Chapter 5

Linear Identities and Polynomials Norms

We study nontrivial linear identities such as

$$\sum_{k=0}^{m} a_k \|c_k(0)x_0 + \cdots + c_k(n)x_n\|^p = 0 \qquad (1)$$

for all elements x_i on a Banach space X. If X is a two-dimensional real space in $\wp_4$, then it is embeddable in L_4. This is not necessarily true for spaces with more dimensions or for $\wp_{2n}, n \geq 3$. The question of embeddability is equivalent to the classical moment problem. All spaces in $\wp_{2n}$ are uniformly convex and Uniformly smooth and thus reflexive. They obey generally weaker versions of the Holder and Clarkson inequalities. Krivine's inequalities, shown to determine embeddability into $L_p, p \neq 2n$, fail in the even case. We lead to a unified treatment and generalization of some classical results on linear identities and polynomial characterizations due to Frechet, Mazur, Orlicz, Reznick, Wilson, and others.

Section 5.1: Linear Identities In Banach Spaces

A necessary condition for (1) to hold in X is that $\|x + ty\|^{\nu}$ must be a polynomial in t for all choices of elements x and y. A sufficient condition for (2) to hold in X is that (2) must hold in the field of scalars. Specific identities are presented including a generalized parallelopiped law first observed by Koehler, and some isometric results are stated.

Thus the parallelogram law revisited. In 1909 [186], Frechet proved the following result.

Lemma (5.1.1)[185]: (Fréchet). If g is continuous function on R and, for all real r and s, equation (3) holds, then g is a polynomial with degree less than N.

$$\sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} g(r + ks) = 0 \qquad (3)$$

Proof. It is well-known that any sequence $\{a_n\}$ satisfying $\sum_{k=0}^{N}(-1)^{N-k}\binom{N}{k}a_{k+M} = 0$ for all M is generated by a polynomial; that is, there is a polynomial P with degree less than N for which $a_n = P(n)$.

In (3), put $g(n) = a_n, s = 1$ and let r range over the integers.Then there is a polynomial P with $P(n) = a_n = g(n)$. Now put $g(\frac{n}{2}) = b_n, s = \frac{1}{2}$ and let r range over the half-integers. There is a polynomial Q with $Q(n) = b_n = g(\frac{n}{2})$. Thus $Q(2n) = P(n)$ for all nand $Q(x) = P(\frac{x}{2})$, so $g(n/2) = P(\frac{n}{2})$. A repetition of this argument demonstrates that $P(n2^{-m}) = g(n2^{-m})$ for all integers m and n. By the continuity of $g, P(x) = g(x)$ for all x and the lemma is proved.

The parallelogram law has a second difference nature: $\|x+y\|^2-2\|x\|^2+\|x-y\|^2=2\|y\|^2$, see [187]. Putting successively $x=u+2v$, $y=v$ and $x=u+v, y=v$ and subtracting, we get (4). Fix w and z, elements of any space in which (1) holds.

$$\sum_{k=0}^{3}(-1)^k\binom{3}{k}\|u+kv\|^2=0 \tag{4}$$

Let $u=w+rz, v=sz$ and substitute in (4). Setting $\|w+tz\|^2=g(t)$, we obtain (5). By the triangle inequality, $\left|g^{\frac{1}{2}}(t_0)-g^{\frac{1}{2}}(t_1)\right|\le|t_1-t_0|\cdot\|z\|$, hence g is continuous.

$$\sum_{k=0}^{3}(-1)^k\binom{3}{k}g(r+ks)=0 \tag{4}$$

Applying Lemma (5.1.1) to (5) we see that $g(t)$ is quadratic in t. Indeed, if$\|w+tz\|^2=A(w,z)+2B(w,z)t+C(w,z)t^2$, then clearly $A(w,z)=\|w\|^2$ and $C(w,z)=\|z\|^2$. It is not hard to verify that $B(w,z)$ satisfies the definition of a real inner-product and $B(w,z)+iB(w,iz)$that of a complex inner-product. This provides an alternative proof to the Jordan-von Neumann theorem. We shall return to the parallelogram law as an embarkation point for a series of linear identities which hold in more spaces than Hilbert space. As an appetizer, consider (6), a generalization of (1) to a three-dimensional parallelopiped.

$$\begin{aligned}&\|x+y+z\|^k+\|x+y-z\|^k+\|x-y+z\|^k+\|x-y-z\|^k\\&-2\big(\|x+y\|^k+\|x-y\|^k+\|x+z\|^k+\|x-z\|^k\\&+\|y+z\|^k+\|y-z\|^k\big)\\&+4\big(\|x\|^k+\|y\|^k+\|z\|^k\big)=0\end{aligned} \tag{6}$$

Observe that (6) holds for $k=2$ in Hilbert space and for $k=4$ in Hilbert space and in $L_4(x,\mu)$ for any (x,μ). Indeed, it may be verified that (6) holds in any Banach space in which $\|rx+sy+tz\|^4$ is a homogeneous polynomial in r, s, andtfor fixed elements x, y, and z.This condition turns out to be necessary as well, and the situation will prove to be typical.

Theorem (5.1.2)[185]: Suppose X is a Banach space in which (7) holds for all elements x and y, where $a_k\neq 0, p>0, b_0=0$ and the $b_k{}'s$ aredistinct.

$$\sum_{k=0}^{m}a_k\|x+b_ky\|^p=0 \tag{7}$$

Then for every x andy in X,$\|x+ty\|^p$ is a polynomial in t. In particular, p is an even integer

Proof. Set $A(u,v)=a_0\|u\|^p+\sum_{k=0}^{m}a_k\|u+b_kv\|^p\cdot$ By the hypotheses, $A(u,v)=0$ for all elements u and v and $b_k\neq 0$ for $k\geq 1$. Fix elements x and y; let z be arbitrary. Then $0=A(x+z,y-b_1^{-1}z)-A(x,y)$. Writing this out, we obtain (8). Notice that the term for $k=1$ in the sum in (8) vanishes.

$$0 = a_k(\|x+z\|^p - \|x\|^p)$$

$$+\sum_{k=0}^{m} a_k\left(\|x + b_k y + (1 - b_k b_1^{-1})z\|^p - \|x - b_k y\|^p\right) \qquad (8)$$

We repeat this procedure (due to Wilson [188]) and obtain (9), where the inner sum is taken over all choices of $1 \le i_1 < \cdots < i_j \le m$.

$$0 = \sum_{j=0}^{m}(-1)^j\left(\sum A\left(x + jz, y - \left(b_{i_1}^{-1} + \cdots + b_{i_j}^{-1}\right)z\right)\right) \qquad (9)$$

Equation (9) may be expanded using the definition of $A(u, v)$ as (10), where the inner sum is as before.

$$0 = \sum_{k=0}^{m}(-1)^j\binom{m}{j}\|x + jz\|^p$$

$$+\sum_{k=0}^{m} a_k \sum_{j=0}^{m}(-1)^j\left(\sum\left\|x + b_k y + \left(j - b_k\left(b_{i_1}^{-1} + \cdots + b_{i_j}^{-1}\right)\right)z\right\|^p\right) \qquad (10)$$

For any fixed k, the 2^m subsets of $\{1,\cdots,m\}$ divide into two corresponding classes: if $k \notin I = \{i_1, \dots, i_j\}$ then I and $I \cup \{k\}$ are associated. Using this pairing, the terms

$$(-1)^j\left\|x + b_k y + \left(j - b_k\left(b_{i_1}^{-1} + \cdots + b_{i_j}^{-1}\right)\right)z\right\|^p$$

and $(-1)^{j+1}\left\|x + b_k y + \left(j - b_k\left(b_{i_1}^{-1} + \cdots + b_{i_j}^{-1}\right)\right)z\right\|^p$ cancel out in the triple sum. Hence, (10) reduces to (11).

$$0 = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\|x + jz\|^p \qquad (11)$$

Then we choose u and v arbitrary nonzero elements in X, let r and s be arbitrary reals and set $x = u + rv, y = sv$ and $g(t) = \|u + tv\|^p$ in (11). We obtain (12), and by Lemma(5.1.1), $g(t) = \|u + tv\|^p$ must be a polynomial.

$$0 = \sum_{j=0}^{m}(-1)^j\binom{m}{j}g(r + js) \qquad (12)$$

As $t^{-p}g(t) \to \|v\|^p \neq 0$, g has degree p and as $g(t) \ge 0$, p is an even integer.

Lemma (5.1.3)[185]: Suppose X is a Banach space in which (13) holds for all elements x and y, where $a_k \neq 0, p > 0$ and the $(b_k, c_k)'$s are pairwise linearly independent .

$$\sum_{k=0}^{m} a_k\|b_k x + c_k y\|^p = 0 \qquad (13)$$

Then $\|x + ty\|^p$ is a polynomial in t for all elements x and y.

Proof. We shall reduce (13) to (7). Permute the k's so that $b_0 \leq \cdots \leq b_m$ and let $d_k = {b_k}^{-1} c_k$ then $b_0 \neq 0$then$b_k \neq 0$ if $b_0 = 0$ then $c_0 \neq 0$and $b_k \neq 0$ for $k \geq 1$ and the d_k's are distinct by the linear independence. In the first case, rewrite (13) as (14) and then put into it $x = u - d_0 v, y = v$ where u and v are arbitrary.

$$\sum_{k=0}^{m} a_k |b_k|^p \, ||x + d_k y||^p = 0 \tag{14}$$

We obtain (15) which is in the form of (7).

$$\sum_{k=0}^{m} a_k |b_k|^p \, ||u + (d_k - d_0)v||^p = 0 \tag{15}$$

In the second case. Let s be a number for which $b_k + sc_k$ and put $x = u, y = su + v$ into (13) where u and v are again arbitrary. (If (b_j, c_j) and (b_k, c_k) are independent then so are$(b_j + sc_j, c_j)$ and $(b_k + sc_k, c_k)$). We obtain (16) which now falls under the first case.

$$\sum_{k=0}^{m} a_k \, ||(b_k + sc_k)u + c_k v||^p = 0 \tag{16}$$

Carlsson [189] proved Lemma (5.1.3) for $p = 2$ and a_k, b_k, c_k real.

Theorem (5.1.4)[185]: Suppose X is a Banach space in which (17) holds for all elements x_i, where $a_k \neq 0, p > 0$, the $(c_k(0), \cdots, c_k(n))$'s arepairwise linearly independent $(n + 1) -$ tuples, and for every i there is at least one k with$c_k(i) \neq 0$.

$$\sum_{k=0}^{m} a_k \, ||c_k(0)x_0 + \cdots + c_k(n)x_n||^p = 0 \tag{17}$$

Then, for all xandy, $||x + ty||^p$is a polynomial in t.

Proof. Permute the k's so that $|c_0(0)| \geq |c_1(0)| \geq \cdots \geq |c_m(0)|$, then $c_0(0) \neq 0$. If $c_k(0) = 0$ for $k \geq 1$, put $x_1 = \cdots = x_n = 0$ in (17), then $a_0 ||c_0(0)x_0||^p = 0$ for all x_0, that is, X is trivial. Otherwise, suppose$c_k(0) \neq 0$ for $k \leq j, j \geq 1$ and (17) may be rewritten as (18), where $d_k(i) = c_k(i) c_k(0)^{-1}$,

$$\sum_{k=0}^{j} a_k \, |c_k(0)|^p ||x_0 + d_k(0)x_1 + \cdots + d_k(n)x_n||^p + \sum_{k=j+1}^{m} a_k \, ||c_k(1)x_1 + \cdots + c_k(n)x_n||^p = 0 \tag{18}$$

From the linear independence, it follows that the n-tuples $\{(d_k(1), \dots, d_k(n))\}$ are distinct. Define the sets

$$A_{ik} = \left\{ (s_1, \dots, s_n) \in \mathbb{C}^n \mid \sum_{a=1}^{n} s_a \, d_i(q) = \sum_{a=1}^{n} s_a \, d_k(q) \right\}$$

These sets do not exhaust C_n^n and so we can find r_q for which the b_k, defined by $b_k = \sum_{q=1}^{n} r_q d_k(q)$, are distinct.

For arbitrary x and y, let $x_0 = x, x_k = r_k y$ in (18). We obtain (19) which is in the form of (13).

$$\sum_{k=0}^{j} a_k |c_k(0)|^p \|x + b_k y\|^p + \left(\sum_{k=j+1}^{m} \sum_{i=0}^{n} |c_k(i) r_i|^p \right) \|y\|^p = 0$$

This completes the proof.

To obtain the natural corollary we need a lemma.

Lemma (5.1.5)[185]: Suppose f is a nonnegative function and q and r are positive integers such that f^p and f^r are polynomials. Then f^s is a polynomial, where $s = (q, r)$, the greatest common divisor.

Proof. We have $f^q(x) = A^q \prod_{i=1}^{k} (x - x_i), x_i \neq x_j, A \geq 0$. Thus, $f^r(x) = A^r \prod_{i=1}^{k} (x - x_i)^{\lambda_i \frac{r}{q}}$, and so $q | r\lambda_i$. Write $q = s\bar{q}, r = s\bar{r}, (\bar{q}, \bar{r}) = 1$. As $s\bar{q} | s\bar{r}\lambda_i, \bar{q} | \lambda_i$ and so $f^s = A^s \prod_{i=1}^{k} (x - x_i)^{\frac{\lambda_i}{\bar{q}}}$, is a polynomial.

Corollary (5.1.6)[185]: Suppose X is a Banach space which satisfies (20) for all elements x_i, where $p_k > 0$ and the same restrictions on constants apply as in Theorem (5.1.4).

$$\sum_{k=0}^{m} a_k \|c_k(0) x_0 + \cdots + c_k(n) x_n\|^{p_k} = 0 \tag{20}$$

Then p_k is an even integer for each k and $\|x + ty\|^p$ is a polynomialin t for all x and y, where $p = (p_1, \dots, p_k)$, the greatest common divisor.

Proof. Fix $y_0, \dots, y_n$ and let $b_k = a_k \|c_k(0) y_0 + \cdots + c_k(n) y_n\|^{p_k}$. Let $x_k = \lambda y_k$ for $\lambda > 0$, then by (20), $\sum_{k=0}^{m} b_k \lambda^{p_k}$. Upon collectingequal p_k's as q_j's we find that $\sum_{j=1}^{s} \left(\sum_{p_k = q_j} b_k \right) \lambda^{q_j} = 0$. Hence $\sum_{p_k = q_j} b_k = 0$. Thus X satisfies an equation of form (17) for each p_k and $\|x + ty\|^{p_k}$ is a polynomial in t. We now apply Lemma (5.1.5) to this situation and conclude that p is a polynomial. The Class $\mathcal{P}_{2n}$. The necessary condition of Theorem (5.1.4) suggests the following definition: a Banach space X is polynomial of degree 2n if, for all elements x and y, $\|x + ty\|^{2n}$ is a polynomial of degree 2n in real t. The class $\mathcal{P}_{2n}$ consists of all Banach spaces which are polynomial of degree 2n.

Theorem (5.1.7)[185]:

(i) If X is in $\mathcal{P}_2$, then X is a Hilbert space.

(ii) If m divides n, then $\mathcal{P}_{2m}$, is contained in $\mathcal{P}_{2n}$.

(iii) If $r = (m, n)$, then $\mathcal{P}_{2m} \cap \mathcal{P}_{2n} = \mathcal{P}_{2r}$.

(iv) If k is an integer dividing n, then for all measure spaces (X, μ), $L_{2k}(X, \mu)$ is in $\mathcal{P}_{2n}$.

(v) If p is not an even integer and (X,μ) is not trivial, then $L_p(X,\mu)$ is not in $\mathcal{P}_{2n}$ for any n.

Proof. (i) This is the Jordan-von Neumann theorem.

(ii) If $\|x+ty\|^{2m}$ is a polynomial and $\frac{2n}{2m}$ is an integer, then $\|x+ty\|^{2n}$ is a polynomial.

(iii) Combine (ii) and Lemma (5.1.5).

(iv) It suffices to show that $L_{2n}(X,\mu)$ is in $\mathcal{P}_{2n}$. Pick elements f and g in $L_{2n}(X,\mu)$ then

$$\|f+tg\|^{2n} = \int |f+tg|^{2n} d\mu = \int \left(|f|^2 + t(f\bar{g} + \bar{f}g) + t^2\right)^n d\mu$$

$$= \int \sum c_i(f,g) t^i d\mu$$

where $c_i(f,g)$ is a sum of terms of the form $f^a \overline{f^b} g^c \overline{g^d}$ with $c+d=i, a+b=2n-i$. As $\int |f|^{2n} d\mu < \infty$ and $\int |g|^{2n} d\mu < \infty$, each $c_i(f,g)$ is integrable by Holder's inequality. Thus $\|f+tg\|^{2n} = \sum \int c_i(f,g) d\mu \cdot t^i$ and $L_{2n}(X,\mu)$ is in $\mathcal{P}_{2n}$.

(v) If (X,μ) has two disjoint sets of positive measure, then one may easily construct elements x and y in $L_p(X,\mu)$ with $\|x+ty\|^p = 1+|t|^p$. If p is not an even integer then $(1+|t|^p)^{2n/p}$ is not a polynomial as it is not in $C^{[p]+1}$. See [190]. The embedding properties of $\mathcal{P}_{2n}$ will be described in [191]. We state without proof the following theorem.

Theorem (5.1.8)[185]: (i) If X is a real two-dimensional space in $\mathcal{P}_4$, then it is isometrically isomorphic to a subspace of L_4 [0,1].

(ii) There exists a three-dimensional space in $\mathcal{P}_4$ which isnot isometrically isomorphic to any subspace of $L_4(X,\mu)$ for any (X,μ).

(iii) There exist two-dimensional spaces in $\mathcal{P}_{2n} (n \geq 3)$ which are not isometrically isomorphic to any subspace of $L_{2n}(X,\mu)$ for any (X,μ).

Now we show the sufficient condition that the classes $\mathcal{P}_{2n}$ form the finest possible gradation of Banach spaces according to the linear identities they satisfy: if an identity of type (16) holds with $p=2n$ for one space in $\mathcal{P}_{2n}$ then it holds for all spaces in $\mathcal{P}_{2n}$. We begin with a few preliminaries.

Lemma (5.1.9)[185]: Suppose a function $g(u_1,\ldots,u_n)$ is a polynomial in each of its variables separately, that is, (21) holds for each $r, 1 \leq r \leq n$, where the $g_{k,r}$'s are continuous and a carat over a variable signifies its omission.

$$g(u_1,\ldots,u_n) = \sum_{k=1}^{s_r} g_{k,r}(u_1,\ldots,\hat{u}_r,\ldots,u_n) u_r^k \qquad (21)$$

Then g is in fact a polynomial in the variables together.

Proof. The proof will be by induction on n. The theorem is certainly true for $n = 1$. Suppose it is true if $n = m$. For $n = m + 1$ we have by hypothesis a representation of $g(u_1, \dots, u_{m+1})$ in the form (22). Let $M = \max s_r + 1$.

$$g(u_1, \dots, u_{m+1}) = \sum_{k=1}^{m-1} g_{k,m+1}(u_1, \dots, u_{m+1}) u_{m+1}^k \tag{22}$$

Define the M^th difference $\Delta^M h(r, v)(u_1, \dots, u_s)$ by (23).

$$\Delta^M h(r, v)(u_1, \dots, u_s) = \sum_{i=0}^{M} (-1)^{M-i} \binom{M}{i} h(u_1, \dots, u_r + iv, \dots, u_s) \tag{23}$$

Since Δ^M is certainly linear, we can compute for $1 \leq t \leq m + 1$ $\Delta^M g(t, v)(u_1, \dots, u_{m+1})$ obtaining (24).

$$\Delta^M g(t, v)(u_1, \dots, u_{m+1}) = \sum_{k=1}^{M} \Delta^M g_{k,m+1}(t, v)(u_1, \dots, u_m) u_{m+1}^k \tag{24}$$

Since a representation of form (21) holds fort and the Mth difference annihilates all polynomials with degree less than M, $\Delta^M g(t, v)(u_1, \dots, u_{m+1}) = 0$. Thus, $\Delta^M g_{k,m+1}(t, v)(u_1, \dots, u_m) = 0$for each t. By Lemma (5.1.5),$g_{k,m+1}(u_1, \dots, u_m)$is a polynomial in each u_i separately. The induction hypothesis for $n = m$ ensures that $g_{k,m+1}(u_1, \dots, u_m)$ is a polynomial in $u_1, \dots, u_{m+1}$ together.By (22), $g(u_1, \dots, u_{m+1})$ is therefore a polynomial in $u_1, \dots, u_{m+1}$. This establishes the induction step and completes the proof.

Lemma (5.1.10)[185]: A space X is in $\wp_{2n}$ if and only if, for all m and elements x_i, $\|t_0 x_0 + \cdots + t_m x_m\|^{2n}$ is a polynomial in $t_0, \dots, t_m$.

Proof. It is easy to see that $\|t_0 x_0 + \cdots + t_m x_m\|^{2n}$ must be a homogeneous polynomial if it is a polynomial at all, and that this condition is equivalent to $\|t_0 + t_1 x_1 + \cdots + t_m x_m\|^{2n}$ being a polynomial in $t_1, \dots, t_m$

If $\|t_0 + t_1 x_1 + \cdots + t_m x_m\|^{2n}$ is a polynomial, set $x_0 = x, x_1 = y, x_2 = \cdots, x_m = 0$, arbitrary x and y. We find that X is in $\wp_{2n}$.To prove the converse, define $f(t_1, \dots, t_m) = \|t_0 + t_1 x_1 + \cdots + t_m x_m\|^{2n}$. Fix all variables save t_r, then $f(t_1, \dots, t_m)$ is a polynomial in t_r withcoefficients depending on $t_1, \dots, \hat{t}_r, \dots, t_m$. For any polynomial $p(t) = \sum_{j=0}^{2n} a_k t^k$ of degree 2n we have the formulae $a_k = \sum_{j=0}^{2n} c_{j,k} p(j)$, where$c_{i,k}$ can be found by solving the linear system: $\sum_{j=0}^{2n} a_k j^k = p(j), 0 \leq j \leq 2n$. Now $f(t_1, \dots, t_m)$ can be put into form (25) for each r and by the above, we can write $a_{k,r}(t_1, \dots, \hat{t}_r, \dots, t_m) = \sum_{j=0}^{2n} c_{j,k} f(t_1, \dots, j, \dots, t_m)$.

$$f(t_1, \dots, t_m) = \sum_{j=0}^{2n} a_k(t_1, \dots, \hat{t}_r, \dots, t_m) t_r^k \tag{25}$$

Hence $a_k(t_1, \dots, \hat{t}_r, \dots, t_m)t_r^k$ is a sum of terms $\|t_0 + t_1x_1 + \cdots + jx_r + t_mx_m\|^{2n}$ each of which is continuous in $(t_1, \dots, \hat{t}_r, \dots, t_m)$. We can now apply Lemma (5.1.9) and conclude that $f(t_1, \dots, t_m)$ is a polynomial in $t_1, \dots, t_m$ jointly. As $|f(t_1, \dots, t_m)| \le (\|x_0\| + |t_1|\|x_1\| + \cdots + |t_m|\|x_m\|)^{2n}$ has degree at most $2n$.

Theorem (5.1.11)[185]: If an identity of the form (17) (with restrictions on constants as in Theorem (5.1.4)) holds for one space in $\wp_{2n}$then it holds for all spaces in $\wp_{2n}$. In particular, an identity holds in$\wp_{2n}$if and only if it holds in Hilbert space.

Proof. An ordered p-tuple $(i_0, \dots, i_{p-1})$ of nonnegative integers is a p-partition of d if $i_0 + \cdots + i_{p-1} = d$. For each p and d there are a finite number of such partitions. Suppose (17) holds for X in$\wp_{2n}$. The $i^{th}(m+1)$-partition of t, $(\pi(t,i,0), \dots, \pi(t,i,m))$ will be called $\pi(t,i)$ for short. Write $\pi(n,i) = \pi(i)$. In (17), we restrict x_i to a one-dimensional subspace of X generated by x, $\|x\| = 1$: $x_i = z_ix$, $\|x_i\| = |z_i|$, complex z_i. We obtain (26), which can be rewritten as (27) using $|z_i|^{2n} = (|z_i|^2)^n$.

$$\sum_{k=0}^{r} a_k |c_k(0)z_0 + \cdots + c_k(m)z_m|^{2n} = 0 \qquad (26)$$

$$\sum_{k=0}^{r} a_k \left(\sum_{i=0}^{m} \sum_{j=0}^{m} c_k(i)\bar{c}_k(j)z_i\bar{z}_j \right)^n = 0 \qquad (27)$$

Index the $(m+1)$-partitions of n from 1 to s, then (27) can be written as (28); where $c_k^{\pi_p}$ denotes $c_k(0)^{\pi_{p(0)}} \dots c_k(m)^{\pi_{p(m)}}$, z^{π_p} denotes $z_0^{\pi_{p(0)}} \dots z_0^{\pi_{p(0)}}$, $\bar{c}_k^{\pi_q}$ and $\bar{z}^{\pi_p}$ are defined analogously, where the double sum ranges independently over all pairs of $(m+1)$-partitions of n and where $d_{p,q}$ is the positive multinomial coefficient depending onπ_p and π_q

We now rewrite (28) as (29).

$$\sum_{p=0}^{s} \sum_{q=0}^{s} dp,q \left(\sum_{k=0}^{r} ak\, c_{k^p}^{\pi} \bar{c}_k^{\pi_q} \right) z^{\pi_p} \bar{z}^{\pi_p} = 0 \qquad (29)$$

A polynomial in $z_0, \bar{z}_0, \dots, z_m, \bar{z}_m$ which vanishes identically must have vanishing coefficients. As $d_{p,q} > 0$ we deduce (30) for all partitions π_p and π_q

$$\sum_{k=0}^{r} ak\, c_{k^p}^{\pi} \bar{c}_k^{\pi_q} = 0 \qquad (30)$$

Now let Y be a space in $\wp_{2n}$ and fix elements $x_0, \dots, x_m$ in Y. Write $f(z_0, \dots, z_m) = \|z_0x_0 + \cdots + z_mx_m\|^{2n}$ for complex $z_k = v_k + iw_k$, (v_k, w_kreal). Then $f(z_0, \dots, z_m) = \|v_0x_0 + \cdots + v_mx_m + w_0(ix_0) + \cdots + w_m(ix_m)\|^{2n}$ is a polynomial of degree 2n in $v_0, \dots, v_m, w_0, \dots, w_m$ by Lemma (5.1.10). From $2v_k = z_k + \bar{z}_k, 2iw_k = z_k - \bar{z}_k$it follows that $f(z_0, \dots, z_m)$ is a polynomial in z_k and $\bar{z}_k$ of degree 2n. Rewrite $f(z_0, \dots, z_m)$ in form (31), where the sum is taken over all

$(m+1)$-partitions of t and $2n-t$, indexed by jand k respectively, where the $b_{t,j,k}$'s are the coefficients and the condensed $z^{\pi(t,j)}$ is as before.

$$f(z_0,\dots,z_m)=\sum_{t=0}^{2n}\sum_{\pi(t,j)}\sum_{\pi(2n-t,k)} b_{t,j,k}\, z^{\pi(t,j)}\bar{z}^{\pi(2n-t,k)} \qquad (31)$$

Because $f(z_0,\dots,z_m)=f\left(e^{i\theta}z_0,\dots,e^{i\theta}z_m\right)$, (31) does not depend on θ.

Viewing (32) as a polynomial in $e^{i\theta}$ having as coefficients polynomials in z and $\bar{z}$, it follows that $b_{t,j,k}=0$ unless $t=n$.

$$\sum_{t=0}^{n} e^{(2t-2n)i\theta}\sum_{\pi(t,j)}\sum_{\pi(2n-t,j)} b_{t,j,k}\, z^{\pi(t,j)}\bar{z}^{\pi(2n-t,k)} \qquad (32)$$

We can thus rewrite (31) as (33), and insert this into (17).

$$f(z_0,\dots,z_m)=\sum_{p=0}^{s}\sum_{q=0}^{s} b_{p,q}\, z^{\pi_p}\bar{z}^{\pi_q} \qquad (33)$$

We obtain (34).

$$\sum_{k=0}^{r} a_k\left|c_k(0)z_0+\dots+c_k(m)z_m\right|^{2n}$$
$$=\sum_{k=0}^{r} a_k\left(\sum_{p=0}^{s}\sum_{q=0}^{s} b_{p,q}\, c_k^{\pi_p}\bar{c}_k^{\pi_q}\right)=\sum_{p=0}^{s}\sum_{q=0}^{s} b_{p,q}\sum_{k=0}^{r} a_k\, c_k^{\pi_p}\bar{c}_k^{\pi_q}=0 \qquad (34)$$

This last equality is a consequence of (30).

Never the less the parallelopiped law. Frechet [192] proved that any Banach space which satisfies (35) for all elements x, y, and z is a Hilbert space.

$$\|x+y+z\|^2-\|x+y\|^2-\|x+z\|^2-\|y+z\|^2$$
$$+\|x\|^2+\|y\|^2+\|z\|^2=0 \qquad (35)$$

Jordan and von Neumann reduced (35) to (1) by putting $z=-y$ and proving that the new condition is as strong as the old. The same sort of reduction applied to Theorem (5.1.12) will lead to a generalized parallelopiped law.

Theorem (5.1.12)[185]: If X is in $\mathscr{p}_{2n}$, and $m>2n$, then for all elements $x_1\ \cdots, x_m$, equation (36) holds, where the inner sum is taken over all k-tuples $1\le i_1\le\dots<i_k\le m$.

$$\sum_{i=0}^{m}(-1)^k\sum\left|x_{i_1}+\dots+x_{i_k}\right|^{2n}=0 \qquad (36)$$

Proof. In light of Theorem (5.1.11) it is sufficient to prove the theorem for elements in Hilbert space, indeed, we need only prove that (36) holds for all complex numbers z_j. The sum on the left-hand side is, in any case, a polynomial in the z_j's and $\bar{z}_j$'s.

$$\sum_{i=0}^{m}(-1)^k\sum\left|z_{i_1}+\cdots+z_{i_k}\right|^{2n}=0 \quad (37)$$

A monomial $z_{j_1}^{r_1}\dots z_{j_s}^{r_s}\cdot\bar{z}_{j_1}^{t_s}\dots\bar{z}_{j_s}^{t_s}$ with $\sum r_k=\sum t_k=n, r_k+t_k>0$will occur in $\left|z_{i_1}+\cdots+z_{i_k}\right|^{2n}$either with multinomial coefficient

$$\frac{(n!)^2}{(r_1!)\dots(r_s!)(t_1!)\dots(t_s!)}$$

or not at all depending on whether the j1's are contained in the i c's or not. Because $r_k+t_k>0, s\leq 2n<m$, hence each monomial occurs in (37). Indeed, for $k\geq s$, the j_i's are contained in precisely $\binom{m-s}{k-s}$k-tuples and so, altogether, a monomial $z_{j_1}^{r_1}\dots z_{j_s}^{r_s}\cdot\bar{z}_{j_1}^{t_s}\dots\bar{z}_{j_s}^{t_s}$ will occurin (37) with total coefficient (38). As $\sum_{j=0}^{m-s}(-1)^j\binom{m-s}{j}=0$, thesum in (38) vanishes and so so (37) is proved.

$$\sum_{j=0}^{m-s}(-1)^j\binom{m-s}{k-s}\frac{(n!)^z}{r_1!\dots r_s!\,t_1!\dots t_s!}$$

The identity holds in Hilbert space and hence in all $\wp_{2n}$.

Theorem (5.1.13)[185]: (The parallelepiped law). If X is in $\wp_{2n}$ and $r>n$, then for all elements $x_1,\dots,x_r$, the identity (39) holds in X, where the inner sum is taken over all choices of sign$\varepsilon_i=\pm1$ and all ordered, k-tuples with $1\leq i_1<\cdots<i_k\leq r$.

$$\sum_{k=1}^{r}(-1)^k2^{r-k}\sum\left\|\varepsilon_1x_{i_1}+\cdots+\varepsilon_kx_{i_k}\right\|^{2n}=0 \quad (39)$$

Proof. In Theorem (5.1.12), let $m=2r$, $x_{2k-1}=x'_k$ and $x_{2k}=x'_k$for$1\leq k\leq r$. Each term $\varepsilon_1x'_{i_1}+\cdots+\varepsilon_kx'_{i_k}$ will appear in (37) 2^{r-k} times depending on the joint inclusion or exclusion of the pairs $x_{2j-1}+x_{2j}$ for $j\neq i_l$. If the primes are dropped, (36) becomes (39) and the result follows.

Another identity which is satisfied in $\wp_{2n}$ is the following.

Theorem (5.1.14)[185]: If X is in $\wp_{2n}$ and $4k>2n$, then for all reala_i with $\sum_{i=1}^{m}a_i=0$, equation (40) holds for all elements $x_1,\cdots,x_m$.

$$\sum_{i=1}^{m}a_i\dots a_{i_{2k}}\left\|x_{i_1}\pm x_{i_2}\pm\cdots\pm x_{i_{2k}}\right\|^{2n}=0 \quad (40)$$

In (40), the inner sum is taken over all choices of sign ±1 as the i_j's range independently from 1 to m; the sum has $2^{2k-1}m^{2k}$terms.

The proof is reserved for [191]. Krivine [193] introduced an inequality which determines whether a space Xis isometrically isomorphic to a subspace of L_p. For technical reasons, the proof in [193] fails when $p=2n$. Theorem (5.1.14) illustrates what happens to Krivine's inequality in this case. Further implications will be considered in [191].

In 1970, Koehler [194] defined a G_2 space to be a complex Banach space on which a form $\langle x_1,\cdots,x_{2n}\rangle$ is defined satisfying (41)-(44).

$$\langle x,\cdots,x\rangle = \|x\|^{2n} \tag{41}$$

$$\overline{\langle x_1,\cdots,x_{2n}\rangle} = \langle x_{2n},\cdots,x_1\rangle \tag{42}$$

$$\langle\ \ \rangle \text{ is linear in } x_1,\cdots,x_{2n} \tag{43}$$

$$\langle\ \ \rangle \text{ is conjugate linear in } x_{n+1},\cdots,x_{2n} \tag{44}$$

Various properties of G_2 spaces are then discussed which parallel the development of Hilbert spaces. Among them are a proof that any G_2 space satisfies (39) and the construction through a polarization formula of a form $\langle x_1,\cdots,x_{2n}\rangle$ on any complex Banach space which satisfies (39). Thus, by Theorem (5.1.6), we can identify G_{2n} and $p_{2n}$and deduce the following corollary. (Note that one direction is immediate upon consideration of $\langle x+ty,\cdots,x+ty\rangle$.)

Corollary (5.1.15)[185]: A form $\langle x_1,\cdots,x_{2n}\rangle$ satisfying (41)-(44) can be defined on a complex Banach space if and only if $\|x+ty\|^{2n}$ is a polynomial in t for every xandy in X.

Section (5.2): Polynomials Norms in Banach Spaces:

We shall consider real Banach spaces, and, except where indicated, $L_{2n}(Y,\mu)$ with real-valued functions and real scalars. The phrase "X is embeddable in L_{2n}" is an abbreviation for "X is isometrically isomorphic to a subspace of $L_{2n}(Y,\mu)$ for some (Y,μ)." Although $\wp_{2n}$ was introduced and motivated in [11], that paper and this one are largely independent.

Norm functions. Suppose $X=\langle x_1,\dots,x_m\rangle$ is the real vector space spanned bythe x_i's and $\emptyset$ is a real function of m real variables. Under what circumstances does $\|\sum u_i x_i\| = \emptyset(u_1,\dots,u_m)$ make $(X,\|\cdot\|)$ a Banach space? For $u=(u_1,\dots,u_m)$, let $\emptyset(u)=\emptyset(u_1,\dots,u_m)$. From the standard definition of the norm, it is evident that conditions (i), (ii) and (iii) are necessary and sufficient. (Here, t is an arbitrary real.)

(i) $\emptyset(u)\geq 0$ and $\emptyset(v)=0$ implies $\emptyset(u)\equiv\emptyset(u+tv)$

(ii) $\emptyset(tu)=|t|\emptyset(u)$

(iii)$\emptyset(u)+\emptyset(v)\geq\emptyset(u+v)$

Condition (iii) is cumbersome to verify; the following lemma simplifies matters.

Lemma (5.2.1)[195]: Conditions (i), (ii) and (iii) are equivalent to (i), (ii) and (iii).

(D) $\psi(t)=\phi(u+tv)$ is a convex function in t for all u and v.

Proof. Assume (i), (ii) and (iii) and fix u and v. Then for $0\leq\lambda\leq 1, \lambda\psi(t_0)+(1-\lambda)\psi(t_0) = \lambda\phi(u+t_0v)+(1-\lambda)\phi(u+t_1v) = \phi(\lambda u+\lambda t_0 v)+(\phi(1-\lambda)(u)+\phi(1-\lambda)t_1 v)\geq \phi(u+(\lambda t_0+(1-\lambda)t_1)v) = \psi(\lambda t_0+(1-\lambda)t_1)$. Conversely, assume (A), (B) and (D), then $\emptyset(u)+\emptyset(v)=\psi(0)+\psi(1)\geq 2\psi\left(\frac{1}{2}\right)=\phi(u+v)$.

Observe that it is sufficient to check $\emptyset$ on all two-dimensional subspaces of X. For a discussion of a different condition on two-dimensional subspaces, see Dor [197]. We shall consider spaces X in $\mathcal{P}_{2n}$ for which $p(t) = \|x + ty\|^{2n}$ is a polynomial in t of degree 2n. When p is given in this way, we shall tacitly assume that $\|sx + ty\|^{2n} = s^{2n} p\left(\frac{t}{s}\right)$ for $s \neq 0$ and $\|y\|^{2n} = \lim_{t\to\infty} t^{-2n} p(t)$; that is, (B) is implicit.

Theorem (5.2.2)[195]: Suppose p is a nonnegative polynomial of degree 2n. Let $X = \langle x, y\rangle$ and define $\|\cdot\|$ on X by $(t) = \|x + ty\|^{2n}$. Then $(X, \|\cdot\|)$ is a Banach space if and only if $2np(t)p''(t) - (2n-1)\left(p'(t)\right)^2 \geq 0$ for all t.

Proof. With $\|sx + ty\|^{2n}$ defined as above, we need verify (A) and (D). Suppose $(X, \|\cdot\|)$ is a Banach space, then, $\psi(t) = \|x + ty\| = p(t)^{\frac{1}{2n}} = f(t)$ is convex. If x and y are linearly dependent then $\|x + ty\| = |a + bt|$, and for $p(t) = (a + bt)^{2n}, 2npp'' = (2n-1)(p')^2$. If x and y are linearly independent, then $f(t) > 0$ and f is convex if and only if $f''(t) = (2n)^{-2}\left(f(t)\right)^{1-4n}\left(2np(t)p''(t) - (2n-1)\left(p'^{(t)}\right)^2\right) \geq 0$.

On the other hand, suppose $2np\left((t)p''(t) - (2n-1)\left(p'^{(t)}\right)^2\right) \geq 0$ and $\|\cdot\|$ is defined as above. If $\|sx + ty\| = 0$ for $(s,t) \neq (0,0)$ then either $p(t_0) = 0$ or $\lim t^{-2n}$. As the hypothesized condition is translation-invariant, assume $t_0 = 0$ in the first case. Since $p(t) \geq 0$ we have $p'(0) = 0$; let $p(t) = a_k t^k + o(t^k), a_k \neq 0, k \geq 2$, for small t. Then $2np(t)p''(t) - (2n-1)\left(p'^{(t)}\right)^2 = -a_k^2 k(2n-k)t^{2-1} + o\left(t^{2k-2}\right)$ hence $k = 2n, p(t) = a_{2n}t^{2n}$ and $(X, \|\cdot\|)$ is a valid one-dimensional space. In the second case, let $p(t) = a_k t^k + o(t^k)$ for $k < 2n$, $a_k \neq 0$ and t large.
Then $k = 0$ and $(X, \|\cdot\|)$ is again one-dimensional.

Now suppose $p(t) > 0$. Let $u = dx + by, v = cx + ay$ be given; (D) will be satisfied provided , $\psi(t)$ is convex, where

$$\psi^{2n} = \|dx + by + t(cx + ay)\|^{2n} = |ct + d|^{2n} p\left((at + b)/(ct + d)\right).$$

(If $c = d = 0$, then, y is a constant and so convex). Note that ψ^{2n} is again a positive polynomial of degree 2n so that y" is continuous. It suffices, therefore, to check that $\psi''(t) \geq 0$ for $t \neq \frac{-d}{c}$. As above, $\psi''(t) \geq 0$ provided $2n\psi(t), \psi''(t) - (2n-1)(\psi'(t))^2 \geq 0$. A computation shows that this expression equals $(ad - bc)^2(ct + d)^{4n-4}\,(2np(u)p''(u) - (2n-1)(p'(u))^2)$, where $u = (at + b)/(ct + d)$. Thus, if $2npp'' - (2n-1)(p')^2 \neq 0$ then every ψ is convex and $(X, \|\cdot\|)$ is a Banach space.

It follows from Theorem (5.2.2) that the two-dimensional spaces in $\mathcal{P}_{2n}$ are characterized by $p(t) = \|x + ty\|^{2n}$, and that a study of such polynomials is appropriate. Note also that generators may be chosen to make any computations easier; in general, (D) must be separately verified for each two-dimensional subspace.

The cone P_{2n}. Let P_{2n} consist of all polynomials p of degree 2n for which $p(t) \geq 0$ and $C_{2n}(p(t)) = 2np(t)p''(t) - (2n-1)(p'(t))^2 \geq 0$.

If $p(t) = \sum \binom{2n}{k} a_k t^k$ then

$$C_{2n}\big(p(t)\big) = 4n^2(2n-1)\left(\left(\sum \binom{2n}{k} a_k t^k\right)\left(\sum \binom{2n-2}{k} a_{k+2} t^k\right) - \left(\sum \binom{2n-1}{k} a_{k+1} t^k\right)^2\right)$$

We shall omit the subscript 2n when it is superfluous. As defined, $C_{2n}(p)$ is a polynomial with nominal degree $4n-2$; the coefficients for t^{4n-2} and t^{4n-s} actually vanish identically.

Theorem (5.2.3)[195]: The set P_{2n} is a closed cone.

Proof. Suppose p is in P_{2n}. Then $C(p) \geq 0$ and for $\lambda \geq 0, \lambda p \geq 0$ and $C(\lambda p) = \lambda^2 C(p)$ so λp is in P_{2n}. If p_i is in P_{2n}, then $p_1 + p_2 \geq 0$ and $C(p_1+p_2) = C(p_1) + C(p_2) + 2np_1''p_2 + 2np_1p_2'' - (4n-2)p_1'p_2'$.Since p_ip_i'' we have $(2np_ip_i'')^{\frac{1}{2}} \geq (2n-1)^{\frac{1}{2}}|p_i|$so that $2np_1''p_2 + 2np_1p_2'' - (4n-2)p_1'p_2' = 2n\left((p_1''p_2)^{\frac{1}{2}} - (p_1p_2'')^{\frac{1}{2}}\right)^2 + 4n(p_1p_1'')^{\frac{1}{2}}(p_2p_2'')^{\frac{1}{2}} - (4n-2)(|p_1p_2| - p_1p_2) \geq 0$. Thus, P_{2n} forms a cone.

Associate $p(t) = \sum \binom{2n}{k} a_k t^k$ with the element $(a_0, \ldots, a_{2n})$ in R^{2n+1} and pull back the usual topology. Convergence is then or coefficient-wise. If $\{p_m\}$ is a sequence of polynomials in P_{2n} and $p_m - p$ then $C\big(p_m(t)\big) \to C\big(p(t)\big)$. Hence P_{2n} is closed.

By the proof of Theorem (5.2.2), if $p(t)$ is in P_{2n} then so is

$$(ct+d)^{2n} p\left(\frac{(at+b)}{(ct+d)}\right)$$

For future reference, observe that, if p_1 and p_2 are in P_{2n} and $C\big((p_1+p_2)(t_0)\big) = 0$ then $C\big(p_1(t_0)\big) = C\big(p_2(t_0)\big) = 0$, $p_1''(t_0)p_2(t_0) = p_1(t_0)p_2''(t_0)$and $p_1'(t_0)p_2'(t_0) \geq 0$.

Since P_{2n} is a cone, it is natural to study its extreme elements. For $q(t) = (bt + c)^{2n}, C_{2n}(q) \equiv 0$. Suppose $q = p_1 + p_2$, with p_i in P_{2n} If $b = 0$, then p_1 and p_2 must both be either point-wise nonnegative constants. Suppose $b \neq 0$, then we may normalize $b = 1$ so $q(t) = (t+c)^{2n}$, hence $p_1(-c) = p_2(-c) = 0$. As in the proof of Theorem (5.2.2), it follows that $p_i(t) = r_i(t+c)^{2n}$ so each p_i is a multiple of q. We have proved that $(bt+c)^{2n}$ is an extreme element in P_{2n}. Since P_{2n} is a cone, $\sum(b_kt + c_k)^{2n}$ is in P_{2n} This is to be expected in light of Theorem (5.2.2) applied to the subspace of ℓ_{2n} generated by $(b_1, b_2, \ldots)$ and $(c_1, c_2, \ldots)$. If $2n = 2, C_2(a_2t^2 + 2a_1t + a_0) = 4(a_0a_2 - a_1^2)$ so that $p \geq 0$ implies $C_2(p) \geq 0$. Hence the extreme elements of P_2 are precisely $(bt+c)^2$. Surprisingly enough, the same is true for $2n = 4$.

Theorem (5.2.4)[195]: The extreme functions of P_4, are $(bt+c)^4$; indeed, if p is in P_4 then $p(t)=(b_0t+c_0)^4+(b_1t+c_1)^4+c_2^4$ for some b_i and c_i.

Proof. Write $p(t)=\sum\binom{4}{k}a_kt^k$, then $(48)^{-1}C_4\big(p(t)\big)=(a_2a_4-a_3^2)t^4+(2a_1a_4-2a_2a_3)t^3+(a_0a_4+2a_1a_3-3a_2^2)t^2+(2a_0a_3-2a_1a_2)t+a_0a_2-a_1^2$. If $p(t_0)=0$, then, as before, $p(t)=a_4(t-t_0)^4$. If $C(p(t_0))=0$, then with $q(t)=p(t-t_0), C(q(0))=0$. As the conclusion is invariant under translation, assume $t_0=0$. In this case, since $C(p)\geq 0, a_0a_2=a_1^2$ and $a_0a_3=a_1a_2$. As $a_0=p(0)\neq 0$, let $a_1=ra_0$, then $a_2=r^2a_0$ and $a_3=r^3a_0$. If $a_4=r^4a_0+s$ then $C(p(t))=sa_0t^2(rt+1)^2$, so $s\geq 0$ and $p(t)=a_0(rt+1)^4+st^4$. (In general $p(t)=a_0(r(t-t_0)+1)^4+s(t-t_0)^4$.) If the degree of $C(p(t))$ is less than four, then by a similar argument, $p(t)=a_4(t+r)^4+s, s\geq 0$. Finally, suppose that $C(p(t))$ is a positive quadratic and let $p_2(t)=p(t)-\lambda$, then $C(p_2(t))=C(p(t))-4\lambda p''(t)$. Since p'' is quadratic, and $pp''>0, (4p''(t))^{-1}C(p(t))$ is continuous, goes to infinity quadratically in t, and achieves a minimum $\lambda_0>0$ at $t=t_0$. Thus $p(t)-\lambda_0$ is in P_4, $C\left(p_{\lambda_0}(t_0)\right)=0$; hence $p(t)=\lambda_0+a_0(r(t-t_0)+1)^4+s(t-t_0)^4$, which may be rewritten as in the conclusion.

By considering $(ct+d)^4p\left(\frac{(at+b)}{(ct+d)}\right)$ instead of p, we may replace c_2^4 by $s^4(ct+d)^4$ for any pre-selected c and d. It would be nice if this pattern continued for $2n\geq 6$; unfortunately, this is not the case.

Theorem (5.2.5)[195]: If $n\geq 3$ then there exists a polynomial p in P_{2n} which cannot be written $p(t)=\sum(b_kt+c_k)^{2n}$.

Proof Fix n and let $p(t)=t^{2n}+t^2+1$. A computation shows that $C_{2n}(p(t))=(8n^3-20n^2+12n)t^{2n}+(8n^3-4n^2)t^{2n-2}+(4-4n)t^2+4n$. Since $n\geq 3$, each term bu $(4-4n)t^2$ is positive. For $|t|\leq 1, (4-4n)t^2+4n\geq 0$; for $|t|\geq 1, (8n^3-4n^2)t^{2n-2}+(4-4n)t^2>(8n^3-4n^2-4n)t^2>0$. Thus $C_{2n}(p(t))\geq 0$ and p is in P_{2n}.

Suppose $t^{2n}+t^2+1=\sum(b_kt+c_k)^{2n}$; from the coefficient of t^4 and $t^2, 0=\sum b_k^4c_k^{2n-4}$ and $1=\binom{2n}{2}\sum b_k^2c_k^{2n-2}$. Since $n\geq 3$, the first implies that $b_kc_k=0$ for each k, and this contradicts the second.

The coefficient 1 for t^2 is not the best possible. The following proposition provides a sharp estimate.

Proposition (5.2.6)[195]: If $t^{2n}+\alpha t^{2k}+1$ is in P_{2n} then

$$0\leq\alpha\leq 2n(2n-1)c(k,n)$$

where $\big(c(k,n)\big)^n=(2k)^{-k}(2n-2k)^{k-n}(2k-1)^{n-2k}(2n-2k-1)^{2k-n}$.

We see then that there are extreme functions in $P_{2n}, n\geq 3$, which are not of the form $(bt+c)^{2n}$.

Proposition (5.2.7)[195]: The extreme rays of P_6 are generated by

$$(ct+d)^{2n} f_\lambda\left(\frac{(at+b)}{(ct+d)}\right)$$

where $f_\lambda(t) = t^6 + 6\lambda t^5 + 15\lambda^2 t^4 + 20\lambda^3 t^3 + 15\lambda^2 t^2 + 6\lambda t + 1$ and $|\lambda| \le \frac{1}{2}$ or $|\lambda| = 1$

Proof. As in Theorem (5.2.3), we consider special cases and then subtract various $(ct+d)^6$'s. Then f_λ are those polynomials for which $C_6(f_\lambda(0)) = 0$ and $C_6(f)$ is at most quartic.

As Proposition (5.2.7) is not directly relevant to the rest of this section and its proof is tedious, we omit the details. The general question of finding the extreme rays of P_{2n} for $n \ge 4$ remains open.

Let $\mathcal{Q}_{2n}$ denote the closure of the cone of polynomials of the form $\sum_{j=1}^{R}\left(b_j t + c_j\right)^{2n}$; $\mathcal{Q}_{2n} \subseteq P_{2n}$ with equality if and only if $2n = 2$ or 4. As any $2n+2$ distinct $2n^{\text{th}}$ powers are linearly dependent, we may assume that $R \le 2n+1$. Suppose $q(t) = \sum\binom{2n}{k} a_k t^k$. Then $q = \lim qm$ where $q_m(t) = \sum_{j=1}^{2n+1}\left(b_j^{(m)} t + c_j^{(m)}\right)^{2n}$. Since $\sum\left(b_j^{(m)}\right)^{2n} \to a_{2n}$ and $\sum\left(c_j^{(m)}\right)^{2n} \to a_0$, we may take $\left|b_j^{(m)}\right| < M, \left|c_j^{(m)}\right| < M$. Thus there exists a convergent subsequence with limit b_j and c_j so that one may write $q(t) = \sum_{j=1}^{2n+1}\left(b_j t + c_j\right)^{2n}$ for all q in $\mathcal{Q}_{2n}$. Similar considerations apply for the generalization of $\mathcal{Q}_{2n}$ to several variables.

In [196] we showed that $L_{2n}(Y,\mu)$ is in $\wp_{2n}$, that is, $\|f+tg\|^{2n} = \int |f+tg|^{2n} d\mu$ is a polynomial in t for all f and g. The converse, as we shall see, is false. Suppose that $X = \langle x, y\rangle$ is a two-dimensional space in $\wp_{2n}$ then $p(t) = \|x+ty\|^{2n}$ is in P_{2n}. Suppose that X is embeddable in $L_{2n}(Y,\mu)$, then $p(t) = \sum\binom{2n}{k} a_k t^k = \int\left((f+tg)\right)^{2n} d\mu = \|x+ty\|^{2n}$. By Holder's inequality, since $\int f^{2n}\, d\mu < \infty$ and $\int g^{2n}\, d\mu < \infty$ so that the integral can be broken up and $a_k = \int f^{2n-k} g^k\, d\mu$. Let $Y_0 = \{s \in Y : f(s) = 0\}, Z = Y - Y_0$; let $d\nu = f^{2n} d\mu$ and a_{2n} on Z. Then we have $a_k = \int h^k\, d\nu, 0 \le k \le 2n-1$, and $a_{2n} = \int_Z h^{2n}\, d\nu + \int_{Y_0} g^{2n}\, d\mu$. If $\phi(r) = \nu(h^{-1}\{(-\infty, r]\})$, then $a_k = \int_{-\infty}^{\infty} s^k\, d\emptyset$ for $0 \le k \le 2n-1$ and $a_{2n} \ge \int_{-\infty}^{\infty} s^k\, d\emptyset$.

Conversely, suppose there exists a nonnegative measure $\emptyset$ and a^k's so that $a_k = \int_{-\infty}^{\infty} t^k\, d\emptyset$ and $a_{2n} \ge \int_{-\infty}^{\infty} t^{2n}\, d\emptyset$. Define (Y,μ) as follows: $Y = R \cup \{p_0\}, \mu = \emptyset$ on R and $\mu\{p_0\} = a_{2n} - \int_{-\infty}^{\infty} s^{2n}\, d\emptyset$. Let $\left(f(s), g(s)\right) = (1, s)$ on R and $(0,1)$ on $\{p_0\}$. Then

$$\|f+tg\|^{2n} = \int_{-\infty}^{\infty} (1+st)^{2n}\, d\emptyset + \left(a_{2n} - \int_{-\infty}^{\infty} s^{2n}\, d\emptyset\right) t^{2n}$$

$$= \sum\binom{2n}{k} t^k \int_{-\infty}^{\infty} s^k\, d\emptyset + \left(a_{2n} - \int_{-\infty}^{\infty} s^{2n}\, d\emptyset\right) t^{2n} = \sum\binom{2n}{k} a_k t^k = p(t).$$

Fortunately, this transforms the embedding problem into the classical moment problem, which has been studied extensively. The complete solution is known, see for example see [198], and we may combine this solution with the previous discussion to obtain the following theorem.

Theorem (5.2.8)[195]: Let X be a two-dimensional Banach space in $\mathscr{p}_{2n}$ with generators x and y and let $p(t) = \|x + ty\|^{2n} = \sum \binom{2n}{k} a_k t^k$. Define the $(n+1) \times (n+1)$ matrix $B = (b_{ij})$ by $b_{ij} = a_{i+j}$ for $0 \le i, j \le n$. Then X is embeddable in L_{2n} if and only if the matrix B is positive semidefinite. Further, X is embeddable in L_{2n} if and only if p is in $\mathcal{Q}_{2n}$.

Corollary (5.2.9)[195]: If X is two-dimensional space in $\mathscr{p}_{2n}$ then X is embeddable in L_4. There are two-dintensional spaces in $\mathscr{p}_{2n}, n \ge 3$, which are not embeddable in L_{2n}.

Proof. Combine Theorems (5.2.4), (5.2.5) and (5.2.8).

The case for higher dimensions is less clearcut. Professor J. H. B. Kemperman [199] has pointed out, using techniques from [200] and [201], that the analogous moment problem in more than one variable has a solution which requires knowledge of all polynomials $f(u_1, u_2, \dots, u_p)$ of total degree 2n which are nonnegative for all real u_i.

Specifically, one transforms the polynomial $p(t_1, \dots, t_p) = \|x_0 + tx_1 + \dots + t_p x_p\|^{2n}$ for a space $X = \langle x_0, \dots, x_p \rangle$ into a family of equations $a(m_1, \dots, m_p) = \int \cdots \int t_1^{m_1} \cdots t_p^{m_p} d\mu$; $m_1 + \dots + m_p < 2n$, with inequality if $\sum m_i < 2n$. Suppose $f(u_1, u_2, \dots, u_p) \ge 0$ for all real u, and $f(u_1, \dots, u_p) = \sum b\,(m_1, \dots, m_p) u_1^{m_1} \cdots u_p^{m_p}$, where the sum is taken over all $m_i, \sum m_i < 2n$. Then certainly $\int \cdots \int \left(f(u_1, \dots, u_p)\right) d\mu = \int a(m_1, \dots, m_p)\, b(m_1, \dots, m_p) \ge 0$. It turns out this condition holding for all such f is sufficient for the existence of a measure with the desired property.

Since X is real, it is unreasonable to embed X in an L_{2n} space with complex scalars; one might, however, embed X in an $L_{2n}(Y, \mu)$ space with real scalars but complex-valued functions. This situation is taken care of by the following theorem.

Theorem (5.2.10)[195]: There is an isometry from the space of all complex-functions in $L_{2n}(Y, \mu)$, taken with real scalars, into real $L_{2n}(Z, \nu)$, where (Z, ν) consists of $2n + 1$ copies of (Y, μ).

Proof. It is well known that ℓ_2^2 is embeddable in any infinite dimensional Banach space. Let x and y be orthogonal generators of ℓ_2^2 and let $\bar{x}$ and $\bar{y}$ be their isometric images in ℓ_{2n}. Then $\|tx + uy\|^{2n} = \|t\bar{x} + u\bar{y}\|^{2n} = \sum (b_k t + c_k u)$; by the end, we may say that $(t^2 + u^2)^n = \sum_{k=1}^{2n+1} (b_k t + c_k u)^{2n}$. Define the mapping $\emptyset$ from $L_{2n}(Y, \mu)$ with complex-valued functions to $L_{2n}(Z, \nu)$ as follows: if $f = g + ih$ is the decomposition into real and imaginary parts, then $\emptyset(f) = b_k g + c_k h$ on the k^{th} copy of (Y, μ). For real $\lambda_i, \phi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \phi(f_1) + \lambda_2 \phi(f_2)$; $\|\phi(f)\|^{2n} = \sum_{k=1}^{2n+1} \int_Y (b_k g + c_k h)^{2n} d\mu = \int_Y (g^2 + h^2)^n d\mu = \int_Y |f|^{2n} d = \|f\|^{2n}$ so ϕ is an isometry.

We may actually choose b_k and c_k by: $b_k + ic_k = a(n)\exp(2\pi ki(n+1)^{-1})$, where $a(n) = 2\left(\binom{2n}{k}(2n+1)\right)^{\frac{-1}{2n}}$. Hilbert has proved that b_k and c_k may be chosen to be rational; see Ellison [3] for an extended discussion. In any case; it suffices to consider embeddings into real L_{2n}.

A counterexample is the remaining case for embedding is the three-dimensional one for $\wp_4$. We shall construct a three-dimensional space in $\wp_4$, which is not embeddable in L_4. Consequently, there are spaces with arbitrarily large dimensions which are not embeddable in L_4. This example is drastically simplified from the one appearing in [196].

Suppose $X = \langle x, y, z\rangle$ and a polynomial $p(u,v)$ with total degree 4 is given. Let $\|\cdot\|$ be defined on X by $\|x + uy + vz\|^4 = p(u,v)$; $\|x + uy + vz\|^4$; for $t \neq 1$ is defined in the usual way. In view of Lemma (5.2..1), we need check (A), (B) and (D) on every two-dimensional.

Subspace of X. Conditions (A) and (B) will be automatic. A two- dimensional subspace of X is either $\langle y, z\rangle$ or $\langle x + ay + cz, by + dz\rangle$ for some a, b, c, d. Thus, for $f(u,v) = (p(u,v))^{\frac{1}{4}}$, it suffices to show that $\psi(t) = f(a + bt, c + dt)$ is convex for all a, b, c, d. (We consider $\langle y, z\rangle$ separately.) Adopt the usual convention that $f_1(u,v) = \left(\frac{\partial}{\partial u}\right) f(u,v), f_{22}(u,v) = \left(\frac{\partial^2}{\partial v^2}\right) f(u,v)$, etc. Then $\psi''^{(t)} = (b^2 f_{11} + 2bdf_{12} + d^2 f_{22})(a + bt, c + dt)$. Hence it suffices to show that $f_{11} \geq 0, f_{22} \geq 0$ and, $f_{11}\, f_{22} \geq f_{12}^2$ at all points in the plane. If we can verify this for $f = p^{\frac{1}{4}}$ then $(X, \|\cdot\|)$ will be a Banach space.

Theorem (5.2.11)[195]: For $X = \langle x, y, z\rangle$, let $\|tx + uy + vz\|^4 = t^4 + 6t^2(u^2 + v^2) + (u^2 + v^2)^2$. Then $(X, \|\cdot\|)$ is a Banach space which is not embeddable in L_4.

Proof. Note that $\|tx + uy + vz\|^4 > 0$ unless $t = u = v = 0$ so that (A) is satisfied. On $\langle y, z\rangle$, $\|uy + vz\| = (u^2 + v^2)^{\frac{1}{2}}$ so $\langle y, z\rangle$ is isometric to ℓ_2^2 and (D) is satisfied. In general, let $f = p^{\frac{1}{4}}$, then $16f_{11} = p^{\frac{-7}{4}}(4pp_{11} - 3p_1^2), 16f_{22} = p^{\frac{-7}{4}}(4pp_{22} - 3p_2^2)$ and $16f_{12} = p^{\frac{-7}{4}}(4pp_{12} - 3p_1p_2)$. We must show that $4pp_{ii} - 3p_i^2 \geq 0$ and that

$$\begin{aligned}&(4pp_{11} - 3p_1^2)(4pp_{22} - 3p_2^2) - (4pp_{12} - 3p_1p_2)^2\\&\quad = 4p(4p(p_{11}p_{22} - p_{12}^2) - 3p_1^2p_{22} + 6p_1p_2p_{12} - 3p_2^2p_{11})\\&\quad = 4pD(p) \geq 0\end{aligned}$$

For $p(u,v) = \|x + uy + vz\|^4 = 1 + 6(u^2 + v^2) + (u^2 + v^2)^2$ let $w = u^2 + v^2$, then $p = 1 + 6w + w^2$, $P_1 = 4u(3 + w), p_2 = 4v(3 + w)$, $P_{11} = 4(3 + w + 2u^2)$, $P_{12} = 4(3 + w + 2v^2)$. Hence

$$4pp_{11} - 3p_1^2 = 16(3(1 - u^2)^2 + v^2(19 + 12u^2 + u^4) + v^4(9 + 2u^2) + v^6) \geq 0$$

and similarly $4pp_{22} - 3p_2^2 \geq 0$. Further, $p_{11}p_{22} - p_{12}^2 = 48(w + 3)(w + 1)$ and $p_1^2p_{22} - 2p_1p_2p_{12} + p_2^2p_{11=}64w(w + 3)^8$, hence

$$D(p) = 192(w + 3)(w + 1)(w^2 + 6w + 1) - 192w(w + 3)^8$$

$$= 192(w+3)(w-1)^2 \geq 0.$$

Thus $(x, \|\cdot\|)$ is a Banach space.

If X were embeddable in L_4 then for some f, g and h, $t^4 + 6t^2(u^2+v^2) + (u^2+v^2)^2 = \int_Y (tf+ug+vh)^4 d\mu$, so $\int f^4 = \int g^4 = \int h^4 = \int f^2 g^2 = \int f^2 h^2 = 1$, $\int g^2 h^2 = \frac{1}{3}$. The first five equations imply that $f^2 = g^2$ and $f^0 = h^2\mu$ - a.e.; this is contradicted by the sixth. Alternatively, in the spirit of the moment problem, $\int_Y (f^2 - g^2 - h^2)^2 d\mu = -\frac{1}{3}$. Either proof shows that X is not embeddable in L_4.

One can make a lengthy plausibility argument that the set of polynomials $p(t,u,v) = \|tx+uy+vz\|^4$ has 15 degrees of freedom for spaces in $\mathcal{P}_4$, and 14 for spaces in L_4. The last degree of freedom manifests itself here as the coefficient of u^2v^2.

Now we show other properties of $\mathcal{P}_{2n}$. Since $\mathcal{Q}_{2n} \subseteq P_{2n}$, with strict inclusion for $n \geq 3$, it is not obvious that spaces in $\mathcal{P}_{2n}$ are necessarily as "nice" as spaces in L_{2n}. For example, $L_{2n}(Y,\mu)$, is uniformly convex and uniformly smooth (see Lindenstrauss and Tzafriri [203] for definition) and hence reflexive. Holder's inequality says that, if $\int f^{2n} = \int g^{2n} = 1$ then $\left|\int f^k g^{2n-k}\right| \leq 1$ for $0 \leq k \leq 2n$. Thus if $q(t) = 1 + \sum_{k=1}^{2n-1}\binom{2n}{k}a_k t^k$ is in $\mathcal{Q}_{2n}$, then $|a_k| \leq 1$; indeed, $1 \geq a_k \geq r(k)$, where $r(2j) = 0, r(2j+1) = -1$. Clarkson's inequality states that $\|f+g\|^{2n} + \|f-g\|^{2n} \geq 2(\|f\|^{2n} + \|g\|^{2n})$ if $q(t) = \sum_{k=0}^{2n}\binom{2n}{k}a_k t^k$ is in $\mathcal{Q}_{2n}$, then $q(1) + q(-1) \geq 2(q(0) + a_{2n})$. As a whole, these properties extend to $\mathcal{P}_{2n}$, although numerical constants are generally weaker.

Koehler [204] defined a G_{2n} space to be a Banach space on which a $2n$-fold inner product $\langle x_1, \dots, x_{2n}\rangle$ is defined, satisfying certain regularity conditions. In [196] it was shown that G_{2n} spaces and $\mathcal{P}_{2n}$ spaces coincide. Koehler [205] proved that G_{2n} spaces are uniformly convex. That is, $\mathcal{P}_{2n}$ spaces are uniformly convex and thus reflexive. To prove uniform smoothness and the other regularity conditions we need the analogue to Holder's inequality.

Theorem (5.2.12)[195]: If $p(t) = 1 + \sum_{k=1}^{2n-1}\binom{2n}{k}a_k t^k + t^{2n}$ is in P_{2n} then there are constants so that $m(k,2n) \leq a_k \leq M(k,2n)$.

Proof. Since $p^{\frac{1}{2}}(t)$ is convex, by the triangle inequality on the space induced by p, $(1-|t|)^{2n} \leq (1+|t|)^{2n}$, so for $t \geq 0, (t-1)^{2n} \leq p(t) \leq (t+1)^{2n}$. The set of $2n-1$ equations $\sum_{k=1}^{2n-1}\binom{2n}{k}a_k j^k = p(j) - 1 - j^{2n}, 1 \leq j \leq 2n-1$, has a V and ermonde determinant, hence $\binom{2n}{k}a_k$ may be expressed in terms of $p(j) - 1 - j^{2n}$. Since $p(j)$ is bounded one obtains bounds on a_k which are, in general, wildly generous.

Alternatively, a sequence of polynomials with unbounded a_k's has a subsequence from which can be deduced the existence of $\bar{p}$ in P_{2n}, $\bar{p}(t) = \sum_{k=1}^{2n-1}\binom{2n}{k}\bar{a}_k t^k$, not all a_k's equal to zero. This yields a contradiction.

It follows that the set of all points $(a_1, \dots, a_{2n-1})$, A, in R^{2n-1} so that $1 + \sum_{k=1}^{2n-1} \binom{2n}{k} a_k t^k + t^{2n}$ is in P_{2n} forms a closed by Theorem (5.2.3) and bounded set by Theorem (5.2.12). Thus functionals, such as $p(1)$, achieve maxima and minima on A.

The actual values of $m(k, 2n)$ and $M(k, 2n)$ can be found in a few instances. Since $p(t)$ in P_{2n} implies $p(-t)$ and $t^{2n} p\left(\frac{1}{t}\right)$ are in $P_{2n}, m(2j+1, 2n) = -M(2j+1, 2n), m(2n-k, 2n) = m(k, 2n)$ and $m(2n-k, 2n) = m(k, 2n)$. As L_{2n} spaces are in $\mathcal{P}_{2n}, M(k, 2n) \geq 1$ and $m(k, 2n) \leq r(k)$. These coefficients are a two-dimensional property; consequently $m(k, 2n)$ and $M(k, 2n)$ are already determined for $2n = 2$ or 4.

In any case, $a_1 = \lim_{t\to\infty} t^{-1}(\|x+ty\| - \|x\|)$, so $\|a_1\| \leq 1$ and $M(1,2n) = -m(1,2n) = 1$. Further, $C\big(p(0)\big) = (2n)^2(2n-1)(a_0 a_2 - a_1^2)$ so $a_2 \geq 0$ and $m(2,2n) = 0$. The condition in Theorem (5.2.12) is, for general p in $P_{2n}, a_k \leq M(k, 2n) a_0^{t-\alpha} a_{2n}^{\alpha}$, where $\alpha = \frac{k}{2n}$. From the convexity of x^{α}, extreme values are attained on extreme elements in P_{2n}. In this way, considering Proposition (5.2.7), one can show that $M(3,6) = -m(3,6) = 1$ and $M(2,6) = 5^{\frac{-5}{3}}\left(1565 + 496\sqrt{10}\right)^{\frac{1}{3}} \cong 1.000905$. The general problem remains open.

Theorem (5.2.13)[195]: If X is in $\mathcal{P}_{2n}$ then X is uniformly convex, uniformly smooth and so is reflexive.

Proof. The uniform convexity follows from Koehler, or by noting that $\|x\| = \|y\| = 1, \|x+y\| = 2$ implies $\|x+ty\| = 1+t$ for $t \geq 0$ so $p(t) = (1+t)^{2n}$ and $\|x-y\| = 0$. Since the set of coefficients A_ε, for which $\|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon$ is compact, $\|x+y\|$ achieves a maximum, which is strictly less than 2.

For uniform smoothness, let $\|x\| = \|y\| = 1$. For $t \leq \tau$, by Taylor's theorem, $\|x+ty\| + \|x-ty\| = 2 + (2n-1)(a_2 - a_1^2)t^2 + o(t^2)$. Thus $\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 \leq c\tau^2 + o(\tau^2)$ so X is uniformly smooth.

If X is any Banach space, suppose $t = \|y\| \geq \|x\| = 1$ and $u = \|x+y\| \geq \|x-y\| = v$. Then $u+v \geq 2t$ so $u^p + v^p \geq u^p + (2t-u)^p \geq 2t^p \geq t^p + 1$. That is, $\|x+y\|^p + \|x-y\|^p \geq \|x\|^p + \|y\|^p$ with equality if and only if $\|x\| = \|y\| = \|x+y\| = \|x-y\| = 1$. In this case, by the triangle inequality, $\|x+y\| \equiv 1$ for $|r| \leq 1$ so X cannot be in $\mathcal{P}_{2n}$. Thus, by the compactness of A, $\|x+y\|^{2n} + \|x-y\|^{2n} \geq c(n)(\|x\|^{2n} + \|y\|^{2n})$ for x and y in X in $\mathcal{P}_{2n}$. Taking $x = 0, c(n) \leq 2$.

Theorem (5.2.14)[195]: If X is in $\mathcal{P}_{2n}$ for $n \leq 3$ then $\|x+y\|^{2n} + \|x-y\|^{2n} \geq 2(\|x\|^{2n} + \|y\|^{2n})$, but this is not necessary true for $n \geq 4$.

Proof. For $n \le 2$, X is embeddable in L_{2n}. For $n = 3$, let $\|x + y\|^6 = \sum_{k=0}^{6}\binom{6}{k}a_k t^k$ then $\|x + y\|^6 + \|x - y\|^6 - 2\|x\|^6 - 2\|y\|^6 = 30(a_2 + a_4) \ge 0$ since $m(2,6) = m(4,6) = 0$.

Fix $n \ge 4$ and set $p_\varepsilon(t) = 1 + \varepsilon(t^2 - 3t^4 + t^6) + t^{2n}$ and $\|x + y\|^{2n} = p_\varepsilon(t)$, then $\|x + y\|^{2n} + \|x - y\|^{2n} - (\|x\|^{2n} + \|y\|^{2n}) = -2\varepsilon > 0$ for $\varepsilon > 0$. A computation shows that $C_{2n}(p_\varepsilon(t)) = 4n^2(2n-1)t^{2n-2} + \varepsilon(g(t) + \varepsilon h(t))$, where $g(t) = 4n^2(2n-1)t^{2n-2}(t^2 - 3t^4 + t^6) + 2n(1 + t^{2n})(2 - 36t^2 - 30t^4) - 4n^2(2n-1)t^{2n-1}(2t - 12t^3 + 6t^5)$ and $h(t) = 2n(t^6 - 3t^4 + t^2)(30t^4 - 36t^2 + 2) - (2n-1)(6t^5 - 12t^3 + 2t)^2$.

As $n \ge 4$, the highest order term of $g + \varepsilon h$ is

$$2n(4n^2 - 26n + 42)t^{2n+4},$$

there exist ε_0 and R so that for $0 \le \varepsilon \le \varepsilon_0$ and $|t| > R, (g + \varepsilon h)(t) \ge 0$ and thus $C_{2n}(p_\varepsilon(t)) > 0$. As $(g + \varepsilon h)(0) = 4n$, for $0 \le \varepsilon \le \varepsilon_0$ and $|t| < \delta$ or $|t| > R, C_{2n}(p_\varepsilon(t)) > 0$. On the remaining (compact) set, t^{2n-2} is positive and $|g| + \varepsilon_0|h|$ is bounded, so for some further reduced range of ε, $C_{2n}(p_\varepsilon) > 0$ and p_ε is in P_{2n}.

For $n = 4$ take $\varepsilon = .04$, then $p_\varepsilon(t) = t^8 + .04t^6 - .12t^4 + .04t^2 + 1$. A direct computation shows that $C_8(p_\varepsilon(t)) = 64(t^{12} + 1) + 11.5392(t^{10} + t^2) + 9.68(t^8 + t^4) + 447.9104t^6$. If we factor out $.64t^6$ and let $u = t^2 + t^{-2}$, then we obtain $u^3 - 18.03u^2 + 12u + 735.92 = q(u)$. (The range for $t^2 + t^{-2}$ is $u \ge 2$.) Clearly $q(2) > 0$, and q achieves its minimum when $u = u_0 = 6.01 + \sqrt{32.1201} \cong 11.67$. Since $q(u_0) \cong 9.79 > 0, C(4) \le 1.96$. This bound is not sharp. This example also shows that $m(4,8) < 0$.

The question of describing spaces dual to spaces in $\mathcal{P}_{2n}$ also remains open. Indeed it is false, in general, that the dual space to a subspace of $L_p(Y, \mu)$ is necessarily embeddable in $L_q, p^{-1} + q^{-1} = 2$. For example, if $p = \frac{2n}{(2n-1)}, x = (1,1,0), y = (1,0,1)$ and X is the subspace of ℓ_p^3 generated by x and y, then X^* is not even in $\mathcal{P}_{2n}$, let alone L_{2n}. We omit the proof.

Krivine inequalities. Krivine [206] has described necessary and sufficient conditions for a space to be embeddable in L_p provided p is not an even integer. Krivine's proof does not apply when $p = 2n$ because it involves the Taylor series remainder of $\cos(x)$. Theorem (5.2.16) discusses this case and provides an underlying reason for this failure when viewed in conjunction with Corollary(5.2.9).

Theorem (5.2.16)[195]: (Krivine). If $2r - 2 < p < 2r \le 4k$ then a necessary and sufficient condition for X to be embeddable in L_p is that (45) holds for all elements x_i and all choices of real scalars r_i with $\sum r_i = 0$. The sum is taken as the i_j's range independently from 1 to m and as the ε_j's range over all choices of sign ± 1. The sum has $m^{2k}2^{2k-1}$ terms.

$$(-1)^r \sum_{i_1=1}^{m} \cdots \sum_{i_{2k}=1}^{m} r_{i_1} \cdots r_{i_{2k}} \sum_{\varepsilon_j} \left\| x_{i_1} + \varepsilon_2 x_{i_2} + \cdots + \varepsilon_{2k} x_{i_{2k}} \right\|^p \geq 0 \qquad (45)$$

Theorem (5.2.17)[195]: If $4k > 2n$ and X is in $\wp_{2n}$, then the sum in (45), taken with $p = 2n$, is identically zero.

Proof. By (Theorem 11 in [196]), it suffices to verify any linear identity on one space in $\wp_{2n}$, say C. Since in (45) all elements are combined with real coefficients, by Theorem (5.2.10), we may embed C isometrically in R. It therefore suffices to check that (46) holds in R.

$$\sum_{i_1=1}^{m} \cdots \sum_{i_{2k}=1}^{m} r_{i_1} \cdots r_{i_{2k}} \sum_{\pm} \left(t_{i_1} + \varepsilon_2 t_{i_2} + \cdots + \varepsilon_{2k} t_{i_{2k}}\right)^{2n} = 0 \qquad (46)$$

Because of the signs in the inner sum, we may rewrite this in the form $\sum_j d_j t_{i_1}^{\pi_j(1)} \cdots t_{i_{2k}}^{\pi_j(2k)}$, where j indexes all partitions of 2n into 2k even integers and dj is the positive multinomial coefficient. If we now exchange the order of summation, then (46) becomes (47).

$$\sum_j d_j \prod_{s=1}^{2k} \left(\sum_{i_s=1}^{m} r_{i_s} t_{i_s}^{\pi_j(s)} \right) = 0 \qquad (47)$$

Fix j; since $4k > 2n$, at least one of the $\pi_j(s)$'s is zero. Thus, one term in the product is $\sum r_i = 0$, each term in the sum vanishes and (47) is verified.

For $2n \geq 4$, there are spaces in $\wp_{2n}$ which are not embeddable in L_4 so that Krivine's inequalities do not extend. For $4k = 2n$ and $X = L_{2n}(Y, \mu)$, it is not hard to show that the left hand side of (45) becomes $(\int \sum r_i x_i^2 \, d\mu)^{2k}$ which is nonnegative. If, on the other hand, X is the space in Theorem (5.2.11), $x_1 = x, x_2 = y, x_3 = z, r_1 = -2, r_2 = r_3 = 1$, then

$$\sum_{1}^{3} \sum_{1}^{3} r_i r_j \sum \left\| x_i \pm x_j \right\|^4 = -16.$$

It is possible that a careful study of Krivine's inequality for such borderline cases could lead to an embedding theorem for $L_p, p = 2n$.

Section (5.3): Real Banach Spaces with Polynomials and Identities:

We study linear identities via the duality theory for real polynomials and functions on Banach spaces, which allows for a unified treatment and generalization of some classical results in the area. The basic idea is to exploit point evaluations of polynomials, as e.g. in [208]. As a by-product we also obtain a curious variant of the well-known Hilbert lemma on the representation of the even powers of the Hilbert norm as sums of powers of functional . In (generalizing [209] and [210]) we prove that under certain natural assumptions identities derived from point evaluations can be satisfied only by polynomials. We apply the Lagrange interpolation theory in order to create a machinery allowing the creation of linear identities which characterize spaces of polynomials of prescribed degrees. We elucidate the special situation when all the evaluation points are collinear .

The work is based on the theory of functional equations in the complex plane due to Wilson [209] and Reznick (in the homogeneous case) [210,211], and the classical characterizations of polynomials due to Frechet [212,213], and Mazur and Orlicz [214,215], which can be summarized in the following theorem.

Theorem (5.3.1)[207]: Let X, Y be real Banach spaces, $f: X \to Y$ be continuous, $n \in N \cup \{0\}$. TFAE

(i) $f \in p^n(X; Y)$.

(ii) $\Delta^{n+1} f(x; h_1, \ldots, h_{n+1}) = 0$ for all $x, h_i \in X$.

(iii)$f\restriction_E$ is a polynomial of degree at most n for every affine one-dimensional subspace E of X.

(iv)

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} P(x+kh) = 0 \quad \text{for all } x, h \in X$$

Here we use the higher order differences defined as follows.

$$\Delta^k f(x: h_1, \ldots, h_k) = \sum_{j=0}^{k} \sum_{A\subset\{1,\ldots,k\},|A|=j} (-1)^{k-j} f\left(\sum_{l\in A} h_l\right)$$

In particular,

$$\Delta^k f(x: h_1, \ldots, h_k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+ih).$$

The theory of linear identities for Banach space norms was developed. Its first and well-known result is a theorem of Jordan and von Neumann.

Theorem (5.3.2)[207]: (See [216]) . Let $(X, \|.\|)$ be a Banach space such that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in X.$$

Then X is isometric to a Hilbert space.

Note that a real Banach space X is isometric to a Hilbert space iff $\|.\|^2$ is a 2-homogeneous polynomial. Theorem (5.3.2) has been the basis of subsequent development with the aim of using similar identities in order to characterize the Hilbert spaces, or the classes of Banach spaces allowing the polynomial norms, e.g. Carlsson [217], Day [218,219], Giles [220], Johnson [221], Koehler [222,223], Lorch [224], Reznick [210,211] and Senechalle [225]. This theory is closely related to the isometric Banach space theory, see e.g. Koldobsky and Konig [226]. We develop an abstract approach to the theory of linear identities, generalizing Wilson's and Reznick's works. The novelty lies in giving a new functional-analytic meaning to these identities, finding the link to the Lagrange interpolation, and finding a general method for establishing new identities with prescribed properties.

Let X, Y be real Banach spaces. We denote by $p(d\, X;\ Y)$ (resp. $P^d(X;\ Y)$) the Banach space of continuous d-homogeneous polynomials from X to Y (resp. continuous polynomials of degree at most d).

Let $n \in \mathbb{N}, d \in \mathbb{N} \cup \{0\}$. We are go in$\mathbb{R}^n)^* = \mathbb{R}^n$, using the dot product. For simplicity of notation, we put $F_{n,d} = p(d\mathbb{R}^n; \mathbb{R})$. Denote the set of multi-indices by

$$\mathcal{J}(n, d) = \left\{\alpha :\ \{1, \dots, n\} \to \{0, \dots, d\}: |\alpha| = \sum_{i=1}^{n} \alpha(i) = d\right\}$$

One gets dim $F_{n,d} = |\mathcal{J}(n, d)| = \binom{n + d - 1}{n - d}$ Further, we put $\Pi_{n,d} = p^d(\mathbb{R}^n; \mathbb{R})$. Let $\mathcal{J}(n, d) = \bigcup_{1=0}^{d} \mathcal{J}(n, l)$ be the set of all multi-indices of degree at most d. Clearly, for every $P \in \Pi_{n,d}$ there exists a uniquely determined representation $P(x) = \sum_{\alpha \in \mathcal{J}(n,d)} a_\alpha x^\alpha$, where $\alpha = \prod_{i=1}^{n} x_i^{\alpha(i)}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Fact (5.3.3)[207]:

$$\dim \prod\nolimits_{n,d} = \sum_{1=0}^{d} \binom{n + l - 1}{n - 1} = \binom{n + d}{n} = \dim F_{n+1,d}\,.$$

Moreover, there is a natural linear isomorphism $i : F_{n+1,d} \to \prod_{n,d}$, given by the restriction $i(P) = P\,\Gamma_E$, where $E = \{x \in \mathbb{R}^{n+1}: x_{n+1} = 1\}$ is an affine hyperplane. In other words, performing i on a d-homogeneous polynomial means replacing the $n + 1 - st$ coordinate by the constant 1.

Let $C(\mathbb{R}^n)$ be the space of all continuous functions on $(\mathbb{R}^n)$ Point evaluations at $x \in (\mathbb{R}^n)$ belong to the linear dual of $C(\mathbb{R}^n)$. Point evaluations separate elements of $C(\mathbb{R}^n)$. For $z \in \mathbb{R}^n$ we are going to use the notation $\mathbf{z} = 1z \in C(\mathbb{R}^n)^*$ where $\mathbf{z}(f) = f(z), f \in C(\mathbb{R}^n)$, and we will call these evaluation functionals nodes. To simplify the language, we will occasionally identify $z \in \mathbb{R}^n$ with its corresponding node **z**, calling the elements of $\mathbb{R}^n$themselves nodes. We are going to introduce an abstract formalism suitable for working with nodes and their linear combinations. Consider the linear space $\mathcal{F}(\mathbb{R}^n)$ of all formal finite linear combinations of nodes. It is important to note that a linear multiple ξ**z** of the node **z** is not the same element as the node corresponding to the point $\xi z \in \mathbb{R}^n$. Informally, whenever we write ξz as an element of $\mathcal{F}(\mathbb{R}^n)$), it is understood that we are dealing with the element ξ**z**. In order to distinguish the usual vector summation from the space $\mathbb{R}^n$ from the formal summation of the nodes we will introduce the new summation symbol ⊞. So for every $\mathbf{x} \in \mathcal{F}(\mathbb{R}^n)$there exist $a_i \in \mathbb{R}, x_i \in \mathbb{R}^n$ so that

$$\mathbf{x} = a_1 x_1 \boxplus \cdots \boxplus a_k x_k = \boxplus - \sum_{i=1}^{k} a_i x_i\,.$$

The previous expression is unique if x_i are assumed pairwise distinct and $a_i \neq 0, i = 1, \dots, k$.

The operation ⊞ formally acts on $\mathbf{x} = \boxplus - \sum_{i=1}^{k} a_i x_i$ and $\mathbf{y} = \boxplus - \sum_{i=1}^{1} b_i\, y_i$ as

$$\mathbf{x} \boxplus \mathbf{y} = \left(\boxplus - \sum_{i=1}^{k} a_i x_i \right) \boxplus \left(\boxplus - \sum_{i=1}^{l} b_i y_i \right).$$

Similarly, we define the scalar multiplication of $\xi \in R$ and $\mathbf{x}$ as

$$\xi \mathbf{x} = \boxplus - \sum_{i=1}^{k} (\xi a_i) x_i .$$

With these operations $\mathcal{F}(\mathbb{R}^n)$ is a linear space. Then $\langle C(\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n) \rangle$form a dual pair [227] with the evaluation

$$\langle f, x \rangle = \sum_{i=1}^{k} a_i f(x_i)$$

Restricting this dual pairing to subspaces $F_{n,d}$ (resp. $\Pi_{n,d}$) of $C(\mathbb{R}^n)$ leads to a dual factorization of the action of $\boxplus$ on $\mathcal{F}(\mathbb{R}^n)$so that $\mathbf{x_d} = \boxplus_d - \sum_{i=1}^{k} a_i x_i \left(\text{res.}\, x^d = \boxplus^d - \sum_{i=1}^{k} a_i x_i\right)$ and

$$x_d = \boxplus_d - \sum_{i=1}^{k} a_i x_i = y_d = \boxplus_d - \sum_{i=1}^{l} b_i y_i$$

iff

$$\langle f, \mathbf{x} d \rangle = \langle f, \mathbf{y} d \rangle \quad \text{holds for all } f \in F_{n,d}$$

(and the resp. case of $\Pi_{n,d}$).

Thus we have a (non-unique) representation of the elements of $F^*_{n,d}$ (resp. $\Pi^*_{n,d}$) as elements in $\mathcal{F}(\mathbb{R}^n)$, given by

$$\langle P, \mathbf{x} \rangle = \langle P, \boxplus - \textstyle\sum_{i=1}^{k} a_i x_i \rangle = \displaystyle\sum_{I-1}^{K} a_i P(x_i).$$

$P \in F_{n,d}$ (resp. $\Pi_{n,d}$), $\mathbf{x} = \boxplus - \sum_{i=1}^{k} a_i x_i$. We let $\kappa_d \to \mathcal{F}(\mathbb{R}^n)$ be the subspace consisting of all elements for which

$$\langle P, \boxplus - \sum_{i=1}^{k} a_i x_i \rangle = 0 \text{ holds for all } P \in \Pi_{n,d}$$

Then $\Pi^*_{n,d} = F(R^n)/K_d$. Suppose $A = \{y_1, \ldots, y_r\} \subset R^n$. We say that the corresponding set of nodes $A = \{y_1, \ldots, y_r\}$ is $F_{n,d}$ independent if the nodes are linearly independent as elements of $F^*_{n,d}$. For simplicity, if the space $F^*_{n,d}$. is understood, we will often drop the boldface notation and say that A is a set of nodes, and that A is $F_{n,d}$. -independent. It is clear from basic linear algebra that A is $F_{n,d}$-independent iff there exist dual elements $\{h_1, \ldots, h_r\} \subset F_{n,d}$ so that $h_j(y_K) = \delta_j^k$, where δ is the Kronecker delta. If $\{y_1, \ldots, y_r\}$ are $F_{n,d}$ -independent then $r = |\mathcal{J}(n,d)|$. In case of $r = |\mathcal{J}(n,d)|$, $F^*_{n,d} = \text{span}(\{y_K\}_{k=1}^{r})$ and we call $\{y_K\}_{k=1}^{r}$ a basic set of nodes for $F_{n,d}$ A classical example of a basic set of nodes for $F_{n,d}$ is the set $\mathcal{J}(n,d)$ (Biermann, see [208]). The following result is immediate.

Proposition (5.3.4)[207]: Let $r = |\mathcal{J}(n,d)|$. If $\{y_k\}_{k=1}^r$ is a basic set of nodes for $F_{n,d}$ and $\{h_k\}_{k=1}^r \subset F_{n,d}$ is its dual basis, then for all $P \in F_{n,d}$

$$P(x) = \sum_{k=1}^{r} P(y_k)h_k(x), x \in \mathbb{R}^n.$$

The following is a general characterization of basic sets of nodes [228,208].

Theorem (5.3.5)[207]: Let $r = |\mathcal{J}(n,d)|, \mathcal{J}(n,d) = \{\alpha_1, \dots, \alpha_r\}$. Let $\{y_k\}_{k=1}^r \subset \mathbb{R}^n$. Then $\{y_k\}_{k-1}^r$ is a basic set of nodes for $F_{n,d}$ iff it holdsdet

$$\det\begin{pmatrix} y_1^{\alpha_1} & y_1^{\alpha_2} \cdots & y_1^{\alpha_r} \\ y_2^{\alpha_1} & \ddots\, y_2^{\alpha_2} & y_2^{\alpha_r} \\ & \vdots & \\ y_r^{\alpha_1} & \cdots y_r^{\alpha_2} & y_r^{\alpha_r} \end{pmatrix} \neq 0.$$

Moreover, if $\{y_k\}_{k=1}^r$ is a basic set of nodes for $F_{n,d}$, then every $P \in F_{n,d}$ can be written uniquely as $P(x) = \sum_{k=1}^r \langle a_k y_k, x\rangle^d$.

The same notation and terminology applies to the case of $\Pi_{n,d}$ spaces. Analogously, for $r = |\mathcal{E}(n,d)|$, we say that $\{y_k\}_{k=1}^r \subset \mathbb{R}^n$ is a basic set of nodes for $\Pi_{n,d}$ if these elements form a linear basis of $\Pi_{n,d}^*$. Observe that basic sets of nodes exist, as the pointwise evaluations form a separating set of functionals for $\Pi_{n,d}$. The following is a general characterization of basic sets of nodes for $\Pi_{n,d}$, analogous to Theorem (5.3.5), see [228].

Theorem (5.3.6)[207]:L Let $r = |J(n,d)|, J(n,d) = \{\alpha_1, \dots, \alpha_r\}$. Let $\{y_k\}_{k=1}^r r \subset \mathbb{R}^n$. Then $\{y_k\}_{k=1}^r$ is a basic set of nodes for $\Pi_{n,d}$ iff it holds

$$\det\begin{pmatrix} y_1^{\alpha_1} & y_1^{\alpha_2} \cdots & y_1^{\alpha_r} \\ y_2^{\alpha_1} & \ddots\, y_2^{\alpha_2} & y_2^{\alpha_r} \\ & \vdots & \\ y_r^{\alpha_1} & \cdots y_r^{\alpha_2} & y_r^{\alpha_r} \end{pmatrix} \neq 0.$$

Moreover, if $\{y_k\}_{k=1}^r$ is a basic set of nodes then every node $y \in \mathbb{R}^n \hookrightarrow \Pi_{n,d}^*$ can be written uniquely as a linear combination of the elements in $\{y_k\}_{k=1}^r$. More precisely, $\mathbf{y} = \boxplus^d - \sum_{k=1}^r a_k\, y_k$ iff $\{a_k\}_{k=1}^r$ form a solution of the system of linear equations

$$\sum_{k=1}^{r} a_k y_k^\alpha = y^\alpha, \alpha \in \mathcal{J}(n,d)$$

The generalized Lagrange formula is an expression of linear dependence of nodes in the dual of $\Pi_{n,d}$.

Theorem (5.3.7)[207]: (Generalized Lagrange formula). Let $r = |J(n,d)|, \{y_k\}_{k=1}^r$ be a basic set of nodes for $\Pi_{n,d}$. Then for every $z \in \mathbb{R}^n \setminus \{y_k\}_{k=1}^r$ there exists a unique set of coefficients $a_k(z) \in \mathbb{R}$ such that $\mathbf{z} = \boxplus^d \sum_{k=1}^r a_k(z) y_k$. The functions $z \to a_k(z)$ are polynomials of degree at most d, given by the formula

$$ak(z) = \frac{\det\begin{pmatrix} y_1^{\alpha_1} & y_1^{\alpha_2} \cdots & y_1^{\alpha_r} \\ y_2^{\alpha_1} & \ddots\, y_2^{\alpha_2} & y_2^{\alpha_r} \\ & \vdots & \\ z^{\alpha_1} & \cdots z^{\alpha_2} & z^{\alpha_r} \\ & \vdots & \\ y_r^{\alpha_1} & y_r^{\alpha_2} & y_r^{\alpha_r} \end{pmatrix}}{\det\begin{pmatrix} y_1^{\alpha_1} & y_1^{\alpha_2} \cdots & y_1^{\alpha_r} \\ y_2^{\alpha_1} & \ddots\, y_2^{\alpha_2} & y_2^{\alpha_r} \\ & \vdots & \\ y_k^{\alpha_1} & \cdots y_k^{\alpha_2} & y_k^{\alpha_r} \\ & \vdots & \\ y_r^{\alpha_1} & y_r^{\alpha_2} & y_r^{\alpha_r} \end{pmatrix}}$$

Then $\{a_k, y_k\}_{k=1}^r$ is a biorthogonal system in $\Pi_{n,d} \times \Pi_{n,d}^*$ and the formula

$$P(z) = \sum_{k=1}^{r} a_k(z) P(y_k)$$

is valid for $P \in \Pi_{n,d}$.

We remark that the problem of characterizing geometrically basic sets of nodes for $\Pi_{n,d}$, when $n \geq 2$, is open, and it is important for approximation theory and its applications in numerical mathematics. We refer to [228,229,230] for more results and references. An interesting special case is due to Chung and Yao [229], for certain implicitly described sets of nodes. Let us briefly describe this elegant result.

Let $x_1, \ldots, x_k \in \mathbb{R}^n, k \geq n$, be such that every affine hyperplane in $\mathbb{R}^n$ contains at most n points of $0, x_1, \ldots, x_k$. Then for every $I \subset \{1, \ldots, k\}$ such that $\#I = n$ there exists a unique point $z_I \in \mathbb{R}^n$ such that $\langle z_I, x_i \rangle = -1$ for every $i \in I$ and $\langle z_I, x_i \rangle \neq -1$ for every $i \notin I$. Indeed, by the hypothesis the points $x_i, i \in I$, lie in an affine hyperplane H not containing 0, and $x_i \notin H$ for every $i \notin I$. Define

$$h_I(x) := \prod_{i=1, i \notin I}^{k} \frac{1 + \langle x, x_i \rangle}{1 + \langle z_I, x_i \rangle} \text{ for } x \in \mathbb{R}^N.$$

Then hI is well defined and $h_I \in \Pi_{n,k-n}$. Further, if $J \subset \{1, \ldots, k\}$ is such that $\# J = n$, then $h_I(z_J) = \delta_{I,J}$ (δ is the Kroneckerdelta). Hence the set $\{x_I : I \subset \{1, \ldots, k\}, \#I = n\}$ is a basic set of nodes for $\Pi_{n,k-n}$ (since the cardinality of this set is $\binom{k}{n} = \dim \Pi_{n,k-n}$).

Let $\mathcal{L} \in L(\mathbb{R}^N; \mathbb{R}^M)$. We let $\tilde{L} \in \mathcal{L}(\mathcal{F}(\mathbb{R}^N); \mathcal{F}(\mathbb{R}^M))$ be defined as

$$\tilde{L}\left(\boxplus-\sum_{i=1}^{K} a_i L(X_i)\right)=\boxplus-\sum_{i=1}^{K} a_i L(X_i)$$

We introduce a partial ordering for elements of $\bigcup_{n=1}^{\infty}\mathcal{F}(\mathbb{R}^n)$ by setting for $\mathbf{x} = a_1x_1 \boxplus\cdots\boxplus a_nx_n \in \mathcal{F}(\mathbb{R}^N)$ and $\mathbf{y} = b_1 y_1 \boxplus\cdots\boxplus b_m y_m \in F\mathcal{F}(\mathbb{R}^M)$

$$\mathbf{x} \succ \mathbf{y} \text{ iff } \tilde{L}\,\mathbf{x} = \mathbf{y} \text{ for some } L \in \mathcal{L}\,(\mathbb{R}^N;\mathbb{R}^M).$$

Definition (5.3.8)[207]: We say that a polynomial $P \in \Pi_{n,d}$ is compatible with $x \in \mathcal{F}(\mathbb{R}^m)$if

$$P \circ L, \mathbf{x} = P, \tilde{L}\mathbf{x} = 0 \text{ for all } L \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$$

Let X, Y be Banach spaces and $f: X \to Y$ be a continuous mapping. Then we say that f is compatible with $\mathbf{x} \in \mathcal{F}(\mathbb{R}^M), x =\boxplus-\sum_{i=1}^{k} a_i x_i$ if

$$\langle f\circ L, X\rangle = \sum_{i=1}^{k} a_i x_i\, f(Lx_i) = 0 \text{ for all } L \in \mathcal{L}(\mathbb{R}^m;X)$$

Remark (5.3.9)[207]: Clearly, if X, Y are Banach spaces, then a continuous mapping $f: X \to Y$ is compatible with $\mathbf{x} = a_1x_1 \boxplus\cdots\boxplus a_nx_n \in \mathcal{F}(\mathbb{R}^m)$, where $x_k = (x_k^1,\ldots,x_k^m)$, iff

$$\sum_{k=1}^{n} a_k f\left(\sum_{i-1}^{m} x_k^i z_i\right) = 0 \quad \text{forevery } z_1,\ldots,z_m \in X. \tag{48}$$

The expression (48) is called a linear identity. In particular, Frechet Theorem (5.3.1) is equivalent to saying that f is a polynomial of degree at most n iff f is compatible with an element $\mathbf{X}_{\mathbf{M},\mathbf{n}} \in \mathcal{F}(\mathbb{R}^{n+2})$(resp. $\mathbf{X}_{\mathbf{F},\mathbf{n}} \in \mathcal{F}(\mathbb{R}^2)$ where

$$\mathbf{x}_{\mathbf{M},n} =\boxplus-\sum_{j=0}^{n+1}\sum_{A\subset\{1,\ldots,n+1\},|A|=j}(-1)^{n+1-j}\left(e_0+\sum_{I\in A} e_l\right)$$

$$\mathbf{x}_{\mathbf{F},n} =\boxplus-\sum_{j=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}(1,k) \in \mathcal{F}(\mathbb{R}^2) \tag{49}$$

Moreover, the linear operator $L: \mathbb{R}^n + 2 \to \mathbb{R}^2$ defined by

$$L(x_0, x_1,\ldots,x_{n+1}) = \sum_{i=1}^{n+1} x_i$$

satisfies $\tilde{L}(\mathbf{x}_{\mathbf{M},n}) = \mathbf{x}_{\mathbf{F},n}$, so in particular $\mathbf{x}_{\mathbf{M},n} \succ \mathbf{x}_{\mathbf{F},n.}$ It is easy to see that $L: \mathbb{R}^N \to \mathbb{R}^M$ leads to a linear mapping $L^*: \Pi_{M,d} \to \Pi_{N,d}$ defined as $L^*(P) = P\circ oL$. The adjoint linear operator $L^{**}: \Pi_{N,d}^* \to \Pi_{M,d}^*$ coincides with $\tilde{L}$ (if the duals are represented using the canonical evaluations). The following is a simple consequence of the definitions.

Fact (5.3.10)[207]: Let $\mathbf{x} \in \mathcal{F}(\mathbb{R}^m), \mathbf{y} \in \mathcal{F}(\mathbb{R}^m), X, Y$ be Banach spaces and $f: X \to Y$ be continuous. Suppose that $\mathbf{x} \succ \mathbf{y}$. Then the compatibility of f with $\mathbf{x}$ implies the compatibility of f with $\mathbf{y}$. Consequently, if $\tilde{L}\mathbf{x} = \mathbf{y}$ for some bijection $L \in \mathcal{L}(\mathbb{R}^m)$, then f is compatible with $\mathbf{x}$ iff f is compatible with $\mathbf{y}$.

The implication in Fact (5.3.10) cannot be reversed. For example, let $n \in N$ and let $\mathbf{x}, \mathbf{y} \in \mathcal{F}(\mathbb{R}^3)$ be defined by

$$\mathbf{x} = (-1)^{n+1}\left(\boxplus - \sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}(1,k,0)\right)$$

$$y = \boxplus - \sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}(1,k,0)$$

($\mathbf{x}$ and $\mathbf{y}$ differ only in the third coordinate of the first node). Then clearly $\mathbf{x} \succ \mathbf{y}$. It is also clear that the compatibility of a continuous $f: \mathbb{R}^3 \to \mathbb{R}$ with $\mathbf{y}$ is equivalent to the compatibility of f with $\mathbf{x}_{\mathbf{F},n}$ from (49), and therefore the space of those continuous $f: \mathbb{R}^3 \to \mathbb{R}$ compatible with $\mathbf{y}$ is $\Pi_{3,n}$. On the other hand, if $P \in \Pi_{3,1}$ is defined as $P:(x,y,z) \mapsto z$, then $\langle P, \mathbf{x}\rangle = (-1)^{n+1} \neq 0$, and therefore P is not compatible with $\mathbf{x}$. In fact, it will follow from Theorem (5.3.22) that the only continuous functions on $\mathbb{R}^3$ compatible with $\mathbf{x}$ are the constant functions.

In this section, we establish basic results concerning compatibility and show that, under some natural assumptions, polynomials are the only continuous mappings satisfying linear identities.

Lemma (5.3.11)[207]: Let $\mathbf{x} \in \mathcal{F}(\mathbb{R}^m)$. TFAE

(i) For every Banach spaces X, Y every $P \in \mathcal{P}(^d X;\ Y)$ is compatible with $\mathbf{x}$.

(ii) Every $P \in F_{m,d}$ is compatible with $\mathbf{x}$.

(iii) $\langle P, \mathbf{x}\rangle = 0$ for every $P \in F_{m,d}$.

Proof. The implications (i)⇒(ii)⇒(iii) are clear.

(iii)⇒(ii): Suppose that (iii) holds, and let $P \in F_{m,d}$. If $L \in \mathcal{L}(\mathbb{R}^m)$, then $P \circ L \in F_{m,d}$, hence $P \circ L, \mathbf{x} = 0$, and therefore P is compatible with $\mathbf{x}$.

(ii) ⇒ (i): Suppose that every $P \in F_{m,d}$ is compatible with $\mathbf{x}$. Let X, Y be Banach spaces and $P \in \mathcal{P}(^d X;\ Y)$. Let $L \in \mathcal{L}(\mathbb{R}^m X)$ and choose $\phi \in Y^*$ arbitrary. Then $\varphi \circ P \circ L \in F_{m,d}$, and therefore $0 = \langle \varphi\ P \circ L, \mathbf{x}\rangle = \varphi(P \circ L, \mathbf{x})$. Since φ was arbitrary, we conclude that $P o L, \mathbf{x} = 0$.

Lemma (5.3.12)[207]: Let X, Y be Banach spaces and let $P = \sum_{k=0}^{d} \in \mathcal{P}^d(X;\ Y)$, where $P_k \in \mathcal{P}(^k X;\ Y)$ are k-homogeneous summands. If P is compatible with $\mathbf{x} \in \mathcal{F}(\mathbb{R}^m)$, then each nonzero summand P_k is compatible with $\mathbf{x}$.

Proof. By assumption,

$$\langle P \circ L, \mathbf{x}\rangle = \sum_{k=0}^{d} P_k\ L, \mathbf{x} = 0 \quad \text{for all } L \in \mathcal{L}(\mathbb{R}^m; X).$$

In particular, fixing L, composing $L \circ (t\, Id_{\mathbb{R}^m})$, and using the homogeneity of P_k we obtain

$$0 = \langle P \circ \left(L \circ (t\, Id_{\mathbb{R}^m})\right), x\rangle = \sum_{k=0}^{d} P_k\ L, \mathbf{x} = 0 \quad \text{for all } L \in \mathcal{L}(\mathbb{R}^m;\ X).$$

The right hand side, for a fixed L, is a Y -valued polynomial in t. Thus each $\langle P_k \circ L, \mathbf{x}\rangle = 0$, otherwise for some t the total value could not be zero.

The following result was proved by Reznick. We give a proof using our formalism.

Lemma (5.3.13)[207]: Let X, Y be Banach spaces and let $0 \neq P \in \mathcal{P}(^d X; Y), \mathbf{x} \in \mathcal{F}(\mathbb{R}^m)$. Then P is compatible with **x** iff the polynomial $t \to t^d$ from $F_{1,d}$ is compatible with **x**.

Proof. On one hand, there exists a one-dimensional subspace $E \hookrightarrow X$ such that $P \restriction_E = at^D, a \neq 0$. So for every $L : \mathbb{R}^m \to E$ we have that $\langle P \circ L, \mathbf{x}\rangle = 0$. Consequently, t^d is compatible with **x**. On the other hand, if td is compatible with **x**, then so is every $\phi^d(y)$, where $\phi \in (\mathbb{R}^m)^*$. Indeed, $\phi^d(y)$ is a composition of a linear projection of $\mathbb{R}^m$ onto a one-dimensional subspace $F \hookrightarrow \mathbb{R}^m$, and the polynomial t^d defined on $F = \mathbb{R}$. If $Q \in F_{m,d,}$ then by Theorem (5.3.5) $Q(y) = \sum a_K \emptyset_k^d(y)$, so Q is compatible with **x**, being a sum of finitely many polynomials compatible with **x**. Lemma (5.3.11) then finishes the proof.

Corollary (5.3.14)[ ]: An element $\mathbf{x} = a_1x_1 \boxplus \cdots \boxplus a_nx_n \in \mathcal{F}(\mathbb{R}^m)$ is compatible with $t \to t^d$ (or any other nonzero d-homogeneous polynomial)
iff $a_1x_1 \boxplus \cdots \boxplus a_nx_n = 0$ in $F^*_{m,d}$

Corollary (5.3.15)[207[: Let $0 \neq P \in F_{n,d}$. Then for any $Q \in F_{n,d}$ there exist a finite collection of linear $L_k \in \mathcal{L}(\mathbb{R}^n)$ and $a_k \in \{\pm 1\}, k = 1,\ldots,r = |\mathcal{I}(n,d)|$, such that $Q = \sum_{k=1}^r a_k P \circ L_k$.

Proof. Suppose, by contradiction, that the linear span $H = \text{span}\{P \circ L: L \in \mathcal{L}(\mathbb{R}^n)\}$ in the space $F_{n,d}$ is a proper subspace, i.e. there exist some $Q \in F_{n,d} \setminus H$ and a linear functional **x** which is zero on H and nonzero on Q . Thus P is compatible with **x**, but Q is not. This contradicts Lemma (5.3.13). Hence $H = F_{n,d}$ and Q is a finite linear combination of elements of the form $P \circ L_k$. By Caratheodory lemma [208], we infer that the number of summands can be chosen not to exceed the dimension of the space $F_{n,d}$.

The above corollary is analogous to the celebrated Hilbert lemma, which claims that for given $l, n \in \mathbb{N}$ there exists a finite collection $\{\emptyset_1,\ldots,\emptyset_n\} \subset (\mathbb{R}^n)^*$ such that

$$\|X\|_{\ell_2}^{2I} = \sum_{i=1}^{N} \emptyset_i^{2I}(x), x \in \mathbb{R}^n.$$

The difference lies in the value of coefficients a_K, which in the Hilbert case can be chosen to be positive. Such conclusion is false in our setting, by easy examples when Q is non-positive or non-convex and $P(x) = \emptyset^n(x)$. Much subtler counterexamples follow from the work of Neyman [231], who proved that there exists a finite dimensional Banach space whose norm taken to $n-$th power is an n-homogeneous polynomial Q but the space is not isometric to a subspace of ℓ_n space. It follows that the polynomial $Q \in F_{n,d}$ may be convex and non-negative and yet it admits no formula with all $a_k \geq 0$.

Next, we investigate the properties of $\mathbf{x} = a_1x_1 \boxplus \cdots \boxplus a_nx_n$, which lead to compatibility. We restrict our attention to the case when there is k such that for every $i \neq k$ the vectors x_k and x_i are linearly independent. (This assumption is natural, since if the vectors xi are nonzero and fail this condition, then there clearly exist nonzero a_i such that all d homogeneous continuous functions are compatible with **x**.) An interesting example in this direction is derived from the polarization formula below. The second part (50) is an easy observation of the present authors, which follows by inspection of the classical proof (e.g. [232, p. 8]).

Proposition (5.3.16)[207]: (Polarization formula). (See [233,214].) For every $P \in \mathcal{P}(^n X; Y)$, where X, Y are Banach spaces, there exists a unique symmetric n-linear form $\widetilde{P} \in \mathcal{L}^s(^n X; Y)$ such that $P(x) = \widetilde{P}(x, \ldots, x)$. The following formula holds.

$$\widetilde{P}(x_n, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_i \cdots \varepsilon_n P\left(\sum_{i=1}^{n} \varepsilon_i x_i\right).$$

On the other hand, for every $0 \neq P \in P(^kX; Y), k < n$, or $k - n$ odd and positive the following formula holds.

$$\sum_{\varepsilon_i = \pm 1} \varepsilon_i \cdots \varepsilon_n P\left(\sum_{i=1}^{n} \varepsilon_i x_i\right) = 0, \qquad x_i \in X \tag{50}$$

In the remaining case when $k > n$ and $k - n$ is even, there exists $x \in X$ such that the left hand side in (50) for $x_i = x, i = 1, \ldots, n$, is nonzero.

Translated into our language, we see that $\mathbf{x_B} = \boxplus - \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots (\varepsilon_n \sum_{i=1}^{n} \varepsilon_i e_i) \in \mathcal{F}(\mathbb{R}^n)$ is compatible with k-homogeneous polynomials iff either $k < n$ or $k - n$ is a positive odd number. Note that to each summand involved in the definition of $\mathbf{x_n}$ we can find some other summand which is its multiple. Under the assumption that $\{x_i\}$ contains a vector which is linearly independent with any other vector from $\{x_i\}$, we will prove in Theorem 3.8 that every continuous mapping which is compatible with $\mathbf{x} = a_0x_0 \boxplus \cdots \boxplus a_{n+1xn+1} \in \mathcal{F}(\mathbb{R}^m)$ is necessarily a polynomial of degree at most n.

In particular, the Jordan–von Neumann Theorem (5.3.2) follows immediately from this statement. The result was proved by Reznick under the assumption that the continuous function f is homogeneous.

In the proof of Theorem (5.3.18) we will use the following result due to Wilson [209].The original statement in [209] is for functions on $\mathbb{R}^2$, but the proof works with no change for arbitrary mappings between Banach spaces.

Theorem (5.3.17)[207]: (See [209].) Let X, Y be Banach spaces, $f: X \to Y$ be a continuous mapping and let $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_n + 1 x_n + 1 \in \mathcal{F}(\mathbb{R}^2), n \in \mathbb{N} \cup \{0\}$. Suppose that for every $k \neq n+1$ the vectors x_k and x_{n+1} are linearly independent, and that $a_{n+1} \neq 0$. Let $p+2$ be the number of distinct directions determined by the vectors $x_0, \ldots, x_{n+1}$. If f is compatible with $\mathbf{x}$, then f is compatible with $\mathbf{x_{F,p}}$, from (49).

Proof. Let $x_i = (r_i, s_i), i = 0, \ldots, n+1$. By Fact (5.3.10) we may suppose wlog that $r_{n+1} = 0$ and $s_{n+1} = 1$. Then $r_i \neq 0$ for every $i \neq n+1$. By Remark (5.3.9) the mapping f for every $x, y \in X$ satisfies

$$\sum_{i=0}^{n} a_i f(r_i x + s_i y) + a_n + 1 f(y) = 0 \tag{51}$$

First, let us suppose that $x_0, \ldots, x_{n+1}$ are pair wise linearly independent, i.e. $p = n$. Put $\Delta_{j,i} = s_i - r_i s \frac{s_j}{r_j}$ for $i, j \in \{0, \ldots, n\}$. Then $\Delta_{j,i} = 0$ iff $j = i$.

In the first step, we subtract (4) from the equation derived from (4) by replacing x by $x - \frac{s_0}{r_0} x$ and y by $y + x$. We obtain

$$\sum_{i=1}^{n} a_i \left(f(r_i x + s_i y + \Delta_0, i^x) - f(r_i x + s_i y)\right) + a_{n+1}(f(y+x) - f(y)) = 0. \tag{52}$$

Note that since $\Delta_{0,0} = 0$, we have eliminated the terms with $i = 0$.

In the second step, we subtract (52) from the equation derived from (5) by replacing x by $x - \frac{s_I}{r_I}$ and y by $y + x$. Weobtain

$$\sum_{i=1}^{n} a_i \left(f(r_i x + s_i y + (\Delta_1, i + \Delta_0, i)x) - f(r_i x + s_i y + \Delta_1, ix)\right.$$
$$\left. - f(r_i x + s_i y + \Delta_0, ix) + f(r_i x + s_i y)\right)$$
$$+ a_{n+1}(f(y + 2_x) - 2f(y+x) + f(y) = 0)$$

In this step we have eliminated the terms with $i = 1$.

We continue in this manner. In the $k - th$ step we subtract the last equation from the equation derived from the last one by replacing x by $x - \frac{s_k - 1}{r_k - 1}$ and y by $y + x$. Since the substitutions replace $r_i x + s_i y$ by $r_i x + s_i y + \Delta_{k-1}, i^x$ and since $\Delta_{k-1,k-1} = 0$, the subtraction eliminates the terms with $i = k-1$. After $n+1$ steps we arrive at

$$a_{n+1} \sum_{k=0}^{p+1} (-1)^{p+1-k} \binom{P+1}{k} f(y + kx) = 0,$$

and since $a_{n+1} \neq 0$, we see that f is compatible with $\mathbf{x_{F,n}}$.

Now consider the case when some pairs of the vectors $x_0, \ldots, x_n$ are linearly dependent. Then in some steps we eliminate terms corresponding to more than one value of i. It is easy to see that after $p + 1$ steps we arrive at

$$a_{n+1} \sum_{k=0}^{p+1} (-1)^{p+1-k} \binom{p+1}{k} f(y + kx) = 0,$$

and therefore f is compatible with $\mathbf{x}_{\mathbf{F},p}$.

Theorem (5.3.18)[207]: Let X, Y be Banach spaces, $f : X \to Y$ be a continuous mapping and $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_{n+1xn+1} \in \mathcal{F}(\mathbb{R}^m), m \geq 2, n \in \mathbb{N} \cup \{0\}$. Suppose that for every $k \neq n + 1$ the vectors x_k and x_{n+1} are linearly independent, and that $a_n + 1 \neq 0$. Let $p + 2$ be the number of distinct directions determined by the vectors $x_0, \ldots, x_n + 1$. If f is compatible with **x**, then f is a polynomial of degree at most p.

Proof. First let m = 2. Since f is compatible with **x**, it is compatible with $\mathbf{x}_{\mathbf{F},p}$ by Theorem (5.3.17). By Theorem (5.3.1) the mapping f is a polynomial of degree at most p.

The case $m > 2$. By Fact (5.3.10) it is enough to find a $\mathbf{y} \in \mathcal{F}(\mathbb{R}^2)$, such that $\mathbf{x} \succ \mathbf{y}$, and **y** satisfies the assumptions of the previous case. So we claim that there exists a linear operator $T : \mathbb{R}^m \to \mathbb{R}^2$ such that the couple $T(x_k)$ and $T(x_{n+1})$ is linearly independent for all $k \neq n + 1$. (The number of distinct directions determined by $T(x_0), \ldots, T(x_{n+1})$ is clearly less or equal to p + 2.) This is easily seen as follows. Let $E_k = \operatorname{span}\{x_k, x_n + 1\} \hookrightarrow \mathbb{R}^m, k \in \{0, \ldots, n\}$, be a system of 2-dimensional subspaces of $\mathbb{R}^m$. There exists an (m−2)-dimensional subspace $F \hookrightarrow \mathbb{R}^m$ such that $F \cap E_k = \{0\}, k \in \{0, \ldots, n\}$. (Equivalently, $F + E_k = \mathbb{R}^m$.) Then the orthogonal projection T in $\mathbb{R}^m$, with kernel F and two-dimensional range $F^\perp \to \mathbb{R}^m$, clearly satisfies the condition.

If $\mathbf{x} \in \mathcal{F}(\mathbb{R}^m)$ and X, Y are Banach spaces, then the set of all continuous mappings from X to Y which are compatible with **x** is clearly a linear space. We are now ready to describe this space more precisely.

Theorem (5.3.19)[207]: Let $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_{n+1} x_{n+1} \in \mathcal{F}(\mathbb{R}^m), m \geq 2, n \in \mathbb{N} \cup \{0\}$. Suppose that for every $k \neq n + 1$ the vectors xk and x_{n+1} are linearly independent, and that $a_{n+1} \neq 0$. Let $p + 2$ be the number of distinct directions determined by the vectors $x_0, \ldots, x_n + 1$.

Then there exists $A \subset \{0, \ldots, p\}$ such that if X, Y are Banach spaces and $f : X \to Y$ is a continuous mapping, then f is compatible with **x** iff $f \sum_{k \in A} P_k$ for some $P_K \in \mathcal{P}({}^k X; Y)$ (if A is empty, the sum is understood to be equal to 0).

Proof. Let A be the set of all $k \in \{0, \ldots, p\}$ for which there exist Banach spaces X, Y and a nonzero polynomial from $\mathcal{P}({}^k X; Y)$ which is compatible with **x**. By Lemma (5.3.13), if $k \in A$, then for every Banach spaces X, Y every polynomial from $\mathcal{P}({}^k X; Y)$ is compatible with **x**, and the same holds also for their linear combinations. Let now X, Y be Banach spaces and $f : X \to Y$ be a continuous mapping compatible with **x**. By Theorem (5.3.18) the mapping f is a polynomial of degree at most p.

Say $f = \sum_{k=0}^{n} P_k$, where $P_k \in \mathcal{P}({}^k X; Y)$. If $P_k \neq 0$ for some $k \in \{0,\dots,p\}$, then it follows from Lemma (5.3.12) that $k \in A$. Hence $f = \sum_{k\in A} P_k$.

It may happen that the set A from the above theorem contains some gaps. In fact, we have even the following.

Theorem (5.3.20)[207]: Let $0 \le d_1 < d_2 < \cdots < d_k \le d$ be given integers and let ≥ 2. Then there exists $\mathbf{x} = a_1x_1 \boxplus \cdots \boxplus a_nx_n \in \mathcal{F}(\mathbb{R}^m), n \ge 2,$ where $x_1,\dots,x_n$ are pairwise linearly independent vectors and $ai \neq 0$ for $i = 1,\dots,n$, such that **x** is compatible with $\to t^l, l \le d,$ iff $l \in \{d_1, d_2, \dots, d_k\}$.

Proof. Consider the linear subspace E of Πm,d generated by $\cup_{i=1}^{k} F_{m,di}$. Let $M = \{0,\dots,d\} \setminus \{d_1, d_2,\dots,d_k\}$. Choose for every $l \in M$ some nonzero l-homogeneous polynomial $P_l \in F_{m,di}$. Then $P_l \notin E$ for every $l \in M$.

Now, let $x_1,\dots,x_n \in \mathbb{R}^m$ be pairwise linearly independent vectors such that the restriction map $\Phi : \Pi_{m,d} \to C(\{x_1,\dots,x_2\}) = \mathbb{R}^n$ defined by

$$\Phi(P) = P \restriction_{\{x_1,\dots,x_n\}}, P \in \Pi_{m,d},$$

is one-to-one and not surjective (for example, take a pairwise linearly independent basic set of nodes for $\Pi_{m,d}$ and add one point which is not a multiple of any of the nodes). Then $\Phi(P_l) \notin \Phi(E)$ for every $l \in M$ and $\Phi(\Pi_{m,d})$ is a proper subspace of $\mathbb{R}^N$. It is easy to see that there exists $f = (a_1,\dots,a_n) \in \mathbb{R}^{n^*} \setminus \{0\}$ such that $f(\Phi(E)) = 0$ and $f(\Phi(Pl)) \neq 0$ for every $l \in M$. It is clear that if $\mathbf{x} = a_1x_1 \boxplus \cdots \boxplus a_nx_n \in \mathcal{F}(\mathbb{R}^m)$, then **x** is not compatible with P_l for every $l \in M$, but it is compatible with members of E by Lemma (5.3.11). We may of course suppose that $a_i \neq 0$ for $i = 1,\dots,n$. Lemma (5.3.13) then concludes the proof.

More can be said if the points $x_0,\dots,x_{n+1}$ lie in an affine hyperplane not containing 0.

Lemma (5.3.21)[207]: Let $\mathbf{x} = a_0x_0 \boxplus \cdots \boxplus a_{n+1}x_{n+1} \in \mathcal{F}(\mathbb{R}^m), m \ge 2, n \in N \cup \{0\},$ where $x_0,\dots,x_{n+1}$ are distinct and lie in an affine hyperplane not containing 0. If every polynomial from $F_{m,d}$ is compatible with **x**, then the same holds for every polynomial from $\Pi_{m,d}$.

Theorem (5.3.22)[207]: Let $\mathbf{x} = a_0x_0 \boxplus \cdots \boxplus a_{n+1}x_{n+1} \in \mathcal{F}(\mathbb{R}^m), m \ge 2, n \in N \cup \{0\},$ where $x_0,\dots,x_{n+1}$ are distinct and lie in an affine hyperplane not containing 0, and $a_k \neq 0$ for $k = 0,\dots,n+1$. If $\sum_{k=0}^{n+1} a_k = 0$, then there exists $l \in \{0,\dots,n\}$ such that if X, Y are Banach spaces and $f : X \to Y$ is a continuous mapping, then f is compatible with **x** iff f is a polynomial of degree at most l. If $\sum_{k=0}^{n+1} a_k \neq 0$, then there is no nonzero mapping compatible with **x**.

Proof. Since $x_0,\dots,x_{n+1}$ are pairwise linearly independent, Theorem (5.3.19) applies. Let $A \subset \{0,\dots,n\}$ be a set whose existence is ensured by Theorem (5.3.19). If$\sum_{k=0}^{n+1} a_k = 0$, then $t \hookrightarrow 1, t \in \mathbb{R}$, is compatible with $\mathbf{x}$ and therefore A is nonempty. Let $l \in A$ be maximal. Since every polynomial from $F_{m,l}$ is compatible with $\mathbf{x}$, by Lemma (5.3.21) every polynomial from $\Pi_{m,l}$ is also. Hence $A = \{0,\dots,l\}$. This argument also shows that if A is nonempty, then $t \hookrightarrow 1$ is compatible with $\mathbf{x}$, and consequently If$\sum_{k=0}^{n+1} a_k = 0$. Hence If$\sum_{k=0}^{n+1} a_k \neq 0$, then there is no nonzero mapping compatible with $\mathbf{x}$.

Some information on the exact value of l can be derived from the geometrical properties of the set $\{x_0,\dots,x_{n+1}\}$. Clearly there is no lower bound on l, since to each $x_0,\dots,x_{n+1}$ we may take $a_0,\dots,a_{n+1}$such that If $\sum_{k=0}^{n+1} a_k \neq 0$, and then there is no nonzero mapping compatible with $\mathbf{x}$. Even if we demand that If$\sum_{k=0}^{n+1} a_k = 0$, it is easy to find such $a_0,\dots,a_{n+1}$ so that some $P \in F_{m,1} = (\mathbb{R}^n)^*$ is not compatible with $\mathbf{x}$. Indeed, take $P \in F_{m,1}$ which is not constant on $x_0,\dots,x_{n+1}$ and then find $a_0,\dots,a_{n+1}$ such that If $\sum_{k=0}^{n+1} a_k P(X_K) \neq 0$. However, there is a simple upper bound in terms of the dimension of the affine hull of the points $x_0,\dots,x_{n+1}$. It will be given in Corollary (5.3.26). In the proof of Lemma (5.3.24) we will use the following simple fact.

Fact (5.3.23)[207]: If $M \subset \mathbb{R}^2$ is a union of n distinct lines containing 0, then M is a null space of an n-homogeneous polynomial $P : \mathbb{R}^2 \to \mathbb{R}$. Indeed, let $P(x) = \Pi_{i=1}^{n} \emptyset_i(x)$, where $\emptyset \in (\mathbb{R}^2)^*$ are chosen so that their kernels coincide with the given lines.

If $M \subset \mathbb{R}^m$, we denote by aff(M) the affine hull of M.

Lemma (5.3.24)[207]: Let $x_0,\dots,x_{n+1} \in \mathbb{R}^m, n \in \mathbb{N} \cup \{0\}$, be distinct and denote by d the dimension of aff $(\{x_1,\dots,x_{n+1}\})$. Then there exist $k_0 \in \{0,\dots,n+1\}$ and a polynomial $P : \mathbb{R}^m \to \mathbb{R}$ of degree at most $n + 2 - d$ such that $P(x_{k0}) \neq 0$ and $P(x_k) = 0$ for every $k \in \{0,\dots,n+1\} \setminus \{k_0\}$.

Proof. We may wlog suppose that m =d. The case d = 1 is trivial. Let $d \geq 2$. We may further suppose wlog that $x_0,\dots,x_{d-1}$ are affinely independent, that $M = \text{aff}(\{x_0,\dots,x_{d-1}\})$ is a hyperplane in $\mathbb{R}^d$ (i.e. it is a subspace), and that $x_{n+1} \notin M$. Using a similar argument as in the proof of Theorem (5.3.18) we construct a linear mapping $L : \mathbb{R}^d \to \mathbb{R}^2$ such that $L(x_0),\dots,L(x_{d-1})$ lie on a line $p \subset \mathbb{R}^2, L(x_{d-1}) \notin p$ and $L(x_{d-1}) \neq L(x_k)$ for all $k \neq n+1$.

Now, there exists $z \in p$ such that the line $q \subset \mathbb{R}^2$ which contains z and $L(x_{n+1})$ does not contain $L(x_k)$ for all $k \neq n+1$. Let $p_1,\dots,p_r$ be distinct lines which contain z and some $L(x_k), k \neq n+1$. Then $r \leq +2-d$. By Fact (5.3.23) (since a translation of a polynomial of degree r is again a polynomial of degree r) there exists a polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ of degree $r \leq n+2-d$ such that the nullspace of $Q \cup_{i-1}^{r}$ pi. Then $P = Q \circ L \in \Pi_{d,n+2-d}$ is the desired polynomial.

Proposition (5.3.25)[207]: Let $\mathbf{x} = a_0x_0 \boxplus\cdots\boxplus a_{n+1}x_{n+1} \in \mathcal{F}(\mathbb{R}^m), n \in \mathbb{N} \cup \{0\}$, where $x_0,\ldots,x_{n+1}$ are distinct and ak $\neq 0$ for $k = 0,\ldots,n+1$, and denote by d the dimension of aff $(\{x_0,\ldots,x_{n+1}\})$. If every $P \in \Pi_{m,k}$ is compatible with $\mathbf{x}$, then $k \le n + 1 - d$.

Proof. By Lemma (5.3.24) there exist $k_0 \in \{0,\ldots,n+1\}$ and a polynomial $P : \mathbb{R}^m \to \mathbb{R}$ of degree at most n + 2 − d such that $P(x_{k_0}) \neq 0$ and $P(x_k) = 0$ for every $k \in \{0,\ldots,n+1\} \setminus \{k_0\}$. Then P cannot be compatible with **x**, since otherwise we would have

$$0 = \langle P, \mathbf{x} \rangle = \sum_{k=0}^{n+1} a_k P(x_k) = a_{k_0} P(x_{k_0}),$$

and therefore $a_{k_0} = 0$, a contradiction. Hence $k \le n + 1 - d$.

Corollary (5.3.26)[207]: Let $\mathbf{x} = a_0x_0 \boxplus\cdots\boxplus a_{n+1}x_{n+1} \in \mathcal{F}(\mathbb{R}^m), m \ge 2, n \in \mathbb{N} \cup \{0\}$, where $x_0,\ldots,x_{n+1}$ are distinct and lie in an affine hyperplane not containing $0, a_k \neq 0$ for $k = 0,\ldots,n+1$ and $\sum_{k=0}^{n+1} a_k = 0$. Let l be as in Theorem4.4 and denote by d the dimension of aff $(\{x_0,\ldots,x_{n+1}\})$. Then $l \le n + 1 - d$.

For example, if in Corollary (5.3.26) the points $x_0,\ldots,x_{n+1}$ are affinely independent, then d = n + 1 and there fore $l = 0$. Corollary (5.3.26) also shows that in order to achieve the maximal possible value of l in Theorem (5.3.22) (i.e. l =n), it is necessary that $x_0,\ldots,x_{n+1}$ be collinear; see Theorem (5.3.30) for more general result.

In order to generate linear identities, we can use Theorem (5.3.7) on the generalized Lagrange formula. In fact, the Lagrange formula is an expression of linear dependence of functional in the dual of $\Pi_{m,d}$. Let $\{x_k\}_{k=1}^r \subset \mathbb{R}^m$ be a basic set of nodes for $\Pi_{m,d}$ and let $\{x_k\}_{k=1}^r \subset \Pi_{m,d}$ be its dual basis. Given $z \in \mathbb{R}^m \setminus \{x_k\}_{k=1}^r$, there exists a unique set of coefficients $a_k = a_k(z) \in \mathbb{R}$ such that $P(z) = \sum_{k=1}^r a_k(z)P(x_k)$ for every $P \in \Pi_{m,d}$, and $a_k(z) = h_k(z), k = 1,\ldots,r$. Then every $P \in \Pi_{m,d}$ is compatible with $a_1(z)x_1 \boxplus\cdots\boxplus a_r(z)x_r \boxplus (-1)z$.

Lemma (5.3.27)[207]: Let $\{x_k\}_{k=1}^r \subset \mathbb{R}^m$ be a basic set of nodes for $\Pi_{m,d}, z \in \mathbb{R}^m \setminus \{x_k\}_{k=1}^r$, and let $\mathbf{x} = a_1(z)x_1 \boxplus\cdots\boxplus a_r(z)x_r \boxplus (-1)z$. If every $P \in \Pi_{m,l}$ is compatible with **x**, then $l \le d$.

Proof. Assume wlog that $a_1(z) \neq 0$. Considering the dual basis of $\{x_k\}_{k=1}^r$we see that there exists $Q \in \Pi_{m,d}$ such that $Q(x1) \neq 0$ and $Q(x_k) = 0$ for $k = 2,\ldots,r$. Further, it is clear that there exists $R \in \Pi_{m,1}$ (these are the affine functions on $\mathbb{R}^m$) such that $R(x_1) \neq 0$ and $R(z) = 0$. Then clearly $P = QR \in \Pi_{m,d+1}, P(x_1) = 0, P(x_k) = 0$ for $k = 2,\ldots,r$ and $P(z) = 0$. But then $\langle P, \mathbf{x} \rangle = a_1(z)P(x_1) \neq 0$, hence P is not compatible with **x**, and therefore $l \le d$.

The following theorem describes a method of generating linear identities which, for prescribed d, characterize polynomials of degree at most d.

Theorem (5.3.28)[207]: Let $\{x_k\}_{k=1}^r \subset \mathbb{R}^m$ be a basic set of nodes for $\Pi_{m,d}, z \in \mathbb{R}^m \setminus \{x_k\}_{k=1}^r$, and let $\mathbf{x} = a_1(z)x_x \boxplus \cdots \boxplus a_r(z)x_r \boxplus (-1)z$. Let $T : Rm \to \mathbb{R}^n, n > m$, be an affine one-to-one mapping such that $0 \notin T(\mathbb{R}^m)$. Then $a_1 = a_1(z), \ldots, a_r = a_r(z)$ are the unique coefficients with the following property. Let $\mathbf{y} = a_I T(x_1) \boxplus \cdots \boxplus a_r T(x_r) \boxplus (-1)T(z)$. If X, Y are Banach spaces and f : X →Y is continuous, then f is compatible with $\mathbf{y}$ iff f is a polynomial of degree at most d.

Proof. Since $T(x_1), \ldots, T(x_r), T(z)$ lie in an affine hyperplane not containing 0, Theorem (5.3.22) applies. It follows that it suffices to prove the theorem for $X = \mathbb{R}^n$ and $Y = \mathbb{R}$, and it also follows that the space of those continuous $f : \mathbb{R}^n \to \mathbb{R}$ which are compatible with $\mathbf{y}$ is $\Pi_{n,l}$ for some $l \in N \cup \{0\}$ or a trivial space. If $P \in \Pi_{n,d}$, then $P \circ T \in \Pi_{m,d}$, so $P \circ T$ is compatible with $\mathbf{x}$, and therefore $\langle P, \mathbf{y} \rangle = 0$. By Lemma (5.3.11) every member of $\Pi_{n,d}$ is compatible with $\mathbf{y}$. Hence the space of compatible functions is nontrivial and $l \geq d$. On the other hand, if $P \in \Pi_{m,l}$, then $P \circ T^{-1} : T(\mathbb{R}^m) \to R$ can be extended to a member of $\Pi_{n,l}$, which is compatible with $\mathbf{y}$ by the definition of l. It follows from Lemma (5.3.11) that every polynomial from $\Pi_{m,l}$ is compatible with $\mathbf{x}$. By Lemma (5.3.27) we conclude that $l \leq d$. Theorem (5.3.7) then yields the uniqueness part.

A special case of Theorem (5.3.28) in dimension one corresponds to the classical Lagrange interpolation polynomial.

Theorem (5.3.29)[207]: (Classical Lagrange interpolation). Let $x_1, \ldots, x_{n+1} \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$, be distinct. Then there exists a unique set of coefficients $a_{,0} \ldots, a_n \in \mathbb{R} \setminus \{0\}$, such that every $P \in \Pi_{1,n}$ is compatible with $a_0x_0 \boxplus \cdots \boxplus a_nx_n \boxplus (-1)x_{n+1}$. Moreover,

$$a_k = \prod_{i=0, i\neq k}^{n} \frac{x_{n+1} - x_i}{x_k - x_i}, k = 0, \ldots, n$$

The following theorem characterizes those $a_0x_0 \boxplus \cdots \boxplus a_{n+1}x_{n+1} \in \mathcal{F}(\mathbb{R}^m)$ which can be used to characterize polynomials of degree at most n, the highest possible degree. It is a generalization of the equivalence of the conditions (i) and (iv) in Theorem (5.3.1).

Theorem (5.3.30)[207]: Let $x_0, \ldots, x_{n+1} \in \mathbb{R}^m, m \geq 2, n \in \mathbb{N}$, be distinct points. TFAE

(i) The points $x_0, \ldots, x_{n+1}$ lie on a line not containing 0.

(ii) There exist $a_0, \ldots, a_n \in \mathbb{R} \setminus \{0\}$ such that if X, Y are Banach spaces and $f : X \to Y$ is a continuous mapping, then f is compatible with $\mathbf{x} = a_0x_0 \boxplus \cdots \boxplus a_nx_n \boxplus (-1)x_{n+1}$ iff f is a polynomial of degree at most n.

Moreover, the coefficients $a_0, \ldots, a_n$ from (ii) are uniquely determined, and if $T : \mathbb{R} \to \mathbb{R}^m$ is an affine one-to-one map and $y_k \in \mathbb{R}, k = 0, \ldots, n + 1$, are such that $T(y_k) = x_k$, then

$$a_k = \prod_{i=0, i\neq k}^{n} \frac{y_{n+1} - y_i}{y_k - y_i}, \qquad k = 0, \dots, n$$

Proof. (i)⇒(ii): Suppose that (i) holds. Since $x_0, \dots, x_{n+1}$ lie on a line not containing 0, there exists an affine one-to-one map $T : \mathbb{R} \to \mathbb{R}^m$ and $y_k \in \mathbb{R}, k = 0, \dots, n+1,$ such that $T(y_k) = x_k$ and $0 \notin T(\mathbb{R})$. Combining Theorem (5.3.28) with Theorem (5.3.29) gives (ii) and also the moreover part.

(ii)⇒ (i): Denote by d the dimension of $\mathrm{aff}(\{x_0, \dots, x_{n+1}\})$. If (ii) holds, then it follows from Proposition (5.3.25) that $n \leq n + 1 - d$, and therefore $x_0, \dots, x_{n+1}$ are collinear.

Suppose by contradiction that $x_0, \dots, x_{n+1}$ lie on a line containing 0. It is easy to construct a continuous function $f : \mathbb{R}^m \to \mathbb{R}$ which is not a polynomial but it is linear on every one-dimensional subspace of $\mathbb{R}^m$. Let $L \in L(\mathbb{R}^m)$. As $x_0, \dots, x_{n+1}$ lie in a one-dimensional subspace, the same holds for $L(x_0), \dots, L(x_{n+1})$. Hence there exists $P \in \Pi_{m,1}$ such that $P\big(L(x_k)\big) = f\big(L(x_k)\big)$ for all k. Since P is compatible with **x**, we obtain $0 = P \circ L, \mathbf{x} = f \circ L, \mathbf{x}$. Hence f is compatible with **x**. But this is a contradiction, since f is not a polynomial.