Chapter 1
Distortable Banach Space with Operation

In fact, we make use of just one consequence of the axiom $\lozenge_{N_1}$ shown by Jensen, which is widely used by mathematical logicians. We construct a "Tsirelson like Banach space" which is arbitrarily distortable.

Section (1.1): Few Operations in Banach Spaces :

Let i, j, be ordinals, ω the first infinite ordinal, ω_1 the first uncountable ordinal. Let k, l, m, n, p be natural numbers, and let a,b,c,d be reals, and x, y, z elements of a (vector, or norm, or Banach space).

Theorem (1.1.1)[1]: Assume the axiom V = L holds. Then there is a Banach space $\bar{z}$, and an element of the space z_1 $(i<\omega_1)$ such that:

(i) span$\{z_1: i <\omega_1\}$ is dense in $\bar{z}$, $\|z_1\| = 1$, and there are projections p_α,$(\alpha <\omega_1)$ of norm 1 of $\bar{z}$, into itself, $p_\beta(z_i) = 0$ for $i \geq \beta, p_\beta(z_i) = z_i$, for $i < \beta$ .So the density character of $\bar{z}$ is ω_1, and it has a basis $\{z_i : i < \omega_1\}$.

(ii) If T: B→ B is (linear, bounded) operator, then for some real a, Tz_i=a z_i for all but countably many i 's. So T-aI is an operator with a separable range.

Stage (1.1.2)[1]: Let $\{z_i: i <\omega_1\}$ generate freely a vector space H over Q (the rationals). For a set I of ordinals let H_1 and also H(I) denote span$\{z_i: i \in I\}$ (= the subvector space spanned by $z_i, i \in I$). As an ordinal i is $\{j : j < i\}$, H_i is the vector space spanned by $\{ z_j : j < i \}$. Let I_i^m(' (m <ω, <i < ω1) be finite subsets of i, increasing with m, and $i = \cup_m I_i^m$ For subsets $A_1, A_2, \ldots$ of H, $\langle A_1, A_2, \ldots \rangle_H$ is the span of $A_1 \cup A_2 \cup \ldots$ We usually omit H and write y instead of { y }.

Stage (1.1.3)[1]: A subset of ω_1 is called closed if for each limit ordinal $i < \omega_1$ which satisfies $(\forall j < i)\ (\exists\alpha)\ (j < \alpha < i \wedge \alpha \in I)\ (\exists j < \omega_1)$ belong to I. I is unbounded if $(\forall i \in \omega_1)\ (\exists j < \omega_1)\ (i < j \wedge j \in I)$ A set of $I \subseteq \omega_1$ is called stationary if it has a non-empty intersection with every closed unbounded subset of ω_1.

Stage (1.1.4)[1]: By Jensen [2], if V = L then there are sets D_i functions f_i $(i < \omega_1)$ and $r_i \in \{0, 1\}$ such that

(i) f_i is a two-place function from H_i into the reals, D_i a subset of i.

(ii) For every subset D of ω_1 and two-place function from H into the reels, and $r \in \{0, 1\}$, $\{ i < \omega_1 : D \cap i = D_i, f/H_i = f_i, r_i = r \}$ is a stationary subset of ω_1. From now on f_i are as above.

Stage (1.1.5)[1]: In a norm space Z, for $z \in Z, X \subseteq Z$, we say z is good over X if $(\forall x \in X)\ \|z + x\| \geq \|z\|, \|x\|$ and $\|z\| = 1$.

If $z_0 \ldots z_k \in Z, X \subseteq Z$ we say $(z_0 \ldots z_k)$ is good over X if $\|z_1\| = 1$ and for any reals a_1 and $x \in X$

$$\left\|\sum_{i=0}^{k} a_i z_i + x\right\| \geq \left\|\sum_{i=0}^{k} a_i z_i\right\| , \|x\| .$$

Note that

(i) (z_0) is good over X iff z_0 is good over X;

(ii) if $(z_0 \ldots z_k)$ is good over X then so is every sequence from $(z_0 \ldots z_k)$

Stage (1.1.6)[1]: Suppose Y, Z are norm spaces, $Y \cap Z = X$, and let W be a vector space such that Y, Z are subspaces of it, and $W = Y + Z$ (as vector spaces). We can define a norm on W which extends the norms on Y and Z, and get a norm space, as follows:

$$\|w\| = \inf\{\|y\| + \|z\| : y \in Y; z \in Z, w = y + z\}.$$

In this case the unit ball of W is the convex hull of the unit balls of Y and Z. We call this N_1 -amalgamation. Note that

(i) if $y \in Y$ is good over X,it will be good over Z; and

(ii) if also $z \in Z$ is good over X then $\|y + z\| = 2$.

Stage (1.1.7)[1]: Suppose that in stage (1.1.6)$Y = \langle X, y_0 \dots\dots y_k\rangle$, $Z = \langle X, z_0 \dots\dots z_i\rangle$ $(y_0 \dots\dots y_k)$, $(z_0 \dots\dots z_i)$ are good over X. Then there is another way to define a norm on W extending the norms on Y and Z: for $x \in X$

$$\left\|\sum_{n=0}^{k} b_n y_{n+} \sum_{n=0}^{i} c_n z_n + x\right\| = \max\left\{\left\|\sum_{n=0}^{k} b_n y_{n+} x\right\|, \left\|\sum_{n=0}^{i} c_n z_{n+} x\right\|\right\}$$

We call this N_∞-amalgamation (unlike N_1-amalgamation, it apparently does not depend only on Y and Z, but also on span $\{y_0 \dots\dots y_k\}$, and span $\{y_0 \dots\dots y_k\}$.

Note that

(i) $(z_0 \dots\dots z_k)$, is good over Y,

(ii) for $n \le l, m \le k, z_n + y_n$, is good over X and in particular $\|z_n + y_n\| = 1$,

(iii) if $l(1) < l$, and we first amalgamate $X, \langle X, y_0 \dots\dots y_k\rangle, \langle X, z_0, \dots\dots, z_{1(1)}\rangle$ in the above-mentioned way and then amalgamate $\acute{X} = \langle X, z_0, \dots\dots, z_{1(1)}\rangle$, $\langle \acute{X}, y_0, \dots\dots, y_k\rangle$, $\langle \acute{X}, z_{i(1)+1}, \dots\dots, z_i\rangle$, we get the same norm.

Stage (1.1.8)[1]: We shall define by induction on $i < \omega_1$ norm spaces Z_i, increasing with i such that Z_i as a+ vector space is H_i and for some i's, infinite sets $S_i \subset \omega$ and elements $y_i^m y_i^m \in H_i$ (for $m < \omega$) when $r_i = 0$, and $y_{i,j}^m$ $(m < \omega, 1 \le l \le p(m, i)$ when $r_i = 1$, such that (not distinguishing strictly between subspaces of H_i and of Z_1)

(i) if $\gamma \le \alpha_0 < \alpha_1 < \dots < \alpha_k \le i, \omega \le i, k$, a natural number, $r_\gamma = 0, y_\gamma^0$, is defined, then for infinitely many $m \in S_\gamma$

(í) the amalgamation of the triple $H(I_\gamma^m)$,$\langle H(I_\gamma^m), y_\gamma^m\rangle$, $\langle H(I_\gamma^m), z_{a0}, \dots z_{ak}\rangle$ is by the N_∞-amalgamation, i.e., for $x \in H(I_i^m)$

$$\left\|a y_i^m + \sum_{i=0}^{k} b_i z_{\alpha 1+} x\right\| = \max\left\{\|a y_i^m + x\|, \left\|\sum_{i=0}^{k} b_i z_{\alpha 1} + x\right\|\right\};$$

(ΐ) the amalgamation of the triple $\langle H(I_\gamma^m), y_\gamma^m\rangle$, $\langle H(I_\gamma^m), y_\gamma^m, y_\gamma^m\rangle$, $\langle H(I_\gamma^m), y_\gamma^m, z_{a0}, \dots z_{ak}\rangle$ is by the N_1-amalgamation. So in particular

(ii) z_γ is good over H_γ , and if $\gamma \le \alpha_0 \le \alpha_1 \le \dots . \le \alpha_k$ then $(z_{a0}, \dots z_{ak})$is good over H_γ .We also demand

(iii) if, $\gamma \le \alpha_0 < \alpha_1 < \dots . \le \alpha_k \le i$, $\omega \le i$, k a natural number, $r_\gamma = 1$, then for infinitely many $m < \omega$ the amalgamation of the triple

$$H(I_\gamma^m), \langle H(I_\gamma^m), y_{i,1}^m, \dots\dots, y_{i,p(m,\gamma)}^m\rangle, \langle H(I_\gamma^m), y_\gamma^m, z_{a0}, \dots z_{ak}\rangle$$

Is by N_∞ amalgamaon for i limit $z_i = \bigcup_{j<i} z_i$ for $i < \omega$, $\|\sum_{l<i} a_l z_l\| = \max_{l<\omega} |a_l|$.

Stage (1.1.9)[1]: Now we do the induction step, so we suppose the norm on H_i is defined, $i \geq \omega$ and we call the norm space Z_i In this stage we shall define $y_i^m y_i^m$ ($m < \omega$) and S_i, and in the next stage we shall define the norm on H_{i+1}. Remember that f_i is a two place function from H_i to R given by the Jensen diamond (see Stage (1.1.3)).

If there is a (bounded) operator T on $\overline{Z_l}$ such that for every x, y, $\in H_i$, $f_i(x,y) = \|Tx - y\|$, it is unique, and we call it T_i.

If T_i is not defined we do not define S_i, y_i^m, y_i^m So suppose T_i is defined.

(i) If Y is a Banach space, T an operator on $Y, H \subseteq Y$ a subspace, then let $C(H,T,Y) = \sup\{d(Ty, \langle H, y\rangle) : y \in Y, y$ good over $H\}$, where $d(y_1, H_1)$ is the distance between y_1 and H_1 i.e., $\inf\{\|y_1 - x\| : x \in H_1\}$, and let

$$c_\varepsilon(H,T,Y) = \sup\{d(Ty,H) : d(Ty, \langle H, y\rangle) \geq c(H,T,Y) - \varepsilon\}$$

and y is good over H.

Note that $c(H,T,Y) \leq \|T\|$ and it decreases with H.

Now if $r_1 = 0$, choose $y_i^m y_i^m$ in H_1 such that:

(ii) (i) $d(Ty_i^m, H(I_i^m, y_i^m) \geq c(H(I_i^m), T_i, \overline{Z}_l) - 1/m$,

(ii) y_i^m is good over $H(I_i^m)$,

(iii) $d(Ty_\gamma^m, H(I_\gamma^m) \geq c_{1/m}(H(I_i^m), T_i, \overline{Z}_l) - 1/m$,

(iv) $\|Ty_i^m - y_i^m\| < 1/m$.

Clearly $c_{1/m}(H(I_i^m), T_i, \overline{Z}_l)$ is a real number of absolute value $< \|T\|$, hence there is an infinite set $S_i \subseteq \omega$ such that

(iii) for $k < m \neq n$ in S_i, $1/k > \left|c_{1/m}\left((H(I_i^m), T_i, \overline{Z}_l)\right) - c_{1/n}((H(I_i^m), T_i, \overline{Z}_l))\right|$.

(iv) If $r_i = 1$ choose a $p = p(m,i) < \omega$ and $y_{i,1}^m \in \{z_\alpha : \max I_i^m < \alpha < i, \alpha \in D_1\}$ such that:

(v) for every $x \in H(I_i^m)$,

$$\left\|\sum_{i=0}^{p(m)} a_1 y_{i,1}^m + x\right\| = \sup\left\|a_1 y_{i,1}^m + x\right\|$$

(notice each $y_{i,1}^m$ is good over $H(I_i^m)$), (ii) if among the p's satisfying (i) there is a maximal one, this will be our p; otherwise choose $p = m$.

Stage (1.1.10)[1]: Now we have to define the norm on H_{i+1}(after we have defined it on H_i), and define, if necessary, $y_i^m\ y_1^m$ ($m < \omega$) or $y_{i,1}^m$.

We have to satisfy the requirements (i) and (ii)from Stage (1.1.8); when $a_k < i$ they are satisfied by the induction hypothesis. Clearly there are only countably many appropriate requirements, so we can find a list of them of length w, each appearing infinitely many times.

Let $\{\beta_n : n < \omega\}$ be a list of $i = \{j : j < i\}$. Now we define by induction on $n < \omega$ a finite set $J_n \subseteq i$, and a norm space Z_i^m which as a vector space is $H(J_n \cup \{i\})$(we shall not distinguish) such that

(i) $J_n \quad J_{n+1}$,

(ii) Z_i^m is a subspace of Z_i^{n+1}

(iii) $i = \bigcup_{n<\omega} J_n$

(iv) in Z_i^m z_i is good over $H(J_n)$.

For n =0 let H_0 be the empty set, and the norm Z_i^0 is $\|az_i\| = |a|$.

Suppose we have defined Z_i^n for n, and let us define Z_i^{n+1}. Let $\langle k, \gamma, a_{0,\ldots\ldots,} a_{k-1}\rangle$ be the n-th in the list of cases of (i)and (ii) from Stage (1.1.8). Assume for now that $r_\gamma = 0$ (the case $r_\gamma = 1$ is just simpler). If $\{a_{0,\ldots\ldots,} a_{k-1}\} \not\subset J_n$, we let $J_{n+1} = J_n \cup \{\beta_n\}$ and we define the norm of $Z_i^{n+1}$by $N_1 -$ amalgamation of $H(J_n), Z_i^n, H(J_{n+1}),$) (see Stage (1.1.7). Now if $\{a_{0,\ldots\ldots,} a_{k-1}\} \subseteq J_n$, let $J_n - \gamma = \{\beta_{0,\ldots\ldots,} \beta_i\}$ (as $\gamma \leq \alpha_0 < \cdots$ necessarily $\{a_{0,\ldots\ldots,} a_{k-1}\} \subseteq \{\beta_{0,\ldots\ldots,} \beta_i\}$ By the induction hypothesis, (i) of Stage (1.1.9) holds for $\gamma \leq \beta_0 \leq \cdots \leq \beta_i$ hence there is an $m \in S_\gamma$ satisfying

(i) $J_n \cap \gamma \subseteq I_\gamma^m$ (possible as (i) says "for infinitely many m's" and $\gamma = \cup_m I_\gamma^m, I_\gamma^m$; increase with m, and J_n is finite),

(ii) the amalgamation of the triple $H(I_\gamma^m), \langle H(I_\gamma^m), y_\gamma^m\rangle, \langle H(I_\gamma^m), z_{\beta_0}, \ldots\rangle$is an N_∞-amalgamation.

(iii) the amalgamation of the triple

$$H(I_\gamma^m), \langle H(I_\gamma^m), y_\gamma^m, y_\gamma^m\rangle, \langle H(I_\gamma^m), y_\gamma^m, y_\gamma^m, z_{\beta_0}, \ldots\rangle$$

is an N_1-amalgamation. We choose a finite J_{n+1} such that $J_n \subseteq J_{n+1} \subseteq i, \beta_n \in J_{n+1}$ and $y_\gamma^m, y_\gamma^m \in H(J_{n+1})$ (this is trivial). Now we define Z_i^{n+1} by successive amalgamation.

(γ) We make an N_1 -amalgamation of the triple $H(J_n), Z_i^n, H(J_n \cup I_\gamma^m)$:$z_i$, is good (in it) over $H(J_n \cup I_\gamma^m)$ by (a) of Stage (1.1.6). $H(J_n \cup I_\gamma^m \cup \{i\})$ (defined in α), and $\langle H(J_n \cup I_\gamma^m), y_\gamma^m\rangle$ (possible as $z_i$is good over $H(J_n \cup I_\gamma^m)$ by (α) and y_γ^m is good over $H(J_n \cup I_\alpha^m)$, by the choice of m to satisfy (ii) and (i) of Stage (1.1.7)). By (i) of Stage (1.1.7), z_i is good over $\langle H(J_n \cup I_\gamma^n), y_\gamma^n\rangle$ in the amalgamated space we have just defined.

(γ) We make the N_1-amalgamation$\langle H(J_n \cup I_\gamma^m), y_\gamma^m\rangle, H(J_{n+1}), \langle H(J_n \cup I_\gamma^m \cup \{i\}), y_\gamma^m\rangle$

(with the norm defined in (β)and call it Z_i^{n+1}By (i) of Stage (1.1.7) z_i is good over $H(J_{n+1})$ in Z_i^{n+1}

It is easy to check that (i) and (ii) of (a) hold for $\gamma, \alpha_0, \ldots, \alpha_k$ and m (by (iii) of Stage (1.1.7).

So Z_i^n is defined for every i, and let $Z_{i+1} = \cup_{n<\omega} Z_i$.

Clearly Z_{i+1}as a vector space is H_{i+1}(as $\beta_n \in J_{i+1}$). Each requirement $\gamma, \alpha_0, \ldots, \alpha_k =$ i appears in our list infinitely many times so for every n big enough $\{\alpha_0, \ldots, \alpha_k\} \subseteq J_n$ so clearly (i) holds for i + 1.

Stage (1.1.11)[1]: We have defined Z_i for $i < \omega_1$. Let $Z = \cup_{n<\omega_1} Z_i$ (so as a vector space it is H), and $\bar{Z}$, its completion, is the Banach space which exemplifies our theorem.

So let T be an operator on Z and we shall prove it is as mentioned in the theorem, i.e., for some a, for every large enough i, $Tz_1 = az_1$ .We define a two place function f from H into R :

$$f(x, y) = \|Tz - y\|$$

By Stage (1.1.3)

$$I = \{i < \omega_1 : f/H_i, r_i = 0\}$$

is a stationary subset of ω_1 (see Stage(1.1.3)).

Stage (1.1.12)[1]: For each finite-dimensional subspace G of Z and $m < \omega$ there is y_G^m good over G such that
$d(Ty_G^m, G, y_G^m) \geq c(G, T, \bar{Z})(1 - 1/m)$ $d(T_G^m y. H) \geq c_{1/m}(G, T, \bar{Z}) - 1/m$.
For each $x \in Z$ there is $i(x) < \omega_1$ such that $x, Tx, \in \bar{Z}_{i(x)}$ Now for each $\alpha < \omega_1, = \{i(x): x \in H_\alpha \text{ or } x = y_G^m \text{ for some finite dimensional } G \subseteq H_m, m < \omega\}$ is countable, hence $i(\alpha) = \sup A_\alpha A_\alpha < \omega_1$. Now $A = \{j < \omega_1 : (\forall \alpha < j) i(\alpha) < j\}$ is a closed unbounded subset of ω_1 (closed-trivially by the definition, unbounded because $i(\alpha)$ increases with α, so if $j_0 = j, j_{n+1} = i(j_n)$ then $j_0 \leq \cup_n j_n < \omega_1$ and $\cup_n j_n$ is in this set). As I is stationary (see Stage (1.1.3) for definition, and Stage (1.1.11) for the fact) there is $\gamma \in A \cap I$ (I from Stage (1.1.11)). Clearly T maps $\overline{Z_\gamma}$ into Z_γ hence it maps $\overline{Z_\gamma}$ into $\overline{Z_\gamma}$, and

$$c(H(I_\gamma^m), T, \bar{Z}) = c(H(I_\gamma^m), T_\gamma, \overline{Z_\gamma})$$
$$c_{1/m}(H(I_\gamma^m), T, \bar{Z}) = c_{1/m}(H(I_\gamma^m), T_\gamma, \overline{Z_\gamma})$$
$$(\text{as } \gamma \in A \text{ and } T_\gamma = T/Z_\gamma (\text{as } \gamma \in I).$$

Stage (1.1.13)[1]: Now we shall prove that for every $i > \gamma$, $Tz_i \in \langle Z_\gamma, z_i \rangle$ (γ is as chosen at the end of Stage (1.1.12)[2], and will remain fixed).

For this it suffices to prove that for any real $\varepsilon > 0$, $d(Tz_i, \langle H(I_\gamma^m), z_i \rangle) \leq 5\varepsilon$) for some $m < \omega$. So let $\varepsilon > 0$ be given. Now Tz_i is in the closure of $Z = \text{span}\{z_\alpha : \alpha < \omega_1\}$ so for some $l(0) < \omega$ and $a_l \in A$, and distinct $\beta(l) < \omega_1$, (for $l < l(0)$:

(i) $\left\| Tz_i - \sum_{l<l(0)} a_l z_{\beta(l)} \right\| < \varepsilon$

So we can choose $k < \omega$, and $\alpha_0 < \cdots < \alpha_k < \omega_1$. $\gamma \leq \alpha_0$ such that
$\{i, \beta(0), \ldots . \beta(l(0))\} - \gamma \subseteq \{\alpha_0 < \cdots < \alpha_k\}$.
Now by (i) (from Stage (1.1.8), for infinitely many $m \in S_\gamma$, i and ii from (i) hold (for our $k, \gamma, \alpha_0, \ldots, \alpha_k$). So we can choose some m for which $\{\beta(0), \ldots . \beta(l(0))\} \cap \gamma \subseteq I_\gamma^m$; and $1/m < \varepsilon$ .Clearly

(ii) $\sum_{l \leq l(0)} a_l z_{\beta(l)} \in H(I_\gamma^m \cup \{\alpha_0, \ldots, \alpha_k\}$

and by I of (i) and Stage (1.1.7)

(iii) $z_\gamma + y_\gamma^m$ is good over H_i^m. Now we shall write a series of inequalities which will prove $d(Tz_i, \langle H(I_\gamma^m), z_i \rangle \leq 5\varepsilon)$; for notational convenience let x range over $H(I_\gamma^m)$, and a,b range over R.

(iv) $c(H(I_\gamma^m), T_i, \bar{Z}_i) =$ [as $\gamma \in A$, see stage (1.1.12)]

$c(H(I_y^m). T, \bar{Z}) \geq$ [by c`s definition, and (iii)above]

$d(T((z_i + y_\gamma^m), \langle H_\gamma^m, z_i + y_\gamma^m \rangle \geq$

$\inf_{a,x} \| T((z_i + y_\gamma^m) + a(z_i + y_\gamma^m) + x \| \geq$ [as $\|Tz_1 - z_i\| < \varepsilon, Ty_\gamma^m = T_i y_\gamma^m$ and

$\|T_1 y_\gamma^m - y_\gamma^m\| \leq 1/m$ as mentioned in (ii)

of stage (1.1.9)

$\inf_{a,x} \| z_i + y_\gamma^m + az_i + ay_\gamma^m + x \| - 1/m - \varepsilon \geq$ [by (ii) of (í)]

$\inf_{a,b,x,x_1} (\| y_\gamma^m + ay_\gamma^m + x + (by_\gamma^m + x_1) \| +$

$+\|z_i + az_i - (by_\gamma^m + x_1)\| - 1/m - \varepsilon) = \inf_{a,b,x_1,x_2} (\|y_\gamma^m + ay_\gamma^m + by_\gamma^m + x_1\| +$

$+\|z_i + az_i - by_\gamma^m + x_2\| - 1/m - \varepsilon) \geq$

$\inf_{a,b,x,x_1} \|y_\gamma^m + ay_\gamma^m + by_\gamma^m + x_1\| + \inf_{a,b,x,x_1} \|z_i + az_i - by_\gamma^m + x_2\| - 1/m - \varepsilon \geq$

$[\text{as } \|Ty_\gamma^m - y_\gamma^m\| < 1/m, \|Tz_i - z_i\| < \varepsilon]$

$\inf_{a,b,x_1} \|Ty_\gamma^m - ay_\gamma^m + by_\gamma^m + x_1\| - 1/m + \inf_{a,b,x_2} \|Tz_i + az_i - by_\gamma^m + x_2\| - \varepsilon - 1/m - \varepsilon$

$\geq$ [by d's definition]

$d(Ty_\gamma^m, \langle y_\gamma^m \rangle) + \inf_{a,b,x} \|Tz_i + az_i - by_\gamma^m + x\| - 2/m - 2\varepsilon \geq$

[by (ii) of stage (1.1.9)]

$c(H(y_\gamma^m), T_i, \overline{Z}_\imath) - 1/m + \inf_{a,b,x} \|Tz_i + az_i - by_i^m + x\| - 2/m - 2\varepsilon.$

Comparing the first and last elements we see that

(v) $\inf_{a,b,x} \|Tz_i + az_i - by_\gamma^m + x\| \leq 3/m + 2\varepsilon.$

Now by the choic of m

(vi) $1/m < \varepsilon.$

Combining we get $d(Tz_i, \langle H_\gamma^{m+1}, z_i \rangle \leq d(Tz_i, \langle H_\gamma^m, y_\gamma^m \rangle \leq 3/m + 2\varepsilon < 5\varepsilon.$

Stage (1.1.14)[1]: For each $\beta < \omega_1$ we define an operator P_β on $\overline{Z}$: $P_\beta(z_i) = 0$ for $i \geq \beta$, and $P_\beta(z_i) = z_i$ for $i < \beta$.

it is easy to check that:

(i) P_β is well defined and is a projection with norm 1 onto Z_β;

(ii) for $\beta < \alpha, P_\beta P_\alpha = P_\alpha P_\beta = P_\beta$.

(iii) if $P_\alpha(x) \neq 0$, αlimit, then for sorne $\beta < \alpha$, $P_\beta(x) \neq 0$.

Stage (1.1.15)[1]: Let T, γ be as in Stage (1.1.13). So for every $i \geq \delta, Tz_i \in \langle \overline{Z}, z_i \rangle$, so $Tz_i = d_i z_i + x_i^0$, $x_i^0 \in \overline{Z_\gamma}$.

We shall prove that for some $\delta, \gamma \leq \delta < \omega_1$, and for every $i \geq \delta$, $x_i^0 = 0$.. Suppose not, so $A_1 = \{i < \omega_1 : i \geq \gamma, \|x_i^0\| \neq 0\}$ uncountable. For each $i \in A_1$ choose aminimal $\beta_i \leq \gamma$such that $P_{\beta_i}(x_i^0) \neq 0$ (it exists as $P_\gamma(x_i^0) = x_i^0$, because $x_i^0 \in \overline{Z_\gamma}$).

By (iii) of Stage (1.1.14) β_i is a successor ordinal, so for some $\beta < \gamma$, $A_2 = \{i \in A_1 : \beta_i = \beta + 1\}$is uncountable. So for each $i \in A_2$, for some real $d_i^1 \neq 0, P_{aq_i}(x_i^0) = d_i^1 z_\beta$. So for some $a > 0$ and $s \in \{1, -1\}$, and s$\in A_3 = \{i \in A_2 : sd_i^1 > a\}$ is uncountable. So for each $i \in A_3$, $P_\beta T z_i = d_i^1 x_\beta, sd_i^1 > a$.

By Stage (1.1.3), $\acute{I} = \{i < \omega_1 : r_i = 1, f/H_i = f_i, A_3 \cap i = D_i\} = D;\}$ is a stationary subset of ω_1 Let

$A = \{i < \omega_1$: i is limite, $i > \gamma$,and $A_3 \cap i$ is unbounded below i and in (iv) of stage (1.1.9) if we ask $y_{i,j}^m$ in $\{z_\alpha : \max I_i^m < \alpha, \alpha \in A_3\}$ the value of $p = p(m, i)$ does not change$\}$.

As in Stage (1.1.12), we can prove A is closed and unbounded so $I \cap A \neq \emptyset$, and choose in it an element δ. Now for infinitely many $m < \omega$, $p(m,\delta) \geq m$. Otherwise choose $m_0 < \omega$ such that

(i) $m \geq m_0 \Rightarrow p(m, \delta) < m$

and choose $i \in A_2, i > \delta$.By (ii) of Stage(1.1.9), for some $m > m_0, H(I_\delta^m)$, $\langle H(I_\delta^m), z_i\rangle, \langle H(I_\delta^m), y_{\delta,i}^m, y_{\delta,p(m)}^m\rangle$ have N_∞-amalgamation. Now checking (ii) of Stage (1.1.9), we see that z_1, was an appropriate candidate for being $y_{\delta,p(m,\delta)+1}^m$ hence $p(m,\delta)$, contradiction.

So for $m, l, y_{\delta,l}^m \in \{z_\alpha: \alpha \in A_2\}$ hence $P_\beta T y_{b\delta,l}^m \in \{sbx_\beta: b > a\}$. Now for every m, (see(ii) of stage (1.1.8)

$$\left\|\sum_{l=1}^{b(m,\delta)} y_{\delta,l}^m\right\| = \max_i \|y_{\delta,l}^m\| = 1,$$

$$\left\|T\left(\sum_{l=1}^{b(m,\delta)} y_{\delta,l}^m)\right)\right\| \geq \left\|P_{\beta+1}T\left(\sum_{l=1}^{b(m,\delta)} y_{\delta,l}^m)\right)\right\|$$

$[\text{as } \|P_{\beta+1}\| = 1 \text{ by stage } (1.1.14)]$

$$= \left\|\sum_{i=1}^{P(m,\delta)} P_{\beta+1} T y_{\delta,l}^m\right\|$$

$[\text{as } P_{\beta+1} T y_{\delta,l}^m \in \{sbx_\beta: b > a\}]$

$$= \sum_{L=1}^{P(m,\delta)} \|P_{\beta+1} T y_{\delta,l}^m\|$$

$$\geq P(m,\delta)a$$

$$\geq ma.$$

Hence $\|T\| \geq ma, \text{as } a > 0, m(m < \omega)$, as a > 0, m (m < w) arbitrarily large, we get a contradiction.

Stage (1.1.16)[1]: (we omit O as a stage). We now want to show that $d_i(i < \omega_1)$ is eventually constant. Otherwise there are distinct reals $d^0 d^1$ such that

(i) for l = 1,2, 2 and $\alpha < \omega_1$ and $\varepsilon > 0$ there is $i, \alpha < i < \omega_1$, and $|d_i - d^l| < \varepsilon$; w.l.o.g. $d^0 = 0, d^1 = 1$, (otherwise, we look at the operator $1/(d_i - d^l)(T - d^0 I)$ (I -the identity operator).

Let $\varepsilon > 0$ be arbitrary, $\varepsilon < 1/100$ Choose $\alpha < \beta < \delta\ (\geq \gamma), |d_\alpha| < \varepsilon, |1 - d_\beta| < \varepsilon$ By (i) of Stage(1.1.8)[2] , for $k = 1, \alpha_0 = \alpha, \alpha_1 = \beta, i = \gamma$ we can find $m(1) < m$ in S_γ such that (i) and (ii) of (a) holds for m and for m(1) and

$$1/m(1) < \varepsilon, \quad 12\, m(1) < m.$$

We now try to get a contradiction to the choice of y_γ^m. We repeat Stage (1.1.13) with z_α for z_i so (ii), (iii), (iv) holds ((i) is trivialized-we know better), but we want to deviate in the middle of (iv):

$$c(H(I_\gamma^m), T, \bar{Z}_l) \geq \inf_{a,b,x,x_1}(\|y_\gamma^m + a y_\gamma^m + b y_\gamma^m + x_1\|$$
$$+\|T z_\alpha + a z_\alpha - b y_\gamma^m + x_2\| - 1/m).$$

So for some a, b, x_1, x_2 we get this infimum up to 1/m, so

$$c(H(I_\gamma^m), T_i, \bar{Z}_l) + 2/m \geq \|y_\gamma^m + a y_\gamma^m + b y_\gamma^m + x_1\| + \|T z_\alpha + a z_\alpha - b y_\gamma^m + x_2\| \geq$$

$[\text{as } \|T y_\gamma^m - y_\gamma^m\| < 1/m) \text{ and } T z_\alpha = d_\alpha z_\alpha]$

$$\left\|Ty_\gamma^m + (a+b)y_\gamma^m + x_1\right\| + \left\|Tz_\alpha + az_\alpha - by_\gamma^m + x_2\right\| - 1/m = \qquad \text{[by (i) of (i)]}$$

$$\left\|Ty_\gamma^m + (a+b)y_\gamma^m + x_2\right\| + \max\{\|(d_\alpha + a)z_\alpha + x_2\|, \left\|-by_\gamma^m + x_2\right\|\} - 1/m$$

$$\left[\text{as } z_\alpha, y_\gamma^m \text{ are good over } H(I_\gamma^m)\right]$$

$$\geq \left\|Ty_\gamma^m + (a+b)y_\gamma^m + x_1\right\| + \max\{|d_\alpha + a|, |b|\} - 1/m$$

$$\geq d\left(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m\rangle\right) + \max\{|d_\alpha + a|, |b|\} - 1/m$$

$$\geq c\left(H(I_\gamma^m), T_\gamma, \overline{Z_\gamma}\right) + \max\{|d_\alpha + a|, |b|\} - 2/m.$$

We can conclude that

(ii) $|b|, |d_\alpha + a| < 4/m$,

(iii) $\left\|Ty_\gamma^m + (a+b)y_\gamma^m + x_1\right\| \leq d\left(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m\rangle\right) + 4/m$

(for (ii) look at the first and last terms in our series of inequalities, for (iii), if it fails use this in the passage from the fifth term to the sixth term, and we shall get a contradiction).

Combining (ii) and (iii) we get

(iv) $\left\|Ty_\gamma^m - d_\alpha y_\gamma^m + x_1\right\| \leq d\left(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m\rangle\right) + 12/m$. Now remember $|d_\alpha| < \varepsilon, 1/m < \varepsilon$ hence

(v) $n\left\|Ty_\gamma^m + x_1\right\| \leq d\left(Ty_\gamma^m, \langle H(I_\gamma^m), y_\gamma^m\rangle\right) + 13\varepsilon$.

Similarly for β instead α (d) holds, but $\left|1 - d_\beta\right| < \varepsilon$ hence for some $x_1' \in H(I_\gamma^m)$

(vi) $\left\|Ty_\gamma^m - y_\gamma^m + x_1'\right\| \leq d\left(Ty_\gamma^m, \langle, y_\gamma^m\rangle\right) + 13\varepsilon$.

By the version of (iv) for (β) for $y = y_\gamma^m + z_\beta$ and the choice of y_γ^m in Stage (1.1.10)

(vii) $d\left(Ty, \langle H(I_\gamma^m), y\rangle\right) > c\left(H(I_\gamma^m), T_i, \overline{Z_i}\right) - 1/m\,(1)$. Now z_β is good over $\langle H(I_\gamma^m), y_\gamma^m, T\rangle$ hence

(viii) $d\left(Ty, H(I_\gamma^m)\right) = \inf_x \left\|Ty_\gamma^m - d_\beta z_\beta + x\right\|$

$$= \inf_{a,,x,x_1}\left[\left\|Ty_\gamma^m - ay_\gamma^m + +x_1\right\| + \left\|d_\beta z_\beta - ay_\gamma^m + x_2\right\|\right]$$

$$\geq d\left(Ty_\gamma^m, \langle, y_\gamma^m, y_\gamma^m\rangle\right) + 1 - \varepsilon \geq$$

[by (v)]

$$\geq d\left(Ty_\gamma^m, H(I_\gamma^m)\right) + 1 - 14\varepsilon$$

So y contradicts the definition of $c_{1/m(1)}(H(I_\gamma^m), \overline{T_\gamma}, \overline{Z_\gamma})$ and the choice of y_γ^m.

Section (1.2): Distortable Banach Space:

We consider the following notions.

Definition (1.2.1)[3]: Let X be an infinite dimensional Banach space, and $\|\cdot\|$ its norm. If $|\cdot|$ is an equivalent norm on X and $\lambda > 1$ we say $|\cdot|$ is a λ-distortion of X if for each infinite dimensional subspace Y of X we have

$$\sup\left\{\frac{y_1}{y_2} : \ y_1, y_2 \in Y \ \ \|y_1\| = \|y_2\| = 1\right\} \geq \lambda.$$

X is called λ-distortable if there exists a λ-distortion on X . X is called distortable if X is λ-distortable for some $\lambda > 1$, and X is called arbitrarily distortable if X is λ-distortable for all $\lambda > 1$.

From the proof of [5, Theorem 5.2, p.145] it follows that each infinite dimensional uniform convex Banach space which does not contain a copy of l_p, $1 < p < \infty$, has a distortable subspace. In [6] this result was generalized to any infinite dimensional Banach space which does not contain a copy of l_p, $1 \leq p < \infty$, or c_0 .

We construct a Banach space X which is arbitrarily distortable. We first want to mention the following questions which are suggested by the existence of such a space.

Problem. Is every distortable Banach space arbitrarily distortable? Is, for example, Tsirelson's space T (as presented in [7, Example 2.e.1]) arbitrarily distortable?

We first want to introduce some notations.

The vector space of all real valued sequences (x_n) whose elements are eventually zero is denoted by c_{00}, (e_i) denotes the usual unit vector basis of c_{00} i.e., $e_i(j)=1$ if $i = j$ and $e_i(j)= 0$ if $i \neq j$.

For $x = \sum_{i=1}^{\infty} \alpha_i e_i \in c_{00}$ the set $\text{supp}(x)=\{i \in N: a_i \neq 0\}$ is called the support of x. If E and F are two finite subsets of N we write $E < F$ if $\max(E) < \min(F)$, and for $x, y \in c_{00}$ we write $x < y$ if $\text{supp}(x) < \text{supp}(y)$.

For $E \subset N$ and $x=\sum_{i=1}^{\infty} x_i e_i \in c_{00}$ we put $E(x) := \sum_{i \in E} x_i e_i$. For the construction of X we need a function : $[1, \infty) \to [1, \infty)$ having the properties (f_1) through (f_5) as stated in the following lemma. The verification of (i), (ii), and (iii) are trivial while the verification of (iv) and (v) are straightforward.

Lemma (1.2.2)[3]: Let $f(x) = \log_2(x+1)$, for $x \geq 1$. Then f has the following properties:

(i) $f(1) = 1$ and $f(x) < x$ for all $x > 1$,

(ii) f is strictly increasing to ∞,

(iii) $\lim_{x\to\infty} \frac{f(x)}{x^q} = 0$ for all $q < o$,

(iv) the function $g(x) = \frac{x}{f(x)}, x \geq 1$ is concave, and

(v) $f(x) \cdot f(y) \geq f(x \cdot y)$ for $x, y \geq 1$.

For the sequel we fix a function f having the properties stated in Lemma (1.2.2) On c_{00} we define by induction for each $k \in N_0$ a norm $|\cdot|_k$. For $x = \sum x_n \cdot e_n \in c_{00}$.

Let $|x|_0 = \max_{n \in N} |x_n|$. Assuming that $|x|_k$ is defined for some $k \in N_0$ we put

$$|x|_{k+1} = \max_{\substack{l \in N \\ E_1 < E_2 < \cdots < E_l \\ E_i \subset N}} \frac{1}{f(l)} \sum_{i=1}^{l} |E_i(x)|_k$$

Since $f(1) = 1$ it follows that $(|x|_k)$ is increasing for any $x \in c_{00}$ and since $f(l) > 1$ for all $l \geq 2$ it follows that $|e_i|_k = 1$ for any $i \in N$ and $k \in N_0$. Finally, we put for $x \in c_{00}$

$$\|x\| = \max_{k \in N} |x|_k.$$

Then $\|\cdot\|$ is a norm on c_{00} and we let X be the completion of c_{00} with respect to $\|\cdot\|$. The following proposition states some easy facts about X .

Proposition (1.2.3)[3]:

(i) (e_i) is a 1-subsymmetric and 1-unconditional basis of X; i.e, for any $x = \sum_{i=1}^{\infty} x_i \cdot e_i \in X$ and strictly increasing sequence $(n_i) \subset N$ and any $(\varepsilon_i)_{i \in N} \subset \{-1,1\}$ it follows that

$$\left\| \sum_{i=1}^{\infty} x_i \cdot e_i \right\| = \left\| \sum_{i=1}^{\infty} \varepsilon_i x_i \cdot e_i \right\|.$$

(ii) For $x \in X$ it follows that

$$\|x\| = \max\left\{ |x|_0, \sup_{\substack{l \geq 2 \\ E_1 < E_2 < \cdots < E_l}} \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\| \right\}$$

(Where $|x|_0 = \sup_{n\in N}|x_n| \text{for} \sum_{i=1}^{\infty} x_i \cdot e_i \in X$)

Proof. Part (i) follows from the fact that (e_i) is a 1-unconditional and 1-subsymmetric basis of the completion of c_{00} with respect to $|\cdot|_k$ for any $k \in N_0$, which can be verified by induction for every k $\in$N. Since c_{00} is dense in X it is enough to show the equation in (ii) for an $x \in c_{00}$. If $\|x\| = |x|_0$ it follows for all $l \geq 2$ and finite subsets $E_1, E_2, \dots E_l$ of N with $E_1 < E_2 < \cdots < E_l$

$$\frac{1}{f(l)}\sum_{i=1}^{l}\|E_i(x)\| = \max_{k\geq 0}\frac{1}{f(l)}\sum_{i=1}^{l}|E_i(x)|_k \leq \max_{k\geq 1}|x|_k \leq \|x\|,$$

which implies the assertion in this case.

If $\|x\| = |x|_k > |x|_{k-1} \geq |x|_0$ for somek ≥ 1 there are , $\acute{l} \in N, l \geq 2$ finite subs of $E_1, E_2, \dots E_l$ and $\acute{E_1}, \acute{E_2}, \dots \acute{E_{\acute{l}}}$ with $E_1 < E_2 < \cdots E_l$ and $\acute{E_1}, < \acute{E_2} < \cdots < \acute{E_{\acute{l}}}$, and a$\acute{k} \in N$ so that

$$\begin{aligned}\|x\| &= |x|_k \\ &= \frac{1}{f(l)}\sum_{i=1}^{l}|E_i(x)|_{k-1} \\ &\leq \frac{1}{f(l)}\sum_{i=1}^{l}\|E_i(x)\| \\ &\leq \sup_{\substack{2\leq\tilde{l} \\ \widetilde{E_1}<\widetilde{E_2}<\cdots<\widetilde{E_{\tilde{l}}}}} \frac{1}{f(\tilde{l})}\sum_{i=1}^{\tilde{l}}\|\widetilde{E_\imath}(x)\| \\ &= \frac{1}{f(\acute{l})}\sum_{i=1}^{\tilde{l}}\|\acute{E_\imath}(x)\| \\ &= \frac{1}{f(\acute{l})}\sum_{i=1}^{\tilde{l}}|\acute{E_i}(x)|_{\acute{k}} \\ &\leq |x|_{\acute{k}+1} \leq \|x\|,\end{aligned}$$

which implies the assertion.

Remark (1.2.4)[3]:

(i) The equation in Proposition (1.2.3). determines the norm $\|\cdot\|$, in the following sense: If $\|\cdot\|$ is a norm on C_{00} with $\|e_i\|$=1 for all $i \in N$ and with the property that

$$\|x\| = \max\left\{|x|_0, \sup_{\substack{l>2 \\ E_1<E_2<\cdots<E_l}} \frac{1}{f(l)}\sum_{i=1}^{l}|\|E_i(x)\||\right\}$$

for all x$\in C_{00}$, then it follows that $\|\cdot\|$, and $|\|\cdot\||$, are equal. Indeed one easily shows by induction for each m $\in$ IN and each x$\in C_{00}$ with #supp(x)= m that $\|x\| = |\|x\||$.

(ii) The equation in Proposition (1.2.3) is similar to the equation which defines Tsirelson's space T [7, Example 2.e.1]. Recall that T is generated by a norm $\|\cdot\|_T$ on C_{00} satisfying

$$\|x\|_T = \max\left\{|x|_0, \sup_{\substack{l\in N \\ l\leq E_1<\cdots E_l}} \frac{1}{2}\sum_{i=1}^{l}\|E_i(x)\|_T\right\}$$

(where $\ell \le E_1$ means that $\ell \le \min E_1$). Note that in the above equation the supremum is taken over all "admissible collections" $E_1 < E_2 < \cdots < E_l$ (meaning that $\ell \le E_1$) while the norm on X is computed by taking all collections $E_1 < E_2 < \cdots < E_l$. This forces the unit vectors in T to be not subsymmetric, unlike in X. The admissibility condition, on the other hand, is necessary in order to imply that T does not contain any l_p, $1 \le p < \infty$, or C_0, which was the purpose of its construction. We will show that X does not contain any subspace isomorphic to l_p, $1 < p < \infty$, or C_0 and secondly that X is distortable, which by [4] implies that it cannot contain a copy of l_1, either. Thus, in the case of X, the fact that X does not contain a copy of l_1 is caused by the factor $\frac{1}{f(l)}$(replacing the constant factor $\frac{1}{2}$ in T) which decreases to zero for increasing ℓ.

In order to state the main result we define for $l \in N$, $l \ge 2$, and $x \in X$

$$\|x\|_l := \left\{ \sup_{E_1<E_2<\cdots E_l} \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\| \right\}.$$

For each $l \in N$, $\|\cdot\|_l$ is a norm on X and it follows that

$$\frac{1}{f(l)} \|x\| \le \|x\|_l \le \|x\|, \quad \text{for } x \in X$$

Theorem (1.2.5)[3]: For each $l \in N$, each $\varepsilon > 0$, and each infinite dimensional subspace Z of X there are $z_1, z_2 \in Z$ with $\|z_1\| = \|z_1\| = 1$ and in particular, $\|\cdot\|_l$is an f(l)-distortion for each $l \in N$.

Proof. Let Z be an infinite dimensional subspace of X and $\varepsilon > 0$. By passing to a further subspace and by a standard perturbation argument we can assume that Z is generated by a block of (e_i)

Choice of z_1 :

By Lemma (1.2.8) and Lemma (1.2.9) one finds $(y_i)_{i=1}^{l} \subset Y$, with $y_1 < y_2 < \cdots < y_l$ so that $\|y_i\| \ge 1 - \varepsilon, 1 \le i \le l$, and so that

$$\left\| \sum_{i=1}^{l} y_i \right\| \le \frac{l}{g(l)}.$$

Thus, choosing

$$z_1 = \sum_{i=1}^{l} y_i \Big/ \left\| \sum_{i=1}^{l} y_i \right\|$$

it follows that

$$\|z_1\|_l \ge \frac{1}{g(l)} \sum_{i=1}^{l} \|y_i\| \Big/ \left\| \sum_{i=1}^{l} y_i \right\| \begin{bmatrix} \text{choose} E_i = \operatorname{supp}(y_i) \\ \text{for } i = i, \dots \dots, l \end{bmatrix}$$

which shows the desired property of z_1.

Choice of z_2:

Let $n \in N$ so that $\frac{4l}{n} \le \varepsilon$ and choose according to Lemma (1.2.8) normalized elements$x_1 < x_2 < \cdots < x_n$of Z so that $(x_i)_{i=1}^{n}$ is $(1 + \varepsilon/2)$-equivalent to the unit basis of l_1^n and put $z_2 = \sum_{i=1}^{n} x_i / \left\| \sum_{i=1}^{l} x_i \right\|$.

Now let $E_1, \dots, E_l$ be finite subsets of N so that $E_1 < E_2 < \dots < E_l$ and so that

$$\|z_2\|_l = \frac{1}{f(l)}\sum_{i=1}^{l}\|E_i(z_2)\|.$$

We can assume that E_i is an interval inIN for each $i \le l$. For each $i \in N$ there are at most two elements $j_1, j_2 \in \{1, \ldots, n\}$ so that $supp(x_{js}) \cap E_i \neq \emptyset$ and $supp(x_{js})/E_i \neq \emptyset, s = 1,2$. Putting for i = 1, 2, . . . , l

$$\widetilde{E_i} = \cup\left\{supp(x_j) : j \le n \text{ and } supp(x_j) \subset E_i\right\}$$

it follows that $\|E_i(z_2)\| \le \left\|\widetilde{E_i}(z_2)\right\| + \frac{2}{n}$, and, thus, from the fact that $(\widetilde{E_i}(z_2)$: i =1,2,….., l) is a block of a sequence which is (1 + ε/2)-equivalent to the l_1^n unit basis, it follows that

$$\|z_2\|_l \le \frac{l}{2n} + \frac{1}{f(l)}\sum_{i=1}^{l}\left\|\widetilde{E_i}(z_2)\right\| \le \frac{\varepsilon}{2} + \frac{1+\varepsilon/2}{f(l)}\left\|\sum_{i=1}^{l}\widetilde{E_i}(z_2)\right\| \le \varepsilon + \frac{1}{f(l)}.$$

which verifies the desired property of z_2.

Remark (1.2.7)[3]: Considering for n∈ N the space $T_{1/n}$ (see for example [8]) which is the completion of C_{00} under the norm$\|\cdot\|_{(T,1/n)}$, satisfying the equation

$$\|x\|_{(T,1/n)} = \max\left\{|x|_0, \sup_{l\le E_1<E_2\ldots\ldots E_l}\frac{1}{n}\cdot\sum_{i=1}^{l}\|E_i\|_{(T,1/n)}\right\}$$

for all x $\in C_{00}$and putting for x $\in T_{1/n}$

$$|\|x\||_{(T,1/n)} = \sup_{E_1<E_2\ldots\ldots E_n}\sum_{i=1}^{l}\|E_i\|_{1/n}$$

E. Odell [5] observed that $|\|x\||_{(T,1/n)}$ is a c · n distortion of $T_{1/n}$ (where c is a universal constant).

In order to show Theorem (1.2.6) we will state the following three lemmas, and leave their proof for the next section.

Lemma (1.2.8)[3]: For n ∈ IN it follows that

$$\left\|\sum_{i=1}^{n} e_i\right\| = \frac{n}{f(n)}$$

For the statement of the next lemma we need the following notion. If Y is a Banach space with basis (y_i) and if $1 \le p \le \infty$ we say that l_p is finitely block represented in Y if for any ε > 0 and any n∈ N there is a normalized block $(z_i)_{i=1}^n$ of length n of (y_i), which is (1 + ε)-equivalent to the unit basis of l_p^n and we call (z_i) a block of (y_i) if $z_i = \sum_{j=k_{i-1}+1}^{k_i}\alpha_j y_j$ for i = 1, 2, . . . and some $0 = k_0 < k_1 < \cdots$ in IN_0 and $(\alpha_j) \subset R$.

Proof. By induction we show for each n∈N that $\left\|\sum_{i=1}^{l} e_i\right\| = \frac{n}{f(n)}$. If n = 1 the assertion is clear. Assume that it is true for all ñ< n, where n ≥ 2. Then there is an l∈N, 2 ≤ l ≤ n, and there are finite subsets of N, $E_1 < E_2 < \ldots < E_l$, so that

$$\left\|\sum_{i=1}^{l} e_i\right\| = \frac{1}{f(l)}\sum_{i=1}^{l}\left\|E_j\left(\sum_{i=1}^{l} e_i\right)\right\|$$

$$= \frac{1}{f(l)}\sum_{i=1}^{l}\frac{n_i}{f(n_i)} \qquad [\text{where } n_i = \#E_i, \text{and } \sum n_i = n]$$

$$= \frac{l}{f(l)} \sum_{i=1}^{l} \frac{1}{l} \cdot \frac{n_i}{f(n_i)}$$

$$\leq \frac{l}{f(l)} \frac{\frac{n}{l}}{f(\frac{n}{l})} \qquad \text{[Property (iv) of Lemma (1.2.3)]}$$

$$= \frac{n}{f(l) \cdot f(\frac{n}{l})}$$

$$\leq \frac{n}{f(n)} \qquad \text{[Property (v)of Lemma (1.2.3)]}$$

Since it is easy to see that $\left\|\sum_{i=1}^{l} e_i\right\| \geq \frac{n}{f(n)}$, the assertion follows.

Lemma (1.2.9)[3]: l_i is finitely block represented in each infinite block of (e_i).

Proof . The statement of Lemma (1.2.9) will essentially follow from the Theorem of Krivine ([9] and [10]). It says that for each basic sequence (y_n) there ia a $1 \leq p \leq \infty$ so that l_p is finitely block represented in (y_i). Thus, we have to show that l_p, $1 < p \leq \infty$, is not finitely represented in any block basis of (e_i). This follows from the fact that for any $1 < p \leq \infty$, any $n \in N$ and any block basis $(x_i)_{i=1}^{n}$ of (e_i) we have

$$\left\|\frac{1}{n^{1/p}} \sum_{i=1}^{l} x_i\right\| \geq \frac{1}{n^{1/p}} \frac{n}{f(n)} = \frac{n^{1-1/p}}{f(n)}$$

and from (iii).

Lemma (1.2.10)[3]: Let (y_n) be a block basis of (e_i) with the following property: There is a strictly increasing sequence $(k_n) \subset N$, a sequence $(\varepsilon_n) \subset R_+$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and for each n a normalized block basis $\big((n,i)\big)_{i=1}^{k_n}$ which is $(1 + \varepsilon_n)$-equivalent to the $l_1^{\varepsilon_n}$-unit basis so that

$$y_n = \frac{1}{k_n} \sum_{i=1}^{k_n} y(n,i).$$

then it follows for all $l \in N$

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \dots \lim_{n_l\to\infty} \left\|\sum_{i=1}^{l} y_{ni}\right\| = \frac{l}{g(l)}.$$

Proof. Let $y_n = \frac{1}{k_n} \sum_{i=1}^{k_n} y(n,i)$, for $n \in N$ and $\big(y(n,i)\big)_{i=1}^{k_n} (1 + \varepsilon_m)$-equivalent to the $\ell_1^{k_n}$ unit basis.

For $x, \tilde{x} \in c_{00}$ and $m \in N$ with $x < e_m < \tilde{x}$ we will show that

$$\lim_{n\to\infty} \left\|x + y_n + \tilde{x}^{(n)}\right\| = \|x + e_m + \tilde{x}\|, \qquad (1)$$

where

$$\tilde{x}^{(n)} = \sum_{i=m+1}^{\infty} \tilde{x}_i \cdot e_{i+s_n} \left(\tilde{x} = \sum_{i=m+1}^{\infty} \tilde{x}_i \cdot e_i\right)$$

and $s_n \in N$ is chosen big enough so that $y_n < \tilde{x}^{(n)}$.

This would, together with Lemma (1.2.7), imply the assertion of Lemma (1.2.9). Indeed, for $l \in N$ it follows from (1) that

$$\frac{l}{f(l)} = \left\|\sum_{i=1}^{l} e_i\right\| \qquad \text{(Lemma (1.2.7))}$$

$$= \lim_{n\to\infty}\left\|e_1 + \sum_{i=2}^{l} e_{i+n}\cdot\right\| \qquad \text{[sub symmetry]}$$

$$= \lim_{n_1\to\infty}\lim_{n\to\infty}\left\|y_{n_1} + \sum_{i=2}^{l} e_{i+n}\right\|$$

$$= \lim_{n_1\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\left\|y_{n_1} + e_n + \sum_{i=3}^{l} e_{i+m}\right\|$$

$$= \lim_{n_1\to\infty}\lim_{n_2\to\infty}\lim_{m\to\infty}\left\|y_{n_1} + y_{n_2} + \sum_{i=3}^{l} e_{i+m}\right\|$$

$$\vdots$$

$$= \lim_{n_1\to\infty}\lim_{n_2\to\infty}\ldots\lim_{n_l\to\infty}\left\|\sum_{i=3}^{l} y_{n_i}\right\|.$$

In order to prove (1) we show first the following

Claim (1.2.10)[3]: For $x, y \in c_{00}$, and $n \in N$, with $x < e_n < y$ and $\alpha, \beta \in R_0^+$ it follows that

$$\|x + \alpha e_n\| + \|\beta e_n + y\| \le \max\{\|x + (\alpha+\beta)e_n\| + \|y\|, \|x\| + \|(\alpha+\beta)e_n + y\|\}.$$

We show by induction for all $k \in N_0$, all $x, y \in c_{00}$, and $n \in N$, with #supp(x) +#supp(y) $\le k$, and $x < e_n < y$ and all $q_1, q_2, \alpha, \beta \in R_0^+$ that

$$q_1\|x + \alpha t_n\| + q_2\|\beta e_n + y\|$$
$$\le \max\{q_1\|x + (\alpha+\beta)e_n\| + q_2\|y\|, q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\|\}.$$

For k= 0 the assertion is trivial. Suppose it is true for somek≥ 0 and suppose $x, y \in c_{00}$, $x < e_n < y$ and #supp (x) + #supp(y) = k + 1. We distinguish between the following cases.

Case (i). $\|x + \alpha e_n\| = |x + \alpha e_n|_0$ and $\|\beta e_n + y\| = |\beta e_n + y|_0$

If $\|x + \alpha e_n\| = |x|_0$, then

$$q_1\|x + \alpha t_n\| + q_2\|\beta e_n + y\| = q_1\|x\| + q_2\|\beta e_n + y\|$$
$$\le q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\|.$$

If $\|\beta e_n + y\| = |y|_0$ we proceed similarly and if $\|x + \alpha e_n\| = \alpha$, and $\|\beta e_n + y\| = \beta$ if w.l.o.g., $q_1 \le q_2$, it follows that

$$q_1\|x + \alpha t_n\| + q_2\|\beta e_n + y\| = q_1\alpha + q_2\,\beta \le q_2(\alpha+\beta)$$
$$\le q_1\|x\| + q_2\|e_n(\alpha+\beta) + y\|.$$

Case (ii). $\|x + \alpha e_n\| \ne |x + \alpha e_n|_0$.

Then we find $l \ge 2$ and $E_1 < E_2 < \ldots < E_l$ so that $E_i \cap \mathrm{supp}(x) \ne \emptyset$ for i = 1, . . . , l and

$$q_1\|x\| + q_2\|e_n(\alpha+\beta) + y\|$$

$$= \frac{q_1}{f(l)}\left[\sum_{i=1}^{l-1} \|E_i(x)\| + \|E_l(x+\alpha e_n)\|\right] + q_2\|\beta e_n + y\|$$

$$\leq \frac{q_1}{f(l)}\sum_{i=1}^{l-1} \|E_i(x)\| + \begin{cases} \frac{q_1}{f(l)}\|E_l(x) + (\alpha+\beta)e_n\| + q_2\|y\| \\ \text{or} \\ \frac{q_1}{f(l)}\|E_l(x)\| + q_2\|(\alpha+\beta)e_n + y\| \end{cases}$$

[By the induction hypothesis]

$$\leq \max\{q_1\|x + (\alpha+\beta)e_n\| + q_2\|y\|, q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\|\},$$

which shows the assertion in this case.

In the case $\|\beta e_n + y\| \neq |\beta e_n + y|_0$ we proceed like in Case (ii).

In order to show the equation (1) we first observe that for all $k \in N_0$, $|x + e_m + \tilde{x}|_k \leq \left\|x + y_n + \tilde{x}^{(n)}\right\|$ (which easily follows by induction for each k∈N) and, thus, that $\liminf_{n\to\infty}\left\|x + y_n + \tilde{x}^{(n)}\right\| \geq \|x + e_m + \tilde{x}\|$. Since every subsequence of (y_n) still satisfies.the assumptions of Lemma (1.2.9) it is enough to show that

$$\liminf_{n\to\infty}\left\|x + y_n + \tilde{x}^{(n)}\right\| \leq \|x + e_m + \tilde{x}\|.$$

This inequality will be shown by induction for each $k \in N_0$ and all $x < e_m < \tilde{x}$ with #supp(x)+#supp($\tilde{x}$) ≤ k. For k = 0 the assertion is trivial. We assume the assertion to be true for some k ≥ 0 and we fix x, $\tilde{x} \in c_{00}$ with $x<e_m<\tilde{x}$ and #supp(x)+#supp($\tilde{x}$) = k + 1

We consider the following three cases:

Case (i). $\|x + y_n + \tilde{x}\| = |x + y_n + \tilde{x}|_0$ for infinitely many n ∈N. Since

$\left|x + y_n + \tilde{x}^{(n)}\right|_0 \leq |x + y_n + \tilde{x}|_0$, n∈N ,

the assertion follows.

Case (ii). For a subsequence (y'_n) of (y_n) we have

$$\|x + y'_n + \tilde{x}\| = \frac{1}{f(l_n)}\sum_{i=1}^{l_n}\left\|E_i^{(n)}(x + y'_n + \tilde{x})\right\|$$

where $l_n \uparrow \infty$ and $E_1^{(n)} < E_2^{(n)} < \cdots < E_{l_n}^{(n)}$ are finite subsets of N. Since $f(l_n) \to \infty$ for $n \to \infty$ it then follows that

$$\liminf_{n\to\infty}\left\|x + y_n + \tilde{x}^{(n)}\right\| = 1 \leq \|x + e_n + \tilde{x}\|.$$

Assume now that neither Case i nor Case ii occurs. By passing to a subsequence we can assume

Case (iii). There is an l ≥ 2 so that

$$\lim_{n\to\infty}\left(\left\|x + y_n + \tilde{x}^{(n)}\right\| - \frac{1}{f(l)}\sum_{i=1}^{l}\left\|E_i^{(n)}(x + y_n + \tilde{x}^{(n)})\right\|\right) = 0$$

where $E_1^{(n)} < \cdots < E_l^{(n)}$ are finite subsets of IN with the following properties:

(i) $\operatorname{supp}\left(x + y_n + \tilde{x}^{(n)}\right) \cap E_i^{(n)} \neq \emptyset$, $i \leq l$, and $\operatorname{supp}(x + y_n + \tilde{x}^{(n)}) \subset \cup_{i=1}^{l} E_i^{(n)}$

(ii) The set $\operatorname{supp}(x) \cap E_i^{(n)}$, $i = 1, \ldots, l$ does not depend on n (note that supp(x) <∞), and we denote it by $E_i^{(n)}$

(iii) There are subsets e $\widetilde{E_1} < \widetilde{E_2} < \cdots < \widetilde{E_l}$ of $\mathrm{supp}(\tilde{x})$ and integers r_n so that $\mathrm{supp}(\tilde{x}^{(n)}) \cap {E_i}^{(n)} = \widetilde{E_l} + r_n$ for $n \in N$, (we use the convention that $\emptyset <$ E for any finite $E \subset N$),

(iv) for $i \leq l$ and $1 \leq j \leq k_n$ we have either $\mathrm{supp}(y(n,j)) \subset {E_i}^{(n)}$ or $\mathrm{supp}\big(y(n,j)\big) \cap {E_i}^{(n)} = \emptyset$. Indeed, letting for $i \leq l$

$$\widetilde{E_l}^{(n)} = \begin{cases} {E_i}^{(n)} \text{if } {E_i}^n \cap \ \mathrm{supp}(y_n) = \emptyset \\ \mathrm{supp}(y(n,s) \cup {E_i}^{(n)} \backslash \mathrm{supp}(y(n,s) \\ \ \text{where } s = \min\{\tilde{s}: \mathrm{supp}\big(y(n,\tilde{s})\big) \cap {E_i}^{(n)} \neq \emptyset\} \\ \ \text{an} + \text{d } t = \max\{\tilde{s}: \mathrm{supp}\big(y(n,\tilde{t})\big) \cap {E_i}^{(n)} \neq \emptyset\} \end{cases}$$

The value $\sum_{i=1}^{l} \|{E_i}^{(n)}(x + y_n + \tilde{x}^{(n)})\|$ differs from $\sum_{i=1}^{l} \left\|\widetilde{E_l}^{(n)}(x + y_n + \tilde{x}^{(n)})\right\| \ {}^{2l}/_{k_n}$, which shows that (iv)) can be assumed.

(v) For $i \leq l$ the value

$$q_i = \lim_{n\to\infty} \frac{\{i \leq k_n, \mathrm{supp}(y(n,j)) \subset {E_i}^{(n)}\}}{k_n}$$

exists.Now we distinguish between the following subcases.

Case (iii) í). There are $l_1, l_2 \in$ N, so that $1 \leq l_1, \leq l_2 - 2 < l_2 \leq l$ and

$$\|x + y_n + \tilde{x}^{(n)}\| = \frac{1}{f(l)} = \left\{\sum_{i=1}^{l_1 -1} \|{E_i}^{(n)}(x)\| + \left\|E_{l_1}^{(n)}(x + y_n)\right\| + \sum_{i=l_1 +1}^{l_2 -1} \|{E_i}^{(n)}(y_n)\| \right.$$

$$\left. + \left\|E_{l_2}^{(n)}(y_n + \tilde{x}^{(n)})\right\| + \sum_{i=l_2 +1}^{l} \|{E_i}^{(n)}(\tilde{x}^{(n)})\| \right\}$$

In this case it follows that

$$\|x + y_n + \tilde{x}^{(n)}\| \leq \frac{1}{f(l)} \left[\sum_{i=1}^{l_1} \|{E_i}^{(n)}(x)\| + \sum_{i=l_1}^{l_2} \|{E_i}^{(n)}(y_n)\| + \sum_{i=l_2}^{l} \|{E_i}^{(n)}(\tilde{x}^{(n)})\|\right]$$

$$\leq \frac{1}{f(l)} \left[\sum_{i=1}^{l_1} \|{E_i}^{(n)}(x)\| + 1 + \varepsilon_n + \sum_{i=l_2}^{l} \|{E_i}^{(n)}(\tilde{x}^{(n)})\|\right]$$

[By (iv)) and the fact that $\big(y(j,n)\big)_{j=1}^{k_n}$ is $(1 + \varepsilon_n)$-equivalent to the $l_1^{k_n}$-unit basis]

$$\leq \|x + e_m + \tilde{x}\| + \varepsilon_n \,,$$

Note that

$$[l_1 + 1 + (l - l_2 + 1) \leq l]$$

which implies the assertion in this case.

Case (iii) íı). There is an $1 \leq l_1 \leq l$ so that

$$\|x + y_n + \tilde{x}^{(n)}\| = \frac{1}{f(l)} \left[\sum_{i=1}^{l_1 -1} \|{E_i}^{(n)}(x)\| + \|{E_{l_1,}}^{(n)}(x + y_n + \tilde{x}^{(n)})\|\right.$$

$$+\sum_{i=l_1+1}^{l}\left\|E_i^{(n)}(x^{(n)})\right\|\Bigg].$$

Then the assertion can be deduced from the induction hypothesis (note, that by a) and the fact that $l \geq 2$ we have that $\#\mathrm{supp}\left\|E_{l_1,}{}^{(n)}(x+\tilde{x}^{(n)})\right\| < \#\mathrm{supp}(x+\tilde{x}^{(n)})$).

Case (iii) ííı)... There is an $l_1 < l$ so that

$$\left\|x+y_n+\tilde{x}^{(n)}\right\| =$$

$$= \frac{1}{f(l)}\Bigg[\sum_{i=1}^{l_1-1}\left\|E_i^{(n)}(x)\right\| + \left\|E_{l_1,}{}^{(n)}(x+y_n)\right\| + \left\|E_{l_1+1}{}^{(n)}(y_n+\tilde{x}^{(n)})\right\|$$

$$+\sum_{i=l_2+2}^{l}\left\|E_i^{(n)}(\tilde{x}^{(n)})\right\|\Bigg]$$

We can assume that $\mathrm{supp}(x) \neq 0$ and $\mathrm{supp}(\tilde{x}) \neq 0$ (otherwise we are in case 3b). If ql_1 (as defined in v) vanishes it follows that

$$\lim_{n\to\infty}\left\|E_i^{(n)}(x+y_n)\right\| = \left\|E_{l_1}{}^{x}(x)\right\|.$$

Otherwise there is a sequence $(j_n) \subset N$ with $\lim_{n\to\infty} j_n = \infty$ so that

$$E_{l_1}^{(n)}(y_n) = \frac{1}{k_n}\sum_{j=1}^{j_n} y(n,j)$$

and so that

$$\lim_{n\to\infty}\frac{j_n}{k_n} = ql_1 > 0.$$

Since the sequence $(E_{l_1}{}^{(n)}(y_n)/ql_1)_{n\in N}$ is asymptotically equal to the sequence $(\widetilde{y_n})$ with $\widetilde{y_n} = \frac{1}{j_n}\sum_{j=1}^{j_n} y(n,j)$ (note that $(\widetilde{y_n})$ satisfies the assumption of the lemma) we deduce from the induction hypothesis for some infinite $N \subset \mathrm{IN}$ that

$$\lim_{\substack{n\to\infty\\ n\in N}}\left\|E_{l_1}{}^{(n)}(x+y_n)\right\| = ql_1 \lim_{n\to\infty}\left\|E_i{}^{x}\left(\frac{x}{ql_1}+\widetilde{y_n}\right)\right\|$$

$$\leq ql_1\left\|E_i{}^{x}\left(\frac{x}{ql_1}\right)+e_m\right\|$$

$$= \left\|E_i{}^{x}\left(\frac{x}{ql_1}\right)+ql_1\, e_m\right\|.$$

Similarly we show for some infinite $M \subset N$, that

$$\lim_{\substack{n\to\infty\\ n\in N}}\left\|E_{l_{1+1}}{}^{(n)}(y_n+\tilde{x}^{(n)})\right\| \leq \left\| ql_{1+1}\, e_m + \widetilde{E_{l_{1+1}}}(\tilde{x})\right\|.$$

From the claim at the beginning of the proof we deduce now that

$$\liminf_{n\to\infty}\left\|x+y_n+\tilde{x}^{(n)}\right\|$$

$$\leq \frac{1}{f(l)}\sum_{i=1}^{l_1-1}\left\|E_i^{(n)}(x)\right\| + \left\|E_{l_1}{}^{x}(x)+ql_1\, e_m\right\|$$

$$+\left\| ql_{1+1} + e_m + \widetilde{E_{l_{1+1}}}(\tilde{x}) \right\| + \sum_{i=l_1+2}^{l} \left\| \widetilde{E_\iota}(\tilde{x}) \right\|$$

$$\leq \frac{1}{f(l)} \left[\sum_{i=1}^{l_1-1} \|E_i{}^x(x)\| + \sum_{i=l_1+2}^{l} \left\| \widetilde{E_\iota}(\tilde{x}) \right\| \right.$$

$$\left. + \max\left\{ \left\| E^x_{l_1}(x) + e_m \right\| + \left\| \widetilde{E}_{l_{1+1}}(\tilde{x}) \right\|, \left\| E^x_{l_1}(x) \right\| + \left\| e_m + \widetilde{E}_{l_{1+1}}(\tilde{x}) \right\| \right\} \right]$$

$$[ql_1 + ql_{1+1} = 1]$$

$$\leq \|x + e_m + \tilde{x}\|,$$

which shows the assertion in this case and finishes the proof of the Lemma.

Corollary (1.2.11)[257]. i) (e_i) is a 1-subsymmetric and 1-unconditional basis of X; i.e; for any $x_r = \sum_{i=1}^{\infty}(x_r)_i \cdot e_i \in X$ and strictly increasing sequence $(n_i) \subset N$ and any $(\varepsilon_i)_{i\in N}\{-1,1\}$ it follows that

$$\left\| \sum_{i=1}^{\infty} (x_r)_i \cdot e_i \right\| = \left\| \sum_{i=1}^{\infty} \varepsilon_i (x_r)_i \cdot e_{n_i} \right\| .$$

ii) For $x_r \in X$ it follows that

$$\|x_r\| = \max\left\{ |x_r|_0, \sup_{\substack{\epsilon\geq 0 \\ r<r+1<\cdots<(r-1)_{\epsilon+2}}} \frac{1}{f(\epsilon+2)} \sum_{i=1}^{\epsilon+2} \|(r-1)_i(x_r)\| \right\}$$

(Where $|x_r|_0 = \sup_{n\in N}|(x_r)_n|$ for $\sum_{i=1}^{\infty}(x_r)_i \cdot e_i \in X$)

Proof. Part (i) follows from the fact that (e_i) is a 1-unconditional and 1-subsymmetric basis of the completion of c_{00} with respect to $|\cdot|_k$for any $k \in N_0$, which can be verified by induction for every $k \in N$. Since c_{00} is dense in X it is enough to show the equation in (ii) for an $x_r \in c_{00}$.If $\|x_r\| = |x_r|_0$it follows for all $\epsilon \geq 0$ and finite subsets $r, r+1, \ldots (r-1)_{\epsilon+2}$ of N with $r < r+1 < \cdots < (r-1)_{\epsilon+2}$

$$\frac{1}{f(\epsilon+2)} \sum_{i=1}^{\epsilon+2} \|(r-1)_i(x_r)\| = \max_{k\geq 0} \frac{1}{f(\epsilon+2)} \sum_{i=1}^{\epsilon+2} |(r-1)_i(x_r)|_k \leq \max_{k\geq 1} |x_r|_k \leq \|x_r\|,$$

which implies the assertion in this case.

If $\|x_r\| = |x_r|_k > |x_r|_{k-1} \geq |x_r|_0$ for some $k \geq 1$ there are $\epsilon + 2, \acute{\imath} \in N, \epsilon \geq 0$ finite subs of N , $r, r+1, \ldots (r-1)_{\epsilon+2}$and $\acute{r}, \acute{r}+1, \ldots, \acute{r}_{\acute{\imath}}$with $r < r+1 < \cdots < (r-1)_{\epsilon+2}$and $r' < \acute{r} + 1 < \cdots . < \acute{r}_{\acute{\imath}}$, and a$\acute{k} \in N$ so that

$$\|x_r\| = |x_r|_k$$

$$= \frac{1}{f(\epsilon+2)} \sum_{i=1}^{\epsilon+2} |(r-1)_i(x_r)|_{k-1} \leq \frac{1}{f(\epsilon+2)} \sum_{i=1}^{\epsilon+2} \|(r-1)_i(x_r)\|$$

$$\leq \sup_{\substack{2\leq \tilde{\imath} \\ \tilde{r}<\tilde{r}+1<\cdots<(\tilde{r}-1)_{\tilde{\imath}}}} \frac{1}{f(\tilde{\imath})} \sum_{i=1}^{\tilde{\imath}} \|(\tilde{r}-1)_i(x_r)\|$$

$$= \frac{1}{f(\acute{I})} \sum_{i=1}^{\tilde{I}} \|(\tilde{r}-1)_i(x_r)\|$$
$$= \frac{1}{f(\acute{I})} \sum_{i=1}^{\tilde{I}} |(\tilde{r}-1)_i(x_r)|_{\acute{k}}$$
$$\leq |x_r|_{\acute{k+1}} \leq \|x_r\|,$$
which implies the assertion.

Corollary (1.2.12)[257]. For each $\epsilon + 2 \in N$, each $\varepsilon > 0$, and each infinite dimensional subspace Z of X there are x_{r+3}, $z_{r+4} \in Z$ with $\|x_{r+3}\| = \|x_{r+4}\| = 1$and $\|x_{r+3}\|_{\epsilon+2} \geq 1 - \varepsilon$ and $\|x_{r+4}\|_{\epsilon+2} \leq \frac{1+\varepsilon}{f(\epsilon+2)}$. In particular, $\|\cdot\|_{\epsilon+2}$is an $f(\epsilon+2)$- small distortion for each $\epsilon + 2 \in N$.

Proof.(see [3]). Let Z be an infinite dimensional subspace of X and $\varepsilon > 0$. By passing to a further subspace and by a standard perturbation argument we can assume that Z is generated by a block of (e_i)

Choice of x_{r+3} :

By Corollary (1.2.14) and Corollary (1.2.15) one finds $((x_{r+1})_i)_{i=1}^{\epsilon+2} \subset Y$, with $x_{r+2} < x_{r+3} < \cdots < (x_{r+1})_{\epsilon+2}$ so that $\left\|(\sum_{i=1}^{\epsilon+2}(x_{r+1})_i)\right\| \geq 1 - \varepsilon, 1 \leq i \leq \epsilon + 2$, and so that$\left\|\sum_{i=1}^{\epsilon+2}(x_{r+1})_i\right\|x_{r+1} \leq \frac{\epsilon+2}{g(\epsilon+2)}$.Thus, choosing
$$x_{r+3} = \sum_{i=1}^{\epsilon+2}(x_{r+1})_i \Big/ \left\|\sum_{i=1}^{\epsilon+2}(x_{r+1})_i\right\|$$
It follows that
$$\|x_{r+3}\|_{\epsilon+2} \geq \frac{1}{g(\epsilon+2)} \sum_{i=1}^{\epsilon+2} \|(x_{r+1})_i\| \Big/ \left\|\sum_{i=1}^{\epsilon+2}(x_{r+1})_i\right\| \ \begin{bmatrix}\text{choose}(x_{r+1})_i = \text{supp}(x_{r+1})_i \\ \text{for } i = 1, \dots\dots, \epsilon + 2\end{bmatrix}$$
Which shows the desired property of x_{r+3}

Choice of x_{r+4}:

Let $n \in N$ so that $\frac{4(\epsilon+2)}{n} \leq \varepsilon$ and choose according to Lemma(1.2.8) normalized elements $x_{r+1} < x_{r+2} < \cdots < (x_r)_n$of Z so that $((x_r)_i)_{i=1}^n$ is $(1 + \varepsilon/2)$-equivalent to the unit basis of l_1^n and put
$$x_{r+4} = \sum_{i=1}^{n}(x_r)_i \Big/ \left\|\sum_{i=1}^{\epsilon+2}(x_r)_i\right\|.$$
Now let $r, \dots, (r-1)_{\epsilon+2}$ be finite subsets of N so that
$r < r + 1 < \dots < (r-1)_{\epsilon+2}$
And so that $\|x_{r+4}\|_{\epsilon+2} = \frac{1}{f(\epsilon+2)} \sum_{i=1}^{\epsilon+2} \|(r-1)_i(x_{r+4})\|$.

We can assume that $(r-1)_i$ is an interval in IN for each $i \leq \epsilon + 2$.

For each $i \in N$ there are at most two elements $j_1, j_2 \in \{1,\dots,n\}$so that $\text{supp}(x_r)_{js} \cap (r-1)_i \neq \emptyset$ and $\text{supp}(x_r)_{js}/(r-1)_i \neq \emptyset$, $s = 1,2$. Putting for $i = 1, 2\dots, (\epsilon+2)(\tilde{r}-1)_i = \cup \{\text{supp}(r-1)_j : j \leq n \text{ and } \text{supp}(x_r)_j \subset (r-1)_i\}$ it follows that $\|(r-1)_i(x_{r+4})\| \leq$

$\|(\tilde{r}-1)_i(x_{r+4})\| + \frac{2}{n}$, and, thus, from the fact that $((\tilde{r}-1)_i(x_{r+4})$: i =1,2,….., $\epsilon + 2)$ is a block of a sequence which is (1 + ε/2)-equivalent to the l_1^n unit basis, it follows that

$$\|x_{r+4}\|_{\epsilon+2} \leq \frac{\epsilon+2}{2n} + \frac{1}{f(\epsilon+2)}\sum_{i=1}^{\epsilon+2}\|(\tilde{r}-1)_{\epsilon+2}(x_{r+4})\| \leq \frac{\varepsilon}{2} + \frac{1+\varepsilon/2}{f(\epsilon+2)}\left\|\sum_{i=1}^{\epsilon+2}(\tilde{r}-1)_{\epsilon+2}(x_{r+4})\right\|$$
$$\leq \varepsilon + \frac{1}{f(\epsilon+2)}.$$

Which verifies the desired property of x_{r+4}.

Corollary (1.2.13)[257]. For n ∈IN it follows that

$$\left\|\sum_{i=1}^{n} e_i\right\| = \frac{n}{f(n)}$$

For the statement of the next lemma we need the following notion. If Y is a Banach space with basis $(x_{r+1})_i$ and if $\epsilon \geq 0$ we say that $l_{\epsilon+1}$ is finitely block represented in Y if for any ε > 0 and any n ∈ N there is a normalized block $((x_{r+2})_i)_{i=1}^n$ of length n of $(x_{r+1})_i$,which is (1 + ε)-equivalent to the unit basis of $l_{\epsilon+1}^n$ and we call $(x_{r+2})_i$ a block of $(x_{r+1})_i$ if $(x_{r+2})_i = \sum_{j=k_{i-1}+1}^{k_i}(\alpha^r)_j(x_{r+2})_j$ for i = 1, 2, . . . and some $0= k_0 < k_1 < \cdots$ in IN_0 and $(\alpha^r)_j \subset R$.

Proof

By induction we show for each n ∈N that $\left\|\sum_{i=1}^{\epsilon+2} e_i\right\| = \frac{n}{f(n)}$. If n = 1 the assertion is clear. Assume that it is true for all $\tilde{n} < n$, where $n \geq 2$. Then there is an $\epsilon + 2 \in N$, $0 \leq \epsilon \leq n-2$, and there are finite subsets of N, $r < r+1 < \cdots < (r-1)_{\epsilon+2}$, so that

$$\left\|\sum_{i=1}^{\epsilon+2} e_i\right\| = \frac{1}{f(\epsilon+2)}\sum_{i=1}^{\epsilon+2}\left\|(r-1)_j\left(\sum_{i=1}^{\epsilon+2} e_i\right)\right\|$$
$$= \frac{1}{f(\epsilon+2)}\sum_{i=1}^{\epsilon+2}\frac{n_i}{f(n_i)} \text{ [where } n_i = (r-1)_i, \text{and } \sum n_i = n]$$
$$= \frac{\epsilon+2}{f(\epsilon+2)}\sum_{i=1}^{\epsilon+2}\frac{1}{\epsilon+2}\cdot\frac{n_i}{f(n_i)}$$
$$\leq \frac{\epsilon+2}{f(\epsilon+2)}\frac{\frac{n}{\epsilon+2}}{f\left(\frac{n}{\epsilon+2}\right)} \text{ [Property (iv) of Corollary (1.2.13)]}$$
$$= \frac{n}{f(\epsilon+2)\cdot f\left(\frac{n}{\epsilon+2}\right)}$$
$$\leq \frac{n}{f(n)} \qquad \text{[Property(v) of Corollary (1.2.11)]}$$

Since it is easy to see that $\left\|\sum_{i=1}^{\epsilon+2} e_i\right\| \geq \frac{n}{f(n)}$, the assertion follows.

Corollary (1.2.14)[257]. l_i is finitely block represented in each infinite block of (e_i).

Proof .

The statement of Lemma(1.2.8) will essentially follow from the Theorem of Krivine ([3] and [4]). It says that for each basic sequence $(x_{r+1})_n$ there is $\epsilon \geq 0$ so that $l_{\epsilon+1}$ is finitely block represented in $(x_{r+1})_i$.Thus, we have to show that $l_{\epsilon+1}$, $\epsilon > 0$ is not finitely represented in any block basis of (e_i). This follows from the fact that for any $0 \leq \epsilon \leq \infty$, any n∈N and any block basis $((x_r)_i)_{i=1}^{n}$ of (e_i) we have

$$\left\| \frac{1}{n^{1/\epsilon+1}} \sum_{i=1}^{n} (x_r)_i \right\| \geq \frac{1}{n^{1/\epsilon+1}} \frac{n}{f(n)} \;\; = \frac{n^{1-1/\epsilon+1}}{f(n)}$$

and from (iii).

Corollary (1.2.15)[257]. Let $(x_{r+1})_n$ be a block basis of (e_i) with the following property: There is a strictly increasing sequence $(k_n)\subset N$, a sequence $(\varepsilon_n)\subset R_+$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and for each n a normalized block basis $\left(x_{r+1}(n,i)\right)_{i=1}^{k_n}$ which is $(1+\varepsilon_n)$-equivalent to the$l_1^{\epsilon_n}$-unit basis so that

$$(x_{r+1})_n = \frac{1}{k_n} \sum_{i=1}^{k_n} x_{r+1}(n,i).$$

then it follows for all $\epsilon + 2 \in N$

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \dots \lim_{n_{\epsilon+2}\to\infty} \left\| \sum_{i=1}^{\epsilon+2} (x_{r+1})_{n_i} \right\| = \frac{\epsilon+2}{g(\epsilon+2)}.$$

Proof .

Let $(x_{r+1})_n = \frac{1}{k_n} \sum_{i=1}^{k_n} x_{r+1}(n,i)$, for $n \in N$ and $\left(x_{r+1}(n,i)\right)_{i=1}^{k_n}(1+\varepsilon_m)$-equivalent to the $l_1^{k_n}$ unit basis. For $x_r, \widetilde{(x_r)} \in c_{00}$ and $m \in N$ with $x_r < e_m < \widetilde{(x_r)}$ we will show that

$$\text{(2)} \qquad \lim_{n\to\infty} \left\| x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)} \right\| = \left\| x_r + e_m + \widetilde{(x_r)} \right\|,$$

where

$$\widetilde{(x_r)}^{(n)} = \sum_{i=m+1}^{\infty} \widetilde{(x_r)}_i \cdot e_{i+s_n} \left(\widetilde{(x_r)} = \sum_{i=m+1}^{\infty} \widetilde{(x_r)}_i \cdot e_i \right)$$

and s_n∈N is chosen big enough so that $(x_{r+1})_n < \widetilde{(x_r)}^{(n)}$.

This would, together with Lemma 4, imply the assertion of Corollary(1.2.15) **.** Indeed, for $\epsilon + 2$ ∈N it follows from (2) that

$$\frac{\epsilon+2}{f(\epsilon+2\,)} = \left\| \sum_{i=1}^{\epsilon+2} e_i \right\| \quad \text{(Corollary (1.2.13))}$$

$$= \lim_{n\to\infty} \left\| e_1 + \sum_{i=2}^{\epsilon+2} e_{i+n} \cdot \right\| \qquad \text{[sub symmetry]}$$

$$= \lim_{n_1\to\infty} \lim_{n\to\infty} \left\| (x_{r+1})_{n_1} + \sum_{i=2}^{\epsilon+2} e_{i+n} \right\|$$

$$= \lim_{n_1\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \left\| (x_{r+1})_{n_1} + e_n + \sum_{i=3}^{\epsilon+2} e_{i+m} \right\|$$

$$= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \lim_{m\to\infty} \left\| (x_{r+1})_{n_1} + (x_{r+1})_{n_1} + \sum_{i=3}^{\epsilon+2} e_{i+m} \right\|$$

$$\vdots$$

$$= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \ldots\ldots \lim_{n_{\epsilon+2}\to\infty} \left\| \sum_{i=3}^{\epsilon+2} (x_{r+1})_{n_i} \right\|.$$

In order to prove (2) we show first the following

Claim. For $x_r, x_{r+1} \in c_{00}$, and $n \in N$, with $x_r < e_n < x_{r+1}$ and $\alpha^r, \beta \in R_0^+$ it follows that $\|x_r + \alpha^r e_n\| + \|\beta e_n + x_{r+1}\| \leq$

$$\max\{\|x_r + (\alpha^r + \beta)e_n\| + \|x_{r+1}\|, \|x_r\| + \|(\alpha^r + \beta)e_n + x_{r+1}\|\}.$$

We show by induction for all $k \in N_0$, all $x_r, x_{r+1} \in c_{00}$, and $n \in N$, with $\#supp(x_r)$ $+\#supp(x_{r+1}) \leq k$, and $x_r < e_n < x_{r+1}$ and all q_1, q_2, $\alpha^r, \beta \in R_0^+$ that

$$q_1\|x_r + \alpha^r t_n\| + q_2\|\beta e_n + x_{r+1}\|$$
$$\leq \max\{q_1\|x_r + (\alpha^r + \beta)e_n\| + q_2\|x_{r+1}\|, q_1\|x_r\| + q_2\|(\alpha^r + \beta)e_n + x_{r+1}\|\}.$$

For k= 0 the assertion is trivial. Suppose it is true for some $k \geq 0$ and suppose $x_r, x_{r+1} \in c_{00}$, $x_r < e_n < x_{r+1}$ and $\#supp(x_r) + \#supp(x_{r+1}) = k+1$. We distinguish between the following cases.

Case (i). $\|x_r + \alpha^r e_n\| = |x_r + \alpha^r e_n|_0$ and $\|\beta e_n + x_{r+1}\| = |\beta e_n + x_{r+1}|_0$

If $\|x_r + \alpha^r e_n\| = |x_r|_0$, then

$$q_1\|x_r + \alpha^r e_n\| + q_2\|\beta e_n + x_{r+1}\| = q_1\|x_r\| + q_2\|\beta e_n + x_{r+1}\|$$
$$\leq q_1\|x_r\| + q_2\|e_n(\alpha^r + \beta) + x_{r+1}\|.$$

If $\|\beta e_n + x_{r+1}\| = |x_{r+1}|_0$ we proceed similarly and if $\|x_r + \alpha^r e_n\| = \alpha^r$, and $\|\beta e_n + x_{r+1}\| = \beta$, and if w.l.o.g., $q_1 \leq q_2$, it follows that

$$q_1\|x_r + \alpha^r e_n\| + q_2\|\beta e_n + x_{r+1}\| = q_1\alpha^r + q_2\beta \leq q_2(\alpha^r + \beta)$$
$$\leq q_1\|x_r\| + q_2\|e_n(\alpha^r + \beta) + x_{r+1}\|.$$

Case (ii). $\|x_r + \alpha^r e_n\| \neq |x_r + \alpha^r e_n|_0$.

Then we find $\epsilon \geq 0$ and $r < r+1 < \ldots < (r-1)_{\epsilon+2}$ so that $(r-1)_i \cap supp\,(x_r) \neq \emptyset$ for i = 1,. . ., $\epsilon + 2$ and

$$q_1\|x_r + \alpha^r e_n\| + q_2\|\beta e_n + x_{r+1}\|$$
$$= \frac{q_1}{f(\epsilon+2)}\left[\sum_{i=1}^{\epsilon+1} \|(r-1)_i(x_r)\| + \|(r-1)_{\epsilon+2}\,(x_r + e_n)\|\right] + q_2\|\beta e_n + x_{r+1}\|$$
$$\leq \frac{q_1}{f(\epsilon+1)}\sum_{i=1}^{\epsilon+1} \|(r-1)_i(x)\|$$
$$+\begin{cases} \frac{q_1}{f(\epsilon+2)}\|(r-1)_{\epsilon+2}\,(x_r) + (\alpha^r + \beta)e_n\| + q_2\|x_{r+1}\| \\ \quad\text{or} \\ \frac{q_1}{f(\epsilon+2)}\|(r-1)_{\epsilon+2}\,(x_r)\| + q_2\|(\alpha^r + \beta)e_n + x_{r+1}\| \end{cases}$$

[By the induction hypothesis]

$$\leq \max\{q_1\|x_r + (\alpha^r + \beta)e_n\| + q_2\|x_{r+1}\|, q_1\|x_r\| + q_2\|(\alpha^r + \beta)e_n + x_{r+1}\|\},$$

which shows the assertion in this case.

In the case $\|\beta e_n + x_{r+1}\| \neq |\beta e_n + x_{r+1}|_0$ we proceed like in Case (ii).

In order to show the equation (2) we first observe that for all $k \in N_0$, $\left|x_r + e_m + \widetilde{(x_r)}\right|_k \leq \left\|x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)}\right\|$ (which easily follows by induction for each k∈N) and, thus, that $\liminf_{n\to\infty}\left\|x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)}\right\| \geq \left\|x_r + e_m + \widetilde{(x_r)}\right\|$. Since every subsequence of $((x_{r+1})_n)$ still satisfies. the assumptions of Corollary (1.2.15) it is enough to show that

$$\liminf_{n\to\infty}\left\|x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)}\right\| \leq \left\|x_r + e_m + \widetilde{(x_r)}\right\|.$$

This inequality will be shown by induction for each $k \in N_0$ and all $x_r < e_m < \widetilde{(x_r)}$ with #supp(x_r)+#supp($\widetilde{(x_r)}$) ≤ k. For k = 0 the assertion is trivial. We assume the assertion to be true for some k ≥ 0 and we fix $x_r, \widetilde{(x_r)} \in c_{00}$ with $x_r < e_m < \widetilde{(x_r)}$ and #supp(x_r)+#supp($\widetilde{(x_r)}$) = k + 1

We consider the following three cases:

Case (i). $\left\|x_r + (x_{r+1})_n + \widetilde{(x_r)}\right\| = \left|x_r + (x_{r+1})_n + \widetilde{(x_r)}\right|_0$ for infinitely many n ∈N. Since $\left|x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)}\right|_0 \leq \left|x_r + (x_{r+1})_n + \widetilde{(x_r)}\right|_0$, n∈N ,the assertion follows.

Case (ii). For a subsequence $(x'_{r+1})_n$ of $(x_{r+1})_n$ we have

$$\left\|x_r + (x'_{r+1})_n + \widetilde{(x_r)}\right\| = \frac{1}{f(l_n)}\sum_{i=1}^{l_n}\left\|(r-1)_i^{(n)}(x_r + (x'_{r+1})_n + \widetilde{(x_r)})\right\|$$

where $l_n \uparrow \infty$ and $r^{(n)} < (r+1)^{(n)} < \cdots < (r-1)_{l_n}^{(n)}$ are finite subsets of IN. Since $f(l_n) \to \infty$ for $n \to \infty$ it then follows that

$$\liminf_{n\to\infty}\left\|x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)}\right\| = 1 \leq \left\|x_r + e_n + \widetilde{(x_r)}\right\|.$$

Assume now that neither Case (i) nor Case (ii) occurs. By passing to a subsequence we can assume

Case (iii). There is an $\epsilon \geq 0$ so that

$$\lim_{n\to\infty}\left(\left\|x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)}\right\| - \frac{1}{f(\epsilon+2)}\sum_{i=1}^{\epsilon+2}\left\|(r-1)_i^{(n)}(x_r + (x_{r+1})_n + \widetilde{(x_r)}^{(n)})\right\|\right) = 0$$

where $r^{(n)} < \cdots < ((r-1)_{\epsilon+2})^{(n)}$ are finite subsets of IN with the following properties:

(i) $\mathrm{Sup}\left(x_r + (x_{r+1})_n + \tilde{x}^{(n)}\right) \cap (r-1)_i^{(n)} \neq \emptyset$, $i \leq \epsilon + 2$, and $\mathrm{supp}(x_r + (x_{r+1})_n + \tilde{x}^{(n)}) \subset \cup_{i=1}^{\epsilon+2} (r-1)_i^{(n)}$

(ii) The set $\mathrm{supp}(x_r) \cap (r-1)_i^{(n)}$, $i = 1, \ldots, \epsilon+2$ does not depend on n (note that $\mathrm{supp}(x_r) < \infty$), and we denote it by $((r-1)_i)^{(n)}$

(iii) There are subsets $\tilde{r} < \tilde{r} + 1 < \cdots < (\tilde{r} - 1)_{\epsilon+2}\mathrm{supp}(\widetilde{x_r})$ and integers r_n so that supp $(\widetilde{x_r})^{(n)}) \cap (r-1)_i^{(n)} = (\tilde{r}-1)_i + r_n$ for n ∈N,(we use the convention that $\emptyset < r-1$ for any finite $(r-1) \subset N$),

(iv) for $i \leq \epsilon + 2$ and $1 \leq j \leq k_n$ we have either $\mathrm{supp}\left(x_{r+1}(n,j)\right) \subset (r-1)_i^{(n)}$ or $\mathrm{supp}(x_{r+1}(n,j)) \cap (r-1)_i^{(n)} = \emptyset$.

Indeed, letting for $i \leq \epsilon + 2$

$$((\tilde{r}-1)_{\epsilon+2})^{(n)}=\begin{cases}(r-1)_i^{(n)} \text{if } (r-1)_i^{(n)}\cap\ \mathrm{supp}(y_n)=\emptyset\\ \mathrm{supp}(x_{r+1}(n,s)\cup(r-1)_i^{(n)}/\mathrm{supp}(x_{r+1}(n,s))\\ \text{where } s=\min\left\{\tilde{s}:\mathrm{supp}\left(x_{r+1}(n,\tilde{s})\right)\cap(r-1)_i^{(n)}\neq\emptyset\right\}\\ \text{an} + \text{d } t=\max\left\{\tilde{s}:\mathrm{supp}\left(x_{r+1}(n,\tilde{t})\right)\cap(r-1)_i^{(n)}\neq\emptyset\right\}\end{cases}$$

The value$\sum_{i=1}^{\epsilon+2}\left\|(r-1)_i{}^{(n)}(x_r+(x_{r+1})_n+\widetilde{x_r}^{(n)})\right\|$ differs from

$$\sum_{i=1}^{\epsilon+2}\left\|(\tilde{r}-1)_{\epsilon+2}^{(n)}\left(x_r+(x_{r+1})_n+\widetilde{x_r}^{(n)}\right)\right\|$$

at most by $2(\epsilon+2)/k_n$which shows that iv) can be assumed.

v) For $i\leq\epsilon+2$ the value

$$q_i=\lim_{n\to\infty}\frac{\left\{i\leq k_n,\mathrm{supp}(x_{r+1}(n,j))\subset(r-1)_i^{(n)}\right\}}{k_n}$$

exists. Now we distinguish between the following subcases.

Case (iii)a. There are $l_1, l_2\in N$, so that $1\leq l_1,\leq l_2-2<l_2\leq\epsilon+2$ and$\left\|x_r+(x_{r+1})_n+\widetilde{x_r}^{(n)}\right\|=\frac{1}{f(\epsilon+2)}\left[\sum_{i=1}^{l_1-1}\left\|(r-1)_i^{(n)}(x_r)\right\|+\left\|(r-1)_{l_1}^{(n)}(x_r+(x_{r+1})_n)\right\|+\sum_{i=l_1+1}^{l_2-1}\left\|(r-1)_i^{(n)}(x_{r+1})_n\right\|+\left\|(r-1)_{l_2}^{(n)}\left((x_{r+1})_n+\widetilde{x_r}^{(n)}\right)\right\|+\sum_{i=l_2+1}^{\epsilon+2}\left\|(r-1)_i^{(n)}\left(\widetilde{x_r}^{(n)}\right)\right\|\right]$. In this case it follows that

$$\begin{aligned}&\left\|x_r+(x_{r+1})_n+\widetilde{x_r}^{(n)}\right\|\\ &\leq\frac{1}{f(\epsilon+2)}\left[\sum_{i=1}^{l_1}\left\|(r-1)_i^{(n)}(x_r)\right\|\right.\\ &\left.+\sum_{i=l_1}^{l_2}\left\|(r-1)_i^{(n)}((x_{r+1})_n)\right\|+\sum_{i=l_2}^{\epsilon+2}\left\|(r-1)_i^{(n)}(\widetilde{x_r}^{(n)})\right\|\right]\\ &\leq\left[\sum_{i=1}^{l_1}\left\|(r-1)_i^{(n)}(x_r)\right\|+1+\varepsilon_n+\sum_{i=l_2}^{l}\left\|(r-1)_i^{(n)}(\widetilde{x_r}^{(n)})\right\|\right]\end{aligned}$$

[By iv) and the fact that $\left(x_{r+1}(j,n)\right)_{j=1}^{k_n}$ is (1+ε_n)-equivalent to the $l_1^{k_n}$-unit basis] $\leq\|x_r+e_m+\tilde{x}\|+\varepsilon_n$,Note that [$l_1+2\leq l_2$] which implies the assertion in this case.

Case (iii)b. There is an $1\leq l_1\leq\epsilon+2$ so that

$$\begin{aligned}&\left\|x_r+(x_{r+1})_n+\widetilde{x_r}^{(n)}\right\|\\ &=\left[\frac{1}{f(\epsilon+2)}\sum_{i=1}^{l_1-1}\left\|(r-1)_i^{(n)}(x_r)\right\|+\left\|(r-1)_{l_1}^{(n)}(x_r+(x_{r+1})_n+\widetilde{x_r}^{(n)})\right\|\right.\\ &\left.+\sum_{i=l_1+1}^{\epsilon+2}\left\|(r-1)_i^{(n)}(x_r{}^{(n)})\right\|\right].\end{aligned}$$

Then the assertion can be deduced from the induction hypothesis (note, that by i) and the fact that $\epsilon \geq 0$ we have that $\#\mathrm{supp}\left\| (r-1)_{l_1}^{(n)}(x_r + \widetilde{x_r}^{(n)}) \right\| < \#\mathrm{supp}(x_r + \widetilde{x_r}^{(n)})$).

Case (iii)c**.** There is an $l_1 < \epsilon + 2$ so that

$$\begin{aligned}
&\left\| x_r + (x_{r+1})_n + \widetilde{x_r}^{(n)} \right\| \\
&\quad = \left[\frac{1}{f(\epsilon+2)} \sum_{i=1}^{l_1 - 1} \left\| (r-1)_i^{(n)}(x_r) \right\| + \left\| (r-1)_{l_1}^{(n)}(x_r + (x_{r+1})_n) \right\| \right. \\
&\quad \left. + \left\| \left((r-1)_{l_1+1}\right)^{(n)}((x_{r+1})_n + \widetilde{x_r}^{(n)}) \right\| + \sum_{i=l_2+2}^{\epsilon+2} \left\| (r-1)_i^{(n)}(\widetilde{x_r}^{(n)}) \right\| \right].
\end{aligned}$$

We can assume that $\mathrm{supp}(x_r) \neq 0$ and $\mathrm{supp}(\widetilde{x_r}) \neq 0$ (otherwise we are in case (iii)b). If q_{l_1} (as defined in e)) vanishes it follows that $\lim_{n\to\infty} \left\| (r-1)_i^{(n)}(x_r + (x_{r+1})_n) \right\| = \left\| (r-1)_{l_1}^{x_r}(x_r) \right\|$. Otherwise there is a sequence $(j_n) \subset N$ with $\lim_{n\to\infty} j_n = \infty$ so that

$$(r-1)_{l_1}^{(n)}(x_{r+1})_n = \frac{1}{k_n} \sum_{j=1}^{j_n} x_{r+1}(n,j)$$

and so that

$$\lim_{n\to\infty} \frac{j_n}{k_n} = q_{l_1} > 0.$$

Since the sequence $\left((r-1)_{l_1}^{(n)}(x_{r+1})_n / q_{l_1}\right)_{n\in N}$ is asymptotically equal to the sequence $(\widetilde{x_{r+1}})_n$ with $(\widetilde{x_{r+1}})_n = \frac{1}{j_n}\sum_{j=1}^{j_n} x_{r+1}(n,j)$. (note that $((\widetilde{x_{r+1}})_n)$ satisfies the assumption of the lemma) we deduce from the induction hypothesis for some infinite $N \subset IN$ that

$$\begin{aligned}
&\lim_{\substack{n\to\infty \\ n\in N}} \left\| (r-1)_{l_1}^{(n)}(x_r + (x_{r+1})_n) \right\| \\
&\quad = q_{l_1} \lim_{n\to\infty} \left\| (r-1)_i^{x_r}(q_{l_1} + (\widetilde{x_{r+1}})_n) \right\| \leq q_{l_1} \left\| (r-1)_i^{x_r}\left(\frac{x_r}{q_{l_1}}\right) + e_m \right\| \\
&\quad = \left\| ((r-1)_i)^{x_r}\left(\frac{x_r}{q_{l_1}}\right) + q_{l_1} e_m \right\|.
\end{aligned}$$

Similarly we show for some infinite $M \subset N$, that

$$\begin{aligned}
&\lim_{\substack{n\to\infty \\ n\in N}} \left\| (r-1)_{l_{1+1}}^{(n)}\left((x_{r+1})_n + (\widetilde{x_r})^{(n)}\right) \right\| \\
&\quad \leq \left\| q_{l_{1+1}} e_m + (\tilde{r}-1)_{l_{1+1}}(\widetilde{x_r}) \right\|.
\end{aligned}$$

From the claim at the beginning of the proof we deduce now that

$$\lim_{n\to\infty}\left\|x_r + (x_{r+1})_n + \widetilde{x_r}^{(n)}\right\|$$

$$\leq \left[\frac{1}{f(\epsilon+2)}\sum_{i=1}^{l_1-1}\left\|(r-1)_i^{x_r}(x_r)\right\| + \left\|(r-1)_{l_1}^{x_r}(x_r) + q_{l_1}\, e_m\right\|\right.$$

$$\left. + \left\|\, q_{l_1+1} + e_m + (\tilde{r}-1)_{l_{1+1}}\,(\widetilde{x_r})\,\right\| + \sum_{i=l_1+2}^{\epsilon+2}\left\|\,(\tilde{r}-1)_i(\widetilde{x_r})\right\|\right]$$

$$\leq \frac{1}{f(\epsilon+2)}\left[\sum_{i=1}^{l_1-1}\left\|(r-1)_i^{x_r}(x_r)\right\|\right.$$

$$+ \sum_{i=l_1+2}^{\epsilon+2}\left\|(\tilde{r}-1)_i(\widetilde{x_r})\right\|$$

$$+ \max\left\{\left\|(r-1)_{l_1}^{x_r}(x_r) + e_m\right\| + \left\|\,(\tilde{r}-1)_{l_1+1}(\widetilde{x_r})\,\right\|, \left\|(r-1)_{l_1}^{x_r}(x_r)\right\|\right.$$

$$\left.\left. + \left\|\, e_m + (\tilde{r}-1)_{l_{1+1}}\,(\widetilde{x_r})\,\right\|\right\}\right]\left[q_{l_1+1} + q_{l_1} = 1\right] \leq \left\|x_r + e_m + \widetilde{x_r}\right\|,$$

which shows the assertion in this case and finishes the proof of the Corollary