Chapter 4

The Big Slics Phenomena and Weakly Open Sets in The Unit Ball

The result obtained for L-embedded spaces can be applied to show that the above property is satisfied for every predual of an atomless real JBW*-triple. As a consequence, a characterization of the Radon-Nikodym property is obtained in this setting, showing that a predual of a real JBW*-triple E verifies the Radon-Nikodym property if, and only if, E is the l_∞ sum of real type I triple factors.In the case of the existence of a pre-dual, appropriate specializations of these characterizations are also reviewed.We provide examples of classical Banach spaces satisfying the assumptions of the results. If K is any infinite compact Hausdorff topological space, then $C(K)\,\widehat{\otimes}_\pi Y$ has the diameter two property for any nonzero Banach space Y . We also provide a result on the diameter two property for the injective tensor product.

Section (4.1): M- embedded And L-embedded Spaces:

The nonexistence of denting points in the unit ball of some function spaces has been the subject of several recent papers [67], [68]. A point x_0 in the sphere of a Banach space X, S_X, is a denting point of the unit ball in X, B_X, if there are slices, that is, subsets defined as

$$S(x^*,\alpha)=\{x\in B_X : x^*(x)>\|x^*\|-\alpha\},\ \ x^*\in X^*,\alpha\in\mathbb{R},$$

containing x_0, with diameter arbitrarily small. From [69], x_0 is a denting point of the unit ball of X if, and only if, x_0 is an extreme point in B_X and x_0 is a point of weak-norm continuity, that is, a point of continuity for the identity map from (B_X, w) onto (B_X, n), where w and n denote the weak and the norm topology, respectively. In particular, the existence of denting points in the unit ball of a Banach space X implies the existence of nonempty relatively weakly open subsets of the unit ball in X with diameter arbitrarily small. Then the extreme opposite property to the existence of denting points in the unit ball of a Banach space is that every nonempty relatively weakly open subset of the unit ball has diameter 2. This is the case, for example, for infinite-dimensional C^*-algebras [70], uniform algebras [71], non-hilbertizable real JB*-triples [72] and for some Banach spaces of vector valued functions and some spaces of operators [73].

We study when every nonempty relatively weakly open subset of the unit ball of an M-embedded or L-embedded space has diameter 2.We obtain sufficient conditions in order to assure the above property in the M-ideals case, when only the original norms are considered, by improving the results in [68]. After this, it is shown that every Banach space containing c_0 can be equivalently renormed so that every nonempty relatively weakly open subset of its unit ball has diameter 2, and then the same is true for proper M-ideals.

The result for the L-embedded case ,where a sufficient condition to have diameter 2 for all nonempty relatively weakly open subsets of the unit ball of an L-embedded space is obtained. This condition works in the setting of preduals of real JBW^*-triples and, as a consequence, we prove that every nonempty relatively weakly open subset of the unit ball of the predual of an atomless real JBW^* -triple has diameter 2. Then the same holds for preduals of atomless Von Neumann algebras. Finally an easy characterization of the Radon- Nikodym property is given , where it is shown that the predual of a real JBW^* -triple E satisfies the Radon-Nikodym property if, and only if, E is the l_∞-sum of type I real triple factors and then, the predual B of a Von Neumann algebra A satisfies the Radon-Nikodym property if, and only if, B is the c_0-sum of trace class operators on a complex Hilbert space. The above characterizations of Radon-Nikodym property can be found in [74] and [75] for preduals of complex JBW^* -triples and in [76] for preduals of Von Neumann algebras. Also the relation between L-embedded spaces and the Radon-Nikodym propety was studied in [77].

Finally, in order to obtain that every nonempty relatively weakly open subset of the unit ball of L_1/H^1 has diameter 2, where L_1 and H^1 denote the classical Lebesgue space and Hardy space, respectively, on the unit interval $[0,1]$.

Let X be a real or complex Banach space. We denote by S_X, B_X, and X^* the unit sphere, the closed unit ball and the topological dual, respectively, of X. We denote by w the weak topology of X, and by w^* the weak* topology of X^*. Given a subspace M of X, we denote by M° the polar or annihilator subspace of M in X^*. An L-projection (resp. M-projection) on X is a linear projection p on X satisfying $\|x\| = \|p(x)\| + \|x - p(x)\|$ (resp. $\|x\| = \max\{\|p(x)\|, \|x - p(x)\|\}$) for all $x \in X$. A subspace M of X is said to be an L-summand (resp. M-summand) of X if it is the range of an L-projection (resp. M-projection) on X, and an M-ideal of X if M° is an L-summand of X^*. X is said to be L-embedded (resp. M-embedded) whenever X is an L-summand (resp. M-ideal) of X^{**} (see [78]).

We have for the main results.

Lemma (4.1.1)[66]: Let X be a Banach space such that every nonempty relatively weakly open subset of B_X has diameter 2. Then every nonempty relatively weakly open subset of $B_{X\oplus_\infty}Y$ has diameter 2, where Y is an arbitrary Banach space.

Proof. We call $Z = X \oplus_\infty Y$ and let $P : Z \to X$ be the projection from Z onto X, which is weak open. It is clear that $B_Z = B_X \times B_Y$ and $\|P\| = 1$. Then if W is a weakly open subset of Z such that $W \cap B_Z \neq \emptyset$, one has that $V = P(W \cap B_Z)$ is a nonempty weak open set relative to B_X, and so $\mathrm{diam}(V) = 2$. Hence $\mathrm{diam}(W \cap B_Z) = 2$.

The following is a w^* version of the above lemma. We omit the proof, since it is similar to the one above.

Lemma (4.1.2)[66]: Let X be a Banach space. Assume that $X^{**} = Y^{\circ\circ} \oplus_\infty Z^\circ$, where Y is a closed subspace of X and Z is a closed subspace of X^*. Assume that every nonempty relatively $w^*(Y^{\circ\circ})$-open subset of $B_{Y^{\circ\circ}}$ has diameter 2. Then every nonempty relatively $w^*(X^{**})$-open subset of $B_{X^{**}}$ has diameter 2.

For a Banach space X and a subspace Z of X^* given, we denote by $\sigma(X, Z)$ the weak topology on X endowed by the dual pair (X, Z), that is, the smallest vector topology on X such that every element of Z is a continuous map.

The following result shows that the size of many nonempty relatively weakly open subsets of the unit ball of an M-ideal have diameter 2.

Proposition (4.1.3)[66]: Let X be a Banach space and let Y be a closed and proper subspace of X. Assume that Y is an M-ideal of X (that is, there is an L-projection from X^* onto some subspace Z of X^*, with kernel Y°). Then every nonempty relatively $\sigma(X, Z)$-open subset of B_X which intersects B_Y has diameter 2.

Proof. Let U be a $\sigma(X, Z)$-open subset of X and assume that $U \cap B_Y \neq \emptyset$. Choose some $x_0 \in U \cap B_Y$.

As Y is a proper subspace of X, given $\varepsilon > 0$, there is an $x \in S_X$ such that $\|x + Y\| > 1 - \varepsilon$, where $x + Y$ denotes the class of the element x in the quotient X/Y.

By [79, Proposition (4.1.3)], there exists a net $\{x_\alpha\}$ of elements of Y converging to x in the $\sigma(X, Z)$-topology and satisfying

$$\limsup_\alpha \|x_0 \pm (x - x_\alpha)\| \leq 1.$$

Then, for $0 < \lambda < 1$ given, we can choose α_0 so that $\lambda\big(x_0 \pm (x - x_\alpha)\big) \leq 1$, whenever $\alpha \geq \alpha_0$.

Furthermore, for λ close enough to 1, we can assume that $\lambda\big(x_0 \pm (x - x_\alpha)\big) \in U$ for each $\alpha \geq \alpha_0$, since the net $\big\{\lambda\big(x_0 \pm (x - x_\alpha)\big)\big\}$ converges to λx_0 in the $\sigma(X, Z)$-topology and $x_0 \in U \cap S_Y$.

Then $\lambda\big(x_0 \pm (x - x_\alpha)\big) \in U \cap B_X$, whenever $\alpha \geq \alpha_0$ and $0 < \lambda < 1$ is close enough to 1. Hence

$$\mathrm{diam}(U \cap B_X) \geq 2\lambda\|x - x_\alpha\| \geq 2\lambda\|x + Y\| > 2\lambda(1 - \varepsilon)$$

whenever $\alpha \geq \alpha_0$ and $0 < \lambda < 1$ are close enough to 1. Now, it is enough to take the limit when λ tends to 1 and ε to 0, to obtain that U has diameter 2.

The following is the main result in the M-ideals setting, which improves the results in [68], where only the nonexistence of strongly exposed points is deduced with an extra hypothesis.

Theorem (4.1.4)[66]: Let X be a Banach space and let Y be a closed and proper subspace of X. Assume that Y is an M-ideal of X (that is, there is an L-projection on X^* onto some subspace Z of X^*, with kernel Y°). If B_Z is weak-$*$ dense in B_{X^*}, then every nonempty relatively weakly open subset of B_X and B_Y has diameter 2.

Proof. As B_Z is weak-$*$ dense in B_{X^*}, then $\|x\| = \sup_{z \in B_Z} |z(x)|\ \forall x \in X$, and so the norm of X is $\sigma(X, Z)$-lower semi-continuous.

Let U be a nonempty relatively weakly open subset of B_Y. Then, there are $z_1, \cdots, z_n \in Z$ and $y_0 \in B_Y$ such that

$$U_0 = \{y \in B_Y : |z_i(y - y_0)| < 1, 1 \le i \le n\} \subset U,$$

since the $\sigma(X, Z)$-topology on Y is just the weak topology of Y.

Setting $V = \{x \in B_X : |z_i(x - y_0)| < 1, 1 \le i \le n\}$, we have that V is a nonempty relatively $\sigma(X, Z)$ open subset of B_X intersecting B_Y. By Proposition (4.1.3), we obtain that $\mathrm{diam}(V) = 2$. Now, we claim that U_0 is dense in the topological space $(V, \sigma(X, Z))$.

Indeed, $X^{**} = Y^{\circ\circ} \oplus_\infty Z^\circ$; hence every $x \in B_X$ can be written as $x = u + v$ with $u \in B_{Y^{\circ\circ}}$ and $v \in B_{Z^\circ}$. There exists a net of elements $y_\alpha \in B_Y$ which converges to u in the weak-$*$ topology. Hence, for every $z \in Z$, we have $z(x) = (u + v)(z) = u(z) = \lim_\alpha z(y_\alpha)$. Now it is clear that the assumption $|z_i(x - y_0)| < 1, 1 \le i \le n$, implies that, for some α_0 and $\alpha \ge \alpha_0, |z_i(y_\alpha - y_0)| < 1, 1 \le i \le n$. That proves the $\sigma(X, Z)$-density of U_0 in V. Moreover, as the norm of X is $\sigma(X, Z)$-lower semicontinuous, we have that $\mathrm{diam}(U_0) = 2$ and so, $\mathrm{diam}(U) = 2$. Then we have proved that every nonempty relatively weakly open subset of B_Y has diameter 2.

As every nonempty relatively weak-$*$ open subset of $B_{Y^{**}}$ contains a nonempty relatively weakly open subset of B_Y, and B_Y is weak-$*$ dense in $B_{Y^{**}}$, we deduce that every nonempty relatively weak-$*$ open subset of $B_{Y^{**}}$ also has diameter 2. Now, as Y is an M-ideal of X, we have $X^* = Y^\circ \oplus_1 Z$ and then $X^{**} = Y^{\circ\circ} \oplus_\infty Z^\circ$, where, $\oplus_1$ and $\oplus_\infty$ denote the ℓ_1 and ℓ_∞ sum, respectively. Hence, by Lemma (4.1.2), we have that every nonempty relatively weak-$*$ open subset of $B_{X^{**}}$ has diameter 2, and then every nonempty relatively weakly open subset of B_X also has diameter 2, since from the weak-$*$ density of B_X in $B_{X^{**}}$, every nonempty relatively weakly open subset of B_X is weak-$*$ dense in some nonempty relatively weak-$*$ open subset of $B_{X^{**}}$ and the norm in X^{**} is weak-$*$ lower semi-continuous.

For M-embedded Banach spaces X, we have $X^{***} = X^* \oplus_1 X^\circ$, and so we have automatically the weak-$*$ density of B_{X^*} in $B_{X^{***}}$. Then we obtain the following

Corollary (4.1.5)[66]: Let X be a non-reflexive M-embedded Banach space, and let Y be a closed subspace of X^{**} containing X. Then every nonempty relatively weakly open subset of B_Y has diameter 2.

Proof. If Y is a closed subspace of X^{**} containing X, then X is an M-ideal of Y. In fact, if p is the L-projection in X^{***} with kernel X° and image X^{*}, we identify Y^{*} with the quotient X^{***}/Y° and define $\pi : Y^{*} \to Y^{*}$ by $\pi(x^{***} + Y^{\circ}) = p(x^{***}) + Y^{\circ}$. Now π is an L-projection whose kernel is the anihilator of X in Y^{*}.

Finally, as $B_{X^{*}}$ is weak-$*$ dense in $B_{X^{***}}$, given $x^{***} \in X^{***}$ with $\|x^{***} + Y^{\circ}\| \le 1$, we choose $y^{\circ} \in Y^{\circ}$ such that $\|x^{***} + y^{\circ}\| \le 1$, and then there exists a net $\{z_{\lambda}\}$ of elements of $B_{X^{*}}$ converging to $x^{***} + y^{\circ}$ in the weak-$*$ topology on X^{***}. Then $\{z_{\lambda} + Y^{\circ}\}$ is a net of elements of $B_{\pi(Y^{*})}$ converging to $x^{***} + Y^{\circ}$ in the weak-$*$ topology on Y^{*}. Then we have proved that the closed unit ball of $\pi(Y^{*})$ is weak-$*$ dense in $B_{Y^{*}}$. It is enough to apply Theorem (4.1.4) to obtain that every nonempty relatively weakly open subset of B_Y has diameter 2.

Note that, since the property of being M-embedded is hereditary and stable by quotients, the same result is true when Y is a non-reflexive closed subspace of X or a non-reflexive quotient of X.

In particular, from the above corollary, we deduce that the unit closed ball of every closed and non-reflexive subspace of an M-embedded space has no continuity points and so has no strongly exposed points, and the same is true for non-reflexive quotients of an M-embedded spaces. Roughly speaking, this fact shows that every subspace and every quotient of an M-embedded space fails the Radon-Nikodym property in a very strong way, if it is not reflexive. As a consequence of the above corollary, it is worth mentioning some interesting examples. As c_0 is an M-embedded space, not only every infinite-dimensional subspace or quotient of c_0 verifies the conclusion of Corollary (4.1.5), but also every subspace of ℓ_{∞} containing c_0. If H is a Hilbert space, and $K(H)$ and $L(H)$ stand for the space of compact operators on H and the space of all bounded operators on H, respectively, it is well known that $K(H)$ is an M-embedded space. Again, not only every subspace or quotient of $K(H)$ satisfies the conclusion of Corollary (4.1.5), but also every subspace of $L(H)$ containing $K(H)$, since $K(H)^{**}$, the bidual space of $K(H)$, is isometrically isomorphic to $L(H)$.

As the failure of the Radon-Nikodym property in non-reflexive M-embedded spaces is well known, since every non reflexive M-embedded space contains an isomorphic copy of c_0, it is natural to ask for the behavior of relatively weakly open subsets of the unit ball of a Banach space containing c_0-copies. Of course, not every Banach space containing c_0-copies lacks a point of continuity in its unit ball. For this, it is enough to consider $X = c_0 \oplus_1 \ell_1$. It is easy to see that $(0, e_1)$ is a denting point of B_X, where e_1 denotes the first vector of the unit vector basis in ℓ_1. However the following result shows that, up to renorming, the above situation cannot happen.

Proposition (4.1.6)[66]: Let X be a Banach space containing a subspace isomorphic to c_0. Then there exists an equivalent norm in X so that every nonempty relatively weakly open subset of the new unit ball has diameter 2.

Proof. As the conclusion is isomorphic in nature, we can suppose that X contains a subspace Y isometric to c_0. Now, by [78, Proposition II.2.10], there exists an equivalent norm $|\,|$ on X which agrees with the original norm on Y so that Y becomes M-ideal in X.

Then we have $(X, |\,|^*)^{**} = Z \oplus_\infty Y^{\circ\circ}$ for some subspace Z of X^{**}. Finally, as $Y^{\circ\circ}$ is isometric to ℓ_∞ and every nonempty relatively weakly open subset of B_{ℓ_∞} has diameter 2, it is enough to apply Lemma (4.1.1) to obtain that every nonempty relatively weakly open subset of $(B_X, |\,|)$ has diameter 2.

As every proper M-ideal, that is, an M-ideal which is not an M-summand, contains an isomorphic copy of c_0, we deduce the following

Corollary (4.1.7)[66]: Let X be a proper M-ideal of a superspace Y. Then there is an equivalent norm in Y so that every nonempty relatively weakly open subset of the new unit ball of X and Y has diameter 2.

Now we pass to study the size of nonempty relatively weakly open subsets of the unit ball of an L-embedded space. The result is the following

Theorem (4.1.8)[66]: Let X be an L-embedded Banach space, that is, $X^{**} = X \oplus_1 Z$ for some subspace Z of X^{**}. If B_Z is weak-$*$ dense in $B_{X^{**}}$, then every nonempty relatively weakly open subset of B_X has diameter 2.

Proof. Let U be a nonempty relatively weakly open subset of B_X. As X is infinite dimensional, there is an $x_0 \in U \cap S_X$ and then there exist $f_1, \dots, f_n \in X^*$ such that

$$U_0 = \{x \in B_X : |f_i(x - x_0)| < 1, 1 \le i \le n\} \subset U.$$

Let $V = \{x^{**} \in B_{X^{**}} : |f_i(x^{**} - x_0)| < 1, 1 \le i \le n\}$. Then V is a nonempty relatively weak-$*$ open subset of $B_{X^{**}}$ such that $x_0 \in U_0 \subset V$.

From the weak-$*$ density of B_Z in $B_{X^{**}}$ we can choose a net $\{z_\lambda\}$ of elements in B_Z converging to x_0 in the weak-$*$ topology of X^{**}, hence there is a λ_0 such that $z_\lambda \in V$, whenever $\lambda \ge \lambda_0$. Furthermore, as the norm of X^{**} is weak-$*$ lower semi-continuous, we have $\liminf_\lambda \|z_\lambda\| \ge \|x_0\| = 1$. Then, given $\varepsilon > 0$ then there is a $\mu \ge \lambda_0$ such that $\|z_\mu\| > 1 - \varepsilon$, and $z_\mu \in V$. Now, we deduce

$$\operatorname{diam}(V) \ge \|x_0 - z_\mu\| = \|x_0\| + \|z_\mu\| > 1 + 1 - \varepsilon = 2 - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude that $\operatorname{diam}(V) = 2$.

Finally, from the weak-$*$ density of B_X in $B_{X^{**}}$ we obtain that U_0 is relatively dense in the topological space V endowed with the weak-$*$ topology of X^{**}. Now, the weak-$*$ lower semicontinuity of the norm in X^{**} allows us to assure that $\operatorname{diam}(U) \geq \operatorname{diam}(U_0) = \operatorname{diam}(V) = 2$.

In order to show examples where Theorem (4.1.8) works, we denote by H^1 and L_1 the Hardy space and the Lebesgue space on the interval $[0,1]$. Also, H_0^1 stands for the subspace of H^1 of functions in H^1 vanishing at 0. From [80, p. 27], the unit ball of L_1/H_0^1 lacks extreme points, and it is well known that L_1/H_0^1 is an L-embedded space. Then we can apply Theorem (4.1.8) as in Theorem (4.1.12) to obtain the following

Corollary (4.1.9)[66]: Every nonempty relatively weakly open subset of the unit ball of $L_1/H_0^1 h$ as diameter 2.

The same is true for L_1/H^1 instead L_1/H_0^1 since they are isometric.

Let X be a Banach space. For u in S_X, we define the roughness of X **at** $u, \eta(X,u)$, by the equality

$$\eta(X,u) := \limsup_{\|h\|\to 0} \frac{\|u+h\| + \|u-h\| - 2}{\|h\|}.$$

We remark that the absence of roughness of X at u (i.e., $\eta(X,u) = 0$) is nothing other than the Fréchet differentiability of the norm of X at u [81, Lemma I.1.3]. Given $\epsilon > 0$, the Banach space X is said to be **ϵ-rough** if, for every u in S_X, we have $\eta(X,u) \geq \epsilon$. We say that X is rough whenever it is ϵ-rough for some $\epsilon > 0$, and extremely rough whenever it is 2-rough.

Assume that X is a Banach space such that every nonempty relatively weakly open subset of B_X has diameter 2. Then, by [81, Proposition I.1.11], the dual X^* of X (resp. the predual X_* of X, if this exists) is extremely rough.

Then, we have the following consequences (see Theorem (4.1.12) and Corollary (4.1.13) for part ii)):

Corollary (4.1.10)[66]: The following Banach spaces are extremely rough:

i) The dual of any nonreflexive M-embedded space.

ii) The real atomless JBW^*-triples, and so the atomless Von Neumann algebras.

Corollary (4.1.11)[66]: Every Banach space X containing an isomorphic copy of c_0 can be equivalently renormed so that X_* becomes extremely rough.

In order to show a new application of Theorem (4.1.8) we introduce some notation.

We recall that a complex JB^*-triple is a complex Banach space X with a continuous triple product $\{\cdots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables and conjugate-linear in the middle variable, and satisfies:

(i) For all $x \in X$, the map $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.

(ii) $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ for all $a, b, x, y, z \in X$.

(iii) $\|\{xxx\}\| = \|x\|^3$ for all $x \in X$.

We also recall that a bounded linear operator T on a complex Banach space X is said to be hermitian if $\|exp(irT)\| = 1$ for every real number r.

Following [82], we define real JB^*-triples as norm-closed real subtriples of complex JB^* -triples. Here, by a subtriple we mean a subspace which is closed under triple products of its elements. A triple ideal of a real or complex JB^*-triple X is a subspace M of X such that $\{XXM\} + \{XMX\} \subseteq M$. Real JBW^*-triples where first introduced as those real JB^*-triples which are dual Banach spaces in such a way that the triple product becomes separately w^*-continuous (see [82, Definition 4.1 and Theorem 4.4]). Later, it was shown in [83] that the requirement of separate w^*-continuity of the triple product is superabundant.

Finally, we recall that an element x of a real JB^*-triple E is said to be a tripotent if $\{xxx\} = x$. Given x, y tripotents in E, we say that x and y are orthogonal if $\{uuv\} = 0$ and we say that $x \geq y$ if $x - y$ and y are orthogonal tripotents. Then a minimal tripotent will be a tripotent which is minimal in the partial order defined above.

Examples of real JB^*-triples are the spaces $L(H, K)$, for arbitrary real, complex, or quaternionic Hilbert spaces H and K, under the triple product $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$. Also, the corresponding spaces of all symmetric, $S(H)$, and skew, $A(H)$, bounded linear operators on H can be considered real JB^*-triples. The above examples become particular cases of those arising by considering either the so-called complex Cartan factors (regarded as real JB^*-triples) or real forms of complex Cartan factors [84]. We recall that real forms of a complex Banach space X are defined as the real closed subspaces of X of the form $X^\tau := \{x \in X : \tau(x) = x\}$, for some conjugation (i.e., conjugate-linear isometry of period two) on X. We note that, if X is a complex JB^*-triple, then every real form of X is a real JB^*-triple (since conjugations on X preserve triple products [85]). Among complex Cartan factors, the so-called complex spin factors become especially relevant for our present approach. They are built from an arbitrary complex Hilbert space $(H, (\cdot \mid \cdot))$ of hilbertian dimension ≥ 3 by taking a conjugation σ on H, and then by defining the triple product and the norm by

$$\{xyz\} := (x|y)z + (z|y)x - \big(x\big|\sigma(z)\big)\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - \left|\big(x\big|\sigma(x)\big)\right|^2},$$

respectively, for all x, y, z in H. Following [86], we say that a real JB^*-triple is a generalized real spin factor if it is either a complex spin factor (regarded as a real JB^*-triple) or a real form of a complex spin factor.

It is well known that every complex JBW^*-triple E has a unique isometric predual V, and this is an L-embedded space. This is also the case for preduals of real JBW^*-triples as done in [72].

Now we are ready to show the application of Theorem (4.1.8).

Theorem (4.1.12)[66]: Let A be a real JBW^*-triple and let A_* be its predual. If A is atomless, that is, A lacks minimal tripotentes, then every nonempty relatively weakly open subset of B_{A_*} has diameter 2.

Proof. As A is atomless, according to [87, Corollary 2.1], we deduce that B_{A_*} lacks extreme points. Now, A_* is an L-embedded space by [72, Proposition 2.2]; then we have $A_* = A_* \oplus_1 Z$, for some subspace Z of A_*. Let us see that B_Z is weak$*$-dense in B_{A_*}.

As B_{A_*} lacks extreme points and the set of extreme points of B_{A_*} is the union of the sets of extreme point of B_{A_*} and B_Z, we obtain that $ext\big(B_{A_*}\big) = ext(B_Z)$, where $ext(K)$ denotes the set of extreme points of K. Now, the Krein-Milman theorem applied to B_{A_*} gives us that $B_{A_*} = \overline{co}^{w^*}\big(ext(B_Z)\big)$, and then the desired conclusion.

Finally, it is enough to apply Theorem (4.1.8) to finish the proof.

In the setting of C^*-algebras, the concept of mininal tripotents is exactly the well-known notion of minimal projections. As a Von Neumann algebra is also a real JBW^*-triple, we obtain the following

Corollary (4.1.13)[66]: Let A be an atomless Von Neumann algebra, that is, A lacks minimal projections, and A_* stands for its predual. Then every nonempty relatively weakly open subset of B_{A_*} has diameter 2.

Finally, we show a characterization of Radon-Nikodym property in the setting of the preduals of real JBW^*-triples and, as a consequence, in the setting of the preduals of Von Neumann algebras, too.

Theorem (4.1.14)[66]: Let A be real JBW^*-triple and A_* its preduals. Then:

i) A_* satisfies the Radon-Nikodym property if, and only if, A is purely atomic, that is, A is the weak-$*$ closed linear span of its minimal tripotents. Furthermore, in this case, A is the ℓ_∞-sum of weak-$*$ closed simple ideals which are either finite-dimensional, infinite-dimensional generalized real spin factors or of the form $L(H, K), S(H)$ or $A(H)$ for some real, complex or quaternionic infinite-dimensional Hilbert spaces H and K.

ii) A_* fails the Radon-Nikodym property if, and only if, A_* can be equivalently renormed so that every nonempty relatively weakly open subset of B_{A_*} has diameter 2.

iii) A_* satisfies the Radon-Nikodym property if, and only if, A_* satisfies the Krein-Milman property.

Proof. i) Assume that A_* verifies the Radon-Nikodym property. By [87, Theorem 3.6], we have $A = B \bigoplus_\infty C$, where B and C are weak$*$ closed real triple ideals of A, such that B is purely atomic and C is atomless. By the above corollary $C = 0$, since the Radon-Nikodym property is hereditary. Then $A = B$ is purely atomic.

Now, in order to describe the preduals of real JBW^* triples satisfiyng the Radon-Nikodym property, assume that A is purely atomic. We denote by $\hat{A} = A \bigoplus iA$ the complexification of A. By [87], $\hat{A}$ is a purely atomic complex JBW^* triple and then, by [88], $\hat{A}$ is the ℓ_∞ sum of type I Cartan factors, that is, the ℓ_∞ sum of w^*-closed simple ideals which are either finite-dimensional, infinite-dimensional complex spin factors or of the form $L(H,K), S(H)$ or $A(H)$ for some complex Hilbert spaces H and K. Taking into account that the conjugation τ preserves the triple product and is w^*-continuous, it is enough to apply [84] to deduce that A is the ℓ_∞ sum of w^*-closed simple ideals which are either finite-dimensional, infinite-dimensional generalized real spin factors or of the form $L(H,K), S(H)$ or $A(H)$ for some real, complex or quaternionnic Hilbert spaces H and K. Finally, as the Radon-Nikodym property is stable by ℓ_1 sums and the preduals of the above spaces satisfy the Radon-Nikodym property (see [89]) we deduce that A_* verifies the Radon-Nikodym property.

ii) If A_* fails the Radon-Nikodym property, as in the above paragraph, we set $A = B \bigoplus_\infty C$. Now, as B and C are w^*-closed real triple ideals of A, we have $A_* = D \bigoplus_1 E$, where E is the predual of the atomless real JBW^*-triple C. By the above corollary, every nonempty relatively weakly open subset of B_E has diameter 2. Now it is enough to apply Lemma (4.1.1), to see that A_* can be equivalently renormed so that every nonempty relatively weakly open subset of B_{A_*} has diameter 2. The converse implication is trivial.

iii) It is well known that every Banach space satisfying the Radon-Nikodym property also verifies the Krein-Milman property. In order to prove the converse, assume that A_* fails the Radon-Nikodym property. Then, by i), A is not purely atomic. Now, as in ii), $A = B \bigoplus_\infty C$, where B and $C \neq 0$ are weak* closed real triple ideals of A, such that B is purely atomic and C is atomless. Then, $A_* = D \bigoplus_1 E$, where E is the predual of the atomless real JBW^*-triple C. By [87, Corollary 2.1], we deduce that B_E lacks extreme points and hence A_* fails the Krein-Milman property.

Corollary (4.1.15)[66]: Let A be a Von Neumann algebra and A_* its predual. Then:

i) A_* satisfies the Radon-Nikodym property if, and only if, A is purely atomic. Furthermore, in this case, there exists $\{H_i\}$ a family of infinite-dimensional complex Hilbert spaces, such that $A = \ell_\infty - \sum_i L(H_i)$ and $A_* = \ell_1 - \sum_i N(H_i)$, where $N(H)$ denotes the space of all nuclear operators on H.

ii) A_* fails the Radon-Nikodym property if, and only if, A_* can be equivalently renormed so that every nonempty relatively weakly open subset of BA_* has diameter 2.

iii) A_* satisfies the Radon-Nikodym property if, and only if, A_* satisfies the Krein-Milman property.

Section (4.2): Characterizations of Unitaries:

C^*-algebras have inspired many notions and results in the theory of Banach spaces. Concerning the interest of the present paper, the starting point could be dated in the early Bohnenblust–Karlin [91], where it is shown that the unit of every norm-unital complex Banach algebra is a vertex of the closed unit ball of the algebra (a purely Banach space notion), and that vertices of the closed unit ball of a unital C^*-algebra are precisely the unitary elements of the algebra .Since the Bohnenblust–Karlin paper, many authors have been interested in the geometry of the Banach spaces of norm-unital Banach algebras around their units, and the relationship between this geometry and the algebraic structure. The results obtained in this line until 1973 are nicely collected in the Bonsall–Duncan monograph [92,93], and culminate in the celebrated Vidav–Palmer theorem characterizing unital C^*-algebras among norm-unital complex Banach algebras in terms involving only the Banach space of the algebra, and the unit [92].

Another aspect where C^*-algebras inspired deep developments in the theory of Banach spaces is that of the infinite dimensional holomorphy. Indeed, the open unit ball of any C^*-algebra is a bounded symmetric domain [94]. Moreover, if the open unit ball of a complex Banach space is a bounded symmetric domain, then the Banach space itself is "almost" a C^*-algebra, and there is an intrinsically defined triple product $\{\cdots\}$ on it which behaves algebraically and geometrically like the one obtained from the binary product of a C^*-algebra by taking $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$. The resulting mathematical creature, called a JB^*-triple, has been feverishly studied in last years, and the star result in the theory, due to W. Kaup [95], asserts that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable JB^*-triple. Let us mention also that, by combining the theory of JB^*-triples with the literature originated in the Bohnenblust–Karlin paper [91], it has been possible to characterize (possibly non-unital) C^*-algebras as those complex Banach algebras having an approximate unit bounded by 1, and whose open unit ball is a bounded symmetric domain [96].

We review in detail the already commented Bohnenblust–Karlin Banach space characterization of unitaries in C^*-algebras ,as well as some results, both in the theory of Banach algebras [97,92,93,98,99,100,101] and in the one of Banach spaces [102,103,104,105,106], originated in that characterization. It is worth mentioning that the Bandyopadhyay–Jarosz–Rao paper [102] is motivated by the recent Akemann–Weaver rediscovery [107] of the Bohnenblust–Karlin characterization, and that, in its turn, some results in [102] become rediscoveries of previous ones in [14,106].

We show how, thanks to [108,97,109], the Bohnenblust–Karlin Banach space characterization of unitaries in C^*-algebras remains true in the more general setting of JB^*-triples , and note that a JB^*-triple has unitary elements if and only if it is the JB^*-triple underlying a suitable unital JB^*-algebra [108]. JB^*-algebras are Jordan–Banach complex algebras with involution, which behave like C^*-algebras endowed with the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ [110].

We devoted to the study of unitaries in real JB^*-triples (i.e., closed real subtriples of JB^*-triples [111]). Since a Banach space characterization of unitaries in real JB^*-triples cannot be found and such a characterization could be found [112], but remains unknown to date, in the particular case of real JB^*-algebras (i.e., closed $*$-invariant real subalgebras of JB^*-algebras), we center our attention in the still more particular case of JB-algebras. JB-algebras are Jordan– Banach real algebras which behave like the self-adjoint parts of C^*-algebras endowed with the Jordan product [113]. We collect a Banach space characterization of unitaries in JB-algebras, essentially due to J.D.M. Wright and M.A. Youngson [114], and review the recent Leung–Ng–Wong determination of vertices of closed unit balls of JB-algebras [115].

We involving in our development the notions of norm-norm and norm-weak upper semi-continuity of the duality mapping of a Banach space at a point [116]. Although these notions were not motivated by any Banach algebra question or result, they have shown useful in the study of norm-unital Banach algebras [99,106], mainly in the case that these algebras are in addition dual Banach spaces [117]. Then, we invoke deep results in the Godefroy–Indumathi and Godefroy–Rao papers [118,119], about the norm-weak upper semi-continuity of the pre-duality mapping of a dual Banach space, and derive three previously unnoticed relevant facts [120],[121].

The celebrated of R.V. Kadison [122], on surjective linear isometries of C^*-algebras, implicitly contains a Banach space characterization of unitary elements in unital C^*-algebras. Actually, an explicit characterization is given by the following.

Theorem (4.2.1)[90]: Let A be a unital C^*-algebra, and let u be a norm-one element of A. Then the following conditions are equivalent:

(i) u is unitary.

(ii) The dual space A^* is the linear hull of the set of states of u.

(iii)u is a vertex of the closed unit ball of A.

We recall that, given a norm-one element u of a Banach space X, the states of u (relative to X) are defined as those norm-one elements f of the dual space X^* satisfying $f(u) = 1$. We also recall that vertices of the closed unit ball of a Banach space X are defined as those norm-one elements u of X such that the set of states of $u, D(X, u)$, separates the points of X.

The proof of the implication $(i) \Rightarrow (ii)$ is really easy. Indeed, since it is well known that Condition (ii) is fulfilled in the case that u equals the unit **1** of A, it also remains fulfilled for every unitary u because unitary elements of A lie in the orbit of **1** under the group of all surjective linear isometries on A. By the way, an alternative proof of $(i) \Rightarrow (ii)$ can be given by noticing that, if u is a unitary element of A, then A becomes a C^*-algebra with unit u under the product $xy := xu^*y$ and the involution $x \to ux^*u$. On the other hand, the implication $(ii) \Rightarrow (iii)$ is clear. Therefore, the core of the theorem is the implication $(iii) \Rightarrow (i)$, which is proved in Example 4.1 of the early paper of H.F. Bohnenblust and S. Karlin [91], and is collected in Theorem 9.5.16(c) of Palmer's book [123]. As pointed out by Bohnenblust and Karlin in [91], the equivalence $(i) \Leftrightarrow (iii)$ in the above theorem drastically simplifies Kadison's original arguments in [122].

Theorem (4.2.1) remained forgotten by many people until it has been rediscovered by C. Akemann and N. Weaver [107, Theorem 2], who only formulate the equivalence $(i) \Leftrightarrow (ii)$ in that theorem.

We list in Remark (4.2.2) immediately below some other known results related to Theorem (4.2.1). To be short, norm-one elements u of a Banach space X such that X^* is the linear hull of $D(X, u)$ will be called geometrically unitary elements of X.

Remark (4.2.2)[90]: (i) Let X be a Banach space, let u be a norm-one element of X, and define the numerical index, $n(X, u)$, of X at u by

$$n(X, u) := \inf_{\|x\|=1} \sup_{f \in D(X,u)} f(x).$$

Then u is a geometrically unitary element of X if and only if $n(X, u) > 0$ [106, Theorem 3.2]. We note that $n(X, u)$ can be equivalently defined as the maximum nonnegative real number k satisfying

$$k \sup_{f \in D(X,u)} |f(x)| \leq \|x\|$$

for every $x \in X$.

(ii) The above result becomes an abstract version of the Moore–Sinclair theorem [124,125] (see also [93]) that, if A is a complex Banach algebra with a norm-one unit $\mathbf{1}$, then $\mathbf{1}$ is a geometrically unitary element of A. Indeed, the Moore–Sinclair theorem follows from (i) by keeping in mind the Bohnenblust–Karlin theorem that $n(A, \mathbf{1}) \geq \frac{1}{e}$ [91] (see also [92]). Part (i) of the present remark also implies that the requirement of associativity of A in the Moore–Sinclair theorem can be altogether removed because, as pointed out in [106], the Bohnenblust–Karlin theorem remains true in the non-associative context.

(iii) Let A be a complex Banach algebra with a norm-one unit $\mathbf{1}$ (associativity of A is now required), and let u be an algebraically unitary element of A (i.e., an invertible element satisfying $\|u\| = \|u^{-1}\| = 1$). Since the operator of left multiplication by u on A is a surjective linear isometry taking $\mathbf{1}$ to u, it follows from the Moore–Sinclair theorem that u is a geometrically unitary element of A.

(iv) Let X be a Banach space, and let u be a norm-one element of X. Then we have $n(X^{**}, u) = n(X, u)$ [106]. Consequently, by Part (a) of the present remark, u is a geometrically unitary element of X^{**} if and only if it is a geometrically unitary element of X [102].

(v) Let A be a unital C^*-algebra. It follows from [98] that, for a norm-one element u in A, each of Conditions (i) to (iii) in Theorem (4.2.1) is equivalent to (iv) $n(A, u)$ is equal to 1 or $\frac{1}{2}$.

Moreover, the existence of a norm-one element u of A with $n(A, u) = 1$ is equivalent to the commutativity of A.

(vi) It is straightforward that vertices of the closed unit ball, B_X, of a Banach space X are extreme points of B_X. By the way, the closed unit ball of a C^*-algebra A has extreme points if and only if A has a unit, and, in this case, the extreme points of B_A are the elements $u \in A$ such that $(\mathbf{1} - uu^*)A(\mathbf{1} - u^*u) = 0$ [126].

(vii) Let us say that a norm-one element u of a Banach space X is a strongly extreme point of B_X if, whenever (x_n) and (y_n) are sequences in B_X with $lim_{n\to\infty} \frac{1}{2}(x_n + y_n) = u$, we have $lim_{n\to\infty}(x_n - y_n) = 0$. According to [102], geometrically unitary elements of X are strongly extreme points of B_X.

(viii) The above result becomes an abstract version of the previously known fact that, if A is a complex Banach algebra with a norm-one unit $\mathbf{1}$, then $\mathbf{1}$ is a strongly extreme point of B_A [92]. Indeed, this fact follows from (vii) and the Moore–Sinclair theorem. Now note that the associativity of A is not needed in the Moore–Sinclair theorem (see Part (ii) of the present remark), and that, if A is a (possibly non-associative) real Banach algebra with a norm-one unit $\mathbf{1}$, then the projective tensor product $\mathbb{C}_{\otimes\pi}A$ is a complex Banach algebra with unit $1 \otimes \mathbf{1}$. It follows that, if A is a real or complex (possibly non-associative) Banach algebra with a norm-one unit **1**, then **1** is a strongly extreme point of B_A. For quantitative versions of this fact, the reader is referred to [127].

(ix) Let X be a Banach space (with unit sphere S_X), and let u be in S_X. The element u is said to be a strongly exposed point of B_X if there exists $g \in S_{X^*}$ with the property that, whenever (x_n) is a sequence in B_X such that $\big(g(x_n)\big) \to 1$, we have $(x_n) \to u$. It is well known that, if u is a strongly exposed point, then u is a denting point of B_X, which means that there are slices of B_X of arbitrarily small diameter which contain u. On the other hand, if u is a denting point of B_X, then u is a strongly extreme point of B_X [128]. Now, let A be a (possibly non-associative) real or complex Banach algebra with a norm-one unit **1**. Then there exists an equivalent algebra norm $|||\cdot|||$ on A arbitrarily close to $\|\cdot\|$, satisfying $|||\mathbf{1}||| = 1$, and such that **1** becomes a strongly exposed point of $B_{(A,|||\cdot|||)}$ [100]. Moreover, if in addition A is a dual Banach space, then the norm $|||\cdot|||$ above can be chosen among the dual norms on A, and in such a way that **1** becomes in fact a w^*-strongly exposed point of $B_{(A,|||\cdot|||)}$. We note that, even if A is associative, **1** need not be a denting point (much less a strongly exposed point) of B_A. Indeed, the Banach algebra of all bounded linear operators on any infinite-dimensional complex Hilbert space has no denting point in its closed unit ball [129]. More generally, the closed unit ball of a C^*-algebra A is dentable (i.e., there are slices of B_A of arbitrarily small diameter) if and only if A is finite-dimensional [130].

(x) By a unitary Banach algebra we mean an associative Banach algebra A with a norm-one unit **1**, and such that B_A equals the closed convex hull of the set of all algebraically unitary elements of A (in the sense of Part (iii) of the present remark). Relevant examples of unitary Banach algebras are all unital C^*-algebras [93] and the discrete group algebras $\ell_1(G)$ for every group G. Let A be a unitary Banach algebra. If A is complex, then the product of A becomes a geometrically unitary element in the Banach space of all continuous bilinear mappings from $A \times A$ to A [97]. On the other hand, if B_A is dentable, then, according to [97], for a norm-one element u of A, the following assertions are equivalent:

($\acute{\imath}$) u is an algebraically unitary element of A.

($\acute{\imath}\imath$) u is a denting point of B_A.

($\acute{\imath}\imath\imath$) B_A equals the closed convex hull of the orbit of u under the group of all surjective linear isometries on A.

We do not know if Theorem (4.2.1) remains true whenever "unital C^*-algebra" is replaced with "unitary complex Banach algebra", nor even if algebraic unitaries in unitary complex Banach algebras can be geometrically characterized. Unitary Banach algebras originated in Cowie's PhD thesis [131], and were reconsidered, without noticing Cowie's forerunner, in the Hansen–Kadison [132]. For later development of the theory, see [133,134,135,97] and references therein.

(xi) Let X be a Banach space. By a product on X we mean a continuous bilinear mapping from $X \times X$ to X. We denote by $\mathcal{P}(X)$ the Banach space of all products on X. Now, assume that X is complex. By a C^*-product on X we mean an element $p \in \mathcal{P}(X)$ such that X, endowed with the product p and a suitable involution, becomes a C^*-algebra. Every C^*-product p on X is both a geometrically unitary element of $\mathcal{P}(X)$ [99] and an "approximately norm-unital product" (which means that (X, p) is a Banach algebra having an approximate unit bounded by 1). In the case that X is the Banach space underlying a C^*-algebra, the converse is also true. More precisely, if there are C^*-products on X, and if p is both an extreme point of $B_{\mathcal{P}(X)}$ and an approximately norm-unital product (associativity of p is not assumed), then p is an (automatically associative) C^*-product on X [101].

(xii) Vertices of the closed unit ball of a Banach space X need not be geometric unitaries [105,], nor even strongly extreme points of B_X [102]. The vertex in [105] is in fact the unit of a real norm-unital Banach algebra (say X), and hence, by Part (h) of the present remark, it is a strongly extreme point of B_X.

(xiii) Let X be a complex Banach space. As a consequence of Part (b) of the present remark, the identity mapping on X (say **1**) is a vertex of the closed unit ball of the Banach space $L(X)$ of all bounded linear operators on X. Actually, **1** remains a vertex of the closed unit ball of a much larger Banach space. More precisely, by passing to restrictions to B_X, the space $L(X)$ can be identified with a closed subspace of the sup-normed Banach space $A_0(X)$ of all bounded continuous functions from B_X to X which are holomorphic in the open unit ball of X and vanish at zero, and it follows straightforwardly from [103] that **1** becomes a vertex of $B_{A_0(X)}$. In fact, minor changes to the proof of [102] allow us to realize that **1** is a vertex of $B_{A(X)}$, where $A(X)$ stands for the Banach space of all bounded continuous functions from B_X to X which are holomorphic in the open unit ball of X.

(xiv) As pointed out in [102], a bounded linear operator T on a Banach space X is a geometrically unitary element of $L(X)$ whenever the adjoint operator T^* is a geometrically unitary element of $L(X^*)$. The converse is far from being true. Indeed, as a consequence of [104] (with E equal to the two-dimensional Euclidean real space), there is a real Banach space X (namely, the space $X(E)$ in that theorem) such that the identity mapping **1** on X is a geometrically unitary element of $L(X)$ (with $n(L(X), \mathbf{1}) = 1$), whereas the identity mapping on X^* is not even a vertex of $B_{L(X^*)}$.

We note that Parts (i) and (iii) of the above remark have been recently rediscovered (see [102] and [102], respectively).

JB^*-triples are defined as those complex Banach spaces X endowed with a continuous triple product $\{\cdots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(i) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a Hermitian operator on X (in the sense of [92]) and has nonnegative spectrum.

(ii) The equality

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.

(iii) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Every C^*-algebra becomes a JB^*-triple under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

More generally, C^*-algebras are JB^*-algebras under the product

$$x \circ y := \frac{1}{2}(xy + yx), \tag{1}$$

and JB^*-algebras become JB^*-triples under the triple product

$$\{xyz\} := x \circ (y^* \circ z) + z \circ (y^* \circ x) - (x \circ z) \circ y^* \tag{2}$$

(see [108,136]). We recall that JB^*-algebras are defined as those complete normed Jordan complex algebras A endowed with a conjugate-linear algebra-involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in A, where, for x in A, the operator $U_x : A \to A$ is defined by $U_x(y) = 2x \circ (x \circ y) - x^2 \circ y$.

The main interest of JB^*-triples relies on the fact that, up to biholomorphic equivalence, there are no bounded symmetric domains in complex Banach spaces others than the open unit balls of JB^*- triples [95]. Unitary elements of a JB^*-triple X are defined as those elements u of X satisfying $\{u\{uxu\}u\} = x$ for every $x \in X$. It is easily seen that, if a C^*-algebra A has a unitary element in the JB^*-triple sense, then A has a unit, and unitary elements in the JB^*-triple sense coincide with unitary elements in the usual C^*-algebra meaning.

According to the celebrated Kaup's version [95] of the Banach–Stone theorem, the surjective linear isometries between JB^*-triples are precisely the triple-isomorphisms. Therefore, implicitly, unitary elements of a JB^*-triple are determined by the Banach space structure. An explicit determination is provided by the following.

Theorem (4.2.3)[90]: For a norm-one element u in a JB^*-triple X, the following conditions are equivalent:

(i) u is unitary.

(ii) $n(X,u)$ is equal to 1 or $\frac{1}{2}$.

(iii) u is geometrically unitary.

(iv) u is a vertex of the closed unit ball of X.

Proof: If Condition (1) is fulfilled, then X, endowed with the product $x \circ y := \{xuy\}$ and the involution $x^* := \{uxu\}$, becomes a unital JB^*-algebra whose unit is precisely u (see [108]), and hence (2) holds by [109, Theorem 26] (see also [137, Theorem 4]). On the other hand, the implication (2) ⇒ (3) follows from Remark (4.2.2)(a), and the one (3) ⇒ (4) is clear. Finally, the implication (4) ⇒ (1) follows from [96, Lemma 3.1] and [108, Theorem 4.4]. An alternative proof of (4) ⇒ (1) can be given by keeping in mind that vertices of the closed unit ball of a Banach space are extreme points, that extreme points of the closed unit ball of a JB^*-triple are well understood [138, Proposition 3.5], and then by selecting (with the help of [139, Proposition 1(a)]) those extreme points which are in fact vertices.

In relation to the above proof, it is worth mentioning that, contrarily to what happens in the case of C^*-algebras, the group of all surjective linear isometries on a JB^*-triple X need not act transitively on the set of all unitary elements of X [108, Example 5.7]. Let us also notice that, by the references applied above, the existence in a JB^*-triple X of a geometrically unitary element (respectively, of a norm-one element u with $n(X,u) = 1$) is equivalent to the fact that X is triple-isomorphic to a unital JB^*-algebra (respectively, to a unital commutative C^*-algebra). Therefore, as in the case of C^*-algebras (see Remark (4.2.2)(j)), if a JB^*-triple X has a unitary element, then B_X equals the closed convex hull of the set of all unitaries in X [140].

We list in Remark (4.2.4) immediately below some other known results related to Theorem (4.2.3).

Remark (4.2.4)[90]:(i) The closed unit ball of a JB^*-algebra A has extreme points if and only if A has a unit [136]. Therefore, by Remarks (4.2.2)(b) and (4.2.2) (f), a JB^*-algebra A has geometrically unitary elements if and only if B_A has vertices, if and only if B_A has extreme points. A similar situation need not hold for general JB^*-triples. Indeed, every complex Hilbert space H becomes a JB^*-triple under the triple product $\{xyz\} := \frac{(x|y)z+(z|y)x}{2}$, and however every element in S_H is an extreme point of B_H whereas, if $dim(H) \geq 2, B_H$ has no vertex.

(ii) Let A be a Jordan algebra with a unit $\mathbf{1}$, and let x be an element of A. Following [141], we say that x is invertible in A if there exists $y \in A$ such that the equalities $x \circ y = \mathbf{1}$ and $x^2 \circ y = x$ hold. If x is invertible in A, then the element y above is unique, it is called the inverse of x, and is denoted by x^{-1}. Now assume that A has an involution (respectively, that A is normed). Then we say that x is $*$-unitary (respectively, algebraically unitary) if it is inversible in A with $x^{-1} = x^*$ (respectively, with

$\|x\| = \|x^{-1}\| = 1$). According to [108], if A is in fact a unital JB^*-algebra, then x is a unitary element of A in the JB^*-triple sense if and only if it is $*$-unitary, if and only if it is algebraically unitary.

(iii) The bidual of a JB^*-algebra A, endowed with a suitable product and a suitable involution, is a unital JB^*-algebra containing A as a JB^*-subalgebra [136].

(iv) Unitaries in JB^*-triples are examples of tripotents (i.e., elements x such that $\{xxx\} = x$). In the particular case of C^*-algebras, tripotents are usually called partial isometries. As a consequence of [142] and [143], tripotents in a JB^*-triple X are precisely those elements in X which are centers of symmetry of some w^*-closed face of the closed unit ball of X^{**}.

(v) A more recent Banach space characterization of tripotents in JB^*-triples, generalizing a previous result for C^*- algebras [107], is proved in [121]. Indeed, a norm-one element u of a JB^*-triple X is a tripotent if and only if the sets

$$\{x \in X: \text{there exists } \alpha > 0 \text{ with } \|u + \alpha x\| = \|u - \alpha x\| = 1\}$$

and

$$\{x \in X: \|u + \beta x\| = max\{1, \|\beta x\|\} \text{ for all } \beta \in \mathbb{C}\}$$

coincide [121].

JB-algebras are defined as those complete normed Jordan real algebras A satisfying $\|x\|^2 \leq \|x^2 + y^2\|$ for all x, y in A. The basic reference for the theory of JB-algebras is the book of H. Hanche-Olsen and E. Stormer [113]. An element u of a unital JB-algebra is said to be a symmetry if $u^2 = \mathbf{1}$. The space of all self-adjoint bounded linear operators on a complex Hilbert space, endowed with the product $x \circ y := \frac{1}{2}(xy + yx)$, becomes an illustrative example of a JB-algebra. As in this particular case, every unital JB-algebra A is endowed with an order with the property that B_A is affinely isomorphic to its positive part (say B_A^+) via the mapping $x \to \frac{1}{2}(x + \mathbf{1})$. It is proved in the Wright–Youngson early paper [114] that the extreme points of B_A^+ are precisely the idempotents of A. Therefore, as codified explicitly in [144], we have the following.

Theorem (4.2.5)[90]: Let A be a unital JB-algebra. Then the extreme points of the closed unit ball of A are precisely the symmetries of A.

By a real JB^*-algebra we mean a closed $*$-invariant real subalgebra of a (complex) JB^*-algebra. In the case of the existence of a unit, real JB^*-algebras were introduced (under the name of J^*B-algebras) by K. Alvermann [145], who provided a system of intrinsic axioms for them. By a non-unital version of [111] (see [113]), JB-algebras are precisely those real JB^*-algebras whose involution is the identity mapping. Real C^*-algebras (i.e., closed $*$-invariant real subalgebras of C^*-algebras), equipped with the Jordan product defined by the equality (1), become also relevant examples of real JB^*-algebras. According to Proposition (4.2.6) below, our notion of a real JB^*-algebra coincides with the apparently stronger one in [111, p. 321], where this concept was introduced by the first time.

Following [111], by a real JB^*-triple we mean a closed real subtriple of a (complex) JB^*-triple. Clearly, every real JB^*-algebra becomes a real JB^*-triple under the triple product defined by the equality (2). If A is a real JB^*-triple, then its complexification has a natural triple product, namely the one obtained by extending that of A by complex linearity in the outer variables, and by conjugate-linearity in the middle variable. Analogously, if A is a real JB^*-algebra, we can extend both the product of A (by complex bilinearity) and the involution of A (by conjugate-linearity) to the complexification of A. Actually, we have the following.

Proposition (4.2.6)[90]: Let $(X, \|\cdot\|)$ be a JB^*-triple (respectively, a JB^*-algebra), and let A be a closed real subtriple (respectively, a closed $*$-invariant real subalgebra) of X. Then the complexification $A \oplus iA$ of A, endowed with its natural triple product (respectively, with its natural product and involution) and the norm $|||a \oplus ib||| \coloneqq max\{\|a + ib\|, \|a - ib\|\}$, becomes a JB^*-triple (respectively, a JB^*-algebra).

The bracket-free version of the above proposition is shown in [111]. The bracket version is proved in an analogous way. Indeed, consider a set copy $\hat{X}$ of X with sum, product by scalars, product, involution, and norm defined by $\hat{x} + \hat{y} \coloneqq \widehat{x + y}, \lambda\hat{x} \coloneqq \widehat{\bar{\lambda}x}, \hat{x} \circ \hat{y} \coloneqq \widehat{x \circ y}, \hat{x}^* \coloneqq \widehat{x^*}$, and $\|\hat{x}\| \coloneqq \|x\|$, respectively. Then $\hat{X}$ becomes a JB^*-algebra, and hence $X \times \hat{X}$ is also a JB^*-algebra under the norm $\|(x, \hat{y})\| \coloneqq max\{\|x\|, \|y\|\}$. Now, the mapping $a \oplus ib \to (a, \hat{a}) + i(b, \hat{b}) = (a + ib, \widehat{a - ib})$ becomes an algebra $*$-isomorphism from the complexification of A onto a JB^*-subalgebra of $\times \hat{X}$.

Let X and A be as in Proposition (4.2.6). It is proved in [111] that extreme points of B_A remain extreme in $B_{(A \oplus iA, |||\cdot|||)}$. Therefore, invoking Remark (4.2.4)(a), we deduce the following.

Corollary (4.2.7)[90]: Let A be a real JB^*-algebra. Then the closed unit ball of A has extreme points if and only if A has a unit.

It follows from Corollary (4.2.7) and Remark (4.2.2)(h) that the closed unit ball of a real JB^*-algebra has extreme points if and only if it has strongly extreme points. On the other hand, keeping in mind Proposition (4.2.6) and Remark (4.2.4)(c), an easy argument (like the one in the proof of [111,]) leads to the following.

Corollary (4.2.8)[90]: Let A be a real JB^*-algebra. Then the bidual of A, endowed with a suitable product and a suitable involution, is a unital real JB^*-algebra containing A as a real JB^*-subalgebra.

According to [111], surjective linear isometries between real JB^*-triples are precisely those linear bijections preserving the cube mapping $x \to \{xxx\}$. As a consequence, surjective triple-isomorphisms between real JB^*-triples are isometries. In the case of real JB^*-algebras, we have the following converse.

Corollary (4.2.9)[90]: Surjective linear isometries between real JB^*-algebras are in fact triple-isomorphisms.

The above corollary is proved by F.J. Fernández-Polo, J. Martínez, and A.M. Peralta, under the additional assumption that the real JB^*-algebras under consideration have units (see [112]). To remove this additional assumption, it is enough to pass to biduals, to keep in mind Corollary (4.2.8), and to apply the original result in [112] to the bitranspose mapping of the given isometry. In fact, the proof itself in [112] works verbatim in our more general situation as soon as Corollary (4.2.8) is kept in mind.

As in the complex case, unitary elements of a real JB^*-triple X are defined as those elements u of X satisfying $\{u\{uxu\}u\} = x$ for every $x \in X$. Unitary elements of a real JB^*-triple X are strongly extreme points of B_X because, as in the complex case, if u is a unitary element of X, then X becomes a real JB^*-algebra with unit u (for a suitable product and a suitable involution), and Remark (4.2.2)(h) applies. Keeping in mind Proposition (4.2.6), Corollary (4.2.7), and Remark (4.2.4)(b), we easily realize that, if a real JB^*-algebra A has a unitary element in the real JB^*-triple sense, then A has a unit, and unitary elements of A in the real JB^*-triple sense coincide with $*$-unitary elements, as well as with algebraically unitary elements. In particular, if a JB-algebra A has a unitary element in the real JB^*-triple sense, then A has a unit, and unitary elements in the real JB^*-triple sense coincide with symmetries in the usual JB-algebra meaning.

A consequence of Corollary (4.2.9) is that, in the setting of real JB^*-algebras, the notion of a unitary element is actually a Banach space notion. In the particular case of JB-algebras, an explicit Banach space characterization of unitaries is provided by Theorem (4.2.5). For a general real JB^*-algebra, we do not know any explicit Banach space characterization of its unitary elements. Could they coincide with the strongly extreme points of its closed unit ball? Anyway, in the general setting of real JB^*-triples, Banach space characterizations of unitaries cannot be found. Indeed, we have the following.

Example (4.2.10)[90]: Let X stand for the two-dimensional real Euclidean space. Then we can identify X with $\mathbb{C}$, and consider it as a real JB^*-triple under the triple product $\{\lambda\mu\rho\} := \lambda\bar{\mu}\rho$, so that all elements in S_X become unitary elements. However, as it happens with any real Hilbert space, we can also consider X as a real JB^*-triple under the triple product $\{xyz\} := \frac{(x|y)z+(z|y)x}{2}$, so that no element in S_X becomes unitary.

According to [111], linear isometries from JB-algebras onto real JB^*-triples are in fact triple-isomorphisms. It follows from Example (4.2.10) that the result just reviewed does not remain true if we replace JB-algebras with real JB^*- algebras, nor even if we replace JB-algebras with commutative real C^*-algebras.

Geometrically unitary elements of real JB^*-triples are considered in F.J. Fernández-Polo, J. Martínez, and A.M. Peralta [121], where the following result is proved.

Proposition (4.2.11)[90]: (See [121, Proposition 2.8].) Let X be a real JB^*-triple, and let u be a norm-one element in X. Then the following conditions are equivalent:

(i) u is a geometrically unitary element of X.

(ii) u is a vertex of the closed unit ball of X.

(iii) The Banach space of X, endowed with the product $x \circ y := \{xuy\}$, becomes a JB-algebra with unit u.

It follows from the above proposition that the existence in a real JB^*-triple X of a geometrically unitary element is equivalent to the fact that X is triple-isomorphic to a unital JB-algebra. Therefore, the study of geometrically unitary elements in real JB^*-triples is concluded with the following.

Theorem (4.2.12)[90]: Let A be a unital JB-algebra, and let u be a norm-one element of A. Then the following conditions are equivalent:

(i) u is a central symmetry of A.

(ii) $n(A, u) = 1$.

(iii) u is a geometrically unitary element of A.

(iv) u is a vertex of the closed unit ball of A.

The proof of the implication $(i) \Rightarrow (ii)$ is easy. Indeed, since Condition (ii) is fulfilled in the case that u equals the unit **1** of A [113], it also remains fulfilled in the case that u is any central symmetry because, in that case, the mapping $x \to u \circ x$ is a surjective linear isometry on A taking **1** to u. On the other hand, the implication $(ii) \Rightarrow (iii)$ follows from Remark (4.2.2)(i), and the one $(iii) \Rightarrow (iv)$ is clear. Therefore, the core of the theorem is the implication $(iv) \Rightarrow (i)$, which is in fact the main result in the Leung–Ng–Wong [115].

The proof of the implication $(iv) \Rightarrow (i)$ in Theorem (4.2.12), given in [115], combines order theoretical arguments with the previously known fact that central symmetries of a unital JB-algebra A are precisely the isolated points of the set of all extreme points of B_A ([144] together with Theorem (4.2.5)). As pointed out in the introduction of [115], an alternative (and quicker) proof of the implication $(iv) \Rightarrow (i)$ in Theorem (4.2.12) can be provided by invoking Proposition (4.2.11) and a result taken from [144]. Indeed, if u is a vertex of A, then, by Proposition (4.2.11), the Banach space of A, endowed with a suitable product, becomes a JB-algebra (say A_u) with unit u. Now the mapping $a \to a$ is a surjective linear isometry from A_u to A, and hence, as a consequence of [144], there exists a central symmetry b in A, together with a surjective algebra isomorphism $\Phi : A_u \to A$, such that we have $a = b \cdot \Phi(a)$ for every $a \in A$. This implies, by taking $a = u$ and noticing that $\Phi(u) = \mathbf{1}$, that $u = b$ is indeed a central symmetry in A.

By Corollary (4.2.7) and Theorem (4.2.12), the closed unit ball of a JB-algebra A has extreme points if and only if it has vertices, if and only if A has geometrically unitary elements.

To conclude this section, let us mention that, as a consequence of [111] and [146], the Banach space characterization of tripotents of (complex) JB^*-triples, collected in Remark (4.2.4)(d), remains true verbatim in the caseof real JB^*-triples. In its turn, the natural real variant of the geometric characterization of tripotents of (complex) JB^*-triples, reviewed in Remark (4.2.4)(e), determines tripotents in real JB^*-triples [121].

Let σ be a topology on a set E, let τ be a vector space topology on a vector space Y over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), let f be a function from E to 2^Y (empty values for f are allowed), and let u be in E. We say that f is σ-τ upper semi-continuous (in short, σ-τ usc) at u if, for every τ-neighborhood V of zero in Y, there exists a σ-neighborhood U of u in E such that $f(x) \subseteq f(u) + V$ whenever x lies in U. Now, let X be a Banach space. The duality mapping of X is defined as the function $x \to D(X, x)$ from the unit sphere of X to 2^{X^*}. These notions are related to the material previously reviewed in this survey because of the following.

Fact (4.2.13)[90]: (See [106, Proposition 4.5].) Let A be a (possibly non-associative) real or complex Banach algebra with a norm-one unit $\mathbf{1}$. Then the duality mapping of A is norm-norm usc at $\mathbf{1}$.

Keeping in mind that, for a norm-one element u in a Banach space X, we have $0 \leq n(X, u) \leq 1$, and that such an element u is geometrically unitary if and only if $n(X, u) > 0$ (by Remark (4.2.2)(a)), the requirement $n(X, u) = 1$ can be read as that u is a geometrically unitary element of the "best possible type". Such a special goodness of u reflects into the following easy consequence of Fact (4.2.13).

Fact (4.2.14)[90]: (See [117, Corollary (4.2.21)].) Let X be a Banach space, and let u be a norm-one element of X such that $n(X, u) = 1$. Then the duality mapping of X is norm-norm usc at u.

Fact (4.2.14) does not remain true if the assumption that $n(X, u) = 1$ is relaxed to the one that u is a geometrically unitary element of X. Indeed, for every real number ρ with $0 < \rho < 1$, we can find a couple (X, u), where X is a Banach space, and u is a norm-one element of X satisfying $n(X, u) = \rho$ and such that the duality mapping of X is not norm-weak usc at u [117]. A much more recent result in the same direction is the one asserting that, given any non-reflexive Banach space X, and any non-zero element $u \in X$, there exists an equivalent norm on X such that, in the new norm, u is a geometrically unitary element but not a point of norm-weak upper semi-continuity for the duality mapping [119].

Another result, whose proof involves Fact (4.2.13), is the following.

Fact (4.2.15)[90]: (See [99, Corollary 1.3].) Let A be a C^*-algebra. Then the product of A becomes a point of norm-norm upper semi-continuity for the duality mapping of the Banach space of all continuous bilinear functions from $A \times A$ to A.

The above fact remains true if we replace C^*-algebra with unitary real or complex Banach algebra (in the sense of Remark (4.2.2)(x)) [97].

Let X be a Banach space, and let Y be a closed subspace of X. We say that Y is a semi-L-summand of X if, for each $x \in X$, there exists a unique $y \in Y$ such that $\|x - y\| = \|x + Y\|$, and moreover this y satisfies $\|x\| = \|y\| + \|x - y\|$. This happens in particular if Y is an L-summand of X (which means that Y is the range of a linear projection P on X such that $\|x\| = \|P(x)\| + \|x - P(x)\|$ for every $x \in X$). The following result is folklore. Indeed, it follows for example from [147].

Fact (4.2.16)[90]: Let X be a Banach space over $\mathbb{K}$, and let u be a norm-one element of X such that $\mathbb{K}u$ is a semi-L-summand of X. Then $n(X, u) = 1$.

Now, let u be a non-zero element in a Banach space X over $\mathbb{K}$, and note that X can be equivalently renormed in such a way that, in the new norm, u has norm one, and $\mathbb{K}u$ becomes an L-summand of X. It follows from Remark (4.2.2)(a) and Facts (4.2.14) and (4.2.16) that, up to such a renorming, u becomes both a geometrically unitary element and a point of norm-norm upper semi-continuity for the duality mapping (compare Proposition 2.1 of [119], and the comments following it).

The different notions of upper semi-continuity of the duality mapping of a Banach space were introduced and studied by J.R. Giles, D.A. Gregory, and B. Sims [116] who, among other results, proved the following.

Fact (4.2.17)[90]: (See [116, Theorem (4.2.3)].) Let X be a Banach space, and let u be a norm-one element of X. Then the duality mapping of X is norm-weak usc at u if and only if $D(X, u)$ is weak*-dense in $D(X^{**}, u)$.

The notion of strong sub differentiability of the norm of a Banach space X at a norm-one element $u \in S_X$ (the meaning of which will not be specified here) was introduced by D.A. Gregory [149], and was rediscovered later in [117] under the name that (X, u) is a strong numerical range space. As proved in [148], the strong sub differentiability of X at u is equivalent to the norm-norm upper semi-continuity of the duality mapping of X at u. It is worth mentioning that, after Gregory, most have preferred the terminology of strong sub differentiability of the norm instead of that of norm-norm upper semi-continuity of the duality mapping. This happens where the points of norm-norm upper semi-continuity for the duality mapping of C^*-algebras, JB^*-triples, and real JB^*-triples are determined (see [149], [150], and [151]).

Let X be a Banach space having a (complete) pre-dual X_*. We define the pre-duality mapping of X as the function $x \to D(X,x) \cap X_*$ from the unit sphere of X to 2^{X_*}. According to Godefroy–Indumathi [118], we have the following.

Fact (4.2.18)[90]: Let X be a dual Banach space, and let u be a norm-one element of X. Then we have:

(i) The pre-duality mapping of X is norm-weak usc at u if and only if $D(X,u) \cap X_*$ is weak*-dense in $D(X,u)$.

(ii) If the duality mapping of X is norm-weak usc at u, then so is the pre-duality mapping of X.

As pointed out in [118], a consequence of Facts (4.2.17) and (4.2.18)(i) is that, given a norm-one element u of an arbitrary Banach space X, the duality mapping of X is norm-weak usc at u if and only if so is the pre-duality mapping of X^{**}. Now, let u be a norm-one element in a dual Banach space X over $\mathbb{K}$. An early forerunner of Fact (4.2.18)(2) is [117], which, with our present terminology, asserts that the pre-duality mapping of X is norm-weak usc at u as soon as the duality mapping of X is norm-norm usc at u. We note also that Remark (4.2.2)(iv), together with Facts (4.2.14) and (4.2.18)(ii), implies that, if $n(X,u) = 1$, then the pre-duality mapping of X, as well as the pre-duality mapping of any dual of X of even order, is norm-weak usc at u [152]. As a consequence, by Fact (4.2.16), the same conclusion holds whenever $\mathbb{K}u$ is a semi-L-summand of X [152].

Again, let X be a dual Banach space, and let u be a norm-one element of X. We say that u is a w^*-vertex of B_X if $D(X,u) \cap X_*$ separates the points of X, and that u is a w^*-unitary element of X if X_* equals the linear hull of $D(X,u) \cap X_*$. It is obvious that w^*-vertices are vertices, and it is not so obvious but true that w^*-unitaries are geometric unitaries [102]. On the other hand, if u is a vertex of B_X, and if the pre-duality mapping of X is norm-weak usc at u, then, by Fact (4.2.18)(1), u is a w^*-vertex of B_X. Now, we can complete the picture with the following result, due to G. Godefroy and T.S.S.R.K. Rao.

Theorem (4.2.19)[90]: (See [119, Proposition 2.2].) Let X be a dual Banach space, and let u be a geometrically unitary element of X such that the pre-duality mapping of X is norm-weak usc at u. Then u is w^*-unitary.

Since Theorem (4.2.19) is proved in [119] only for real spaces, and we are interested also in complex spaces, we reproduce here the proof, introducing the appropriate changes to cover both the real and complex cases. We begin by claiming that, if S is a bounded, closed, and convex subset of a Banach space Y such that its closed absolutely convex hull is a neighborhood of zero in Y, then Y is the linear hull of S. Indeed, putting us in the more complicate case that X is complex, and noticing that $|\mathrm{co}|(S) \subseteq \sqrt{2}\mathrm{co}(S \cup -S \cup iS \cup -iS)$ (where $\mathrm{co}(\cdot)$ and $|\mathrm{co}|(\cdot)$ means convex and absolutely convex hull, respectively), and that, by [153], $\mathrm{co}(S \cup -S \cup iS \cup -iS)$ is a CS-closed subset of Y in the sense of [153,], the claim follows from [153]. Now, let X and u be as in Theorem (4.2.19), and, for x in X, put

$$v(x) := sup\{|f(x)|: f \in D(X,u)\}.$$

Then, by Remark (4.2.2)(i), $v(\cdot)$ is an equivalent norm on X, and, by Fact (4.2.18)(i), we have

$$v(x) = sup\{f(x): f \in D(X,u) \cap X_*\}$$

for every x, which implies that the closed unit ball of $(X, v(\cdot))$ is the absolute polar of $D(X,u) \cap X_*$ in X. It follows from the bipolar theorem that the closed absolutely convex hull of $D(X,u) \cap X_*$ is the closed unit ball of X_* for some equivalent norm on X_*. Finally, by the claim, X_* equals the linear hull of $D(X,u) \cap X_*$, i.e., u is w^*-unitary, as required.

Keeping in mind Remark (4.2.2)(ii) and Facts (4.2.13) and (4.2.18)(ii), Theorem (4.2.19) implies the following result, which, as far as we know, has not been noticed previously.

Corollary (4.2.20)[90]: Let A be a (possibly non-associative) complex Banach algebra with a norm-one unit **1**, and assume that A is also a dual Banach space. Then **1** is w^*-unitary. Moreover, if A is in fact associative, then all algebraically unitary elements of A (in the sense of Remark (4.2.2)(c)) are also w^*-unitary elements of A.

Analogously, invoking Remark (4.2.14)(a) and Facts (4.2.14) and (4.2.18)(2), Theorem (4.2.19) implies the following.

Corollary (4.2.21)[90]: Let X be a dual Banach space, and let u be a norm-one element of X satisfying $n(X,u) = 1$. Then u is w^*-unitary.

We note that, in the case of real spaces, Corollary (4.2.21) can be easily derived from the notions and results in the theory of the so-called unit-order spaces [154]. Looking at the proof of Theorem (4.2.3), and applying Corollary (4.2.20), we realize that, if X is a JBW^*-triple (i.e., a JB^*-triple which is also a dual Banach space), then unitary elements of X, in the JB^*-triple sense, coincide with w^*-unitary elements of X, and also with w^*-vertices of B_X. In particular, for von Neumann algebras, unitaries (in the C^*-algebra sense), w^*-unitaries, and w^*-vertices are the same [108]. On the other hand, it follows from Theorem (4.2.12) and Corollary (4.2.21) that, if A is a JBW-algebra (i.e., a JB-algebra which is also a dual Banach space), then central symmetries of A coincide with w^*-unitary elements of A, as well as with w^*-vertices of B_A [117].

We recall that, if X is any dual Banach space, then the Banach space $P(X)$, of all continuous bilinear mappings from $X \times X$ to X, becomes naturally a dual Banach space. Now, keeping in mind Remark (4.2.2)(k) and Facts (4.2.15) and (4.2.18)(ii), Theorem (4.2.19) implies the following.

Corollary (4.2.22)[90]: Let A be a von Neumann algebra. Then the product of A is a w^*-unitary element of $\mathcal{P}(A)$.

To conclude our discussion, let us note that Theorem (4.2.19) does not remain true if the assumption that the pre-duality mapping of X is norm-weak usc at u is removed. Indeed, given any non-reflexive separable dual Banach space X, there exists $u \in X$ such that, up to a suitable equivalent dual renorming, u becomes geometrically unitary, but is not w^*-unitary, nor even a w^*-vertex (since, in fact, we have $D(X,u) \cap X_* = \emptyset$ in the renorming) [119, Theorem 2.4].

Section (4.3): Projective Tensor Product of Banach Spaces:

Nygaard and Werner showed that some of the classical Banach spaces without the Radon–Nikodm property actually fail much weaker requirements. Indeed they proved that for any infinite-dimensional uniform algebra, every non-empty relatively weakly open subset of its closed unit ball has diameter equal to two [156]. If a Banach space satisfies the above condition, we will say that it has the diameter two property [157]. As a consequence of the mentioned result, every infinite dimensional real or complex $C(K)$ satisfies the diameter two property. The result for $C(K)$ was extended to real JB^*-triples (in the sense of [158]) whose Banach space is not reflexive by Becerra, López, Peralta and Rodríguez [159]. Hence every infinite dimensional C^*-algebra satisfies the diameter two property (see also [160]).

Becerra and López proved that for every atomless measure μ and for every compact Hausdorff topological space K, the spaces $L_1(\mu, X)$ and $C(K, X)$ have the diameter two property for every nonzero Banach space X [161]. López obtained positive results for L-embedded and M-embedded Banach spaces under some additional assumptions [162].

Recently the results of [160,159,161] have been generalized and unified in [157] by proving that every Banach spaces whose centralizer is infinite-dimensional satisfies the diameter two property.

It is also known that every Banach space with the Daugavet property has the diameter two property [163]. However, there are spaces without the Daugavet property that enjoy some of its consequences. For instance, in [164] it was proved that the interpolation spaces $L_1 \cap L_\infty$ and $L_1 + L_\infty$

satisfy that every slice of the unit ball have diameter two. For the symmetric projective tensor of some classical Banach spaces some results along the same line can be found in [165,166,167].

If a Banach space satisfies that every slice of the unit ball has diameter two, then it is immediate that the same property is satisfied by its projective tensor product with any other nonzero Banach space. For the diameter two property, it is not clear the behavior.

The complete projective tensor product of two Banach spaces whose centralizers are infinite-dimensional, has the diameter two property. The previous result can be applied, for instance, to every infinite-dimensional C^*-algebra, or to any space $C(K,X)$, for every infinite compact Hausdorff space K and for any nonzero Banach space X. we obtain a result along the same line assuming that the centralizer of one of the Banach spaces is infinite-dimensional and the unit sphere of the dual of the other contains an element of numerical index one. In order to show this statement, we need results on the numerical index that are interesting by themselves. The class of the spaces satisfying the assumption on the numerical index contains the so-called CL-spaces. For instance, the spaces $L_1(\mu)$ and $C(K)$ are CL-spaces. For the case of $C(K)$ spaces we obtain a refinement. The projective tensor product of every infinite-dimensional $C(K)$ and any nonzero Banach space satisfies the diameter two property. Finally the section contains one result stating the diameter two property for the injective tensor product under certain assumptions on the Banach spaces. We do not know in general if the projective tensor product of a Banach space with the diameter two property and any other non-trivial Banach space also satisfies the diameter two property.

Throughout, X will be a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). As usual, S_X, B_X and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X.

We recall that a function module is (the third coordinate of) a triple $(K,(X_t)_{t\in K},X)$, where K is a non-empty compact Hausdorff topological space (called the base space), $(X_t)_{t\in K}$ a family of Banach spaces, and X a closed $C(K)$-submodule of the $C(K)$-module $\prod_{t\in K}^{\infty} X_t$ (the ℓ_∞-sum of the spaces X_t) such that the following conditions are satisfied:

(i) For every $x\in X$, the function $t\to\|x(t)\|$ from K to $\mathbb{R}$ is upper semi-continuous.

(ii) For every $t\in K$, we have $X_t=\{x(t): x\in X\}$.

(iii)The set $\{t\in K: X_t\neq 0\}$ is dense in K.

We follow the notation of [168], where the basic results on function modules can be found.

Lemma (4.3.1)[155]: (See [10, Lemma 2.1].) Let $(K,(X_t)_{t\in K},X)$ be a function module, and let x be an extreme point of B_X. Then, for every $t\in K$ we have $\|x(t)\|=1$.

Let X be a Banach space over K and $L(X)$ the space of all bounded and linear operators on X. By a multiplier on X we mean an element $T \in L(X)$ such that every extreme point of B_{X^*} becomes an eigenvector for T^*. Thus, given a multiplier T on X, and an extreme point p of B_{X^*}, there exists a unique number $a_T(p)$ satisfying $T^*(p) = a_T(p)p$. The centralizer of X (denoted by $Z(X)$) is defined as the set of those multipliers T on X such that there exists a multiplier S on X satisfying $a_S(p) = a_T(p)$ for every extreme point p of B_{X^*}. Thus, if $\mathbb{K} = \mathbb{R}$, then $Z(X)$ coincides with the set of all multipliers on X. In all cases, $Z(X)$ is a closed subalgebra of $L(X)$ isometrically isomorphic to $C(K_X)$, for some compact Hausdorff topological space K_X (see [168]). Moreover X can be seen as a function module whose base space is precisely K_X, and such that the elements of $Z(X)$ are precisely the operators of multiplication by the elements of $C(K_X)$ (see [168]).

If X and Y are Banach spaces over the same scalar field ($\mathbb{K}$), we will denote by $B(X \times Y)$ the space of bounded bilinear forms on $X \times Y$. We recall that the projective tensor product of X and Y, denoted by $X \widehat{\otimes}_\pi Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\| \coloneqq inf\left\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\,, n \in \mathbb{N}, x_i \in X, y_i \in Y, \forall 1 \le i \le n\right\}.$$

We recall that the space $B(X \times Y)$ is linearly isometric to the topological dual of $X \widehat{\otimes}_\pi Y$. Under this identification, for every $A \in B(X \times Y)$, we will denote by $\tilde{A}$ the corresponding linear functional on $X \widehat{\otimes}_\pi Y$. It is satisfied that

$$\tilde{A}(x \otimes y) = A(x, y) \quad \forall (x, y) \in X \times Y.$$

Lemma (4.3.2)[155]: Let X and Y be Banach spaces and assume that B_X contains some extreme point and $Z(X)$ is infinite-dimensional. Let W be an open set in $\left(B_{X \widehat{\otimes}_\pi Y}, w\right)$ and $z_0 \in W$ such that

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j\,,$$

where $\alpha_j \in \mathbb{R}, \alpha_j > 0, \sum_{j=1}^{k} \alpha_j = 1, x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \ldots, k$. Then there are elements $\varphi \in B_{X^*}$ and u_j, v_j in B_X for $1 \le j \le k$ such that

$$\sum_{j=1}^{k} \alpha_j u_j \otimes y_j\,, \sum_{j=1}^{k} \alpha_j v_j \otimes y_j \in W$$

and $\varphi(u_j) = 1 = -\varphi(v_j)$ for all $j = 1, \ldots, k$.

Proof. By a previous remark, we can assume that X is a function module with base space equal to some compact $K \coloneqq K_X$, and such that $Z(X)$ coincide with the set of operators of multiplication by elements of $C(K)$. Since $Z(X)$ is infinitedimensional K is infinite. Hence there is a sequence $\{O_n\}$ of non-empty pair-wise disjoint open subsets of K. For $n \in \mathbb{N}$, take $t_n \in O_n$, and apply Urysohn's Lemma to pick f_n in $C(K)$ with $0 \leq f_n \leq 1$, $f_n(t_n) = 1$ and $f_n(t) = 0$ whenever $t \in K/O_n$. Since the bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to 0, it converges weakly to 0 in $C(K)$, and hence $\{f_n x\}_{n\in\mathbb{N}}$ converges weakly to zero and $\{(1 - f_n)x\}_{n\in\mathbb{N}}$ converges weakly to x in X for every element x in X. By the assumption there is an extreme point p of B_X. For each $j = 1, \ldots, k$, we define the sequences $\{u_n^j\}$ and $\{v_n^j\}$ in B_X by

$$u_n^j = (1 - f_n)x_j + f_n p,$$

and

$$v_n^j = (1 - f_n)x_j - f_n p.$$

Let us notice that for every $s \in K$, one has

$$\big(1 - f_n(s)\big)\|x_j(s)\| + f_n(s)\|p(s)\| \leq 1,$$

hence $\|u_n^j\| \leq 1$ and $\|v_n^j\| \leq 1$.

Since the sequences $\{u_n^j\}_n$ and $\{v_n^j\}_n$ converge weakly to x_j for all $j = 1, \ldots, k$, we deduce that $\{u_n^j \otimes y_j\}$ and $\{v_n^j \otimes y_j\}$ converge weakly to $x_j \otimes y_j$ in $B_{X\widehat{\otimes}_\pi Y}$, for all $j = 1, \ldots, k$. Then there exists n such that

$$\sum_{j=1}^{k} \alpha_j u_n^j \otimes y_j \,, \sum_{j=1}^{k} \alpha_j v_n^j \otimes y_j \in W,$$

we define $u_j := u_n^j$ and $v_j := v_n^j$ for all $j = 1, \ldots, k$. We know that there are elements $t_n \in K$ satisfying $f_n(t_n) = 1$. Since p is an extreme point of B_X, by Lemma (4.3.1), $\|p(t)\| = 1$ for every $t \in K$. Let $\varphi_{t_n} \in B\big(X(t_n)\big)^*$, such that $\varphi_{t_n}\big(p(t_n)\big) = 1$. So the functional defined by $\varphi(x) := \varphi_{t_n}\big(x(t_n)\big)$ $(x \in X)$ belongs to B_{X^*}. We have that $\varphi(u_j) = 1 = -\varphi(v_j)$ for every $j = 1, \ldots, k$.

Every (bounded) bilinear form $A : X \times Y \to \mathbb{K}$ can be identified with an operator $T : X \to Y^*$ by the formula $T(x)(y) = A(x, y)$ for $(x, y) \in X \times Y$. We will denote by $\hat{A}^{(2}$ the bilinear form on $X^{**} \times Y$ associated to the w^*-continuous operator $S := (J_Y)^* \circ T^{**}$, where $J_Y : Y \to Y^{**}$ is the canonical injection of Y in its bidual. Since T^{**} is an extension of T, then $\hat{A}^{(2}$ is an extension of A. Indeed $\hat{A}^{(2}$ is the restriction to $X^{**} \times Y$ of the Arens extension of A. Since S is w^*-continuous, then $\hat{A}^{(2}$ satisfies

$$\hat{A}^{(2}(x^{**}, y) = \lim_{\alpha} A(x_\alpha, y) \quad \forall (x^{**}, y) \in X^{**} \times Y, \tag{3}$$

for every net (x_α) in X that converges to x^{**} in the w$*$-topology of X^{**}.

Lemma (4.3.3)[155]: Let X and Y be Banach spaces and assume that $Z(X)$ is infinite-dimensional. Let W be an open set in $\left(B_{X\widehat{\otimes}_\pi Y}, w\right)$ and $z_0 \in W$ that can be written as

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j ,$$

where $\alpha_j \in \mathbb{R}, \alpha_j > 0, \sum_{j=1}^{k} \alpha_j = 1, x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \dots, k$. Then for every $\varepsilon > 0$, there exists u_j, v_j in B_X and $\varphi \in B_{X^*}$ satisfying

$$\sum_{j=1}^{k} \alpha_j u_j \otimes y_j , \sum_{j=1}^{k} \alpha_j v_j \otimes y_j \in W$$

and $\left|\varphi\left(u_j\right) - 1\right| < \varepsilon$ and $\left|\varphi\left(v_j\right) + 1\right| < \varepsilon$ for every $j = 1, \dots, k$.

Proof. Since $z_0 \in W$, we can assume that there are $\eta > 0$ and $A_1, \dots, A_m$ in $B(X \times Y)$ such that

$$W \coloneqq \left\{z \in B_{X\widehat{\otimes}_\pi Y} : \left|\tilde{A}_i(z) - \tilde{A}_i(z_0)\right| < \eta, \ \forall 1 \le i \le m\right\}.$$

We write, for each $i \in \{1, \dots, m\}, B_i \coloneqq \hat{A}_i^{(2}$, the extension of A_i to $X^{**} \times Y$ described above. Now, we consider the weakly open set of $B_{X^{**}\widehat{\otimes}_\pi Y}$ given by

$$\widehat{W} \coloneqq \left\{\hat{z} \in B_{X^{**}\widehat{\otimes}_\pi Y} : \left|\tilde{B}_i(z) - \tilde{B}_i(z_0)\right| < \eta, \forall 1 \le i \le m\right\}.$$

It is clear that $W \subset \widehat{W}$, so $z_0 \in \widehat{W}$. By assumption, $Z(X)$ is infinite-dimensional, so $Z(X^{**})$ is also infinite-dimensional in view of [169, Corollary I.3.15]. Since $B_{X^{**}}$ has extreme points, we can apply Lemma (4.3.2) to the element $z_0 \in \widehat{W}$. We have that $\widehat{W}$ contains elements $\hat{z}_1, \hat{z}_2$ that can be expressed as

$$\hat{z}_1 = \sum_{j=1}^{k} \alpha_j u_j^{**} \otimes y_j ,$$

and

$$\hat{z}_2 = \sum_{j=1}^{k} \alpha_j v_j^{**} \otimes y_j ,$$

where $u_j^{**}, v_j^{**} \in S_{X^{**}}$, and there exist $\varphi \in B_{X^{***}}$ such that $\varphi\left(u_j^{**}\right) = 1 = -\varphi\left(v_j^{**}\right)$ for all $j = 1, \dots, k$. Given $\varepsilon > 0$, since B_{X^*} is w^*-dense in $B_{X^{***}}$, we can assume that $\varphi \in B_{X^*}$ and $\left|\varphi\left(u_j^{**}\right) - 1\right| < \varepsilon$ and $\left|\varphi\left(v_j^{**}\right) + 1\right| < \varepsilon$ for all $j = 1, \dots, k$.

Since S_X is w^*-dense in $S_{X^{**}}$ and each $\hat{A}_i^{(2}$ is w^*-continuous on the first variable for every $1 \le i \le m$, there are $u_j, v_j \in S_X$ such that $\left|\varphi\left(u_j\right) - 1\right| < \varepsilon$ and $\left|\varphi\left(v_j\right) + 1\right| < \varepsilon$ for all $j \in \{1, \dots, k\}$ and the elements

$$z_1 := \sum_{j=1}^{k} \alpha_j u_j \otimes y_j , \qquad z_2 = \sum_{j=1}^{k} \alpha_j v_j \otimes y_j$$

satisfy that $z_1, z_2 \in W$.

Proposition (4.3.4)[155]: (See [166, Proposition 3.1].) Let X be a Banach space. Then $B_{(X^*)^{(\infty}}$ is w^*-dense in $B_{(X^{(\infty)})^*}$.

For every Banach spaces X and Y, we will show that there is a natural embedding $A \to \tilde{A}$ from $B(X \times Y)$ to $B(X^{(\infty} \times Y)$. Let us recall that we denoted by $\hat{A}^{(2}$ the extension of A to $X^{**} \times Y$ that we described before. We know that this canonical extension satisfies $\|\hat{A}^{(2}\| = \|A\|$. We denote by $\hat{A}^{(2n}$ the extension of $\hat{A}^{(2n-2}$ to $X^{(2n} \times Y$ defined in (3). We have that $\|\hat{A}^{(2}\| = \|A\|$ for all $n \in \mathbb{N}$.

In this way we have the following chain of embeddings

$$B(X \times Y) \hookrightarrow B(X^{**} \times Y) \hookrightarrow B(X^{(4} \times Y) \hookrightarrow \cdots \hookrightarrow B(X^{(2n} \times Y) \hookrightarrow \cdots ,$$

where each arrow means the corresponding extension.

Hence we can complete the above chain as follows

$$B(X \times Y) \hookrightarrow B(X^{**} \times Y) \hookrightarrow \cdots \hookrightarrow B(X^{(2n} \times Y) \hookrightarrow \cdots \hookrightarrow B(X^{(\infty} \times Y),$$

and the embedding $A \to \tilde{A}$ from $B(X \times Y)$ to $B(X^{(\infty} \times Y)$ is an isometry.

Lemma (4.3.5)[155]; Let X and Y be Banach spaces and assume that $Z(X^{(\infty})$ is infinite-dimensional. Let W be an open set in $(B_{X \widehat{\otimes}_\pi Y}, w)$ and $z_0 \in W$ such that

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j ,$$

where $\alpha_j \in \mathbb{R}, \alpha_j > 0, \sum_{j=1}^{k} \alpha_j = 1, x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \dots, k$. Then for every $\varepsilon > 0$, there exists u_j, v_j in B_X and $\varphi \in B_{X^*}$ such that

$$\sum_{j=1}^{k} \alpha_j u_j \otimes y_j , \qquad \sum_{j=1}^{k} \alpha_j v_j \otimes y_j \in W$$

and $|\varphi(u_j) - 1| < \varepsilon$ and $|\varphi(v_j) + 1| < \varepsilon$ for every $1 \le j \le k$.

Proof. Since $z_0 \in W$, we can assume that there is $\eta > 0$ and $A_1, \dots, A_m$ in $B(X \times Y)$ such that

$$W := \{z \in B_{X \widehat{\otimes}_\pi Y} : |\hat{A}_i(z) - \hat{A}_i(z_0)| < \eta, \forall 1 \le i \le m\}.$$

Let us consider, for each $i \in \{1, \dots, m\}$, the extension $\hat{A}_i$ of A_i to $X^{(\infty} \times Y$. We denote by L_i the linear functional on $X^{(\infty} \widehat{\otimes}_\pi Y$ associated to the bilinear form $\tilde{A}_i$ for $1 \le i \le m$. Now we define the weakly open set in the unit ball of $X^{(\infty} \widehat{\otimes}_\pi Y$ by L_i

$$\widehat{W} := \left\{\hat{z} \in B_{X^{(\infty}\widehat{\otimes}_\pi Y} : |L_i(\hat{z}) - L_i(z_0)| < \eta, \forall 1 \leq i \leq m\right\}.$$

We know that $z_0 \in \widehat{W}$. By assumption, $Z\left(X^{(\infty}\right)$ is infinite-dimensional, so we can apply Lemma (4.3.3) to $z_0 \in \widehat{W}$. We obtain that $\widehat{W}$ contain elements $\hat{z}_1, \hat{z}_2$ that can be expressed as

$$\hat{z}_1 = \sum_{j=1}^{k} \alpha_j u_j^{(\infty} \otimes y_j ,$$

and

$$\hat{z}_2 = \sum_{j=1}^{k} \alpha_j v_j^{(\infty} \otimes y_j ,$$

where $u_j^{(\infty}, v_j^{(\infty} \in S_{X^{(\infty}}$, and there exists $\varphi \in B_{\left(X^{(\infty)}\right)^*}$ such that $\left|\varphi\left(u_j^{(\infty}\right) - 1\right| < \varepsilon$ and $\left|\varphi\left(v_j^{(\infty}\right) + 1\right| < \varepsilon$ for all $j = 1, \dots, k$. By Proposition (4.3.4) we know that $B_{(X^*)^{(\infty}}$ is w^*-dense in $B_{(X^*)^{(\infty}}$, so by using the definition of $(X^*)^{(\infty}$ we can assume that there exists $p \in \mathbb{N}$ such that $\varphi \in B_{(X^*)^{(2p}}$. Now, by the definition of $X^{(\infty}$, we can assume that there exists $q \in \mathbb{N}$ such that $u_j^{(\infty}, v_j^{(\infty} \in B_{X^{(2q}}$, for all $j = 1, \dots, k$. This implies that there exists $n \in \mathbb{N}$ such that $u_j^{(\infty}, v_j^{(\infty} \in B_{X^{(2n}}$, for all $j = 1, \dots, k$, and $\varphi \in B_{(X^*)^{(2n}}$. If we proceed as in the last part of the proof of Lemma (4.3.3), after a finite number of steps, we conclude the proof.

Theorem (4.3.6)[155]: Let X and Y be Banach spaces such that $Z\left(X^{(\infty}\right)$ and $Z\left(Y^{(\infty}\right)$ are infinite-dimensional. Then the space $X \widehat{\otimes}_\pi Y$ has the diameter two property.

Proof. Let W be a non-empty open set in $(B_{X\widehat{\otimes}_\pi Y}, w)$. We can clearly assume that there is $\eta > 0, A_1, \dots, A_m$ in $B(X \times Y)$, and $z_0 \in B_{X\widehat{\otimes}_\pi Y}$ such that

$$W := \left\{z \in B_{X\widehat{\otimes}_\pi Y} : \left|\hat{A}_i(z) - \hat{A}_i(z_0)\right| < \eta, \forall 1 \leq i \leq m\right\}.$$

Since every weakly open set is norm open set, we suppose that

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j ,$$

where $\alpha_j \in \mathbb{R}, \alpha_j > 0, \sum_{j=1}^{k} \alpha_j = 1, x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \dots, k$. Given $\varepsilon > 0$, by applying Lemma (4.3.5) to the Banach space Y, there are elements w_j in B_Y for $1 \leq j \leq k$ and $\psi \in B_{Y^*}$ such that

$$\sum_{j=1}^{k} \alpha_j x_j \otimes w_j \in W$$

and $|\psi(w_j)-1|<\varepsilon$ for all $j=1,\ldots,k$. If we apply Lemma (4.3.5) to X and $\sum_{j=1}^{k}\alpha_j x_j\otimes w_j$, there are u_j,v_j in B_X $(1\le j\le k)$ and $\varphi\in B_{X^*}$ such that

$$z_1=\sum_{j=1}^{k}\alpha_j u_j\otimes w_j,\quad z_2=\sum_{j=1}^{k}\alpha_j v_j\otimes w_j\in W$$

and $|\varphi(u_j)-1|<\varepsilon$ and $|\varphi(v_j)-1|<\varepsilon$ for all $j=1,\ldots,k$. We consider the bilinear map $\phi: X\times Y\to\mathbb{K}$ given by $\phi(x,y):=\varphi(x)\psi(y)$, which is bounded and satisfies $\|\phi\|\le 1$. Then

$$\|z_1-z_2\|\le|\bar{\phi}(z_1-z_2)|=\left|\sum_{j=1}^{k}\alpha_j\varphi(u_j)-\varphi(v_j)\psi(w_j)\right|$$

$$\ge\sum_{j=1}^{k}\alpha_j(2-2\varepsilon)(1-\varepsilon)=(2-2\varepsilon)(1-\varepsilon).$$

We conclude that

$$(2-2\varepsilon)(1-\varepsilon)\le\|z_1-z_2\|\le\operatorname{diam}W\le 2,$$

for every $\varepsilon>0$. Hence diamW = 2 as we wanted to show.

Now we will provide examples of spaces where the previous results can be applied. Given a Banach space X, there exists a natural embedding $T\to\tilde{T}$ from $\mathcal{L}(X)$ to $\mathcal{L}(X^{(\infty})$. Indeed, let T be in $\mathcal{L}(X)$. Given $\alpha\in\bigcup_{n=0}^{\infty}X^{(2n}$, there exists $m\in\mathbb{N}$ such that α belongs to $X^{(2m}$, allowing us to consider the element $T^{(2m}(\alpha)$ of $\bigcup_{n=0}^{\infty}X^{(2n}$, which does not depend on m. In this way we are provided with a natural extension of T to $\bigcup_{n=0}^{\infty}X^{(2n}$, which extends uniquely by continuity to $X^{(\infty}$, giving rise to an element $\tilde{T}$ of $\mathcal{L}(X^{(\infty})$. It is known that for every T in $Z(X), T^{**}$ lies in $Z(X^{**})$ [169]. Hence we already are aware of the chain of embeddings

$$Z(X)\hookrightarrow Z(X^{**})\hookrightarrow Z(X^{****})\hookrightarrow\cdots\hookrightarrow Z(X^{(2n})\hookrightarrow\cdots,$$

where each arrow means double transposition.

Indeed, it is known that the image of $Z(X)$ under this embedding is contained in $Z(X^{(\infty})$ (see [157]). Hence we can complete the above chain as follows

$$Z(X)\hookrightarrow Z(X^{**})\hookrightarrow Z(X^{****})\hookrightarrow\cdots\hookrightarrow Z(X^{(2n})\hookrightarrow\cdots\hookrightarrow Z(X^{(\infty}).$$

For a Banach space X, an L-projection on X is a (linear) projection $P: X\to X$ satisfying $\|x\|=\|P(x)\|+\|x-P(x)\|$ for every $x\in X$. In such a case, we will say that the subspace $P(X)$ is an L-summand of X. Let us notice that the composition of two L-projections on X is an L-projection [168], so the closed linear subspace of $\mathcal{L}(X)$ generated by all L-projections on X is a subalgebra of $\mathcal{L}(X)$, the space of all bounded and linear operators on X. This algebra, denoted by $C(X)$, is called the Cunningham algebra of X. It is known that $C(X)$ is linearly isometric to $Z(X^*)$ (see [168]).

The following Banach spaces X satisfy that $sup\{dim\, Z(X^{(2n)}) : n \in \mathbb{N}\} = \infty$, so $Z(X^{(\infty)})$ is infinite-dimensional:

(i) Every non-reflexive Banach space X such that X^* is L-embedded [166]. For instance, an infinitedimensional predual of an L_1-space or a (real or complex) infinite-dimensional C^*-algebra belongs to this class.

(ii) The space $C(K,(X,\tau))$ where K is an infinite compact topological space, X is a non-null Banach space and τ is a topology such that the weak topology is contained in τ and the norm topology is finer than τ [157].

(iii) $L(X,Y)$ (the space of all bounded and linear operators from X to Y) for every Banach spaces X, Y such that either $C(X)$ is infinite-dimensional or $Z(Y)$ is infinite-dimensional (see [169,]). For instance, any infinite-dimensional space $L_1(\mu)$ satisfies that its Cunningham algebra is infinite-dimensional.

Under some isomorphic condition, Banach spaces have an equivalent norm satisfying the assumption of Theorem (4.3.6). The diameter two property for an equivalent norm under the next assumption was previously obtained in [162] by the same procedure.

Lemma (4.3.7)[155]: (See [170, Theorem 22.4].) In a Banach space, a CS-closed set and its closure have the same interior.

For a subset S of a Banach space, $\mathrm{co}(S)$, $|\mathrm{co}|(S)$, and $|\overline{\mathrm{co}}|(S)$ will denote the convex, absolutely convex, and closed absolutely convex hull of S, respectively.

Corollary (4.3.8)[155]: Let S be a CS-closed set in a Banach space X. Then $|\mathrm{co}|(S)$ and $|\overline{\mathrm{co}}|(S)$ have the same interior in X.

Proof. If X is real, then $|\mathrm{co}|(S) = \mathrm{co}(S \cup -S)$ is a CS-closed set (by [170, 22.2 and 22.3]), and the result follows from Lemma (4.3.7). Assume that X is complex. Let $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $B_{\mathbb{C}} \subseteq (1+\varepsilon)\,\mathrm{co}(\{z_1, \dots, z_n\})$, where $z_1, \dots, z_n$ are the n-th roots of 1 in $\mathbb{C}$. Then we have

$$|\mathrm{co}|(S) = \mathrm{co}(B_{\mathbb{C}} S) \subseteq (1+\varepsilon)\,\mathrm{co}\left(\bigcup_{i=1}^{n} z_i S\right) \subseteq (1+\varepsilon)|\mathrm{co}|(S).$$

By keeping in mind that $\mathrm{co}(\bigcup_{i=1}^{n} z_i S)$ is a CS-closed set, and applying Lemma (4.3.7), we deduce that $T \subseteq (1+\varepsilon)|\mathrm{co}|(S)$, where T stands for the interior of $|\mathrm{co}|(S)$. Therefore, since T is open, we have $T \subseteq \bigcup_{\varepsilon>0} \frac{1}{1+\varepsilon} T \subseteq |\mathrm{co}|(S)$. _

Let X be a Banach space, and let u be a norm-one element in X. We put

$$D(X,u) := \{f \in B_{X^*} : f(u) = 1\}.$$

Now, assume that X has a (complete) predual X^*, and put

$$D^{w^*}(X,u) := D(X,u) \cap X^*.$$

If $D^{w^*}(X,u) = \emptyset$, then we define $n^{w^*}(X,u) := 0$. Otherwise, we define $n^{w^*}(X,u)$ as the largest non-negative real number k satisfying

$$k\|x\| \leq v^{w^*}(x) := sup\{|f(x)|: f \in D^{w^*}(X,u)\}$$

for every $x \in X$. We say that u is an w^*-unitary element of X if the linear hull of $D^{w^*}(X,u)$ equals the whole space X^*.

Proposition (4.3.9)[155]: Let u be a norm-one element in a dual Banach space X. Then u is w^*-unitary in X if and only if $n^{w^*}(X,u) > 0$. Moreover, we have

$$n^{w^*}(X,u) int(B_{X_*}) \subseteq |co| D^{w^*}(X,u). \tag{4}$$

Proof. Assume that u is w^*-unitary. Then $|\overline{\text{co}}|\left(D^{w^*}(X,u)\right)$ is a barrel in X_*. Since barrels in a Banach space are neighborhoods of zero, there exists $k > 0$ such that $kB_{X_*} \subseteq |\overline{\text{co}}|\left(D^{w^*}(X,u)\right)$. This implies that $k\|x\| \leq v^{w^*}(x)$ for every $x \in X$, and hence that $n^{w^*}(X,u) > 0$.

Now we can clearly assume that $n^{w^*}(X,u) > 0$. Then, in the duality (X, X_*), the set

$$B := \left\{x \in X: v^{w^*}(x) \leq 1\right\}$$

is the absolute polar of $D^{w^*}(X,u)$, and the inclusion $B \subseteq \frac{1}{n^{w^*}(X,u)} B_X$ holds. It follows from the bipolar theorem that $n^{w^*}(X,u) B_{X_*} \subseteq |\overline{\text{co}}|\left(D^{w^*}(X,u)\right)$. By applying Corollary (4.3.8), the inclusion (4) follows. Clearly, that inclusion implies that u is w^*-unitary.

The first paragraph in the above proof is taken from the proof of [171, Corollary (4.3.8)].

Now, let X be an arbitrary Banach space, and let u be a norm-one element in X. We define $n(X,u)$ as the largest non-negative real number k satisfying

$$k\|x\| \leq v(x) := sup\{|f(x)|: f \in D(X,u)\}$$

for every $x \in X$, and we say that u is an unitary element of X if the linear hull of $D(X,u)$ equals the whole space X_*.

Let σ be a topology on a set E, let τ be a vector space topology on a vector space Y over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), let f be a function from E to $2Y$ (empty values for f are allowed), and let u be in E. We say that f is $\sigma - \tau$ upper semi-continuous (in short, $\sigma - \tau$ usc) at u if, for every τ -neighborhood V of zero in Y, there exists a σ-neighborhood U of u in E such that $f(x) \subseteq f(u) + V$ whenever x lies in U.

Now, let X be a dual Banach space. We define the pre-duality mapping of X as the function $x \to D^{w^*}(X,x)$ from the unit sphere of X to 2^{X_*}. Let u be a norm-one element in X. Since, clearly, $n^{w^*}(X,u) \leq n(X,u)$, it follows from Proposition (4.3.9) and [172, Theorem 3.1] that, if u is w^*-unitary, then u is unitary [171, Corollary (4.3.8)]. Under the requirement that the preduality mapping of X is norm-weak usc at u, the converse is also true. This result is due to G. Godefroy and T.S.S.R.K. Rao [173, Proposition 2.2] in the real case, and its proof is clarified and adapted to the complex case in [174]. Now we are ready to formulate and prove a quantification of the result just quoted.

Theorem (4.3.10)[155]: Let u be an unitary element in a dual Banach space X, and assume that the pre-duality mapping of X is norm-weak usc at u. Then u is w^*-unitary. More precisely, we have $n(X,u) = n^{w^*}(X,u) > 0$ and

$$n(X,u)\,\mathrm{int}\left(B_{X_*}\right) \subseteq |\mathrm{co}| D^{w^*}(X,u).$$

Proof. According to [175, Lemma (4.3.2)], the assumption that the pre-duality mapping of X is norm-weak usc at u is equivalent to the fact that $D^{w^*}(X,u)$ is w^* -dense in $D(X,u)$. Therefore we have $v(x) = v^{w^*}(x)$ for every $x \in X$, and, consequently, the equality $n(X,u) = n^{w^*}(X,u)$ holds. Now apply Proposition (4.3.9).

Now, let X be an arbitrary Banach space. The duality mapping of X is defined as the function $x \to D(X,x)$ from the unit sphere of X to 2^{X^*}. Let u be a norm-one element in X. If $n(X,u) = 1$, then u is both an unitary element of X (by [172, Theorem 3.1]) and a point of norm–norm upper semi-continuity of the duality mapping of X [176, Corollary 5.9]. On the other hand, in the case that X is in fact a dual Banach space, the mere norm-weak upper semi-continuity of the duality mapping of X at u implies the norm-weak upper semi-continuity of the pre-duality mapping of X at u [175, Theorem 2.3]. Therefore, by invoking Theorem (4.3.10), we get the following.

Corollary (4.3.11)[155]: Let X be a dual Banach space, and let u be a norm-one element in X such that $n(X,u) = 1$. Then we have

$$\mathrm{int}\left(B_{X_*}\right) \subseteq |\mathrm{co}|\left(D^{w^*}(X,u)\right).$$

As a consequence

$$B_{X_*} = |\overline{\mathrm{co}}|\left(D^{w^*}(X,u)\right).$$

We will deduce some consequences on the diameter two property.

We will provide some examples of spaces satisfying the assumption of the last statement. If μ is a σ-finite measure, then the space

(i) $Y = L_1(\mu)$, by taking the unit of $L_\infty(\mu)$ as f,

(ii) $Y = L_1(\mu) \widehat{\otimes}_\pi L_\infty(\mu)$, so Y^* can be identified with $\mathcal{L}(L_\infty(\mu), L_\infty(\mu))$, and f is the identity operator on $L_\infty(\mu)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$),

(iii) in the complex space, $Y = L_1(\mu) \widehat{\otimes}_\pi L_\infty(\mu) \widehat{\otimes}_\pi L_\infty(\mu)$, so Y^* can be identified with the space of continuous bilinear mappings on $L_\infty(\mu) \times L_\infty(\mu)$ with values in $L_\infty(\mu)$, and f is the usual product on $L_\infty(\mu)$,

satisfies that $n(Y^*, f) = 1$. In the first and second cases it can be checked directly. Indeed the second example is a consequence of [177]. The third one can be found in [1781]. Indeed, every CL-space (see [179] or [180] for the definitions) or more generally an almost CL-space satisfies that the unit sphere of its dual contains points where the numerical index is one. This class contains the spaces $L_1(\mu)$ and $C(K)$. It is also known that $C(K, X)$ is an almost CL-space if X is an almost CL-space [180].

Let us also observe that every Banach space has an equivalent norm for which the unit sphere of the dual has an element with numerical index one. If X is a Banach space, $x_0 \in S_X$ and M is a closed subspace of X such that $X = M \oplus \mathbb{K}x_0$, we consider the norm in X given by

$$|\|m + \lambda x_0\|| := max\{\|m\|, |\lambda|\}.$$

So $X^* = M^* \oplus_1 \mathbb{K}x_0^*$ for some functional $x_0^* \in S_{X^*} \cap M^0$. Then it is immediate that for the norm $|\| \, \||$ we have $n(X^*, x_0^*) = 1$.

On Theorems (4.3.6) and (4.3.10) we assumed conditions on both spaces in order to obtain the diameter two property for their projective tensor product. The next result shows that it is not needed any extra assumption in the case that one of the spaces is any infinite-dimensional $C(K)$.

Theorem (4.3.12)[155]: Let K be any infinite compact Hausdorff topological space and X a non-null Banach space. Then the space $Y := C(K) \widehat{\otimes}_\pi X$ satisfies the diameter two property.

Proof. Assume that W is a non-empty open set in (B_Y, w). Since K is infinite and $X \neq \{0\}$, then Y is infinite-dimensional and so $W \cap S_Y \neq \emptyset$. Since W is weakly open in B_Y, it is open in B_Y for the norm topology. Hence, for every $\varepsilon > 0$, there are $m \in \mathbb{N}, f_1, \ldots, f_m \in S_{C(K)}, x_1, \ldots, x_m \in S_X$ and positive real numbers t_i $(1 \leq i \leq m)$ with $\sum_{i=1}^m t_i = 1$ such that

$$y_0 := \sum_{i=1}^m t_i \, f_i \otimes x_i \in W$$

and $\|y_0\| > 1 - \varepsilon^2$. Hence there is $y_0^* \in S_{Y^*}$ satisfying that $y_0^*(y_0) = \operatorname{Re} y_0^*(y_0) > 1 - \varepsilon^2$. Since Y^* is linearly isometric to $\mathcal{L}(C(K), X^*)$, then there is $T_0 \in S_{\mathcal{L}(C(K), X^*)}$ such that T_0 is the operator associated to the functional y_0^*. We consider the sets

$$G := \{i \in \{1, \ldots, m\}: \operatorname{Re}T_0(f_i)(x_i) > 1 - \varepsilon\}, \qquad P := \{1, \ldots, m\} \backslash G.$$

We know that

$$1-\varepsilon^2<\operatorname{Re} y_0^*(y_0)=\sum_{i=1}^{m} t_i \operatorname{Re} T_0(f_i)(x_i)=\sum_{i\in G} t_i \operatorname{Re} T_0(f_i)(x_i)+\sum_{i\in P} t_i \operatorname{Re} T_0(f_i)(x_i)$$
$$\leq \sum_{i\in G} t_i+\sum_{i\in P} t_i(1-\varepsilon)=1-\varepsilon\sum_{i\in P} t_i.$$

Hence $\sum_{i\in P} t_i<\varepsilon$ and so

$$\sum_{i\in G} t_i>1-\varepsilon. \tag{5}$$

Now we restrict the operator $T_0^{**}: C(K)^{**}\to X^{***}$ to the linear space $B(K)$ of bounded measurable functions on K, that clearly contains $C(K)$. Let us remark that the restriction of the norm of $C(K)^{**}$ to $B(K)$ is just the supremum norm, that is,

$$\|f\|:=\sup_{t\in K}|f(t)| \qquad \forall\, f\in B(K).$$

Since the linear space of simple measurable functions on K is dense in $B(K)$, there are $k\in \mathbb{N}, A_1,\ldots,A_k\subset K$ measurable sets, non-empty and pairwise disjoint such that

$$\left\|\sum_{s=1}^{k}\beta_s^i\chi_{A_s}-f_i\right\|<\varepsilon \quad \forall 1\leq i\leq m \tag{6}$$

for convenient scalars $\{\beta_s^i: 1\leq i\leq m, 1\leq s\leq k\}$ satisfying $|\beta_s^i|\leq 1, \forall i,s$. Since the subsets $\{A_s: 1\leq s\leq k\}$ are non-empty and pairwise disjoint, the space M generated by $\{\chi_{A_s}: 1\leq s\leq k\}$ is a subspace of $B(K)$ is isometric to ℓ_∞^k. Indeed the unique linear mapping $\Psi:\ell_\infty^k\to M$ that satisfies

$$\Psi(e_s)=\chi_{A_s} \quad \text{for every } 1\leq s\leq k \tag{7}$$

is a linear isometry.

We write

$$s_i:=\sum_{s=1}^{k}\beta_s^i\chi_{A_s} \quad (1\leq i\leq m). \tag{8}$$

We know that $s_i\in M$ and by (6) it holds that $\|s_i-f_i\|<\varepsilon$ for each i. Since it is satisfied that Re $T(f_i)(x_i)>1-\varepsilon$ for $i\in G$ and $\|T\|=1$, then we deduce that

$$\operatorname{Re} T^{**}(s_i)(x_i)>1-2\varepsilon \qquad \forall i\in G. \tag{9}$$

Since K is infinite, there are sequences $\{g_n\}$ and $\{h_n\}$ in $C(K)$ satisfying the conditions of Lemma 2.1.i) in [165]. That is, there are sequences of non-empty open subsets $\{U_n\}$ and $\{V_n\}$ of K such that

$$V_n\subset U_n \quad \forall n\in\mathbb{N}, \quad U_n\cap U_m=\emptyset \quad \text{if } n\neq m,$$
$$\operatorname{supp} h_n\subset V_n, \qquad \operatorname{supp} g_n\subset U_n \quad \forall n, \tag{10}$$

and

$$g_n|_{V_n} \equiv 1, \qquad 0 \le g_n, h_n \le 1 \forall n, \quad \|h_n\| = 1 \ \ \forall n. \tag{11}$$

Since the functions $\{h_n : n \in \mathbb{N}\}$ have disjoint supports, then its linear span is isometric to c_0. Indeed there is a linear isometry from $\{h_n : n \in \mathbb{N}\}$ onto c_0 that maps $\{h_n\}$ into the usual Schauder basis of c_0. The same argument also holds for $\{g_n : n \in \mathbb{N}\}$. Then $\{g_n\}$ and $\{h_n\}$ are weakly null sequences in $C(K)$ and so for each $1 \le i \le m$, the sequence $\{u_{i,n}\}$ given by

$$u_{i,n} := f_i \prod_{j=nk}^{(n+1)k-1} \left(1 - g_j\right) + \sum_{j=1}^{k} \beta_j^i h_{nk+j-1}\,.$$

converges weakly to f_i in $C(K)$. We will also check that $\|u_{i,n}\| \le 1$ for each $1 \le i \le m$ and $n \in \mathbb{N}$. Let us fix n and i and choose $t \in K$. If $t \notin U_s$ for every $s \in \{j\mathbb{N} \in : nk \le j \le (n+1)k - 1\}$, then in view of (10) we have that $g_s(t) = h_s(t) = 0$ for every $nk \le s \le (n+1)k - 1$ and so $u_{i,n}(t) = f_i(t)$. Assume now that there is some $j_0 \in [1, k]$ such that $t \in U_{s_0}$, where $s_0 = nk + j_0 - 1$. If $t \in V_{s_0}$ then we have

$$u_{i,n}(t) = \beta_{j_0}^i h_{s_0}(t).$$

On the other hand, if $\notin V_{s_0}$, then we obtain

$$u_{i,n}(t) = f_i(t)\left(1 - g_{s_0}(t)\right),$$

and in any case $|u_{i,n}(t)| \le 1$. A similar argument proves that for every $1 \le i \le m$, the sequence $\{v_{i,n}\}$ given by

$$v_{i,n} := f_i \prod_{j=n}^{(n+1)k-1} k\left(1 - g_j\right) - \sum_{j=1}^{k} \beta_j^i h_{nk+j-1}$$

It is known that every space with the Daugavet property satisfies the diameter two property [163, Lemma 3]. In [181, Corollary 4.3] the authors provided an example of a two-dimensional complex normed space F such that $L_\infty^{\mathbb{C}}[0,1] \widehat{\otimes}_\pi F$ fails the Daugavet property. However our result can be applied to the previous space. Besides $C(K), L_1(\mu)$ also satisfies the diameter two property if the measure μ is atomless. If we consider $L_1(\mu)$ (for an atomless measure μ) instead of $C(K)$, Theorem (4.3.12) still holds. Indeed $L_1(\mu, X)$ does have the Daugavet property (see [182, Example, p. 858]) if μ is an atomless measure.

We considered results stating the diameter two property for projective tensor products of Banach spaces. In the case of the injective tensor product, it will be enough to assume one restriction only to one of the spaces in order to obtain a positive result.

Lemma (4.3.13)[155]; (See [157, Proposition 4.1].) Let X be a Banach space failing the diameter 2 property. Then $X^{(\infty}$ also fails this property.

Lemma (4.3.14)[155]: (See [157, Theorem 4.4].) Let X be a Banach space failing the diameter two property. Then there exists $m \in \mathbb{N}$ such that $dim\left(Z\left(X^{(2n)}\right)\right) \leq m$ for every $n \in \mathbb{N}$.

If X and Y are Banach spaces over the same scalar field ($\mathbb{K}$), we recall that the injective tensor product of X and Y, denoted by $X \widehat{\otimes}_\varepsilon Y$, is the completion of $X \otimes Y$ under the norm given by
$\|u\| \coloneqq$.

$$sup\left\{\sum_{i=1}^{n}|x^*(x_i)y^*(y_i)| : u = \sum_{i=1}^{n} x_i \otimes y_i\,, n \in \mathbb{N}, x_i \in X, y_i \in Y, \forall 1 \leq i \leq n, x^* \in S_{X^*}, y^* \in S_{Y^*}\right\}$$

Theorem (4.3.15)[155]: Let X be a Banach space over $\mathbb{K}$, such that $sup\{dim\ Z\left(X^{(2n}\right) : n \in \mathbb{N}\} = \infty$. Then the space $X \widehat{\otimes}_\varepsilon Y$ satisfies the diameter two property, for every non-null Banach space Y.

Proof. Since $X^{**} \widehat{\otimes}_\varepsilon Y$ can be seen as a subspace of $\left(X \widehat{\otimes}_\varepsilon Y\right)^{**}$ containing $X \widehat{\otimes}_\varepsilon Y$ [183, Lemma 1], we have that

$$\left(X \widehat{\otimes}_\varepsilon Y\right)^{**} \subseteq X^{**} \widehat{\otimes}_\varepsilon Y^{**} \subseteq \left(X \widehat{\otimes}_\varepsilon Y\right)^{(4}.$$

By applying again the mentioned result to $X^{**} \widehat{\otimes}_\varepsilon Y$, we obtain

$$X^{**} \widehat{\otimes}_\varepsilon Y \subseteq X^{(4} \widehat{\otimes}_\varepsilon Y \subseteq X^{**} \widehat{\otimes}_\varepsilon Y^{**}.$$

We conclude that

$$X \widehat{\otimes}_\varepsilon Y \subseteq X^{**} \widehat{\otimes}_\varepsilon Y \subseteq X^{(4} \widehat{\otimes}_\varepsilon Y \subseteq \left(X \widehat{\otimes}_\varepsilon Y\right)^{(4}.$$

By induction we prove that for every $n \in \mathbb{N}$

$$X \widehat{\otimes}_\varepsilon Y \subseteq X^{(2n} \widehat{\otimes}_\varepsilon Y \subseteq \left(X \widehat{\otimes}_\varepsilon Y\right)^{(2n}.$$

We fix $n \in \mathbb{N}$, then for $m \in \mathbb{N}$ we have that

$$\left(X \widehat{\otimes}_\varepsilon Y\right)^{(2M} \subseteq \left(X^{(2n} \widehat{\otimes}_\varepsilon Y\right)^{(2m} \subseteq \left(\left(X \widehat{\otimes}_\varepsilon Y\right)^{(2n}\right)^{(2m}.$$

This implies that

$$\left(X \widehat{\otimes}_\varepsilon Y\right)^{(\infty} \subseteq \left(X^{(2n} \widehat{\otimes}_\varepsilon Y\right)^{(\infty} \subseteq \left(\left(X \widehat{\otimes}_\varepsilon Y\right)^{(2n}\right)^{(\infty}.$$

For every Banach space Z and every $p \in \mathbb{N}$, we have $\left(Z^{(2p}\right)^{(\infty} = Z^{(\infty}$. It follows that

$$\left(X \widehat{\otimes}_\varepsilon Y\right)^{(\infty} = \left(X^{(2n} \widehat{\otimes}_\varepsilon Y\right)^{(\infty}.$$

By Lemma (4.3.15), if there is a relatively weakly open subset of $B_{X \widehat{\otimes}_\varepsilon Y}$ whose diameter is less than two, the same happens for the space $\left(X \widehat{\otimes}_\varepsilon Y\right)^{(\infty}$. By Lemma (4.3.16), there exists $m \in \mathbb{N}$ such that

$dim\left(Z\left(X^{(2n}\,\widehat{\otimes}_\varepsilon\, Y\right)\right) \leq m$. Hence, for all $n \in \mathbb{N}$. As a consequence, $dim\left(Z\left(X^{(2n}\,\widehat{\otimes}_\varepsilon\, Y\right)^{(\infty}\right) \leq m$. for $n \in \mathbb{N}$. Since $Z\left(X^{(2n}\,\widehat{\otimes}_\varepsilon\, Y\right)$ contains a copy of $Z\left(X^{(2n)}\right) \otimes Z(Y)$ for all $n \in N$ (see [184] and also [168, pp. 129 and 171]) and $Z(Y) \neq \{0\}$ we conclude that $dim\left(Z\left(X^{(2n)}\right)\right) \leq M$ for all $n \in \mathbb{N}$. This contradicts the assumption.