Chapter 3

Renormings and Quantification

We construct an exposed point in the unit ball of a Banach space X that remains exposed in the unit ball of $X^{(4)}$ but is not extreme in the unit ball of $X^{(6)}$.

We show some examples witnessing natural limits of our positive results, in particular, we construct a separable Banach space X with the Schur property that cannot be renormed to have a certain quantitative form of weak sequential completeness, thus providing a partial answer to a question of G. Godefroy.

Section (3.1): Extermal Structrues:

We attempt to understand several new extremal structures of Banach spaces that have been recently studied in [43], [44] and [45]. For a Banach space X let X_1 denote the closed unit ball. The authors of [43] have introduced a new class of extreme points by calling a unit vector $u \in X$ whose state space $S = \{x^* \in X_1^* : x^*(u) = 1\}$ spans X^*, a unitary. When u belongs to a dual space X^*, if $S' = \{x \in X_1 : u(x) = 1\}$ spans X then u is called a weak*-unitary. These notions are the abstract analogues of the corresponding notion of a unitary in a unital C^*-algebra. It follows from Theorem 9.5.16 of [46] (see also [47]) that a vector u in a C^* algebra is a unitary in this sense if and only if it is unitary in the (usual) algebraic sense. A unitary is in particular a strongly extreme point (see [43]).

It was shown in [43] that any weak*-unitary of X^* is a unitary. The converse holds in several natural situations: it follows from Theorem 3 in [47] that when X^* is a von Neumann algebra every unitary in X^* is a weak*-unitary. More generally, this is true under an upper semi-continuity assumption on the duality map .However ,we show below that this converse fails for general Banach spaces: indeed, if X is non-reexive and X^* is separable, there is a renorming of X such that the dual space X^* attain its norm: hence, while the subset K of the bidual unit ball where u attains its norm is so large that it spans X^{**}, this set K does not meet X.

Any unitary u is clearly a weak*-unitary (under the canonical embedding) of the bidual. Thus a natural question is the following: When is $\{x^* \in X_1^* : x^*(u) = 1\}$ weak*-dense in $\{\tau \in X_1^{***} : \tau(u) = 1\}$? It turns out that this density condition is frequently satisfied for natural examples of unitaries. For instance, when X is a C^*-algebra, a unitary in X remains a unitary of the enveloping von Neumann algebra; thus it follows from Proposition 3.3 in [43] that the density condition holds in this case. Following the terminology of Theorem 3.1 in [48], when this density condition is satisfied, we call u a point of norm-weak upper semi-continuity (norm-weak usc) for the duality map $x \to \{f \in X_1^* : f(x) = \|x\|\}$. We show that any non-reflexive Banach space with a unitary u can be renormed so that in the new norm u is still a unitary, but is not a point of norm-weak usc for the duality map.

For a convex set K we denote by $\partial_e K$ its set of extreme points. We always consider a Banach space as canonically embedded in its bidual. For $n > 3$ we denote by $X(n)$ the n-th dual of X. It is easily seen that a unitary $u \in X$ is extreme in all dual unit balls $B_{X^{(2n)}}$.

For a non- reflexive space $X, x \in \partial_e X_1$ is said to be a weak*-extreme point if it is also an extreme point of X_1^{**}. In [5] the authors gave an example of a space with a smooth dual, whose unit vectors are all extreme points of X_1^{**} but none is an extreme point of $X_1^{(4)}$. By a renorming result that is applicable, e.g., to certain separable Asplund spaces, we construct (Theorem (3.1.5)) a point $x \in X \cap \partial_e X_1^{(4)}$ that is not an extreme point of $X_1^{(6)}$. Such counterexamples suggest that it is hopeless to find a condition which ensures that an extreme point of the unit ball of a Banach space remains extreme in all duals of even order, and which boils down to usual extremality in reexive spaces (see [45]).

We denote by $CO(E)$ the convex hull of a set E. When E is a subset of a dual space, the weak*-closure of E is denoted E^-, and $CO^-(E)$ is the weak*-closed convex hull. We denote by $CO^=(E)$ the norm closed convex hull of E.

We first observe that the simplest way to obtain unitaries through renorming techniques actually provides unitaries which are also points of norm-weak usc for the duality map.

Proposition (3.1.1)[42]: Let X be a Banach space and $0 \neq u \in X$. There is a renorming on X in which u is a unitary and a point of norm-weak usc of the duality map.

Proof. As remarked in [43], given a non-zero vector u in a Banach space we can renorm the space so that u is a unitary in the new norm. It suffices indeed to consider the unit ball B_0 of some equivalent norm such that $u \notin B_0$, and to consider the equivalent norm whose unit ball is $B_1 = CO(B_0 \cup \{\pm u\})$. So we may and do assume that u is a unitary. Put $B^* = CO(S \cup -S)$. Then B^* is the dual unit ball of an equivalent norm on X whose state space is S and clearly u is still a unitary. It is easy to see that the unit ball of the triple dual is given by $B' = CO(S^- \cup -S^-)$ (where the closure is taken with respect to the weak*-topology). Now if $\tau \in B'$, then $\tau(u) = 1$ if and only if $\tau \in S^-$. Thus u is a point of norm-weak usc for the duality map in this norm.

In the above proof, the point u is a (QP) point (in the sense of [49]) of X when this space is equipped with the norm whose unit ball is B_1. Hence it is in particular, for this norm as well, a point where the duality map is even norm-to-norm upper semi-continuous.

The following simple proposition shows that in the presence of norm-weak upper semi-continuity unitaries in dual spaces are weak*-unitaries. We refer to [44] for the definition and basic statements on norm-weak upper semi-continuity of the pre-duality map, $x^* \to \{x \in X_1 : x^*(x) = \|x^*\|\}$.

Proposition (3.1.2)[42]: Let $x_0^* \in X^*$ be a unitary. If x_0^* is a point of norm-weak usc for the pre-duality map then x_0^* is a weak*-unitary. In particular, if the duality map is norm-weak usc at x_0^*, then x_0^* is a weak*-unitary.

Proof. By Lemma 2.1 in [44], the functional x_0^* is norm-attaining. Let $S' = \{x \in X_1 : x_0^*(x) = 1\}$. To show that x_0^* is a weak* -unitary we shall show that $B = CO^{=}(S' \cup -S')$ is the unit ball of an equivalent norm on X. By Corollary 3.2 in [43] the conclusion follows. Note that $B^- = CO(S'^- \cup -S'^-)$ (where the closure is taken in the weak*-topology of X^{**}). Thus by our assumption of upper semi-continuity, B^- is also the absolute convex hull of $\{\tau \in X_1^{**} : \tau(x_0^*) = 1\}$. As x_0^* is a unitary, by Theorem 3.1 in [43] we have that B^- is the unit ball of an equivalent dual norm on X^{**}. Thus by the bipolar theorem we see that B is an equivalent norm on X. The last part follows from Theorem (3.1.3) in [44], since if a norm attaining functional is a point of norm-weak usc for the duality map then it is also a point of norm-weak usc for the pre-duality map.

We will now use finer renorming techniques for exhibiting unitaries that are not points of norm-weak usc for the duality map.

Theorem (3.1.3)[42]: Let X be a non-reexive Banach space, and u a non-zero vector in X. Then X can be renormed so that in the new norm u is a unitary but not a point of norm-weak usc for the duality map.

Proof. As shown in [43], we may and do assume that u is a unitary in the original norm of X. We first consider the case of a separable Banach space X. Let K denote the state space of u. Let $x^{**} \in X^{**} \setminus X$. Let $\alpha = \sup\{|x^{**}(x^*)|: x^* \in K\}$. Since $x^{**}|\ker(u)$ is not weak*-sequentially continuous, we can choose a sequence $\{y_n^*\}$ with $x^{**}(y_n^*) > 3 + \alpha, y_n^*(u) = 1$ and $y_n^* \to x^* \in K$ (with respect to the weak*-topology, here and in the rest of the proof).

Let $z_n^* = (1 - 1/n)y_n^*$ and let $B = CO^-(X_1^* \cup \pm\{z_n^*\})$. The convex set B is the dual unit ball for an equivalent norm on X. As before it is easy to see that u is still a unitary with respect to this norm with the same state space K.

We now show that u is not a point of norm-weak usc of the duality map of this norm. We show that the criterion in Theorem 2.1 of [48] is violated. Let $V = \{x^* : |x^{**}(x^*)| < 1\}$. Pick any $\delta > 0$. Since $z_n^* \to x^* \in K$, these functionals are eventually in the set $\{f \in B : f(u) > 1 - \delta\}$. It follows that this set is not contained in $(K + V)$. Indeed, for $k \in K$ and $v \in V$, $x^{**}(k + v) < \alpha + 1$, while $x^{**}(z_n^*) > 2 + \alpha$ for n large enough. Thus for n large enough we get that $z_n^* \in \{f \in B : f(u) > 1 - \delta\}$ but $z_n^* \notin (K + V)$.

The general case follows easily from the separable one. Let $u \in Y \subset X$ and Y be separable non-reexive. We construct z_n^* in Y^*as above, and we denote by z_n^* norm preserving extensions to X of the functionals z_n^* obtained above. Let $B = CO^-(X_1^* \cup \pm\{z_n'^*\})$. It is easily checked that u is still a unitary in the new norm but is not a point of norm-weak usc for the duality map.

Theorem (3.1.4)[42]: Let X be a non-reexive Banach space such that X^* is separable. Then X can be equivalently renormed so that, in the new dual norm, X^* contains a unitary which fails to attain its norm.

Proof. Since X^* is separable, we may assume (see [50], Theorem II.7.1) that X is equipped with an equivalent norm such that X^{**} is strictly convex. Since X is not reexive, by James' theorem there exists a unit vector x_0^* that is not norm attaining. Let $\|x_0^{**}\| = x_0^{**}(x_0^*) = 1$. Clearly $x_0^{**} \notin X$. Let $d = d(x_0^{**}; X)$. Let $K = \{x^{**} \in X^{**} : x^{**}(x_0^{**}) = 1; \|x^{**} - x_0^{**}\| \le d/2\}$. Clearly span $K = X^{**}$ and $K \cap X = \emptyset$.

We now renorm X such that K is the state space of x_0^* in this norm. It will clearly follow that x_0^* is a unitary that does not attain its norm.

As X^* is separable and K is bounded, let $K = \{x_n^{**}\}^-$. We can write $K = \cap W_l = \cap W_l^-$ for a sequence $\{W_l\}$ of weak*-open subsets of $(1 + d/2)X_1^{**}$. For each n, we can choose a sequence $\{x_{n,k}\} \subset (1 + d/2)X_1$ such that $x_{n,k} \to x_n^{**}$ in the weak*-topology and such that $|x_0^*(x_{n,k})| < 1, x_{n,k} \in W_l$ for all $n \ge l$ and for all k.

Let $B' = CO^=(X_1 \cup \pm\{x_{n,k}\})$. Let $\|\cdot\|'$ denote the equivalent norm on X whose unit ball is B'. As $|x_0^*(x_{n,k})| \le 1$, we have that x_0^* is a unit vector with respect to the new norm.

Also in the new norm the bidual unit ball is given by $B'^{**} = CO\left(X_1^{**} \cup CO^-(\pm\{x_{n,k}\})\right)$. Now suppose $x^{**} = \lambda x_1^{**} + (1-\lambda)x_2^{**}$ for some $x_1^{**} \in X_1^{**}$, $x_2^{**} CO^-(\pm\{x_{n,k}\}), \lambda \in [0,1]$ and $x^{**}(x_0^*) = 1$. Then $1 = x_1^{**}(x_0^*) = x_2^{**}(x_0^*)$. Since X^{**} is strictly convex, $x_1^{**} = x_0^{**} \in K$.

Since for any $l \ge 1$ all $x_{n,k}$'s but a finite number are contained in W_l, we have that $(\pm\{x_{n,k}\})^- = \pm(K \cup \{x_{n,k}\})$. We now claim that $x_2^{**} \in K$. To see this we use the description of $CO^-(\pm\{x_{n,k}\})$ in terms of barycenters (Proposition 1.2 in [51]). Thus there is a probability measure μ with $\mu((\pm\{x_{n,k}\})^-) = 1$ and x_2^{**} is the barycenter of μ. Then $1 = x_2^{**}(x_0^*) = \int x_0^* d\mu$. Since $(\pm\{x_{n,k}\})^- = \pm(K \cup \{x_{n,k}\})$ and by the choice of $x_{n,k}$, this implies $\mu(K) = 1$. As K is a weak*-compact convex set by Proposition 1.2 of [51] again, we have $x_2^{**} \in K$. Hence $x^{**} \in K$ and so K is also the state space for this norm.

Since $K \cap X = \emptyset$, it follows that x_0^* does not attain its supremum on B', in other words, that x_0^* fails to attain its norm.

A unitary in X remains unitary in X^{**}, and it follows through an obvious induction that a unitary is in particular an extreme point in the unit ball of every dual space $X^{(2n)}$. It is well known that such a stability fails for general extreme points. Our last result shows that, even if stability holds to begin with, it may fail afterwards [45], [52].

Theorem (3.1.5)[42]: Let X be a separable space such that X^{***}/X^{*} is separable and non-reexive. There is an equivalent norm on X and a vector $f \in X$ of norm one, which is an exposed point of the unit ball of the fourth dual but is not an extreme point of the unit ball of the sixth dual.

Proof. The proof below is a modification of the proof of Proposition 4.1 in [44] that the reader is invited to consult before dwelling upon this proof. Our strategy is to adjust what has been done for proving [44, Prop. 4.1], in such a way that the renormed space is actually a dual space. In order to keep the notation of the proof of [44, Prop. 4.1], we will denote by x a smooth point of X^{*} and by $f \in X$ the corresponding differential.

We first note that the original dual norm is Fréchet smooth on a dense set since X^{**} is separable. Indeed, the dual of X^{**}/X is isomorphic to X^{***}/X^{*}. Let $x \in X^{*}$ be a unit vector where the norm is Fréchet differentiable. It is an easy consequence of Smulyan's lemma that when a dual norm is Fréchet differentiable at a given point x, the differential f at this point belongs to the predual. Hence, let $f \in X$ be such that $x(f) = \|f\| = 1$. Let $\{\phi_j\}_{j\geq 1} \subset X^{***}$ be such that $\{\phi_{2j}\}_{j\geq 1}$ is norm dense in X^{*} and $\{\phi_{2j+1}\}_{j\geq 0}$ is norm dense in $X^{\perp}$.

Since X^{***}/X^{*} is separable and non-reexive, we can choose a sequence of unit vectors $\{t_n\}_{n\geq 1} \subset (X^{*})^{\perp} \subset X^{(4)}$ such that $t_n \to 0$ in the weak*- topology of $X^{(4)}$ and there exists $0 \neq F \in X^{(6)}$ with $\{\pm F\} \subset \{t_n\}^{-}$ (where the closure is taken with respect to the weak*-topology of $X^{(6)}$).

Again by separability there exist sequences $\{f_{n,k}\} \subset X_1^{**}$ such that $f_{n,k} \to t_n$ for each n, in the weak*-topology of $X^{(4)}$. Note that this in particular implies that for fixed n and k tending to infinity, $f_{n,k} \to 0$ in the weak*-topology of X^{**} as $\{t_n\} \subset (X^{*})^{\perp}$. Without loss of generality we may assume that:

(i) $|t_n(\phi_j)| < 1/2^n$ for $n > j$;

(ii) $|\phi_j(f_{n,k})| < 1/2^n$ for $n > j$ and for all k;

(iii) $|f_{n,k}(x)| < 1/2^n$ for all n, k.

We finally choose a sequence $\{z_{n,k,l}\} \subset X_1$ such that $z_{n,k,l} \to f_{n,k}$ in the weak*-topology of X^{**} and assume again without loss of generality:

(i) $|\phi_{2j}(z_{n,k,l})| < 1/2^n$ for $n > 2j$, for all k, l;

(ii) $|\phi_{2j}(z_{n,k,l} - f_{n,k})| < 1/2^k$ for $k > j$, for all n, l;

(iii) $|x(z_{n,k,l})| < 1/2^n$ for all n, k, l.

As in the previous renormings, we now let $z'_{n,k,l} = z_{n,k,l} + (1 - 1/n)f$ and take $B' = CO^{=}(X_1 \cup \pm\{z'_{n,k,l}\})$ as the new unit ball.

As before, the unit ball $B'_{X^{**}}$ of the bidual in the new norm is given by $B'_{X^{**}} = \mathrm{CO}^{-}(X_1^{**} \cup L^{-})$ with $L = \left(\pm\{z'_{n,k,l}\}\right)$, where the closures are taken in the weak*-topology of X^{**}. By our choice of these sequences, we have:

(i) $z'_{n,k,l} \to f$ in (X, w) as $n \to \infty$, for any $k = k(n)$ and $l = l(n)$;

(ii) for any $n_0, z'_{n_0,k,l} \to (1 - 1/n_0)f$ in (X, w) for $k \to \infty$ and any $l = l(k)$;

(iii) for any $n_0, k_0, z'_{n_0,k_0,l} \to f_{n_0,k_0} + (1 - 1/n_0)f$ in (X^{**}, w^*) for $l \to \infty$.

Therefore $L^{-} = L \cup \pm\{f_{n,k} + (1 - 1/n)f\}_{n,k\geq 1} \cup \pm\{(1 - 1/n)f\}_{n\geq 1} \cup \pm\{f\}$. Now since X^{**} has the Radon-Nikodym property, we have (see page 327 of [44]) that the weak*-closed convex hull of any weak*-compact subset of X^{**} coincides with its norm-closed convex hull. Therefore $B'_{X^{**}} =$ $\mathrm{CO}\{X_1^{**} \cup \mathrm{CO}^{=}(L^{-})\}$ with

$$L^{-} = \pm\left\{z_{n,k,l} + \left(1 - \frac{1}{n}\right)f\right\}_{n,k,l\geq 1} \cup \pm\left\{f_{n,k} + \left(1 - \frac{1}{n}\right)f\right\}_{n,k\geq 1}$$
$$\cup \pm\left\{\left(1 - \frac{1}{n}\right)f\right\}_{n\geq 1} \cup \pm\{f\}.$$

We now show that x is a smooth point of the third dual in the new norm (in other words, a very smooth point, with the notation used in [44], of the dual), and thus f is an exposed point (and so in particular an extreme point) of the fourth dual unit ball.

To achieve this we first note that x is a smooth point of X^* equipped with the new dual norm, with derivative f. Indeed, we clearly have

$$L^{-} \cap x^{-1}(\{1\}) = \{f\},$$

and since x is a smooth point of the original dual norm,

$$X_1^{**} \cap x^{-1}(\{1\}) = \{f\},$$

and our claim easily follows.

Also by the choice of the sequences we have:

(i) $|\phi_j(f_{n,k})| < 1/2^n$ for $n > j$, for all k;

(ii) $|\phi_{2j}(z_{n,k,l})| < 1/2^n$ for $n > 2j$, for all k, l;

(iii) $\phi_{2j+1}(z_{n,k,l}) = 0$ for all j, n, k, l.

Therefore when n goes to infinity, the sequences $(f_{n,k})$ and $(z_{n,k,l})$ converge weakly to 0 in X^{**} regardless of k and l. Now, using convex combinations as in the proof of Fact 6 in [44], we can show that f is a point of weak*- weak continuity for the identity map on the unit ball of the bidual. Thus by Remark 3.1 in [44] we get that x is a smooth point of the third dual, and thus f is exposed by x in the new unit ball $B'_{X^{(4)}}$ of $X^{(4)}$.

We finally invoke the choice of $F \in X^{(6)}$ we made at the start of the proof. By our construction we have that

$$f_{n,k} + \left(1 - \frac{1}{n}\right) f \in B'_{X^{**}}$$

for every n and k, and thus

$$t_n + \left(1 - \frac{1}{n}\right) f \in B'_{X^{(4)}}$$

for every n, and finally $f + F \in B'_{X^{(6)}}$ and $f - F \in B'_{X^{(6)}}$. Therefore f is not an extreme point of the sixth dual unit ball.

Section (3.2): Weak Sequential Completeness:

If X is a Banach space, we recall that it is weakly sequentially complete if any weakly Cauchy sequence in X is weakly convergent. In the present paper we investigate quantitative versions of this property. To this end we use several quantities related to a given bounded sequence (x_k) in X.

Let $\text{clust}_{X^{**}}(x_k)$ denote the set of all weak* cluster points of (x_k) in X^{**}. By $\delta(x_k)$ we will denote the diameter of $\text{clust}_{X^{**}}(x_k)$ (see also (4) below). Further, if A, B are nonempty subsets of a Banach space X, then

$$d(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}$$

denotes the usual distance between A and B and the Hausdorff non-symmetrized distance from A to B is defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

Note that a space X is weakly sequentially complete if for each bounded sequence (x_k) in X satisfying $\delta(x_k) = 0$ (this just means that the sequence is weakly Cauchy) we have $\hat{d}(\text{clust}_{X^{**}}(x_k), X) = 0$ (i.e., all the weak$*$ cluster points are contained in X, which for a weakly Cauchy sequence means that it is weakly convergent). It is thus natural to ask which Banach spaces satisfy a quantitative version of weak sequential completeness, i.e., the inequality

$$\hat{d}(\text{clust}_{X^{**}}(x_k), X) \le C \cdot \delta(x_k) \quad (1)$$

for all bounded sequences (x_k) in X and for some $C > 0$. The starting point of our investigation was the following remark made by G. Godefroy in [57]:

If X is complemented in X^{**} by a projection P satisfying

$$\|x^{**}\| = \|Px^{**}\| + \|x^{**} - Px^{**}\|, x^{**} \in X^{**}, \quad (2)$$

then X is weakly sequentially complete and

$$\hat{d}(\text{clust}_{X^{**}}(x_k), X) \le \delta(x_k) \quad (3)$$

for any sequence (x_k) in X.

It can be easily seen that

$$\delta(x_k) = \sup_{x^* \in B_{X^*}} \left(\limsup_{k\to\infty} x^*(x_k) - \liminf_{k\to\infty} x^*(x_k) \right)$$
$$= \sup_{x^* \in B_{X^*}} \lim_{n\to\infty} \sup\left\{ \left| x^*(x_l) - x^*\left(x_j\right) \right| : l, j \geq n \right\}. \quad (4)$$

The first formula of (4) is used in [58], the second one in [57].

Banach spaces satisfying assumption (2) above are called L-embedded, see [59]. The proof of (3) can be found in [60].

By what has been said above, inequality (3) is a quantitative form of weak sequential completeness.

In [57] G. Godefroy mentions that it is not clear which weakly sequentially complete spaces can be renormed to have such a quantitative form of weak sequential completeness.

On the one hand we show that the answer to G. Godefroy's question cannot be positive for all weakly sequentially complete Banach spaces, more precisely we construct a weakly sequentially complete space that cannot be renormed in such a way that (3) holds, see Example (3.2.5) below.On the other hand we put inequality (3) into context by studying some modifications and possible converses, see the following theorem. In particular, we slightly improve inequality (3) – see (6) in the theorem – but such that now the additional factor 2 is optimal.

We will use one more quantity (cf. [61] but appearing implicitly in [58]) which in some situations is more natural than the quantity δ, namely

$$\tilde{\delta}(x_k) = \inf\left\{ \delta\left(x_{k_j}\right) : \left(x_{k_j}\right) \text{ is a subsequence of } (x_k) \right\}.$$

The negative partial answer to the mentioned question of G. Godefroy is given by the following example. In fact, we obtain a slightly stronger result. Not only there is a weakly sequentially complete Banach space not satisfying (1) for all bounded sequences and some $C > 0$, but we get even a weakly sequentially complete space not satisfying a weaker form of (1) – with d in place of $\hat{d}$.

We remark that a separable space with the Schur property belongs to the class of so-called strongly weakly compactly generated spaces (see [62, Examples 2.3]) and thus such a quantitative form of weak sequential completeness does not hold even for this class of spaces.

Lemma (3.2.1)[56]: Let X be a Banach space and (x_n) be a bounded sequence in X. Suppose that $c > 0$ is such that

$$\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| \geq c \sum_{j=1}^{n} \left| \alpha_j \right|$$

whenever $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n$ are real numbers. Then

(i) $\delta(x_n) \geq 2c$,

(ii) $d(\mathrm{clust}_{X^{**}}(x_k), X) \geq c$.

Proof. (i) It is clear that the sequence (x_n) is linearly independent. Hence there is a unique linear functional defined on its linear span whose value is c at x_{2k-1} and $-c$ at x_{2k} for each $k \in \mathbb{N}$. By the assumption, the norm of this functional is at most 1. Let $x^* \in B_{X^*}$ be its Hahn–Banach extension. Then x^* witnesses that $\delta(x_n) \geq 2c$.

(ii) Let x^{**} be any weak* cluster point of the sequence (x_n) in X^{**} and $x \in X$ be arbitrary. It follows from [63, Proposition 4.2] that there is an index $m \in \mathbb{N}$ such that

$$\left\|\sum_{j=m}^{\infty} \alpha_j (x_j - x)\right\| \geq c \sum_{j=m}^{\infty} |\alpha_j|$$

for every sequence $(\alpha_j)_{j=m}^{\infty}$ with finitely many nonzero elements. In particular, it follows that the vectors $x_j - x, j \geq m$, are linearly independent. So, there is a unique linear functional on their linear span whose value at each $x_j - x$ is equal to c. By the above inequality, the norm of this functional is at most one. Let $x^* \in X^*$ be its Hahn–Banach extension. Then we have

$$\|x^{**} - x\| \geq (x^{**} - x)(x^*) \geq \liminf_{j\to\infty} x^*(x_j - x) = c.$$

This completes the proof of the lemma.

Theorem (3.2.2)[56]: Let X be a Banach space and (x_k) be a bounded sequence in X. Then

$$\tilde{\delta}(x_k) \leq 2\hat{d}(\mathrm{clust}_{X^{**}}(x_k), X). \tag{5}$$

If the space X is L-embedded, then also the following inequalities hold:

$$2\hat{d}(\mathrm{clust}_{X^{**}}(x_k), X) \leq \delta(x_k), \tag{6}$$

$$2\hat{d}(\mathrm{clust}_{X^{**}}(x_k), X) \leq \tilde{\delta}(x_k). \tag{7}$$

Since we have trivially that $\tilde{\delta} \leq \delta$ and $d \leq \hat{d}$ it is natural to ask whether one of these quantities can be replaced by a sharper one in the inequalities of the theorem. The following remark and Example 3 show that this cannot be done in any of the inequalities (5)–(7).

Proof.

We start by proving (5): Let (x_k) be a bounded sequence in X. We assume that $\delta(x_k) > 0$ because otherwise (5) holds trivially. Let $c \in \left(0, \tilde{\delta}(x_k)\right)$ be arbitrary. The key ingredient is provided by a result of E. Behrends (see [58, Theorem 3.2]) that yields a subsequence $\left(x_{n_k}\right)$ such that

$$\left\|\sum_{i=1}^{k}\alpha_i x_{n_i}\right\| \geq \frac{c}{2}\sum_{i=1}^{k}|\alpha_i|$$

whenever $k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. By Lemma (3.2.3)(ii) we get $d\big(\mathrm{clust}_{X^{**}}(x_{n_k}), X\big) \geq \frac{c}{2}$, hence $\hat{d}(\mathrm{clust}_{X^{**}}(x_k), X) \geq \frac{c}{2}$. As $c \in \big(0, \tilde{\delta}(x_k)\big)$ is arbitrary, (5) follows.

We continue by proving (6): We set $c = \hat{d}(\mathrm{clust}_{X^{**}}(x_k), X)$ and assume that $c > 0$ because otherwise (6) holds trivially. Let $\varepsilon \in (0, c)$ be arbitrary and let x^{**} be a weak$*$ cluster point of the sequence (x_k) in X^{**} such that $d(x^{**}, X) > c - \frac{\varepsilon}{2}$. Set $x = Px^{**}$ and $x_s = x^{**} - x$ where P denotes the projection on X as in (2). Then $d(x^{**}, X) = \|x_s\|$. We claim that there is a subsequence $\big(x_{k_n}\big)$ such that

$$\sum_{i=1}^{n}\alpha_i\big(x_{k_i} - x\big) \geq (c - (1 - 2^{-n})\varepsilon)\sum_{i=1}^{n}|\alpha_i| \tag{8}$$

for all $n \in \mathbb{N}$ and all $(\alpha_i)_{i=1}^{n}$ in $\mathbb{R}^n$. This will be proved by G. Godefroy's 'ace of ♢ argument' [59, p. 170], cf. the proof of [59, Proposition IV.2.5]. Since x_s is a weak* cluster point of the sequence $(x_k - x)$, there is k_1 such that $\|x_{k_1} - x\| > c - \frac{\varepsilon}{2}$ which settles the first induction step.

Suppose we have constructed $x_{k_1}, \dots, x_{k_n}$. Let $\big(\alpha^l\big)_{l=1}^{L}$ be a finite sequence of elements of the unit sphere of ℓ_1^{n+1} such that $\alpha_{n+1}^l \neq 0$ for all $l \in \{1, \dots, L\}$ and such that for each α in the unit sphere of α_{n+1}^l there is an element α^l such that

$$\|\alpha - \alpha^l\|_{\ell_1^{n+1}} < \frac{\varepsilon}{2^{n+2}\ \sup\limits_k\|x_k\|}.$$

Let $l \in \{1, \dots, L\}$ be arbitrary. Then $\sum_{i=1}^{n}\alpha_i^l\big(x_{k_i} - x\big) + \alpha_{n+1}^l x_s$ is a weak* cluster point of the sequence $\big(\sum_{i=1}^{n}\alpha_i^l\big(x_{k_i} - x\big) + \alpha_{n+1}^l(x_k - x)\big)_{k=1}^{\infty}$ and for its norm we have

$$\begin{aligned}\left\|\sum_{i=1}^{n}\alpha_i^l\big(x_{k_i} - x\big) + \alpha_{n+1}^l x_s\right\| &= \left\|\sum_{i=1}^{n}\alpha_i^l\big(x_{k_i} - x\big)\right\| + \|\alpha_{n+1}^l x_s\| \\ &\geq (c - (1 - 2^{-n})\varepsilon)\sum_{i=1}^{n}|\alpha_i^l| + |\alpha_{n+1}^l|\left(c - \frac{\varepsilon}{2}\right) \\ &> (c - (1 - 2^{-n})\varepsilon)\sum_{i=1}^{n+1}|\alpha_i^l| = c - (1 - 2^{-n})\varepsilon.\end{aligned}$$

It follows that there is $k_{n+1} > k_n$ such that

$$\left\|\sum_{i=1}^{n+1}\alpha_i^l\big(x_{k_i} - x\big)\right\| > c - (1 - 2^{-n})\varepsilon$$

for all $l \in \{1, \dots, L\}$. By a straightforward calculation using the choice of the α^l and the triangle inequality we get that inequality (8), with $n+1$ instead of n, holds for all α in the unit sphere of ℓ_1^{n+1} and hence for all elements of $\mathbb{R}^{n+1}$.

This finishes the construction. By Lemma (3.2.1)(i) we get

$$\delta(x_{k_n} - x) \geq 2(c - \varepsilon),$$

hence clearly

$$\delta(x_k) \geq \delta(x_{k_n}) = \delta(x_{k_n} - x) \geq 2(c - \varepsilon).$$

As $\varepsilon \in (0, c)$ is arbitrary, we get (6).

Finally, let us prove (7): We take any subsequence (x_{k_n}) and observe that

$$2d(\mathrm{clust}_{X^{**}}(x_k), X) \leq 2\hat{d}(\mathrm{clust}_{X^{**}}(x_{k_n}), X) \leq \delta(x_{k_n})$$

by (6). Then we can pass to the infimum over all (x_{k_n}). This finishes the proof of the theorem.

Example (3.2.3)[56]: There is an L-embedded space X and a bounded sequence (x_k) in X such that $\tilde{\delta}(x_k) = 2$ and $d(\mathrm{clust}_{X^{**}}(x_k), X) = 0$.

Proof. For $n \in \mathbb{N}$ set $X_n = \ell_\infty^n$ and let X be the ℓ_1-sum of all the spaces $X_n, n \in \mathbb{N}$. Then X is L-embedded by [59, Proposition IV.1.5].

Further, let $e_{n_1}, \dots, e_{n_n}$ be the canonical basic vectors of X_n and let (x_k) be the sequence in X containing subsequently these basic vectors, i.e., the sequence

$$e_1^1, e_1^2, e_2^2, e_1^3, e_2^3, e_3^3, e_1^4, \dots, e_4^4, \dots.$$

Then we have $\tilde{\delta}(x_k) = 2$ as each subsequence of (x_k) contains a further subsequence isometrically equivalent to the canonical basis of ℓ_1.

It remains to show that $d(\mathrm{clust}_{X^{**}}(x_k), X) = 0$. To do so, it is enough to prove that 0 is a weak cluster point of the sequence (x_k). To verify this, we fix $g_1, \dots, g_m \in X^*$ and $\varepsilon > 0$. Let $K = \max\{\|g^1\|, \dots, \|g^m\|\}$.

The dual X^* can be canonically identified with the ℓ_∞-sum of the spaces $X_n^*, n \in \mathbb{N}$. Moreover, X_n^* is canonically isometric to ℓ_1^n. Thus each $g \in X^*$ can be viewed as a bounded sequence $(g_n)_{n\in\mathbb{N}}$, where $g_n = (g_{n,j})_{j=1}^n \in \ell_1^n$ for each $n \in \mathbb{N}$.

We find $N \in \mathbb{N}$ such that $\frac{K}{N} < \varepsilon$ and let $n \in \mathbb{N}$ be such that $n > mN$. Let $k \in \{1, \dots, m\}$ be arbitrary. We have $\|g_n^k\| \leq \|g^k\| \leq K$. As $\|g_n^k\| = \sum_{j=1}^n |g_{n,j}^k|$, the set

$$\left\{j \in \{1, \dots, n\}: |g_{n,j}^k| \geq \frac{K}{N}\right\}$$

has at most N elements. It follows that the set

$$\left\{j \in \{1, \ldots, n\}: \left(\exists k \in \{1, \ldots, m\}, \left|g_{n,j}^{k}\right| \geq \frac{K}{N}\right)\right\}$$

has at most mN elements. As $n > mN$, there is some $j \in \{1, \ldots, n\}$ such that $\left|g_{n,j}^{k}\right| < \frac{K}{N} < \varepsilon$ for each $k \in \{1, \ldots, m\}$. It means that $\left|g^{k}\left(e_j^n\right)\right| < \varepsilon$ for each $k \in \{1, \ldots, m\}$.

Since e_j^n is an element of the sequence (x_k), this completes the proof that 0 is in the weak closure of the sequence, hence 0 is a weak cluster point (as the sequence (x_k) does not contain 0).

Example (3.2.4)[56]: There exists a separable Banach space X with the Schur property – in particular, X is weakly sequentially complete – which is 1-complemented in its bidual, such that there is no constant $C > 0$ satisfying

$$d(\text{clust}_{X^{**}}(x_k), X) \leq C \cdot \delta(x_k)$$

for every bounded sequence (x_k) in X.

Proof. We recall that $\beta\mathbb{N}$ is the Čech–Stone compactification of $\mathbb{N}$ and $M(\beta\mathbb{N})$ is the space of all signed Radon measures on $\beta\mathbb{N}$ considered as the dual of ℓ_∞.

Let us fix $\alpha > 0$ and consider the space

$$Y_\alpha = (\ell_1, \alpha\|\cdot\|_1) \oplus_1 (C[1, \omega], \|\cdot\|_\infty).$$

Here $\|\cdot\|_1$ denotes the usual norm on ℓ_1, ω is the first infinite ordinal, $C[1, \omega]$ stands for the space of all continuous functions on the ordinal interval $[1, \omega]$ and $\|\cdot\|_\infty$ is the standard supremum norm. Note that we have the following canonical identifications:

$$Y_\alpha^* = \left(\ell_\infty, \frac{1}{\alpha}\|\cdot\|_\infty\right) \oplus_\infty (\ell_1[1, \omega], \|\cdot\|_1), \quad \text{and}$$

$$Y_\alpha^{**} = \left(M(\beta\mathbb{N}), \alpha\|\cdot\|_{M(\beta\mathbb{N})}\right) \oplus_1 (\ell_\infty[1, \omega], \|\cdot\|_\infty).$$

For $k \in \mathbb{N}$, let $x_k = \left(e_k, \chi_{[k,\omega]}\right) \in Y_\alpha$, where e_k denotes the k-th canonical basic vector in ℓ_1 and $\chi_{[k,\omega]}$ is the characteristic function of the interval $[k, \omega]$. Let X_α be the closed linear span of the set $\{x_k : k \in \mathbb{N}\}$. We observe that

$$X_\alpha = \left\{\left((\eta_k), f\right) \in Y_\alpha : f(n) = \sum_{k=1}^{n} \eta_k \text{ for all } n \in \mathbb{N}\right\}. \tag{9}$$

Indeed, the set on the right-hand side is a closed linear subspace of Y_α containing x_k for each $k \in \mathbb{N}$. This proves the inclusion '$\subset$'. To prove the converse one, let us take any point $\left((\eta_k), f\right)$ in the set on the right-hand side. Since $(\eta_k) \in \ell_1$, we get

$$\left((\eta_k), f\right) = \sum_{k=1}^{\infty} \eta_k x_k \in X_\alpha$$

as the series is absolutely convergent.

It follows that for each $\big((\eta_k), f\big) \in X_\alpha$ we have

$$\alpha\|(\eta_k)\|_1 \le \big\|\big((\eta_k), f\big)\big\| \le (\alpha+1)\|(\eta_k)\|_1,$$

hence X_α is isomorphic to ℓ_1. More precisely, the projection on the first coordinate is an isomorphism onto ℓ_1. In particular, X_α has the Schur property (and thus it is weakly sequentially complete).

We further observe that X_α^{**} is canonically identified with the weak* closure of X_α in Y_α^{**}, thus

$$X_\alpha^{**} = \{(\mu, f) \in M(\beta\mathbb{N}) \times \ell_\infty[1,\omega]: \\ (\forall n \in \mathbb{N}: f(n) = \mu\{1,\dots,n\}) \text{ and } f(\omega) = \mu(\beta\mathbb{N})\}. \qquad (10)$$

Indeed, the set on the right-hand side is a weak* closed linear subspace of Y_α^{**} containing X_α, which proves the inclusion '$\subset$'. To prove the converse one let us fix (μ, f) in the set on the right-hand side. Take a bounded net (u_τ) in ℓ_1 which weak* converges to μ. For each τ there is a unique $f_\tau \in C[1,\omega]$ such that $(u_\tau, f_\tau) \in X_\alpha$. Then (f_τ) is clearly a bounded net in $\ell_\infty[1,\omega]$. Moreover, we will show that (f_τ) weak* converges to f. Since the weak* topology on bounded sets coincides with the topology of pointwise convergence, it suffices to show that f_τ pointwise converge to f. Indeed,

$$f_\tau(n) = \sum_{k=1}^{n} u_\tau(k) \to \mu\{1,\dots,n\} = f(n), \quad \text{for each } n \in \mathbb{N},$$

$$f_\tau(\omega) = \sum_{k=1}^{\infty} u_\tau(k) \to \mu(\beta\mathbb{N}) = f(\omega).$$

It follows that X_α is 1-complemented in its bidual. To show that we set

$$P(\mu, f) = \big((\mu\{k\}), f - \mu(\beta\mathbb{N}\setminus\mathbb{N}) \cdot \chi_{\{\omega\}}\big), \qquad (\mu, f) \in X_\alpha^{**}.$$

Then P is a projection of X_α^{**} onto X_α of norm one. Indeed, if $(\mu, f) \in X_\alpha$, then $\mu(\beta\mathbb{N}\setminus\mathbb{N}) = 0$ and hence $P(\mu, f) = (\mu, f)$. Further, by (9) and (10) we get that $P(\mu, f) \in X_\alpha$ for each $(\mu, f) \in X_\alpha^{**}$. Thus P is a projection onto X_α. To show it has norm one, it is enough to observe that, given $(\mu, f) \in X_\alpha^{**}$, we have $\|(\mu\{k\})\|_{\ell_1} \le \|\mu\|$, and that $f - \mu(\beta\mathbb{N}\setminus\mathbb{N}) \cdot \chi_{\{\omega\}}$ is a continuous function on $[1,\omega]$ coinciding on $[1,\omega)$ with f and so $\big\|f - \mu(\beta\mathbb{N}\setminus\mathbb{N}) \cdot \chi_{\{\omega\}}\big\|_\infty \le \|f\|_\infty$.

Further, for the sequence (x_k), its weak* cluster points in X_α^{**} are equal to

$$\big(\varepsilon_t, \chi_{\{\omega\}}\big): t \in \beta\mathbb{N}\setminus\mathbb{N},$$

where ε_t denotes the Dirac measure at a point $t \in \beta\mathbb{N}$.

We claim that, for our sequence (x_k), we have

$$d\big(\mathrm{clust}_{X_\alpha^{**}}(x_k), X_\alpha\big) \ge \frac{1}{2} \ \text{ and } \ \delta(x_k) = 2\alpha. \qquad (11)$$

To see the first inequality, we use the fact that the distance of any weak* cluster point of (x_k) from X_α is at least $d(\chi_{\{\omega\}}, C[1,\omega]) = \frac{1}{2}$. On the other hand, if $t, t' \in \beta\mathbb{N}\setminus\mathbb{N}$ are distinct, then

$$\left\|(\varepsilon_t, \chi_{\{\omega\}}) - (\varepsilon_{t'}, \chi_{\{\omega\}})\right\|_{X_\alpha^{**}} = \|\varepsilon_t - \varepsilon_{t'}, 0\|_{X_\alpha^{**}} = \alpha\|\varepsilon_t - \varepsilon_{t'}\|_{M(\beta\mathbb{N})} = 2\alpha.$$

This verifies (11).

Now we use the described procedure to construct the desired space X. For $n \in \mathbb{N}$, let $\alpha_n = \frac{1}{n}$ and let $X_{\frac{1}{n}}$ be the space constructed for α_n. Let

$$X = \left(\sum_{n=1}^{\infty} X_{\frac{1}{n}}\right)_{\ell_1}$$

be the ℓ_1-sum of the spaces $X_{\frac{1}{n}}$. We claim that X is the required space.

First, since each $X_{\frac{1}{n}}$ has the Schur property, X, as their ℓ_1-sum, possesses this property as well (this follows by a straightforward modification of the proof that ℓ_1 has the Schur property, see [64, Theorem 5.19]). Hence X is weakly sequentially complete.

Further, observe that

$$X^* = \left(\sum_{n=1}^{\infty} X_{\frac{1}{n}}^*\right)_{\ell_\infty} \quad \text{and} \quad X^{**} \supset \left(\sum_{n=1}^{\infty} X_{\frac{1}{n}}^{**}\right)_{\ell_1}.$$

Note that the latter space is not equal to X^{**} but it is 1-complemented in X^{**} (cf. the proof of [59, Proposition IV.1.5]). Now it follows that X is 1-complemented in X^{**}.

Finally, fix $n \in \mathbb{N}$. We consider a sequence $\hat{x}_k = \left(0, \ldots, 0, \overset{\text{n-th}}{x_k}, 0, \ldots\right)$, where the elements $x_k \in X_{\frac{1}{n}}$, $k \in \mathbb{N}$, are defined above. Let $y = \left(0, \ldots, 0, \overset{\text{n-th}}{(\varepsilon_t, \chi_{\{\omega\}})}, 0, \ldots\right)$, where $t \in \beta\mathbb{N}\setminus\mathbb{N}$, be a weak* cluster point of $(\hat{x}_k)$ in X^{**}. Then, for any $z = (z(1), z(2), \ldots) \in X$,

$$\|y - z\|_{X^{**}} \geq \left\|(\varepsilon_t, \chi_{\{\omega\}}) - z(n)\right\|_{X_{\frac{1}{n}}^{**}} \geq \frac{1}{2}$$

by (11). Hence

$$d(\operatorname{clust}_{X^{**}}(\hat{x}_k), X) \geq \frac{1}{2}.$$

On the other hand,

$$\delta(\hat{x}_k) = \delta(x_k) = \frac{2}{n},$$

again by (11). From this observation the conclusion follows.

Even though the second part of Theorem (3.2.2) is formulated for L-embedded spaces, using results of A.S. Granero and M. Sánchez we can prove the following variant of Theorem (3.2.2).

Let X be a subspace of an L-embedded Banach space Y and (x_k) be a bounded sequence in X. Then

$$\hat{d}(\mathrm{clust}_{X^{**}}(x_k), X) \leq \delta(x_k) \quad \text{and} \quad d(\mathrm{clust}_{X^{**}}(x_k), X) \leq \tilde{\delta}(x_k). \tag{12}$$

To verify the first inequality, we consider $x^{**} \in \mathrm{clust}_{X^{**}}(x_k)$. Since $\mathrm{clust}_{X^{**}}(x_k) = \mathrm{clust}_{Y^{**}}(x_k)$, using [65, Lemma 2.2] (with X, Y for D, X) and Theorem (3.2.2), we obtain

$$d(x^{**}, X) \leq 2d(x^{**}, Y) \leq 2\hat{d}(\mathrm{clust}_{Y^{**}}(x_k), Y) \leq \delta(x_k).$$

This proves the first statement because $x^{**} \in \mathrm{clust}_{X^{**}}(x_k)$ was arbitrary. The second one can be deduced from the first one just as in the proof of Theorem (3.2.2).

However, we do not know whether it is possible to obtain not only (12) but (6) and (7) of Theorem (3.2.2) for subspaces of L-embedded spaces.

Up to now we have tacitly assumed that we are dealing with real Banach spaces. In fact, our proofs work for real spaces but all the results can be easily transferred to complex spaces as well. Let us indicate how to see this.

Let X be a complex Banach space. Denote by X_R the same space considered over the field of real numbers (i.e., we just forget multiplication by imaginary numbers). Let $\phi : X^* \to (X_R)^*$ be defined by

$$\phi(x^*)(x) = \mathrm{Re}\, x^*(x), \qquad x^* \in X^*, x \in X.$$

It is well known that ϕ is a real-linear isometry of X^* onto $(X_R)^*$. Let us define a mapping $\psi : X^{**} \to (X_R)^{**}$ by the formula

$$\psi(x^{**})(y^*) = \mathrm{Re}\, x^{**}\big(\phi^{-1}(y^*)\big), \qquad x^{**} \in X^{**}, y^* \in (X_R)^*.$$

It is easy to check that the mapping ψ satisfies the following properties:

(i) ψ is a real-linear isometry of X^{**} onto $(X_R)^{**}$.

(ii) ψ is a weak*-to-weak* homeomorphism.

(iii) $\psi(X) = X_R$.

It follows that for any sequence in X all the quantities in question (i.e., $\delta, \tilde{\delta}$, d and $\tilde{d}$) are the same with respect to X and with respect to X_R. (Recall that δ is defined as the diameter of weak* cluster points, which has good sense in a complex space as well, even though in the complex case only the second formula of (4) works.) If, moreover, we observe that X_R is L-embedded whenever X is L-embedded, we conclude that Theorem (3.2.2) is valid for complex spaces as well.

As for Examples (3.2.3) and (3.2.4), it is clear that they work also in the complex setting – we can just consider complex versions of the respective spaces.

We finish by recalling that G. Godefroy's question, for which Banach spaces (3) holds, remains open. In particular, the following question seems to be open.

Question (3.2.5)[56]: Let X be a Banach space which is a u-summand in its bidual, i.e., there is a projection $P : X^{**} \to X$ with $\|I - 2P\| = 1$. Does (1) hold for X for some $C > 0$?

We conjecture that the space from Example (3.2.4), although it is 1-complemented in its bidual, is not a u-summand. At least the projection we have constructed does not work.

Corollary (3.2.6)[257]. Let X be a Banach space and $({x^2}_n)$ be a square bounded sequence in X. Suppose that $c > 0$ is such that

$$\left\|\sum_{j=1}^{n}(\alpha^2-\alpha+1)_j {x^2}_j\right\| \geq c\sum_{j=1}^{n}\left|(\alpha^2-\alpha+1)_j\right|$$

When ever$n \in \mathbb{N}$ and$(\alpha^2-\alpha+1)_1, \dots, (\alpha^2-\alpha+1)_n$are real numbers. Then

(i) $\left(\tilde{\delta}+\epsilon\right)({x^2}_n) \geq 2c,$

(ii) $\mathrm{d}(\mathrm{clust}_{X^{**}}({x^2}_k), X) \geq c.$

Proof.(i) It is clear that the square sequence $({x^2}_n)$is linearly independent. Hence there is a unique linearfunctional defined on its linear span whose value is c at ${x^2}_{2k-1}$ and $-c$ at ${x^2}_{2k}$for each$k \in \mathbb{N}$. Bythe assumption, the norm of this functional is at most 1. Let $(x^2)^* \in B_{X^*}$ be its Hahn–Banach extension. Then $(x^2)^*$ witnesses that

$\left(\tilde{\delta}+\epsilon\right)({x^2}_n) \geq 2c.$

(ii) Let $(x^2)^{**}$ be any weak* cluster point of the square sequence $({x^2}_n)$in X^{**} and $x^2 \in X$ be arbitrary. It follows from [7, Proposition 4.2] that there is an index $m \in \mathbb{N}$ such that

$$\left\|\sum_{j=m}^{\infty}(\alpha^2-\alpha+1)_j\left({x^2}_j - x^2\right)\right\| \geq c\sum_{j=m}^{\infty}\left|(\alpha^2-\alpha+1)_j\right|$$

for every quadratic sequence $\left((\alpha^2-\alpha+1)_j\right)_{j=m}^{\infty}$with finitely many nonzero elements. In particular, it follows thatthe vectors${x^2}_j - x^2, j \geq m$, are linearly independent. So, there is a unique linear functional ontheir linear span whose value at each ${x^2}_j - x^2$is equal to c. By the above inequality, the norm ofthis functional is at most one. Let $(x^2)^* \in X^*$ be its Hahn–Banach extension. Then we have

$$\|(x^2)^{**} - x^2\| \geq ((x^2)^{**} - x^2)((x^2)^*) \geq \liminf_{j\to\infty}(x^2)^*\left({x^2}_j - x^2\right) = c.$$

This completes the proof of the lemma.

Corollary (3.2.7)[257]. Let X be a Banach space and $({x^2}_k)$ be a square bounded sequence in$X$. Then

$$\tilde{\delta}({x^2}_k) \leq 2(d+\epsilon)(\mathrm{clust}_{X^{**}}({x^2}_k), X). \tag{13}$$

If the space X is L-embedded , then also the following inequalities hold:

$$2(d+\epsilon)(\text{clust}_{X^{**}}(x^2{}_k),X) \le \big(\tilde{\delta}+\epsilon\big)(x^2{}_k), \tag{14}$$

$$2d(\text{clust}_{X^{**}}(x^2{}_k),X) \le \tilde{\delta}(x^2{}_k). \tag{15}$$

Proof .

We start by proving (13): Let $(x^2{}_k)$be a square bounded sequence in X. We assume that$\big(\tilde{\delta}+\epsilon\big)(x^2{}_k) > 0$ because otherwise (15) holds trivially. Let $c \in \Big(0, \tilde{\delta}(x^2{}_k)\Big)$be arbitrary. The key ingredient is provided by a result of E.Behrends (see [1, Theorem 3.2]) that yields a square subsequence $\big(x^2{}_{n_k}\big)$ such that

$$\left\|\sum_{i=1}^{k}(\alpha^2-\alpha+1)_i x^2{}_{n_i}\right\| \ge \frac{c}{2}\sum_{i=1}^{k}|(\alpha^2-\alpha+1)_i|$$

Whenever $k \in \mathbb{N}$ and$(\alpha^2-\alpha+1)_1, \dots, (\alpha^2-\alpha+1)_n \in \mathbb{R}$. By corollary (3.2.6)(ii) we get $\text{d}\big(\text{clust}_{X^{**}}\big(x^2{}_{n_k}\big),X\big) \ge \frac{c}{2}$, hence$(d+\epsilon)(\text{clust}_{X^{**}}(x^2{}_k),X) \ge \frac{c}{2}$. As $c \in \Big(0, \tilde{\delta}(x^2{}_k)\Big)$ is arbitrary, (13) follows.

We continue by proving (14): We set $c = (d+\epsilon)(\text{clust}_{X^{**}}(x^2{}_k),X)$and assume that $c > 0$ because otherwise (14) holds trivially. Let $\varepsilon \in (0,c)$be arbitrary and let $(x^2)^{**}$ be a weak* cluster point of the square sequence $(x^2{}_k)$in X^{**} such that $\text{d}((x^2)^{**},X) > c - \frac{\varepsilon}{2}$. Set $x^2 = P(x^2)^{**}$ and $x^2{}_s = (x^2)^{**} - x^2$where P denotes the projection on Xas in (2). Then $\text{d}((x^2)^{**},X) = \|x^2{}_s\|$.We claim that there is a square subsequence$\big(x^2{}_{k_n}\big)$such that

$$\sum_{i=1}^{n}(\alpha^2-\alpha+1)_i\big(x^2{}_{k_i}-x^2\big) \ge (c-(1-2^{-n})\varepsilon)\sum_{i=1}^{n}|(\alpha^2-\alpha+1)_i| \tag{16}$$

for all $n \in \mathbb{N}$ and all $((\alpha^2-\alpha+1)_i)_{i=1}^n$ in $\mathbb{R}^n$. This will be proved by G. Godefroy's 'ace of ♢ argument' [6,p. 170], cf. the proof of [6, Proposition IV.2.5]. Since $x^2{}_s$is a weak* cluster point of the square sequence$(x^2{}_k - x^2)$,there is k_1 such that $\big\|x^2{}_{k_1} - x^2\big\| > c - \frac{\varepsilon}{2}$which settles the first induction step.

Suppose we have constructed $x^2{}_{k_1}, \dots, x^2{}_{k_n}$. Let $((\alpha^2-\alpha+1)^l)_{l=1}^L$ be a finite sequence of elements of theunit sphere ofℓ_1^{n+1} such that $(\alpha^2-\alpha+1)_{n+1}^l \ne 0$ for all $l \in \{1, \dots, L\}$ and such that for each $(\alpha^2-\alpha+1)$in the unitsphere of $(\alpha^2-\alpha+1)_{n+1}^l$ there is an element $(\alpha^2-\alpha+1)^l$such that

$$\|(\alpha^2-\alpha+1)-(\alpha^2-\alpha+1)^l\|_{\ell_1^{n+1}} < \frac{\varepsilon}{2^{n+2}\sup\limits_k\|x^2{}_k\|}.$$

Let $l \in \{1, \dots, L\}$ be arbitrary. Then

$$\sum_{i=1}^{n}(\alpha^2-\alpha+1)_i^l\left(x^2{}_{k_i}-x^2\right)+(\alpha^2-\alpha+1)_{n+1}^l x^2{}_s$$

is a weak* cluster point ofthe square sequence

$$\left(\sum_{i=1}^{n}(\alpha^2-\alpha+1)_i^l\left(x^2{}_{k_i}-x^2\right)+(\alpha^2-\alpha+1)_{n+1}^l(x^2{}_k-x^2)\right)_{k=1}^{\infty}$$

and for its norm we have

$$\left\|\sum_{i=1}^{n}(\alpha^2-\alpha+1)_i^l\left(x^2{}_{k_i}-x^2\right)+(\alpha^2-\alpha+1)_{n+1}^l x^2{}_s\right\|$$

$$=\left\|\sum_{i=1}^{n}(\alpha^2-\alpha+1)_i^l\left(x^2{}_{k_i}-x^2\right)\right\|+\left\|(\alpha^2-\alpha+1)_{n+1}^l x^2{}_s\right\|$$

$$\geq (c-(1-2^{-n})\varepsilon)\sum_{i=1}^{n}\left|(\alpha^2-\alpha+1)_i^l\right|+\left|(\alpha^2-\alpha+1)_{n+1}^l\right|\left(c-\frac{\varepsilon}{2}\right)$$

$$>(c-(1-2^{-n})\varepsilon)\sum_{i=1}^{n+1}\left|(\alpha^2-\alpha+1)_i^l\right|=c-(1-2^{-n})\varepsilon.$$

It follows that there is $k_{n+1}>k_n$such that

$$\left\|\sum_{i=1}^{n+1}(\alpha^2-\alpha+1)_i^l\left(x^2{}_{k_i}-x^2\right)\right\|>c-(1-2^{-n})\varepsilon$$

for all $l\in\{1,\ldots,L\}$.By a straight forward calculation using the choice of the $(\alpha^2-\alpha+1)^l$and the triangle inequality we get that inequality (16), with $n+1$ instead of n, holds for all $(\alpha^2-\alpha+1)$in the unit sphere of$\ell_1^{n+1}$and hence for all elements of $\mathbb{R}^{n+1}$.

This finishes the construction. By Corollary (3.2.7)(i) we get

$$\left(\tilde{\delta}+\epsilon\right)\left(x^2{}_{k_n}-x^2\right)\geq 2(c-\varepsilon),$$

Hence clearly

$$\left(\tilde{\delta}+\epsilon\right)(x^2{}_k)\geq\left(\tilde{\delta}+\epsilon\right)\left(x^2{}_{k_n}\right)=\left(\tilde{\delta}+\epsilon\right)\left(x^2{}_{k_n}-x^2\right)\geq 2(c-\varepsilon).$$

As $\varepsilon\in(0,c)$is arbitrary, we get (14).

Finally, let us prove (15): We take any square subsequence $\left(x^2{}_{k_n}\right)$and observe that

$$2\mathrm{d}(\mathrm{clust}_{X^{**}}(x^2{}_k),X)\leq 2(d+\epsilon)\left(\mathrm{clust}_{X^{**}}\left(x^2{}_{k_n}\right),X\right)\leq\left(\tilde{\delta}+\epsilon\right)\left(x^2{}_{k_n}\right)$$

by (14). Then we can pass to the infimum over all $\left(x^2{}_{k_n}\right)$. This finishes the proof of the theorem.

Corollary (3.2.8)[257]. There is an L-embedded space X and square bounded sequence $(x^2{}_k)$in X such that$\tilde{\delta}(x^2{}_k)=2$ and $\mathrm{d}(\mathrm{clust}_{X^{**}}(x^2{}_k),X)=0$.

Proof.

For $n \in \mathbb{N}$ set $X_n = \ell_\infty^n$ and let X be the ℓ_1-sum of all the spaces$X_n, n \in \mathbb{N}$.

Then X is L-embedded by [6, Proposition IV.1.5].

Further, let $e_{n_1}, \dots, e_{n_n}$be the canonical basic vectors of $X_n$and let $({x^2}_k)$be the square sequence in X containing subsequently these basic vectors, i.e., the sequence

$$e_m^m, e_m^{m+1}, e_{m+1}^{m+1}, e_m^{m+2}, e_{m+1}^{m+2}, e_{m+2}^{m+2}, e_m^{m+3}, \dots, e_{m+3}^{m+3}, \dots.$$

Then we have $\tilde{\delta}({x^2}_k) = 2$ as each square subsequence of $({x^2}_k)$contains a further square subsequence isometrically equivalent to the canonical basis ofℓ_1.

It remains to show that $\mathrm{d}(\mathrm{clust}_{X^{**}}({x^2}_k), X) = 0$. To do so, it is enough to prove that 0 is a weak cluster point of the square sequence$({x^2}_k)$. To verify this, we fix $g^1, \dots, g^m \in X^*$and $\varepsilon > 0$. Let $K = \max\{\|g^1\|, \dots, \|g^m\|\}$.

The dual X^* can be canonically identified with the ℓ_∞-sum of the spaces $X_n^*, n \in \mathbb{N}$. Moreover, $X_n^*$is canonically isometric to ℓ_1^n. Thus each $g \in X^*$ can be viewed as a bounded sequence$(g_n)_{n\in\mathbb{N}}$, where $g_n = \left(g_{n,j}\right)_{j=1}^n \in \ell_1^n$for each $n \in \mathbb{N}$.

We find $N \in \mathbb{N}$ such that $\frac{K}{N} < \varepsilon$ and let $n \in \mathbb{N}$ be such that $n > mN$. Let $k \in \{1, \dots, m\}$be arbitrary. We have$\|g_n^k\| \le \|g^k\| \le K$. As $\|g_n^k\| = \sum_{j=1}^n \left|g_{n,j}^k\right|$,the set

$\left\{j \in \{1, \dots, n\}: \left|g_{n,j}^k\right| \ge \frac{K}{N}\right\}$ Has at most N elements. It follows that the set

$$\left\{j \in \{1, \dots, n\}: \left(\exists\, k \in \{1, \dots, m\}, \left|g_{n,j}^k\right| \ge \frac{K}{N}\right)\right\}$$

has at most mN elements. As $n > mN$, there is some $j \in \{1, \dots, n\}$ such that $\left|g_{n,j}^k\right| < \frac{K}{N} < \varepsilon$ for each $k \in \{1, \dots, m\}$. It means that $\left|g^k\left(e_j^n\right)\right| < \varepsilon$ for each $k \in \{1, \dots, m\}$.

Since $e_j^n$is an element of the square sequence$({x^2}_k)$, this completes the proof that 0 is in the weak closure of the square sequence, hence 0 is a weak cluster point (as the square sequence $({x^2}_k)$does not contain 0).

Corollary (3.2.9)[257]. There exists a separable Banach space X with the Schur property–in particular, X is weakly sequentially complete – which is 1-complemented in its bidual, such that there is no constant $C > 0$ satisfying

$$\mathrm{d}(\mathrm{clust}_{X^{**}}({x^2}_k), X) \le C\left(\tilde{\delta} + \epsilon\right)({x^2}_k)$$

for every square bounded sequence $({x^2}_k)$in X.

Proof

We recall that $\beta\mathbb{N}$ is the Čech–Stone compactification of $\mathbb{N}$ and $M(\beta\mathbb{N})$is the space of all signed Radon measures on $\beta\mathbb{N}$ considered as the dual ofℓ_∞.

Let us fix $(\alpha^2-\alpha+1)>0$ and consider the space

$$Y_{(\alpha^2-\alpha+1)} = (\ell_1,(\alpha^2-\alpha+1)\|\cdot\|_1) \oplus_1 (C[1,\omega],\|\cdot\|_\infty).$$

Here $\|\cdot\|_1$ denotes the usual norm on ℓ_1, ω is the first infinite ordinal, $C[1,\omega]$ stands for the spaceof all continuous functions on the ordinal interval $[1,\omega]$ and $\|\cdot\|_\infty$ is the standard supremum norm. Note that we have the following canonical identifications:

$$Y^*_{(\alpha^2-\alpha+1)} = \left(\ell_\infty, \frac{1}{(\alpha^2-\alpha+1)}\|\cdot\|_\infty\right) \oplus_\infty (\ell_1[1,\omega],\|\cdot\|_1), \quad \text{and}$$

$$Y^{**}_{(\alpha^2-\alpha+1)} = \left(M(\beta\mathbb{N}),(\alpha^2-\alpha+1)\|\cdot\|_{M(\beta\mathbb{N})}\right) \oplus_1 (\ell_\infty[1,\omega],\|\cdot\|_\infty).$$

For $k\in\mathbb{N}$, let ${x^2}_k = \left(e_k,\chi_{[k,\omega]}\right)\in Y_{(\alpha^2-\alpha+1)}$, where $e_k$denotes the k-th canonical basic vector in $\ell_1$and $\chi_{[k,\omega]}$ is the characteristic function of the interval $[k,\omega]$. Let $X_{(\alpha^2-\alpha+1)}$be the closed linear span ofthe set$\{{x^2}_k : k\in\mathbb{N}\}$. We observe that

$$X_{(\alpha^2-\alpha+1)} = \left\{\left((\eta_k),f\right)\in Y_{(\alpha^2-\alpha+1)} : f(n) = \sum_{k=1}^{n}\eta_k \text{ for all } n\in\mathbb{N}\right\}. \tag{17}$$

Indeed, the set on the right-hand side is a closed linear subspace of $Y_{(\alpha^2-\alpha+1)}$containing ${x^2}_k$for each $k\in\mathbb{N}$. This proves the inclusion '⊂'. To prove the converse one, let us take any point $\left((\eta_k),f\right)$in the set on the right-hand side. Since$(\eta_k)\in\ell_1$, we get

$$\left((\eta_k),f\right) = \sum_{k=1}^{\infty}\eta_k {x^2}_k \in X_{(\alpha^2-\alpha+1)}$$

as the series is absolutely convergent.

It follows that for each $\left((\eta_k),f\right)\in X_{(\alpha^2-\alpha+1)}$we have

$$(\alpha^2-\alpha+1)\|(\eta_k)\|_1 \le \left\|\left((\eta_k),f\right)\right\| \le \left((\alpha^2-\alpha+1)+1\right)\|(\eta_k)\|_1,$$

hence$X_{(\alpha^2-\alpha+1)}$is isomorphic to ℓ_1. More precisely, the projection on the first coordinate is an isomorphismonto ℓ_1. In particular, $X_{(\alpha^2-\alpha+1)}$has the Schur property (and thus it is weakly sequentially complete).

We further observe that $X^{**}_{(\alpha^2-\alpha+1)}$is canonically identified with the weak* closure of $X_{(\alpha^2-\alpha+1)}$in$Y^{**}_{(\alpha^2-\alpha+1)}$,thus

$$X^{**}_{(\alpha^2-\alpha+1)} = \{(\mu,f)\in M(\beta\mathbb{N})\times\ell_\infty[1,\omega]:$$

$$(\forall n \in \mathbb{N}: f(n) = \mu\{1,\dots,n\}) \text{and } f(\omega) = \mu(\beta\mathbb{N})\}. \quad (18)$$

Indeed, the set on the right-hand side is a weak* closed linear subspace of $Y^{**}_{(\alpha^2-\alpha+1)}$ containing $X_{(\alpha^2-\alpha+1)}$,which proves the inclusion '⊂'. To prove the converse one let us fix (μ, f)in the set on theright-hand side. Take a bounded net (u_τ)in $\ell_1$which weak* converges to μ. For each τ there isa unique $f_\tau \in C[1,\omega]$ such that$(u_\tau, f_\tau) \in X_{(\alpha^2-\alpha+1)}$. Then $(f_\tau)$is clearly a bounded net in $\ell_\infty[1,\omega]$.Moreover, we will show that (f_τ) weak* converges to$f$. Since the weak* topology on bounded sets coincides with the topology of point wise convergence, it suffices to show that $f_\tau$point wise converge to$f$. Indeed,

$$f_\tau(n) = \sum_{k=1}^{n} u_\tau(k) \to \mu\{1,\dots,n\} = f(n), \quad \text{for each } n \in \mathbb{N},$$

$$f_\tau(\omega) = \sum_{k=1}^{\infty} u_\tau(k) \to \mu(\beta\mathbb{N}) = f(\omega).$$

It follows that $X_{(\alpha^2-\alpha+1)}$is 1-complemented in its bidual. To show that we set

$$P(\mu, f) = \left((\mu\{k\}), f - \mu(\beta\mathbb{N}\backslash\mathbb{N}) \cdot \chi_{\{\omega\}}\right), (\mu, f) \in X^{**}_{(\alpha^2-\alpha+1)}.$$

Then P is a projection of $X^{**}_{(\alpha^2-\alpha+1)}$onto $X_{(\alpha^2-\alpha+1)}$of norm one. Indeed, if$(\mu, f) \in X_{(\alpha^2-\alpha+1)}$, then $\mu(\beta\mathbb{N}\backslash\mathbb{N}) = 0$ and hence $P(\mu, f) = (\mu, f)$. Further, by (9) and (10) we get that $P(\mu, f) \in X_{(\alpha^2-\alpha+1)}$foreach$(\mu, f) \in X^{**}_{(\alpha^2-\alpha+1)}$. Thus P is a projection onto $X_{(\alpha^2-\alpha+1)}$. To show it has norm one, it is enough to observe that, given$(\mu, f) \in X^{**}_{(\alpha^2-\alpha+1)}$, we have$\|(\mu\{k\})\|_{\ell_1} \le \|\mu\|$, and that $f - \mu(\beta\mathbb{N}\backslash\mathbb{N}) \cdot \chi_{\{\omega\}}$ isa continuous function on $[1,\omega]$ coinciding on $[1,\omega)$with $f$and so$\left\|f - \mu(\beta\mathbb{N}\backslash\mathbb{N}) \cdot \chi_{\{\omega\}}\right\|_\infty \le \|f\|_\infty$.

Further, for the square sequence$({x^2}_k)$, its weak* cluster points in $X^{**}_{(\alpha^2-\alpha+1)}$are equal to $(\varepsilon_t, \chi_{\{\omega\}})$: $t \in \beta\mathbb{N}\backslash\mathbb{N}$, where ε_t denotes the Dirac measure at a point $t \in \beta\mathbb{N}$.

We claim that [10], for our square sequence$({x^2}_k)$,we have

$$\mathrm{d}\left(\mathrm{clust}_{X^{**}_{(\alpha^2-\alpha+1)}}({x^2}_k), X_{(\alpha^2-\alpha+1)}\right) \ge \frac{1}{2} \text{and} \left(\tilde{\delta} + \epsilon\right)({x^2}_k) = 2(\alpha^2 - \alpha + 1). \qquad (19)$$

To see the first inequality, we use the fact that the distance of any weak* cluster point of $({x^2}_k)$ from $X_{(\alpha^2-\alpha+1)}$is at least $\mathrm{d}\left(\chi_{\{\omega\}}, C[1,\omega]\right) = \frac{1}{2}$. On the other hand, if $t, t' \in \beta\mathbb{N}\backslash\mathbb{N}$ are distinct, then

$$\left\|(\varepsilon_t, \chi_{\{\omega\}}) - (\varepsilon_{t'}, \chi_{\{\omega\}})\right\|_{X^{**}_{(\alpha^2-\alpha+1)}} = \|\varepsilon_t - \varepsilon_{t'}, 0\|_{X^{**}_{(\alpha^2-\alpha+1)}} = (\alpha^2 - \alpha + 1)\|\varepsilon_t - \varepsilon_{t'}\|_{M(\beta\mathbb{N})}$$

$$= 2(\alpha^2 - \alpha + 1).$$

This verifies (19).

Now we use the described procedure to construct the desired space X(see [10]).For $n \in \mathbb{N}$, let $(\alpha^2 - \alpha + 1)_n = \frac{1}{n}$and let $X_{\frac{1}{n}}$be the space constructed for $(\alpha^2 - \alpha + 1)_n$.

Let $X = \left(\sum_{n=1}^{\infty} X_{\frac{1}{n}}\right)_{\ell_1}$ be the ℓ_1-sum of the spaces $X_{\frac{1}{n}}$. We claim that X is the required space.

First, since each $X_{\frac{1}{n}}$has the Schur property, X, as their ℓ_1-sum, possesses this property aswell (this follows by a straight forward modification of the proof that $\ell_1$has the Schur property,see [2, Theorem 5.19]).Hence X is weakly sequentially complete.

Further, observe that

$$X^* = \left(\sum_{n=1}^{\infty} X^*_{\frac{1}{n}}\right)_{\ell_\infty} \text{ and } X^{**} \supset \left(\sum_{n=1}^{\infty} X^{**}_{\frac{1}{n}}\right)_{\ell_1}.$$

Note that the latter space is not equal to X^{**} but it is 1-complemented in X^{**}(cf. the proof of [6, Proposition IV.1.5]). Now it follows that X is 1-complemented in X^{**}.

Finally, fix $n \in \mathbb{N}$. We consider a square sequence $\hat{x}^2{}_k = \left(0, \dots, 0, \underset{x_k}{n\text{-th}}, 0, \dots\right)$, where the elements$x^2{}_k \in X_{\frac{1}{n}}, k \in \mathbb{N}$, are defined above. Let $y^2 = \left(0, \dots, 0, \underset{(\varepsilon_t, \chi_{\{\omega\}})}{n\text{-th}}, 0, \dots\right)$, where $t \in \beta\mathbb{N}\backslash\mathbb{N}$, be a weak* cluster point of $(\hat{x}^2{}_k)$in X^{**}.Then, for any

$$z^2 = (z^2(1), z^2(2), \dots) \in X, \|y^2 - z^2\|_{X^{**}} \geq \left\|(\varepsilon_t, \chi_{\{\omega\}}) - z^2(n)\right\|_{X^{**}_{\frac{1}{n}}} \geq \frac{1}{2}$$

By (19). Hence

$$\mathrm{d}(\mathrm{clust}_{X^{**}}(\hat{x}^2{}_k), X) \geq \frac{1}{2}.$$

On the other hand,

$$(\tilde{\delta} + \epsilon)(\hat{x}^2{}_k) = (\tilde{\delta} + \epsilon)(x^2{}_k) = \frac{2}{n},$$

again by (19). From this observation the conclusion follows.