Sudan University of Science and Technology

College of Graduate Studies

Common Fixed Point Theorems and Defect Indices with Failure of Rational Dilation

مبرهنات النقطة الثابتة المشتركة وأدلة الخلل مع فشل التمدد النسبي

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A Thesis Submitted in partial Fulfillment for the Requirements of the M.Sc Degree in Mathematics

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2015

Dedication

To my mother

Unknown soldier in our home

To my father

It's the greatest love that he holds

To my brothers and sisters

Acknowledgements

Before of all the praise and thanks be to Allah whom to be a scribed all perfection and majesty. The thanks after Allah must be to my Supervisor Prof. Shawgy Hussein Abdalla who supervised this research and guide me in patience until the result of this research are obtained. I wish to express my thanks to the Sudan University of Science and Technology, and Omdurman Islamic University, Faculty of Science & Technology, department of mathematics. My thanks also must be sent to all my friends and classmates for any support that make me complete this research.

Abstract

We generalize results on common fixed points in ordered cone metric spaces by weakening the respective contractive condition. Then, the notions of quasicontraction and g-quasicontraction are introduced in the setting of ordered cone metric spaces and respective (common) fixed point Theorems are shown. We show that the equality holds for unitary or the eigen values are all in the open unit disk. We also consider the defect index for a finite Blaschke product. We study common fixed points for the self and non-self type maps in cone metric spaces. For particular class of $\mathbb{E}$-contractions, we show it necessary for the existence of rational dilation that the corresponding fundamental operators satisfy certain conditions. Then we construct an $\mathbb{E}$-contraction from that particular class which fails to satisfy the certain condition. We produce a concrete functional model for pure $\mathbb{E}$-isometries and a class of E-contractions analogous to the pure isometries in one variable.

الخلاصة

عممنا نتائج على النقاط الثابتة المشتركة في الفضاءات المترية المخروطية المنظمة بواسطة إضعاف شرط الإنكماش المختص. أوضحنا المفاهيم لشبه الإنكماش وشبه الإنكماش-**g** تم إدخالها في ضبط الفضاءات المترية المخروطية المنظمة ومبرهنات النقطة الثابتة المشتركة المختصة. تم إيضاح أن المتساوية تحقق لأجل الواحدية أو القيم الذاتية وكلها في قرص الوحدة المفتوح. إيضاً إعتبرنا دليل الخلل لأجل ضرب بلاشيك المنتهي. درسنا النقاط الثابتة المشتركة لأجل رواسم النوع الذاتي وغير الذاتي في الفضاءات المترية المخروطية. لأجل العائلة الخاصة لإنكماشات-$\mathbb{E}$ وأوضحنا ضروريتها لأجل وجود التمدد النسبي حيث المؤثرات الأساسية المقابلة تحقق شروط مؤكدة. تم بناء الإنكماش-$\mathbb{E}$ من هذه العائلة الخاصة والتي تفشل لتحقق الشرط المؤكد. أدخلنا النموذج الذاتي المحدد لأجل متساويات المسافة-$\mathbb{E}$ البحتة وعائلة إنكماشات-E المماثلة إلي متساويات المسافة-E في المتغير الواحد.

The Contents

Chapter1

Ordered Contractions and Quasi Contraction in Ordered Cone Metric Space

In such a way known results on quasicontractions and g-quasicontraction in metric spaces and cone metric spaces are extended to the setting of ordered cone metric spaces. Examples show that there are cases when new results can be applied, while old ones cannot.

Section (1.1): Common Fixed Points of Weakly Increasing Mappings

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton's approximation method and in optimization theory. Numerous generalizations of the Banach contraction principle in the setting of metric spaces were given by many authors. Abstract (cone) metric spaces were studied by Huang and Zhang.

The existence of fixed points in partially ordered sets was investigated, e.g., by Ran and Reurings, and then by Nieto and Lopez. The following two versions of the fixed point theorem were proved, among others in this chapter.

Theorem (1.1.1)[1]. Let $(X, \sqsubseteq)$ be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space, let $f: X \to X$ be a nondecreasing map with respect to $\sqsubseteq$. Suppose that the following condition hold:

(i) there exist $k \in (0,1)$such that $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) there exist $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;

(iii) f is continuous, or

(iv) if a nondecreasing sequence $\{x_n\}$ converges to $x \sqsubseteq X$, then $x_n \sqsubseteq x$ for all n.

Then f has a fixed poin $x^* \in X$

Fixed point results in ordered cone metric spaces were obtained by Altun and Durmaz, as well as by Altun Damnjanović and Djorić.

Theorem (1.1.2)[1]. *L*et $(X, \sqsubseteq)$ be a partially ordered set and let d be a cone metric on X (defined over a normal cone P with the normal constant k) such that (X, d) is a complete cone metric space. Let $f: X \to X$ be a continuous and nondecreasing map with respect to $\sqsubseteq$.

Suppose that the following condition hold:

(i) there exist $k \in (0,1)$ such that $d(fx, fy) \le kd(x, y)$ for all $x, y \in X$ with $y \sqsubseteq x$;

(ii) there exist $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.

Then f has a fixed point $x^* \in X$.

In some generalizations of the previous Theorem were proved, including the case when the underlying cone P is not normal. Also, some common fixed point Theorems were obtained. We state the following theorem which is an "ordered" variant of a result of Abbas and Rhoades.

Theorem (1.1.3)[1]. let $(X, \sqsubseteq)$ be a partially ordered set and let d be a cone metric on X (defined over a cone P with int$P \neq \varphi$) such that (X, d) is a complete cone metric a space. Let $f, g: X \to X$ be self-maps such that (f, g) is a weakly increasing pair with respect to $\sqsubseteq$. Suppose that the following conditions hold:

(i) there exest $\alpha, \beta, \gamma \ge 0$ such that $\alpha + 2\beta + 2\gamma < 1$ and

$$d(fx, gy) \le \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)] \quad (1.1)$$

*F*or all comparable $x, y \in X$;

(ii) f or g is continuous, or

(iii) if a nondecreasing sequnce $\{x_n\}$ converegs to $x \in X$, then $x_n \sqsubseteq x$ for all n.

Then f and g have a common fixed point $x^* \in X$.

Note that a pair (f, g) of self- maps on a partially ordered set $(X, \sqsubseteq)$ is said to be weakly increasing if $fx \sqsubseteq \mathrm{g}fx$ and $\mathrm{g}x \sqsubseteq f\mathrm{g}x$ for all $x \in X$.There are examples when neither of such mappings f, g is nondecreasing w.r.t. $\sqsubseteq$. In particular, the pair(f, i_X) $(i_X$- the identity function) is weakly increasing if and only if $x \sqsubseteq fx$ for each $x \in X$.

We show by the following simple example that a mapping on an ordered cone metric space can be an "ordered" contraction, while it is not a contraction in the classical sense.

Example (1.1.4)[1]. Let $X = \{1,2,4\}, \sqsubseteq = \{(1,1),(2,2),(4,4),(1,4)\}; E = \mathbb{R}^2, P = \{(a,b): a,b \geq 0\},$ $d(x,y) = (|x-y|, 2|x-y|),$ and let $f: X \to X, f1 = 1, f2 = 2, f4 = 1.$

The mapping f is a (Banach-type) contraction in the ordered cone metric space $(X, \sqsubseteq, d)$, i.e.,

$$d(fx, fy) \leq \lambda d(x,y), y \sqsubseteq x, \tag{1.2}$$

for some $\lambda \in [0,1)$. indeed, we have only to check validity of (1.2) for $y = 1, x = 4$. But it is equivalent to $|f4 - f1| \leq \lambda|4-1|$, i.e., $|1-2| \leq \lambda|4-1|$ which is satisfied if (and only if) $\lambda \in \left[\frac{1}{3}, 1\right)$.

On the other hand, f is not a contraction in the (non-ordered) cone metric space (X, d). indeed, for $x = 2, y = 1$ we have that

$$|f2 - f1| \leq \lambda|2-1| \Leftrightarrow 1 \leq \lambda \cdot 1 \Leftrightarrow \lambda \geq 1.$$

It also means that f is not a contraction in the metric space (X, d_1) where d_1 is the usual metric $d_1(x,y) = |x-y|$ on $\mathbb{R}$. We need the following definitions and results. Let E be a real Banach space. A subset P of E is a cone if:

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}, a, b \geq 0,$ and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define the partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int } P$ (the interior of P).

A cone $P \subset E$ is called normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\| \quad (1.3)$$

Or, equivalently, if $x_n \preceq y_n \preceq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x \quad (1.4)$$

The least positive number K satisfying (1.3) is called the normal constant of P. it is clear that $k \geq 1$. Most of ordered Banach spaces used in applications posses a cone with the normal constant $K = 1$, and if this is the case, proofs of the corresponding results are much alike as in the metric setting. If $k > 1$, this is not the case.

Example(1.1.5)[1]. Let $E = \{C^1_{\mathbb{R}}[0,1] \; with \; \|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0,1]\}$. This cone isn't normal.

Consider, for example $x_n(t) = \frac{t^n}{n}, y_n(t) = \frac{1}{n}$. Then $0 \preceq x_n \preceq y_n$ and $\lim_{n\to\infty} y_n = 0$, but

$$\|x_n\| = \max_{t\in[0,1]} \left|\frac{t^n}{n}\right| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1;$$

Hence (x_n) does not converge to zero. It follows by (1.2) that P is a non normal cone.

Definition (1.1.6)[1] Let X be a nonempty set and, P a cone in a Banach space E. suppose that a mapping $d : X \times X \to E$ satisfies:

(i) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general then that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

The following remark will be useful in the sequel.

Remark (1.1.7)[1].

(i) if $u \preceq v$ and $v \ll w$, then $u \ll w$.

(ii) if $0 \preceq u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.

(iii) if $a \preceq b + c$ for each $c \in \operatorname{int} P$ then $a \preceq b$.

(iv) if $0 \preceq x \preceq y$, and $0 \preceq a$, then $0 \preceq ax \preceq ay$.

(v) if $0 \preceq x_n \preceq y_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \preceq x \preceq y$.

(vi) if $0 \preceq d(x_n, x) \preceq b_n$ and $b_n \to 0$, then $d(x_n, x) \ll c$ where x_n, x are, respectively, a sequence and a given point in X.

(vii) if E is a real Banach space with a cone P and if $a \preceq \lambda a$ where $a \in$P and $0 < \lambda < 1$,then $a = 0$.

(viii) if $c \in \operatorname{int} P$, $0 \preceq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

In the rest of the chapter $(X, \sqsubseteq, d)$will always be an ordered cone metric space, i.e., $\sqsubseteq$ will be a partial order on the set X, and d will be a cone metric on X with always the underlying cone P such that $\operatorname{int} P \neq \varphi$ (such a cone will be called solid). Normality of the cone is not assumed.

Theorem (1.1.8)[1]. Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space. Let (f, g) be a weakly increasing pair of self-maps on X with respect to $\sqsubseteq$. Suppose that the following conditions hold:

(i) there exist $p, q, r, s, t \geq 0$ satisfing $p + q + r + s + t < 1$ and $q = r$ or $s = t$, such that

$$d(fx, gy) \preceq pd(x, y) + qd(x, fx) + rd(y, gy) + sd(x, gy) + td(y, fx)$$

for all comparable $x, y \in X$; (1.5)

(ii) f or g is continuous, or

(iii) If a non decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

then f and g have a common fixed point $x^* \in X$.

Proof: Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}_0$. Using that the pair of mappings (f, g) is weakly increasing, it can be easily shown that the sequence $\{x_n\}$ is nondecreasing w.r.t $\sqsubseteq$, i.e., $x_0 \sqsubseteq x_1 \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$. In particular, x_{2n} and x_{2n+1} are comparable, so we can apply relation (1.5) to obtain

$$d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1})$$

$$\preceq pd(x_{2n}, x_{2n+1}) + qd(x_{2n}, x_{2n+1}) + rd(x_{2n+1}, x_{2n+2}) + sd(x_{2n}, x_{2n+2}) + td(x_{2n+1}, x_{2n+1})$$

$$\preceq pd(x_{2n}, x_{2n+1}) + qd(x_{2n}, x_{2n+1}) + rd(x_{2n+1}, x_{2n+2}) + s[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})].$$

It follows that

$$(1 - r - s)d(x_{2n+1}, x_{2n+2}) \preceq (p + q + s)d(x_{2n}, x_{2n+1}),$$

i.e., $d(x_{2n+1}, x_{2n+2}) \preceq \frac{p+q+s}{1-(r+q)} d(x_{2n}, x_{2n+1}).$ (1.6)

In a similar way one obtains that

$$d(x_{2n+2}, x_{2n+3}) \preceq \frac{p+q+t}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)} d(x_{2n}, x_{2n+1}). \quad (1.7)$$

Now, from (1.6) and (1.7), by induction, we obtain that

$$d(x_{2n+1}, x_{2n+2}) \preceq \frac{p+q+s}{1-(r+s)} d(x_{2n}, x_{2n+1})$$

$$\preceq \frac{p+q+s}{1-(r+s)} \cdot \frac{p+r+s}{1-(q+t)} \; d(x_{2n-1}, x_{2n})$$

$$\preceq \frac{p+q+s}{1-(r+s)} \cdot \frac{p+r+s}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)} d(x_{2n-2}, x_{2n-1})$$

$$\preceq \cdots \preceq \frac{p+q+s}{1-(r+s)} \; \left(\frac{p+r+t}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)}\right)^n \; d(x_0, x_1),$$

And

$$d(x_{2n+2}, x_{2n+3}) \preceq \frac{p+r+t}{1-(q+t)} \; d(x_{2n+1}, x_{2n+2})$$

$$\preceq \cdots \preceq \left(\frac{p+r+t}{1-(q+t)} \cdot \frac{p+q+s}{1-(r+s)}\right)^{n+1} \; d(x_0, x_1).$$

Let

$$A = \frac{p+q+s}{1-(r+s)} \; , \qquad B = \frac{p+r+t}{1-(q+t)}.$$

In the case $q = r$,

$$AB = \frac{p+q+s}{1-(q+s)} \cdot \frac{p+r+t}{1-(q+t)} = \frac{p+q+s}{1-(q+t)} \cdot \frac{p+r+t}{1-(r+s)} < 1 \cdot 1 = 1,$$

And if $s = t$,

$$AB = \frac{p+q+s}{1-(r+s)} \cdot \frac{p+r+s}{1-(q+t)} < 1 \cdot 1 = 1.$$

Now, for $n < m$ we have

$$d(x_{2n+1}, x_{2m+1}) \preceq d(x_{2n+1}, x_{2n+2}) + \cdots + d(x_{2n}, x_{2m+1})$$

$$\preceq \left(A \sum_{i=1}^{m-1} (AB)^i + \sum_{i=n+1}^{m} (AB)^i\right) d(x_0, x_1)$$

$$\preceq \frac{A(AB)^n}{1-AB} + \frac{(AB)^{n+1}}{1-AB} \; d(x_0, x_1)$$

$$= (1+B)\frac{A(AB)^n}{1-AB} \; d(x_0, x_1).$$

Similarly, we obtain

$$d(x_{2n}, x_{2m+1}) \preceq (1 + A)\, \frac{(AB)^n}{1-AB}\, d(x_0, x_1),$$

$$d(x_n, x_{2m}) \preceq (1 + A) \frac{(AB)^n}{1-AB}\, d(x_0, x_1)$$

And

$$d(x_{2n+1}, x_{2m}) \preceq (1 + B)\, \frac{A(AB)^n}{1-AB}.$$

Hence, for $n < m$

$$d(x_n, x_m) \preceq \max\left\{(1 + B)\, \frac{A(AB)^n}{1-AB}, (1 + A)\, \frac{(AB)^n}{1-AB}\right\} d(x_0, x_1)$$

$$= \lambda_n d(x_0, x_1),$$

where $\lambda_n \to 0$, as $n \to \infty$.

Now, using (viii) and (i) of Remark (1.1.7) and only the assumption that the underlying cone is solid, we conclude that $\{x_n\}$ is a Cauchy sequence. Since the space (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*(n \to \infty)$.

Suppose that, for example, f is a continuous mapping, then we have that $fx_n \to fx^*$, which (taking n even) implies that $fx^* = x^*$. Now, since $x^* \sqsubseteq x^*$, taking $x = y = x^*$ in relation (1.1.8), we obtain that

$$d(fx^*, \mathrm{g}x^*) \preceq pd(x^*, x^*) + qd(x^*, fx^*) + rd(x^*, \mathrm{g}x^*) + sd(x^*, \mathrm{g}x^*) + td(x^*, fx^*),$$

i.e., since $fx^* = x^*$, $d(x^*, \mathrm{g}x^*) \preceq (r + s)d(x^*, \mathrm{g}x^*)$.

Since $r + s < 1$, using Remark (1.1.7) (vii), it follows that $\mathrm{g}x^* = x^*$, and x^* is a common fixed point of f and g.

The proof is similar when g is a continuous mapping. Consider now the case when condition (iii) is satisfied. For the sequence $\{x_n\}$ we have $x_n \to x^* \in X(n \to \infty)$ and $x_n \sqsubseteq x^*$ $(n \in \mathbb{N})$. By the construction, $fx_n \to x^*$ and $\mathrm{g}x_n \to x^*(n \to \infty)$. Let us prove that x^* is a common fixed point of f and g. Putting $x = x^*$ and $y = x_n$ in (1.5) (since they are comparable) we get $d(fx^*, \mathrm{g}x_n) \preceq pd(x^*, x_n) + qd(x^*, fx^*) + rd(x_n, \mathrm{g}x_n) + sd(x^*, \mathrm{g}x_n) + td(x_n, fx^*)$. For the first and fourth term on the right-hand side we have

$d(x_n, x^*) \ll c$ and $d(x^*, \mathrm{g}x_n) \ll c$ (for $c \in \operatorname{int} P$ a rbitrary and $n \geq n_0$). For the second term, $d(x^*, fx^*) \preceq d(x^*, x_n) + d(x_n, \mathrm{g}x_n) + d(\mathrm{g}x_n, fx^*)$ (again the first term on the right can be neglected), and for the fifth term $d(x_n, fx^*) \preceq d(x_n, \mathrm{g}x_n) + d(\mathrm{g}x_n, fx^*)$. It follows that

$$(1 - q - t)d(fx^*, \mathrm{g}x_n) \preceq (q + r + t)d(x_n, \mathrm{g}x_n).$$

But, $x_n \to x^*$ and $\mathrm{g}x_n \to x^*$ implies that $d(x_n, \mathrm{g}x_n) \ll c$, which means that also $d(fx^*, \mathrm{g}x_n) \ll c$, i.e. $\mathrm{g}x_n \to fx^*$. it follows that $fx^* = x^*$ and, in a symmetric way (using that $x^* \sqsubseteq x^*$), $\mathrm{g}x^* = x^*$.

Example (1.1.9)[1].

Let $X = \{1,2,3\}, \sqsubseteq = \{(1,1), (2,2), (3,3), (2,3), (3,1), (2,1)\}$ and $d: X \times X \to C^1_{\mathbb{R}}[0,1]$ be defined by $d(x, y)(t) = 0$ for all $x = y$ and

$$d(1,2)(t) = d(2,1)(t) = 6e^t, d(1,3)(t) = d(3,1)(t) = \frac{30}{7}e^t,$$

$d(2,3)(t) = d(3,2)(t) = \frac{24}{7}e^t$. Further, let $fx = 1, x \in X$, and $\mathrm{g}1 = \mathrm{g}3 = 1, \mathrm{g}2 = 3$. We have that $d(f3, \mathrm{g}2)(t) = d(1,3)(t) = \frac{30}{7}e^t$. But, the right-hand side of (1) for $x = 3, y = 2$ has the form

$$\alpha d(3,2) + \beta[d(3, f3) + d(2, \mathrm{g}2)] + \gamma[d(3, \mathrm{g}2) + d(2, f3)]$$
$$= \alpha \frac{24}{7}e^t + \beta\left(\frac{30}{7}e^t + \frac{24}{7}e^t\right) + \gamma(0 + 6e^t) = \frac{24\alpha}{7}e^t + \frac{54\beta}{7}e^t + 6\gamma e^t,$$

Which is less than $\frac{30}{7}e^t$ for arbitrary α, β, γ satisfying the condition $\alpha + 2\beta + 2\gamma < 1$. Indeed, $\frac{24}{7}\alpha + \frac{54}{7}\beta + 6\gamma < \frac{30}{7}$ follows from $\frac{24}{30}\alpha + \frac{54}{30}\beta + \frac{42}{30}\gamma < \alpha + 2\beta + 2\gamma < 1$. Hence, the conditions of Theorem (1.1.3) are not fulfilled and this Theorem cannot be used to conclude that f and g have a common fixed point. On the other hand, taking $p = q = r = s = 0, t = \frac{5}{7}$ all the conditions of Theorem (1.1.8) are fulfilled. Indeed, Since $f1 = \mathrm{g}1 = f3 = \mathrm{g}3 = 1$, we have only to check that

$df3, g2)(t) \preceq 0 \cdot d(3.2)(t) + 0 \cdot d(3, f3)(t) + 0 \cdot d(2, g2)(t) + 0 \cdot d(3, \mathrm{g}2)(t) + \frac{5}{7}\, d(2, f3)(t),$

Which is equivalent to

$$\frac{30}{7} e^t \preceq \frac{5}{7}\, d(2, f3)(t) = \frac{5}{7}\, d(2,1)(t) = \frac{5}{7} \cdot 6e^t = \frac{30}{7} \cdot e^t.$$

Thus, we can apply Theorem (1.1.8) and conclude that the mappings f and g have a (unique) common fixed point $u = 1$.

The next example shows that the condition $p + q + r + s + t < 1$ alone is not sufficient to obtain the conclusion of Theorem (1.1.8). We shall stay in the setting of metric spaces-it would be easy to adapt it to the setting of ordered cone metric spaces.

Example (1.1.10)[1]. Let $X = \{x, y, u, v\}$, whrer $x = (0,0,0)$, $y = (4,0,0)$, $u = (2,2,0)$, $v = (2, -2,1)$, and let d be the Euclidean metric in $\mathbb{R}^3$. Consider the mappings

$$f = \begin{pmatrix} x & y & u & v \\ u & v & v & u \end{pmatrix}, \qquad g = \begin{pmatrix} x & y & u & v \\ y & x & y & x \end{pmatrix}.$$

By a careful computation it is easy to obtain that

$$d(fa, \mathrm{g}b) \leq \frac{3}{4} \max\{d(a, b), d(a, fa), d(b, \mathrm{g}b), d(a, \mathrm{g}b), d(b, fa)\}, \qquad (1.8)$$

for all $a, b \in X$. We shall that f and g satisfy the following contractive condition: there exist $p, q, r, s, t \geq 0$ with $p + q + r + s + t < 1$ and $q \neq r, s \neq t$ such that

$$d(fa, \mathrm{g}b) \leq pd(a, b) + qd(a, fa) + rd(b, \mathrm{g}b) + sd(a, \mathrm{g}b) + td(b, fa) \qquad (1.9)$$

holds true for all $a, b \in X$. Obviously, f and g do not have a common fixed point. Taking (1.8) into account, we have to consider, the following cases:

(i) $d(fa, \text{g}b) \leq \frac{3}{4} d(a,b)$. Then (1.9) holds for $q = \frac{3}{4}, r = t = 0$ and $q = s = \frac{1}{9}$.

(ii) $d(fa, \text{g}b) \leq \frac{3}{4} d(a, fa)$. Then (1.9) holds for $q = \frac{3}{4}, p = r = t = 0$ and $s = \frac{1}{5}$.

(iii) $d(fa, \text{g}b) \leq \frac{3}{4} d(b, \text{g}b)$. Then (1.9) holds for $r = \frac{3}{4}, p = q = t = 0$ and $s = \frac{1}{5}$.

(iv) $d(fa, \text{g}b) \leq \frac{3}{4} d(a, \text{g}b)$. Then (1.9) holds for $s = \frac{3}{4}, p = r = t = 0$ and $q = \frac{1}{5}$.

(v) $d(fa, \text{g}b) \leq \frac{3}{4} d(b, fa)$. Then (1.9) holds for $t = \frac{3}{4}, p = r = s = 0$ and $q = \frac{1}{5}$.

Corollary (1.1.11)[1]. Let $(X, \sqsubseteq, d)$ be an ordered cone metric space. Let $f: X \to X$ be a self- map such that $x \sqsubseteq fx$ for all $x \in X$. Suppose that the following conditions hold:

(i) There exist $p, q, r, s, t \geq 0$ satisfying $p + q + r + s + t < 1$ and $q = r$ or $s = t$, such that

$$d(f^m x, f^n y) \preceq pd(x,y) + qd(x, f^m x) + rd(y, f^n y) + sd(x, f^n y) + td(y, f^m x)$$

for all $m, n \in \mathbb{N}, m \leq n$ and all comparable $x, y \in X$;

(ii) f is continuous.

Then f has a fixed point $x^* \in X$.

Proof: Follows from Theorem (1.1.8) by putting $f^m \equiv f, f^n \equiv \text{g}$.Taking $m = n = 1$ in the previous corollary, one obtains

Corollary (1.1.12)[1]. Let $(X,\sqsubseteq,d)$ be an ordered complete cone metric space. Let $f: X \to X$ be a self-map such that $x \sqsubseteq fx$, for all $x \in X$. suppose that the following condition hold:

(i) There exist $p,q,r,s,t \geq 0$ such that $p+q+r+s+t<1$ and

$$d(fx,fy) \leq pd(x,y)+qd(x,fx)+rd(y,y)+sd(x,fy)+td(y,fx) \quad (1.10)$$

for all comparable $x,y \in X$;

(ii) f is continuous.

Then f has a fixed point $x^* \in X$.

Note that here (when just one function f is considered) there was no need for additional assumptions on coefficients p,q,r,s,t.

Section (1.2): Fixed Points of Quasicontractions on Ordered Cone Metric Space

The notion of aquasicontractions in a metric space was first used by Ćirić and Das and Naik. Cone metric version of this notion was considered by Ilić and Rakočević, as well as Kadelburg, Radenović and Rakočević and Pathak and Shahzad. Generalized g-quasicontractions in cone metric spaces were investigated. We shall introduce here the notion of an ordered g-quasi contraction in an ordered cone metric space and prove the respective common fixed point Theorem.

Let (f,g) be a pair of self-maps on an ordered cone metric space $(X,\sqsubseteq,d)$ such that $f(X) \subset g(X)$. Let the mapping f be g-nondicreasing, i.e., let for each $x,y \in X$, $gx \sqsubseteq gy$ implies $fx \sqsubseteq fy$. Suppose also that there is a point $x_0 \in X$ such that $gx_0 \sqsubseteq fx_0$. Then it is possible to construct a so called Jungck sequence in the following way: starting with given x_0, choose $x_1 \in X$ such that $fx_0 = gx_1$(which is possible since $fX \subset gX$). Now it is $gx_0 \sqsubseteq gx_1$ which implies that $fx_0 \sqsubseteq fx_1$.Then there exists $x_2 \in X$ such that $fx_1 = gx_2$, and again $fx_0 \sqsubseteq fx_1$ implies that $gx_1 \sqsubseteq gx_2$ and $fx_1 \sqsubseteq fx_2$. Continuing this procedure, we obtain:

$$fx_0 \sqsubseteq fx_1 \sqsubseteq fx_2 \sqsubseteq \cdots \sqsubseteq fx_n \sqsubseteq fx_{n+1} \sqsubseteq \cdots$$

and

$$gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_{n+1} \sqsubseteq gx_{n+2} \sqsubseteq \cdots.$$

Definition (1.2.1)[1]. The mapping f is called an ordered g-quasicontraction if there exists $\lambda \in [0, 1/2)$ such that for each $x, y \in X$ satisfing $gy \sqsubseteq gx$, there exists

$$u \in M_0^{f,g}(x,y) = \begin{Bmatrix} d(gx, gy), d(gx, fx), d(gy, fy), \\ d(gx, fy), d(gy, fx) \end{Bmatrix} \qquad (1.11)$$

such that $d(fx, fy) \preceq \lambda \cdot u$ holds.

Theorem (1.2.2)[1] Let (f, g) be a pair of self-maps on a complete ordered cone metric space $(X, \sqsubseteq, d)$ such that $f(X) \subset g(X)$ and such that there is a point $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$. Suppose that

(i) f is an ordered g-quasicontraction;

(ii) $g(X)$ is closed in X;

(iii) f is g-nondecreasing;

(iv) if $\{g(x_n)\} \subset X$, is a nondecreasing sequence, converging to some gz, then $g(x_n) \sqsubseteq gz$ and $gz \sqsubseteq ggz$.

Then f and g have a coincidence point, i.e., there exists $z \in X$ such that $fz = gz$. If, further, f and g are weakly compatible, then they a common fixed point. Recall that the mappings f and g are said to be weakly compatible if, for each $x \in X, fx = gx$ implies $fgx = gfx$.

Proof: Starting with given x_0 construct the Jungck sequence $fx_{n-1} = gx_n$ of the pair (f, g), with the initial pointx_0. We shall prove that it is a Cauchy sequence in X. Let us prove first that

$$d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-1}, fx_n) \qquad (1.12)$$

For all $n \geq 1$. Indeed, since $gx_n \sqsubseteq gx_{n+1}$,we can apply condition (i) to obtain

$$d(fx_n, fx_{n+1}) \preceq \lambda u_n, \qquad (1.13)$$

Where

$$u_n \in \left\{ \begin{matrix} d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), d(gx_n, fx_{n+1}), \\ d(gx_{n+1}, fx_n) \end{matrix} \right\}$$

$$= \{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1}), 0\}.$$

There are four possible cases:

(i) $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-1}, fx_n)$ since $\lambda \leq \frac{\lambda}{1-\lambda}$;

(ii) $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_n, fx_{n+1})$; it follows that $d(fx_n, fx_{n+1}) = 0$. hence,(1.12) holds true;

(iii) $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n) + \lambda d(fx_n, fx_{n+1})$; hence, (1.12) holds true;

(iv) $d(fx_n, fx_{n+1}) \preceq \lambda \cdot 0 = 0$ and so $d(fx_n, fx_{n+1}) = 0$ and again (1.12) holds. Put $h = \frac{\lambda}{1-\lambda}$. Then it follows from (1.12) that

$$d(fx_n, fx_{n+1}) \preceq hd(fx_{n-1}, fx_n) \preceq \cdots \preceq h^n d(fx_0, fx_1),$$

For all $n \geq 1$. Now we have for all $n, m \in \mathbb{N}, n > m$ that

$$\begin{aligned} d(fx_n, fx_m) &\preceq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots + d(fx_{m+1}, fx_m) \\ &\preceq (h^{n-1} + h^{n-2} + \cdots + h^m) d(fx_0, fx_1) \\ &\preceq \frac{h^m}{1-h} d(fx_0, fx_1) \to 0 \ as\ m \to \infty. \end{aligned}$$

According to Remark (1.1.7)(i) and (viii), $\{fx_n\}$, i.e., $\{gx_n\}$ is a Cauchy sequence and, since X is complete and gX is closed, there exists $z \in X$ such that

$$gx_n \to gz \quad \text{i.e.}, fx_n \to gz \qquad as\ n \to \infty.$$

We will prove that $fz = gz$.

Since $gx_n \sqsubseteq gz$ (condition (iv)) putting $x = x_n, y = z$ in (1.10), we get

$$d(fx_n, fz) \preceq \lambda \cdot u_n \tag{1.14}$$

Where $u_n \in \left\{d(gx_n, gz), d(gx_n, fx_n), \underline{d(gz, fz)}, d(gz, fx_n), \underline{d(gz, fz)}\right\}$.

observe that $\underline{d(gz, fz)} \preceq d(gz, fx_n) + d(fx_n, fz)$ and

$\underline{d(gz, fz)} \preceq d(gz, fx_n) + d(fx_n, fz)$. Now let $0 \ll c$ be given. In all of the possible five cases there exists $n_0 \in \mathbb{N}$ such that (using (14)) one obtains that $d(fx_n, fz) \ll c$:

(i) $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, gz) \ll \lambda \frac{c}{\lambda} = c$;

(ii) $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fx_n) \ll \lambda \frac{c}{\lambda} = c$;

(iii) $d(fx_n, fz) \preceq \lambda \cdot d(gz, fz) \preceq \lambda d(gz, fx_n) + \lambda d(fx_n, fz)$; it follows

that $d(fx_n, fz) \preceq \frac{\lambda}{1-\lambda} d(gz, fx_n) \ll \frac{\lambda}{1-\lambda} \frac{(1-\lambda)c}{\lambda} = c$;

(iv) $d(fx_n, fz) \preceq \lambda \cdot d(gz, fx_n) \ll \lambda \frac{c}{\lambda} = c$;

(v) $d(fx_n, fz) \preceq \lambda \cdot d(gx_n, fz) \preceq \lambda d(gx_n, fx_n) + \lambda d(fx_n, fz)$; it follows that $d(fx_n, fz) \preceq \frac{\lambda}{1-\lambda} d(gx_n, fx_n) \ll \frac{\lambda}{1-\lambda} \frac{(1-\lambda)c}{\lambda} = c$.

It follows that $fx_n \to fz$ $(n \to \infty)$. The uniqueness of limit in a cone metric space implies that $fz = gz = t$, Thus, in the terminology, z is a coincidence point of the pair (f, g), and t is a point of coincidence. Suppose now that f and g are weakly compatible. By the assumption (iv), $gz \sqsubseteq ggz$ and hence we obtain that $fgz = gfz = ffz = ggz$.

Suppose that it is not $fz = ffz$. Then, the contractibility condition (10) for $x = z, y = fz$ implies that $d(fx, fy) = d(fz, ffz) \preceq \lambda u$, where

$$\begin{aligned} u &\in \{d(gz, gfz), d(gz, fz), d(gfz, ffz), d(gfz, fz), d(gz, ffz)\} \\ &= \{d(fz, ffz), 0, d(ffz, ffz), d(ffz, fz), d(fz, ffz)\} \\ &= \{0, d(fz, ffz)\}, \end{aligned}$$

So we have only two possibilities:

(i) $d(fz, ffz) \preceq \lambda \cdot 0 = 0 \Rightarrow d(fz, ffz) = 0 \Rightarrow fz = ffz$;

(ii) $d(fz, ffz) \preceq \lambda d(fz, ffz) \Rightarrow$ (by Remark (1.1.7)) $d(fz, ffz) = 0$, i.e., $fz = ffz$. In other words, $fz = gz$ is a common fixed point of the mappings f and g. Taking $g = i_X$ (the identity function) in Theorem (1.2.2) we obtain a result for ordered quasicontractions in ordered cone metric spaces.

Corollary (1.2.5)[1]. Let f be a self- map on a complete ordered cone metric space $(X, \sqsubseteq, d)$ such that there is a point $x_0 \in X$ with $x_0 \in X$. With $x_0 \sqsubseteq fx_0$. Suppose that

(i) f is an ordered quasicontraction, i.e., there exists $\lambda \in [0, 1/2)$ such that for each $x, y \in X$ satisfying $y \sqsubseteq x$, there exists

$$u \in \{d(x,y), d(x,fx), d(y,fy), d(y,fx)\}, \tag{1.15}$$

such that $d(fx, fy) \preceq \lambda \cdot u$ holds;

(ii) f is nondecreasing;

(iii) $if\{x_n\} \subset X$ is a nondecreasing sequence, converging to some z, then $x_n \sqsubseteq z$.

Then f has a fixed point in X.

Remark (1.2.6)[1]. If, in the Definition (1.2.1) of an ordered g-quasicontractions, we use the set $\{d(gx, gy), d(gx, fx), d(gy, fy)\}$, instead of $M_0^{f,g}(x,y)$, then it can be proved in a similar way that Theorem (1.2.2) holds even with $\lambda \in [0,1)$. If we further reduce this set to $\{d(gx, fx), d(gy, fy)\}$, then an ordered version of the known Bianchini's result is obtained.

Finally, if we take a singleton $\{d(gx, gy)\}$, we obtain an ordered version of a result of Jungck which is a direct generalization of the Banach's principle.

In the sequel, we shall modify condition of ordered g-quasicontraction by considering, together with $M_0^{f,g}(x,y)$, the following sets:

$$M_1^{f,g}(x,y) = \left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2}\right\},$$

$$M_2^{f,\mathrm{g}}(x,y) = \left\{d(\mathrm{g}x,\mathrm{g}y), \frac{d(\mathrm{g}x,fx)+d(\mathrm{g}y,fy)}{2}, \frac{d(\mathrm{g}x,fy)+d(\mathrm{g}y,fx)}{2}\right\}.$$

In the setting of cone metric spaces, they were used, for example (where non-self-mappings were considered) and (when considering strict contractive conditions). We shall prove here two related results in the setting of ordered cone metric spaces.

Theorem (1.2.7)[1]. Let(f,g) be a pair of self- maps on a complete ordered cone metric space $(X,\sqsubseteq,d)$such that $f(X)\subset \mathrm{g}(X)$ and such that there is a pair $x_0\in X$ with $\mathrm{g}x_0 \sqsubseteq fx_0$. suppose that

(i) There exists $\lambda\in[0,1)$ such that for each $x,y\in X$ satisfing $\mathrm{g}y\sqsubseteq \mathrm{g}x$,

there exist $u\in M_1^{f,\mathrm{g}}(x,y)$, such that $d(fx,fy)\preceq \lambda\cdot u$ hold.

(ii) $\mathrm{g}(X)$ is closed in X;

(iii) f is g-nondecreasing;

(iv) if $\{\mathrm{g}(x_n)\}\subset X$ is a nondecreasing sequence, converging to some $\mathrm{g}z$,

then $\mathrm{g}x_n\sqsubseteq \mathrm{g}z$, and $\mathrm{g}z\sqsubseteq \mathrm{gg}z$.

Then f and g have a coincidence point. Moreover, if f and g are weakly compatible, then they have a common fixed point.

Proof: Starting from the given x_0, construct the Jungch sequence as in the proof of Theorem (1.2.2)

$$fx_0\sqsubseteq fx_1\sqsubseteq fx_2\sqsubseteq\cdots\sqsubseteq fx_n\sqsubseteq fx_{n+1}\sqsubseteq\cdots,$$

$$\mathrm{g}x_1\sqsubseteq \mathrm{g}x_2\sqsubseteq\cdots\sqsubseteq \mathrm{g}x_{n+1}\sqsubseteq \mathrm{g}x_{n+2}\sqsubseteq\cdots,$$

First we prove that

$$d(fx_n,fx_{n+1})\preceq \lambda d(fx_{n-1},fx_n)\quad \text{for } n\geq 1 \tag{1.16}$$

Since $\mathrm{g}x_n\sqsubseteq \mathrm{g}x_{n+1}$, it is

$$d(fx_n,fx_{n+1})\preceq \lambda\cdot u,$$

Where

$$u \in \left\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}), \frac{d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)}{2}\right\}$$
$$= \left\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1})}{2}\right\}.$$

Now we have to consider the following three cases:

(i) If $u = d(fx_{n-1}, fx_n)$ then cleary (1.16) hold.

(ii) If $u = d(fx_n, fx_{n+1})$ then according to Remark (1.1.7)(vii)

$d(fx_n, fx_{n+1}) = 0$, and (1.16) is immediate.

(iii) Finally, suppose $u = \frac{d(fx_{n-1}, fx_{n+1})}{2}$. Now

$$d(fx_n, fx_{n+1}) \preceq \lambda \frac{d(fx_{n-1}, fx_{n+1})}{2} \preceq \frac{\lambda}{2} d(fx_{n-1}, fx_n) + \frac{1}{2}(fx_n, fx_{n+1}).$$

Hence $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n)$, and we have proved (1.16).

Now, we have

$$d(fx_n, fx_{n+1}) \preceq \lambda^n d(fx_0, fx_1).$$

We shall show that $\{ f_n\}$ is a Cauchy sequence. For $m, n \in \mathbb{N}, n > 0$ we have

$d(fx_n, fx_m) \preceq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots + d(fx_{m+1}, fx_m)$,

and we obtain

$$d(fx_n, fx_m) \preceq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) d(fx_0, fx_0)$$
$$\preceq \frac{\lambda^m}{1-\lambda} d(fx_0, fx_1) \to 0 \; as \; m \to \infty.$$

From Remark (1.1.7)(viii) it follow that for $0 \ll c$ and m sufficiently large, $\lambda^m (1-\lambda)^{-1} d(fx_0, fx_1) \ll c$; then also $d(fx_n, fx_m) \ll c$. hence, $\{ f_n\}$ is a Cauchy sequence.

Since $f(X) \subset g(X), g(X)$ is closed, and X is complete, there exists $u \in g(X)$ such that $g(x_n) \to u \; as \; n \to \infty$. Consequently, we can find $z \in X$ such that $gz = u$.

Let us show that $fz = u$. For this we have (because of $gx_n \sqsubseteq gz$)

$$d(fz, u) \preceq d(fz, fx_n) + d(fx_n, u) \preceq \lambda \cdot u_n + d(fx_n, u),$$

Where

$$u \in \left\{ d(\mathrm{g}x_n, \mathrm{g}z), d(fx_n, gx_n), d(fz, \mathrm{g}z), \frac{d(fx_n, \mathrm{g}z) + d(fz, \mathrm{g}z)}{2} \right\}.$$

Let $0 \ll c$ be given. Since $\mathrm{g}x_n \to \mathrm{g}z$, in each of the following cases there exists n_0 such that for $n \geq n_0$ we have $d(fz, u) \ll c$.

(i) $d(fz, u) \preceq \lambda \cdot d(\mathrm{g}x_n, \mathrm{g}z) + d(fx_n, u) \ll \lambda \cdot \frac{c}{2\lambda} + \frac{c}{2} = c.$

(ii) $d(fz, u) \preceq \lambda \cdot d(fx_n, \mathrm{g}x_n) + d(fx_n, u) \preceq \lambda \cdot d(fx_n, u) + \lambda \cdot d(u, \mathrm{g}x_n) + d(fx_n, u) = (\lambda + 1) \cdot d(fx_n, u) + \lambda \cdot d(u, \mathrm{g}x_n) \ll (\lambda + 1) \cdot \frac{c}{2(\lambda+1)} + \lambda \cdot \frac{c}{2\lambda} = c.$

(iii) $d(fz, u) \preceq \lambda \cdot d(fz, u) + d(fx_n, u)$, i.e., $d(fz, u) \ll \frac{1}{1-\lambda} \cdot (1-\lambda)c = c.$

(iv) $d(fz, u) \preceq \lambda \cdot \frac{d(fx_n, \mathrm{g}z) + d(fz, \mathrm{g}z)}{2} + d(fx_n, u) \preceq \frac{\lambda d(fx_n, \mathrm{g}z)}{2} + \frac{1}{2} d(fz, \mathrm{g}z) + d(fx_n, u)$, i.e., $d(fz, u) \preceq (\lambda + 2) d(fx_n, u) \ll (\lambda + 2) \frac{c}{(\lambda+2)} = c.$

Using Remark (1.1.7)(ii) we conclude that $d(fz, u) = 0$, i.e., $fz = u$.

Hence, we have proved that f and g have a coincidence point $z \in X$ and a point of coincidence $u \in X$ such that $u = f(z) = \mathrm{g}(z)$. if they are weakly compatible, then $\mathrm{gg}z = \mathrm{g}fz = f\mathrm{g}z = ffz$.

We shall prove that $fz = \mathrm{g}z$ is a common fixed point of the mapping f and g. using $\mathrm{g}z \sqsubseteq \mathrm{gg}z$ (condition (iv)), we obtain from condition (i) that

$$d(fz, ffz) \preceq \lambda \cdot u,$$

Where

$$u \in \left\{ d(\mathrm{g}z, \mathrm{g}fz), d(fz, \mathrm{g}z), d(ffz, \mathrm{g}fz), \frac{d(fz, \mathrm{g}fz) + d(ffz, \mathrm{g}z)}{2} \right\}$$

$$= \left\{ d(fz, ffz), 0, \frac{d(fz, ffz) + d(fzz, fz)}{2} \right\} = \{0, d(fz, ffz)\}.$$

Hence, be Remark (1.1.7) $d(fz, ffz) = 0$, i.e., $fz = ffz$. Similarly, $\mathrm{g}z = \mathrm{gg}z$ and the Theorem is proved.

Theorem (1.2.8)[1]. Let (f, g) be a pair of self-maps on a complete ordered cone metric space $(X, \sqsubseteq, d)$ such that $f(X) \subset g(X)$ and such that there is a point $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$. suppose that

(i) There exist $\lambda \in [0,1)$ such that for each $x, y \in X$ satisfying $gy \sqsubseteq gx$,

there exist $u \in M_2^{f,g}(x, y)$, Such that $d(fx, fy) \preceq \lambda \cdot u$ holds.

(ii) $g(X)$ is closed in X;

(iii) f is g-nondecreasing;

(iv) if $\{g(x_n)\} \subset X$ is a nondecrasing sequence, converging to some gz,

then $gx_n \sqsubseteq gz$ and $gz \sqsubseteq ggz$.

Then f and g have a coincidence point.

Moreover, if f and g are weakly compatible, then they have a common fixed point. The proof is similar, and so is omitted.

Note that conditions (i) of Theorem (1.2.2),(1.2.7) and (1.2.8) are incomparable in the cone metric settings (to the contrary with the situation in metric settings), since for $a, b \in P$, if a and b are incomparable, then also $\frac{a+b}{2}$ is incomparable, both with a and with b.

Remark (1.2.9)[1]. Putting $E = \mathbb{R}$, $P = [0, +\infty)$ in Theorem (1.2.7) and (1.2.8), one obtains the respective common fixed point Theorem in ordered metric spaces (we could not find explicit formulations for some of these assertions in literature). For example, taking $u = d(gx, gy), g = i_X$, a result of Abbas and Jungck is obtained; then, taking $E = \mathbb{R}, P = [0, +\infty)$ the respect result in the setting of ordered metric spaces follows. If we take $u = \frac{1}{2}\big(d(gx, fx) + d(gy, fy)\big), g = i_X$, we obtain an ordered cone metric version of Kannaan's Theorem (State that for any fixed k there exists alanguage L in $\sum_2$, which is not in size (n^k) (this is different statement than $\sum_2 \not\subseteq p/poly$, which is currently open and state that there exists asingle language that is not in size$(n^k)$for any k).it is asimple Circuitlower bound).[5] ; ordered metric version of this theorem follows immediately. The same applies for the known Zamfiresu's result.

We conclude with an example showing that our Theorem (1.2.2),(1.2.7) and (1.2.8) are proper extensions of the respective results from the setting of cone metric space. Namily, we shall construct an example of a mapping which is an ordered g-quasicontraction (where from the existence of common fixed point of f and g follows), while it is not a g-quasicontraction in cone metric sence. Similar conclusion then applies for relationship between contractive conditions in ordered metric spaces and simple metric spaces.

Example (1.2.10)[1]. Let $X = [0, +\infty)$ and let order relation $\sqsubseteq$ be defined by $x \sqsubseteq y \Leftrightarrow \{(x = y) \text{or } (x, y \in [0,1] \text{ with } x \leq y)\}$.

Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E: x(t) \geq 0 \text{ on } [0,1]$ (this cone is not normal). Define $d: X \times X \to E$ by $d(x, y) = |x - y|\varphi$ where $\varphi: [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. it is easy to see that d is a cone metric on X. consider the mappings

$$fx = \begin{cases} \frac{x}{4}, & 0 \leq x \leq 1, \\ 4x - \frac{15}{4}, & x > 1; \end{cases} \qquad gx = \begin{cases} x, & 0 \leq x \leq 1, \\ \frac{3}{4}x, & x > 1. \end{cases}$$

Then, for $y \sqsubseteq x$ we have that

$$d(fx, fy)(t) = |fx - fy|e^t = \frac{1}{4}|x - y|e^t \leq \lambda|x - y|e^t, \ \ \forall\, t \in [0,1] \Leftrightarrow$$

$\lambda \in \left[\frac{1}{4}, 1\right)$,

While for $x, y > 1$

$$d(fx, fy)(t) = |fx - fy|e^t = 4|x - y|e^t \leq \lambda \frac{3}{4}|x - y|e^t, \ \ \forall\, t \in [0,1] \Leftrightarrow$$

$\lambda \in \left[\frac{16}{3}, +\infty\right)$,

And, checking all other conditions, one concludes that f is an ordered g-quasicontraction, while it is not a g-quasicontraction in a (non-ordered) cone metric sense. Obviously, $f(0) = g(0) = 0$.

Similar conclusions apply to conditions of Theorems (1.2.7) and (1.2.8).

Chapter 2

Powers of a Contraction

Let A be a contraction on a Hilbert space H. The defect index d_A of A is by definition, the dimension of the closure of the range of $1 - A^*A$ .We show that (i) $d_{A^n} \leq nd_A$ for all n ≥ 0, (ii)if, in addition, A^n converges to 0 im the strong operator topology and $d_A = 1$, then $d_{A^n} = n$ for all finite n, $0 \leq n \leq dimH$, and (iii) $d_A = d_{A^*}$ implies $d_{A^n} = d_{A^{n*}}$ for all n ≥ 0. The norm-one index K_A of A is defined as $\sup\{n \geq 0: \| A^n \| = 1\}$. When dim H = m < ∞, a lower bound for K_A was obtained before: $K_A \geq (m/d_A) - 1$

Section (2.1). Powers of Contraction

Let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$, and let A be a contraction ($\|A\| \equiv \sup\{\|Ax\|: x \in H, \|x\| = 1\} \leq 1$) on H. the defect index of A is, by definition, $\overline{\operatorname{ran}(I - A^*A)}$ of $1 - A^*A$. It is a measure of how far A is from the isometries, and plays a prominent role in the Sz.-Nagy-Foias Theory of canonical model for contractions.

In this chapter, we are concerned with the defect indices of powers of a contraction. We show that, for a contraction A, d_{A^n} is at most nd_A for any $n \geq 0$. they are in general not equal. The equality does hold in certain cases. For example, if A^n converges to 0 in the strong operator topology and $d_A = 1$, then $d_{A^n} = n$ for all finite n, $0 \leq n \leq \dim H$. The equality (for some n's) also arises in another situation, namely, in relation to the norm- one index. Recall that the norm-one index K_A of a contraction A is defined as $\sup\{n \geq 0: \|A^n\| = 1\}$. it was proven that if A acts on an m-dimensional space, then $K_A \geq (m/d_A) - 1$. Here we complement this result by characterizing all the m-dimensional A with $K_A = (m/d_A) - 1$; this is case if and only if either A is unitary or the eigenvalues of A are all in the open unit disc $\mathbb{D}(\equiv \{z \in \mathbb{C}: |z| < 1\})$, d_A divides m and $d_{A^n} = nd_A$ for all $n, 1 \leq n \leq m/d_A$. we consider contractive analytic functions of a contraction, instead of just its powers. Among other things, we show that if f is a Blaschke product with n zeros, then $d_{f(A)} = d_{A^n}$.

We start with some basic properties for the defect indices of powers of a contraction. These include a "triangle inequality" and their increasingness.

Lemma (2.1.1): Let $A = BC$, where B and C are contractions. Then $d_C \leq d_A \leq d_B + d_C$. if B and C commute, then we also have $d_B \leq d_A$.

Proof : Since

$$I - A^*A = I - C^*B^*BC \geq I - C^*C \geq 0,$$

Where we used $C^*B^*BC \leq CC^*$ because $B^*B \leq 1$, we obtain $\overline{\text{ran}(I - A^*A)} \supseteq \overline{\text{ran}(I - C^*C)}$ and thus $d_A \geq d_C$. If B and C commute, then $A = CB$ and, Therefore, $d_B \leq d_A$ follows from above.

On the other hand, since

$$I - A^*A = 1 - C^*B^*BC = (I - C^*C) + C^*(I - B^*B)C,$$

We have

$$\text{ran}(I - A^*A) \subseteq \text{ran}(I - C^*C) + \text{ran}C^*(I - B^*B)C.$$

Thus

$$\begin{aligned} d_A &\leq d_C + \text{rank } C^*(I - B^*B)C \\ &\leq d_C + \text{rank } (I - B^*B)C \\ &\leq d_C + d_B, \end{aligned}$$

Completing the proof.

For any contraction A, let $H_n = \overline{\text{ran}(I - A^{n*}A^n)}$ for $n \geq 0$ and $H_\infty = \bigvee_{n=0}^{\infty} H_n$. In the following, we will frequently use the fact that, for a contraction A, x is in ker $(I - A^*A)$ if and only if $\|Ax\| = \|x\|$.

Note that $d_B \leq d_A$ may not hold without the commutativity of B and C, For example, if $A = I$, $B = S^*$ and $C = S$, where S denotes the (simple) unilateral shift, then $A = BC$, $d_A = 0$ and $d_B = 1$.

Theorem (2.1.2)[2]. Let A be a contraction on H.

(i) The inequality $d_{A^{m+n}} \leq d_{A^m} + d_{A^n}$ holds for any $m, n \geq 0$. in particular, $d_{A^n} \leq n d_{A^n}$ for $n \geq 0$.

(ii) The sequence $\{d_{A^n}\}_{n=0}^{\infty}$ is increasing in n, Moreover, if $d_{A^n} = d_{A^{n+1}} < \infty$ for some $n, 0 \leq n \leq \dim H$, then $d_{A^k} = d_{A^n}$ for all $k \geq n$.

The proof depends on the following more general lemma.

Proof :(i) and the increasingness of the $d_{A^n}{}'S$ in (ii) follow immediately from lemma (2.1.1). To prove the remaining part of (ii), we check that $H_n = \bigvee_{k=0}^{n-1} A^{k*} H_1$ for $n \geq 1$. Indeed, if $x = (I - A^{n*}A^n)y$ for some y in H, then $x = \sum_{k=0}^{n-1} A^{k*}(I - A^*A)A^k y$, which shows that x is in $\bigvee_{k=0}^{n-1} A^{k*} H_1$. For the converse containment, note that A maps $\ker(I - A^{k+1*}A^{k+1})$ to $\ker(I - A^{k*}A^k)$ isometrically for each $k \geq 0$. Indeed, if x is in the former, then

$$\|x\| = \|A^{k+1}x\| \leq \|Ax\| \leq \|x\|.$$

Hence we have the equalities throughout and, in particular, $\|A^k(Ax)\| = \|Ax\|$ and $\|Ax\| = \|x\|$. The former implies that $Ax \in \ker(I - A^{k*}A^k)$. Together with the latter, this proves our assertion. Therefore, A^* maps H_k to H_{k+1} for $k \geq 0$. by iteration, we have that A^{k*} maps H_1 to H_{k+1} for all $k \geq 1$. Arguing as above, we also obtain $\ker(I - A^{k+1*}A^{k+1}) \subseteq \ker(I - A^{k*}A^k)$ $H_k \subseteq H_{k+1}$ for $k \geq 0$. Therefore, A^{k*} maps H_1 to H_n for all $k, 0 \leq k \leq n-1$. This proves $\bigvee_{k=1}^{n-1} A^{k*} H_1 \subseteq H_n$ and hence our assertion on their equality.

If $d_{A^n} = d_{A^{n+1}} < \infty$ for some n, then $H_n = H_{k+1}$. hence

$$H_{n+2} = \bigvee_{k=0}^{n+1} A^{k*} H_1 = \bigvee_{k=0}^{n} A^{k*} H_1 \vee (A^{n+1*} H_1)$$
$$\subseteq H_{n+1} \vee (A^* H_{n+1}) = H_{n+1} \vee (A^* H_n)$$
$$\subseteq H_{n+1} \vee H_{n+1} = H_{n+1} \subseteq H_{n+2}.$$

Therefore, we have equalities throughout. This implies that $d_{n+1} = d_{n+2}$. Repeating this argument gives us $d_{A^k} = d_{A^n}$ for all $k \geq n$.

Note that, in Theorem (2.1.2)(i), $d_{A^{m+n}} < d_{A^m} + d_{A^n}$ may happen even for for $m = n = 1$. For example, if

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Then $d_A = 2$ and $d_{A^2} = 3$, Thus $d_{A^2} < d_A + d_A$.

The following corollary is an easy consequence of Theorem (2.1.2)(ii).

Corollary (2.1.3)[2]. if A is a contraction with A^n isometric (resp., unitary), then A itself is isometric (resp., unitary). The next Theorem says that the equalities $d_{A^n} = nd_A, n \geq 0$, do hold for certain contractions A.

Theorem (2.1.4)[2]. If A is a contraction on H with A^n converging to 0 in the strong operator topology and $d_A = 1$, then $d_{A^n} = n$ for all finite $n, 0 \leq n \leq \dim H$.

Proof: Under our assumption that $d_A = 1$, we have $d_{A^n} \leq n$ for all $n \geq 0$ by Theorem (2.1.2)(i). Assume that $d_{A^{n_0}} < n_0$ for some finite $n_0, 1 < n_0 \leq \dim H$. since d_{A^n} increases in n, the pigeonhole principle(states that if n items are put into m containers, with n>m, then at least one container must contain more than one items)[6]. and Theorem (2.1.2)(ii) yield that $d_{A^{n_0-1}} = d_{A^{n_0}} = d_{A^n} < n_0 < \infty$ for all $n \geq n_0$. hence

$$\ker(I - A^{n_0*}A^{n_0}) = \overline{\operatorname{ran}(I - A^{n_0*}A^{n_0})}^{\perp} = \overline{\operatorname{ran}(I - A^{n*}A^{n})}^{\perp} = \ker(I - A^{n*}A^n)$$ for all $n \geq n_0$. Let K denote this common subspace. For x in K, we have $\|A^n x\| = \|x\|$ for all $n \geq n_0$. On the other hand, the assumption that $A^n \to 0$ in the strong operator topology yields that $\|A^n x\| \to 0$ as $n \to \infty$. From these, we conclude that $x = 0$ and hence $K = \{0\}$. This is the same as $\ker(I - A^{n_0*}A^{n_0}) = \{0\}$ or $\overline{\operatorname{ran}(I - A^{n_0*}A^{n_0})} = H$. Thus $\dim H = d_{A^{n_0}} < n_0$. which is a contradiction. Therefore, we must have $d_{A^n} = n$ for all finite $n, 0 \leq n \leq \dim H$. Let A a contraction on H. Since A^* maps H_n to H_{n+1} for $n \geq 0$ as shown in the proof of Theorem (2.1.2)(ii), we have $A^* H_\infty \subseteq H_\infty$. Hence

$$A = \begin{bmatrix} A' & 0 \\ B & V \end{bmatrix} \text{ on } H = H_\infty \oplus H_\infty^{\perp}.$$

Note that, for any x in $H_\infty^\perp = \cap_{n=0}^\infty \ker(I - A^{n*}A^n)$, we have $A^*Ax = x$, which implies that $\|Vx\| = \|Ax\| = \|x\|$. Thus V is isometric on $H_\infty^\perp$. Recall that a contraction is completely nonunitary (c.n.u.) if it has no nontrivial reducing subspace on which it is unitary. A can be uniquely decomposed as $A_1 \oplus U$ on $K \oplus K^\perp$, where A_1 is c.n.u. on K and U is unitary on $K^\perp = \cap_{n=0}^\infty (\ker(I - A^{n*}A^n) \cap \ker(I - A^n A^{n*}))$. Thus the above decomposition can be further refined as

$$A = \begin{bmatrix} A' & 0 & 0 \\ B_1 & S_m & 0 \\ 0 & 0 & U \end{bmatrix},$$

Where S_m denotes the unilateral shift with multiplicity $m (0 \le m \le \infty)$, $A_1 = \begin{bmatrix} A' & 0 \\ B_1 & S_m \end{bmatrix}$ is c.n.u., and $V = S_m \oplus U$ corresponds to the Wold decomposition of V.

Corollary (2.1.5)[2]. If A is a contraction on a finite-dimensional space with $d_A = 1$, then

$$d_{A^n} = \begin{cases} n & if\ 0 \le n \le n_0, \\ n_0 & if\ n > n_0, \end{cases}$$

Where $n_0 = \dim H_\infty$.

Proof: On a finite-dimensional space, the above representation of A becomes $A = A' \oplus V$ on $H = H_\infty \oplus H_\infty^\perp$ with V unitary. It is easily seen that A' has no eigenvalue of modulus one. Hence A'^n converges to 0 in norm. Our assertion on d_{A^n} then follows from Theorems (2.1.4) and (2.1.2)(ii).

The next theorem characterizes those contractions A for which $d_{A^n} = n$ for finitely many n's or for all $n \ge 0$. it generalizes Corollary (2.1.5).

Recall that an operator A on an n-dimensional space is said to be of class S_n if A is a contraction, its eigenvalues are all in $\mathbb{D}$ and $d_A = 1$. The n-by-n Jordan block

$$J = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

Is one example, Such operators and their infinite-dimensional analogues $S(\varphi)$ (φ an inner function) are first studied by Sarason. They play the role of the building blocks of the Jordan model for C_0 contractions.

Theorem (2.1.6)[2]. Let A be a contraction on H.

(i) Let n_0 be a nonnegative integer. Then

$$d_{A^n} = \begin{cases} n & if\ 0 \le n \le n_0, \\ n_0 & if\ n > n_0 \end{cases}$$

If and only if $P_{H_\infty} A|H_\infty$, the compression of A to H_∞, is of class S_{n_0}. In this case, $\dim H_\infty = n_0$.

(ii) $d_{A^n} = n$ for all $n, 0 \le n < \infty$, *if and only if* $d_A = 1$ and $\dim H_\infty = \infty$.

Proof: (i) let

$$A = \begin{bmatrix} A' & 0 \\ B & V \end{bmatrix} \text{ on } H = H_\infty \oplus H_\infty^\perp,$$

Where V is isometric, First assume that the $d_{A^n}'s$ are as asserted. We need to show that $A' = P_{H_\infty} A|H_\infty$ is of class S_{n_0}. Our assumption on d_{A^n} implies $H_\infty = H_{n_0}$ is of dimension n_0. Moreover, for any $n \ge 0$, we have

$$I - A^{n*}A^n = I - \begin{bmatrix} A'^{n*} & B_n^* \\ 0 & V^{n*} \end{bmatrix} \begin{bmatrix} A'^n & 0 \\ B_n & V^n \end{bmatrix}$$

$$= \begin{bmatrix} I - A'^{n*}A'^n - B_n^* B_n & -B_n^* V^n \\ -V^{n*}B_n & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I - A'^{n*}A'^n - B_n^* B_n & 0 \\ 0 & 0 \end{bmatrix},$$

Where the last equality holds because $I - A^{n*}A^n \ge 0$. Hence

$$n = d_{A^n} = \text{rank}\,(I - A'^{n*}A'^n - B_n^* B_n) \le \text{rank}\,(I - A'^{n*}A'^n) = d_{A'^n}$$

For $1 \le n \le n_0$. If $n_1 < d_{A'^{n_0}}$ for some $n_1, 1 \le n_1 \le n_0$, then the pigeonhole principle and Theorem (2.1.2)(ii) yield $d_{A^n} = d_{A^{n_0}} < n_0$ for all that $d_{A'^{n_0-1}} = d_{A'^{n_0}}$. and the fact that A' has no eigenvalue of modulus one, we conclude that

$I - A'^{n_0-1*}A'^{n_0-1}$ is one-to-one and hence $d_{A'^{n_0}} = n_0$, contradicting our assumption. Hence $d_{A'^n} = n$ for all n, $1 \le n \le n_0$. implies that A' is of class S_{n_0}. this proves one direction. For the converse, we derive as above to obtain $I - A^{n*}A^n = (I - A'^{n*}A'^n - B_n^* B_n) \oplus 0$ on $H = H_\infty \oplus H_\infty^\perp$ and

$$d_{A^n} \le d_{A'^n} = \begin{cases} n & if\ 1 \le n \le n_0 \\ n_0 & if\ n > n_0 \end{cases} \quad (1.1)$$

Assume that $d_{A^{n_1}} < n_1$ for some $n_1, 1 \le n_1 \le n_0$. then the pigeonhole principle and Theorem (2.1.2)(ii) yields $d_{A^n} = d_{A^{n_0}} < n_0$ for all $n \ge n_0$. This implies that $H_n = H_{n_0}$ for all $n \ge n_0$. Therefore, $H_\infty = H_{n_0}$ has dimension strictly less than n_0, which contradicts the fact that $\dim H_\infty = d_{A'^{n_0}} = n_0$. Hence we have $d_{A^n} = n$ for all $n_1, 1 \le n \le n_0$. If $n > n_0$, then $d_{A^n} \ge d_{A^{n_0}} = n_0$ by Theorem (2.1.2)(ii) and what we have just proven. This, together with (1.1), yields, $d_{A^n} = n_0$ for $n > n_0$.

(ii) Since $\dim H_\infty \ge d_{A^n}$ for all n, the necessity is obvious. Conversely, assume that $d_A = 1$ and $\dim H_\infty = \infty$. Then $d_{A^n} \le n d_A = n$ by Theorem (2.1.2)(i). If $d_{A^{n_1}} \le n_1$ for some $n_1 \ge 2$, then an argument analogous to the one for the second half of (i) yields that $H_\infty = H_{n_1}$ is of dimension less than n_1. This contradicts our assumption. Hence we must have $d_{A^n} = n$ for all n.

We now proceed to consider contractions A with $d_A = d_{A^*}$ and start with the following lemma giving conditions of the equality of d_A and d_{A^*} for an arbitrary operator A. Note that, in this case, the definition of the defect index still makes sense.

Lemma (2.1.7)[2]. Let A be an operator on H.

(i) If $\dim \ker A = \dim \ker A^*$, then $d_A = d_{A^*}$. In particular, if A acts on a finite-dimensional space, then $d_A = d_{A^*}$.

(ii) If d_A is finite, then the following conditions are equivalent:

1^0. $d_A = d_{A^*}$;

2^0. $\dim \ker A = \dim \ker A^*$;

3^0. A^*A and AA^* are unitarily equivalent;

4^0. A is the sum of a unitary opereter and a finite-rank operator.

Proof: (i) if $\dim \ker A = \dim \ker A^*$, then $A = U(A^*A)^{1/2}$ for some unitary operator U. Hence $A^*A = U(A^*A)U^*$ is unitarily equivalent to A^*A. Then the same is true for $I - A^*A$ and $I - AA^*$. Thus $d_A = d_{A^*}$.

(ii) It was proven that if $A^*A = A_1 \oplus 0$ (resp., $AA^* = A_2 \oplus 0$) on $H = \overline{\operatorname{ran} A^*} \oplus \ker A$ (resp., $H = \overline{\operatorname{ran} A^*} \oplus \ker A^*$), then A_1 and A_2 are unitarily equivalent. If $d_A = d_{A^*} < \infty$, then

$$\operatorname{rank}(I - A_1) + \dim \ker A = \operatorname{rank}(I - A^*A) = \operatorname{rank}(I - AA^*)$$
$$= \operatorname{rank}(I - A_2) + \dim \ker A^*$$

And hence $\dim \ker A = \dim \ker A^*$. This proves that 1^0 implies 2^0 If 2^0.hold, then the unitary equivalence of A_1 and A_2 implies the same for A^*A and AA^*, that is, 2^0 implies 3^0. Now assume that 3^0 holds. Since $\ker A^*A = \ker A$ and $\ker AA^* = \ker A^*$, the unitary equivalent of A^*A and AA^* implies that $\dim \ker A = \dim \ker A^*$. Hence $d_A = d_{A^*}$ by (i), that is, 1^0 holds. Finally, the equivalence of $1^0$and $4^0$was proven.

Note that, in the preceding lemma, $d_A = d_{A^*} = \infty$ does not imply $\dim \ker A = \dim \ker A^*$in general, For example, if

$A = \operatorname{diag}(1, 1/2, 1/3, \cdots) \oplus S$, where S is the (simple) unilateral shift, then $d_A = d_{A^*} = \infty$, $\dim \ker A = 0$ and $\dim \ker A^* = 1$.

Theorem (2.1.8)[2]. Let A be a contraction with $d_A = d_{A^*} < \infty$. Then $\dim H_\infty < \infty$ if and only if the completely nonunitary part of A acts on a finite-dimentional space.

Proof: Assume that $\dim H_\infty < \infty$ and let

$$A = \begin{bmatrix} A' & 0 & 0 \\ B & S_m & 0 \\ 0 & 0 & U \end{bmatrix}, \text{on } H = H_\infty \oplus K_1 \oplus K_2,$$

Where S_m denotes the unilateral shift with multiplicity $m, 0 \le m \le \infty$, and U is unitary. We need to show that S_m does not appear in this representation of A or, equivalently, $m = 0$. We first prove that m, is finite. Indeed, since

$$I - AA^* = \begin{bmatrix} I - A'A'^* & -A'B^* & 0 \\ -BA'^* & I - BB^* - S_m S_m^* & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

We have

$$\begin{aligned} m = \text{rank}\,(I - S_m S_m^*) &\leq \text{rank}\,(I - BB^* - S_m S_m^*) + \text{rank}\,BB^* \\ &\leq \text{rank}\,(I - AA^*) + \text{rank}\,BB^* \\ &\leq d_{A^*} + \dim H_\infty < \infty \end{aligned}$$

As asserted. Now to show that $m = 0$, consider S_m as

$$J = \begin{bmatrix} 0 & & & \\ I_m & 0 & & \\ & I_m & 0 & \\ & & \ddots & \ddots \end{bmatrix}$$

Then B is of the form $[B' \quad 0 \quad 0 \quad \cdots]^T$. Let $\tilde{A} = \begin{bmatrix} A' & 0 \\ B' & 0 \end{bmatrix}$. Since $\tilde{A}$ acts on a finite-dimensional space, we have $d_{\tilde{A}} = d_{\tilde{A}^*}$ by Lemma (2.1.7)(i). Then

$$\begin{aligned} d_{A^*} &= \text{rank}\,(I - A^*A) \\ &= \text{rank}\begin{bmatrix} I - A'A'^* & -A'B^* \\ -BA'^* & I - BB^* - S_m S_m^* \end{bmatrix} \\ &= d_{\tilde{A}^*} = d_{\tilde{A}} = \text{rank}\begin{bmatrix} I - A'^*A' - B'^*B' & 0 \\ 0 & I_m \end{bmatrix} \\ &= m + \text{rank}\,(I - A'^*A' - B'^*B') \\ &= m + \text{rank}\,(I - A'^*A' - BB^*) \\ &= m + \text{rank}\begin{bmatrix} I - A'^*A' - BB^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= m + \text{rank}\,(I - A^*A) = m + d_A. \end{aligned}$$

We infer from the assumption $d_A = d_{A^*} < \infty$ then $m = 0$. Thus $A = A' \oplus U$, where A' is the c.n.u. part of A acting on the finite-dimensional space H_∞.

The converse is trivial. The next two results are valid for any operators.

Proposition (2.1.9)[2]. If A is an operator with $d_A = d_{A^*}$, then $d_{A^n} = d_{A^{n*}}$ for all $n \geq 1$.

Proof: If $d_A = d_{A^*} < \infty$ then $A = U + F_n$ where U is unitary and F has finite rank, by Lemma (2.1.7)(ii). For any $n \geq 1$, we have $A^n = U^n + F_n$ where F_n is some finite-rank operator. By Lemma (2.1.7)(ii) again, this implies that $d_{A^n} = d_{A^{n*}}$. On the other hand, if $d_A = d_{A^*} = \infty$, then $d_{A^n} = d_{A^{n*}} = \infty$, for any $n \geq 1$ by Theorem (2.1.1)(ii). This completes the proof.

Two operators A on H and B on K are said to be quasi-similar if there operators $X: H \to K$ and $Y: K \to H$ which are one and have dense range such that $XA = BX$ and $YB = AY$. We conclude this section with the following result on quasi-similar operators.

Proposition (2.1.10)[2]. Let A and B be quasi-similar operators. If $d_A = d_{A^*} < \infty$, then $d_A = d_{B^*}$.

Proof. Our assumption of $d_A = d_{A^*} < \infty$ implies, by Lemma (2.1.7)(ii), that $\dim \ker A = \dim \ker A^*$. The quasi-similarity of A and B then yields

$$\dim \ker B = \dim \ker A = \dim \ker A^* = \dim \ker B^*$$

Then $d_B = d_{B^*}$ by lemma (2.1.7)(i).

Note that the preceding proposition is false if $d_A = d_{A^*} = \infty$.

Example (2.1.11)[2]. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of distinct complex numbers in $\mathbb{D}$ with $\sum_n (1 - |a_n|) < \infty$. Let $A = \operatorname{diag}(a_1, a_2, \cdots) \oplus S$, where S denotes the (simple) unilateral shift. Let $\emptyset$ be the Blaschke product with zeros a_n: $\phi(z) = \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{z - a_n}{1 - \overline{a_n} z}$, $z \in \mathbb{D}$,

And let $B = S(\phi) \oplus S$, where $S(\phi)$ denotes the compression of the shift

$$S(\phi) f = P(zf(z)), \quad f \in H^2 \ominus \phi H^2,$$

P being the (orthogonal) projection from H^2 onto $H^2 \ominus \phi H^2$. It is known that $\operatorname{diag}(a_n)$ is itself a C_0 contraction which is quasi-similar to $S(\phi)$. Thus A is quasi-similar to B. But $d_A = d_{A^*} = \infty$, $d_B = 1$ and $d_{B^*} = 2$

Section (2.2): Norm-one Index and Contractive Functions of a Contraction

As defined the norm-one index of a contraction A on H is $K_A \equiv \sup(n \geq 0: \|A^n\| = 1\}$. This number is to measure how far the powers of A remain to have norm one. It is easily seen that (i) $0 \leq K_A \leq \infty$, (ii) $K_A = 0$ if and only if $\|A\| < 1$, and (iii) $K_A = \infty$ if and only if $\sigma(A) \cap \partial\mathbb{D} \neq \emptyset$. The main result say that if $\dim H = m < \infty$, then (iv) $0 \leq K_A \leq m-1$ or $K_A = \infty$ or (v) $K_A = m-1$ if and only if A is of class S_m, and (vi) $K_A \geq (m/d_A) - 1$. The purpose of this chapter is to determine when the equality holds in (vi).

Theorem (2.2.1)[2]. Let A be a contraction on an m-dimensional space. Then $K_A = (m/d_A) - 1$ if and only if one of the following holds:

(i) A is unitary,

(ii) $\sigma(A) \subseteq \mathbb{D}$, d_A divides m, and $d_{A^n} = nd_A$ for all n, $1 \leq n \leq m/d_A$.

Proof: Assume that $K_A = (m/d_A) - 1$. If $\sigma(A) \cap \partial\mathbb{D} \neq \emptyset$, then $(m/d_A) - 1 = \infty$, which implies that $d_A = 0$ or A is unitary. Hence we may assume that $(A) \subseteq \mathbb{D}$. Then $K_A < \infty$. From $K_A = (m/d_A) - 1$, we have$(d_A|m)$. By the pigeonhole principle and theorem (2.1.2)(ii), there is a smallest integer l, $1 \leq l \leq m$, such that $d_{A^l} = d_{A^{l+1}}$. since A has no unitary part, this is equivalent to $I - A^{l*}A^l$ being one-to-one or $\|A^l\| < 1$. As l is the smallest such integer, we obtain $K_A = l - 1$. From $K_A = (m/d_A) - 1$, we have $m/d_A = l$. Note that $d_{A^n} \leq nd_A \; for \; 1 \leq n \leq l$ by Theorem (2.1.2)(i). If $d_{A^{n_0}} < n_0 d_A$ for some n_0, $1 \leq n_0 \leq l$, then

$$d_{A^l} \leq d_{A^{n_0}} + d_{A^{l-n_0}} < n_0 d_A + (l - n_0)d_A = ld_A = m$$

Again by Theorem (2.1.2)(i). This contradicts the fact that $I - A^{l*}A^{l}$ is one-to-one. Hence we must have $d_{A^n} < nd_A$ for $1 \leq n \leq m/d_A$. This prove (ii).

Conversely, if (i) holds, that is, if A is unitary, then $K_A = \infty$ and $d_A = 0$. hence $K_A = (m/d_A) - 1$.

Now assume that (ii) holds. If $l = m/d_A$, then our assumptions imply that $1 \leq d_A < d_{A^2} < \cdots < d_{A^l} = m$. Hence $I - A^{l*}A^{l}$ is one-to-one, but $I - A^{l-1*}A^{l-1}$ is not. Thus $\|A^l\| < 1$ and $\|A^{l-1}\| = 1$. This yields $K_A = l - 1 = (m/d_A) - 1$ as required.

On an m-dimensional space, other than unitary operators, S_m-operators and strict contractions (operators with norm strictly less than one), which correspond to $d_A = 0,1$ and m, respectively, there are other contractions A satisfying $K_A = (m/d_A) - 1$. For example, if $A = \underbrace{J_l \oplus \cdots \oplus J_l}_{m/l}$ where l divides, then $K_A = l - 1 = (m/d_A) - 1$. The same is true for the more general $B = \underbrace{A_1 \oplus \cdots \oplus A_1}_{m/l}$, where A_1 is an S_1-operator. Another generalization of the a contraction A is

$$C = \begin{bmatrix} 0 & a_1 & & & \\ & 0 & \ddots & & \\ & & \ddots & & \\ & & & & a_{m-1} \\ & & & & 0 \end{bmatrix},$$

Where $|a_j| < 1$ for $j = kl, 1 \leq k \leq (m/l) - 1$, and $|a_j| = 1$ for all other $j's$. In this case, it is easily seen that d_c equals m minus number of $j's$ for which $|a_j| = 1$ and hence $d_c = m/l$. On the other hand, K_c equals the maximum number of consecutive $j's$ whith $|a_j| = 1$, and thus $K_c = l - 1$. Therefore, $K_A = (m/d_A) - 1$ holds.

In this chapter, we consider the defect indices of contractive functions of a contraction, instead of just its powers. The first one is Blaschke products:

$$f(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\overline{a}_j z}, \quad z \in \mathbb{D}, \text{ where } |a_j| < 1 \text{ for all } j,$$

Theorem (2.2.2)[2]. If A is a contraction on H and f is a Blaschke product with n zeros (counting multiplicity), then $d_{f(A)} = d_{A^n}$.

Proof . Let f be as above and let $f_j(z) = (z - a_j)/(1 - \overline{a}_j A), z \in \mathbb{D}$, for each j. Let $X = \prod_{j=1}^{n}(I - a_j A)$, $K_1 = \ker(I - A^{n*}A^n)$, and $K_2 = \ker(I - f(A)^* f(A))$. We first show that $XK_1 \subseteq K_2$. indeed, if x is in K_1, then $\|A^n x\| = \|x\|$. Applying once (with ϕ_1 there as f_1 and the remaining $\phi_j's$ given by $\phi_j(z) = z$) yields $\|f_1(A)A^{n-1}(I - \overline{a_1}A)x\| = \|(I - \overline{a_1}A)x\|$. We then apply repeatedly to obtain $\|f_1(A) \cdots f_n(A)Xx\| = \|Xx\|$. This means that Xx is in K_2. Hence we have $XK_1 \subseteq K_2$ as asserted. Since X is invertible, if

$$X = \begin{bmatrix} x_1 & * \\ 0 & x_2 \end{bmatrix} : H = K_1 \oplus K_1^{\perp} \to H = K_2 \oplus K_2^{\perp},$$

then X_2 has dense range. Thus $X_2^*: K_2^{\perp} \to K_1^{\perp}$ is one-to-one. Therefore,

$$d_{f(A)} = \dim K_2^{\perp} \leq \dim K_1^{\perp} = d_{A^n}$$

In a similar fashion, if $Y = \prod_{j=1}^{n}(I + \overline{a}_j A)$, then successive applications of also yield $YK_2 \subseteq K_1$. We can then infer as above that $d_{A^n} \leq d_{f(A)}$. This proves their equality.

For more general functions, we use the Sz.-Nagy-Foias functional calculus for contractions. For any absolutely continuous contraction A (this means that A has no nontrivial reducing subspace on which A is a singular unitary operator) and any function $f \ in \ H^\infty$ with $\|f\|_\infty \leq 1$, the operator $f(A)$ can be defined and is again a contraction. Not that every function in H^∞ can be factored as the product of an inner and an outer function, and every inner function is the product of a Blaschke product and a singular inner function.

Note that if f is as above, then $f(A) = \prod_{j=1}^{n}(A - a_j I)(I - \overline{a}_j A)^{-1}$ is also a contraction **.**

Theorem (2.2.3)[2]. Let A be an absolutely continuous contraction on H and f be a function in H^{∞} with $\|f\|_{\infty} \leq 1$.

(i) If f has an infinite Blaschke product factor, then $d_{f(A)} \geq \sup\{d_{A^n}: n \geq 0\}$.

(ii) If f is a (nonconstant) inner function, then $d_{f(A)} \leq \sup\{d_{A^n}: n \geq 0\}$. In particular, if f is an inner function with an infinite Blaschke product factor, then $d_{f(A)} = \sup\{d_{A^n}: n \geq 0\}$.

Proof: (i) For each $n \geq 1$, let $f = f_n g_n$, where f_n is a finite Blaschke product with n zeros and g_n is in H^{∞}. Then $f(A) = f_n(A) g_n(A)$. Theorem (2.2.2) and Lemma (2.1.1) imply that $d_{A^n} = d_{f_n(A)} \leq d_{f(A)}$ for all $n \geq 1$. thus $d_{f(A)} \geq \sup\{d_{A^n}: n \geq 0\}$.

(ii) We may assume that $n_0 \equiv \sup\{d_{A^n}: n \geq 0\} < \infty$. This means that $\dim H_{\infty} = n_0$ is finite. Let

$$A = \begin{bmatrix} A' & 0 & 0 \\ B & S_m & 0 \\ 0 & 0 & U \end{bmatrix} \text{ on } H = H_{\infty} \oplus K_1 \oplus K_2,$$

Where S_m is the unilateral shift with multiplicity m, $0 \leq m \leq \infty$, and U is unitary. Then

$$f(A) = \begin{bmatrix} f(A') & 0 & 0 \\ C & f(S_m) & 0 \\ 0 & 0 & f(U) \end{bmatrix}.$$

Note that $f(S_m)$ is itself a unilateral shift, say, $S_l (0 \leq l \leq \infty)$ and $f(U)$ is unitary because f is inner. Hence

$$I - f(A)^* f(A) = \begin{bmatrix} I - f(A')^* f(A') - C^* C & -C^* S_l & 0 \\ -S_l^* C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I - f(A')^* f(A') - C^* C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $I - f(A)^* f(A) \geq 0$, Therefore,

$$d_{f(A)} = \text{rank}\,(I - f(A')^* f(A') - C^*C) \leq \text{rank}\,(I - f(A')^* f(A'))$$
$$= d_{f(A')} \leq n_0.$$

This completes the proof.

Note that Theorem (2.2.3)(i) is in general false if f is a finite Blaschke product. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $f(z) = z$, then $d_{f(A)} = d_A = 1$, but $\sup\{d_{A^n} : n \geq 0\} = 2$. Theorem (2.2.3)(ii) is also false for general f in H^∞ with $\|f\|_\infty \leq 1$. As an example, let A be the (simple) unilateral shift. Then $\sup\{d_{A^n} : n \geq 0\} = 0$. On the other hand, $f(A)$ is an analytic Toeplitz operator with symbol f, which is an isometry if and only if f is inner. Thus $d_{f(A)} = 0$ can happen only when f is inner. The next corollary generalizes Proposition (2.1.9).

Corollary (2.2.4)[2]. if A is an absolutely continuous contraction and f is either a finite Blaschke product or an inner function with an infinite Blaschke product factor, then $d_{f(A)} = d_{f(A)^*}$.

Proof: since $f(A)^* = \tilde{f}(A^*)$, where $\tilde{f}(z) = \overline{f(\bar{z})}$ for $z \in \mathbb{D}$, the assertion follows easily from Theorems (2.2.2) and (2.2.3).

Chapter 3

Type Maps in Cone Metric Space

Results are related to the cases when g is f quasi contraction in a sense of Das and Naik, and the cone need not be normal. These results generalize several well known comparable results.

Section (3.1): Cone Metric Space and Self-mappings

In 1922, Banach proved the following famous fixed point theorem. Let (X, d) be a complete metric space. Let g be a contractive mapping on X, that is, there exists $\lambda \in [0,1)$ satisfying

$$d(\mathrm{g}x, \mathrm{g}y) \leq \lambda \cdot d(x, y). \tag{1.1}$$

for all $x, y \in X$, then there exists a unique fixed point $x_0 \in X$ of g. This Theorem, called the Banach contraction principle, is a forceful tool in nonlinear analysis. This principle has many applications an is extended by several authors.

The study of common fixed points of mappings satisfying certain contractive conditions has many applications and has been at the center of various research activity. For the convenience of the reader, let us recall the following results.

Theorem (3.1.1)[3]. Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f, g satisfy

$$\mathrm{g}(X) \subset f(X) \tag{1.2}$$

and there exists a constant $\lambda \in (0,1)$ such that for evry $x, y \in X$

$$d(\mathrm{g}x, \mathrm{g}x) \leq \lambda \cdot d(fx, fy). \tag{1.3}$$

then f and g have a unique common fixed point.

If f and g satisfy (1.2) and $x_0 \in X$, let us define $x_1 \in X$ such that $\mathrm{g}(x_0) = f(x_1)$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $\mathrm{g}(x_n) = f(x_{n+1})$. Set $y_n = \mathrm{g}(x_n), n = 0,1,2, \ldots$. This procedure was essentially introduced by Jungch, and is Picard iteration procedure when $f = I_X$ is the identity map on X.

Theorem (3.1.2)[3]. Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f, g satisfy (1.2) and there exists a constant $\lambda \in (0,1)$ such that for evry $x, y \in X$

$$d(gx, gx) \leq \lambda \cdot M(x, y), \tag{1.4}$$

Where

$$M(x, y) = \max \left\{ \begin{matrix} d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gy), \\ d(fy, gx) \end{matrix} \right\}. \tag{1.5}$$

Then f and g have a unique common fixed point.

Let us mention that if $f = I_X$ is identity map on X, and g satisfies (1.5), than g is called quasi contraction.Ćirić introduced and studied quasicontraction as one of the most general contractive type map. The well known Ćirić$'s$ result is that quasicontraction g possesses a unique fixed point.

There exist a lot fixed- point theorems for self-mappings defined on closed subset of Banach space. However, for applications (numerical analysis, optimization, etc.) it is important to consider functions that are not self-mappings. And it is natural to search for sufficient conditions which would guarantee the existence of fixed points for such mappings. The study of fixed point Theorems for non-self mappings in metrically convex spaces was initiated by Assad and Kirk which proved productive as metrically convex spaces offer a natural setting for proving such results. In recent years this technique has been exploited by many authors and by now there exists considerable literature on this topic. To mention a few, and let us recall the next result.

Theorem (3.1.3)[3]. let X be a Banach space, C a nonempty closed subset of X, and ∂C the boundary of C. Let $T: C \to X$ be a nonself mapping such that for some constant $\lambda \in (0,1)$ and for every $x, y \in C$

$d(Tx, Ty) \leq$

$\lambda \cdot$

$$\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \qquad (1.6)$$

Suppose that

$$T(\partial C) \subset C \qquad (1.7)$$

then T has a unique fixed point in C.

Let us remark that to extend the known fixed point theorem for self quasi contraction $T: C \to C$ to corresponding non self mapping $T: C \to X, C \neq X$, was open more than 20 yr.

Definition (3.1.4)[3]. Let X be a linear space. Suppose that the mapping $\|\cdot\|: X \to E$ satisfies:

(i) $\|x\| \succeq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X, \lambda \in \mathbb{C}$;

(iii) $\|x + y\| \preceq \|x\| + \|y\|$ for all $x, y \in X$.

Then $\|\cdot\|$ is called a cone norm on X and $(X, \|\cdot\|)$ is called a cone normed space.

Each cone normed space X is a cone metric space with cone metric in X defined by means of the formula

$$d(x,y) = \|x - y\|, \quad x, y \in X. \qquad (1.8)$$

Definition (3.1.5)[3]. Let (X,d) be a cone metric space, $x \in X$ and $\{X_n\}_{n\geq 1}$ as a sequence in X. Then

(i) $\{X_n\}_{n\geq 1}$ converge to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{X_n\}_{n\geq 1}$ Is a Cauchy sequence if for every c in E with $0 \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X,d) Is a complete cone metric space if every Cauchy sequence is convergent.

(iv) let $f: X \to X$ and $x_0 \in X$. Function f is continuous at x_0 if any sequence $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.

Example (3.1.6)[3]. Let $X = \mathbb{R}, E = \mathbb{R}^n$ and $P = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. it is easy to see that $d: X \times X \to E$ defined by $d(x, y) = (|x - y|, k_1|x - y|, \cdots, k_{n-1}|x - y|)$ is a cone metric on X, where $k_i \geq 0$ for all $i \in \{1, \cdots, n - 1\}$.

Example (3.1.7)[3]. Let $E = l^1, P = \{\{x_n\}_{n\geq1} \in E: x_n \geq 0; \text{for all } n\}$, (X, ρ) be a metric space and $d: X \times X \to E$ defined by $d(x, y) = \left\{\frac{\rho(x,y)}{2^n}\right\}_{n\geq1}$.Then (X, d) is a cone metric space.

Example (3.1.8)[3]. Let $E = C^1_{\mathbb{R}}([0,1])$ with norm $\|f\| = \|f\|_\infty + \|f^2\|_\infty$. the cone $P = \{f \in E: f \geq 0\}$ is a non-normal cone.

We need the following lemma in the sequel.

Lemma (3.1.9)[3]. Let (X, d) be a cone metric space. Then, the following statements hold.

(i) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

(ii) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(iii) If $u \ll v$ and $v \ll w$, then $u \ll w$.

(iv) If $0 \preceq u \ll c$, for each $c \in \text{int } P$ then $u = 0$.

(v) Let $x \in X, \{x_n\}_{n\geq1}$ and $\{b_n\}_{n\geq1}$ two sequences in X, $0 \ll c$ and $0 \preceq d(x_n, x) \preceq b_n$ for all $n \geq 1$. if $b_n \to 0$, then there exists a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.

Definition (3.1.10)[3]. Let (X, d) be a cone metric space, and let $\text{g}, f: X \to X$, then g is called f-quasi contraction if for some constant $\lambda \in (0,1)$ and for every $x, y \in X$, there exists

$$u \in C(f; x, y) = \{d(fx, fy), d(fx, \text{g}x), d(fx, \text{g}y), d(fy, \text{g}y), d(fy, \text{g}x)\},$$

such that

$$d(\text{g}x, \text{g}y) \preceq \lambda \cdot u. \tag{1.9}$$

since, in the case of the cone metric spaces, the set $C(f; x, y)$ need not even have the sup in ordered Banach space E, then, we use "ϵ". It clear that " ϵ" can be used in metric spaces, while"$\preceq$ " cannot be used, in general, in cone metric spaces.

Theorem (3.1.11)[3]. Let (X, d) be a cone metric space, and P a normal cone. let $g, f: X \to X, f$ commutes with g, f or g is continuous, and satisfy (1.2) and (1.9). Let $\{y_n\}$ be the sequence defined by procedure introduced by Jungck. Sequence $\{y_n\}$ is a Cauchy and $\lim_n\{y_n\} = y \in X$. then f and g have a common unique fixed point u in X. In the case when f is continuous, then $u = gy = fy$; if g is continuous, then $u = y$.

In this chapter we study common fixed points for the self and non-self (g, f) type maps in cone metric spaces. Our main results are related to the cases when g is f quasi contraction in a sense of Das and Naik, and cone need not be normal. These results generalize several well known comparable results in the literature. In this chapter we study quasi contraction type self mappings on cone metric spaces. The intention is to prove previous results in the frame of cone metric spaces in which the cone need not be normal. We begin with the following result.

Theorem (3.1.12)[3]. Let (X, d) be a complete cone metric space. Let f a continuous self-map on X and g be any self-map on X that commutes with$f$. Further let f and g satisfy

$$gX = fX \tag{1.10}$$

and there exists a constant $\lambda \in (0, 1/2)$ such that for every $x, y \in X$, there exists

$$u(x, y) \in C(f, x, y) = \{d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gy), d(fy, gx)\}$$

Such that

$$d(gx, gy) \preceq \lambda u(x, y). \tag{1.11}$$

Then f and g have the unique common fixed point.

Note that the corresponding result in the case when $f = i_X$ and $\lambda \in (0,1)$.

Proof: Let us remark that the condition (1.10) implies that starting with an arbitrary $x_0 \in X$, we can construct a sequence $\{y_n\}$ of points in X such that $y_n = gx_n = fx_{n+1}$, for all $n \geq 0$. we shall prove that $\{y_n\}$ is a Cauchy sequence. First, we show that

$$d(y_n, y_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(y_{n-1}, y_n) \tag{1.12}$$

for all $n \geq 1$. indeed,

$$d(y_n, y_{n+1}) = d(gy_n, gy_{n+1}) \preceq \lambda u_n, \tag{1.13}$$

where

$$u_n \in \left\{ \begin{matrix} d(fx_n, fx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}), d(fx_n, gx_{n+1}), \\ d(fx_{n+1}, gx_n) \end{matrix} \right\}$$

$$= \{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), d(y_n, y_n)\}$$

$$= \{d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), \theta\}.$$

From (1.13) it follows four cases:

(i) $d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n) \preceq \dfrac{\lambda}{1-\lambda} d(y_{n-1}, y_n)$.

(ii) $d(y_n, y_{n+1}) \preceq \lambda d(y_n, y_{n+1})$ And so $d(y_n, y_{n+1}) = 0$. In this case, (1.12) follow immediately, because$\lambda < \frac{\lambda}{1-\lambda}$

(iii) $d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n) + \lambda d(y_n, y_{n+1})$. It follows that (12) holds.

(iv) $d(y_n, y_{n+1}) \preceq \lambda \cdot 0 = 0$ and so $d(y_n, y_{n+1}) = 0$. Hence, (1.12) holds.

Thus by putting $h = \frac{\lambda}{1-\lambda}$, $d(y_n, y_{n+1}) \preceq hd(y_{n-1}, y_n)$. now, by using (1.12) we have

$$d(y_n, y_{n+1}) \preceq hd(y_{n-1}, y_n) \preceq \cdots \preceq h^n d(y_0, y_1),$$

For all $n \geq 1$. Now, $n > m$ we have

$$\begin{aligned} d(y_n, y_m) &\preceq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m) \\ &\preceq (h^{n-1} + h^{n-2} + \cdots + h^m) d(y_0, y_1) \\ &\preceq \frac{h^m}{1-h} d(y_0, y_1) \to \theta \text{ as } m \to \infty. \end{aligned}$$

By Lemma (3.1.10) (v) and (i), $\{y_n\}$ is a Cauchy sequence. Therefore, there exists $z \in X$ such that

$$y_n = \mathrm{g}x_n = fx_{n+1} \to z.$$

Now we show that $fz = \mathrm{g}z = z$. In this way, note that

$$d(fz, \mathrm{g}z) \preceq d(fz, \mathrm{g}fx_n) + d(\mathrm{g}fx_n, \mathrm{g}z),$$

for all $n \geq 1$. Also we have $d(\mathrm{g}fx_n, \mathrm{g}z) \preceq \lambda u_n$ for all $n \geq 1$, where

$$u_n \in \{d(f^2x_n, fz), d(f^2x_n, \mathrm{g}fx_n), d(fz, \mathrm{g}z), d(f^2x_n, \mathrm{g}z), d(fz, \mathrm{g}fx_n)\}.$$

Let $0 \ll c$. Since $\mathrm{g}fx_n = f\mathrm{g}x_n \to fz$ and $f^2x_n \to fz$, choose a natural number n_0 such that for all $n \geq n_0$ we have $d(fz, \mathrm{g}fx_n) \ll \frac{(1-\lambda)c}{2}$ and $d(f^2x_n, fz) \ll \frac{(1-\lambda)c}{2\lambda}$. Thus, we obtain the following cases:

(i) $d(fz, \mathrm{g}z) \preceq d(fz, \mathrm{g}fx_n) + \lambda d(f^2x_n, fz) \ll \frac{c}{2} + \lambda\frac{c}{2\lambda} = c$.

(ii) $d(fz, \mathrm{g}z) \preceq d(fz, \mathrm{g}fx_n) + \lambda d(f^2x_n, \mathrm{g}fx_n)$

$$\begin{aligned} &\preceq d(fz, \mathrm{g}fx_n) + \lambda\big(d(f^2x_n, fz) + d(fz, \mathrm{g}fx_n)\big) \\ &= (1+\lambda)d(fz, \mathrm{g}fx_n) + \lambda d(f^2x_n, fz) \\ &\ll (1+\lambda)\frac{(1-\lambda)c}{2} + \lambda\frac{(1-\lambda)c}{2\lambda} < \frac{c}{2} \ll c \end{aligned}$$

(iii) $d(fz, \mathrm{g}z) \preceq d(fz, \mathrm{g}fx_n) + \lambda d(fz, \mathrm{g}z)$.

Hence, $d(fz, \mathrm{g}z) \preceq \frac{1}{1-\lambda}d(fz, \mathrm{g}fx_n) \ll \frac{1}{1-\lambda}\frac{(1-\lambda)c}{2} \ll c$

(iv) $d(fz, \mathrm{g}z) \preceq d(fz, \mathrm{g}fx_n) + \lambda d(f^2x_n, \mathrm{g}z) \preceq d(fz, \mathrm{g}fx_n) + \lambda\big(d(f^2x_n, fz) + d(fz, \mathrm{g}z)\big)$.

Hence,

$$d(fz, \mathrm{g}z) \preceq \frac{1}{1-\lambda}d(fz, \mathrm{g}fx_n) + \frac{1}{1-\lambda}d(f^2x_n, \mathrm{g}z) \ll \frac{1}{1-\lambda}\frac{(1-\lambda)c}{2} + \frac{\lambda}{1-\lambda}\frac{(1-\lambda)c}{2\lambda} = c$$

(v) $d(fz, \mathrm{g}z) \preceq d(fz, \mathrm{g}fx_n) + \lambda d(fz, f\mathrm{g}x_n) = (1+\lambda)d(fz, \mathrm{g}fx_n)$

$$\ll (1+\lambda)\frac{(1-\lambda)c}{2} < \frac{c}{2} \ll c.$$

Therefore, $d(fz, gz) \ll c$ for all $0 \ll c$. By Lemma (3.1.10)(iv), $d(fz, gz) = 0$ and so $fz = gz$. Thus,

$$d(fz, z) \preceq d(gz, gx_n) + d(gx_n, z) \preceq d(gx_n, z) + \lambda v_n,$$

Where

$$v_n \in \{d(fx_n, fz), d(fx_n, gx_n), d(fz, gz), d(fx_n, gz), d(fz, gx_n)\}$$
$$= \{d(fx_n, fz), d(fx_n, gx_n), 0, d(fz, gx_n)\}.$$

Let $0 \ll c$ be given. Choose a natural number n_0 such that for all $n \geq n_0$ we have $d(fx_n, z) \ll \frac{(1-\lambda)c}{2\lambda}$ and $d(gx_n, z) \ll \frac{(1-\lambda)c}{2}$.

Again, we have the following cases:

(i) $d(fz, z) \preceq d(gx_n, z) + \lambda d(fx_n, fz) \preceq d(gx_n, z) + \lambda d(fx_n, fz) + \lambda d(z, fz)$.

Hence,

$$d(fz, z) \preceq \frac{1}{1-\lambda} d(gx_n, z) + \frac{\lambda}{1-\lambda} d(fx_n, fz) \ll \frac{1}{1-\lambda}\frac{(1-\lambda)c}{2} + \frac{\lambda}{1-\lambda}\frac{(1-\lambda)c}{2\lambda} = c$$

(ii) $d(fz, z) \preceq d(gx_n, z) + \lambda d(fx_n, gx_n) \preceq d(gx_n, z) + \lambda d(fx_n, fz) + \lambda d(z, gx_n) = (1 + \lambda) d(z, gx_n) + \lambda d(fx_n, fz)$

$$\ll (1 + \lambda)\frac{(1-\lambda)c}{2} + \lambda \frac{(1-\lambda)c}{2\lambda} = \frac{2 - \lambda^2 - \lambda}{2} c \ll c.$$

(iii) $d(fz, z) \preceq d(gx_n, z) + \lambda \cdot \theta = d(gx_n, z) \ll c$.

(iv) $d(fz, z) \preceq d(gx_n, z) + \lambda\, d(fz, gx_n) \preceq d(gx_n, z) + \lambda d(fz, z) + \lambda d(z, gx_n)$.

Hence,

$$d(fz, z) \preceq \frac{1+\lambda}{1-\lambda} d(gx_n, z) \ll \frac{1+\lambda}{1-\lambda} + \frac{(1-\lambda)c}{2} \ll c.$$

Therefore, $d(fz, z) \ll c$ for all $0 \ll c$. By Lemma (3.1.9).(iv) $fz = gz = z$ is a common fixed point for f and g. Uniqueness follows easily from (1.11).

From Theorem (3.1.12), as corollaries, among other things, we recover and generalize the results of Huang and Zhang, and Rezapour and HamIbarani. As consequences, we also obtain cone metric versions, Finally, in the next corollary, we extend the well known Jungck result (Theorem (3.1.1)).

Corollary (3.1.13)[3]. Let (X, d) be a complete cone metric space. Let f a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy $gX \subset fX$ and that for some constant $\lambda \in (0,1)$ and every $x, y \in X$,

$$d(gx, gy) \preceq \lambda \cdot d(fx, fy).$$

Then f and g have the unique common fixed point. Now, we prove a result analogue to Theorem (3.1.2) in the frame of cone metric space when the cone need not be normal, i.e., cone version of Das and Naik.

Theorem (3.1.14)[3]. Let (X, d) be a complete cone metric space. Let f^2 a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy

$$gfX \subset f^2X \tag{1.14}$$

and there exists a constant $\lambda \in (0, 1/2)$ such that for every $x, y \in X$, there exists

$$u(x, y) = \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\},$$

Such that

$$d(gx, gy) \preceq \lambda u(x, y). \tag{1.15}$$

Then f and g have the unique common fixed point.

Proof: By (1.14) starting with an arbitrary $x_0 \in fX$, we can construct a sequence $\{x_n\}$ of points in fX such that $y_n = gx_n = fx_{n+1}, n \geq 0$ (as in Theorem (1.12). Now $fy_{n+1} = fgx_{n+1} = gfx_{n+1} = gy_n = z_n, n \geq 1$. As in the Theorem (1.13). We prove that $\{z_n\}$ is a Cauchy sequence and hence convergent to some $z \in X$. Further, we shall show that $f^2z = gfz$. Since

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n$$
$$= \lim_{n\to\infty} gy_n = \lim_{n\to\infty} z_n = z,$$ it follows that

$$\lim_{n\to\infty} f^4x_n = \lim_{n\to\infty} f^3\mathrm{g}x_n = \lim_{n\to\infty} \mathrm{g}f^3x_n = f^2 z,$$

Because f^2 is continuous. Now, we obtain

$$d(f^2z, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + d(f^3\mathrm{g}x_n, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda \cdot u_n,$$

Where

$$u_n \in \left\{ \begin{matrix} d(f^4x_n, f^2z), d(f^4x_n, \mathrm{g}f^3x_n), d(f^2z, \mathrm{g}fz), d(f^4x_n, \mathrm{g}fz), \\ d(f^2z, \mathrm{g}f^3x_n) \end{matrix} \right\}.$$

Let $0 \ll c$ be given. Since $f^3\mathrm{g}x_n \to f^2z$ and $f^4x_n \to f^2z$, choose a natural number n_0 such that for all $n \geq n_0$ we have $d(f^2z, f^3\mathrm{g}x_n) \ll \frac{c(1-\lambda)}{2}$ and $d(f^4x_n, \mathrm{g}f^3x_n) \ll \frac{(1-\lambda)c}{2\lambda}$. Again, we have the following cases:

(i) $d(f^2z, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^4x_n, f^2z) \ll \frac{c}{2} + \lambda\frac{c}{2\lambda} = c.$

(ii) $d(f^2z, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^4x_n, \mathrm{g}f^3x_n)$

$$\begin{aligned} &\preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^4x_n, f^2z) + \lambda d(f^2z, \mathrm{g}f^3x_n) \\ &= (1+\lambda)d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^4x_n, f^2z) \\ &\ll (1+\lambda)\frac{c(1-\lambda)}{2} + \lambda\frac{(1-\lambda)c}{2\lambda} \ll c. \end{aligned}$$

(iii) $d(f^2z, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^2z, \mathrm{g}fz).$

Hence,

$$d(f^2z, \mathrm{g}fz) \preceq \frac{1}{1-\lambda} d(f^2z, f^3\mathrm{g}x_n) \ll \frac{1}{1-\lambda}\frac{c(1-\lambda)}{2} = c.$$

(iv) $d(f^2z, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^4x_n, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^4x_n, f^2z) + d(f^2z, \mathrm{g}fz).$

Hence,

$$\begin{aligned} d(f^2z, \mathrm{g}fz) &\preceq \frac{1}{1-\lambda} d(f^2z, f^3\mathrm{g}x_n) + \frac{\lambda}{1-\lambda} d(f^4x_n, f^2z) \\ &\ll \frac{1}{1-\lambda}\frac{c(1-\lambda)}{2} + \frac{\lambda}{1-\lambda}\frac{(1-\lambda)c}{2\lambda} = c. \end{aligned}$$

(v) $d(f^2z, \mathrm{g}fz) \preceq d(f^2z, f^3\mathrm{g}x_n) + \lambda d(f^2z, \mathrm{g}f^3x_n) \ll \frac{c}{2} + \lambda\frac{c}{2\lambda} = c.$

Therefore, $d(f^2z, \mathrm{g}fz) \ll c$ for all $0 \ll c$. By Lemma (3.1.9).(iv), $f^2z = \mathrm{g}fz$ and so $\mathrm{g}fz$ is a common fixed point for f and g. Indeed, putting in (1.15)

$x = \mathrm{g}fz, y = fz$ we get $\mathrm{g}(\mathrm{g}fz) = \mathrm{g}fz$. Because $f^2z = \mathrm{g}fz$; i.e., $f(fz) = \mathrm{g}(fz)$, we have $f(\mathrm{g}fz) = \mathrm{g}f^2z = \mathrm{g}(\mathrm{g}fz) = \mathrm{g}fz$.

Section (3.2): The Non-Self Maps

In this chapter we consider quasi contraction and f-quasi contractions as non-self mappings in the frame of cone metric spaces in which the cone need not be normal.

Definition (3.2.1)[3]. Let (X, d) be a cone metric space, C a nonempty closed subset of X, and $\mathrm{g}, f: C \to X$. If for some $\lambda \in \left(0, \frac{3-\sqrt{5}}{2}\right)$ and for all $x, y \in C$ there exists

$$u(x, y) \in \{ d(fx, fy), d(\mathrm{g}x, fx), d(\mathrm{g}y, fy), d(\mathrm{g}x, fy), d(\mathrm{g}y, fx)\},$$

Such that

$$d(\mathrm{g}x, \mathrm{g}y) \preceq \lambda u(x, y), \tag{1.16}$$

Then g is called f-quasicontractive mapping from C into X.

Theorem (3.2.2)[3]. Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that

$$d(x, z) + d(z, y) = d(x, y). \tag{1.17}$$

Suppose that $\mathrm{g}, f: C \to X$ are such that g is f-quasicontractive mapping of C into X and

(i) $\partial C \subseteq fC, \mathrm{g}C \cap C \subset fC$,

(ii) $fx \in \partial C \Rightarrow \mathrm{g}x \in C$,

(iii) fC is closed in X.

Then, there exists a coincidence point z in C. Moreover, if (g, f) is a coincidentally commuting, then z is the unique common fixed point of f and g.

Proof: First of all, we construct two sequence:$\{x_n\}$ in C and the sequence $\{y_n\}$ in $fC \subset X$ in the following way.

Let $x \in \partial C$ be arbitrary. There exists a point $x_0 \in C$ such that $x = fx_0$ as $\partial C \subset fC$. Since $fx_0 \in \partial C$ then $gx_0 \in C$, we conclude that $gx_0 \in C \cap gC \subset fC$. Let $x_1 \in C$ be such that $y_1 = fx_1 = gx_0 \in C$. Let $y_2 = gx_1$. Suppose $y_2 \in C \cap gC \subset fC$, which implies that there exists a point $x_2 \in C$ such that $y_2 = fx_2$. Suppose $y_2 \notin C$. Then there exists a point $p \in \partial C$ such that

$$d(fx_1, p) + d(p, y_2) = d(fx_1, y_2).$$

Since $p \in \partial C \subset fC$, there exists a point $x_2 \in C$ such that $p = fx_2$, so that the equation above takes the form

$$d(fx_1, fx_2) + d(fx_2, y_2) = d(fx_1, y_2).$$

Put $y_3 = gx_2$. In this way, repeating the following arguments, one obtains two sequences:$\{x_n\} \subset C$ and $\{y_n\} \subset gC \subset X$ such that:

(i) $y_{n+1} = gx_n$, for $n = 0,1,2, \dots$;

(ii) if $y_n \in C$, then $y_n = fx_n = gx_{n-1}$;

(iii) if $y_n \notin C$, then $fx_n \in \partial C$ and

$$d(fx_{n-1}, y_n) + d(fx_n, y_n) = d(fx_{n-1}, y_n).$$

Put

$$S = \{fx_i \in \{fx_n\}: fx_i = y_i\}, \quad Q = \{fx_i \in \{fx_n\}: fx_i \neq y_i\}.$$

Obviously, two consecutive terms cannot lie in Q. Now we wish to estimate $d(fx_n, fx_{n+1})$.

Note that the estimate of $d(fx_n, fx_{n+1})$ in this cone version. In the case of convex metric space it can be used that, for each $x, y, u \in X$ and each $\lambda \in [0,1]$, it is $\lambda d(u,x) + (1-\lambda)d(u,y) \leq \max\{d(u,x), d(u,y)\}$. In cone spaces the maximum of the set $\{d(u,x), d(u,y)\}$ need not exist. Therefore, besides (1.17) we have to use here the relation "$\in$ " and to consider several cases. In cone metric spaces as well as in metric spaces the key step is the Assad-Kirk's induction.

If $d(fx_n, fx_{n+1}) = 0$ for some n, then it is easy to show that $d(fx_n, fx_{n+1}) = 0$ for all $k \geq 1$.

Suppose that $d(fx_n, fx_{n+1}) \succ 0$ for all n. From the above construction we conclude that there are three possibilities:

Case 1^0. If $fx_n \in S$ and $fx_{n+1} \in S$, then according to (i), (ii) and (1.16) we have:

$$d(fx_n, fx_{n+1}) = d(y_n, y_{n+1}) = d(gx_{n-1}, gx_n) \preceq \lambda u_n,$$

Where

$$u \in \left\{ \begin{matrix} d(fx_{n-1}, fx_n), d(gx_{n-1}, fx_{n-1}), d(gx_n, fx_n), d(gx_{n-1}, fx_n), \\ d(gx_n, fx_{n-1}) \end{matrix} \right\},$$

$$= \{ d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \theta, d(fx_{n+1}, fx_{n-1})\}.$$

Clearly, there are infinite many n such that at least one of the following cases holds:

(i) $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n)$.

(ii) $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n+1}, fx_n)$ and so $d(fx_n, fx_{n+1}) = 0$. But we suppose that $d(fx_n, fx_{n+1}) \succ 0$ for each n.

(iii) $d(fx_n, fx_{n+1}) \preceq \lambda \cdot 0 = 0$ and so $d(fx_n, fx_{n+1}) = 0$. But we suppose that $d(fx_n, fx_{n+1}) \succ 0$ for each n.

(iv) $d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n+1}, fx_{n-1}) \preceq \lambda d(fx_{n-1}, fx_n)\ \lambda\, d(fx_n, fx_{n+1})$ and so $d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda}\, d(fx_{n-1}, fx_n)$.

From (i), (ii), (iii) and (iv) it follows that

$$d(fx_n, fx_{n+1}) \preceq \max\left\{\lambda, \frac{\lambda}{1-\lambda}\right\} d(fx_{n-1}, fx_n)$$
$$= \frac{\lambda}{1-\lambda}\, d(fx_{n-1}, fx_n). \tag{1.18}$$

Case 2^0. Let $fx_n \in S, fx_{n+1} \in Q$. Then $y_n = fx_n, y_{n+1} \notin C, fx_{n+1} \in \partial C$ such that

$$d(y_{n+1}, fx_{n+1}) + d(fx_{n+1}, fx_n) = d(y_{n+1}, fx_n).$$

Note then from this and (1.18), we get

$$d(fx_{n+1}, fx_n) = d(y_n, fx_{n+1}) = d(y_n, y_{n+1}) - d(y_{n+1}, fx_{n+1}) \prec d(y_n, y_{n+1}) \tag{1.19}$$

That is, according to (i) and (1.17) $(y_n, y_{n+1}) = d(\mathrm{g}x_{n-1}, \mathrm{g}x_n) \preceq \lambda u_n$, where

$$u_n \in \left\{ \begin{array}{c} d(fx_{n-1}, fx_n), d(\mathrm{g}x_{n-1}, fx_{n-1}), d(\mathrm{g}x_n, fx_n), d(\mathrm{g}x_{n-1}, fx_n), \\ d(\mathrm{g}x_n, fx_{n-1}) \end{array} \right\}$$

$$= \{ d(fx_{n-1}, fx_n), d(\mathrm{g}x_n, fx_n) = d(y_n, y_{n+1}), \theta, d(\mathrm{g}x_n, fx_{n-1})\}.$$

Again, we obtain the following four cases:

(v) $d(y_n, y_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n)$;

(vi) $d(y_n, y_{n+1}) \preceq \lambda d(y_n, y_{n+1})$ and so $d(y_n, y_{n+1}) = 0$, contradicting the assumption that $d(fx_n, fx_{n+1}) \succ 0$ for each n.

(vii) $d(y_n, y_{n+1}) \preceq \lambda \cdot 0 = 0$ and so $d(y_n, y_{n+1}) = 0$, that is $d(fx_n, fx_{n+1}) = 0$, contradicting the assumption that $d(fx_n, fx_{n+1}) \succ 0$ for each n.

(viii) $d(y_n, y_{n+1}) \preceq \lambda d(y_{n+1}, fx_{n-1}) \preceq \lambda d(y_n, y_{n+1}) + \lambda d(fx_{n-1}, fx_n)$ and so $d(y_n, y_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-1}, fx_n)$.

From (1.19), (v), (vi), (vii) and (viii) we have

$$d(fx_n, fx_{n+1}) \preceq \mu d(fx_{n-1}, fx_n),$$

Where $\mu = \max\left\{\lambda, \frac{\lambda}{1-\lambda}\right\} \frac{\lambda}{1-\lambda}$. (1.20)

Case 3^0. Let $fx_n \in Q, fx_{n+1} \in S$. Then $y_{n+1} = \mathrm{g}x_n = fx_{n+1} \in C, y_n \notin C$ and $fx_n \in \partial C$, such that

$$d(y_{n-1}, fx_n) + d(fx_n, y_n) = d(y_{n-1}, y_n).$$

From this we get

$$d(fx_n, fx_{n+1}) = d(fx_n, y_{n+1}) \preceq d(fx_n, y_n) + d(y_n, y_{n+1})$$

$$= d(y_{n-1}, y_n) - d(fx_{n-1}, fx_n) + d(y_n, y_{n+1}). \tag{1.21}$$

we shall estimate $d(y_{n-1}, y_n)$ and $d(y_n, y_{n+1})$. since $y_{n-1} = fx_{n-1}$, by using 2^0, one can

$$d(y_{n-1}, y_n) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}). \tag{1.22}$$

Further,

$$d(y_n, y_{n+1}) = d(gx_{n-1}, gx_n) \preceq \lambda u_n, \tag{1.23}$$

Where

$$u_n \in \left\{ \begin{matrix} d(fx_{n-1}, fx_n), d(\mathrm{g}x_{n-1}, fx_{n-1}), d(\mathrm{g}x_n, fx_n), d(\mathrm{g}x_{n-1}, fx_n), \\ d(\mathrm{g}x_n, fx_{n-1}) \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} d(fx_{n-1}, fx_n), d(y_{n-1}, y_n), d(fx_n, fx_{n+1}), d(y_n fx_n,), \\ d(fx_{n-1}, fx_{n+1}) \end{matrix} \right\}.$$

Because

$$d(fx_{n-1}, fx_{n+1}) \preceq d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})$$

And $d(y_{n-1}, y_n) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1})$, we have

$$d(y_n, y_{n+1}) \preceq \lambda u_n, \tag{1.24}$$

Where

$$u_n \in \left\{ \begin{matrix} d(fx_{n-1}, fx_n), \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}), d(fx_n, fx_{n+1}), d(y_n fx_n,), \\ d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}) \end{matrix} \right\}.$$

By substituting (1.22) and (1.24) in (1.21) we get

$$d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) - d(fx_{n-1}, fx_n) + \lambda u_n. \tag{1.25}$$

Hence, we get the following cases

(i) $d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) - d(fx_{n-1}, fx_n) + \lambda d(fx_{n-1}, fx_n) = \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) - (1-\lambda) d(fx_{n-1}, fx_n) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1})$.

(ii) $d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) - d(fx_{n-1}, fx_n) + \frac{\lambda^2}{1-\lambda} d(fx_{n-2}, fx_{n-1}) \preceq \frac{\lambda+\lambda^2}{1-\lambda} d(fx_{n-2}, fx_{n-1})$.

(iii) $d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) - d(fx_{n-1}, fx_n) +$

$\lambda d(fx_n, fx_{n+1})$.

Hence,

$$d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda^2} d(fx_{n-2}, fx_{n-1}).$$

(iv) $d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) - d(fx_{n-1}, fx_n) +$

$\lambda(d(y_{n-1}, y_n) - d(fx_{n-1}, fx_n)$

$$\preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) + \frac{\lambda^2}{1-\lambda} d(fx_{n-2}, fx_{n-1})$$

$$= \frac{\lambda+\lambda^2}{1-\lambda} d(fx_{n-2}, fx_{n-1}).$$

(v) $d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{1-\lambda} d(fx_{n-2}, fx_{n-1}) - d(fx_{n-1}, fx_n) +$

$\lambda d(fx_{n-1}, fx_n) + \lambda d(fx_n, fx_{n+1})$.

Hence,

$$d(fx_n, fx_{n+1}) \preceq \frac{\lambda}{(1-\lambda)^2} d(fx_{n-2}, fx_{n-1}) - d(fx_{n-1}, fx_n)$$

$$\preceq \frac{\lambda}{(1-\lambda)^2} d(fx_{n-2}, fx_{n-1}).$$

From (i), (ii), (iii), (iv) and (v) we have

$$d(fx_n, fx_{n+1}) \preceq \mu d(fx_{n-2}, fx_{n-1}),$$

Where,

$$\mu = \max\left\{\frac{\lambda}{1-\lambda}, \frac{\lambda+\lambda^2}{1-\lambda}, \frac{\lambda}{(1-\lambda)^2}\right\} = \frac{\lambda}{(1-\lambda)^2}.$$

Thus, in all cases $1^0 - 3^0$

$$d(fx_n, fx_{n+1}) \preceq hw_n,$$

Where $w_n \in \{d(fx_{n-2}, fx_{n-1}), d(fx_n, fx_{n+1})\}$ and

$$h = \max\left\{\frac{\lambda}{1-\lambda}, \frac{\lambda}{(1-\lambda)^2}\right\} = \frac{\lambda}{(1-\lambda)^2}.$$

Since, $0 < \lambda < \frac{3-\sqrt{5}}{2}$, $\frac{\lambda}{(1-\lambda)^2} = h < 1$.

Following the procedure of Assad and Kirk, it can easily be shown by induction that, for $n > 1$,

$$d(fx_n, fx_{n+1}) \preceq h^{\frac{n-1}{2}} w_2, \tag{1.26}$$

Where $w_2 \in \{d(fx_0, fx_1), d(fx_1, fx_2)\}$.

By the triangle inequality, for $n > m$ we have:

$$d(fx_n, fx_m) \preceq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots + d(fx_{m+1}, fx_m)$$

$$\preceq \left(h^{\frac{n-1}{2}} + h^{\frac{n-2}{2}} + \cdots + h^{\frac{m-1}{2}}\right) w_2 \preceq \frac{\sqrt{h^{m-1}}}{1-\sqrt{h}} w_2 \to \theta, \quad \text{as } m \to \infty.$$

By Lemma (3.1.9). (v) and (i), $\{fx_n\}$ is a Cauchy sequence.

Since $fx_n \in C \cap fC$ and $C \cap fC$ is complete, there is some point $z \in C \cap fC$ such that $fx_n \to z$. let w in C be such that $fw = z$. by the construction of $\{fx_n\}$, there is a subsequence $\{fx_{n(k)}\}$ such that $fx_{n(k)} = y_{n(k)} = \mathrm{g}x_{n(k)-1}$ and so $\mathrm{g}x_{n(k)-1} \to z$. But, we have

$$d(\mathrm{g}w, z) \preceq d(\mathrm{g}w, \mathrm{g}x_{n(k)-1}) + d(\mathrm{g}x_{n(k)-1}, z)$$

$$\preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda u_{n(k)},$$

Where

$$u_{n(k)} \in \left\{\begin{matrix} d(fx_{n(k)-1}, fw), d(\mathrm{g}x_{n(k)-1}, fx_{n(k)-1}), d(\mathrm{g}w, fw), \\ d(\mathrm{g}x_{n(k)-1}, fw), d(\mathrm{g}w, fx_{n(k)-1}) \end{matrix}\right\}.$$

Let $0 \ll c$ be given. Since $\mathrm{g}x_{n(k)-1} \to z = fw$ and $fx_{n(k)-1} \to z = fw$, choose a natural number k_0 such that for all $k \geq k_0$ we have $d(\mathrm{g}x_{n(k)-1}, z) \ll \frac{(1-\lambda)c}{2}$ and $d(fx_{n(k)-1}, z) \ll \frac{(1-\lambda)c}{2\lambda}$. Thus, we get the following cases:

(i) $d(\mathrm{g}w, z) \preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(fx_{n(k)-1}, fw) \ll \frac{(1-\lambda)c}{2} + \lambda \frac{(1-\lambda)c}{2\lambda} \ll c$.

(ii) $d(\mathrm{g}w, z) \preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(\mathrm{g}x_{n(k)-1}, fx_{n(k)-1})$

$$\preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(z, fx_{n(k)-1})$$

$$= (1+\lambda) d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(z, fx_{n(k)-1})$$

$$\ll (1+\lambda)\frac{(1-\lambda)c}{2} + \lambda\frac{(1-\lambda)c}{2\lambda} \ll c.$$

(iii) $d(\mathrm{g}w, z) \preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(\mathrm{g}w, fw)$.

Hence,

$$d(\mathrm{g}w, z) \preceq \frac{1}{(1-\lambda)} d(\mathrm{g}x_{n(k)-1}, z) \ll \frac{1}{1-\lambda}\frac{(1-\lambda)c}{2} \ll c.$$

(iv) $d(\mathrm{g}w, z) \preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(\mathrm{g}x_{n(k)-1}, fw)$

$$= (1+\lambda) d(\mathrm{g}x_{n(k)-1}, z) \ll (1+\lambda)\frac{(1-\lambda)c}{2} \ll c.$$

(v) $d(\mathrm{g}w, z) \preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(\mathrm{g}w, fx_{n(k)-1})$

$$\preceq d(\mathrm{g}x_{n(k)-1}, z) + \lambda d(\mathrm{g}w, z) + \lambda d(z, fx_{n(k)-1}).$$

Hence,

$$d(\mathrm{g}w, z) \preceq \frac{(1-\lambda)}{2} d(\mathrm{g}x_{n(k)-1}, z) + \frac{\lambda}{1-\lambda} d(fx_{n(k)-1}, z) \ll \frac{1}{1-\lambda}\frac{(1-\lambda)c}{2} + \frac{\lambda}{1-\lambda}\frac{(1-\lambda)c}{2\lambda} = c.$$

Therefore, $d(\mathrm{g}w, z) \ll c$ for all $0 \ll c$. By Lemma (3.1.9). (iv), $d(\mathrm{g}w, z) = 0$ and so $\mathrm{g}w = z = fw$ which show that w is a point of coincidence for g and f.

Suppose now that g and f are coincidentally commuting. Then

$$z = \mathrm{g}w = fw \Rightarrow \mathrm{g}z = \mathrm{g}fw = f\mathrm{g}w = fz.$$

Then again from (1.16), $d(\mathrm{g}z, z) = d(\mathrm{g}z, \mathrm{g}w) \preceq \lambda u$, where

$$u \in \{d(fz, fw), d(\mathrm{g}z, fw), d(\mathrm{g}w, fw), d(\mathrm{g}z, fw), d(\mathrm{g}w, fz)\}$$

$$= \{d(\mathrm{g}z, z), d(\mathrm{g}z, z), d(z, z), d(\mathrm{g}z, z), d(z, \mathrm{g}z)\} = \{\theta, d(\mathrm{g}z, z)\}.$$

Hence, we get $d(\mathrm{g}z, z) \preceq \lambda \cdot 0 = 0$ and $d(\mathrm{g}z, z) \preceq \lambda d(\mathrm{g}z, z)$, from which it follows that $d(\mathrm{g}z, z) = 0$, that is z is a common fixed point of g and f. Uniqueness of the common fixed point follows easily from (1.16).Setting $f = I_X$, the identity mapping of X in Theorem (3.2.2), we obtain the following result:

Corollary (3.2.3)[3]. Let (X, d) be a complete cone metric space, C a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that $d(x, z) + d(z, y) = d(x, y)$.

Suppose that $g: C \to X$, such that for some $\lambda \in (0, \frac{3-\sqrt{5}}{2})$ and for all $x, y \in C$, there exists $u(x,y) \in \{d(x,y), d(x, gx), d(y, gy), d(x, gy), d(y, gx)\}$,

So that $d(gx, gy) \preceq \lambda u(x,y)$.

Also, suppose that g has additional property that for each $x \in \partial C, gx \in C$, then g has a unique fixed point. Setting $E = \mathbb{R}$, $P = [0, +\infty)$, $\|\cdot\| = |\cdot|$ in the Corollary (3.2.3).

Theorem (3.2.4)[3]. Let (M, d) be a complete convex metric space with convex structure W which is continuous on the third variable, C be a nonempty closed subset of M and $T: C \to M$ be a nonself mapping satisfying the contractive type condition (ast), that is: there exists $q \in (0,1)$ such that for every $x, y \in C$

$$d(Tx, Ty) \leq q \cdot \max \{d(x,y), d(x, Tx), d(y, Ty), d(y, Tx)\}. \qquad (1.27)$$

If T has the additional property $T(\partial C) \subset C$ then T has a unique fixed point in C.

We present now two examples showing that Theorem (3.2.2) is a proper extension of the known results. In both examples, the conditions of Theorem (3.2.2) are fulfilled, but in the first one (because of non-normality of the cone) the main theorems cannot be applied. This shows that Theorem (3.2.2) is more general, i.e., the main Theorems can be obtained as its special cases (for $0 < \lambda < \left(0, \frac{3-\sqrt{5}}{2}\right)$ taking $\|\cdot\| = |\cdot|, E = \mathbb{R}$, and $P = [0, +\infty[$.

Example (3.2.5)[3]. ((The case of a non-normal cone)). Let $X = \mathbb{R}, C = [0,1], E = C^1_{\mathbb{R}}[0,1], P = \{\varphi \in E: \varphi(t) \geq 0, t \in [0,1]\}$. The mapping $d: X \times X \to E$ is defined in the following way: $d(x,y) = \|x - y\|\varphi$, which $\varphi \in P$ is a fixed function, e.g., $\varphi(t) = 2^t$.

Take functions $gx = ax, fx = bx, 0 < a < 1 < b$, so that $\frac{a}{b} \leq \lambda < \frac{3-\sqrt{5}}{2}$, which map the set $C = [0,1]$ into $\mathbb{R}$. We have that (X, d) is a complete cone metric space with a non-normal cone having the nonempty interior. For example, one easily checks the condition (1.17) that for $x \in [0,1], y \notin [0,1]$ the following holds

$$d(x,1) + d(1,y) = d(x,y) \Leftrightarrow |1-x|\varphi + |y-1|\varphi$$
$$= |y-x|\varphi \Leftrightarrow (1-x)\varphi + (y-1)\varphi = (y-x)\varphi.$$

The mappings g and f are weakly compatible, i.e., they commute in their fixed point $x = 0$. All the conditions of Theorem (3.2.2) are fulfilled, and so the non-self mappings g and f have a unique common fixed point $x = 0$.

Example (3.2.6)[3]. ((The case of a normal cone)). Let $X = [0, +\infty[, C = [0,1] \subset X, E = \mathbb{R}^2, P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$. the mapping $d: X \times X \to E$ is defined in the following way: $d(x,y) = (\|x-y\|, \alpha\|x-y\|, \alpha \geq 0$.Take the function $gx = ax, fx = bx, 0 < a < 1 < b$, so that $\frac{a}{b} \leq \lambda < \frac{3-\sqrt{5}}{2}$ which map the set $C = [0,1]$ into$\mathbb{R}$. We have that (X,d) is a complete cone metric space with a normal cone having the normal coefficient $K = 1$, whose interior is obviously nonempty. All the conditions of Theorem (3.2.2) are fulfilled. We check again the condition (1.17), i.e., that for $x \in C = [0,1], y \notin C = [0,1]$ the following holds

$$d(x,1) + d(1,y) = d(x,y) \Leftrightarrow (|1-x|, \alpha|1-x| + (|y-1|, \alpha|1-y|)$$
$$= (|y-x|, \alpha|y-1|) \Leftrightarrow (1-x) + (y-1) = (y-x).$$

And

$$\alpha|1-x| + \alpha|y-1| = \alpha(y-x).$$

The mappings g and f are weakly compatible, i.e., they commute in their fixed point $x = 0$. All the conditions of Theorem (3.2.2) are again fulfilled. The point $x = 0$ is the unique common fixed point for non-self mappings g and f.

Chapter 4

Rational Dilation on the Tetrablock

We show by a counter example the failure of rational dilation on the tetrablock , a polynomially convex and non-convex domain in $\mathbb{C}^3$, defined as

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zwx_3 \neq 0 \; when\; ever\; |z| \leq 1, |w| \leq 1\}.$$

A commuting triple of operators (T_1, T_2, T_3) for which the closed tetrablock $\overline{\mathbb{E}}$ is a spectral set, is called an $\mathbb{E}$ –contraction. For an $\mathbb{E}$ –contraction(T_1, T_2, T_3), the two operator equations $T_1 - T_2^* T_3 = D_{T_3} X_1 D_{T_3}$ and $T_2 - T_1^* T_3 = D_{T_3} X_2 D_{T_3}$,$D_{T_3} = (I - T_3^* T_3)^{1/2}$, have unique solutions $A_1, A_2 \; on \; D_{T_3} = \overline{Ran} D_{T_3}$ and they are called the fundamental operators of (T_1, T_2, T_3).

Section (4.1): Functional Model for Pure ε-isometries

Let X be a compact subset of $\mathbb{C}^n$ and let $\mathcal{R}(X)$ denote the algebra of all rational functions on X, that is, all quotients p/q of polynomials p, q for which q has no zeros in X. The norm of an element f in $\mathcal{R}(X)$ is defined as

$$\|f\|_\infty, X = \sup\{\, |f(\xi)| : \xi \in X\}.$$

Also for each $k \geq 1$, let $\mathcal{R}_k(X)$ denote the algebra of all $\mathcal{K} \times \mathcal{K}$ matrices over $\mathcal{R}(X)$. Obviously each element in $\mathcal{R}_k(X)$ is a $\mathcal{K} \times \mathcal{K}$ matrix of rational functions $F = (f_{i,j})$ and we can define a norm on $\mathcal{R}_k(X)$ in the canonical way

$$\|F\| = \sup\{\, |F(\xi)| : \xi \in X\},$$

There by making $\mathcal{R}_k(X)$ into a non-commutative normed algebra. Let $\underline{T} = (T_1, \cdots, T_n)$ be an n-tuple of commuting operators on a Hilbert space $\mathcal{H}$. X is said to be a spectral set for $\underline{T}$ if the Taylor joint spectrum $\sigma(\underline{T})$ of $\underline{T}$ is a subset of X and

$$\|f(\underline{T})\| \leq \|f\|_{\infty, X}, for\; every\; f \in \mathcal{R}(X). \quad (1.1)$$

Hence $f(\underline{T})$ can be interpreted as $p(\underline{T})q(\underline{T})^{-1}$when $f = p/q$. Moreover, X is said to be a complete spectral set if $\|F(\underline{T})\| \leq \|F\|$ = for every F in$\mathcal{R}_k(X)$, $k = 1,2,\cdots$.

Let $\mathcal{A}(\underline{T})$ be the algebra of continuous complex-valued functions on X which separates the points of X. A boundary for $\mathcal{A}(X)$ is a closed subset F of X such that every function in $\mathcal{A}(X)$ attains its maximum modulus on F. It follows from the theory of uniform algebras that if bX is the intersection of all the boundaries of X then bX is a boundary for $\mathcal{A}(X)$. This smallest boundary bX is called the Šilov boundary relative to the algebra $\mathcal{A}(X)$.

A commuting n-tuple of operators $\underline{T}$ that has X as a spectral set, is said to have a rational dilation or normal bX-dilation if there exists a Hilbert space $\mathcal{K}$, an isometry $V: \mathcal{H} \to \mathcal{K}$ and an n-tuple of commuting normal operators $\underline{N} = (N_1, \cdots, N_n)$ on $\mathcal{K}$ with $\sigma(\underline{N}) \subseteq bX$ such that

$$f(\underline{T}) = V^* f(\underline{T}) V, \text{ for every } f \in \mathcal{R}_k(X). \tag{1.2}$$

One of the important discoveries in operator theory is $S_{z.}$-Nagy's unitary dilation for a contraction, which opened a new horizon by announcing the success of rational dilation on the closed unit disk of $\mathbb{C}$. Since then one of the main aims of operator theory has been to determine the success or failure of rational dilation on the closure of a bounded domain in $\mathbb{C}^n$. It is evident from the definitions that if X is a complete spectral set of $\underline{T}$ then X is a spectral set for $\underline{T}$. A celebrated theorem of Arveson states that $\underline{T}$ has a normal bX-dilation if and only if X is a complete spectral set of $\underline{T}$. Therefore, the success or failure of rational dilation is equivalent to asking whether the fact that X is spectral set for $\underline{T}$ automatically turns X into a complete spectral set of $\underline{T}$. History witnessed an affirmative answer to this question given by Alger when X is an annulus and by Ando when $X = \overline{\mathbb{D}^2}$. Agler, Harland and Raphael have produced an example of a triply connected domain in $\mathbb{C}$ where the answer is negative. Dritschel and M^cCullough also gave a negative answer to that question when X is an arbitrary triply connected domain. Parrot showed by a counter example that rational dilation fails on the closed tridisc $\overline{\mathbb{D}^3}$. Also recently we have success of rational dilation on the closed symmetrized bidisc Γ, where Γ is defined as

$$\Gamma = \{(z_1 + z_2, z_1 z_2): |z_1| \leq 1, |z_2| \leq 1\}. \quad (1.3)$$

In this chapter, we show that rational dilation fails when X is the closure of the tetrablock $\mathbb{E}$(A triple (A,B,P) of commuting bounded operators on a Hilbert space $\mathcal{H}$is called atetrablock contraction if E is a spectral set for (A,B,P) i.e. the Taylor joint spectrum of (A,B,P) is contained in $\bar{E}$ and $\|f(A,B,P)\| \leq \|f\|_{\infty,E} = \sup\{|f(x_1,x_2,x_3)| : x_1,x_2,x_3 \in \bar{E}\}$ for any polynomial f in three variables)[7]. a polynomial convex, non-convex and inhomogeneaous domain in $\mathbb{C}^3$, defined a

$$\mathbb{E} = \{(x_1,x_2,x_3) \in \mathbb{C}^3: 1 - zx_1 - wx_2 + zwx_3 \neq 0 \text{ whernever } |z_1| \leq 1, |w| \leq 1\}.$$

This domain has been a center of attraction in past one decade to a number of mathematicians because of its relevance to μ-synthesis and H^∞control theory. To get clear with the geometric location of the domain.

Theorem (4.1.1)[4]. A point $(x_1,x_2,x_3) \in \mathbb{C}^3$ is in $\overline{\mathbb{E}}$ if and only if $|x_3| \leq 1$ and there exist $\beta_1\beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| \leq 1$ and $x_1 = \beta_1 + \overline{\beta_2}x_3$, $x_2 = \beta_2 + \overline{\beta_2}x_3$.

It is evident from the above result that the tetrablock lives inside the tridisc $\mathbb{D}^3$. The distinguished boundary (which is same the Šilov boundary) of the teterablock to be the set

$$b\mathbb{E} = \{(x_1,x_2,x_3) \in \mathbb{C}^3: x_1 = \overline{x_2}x_3, |x_2| \leq 1, |x_3| = 1\}.$$
$$= \{(x_1,x_2,x_3) \in \overline{\mathbb{E}}, |x_3| = 1\}.$$

In Bhattacharyya introduced the study of commuting operator triples that have $\overline{\mathbb{E}}$ as a spectral set. There such a triple was called a tetrablock contraction.

As a notation is always convenient, we shall such a triple an $\mathbb{E}$-contraction. So we are to led the following definition:

Definition (4.1.2)[4]. A triple of commuting operators (T_1, T_2, T_3) on a Hilbert space $\mathcal{H}$ for which $\overline{\mathbb{E}}$ is a spectral set is called an $\mathbb{E}$-contraction.

Since the tetrablock lives inside the tridisk, an $\mathbb{E}$-contraction consists of commuting contractions. Evidently (T_1^*, T_2^*, T_3^*) is an $\mathbb{E}$-contraction when (T_1, T_2, T_3) is an $\mathbb{E}$-contraction. We briefly recall from the literature the special classes of an $\mathbb{E}$-contraction which are analogous to uniteries, isometries and co-isometries in one variable operator theory.

Definition (4.1.3)[4]. Let T_1, T_2, T_3 be commuting operators on a Hilbert space $\mathcal{H}$. We say that (T_1, T_2, T_3) is

(i) an $\mathbb{E}$-unitary if T_1, T_2, T_3 are normal operators and the joint spectrum $\sigma_T(T_1, T_2, T_3)$ is contained in $\overline{\mathbb{E}}$;

(ii) an $\mathbb{E}$-isometry if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an $\mathbb{E}$-unitary $(\tilde{T}_1, \tilde{T}_2, \tilde{T}_3)$ on $\mathcal{K}$ such that $\mathcal{H}$ is a common invariant subspace of T_1, T_2, T_3 and that $T_i = \tilde{T}_i|_{\mathcal{H}}$ for $i = 1,2,3$;

(iii) an $\mathbb{E}$-co-isometry if (T_1^*, T_2^*, T_3^*) is an $\mathbb{E}$-isometry.

Moreover, an $\mathbb{E}$-isometry (T_1, T_2, T_3) is said to be pure isometry, i.e., if $T_3^{*n} \to 0$ strongly as $n \to \infty$. It clear that a rational dilation of an $\mathbb{E}$-contraction (T_1, T_2, T_3) is nothing but an $\mathbb{E}$-unitary dilation of (T_1, T_2, T_3), that is, an $\mathbb{E}$-unitary $N = (N_1, N_2, N_3)$ that dilates T by satisfying (1.2). Similarly an $\mathbb{E}$-isometry dilation of $T = (T_1, T_2, T_3)$ is an $\mathbb{E}$-isometry $V = (V_1, V_2, V_3)$ that satisfies (1.2) an explicit $\mathbb{E}$-isometric dilation was constructed for a particular class of $\mathbb{E}$-contraction and that dilation involves two unique operators A_1, A_2 from $\mathcal{L}(\mathcal{D}_{T_3})$ which are the unique solutions of the operator equations

$$T_1 - T_2^* T_3 = D_{T_3} X_1 D_{T_3}, T_2 - T_1^* T_3 = D_{T_3} X_2 D_{T_3}$$

Respectively, Here $D_{T_3} = (I - T_3^* T_3)^{\frac{1}{2}}$ and $\mathcal{D}_{T_3} = \overline{Ran D_{T_3}}$ and $\mathcal{L}(\mathcal{H})$, for a Hilbert space $\mathcal{H}$, always denotes the algebra of bounded operators on $\mathcal{H}$. For their pivotal role in the dilation, A_1 and A_2 were called the fundamental operators of (T_1, T_2, T_3). In this chapter, we produce a set of necessary conditions for the existence of rational dilation for a class of $\mathbb{E}$-contraction. Indeed, in Proposition (4.2.5), we show that if (T_1, T_2, T_3) is an $\mathbb{E}$-contraction on $\mathcal{H}_1 \oplus \mathcal{H}_1$ for some Hilbert space $\mathcal{H}_1$, satisfying

(i) $\operatorname{Ker}(D_{T_3}) = \mathcal{H}_1 \oplus \{0\}$ and $\mathcal{D}_{T_3} = \{0\} \oplus \mathcal{H}_1$

(ii) $T_3(\mathcal{D}_{T_3}) = \{0\}$ and $T_3 \operatorname{Ker}(D_{T_3}) \subseteq \mathcal{D}_{T_3}$

And if A_1, A_2 are the fundamental operators (T_1, T_2, T_3), then for the existence of an $\mathbb{E}$-isometric dilation of (T_1^*, T_2^*, T_3^*) it is necessary that

$$[A_1, A_2] = 0 \text{ and } [A_1^*, A_1] = [A_2^*, A_2]. \tag{1.4}$$

Here $[S_1, S_2] = S_1 S_2 - S_2 S_1$, for any two operators S_1, S_2. we construct an example of an $\mathbb{E}$-contraction that satisfies the hypotheses of Proposition (4.2.5) but fails to satisfy (1.4). This concludes the failure of rational dilation on the tetrablock. The proof of Proposition (4.2.5) depends heavily upon a functional model for pure $\mathbb{E}$-isometries which provide in Theorem (4.1.8). There is an Wold type decomposition for an $\mathbb{E}$-isometry that splits an $\mathbb{E}$-isometry into two parts of which one is an $\mathbb{E}$-unitary and the other is a pure $\mathbb{E}$-isometry . Again Theorem (4.1.4) describes the structure of an $\mathbb{E}$-unitary. Therefore, a concrete model for pure $\mathbb{E}$-isometries gives a complete vision of an $\mathbb{E}$-isometry. In Theorem (4.1.8), we show that a pure $\mathbb{E}$-isometry $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$ can be modeled as a commuting triple of Toeplitz operators $(T_{A_1^* + A_2 Z}, T_{A_2^* + A_1 Z}, T_Z)$ on the vectorial Hardy space $H^2(\mathcal{D}_{\hat{T}_3^*})$, where A_1 and A_2 are the fundamental operators of the $\mathbb{E}$-co-isometry $(\hat{T}_1{}^*, \hat{T}_2{}^*, \hat{T}_3{}^*)$. The converse is also true, that is, every such triple of commuting contractions $(T_{A+Bz}, T_{B^* + A^* Z}, T_z)$ on a vectorial Hardy space is a pure $\mathbb{E}$-isometry. We begin with a Lemma that simplifies the definition of $\mathbb{E}$-

contraction

Lemma (4.1.3)[4]. A commuting triple of bounded operators (T_1, T_2, T_3) is an $\mathbb{E}$-contraction if and only if $\|f(T_1, T_2, T_3)\| \leq \|f\|_{\infty,\bar{E}}$ for any holomorphic polynomial f in three variables.

This actually follows from the fact that $\bar{E}$ is polynomially convex. The following theorem gives a set of characterization for $\mathbb{E}$-unitaries.

Theorem (4.1.4)[4]. Let $\underline{N} = (N_1, N_2, N_3)$ be a commuting triple of bounded operators. Then the following are equivalent.

(i) N is an $\mathbb{E}$-unitary.

(ii) N_3 is a unitary, N_2 is a contraction and $N_1 = N_2^* N_3$,

(iii) N_3 is a unitary and $\underline{N}$ is a $\mathbb{E}$-contraction.

Here is a structure Theorem for the $\mathbb{E}$-isometries.

Theorem (4.1.5)[4]. Let $\underline{V} = (V_1, V_2, V_3)$ be a commuting triple of bounded operators. Then the following are equivalent.

(i) $\underline{V}$ is an $\mathbb{E}$-isometry.

(ii) $\underline{V}$ is an $\mathbb{E}$-contraction and V_3 is an isometry.

(iii) V_3 is an isometry, V_2 is a contraction and $V_1 = V_2^* V_3$.

(iv) (Wold decomposition) $\mathcal{H}$ has a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into reducing subspace of V_1, V_2, V_3 such that $(V_1|_{\mathcal{H}_1}, V_2|_{\mathcal{H}_1}, V_3|_{\mathcal{H}_1}|)$ is an $\mathbb{E}$-unitary and $(V_1|_{\mathcal{H}_2}, V_2|_{\mathcal{H}_2}, V_3|_{\mathcal{H}_2}|)$ is a pure $\mathbb{E}$-isometry. Let us recall that the numerical radius of an operator T on a Hilbert space $\mathcal{H}$ is defined by

$$\omega(T) = \sup\{\langle Tx, x\rangle : \ \|x\|_{\mathcal{H}} = 1\}.$$

It is well known that

$$r(T) \leq \omega(T) \leq \|T\| \text{ and } \frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, \tag{1.5}$$

Where $r(T)$ is the spectral radius of T. We state a basic Lemma on numerical radius and give a proof because of lack of an appropriate reference. We shall use this Lemma in sequel.

Lemma (4.1.6)[4]. The numerical radius of an operator T is not greater than one if and only if $Re\beta T \leq I$ for all complex numbers β of modulus 1.

Proof: Let $\omega(T) \leq 1$. For a unit vector h and a complex number β of unit modulus, we have

$$\langle [2I - (\beta T + \overline{\beta}T^*)h, h\rangle = 2 - \langle(\beta T + \overline{\beta}T^*)h, h\rangle = 2 - \langle \beta Th, h\rangle - \langle \overline{\beta}T^*h, h\rangle \geq 0,$$

Since $\omega(T) \leq 1$. Therefore, $\beta T + \overline{\beta}T^* \leq 2I$ and hence $Re\beta T \leq I$.

Again by hypothesis, $\langle Re\beta Th, h\rangle \leq 1$, for a unit vector h and for all β of modulus. Note that $\langle Re\beta Th, h\rangle = Re\beta\langle h, h\rangle$. Write $\langle Th, h\rangle = e^{i\varphi_h}|\langle Th, h\rangle|$ for some real number φ_h, and then choose $\beta = e^{-i\varphi_h}$. Then we get $|\langle Th, h\rangle| \leq 1$.

Theorem (4.1.7)[4]. Let (T_1, T_2, T_3) be an $\mathbb{E}$-contraction. Then there are two unique operators A_1, A_2 in $\mathcal{L}(D_{T_3})$ such that

$$T_1 - T_2^*T_3 = D_{T_3}A_1D_{T_3} \text{ and } T_2 - T_1^*T_3 = D_{T_3}A_2D_{T_3}. \quad (1.6)$$

Moreover, $\omega(A_1 + zA_2) \leq 1$ for all $z \in \overline{\mathbb{D}}$.

These two unique operators A_1, A_2 are called the fundamental operators of (T_1, T_2, T_3). The following Theorem gives a concrete model for pure $\mathbb{E}$-isometries in terms of Toeplitz operators on vectorial Hardy space.

Theorem (4.1.8)[4]. Let $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$ be a commuting triple of operators on a Hilbert space $\mathcal{H}$. If $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$ is a pure $\mathbb{E}$-isometry then there is a unitary operator $U: \mathcal{H} \to H^2(D_{\hat{T}_3{}^*})$ such that

$$\hat{T}_1 = U^*T_\varphi U, \quad \hat{T}_2 = U^*T_\varphi U \quad \text{and} \quad \hat{T}_3 = U^*T_zU,$$

Where $\varphi(z) = A_1^* + A_2z$, $\psi(z) = A_2^* + A_1z$, $z \in \mathbb{D}$ and A_1, A_2 are the fundamental operators of $(\hat{T}_1{}^*, \hat{T}_2{}^*, \hat{T}_3{}^*)$ satisfying

(i) $[A_1, A_2] = 0$ and $[A_1^*, A_1] = [A_2^*, A_2]$

(ii) $\|A_1 + A_2z\|_{\infty,\overline{\mathbb{D}}} \leq 1$.

Conversely, if $A_1$and A_2 are two boundary operators on a Hilbert space E satisfying the above two conditions, then $\left(T_{A_1^*+A_2Z}, T_{A_2^*+A_1Z}, T_z\right)$ on $H^2(E)$ is a pure $\mathbb{E}$-isometry.

Proof: suppose that $\left(\hat{T}_1, \hat{T}_2, \hat{T}_3\right)$ is a pure $\mathbb{E}$-isometry. Then $\hat{T}_3$ is a pure isometry and it can be identified with the Toeplitz operator T_z on $H^2(D_{\hat{T}_3}{}^*)$. Therefore, there is a unitary U from $\mathcal{H}$ onto $H^2(D_{\hat{T}_3}{}^*)$ such that $\hat{T}_3 = U^*T_zU$. since for $i = 1,2$, $\hat{T}_i$ is a commutant of $\hat{T}_3$, there are two multipliers φ, ψ in $H^\infty(\mathcal{L}(D_{\hat{T}_3}{}^*))$ such that $\hat{T}_1 = U^*T_\varphi U$ and $\hat{T}_2 = U^*T_\psi U$.

Claim. If (V_1, V_2, V_3) on a Hilbert space $\mathcal{H}_1$ is an $\mathbb{E}$-isometry then $V_2 = V_1^*V_3$.

Proof of claim. Let (V_1, V_2, V_3) be the restriction of an $\mathbb{E}$-isometry (N_1, N_2, N_3) to the common invariant subspace $\mathcal{H}_1$.. By part-(ii) of Theorem (4.1.4), $N_1 = N_2^*N_3$ and hence $N_2 = N_1^*N_3$ by an application of Fugled's theorem, which states that if a normal operator N commutes with a bounded operator T then it commutes with T^* too. Taking restriction to the common invariant subspace $\mathcal{H}_1$we get $V_2 = V_1^*V_3$. We apply this claim and part-(iii) of Theorem (4.1.5) to the $\mathbb{E}$-isometry (T_φ, T_ψ, T_Z). So $T_\varphi = T_\psi^*T_Z$ and $T_\psi = T_\varphi^*T_Z$ and by these two relations we have tha $\varphi(z) = G_1+G_2z$ and $\psi(z) = G_2^* + G_1^*z$ for some $G_1,G_2 \in \mathcal{L}\left(D_{\hat{T}_3}{}^*\right)$. Setting $A_1 = G_1^*$ and $A_2 = G_2$ and by the commutativity of $\varphi(z)$ and $\psi(z)$ we obtain $[A_1, A_2] = 0$ and $[A_1^*, A_1] = [A_2^*, A_2]$.

We now compute the fundamental operators of the $\mathbb{E}$-co-isometry $\left(T^*_{A_1^*+A_2z}, T^*_{A_2^*+A_1z}, T_z^*\right)$.

Clearly $I - T_zT_z^*$ is the projection onto the space $D_{T_z^*}$. Now

$$T^*_{A_1^*+A_2z} - T^*_{A_2^*+A_1z}T_z^* = T_{A_1+A_2^*\bar{z}} - T_{A_2^*+A_1z}T_{\bar{z}} = T_{A_1}$$

$$= (I - T_zT_z^*)A_1(I - T_zT_z^*).$$

Similarly

$$T^*_{A_2^*+A_1z} - T^*_{A_1^*+A_2z}T^*_z = (I - T_zT^*_z)A_2(I - T_zT^*_z).$$

Therefore, A_1, A_2 are the fundamental operators of $(T^*_{A_1^*+A_2z}, T^*_{A_2^*+A_1z}, T^*_z)$ and $\|A_1 + A_2z\|_{\infty,\overline{\mathbb{D}}} \leq 1$.

For the converse, we first prove that the triple of multiplication operators $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}M_z)$ on $L^2(E)$ is an $\mathbb{E}$-unitary when A_1, A_2 satisfy the given conditions. It is evident that $M_{A_1^*+A_2z}, M_{A_2^*+A_1z}M_z)$ is a commuting triple of normal operators when $[A_1, A_2] = 0$ and $[A_1^*, A_1] = [A_2^*, A_2]$. Also $M_{A_1^*+A_2z} = M_{A_2^*+A_1z}M_z$ and M_z on $L^2(E)$ is unitary. Therefore, by part-(ii) of Theorem (4.1.4) $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}M_z)$ becomes an $\mathbb{E}$-unitary if we prove that $\|M_{A_2^*+A_1z}\| \leq 1$.

We have that $\omega(A_1 + zA_2) \leq 1$ for every $z \in \mathbb{T}$, which is same as saying that $\omega(z_1A_1 + z_2A_2) \leq 1$ for all complex numbers z_1, z_2 of unit modulus. Thus by Lemma (4.1.6)

$$(z_1A_1 + z_2A_2) + (z_1A_1 + z_2A_2)^* \leq 2I,$$

That is

$$(z_1A_1 + \bar{z}_2A_2^*) + (z_1A_1 + \bar{z}_2A_2^*)^* \leq 2I.$$

Therefore, $\bar{z}_2(A_2^* + zA_1) + z_2(A_2^* + zA_1)^* \leq 2I$ for all $z,\ z_2 \in \mathbb{T}$. this is same as saying that

$$Re\, z_2(A_2^* + zA_1) \leq 1 \text{ for all } z,\ z_2 \in \mathbb{T}.$$

Therefore, by Lemma (4.1.6) again $\omega(A_2^* + zA_1) \leq 1$for any $z \in \mathbb{T}$. Since $M_{A_2^*+zA_1}$ is a normal operator we have that $\|M_{A_2^*+zA_1}\| = \omega(M_{A_2^*+zA_1})$ and thus $\|M_{A_2^*+zA_1}\| \leq 1$. Therefore, $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}, M_z)$ on $L^2(E)$ is an $\mathbb{E}$-unitary and hence $(T_{A_1^*+A_2z}, T_{A_2^*+A_1z}, T_z)$, being the restriction $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}, M_z)$ to the common invariant subspace $H^2(E)$, is an $\mathbb{E}$-isometry. Also T_z on $H^2(E)$ is a pure isometry. Thus we conclude that $(T_{A_1^*+A_2z}, T_{A_2^*+A_1z}, T_z)$ is a pure $\mathbb{E}$-isometry.

Section (4.2): Necessary Condition for the Existence of Dilation with a Counter Example

Show the definitions of the $\mathbb{E}$-isometric and $\mathbb{E}$-unitary dilations of an $\mathbb{E}$-contraction. In fact they can be defined in a simpler way by involving polynomials only. This is because the polynomials are dense in the rational functions.

Definition (4.2.1)[4]. Let (T_1, T_2, T_3) be a $\mathbb{E}$-contraction on $\mathcal{H}$. A commuting tuple (Q_1, Q_2, V) on $\mathcal{K}$ is said to be an $\mathbb{E}$-isometric dilation of (T_1, T_2, T_3) if $\mathcal{H} \subseteq \mathcal{K}$, (Q_1, Q_2, V) is an $\mathbb{E}$-isometry and

$$P_{\mathcal{H}}(Q_1^{m_1}, Q_2^{m_2}, V^n)\big|_{\mathcal{H}} = T_1^{m_1} T_2^{m_2} T_3^n, \text{ For all non-negative integers}$$

m_1, m_2, n.

Here $P_{\mathcal{H}}: \mathcal{K} \to \mathcal{H}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Moreover, the dilation is called minimal if $\mathcal{K} = \overline{\text{span}}\{Q_1^{m_1}, Q_2^{m_2}, V^n h\colon\ h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N}\cup\{0\}\}$.

Definition (4.2.2)[4]. A commuting tuple (R_1, R_2, U) on $\mathcal{K}$ is said to be an $\mathbb{E}$-unitary dilation of (T_1, T_2, T_3) if $\mathcal{H} \subseteq \mathcal{K}$, (R_1, R_2, U) is an $\mathbb{E}$-unitary and

$$P_{\mathcal{H}}(R_1^{m_1} R_2^{m_2} U^n)\big|_{\mathcal{H}} = T_1^{m_1} T_2^{m_2} T_3^{m_n}, \text{ for all non-negative integers}$$

m_1, m_2, n.

Moreover, the dilation is called minimal if $\mathcal{K} = \overline{\text{span}}\{R_1^{m_1} R_2^{m_2} U^n\colon\ h \in \mathcal{H}$ and $m_1, m_2, n \in \mathbb{Z}\}$. Here $R_i^{m_i} = R_i^{*-m_i}$ for $i = 1,2$ and $U^n = U^{*-n}$ when m_i and n are negative integers.

Proposition (4.2.3)[4]. If a $\mathbb{E}$-contraction (T_1, T_2, T_3) defined on $\mathcal{H}$ has a $\mathbb{E}$-isometric dilation, then it has a minimal $\mathbb{E}$-isometric dilation.

Proof: Let (Q_1, Q_2, V) on $\mathcal{K} \supseteq \mathcal{H}$ be a $\mathbb{E}$-isometric dilation of (T_1, T_2, T_3). Let $\mathcal{K}_0$ be space defined as

$$\mathcal{K}_0 = \overline{\text{span}}\{Q_1^{m_1} Q_2^{m_2} V^n h\colon\ h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N}\cup\{0\}\}.$$

Cleary $\mathcal{K}_0$ is invariant under $Q_1^{m_1}, Q_2^{m_2}$ and V^n, for any non-negative integer m_1, m_2, n. Therefore if we denote the restrictions of Q_1, Q_2 and V to the common invariant subspace $\mathcal{K}_0$ by Q_{11}, Q_{22} and V_1 respectively, we get $Q_{11}^{m_1}k = Q_1^{m_1}k$, $Q_{12}^{m_2}k = Q_2^{m_2}k$, and $V_1^n k = V^n k$, for any $k \in \mathcal{K}_0$. Hence

$$\mathcal{K}_0 = \overline{\text{span}}\{Q_{11}^{m_1}Q_{12}^{m_2}V_1^n h:\ h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N}\cup\{0\}\}.$$

Therefore, for any non-negative integer m_1, m_2 and n we have

$$P_{\mathcal{H}}\left(Q_{11}^{m_1}Q_{12}^{m_2}V_1^n\right) = P_{\mathcal{H}}\left(Q_1^{m_1}Q_2^{m_2}V^n\right)h \quad \text{for all } h \in \mathcal{H}.$$

Now (Q_{11}, Q_{22}, V_1) is an $\mathbb{E}$-contraction by being the restriction of an $\mathbb{E}$-contraction (Q_1, Q_2, V) to a common invariant subspace $\mathcal{K}_0$. Also V_1, being the restriction of an isometry to an invariant subspace, is also an isometry. Therefore by Theorem (4.1.5)-part (ii), (Q_{11}, Q_{22}, V_1) is an $\mathbb{E}$-isometry. Hence (Q_{11}, Q_{22}, V_1) is a minimal $\mathbb{E}$-isometry dilation of (T_1, T_2, T_3).

Proposition (4.2.4)[4]. Let (Q_1, Q_2, V) on $\mathcal{K}$ be an $\mathbb{E}$-isometric dilation of an $\mathbb{E}$-contraction (T_1, T_2, T_3) on $\mathcal{H}$. If (Q_1, Q_2, V) is minimal, then (Q_1^*, Q_2^*, V^*) is an $\mathbb{E}$-co-iosmetric extension of (T_1^*, T_2^*, T_3^*).

Proof: We first prove that $T_1P_{\mathcal{H}} = P_{\mathcal{H}}Q_1$, $T_2P_{\mathcal{H}} = P_{\mathcal{H}}Q_2$ and $T_3P_{\mathcal{H}} = P_{\mathcal{H}}V$. Clearly

$$\mathcal{K} = \overline{\text{span}}\{Q_1^{m_1}Q_2^{m_2}V^n h:\ h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N}\cup\{0\}\}.$$

Now for $h \in \mathcal{H}$ we have that

$$\begin{aligned} T_1P_{\mathcal{H}}\left(Q_1^{m_1}Q_2^{m_2}V^n h\right) &= T_1\left(T_1^{m_1}T_2^{m_2}T_3^n h\right) = T_1^{m_1+1}T_2^{m_2}T_3^n h \\ &= P_{\mathcal{H}}\left(Q_1^{m_1+1}Q_2^{m_2}V^n h\right) = P_{\mathcal{H}}\left(Q_1^{m_1}Q_2^{m_2}V^n h\right). \end{aligned}$$

Thus we have that $T_1P_{\mathcal{H}} = P_{\mathcal{H}}Q_1$ and similarly we can prove that $T_2P_{\mathcal{H}} = P_{\mathcal{H}}Q_2$ and $T_3P_{\mathcal{H}} = P_{\mathcal{H}}V$. Also for $h \in \mathcal{H}$ and $k \in \mathcal{K}$ we have that

$$\langle T_1^*h, k\rangle = \langle P_{\mathcal{H}}T_1^*h, k\rangle = \langle T_1^*h, P_{\mathcal{H}}k\rangle = \langle h, T_1P_{\mathcal{H}}k\rangle = \langle h, P_{\mathcal{H}}Q_1k\rangle = \langle Q_1^*h, k\rangle.$$

Hence $T_1^* = Q_1^*|_{\mathcal{H}}$ and similarly $T_2^* = Q_2^*|_{\mathcal{H}}$ and $T_3^* = V^*|_{\mathcal{H}}$. Therefore, (Q_1^*, Q_2^*, V^*) is an $\mathbb{E}$-co-iosmetric extension of (T_1^*, T_2^*, T_3^*).

Proposition (4.2.5)[4]. Let $\mathcal{H}_1$ be a Hilbert space and let (T_1, T_2, T_3) be an $\mathbb{E}$-contraction on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ with fundamental operators A_1, A_2. let

(i) $Ker(D_{T_3}) = \mathcal{H}_1 \oplus \{0\}$ and $D_{T_3} = \{0\} \oplus \mathcal{H}_1$;

(ii) $T_3(D_{T_3}) = \{0\}$ and $T_3 \mathrm{Ker}(D_{T_3}) \subseteq D_{T_3}$.

If (T_1^*, T_2^*, T_3^*) has an $\mathbb{E}$-isometric dilation then

(1^0) $A_1 A_2 = A_2 A_1$,

(2^0) $A_1^* A_1 - A_1 A_1^* = A_2^* A_2 - A_2 A_2^*$.

Proof: Let (Q_1, Q_2, V) on a Hilbert space $\mathcal{K}$ be a minimal $\mathbb{E}$-isometric dilation of (T_1^*, T_2^*, T_3^*) (such a minimal $\mathbb{E}$-isometric dilation exist by Proposition (4.2.3)) so that (Q_1^*, Q_2^*, V^*) is an $\mathbb{E}$-co-iometric extension of (T_1, T_2, T_3) by Proposition (4.2.4). Since (Q_1, Q_2, V) on $\mathcal{K}$ is an $\mathbb{E}$-isometry, by part-(iv) of Theorem (4.1.5), $\mathcal{K}$ has decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ into reducing subspace $\mathcal{K}_1, \mathcal{K}_2$ of Q_1, Q_2, V such that $(Q_1|_{\mathcal{K}_1}, Q_2|_{\mathcal{K}_1}, V|_{\mathcal{K}_1}) = (Q_{11}, Q_{12}, U_1)$ is an $\mathbb{E}$-unitary and $(Q_1|_{\mathcal{K}_2}, Q_2|_{\mathcal{K}_2}, V|_{\mathcal{K}_2}) = (Q_{21}, Q_{22}, V_1)$ is a pure $\mathbb{E}$-isometry. Since (Q_{21}, Q_{22}, V_1) on $\mathcal{K}_2$ is a pure $\mathbb{E}$-isometry, by Theorem (4.1.8), $\mathcal{K}_2$ can be identified with $H^2(E)$, where $E = D_{V_1^*}$ and Q_{21}, Q_{22}, V_1 can be identified with T_φ, T_ψ, T_z respectively on $H^2(E)$, where $\varphi(z) = A + Bz$ and $\psi(z) = B^* + A^* z, z \in \mathbb{D}$. Here A^*, B are the fundamental operators of $Q_{21}^*, Q_{22}^*, V_1^*$. Again $H^2(E)$ can be identified with $l^2(E)$ and T_φ, T_ψ, T_z on $H^2(E)$ can be identified with the multiplication operators M_φ, M_ψ, M_z on $l^2(E)$ respectively. So without loss of generality we can assume that $K_2 = l^2(E)$ and $Q_{21} = M_\varphi, Q_{22} = M_\psi$ and $V_1 = M_z$ on $l^2(E)$.

The block matrices of M_φ, M_ψ, M_z are given by

$$M_\varphi = \begin{bmatrix} A & 0 & 0 & \cdots \\ B & A & 0 & \cdots \\ 0 & B & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, M_\psi = \begin{bmatrix} B^* & 0 & 0 & \cdots \\ A^* & B^* & 0 & \cdots \\ 0 & A^* & B^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \text{ and } M_z = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

From now onward we shall consider $\mathcal{H}$ as a subspace of $\mathcal{K}$ and T_1, T_2, T_3 on $\mathcal{H}$ as the restrictions of Q_1^*, Q_2^*, V^* respectively to $\mathcal{H}$.

Claim 1. $D_{T_3} \subseteq E \oplus \{0\} \oplus \{0\} \oplus \cdots \subseteq l^2(E)$.

Proof of claim. Let $h = h_1 \oplus h_2 \in D_{T_3} \subseteq \mathcal{H}$, where $h_1 \in \mathcal{K}_1$ and $h_2 = (c_0, c_1, c_2, \cdots)^T \in l^2(E)$. Here $(c_0, c_1, c_2, \cdots)^T$ denotes the transpose of the vector $(c_0, c_1, c_2, \cdots)$. Since $T_3(D_{T_3}) = \{0\}$, we have that

$$T_3 h = V^* h = V^*(h_1 \oplus h_2) = U_1^* h_1 \oplus M_2^* h_2 = U_1^* h_1 \oplus (c_1, c_2, \cdots)^T = 0$$

Which implies that $h_1 = 0$ *and* $c_1 = c_2 = \cdots = 0$. This completes the proof of claim1. Claim 2**.** $\mathrm{Ker}(D_{T_3}) \subseteq \{0\} \oplus E \oplus \{0\} \oplus \{0\} \oplus \cdots \subseteq l^2(E)$.

Proof of claim**.** For $h = h_1 \oplus h_2 \in \mathrm{Ker}(D_{T_3}) \subseteq \mathcal{H}$, where $h_1 \in \mathcal{K}_1$ and $h_2 = (c_0, c_1, c_2, \cdots)^T \in l^2(E)$, we have that

$$D_{T_3}^2 h = (I - T_3{}^* T_3)h = P_{\mathcal{H}}(I - VV^*)h = P_{\mathcal{H}}(h_1 \oplus h_2 - h_1 \oplus M_z M_z^* h_2) = 0$$

Which implies that $P_{\mathcal{H}}(h_1 \oplus h_2) = P_{\mathcal{H}}(h_1 \oplus M_z M_z^* h_2)$. Therefore, $h_1 \oplus (c_0, c_1, \cdots)^T = P_{\mathcal{H}}(h_1 \oplus (0, c_1, c_2, \cdots)^T)$ which further implies that $\|h_1 \oplus (0, c_1, c_2, \cdots)^T\| \geq \|h_1 \oplus (c_0, c_1, c_2, \cdots)^T\|$. Thus $c_0 = 0$.

Again $T_3 \mathrm{Ker}(D_{T_3}) \subseteq D_{T_3}$. Therefore, for $h_1 \oplus (0, c_1, c_2, \cdots)^T \in \mathrm{Ker}(D_{T_3})$, we have that $T_3(h_1 \oplus (0, c_1, c_2, \cdots)^T) = U_1^* h_1 \oplus M_z^*(0, c_1, c_2, \cdots)^T = U_1^* h_1 \oplus (c_1, c_2, \cdots)^T \in D_{T_3}$.

Then by claim 1, $h_1 = 0$ and $c_2 = c_3 = \cdots = 0$. Hence claim 2 is established.

Now since $\mathcal{H} = D_{T_3} \oplus \mathrm{Ker}(D_{T_3})$, we can conclude that $\mathcal{H} \subseteq E \oplus E \oplus \{0\} \oplus \{0\} \oplus \cdots \subseteq l^2(E) = \mathcal{K}_2$. Therefore $(M_\varphi^*, M_\psi^*, M_z^*)$ on $l^2(E)$ is an $\mathbb{E}$-co-isometric extension of (T_1, T_2, T_3).We now compute the fundamental operators of $(M_\varphi^*, M_\psi^*, M_z^*)$.

$$M_\varphi^* - M_\psi M_z^* = \begin{bmatrix} A^* & B^* & 0 & \cdots \\ 0 & A^* & B^* & \cdots \\ 0 & 0 & A^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} B^* & 0 & 0 & \cdots \\ A^* & B^* & 0 & \cdots \\ 0 & A^* & B^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & I & 0 & \cdots \\ 0 & 0 & I & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} A^* & B^* & 0 & \cdots \\ 0 & A^* & B^* & \cdots \\ 0 & 0 & A^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} 0 & B^* & 0 & \cdots \\ 0 & A^* & B^* & \cdots \\ 0 & 0 & A^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} A^* & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Similarly

$$M_\varphi^* - M_\psi M_z^* = \begin{bmatrix} B & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Also $\quad D_{M_z^*}^2 = I - M_z M_z^*$

$$= \begin{bmatrix} I & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, $D_{M_z^*} = E \oplus \{0\} \oplus \{0\} \cdots$ and $D_{M_z^*}^2 = D_{M_z^*} = I_d$ on $E \oplus \{0\} \oplus \{0\} \cdots$.

If we set

$$\hat{A}_1 = \begin{bmatrix} A^* & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} B & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (1.7)$$

Then

$$M_\varphi^* - M_\psi M_z^* = D_{M_z^*} \hat{A}_1 D_{M_z^*} \text{ and } M_\varphi^* - M_\psi M_z^* = D_{M_z^*} \hat{A}_2 D_{M_z^*}.$$

Therefore, $\hat{A}_1$, $\hat{A}_2$ are the fundamental operators of $(M_\varphi^*, M_\psi^*, M_z^*)$.

Let us denote $\left(M_\varphi^*, M_\psi, M_z^*\right)$ by (R_1, R_2, W). Therefore,

$$R_1 - R_2^* W = D_W \hat{A}_1 D_{W.} \tag{1.8}$$

$$R_2 - R_1^* W = D_W \hat{A}_2 \hat{A}_1 D_{W.}. \tag{1.9}$$

Claim 3. $\hat{A}_i D_W|_{D_{T_3}} \subseteq D_{T_3}$ and $\hat{A}_i{}^* D_W|_{D_{T_3}} \subseteq D_{T_3}\ for\ i = 1,2.$

Proof of claim**.** Clearly $D_W = D_{M_z^*} = I_d$ on D_W. Let $h_0 = (c_0, c_1, c_2, \cdots)^T \in D_{T_3}$. Then $\hat{A}_1 D_W h_0 = (A^* c_0, 0,0, \cdots)^T = M_\varphi^* h_0 = R_1 h_0$. Since $R_1|_{\mathcal{H}} = S_1, R_1 h_0 \in \mathcal{H}$. Therefore $(A^* c_0, 0,0, \cdots)^T \in D_{T_3}$ and $\hat{A}_1 D_W|_{D_{T_3}} \subseteq D_{T_3}$. Similarly we can prove that $\hat{A}_2 D_W|_{D_{T_3}} \subseteq D_{T_3}$.

We compute the adjoint of T_3. Let $(c_0, c_1, c_2, \cdots)^T$and $(d_0, d_1, 0, \cdots)^T$ be two arbitrary elements in $\mathcal{H}$ where $(c_0, c_1, c_2, \cdots)^T$, $(d_0, d_1, 0, \cdots)^T \in D_{T_3}$ and $(c_0, c_1, c_2, \cdots)^T$, $(d_0, d_1, 0, \cdots)^T \in Ker(D_{T_3})$. Now

$$\begin{aligned}\langle T_3^*(c_0, c_1, c_2, \cdots)^T, (d_0, d_1, 0, \cdots)^T \rangle &= \langle (c_0, c_1, c_2, \cdots)^T, T_3\ (d_0, d_1, 0, \cdots)^T \rangle \\ &= \langle (c_0, c_1, c_2, \cdots)^T, W\ (d_0, d_1, 0, \cdots)^T \rangle \\ &= \langle (c_0, c_1, c_2, \cdots)^T, (d_0, d_1, 0, \cdots)^T \rangle \\ &= \langle c_0, d_0 \rangle_E \\ &= \langle (0, c_0, 0, \cdots)^T, (d_0, d_1, 0, \cdots)^T \rangle.\end{aligned}$$

Therefore,

$$T_3^*(c_0, c_1, c_2, \cdots)^T = (0, c_0, 0, \cdots)^T.$$

Now $h_0 = (0, c_0, 0, \cdots)^T \in D_{T_3}$, implies that $T_3^* h_0 = (0, c_0, 0, \cdots)^T \in \mathcal{H}$ and $M_\psi^*(0, c_0, 0, \cdots)^T = R_2(0, c_0, 0, \cdots)^T = (Ac_0, 0,0, \cdots)^T \in \mathcal{H}$. In particular, $(Ac_0, 0,0, \cdots)^T \in D_{T_3}$. Therefore $\hat{A}_1{}^* D_W h_0 = (Ac_0, 0,0, \cdots)^T \in D_{T_3}$ and $\hat{A}_2{}^* D_W|_{D_{T_3}} \subseteq D_{T_3}$. Similarly we can prove that $\hat{A}_2{}^* D_W|_{D_{T_3}} \subseteq D_{T_3}$. Hence claim 3 is proved.

Claim 4. $\hat{A}_i D_W|_{D_{T_3}} = A_i$ and $\hat{A}_i^*|_{D_{T_3}} = A_i^*$ for $i = 1,2$.

Proof of claim. It is obvious that $D_{T_3} \subseteq D_W = E \oplus \{0\} \oplus \{0\} \cdots$. Now since $W|_{\mathcal{H}} = T_3$ and D_W is projection onto D_W, we have that $D_W|_{\mathcal{H}} = D_W^2|_{\mathcal{H}} = D_W^2|_{D_{T_3}} = D_{T_3}^2$. Therefore, $D_{T_3}^2$ is a projection onto D_{T_3} and $D_{T_3}^2 = D_{T_3}$. From (1.8) we have that

$$P_{\mathcal{H}}(R_1 - R_2^* W)|_{\mathcal{H}} = P_{\mathcal{H}}(D_W \hat{A}_1 D_W)\big|_{\mathcal{H}}. \tag{1.10}$$

Since (R_1, R_2, W) is an $\mathbb{E}$-co-isometric extension of (T_1, T_2, T_3), the LES of (1.10) is equal to $T_1 - T_2^* T_3$. Again since A_1, A_2 are the fundamental opereators of (T_1, T_2, T_3), we have that

$$T_1 - T_2^* T_3 = D_{T_3} A_1 D_{T_3}, A_1 \in \mathcal{L}(D_{T_3}). \tag{1.11}$$

It is clear that $T_1 - T_2^* T_3$ is 0 on the ortho-complement of D_{T_3}, that is on $Ker(D_{T_3})$. Therefore

$$T_1 - T_2^* T_3 = (R_1 - R_2^* W)|_{D_{T_3}} = P_{D_{T_3}}(D_W \hat{A}_1 D_W)\big|_{D_{T_3}}. \tag{1.12}$$

Again since $D_W|_{D_{T_3}} = D_{T_3} = I_d$ on D_{T_3}, the RES of (12) is equal to $(D_W \hat{A}_1 D_W)\big|_{D_{T_3}}$ and hence

$$T_1 - T_2^* T_3 = (R_1 - R_2^* W)|_{D_{T_3}} = (D_W \hat{A}_1 D_W)\big|_{D_{T_3}} = D_{T_3} \hat{A}_1 D_{T_3}. \tag{1.13}$$

The last identity follows from the fact (claim 3) that $\hat{A}_1 D_W)\big|_{D_{T_3}} \subseteq D_{T_3}$. By the uniqueness of A_1 we get that $\hat{A}_1\big|_{D_{T_3}} = A_1$. Also since D_{T_3} is invariant under $\hat{A}_1^*$ by claim 3, we have that $\hat{A}_1^*\big|_{D_{T_3}} = A_1^*$. Similarly we can prove that $\hat{A}_2\big|_{D_{T_3}} = A_2$ and $\hat{A}_2^*\big|_{D_{T_3}} = A_2^*$. Thus the proof to claim 4 is complete.

Now since (M_φ, M_ψ, M_z) on $l^2(E)$ is an $\mathbb{E}$-isometry, M_φ and M_ψ compute, that is

$$\begin{bmatrix} A & 0 & 0 & \cdots \\ B & A & 0 & \cdots \\ 0 & B & A & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\begin{bmatrix} B^* & 0 & 0 & \cdots \\ A^* & B^* & 0 & \cdots \\ 0 & A^* & B^* & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} B^* & 0 & 0 & \cdots \\ A^* & B^* & 0 & \cdots \\ 0 & A^* & B^* & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}\begin{bmatrix} A & 0 & 0 & \cdots \\ B & A & 0 & \cdots \\ 0 & B & A & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Which implies that

$$\begin{bmatrix} AB^* & 0 & 0 & \cdots \\ BB^* + AA^* & AB^* & 0 & \cdots \\ BA^* & BB^* + AA^* & AB^* & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} B^*A & 0 & 0 & \cdots \\ AA^* + BB^* & AB^* & 0 & \cdots \\ A^*B & AA^* + BB^* & B^*A & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Comparing both sides we obtain

(i) $A^*B = B^*A$

(ii) $AA^* - A^*A = BB^* - B^*$B.

Therefore from (1.7) we have that

(i) $\hat{A}_1\hat{A}_2 = \hat{A}_2\hat{A}_1$

(ii) $\widehat{A}_1{}^*\hat{A}_1 - \hat{A}_1\hat{A}_1{}^* = \hat{A}_2{}^*\hat{A}_2 - \hat{A}_2\hat{A}_2{}^*$.

Taking restriction of the above two operator identities to the subspace D_{T_3} we get

(i) $A_1A_2 = A_2A_1$

(ii) $A_1^*A_1 - A_1A_1^* = A_2^*A_2 - A_2A_2^*$.

The proof is now complete.

Let $\mathcal{H}_1 = l^2(E) \oplus l^2(E), E = \mathbb{C}^2$ and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$. Let $T_1 = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}$, $T_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $T_3 = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix}$ on $\mathcal{H}_1 \oplus \mathcal{H}_1$, where $J = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & V \\ I & 0 \end{bmatrix}$ on $\mathcal{H}_1 = l^2(E) \oplus l^2(E)$. Here $V = M_z$ and $I = I_d$ on $l^2(E)$ and F on $l^2(E)$ is defined as

$$F: l^2(E) \to l^2(E)$$

$$(c_0, c_1, c_2, \cdots)^T \to (c_0, c_1, c_2, \cdots)^T,$$

where we choose F_1 on E to be a non-normal contraction such that $F_1^2 = 0$. For example we can choose $F_1 = \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix}$ for some $\eta > 0$. Clearly $F^2 = 0$ and $F^*F \neq FF^*$. Since $FV = 0$, $JY = 0$ and thus the product of any two of T_1, T_2, T_3

is equal to 0. Now we unfold the operators T_1, T_2, T_3 and write their block matrices with respect to the decomposition $\mathcal{H}_1 = l^2(E) \oplus l^2(E) \oplus l^2(E) \oplus l^2(E)$:

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}.$$

We shall prove later that (T_1, T_2, T_3) is an $\mathbb{E}$-contraction and let us assume it for now. Here

$$D_{T_3}^2 = I - T_3^* T_3 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & V^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = D_{T_3}$$

Clearly $D_{T_3} = \{0\} \oplus \{0\} \oplus l^2(E) \oplus l^2(E) = \{0\} \oplus \mathcal{H}_1$ and $Ker(D_{T_3}) = l^2(E) \oplus l^2(E) \oplus \{0\} \oplus \{0\} = \mathcal{H}_1 \oplus \{0\}$. also for a vector $\mathcal{K}_0 = (h_0, h_1, 0,0)^T \in Ker(D_{T_3})$ and for a vector $\mathcal{K}_1 = (0,0, h_2, h_3)^T \in D_{T_3}$,

$$T_3 \mathcal{K}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} (h_0, h_1, 0,0)^T = (0,0, V h_1, h_0)^T \in D_{T_3}.$$

And

$$T_3 \mathcal{K}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} (0,0, h_2, h_3)^T = (0,0,0,0)^T.$$

Thus (T_1, T_2, T_3) satisfies all the conditions of Proposition (4.2.5). We now compute the fundamental operators A_1, A_2 of (T_1, T_2, T_3).

$$T_1 - T_3^*T_3 = T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D_{T_3}A_1D_{T_3}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} A_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

By the uniqueness of A_1 we conclude that $A_1 = 0 \oplus \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ on D_{T_3}. Again $T_1^*T_3 = 0$ as $X^*V = 0$ and therefore $T_2 - T_1^*T_3 = 0$. This show that the fundamental operator A_2, for which $T_2 - T_1^*T_3 = D_{T_3}A_2D_{T_3}$ holds, has to be equal to 0. Clearly

$$A_1^*A_1 - A_1^*A_1 = 0 \oplus \begin{bmatrix} F^*F - FF^* & 0 \\ 0 & 0 \end{bmatrix} \neq 0 \text{ as } F^*F \neq FF^*$$

But $A_2^*A_2 - A_2^*A_2 = 0$. This violets the conclusion of Proposition (4.2.5) and it is guaranteed that the $\mathbb{E}$-contraction (T_1^*, T_2^*, T_3^*) does not have an $\mathbb{E}$-isometric dilation, (T_1^*, T_2^*, T_3^*) does not have an $\mathbb{E}$-unitary dilation.

Now we prove that (T_1, T_2, T_3) is an $\mathbb{E}$-contraction. By Lemma (4.1.3), It suffices to show that $\|p(T_1, T_2, T_3)\| \leq \|p\|_{\infty,\overline{\mathbb{E}}}$, for any polynomial. $p(x_1, x_2, x_3)$ in the co-ordinates of $\mathbb{E}$. Let

$$p(x_1, x_2, x_3) = a_0 + \sum_{i=1}^{3} a_i x_i + q(x_1, x_2, x_3),$$

where q is a polynomial containing only terms of second or higher degree. Now

$$p(T_1, T_2, T_3) = a_0\mathrm{I} + a_1T_1 + a_3T_3 = \begin{bmatrix} a_0\mathrm{I} & 0 \\ a_3Y & a_0\mathrm{I} + a_1\mathrm{J} \end{bmatrix}$$

Since $Y \text{ and } J$ are contractions, it is obvious that

$$\left\| \begin{bmatrix} a_0\mathrm{I} & 0 \\ a_3Y & a_0\mathrm{I} + a_1\mathrm{J} \end{bmatrix} \right\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{pmatrix} \right\|, b = \|a_1\mathrm{J}\|.$$

We first show that $\left\| \begin{bmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{bmatrix} \right\| \le \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \right\|$ when b is very small. Let $\binom{\epsilon}{\delta}$ be a unit vector in $\mathbb{C}^2$ such that $\left\| \begin{bmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{bmatrix} \right\| = \left\| \begin{bmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{bmatrix} \binom{\epsilon}{\delta} \right\|$.

It suffices to show that

$$\left\| \begin{bmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{bmatrix} \binom{\epsilon}{\delta} \right\| \ge \left\| \begin{bmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{bmatrix} \binom{\epsilon}{\delta} \right\|.$$

We have

$$\left\| \begin{bmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{bmatrix} \binom{\epsilon}{\delta} \right\|^2$$

$= |a_0|^2|\epsilon|^2 + [|a_3|\epsilon + (|a_0| + b)\delta]\left[|a_3|\bar{\epsilon} + (|a_0| + b)\bar{\delta}\right]$

$= (|a_0|^2+|a_3|^2)|\epsilon|^2 + (|a_0| + b)^2|\delta|^2 + |a_3|(|a_0| + b)\left(\epsilon\bar{\delta} + \delta\bar{\epsilon}\right)$

$= |a_0|^2 + |a_3|^2|\epsilon|^2 + (b^2 + 2|a_0|b)|\delta|^2 + |a_0a_3|\left(\epsilon\bar{\delta} + \delta\bar{\epsilon}\right) + |a_3|b\left(\epsilon\bar{\delta} + \delta\bar{\epsilon}\right)$, Since $|\epsilon|^2 + |\delta|^2 = 1$. (1.14)

Also

$$\left\| \begin{bmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{bmatrix} \binom{\epsilon}{\delta} \right\|^2$$

$= |a_0|^2|\epsilon|^2 + [(|a_1| + |a_3|)\epsilon + |a_0|\delta][(|a_1| + |a_3|)\bar{\epsilon} + |a_0|\bar{\delta}]$

$= |a_0|^2 + (|a_1| + |a_3|)^2|\epsilon|^2 + |a_0|(|a_1| + |a_3|)\left(\epsilon\bar{\delta} + \delta\bar{\epsilon}\right)$

$= |a_0|^2 + |a_3|^2|\epsilon|^2 + (|a_1|^2+2|a_1a_3|)\,|\epsilon|^2 + |a_0a_3|\left(\epsilon\bar{\delta} + \delta\bar{\epsilon}\right) + |a_0a_1|\left(\epsilon\bar{\delta} + \delta\bar{\epsilon}\right)$ (1.15)

Now $\begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{pmatrix}$ attains its norm $\binom{\epsilon}{\delta}$. So without loss of generality we can assume that $\epsilon\bar{\delta} + \delta\bar{\epsilon}$ is a positive (non-negative) real number because otherwise altering the sign of one of ϵ or δ we can have $\epsilon\bar{\delta} + \delta\bar{\epsilon}$ to be positive (non-negative) which increases the norm of $\begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{pmatrix}$, a contradiction. Therefore, $\epsilon\bar{\delta} + \delta\bar{\epsilon}$ is positive (non-nagative). It is evident from (1.14) and (1.15) that we can choose b so small that $(b^2 + 2|a_0|b)|\delta|^2$ and $|a_3|b(\epsilon\bar{\delta} + \delta\bar{\epsilon})$ become lesser than $(|a_1|^2 + 2|a_1a_3|)\,|\epsilon|^2$ and $|a_0a_3|\left(\epsilon\bar{\delta} + \delta\bar{\epsilon}\right)$ respectively. Such a choice of b is possible because we can choose η in the definition of J to be very small positive number. As a consequence we get

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\| \geq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|.$$

Therefore,

$$\|p(T_1, T_2, T_3)\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + b \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \right\|.$$

A classical result of Caratheodory and Fejer states that

$$\inf \|b_0 + b_1 z + r(z)\|_{\infty,\bar{\mathbb{D}}} = \left\| \begin{pmatrix} b_0 & b_1 \\ b_1 & b_0 \end{pmatrix} \right\|,$$

Where the infimum is taken over all polynomials $r(z)$ in one variable which contain only terms of degree two or higher. For an elegant proof to this result, where the result is derived as a consequence of the classical commutant lifting theorem of Sz.- Nagy and Foias. Using this fact we have

$$\|p(T_1, T_2, T_3)\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \right\|$$

$$= \inf \||a_0| + (|a_1| + |a_3|)z + r(z)\|_{\infty,\bar{\mathbb{D}}}$$

$$\leq \inf \||a_0| + |a_1|x_1 + |a_3|x_3 + r_1(x_1, x_2, x_3)\|_{\infty,\Lambda} \qquad (1.16)$$

$$\leq \inf \||a_0| + |a_2| + |a_1|x_1 + |a_3|x_3 + r_1(x_1, x_2, x_3)\|_{\infty,\Lambda} \qquad (1.17)$$

$$= \inf \||a_0| + |a_1|x_1 + |a_2|x_2 + |a_3|x_3 + r_1(x_1, x_2, x_3)\|_{\infty,\Lambda}$$

$$\leq \|a_0 + a_1x_1 + a_2x_2 + a_3x_3 + q(x_1, x_2, x_3)\|_{\infty,\Lambda} \qquad (1.18)$$

$$\leq \|a_0 + a_1x_1 + a_2x_2 + a_3x_3 + q(x_1, x_2, x_3)\|_{\infty,\overline{\mathbb{E}}}$$

$$= \|p(x_1, x_2, x_3)\|_{\infty,\overline{\mathbb{E}}}.$$

Here $\Lambda = \{(x, 1, x) : x \in \overline{\mathbb{D}}\} \subseteq \overline{\mathbb{E}}$ (by choosing $\beta_1 = 0$, $\beta_1 = 1$ in Theorem (1.16) and $r(z)$ and $r_1(x_1, x_2, x_3)$ range over polynomials of degree two or higher. The inequality (1.16) was obtained by putting $x_1 = x_3 = z$ and $x_2 = 1$ which makes the set of polynomials $|a_0| + |a_1|x_1 + |a_3|x_3 + r_1(z_1, z_2, z_3)$, a subset of the set of polynomials $|a_0| + (|a_1| + |a_3|)z + r(z)$. The infimum taken over a subset is always bigger than or equal to the taken over the set itself. We obtained the inequality (1.17) by applying a similar argument because we can extract the polynomial $|a_2|x_2^2$ from the set $r_1(x_1, x_2, x_3)$ and $|a_2|x_2^2 = |a_2|$ when $x_2 = 1$. the equality (1.18) was obtained by choosing $r_1(x_1, x_2, x_3)$ in particular to be equal to

$$(a_0 - |a_0| + a_2 - |a_2|)x_2^2 + (a_1 - |a_1|)x_1x_2 + (a_3 - |a_3|)x_2x_3 + q(x_1, x_2, x_3).$$

List of Symbols

Symbol	Page
Max: maximum	4
int: interior	8
dim: dimension	22
sup : supemum	22
ran : range	22
ker: kernel	23
c.n.u : contraction is completely nonunitary	26
$\bigoplus$: direct sum	26
diag : diagonal	29
$\ominus$: direct difference	31
H^2: Hardy space	31
Re : real	62
L^2 : Helbert space	65

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