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Stability and Perturbed Metric-Preserved Mappings with Universal Theorem

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By

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Dedication

To my mother
Unknown soldier in our home

To my father
It's the greatest love that he holds

To my brothers and sisters

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Before of all the praise and thanks be to Allah whom to be a scribed all perfection and majesty. The thanks after Allah must be to my Supervisor Prof. Shawgy Hussein Abdalla who supervised this research and guide me in patience until the result of this research are obtained. I wish to express my thanks to the Sudan University of Science and Technology, and Omdurman Islamic University, Faculty of Science & Technology, department of mathematics. My thanks also must be sent to all my friends and classmates for any support that make me complete this research.

Abstract

If Y is Gateaux smooth, strictly convex and admitting the Kadec- Klee property, then we has the following sharp estimate $\| Tf(x) - x \| \leq 2\varepsilon$, for all $x \in X$. Let X, Z be two real Banach spaces and ε ≥ 0, we show that if there is a mapping f: X→ Z with $f(0) = 0$ satisfying
$| \| f(x) - f(y) \| - \| x - y \| | \leq \varepsilon \; for \; all \; x, y \in X$, then we can define a linear surjective isometry $U: X^* \to Z^* / N$ for some closed subspace N of Z^* by an invariant mean of X. There is a linear surjective operator
$T: Y \to X$ of norm one such that $\| Tf(x) - x \| \leq 2\varepsilon, for \; all \; x \in X$; when the ε-isometry f is surjective, it is equivalent to Omladič - Šemrl Theorem: There is a surjective linear isometry $U: X \to Y$ so that
$\| f(x) - Ux \| \leq 2\varepsilon, for \; all \; x \in X$.

الخلاصة

اذا كان Y هو ملسان جاتوكس و التحدب التام و يقبل خاصية كاديك- كلى و اذا ينتج لنا التقدير القاطع اللآتى $\| Tf(x) - x \| \leq 2\varepsilon$ لاجل كل $x \in X$. ليكن X, Z فضائى باناخ الحقيقيين و$0 \geq \varepsilon$ اوضحنا انه اذا كان يوجد راسم $f: X \to Z$ مع $f(0) = 0$ يحقق $| \| f(x) - f(y) \| - \| x - y \| | \leq \varepsilon$ لاجل كل $x, y \in X$ اذا يمكننا ان نعرف الايزومترى الشامل الخطى $U: X^* \to Z^* / N$ لاجل بعض الفضاء الجزئى المغلق N الى Z^* بواسطة وسط ثابت الى X . يوجد مؤثر شامل خطى $T: Y \to X$ له نظيم واحد حيث ان $\| Tf(x) - x \| \leq 2\varepsilon$ لاجل كل $x \in X$ عندما f الايزومترى-ε هو شامل وهو مكافئ لمبرهنة أملديك- سيمرل: يوجد ايزومترى خطى شامل $U: X \to Y$ عليه ان $\| f(x) - Ux \| \leq 2\varepsilon$ لاجل كل $x \in X$.

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Chapter 1
Non linear Non surjectiveε–isometries of Banach Spaces

Let X, Y be two Banach spaces, $\varepsilon \geq 0$. And let $f: X \to Y$ be an ε- isometry with $f(0) = 0$. We show first that for every $x^* \in X^*$, there exists $\emptyset \in Y^*$,with $\| \emptyset \| = \| x^* \| \equiv r$ such that $|\langle \emptyset, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r, for\ all\ x \in X$. We Show that if Y is reflexive and if $E \subset Y$ is α-complemented in Y, then there is a bounded linear operator $T: Y \to X$ with $\| T \| \leq \alpha$ such that $\| Tf(x) - x \| \leq 4\varepsilon, for\ all\ x \in X$.

Section(1.1) Applications And Stability Version of Reflexive Banach Space

Let X, Y be two Banach Spaces and $\varepsilon \geq 0$. A mapping $f: X \to Y$ is said to be an ε–isometry provided

$$\big| \| f(x) - f(y) \| - \| x - y \| \big| \leq \varepsilon, \text{ for all } x, y \in X.$$

If $\varepsilon = 0$, then the mapping f is simply called an isometry; and it is said to be A surjectiveε–isometry if, in addition, $f(X) = Y$. The study of ε–isometry has been divided into four cases:

(i) f is surjective and $\varepsilon = 0$;

(ii) f is non–surjective and $\varepsilon = 0$;

(iii) f is surjective and $\varepsilon \neq 0$; and

(iv) f is non–surjective and $\varepsilon \neq 0$.

A celebrated result, known as the Mazur–Ulam Theorem is a perfect answer to case (i).

Theorem (1.1.1)[1] (Mazer-Ulam). Suppose that $f: X \to Y$ is a surjective isometry with $f(0) = 0$. Then f is linear.

The following mapping $f: \mathbb{R} \to \ell_\infty^2$ defined for $t \in \mathbb{R}$ by $f(t) = (t, sint)$ shows that an into isometry f with $f(0) = 0$ is not necessarily linear. While a remarkable result about non-surjective isometry (i.e.,Case(ii)) was given by Figiel in 1967,which plays an important role in the study of isometric embedding and of Lipschitz-free Banach space. Godefro and Kalton show some deep relationship between isometry and linear isometry.

Theorem(1.1.2)[1](Figiel). Suppose that $f: X \to Y$ is an isometry with $f(0) = 0$. Then there exists a linear operator $F: L(f) \equiv \overline{span} f(x) \to X$ with $\| F \| = 1$ such that $Fof = I$ (the identity) on X. We call the operator F in the Theorem above Figiel's operator. We refer the reader for more detailed discussions of geometric embedding and related topics.

In1945, Hyers and Ulam proposed the following question: whether for every surjective ε–isometry $f: X \to Y$ with $f(0) = 0$ where exists a surjective linear isometry $U: X \to Y$ and $\gamma > 0$ $such$ $that$

$$\| f(x) - Ux \| \leq \gamma\varepsilon, \ for\ all\ x \in X. \tag{1}$$

After many years efforts of a number of mathematicians, the following sharp estimate was finally obtained by Omladič and Šemrl

Theorem(1.1.3)[1] (Omladič–Šemrl). If $f: X \to Y$ is a surjective ε–isometry with $f(0) = 0$, then there is a surjective linear isometry $U: X \to Y$ such that

$$\| f(x) - Ux \| \leq 2\varepsilon, \text{for all } x \in X.$$

Therefore, answers to the first three cases are perfect. The study of non-surjective ε–isometry (i.e.,case(iv)) has also brought to mathematicians′ attention. First proposed the following problem in 1995, and then he showed that the answer is affirmative if both X and Y are L_p $spaces$.

Šemrl and Väisälä further presented a sharp estimate of (2) With $\Upsilon = 2$ if both X and Y are L^p–spaces for $1 < p < \infty$.

Problem(1.1.4)[1]. Whether there exists a constant $\gamma > 0$ depending only on X and Y with the following property: For each ε–isometry $f: X \to Y$ with $f(0) = 0$ there is abounded linear operator $T: L(f) \to X$ such that

$$\| Tf(x) - x \| \leq \gamma\varepsilon, \ for\ all\ x \in X. \tag{2}$$

As we have known, the answer to Problem (1.1.4) is affirmative for L^p –spaces with $1< p < \infty$.

However, Qian presented the following simple counterexample.

Example(1.1.5)[1] (Qian). Given $\varepsilon > 0$, and let Y be a separable Banach space admitting an uncomplemented closed subspace X. Assume that g is a bijective mapping from X onto the closed unit ball B_y of Y with $g(0) = 0$. We define a map $f: X \to Y$ by $f(x) = x + \varepsilon g(x) / 2$ for all $x \in X$. Then f is ε–isometry with $f(0) = 0$ and $L(f) = Y$. But there are no such T and ϒ satisfying (2). Qian's counterexample, incorporating of an early result of Lindenstrauss and Tzafriri (a Banach space satisfying that every closed subspace is complemented is isomorphic to a Hilbert space) entails the following result.

Theorem (1.1.6).[1] A Banach space Y satisfying that for every closed subspace X ⊂Y and every ε–isometry $f: X \to Y$ with $f(0) = 0$ there exist bounded linear operator $T: L(f) \to X$ and $\Upsilon > 0$ such that (2) holds if and only if Y is isomorphic to a Hilbert space.

This disappointment makes us to search for(i) some weaker stability version satisfied by every ε–isometry, and (ii) some appropriate complementability assumption on some subspaces of Y associated with the mapping such that the strong stability result (2) holds. For an ε–isometry $f: X \to Y$ with $f(0) = 0$, we introduce the following subspace E of Y associated with the mapping f, which will play an important part in the sequel. Let $F = \{y^* \in Y^*: y^*$is bounded on $C(f) \equiv \overline{co}(f(X), -f(X))\}$ $E \subset Y$ is defined as the annihilator of the subspace $F \subset Y^*, i.e.$

$$E = \{y \in Y: \langle y^*, y\rangle = 0, for\ all\ y^* \in F\} \quad (3)$$

From Qian's counterexample we can observe that for every Banach space Y containing an uncomplemented closed subspace X, and for every $\varepsilon > 0$, there exist an ε–isometry f from X to Y with $f(0) = 0$ and with $E = X$ such That (2) of Problem (1.1.4) fails for f. In other words, the assumption that E is complemented in Y is essential for the study of the stability property (2) of an ε–isometry f. Before describing the main results of this chapter, we first introduce same notations to be used in the sequel. The letter X will always be a Banach space, and X^* its dual. We denote by B_x (resp.,S_x) the closed unit ball (resp., the unit sphere) of X. For a sub set $A \subset X$, $\bar{A}$ stands for the closure of A, and $co\ A$ ($\overline{co}\ A$) for the (closed) convex hull of A. Let $f: X \to Y$ be an ε–isometry for some $\varepsilon \geq 0$ with $f(0) = 0$; $L(f)$ = the closure of the linear span of $f(x)$;

$$M_\varepsilon = \{\emptyset \in Y^* is\ bounded\ by\ \beta_\emptyset \varepsilon\ on\ C(f) for\ some\ \beta_\emptyset > 0\};$$

M =the closure of $M\varepsilon$. We should mention here that the set $M_\varepsilon = \{\emptyset \in Y^*$is bounded on $C(f)$ if $\varepsilon \geq 0$; = $C(F)^\perp$, the annihilator of $C(f)$, if $\varepsilon = 0$.

This chapter is organized as follows. In the second section, after giving an improvement of a one –dimensional lemma which is presented in Qian, we show the following result, which can be understood as a weak stability version; on the other hand, because it plays a central rule and is used frequently in every section of this chapter ,we call it the Main Lemma.

Lemma (1.1.7)[1] (Main lemma). Let X and Y be Banach spaces, and let $f: X \to Y$ be an ε-isometry for some $\varepsilon \geq 0$ with $f(0) = 0$. Then for every $x^* \in X^*$, there exists $\emptyset \in Y^*$ with $\| \emptyset \| = \| x^* \| \equiv r$ such that

$$| \langle \emptyset, f(x) \rangle - \langle x^*, x \rangle | \leq 4\varepsilon r,\ \ for\ all\ x \epsilon X. \tag{4}$$

We present three examples of simple applications of the Main Lemma: the first one is , motivated by Dutrieux and Lancien's observation –an equivalence Theorem of Figiel's Theorem, a generalization of the equivalence theorem from isometry to ε–isometry; and the second one is that if $Y = \ell_\infty(\Gamma)$ for a non-empty setΓ, then the answer to Problem (1.1.4) is positive with $\Upsilon = 4$; and the third one is the for an ε–isometry from an n-dimensional space to a Banach space, the answer to Problem (1.1.4) is always affirmative with $\Upsilon = 4n$.For each ε–isometry f, making use of the

Main Lemma, we define first a set- valued "linear" mapping V associated with f, we discuss then the properties of the operators V and Q:$X^* \to Y^*/M$ define by $Qx^* = Vx^* + M$. We show finally the following stability version in reflexive spaces.

Theorem (1.1.8)[1]: Let X, Y be Banach spaces and Y be reflexive, and let $f: X \to Y$ be an ε–isometry with $f(0) = 0$. If E is α–complemented in Y, then there is a bounded linear operator $T: Y \to X$ with $\|T\| \leq \alpha$ such that

$$\| Tf(x) - x \| \leq 4\varepsilon, for\ all\ x \in X.$$

It is shown by Šemrl and Väälä that for the ε–isometry $f: X \to Y$ if Y is uniformly convex then the following limit always exists and defines a linear isometry T : $X \to Y$, $Tx \equiv \lim_{t\to\infty}(f(tx))/t$, $for\ all\ x \in X$.

Motivated by the result above, we discuss existence of such limits in general reflexive Banach spaces. As a result, we show the following result.

Theorem (1.1.9)[1]. Suppose that X, Y are Banach spaces and that Y is reflexive, and suppose that $f: X \to Y$ is an ε–isometry for some $\varepsilon \geq 0$ with $f(0) = 0$. If, in addition, the subspace $E \subset Y$ is strictly convex, then for all $x \in X$

$$Tx = w - \lim_{\lambda\to+\infty} f(\lambda x)/\lambda$$

Exist and $T: X \to E$ is a linear isometry.

Theorem (1.1.10)[1]. Suppose that X is a Banach space and that Y is a reflexive, Gateaux smooth and strictly convex Banach space admitting the Kadec–Klee property (A Banach space is said to have the Kadec-Klee property or (H-property) if weakly convergent sequence on the unit sphere is convergent in norm. Recall that sequence $\{x_n\} \subset X$ is said to be ε-separated sequence for some $\varepsilon > 0$ if $sep(x_n) = \inf\{\| x_n - x_m \| : n \neq m\} > \varepsilon$. A Banach space is said to have uniform Kadec-Klee property (UKK) if for every $\varepsilon > 0$ there exist $\delta > 0$ such that for every sequence (x_n) in $S(x)$ with sep$(x_n) > \varepsilon$ and $x_n \overset{w}{\to} x$ we have $\| x \| < 1 - \delta$**)[5].** Suppose that $f: X \to Y$

is an ε–isometry with $f(0) = 0$, and that the subspace $E \subset Y$ associated with f is α–complemented in Y. Then there is a linear operator $T: Y \to X$ with $\| T \| \leq \alpha$ such that $\| Tf(x) - x \| \leq 2\varepsilon$, $x \in X$.The following Lemma is an improvement of a result of Qian from 5ε to 3ε.

Lemma (1.1.11)[1]. Let Y be a Banach space, and let $f: \mathbb{R} \to Y$ be an ε– isometry with $f(0) = 0$. Then there is a linear functional $\emptyset \in Y^*$ with $\| \emptyset \| = 1$ such that

$$|\langle \emptyset, f(t) \rangle - t| \leq 3\varepsilon \ \ for\ all\ t \in \mathbb{R}. \quad (5)$$

Proof: Given $n \in \mathbb{N}$, let $\emptyset_n \in Y^*$ with$\| \emptyset_n \| = 1$ such that

$$2n - \varepsilon \leq \| f(n) - f(-n) \| = \langle \emptyset_n, f(n) - f(-n) \rangle \leq 2n + \varepsilon.$$

Then,

$$n + \varepsilon \geq \langle \emptyset_n, f(n) \rangle = \| f(n) - f(-n) \| + \langle \emptyset_n, f(-n) \rangle \geq n - 2\varepsilon,$$

And

$$-n - \varepsilon \leq \langle \emptyset_n, f(-n) \rangle = \langle \emptyset_n, f(n) \rangle - \| f(n) - f(-n) \| \leq -n + 2\varepsilon.$$

Note that for every t $\in [0, n]$,

$\langle \emptyset_n, f(t) \rangle = \langle \emptyset_n, f(n) \rangle - \langle \emptyset_n, f(n) - f(t) \rangle \geq (n - 2\varepsilon) - (n - t + \varepsilon) = t - 3\varepsilon$. We have

$$t - 3\varepsilon \leq \langle \emptyset_n, f(t) \rangle \leq t + \varepsilon.\ \ for\ all\ t \in [0, n] \quad (6)$$

On the other hand, for every t $\in [-n, 0]$,

$$t - \varepsilon \leq \langle \emptyset_n, f(t) \rangle = \langle \emptyset_n, f(-n) \rangle + \langle \emptyset_n, f(t) - f(-n) \rangle$$
$$\leq (-n + 2\varepsilon) + (t + n + \varepsilon) = t + 3\varepsilon,$$

That is,

$$t - \varepsilon \leq \langle \emptyset_n, f(t) \rangle \leq t + 3\varepsilon,\ \ for\ all\ t \in [-n, 0]. \quad (7)$$

Combining (6) with (7), we obtain

$$|\langle \emptyset_n, f(t) \rangle - t| \leq 3\varepsilon,\ \ for\ all\ n \in \mathbb{N}\ and\ t \in [-n, n]. \quad (8)$$

Note that$\| \emptyset_n \| = 1$ for all n. Alaoglu's Theorem implies that there is a net $(\emptyset_\alpha)$ in $(\emptyset_n)$ w^*–converging to a functional $\emptyset \in B_{Y^*}$. This and (8) entail that $-3\varepsilon \leq \langle \emptyset, f(t) \rangle \leq t + 3\varepsilon,\ \ for\ all\ t \in \mathbb{R}$. And clearly, $\| \emptyset \| = 1$.

To show the Main Lemma of this chapter, we need some Gateaux differentiability results about norm of Banach space. Recall that a Banach space X is said to be Gateaux differentiability space (GDS) provided every continuous convex function on X is densely Gateaux differentiable. This is equivalent to that every equivalent norm on X is somewhere Gateaux differentiable. A point x^* in a w^* −closedconvex set $C \subset X^*$ is said to be a w^*- exposed point of C provided there exists a point $x \in X$ such that $\langle x^*, x\rangle > \langle y^*, x\rangle\ for\ all\ y^* \in C\ with\ y^* \neq x^*$. In this case, the point x is called a w^* –exposing functional of C and exposing C at x^* . We denote by w^* –exp C the set of all w^*– exposed points of C. For a convex function f defined on a Banach space X, its sub differential mapping $\partial f: X \to 2^{X^*}$is defined for $x \in X$ by $\partial f(x) = \{x^* \in X^*: f(y) - f(x) \geq \langle x^*, y - x\rangle$, for all $x \in X$ $\}$. It is easy to observe that if $f = \|.\|$ (the norm of X), then $\partial \| x \|$ $(x \neq 0)$ is always non-empty and $x^* \in \partial \| x \|$ if and only if $\langle x^*, x\rangle = \| x \|$ $with$ $\| x^* \| = 1$. The following result s are classical.

Proposition (1.1.12)[1]. Suppose that X is a Banach space and that $C \subset X^*$ is a non-empty w^*-compact convex set (Z is dual Banach space the closed unit ball of Z is weak star compact)**[6]** . Then $x^* \in C$ is a w^*–exposed point of C, and is w^*–exposed by $x \in X$ if and only if $\sigma c \equiv \sup c$ is Gateaux differentiable at x and with Gateaux derivative $d\sigma c(x) = x^*$.

Theorem (1.1.13)[1]. A Banach space X is a Gateaux differentiability space if and only if every non- empty w^*–compact convex set of its dual (of cause, including the closed unit ball of its dual) is the w^*–closed convex hull of its w^*–exposed points.

Lemma (1.1.14)[1] (Main lemma). Let X, Y be Banach spaces, $\varepsilon \geq 0$, and let $f: X \to Y$ be an ε–isometry with $f(0) = 0$. Then for every $x^* \in X^*$ there is a linear functional $\emptyset \in Y^*$ with $\| \emptyset \| = \| x^* \| = r$ such that

$$|\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon r,\ for\ all\ x \in X. \quad (9)$$

Proof: The proof shall be divided into two parts. In the first part we show that it is true if X is finite –dimensional. Then, making use of this result we show in the second part that the Lemma holds for a general Banach space X. Assume that dim $X < \infty$. Note X is a GDS. Then the closed unit ball B_{X^*} of X^* is the w^*–closed convex hull of its w^*–exposed points (Theorem (1.1.13)). Without loss of generality we can assume that r $= 1$. We show first that (9) is valid for some $\emptyset \in S_{Y^*}$, if $x^* \in S_{X^*}$ is a w^*-exposed point of B_{X^*}. By Proposition (1.1.12), there is a Gateaux differentiability point $x_0 \in S_X$ such that $\| x_0 \| = x^*$. Therefore, for every $x \in X$,

$$\text{Lim}_{t\to+\infty}(\| tx_0 + x \| - t) = \lim_{t\to+\infty} \frac{\|x_0 + \left(\frac{1}{t}\right)x\| - \|x_0\|}{\frac{1}{t}} = \langle x^*, x\rangle. \quad (10)$$

Let $g: \mathbb{R} \to Y$ be defined for t by $g(t) = f(tx_0)$. Then g is an ε-isometry with $g(0) = 0$. By Lemma (1.1.11), there is a linear functional $\emptyset \in S_{Y^*}$ such that

$$|\langle \emptyset, f(tx_0)\rangle - t| \le 3\varepsilon, \ \ for\ all\ t \in \mathbb{R}. \quad (11)$$

It entails that

$$t - 3\varepsilon - \langle \emptyset, f(x)\rangle \le \langle \emptyset, f(tx_0)\rangle - \langle \emptyset, f(x)\rangle \le \| f(tx_0) - f(x) \| \le$$
$$\| tx_0 - \mathrm{x} \| + \varepsilon.$$

Therefore, for all $t > 0$,

$$\| tx_0 - x \| - t + \langle \emptyset, f(x)\rangle \ge -4\varepsilon.$$

Let t$\to +\infty$ in the inequality above. Then (10) yields

$$\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle \le 4\varepsilon. \quad (12)$$

On the other word, we substitute –t for t in (11). Then

$$t - 3\varepsilon + \langle \emptyset, f(x)\rangle \le -\langle \emptyset, f(-tx_0)\rangle + \langle \emptyset, \mathrm{f(x)}\rangle \le \| f(-tx_0 - \mathrm{f(x)} \| \le$$
$$\| tx_0 + \mathrm{x} \| + \varepsilon.$$

Consequently,

$$\| tx_0 + x \| - t - \langle \emptyset, f(x)\rangle \ge -4\varepsilon.$$

Let t tend to $+\infty$ in the inequality above. Then (10) again implies that

$$\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle \ge -4\varepsilon. \quad (13)$$

Inequality (9) follows immediately from (12) and (13).

Next, we show that for every $x^* \in S_{X^*}$ there exist $\emptyset \in S_{Y^*}$satisfying (9). Let $x^* \in S_{X^*}. Since\ co(w^* - \exp B_{X^*})$ is dense in B_{X^*} (by Theorem (1.1.13) and noting dim X< ∞), there is a sequence$(x_n) \subset co(w^* - expB_{X^*})$ converging to x^* . Note that for every x_n^* there exist m w^*–exposed points $(x_{n_1}^*, x_{n_2}^*, \dots, x_{n_m}^*)$ and m non–negative numbers $(\lambda_{n_1}, \lambda_{n_2}, \dots, \lambda_{n_m})$ with $\sum_{j=1}^{m} \lambda_{n_j} = 1$ for some $m \in \mathbb{N}$ such that $x_n^* = \sum_{j=1}^{m} \lambda_{n_j} x_{n_j}^*$. Then by the fact we have just proven that there exist m functional $(\phi_{n_1}, \phi_{n_2}, \dots, \phi_{n_m}) \subset S_{Y^*}$ satisfying

$$\left|\langle \emptyset_n, f(x)\rangle - \langle x_{n_j}^*, x\rangle\right| \le 4\varepsilon, \tag{14}$$

For all$x \in X$, and $1 \le j \le m$. Let $\psi_n = \sum_{j=1}^{m} \lambda_{n_j} \emptyset_{n_j}$. Then $\| \psi_n \| \le 1$, and

$$|\langle \psi_n, f(x)\rangle - \langle x_n^*, x\rangle| \le 4\varepsilon, \qquad for\ all\ x \in X. \tag{15}$$

Since (ψ_n) is relatively w^*–compact, there must be a subset of (ψ_n) w^*–converging to some $\emptyset \in B_{Y^*}$. This, (15) and (x_n^*) being convergent to x^* together imply the following inequality

$$|\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \le 4\varepsilon, \quad for\ all\ x \in X.$$

Clearly, $\| \emptyset \| = \| x^* \| = 1$. Thus we have shown (9) for every finite-dimensional space X. We will finally show that (9) holds for a general Banach space X. Recall that Bishop–Phelps' Theorem states that norm–attaining functionals are always dense in the dual X^* of X. According to this Theorem, it suffices to show that (9) is true for every norm–attaining functional $x^* \in X^*$ with $\| x^* \| = 1$. (Indeed, suppose that (9) holds for every norm–attaining functional, i.e. for every norm–attaining functional $x^* \in X^*$ with $\| x^* \| = 1$, there is $\emptyset \in Y^*$ with $\| \emptyset \| = \| x^* \| = 1$ such that (9) holds. Then for every (general) $x^* \in X^*$with $\| x^* \| = 1$,by the Bishop–Phelps Theorem there is a sequence $(x_n^*) \subset X^*$ of norm–attaining functionals with$\| x_n^* \| = 1$ such that $x_n^* \to x^*$. For each n ∈ ℕ , let $\emptyset_n \in Y^*$ with $\| \emptyset_n \| = 1$ be the functional corresponding to x_n^* such that $|\langle \emptyset_n, f(x)\rangle - \langle x_n^*, x\rangle| \leq 4\varepsilon, \quad for\ all\ x \in X$. Then w^*–relative compactness of $(\emptyset_n)$ entails that there is a w^*–cluster point $\emptyset \in Y^*$ of $(\emptyset_n)$. It is easy to see that (9) holds again for such the functionals x^* and $\emptyset$ .Given such norm–attaining functional $x^* \in X^*$, let $x_0 \in S_X$ such that $\langle x^*, x_0\rangle = 1$, and let $\mathcal{F}$ be the collection of all finite–dimensional subspace of X containing x_0 .Since every $F \in \mathcal{F}$ is a GDS by (9) we have just proven, there exists $\emptyset_F \in S_{Y^*}$ such that

$$|\langle \emptyset_F, f(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon, \quad for\ all\ x \in F. \tag{16}$$

Let $\Phi_F = \{\emptyset_F \in Y^*\ satisfies\ (16)\ with\ \| \phi_F \| \leq 1\}$,and let

$$\Phi = \{\Phi_F : F \in \mathcal{F}\}.$$

It is clear that for every $F \in \mathcal{F}$, Φ_F , is a non-empty w^*–compact convex set of Y^* . Since for all $F, G \in \mathcal{F}, \Phi_F \cap \Phi_G \supset \Phi_H$, (where H = span {F, G}, they have the finite intersection property .Since every Φ_F is w^*-compact, they have a non- empty intersection, and any element $\emptyset$ of this intersection is clearly a solution of (9). The following result was first noticed by Dutrieux and Lancien, and it is equivalent to Figiel's Theorem.

Theorem (1.1.15)[1]. Let $f: X \to Y$ be an isometry with $f(0) = 0$.Then for all $x_1, x_2, \dots, x_n \in X$, and for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{j=1}^{n} |\lambda_j| = 1$, we have

$$\| \sum_{i=1}^{n} \lambda_i f(x_i) \| \geq \| \sum_{i=1}^{n} \lambda_i x_i \|.$$

Motivated by the theorem above, as an application of Lemma (1.1.14), we will show an analogous result of the theorem for ε–isometry.

Theorem (1.1.16)[1]. Let X and Y be Banach spaces, and let $f: X \to Y$ be an ε– isometry with $f(0) = 0$. Then

$$\| \sum_{i=1}^{n} \lambda_i f(x_i) \| + 4\varepsilon \geq \| \sum_{i=1}^{n} \lambda_i x_i \|. for\ all\ x_1, x_2, \dots, x_n \in X$$

And for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ satisfying $\sum_{i=1}^{n} |\lambda_i| = 1$.

Proof: Given $x_1, x_2, \dots, x_n \in X$, let $X_n = span(x_1, x_2, \dots, x_n)$. By Lemma (1.1.14) for every $x^* \in X_n^*$ there is a linear functional $\emptyset_{x^*} \in S_{Y^*}$ such that

$$|\langle \emptyset_{x^*}, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon, \quad for\ all\ x \epsilon X_n.$$

It entails that $\| \sum_{i=1}^{n} \lambda_i f(x_i) \| \geq sup_{x^* \in S_{X_n^*}} \langle \emptyset_{x^*}, \sum_{i=1}^{n} \lambda_i f(x_i) \rangle =$

$sup_{x^* \in S_{X_n^*}} (\langle \emptyset_{x^*}, \sum_{i=1}^{n} \lambda_i f(x_i) \rangle - \langle x^*, \sum_{i=1}^{n} \lambda_i x_i \rangle + \langle x^*, \sum_{i=1}^{n} \lambda_i x_i \rangle) \geq$

$sup_{x^* \in S_{X_n^*}} |\langle x^*, \sum_{i=1}^{n} \lambda_i x_i \rangle| - \sum_{i=1}^{n} |\lambda_i| |\langle \phi_{x^*}, f(x_i) \rangle - \langle x^*, x_i \rangle| \geq$

$sup_{x^* \in S_{X_n^*}} |\langle x^*, \sum_{i=1}^{n} \lambda_i x_i \rangle| - 4\varepsilon = \| \sum_{i=1}^{n} \lambda_i x_i \| - 4\varepsilon, for\ all\ \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}\ satisfying\ \sum_{i=1}^{n} |\lambda_i| = 1$. Consequently,

$\| \sum_{i=1}^{n} \lambda_i f(x_i) \| + 4\varepsilon \geq \| \sum_{i=1}^{n} \lambda_i x_i \|$.

The following Theorems are also simple applications of Lemma (1.1.14).

Theorem (1.1.17) [1]. For any setΓ, let $X = \ell^\infty(\Gamma)$ and Y be a Banach space. If $f: X \to Y$ is and ε–isometry for some $\varepsilon > 0$, then there exists an operator $S: Y \to X$ with $\| S \| = 1$ such that

$$\| Sf(x) - x \| \leq 4\varepsilon, \ for\ all\ x \in X \tag{17}$$

Proof: Since Fréchet differentiability points are dense in $\ell^\infty(\Gamma), B_{X^*}$
Is the w^*–closed convex hull of its w^* –strongly exposed points (in fact, the set of all w^*–strongly exposed points of B_{X^*} is just$(e_\Upsilon)_{\gamma \in \Gamma,}$all of the standard unit vectors of $\ell_1(\Gamma)$. Given any $\gamma \in \Gamma$, let $\delta_\gamma \in S_{X^*}$ by defined for y$\in Y$ by $\delta_\gamma(x) = x(\gamma), \ for\ all\ x = (x(\gamma))_{\gamma \in \Gamma} \in X$.

Then by Lemma (1.1.14) there exists $\emptyset_\gamma \in S_{Y^*}$ such that $|\langle \emptyset_\gamma, f(x) \rangle - \langle \delta_\gamma, x \rangle| \leq 4\varepsilon, \ for\ all\ x \in X$. Now, let $S: Y \to X$ be defined by $S(y) = (\langle \emptyset_\gamma, y \rangle e_\gamma)_{\gamma \in \Gamma}$. Clearly, $\| S \| = 1$ and $\| Sf(x) - x \| = sup_{\gamma \in \Gamma} |\langle \phi_\Upsilon, f(x) \rangle - \langle \delta_\gamma, x \rangle| \leq 4\varepsilon$.

Theorem (1.1.18)[1]. Suppose that $X.Y$ are Banach spaces with $dim\, X = n$, and suppose that $f: X \to Y$ is an ε–isometry with $f(0) = 0$. Then there is a continuous linear operator $S: Y \to X$ with $\| S \| \leq n$ such that

$$\| Sf(x) - x \| \leq 4n\varepsilon, \qquad for\ all\ x \in X.$$

Proof: Since $dim\, E = n$, by Auerbach's Theorem, there exist n vector $(x_i)_{i=1}^{n} \subset S_X$ and n vectors $(x_i^*)_{i=1}^{n} \subset S_{X^*}$ such that $\langle x_j^*, x_i \rangle = \delta_{ij}$. By Lemma (1.1.14), there exist n linear functional $(\emptyset_i)_{i=1}^{n} \subset S_{Y^*}$ such that for all $1 \leq i \leq n$

$$|\langle \emptyset_i, f(x) \rangle - \langle x_i^*, x \rangle| \leq 4\varepsilon, \ for\ all\ x \in X. \tag{18}$$

We define $S: Y \to X$ for $y \in Y$ by $S_y = \sum_{i=1}^{n} \langle \emptyset_i, y \rangle x_i$. Then $\| S \| \leq n$ and (18) yields

$$\begin{aligned} \| x - Sf(x) \| &= \left\| \sum_{i=1}^{n} (\langle x_i^*, x \rangle - \langle \emptyset_i, f(x) \rangle) x_i \right\| \\ &\leq \sum_{i=1}^{n} |\langle x_i^*, x \rangle - \langle \emptyset_i, f(x) \rangle| \leq 4n\varepsilon. \end{aligned}$$

We shall deal with ε–isometry between two Banach spaces, and show a stability version of reflexive Banach spaces. To begin with, we recall some notations. For a sub set $\subset X$, we denote by $G^\circ = \{x^* \in X^*: \langle x^*, x \rangle \leq 1$, for all $x \in G$, the polar of G, and ${}^\circ G^\circ$ of G is defined by ${}^\circ G^\circ = \{x \in X: \langle x^*, x \rangle \leq 1,\ for\ all\ x^* \in G^\circ\}$. $G^\perp$ stands for the annihilator of G. i.e.$G^\perp = \{x^* \in X^*: \langle x^*, x \rangle = 0,\ for\ all\ x \in G\}$. Analogously, ${}^\perp G^\perp = \{x \in X: \langle x^*, x \rangle = 0\}$ for all $x^* \in G^\perp$ .The following results are either classical, or, easily to be verified.

Proposition (1.1.19)[1]. Suppose that G is a subset of a Banach space X. Then

(i) G° is aw^* –closed convex set and $G^\perp$is a w^*–closed subspace in X^*;

(ii)$^\circ G^\circ = \overline{co}(G \cup \{0\}), and\ {}^\perp G^\perp = \overline{span} G$;

(iii) $G^\circ = G^\perp$ if G is a subspace;

(iv) IF $M \subset X^*$ isa w^*-closed subspace, then $({}^\perp G)^\perp = M$.

Recall that for an ε–isometry $f: X \to Y$ with $f(0) = 0$ and $\varepsilon \geq 0. C(f)$ denotes the closed absolutely convex hull of $f(X)$, E the annihilator of the subspace $F \subset Y^*$ consisting of all functional bounded on $C(f)$,and

$$M_\varepsilon = \{\phi \in Y^* : \emptyset is bounded\ by\ \beta_\varepsilon for some \beta > 0\ on C(f)\}.$$

Note that the set $M_\varepsilon = \{\emptyset \in Y^*: \emptyset$ is bounded on $C(f)\}\ if\ \varepsilon > 0$; = $C(F)^\perp$ the annihilator of $C(f)$, if $\varepsilon = 0$, Since $C(f)$ is symmetric, M_ε is a linear subspace of Y^* with $M_\varepsilon = \cup_{n=1}^{\infty} nC(f)^\circ$. Therefore,

$E = \cap \{\ker \emptyset : \emptyset \in M_\varepsilon\} = {}^\perp M_\varepsilon$.

Lemma (1.1.20)[1]. With the notions as the same as above, then the following assertions are equivalent.

(i)$C(f) \subset E + B\ for$ some bounded set B $\subset Y$;

(ii) M_ε is w^*–closed;

(iii) M_ε is closed.

Proof: (i)⟹ (ii). Since B is bounded $M_\varepsilon = E^\circ$.Therefore, it is w^*-closed.

(ii)⟹ (iii) is trivial.

(iii) ⟹ (ii). Since M_ε is closed in Y^*, it is a Banach space. Since $C(f)^\circ$ is w^*–closed in Y^*, it is necessarily closed. Note that $M_\varepsilon = \cup_{n=1}^{\infty} nC(f)^\circ$. Baire's Category Theorem implies that $C(f)^\circ$ is a (norm) neighborhood of 0 in M_ε. This and w^*-closedness of $C(f)^\circ$ entail that M_ε is w^*–closed.

(ii) ⟹ (i). Since $M_\varepsilon^\circ = M_\varepsilon^\perp$ is aw^*–closed subspace, and since${}^\perp M_\varepsilon = E$, according to Proposition (1.1.19) $(Y / E)^* = E^\perp = ({}^\perp M_\varepsilon)^\perp = M_\varepsilon$. By the Banach Steinhauss Theorem we see that $C\ (f)/E$ is a bounded subset of the quotient space Y/E, or equivalently, $C(f) \subset E + B$ for some bounded set $B \subset Y$.

For every ε–isometry f, we will define a set -valued mapping $\ell: X^* \to 2^{Y^*}$. Inequality (9) of Lemma (1.1.14) says that for $x^* \in X^*$, there exist $\emptyset \in Y^*$ and $\beta > 0$ such that

$$|\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \leq \beta\varepsilon,\ \ for\ all\ x \in X. \tag{19}$$

$$Let\ \ell x^* = \{\emptyset \in Y^* satisfies (19) for\ some\ \ \beta > 0\}. \tag{20}$$

Lemma (1.1.21)[1]. With the mapping ℓ as the same as above, then (i) ℓ is non-empty convex-valued and with

$$\| x^* \| = \inf\{\| \emptyset \| : \emptyset \in \ell x^*\} = \min\{\| \emptyset \| : \emptyset \in \ell x^*\}, \ for\ all\ x^* \in X^* \quad (21)$$

(ii) ℓ satisfies that for all $x^*, y^* \in X^*$ and $\alpha \in \mathbb{R}$,

$$\ell(\alpha x^*) = \alpha \ell x^* \text{ and } \ell(x^* + y^*) = \ell x^* + \ell y^*;$$

(iii) $\ell 0 = M_\varepsilon and\ \ell x^* = \emptyset_{x^*} + M_\varepsilon$, where $\emptyset = \emptyset_{x^*}$ satisfies (9);

(iv) ℓ is properly injective, i.e. if $x^* \neq y^*$, then $\ell x^* \cap \ell y^* = \emptyset$.

Proof: (i) Non-emptiness and convexity of ℓx^*,and the inequality$\| x^* \| \geq \inf\{\| \emptyset \| : \emptyset \in \ell x^*\}$ follow from (9) of Lemma (1.1.14) .To show$\| x^* \| \leq \inf\{\| \emptyset \| : \emptyset \in \ell x^*\}$, let $\emptyset \in \ell x^*$ .Then by definition of ℓ ,there exists $\beta > 0$ such that

$$|\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \leq \beta\varepsilon, \ \ for\ all\ x \in X.$$

Given $\delta > 0$, we choose $x_0 \in S_X$ such that $\langle x^*, x_0\rangle > \| x^* \| - \delta$, and substitute nx_0 for x in the inequality above . Then we obtain that for all $n \in \mathbb{N}$,

$$\left|\langle \emptyset, \frac{f(nx_0)}{n}\rangle - \langle x^*, x_0\rangle\right| \leq \frac{\beta\varepsilon}{n}.$$

Note that $n - \varepsilon \leq \| f(nx_0) \| \leq n + \varepsilon$. By letting $n \to \infty$ in the inequality above, we observe that

$$\| \emptyset \| \geq \lim_n sup\langle \emptyset, (f(nx_0))/n\rangle = \langle x^*, x_0\rangle > \| x^* \| - \delta.$$

Arbitrariness of δ entails that $\| \emptyset \| \geq \| x^* \|$.

(ii) Homogeneity of ℓ and the one side inclusion $\ell(x^* + y^*) \supset \ell x^* + \ell y^*$ immediately follow from definition of ℓ .To show $\ell(x^* + y^*) \subset \ell x^* + \ell y^*$, let $\psi \in \ell(x^* + y^*)$ and $\emptyset \in \ell x^*$ .Then there exist $\beta_1, \beta_2 > 0$ such that for all $x \in X$,

$$|\langle \psi, f(x)\rangle - \langle x^* + y^*, x\rangle| \leq \beta_1 \varepsilon and |\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \leq \beta_2 \varepsilon$$

Let $\mu = \psi - \phi$ .Then

$$|\langle \mu, f(x)\rangle - \langle y^*, x\rangle| = |(\langle \psi, f(x)\rangle - \langle x^* + y^*, x\rangle) - (\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle)|$$
$$\leq (\beta_1 + \beta_2)\varepsilon,$$

And this says that $\mu \in \ell y^*$and $\psi = \phi + \mu \in \ell x^* + \ell y^*$.

(iii) If $\varepsilon = 0$, then $\ell 0 = f(x)^\perp = C(f)^\perp = M_0 = M$. If > 0 , then

$$\ell 0 = \{\emptyset \in Y^* : |\emptyset| is\ bounded\ on\ f(x)\}$$
$$= \{\emptyset \in Y^* : \emptyset\ is\ bounded\ above\ on\ C(f)\} = M_\varepsilon.$$

Given $x^* \in X^*$, and $\emptyset \in \ell x^*$, by (ii) we have just proven,

$$\ell x^* = \ell(x^* + 0) = \ell x^* + \ell 0 \supset \emptyset + \ell 0 = \emptyset + M_\varepsilon.$$

Conversely, for any $\emptyset, \psi \in \ell(x^*)$, let $\beta_1, \beta_2 \in \mathbb{R}^+$ such that for all $x \in X$,

$$|\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \leq \beta_1\varepsilon, and\ \ |\langle \psi, f(x)\rangle - \langle x^*, x\rangle| \leq \beta_2\varepsilon. Then,$$
$$|\langle \emptyset - \psi, f(x)\rangle| \leq (\beta_1 + \beta_2)\varepsilon,\ \ for\ all\ x \in X,$$

And this is equivalent to $\emptyset - \psi \in \ell 0$. Thus, (iii) has been proven.

(iv) According to (ii), it suffices to show $\ell x^* \cap \ell 0 = \emptyset$ that for every $x^* \in X^* \backslash \{0\}$. Given$x^* \in X^*$ with $x^* \neq 0$, let $\emptyset \in \ell x^*$. Then there exists $\beta \in \mathbb{R}^+$ such that

$$|\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \leq \beta\varepsilon, \qquad for\ all\ x \in X.$$

If $\emptyset \in \ell 0$, then there is $\beta_1 > 0$ such that $|\langle \emptyset, f(x)\rangle| \leq \beta_1\varepsilon$, for all $x \in X$. Thus $|\langle x^*, x\rangle| \leq (\beta + \beta_1)\varepsilon$,for all $x \in X$ .This is impossible, since $x^* \neq 0$.

Theorem (1.1.22)[1]. Let X, Y be Banach spaces, $f: X \to Y$ be an ε–isometry with $f(0) = 0$, and let ℓ be defined as in Lemma (1.1.21), and $M = \bar{\ell}0$. Then:

(i) $Q = X^* \to Y^*/M$ Defined by $Qx^* = \ell x^* + M$ is a linear isometry.

(ii) If M is w^*–closed, then Q is the conjugate operator of a surjective operator U from E onto X with $\| U \| = 1$.

(iii) In particular, if $\varepsilon = 0$, then U is just Figiel's operator.

Proof: (i) According to Lemma (1.1.21), it is clear that Q is single-valued and linear. Note that and $M = \overline{\ell 0} = \overline{M_\varepsilon}$. For every $x^* \in X^*$,due to (21) of Lemma (1.1.21),

$\| Qx^* \| = \inf \{\| \emptyset - m \| : \emptyset \in \ell x^*, m \in M\}$

$= \inf\{\| \emptyset - m \| : \emptyset \in \ell x^*, m \in M_\varepsilon\} = \inf \{\| \emptyset \| : \emptyset \in \ell x^*\} = \| x^* \|$.

(ii) Suppose that M is w^*–closed in Y^*. Then, by Proposition (1.1.19), $M = ({}^\perp M)^\perp = E^\perp$. Therefore, $Y^*/M = Y^* / E^\perp = E^*$. We claim first that Q is $w^* - to{-}w^*$ continuous (hence, it is a conjugate operator) (an operator $T: Y^* \to X^*$is $weak^* - weak^*$ continuous if and only if it is of the form $T = S^*$ for some bounded operator $S: X \to Y$ in particular T must be bounded**) [7].** By the Krein-Smulian Theorem, it suffices to show that it is $w^* - to{-}w^*$ continuous on Bx^*, the unit ball of X^* . Let$(x_\alpha^*) \subset B_{X^*}$be a net is w^*–converging to $x^* \in X^*$. Then by Lemma (1.1.14) there is a net $(\emptyset_\alpha) \subset Y^*$ with $\| \emptyset_\alpha \| = \| x_\alpha^* \| \equiv r_\alpha \leq 1$ such that

$$|\langle \emptyset_\alpha, f(x)\rangle - \langle x_\alpha^*, x\rangle| \leq 4\varepsilon r_\alpha for all\ x \in X.$$

w^* –Relative compactness of $(\emptyset_\alpha)$ implies that there is a w^*–cluster point $\emptyset \in Y^*$ of $(\emptyset_\alpha)$ such that $|\langle \emptyset , f(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon r for some\ 0 \leq r \leq \lim_\alpha \sup r_\alpha$. Clearly, $\emptyset \in \ell x^*$. Since every w^*– cluster point of(x_α^*) is in $\ell x^*, Qx_\alpha^* = \ell x_\alpha^* + M = \emptyset_\alpha + M$, and which further entails that (Qx_α^*) is w^*–convergent to $\emptyset + M = Qx^* in\ Y^* / M = E^*$ .Hence,$Q: B_{X^*} \to E^*$ is $w^* - to - w^*$ continuous. Let $U: E \to X$ be a linear operator such that $U^* = Q$ .Clearly, U is a surjective mapping with $\| U \| = 1$, since $Q = U^*: X^* \to E^*$ is a linear isometry.

(iii) If ε= 0, then $M = M_\varepsilon = M_0 = C(f)^\perp$ is w^*–closed and $E = {}^\perp M = L(f)$. According to (ii) we have just proven, there exists $U: L(f) \to X$ such that $U^* = Q$. And in this case, it is easy to observe that

$$\langle Qx^*, f(x)\rangle = \langle x^*, x\rangle, for\ all\ x \in X\ and\ x^* \in X^*.$$

Let F be Figiel's operator from $L(f)$ to X such that $F \circ f = I$ on X. Then its conjugate operator $F^*: X^* \to L(f)^* = Y^*/L(f)^{\perp} = Y^*/M$satisfies

$\| F^* \| = \| F \| = 1$. Since $F \circ f = I_X$, by definition of conjugate operator we have for all $x \in X and\ x^* \in X^*$,

$$\langle F^* x^*, f(x) \rangle = \langle x^*, Ff(x) \rangle = \langle x^*, I_X x \rangle = \langle x^*, x \rangle = \langle Qx^*, f(x) \rangle$$

Therefore,$U^* = Q$ that is, $U = F$.

Corollary (1.1.23)[1]. With the notations as the same as in Theorem (1.1.22), then Q is a conjugate operator if one of the following conditions holds.

(i)$C(f) \subset E + B$ For some bounded set $B \subset Y$;

(ii) $M_\varepsilon$is closed;

(iii) Y is reflexive.

Proof: According to Theorem (1.1.22), it suffices to show that M is w^*–closed. By Lemma (1.1.14), both (i) and (ii) imply that $M_\varepsilon$is w^*–closed. Therefore, $M = \overline{M_\varepsilon}$ is certainly w^*-closed. Note that M is always weakly closed. If Y is reflexive, then M is w^*-closed.

Definition (1.1.24)[1]. Let X be a Banach space and $0 \le \alpha < +\infty$. A closed subspace $M \subset X$ is said to be α–complemented provided there exists a closed subspace $N \subset X$ with $M \cap N = \{0\}$ and a projection P: X⟶ M along N such that $X = M + N$ and $\| P \| \le \alpha$.

Theorem (1.1.25)[1]. Suppose that X, Y are Banach spaces and Y is reflexive, and suppose f: X⟶ Y is an ε-isometry with f (0) = 0. If E is α–complemented in Y, then there is a bounded linear operator T: Y⟶ X with $\| T \| \le \alpha$ such that

$$\| Tf(x) - x \| \le 4\varepsilon, \quad for\ all\ x \in X. \tag{22}$$

Proof: Since Y is reflexive, by Theorem (1.1.22) and Corollary (1.1.23), there is a surjective operator $U: E \to X$ with$\| U \| = 1$ such that Q = U^* .

Since E is α–complementedin Y, there is a closed (complemented) sub space F of Y with $E \cap F = \{0\}$ such that $E + F = Y$ and the projection $P: Y \to E$

along F satisfies $\| P \| \leq \alpha$. Let $T = U \circ P$. Then $\| T \| \leq \alpha$. In the following we will show that T satisfies (26). Note that $Y^*/M \quad = Y^*/E^{\perp} = E^* = F^{\perp}$. We have

$$\langle Qx^*, Py\rangle = \langle Qx^*, y\rangle, \qquad for\ all\ x^* \in X^*\ and\ y \in Y. \qquad (23)$$

Therefore,

$$|\langle Qx^*, Pf(x)\rangle - \langle x^*, x\rangle| = |\langle Qx^*, f(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon \| x^* \|,$$

For all $x \in X\ and\ x^* \in X^*$. (24)

By definition of conjugate operator, we Observe that for all $x \in X$ and $x^* \in X^*$,

$$\langle Qx^*, Pf(x)\rangle = \langle x^*, (U \circ P)f(x)\rangle = \langle x^*, Tf(x)\rangle. \qquad (25)$$

(24) and (25) together entail that

$$|\langle x^*, Tf(x) - x\rangle| \leq 4\varepsilon \| x^* \| , \quad for\ all\ x \in X\ and\ x^* \in X^*$$

Or, equivalently,

$$\| Tf(x) - x \| \leq 4\varepsilon, \quad for\ all\ \ x \in X.$$

Section (1.2): ε- isometries in Reflexive Spaces and Sharp Estimates of a Certain Class of Reflexive Spaces:

In this section, we shall continue to deal with ε-isometry in reflexive Banach spaces. This is also preparation for showing a sharp estimate.

Definition (1.2.1)[1]. Suppose that X, Yare two Banach spaces, and that S is a (set-valued) mapping from X to 2^Y. S is said to be β–Lipschitz for some $\beta > 0$ provided for all $x_1, x_2 \in X, Sx_1 \subset Sx_2 + \beta \| x_1 - x_2 \| B_Y$.

It is clear that if $S: X \to 2^Y$ is β–Lipschitz then $T: X \to 2^Y$ defined for $x \in X$ by $Tx = \overline{co}(Sx)$ is also β –Lipschitz.

Theorem (1.2.2)[1]. Suppose that X, Yare Banach spaces and that Y is reflexive, and suppose that $f: X \to Y$ is an ε-isometry for some $\varepsilon \geq 0$ with $f(0) = 0$. Let U be the pre-conjugate operator of Q defined as in Theorem (1.1.22). If there is a closed subspace $F \subset Y$ with $E \cap F = \{0\}$ such that

$E + F = Y$, then there is a non-empty weakly compact convex-valued 1-Lipschitz mapping V: X→2^E such that $U \circ V = I_X$ on X.

Proof: Let

$$\wedge = \{\lambda = (\lambda_n) \subset \mathbb{R}^+ with\ \lambda_n \nearrow \infty\}.$$

We define then a set-valued mapping W from X to $the\ 2^Y$ for $x \in X$ by

$$Wx = \{u \in Y : \exists \lambda \in \wedge\ such\ that\ u = w - \lim_n f(\lambda_n x) / \lambda_n\}. \quad (26)$$

We show first that W is everywhere non-empty valued with $W(x) \subset E$, and with $\| u \| = \| x \|$ for all $u \in Wx$. Since f is an ε-isometry, $\lim_{\lambda \to \infty} \| f(\lambda x) / \lambda \| \to 1$. Boundedness of $(f(\lambda x)/\lambda)_{\lambda \geq 1}$ and reflexivity of Y entail that $(f(\lambda x)/\lambda)_{\lambda \geq 1}$ is relatively weakly compact. Consequently, $Wx \neq \emptyset$ for all $x \in X$. Note $E = {}^{\perp}M$. Given $x \in X$ and $u \in Wx$, Let $\lambda \in \wedge$ satisfy $u = w - \lim_n f(\lambda_n x)/\lambda_n$. Without loss of generality, we can assume $x \neq 0$. By definition of M_ε, for every $\phi \in M_\varepsilon$ $there\ is\ a\ \beta > 0$ such that

$$|\langle \emptyset, f(\mathrm{z}) \rangle| \leq \beta\varepsilon, \qquad for\ all\ \mathrm{z} \in X. \quad (27)$$

Substituting $\lambda_n x$ for z in (27), and dividing the both sides of the inequality by λ_n, then we obtain

$$|\langle \phi, f(\lambda_n x) / \lambda_n \rangle| \leq \beta\varepsilon/\lambda_n, \quad for\ all\ n \in \mathbb{N}. \quad (28)$$

Let $n \to \infty$. Then $\langle \emptyset, u \rangle = 0$. Therefore, $u \in {}^{\perp}M_\varepsilon = {}^{\perp}M = E$. To show $\| u \| = \| x \|$, let $x^* \in \partial \| x \|$. Then $x^* \in S_{X^*}$ and $\langle x^*, x \rangle = \| x \|$. According to (9) of Lemma (1.1.14), there exists $\emptyset_{x^*} \in Y^*$ with $\| \emptyset_{x^*} \| = \| x^* \| = 1$ such that

$$|\langle \emptyset_{x^*}, f(z) \rangle - \langle x^*, z \rangle| \leq 4\varepsilon, \qquad forall\ z \in X. \quad (29)$$

Substituting $\lambda_n x$ for z in the inequality above, and dividing the both sides of the inequality by λ_n, then we get

$$|\langle \emptyset_{x^*}, f(\lambda_n x) / \lambda_n \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r/\lambda_{n,} \text{ for all } n \in \mathbb{N}. \quad (30)$$

Let $n \to \infty$. Then (30), weakly lower semi-continuity of the norm of X and $u = w - \lim_n f(\lambda_n x) / \lambda_n$ together yield that

$$\| x \| = \lim_n \inf \| f(\lambda_n x)/\lambda_n \| \geq \| u \| \geq \langle \phi_{x^*}, u \rangle = \langle x^*, x \rangle = \| x \|. \quad (31)$$

$hence$ $\| u \| = \| x \|$. Note that (31) entails that for every $x \in X$ and for every $x^* \in S_{X^*}$ with $\langle x^*, x \rangle = \| x \|$ (i.e.$x^* \in \partial \| x \|$) there exists $\emptyset \in Qx^*$ (acting as a subset of Y^*) with$\| \phi \| = \| x^* \| = 1$ such that

$$Wx \subset \{u \in E : \langle \emptyset, u \rangle = \| u \| = \| x \|\}. \quad (32)$$

We show next that W is positively homogenous. Let $u \in Wx$ and $\lambda \in \Lambda$ such that $u = w - \lim_n f(\lambda_n x) / \lambda_n$. For any $a \in \mathbb{R}^+$, let $\lambda^a = \frac{1}{a}\lambda$. Then

$$u = w - \lim_n f(\lambda_n x)/\lambda_n = w - \lim_n f\left(\lambda_n^a (ax)\right)/\lambda_n$$

$$= \frac{1}{a}(w - \lim_n f\left(\lambda_n^a (ax)\right)/\lambda_n^a).$$

This says that $aWx \subset W(ax)$ for $a > 0$. Consequently,

$Wx = W(\frac{1}{a}(ax)) \supset \frac{1}{a}W(ax)$. Thus, $W(ax) = aWx$ for all $x \in X$ and $a \in \mathbb{R}^+$.

In the following, we show that W is 1-Lipschitz. We want to prove that given, $y \in X$, and $u \in Wx$, there exists $v \in Wy$ such that $\| v - u \| \leq \| y - x \|$. Indeed, by definition of Wx there exists $\lambda \in \Lambda$ such that $f(\lambda_n x) / \lambda_n \to u$ in the weak topology. Relatively weak Compactness of$f(\lambda_n y) / \lambda_n$entails that there is $\lambda_s \equiv (\lambda_{n_k}) \in \Lambda$ such That $(f\left(\lambda_{n_k} y\right) \ /\lambda_{n_k})$ weakly converges to some $v \in Wy$. Weakly lower semi-continuity of the norm$\|.\|$ on Y entails

$$\| v - u \| \leq \liminf_k \| f\left(\lambda_{n_k} y\right)/\lambda_{n_k} - f(\lambda_{n_k} x) / \lambda_{n_k} \|$$

$$\leq \liminf_k \left(\| \lambda_{n_k} y\right) - \lambda_{n_k} x \| + \varepsilon) \quad /\lambda_{n_k} = \| y - x \|.$$

Therefore, W is 1-Lipschitz.

Next, we will show that $UoW = I_X$ on X. Note that both E and F are complemented subspaces of Y. The projection $P: Y \to E$ along F is bounded, and $U^* = Q$ is actually $X^* \to F^{\perp}$. Given x $\in X$, and $u \in Wx$,let $\lambda \in \Lambda$ such that $u = w - \lim_n f(\lambda_n x)/\lambda_n$ .This and definition of conjugate operator imply that

$$\langle Uu, x^* \rangle = \langle u, Qx^* \rangle = \lim_n \langle f(\lambda_n x/\lambda_n, Qx^* \rangle = \langle x, x^* \rangle.$$

This says that $Uu = x$ for all $u \in Wx$, or equivalently, $UoW = I$. Therefore, W is a (set-valued) positively homogenously 1- Lipschitz mapping and satisfies $UoW = I_X$.

Finally, let $Vx = \overline{co}(Wx)$ for all $x \in X$. Then V is again a non-empty w-compact convex-valued 1-Lipschitz and positively homogenous mapping. Since $UoWx = x$ for all $x \in X$, linearity and w-continuity of U together entail that $UoVx = U(\overline{co}Wx) = x$, that is, $UoV = I_X$.

Theorem (1.2.3)[1]. Suppose that X, Y are Banach spaces and that Y is reflexive, and suppose that f: X→Y is and ε -isometry for some $\varepsilon \geq 0$ with $f(0) = 0$. Suppose that the subspaces E and F, the operators U, V, P and Q associated with f and F, are as same as in Theorem (1.1.18). If, in addition, the subspace $E \subset Y$ is strictly convex, then $V = W: X \to E$ is a (single-valued) linear isometry satisfying

$$Vx = w - \lim_{\lambda \to +\infty} f(\lambda x)/\lambda, \quad for\ all\ x \in X$$

Therefore, $V^* \circ Q = (U \circ V)^* = I_{X^*}$, and X is reflexive and strictly convex.

Proof: Suppose that E is strictly convex. Then $E^*$is smooth. According to Theorem (1.1.13), each $\emptyset \in E^*$ with $\emptyset \neq 0$ has a unique support functional $u \in S_E$ .This, incorporating (32) entails that Wx $(hence, Vx)$ is a singleton, which in turn implies that

$$Vx = Wx = w - \lim_{\lambda \to +\infty} f(\lambda x)/\lambda, forallx \in X \tag{33}$$

And V is single-valued 1-Lipschitz mapping. On the other hand, given, $x_1, x_2 \in X$, let $x^* \in X^*$ with$\| x^* \| = 1$ such that

$$\langle x^*, x_1 - x_2 \rangle = \| x_1 - x_2 \|$$

By a simple discussion similar to that from (29) to (31), there is $\emptyset \in S_{Y^*}$ corresponding to x^*satisfying the following equalities

$$\langle \emptyset, Vx_1 \rangle = \langle x^*, x_1 \rangle \; and \; \langle \emptyset, Vx_2 \rangle = \langle x^*, x_2 \rangle. \tag{34}$$

Therefore,

$$\| x_1 - x_2 \| \geq \| Vx_1 - Vx_2 \| \geq \langle \emptyset, Vx_1 - Vx_2 \rangle = \langle x^*, x_1 - x_2 \rangle = \| x_1 - x_2 \| \tag{35}$$

We have proven that V is a positively homogenous isometry.

We show finally that V is linear. For any $x \in X, x \neq 0$, let $x_1 = x, x_2 = -x$, and $x^* \in S_{X^*}$ with $\langle x^*, x \rangle = \| x \|$, and let $\emptyset \in E^*$ with $\| \emptyset \| = 1$ be the functional corresponding to x^* satisfying (34). Then

$$\| Vx - V(-x) \| = \langle \emptyset, Vx - V(-x) \rangle = \langle x^*, x - (-x) \rangle = 2 \| x \|.$$

This and strict convexity of E yield $V(-x) = -Vx$. Thus, V is a homogenously symmetrical isometry. It remains to show additivity of V. For any $x, y \in X, let x^* \in S_{X^*} satisfy \; \langle x^*, x + y \rangle = \| x + y \| \; and, let \; \emptyset \in S_{E^*}$, be a functional corresponding to x^* such that

$$\langle \emptyset, Vx \rangle = \langle x^*, x \rangle \; and \; \langle \emptyset, Vy \rangle = \langle x^*, y \rangle.$$

Then $\| x + y \| = \| Vx - V(-y) \| = \| Vx + Vy \| \geq \langle \emptyset, Vx + Vy \rangle = \langle x^*, x + y \rangle = \| x + y \| = \| V(x + y) \| = \langle \emptyset, V(x + y) \rangle$

Therefore,

$$\langle \emptyset, V(x + y) \rangle = \| V(x + y) \| = \| Vx + Vy \| = \langle \emptyset, Vx + Vy \rangle.$$

This says that both $V(x + y)$ and $Vx + Vy$ are support functionals (with same norm) of B_{E^*} and supporting $B_{E^*} \; at \; \emptyset$. Smoothness of E^* implies that $V(x + y) = V(x) + V(y)$.

Theorem (1.2.4)[1]. Suppose that X is a Banach space and that Y is a reflexive, Gateaux smooth and strictly convex Banach space admitting the Kadec-Klee property. Suppose that $f: X \to Y$ is an ε-isometry with $f(0) = 0$, and that the subspace $E \subset Y$ associated with f is α -complemented in Y. Then there is a linear operator $T: Y \to X$ with $\| T \| \leq \alpha$ such that

$$\| Tf(x) - x \| \leq 2\varepsilon, \quad x \in X. \tag{36}$$

Proof: Let the subspace F, and the operators P, Q, U, V associated with f and E be as the same as in Theorem (1.2.3). According to Theorem (1.2.3), X is the reflexive strictly convex and Gateaux smooth; and $V: X \rightarrow E$ satisfying

$$V_{X=} w - \lim_{\lambda\rightarrow\infty} f(\lambda x)/\lambda = w - \lim_{n\rightarrow\infty} f(nx)/n \quad (37)$$

Is a linear isometry with $U \circ V = I$ on X. The Kadec- Klee property of Y implies that

$$Vx = w - \lim_{\lambda\rightarrow\infty} f(\lambda x)/\lambda = \lim_{n\rightarrow\infty} f(nx)/n\,. \quad (38)$$

Note that the closed subspace F of Y satisfies $E \cap F = \{0\}$ *and* $E + F = Y$, and the projection $P: Y \rightarrow E$ along F satisfies $\| P \| \leq \alpha$. Since $Y(X)$ is smooth, we get that $\partial \| u \| = d \| u \|$ is unique for all $u \neq 0$ *in* $Y(X)$.

Let $T \equiv U \circ P$. Then $\| T \| \leq \alpha$ we want to prove that T satisfies (36). Given $x \in X$, without loss of generality, we assume that $x \neq Tf(x)$. For every n ∈ ℕ , let $\beta = \| x - Tf(x) \|$, $z = (x - Tf(x))/\beta$, $q_n(x) = f(x + nz)$, $r_n(x) = f(x + nz)/n$ *and* $\emptyset_n = d \| r_n(x) \|$

Note that for any $\gamma > 0$ and for any $u \in Y$ with $u \neq 0, d \| \gamma u \| = d \| u \|$

Let $\emptyset_n = d \| r_n(x) \|$. Then

$$\| f(x + nz) \| = \langle \emptyset_n, f(x + nz) \rangle \leq \langle \emptyset_n, f(x) \rangle + \| f(x + nz) - f(x) \|$$
$$\leq \langle \emptyset_n, f(x) \rangle + n + \varepsilon. \quad (39)$$

By (38), $r_n(x) \rightarrow Vz$ .Gateaux smoothness and reflexivity of Y together entail that $\emptyset_n \rightarrow \emptyset \equiv d \| V(z) \|$ in the weak topology of Y. Consequently,

$$\lim_n(\| f(x + nz) \| - n) \leq \langle \emptyset, \mathrm{f(x)} \rangle + \varepsilon. \quad (40)$$

On the other hand, let $x^* = d \| z \|$. Since

$$\| f(x + nz) \| - n \geq \| x + nz \| - n - \varepsilon,$$

We have

$$\lim_n(\| f(x + nz) \| - n) \geq \lim_n(\| x + nz \| - n) - \varepsilon =$$

$$\lim_n \frac{\|z + \frac{1}{n}x\| - \|z\|}{\frac{1}{n}} - \varepsilon = \langle x^*, x \rangle - \varepsilon \quad (41)$$

(40) and (41) yield

$$\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle \le 2\varepsilon. \tag{42}$$

According to (9) of the Main Lemma, we have

$$|\langle Qx^*, f(y)\rangle - \langle x^*, y\rangle| \le 4\varepsilon \parallel x^* \parallel, \quad for\ all\ y \in X.$$

We substitute nz for y, n=1,2,……, Then

$$|\langle Qx^*, f(nz)\rangle - \langle x^*, nz\rangle| \le 4\varepsilon \parallel x^* \parallel$$

Consequently,

$$|\langle Qx^*, f(nz)/n\rangle - \langle x^*, z\rangle| \le 4\varepsilon \parallel x^* \parallel /n.$$

Letting $n \to \infty$, incorporating Theorem (1.2.3), we obtain

$$\langle Qx^*, Vz\rangle - \langle V^* Qx^*, z\rangle = \langle x^*, z\rangle = \parallel z \parallel.$$

$\parallel Qx^* \parallel = \parallel x^* \parallel = \parallel \emptyset \parallel = 1$ and smoothness of Y together imply $Qx^* = \emptyset$. Now, we turn to prove that

$$\emptyset \circ V = d \parallel z \parallel. \tag{43}$$

In fact, let $z^* = \emptyset \circ V$.Then $\parallel z^* \parallel \le 1$ and

$$\parallel z \parallel = \parallel Vz \parallel = \langle \emptyset . Vz\rangle = \langle z^*, z\rangle.$$

Therefore, $z^* \in \partial \parallel z \parallel = d \parallel z \parallel = x^*$.

Note $x^* = \emptyset \circ V = d \parallel z \parallel$, $T = U \circ P\ and\ U \circ V = I_X$, we have

$$\langle x^*, Tf(x)\rangle = \langle \emptyset, (V \circ U \circ P)f(x)\rangle = \langle Qx^*, (V \circ U \circ P)f(x)\rangle$$
$$= \langle x^*, U \circ (V \circ U \circ P)f(x)\rangle = \langle x^*, (U \circ P)f(x)\rangle$$
$$= \langle Qx^*, Pf(x)\rangle = \langle \emptyset, Pf(x)\rangle = \langle \emptyset, f(x)\rangle.$$

Since$1 = \langle x^*, z\rangle = \frac{1}{\beta}\langle x^*, x - Tf(x)\rangle$,

$$\beta = \langle x^*, \big(x - Tf(x)\big)\rangle = \langle x^*, x\rangle - \langle x^*, Tf(x)\rangle = \langle x^*, x\rangle - \langle \emptyset, f(x)\rangle \le 2\varepsilon.$$

Corollary (1.2.5)[1]: Suppose that X, Y are Banach spaces and that Y is reflexive, Gateaux smooth and locally uniformly convex. Suppose that $f: X \to Y$ is an ε -isometry with $f(0) = 0$, and that the subspace $E \subset Y$ is α –complemented in Y. Then there is a linear operator $T: Y \to X$ with $\parallel T \parallel \le \alpha$ such that

$$\parallel Tf(x) - x \parallel \le 2\varepsilon, \quad x \in X$$

Proof: According to Theorem (1.2.14), it suffices to note that locally uniform convexity implies both the strict convexity and the KKP.

Corollary (1.2.6)[1] (Semrl and Väiälä). Let $1 < p < \infty$ if X *and* Y are L^p-spaces, and if $f: X \to Y$ is an ε-isometry with $f(0) = 0$, then there is a linear operator $T : Y \to X$ with $\| T \| = 1$ such that

$$\| Tf(x) - x \| \leq 2\varepsilon \quad x \in X$$

Proof: Assume that both X and Y are L^p–spaces with $1 < p < \infty$. Then they are both (super) reflexive, uniformly convex and uniformly smooth. Suppose that $\varepsilon \geq 0$, and that $f: X \to Y$ is an ε-isometry with $f(0) = 0$. Then by Theorem (1.2.3), there exists a linear isometry $V: X \to Y$. By Theorem (1.2.4), it suffices to note that for any fixed $1 < p < \infty$, if one L^p–spaces X is linearly isometric to sub space of another L^p–space Y, then X is 1-complemented in Y.

Chapter 2
Stability characterization

We show that the subspace N plays a crucial role. For example, (i) $U^*: N^\perp \to X^{**}$ *is* $w^* - to - w^*$ continuous surjective isometry; and, in particular, if $Y \equiv \overline{span} f(X)$ is surjective, then the mapping f is stable, if and only if $N^\perp$ is complemented in Y; (ii) if Y is reflexive and $N^\perp$ is complemented in Y, then for any projection $P: Y \to N^\perp$, the operator $T = U^*P$ satisfy $\| Tf(x) - x \| \leq 4\varepsilon, for\ all\ x \in X$; and (iii) if, in addition, Y is Gateaux smooth and locally uniformly convex, then $T = U^*P$ satisfies the sharp estimate $\| Tf(x) - x \| \leq 2\varepsilon, for\ all\ x \in X$. We present similar results for such mappings on general Banach spaces.

Section (2.1) ε-isometry and linear isometry with Stability

Characterization of ε-isometry or Reflexive Spaces

A mapping f from a Banach space X to another Banach space Y is said to be perturbed metric –preserved provided there exists $\varepsilon \geq 0$ such that

$$| \| f(x) - f(y) \| - \| x - y \| | \leq \varepsilon, \text{ for all } x, y \in X. \quad (1)$$

The mapping f is also called an ε- isometry. There are many different names for this notion such as approximate isometry, non-linear perturbation of isometry and nearisometry The mapping f is said to be standard if $f(0) = 0$. A standard ε-isometry $f: X \to Y$ is said to be (α, γ) –stable for some $\alpha \geq 0$ and $\gamma \geq 2$, if there exists $T \in B(L(f), X)$ with $\| T \| \leq \alpha$ such that

$$\| Tf(x) - x \| \leq \gamma\varepsilon\ for\ all\ x \in X, \quad (2)$$

Where $L(f)$ denotes the closure of span $f(X)$ in Y. We call a 0-isometry an isometry, and if no confusion arises, we simply call $(\alpha, \gamma) -$ stable "stable". In this chapter, we consider the two questions:

(I) Is there a stability characterization for a general ε-isometry $f: X \to Y$?

(II) Is there an isometric copy of X in Y if an ε-isometry $f: X \to Y$ exists?

ε-isometry, isometry and linear isometry. The study of properties of isometries between Banach spaces and their generalizations has continued for 80 years. The first celebrated result is due to Mazur and Ulam: Every surjective isometry between two Banach spaces is necessarily affine. But the simple example: $f : \mathbb{R} \to \ell_\infty^2$ defined for t by $f(t) = (t, sint)$ shows that it is not true if an isometry is not surjective. For non-surjective isometries, Figiel showed the following remarkable result: Every standard isometry admits a linear left- inverse of norm one. Godefroy and Kalton studied the relationship between isometries and linear isometries , and showed the following deep Theorem, which resolves a long-standing problem whether the existence of an isometry implies the existence of a linear isometry : If X is separable and there is an isometry $f: X \to Y$, then Y contains a linear isometric copy of X; and for every nonseparable weakly compactly generated space X there exist a Banach space Y and an isometry $f : X \to Y$ so that X is not linearly isomorphic to any subspace of Y.

ε-isometry and stability. In 1945, Hyers and Ulam proposed the following question: whether for every surjective ε-isometry $f: X \to Y$ $with$ $f(0) = 0$ there exist a surjective linear isometry $U: X \to Y$ and $\gamma > 0$ such that

$$\| f(x) - Ux \| \leq \gamma\varepsilon, \quad for\ all\ x \in X. \tag{3}$$

After many years of efforts of a number of mathematicians, the sharp estimate $\gamma = 2$ was finally obtained by Omladič and Šemrl.

The study of non surjective ε- isometry has also brought to mathematicians attention since 90s of the last century. First proposed the following Problem in 1995.

Problem (2.1.1)[2]. Given two Banach spaces X and Y, whether there exists a constant $\gamma > 0$ Such that for every standard ε-isometry $f: X \to Y$ there is $\alpha > 0$ so that f is (α, γ)-stable?

Then he showed that the answer is affirmative if both X and Y are L_p $spaces$ $(1 < p < \infty)$. Further presented a sharp estimate of (2) with $\gamma = 2$ if both X and Y areL^p $spaces$ for $1 < p < \infty$. However, Qian gave a counterexample showing if the space Y admits an uncomplemented subspace

X then for all $\varepsilon > 0$ there is an unstable standard ε-isometry from X to Y. Since a Banach space satisfying that every closed subspace is complemented in it must be linearly isomorphic to a Hilbert space, this, incorporating of Qian's counterexample, entails that if a Banach space Y satisfies that every standard ε-isometry $f : X \to Y$ is stable for every Banach space X, then Y is linearly isomorphic to a Hilbert space . This disappointment makes us to search for some weaker stability version and some appropriate complementability assumption on some subspaces of Y associated with the mapping. Recently, Cheng, Dong and Zhang gave the two questions above some affirmative answers.

Now we know that there are many remarkable results about ε-isometries on Banach spaces in the past eight decades. However, there are still many questions deserving consideration. For example, what classes of non-separable Banach spaces can guarantee that every isometric mapping from a Banach space to a space in this class always induces a linear isometry? Is there a stability characterization for ε- isometries? If there is an ε-isometry f from a Banach space X to another Banach space Y, is there an isometry (not necessarily linear) $g: X \to Y$? The propose of this chapter is to consider the first two questions.

This chapter is organized as follows. In this section , making use of Cheng, Dong and Zhang's Lemma and invariant means of $\ell_\infty(X)$,we show that every ε-isometry $f : X \to Y$ induces a closed subspace N of Y^* and a linear surjective isometry $U: X^* \to Y^* / N$ hence, $U^* : N^\perp \to X^{**}$,the conjugate operator of U is a $w^* to\ w^*$ continuous linear surjective isometry. In particular, if Y is reflexive, then $N^\perp$ is just a linear isometric copy of X in Y. And we show that if Y is reflexive, then a standard ε-isometry $f: X \to Y$ is $(\alpha, 4)$ $-$stable if and only if the subspace $N^\perp \subset Y$ is α-complemented in $L(f)$, the closure of span $f(X)$ in Y. And if $N^\perp$ is α-complemented in $L(f)$, then f is $(\alpha, 4)$ $-$stable with $T = U^*P$ for every projection $P: L(f) \to N^{\perp}$ with $\| P \| \leq \alpha$. If, in addition, Y is smooth and locally uniformly convex (or, more general, strictly convex and admitting the Kadec-Klee property), then f is $(\alpha, 2)$ $-$stable. In another section we show that a standard ε-isometry $f: X \to Y\ is\ (\alpha, \gamma)$ $-$stable in the sense that there exists a bounded linear operator $T: L(f)^{**} \to X^{**}$ with $\| T \| \leq \alpha$ such that

$$\| Tf(x) - x \| \leq \gamma\varepsilon,\ \ for\ all\ x \in X \qquad (4)$$

If and only if $N^{\perp} is\ w^* - \alpha$-complemented in $L(f)^{**}$ (where$L(f)^{**}$ denotes the second dual of the space $L(f)$); and an (α, γ) $-$stable ε-isometry is always $(\alpha, 4)$ $-$stable.

In this chapter, all notations are standard. The letter X will always be a real Banach space andX^*its dual. $B_X\ and\ S_X$ denote the closed unit ball and the unit sphere of X, respectively. For a subspace $M \subset X, M^{\perp}$ presents the annihilator of M, i.e.$M^{\perp} = \{x^* \in X^*: \langle x^*, x\rangle = 0\ for\ all\ x \in M\}$. If $M \subset X^*$ then $^{\perp}M$, the pre-annihilator of M is defined as $^{\perp}M = \{x \in X: \langle x^*, x\rangle = 0\ \ forall x^* \in M\}$ .Geven a bounded linear operator $T: X \to Y, T^*: Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X(X^*), \bar{A}, (w^* - \overline{A})$ and $co(A)$ present the closure (the w^*–closure), and the convex hull of A, respectively. For simplicity, we also use A^{**} to denote the w^*–closure of $A \subset X\ in\ X^{**}$.

Assume that X, Y are Banach spaces, and $f: X \to Y$ is anε-isometry. In this section, we shall use an invariant mean of X to define a bounded surjective operator $R: Y^* \to X^*$. With the aim of the kernel N of R and the operator Q, we further define a linear surjective isometry $U : X^* \to Y^*/N$. To begin with, we recall definition of invariant mean of a semi group and some related result.

Definition (2.1.2)[2]. Let G be a semi group. A left-invariant mean on G is a linear functional μ on $\ell_\infty(G)$ such that:

(i)$\mu(f) = 1$,

(ii)$\mu(f) \geq 0$ for every $f \geq 0$,

(iii) $\forall f \in l_\infty(G), \forall g \in G, \mu(f_g) = \mu(f)$, where f_g is the left-translation of f by g ; i.e. $f_g(h) = f(gh), \forall h \in G$.

(iv) A analogously, we can define right-invariant mean of G. an invariant mean is a linear functional on $\ell_\infty(G)$ which is both left- invariant and right-invariant. Note that (i) and (ii) are equivalent to (i) and $\| \mu \| = 1$.

Lemma (2.1.3)[2]. Every abelian semigroup G (in particular, every linear space) has an Invariant mean.

Theorem (2.1.4)[2]. Suppose that X, Y are Banach spaces, and that $f: X \to Y$ is an ε-isometry for some $\varepsilon \geq 0$. Then there exist a closed subspace $N \subset Y^*$, two linear isometries $U : X^* \to Y^*/N$ and $V : Y^*/N \to X^*$ such that

$$VU = I_{X^*} \text{ and } V = U^{-1}$$

Proof: Without loss of generality, we can assume that f is standard; otherwise, we substitute g $\equiv f - f(0)$ $for\ f.\ let\ C(f) = \overline{co}(f(X) \cup -f(X))$.

We first define a linear operator $R : Y^* \to X^* with \| R \| \leq 1$. Note X is an Abelian group with respect to the vector addition of X. by Lemma (2.1.3), there exists a translation invariant mean μ on $\ell_\infty(X)$ .Fix any $x \in X$ since f is an ε-isometry,

$$g_x(z) = f(x + z) - f(z), \quad for\ all\ z \in X \tag{5}$$

Define a bounded mapping $g_x : X \to Y$. Therefore, $\langle \emptyset, g_x \rangle \in \ell_\infty(X)$ for every $\emptyset \in Y^*$. We also dente the invariant mean by μ_z or $\mu_z(.)$, emphasizing that the mean is taken with respect to the variable z.

Now we define the linear mapping $R : Y^* \to \mathbb{R}^X for\ \emptyset \in Y^*$ by

$$\langle R\emptyset, x \rangle = \mu_z(\langle \emptyset, g_x \rangle), \quad for\ all\ x \in X. \tag{6}$$

We claim that:

(i) $R\emptyset \in X^*\ for\ \ every\ \emptyset \in Y^*$;

(ii) $\| R\emptyset \| \leq \| \emptyset \|$ $for\ every\ \emptyset \in Y^*$;

(iii) If $\emptyset \in Y^*$ $is\ bounded\ on\ C(f), then\ \emptyset \in \ker R$.

Fix any $\emptyset \in Y^*$ and let $u, v \in X$. Then

$$\langle R\emptyset, u + v\rangle = M(\langle \emptyset, g_{u+v}\rangle) = \mu_z(\langle \emptyset, f(u + v + z) - f(z)\rangle) =$$
$$\mu_z(\langle \emptyset, f(u + v + z) - f(v + z)\rangle) + \mu_z(\langle \emptyset, f(v + z) - f(z)\rangle) =$$
$$\mu_z(\langle \emptyset, f(u + z) - f(z)\rangle) + \mu_z(\langle \emptyset, f(v + z) - f(z)\rangle) =$$
$$\mu(\langle \emptyset, g_u\rangle) + \mu(\langle \emptyset, g_v\rangle) = \langle R\emptyset, u\rangle + \langle R\emptyset, v\rangle. \quad (7)$$

Therefore, additivity of $R\phi$ has been shown. It follows from additivity of,

$$\langle R\phi, \lambda u\rangle = \lambda\langle R\phi, u\rangle \text{ for all rational number } \lambda. \quad (8)$$

Note $\| \mu \| = 1$ and note that f is an ε-isometry. Therefore, (5), (6) and (8) imply for all $u, v \in X$ and $k \in \mathbb{N}$,

$$|\langle R\emptyset, u\rangle - \langle R\emptyset, v\rangle| = \frac{1}{k}|\langle R\emptyset, ku\rangle - \langle R\emptyset, kv\rangle|$$
$$= \left|\mu_z\left(\langle \emptyset, \frac{f(ku + z) - f(z)}{k}\rangle\right) - \mu_z(\langle \emptyset, \frac{f(kv + z) - f(z)}{k}\rangle)\right|$$
$$= \left|\mu_z\left(\langle \emptyset, \frac{f(ku + z) - f(kv + z)}{k}\rangle\right)\right| \leq \| \mu \| \| \emptyset \|$$
$$\| \frac{f(ku + z) - f(kv + z)}{k} \| \leq \| \mu \| \| \emptyset$$
$$\| \frac{\| (ku + z) - (kv + z) \| + \varepsilon}{k} = \| \emptyset \| \| u - v \| + \frac{\varepsilon}{k} \to \| \emptyset \|$$
$$\| u - v \|, \qquad as\ k \to \infty.$$

Hence

$$|\langle R\emptyset, u\rangle - \langle R\emptyset, v\rangle| \leq \| \emptyset \| \| u - v \| \ for\ all\ u, v \in X. \quad (9)$$

We have proven that $R\emptyset$ is 1-Lipschitz on X. this and (8) together entail that $R\emptyset$ is linear and with $\| R\emptyset \| \leq \| \emptyset \|$ .Therefore, (i) and (ii) hold, and $R: Y^* \to X^*$ is a linear operator with $\| R \| \leq 1$.

To show (iii), let M_ε associated the ε-isometry f be defined by

$$M_\varepsilon = \{\psi \in Y^* is\ bounded\ on\ C(f)\}, \ if \varepsilon > 0; = C(f)^\perp, if \varepsilon = 0, \quad (10)$$

$M = \overline{M_\varepsilon}$, the closure of M_ε. Given $\emptyset \in M_\varepsilon$, definition of M_ε implies that

$$\langle \emptyset, f\rangle \in \ell_\infty(X).$$

For every $x \in X$ it follows from the translation invariance of μ,

$$\langle R\emptyset, x\rangle = \mu_z(\langle \emptyset, f(x+z)) - f(z)\rangle) = \mu_z(\langle \emptyset, f(x+z)\rangle) - \mu_z(\langle \emptyset, f(z)\rangle) = 0.$$

Consequently, (iii) holds.

The following two mappings $\ell: X^* \to 2^{Y^*} and Q: X^* \to Y^* / M$ are defined in Cheng, Dong and Zhang:

$$\ell(x^*) = \{\emptyset \in Y^*: \langle \emptyset, f\rangle - x^* is\ bounded\ on\ X\},\ \ if \varepsilon > 0; \qquad (11)$$

$$\ell(x^*) = \{\emptyset \in Y^*: \langle \emptyset, f\rangle = x^*\},\ \ if \varepsilon = 0, \qquad (12)$$

And

$$Q(x^*) = \ell(x^*) + M,\quad for\ all\ x^* \in X^*. \qquad (13)$$

By Theorem (1.1.22), Q is a linear isometry.

Next, we show $RQ = I_{X^*}$.By the fact (iii) we have just proven, $M_\varepsilon \subset kerR$. Continuity of R implies $M \subset kerR$.Thus, R is eventually a linear operator from $Y^*/M\ to\ X^*$with $\| R \| \le 1$. Note for each $\emptyset \in \ell(x^*)$, we have

$$|\langle \emptyset, f(z)\rangle - \langle x^*, z\rangle| \le \beta\varepsilon,\quad for\ some\ \beta > 0\ and\ for\ all\ z \in X.$$

Since μ is a positive functional on $\ell_\infty(X)\ with\ \| \mu \| = 1 = \mu(1)$, for all $x \in X$,

$$\begin{aligned}\langle R\emptyset, x\rangle &= \mu_z(\langle \emptyset, f(x+z) - f(z)\rangle) \\ &= \mu_z\{(\langle \emptyset, f(x+z)\rangle - \langle x^*, x+z\rangle) - (\langle \emptyset, f(z)\rangle - \langle x^*, z\rangle) \\ &\quad + \langle x^*, x\rangle\} \le \mu(\beta\varepsilon) + \mu(\beta\varepsilon) + \mu_z(\langle x^*, x\rangle) = 2\beta\varepsilon + \langle x^*, x\rangle\end{aligned}$$

Or, equivalently,

$$\langle R\emptyset - x^*, x\rangle \le 2\beta\varepsilon,\ \ for\ all\ x \in X.$$

Therefore, $R\emptyset - x^* = 0\ \ for\ all\ \emptyset \in \ell(x^*)$. Consequently,

$$RQ(x^*) = R\ell(x^*) = x^*,\quad for\ all\ x^* \in X^*. \qquad (14)$$

Finally, let $N = kerR$ (please keep this in mind! this subspace and its annihilator $N^\perp$ will play an important part for discussion of stability in this section). We define the operators $U: X^* \to Y^*/N\ \ and\ V: Y^* / N \to X^*$

As follows:

$$Ux^* = Qx^* + N, \text{ for all } x^* \in X^*, \qquad (15)$$

And

$$V(\emptyset + N) = R(\emptyset), \quad for\ all\ \emptyset \in Y^*. \tag{16}$$

Since $N = kerR$, clearly, V is well-defined with $\| V \| = \| R \| \leq 1$. Since $Q: X^* \to Y^* / M$ is an isometry, and since $M \subset kerR = N$, U is also well-defined and with $\| U \| = \| Q \| \leq 1$. It is easy to observe that $VU = RQ = I_{X^*}$. On the other hand, $RQ = I_{X^*}$ implies that $R: Y^* \to X^*$ is surjective, which in turn implies that $V: Y^*/kerR = Y^*/N \to X^*$ is an isomorphism. This and $VU = I_{X^*}$ further entail $U: X^* \to Y^* / N$ is an isomorphism, while for each $x^* \in X^*$

$$\| x^* \| = \| (VU)x^* \| = \| V(Ux^*) \| \leq \| V \| \| Ux^* \| \leq \| x^* \|$$

Implies $\| Ux^* \| = \| x^* \|$ and $\| V\emptyset \| = \| \emptyset \|$ $fore\ ach\ \emptyset \in Y^*/N$. Hence, both U and V are surjective isometries and with $V = U^{-1}$.

Remark (2.1.5)[2]. By (14) and the proof of Theorem (2.1.4), we have $VU = RQ = I_{X^*}$ we often blur the distinction $V: Y^*/N \to X^*$ and $V: Y^* \to X^*$ i.e. the operator V is acting either on the quotient space Y^*/N, or, on Y^*, if no confusion arises. Therefore,

$$V\emptyset = x^* \ \ for\ every\ x^* \in X^* \ \ and\ every\ \emptyset \in Qx^* + N.$$

Corollary (2.1.6)[2]. Suppose that X, Y are Banach spaces, and $f: X \to Y$ is a standard ε-isometry for some $\varepsilon \geq 0$. With the subspace N and the operators U and V associated with f as in Theorem (2.1.4), we have:

(i) V^* $is\ a w^* - to - w^*$Continuous surjective isometry from $X^{**} to\ N^{\perp} \subset C^{**}(f)$, where $C^{**}(f)$ denotes the w^*-closure of $C(f)$ in Y^{**}.

(ii) If, in addition, Y is reflexive, then V^* is actually a surjective isometry from X to the subspace $N^{\perp} \subset C(f) \subset Y$.

Proof: (i). Let the operator V and the subspace N associated with f be defined as in Theorem (2.1.4). Then $V: Y^* / N \to X^*$is a linear surjective isometry. Therefore, $V^* : X^{**} \to N^{\perp}$ is a linear surjective $w^* - to - w^*$

Continuous isometry. Next, we show $N^{\perp} \subset C^{**}(f)$. Suppose, to the contrary, that there is $u \in N^{\perp} \setminus C^{**}(f)$. Note that both $C^{**}(f)$ and {u} are non-empty w^{*}–closed convex sets. Then by separation Theorem of convex sets in locally convex spaces, there is $\emptyset \in Y^{*}$ such that

$\langle \emptyset, u \rangle > sup_{v \in C^{**}(f)} \langle \emptyset, v \rangle$. Hence, $\emptyset \in M_{\varepsilon} \subset N$ and this is a contradiction!

(ii). Suppose, in addition, that Y is reflexive. Because $V^{*}\ is\ w^{*} - to - w^{*}$ continuous isometry from X^{**} onto the subspace $N^{\perp}\ of\ Y^{**} = Y$,we see that X is also reflexive and V^{*} is surjective from X to $N^{\perp} \subset C^{**}(f) = C(f) \subset Y$.

Remark (2.1.7)[2]. For an ε-isometry f from a reflexive Banach space X to another Banach space Y, though Corollary (2.1.6) follows that X is isometric to a reflexive subspace of Y^{**}, we cannot claim that X is isomorphic to a subspace of Y, even if X is a Hilbert space and f is simply an isometry. For example, let H be a non-separable Hilbert space. Then by Godefroy-kalton's Theorem, there exist a Banach space Y and an isometry $f: H \rightarrow Y$, so that H is not even linear isomorphic to a subspace of Y.

Let X and Z be Banach spaces and $f: X \rightarrow Z$ be a standard ε-isometry for some $\varepsilon \geq 0$. We use $Y \equiv L(f)$ to denote the closure of span $f(X)\ in\ Z$ and $C(f)(C^{**}(f), resp.)$ again the absolute $(w^{*}-, resp.)$ closed convex hull of $f(X)$. Note that f is actually from X to Y. Let the subspace $M_{\varepsilon}, M \subset Y^{*}$, the operators $R: Y^{*}(Y^{*}/M) \rightarrow X^{*}\ and\ Q : X^{*} \rightarrow Y^{*} / M$ and be defined in the proof of Theorem (2.1.4). The subspace N= ker $(R) \subset Y^{*}$, the linear isometries

$$U: X^{*} \rightarrow Y^{*}/N\ \ and\ V = U^{-1}: Y^{*}/N \rightarrow X^{*}$$

Associated with f are also defined as in Theorem (2.1.4), and

$$U^{*} : N^{\perp} \rightarrow X^{**}\ \ and\ \ V^{*} = (U^{*})^{-1}: X^{**} \rightarrow N^{\perp}$$

Are their conjugate operators. We further assume that Y is reflexive.

Recall that a standard ε-isometry $f: X \rightarrow\ Z\ is\ (\alpha, \gamma)$ −stable if there exist a positive number γ and $T \in\ B(Y, X)$ with $\| T \| \leq \alpha$ such that

$$\| Tf(x) - x \| \leq \gamma\varepsilon, \ for\ all\ x \in X, \tag{17}$$

Where $Y \equiv L(f)$.

Theorem (2.1.8)[2]. Let X, Z be Banach spaces and $f: X \to Z$ be a standard ε-isometry. Suppose that $Y = L(f)$ is reflexive and f is (α, γ) $-$stable . Then

(i) $N^{\perp} \subset Y$ is α-complemented in Y;

(ii) $P \equiv V^*T: Y \to N^{\perp}$is a projection with $\| P \| \leq \alpha$;

(iii) $T|_{N^{\perp}}: N^{\perp} \to X$ Is an isomorphism with $\| T_{N^{\perp}} \| \leq \alpha$; and

(iv) $TV^* = I_X,\ and\ V^*T|_{N^{\perp}} = I_{N^{\perp}}$

Proof: Since Y is reflexive, by Corollary (2.1.6), $V^* = (U^*)^{-1}$ is a surjective isometry from X to $N^{\perp} \subset C(f)$. (iv). clearly, it suffices to show $TV^* = I_X$. Indeed $TV^* = I_X$ entails

$$I_{Z^{\perp}} = V^* I_X U^* = V^*(TV^*)U^* = (V^*T)(V^*U^*) = (V^*T)I_{Z^{\perp}} = V^*T_{Z^{\perp}},$$

That is, $V^*T|_{N^{\perp}} = I_{Z^{\perp}}. Note\ V^*(X) = N^{\perp} \subset C(f) = \overline{co}\big(f(X) \cup -f(X)\big)$. For every $x_0 \in X$, let $(y_n) \subset C(f)$ be a sequence such that

$$y_n = \sum_{j \in J_n} \lambda_j^n f(x_j^n) \to Vx_0, as\ n \to \infty, \tag{18}$$

For some finite sets $J_n \subset \mathbb{N},\ (x_j^n)_{j \in J_n} \subset X\ and\ (\lambda_j^n)_{j \in J_n} \subset \mathbb{R}$ with $\sum_{j \in j_n} |\lambda_j^n| = 1$. For each $x^* \in X^*$, by the Main Lemma there exists $\emptyset \in Y^* with \| \emptyset \| = \| x^* \| \equiv r$ such that

$$|\langle \emptyset, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r,\ for\ all\ x \in X. \tag{19}$$

Let $x_n = \sum_{j \in J_n}^{n} \lambda_j^n x_j^n$. Then, Remark (2.1.5), (18), (19) and $\sum_{j \in J_n} |\lambda_j^n| = 1$ together entail $|\langle \emptyset, V^*x_0 - y_n \rangle| = |\langle \emptyset, V^*x_0 \rangle - \langle \emptyset, y_n \rangle| = |\langle V\emptyset, x_0 \rangle - \langle \emptyset, y_n \rangle| = |\langle x^*, x_0 - x_n \rangle - (\langle \emptyset, y_n \rangle - \langle x^*, x_n \rangle)| \geq |\langle x^*, x_0 - x_n \rangle| - |\langle \emptyset, y_n \rangle - \langle x^*, x_n \rangle| =$
$|\langle x^*, x_0 - x_n \rangle| -$
$|\sum_{j \in J_n} \lambda_j^n \big(\langle \emptyset, f(x_j^n) \rangle - \langle x^*, x_j^n \rangle\big)| \geq |\langle x^*, x_0 - x_n \rangle| - 4\varepsilon \| x^* \|.$

Note $\| \emptyset \| = \| x^* \|$ for all $x^* \in X^*$. Consequently,

$$|\langle x^*, x_0 - x_n\rangle| \le |\langle \emptyset, V^*x_0 - y_n\rangle| + 4\varepsilon \parallel x^* \parallel \le (\parallel V^*x_0 - y_n \parallel$$
$$+4\varepsilon) \parallel x^* \parallel \to 4\varepsilon \parallel x^* \parallel. \qquad (20)$$

Suppose that $T: Y \to X$ satisfies $\parallel T \parallel \le \alpha$ and

$$\parallel Tf(x) - x \parallel \le \gamma\varepsilon, \quad for \ all \ x \in X, \qquad (21)$$

For some $\gamma > 0$. Then for every $x^* \in X^*$

$$\gamma\varepsilon \parallel x^* \parallel \ge \langle x^*, Tf(x) - x\rangle = |\langle T^*x^*, f(x)\rangle - \langle x^*, x\rangle|, for \ all \ x \in X \qquad (22)$$

This implies that the function $|\langle T^*x^*, f\rangle - x^*|$ defined on X is bounded by $\gamma\varepsilon \parallel x^* \parallel. By (11), \emptyset_{x^*} \equiv T^*x^* \in \ell x^*. \ Let x^* \in S_{X^*}$ be such that

$$\langle x^*, TV^*(x_0) - x_0\rangle = \parallel TV^*x_0 - x_0 \parallel.$$

Then (19)-(22) together imply

$$\parallel TV^*x_0 - x_0 \parallel = |\langle x^*, TV^*x_0 - x_0\rangle| = |\langle x^*, TV^*x_0\rangle - \langle x^*, x_0\rangle|$$
$$= |\langle T^*x^*, V^*x_0\rangle - \langle x^*, x_0\rangle| = |\langle \emptyset_{x^*}, V^*x_0\rangle - \langle x^*, x_0\rangle|$$
$$\le |\langle \emptyset_{x^*}, V^*x_0 - y_n\rangle| + |\langle \emptyset_{x^*}, y_n\rangle - \langle x^*, x_n\rangle| + |\langle x^*, x_0 - x_n\rangle|$$
$$\le 2|\langle \emptyset_{x^*}, V^*x_0 - y_n\rangle|$$
$$+ \sum_{j\in J_n} \left|\lambda_j\left(\langle \emptyset_{x^*}, f(x_j^n)\rangle - \langle x^*, x_j^n\rangle\right)\right| + 4\varepsilon \le 2|\langle \emptyset_{x^*}, V^*x_0 - y_n\rangle|$$
$$+ \sum_{j\in J_n} |\lambda_j| \parallel Tf(x_j^n) - x_j^n \parallel + 4\varepsilon$$
$$\le 2|\langle \emptyset_{x^*}, V^*x_0 - y_n\rangle| + \gamma\varepsilon + 4\varepsilon \to (4+\gamma)\varepsilon.$$

Therefore,

$$\parallel TV^*(x_0) - x_0 \parallel \le (\gamma + 4)\varepsilon.$$

Arbitrariness of $x_0 \in X \ entails \ \ TV^* = I_X$, hence, (iv).

(iii). It immediately follows from $TV^* = I_X$ we have just proven, since $V^* : X \to N^\perp$ is a surjective isometry.

(ii). It follows from $(V^*T)(Y) = V^*(X) = N^\perp \subset Y$.

(i). It follows from (ii).

Theorem (2.1.9)[2]. Let X, Z be Banach spaces and $f: X \to Z$ be standard ε-isometry. Suppose that $Y = L(f)$ is reflexive. If $N^\perp$ is α-complemented in Y, then for every projection $P : Y \to N^\perp$ with $\parallel P \parallel \le \alpha$, $T = U^*P$ satisfies (i)$\| T \| \leq \alpha$;

(ii)$TV^* = I_X$, $theidentityonX$;

(iii) $\| Tf(x) - x \| \leq 4\varepsilon$, $for\ all\ x \in X$.

Proof: Let $W \subset Y$ be a closed subspace of Y with $N^\perp \cap W = \{0\}$ and with $N^\perp + W = Y$ such that the projection $P : Y \to N^\perp$ along W satisfies $\| P \| \leq \alpha$; , and let $T = U^*P$. $Then$ $\| T \| \leq \| U^* \| \| P \| \leq \alpha$. Therefore, (i) follows. (ii). since $V^* = (U^*)^{-1} : X \to N^\perp$, we have

$$TV^* = (U^*P)V^* = U^*(PV^*) = U^*V^* = I_X.$$

It remains to show that T satisfies (iii). Note that $U : X^* \to Y^* / N$ is defined by $Ux^* = \emptyset_{x^*} + N$, $and\ Y^*/N = [(N^\perp)^\perp \oplus W^\perp]/N = W^\perp$, $where\ \emptyset_{x^*}(\equiv \emptyset) \in Y^*$satisfies (19) for every $x^* \in X^*$. Therefore,

$$\langle x^*, Ty\rangle = \langle Ux^*, Py\rangle = \langle Ux^*, y\rangle = \langle \emptyset_{x^*}, y\rangle, for\ all\ x^* \in X^*, y \in Y \qquad (23)$$

Thus, for every fixed $x \in X$, we choose $x^* \in Y^*$ $with$ $\| x^* \| = 1$ such that

$$|\langle x^*, Tf(x) - x\rangle| = \| Tf(x) - x \|. \qquad (24)$$

It follows from definition of T, (23) and (24)

$$\| Tf(x) - x \| = |\langle x^*, Tf(x) - x\rangle| = |\langle x^*, Tf(x)\rangle - \langle x^*, x\rangle| = \left|\langle x^*, U^*\big(Pf(x)\big)\rangle - \langle x^*, x\rangle\right| = |\langle Ux^*, Pf(x)\rangle - \langle x^*, x\rangle| = |\langle \mathrm{U}x^*, \mathrm{f}(x)\rangle - \langle x^*, x\rangle| = |\langle \emptyset_{x^*}, \mathrm{f}(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon. \qquad (25)$$

Therefore, we have shown (iii), and which completes our proof.

Corollary (2.1.10)[2]. Let X, Y be Banach spaces and $f: X \to Y$ be a standard ε- isometry with reflexive $Y = L(f)$. Then

(i) f is (α,γ)-stable if and only if the subspace $N^\perp$ associated with f is α-complemented in Y.

(ii) If the subspace $N^\perp$ associated with f is α-complemented in Y, then f is (α, 4)-stable.

Proof: (i). Sufficiency follows from Theorem (2.1.8), while Theorem (2.1.9) implies necessity. (ii). Suppose that the subspace $N^\perp$ associated with f is α-complemented in Y Theorem (2.1.9) implies that f is (α, 4)-stable.

For a (continuous) convex function g defined on a Banach space X, its sub differential mapping $\partial g : X \to 2^{X^*}$ is defined for $x \in X$ by

$$\partial g(x) = \{x^* \in X^* : g(y) - g(x) \geq \langle x^*, y - x\rangle, \quad for\ all\ x \in X\}.$$

The convex function g is Gateaux differentiable at$x \in X$ if and only if ∂g is single-valued at x, and in this case, $\partial g(x) = \{dg(u)\}$ .The following property is a direct consequence of Proposition in Phelps'

Proposition (2.1.11)[2]. A Banach space X is Gateaux smooth if and only if the sub differential mapping $\partial \|.\|: X \to 2^{X^*}$ is single-valued and norm –to-w^* continuous at each point $x \neq 0$.

For a standard ε- isometry $f: X \to Y$, letM_ε be defined by (10), and let

$$E = \{y \in Y : \langle \emptyset, y\rangle = 0 \ for\ all\ \emptyset \in M_\varepsilon\}. \quad (26)$$

We have the following Theorem

Theorem (2.1.12)[2]. Let X, Y be Banach spaces and $f: X \to Y$ be standard ε –isometry with reflexive $Y = L(f)$. If, in addition, Y is Gateaux smooth, strictly convex and possessing the Kadec-Klee property (in particular, locally uniformly convex), then f is (α, 2)-stable if and only if the subspace $N^\perp$ is α-complemented in Y.

Proof: Sufficiency. Since Y is reflexive and strictly convex, by Theorem (1.2.3) and the Kadec-Klee property of Y, the operator $\Psi : X \to Y$defined by

$$\Psi x = w - \lim_{\lambda\to\infty} f(\lambda x)/\lambda = \lim_{n\to\infty} f(nx)/n \quad (27)$$

Is a linear isometry. (19) further implies $\Psi\ (X) \subset E$. According to Theorem (2.1.4), X is also reflexive, strictly convex and Gateaux smooth. Since the closed subspace$N^\perp$ is α-complemented, there is a closed subspace W of Y satisfying $N^\perp \cap W = \{0\}\ and\ N^\perp + W = Y$ such that the projection $P: Y \to N^\perp$ along W satisfies $\| P \| \leq \alpha$. Since $Y(X)$ is reflexive and Gateaux smooth, by Proposition (2.1.11), $d \| u \|$ exist for all $u \neq 0$ in $Y(X)$ and $d \|.\|$ is norm –to-weak continuous at each point

$u \neq 0$. $Let\ U : Y^* / N \to X^*$ be defined by (15) and let $T = U^*P$. Then $\| T \| \leq \alpha$. We want to prove that T satisfies

$$\| Tf(x) - x \| \leq 2\varepsilon,\ for\ all\ x \in X. \qquad (28)$$

Given $x \in X$, without loss of generality, we assume that $x \neq Tf(x)$. Let

$$\beta = \| x - Tf(x) \| , \quad z = x - Tf(x)/\beta, \qquad (29)$$

And let $x^* = d \| z \| (\in S_{X^*}). Then\ \langle x^*, x - Tf(x)\rangle = \| x - Tf(x) \|$ by (23)

$$\beta = \langle x^*, x - Tf(x)\rangle = \langle x^*, x\rangle - \langle \emptyset_{x^*}, f(x)\rangle \leq 4\varepsilon, \qquad (30)$$

Where $\emptyset_{x^*} \in S_{Y^*}$ satisfies

$$|\langle \emptyset_{x^*}, f(y)\rangle - \langle x^*, y\rangle| \leq 4\varepsilon,\ \ for\ all\ y \in X. \qquad (31)$$

Substituting nz for y and dividing the both sides of (31) by $n \in \mathbb{N}$,we obtain

$$|\langle \emptyset_{x^*}, f(nz)/n\rangle - \langle x^*, z\rangle| \leq (4/n)\varepsilon. \qquad (32)$$

Let $n \to \infty$ .Then, by (27)

$$\langle \emptyset_{x^*}, \Psi z\rangle = \langle x^*, z\rangle = d \| z \| (z) = \| z \| = 1.$$

$\langle \emptyset_{x^*}, \psi z\rangle = 1 = \| \phi_{x^*} \| \| \psi z \|$ and smoothness of Y further entail $\phi_{x^*} = d \| \psi(z) \|$ .On the other hand, let

$$q_n(x) = f(x + nz),\ \ r_n(x) = f(x + nz)/n\ \ and\ \emptyset_n = d \| r_n(x) \| \qquad (33)$$

Note that for any $\gamma > 0$ and for any u ∈ Y with $u \neq 0$, $d \| \gamma u \| = \| u \|$. Then

$$\| f(x + nz) \| = \langle \emptyset_n, f(x + nz)\rangle \leq \langle \emptyset_n, f(x)\rangle + \| f(x + nz) - f(x) \|$$
$$\leq \langle \emptyset_n, f(x)\rangle + n + \varepsilon. \qquad (34)$$

By (27) again, $r_n(x) \to \Psi z$ .Gateaux smoothness, reflexivity of Y and Proposition (2.1.11) together entail $\phi_n \to d \| \Psi(z) \| = \phi_{x^*}$ in the weak topology of Y. Consequently,

$$\lim_n \sup(\| f(x + nz) \| - n) \leq \langle \emptyset_{x^*}, f(x)\rangle + \varepsilon. \qquad (35)$$

Note$x^* = d \| z \|$. Since

$$\| f(x + nz) \| - n \geq \| x + nz \| - n - \varepsilon,$$

We have

$\lim_n \inf(\| f(x+nz) \| - n) \geq \lim_n(\| x+nz \| - n) - \varepsilon =$

$$\lim_n \frac{\|z+\frac{1}{n}x\| - \|z\|}{\frac{1}{n}} - \varepsilon = \langle x^*, x\rangle - \varepsilon. \quad (36)$$

(29), (35) and (36) yield

$$\| x - Tf(x) \| = \langle x^*, x\rangle - \langle \emptyset_{x^*}, f(x)\rangle \leq 2\varepsilon.$$

Necessity. If there exist a linear operator $T: Y \to X$ with $\| T \| \leq \alpha$ and $\gamma > 0$ satisfying

$$\| Tf(x) - x \| \leq \gamma\varepsilon, \quad for\ all\ x \in X,$$

Then by Theorem (2.1.8), $N^{\perp}$ is complemented and $S = U^*T : Y \to N^{\perp}$ is a projection with $\| S \| \leq \| U^* \| \| T \| = \| T \| \leq \alpha$.

Section (2.2): Stability Properties in General Banach Spaces

Let Z be a Banach space. A w^*–closed subspace E of the dual space $Z^*$is said to be $w^* - \alpha$-complemented, if there exists a w^*–closed subspace F of Z^* $with\ E \cap F = \{0\}$ $and\ with\ E + F = Z^*$, such that the project ion $P : Z^* \to E$ along F satisfies $\| P \| \leq \alpha$.

Theorem (2.2.1)[2]. Let X, Z be Banach spaces, $f: X \to Z$ be a standard ε-isometry with $Y \equiv L(f)$ and let f be (α,γ)-stable. Then

(i) $N^{\perp} \subset C^{**}(f) \subset Y^{**}\ is\ w^* - \alpha$ -complemented in Y^{**};

(ii)$V^*T^{**}: Y^{**} \to N^{\perp}\ is\ w^* - to - w^*$Continuously surjective and with $\| V^*T^{**} \| \leq \alpha$;

(iii) $T^{**}|_{N^{\perp}} : N^{\perp} \to X^{**} is\ anisomorphism; and$

(iv) $T^{**}V^* = I_{X^{**}},\ and V^*T^{**}|_{N^{\perp}} = I_{Z^{\perp}}$.

Proof: Recall that $V^* = (U^*)^{-1}\ is\ a\ w^* - to - w^*$ continuous surjective isometry from $X^{**} to\ N^{\perp} \subset C^{**}(f)$ (Corollary (2.1.6) (i))

(iv) Clearly, it suffices to show $T^{**}V^* = I_{X^{**}}. Indeed, T^{**}V^* = I_{X^{**}}$ entails $I_{Z^{\perp}} = V^*I_{X^{**}}U^* = V^*(T^{**}V^*)U^* = (V^*T^{**})(V^*U^*) = (V^*T^{**})I_{Z^{\perp}} = V^*T^{**}_{Z^{\perp}},$

That is, $V^*T^{**}|_{N^\perp} = I_{Z^\perp}. Note V^*(X^{**}) = N^\perp \subset C^{**}(f) = w^* - \overline{co}(f(X) \cup -f(X))$. For every $x_0 \in X^{**}$, there exists net $(y_\alpha) \subset co(f(X) \cup -f(X))$ of the form: for each α, there exist three finite sets $J_\alpha, (\lambda_j^\alpha)_{j\in J_\alpha} \subset \mathbb{R}\, with\ \sum_{j\in J_\alpha} |\lambda_j^\alpha| = 1, and\ (x_j^\alpha)_{j\in J_\alpha} \subset X$ such that

$$y_\alpha = \sum_{j\in J_\alpha} \lambda_j^\alpha f(x_j^\alpha) \to V^* x_0,\ in\ the\ w^* - topology\ of\ Y^{**}. \tag{37}$$

For each $x^* \in X^*$, there exists $\emptyset \in Y^* with \parallel \emptyset \parallel = \parallel x^* \parallel \equiv r$ such that

$$|\langle \emptyset, f(x)\rangle - \langle x^*, x\rangle| \le 4\varepsilon r,\quad \text{For all x} \in X. \tag{38}$$

Let $x_\alpha = \sum_{j\in J_\alpha}^n \lambda_j^\alpha x_j^\alpha$. Therefore, (37), (38) and $\sum_{j\in J_\alpha} |\lambda_j^\alpha| = 1$ together entail

$$\begin{aligned}
|\langle \phi, V^*x_0 - y_\alpha\rangle| &= |\langle \phi, V^*x_0\rangle - \langle \phi, y_\alpha\rangle| = |\langle V\phi, x_0\rangle - \langle \emptyset, y_\alpha\rangle| \\
&= |\langle x^*, x_0 - x_\alpha\rangle - (\langle \phi, y_\alpha\rangle - \langle x^*, x_\alpha\rangle)| \\
&\ge |\langle x^*, x_0 - x_\alpha\rangle| - |\langle \phi, y_\alpha\rangle - \langle x^*, x_\alpha\rangle| \\
&= |\langle x^*, x_0 - x_\alpha\rangle| - |\sum_{j=1}^n \lambda_j \big(\langle \emptyset, f(x_j^\alpha)\rangle - \langle x^*, x_j^\alpha\rangle\big)\ | \\
&\ge |\langle x^*, x_0 - x_\alpha\rangle| - 4\varepsilon \parallel x^* \parallel.
\end{aligned}$$

Consequently,

$$|\langle x^*, x_0 - x_\alpha\rangle| \le 4\varepsilon \parallel x^* \parallel + |\langle \phi, V^*x_0 - y_\alpha\rangle|,\quad for all\ x^* \in X^*. \tag{39}$$

Since $T \in B(Y, X) with\ \parallel T \parallel \le \alpha$ satisfies

$$\parallel Tf(x) - x \parallel \le \gamma\varepsilon,\quad for\ all\ x \in X, \tag{40}$$

For every $x^* \in X^*$,

$$|\langle T^*x^*, f(x)\rangle - \langle x^*, x\rangle| = |\langle x^*, Tf(x) - x\rangle| \le \gamma\varepsilon \parallel x^* \parallel. \tag{41}$$

Given $\delta > 0, let x^* \in S_{X^*} be\ such\ that\ \langle x^*, T^{**}V^*x_0 - x_0\rangle \ge \parallel T^{**}V^*x_0 - x_0 \parallel - \delta$. Then (37)- (41) together imply $\parallel T^{**}V^*x_0 - x_0 \parallel - \delta \le |\langle x^*, T^{**}V^*x_0 - x_0\rangle| \le |\langle x^*, T^{**}V^*x_0 - T^{**}y_\alpha\rangle| + |\langle x^*, T^{**}y_\alpha - x_\alpha\rangle| + |\langle x^*, x_\alpha - x_0\rangle| = |\langle T^*x^*, V^*x_0 - y_\alpha\rangle| + |\langle T^*x^*, y_\alpha\rangle - \langle x^*, x_\alpha\rangle| + |\langle x^*, x_\alpha - x_0\rangle| \le |\langle T^*x^*, V^*x_0 - y_\alpha\rangle| + \sum_{j\in J_\alpha} |\lambda_j^\alpha \big(\langle T^*x^*, f(x_j^\alpha)\rangle -$

$\langle x^*, x_j^{\alpha}\rangle)| + |\langle x^*, x_\alpha - x_0\rangle| \leq |\langle T^*x^*, V^*x_0 - y_\alpha\rangle| + \gamma\varepsilon + |\langle x^*, x_\alpha - x_0\rangle| \leq$
$|\langle T^*x^*, V^*x_0 - y_\alpha\rangle| + |\langle \emptyset, V^*x_0 - y_\alpha\rangle| + (4+\gamma)\varepsilon \to (4+\gamma)\varepsilon.$

Therefore, $\| T\, V^*x_0 - x_0 \| \leq (\gamma + 4)\varepsilon$. Arbitrariness of $x_0 \in X^{**}$ entails

$$\| (T\, V^* - I_{X^{**}})x \| \leq (\gamma + 4)\varepsilon, \qquad for all x \in X^{**},$$

Which, in turn, implies $TV^* = I_{X^{**}}$. Hence, we have proven (iv).

(iii). It immediately follows from $T^{**}V^* = I_{X^{**}}$ we have just proven, since $V^* : X^{**} \to N^\perp$ is a $w^* - to - w^*$ continuous surjective isometry.

(ii). since $(V^*T^{**})(Y^{**}) = V^*(X^{**}) = N^\perp \subset Y^{**}$, and since both V^* and T^{**} are $w^* - to - w^*$ continuous and with $\| V^*T^{**} \| \leq \| V^* \| \| T^{**} \| = \| T \| \leq \alpha$, (ii) follows. (i). It directly follows from (ii).

Theorem (2.2.2)[2]. Let X, Z be Banach spaces and $f: X \to Z$ be a standard ε-isometry. If $N^\perp$ is $w^* - \alpha$–complemented in Y^{**}, then there is a linear operator $T: Y^{**} \to X^{**}$ such that

(i) T is $w^* - to - w^*$ continuously surjective and with $\| T \| \leq \alpha$. ;

(ii) $T = U^*P$, for some $w^* - to - w^*$ continuous projection $P : Y^{**} \to N^\perp$;

(iii) $TV^* = I_{X^{**}}$, the identity on X^{**} ; and

(iv) $\| Tf(x) - x \| \leq 4\varepsilon$, for all $x \in X$.

Proof: Let W be a w^*-closed subspace of Y^{**} and with $N^\perp \cap W = \{0\}$ and with $N^\perp + W = Y^{**}$ such that the projection $P : Y^{**} \to N^\perp$ along W is $w^* - to - w^*$ continuous and satisfies $\| P \| \leq \alpha$, and let T $= U^*P$. Since both U^* and P are $w^* - to - w^*$ continuous, T is $w^* - to - w^*$ continuous with $\| T \| \leq \| U^* \| \| P \| \leq \alpha$. Therefore, (i) and (ii) have been shown. (iii) is trivial by definition of T and properties of U and V. It remains to show that T satisfies (iv). Since $N^\perp$ is $w^* - \alpha$–complemented in Y^{**} there are two closed sub spaces $C, D \subset Y^*$ with $C \oplus D = C + D = Y^*$ such that $D^\perp = N^\perp$ and $C^\perp = W$. Thus $D = {}^\perp(D^\perp) = {}^\perp(N^\perp) = N$ and $C = {}^\perp(C^\perp) = {}^\perp W$. Note that $U : X^* \to Y^*/N$ is defined by $Ux^* = \emptyset_{x^*} + N$ and $Y^*/N =$

$[N \oplus {}^{\perp}W]/N = {}^{\perp}W, where\ \emptyset_{x^*}(\equiv \emptyset) \in Y^*$ satisfies (38) for each $x^* \in X^*$. Therefore,

$$\langle x^*, Ty\rangle = \langle Ux^*, Py\rangle = \langle Ux^*, y\rangle = \langle \emptyset_{x^*}, y\rangle, x^* \in X^*, y \in Y^{**} \qquad (42)$$

Thus, given $\delta > 0$, for every $x \in X$,we choose $x^* \in X^* with \parallel x^* \parallel = 1$ such that

$$\parallel Tf(x) - x \parallel -\delta \le |\langle x^*, Tf(x) - x\rangle| = |\langle x^*, Tf(x)\rangle - \langle x^*, x\rangle| = \left|\langle x^*, U^*\big(Pf(x)\big)\rangle - \langle x^*, x\rangle\right| = |\langle Ux^*, Pf(x)\rangle - \langle x^*, x\rangle| = |\langle Ux^*, f(x)\rangle - \langle x^*, x\rangle| = |\langle \emptyset_{x^*}, f(x)\rangle - \langle x^*, x\rangle| \le 4\varepsilon. \qquad (43)$$

Arbitrariness of δ implies (iv), and which completes our proof.

Theorem (2.2.3)[2]. Let X and Z be Banach spaces and f : X→ Z be a standard ε-isometry with L(f) = Y. Then there is a $w^* - to - w^*$ continuous linear surjective operator $T : Y^{**} \to X^{**}$ such that

$$\parallel Tf(x) - x \parallel \le \gamma\varepsilon,\ for\ some\ \gamma > 0\ and\ for\ all\ x \in X, \qquad (44)$$

If and only if the subspace $N^{\perp} is\ w^*$–complemented in Y^{**}.

Proof: Sufficiency follows from Theorem (2.2.2) (iv).

Necessity. It follows from the proof of Theorem (2.2.1), for every $x_0 \in X^{**}$, there exists a net $(y_\alpha) \subset co(f(X) \cup -f(X))$ of the form : for each α, there exist finite sets

$J_\alpha, (\lambda_j^\alpha)_{j\in J_\alpha} \subset \mathbb{R}\ with\ \sum_{j\in J_\alpha} |\lambda_j^\alpha| = 1, and\ (x_j^\alpha)_{j\in J_\alpha} \subset X$, suth that

$$y_\alpha = \sum_{j\in J_\alpha} \lambda_j^\alpha f\big(x_j^\alpha\big) \to V^* x_0, in\ the\ w^* - topology\ of\ Y^{**}. \qquad (45)$$

And such that

$$|\langle x^*, x_0 - x_\alpha\rangle| \le 4\varepsilon \parallel x^* \parallel + |\langle \emptyset, V^* x_0 - y_\alpha\rangle|,\ for\ all\ x^* \in X^*. \qquad (46)$$

Suppose that $T : Y^{**} \to X^{**}$ is a bounded $w^* - to - w^*$continuous linear surjective operator satisfying

$$\parallel Tf(x) - x \parallel \le \gamma\varepsilon, for\ all\ x \in X, \qquad (47)$$

For some $\gamma > 0$ then ker T is w^*–closed in Y^{**} and $T: Y^{**}/\ker\ T \to X^{**}$ is a $w^* - to - w^*$continuous isomorphism. Therefore, there is a bounded linear isomorphism $S: X^* \to {}^{\perp}kerT$ such that $S^* = T$. For every $x^* \in X^*$, by (47) we observe

$$|\langle Sx^*, f(x)\rangle - \langle x^*, x\rangle| = |\langle x^*, S^*f(x)\rangle - \langle x^*, x\rangle| = |\langle x^*, Tf(x)\rangle - \langle x^*, x\rangle| \leq \| x^* \| \| Tf(x) - x \| \leq \gamma\varepsilon \| x^* \|. \quad (48)$$

Hence, the function $|\langle Sx^*, f(.)\rangle - x^*|$ is bounded by $\gamma\varepsilon \| x^* \|$ on X. By (47) and (48), $Sx^* \in \ell x^*. Let x_\alpha = \sum_{j \in J_\alpha} \lambda_j^\alpha x_j^\alpha . Given\ \delta > 0, let\ x^* \in S_{X^*}$ be such that

$$\langle x^*, TV^*x_0 - x_0\rangle \geq \| TV^*x_0 - x_0 \| - \delta.$$

Then by (45)-(48)

$$\begin{aligned}
\| TV^*x_0 - x_0 \| - \delta &\leq |\langle x^*, TV^*x_0 - x_0\rangle| \\
&\leq |\langle x^*, TV^*x_0 - Ty_\alpha\rangle| + |\langle x^*, Ty_\alpha - x_\alpha\rangle| + |\langle x^*, x_\alpha - x_0\rangle| \\
&= |\langle Sx^*, V^*x_0 - y_\alpha\rangle| + |\langle Sx^*, y_\alpha\rangle - \langle x^*, x_\alpha\rangle| + |\langle x^*, x_\alpha - x_0\rangle| \\
&= |\langle \emptyset, V^*x_0 - y_\alpha\rangle| \\
&\quad + \sum_{j \in J_\alpha} \left|\lambda_j^\alpha\left(\langle Sx^*, Tf(x_j^\alpha)\rangle - \langle x^*, x_j^\alpha\rangle\right)\right| + |\langle x^*, x_\alpha - x_0\rangle| \\
&\leq |\langle Sx^*, V^*x_0 - y_\alpha\rangle| + \gamma\varepsilon + |\langle x^*, x_\alpha - x_0\rangle| \\
&\leq |\langle Sx^*, V^*x_0 - y_\alpha\rangle| + |\langle \emptyset, V^*x_0 - y_\alpha\rangle| + (4 + \gamma)\varepsilon \\
&\to (4 + \gamma)\varepsilon.
\end{aligned}$$

Arbitrariness of δ and x_0 implies

$$\| (T \quad V^* - I_{X^{**}})x \| \leq (\gamma + 4)\varepsilon, \qquad for\ all\ x \in X^{**}.$$

Therefore, $TV^* = I_{X^{**}}$ .Consequently, $V^*T|_{N^\perp} = I_{N^\perp}$.Thus, $p = V^*T: Y^{**} \to N^\perp$ is a $w^* - to - w^*$ continuous projection. Suppose that X, Z are Banach spaces and $f: X \to Z$ is a standard ε-isometry for some $\varepsilon \geq 0$ with $L(f) = Y$. Let the subspace $M \subset Y^*$ and the operator $Q : X^* \to Y^* / M$ be defined by (10) and (13), respectively. Then, analogous to Theorem (1.1.25), we have also the following Theorem.

Theorem (2.2.4)[2]. Let X, Z be Banach spaces and $f: X \to Z$ be a standard

ε-isometry. If $M^{\perp}$ $is\ w^* - \alpha$-complemented in Y^{**}, then there is a linear operator $T: Y^{**} \to X^{**}$ such that

(i) T is $w^* - to - w^*$continuously surjective and with $\| T \| \leq \alpha$;

(ii)$T = Q^* P$, for some $w^* - to - w^*$ continuous projection $P: Y^{**} \to M^{\perp}$;

(iii) $\| Tf(x) - x \| \leq 4\varepsilon, for\ all\ x \in X.$

Proof: Let W be a w^*–closed subspace of Y^{**} with $N^{\perp} \cap W = \{0\}$ and with $N^{\perp} + W = Y^{**}$ such that the projection $P: Y^{**} \to N^{\perp}$ along W (is $w^* - to - w^*$ continuous and) satisfies $\| \alpha \| \leq \alpha$; ,and let $T = Q^* P$. Since both $Q^* and\ P\ are\ w^* - to - w^*$ continuous, T is $w^* - to - w^*$ continuous with $\| T \| \leq \| Q^* \| \| P \| \leq \alpha$. Therefore, (i) and (ii) have been shown. It remains to show that T satisfies (iii). Let $C, D \subset Y^*$ be two closed subspaces with $C \oplus D = C + D = Y^*$ such that $D^{\perp} = M^{\perp}$ and $C^{\perp} = W$. Then $D = {}^{\perp}(D^{\perp}) = {}^{\perp}(M^{\perp}) = M$ and $C = {}^{\perp}(C^{\perp}) = {}^{\perp}W$. Note that $Q = X^* \to Y^* / M$ is defined by $Qx^* = \emptyset_{x^*} + M$, and $Y^*/M = [M \oplus {}^{\perp}W]/M = {}^{\perp}W$, where $\emptyset_{x^*} (\equiv \emptyset) \in Y^*$ satisfies (38) for each $x^* \in X^*$. Therefore,

$$\langle x^*, Ty \rangle = \langle Qx^*, Py \rangle = \langle Qx^*, y \rangle = \langle \emptyset_{x^*}, y \rangle, \quad x^* \in X^*, y \in Y^{**}. \tag{49}$$

Given $\delta > 0$, for every$x \in X$ we choose$x^* \in X^* with\ \| x^* \| = 1$such that

$$\| Tf(x) - x \| - \delta \leq |\langle x^*, Tf(x) - x \rangle| = |\langle x^*, Tf(x) \rangle - \langle x^*, x \rangle| =$$
$$\left|\langle x^*, Q^*\big(Pf(x)\big) \rangle - \langle x^*, x \rangle\right| = |\langle Qx^*, Pf(x) \rangle - \langle x^*, x \rangle| =$$
$$|\langle Qx^*, f(x) \rangle - \langle x^*, x \rangle| = |\langle \emptyset_{x^*}, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon. \tag{50}$$

Then $\| Tf(x) - x \| \leq 4\varepsilon + \delta\ forall\ \delta > 0$. So that (iii) holds.

Please note the following example which says that E and F are not linearly isometric in general.

Example (2.2.5)[2]. Let $X = \mathbb{R}, Y = \ell_{\infty}^2\ and\ f: X \to Y$ be defined by

$$f(t) = (t, \ln(1 + t)), \quad if\ t \geq 0; \quad = (t, 0), if\ t < 0.$$

Clearly, f is a standard isometry with $E = C(f) = Y$, so that $N^{\perp} \subset Y$ is isometric to $X = \mathbb{R}$.

Chapter 3
Stability of ε-isometries of Banach Spaces

Let X, Y be two Banach spaces, and $f: X \to Y$ be standard ε-isometry for some $\varepsilon \geq 0$. We show the following sharp weak stability inequality of f: for every $x^* \in X^*$ there exists $\emptyset \in Y^*$ with $\| \emptyset \| = \| x^* \| \equiv r$ such that $|\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle| \leq 2\varepsilon r$ for all $x \in X$. It is not only a sharp quantitative extension of Figiel's Theorem but it also unifies, generalizes and improves a series of known results about stability of ε-isometries. For example, if the mapping f satisfies $C(f) \equiv \overline{co}[f(X) \cup -f(X)] = Y$, thenit is equivalent to the sharp stability Theorem.

Section (3.1): Sharp Inequality of Weak Stability of ε-isometries

Assume that X, Y are Banach spaces. A mapping $f: X \to Y$ is said to be anε-isometry for some $\varepsilon \geq 0$ provided

$$| \| f(x) - f(y) \| - \| x - y \| | \leq \varepsilon, \ for\ all\ x, y \in X. \quad (1)$$

The mapping f is called an isometry if $\varepsilon = 0$. f is standard if $f(0) = 0$. In this case we use Y_f to denote the subspace $\overline{span} f(X)$ of Y.

The study of properties of isometries and ε-isometries between Banach spaces has continued for over eighty years since the Mazur-Ulam celebrated Theorem: every surjective isometry between two Banach spaces is necessarily affine. For general isometries, Figiel showed the remarkable result in 1968: Every standard isometry from a Banach space to another Banach space admits a linear left-inverse of norm one. Godefroy and Kalton resolved a long standing problem about the relation between the existence of isometries and linear isometries.

Hyers and Ulam first studied ε-isometries and proposed a problem, which can be reformulated as follows: Given two Banach spaces X, Y, whether isometry $f: X \to Y$ there is a linear surjective isometry $U \in B(X, Y)$ so that

$f - U$ is uniformly bounded by $\gamma\varepsilon$ on X. After 50 years efforts of a number of mathematicians, a positive answer with the estimate $\gamma = 2$ was finally achieved by omladič and Šemrl. They gave an example of a standard surjective ε- isometry $f: \mathbb{R} \to \mathbb{R}$ showing that $\gamma = 2$ is optimal. Thus, Omladič-Šemrl's Theorem can be regarded as a sharp quantitative extension of the Mazur –Ulam Theorem.

The study of properties of non-surjective ε-isometries has been active since 90's of the last century. The question, if every standard ε-isometry $f: X \to Y$ admits a linear quasi-left inverse, that is, if there exists $T \in B(Y_f, X) so\ that\ Tf - Id$ is uniformaly bounded on X seems to be very natural. However, Qian showed that for all $\varepsilon > 0$ every separable Banach space Y admitting an uncomplemented subspace X has an unstable standard ε-isometry from X to Y. Therefore, an affirmative answer for the question would imply that Y is, up to an isomorphism, a Hilbert space. This disappointment makes us to search for some weaker stability version and some appropriate complementability assumption on some subspaces of Y associated with the mapping. Recently, Cheng, Dong and Zhang gave a weak stability Theorem (Lemma (1.1.14)), which can be regarded as a quantitative extension of Figiel's Theorem: Suppose that $f: X \to Y\ is$ a standard ε-isometry. Then for every $x^* \in X^* there\ exists\ \emptyset \in Y^* with\ \| \emptyset \| = \| x^* \| \equiv r$ so that

$$|\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle| \leq 4\varepsilon r\ for\ all\ x \in X. \qquad (2)$$

It has played an important role in the study of stability properties of ε-isometries. Making use of it, Cheng and Zhou further presented a stability characterization of ε-isometries.

Since Figiel's Theorem says that every standard isometry is stable, without loss of generality, we can always assume an ε-isometry is standard and with $\varepsilon > 0$. This chapter is organized as follows. In this Section, we first show a sharp version of Cheng-Dong-Zhang's weak stability Theorem (Theorem (3.1.3)), i.e. constant "4" in (2) is replaced by "2". Motivated by Omladič-Šemrl, we show the constant "2" in the estimate above is optimal (Theorem (3.1.4)). And we showed that if the ε-isometry f satisfies that $f(X)$ contains a sublinear growth net of Y, then Theorem (3.1.3) is equivalent to the following generalized Omladič-Šemrl's Theorem: There is a surjective linear isometry $U: X \to Y$ so that

$$\| f(x) - Ux \| \leq 2\varepsilon \ for\ all\ x \in X. \tag{3}$$

We show the constant "2" in the estimate above is optimal in the classical sense; and if $C(f) \equiv \overline{co}[f(X) \cup -f(X)] = Y$, then it is equivalent to the following sharp stability Theorem: there is a linear surjective operator $T : Y \to X$ of norm one such that

$$\| Tf(x) - x \| \leq 2\varepsilon forall\ x \in X. \tag{4}$$

The letter X will be a real Banach space and X^* its dual. B_X and S_X, $resp.$, denote the closed unit ball and the unit sphere of X, $resp. B(X, Y)$ stands for the space of all bounded operators from X to Y, and $\partial \|.\|: X \to 2^{X^*}$ for the sub differential mapping of the norm $\|.\|$. For a subspace $M \subset X, M^{\perp}$ presents the annihilator of M, i.e.$M^{\perp} = \{x^* \in X^*: \langle x^*, x \rangle = 0$ for all $x \in M\}$. If $M \subset X^*\ then\ {}^{\perp}M$ the pre-annihilator of M is defined as ${}^{\perp}M = \{x \in X : \langle x^*, x \rangle = 0\ for\ all\ x^* \in M\}$. Given a bounded linear operator $T: X \to Y, T^*: Y^* \to X^*$stands for its conjugate operator. For a subset $A \subset X(X^*), \bar{A}, (w^* - \bar{A})$ and $co(A)$ presents the closure (the w^* –closure), and the convex hull of A, respectively. For simplicity, we also use A^{**} to denote the w^* –closure of $A \subset X\ in\ X^{**}$.

We will show the sharp weak stability version of Cheng-Dong-Zhang's Lemma. Before doing this, we first establish the following Lemma about ε-isometries.

Recall that for a non-empty set Ω, a family $\mathcal{U}$ of subsets of Ω is said to be a free ultrafilter provided (i) $\emptyset \notin \mathcal{U}$, and $\cap \{U \in u\} = \emptyset$; $(ii) U, V \in \mathcal{U} \Longrightarrow U \cap V \in \mathcal{U}$; $(iii) U \in \mathcal{U} and U \subset V \subset \Omega \Longrightarrow V \in \mathcal{U}$; $and (iv) A \subset \Omega \Longrightarrow either A \in \mathcal{U}, or, \Omega \backslash A \in \mathcal{U}. let \mathcal{U}\ be a free\ ultrafilter$, and K be a Hausdorff space. A mapping $g: \Omega \to K$ is said to be $\mathcal{U}$ -convergent to $k \in K$ provided for any neighborhood W of k, we have $g^{-1}(W) \in \mathcal{U}$. In this case, we write $\lim_u g = k$. Please note that if K is compact then every mapping $g: \Omega \to K$ is $\mathcal{U}$ -convergent.

Lemma (3.1.1)[3]. Suppose that $f: X \to Y$ is an ε-isometry, and $\mathcal{U}$ is a free ultrafilter on ℕ Then

$$\Phi(x) = w^* - \lim_u \frac{f(nx)}{n}, \forall x \in X, \tag{5}$$

Defines an isometry $\Phi: X \to Y^{**}$.

Proof: Clearly,Φ is well –defined since for every $x \in X$, the bounded sequence $(\frac{f(nx)}{n})$ is relatively w^*–compact in Y^{**}. Given$x, y \in X$, w^* –lower semi-continuity of the dual norm $\|.\|\ of\ Y^{**}$ implies

$$\| \Phi(x) - \Phi(y) \| = \| w^* - \lim_u (\frac{f(nx)}{n} - \frac{f(ny)}{n}) \|$$

$$\leq \lim_u \| \frac{f(nx)}{n} - f(ny)/n \| = \| x - y \|.$$

On the other hand, according to the weak stability Theorem (Lemma (1.1.14)), for any $x^* \in \partial \| x - y \|, there\ is\ \emptyset \in Y^* with\ \| \emptyset \| = \| x^* \| = 1$ such that

$$|\langle \emptyset, f(z) \rangle - \langle x^*, z \rangle| \leq 4\varepsilon, \forall z \in X.$$

We substitute nx for z in the inequality above, and divide its both sides by n. then

$$|\langle \emptyset, f(nx)/n \rangle - \langle x^*, x \rangle| \leq 4\varepsilon/n.$$

Therefore, w^*-continuity of$\emptyset\ on\ Y^{**}$entails

$$\langle x^*, x\rangle = \lim_u \langle \emptyset, \frac{f(nx)}{n}\rangle = \langle \emptyset, \Phi(x)\rangle.$$

Analogously, we obtain

$$\langle x^*, y\rangle = \lim_u \langle \emptyset, \frac{f(ny)}{n}\rangle = \langle \emptyset, \Phi(y)\rangle.$$

Thus,

$$\| \Phi(x) - \Phi(y) \| \geq \langle \emptyset, \Phi(\mathrm{x}) - \Phi(\mathrm{y})\rangle = \langle \mathrm{x}^*, \mathrm{x} - \mathrm{y}\rangle = \| x - y \|.$$

So that

$$\| \Phi(x) - \Phi(y) \| = \| x - y \|, \qquad \forall x, y \in X.$$

Note that if $f: X \to Y$ is an ε-isometry, then $g = -f(-\cdot)$ is also an ε-isometry. We can obtain another isometry $\Psi: X \to Y^{**}$ defined by

$$\Psi(x) = w^* - \lim_u -\frac{f(-nx)}{n}, \ for\ all\ x \in X. \tag{6}$$

Lemma (3.1.2)[3]. Let X, Y be Banach spaces, $f: X \to Y$ be a standard ε-isometry, and let $\Phi: X \to Y^{**}$ be defined by (5). If the norm $\|.\|$ of X is Gateaux differentiable at $z \in X$and with $d \parallel \mathrm{z} \parallel = x^*$, then there exists $\phi \in \partial \parallel \Phi(z) \parallel \cap Y^*$such that

$$|\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle| \leq 2\varepsilon, \ for\ all\ x \in X. \tag{7}$$

Proof: Let $x^* \in X^* and\ z \in X\ satisfy\ d \parallel z \parallel = x^*$,i.e.

$$\lim_{t\to\infty}(\| x + tz \| - t) = \lim_{t\to 0^+} \| z + tx \| - \| z \| / t = \langle x^*, x\rangle, \qquad for\ all\ x \in X. \tag{8}$$

Then $\| x^* \| = 1$. We can assume $\| z \| = 1$.Given x $\in X$, let $u_n(x) = f(x + nz)$, and let $\emptyset_n \in Y^*$ with $\| \emptyset_n \| = 1$ such that $\langle \emptyset_n, u_n(x)\rangle = \| u_n(x) \|$. Then

$$\| u_n(x) \| = \langle \emptyset_n, f(x + nz)\rangle = \langle \emptyset_n, f(x)\rangle + \langle \emptyset_n, f(x + nz) - f(x)\rangle \leq \langle \emptyset_n, f(x)\rangle + \| f(x + nz) - f(x) \| \leq \langle \emptyset_n, f(x)\rangle + n + \varepsilon. \tag{9}$$

Thus, for any w^*–cluster point $\emptyset of (\emptyset_n)\ we\ have \parallel \emptyset \parallel \leq 1$ and

$$\lim_n \inf(\| u_n(x) \| - n) \leq \langle \emptyset, f(x)\rangle + \varepsilon. \tag{10}$$

On the other hand, by definition of ε-isometry we have

$$\liminf_n (\| u_n(x) \| - n)$$
$$\geq \liminf_n (\| x + nz \| - n) - \varepsilon$$
$$= \liminf_n \frac{(\| z + n^{-1}x \| - \| z \|)}{n^{-1}} - \varepsilon = \langle x^*, x \rangle - \varepsilon.$$

Therefore,

$$\lim_n \inf(\| u_n(x) \| - n) \geq \langle x^*, x \rangle - \varepsilon. \quad (11)$$

This combined with (10) entails

$$\langle x^*, x \rangle - \langle \emptyset, f(x) \rangle \leq 2\varepsilon. \quad (12)$$

Next, we show that the functional $\emptyset$ in the inequality above is independent of x. in fact for any $t \geq 0$,

$$t + \varepsilon \geq \| f(tz) \| \geq \langle \emptyset_n, f(tz) \rangle$$
$$= \langle \emptyset_n, f(x + nz) \rangle - \langle \emptyset_n, f(x + nz) - f(tz) \rangle \geq \| f(x + nz)$$
$$\| - \| f(x + nz) - f(tz) \| \geq (\| x + nz \| - \varepsilon) - \| x + (n - t)z$$
$$\| - \varepsilon \geq t - 2(\| x \| + \varepsilon).$$

Therefore,

$$t + \varepsilon \geq \langle \emptyset, f(tz) \rangle \geq t - 2(\| x \| + \varepsilon), for\ all\ t \geq 0.$$

Divide the inequality above by t > 0. Then $\lim_{t\to\infty} \langle \emptyset, f(tz)/t \rangle = 1$. Thus for any w^*-cluster point $z^{**} \in Y^{**} of\ (f(nz)/n)_{n\in\mathbb{N}} \big(say, \Phi(z)\big)$, we obtain $\langle \emptyset, z^{**} \rangle = 1$. Note $\| z^{**} \| \leq 1$ and $\| \emptyset \| \leq 1$. We have $\emptyset \in \partial \| z^{**} \|$ and $z^{**} \in \partial \| \emptyset \|$. In particular,

$$\emptyset \in \partial \| \Phi(z) \|\ and\ \Phi(z) \in \partial \| \emptyset \|. \quad (13)$$

Since z^{**} is independent of x, $\emptyset$ is necessarily independent of x. Thus, we have shown

$$\langle x^*, x \rangle - \langle \emptyset, f(x) \rangle \leq 2\varepsilon,\ for\ all\ x \in X. \quad (14)$$

Note that, in the proof of the inequality (14), for the Gateaux differentiability point z $\in X$, and for any fixed $x \in X$, the functional $\emptyset$ can be chosen to be any w^*-cluster point of $(\emptyset_n)$ satisfying

$$\langle \emptyset_n, f(x + nz) \rangle = \| f(x + nz) \|, \qquad for\ all\ n \in \mathbb{N}.$$

Since $\emptyset$ is independent of x, by putting $x = 0$, $\emptyset$ can be any w^*–cluster point of $(\emptyset_n)$ satisfying

$$\langle \emptyset_n, f(nz)\rangle = \| f(nz) \|, \ for\ all\ n \in \mathbb{N}. \quad (15)$$

In the following we show

$$\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle \geq -2\varepsilon, \ for\ all\ x \in X. \quad (16)$$

Given $x \in X$, $let\ \psi_n \in Y^* with\ \| \psi_n \| = 1$ such that

$$\langle \psi_n, f(x+nz) - f(x)\rangle = \| f(x+nz) - f(x) \|.$$

Then

$$\| u_n(x) \| \geq \langle \psi_n, u_n(x)\rangle = \langle \psi_n, f(x+nz) - f(x)\rangle + \langle \psi_n, f(x)\rangle =$$
$$\| f(x+nz) - f(x) \| + \langle \psi_n, f(x)\rangle \geq (\| nz \| - \varepsilon) + \langle \psi_n, f(x)\rangle$$
$$= n - \varepsilon + \langle \psi_n, f(x)\rangle.$$

Since

$$\| u_n(x) \| - n \leq (\| x+nz \| + \varepsilon) - n = (\| x+nz \| - \| nz \|) + \varepsilon$$
$$= \frac{\| z + n^{-1}x \| - \| z \|}{n^{-1}} \to \langle x^*, x\rangle + \varepsilon,$$

For any w^*–cluster point $\psi of\ (\psi_n)$ we have

$$\langle x^*, x\rangle - \langle \psi, f(x)\rangle \geq -2\varepsilon.$$

Note

$$t + \varepsilon \geq \| f(tz) \| \geq \langle \psi_n, f(tz)\rangle$$
$$= \langle \psi_n, f(x+nz) - f(x)\rangle - \langle \psi_n, f(x+nz) - f(tz)\rangle$$
$$+ \langle \psi_n, f(x)\rangle \geq \| f(x+nz) - f(x) \| - \| f(x+nz) - f(tz)$$
$$\| - \| f(x) \|$$
$$\geq (\| nz \| - \varepsilon) - (\| x + (n-t)z \| - \varepsilon) - (\| x \| + \varepsilon) \geq t - 2$$
$$\| x \| - 3\varepsilon.$$

Therefore,

$t + \varepsilon \geq \langle \psi, f(tz)\rangle \geq t - 2 \| x \| - 3\varepsilon, \ for\ all\ t \geq 0$. Divide the inequality above by t > 0. Then $\lim_{t\to\infty}\langle \psi, f(tz)/t\rangle = 1$. Thus for any w^*–cluster point

$z^{**} \in Y^{**} of (f(nz)/n)_{n\in\mathbb{N}}$, we obtain again $\langle \psi, z^{**} \rangle = 1$ $Note \parallel z^{**} \parallel \leq 1$ $and \parallel \psi \parallel \leq 1$. $We\ have\ \psi \in \partial \parallel z^{**} \parallel andz^{**} \in \partial \parallel \psi \parallel$. In particular,

$$\psi \in \partial \parallel \Phi(z) \parallel and\ \Phi(z) \in \partial \parallel \psi \parallel. \tag{17}$$

Since z^{**} is independent of x, ψ is necessarily independent of x. Consequently,

$$\langle x^*, x \rangle - \langle \psi, f(x) \rangle \geq -2\varepsilon,\ forall\ x \in X. \tag{18}$$

Note that, in the proof of the inequality (18), for the Gateaux differentiability point z $\in X$, and for any fixed $x \in X$, the functional ψ can be chosen to be any w^*–cluster point of (ψ_n) satisfying $\parallel \psi_n \parallel = 1$ and

$$\langle \emptyset_n, f(x+nz) - f(x) \rangle = \parallel f(x+nz) - f(x) \parallel, \qquad for\ all\ n \in \mathbb{N}.$$

Since ψ is independent of x, by putting $x = 0$, ψ can be any w^*–cluster point of (ψ_n) satisfying

$$\langle \psi_n, f(nz) \rangle = \parallel f(nz) \parallel,\ for\ all\ n \in \mathbb{N}. \tag{19}$$

(15) and (19) together imply that we can take $\phi = \psi$ in the Inequalities (14) and (16). Hence, our proof is complete.

A Banach space X is said to be a Gateaux differentiability space provided every continuous convex function defined on a nonempty open convex set D ⊂ X is densely Gateaux differentiable in D. Please note that every separable Banach space is a Gateaux differentiability space. A nice characterization for a Banach space X to be a Gateaux differentiability space is that every nonempty w^*-compact convex set C ⊂ X^* is the $w^* - closed$ convex hull of it is w^*-exposed points. (Notes that $x^* \in X^*$ is aw^*-exposed point of dual unit ball B_{X^*}if and only if $x^* = d \parallel x \parallel$ for some Gateaux differentiability point $x \in S_X$.) In particular, a Gateaux differentiability space X satisfies that the closed unit ball $B_{X^*}$of X^* is the $w^* - closed$ convex hull of all Gateaux derivatives {d‖z‖: z is a Gateaux differentiability point on the norm $\parallel . \parallel$}. The following Theorem is main result of this chapter.

Theorem (3.1.3)[3]. Let X, Y be Banach space, and $f: X \to Y$ be a standard ε-isometry. Then for each $x^* \in X^*$ there exists $\emptyset \in Y^*$with. $\| \emptyset \| = \| x^* \| \equiv r$ such that

$$|\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle \leq 2r\varepsilon, for\ all\ x \in X. \qquad (20)$$

Proof: Our proof is divided into three steps.

Step I. we first show that its true if X is a Gateaux differentiability space. Given a Gateaux differentiability point z $\in S_x$, let x^* = d$\| z \|$.

Then by Lemma (3.1.2) there exists $\emptyset \in \partial \| \Phi(z) \|$ such that (20) holds with $r = 1$. Since X is a Gateaux differentiability space, for any $x^* \in S_{x^*}$ there exist a directed set I, and a net $(x^*_\alpha)_{\alpha \in I} \subset B_{X^*}$ of the form:

$$x^*_\alpha = \sum_{j \in J_\alpha} \lambda^\alpha_j z^{*\alpha}_j, \qquad for\ each\ \alpha \in I,$$

Such that $x^*_\alpha \xrightarrow{w^*} x^*$; $where J_\alpha \subset \mathbb{N}\ is\ a finite\ set, \lambda^\alpha_j \geq 0 (j \in J_\alpha)$ satisfy $\sum_{j \in J_\alpha} \lambda^\alpha_j = 1$; and $(z^{*\alpha}_j)_{j \in J_\alpha}$ are w^* -exposed points of B_{X^*}. Let $z^\alpha_j \in S_X$be Gateaux differentiability point so that $d \| z^\alpha_j \| = z^{*\alpha}_j$. Then there exists $\xi^\alpha_j \in \partial \| \Phi(z^\alpha_j) \| \cap Y^*$such that

$$\left|\langle z^{*\alpha}_j, x\rangle - \langle \xi^\alpha_j, f(x)\rangle\right| \leq 2\varepsilon,\ for\ all\ x \in X. \qquad (21)$$

Let $\phi_\alpha = \sum_{j \in J_\alpha} \lambda^\alpha_j \xi^\alpha_j$. Then we obtain

$$|\langle x^*_\alpha, x\rangle - \langle \emptyset_\alpha, f(x)\rangle| \leq 2\varepsilon,\ for\ all\ x \in X. \qquad (22)$$

Consequently, for any w^*–cluster point $\phi of\ (\emptyset_\alpha)$ we have

$$\left|\langle x^*\ , x\rangle - \langle \emptyset\ \ , f(x)\rangle\right| \leq 2\varepsilon,\ for\ all\ x \in X. \qquad (23)$$

Note $\| x^* \| = 1\ and\ \| \emptyset \| \leq 1$. It is not difficult to observe$\| \emptyset \| = 1$. In fact, since $\langle x^*, z\rangle = \| z \| = 1$, by substituting nz for x in the inequality above, and dividing it's both sides by n, we obtain

$$|1 - \langle \emptyset, f(nz)\ /\ n\rangle| \leq 2\varepsilon/n.$$

This says $\| \emptyset \| \geq \lim_n |\langle \emptyset, f(nz) / n\rangle| = 1$. Positive homogeneity of (23) implies (20). Thus, we have shown the Theorem in assuming that X is a Gateaux differentiability space.

Step II. In the case that X is a general Banach space, we will show that (20) is true for every norm-attaining functional $x^* \in X^*$. Positive homogeneity of (20) allows us, without loss of generality, to assume $\| x^* \| = 1$. Let $x_0 \in S_X$ *be such that* $\langle x^*, x_0\rangle = 1$.

Let $\mathcal{F} = \{F \subset X$ is a finite dimensional subspace containing x_0 }. Then every element $F \in \mathcal{F}$ is a Gateaux differentiability space and x^* (restricted to F) is again a norm-attaining functional with $\| x^*|_F \| = \| x^* \| = \langle x^*, x_0\rangle = 1$. Given $F \in \mathcal{F}$, by the fact we have just proven in Step 1, there exists $\emptyset \in S_{Y^*}$ such that

$$\left|\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle\right| \leq 2\varepsilon, \ for\ all\ x \in F. \tag{24}$$

Fix any $F \in \mathcal{F}$ and let

$$\Phi_F = \{\emptyset \in B_{Y^*}\ satisfying\ (24)\}.$$

Then it is easy to observe that Φ_F is a non empty w^*–compact convex set. Indeed, nonemptiness of Φ_F has been proven by step I, since F is a Gateaux differentiability space; convexity and w^*–compactness of Φ_F are trivial by its definition. Note $\Phi_F \cap \Phi_G \supset \Phi_{span(F\cup G)} for\ all\ F, G \in \mathcal{F}$. We obtain that $\cap_{F\in\mathcal{F}} \Phi_F \neq \emptyset$. Clearly, any $\phi \in \cap_{F\in\mathcal{F}} \Phi_F$ is a solution of (20) with $\| \emptyset \| = \| x^* \| = 1$. In fact, given $\phi \in \cap_{F\in\mathcal{F}} \Phi_F$, we have $\| \emptyset \| \leq 1$ and

$$\left|\langle x^*, x\rangle - \langle \emptyset, f(x)\rangle\right| \leq 2\varepsilon, \quad for\ all\ x \in X.$$

On the other hand, we replace x by nx_0 in the inequality above, divide the two sides by n and notice $\langle x^*, x_0\rangle = 1$. Then we obtain $\| \emptyset \| \geq 1$. Therefore, $\| \emptyset \| = 1$.

Step III. Finally, we show that the inequality (20) holds for every functional $x^* \in X^*$. We can assume again $\| x^* \| = 1$. By the Bishop-Phelps Theorem, there is a sequence $(x_n^*) \subset S_{X^*}$ of norm-attaining functionals such that $x_n^* \to x^*$. By the fact we have just proven in Step II, for each $n \in \mathbb{N}$, there exists $\emptyset_n \in S_{Y^*}$ so that

$$|\langle x_n^*, x\rangle - \langle \emptyset_n, f(x)\rangle| \le 2\varepsilon, \ for\ all\ x \in X. \quad (25)$$

Since $x_n^* \to x^*$, for any w^*–cluster point $\emptyset of\ (\emptyset_n)$, we have

$$\left|\langle x^*, x\rangle - \langle \emptyset \ , f(x)\rangle\right| \le 2\varepsilon, \ for\ all\ x \in X. \quad (26)$$

Clearly, $\| \emptyset \| \le 1$. Conversely, let $(z_n) \subset S_X$ satisfy

$1 = \| x^* \| = \lim_n \langle x^*, z_n\rangle$ By substituting nz_n of x in (26) we obtain

$$\left|\langle x^*, z_n\rangle - \langle \emptyset \ , f(nz_n) / n\rangle\right| \le 2\varepsilon/n, \qquad for\ all\ n \in \mathbb{N}.$$

Thus,

$$\| \emptyset \| \ge \lim_n \langle \emptyset, f(nz_n) / n\rangle = 1.$$

Consequently, $\| \emptyset \| = 1$.

The following result, says that the constant $\gamma = 2$ is optimal.

Theorem (3.1.4)[3]. Let X, Y be Banach spaces. If there is a standard ε-isometry $g: X \to Y$ for some $\varepsilon > 0$, there for every $\delta > 0$ there is a standard $(\varepsilon + \delta)$ –isometry $f: X \to Y$ such that the following assertion holds: there exist $x^* \in S_{X^*}\ and\ \emptyset \in S_{Y^*}$ so that

$$\left|\langle x^*, x\rangle - \langle \emptyset \ , f(x)\rangle\right| \le 2\varepsilon + \delta, \ for\ all\ x \in X. \quad (27)$$

And

$$sup_{x \in X}\left|\langle x^*, x\rangle - \langle \emptyset \ , f(x)\rangle\right| > 2\varepsilon - \delta. \quad (28)$$

Proof: Note that for every $n \in \mathbb{N}$. $g_n: X \to Y$ defined by $g_n(x) = g(nx) / n$ is a standard ε / n –isometry. Given $0 < \delta < \varepsilon$, let $m \in \mathbb{N}$ such that $h \equiv g_m$ is a standard $\delta / 2$ –isometry. By Theorem (3.1.3), for every $x^* \in S_{X^*}$ there exists $\emptyset \in S_{Y^*}$ so that

$$\left|\langle x^*, x\rangle - \langle \emptyset \ , h(x)\rangle\right| \le \delta, \ for\ all\ x \in X. \quad (29)$$

We fix any point $x_0 \in \varepsilon S_X. Let x_0^* \in \partial \| x_0 \|$. By Theorem (3.1.3) again, there is $\emptyset_0 \in S_{Y^*}$ so that

$$\left|\langle x_0^*, x\rangle - \langle \emptyset_0 \ , h(x)\rangle\right| \le \delta, \ for\ all\ x \in X. \quad (30)$$

Since $\langle x_0^*, x_0\rangle = \varepsilon, \varepsilon + \delta \geq \langle \emptyset_0, h(x_0)\rangle \geq \varepsilon - \delta$. We define $f\colon X \to Y$ for $x \in X$ by

$$f(x) = \begin{cases} -3th(x_0), & if\ x = tx_0, t \in \left[0, \frac{1}{2}\right]; \\ (t-1)h(x_0), & if\ x = tx_0, t \in \left(\frac{1}{2}, 1\right] \\ h(x), & otherwise. \end{cases} \tag{31}$$

Then, it is easy to observe that f is a standard $(\varepsilon + \delta)$ $-$isometry, and the functional x_0^* and $\emptyset_0$ satisfy

$$\left|\langle x_0^*, x\rangle - \langle \emptyset_0\ , f(x)\rangle\right| \leq 2\varepsilon + \delta,\ for\ all\ x \in X. \tag{32}$$

Let $z = (1/2)x_0$ in (32). Then

$$\left|\langle x_0^*, z\rangle - \langle \emptyset_0\ , f(z)\rangle\right| > 2\varepsilon - \delta\ . \tag{33}$$

Remark (3.1.5)[3]. Figiel's Theorem states that every standard isometry from a Banach space X to another Banach space Y has a linear left-inverse F of norm one. If $\varepsilon = 0$, then Theorem (3.1.3) deduces for all $x^* \in X^*$, there exists $\emptyset \in Y^* with \parallel \emptyset \parallel = \parallel x^* \parallel$ such that

$$\langle x^*\ , x\rangle = \langle \emptyset\ , f(x)\rangle,\quad for\ all\ x \in X. \tag{34}$$

The following result says that Theorem (3.1.3) can be regarded as a sharp quantitative extension of Figiel's Theorem.

Theorem (3.1.6)[3]. Suppose that X, Y are two Banach spaces, and $f\colon X \to Y$ is a standard isometry. Let $Y_f = \overline{span} f(X)$, and the correspondence $K : X^* \to Y_f^*$ be defined by (34), i.e. $Kx^* = \emptyset$, where x^* and $\emptyset$ satisfy (34). Then K is a$w^* - to - w^*$ continuous linear isometry, which is just the conjugate operator of Figiel's operator F associated with f.

Proof: We first claim that the correspondence $K : X^* \to Y_f^*$ is one-to-one.

Given $x^* \in X^*$, assume $\emptyset, \psi \in Y^*$ such that

$$\langle \emptyset, f(x)\rangle = \langle x^*, x\rangle = \langle \psi\ , f(x)\rangle,\quad for\ all\ x \in X. \tag{35}$$

Then

$$\langle \phi - \psi, f(x)\rangle = 0,\quad for\ all\ x \in X,$$

Or, equivalently, $\phi = \psi$ on Y_f . It is clear the correspondence K defined by (34) is homogeneous and additive, i.e. K is linear. Consequently, K is a linear isometry since it is norm-preserving. To show $w^* - to - w^*$ continuity of K, let $F : Y_f \to X$ be Figiel's operator associated with the isometry f, i.e. the left-inverse of f with $\| F \| = 1$. Then for every $x^* \in X^*$,

$$\langle x^*, x\rangle - \langle x^* , F(f(x))\rangle = \langle F^*(x^*), f(x)\rangle, \quad for\ all\ x \in X. \tag{36}$$

This and (34) together imply

$$\langle K(x^*), f(x)\rangle = \langle F^*(x^*), f(x)\rangle, \quad for\ all\ x \in X, x^* \in X^*. \tag{37}$$

Thus, $K = F^*$, which entails that K is $w^* - to - w^*$ continuous.

Section (3.2) Sharp Stability Results of a Certain Class of ε-isometries:

In this section, we shall see that Theorem (3.1.3) is useful in the study of stability of ε-isometries. It is not only an extension of Figiel's Theorem, but also a generalized version of the Omladič-Šemrl Theorem.

A subset N in a metric space (Ω, ϱ) is said to be a sublinear growth net in metric ϱ provided for any fixed $\omega_0 \in \Omega$,

$$\lim_{\varrho(\omega,\omega_0)\to\infty} \frac{\varrho(\omega, N)}{\varrho(\omega,\omega_0)} = 0. \tag{38}$$

For example, let $m: \mathbb{R} \to \mathbb{Z}$ be defined by $m(x) = [x] + (signx)[x]p$, where $[\cdot]$ denotes the floor function and $[\cdot]_p$ denotes the cardinality of the prime number set $P \equiv P(x) = \{p \in \mathbb{N}\ isaprimenumberwith\ p \leq |x|\}. Then\ N \equiv \{m(x): x \in \mathbb{R}\}$ is a sublinear growth net of $\mathbb{R}$. In fact, since $\lim_{x\to+\infty}[x]_p / \ln x = 1$, for any fixed $x_0 \in \mathbb{R}$,

$$\frac{\varrho(x, N)}{|x - x_0|} \leq \frac{[x]_p + 1}{|x - x_0|} \to 0, as\ |x - x_0| \to \infty.$$

Theorem (3.2.1)[3]. Let X, Y be Banach spaces, and $f: X \to Y$ be a standard ε-isometry for some $\varepsilon \geq 0$. Suppose that $f(X)$ contains a sublinear growth net of Y. Then there is a linear surjective isometry $U: X \to Y$ such that

$$\| f(x) - Ux \| \leq 2\varepsilon, \ for\ all\ x \in X. \tag{39}$$

Proof: Note that if f (X) contains a sublinear growth net of Y, then$Y_f = Y$.

We first show that this is true if ε= 0, i.e. when f is an isometry. According to Theorem (3.1.6), the operator $K : X^* \to Y_f^*$ defined by $Kx^* = \emptyset$, where$x^* and\ \emptyset$ satisfy (34), is just the conjugate of Figiel's operator, hence, a $w^* - to - w^*$ continuous linear isometry. We claim K is surjective. Otherwis, $Z \equiv K(X^*)\ is\ a\ w^*$-closed proper subspace of $Y_f^* = Y^*$. In fact, since $K : X^* \to Z$ is a linear surjective isometry, $KB_{X^*} = B_Z. w^*$-compactness of $B_{X^*} and\ w^*$-continuity of K deduce that $B_Z is w^*$-compact in Y^*. Consequently, $Z = \cup_{n\in\mathbb{N}} nB_Z is w^*$-closed in$Y^*$. By separation Theorem, there exist $\psi \in S_{Y^*} \setminus K(X^*) and y \in S_Y$such that

$$\langle \psi, y \rangle = \| y \| = 1, and\ \langle \phi, y \rangle = 0, \ for\ all\ \phi \in K(X^*). \tag{40}$$

Let$y_n = ny\ for\ all\ n \in \mathbb{N}$. Since $f(X)$ contains a sublinear growth net of Y, for the sequence$(y_n)_{n\in\mathbb{N}} \subset Y$, there is a sequence$(x_n)_{n\in\mathbb{N}} \subset X$ so that

$$\lim_n \frac{\|y_n - f(x_n)\|}{n} = 0. \tag{41}$$

Choose any $x_n^* \in \partial \| x_n \|, and\ let\ \emptyset_n = Kx_n^*$. Then

$$0 = |\langle x_n^*, x_n \rangle - \langle \emptyset_n, f(x_n) \rangle| = |\langle x_n^*, x_n \rangle - \langle \emptyset_n, f(x_n) - y_n \rangle| \geq \| x_n \| - \| f(x_n) - y_n \| = \| f(x_n) \| - \| f(x_n) - y_n \| \geq \| y_n \| - 2 \| f(x_n) - y_n \| = n\left(1 - 2\, \frac{\| f(x_n) - y_n \|}{n}\right) \to \infty.$$

This is a contradiction.

We have shown that $K : X^* \to Y^* is\ a\ w^* - to - w^*$ continuous linear surjective isometry. Therefore, its pre-conjugate operator $F: Y \to X$ is also a linear surjective isometry satisfying $F \circ f = Id$. We are done by letting $U = F^{-1}$.

Next, suppose ε> 0. Let $\ell : X^* \to 2^{Y^*}$ be defined for $x^* \in X^*$ by

$$\ell x^* = \{\emptyset \in Y^*: |x^* - \emptyset \circ f| \text{ is bounded on } X\}; \tag{42}$$

$$M = \overline{\ell 0} = \overline{\{\emptyset \in Y^* : |\emptyset \circ f| \iota s bounded on X\}}, \tag{43}$$

And let $Q : X^* \to Y^* / M$ be defined by

$$Qx^* = \ell x^* + M. \tag{44}$$

Then, due to Theorem (1.1.22), Q is a linear isometry. Since $f(X)$ admits a sublinear growth net of Y, $co\ (f(X))$ is dense in Y. Consequently, M = {0}.

Note that if $\phi, \psi \in \ell x^*$ for some $x^* \in X^*, then \emptyset - \psi \in \ell 0 \subset M$.

Thus,$Q: X^* \to Y^*$, is actually a single-valued linear isometry. This and Theorem (3.1.3) together entail

$$|\langle x^*, x\rangle - \langle Qx^*, f(x)\rangle| \le 2 \parallel x^* \parallel \varepsilon, for\ all\ x \in X, x^* \in X^*, \tag{45}$$

And which further implies that $Q: X^* \to Y^*, is\ a\ w^* - to - w^*$ continuous linear isometry. Hence, it is a conjugate operator of norm one. Let $T: Y \to X$ be a linear operator so that $T^* = Q$. This and (44) entail

$$|\langle x^*, x\rangle - \langle x^*, Tf(x)\rangle| \le 2 \parallel x^* \parallel \varepsilon, for\ all\ x \in X, x^* \in X^*, \tag{46}$$

Or, equivalently,

$$\parallel Tf(x) - x \parallel \le 2\varepsilon, \quad for\ all\ x \in X. \tag{47}$$

Clearly, T is surjective. In order to show that T is a linear isometry, it suffices to prove that Q is surjective. Suppose, to the contrary, That $Q(X^*)$ is a proper subspace of $Y.^*$ w^*–closedness of $Q(X^*)$ implies that there exist $\psi \in S_{Y^*} \backslash (Q(X^*))\ and y \in S_Y$ such that

$$\langle \psi, y\rangle = \parallel y \parallel = 1, and\ \langle \phi, y\rangle = 0, for\ all\ \phi \in Q(X^*). \tag{48}$$

Let again $y_n = ny\ for\ all\ n \in \mathbb{N}$. Since $f(X)$ contains a sublinear growth net of Y, for the sequence$(y_n)_{n\in\mathbb{N}} \subset Y$, there is again a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ so that $\lim_n \frac{\parallel y_n - f(x_n) \parallel}{n} = 0$. Choose any $x_n^* \in \partial \parallel x_n \parallel$, and let $\emptyset_n = Qx_n^*$.

Then

$$2\varepsilon \ge |\langle x_n^*, x_n\rangle - \langle \emptyset_n, f(x_n)\rangle| = |\langle x_n^*, x_n\rangle - \langle \emptyset_n, f(x_n) - y_n\rangle| \ge \parallel x_n \parallel - \parallel f(x_n) - y_n \parallel \ge (\parallel f(x_n) \parallel - \varepsilon) - \parallel f(x_n) - y_n \parallel \ge \parallel y_n \parallel - 2 \parallel f(x_n) - y_n \parallel - \varepsilon = n\left(1 - 2\ \frac{\parallel f(x_n) - y_n \parallel}{n}\right) - \varepsilon \to \infty.$$

This contradiction says that Q is surjective. Therefore, we have proven that $T: Y \to X$ is a surjective linear isometry. We finish the proof by letting $U = T^{-1}$. The following theorem tells us that the estimate in Theorem (3.2.1) is sharp.

Theorem (3.2.2)[3]. Given a pair of Banach spaces , Y , if there is a standard ε-isometry $g : X \to Y$ satisfying that $g(X)$ contains a sublinear growth net of Y, then there exists a standard ε-isometry $f : X \to Y$ with $f(X)$ containing a sublinear growth net of Y such that for every linear surjective isometry $U: X \to Y$ we have

$$sup_{x \in X} \| f(x) - Ux \| \geq 2\varepsilon. \quad (49)$$

Proof: Since there is a standard ε-isometry $g: X \to Y$ with $g(X)$ containing a sublinear growth net of Y, by Theorem (3.2.1), there exist a linear isometry $U_0: X \to Y$. We fix any point $x_0 \in \varepsilon S_X$, and defined $f: X \to Y$ for $x \in X$ by

$$f(x) = \begin{cases} -3tU_0(x_0), & if x = tx_0, t \in \left[0, \frac{1}{2}\right]; \\ (t-1)U_0(x_0), & if x = tx_0, t \in \left(\frac{1}{2}, 1\right] \\ U_0(x), & otherwise. \end{cases} \quad (50)$$

Then, it is not difficult to see that f is a standard ε-isometry with $f(X)$ containing a sublinear growth net of Y, which satisfies that for every linear surjective isometry $U: X \to Y$,

$$sup_{x \in X} \| f(x) - Ux \| \geq sup_{x \in X} \| f(x) - U_0 x \| \geq sup_{x \in \mathbb{R}x_0} \| f(x) - U_0 x \| = 2\varepsilon.$$

Theorem (3.2.3)[3]. Let X, Y be Banach spaces, and $f: X \to Y$ be a standard ε-isometry. If $C(f) \equiv \overline{co}\big(f(X) \cup -f(X)\big) = Y$, then there is a linear operator $T : Y \to X$ with $\| T \| = 1$ such that

$$\| Tf(x) - x \| \leq 2\varepsilon, \quad for\ all\ x \in X. \quad (51)$$

Proof: Let the mapping $\ell: X^* \to 2^{Y^*}$ and the subspace $M \subset Y^*$ be defined as (42) and (43). Then $C(f) = Y$ implies $M = \{0\}$. Indeed, given $\emptyset \in \ell 0, |\phi o f|$

is bounded by some $\beta > 0$ on X. Then is equivalent to that $|\phi|$ is bounded by β on $C(f) = Y$. consequently, $\phi = 0$. Therefore, Q= $\ell : X^* \rightarrow Y^*$ is a w^* to w^*continuous linear isometry, where Q is defined by (44). This, incorporating Theorem (3.1.3), further entails

$$|\langle x^*, x\rangle - \langle Qx^*, f(x)\rangle| \leq 2 \parallel x^* \parallel \varepsilon, for\ all\ x \in X, x^* \in X^*, \qquad (52)$$

Let $T: Y \rightarrow X$ be the pre-conjugate operator of Q. Then we obtain$\parallel T \parallel = 1$ and

$$\langle x^*, Tf(x)\rangle - \langle Qx^*, f(x)\rangle, \quad for\ all\ x \in X, x^* \in X^*. \qquad (53)$$

Therefore,

$$|\langle x^*, x\rangle - \langle x^*, Tf(x)\rangle| \leq 2 \parallel x^* \parallel \varepsilon, for\ all\ x \in X, x^* \in X^*, \qquad (54)$$

The inequality above is apparently equivalent to (51).

Remark (3.2.4)[3]. The assumption that $C(f) = Y$ cannot guarantee the operator T is invertible in Theorem (3.2.3), even if f is an isometry. For example, let $f : X = \mathbb{R} \rightarrow \ell_\infty^2 = Y$ be defined by $f(x) = (x, \ln(1 + x)),\ if x \geq 0; = (x, 0), if x < 0$.Then f is a standard isometry with C $(f) = \ell_\infty^2$, so that there is no linear surjective isometry $U: X \rightarrow Y$.

Chapter 4

Almost Surjective ε- isometries Of Banach Spaces

We show that for every pair of Banach spaces X and Y and for every ε-isometry $f: X \to Y$ with $sup_{y \in S_Y} lim_{|t| \to \infty} \inf dist(ty, f(X)) / |t| < 1/2$ there exists an affine surjective isometry $V: Y \to X$ such that $\| f(x) - Vx \| \leq 2\varepsilon \; for \; all \; x \in X$.

Section (4.1): Almost Surjective ε- isometries Of Banach Spaces

The classical Theorem of Mazur and Ulam as serts that a surjective isometry between real normed spaces is affine. Note that it is not valid for complex normed spaces (just consider complex conjugation on C). The hypothesis that an isometry is surjective is essential in general, but can be dropped if the target space is strictly convex. As real-world observations have always some minimal error, one may not be able to deduce from measurements whether a given mapping is really isometric or surjective. Thus it is natural to ask if a mapping, which only nearly preserves distances and only almost covers the target space, can be well approximated by a surjective (affine) isometry. In this chapter we deal with ε-isometries of one Banach space X into another Y which almost cover (in some sense) the target space. Throughout the chapter X and Y denote real Banach spaces.

Definition (4.1)[4]. Let $\varepsilon \geq 0$. A map $f: X \to Y$ is called an ε-isometry if

$$| \| f(y) - f(x) \| - \| y - x \| | \leq \varepsilon$$

For all $x, y \in X$.

There is an extensive literature on such mappings starting with the influential of Hyers and Ulam. They proved that every surjective ε-isometry between real Hilbert spaces can be uniformly approximated to within 10ε by an affine surjective isometry. Later this result has been extended to all pairs of real Banach spaces, and the constant 10 has been reduced to 2 which is sharp. Dilworth showed that the subjectivity condition can be dropped if both above Banach spaces are of the same finite dimension. However, the example of the map $x \mapsto (x, \sqrt{2\varepsilon \parallel x \parallel})$ $from\ l_2^n\ to\ l_2^{n+1}$ (which is ε-isometric, but far from any affine map and thus from any isometry) shows that the surjectivity assumption is indispensable in this Theorem even for Euclidean spaces. Šemrl and Väisälä showed that every ε-isometry $f: X \to Y$ can be uniformly approximated to within 2ε by an affine surjective isometry provided

$$sup_{y \in Y}\{dist(y, f(X))\} < \infty.$$

We show that this result remains true when the above almost surjectivity condition is further relaxed and replaced by

$$sup_{y \in S_Y} \liminf_{|t| \to \infty} dist(ty, f(X))/|t| \quad < \frac{1}{2}.$$

Namely, we give the following Theorem. Given a nonempty Q⊂Y and y∈S_Y, we denote

$$\varrho(y, Q) = \liminf_{|t| \to \infty} d(ty, Q)/|t|, \qquad \tau(Q) = sup_{y \in S_Y} \varrho(u, Q).$$

Given a map $f: X \to Y$, we abbreviate $\varrho(y, f(X))$ and $\tau(f(X))$ by $\varrho(y, f)$ and $\tau(f)$. We also abbreviate $co(f(X) \cup -f(X)) by\ C(f)$.

Väisälä has posed the following problem: Whether an ε-isometry $f : X \to Y$ with $\tau(f) = 0$ can be approximated by a surjective isometry? Theorem (4.4)(iii) answers this question in affirmative even for $\tau(f) < 1/2$. Note that in the case when Y is uniformly convex, the weaker condition $\tau(f) < 1$ implies the existence of such an approximating isometry. It is shown in this chapter. We do not know whether the condition $\varrho(y, f) < 1/2$ for every $y \in S_Y$ is enough to guarantee the existence of an approximating isometry. However, if such an approximating isometry exists, it is necessarily linear and surjective.

Proposition (4.2)[4]. Let $f: X \to Y$ be an ε-isometry with $f(0) = 0$ and $\varrho(y, f) < 1/2$ for every $\in S_Y$. Let $U : X \to Y$ be an isometry such that

$U(0) = 0$ and $\| U(x) - f(x) \| = o(\| x \|)$ as $\| x \| \to \infty$ uniformly. Then U is a surjective linear isometry and

$$\| f(x) - Ux \| \leq 2\varepsilon, \qquad x \in X$$

Proof: Let $y \in S_Y$. By our assumptions, there are sequences $\{t_n\} \subset R$ and $\{x_n\} \subset X$ such that $|t_n| \to \infty and$ $\| y - U(x_n) / t_n \| < 1/2$ for all n. Therefore by the Theorem of Figiel, Šemrl and Väisälä, U is surjective and linear.

In what follows, we shall use some results and notation from chapter1. For an ε-isometry $f : X \to Y$ with $f(0) = 0$, we denote by M_ε the subspace of Y^* consisting of all functional bounded on $C(f)$ and by E the annihilator of M_ε. Let $\alpha \geq 0$. A closed subspace $M \subseteq X$ is said to be α-complemented provided there exist a closed subspace $N \subseteq X$ $with$ $M \cap N = \{0\}$ and a projection $P: X \to M$ along N such that $X = M + N$ and $\| P \| \leq \alpha$.

It follows from Remark (1.1.27) and a quick inspection of the proof of Theorem (1.1.26).

Theorem (4.3)[4]: Let $f: X \to Y$ be anε-isometry with $f(0) = 0$. Let E be α-complemented in Y and P be a projection $P: X \to M$ along N such that $X = M + N$ and $\| P \| \leq \alpha$. Let $co\big(f(X) \cup -f(X)\big) \subset E + B$ for some bounded set $B \subset Y$. Then there is a surjective norm-one linear operator $U: E \to X$ such that

$$\| UPf(x) - x \| \leq 4\varepsilon, \qquad x \in X$$

Theorem (4.4)[4]. Let $f: X \to Y$ be an ε-isometry with $f(0) = 0$.

(i)If $\tau(C(f)) < 1$, then there is a surjective norm-one linear operator $U: Y \to X$ such that

$$\| Uf(x) - x \| \leq 4\varepsilon, \ x \in X. \qquad (1)$$

(ii) If $\varrho(y, f) < 1/2$ for every $y \in S_Y$, then U is surjective. We denote its inverse by

$$V := U^{-1}: X \to Y.$$

(iii)If $\tau(f) < 1/2$, then V is a surjective linear isometry satisfying

$$\| f(x) - Vx \| \leq 2\varepsilon, \qquad x \in X$$

Proof : (i) We prove that $M_\varepsilon = \{0\}$. Then $E = Y$, and (1) follows by Theorem (4.3) with $P = I$ (the identity).

Choose $\tau(C(f)) < q' < q'' < 1$. Suppose that there is a norm-one $\varphi \in M_\varepsilon$. Then there is $y \in S_Y$ such that $\langle \varphi, y \rangle > q''$. By the definition of M_ε, there is $r > 0$ such that $|\langle \varphi, u \rangle| < r$ for all $u \in C(f)$. Since $\tau(C(f)) < q'$, there are sequences $\{t_n\} \subset R$ and $\{u_n\} \subset C(f)$ such that $|t_n| \to \infty$ and $\| y - u_n / t_n \| < q'$ for all n. Since $\| \varphi \| = 1$, it follows that

$$q' > |\langle \varphi, y - u_n/t_n \rangle| \geq |\langle \varphi, y \rangle| - |\langle \varphi, u_n/t_n \rangle| > q'' - r/|t_n|,$$

Which implies $q' \geq q''$–a contradiction.

(ii) Let $y \in S_Y$. We show that

$$\| Uy \| \geq 1 - 2\,\varrho(y, f), \tag{2}$$

which implies injectivity of U.

let $\varrho(y, f) < q' < \frac{1}{2}$. Then there are sequences $\{t_n\} \subset$R and $\{x_n\} \subset X$ such that $|t_n| \to \infty$ and $\| y - f(x_n)/t_n \| < q'$for all n. Hence $\| f(x_n) \| > (1 - q')|t_n|$ and then $\| x_n \| \geq \| f(x_n) \| - \varepsilon > (1 - q')|t_n| - \varepsilon$. On the other hand, by (1) and $\| U \| \leq 1$,

$$q'|t_n| > \| t_n Uy - Uf(x_n) \| \geq \| x_n \| - |t_n| \, \| Uy \| - \| Uf(x_n) - x_n \| \geq \| x_n \| - |t_n| \, \| Uy \| - 4\varepsilon,$$

Which implies $\| x_n \| < (q' + \| Uy \|)|t_n| + 4\varepsilon$. thus, $(1 - q')|t_n| - \varepsilon < (q' + \| Uy \|)|t_n| + 4\varepsilon$ for all n. Thus $\| Uy \| \geq 1 - 2q'$. As q' was arbitrary in interval $(\varrho(y, f), \frac{1}{2})$, (2) holds.

(iii) In this case, U is bijective and $\| Uy \| \geq 1 - 2\tau(f)$. Hence its inverse V is bijective and bounded with $\| V \| \leq 1 / (1 - 2\tau(f))$. By (1), for every t>0

$$\| tVx - f(tx) \| \leq \| V \| \| tx - Uf(tx) \| \leq 4\varepsilon \,/ (1 - 2\tau(f)), \quad x \in X.$$

Hence

$t \parallel Vx \parallel \leq \parallel f(tx) \parallel +4\varepsilon\ /(1-2\tau(f)) \leq t \parallel x \parallel + \left(1+\frac{4}{1-2\tau(f)}\right)\varepsilon, x \in X,$

which implies $\parallel V \parallel \leq 1$. This along with $\parallel U \parallel \leq 1$ gives that both U and V are isometries. The result follows now by Proposition (4.2).

Note that conditions in Theorem (4.4) are rather sharp. The sharpness of the condition in (i) follows from the next two facts:

a. For any mapping $f: X \to Y$ with $f(0) = 0$, we have $\tau(C(f)) \leq 1$.

b. There exists an ε-isometry with $f(0) = 0$ such that for any bounded linear operator $T : Y \to X$

$$sup_{x\in X} \parallel Tf(x) - x \parallel = \infty$$

The sharpness of the conditions in (ii) and (iii) is shown in the following simple example.

Example (4.5)[4]. Let $X = R\ and\ Y = l_\infty^2. define\ f: R \to l_\infty^2$ by the formula $f(x) = (x, |x|)$. Then f is a nonlinear isometry and yet, if $y \neq 0 \in l_\infty^2\ then\ \parallel \theta_1 ty - f\left(\frac{1}{2}\theta_2 \parallel ty \parallel\right) \parallel \leq \parallel ty \parallel\ /2$ for some $\theta_1, \theta_2 \in \{-1,1\}$ and for all $t \in R$.

Concerning Theorem (4.3), Cheng and Zhou have posed the following problem: Given $\varepsilon > 0$, whether $f(X)$ is always contained in $E + B$ (for some bounded subset $B \subset Y$) for every ε-isometry?

The following Lemma gives a negative answer to this question.

Lemma (4.6)[4]. For every 1< p<∞ and $\varepsilon > 0$, there exists a continuous ε-isometry $f: R \to l_p$ such that $sup_{x\in R} dist\ (f(x), E) = \infty$.

Proof : Let $\{e_i\}_{i=0}^\infty and\ \{e_i'\}_{i=0}^\infty$ be the canonical bases of l_p and $l_{p/(p-1)}$, respectively. Define real functions by

$$g_i(x) = \min\left\{\sqrt[p]{\frac{p\varepsilon|x|^{(p-1)}}{2^i}}, i\right\}.$$

Note that $|g_i(x) - g_i(y)|^p \leq \frac{p\varepsilon\left||x|^{(p-1)/p} - |y|^{(p-1)/p}\right|^p}{2^i}$.

Define$f: R \to l_p$ by $f(x) = xe_0 + \sum_{i=1}^{\infty} g_i(x)e_i$. Then

$$|x-y|^p \leq \| f(x) - f(y) \|^p$$

$$= |x-y|^p + \sum_{i=1}^{\infty} |g_i(x) - g_i(y)|^p$$

$$\leq |x-y|^p + p\varepsilon\left||x|^{(p-1)/p} - |y|^{(p-1)/p}\right|^p$$

$$\leq |x-y|^p + p\varepsilon|x-y|^{p-1} < (|x-y| + \varepsilon)^p.$$

Thus, f is an ε-isometry. For every $i > 0$ and real x, $|\langle e_i', f(x)\rangle| = g_i(x) \leq i. Hence\ e_i' \in M_\varepsilon$. It follows that E=spane_0. But for every x, n with,$|x| \geq \left(n^p 2^n / (p\varepsilon)\right)^{1/(p-1)}$,

$$dist(f(x), E) > \sum_{i=1}^{n} |g_i(x)| = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Thus, f is a desired mapping.

List of Symbols

Symbol	page
l_p^2 : Hilbert space	1
L^p : Lebesgue space	2
$\overline{co}$: closed convex hull	4
sep : separated	5
inf : infimum	5
UKK : Uniform Kadec-Klee	5
GDS : Gateaux Differentiability Space	7
Sup : Supremum	7
dim : dimension	9
Ker : Kernal	13
Min : Minimal	14
$\oplus$: Direct Sum	37
Dist : Distance	63
l_p : Hilbert Space	67

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