\[ \mu(s(Q_0, h)) = \int d\mu(w) \leq Ch(w)Ch^\beta \int \frac{d\mu(w)}{|1-z_0w|^\beta} \leq C p(h)h^{\beta-q} \]

Conversely let us take \( z_0 = \varepsilon C^0 \) with \( \varepsilon > \frac{3}{4} \), and consider

\[ E_n = S(Q_0, 2^{-n}(1-|z_0|)) \]

An elementary computation shows that for \( n \to \infty \), we have

\[
\int \frac{d\mu(w)}{|1-z_0w|^\beta} \leq \sum_{n=0}^\infty \mu(S(Q_0,(1-|z_0|)2^{-n})) \leq C \sum_{n=0}^\infty \frac{P(2^{-n}(1-|z_0|))}{(2^{-n})^\beta(1-|z_0|)^q}
\]

under the pairing duality given by

\[
\langle f, \phi \rangle = \sum_{n=0}^\infty \alpha_n \phi(x^n)
\]

where

\[
f(x) = \sum_{n=0}^\infty \alpha_n x^n, \quad \phi(x) = \sum_{n=0}^\infty \alpha_n x^n
\]

Theorem (2-2-3)[1] :-

Let \( \mu \) be a finitely Borel measure on the disc and a Dini weight such that \( P = P_0 \). The following are equivalent

(i) \( \mu \) is a Carleson measure

(ii) \( |P(\mu)(x)| = O \left( \frac{p(1-|x|)}{1-|x|^\beta} \right) \)

(iii) \( B_p = L_p(D) \) with continuity.

Proof: Lemma (2.2.2) gives (I) of and only if (ii).

The equivalence between (ii) and (iii) follows from theorem (2.1.11).

Corollary (2.2.4) [1]: Let \( \frac{1}{3} < \frac{1}{p} \) be a finite Borel measure on \( D \), and \( \alpha = \frac{1}{p} \). The following are equivalent:

(i) \( H_p(D) \subset L_p(D) \) with continuity.
(ii) is $\varepsilon$- Carleson measure.

Proof: use Remark (2.1.11) and observe that $p(\varepsilon) \to \varepsilon^{1-p}$.

Lemma (2.2.5)[1]: Let be a Dini weight such that $p = \varepsilon^{1-p}$. Let be an analytic function with continuous extension at the boundary. The following are equivalent.

(i) \[ |g(x)| = O\left( \frac{p(1-|z|)}{1-|z|} \right) \quad (|z| \to 0). \]

(ii) \[ \int_{|z|=1} \frac{|g(z)|}{|1-\overline{z}|} \, d\overline{z} = O\left( \frac{p(1-|z|)}{1-|z|} \right) \quad (|z| \to \infty). \]

Proof:

(ii) (I) obvious from the Cauchy formula.

\[ g(x) = \int_{|z|=1} \frac{g(z(1-e^{2i\varepsilon}))-g(z)}{(1-\overline{z})^2} \, \frac{dz}{|z|}. \]

For the converse, take $z = e^{2i\varepsilon}$ and $z = |z|e^{i\varepsilon}.

Let us first estimate

\[ |g(z)| = \frac{g(z(1-e^{2i\varepsilon}))-g(z)}{(1-\overline{z})^2} \, \frac{dz}{|z|} \leq C \int_{|z|=1} \frac{p(1-|z|)}{1-|z|} \, ds \leq C p(1-|z|). \]

On the one hand, using the Dini condition, we have

\[ |g(z)| \leq \frac{p(1-|z|)}{1-|z|} ds \leq C |z| \int_{|z|=1} \frac{p(1-|z|)}{1-|z|} \, ds \leq C p(1-|z|). \]

On the other hand, we use Lemma (2.1.7) with $\varepsilon \to \infty$ and $e^{-i\varepsilon} \to \infty$ to get

\[ |g(z)| \leq C p(1-|z|). \]

Therefore

\[ \int_{|z|=1} \frac{|g(z)|}{1-\overline{z}} \, d\overline{z} = C \int_{|z|=1} \frac{p(1-|z|)}{1-|z|} \, ds \leq C p(1-|z|). \]

Let us finally use the facts that $p(\varepsilon)$ is nondecreeasing and belongs to $p(\varepsilon)$ to estimate.

\[ \int_{|z|=1} \frac{p(\varepsilon)}{(1-|z|)^2 + 2|z|\sin^2(\varepsilon \pi)} \, dt \leq C \int_{|z|=1} \frac{p(\varepsilon)}{(1-|z|)^2 + C \varepsilon^2} \, dt \]

\[ \leq C \int_{|z|=1} \frac{p(\varepsilon)}{(1-|z|)^2 + C \varepsilon^2} \, dt \leq C \int_{|z|=1} \frac{p(\varepsilon)}{1 + \varepsilon^2} \, dt. \]

\[ \leq C \int_{|z|=1} \frac{p(\varepsilon)}{1 + \varepsilon^2} \, dt \leq C \int_{|z|=1} \frac{p(\varepsilon)}{1 + \varepsilon^2}. \]

\[ \leq C p(1-|z|) \left( \int_{1-|z|}^{1-|z|} \frac{1}{1+|z|^2} \, ds \right) \leq C p(1-|z|) \]

\[ \leq C p(1-|z|) \left( \int_{1-|z|}^{1-|z|} \frac{1}{1+|z|^2} \, ds \right) \leq C p(1-|z|) \]

\[ \leq C p(1-|z|) \left( \int_{1-|z|}^{1-|z|} \frac{1}{1+|z|^2} \, ds \right) \leq C p(1-|z|) \]
Theorem (2.2.6) Let \( p \) be a Dini weight such that \( p = b \) . Let 

The following are equivalent

(i) \( H_{p} \to H_{b} \) is bounded .

(ii) \( |b'(z)| = O\left(\frac{p(1-|z|)}{1-|z| \log \frac{1}{1-|z|}}\right) \) \(|z| \to 1\).

Proof : 
Denote \( F(z) = H_{b}(z) \), and use definition (1-3) to write 

\[
F'(z) = \frac{F(z) - b(z)}{z} = \frac{b(z)}{1 - \frac{z}{z'}}
\]

(12)

Let us assume (i) . Applying Corollary (2.1.14) we have 

\[
\|F'(z)\|_{b'} = O\left(\frac{p(1-|z|)}{1-|z|}\right)
\]

(13)

Now \( H_{b}(f) = \int_{a}^{b} f \), so the boundedness of \( b \) implies 

\[
\left\| \frac{1}{b'(z)} \right\|_{L_{1}} \left\| H_{b}(f) \right\|_{b'} \leq C \|f\|_{L_{b}(p)}
\]

This implies \( b = (b_{b}(p))^{-1} \), which coincides with \( b \).

According to Corollary (2.1.14), we can apply Lemma (2.2.5) to obtain 

\[
\int_{a}^{b} \frac{|b(z) - b(z)|}{|1 - \frac{z}{z'}|} dz = O\left(\frac{1}{\log \frac{1}{1-|z|}}\right)
\]

(14)

from we have 

\[
|b'(z)| \int_{a}^{b} \frac{d\xi}{|1 - \xi|} = \|F'(z)\|_{b'} \int_{a}^{b} \frac{|b(z) - b(z)|}{|1 - \frac{z}{z'}|} dz
\]

using 

\[
\int_{a}^{b} \frac{d\xi}{|1 - \xi|} = O\left(\frac{1}{\log \frac{1}{1-|z|}}\right)
\]

(13) and (14) we get (ii).

Let us now assume (ii), From Theorem (2-1-11) we have to show (13)- using (12) again we have :

\[
\|F'(z)\|_{b'} \leq |b'(z)| \int_{a}^{b} \frac{d\xi}{|1 - \xi|} + \int_{a}^{b} \frac{|b(z) - b(z)|}{|1 - \frac{z}{z'}|} dz
\]
Now we establis (13) follows easyly by using (ii) and Lemma (2-2-5).

Corollary (2-2-7) [1]:

Let \( \frac{1}{2} < p < 1 \) and let \( \phi \) be analytic. Then \( \mu_n \rightarrow \mu \) if and only if

\[
|p\phi(z)| = O\left(\frac{1}{(1+|z|)^{2\log(1+|z|)}}\right)
\]

Proposition (2-2-8) [1]: let \( \phi \) be a weight function. Let \( \phi \) be analytic and \( \phi \rightarrow \phi \), then

\[
\int_0^\infty \frac{p(1-|z|)}{1+|z|} |f(\phi(z))| dA(z) \leq \int_0^\infty \frac{(1+|\phi(z)|)}{1+|z|} \int_0^\infty p(1-|z|) |f(z)| dA(z) .
\]

Proof:

Let \( \phi \rightarrow \infty \) and consider \( \phi = \infty \) where

\[
\phi(w) = \frac{w-\alpha}{1-\alpha w}
\]

\( \phi \rightarrow \infty \), \( \phi \rightarrow 0 \) and \( \phi \rightarrow \phi \), we can we little wood.