CHAPTER 1

Weighted Composition Operators on Hardy space with Complex Symmetric

We obtain several examples for non normal complex symmetric operators. In addition, we give spectral properties of complex symmetric weighted composition operators.We examine eigenvalues and eigenvectors of such operators and find some conditions for which a complex symmetric weighted composition operator is Hilbert-Schmidt. Finally, we consider cyclicity, hypercyclicity and the single-valued extension property for complex symmetric weighted composition operators.

Section(1.1) Properties of ψ &φ:

In this section, we provide some characterizations of ψ and φ when a weighted composition operator $W_{\psi,\varphi}$ is complex symmetric. We give an equivalent condition for weighted composition operators to be complex symmetric with a special conjugation. As some applications, we obtain several examples for non normal complex symmetric operators.

Let L(H) be the algebra of all bounded linear operators on a separable complex Hilbert space H. If T ∈ L(H), we write ρ(T), σ(T), σ_{su}(T), σ_p(T), and σ_{ap}(T) for the resolvent set, the spectrum, the surjective spectrum, the In this section, we provide some characterizations of ψ and φ when a weighted composition operator $W_{\psi,\varphi}$ is complex symmetric. We give an equivalent condition for weighted composition operators to be complex symmetric with a special conjugation. As some applications, we obtain several examples for non normal complex symmetric operators.

point spectrum, and the approximate point spectrum of T, respectively, while r(T) denotes the spectral radius of T.

A conjugation on H is an anti linear operator C: H → H which satisfies (Cx, Cy) = (y, x) for all x, y ∈ H and C^2 = I. For any conjugation C, there is an orthonormal basis $\{e_n\}_n^\infty = 0$ for H such that $C_{e_n} = e_n$ for all n. An operator T ∈L(H) is said to be complex symmetric if there exists a conjugation C on H such that T = CT^*C. In this case, we say that T is complex symmetric with conjugation C. This concept is due to the fact that T is a complex symmetric operator if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension. The class of complex symmetric operators is unexpectedly large. This class includes all normal operators, Hankel operators, compressed Toeplitz operators, and the Volterra integration operator.We also remark that there is no difference between p-hyponormality and normality in this class. For $0 < p \leq 1$, an operator T ∈ L(H) is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. In particular, if p = 1, then T is called hyponormal. If T ∈ L(H) is complex symmetric, then it turns out that T is p-hyponormal if and only if it is normal .

Let D denote the open unit disk in the complex plane C. The space $H^2(D)$ consists of all the analytic functions on D having power series representation with square summable complex coefficients. The space $H^\infty(D)$ consists of all the functions that are analytic and bounded on D. If φ is an analytic mapping from D into itself, the composition operator C_φ on $H^2(D)$ is defined by $C_\varphi f = f \circ \varphi$ for all $f \in H^2(D)$. It is a well-known fact that the composition operator C_φ is bounded on $H^2(D)$ by Little wood subordination theorem. Moreover, the composition operator C_φ defined on $H^2(D)$ is normal if and only if $\varphi(z) = \gamma z$ where $|\gamma| \leq 1$. Hence we observe that $C_{\gamma z}$ is complex symmetric whenever $|\gamma| \leq 1$. Moreover, if φ is an automorphism of D given by $\varphi(z) = \frac{a-z}{1-\bar{a}z}$ for some a ∈ D, then $(\varphi \circ \varphi)(z) = z$ and so $C_\varphi^2 = C_{\varphi\circ\varphi} = I$. Thus C_φ is complex symmetric .

For an analytic function ψ on D and an analytic self map φ of D, the operator $W_{\psi,\varphi}: H^2(D) \to H^2(D)$ given by

$$W_{\psi,\varphi}f = \psi.(f \circ \varphi)$$

is called a weighted composition operator. If ψ is bounded on D, then $W_{\psi,\varphi}$ is clearly bounded on $H^2(D)$. For $\psi \in H^\infty(D)$, the multiplication operator on $H^2(D)$ is given by $M_\psi f = \psi f$ for all $f \in H^2(D)$. Remark that $W_{\psi,\varphi}$ can be written by $W_{\psi,\varphi} = M_\psi C_\varphi$ if $\psi \in H^\infty(D)$. In particular, C.C. Cowen and E. Ko have characterized self-adjoint weighted composition operators on $H^2(D)$. Beyond that, there are many examples for complex symmetric weighted composition operators on $H^2(D)$ which are not normal .

Recently, S.R. Garcia and C. Hammond provided properties of complex symmetric weighted composition operator on weighted Hardy spaces. In particular, they gave explicit forms of complex symmetric weighted composition operators with a specific conjugation on weighted Hardy spaces.

We study complex symmetric weighted composition operators on the Hardy space. We provide some characterizations of ψ and φ when a weighted composition operator $W_{\psi,\varphi}$ is complex symmetric. We investigate which combinations of weights ψ and maps of the open unit disk φ give rise to complex symmetric weighted composition operators with a special conjugation. As some applications, we obtain several examples for non normal complex symmetric operators. In addition, we give spectral properties of complex symmetric weighted composition operators. We examine eigenvalues and eigenvectors of such operators and find some conditions for which a complex symmetric weighted composition operator is Hilbert–Schmidt. Finally, we consider cyclicity, hypercyclicity, and the single-valued extension property for complex symmetric weighted composition operators.

In this section, we recall basic definitions needed for our program. Let ∂D denote the unit circle. Consider the Hilbert space

$\widetilde{H}^2(D) := \tilde{f}: \partial D \to C : \tilde{f}(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ with $\sum_{n=0}^{\infty} |a_n|^2 e^{in\theta} < \infty$

endowed with the norm $\|\tilde{f}\|_{\widetilde{H}^2} = (\sum_{n=0}^{\infty} |a_n|^2)^{½}$. It is obvious that the Hardy space $H^2(D)$ is isometrically isomorphic to $\widetilde{H}^2(D)$ by the isomorphism sending $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(D)$ to $\tilde{f}(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \in \widetilde{H}^2(D)$. Every function $f \in H^2(D)$ satisfies that $\lim_{r} \to 1- f(re^{i\theta}) = \tilde{f}(e^{i\theta})$ for almost every θ. When a function $\varphi \in H^2(D)$ satisfies that $|\tilde{\varphi}(e^{i\theta})| = 1$ for almost every θ, we say that φ is an inner function. A function $F \in H^2(D)$ is said to be outer if F is a cyclic vector for the unilateral shift M_z, i.e., $\bigvee_{n=0}^{\infty}$

$\{M_z^n F\} = H^2(D)$.

For each $\beta \in D$, the function $K_\beta(z) = \frac{1}{1-\bar{\beta}z} \in H^2(D)$, called the reproducing kernel for $H^2(D)$ at β, has the property that $\langle f, K_\beta \rangle = f(\beta)$ for every $f \in H^2(D)$ and $\beta \in D$.

Moreover, it is well known that the linear span of the reproducing kernels $\{K_\beta : \beta \in D\}$ is dense in $H^2(D)$. C.C. Cowen gave an adjoint formula of a composition operator whose symbol is a linear fractional selfmap of D. If $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional selfmap of D, then $C_\varphi^* = M_g C_\sigma M_h^*$ where $g(z) = \frac{1}{-\bar{b}z+\bar{d}}$, $\sigma(z) = \frac{\bar{a}z+\bar{c}}{-\bar{b}z+\bar{d}}$, and $h(z) = cz + d$. It follows that σ is a self map of D and $g \in H^\infty(D)$. Notice that $W_{\psi,\varphi}^* K_\beta = \overline{\psi(\beta)} K_{\varphi(\beta)}$ when $W_{\psi,\varphi}$ is bounded on $H^2(D)$ and $\beta \in D$; in fact, for any $\beta \in D$ and $f \in H^2(D)$

$\langle f, W^*_{\psi,\varphi}K_\beta\rangle = \langle \psi.(f \circ \varphi), K_\beta\rangle = \psi(\beta)f(\varphi(\beta)) = \langle f, \overline{\psi(\beta)}K_{\varphi(\beta)}\rangle$.

In particular, $C^*_\varphi K_\beta = K_{\varphi(\beta)}$ because $C_\varphi = W_{1,\varphi}$.

For any self map φ of D and each positive integer n, we write $\varphi_1 := \varphi$ and $\varphi_{n+1} := \varphi \circ \varphi_n$, which is called the iterate of φ for n. When φ is any analytic selfmap of D, we call $a \in \overline{D}$ a fixed point of φ if $\lim_{r\to1^-} \varphi(ra) = a$. It is well known that for an analytic function $\varphi: D \to D$, if φ is neither the identity map nor an elliptic automorphism of D, then there is a point a of $\overline{D}$ so that the iterates of φ converges uniformly to a on compact subsets of D. Moreover, a is the unique fixed point of φ in $\overline{D}$ for which $|\varphi'(a)| \leq 1$. We say that the unique fixed point a is the Denjoy–Wolff point of φ (is a theorm in complex analysis and dynamical systems concerning fixed points and iterations of holomophic mappings of the unit disc in the complex numbers into itself see [5]).

Let φ be an automorphism of D. Then φ is of the form

$$\varphi(z) = \frac{a_z + \overline{b}}{b_z + \overline{a}}$$

for all $z \in D$, where a and b in C with $|a|^2 - |b|^2 = 1$.When $b \neq 0$, it is easy to calculate that

$$\frac{i\,\mathrm{Im}(a) \pm \sqrt{|b|^2 - (\mathrm{Im}(a))^2}}{b}$$

are the fixed points of φ. If $|\mathrm{Im}(a)| = |b|$, then φ is called parabolic, and we say that φ is hyperbolic if $|\mathrm{Im}(a)| < |b|$. Remark that φ is parabolic if and only if it has one fixed point lying on ∂D, while φ is hyperbolic if and only if it has two fixed points lying on ∂D. If $|\mathrm{Im}(a)| > |b|$, then φ is said to be elliptic. We note that φ is elliptic if and only if one of its fixed points is inside D and another is outside D. In this sense, this type also includes the case when b = 0, i.e., when 0 and ∞ are the fixed points of φ.

For an operator $T \in L(H)$, a vector $x \in H$ is said to be cyclic if the linear span of the orbit $O(x, T) := \{Tn_x\}_{n=0}^{\infty}$ is norm dense in H. If there is a cyclic vector x for T, then we say that T is a cyclic operator. If the orbit O(x, T) is normdense in H for some $x \in H$, i.e., $\overline{O(x, T)} = H$, then T is called hypercyclic operator and x is called a hypercyclic vector. It is obvious that every hypercyclic operator is cyclic operator. An operator $T \in L(H)$ is said to have the single-valued extension property at z_0 if for every neighborhood G of z_0 and any H-valued analytic function f on G such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G.

Lemma (1.1.1)[1]: Let φ be a nonconstant analytic self map of D and let $\psi \in H^{\infty}(D)$. If $W_{\psi,\varphi}$ is a complex symmetric operator with conjugation C, then the following hold:

(i) Either $\psi \equiv 0$ or ψ never vanishes on D.

(ii) If ψ is not identically zero, then φ is univalent and

$$\psi = \frac{\psi(\lambda) CK_{\varphi(\lambda)}}{(CK_{\lambda}) \circ \varphi} \text{ for any } \lambda \in D.$$

Proof: Note that

$$W^{*}_{\psi,\varphi} K_{\beta} = \overline{\psi(\beta)} K_{\varphi(\beta)} \qquad (1)$$

for any $\beta \in D$.

(i) Assume that ψ is not identically zero on D and $\psi(\beta) = 0$ for some β in D. Then it suffices to assume that $\psi(z) \neq 0$ for all z in a deleted neighborhood of β. By Eq. (1), $W^{*}_{\psi,\varphi} K_{\beta} = \overline{\psi(\beta)} K_{\varphi(\beta)} = 0$. Since $W_{\psi,\varphi} C = CW^{*}_{\psi,\varphi}$ and C is an isometry, it follows that

$$\|W_{\psi,\varphi} CK_{\beta}\| = \|W^{*}_{\psi,\varphi} K_{\beta}\| = 0,$$

and so $W_{\psi,\varphi} CK_{\beta}(z) = 0$ for all $z \in D$. That is, $\psi(z)(CK_{\beta})(\varphi(z)) = 0$ for all $z \in D$, which gives that $(CK_{\beta})(\varphi(z)) = 0$ for all z in a deleted neighborhood of β. This implies that $CK_{\beta} \equiv 0$ on D by the identity theorem, but it is a contradiction. Thus ψ does not vanish on D.

(ii) Suppose that φ is not univalent. Then there exist two distinct points z_1 and z_2 in D so that $\varphi(z_1) = \varphi(z_2)$. Since ψ is not identically zero, it follows from the assertion (i) that $\psi(z_1) \neq 0$ and $\psi(z_2) \neq 0$. Set $f = \frac{Kz_1}{\psi(z_1)} - \frac{Kz_2}{\psi(z_2)}$. Then f is a nonzero vector in $H^2(D)$ and

$$W^{*}_{\psi,\varphi} f = \frac{1}{\overline{\psi(z_1)}} W^{*}_{\psi,\varphi} K_{z1} - \frac{1}{\overline{\psi(z_2)}} W^{*}_{\psi,\varphi} K_{z_2} = K\varphi(z_1) - K\varphi(z_2) = 0$$

From(1). The identity $W_{\psi,\varphi} C = CW^{*}_{\psi,\varphi}$ implies that $\|W_{\psi,\varphi} Cf\| = W^{*}_{\psi,\varphi} f = 0$. So we have $\psi(z)(Cf)(\varphi(z)) = 0$ for all $z \in D$. Since ψ does not vanish on D by (i) and φ is a non constant analytic map, it follows that $Cf \equiv 0$ on D. Since $C^2 = I$, we get the contradiction $f \equiv 0$ on D. Hence φ is univalent.

Let $\lambda \in D$ be given. Then it follows from Eq. (1) that

$$\psi(\lambda) CK_{\varphi(\lambda)} = C(\overline{\psi(\lambda)} K_{\varphi(\lambda)}) = CW^{*}_{\psi,\varphi} K_{\lambda} = W_{\psi,\varphi} CK_{\lambda} = \psi.((CK_{\lambda}) \circ \varphi).$$

We note that $(CK_\lambda) \circ \varphi$ does not vanish on D. Thus we obtain that $\psi = \frac{\psi(\lambda)CK_{\varphi(\lambda)}}{(CK_\lambda)o\varphi}$ for any $\lambda \in D$.

Recall that an operator $X \in L(H)$ is said to be a quasiaffinity if it has trivial kernel and dense range. It is well known that ker $(W^*_{\psi,\varphi})$ is not trivial in general. But the next result shows that ker $(W^*_{\psi,\varphi})$ is trivial if $W_{\psi,\varphi}$ is complex symmetric.

Proposition (1.1.2)[1]: Let φ be a non constant analytic self map of D and let $\psi \in H^\infty(D)$ be not identically zero on D. If $W_{\psi,\varphi}$ is complex symmetric, then the following statements hold.

(i) $W_{\psi,\varphi}$ is a quasiaffinity.

(ii) If φ is inner, then it is an automorphism of D.

Proof: (i) If $f \in \ker(W_{\psi,\varphi})$, then $\psi(z)f(\varphi(z)) \equiv 0$ on D. Since ψ is not identically zero, it follows that ψ never vanishes on D, and so we get that $f(\varphi(z)) \equiv 0$ on D. Since φ(D) is open, we have $f(z) \equiv 0$ on D by the identity theorem. Thus ker $(W_{\psi,\varphi}) = \{0\}$. Suppose that $W_{\psi,\varphi}$ is complex symmetric with conjugation C. If $f \in$ ker $(W^*_{\psi,\varphi})$, then it holds that $W_{\psi,\varphi} Cf = CW^*_{\psi,\varphi}f = 0$. Since ker $(W_{\psi,\varphi}) = \{0\}$, we obtain that $f = 0$, which means ker $(W^*_{\psi,\varphi}) = \{0\}$. Hence $W_{\psi,\varphi}$ is a quasiaffinity.

(ii) By Lemma (1.1.1), φ is univalent. Since a univalent inner function should be an automorphism of D , we complete our proof.

Next we investigate which combinations of weights ψ and maps of the disk φ give rise to complex symmetric weighted composition operators with the special conjugation J where $(J f)(z) = \overline{f(\overline{z})}$. With the fixed conjugation J, the complex symmetry significantly restricts the possible symbols for the weighted composition operators. These symbols are different from those symbols obtained from the self-adjointness. The following theorem provides many examples for complex symmetric weighted composition operators.

Theorem(1.1.3)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be not identically zero. If the weighted composition operator $W_{\psi,\varphi}$ is complex symmetric with conjugation J where $(J f)(z) = \overline{f(\overline{z})}$., then

$\psi(z) = \frac{b}{1 - a_0 z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$

Where $a_0 = \varphi(0)$, $a_1 = \acute{\varphi}\_(0)$, and $b = \psi(0)$.

Conversely, let $a_0 \in$ D. If $\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$ maps the unit disk into itself and $\psi(z) = \frac{b}{1 - a_0 z}$, then the weighted composition operator $W_{\psi,\varphi}$ is complex symmetric with conjugation J .

Proof: From Eq. (1), we obtain that

$$\begin{cases} (W_{\psi,\varphi} J)K_w(z) = W_{\psi,\varphi}(K_{\overline{w}}(z)) = \psi(z)K_{\overline{w}}(\varphi(z)) \text{ and} \\ (JW^*_{\psi,\varphi})K_w(z) = \psi(w)(JK_{\varphi(w)})(z) = \psi(w)K_{\overline{\varphi w}}(z) \end{cases} \quad (2)$$

for all z,w ∈ D. If $W_{\psi,\varphi}$ is complex symmetric with conjugation J , then

$$\psi(z)K_{\overline{w}}(\varphi(z))= \psi(w)K_{\overline{\varphi w}}(z) \quad (3)$$

for all z,w ∈ D. If we put w = 0 in Eq. (3), then

$$\psi(z) = \psi(0)K_{\overline{\varphi(0)}}(z) = \frac{\psi(0)}{1 - \varphi(0)z} \quad (4)$$

for all z ∈D. Since ψ is not identically zero on D, ψ (0) ≠ 0. Hence it follows from Eqs. (3) and (4) that

$$K_{\overline{\varphi(0)}}(z)K_{\overline{w}}(\varphi(z))= K_{\overline{\varphi(0)}}(w)K_{\overline{\varphi(w)}}(z)$$

for all z,w ∈ D. This implies that

$$(1 - \varphi(0)z)(1 - w\varphi(z))= (1 - \varphi(0)w)(1 - \varphi(w)z)$$

for all z,w∈D. By taking the derivative with respect to z, we have

$$-\varphi(0)\,(1 - w\varphi(z))+ (1 - \varphi(0)z)(-w\acute{\varphi}((z))=(1 - \varphi(0)w)(-\varphi(w))$$

for all z,w ∈ D. Putting z = 0, we get that

$$\varphi(w) = \varphi(0) + \frac{\acute{\varphi}(0)w}{1 - \varphi(0)w} = a_0 + \frac{a_1 w}{1 - a_0 w}$$

where $a_0 = \varphi(0)$ and $a_1 = \acute{\varphi}(0)$.

Conversely, if $\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$ is a self-map of D and $\psi(z) = \frac{b}{1 - a_0 z}$, then we can obtain from direct computations that

$$\psi(z)K_{\overline{w}}(\varphi(z)) = \frac{1}{(1 - a_0 w) - z[(a_1 - a_0^2)w + a_0]} = \psi(w)K_{\overline{\varphi w}}(z)$$

for all z,w ∈ D. Hence, implies that $W_{\psi,\varphi}$ is complex symmetric with conjugation J.

The following corollary explains how to construct complex symmetric operators by using unitary equivalence.

Corollary (1.1.4)[1]: Let $a_0 \in D$. If $\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$ maps the unit disk into itself and $\psi(z) = \frac{b}{1 - a_0 z}$, then $W_{\tilde{\psi},\tilde{\varphi}}$ is complex symmetric where $\tilde{\varphi}(z) = a_0 e^{i\theta} + \frac{a_1 z}{1 - a_0 e^{i\theta} z}$ and $\psi(z) = \frac{b}{1 - a_0 e^{-i\theta} z}$ for any fixed real number θ.

Proof: We know from Theorem (1.1.3) that $W_{\tilde{\psi},\tilde{\varphi}}$ is complex symmetric with conjugation J. In addition, since $\tilde{\psi}(z) = \psi(e^{-i\theta}z)$ and $\tilde{\varphi}(z) = e^{i\theta}\varphi(e^{-i\theta}z)$, we obtain that $W_{\tilde{\psi},\tilde{\varphi}} = U_\theta^* W_{\psi,\varphi} U_\theta$ where U_θ is the unitary operator defined by $(U_\theta f)(z) = f(e^{i\theta}z)$. Setting $J_\theta := U_\theta^* J U_\theta$, we get that J_θ is a conjugation and

$$J_\theta W_{\tilde{\psi},\tilde{\varphi}} J_\theta = U_\theta^* J W_{\psi,\varphi} J U_\theta = U_\theta^* W_{\psi,\varphi}^* U_\theta = W_{\tilde{\psi},\tilde{\varphi}}^*$$

as pointed out . Hence $W_{\tilde{\psi},\tilde{\varphi}}$ is complex symmetric with conjugation J_θ.

Corollary (1.1.5)[1]: Let φ be a non constant analytic self map of D and let $\psi \in H^\infty$ be not identically zero on D. If φ has a fixed point $\lambda \in D$ with $Im(\lambda) = 0$ and $W_{\psi,\varphi}$ is complex symmetric with conjugation J , then $C_{\tilde{\varphi}}$ is a complex symmetric operator with conjugation $(W_{\psi\lambda,b\lambda})*J\,(W_{\psi\lambda,b\lambda})$ where $\tilde{\varphi} = b_\lambda \circ \varphi \circ b_\lambda$, $\psi_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$, and $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$.

Proof: Set $\psi_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$ and $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$. Since b_λ is an automorphism of D and $b_\lambda(\lambda) = 0$, it follows that $W_{\psi_\lambda,b_\lambda}$ is a unitary operator. Put $T := (W_{\psi_\lambda,b_\lambda})^*(W_{\psi,\varphi})(W_{\psi_\lambda,b_\lambda})$. Then T is complex symmetric with conjugation $(W_{\psi_\lambda,b_\lambda})^* J (W_{\psi_\lambda,b_\lambda})$. It is well known from that $C_{b_\lambda}^* = M g C_{b_\lambda} M_h^*$ where $g(z) = (1 - \tilde{\lambda} z)^{-1}$ and $h(z) = 1 - \tilde{\lambda} z$. Since $M_h^* M_{\psi_\lambda}^* = (M_{\psi_\lambda} M_h)^* = \frac{1}{\|K_\lambda\|} I$, we have

$$T = C_{b_\lambda}^* M_{\psi_\lambda}^* M_\psi C_\varphi M_{\psi\lambda} C_{b_\lambda}$$

$$= M_g C_{b_\lambda} M_h^* M_{\psi_\lambda}^* M_\psi C_\varphi M_{\psi_\lambda} C_{b_\lambda}$$

$$= \frac{1}{\|K_\lambda\|} M_g C_{b_\lambda} M_\psi C_\varphi M_{\psi_\lambda} C_{b_\lambda}$$

$$= \frac{1}{\|K_\lambda\|} M_g M_{\psi o b_\lambda} M_{\psi_\lambda o \varphi o b_\lambda} C_{b_\lambda o \varphi o b_\lambda}.$$

Thus $T = W_{\tilde{\psi},\tilde{\varphi}}$ where $\tilde{\varphi} = b_\lambda \circ \varphi \circ b_\lambda$ and $\tilde{\psi} = \frac{1}{\|K_\lambda\|} g \cdot (\psi \circ b_\lambda)\,(\psi_\lambda \circ \varphi \circ b_\lambda)$. Since φ is non constant, $\tilde{\varphi}$ is also non constant and $\tilde{\varphi}(0) = 0$. Therefore, we see that

$$\widetilde{\psi} = \frac{\widetilde{\psi}(0)(W_{\psi_\lambda}, b_\lambda)*J\,(W_{\psi_\lambda}, b_\lambda)K_0}{((W_{\psi_\lambda}, b_\lambda)*J\,(W_{\psi_\lambda}, b_\lambda)K_0)\ o\widetilde{\varphi}} \qquad (5)$$

From Lemma (1.1.1) Since $b_\lambda(\bar{\lambda}) = \frac{2i\,\mathrm{Im}(\lambda)}{1-\bar{\lambda}2} = 0$, we get that

$(W_{\psi_\lambda}, b_\lambda)*J\,(W_{\psi_\lambda}, b_\lambda)K_0 = W^*_{\psi\lambda,b\lambda}J\,(\psi_\lambda) = \frac{1}{\|K_\lambda\|}W^*_{\psi\lambda,b\lambda}K_{\bar{\lambda}} = \frac{\overline{\psi\lambda(\bar{\lambda})}}{\|K_\lambda\|}K_0,$

which implies with (5) that $\widetilde{\psi} \equiv \widetilde{\psi}(0)$ on D, and so $T = \widetilde{\psi}(0)C_{\widetilde{\psi}}$. Since ψ as well as g and ψ_λ cannot vanish on D from Lemma (1.1.1), it follows that $\widetilde{\psi}(0)\neq 0$. Hence, we conclude that $C_{\widetilde{\psi}} = \frac{1}{C_{\widetilde{\psi}}(0)}T$ is complex symmetric with conjugation $(W_{\psi_\lambda}, b_\lambda)*J\,(W_{\psi_\lambda}, b_\lambda)$.

Let $\psi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ map D into itself where $a_1\neq 0$, and let $\psi(z) = \frac{b}{1-a_0 z}$ where $b\neq 0$. Then $W_{\psi,\varphi}$ is a complex symmetric weighted composition operator with conjugation J by Theorem (1.1.3) If φ has a fixed point $\lambda \in D$ with $\mathrm{Im}(\lambda) = 0$, by Corollary (1.1.5) we can get a complex symmetric composition operator $C_{\widetilde{\varphi}}$ with conjugation $(W_{\psi_\lambda}, b_\lambda)*J\,(W_{\psi_\lambda}, b_\lambda)$ where $\widetilde{\varphi} = b_\lambda\circ\varphi\circ b_\lambda$, $\psi_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$, and $b_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$. In fact, from direct computations we have

$$\widetilde{\varphi}(z) = \frac{\lambda - \varphi(b_\lambda(z))}{1-\bar{\lambda}\varphi(b_\lambda(z))}$$

$$= \frac{-(|\lambda|^2 - a_0(\lambda+\bar{\lambda}) + a_0^2 - a_1)z}{(a_0\bar{\lambda}^2 - (1 + a_0^2 - a_1)\bar{\lambda} + a_0)z + (a_0^2 - a_1)|\lambda|^2 - a_0(\lambda+\bar{\lambda}) + 1} \qquad (6)$$

for $z \in D$. From Eq. (6) we obtain a complex symmetric composition operator which is not hyponormal.

Example(1.1.6)[1]: If $\varphi(z) = \frac{-1}{2\sqrt{2}+z}$ and $\psi(z) = \frac{2}{2\sqrt{2}+z}$(i.e., $a_0 = -\frac{1}{2\sqrt{2}}$, $a_1 = \frac{1}{8}$, and $b = \frac{1}{2}$), then we know that φ is an analytic self map of D from Thus, $W_{\psi,\varphi}$ is complex symmetric with conjugation J by Theorem (1.1.3) In addition, φ has a fixed point $\lambda = \sqrt{2} - 1 \in D$ with $\mathrm{Im}(\lambda) = 0$. Under the same notations as in Corollary (1.1.5), we get from Eq. (6) that $C_{\widetilde{\varphi}}$ is complex symmetric with conjugation $(W_{\psi_\lambda}, b_\lambda)*J\,(W_{\psi_\lambda}, b_\lambda)$, where

$$\varphi(z) = \frac{(5-4\sqrt{2})z}{(-4+2\sqrt{2})z + (-1+2\sqrt{2})}$$

It is evident that $C_{\widetilde{\varphi}}$ is not normal. Indeed, it is not hyponormal by rewriting $\widetilde{\varphi}(z) = \frac{z}{uz+v}$ where $u = \frac{4+6\sqrt{2}}{7}$ and $v = -\frac{11+6\sqrt{2}}{7}$.

As an application for Theorem (1.1.3), we find an equivalent condition for a complex symmetric weighted composition operator with conjugation J to be normal.

Corollary (1.1.7)[1]: Let φ be an analytic self map of D and let $\psi \in H^{\infty}(D)$ be not identically zero on D. Suppose that $W_{\psi,\varphi}$is a complex symmetric operator with conjugation J. Then $W_{\psi,\varphi}$ is normal if and only if $\varphi(\overline{\varphi(0)})$ is real. In particular, if φ(0) and ψ(0) are nonzero real numbers, then $W_{\psi,\varphi}$ is normal if and only if it is self-adjoint.

Proof: Since $W_{\psi,\varphi}$ is complex symmetric with conjugation J , it follows from Theorem (1.1.3) that

$\psi(z) = \frac{b}{1-a_0 z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$

where $a_0 = \varphi(0)$, $a_1 = \acute{\varphi}(0)$, and $b = \psi(0)$. For any z,w ∈ D, we have

$$W^*_{\psi,\varphi}W_{\psi,\varphi}K_w(z) = W^*_{\psi,\varphi}\frac{\psi(z)}{1-\overline{w}\varphi(z)} = W^*_{\psi,\varphi}\frac{\frac{\psi(0)}{1-a_0 z}}{1-\overline{w}\left(a0+\frac{a_1 z}{1-a_0 z}\right)}$$

$$= bW^*_{\psi,\varphi}\frac{1}{1-a_0\overline{w}-[a_0-a_0^2\overline{w}+a_{1\,\overline{w}}]z}$$

$$= \frac{b}{1-a_0\overline{w}}W^*_{\psi,\varphi}K_{\frac{\overline{a_0}-\overline{a_0}^2 w+\overline{a_1}w}{1-\overline{a_0}w}}(z)$$

$$= \frac{b}{1-a_0\overline{w}}\psi\left(\frac{\overline{a_0}-\overline{a_0}^2 w+\overline{a_1}w}{1-\overline{a_0}w}\right)K_{\varphi\left(\frac{\overline{a_0}-a_0{}^2 w+\overline{a_1}w}{1-\overline{a_0}w}\right)}(z) \qquad (7)$$

The result obtained by expanding (7) is that

$$\frac{|b|^2}{[(-a_0-\overline{a_0}(a_1-a_0^2))\overline{w}+1-|a_0|^2]-[(|a_1-a_0^2|^2-|a_0|^2)\overline{w}+(\overline{a_0}+a_0(\overline{a_1}-\overline{a_0}^2))]z} \qquad (8)$$

for any z,w ∈ D. On the other hand, it holds for any z,w ∈ D that

$$W_{\psi,\varphi}W^*_{\psi,\varphi}K_w(z) = W_{\psi,\varphi}\overline{\psi(w)}K_{\varphi(w)}(z)$$

$$= M_\psi C_\varphi\left(\frac{\overline{\psi(w)}}{1-\overline{\varphi(w)}z}\right)$$

$$= \frac{\overline{\psi(w)}\,\psi(z)}{1-\overline{\varphi(w)}\,\varphi\, z} \qquad (9)$$

By expanding (9), we get that

$$\frac{|b|^2}{[(-\overline{a_0}-a_0(\overline{a_1}-\overline{a_0}^2))\overline{w}+1-|a_0|^2]-[(|a_1-a_0^2|^2-|a_0|^2)\overline{w}+(a_0+\overline{a_0}(a_1-a_0^2))]z} \quad (10)$$

for any z,w ∈ D. Since ψ is not identically zero on D, we have b = ψ(0)≠ 0. Hence it follows from (8) and (10) that $W_{\psi,\varphi}$ is normal if and only if $a_0 + \overline{a_0}(a_1 - a_0^2)$ is real. Since $\varphi(\overline{\varphi(0)}) = \frac{a_0+\overline{a_0}(a_1-a_0^2)}{1-|a_0|^2}$, we conclude that $W_{\psi,\varphi}$ is normal if and only if $\varphi(\overline{\varphi(0)})$ is real. In particular, assume that φ(0) and ψ(0) are real. If $W_{\psi,\varphi}$ is normal, then $a_0(1+a_1 - a_0^2)$ is real, or equivalently, a1 is real. Therefore $W_{\psi,\varphi}$ is self-adjoint.

As some applications of Corollary (1.1.7), we can find many examples which are complex symmetric weighted composition operators but not normal.

Example(1.1.8)[1]: Let $\psi(z) = \frac{\frac{1}{2}}{1-\frac{1}{2}z}$ and $\varphi(z) = \frac{\frac{i}{2}}{1-\frac{i}{2}z} = \frac{i}{2} + \frac{-\frac{1}{4}z}{1-\frac{i}{2}z}$. Then $\psi \in H^\infty(D)$ and φ is an analytic selfmap of D. By Theorem (1.1.3), we have $W_{\psi,\varphi}$ is complex symmetric with conjugation *J*. But $W_{\psi,\varphi}$ is not normal from Corollary (1.1.7), since $\varphi(\overline{\varphi(0)}) = \frac{2i}{5}$ is not real.

Example (1.1.9)[1]: Let $\psi(z) = \frac{\frac{1}{4}}{1-\left(\frac{\sqrt{3}}{8}+\frac{1}{8}i\right)z}$ and

$$\varphi(z) = \frac{\frac{\sqrt{3}}{8}+\frac{1}{8}i}{1-\left(\frac{\sqrt{3}}{8}+\frac{1}{8}i\right)z} = \frac{\sqrt{3}}{8} + \frac{1}{8}i + \frac{\left(\frac{1}{32}+\frac{\sqrt{3}}{32}i\right)z}{1-\left(\frac{\sqrt{3}}{8}+\frac{1}{8}i\right)z}.$$

Then $\psi \in H^\infty(D)$ and φ is analytic. In addition, we know that φ maps D into itself by Lemma (1.2.8). Thus, Theorem (1.1.3) implies that $W_{\psi,\varphi}$ is complex symmetric with conjugation *J* . However, $\varphi(\overline{\varphi(0)}) = \frac{2}{15}(\sqrt{3} + i)$ is not real and so $W_{\psi,\varphi}$ is not normal from Corollary (1.1.7) In the following corollary, we explain that every complex symmetric composition operators with conjugation *J* must be normal.

Corollary(1.1.10)[1]: Let φ be an analytic selfmap of D. Then C_φ is a complex symmetric operator with conjugation J if and only if C_φ is normal.

Proof: Suppose that C_φ is complex symmetric with conjugation *J*. Since $C_\varphi = W_{\psi,\varphi}$ where $\psi \equiv 1$, we can set $\varphi(z) = az$ where $|a| \le 1$ from Theorem (1.1.3) Thus C_φ is normal.

On the other hand, if C_φ is normal, then $\varphi(z) = az$ where $|a| \leq 1$. Therefore we have

$$\begin{cases} (C_\varphi J K_\alpha)(z) = K_{\bar{\alpha}}(\varphi(z)) = \dfrac{1}{1-\alpha a z} \\ (J\, C_\varphi^* K_\alpha((z) = K_{\overline{\varphi(\alpha)}}(z) = \dfrac{1}{1-\alpha a z} \end{cases}$$

for all $z \in D$, and so C_φ is complex symmetric with conjugation J .

Lemma (1.1.11)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be not identically zero on D. Suppose that $W_{\psi,\varphi}$ is a complex symmetric operator with conjugation J . If φ(0) = 0, then ψ is constant, $\varphi(z) = \acute{\varphi}(0)z$, and C_φ is complex symmetric with conjugation J .

Proof: The proof follows from Theorem (1.1.3).

Theorem (1.1.12)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be not identically zero on D. Suppose that $W_{\psi,\varphi}$ is a complex symmetric operator with conjugation J. If C_φ is p-hyponormal, then $W_{\psi,\varphi}$ is normal.

Proof: If C_φ is p-hyponormal, then C_φ is normaloid (i.e., $r(C_\varphi) = C_\varphi$). By some applications, we obtain that c = 0 or |c| = 1 and $\acute{\varphi}(c) < 1$, where c is the Denjoy–Wolff point of φ. Assume that |c| = 1 and $\acute{\varphi}(c) < 1$. $\sigma(C_\varphi)$ includes an open annulus of eigenvalues. If C_φ is p-hyponormal, then

$$\ker(C_\varphi - \lambda) \subset \ker(C_\varphi^* - \bar{\lambda})$$

for any $\lambda \in C$. Hence $\ker(C_\varphi - \lambda)$ is a reducing subspace for C_φ. Therefore, each of these eigenvalues corresponds to a reducing subspace of $H^2(D)$. But since $H^2(D)$ is separable, it is a contradiction. Thus c = 0. Since c is a fixed point of φ, we have φ(0) = 0. Hence Lemma (1.1.11) implies that ψ is a constant function and $\varphi(z) = \acute{\varphi}(0)z$ on D, and so $W_{\psi,\varphi}$ is normal.

The converse of Theorem (1.1.12) does not hold. Indeed, in the first part of Example (1.1.8), $W_{\psi,\varphi}$ is normal, but C_φ is not p-hyponormal since $\varphi(0) \neq 0$ (see the proof of Theorem 1.1.12).

Section (1.2) Spectral Theory and Cyclicity with Hypercyclicity and Extension Property:

In this section, we give spectral properties of complex symmetric weighted composition operators. In particular, we examine eigenvectors and eigenvalues of a complex symmetric weighted composition operator and consider some conditions for which a complex symmetric weighted composition operator is Hilbert–Schmidt. First, we provide some invariant subspaces of a complex symmetric weighted composition operator. For a positive integer j and a ∈ D, the jth derivative evaluation kernel for $H^2(D)$ at a, denoted by $K_a^{[j]}$, is the function in $H^2(D)$ such that $\langle f, K_a^{[j]}\rangle = f^{(j)}(a)$ for all $f \in H^2(D)$.

Proposition (1.2.1)[1]: Let $W_{\psi,\varphi}$ be a complex symmetric operator with conjugation C where φ is an analytic selfmap of D and ψ is in $H^\infty(D)$. Then the following statements hold.

(i) The set $\{CK_a : |a| < 1\}$ is linearly independent.

(ii) For a ∈ D, the set $\{CK_a^{[j]} : j = 0, 1, 2, 3, \cdots\}$ is linearly independent, where $K_a^{[0]}$ denotes the reproducing kernel K_a.

(iii) If m is a positive integer and φ has a fixed point a ∈ D, then the span of $\{CK_a, CK_a^{[1]}, \cdots, CK_a^{[m]}\}$ is an invariant subspace of $W_{\psi,\varphi}$.

Proof: (i) Suppose that $a_1, a_2, \cdots, a_n$ are distinct points in D. If there exist complex numbers $c_1, c_2, \cdots, c_n$ such that $\sum_{j=1}^{n} c_j CK_{a_j} = 0$, then $\sum_{j=1}^{n} \bar{c}_j CK_{a_j} = 0$. For j =1, 2, $\cdots$, n, define

$$f_j(z) = \prod_{\substack{i=1 \\ i\neq j}}^{n} (z - a_i)$$

Then we obtain that

$$0 = \langle f_j, \sum_{k=1}^{n} \bar{c}_k K_{a_k}\rangle = c_j f_j(a_j).$$

Since $f_j(a_j) \neq 0$, we have $c_j = 0$ for j = 1, 2, $\cdots$, n.

(ii) Let $m_0, m_1, \cdots, m_n$ be arbitrary distinct nonnegative integers. Assume that $c_0, c_1, \cdots, c_n$ are complex numbers such that $\sum_{j=0}^{n} c_j CK_a^{[m_j]} = 0$. Then we have $\sum_{j=0}^{n} \bar{c}_j CK_a^{[m_j]} = 0$. Setting $g_j(z) = \frac{1}{m_j!}(z - a)^{m_j}$ for j = 0, 1, 2, $\cdots$, n, we obtain that

$$0 = \langle g_j, \sum_{k=0}^{n} \bar{c}_k K_a^{[m_j]}\rangle = \sum_{k=0}^{n} c_k g_j^{[m_j]}(a) = c_j$$

for j = 0, 1, 2, $\cdots$, n.

(iii) For any $f \in H^2(D)$ and each positive integer n, it holds that

$$\langle f, W_{\psi,\varphi}^* K_a^{[n]}\rangle = \langle W_{\psi,\varphi} f, K_a^{[n]}\rangle$$

$=\frac{d^n}{dz^n}[\psi(z) f(\varphi(z))\Big|_{z=a}$

$$= \sum_{i=0}^{n-1} \alpha_i(a) f^{(i)}(a) + \psi(a)\acute{\varphi}(a)^n f^{(n)}(a)$$

$$= \langle f, \sum_{i=0}^{n-1} \overline{\alpha_\imath(a)} K_a^{[i]} + \overline{\psi(a)\acute{\varphi}(a)^n f^{(n)}} K_a^{[n]} \rangle \qquad (11)$$

where each α_i is a function consisting of products of derivatives of ψ and φ. Thereforewe have

$$W_{\psi,\varphi}^* K_a^{[n]} = \sum_{i=0}^{n-1} \overline{\alpha_\imath(a)} K_a^{[i]} + \overline{\psi(a)\acute{\varphi}(a)^n}\, K_a^{[n]} \qquad (12)$$

Since $W_{\psi,\varphi}$ is complex symmetric with conjugation C, we get that

$W_{\psi,\varphi} K_a^{[n]} = W_{\psi,\varphi}^* K_a^{[i]}$

$= \sum_{i=0}^{n-1} \alpha_i(a) C K_a^{[i]} + \psi(a)\acute{\varphi}(a)^n C K_a^{[n]}$

Moreover $W_{\psi,\varphi} CK_a = C W_{\psi,\varphi}^* K_a^{[i]} = \psi(a) K_a$. Thus we conclude that for any positive integer m, span $\{CK_a, K_a^{[1]}, \cdot\cdot\cdot, CK_a^{[m]}\}$ is an invariant subspace of $W_{\psi,\varphi}$.

Corollary (1.2.2)[1]: Let $W_{\psi,\varphi}$ be a complex symmetric operator with conjugation C where φ is an analytic selfmap of D and $\psi \in H^\infty(D)$. If φ has a fixed point $a \in D$ and $M_m := \text{span}\{CK_a, CK_a^{[1]}, \cdot\cdot\cdot, CK_a^{[m]}\}$ for a positive integer m, then

$$\sigma(W_{\psi,\varphi}) = \sigma(W_{\psi,\varphi}|M_m) \cup \sigma(B)$$

where $B = (I - P)W_{\psi,\varphi}(I - P)|M_m^\perp$ and P denotes the orthogonal projection of $H^2(D)$ onto M_m.

Proof: Since Mm is finite dimensional, it is a closed subspace of $H^2(D)$. Hence $H^2(D) = M_m \oplus M_m^\perp$. Since Mm is an invariant subspace for $W_{\psi,\varphi}$ we can write

$$W_{\psi,\varphi} = \begin{pmatrix} W_{\psi,\varphi}|M_m & A \\ 0 & B \end{pmatrix} \text{on } M_m \oplus M_m^\perp$$

where B = (I − P)$W_{\psi,\varphi}$(I − P)$|M_{M_m}^{\perp}$ and P denotes the orthogonal projection of $H^2(D)$ onto Mm. Since Mm is finite dimensional, we obtain that $\sigma(W_{\psi,\varphi}) = \sigma(W_{\psi,\varphi}|M_m) \cup \sigma(B)$.

In the following theorem, we find the eigenvalues of a complex symmetric weighted composition operator $W_{\psi,\varphi}$ when φ has the Denjoy–Wolff point inside the unit circle.

Theorem (1.2.3)[1]: Let $W_{\psi,\varphi}$ be a complex symmetric operator with conjugation C where φ is an analytic selfmap of D and $\psi \in H^{\infty}(D)$. If φ has the Denjoy–Wolff point a ∈ D, then $\sigma_p(W_{\psi,\varphi}) = \{\psi(a)\bar{\varphi}(a)_j : j = 0, 1, 2, \cdots\}$.

Proof: The inclusion $\sigma_p(W_{\psi,\varphi}) \supset \{\psi(a)\bar{\varphi}(a)_j : j = 0, 1, 2, \cdots\}$ holds . Indeed, we know from the proof of Proposition (1.2.1) that

$$\begin{cases} W_{\psi,\varphi}K_a = W_{\psi,\varphi}^{*}K_a = \psi(a)K_a \\ W_{\psi,\varphi}^{*}K_a^{[n]} = \sum_{i=0}^{n-1} \alpha_i(a)CK_a^{[i]} + \psi(a)\acute{\varphi}(a)^{n}CK_a^{[n]} \end{cases}$$

for each positive integer n, and span$\{CK_a^{[i]} : i = 0, 1, 2, \cdots, j\}$ is invariant for $W_{\psi,\varphi}$. Then it follows that the matrix of $W_{\psi,\varphi}$ restricted to span$\{CK_a^{[i]} : i = 0, 1, 2, \cdots, j\}$ is representable as the upper triangular matrix with the diagonal values $\psi(a)\acute{\varphi}(a)j$.Since the dimension of span$\{CK_a^{[i]} : i = 0, 1, 2, \cdots, j\}$ is finite, we conclude that $\psi(a)\acute{\varphi}(a)j$ is an eigenvalue of $W_{\psi,\varphi}$ with an eigenvector in span$\{CK_a^{[i]} : i = 0, 1, 2, \cdots, j\}$. Conversely, let $\lambda \in \sigma_p(W_{\psi,\varphi})$. Then there exists a nonzero element $f \in H^2(D)$ such that $W_{\psi,\varphi}f = \lambda f$, i.e.,

$$\psi(z)f(\varphi(z)) = \lambda f(z) \text{ for } z \in D \qquad (13)$$

Since φ(a) = a, we have ψ(a)f(a) = λf(a). If f has no zero at a, then λ = ψ(a). Suppose that f has a zero at a of order j for some positive integer j. Differentiating Eq. (13), we obtain from Eq. (11) that

$$\lambda f^{(j)}(a) = \sum_{i=0}^{i-1} \alpha_i(a)f^{(i)}(a) + \psi(a)\acute{\varphi}(a)^{i}f^{(i)}(a)$$

$$= \psi(a)\acute{\varphi}(a)^{j} f^{(i)}(a)$$

where each α_i is a function consisting of products of derivatives of ψ and φ. Since $f^{(i)}(a) \neq 0$, it follows that $\lambda = \psi(a)\acute{\varphi}(a)^j f^{(i)}$

Now we deal with complex symmetric weighted composition operators with a special conjugation. The following proposition gives an explicit form of an eigenvector for such an operator.

Proposition (1.2.4)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^{\infty}(D)$ be not identically zero on D where $\varphi(0) \neq 0$, $\acute{\varphi}(0) \neq 0$, and $\varphi(\lambda) = \lambda$ for some λ in D. If $W_{\psi,\varphi}$ is complex symmetric with conjugation J where $(J f)(z) = \overline{f(\bar{z})}$, then

$$g_j(z) := \frac{1}{1-\lambda z}\left(\frac{\lambda - z}{1-\lambda z}\right)^j$$

is an eigenvector of $W_{\psi,\varphi}$(which is not necessarily orthogonal) with respect to the eigenvalue $\psi(\lambda)\acute{\varphi}(\lambda)^j f^{(i)}$ for each nonnegative integer j.

Proof: At first, note that $g_j \in H^{\infty}(D)$ for each nonnegative integer j. We know from Theorem (1.1.3) that $\psi(z) = \frac{b}{1-a_0 z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ where $a_0 = \varphi(0) \neq 0$, $a_1 = \acute{\varphi}(0) \neq 0$, and $b = \psi(0)$. Since $\varphi(\lambda) = \lambda$ for some λ in D, we get that

$$(1 - a_0\lambda)\lambda = (a_1 - a_0^2)\,\lambda + a_0 \qquad (14)$$

Hence we have

$$\lambda - a_0 = \lambda\,(a_1 - a_0^2 + a_0\lambda) \qquad (15)$$

Since $a_0 \neq 0$, λ should be nonzero. If $a_1 - a_0^2 + a_0\lambda = 0$, then $\lambda = a_0$, and so $a_1 = 0$, which is a contradiction. Thus $a_1 - a_0^2 + a_0\lambda \neq 0$. Since $\varphi(z) - a_0 = \frac{a_1 z}{1-a_0 z}$, it follows that

$$\lambda - a_0 = \varphi(\lambda) - a_0 = \frac{a_1\lambda}{1-a_0\lambda} \qquad (16)$$

From Eqs. (15) and (16), we get that

$$\frac{a_1}{1-a_0\lambda} = \frac{\lambda - a_0}{\lambda} = a_1 - a_0^2 + a_0\lambda \qquad (17)$$

Therefore Eq. (17) implies that

$$\acute{\varphi}(\lambda) = \frac{a_1}{(1-a_0\lambda)^2} = \left(\frac{a_1}{1-a_0\lambda}\right)\left(\frac{1}{1-a_0\lambda}\right) = \frac{a_1 - a_0^2 + a_0\lambda}{1-a_0\lambda} \qquad (18)$$

From Eq. (14) we have

$$1-\lambda\varphi(z) = 1-\lambda\frac{(a_1-a_0^2)z+a_0}{1-a_0z} = \frac{(1-a_0\lambda)-(a_{1\lambda}-a_0^2\lambda+a_0)z}{1-a_0z}$$

$$= \frac{(1-a_0\lambda)-(1-a_0\lambda)\lambda z}{1-a_0z} = \frac{(1-a_0\lambda)(1-\lambda z)}{1-a_0z} \qquad (19)$$

It follows from Eq. (19) that

$$\frac{\lambda-\varphi(z)}{1-\lambda\varphi(z)} = \frac{(1-a_0z)(\lambda-\varphi(z))}{(1-a_0\lambda)(1-\lambda z)}$$

$$= \frac{(1-a_0z)\left(\lambda-a_0-\frac{a_1z}{1-a_0z}\right)}{(1-a_0\lambda)(1-\lambda z)} = \frac{(\lambda-a_0)-z(a_1-a_0^2+a_0\lambda)}{(1-a_0\lambda)(1-\lambda z)}$$

Using Eqs. (15) and (18), we get that

$$\frac{\lambda-\varphi(z)}{1-\lambda\varphi(z)} = \frac{\lambda(a_1-a_0^2+a_0\lambda)-z(a_1-a_0^2+a_0\lambda)}{(1-a_0\lambda)(1-\lambda z)}$$

$$= \left(\frac{a_1-a_0^2+a_0\lambda}{(1-a_0\lambda)}\right)\left(\frac{\lambda-z}{1-\lambda z}\right) = \acute{\varphi}(\lambda)\frac{\lambda-z}{1-\lambda z}$$

Thus it holds that

$(W_{\psi,\varphi}g_j)(z) = \psi(z)g_j(\varphi(z))$

$$= \frac{\psi(0)}{(1-a_0z)(1-\lambda\varphi(z))}\left(\frac{\lambda-\varphi(z)}{1-\lambda\varphi(z)}\right)^j$$

$$= \frac{\psi(0)}{(1-a_0z)(1-\lambda z)}\left(\acute{\varphi}(\lambda)\frac{\lambda-z}{1-\lambda z}\right)^j$$

$$= \left(\frac{\psi(0)}{1-a_0\lambda}\acute{\varphi}(\lambda)^i\right)\frac{1}{1-\lambda z}\left(\frac{\lambda-z}{1-\lambda z}\right)^i = \left(\psi(\lambda)\acute{\varphi}(\lambda)^j\right)g_j(z)$$

for $z \in D$ and each nonnegative integer j. Hence we complete our proof.

Next we give a lower bound and an upper bound for the spectral radius of a complex symmetric weighted composition operator with conjugation J .

Lemma (1.2.5)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be non constant. Suppose that $W_{\psi,\varphi}$ is a complex symmetric operator with conjugation J where $(Jf)(z) = \overline{f(\overline{z})}$. If $\varphi\overline{(\varphi(0))}$ is real, then the following properties hold:

(i) $\|\psi\| = \frac{|\psi(0)|}{\sqrt{1-|\varphi(0)|^2}}$ and $0 < |\varphi(0)| < 1$.

(ii) $\frac{|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))|}{\sqrt{1-|\varphi_n(0)|^2}} \le \left\|W^n_{\psi,\varphi}\right\| \le \frac{2\|\psi\|^n_\infty}{\sqrt{1-|\varphi_n(0)|^2}}$ for each positive integer n, where φ_0 denotes the identical function on D.

Proof: (i) Note that

$$\begin{cases} \|W_{\psi,\varphi}1\| = \|M_\psi 1\| = \|\psi\| \text{ and} \\ \|W^*_{\psi,\varphi}1\| = \|\overline{\psi(0)}K_\varphi(0)\| = \dfrac{|\psi(0)|}{\sqrt{1-|\varphi(0)|^2}} \end{cases}$$

By Corollary (1.1.7), Wψ,φ is normal and it gives that

$$\|\psi\| = \frac{|\psi(0)|}{\sqrt{1-|\varphi(0)|^2}} .$$

Since φ is an analytic selfmap of D, it ensures that $|\varphi(0)| < 1$. If $\varphi(0) = 0$, then Theorem (1.1.3) implies that $\psi(z) \equiv \psi(0)$ on D, which is a contradiction. Thus we have $|\varphi(0)| > 0$. There fore $\|\psi\| = \frac{|\psi(0)|}{\sqrt{1-|\varphi(0)|^2}}$ and 0 $|\varphi(0)| < 1$.

(ii) It follows that

$$\|W^n_{\psi,\varphi}\| = \|W_{\psi\cdot(\psi o\varphi)\cdots(\psi o\varphi n-1),\varphi n}\| \le \|M_{\psi\cdot(\psi o\varphi)\cdots(\psi o\varphi n-1)}\|C_{\varphi n}$$
$$\le \|\psi^n_\infty\|\frac{1+|\varphi_n(0)|}{\sqrt{1-|\varphi_n(0)|^2}} \le \frac{2\|\psi^n_\infty\|}{\sqrt{1-|\varphi_n(0)|^2}}$$

Since $W_{\psi,\varphi}$ is normal by Corollary (1.1.7), so is $W^n_{\psi,\varphi}$. This implies that

$$\|W_{\psi,\varphi}1\| = \left\|W^*_{\psi\cdot(\psi o\varphi)\cdots(\psi o\varphi_{n-1}),\varphi_n}K_0\right\|$$
$$= \left|\psi(0)\psi(\varphi(0))\cdots\psi\left(\varphi_{n-1}(0)\right)\right|\left\|K_{\varphi_n(0)}\right\|$$
$$= \frac{|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))|}{\sqrt{1-|\varphi_n(0)|^2}}$$

Hence we obtain that $\frac{|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))|}{\sqrt{1-|\varphi_n(0)|^2}} \le \|W^n_{\psi,\varphi}\|$ for each positive integer n. So we complete the proof.

Theorem (1.2.6)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be non constant.

Suppose that $W_{\psi,\varphi}$ is a complex symmetric operator with conjugation J where $(Jf)(z) = \overline{f(\overline{z})}$, $\varphi\overline{(\varphi(0))}$ is real, and a is the Denjoy–Wolff point of φ.

(i) If $a \in D$, then $|\psi(a)| \le r(W_{\psi,\varphi}) \le \|\psi\|\infty$.

(ii) If $a \in \partial D$, then $\acute{\varphi}(a)^{-\frac{1}{2}}|\psi(a)| \le r(W_{\psi,\varphi}) \le \acute{\varphi}(a)^{-\frac{1}{2}}\|\psi\|_\infty$.

Proof: By Lemma (1.2.5) we have

$$\frac{|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))|^{\frac{1}{n}}}{(1-|\varphi_n(0)|^2)^{\frac{1}{2n}}} \le \|W^n_{\psi,\varphi}\|^{\frac{1}{n}} \le \frac{2^{\frac{1}{n}}\|\psi\|_\infty}{(1-|\varphi_n(0)|^2)^{\frac{1}{2n}}}$$

for each positive integer n where φ_0 denotes the identical function on D. Let $\varepsilon > 0$ be given, and put $x_n = \log|\psi(\varphi_n(0))|$ for each nonnegative integer n. Since $\psi(z) = \frac{b}{1-a_0 z}$ with $a_0 \in D$ from Theorem (1.1.3), ψ never vanishes on $\overline{D}$. In addition, $lim_{n\to\infty} x_n = \log|\psi(a)|$ and thus we can choose a positive integer N such that $|x_n - \log|\psi(a)|| < \frac{\varepsilon}{2}$ whenever $n > N$. Moreover, there is a real number $M > 0$ with $|x_n - \log|\psi(a)|| \le M$ for each nonnegative integer n. Hence it holds for all $n > N$ that

$$\left|\log\left(\left|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))\right|^{\frac{1}{n}}\right) - \log|\psi(a)|\right|$$
$$\le \frac{1}{n}\sum_{j=0}^{N-1}|x_j - \log|\psi(a)|| + \frac{1}{n}\sum_{j=N}^{N-1}|x_j - \log|\psi(a)||$$
$$\le \frac{MN}{n} + \frac{(n-N)\varepsilon}{2n} < \frac{MN}{n} + \frac{\varepsilon}{2}$$

If we select a positive integer $\acute{N}$ such that $\acute{N} > \max\{\frac{2MN}{\varepsilon}, N\}$, then for all $n > \acute{N}$ we obtain that

$$\left|\log\left(\left|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))\right|^{\frac{1}{n}}\right) - \log|\psi(a)|\right| < \frac{MN}{\acute{N}} + \frac{\varepsilon}{2} < \varepsilon$$

Therefore we have $\lim_{n\to\infty}\log\left(\left|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))\right|^{\frac{1}{n}}\right) = \log|\psi(a)|$, which gives that

$$\lim_{n\to\infty}\left|\psi(0)\psi(\varphi(0))\cdots\psi(\varphi_{n-1}(0))\right|^{\frac{1}{n}} == |\psi(a)| \tag{20}$$

On the other hand, since it holds for all n that

$$\frac{1}{(1-|\varphi_n(0)|^2)^{\frac{1}{2n}}} \le \|C^n_\varphi\|^{\frac{1}{n}} \le \frac{2^{\frac{1}{n}}}{(1-|\varphi_n(0)|^2)^{\frac{1}{2n}}}$$

it ensures that

$$r(C_\varphi) = \lim_{n\to\infty} \frac{1}{(1-|\varphi_n(0)|^2)^{\frac{1}{2n}}} \qquad (21)$$

Hence we obtain from (20) and (21) that $r(C_\varphi)|\psi(a)| \le r(W_{\psi,\varphi}) \le r(C_\varphi)\|\psi\|_\infty$, and so the proof follows from Theorem (1.1.9).

Remark(1.2.7)[1]: In Theorem (1.2.6) ψ and φ have the following form:

$$\psi(z) = \frac{b}{1-a_0 z} \text{ and } \varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z} \qquad (22)$$

from Theorem (1.1.3) Since $\|\psi\|_\infty = \frac{|b|}{1-|a_0|} = \frac{|\psi(0)|}{1-|\varphi(0)|}$, we obtain from Theorem (1.2.6) that

$$r(C_\varphi)\frac{|\psi(0)|}{1-|\varphi(0)|} \le r(W_{\psi,\varphi}) \le r(C_\varphi)\frac{|\psi(0)|}{1-|\varphi(0)|} \quad (23)$$

Where a is the Denjoy–Wolff point of φ. For example, let ψ and φ be defined as in (22).Take $0 < a_0 < 1$ and $a_1 = (1-a_0)^2$. Since $2|a_0 + \overline{a_0}(a_1 - a_0^2)| = 4(a_0 - a_0^2) = 1 - |a_1 - a_0^2|^2$, it follows from Lemma (1.2.8) that φ is a selfmap of D. Moreover, the only fixed point of φ is 1 by a simple computation with formula (28), and so (1) is the Denjoy–Wolff point of φ.

Since$r(W_{\psi,\varphi}) = r(C_\varphi)\frac{|\psi(0)|}{1-\varphi(0)}$ from inequality (23) and

$r(C_\varphi) = \acute{\varphi}(1)^{\frac{-1}{2}} = 1$ we have $r(W_{\psi,\varphi}) = r(C_\varphi)\frac{|\psi(0)|}{1-\varphi(0)}$

If $\|Tx\| = \|T\|\|x\|$ holds for some non-zero $x \in H$, then $T \in L(H)$ is said to *attain its norm on x*. Remark that T attains its norm on x if and only if $T^*Tx = \|T\|^2 x$ Moreover, if T is complex symmetric with conjugation C, then it is easy to show that T attains its norm on x if and only if T* attains its norm on Cx. Next we consider the case of a weighted composition operator that attains its norm on the normalized reproducing kernel $k_w := \frac{k_w}{\|Kw\|}$.

Proposition (1.2.8)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be not identically zero on D. Suppose that $W_{\psi,\varphi}$ is complex symmetric with conjugation J where $(Jf)(z) = f(\bar{z})$If $W_{\psi,\varphi}$ attains its norm on kw for some w ∈ D, then the following statements hold.

(i) $W_{\psi,\varphi} = \frac{|\psi(0)|}{\sqrt{1-|\varphi(0)|^2}} \cdot \frac{1}{\sqrt{1-\overline{w}\varphi(\overline{\varphi(0)})}}$

(ii) If $W_{\psi,\varphi}$ is normal and $\varphi(0) \ne 0$, then w is real and $0 \le w\,\varphi(\overline{\varphi(0)}) < 1$.

Proof: (i) Since $W_{\psi,\varphi}$ is complex symmetric with conjugation J , it follows from Theorem(1.1.3) that $\psi(z) = \frac{b}{1-a_0 z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$ where $a_0 = \varphi(0)$, $a_1 = \acute{\varphi}(0)$, and $b = \psi(0)$. Since $W_{\psi,\varphi}$ attains its norm on k_w, we have

$$\|W_{\psi,\varphi}\|^2 = \langle \|W_{\psi,\varphi}\|^2 K_w, K_0\rangle = \langle W^*_{\psi,\varphi}, W_{\psi,\varphi}K_w, K_0\rangle = (W^*_{\psi,\varphi}, W_{\psi,\varphi}K_w)(0) \quad (24)$$

From Eq. (8) in Corollary (1.1.7) with $z = 0$ and (24), we get the following;

$$\|W_{\psi,\varphi}\|^2 = \frac{|b|^2}{[-a_0 - \overline{a_0}(a_1 - a_0^2)]\overline{w} + 1 - |a_0|^2}$$

$$= \frac{|b|^2}{1-|a_0|^2} \cdot \frac{1}{1-\overline{w}\left(\frac{a_0+\overline{a_0}(a_1-a_0^2)}{1-|a_0|^2}\right)}$$

$$= \frac{|\psi(0)|^2}{1-|\varphi(0)|^2} \cdot \frac{1}{1-\overline{w}\,\varphi(\overline{\varphi(0)})}$$

(ii) If $W_{\psi,\varphi}$ is normal, then $\varphi(\overline{\varphi(0)})$ is real by Corollary (1.1.7) Thus we obtain from Lemma (1.2.5) that

$$\frac{|\psi(0)}{\sqrt{1-|\varphi(0)|^2}} \le \|W_{\psi,\varphi}\| = \frac{|\psi(0)}{\sqrt{1-|\varphi(0)|^2}} \cdot \frac{1}{\sqrt{1-\overline{w}\,\varphi(\overline{\varphi(0)})}}$$

which implies that $0 \le \overline{w}\varphi(\overline{\varphi(0)}) < 1$. In particular, $\overline{w}\,\varphi(\overline{\varphi(0)})$ is real, and so is w.

In Theorem (1.1.3) we assume that the function φ maps the disk into itself. We next consider which combinations of a 0 and a1 provide a mapping of the disk into itself.

Lemma (1.2.9)[1]: Let $\varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}$. Then φ maps the open unit disk into itself if and only if $|a_0| < 1$ and $2|a_0 + a_0(a_1 - a_0^2)| \le 1 - |a_1 - a_0^2|^2$, i.e.,

$$|\varphi(\overline{a_0})| \le \frac{1-|a_1-a_0^2|^2}{2(1-|a_0|^2)} .$$

In particular, when $a_1 = a_0^2 \ne 0$, φ maps the open unit disk into itself if and only if $|a_0| \le \frac{1}{2}$, and when $a_1 - a_0^2 = \pm 1$, φ maps the open unit disk into itself if and only if a_0 is either real or purely imaginary.

Proof: If $a_0 = 0$, then $\varphi(z) = a_1 z$. Hence it is clear that φ is a selfmap of D if and only if $|a_1| \le 1$. If $a_1 = 0$, then $\varphi(z) = a0$, and so φ is a selfmap of D if and only if $|a_0|<1$. In addition, the inequality $2|a_0+\overline{a_0}(a_1 - a_0^2)| \le 1-|a_1-a_0^2|^2$ holds.

Thus we may assume that $a_0 \neq 0$ and $a_1 \pm 0$. Note that $|\varphi(z)|<1$ for each $z \in D$ if and only if

$$|a_0 + (a_1 - a_0^2)\, z|^2 < |1 - a_0 z|^2$$

Which is equivalent to

$$(a_0 + (a_1 - a_0^2)\, z)\, \overline{(a_0 + (a_1 - a_0^2)} < (1 - a_0 z)\,(1 - \overline{a_0}\, z)$$

For all $z \in D$. This means that

$$(|(a_1 - a_0^2|^2 - |a_0|^2)|z|^2 + 2\mathrm{Re}\{\overline{a_0}(a_1 - a_0^2) + a_0)z\} + |a_0|^2 - 1 < 0.$$

If $a_0 + \overline{a_0}(a_1 - a_0^2) = 0$, then $\varphi(z) = \frac{a_0}{\overline{a_0}} \frac{\overline{a_0} - z}{1 - a_0 z}$ is a selfmap of D and $2|a_0 + \overline{a_0}(a_1 - a_0^2)| = 0 = 1 - |a_1 - a_0^2|^2$. Hence we may assume that $a_0 + \overline{a_0}(a_1 - a_0^2) \neq 0$. Choose $\theta \in R$ so that $[a_0 + \overline{a_0}(a_1 - a_0^2)]e^{i\theta} > 0$. Set $A = [a_0 + \overline{a_0}(a_1 - a_0^2)]e^{i\theta} = |a_0 + \overline{a_0}(a_1 - a_0^2)|$ and define

$$\tilde{\varphi}(z) = e^{-i\theta}\varphi(e^{i\theta}z) = \frac{(a_1 - a_0^2)z + a_0 e^{-i\theta}}{1 - a_0 e^{i\theta} z}.$$

Then it is trivial that $\varphi(D) \subset D$ if and only if $\tilde{\varphi}(D) \subset D$.

Claim. $\tilde{\varphi}(D) \subset D$ if and only if $|a_0| < 1$ and

$$|\tilde{\varphi}(\zeta)| \leq 1 \text{ for all } \zeta \in \partial D \qquad (25)$$

Suppose that $\tilde{\varphi}(D) \subset D$ and let $\zeta \in \partial D$. Then $|a_0| = |\tilde{\varphi}(0)| < 1$. In addition, since there is a sequence $\{z_n\} \subset D$ such that $\lim_{n\to\infty} z_n = \zeta$, we get that

$$|\tilde{\varphi}(\zeta)| = \lim_{n\to\infty} |\tilde{\varphi}(z_n)| \leq 1.$$

Conversely, assume that $|a_0| < 1$ and $|\tilde{\varphi}(\zeta)| \leq 1$ for all $\zeta \in \partial D$. Since $|a_0| < 1$ and $a1 \neq 0$, it follows that $\tilde{\varphi}$ is nonconstant and analytic on $\overline{D}$. Hence it holds for any $z \in D$ that

$$|\tilde{\varphi}(z)| \leq \max_{\zeta \in \overline{D}} |\tilde{\varphi}(\zeta)| = \max_{\zeta \in \partial D} |\tilde{\varphi}(\zeta)| \leq 1,$$

and so $|\tilde{\varphi}(z)| < 1$ by the open mapping theorem, which completes the proof of our claim.

From the above claim, it suffices to show that inequality (25) holds if and only if $2|a_0 + \overline{a_0}(a_1 - a_0^2)| \leq 1 - |a_1 - a_0^2|^2$. By a simple calculation, we obtain the following condition equivalent to inequality (25):

$$2\,\mathrm{Re}\,\{[a_0 + \overline{a_0}(a_1 - a_0^2)]e^{i\theta}\zeta\} + |a_1 - a_0^2|^2 - 1 \leq 0 \qquad (26)$$

for all $\zeta \in \partial D$. Since $A = [a_0 + \overline{a_0}(a_1 - a_0^2)]\, e^{i\theta} > 0$, we can replace (26) by the following inequality:

$$\mathrm{Re}(\zeta) \le \frac{1 - |a_1 - a_0^2|^2}{2A} \text{ for any } \zeta \in \partial D \qquad (27)$$

If $2|a_0 + \overline{a_0}(a_1 - a_0^2)| \le 1 - |a_1 - a_0^2|^2$, then $\frac{1 - |a_1 - a_0^2|^2}{2A} \ge 1$. Thus inequality (27) is clearly true. Otherwise, we have $\frac{1 - |a_1 - a_0^2|^2}{2A} < 1$, and so (27) does not hold for $\zeta \in \partial D$ with $\frac{1 - |a_1 - a_0^2|^2}{2A} < \mathrm{Re}(\zeta) \le 1$. So we complete our proof.

Lemma (1.2.10)[1]: Let $\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$ be a selfmap of D where a_0 and a_1 are nonzero complex numbers. If $a_1 - a_0^2 - 1 = 2\gamma a_0$ for some real number γ where $|\gamma| > 1$, then φ has two fixed points inside and outside the unit circle ∂D. Moreover, if $a_1 - a_0^2 = \pm 2a_0$, then φ has one fixed point lying on ∂D. In both cases, we have $r(C_\varphi) = 1$.

Proof: Note that the fixed points of φ are

$$\frac{-(a_1 - a_0^2 - 1) \neq \sqrt{(a_1 - a_0^2 - 1)^2 - 4a_0^2}}{2a_0} \qquad (28)$$

If $a_1 - a_0^2 - 1 = 2\gamma a_0$ for some real number γ where $|\gamma| > 1$, then

$$\frac{-(a_1 - a_0^2 - 1) \neq \sqrt{(a_1 - a_0^2 - 1)^2 - 4a_0^2}}{2a_0} = -\gamma \pm \sqrt{\gamma 2 - 1}$$

are two distinct fixed points of φ. Note that $(-\gamma + \sqrt{\gamma 2 - 1})(-\gamma - \sqrt{\gamma 2 - 1}) = 1$. If $\gamma > 1$, then $-\gamma - \sqrt{\gamma 2 - 1} < -1$, and so $-1 < -\gamma + \sqrt{\gamma 2 - 1} = \frac{1}{-\gamma - \sqrt{\gamma 2 - 1}} < 0$. If $\gamma < -1$, then $-\gamma + \sqrt{\gamma 2 - 1} > 1$, and so $0 < -\gamma - \sqrt{\gamma 2 - 1} = \frac{1}{-\gamma + \sqrt{\gamma 2 - 1}} < 1$. Hence it follows that φ has two fixed points inside and outside ∂D and $r(C_\varphi) = 1$.

Suppose that $a_1 - a_0^2 - 1 = \pm 2a_0$. If $a_1 - a_0^2 - 1 = 2a_0$, then we obtain that

$$\frac{-(a_1 - a_0^2 - 1) \neq \sqrt{(a_1 - a_0^2 - 1)^2 - 4a_0^2}}{2a_0} = -1$$

is the Denjoy–Wolff point of φ and so $r(C_\varphi) = \acute{\varphi}(-1)^{-½} = 1$. similarly, if $a_1 - a_0^2 - 1 = -2a_0$, then we get that 1 is the Denjoy–Wolff point of φ and $r(C_\varphi) = \acute{\varphi}(1)^{-½} = 1$.

Theorem (1.2.11)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^{\infty}(D)$ be not identically zero on D. If $W_{\psi,\varphi}$ is complex symmetric with conjugation J and

$$2|a_0 + \overline{a_0}(a_1 - a_0^2)| < 1 - |a_1 - a_0^2|^2$$

where $a_0 = \varphi(0)$ and $a_1 = \varphi'(0)$, then $W_{\psi,\varphi}$ is Hilbert–Schmidt.

Proof: Since $W_{\psi,\varphi}$ is complex symmetric with conjugation J , from Theorem (1.1.3) we can write $\psi(z) = \frac{b}{1-a_0 z}$ and $\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$ where $a_0 = \varphi(0)$, $a_1 = \varphi'(0)$, and $b = \psi(0)$. If $a_0 = 0$, then $\varphi(z) = a_1 z$ and $|a_1| < 1$, which implies that $\overline{\varphi(D)} \subset D$.

we obtain that $W\psi,\varphi$ is Hilbert–Schmidt. Now assume $a_0 \neq 0$. If $a_0 + \overline{a_0}(a_1 - a_0^2) = 0$, then $|a_1 - a_0^2| = |-\frac{a_0}{\overline{a_0}}| = 1$, which gives a contradiction such as $2|a_0 + \overline{a_0}(a_1 - a_0^2)| < 1 - |a_1 - a_0^2|^2 = 0$. Hence $a_0 + \overline{a_0}(a_1 - a_0^2) \neq 0$. Note that $\varphi(\partial D) \subset D$ if and only if $\tilde{\varphi}(\partial D) \subset D$ where $\tilde{\varphi} z) = e^{-i\theta}\varphi(e^{i\theta} z) = \frac{(a_1 - a_0^2)z + a_0 e^{-i\theta}}{1 - a_0 e^{i\theta} z}$ and the real number θ is taken so that $[a_0 + \overline{a_0}(a_1 - a_0^2)]e^{i\theta} = |a_0 + \overline{a_0}(a_1 - a_0^2)| > 0$. Moreover, the inclusion $\tilde{\varphi}(\partial D) \subset D$ means that $|(a_1 - a_0^2)\zeta + a_0 e^{-i\theta}| < |1 - a_0 e^{i\theta}\zeta|$ for all $\zeta \in \partial D$, which is equivalent to

$$\mathrm{Re}(\zeta) < \frac{1 - |a_1 - a_0^2|^2}{2|a0 + \overline{a_0}(a_1 - a_0^2)|} \text{ for all } \zeta \in D \qquad (29)$$

as in the proof of Lemma (1.2.8) Since $\frac{1 - |a_1 - a_0^2|^2}{2|a_0 + \overline{a_0}(a_1 - a_0^2)|} > 1$, inequality (29) holds. Thus we get that $\varphi(\partial D) \subset D$. Since φ is a linear fractional selfmap of D, it ensures that $\varphi(\partial D)$ is a circle contained in D, and $\varphi(D)$ is the open disk whose boundary is $\varphi(\partial D)$. Hence it follows that $\overline{\varphi(D)} \subset D$. Therefore we conclude that $W_{\psi,\varphi}$ is Hilbert–Schmidt .

Corollary(1.2.12)[1]: Under the same hypotheses as in Theorem (1.2.10), the following assertions hold.

(i) $W_{\eta,\varphi}$ is Hilbert–Schmidt for any $\eta \in H^{\infty}(D)$.

(ii) If φ has the Denjoy–Wolff point a in D, then we have

$$\sigma(W_{\psi,\varphi}) = \{ 0, \psi(a)\varphi'(a)^j : j = 0, 1, 2, \cdots \}$$

(iii) $(0 \in \sigma_{ap}(W_{\psi,\varphi}) \cap \sigma_{su}(W_{\psi,\varphi})$.

Proof: (i) We know from the proof of Theorem (1.2.10) that $\overline{\varphi(D)} \subset D$. Therefore the proof follows .

(ii) Since $W_{\psi,\varphi}$ is compact by Theorem (1.2.10), the Fredholm alternative theorem and Theorem (1.2.3) imply that $\sigma(W_{\psi,\varphi}) = \{ 0, \psi(a)\varphi'(a)^j : j = 0, 1, 2, \cdots \}$

(iii) The proof follows from the statement (ii).

We study cyclic weighted composition operators which are complex symmetric. The concept of cyclicity is closely related to the invariant subspace problem. Indeed, T has a noncyclic vector if and only if it has a ontrivial invariant subspace. Similarly, T has a non hypercyclic vector if and only if it has a nontrivial invariant closed subset. We start our program with the following theorem.

Theorem (1.2.13)[1]: Let φ be a non constant analytic selfmap of D with φ(a) = a for some a ∈ D, and let $\psi \in H^\infty(D)$ be not identically zero. If $W_{\psi,\varphi}$ is complex symmetric, then the following assertions are valid.

(i) If φ is not an elliptic automorphism, then both $W_{\psi,\varphi}$ and $W^*_{\psi,\varphi}$ are cyclic operators.

(ii) Neither $W_{\psi,\varphi}$ nor $W^*_{\psi,\varphi}$ is hypercyclic.

Proof: (i) Let $z_0 \in D$ be an arbitrary point with $z_0 \neq a$, and let $g \in H^2(D)$ be such that $g \perp V_{n=0}^{\infty} \{(W^*_{\psi,\varphi})^n K_{z_0}\}$. Then it is clear that $0 = \langle g, K_{z_0}\rangle$ $= g(z_0)$. Furthermore, since it holds that for any positive integer n

$$W^n_{\psi,\varphi} = W_{\psi\cdot(\psi o\varphi)\cdot(\psi o\varphi_2)\cdots(\psi o\varphi_{n-1}),\varphi_n}$$

Where φ_0 is the identical function on D, we obtain that

$$0 = \langle g, (W^*_{\psi,\varphi})^n K_{z_0}\rangle = \langle W^n_{\psi,\varphi}\, g, \; K_{z_0}\rangle$$

$$= \psi(z_0)\psi(\varphi(z_0))\, \psi(\varphi_2(z_0))\ldots\, \psi(\varphi_{n-1}(z_0))\, g(\varphi_n(z_0))$$

for any positive integer n. Since ψ has no zeros in D by Lemma (1.1.1), it follows that $g(\varphi_n(z_0)) = 0$ for any positive integer n. Notice that the sequence $\{\varphi_n(z_0)\}_{n=0}^{\infty}$ consists of pairwise distinct points in D which converges to a. Thus $g \equiv 0$ by the identity theorem, and so we have $V_{n=0}^{\infty} \{(W^*_{\psi,\varphi})^n K_{z_0}\} = H^2(D)$. Since $W_{\psi,\varphi}$ is complex symmetric, there is a conjugation C such that $W_{\psi,\varphi}C = CW^*_{\psi,\varphi}$, which implies that

$$V_{n=0}^{\infty} \{W^n_{\psi,\varphi}(CK_{z_0})\} = CV_{n=0}^{\infty} \{(W^*_{\psi,\varphi})^n K_{z_0}\} = CH^2(D) = H^2(D).$$

Hence both $W_{\psi,\varphi}$ and $W^*_{\psi,\varphi}$ are cyclic.

(ii) Suppose that $W_{\psi,\varphi}$ is a complex symmetric operator with conjugation C. Then we have

$$W_{\psi,\varphi}(CK_a) = CW^*_{\psi,\varphi}K_a = C(\overline{\psi(a)}K_{\varphi(a)}) = \psi(a)CK_a.$$

Since $C^2 = I$ and $K_a \neq 0$, we have $CK_a \neq 0$, and so $\psi(a) \in \sigma_p(W_{\psi,\varphi})$. Hence $W^*_{\psi,\varphi}$ is not hypercyclic. Since $W_{\psi,\varphi}$ is complex symmetric, we conclude that $W_{\psi,\varphi}$ is not hypercyclic.

From the following example, we observe that Theorem (1.2.12) provides some criteria for a weighted composition operator to be complex symmetric operators.

Example(1.2.14)[1]: Let $\varphi(z) = \frac{z}{2-z}$. Then it is clear that φ is a nonconstant analytic selfmap of D with φ(0) = 0 and is not an elliptic automorphism. We know that $C^*_\varphi$is cyclic, but C_φ is not cyclic. Since $C_\varphi = W_{1,\varphi}$, we obtain from Theorem (1.2.12) that C_φ is not a complex symmetric operator.

Next we give an example for the assertion (ii) of Theorem (1.2.12).

Example(1.2.15)[1]: Let $a_0 \in$ D. If $\varphi(z) = a_0 + \frac{a_1 z}{1 - a_0 z}$ maps the unit disk into itself and $\psi(z) = \frac{b}{1 - a_0 z}$ where $a_1 = a_0^2 - 1$ and $b = \sqrt{1 - a_0^2}$, then $W_{\psi,\varphi}$ is complex symmetric

from Theorem (1.1.3) and ψ is not identically zero by our assumption. In particular, if $a_0 = 0$, then $W_{\psi,\varphi} = bC_{a1z}$ and so $W_{\psi,\varphi}$ is normal. Thus it is not hypercyclic and then $W^*_{\psi,\varphi}$ is not hypercyclic .Now assume $a_0 \neq 0$. We note that $\frac{1 \pm b}{a_0}$ are the fixed points of φ and $\frac{1-b}{a_0} \in$ D since a_0 is real or purely imaginary from Lemma (1.2.8) Thus neither $W_{\psi,\varphi}$ nor $W^*_{\psi,\varphi}$ is hypercyclic from Theorem (1.2.12) (ii).

Next we consider some relations between complex symmetry and hypercyclicity of weighted com

position operators $W_{\psi,\varphi}$ when φ has no fixed points in D.

Proposition (1.2.16)[1]: Let φ be a non constant univalent analytic selfmap of D with no fixed points in D, and let $\psi \in H^\infty$ be not identically zero. If there exists an outer function $\in H^\infty$ such that $W_{\psi,\varphi}g = \lambda g$ for some complex number $\lambda \in \partial D$, then $W_{\psi,\varphi}$ is hypercyclic, but $W^*_{\psi,\varphi}$ is not. Hence $W_{\psi,\varphi}$ is not complex symmetric.

Proof: Since $W_{\psi,\varphi}g = \lambda g$ for some $\lambda \in \partial D$, we obtain that

$$W_{\psi,\varphi}M_g f = \psi \cdot (g \circ \varphi) \cdot (f \circ \varphi) = M_g(\lambda C_\varphi)f$$

for all $f \in H^2$, i.e., $W_{\psi,\varphi}M_g = Mg(\lambda C_\varphi)$. If $f \in \ker(M_g)$, then we get that $g(z)f(z) \equiv 0$ on D. Since g never vanishes on D, it ensures that $f(z) \equiv 0$ on D, and so $\ker(M_g) = \{0\}$. Since g is outer, $\ker(M^*_g) = (gH^2)^\perp = (H^2)^\perp = \{0\}$. Since φ is a univalent map without fixed points in D, it follows that λC_φ is hypercyclic. If $F \in H^2$is a hypercyclic vector for λC_φ, then

$$\overline{O(M_g F, W_{\psi,\varphi})} = \overline{M_g O(F, \lambda C_\varphi)} = \overline{M_g H^2} = H^2.$$

Therefore $W_{\psi,\varphi}$ is hypercyclic. On the other hand, since $\lambda \in \sigma_p(W_{\psi,\varphi})$, we obtain that $W^*_{\psi,\varphi}$ is not hypercyclic, and so $W_{\psi,\varphi}$ is not complex symmetric .

Finally, we consider local spectral properties of complex symmetric weighted composition operators.

Theorem(1.2.17)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be not identically zero. Then $W_{\psi,\varphi}$ has the single-valued extension property at 0.

Proof: Suppose that G is any neighborhood of 0 and $f : G \to H^2(D)$ is an analytic function such that

$$(W_{\psi,\varphi} - \lambda)f(\lambda) = 0$$

for any $\lambda \in G$. Then it holds that

$$\psi \cdot C_\varphi(f(\lambda)) = \lambda f(\lambda) \tag{30}$$

for any $\lambda \in G$. Since $f(\lambda) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \lambda^n$ on G, it suffices to show that

$$f^{(n)}(0) \equiv 0 \text{ on} D$$

for all nonnegative integers n. Taking $\lambda = 0$ in Eq. (30), we get that

$$\psi(z) \cdot C_\varphi(f(0))(z) = \psi(z)f(0)(\varphi(z)) = 0$$

for any $z \in D$. Since ψ is not identically zero, there is a nonempty open subset U of D so that $\psi(z) \neq 0$ for $z \in U$, and thus $f(0)(\varphi(z)) = 0$ for $z \in U$. Since $\varphi(U)$ is a nonempty open set by the open mapping theorem, it follows from the identity theorem that

$$f(0) \equiv 0 \text{ on } D.$$

In order to use induction, suppose that $f^{(k)}(0) \equiv 0$ on D for some nonnegative integer k. Differentiating Eq. (30) with respect to λ, we obtain that

$$\psi \cdot C_\varphi(f^{(k+1)}(\lambda)) = (k+1)f^{(k)}(\lambda) + \lambda f^{(k+1)}(\lambda)$$

for $\lambda \in G$. Thus the induction hypothesis implies that $\psi(z)f^{(k+1)}(0)(\varphi(z)) \equiv 0$ on D. As the above argument, we have that

$$f^{(k+1)}(0) \equiv 0 \text{ on } D.$$

Hence $W_{\psi,\varphi}$ has the single-valued extension property at 0.

Corollary (1.2.18)[1]: Let φ be an analytic selfmap of D and let $\psi \in H^\infty(D)$ be not identically zero. If $W_{\psi,\varphi}$ is complex symmetric, then $W^*_{\psi,\varphi}$ has the single-valued extension property at 0.

Proof: Let G be any neighborhood of 0 and let $f : G \to H^2(D)$ be an analytic function such that $(W^*_{\psi,\varphi} - \lambda)f(\lambda) = 0$ for all $\lambda \in G$. Suppose that $W_{\psi,\varphi}$ is complex symmetric with conjugation C. Since $CW^*_{\psi,\varphi} = W_{\psi,\varphi}C$, we get that

$$(W_{\psi,\varphi} - \bar{\lambda})\, Cf(\lambda) = C\,(W^*_{\psi,\varphi} - \lambda)\, f(\lambda) = 0$$

for all $\lambda \in G$. This means that $(W_{\psi,\varphi} - \omega)Cf(\bar{\omega}) = 0$ for all $\omega \in G^*$, where $G^* := \{\bar{\lambda} : \lambda \in G\}$. Fix any $\omega_0 \in G^*$. Since f is analytic at $\overline{\omega_0}$, then we write $f(\lambda) = \sum_{n=0}^{\infty}(\lambda - \overline{\omega_0})^n f_n$ for all λ in some neighborhood of $\overline{\omega_0}$ and $f_n \in H^2(D)$. Thus for all ω in some neighborhood of ω_0,

$$Cf(\bar{\omega}) = C\left(\sum_{n=0}^{\infty}(\bar{\omega} - \overline{\omega_0})^n f_n\right) = \sum_{n=0}^{\infty}(\bar{\omega} - \overline{\omega_0})^n\, Cf_n,$$

which means that $Cf(\bar{\omega})$ is analytic at ω_0. Since $W_{\psi,\varphi}$ has the single-valued extension property at 0, it follows from Theorem (1.2.16) that $Cf(\bar{\omega}) = 0$ or all $\omega \in G^*$, that is, $Cf(\lambda) = 0$ for all $\lambda \in G$. Since $C^2 = I$, it ensures that $f(\lambda) = 0$ for all $\lambda \in G$. Hence $W^*_{\psi,\varphi}$ has the single-valued extension property at 0.

Corollary (1.2.19)[1]: Let φ be a non constant analytic selfmap of D and let $\psi \in H^\infty(D)$ be not identically zero. If $W_{\psi,\varphi}$ is complex symmetric, then the following properties hold.

(i) Either $0 \in \rho(W_{\psi,\varphi})$ or $0 \in \sigma_{ap}(W_{\psi,\varphi}) \cap \sigma_{su}(W_{\psi,\varphi})$.
(ii) If $W_{\psi,\varphi}$ has closed range, then it is invertible.

Proof: (i) Since both $W_{\psi,\varphi}$ and $W^*_{\psi,\varphi}$ have the single-valued extension property at 0 follows that $0 \notin \sigma(W_{\psi,\varphi}) \setminus \sigma_{ap}(W_{\psi,\varphi})$ and $0 \notin \sigma(W_{\psi,\varphi}) \setminus \sigma_{su}(W_{\psi,\varphi})$. Hence we obtain that $0 \notin \sigma(W_{\psi,\varphi}) \setminus [\sigma_{ap}(W_{\psi,\varphi}) \cap \sigma_{su}(W_{\psi,\varphi})]$, which completes the proof.

(ii) Since $W_{\psi,\varphi}$ has closed range and $\ker(W_{\psi,\varphi}) = \{0\}$ from Proposition (1.1.2), we have $0 \notin \sigma_{ap}(W_{\psi,\varphi})$. Thus $0 \in \rho(W_{\psi,\varphi})$ by the statement (i).

Corollary (1.2.20)[1]: Suppose that $W_{\psi,\varphi}$ is complex symmetric where φ is an analytic selfmap of D and $\psi \in H^{\infty}(D)$ is not identically zero. If S is a bounded linear operator satisfying that $\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} W^*_{\psi,\varphi} S^{k-j} = 0$ for some positive integer k, then S has the single-valued extension property at 0.

Proof: Let G be a open neighborhood of 0. If $f : G \to H^2(D)$ is an analytic function such that $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in G$, then we note that

$$
\begin{aligned}
(W^*_{\psi,\varphi} - \lambda)^k f(\lambda) &= \left((W^*_{\psi,\varphi} - \lambda\right)^k f(\lambda) - \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} W^*_{\psi,\varphi} S^{k-j} \\
&= (W^*_{\psi,\varphi} - \lambda)^k f(\lambda) - \sum_{j=0}^{k} \binom{k}{j} (W^*_{\psi,\varphi} - \lambda)^j (\lambda - S)^{k-j} f(\lambda) \\
&= -\sum_{j=0}^{k-1} \binom{k}{j} (W^*_{\psi,\varphi} - \lambda)^j (\lambda - S)^{k-j} f(\lambda) \\
&= \sum_{j=0}^{k-1} \binom{k}{j} (W^*_{\psi,\varphi} - \lambda)^j (\lambda - S)^{k-j-1} (S - \lambda) f(\lambda) = 0 \qquad (31)
\end{aligned}
$$

for all $\lambda \in G$. Since $W^*_{\psi,\varphi}$ has the single-valued extension property it follows that $f(\lambda) = 0$ for all $\lambda \in G$. Therefore S has the single-valued extension property at 0.

CHAPTER 2

A Spectral Radius for Approximation Number of Composition Operators

We show the approximation numbers of composition operators on weighted analytic Hilbert spaces, including the hardy, Bergman and Dirichlet cases with symbol of uniform norm .

Section (2.1) Background and Framework:

The determination of the approximation numbers of composition operators on Hilbert spaces of analytic functions on the unit disk (Hardy space, weighted Bergman space, Dirichlet space) is a difficult problem. Some partial results show that no simple answer may be expected. However we proved that these approximation numbers cannot decay faster than geometrically: we always have $a_n(C_\varphi) \geq c\, r^n$ or some constant c > 0 andsome $0 < r < 1$. Moreover, we showed in those papers that $\lim_{n\to\infty}[a_n(C_\varphi)]^{1/n} = 1$if and only if $\|\varphi\|_\infty = 1$.

The quantity $\lim_{n\to\infty}[a_n(C_\varphi)]^{1/n} = 1$ looks like a spectral radius formula for theapproximation numbers. Recall that if T is a bounded operator on a complex Hilbert space H, with spectrum σ(T), the classical spectral radius formula tells that for the spectral radius$r\,(T) := \sup_{\lambda\,\in\,\sigma(T)}|\lambda|$, one has the formula:

$$r(T) = \lim_{n\to\infty}\|T^n\|^{1/n}$$

(the existence of the limit being part of the conclusion).

Now, if $a_n = a_n(T)$ is the n-th approximation number of a bounded operator T on a Hilbert space H, it was shown by taking a rank-one perturbation of an n-dimensional shift, that, given $0 < \sigma < 1$, we can have $a_1 = \ldots = a_{n-1} = 1$, and $a_n = \sigma$. Using orthogonal blocks of such normalized operators, one easily builds examples of compact operators T for which the quantity $[a_n(T)]^{1/n}$has no limit as n goes to infinity, and indeed satisfies:

$$\liminf_{n\to\infty}[a_n(T)]^{1/n} = 0\ ,\ \limsup_{n\to\infty}[a_n(T)]^{1/n} = 1\,.$$

We might as well use a diagonal operator with non-increasing positive diagonal entries ε_n such that $\liminf_n \varepsilon_n^{1/n} = 0$ and $\limsup_n \varepsilon_n^{1/n} = 1$. Nevertheless, the parameters

$$\beta^-(T) = \liminf_{n\to\infty}[a_n(T)]^{1/n}, \ \beta^+(T) = \limsup_{n\to\infty}[a_n(T)]^{\frac{1}{n}} \qquad (1)$$

Which satisfy $0 \le \beta^-(T) \le \beta^+(T) \le 1$ are similar to the term $\lim_{n\to\infty}\|T^n\|^{1/n}$in the spectral radius formula. When the limit exists we will denote it by:

$$\beta(T) = \lim_{n\to\infty}[a_n(T)]^{\frac{1}{n}} \qquad (2)$$

These parameters were shown to play an important role in the study of composition operators. As said above, the following was proved in these section.

Theorem (2.1.1)[2]: Let H be a weighted Bergman space B_α (in particular the Hardy space H^2) or the Dirichlet space D and φ: D → D inducing a composition operator $C_\varphi : H \to H$. Then:

(i) if $0 < \|\varphi\|_\infty < 1$, one has $0 < \beta^-(C_\varphi) \le \beta^+(C_\varphi) < 1$;

(ii) if $\|\varphi\|_\infty = 1$, one has $\beta(C_\varphi) = 1$.

The aim of this work is to complete this result by showing that $\beta(C_\varphi)$ existsas well when $\|\varphi\|_\infty < 1$ and to give a closed formula for this $\beta(C_\varphi)$ in termsof a Green capacity, relying on a basic work in the above theorem.

We end the paper with some words on the H^p case for $1 \le p < \infty$.We begin by giving notations, definitions and facts which will be used throughout this work.

Recall that if X and Y are two Banach spaces of analytic functions on the unit disk D, and φ: D → D is an analytic self-map of D, one says that φ induces a composition operator $C_\varphi: X \to Y$ if $f \circ \varphi \in Y$ for every $f \in X$; φ is then called the symbol of the composition operator. One also says that φ is a symbol for X and Y if it induces a composition operator $C_\varphi: X \to Y$.

For an operator T:X → Y between Banach spaces X and Y , its approximation numbers are defined, for n ≥ 0, as:

$$a_n(T) = \inf_{\operatorname{rank} R < n} \|T - R\| \qquad (3)$$

One has $\|T\| = a_1(T) \ge a_2(T) \ge \cdot\cdot\cdot \ge a_n(T) \ge a_{n+1}(T) \ge \cdots$, and (assuming that Y has the Approximation Property), T is compact if and only if $a_n(T) \xrightarrow[n\to\infty]{} 0$.

The n-th Kolmogorov number $d_n(T)$ of T is defined:

$$d_n(T) = \inf_{\substack{E \subseteq Y \\ \dim E < n}} \left[\sup_{x\in B_X} \operatorname{dist}(Tx, E)\right] = \inf_{\substack{E \subseteq Y \\ \dim E < n}} \|Q_E T\|_{Y/E} \qquad (4)$$

where $Q_E : Y \to \frac{Y}{E}$ is the quotient map. One always has $a_n(T) \geq d_n(T)$ and,when X and Y are Hilbert spaces, one has $a_n(T) = d_n(T)$.As usual, the notation $A \lesssim B$ means that there is a constant c such that $A \leq C B$.

An analytic Hilbert space H on D is a Hilbert space $H \subset Hol(D)$, the analytic functions on the unit disk D, for which the evaluations $f \mapsto f(a)$ are continuous on H for all $a \in D$ and therefore given by a scalar product:

$$f(a) = \langle f, K_a \rangle, \quad K_a \in H.$$

Since weakly convergent sequences of H are norm-bounded, the reproducing kernels K_a are automatically norm-bounded on compact subsets of D, that is:

$$L_r := \sup_{|a|\leq r} \|K_a\| < \infty, \quad \text{for all } r < 1 \qquad (5)$$

We will be slightly less general here, and adopt the framework. Let

$\omega : [0, 1) \to (0,\infty)$ be a continuous, positive, and Lebesgue-integrable function.

We extend this function to a radial weight on D by setting $\omega(z) = \omega(|z|)$. Wedenote by H_ω the space of analytic functions on D such that

$$\|f\|_\omega^2 := |f(0)|^2 + \int_D |f'(z)|^2 \, \omega(z) \, dA(z) < +\infty,$$

where dA stands for the normalized area measure on D. We will often omit the subscript ω and write $\|.\|$ for $\|.\|_\omega$.

If $f(z) = \sum_{n=0}^{\infty} b_n z^n$, a computation in polar coordinates shows that:

$$\|f\|^2 = \sum_{n=0}^{\infty} |b_n|^2 \omega_n \qquad (6)$$

where:

$$\omega_0 = 1 \text{ and } \omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) \, dr, \qquad n \geq 1 \qquad (7)$$

Observe that there is a constant $C = C(\omega) \geq 1$ and, for each $\varepsilon > 0$, a $\delta_\varepsilon > 0$ such that:

$$\delta_\varepsilon e^{-\varepsilon n} \leq w_n \leq C n^2, \qquad n \geq 1 \qquad (8)$$

Indeed, in one side, one has $w_n \leq 2n^2 \int_0^1 \omega(r)\, dr$, and, on the other side, foreach $0 < \delta < 1$, setting $c_\delta = \inf_{0\leq r\leq\delta} \omega(r)$, we have $c_\delta > 0$ and:

$$w_n \geq 2n^2 c_\delta \int_0^\delta r^{2n-1}\, dr = c_\delta\, n\, \delta^{2n},$$

giving (8). This shows in passing that H_ω is an analytic Hilbert space, and we call it a weighted analytic Hilbert space. This framework is sufficiently general for our purposes and includes for example the case of (weighted) Bergman, Hardy, and Dirichlet spaces, corresponding to $\omega(r) = (1 - r^2)^{\alpha}, \alpha > -1$, that is $w_n \approx n^{1-\alpha}$. The standard Bergman, Hardy, Dirichlet spaces correspond to the respective values α = 2, 1, 0.

The following simple fact will be used. Let $a \in D$ and $j \geq 0$; the

$f \mapsto$

is a continuous linear form on H.

This holds for any analytic Hilbert space on D, and here can also be viewed as a consequence of (8).

An analytic self-map $\varphi : D \to D$ which induces a composition operator C_φ: $H \to H$ will be called a symbol for $H = H_\omega$. In our space H, we have a quite easy case for deciding if some φ is a symbol.

Lemma (2.1.2)[2]: If $\|\varphi\|_\infty < 1$, then φ is a symbol if and only if $\varphi \in H$. Equivalently, if and only if the positive measure $\mu = |\varphi'|^2 \omega\, dA$ is finite. In that case, we moreover have $\|\varphi^k\| \leq C\, k\, \|\varphi\|_\infty^k$ for every $k \geq 1$.

Proof: If φ is a symbol, then $\varphi = C_\varphi(z) \in H$. Conversely, let $\rho = \|\varphi\|_\infty < 1$. We first note that, if $\varphi \in H$, we have for any integer $k \geq 1$:

$$\|\varphi^k\|^2 = |\varphi(0)|^{2k} + \int_D \omega(z) k^2 |\varphi(z)|^{2(k-1)} |\varphi'(z)|^2 \, dA(z) \leq \rho^{2k}(1 + k^2\rho^{-2})\, \|\varphi\|^2 \qquad (10)$$

Now, let $\varepsilon > 0$ be such that $\rho e^{\varepsilon} < 1$. If $f(z) = \sum b_k z^k \in B_H$, the unit ball of H, we have by (8): $|b_k| \leq w_k^{-1/2} \leq C_\varepsilon e^{k\varepsilon}$, so that, using (10), we see that the series $\sum b_k \varphi^k = f \circ \varphi$ converges absolutely in H, which proves that C_φ is compact (and even nuclear).

The Green function $g: D \times D \to (0,\infty]$ of the unit disk D is defined as:

$$g(z, w) = log \left|\frac{1-\bar{w}z}{z-w}\right| \quad (11)$$

If μ is a finite positive Borel measure on D with compact support in D, its Green potential is:

$$G_\mu(z) = \int_D g(z, w)\, d\mu(w) \quad (12)$$

and its energy integral is:

$$I(\mu) = \iint_{D\times D} g(z, w)\, d\mu(z)\, d\mu(w) \quad (13)$$

Of course,

$$I(\mu) = \int_D G_\mu(z)\, d\mu(z) \quad (14)$$

For any subset E of D, one sets:

$$V(E) = \inf_\mu I(\mu) \quad (15)$$

where the infimum is taken over all probability measures μ supported by a compact subset of E. Then the Green capacity[1] of E in D is:

$$Cap\,(E) = 1/V\,(E) \quad (16)$$

If K ⊆ D is compact, the infimum in (15) is attained for a probability measure μ_0. If moreover V (K) <∞ (i.e. Cap (K) > 0), this measure is unique and is called the equilibrium measure of K. One always has V (K) <∞ when K has non-empty interior, since then I(λ) <∞ where λ is the normalized planar measure on some open disk Δ⊆K. It is clear that we have:

$$K \subseteq L \Rightarrow V\,(K) \geq V\,(L) \Rightarrow Cap\,(K) \leq Cap\,(L)\,,$$

i.e. Cap (K) increases with K and:

$$Cap\,(E) = \sup_{K\subseteq E, K\ compact} Cap\,(K)\,.$$

We refer to and the clear presentation for the definition of the Green capacity and its basic properties. Actually, the capacity is defined by another way as follows.

Lemma (2.1.3)[2]: For every compact set K ⊆D, one has:

Cap(K) = $sup\,\{\|\mu\|;\ \mu\ positive\ Borel\ measure\ supported\ by\ K\ and\ G_\mu \leq 1\ on\ D\}$

This is the definition of de la Vallée-Poussin. Since our main result is based on H. Widom,s it must be specified that he also used this definition .

Let us note, though we will not use that, that we also have:

Cap (K) = $\inf\{\|\mu\|;\ \mu$ positive Borel measure on D and $G_\mu \geq 1$ on K$\}$

= $\inf\{\|\mu\|;\ \mu$ positive Borel measure on D and $G_\mu \geq 1$ q. e. on K$\}$,

Where q.e. means: out of a set of null capacity. The equivalence between these two definitions is shown.

An important fact for this chapter is well-known to specialists on the (Green) capacity.

Theorem(2.1.4)[2]: For every connected Borel subset E of D whose closure $\bar{E}$ iscontained in D, one has:

$$Cap\ (E)\ =\ Cap\ (\bar{E}) \qquad (17)$$

For sake of completeness, we provide details for the reader. We begin with a definition: a subset E of D is said to be thin at u $\in \bar{E}$ if there exists a function s which is superharmonic in a neighbourhood of u and such that

$$s(u)\ <\ \liminf_{\substack{v\to u\\ v\in E}}\ s(v)\ .$$

We denote by $\tilde{E}$the union of E and of points in $\bar{E}$ at which E is not thin (it is known that $\tilde{E}$ is the closure of E for the fine topology). Then:

Lemma (2.1.5)[2]: If E is a connected Borel subset of D whose closure $\bar{E}$ is contained in D, one has:

$$\tilde{E} = \bar{E}\ .$$

Proof: Lemma (2.1.5) is an immediate consequence of the following result.

Theorem (2.1.6)[2]: (Beurling-Brelot) Let E $\subseteq$ D and u $\in \bar{E}$. If E is thin at u, there exist circles with center u and arbitrarily small radius > 0 which do not intersect E.

Indeed, taking the previous result for granted, suppose that E is thin at u $\in \bar{E}$, u $\notin$E, and let $v_0 \in$ E, with $|v_0 - u| = d > 0$. The function ρ: E → R defined by $\rho(v) = |v - u|$ takes the value d as well as arbitrarily small values since u $\in \bar{E}$. By the intermediate value theorem, it takes every value in (0, d], contradicting. This contradiction shows that $\bar{E} \subseteq \tilde{E}$, there by ending the proof .

Section (2.2) Main Result and the Hardy Case:

The goal of this chapter to prove the following result.

Theorem (2.2.7)[2]: Let H be a weighted analytic Hilbert space with norm ‖ . ‖. Letφ: D → D be a symbol for H, with $\overline{\varphi(D)} \subseteq D$. Then

$$\lim_{n\to\infty}[a_n(C_\varphi)]^{1/n} =: \beta(C_\varphi)$$

Exists and the value of this limit is:

$$\beta(C_\varphi) = e^{-1/Cap\,[\varphi(D)]} \qquad (18)$$

Note that, by Theorem (2.1.4), Cap [φ(D)] = Cap [$\overline{\varphi(D)}$], so Theorem (2.2.7) will follow immediately from Theorem (2.2.14) and Theorem (2.2.17) below.

The proof is based on two results of H. Widom .Though those theorems are in the H^∞ setting, we will be able to transfer them to our Hilbertian setting. Before giving this proof, we will check the result "by hand" with an explicit example.

Before going into the proof of Theorem (2.2.7) we are going to illustrate it in a simple situation.

Let φ be a symbol acting on H = H^2 with $\|\varphi\|_\infty < 1$. We know that $\beta^+(C_\varphi) < 1$, and for very special φ,s we will show directly, without appealing to Widom,s results, that (2.2.7) holds.

We have the following two facts .

Lemma (2.2.8)[2]: Let $L = \bar{\Delta}(w, r)$ be a closed pseudo-hyperbolic disk of pseudohyperbolicradius r. Then:

$$Cap\ (L) = \frac{1}{log(1/r)} \qquad (19)$$

Lemma (2.2.9)[2]: Let u, v: D → D be univalent analytic maps such that u(D) = v(D). Then, u = v ∘ ψ where ∈ Aut (D).

Indeed , by hypothesis u = v ∘ ψ with ψ well-defined and holomorphic for v is injective. Moreover, u(D) = v[ψ(D)] = v(D), whence ψ(D) = D, again because v is injective. Finally ψ is injective since u is.

Theorem (2.2.10)[2]: Let $\varphi(z) = \frac{az+b}{cz+d}$ be a fractional linear function mapping D into D, i.e. :

$$|a|^2 + |b|^2 + 2|\bar{a}b - \bar{c}d| \leq |c|^2 + |d|^2 \quad and \quad |c| \leq |d|.$$

Then $\beta(C_\varphi) = exp\left[-\frac{1}{Cap\,(K)}\right]$.

The example φ(z) = z/(2z + 1) shows that one cannot omit the condition |c| ≤ |d|.

Recall that the pseudo-hyperbolic distance on D is defined by:

$$\rho(z,w) = \frac{z-w}{1-\bar{z}w}, \qquad z,w \in D \qquad (20)$$

We denote by Δ(w, r) = {z ∈ D; ρ(z,w) < r} the open pseudo-hyperbolic disk of center w and radius r.

Proof : We may assume $\|\varphi\|_\infty < 1$. We first consider the Case

φ(z) = az, with |a| < 1. In that case, it is clear that $an(C_\varphi) = |a|^{n-1}$, and hence $\beta(C_\varphi) = |a|\ and\ \overline{\varphi(D)} = \bar{D}(0,|a|) = \bar{\Delta}(0,|a|)$. So that (18) holds in view of (19).

In the general case, one might say that the conformal invariance of Cap and β does the rest. Let us provide some details.

In general, φ(D) is an euclidean disk, therefore a pseudo-hyperbolic disk

Δ(w, r) := {z ; ρ (z,w) < r} = ψ_1[Δ(0, r)], where ρ is the pseudo-hyperbolic

distance and ψ_1∈Aut (D); one has the same radius since automorphisms preserve ρ. If h(z) = rz, one therefore has φ(D) = ψ_1[h(D)] (since $\bar{\Delta}$(0, r) andthe euclidean disk $\bar{D}$(0, r) coincide). From Lemma (2.2.9), $\varphi = \psi_1 \circ h \circ \psi_2$ with ψ_2∈ Aut (D), and so = $C_{\psi_2} C_h C_{\psi_1}$, implying

$$\beta(C_\varphi) = \beta(C_h) = r,$$

by the ideal property. Moreover,

$$Cap\,[\varphi(D)] = Cap\,[h(D)]$$

by conformal invariance. Since we know that the desired equality between $\beta$and Cap holds for h, we get the result.

Let us numerically test the claimed value of $\beta(C_\varphi)$ on the affine exampl $\varphi(z) = \varphi_{a,b}(z) = az + b\ with\ a,b > 0\ and\ a + b < 1$

(note that $C_{\varphi_{a,b}}$ and $C_{\varphi_{|a|,|b|}}$ are unitarily equivalent and have the same approximation numbers a_n, so that there is no loss of generality by assuming a, b > 0). In that case, the $a_n(C_\varphi) = a_n$ were computed exactly by Clifford and Dabkowski .Their result is as follows. One sets:

$$\Delta = (a^2 - b^2 - 1)^2 - 4b^2 \; and \; Q = \frac{1 + a^2 - b^2 - \sqrt{\Delta}}{2a^2} \tag{21}$$

Then, one has $a_n = a^{n-1}Q^{n-1/2}$, and so:

$$\beta(C_\varphi) = aQ \tag{22}$$

The result of the theorem can be tested on that example. Indeed, we have $K := \overline{\varphi(D)} = \overline{D}(b, a)$, so that

$$Cap\ (K) = \frac{1}{log\ \lambda},$$

Where $\lambda > 1$ is the biggest root of the quadratic polynomial

$$P(z) = az^2 - (1 + a^2 - b^2)z + a\,.$$

In explicit terms:

$$e^{-1/Cap\,(K)} = \frac{1}{\lambda} = \frac{1 + a^2 - b^2 - \sqrt{\Delta_0}}{2a^2},$$

with:

$$\Delta_0 = (1 + a^2 - b^2)^2 - 4a^2 \tag{24}$$

To get $\beta(C_\varphi) = e^{-1/Cap\,(K)}$, it remains to compare (22) and (18), using (21)and (24), and to observe that

$$\Delta = \Delta_0 = (1 + a + b)(1 + a - b)(1 - a + b)(1 - a - b)\,.$$

We are going to state widom's results in a form suitable for us. We first quote the following lemma .

Lemma (2.2.11)[2]: (Widom) Let $K \subseteq D$ be compact. Then, given $\varepsilon > 0$, there exists a cycle γ, which is a finite union of disjoint Jordan curves contained in D, and whose interior U contains K, and a rational function R of degree < n, having no zero on and all poles on ∂D, such that, for n large enough

(i) $|R(z)| \geq e^{-\varepsilon n}\ for\ z \notin U;$

(ii) $|R(z)| \leq e^{\varepsilon n} e^{-n/Cap\,(K)}\ for\ z \in K.$

The first theorem of Widom in which C(K) denotes the space of complex, continuous functions on K with the sup-norm, can now be rephrased as follows.

Theorem (2.2.12)[2]: (Widom) Let K ⊆ D be a compact set, and ε> 0. Then, there exist a constant $C_\varepsilon > 0$ and, for every integer n large enough, a rational function R with poles on ∂D and points $\zeta_i \in$ D\K such that for every g∈H^∞, one has:

$$\|g - h\|_{C(K)} \leq C_\varepsilon e^{\varepsilon n} e^{-n/Cap\,(K)} \|g\|_\infty \tag{25}$$

where:

$$h(w) = R(w) \sum_{\substack{i,k \\ 1 \leq k \leq m_i}} c_{i,k}(g)\,(w - \zeta_i)^{-k}\; with \sum_i m_i < n$$

and the maps g ∈$H^\infty \mapsto c_{i,k}(g)$ are linear.

Moreover, if H is a weighted analytic Hilbert space, these maps, restricted to $H^\infty \cap$ H, extend to continuous linear forms on H.

Widom,s theorem precisely says the following. If R and γ are the rational function and cycle of Lemma (2.2.12), let ζ_i be the zeros of R inside γ. Consider , for w∈K, the function

$$G(w) = R(w) \left[\frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{R(\zeta)\,(\zeta - w)}\, d\zeta\right];$$

Then, by the residues theorem,

$$G(w) = g(w) - R(w) \sum_{i,k} c_{i,k}(g)\,(w - \zeta_i)^{-k} = g(w) - h(w),$$

and Widom,s theorem says that $\|G\|_{C(K)} \leq C_\varepsilon e^{2\,\varepsilon n}\,[M(K)]^n \|g\|_\infty$.

The only additional remark made here is that the $c_{i,k}$ are of the form

$$c_{i,k}(g) = \sum_{j \leq k \leq m_i - K} \lambda_{i,j,k}\, g^{(j)}(\zeta_i)$$

where $\lambda_{i,j,k}$ are fixed scalars, so that by (9) they extend to continuous linear forms on H.

Observe that the linear forms $g \mapsto c_{i,k}\,(g')$ are also continuous on H since

$$c_{i,k}\,(g') = \textstyle\sum_{j \leq m_i - K} \lambda_{i,j,k}\, g^{(j+1)}(\zeta_i) \tag{26}$$

This observation will be useful later.

Theorem (2.2.13)[2]: (Widom) Let K be a compact subset of D and C(K) be the space of continuous functions on K with its natural norm. Set:

$$\delta_n(K) = \inf_E \left[\sup_{f \in BH^\infty} dist\,(f, E) \right]$$

where E runs over all (n − 1)-dimensional subspaces of C(K) and dist (f, E)= $inf_{h \in E} \| f - h \|_{C(K)}$. Then

$$\delta_n(K) \geq \alpha\, e^{-n/Cap\,(K)} \qquad (27)$$

for some positive constant α.

Theorem (2.2.14)[2]: Let H be an analytic weighted Hilbert space with norm $\|.\|$. Let

φ: D → D be a symbol for H, such that $\|\varphi\|_\infty = \rho < 1$. Then:

$$\beta^+(C\varphi) := \lim_{n\to\infty} sup[a_n(C_\varphi)]^{1/n} \leq e^{-1/Cap\,[\overline{\varphi(D)}]}.$$

Proof: Fix ε> 0 such that $\rho\, e^\varepsilon < 1$.

If $f(z) = \sum_{k=0}^{\infty} b_k z^k \in H$, $let\ g(z) := S_l f(z) = \sum_{k=0}^{l-1} b_k z^k$, $with\ l = l(n)$ be an integer to be adjusted.

Lemma (2.2.15)[2]: We have:

$$\| f \circ \varphi - g \circ \varphi \| \leq K_\varepsilon \rho^l e^{\varepsilon l}.$$

Proof: For f(z) = $\sum_{k=0} b_k z^k$, we have:

$$\| f \circ \varphi - g \circ \varphi \| = \left\| \sum_{k=l}^{\infty} b_k \varphi^k \right\| \leq \sum_{k=l}^{\infty} |b_k| \|\varphi^k\|$$

$$\leq \left(\sum_{k=l}^{\infty} |b_k|^2 w_k \right)^{\frac{1}{2}} \left(\sum_{k=l}^{\infty} \|\varphi^k\|^2 w_k^{-1} \right)^{\frac{1}{2}} \leq K_\varepsilon \rho^l e^{\varepsilon l},$$

by using Cauchy-Schwarz inequality, the fact that $\|f\| \leq 1$, the inequalities(8), and a geometric progression.

Also, remark that we have, by the Cauchy-Schwarz inequality:

$$\|(S_l f)'\|_\infty \leq \sum_{k=0}^{l-1} k\,|b_k| \leq \left(\sum_{k=0}^{l-1} |b_k|^2 w_k\right)^{\frac{1}{2}} \left(\sum_{k=0}^{l-1} k^2 w_k^{-1}\right)^{\frac{1}{2}} \leq \|f\| \left(\sum_{k=0}^{l-1} k^2 w_k^{-1}\right)^{\frac{1}{2}}$$

Therefore, using (8), we see that the linear map $S_l' : H \to H^\infty$, defined by $S_l'(f) = (S_l f)'$, is continuous with a norm less than $\left(\sum_{k=0}^{l-1} k^2 w_k^{-1}\right)^{\frac{1}{2}} \leq K_\varepsilon\, e^{\varepsilon l}$.

We now use Theorem (2.2.12), with $K = \overline{\varphi(D)} \subseteq D$ (and for n − 1 instead of n).Set, for n ≥ 2, large enough:

$$h_1(w) = R(w) \sum_{\substack{i,k \\ 1 \leq k \leq m_i}} c_{i,k}(\acute{g})\,(w - \zeta_i)^{-k} \; with \sum_i m_i < n - 1 \,.$$

Recall that h_1 is analytic in D. Remark that h_1 depends linearly on f and themap f $\mapsto h_1$ has a rank < n − 1. We denote by $I_1 \in$ Hol (D) the primitive of h_1 taking the value g[φ(0)] at φ(0):

$$I_1(z) = \int_{\varphi(0)}^{z} h_1(u)\,du + g[\varphi(0)] \;.$$

Next, define an operator A of rank < n on H (the continuity of A being justified by (2.2.13) by the formula:

$$A(f) = I_1 \circ \varphi \qquad (28)$$

Note that, even if $I_1 \notin$H, we easily check on the integral representation of the norm that $I_1 \circ$ φ∈H since we assumed φ∈H, i.e. that φ is a symbol.

Assuming for the rest of the proof that $\|f\| \leq 1$, we have the following lemma.

Lemma (2.2.16)[2]: We have:

$$\|g \circ \varphi - I_1 \circ \varphi\| \leq K_\varepsilon e^{\varepsilon(n-1)} e^{\varepsilon l} e^{-(n-1)/Cap\,(K)}.$$

Proof: Since φ∈ H and since $h_1 = I_1'$ approximates g′ uniformly on K and $\|g'\|_\infty = \|(S_l f)'\|_\infty \leq K_\varepsilon e^{l}$, we have, by Theorem (2.2.11):

$$\|g\circ\varphi - I_1\circ\varphi\|^2 = \int_D |g'[\varphi(z)] - h_1[\varphi(z)]|^2|\varphi'(z)|^2\omega(z)\, dA(z)$$
$$\leq K_\varepsilon^2 e^{2\varepsilon(n-1)}[M(K)]^{2(n-1)}\|g'\|_\infty^2 \int_D |\varphi'(z)|^2\omega(z)dA(z)$$
$$\leq C\, K_2^\varepsilon e^{2\varepsilon l} e^{2\varepsilon(n-1)}[M(K)]^{2(n-1)}, (with\ C = \|\varphi\|_\omega^2),$$

hence the lemma, provided that we increase K_ε.

We can now end the proof of Theorem (2.2.13).

Writing:

$$\|C\varphi(f) - A(f)\| = \|f\circ\varphi - I_1\circ\varphi\|$$
$$\leq \|f\circ\varphi - g\circ\varphi\| + \|g\circ\varphi - I_1\circ\varphi\|,$$

we have:

(i) $\|f\circ\varphi - g\circ\varphi\| \leq K_\varepsilon \rho^l e^{\varepsilon l}$ *by Lemma* (2.2.15);
(ii) $\|g\circ\varphi - I_1\circ\varphi\| \leq K_\varepsilon e^{\varepsilon(n-1)}[M(K)]^{n-1} e^{\varepsilon l}$ *by Lemma* (2.2.16).

We therefore get, since $a_n := a_n(C_\varphi) \leq \|C_\varphi - A\|$:

$$a_n \leq K_\varepsilon \rho^l e^{\varepsilon l} + K_\varepsilon e^{\varepsilon l} e^{\varepsilon(n-1)}[M(K)]^{n-1}.$$

Next, since $(a + b)^{1/n} \leq a^{1/n} + b^{1/n}$, we infer that:

$$a_n^{1/n} \leq (K_\varepsilon)^{1/n}(\rho\, e^\varepsilon)^{l/n} + K_\varepsilon^{\frac{1}{n}} e^{\frac{\varepsilon l}{n}} e^{\frac{\varepsilon(n-1)}{n}} M(K)^{\frac{n-1}{n}} \qquad (29)$$

We now adjust l = Nn, where N is a fixed positive integer, and pass to the upper limit with respect to n in (28). We get:

$$L := \lim\sup a_n^{1/n} \leq [\rho\, e^\varepsilon]^N + e^\varepsilon e^{\varepsilon N} M(K).$$

Letting ε go to 0, we get $L \leq \rho^N + M(K)$.Finally, letting N tend to infinity, we get L ≤ M(K) as claimed, and that ends the proof of Theorem (2.2.13).

Lemma (2.2.17)[2]: For every Hilbert space H and every compact operator T:H→H,one has, BH denoting the unit ball of H:

$$d_n(T) = \inf_{dimE<n}\left[\sup_{f\in B_H} dist\,(T\,f, T\,(E))\right] \qquad (30)$$

Proof: Indeed, if ε_n(T) denotes the right hand side in (29), we clearly have $d_n(T) \leq \varepsilon_n(T)$. *Now, let*:

$$Tf = \sum_{j=1}^{\infty} a_j(T)\,\langle f, v_j\rangle u_j ,$$

with (u_j) and (v_j) two orthonormal sequences, be the Schmidt decomposition of T . Let E_0 be the span of $v_1, \ldots, v_{n-1}$. Observe that $u_j = T(a_j^{-1}v_j) \in T(E_0)$for j < n. Now, if f $\in B_H$, one has:

$$dist\,(T\,f, T\,(E_0))^2 = \left\|\sum_{j=n}^{\infty} a_j(T)\,\langle f, v_j\rangle u_j\right\|^2 = \sum_{j=n}^{\infty}\left[a_j(T)\right]^2\left|\langle f, v_j\rangle\right|^2$$

$$\leq [a_n(T)]^2 \sum_{j=n}^{\infty}\left|\langle f, v_j\rangle\right|^2 \leq [a_n(T)]^2;$$

so that $\varepsilon_n(T) \leq sup_{f\in B_H}\, dist\,(T\,f, T\,(E_0)) \leq a_n(T) = d_n(T)$.

Theorem (2.2.18)[2]: Let H be a weighted analytic Hilbert space and $\varphi \in$H such that $\|\varphi\|_\infty < 1$. Then:

$$\beta^-(C\varphi) := \lim_{n\to\infty} inf[a_n(C\varphi)]^{1/n} \geq e^{-1/Cap\,[\varphi(D)]} .$$

It will be convenient to work with the Kolmogorov numbers $d_n(C\varphi)$ instead of the approximation numbers $a_n(C\varphi)$. Recall that, for Hilbert spaces, one has$d_n(C\varphi) = a_n(C\varphi)$. We begin with a simple lemma, undoubtedly well known to experts, on approximation numbers of an operator T on a Hilbert space H.

Proof : Let $0 < r_j < 1$, $r_j \to 1$ and ψ_j : D → D be given by $\psi_j(z) = r_j$ z. Set $K_j = \overline{\varphi \circ \psi_j(D)} = \overline{\varphi(r_j D)}$. Let E be a subspace of H of dimension < n. By restriction, E can be viewed as a subspace of C(K_j). By the second result of Widom (Theorem 2.2.13), we can find $f \in B_{H^\infty}$,

$f(z) = \sum_{k\geq 0} b_k z^k$, such that:

$$\|f - h\|_{C(k_j)} \geq 2\alpha\,[M(K_j)]^n , \qquad \forall h \in E ,$$

Where $\alpha > 0$ is an absolute constant. If H^∞contractively embeds into H, wecan continue with this f. In the general case, we have to correct f in order to be in B_H, the unit ball of H. To that effect, we simply consider a partial sum:

$$g(z) = \sum_{k\geq 0}^{l-1} b_k z^k$$

and we note that, setting $\rho_j = \sup_{w\in K_j}|w|$, one has $\rho_j < 1$ and:

$$\|f - g\|_{C(Kj)} \leq \frac{\rho_j^l}{(1-\rho_j^2)^{1/2}} \quad (31)$$

$$\|g\|_H \leq C\, l \quad (32)$$

Where C = C(ω) ≥ 1 is the constant appearing in (8).

Indeed, we have $\|f - g\|_{C(Kj)} \leq \sum_{k=l}^{\infty} |b_k|\, \rho_j^k$ and then (31) follows from Cauchy-Schwarz,s inequality and the fact that $\sum_{k\geq 0} |b_k|^2 \leq 1$ since f $\in H^\infty$. For (32), we simply use that, by (8), the weight w satisfies $w_k \leq C\,(k + 1)^2$ and get:

$$\|g\|_H^2 = k = 0\sum_{k=0}^{l-1} |b_k|^2 w_k \leq C\, l^2 \sum_{k=0}^{l-1} |b_k|^2 \leq C\, l^2 \leq C^2 l^2.$$

We then notice that (30) gives, for every h $\in$ E:

$$\|g - h\|_{C(K_j)} \geq \|f - h\|_{C(K_j)} - \|f - g\|_{C(K_j)}$$

$$\geq 2\alpha\, [M(K_j)]^n - \frac{\rho_j^l}{(1-\rho_j^2)^{\frac{1}{2}}} \geq \alpha\, [M(K_j)]^n \quad (33)$$

if we take $l = A_j n$ where A_j is a large positive integer depending only on j. Explicitly:

$$A_j > \frac{log\left[1/\alpha\,(1 - \rho_j^2)^{1/2}\right]}{log(1/\rho_j)} + \frac{log[1/M(K_j)]}{log(1/\rho_j)}\ .$$

Finally, set F = g/CL. Then F $\in B_H$. Since E is a vector space, (31) and (32) imply:

$$\|F - h\|_{C(K_j)} = \frac{1}{C\, l} \|g - C\, l\, h\|_{C(K_j)} \geq \frac{1}{C\, l} \alpha\, [M(K_j)]^n.$$

But we also know that:

$$\|F - h\|_{C(K_j)} = \left\|F_{\circ}\, \varphi_{\circ}\, \psi_j - h_{\circ}\, \varphi_{\circ}\, \psi_j\right\|_\infty \leq L_{r_j} \|F_{\circ}\, \varphi - h_{\circ}\, \varphi\|_H\,,$$

So we are left with (recall that l = A_jn):

$$\left\|C_\varphi F - C_\varphi h\right\|_H \geq \frac{\alpha}{C\, L_{r_j} A_j} \frac{M(K_j)^n}{n}, \qquad \forall h \in E,$$

Implying by Lemma (2.2.17):

$$a_n(C_\varphi) = d_n(C_\varphi) \geq \frac{\alpha}{C\, L_{r_j} A_j} \frac{M(K_j)^n}{n} .$$

Now, taking n-th roots and passing to the lower limit, we get:

$$\beta^-(C_\varphi) \geq M(K_j) \tag{34}$$

It remains now to let j → ∞. Observe that the compact subsets $K_j \subseteq \varphi(D)$ form an exhaustive sequence of compact subsets of φ(D). Let then L $\subseteq \varphi$(D) be compact; we have L $\subseteq K_{j0}$ for some j_0, and using (34), we get $\beta^-(C_\varphi) \geq M(K_{jo}) \geq M(L)$. Passing to the supremum on L, we get $\beta^-(C_\varphi) \geq M[\varphi(D)]$, and this ends the proof of Theorem (2.2.18).

As said in the Introduction, for weighted Bergman spaces (including the Hardy space), and for the Dirichlet, that $\beta^-(C_\varphi) = 1$ if $\|\varphi\|_\infty = 1$ for every φ inducing a composition operator on one of those spaces.

In this section, we use Theorem (2.2.14) to generalize this result to all composition operators C_φ on weighted analytic Hilbert spaces, with another, and simpler, proof.

For that, it suffices to use the following result, which is certainly well-known to specialists. The pseudo-hyperbolic metric ρ on D is defined in (2.2.15) and we denote by diam_ρ the diameter for this metric.

The following proof of Theorem (2.2.19) was kindly shown to the second-named author by E.

It make use of the following alternative definition of Green capacity, where

$C_0^\infty(D)$ is the space of infinitely differentiable functions on D which are null on ∂D, and dz = dxdy is the usual 2-dimensional Lebesgue measure.

Theorem (2.2.19)[2]: Let K be a compact and connected subset of D. Then, for 0 < ε < 1:

$$\mathrm{diam}_\rho K > 1 - \varepsilon \Rightarrow \mathrm{Cap}\,(K) \geq c \log 1/\varepsilon,$$

For some absolute positive constant c.

Hence, the Green capacity of K tends to ∞ as its pseudo-hyperbolic diameter tends to 1.

Before proving that, let us give two suggestive examples.

(i) Let K = D(0, r); then:

$$diam_{\rho}\, K \;=\; \frac{2r}{1+r^2} \quad and \quad Cap\,(K) \;=\; \frac{1}{log\, 1/r} \;\cdot$$

One sees that r goes to 1 when diam_ K goes to 1, and hence Cap (K) tends to infinity.

(ii) Let K = [0, h], with 0 < h < 1. Then:

$$diam_{\rho}\, K \;=\; h \quad and \qquad Cap\,(K) \;=\frac{1}{\pi}\frac{I'}{I},$$

where I and I′ are the elliptic integrals:

$$I \;=\; \int_0^1 \frac{1}{\sqrt{(1\,-\,t^2)(1\,-\,k^2\,t^2)}}\,dt \quad and \quad I' \;=\; \int_0^1 \frac{1}{\sqrt{(1\,-\,t^2)(1\,-\,k^2\,t^2)}}\,dt\,,$$

With $k \;=\; \frac{1-h}{1+h}$ $\quad and \quad$ $k'^2 \;=\; 1\,-\,k^2$.

$If\ 0 \le\; a < b \le\; h, then\ b-a+hab \le\; h-a+ah^2 \;=\; h-a(1-h^2 \le h,$ so that $\rho(a, b) \;\le\; h$ Therefore, in this example again, the assumption $diam_{\rho}\, K$ −→ 1 implies successively that h → 1, k → 0, k′ → 1, I → _/2, I′ → ∞, and at last Cap (K) → ∞.

This example shows that Theorem (2.2.19) is optimal since

$$\int_0^1 \frac{dt}{\sqrt{(1\,-\,t^2)(1\,-\,k^2\,t^2)}} \;\approx\; log\frac{1}{1\,-\,k'2} \approx\; log\frac{1}{1\,-\,h}$$

as h (and hence k′) goes to 1.

Proof : If $diam_{\rho}\, K$ > 1 − ε and K is connected, it contains two points z_1 and z_2 such that $\rho(z_1, z_2) \;=\; 1 -$ ε. By the invariance of the green capacity and of ρ under automorphisms of the disk, we can assume that z_1 = 0 and z_2 = 1 − ε. Take ε < r < 1. Denote by Δ_r the intersection of the closed disk with center 1 and radius r with the closed unit disk. We observe that K meets the exterior of Δ_r at 0 and its interior at 1 − ε. The connectedness of K implies that K meets the boundary of Δ_r: there is b ∈ K such that |b − 1| = r. Write $b \;=\; 1 \;+\; re^{i\theta}$. Take now a = 1 + $re^{i\theta}$ with |a| = 1 and 0 ≤ θ ≤ ϑ ≤ 2_.

Since u(a) = 0 and u(b) ≥ 1, we get, by the fundamental theorem of calculus, that:

$$1 \le u(b) - u(a) = \int_{\theta}^{\vartheta} ire^{it}\nabla u(1 + re^{it})\,dt = \left|\int_{\theta}^{\vartheta} ire^{it}\nabla u(1 + re^{it})\,dt\right|$$
$$\le r\int_{\theta}^{\vartheta} |\nabla u(1 + re^{it})|\,dt \le r\int_{\theta}^{2\pi} |\nabla u(1 + re^{it})|\,dt.$$

Now, Cauchy-Schwarz inequality gives:

$$\int_{\theta}^{2\pi} |\nabla u(1 + re^{it})|^2\,dt \ge \frac{1}{2\pi r^2}$$

Integrating in polar coordinates centered at 1 and remembering that u = 0 outside D, we get:

$$\int_D |\nabla u(z)|^2\,dz \ge \int_{\varepsilon<|z-1|<1} |\nabla u(z)|^2\,dz$$
$$= \int_{\varepsilon}^{1}\left[\int_{\theta}^{2\pi} |\nabla u(1 + re^{it})|^2\,dt\right] rdr$$
$$\ge \frac{1}{2\pi}\int_{\varepsilon}^{1}\frac{dr}{r} = \frac{1}{2\pi}\log\frac{1}{\varepsilon}\ .$$

In view of (32), this ends the proof of Theorem (2.2.19).

Lemma (2.2.20)[2]: For every compact subset K of D, one has:

$$Cap\,(K) = \inf\left\{\frac{1}{2\pi}\int_D |\nabla u(z)|^2\,dz\,;\ u \in C_0^{\infty}(D)\ and\ u \ge 1\ on\ K\right\}$$

Proof : Though this result is often considered as “well-known“, we were not able to find anywhere an explicit reference. Since the average reader (if any!) of this paper will not be a specialist in Potential theory, we give a proof here.

(i) We first prove that the capacity of the compact K is less than the right hand side (though we only need that it is greater). We shall use Lemma (2.1.3).

We know that for every measure μ on D supported by K, one has $\Delta G_\mu = -2\pi\mu$, where G_μ is seen as a distribution. Hence, for every function $u \in C_0^{\infty}(D)$ such that u ≥ 1 on K and every positive measure μ supported by K such that $G_\mu \le 1$ on D, one has:

$$\mu(K) = \int_K d\mu \leq \int_D u d\mu = -\frac{1}{2\pi}\int_D u(z)\Delta G_\mu(z)\, dz \ .$$

Then, by definition of the Laplacian of a distribution, we get:

$$\mu(K) \leq -\frac{1}{2\pi}\int_D \Delta u(z)\Delta G_\mu(z)\, dz$$

But for every real Borel measures υ_1 and υ_2 with finite energy (meaning that their positive and negative parts have finite energy), this energy is positive and one has the Cauchy-Schwarz inequality for the Dirichlet space :

$$\left|\int_D G\upsilon_1\, d\upsilon_2\right| \leq \left(\int_D G\upsilon_1\, d\upsilon_1\right)^{1/2}\left(\int_D G\upsilon_2\, d\upsilon_2\right)^{1/2}.$$

Applying this to the measures $\upsilon_1 = \mu$ and $\upsilon_2 = \upsilon = \Delta u.dz$, we get, since $G_\mu \leq 1$:

$$\mu(K) \leq \frac{1}{2\pi}\left(\int_D G_\mu(z)d\mu(z)\right)^{1/2}\left(\int_D G_\upsilon(z)\Delta u(z)\, dz\right)^{1/2}$$

$$\leq \frac{1}{2\pi}[\mu(K)]^{1/2}\left(\int_D G_\upsilon(z)\Delta u(z)\, dz\right)^{1/2}$$

$$= \frac{1}{2\pi}[\mu(K)]^{1/2}\left(\int_D G_\upsilon\, d\upsilon\right)^{1/2}$$

Now, since $u \in C_0^\infty(D)$, one has G.

$$\int_D G_\upsilon\, d\upsilon = 2\pi\int_D |\nabla u(z)|^2\, dz \ .$$

Therefore, we get:

$$\mu(K) \leq \frac{1}{2\pi}\int_D |\nabla u(z)|^2\, dz \ .$$

Taking the supremum on μ of the left-hand side and the infimum on u of the right-hand side, we obtain:

$$Cap\,(K) \leq inf\ \left\{\frac{1}{2\pi}\int_D |\nabla u(z)|^2\, dz\ ;\ u \in C_0^\infty(D)\ and\ u \geq 1\ on\ K\right\}.$$

(ii) Let ε> 0.

Let $K_j = \{z \in C;\ dist\,(z, K) \leq 1/j\}$, $j \geq 1$. Each K_j is compact and is contained in D for j large enough, say $j \geq j_0$. Since $K = \cap_{j\geq j_0} Kj$ (and the sequence is decreasing), one has $Cap\,(Kj) \underset{j\to\infty}{\longrightarrow}$ Cap (K); note that though this proposition is stated for the logarithmic capacity, the proof clearly works also for the Green capacity). Hence, there is some $j \geq j_0$ such that, for K′ = K_j, one has (1 + ε) Cap (K) ≥ Cap (K′).

Let μ_0 be an equilibrium measure of K′. One has μ_0 (K′) = 1, I(μ_0) = V (K′), $G_{\mu_0} \leq$ V (K′) on D. one has G_{μ_0} = V (K′) on int (K′), hence on K. Let μ = Cap (K′) μ_0 . Then $\mu(K') = Cap\,(K'), I(\mu) = [Cap\,(K')]^2 I(\mu_0) = Cap\,(K')$, and, since $G_\mu = Cap\,(K')G_{\mu_0}$, one has also $G_\mu \leq 1$ on D and G_μ = 1 on K.

By a theorem of G. we can find, by regularization. since an increasing sequence of positive infinitely differentiable functions v_n on D which

$$\int_D |\nabla v_n(z)|^2\, dz \underset{n\to\infty}{\longrightarrow} \int_D |\nabla G_\mu(z)|^2\, dz\ .$$

Since $(v_n)_n$ is increasing and converges point wise to 1 on the compact set K, Dini's theorem tells that one has uniform convergence(Dini's theorem says that if a monotone sequence of continuous functions converges on a compact space and if the limit function is also continuous, then the convergence is uniform see [6]). Hence, we can find some $v = v_n$ such that $v \geq (1 + ε)^{-1}$ on K and

$$\int_D |\nabla v_n(z)|^2\, dz \leq (1 + \varepsilon) \int_D |\nabla G_\mu(z)|^2\, dz\,.$$

Note that v = 0 on ∂D since 0 ≤ v ≤ G_μ, which is equal to 0 on ∂D.

Putting u = (1 + ε)v, one has u $\in C_0^\infty$(D), u ≥ 1 on K and

$$\int_D |\nabla v_n(z)|^2\, dz \leq (1 + \varepsilon)^3 \int_D |\nabla G_\mu(z)|^2\, dz\,.$$

But we know by G. C. Evans,s theorem that:

$$I(\mu) = \frac{1}{2\pi} \int_D |\nabla G_\mu(z)|^2\, dz\ .$$

We get hence:

$$(1 + \varepsilon)\, Cap\,(K) \geq Cap\,(K') = I(\mu) = \frac{1}{2\pi}\int_D |\nabla G_\mu(z)|^2\, dz$$
$$\geq \frac{1}{(1 + \varepsilon)^3}\frac{1}{2\pi}\, ZD\, |\nabla u(z)|2\, dz$$

Since ε > 0 was arbitrary, we get:

$$Cap\,(K) \leq inf\ \left\{\frac{1}{2\pi}\int_D |\nabla u(z)|^2\, dz\ ;\ u \in C_0^\infty\,(D)\ and\ u \geq 1\ on\ K\right\}.$$

And that ends the proof.

As in the above proof, we may assume that 0 and 1−ε belong to K. Consider $K^* = \{|z|\ ;\ z \in K\}$. Since K is connected, the same holds for K^*. Hence the interval [0, 1− ε] is contained in K^*. It follows that Cap ([0, 1− ε]) ≤ Cap (K^*). But we saw that Cap ([0, 1 − ε]) ≈ log (1/ε); hence Cap (K^*) & log (1/ε). It remains to use that the map α: z ↦ |z| is a contraction for the pseudo-hyperbolic metric and hence Cap (K^*) ≤ Cap (K). In fact, if υ is any probability measure supported by K^*, there exists a probability measure μ on K such that α(μ) = υ. Hence:

$$\begin{aligned} V\,(K) \leq I_K(\mu) &= \iint_{D\times D} g(z,w)\, d\mu(z)\, d\mu(w) \\ &= \iint_{D\times D} log\frac{1}{\rho(z,w)}\, d\mu(z)\, d\mu(w) \\ &\leq \iint_{D\times D} log\frac{1}{\rho(|z|,|w|)}\, d\mu(z)\, d\mu(w) \\ &= \iint_{D\times D} log\frac{1}{\rho(z,w)}\, dv(z)\, dv(w) = I_{K^*}(v)\,. \end{aligned}$$

Taking the infimum over all υ, we get V (K) ≤ V (K^*).

As a corollary of Theorem (2.2.19), we get a new proof .

Theorem (2.2.21)[2]: There exists an absolute constant c > 0 such that, for any symbol φ on a weighted analytic space H, one has:

$$diam_\rho\,[\varphi(D)] > r \Rightarrow \beta(C_\varphi) \geq exp\ \left[-\frac{c}{log\, 1/(1-r)}\right].$$

In particular:

$$\|\varphi\|_\infty = 1 \Rightarrow \beta(C_\varphi) = 1\,.$$

Proof: The first statement is a direct consequence of Theorem (2.2.7), modulo Theorem (2.1.4) and Theorem (2.2.19), applied to φ (D) and its closure.

One cannot replace $diam_\rho\, [\varphi(D)] > r$ by $\|\varphi\|_\infty > r$ in this first statement as indicated by the following example:

$$\varphi(z) = \frac{a - (z/2)}{1 - a(z/2)} = \Phi_a[h(z)],$$

Where $\Phi_a(z) = \frac{a-z}{1-\bar{a}z}$ with a ∈ D and h(z) = z/2 is the dilation with ratio 1/2.

Then $\|\varphi\|_\infty \geq |\Phi_a(0)| = |a|$ and $\beta(C_\varphi) = \beta(C_h) = 1/2$.

However, one can do so if moreover φ(0) = 0 because then, clearly:

$$\|\varphi\|_\infty > r \Rightarrow diam_\rho\, [\varphi(D)] > r\,.$$

This is enough for the second statement since, putting a = φ(0), we have, due to the fact that Φ_a is unimodular on the whole unit circle: $\|\Phi_a \circ \varphi\|_\infty = \|\varphi\|_\infty = 1$, $(\Phi_a \circ \varphi)(0) = 0$ *and* $\beta(C_\varphi) = \beta(C_{\Phi_a \circ \varphi})$.

Here, we consider the case of composition operators on Hp for $1 \leq p < \infty$.

For every a ∈ D, we denote by $e_a \in (H^p)^*$ the evaluation map at a, namely:

$$e_a(f) = f(a), \quad f \in H^p \qquad (35)$$

We know that :

$$\|e_a\| = \left(\frac{1}{1-|a|^2}\right)^{1/p} \qquad (36)$$

and the mapping equation

$$C_\varphi^*(e_a) = e_{\varphi(a)} \qquad (37)$$

Still holds

Throughout this section we denote by $\|.\|$, without any subscript, the norm in the dual space $(Hp)^*$.

Let us stress that this dual norm of $(H^p)^*$ is, for $1 < p < \infty$, equivalent, but not equal, to the norm $\|.\|_q$ of H^q, and the equivalence constant tends to infinity when p goes to 1 or to ∞.

With this preliminaries , we are going to see that Theorem (2.2.7) remains true.

We begin with the following lemma, which extends Lemma (2.2.17).

Lemma (2.2.22)[2]: Let X be a Banach space, and T : X → X be a compact operator. Let us set:

$$\varepsilon_n(T) = \inf_{dimE<n} \left[\sup_{x\in BX} dist\,(T\,x, TE)\right] \qquad (38)$$

Then $\varepsilon_n(T) \leq 2\sqrt{n}\, c_n(T)$.

Proof: Let ε > 0, and let F be a subspace of X of codimension < n such that $\|T_{|F}\| \leq c_n(T) + \varepsilon$. Let Q: X → F be an onto projection of norm $\|Q\| \leq \sqrt{n} + \leq 2\sqrt{n}$, and let R = T (I − Q). Then E = (I − Q)X satisfies dim E < n. If x ∈ B_X, the closed unit ball of X, then:

$$dist\,(T\,x, TE) \leq \|Tx - Rx\| = \|TQx\| \leq \|T_{|F}\|\,\|Qx\| \leq (c_n(T) + \varepsilon)\,2\sqrt{n}\ .$$

This implies $\varepsilon_n(T) \leq 2\sqrt{n}\,(c_n(T) + \varepsilon)$.

The result follows since ε was arbitrary.

Theorem (2.2.23)[2]: Let $1 \leq p < \infty$ and $C_\varphi : H^p \to H^p$.

(i) If $\overline{\varphi(D)} \subseteq D$, then:

$$\beta(C_\varphi) = e^{-1/Cap\,[\varphi(D)]}\ .$$

(ii) One has:

$$\|\varphi\|_\infty = 1 \Rightarrow \beta(C_\varphi) = 1\ .$$

Proof : (i) a) We first prove that $\beta^-(C_\varphi) \geq e^{-1/Cap\,[\varphi(D)]}$.

Let $\tilde{L}_r = sup_{|a|\leq r}\|e_a\| = \left(\frac{1}{1-r^2}\right)^{1/p}$, for $0 < r < 1$. Using the same notations and estimations as in Theorem (2.2.18), up to the replacement of L_r by $\tilde{L}_r$, we get:

$$\varepsilon_n(T) \geq (1 - \varepsilon)\,\tilde{L}_{r_j}^{-1}\,\alpha\,[M(K_j)]^n\ .$$

Lemma (2.2.22) now implies:

$$a_n(T) \geq c_n(T) \geq \alpha\frac{1 - \varepsilon}{2\sqrt{n}}\tilde{L}_{r_j}^{-1}\,[M(Kj)]^n\ .$$

The rest of the proof is unchanged, since the presence of the factor $1/\sqrt{n}$ does not affect the result.

b) The upper bound is even simpler since $H^{\infty} \subseteq H^{p}$. For example, setting $A(f) = h \circ \varphi$, we can replace Lemma (2.2.16) by

$$\|g \circ \varphi - h \circ \varphi\|_{p} \leq \|g \circ \varphi - h \circ \varphi\|_{\infty} = \|g - h\|_{C(K)} ,$$

Where $K = \overline{\varphi(D)}$.

(ii) That follows from Theorem (2.2.20).

CHAPTER 3

Weighted Composition Operators Between Hilbert Spaces in the Operator Norm and Hilbert-Schmidt Norm Topologies

We will consider the operator norm topology and the Hilbert-Schmidt norm topology respectively. These results will be involved in the investigation for the explicit cases of the classical Hardy-Hilbert space, the weighted bergman spaces and the Dirichlet space. Furthermore we will estimate the Hilbert-Schmidt norms of difference of two composition operators acting from the Dirichlet space to the Hardy and the weighted bergman spaces.

Section(3.1)Path connectedness of $C_w(H_1, H)$ and $C_wH_S(H_1, H)$:

Let H(D)be the space of analytic functions on the open unit disk D:={|z| <1} and H^{∞} the space of bounded analytic functions on D with the supremum norm $\|\cdot\|_{\infty}$. Let S(D) be the set of analytic self-maps of D. Denote by H a Hilbert space of analytic functions on D with the norm $\|\cdot\|_H$ satisfying the following conditions:

(#1) For any $\alpha \in$ D, the point evaluation τ_α: H $\ni$ f→f(α) is a bounded linear functional on H, and $\sup_{|\alpha|\le r}\|\tau_\alpha\|_H < \infty$ for every 0 < r < 1.

(#2) $H^{\infty}\cdot H \subset H$ and $\|fg\|_H \le \|f\|_{\infty}\|g\|_H$ for every f ∈ H^{∞} and g ∈ H.

(#3) $\|1\|_H$=1and $\{z^n/\|z^n\|_H : n \ge 0\}$is an orthonormal basis in H.

(#4) For every f ∈ Hand 0 ≤ r ≤1, we have $f_r(z) := f(r_z) \in$ H.

By (#3), H contains all analytic polynomials. By (#2), for f ∈ H^{∞} we have $\|f\|_H = \|f\cdot 1\|_H \le \|f\|_{\infty}\|1\|_H = \|f\|_{\infty}$. Many classical Hilbert spaces of analytic functions on D satisfy conditions (#1)–(#4).

For φ ∈ S(D), we define the composition operator C_φ: H → H(D) by $C_\varphi f = f\circ\varphi$ for f ∈ H. If $C_\varphi f \in$ H for every f ∈ H, then C_φ: H→H is a bounded linear operator. We denote by C(H) the space of bounded composition operators C_φ: H→H with the operator norm topology. for an overview of composition operators.

Let φ ∈ S(D) and u ∈ H. We may define the weighted composition operator M_uC_φ: H→H (D) by $M_uC_\varphi f = u\cdot(f\circ\varphi)$ for every f ∈ H. If $u\cdot(f\circ\varphi) \in$ H for every f ∈ H, then M_uC_φ: H→H is bounded and we denote by$\|M_uC_\varphi\|_H$ its operator norm. We note that $M_uC_\varphi=0$ if and only if u = 0. Let $C_w(H)$ be the space of non zero bounded weighted composition operators on H with the operator norm topology, that is,

$$C_w(H) = \{M_uC_\varphi : M_uC_\varphi : H \to H \text{ is bounded}, u \neq 0\}.$$

In the study of (weighted) composition operators, one of the main subjects is determining the set $\{u \in H: M_uC_\varphi \in C_w(H)\}$ for a given $\varphi \in S(D)$and the other is determining the topological structure in $C_w(H)$.By (#2), if $C_\varphi \in C_w(H)$then $M_uC_\varphi \in C_w(H)$for every $u \in H^\infty$ with $u \neq 0$. The boundedness of M_u on range of a composition operator was investigated. The boundedness and compactness of weighted composition operators on the Hardy and Bergman spaces have been characterized. About the topological structure, first studied the component structure of the set of all composition operators on the Hardy–Hilbert space H^2 in the topology induced by the operator norm. Further investigated and the latter authors raised the problems on the component structure in the topologies induced by the operator norm and the essential operator norm and explicitly gave the conjecture that two composition operators would lie in the same component if and only if they have compact difference, that is, the difference of the two composition operators is compact. This conjecture was answered in the negative by and Bourdon provided an example of two composition operators induced by linear fractional self-maps of D which are in the same component but do not have compact difference. In general, it seems fairly difficult to describe all path connected components in $C_w(H)$. We would like to mention that $C_w(H) \cup\{0\}$is a path connected set. The reason is that for $M_uC_\varphi \in C_w(H)$, we have $M_{tu}C_\varphi \in C_w(H) \cup\{0\}$ and

$$\|M_{t_0u}C_\varphi - M_{tu}C_\varphi\|H = |t_0 - t|\|M_uC_\varphi\|_H$$

for every $0 \le t_0, t \le 1$, so M_uC_φ and 0 are in the path connected set in $C_w(H) \cup\{0\}$. By this fact, the condition $0 \notin C_w(H)$ is a key to study path connected components in $C_w(H)$.By condition (#3), $M_uC_\varphi \in C_w(H)$is Hilbert–Schmidt if and only if

$$\|M_uC_\varphi\|^2_{H,HS} := \sum_{n=0}^{\infty} \frac{\|u\varphi^n\|^2_H}{\|z^n\|^2_H} < \infty.$$

We denote by $C_{w,HS}$ (H) the space of Hilbert–Schmidt operators $M_uC_\varphi$in$C_w(H)$ with the Hilbert–Schmidt norm topology. The topology on

$C_{w,HS}(H)$ is stronger than the operator norm one. So a path connected set in $C_{w,HS}(H)$ is so in $C_w(H)$.

Let H_1be another Hilbert space of analytic functions on D with $H_1 \subset H$ satisfying conditions (#1), (#3) and (#4). We note that H_1 needs not satisfy (#2). Furthermore we assume that

(#5)$\|f\|_H \leq \|f\|_{H_1}$ for every $f \in H_1$.

For a bounded linear operator $T{:}H_1 \rightarrow H$, we write$\|T\|_{H_1,H}$ its operator norm. For $M_uC_\varphi \in C_w(H)$, we have

$$\|M_uC_\varphi f\|_H \leq \|M_uC_\varphi\|_H \|f\|_H \leq \|M_uC_\varphi\|_H \|f\|_{H_1}$$

for every $f \in H_1$. Hence $M_uC_\varphi{:}H_1 \rightarrow H$ is bounded and

$$\|M_uC_\varphi\|_{H_1,H} \leq \|M_uC_\varphi\|_H \text{ for every } M_uC_\varphi \in C_w(H) \qquad (1)$$

Restricting $M_uC_\varphi \in C_w(H)$on H_1, we may consider that M_uC_φ is also a bounded linear operator from $H_1$to H. We denote by $C_w(H_1, H)$ the space of $M_uC_\varphi{:}\ H_1 \rightarrow H$, $M_uC_\varphi \in C_w(H)$, with the operator norm topology. For the non-weighted case, we write C(H1, H). We note that

$$C_w(H_1, H) = C_w(H)$$

as sets, so if $M_uC_\varphi \in C_w(H_1, H)$, then $u \in H$. By (1), the topology of $C_w(H)$ is stronger than the one of $C_w H_1, H$. Hence a path connected set in $C_w(H)$is so in $C_w(H_1, H)$.

We have that $M_uC_\varphi \in C_w(H_1, H)$ is Hilbert–Schmidt if and only if

$$\|M_uC_\varphi\|^2_{H_1,H,HS} := \sum_{n=0}^{\infty} \frac{\|u\varphi^n\|^2_H}{\|z^n\|^2_{H_1}} < \infty.$$

We denote by $C_{w,HS}(H_1, H)$ the space of $M_uC_\varphi \in C_w(H_1, H)$ which are Hilbert–Schmidt. We consider the Hilbert–Schmidt norm topology on $C_{w,HS}(H_1, H)$. The topology on $C_{w,HS}(H_1, H)$ is stronger than the operator norm one. So a path connected set in $C_{w,HS}(H_1, H)$ is so in $C_w(H_1, H)$. Since

$$\left\|M_u C_\varphi\right\|^2_{H_1,H,HS} \le \sum_{n=0}^{\infty} \frac{\|u\varphi^n\|^2_H}{\|z^n\|^2_{H_1}} = \left\|M_u C_\varphi\right\|^2_{H,HS} \text{ by (\#5)},$$

we have $C_{w,HS}(H) \subset C_{w,HS}(H_1, H)$

This chapter is organized as follows. In Section (3.2), we shall prove that if $\frac{\|z^n\|_H}{\|z^n\|_{H_1}} \to 0$ as $n \to \infty$ then $C_w(H_1, H)$ is a path connected space. In Section (3.1), we shall prove that $C_{w,HS}(H_1, H)$ is a path connected space. As applications of these results, and we study the cases that H is either the classical Hardy–Hilbert space H^2 or the weighted Bergman spaces L^2_α, $-1< \alpha <\infty$, on D, and H_1 is either H_2 or L^2_α or the Dirichlet space D on D. we study the Hilbert–Schmidt norms of differences of composition operators in C(D, H^2) and C(D, L^2_α) for $-1 < \alpha < \infty$. We shall show that $C_{HS}(D, L^2_\alpha) = \{C_\varphi : \varphi \in S(D)\}$as sets. For φ, ψ∈S(D), let $\varphi_t = t_\varphi + (1-t)\psi$ for $0 \le t \le 1$. We also prove that $\{C_{\varphi t}: 0 \le t \le 1\}$ is a continuous path in $C_{HS}(D, L^2_\alpha)$

Let H and H_1 be the spaces satisfying conditions given in the introduction.

Lemma (3.1.1)[3]: If $\varphi \in S(D)$and $\|\varphi\|_\infty<1$, then $C_\varphi f \in H^\infty$ for every $f \in H$ and

$$\left\|C_\varphi f\right\|_\infty \le \|f\|_H \sup_{|\alpha| \le \|\alpha\|_\infty} \|\tau_\alpha\|_H.$$

Proof : For $f \in H$ and $z \in D$, by (#1) we have so we get the assertion.

$$\left|(C_\varphi f)(z)\right| = \left|f\left(C_\varphi(z)\right)\right| \le \|f\|_H \left\|\tau_{\varphi(z)}\right\|_H \le \|f\|_H \sup_{|\alpha| \le \|\alpha\|_\infty} \|\tau_\alpha\|_H,$$

so we get the assertion.

Theorem(3.1.2)[3]: If$\|z^n\|_H / \|z^n\|_{H_1} \to 0$ as $n \to \infty$ then $C_w(H_1, H)$ is a path connected space.

Proof: Let $M_u C_\varphi \in C_w(H_1, H)$. Since $C_w(H_1, H) = C_w(H)$ as sets, we have $u \in H$ and $M_u C_\varphi H<\infty$. Let $0 \le r<1$. For $f \in H$, by Lemma (3.1.1) we have for $\varphi \in H^\infty$ and

$$\left\|M_u C_{r\varphi} f\right\|_H = \|u(f \circ r\varphi)\|_H$$

$$\le \|f \circ r\varphi\|_\infty \|u\|_H \text{ by (\#2)}$$

$$\leq \|f\|_H \|u\|_H \sup_{|\alpha|\leq r} \|\tau_\alpha\|_H$$

By (#1), $M_u C_{r\varphi} \in C_w(H)$, so $M_u C_{r\varphi} \in C_w(H_1, H)$.

We shall show that $\{M_u C_{r\varphi} : 0 \leq r \leq 1\}$ is a path connected set in $C_w(H_1, H)$. Fix $0 \leq r_0 \leq 1$. It is sufficient to show that $\left\|M_u C_{r_0\varphi} - M_u C_{r\varphi}\right\|_{H_1,H} \to 0$ as $r \to r_0$. Let $g = \sum_{n=0}^{\infty} a_n z^n \in H_1$. For each $0 \leq r \leq 1$, let

$$g_{[r]}(z) = \sum_{n=1}^{\infty} a_n (r_0^n - r^n) z^n.$$

Since $H_1 \subset H$ and H satisfies (#4), we have $g_{[r]} \in H$. Hence

$$\left\|(M_u C_{r_0\varphi} - M_u C_{r\varphi})g\right\|_H^2 = \left\|u \sum_{n=1}^{\infty} a_n (r_0^n - r^n)\varphi^n\right\|_H^2$$

$$= \left\|M_u C_\varphi g_{[r]}\right\|_H^2 \leq \left\|M_u C_\varphi\right\|_H^2 \left\|g_{[r]}\right\|_H^2$$

$$= \left\|M_u C_\varphi\right\|_H^2 \sum_{n=1}^{\infty} |a_n|^2 |r_0^n - r^n|^2 \|z^n\|_H^2 \quad \text{by (\#3)}$$

$$\leq \left\|M_u C_\varphi\right\|_H^2 \sup_{k\geq 1} \left(\left|r_0^k - r^k\right|^2 \frac{\left\|z^k\right\|_H^2}{\|z^k\|_{H_1}^2} \right) \sum_{n=1}^{\infty} |a_n|^2 \|z^n\|_{H_1}^2$$

$$\leq \left\|M_u C_\varphi\right\|_H^2 \sup_{k\geq 1} \left(\left|r_0^k - r^k\right| \frac{\left\|z^k\right\|_H}{\|z^k\|_{H_1}} \right)^2 \|g\|_{H_1}^2 .$$

Then

$$\left\|(M_u C_{r_0\varphi} - M_u C_{r\varphi})\right\|_{H_1,H} \leq \left\|M_u C_\varphi\right\|_H \sup_{k\geq 1} \left(\left|r_0^k - r^k\right| \frac{\left\|z^k\right\|_H}{\left\|z^k\right\|_{H_1}} \right)$$

For any positive integer n, we have

$$\sup_{k \geq 1} \left(\left|r_0^k - r^k\right|^2 \frac{\left\|z^k\right\|_H}{\|z^k\|_{H_1}} \right) \leq \sum_{k=1}^{n-1} \left(\left|r_0^k - r^k\right|^2 \frac{\left\|z^k\right\|_H}{\|z^k\|_{H_1}} \right) + \sup_{k \geq 1} \left(\frac{\left\|z^k\right\|_H}{\|z^k\|_{H_1}} \right)$$

Hence

$$\limsup_{r \to r_0} \|(M_u C_{r_0\varphi} - M_u C_{r\varphi})\|_{H_1,H} \leq \|M_u C_\varphi\|_H \sup_{k\geq n} \frac{\|z^k\|_H}{\|z^k\|_{H_1}}$$

Therefore by the assumption, we get $M_u C_{r\varphi} \to M_u C_{r_0\varphi}$ as $r \to r_0$ in $C_w(H_1, H)$. This shows that $\{M_u C_{r\varphi}: 0 \leq r \leq 1\}$ is a path connected set in $C_w(H_1, H)$. Thus $M_u C_\varphi$ and $M_u C_0$ are in the same path connected set in $C_w(H_1, H)$.

Let $M_u C_\varphi, M_v C_\psi \in C_w(H_1, H)$ We have

$$\|(M_u C_0 - M_v C_0)f\|_H \leq \|u - v\|_H |f(0)| \leq \|u - v\|_H \|\tau_0\|_{H_1} \|f\|_{H_1}$$

For every $f \in H_1$. Hence

$$\|(M_u C_0 - M_v C_0)\|_{H_1,H} \leq \|u - v\|_H \|\tau_0\|_{H_1}.$$

It is not difficult to show that there is a continuous path $\{u_t: 0 \leq t \leq 1\}$ in H such $u_0=u$, $u_1=v$ and $u_t \neq 0$ for every $0 \leq t \leq 1$. For $0 \leq t_0 \leq 1$, we have

$$\left\|(M_{u_{t_0}} C_0 - M_{u_t} C_0)\right\|_{H_1,H} \leq \left\|u_{t_0} - u_t\right\|_H \|\tau_0\|_{H_1}.$$

Letting $t \to t_0$, we have $M_{u_t} C_0 \to M_{u_{t_0}} C_0$ in $C_w(H_1, H)$. Hence $M_u C_0$ and $M_v C_0$are in the same path connected component in $C_w(H_1, H)$.. Thus by the last paragraph, $C_w(H_1, H)$is a path connected space.

Lemma (3.1.3)[3]: If $\|\phi\|_\infty$<1and u $\in$ H, then $M_u C_\varphi \in C_w(H)$ and is compact.

Proof: By the first paragraph of the proof of Theorem (3.1.1) , we have $M_u C_\varphi \in C_w(H)$.

To show that $M_u C_\varphi$is compact, let $\{f_n\}_n$be a sequence in H such that there is a positive constant K satisfying $\|f_n\|_H < K$for every n. By (#1), we may assume that $f_n$converges to some f $\in$ H(D) uniformly on any compact subset of D. By the assumption, $f_n \circ \varphi \to f \circ \varphi$ in H^∞ .Hence by (#2), $u(f_n \circ \varphi)$, $u(f \circ \varphi) \in$ H and

$$\left\|M_u C_\varphi f_n - u(f\, o\varphi)\right\|_H \leq \|u\|_H \|f_n o\varphi - f\, o\varphi\|_\infty \to 0, n \to \infty$$

Thus $M_u C_\varphi \in C_w(H)$.is compact.

Corollary(3.1.4)[4]: *If* $\|z^n\|_H / \|z^n\|_{H_1} \to 0$ as $n \to \infty$ then any $M_u C_\varphi \in C_w(H_1, H)$ *is compact.*

Proof: For $0 < r < 1$, by Lemma (3.1.2) $M_u C_{r\varphi} \in C_w(H)$ is compact. By (#5), id $: H_1 \to H$ is bounded. Hence $M_u C_{r\varphi} : H_1 \to H$ is compact. By the proof of Theorem (3.1.1) we get the assertion.

Let H and H_1 be the spaces satisfying conditions given in the introduction. We note that $C_{w,HS}(H_1, H) \subset C_w(H_1, H) = C_w(H)$ as sets and the topology on $C_{w,HS}(H_1, H)$ is induced by the Hilbert–Schmidt norm.

Lemma (3.1.5)[3]:

(i) $\{M_u C_\varphi : \varphi \in H^\infty, \|\varphi\|_\infty < 1, u \in H, u \neq 0\} \subset C_{w,HS}(H)$.

(ii) $C_{w,HS}(H) \subset C_w(H_1, H)$.

Proof: (i) Let $\varphi \in H^\infty$ with $\|\varphi\|_\infty < 1$ and $u \in H$ with $u \neq 0$. We have

$$\left\|M_u C_\varphi\right\|_{H_1,H}^2 = \sum_{n=0}^{\infty} \frac{\|u\varphi^n\|_H^2}{\|z^n\|_H^2} \leq \|u\|_H^2 \sum_{n=0}^{\infty} \frac{\|\varphi\|_\infty^2}{\|z^n\|_H^2} \text{ by (\#2) } = \|u\|_H^2 \left\|C_{\|\varphi\|_\infty}\right\|_{H,HS}^2.$$

Since $C_{\|\varphi\|_\infty}$ is rank one, so is Hilbert–Schmidt. Hence $M_u C_\varphi \in C_{w,HS}(H)$.

(ii) Let $M_u C_\varphi \in C_{w,HS}(H)$. Since id $: H_1 \to H$ is bounded and $M_u C_\varphi|_{H_1} = M_u C_\varphi \cdot \text{id}$, we get (ii).

Theorem (3.1.6)[3]: $C_{w,HS}(H_1, H)$ is a path connected space.

Proof: Let $M_u C_\varphi \in C_{w,HS}(H_1, H)$. By (#3),

$$\sum_{n=0}^{\infty} \frac{\|u\varphi^n\|_H^2}{\|z^n\|_{H_1}^2} = \left\|M_u C_\varphi\right\|_{H_1,H,HS}^2 < \infty \qquad (2)$$

We shall show that $\{M_u C_{r\varphi} : 0 \leq r \leq 1\}$ is a path connected set in $C_{w,HS}(H_1, H)$. By Lemma (3.1.5), $M_u C_{r\varphi} \in C_{w,HS}(H_1, H)$ for every $0 \leq r \leq 1$. Let us fix $0 \leq r_0 \leq 1$. We shall show that $\left\|M_u C_{r_0\varphi} - M_u C_{r\varphi}\right\|_{H_1,H,HS} \to 0$ as $r \to r_0$. For any positive integer N, we have

$$\left\|M_uC_{r_0\varphi} - M_uC_{r\varphi}\right\|^2_{H_1,H,HS} = \sum_{n=0}^{\infty} \frac{\|u(r_0^n - r^n)\varphi^n\|_H^2}{\|z^n\|_{H_1}^2}$$

$$\leq \sum_{n=0}^{N-1} |r_0^n - r^n|^2 \frac{\|u\varphi^n\|_H^2}{\|z^n\|_{H_1}^2} + \sum_{n=0}^{\infty} \frac{\|u\varphi^n\|_H^2}{\|z^n\|_{H_1}^2}$$

Take $\varepsilon > 0$ arbitrarily. Then by (2), we may take N large enough so that

$$\sum_{n=N}^{\infty} \frac{\|u\varphi^n\|_H^2}{\|z^n\|_{H_1}^2} < \varepsilon$$

Hence

$$\left\|M_uC_{r_0\varphi} - M_uC_{r\varphi}\right\|^2_{H_1,H,HS} < \varepsilon + \sum_{n=0}^{N-1} |r_0^n - r^n|^2 \frac{\|u\varphi^n\|_H^2}{\|z^n\|_{H_1}^2}$$

Letting $r \to r_0$, we have

$$\limsup_{r\to r_0} \left\|M_uC_{r_0\varphi} - M_uC_{r\varphi}\right\|^2_{H_1,H,HS} < \varepsilon$$

Thus we get $\left\|M_uC_{r_0\varphi} - M_uC_{r\varphi}\right\|^2_{H_1,H,HS} \to 0$ as $r \to r_0$. Hence M_uC_φ and M_uC_0 are in the same path connected component in $C_{w,HS}(H_1, H)$.

Let $M_vC_\psi \in C_{w,HS}(H_1, H)$ be another operator. By the last paragraph, M_vC_ψ and M_vC_0 are in the same path connected component in $C_{w,HS}(H_1, H)$. Let $\{u_t : 0 \leq t \leq 1\}$ be a continuous path in H such that $u_0=u$, $u_1= v$ and $u_t \neq 0$ for every $0 \leq t \leq 1$. We have

$$\left\|M_{u_{t_0}}C_0 - M_{u_t}\right\|^2_{H1,H,HS} = \left\|u_{t_0} - u_t\right\|^2_H \to 0 \text{ as } t \to t_0.$$

Hence M_uC_0 and M_vC_0 are in the same path connected component in $C_{w,HS}(H_1, H)$. Therefore M_uC_φ and M_vC_ψ are in the same path connected component in $C_{w,HS}(H_1, H)$. This completes the proof.

Section (3.2) Applications:

Let H^2 be the classical Hardy–Hilbert space on D. It is well known that H^2satisfies conditions (#1)–(#4). For each $f \in H^2$, it is known that there is the radial limit f^* almost everywhere on ∂D with respect to the normalized Lebesgue measure m. By Littlewood's subordination theorem (it states that any holomorphic univalent self-mapping of the unit disk in the complex numbers that fixes 0 induces a contractive composition operator on various function spaces of holomorphic functions on the disk see [7]), C_φ: $H^2 \to H^2$is bounded for every $\varphi \in S(D)$ For a given $\varphi \in S(D)$, it is not known the characterization of the set of $u \in H^2$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded. But it is known that $C_w(H^2)$ has many path connected components .

For $-1 < \alpha < \infty$, the weighted Bergman space L^2_α on D is the space of $f \in H(D)$ satisfying

$$\|f\|^2_{L^2_\alpha} := \int_D |f(z)|^2 \, dA_\alpha(z) < \infty,$$

where $dA_\alpha = (\alpha+1)(1 - |z|^2)^\alpha$ dA and A stands for the normalized Lebesgue measure on D. When $\alpha = 0$, L^2_0 is the classical Bergman space. It is known that $l^2_\propto$ satisfies conditions (#1)–(#4) and $C_\varphi : L^2_\alpha \to L^2_\alpha$is bounded for every $\varphi \in S(D)$. We have

$$\|z^n\|^2_{L^2_\alpha} = \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)},$$

where $\Gamma(s)$stands for the usual Gamma function. Then $H^2 \subsetneq L^2_\alpha$ and $\|f\|_{L^2_\alpha} \le \|f\|_{H^2}$ for $f \in H^2$. Also we have that for $-1 < \alpha_1 < \alpha_2 < \infty$, $L^2_{\alpha_1} \subsetneq L^2_{\alpha_2}$ and $\|f\|_{L^2_{\alpha_2}} \le \|f\|_{L^2_{\alpha_1}}$ for $f \in L^2_{\alpha_1}$. For a given $\varphi \in S(D)$, it is not known the characterization of the set of $u \in L^2_\alpha$ such that $M_u C_\varphi : L^2_\alpha \to L^2_\alpha$is bounded. for the boundedness and compactness of weighted composition operators on the weighted Bergman spaces. partially answered the question of when two composition operators lie in the same component of $C_w(L^2_\alpha)$.

Let D be the Dirichlet space on D. For $f \in H(D)$, we have that $f \in D$ if and only if

$$\|f\|^2_D := |f(z)|^2 + \int_D \left|f'(z)\right|^2 dA_\alpha(z) < \infty,$$

It is known that D satisfies conditions (#1), (#3) and (#4). But D does not satisfy condition (#2). For $f = \sum_{n=0}^{\infty} a_n z^n \in D$, we have $\|f\|_D^2 = |a|^2 + \sum_{n=1}^{\infty} n|a_n|^2$. Then $D \subsetneq H^2$ and $\|f\|_{H^2} \leq \|f\|_D$ for $f \in D$. It is also known that there is an analytic self-map $\varphi \in S(D)$ such that $C_\varphi: D \to D$ is not bounded. We note that $\|1\|_D^2=1$ and $\|z^n\|_D^2=n$ for every $n \geq 1$.

We have

$$\frac{\|z^n\|_{H^2}}{\|z^n\|_D} = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

By Stirling's formula, we have

$$\frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} \sim (n+1)^{\lambda-1},\ \lambda > 0.$$

Hence for $-1 < \alpha < \infty$, we have

$$\frac{\|z^n\|_{L_\alpha^2}^2}{\|z^n\|_D^2} = \frac{n!\,\Gamma(2+\alpha)}{n\Gamma(n+2+\alpha)} \sim \frac{1}{n(n+1)^{1+\alpha}} \to 0 \text{ as} n \to \infty$$

and

$$\frac{\|z^n\|_{L_\alpha^2}^2}{\|z^n\|_{H^2}^2} = \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} \sim \frac{1}{(n+1)^{1+\alpha}} \to 0 \text{ as} n \to \infty$$

For $-1 < \alpha_1 < \alpha_2 < \infty$, we also have

$$\frac{\|z^n\|_{L_{\alpha_2}^2}^2}{\|z^n\|_{L_{\alpha_1}^2}^2} = \frac{n!\,\Gamma(2+\alpha_2)\,\Gamma(n+2+\alpha_1)}{\Gamma(n+2+\alpha_2)\,n!\,\Gamma(2+\alpha_1)}$$

$$\sim (n+1)^{-(2+\alpha_2-1)}(n+1)^{2+\alpha_1-1} = (n+1)^{\alpha_1-\alpha_2} \to 0 \text{ as } n \to \infty$$

Hence by Theorem (3.1.1), we have the following.

Corollary (3.2.1)[3]: $C_w(D, H^2)$, $C_w(D, L_\alpha^2)$, $C_w(H^2, L_\alpha^2)$, for $-1 < \alpha < \infty$, and $C_w(L_{\alpha_1}^2, L_{\alpha_2}^2)$, for $-1 < \alpha_1 < \alpha_2 < \infty$ are path connected spaces.

By Theorem (3.1.6), we have the following.

Corollary (3.2.2)[3]: $C_{w,HS}(D, H^2)$, $C_{w,HS}(D, L_\alpha^2)$, $C_{w,HS}(H^2, L_\alpha^2)$ for $-1 < \alpha < \infty$, and $C_{w,HS}(L_{\alpha_1}^2, L_{\alpha_2}^2)$, for $-1 < \alpha_1 < \alpha_2 < \infty$ are path connected spaces.

Since $\{1, z^n/\sqrt{n}: n \geq 1\}$ is an orthonormal basis of D, in the same way as in Shapiro and Taylor we have

$$\|M_u C_\varphi\|^2_{D,H^2,HS} = \|u\|^2_{H^2} + \sum_{n=1}^{\infty} \frac{\|u\varphi^n\|^2_{H^2}}{n} = \|u\|^2_{H^2} + \int_{\partial D} |u^*|^2 \log \frac{1}{1 - |\varphi^*|^2} dm$$

Hence $M_u C_\varphi \in C_w(D, H^2)$ is Hilbert–Schmidt if and only if

$$\int_{\partial D} |u^*|^2 \log \frac{1}{1 - |\varphi^*|^2} dm < \infty$$

Similarly,

$$\|M_u C_\varphi\|^2_{D,L^2_\alpha,HS} = \|u\|^2_{L^2_\alpha} + \sum_{n=1}^{\infty} \frac{\|u\varphi^n\|^2_{L^2_\alpha}}{n} = \|u\|^2_{L^2_\alpha} + \int_{D} |u(z)|^2 \log \frac{1}{1 - |\varphi(z)|^2} \, dA_\alpha(z)$$

and

$$\|M_u C_\varphi\|^2_{D,L^2_\alpha,HS} = \|u\|^2_{L^2_\alpha} + \sum_{n=0}^{\infty} \|u\varphi^n\|^2_{L^2_\alpha} = \int_{D} \frac{|\varphi(z)|^2}{1 - |\varphi(z)|^2} \, dA_\alpha(z)$$

Hence $M_u C_\varphi \in C_w(H^2, L^2_\alpha)$ is Hilbert–Schmidt if and only if.

$$\int_{D} |u(z)|^2 \log \frac{1}{1 - |\varphi(z)|^2} \, dA_\alpha(z) < \infty,$$

and $M_u C_\varphi \in C_w(D, L^2_\alpha)$ is Hilbert–Schmidt if and only if

$$\int_{D} \frac{|u(z)|^2}{1 - |\varphi(z)|^2} \, dA_\alpha(z) < \infty.$$

When $H = H_1$ in Theorem (3.1.6), we have the following.

Corollary (3.2.3)[3]: If H satisfies all (#1)–(#4), then $C_{w,HS}(H)$ is a path connected space.

In this section, we study the Hilbert–Schmidt norms of differences of composition operators in $C(D, H^2)$ and $C(D, L^2_\alpha)$ for $-1 < \alpha < \infty$. We note that

$$C(D, H^2) = C(D, L^2_\alpha) = \{C_\varphi : \varphi \in S(D)\}$$

as sets. By the same way as in the proof of Theorem (3.1.1), $C(D,H^2)$ and $C(D, L^2_\alpha)$ are path connected spaces with respect to the operator norm topologies.

Lemma (3.2.4)[3]:

(i) $C_\varphi \in C_{HS}(D, H^2)$ if and only if

$$\int_{\partial D} \log \frac{1}{1 - |\varphi^*|^2} dm < \infty.$$

(ii) $C_\varphi \in C_{HS}(D, L^2_\alpha)$ if and only if

$$\int_{D} \log \frac{1}{1 - |\varphi(z)|^2} dA_\alpha(z) < \infty.$$

For $\varphi \in S(D)$, we write $E(\varphi) = \{e^{i\theta} \in \partial D : |\varphi^*(e^{i\theta})| = 1\}$. In case (i) of Lemma (3.2.4), we have $m(E(\varphi)) = 0$, we may say that $C_\varphi \in C_{HS}(D, H^2)$ if and only if φ is a non-extreme point of the closed unit ball of H^∞. Applying the same way as in the proof of Theorem (3.1.6), $C_{HS}(D, H^2)$ and $C_{HS}(D, L^2_\alpha)$ are path connected spaces with respect to the Hilbert–Schmidt norm topologies.

Theorem (3.2.5)[3]: $C_{HS}(D, L^2_\alpha) = \{C_\varphi : \varphi \in S(D)\}$ as sets for every $-1 < \alpha < \infty$.

Proof : We have

$$\int_{D} \log \frac{1}{1 - |z|^2} dA_\alpha(z) = \frac{1}{(\alpha + 1)^2} < \infty.$$

Let $\varphi \in S(D)$ with $\varphi(0) = 0$. Then $|\varphi(z)| \le |z|$ on D and

$$\int_{D} \log \frac{1}{1 - |\varphi(z)|^2} dA_\alpha(z) \le \int_{D} \log \frac{1}{1 - |z|^2} dA_\alpha(z) < \infty$$

By Lemma (3.2.4)(i), we have $C_\varphi \in C_{HS}(D, L^2_\alpha)$

Let $\varphi \in S(D)$ with $\varphi(0) \neq 0$. Put $a = \varphi(0)$ and

$$\psi(z) = \frac{\varphi(z) - a}{1 - \bar{a}\varphi(z)}, \qquad z \in D.$$

Then $\psi \in S(D)$ with $\psi(0) = 0$. By the last paragraph,

$$\int_{D} \log \frac{1}{1 - |\psi(z)|^2}\, dA_\alpha(z) < \infty$$

so

$$\int_{D} \log \frac{1}{1 - |\psi(z)|}\, dA_\alpha(z) < \infty$$

We have

$$\varphi(z) = \frac{\psi(z) + a}{1 + \bar{a}\psi(z)}, \qquad z \in D.$$

Since

$$1 - |\varphi(z)| \geq 1 - \frac{|\psi(z)| + |a|}{1 + |a||\psi(z)|} = \frac{(1 - |a|)(1 - |\psi(z)|)}{1 + |a||\psi(z)|}$$

we have

$$\frac{1}{1 - |\varphi(z)|} \leq \frac{2}{(1 - |a|)(1 - |\psi(z)|)}$$

Hence

$$\int_{D} \log \frac{1}{1 - |\varphi(z)|}\, dA_\alpha(z) \leq \int_{D} \log \frac{2}{(1 - |a|)}\, dA_\alpha(z) + \int_{D} \log \frac{1}{1 - |\psi(z)|}\, dA_\alpha(z) < \infty,$$

so, by Lemma(3.2.4)(ii) we have $\varphi(z) \in C_{HS}(D, L^2_\alpha)$.

By Lemma (3.2.4) and Theorem (3.2.5) , we have the following.

Corollary (3.2.6)[3]: $C_{HS}(D, H^2) \subsetneq C_{HS}(D, L^2_\alpha) = \{C_\varphi : \varphi \in S(D)\}$ for every $-1 < \alpha < \infty$.

Corollary (3.2.7)[3]: For any $\varphi, \psi \in S(D)$, $C_\varphi - C_\psi : D \to L^2_\alpha$ is Hilbert–Schmidt for every $-1 < \alpha < \infty$.

For $\varphi, \psi \in S(D)$, we have

$$\|C_\varphi C_\psi\|^2_{D, H^2, HS} = \sum_{n=1}^{\infty} \frac{1}{n} \|\varphi^n - \psi^n\|^2_{H^2} \geq \sum_{n=1}^{\infty} \frac{1}{n} \|\varphi^n - \psi^n\|^2_{L^2_\alpha} = \|C_\varphi C_\psi\|^2_{D, L^2_\alpha, HS}$$

For $z, w \in \overline{D}$ withz $\neq w$, let $\rho(z, w) = |z - w| / |1 - \overline{w} z|$.

Theorem (3.2.8)[3]: For $\varphi, \psi \in S(D)$, we have

$$\|C_\varphi - C_\psi\|^2_{D,L^2_\alpha,HS} = \int_D \log \frac{1}{1 - \rho(\varphi(z), \psi(z))^2} \, dA_\alpha(z)$$

Proof: We have that

$$\|C_\varphi - C_\psi\|^2_{D,L^2_\alpha,HS} = \sum_{n=1}^{\infty} \frac{1}{n} \|\varphi^n - \psi^n\|^2_{L^2_\alpha}$$

$$= \int_D \sum_{n=1}^{\infty} \frac{|\varphi(z)|^{2n} + |\psi(z)|^{2n} - \varphi^n(z)\overline{\psi(z)^n} - \overline{\varphi(z)^n}\psi(z)^n}{n} dA_\alpha(z)$$

$$= \int_D \log \frac{\left|1 - \varphi(z)\overline{\psi(z)}\right|^2}{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)} dA_\alpha(z)$$

$$= \int_D \log \frac{1}{1 - \rho(\varphi(z), \psi(z))^2} \, dA_\alpha(z)$$

Corollary (3.2.9)[3]: For φ , $\psi \in S(D)$, we have

$$\int_D \log \frac{1}{1 - \rho(\varphi(z), \psi(z))^2} \, dA_\alpha(z) < \infty$$

Proof: By Theorem (3.2.5),

$$\|C_\varphi\|_{D,L^2_\alpha,HS} + \|C_\psi\|_{D,L^2_\alpha,HS} < \infty$$

By Theorem (3.2.8),

$$\int_D \log \frac{1}{1 - \rho(\varphi(z), \psi(z))^2} \, dA_\alpha(z) = \|C_\varphi - C_\psi\|^2_{D,L^2_\alpha,HS}$$

$$\leq \left(\|C_\varphi\|_{D,L^2_\alpha,HS} + \|C_\psi\|_{D,L^2_\alpha,HS}\right)^2 < \infty$$

Corollary (3.2.10)[3]: For $\varphi,\psi \in S(D)$ and $0 \leq t \leq 1$, let $\varphi_t = \varphi_t + (1 - t)\psi \in S(D)$. Then $\{C_{\varphi_t} : 0 \leq t \leq 1\}$ is a continuous path in $C_{HS}(D, L^2_\alpha)$ connecting C_φ with C_ψ.

Proof: By Theorem (3.2.5), $C_{\varphi_t} \in C_{HS}(D, L^2_\alpha)$ for every $0 \le t \le 1$. It is sufficient to show that

$$\lim_{t \to 1} \left\| C_{\varphi_t} - C_\varphi \right\|^2_{D, L^2_\alpha, HS} = 0$$

Since $\rho((z), \varphi_t(z)) \le \rho((z), \psi(z))$ for $z \in D$,

$$\log \frac{1}{1 - \rho(\varphi, \varphi_t)^2} \le \log \frac{1}{1 - \rho(\varphi, \psi)^2} \text{ on D.}$$

By Corollary (3.2.9),

$$\int_D \log \frac{1}{1 - \rho(\varphi(z), \psi(z))^2} dA_\alpha(z) < \infty.$$

Since $\rho(\varphi(z), \varphi_t(z)) \to 0$ as $t \to 1$, by the Lebesgue dominated convergence theorem,

$$\int_D \log \frac{1}{1 - \rho(\varphi(z), \varphi_t(z))^2} dA_\alpha(z) \to 0$$

as $t \to 1$. By Theorem (3.2.8), we get the assertion.

Next, we shall study the structure of $C_{HS}(D, H^2)$. For $\varphi, \psi \in S(D)$ with $\varphi \neq \psi$, we have that $\varphi^*(e^{i\theta}) \neq \psi^*(e^{i\theta})$ for almost every $e^{i\theta} \in \partial D$. Hence we may define $\rho(\varphi^*(e^{i\theta}), \psi^*(e^{i\theta}))$ for almost every $e^{i\theta} \in \partial D$. In the same way as in the proof of Theorem (3.2.8), we have the following.

Theorem(3.2.11)[3]: For $\varphi, \psi \in S(D)$ with $\varphi \neq \psi$, we have

$$\left\| C_\varphi C_\psi \right\|^2_{D, H^2, HS} = \int_{\partial D} \log \frac{1}{1 - \rho(\varphi^*, \psi^*)^2} dm$$

Proof: We have that

$$\left\| C_\varphi - C_\psi \right\|^2_{D, H^2, HS} = \sum_{n=1}^{\infty} \frac{1}{n} \|\varphi^n - \psi^n\|^2_{H^2}$$

$$= \int_{\partial D} \sum_{n=1}^{\infty} \frac{|\varphi^*|^{2n} + |\psi^*|^{2n} - \varphi^n \overline{\psi^{*n}} - \overline{\varphi^{*n}} \psi^{*n}}{n} dm$$

$$= \int_{\partial D} \left(\log\frac{1}{1 - |\varphi^*|^2} + \log\frac{1}{1 - |\psi^*|^2} - \log\frac{1}{1 - \varphi^*\overline{\psi^*}} - \log\frac{1}{1 - \overline{\varphi^*}\psi^*} \right) dm$$

$$= \int_{\partial D} \log\frac{|1 - \varphi^*\overline{\psi^*}|^2}{(1 - |\varphi^*|^2)(1 - |\psi^*|^2)}\, dm = \int_{\partial D} \log\frac{1}{1 - \rho(\varphi^*, \psi^*)^2} dm$$

Corollary (3.2.12)[3]: For φ, ψ ∈S(D) with C_φ, $C_\psi \in C_{HS}(D, H^2)$ and $0 \le t \le 1$, let $\varphi_t = t_\varphi + (1-t)\psi \in S(D)$. Then $\{C_{\varphi_t}: 0 \le t \le 1\}$ is a continuous path in $C_{HS}(D, H^2)$ connecting C_φ with C_ψ.

Proof: By the fact mentioned in the below of Lemma (3.2.4), φ and ψ are non-extreme points of the closed unit ball of H^∞ .Hence $C_{\varphi_t} \in C_{HS}(D, H^2)$ Using Theorem (3.2.11), in the same way as in the proof of Corollary (3.2.10) we may prove the assertion.

Theorem (3.2.13)[3]: Let φ ∈ S(D) satisfy $C_\varphi \notin C_{HS}(D, H^2)$Then $\|C_\varphi - C_\psi\|^2_{D,H^2,HS} = \infty$ for every ψ ∈ S(D) with ψ ≠ φ.

Proof: Suppose that $\|C_\varphi - C_\psi\|^2_{D,H^2,HS} < \infty$ for some ψ ∈ S(D) with ψ ≠ φ. By Theorem (3.2.11), we have m(E(φ)) = m(E(ψ)) =0 and

$$\int_{\partial D} \log\frac{1}{1 - \rho(\varphi^*, \psi^*)^2} dm < \infty.$$

Let η=(φ +ψ)/2. We have that ρ(φ*, η*) ≤ ρ(φ*, ψ*) almost everywhere on ∂D. Hence

$$\int_{\partial D} \log\frac{1}{1 - \rho(\varphi^*, \eta^*)^2} dm \le \int_{\partial D} \log\frac{1}{1 - \rho(\varphi^*, \psi^*)^2} dm < \infty.$$

By Theorem (3.2.11) again, $\|C_\varphi - C_\eta\|^2_{D,H^2,HS} < \infty$. Since ηis a non-extreme point of the closed unit ball of H^∞, by Lemma (3.2.4) we have $C_\eta \in C_{HS}(D, H^2)$ Then $C_\varphi \in C_{HS}(D, H^2)$ This is a contradiction. Thus we get the assertion.

Corollary (3.2.14)[3]: Let φ, ψ ∈ S(D). If $\|C_\varphi - C_\psi\|^2_{D,H^2,HS} < \infty$, then both C_φ and C_ψ are in $C_{HS}(D, H^2)$.

CHAPTER 4

Symbol of Universal Covering Map for Compact Composition Operators

We consider , in particular, conditions that determine compactness of such operators and demonstrate a link with the Poincare series of the uniformizing fuchsian group. We show that is compact if if and only if does not not have afinite angular derivative at any point of the unit circle, there by extending the result for univalent and finitely multivalent.

Section (4.1) Introduction and Preliminaries:

Let $D = \{z \in C: |z| < 1\}$ be the unit disk in the complex plane, then the Hardy space $H^p, 1 \le p < 1$, is defined to be the Banach space of functions holomorphic in D with norm

$$|f|_p^p = \lim_{r \to 1} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta < \infty$$

The limit here is guaranteed by the fact that the integral mean is increasing in r. The standard text for the theory of Hardy spaces is .

Given a holomorphic map $\phi: D \to D$ we define the composition operator

$$C_\phi: f \to f \circ \phi$$

The study of composition operators acting on function spaces has received much attention over the last four decades. The central theme of this work is to understand how operator theoretic properties of composition operators are related to geometric or analytic properties of their inducing functions.of central importance in this area is a result, which describes the essential norm of a composition operator in terms of the Nevanlinna counting function of its inducing holomorphic map. The Nevanlinna counting function is known explicitly in a number of situations, for example for inner functions, univalent functions and finitely multivalent functions .In this chapter we study composition operators with symbol a universal covering map of the unit disk onto a finitely connected domain, in this case the Nevanlinna counting function can be estimated precisely by properties of the underlying Fuchsian group (is a discrete subgroup of PSL(2,R) which can be regarded as a group of isometries of the hyperbolic plane, or conformal transformations of the unit disc, or conformal transformations of the upper half plane see[8]).We will provide all the preliminary definitions in section (4.2).

We consider throughout this article domains of the form

$$D = D_0 \backslash \{p_1, \ldots, p_n\} \qquad n \geq 1 \tag{1}$$

where D_0 is a simply connected domain contained in D and $p_1, \ldots, p_n$ are distinct, isolated points in the interior of D_0.We will study composition operators whose symbol ϕ is the universal covering map of D onto D.

For a Fuchsian group Γ we define the limit set Λ (Γ) to be the set of accumulation

points of orbits of points in D by functions in Γ. The Poincare series for Γ of order s is

$$\rho_\Gamma(z, w; s) = \sum_{g \in \Gamma} \exp - s d_D\big(z, g(w)\big) \tag{2}$$

where $d_D(z, w)$ is the hyperbolic distance from z to w in D.

It is known that there is a critical exponent, δ(Γ) such that the Poincare series converges for all s < δ (Γ) but diverges for all s >δ (Γ). For finitely generated Fuchsian groups

$$\delta(\Gamma) = \dim(\Lambda(\Gamma));$$

the Hausdorff dimension of the limit set of Γ.

A simple calculation shows that if Γ is elementary and generated by a parabolic element then

$$\delta(\Gamma) = 1/2:$$

If Γ is non-elementary and contains a parabolic element then Beardon showed in that

$$\delta(\Gamma) > 1/2$$

and if Γ is finitely generated and of the second kind then

$$\delta(\Gamma) < 1. \tag{3}$$

Our first result links the compactness of a composition operator to the growth of the universal covering map with respect to the Poincare series.

Theorem (4.1.1)[4]: Let D be a domain in D defined by (1) and suppose that ϕ is a universal covering map of D onto D.

Let Γ be the Fuchsian group that uniformizes D, then C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if for each $\zeta \in \partial D \, \Lambda(\Gamma)$

$$\lim_{z\to\zeta} \frac{\rho_\Gamma(0,z;1)}{1-|\phi(z)|} = 0 \qquad (4)$$

Note that the hypothesis implies that Γ is finitely generated and so (4) is well defined by (3).

An important geometric quantity that has proved useful in describing compactness of composition operators has been the angular derivative. A holomorphic mapping ϕ: D →D has a finite angular derivative $|\acute{\phi}(\zeta)|$ for $\zeta \in \partial D$ if

$$\liminf_{z\to\zeta} \frac{1-|\phi(z)|}{1-|z|} < \infty .$$

The existence of a finite angular derivative implies a number of well behaved mapping properties of ϕ near ζ, a good reference for this. Note that if an angular derivative exists then, in particular, $\lim_{z\to\zeta} |\phi(z)| = 1$, where the limit is non-tangential.

To appreciate the importance of this quantity it is known that, for ϕ univalent, C_ϕ is compact if, and only if

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = \infty \qquad (5).$$

or, equivalently, ϕ does not have a finite angular derivative at any point on ∂D. For arbitrary ϕ it was shown that if C_ϕ is compact then ϕ does not have a finite angularderivative at any point on ∂D, however in general it is not difficult to find counter examples to the converse. For example no inner function induces a compact operator but there are inner functions with no angular derivative at any point on ∂D .

We generalize the above result to the current setting.

In this section we will state and discuss Shapiro's characterisation of compact composition operators, followed by an short introduction to the relevant theory of universal covering maps and Fuchsian groups.

Recall the Calkin algebra for H^p is the algebra $B(H^p)/B_0(H^p)$ where $B(H^p)$ is the algebra of bounded linear operators mapping H^p to H^p, and $B_0(H^p)$ is the corresponding ideal of compact operators in $B(H^p)$. The essential norm of an operator T, written $\|T\|_e$ is the norm of T in the Calkin algebra. The essential norm measures the distance, in the norm induced metric, to the compact operators,

$$\|T\|_e = \inf_{K \in B_0(H^p)} \|T - K\| \quad .$$

provides a formula for the essential norm of C_ϕ that describes precisely its relationship with the inducing function ϕ. In order to state Shapiro's result we define the Nevanlinna counting function for ϕ to be

$$N_\phi(w) = \begin{cases} \sum_{z:\phi(z)=w} \log \frac{1}{\|z\|} & w \in \phi(D) \\ 0 & w \in D \setminus \overline{\phi(D)} \end{cases}$$

It is known and relatively easy to estimate N_ϕ when ϕ is finitely valent. Shapiro proved that

$$\|C_\phi\|_e^2 = \lim_{|w| \to 1} \sup \frac{N_\phi(w)}{\log \frac{1}{|w|}} \qquad (6)$$

In particular, C_ϕ is compact on Hp if and only if

$$\lim_{|w| \to 1} \frac{N_\phi(w)}{\log \frac{1}{|w|}} = 0$$

An inner function is a bounded holomorphic function I, on D for which

$$\lim_{r \to 1} \left| I(re^{i\theta}) \right| = 1$$

For almost every $\theta \in [0, 2\pi)$ with respect to Lebesugue measure. It is known that, outsidea set of 2-dimensional Lebesgue measure 0, an inner function I satisfies

$$N_1(w) = \log \left| \frac{I(0) - w}{1 - \overline{I(0)} w} \right|$$

An example of an inner function that is relevant to the current work is the function

$$z \mapsto \exp\left(-\frac{1+z}{1-z}\right)$$

That maps D conformally onto D\{0}. It is notable that the radial limit of this function along the positive real axis is 0, whereas all other radial limits have modulus 1. It has infinite angular derivative at 1 but has finite angular derivative elsewhere on ∂D. This is the universal covering map of D onto D\{0}.

In this chapter we examine how Shapiro's characterisation of compact composition operators may be interpreted when ϕ is a universal covering map. We will cover the prerequisite details required here in order to fix notation and relevant ideas. First recall that the hyperbolic metric, on D is defined by

$$d_D(z, w) = \inf \int_\gamma \frac{2}{1-|z|^2}|dz|$$

Where the infimum is taken over all smooth curves connecting z to w in D. The constant 2 is required to ensure that the Gaussian curvature of the metric is equal to -1 throughout D, it is often omitted in the literature. This metric is so called because it induces Poincare's disk model of hyperbolic space where geodesics are arcs of circles orthogonal to the unit circle or radii. In particular, we have that

$$d_D(0, w) = \log\frac{1+|w|}{1-|w|} \qquad (7)$$

Automorphisms of D are of the form

$$z \mapsto \lambda\frac{a-z}{1-\bar{a}z}$$

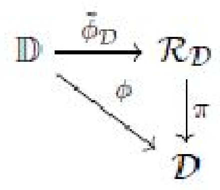
where $|\lambda|= 1$ and a ∈ D, and are isomorphisms in the hyperbolic metric. They are classified as elliptic, parabolic or hyperbolic according to whether they have a fixed point in D, a fixed point in ∂D, or 2 fixed points in ∂D respectively. The theory of automorphisms of D are covered in detail where many of the results concerning Fuchsian groups in this section may be found.

A group Γ of automorphisms of D may be considered a subspace of the topological space $GL_2(C)$, Γ is called a Fuchsian Group if it is discrete in the subspace topology .For any hyperbolic Riemann surface, R, there is a Fuchsian group Γ_R that contains no elliptic elements such that R is homeomorphic to D/Γ.

Given a domain $\mathbb{D} \subset D$ there is a Riemann surface R_D and a covering projection $\pi : R_D \rightarrow D$. Since R_D is conformally equivalent to $\mathbb{D}$ by the uniformization theorem we may find a $\tilde{\phi}_D : \mathbb{D} \rightarrow R_D$ so that the mapping

$$\phi = \pi \circ \tilde{\phi}_D$$

Maps D conformally onto D:



ϕ is the universal covering map of D and is unique up to pre-composition with an automorphism of D. It follows from the construction above that the inverse of $\phi(w)$ for any $w \in D$ is the fiber over w and this is a Γ-orbit, i.e. is of the form $\Gamma(z) = \{g(z) : g \in \Gamma\}$.

A fundamental domain for the action of Γ on D is said to be locally finite if each compact subset of D meets only finitely many Γ-images of $\tilde{F}$. F is locally finite if and only if the mapping

$$\theta : \tilde{F} \cap \Gamma(z) \mapsto \Gamma(z)$$

is a homeomorphism from of $\tilde{F}/\Gamma$ onto D/Γ. Here $\tilde{F}$ represents the relative closure of F in D.

The Dirichlet fundamental polygon for Γ is defined for given $w \in D$ as

$$D(w) = \bigcap_{g \in \Gamma, g \neq id} \{z \in D : d_D(z, w) < d_D(z, g(w))\}.$$

it is locally finite.

Finally for a Fuchsian group of the second kind the set of discontinuity is

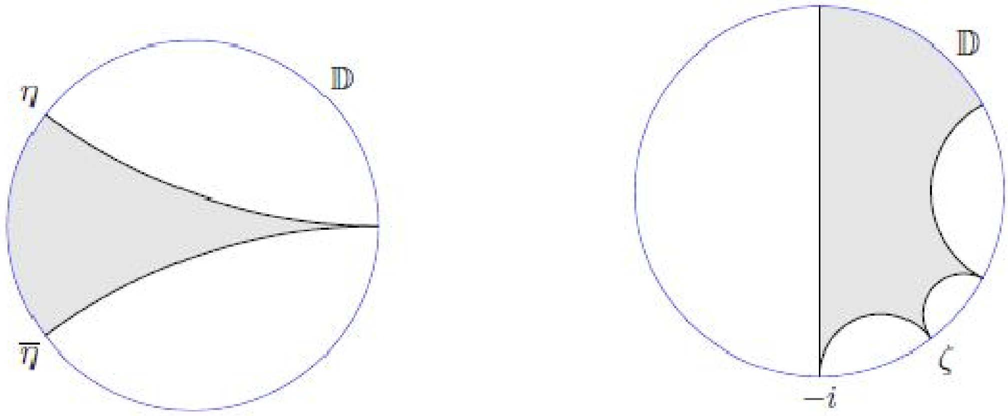
$$\Omega(\Gamma) = \hat{\mathbb{C}} \backslash \Lambda(\Gamma),$$

where $\hat{C} = C \cup \{\infty\}$. This set is connected and the action of Γ can be extended canonically to $\Omega(\Gamma)$ where it acts discontinuously.

Examples:

Figure 1: Fundamental domains for n = 1 and n = 2

n = 1: In the case n = 1 the domain D is uniformized by an elementary



Fuchsian group of the form

$$\Gamma = \langle \varrho \rangle,$$

with ϱ a parabolic disk automorphism. Suppose that ϱ has fixed point 1 then the Dirichlet domain D(0) is shown on the left in Figure 1

The two sides of F in D are equivalent in $D/\langle \varrho \rangle$. The free side of F is homeomorphicto ∂D_0 (note the two end points of the free side are equivalent).

n = 2 :For the case n = 2 the domain D is conformally equivalent to the Riemann surface D/Γ where Γ is generated by two parabolic automorphisms, ϱ1and ϱ2. A fundamental set for Γ is illustrated on the right in Figure 1, here we assume that the fixed points of $\varrho_1$and ϱ_2 are ζ and - i. The point ζ can be determined from the geometry of D, specifically the length of the closed hyperbolic geodesic separating the points p_1 and p_2 from the boundary of D_0. This example is taken where a more detailed discussion is available.

Section (4.2) Proof of Theorems and Concluding Remarks:

We will assume that $\partial D_0 \cap \partial D \neq \emptyset$, the result (and all main results) are trivially true if $\sup_\zeta |\emptyset(\zeta)| < 1$ in which case the angular derivative cannot exist anywhere.

Note first that the points $p_i, i = 1, \dots, n,$ are considered punctures in the Riemann surface and are therefore in one-to-one correspondence with the conjugacy class of parabolic elements in Γ.

Let F be a locally finite fundamental domain for the action of Γ on D. Then F can be chosen to be a finite sided convex polygon with one free side contained in ∂D, forexample we may take F to be a Dirichlet convex fundamental polygon.

Now $\tilde{F}/\Gamma$ is homeomorphic to D/Γ so that we may define a branch of the inverse of $\emptyset$ on a subdomain of D, ψ say, that maps this sub domain univalently onto F.

Let I be the free side of F then as $|w| \to 1$ in D, $z = \psi(w)$ tends to I. To see this we simply need to ensure that z does not converge to other boundary points of F. To this end, suppose that Γ is non-elementary, then I is contained in an interval of discontinuity of Γ on ∂D, γ say. If we let A be the hyperbolic geodesic in D with the same end points as γ then $A \cap \tilde{F}$ is homeomorphic to the closed hyperbolic geodesic in D that separates ∂D_0 from the points $p_1, \dots, p_n$. Therefore A separates I from other corners of F and so as

$$w \to \partial D\{p_1, \dots, p_n\}, \qquad z \to I.$$

One can check the case n = 1 when Γ is elementary directly.

Assume then that $|z| > R > \frac{1}{2}$ for a given R. Then for w ∈ D

$$N_\phi(w) = \sum_{g\in\Gamma} \log \frac{1}{|g(z)|}.$$

Since Γ is discontinuous on D there are only finitely many g ∈Γ with $g(z) \in \{z : |z| \le R\}$. Hence, using the inequality

$$\log\frac{1}{x} \le 1 - x^2 \le 2\log\frac{1}{x}; \qquad 1/2 < x < 1$$

we have that

$$N_\phi(w) \le C\sum_{g\in\Gamma}(1 = |g(z)|)^2$$

$$\le C\sum_{g\in\Gamma}\frac{1-|g(z)|}{1+|g(z)|}$$

$$= C\sum_{g\in\Gamma}\exp -d_D(0,g(z))$$

$$= C_{\rho\Gamma}(0,z;1)$$

Similarly $N_\phi(w) \ge C_{\rho\Gamma}(0,z;1)$.

We have shown thus far that

$$\lim_{|w|\to 1}\frac{N_\phi(w)}{\log\frac{1}{|w|}}$$

If and only if

$$\lim_{z\to 1}\frac{\rho\Gamma(0,z;1)}{1-|\phi(z)|} = 0 \qquad (8)$$

Where the limit takes place in $\tilde{F}$.

To show that this implies our result note that since Γ is discontinuous on Ω(Γ), we have that for any closed arc$J \subset \partial D\setminus\Lambda(\Gamma)$ finitely many images of $\tilde{F}$ under mapping in Γ cover J and we may apply (8) to each without difficulty using the automorphic property of ρΓ. Therefore the limit (4) is zero at any point in ζ∈ J and, in particular at any point in $\partial D\setminus\Lambda(\Gamma)$.

The converse, that (4) implies (8), is, of course, trivial.

In order to prove this result we will require the following quantitative estimate of the Poincare series of index 1.

Lemma (4.2.1)[4]: If Γ uniformizes a domain of the form (1) then for z ∈D(0) with $|z|$close enough to 1.

$$c_1 \exp - d_D(0,z) \le \rho\Gamma(0,z,1) \le c_2 \exp - d_D(0,z)$$

Where c_1 and c_2 are constants depending only on Γ.

Lemma (4.2.2)[4]: Γ is finitely generated if and only if each ζ∈Λ(Γ) is either

(i) A fixed point for a parabolic element of Γ; or

(ii) A point of approximation – i.e. there is a sequence g_n, n = 1, 2,…, of elements of Γ such that $g_n(0) \to \zeta$ non-tangentially.

Theorem (4.2.3)[4]: Suppose that D is defined by (1) and ϕ is a universal covering of D onto D. Then C_ϕ is compact on H^p, $1 \le p < \infty$, if and only if

$$\lim_{z\to\zeta} \frac{1-|\phi(z)|}{1-|z|} = \infty$$

For all ζ ∂D.

It follows that the counter examples to Shapiro and Taylor's result cannot come from universal covering maps of finitely connected domains.

This result begins to demonstrate the link between the compactness of C_ϕ and the geometry of the image domain. In fact, as a consequence of the previous theorem and properties of inner functions that we will discuss later, we can develop this idea further.

Proof : Let ζ∈∂D be arbitrary. If ζ is a parabolic fixed point, then $\phi(z) \to p_j$ for some j when z →ζ, .Since $|p_j| < 1$ it follows that ϕ has infinite angular derivative there.

Similarly if ζ is a point of approximation then, with g_n a suitable sequence such that $g_n(0) \to \zeta$ as $n \to \infty$, we have that

$$|\phi(g_n(0))| = |\phi(g_0(0))| < 1$$

and since $g_n(0)$ converges non-tangentially, it follows from the Julia-Caratheodory theorem(states that if U is a simply connected open subset of the complex plane C, whose boundary is Jordan curve Γ then the Riemann map complex plane C , Whose boundary is Jordan curve Γ then the Riemann map f:U→D from U to the unit disk D extends continuously to the boundary, giving a homeomorphism F:Γ→S^1 from Γ to the unit circle S^1 see [9])that the angular derivative at ζ is

$$\left|\acute{\phi}(\zeta)\right| = \lim_{n\to\infty}\frac{1-|\phi(g_n(0))|}{1-|z|} = \lim_{n\to\infty}\frac{1-|\phi(g_0(0))|}{1-|z|} = \infty.$$

From Lemma (4.2.2) all other points in ∂D are in the complement of the limit set of Γ.

Suppose first that ϕ has infinite angular derivative at all points ζ∈ ∂D\ Λ(Γ). Then from Lemma (4.2.1)

$$\lim_{z\to\zeta}\frac{1-|\phi(z)|}{\rho\Gamma(0,z;\,1)} \geq c\,\lim_{z\to\zeta}\frac{1-|\phi(z)|}{1-|z|} = \infty:$$

The compactness of C_ϕ now follows from Theorem (4.1.1).

Suppose, conversely, that C_ϕ is compact. Let ζ ∈∂D\ Λ(Γ) and I⊂∂D(0) be the free edge of D(0). There is a h ∈Γ such that ζ∈h(I) and we may suppose without loss of generality that z →ζ inside

$$h\big(D(0)\big) = D\big(h(0)\big).$$

Then by continuity of h^{-1}, as z → ζ,

$$z^* = h^{-1}(z) \to h^{-1}(\zeta) = \zeta^*$$

And $\zeta^*\in I$.

From Lemma (4.2.2) we thus have

$$\liminf_{z\to\zeta}\frac{1-|\phi(z)|}{1-|z|} = \liminf_{z^*\to\zeta^*}\frac{1-|\phi(z^*)|}{1-|z^*|}\cdot\frac{1-|z^*|}{1-|z|}$$

$$= \frac{1}{\left|\acute{h}(\zeta)\right|}\liminf_{z^*\to\zeta^*}\frac{1-|\phi(z^*)|}{1-|z^*|} \geq c\liminf_{z^*\to\zeta^*}\frac{1-|\phi(z^*)|}{\rho\Gamma(0,z^*;\,1)} = 1$$

Therefore ϕ has infinite angular derivative at each ζ∈∂D as required

Theorem (4.1.1) follows almost immediately from Theorem (4.2.3) given certain properties of inner functions. First, for a given inner function I, a singular point is a point η∈∂D such that I cannot be extended to be analytic in a neighbourhood of η. The set of singular points of a universal cover of D onto D\{p_1,…,p_n} is easily seen to be the limit set of the uniformizing Fuchsian group. In fact if η∈Λ(Γ) then in each neighborhood there are infinitely many zeros of I- a for any a ≠ p_i, i = 1,…,n and so η is singular. To prove the contrapositive recall that Γ acts discontinuously on the larger set Ω(Γ). Therefore I may be extended to a holomorphic function on Ω(Γ)by considering the universal covering map of Ω(Γ)onto the so-called Schottky double Ω(Γ)/Γ. It follows that if $\eta \notin \Lambda(\Gamma)$ then η is not singular. Note in this case $\acute{I}$ exists inthe normal sense on $\partial D \setminus \Lambda(\Gamma)$ and is non-zero there since it is conformal, furthermoresince I is inner the absolute value of $\acute{I}(\eta)$ coincides with the angular derivative.

Theorem (4.2.4)[4]: Suppose that D is defined by (1), ϕ is a universal covering of D onto D, and ψ is the univalent Riemann mapping of D onto D_0. Then C_{ϕ} is compact on H^p,$1 \leq p < \infty$, if and only if C_{ψ} is.

There are a number of geometric interpretations of the existence of an angular derivative for univalent functions that can now be applied to D_0 that will ensure compactness of C_{ϕ}. We will not list these here but many of these can be found and throughout the literature on compact composition operators.

Proof: The proof of this result follows from the properties above and the Julia-Caratheodory theorem , we will merely sketch the details here.

First note that

$$\psi^{-1} o\ \phi : D \to D\{\acute{p}_1, \dots, \acute{p}_n\}$$

For $\acute{p}_i = \psi(\acute{p}_i), i = 1, \dots, n$. It follows from uniqueness that $I = \psi^{-1} o\ \phi$ is theuniversal covering map of D onto $D \setminus \{\acute{p}_1, \dots, \acute{p}_n\}$. Clearly it is also an inner functionso that the remarks above apply.

Suppose first that C is compact, then

$$C_{\phi} = C_I C_{\psi} \tag{9}$$

Since the compact operators form a left ideal in the algebra of bounded operators this means that C_{ϕ} is compact.

To prove the converse, suppose C_{ϕ} is compact. Then ϕ has infinite angular derivative at all points of ∂D by Theorem (4.2.3). Suppose now that $\eta \in \partial D$ is a point at which

$$|\acute{\psi}(\eta)| < \infty.$$

Let F be locally finite fundamental domain for the uniformizing group Γ of $D \setminus \{\acute{p}_1, \dots, \acute{p}_n\}$,then we may find $\zeta \in \partial F$ such that

$$\lim_{r \to 1} I(r\zeta) = \eta;$$

Furthermore ζ is in the free side of F and hence, that $\left|\acute{I}(r\zeta)\right| < \infty 1$ and that the limit above is non-tangential. It follows that

$$\lim_{r\to 1} \acute{\phi}(r\zeta) = \lim_{r\to 1} \acute{\psi}\big(I(r\zeta)\big)\,.\,\acute{I}(r\zeta) = \lambda\acute{\psi}(\eta)\left|\acute{I}(\eta)\right|$$

for some $|\lambda| = 1$. Since the right hand side above is finite we have a contradiction and hence $\left|\acute{\psi}(\eta)\right| = \infty$.

List of Symbols

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