Chapter 2

Subsymmtric Sequences and Minimal Subspaces

If a Banach space is saturated with subspaces with a Schauder basis, which embed into the linear span of any subsequence of their basis ,then it contains a minimal subspace.

Section (2.1): Minimal Spaces:

W. T. Gowers proved in [13] the celebrated dichotomy theorem concerning un- conditional basic sequences and hereditarily indecomposable spaces using Ramsey- type arguments. In [14] the reasoning was generalized and, as an application, a dichotomy concerning quasi-minimal spaces, i.e. those spaces for which any two infinite dimensional subspaces contain two further infinite dimensional subspaces which are isomorphic, was obtained. Putting these results together Gowers obtained the following "classification" theorem .

Theorem (2.1.1)[12]: ([14]). Let E be an infinite dimensional Banach space. Then E has an infinite dimensional subspace G with one of the following properties. The properties are mutually exclusive and all can and do occur:

(i) G is a hereditarily indecomposable space,

(ii) G has an unconditional basis and every isomorphism between block subspaces of G is a strictly singular perturbation of the restriction of some invertible diagonal operator on G,

(iii) G has an unconditional basis and is strictly quasi-minimal (i.e. is quasi- minimal and does not contain a minimal subspace),

(iv) G has an unconditional basis and is minimal.

In this section we show that every Banach space saturated with sub symmetric basic sequences contains a minimal subspace. It follows that the class (iii) can be restricted to strictly quasi-minimal spaces not containing sub symmetric basic sequences and one could split (iv) into minimal spaces with a sub symmetric basis or minimal spaces not containing a sub symmetric basic sequences. An example of a minimal space not containing any sub symmetric sequence is the dual to Tsirelson's space ([15], [16]), whereas Tsirelson's example is a strictly quasi-minimal space ([17]).

The method used here extends the technique applied in [18], which reflects the technique of Maurey's proof of Gowers' dichotomy theorem for unconditional se- quences and HI spaces ([19]). The same method also provides extensions in the class (i) by examining unconditional-like sequences introduced in [20] ([18]). Let E be a Banach space. Given a set $A \subseteq E$ by $\langle A \rangle$ denote the vector subspace spanned by A. We will denote by $\ominus$ the origin in the space E in order to distinguish it from the number zero.

For standard Banach space we refer to [15]. We say that two Banach spaces E_1, E_2 are C-isomorphic, for $C \geq 1$, if there is an isomorphism $T: E_1 \longrightarrow E_2$ satisfying $\frac{1}{C}\|x\| \leq \|Tx\| \leq C\|x\|$ for $x \in E_1$. We say that sequences $\{x_n\}_n, \{y_n\}_n$n of vectors of a Banach space are C-equivalent, for $C \geq 1$, if for any scalars $a_1, \dots, a_n$ and $n \in \mathbb{N}$ we have

$$\frac{1}{C}\|a_1y_1 + \cdots + a_ny_n\| \leq \|a_1x_1 + \cdots + a_nx_n\| \leq C\|a_1y_1 + \cdots + a_ny_n\|.$$

Assume now that E is a Banach space with a basis $\{e_n\}_{n=1}^{\infty}$

The support of a vector $x = \sum_{n=1}^{\infty} x_n e_n$is the set supp $x = \{n \in \mathbb{N}: x_n \neq 0\}$. We use the notation $x < y$ for vectors $x, y \in E$, if every element of supp x is smaller than every element of supp y. We write $x < A$ for a vector $x \in E$ and $A \subseteq E$, if $x < y$ for all $y \in A$, and so forth in this manner. A block sequence with respect to $\{e_n\}$ is any sequence of non-zero finitely supported vectors $x_1 < x_2 < \dots$ A block subspace is a closed subspace spanned by a block sequence. We will use letters $x, y, z, \dots$ to denote vectors of a Banach space, letters $x, y, z, \dots$ to denote finite block sequences and capital letters W, X, Y, Z, ... for infinite block sequences. For any finite block sequence x, by lxl we denote the length of x, i.e. the number of elements of x. Given any two block sequences $\{x_1, \dots, x_2\} < \{y_1, y_2, \dots\}$ let

$$\{x_1, \dots, x_1\} \cup \{y_1, y_2 \dots\} = \{x_1 \dots , x_n, y_1, y_2, \dots\}$$

For convenience in the reasoning presented in the next sections we will treat $\{\ominus\}$ as a block sequence and adopt the following convention: $|\{\ominus\}| = 0$, $\ominus < x$, for any $x \neq \ominus, \{\ominus\} \cup \{y_1, y_2, \dots\} = \{y_1, y_2, \dots\}$ for any block sequence $\{y_1, y_2, \dots\}$. While restricting our consideration to the family of block sequences we will use the following fact (see e.g. [15], l.a.12). Recall that a sequence $\{x_n\}_n$of vectors of a Banach space is called seminormalized if $0 < \inf \|x_n\|$ and $\sup \|x_n\| < \infty$.

Lemma (2.1.2)[12]: Let E be a Banach space with a basis $\{e_i\}_i$. Let $\{\{x_n\}_n \subseteq E$be a seminormalized sequence satisfying $\lim_{n\to\infty} e_i^*(x_n) = 0, i \in \mathbb{N}$, where $\{e_i^*\}_i$ is the sequence of biorthogonal functional of $\{e_i^*\}_i$.Then for any $\varepsilon > 0$ there is a block sequence $\{y_n\}_n$which is $(1 + \varepsilon)$-equivalent to some subsequence of the sequence $\{x_n\}_n$.

We present some more terminology and a stabilizing lemma. It reflects some combinatorial techniques used in [19], [21], [22] and others.

First we need some more notation. Let E be a Banach space with a basis $\{e_n\}$. Let Q denote the set of all vectors of the form $\sum_{i=1}^{n} a_i e_i$ for $n \in \mathbb{N}, \{a_i\}_i^n \subseteq \mathbb{Q}$where $\mathbb{Q}$ denotes rationals. Thus $\mathbb{Q}$ is a countable vector space over $\mathbb{Q}$ and $\mathbb{Q}$ is dense in E. Most of our arguments shall take place in $\mathbb{Q}$.

If Z and W are block sequences of $\{e_n\}$ in $\mathbb{Q}$ we write $Z \leq W$ if Z is a block sequence of W and $Z \preceq W$ if except for finitely many vectors, Z is a block sequence of W. $Z = W$ denotes $Z \preceq W$ and $Z \preceq W$. Given a block sequence W in $\mathbb{Q}$ let $\sum(W)$ (resp. $\sum_f(W)$ be the set of all infinite (resp. finite) block sequences of W in $\mathbb{Q}$. We let $\sum$ denote the set of all infinite block sequences of $\{e_n\}$ in $\mathbb{Q}$ and let $\sum_f$ denote the set of all finite block sequences of $\{e_n\}$ in $\mathbb{Q}$.

Lemma (2.1.3)[12]: ([18]). Let E be a Banach space with a basis $\{e_n\}$ and let $\sum$ be as defined above w.r.t. $\{e_n\}$. Let A be a countable set and let t: $\sum \to 2^A$ be monotone $\mathrm{w.r.t.} \preceq$ and inclusion, i.e

$$\forall Z, W \in \textstyle\sum, Z \preceq W \Rightarrow \mathcal{T}(Z) \subseteq \mathcal{T}(W)$$

$$\text{or } \forall\, Z, W \in \textstyle\sum, Z \preceq W \Rightarrow \mathcal{T}(Z) \supseteq \mathcal{T}(W).$$

Then there exists $W_0 \in \sum$ so that

$$\forall\, W \in \textstyle\sum(W_0), \mathcal{T}(W) = \mathcal{T}(W_0).$$

Proof. Without loss of generality we may assume that r is increasing (otherwise consider $\hat{\mathcal{T}}$(W)=A/$\mathcal{T}$ (W)). If the conclusion is false, then by transfinite induction and diagonalization we can construct $\{W_\alpha\}_{\alpha<\omega_1}$ so that if $\alpha < \beta$, then $W_\alpha < W_\beta$ and $\mathcal{T}(W_\alpha) \subsetneq (\mathcal{T}(W_\beta)$. But this is impossible since A is countable.

Definition (2.1.4)[12]: A Banach space E is called C-minimal, for $C \geq 1$ if any infinite dimensional closed subspace of E contains a subspace which is C-isomorphic to E.

A Banach space E is called minimal if any infinite dimensional closed subspace of E contains a subspace which is isomorphic to E.

Definition (2.1.5)[12]: Abasic sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ is called C-sub symmetric, for $C \geq 1$, if it is C-equivalent to any of its infinite subsequences.

A basic sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ is called sub symmetric, if it is C-sub symmetric, for some $C \geq 1$.

However if $\{e_n\}$ is a bounded sub symmetric sequence, then it is either equivalent to the unit vector basis of ℓ_1 by [23] or it is weak Cauchy and hence $\{e_1 - e_2, e_3 - e_4, \dots\}$ is sub symmetric and unconditional. Thus we prefer to use the definition above.

Lemma (2.1.6)[12]: Let E be a Banach space with a basis. If E contains a C-sub- symmetric basic sequence, for some constant $C \geq 1$, then for any $\delta > 0$ the space E contains a $(C + \delta)$-sub symmetric block sequence.

Proof. Let $\{e_i\}_i$be a basis for E. By $\{e_i^*\}_i$ denote the biorthogonal functionals for $\{e_i\}_i$. Let $\{x_n\}_n \subseteq E$ be a C-sub symmetric basic sequence, for $C \geq 1$. We can assume, picking a subsequence of $\{x_n\}_n$ if needed by diagonalization, that for some scalars $\{a_i\}_i$ we have $\lim_{n\to\infty} e_i^*(x_n) = a_i, i \in \mathbb{N}$, put $z_n = x_{2n} - x_{2n-1}$ for $n \in \mathbb{N}$. Then $\{z_n\}_n$ is clearly C-sub symmetric.

Fix $\delta > 0$. Pick $\eta > 0$ satisfying $(1+\eta)^2 2C < C + \delta$. Since $\lim_{n\to\infty} e_i^*(z_n) = 0$ $i \in \mathbb{N}$, by Lemma (2.1.2) there is a block sequence $\{y_n\}_n$ which is $(1+\eta)$-equivalent to some subsequence of $\{z_n\}_n$. Thus by the choice of η the sequence $\{y_n\}_n$ is $(C+\delta)$-sub symmetric.

We say that a Banach space is saturated with sequences of a given type, if every infinite dimensional subspace contains a sequence of this type. Now we present the main results:

Theorem (2.1.7)[12]: Let E be a Banach space saturated with C-subsymmetric basic sequences, for some constant $C \geq 1$. Then for any $\varepsilon > 0$, the space E contains a $(C^2 + \varepsilon)$ −minimal subspace.

Proof. We can assume that E is a Banach space with a basis. We will use below the notation introduced above. Assume that E is saturated with C-subsymmetric sequences, for some $C \geq 1$, and fix $\varepsilon > 0$.

Corollary (2.1.8)[12]: A Banach space saturated with subsymmetric basic sequences con- tains a minimal space.

Proof. We may assume that E is a Banach space with a basis. It sufficies to show that for some constant $C \geq 1$ there exists a block subspace so that all further block subspaces contain a C-subsymmetric block sequence. If not, one can construct a block sequence $\{x_i\}_{i=1}^{\infty}$ so that for all n no block sequence of $\{x_i\}_{i=n}^{\infty}$ is n-subsymmetric. But then no block sequence of $\{x_i\}_{i=1}^{\infty}$is subsymme- tric. Thus by Lemma (2.1.6), the block subspace spanned by $\{x_i\}_{i=1}^{\infty}$does not contain a subsymmetric basic sequence.

Notice that we proved above that a Banach space saturated with subsymme- tric sequences contains a "uniformly" minimal subspace, i.e. C-minimal for some constant $C \geq 1$.

Pick a scalar $\delta > 0$ satisfying $(C+\delta)^2(1+\delta) < c^2 + \varepsilon$. By Lemma (2.1.6) and the density of **Q** in E the space E is saturated with $(C+\delta)$ -subsymmetric block sequences from the family $\sum$. We shall produce $Z_0 \in \sum$ so that for any $W \in \sum(Z_0)$ there exists $Y \in \sum(W)$ which is $(C+\delta)^2$ -equivalent to Z_0. By the choice of 6 this will finish the proof of Theorem (2.1.7).From now on, unless otherwise stated, we work in **Q**. Thus e.g. (W) will denote $\langle W\rangle \cap$ **Q**. Put $c = C + \delta$.

Recall that a tree $\mathcal{T}$ on an arbitrary set A is a subset of the set $\bigcup_{n=}^{\infty} A^n$An such that $\{a_1, \ldots, a_n\} \in \mathcal{T}$ whenever $\{a_1,, \ldots, a_n, a_{n+}\} \in \mathcal{T}$.

A branch of a tree $\mathcal{T}$ is an infinite sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\{a_1, \ldots, a_n\} \in \mathcal{T}$ for any $n \in$ N N.

Now we introduce some important terminology. We call a tree $\mathcal{T}$ on **Q** a block tree if $\mathcal{T} \subseteq \sum_f$ and for any $x \in \mathcal{T}$ the set $\mathcal{T}(x) = \{x \in E : x \cup \{x\} \in \mathcal{T}\}$ contains an infinite block sequence in **Q**. Any branch of a block tree is a block sequence.

Moreover, since for any $x \in \mathcal{T}$ we have $\mathcal{T}(x) \neq \emptyset$, every element $x \in \mathcal{T}$ is a part of some branch of $\mathcal{T}$.

Definition (2.1.9)[12]: Given sequences $x, y \in \sum_f \cup\{\ominus\}, |x| \geq |y|$, a block sequence $Z \in \sum$ and a block tree $\mathcal{T}$ on Q we write $(x;\ Z) \sim (y;\ \mathcal{T})$ if $\mathcal{T} = \cup\{\mathcal{T}_X : X \in \sum(Z), X > x\}$, where $\{\mathcal{T}_X\}$ are block trees on Q satisfying the following conditions:

(i) for every block sequence $X \in \sum(Z), X > x$, and every branch Y of $\mathcal{T}_X$ we have that $Y > y$ and the sequences $x \cup X, y \cup Y$ are c-equivalent,

(ii) for any block sequences $X_1, X_2 \in \sum(Z), X_1 > x, X_2 > x$, and $n \in \mathbb{N} \cup \{0\}$, if $X_1 \cap E^n = X_2 \cap E^n$, then $\mathcal{T}_{X_1} \cap E^{n+|x|-|Y|} = \mathcal{T}_{X_2} \cap E^{2+|x|-|Y|}$, where $E^0 = \{\ominus\}$

This means that a tree of block sequences of Z beginning with a finite sequence x can be represented in T in a special way. In fact we will use the relation defined above only in the case when $|x| = |y|$ or $|x| = |y| + 1$.

Claim (2.1.10)[12]: Let $x, y \in \sum_f \cup \{\ominus\}, |X| \geq |y|, Z \in \sum$, and let $\mathcal{T}$ be a block tree on **Q**. Assume $(x; Z) \sim (y;\ \mathcal{T})$.

(i) Let $x_0 \in (Z), x < x_0$. Then there exists a block subtree $\mathcal{T}' \subseteq T$ which satisfies $(x \cup \{x_0\};\ Z) \sim (y; \mathcal{T}')$.

(ii) Let $|x| > |y|, y_0 \in \mathcal{T} \cap E$. Then $\mathcal{T}[y_0] = \{\{y_1, \dots, y_n\} : \{y_0, y_1, \dots, y_n\} \in \mathcal{T}\}$ is a block tree and $(x;\ Z) \sim (y \cup \{y_0\}; \mathcal{T}[y_0])$.

Proof. For the first case, in the situation as above define $\mathcal{T}'$ by putting

$$\mathcal{T}'_X = T_{\{x_0\} \cup X}, \text{for } X \in \sum(Z), X > x_0.$$

The second case is obvious by the definition of the relation $\sim$, since we can put $(\mathcal{T}[y_0])x = (\mathcal{T}_X)[y_0]$ for any $X \in \sum(Z)$.

Given $W \in \sum(Z)$ put

$$\mathcal{T}(W_1) = \left\{(x, y) \in \sum\nolimits_f \cup\{\ominus\}\right)^2 : |X| \geq |y|, \exists Z \in \sum, Z \preceq W,$$

$$\text{and } \exists \text{ a block tree } \mathcal{T} \text{ on } W \text{ with} (x; Z) \sim (y;\ \mathcal{T})\Bigg\}.$$

Take $W_1 \preceq W_2$ and a pair $(x, y) \in \mathcal{T}(W_1)$. Then there exists $Z \in \sum, Z \preceq W_1$ and a block tree $\mathcal{T}_1$ on W_1, such that $(x;\ Z) \sim (y;\ \mathcal{T}_1)$. Put $\mathcal{T}_2 = T_1 \cap \bigcup_{n \in \mathbb{N}} (W_2)^n$ (this means cutting off from $\mathcal{T}_1$ sequences containing vectors lying outside W_2). Then $\mathcal{T}_2$ is also a block tree (since $W_1 \preceq W_2$) satisfying $(x;\ Z) \sim (y;\ \mathcal{T}_2)$. One only has to realize that for any sequence $X \in \sum(Z)$, the tree $(\mathcal{T}_2)_X = T_X \cap \bigcup_{n \in \mathbb{N}} (W_2)^n$ satisfies the definition.

Therefore we have shown that the mapping r is monotone, i.e. if $W_1 \preceq W_2$, then $\mathcal{T}(W_1) \subseteq \mathcal{T}(W_2)$. Hence, on the basis of Lemma (2.1.3), there exists $W_0 \in \sum$ which is stabilizing for $\mathcal{T}$.

Claim (2.1.11)[12]: Let $(x, y) \in \tau(W_0), |x| > |y|$. Then for any $W \in \sum(W_0)$ there is a vector yo $\in$ (W) such that $(x, y\, |U\, \{y_0\}) \in \tau(W)$.

Proof. In the situation as above, by the stabilization property, for some $Z \in \sum(W)$ and a block tree $\mathcal{T}$ on W we have $(x;\ Z) \sim (y;\ \mathcal{T})$ and Claim (2.1.10) finishes the proof of Claim (2.1.11). Given $W \in \sum(W_0)$ let

$$\rho(W) = \Big\{(X, y) \in \left(\sum{}_f \cup \{\ominus\}\right)^2 : |X| \geq |y|, \exists\, Z \in \sum, Z \doteq W,$$

$$\text{and } \exists \text{ a block tree } \mathcal{T} \text{ on } W_0 \text{ with } (x;\ Z) \sim (y;\ \mathcal{T})\Big\}.$$

Let $W_1 \preceq W_2$ and $(x, y) \in \rho(W_1)$. There exists $Z_1 \in \sum, Z_1 \doteq W_1$ and a block tree T_1 on W_0 such that $(x;\ Z_1) \sim (y; \mathcal{T}_1)$ .Put

$$Z_2 = \langle Z_1 \rangle \cup W_2, \qquad \mathcal{T}_2 = \bigcup \{(\mathcal{T}_1)_X : X \in \sum(Z_2)\},$$

Then obviously $Z_2 \doteq W_2$ and $(x;\ Z_2) \sim (y;\ \mathcal{T}_2)$, hence $(x, y) \in \rho(W_2)$. Therefore the mapping ρ is monotone. Let $W_{00} \in \sum(W_0)$ be stabilizing for ρ, chosen on the basis of Lemma (2.1.3).

Claim (2.1.12)[12]: For any $W, Z \in \sum(W_{00})$ we have $\mathcal{T}(W) = \rho(Z)$.

Proof. By the stabilization property it is enough to prove that $\mathcal{T}(W_{00}) = \rho(W_{00})$. By definition and the stabilization property

$$\rho(W_{00}) \subseteq \mathcal{T}(W_0) = \mathcal{T}(W_{00}).$$

Now, if $(x, y) \in \mathcal{T}(W_{00})$, then $(x, y) \in \rho(W)$ for some $W \leq W_{00}$, hence again by the stabilization $(x, y) \in \rho(W_{00})$.

By the assumption and Lemma (2.1.6) there is a c-subsymmetric block sequence $Z_0 = \{z_n\}_{n=1}^{\infty} =\in \sum(W_{00})$.

Claim (2.1.13)[12]: $(\{\ominus\};\ Z_0) \sim (\{\ominus\};\ \sum_f(Z_0))$, in particular $(\{\ominus\}, \{\ominus\}) \in \mathcal{T}(W_{00})$

Proof. Take any block sequence X $\{z_n\}_{n=1}^{\infty} \in \sum(Z_0)$.Then

$$x_n = \sum_{i=i_n}^{i_{n+1}-1} a_i z_i, n \in \mathbb{N},$$

for some scalars $\{a_i\}_{i \in \mathbb{N}}$ and some sequence $\{i_n\}_{i \in \mathbb{N}} \subseteq \mathbb{N}$ let

$$\mathcal{T}_X = \left\{ \left\{ \sum_{i=i_1}^{i_2-1} a_i z_{\emptyset(i)}, \dots, \sum_{i=i_n}^{i_{n+1}-1} a_i z_{\emptyset(i)} \right\} \middle| \emptyset : \mathbb{N} \to \mathbb{N} \text{ stricyly increasing } n \in \mathbb{N} \right\}$$

Obviously every set $\mathcal{T}_X$ is a block tree. Moreover, $\sum_f(Z_0) = U\{\mathcal{T}_x : X \in \sum(Z_0)\}$ and, by the c-subsymmetry of the sequence $\{Z_n\}_n$, for any $X \in \sum(Z_0)$ every infinite branch of $\mathcal{T}_x$ is c-equivalent to X. The "uniqueness" condition is also satisfied.

We will show that for any block sequence $W \in \sum(Z_0)$ there exists $Y \in \sum(W)$ which is c^2-equivalent to Z_0 which will finish the proof of the Theorem.

Let $W \in \sum(Z_0)$. We will pick by induction block sequences $\{Z_{k_n}\}$ and $\{y_n\} \leq W$ such that $(Z_n, Y_n) \in \mathcal{T}(W)$ for $n \in \mathbb{N}$, where $Z_n = \{Z_{k_1}, \ldots Z_{k_n}\}$ and $y_n = \{y_1, \ldots, y_n\}$, $n \in \mathbb{N}$.This implies in particular that for any $n \in \mathbb{N}$ sequences $\{Z_{k_1}, \ldots Z_{k_n}\}$ and $\{y_1, \ldots, y_n\}$ are c-equivalent, thus also sequences $\{Z_{k_n}\}_{n \in \mathbb{N}}$ and $\{y_{k_n}\}_{n \in \mathbb{N}}$are c-equivalent. By the c-subsymmetry of the sequence $\{Z_n\}_n$the sequences $\{Z_n\}_n$and $\{y_n\}$n are c^2-equivalent, hence Z_0 works.

Put $k_1 = 1$. By Claims (2.1.13) and 1 $(z_1, \{\ominus\}) \in \mathcal{T}(W_0) = \mathcal{T}(W)$. By Claim (2.1.11) there is a vector $y_1 \in W$ such that $(z_1, y_1) \in \mathcal{T}(W)$.

Assume now that we have picked vectors $Z_{k_1}, \ldots Z_{k_n}$ and $y_1, \ldots, y_n \in (W)$ such that $(z_n, y_n) \in \mathcal{T}(W)$. By Claim (2.1.12) $(Z_n, y_n) \in \rho(Z_0)$. Therefore for some $Z \doteq Z_0$ there is a tree T on W_0 such that $(Z_n; Z) \sim ((y_n; \mathcal{T})$. Let k_{n+1} be such that $Z_{k_{n+1}} > Z_{k_n}$ and $k_{n+1} \in (Z)$. Then by Claim (2.1.10) $(z_{n+1}, y_n) \in \mathcal{T}(W_0)$. Hence by Claim (2.1.11) there is a vector y_{n+1} c (W) such that $(z_{n+1}, y_{n+1}) \in \mathcal{T}(W)$, which finishes the inductive step and the proof of Theorem (2.1.7).

We should point out that there exist minimal spaces with a subsymmetric basis which do not contain any isomorph of c_0 or any ℓ_ρ $(1 \ p \leq \infty)$. One such space is due to Th. Schlumprecht [24]. We do not know if this is the case for symmetric bases.

Section (2.2): Isomorphically Homogeneous Sequences in A Banach Space :

The starting point is the solution to the Homogeneous Banach Space Problem given by W. T. Gowers [26] and R. Komorowski - N. Tomczak-Jaegermann [27]. A Banach space is said to be homogeneous if it is isomorphic to its infinite-dimensional closed subspaces; these proved that a homogeneous Banach space must be isomorphic to ℓ_2.

Gowers proved that any Banach space must either have a subspace with an unconditional basis or a hereditarily indecomposable subspace. By properties of hereditarily indecomposable Banach spaces, it follows that a homogeneous Banach space must have an unconditional basis (see [26]). Komorowski and Tomczak-Jaegermann proved that a Banach space with an unconditional basis must contain a copy of 12 or a subspace with a successive finite-dimensional decomposition on the basis (2-dimensional if the space has finite cotype) which does not have an unconditional basis. It follows that a homogeneous Banach space must be isomorphic to ℓ_2.

While Gowers' dichotomy theorem is based on a general Ramsey-type theorem for block-sequences in a Banach space with a Schauder basis, the subspace with a finite-dimensional decomposition constructed in Komorowski and Tomczak- Jaegermann's theorem can never be isomorphic to a block-subspace. If one restricts one's attention to block-subspaces, the standard homogeneous examples become the sequence spaces $c_0$and $\ell_p, 1 \leq p < +\infty$, with their canonical bases; these spaces are well-known to be isomorphic to their block-subspaces. Furthermore, there are classical theorems which characterize c_0 and $\ell_p, 1 \leq p < +\infty$, by means of their block-subspaces. An instance of this is Zippin's theorem ([28]): a normalized basic sequence is perfectly homogeneous (i.e. equivalent to all its normalized block-sequences) if and only if it is equivalent to the canonical basis of co or some ℓ_p. See also [28].

So it is very natural to ask what can be said on the subject of (isomorphic) homogeneity restricted to block-subspaces of a given Banach space with a Schauder basis:

Question (2.2.1)[25]: If a Banach space X with a Schauder basis $(e_n)_{n\in\mathbb{N}}$~ is isomorphic to its block-subspaces, does it follow that X is isomorphic to C_0 or $\ell_p, 1 \leq p < +\infty$?

Note that such a basis is not necessarily equivalent to the canonical basis of c_0 or some gp; take ℓ_2 with a conditional basis.

In the other direction, if a Banach space is not homogeneous, then how many non-isomorphic subspaces must it contain? This question may be asked in the setting of the classification of analytic equivalence relations on Polish spaces by Borel reducibility. This area of research originated from the works of H. Friedman and L. Stanley [29] and independently from the works of L. A. Harrington, A. S. Kechris and A. Louveau [30], and may be thought of as an extension of the notion of cardinality in terms of complexity, when one compares equivalence relations.

If R (resp. S) is an equivalence relation on a Polish space E (resp. F), then it is said that (E,R) is Borel reducible to (F,S) if there exists a Borel map $f: E \rightarrow F$ such that $\forall x, y \in E, xRy \Leftrightarrow f(x)Sf(y)$. An important equivalence relation is the relation E_0: it is defined on $2^\omega$by

$$\alpha E_0 \beta \; \exists m \in \mathbb{N} \; \forall n \geq m, \alpha(n) = \beta(n).$$

The relation E_0 is a Borel equivalence relation with 2^ω classes and which, furthermore, does not admit a Borel classification by real numbers, that is, there is no Borel map f from 2^ω into $\mathbb{R}$ (equivalently, into a Polish space), such that $\alpha E_0 \beta \Leftrightarrow f(\alpha) = f(\beta)$; such a relation is said to be non-smooth. In fact E_0 is the$\leq_B$ minimal non-smooth Borel equivalence relation [30].

There is a natural way to equip the set of subspaces of a Banach space X with a Borel structure, and the relation of isomorphism is analytic in this setting [31]. The relation E_0 appears to be a natural threshold for results about the relation of isomorphism between separable Banach spaces [32], [33], [34], [35]. ABanach space X is said to be ergodie if E_0 is Borel reducible to isomorphism between subspaces of X; in particular, an ergodic Banach space has continuum many non-isomorphic subspaces, and isomorphism between its subspaees is nonsmooth. The results in [31], [32], [33], [34], [35] suggest that every Banach space non-isomorphic to 62 should be ergodic, and we also refer to these articles for an introduction to the classification of analytic equivalence relations on Polish, spaces by Borel reducibility, and more specifically to complexity of isomorphism+ between Banach spaces.

Restricting our attention to block-subspaces, the natural question becomes the following:

Question (2.2.2)[25]: If X is a Banach space with a Sehauder basis, is it true that either X is isomorphic to its block-subspaces or E_0 is Bore1 reducible to isomorphism between the block-subspaces of X ?

Let us provide some ground for this conjecture by noting that, if we replace isomorphism by equivalence of the corresponding basic sequences, it is completely solved by a result of the author and C. Rosendal using the theorem of Zippin: if X is a Banach space with a normalized basis $(e_n)_{n\in\mathbb{N}}$ then either $(e_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of c_0 or $\ell_p, 1 \leq p < +\infty,$ or E_0 is Borel reducible to equivalence between normalized block-sequences of X.

A. M. Pelczar has proved that a Banach space which is saturated with subsymmetric sequences contains a minimal subspace [36]. The aim is to prove the isomorphic counterpart of her theorem. The natural generalization is to replace subsymmetric sequences by sequences which are isomorphieally homogeneous, i.e. such that all subspaces spanned by subsequences are isomorphic. However, it will be enough and more natural with our methods to consider embeddings instead of isomorphisms, which leads us to a stronger result: if a Banach space X is saturated with basic sequences whose linear span embeds in the linear span of any subsequenee, then X contains a minimal subspace .

In combination with a result of C. Rosendal [35], it follows that if X is a Banach space with a Schauder basis, then either E_0 is Borel reducible to isomorphism between block-subspaces of X, or X contains a block-subspace which is block-minimal.This improves a result of [34] which states that a Banach space contains continuum many non-isomorphic subspaces or a minimal subspace [37].

Combinatorial methods about subsequences or about block-sequences are often used in Banach space theory; but they are less frequently combined. Hopefully, our methods could lead to other applications in that area.

Let X be a Banach space with a Schauder basi $(e_n)_{n\in\mathbb{N}}$. If $(x_n)_{n\in J}$is a finite or infinite block-sequence of X then $(x_n)_{n\in J}$ will stand for its closed linear span. We shall also use some standard notation about finitely supported vectors on $(e_n)_{n\in\mathbb{N}}$, for example, we shall write $x < y$ and say that x and y are successive when $\max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y))$. The set of normalized block-sequences in X, i.e. sequences of successive blocks in X, is denoted bb(X).

Let Q(X) be the set of non-zero blocks of the basis (i.e. finitely supported $\mathbb{Q} + i\mathbb{Q}$ if we deal with a complex Banach space). We denote by $bb_{\mathbb{Q}}$ (X) the set of block-bases of vectors in $\mathbb{Q}(X)$, and by $G_{\mathbb{Q}}(X)$ the corresponding set of block-subspaces of X.

The notation $bb_{\mathbb{Q}}^{<\omega}(X)$ (resp. $bb_{\mathbb{Q}}^{n}(X)$) will be used for the set of finite (resp. $(e_n)_{n\in\mathbb{N}}$ length n) block-sequences with vectors in $\mathbb{Q}(X)$; the set of finite block-subspaces generated by block-sequences in $bb_{\mathbb{Q}}^{<\omega}(X)$ will be denoted by Fin $_{\mathbb{Q}}(X)$.

We shall consider $bb_{\mathbb{Q}}(X)$ as a topological space, when equipped with the product of the discrete topology on $\mathbb{Q}X)$. As $\mathbb{Q}(X)$ is countable, this turns $bb_{\mathbb{Q}}(X)$ into a Polish space. Likewise, $\mathbb{Q}(X)^{\omega}$ is a Polish space.

For a finite block sequence $\tilde{x} = (x_1, \ldots, x_n) \in bb_{\mathbb{Q}}^{<\omega}(X)$, we denote by $N_{\mathbb{Q}}(\tilde{x})$ the set of elements of $bb_{\mathbb{Q}}^{<\omega}(X)$ whose first n vectors are $(x_1, \ldots, x_n)$; this is the basic open set associated to $\tilde{x}$.

The set $[w]^{\omega}$ is the set of increasing sequences of integers, which we sometimes identify with infinite subsets of ω. It is equipped with the product of the discrete topology on ω. The set $[\omega]^{<\omega}$ is the set of finite increasing sequences of integers. If $a = (a_1, \ldots, a_k) \in [\omega]^{<\omega}$ then [a] stands for the basic open set associated to a, that is the set of increasing sequences of integers of the form $\{a_1, \ldots, a_n, n_{k+1} n_{k+2}, \ldots\}$ If $\in [\omega]^{\omega}$, then $[A]^{\omega}$ is the set of increasing sequences of integers in A (where A is seen as a subset of ω).

We recall that two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be equivalent if the map $T\colon (x_n)_{n\in\mathbb{N}} \to (y_n)_{n\in\mathbb{N}}$ defined by $T(x_n) = y_n$ for all $n \in \mathbb{N}$ is an isomorphism. For $C \geq 1$, they are C-equivalent if $\|T\|\,\|T^{-1}\| \leq C$. A basic sequence is said to be (C-)subsymmetric if it is (C-)equivalent to all its subsequences.

We shall sometimes use "standard perturbation arguments" without being explicit. This expression will refer to one of the following well-known facts about block-subspaces of a Banach space X with a Schauder basis. Any basic sequence (resp. block-basic sequence) in X is an arbitrarily small perturbation of a basic sequence in $\mathbb{Q}(X)^{\omega}$ (resp. block-basic sequence in $bb_{\mathbb{Q}}(X)$), and in particular is $1+\epsilon$-equivalent to it, for arbitrarily small $\epsilon > 0$. Any subspace of X has a subspace which is an arbitrarily small perturbation of a block-subspace of X (and in particular, with $1 + \epsilon$-equivalence of the corresponding bases, for arbitrarily small $\epsilon > 0$). If X is reflexive, then any basic sequence in X has a subsequence which is a perturbation of a block-sequence of X (and, in particular, is $1 + \epsilon$-equivalent to it, for arbitrarily small $\epsilon > 0$).

We shall also use the fact that any Banach space contains a basic sequence.

Finally, we recall the definition of unconditionality for basic sequences: a Schauder basis $(e_n)_{n\in\mathbb{N}}$ of a Banach space X is said to be unconditional if there is some $C \geq 1$ such that for any $I \subset \mathbb{N}$, any norm 1 vector $x = \sum_{n\in\mathbb{N}} a_n e_n$, $\|\sum_{n\in\mathbb{N}} a_n e_n\| \leq C$.

We recall different notions of minimality for Banach spaces. A Banach space X is said to be (C-) minimal if it (C-)embeds into any of its subspaces. If X has a Schauder basis $(e_n)_{n\in\mathbb{N}}$, then it is said to be block-minimal if every block-subspace of X has a further block-subspace which is isomorphic to X, and is said to be equivalence block-minimal if every block-sequence of $(e_n)_{n\in\mathbb{N}}$ has a further block-sequence which is equivalent to $(x_n)_{n\in\mathbb{N}}$.

The theorem of Pelczar [36] states that a Banach space which is saturated with subsymmetric sequences must contain an equivalence block-minimal subspace with a Schauder basis.

A basic sequence embeds (resp. C-embeds) into its subsequences if its linear span embeds (resp. C-embeds) into the linear span of any of its subsequences. We may now state our isomorphic version of Pelczar's theorem:

Theorem (2.2.3)[25]: A Banach space which is saturated with basic sequences which embed into their subsequences contains a minimal subspace.

We first prove two uniformity lemmas. For $N \in \mathbb{N}$ let $d_c(N)$ denote an integer such that if X is a Banach space with a basis $(e_n)_{n\in\mathbb{N}}$with basis constant c, and $((x_n)_{n\in\mathbb{N}}$and $((y_n)_{n\in\mathbb{N}}$ are normalized block-basic sequences of X such that $x_n : y_n$ for all $n > N$, then $(x_n)_{n\in\mathbb{N}}$and $(y_n)_{n\in\mathbb{N}}$ are $d_c(N)$ $-$equivalent. We leave as an exercise to the reader to check that such an integer exists.

Lemma (2.2.4)[25]: Let $(x_n)_{n\in\mathbb{N}}$be a basic sequence in a Banach space which embeds into its subsequences. Then there exists $C \geq 1$ and a subsequence of $(x_n)_{n\in\mathbb{N}}$ which C-embeds into its subsequences.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a basic sequence which embeds into its subsequences, and let c be its basis constant. It is clearly enough to find a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $C \geq 1$ such that $(x_n)_{n\in\mathbb{N}}$ C-embeds into any subsequence of $(y_n)_{n\in\mathbb{N}}$ (with the obvious definition). Assuming the conclusion is false, we construct by induction a sequence of subsequences $(x_n^k)_{n\in\mathbb{N}}$of $(x_n)_{n\in\mathbb{N}}$, such that for all k $\mathbb{N}$, $(x_n^k)_{n\in\mathbb{N}}$ is a subsequence of $(x_n^{k-1})_{n\in\mathbb{N}}$n ew such that $(x_n)_{n\in\mathbb{N}}$N does not $kd_c(k)$-embed into $(x_n^k)_{n\in\mathbb{N}}$.

Let $(y_n)_{n\in\mathbb{N}}$be the diagonal subsequence of $(x_n)_{n\in\mathbb{N}}$ defined by $y_n = x_n^n$ Then $(x_n)_{n\in\mathbb{N}}$ does not $kd_c(k)$-embed into $x_1^k, \ldots, x_{k-1}^k$, y_k , $y_k + 1, \ldots$) So $(x_n)_{n\in\mathbb{N}}$does not k-embed in $(y_n)_{n\in\mathbb{N}}$. Now k was arbitrary, so this contradicts our hypothesis.

Lemma (2.2.5)[25]: Let X be a Banach space which is saturated with basic sequences which embed into their subsequences. Then there exists a subspace Y of X with a Schauder basis, and a constant $C \geq 1$ such that every block-sequence of Y (resp. in $bb_{\mathbb{Q}}(Y)$) has a further block-sequence (resp. in $bb_{\mathbb{Q}}(Y)$) which C-embeds into its subsequences.

Proof. A space which is spanned by a basic sequence which embeds into its subsequences must in particular embed into its hyperplanes, so is isomorphic to a proper subspace; by [38] Corollary 19 and Theorem 21, such a space cannot be hereditarily indecomposable. Thus X does not contain a hereditarily indecomposable subspace; otherwise, some further subspace would be hereditarily indecomposable (since the H.I. property is hereditary) and spanned by a basic sequence which embeds in its subsequences (by the saturation property).

So by Cowers' dichotomy theorem, we may assume X has an unconditional basis (let c be its basis constant). If c_0 or ℓ_1 embeds into X then we are done, so by the classical theorem of James, we may assume X is reflexive. Thus by standard perturbation arguments, every normalized block-sequence in X has a further normalized block-sequence in X which embeds into its subsequences (here we also used the obvious fact that if a basic sequence $(X_n)_{n\in\mathbb{N}}$ embeds into its subsequences, then so does any subsequence of $(X_n)_{n\in\mathbb{N}}$.

Assuming the conclusion is false, we construct by induction a sequence of block-sequences $(x_n^k)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$, such that for all $k \in \mathrm{N}$, $(x_n^k)_{n\in\mathbb{N}}$ is a block-sequence of $(x_n^{k-1})_{n\in\mathbb{N}}$ such that no block-sequence of $(x_n^k)_{n\in\mathbb{N}}$ $kd_c(k)^2$-embeds into its subsequenees.

Let $(y_n)_{n\in\mathbb{N}}$ be the diagonal block-sequence of $(x_n)_{n\in\mathbb{N}}$ defined by $y_n = x_n^n$, and let $(z_n)_{n\in\mathbb{N}}$ be an arbitrary block-sequence of $(y_n)_{n\in\mathbb{N}}$.

Then $(x_1^k, \ldots, x_{k-1}^k, z_k, z_{k+1}, \ldots)$ is a block-sequence of $(x_n^k)_{n\in\mathbb{N}}$ and so does not $kd_c(k)^2$-embed into its subsequences. So $(z_n)_{n\in\mathbb{N}}$ does not k-embed into its subsequences - this is true as well of its subsequences. As k was arbitrary, we deduce from Lemma (2.2.4) that $(z_n)_{n\in\mathbb{N}}$ does not embed into its subsequences. As $(z_n)_{n\in\mathbb{N}}$ was an arbitrary block-sequence of $(y_n)_{n\in\mathbb{N}}$, this contradicts our hypothesis.

By standard perturbation arguments, we deduce from this the stated result with block-sequences in $bb_{\mathbb{Q}}(Y)$.

Recall that $\mathbb{Q}(X)^\omega$ is equipped with the product of the discrete topology on $\mathbb{Q}(X)$, which turns it into a Polish space.

Definition (2.2.6)[25]: Let X be a Banach space with a Schauder basis, and let $(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}(X)^\omega$. We shall say that $(x_n)_{n\in\mathbb{N}}$continuously embeds (resp. C-continuously embeds) into its subsequences if there exists a continuous map $\emptyset[w]^{\omega} \rightarrow \mathbb{Q}(X)^{\omega}$ such for all $A \in [w]^{\omega}, \emptyset(A)$, is a sequence of vectors in $(z_n)_{n\in A} \cap \mathbb{Q}$ (X) which is equivalent (resp. C-equivalent) to $(x_n)_{n\in\mathbb{N}}$.

This definition depends on the Banach space X in which we pick the basic sequence $(x_n)_{n\in\mathbb{N}}$; this will not cause us any problem, as it will always be clear which is the underlying space X.

The interest of this notion stems from the following lemma, which was essentially obtained by Rosendal as part of the proof of [35], Theorem 11. To prove it, we shall need the following fact, which is well-known to descriptive set theoricians. The algebra $\sigma(\sum_1^1)$ is the σ -algebra generated by analytic sets.For any $\sigma(\sum_1^1)$ -measurable function from $[\omega]^{\omega}$into a metric space, there exists $B \in \omega$ such that the restriction of f to $[B]^{\omega}$is continuous.

Indeed, by Silver's Theorem 21.9 in [39], any analytic set in $[\omega]^{\omega}$is completely Ramsey, and so any $\sigma(\sum_1^1)$ set in $[\omega]^{\omega}$ is (completely) Ramsey as well (use, for example, [39] Theorem 19.14). One concludes using the proof of ([40] Theorem 9.10) which only uses the Ramsey-measurability of the function.

Lemma (2.2.7)[25]: Let X be a Banach space with a Schauder basis, let $(x_n)_{n\in\mathbb{N}} n \in bb_{\mathbb{Q}}(X)$ be a block-sequence which C-embeds into its subsequences, and let ε be positive. Then some subsequence of$(x_n)_{n\in\mathbb{N}}$ $C + \varepsilon$-continuously embeds into its subsequences.

Proof. By standard perturbation arguments, we may find for each $A \in [w]^{\omega}$ a sequence $(y_n)_{n\in\mathbb{N}} \in \mathbb{Q}(X)^{\omega}$ such that $y_n \in [x_k]_{k\in A}$ for all $n \in \mathbb{N}$, and such that the basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$are C + e-equivalent. The set $P \subset [w]^{\omega} \times \mathbb{Q}(X)^{\omega}$ of couples $(A, (y_n))$ with this property is Borel (even closed), so by the Jankov-von Neumann Uniformization Theorem (Theorem 18.1 in [39]), there exists a C-measurable selector $f: [w]^{\omega} \rightarrow \mathbb{Q}(X)^{\omega}$ for P. By the fact before this lemma, there exists $B \in [w]^{\omega}$ such that the restriction of f to $[B]^{\omega}$ is continuous. Write $B = (b_k)_{k\in\mathbb{N}}$ where $(b_k)_k$ is increasing. By composing f with the obviously continuous maps $\Psi B: [w]^{\omega} \rightarrow [B]^{\omega}$ defined by $\Psi B((n_k)_{n\in\mathbb{N}}) = (b_{n_k})_{n\in\mathbb{N}}$, and $\mu B: \mathbb{Q}(X)^{\omega} \rightarrow \mathbb{Q}(X)^{\omega}$, defined by $\mu B((n_k)_{n\in\mathbb{N}} = (y_{n_k})_{n\in\mathbb{N}}$, we obtain a continuous map $\emptyset\ [w]^{\omega} \rightarrow \mathbb{Q}(X)^{\omega}$ which indicates that $(y_{n_k})_{n\in\mathbb{N}} C + \varepsilon$ e-continuously embeds into its subsequences.

We now start the proof of Theorem (2.2.3). So we consider a Banach space X which is saturated with basic sequences which embed into their subsequences and wish to find a minimal subspace in X.

By Lemma (2.2.5) and Lemma (2.2.7), we may assume that X is a Banach space with a Schauder basis and that there exists $C \geq 1$ such that every block-sequence in $bb_{\mathbb{Q}}(X)$ has a further block-sequence in $bb_{\mathbb{Q}}(X)$ which C-continuously embed into its subsequences.

For the rest of the proof X and $C \geq 1$ are fixed with this property. Recall that the set of block-subspaces of X which are generated by block-sequences in $bb_{\mathbb{Q}}(X)$ is denoted by $G_{\mathbb{Q}}(X)$; the set of finite block-subspaces which are generated by block-sequences in $bb_{\mathbb{Q}}^{<\omega}$ (X) is denoted by $Fin_{\mathbb{Q}}(X)$. If $n \in \mathbb{N}$ and $F \in Fin_{\mathbb{Q}}(X)$, we write $n \leq F$ to mean that $n \leq \min((supp(x))$ for all $x \in F$.

We first express the notion of continuous embedding in terms of a game. For $L = (l_{n_k})_{n\in\mathbb{N}}$ with $(l_{n_k})_{n\in\mathbb{N}} \in bb_{\mathbb{Q}}(X)$, we define an "asymptotic" game H_L as follows. A k-th move for Player 1 is some $n_k \in \mathbb{N}$. A k-th move for Player 2 is some $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$, with $n_k \leq F_k \subset L$ and $y_k \in \sum_{j=1}^{k} F_j$.

PlayerII wins the game H_L if $(y)_{n\in\mathbb{N}}$ is C-equivalent to $(l_n)_{n\in\mathbb{N}}$.

We claim the following:

Lemma (2.2.8)[25]: Let X be a Banach space with a Schauder basis, and let $(l_n)_{n\in\mathbb{N}} \in bb_{\mathbb{Q}}(X)$ be a block-sequence which C-continuously embeds into its subsequences. Let $L = (l_n)_{\in\mathbb{N}}$. Then Player 2 has a winning strategy in the game H_L.

Proof. Let $\emptyset$ be the continuous map in Definition (2.2.6). We describe a winning strategy for Player II by induction.

We assume that Player I's moves $(n_i)_{i\leq k-1}$and that the k-1 first moves prescribed by the winning strategy for Player II were $(F_i, y_i)_{i\leq k-1}$, with Fi of the form $[l_{n_i}, \dots, l_{m_i}], n_i \leq m_i$, for all $i \leq k-1$; letting

$$a_{k-1} = [n_1, m_1] U .. U [n_k - 1, n_k - 1] \in [\omega]^{<\omega},$$

we also assume that $\emptyset([a_k - 1]) \subset N_{\mathbb{Q}}(y_1, \dots y_k - 1)$. We now describe the k-th move of the winning strategy for Player II.

Let n_k be a k-th move for Player I. We may clearly assume that $n_k > m_{k-1}$. Let $A_k = \cup_{i\leq k-1} [n_i, m_i] U [n_k, +\infty) \in [w]^{\omega}$. The sequence $\emptyset$ is of the form $(y_1, \dots, y_{k-1}, y_k, z_{k-1} \dots)$ for some $y_k, Z_{k-1}, \dots$ in $\mathbb{Q}(X)$. By continuity of $\emptyset$ in A_k there exists $m_k > n_k$ such that, if $a_k = [n_1, m_1] U \dots U [n_k m_k] \in [\omega]^{\omega}$, then $\emptyset([a_k]) \subset N_{\mathbb{Q}}(y_1, \dots, y_k)$. We may assume that $\max(supp(l_{m_k})) \geq k \max(supp(y_k))$so as $y_k \in [l_i] i \in A_k$, we have that $y_k \in \bigoplus_{j=1}^{k} [l_i]_{i\in[n_j, m_j]}$. So $(F_k, y_k) = ([l_{nk} \dots l_{mk}], y_k)$ is an admissible k-th move for Player 2 for which the induction hypotheses are satisfied.

Repeating this by induction we obtain a sequence $(y_n)_{n\in\mathbb{N}}$ which is equal to $\emptyset(A)$ where $A = U_{k\in N}[n_k, m_k]$, and so which is, in particular, C-equivalent to $(l_n)_{n\in\mathbb{N}}$.

Definition (2.2.9)[25]: Given L,M two block-subspaces in $G_{\mathbb{Q}}(X)$, define the game $G_{L,M}$ as follows. A k-th move for Player I is some $(x_n, n_k) \in \mathbb{Q}(X) \times \mathbb{N}$, with $x_k \in L$, and $x_k > x_{k-1}$ if $k \geq 2$. A k-th move for Player II is some $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$ with $n_k \geq F_k \subset M$ and $y_k \in F_1 \oplus \dots \oplus F_k$ for all $k \in N$.

$$\begin{array}{llll} & \text{I:} \quad x_1, n_1 & x_2, n_2 & \ldots\ldots \\ G_{L,M} & & & \\ & \text{II:} \quad F_1, y_1 & F_2, y_2 & \ldots\ldots \end{array}$$

Player II wins GL,M if $(y_n)_{n\in\mathbb{N}}$ is C-equivalent to $(x_n)_{n\in\mathbb{N}}$.

The following easy fact will be needed in the next lemma: if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are C-equivalent basic sequences, then any block-sequence of $(x_n)_{n\in\mathbb{N}}$of the form $\left(\sum_{i\in 1_n}\lambda_i x_i\right)$is C-equivalent to $\left(\sum_{i\in 1_n}\lambda_i y_i\right)_{n\in\mathbb{N}}$.

Lemma (2.2.10)[25]: Assume $(l_n)_{n\in\mathbb{N}}$ is a block-sequence in $bb_{\mathbb{Q}}(X)$ which C-continuously embeds into its subsequences, and let $L = [l_n, n \in \mathbb{N}]$. Then Player II has a winning strategy in the game $G_{L,L}$.

Proof. Assume the first move of Player I was (x_1, n_1); write $x_1 = \left(\sum_{i\in 1_n}\lambda_i x_i\right)$. Letting in the game H_L Player I play the integer n_1, k_1 times, the winning strategy of Lemma (2.2.8) provides moves $(F_1^1, z_1), \ldots, (F_1^{k_1}, z_{k_1})$ for Player II in that game. We let $y_1 = \sum_{j\le k_1}\lambda_i x_i$, and $F_1 = \sum_j^{k_1} F_1^j$. In particular, $n_1 \le F_1 \subset L$ and $y_1 \in F_1$.

We describe the choice of F_p and y_p at step p. Assuming the p-th move of Player I was (x_p, n_p), we write $x_p = \sum_{k_{p-1}<j\le k_p}\lambda_j l_j$. Letting in the game H_L Player I play $k_p - k_{p-1}$ times the integer n_p, the winning strategy of Lemma (2.2.8) provides moves $(F_p^{k_{p-1}+1}, zk_{p-1} + 1), \ldots, (F_p^{k_p}, Z_{k_p})$ for Player II in that game.

We let $y_p = \sum_{k_{p-1}<j\le k_p}\lambda_j z_j$, and $F_p = \sum_{k_{p-1}<j\le k_p} F_p^j$ particular, $n_p < F_p \subset L$ and $y_p \in F_{j=1}^p F_j$.

Finally, by construction, $(Z_n)_{n\in\mathbb{N}}$ is C-equivalent to $(l_n)_{n\in\mathbb{N}}$. It follows that $(y_p)_{p\in\mathbb{N}}$ is C-equivalent to $(x_p)_{p\in\mathbb{N}}$.

The non-trivial Lemma (2.2.10) will serve as the first step of a final induction which is on the model of the demonstration of Pelczar in [36] (note that there, the first step of the induction was straight forward). The rest of our reasoning will now be along the lines of her work, with the difference that we chose to express the proof in terms of games instead of using trees, and that we needed the moves of Player II to include the choice of finite-dimensional subspaces F_n's in which to pick the vectors y_n's. This is due to the fact that the basic sequence which witnesses the embedding of X into a given subspace generated by a subsequence is not necessarily successive on the basis of X.

Let L, M be block-subspaces in $G_{\mathbb{Q}}(X)$. Let

$$a \in b_{\mathbb{Q}}^{<\omega} b(X) \text{ and } b \in (\mathbb{Q}(X) \times \mathbb{Q}(X))^{<\omega}$$

be such that $|a| = |b|$ or $|a| = |b| + 1$ (here $|x|$ denotes as usual the length of the finite sequence x). Such a couple (a, b) will be called a state of the game $G_{L,M}$ and the set of states will be written St(X). It is important to note that St(X is countable. The empty sequence in $b_{\mathbb{Q}}^{<\omega}(X)$ (resp. $(Fin_{\mathbb{Q}}(X) \text{ x } \mathbb{Q}(X))^{<\omega}$ will be denoted by $\emptyset$.

We define $G_{L,M}(a, b)$ intuitively as "the game $G_{L,M}$ starting from the state (a,b)". Precisely, if $|a| = |b|$, then write $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p)$, with $b_i = (B_i, \beta_i)$ for $i \leq p$.

A k-th move for Player I is $(x_k, n_k) \in \mathbb{Q}(X) \times \mathbb{N}$, with $x_k \in L, x_i > a_p$ if $k = 1$ and $a \neq 0$, and $x_k > x_{k-1}$ if $k \geq 2$. Ak-th move for Player II is $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$ with $n_k \leq F_k \subset M$ and $y_k \in B_1 \oplus \ldots \oplus B_p \oplus F_1 \oplus \ldots \oplus F_k$ for all k.

	I: x_1, n_1		x_2, n_2	....
$G_{L,M}(A, B)$ $\lvert a\rvert = \lvert b\rvert$				
	II:	F_1, y_1		F_2, y_2 ....

Player IIwins $G_{L,M}(a, b)$ if the sequence $(\beta_1, \ldots \beta_p, y_1, y_2, \ldots)$ is C-equivalent to the sequence $(a_1, \ldots, a_p, x_1, x_2, \ldots)$.

Now if $|a| = |b| + 1$, then write $a = (a_1, \ldots, a_p + l)$ and $b = (b_1, \ldots, b_p)$, with $b_i = (B_i, \beta_i)$ for $i \leq p$.

A first move for Player II is $n_1 \in \mathbb{N}$. A first move for Player II is $(F_1, y_1) \in Fin_{\mathbb{Q}}(X).Q(X)$ with $n_1 \leq F_1 \subset M$ and $y_i \in B_1 \oplus \ldots \oplus B_p \oplus F_1$.

For $k \geq 2$, a k-th move for Player I is $(x_k, n_k) \in \mathbb{Q}(X) \times \mathbb{N}$, with $x_k \in L$, $x_2 > a_{p+l}$ if $k = 2$, and $x_k > x_{k-1}$ if $k > 2$; a k-th move for Player II is $(F_k, y_k) \in Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X)$with $n_k < F_k \subset M$ and $y_k \in B_1 \oplus \ldots \oplus B_p \oplus F_1 \oplus \ldots \oplus F_k$.

	I: n_1		x_2, n_2	....
$G_{L,M}(a, b)$ $\lvert a\rvert = \lvert b\rvert + 1$				
	II:	F_1, y_1		F_2, y_2 ....

Player 2 wins $G_{L,M}(a, b)$ if the sequence $(\beta_1, \ldots, \beta_p, y_1, y_2, \ldots)$ is C-equivalent to the sequence $(a_1, \ldots, a_p, a_p + 1, x_2, \ldots)$.

We shall use the following classical stabilization process, called "zawada" in [36]; see also the proof by B. Maurey of Gowers' dichotomy theorem [37]. We define the following order relation on $G_{\mathbb{Q}}(X)$: for $M, N \in G_{\mathbb{Q}}(X)$, with $M = | m_i|_{i \in \mathbb{N}}, (m_i) \in bb_{\mathbb{Q}}(X)$, write $M \subset^* N$ if there exists $p \in \mathbb{N}$ such that $m_i \in N$ for all $i \geq p$.

Let T be a map defined on $\mathcal{G}_{\mathbb{Q}}(X)$ with values in the set 2^{Σ} of subsets of some countable set $\sum$. Assume the map $\mathcal{T}$- is monotonous with respect to $\subset^*$ on $G_{\mathbb{Q}}(X)$ and to inclusion on 2^{Σ}. Then by ([36] Lemma 2.1), there exists a block-subspace $M \in G_{\mathbb{Q}}(X)$ which is stabilizing fort i.e. $\mathcal{T}(N) = \mathcal{T}(M)$ for every N C^* M.

We now define a map $\mathcal{T}: G_{\mathbb{Q}}(X) \to 2^{st(x)}$ by $(a, b) \in \mathcal{T}(M)$ iff there exists $L \subset^* M$ such that Player II has a winning strategy for the game $G_{L,M}(a, b)$.

Lemma (2.2.11)[25]: Let M' and M be in $\mathcal{G}_{\mathbb{Q}}(X)$. If $M' \subset^* M$ then$\mathcal{T}(M') \subset \mathcal{T}(M)$.

Proof. Let $M' \subset^* M$, let $(a, b) \in \mathcal{T}(M')$, and let $L \subset^* M'$ be such that Player 2 has a winning strategy in $G_{L,M'}(a, b)$. Let m be an integer such that for any $x \in \mathbb{Q}(X), x \in M'$ and $\min(\text{supp}(x)) \geq m$ implies $x \in M$. We describe a winning strategy for Player II in the game $G_{L,M}$: assume Player I's p-the move was (n_p, x_p) (or just n_i if it was the first move and $|a||b| + 1$); without loss of generality $n_p \geq m$. Let (F_p, y_p) be the move prescribed by the winning strategy for Player II in $G_{L,M'}(a, b)$.Then $F_p > n_p \leq m$ and $F_p M'$, so $F_p \subset M$. The other conditions are satisfied to ensure that we have described the p -th move of a winning strategy for Player II in the game $G_{L,M}(a, b)$. It remains to note that $L \subset^* M$ as well as to conclude that $(a, b) \in t(M)$.

By the stabilization lemma, there exists a block-subspace $M_0 \in G_{\mathbb{Q}}(X)$ such that for any $M \subset^* M_0, \mathcal{T}(M) = \mathcal{T}(M_0)$.

For $L, M \in G_{\mathbb{Q}}(X)$ we write $L =^* M$ if $L =^* M$ and $M =^* L$.

We now define a map $\rho: \mathcal{G}_{\mathbb{Q}}(X) \to 2^{St(X)}$ by $(a, b) \in \rho(M)$ iff there exists $L =^* M$ such that Player II has a winning strategy for the game $G_{L,M_0}(a, b)$.

Lemma (2.2.12)[25]: Let M' and M be in $G_{\mathbb{Q}}(X)$. If $M' \subset^* M$ then $\rho(M') \supset \rho(M)$.

Proof. Let $M' \subset^* M$, let $(a, b) \in \rho(M)$, and let $L =^*$ be such that Player 2 has a winning strategy in $G_{L,M_0}(a, b)$. Define $\acute{L} = \acute{M} \cap L$. As $\grave{L} \subset L$, it follows immediately that Player 2 has a winning strategy in the game $G_{L',M_0}(a, b)$. It is also clear that $L' =^* M'$, so $(a, b) \in \rho(M')$.

So there exists a block-subspace $M_{00} \in G_{\mathbb{Q}}(X)$ of M_0 which is stabilizing for p, i.e. for any $M \subset^* M_{00}, \rho(M) = \rho(M_{00})$.

Lemma (2.2.13)[25]: $\rho(M_{00}) = \mathcal{T}\ t(M_{00}) = \mathcal{T}(M_0)$.

Proof. First it is obvious by definition of M_0 that $\mathcal{T}(M_{00}) = \mathcal{T}(M_0)$.

Let $(a,b) \in \rho(M_{00})$. There exists $L =^* M_{00}$ such that Player II has a winning strategy in $G_{L,M_0}(a,b)$; as $L \subset^* M_0$, this implies that $(a,b) \in \mathcal{T}(M_0)$.

Let (a, b)$\in \mathcal{T}(M_{00})$There exists $L \subset^* M_{00}$ such that Player II Ihas a winning strategy in $G_{L,M_{00}}(a,b)$. As $M_{00} \subset M_0$, this is a winning strategy for $G_{L,M_0}(a,b)$ as well. This implies that $(a,b) \in \rho(L)$ and, by the stabilization property for ρ $(a,b) \in \rho(M_{00})$.

We now turn to the concluding part of the proof of Theorem (2.2.3). By our assumption about X just before Definition (2.2.9), there exists a block-sequence $(l_n)_{n\in\mathbb{N}}$ of $bb_{\mathbb{Q}}(X)$ which is contained in M_{00}, and C-continuously embeds into its subsequences, and without loss of generality assume that $L_0 := [l_n, n \in \mathbb{N}] = M_{00}$. We fix an arbitrary block-subspace M of L_0 generated by a block-sequence in $bb_{\mathbb{Q}}(X)$ and we shall prove that L_0 embeds into M. By standard perturbation arguments this implies that L_0 is minimal.

We construct by induction a subsequence $(a_n)_{n\in\mathbb{N}}$ of $(l_n)_{n\in\mathbb{N}}$, a sequence $b_n = (F_n, y_n) \in (Fin_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^{\omega}$ such that $F_n \subset M$ and $y_n \in F_1 \oplus ... \oplus F_n$ for all $n \in \mathbb{N}$, and such that $((a_n)_{n\leq\rho}, (F_n, y_n)n \leq \rho) \in \rho(L_0)$ for all $\rho \in \mathbb{N}$.

By Lemma (2.2.10), Player 2 has a winning strategy in G_{L_0,L_0} and so in particular$(\emptyset, \emptyset) \in \rho(L_0)$ (recall that $\emptyset$ denotes the empty sequence in the sets corresponding to the first and second coordinates). This takes care of the first step of the induction.

Assume $(a,b) = ((a_n)_{n\leq p-1}, (F_n, y_n)_{n\leq p-1}$ is a state such that $(a_n)_{n\leq p-1}$ is a finite subsequence of $(l_n)_{n\in\mathbb{N}}$,such that $F_n \subset M$ and $y_n \in F_1 \oplus ... \oplus F_n$ for all $n \leq p - 1$, and such that $(a,b) \in \rho(L_0)$.

As (a, b) belongs to $\rho(L_0)$, there exists $L =^* L_0$ such that Player 2 has awinning strategy in the game $G_{L,M_0}(a,b)$. In particular, $L_0 \subset^* L$ so we may choose m_p large enough such that $l_{m_p} > a_{p-1}$ and $l_{m_p} \in L$; we let Player I play $a_p = I_{m_p}$. Player 2 has a winning strategy in the game $G_{L,M_0}(a', b)$, where $a' = (a_n)_{n\leq p}$. In other words, (a', b) belongs to $\rho(L_0)$. Now $\rho(L_0) = \mathcal{T}(M)$, so there exists $L \subset^* M$ such that Player II has a winning strategy in the game $G_{L,M}(a', b)$. Let Player I play any integer n_p, and (F_p, y_p) with $F_p \subset M$ and $y_p \in F_1 \oplus ... \oplus F_p$ be a move for Player II prescribed by that winning strategy in response to n_p. Once again, Player II has a winning strategy in $G_{L,M}(a', b')$, with $b' = (F_n, y_n)_{n<p;}$ i.e. $(a', b') \in \mathcal{T}(M) = \rho(L_0)$.

To conclude, note that $(a_n, b_n)_{n\leq p} \in \rho(L_0)$ implies in particular that $(a_n)_{n\leq p}$ and $(y_n)_{n\leq p}$are C-equivalent, and this is true for any $p \in \mathbb{N}$, so $(a_n)_{n\leq p}$ and $(y_n)_{n\in\mathbb{N}}$ are C-equivalent. Hence $(a_n)_{n\in\mathbb{N}}$ C-embeds into M. Now $(a_n)_{n\in\mathbb{N}}$is a subsequence of $(l_n)_{n\in\mathbb{N}}$, so by our hypothesis, L_0 C-embeds into$(a_n)_{n\in\mathbb{N}}$ and thus C^2-embeds in M, and this concludes the proof of Theorem (2.2.3).

As a consequence of our proof we obtain a uniform version of Theorem (2.2.3).

Theorem (2.2.14)[25]: Let $C \geq 1$ $and\ let\ \omega > 0$. If a Banach space is satlzrated with basic sequences which C-embed into their subsequences, then it contains a $C^2 + \epsilon$-minimal subspace.

D. Kutzarova drew our attention to the dual T^* of Tsirelson's space; it is minimal [41], but contains no block-minimal block-subspace (use, e.g., [41] Proposition 2.4 and Corollary 7. b.3 in their T^* versions, with Remark 1 after [41] Proposition 1.16). So Theorem (2.2.3) applies to situations where Pelczar's theorem does not. On the other hand, we do have (recall that a basic sequence $(x_n)_{n\in\mathbb{N}}$ is isomorphically homogeneous if all snbspaces spanned by subsequences of $(x_n)_{n\in\mathbb{N}}$are isomorphic):

Corollary (2.2.15)[25]: A Banach space with a Schauder basis which is saturated with isomorphically homogeneous basic sequences contains a block-minimal block-subspace.

Proof. Let X have a Schauder basis and be saturated with isomorphically homogeneous basic sequences. By the beginning of the proof of Lemma (2.2.5), we may assume X is reflexive. By Theorem (2.2.3), there exists a minimal subspace Y in X, which is a block-subspace if you wish; passing to a further block-subspace assume furthermore that Y has an isomorphically homogeneous basis. Take any block-subspace $Z\ of\ Y = [y_n]_{n\in\mathbb{N}}$; then Y embeds into Z. By reflexivity and standard perturbation results, some subsequence of $(y_n)_{n\in\mathbb{N}}$ spans a subspace which embeds as a block-subspace of Z. As $(y_n)_{n\in\mathbb{N}}$ is isomorphically homogeneous, this means that Y embeds as a block-subspace of Z.

We recall that a Banach space is said to be ergodic if the relation E_0 is Borel reducible to the relation of isomorphism between its subspaces.

Corollary (2.2.16)[25]: A Banach space is ergodic or contains a minimal subspace.

Proof. We prove the stronger result that if X is a Banach space with a Schauder basis, then either E_0 is Borel reducible to isomorphism between block-subspaces of X or X contains a block-minimal block-subspace.

Assume E_0 is not Borel reducible to isomorphism between block-subspaces of X. By [35] Theorem 19, any block-sequence in X has an isomorphic ally homogeneous subsequence. In particular, X is saturated with isomorphically homogeneous sequences, so apply Corollary (2.2.15).

Corollary (2.2.17)[25]: A Banach space X contains a minimal subspace or the relation E_0 is Bore1 reducible to the relation of biembeddability between subspaces of X.

Proof. Note that the relation $\sim^{emb}$of biembeddability between subspaces of X is analytic. By [35] Theorem 15, if E_0 is not Borel reducible to biembeddability between subspaces of X, then every basic sequence in X has a subsequence $(x_n)_{n\in\mathbb{N}}$ which is homogeneous for the relation between subsequences corresponding to$\sim^{emb}$, that is, for any subsequence $(x_n)_{n\in 1}$ of $((x_n)_{n\in\mathbb{N}}, [x_n]_{n\in 1} \sim^{emb} (x_n)_{n\in\mathbb{N}}$. This means that $(x_n)_{n\in\mathbb{N}}$embeds into its subsequences. So X is saturated with basic sequences which embed into their subsequences.

We conclude with a remark about the proof of Theorem (2.2.3). The sequences $\left(m_p\right)_{p\in\mathbb{N}} \in |\omega|^{\omega}$ (associated to a subsequence of $(l_n)_{n\in\mathbb{N}}$ and $b_p = (F_p, y_p) \in (\mathrm{Fin}_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^{\omega}$ (with $(y_p)_{p\in\mathbb{N}}$ $C-$equivalent to $(I_{m_p})_{p\in\mathbb{N}}$) in our final induction may clearly be chosen with $F_p \subset M_p$ for all p, for an arbitrary sequence $(M_p)_{p\in\mathbb{N}}$ of block-subspaces of L_0. Also, $(l_n)_{n\in\mathbb{N}}$ C-continuously embeds into its subsequences, i.e. there is a continuous map $f\colon [\omega]^{\omega}\, bb_{\mathbb{Q}}(X)$ such that f(A) is C-equivalent to $(l_n)_{n\in\mathbb{N}}$for all $A \in bb_{\mathbb{Q}}(X)$.

By combining these two facts, it is easy to see that Player II has a winning strategy to produce a sequence $(y_n)_{n\in\mathbb{N}}$ which is C^2-equivalent to $(l_n)_{n\in\mathbb{N}}$, in a "modified" Gowers' game, where a p-th move for Player 1 is a block-subspace $Y_p \in \mathcal{G}_{\mathbb{Q}}(X)$ with $Y_p \subset L_0$, and a p-th move for Player II is a couple $(F_p, y_p) \in (\mathrm{Fin}_{\mathbb{Q}}(X) \times \mathbb{Q}(X))^{\omega}$ with $F_p \subset Y_p$ and $y_p \in F_1 \oplus \ldots \oplus F_p$.

This is an instance of a result with a Cowers-type game where Player II is allowed to play sequences of vectors which are not necessarily block-basic sequences.