sqsubseteq Y$ if X embeds isomorphically into Y and will always suppose the spaces we are working with to be separable infinite dimensional. Then $\sqsubseteq$ restricted to the standard Borel space of sub- spaces of some separable Banach space becomes an analytic quasi-order, i.e., transitive and reflexive. So the result above amounts to saying that either $\sqsubseteq$ has a minimal element or a perfect antichain.

Suppose (e_i) is a normalised basic sequence with norm denoted by $\|\cdot\|$. We call a normalised block vector x with finite support rational if it is a scalar multiple of a finite linear combination of (e_i) with rational coordinates.Notice that there are only countably many rational (finite) block vectors, which we can gather in a set Q and give it the discrete topology. Let $bb_Q(e_i)$ be the set of block bases of (e_i) consisting of rational normalised block vectors, which is easily seen to be a closed subspace of $Q^{\mathbb{N}}$, which is itself a Polish space. Moreover the canonical function sending $X \in bb_Q(e_i)$ to its closed span in $\mathfrak{P}$ is Borel, so the relations of isomorphism, etc.,become analytic $bb_Q(e_i)$.

We recall the following classical facts: Any infinite dimensional Banach space contains an infinite normalised basic sequence (e_i). Moreover, if Y is any subspace of $[e_i]$, then it contains an isomorphic perturbation of a block basic sequence of (e_i). Again any block basic sequence is equivalent to some member of $bb_Q(e_i)$. So this explains why we can concentrate on $bb_Q(e_i)$ if we are only looking for minimal subspaces.

For $X, Y \in bb_Q(e_i)$, let $X \leq Y$ if X is a blocking of Y, i.e., if any element of X is a linear combination over . Note that this does not imply that they are rational block vectors over Y, but only over (e_i). Moreover, if $Y = (y_i), X = (x_i) \in bb_Q(e_i)$, put $Y \leq^* X$ if for some $k, (y_i)i \geq k \leq X$. Also, for $\Delta = (\delta_i)$ an infinite sequence of strictly positive reals write $d(X, Y) < \Delta$ if $\forall i \|x_i - y_i\| < \delta_i$.

Put $X \approx Y$ if the bases are equivalent and $X \cong Y$ if they span isomorphic spaces. Then a classical perturbation argument shows that there is some Δ depending only on the constant of the basis, such that for any $X, Y \in bb_Q(e_i)$ if $d(X, Y) < \Delta$, then $X \approx Y$ and in particular $X \cong Y$. Put also $X = (x_i) \doteq Y = (y_i)$ if $\exists k \forall i \geq k x_i = y_i$. Then evidently $X \doteq Y$ implies $X \approx Y$.

For a subset $A \subset bb_Q(e_i)$let $A^* = \{Y \in bb_Q(e_i) | \exists X \in A\, X \doteq Y\}$ and $A_\Delta = \{Y \in bb_Q(e_i) | \exists X \in A\, d(X, Y) < \Delta\}$. Notice that if A is analytic so are both A^* and A_Δ. Again $[Y] = \{X \in bb_Q(e_i) | X \leq Y\}$. Such an A is said to be large in $[Y]$ if for any $X \in [Y]$ we have $[X] \cap A \neq \emptyset$.

For (e_i) a given normalised basis, $A \subset bb_Q(e_i)$ and $X \in bb_Q(e_i)$, the Gowers game G_X^A is defined as follows: Player I plays in the k'th move of the game a rational normalised block vector yk of (e_i) such that $y_{k-1} < y_k$ and y_k is a block on X. Player II responds by either doing nothing or playing a rational normalised block vector x such that $x \in [y_{l+1}, \ldots, y_k]$ where l was the last move where II played a vector. So player II wins the game if in the end she has produced an infinite rational block basis $X = (x_i) \in A$. This is an equivalent formulation due to J. Bagaria and J. Lopez-Abad (see [243]) of Gowers' original game.

Gowers [235] proved that if $A \subset bb_Q(e_i)$ is analytic, large in $[Y]$ and Δ is given, then for some $X \in [Y]$ II has a winning strategy in the game $G_X^{A_\Delta}$.

We mention also a result of Odell and Schlumprecht [244] obtained from their solution to the distortion problem: If E is an infinite dimensional Banach space not containing c0, there are an infinite dimensional subspace F and $A, B \subset S_F$ of positive distance such that any infinite dimensional sub-space of F intersects both A and B.

The following was shown in [242]:

Lemma (6.1.5)[234]: (MA) Let $A \subset bb_Q(e_i)$ be linearly ordered under $\leq^*$ of cardinality strictly less than the continuum. Then there is some $X \in bb_Q(e_i)$ such that $X \leq^* Y$ for all $Y \in A$.

From this lemma one gets the following:

Lemma (6.1.6)[234]: (MA + ¬CH) Suppose $W \subset bb_Q(e_i)$ is a $\sum_2^1$ set, large in some $[Y]$ and $\Delta > 0$. Then II has a winning strategy in $G_X^{W_\Delta}$ for some $X \in [Y]$.

Proof : Let $W = \bigcup_{\omega_1} V_\xi$ be a decomposition of W as an increasing union of $\mathcal{N}_1$ Borel sets. We claim that some V_ξ^* is large in $[Z]$ for some $Z \in [Y]$, which by Gowers' theorem will be enough to prove the lemma. So suppose not and find $Y_0 \in [Y]$ such that $[Y0] \cap V_0^* = \emptyset$. Repeating the same process and diagonalising at limits, we find $Y_\xi \in [Y]$ for $\xi < \omega_1$ such that $[Y_\xi] \cap V_\xi^* = \emptyset$ and $Y_\xi \leq^* Y_\eta$ for $\eta < \xi$. By the above lemma there is some $Y_\infty = (y_i) \in [Y]$ with $Y_\infty \leq^* Y_\xi$ for all $\xi < \omega_1$.

We claim that $[(y_{2i})] \cap W = \emptyset$. Otherwise, for $Z = (z_i) \in [(y_{2i})] \cap W$ find $\xi < \omega_1$ such that $Z \in V_\xi$. Now as $(y_i) \leq^* Y_\xi$ there is some k with $(y_i)i \geq k \leq Y_\xi$, but then

$$(y_{2i}) \asymp (y_k, y_{k+1}, y_{k+2}, \ldots, y_{2k-1}, y_{2k}, y_{2(k+1)}, y_{2(k+2)}, \ldots) \leq Y_\xi$$

One now easily sees that there is some (x_i) with

$$(z_i) \asymp (x_i) \leq ((y_k, y_{k+1}, y_{k+2}, \ldots, y_{2k-1}, y_{2k}, y_{2(k+1)}, y_{2(k+2)}, \ldots)$$

whereby $(x_i) \in V_\xi^*$ contradicting $V_\xi^* \cap [Y_\xi] = \emptyset$.

Therefore $[(y_{2i})] \cap W = \emptyset$, again contradicting the largeness of W.

Lemma (6.1.7)[234]: (MA + ¬CH) Suppose that (e_i) is a basic sequence such that $[e_i]$ does not contain a minimal subspace. Then for any $Y \in bb_Q(e_i)$ there are a $Z \in [Y]$ and a Borel function $g : [Z] \to [Z]$, with $g(X) \leq X$ and $X \not\sqsubseteq g(X)$ for all $X \in [Z]$.

Proof : As c_0 is minimal, $[e_i]$ does not contain c_0. Therefore, by the solution to the distortion problem by Odell and Schlumprecht, we can by replacing (e_i) by a block suppose that we have two positively separated sets F_0, F_1 of the unit sphere, such that for any $X \in bb_Q(e_i)$ there are rational normalized blocks x, y on X with $x \in F_0$ and $y \in F_1$. We call such sets inevitable.

Let $D = \{X = (x_i) \in bb_Q(e_i) \mid \forall i\, x_i \in F_0 \cup F_1\}$ and for $X \in D$ let $\alpha(X) \in 2^{\mathbb{N}}$ be defined by $\alpha(X)(i) = 0 \leftrightarrow x_i \in F_0$. Then D is easily seen to be a closed subset of $bb_Q(e_i)$ and $\alpha: D \to 2^{\mathbb{N}}$ to be continuous. Furthermore by the inevitability of F_0 and F_1 we have that D is large in every $[Y]$.

Let $Q_*^{<N}$ be the set of finite non identically zero sequences of rational numbers given the discrete topology. Then $(Q_*^{<N})^{\mathbb{N}}$ is Polish. Define for any $Y \in bb_Q(e_i)$ and $(\lambda_i) \in (Q_*^{<N})^{\mathbb{N}}$ the block basis $(\lambda_i) \cdot Y$ of Y in the obvious way, by taking the linear combinations given by (λ_i).

Fix also some perfect set P of almost disjoint subsets of $\mathbb{N}$ seen as a subset of $2^{\mathbb{N}}$ and let $\beta : P \leftrightarrow (Q_*^{<N})^{\mathbb{N}}$ be a Borel isomorphism.

Again $E = \{X \in D \mid \alpha(X) \in P\}$ in large and closed in $bb_Q(e_i)$.

Then the set

$$W = \{X = (x_i) \in bb_Q(e_i) \mid (x_{2i}) \in E \wedge (x_{2i+1}) \not\sqsubseteq \beta \circ \alpha((x_{2i})) \cdot (x_{2i+1})\}$$

is coanalytic. We claim moreover that it is large in $bb_Q(e_i)$.

To see this, let $Y \in bb_Q(e_i)$ be given and take by inevitability of F_0 and F_1 some $(z_i) \in [Y]$ with $z_{3i} \in F_0$ and $z_{3i+1} \in F_1$. As $[z_{3i+2}]$ is not minimal there is some $X \leq (z_{3i+2})$ such that $(z_{3i+2}) \not\sqsubseteq X$. Take some $(\lambda_i) \in (Q_*^{<N})^{\mathbb{N}}$ such that $(\lambda_i) \cdot (z_{3i+2}) \approx X$ and $(z_{3i+2}) \not\sqsubseteq (\lambda_i) \cdot (z_{3i+2})$. We can now define some (v_i) such that either $v_{2i} = z_{3i}$ or $v_{2i} = z_{3i+1}$, $\beta \circ \alpha((v_{2i})) = (\lambda_i)$ and $v_{2i+1} = z_{3i+2}$. This ensures that $(v_i) \in W$. So as $(v_i) \leq (z_i)$ it is in $[Y]$ and W is indeed large.

Take now some $\Delta = (\delta_i)$ depending on the basic constant as above with $\delta_i < \frac{1}{2} d(F_0, F_1)$. By the preceding lemma we can find a $Y \in bb_Q(e_i)$ such that II has a winning strategy σ in the game $G_X^{W_\Delta}$.

Suppose that $= (x_i)$ has been played by II according to the strategy σ as a response to Z played by I. As σ is winning, $X \in W_\Delta^*$. Define $\gamma(X) \in 2^{\mathbb{N}}$ by $\gamma(X)(i) = 0$ if $d(x_{2i}, F_0) < \delta_{2i}$ and $\gamma(X)(i) = 1$ otherwise. Then γ is Borel from W_Δ^* to $2^{\mathbb{N}}$, and furthermore there is a unique $\gamma^*(X) \in P$ such that $\exists k \, \forall i \geq k \, \gamma(X)(i) = \gamma^*(X)(i)$. This is because P was chosen to consist of almost disjoint subsets of ℕ. Again $X \mapsto \gamma^*(X)$ is Borel.

Take some $U = (u_i) \in W$ such that $\forall^\infty n \, \|u_n - x_n\| < \delta_n$. Then $\alpha(U) = \gamma^*(X), (u_{2i+1}) \approx (x_{2i+1})$ and $(u_{2i+1}) \not\sqsubseteq \beta \circ \alpha(U) \cdot (u_{2i+1})$. So due to the equivalence invariance of the basis by Δ perturbations we have $(x_{2i+1}) \not\sqsubseteq \beta \circ \gamma^*(X) \cdot (x_{2i+1})$.

Let $V \in [X]$ be the normalisation of $\beta \circ \gamma^*(X) \cdot (x_{2i+1})$. The function

$g : Z \mapsto V$ is Borel and obviously $V \approx \beta \circ \gamma^*(X) \cdot (x_{2i+1}) \leq (x_{2i+1}) \leq Z$ and as $(x_{2i+1}) \not\sqsubseteq \beta \circ \gamma^*(X) \cdot (x_{2i+1})$ also $Z \not\sqsubseteq V$.

A Banach space is called quasi-minimal if any two subspaces have further isomorphic subspaces. The following is a standard observation.

Lemma (6.1.8)[234]: Suppose $[e_i]$ is quasi-minimal. Then ⊑ is downwards σ-directed on $bb_Q(e_i)$, i.e., any countable family has a common minorant.

Proof : Suppose that $Y_i \in bb_Q(e_i)$ are given, then define inductively $Z_i \in [Y_0]$ such that $Z_i \sqsubseteq Y_i$ and $Z_{i+1} \leq Z_i$. Take some $Z = (z_i) \leq^* Z_n$ for all n and notice as in the proof of Lemma (6.1.6) that $(z_{2i}) \sqsubseteq Z_n$ for all n.

Lemma (6.1.9)[234]: If R is a downwards σ-directed Borel quasi-order on a standard Borel space X. Then either R has a perfect antichain or a minimal element.

Proof : This is a simple consequence of the results of L. Harrington, D. Marker and S. Shelah [240], as we will see. Suppose that R did not have a perfect antichain, then by their results there is a countable partition $X = \bigcup X_n$ into Borel sets, so that R is total on each piece, i.e., R can be written as a countable union of R-chains.

Applying another of their results this implies that for some countable ordinal α there are Borel functions $f_n : X_n \to 2^\alpha$, such that for any $x, y \in X_n$:

$$yRx \Leftrightarrow x \leq_{lex} y$$

Where $\leq_{lex}$ is the usual lexicographical ordering. In their terminology, R is linearisable on each X_n.

One can easily check that any subset of 2^α has a countable subset cofinal with respect to $\leq_{lex}$, so pulling it back by fn it becomes coinitial in $R \upharpoonright Xn$. Putting all these sets together one gets a countable subset of X coinitial with respect to R. So by downwards σ-directedness there is therefore a minimal element in X.

After this series of lemmas we can now prove the theorem:

Theorem (6.1.10)[234]: Let X be an infinite dimensional Banach space. Then X contains either a minimal subspace or a continuum of pairwise incomparable subspaces.

Proof : By Gowers' quadrichotomy X contains either a quasi-minimal sub- space or a subspace with a basis such that any two disjointly supported subspaces are totally incomparable (see Gowers [235] theorem 7.2 and the fact that H.I. spaces are quasi-minimal). In the latter case any perfect set of almost disjoint subsets of N will give rise to subsequences of the basis spanning totally incomparable spaces, which would prove the theorem. So we can suppose that $X = [e_i]$ is quasi-minimal for some basis (e_i). If X does not contain a minimal subspace, we can choose $Z \in bb_Q(e_i)$ and the Borel function as above (under MA + ¬CH of course). So define the following property on subsets A, B of $[Z]^2$:

$$\Phi(A, B) \Leftrightarrow$$

$$\forall Y, V, W \in [Z]\, [(Y, V) \notin A \vee (V, W) \notin A \vee (Y, W) \notin B] \wedge \forall Y \in [Z] (Y, g(Y)) \notin A$$

We see that Φ is $\prod_1^1$ on $\sum_1^1$, hereditary and continuous upwards in the second variable. Furthermore, $\Phi(\sqsubseteq, \not\sqsubseteq)$, so by the second reflection theorem (see Kechris [245] theorem (35.16)) there is some Borel set R containing $\sqsubseteq$ such that $\Phi(R, \complement R)$. But then R is a Borel quasi-order, downwards σ-directed, as it contains $\sqsubseteq$, and without a minimal element, as witnessed by g. So R has a perfect antichain by the previous lemma, which then is an antichain for $\sqsubseteq$ too.

The statement is therefore proved under the additional hypothesis of Martin's axiom and the negation of the continuum hypothesis. We will see that this is in fact sufficient to prove the theorem. By standard metamathematical facts and Shoenfield's absoluteness theorem it is enough to show that the statement we wish to prove is $\sum_2^1$.

It was proved by Ferenczi and the author in [242] that the property of having a block minimal subspace was $\sum_2^1$. For using Gowers' determinacy result and codings as above, one can continuously find an isomorphism between the space and a certain subspace to testify the minimality. This proof can trivially be modified to show that the property of having a minimal (i.e., not necessarily block minimal) subspace is also $\sum_2^1$. For now we only have to choose not a code for a subspace and an isomorphism, but a code for a subspace and an embedding. For the convenience of the reader, we have included the proof of this in(see[234]).

On the other hand, the property of having a perfect antichain is obviously $\sum_2^1$ by just counting quantifiers. So these remarks finish the proof.

We will show two Ramsey type results and afterwards some applications to Banach space theory.

It is well known that there are no nice Ramsey properties for the product space $[\mathbb{N}]^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}$ in contradistinction to the simple Ramsey space $[\mathbb{N}]^{\mathbb{N}}$. That is, there are even quite simple relations not admitting a square $[A]^{\mathbb{N}} \times [B]^{\mathbb{N}}$ that is either included in or disjoint from the relation. An example of this is the oscillation relation $\mathcal{O}$ defined by

$$(a_n)\mathcal{O}(b_n) \iff$$
$$\exists N \, \forall n[\, \#(k \,| a_n < b_k < a_{n+1}) \leq N \wedge \#(k \,|\, b_n < a_k < b_{n+1}) \leq N]$$

Where $[\mathbb{N}]^{\mathbb{N}}$ is seen as the space of strictly increasing sequences of integers (a_i).

The situation is very different if one replaces one of the factor spaces by other Ramsey spaces and there are now very deep positive theorems on so called polarised partition relations.

We are interested in the case when the relation on the product is in fact a definable equivalence relation. Here the right question seems to be when there is a cube $[A]^{\mathbb{N}}$ contained in one class. Now if one lets two subsets of $\mathbb{N}$ be equivalent iff they have the same minimal element, then the relation has exactly $\mathcal{N}_0$ classes and does not admit a homogeneous set.

On the other hand if the relation is invariant under finite changes, such as E_0, then there are bigger chances that it should have a homogeneous set. We will show that in the case of analytic equivalence relations, E_0 is in fact the minimal counterexample to the Ramsey property, in the sense that, if an analytic equivalence relation is invariant under finite changes and does not admit a homogeneous set, then it Borel reduces E_0. In the same vein it is shown that if an analytic equivalence relation does not admit a cube on which it has only countably many classes, then it has at least a perfect set of classes. We notice that both of these results are relatively direct consequences of the Silver and Glimm-Effros dichotomies in the case of the equivalence relation being Borel. But our results are motivated by applications to isomorphism of separable Banach spaces, which is true analytic, and the dichotomies are known not to hold in this generality.

The following result was also found independently by S. Todorcevic, albeit with a somewhat different proof:

Theorem (6.1.11)[234]: Let E be an analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$. Then either
E has a continuum of classes or there is some $A \in [\mathbb{N}]^{\mathbb{N}}$ such that E only has a countable number of classes on $[A]^{\mathbb{N}}$.

Moreover, A's E_0 class will be a complete section for E on $[A]^{\mathbb{N}}$.

Proof : We will prove the theorem under MA + ¬CH. By Burgess' theorem (Exercise (35.21) in [238]) we can suppose that E has at most $\mathcal{N}_1$ classes $(C_\xi)_{\omega_1}$. Define $P_\xi(A) \leftrightarrow [A]_{E_0} \cap C_\xi \neq \emptyset$ and notice that this an analyti1c E_0-invariant property. We can by simple diagonalisation find $(A_\xi)_{\omega_1}, A_\xi \subset^* A_\eta$ for $\eta < \xi < \omega_1$ such that $\forall \xi < \omega_1$ either $[A_\xi]^{\mathbb{N}} \subset P_\xi$ or $[A_\xi]^{\mathbb{N}} \subset CP_\xi$. And by MA + ¬CH there is an $A \subset^* A_\xi, \forall \xi < \omega_1$.

Notice now that by E_0-invariance of P_ξ, if $B \subset^* A$ and $[A]^{\mathbb{N}} \subset P_\xi$ or $[A]^{\mathbb{N}} \subset CP_\xi$ then also $[B]^{\mathbb{N}} \subset P_\xi$, respectively $[B]^{\mathbb{N}} \subset CP_\xi$. So therefore $\forall \xi < \omega_1 [A]^{\mathbb{N}} \subset P_\xi$ or $[A]^{\mathbb{N}} \subset CP_\xi$.

Suppose now that $B \in [A]^{\mathbb{N}}, B \in C_\xi$ then $P_\xi(B)$ and therefore $[A]^{\mathbb{N}} \subset P_\xi$ and $P_\xi(A)$, i.e., $\exists A' E_0 A \; A' E B$. This means that $[A]_{E_0}$ is a complete section for E on $[A]^{\mathbb{N}}$.

Let us now see that the statement of the theorem is absolute. Saying that E has a continuum of classes is equivalent to saying that there is a compact perfect set $K \subset [\mathbb{N}]^{\mathbb{N}}$ consisting of pairwise E-inequivalent points:

$$\exists K \subset [\mathbb{N}]^{\mathbb{N}} \text{ compact, perfect } \forall x, y \in K (x = y \vee xEy)$$

This is obviously a $\sum_2^1$ statement.

For the other case, notice that as $[A]_{E_0}$ is a complete section for E on $[A]^{\mathbb{N}}$ there is by the Jankov-von Neumann selection theorem a C-measurable selector $f : [A]^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and a Borel set $B \subset [\mathbb{N}]^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $E = \pi_{[\mathbb{N}]^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}} B$, such that for $D \in [A]^{\mathbb{N}}$ there is an $A' E_0 A$ with $B(D, A', f(D))$. That is, we can choose a witness to D being E equivalent to some $A' E_0 A$ in a C-measurable way. But any C-measurable function can, using Ellentuck's theorem, be rendered continuous on a cube, i.e., there is some $B \in [A]^{\mathbb{N}}$ such that f's restriction to $[B]^{\mathbb{N}}$ is continuous. So by the proof above the E-classes on $[A]^{\mathbb{N}}$ are the same as the E-classes on $[B]^{\mathbb{N}}$ and the other possibility can be written as:

$$\exists B \, \exists f : [B]^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \text{ continuous } \forall D \in [B]^{\mathbb{N}} \, \exists B' E_0 B \; B(D, B', f(D))$$

This statement is $\sum_2^1$ as the quantifier $\exists B' E_0 B$ is over a countable set, so by Shoenfield absoluteness and standard metamathematical facts it is enough to prove the result under MA + ¬CH.

The next results render explicit the connection with the Borel reducibility ordering.

Definition (6.1.12)[234]: For $A, B \subset \mathbb{N}$ set $A E_0' B$ iff $\exists n \, |A \cap n| = |B \cap n| \wedge A \backslash n = B \backslash n$

It is easy to see that the equivalence class of any infinite-coinfinite subset of N is dense in $[\mathbb{N}]^{\mathbb{N}}$ and in fact the equivalence relation is generically ergodic. Moreover, E_0' is just a refinement of E_0.

Lemma (6.1.13)[234]: E_0' is generically ergodic (i.e., any invariant set with the Baire property is either meagre or comeagre) and all classes $[A]_{E_0'}$, for A infinite- coinfinite, are dense.

Proof : Since $[\mathbb{N}]^{\mathbb{N}}$ is cocountable in $2^{\mathbb{N}}$ we can restrict our attention to it. Suppose that some invariant set A is non meagre, then there is some $a \subset [0,n]$ such that A is comeagre in $D_{a,n} = \{A \in [\mathbb{N}]^{\mathbb{N}} | A \cap [0,n] = a\}$. So for any $D_{b,m}$ there are $c, d \in [0,k]$; $\max(n,m) < k$ such that $a \subset c, b \subset d, |c| = |d|$. Now for any $A \in [\{k+1, k+2, \dots\}]^{\mathbb{N}}$ we have $\phi(c \cup A) = (d \cup A) E_0' (c \cup A)$ and ϕ is a homeomorphism of $D_{c,k} \subset D_{a,n}$ with $D_{d,k}$. But that means that the image of A is comeagre in $D_{d,k} \subset D_{b,m}$ and is included in the saturation of A, which is A. So A is comeagre in the space.

If A is infinite-coinfinite, then for any $D_{a,n}$ there are $b, c \subset [0,k]$; $b \supset a, n < k, A \cap [0,k] = c, b \cap [0,n] = a$ and $|b| = |c|$. So $A = (c \cup A/k) E_0' (b \cup A/k) \in D_{b,k} \subset D_{a,n}$. And its class is dense.

Proposition (6.1.14)[234]: Let E be a meagre equivalence relation on $2^{\mathbb{N}}$ containing E_0'. Then $E_0 \leq_B E$.

Proof. Let (D_n) be a decreasing sequence of dense open sets, such that $E \cap \bigcap_n D_n = \emptyset$.

We will inductively construct sequences $b_0^n, b_1^n \in 2^{<\mathbb{N}}$ for $n \in \mathbb{N}$ such that for all $n, |b_0^n| = |b_1^n|, \overline{\overline{b_0^n}} = \overline{\overline{b_1^n}} := \#\{k \,|\, b_1^n(k) = 1\}$. And if $a_s = b_{s(0)}^0 {}^\frown \dots {}^\frown b_{s(|s|-1)}^{|s|-1}$ for all $s \in 2^{<N}$, then for any $s, t \in 2^n, N_{a_{s^\frown 0}} \times N_{a_{t^\frown 1}} \subset D_{n+1}$.

Suppose that this can be done. Then define $\alpha \longmapsto \cup_n a_{\alpha\lceil n} = a_\alpha$. This is clearly continuous. If now $\neg \alpha E_0 \beta$, then for infinitely many $n, \alpha(n) \neq \beta(n)$. So for these n $(a_\alpha, a_\beta) \in N_{a_{\alpha\restriction n+1}} \times N_{a_{\beta\restriction n+1}} \subset D_{n+1}$, which implies that $(a_\alpha, a_\beta) \in \bigcap_k D_k \subset \complement E$.

Conversely, if $\alpha E_0 \beta$, then for some N, we have $\forall n \geq N \, \alpha(n) = \beta(n)$. But then easily $a_\alpha = a_{\alpha\restriction N} {}^\frown b_{\alpha(N)}^N {}^\frown b_{\alpha(N+1)}^{N+1} \dots$ and $a_\beta = a_{\alpha\restriction N} {}^\frown b_{\alpha(N)}^N {}^\frown b_{\alpha(N+1)}^{N+1} \dots$, so by the construction, $a_\alpha E_0' \beta$.

Now for the constructinon: Suppose that b_0^n, b_1^n have been chosen for $\forall m < n$, enumerate $2^n \times 2^n$ by $(s_0, t_0), \dots, (s_k, t_k)$ and take $c_0^0, c_1^0 \in 2^{<N}$ such that $Na_{s_0{}^\frown c_0^0} \times Na_{t_0{}^\frown c_1^0} \subset D_n$. This can be done as D_n is dense and open in the product.

Prolong c_0^0, c_1^0 to c_0^1, c_1^1 respectively in such a way that $Na_{s_1{}^\frown c_0^1} \times Na_{t_1{}^\frown c_1^1} \subset D_n$.

Again, prolong c_0^1, c_1^1 to c_0^2, c_1^2 respectively in such a way that $Na_{s_2{}^\frown c_0^2} \times Na_{t_2{}^\frown c_1^2} \subset D_n$, etc.

Finally, prolong c_0^k, c_1^k to b_0^n, b_1^n respectively, such that $|b_0^n| = |b_1^n|, \overline{\overline{b_0^n}} = \overline{\overline{b_1^n}}$.

This finishes the construction.

For the following, we recall that $\forall^* x\, R(x)$ means that the set $\{x|R(x)\}$ is comeagre, where x varies over some Polish space.

Theorem (6.1.15)[234]: Let E be an analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ such that $E_0' \subset E$, i.e., E is E_0' -invariant. Then either $E_0 \leq_c cE$ or there is some $A \in [\mathbb{N}]^{\mathbb{N}}$ such that E only has one class on $[A]^{\mathbb{N}}$.

Proof : By corollary 3.5 of [10], if $E_0 \nleq_c E$, then E will be a decreasing intersection of $\mathcal{N}_1$ smooth equivalence relations:

$$E = \bigcap_{\omega_1} E_\xi \ , E_\xi \subset E_\eta \ , \eta < \xi < \omega_1$$

Let f_ξ: $[\mathbb{N}]^{\mathbb{N}} \to R$ be a Borel reduction of E_ξ to identity on $\mathbb{R}$. Then for any $A \in [\mathbb{N}]^{\mathbb{N}}$, there is a $B \in [A]^{\mathbb{N}}$ such that $f \upharpoonright [B]^{\mathbb{N}}$ is continuous. But since there is a dense E_ξ-class the function has to be constant, that is, there is only one class.

We construct inductively a $\subset^*$-decreasing sequence $(A_\xi)_{\omega_1}$ of infinite sub- sets of $\mathbb{N}$, with each A_ξ being homogeneous for E_ξ. Under MA + ¬CH such a sequence can be diagonalised to produce an infinite $A_\infty \not\subseteq^* A_\xi, \forall \xi < \omega_1$. Now as $A_\infty \not\subseteq^* A_\xi$ it is easily seen that A_∞ is E_0' -equivalent with some subset of A_ξ and therefore also E_ξ-equivalent with A_ξ itself. Furthermore, the same holds for any infinite subset of A_∞, so A_∞ is homogeneous for all of the E_ξ and therefore for E too.

As before one sees that the property of having a homogeneous set is $\sum_2^1$, so we need only check that continuously reducing E_0 is $\sum_2^1$. But this can be written as:

$$\exists f : [N]^N \to [N]^N \text{ continuous } [\ \forall^* \alpha \in [N]^N\ \forall \beta E_0 \alpha\ \forall \gamma E_0 \alpha\ f(\beta) E f(\gamma)$$
$$\wedge\ \forall \alpha, \beta \in [N]^N \{\alpha E_0 \beta \vee \neg f(\alpha) E f(\beta)\}]$$

So as the quantifier $\forall \beta E_0 \alpha$ is over a countable set and that the category quantifier $\forall *$ preserves analyticity (see Theorem (29.22) in [238]), the statement is $\sum_2^1$.

Let (e_i) be some basic sequence in a Banach space X and define the following equivalence relation on $[N]^N$: $A \cong B \iff [e_i]_A \cong [e_i]_B$. Then $\cong$ is analytic and extends E_0' . For suppose that $A E_0' B$. Then $[e_i]_A$ and $[e_i]_B$are spaces of the same finite codimension in $[e_i]_{A \cup B}$ and are therefore isomorphic. So, using the proposition, one sees that if $E_0 \nleq B \cong$, then $\cong$ must be non meager and therefore by Kuratowski-Ulam have a non meagre class, which again by the lemma is comeagre.

To avoid trivialities, let us in the following suppose that all Banach spaces considered are separable, infinite dimensional.

Gowers showed the following amazing result about the structure of sub-spaces of a Banach space: if X is a Banach space, then it contains either an unconditional basic sequence or an H.I. subspace [235].

Here an H.I. (hereditarily indecomposable) space Y is one in which no two infinite dimensional subspaces form a direct sum. This property, which passes to subspaces, insures that Y cannot be isomorphic to any of its sub- spaces and cannot contain any unconditional basic sequence. Therefore in the classification of the subspaces of a Banach space one can always suppose to be dealing with an H.I. space or a space with an unconditional basis.

Proposition (6.1.16)[234]: Let (e_i) be a basic sequence in a Banach space. Then either E_0 Borel reduces to isomorphism of spaces spanned by subsequences of the basis or there will be some infinite $A \subset \mathbb{N}$, such that for any infinite $B \subset A$: $[e_i]_A \cong [e_i]_A$.

Example (6.1.17)[234]: Hereditarily Indecomposable spaces.

Suppose that we are given a hereditarily indecomposable space X. Then as any Banach space contains a (conditional) basic sequence, we can suppose that we have a basis (e_i). By the above proposition, if E_0 does not reduce, there would be a subsequence spanning a space isomorphic to some proper subspace in contradiction with the properties of H.I. spaces. So E_0 reduces to isomorphism of its subspaces. The same reasoning shows, using the first theorem, that it has a continuum of incomparable subspaces.

A recent result due to Ferenczi and Galego [237] says that E_0 Borel reduces to the isomorphism relation between subspaces of c_0 and ℓ_1. So if E_0 does not reduce to isomorphism between the subspaces of an Banach space, then using Gowers' dichotomy we can find a subspace with an unconditional basis. Therefore by James' characterisation of reflexivity this basis must span a reflexive space. All in all this gives us the following:

Theorem (6.1.18)[234]: Let X be an Banach space such that the isomorphism relation between its subspaces does not reduce E_0. Then X contains a reflexive sub- space with an unconditional basis, all of whose subsequences span isomorphic spaces.

Let us notice that if a basis (e_i) has the property that no two disjointly supported block basic sequences are equivalent, then one can easily show that this basis has the Casazza property and moreover that it satisfies

$$(e_i)_A \approx (e_i)_B \iff (e_i)_A \cong (e_i)_B \iff A E_0' B$$

See the work of Gowers and Maurey, [247], for unconditional examples of such bases. So as $E_0'$and E_0 are Borel bi-redicible, there are bases on which both equivalence and isomorphism between subsequences are exactly of complexity E_0.

We will now see an extension of some results by Ferenczi and the author, [248], and Kalton, [245].

Theorem (6.1.19)[234]: Let (e_i) be an unconditional basic sequence. Then either E_0 Borel reduces to isomorphism of spaces spanned by subsequences of the basis or any space spanned by a subsequence is isomorphic to its square and its hyperplanes. And there is some infinite $A \subset N$ such that for any infinite

$$B \subset A, [e_i]_A \cong [e_i]_B.$$

Proof. As before we can suppose we have some comeagre class $A \subset [N]^N$. But then A is also comeagre in $2^{\mathbb{N}}$ and there is therefore a partition A_0, A_1 of $\mathbb{N}$ and subsets $B_0 \subset A_0, B_1 \subset A_1$ such that for any $C \subset N$, if $C \cap A_0 = B_0$ or $C \cap A_1 = B_1$, then $C \in A$. In particular, $B_0, B_1, B_0 \cup B_1 \in A$. Moreover, as the complement operation is a homeomorphism of $2^{\mathbb{N}}$ with itself, there is some C such that $C, CC \in A$. So identifying subsets of $\mathbb{N}$ with the Banach spaces they generate and using the fact that the basis is unconditional, and therefore that disjoint subsets form direct sums, we can calculate:

$$N = C \cup CC \cong C \oplus CC \cong B_0 \oplus B_0 \cong B_0 \oplus B_1 \cong B_0 \cup B_1 \cong B_0$$

So $N \in A$ and A consists of spaces isomorphic to their squares. Now for any $D \subset N$:

$$N \oplus D \cong B_0 \oplus B_1 \oplus D \cong [B_0 \cup (D \cap A_1)] \oplus [B_1 \cup (D \cap A_0)] \cong N \oplus N \cong N$$

This in particular shows that $[e_i]_N$ is isomorphic to its hyperplanes.

We notice now that the argument is quite general, in the sense that we could have begun from any $[e_i]_A$instead of $[e_i]_N$, and therefore the results hold for any space spanned by a subsequence.

Kalton [245] showed that in case an unconditional basis only has a countable number of isomorphism classes on the subsequences of the basis, then the space spanned is isomorphic to its square and hyperplanes. The above result is along the same lines and we should mention that one can get uniformity results with a bit of extra care in the proof, see the article by Ferenczi,[248], for this.

Notice that permutative equivalence between subsequences of a basis in- duces an analytic equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$.

Theorem (6.1.20)[234]: (Bourgain, Casazza, Lindenstrauss, Tzafriri) If $(e_i)_N$ is an un- conditional basic sequence permutatively equivalent to all of its subsequences, then there is a permutation π of N such that $(e_{\pi(i)})_N$ is subsymmetric.

Their statement of the theorem is slightly more general, but the general case is easily seen to follow from the infinite dimensional Ramsey theorem.

Proposition (6.1.21)[234]: Let $(e_i)_A$ is an unconditional basic sequence. Then either E_0 reduces to the relation of permutative equivalence of the subsequences of the basis or there is some $A \in [N]^N$ such that $(e_i)_A$ is subsymmetric.

Proof. Notice that $\sim_p$ (permutative equivalence) on [N]N is E_0'-invariant, so applying the Ramsey result we can suppose that there is some $B \in [N]^N$ such that all $C \in [B]^N$ are $C \sim_p B$. Now there is some permutation π of B such that $(e_{\pi(i)})_B$ is subsymmetric. Again choosing a strictly increasing sequence $A = \{n_0, n_1, n_1, \ldots\} \subset B$ such that $\pi(n_0) < \pi(n_1) < \pi(n_2) < \ldots$, we get a subsymmetric $(e_{(i)})_A$.

Section (6.2): α-Minimal Banach Space:

Suppose $\mathcal{W}$ is a separable, infinite-dimensional Banach space. We say that $\mathcal{W}$ is minimal if $\mathcal{W}$ isomorphically embeds into any infinite-dimensional subspace $Y \subseteq \mathcal{W}$ (and write $\mathcal{W} \subseteq \mathcal{W}$ to denote that $\mathcal{W}$ embeds into Y). The class of Banach spaces without minimal subspaces was studied by V. Ferenczi and the author in [251], extending work of W.T. Gowers [252] and A.M. Pelczar [253], in which a dichotomy was proved characterising the presence of minimal subspaces in an arbitrary infinite-dimensional Banach space.

The dichotomy hinges on the notion of tightness, which we can define as follows. Assume that $\mathcal{W}$ has a Schauder basis (e_n) and suppose $Y \subseteq \mathcal{W}$ is a subspace. We say that Y is tight in the basis (e_n) for $\mathcal{W}$ if there are successive finite intervals of ℕ,

$$1_0 < 1_1 < 1_2 < \cdots \subseteq N,$$

such that for any isomorphic embedding $T : Y \to \mathcal{W}$, if P_{I_m} denotes the canonical projection of $\mathcal{W}$ onto $[e_n]_n \in I_m$, then

$$\lim_{m\to\infty} \inf\|P_{I_m}T\| > 0$$

Alternatively, this is equivalent to requiring that whenever $A \subseteq N$ is infinite, there is no embedding of y into$[e_n | n \notin \cup_{mA} I_m]$. Also, the basis (e_n) is tight if any infinite-dimensional subspace $Y \subseteq \mathcal{W}$ is tight in (e_n) and a space is tight in case it has a tight basis. We note that if $\mathcal{W}$ is tight, then so is any shrinking basic sequence in $\mathcal{W}$.

Tightness is easily seen to be an obstruction to minimality, in the sense that a tight space cannot contain a minimal subspace. In [251] the following converse is proved: any infinite-dimensional Banach space contains either a minimal or a tight subspace.

J. Bourgain introduced in [254] an ordinal index that gives a quantitative measure of how much one Banach space with a basis embeds into another. Namely, suppose $\mathcal{W}$, is a space with a Schauder basis $(e_n)$and y is any Banach space. We let $T((e_n), y, K)$be the tree of all finite sequences $(y_0, y_1 \ldots, y_k)$in Y, including the empty sequence $\emptyset = (\ )$, such that

$$(y_0, \ldots, y_k) \sim k(e_0, \ldots e_k)$$

Here, whenever (x_i) and $(y_i)$are sequences of the same (finite or infinite) length in Banach spaces χ and y, we write

$$(x_i)\sim k(y_i)$$

if for all $a_0, \dots, a_k \in \mathbb{R}$

$$\frac{1}{K}\left\|\sum_{i=0}^{k} a_i x_i\right\| \leq \left\|\sum_{i=0}^{k} a_i y_i\right\| \leq K\left\|\sum_{i=0}^{k} a_i x_i\right\|.$$

We notice that $T\big((e_n), y, K\big)$is ill-founded, i.e., admits an infinite branch, if and only if $\mathcal{W} = [e_n]$ embeds with constant K into y.

The rank function ρTon a well-founded tree T, i.e., without infinite branches, is defined by $\rho T^{(S)} = 0$ if s ∈ T is a terminal node and

$$\rho T^{(S)} = \sup\{\rho T^{(t)} + 1 | s \prec t, t \in T\}$$

Otherwise. Then, the rank of T is defined by

$$\text{rank}(T) = \sup\{\rho T^{(S)} + 1 |\ s \epsilon T\},$$

Whence rank(T)= $\rho T^{(\emptyset)}$ + 1 if T is non-empty. Moreover, if T is ill-founded, we let rank$(T) = \infty$, with the stipulation that $\alpha < \infty$for all ordinals α.

Then, rank$(T(e_n), y, K)$ measures the extent to which $\mathcal{W} = [e_n]$ K-embeds into y and we therefore define the embed ability rank of $\mathcal{W} = [e_n]$ into y by

$$\text{Emb}((e_n), y) = \sup_{k \geq 1} \text{rank}\,(T(e_n), y, K)$$

Since (e_n) is a basic sequence, there is for any K ≥ 1 a sequence $\Delta = (\delta_n)$ of positive real numbers, such that if $y_n, z_n \in y, \|y_n - z_n\| < \delta_n$and $(y_0 \dots, y_k)\sim k(e_0 \dots, e_k)$then also$(z_0 \dots, z_k)\sim k + 1(e_0 \dots, e_k)$. Therefore, to calculate the embed ability rank, Emb$\big((e_n), y\big)$, it suffices to consider the trees of all finite sequences $(y_0 \dots, y_k)$ with $(y_0 \dots, y_k)\sim k(e_0 \dots, e_k)$, where, moreover, we require the $y_n$to belong to some fixed dense subset of y. We shall use this repeatedly later on, where we replace y by a dense subset of itself. This comment also implies that Emb$\big((e_n), y\big)$is either ∞, if $\mathcal{W} \subseteq y$, or an ordinal < density$(y)^+$, if $\mathcal{W} \nsubseteq y$. In particular, if y is separable, then Emb$\big((e_n), y\big)$ is either or a countable ordinal. Also, note that the embeddability rank depends not only on the space $\mathcal{W}$, but also on the basis (e_n). However, if y is separable and $\mathcal{W} \nsubseteq y$, then by the Boundedness Theorem for coanalytic ranks (see [255]), the supremum of Emb$\big((e_n), y\big)$ over all bases (e_n). for w is a countable ordinal. In case Emb$\big((e_n), y\big)$ ≥α, we say that $\mathcal{W} = [e_n]$ α-embeds into y.

Since minimality is explicitly expressed in terms of embeddability, it is natural to combine it with Bourgain's embeddability index in the following way.

Definition (6.2.1)[250] Let α be a countable ordinal. A Banach space $\mathcal{W}$ with a Schauder basis (e_n) is α- minimal if any block subspace $Z = [\mathcal{Z}_n] \subseteq W$ α-embeds into any infinite-dimensional subspace $y \subseteq \mathcal{W}$.

It is easy to check that if $\mathcal{W} = [e_n]$ is a space with a basis and $\mathcal{X}=[x_n]$ and $y = [y_n]$ are block subspaces of $\mathcal{W}$ such that $x_n \in y$ for all but finitely many n, which we denote by $\mathcal{X} \subseteq^* y$, then if y is α-minimal, so is $\mathcal{X}$. In particular, α-minimality is preserved by passing to block subspaces. Similarly, we can combine tightness with the embeddability index.

Definition (6.2.2)[250]: Let α be a countable ordinal and $\mathcal{W}$ a Banach space with a Schauder basis (e_n). We say that $\mathcal{W} = [e_n]$ is α-tight if for any block basis (y_n)in $\mathcal{W}$ there is a sequence of intervals of $\mathbb{N}$.

$$1_0 < 1_1 < 1_2 < \cdots \subseteq N$$

such that for any infinite set $A \subseteq \mathbb{N}$,

$$\text{Emb}\left((y_n), \left[e_n | n \notin \bigcup_{j\in A} I_j\right]\right) \leq \alpha.$$

In other words, if $y = [y_n]$ $(\alpha + 1)$-embeds into some subspace $Z \subseteq W$, then

$$\liminf_{k\to\infty} \|P_{I_k}|z\| > 0$$

Again, it is easy to see that if $\mathcal{W} = [e_n]$ is α-tight, then so is any block subspace of $\mathcal{W}$. Also, if $\mathcal{W} = [e_n]$ is α-tight, then no block subspace, $y = [y_n]$, is β-minimal for $\alpha < \beta$. And, if $y = [y_n]$ is minimal, then $y = [y_n]$ is α-minimal for any $\alpha < \omega_1$. It follows from this that if $\mathcal{W} = [e_n]$ is α-tight, then $\mathcal{W} = [e_n]$ admits no minimal block subspaces, and thus, as any infinite-dimensional subspace contains a block subspace up to a small perturbation $\mathcal{W}$, contains no minimal subspaces either.

Our first result says that tightness can be reinforced to α-tightness.

Theorem (6.2.3)[250]: Let $\mathcal{W}$ be a Banach space with a Schauder basis. Then $\mathcal{W}$ has a minimal subspace or a block subspace $\mathcal{X} = [x_n] \subseteq \mathcal{W}$ that is α-minimal and ω_α-tight for some countable ordinal α.

Proof. Suppose that $\mathcal{W}$ has no minimal subspace and pick by Theorem (6.2.3) some block subspace $\mathcal{W}_0 \subseteq \mathcal{W}$ that is β-tight for some $\beta < \omega_1$. So no block subspace of $\mathcal{W}_0$ is $(\beta + 1)$-minimal.

Let now α be the supremum of all ordinals y such that W_0 is saturated with y -minimal block subspaces and pick a block subspace $\mathcal{W}_1 \subseteq \mathcal{W}_0$ not containing any $(\alpha + 1)$-minimal subspace.

We claim that $\mathcal{W}_1$ contains an α-minimal block subspace $\mathcal{W}_\infty$.If α is a successor ordinal, this is obvious, so suppose instead that α is a limit. Then we can find ordinals $\gamma 2 < \gamma 3 < \ldots$ with supremum α. We then inductively choose block subspaces $\mathcal{W}_1 \supseteq \mathcal{W}_2 \supseteq \mathcal{W}_3 \supseteq \ldots$ such that $\mathcal{W}_n$is y_n-minimal. Letting

$\mathcal{W}_\infty \subseteq \mathcal{W}_1$ be a block subspace such that $\mathcal{W}_\infty \subseteq^* \mathcal{W}_n$ for all n, we see that $\mathcal{W}_\infty$ is γ_n-minimal for all n, which means that for any block sequence $(z_m) \subseteq \mathcal{W}_\infty$ and infinite-dimensional subspace $y \subseteq \mathcal{W}_\infty$, we have

$$\mathrm{Emb}((z_m), y) \geq y_n$$

for all n, whence $\mathrm{Emb}((z_m), y) \geq \sup_n y_n = \alpha$. So $\mathcal{W}_\infty$ is α-minimal and so are its subspaces. Now, $\mathcal{W}_\infty$ has no (α + 1)-minimal subspace, so, by Theorem4, $\mathcal{W}_\infty$ contains an $\mathcal{W}_\alpha$-tight block subspace $\mathcal{X}$, which simultaneously is α-minimal.

Since any two Banach spaces of the same finite dimension are isomorphic, one easily sees that any space $\mathcal{W}$ with a Schauder basis (e_n) is ω-minimal. On the other hand, in [251], a space $\mathcal{W} = [e_n]$ is defined to be tight with constants if for any block subspace $y = [y_n]$ there are intervals $1_0 < 1_1 < 1_2 < \cdots$ such that for any integer constant K,

$$[y_n]_{n \in 1_k} \not\sqsubseteq k[e_n]_{n \in 1_k}$$

Where ⊑ denotes the embeddability relation and $\sqsubseteq_K$ denotes embeddability with constant K. In this case, it follows that for any infinite set A⊆N and any K ∈ A,

$$\mathrm{rank}\left(T\left((y_n), \left[e_n | n \notin \bigcup_{j \in A} 1_j\right], k\right)\right) \leq \max 1_k,$$

and hence

$$\mathrm{Emb}\left((y_n), \left[e_n | n \notin \bigcup_{j \in A} 1_j\right]\right) = \lim_{K \in A}\left(T\left((y_n), \left[e_n | n \notin \bigcup_{j \in A} 1_j\right], k\right)\right) \leq \omega$$

So, if $\mathcal{W} = [e_n]$ is tight with constants, we see that $\mathcal{W} = [e_n]$ is ω-tight and ω-minimal.Following [251], we also define a space $\mathcal{W}$ to be locally minimal if there is a constant K ≥ 1 such that $\mathcal{W}$ is K-crudely finitely representable in any infinite-dimensional subspace, i.e., if for any finite-dimensional $F \subseteq \mathcal{W}$ and infinite-dimensional $y \subseteq \mathcal{W}$, $\mathcal{F} \sqsubseteq_K y$. Let us first see local minimality in terms of α-minimality.

Proposition (6.2.4)[250]: Suppose $\mathcal{W}$ is a locally minimal Banach space with a Schauder basis (e_n) Then $\mathcal{W} = [e_n]$ is ω^2-minimal.

Proof. Let K be the constant of local minimality. For any infinite-dimensional subspace $\subseteq \mathcal{W}$, block sequence $(w_i) \subseteq \mathcal{W}$ and $\alpha < \omega^2$, we need to show that $\mathrm{Emb}((w_i), y) > \alpha$. So choose n such that $\alpha > \omega \cdot n$ and find some constant C such that if $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$ are finite block sequences of (e_i) such that $\frac{1}{k}\|x_i\| \leq \|y_i\| \leq k\|x_i\|$, then $(x_i) \sim c(y_i)$. we claim that

$$\operatorname{rank}\left(T((w_i), y, 2C)\right) \geq \omega . n$$

To see this, find some block subspace x such that $\mathcal{X} \sqsubseteq_2 y$. It suffices to prove that

$$\operatorname{rank}\left(T((w_i), \mathcal{X}, c)\right) \geq \omega . n$$

Let k_1 be given.We shall see that $\emptyset$ has rank≥ $\omega(n-1)+k_1-1$ in $\left(T((w_i), y, C)\right)$.So choose by local K-minimality some $z_0,\ldots,z_{k_1-1} \in \mathcal{X}$ such that

$$(w_0 \ldots, w_{k_1-1}) \sim k(z_0, \ldots, z_{k_1-1})$$

It then suffices to show that $(z_0, \ldots, z_{k_1-1})$has rank $\geq \omega(n-1)$ in $T((w_i), y, C)$. or, equivalently, that for any k_2, it has rank $\geq \omega(n-1) + k_2-1$. So choose$z_{k_1}, \ldots, z_{k_1+k_2-1}$ in $\mathcal{X}$ with support after all of $z_0,\ldots,z_{k_1-1}$ such that

$$(w_{k_1} \ldots, w_{k_1+k_2-1}) \sim k(z_{k_1}, \ldots, z_{k_1+k_2-1})$$

Again, it suffices to show that

$$(z_0, \ldots, z_{k_1-1}, z_{k_1}, \ldots, z_{k_1+k_2-1})$$

has rank $\geq \omega(n-1)$ in $T((w_i), y, C)$. Et cetera.

Eventually, we will have produced

$$z_0, \ldots, z_{k_1-1} < z_{k_1}, \ldots, z_{k_1+k_2-1} < \cdots < z_{k_1}, \ldots, z_{k_n-1}, \ldots z_{k_1+\cdots k_n-1}$$

Such that for each l.

$$(w_{k_1+\cdots+k_{l-1}}, \ldots, w_{k_1+\cdots+k_{l-1}}) \sim k(z_{k_1+\cdots+k_{l-1}}, \ldots,, \ldots z_{k_1+\cdots+k_{l-1}})$$

Since we have chosen the successive sections of (z_i) successively on the basis, we have, by the choice of C, that

$$(w_0 \ldots, w_{k_1+\cdots k_n-1}) \sim c(z_0, \ldots, z_{k_1+\cdots k_n-1})$$

Where by$(z_0, \ldots, z_{k_1+\cdots k_n-1}) \in T((w_i), \mathcal{X}, C)$. and hence has rank $\geq 0 = \omega(n-n)$ in $T((w_i), \mathcal{X}, C)$. This finishes the proof.

In [251], another dichotomy was proved stating that any infinite-dimensional Banach space contains a subspace with a basis that is either tight with constants or is locally minimal. In particular, we have the following dichotomy.

Theorem (6.2.5)[250]: (See V. Ferenczi and C. Rosendal [251].) Any infinite-dimensional Banach space contains an infinite-dimensional subspace with a basis that is either ω-tight or is ω^2-minimal.

One problem that remains open is to exhibit spaces that are α-minimal and ωα-tight for unbounded $\alpha < \omega_1$. We are not aware of any construction in the literature that would produce this, but remain firmly convinced that such spaces must exist, since otherwise there would be a universal $\beta <$

$\omega_1$such that any Banach space would either contain a minimal subspace or a β-tight subspace, which seems unlikely.

Problem (6.2.6)[250]: Show that there are α-minimal, ωα-tight spaces for unboundedly many $\alpha < \omega_1$.

The main result, Theorem (6.2.7), allows us to refine the classification scheme developed in [252] and [251], by further differentiating the class of tight spaces into α-minimal, ωα-tight for $\alpha < \omega_1$.

Currently, the most interesting direction for further results would be to try to distinguish between different classes of minimal spaces, knowing that these pose particular problems for applying Ramsey Theory.

We follow A.S. Kechris [255] presentation of trees and games, except that we separate a game from its winning condition and thus talk about players having a strategy to play in a certain set, rather than having a strategy to win.

For the proof of Theorem (6.2.9), we will need to replace Banach spaces with the more combinatorial setting of normed vector space over countable fields, which we will be using throughout (cf. [256]). So suppose W is a Banach space with a Schauder basis (e_n). By a standard Skolem hull construction, we find a countable subfield $\mathfrak{F}\subseteq$ R such that for any $\mathfrak{F}$ -linear combination $\sum_{n=0}^{m} a_n e_n$ the norm$\|\sum_{n=0}^{m} a_n e_n\|$ belongs to $\mathfrak{F}$. Let also W be the countable-dimensional F-vector space with basis (e_n). In the following, we shall exclusively consider the F-vector space structure of W, and thus subspaces etc. refer to $\mathfrak{F}$ -vector subspaces. We equip W with the discrete topology, whereby any subset is open, and equip its countable power $W^{\mathbb{N}}$with the product topology. Since W is a countable discrete set, $W^{\mathbb{N}}$is a Polish, i.e., separable and completely metrisable, space. Notice that a basis for the topology on $W^{\mathbb{N}}$ is given by sets of the form

$$N(x_o, \dots, x_k) = \{(y_n) \in w^{\mathbb{N}} | y_0 = x_0 \& \dots \& y_k = x_k\},$$

where $x_o, \dots, x_k \in$ W. Henceforth, we let x, y, z, v be variables for non-zero elements of W. if $x = \sum a_n e_n \in W$, we define the support of x to be the finite, non-empty set supp(x) = $\{n | a_n \neq 0\}$ and set for x,y ∈W,

$$x < y| \Leftrightarrow \forall n \in \operatorname{supp}(x)\, \forall m \in \operatorname{supp}(y), n < m.$$

Similarly, if k is a natural number, we set

$$k < x \Leftrightarrow \forall n \in \operatorname{supp}(x), k < n.$$

Analogous notation is used for finite subsets of N and finite-dimensional subspaces of W. A finite or infinite sequence $(x_0, x_1, x_2, x_3, \dots)$of vectors is said to be a block sequence if for all n, $x_n < x_n + 1$

Note that, by elementary linear algebra, for all infinite-dimensional subspaces X ⊆W there is a subspace Y ⊆ X spanned by an infinite block sequence, called a block subspace. Henceforth, we use variables X, Y,Z,V to denote infinite-dimensional block subspaces of W. Also, denote finite sequences of non-zero vectors by variables $\vec{x}, \vec{y}, \vec{z}, \vec{v}$. Finally, variables E,F are used to denote finite-dimensional subspaces of W.

Theorem (6.2.7)[250]: Let $\mathcal{W}$ be a Banach space with a Schauder basis and having no minimal subspaces.Then there is a block subspace $\mathcal{X}= [x_n]$ that is α-tight for some countable ordinal α.

The main results, however, provide us with more detailed structural information.

Proof. We should first recall a natural strengthening of tightness from [251]. Suppose W is a Banach space with a Schauder basis (e_n). Let also bb$(e_n) \subseteq W^{\mathbb{N}}$be the closed set of all block sequences in $W^{\mathbb{N}}$. Let I be the countable set of all non-empty finite intervals {n, n + 1, . . . , m} ⊆ N and give $I^{\mathbb{N}}$the product topology, where I is taken discrete. We say that W = $[e_n]$ is continuously tight if there is a continuous function

$$f: bb(e_n) \to I^{\mathbb{N}}$$

such that for any block sequence $(y_n)\in w^{\mathbb{N}}$, f $((y_n))=(I_n)\in I^{\mathbb{N}}$ is a sequence of intervals such that $1_0 < 1_1 < 1_2 < \cdots$and such that whenever

A ⊆ N is infinite,

$$[y_n] \nsubseteq \left[e_n | n \notin \bigcup_{k\in A} 1_k\right].$$

In other words, f continuously chooses the sequence of intervals witnessing tightness. As in the case of Banach spaces, for any $k \geq 1$, block subspace Y ⊆W, and block sequence (x_n) of (e_n)we define T $((x_n), Y, K)$to be the non-empty tree consisting of all finite sequences $(y_0, \dots, y_k)$in Y such that

$$(y_0, \dots, y_k) \sim k(x_0, \dots, x_k)$$

Similarly define the embeddability index of (x_n) in Y by

$$\mathrm{Emb}((x_n), Y) = \sup_{k\geq 1} \mathrm{rank}(T(x_n), Y, k)$$

Then, if y denotes the closed R-linear subspace of $\boldsymbol{\mathcal{W}}$ spanned by Y,we have, as was observed earlier, that

$$\mathrm{Emb}((x_n), y) = \mathrm{Emb}((x_n), Y)$$

We recall the statement of Theorem (6.2.3).

Theorem (6.2.8)[250]: Let $\boldsymbol{\mathcal{W}}$ be a Banach space with a Schauder basis (e_n) and having no minimal subspaces.

Then there is a block subspace $\mathcal{X} = [x_n]$ that is α-tight for some countable ordinal α.

Proof. By the results of [251], we have that, as $\mathcal{W}$ has no minimal subspaces, there is a block subspace $\mathcal{X}=[x_n]$ of $\mathcal{W}=[x_n]$ that is continuously tight as witnessed by a function f. So it suffices to show that for some $\alpha<\omega_1$ and any block sequence (y_n) of (x_n), if $x_n = f((y_n))$, then

$$\mathrm{Emb}\left((y_n),\left[x_n|n\notin\bigcup_{k\in A} I_K\right]\right)\leq\alpha,$$

for any infinite set A ⊆ N.

Note that if D is any countable set, we can equip the power set P(D) with the compact metric topology obtained from the natural identification with 2^D. Let [ℕ] denote the space of subsets of ℕ equipped with the Polish topology induced from P(ℕ). We define a Boral measurable function between Polish spaces

$$T: bb(y_n)\times\mathbb{N}\times[\mathbb{N}]\to p(X^{<N})$$

by setting

$$T((y_n),A,K) = T\left((y_n),\left[x_n|n\notin\bigcup_{j\in A} I_j\right],k\right),$$

where $(I_n) = f((y_n))$.By assumption, the image of T is an analytic set of well-founded trees on X. So, by the Boundedness Theorem for analytic sets of well-founded trees, there is some $\alpha<\omega_1$ such that

$$\sup_{((y_n),A,K)\in bb(X_n)\times[\mathbb{N}]\times\mathbb{N}} \mathrm{rank}\left(T\left((y_n),\left[x_n|n\notin\bigcup_{k\in A} I_k\right],k\right)\right)\leq\alpha,$$

whereby, for any block sequence (y_n) of (x_n) and any infinite subset A ⊆ N,

$$\mathrm{Emb}\left((y_n),\left[x_n|n\notin\bigcup_{k\in A} I_k\right],k\right)\leq\alpha,$$

showing that $\mathcal{X}$ is α-tight.

Theorem (6.2.9)[250]: Let $\mathcal{W}$ be Banach space with a Schauder basis and suppose $\alpha<\omega_1$. Then there is a block subspace $\mathcal{X}=[x_n]\subseteq\mathcal{W}$ that is either ωα-tight or (α +1)-minimal. Finally, combining Theorems (6.2.7) and (6.2.9), we have the following refinement of Theorem (6.2.7).

Proof. Suppose X ⊆ W and α is a countable ordinal number. We define the generalised Gowers α-game below X, denoted G_X^{α}, between two players I and II as follows:

$$\begin{array}{llll} 1 & \begin{matrix} Y_0 \\ \xi_0<\alpha \end{matrix} & \begin{matrix} Y_1 \\ \xi_1<\xi_0 \end{matrix} & \begin{matrix} Y_k \\ \xi_k<\xi_{k-1} \end{matrix} \end{array}$$

…..

II	$F_0 \subseteq Y_0$ $x_0 \in F_0$	$n_1 \subseteq Y_1$ $x_1 \in F_0 + F_1$	$n_k \subseteq F_k$ $x_k \in F_0 + \cdots + F_k$

Here $\alpha > \xi_0 > \xi_1 > \cdots > \xi_k = 0$ is a strictly decreasing sequence of ordinals,$Y_l \subseteq X$ are block subspaces, the $F_l \subseteq Y_l$ are finite-dimensional subspaces, and $x_l \in F_0 + F_1 + \ldots + F_l$ non-zero vectors. Since I plays a strictly decreasing sequence of ordinals, the game will end once $\xi_K = 0$
has been chosen and II has responded with some x_k. We then say that the sequence $(x_0, \ldots, x_k)$ of non-zero vectors is the outcome of the game.

Similarly, we can define the asymptotic α-game below X, F_X^α, as follows

I	n_0 $\xi_0 < \alpha$	n_0 $\xi_1 < \xi_0$	n_k $\xi_k < \xi_{k-1}$
		…	
II	$n_0 \subseteq F_0$ $x_0 \in F_0$	$n_1 \subseteq F_1$ $x_1 \in F_0 + F_1$	$n_k \subseteq F_k$ $x_1 \in F_0 + \cdots + F_k$

Here again, $\alpha > \xi_0 > \xi_1 > \cdots > \xi_k = 0$ is a strictly decreasing sequence of ordinals, nl natural numbers, the F_l are finite-dimensional subspaces of $[e_i]_{i=n_l+1}^{\infty}$and $x_l \in F_0 + \cdots + F_l$ nonzero vectors. The game ends once I has played ξ_k=0 and II has responded with some x_k. The outcome is the sequence of non-zero vectors $(x_0, \ldots, x_k)$. If $\vec{x}$ is a finite sequence of non-zero vectors, we define the games $G_X^\alpha(\vec{X}), F_X^\alpha(\vec{x})$as above, except that the outcome is now $\vec{x}\,\hat{}\,(z_0, \ldots, z_k)$. We also define adversarial α-games by mixing the games above. For this, suppose E, F are finite-dimensional subspaces of W and $\vec{z}$ is an even-length sequence of non-zero vectors. We define $A_X^\alpha(\vec{z}, E, F)$by

I		$n_0 < E_0$ x_0 Y_0 ξ_0	$n_1 < E_1$ x_1 y_1 ξ_1	$n_k < E_k$ x_k y_k ξ_k
			…	
II	n_0	n_1 $F_0 < Y_0$ y_0	n_2 $F_1 < Y_1$ y_1	$F_k < Y_k$ y_k

$B_X^\alpha(\vec{z}, E, F)$by

I		$E_0 < Y_0$ x_0 n_0 ξ_0	$n_1 < E_1$ x_1 n_1 ξ_1	$n_k < E_k$ x_k n_k ξ_k
			…	
II	Y_0	Y_1 $F_0 < Y_0$ y_0	Y_2 $F_1 < Y_1$ y_1	$n_k < F_k$ y_k

Where $\alpha > \xi_0 > \xi_1 > \cdots > \xi_k = 0$

is a decreasing sequence of ordinals, $Y_l \subseteq X$ are block subspaces, and n_l natural numbers. Moreover ,in $A_X^{\alpha}(\vec{Z}, E, F)$,

$$E_l \subseteq X \cap [e_i]_{i=n_l+1}^{\infty} 1 \text{ and } F_l \subseteq Y_l$$

are finite-dimensional subspaces, while in$B_X^{\alpha}(\vec{Z}, E, F)$

$$F_l \subseteq X \cap [e_i]_{i=n_l+1}^{\infty} \text{ and } E_l \subseteq Y_l$$

are finite-dimensional subspaces. Finally, the non-zero vectors xl and yl are chosen such that

$$x_l \in E + E_0 + \cdots + E_l,$$

while

$$y_l \in F + F_0 + \cdots + F_l.$$

Both games terminate once I has played $\xi_k = 0$ and II has responded with some y_k. The outcome is then the finite sequence of non-zero vectors

$$\vec{z}(x_0, y_0, x_1, y_1, \ldots x_k, y_k).$$

Now suppose instead that $\vec{z}$ is an odd-length sequence of non-zero vectors. We then define $A_X^{\alpha}(\vec{Z}, E, F)$, by

		$n_1 < E_1$		$n_2 < E_2$		$n_k < E_k$
I	y_0	x_1 Y_1 ξ_1		x_2 Y_2 ξ_2	…	x_k Y_k ξ_k
II		n_1 $F_0 \subseteq Y_0$ y_0	n_2 $F_1 \subseteq Y_1$ y_1		$F_k \subseteq Y_k$ y_k	

And $B_X^{\alpha}(\vec{Z}, E, F)$, by

		$n_1 \subseteq y_1$		$E_2 \subseteq y_2$		$E_k \subseteq y_k$
I	n_0	x_1 n_1 ξ_1		x_2 n_2 ξ_2	….	x_k n_k ξ_k
II	y_1 $n_0 \subseteq F_0$ y_0		n_2 $n_1 \subseteq F_1$ y_1			$n_k \subseteq F_K$ y_k

Where

$$\alpha > \xi_1 > \cdots > \xi_k = 0$$

is a decreasing sequence of ordinals,

$x_l \epsilon E + E_1 + \cdots + E_l,$

$$y_l \in F + F_0 + \cdots + F_{l,}$$

and otherwise the games are identical to those above. The outcome is now the finite sequence $\vec{z}^\frown(y_0, x_1, y_1, \ldots, x_k, y_k)$

If $\vec{z}=\emptyset$ and E = F = {0}, we shall write B_X^α and A_X^α instead of $A_X^\alpha(\vec{z}, E, F)$by respectively $B_X^\alpha(\vec{z}, E, F)$ Thus, in both gamesA_X^α and B_X^α one should remember that I is the first to play a vector. And in A_X^αX, I plays block subspaces and II plays integers, while in B_X^α II takes the role of playing block subspaces and I plays integers.We should also mention the degenerate case when α = 0. The games$G_X^\alpha(\vec{Z})$ and $F_X^\alpha(\vec{Z})$ then terminate immediately with outcome $\vec{Z}$ and, if $\vec{Z}$ is of even length, the same holds for the games $A_X^\alpha(\vec{z}, E, F)$ and $B_X^\alpha(\vec{z}, E, F)$On the other hand if $\vec{Z}$is of odd length in $A_X^\alpha(\vec{z}, E, F)$ and $B_X^\alpha(\vec{z}, E, F)$,I will play respectively Y_0 and Y_0 and II respond with a single y0 according to the rules, whereby the outcome is now $\vec{z}\, y_0$.If X and Y are subspaces, where Y is spanned by an infinite block sequence $(y_0, y_1, y_2, y_3 \ldots.)$,we write Y ⊆*X if there is n such that $y_m \in X$ for all m≥n. A simple diagonalisation argument shows that if $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ is a decreasing sequence of block subspaces, then there issome Y ⊆X_0 such that Y ⊆* $X_n$for all n. The aim of the games above is for each of the players to ensure that the outcome lies in some predetermined set depending on the player. By the asymptotic nature of the game, it is easily seen that if $T \subseteq W^{<N}$and $Y \subseteq^* X$, then if II has a strategy in G_X^α or $A_X^\alpha(\vec{z}, E, F)$to play in T , i.e., to ensure that the outcome is in T , then II will have a strategy in G_Y^α respectively$A_Y^\alpha(\vec{z}, E, F)$to play in T too. Similarly, if I has a strategy in F_X^α or $B_X^\alpha(\vec{z}, E, F)$ To play in T,then I also has a strategy in F_Y^α , respectively in $B_X^\alpha(\vec{z}, E, F)$ to play in T.We are now ready to prove the basic determinacy theorem for adversarial α-games, which can be seen as a refinement of the determinacy theorem for open adversarial games (see Theorem 12 in [256]).

Theorem (6.2.10)[250]: Suppose $\alpha<\omega_1$ and T⊆$W^{<N}$.Then for any $X \subseteq W$there is Y ⊆ X such that either

(i) II has a strategy in $A_Y^\alpha$to play in T , or

(ii) I has a strategy in B_Y^α to play in ∼ T .

Proof. We say that

(i) $(\vec{x}, E, F, \beta, X)$ is good if II has a strategy in $A_X^\beta(\vec{x}, E, F,)$ to play in T .

(ii) $(\vec{x}, E, F, \beta, X)$ is bad if ∀Y ⊆ X, $(\vec{x}, E, F, \beta, X)$is not good.

(iii) $(\vec{x}, E, F, \beta, X)$ is worse if it is bad and either

(a) $|\vec{X}|$is even and β = 0, or

(b) $|\vec{X}|$ is even, β >0, and $\forall Y \subseteq X\, \exists E_0 \subseteq Y \exists x_0 \in E + E_0\, \exists \gamma < \beta\ (x^\frown x_0, E + E0, F, \gamma, X)$ is bad,

or

(c) $|\vec{x}|$is odd and

$\exists n\ \forall n<F_0 \subseteq X\ \forall y_0 \in F((x\hat{\ }y_0, E, F + F_0, \beta, X)$ is bad,

(iv) $(\vec{x}, E, F, \beta, X)$ is wicked if $\forall y_0 \in F((x\hat{\ }y_0, E, F + F_0, \beta, X)$is bad.

One checks that good, bad and wicked are all $\subseteq^*$-hereditary in the last coordinate, that is, if $(\vec{x}, E, F, \beta, X)$is good and $Y \subseteq^* X$,then also $(\vec{x}, E, F, \beta, X)$is good, etc. So, by diagonal sing over the countable many tuples of $\vec{x}$,E, F, and $\beta \leq \alpha$, we can find some $Y \subseteq X$ such that for all

$$\vec{x}, E, F \text{ and } \beta \leq \alpha,$$

(i)($\vec{x}, E, F, \beta, Y$) is either good or bad, and

(ii) if there is some $Y_0 \subseteq Y$ such that for all $F_0 \subseteq Y_0$, $(\vec{x}, E, F + F_0, \beta, Y)$is wicked, then there is some n such that for all $n<F_0 \subseteq Y$, $(\vec{x}, E, F + F_0, \beta, Y)$is wicked.

Lemma (6.2.11)[250]: If $(\vec{x}, E, F, \beta, Y\ )$is bad, then it is worse.

Proof. Assume first that $|\vec{x}|$is even. The case when $\beta = 0$ is trivial, so assume also $\beta > 0$. Since $(\vec{x}, E, F, \beta, Y)$is bad, we have $\forall V \subseteq Y$ II has no strategy in $A_V^{\beta}(\vec{x}, E, F)$ to play in T.

Referring to the definition of the game $A_V^{\beta}(\vec{x}, E, F)$ this implies that

$$\forall V \subseteq Y\ \exists E_0 \subseteq V\ \exists x_0 \in E + E_0\ \exists \gamma < \beta$$

II has no strategy in $A_V^{\gamma}(\vec{x}\hat{\ }x_0, E + E_0, F)$ to play in T

(note that the subspace $Y_0 \subseteq V$ also played by I becomes the first play of I in the game $A_V^{\gamma}(\vec{x}\hat{\ }x_0, E + E_0, F)$ But if $V \subseteq Y$ and II has no strategy in$A_V^{\gamma}(\vec{x}\hat{\ }x_0, E, E_0, F)$ to play in T , then $(\vec{x}\hat{\ }x_0, E + E_0, F, y, V)$ is not good and hence must be bad. Thus,

$$\forall V \subseteq Y\ \exists E_0 \subseteq V\ \exists x_0 \in E + E_0\ \exists \gamma < \beta\ (\vec{x}\hat{\ }x_0, E + E_0, F, \gamma, V\ ) \text{ is bad,}$$

which is just to say that ($\vec{x}$,E,F,β,Y) is worse. Now suppose instead that $|\vec{x}|$ is odd. As $(\vec{x}, E, F, \beta, Y)$ is bad, it is not good and so II has no strategy in $A_V^{\beta}((\vec{x}, E, F)$to play in T . Therefore, for some $Y_0 \subseteq Y$, we have $\forall F_0 \subseteq Y_0\ \forall y_0 \in F + F_0$ II has no strategy in $A_Y^{\beta}(\vec{x}\hat{\ }y_0, E, F + F_0)$to play in T i.e., $\forall F_0 \subseteq Y_0\ \forall y_0 \in F + F_0\ (x\hat{\ }y_0, E, F + F_0, \beta, Y)$ is not good and hence is bad. In other words, $\forall F_0 \subseteq Y_0\ (\vec{x}, E, F + F_0, \beta, Y)$ is wicked.

So by (ii) we have $\exists n\ \forall n < F_0 \subseteq Y\ (\vec{x}, E, F + F_0, \beta, Y)$ is wicked, that is $\exists n\ \forall n < F_0 \subseteq Y\ \forall y_0 \in F + F_0\ (\vec{x}\hat{\ }y_0, E, F + F_0, \beta, Y)$ is bad, showing that ($\vec{x}$,E,F,β,Y) is worse. If $(\emptyset, \{0\}, \{0\}, \alpha, Y)$ is good, the first possibility of the statement of the theorem holds. So suppose instead $(\emptyset, \{0\}, \{0\}, \alpha, Y)$ is bad and hence worse. Then, using the lemma and unraveling the definition of worse, we see that I has a strategy to play the game B_Y^{α}such that at any point in the game, if

$$\vec{x} = (x_0, y_0, x_1, y_1, \ldots, x_l, y_l),$$

$$E_0, F_0, E_1, F_1, \ldots, E_l, F_l ,$$
$$\alpha > \xi_0 > \xi_1 > \cdots > \xi_l ,$$
respectively,
$$\vec{\vec{y}} = (x_0, y_0, x_1, y_1, \ldots, y_{l-1}, x_l),$$
$$E_0, F_0, E_1, F_1, \ldots, F_{l-1}, E_l ,$$
$$\alpha > \xi_0 > \xi_1 > \cdots > \xi_l ,$$
have been played, then
$$(\vec{x}, E_0 + \cdots + E_l, F_0 + \cdots + F_l, \xi_l, Y),$$
respectively
$$(\vec{y}, E_0 + \cdots + E_l, F_0 + \cdots + F_l - 1, \xi_l, Y),$$
is worse. Since $\alpha > \xi_0 > \xi_1 \ldots$, we eventually have $\xi_k = 0$, that is, the game terminates with some worse
$$(\vec{z}, E_0 + \cdots + E_k, F_0 + \cdots + F_k, 0, Y),$$
where by the outcome $\vec{z}$ lies in $\sim T$.

We first need a lemma ensuring us a certain uniformity.

Lemma (6.2.12)[250]: Let $\beta < \omega_1$ and suppose that for every $X \subseteq W$ there are $K \geq 1$ and a block sequence $(y_n) \subseteq X$ such that II has a strategy in F_X^β to play $(x_0, x_1, \ldots, x_k)$ satisfying
$$(x_0, x_1, \ldots, x_k) \sim K\,(y_0, y_1, \ldots, y_k).$$
Then there are $K \geq 1$ and $Y \subseteq W$ such that for all $X \subseteq Y$ there is a block sequence $(y_n) \subseteq X$ such that II has a strategy in F_X^β to play $(x_0, x_1, \ldots, x_k)$ satisfying
$$(x_0, x_1, \ldots, x_k) \sim K\,(y_0, y_1, \ldots, y_k).$$
In other words, $K \geq 1$ can be chosen uniformly for all $X \subseteq Y$.

Proof. Assume toward a contradiction that the conclusion fails. Then, as the games F_X^β to play in any set $T \subseteq W^{<N}$ are determined, i.e., either I or II has a winning strategy, we can inductively define $W \supseteq Y_0 \supseteq Y_1 \supseteq \cdots$ such that for any block sequence (y_n) in Y_K, I has a strategy in F_{Yk}^β to play $(x_0, x_1, \ldots, x_k)$ satisfying
$$(x_0, x_1, \ldots, x_k) \nsim K\,(y_0, y_1, \ldots, y_k).$$
For each $N \in \mathbb{N}$, let $c(N)$ be a constant such that if $(v_0, v_1, \ldots, v_{N-1}, v_N, v_{N+1}, \ldots)$ and $(u_0, u_1, \ldots, u_{N-1}, v_N, v_{N+1}, \ldots)$ are two normalised block sequences of (e_n), then
$$(v_0, v_1, \ldots, v_{N-1}, v_N, v_{N+1}, \ldots) \sim c(N)\,(u_0, u_1, \ldots, u_{N-1}, v_N, v_{N+1}, \ldots).$$
Now choose a block sequence $(x_0, x_1, x_2, \ldots)$ such that for every N there are normalised $v_0, v_1, \ldots, v_{N-1} \in Y_{N \cdot c(N)}$ with
$$v_0 < v_1 < \cdots < v_{N-1} < x_N < x_{N+1} < \cdots$$

and, moreover, such that $x_N, x_{N+1}, \ldots \in Y_{N\cdot c(N)}$. Set also X = $[x_n]$.

By the assumptions of the lemma, we can find some constant $N \in \mathbb{N}$ and a normalised block sequence $(y_0, y_1, \ldots)$ in X such that II has a strategy in F_X^β to play $(w_0, w_1, \ldots, w_k)$ with

$$(w_0, w_1, \ldots, w_k) \sim N\ (y_0, y_1, \ldots, y_k).$$

Since min supp$(x_N) \le$ min supp(y_N), it follows by the choice of (x_n) that there are normalized $v_0, v_0, \ldots, v_{N-1} \in Y_N \cdot c(N)$ such that

$$v_0 < v_1 < \cdots < v_{N-1} < y_N < y_{N+1} < \cdots .$$

Moreover, by the definition of c(N), we have

$$(v_0, v_1, \ldots, v_{N-1}, y_N, y_{N+1}, \ldots) \sim c(N)\ (y_0, y_1, \ldots, y_{N-1}, y_N, y_{N+1}, \ldots).$$

Thus, if we let $v_n = y_n$ for all $n \ge N$, we see that II has a strategy in F_X^β to play $(w_0, w_1, \ldots, w_k)$ with

$$(w_0, w_1, \ldots, w_k) \sim N\ (y_0, y_1, \ldots, y_k) \sim c(N)\ (v_0, v_1, \ldots, v_k).$$

But $X \subseteq^* Y_{N.c(N)}$, so II has a strategy in $F^\beta_{Y_{N.c(N)}}$ to play $(w_0, w_1, \ldots, w_k)$ with

$$(w_0, w_1, \ldots, w_k) \sim N.c(N)\ (v_0, v_1, \ldots, v_K).$$

On the other hand, $(v_n) \subseteq Y_{Y_{N.c(N)}}$ and so I has a strategy in $F^\beta_{Y_{N.c(N)}}$ to play $(w_0, w_1, \ldots, w_k)$ such that

$$(w_0, w_1, \ldots, w_k) \backsim\!\backsim (v_0, v_1, \ldots, v_K),$$

which is absurd. This contradiction proves the lemma.

Lemma (6.2.13)[250]: Suppose $X \subseteq W$, $(y_0, y_1, y_2, \ldots)$ is a sequence of vectors in W, $\alpha < \omega_1$ and $K \ge 1$.

Assume that II has a strategy in $F_X^{\omega\cdot\alpha}$ to play $(x_0, x_1, \ldots, x_K)$ such that

$$(x_0, x_1, \ldots, x_K) \sim K\ (y_0, y_1, \ldots, y_K).$$

Then II has a strategy in B_X^α to play $(u_0, v_0, u_1, v_1, \ldots, u_K, v_k)$ such that

$$(u_0, u_1, \ldots, u_K) \sim K\ (v_0, v_1, \ldots, v_k).$$

Proof. We shall describe the strategy for II in the game B_X^α, the idea being that, when playing the game B_X^α, II will keep track of an auxiliary run of $F_X^{\omega-\alpha}$, using his strategy there to compute his moves in B_X^α.

Now, in F_X^α, II will play subspaces $Y_0, Y_1, \ldots$ all equal to $Y = [y_n]$, whereby the subspaces $Y_0, Y_1, \ldots$ and $E_0, E_2, \ldots$ lose their relevance and we can eliminate them from the game for simplicity of notation. We thus have the following presentation of the game B_X^α

I	$u_0 \epsilon Y$ n_0 $\xi_0 < \alpha$	$u_1 \epsilon Y$ n_1 $\xi_1 < \xi_0$	$u_k \epsilon Y$ n_k $\xi_k < \xi_{k-1}$	
			$\ldots$	
II	$n_0 < F_0$ $v_0 \epsilon F_0$	$n_1 < F_1$ $v_1 \epsilon F_0 + F_1$	$n_k < F_k$ $v_k \epsilon F_0 + \cdots + F_k$	

So suppose $u_0, u_1, \ldots$ is being played by I in B_X^α. To compute the answer $v_0, v_1, \ldots$, II follows his strategy $F_X^{\omega-\alpha}$ to play $(z_0, z_1, \ldots, z_k) \sim K(y_0, y_1, \ldots, y_k)$ as follows. First, as $u_0, u_1, \ldots \in Y = [y_n]$, we can write each u_i as

$$u_i = \sum_{j=0}^{m_i-1} \lambda_j^i y_j,$$

where we, by adding dummy variables, can assume that $m_0 < m_1 < m_2 < \cdots$. So to compute v_0 and F_0 given u_0, n_0 and ξ_0, II first runs an initial part of $F_X^{\omega-\alpha}$ as follows

I	n_0 $\omega\xi_0 + m_0 - 1$	n_0 $\omega\xi_0 + m_0 - 2$	n_0 $\omega\xi_0$	
II		$n_0 < F_1^0$ $x_0 \epsilon F_1^0$	$n_0 < F_2^0$ $x_1 \epsilon F_1^0 + F_2^0$	$n_0 < F_{m_0}^0$ $x_{m_0-1} \epsilon F_1^0 + \cdots + F_{m_0}^0$

He then plays $F_0 = F_1^0 + \cdots + F_{m_0}^0$ and $v_0 = \sum_{j=0}^{m_0-1} \lambda_j^0 x_j \epsilon F_0$, in B_x^α Next, I will play some u_1, n_1 and ξ_1, and, to compute v_1 and F_1 II will continue the above run of $F_X^{\omega-\alpha}$ with

I	n_0 $\omega\xi_1 + m_1 - 1$	n_1 $\omega\xi_1$	
		$\ldots$	
II		$n_1 < F_1^1$ $x_{m_0} \epsilon F_0 + F_1^0$	$n_1 < F_{m_1}^1$ $x_{m_1-1} \epsilon F_0 + F_1^1 + \cdots + F_{m_1}^1$

He then plays $F_1 = F_1^1 + \ldots + F_{m_1}^1$ and

$$v_1 = \sum_{j=0}^{m_1-1} \lambda_j^1 x_j \epsilon F_0 + F_1$$

in B_X^α So at each stage, II will continue his run of $F_X^{\omega-\alpha}$ a bit further until eventually I has played some $\xi_k = 0$. Thus, in the game $F_X^{\omega \cdot \alpha}$, I will play ordinals

$$\alpha > \omega\xi_0 + m_0 - 1 > \omega\xi_0 + m_0 - 2 > \cdots > \omega\xi_0 > \omega\xi_1 + m_1 - 1 > \cdots > \omega\xi_k = 0$$

and integers $n_0 \geq n_0 \geq \cdots \geq n_0 \geq n_1 \geq \cdots \geq n_K$, while II will use his strategy to play $(x_0, x_{,1} \ldots, x_{m_k-1})$ such that

$$(x_0, x_0, \ldots, x_{mk}) \sim K(y_0, y_1, \ldots, y_{m_k-1})).$$

Since the v_i and u_i have the same coefficients over respectively (x_n) and (y_n), it follows that

$$(u_0, u_1, \ldots, u_k) \sim K(v_0, v_1, \ldots, v_k).$$

By a similar argument, we have the following lemma.

Lemma (6.2.14)[250]: Suppose $X \subseteq W, (y_0, y_1, y_2, \ldots)$ is a block sequence in $W, \alpha < \omega_1$ and $K \geq 1$. Assume that II has a strategy in $F_X^{\omega-\alpha}$ to play $(x_0, x_1, \ldots, x_k)$ such that $(x_0, x_1, \ldots, x_k \sim K\,(y_0, y_1, \ldots, y_k)$.

Then for any block sequence (z_n) in $[y_n]$, II has a strategy in F_X^{α} to play $(v_0, v_1, \ldots, v_k)$ such that $(v_0, v_1, \ldots, v_k) \sim K\,(z_0, z_1, \ldots, z_k)$.

Proof. First, as (z_n) is a block sequence in $[y_n]$, we can write each z_i as

$$zi = \sum_{j=m_{i-1}}^{m_j-1} \lambda_j y_j \,,$$

where $-1 = 0 < m_0 < m_1 < m_2 < \cdots$.

As before, when playing F_X^{α}, II will keep track of an auxiliary run of $F_X^{\omega\alpha}$, using his strategy there to compute his moves in F_X^{α}. So the game F_X^{α} runs as follows:

I	n_0 ξ_0	n_1 ξ_1	n_k ξ_k
		$\cdots$	
II	$n_0 < F_0$ $v_0 \epsilon F_0$	$n_1 < F_1$ $v_1 \epsilon F_0 + F_1$	$n_k < F_k$ $v_k \epsilon F_0 + \cdots + F_k$

To compute v_0, II first runs an initial part of $F_X^{\omega\alpha}$ as follows

I	n_0 $\omega\xi_0 + m_0 - 1$	n_0 $\omega\xi_0 + m_0 - 2$	n_0 $\omega\xi_0$
		$\cdots$	
II	$n_0 < F_1^0$ $x_0 \epsilon F_1^0$	$n_0 < F_2^0$ $x_1 \epsilon F_1^0 + F_2^0$	$n_0 < F_{m_0}^0$ $x_{m_0} \epsilon F_1^0 + \cdots + F_{m_0}^0$.

He then plays $F_0 = F_1^0 + \cdots + F_{m_0}^0$ and

$$v_0 = \sum_{j=m-1}^{m_0-1} \lambda_j x_j \in F_0$$

In F_X^{α}

Next, I will play some ξ_1 and n_1 and to compute v_1 and F_1, II will continue the above run of $F_X^{\omega\alpha}$ with

I	n_1 $\omega\xi_1 + m_1 - m_0 - 1$	n_1 $\omega\xi_1$
	$\ldots.$	
II	$n_1 < F_1^1$ $x_{m_0} \epsilon F_0 + F_1^0$	$n_1 < F_{m_1-m_0}^1$ $x_{m1} - 1 \epsilon F_0 + F_1^1 + \cdots + F_{m_1-m_0}^1$

He then plays $F_1 = F_1^1 + \cdots + F_{m_1-m_0}^1$ and

$$v_1 = \sum_{j=m_0}^{m_j-1} \lambda_j x_j \in F_0 + F_1$$

In F_X^{α}

So at each stage, II will continue his run of In $F_X^{\omega\alpha}$ a bit further until eventually I has played some $\xi_k = 0$. Thus, in the game $F_X^{\omega\alpha}$, I will play ordinals

$$\alpha > \omega\xi_0 + m_0 - 1 > \omega\xi_0 + m_0 - 2 > \cdots > \omega\xi_0 > \omega\xi_1 + m_1 - m_0 - 1 > \cdots > \omega\xi_k = 0$$

and integers $n_0 \geq n_0 \geq \cdots \geq n_0 \geq n_1 \ldots \geq n_k$, while II will use his strategy to play $(x_0, x_1, \ldots, x_{m_k-1})$ such that

$$(x_0, x_1, \ldots, x_{m_k-1}) \sim K\,(y_0, y_1, \ldots, y_{m_k-1}).$$

Since the v_i and z_i have the same coefficients over respectively (x_n) and (y_n), it follows that

$$(v_0, v_1, \ldots, v_k) \sim K\,(z_0, z_1, \ldots, z_k).$$

Lemma (6.2.15)[250]: Suppose X⊆W, (y_n) is a block sequence in $W, \alpha < \omega_1$, and $K, C \geq 1$. Assume that

(a) II has a strategy in F_X^{α} to play $(x_0, \ldots, x_K)$ such that

$$(x_0, x_1, \ldots, x_k) \sim K\,(y_0, y_1, \ldots, y_k),$$

and

(b) II has a strategy in A_X^{α} to play $(u_0, v_0, \ldots, u_K, v_K)$ such that

$$(u_0, u_1, \ldots, u_k) \sim C\,(v_0, v_1 \ldots, v_k).$$

Then II has a strategy in G_X^{α} to play $(v_0, \ldots, v_k)$ such that

$$(v_0, v_1, \ldots, v_K) \sim KC\,(y_0, y_1, \ldots, y_k).$$

Proof. To compute his strategy in G_X^{α}, II will play auxiliary runs of the games A_X^{α} and F_X^{α} in which he is using the strategies described above. Information is then copied between the games as indicated in the diagrams below. The game G_X^{α}

I	y_0 ξ_0	y_1 ξ_1	….	y_k ξ_k
II	$F_0 \subseteq Y_0$ $v_0 \epsilon F_0$	$F_1 \subseteq Y_1$ $v_1 \epsilon F_0 + F_1$		$F_k \subseteq Y_k$ $v_k \epsilon F_0 + \cdots + F_k$

The game F_X^{α}

I	n_0 ξ_0	n_1 ξ_1	….	n_k ξ_k
II	$n_0 < E_0$ $x_0 \epsilon E_0$	$n_1 < E_1$ $x_1 \epsilon E_0 + E_1$		$n_k < E_k$ $x_k \epsilon E_0 + \cdots + E_k$

The game A_X^α

I	$n_0 < E_0$ $x_0 \in E_0$ Y_0 ξ_0	$n_1 < E_1$ $x_1 \in E_0 + E_1$ Y_1 ξ_1	…	$n_k < E_k$ $x_k \in E_0 + \cdots + E_k$ Y_k ξ_k
II	n_0	n_1 $F_0 \subseteq Y_0$ $v_0 \in F_0$		$F_k \subseteq Y_k$ $v_k \in F_0 + \cdots + F_k$

By chasing the diagrams, one sees that this fully determines how II is to play in G_X^α.Moreover, since II follows his strategy in F_X^α, we have

$$(x_0, x_1, \dots, x_k) \sim K\,(y_0, y_1, \dots, y_k),$$

while the strategy in A_X^α ensures that

$$(x_0, x_1, \dots, x_k) \sim C\,(v_0, v_1, \dots, v_k),$$

from which the conclusion follows.

Theorem (6.2.16)[ ]: Suppose $\alpha < \omega_1$. Then there is X ⊆W such that one of the following holds.

(1) For every block sequence (y_n) in X and $K \geq 1$, I has a strategy in $F_X^{\omega\alpha}$ to play $(x_0, x_1, \dots, x_K)$ satisfying

$$(x_0, x_1, \dots, x_k) \nsim K\,(y_0, y_1, \dots, y_k).$$

(2) For some $K \geq 1$ and every block sequence (z_n)⊆X, II has a strategy in G_X^α to play $(x_0, x_1, \dots, x_k)$ satisfying

$$(x_0, x_1, \dots, x_k) \sim K\,(z_0, z_1, \dots, z_k).$$

Proof. Suppose that there is no $X \subseteq W$ for which (1) holds. Then, using that the game $F_X^{\omega\alpha}$ is determined, for every $X \subseteq W$ there is a block sequence (y_n) in X and some $K \geq 1$ such that II has a strategy in $F_X^{\omega\alpha}$ to play $(x_0, x_1, \dots, x_k)$ satisfying

$$(x_0, x_1, \dots, x_K) \sim K\,(y_0, y_1, \dots, y_k).$$

So, by Lemma (6.2.12), there is some $K \geq 1$ and Y ⊆ W such that for all X ⊆ Y there is some block sequence (y_n) in X such that II has a strategy in $F_X^{\omega\alpha}$ to play $(x_0, x_1, \dots, x_K)$ satisfying

$$(x_0, x_1, \dots, x_k) \sim K\,(y_0, y_1, \dots, y_k)$$

It thus follows from Lemma (6.2.13) that for all X ⊆ Y, II has a strategy in B_X^α to play $(u_0, v_0, u_1, v_1, \dots, u_k, v_k)$ such that

$$(u_0, u_1, \dots, u_k) \sim K\,(v_0, v_1, \dots, v_k).$$

Therefore, there is no X⊆Y such that I has a strategy in B_X^α to play a sequence $(u_0, v_0, u_1, v_1, \dots, u_k, v_k)$ satisfying

$$(u_0, u_1, \ldots, u_k) \nsim K\,(v_0, v_1, \ldots, v_k),$$

and thus, by Theorem (6.2.10), we can find some $X \subseteq Y$ such that II has a strategy in A_X^α to play $(u_0, v_0, u_1, v_1, \ldots, u_k, v_k)$ satisfying

$$(u_0, u_1, \ldots, u_k) \sim K\,(v_0, v_1, \ldots, v_k).$$

Let (y_n) be the block sequence in X such that II has a strategy in $F_X^{\omega\alpha}$ to play $(x_0, x_1, \ldots, x_K)$ satisfying

$$(x_0, x_1, \ldots, x_k) \sim K\,(y_0, y_1, \ldots, y_k).$$

Then, using Lemma (6.2.14), we see that for any block sequence $(z_n) \subseteq [y_n]$, II has a strategy in F_X^α to play $(x_0, x_1, \ldots, x_k)$ such that

$$(x_0, x_1, \ldots, x_k) \sim K\,(z_0, z_1, \ldots, z_k).$$

In other words, there is some block sequence (y_n) in X such that for any block sequence $(z_n) \subseteq [y_n]$

(a) II has a strategy in F_X^α to play $(x_0, \ldots, x_k)$ satisfying

$(x_0, x_1, \ldots, x_k) \sim K\,(z_0, z_1, \ldots, z_k)$, and

(b) II has a strategy in A_X^α to play $(u_0, v_0, \ldots, u_k, v_k)$ satisfying

$$(u_0, u_1, \ldots, u_k) \sim K\,(v_0, v_1, \ldots, v_k).$$

So finally, by Lemma (6.2.15), for any block sequence $(z_n) \subseteq [y_n]$, II has a strategy in G_X^α to play $(v_0, \ldots, v_k)$ such that

$$(u_0, u_1, \ldots, u_k) \sim K^2\,(z_0, z_1, \ldots, z_k).$$

Replacing X by the block subspace $[y_n] \subseteq X$ and K by K^2, we get (2).

Lemma (6.2.17)[250]: Suppose $\alpha < \omega_1, K \geq 1, X \subseteq W$ and $(z_n) \subseteq W$ is a block sequence such that II has a strategy in G_X^α to play $(y_0, \ldots, y_k)$ satisfying

$$(y_0, \ldots, y_k) \sim K\,(z_0 \ldots, z_k).$$

Then for any subspace $Y \subseteq X$, rank $(T((z_n), Y, K)) > \alpha$.

Proof. Let $Y \subseteq X$ and suppose toward a contradiction that rank $(T((z_n), Y, K)) = \xi_0 + 1 \geq \alpha$, where ξ_0 is the rank of the root $\emptyset$ in T $(T((z_n), Y, K)$. Now, let I play Y, ξ_0 in G_X^α and let II respond using his strategy

I	Y ξ_0
II	$E_0 \subseteq Y$ $y_0 \in E_0$

Then the rank of $(y_0) \in T((z_n), Y, K)$ is some ordinal $\xi_1 < \xi_0$, so in G_X^α, I continues by playing Y, ξ_1 and II responds according to his strategy

I	Y ξ_0	Y ξ_1

II	$E_0 \subseteq Y$ $y_0 \in E_0$	$E_1 \subseteq Y$ $y_1 \in E_0 + E_1$

Again, the rank of $(y_0, y_1) \in T((x_n), Y, K)$ is some ordinal $\xi_2 < \xi_1$, so in G_X^α, I continues by playing Y, ξ_2 and II responds according to his strategy

I	Y ξ_0	Y ξ_1	Y ξ_2
	…		
II	$E_0 \subseteq Y$ $y_0 \in E_0$	$E_1 \subseteq Y$ $y_1 \in E_0 + E_1$	$E_2 \subseteq Y$ $y_2 \in E_0 + E_1 + E_2$

Etc.

Eventually, we will have constructed some $(y_0, y_1, \ldots, y_{k-1})$ whose

T $((z_n), Y, K)$-rank is $\xi_k = 0$, while

I	Y ξ_0	Y ξ_{k-1}
	…	
II	$E_0 \subseteq Y$ $y_0 \in E_0$	$E_{k-1} \subseteq Y$ $y_{k-1} \in E_0 + E_{k-1}$

has been played according to the strategy of II.

It follows that if I continues the game by playing $Y, \xi_k = 0$,

I	Y ξ_0	Y ξ_{k-1}	Y $\xi_k = 0$
	…		
II	$E_0 \subseteq Y$ $y_0 \in E_0$	$E_{k-1} \subseteq Y$ $y_{k-1} \in E_0 + \cdots + E_{k-1}$	

using his strategy, II must be able to respond with some E_k and

$y_k \in E_0 + \ldots + E_k$

I	Y ξ_0	Y ξ_{k-1}	Y $\xi_k = 0$
	…		
II	$E_0 \subseteq Y$ $y_0 \in E_0$	$E_{k-1} \subseteq Y$ $y_{k-1} \in E_0 + \cdots + E_{k-1}$	$E_k \subseteq Y$ $y_0 \in E_0 + \cdots + E_k$

Since II played according to his strategy, we have $(y_0, y_1, \ldots, y_k) \sim K(z_0, z_1, \ldots, z_k)$ and thus $(y_0, y_1, \ldots, y_k) \in T((z_n), Y, K)$, contradicting that $(y_0, \ldots, y_k - 1)$ has $T((z_n), Y, K)$-rank 0 and hence is a terminal node.

Lemma (6.2.18)[250]: Suppose $(x_n) \subseteq W$ is a block sequence, $\beta < \omega_1$ and that for every normalised block sequence (y_n) in X = $[x_n]$ and $K \geq 1$, I has a strategy in F_X^β to play $(z_0, z_1, \ldots, z_k)$ such that

$$(z_0, z_1, \ldots, z_K) \nsim K\,(y_0, y_1, \ldots, y_k).$$

Then, for every normalised block sequence (y_n) in X and $K \geq 1$, there is a sequence (J_m) of intervals of $\mathbb{N}$ with min $J_m \to \infty$, such that if A ⊆ N is infinite, contains 0 and $Z = [x_j \,|\, j \notin \cup_{m\in A} J_m$ then

$$\mathrm{Rank}\Big(T\big((y_n), z, k\big)\Big) \leq \beta.$$

Proof. We relativise the notions of support of vectors et cetera to the basis (x_n) for X. So the reader can assume that (x_n) is the original basis (e_n) and X =W. Assume (y_n) is a normalised block sequence in X and $K \geq 1$. Let also $\Delta = (\delta_j)$ be a sequence of positive real numbers such that whenever $z_j, v_j \in X$, $\|z_j - v_j\| < \delta_j$, and

$$(v_0, \ldots, v_k) \sim K\,(y_0, \ldots, y_k),$$

then

$$(z_0, \ldots, z_k) \sim 2K\,(y_0, \ldots, y_k).$$

We choose sets $\mathbb{D}_i \subseteq X$ such that for each finite set $d \subseteq \mathbb{N}$, the number of $z \in \mathbb{D}_i$ such that supp(z)=d is finite, and for every v∈X with $\|v\| \leq K$ there is some $z \in \mathbb{D}_i$ with supp(z)= supp$(v)$and $\|z - v\| < \delta_i$. This is possible since the K-ball in $[xj]_{j\in d}$ is totally bounded for all finite d ⊆N.

The strategy for I in F_X^β in the game for (y_n) with constant 2K can be seen as a pair of functions ξ and n that to each legal position $(z_0, E_0, \ldots, z_j, E_j)$ of II in F_X^β provide the next play $\xi(z_0, E_0, \ldots, z_j, E_j) \in$Ord and $n(z_0, E_0, \ldots, z_j, E_j) \in \mathbb{N}$ by I.

We define a function $p\colon \mathbb{N} \to \mathbb{N}$ by letting $p(m)$ be the maximum of m and

$$\max\Big(n\big(z_0, [x_l]_{l\in d_0}, \ldots . z_i, [x_l]_{l\in d_i}\big) \big| d_j \subseteq [0, m-1] z_j \in [x_l]_{l\in d_0 \cup \ldots \cup d_j} \cap \mathbb{D}_j\Big)$$

By assumption on the sets $\mathbb{D}_j$, p is well-defined and so we can set

$$J_m = [m, p(m)] \subseteq N.$$

We claim that if A ⊆ N is an infinite set containing 0 and

$$Z = \left[x_n \middle| n \notin \bigcup_{m\in A} J_m\right],$$

Then

$$\mathrm{rank}\,(T(y_n), Z, K)) \leq \beta.$$

To see this, we define a monotone function φ, i.e., $\vec{v} \prec \vec{w} \Rightarrow \varphi(\vec{v}) \prec \varphi(\vec{w})$, associating to each $\vec{v} = (v_0, v_1, \ldots, v_i) \in T\,((y_n), Z, K)$ some

$$\varphi(\vec{v}) = (z_0, z_1, \ldots, z_i) \in \mathbb{D}_0 \times \mathbb{D}_1 \times \cdots \times \mathbb{D}_i$$

such that for all $j \leq i$, $\|z_j - v_j\| < \delta_j$ and supp(z_j) = supp(v_j), whereby, in particular, $z_j \in Z$.

Also set $T = \emptyset[T\,((y_n), Z, K)]$ and note that T is a subtree of $Z^{<N}$ with

$$\mathrm{rank}(T) \geq \mathrm{rank}(T(yn), Z, K))$$

Suppose toward a contradiction that rank $(T) > \beta$, whereby the rank of $\emptyset$ in T is $\geq \beta$. We describe how II can play against the strategy for I in F_X^{β} to play $(z_0, \dots, z_k)$ such that

$$(z_0, \dots, z_k) \sim 2K\,(y_0, \dots, y_k),$$

which will contradict the assumption on the strategy for I. The case $\beta = 0$ is trivial, so we assume that $\beta > 0$.

First, I plays $\xi(\emptyset) < \beta$ and $n(\emptyset)$. Since, $a_0 = 0 \in A$, we have $n(\emptyset) \leq p(a_0) = \max J_{a_0} < Z$ and thus there is some $n(\emptyset) < z_0 \in T$ whose rank in T is $\geq \xi(\emptyset)$. Find also $a_1 \in A$ such that $z_0 < J_{a_1}$ and let $E_0 = [x_j \mid J_{a_0} < x_j < J_{a_1}]$ So let II respond by

I	$n(\emptyset)$ $\xi(\emptyset)$
II	$n(\emptyset) < E_0$ $z_0 \in E_0$

Now, by his strategy, I will play some $\xi(z_0, E_0) < \xi(\emptyset)$ and $n(z_0, E_0) \leq p(a_1) = \max J_{a_1}$. So find some z_1 such that $(z_0, z_1) \in T$ and has rank$\geq \xi(z_0, E_0)$ in T . Find also $a_2 \in A$ such that $z_1 < J_{a_2}$ .Then, as $a_0, a_1 \in A$, if we set $E_1 = [x_j \mid J_{a_1} < x_j < J_{a_2}]$, we have $z_1 \in E_0 + E_1$, so we let II respond by

I	$n(\emptyset)$ $\xi(\emptyset)$		$n(z_0, E_0)$ $\xi(z_0, E_0)$	
II		$n(\emptyset) < E_0$ $z_0 \in E_0$		$n(z_0, E_0) < E_1$ $z_1 \in E_0 + E_1$

Et cetera. It follows that at the end of the game,

I	$n(\emptyset)$ $\xi(\emptyset)$		$n(z_0, E_0, \dots\, z_k - 1, E_k - 1)$ $\xi(z_0, E_0, \dots\, z_k - 1, E_k - 1) = 0$	
		….		
II		$n(\emptyset) < E_0$ $z_0 \in E_0$		$n(z_0, E_0, \dots\, z_k - 1, E_k - 1) < E_k$ $z_k \in E_0 + \cdots .. + E_k$

II will have constructed a sequence $(z_0, \dots, z_k) \in T$. So, by the definition of T,there is some $(v_0, \dots, v_k) \in T((y_n), Z, K)$ such that $\emptyset(v_0, \dots, v_k) = (z_0, \dots, z_k)$ and hence $\|z_j - v_j\| < \delta_j$ for all j . Thus,

$$(v_0, \dots, v_k) \sim K\,(y_0, \dots, y_k),$$

and hence

$$(z_0, \dots, z_k) \sim 2K\,(y_0, \dots, y_k).$$

Since II cannot have such a strategy, it follows instead that

$$\operatorname{rank}(T(n_y), Z, K) \le \operatorname{rank} \le \beta,$$

which proves the lemma.

Lemma (6.2.19)[250]: Suppose $(x_n) \subseteq W$ is a normalised block sequence, $\beta < \omega_1$, and that for every normalised block sequence (y_n) in $X = [x_n]$and

$K \ge 1$, I has a strategy in F_X^β to play $(z_0, z_1, \dots, z_k)$ such that

$$(z_0, z_1, \dots, z_k) \nsim K\,(y_0, y_1, \dots, y_k).$$

Then, for every normalised block sequence (y_n) in X there is a sequence

$$I_0 < I_1 < I_2 < \cdots$$

of intervals of N, such that if $A \subseteq \mathbb{N}$ is infinite and $Z = [x_j \,|j \notin \bigcup_{m\in A} I_m]$, then

$$\operatorname{Emb}((y_n), z) \le \beta.$$

Proof. Fix a normalised block sequence (y_n) in X and relativise again all notions of support et cetera to the block basis (x_n). By Lemma (6.2.18), we can for every K find a sequence (J_n^k) of intervals of N with $\min_{n\to\infty} J_n^k \to \infty$ such that for any infinite set $A \subseteq \mathbb{N}$ containing 0, we have

$$\operatorname{rank}\left(T\left((y_n), \left[x_j \,|j \notin \textstyle\bigcup_{n\in A} J_n^K\right], K\right)\right) \le \beta.$$

Also, for every N, we let $c(N) \in \mathbb{N}$ be a constant such that any two subsequences of (x_j)differing in at most N terms are $c(N)$ −equivalent.

We construct intervals $I_0 < I_1 < I_2 < \cdots$ such that each I_n contains an interval from each of the families $(J_i^1), \dots, (J_i^n)$ and, moreover,

$$\min 1_n < \max 1_n - \max J_0^{n\cdot c(\min 1_n)}.$$

We claim that if $A \subseteq \mathbb{N}$ is infinite and $Z = [x_j \,|j \notin \bigcup_{m\in A} I_m]$, then

$$\operatorname{Emb}((y_n), Z) \le \beta.$$

Suppose towards a contradiction that this fails for some A and pick some N such that rank $(T((y_n), Z, N)) > \beta$. Choose a ∈ A such that $a \ge N$ and note that

$$\min I_a < \max I_a - \max J_0^{a.c(\min 1_a)}.$$

Thus, by changing only the terms x_j for $j < \min I_a$ of the sequence

$$\left(x_j \middle| j \notin \bigcup_{m\in A} 1_m\right) = \left(x_j \middle| j \notin \bigcup_{m\in A} 1_m \,\&j < \min 1_a\right) \cup \left(x_j \middle| j \notin \bigcup_{m\in A} 1_m \,\&\, j < \max 1_a\right),$$

we find a subsequence of

$$\left(x_j \middle| \max j_0^{a.c(\min 1_a)} < j \le \max 1_a\right) \cup \left(x_j \middle| j \notin \bigcup_{m\in A} 1_m \,\&\, j > \max 1_a\right)$$

that is $c(\min I_a)$ –equivalent with

$$\left(x_j \middle| j \notin \bigcup_{m\in A} 1_m\right)$$

Since N. $c(\min I_a) \le a.c(\min I_a)$, it follows that if

$$Y = \left[x_j \middle| \max j_0^{a.c(\min 1_a)} < j \le \max 1_a\right] + \left[x_j \middle| j \notin \bigcup_{m\in A} 1_m \,\&\, j > \max 1_a\right]$$

then $\sqsubseteq_{c(\min I_a)} Y$, and so

$$\beta < \operatorname{rank}(T(y_n), Z, N)) \le \operatorname{rank}(T(y_n), y, a.c(\min 1_a))).$$

But, by the choice of the 1_n, we see that there is an infinite subset $B \subseteq \mathbb{N}$ containing 0 such that Y is outright a subspace of

$\left[x_j \middle| j \notin \bigcup_{m\in B} 1_m\, J_m^{a.c\,(\min 1_a)}\right]$ where by, by choice of the intervals $J_m^{a.c\,(\min 1_a)}$, we have

$$\operatorname{rank}(T(y_n), y, a.c(\min 1_a))) \le \beta$$

which is absurd. This contradiction shows that the intervals I_n fulfill the conclusion of the lemma.

By combining Theorem (6.2.16) and Lemmas (6.2.17) and (6.2.19), we obtain

Lemma (6.2.20)[250]: Suppose $\alpha < \omega_1$.Then there is a block subspace $X = [x_n] \subseteq W$ such that one of the following holds.

(i) For every normalised block sequence (y_n) in X there is a sequence

$$I_0 < I_1 < I_2 < \cdots$$

of intervals of $\mathbb{N}$, such that if A$\subseteq \mathbb{N}$ is infinite, then

$$\operatorname{Emb}\left((y_n), \left[x_j \middle| j \notin \bigcup_{m\in A} 1_m\right]\right) \le \alpha\omega$$

(ii) For any subspace $Y \subseteq X$ and any block sequence $(z_n) \subseteq X$,

$$\operatorname{Emb}((y_n), Y) > \alpha$$

And by replacing the normed ῂ-vector subspaces X and Y in Lemma (6.2.20) by their closures $\mathcal{X}$and y in $\boldsymbol{\mathcal{W}}$, we obtain Theorem (6.2.4).

Theorem (6.2.21)[250]: Let $\boldsymbol{\mathcal{W}}$ be Banach space with a Schauder basis and suppose $\alpha < \omega_1$. Then there is a block subspace $\mathcal{X} = [x_n] \subseteq \boldsymbol{\mathcal{W}}$ that is either ωα-tight or $(\alpha + 1)$ –minimal.

List of Symbols

$Symbol$		$Page$
inf	infimum	2
max	maximum	2
Sup	Supremum	3
I_p	Banach space	8
min	minimum	9
$Supp$	Support	9
l_p	Lebesgue space	33
$\bigoplus$	Direct sum	40
emb	embeddability	45
Co	Convex	48
$Clust$	cluster	53
l_∞	Lebesgue space	57
l_1	Lebesgue space	57
$\otimes$	Tensor product	69
$diam$	diameter	71
ext	extreme	77
L_∞	Lebesgue space	95
L_1	Lebesgue space	95
dim	dimension	103
int	interior	105
Re	Real	106
L_p	Lebesgue measure	111
$a.e$	almost everywhere	128
$\boxplus$	Summation symbol	133
det	determinant	135
aff	affine	144
$H.I$	Hereditarily indecomposable	162