Sudan University of Science and Technology
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Asymptotic Toeplitz Operators and Coburn Type Theorem with Brown–Halmos Theorem on Hardy Spaces

مؤثرات تبوليتز المقاربة ومبرهنة نوع كوبيرن مع مبرهنة بروين- هالموس على فضاءات هاردي

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

Acknowledgements

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Abstract

We study the Toeplitz and asymptotic Toeplitz operators on Hardy space of the multidisk. The commuting and products of Toeplitz operators on the polydisk are determined. The Coburn-Simonenko theorem for Toeplitz operators acting between Hardy type subspaces of different Banach function spaces with Toeplitness of composition operators in several variables and Toeplitz projections with essential commutants are explained. The operator theory, the Berger-Shaw theorem and a Coburn type theorem in the Hardy space and module over the bidisk are given. The pointwise multipliers of Orlicz function spaces and factorization are introduced. The Toeplitz operators with the density of analytic polynomials and Brown-Holmos theorem for a pair of abstract Hardy spaces are dealt with.

الخلاصة

قمنا بدراسة مؤثرات تبوليتز ومؤثرات تبوليتز المقاربة على فضاء هاردي للقرص المتعدد. تم تحديد مؤثرات تبوليتز التبديلية والضربية على القرص المتعدد. قمنا بشرح مبرهنة كوبيرن – سيمونينكو لمؤثرات تبوليتز الممثلة بين الفضاءات الجزئية نوع هاردي لفضاءات دالة باناخ المختلفة مع مؤثرات تبوليتز ومؤثرت التركيب في المتغيرات المتعددة ومساقط تبوليتز مع المبدلات الأساسية. قمنا بإعطاء نظرية المؤثر ومبرهنة بيرجير – شو والمبرهنة نوع كوبيرن في فضاء هاردي والمقياس فوق القرص الثنائي. تم إدخال المضاريب النقطية لفضاءات دالة أورليش والتحليل الى عوامل. تعاملنا مع مؤثرات تبوليتز مع الكثافة لكثيرات الحدود التحليلية ومبرهنة بروين – هالموس لأجل زوج من فضاءات هاردي المجردة.

Introduction

An asymptotic Toeplitz is an operator T such the sequence $\{U^{*n}TU^n\}$ is strongly convergent, where U is the unilateral shift. Every element of the norm-closed algebra generated by all Toeplitz and Hankel opertors together is an asymptotic Toeplitz operator. We obtain characterizations of (essentially) commuting Toeplitz operators with pluriharmonic symbols on the Bergman space of the polydisk. We study products of Toeplitz operators on the Hardy space of the polydisk. We show that $T_fT_g = 0$ if and only if T_fT_g is a finite rank if and only if T_f or T_g is zero.

We identify the vector valued Hardy space with the Hardy space over the Bidisk and construct a universal model for the contractive analytic functions. It is well known that the Hardy space over the bidisk $\mathbb{D}^2$ is an $A(\mathbb{D}^2)$ module and that $A(\mathbb{D}^2)$ is contained in $H^2(\mathbb{D}^2)$. Suppose $(h) \subset A(\mathbb{D}^2)$ is the principal ideal generated by a polynomial h, then its closure $[h](\subset H^2(\mathbb{D}^2))$ and the quotient $H^2(\mathbb{D}^2) \ominus [h]$ are both $A(\mathbb{D}^2)$ modules. We let R_z, R_w be the actions of the coordinate functions z and w on $[h]$, and let S_z, S_w be the actions of z and w on $H^2(\mathbb{D}^2) \ominus [h]$.

For X be a Banach function space over the unit circle $\mathbb{T}$ and let $[X]$ be the abstract Hardy space built upon X. If the Riesz projection P is bounded on X and $a \in L^\infty$, then the Toeplitz operator $T_af = (af)$ is bounded on $[X]$. We extend well-known results by Brown and Halmos for $X = L^2$. For Γ be a rectifiable Jordan curve, let X and Y be two reflexive Banach function spaces over Γ such that the Cauchy singular integral operator S is bounded on each of them, and let $M(X,Y)$ denote the space of pointwise multipliers from X to Y. Consider the Riesz projection $P = (I + S)/2$, the corresponding Hardy type subspaces PX and PY, and the Toeplitz operator $T(a) : PX \to PY$ defined by $T(a)f = P(af)$ for a symbol $a \in M(X,Y)$. We show that if $X \hookrightarrow Y$ and $a \in M(X,Y)\backslash\{0\}$, then $T(a) \in \mathcal{L}(PX, PY)$ has a trivial kernel in PX or a dense image in PY.

Motivated by the work of Nazarov and Shapiro on the unit disk, we study asymptotic Toeplitzness of composition operators on the Hardy space of the unit sphere in $\mathbb{C}^n$. We construct a Toeplitz projection for every regular A-isometry $T \in B(\mathcal{H})^n$ on a complex Hilbert space Hand use it to determine the essential commutant of the set of all analytic Toeplitz operators formed with respect to an essentially normal regular A-isometry. We show that the Toeplitz projection annihilates the compact operators if and only if T possesses no joint eigenvalues. We initiate a study of Toeplitz operators and asymptotic Toeplitz operators on the Hardy space $H^2(\mathbb{D}^n)$ (over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$). Our main results on Toeplitz and asymptotic Toeplitz operators can be stated as follows: Let T_{z_i} denote the multiplication operator on $H^2(\mathbb{D}^n)$ by the i^{th} coordinate function $z_i, i = 1, \ldots, n$, and let T be a bounded linear operator on $H^2(\mathbb{D}^n)$.

We show a continuation of a project of developing a systematic operator theory in $H^2(D^2)$. A large part of it is devoted to a study of evaluation operator which is a very useful tool in the theory. A number of elementary properties of the evaluation operator are exhibited, and these properties are used to derive results in other topics such as interpretation of characteristic operator function in $H^2(D^2)$, spectral equivalence, compactness and compressions of shift operators. A famous theorem of Coburn says that a nonzero Toeplitz operator on the Hardy space of the unit disk is injective or its adjoint operator is injective. We study the corresponding problem on the Hardy space of the bidisk.

We show that the space of pointwise multipliers between two distinct Musielak–Orlicz spaces is another Musielak–Orlicz space and the function defining it is given by an appropriately generalized Legendre transform. Let $\{F_n\}$ be the sequence of the Fejér kernels on the unit circle $\mathbb{T}$. It was proved that if X is a separable Banach function space on $\mathbb{T}$ such that the Hardy-Littlewood maximal operator M is bounded on its associate space X', then $\|f * F_n - f\|_X \to 0$ for every $f \in X$ as $n \to \infty$. This implies that the set of analytic polynomials $\mathcal{P}_A$ is dense in the abstract Hardy space $H[X]$ built upon a separable Banach function space X such that M is bounded on X'. Let $H[X]$ and $H[Y]$ be abstract Hardy spaces built upon Banach function spaces Xand Yover the unit circle $\mathbb{T}$. We prove an analogue of the Brown–Halmos theorem for Toeplitz operators T_a acting from $H[X]$ to $H[Y]$ under the only assumption that the space X is separable and the Riesz projection P is bounded on the space Y.

The Contents

Chapter 1

Asymptotic with Commuting and Products of Toeplitz Operators on the Polydisk

We study the relations among Hankel algebra, the classical Toeplitz algebra, the set of all asymptotic Toeplitz operators, and the essential commutant of the unilateral shift. They offer several examples of operators in some of these classes but not in others. We show that commuting and essential commuting properties are the same for dimensions bigger than 2, while they are not for dimensions less than or equal to 2. Also, the corresponding results for semi-commutators are obtained. We show that the product $T_f T_g$ is still a Toeplitz operator if and only if there is a $h \in L^\infty(T^n)$ such that $T_f T_g - T_h$ is a finite rank operator. We also show that there are no compact semi-commutators with symbols pluriharmonic on the polydisk.

Section (1.1): Asymptotic Toeplitz Operators:

What is the essential commutant of the unilateral shift? The experts are convinced that, whatever it is, it is huge. The purpose of this paper is to call attention to an asymptotic property of some operators, use that property to show that certain concrete operators that do not belong to the Toeplitz algebra do belong to the essential commutant of the shift, discuss some related examples, and pose a few unsolved problems.

The underlying Hilbert space is $\boldsymbol{H}^2$ of the unit circle. The unilateral shift U is defined on $\boldsymbol{H}^2$ by $Uf(z) = zf(z)$. The essential commutant of U is, by definition, the set $\boldsymbol{E}$ of all those operators T on $\boldsymbol{H}^2$ for which $UT - TU \in \boldsymbol{K}$ (where $\boldsymbol{K}$ is the ideal of all compact operators on $\boldsymbol{H}^2$).

Since U is essentially unitary (i.e., both U^*U and UU^* are congruent to 1 mod $\boldsymbol{K}$), it follows that $T \in \boldsymbol{E}$ if and only if $U^*TU - T \in \boldsymbol{K}$. This reformulation of the definition of $\boldsymbol{E}$ is convenient in matrix calculations. (For operators on $\boldsymbol{H}^2$, all matrices in the sequel will be formed with respect to the basis $\{e_0, e_1, e_2, \ldots\}$ defined by $e_n(z) = z^n, n = 0,1,2,\ldots$) Since, in terms of the Kronecker delta, the matrix of U is $(\delta_{i,j+1})$, the matrix of a product TU is obtained from the matrix of T by erasing the first column, and the matrix of U^*T is obtained from that of T by erasing the first row. (Caution: "erase" means literally what it says; it does not mean "replace by 0's".) The matrix of U^*TU, therefore, is obtained from that of T by "moving one step to the southeast"; to say that $T \in \boldsymbol{E}$ is the same as to say that, mod $\boldsymbol{K}$, the matrix is not changed by the move.

The essential commutant of every operator is a norm-closed algebra. Since $\boldsymbol{E}$ contains every Toeplitz operator (recall a possible definition: $U^*TU = T$), it follows that the Toeplitz algebra (the norm-closed algebra $\boldsymbol{T}$ generated by the set of all Toeplitz operators) is included in $\boldsymbol{E}$. Question, with a not immediately obvious answer: is $\boldsymbol{E}$ equal to $\boldsymbol{T}$? The experts' conviction ($\boldsymbol{E}$ is huge) means, among other things, that the answer is no; some concrete examples of operators in $\boldsymbol{E}$ but not in $\boldsymbol{T}$ will become visible presently. (The most important earlier work on a closely related problem is [3].)

In view of the role that $\boldsymbol{K}$ plays in the definition of essential commutativity, the relation $\boldsymbol{K} \subset \boldsymbol{E}$ is even more obvious than the relation $\boldsymbol{T} \subset \boldsymbol{E}$. It is not only obvious: it contains no new information. Reason: $\boldsymbol{K} \subset \boldsymbol{T}$. This inclusion can be inferred from a sophisticated fact about irreducible C^*-algebras [4, p. 141], or can be proved directly. [Note that since U is essentially unitary, it follows that $\boldsymbol{E}$ is closed under the formation of adjoints and is therefore a C^*-algebra. Since U is irreducible and $U \in \boldsymbol{E}$, it follows that $\boldsymbol{E}$ is irreducible.] Here is an elementary direct proof. Since $U \in \boldsymbol{T}$, therefore $E = 1 - UU^* \in \boldsymbol{T}$; the operator E is, in fact, the projection $e_0 \otimes e_0$ of rank 1. For arbitrary operators S and T, the product $S(e_0 \otimes e_0)T$ is equal to $(Se_0) \otimes (T^*e_0)$; it follows that if S and T are in $\boldsymbol{T}$, then so is $(Se_0) \otimes (T^*e_0)$. If, in particular, p and q are arbitrary polynomials, and if $S = p(U)$ and $T = q(U)^*$, then $(p(U)e_0) \otimes (q(U)e_0) \in \boldsymbol{T}$. Since the set of all vectors obtained by applying a polynomial in U to e_0 is dense in $\boldsymbol{H}^2$, it follows that every operator of rank 1 is in $\boldsymbol{T}$, and so therefore is every compact operator.

If $\varphi \in L^\infty$ of the unit circle, write M_φ for the multiplication operator defined on L^2 by $M_\varphi f = \varphi f$, and T_φ for the compression defined on $\boldsymbol{H}^2$ by $T_\varphi f = PM_\varphi f$ (where P is the projection from $\boldsymbol{L}^2$ onto $\boldsymbol{H}^2$). The compression T_φ is a Toephtz operator, and every Toeplitz operator is obtained this way. If M_φ is expressed as an operator matrix with respect to the decomposition $\boldsymbol{L}^2 = {H^2}^\perp \oplus \boldsymbol{H}^2$, the result is of the form

$$M_\varphi = \begin{pmatrix} T_{\tilde{\varphi}} & H_\varphi \\ H_{\tilde{\varphi}} & T_\varphi \end{pmatrix},$$

where $\tilde{\varphi}(z) = \varphi(z^*)$, the diagonal entries are Toeplitz operators, and the others are Hankel operators. (The latter can be defined by this remark; alternatively a Hankel operator H is one for which $U^*H = HU$.) If φ and ψ are in L^∞, then $M_{\varphi\psi} = M_\varphi + M_\psi$, and therefore (mulitply matrices and compare lower right corners)

$$T_{\varphi\psi} = T_\varphi T_\psi + H_{\tilde{\varphi}} H_\psi. \tag{1}$$

What is most important about this equation is that the product of two Toeplitz operators differs from a Toeplitz operator by the product of the two Hankel operators, and every product of two Hankel operators arises in this way. A related formula with a related proof (compare upper right corners) can also be useful:

$$H_{\varphi\psi} = T_{\tilde{\varphi}} H_\psi + H_\varphi T_\psi. \tag{2}$$

Hankel operators are an essential part of Toeplitz theory. An effective way to welcome them is to consider the Hankel algebra (the norm-closed algebra $\boldsymbol{T}^+$ generated by all Toeplitz operators and all Hankel operators together).

It is natural to define an asymptotic Toeplitz operator as an operator T such that the sequence $\{U^{*n}TU^n\}$ is strongly convergent. The limit is clearly a Toeplitz operator, and hence of the form T_φ for some φ in L^∞. The function φ will be called the symbol of T and will be denoted by $\sigma(T)$. The simplest examples are the Toeplitz operators; the next simplest the Hankel operators.

Lemma (1.1.1)[1]: If H is a Hankel operator, then $HU^n \to 0$ (strong).

Proof. From the matrix point of view the statement is almost obvious: the matrix of HU^n is obtained from that of H by erasing the first n columns. [Note that each entry occurs in a Hankel matrix only a finite number of times.] Alternatively, $HU^n = U^{*n}H$, and $U^{*n} \to 0$ (strong).

Theorem (1.1.2)[1]: Every element of the Hankel algebra is an asymptotic Toeplitz operator.

Proof. The main step is to show that if $\varphi_1, \dots, \varphi_k$ are in $\boldsymbol{L}^\infty$, if $T = T_{\varphi_1}, \dots, T_{\varphi_k}$, and if $\varphi = \varphi_1, \dots, \varphi_k$, then $U^{*n}TU^n \to T_\varphi$ (strong). The argument is based on a telescoping sum:

$$T_{\varphi_1}, \dots, T_{\varphi_k} - T_{\varphi_1,\dots,\varphi_k} = T_{\varphi_1} T_{\varphi_2,\dots,\varphi_k} - T_{\varphi_1(\varphi_2,\dots,\varphi_k)}$$

$$
\begin{aligned}
&+T_{\varphi_1}\left(T_{\varphi_2}T_{\varphi_3,\dots,\varphi_k}-T_{\varphi_2(\varphi_3,\dots,\varphi_k)}\right)\\
&+T_{\varphi_1}T_{\varphi_2}\left(T_{\varphi_3}T_{\varphi_4,\dots,\varphi_k}-T_{\varphi_3(\varphi_4,\dots,\varphi_k)}\right)\\
&+\cdots\\
&+T_{\varphi_1}T_{\varphi_2}\dots T_{\varphi_{k-2}}\left(T_{\varphi_{k-1}}T_{\varphi_k}-T_{\varphi_{k-1}\varphi_k}\right).
\end{aligned}
$$

In view of this, equation (1) implies that

$$T-T_{\varphi}=HH+\ THH\ +\ TTHH\ +\cdots+\ TT\dots T\ HH,$$

Where each T on the right side indicates a Toeplitz operator and each H a Hankel operator; since the actual subscripts are useless, they are omitted. Multiply by U^{*n} on the left and U^{n} on the right; since T_{φ} is invariant under that operation, and since (by Lemma (1.1.1)) the right side converges strongly to 0 as $n\to\infty$, the main step is complete.

Consider next a finite product all whose factors are either Toeplitz or Hankel operators, with at least one Hankel factor present. If the rightmost factor is a Hankel operator, the asserted strong convergence (to 0) follows from Lemma (1.1.1). In the remaining cases, the first Hankel factor from the right occurs in a context HT, where, as before, the symbols H and T indicate generic Hankel and Toeplitz operators respectively. In such a case, use (2) to replace HT by $H-TH$ (subscripts still omitted), and thus replace the given operator by two others, in each of which the rightmost Hankel factor is one step nearer to the right end; the desired convergence now follows by induction.

The rest is easy. Let $\boldsymbol{T}_0^{+}$ be the (unclosed) algebra consisting of all finite sums of finite products of Toephtz and Hankel operators. If $T\in\boldsymbol{T}_0^{+}$, convergence follows from the strong continuity of operator addition. For norm limits of operators in $\boldsymbol{T}_0^{+}$, convergence follows from the standard techniques of "$\frac{\varepsilon}{3}$" analysis.

Corollary (1.1.3)[1]: The restriction of the symbol map a to the Hankel algebra is a contractive *-homomorphism from $\boldsymbol{T}^{+}$ onto $\boldsymbol{L}^{\infty}$.

Proof. That a is a contraction is immediate from the strong lower semicontinuity of norm: if $U^{*n}TU^{n}\to T_{\varphi}$ (strong), then

$$\|\sigma(T)\|_{\infty}=\|\varphi\|_{\infty}=\left\|T_{\varphi}\right\|\leqslant\lim_{n}\inf\|U^{*n}TU^{n}\|\leqslant\|T\|.$$

That σ preserves sums and products in $\mathbf{T}_0^{+}$ follows from the main step in the preceding proof; that it preserves sums and products for all operators in the Hankel algebra follows from the (norm) continuity of operator addition and multiphcation and the (just proved) continuity of σ. As for adjoints, there seems to be a difficulty; adjunction is not strongly continuous. Suppose, however, that $T\in\boldsymbol{T}^{+}$ and $U^{*n}TU^{n}\to T_{\varphi}$ (strong); the weak continuity of adjunction imphes that $U^{*n}T^{*}U^{n}\to T_{\varphi}^{*}=T_{\varphi^{*}}$. (weak). Since $T^{*}\in\boldsymbol{T}^{+}$, the sequence $\{U^{*n}T^{*}U^{n}\}$ converges strongly to something, say T_{ψ}. Conclusion: $T_{\psi}=T_{\varphi^{*}}$, and therefore $\sigma(T^{*})=\sigma(T)^{*}$.

The symbol map was originally defined for Toeplitz operators only; the existence of a homomorphic extension to the entire Hankel algebra yields a slight improvement of a curious result of Douglas [5, p. 9].

Corollary (1.1.4)[1]: If a finite sum of finite products of Toeplitz or Hankel operators is compact, then the corresponding finite sum of finite products of their symbols is zero almost everywhere.

Proof. If K is compact, then $KU^n e_j = Ke_{j+n} \to 0$ as $n \to \infty$ and therefore $\sigma(K) = 0$; in other words $\boldsymbol{K} \subset \ker\sigma$.

An important part of Toephtz theory concerns the commutator ideal $\boldsymbol{Q}$ of the algebra $\boldsymbol{T}$ (see [4, p. 181]); the following characterization of $\boldsymbol{Q}$ might be useful.

Theorem (1.1.5)[1]: An operator T in the Toeplitz algebra $\boldsymbol{T}$ belongs to the commutator ideal $\boldsymbol{Q}$ of $\boldsymbol{T}$ if and only if $U^{*n}TU^n \to 0$ (strong); equivalently $\boldsymbol{Q} = \ker\sigma$.

Proof. Suppose first that $\varphi_1, \dots, \varphi_k$ are in $\boldsymbol{L}^\infty, T = T_{\varphi_1}, \dots, T_{\varphi_k}$, and $\psi = \varphi_1, \dots, \varphi_k$.

Assertion: $T - T_\psi \in \boldsymbol{Q}$. The proof is induction on k. For $k = 1$, the assertion is trivial. To pass from $k-1$ to k assume, temporarily, that $\varphi_k = \alpha^*\beta$ where a and β are in $\boldsymbol{H}^\infty$; then

$$\begin{aligned}
T - T_\psi &= T_{\varphi_1}, \dots, T_{\varphi_{k-1}} T_{\alpha^*\beta} - T_{\varphi_1,\dots,\varphi_{k-1}\alpha^*\beta} \\
&= T_{\varphi_1}, \dots, T_{\varphi_{k-1}} T_{\alpha^*} T_\beta - T_{\alpha^*} T_{\varphi_1,\dots,\varphi_{k-1}} T_\beta \\
&= \left(T_{\varphi_1}, \dots, T_{\varphi_{k-1}} T_{\alpha^*} - T_{\alpha^*} T_{\varphi_1,\dots,\varphi_{k-1}}\right) T_\beta \\
&= \left(\left[T_{\varphi_1}, \dots, T_{\varphi_{k-1}} T_{\alpha^*} - T_{\alpha^*} T_{\varphi_1}, \dots, T_{\varphi_{k-1}}\right] + \left[T_{\alpha^*} T_{\varphi_1}, \dots, T_{\varphi_{k-1}} - T_{\alpha^*} T_{\varphi_1,\dots,\varphi_{k-1}}\right]\right) T_\beta.
\end{aligned}$$

The first square bracket is a commutator, and therefore belongs to $\boldsymbol{Q}$**.** The second square bracket is T_{α^*}. times an operator of the same form as $T - T_\psi$ except with $k-1$ instead of k, and, consequently, (by the induction hypothesis) it too belongs to $\boldsymbol{Q}$. At this point it seems necessary to use a relatively deep tool, namely the approximation theorem [4, p. 163] according to which functions of the form $\alpha^*\beta$ are dense in $\boldsymbol{L}^\infty$**.** With the use of that theorem the proof of the assertion is obviously complete; if $T - T_\psi \in \boldsymbol{Q}$ whenever $\varphi_k = \alpha^*\beta$, then $T - T_\psi \in \boldsymbol{Q}$ for all φ_k.

The preceding paragraph implies that if T belongs to the (unclosed) algebra $\boldsymbol{T}_0$ consisting of all finite sums of finite products of Toeplitz operators, and if $\psi = \sigma(T)$, then $T - T_\psi \in \boldsymbol{Q}$. Indeed, suppose that $T = T_1 + \dots + T_m$, where each T_j is a finite product of Toeplitz operators. It follows that $\psi = \psi_1 + \dots + \psi_m$, where $\psi_j = \sigma(T_j), j = 1, \dots, m$, and hence that $T - T_\psi = (T_1 - T_{\psi_1}) + \dots + (T_m - T_{\psi_m}) \in \boldsymbol{Q}$.

Suppose now that T is an arbitrary operator in $\boldsymbol{T}$ with $\sigma(T) = 0$. Let $\{T_n\}$ be a sequence, each term of which is an operator in $\boldsymbol{T}_0$, such that $T_n \to T$ (norm). If $\psi_n = \sigma(T_n)$, then $\psi_n \to 0$ in $\boldsymbol{L}^\infty$ (because $\sigma(T) = 0$), and therefore $T_n - T_{\psi_n} \to T$ (norm). Since $T_n - T_{\psi_n} \in \boldsymbol{Q}$ for each n (by the preceding paragraph), it follows that $T \in \boldsymbol{Q}$.

What was proved so far was that $\ker\sigma \subset \boldsymbol{Q}$. Since $T/\ker\sigma$ is commutative, the reverse inclusion is trivial.

The condition $U^*TU - T \in \boldsymbol{K}$ is (necessary and) sufficient for $T \in \boldsymbol{E}$; the condtion that the sequence $\{U^{*n}TU^n\}$ be strongly convergent is necessary for $T \in \boldsymbol{T}$. Are these conditions sharp enough to distinguish between $\boldsymbol{E}$ and $\boldsymbol{T}$?

Example (1.1.6)[1]: The Hankel operator H whose matrix is $(1/(i + j + I)), i, j = 0,1,2, \dots$, (usually known as the Hilbert matrix) is a famous one; it is quite easy to see that it belongs to $\boldsymbol{E}$. Indeed, the matrix of U^*HU is $(1/(i + j + 3))$; the difference $U^*HU - H$ has matrix

$$\left(\frac{-2}{(i + j + 1)(i + j + 3)}\right).$$

Elementary analysis shows that the sum of the squares of all the entries in this difference is finite; in other words, $U^*HU - H$ is a Hilbert-Schmidt operator.

Conclusion: $U^*HU - H \in \boldsymbol{K}$, so that $H \in \boldsymbol{E}$.

Is H an asymptotic Toeplitz operator? The answer is yes, and the proof is easy.

The necessary convergence condition is satisfied, and, for all that is visible at this stage, it could be that $H \in \boldsymbol{T}$.

The fact is that H does belong to the Toeplitz algebra; the proof goes as follows.

Since $1/(i+j+1) = \int_0^1 x^i x^j \, dx$, the matrix of H is a Gramian and therefore positive. The operator H^2, being the product of two Hankel operators, belongs to $\boldsymbol{T}$ (by (1)). Since $\boldsymbol{T}$ is a C^*-algebra, it contains the unique positive square root of each of its positive elements, and therefore, in particular, $\boldsymbol{T}$ contains the positive square root H of H^2.

The Hilbert matrix is an illuminating example, but in an attempt to get new information about $\boldsymbol{E}$ and $\boldsymbol{T}$, it turned out to be a failure. It is, however, not a trivial failure. It belongs to $\boldsymbol{T}$, to be sure (and hence to $\boldsymbol{E}$), but not for the trivial reason; it doesn't belong to $\boldsymbol{K}$.

Proof. if f_k is the vector in $\boldsymbol{H}^2$ whose first k coordinates are $1/\sqrt{k}$ and all other coordinates are 0 $(k = 1,2,3,\ldots)$, then f_k is a unit vector and $f_k \to 0$ (weak). Since elementary estimates show that $(Hf_k, f_k) \geqslant \frac{1}{2}$, be compact. the operator H cannot

Example (1.1.7)[1]: There are some near relatives of the Hilbert matrix that deserve examination. For each complex number a of absolute value 1, let H_α be the operator with

$$\left(\frac{\alpha^{i+j}}{i+j+1}\right).$$

If f_k is the vector whose initial coordinates are $1/(\alpha^j\sqrt{k})$ $(j = 0,\ldots,k-1)$ and all other coordinates are 0 $(k = 1,2,3,\ldots)$ then, as before, f_k tends to 0 weakly but $H_\alpha f_k$ does not tend to 0 strongly; the operator H_α is not compact. Does it belong to $\boldsymbol{E}$? The answer depends on α. If $\alpha = \pm 1$, then $H_\alpha \in \boldsymbol{T}$; otherwise H_α doesn't even belong to $\boldsymbol{E}$. Reason: straightforward computation shows that $U^*H_\alpha U - H_\alpha$ is a scalar multiple of H_α plus a compact operator. Consequence: $U^*H_\alpha U - H_\alpha$ is just as non-compact as H_α.

Example (1.1.8)[1]: The classically important Cesàro operator C is defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ & & & \ddots \end{pmatrix}$$

Is C in $\boldsymbol{E}$? Yes, it is. Proof (straightforward computation): $U^*CU - C$ is a Hilbert-Schmidt operator.

Since C is known to be hyponormal [2] and, in fact, subnormal [7], it follows that C is not compact. Question: is C in $\boldsymbol{T}_0$? Answer: no. Reason: if $T \in \boldsymbol{T}_0$, then $U^*TU - T$

has finite rank, but $U^*CU - C$ has a triangular matrix with all diagonal entries different from 0, and therefore has infinite rank.

The preceding two comments are evidence, however weak, that C does not belong to $\boldsymbol{T}$. There is a bit of evidence that C does not belong to $\boldsymbol{T}$, namely that C is an asymptotic Toeplitz operator. (In fact $\sigma(C) = 0$, which shows incidentally that $\ker\sigma \neq \boldsymbol{K}$.

Example (1.1.9)[1]: Which diagonal operators are in $\boldsymbol{E}$? Which ones are in $\boldsymbol{T}$? (In this context a diagonal operator is not just one that can be diagonalized, but one whose matrix with respect to the standard basis is diagonal.)

The answers are easy. If $T = \operatorname{diag}(\alpha_0, \alpha_1, \alpha_2, \ldots)$, then

$$U^*TU - T = \operatorname{diag}(\alpha, -\alpha_0, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \ldots),$$

and therefore a necessary and sufficient condition that $T \in \boldsymbol{E}$ is that $\alpha_{n+1} - \alpha_n \to 0$.

Since $U^{*n}TU^n = \operatorname{diag}(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots)$, it follows that T is an asymptotic Toeplitz operator if and only if the sequence $\{\alpha_n\}$ is convergent. Note: if $\{\alpha_n\}$ is convergent, then $T \in \boldsymbol{T}$. Proof: if an $\alpha_n \to \alpha$, then

$$T = \alpha + \operatorname{diag}(\alpha_0 - \alpha, \alpha_1 - \alpha, \alpha_2 - \alpha, \ldots),$$

and the diagonal summand is compact. Consequence: a diagonal operator is an asymptotic Toeplitz operator if and only if it belongs to the Toeplitz algebra.

Conclusion: $T \in \boldsymbol{T}$ if and only if $\{\alpha_n\}$ is convergent.

Here at last is a source of decisive examples: to get an operator that is in $\boldsymbol{E}$ but not in $\boldsymbol{T}$, just construct a sequence that does not converge but whose first differences tend to 0. That is easy, of course; form a sequence that oscillates between 0 and 1 more and more slowly. Concrete example:

$$0, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}, 0, \ldots$$

Example (1.1.10)[1]: Is the adjoint of an asymptotic Toeplitz operator another one? No, not necessarily.

Consider an isometry S defined on $\boldsymbol{H}^2$ by $Se_n = e_{2n}, n = 0,1,2,\ldots$, and write $T = S^*$. It follows that $Te_{2n} = e_n$ and $Te_{2n+1} = 0, n = 0,1,2, \ldots$ Consequence: for each k, the result of applying the "far southeast corner" $U^{*n}TU^n$ to e_k results in the zero vector. Precisely, $U^{*n}TU^n e_k = 0$ as soon as $n > k$. Conclusion: $U^{*n}TU^n \to 0$ (strong), so that T is an asymptotic Toeplitz operator. The adjoint $T^*(= S)$ is not.

Reason: $U^{*n}SU^n e_0 = U^{*n}Se_n = U^{*n}e_{2n} = e_n$, and the sequence $\{e_n\}$ is not strongly convergent.

Example (1.1.11)[1]: Is the product of two asymptotic Toephtz operators another one? No, not necessarily. An example can be obtained by modifying Example (1.1.9); the first such modification was suggested by C. Foiaş.

Let S_k be the square matrix of size $2k$ defined as follows: all entries are 0 except the first k in the last row, and they are equal to $1/\sqrt{k}(k = 1,2,3,\ldots)$. Let S be the operator whose matrix is the direct sum of all the S_k's, and let T be the adjoint of S.

Since $Se_n \to 0$ as $n \to 0$, it follows that $\|SU^n e_k\| = \|Se_{n+k}\| \to 0$ as $n \to \infty$, and hence that $\|U^{*n}SU^n e_k\| \to 0$ as $n \to \infty$ (for each k). This in turn implies that S is an asymptotic Toeplitz operator (with $\sigma(S) = 0$). So far the exact sizes of the boxes S_k are irrelevant.

Consider next the matrix of the operator T. Since the only non-zero entry of S_1^* is in the first row of S_1^*, it follows that both T and U^*TU begin with a column of 0's, and, in fact, so does $U^{*n}TU^n$ whenever $n \geqslant 0$. Since the only non-zero entries of S_2^* are in the first two rows of S_2^* it follows that $U^{*2}TU^2$ begins with two columns of 0's, and so does $U^{*n}TU^n$ whenever $n \geqslant 2$. Inductively: $U^{*n}TU^n$ begins with k columns of 0's whenever $n \geqslant k(k+1)$. Consequence: $U^{*n}TU^n e_k = 0$ as soon as $n \geqslant k(k+1)$ (usually sooner—the estimates are generous), so that T is an asymptotic Toeplitz operator.

The product ST is not an asymptotic Toeplitz operator. Reason: the diagonal entries of ST are 0 most of the time, but 1 infinitely often. This implies that $U^{*n}(ST)U^n e_0 = 0$ most of the time but e_0 infinitely often, and, consequently, that the sequence $\{U^{*n}(ST)U^n\}$ is not strongly convergent.

Example (1.1.12)[1]: Typically a projection has a diagonal matrix with diagonal entries equal to 0 or 1. Such a matrix can correspond to an asymptotic Toeplitz operator only if its rank is finite or cofinite. Are there any other asymptotic Toeplitz projections?

Yes, there are. If $\boldsymbol{M}$ is a subspace of $\boldsymbol{H}^2$ invariant under U, then the projection from $\boldsymbol{H}^2$ onto $\boldsymbol{M}$ is in the Toeplitz algebra. Reason: by Beurhng's theorem [6, Problem 125] there exists an inner function φ such that $\boldsymbol{M} = \operatorname{ran} T_\varphi$, it follows that the projection in question is the product $T_\varphi T_\varphi^*$. (This observation is due to Sheldon Axler.)

There are asymptotic Toeplitz projections that do not seem to arise in the natural ways described in the preceding two paragraphs. Here is one. Let T_k be the matrix of size k all whose entries are equal to $\frac{1}{k}$, and form the matrix

$$\begin{bmatrix} T_1 & & & & & \\ & 0 & & & & \\ & & T_2 & & & \\ & & & 0 & & \\ & & & & T_3 & \\ & & & & & \ddots \end{bmatrix}$$

that is the direct sum of the sequence obtained by interlacing a sequence of 0's (of size 1) with the T_k's. Clearly the operator T with that matrix is a projection.

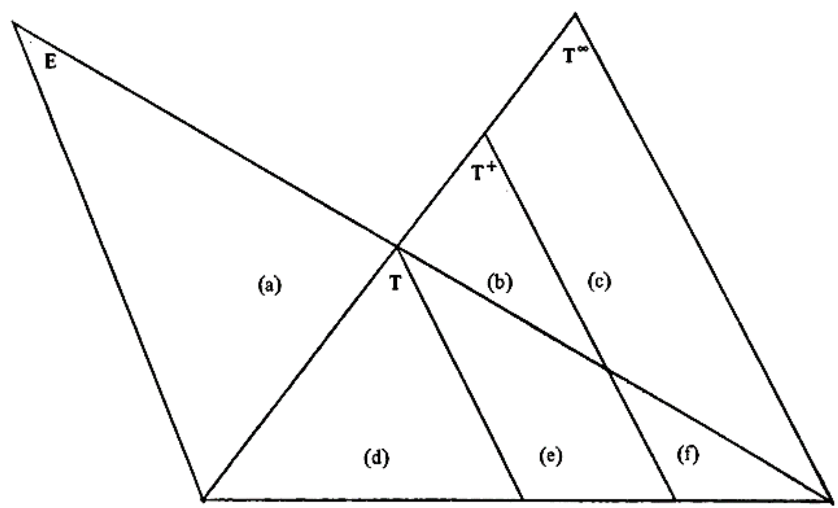
Assertion: it is an asymptotic Toeplitz projection, with $\sigma(T) = 0$. Reason: if the integer n is such that the nth column of T contains the first column of T_k, then $\|Te_n\| = \sqrt{k/k^2} = 1\sqrt{k}$; for all larger n, the norm $\|Te_n\|$ is even smaller.

The reason the 0's were inserted into T was to make it easier to compute $U^*TU - T$. The computation has no virtues other than being easy to carry out. The result is that $U^*TU - T$ is block diagonal, and that the Hilbert-Schmidt norm of the nth block is of the order $1/\sqrt{n}$. Conclusion: $U^*TU - T$ is not a Hilbert-Schmidt operator, but it is at least compact, and therefore $T \in \boldsymbol{E}$. Does T belong to $\boldsymbol{T}$ or to $\boldsymbol{T}^+$? Nobody knows.

Questions (1.1.13)[1]: Two unsolved test problems have been posed already (see Examples (1.1.9) and (1.1.2)); each of them asks whether a certain operator belongs to $\boldsymbol{T}$. That seems to be the crux of the matter in much of this subject. The important question is not "what is $\boldsymbol{E}$?" but "what is $\boldsymbol{T}$?". There is, after all, a way to decide whether or not an operator T belongs to $\boldsymbol{E}$; just form $U^*TU - T$ and see whether it is compact. It's debatable whether this should be called an algorithm, but not even anything as good as that is known for $\boldsymbol{T}$. The Hilbert matrix yields essentially the only non-trivial known example of an operator in $\boldsymbol{T}$; all others are either in $\boldsymbol{T}_0$, or compact, or both.

Other non-trivial examples are easy enough to construct (e.g. non-trivial continuous functions of operators in $\boldsymbol{T}_0$), but the experts seem to agree that the algebra $\boldsymbol{T}$ is far from well understood. The four questions below are special cases or reformulations of the general problem of characterizing the Toeplitz algebra.

The important classes discussed above are: the essential commutant $\boldsymbol{E}$, the Toeplitz algebra $\boldsymbol{T}$, the Hankel algebra $\boldsymbol{T}^+$, and the set $\boldsymbol{T}^\infty$ of all asymptotic Toeplitz operators. The inclusion relations among them can be summarized by the Venn diagram blow.



Operators corresponding to four of the indicated regions are known to exist; namely, (a) Example (1.1.9), (b) Example (1.1.7), (c) Example (1.1.10), and, for (d), any Toeplitz operator. Till now, however, no operators have been proved to belong to the classes (e) and (f).

Questions (1.1.14)[1]: Is there an operator in $\boldsymbol{E} \cap \boldsymbol{T}^+$ that is not in $\boldsymbol{T}$?

Questions (1.1.15)[1]: Is there an operator in $\boldsymbol{E} \cap \boldsymbol{T}^\infty$ that is not in $\boldsymbol{T}^+$?

For each operator T in the Toeplitz algebra, consider the difference $U^*TU - T$, and let $\boldsymbol{D}$ be the set of all such differences. Since $\boldsymbol{T} \subset \boldsymbol{E}$, it follows that $\boldsymbol{D} \subset \boldsymbol{K}$.

Questions (1.1.16)[1]: Which compact operators belong to $\boldsymbol{D}$?

The reason the question is interesting is that it is a reformulation of the question "which operators belong to $\boldsymbol{T}$?". That is, the set $\boldsymbol{D}$ characterizes $\boldsymbol{T}$. More clearly said, an operator S belongs to $\boldsymbol{T}$ if and only if $U^*SU - S$ belongs to $\boldsymbol{D}$. Indeed, if $S \in \boldsymbol{T}$, then $U^*SU - S \in \boldsymbol{D}$ by definition. If, conversely, $U^*SU - S \in \boldsymbol{D}$, then, by definition, there exists an operator T in $\boldsymbol{T}$ such that $U^*SU - S = U^*TU - T$. It follows that $U^*(S - T)U = S - T$, hence that $S - T$ is a Toephtz operator, and hence that $S - T \in \boldsymbol{T}$. Conclusion: $S \in \boldsymbol{T}$.

Example (1.1.12) describes a projection in $\boldsymbol{T}^\infty$, and asks if it is in $\boldsymbol{T}$. It would be good to know the facts in the general case.

Questions (1.1.17)[1]: Which projections belong to $\boldsymbol{T}$?

Problems frequently become more manageable, not less, if they are embedded in a suitable enlarged context. The last question to be raised here is vague; it isn't easy to formulate a crisp, yes-or-no subquestion, but it might give a hint to a suitably general context in which Toeplitz theory can be embedded.

Begin with the observation that Toephtz operators are the solutions of the equation $U^*XU = X$. This suggests consideration of the mapping Γ from operators to operators defined by

$$\Gamma(X) = U^*XU$$

Toeplitz operators are the "Eigen operators" of Γ corresponding to the eigenvalue 1.

Vague question: what is the spectral theory of Γ? What, in particular, can be said about eigenoperators T (generalized Toeplitz operators), $U^*TU = \lambda T$, corresponding to eigenvalues λ other than 1? What algebraic properties do they have, and what can be said about algebras generated by such operators?

Section (1.2): Commuting Toeplitz Operators:

For D be the unit disk in the complex plane $\mathbb{C}$. For a fixed positive integer n, the unit polydisk D^n is the cartesian product of n copies of D. Let $L^p = L^p(D^n)$ denote the usual Lebesgue space with respect to the volume measure V on D^n normalized to have total mass 1. The Bergman space A^2 is then the closed subspace of L^2 consisting of all holomorphic functions on D^n. Let P be the Bergman projection from L^2 onto A^2. For a function $u \in L^\infty$, the Toeplitz operator T_u with symbol u is defined by

$$T_u f = P(uf)$$

for $f \in A^2$. It is clear that $T_u: A^2 \to A^2$ is a bounded linear operator.

We consider the problem of when two Toeplitz operators with pluriharmonic symbols commute or essentially commute. Recall that a complexvalued function $u \in C^2(D^n)$ is said to be pluriharmonic if its restriction to an arbitrary complex line that intersects D^n is harmonic as a function of single complex variable. So, the notions of harmonicity and pluriharmonicity coincide on D. It turns out that every pluriharmonic function on D^n can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function. See Chapter 2 of [15] for details. Also, recall that two bounded linear operators S_1, S_2 on a Hilbert space X are said to be essentially commuting on X if the commutator $S_1S_2 - S_2S_1$ is compact on X.

The problem of characterizing commuting Topelitz operators has been studied on various settings. Axler and Cuckovic [9] first obtained a complete description of harmonic symbols of commuting Toeplitz operators on D: If two Toeplitz operators with harmonic symbols commute, then either both symbols are holomorphic, or both symbols are antiholomorphic, or a nontrivial linear combination of the symbols is constant (the converse implication is also true and trivial). Later, some extensions of this characterization were obtained on higher-dimensional balls as in [11], [17] or [25]. Also, the same problem was considered on the annulus [14] and for more general symbols [10]. For related results on the (pluri)harmonic Bergman space, see [13] and [18].

For essentially commuting Toeplitz operators, Stroetho [21] obtained characterizations of harmonic symbols on D. Choe and Lee [12] extended the result of Stroetho to pluriharmonic symbols on the ball. On the other hand, the polydisk case was studied by Sun and Zheng [22]. However, Sun and Zheng considered holomorphic or antiholomorphic symbols only. They proved that given $f, g \in H^\infty$, the following three conditions are equivalent for $n > 1$: (i) T_f and $T_{\bar{g}}$ are commuting, (ii) T_f and $T_{\bar{g}}$ are essentially commuting, (iii) for each j, either $\partial_j f = 0$ or $\partial_j g = 0$. Here, H^∞ denotes the class of bounded holomorphic functions on D^n and ∂_j denotes the partial differential operator with respect to the j-th variable.

Our results obtained characterizations of general pluriharmonic symbols of commuting or essentially commuting Toeplitz operators. For $n \geq 3$, as in the result of Sun and Zheng mentioned above, our results show that the commuting property and the essential commuting property are the same for Toeplitz operators with pluriharmonic symbols. However, they are different for $n = 2$. Our method, whose main idea is adapted from [12], is entirely different from that of Sun and Zheng.

Following [19], we say that a complex-valued function $u \in C^2(D^n)$ is n −harmonic if u is harmonic in each variable separately. More explicitly, u is n −harmonic if

$$\partial_j \bar{\partial}_j u = 0, \quad j = 1,2, \dots, n.$$

For a characterization of pluriharmonic symbols of commuting Toeplitz operators, we have the following. In what follows $H(D^n)$ denotes the class of all holomorphic functions on D^n.

In addition, we obtain characterizations of functions $f, g, h, k \in H(D^n)$ for which $f\bar{k} - h\bar{g}$ is n −harmonic. Before stating our result.

Let $I = \{1,2, \dots, n\}$. For $J \subset I$, we write $H(J)$ for the set of all holomorphic functions independent of variables z_j with $j \in I \backslash J$. Also, for $J_1 \subset J_2 \subset I$, we write $H(J_2) = H(J_1)$ for the set of all holomorphic functions in $H(J_2)$ whose power series (at the origin) do not contain any nonzero terms in $H(J_1)$.

Our next result is the essential version of Theorem (1.2.14). To state it, we need some more notation. First, we let

$$\widetilde{\Delta}_j u(z) = \left(1 - |z_j|^2\right) \partial_j \bar{\partial}_j u(z)$$

for $j = 1, \dots, n$ and $u \in C^2(D^n)$. Here and elsewhere, z_j denotes the j −th component of $z \in D^n$. Note that u is n −harmonic if and only if u is annihilated by all $\widetilde{\Delta}_j$.
Thus, we will say that u is boundary n −harmonic if $\lim_{a \to \partial D^n} \widetilde{\Delta}_j u(a) = 0$ for all j.

Here, ∂D^n denotes the topological boundary of D^n. Also, we let Φ denote a class of functions related to the maximal ideal space of H^∞.

Theorem (1.2.1)[8]: Let $u, v \in L^\infty$ be pluriharmonic symbols and assume $u = f + \bar{g}, v = h + \bar{k}$ for some $f, g, h, k \in H(D^n)$. Then the following statements are equivalent:

(a) T_u and T_v are essentially commuting on A^2.

(b) $T_{u \circ \varphi} T_{v \circ \varphi} = T_{v \circ \varphi} T_{u \circ \varphi}$ on A^2 for every $\varphi \in \Phi$.

(c) $f\bar{k} - h\bar{g}$ is boundary n −harmonic.

Finally, only for $n \geq 3$, we show that the commuting property and the essential commuting property of Toeplitz operators with pluriharmonic symbols are equivalent. This will follow from Theorem (1.2.11) which asserts that the n −harmonicity and the boundary n −harmonicity are equivalent for functions of the form $f\bar{k} - h\bar{g}$ under consideration.

Theorem (1.2.2)[8]: $(n \geq 3)$. Let $u, v \in L^\infty$ be pluriharmonic symbols. Then the following statements are equivalent:

(a) $T_u T_v = T_v T_u$ on A^2.

(b) T_u and T_v are essentially commuting on A^2.

We arranged as follows. We collect basic materials which we need. We prove Theorem (1.2.8). Also, we show that the n −harmonicity and the boundary n −harmonicity of functions of certain forms are equivalent for $n \geq 3$. We prove Theorem (1.2.14). As an application we obtain a characterization of normal Toeplitz operators with pluriharmonic symbols.

We prove Theorem (1.2.1). As a consequence we obtain Theorem (1.2.2). As an application we obtain a characterization of essentially normal Toeplitz operators with pluriharmonic symbols. As another application we recover the result of Sun and Zheng [22] mentioned above. We modify our arguments used in previous to obtain (essentially) semi-commuting Toeplitz operators with pluriharmonic symbols. It turns out that the semi-commuting property and the essential semi-commuting property are equivalent for $n \geq 2$.

We collect several basic facts which we need.

Since every point evaluation is a bounded linear functional on A^2, there corresponds to every $a \in D^n$ a unique function $K_a \in A^2$ which has the following reproducing property:

$$f(a) = \langle f, K_a \rangle, \quad f \in A^2, \tag{3}$$

where the notation $\langle , \rangle$ denotes the inner product in L^2 with respect to the measure V. The function K_a is the well-known Bergman kernel and its explicit formula is given by

$$K_a(z) = \prod_{j=1}^{n} \frac{1}{\left(1 - \bar{a}_j z_j\right)^2}, \quad z, a \in D^n.$$

The Bergman projection P is the orthogonal projection from L^2 onto A^2. Thus, by the reproducing property (3), the projection P can be represented by

$$P\psi(a) = \int_{D^n} \psi \bar{K}_a \, dV, \quad a \in D^n,$$

for functions $\psi \in L^2$. It follows that P naturally extends via the above formula to an integral operator from L^1 into $H(D^n)$. Moreover, we have $Pf = f$ for functions $f \in A^1$. Here, $A^p = L^p \cap H(D^n)$. Also, it is well known that $P: L^p \to A^p$ is bounded for $p > 1$. See, for example, Theorem 4.2.3 of [26] for details on the disk.

The same proof works on D^n.

For each $a = (a_1, \dots, a_n) \in D^n$, we let $\varphi_a(z) = \left(\varphi_{a_1}(z_1), \dots, \varphi_{a_n}(z_n)\right)$, where each φ_{a_i} is the usual Mobius map on D given by

$$\varphi_{a_i}(z_i) = \frac{a_i - z_i}{1 - \bar{a}_i z_i}, \quad z_i \in D.$$

Then $\varphi_a \in Aut(D^m)$, the set of all automorphisms of D^n. Moreover, $\varphi_a \circ \varphi_a$is the identity on D^n. Now, it is clear that if u is n −harmonic, then so is $u \circ \varphi_a$ for each $a \in D^n$. Therefore, every n −harmonic function $u \in L^1$ satisfies the invariant mean value property

$$\int_{D^n} (u \circ \varphi_a) dV = u(a), \quad a \in D^n.$$

However, the converse of the invariant mean value property is known to hold only for $n = 1$. See [16]. The converse turns out to be true in general with a certain additional hypothesis. To state it, we associate with each $u \in L^1$ its so-called radialization $\mathcal{R}u$ defined by

$$\mathcal{R}u(z) = \int_{T^n} u(z_1\zeta_1, \dots, z_n\zeta_n) d\sigma(\zeta_1, \dots, \zeta_n)$$

for $z = (z_1, \dots, z_n) \in D^n$. Here and elsewhere, T^n denotes the cartesian product of n copies of the unit circle T and $\sigma = \sigma_n$ is the normalized Haar measure on T^n.

The following is taken from Corollary 3.7 of [16].

Proposition (1.2.3)[8]: Let $u \in L^1$. Then u is n −harmonic on D^n if and only if

$$\int_{D^n} (u \circ \varphi_a)\, dV = u(a)$$

and $\mathcal{R}(u \circ \varphi_a) \in L^\infty$ for every $a \in D^n$.

We let k_a denote the normalized kernel, namely,

$$k_a = K_a \prod_{j=1}^{n} \left(1 - |a_j|^2\right).$$

First, we mention that the set $\{k_a : a \in D^n\}$ spans a dense subset of A^2, because its orthogonal complement is $\{0\}$ by (3). Next, since the real Jacobian of φ_a is given by $|k_a|^2$, we have a change-of-variable formula,

$$\int_{D^n} (h \circ \varphi_a) dV = \int_{D^n} h|k_a|^2 dV\, , a \in D^n, \tag{4}$$

whenever the integrals make sense. In particular, we have by the mean value property

$$\int_{D^n} f|k_a|^2 dV = f(a), \quad a \in D^n, \tag{5}$$

for functions $f \in A^1$.

Given $p > 0$, the Hardy space $H^p = H^p(D^n)$ is the space of all $f \in H(D^n)$ for which

$$\|f\|_{H^p}^p = \sup_{0 \le r < 1} \int_{T^n} |f(r\zeta)|^p\, d\sigma(\zeta) < \infty.$$

By an integration in polar coordinates using n −subharmonicity, we have $H^p \subset A^p$.

It is well known that if $f \in H^p$, then $f(\zeta) = \lim_{r \to 1} f(r\zeta)$ exists at almost all points $\zeta \in T^n$. Moreover, we have $\log|f| \in L^1(T^n)$ for any nontrivial $f \in H^p$. In particular, if the boundary function of $f \in H^p$ vanishes on a set of positive measure in T^n, then f itself must be identically 0 on D^n. See Theorem 3.4.2 of [19].

From the above definition, one can easily verify

$$\mathcal{R}(f\bar{g}) \in L^\infty, \qquad f, g \in H^2. \tag{6}$$

Also, by using the L^p −boundedness of the Cauchy projection, one can easily verify the following.

Proposition (1.2.4)[8]: Let $f, g \in H(D^n)$ and assume $f + \bar{g} \in L^\infty$. Then we have $f, g \in H^p$ for all $p > 0$.

Let M be the maximal ideal space of H^∞ which is defined to be the set of all multiplicative linear functionals on H^∞. As is well known, the space M becomes a compact Hausdor space as a subset of the dual of H^∞ with weak-star topology. See Theorem 11.9 of [20] for details. Identifying $z \in D^n$ with the multiplicative evaluation functional $f \longmapsto f(z)$, we can regard D^n as a subset of M.

Given $z \in D^n$, since D^n is a subset of M, we can think of φ_z as a map from D^n to M. In other words, $\varphi_z \in M^{D^n}$. Equipped with product topology, the function space M^{D^n} is compact by Tychono's theorem. Hence, for any net $\{\varphi_{z_\alpha}\}$ of automorphisms, there is a subnet $\{\varphi_{z_\beta}\}$ of $\{\varphi_{z_\alpha}\}$ such that φ_{z_β} converges (pointwise) to a map $\varphi: D^n \to M$. Now, we let

$$\Phi = \text{closure}\{\varphi_z : z \in D^n\} \backslash \{\varphi_z : z \in D^n\}$$

where the closure is taken in M^{D^n}.

We will use a couple of basic facts concerning the maximal ideal space M and the class Φ. First, note that $H^\infty \subset C(M)$ via the Gelfand transform. For bounded pluriharmonic functions, we have the following.

Proposition (1.2.5)[8]: Each bounded pluriharmonic function on D^n extends to a continuous function on M.

We will use the same notation for a bounded pluriharmonic function and its continuous extension on M.

Proposition (1.2.6)[8]: If a net $\{\varphi_{z_\alpha}\}$ of automorphisms converges to some $\varphi \in \Phi$, then for any pluriharmonic function $u \in L^\infty$, the function $u \circ \varphi_{z_\alpha}$ converges to $u \circ \varphi$ uniformly on every compact subset of D^n. So, $u \circ \varphi \in L^\infty$ is also pluriharmonic on D^n.

The above two prsopositions are proved in [24] on the ball. The same proofs work on the poly disk and thus proofs are omitted.

We prove Theorem (1.2.8), which will play an essential role in the proof of Theorem (1.2.14). We begin with a simple lemma.

Lemma (1.2.7)[8]: Suppose J_0, J_1, J_2 are pairwise disjoint subsets of I. Let $p_j \in H(J_0 \cup J_j)/H(J_0)$ for $j = 1,2$ and assume $p_1 + p_2 = 0$. Then $p_1 = p_2 = 0$.

Proof. Let $z^j = (z_r)_{r \in J_j}$ for $j = 0,1,2$. Changing the coordinate system if necessary, we may write $p_j = p_j(z^0, z^j)$ for $j = 1,2$. Now, taking $z^2 = 0$, we have $p_1(z^0, z^1) = p_2(z^0, 0)$ and thus $p_1 \in \frac{H(J_0 \cup J_j)}{H(J_0)} \cap H(J_0) = \{0\}$. So, we have $p_1 = p_2 = 0$. The proof is complete.

We are now ready to prove Theorem (1.2.8).

Theorem (1.2.8)[8]: Let $f, g, h, k \in H(D^n)$. Then the following statements are equivalent:

(a) $f\bar{k} - h\bar{g}$ is n −harmonic.

(b) There are pairwise disjoint sets $I_0, \dots, I_m$ with $\bigcup_{j=0}^{m} I_j = I$ for some nonnegative integer $m \leq n$, functions $f_0, h_0, g_0, k_0, p_1, \dots, p_m, q_1, \dots, q_m$ holomorphic on D^n, and constants $\alpha_1, \dots, \alpha_m$ with the following properties:

(b1) $f_0, h_0, g_0, k_0 \in H(I_0)$ and $p_j, q_j \in H(I_0 \cup I_j)/H(I_0)$ for each j.

(b2) We have

$$f = f_0 + \sum_{j=1}^{m} p_j, \quad h = h_0 + \sum_{j=1}^{m} \alpha_j p_j,$$
$$g = g_0 + \sum_{j=1}^{m} q_j, \quad k = k_0 + \sum_{j=1}^{m} \bar{\alpha}_j q_j,$$

(b3) For each $r \in I_0$, one of the following four cases holds:

(i) $\partial_r f_0 = \partial_r h_0 = 0$ and $\partial_r p_j = 0$ for all j.

(ii) $\partial_r k_0 = \partial_r g_0 = 0$ and $\partial_r q_j = 0$ for all j.

(iii) $\partial_r f_0 = \partial_r g_0 = 0$ and $\partial_r p_j = \partial_r q_j = 0$ for all j.

(iv) $\partial_r k_0 = \partial_r h_0 = 0$ and $\partial_r p_j = \partial_r q_j = 0$ for all j.

(c) There are subsets $J_1, \dots, J_\ell$ of I for some integer $\ell \geq 1$, and holomorphic functions $A_1, \dots, A_\ell, B_1, \dots, B_\ell$ with $A_i \in H(J_i)$ and $B_i \in H(I \backslash J_i)$ for each i such that

$$f\bar{k} - h\bar{g} = \sum_{i=1}^{\ell} A_i \bar{B}_i$$

Proof. First suppose (a) and show (b). So, assume that the function $f\bar{k} - h\bar{g}$ is n −harmonic. Then, for each $r \in I$, we have $\partial_r \bar{\partial}_r (f\bar{k}) - \partial_r \bar{\partial}_r (h\bar{g})$ and thus

$$(\partial_r f)(\overline{\partial_r k}) = (\partial_r h)(\overline{\partial_r g}). \tag{7}$$

Let I_0 be the set of all $r \in I$ with $(\partial_r f)(\partial_r h)(\partial_r g)(\partial_r k) = 0$. Then, for each $r \notin I_0$, we have

$$\frac{\partial_r h}{\partial_r f} = \overline{\left(\frac{\partial_r k}{\partial_r g}\right)}$$

and therefore there exists a constant $\beta_r \neq 0$ such that

$$\partial_r h = \beta_r \partial_r f, \quad \partial_r k = \overline{\beta}_r \partial_r g.$$

Now, define an equivalence relation on $I \backslash I_0$ by $r \sim s$ if and only if $\beta_r = \beta_s$ and let $I_1, \dots, I_m$ be the equivalence classes induced by $\sim$. It follows that there are nonzero constants $1, \dots, m$ such that

$$\partial_r h = \alpha_j \partial_r f, \quad \partial_r k = \overline{\alpha}_j \partial_r g, \quad r \in I_j, \tag{8}$$

for each $j \geq 1$.

Note that, for $r \in I_i, s \in I_j$ with $i, j \geq 1$ and $i \neq j$, we have

$$(\alpha_i - \alpha_j)\partial_r \partial_s f = \alpha_i \partial_r \partial_s f - \alpha_j \partial_r \partial_s f = \partial_s \partial_r h - \partial_r \partial_s h = 0$$

and thus $\partial_r \partial_s f = 0$. This means that the power series of f cannot contain any terms involving both z_r and z_s with $r \in I_i, s \in I_j$ whenever $i, j \geq 1$ and $i \neq j$.

Similarly, the same is true for g. Also, since α_j's are nonzero, the same holds for h and k. Therefore, we may decompose functions f, g, h and k as

$$f=\sum_{j=0}^{m} f_j\,,\quad g=\sum_{j=0}^{m} g_j\,,\quad h=\sum_{j=0}^{m} h_j\,,\quad k=\sum_{j=0}^{m} k_j,$$

where $f_0, g_0, h_0, k_0 \in H(I_0)$ and $f_j, g_j, h_j, k_j \in H(I_0 \cup I_j)$ for $j \geq 1$. Since we have $f_0, g_0, h_0, k_0 \in H(I_0)$, we may further assume $f_j, g_j, h_j, k_j \in H(I_0 \cup I_j)/H(I_0)$ for $j \geq 1$.

Now, we prove (b1) and (b2). Fix $j \geq 1$. Note that we have by(8)

$$\partial_r(h_j - \alpha_j f_j) = \partial_r(h - \alpha_j f) = 0$$

for all $r \in I_j$. It follows that $h_j - \alpha_j f_j \in H(I_0 \cup I_j) = H(I_0) \cap H(I_0) = \{0\}$ and thus $h_j = \alpha_j f_j$. Similarly, we have $k_j = \overline{\alpha}_j g_j$. Thus, (b1) and (b2) hold with $p_j = f_j$ and $q_j = g_j$ for $j \geq 1$.

Finally, we prove (b3). Let $r \in I_0$. Then by(7), one of the following four cases should occur:

(i)$'$ $\partial_r f = \partial_r h = 0,$ (ii)$'$ $\partial_r k = \partial_r g = 0,$
(iii)$'$ $\partial_r f = \partial_r g = 0,$ (iv)$'$ $\partial_r k = \partial_r h = 0.$

In the case (i)$'$, we have by (b2)

$$\partial_r f_0 + \sum_{j=1}^{m} \partial_r p_j = \partial_r h_0 + \sum_{j=1}^{m} \alpha_j \partial_r p_j = 0.$$

Note that, for each j, we have $\partial_r p_j \in H(I_0 \cup I_j)/H(I_0)$, because $p_j \in H(I_0 \cup I_j)/H(I_0)$. Also, we have $\partial_r f_0, \partial_r h_0 \in H(I_0)$. Thus, by repeated applications of Lemma (1.2.7), we conclude $\partial_r f_0 = \partial_r h_0 = 0$ and $\partial_r p_j = 0$ for each j, which is just the case (i). Similarly, the remaining cases (ii)$'$, (iii)$'$ and (iv)$'$ correspond to the cases (ii), (iii), and (iv), respectively. Therefore, we have (b).

Now, suppose (b) and show (c). By (b3), given $r \in I_0$, we have either $\partial_r f_0 = \partial_r p_j = 0$ for all j or $\partial_r k_0 = \partial_r q_j = 0$ for all j. This means that we can decompose $I_0 = J_1 \cup J_2$ where $J_1 \cap J_2 = \emptyset$ such that $f_0 \in H(J_1), k_0 \in H(J_2), p_j \in H(J_1 \cup I_j), q_j \in H(J_2 \cup I_j)$ for all j. Since

$$f\bar{k} - \sum_{j=1}^{m} \alpha_j p_j \bar{q}_j = f_k \bar{k}_0 + \sum_{j=1}^{m} \alpha_j f_0 \bar{q}_j + \sum_{j=1}^{m} p_j \bar{k}_0 + \sum_{i \neq j}^{m} \alpha_j p_j \bar{q}_j$$

by (b2), we see the function $f\bar{k} - \sum \alpha_j p_j \bar{q}_j$ can be written as a finite sum of functions of desired form. Also, a similar argument shows that the same is true for $h\bar{g} - \sum \alpha_j p_j \bar{q}_j$. Hence, we conclude (c).

Finally, it is trivial that (c) implies (a). The proof is complete.

As a special case of Theorem (1.2.8), we have the following consequence.

Corollary (1.2.8)[8]: Let $f, g \in H(D^n)$. Then the following statements are equivalent:

(a) $|f|^2 - |g|^2$ is $n-$harmonic.

(b) There are pairwise disjoint sets $I_1, \dots, I_m$ with $\cup_{j=1}^{m} I_j = I$ for some positive integer $m \leq n$, functions $p_1, \dots, p_m$ with $p_j \in H(I_j)$ for each j, and unimodular constants $\alpha_1, \dots, \alpha_m$ such that

$$f=\sum_{j=1}^{m} p_j\,,\quad g=\sum_{j=1}^{m} \alpha_j p_j+\lambda$$

for some constant λ.

Proof. By Theorem (1.2.8) with $h=g$ and $k=f$, we see that $|f|^2-|g|^2$ is n −harmonic if and only if there are pairwise disjoint sets $I_1,\dots,I_m$ with $\bigcup_{j=1}^{m} I_j=I$ for some nonnegative integer $m\le n$, functions $f_0,g_0\in H(I_0)$, constants $\alpha_1,\dots,\alpha_m$, and $p_1,\dots,p_m$ with $p_j\in H(I_0\cup I_j)/H(I_0)$ for each j, such that

$$f=f_0+\sum_{j=1}^{m} p_j\,,\quad g=g_0+\sum_{j=1}^{m} \alpha_j p_j$$

and

$$|\alpha_j|^2 p_j=p_j,\quad \partial_r f_0=\partial_r g_0=\partial_r p_j=0$$

for all $r\in I_0$ and $j\ge 1$. By the first equation of the above, we may take $|\alpha_j|=1$.

By the second equation, we see that functions f_0,g_0 are constant. Thus, we may take $f_0=0, I_0=\emptyset$ and $p_j\in H(I_j)$ for all $j\ge 1$. The proof is complete.

For functions of the form $f\bar{k}-h\bar{g}$ with a certain regularity, the n −harmonicity and the boundary n −harmonicity turn out to be equivalent in the case $n\ge 3$, while they are different for $n=2$. In order to see this, we need the following lemma which might be known. A proof is included here for completeness.

Lemma (1.2.9)[8]: Let m,ℓ be integers with $1\le m,\ell\ge 0$ and assume $f\in H^2$. Then, $\partial_1^\ell f(z,\cdot)\in H^2(D^{n-m})$ for each $z\in D^m$. Furthermore, there exists a set $E\subset T^{n-m}$ with $\sigma_{n-m}(E)=1$ with the following properties:

(a) $\partial_1^\ell f(z,\eta)=\lim_{r\to 1}\partial_1^\ell f(z,r\eta)$ exists for each $z\in D^m$ and $\eta\in E$.

(b) The function $\partial_1^\ell f(\cdot,\eta)$ is holomorphic on D^m for each $\eta\in E$.

In the proof below we will use well-known facts about maximal functions. For a measurable function ψ on D^m, let $N\psi$ be the nontangential maximal function of with respect to nontangential approach region of a fixed aperture. Also, given $u\in L^1(T^{n-m})$, let Mu be the Hardy-Littlewood maximal function of u. As is well-known, the operator M is bounded on $L^2(T^{n-m})$. Also, it is well known that if is the Poisson integral of some $u\in L^1(T^{n-m})$, then $N\psi\le CMu$ for some constant C independent of u.

Proof. Let $z=(z^1,\dots,z^m)\in D^m$ and $\eta\in T^{n-m}$. Let $\max_{1\le j\le m}|z_j|<t<1$..

Then, for arbitrary $0<r<1$, we have by the Cauchy integral formula

$$\partial_1^\ell f(z,r\eta)=\ell!\int_{T^m}\frac{f(t\zeta,r\eta)}{(1-t^{-1}z_1\bar{\zeta}_1)^\ell\prod_{j=1}^{m}(1-t^{-1}z_j\bar{\zeta}_j)}(t^{-1}\bar{\zeta}_1)^\ell d\sigma_m(\zeta)$$

And therefore

$$\int_{T^{n-m}}\left|\partial_1^\ell f(z,r\eta)\right|^2 d\sigma_{n-m}(\eta)$$

$$\leq \frac{C_\ell}{(t-|z_1|)^{2\ell}\prod_{j=1}^m (t-|z_j|)^2} \int_{T^{n-m}}\int_{T^m} |f(t\zeta, r\eta)|^2 d\sigma_m(\zeta) d\sigma_{n-m}(\eta)$$

$$\leq \frac{C_\ell}{(t-|z_1|)^{2\ell}\prod_{j=1}^m (t-|z_j|)^2} \|f\|_{H^2}^2.$$

Hence, $\partial_1^\ell f(z,\cdot) \in H^2(D^{n-m})$.

Now, pick a sequence of positive numbers $\{t_j\}$ increasing to 1 and let $K_j = \{t_j z : z \in \bar{D}^m\}$ for $j \geq 1$. Fix j and $t_j < t < 1$. Then the above estimate shows

$$\left|\partial_1^\ell f(z, rw)\right|^2 \leq C_j \int_{T^m} |f(t\zeta, rw)|^2 d\sigma_m(\zeta) \tag{9}$$

for all $z \in K_j, 0 < r \leq 1$ and $w \in D^{n-m}$. Here and in what follows, the letter $C_j = C_j(t)$ denotes various constants independent of f, z, r and w. Let $z \in D^m$ and $0 < r < 1$. Then, it follows from(9) that

$$\left(\sup_{z\in K_j} \mathcal{N}\partial_1^\ell f_{z,r}(\eta)\right)^2 \leq C_j \int_{T^m} \mathcal{N} f_{t\zeta,r}(\eta)^2 d\sigma_m(\zeta)\,, \quad \eta \in T^{n-m},$$

where we use the notation $h_{z,r}(w) = h(z, rw)$ for holomorphic functions h on D^n.

Integrating both sides of the above on T^{n-m}, we have

$$\int_{T^{n-m}} \left(\sup_{z\in K_j} \mathcal{N}\partial_1^\ell f_{z,r}(\eta)\right)^2 d\sigma_{n-m}(\eta)$$

$$\leq C_j \int_{T^m}\int_{T^{n-m}} N f_{t\zeta,r}(\eta)^2 d\sigma_{n-m}(\eta) d\sigma_m(\zeta)$$

$$\leq C_j \int_{T^m}\int_{T^{n-m}} M f_{t\zeta,r}(\eta)^2 d\sigma_{n-m}(\eta) d\sigma_m(\zeta)$$

$$\leq C_j \int_{T^m}\int_{T^{n-m}} \left|f_{t\zeta,r}(\eta)\right|^2 d\sigma_{n-m}(\eta) d\sigma_m(\zeta)$$

$$= C_j \int_{T^m}\int_{T^{n-m}} |f(t\zeta, r\eta)|^2 d\sigma_{n-m}(\eta) d\sigma_m(\zeta)$$

$$\leq C_j \|f\|_{H^2}^2.$$

Thus, by Fatou's lemma, we have

$$\int_{T^{n-m}} \left(\sup_{z\in K_j} \mathcal{N}\partial_1^\ell f_{z,1}(\eta)\right)^2 d\sigma_{n-m}(\eta) \leq C_j \|f\|_{H^2}^2 \tag{10}$$

Having the above inequality, one may now follow the well-known proof of Fatou's theorem to conclude that there exists a set $E_j \subset T^{n-m}$ with $\sigma_{n-m}(E_j) = 1$ such that nontangential limits of $\partial_1^\ell f(z,\cdot)$ exist at all points in E_j for each $z \in K_j$. Let $E = \bigcap_{j=1}^\infty E_j$. Then we still have $\sigma_{n-m}(E) = 1$ and nontangential limits of $\partial_1^\ell f(z,\cdot)$ exist at all points in E for each $z \in D^m$. This proves (a).

Note that (10) yields

$$\sup_{z\in K_j} \mathcal{N}\partial_1^\ell f_{z,1}(\eta) < \infty$$
for almost all points η in T^{n-m}. We may assume the above holds for $\eta \in E_j$. Thus, given a compact set $K \subset D^m$, we have
$$\sup_{z\in K} \mathcal{N}\partial_1^\ell f_{z,1}(\eta) < \infty$$
for each $\eta \in E$. In particular, given $\eta \in E$, we see that functions $\partial_1^\ell f(\cdot, r\eta)$ form a normal family and thus (b) holds. The proof is complete.

The following is taken from Lemma 9 of [11].

Lemma (1.2.10)[8]: Let Ω be a given connected open subset of $\mathbb{C}^n$. If A_i and $B_i (1 \le i \le \ell)$ are holomorphic functions such that $\sum_{i=1}^{\ell} A_i \overline{B}_i = 0$ on Ω, then $\sum_{i=1}^{\ell} A_i(z)\overline{B}_i(w) = 0$ for all $z, w \in \Omega$.

Now, we prove the following theorem, which does not extend to $n \le 2$. For $n = 1$, it is not hard to find counterexamples. For $n = 2$, we have a counterexample:
$$f = -g = (1 - z_1)(1 - z_2), \quad h = k = (1 + z_1)(1 + z_2). \tag{11}$$
Theorem (1.2.11)[8]: $(n \ge 3)$. Let $f, h, k, g \in H^2$. Then the following statements are equivalent:

(a) $f\overline{k} - h\overline{g}$ is n −harmonic.

(b) $f\overline{k} - h\overline{g}$ is boundary n −harmonic.

Proof. The implication (a) $\Longrightarrow$ (b) is trivial. We prove (b) $\Longrightarrow$ (a). So, assume (b).

By symmetry we only need to prove
$$(\partial_1 f)\left(\overline{\partial_1 k}\right) = (\partial_1 h)\left(\overline{\partial_1 g}\right). \tag{12}$$
First, let us introduce some notation. For simplicity, put $F = \partial_1 f, G = \partial_1 g, H = \partial_1 h$ and $K = \partial_1 k$. Then, by Lemma (1.2.9), there exists a set $E \subset T$ with $\sigma_1(E) = 1$ such that, given $\eta \in E$, the functions $F(\cdot,\eta), G(\cdot,\eta), H(\cdot,\eta)$ and $K(\cdot,\eta)$ are holomorphic on D^{n-1}. Also, we may assume that, given $\zeta \in E$, the functions $F(.,\zeta,\cdot), G(\cdot,\zeta,\cdot), H(\cdot,\zeta,\cdot)$ and $K(\cdot,\zeta,\cdot)$ are holomorphic on $D^{n-1} = D^{n-2} \times D$.

Now, since we have
$$\lim_{a\to\partial D^n} \widetilde{\Delta}_1\left(f\bar{k} - h\bar{g}\right)(a) = 0$$
by assumption, it follows that
$$\lim_{t\to 1}(F\overline{K} - H\bar{G})(z, t\eta) = 0$$
for all $z \in D^{n-1}$ and $\eta \in T$. In particular, we obtain
$$(F\overline{K} - H\bar{G})(z,\eta) = 0$$
for all $z \in D^{n-1}$ and $\eta \in E$. Thus, we have by Lemma (1.2.10)
$$F(z,\eta)\overline{K(w,\eta)} = H(z,\eta)\overline{G(w,\eta)} \tag{13}$$
for all $z, w \in D^{n-1}$ and $\eta \in E$.

Let E_f be the set of all $\eta \in E$ such that $F(\cdot,\eta) = 0$. Define the sets E_g, E_h and E_k in a similar way. First, consider the case where one of the sets E_f, E_g, E_h and E_k is of positive σ_1 −measure. Without loss of generality, assume $\sigma_1\left(E_f\right) > 0$. Let $z \in D^{n-1}$. Note that $F(z,\cdot) \in H^2(D)$ by Lemma (1.2.9). Since the boundary function of $F(z,\cdot)$ vanishes on a set of positive σ_1 −measure, we have $F(z,\cdot) = 0$ on D. It follows that $F = 0$ on D^n. Thus, we see from (13) that

$$H(z,\eta)G(z,\eta) = 0$$
for all $z \in D^{n-1}$ and $\eta \in E$. This means that, for each $z \in D^{n-1}$, the boundary function of $H(z,\cdot)G(z,\cdot) \in H^1(D)$ vanishes on E. Hence, $HG = 0$ on D^n. So, we have (12).

Now, assume that all the sets E_f, E_g, E_h and E_k are of σ_1 −measure 0. We may further assume that, for each $\eta \in E$, all the functions $F(\cdot,\eta), G(\cdot,\eta), H(\cdot,\eta)$ and $K(\cdot,\eta)$ are not identically 0 on D^{n-1}. Thus, we see from (13) that, for each $\eta \in E$, there exists a constant $\alpha(\eta)$ such that
$$F(b,\lambda,\eta) = \alpha(\eta)H(b,\lambda,\eta) \tag{14}$$
for all $b \in D^{n-2}$ and $\lambda \in D$.

Repeating exactly the same argument as above, we may assume that, given $\zeta \in E$, the functions $F(\cdot,\zeta,\cdot), H(\cdot,\zeta,\cdot)$ are not identically 0 on D^{n-1} and there is a constant $\beta(\zeta)$ such that
$$F(b,\zeta,\lambda) = \beta(\zeta)H(b,\zeta,\lambda) \tag{15}$$
for all $b \in D^{n-2}$ and $\lambda \in D$.

Now, choose $b_0 \in D^{n-2}$ for which $F(b_0,\cdot)$ and $H(b_0,\cdot)$ are not identically 0. By Lemma (1.2.9), we have $F(b_0,\cdot), H(b_0,\cdot)H^2(D^2)$ (it is in this step where we use the hypothesis $n \geq 3$). Hence, $F(b_0,\cdot)$ and $H(b_0,\cdot)$ have nonzero boundary values at almost all points of T^2. Therefore, we may further assume that $F(b_0,\cdot)$ and $H(b_0,\cdot)$ have nonzero boundary values on $E \times E$. Thus, given $\eta, \zeta \in E$, we obtain $\alpha(\eta) = \beta(\zeta)$ by (14) and (15). It follows that $\alpha(\eta) = \alpha$ is also independent of η.

We now have
$$F(z,\eta) = \alpha H(z,\eta)$$
for all $z \in D^{n-1}$ and $\eta \in E$. This yields $F = \alpha H$ on D^n as before. Similarly, we have $G = \bar{\alpha}K$. So, (12) holds. The proof is complete.

Now, Theorem (1.2.2) follows from Theorem (1.2.14), Theorem (1.2.1) and Theorem (1.2.11).

Also, as a corollary of the proof of Theorem (1.2.11), we have the following.

Corollary (1.2.12)[8]: $(n \geq 2)$. Let $f, g \in H^2$. Then the following statements are equivalent:

(a) $f\bar{g}$ is n −harmonic.

(b) $f\bar{g}$ is boundary n −harmonic.

We prove Theorem (1.2.14). The following fact is very useful for our purpose.

Proposition (1.2.13)[8]: Let $f, g \in A^2$. If $\partial_j f = 0$ or $\partial_j g = 0$ for each j, then we have
$$P(f\bar{g}K_a) = f\overline{g(a)}K_a \tag{16}$$
for $a \in D^n$. The converse also holds for $f, g \in H^2$.

Proof. Suppose $\partial_j f = 0$ or $\partial_j g = 0$ for each j. Then there are disjoint sets J_1, J_2 with $J_1 \cup J_2 = I$ such that $f \in H(J_1)$ and $g \in H(J_2)$. Write $z^j = (z_r)_{r \in J_j}$ for $j = 1,2$. By changing the coordinate system if needed, we may write $z = (z^1, z^2)$ for $z \in D^n$. By assumption, we may regard f and g as functions holomorphic on lower-dimensional polydisks. That is, we may write $f(z) = f(z^1)$ and $g(z) = g(z^2)$ for $z \in D^n$. Also, note $K_a(z) = K_{a^1}(z^1)K_{a^2}(z^2)$ for $z, a \in D^n$. Here, we abuse the notation K_{a^1} and K_{a^2} for the kernel functions on the corresponding lower dimensional polydisks. Thus, for every $a, z \in D^n$, we have

$$\begin{aligned}\langle fK_a, gK_z\rangle &= \langle fK_{a^1}K_{a^2}, gK_{z^1}K_{z^2}\rangle \\ &= \langle fK_{a^1}K_{z^1}\rangle\overline{\langle K_{a^2}, gK_{z^2}\rangle} \\ &= \langle fK_{a^1}K_{z^1}\rangle\overline{\langle gK_{z^2}, K_{a^2}\rangle} \\ &= f(z^1)K_{a^1}(z^1)\overline{g(a^2)}\,\overline{K_{z^2}(a^2)} \\ &= f(z)\overline{g(a)}K_a(z)\end{aligned}$$

and therefore

$$P(f\bar{g}K_a)(z) = \langle f\bar{g}K_a, K_z\rangle = \langle fK_a, gK_z\rangle = f(z)\overline{g(a)}K_a(z)$$

for every $a, z \in D^n$.

Now, let $f, g \in H^2$ and assume (16) holds. Let $a \in D^n$ be an arbitrary point.

Then, by (5) we have

$$\langle P(f\bar{g}K_a), k_a\rangle = \langle f\overline{g(a)}k_a, k_a\rangle = f(a)\overline{g(a)}.$$

On the other hand, we have

$$\langle P(f\bar{g}K_a), k_a\rangle = \langle f\bar{g}K_a, k_a\rangle = \int_{D^n}(f\bar{g})\circ\varphi_a\,dV$$

by (4). It follows that

$$\int_{D^n}(f\bar{g})\circ\varphi_a\,dV = f(a)\overline{g(a)}.$$

Note $\mathcal{R}(f\bar{k}) \in L^\infty$ by (6). Now, by Proposition (1.2.3), we conclude that $f\bar{g}$ is n −harmonic and therefore $\partial_j f = 0$ or $\partial_j g = 0$ for each j. This completes the proof.

We now turn to the proof of Theorem (1.2.14).

Theorem (1.2.14)[8]: Let $u, v \in L^\infty$ be pluriharmonic symbols and assume $u = f + g, v = h + \bar{k}$ for some $f, g, h, k \in H(D^n)$. Then the following statements are equivalent:

(a) $T_uT_v = T_vT_u$ on A^2.

(b) $f\bar{k} - h\bar{g}$ is n −harmonic.

Proof. We first prove (a)⟹ (b). So, assume $T_uT_v = T_vT_u$ on A^2.

Since u and v are bounded, the functions f, g, h and k are all in H^2 and hence in A^2 by Proposition (1.2.4). Fix a point $a \in D^n$. By Proposition (1.2.13), we have $P(\bar{g}k_a) = \overline{g(a)}k_a$ and hence

$$T_{f+\bar{g}}k_a = P[(f + \bar{g})k_a] = [f + \overline{g(a)}]k_a.$$

Therefore, we have

$$\begin{aligned}T_{h+\bar{k}}T_{f+\bar{g}}k_a &= P\big[(h + \bar{k})(f + \overline{g(a)})k_a\big] \\ &= fhk_a + h\overline{g(a)}k_a + \overline{g(a)}\,\overline{k(a)}k_a + P\big(f\bar{k}k_a\big).\end{aligned}$$

Similarly, we also have

$$T_{f+\bar{g}}T_{h+\bar{k}}k_a = fhk_a + f\overline{k(a)}k_a + \overline{g(a)}\,\overline{k(a)}k_a + P(h\bar{g}k_a).$$

It follows that

$$\big[T_{f+\bar{g}}T_{h+\bar{k}} - T_{h+\bar{k}}T_{f+\bar{g}}\big]k_a = \big[f\overline{k(a)} - h\overline{g(a)}\big]k_a - P\big[(f\bar{k} - h\bar{g})k_a\big]. \quad (17)$$

Since $T_{f+\bar{g}}T_{h+\bar{k}} - T_{h+\bar{k}}T_{f+\bar{g}}$ by assumption, we get

$$P\big[(f\bar{k} - h\bar{g})k_a\big] = \big[f\overline{k(a)} - h\overline{g(a)}\big]k_a.$$

Now, as in the proof of the second part of Proposition (1.2.13), the above leads to the n −harmonicity of $f\bar{k} - h\bar{g}$.

Next, we prove (b)⟹(a). So, assume $f\bar{k} - h\bar{g}$ is n −harmonic. Note that the set $\{k_a : a \in D^n\}$ spans a dense subset of A^2. Thus, in order to prove $T_uT_v = T_vT_u$, it is sufficient to show

$$P\left[\left(f\bar{k} - h\bar{g}\right)k_a\right] = \left[f\overline{k(a)} - h\overline{g(a)}\right]k_a, \ \ a \in D^n, \tag{18}$$

by (17).

Now, write $f\bar{k} - h\bar{g} = \sum A_i\bar{B}_i$ where A_i, B_i are functions as in (c) of Theorem (1.2.8). Let $z, a \in D^n$. Then, by Lemma (1.2.10) we have

$$f(z)\overline{k(a)} - h(z)\overline{g(a)} = \sum A_i(z)\overline{B_i(a)}.$$

We may assume $A_i, B_i \in A^2$. It follows from Proposition (1.2.13) that

$$\begin{aligned} P\left[\left(f\bar{k} - h\bar{g}\right)k_a\right](z) &= \sum P(A_i\bar{B}_ik_a)(z) \\ &= \sum A_i(z)\overline{B_i(a)}\, k_a(z) \\ &= \left[f(z)\overline{k(a)} - h(z)\overline{g(a)}\right]k_a(z). \end{aligned}$$

So, we conclude (18), as desired. The proof is complete.

Note that the adjoint of T_u is $T_{\bar{u}}$. It follows that T_u is normal if and only if $T_uT_{\bar{u}} = T_{\bar{u}}T_u$. Thus, by Theorem (1.2.14) and Corollary (1.2.8), we have the following.

Corollary (1.2.15)[8]: Let $u \in L^\infty$ be a pluriharmonic symbol. Then the following statements are equivalent:

(a) T_u is normal on A^2.

(b) There are pairwise disjoint sets $I_1, \dots, I_m$ with $\bigcup_{j=1}^m I_j = I$ for some positive integer $m \le n$, functions $p_1, \dots, p_m$ with $p_j \in H(I_j)$ for each j, and unimodular constants $\alpha_1, \dots, \alpha_m$ such that

$$u = \sum_{j=1}^{m} \alpha_j\left(p_j + \bar{p}_j\right) + \lambda \tag{19}$$

for some constant λ.

We prove Theorem (1.2.1). The proof will be completed by proving the following sequence of implications:

(a) ⟹ (b) ⟹ (c) ⟹ (b) ⟹ (a).

Since proofs are somewhat long, we will prove each implication separately.

For the proof of the implication (a) ⟹ (b), we introduce some notation. For each $a \in D^n$, we define a linear operator U_a on L^2 by

$$U_a\psi = (\psi \circ \varphi_a)k_a$$

for $\psi \in L^2$. One can readily see that U_a is an isometry taking A^2 onto itself. Also, since $\varphi_a \circ \varphi_a$ is the identity on D^n, one can see that U_aU_a is the identity. Moreover, for $u \in L^\infty$, we have

$$T_{u\circ\varphi_a} = U_aT_uU_a, \ \ \ a \in D^n. \tag{20}$$

This is proved in [9] on D and the same proof works on D^n.

Proof of (a) ⟹ (b). Let $\varphi \in \Phi$. Since the set $\{k_a : a \in D^n\}$ spans a dense subset of A^2, it is sufficient to show that

$$\left(T_{u\circ\varphi}T_{v\circ\varphi}-T_{v\circ\varphi}T_{u\circ\varphi}\right)k_a=0,\quad a\in D^n. \tag{21}$$

Choose a net $\{w_\alpha\}$ in D^n such that $\varphi_{w_\alpha}\to\varphi$. First, note $u\circ\varphi_{w_\alpha}\to u\circ\varphi$ and $v\circ\varphi_{w_\alpha}\to v\circ\varphi$ uniformly on every compact subset of D^n by Proposition (1.2.6).

Moreover, since u and v are bounded, we have

$$u\circ\varphi_{w_\alpha}\to u\circ\varphi,\quad v\circ\varphi_{w_\alpha}\to v\circ\varphi\quad\text{in } L^2.$$

Fix $a\in D^n$. Then it follows from the above that $\left(v\circ\varphi_{w_\alpha}\right)k_a\to(v\circ\varphi)k_a$ in L^2.

So, $P\left[\left(v\circ\varphi_{w_\alpha}\right)k_a\right]\to P[(v\circ\varphi)k_a]$ in L^2. Since $u\circ\varphi_{w_\alpha}$ is bounded and converges point wise to $u\circ\varphi$, it is not hard to see

$$P\left[u\circ\varphi_{w_\alpha}P\left(v\circ\varphi_{w_\alpha}k_a\right)\right]\to P[u\circ\varphi P(v\circ\varphi k_a)]\quad\text{in } L^2.$$

In other words,

$$T_{u\circ\varphi_{w_\alpha}}T_{v\circ\varphi_{w_\alpha}}k_a\to T_{u\circ\varphi}T_{v\circ\varphi}k_a\quad\text{in } L^2.$$

Similarly, we have

$$T_{v\circ\varphi_{w_\alpha}}T_{u\circ\varphi_{w_\alpha}}k_a\to T_{v\circ\varphi}T_{u\circ\varphi}k_a\quad\text{in } L^2.$$

It follows from (20) that

$$\begin{aligned}\left\|\left(T_{u\circ\varphi}T_{v\circ\varphi}-T_{v\circ\varphi}T_{u\circ\varphi}\right)k_a\right\|_2&=\lim_\alpha\left\|\left(T_{u\circ\varphi_{w_\alpha}}T_{v\circ\varphi_{w_\alpha}}-T_{v\circ\varphi_{w_\alpha}}T_{u\circ\varphi_{w_\alpha}}\right)k_a\right\|_2\\&=\lim_\alpha\left\|U_{w_\alpha}(T_uT_v-T_vT_u)U_{w_\alpha}k_a\right\|_2\\&=\lim_\alpha\left\|(T_uT_v-T_vT_u)U_{w_\alpha}k_a\right\|_2\end{aligned}$$

where $\|\ \|_2$ denotes the L^2 −norm. It is easy to see that $U_{w_\alpha}k_a$ converges to 0 weakly in A^2. Hence, the compactness of $T_uT_v-T_vT_u$ yields (21). This completes the proof.

For the proof of equivalence (b) ⟺ (c), we first prove the following lemma. In the proof below, we will use the well-known fact that

$$\widetilde{\Delta}_j(u\circ\varphi)=\left(\widetilde{\Delta}_ju\right)\circ\varphi,\quad j=1,\dots,n, \tag{22}$$

for all $\varphi\in Aut(D^n)$ and $u\in C^2(D^n)$.

Lemma (1.2.16)[8]: Let $u=f+\bar g, v=h+\bar k$ be as in the hypothesis of Theorem (1.2.1).

Suppose $\{\varphi_{w_\alpha}\}$ is a net such that $\varphi_{w_\alpha}\to\varphi\in\Phi$. If $u\circ\varphi=F+\bar G, v\circ\varphi=H+\bar K$ where $F,G,H,K\in H(D^n)$, then

$$\widetilde{\Delta}_j\left(f\bar k-h\bar g\right)\circ\varphi_{w_\alpha}\to\widetilde{\Delta}_j\left(F\bar K-H\bar G\right)$$

for each j.

Proof. Fix j. Put $f_\alpha=f\circ\varphi_{w_\alpha}-f(w_\alpha)$ and $k_\alpha=k\circ\varphi_{w_\alpha}-k(w_\alpha)$ for simplicity.

First, by (22), we have

$$\widetilde{\Delta}_j\left(f_\alpha\bar k_\alpha-h_\alpha\bar g_\alpha\right)=\widetilde{\Delta}_j\left[\left(f\bar k-h\bar g\right)\circ\varphi_{w_\alpha}\right]=\widetilde{\Delta}_j\left(f\bar k-h\bar g\right)\circ\varphi_{w_\alpha}$$

Thus, it remains to show

$$\widetilde{\Delta}_j\left(f_\alpha\bar k_\alpha-h_\alpha\bar g_\alpha\right)\to\widetilde{\Delta}_j\left(F\bar K-H\bar G\right) \tag{23}$$

Note that

$$u\circ\varphi_{w_\alpha}-u(w_\alpha)\to u\circ\varphi-u\circ\varphi(0)$$

uniformly on every compact subset of D^n. In particular, since u and v are bounded,

$$u\circ\varphi_{w_\alpha}-u(w_\alpha)\to u\circ\varphi-u\circ\varphi(0)\text{ in } L^2.$$

Now, using the L^2 −boundedness of the Bergman projection P, we have

$$P\left[u\circ\varphi_{w_\alpha}-u(w_\alpha)\right]\to P[u\circ\varphi-u\circ\varphi(0)]\text{ in } L^2.$$

Note that an application of Proposition (1.2.13) yields

$$P[u \circ \varphi_{w_\alpha} - u(w_\alpha)] = f_\alpha, \quad P[u \circ \varphi - u \circ \varphi(0)] = F - F(0).$$

Hence, $f_\alpha \to F - F(0)$ in L^2. It follows that $f_\alpha \to F - F(0)$ uniformly on every compact subset of D^n and therefore $\partial_j f_\alpha \to \partial_j F$. Applying the same reasoning to $\bar{v}$, we have $\partial_j k_\alpha \to \partial_j K$. Since

$$\widetilde{\Delta}_j(f_\alpha \bar{k}_\alpha)(z) = \left(1 - |z_j|^2\right)^2 \partial_j f_\alpha \overline{\partial_j k_\alpha},$$

it follows that

$$\widetilde{\Delta}_j(f_\alpha \bar{k}_\alpha) \to \left(1 - |z_j|^2\right)^2 \partial_j F \overline{\partial_j K_\alpha} = \widetilde{\Delta}_j(F\bar{K}).$$

Similarly, we have $\widetilde{\Delta}_j(h_\alpha \bar{g}_\alpha) \to \widetilde{\Delta}_j(H\bar{G})$ Hence, (23) holds. This completes the proof.

We now prove that (b) implies (c) and vice versa.

Proof of (b) ⟹ (c). It is sufficient to show that, for a given net $\{w_\alpha\}$ such that $\varphi_{w_\alpha} \to \varphi$ for some $\varphi \in \Phi$,

$$\widetilde{\Delta}_j(f\bar{k} - h\bar{g})(w_\alpha) \to 0 \tag{24}$$

holds for each j. So, fix a net $\{w_\alpha\}$ such that $\varphi_{w_\alpha} \to \varphi$ for some $\varphi \in \Phi$ and let F, G, H, K be as in Lemma (1.1.16). Since $T_{u\circ\varphi}$ and $T_{v\circ\varphi}$ commute by assumption, Theorem (1.2.14) shows the function $F\bar{K} - H\bar{G}$ is n −harmonic. Hence $\widetilde{\Delta}_j(F\bar{K} - H\bar{G}) = 0$ for each j. Consequently, by Lemma (1.1.16) with evaluation at the origin, we have (24) as desired. The proof is complete.

Proof of (c) ⟹ (b). Let $\varphi \in \Phi$ and assume $\varphi_{w_\alpha} \to \varphi$. Let $u \circ \varphi = F + \bar{G}$ and $v = H + \bar{K}$ as before. Fix an arbitrary point $a \in D^n$ and put $z_\alpha = \varphi_{w_\alpha}(a)$.

Since $\varphi_a \circ \varphi_{w_\alpha} \circ \varphi_{z_\alpha} Aut(Dn)$ fixes the origin, it is a unitary transformation, say $U_{a,\alpha}$. Thus we have

$$\varphi_{z_\alpha} = \varphi_{w_\alpha} \circ \varphi_a \circ U_{a,\alpha}. \tag{25}$$

Since the set of all unitary transformations is compact, we may assume $U_{a,\alpha}$ converges to some unitary transformation U_a. Now, for a given function $\psi \in H^\infty$, since $\psi \circ \varphi_{w_\alpha} \to \psi \circ \varphi$ uniformly on every compact subset of D^n and $\varphi_a \circ U_{a,\alpha} \to \varphi_a \circ U_a$, we see that $\psi \circ \varphi_{w_\alpha} \circ \varphi_a \circ U_{a,\alpha} \to \psi \circ \varphi \circ \varphi_a \circ U_a$. This, together with (25), shows $\varphi_{z_\alpha} \to \tilde{\varphi}$ where $\tilde{\varphi} = \varphi \circ \varphi_a \circ U_a$. By the same argument, we have $u \circ \varphi_{z_\alpha} \to u \circ \tilde{\varphi}$ and $v \circ \varphi_{z_\alpha} \to v \circ \tilde{\varphi}$ uniformly on every compact subset of D^n. Note that $\varphi \in \Phi$ implies $w_\alpha \to \partial D^n$ and thus $z_\alpha \to \partial D^n$. So, $\tilde{\varphi} \in \Phi$.

Now, since $u \circ \tilde{\varphi} = F \circ \varphi_a \circ U_a + \overline{G \circ \varphi_a \circ U_a}$ and $v \circ \tilde{\varphi} = H \circ \varphi_a \circ U_a + \overline{K \circ \varphi_a \circ U_a}$, it follows from Lemma (1.1.16) and (22) that

$$\begin{aligned} 0 &= \lim_\alpha \widetilde{\Delta}_j\left[f\bar{k} - h\bar{g}\right](z) \\ &= \widetilde{\Delta}_j[(F\bar{K} - H\bar{G}) \circ \varphi_a \circ U_a](0) \\ &= \widetilde{\Delta}_j[F\bar{K} - H\bar{G}]\left(\varphi_a \circ U_a(0)\right) \\ &= \widetilde{\Delta}_j[F\bar{K} - H\bar{G}](a) \end{aligned}$$

for all j. So, the function $F\bar{K} - H\bar{G}$ is n −harmonic. Thus, $T_{u\circ\varphi}$ and $T_{v\circ\varphi}$ commute by Theorem (1.2.14). The proof is complete.

For the proof of the implication (b) ⟹ (a), we introduce some notation. Given a pair of bounded pluriharmonic symbols $u = f + \bar{g}$ and $v = h + \bar{k}$ where $f, g, h, k \in H(D^n)$, we let

$$R_{u,v}(z,a) = \big(f(z) - f(a)\big)\big(\overline{k(z)} - \overline{k(a)}\big) - \big(h(z) - h(a)\big)\big(\overline{g(z)} - \overline{g(a)}\big)$$

for $z, a \in D^n$. The significance of the function $R_{u,v}$ lies in the fact that the commutator $T_uT_v - T_vT_u$ can be expressed as an integral operator given by

$$(T_uT_v - T_vT_u)(a) = \int_{D^n} R_{u,v}(z,a)\overline{k(a)}\psi(z)\, dV(z) \tag{26}$$

for $\psi \in A^2$ and $a \in D^n$. Recall that the functions f, g, h and k are all in H^p and hence in A^p for all $p > 0$ by Proposition (1.2.4). In particular, we have $R(\cdot, a) \in L^2$ for each fixed $a \in D^n$. Thus, the above integral is well defined. The above representation is well known. See, for example, [12] for details on the ball. The same proof works on D^n.

Finally, we prove that (b) implies (a).

Proof of (b) ⟹ (a). Put $R = R_{u,v}$ and $R_\varphi = R_{u\circ\varphi, v\circ\varphi}$ for $\varphi \in \Phi$. First, we claim the following:

$$\lim_{a\to\partial D^n} \inf_{\varphi\in\Phi} \int_{D^n} \big|R(\varphi_a(z), a) - R_\varphi(z,a)\big|^{14} dV(z) = 0. \tag{27}$$

Suppose not. Then there exists a net $\{w_\alpha\}$ such that $w_\alpha \to \partial D^n$ and

$$\inf_{\varphi\in\Phi} \int_{D^n} \big|R\big(\varphi_{w_\alpha}(z), w_\alpha\big) - R_\varphi(z,a)\big|^{14} dV(z) > 0 \tag{28}$$

for all w_α. Now, by taking a subnet if necessary, we may assume $\varphi_{w_\alpha} \to \varphi$ for some $\varphi \in \Phi$. Note that

$$u \circ \varphi_{w_\alpha} - u(w_\alpha) \to u \circ \varphi - u \circ \varphi(0)$$

uniformly on every compact subset of D^n. In particular, since u and v are bounded,

$$u \circ \varphi_{w_\alpha} - u(w_\alpha) \to u \circ \varphi - u \circ \varphi(0) \text{ in } L^{28}.$$

Now, using the L^p −boundedness of the Bergman projection P for $p > 1$, we have in particular

$$P\big[u \circ \varphi_{w_\alpha} - u(w_\alpha)\big] \to P[u \circ \varphi - u \circ \varphi(0)] \text{ in } L^{28}.$$

On the other hand, letting $u \circ \varphi = F + \bar{G}$ and $v \circ \varphi = H + \bar{K}$, we see that an application of Proposition (1.2.13) yields

$$P\big[u \circ \varphi_{w_\alpha} - u(w_\alpha)\big] = f \circ \varphi_{w_\alpha} - f(w_\alpha), \quad P[u \circ \varphi - u \circ \varphi(0)] = F - F(0).$$

Hence, $f \circ \varphi_{w_\alpha} - f(w_\alpha) \to F - F(0)$ in L^{28}. Similarly, we have $k \circ \varphi_{w_\alpha} - k(w_\alpha) \to K - K(0)$ in L^{28}. It follows that

$$\big[f \circ \varphi_{w_\alpha} - f(w_\alpha)\big]\overline{\big[k \circ \varphi_{w_\alpha} - k(w_\alpha)\big]} \to [F - F(0)]\overline{[K - K(0)]} \quad \text{in } L^{14}.$$

Also, the same is true for functions h, g. Hence, we have

$$R\big(\varphi_{w_\alpha}(\cdot), w_\alpha\big) - R_\varphi(\cdot\,, 0) \text{ in } L^{14},$$

which is a contradiction to (28). Thus, we have (27).

Fix $\varphi \in \Phi$ and $\psi \in A^2$. Then $T_{u\circ\varphi}$ and $T_{v\circ\varphi}$ are commuting by assumption. Hence, a simple application of (20) yields $T_{u\circ\varphi\circ\varphi_a}T_{v\circ\varphi\circ\varphi_a} = T_{v\circ\varphi\circ\varphi_a}T_{u\circ\varphi\circ\varphi_a}$ for $a \in D^n$. In particular, we have

$$\left(T_{u\circ\varphi\circ\varphi_a}T_{v\circ\varphi\circ\varphi_a} - T_{v\circ\varphi\circ\varphi_a}T_{u\circ\varphi\circ\varphi_a}\right)\psi(a) = 0, \quad a \in D^n.$$

Note that

$$R_{u\circ\varphi\circ\varphi_a, v\circ\varphi\circ\varphi_a}(z,a) = R_{u\circ\varphi, v\circ\varphi}\left(\varphi_a(z), \varphi_a(a)\right) = R_{u\circ\varphi, v\circ\varphi}(\varphi_a(z), 0)$$

Thus, (26) shows

$$\int_{D^n} \frac{R_\varphi(\varphi_a(z),0)}{\prod_{j=1}^n \left(1 - a_j\bar{z}_j\right)^2}\psi(z)dV(z) = 0.$$

It follows from (26) again that

$$(T_uT_v - T_vT_u)\psi(a) = \int_{D^n} \frac{R(z,a) - R_\varphi(\varphi_a(z),0)}{\prod_{j=1}^n \left(1 - a_j\bar{z}_j\right)^2}\psi(z)dV(z) \qquad (29)$$

for $a \in D^n$.

For each $\rho \in (0,1)$, let $M_\rho: L^2 \to L^2$ be the multiplication operator by the characteristic function of ρD^n. Here, $\rho D^n = \{\rho z: z \in D^n\}$. Then M_ρ is compact when restricted to A^2. Thus, the operator $M_\rho(T_uT_v - T_vT_u)$ is also compact. Put

$$S_\rho = \left(1 - M_\rho\right)(T_uT_v - T_vT_u)$$

for simplicity. We note from (29) that

$$S_\rho\psi(a) = \chi_\rho(a) \int_{D^n} \frac{R(z,a) - R_\varphi(\varphi_a(z),0)}{\prod_{j=1}^n \left(1 - a_j\bar{z}_j\right)^2}\psi(z)dV(z), a \in D^n,$$

Where $\chi_\rho = \chi_{D^n \setminus \rho D^n}$.

By (4) and simple manipulations, one obtains

$$\int_{D^n} \frac{\left|R(z,a) - R_\varphi(\varphi_a(z),0)\right|^2}{\prod_{j=1}^n \left|1 - a_j\bar{z}_j\right|^2 \sqrt{1 - \left|z_j\right|^2}} dV(z)$$

$$= \int_{D^n} \frac{\left|R(\varphi_a(z),a) - R_\varphi(z,0)\right|^2 |k_a(z)|^2}{\prod_{j=1}^n \left|1 - a_j\overline{\varphi_{a_j}(z_j)}\right|^2 \sqrt{1 - \left|\varphi_{a_j}(z_j)\right|^2}} dV(z)$$

$$= \frac{1}{\prod_{j=1}^n \sqrt{1 - \left|a_j\right|^2}} \int_{D^n} \frac{\left|R(\varphi_a(z),a) - R_\varphi(z,0)\right|^2}{\prod_{j=1}^n \left|1 - a_j\bar{z}_j\right| \sqrt{1 - \left|z_j\right|^2}} dV(z)$$

$$\leq \frac{1}{\prod_{j=1}^n \sqrt{1 - \left|a_j\right|^2}} \left(\int_{D^n} \left|R(\varphi_a(z),a) - R_\varphi(z,0)\right|^{14} dV(z)\right)^{1/7}$$

$$\times \left(\int_{D^n} \frac{dV(z)}{\prod_{j=1}^n \left|1 - a_j\bar{z}_j\right|^{7/6} \left(1 - \left|z_j\right|^2\right)^{7/12}}\right)^{6/7},$$

where the inequality holds by Hölder's inequality with the conjugate exponents 7=6 and 7. On the other hand, by an application of Lemma 4.2 of [26], we can see

$$\int_{D^n} \prod_{j=1}^{n} \frac{1}{\left|1 - a_j \bar{z}_j\right|^{7/6} \left(1 - \left|z_j\right|^2\right)^{7/12}} dV(z) \leq C$$

for some constants C. Here and in the rest of the proof, we use the same letter C for various constants depending only on n. It follows that

$$\int_{D^n} \frac{\left|R(z,a) - R_\varphi(\varphi_a(z),0)\right|^2}{\prod_{j=1}^{n}\left|1 - a_j \bar{z}_j\right|^2 \sqrt{1 - \left|z_j\right|^2}} dV(z)$$

$$\leq \frac{C}{\prod_{j=1}^{n} \sqrt{1 - \left|a_j\right|^2}} \left(\int_{D^n} \left|R(\varphi_a(z),a) - R_\varphi(z,0)\right|^{14} dV(z) \right)^{1/7}$$

Now, the Cauchy-Schwarz inequality yields

$$\left|S_\rho \psi(a)\right|^2 \leq \left(\chi_\rho(a) \int_{D^n} \frac{\left|R(z,a) - R_\varphi(\varphi_a(z),0)\right|^2}{\prod_{j=1}^{n}\left|1 - a_j \bar{z}_j\right|^2} |\psi(z)| dV(z) \right)^2$$

$$\leq \left(\int_{D^n} \frac{\chi_\rho(a)\left|R(z,a) - R_\varphi(\varphi_a(z),0)\right|^2}{\prod_{j=1}^{n}\left|1 - a_j \bar{z}_j\right|^2 \sqrt{1 - \left|z_j\right|^2}} dV(z) \right)$$

$$\times \left(\int_{D^n} \prod_{j=1}^{n} \frac{\sqrt{\left(1 - \left|z_j\right|^2\right)}}{\left|1 - a_j \bar{z}_j\right|^2} |\psi(z)|^2 dV(z) \right)$$

$$\leq C \frac{\chi_\rho(a)}{\prod_{j=1}^{n} \sqrt{1 - \left|a_j\right|^2}} \left(\int_{D^n} \left|R(\varphi_a(z),a) - R_\varphi(z,0)\right|^{14} dV(z) \right)^{1/7}$$

$$\times \left(\int_{D^n} \prod_{j=1}^{n} \frac{\sqrt{\left(1 - \left|z_j\right|^2\right)}}{\left|1 - a_j \bar{z}_j\right|^2} |\psi(z)|^2 dV(z) \right),$$

It follows from Fubini's theorem that

$$\int_{D^n} \left|S_\rho \psi\right|^2 dV \leq C \sup_{a \in D^n \setminus \rho D^n} \left(\int_{D^n} \left|R(\varphi_a(z),a) - R_\varphi(z,0)\right|^{14} dV(z) \right)^{1/7}$$

$$\times \int_{D^n} \prod_{j=1}^{n} \sqrt{1 - \left|z_j\right|^2} |\psi(z)|^2 \int_{D^n} \frac{dV(a)}{\prod_{j=1}^{n}\left|1 - a_j \bar{z}_j\right|^2 \sqrt{1 - \left|a_j\right|^2}} dV(z).$$

Moreover, by an application of Lemma 4.2.2 of [26], we have

$$\int_{D^n} \frac{dV(a)}{\prod_{j=1}^{n} |1 - a_j \bar{z}_j|^2 \sqrt{1 - |a_j|^2}} \le \frac{C}{\prod_{j=1}^{n} \sqrt{1 - |z_j|^2}}, \qquad z \in D^n,$$

and therefore, we have

$$\int_{D^n} |S_\rho \psi|^2 dV \le C \sup_{a \in D^n \backslash \rho D^n} \left(\int_{D^n} |R(\varphi_a(z), a) - R_\varphi(z, 0)|^{14} dV(z) \right)^{1/7} \int_{D^n} |\varphi|^2 dV.$$

Note that the above holds for all $\varphi \in \Phi$ and $\psi \in A^2$. So, we finally have

$$\|S_\rho\| \le C \sup_{a \in D^n \backslash \rho D^n} \left(\sup_{\varphi \in \Phi} \int_{D^n} |R(\varphi_a(z), a) - R_\varphi(z, 0)|^{14} dV(z) \right)^{1/14}.$$

Now, taking the limit $\rho \to 1$, we conclude $S_\rho \to 0$ in the operator norm by (27).

Hence, $T_u T_v - T_v T_u$ can be approximated by compact operators, so it is compact, as desired. The proof is complete.

We say that a bounded linear operator L on a Hilbert space is essentially normal if L and its adjoint operator are essentially commuting. As a consequence of Theorem (1.2.1) and Corollary (1.2.15), we have the following.

Corollary (1.2.17)[8]: Let $u \in L^\infty$ be a pluriharmonic symbol and assume $u = f + \bar{g}$ for some $f, g \in H(D^n)$. Then, the following statements are equivalent:

(a) T_u is essentially normal on A^2.

(b) $|f|^2 - |g|^2$ is boundary n −harmonic.

(c) For each $\varphi \in \Phi$, there are pairwise disjoint sets $I_1, \dots, I_m$ with $\bigcup_{j=1}^{m} I_j = I$ for some positive integer $m \le n$, functions $p_1, \dots, p_m$ with $p_j \in H(I_j)$ for each j, and unimodular constants $\alpha_1, \dots, \alpha_m$ such that

$$u \circ \varphi = \sum_{j=1}^{m} \alpha_j \left(p_j + \bar{p}_j\right) + \lambda$$

for some constant λ.

As a consequence of Theorem (1.2.2), Corollary (1.2.15) and Corollary (1.1.17), we obtain the following.

Corollary (1.2.18)[8]: $(n \ge 3)$. Let $u \in L^\infty$ be a pluriharmonic symbol and assume $u = f + \bar{g}$ for some $f, g \in H(D^n)$. Then, the following statements are equivalent:

(a) T_u is normal on A^2.

(b) T_u is essentially normal on A^2.

(c) $|f|^2 - |g|^2$ is n −harmonic.

(d) $|f|^2 - |g|^2$ is boundary n −harmonic.

Also, as an immediate consequence of Theorem (1.2.14), Theorem (1.2.1) and Corollary (1.2.12), we recover the result of Sun and Zheng [22] mentioned in the Introduction.

Corollary (1.2.19)[8]: $(n \ge 2)$. Let $f, g \in H^\infty$. Then, the following statements are equivalent:

(a) $T_f T_{\bar{g}} = T_{\bar{g}} T_f$ on A^2.

(b) For each j, we have either $\partial_j f = 0$ or $\partial_j g = 0$.

(c) T_f and $T_{\bar{g}}$ are essentially commuting on A^2.

Example (1.2.20)[8]: Corollary (1.1.18) does not extend to $n = 2$, either. To see an example, let

$$f(z) = \sum_{\ell=0}^{\infty} a_\ell \left(\sum_{i+j=\ell} z_1^i z_2^j \right), \quad z = (z_1, z_2),$$

where coefficients $a_\ell \neq 0$ are chosen so that the series converges on all of $\mathbb{C}^2$. Then, a little manipulation yields

$$z_1 f(z) = \sum_{\ell=0}^{\infty} a_{\ell+1} \left(\sum_{i+j=\ell} z_1^{i+1} z_2^{j+1} \right) + \psi(z_1),$$

Where $\psi(\lambda) = \sum_{\ell=0}^{\infty} a_\ell \lambda^{\ell+1}$. Define $g(z) = z_1 f(z) - \psi(z_1)$. By symmetry, we have $g(z) = z_2 f(z) - \psi(z_2)$. Hence, we have $\partial_1 g(z) = z_2 \partial_1 f(z), \partial_2 g(z) = z_1 \partial_2 f(z)$ and thus

$$\widetilde{\Delta}_j(|f|^2 - |g|^2)(z) = \left(1 - |z_j|^2\right)(1 - |z_1|^2)(1 - |z_2|^2)\left|\partial_j f(z)\right|^2$$

for $j = 1, 2$. Consequently, $|f|^2 - |g|^2$ is boundary 2-harmonic, but not 2-harmonic.

For Toeplitz operators T_u and T_v, we call $T_u T_v - T_{uv}$ the semi-commutator. For Toeplitz operators with pluriharmonic symbols, the commuting property is very closely related to the semi-commuting property.

To see what is going on, let us begin with functions $f, g, h, k \in H^\infty$. Put $u = f + \bar{g}$ and $v = h + \bar{k}$. Then, one can easily verify that

$$T_u T_v - T_{uv} = T_f T_{\bar{k}} - T_{\bar{k}} T_f.$$

Hence, the semi-commuting problem of T_u and T_v simply reduces to the commuting problem of T_f and $T_{\bar{k}}$. Thus, for $n \geq 2$, the essentially semi-commuting property is the same as the semi-commuting property by Corollary (1.1.19).

For general pluriharmonic symbols, our arguments used can be easily modified to conclude the same. Lemma (1.2.21) and Lemma (1.2.22) below are valid even for $n = 1$. For other characterizations on the disk and ball, see [23] and [24].

Lemma (1.2.21)[8]: Let $u, v \in L^\infty$ be pluriharmonic symbols and assume $u = f + \bar{g}, v = h + \bar{k}$ for some $f, g, h, k \in H(D^n)$. Then, the following statements are equivalent:

(a) $T_u T_v = T_{uv}$ on A^2.

(b) For each j, we have either $\partial_j f = 0$ or $\partial_j k = 0$.

Proof. As in the proof Theorem (1.2.14), one obtains

$$(T_u T_v - T_{uv}) k_a = \overline{f k(a)} k_a - P\left(f \bar{k} k_a\right).$$

Thus, $T_u T_v = T_{uv}$ if and only if

$$P\left(f \bar{k} k_a\right) = \overline{f k(a)} k_a,$$

which is in turn equivalent to the fact that $\partial_j f = 0$ or $\partial_j k = 0$ for each j by Proposition (1.2.13), because $f, k \in H^2$ by Proposition (1.2.4). The proof is complete.

For essentially semi-commuting Toeplitz operators, we also have the following.

Lemma (1.2.22)[8]: Let $u, v \in L^\infty$ be pluriharmonic symbols and assume $u = f + \bar{g}, v = h + \bar{k}$ for some $f, g, h, k \in H(D^n)$. Then, the following statements are equivalent:

(a) $T_u T_v - T_{uv}$ is compact on A^2.

(b) $T_{u\circ\varphi} T_{v\circ\varphi} - T_{(uv)\circ\varphi}$ on A^2 for every $\varphi \in \Phi$.

(c) $f\bar{k}$ is boundary n −harmonic.

Proof. As in (26), we have the following representation:

$$(T_u T_v - T_{uv})\psi(a) = \int_{D^n} \big(f(z) - f(a)\big)\big(\overline{k(z)} - \overline{k(a)}\,\big)\overline{k_a(z)}\psi(z)dV(z)$$

for $a \in D^n$ and $\psi \in A^2$. Hence, one can easily modify the proof of Theorem (1.2.1) to conclude the theorem. The proof is complete.

Now, combining Corollary (1.2.12), Lemma (1.2.21) and Lemma (1.2.22), we see that the essentially semi-commuting property is the same as the semi-commuting property for $n \geq 2$.

Theorem (1.2.23)[8]: $(n \geq 2)$. Let $u, v \in L^\infty$ be pluriharmonic symbols and assume $u = f + \bar{g}, v = h + \bar{k}$ for some $f, g, h, k \in H(D^n)$. Then, the following statements are equivalent:

(a) $T_u T_v = T_{uv}$ on A^2.

(b) For each j, we have either $\partial_j f = 0$ or $\partial_j k = 0$.

(c) $T_u T_v - T_{uv}$ is compact on A^2.

(d) $T_{u\circ\varphi} T_{v\circ\varphi} - T_{(uv)\circ\varphi}$ on A^2 for every $\varphi \in \Phi$.

(e) $f\bar{k}$ is boundary n −harmonic.

Section (1.3): Products of Toeplitz Operators:

For D be the open unit disk in the complex plane C. Its boundary is the unit circle T. The polydisk D^n and the torus T^n are the subsets of C^n which are Cartesian products of n copies D and T, respectively. Let $d\sigma(z)$ be the normalized Haar measure on T^n. The Hardy space $H^2(D^n)$ is the closure of the polynomials in $L^2(T^n, d\sigma)\big(\text{or } L^2(T^n)\big)$. Let P be the orthogonal projection from $L^2(T^n)$ onto $H^2(D^n)$. The Toeplitz operator with symbol f in L^∞ is defined by $T_f h = P(fh)$, for all $h \in H^2(D^n)$ and the Hankel operator with symbol f is defined by $H_f h = (I - P)fh$, for all $h \in H^2(D^n)$. We consider the problem when the product $T_f T_g$ of two Toeplitz operators T_f and T_g is zero on the Hardy space. Also we will characterize when the product $T_f T_g$ of two Toeplitz operators T_f and T_g on the Hardy space $H^2(D^n)$ is still a Toeplitz operator. Furthermore we will see that there are no compact semi-commutator $T_f T_g - T_{fg}$ of two Toeplitz operators with bounded pluriharmonic symbols. As is well known, for f and g in L^∞, Brown and Halmos [29] have shown that $T_f T_g$ is a Toeplitz operator if and only if either $\bar{f} \in H^\infty$ or $g \in H^\infty$. In other words, either $H_{\bar{f}}$ or H_g is zero. It was shown in [31] that for Toeplitz operator T_f and T_g on $H^2(D), T_f T_g - T_{fg}$ is a finite rank if and only if either $\bar{f}$ or g is an analytic function plus a rational function.

Since the function theory on the polydisk D^n is quite different from the function theory on the unit disk [32], there exist some differences in Toeplitz operator theory between on the polydisk and on the disk ([33], [30]).

Now we give some preliminaries for our main results.

$K_{z_1}(w_1) = \frac{1}{(1-\bar{z}_1 w_1)}$ is called the reproducing kernel of at the point z_1 in D and $K_{z_1}(w_1) = \frac{(1-|z_1|^2)^{1/2}}{(1-\bar{z}_1 w_1)}$ the normalized reproducing kernel of $H^2(D)$ at the point z_1 in D. It is easy to check that the reproducing kernel of $H^2(D^n)$ at the point z in D^n is the product $K_z(w) = \prod_1^n K_{z_i}(w_i)$. So the normalized reproducing kernel k_z of $H^2(D^n)$ at the point z in D^n is an also the product $k_z(w) = \prod_1^n K_{z_i}(w_i)$. We observe that k_z weakly converges to zero in $H^2(D^n)$ as z tends to the boundary of D^n.

We denote by $Aut(D^n)$ the group of all biholomorphic automorphisms of D^n. The automorphisms of D^n for $n \geq 2$ are generated by the following three subgroups: rotations in each variable separately $R_\theta(z) = (e^{i\theta_1} z_1, \ldots\ldots, e^{i\theta_n} z_n)$, where Mobius transformations are in each variable separately $\Psi_w(z) = \left(\Psi_{w_1}(z_1), \ldots, \Psi_{w_n}(z_n)\right)$, and the coordinate permutations. Here $\theta \in [0, 2\pi]^n$ and $w \in D^n$ are fixed. Mobius transformations are in the form $\Psi_w(z) = \frac{w-z}{1-\bar{w}z}$ $(w \in D, z \in D)$. Thus an arbitrary $\Psi \in Aut(D^n)$ can be written in the form

$$\Psi(z) = \left(e^{i\theta_1} \Psi_{w_1}\left(z_{\sigma(1)}\right), \ldots, e^{i\theta_n} \Psi_{w_n}\left(z_{\sigma(n)}\right)\right)$$

for some $w = (w_1, \ldots, w_n) \in D^n, \theta = (\theta_1, \ldots, \theta_n) \in [0, 2\pi]^n$, and σ is a coordinate permutations. The Poisson integral of $f \in L^1(T^n)$ is

$$P[f](z) = \int_{T^n} f(\zeta) \prod_1^n \frac{1 - |z_j|^2}{|1 - z_j \bar{\zeta}_j|^2} d\sigma(\zeta) = \int_{T^n} f(\zeta) |k_z(\zeta)|^2 d\sigma(\zeta).$$

Lemma (1.3.1)[27]: Let $f \in L^1(T^n), \Psi \in Aut(D^n)$, then

$$P[f \circ \Psi](z) = P[f] \circ \Psi(z),$$

Where $\Psi_z(w) = \left(\Psi_{z_1}(w_1), \ldots, \Psi_{z_n}(w_n)\right), \Psi_{z_i}(w_i) \in Aut(D)$ (see [4]).

Corollary (1.3.2)[27]: For any $z = (z_1, z_2, \ldots, z_n) \in D^n$, we have

$$\int_{T^n} f(\zeta) |k_z(\zeta)|^2 d\sigma(\zeta) = \int_{T^n} f \circ \Psi_z(\zeta) d\sigma(\zeta).$$

Proof. In fact, by Lemma (1.3.1), $P[f \circ \Psi_z](0) = P[f] \circ \Psi_z(0)$ and $\Psi_z(0) = z$, it follows that

$$\int_{T^n} f(\zeta) |k_z(\zeta)|^2 d\sigma(\zeta) = \int_{T^n} f \circ \Psi_z(\zeta) d\sigma(\zeta).$$

Let Z denote the set of all integers, Z_+ denote the set of all nonnegative integers and Z_- denote the set of all negative integers. We recall that by using multiple Fourier series,

$$L^2(T^n) = \left\{f : f = \sum_{\alpha \in Z^n} \hat{f}(\alpha) \zeta^\alpha, \sum_{\alpha \in Z^n} |\hat{f}(\alpha)|^2 < \infty\right\}.$$

We note that for every $\zeta = (\zeta_1, \ldots, \zeta_n) \in T^n, \alpha = (\alpha_1, \ldots, \alpha_n) \in Z^n, \zeta^\alpha = \zeta^{\alpha_1} \cdots \zeta^{\alpha_n}, \zeta_j^{-\alpha_j} = \bar{\zeta}_j^{\alpha_j}, \zeta_j \bar{\zeta}_j = |\zeta_j|^2 = 1$. So we can write also f as

$$f = f(\zeta, \bar{\zeta}) = \sum_{\alpha \in Z_+^n} \hat{f}(\alpha) \tilde{\zeta}^\alpha,$$

Where $\tilde{\zeta}_j = \zeta_j$ or $\tilde{\zeta}_j = \bar{\zeta}_j$.

Theorem (1.3.3)[27]: Let

$$f = f(\zeta, \bar{\zeta}) = \sum_{\alpha \in Z_+^n} \hat{f}(\alpha)\tilde{\zeta}^{\alpha} \in L^2(T^n),$$

Then

$$PfK_z(w) = f(w, \bar{z})K_z(w) \in H^2(D^n)$$

for every $z \in D^n$.

Proof.

$$PfK_z(w) = \langle fK_z, K_w \rangle = \int_{T^n} f(\zeta, \bar{\zeta})K_z(\zeta)\bar{K}_w(\zeta)d\sigma(\zeta)$$

$$= \int_{T^{n-1}} \int_{T} f(\zeta, \bar{\zeta})K_{z_1}(\zeta_1)\bar{K}_{w_1}(\zeta_1)d\sigma(\zeta_1)K_{z'}(\zeta')\bar{K}_{w'}(\zeta')d\sigma(\zeta'),$$

where $z' = (z_2, \dots, z_n) \in D^{n-1}$. Since f is harmonic in variable ζ_1, we can write f as

$$f = f(\zeta, \bar{\zeta}) = \sum_{j \geq 0} \hat{f}_1(j, \zeta', \bar{\zeta'})\zeta_1^j + \sum_{l \geq 0} \hat{f}_2(l, \zeta', \bar{\zeta'})\zeta_1^l$$

Hence

$$\int_{T^{n-1}} \int_{T} f(\zeta, \bar{\zeta})K_{z_1}(\zeta_1)\bar{K}_{w_1}(\zeta_1)d\sigma(\zeta_1)K_{z'}(\zeta')\overline{K_{w'}}(\zeta')d\sigma(\zeta')$$

$$= \int_{T^{n-1}} \left[= \sum_{j \geq 0} \hat{f}_1(j, \zeta', \bar{\zeta'})w_1^j + \sum_{l \geq 0} \hat{f}_2(l, \zeta', \bar{\zeta'})\bar{z}_1^l \right] K_{z_1}(w_1)K_{z'}(\zeta')\bar{K}_{w'}(\zeta')d\sigma(\zeta')$$

$$= \int_{T^{n-1}} f(w_1, \zeta', \bar{z}_1, \bar{\zeta'})K_{z'}(\zeta')\bar{K}_{w'}(\zeta')d\sigma(\zeta')K_{z_1}(w_1)$$

Furthermore $f(w_1, \zeta', \bar{z}_1, \bar{\zeta'})$ is harmonic in the variables $\zeta_2, \zeta_3, \dots, \zeta_n$, respectively.

In the same way as above, we can obtain that

$$PfK_z(w) = f(w_1, \cdots, w_n, \bar{z}_1, \cdots, \bar{z}_n)K_{z_1}(w_1) \cdots K_{z_n}(w_n).$$

This completes the proof of the theorem.

Note that if $f \in L^2(T^n)$, then the Toeplitz operator T_f is densely defined on $H^2(D^n)$. Next we consider Toeplitz operators with symbol in $L^2(T^n)$.

Theorem (1.3.4)[27]: Let f and g be in $L^2(T^n)$, then for any $z_1 \in D, \mu_1 \in T$, we have

$$\lim_{z_1 \to \mu_1} \int_{T} \langle T_f T_g k_{z_1 e^{i\theta}} k_{z'}, k_{z_1 e^{i\theta}} k_{z'} \rangle e^{im\theta} d\theta$$

$$= \int_{T} \langle T_{f(\mu_1 e^{i\theta} \cdot)} T_{g(\mu_1 e^{i\theta} \cdot)} k_{z'}, k_{z'} \rangle \, e^{im\theta} d\theta,$$

Where $\theta \in [0, 2\pi]$, for all $m \in Z$ and $z' \in D^{n-1}$ are fixed.

Proof. We write f and g as

$$f = f_1(\zeta_1, \zeta', \bar{\zeta'}) + f_2(\bar{\zeta}_1, \zeta', \bar{\zeta'})$$

$$= \sum_{j\geq 0} \hat{f}_1(j,\zeta',\overline{\zeta'})\zeta_1^j + \sum_{j\geq 0} \hat{f}_2(j,\zeta',\overline{\zeta'})\bar{\zeta}_1^j\,,$$

$$g = g_1(\zeta_1,\zeta',\overline{\zeta'}) + g_2(\overline{\zeta_1},\zeta',\overline{\zeta'})$$

$$= \sum_{j\geq 0} \hat{g}_1(j,\zeta',\overline{\zeta'})\zeta_1^j + \sum_{j\geq 0} \hat{g}_2(j,\zeta',\overline{\zeta'})\bar{\zeta}_1^j,$$

By Theorem (1.3.3),

$$\langle T_f T_g k_z, k_z\rangle = \langle T_g k_z, T_{\bar{f}} k_z\rangle = \langle g(\zeta,\bar{z})k_z, \bar{f}(z,\bar{\zeta})k_z\rangle$$

$$= \int_{T^{n-1}} \int_T [g_1(\zeta_1,\zeta',\overline{z'}) + g_2(\overline{z_1},\zeta',\overline{z'})]$$

$$[f_1(z_1,z',\overline{\zeta'}) + f_2(\bar{\zeta}_1,\zeta',\overline{\zeta'})]\left|k_{z_1}(\zeta_1)\right|^2 d\sigma(\zeta_1)|k_{z'}(\zeta)|^2 d\sigma(\zeta')$$

$$= \int_{T^{n-1}} \int_T [g_1(z_1,\zeta',\overline{z'})f_1(z_1,z',\overline{\zeta'}) + g_2(\overline{z_1},\zeta',\overline{z'})f_1(z_1,z',\overline{\zeta'})$$

$$+ g_2(\overline{z_1},\zeta',\overline{z'})f_2(\bar{\zeta}_1,\zeta',\overline{\zeta'})]|k_{z'}(\zeta')|^2 d\sigma(\zeta').$$

$$+ \int_{T^n} g_1(\zeta_1,\zeta',\overline{z'})f_2(\bar{\zeta}_1,z',\overline{\zeta'})|k_z(\zeta)|^2 d\sigma(\zeta)$$

Replacing z_1 by $z_1 e^{i\theta}$ in above equation yields

$$\langle T_f T_g k_{z'z_1e^{i\theta}}, k_{z'z_1e^{i\theta}}\rangle$$

$$= \int_{T^{n-1}} [g_1(z_1e^{i\theta},\zeta',\overline{z'})f_1(z_1e^{i\theta},z',\overline{\zeta'}) + g_2(\bar{z}_1e^{-i\theta},\zeta',\overline{z'})f_1(z_1e^{i\theta},z',\overline{\zeta'})$$

$$+ g_2(\bar{z}_1e^{-i\theta},\zeta',\overline{z'})f_1(\bar{z}_1e^{-i\theta},z',\overline{\zeta'})]\,|k_{z'}(\zeta')|d\sigma(\zeta')$$

$$+ \int_{T^n} g_1(\zeta_1,\zeta',\bar{z}')f_2(\bar{\zeta}_1,z',\overline{\zeta'})\left|k_{z'z_1e^{i\theta}}(\zeta)\right|^2 d\sigma(\zeta).$$

Multiplying by $e^{im\theta}$ in above equation and then integrating with respect to θ imply

$$\int_0^{2\pi} \langle T_f T_g k_{z_1e^{i\theta}}k_{z'}, k_{z_1e^{i\theta}}k_{z'}\rangle e^{im\theta} d\theta =$$

$$\int_0^{2\pi} \int_{T^{n-1}} [g_1(z_1e^{i\theta},\zeta',\bar{z}')f_1(z_1e^{i\theta},z',\bar{\zeta}')$$

$$+ g_2(z_1e^{-i\theta},\zeta',\bar{z}')f_2(z_1e^{-i\theta},z',\bar{\zeta}')]\,|k_{z'}(\zeta')|^2 d\sigma(\zeta')e^{im\theta} d\theta$$

$$+ \int_0^{2\pi} \int_{T^{n-1}} g_2(z_1e^{-i\theta},\zeta',\bar{z}')f_1(z_1e^{i\theta},z',\bar{\zeta}')|k_{z'}(\zeta')|^2 d\sigma(\zeta)e^{im\theta} d\theta$$

$$+ \int_0^{2\pi} \int_{T^n} g_1(\zeta_1,\zeta',\bar{z}')f_2(\bar{\zeta}_1,z',\bar{\zeta}')\left|k_{z_1e^{i\theta}}(\zeta_1)k_{z'}(\zeta')\right|^2 d\sigma(\zeta)e^{im\theta} d\theta.$$

Note that the measure is a rotation-invariant positive Borel measure on T^n. Interchanging the order of the above integration, we have

$$\int_0^{2\pi} \langle T_f T_g k_{z_1 e^{i\theta}} k_{z'}, k_{z_1 e^{i\theta}} k_{z'} \rangle e^{im\theta} d\theta$$

$$= \int_{T^{n-1}} \int_0^{2\pi} \left[g_1(z_1 e^{i\theta}, \zeta', \bar{z}') f_1(z_1 e^{i\theta}, z', \bar{\zeta}') \right.$$

$$\left. + g_2(\bar{z}_1 e^{-i\theta}, \zeta', \bar{z}') f_2(\bar{z}_1 e^{-i\theta}, z', \bar{\zeta}') \right] e^{im\theta} d\theta |k_{z'}(\zeta')|^2 d\sigma(\zeta')$$

$$+ \int_{T^{n-1}} \int_0^{2\pi} g_2(\bar{z}_1 e^{-i\theta}, \zeta', \bar{z}') f_1(z_1 e^{i\theta}, z', \bar{\zeta}') e^{im\theta} d\theta |k_{z'}(\zeta')|^2 d\sigma(\zeta')$$

$$+ \int_{T^n} \int_0^{2\pi} g_1(\zeta_1 e^{i\theta}, \zeta', \bar{z}') f_2(\bar{\zeta}_1 e^{-i\theta}, z', \bar{\zeta}') e^{im\theta} d\theta |k_z(\zeta)|^2 d\sigma(\zeta).$$

Also write

$$g_1(z_1 e^{i\theta}, \zeta', \bar{z}') = \sum_{j \geq 0} \hat{g}_1(j, \zeta', \bar{z}') z_1^j e^{ij\theta},$$

$$g_2(\bar{z}_1 e^{-i\theta}, \zeta', \bar{z}') = \sum_{j \geq 0} \hat{g}_2(j, \zeta', \bar{z}') \bar{z}_1^j e^{-ij\theta},$$

$$f_1(z_1 e^{i\theta}, \zeta', \bar{z}') = \sum_{j \geq 0} \hat{f}_1(j, z', \bar{\zeta}') z_1^j e^{ij\theta},$$

$$f_2(\bar{z}_1 e^{-i\theta}, \zeta', \bar{z}') = \sum_{j \geq 0} \hat{f}_2(j, z', \bar{\zeta}') \bar{z}_1^j e^{-ij\theta}.$$

We let

$$H_{m1}(z_1) = \int_0^{2\pi} \left[g_1(z_1 e^{i\theta}, \zeta', \bar{z}') f_1(z_1 e^{i\theta}, z', \bar{\zeta}') \right.$$

$$\left. + g_2(\bar{z}_1 e^{-i\theta}, \zeta', \bar{z}') f_2(\bar{z}_1 e^{-i\theta}, z', \bar{\zeta}') \right] e^{im\theta} d\theta.$$

If $m > 0$, then

$$H_{m1} = \sum_{j+l=m} \hat{g}_2(j, \zeta', \bar{z}') \hat{f}_2(l, z', \bar{\zeta}') \bar{z}_1^m;$$

If $m = 0$, then

$$H_{m1} = g_1(0, \zeta', \bar{z}') f_1(0, z', \zeta') + g_2(0, \zeta', \bar{z}') f_2(0, z', \bar{\zeta}');$$

If $m < 0$, then

$$H_{m1} = \sum_{j+l=-m} \hat{g}_1(j, \zeta', \overline{z'}) \hat{f}_1(l, z', \bar{\zeta}') \bar{z}_1^{-m}.$$

Since

$$\hat{g}_2(j, \zeta', \overline{\zeta'}) \in L^2(T^{n-1}),$$

$$P\hat{g}_2(j, \zeta', \overline{\zeta'}) K_{z'}(\zeta') = \hat{g}_2(j, \zeta', \bar{z}') K_{z'}(\zeta') \in H^2(T^{n-1}).$$

For any fixed $z' \in D^{n-1}$, we have $\left(K_{z'}(\zeta')\right)^{-1} \in H^\infty(T^n)$, hence $\hat{g}_2(j, \zeta', \bar{z}') \in H^2(T^{n-1})$. Similarly

$$\bar{\hat{f}}_2(l, z', \bar{\zeta}'), \hat{g}_1(j, \zeta', \bar{z}'), \text{and } \bar{\hat{f}}_1(l, z', \bar{\zeta}') \in H^2(T^{n-1}).$$

Thus

$$\sum_{j+l=m} \left|\hat{g}_2(j,\zeta',\bar{z}')\hat{f}_2(l,z',\bar{\zeta}')\right|, \text{and} \sum_{j+l=-m} \left|\hat{g}_1(j,\zeta',\bar{z}')\hat{f}_1(l,z',\bar{\zeta}')\right| \in L^1(T^{n-1}).$$

This implies that $H_{m1}(z_1)$ is continuous in variable z_1 on the closure $\overline{D}$. We take a net $\{z_{1\alpha}\} \subseteq D$ converging to μ_1. For every subsequence $\left\{z_{1\alpha_j}\right\}$ of the net $\{z_{1\alpha}\}$, by the dominated convergence theorem, we thus have

$$\lim_{z_{1\alpha_j}\to\mu_1} \int_{T^{n-1}} H_{m1}\left(z_{1\alpha_j}\right)|k_{z'}(\zeta')|^2 d\sigma(\zeta')$$
$$= \int_{T^{n-1}} H_{m1}(\mu_1)|k_{z'}(\zeta')|^2 d\sigma(\zeta')$$
$$= \int_{T^{n-1}} \int_0^{2\pi} \left[g_1(\mu_1 e^{i\theta},\zeta',\bar{z}')f_1(\mu_1 e^{i\theta},z',\bar{\zeta}') + g_2(\bar{\mu}_1 e^{-i\theta},\zeta',\bar{z}')f_2(\bar{\mu}_1 e^{-i\theta},z',\bar{\zeta}')\right] e^{im\theta} d\theta |k_{z'}(\zeta')|^2 d\sigma(\zeta')$$
$$= \int_0^{2\pi} \int_{T^{n-1}} \left[g_1(\mu_1 e^{i\theta},\zeta',\bar{z}')f_1(\mu_1 e^{i\theta},z',\bar{\zeta}') + g_2(\bar{\mu}_1 e^{-i\theta},\zeta',\bar{z}')f_2(\bar{\mu}_1 e^{-i\theta},z',\bar{\zeta}')\right] |k_{z'}(\zeta')|^2 d\sigma(\zeta') e^{im\theta} d\theta.$$

Let

$$H_{m2}(z_1) = \int_0^{2\pi} g_2(\bar{z}_1 e^{-i\theta},\zeta',\bar{z}')f_1(z_1 e^{i\theta},z',\bar{\zeta}')e^{im\theta} d\theta$$
$$= \sum_{-j+l=-m} \hat{g}_2(j,\zeta',\bar{z}')\hat{f}_1(l,z',\bar{\zeta}')\, \bar{z}_1^j z_1^l,$$

Then

$$|H_{m2}(z_1)| \leq \sum_{-j+l=-m} \left|\hat{g}_2(j,\zeta',\bar{z}')\hat{f}_1(l,z',\bar{\zeta}')\right|$$
$$\leq \left(\sum_{j\geq 0} |\hat{g}_2(j,\zeta',\bar{z}')|^2\right)^{\frac{1}{2}} \left(\sum_{j\geq 0} \left|\hat{f}_1(l,z',\bar{\zeta}')\right|^2\right)^{\frac{1}{2}}.$$

By using the orthogonality of $\{\zeta_1^j\}_j$, we have

$$\|g_2\|^2 = \int_{T^{n-1}} \int_T \left|\sum_{j\geq 0} \hat{g}_2(j,\zeta',\bar{\zeta}')\bar{\zeta}_1^j\right|^2 d\sigma(\zeta_1)d\sigma(\zeta')$$
$$= \int_{T^{n-1}} \sum_{j\geq 0} |\hat{g}_2(j,\zeta',\bar{\zeta}')|^2\, d\sigma(\zeta').$$

Because

$$P\hat{g}_2(j,\zeta',\bar{\zeta}')k_{z'}(\zeta') = \hat{g}_2(j,\zeta',\bar{z}')k_{z'}(\zeta'),$$

$$\hat{g}_2(j,\zeta',\bar{z}') = \frac{1}{k_{z'}(\zeta')} P\hat{g}_2(j,\zeta',\bar{\zeta}')k_{z'}(\zeta')$$
$$= \prod_{j=2}^{n} \frac{\left(1-\bar{z}_j\zeta_j\right)}{\left(1-\left|z_j\right|^2\right)} P\hat{g}_2(j,\zeta',\bar{\zeta}')k_{z'}(\zeta').$$

For fixed $z' \in D^{n-1}$,

$$|\hat{g}_2(j,\zeta',\bar{\zeta}')| \le \prod_{j=2}^{n} \left(\frac{1+\left|z_j\right|}{1-\left|z_j\right|}\right)^{\frac{1}{2}} |P\hat{g}_2(j,\zeta',\bar{\zeta}')k_{z'}(\zeta')|.$$

Thus we have

$$\|\hat{g}_2(j,\zeta',\bar{z}')\|^2 \le \prod_{j=2}^{n} \left(\frac{1+\left|z_j\right|}{1-\left|z_j\right|}\right) \|P\hat{g}_2(j,\zeta',\bar{\zeta}')k_{z'}(\zeta')\|^2$$
$$\le \prod_{j=2}^{n} \left(\frac{1+\left|z_j\right|}{1-\left|z_j\right|}\right) \|\hat{g}_2(j,\zeta',\bar{\zeta}')k_{z'}(\zeta')\|^2$$
$$= \prod_{j=2}^{n} \left(\frac{1+\left|z_j\right|}{1-\left|z_j\right|}\right) \int_{T^n} \left|\hat{g}_2(j,\zeta',\bar{\zeta}') \prod_{j=2}^{n} \frac{\left(1-\left|z_j\right|^2\right)^{\frac{1}{2}}}{\left(1-\bar{z}_j\zeta_j\right)}\right|^2 d\sigma(\zeta)$$
$$\le \prod_{j=2}^{n} \left(\frac{1+\left|z_j\right|}{1-\left|z_j\right|}\right) \int_{T^{n-1}} |\hat{g}_2(j,\zeta',\bar{\zeta}')|^2 \, d\sigma(\zeta')$$

This implies that

$$\left\|\sum_{j\ge 0} |\hat{g}_2(j,\zeta',\bar{z}')|^2\right\| \le \sum_{j\ge 0} \|\hat{g}_2(j,\zeta',\bar{z}')\|^2$$
$$\le \prod_{j=2}^{n} \left(\frac{1+\left|z_j\right|}{1-\left|z_j\right|}\right) \int_{T^{n-1}} \sum_{j\ge 0} |\hat{g}_2(j,\zeta',\bar{\zeta}')|^2 \, d\sigma(\zeta').$$

That is, for any $z' \in D^{n-1}, \sum_{j\ge 0}|\hat{g}_2(j,\zeta',\bar{z}')|^2 \in L^1(T^{n-1})$.

Also $P\bar{\hat{f}}_1(j,\zeta',\bar{\zeta'})k_{z'}(\zeta') = \bar{\hat{f}}_1(j,z',\bar{\zeta'})k_{z'}(\zeta')$ and similarly we have $\sum_{j\ge 0}\left|\hat{f}_1(j,z',\bar{\zeta}')\right|^2$ is in $L^1(T^{n-1})$.

Therefore

$$\left(\sum_{j\ge 0} |\hat{g}_2(j,\zeta',\bar{z}')|^2\right)^{1/2} \left(\sum_{j\ge 0} \left|\hat{f}_1(j,z',\bar{\zeta}')\right|^2\right)^{1/2} \in L^1(T^{n-1}).$$

Thus we conclude that $H_{m2}(z_1)$ is continuous in variable z_1 on the closure $\bar{D}$. For every subsequence $\left\{z_{1\alpha_j}\right\}$ of the net $\{z_{1\alpha}\}$, by the dominated convergence theorem again, we thus have

$$\lim_{z_{1\alpha_j}\to\mu_1}\int_{T^{n-1}} H_{m2}\left(z_{1\alpha_j}\right)|k_{z'}(\zeta')|^2 d\sigma(\zeta')$$

$$=\int_{T^{n-1}} H_{m2}(\mu_1)|k_{z'}(\zeta')|^2 d\sigma(\zeta')$$

$$=\int_0^{2\pi}\int_{T^{n-1}} g_2\left(\bar{\mu}_1 e^{-i\theta},\zeta',\bar{z}'\right) f_1\left(\mu_1 e^{i\theta},z',\bar{\zeta}'\right)|k_{z'}(\zeta')|^2 d\sigma(\zeta') e^{im\theta}\,d\theta.$$

Also let

$$H_{m3}(\zeta_1)=\int_0^{2\pi} g_1\left(\zeta_1 e^{i\theta},\zeta',\bar{z}'\right) f_2\left(\bar{\zeta}_1 e^{-i\theta},z',\bar{\zeta}'\right) e^{im\theta}\,d\theta$$

$$=\sum_{j-l=-m}\hat{g}_1(j,\zeta',\bar{z}')\hat{f}_2(l,z',\bar{\zeta}')\zeta_1^j\bar{\zeta}_1^l,$$

Then

$$|H_{m3}(\zeta_1)|\le\sum_{j-l=-m}\left|\hat{g}_1(j,\zeta',\bar{z}')\hat{f}_2(l,z',\bar{\zeta}')\right|$$

$$\le\left(\sum_{j\ge 0}|\hat{g}_1(j,\zeta',\bar{z}')|^2\right)^{1/2}\left(\sum_{j\ge 0}\left|\hat{f}_2(j,z',\bar{\zeta}')\right|^2\right)^{1/2}.$$

Using the same argument as the proof of $H_{m2}(z_1)$, we have

$$\left(\sum_{j\ge 0}|\hat{g}_1(j,\zeta',\bar{z}')|^2\right)^{1/2}\left(\sum_{j\ge 0}\left|\hat{f}_2(l,z',\bar{\zeta}')\right|^2\right)^{1/2}\in L^1(T^n).$$

It follows that $H_{m3}(\zeta_1)$ is continuous in variable z_1 on the closure $\bar{D}$. For every subsequence $\left\{z_{1\alpha_j}\right\}$ of the net $\{z_{1\alpha}\}$, by the dominated convergence theorem again and Corollary (1.3.2), we have

$$\lim_{z_{1\alpha_j}\to\mu_1}\int_{T^n} H_{m3}(\zeta_1)\left|k_{z_{1\alpha_j}}z'(\zeta)\right|^2 d\sigma(\zeta)$$

$$=\lim_{z_{1\alpha_j}\to\mu_1}\int_{T^n} H_{m3}\left(\Phi_{z_{1\alpha_j}}(\zeta_1)\right)|k_{z'}(\zeta')|^2 d\sigma(\zeta)$$

$$=\int_{T^{n-1}} H_{m3}(\mu_1)|k_{z'}(\zeta')|^2 d\sigma(\zeta')$$

$$=\int_{T^{n-1}}\int_0^{2\pi} g_1\left(\mu_1 e^{i\theta},\zeta',\bar{z}'\right) f_2\left(\bar{\mu}_1 e^{-i\theta},z',\bar{\zeta}'\right) e^{im\theta}\,d\theta\,|k_{z'}(\zeta')|^2 d\sigma(\zeta')$$

$$=\int_0^{2\pi}\int_{T^{n-1}} g_1\left(\mu_1 e^{i\theta},\zeta',\bar{z}'\right) f_2\left(\bar{\mu}_1 e^{-i\theta},z',\bar{\zeta}'\right)|k_{z'}(\zeta')|^2 d\sigma(\zeta')\, e^{im\theta}\,d\theta.$$

Summarizing the statement above, we obtain

$$\lim_{z_{1\alpha_j}\to\mu_1}\int_0^{2\pi}\langle T_fT_gk_{z_{1\alpha_j}e^{i\theta}}k_{z'},k_{z_{1\alpha_j}e^{i\theta}}k_{z'}\rangle\, e^{im\theta}d\theta$$

$$=\int_0^{2\pi}\int_{T^{n-1}}\left[g_1(\mu_1e^{i\theta},\zeta',\bar z')f_1(\mu_1e^{i\theta},z',\bar\zeta')\right.$$
$$\left.+g_2(\bar\mu_1e^{-i\theta},\zeta',\bar z')f_2(\bar\mu_1e^{-i\theta},z',\bar\zeta')\right]|k_{z'}(\zeta')|^2d\sigma(\zeta')e^{im\theta}d\theta$$
$$+\int_0^{2\pi}\int_{T^{n-1}}g_2(\bar\mu_1e^{-i\theta},\zeta',\bar z')f_1(\mu_1e^{i\theta},z',\bar\zeta')|k_{z'}(\zeta')|^2d\sigma(\zeta')e^{im\theta}d\theta$$
$$+\int_0^{2\pi}\int_{T^{n-1}}g_1(\mu_1e^{i\theta},\zeta',\bar z')f_2(\bar\mu_1e^{-i\theta},z',\bar\zeta')|k_{z'}(\zeta')|^2d\sigma(\zeta')e^{im\theta}d\theta$$

$$=\int_0^{2\pi}\langle T_{f(\mu_1e^{i\theta},\cdot)}T_{g(\mu_1e^{i\theta},\cdot)}k_{z'},k_{z'}\rangle\, e^{im\theta}d\theta.$$

Because $\int_0^{2\pi}\langle T_fT_gk_{z_{1\alpha_j}e^{i\theta}}k_{z'},k_{z_{1\alpha_j}e^{i\theta}}k_{z'}\rangle\, e^{im\theta}d\theta$ converges to the same number

$$\int_0^{2\pi}\langle T_{f(\mu_1e^{i\theta},\cdot)}T_{g(\mu_1e^{i\theta},\cdot)}k_{z'},k_{z'}\rangle\, e^{im\theta}d\theta$$

for every subsequence of the net $\{z_{1\alpha}\}$. Hence

$$\lim_{z_1\to\mu_1}\int_0^{2\pi}\langle T_fT_gk_{z_1e^{i\theta}}k_{z'},k_{z_1e^{i\theta}}k_{z'}\rangle e^{im\theta}d\theta$$
$$=\int_0^{2\pi}\langle T_{f(\mu_1e^{i\theta},\cdot)}T_{g(\mu_1e^{i\theta},\cdot)}k_{z'},k_{z'}\rangle\, e^{im\theta}d\theta.$$

This completes the proof of the theorem.

Corollary (1.3.5)[27]: Let $f,g\in L^\infty(T^n)$, if T_fT_g is compact, then $f(\mu)g(\mu)=0$, for almost $\mu\in T^n$.

Proof. Since T_fT_g is compact, so for any $z_1\in D,\mu\in T$,

$$\lim_{z_1\to\mu_1}\langle T_fT_gk_{z_1e^{i\theta}}k_{z'},k_{z_1e^{i\theta}}k_{z'}\rangle=0$$

and

$$\left|\langle T_fT_gk_z,k_z\rangle\right|\le\left\|T_fT_g\right\|.$$

Thus we have

$$\lim_{z_{1\alpha_j}\to\mu_1}\int_0^{2\pi}\langle T_fT_gk_{z_{1\alpha_j}e^{i\theta}}k_{z'},k_{z_{1\alpha_j}e^{i\theta}}k_{z'}\rangle\, e^{im\theta}d\theta=0,$$

by dominated convergence theorem for every converges sequence. By Theorem (1.3.4), we have

$$\int_0^{2\pi} \langle T_{f(\mu_1 e^{i\theta},\cdot)} T_{g(\mu_1 e^{i\theta},\cdot)} k_{z'}, k_{z'} \rangle\, e^{im\theta} d\theta = 0, for\ any\ m \in Z.$$

The injection of the Fourier transformation implies that

$$\langle T_{f(\mu_1 e^{i\theta},\cdot)} T_{g(\mu_1 e^{i\theta},\cdot)} k_{z'}, k_{z'} \rangle = 0, \text{for almost } \theta \in [0, 2\pi].$$

Hence

$$\langle T_{f(\mu_1,\cdot)} T_{g(\mu_1,\cdot)} k_{z'}, k_{z'} \rangle = 0, \text{for almost } \mu_1 \in T.$$

Using Theorem (1.3.4) $n-1$ times, we obtain

$$\langle T_{f(\mu_1,\mu_2,\cdots,\mu_{n-1},\cdot)} T_{g(\mu_1,\mu_2,\cdots,\mu_{n-1},\cdot)} k_{z_n}, k_{z_n} \rangle = 0,$$

for almost $\mu_1 \in T, \mu_2 \in T, \cdots, \mu_{n-1} \in T$.

By well-known fact of Toeplitz operators on $H^2(D)$, we immediately see that $f(\mu)g(\mu) = 0$ for almost $\mu \in T^n$. This completes the proof.

Theorem (1.3.6)[27]: Let $f, g \in L^2(T^n)$. Then $T_f T_g$ is a finite rank operator if and only if either f or g is zero.

Proof. Only we need to prove that "only if" part. Since f and g are functions in n variables, we will show that "only if" part by methods of mathematical induction for variables number.

(a) When $n = 1$, if $T_f T_g$ is a finite rank operator on $H^2(D)$, then by a result in [29], we have that either $f = 0$ or $g = 0$.

(b) Assume $n > 1$ and the result is truth for $n-1$, we will prove that the result must be true for n.

Assume $f, g \in L^2(T^n)$ and $T_f T_g$ is a finite rank operator on $H^2(D^n)$. Thus we know that $T_f T_g$ is a bounded operator on $H^2(D^n)$. It follows that $|\langle T_f T_g k_z, k_z \rangle| \le T_f T_g$. Using Theorem (1.3.4), we easily obtain that, $fg = 0$. This implies that $H_{\bar{f}}^* H_g = T_{fg} - T_f T_g$ is also a finite rank operator on $H^2(D^n)$. We write f, g as

$$\bar{f} = \sum_{i=-\infty}^{\infty} f_i(z') z_1^i, g \sum_{i=-\infty}^{\infty} g_i(z') z_1^i.$$

Let $k, l \in Z_+, \alpha, \beta \in Z_+^{n-1}$. Using the similar methods as in [3], we have

$$\begin{aligned}
&\langle H_{\bar{f}}^* H_g z_1^k z'^{\alpha}, z_1^l z'^{\beta} \rangle = \langle H_g z_1^k z'^{\alpha}, H_{\bar{f}} z_1^l z'^{\beta} \rangle \\
&= \sum_{i \le -(k+1)} \langle g_i(z') z_1^{i+k} z'^{\alpha}, H_{\bar{f}} z_1^l z'^{\beta} \rangle + \sum_{i \ge -k} \langle (I-P) g_i(z') z_1^{i+k} z'^{\alpha}, H_{\bar{f}} z_1^l z'^{\beta} \rangle \\
&= \sum_{i \le -(k+1)} \langle g_i(z') z_1^{i+k} z'^{\alpha}, \bar{f} z_1^l z'^{\beta} \rangle + \sum_{i \ge -k} \langle (I_2 - P_2) g_i(z') z'^{\alpha} z_1^{i+k}, \bar{f} z_1^l z'^{\beta} \rangle \\
&= \sum_{i \le -(k+1)} \langle g_i(z') z'^{\alpha}, f_{(i+k-l)}(z') z'^{\beta} \rangle + \sum_{i \ge -k} \langle (I_2 - P_2) g_i(z') z'^{\alpha}, f_{(i+k-l)}(z') z'^{\beta} \rangle,
\end{aligned}$$

Where I_2 is the identity on $L^2(T^{n-1})$ and P_2 is the projection from $L^2(T^{n-1})$ onto $H^2(D^{n-1})$. Therefore

$$\begin{aligned}
&\langle H_g z_1^k z'^{\alpha}, H_{\bar{f}}\, z_1^l z'^{\beta} \rangle - \langle H_g z_1^{k+1} z'^{\alpha}, H_{\bar{f}} z_1^{l+1} z'^{\beta} \rangle \\
&= \langle\, g_{-(k+1)}(z') z'^{\alpha}, f_{-(l+1)}(z') z'^{\beta} \rangle \\
&= \langle P_2 g_{-(k+1)}(z') z'^{\alpha}, f_{-(l+1)}(z') z'^{\beta} \rangle
\end{aligned}$$

$$= \langle T_{\bar{f}-(l+1)} T_{g-(k+1)} z'^{\alpha}, z'^{\beta} \rangle.$$

Let S_1 denote the multiplication by z_1 on $H^2(D^n)$, i.e., $S_1 h = z_1 h$ for $h \in H^2(D^n)$. The above relation implies that

$$\left(S_1^{*l} H_{\bar{f}}^* H_g S_1^k - S_1^{*(l+1)} H_{\bar{f}}^* H_g S_1^{k+1} \right) h(z') = T_{\bar{f}-(l+1)} T_{g-(k+1)} h(z')$$

for all $h \in H^2(D^{n-1})$.

Therefore $T_{\bar{f}-(l+1)} T_{g-(k+1)}$ is a finite rank operator on $H^2(D^{n-1})$. Assume, as induction hypothesis of $n-1$, that either $f_{-(l+1)} = 0$ or $g_{-(k+1)} = 0$ for all $l, k \in Z_+$.

Hence either $f_{-(l+1)} = 0$ for any $l \geq 0$ or $g_{-(k+1)} = 0$ for any $k \geq 0$. This implies that either $\bar{f}$ or g is analytic in variable z_1. Similarly either $\bar{f}$ or g is analytic in variable $z_j (2 \leq j \leq n)$. Without loss of generality, suppose that $\bar{f}$ is analytic in variables $z_1, \ldots, z_j$ and g is analytic in variables $z_{j+1}, \ldots, z_n$. Since $f(\mu) g(\mu) = 0$ for almost $\mu \in T^n$, so let $E \times F \subseteq T^j \times T^{n-j}$ be zero set of f and E have positive measure in T^j. By the assumption, f is analytic in variables $z_1, \ldots, z_j$ and $f \in L^2(T^n)$, hence for each fixed $(z_{j+1}, \ldots, z_n) \in T^{n-j}$, we have $f \in H^2(T^j)$.

Thus $f = 0$ for almost $(z_1, \ldots, z_j) \in T^j$ and $(z_{j+1}, \ldots, z_n) \in F$, i.e., $E = T^j . f$ is zero on $T^j \times F$. If the measure of F in T^{n-j} is 1, then $f = 0$ on T^n. Assume that the measure of F in T^{n-j} is less than 1, then g is zero on $T^n - T^j \times F = T^j \times (T^{n-j} - F)$ and the measure of $T^{n-j} - F$ is positive. But g is analytic in $z_{j+1}, \ldots, z_n$ and for every fixed $(z_1, \ldots, z_j) \in T^j$, we have $g \in H^2(T^{n-j})$. This implies that $g = 0$ for almost $(z_{j+1}, \ldots, z_n) \in T^{n-j}$. Thus we have $T^{n-j} - F = T^{n-j}$, i.e., $g = 0$ on $T^n = T^j \times T^{n-j}$. Thus we shown that "only if" part holds for n. By the principle of mathematical induction, it follows that "only if" part is true for all $n \geq 1$. This completes the proof of the theorem.

Corollary (1.3.7)[27]: Let $f, g \in L^2(T^n)$. The following are equivalent:

(a) $T_f T_g = 0$.

(b) $T_f T_g$ is a finite rank operator.

(c) Either f or g is zero.

Theorem (1.3.8)[27]: Let f and g be two bounded pluriharmonic function on D^n for $n > 1$. Then $T_f T_g$ is compact if and only if f or g is zero.

Proof. First we write

$$f = f_1 + \bar{f}_2 = \sum_{j \geq 0} f_1(j, z') z_n^j + \sum_{j \geq 0} \overline{f_2(j, z') z_n^j},$$

$$g = g_1 + \bar{g}_2 = \sum_{j \geq 0} g_1(j, z') z_n^j + \sum_{j \geq 0} \overline{g_2(j, z') z_n^j},$$

Where f_i, g_i all in $H^2(D^n)$, $f_i(j, z') \in H^2(D^{n-1})$ and $g_i(j, z') \in H^2(D^{n-1})$ for $i = 1, 2$, $z' = (z_1, \ldots, z_{n-1})$. It is only to prove that "only if" part. In fact, using Theorem (1.3.3), we know

$$\int_0^{2\pi} \langle T_{f(\mu_1 e^{i\theta}, \cdot)} T_{g(\mu_1 e^{i\theta}, \cdot)} k_{z'}, k_{z'} \rangle \, e^{im\theta} d\theta =$$

$$\lim_{z_1\to\mu_1}\int_0^{2\pi}\langle T_fT_gk_{z_1e^{i\theta}}k_{z'},k_{z_1e^{i\theta}}k_{z'}\rangle\ e^{im\theta}d\theta=0.$$

It is follows that $\langle T_{f(\mu_1\cdot)}T_{g(\mu_1\cdot)}k_{z'},k_{z'}\rangle=0$, for almost $\mu_1\in T$. The limits can now proceed. Last we obtain $\langle T_{f(\mu',\cdot)}T_{g(\mu',\cdot)}k_{z_n},k_{z_n}\rangle=0$, for almost $\mu'=(\mu_1,\dots,\mu_{n-1})\in T^{n-1}$ and all $z_n\in D$. That is

$$f_1(\mu',z_n)g_1(\mu',z_n)+\bar{f}_2(\mu',z_n)\bar{g}_2(\mu',z_n)$$
$$+P[g_1\bar{f}_2](z_n)+\bar{g}_2(\mu',z_n)f_1(\mu',z_n)=0$$

for all $z_n\in D$.

Since $f_1(\mu',z_n)g_1(\mu',z_n)+\bar{f}_2(\mu',z_n)\bar{g}_2(\mu,z_n)+P[g_1\bar{f}_2](z_n)$ is harmonic in z_n, we have $(\bar{g}_2(\mu',z_n)-\bar{g}_2(\mu',0))(f_1(\mu',z_n)-f_1(\mu',0))=0$. In addition $f(\mu)g(\mu)=0$ by Theorem (1.3.3), we can see that either $f(\mu',z_n)=0$ or $g(\mu',z_n)=0$ for all $z_n\in D$. Hence there is set $E\subseteq T^{n-1}$ which have positive measure, such that either $f(\mu)$ or $g(\mu)$ is zero on $E\times T$. For explicit, let

$$g=\sum_{j\geq 0}g_1(j,\mu')\mu_n^j+\sum_{j\geq 0}\overline{g_2(j,\mu')\mu_n^j}$$

be zero on $E\times T$. This implies that all $g_1(j,\mu')$ and $g_2(j,\mu')$ are zero on E. But E have positive measure in T^{n-1}, $g_1(j,\mu')$ and $g_2(j,\mu')$ all in $H^2(T^{n-1})$, hence all $g_1(j,\mu')$ and $g_2(j,\mu')$ are zero on T^{n-1}. This implies that $g=0$ on D^n. This completes the proof of the theorem.

Note that when $n=1$, the pluriharmonic function on D is harmonic. Any $f\in L^\infty(T)$ can be extended as harmonic function on D. It is well-known that there are two Toeplitz operators such that their product is compact but none of them is compact. So Theorem (1.3.8) is false when $n=1$.

Theorem (1.3.9)[27]: let f and g be in $L^\infty(T^n)$. If there is a $h\in L^\infty(T^n)$, such that $T_fT_g-T_h$ is a compact operator, then $f(\mu)g(\mu)=h(\mu)$ for almost $\mu\in T^n$.

Proof. Since $T_h=T_1T_h$, using Theorem (1.3.3), we have

$$\lim_{z_1\to\mu_1}\int_0^{2\pi}\langle(T_fT_g-T_h)k_{z_1e^{i\theta}}k_{z'},k_{z_1e^{i\theta}}k_{z'}\rangle e^{im\theta}d\theta$$
$$=\int_0^{2\pi}\langle\left(T_{f(\mu_1e^{i\theta}\cdot)}T_{g(\mu_1e^{i\theta}\cdot)}-T_{h(\mu_1e^{i\theta}\cdot)}\right)k_{z'},k_{z'}\rangle\,e^{im\theta}d\theta.$$

As $T_fT_g-T_h$ is compact, so

$$\lim_{z_1\to\mu_1}\int_0^{2\pi}\langle(T_fT_g-T_h)k_{z_1e^{i\theta}}k_{z'},k_{z_1e^{i\theta}}k_{z'}\rangle e^{im\theta}d\theta=0.$$

It is follows that

$$\int_0^{2\pi}\langle\left(T_{f(\mu_1e^{i\theta}\cdot)}T_{g(\mu_1e^{i\theta}\cdot)}-T_{h(\mu_1e^{i\theta}\cdot)}\right)k_{z'},k_{z'}\rangle\,e^{im\theta}d\theta=0.$$

The injection of the Fourier transformation implies that

$$\langle\left(T_{f(\mu_1 e^{i\theta}\cdot)}T_{g(\mu_1 e^{i\theta}\cdot)} - T_{h(\mu_1 e^{i\theta}\cdot)}\right)k_{z'}, k_{z'}\rangle = 0.$$

Also we have

$$0 = \lim_{z_2\to\mu_2}\int_0^{2\pi} \langle\left(T_{f(\mu_1\cdot)}T_{g(\mu_1\cdot)} - T_{h(\mu_1\cdot)}\right)k_{z_2 e^{i\theta}}k_{z''}, k_{z_2 e^{i\theta}}k_{z''}\rangle e^{im\theta}d\theta$$

$$= \int_0^{2\pi} \langle\left(T_{f(\mu_1,\mu_2 e^{i\theta}\cdot)}T_{g(\mu_1,\mu_2 e^{i\theta}\cdot)} - T_{h(\mu_1,\mu_2 e^{i\theta}\cdot)}\right)k_{z''}, k_{z''}\rangle e^{im\theta}d\theta.$$

Hence

$$\langle\left(T_{f(\mu_1,\mu_2 e^{i\theta}\cdot)}T_{g(\mu_1,\mu_2 e^{i\theta}\cdot)} - T_{h(\mu_1,\mu_2 e^{i\theta}\cdot)}\right)k_{z''}, k_{z''}\rangle e^{im\theta}d\theta = 0$$

for almost $(\mu_1,\mu_2)\in T^2$. Using the above argument, we can obtain

$$\langle\left(T_{f(\mu_1,\mu_2,\dots,\mu_{n-1}\cdot)}T_{g(\mu_1,\mu_2,\dots,\mu_{n-1}\cdot)} - T_{h(\mu_1,\mu_2,\dots,\mu_{n-1}\cdot)}\right)k_{z_n}, k_{z_n}\rangle = 0$$

for almost $(\mu_1,\dots,\mu_{n-1})\in T^{n-1}$.

It implies that $f(\mu)g(\mu)=h(\mu)$ for almost $\mu\in T^n$. This completes the proof of the theorem.

Theorem (1.3.10)[27]: Let f and g be in $L^\infty(T^n)$. The following are equivalent:

(a) There is a $h\in L^\infty$ such that $T_fT_g = T_h$.

(b) There is a $h\in L^\infty$ such that $T_fT_g = T_h$ is finte operator.

(c) The Hankel product $H_{\bar f}^* H_g$ is a finte rank operator.

(d) For every $i(1\le i\le n)$, either $\bar f$ or g is analytic in variable z_i.

Proof. We first show that (d) implies (a). Without loss of generality, assume that $\bar f$ is analytic in $z_1,\dots,z_j$, g is analytic in $z_{j+1},\dots,z_n$. Then by a straightforward computation, for every $h_1,h_2\in H^2(D^n)$, we have

$$(I-P)(gh_1) = \sum_{m=(m_1,m_2)\in Z^j\times Z_+^{n-j}} a_m z^m,$$

$$(I-P)(\bar f h_2) = \sum_{m=(m_1,m_2)\in Z_+^j\times Z^{n-j}} b_m z^m.$$

Thus

$$\langle H_g h_1, H_{\bar f}^* h_2\rangle = 0.$$

It follows that $H_{\bar f}^* H_g = 0$, i.e., $T_fT_g - T_{fg} = -H_{\bar f}^* H_g = 0$. Thus we put $h = fg$, it follows that (a) holds.

Using Theorem (1.3.9), if $T_fT_g - T_h$ is finte rank operator, then $h = fg$, it is easy to see that (b) implies (c). It is obvious that (a) implies (b).

Now we prove that (c) implies (4). We write f and g as

$$\bar f = \sum_{i=-\infty}^{\infty} f_i(z')z_1^i, \text{ and } g = \sum_{i=-\infty}^{\infty} g_i(z')z_1^i.$$

Let $S_1 h = z_1 h$ be multiplication operator on $H^2(D^n)$. Using the same argument as Theorem (1.3.6), we have

$$\left(S_1^{*l}H_{\bar f}^*H_gS_1^k - S_1^{*(l+1)}H_{\bar f}^*H_gS_1^{k+1}\right)h(z') = T_{\bar f_{-(l+1)}}T_{g_{-(k+1)}}h(z')$$

for $h \in H^2(D^{n-1})$. Thus $T_{\bar{f}-(l+1)}T_{g-(k+1)}$ is a finte rank operator on $H^2(D^{n-1})$.

Using Theorem (1.3.6), either $f_{-(l+1)}(z') = 0$ or $g_{-(k+1)}(z') = 0$, for any $l \geq 0, k \geq 0$. Therefore either $f_{-(l+1)}(z') = 0$ for any $l \geq 0$ or $g_{-(k+1)}(z') = 0$ for any $k \geq 0$.

That is either $\bar{f}$ or g is analytic in z_1. This finishes the proof of the theorem.

In [30], Caixing Gu and Dechao Zheng give an example that $T_f T_g - T_{fg}$ is compact but is not zero. But if f and g are two bounded pluriharmonic functions on D^n, this case does not take place.

Theorem (1.3.11)[27]: Let f and g be two bounded pluriharmonic functions on D^n.

The following are equivalent:

(a) $T_f T_g - T_{fg} = 0$.

(b) $T_f T_g - T_{fg}$ is compact.

(c) $H_{\bar{f}}^* H_g$ is compact.

(d) $\|H_{\bar{f}}^* H_g k_z\| \to 0$ (as $z \to \partial D^n$).

(e) $\lim_{z \to \partial D^n} \langle H_{\bar{f}}^* H_g k_z, k_z \rangle = 0$.

(f) For every $z_i (1 \leq i \leq n)$, either f or g is analytic in z_i.

Proof. We only prove that (e) implies (f). Suppose the condition (e) holds, then using Corollary (1.3.2), we have $\langle (T_{f(\mu',\cdot)} T_{g(\mu',\cdot)} - T_{f(\mu',\cdot) g_{\mu',\cdot}}) k_{z_n}, k_{z_n} \rangle = 0$ for almost $\mu' \in T^{n-1}$ and every $z_n \in D$. For fixed μ', both $T_{f(\mu',\cdot)}$ and $T_{g(\mu',\cdot)}$ are Toeplitz operators on $H^2(D)$. It is easy to prove that $T_{f(\mu',\cdot)} T_{g(\mu',\cdot)} = T_{f(\mu',\cdot) g(\mu',\cdot)}$.

By a result in [29], we have that either $\overline{f(\mu', z_n)}$ or $\overline{g(\mu', z_n)}$ is analytic in z_n. This implies that there is positive measure set $E \subseteq T^{n-1}$, such that for every $\mu' \in E$, either $\overline{f(\mu', z_n)}$ or $g(\mu', z_n)$ is analytic in z_n. Since f is bounded pluriharmonic function, we can write

$$f = \sum_{j \geq 0} f_{i1}(z') z_1^i + \sum_{j \geq 0} f_{i2}(\bar{z}') \bar{z}_1^i,$$

where $f_{i1}(z')$ and $\overline{f_{i2}(\bar{z}')}$ are all in $H^2(D^{n-1})$. Thus $f_{i1}(z')$ is zero on E. It follows that $f_{i1}(z')$ is zero on T^{n-1}. Hence for almost $\mu \in T^{n-1}, \bar{f}$ is analytic in z_n. This finishes the proof of the theorem.

Note that when $n = 1$, Theorem (1.3.11) is false. In fact, when $n = 1$, if f or g is in $H^\infty + C$, then $H_{\bar{f}}^* H_g$ is compact.

Chapter 2
The Hardy Space and Module over the Bidisk

We study some elementary properties of the submodules and show, in some cases, how the operator theoretical properties are related to the module theoretical properties. The last part focus on the study of double commutativity of compression operators. We will show that R_z and R_w, as well as S_z and S_w, essentially doubly commute. Moreover, both $[R_w^*, R_z]$ and $[S_w^*, S_z]$ are actually Hilbert-Schmidt.

Section (2.1): Operator Theory:

In operator model theory the vector-valued Hardy space $H^2(E)$ is used to construct models for contractions (of. [38], [47]). Without loss of generality we can let E be the Hardy space over the unit disk and identify $H^2(E)$ with the Hardy space over the bidisk $H^2(D^2)$. This identification, on the one hand, can give us a better understanding of some elements in model theory, while, on the other hand, it brings new techniques into the study of $H^2(D^2)$. We will construct a universal model for contractive analytic functions and give an application to a submodule problem in $H^2(D^2)$. We study some elementary properties of submodules in $H^2(D^2)$. We focuses on the almost donble commutativity of compression operators on a quotient module $H^2(D^2) \ominus M$.

We let $\boldsymbol{C}$ denote the complex plane and $\boldsymbol{C}^2$ be the cartesian product of two copies of $\boldsymbol{C}$.

Thus the points of $\boldsymbol{C}^2$ are the ordered pairs (z, w). We let Z_+ denote the set of nonnegative integers.

The ring of polynomials of z and w will be denoted by $\mathcal{R}$, though sometimes the standard notation $\boldsymbol{C}[z, w]$ is also used to avoid possible confusion. The ideal generated by polynomials $p_1, p_2, \dots, p_n$ is denoted by $(p_1, p_2, \dots, p_n)$.

The unit bidisk in $\boldsymbol{C}^2$ is denoted by D^2 with distinguished boundary T^2, where D is the unit disk and T is the unit circle. The closure of the polynomials over D^2 under the supremum norm will be denoted by $A(D^2)$ and is said to be the bidisk algebra. We let $|dz|$ denote the normalized Lebesgue measure on the unit circle T and $|dz||dw| = dm$ be the product measure on the torus T^2.

The Hardy space $H^2(D^2)$ is the Hilbert space of holomorphic functions over D^2 which satisfy the inequality

$$\sup_{0 \le r < 1} \int_{T^2} |f(rz, rw)|^2 dm < \infty.$$

The norm $\|f\|$ of a function $f \in H^2(D^2)$ is defined by

$$\|f\|^2 := \sup_{0 \le r < 1} \int_{T^2} |f(rz, rw)|^2 dm.$$

The inner product induced by this norm will be denoted by $\langle \cdot, \cdot \rangle$.

By Fatou's theorem, every function in $H^2(D^2)$ has nontangential limits at almost every point of T^2. If we let $\hat{f}$ denote the boundary function of $f \in H^2(D^2)$, then

$$\hat{f} \in H^2(T^2, dm) := \overline{\text{span}}\{z^i w^j : z, w \in T, i, j \in Z_+\},$$

where the closure is taken in $L^2(T^2, dm)$. And it is also well known that each function in $H^2(T^2, dm)$ has a unique analytic extension to D^2 which belongs in $H^2(D)$. For convenience, we identify $H^2(D^2)$ with $H^2(T^2, dm)$ and will use f to denote its boundary value $\hat{f}$ as well.

For any bounded function ϕ in $A(D^2)$, we define the Toeplitz operator T_ϕ mapping $H^2(D^2)$ to itself such that

$$T_\phi(f) = P(\phi f),$$

where P is the orthogonal projection from $L^2(T^2, dm)$ to $H^2(D^2)$.

We let $H^\infty(D^2)$ be the space of all bounded holomorphic functions in D^2 with

$$\|f\|_\infty = \sup\{|f(z,w)|, \qquad (z,w) \in D^2\}.$$

It is easily seen that $H^\infty(D^2)$ is a Banach algebra with pointwise multiplication and addition. The collection of invertible elements in the algebra $H^\infty(D^2)$ is denoted by $\left(H^\infty(D^2)\right)^{-1}$.

It is well known that the space $H^2(D^2)$ is an $A(D^2)$ −module with action defined by point wise multiplication by $A(D^2)$ functions. A closed subspace $M \subset H^2(D^2)$ is said to be w invariant if it is invariant under multiplication by $w. M$ is said to be a submodule if it is invariant under the module action, or equivalently, M is invariant under multiphcation by both z and w.

Restrictions of T_z and T_w to a submodule M will be denoted by R_z and R_w respectively.

For any subset $X \subset H^2(D^2)$, we let

$$[X] := \overline{\text{span}}\{A(D^2)X\}$$

denote the submodule generated by X.

If M is a proper submodule of $H^2(D^2)$ and

$$p: H^2(D^2) \to M, \;\; q: H^2(D^2) \to H^2(D^2) \ominus M$$

are the orthogonal projections, then one checks that the map $S: A(D^2) \to B(H^2(D^2) \ominus M)$ defined by

$$S_{fg} \coloneqq {}_q f_g$$

for $f \in A(D^2)$ and $g \in H^2(D^2) \ominus M$ is a homomorphism which tnrrts $H^2(D^2) \ominus M$ into a quotient $A(D^2)$ −module. One sees that the operators S_z, S_w are compressions of the Toeplitz operators T_z, T_w to $H^2(D^2) \ominus M$.

If E is a separable complex gilbert space with an orthonormal bases $\{\eta_j\}$, we can identify the E −valued Hardy space $H^2(E)$ with $H^2(D^2)$ in the following way:

Let u be the unitary map from E to $H^2(D)$ defined by

$$w\eta_j = z^j, j \geq 0.$$

Then $U = I \otimes u$ is a unitary from $H^2(D) \otimes E$ to $H^2(D) \otimes H^2(D)$ such that

$$U(w^i \eta_j) = z^i w^j, \qquad i, j \geq 0.$$

We will take a look at some facts in model theory in the context of $H^2(D^2)$ in Section 1.

The following family of evaluation operators is very important in our study and will be used often.

Definition (2.1.1)[34]: For $\lambda \in D$, we define the evaluation operator $N(\lambda)$ from $H^2(D^2)$ to $H^2(D)$ by

$$N(\lambda)f(z) = f(z, \lambda), f \in H^2(D^2).$$

It is easy to see using the Cauchy integral formula that $N(\lambda)$ has an integral representation from which we get $\|N(\lambda)\| = (1 - |\lambda|^2)^{-1/2}$.

We will be mainly interested in the restrictions of $N(\lambda)$ to certain subspaces and will use the same notation to denote these restrictions.

The evaluation operator was studied in [48] but later we found that it can be viewed as a universal model for contractive analytic functions.

For any function $f(z,w) = \sum_{j=0}^{\infty} z^j f_j(w)$ in $H^2(D^2)$, we define

$$\Phi_f(w) := \int_T |f(z,w)|^2 |dz| = \sum_{j=0}^{\infty} \left|f_j(w)\right|^2, w \in D.$$

It is easy to check that Φ_f is subharmonie on D, and by the Fubini theorem we have

$$\int_T \Phi_f(w)|dw| = \|f\|^2.$$

Definition (2.1.2)[34]: A function $f \in H^2(D^2)$ will be said to be R −inner if $\Phi_f(w) = 1$ almost everywhere on T.

One sees that if f is R −inner, then by the subharmonicity, $\Phi_f(\lambda) \leq 1$ for all $\lambda \in D$, and $\Phi_f(\lambda) = 1$ for some $\lambda \in D$ if and only if Φ_f is a constant.

If Φ_f is a constant, then

$$0 = \frac{\partial^2 \Phi_f}{\partial w \partial \overline{w}} \sum_{j=0}^{\infty} \left|f_j'(w)\right|^2, w \in D,$$

which implies that the f_j's are all constants and hence f is a function ill z only.

If M is w −invariant, then for every $f \in M \ominus wM$,

$$\begin{aligned}\int_T \Phi_f(w) w^i |dw| &= \int_T \left(\int_T |f(z,w)|^2 |dz| \right) w^i |dw| \\ &= \int_{T^2} w^i f(z,w) \overline{f(z,w)}\, |dz||dw| \\ &= \langle w^i f, f \rangle = 0, \quad i \geq 1.\end{aligned}$$

This implies

$$\Phi_f(w) = \|f\|^2$$

almost everywhere on T since Φ_f is real. The computation above yields the following

Proposition (2.1.3)[34]: If $M \subset H^2(D^2)$ is an invariant subspace for w, then every function in $M \ominus wM$ with norm 1 is R −inner.

Definition (2.1.4)[34]: A $B(E', E)$-valued analytic function $\theta(w)$ on D is called left-inner(inner)if its boundary values on the unit circle T are almost everywhere isometries (unitaries) from E' into E.

If M is w-invariant in $H^2(D^2) = H^2(D) \otimes H^2(D)$, then the Lax-Halmos theorem asserts that

$$M = \theta(w) H^2(E)$$

for some Hilbert space E and a $B\big(E, H^2(D)\big)$ −valued left inner function 0. Proposition (2.1.3) enables us to restate the Lax-Halmos theorem in $H^2(D^2)$.

Corollary (2.1.5)[34]: (Lax-Halmos) If M is any w-invariant subspace of $H^2(D^2)$, then the evaluation operator N is left inner from $M \ominus wM$ to $H^2(D)$ and

$$M = N(w)H^2(E),$$

where $E = M \ominus wM$.

Proof. If $f \in M \ominus wM$, then

$$\|N(\lambda)(f)\|^2 = \int_T |f(z,\lambda)|^2 |dz| = \Phi_f(\lambda)$$

and the corollary follows from the remarks preceeding Proposition (2.1.3) and the fact that

$$M = \oplus_{i=0}^{\infty} w^i (M \ominus wM).$$

If S_w is the compression of multiplication by w to $H^2(D^2) \ominus M$, then it is well known in model theory that the N in Corollary (2.1.5) is equivalent to either the characteristic operator function for S_w or its direct sum with a constant untary. This observation and the spectral relation between a contraction and its characteristic function give us the following

Corollary (2.1.6)[34]: If M is a $w-$invariant subspaee of $H^2(D^2)$ and S_w is the compression of multiplication by w to $H^2(D^2) \ominus M$, then

$$\sigma(S_w) = \sigma(N),$$

where $\sigma(N)$ is the set of points $\lambda \in D$ for which the operator $N(\lambda)$ is not boundedly invertible from $M \ominus wM$ to $H^2(D)$, together with those $\lambda \in T$ not lying on any of the open arcs of T on which $N(\lambda)$ is a unitary operator valued analytic function of λ.

If M is $w-$invariant, then the evaluation operator N is left-inner when restricted to $M \ominus wM$ by Corollary (2.1.5). It turns out that every left-inner function is of this form for some $w-$Invariant subspace $M \subset H^2(D^2)$. Actually a general statement holds which provides a universal model for contractive analytic functions.

Proposition (2.1.7)[34]: If $\left(E, H^2(D), \theta(w)\right)$ is a contractive analytic function, then there is a subspace $H \subset H^2(D^2)$ and a constant contraction S from E to H such that S has dense range and

$$\theta(\lambda) = N(\lambda)S, \forall \ominus \in D.$$

When θ is left-inner, S is a unitary and H is of the form $M \ominus wM$ for some $w-$invariant subspace M.

Proof. We define $S: E \to H^2(D^2)$ by

$$Sx = \theta(w)x, \qquad \forall x \in E$$

and let H be the closure of the range of S. Then

$$N(\lambda)(Sx) = N(\lambda)(\theta x) = \theta(\lambda)x, \quad \forall x \in E,$$

i.e., $\theta(\lambda) = N(\lambda)S$ and S is a contraction follows from the fact that $\theta(w)$ is contractive.

When θ is left-inner, we let M be the $w-$invariant subspace generated by H and one checks that

$$M \ominus wM = \theta E.$$

If $\|N(0)f\| < \|f\|$ for an $f \in M \ominus wM, f \neq 0$, then N is said to be purely contractive and it is well known that in this case N is equivalent to the characteristic operator function for S_w.

For any $f \in H^2(D^2)$, we can write

$$f(z,w) = f(z,0) + wg(z,w)$$
for some $g \in H^2(D^2)$. So $\|N(0)f\| = \|f\|$ if and only if $g = 0$, e.g. f is independent of variable w.

Corollary (2.1.6) has an interesting application which reveals how a module theoretical invariant is related to the operator theoretical properties of the compression operators S_z and S_w.

The proof requires a lemma from [48].

Lemma (2.1.8)[34]: ([48]) If $M \subset H^2(D^2)$ is z −invariant, then $N(\lambda)$ restricted to $M \ominus zM$ is Hilbert-Schmidt for every $\lambda \in D$.

Let us first have an intuitive look at this lemma. If M is z −invariant, then the functions in $M \ominus zM$ depend largely on the variable w and hence they don't vary much if the w variable is fixed. Let us consider an example.

If $M = \phi H^2(D^2)$ for some inner function ϕ, then
$$M \ominus zM = \{\phi(z,w)g(w) : g \in H^2(D)\}$$
and $N(\lambda)(M \ominus zM) = C_\phi(z,\lambda)$.

We now go to the application which we state as

Theorem (2.1.9)[34]: If M is a submodule of $H^2(D^2)$ with $M \ominus (zM + wM)$ infinite dimensional, then
$$\sigma(S_z) = \sigma(S_w) = \bar{D}.$$
Proof. If $\{g_n : n \geq 0\} \subset M \ominus (zM + wM) = (M \ominus zM) \cap (M \ominus wM)$ is anortho normal basis, then for every $\lambda \in D$,
$$\sum_{j=0}^{\infty} \|N(\lambda)g_n\|^2 < \infty$$
by Lemma (2.1.8). In particular, this implies that
$$\lim_{n\to\infty} \|N(\lambda)g_n\| = 0$$
and hence $\lambda \in \sigma(s_w)\sigma$ by Corollary (2.1.6).

On the other hand S_w, is clearly a contraction, so in conclusion
$$\sigma(S_w) = \bar{D}.$$
The proof for S_z is similar.

In the one variable case, if M is an $A(D)$ submodule of $H^2(D)$ or the Bergman space $L_a^2(D)$, respectively, then $M \ominus wM$ is a generating set for M by Beurling's theorem or by the results of Aleman, Richter and Sunderberg in [35]. In $H^2(D^2)$, $M \ominus (zM + wM)$ is a natural analogue of '$M \ominus wM$' in the one variable case. However, $M \ominus (zM + wM)$ is not, in general, a generating set for M. We will give one simple example at the end. Here we show the existence of a submodule in $H^2(D^2)$ which has infinite rank but for which $M \ominus (zM + wM)$ is finite dimensional.

This submodule is constructed by Rudin in [46].

Corollary (2.1.10)[34]: There is a submodule $M \subset H^2(D^2)$ of infinite rank with
$$\dim M \ominus (zM + wM) < \infty.$$
Proof. If M is the collection of all the functions in $H^2(D^2)$ that have a zero of order greater than or equal to n at $(0, \alpha_n) = (0, 1 - n^{-3})$ for $n = 1, 2, 3, \ldots$, then M is a submodule of $H^2(D^2)$ of infinite rank by Rudin ([46, pp 71-72]). We now prove that $\dim\big(M \ominus (zM + wM)\big) < \infty$ by showing

$$\sigma(S_w) = \{1\} \cup \{\alpha_n : n \geq 1\}.$$

For the study of the spectra for compression operators on more general quotient spaces, see [37] and [40].

First of all, for each n every function in $N_{\alpha_n}(M)$ vanishes at $z = 0$ and hence $N_{\alpha_n}(M)$ is a proper subset of $H^2(D)$. Therefore

$$\sigma(S_w) \supset \{1\} \cup \{\alpha_n : n \geq 1\}$$

by Corollary (2.1.5).

If we let

$$B(w) = \prod_{n=1}^{\infty} \left(\frac{w - \alpha_n}{1 - \overline{\alpha_n} w} \right)^n,$$

then $B(w)$ is a Blaschke product and $B \in M$ from the construction of M. If $\lambda \in D$ and $B(\lambda) \neq 0$, then

$$B(w) - B(\lambda) = (w - \lambda) b(w)$$

for some bounded analytic function b and therefore for every $f \in M \ominus wM$

$$\begin{aligned}(S_w - \lambda) S_b f &= q\big((w - \lambda) b f\big) \\ &= q(Bf - B(\lambda) f) \\ &= -B(\lambda) q f = -B(\lambda) f.\end{aligned}$$

Thus we have $\lambda \in \rho(S_w)$.

If $|\lambda| = 1$ and $\lambda \neq 1$, then $B(w)$ extends analytically into a neighborhood of λ and the same argument carries over. In conclusion, we have

$$\sigma(S_w) = \{1\} \cup \{\alpha_n : n \geq 1\}.$$

This implies that $\dim\big(M \ominus (zM + wM)\big) < \infty$ by Theorem (2.1.9) and hence $[M \ominus (zM + wM)]$ is a proper submodule of M.

We showed in Corollary (2.1.10) that for Rudin's submodule $M\, M \ominus (zM + wM)$ is finite dimensional. Here is a question that may have an interesting answer.

Question (2.1.11)[34]: If M is Kudin's submodule as in the proof of Corollary (2.1.10), then what is $\dim\big(M \ominus (zM + wM)\big)$?

Another way to look at the spectrum of S_w in the proof of Corollary (2.1.10) is through the theory of C_0 class operators. Let us give the definition first.

Definition (2.1.12)[34]: A completely non-unitary contraction a is said to be in the class C_0 if there is a non-zero $\phi \in H^\infty(D)$ such that $\phi(a) = 0$.

See [47] and [36] for a detailed treatment of C_0 operators.

From the construction of M, S_w is an operator in the class C_0 with B as its minimal function. Proposition 4.2 in [38] then implies that

$$\sigma(S_w) = \{1\} \cup \{\alpha_n : n \geq 1\}.$$

We finish with a proposition on C_0 operators. It should be a known fact.

Proposition (2.1.13)[34]: If M is a submodule in $H^2(D^2)$, then S_w on $H^2(D^2) \ominus M$ is in C_0 if and only if there is bounded function $\phi(w) \in M$.

Proof. If $\phi(w)$ is a bounded function in M, then it is easy to check that

$$\phi(S_w) = S_\phi = 0.$$

Conversely, if there is a $\phi \in H^\infty(D)$ such that $\phi(S_w) = 0$, then

$$q(\phi) = q\phi(q1) = \phi(S_w)(q1) = 0$$

and hence $\phi(w) \in M$.

Proposition (2.1.13) will be needed to give a necessary condition for $[S_z, S_w^*] = 0$.

We study some elemexttary properties of submodules of $H^2(D^2)$. We first give an estimate of the dimension of the quotient $M \ominus IM$, where M is any submodule of $H^2(D^2)$ and $I \subset R$ is any ideal. Then we will give some applications.

If M is a submodule, then $zM + wM$ is proper in M. More generally, if $I \subset R$ is an ideal whose zero variety $V(I)$ intersects D^2 nontrivially, then $I \cdot M$ is a proper subspace of M.

The following theorem gives an estimate of the dimension of $M \ominus IM$.

Theorem (2.1.14)[34]: If $I \subset R$ is an ideal and $M \subset H^2(D^2)$ is a submodule, then

$$\dim(M \ominus IM) \le \dim(R/I)\,\mathrm{rank}(M).$$

Proof. We assume $\dim(R/I) = m_1 < \infty$ with a basis $\{v_1, v_2, \dots, v_{m_1}\}$ for R/I and $\mathrm{rank}(M) = m_2 < \infty$ with a generating set $\{e_1, e_2, \dots, e_{m_2}\}$ for M.

If $\phi \in M \ominus IM$, then thereis a sequence of polynomials $\{f_j^n : n \ge 0, j = 1,2,3, \dots, m_2\}$ such that

$$\lim_{n\to\infty} \sum_{j=1}^{m_2} f_j^n e_j = \phi$$

in $H^2(D^2)$. For each f_j^n, we write

$$f_j^n = f_{j,I}^n + r_j^n$$

With $f_{j,I}^n \in I$ and $r_j^n \in R/I$. If we Iet $P: M \to M \ominus IM$ be the orthogonal projection, then

$$\phi = P\phi = P\left(\lim_{n\to\infty} \sum_{j=1}^{m_2} f_j^n e_j\right)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{m_2} P\left(f_{j,I}^n e_j + r_j^n e_j\right)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{m_2} P\left(r_j^n e_j\right).$$

Since $\{v_1, v_2, \dots, v_{m_1}\}$ is a basis for R/I we can write

$$r_j^n = \sum_{i=1}^{m_1} c_{j,i}^n v_i,$$

Where $c_{j,i}^n, n \ge 0, 1 \le i \le m_1, 1 \le j \le m_2$ are constants. Then,

$$\phi = \lim_{n\to\infty} \sum_{j=1}^{m_2} \sum_{i=1}^{m_1} c_{j,i}^n P\left(v_i e_j\right).$$

and hence $\phi \in span\{P(v_i e_j), 1 \le i \le m_1, 1 \le j \le m_2\}$. Therefore,

$$\dim(M \ominus IM) \le m_1 m_2 = \dim(R/I)\, rank(M).$$

Corollary (2.1.15)[34]: If $M \subset H^2(D^2)$ is a submodule, then

$$\dim\big(M \ominus (zM + wM)\big) \leq \operatorname{rank}(M).$$

Proof. Let $I = (z, w) \subset R$, then $\dim(R/I) = 1$ and $IM = zM + wM$. The corollary then follows directly from Theorem (2.1.16).

Corollary (2.1.16)[34]: If $I_1, I_2, \dots, I_k$ are ideals in R and we set

$$\hat{I}_j = I_1 I_2 \dots I_{j-1} I_{j+1} \dots I_k, \quad J = \hat{I}_1 + \hat{I}_2 + \cdots + \hat{I}_k,$$

Then

$$\dim\left(\bigcap_{j=1}^{k}[I_j] \ominus \left[\prod_{j=1}^{k} I_j\right]\right) \leq \dim(R/I)\operatorname{rank}\left(\bigcap_{j=1}^{k}[I_j]\right).$$

Proof. We denote $\cap_{j=1}^{k}[I_j]$ by N. For any $\phi \in N$, there is a sequence of polynomials $\{p_j^n : 1 \leq j \leq k, n \geq 0\}$ such that $\{p_j^n : n \geq 0\} \subset I_j$ and

$$\lim_{n\to\infty} p_j^n = \phi$$

for each $1 \leq j \leq k$. If $f_j \in \hat{I}_j, j = 1, 2, \dots, k$, then

$$\left(\sum_{j=1}^{k} f_j\right)\phi = \sum_{j=1}^{k} \lim_{n\to\infty} f_j p_j^n.$$

But for each j, $f_j p_j^n \in \hat{I}_j I_j = I_1, I_2 \dots I_k$, so

$$\left(\sum_{j=1}^{k} f_j\right)\phi \in \left[\prod_{j=1}^{k} I_j\right].$$

This shows $JN \subset \left[\prod_{j=1}^{k} I_j\right]$ and hencs

$$\dim\left(N \ominus \left[\prod_{j=1}^{k} I_j\right]\right) \leq \dim(N \ominus \overline{JN}).$$

The corollary then follows from Theorem (2.1.14).

The equality in Corollary (2.1.6) holds in some cases.

Example (2.1.17)[34]: If $I_1 = I_2 = (z, w)$, then $S = (z, w)$ and hence $\dim(R/J) = 1$. It is also easy to see that $[I_1 I_2] = [(z^2, zw, w^2)]$ and one checks that

$$[(z, w)] \ominus ([(z^2, zw, w^2)]) = \operatorname{span}\{z, w\}.$$

Therefore,

$$\dim[(z, w)] \ominus ([(z^2, zw, w^2)]) = 2 = \dim(R/J)\, rank([(z, w)]).$$

By Corollary (2.1.6), if $J = R$, then $\cap_{j=1}^{k}[I_j] = \left[\prod_{j=1}^{k} I_j\right]$ and we can improve this result a little bit. For simplicity we state the improved result for $k = 2$.

Corollary (2.1.18)[34]: If I_1, I_2 are ideals of R such that $(I_1 + I_2) \cap \big(H^\infty(D^2)\big)^{-1} \neq \emptyset$, then

$$[I_1] \cap [I_2] = [I_1 I_2].$$

Proof. In the proof of Corollary (2.1.6), we see that $(I_1 + I_2)([I_1] \cap [I_2]) \subset [I_1 I_2]$. If $(I_1 + I_2) \cap \big(H^\infty(D^2)\big)^{-1} \neq \emptyset$ then

$$[I_1] \cap [I_2] = (I_1 + I_2)([I_1] \cap [I_2]) \subset [I_1 I_2] \subset [I_1] \cap [I_2].$$

Hence,

$$[I_1] \cap [I_2] = [I_1 I_2].$$

We now use Corollary (2.1.18) to give a simple example to show that $M \ominus (zM + wM)$ is in general not a generating set for M.

If $\alpha = (\alpha_1, \alpha_2) \in D^2$ is not equal to $(0,0)$ and H_α is the collection of all functions in $H^2(D^2)$ which vanish at α, then $H_\alpha = [(z - \alpha_1, w - \alpha_2)]$ and

$$zH_\alpha + wH_\alpha = [(z,w)(z - \alpha_1, w - \alpha_2)].$$

Since (z,w) and $(z - \alpha_1, w - \alpha_2)$ have no common zero, by the Nullstellensatz $(z,w) + (z - \alpha_1, w - \alpha_2) = \boldsymbol{C}[z,w]$ and hence

$$[(z,w)(z - \alpha_1, w - \alpha_2)] = [(z,w)] \cap [(z - \alpha_1, w - \alpha_2)]$$

by Corollary (2.1.18). This means that $zH_\alpha + wH_\alpha$ is the collection of all the functions in $H^2(D^2)$ which vanish at both 0 and a and therefore its codimension in $H^2(D^2)$ is 2. But the codimension of H_α is 1, so $H_\alpha \ominus (zH_\alpha + wH_\alpha)$ is one dimensional and therefore it can not be a generating set for H_α.

It is not clear to us whether $M \ominus (zM + wM)$ is a generating set for M when M has rank 1. This question is raised by T. Nakazi in [44].

The condition $(I_1 + I_2) \cap \left(H^\infty(D^2)\right)^{-1} \neq \emptyset$ in Corollary (2.1.18) means in particular that the two ideals I_1 and I_2 have no common zero in D^2. What happens if they have only a finite number of common zeros?

Question (2.1.19)[34]: If $V(I_1) \cap V(I_2)$ is a finite set, then is $[I_1] \cap [I_2] \ominus [I_1 I_2]$ finite dimensional?

We recall that for any submodule M, R_z and R_w are the restrictions of T_z and T_w to M respectively. In [43] it was shown that if M is a submodule in $H^2(D^2)$, then R_z doubly commutes with R_z on M if and only if M is of the form

$$M = \phi H^2(D^2)$$

for some inner function ϕ, we will study the conditions on M under which S_z doubly commutes with S_w on $H^2(D^2) \ominus M$.

In view of the decomposition

$$H^2(D^2) = (H^2(D^2) \ominus M) \oplus M,$$

we can decompose the Toeplitz operators on $H^2(D^2)$ correspondingly.

If we regard w as a multiplication operator on $H^2(D^2)$, then

$$T_w = \begin{pmatrix} qwq & 0 \\ pwq & pwp \end{pmatrix},$$

$$T_z = \begin{pmatrix} qzq & 0 \\ pzq & pzp \end{pmatrix}$$

Where p and q are the orthogonal projections onto M and $H^2(D^2) \ominus M$ respectively. Therefore

$$T_w^* T_z - T_z T_w^* = \begin{pmatrix} q\bar{w}qzq + q\bar{w}pzq - qzq\bar{w}q & q\bar{w}pzp - qzq\bar{w}p \\ p\bar{w}pzq - pzq\bar{w}q & p\bar{w}pzp - pzq\bar{w}p - pz\bar{w}p \end{pmatrix}.$$

It is well known that $T_w^* T_z - T_z T_w^* = 0$ on $H^2(D^2)$, so we have that

$$q\bar{w}qzq + q\bar{w}pzq - qzq\bar{w}q = 0,$$

or equivalently,

$$S_w^* S_z - S_z S_w^* = -q\bar{w}pzq.$$

We can now state and prove the following

Proposition (2.1.20)[34]: If M is a submodule such that $[S_z S_w^*] = 0$ on $K = H^2(D^2) \ominus M$, then either S_z or S_w is in the class C_0.
Proof. By the identity preceding the statement of Proposition (2.1.20),

$$[S_z S_w^*] = (1-p)\bar{w}pz.$$

So for f, g in K,

$$0 =< [S_z S_w^*]f, g \geq< (1-p)\bar{w}pzf, g >$$
$$=< pzf, pwg >.$$

One also checks that for every $h \in M$,

$$< pzf, zh >=< zf, zh >=< f, h >= 0,$$

i.e. pz maps K into $M \ominus zM$ and similarly pw maps K into $M \ominus wM$. Since

$$(M \ominus zM) \cap (M \ominus wM) = M \ominus (zM + wM) \neq \{0\},$$

either $pz(K)$ is not dense in $M \ominus zM$ or $pw(K)$ is not dense in $M \ominus wM$. We assume $pz(K)$ is not dense in $M \ominus zM$, and then there is a $\phi \in M \ominus zM$ such that

$$< zf, \phi >=< pzf, \phi >= 0,$$

for all $f \in K$. Therefore ϕ is orthogonal to both zM and zK and hence is orthogonal to $zM \oplus zK = z(M \oplus K) = zH^2(D^2)$. So ϕ is a function in w only. Since M is invariant for w, the inner factor of ϕ is also in M and hence the corollary follows from Proposition (2.1.13).
Corollary (2.1.21)[34]: M is a submodule such that $K = H^2(D^2) \ominus M$ is invariant for multiplication by z if and only if

$$M = \phi H^2(D^2)$$

for some inner function ϕ depending on w only.
Proof. If K is invariant for z, then by the proof of Proposition (2.1.20) every function in $M \ominus zM$ depends only on w, and hence $M \ominus zM$ is invariant for multiplication by w. By Beurling's Theorem,

$$M \ominus zM = \phi H^2(D)$$

for some inner function ϕ depending on w only. Hence,

$$M = \oplus_{i=0}^{\infty} z^i(M \ominus zM) = \phi \oplus_{i=0}^{\infty} z^i H^2(D) = \phi H^2(D^2).$$

Conversely, if $M = \phi H^2(D^2)$ for some inner function ϕ depending only on w and f is any function in $K = H^2(D^2) \ominus M$, then obviously

$$< zf, \phi w^j >= 0$$

for $j \geq 0$. For any $i \geq 1$ and $j \geq 0$,

$$\langle zf, \phi z^i w^j \rangle = \langle f, \phi z^{i-1} w^j \rangle = 0.$$

In conclusion, $zf \in K$ and hence $K = H^2(D^2) \ominus M$ is invari~nt under multiplication by z.

If M is generated by a polynomial, then Proposition (2.1.20) gives a characterization of M in the case S_z doubly commutes with S_w.
Corollary (2.1.22)[34]: If h is a polynomial in R, then $[S_z, S_w^*] = 0$ on $H^2(D^2) \ominus [h]$ if and only if

$$[h] = GH^2(D^2)$$

with G a finite Blaschke product depending only on one variable.
Proof. We let $Z(h)$ denote the zero set of h.

First of all if $Z(h) \cap D^2 = \emptyset$ then h is outer(e.g. $[h] = H^2(D^2)$) by [40]. So we assume $Z(h) \cap D^2 \neq \emptyset$.

If $[S_z, S_w^*] = 0$ on $H^2(D^2) = [h]$, then by Proposition (2.1.20) either S_z or S_w, is Co. We now assume S_w is C_0 and therefore by Proposition (2.1.13) there is a non-zero bounded function $\phi(w) \in M$.

If $\{w_j : 1 \leq j \leq N \leq \infty\}$ are the distinct zeros of ϕ in D, then

$$Z(h) \cap D^2 \subset \bigcup_{j=0}^{N} D \times \{w_j\}.$$

We assume $\{w_j : 1 \leq j \leq k\}$ is the set of all the zeros of ϕ such that

$$Z(h) \cap D \times \{w_j\} \neq \emptyset.$$

Since h can't have isolated zeros, we have

$$h(\phi z, w_j) = 0, \ \ \forall z \in D, 1 \leq j \leq k.$$

But since h is a polynomial, $w - w_j$ must be a factor of h which we write as

$$(w - w_j) = 0 | h(z, w), 1 \leq j \leq k,$$

and now it is clear that k must be finite. If for each j, we let

$$n_j \coloneqq \max\left\{n : (w - w_j)^n | h(z, w)\right\},$$

Then

$$h(z, w) = \prod_{j=1}^{k} (w - w_j)^{n_j} p(z, w),$$

For some polynomial p. From the construction above,

$$Z(p) \cap D^2 = \emptyset$$

Which means p is outer. If we let

$$G(w) \coloneqq \prod_{j=1}^{k} \left(\frac{w - w_j}{1 - \overline{w_j} w}\right)^{n_j},$$

Then

$$[h] = \left(\prod_{j=1}^{k} (w - w_j)^{n_j} H^2(D^2)\right) = G H^2(D^2).$$

Conversely, if $[h] = G H^2(D^2)$ with G an inner function depending only on w, then $H^2(D^2) \ominus [h]$ is invariant under multiplication by z by Corollary (2.1.15) and hence

$$[S_z, S_w^*] = -(1 - p)\overline{w} p z = 0.$$

Corollary (2.1.15) actually implies that T_z and T_w cannot have a common reducing subspace which, in the module language, can be stated as

Corollary (2.1.23)[34]: $H^2(D^2)$ can not be decomposed as a direct sum of two proper submodules.

Proof. If M and $K = H^2(D^2) \ominus M$ are both submodules then by Corollary (2.1.15),

$$M = \phi H^2(D^2) = \phi_2 H^2(D^2),$$

for some inner functions ϕ_1 in z and ϕ_2 in w. But this is possible only if ϕ_1 and ϕ_2 are both scalars, hence $M = H^2(D^2)$.

Corollary (2.1.18) is actually true for every submodule of $H^2(D^2)$, not just $H^2(D^2)$ itself, and if we use a result in [42] we can prove more. In fact, no two submodules can even have positive angle. It is an easy consequence of the following

Lemma (2.1.24)[34]: If $M \subset H^2(D^2)$ is a nontrivial submodule, then the joint minimal unitary dilation of z and w on M are the multiplications by z and w respectively on $L^2(T^2, dm)$, where dm is the normalized Lebesgue measure on the torus T^2.

Proof. If we let

$$\widehat{M} := \overline{\text{span}}\{z^i w^j f : f \in M, i, j: \text{intergers}\},$$

where the closure is taken in $L^2(T^2, dm)$, then z and w on $\widehat{M}$ are the joint minimal unitary dilation of z and w respectively on M. Since $\widehat{M} \subset L^2(T^2, dm)$ and $\widehat{M}$ is jointly invariant for the multiplications by z, w and $\bar{z}, \bar{w}$, by Lemma 3 in [42],

$$\widehat{M} = 1_E L^2,$$

for some measurable subset $E \subset T^2$. But $\widehat{M}$ contains M and it is well known that nonzero functions in $H^2(D^2)$ cannot vanish on a subset of T^2 with positive measure. So $m(T^2 \backslash E) = 0$ and therefore $\widehat{M} = L^2$.

Corollary (2.1.25)[34]: No two submodules of $H^2(D^2)$ can have positive angle.

Proof. Since two submodules M, N are said to have positive angle if

$$\sup\{|\langle f, g\rangle| : f \in M, g \in N, \|f\| = \|g\| = 1\} < 1,$$

we need to show that

$$\sup\{|\langle f, g\rangle| : f \in M, g \in N, \|f\| = \|g\| = 1\} = 1.$$

If $f \in M, g \in N$ are any two nonzero functions, then by Lemma (2.1.24),

$$[\hat{f}] = [\hat{g}] = L^2.$$

So for any small positive number e we can find polynomials p_1 and p_2 in four variables $z, w, \bar{z}, \bar{w}$ such that

$$\|p_1 f\| = \|p_2 g\| = 1$$

and

$$\|1 - p_1 f\| \le \epsilon, \;\; \|1 - p_2 g\| \le \epsilon.$$

Then,

$$\begin{aligned} |\langle p_1 f, p_2 g\rangle| &= |\langle 1 + p_1 f - 1, 1 + p_2 g - 1\rangle| \\ &\ge 1 - \|p_1 f - 1\| - \|p_2 g - 1\| - \|p_1 f - 1\|\|p_2 g - 1\| \\ &\ge 1 - 2\epsilon - \epsilon^2 \end{aligned}$$

We now choose a sufficiently large integer n such that $z^n w^n p_1, z^n w^n p_2$ are polynomials in z, w only taking $z\bar{z} = 1$ and $w\bar{w} = 1$. Then $z^n w^n p_1 f \in M$ and $z^n w^n p_2 g \in N$ and

$$|\langle z^n w^n p_1 f, z^n w^n p_2 g\rangle| = |\langle p_1 f, p_2 g\rangle| \ge 1 - 2\epsilon - \epsilon^2.$$

This implies

$$\sup\{|\langle f, g\rangle| : f \in M, g \in N, \|f\| = \|g\| = 1\} = 1$$

since e is arbitrary.

We feel Corollary (2.1.25) is a known result but we were not able to find it in the literature.

We finish by raising a question suggested by Corollary (2.1.16).

Section (2.2): The Berger Show Theorem:

The Berger-Shaw theorem says that the self-commutator of a multicyclic hyponormal operator is trace class ([50]). It is interesting to study the multivariate analogue of this theorem. [55] reformulated the theorem in an algebraic language and showed that if the spectrum of a finite rank hyponormal module is contained in an algebraic curve then the module is reductive. They also gave examples showing that it is generally not the case if the spectrum of the module is of higher dimension. However, many examples show that the cross commutators do not seem to have a close relation with the spectra of modules and are generally "small". This suggests that the following general questions may have positive answers.

Questions. Suppose T_1, T_2 are two doubly commuting operators acting on a separable Hilbert space H and R_1, R_2 are the restrictions of them to a jointly invariant subspace that is finitely generated by T_1, T_2.

(a) Is the cross commutator $[R_1^*, R_2]$ in some Schatten p-class?

(b) Is the product $[R_1^*, R_1]$, $[R_2^*, R_2]$ also small?

(c) What about the compressions of T_1, T_2 to the orthogonal complement of M?

A special case of the first question was studied by Curto, Muhly and Yanin [52]. The second question was raised by R. Douglas. The third one appears naturally from the study of essentially reductive quotient modules. Note that when T_1, T_2 the first tow question are answered positively by the Berger-Shaw theorem.

We will make a study of these questions in the case $H = H^2(\mathbb{D}^2)$, the Hardy space over the bidisk, and T_1, T_2 are the multiplications by the two coordinate functions *z* and *w*. Then a closed subspace of $H^2(\mathbb{D}^2)$ is jointly invariant for T_1 and T_2 if and only if it is an $A(\mathbb{D}^2)$ submodule. We will have a look at the third question first because it turns out to be the easiest. The answer to the second question is a consequence of the answer to the first one. Some related questions will also be studied. We now begin the study by doing some preparations.

We let E', E be two separable Hilbert spaces of infinite dimension and $\{\delta_j' : j \geq 0\}, \{\delta_j : j \geq 0\}$ are orthonormal bases for E' and E respectively. We let $H^2(E)$ denote the E-valued Hardy space, i.e. $H^2(E) := \left\{\sum_{j=0}^{\infty} z^j x_j : |z| = 1, \sum_{j=0}^{\infty} \|x_j\|_E^2 < \infty\right\}$ It is well known that every function in $H^2(E)$ has an analytic continuation to the whole unit disk $\mathbb{D}$. For our convenience, we will not distinguish the functions of $H^2(E)$ from their extensions to $\mathbb{D}$. We let T_z be the Toeplitz operator on $H^2(E)$ such that for any $f \in H^2(E)$,

$$T_z f(z) = z f(z)$$

One sees that T_z is a shift operator of infinite multiplicity. *A* $B(E', E)$-valued analytic function $\theta(z)$ on $\mathbb{D}$ is called left-inner (inner) if its boundary values on the unit circle T are almost everywhere isometries (unitaries) from E' into E. Therefore, multiplication by a left-inner θ defines an isometry from $H^2(E')$ into $H^2(E)$.

A closed subspace $M \subset H^2(E)$ is called invariant if $T_z M \subset M$. The Lax-Halmos Theorem gives a complete discription of invariant subspaces in terms of left-inner functions.

Theorem (2.2.1)[49]: (Lax-Halmos) M is a nontrivial invariant subspace of $H^2(E)$ if and only if there is a closed subspace $E' \subset E$ and a $B(E',E)$-valued left-inner function θ such that

$$M = \theta H^2(E') \tag{1}$$

The representation is unique in the sense that

$$\theta H^2(E') = \theta' H^2(E'') \Leftrightarrow \theta = \theta' V$$

where V is a unitary from E' onto E''.

In order to make a study of the Hardy modules over the bidisk, we identify the space E with another copy of the Hardy space. Then $H^2(E) = H^2(D) \otimes E$ will be identified with $H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) = H^2(\mathbb{D}^2)$. We do this in the following way.

Let u be the unitary map from E to $H^2(\mathbb{D})$ such that

$$u\delta_j = u^j, j \geq 0$$

Then $U = I \otimes u$ is a unitary from $H^2(\mathbb{D}) \otimes E$ to $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ such that

$$U(z^i \delta_j) = z^i w^j, i, j \geq 0$$

It is not hard to see that $M \subset H^2(E)$ is invariant if and only if $M \subset H^2(\mathbb{D})$ is invariant under multiplication by the coordinate function z. This identification enables us to use the Lax-Halmos theorem to study certain properties of sub-Hardy modules over the bidisk which we will do. We will let $d|z|$ denote the normalized Lebesgue measure on the unit circle T and $d|z|d|w|$ be the product measure on the torus T^2.

We prove two technical lemmas and an important corollary.

Suppose θ is left inner with values in B(E',E)- and δ is any fixed element of E. We now define an operator N from $\theta E'$ to the Hardy space $H^2(\mathbb{D})$ over the unit disk as the following:

$$N\left(\theta(z)\sum_{j=0}^{\infty}\alpha_j\delta_j'\right) := \langle\, \theta(z)\sum_{j=0}^{\infty}\alpha_j\delta_j'\rangle_E, \tag{2}$$

Where $\sum_{j=0}^{\infty}\alpha_j\delta_j'$ is any element in E'.

Lemma (2.2.2)[49]: N is Hilbert-Schmidt and

$$\operatorname{tr}(N^*\mathrm{N}) = \int_T \|\theta^*(z)\delta\|_{E'}^2, d|z| \tag{3}$$

Proof. Since θ is left inner, $\{\theta\delta_j \mid j \geq 0\}$ is an orthonormal basis for $\theta E'$. To prove the lemma, one suffices to show that $\sum_{j=0}^{\infty}\langle N^*\mathrm{N}\theta\delta_j', \theta\delta_j'\rangle_{\theta E'}$ is finite. In fact,

$$\begin{aligned}\sum_{j=0}^{\infty}\langle N^*\mathrm{N}\theta\delta_j', \theta\delta_j'\rangle_{\theta E'} &= \sum_{j=0}^{\infty}\langle \mathrm{N}\theta\delta_j', \mathrm{N}\theta\delta_j'\rangle_{H^2} \\ &= \sum_{j=0}^{\infty}\int_T \left|\langle\theta(z)\delta_j', \delta\rangle_E\right|^2 d|z| \\ &= \sum_{j=0}^{\infty}\int_T \left|\langle\delta_j', \theta^*(z)\delta\rangle_{E'}\right|^2 d|z| = \int_T \sum_{j=0}^{\infty}\left|\langle\delta_j', \theta^*(z)\delta\rangle_{E'}\right|^2 d|z|\end{aligned}$$

$$= \int_T \|\theta^*(z)\delta\|_{E'}^2 \, d|z|$$

So in gerenal

$$\text{tr}(N^*\text{N}) \geq \|\delta\|^2,$$

and the equality holds when θ is inner.

Back to the $H^2(\mathbb{D}^2)$ case, this lemma has an important corollary. Let us first introduce some operators.

For any bounded function f we let $T_f := Pf$ be the Toeplitz operator on $H^2(\mathbb{D}^2)$, where P is the projection from $L^2(T^2)$ *to* $H^2(\mathbb{D}^2)$. For every non-negative integer j and $\lambda \in \mathbb{D}$, we let operators N_j and N_λ from $H^2(\mathbb{D}^2)$ to $H^2(\mathbb{D})$ be such that for any

$$f(z,w) = \sum_{k=0}^{\infty} f_k(z)w^k \in H^2(D^2)$$

$$N_j f(z) = f_j(z), \quad N_\lambda f(z) = f(z,\lambda)$$

Then one verifies that N_j is a contraction for each j and $\|N_\lambda\| = (1-|\lambda|^2)^{-1/2}$ Furthermore,

$$\sum_{k=0}^{\infty} T_{w^k} N_k = 1 \quad \text{on} \;\; H^2(\mathbb{D}^2) \tag{4}$$

$$N_\lambda = \sum_{k=0}^{\infty} \lambda^k N_k \tag{5}$$

In what follows we will be mainly interested in the restrictions N_k, N_λ to certain subspaces and will use the same notations to denote these restrictions.

Corollary (2.2.3)[49]: For any $A(\mathbb{D}^2)$ submodule $M \subset H^2(\mathbb{D}^2), N_j$ and N_λ are Hilbert-Schmidt operators restricting on $M \ominus zM$ for each $j \geq 0$ and $\lambda \in \mathbb{D}$ and

$$\text{tr}(N^*\text{N}) \leq 1,$$

$$\left\| p^\perp \frac{1}{1-\bar{\lambda}w} \right\|^2 \leq tr(N_\lambda^*, N_\lambda) \leq (1-|\lambda|^2)^{-1}$$

where $p^\perp$ is the projection from $H^2(\mathbb{D}^2)$ onto $M \ominus zM$.

Proof. Because M is invariant under the multiplication by z, U^*M is invariant under T_z, where U is defined in the last paragraph, and hence

$$U^*M = \theta H^2(E')$$

for some Hilbert space $E'$and a left inner function θ Then

$$U^*(M \ominus zM) = \theta H^2(E') \ominus z\theta H^2(E') = \theta H^2(E') \ominus zH^2(E') = \theta E' .$$

Let us first deal with the operator N_λ. In Lemma (2.2.2), if we choose

$$\delta = \sum_{j=0}^{\infty} \bar{\lambda}^j \delta_j \in E$$

then for any $f(z,w) = \sum_{j=0}^{\infty} f_j(z)w^j$ inside $M \ominus zM$, $U^*f = \sum_{j=0}^{\infty} f_j(z)\delta_j$ is in $\theta E'$. and

$$NU^*f(z) = N\left(\sum_{j=0}^{\infty} f_j(z)\delta_j\right) = \langle\sum_{j=0}^{\infty} f_j(z)\delta_{j,}\delta\rangle = \sum_{j=0}^{\infty} f_j(z)\lambda^j = N_\lambda f(z).$$

So $N_\lambda = NU^*$ hence is Hilbert-Schmidt by Lemma (2.2.2), and

$$\mathrm{tr}(N_\lambda^* N_\lambda) = tr(U^*N^*NU) = tr(N^*N).$$

The inequality $\mathrm{tr}(N_\lambda^* N_\lambda) \leq (1-|\lambda|^2)^{-1}$ comes from the remarks following the proof of Lemma (2.2.2). We now show the inequality

$$\left\|p^\perp \frac{1}{1-\bar{\lambda}w}\right\|^2 \leq tr(N_\lambda^*, N_\lambda)$$

Let $\{g_0, g_1, g_2, \ldots\}$ be an orthonormal basis for $M \ominus zM$. Then

$$N_\lambda g_k(z) = g_k(z,\lambda) = \int_{\mathbb{T}} \frac{g_k(z\,,w)}{1-\lambda\bar{w}}\,d|w|,$$

and therefore

$$tr(N_\lambda^*, N_\lambda) = \sum_{k=0}^{\infty}\int_{\mathbb{T}}\left|\int_{\mathbb{T}}\frac{g_k(z,w)}{1-\lambda\bar{w}}\,d|w|\right|^2 d|z|$$

$$\geq \sum_{k=0}^{\infty}\left|\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{g_k(z,w)}{1-\lambda\bar{w}}\,d|w|d|z|\right|^2 = \sum_{k=0}^{\infty}\left|\langle g_k(1-\bar{\lambda}w)^{-1}\rangle\right|^2$$

$$= \left\|p^\perp \frac{1}{1-\bar{\lambda}w}\right\|^2$$

For operators $N_j, j = 0,1,2,\ldots$, we choose $\delta$to be $\delta_j, j = 0,1,2, \ldots\ldots$ correspondingly in Lemma (2.2.2). Similar calculations will establish the assertion and the inequalities.

If $\mathcal{L}^2$ denotes the collection of all the Hilbert-Schmidt operators acting on some Hilbert space *K*, then for any *a,b* in $\mathcal{L}^2$, $\langle a,b\rangle \stackrel{\text{def}}{=} \mathrm{trace}(b^*a)$ defines an inner product which turns $(\mathcal{L}^2\langle .,.\rangle)$ into a Hilbert space. If $|.|$ is the norm induced from this inner product, then

$$|xay| \leq \|x\|\|y\||a|. \tag{6}$$

for any $a \in \mathcal{L}^2$ and any bounded operators *x* and *y* ([56], p. 79), where $\|.\|$ is the operator norm.

Lemma (2.2.4)[49]: Suppose *A,B* are two contractions such that $[A,B] = AB - BA$ is Hilbert-Schmidt and $f(z) = \sum_{j=0}^{\infty} c_j z^j$ is any holomorphic function over the unit disk such that $\sum_{j=0}^{\infty} j|c_j|$ converges, then $[f(A), B]$ is also Hilbert-Schmidt.

Proof. We observe that for any positive interger n,

$$\begin{aligned}
&[A^n, B]\\
&\quad = A^nB - BA^n\\
&\quad = A^nB - A^{n-1}BA + A^{n-1}BA - BA^n\\
&\quad = A^{n-1}[A,B] + [A^{n-1}, B]A\\
&\quad \vdots\\
&\quad = A^{n-1}[A,B] + A^{n-2}[A,B]A + \cdots + A[A,B]A^{n-2} + [A,B]A^{n-1},
\end{aligned}$$

hence

$$|[A^n, B]| \leq n|[A, B]|$$

by inequality (6) if we let $f_n(z) = \sum_{j=0}^{n} c_j z^j$ $then$ $[f_n(A)]$ is in $\mathcal{L}^2$ and

$$|[f_n(A), B] - [f(A), B]| = \left|[\sum_{j=n+1}^{\infty} c_j A^j, B]\right|$$

$$\leq \sum_{j=n+1}^{\infty} |c_j||[A^j, B]| \leq \sum_{j=n+1}^{\infty} j|c_j||[A, B]|$$

From the assumption on f,

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} j|c_j||[A, B]| = 0$$

hence $[f(A), B]$ is also in $\mathcal{L}^2$, i.e. Hilbert-Schmidt.

Corollary (2.2.3) is crucial for the rest and Lemma (2.2.4) will enable us to get around some technical difficulties.

We will define the compression operators and decompose their cross commutators.

For any $h \in H^2(D^2)$, we let

$$[h] := \overline{A(\mathbb{D}^2)h}^{H^2}$$

denote the submodule generated by h. Here we note that h is called inner if

$$|h(z, w)| = 1 \text{ a.e. on } T^2.$$

It is not hard to see that

$$[h] = hH^2(\mathbb{D}^2)$$

when h is inner. Further, h is called outer in the sense of Helson (H) if

$$[h] = H^2(\mathbb{D}^2).$$

Given any submodule M, we can decompose $H^2(\mathbb{D}^2)$ as $H^2(\mathbb{D}^2) = (H^2(\mathbb{D}^2) \ominus M) \oplus M$, and let

$$p: H^2(\mathbb{D}^2) \to M, \quad q: H^2(\mathbb{D}^2) \to M$$

be the projections. For any $f \in H^\infty(\mathbb{D}^2)$ we let S_f and R_f be the compressions of the operator T_f to $H^2(\mathbb{D}^2) \ominus M$ and M respectively, i.e.

$$S_f = qf_q, \qquad R_f = pf_p.$$

We will prove that when $M = [h]$ with h a polynomial, the cross commutators $[S_w^*, S_z]$ and are $[R_w^*, R_z]$ are both Hilbert-Schmidt. To avoid the technical difficulties, we prove the assertion for the operators $[S_{\varphi_\lambda}^*, S_z]$ and $[R_{\varphi_\lambda}^*, R_z]$ first, where $\varphi_\lambda(w) = \frac{w-\lambda}{1-\lambda w}$ with some $\lambda \in \mathbb{D}$ such that $h(z, \lambda) \neq 0$ for all $z \in \mathbb{T}$ and then apply Lemma (2.2.4).

First we need to have a better understanding of the two cross commutators $[S_w^*, S_z]$ and $[R_w^*, R_z]$ In view of the decomposition

$$H^2(\mathbb{D}^2) = (H^2(\mathbb{D}^2)) \ominus M) \oplus M$$

we can decompose the Toeplitz operators on $H^2(\mathbb{D}^2)$ correspondingly.

If we regard φ_λ as a multiplication operator on $H^2(\mathbb{D}^2)$, then

$$T_{\varphi_\lambda} = \begin{pmatrix} q\varphi_\lambda q & 0 \\ p\varphi_\lambda q & p\varphi_\lambda p \end{pmatrix}, \qquad T_z = \begin{pmatrix} qzq & 0 \\ pzq & pzp \end{pmatrix},$$

and

$$T_{\varphi_\lambda}^* T_z - T_z T_{\varphi_\lambda}^* = \begin{pmatrix} q\bar{\varphi}_\lambda qzq + q\bar{\varphi}_\lambda pzq - qzq\bar{\varphi}_\lambda q & q\bar{\varphi}_\lambda pzp - qzq\bar{\varphi}_\lambda p \\ p\bar{\varphi}_\lambda pzq - pzq\bar{\varphi}_\lambda q & p\bar{\varphi}_\lambda pzp - pzq\bar{\varphi}_\lambda p - pzp\bar{\varphi}_\lambda p \end{pmatrix}$$

It is well known that T_z doubly commutes with T_w on $H^2(D^2)$. Because φ_λ is a function of w only, it is then not hard to verify that

$$T_{\varphi_\lambda}^* T_z - T_z T_{\varphi_\lambda}^* = 0,$$

so we have that

$$q\bar{\varphi}_\lambda qzq + q\bar{\varphi}_\lambda pzq - qzq\bar{\varphi}_\lambda q = 0 \text{, and} \quad p\bar{\varphi}_\lambda pzp - pzq\bar{\varphi}_\lambda p - pzp\bar{\varphi}_\lambda p = 0,$$

i.e.

$$q\bar{\varphi}_\lambda qzq - qzq\bar{\varphi}_\lambda p = -q\bar{\varphi}_\lambda pzq,$$
$$p\bar{\varphi}_\lambda pzp - pzp\bar{\varphi}_\lambda p = pzq\bar{\varphi}_\lambda p,$$

Thus we have a following:

Proposition (2.2.4)[49]::

$$S_{\varphi_\lambda}^* S_z - S_z S_{\varphi_\lambda}^* = -q\bar{\varphi}_\lambda pzq, \tag{7}$$
$$R_{\varphi_\lambda}^* R_z - R_z R_{\varphi_\lambda}^* = pzq\bar{\varphi}_\lambda p, \tag{8}$$

We will prove the essential commutativity of $\boldsymbol{S}_{\boldsymbol{\varphi}_\lambda}^*$ and $\boldsymbol{S}_{\boldsymbol{z}}$ on $H^2(\mathbb{D}^2) \ominus [h]$. when h is a polynomial. As we noted, we first prove the assertion for $\boldsymbol{S}_{\boldsymbol{\varphi}_\lambda}^*$ and $\boldsymbol{S}_{\boldsymbol{z}}$.

We first observe that for any $f \in H^2(\mathbb{D}^2) \ominus [h]$ and any $g \in [h]$

$$\langle pzf, zg\rangle_{H^2} = \langle zf, zg\rangle_{H^2} = \langle f, g\rangle_{H^2}$$

So pz actually maps $H^2(\mathbb{D}^2) \ominus [h]$ into$[h] \ominus z[h]$. Therefore, $S_{\varphi_\lambda}^* S_z - S_z S_{\varphi_\lambda}^*$ can be decomposed as

$$H^2(\mathbb{D}^2) \ominus [h] \xrightarrow{-pz} [h] \ominus z[h] \xrightarrow{q\bar{\varphi}} H^2(\mathbb{D}^2) \ominus [h] \tag{9}$$

This observation has an interesting corollary when h is inner.

Corollary (2.2.5)[49]: If h is inner, then $S_w^* S_z - S_z S_w^*$ is at most of rank 1 on

$$H^2(\mathbb{D}^2) \ominus [h]$$

Proof. First we note that when $\lambda = 0$, $\varphi_\lambda(w) = w$. If h is inner, $[h] = hH^2(\mathbb{D}^2)$, and $\{w^n h | n = 0,1,2, \dots \dots \dots\}$ is an orthonormal basis for $[h] \ominus z[h]$ For any function

$$f(z,w) = \sum_{j=0}^{\infty} c_j w^j h \quad \text{inside} \quad [h] \ominus z[h]$$

$$q\bar{w}f = q\bar{w}c_0 h + q\left(\sum_{j=1}^{\infty} c_j w^{j-1} h\right) = c_0 q\bar{w}h$$

This shows that $q\bar{w}$ is at most of rank one and hence $S_w^* S_z - S_z S_w^* = -q\bar{w}pz$ is at most of rank one.

This corollary enables us to give an operator theoretical proof of an interesting fact first noticed by W. Rudin ([60], p. 123).

Corollary (2.2.6)[49]:: $h(z,w) = z - w$ has no inner-outer (H) factorization.

Proof. As before, we let $S_z, , S_w$ be the compressions of $T_z, , T_w$ to $H^2(\mathbb{D}^2) \ominus [h]$ and set

$$e_n = \frac{1}{\sqrt{n+1}}(z^n + z^{n+1}w + \cdots + zw^{n+1} + w^n),\ n = 0,1,2, \dots \dots \dots \dots$$

One verifies that$\{e_n \mid n = 0, 1, 2, \dots\}$ is an orthonormal basis for $H^2(\mathbb{D}^2) \ominus [z - w]$.

Experts will know that $H^2(\mathbb{D}^2) \ominus [z - w]$ is actually the Bergman space over the unit disk. One then easily checks that

$$S_z = S_w$$
$$S_w e_n = \frac{\sqrt{n+1}}{\sqrt{n+2}} e_{n+1}$$
$$S_w^* e_n = \frac{\sqrt{n}}{\sqrt{n+1}} e_{n-1}, n \geq 1$$

Therefore

$$[S_w^*, S_w] e_n = \frac{1}{n(n-1)}, \qquad n = 0,1,2, \dots \dots ..$$

If $z - w$ had an inner-outer factorization, then $[z - w] = gH^2(\mathbb{D}^2)$ for some inner function g and

$$[S_w^*, S_w] = [S_w^*, S_z]$$

would be at most a rank one operator which conflicts with the above computation.

Similar methods can be used to show that the functions like $z - \mu w^n$, for $|\mu| < 1$ and n a nonnegative integer, have no inner-outer (H) factorization.

We now come to the main theorem.

Theorem (2.2.7)[49]: If $h \in H^2(\mathbb{D}^2)$and there is *a* fixed $\lambda \in D$ and *a* positive constant *L* such that

$$L \leq |h(z, \lambda)| \tag{10}$$

for almost every $z \in \mathbb{T}$ then $S_w^* S_w - S_w^* S_w$ on $H^2(\mathbb{D}^2) \ominus [h]$ is Hilbert-Schmidt.

Proof: We first show that $S_{\varphi_\lambda}^* S_z - S_z S_{\varphi_\lambda}^*$ is Hilbert-Schmidt. From (9), it will be sufficient to show that

$$q\bar{\varphi}_\lambda : [h] \ominus z[h] \to H^2(\mathbb{D}^2) \ominus [h]$$

is Hilbert-Schmidt.

Let us recall that the operator N_λ from $[h] \ominus z[h]$ to $H^2(\mathbb{D})$ is defined by

$$N_\lambda g = g(.\,, \lambda)$$

and it is Hilbert-Schmidt by Corollary (2.2.1). Suppose

$$hf_0, hf_1, hf_2, \dots \dots \dots \dots \dots$$

is an orthonormal basis for $[h] \ominus z[h]$. We first show that $h(z, w) f_k(z, \lambda) \in [h]$ for every k. In fact,

$$\int_{\mathbb{T}} |f_k(z, \lambda)|^2 d|z| \leq L^{-2} \int_{\mathbb{T}} |h(z, \lambda) f_k(z, \lambda)|^2 \, d|z| = L^{-2} \|N_\lambda (hf_k)\|^2 < \infty$$

i.e. $f_k(z, \lambda) \in H^2(\mathbb{D})$ and hence $h(z, \lambda) f_k(z, \lambda) \in [h]$ since h is bounded Furthermore,

$$\|h(.,.) f_k(., \lambda)\|^2 \leq \|h\|_\infty^2 \|f_k(., \lambda)\|^2 \leq \|h\|_\infty^2 L^{-2} \|N_\lambda (hf_k)\|^2 , \tag{11}$$

Next, we observe that

$$q\bar{\varphi}_\lambda h f_k = q\bar{\varphi}_\lambda h\big(f_k - f_k(., \lambda)\big) + q\bar{\varphi}_\lambda h f_k(., \lambda). \tag{12}$$

Since $f_k(z, w) - f_k(z, \lambda)$ vanishes at $w = \lambda$ for every $z \in \mathbb{D}$ it has $\varphi_\lambda(w)$ as afactor, and hence

$$\bar{\varphi}_\lambda h(f_k - f_k(., \lambda)) = 0 \tag{13}$$

Combining (11) and (14)

$$\sum_{k=0}^{\infty} \|q\bar{\varphi}_\lambda h f_k\|_{H^2(\mathbb{D}^2)}^2 = \sum_{k=0}^{\infty} \left\|q\bar{\varphi}_\lambda h\big(f_k - f_k(., \lambda)\big) + q\bar{\varphi}_\lambda h f_k(., \lambda)\right\|_{H^2(\mathbb{D}^2)}^2$$

$$= \sum_{k=0}^{\infty} \|q\bar{\varphi}_\lambda h f_k(.,\lambda)\|^2_{H^2(\mathbb{D}^2)} \le \sum_{k=0}^{\infty} \|h(.,.) f_k(.,\lambda)\|^2_{H^2(\mathbb{D}^2)}$$

$$\le \|h\|^2_\infty L^{-2} \sum_{k=0}^{\infty} \|h(.,.) f_k(.,\lambda)\|^2_{H^2(\mathbb{D}^2)} = \|h\|^2_\infty L^{-2} tr(N^*_\lambda N_\lambda).$$

This shows that $q\bar{\varphi}_\lambda$ and hence $[S^*_{\varphi_\lambda}, S_Z]$is Hilbert-Schmidt.

Assuming $q\widehat{\varphi_\lambda}(w) = \overline{\varphi_\lambda(\overline{w})}$ one verifies that $S^*_{\varphi_\lambda} = \widehat{\varphi_\lambda}(S^*_w)$. The fact that

$$\widehat{\varphi_\lambda}\big(\widehat{\varphi_\lambda}(w)\big) = w$$

and an application of Lemma (2.2.4) with $f = \widehat{\varphi_\lambda}$ then imply that $[S^*_w, S_Z]$ is Hilbert-Schmidt.

In theorem (2.2.7), if h is continuous on the boundary of $\mathbb{D} \times \mathbb{D}$, then the inequality (10) will hold once there is a $\lambda \in \mathbb{D}$ such that $h(z,\lambda)$ has no zero on *T*. This idea leads to the assertion that $S^*_w S_Z - S_Z\, S^*_w$ is Hilbert-Schmidt on $H^2(\mathbb{D}^2) \ominus [h]$ for any polynomial h in two complex variables. But we need to recall some knowlege from complex analysis before we can prove it.

Suppose *G* is a bounded open set in the complex plane *C*. We let $A(G)$ denote the collection of all the functions that are holomorphic on *G* and are continuous to the boundary of *G; Z(f)* denotes the zeros of *f.*

To make a study of zero sets of polynomials, we need a classical theorem in several complex variables.

Theorem (2.2.8)[49]: *Let* $h(z,w) = z^n + a_1(w)z^{n-1} + \cdots + a_n(w)$ be a pseudo polynomial without multiple factors, where the $a_j(w)$'s are all in $A(G)$.

Further let

$$D_h \coloneqq \{w \in G | \Delta_h(w) = 0\}$$

where $D_h(w)$ is the discriminant of h. Then for any $w_0 \in G - D_h$ there exists an open neighborhood of $U(w_0) \subset G - D_h$ and holomorphic functions $f_1, f_2, \dots, f_n$ on *U* with $f_i(w) \ne f_j(w)$ for $i \ne j$ and $w \in U$ such that

$$h(z,w) = (z - f_1(w)(z - f_2(w) \dots \dots . . (z - f_n(w))$$

for all $w \in U$ and all complex number *z*.

This theorem is taken from [57], but similar theorems can be found in other standard books on several complex variables. It reveals some information on the zero sets of polynomials which we state as

Corollary (2.2.9)[49]: For any polynomial $p(z,w)$ not having $z - \lambda$ with $|\lambda| = 1$ as a factor, the set

$$Y_p = \{w \in \mathbb{C} | p(z,w) = 0 \text{ for some } z \in \mathbb{T}\}$$

has no interior.

Proof. We first assume that p is irreducible and write *p(z,w)* as

$$p(z,w) = a_0(w)_{z^n} + a_1(w)_{z^{n-1}} + \cdots \dots \dots . a_n(z)$$

with $a_j(w)$ polynomials of one variable and *a0(w)* not identically zero. Then on $C\backslash Z(a_0)$, we have

$$p(z,w) = a_0(w)(z^n + \frac{a_1(w)}{a_0(w)} z^{n-1} + \cdots \dots + \frac{a_n(w)}{a_0(w)})$$

Let Δ_p be the discriminant (see [57] for the definition) of p. If p is irreducible, Δ_p is not identically zero, and so neither is the discriminant of

$$q(z,w) = z^n + \frac{a_1(w)}{a_0(w)} z^{n-1} + \cdots \ldots + \frac{a_n(w)}{a_0(w)}$$

This implies that the pseudopolynomial $q(z,w)$ has no multiple factor either.

We now prove the corollary for the irreducible polynomial p. We do it by showing that given any open disk $B \subset \mathbb{C}$, there is $a\ w \in B$ which is not in Y_p.

Given any small open disk B and a point w_0 in $B \backslash \{Z(\Delta_p) \cup Z(a_0)\}$, the above theorem shows the existence of an open neighborhood $U \subset B$ of w_0 and holomorphic functions $f_1, f_2, \ldots, f_n$ on U with $f_i(w) \neq f_j(w)$ for $i \neq j$ and $w \in U$ such that

$$p(z,w) = a_0(w)\big(z - f_1(w)\big)\big(z - f_2(w)\big) \ldots \ldots \big(z - f_n(w)\big). \tag{14}$$

for all $z \in \mathbb{C}$. Then $f_1(w)$ can not be a constant λ of modulus 1 because p does not have factors of the form $z - \lambda$ from the assumption. So we can choose a smaller open disk $B_1 \subset U$ such that $f_1(B_1) \cap T$ is empty. Carrying the same argument out for f_2 on B_1, we have an open disk $B_2 \subset B_1$ such that $f_2(B_2) \cap T$ is empty.

Continuing this procedure, we have disks $B_1, B_2, \ldots, B_n$ such that $B_j \subset B_{j-1}$ for $j = 2,3,\ldots,n$. Then for any $w \in B_n, p(z,w)$ will have no zero on T and hence w is not in Y_p.

If p is an arbitary polynomial not having $z - \lambda$ with $|\lambda| = 1$as a factor, we factorize p into a product of irreducible polynomials as

$$p(z,w) = p_1^{d_1} p_2^{d_2} \ldots \ldots . . p_m^{d_m}$$

If we let

$$Y_j = \{w \in \mathbb{C} | p_j(z,w) = 0 \text{ for some } z \in \mathbb{T}\}$$

Then

$$Y_p \subset \bigcup_{j=0}^{m} Y_j$$

hence it has no interior.

We feel it may be interesting to have a closer look at the set Y_p, but that is not the purpose. The result in Corollary (2.2.9) is good enough for us to state

Theorem (2.2.10)[49]: For any polynomial $hS_w^* S_z - S_z S_w^*$is Hilbert-Schmidt on $H^2(D^2) \ominus [h]$.

Proof. Suppose h is any polynomial. If h is of the form $(z - \lambda)g$ for some polynomial g and some λ of modulus 1, then $[h] = [g]$ because $z - \lambda$ is outer (H).

So without loss of generality, we assume that h does not have this kind of factor.

Then from the above corollary, $h(z, \mu)$ has no zeros on T for any $\mu \in D \backslash Y_h$.

Theorem (2.2.10) and the observations immediately after it then imply that $[S_w^* S_z]$ is Hilbert-Schmidt.

For any function $f \in A(\mathbb{D}^2)$, we can define an operator S_f by $S_{f^x} \stackrel{\text{def}}{=} qfx$ for any $x \in H^2(\mathbb{D}^2) \ominus [h]$where q is the projection from $H^2(\mathbb{D}^2)$ onto $H^2(\mathbb{D}^2) \ominus [h]$.

One checks that this turns $H^2(\mathbb{D}^2) \ominus [h]$ into a Hilbert $A(\mathbb{D}^2)$ quotient module.

The module is called essentially reductive if Sf is essentially normal for every

$f \in A(\mathbb{D}^2)$ It is easy to see that $H^2(\mathbb{D}^2) \ominus [h]$ is essentially reductive if and only if both $[S_z^* S_z]$ and $[S_w^* S_w]$ are compact. Currently we do not know how to characterize those functions h for which $H^2(\mathbb{D}^2) \ominus [h]$is essentially reductive, even though some partial results are available. [53] and [54] are good references on this topic. However, if we consider $H^2(\mathbb{D}^2) \ominus [h]$ as a module over the the subalgebra $A(\mathbb{D}) \subset A(\mathbb{D}^2)$.

Corollary (2.2.11)[49]: Assume h is a polynomial. If there is $a\ g \in A(\mathbb{D})$ and $a\ f \in [h] \cap H^\infty(\mathbb{D})$ such that $z = g(w) + f(z, w)$, then $H^2(\mathbb{D}^2) \ominus [h]$ is an essentially reductive module over $A(\mathbb{D})$ with the action defined by $f.x \stackrel{\text{def}}{=} f(S_z)x$ for all $f \in A(\mathbb{D})$ and all $x \in H^2(\mathbb{D}^2) \ominus [h]$

Proof. It suffices to show that Sz is essentially normal. From the assumption on *f*, S_f is equal to *0*. Since $z - g(w) = f(z, w)$, we have that $S_z = S_g = g(S_w)$.

Suppose $\{p_n\}$ is a sequence of polynomials which converges to g in supremum norm, then $[S_z^*, p_n(S_w)]$ is compact for each n and it is also not hard to see that $[S_z^*, p_n(S_w)]$ converges to $[S_z^*, g(S_w)]$in the operator norm, and hence $[S_z^*, S_z] = [S_z^*, g(S_w)]$ is compact.

This corollary shows in particular that $H^2(\mathbb{D}^2) \ominus [h]$ is essentially reductive over $A(\mathbb{D}^2)$ when h is linear.

We proved that the module actions of the two coordinate functions *z,w* on the quotient module $H^2(\mathbb{D}^2) \ominus [h]$ essentially doubly commute when h is a polynomial. It is then natural to ask if there is a similar phenomenon in the case of submodules. A result due to Curto, Muhly and Yan ([52]) answered the question affirmatively in a special case and Curto asked if it is true for any polynomially generated submodules ([51]). Since $C[z, w]$ is Noetherian, one only needs to look at the submodules generated by a finite number of polynomials.

We will answer Curto's question partially and a complete answer will be given.

At first, we thought that the submodule case should be easier to deal with than the quotient module case because *z,w* act as isometries on submodules. But it turns out that the submodule case is more subtle and needs a finer analysis.

Suppose *M* is a submodule and R_w and R_z are the module actions by coordinate functions *z* and *w*. It is obvious R_w and R_z are commuting isometries. In [52], curto, Muhly and Yan made a study of the essential commutativity of operators R_w^*, R_z in the case that M is generated by a finite number of homogeneous polynomials. They were actually able to show that $[R_w^*, R_z]$ is Hilbert-Schmidt. We will show that this is also true when *M* is generated by an arbitrary polynomial. The same result for the case that *M* is generated by a finite number of polynomials is a corollary of this result and will be treated.

We suppose *h* is a polynomial that does not have a factor $z-\mu$ with $|\mu| = 1$.

Then there is a $\lambda \in D$ such that $h(z, \lambda)$ is bounded away from 0 on T. we will see that this is crucial in the development of the proofs.

For a bounded analytic function *f(z,w)* over the unit bidisk, we recall that R_f is the restriction of the Toeplitz operator T_f onto $[h]$ and,

$$R_{\varphi_\lambda}^* R_z - R_z R_{\varphi_\lambda}^* = pzq\bar{\varphi}_\lambda p.$$

We let

$$p_1: H^2(\mathbb{D}^2) \to \varphi_\lambda[h], \qquad q_1: H^2(\mathbb{D}^2) \to [h] \ominus \varphi_\lambda[h]$$

be the projections; then $p = p_1 + q_1$ It is not hard to see that

$$(R^*_{\varphi_\lambda} R_z - R_z\, R^*_{\varphi_\lambda}) p_1 = pzq\bar{\varphi}_\lambda p_1 = 0\,.$$

Moreover,

$$T_z T_{\bar{\varphi}_\lambda} - T_z\, T^*_{\varphi_\lambda} = T^*_{\varphi_\lambda} T_z = T_{\bar{\varphi}_\lambda} T_z$$

and hence,

$$R^*_{\varphi_\lambda} R_z - R_z\, R^*_{\varphi_\lambda} = pzq\bar{\varphi}_\lambda(p_1 + q_1) = pzq\bar{\varphi}_\lambda q_1 = pz(P - p)\bar{\varphi}_\lambda q_1$$
$$= pT_{\bar{\varphi}_\lambda} T_z q_1 - pzp\bar{\varphi}_\lambda q_1 = p\bar{\varphi}_\lambda z q_1 - pzp\bar{\varphi}_\lambda q_1, \qquad (15)$$

where P is the projection from $L^2(T^2)$ *to* $H^2(\mathbb{D}^2)$. For any $f \in [h] \ominus \varphi_\lambda[h]$ and

$g \in [h]$

$$\langle p\bar{\varphi}_\lambda f, g\rangle = \langle f, \varphi_\lambda g\rangle = 0$$

i.e.

$$p\bar{\varphi}_\lambda q_1 = 0 \qquad (16)$$

Combining equations (15) and (16) we have that

$$R^*_{\varphi_\lambda} R_z - R_z\, R^*_{\varphi_\lambda} = pzq\bar{\varphi}_\lambda q_1$$

Furthermore, equation (16) also implies that

$$p\bar{\varphi}_\lambda z q_1 = p\bar{\varphi}_\lambda(p_1 + q_1) z q_1 = p\bar{\varphi}_\lambda p_1 z q_1 + p\bar{\varphi}_\lambda q_1 z q_1 = p\bar{\varphi}_\lambda p_1 z q_1$$

Since $p\bar{\varphi}_\lambda$ acts on $\varphi_\lambda[h]$ as an isometry, the above observations then yield.

Proposition (2.2.12)[49]: $[R^*_{\varphi_\lambda}, R_z]$ is Hilbert-Schmidt on *[h]* if and only if $p_1 z q_1$ is Hilbert-Schmidt and

$$tr([R^*_{\varphi_\lambda}, R_z]^*[R^*_{\varphi_\lambda} R_z]) = tr((p_1 z q_1)^*(p_1 z q_1))(p_1 z q_1)).$$

We further observe that, for any $f \in [h] \ominus \varphi_\lambda[h]$ and $g \in \varphi_\lambda[h]$,

$$\langle p_1 z f, zg\rangle = \langle f, g\rangle = 0$$

So the range of operator $p_1 z q_1$ is a subspace of $\varphi_\lambda[h] \ominus z\varphi_\lambda[h]$ If we let $p_\perp$ be the projection from $\varphi_\lambda[h]$ onto $\varphi_\lambda[h] \ominus z\varphi_\lambda[h]$ then

$$p_1 z q_1 = p_\perp z q_1 \qquad (17)$$

We will prove that $p_\perp z q_1$ is Hilbert-Schmidt after some preparation. Suppose

$$h = \sum_{j=0}^{m} a_j(z) w^j$$

is a polynomial and the

$$|h(z, \lambda)| \geq \varepsilon \qquad (18)$$

for some fixed positive ε and all $z \in T$. Assume $\mathcal{H}$ to be the L^2-closure of span$\{h(z, w) z^j | j \geq 0\}$, then $\mathcal{H} \subset [h]$ and we have the following

Lemma (2.2.13)[49]: $\mathcal{H} = \{h(z, w) f(z) | f \in H^2(\mathbb{D})\} = hH^2(\mathbb{D})$.

Proof. It is not hard to check that $hH^2(\mathbb{D}) \subset \mathcal{H}$.

For the other direction, we assume hf is any function in $\mathcal{H}$ and need to show that $f \in H^2(\mathbb{D})$. In fact, if $p_n(z), n \geq 1$ is a sequence of polynomials such that $h(z, w) p_n(z), n \geq 1$ converges to $h(z, w) f(z, w)$ in $L^2(T^2)$, then $h(z, \lambda) p_n(z), n \geq 1$ converges to $h(z, \lambda) f(z, w)$ in $L^2(T)$ by the boundedness of N_λ.

The assumption on h then implies that $p_n(z), n \geq 1$, converges to $f(z, \lambda)$ in $L^2(T)$, and in particular, $f(z, \lambda) \in H^2(\mathbb{D})$This in turn implies that $h(z, \lambda) p_n(z), n \geq 1$, converges to $h(z, w) f(z, \lambda)$ in $L^2(T^2)$ since h is a bounded function. Hence by the uniqueness of the limit,

$$h(z,w)f(z,w) = h(z,w)f(z,\lambda),$$

and therefore

$$f(z,w) = f(z,\lambda)$$

It is interesting to see from this lemma and Corollary (2.2.9) that $hH^2(\mathbb{D})$ is actually closed in $H^2(\mathbb{D}^2)$ for any polynomial h not having a factor $z-\mu$ with $|\mu| = 1$.

Lemma (2.2.14)[49]: The operator $V: [h] \to \mathcal{H}$ defined by $V(hf) = h(z,w)f(z,\lambda)$ is bounded.

Proof. First of all $h(z,w)f(z,\lambda) = N_\lambda(hf)$ is in $H^2(\mathbb{D})$ and hence so is $f(z,\lambda)$ since $|h(z,\lambda)| \geq \varepsilon$ on T. So V is indeed a map from $[h]$ to $\mathcal{H}$.

Next we choose a number M sufficiently large such that

$$\int_{\mathbb{T}} |h(z,w)|^2 d|w| \leq M\varepsilon^2 \leq M|h(z,\lambda)|^2$$

for all $z \in \mathbb{T}$. Then for any $h(z,w)f(z,w) \in [h]$

$$\|V(hf)\|^2 = \int_{\mathbb{T}^2} |h(z,w)f(z,\lambda)|^2 d|z|d|w| = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |h(z,w)|^2 d|w| \right) |f(z,\lambda)|^2 d\,|z|$$

$$\leq M \int_{\mathbb{T}} |h(z,\lambda)f(z,\lambda)|^2\, d|z| \leq M(1-|\lambda|^2)^{-1}\|hf\|^2$$

This lemma enables us to reduce the problem further.

For any $h(z,w)f(z,w) \in [h] \ominus \varphi_\lambda[h]$,

$$p_\perp zhf = p_\perp zV(hf - Vhf)$$

But

$$zh(z,w)f(z,w) - zV(hf)(z,w) = zh(z,w)f(z,w) - f(z,\lambda)$$

and since $f(z,w) - f(z,\lambda)$ vanishes at $w = \lambda$ for every z, it has φ_λ as a factor, hence $z(hf - V(hf)) \in z\varphi_\lambda[h]$. Therefore by the definition of $p_\perp$,

$$p_\perp zhf = p_\perp zV(hf) + p_\perp z\varphi_\lambda hg = p_\perp zV(hf). \tag{19}$$

To prove that $p_\perp zq_1$ is Hilbert-Schmidt, one then suffices to show that $p_\perp z$ restricted to $\mathcal{H}$ is Hilbert-Schmidt. Before proving it, we make another observation and state a lemma.

Since $h(z,w)$ is a polynomial and

$$\int_{\mathbb{T}} |h(z,w)|^2 d|w| = \sum_{k=0}^{m} |a_k(z)|^2$$

the Riesz-Fejér theorem implies that there is a polynomial $Q(z)$ such that

$$|Q(z)|^2 = \int_{\mathbb{T}} |h(z,w)|^2\, d|w|$$

on $\mathbb{T}$. If Q vanishes at some $\eta \in \mathbb{T}$, then $a_k(\eta) = 0$ for each k, and hence h has a factor $(z-\mu)$. But this contradicts our assumption on h. So we can find a positive constant, say η such that

$$|Q(z)| \geq \eta \tag{20}$$

for all $z \in \mathbb{T}$

Suppose $\{h(z,w)f_n(z) \mid n > 0\}$ is an orthonormal basis for $\mathcal{H}$, then

$$\delta_{i,j} = \int_{\mathbb{T}^2} h(z,w)f_i(z)\overline{h(z,w)f_j(z)}d|z|d|w|$$

$$= \int_{\mathbb{T}} (|h(z,w)|^2 d|w|)f_i(z)\overline{f_j(z)}d|z|$$

$$= \int_{\mathbb{T}} Q(z)f_i(z)\overline{Q(z)f_j(z)}d|z|$$

So $\{Q(z)f_k(z) \mid k > 0\}$ is orthonormal in $H^2(\mathbb{D})$, but of course it may not be complete.

Lemma (2.2.15)[49]: The linear operator $J : \overline{\text{span}\{Qf_k|k \geq 0\}} \to H^2(\mathbb{D})$ define by

$$J(Qf_k) = f_k, \quad k \geq 0$$

is bounded.

Proof. By inequality (20), for any function $Qf \in \overline{\text{span}\{Qf_k|k \geq 0\}}$,.

$$\int_{\mathbb{T}} |f(z)|^2 d|z| \leq \eta^{-2} \int_{\mathbb{T}} |Q(z)f(z)|^2 d|z|$$

Now we are in the position to prove

Proposition (2.2.16)[49]: $p_\perp z$ restricted to $\mathcal{H}$ is Hilbert-Schmidt.

Proof. Assume $\{g_k|k \geq 0\} \subset [h] \ominus z[h]$ is an orthonormal basis and, as above, $\{h(z,w)f_n(z)|n \geq 0\}$ is an orthonormal basis for $\mathcal{H}$. Since φ_λ is inner, $\{\varphi_\lambda(w)g_k(z,w)|k \geq 0\}$ is an orthonormal basis for $\varphi_\lambda[h] \ominus z\varphi_\lambda[h]$. Therefore, by identity (9) and the expression of h,

$$p_\perp zhf_n = \sum_{k=0}^{\infty} \langle zhf_n , \varphi_\lambda g_k \rangle \varphi_\lambda g_k = \sum_{k=0}^{\infty} \langle \sum_{i=0}^{m} za_i w^i f_n, \varphi_\lambda \sum_{j=0}^{\infty} T_{w^j}, N_j g_k \rangle \varphi_\lambda g_k$$

Note that a_i's and f_n are functions of z only, so $\sum_{i=0}^{m} za_i w^i f_n$ is orthogonal to $\sum_{j=m+1}^{\infty} w^i \varphi_\lambda N_j g_k$ because the later has the factor w^{m+1}It then follows that

$$p_\perp zhf_n = \sum_{k=0}^{\infty} \langle \sum_{i=0}^{m} za_i w^i f_n, \varphi_\lambda \sum_{j=0}^{\infty} T_{w^j}, N_j g_k \rangle \varphi_\lambda g_k$$

$$= \sum_{k=0}^{\infty} \langle \sum_{i=0}^{m} za_i w^i f_n, \sum_{j=0}^{m} \varphi_\lambda w^J, N_j g_k \rangle \varphi_\lambda g_k$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{m} \varphi_\lambda g_k \left(\int_{\mathbb{T}} z\, a_i(z) f_n(z) \overline{N_j g_k(z)} d|z| \right) \left(\int_{\mathbb{T}} w^i \overline{\varphi_\lambda(w) w^J}\, d|w| \right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{i,j=0}^{m} c_{ij} \langle f_n, T^*_{za_i} N_j g_k \rangle_{H^2(D)} \right) \varphi_\lambda g_k,$$

Where

$$c_{ij} = \int_{\mathbb{T}} w^i \overline{\varphi_\lambda(w) w^j}\, d|w|$$

If $c := \max\{|c_{ij}|\,|0 \le i,j \le m\}$, then the Cauchy inequality yields

$$\|p_\perp z h f_n\|^2 = \sum_{k=0}^{\infty} \left| \sum_{i,j=0}^{m} c_{ij} \langle f_n, T^*_{za_i} N_j g_k \rangle_{H^2(\mathbb{D})} \right|^2$$
$$\le (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \left| \langle f_n, T^*_{za_i} N_j g_k \rangle_{H^2(\mathbb{D})} \right|^2$$
$$= (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \left| \langle J(Qf_n), T^*_{za_i} N_j g_k \rangle_{H^2(\mathbb{D})} \right|^2$$
$$= (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \left| \langle Qf_n, J^* T^*_{za_i} N_j g_k \rangle_{H^2(\mathbb{D})} \right|^2$$

where J is the operator defined in Lemma (2.2.15). Therefore, by the fact that $\{Qf_n | n \ge 0\}$ is orthogonal in $H^2(\mathbb{D})$ and the fact that N_j is Hilbert-Schmidt on $[h] \ominus z[h]$ for each j,

$$\sum_{n=0}^{\infty} \|p_\perp z h f_n\|^2 \le (mc)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \left| \langle Qf_n, J^* T^*_{za_i} N_j g_k \rangle_{H^2(\mathbb{D})} \right|^2$$
$$= (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \sum_{n=0}^{\infty} \left| \langle Qf_n, J^* T^*_{za_i} N_j g_k \rangle_{H^2(\mathbb{D})} \right|^2$$
$$\le (mc)^2 \sum_{k=0}^{\infty} \sum_{i,j=0}^{m} \left\| J^* T^*_{za_i} N_j g_k \right\|^2_{H^2(\mathbb{D})}$$
$$= (mc)^2 \sum_{i,j=0}^{m} \left\| J^* T^*_{za_i} \right\|^2 \sum_{k=0}^{\infty} \left\| N_j g_k \right\|^2_{H^2(\mathbb{D})}$$
$$= (mc)^2 \sum_{i,j=0}^{m} \left\| J^* T^*_{za_i} \right\|^2 tr(N_j^* N_j) < \infty$$

Theorem (2.2.17)[49]: $[R_w^*, R_z]$ is Hilbert-Schmidt on $[h]$ for any polynomial h.

Proof. If $h = (z-\lambda)h_1$ for some polynomial h_1 and $\lambda \in \mathbb{T}$, then $[h] = [h_1]$.

If h_1 is a nonzero constant then $[h_1] = H^2(\mathbb{D}^2)$ and hence

$$R_w = T_w, \qquad R_z = T_z$$

Therefore $[R_w^*, R_z]$. So without loss of generality, we may assume h does not have a factor $z - \lambda$ for some $\lambda \in \mathbb{T}$. Propositions (2.2.12), (2.2.16) and Equality (17) together imply that $[R_w^*, R_z]$ is Hilbert-Schmidt. An argument similar to that in the end of the proof of Theorem (2.2.7) establishes our assertion.

We are going to use the result of the last section to prove the following:

Theorem (2.2.18)[49]: The operator $[R_z^*, R_z]$ $[R_w^*, R_w]$,is Hilbert-Schmidt on *[h]* when h is a polynomial.

Proof. For the same reason as in the proof of Theorem (2.2.17), we assume that h does not have a factor $z-\mu$ for $\mu \in \mathbb{T}$. Then by Corollary (2.2.9), $h(z,\lambda)$ is bounded away from zero on $\mathbb{T}$ for some $\lambda \in \mathbb{D}$.To make our computations clearer, we assume that *h(z,0)* is bounded away from *0* on *T*. Then one sees that for any $hf \in [h], h(f - f(.,0))$ is a function in $w[h]$. Therefore,

$$\begin{aligned}[R_w^*, R_w]hf &= hf - R_w R_w^* hf \\ &= hf - R_w R_w^* h\big(f - f(.,0) + f(.,0)\big) = hf - h\big(f - f(.,0)\big) - R_w R_w^* hf(.,0) \\ &= hf(.,0) - R_w R_w^* hf(.,0) = [R_w^*, R_w]h(.,.)f(.,0), \qquad (21)\end{aligned}$$

Similarly,

$$\begin{aligned}[R_z^*, R_z]hf(.,0) &= hf(.,0) - R_z R_z^* hf(.,0) \\ &= hf(.,0) - R_z R_z^* h(f(.,0) - f(0,0) + f(0,0) \\ &= hf(.,0) - h(f(.,0) - f(0,0) - R_z R_z^* hf(0,0) \qquad (22) \\ &= hf(0,0) - f(0,0) R_z R_z^* h = f(0,0)[R_z^*, R_z]h\end{aligned}$$

By the essential commutativity of R_z^* andR_z, and Equalities (21), (22),

$$\begin{aligned}[R_z^*, R_z]\ [R_w^*, R_w]hf &= [R_z^*, R_z]\ [R_w^*, R_w]h(.,.)f(.,0) \\ &= [R_w^*, R_w]\ \ [R_z^*, R_z]h(.,.)f(.,0) + khf(.,0), \qquad (23)\end{aligned}$$

where *K* a Hilbert-Schmidt operator from Theorem (2.2.17).If we let A, B be operators from *[h]* to itself such that for any $hf \in [h]$

$$Ahf = f(0,0)h; \qquad Bhf = h(.,.)f(.,0)$$

then the above computation shows that

$$[R_z^*, R_z]\ [R_w^*, R_w] = [R_w^*, R_w]\ [R_z^*, R_z]A + KB,$$

We observe that *A* is a rank one operator with kernel $\overline{z[h] + w[h]}$ and one verifies that $[h] \ominus \overline{(z[h] + w[h])}$ is one dimensional, hence *A* is a bounded. Thus to prove that $[R_z^*, R_z]$ $[R_w^*, R_w]$,is Hilbert-Schmidt, it suffices to check that *B* is bounded, but this is clear from our assumption on h and Lemma (2.2.13).

If $h(z,\lambda)$is bounded away from zero on $\mathbb{T}$ for some non-zero $\lambda \in \mathbb{D}$, then similar computations will show that $[R_z^*, R_z]$ $[R_w^*, R_w]$, is Hilbert-Schmidt. Then applying Lemma (2.2.4) twice will establish the assertion.

One sees that the proof of Theorem (2.2.18) depends heavily on the fact that R_z, R_w are isometries. A corresponding study for the product $[S_z^*, S_z]$ $[S_w^*, S_w]$, is thus expected to be harder and we plan to return to that at a later time.

We will generalize the major theorems obtained so far to the case when $[h]$ is replaced by submodules generated by a finite number of polynomials.

Here we need a fact from commutative algebra which we state in a form that fits into the work. We may find more information in [58].

Lemma (2.2.19)[49]: Suppose $p_1, p_2, \ldots, p_k$ are polynomials in $C[z,w]$ such that the greatest common divisor $GCD(p_1, p_2, \ldots, p_k) = 1$, then the quotient $C[z,w]/(p_1, p_2, \ldots, p_k)$ is finite dimensional.

Proof. First of all, $C[z,w]$ is a Unique Factorization Domain (UFD) of Krull dimension 2.

We denote the ideal $(p_1, p_2, \ldots, p_k)$ by *I* and suppose

$$I = \bigcap_{s=1}^{n} I_s$$

is the irredundant primary representation of *I*. If we let $J_s = \sqrt{I_s}$ be the radical of I_s, $s = 1, 2, \ldots, n$, then each J_s is prime and it is either maximal or minimal since the Krull dimension of *C[z,w]* is 2. In an UFD, every minimal prime ideal is principal ([61], p. 238). Since $\mathrm{GCD}(p_1, p_2, \ldots, p_k) = 1$, the associated prime ideals $J_1, J_2, \ldots, J_s$ must all be maximal and hence each J_s must have the form $(z - z_s, w - w_s)$ with $(z_s, w_s) \in \mathbb{C}^2, s = 1,2, \ldots \ldots . n$, mutually different. Therefore, we can choose an integer, say m, sufficiently large such that

$$J_s^m = (z - z_s, w - w_s)^m \subset I_s$$

for each *s*. Then,

$$\bigcap_{s=1}^{n} J_s^m \subset \bigcap_{s=1}^{n} I_s = I$$

and therefore,

$$\dim(C[z,w] \,/\, I) \leq \dim(C[z,w] \,/\, (\bigcap_{s=1}^{n} J_s^m))$$

By the Nullstellensatz, one easily checks that

$$J_i^m + J_j^m = C[z,w], i \neq j$$

The Chinese Remainder Theorem then implies that

$$(C[z,w] \Big/ \left(\bigcap_{s=1}^{n} J_s^m \right) = \prod_{s=1}^{n} C[z,w] \,/\, J_s^m$$

and hence

$$\dim(C[z,w]/I) \leq \prod_{s=1}^{n} \dim(C[z,w]/J_s^m) = \left(\frac{m(m+1)}{2} \right)^n .$$

It would be interesting to generalize this lemma to polynomial rings of higher Krull dimensions.

If $h_1, h_2, \ldots, h_k$ are polynomials and we set

$$G = GCD(h_1, h_2, \ldots, h_k) \quad \text{and} \quad f_j = h_j \,/\, G \qquad (24)$$

$j = 1, 2, \ldots \ldots . . , k$ then $\mathrm{GCD}(f_1, f_2, \ldots, f_k) = 1$. If $\{e_1, e_2, \ldots, e_m\}$ is a basis for $C[z,w]/(f_1, f_2, \ldots, f_k)$, then for any polynomial $g(z,w)$,

$$g(z,w) = \sum_{i=1}^{m} c_i e_i(z,w) + r(z,w)$$

with $r \in (f_1, f_2, \ldots \ldots . f_k)$ and some constants $c_i, i = 1, 2, \ldots, m$. Therefore,

$$C(z,w) g(z,w) = \sum_{i=1}^{m} c_i G(z,w)\, e_i(z,w) + G(z,w) r(z,w) \qquad (25)$$

It is easy to see that $G(z,w)r(z,w) \in (h_1, h_2, \ldots \ldots , h_k)$ and hence $(G)/(h_1, h_2, \ldots \ldots , h_k)$ is also finite dimensional.

Corollary (2.2.20)[49]: If M is a submodule of $H^2(D^2)$ generated by a finite number of polynomials, then

(i) $[S_z^*, S_w]$ is Hilbert-Schmidt on $H^2(\mathbb{D}^2) \ominus M$;

(ii) $[R_z^*, R_w]$ is Hilbert-Schmidt on M;

(iii) $[R_z^*, R_z][R_w^*, R_w]$ is Hilbert-Schmidt on M.

Proof. Suppose $h_1, h_2, \ldots \ldots, h_k$ are polynomials and $M = [h_1, h_2, \ldots \ldots, h_k]$ is the closed submodule generated by $h_1, h_2, \ldots \ldots, h_k$. We assume $G, f_i, i = 1, 2, \ldots, k,$ and $e_j, j = 1, 2, \ldots, m$ to be as in (24) and (25). Consider the space

$$\mathcal{K} \coloneqq \operatorname{span}\{e_j / j = 1,2, \ldots., m\} + M$$

It is closed because span $\{e_j \mid j = 1, 2, \ldots, m\}$ is finite dimensional. For any polynomial g, identity (25) implies that $G_g \in \mathcal{K}$, and hence $[G] \subset \mathcal{K}$. The inclusion

$$[G] \ominus M \subset \mathcal{K} \ominus M$$

then forces $[G] \ominus M$ to be finite dimensional. We let

$$PG: H^2(\mathbb{D}^2) \to [G], \qquad qG: H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \ominus [G]$$

$$PM: H^2(\mathbb{D}^2) \to M, \quad qm: H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2) \ominus M$$

$$p_\perp: H^2(\mathbb{D}^2) \to [G] \ominus M$$

be the projections. Then $p_\perp$ is of finite rank and

$$pG = PM + p_\perp, \;\; qG = qM - p_\perp$$

One verifies that

$$pGzpG = pMzpM + pMzp_\perp + p_\perp zpM + p_\perp zp_\perp,$$
$$qGzqG = qMzqM - qMzp_\perp - p_\perp zqM + p_\perp zp_\perp,$$

and consequently $pGzpG - pMzpM$, and $qGzqG - qMzqM$ are of finite rank. Similarly, $qGwqG = qMwqM$ and $qGwqG = qMwqM$ are also of finite rank. The assertion in this corollary then follows easily from Theorems (2.2.10), (2.2.17) and (2.2.18).

Chapter 3
Toeplitz Operators with Density and Coburn-Simonenko Theorem

We show that, under certain assumptions on the space X, the Toeplitz operator T_a is bounded (resp., compact) if and only if $a \in L^\infty$ (resp., $a = 0$). Moreover, $\|a\|_{L^\infty} \le \|T_a\|_{\mathfrak{B}(H[X])} \le \|P\|_{\mathfrak{B}(X)}\|a\|_{L^\infty}$.These results are specified to the cases of abstract Hardy spaces built upon Lebesgue spaces with Muckenhoupt weights and Nakano spaces with radial oscillating weights. For X be a separable Banach function space on the unit circle T and $H[X]$ be the abstract Hardy space built upon X. We show that the set of analytic polynomials is dense in $H[X]$ if the Hardy-Littlewood maximal operator is bounded on the associate space X'. In particular, if $1 < q \le p < 1,1/r = 1/q - 1/p$, and $a \in L^r \equiv M(L^p, L^q)$ is a nonzero function, then the Toeplitz operator $T(a)$, acting from the Hardy space H^p to the Hardy space H^q, has a trivial kernel in H^p or a dense image in H^q.

Section (3.1): Abstract Hardy Spaces Built Upon Banach Function Spaces:

The Banach algebra of all bounded linear operators on a Banach space E will be denoted by $B(E)$. Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. For $n \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, a function of the form $p(t) = \sum_{k=-n}^{n} \alpha_k t^k$, where $\alpha_k \in \mathbb{C}$ for all $k \in \{-n, \ldots, n\}$ and $t \in \mathbb{T}$, is called a trigonometric polynomial of order n. The set of all trigonometric polynomials is denoted by P. The Riesz projection is the operator P which is defined on P by

$$P: \sum_{k=-n}^{n} \alpha_k t^k \longmapsto \sum_{k=0}^{n} \alpha_k t^k. \tag{1}$$

For $1 \le p \le \infty$, let $L^p := L^p(\mathbb{T})$ be the Lebesgue space on the unit circle $\mathbb{T}$ in the complex plane. For $f \in L^1$, let

$$\hat{f}(n) := \frac{1}{2\pi} \int_{\pi} f(e^{i\varphi})\, e^{-in\varphi} d\varphi, \ \ n \in \mathbb{Z}, \tag{2}$$

be the sequence of the Fourier coefficients of f. The classical Hardy spaces H^p are given by

$$H^p := \{f \in L^p : \hat{f}(n) = 0 \ \forall n < 0\}. \tag{3}$$

It is well known that the Riesz projection extends to a bounded linear operator on L^p if and only if $1 < p < \infty$.

For $a \in L^\infty$, the Toeplitz operator T_a with symbol a on $H^p, 1 < p < \infty$, is given by

$$T_a f = P(af). \tag{4}$$

Toeplitz operators have attracted the mathematical community for the many decades since by Toeplitz [63]. Brown and Halmos [64, Theorem 4] proved that a necessary and sufficient condition that an operator on H^2 is a Toeplitz operator is that its matrix with respect to the standard basis of H^2 is a Toeplitz matrix, that is, the matrix of the form $(a_{k-j})_{j,k \in \mathbb{Z}_+}$. The norm of T_a on the Hardy space H^2 coincides with the norm of its symbol in L^∞ (actually, this result was already in a footnote of [63]). Brown and Halmos also observed, as a corollary, that the only compact Toeplitz operator on H^2 is the zero operator. We here mention [65, Part B, Theorem 4.1.4] and [66, Theorem (3.1.7).8] for the proof of the Brown-Halmos theorem. An analogue of this result is true for Toeplitz operators acting on $H^p, 1 < p < \infty$ [67, Theorem 2.1.7].

We will consider the so-called Banach function spaces X in place of L^p. As usual, we equip the unit circle $\mathbb{T}$ with the normalized Lebesgue measure $dm(\tau) = |d\tau|/(2\pi)$. Denote by L^0 the set of all measurable complex valued functions on $\mathbb{T}$, and let L^0_+ be the subset of functions in L^0 whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ is denoted by $\mathbb{I}_E$. A mapping $\rho: L^0_+ \to [0, \infty]$ is called a function normif, for all functions $f, g, f_n (n \in \mathbb{N})$ in L^0_+, for all constants $c \geq 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

(a) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(cf) = c\rho(f), \rho(f + g) \leq \rho(f) + \rho(g)$,

(b) $0 \leq g \leq f$ a.e.$\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),

(c) $0 \leq f_n \uparrow f \; a.e. \Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(d) $\rho(\mathbb{I}_E) < \infty, \int_E f(\tau) dm(\tau) \leq C_E \rho(f)$,

with $C_E \in (0, \infty)$ depending on E and ρ but independent of f.When functions differing only on a set of measure zero are identified, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is a Banach space under the norm $\|f\|_X := \rho(|f|)$.

Such a space X is called a Banach function space. If ρ is a function norm, its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} f(\tau)\, g(\tau) dm(\tau) : f \in L^0_+, \rho(f) \leq 1\right\}, g \in L^0_+. \qquad (5)$$

The Banach function space X' determined by the function norm ρ' is called the associate space (or Kothe dual space) of X. The associate space X' is a subspace of the dual space X^*. The simplest examples of Banach function spaces are the Lebesgue spaces $L^p, 1 \leq p \leq \infty$. The class of all Banach function spaces includes all Orlicz spaces, as well as all rearrangement-invariant Banach function spaces (see, e.g., [69, Chap. 3]). We are mainly interested in non-rearrangement-invariant Banach function spaces. Two typical examples of non-rearrangement-invariant Banach function spaces are weighted Lebesgue space and weighted Nakano spaces (weighted variable Lebesgue spaces) considered.

Following [70, p. 877], we will consider abstract Hardy spaces $H[X]$ built upon a Banach function space X over the unit circle $\mathbb{T}$ as follows:

$$H[X] := \{f \in X: \hat{f}(n) = 0 \; \forall n < 0\}. \qquad (6)$$

This definition makes sense because X is continuously embedded in L^1 in view of axiom (d). It can be shown that $H[X]$ is a closed subspace of X. It is clear that if $1 \leq p \leq \infty$, then $H[L^p]$ is the classical Hardy space H^p.

It follows from axiom (d) that $P \subset L^\infty \subset X$. We will restrict ourselves to Banach function spaces X such that the Riesz projection defined initially on P by formula (1) extends to a bounded linear operator on the whole space X. The extension will again be denoted by P. If $a \in L^\infty$ and $P \in B(X)$, then the Toeplitz operator defined by formula (4) is bounded on $H[X]$ and

$$\|T_a\|_{B(H[X])} \leq \|P\|_{B(X)} \|a\|_{L^\infty}. \qquad (7)$$

The Brown-Halmos theorem [64, Theorem 4] was extended by [71, Theorem 4.5] to the case of reflexive rearrangement-invariant Banach function spaces with nontrivial

Boyd indices. Note that the nontriviality of the Boyd indices implies the boundedness of the Riesz projection.

We show that the Brown- Halmos theorem remains true for abstract Hardy spaces $H[X]$ built upon reflexive Banach function spaces X (not necessarily rearrangement-invariant) if $P \in B(X)$. Further, we show that, under mild assumptions on a Banach function space X, a Toeplitz operator T_a is compact on the abstract Hardy space $H[X]$ built upon X if and only if $a = 0$.

These results are specified to the case of Hardy spaces built upon Lebesgue spaces with Muckenhoupt weights and upon Nakano spaces with certain radial oscillating weights. Both classes of spaces in our examples are not rearrangement invariant.

For $f \in X$ and $g \in X'$, we will use the following pairing:

$$\langle f, g\rangle := \int_{\mathbb{T}} f(\tau)\overline{g(\tau)}dm(\tau). \quad (8)$$

For $n \in \mathbb{Z}$ and $\tau \in \mathbb{T}$, put $\chi_n(\tau) = \tau^n$. Then the Fourier coefficients of a function $f \in L^1$ can be expressed by $\hat{f}(n) = \langle f, \chi_n\rangle$ for $n \in \mathbb{Z}$.

We need the notion of a function with absolutely continuous norm to formulate the result on the noncompactness of nontrivial Toeplitz operators. Following [69, Chap. 1, Definition 3.1], a function f in a Banach function space X is said to have absolutely continuous norm in X if $\left\|f\mathbb{I}_{E_n}\right\|_X \to 0$ for every sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable sets satisfying $\mathbb{I}_{E_n} \to \emptyset$ almost everywhere as $n \to \infty$. The set of all functions in X of absolutely continuous norm is denoted by X_a. It is known that a Banach function space X is reflexive if and only if X and X' have absolutely continuous norm (see [69, Chap. 1, Corollary 4.4]).

We contains results on the density of the set of all trigonometric polynomials P (resp., the set of all analytic polynomials P_A) in a Banach function space X (resp., in the abstract Hardy space $H[X]$ built upon X). We also show that the norm of a function f in X can be calculated in terms of $\langle f, p\rangle$, where $p \in P$, under the assumption that X' is separable.

Further, we prove that every bounded linear operator on a separable Banach function space, whose matrix is of the form $\left(a_{k-j}\right)_{j,k\in\mathbb{Z}}$, is an operator of multiplication by a function $a \in L^\infty$ and the sequence of its Fourier coefficients is exactly $\{a_k\}_{k\in\mathbb{Z}}$. Finally, we prove that if the characteristic functions of all measurable sets $E \subset \mathbb{T}$ have absolutely continuous norms in X, then the sequence $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on the abstract Hardy space $H[X]$. We provide proofs of our main results, using auxiliary results from the previous. We specify our main results to the case of Hardy spaces built upon weighted Lebesgue spaces $L^p(w)$ with Muckenhoupt weights w and to the case of weighted Nakano spaces $L^{p(\cdot)}(w)$ with certain radial oscillating weights. In both cases, it is known that the Riesz projection is bounded.

The following statement can be proved by analogy with [72, Lemma 1.3].

Lemma (3.1.1)[62] Let X be a Banach function space over the unit circle $\mathbb{T}$. The following statements are equivalent:

(a) the set P of all trigonometric polynomials is dense in the space X;

(b) the space C of all continuous functions on $\mathbb{T}$ is dense in the space X;

(c) the Banach function space X is separable.

Let $m \in \mathbb{Z}_+$. A function of the form $q(t) = \sum_{k=0}^{m} \alpha_k t^k$, where $\alpha_k \in \mathbb{C}$ for all $k \in \{0, \dots, m\}$ and $t \in \mathbb{T}$, is said to be an analytic polynomial on $\mathbb{T}$. The set of all analytic polynomials is denoted by P_A.

Lemma (3.1.2)[62]: Let X be a separable Banach functions space over the unit circle $\mathbb{T}$. If the Riesz projection P is bounded on X, then the set P_A is dense in $H[X]$.

Proof. If $f \in [X] \subset X$, then by Lemma (3.1.1), there exists a sequence $p_n \in P$ such that $\|f - p_n\|_X \to 0$ as $n \to \infty$. It is clear that $f = Pf$ and $Pp_n \in P_A$. Since $P \in B(X)$, we finally have

$$\|f - Pp_n\|_X = \|Pf - Pp_n\|_X \le \|P\|_{B(X)} \|f - p_n\|_X \to 0 \tag{9}$$

As $n \to \infty$. Thus P_A is dense in X.

Lemma (3.1.3)[62]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the associate space X' is separable, then for every

$$\|f\|_X = \sup\{|\langle f, p\rangle| : p \in P, \|p\|_{X'} \le 1\}. \tag{10}$$

Proof. By [69, Theorem 1.7 and Lemma 2.8], for every $f \in X$

$$\|f\|_X = \sup\{|\langle f, g\rangle| : g \in X', \|g\|_{X'} \le 1\}. \tag{11}$$

By the lattice property of the associate space X', we have $P \subset X'$. Hence, equality (11) implies that for $f \in X$

$$\|f\|_X \ge \sup\{|\langle f, p\rangle| : p \in P, \|p\|_{X'} \le 1\}. \tag{12}$$

Fix $g \in X'$ such that $0 < \|g\|_{X'} \le 1$. Since X' is separable, it follows from Lemma (3.1.1) that there exists a sequence $q_n \in P \backslash \{0\}$ such that $\|q_n - g\|_{X'} \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$, put $p_n := \left(\frac{\|g\|_{X'}}{\|q_n\|_{X'}}\right) q_n \in P$. Then for every $n \in \mathbb{N}$

$$\|p_n\|_{X'} = \|g\|_X \le 1, \tag{13}$$

$$\|g - p_n\|_{X'} \le \|g - q_n\|_{X'} + \|q_n\|_{X'} \left(1 - \frac{\|g\|_{X'}}{\|q_n\|_{X'}}\right). \tag{14}$$

Hence

$$\lim_{n \to \infty} \|g - p_n\|_{X'} = 0. \tag{15}$$

It follows from Hölder's inequality for Banach function spaces (see [69, Chap. 1, Theorem 2.4]) and (15) that

$$\lim_{n \to \infty} |\langle f, g\rangle - \langle f, p_n\rangle| \le \lim_{n \to \infty} \|f\|_X \|g - p_n\|_{X'} = 0. \tag{16}$$

Thus, taking into account (13) and (16), we deduce for every function $g \in X'$ satisfying $0 < \|g\|_{X'} \le 1$ that

$$\begin{aligned} |\langle f, g\rangle| = \lim_{n \to \infty} |\langle f, p_n\rangle| \le \sup_{n \in \mathbb{N}} |\langle f, p_n\rangle| \\ \le \sup\{|\langle f, p\rangle| : p \in P, \|p\|_{X'} \le 1\}. \end{aligned} \tag{17}$$

This inequality and equality (11) imply that

$$\|f\|_X \le \sup\{|\langle f, p\rangle| : p \in P, \|p\|_{X'} \le 1\}. \tag{18}$$

Combining inequalities (12) and (18), we arrive at equality (10).

We start this with the following result by Maligranda and Persson on multiplication operators acting on Banach function spaces.

Lemma (3.1.4)[62]: (see [73, Theorem (3.1.7)]). Let X be a Banach function space over the unit circle $\mathbb{T}$. If $a \in L^0$, then the multiplication operator

$$M_a : X \to X,$$

$$f \longmapsto af, \tag{19}$$
is bounded on X if and only if $a \in L^\infty$ and $\|M_a\|_{B(X)} = \|a\|_{L^\infty}$.
It is easy see that
$$\langle M_a\chi_j, \chi_k\rangle = \langle a, \chi_{k-j}\rangle = \hat{a}(k-j)\ \forall j,k \in \mathbb{Z}. \tag{20}$$
The following lemma shows that every bounded operator with such a property is a multiplication operator.

Lemma (3.1.5)[62]: Let X be a separable Banach functions space over the unit circle $\mathbb{T}$. Suppose $A \in B(X)$ and there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of complex numbers such that
$$\langle A\chi_j, \chi_k\rangle = a_{k-j}\ \forall j,k \in \mathbb{Z}. \tag{21}$$
Then there exists a function $a \in L^\infty$ such that $A = M_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$.

Proof. Put $a := A\chi_0 \in X$. Since $X \subset L^1$, we infer from (21) that
$$\hat{a}(n) = \langle a, \chi_n\rangle = \langle A\chi_0, \chi_n\rangle = a_n, n \in \mathbb{Z}. \tag{22}$$
If $f = \sum_{k=-m}^{m} \hat{f}(k)\chi_k \in P$, then $af \in X \subset L^1$ and the jth Fourier coefficient of af is calculated by
$$(af)\hat{\ }(j) = \sum_{k\in\mathbb{Z}} \hat{a}(j-k)\hat{f}(k) = \sum_{k=-m}^{m} a_{j-k}\hat{f}(k). \tag{23}$$
On the other hand, from (21), we get for $j \in \mathbb{Z}$
$$(Af)\hat{\ }(j) = \langle Af, \chi_j\rangle = \sum_{k=-m}^{m} \hat{f}(k)\langle A\chi_k, \chi_j\rangle$$
$$= \sum_{k=-m}^{m} a_{j-k}\hat{f}(k). \tag{24}$$
By (23) and (24), $(af)\hat{\ }(j) = (Af)\hat{\ }(j)$ for all $j \in \mathbb{Z}$. Therefore, $Af = af$ for all $f \in P$ in view of the uniqueness theorem for Fourier series (see, e.g., [74, Chap. I, Theorem 1.7]). Since the space X is separable, the set P is dense in X by Lemma (3.1.1).

Therefore $Af = af$ for $f \in X$. This means that $A = M_a \in B(X)$. It remains to apply Lemma (3.1.4).

Recall that the annihilator of a subspace S of a Banach space E is the set $S^\perp$ of all linear functionals $\Lambda \in E^*$ such that $\Lambda(x) = 0$ for all $x \in S$ (see, e.g., [75, p. 110]).

Lemma (3.1.6)[62]: If X is a Banach function space such that $\mathbb{I}_E \in X_a$ for every measurable subset $E \subset \mathbb{T}$, then $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on $H[X]$.

Proof. By [75, Theorem 7.1], $([X])^*$ is isometrically isomorphic to $X^*/(H[X])^\perp$. Since $\chi_k \in [X]$ for all $k \geq 0$, in view of the above fact, it is sufficient to prove that $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on the whole space X instead of the subspace $H[X]$.

By [69, Chap. 1, Corollary 3.14], if $\mathbb{I}_E \in X_a$ for every measurable subset $E \subset \mathbb{T}$, then $(X_a)^*$ is isometrically isomorphic to X'. In view of [69, Chap. 1, Theorem (3.1.8).2], X' is a Banach function space, which is continuously embedded into L^1 due to axiom (d) of the definition of a Banach function norm. Thus, for every $\Lambda \in X^*$, there exists a function $g \in X' \subset L^1$ such that $\Lambda(f) = \langle f, g\rangle$ for all $f \in X$. In particular, if $f = \chi_k$ with $k \geq 0$, then
$$\Lambda(\chi_k) = \langle \chi_k, g\rangle = \overline{\langle g, \chi_k\rangle} = \overline{\hat{g}(k)}. \tag{25}$$

By the Riemann-Lebesgue lemma (see, e.g., [74, Chap. I, Theorem 1.8]) and (25), $\Lambda(\chi_k) \to 0$ as $k \to \infty$ for every $\Lambda \in X^*$; that is, $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges weakly to zero on X, which completes the proof.

Theorem (3.1.7)[62]: (main result 1). Let X be a reflexive Banach function space over the unit circle $\mathbb{T}$ such that the Riesz projection P is bounded on X. Suppose $A \in B(H[X])$ and there is a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that

$$\langle A\chi_j, \chi_k\rangle = a_{k-j} \quad \forall j, k \in \mathbb{Z}_+. \tag{26}$$

Then there is a function $a \in L^\infty$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover

$$\|a\|_{L^\infty} \le \|T_a\|_{B(H[X])} \le \|P\|_{B(X)}\|a\|_{L^\infty}. \tag{27}$$

We follow the scheme of the proof of [71, Theorem 4.5] (see also [68, Theorem (3.1.8).7]). Without loss of generality, we may assume that the operator A is nonzero. For $n \in \mathbb{Z}_+$, put $b_n := \chi_{-n}A\chi_n$. Then taking into account Lemma (3.1.4) and that $A \in B(H[X])$, we get

$$\begin{aligned}\|b_n\|_X \le \|\chi_{-n}\|_{L^\infty}\|A\chi_n\|_X &= \|A\chi_n\|_{H[X]}\\ \le \|A\|_{B(H[X])}\|\chi_n\|_{H[X]} &= \|A\|_{B(H[X])}\,\|1\|_X.\end{aligned} \tag{28}$$

Consider the following subset of the associate space:

$$V := \left\{y \in X' : \|y\|_{X'} < \frac{1}{\|A\|_{B(H[X])}\|1\|_X}\right\}. \tag{29}$$

It follows from Hölder's inequality for Banach function spaces (see [69, Chap. 1, Theorem (3.1.8).4]) and (28) and (29) that

$$|\langle b_n, y\rangle| \le \|b_n\|_X\|y\|_{X'} < 1 \;\forall y \in V, n \in \mathbb{Z}_+. \tag{30}$$

Since X is reflexive, in view of [69, Chap. 1, Corollaries 4.3-4.4], we know that X' is canonically isometrically isomorphic to X^*. Applying the Banach-Alaoglu theorem (see, e.g., [76, Theorem 3.17]) to V, X', and $\{b_n\}_{n\in\mathbb{Z}_+} \subset X = X^{**} = (X')^*$, we deduce that there exists a $b \in X$ such that some subsequence $\{b_{n_k}\}_{k\in\mathbb{Z}_+}$ of $\{b_n\}_{n\in\mathbb{Z}_+}$ converges to b in the weak topology on X. In particular

$$\lim_{k\to+\infty}\langle b_{n_k}, \chi_j\rangle = \langle b, \chi_j\rangle \;\forall j \in \mathbb{Z}. \tag{31}$$

On the other hand, the definition of b_n and equality (26) imply that

$$\langle b_{n_k}, \chi_j\rangle = \langle A\chi_{n_k}, \chi_{n_k+j}\rangle = a_j \quad \text{whenever } n_k + j \in \mathbb{Z}_+. \tag{32}$$

It follows from (31) and (32) that

$$\langle b, \chi_j\rangle = a_j \;\forall j \in \mathbb{Z}. \tag{33}$$

Now define the mapping B by

$$\begin{aligned}B: P &\to X,\\ f &\mapsto bf.\end{aligned} \tag{34}$$

Assume that f and g are trigonometric polynomials of orders m and r, respectively. Then

$$f = \sum_{k=-m}^{m} \hat{f}(k)\chi_k,$$

$$g = \sum_{j=-r}^{r} \hat{g}(j)\chi_j. \tag{35}$$

It follows from (26) and (33) that for $n \geq \max\{m, r\}$

$$\begin{aligned}
\langle Bf, g\rangle &= \sum_{k=-m}^{m} \sum_{j=-r}^{r} \hat{f}(k)\hat{g}(j)\left\langle b\chi_k, \chi_j\right\rangle \\
&= \sum_{k=-m}^{m} \sum_{j=-r}^{r} \hat{f}(k)\hat{g}(j)\, a_{j-k} \\
&= \sum_{k=-m}^{m} \sum_{j=-r}^{r} \hat{f}(k)\hat{g}(j)\, \left\langle A\chi_{k+n}, \chi_{j+n}\right\rangle \\
&= \sum_{k=-m}^{m} \sum_{j=-r}^{r} \hat{f}(k)\hat{g}(j)\left\langle \chi_{-n}A(\chi_n\chi_k), \chi_j\right\rangle \\
&= \langle \chi_{-n}A(\chi_n f)\,, g\rangle.
\end{aligned} \tag{36}$$

It is clear that $\chi_n f \in H[X]$ for $n \geq \max\{m, r\}$. Therefore, taking into account Lemma (3.1.4), we see that for $n \geq \max\{m, r\}$

$$\begin{aligned}
\left\|M_{\chi-n}AM_{\chi_n}f\right\|_X &\leq \|\chi_{-n}\|_{L^\infty}\|A\chi_n f\|_{H[X]} \\
&\leq \|A\|_{B(H[X])}\|\chi_n f\|_{H[X]} \\
&\leq \|A\|_{B(H[X])}\|\chi_n\|_{L^\infty}\|f\|_X \\
&= \|A\|_{B(H[X])}\|f\|_X.
\end{aligned} \tag{37}$$

By Hölder's inequality for Banach function spaces (see [69, Chap. 1, Theorem (3.1.8).4]) from (36) and (37), we obtain

$$\begin{aligned}
|\langle Bf, g\rangle| &\leq \limsup_{n\to\infty}\left|\left\langle M_{\chi-n}AM_{\chi_n}f, g\right\rangle\right| \\
&\leq \limsup_{n\to\infty}\left\|M_{\chi-n}AM_{\chi_n}f\right\|_X\|g\|_{X'} \\
&\leq \|A\|_{B(H[X])}\|f\|_X\|g\|_{X'}.
\end{aligned} \tag{38}$$

Since a Banach function space X is reflexive and the Lebesgue measure is separable, it follows from [69, Chap. 1, Corollaries 4.4 and 5.6] that the spaces X and X' are separable. Then Lemma (3.1.3) and inequality (38) yield

$$\begin{aligned}
\|Bf\|_X &= \sup\{|\langle Bf, g\rangle| : g \in P, \|g\|_X \leq 1\} \\
&\leq \|A\|_{B(H[X])}\|f\|_X,
\end{aligned} \tag{39}$$

for all $f \in P$. In view of Lemma (3.1.1), P is dense in X. Then (39) implies that the linear mapping B defined in (34) extends to an operator $B \in B(X)$ such that

$$\|B\|_{B(X)} \leq \|A\|_{B(H[X])}. \tag{40}$$

We deduce from (33) that

$$\left\langle B\chi_j, \chi_k\right\rangle = \left\langle b, \chi_{k-j}\right\rangle = a_{k-j}\ \forall j, k \in \mathbb{Z}. \tag{41}$$

By Lemma (3.1.5), there exists a function $a \in L^\infty$ such that $B = M_a$ and $a_n = \hat{a}(n)$ for all $n \in \mathbb{Z}$. Moreover

$$\|B\|_{B(X)} = \|M_a\|_{B(X)} = \|a\|_{L^\infty}. \tag{42}$$

It follows from the definition of the Toeplitz operator T_a that

$$\left\langle T_a\chi_j, \chi_k\right\rangle = \hat{a}(k-j), \qquad j, k \in \mathbb{Z}_+. \tag{43}$$

Combining this fact with equality (26), we arrive at

$$\langle T_a \chi_j, \chi_k \rangle = a_{k-j} = \langle A\chi_j, \chi_k \rangle, \qquad j,k \in \mathbb{Z}_+. \tag{44}$$

Since $T_a\chi_j, A\chi_j \in H[X] \subset H^1$, by the uniqueness theorem for Fourier series (see, e.g., [74, Chap. I, Theorem (3.1.8).7]), it follows from (44) that $T_a\chi_j = A\chi_j$ for all $j \in \mathbb{Z}_+$. Therefore

$$T_a p = Ap \; \forall p \in P_A. \tag{45}$$

In view of Lemma (3.1.2), the set P_A is dense in $H[X]$. This fact and equality (45) imply that $T_a = A$ and

$$\|T_a\|_{B(H[X])} = \|A\|_{B(H[X])}. \tag{46}$$

Combining inequality (40) with equalities (42) and (46), we arrive at the first inequality in (10). The second inequality in (10) is obvious.

Theorem (3.1.8)[62]: (main result 2). Let X be a Banach function space over the unit circle $\mathbb{T}$ such that $\mathbb{I}_E \in X_a$ for every measurable subset $E \subset \mathbb{T}$. If the Riesz projection P is bounded on X and $a \in L^\infty$, then the Toeplitz operator $T_a \in B(H[X])$ is compact if and only if $a = 0$.

Proof. It is clear that if $a = 0$, then T_a is the zero operator, which is compact. Now assume that T_a is compact. Then it maps weakly convergent sequences in $H[X]$ into strongly convergent sequences in $H[X]$ (see, e.g., [77, Section 7.5, Theorem 4]). Since $\{\chi_k\}_{k\in\mathbb{Z}_+}$ converges to zero weakly on $H[X]$ in view of Lemma (3.1.6), we have

$$\lim_{k\to\infty} \|T_a\chi_k\|_{H[X]} = 0. \tag{47}$$

By [69, Chap. 1, Theorem 2.7 and Lemma 2.8], for $k \in \mathbb{Z}_+$,

$$\|T_a\chi_k\|_{H[X]} = \|T_a\chi_k\|_X = \sup\{|\langle T_a\chi_k, g\rangle| : g \in X', \|g\|_{X'} \le 1\}. \tag{48}$$

Since $L^\infty \subset X'$, there exists a constant $c \in (0,\infty)$ such that

$$c^{-1}\|\chi_m\|_{X'} \le \|\chi_m\|_{L^\infty} = 1, \;\; m \in \mathbb{Z}. \tag{49}$$

For all $n \in$ Z and all $k \in \mathbb{Z}_+$ such that $k + n \in \mathbb{Z}_+$ we have

$$\hat{a}(n) = \langle T_a\chi_k, \chi_{k+n}\rangle. \tag{50}$$

Then from (48)–(50) we obtain for all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}_+$ such that $k + n \in \mathbb{Z}_+$

$$\|T_a\chi_k\|_{H[X]} \ge |\langle T_a\chi_k, c^{-1}\chi_{k+n}\rangle| = c^{-1}|\hat{a}(n)|. \tag{51}$$

Passing in this inequality to the limit as $k \to \infty$ and taking into account (47), we see that $\hat{a}(n) = 0$ for all $n \in \mathbb{Z}$. By the uniqueness theorem for Fourier series (see, e.g., [74, Chap. I, Theorem 2.7]), this implies that $a = 0$ a.e. on $\mathbb{T}$.

A measurable function $w: \mathbb{T} \to [0, \infty]$ is referred to as a weight if $0 < w(\tau) < \infty$ almost everywhere on $\mathbb{T}$. If X is a Banach function space over the unit circle and w is a weight, then

$$X(w) := \{f \in L^0 : fw \in X\} \tag{52}$$

is a normed space equipped with the norm $\|f\|_{X(w)} := \|fw\|_X$.

Moreover, if $w \in X$ and $1/w \in X'$, then $X(w)$ is a Banach function space (see [78, Lemma 2.5]).

Let $1 < p < \infty$ and w be a weight. It is well known that the Riesz projection P is bounded on the weighted Lebesgue space $L^p(w)$ if and only if the weight w satisfies the Muckenhoupt A_p −condition; that is,

$$\sup_{I\subset\mathbb{T}}\left(\frac{1}{m(I)}\int_I w^p(\tau)dm(\tau)\right)^{1/p}$$
$$\cdot\left(\frac{1}{m(I)}\int_I w^{-p'}(\tau)dm(\tau)\right)^{1/p'} < \infty, \qquad (53)$$
where the supremum is taken over all subarcs I of the unit circle $\mathbb{T}$ and $1/p + 1/p' = 1$ (see [79] and also [68, Section 1.46], [65, Section 5.7.3(h)]). In the latter case, we will write $w \in A_p(\mathbb{T})$. It is clear that if $w \in A_p(\mathbb{T})$, then $w \in L^p$ and $1/w \in L^{p'}$. Hence $L^p(w)$ is a Banach function space whenever $w \in A_p(\mathbb{T})$. It is well known that if $1 < p < \infty$, then $L^p(w)$ is reflexive. We denote the corresponding Hardy space by $H^p(w) := H[L^p(w)]$.

Corollary (3.1.9)[62]: Let $1 < p < \infty$ and $w \in A_p(\mathbb{T})$. If $A \in B\big(H^p(w)\big)$ and there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying (26), then there exists a function $a \in L^\infty$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover
$$\|a\|_{L^\infty} \le \|T_a\|_{B(H^p(w))} \le \|P\|_{B(L^p(w))}\,\|a\|_{L^\infty}. \qquad (54)$$
This is an immediate consequence of Theorem (3.1.7). For the weight $w = 1$, it is proved in [68, Theorem (3.1.8).7].

Corollary (3.1.10)[62]: Let $1 < p < \infty$ and $w \in A_p(\mathbb{T})$. If $a \in L^\infty$, then the Toeplitz operator $T_a \in B\big(H^p(w)\big)$ is compact if and only if $a = 0$.

This corollary follows from Theorem (3.1.8).

We denote by $P_C(\mathbb{T})$ the set of all continuous functions $p\colon \mathbb{T} \to (1,\infty)$. For $p \in P_C(\mathbb{T})$, let $L^{p(\cdot)}$ be the set of all functions $f \in L^0$ such that
$$\int_{\mathbb{T}} \left|\frac{f(\tau)}{\lambda}\right|^{p(\tau)} dm(\tau) < \infty, \qquad (55)$$
for some $\lambda = \lambda(f) > 0$. This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm
$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0\colon \int_{\mathbb{T}} \left|\frac{f(\tau)}{\lambda}\right|^{p(\tau)} dm(\tau) \le 1\right\}, \qquad (56)$$
(see, e.g., [80, p. 73] or [81, p. 77]). If p is constant, then $L^{p(\cdot)}$ is nothing but the Lebesgue space L^p. The spaces $L^{p(\cdot)}$ are referred to as Nakano spaces. See Maligranda [82] for the role of Hidegoro Nakano in the study of these spaces.

Since $\mathbb{T}$ is compact, we have
$$1 < \min_{t\in\mathbb{T}} p(t)\,, \max_{t\in\mathbb{T}} p(t) < \infty. \qquad (57)$$
In this case, the space $L^{p(\cdot)}$ is reflexive and its associate space is isomorphic to the space $L^{p'(\cdot)}$, where $1/p(\tau) + 1/p'(\tau) = 1$ for all $\tau \in \mathbb{T}$ (see, e.g., [80, Section 2.8] and [81, Section 3.2]).

Let Sf be the Cauchy singular integral of a function $f \in L^1(\mathbb{T})$ defined by
$$(Sf)(t) := \lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_{\mathbb{T}\setminus\mathbb{T}(t,\varepsilon)} \frac{f(\tau)}{\tau - t}d\tau\,, t \in \mathbb{T}, \qquad (58)$$

Where $\mathbb{T}(t,\varepsilon) := \{\tau \in \mathbb{T}: |\tau - t| < \varepsilon\}$. For a weight $w: T \to [0,\infty]$, consider the weighted Nakano space

$$L^{p(\cdot)}(w) = \{f \in L^0: fw \in L^{p(\cdot)}\}. \tag{59}$$

It follows from [78, Theorem 6.1] that if the operator S is bounded on $L^{p(\cdot)}(w)$, then

$$\sup_{I\subset\mathbb{T}} \frac{1}{m(I)} \|w\chi_I\|_{L^{p(\cdot)}} \|w^{-1}\chi_I\|_{L^{p(\cdot)}} < \infty, \tag{60}$$

where the supremum is taken over all subarcs $I \subset \mathbb{T}$. In particular, in this case, $w \in L^{p(\cdot)}$ and $1/w \in L^{p'(\cdot)}$, whence $L^{p(\cdot)}(w)$ is a Banach function space by [78, Lemma 2.5(b)].

We say that an exponent $p \in P_C(\mathbb{T})$ is locally log-Hölder continuous (cf. [80, Definition 2.2]) if there exists a constant $C_{p(\cdot)} \in (0,\infty)$ such that

$$|p(t) - p(\tau)| \le \frac{C_{p(\cdot)}}{-\log|t-\tau|} \forall t,\tau \in \mathbb{T} \text{ satisfying } |t-\tau| < \frac{1}{2}. \tag{61}$$

The class of all locally log-Hölder continuous exponents will be denoted by $LH(\mathbb{T})$. Notice that some authors also denote this class by $\mathbb{P}^{\log}(\mathbb{T})$ (see, e.g., [83, Section 1.1.4]).

Following [84, Section 2.3], denote by W the class of all continuous functions $\varrho: [0,2\pi] \to [0,\infty)$ such that $\varrho(0) = 0, (x) > 0$, if $0 < x \le 2\pi$, and ϱ is almost increasing; that is, there is a universal constant $C > 0$ such that $\varrho(x) \le C\varrho(y)$ whenever $x \le y$. Further, let $\mathbb{W}$ be the set of all functions $\varrho: [0,2\pi] \to [0,\infty]$ such that $x^\alpha \varrho(x) \in W$ and $x^\beta/\varrho(x) \in W$ for some $\alpha,\beta \in \mathbb{R}$. Clearly, the functions $\varrho(x) = x^\gamma$ belong to $\mathbb{W}$ for all $\gamma \in \mathbb{R}$. For $\varrho \in \mathbb{W}$, put

$$\Phi_\varrho^0(x) := \limsup_{y\to 0} \frac{\varrho(xy)}{\varrho(y)}, x \in (0,\infty). \tag{62}$$

Since $\varrho \in \mathbb{W}$, one can show that the limits

$$m(\varrho) := \lim_{x\to 0} \frac{\log \Phi_\varrho^0(x)}{\log x},$$
$$M(\varrho) := \lim_{x\to\infty} \frac{\log \Phi_\varrho^0(x)}{\log x} \tag{63}$$

exist and $-\infty < m(\varrho) \le M(\varrho) < +\infty$. These numbers were defined under some extra assumptions on ϱ by Matuszewska and Orlicz [85, 23] (see also [87] and [88, Chapter 11]).

We refer to $m(\varrho)$ (resp., $M(\varrho)$) as the lower (resp., upper) Matuszewska-Orlicz index of ϱ. For $\varrho(x) = x^\gamma$, one has $m(\varrho) = M(\varrho) = \gamma$. Examples of functions $\varrho \in \mathbb{W}$ with $m(\varrho) < M(\varrho)$ can be found, for instance, in [88, p. 93]. Fix pairwise distinct points $t_1,\ldots,t_n \in \Gamma$ and functions $w_1,\ldots,w_n \in \mathbb{W}$.

Consider the following weight:

$$w(t) := \prod_{k=1}^{n} w_k(|t-t_k|), t \in \Gamma. \tag{64}$$

Each function $w_k(|t-t_k|)$ is a radial oscillating weight.

This is a natural generalization of the so-called Khvedelidze weights $w(t) = \prod_{k=1}^{n}|t-t_k|^{\lambda_k}$, where $\lambda_k \in \mathbb{R}$ (see, e.g., [68, Section 5.8]).

Theorem (3.1.11)[62]: Let $p \in LH(\mathbb{T})$. Suppose $w_1, \ldots, w_n \in \mathbb{W}$ and the weight$w$is given by (64).The Cauchy singular integral operator S is bounded on $L^{p(\cdot)}(w)$ if and only if for all $k \in \{1, \ldots, n\}$

$$0 < \frac{1}{p(t_k)} + m(w_k), \frac{1}{p(t_k)} + M(w_k) < 1. \qquad (65)$$

The sufficiency portion of Theorem (3.1.11) was obtained by Kokilashvili et al. [84, Theorem 4.3] (see also [83, Corollary 2.109]) for more general finite Carleson curves in place of $\mathbb{T}$. The necessity portion was proved by [89, Corollary 4.3] for Jordan Carleson curves.

Lemma (3.1.12)[62]: Let $p \in LH(\mathbb{T})$. Suppose $w_1, \ldots, w_n \in \mathbb{W}$ and the weight w is given by (64).Then the weighted Nakano space $L^{p(\cdot)}(w)$ is a reflexive Banach function space and the Riesz projection P is bounded on $L^{p(\cdot)}(w)$.

Proof. In view of Theorem (3.1.11), the operator S is bounded on the space $L^{p(\cdot)}(w)$. As was observed above, the boundedness of the operator S on the space $L^{p(\cdot)}(w)$ implies that $w \in L^{p(\cdot)}$ and $1/w \in L^{p'(\cdot)}(w)$ by [78, Theorem 6.1]. Hence $L^{p(\cdot)}(w)$ is a reflexive Banach function space thanks to [78, Lemma 2.5 and Corollary 2.8]. By [78, Lemma 1.4], the operator $P = (I + S)/2$ is bounded on $L^{p(\cdot)}(w)$.

Consider the Hardy space $H^{p(\cdot)}(w) := H\left[L^{p(\cdot)}(w)\right]$ built upon the weighted Nakano space $L^{p(\cdot)}(w)$, where $p \in LH(\mathbb{T})$ and w is a weight as in Theorem (3.1.11).

Theorem (3.1.7) and Lemma (3.1.12) yield the following.

Corollary (3.1.13)[62]: Let $p \in LH(\mathbb{T})$. Suppose $w_1, \ldots, w_n \in \mathbb{W}$ and the weight w is given by (64). If $A \in B\left(H^{p(\cdot)}(w)\right)$ and there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying (26), then there exists a function $a \in L^\infty$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover

$$\|a\|_{L^\infty} \le \|T_a\|_{B\left(H^{p(\cdot)}(w)\right)} \le \|P\|_{B\left(L^{p(\cdot)}(w)\right)} \|a\|_{L^\infty}. \qquad (66)$$

Similarly, Theorem (3.1.8) and Lemma (3.1.12) imply the following.

Corollary (3.1.14)[62]: Let $p \in LH(\mathbb{T})$. Suppose $w_1, \ldots, w_n \in \mathbb{W}$ and the weight w is given by (64). If $a \in L^\infty$, then the Toeplitz operator $T_a \in B\left(H^{p(\cdot)}(w)\right)$ is compact if and only if $a = 0$.

Leśnik posted in [90], where among other results he proved analogues of Theorems (3.1.8) and (3.1.7) for Toeplitz operators acting between abstract Hardy spaces $H[X]$ and $H[Y]$ built upon distinct rearrangement invariant Banach function spaces X and Y.The set of allowed symbols in [90] coincides with the set $M(X, Y)$ of pointwise multipliers from X to Y, which may contain unbounded functions. Thus, his results complement ours in a nontrivial way but are not more general than ours, because Lesnik restricts himself to rearrangement-invariant spaces X and Y only. On the other hand, the main aim is to consider the questions of the boundedness and compactness of Toeplitz operators on an abstract Hardy space $H[X]$ in the case when X is an arbitrary, not necessarily rearrangement invariant, Banach function space.

Section (3.2): Analytic Polynomials in Abstract Hardy Spaces:

For $1 \le p \le \infty$, let $L^p := L^p(\mathbb{T})$ be the Lebesgue space on the unit circle $\mathbb{T} := \{z \in \mathbb{TC} : |z| = 1\}$ in the complex plane $\mathbb{C}$. For $f \in L^1$, let

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) e^{-in\varphi}\, d\varphi, n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of f. The classical Hardy spaces H^p are given by

$$H^p := \{f \in L^p : \hat{f}(n) = 0 \text{ for all } n < 0\}.$$

A function of the form

$$q(t) = \sum_{k=0}^{n} \alpha_k t^k, t \in \mathbb{T}, \alpha_0, \dots, \alpha_n \in \mathbb{C},$$

is said to be an analytic polynomial on $\mathbb{T}$. The set of all analytic polynomials is denoted by P_A. It is well known that that the set P_A is dense in H^p whenever $1 \le p < \infty$ (see, e.g., [93, Chap. III, Corollary 1.7(a)]).

Let X be a Banach space continuously embedded in L^1. Following [107, p. 877], we will consider the abstract Hardy space $H[X]$ built upon the space X, which is defined by

$$H[X] := \{f \in X : \hat{f}(n) = 0 \text{ for all } n < 0\}.$$

It is clear that if $1 \le p \le \infty$, then $H[L^p]$ is the classical Hardy space H^p. The aim is to find sufficient conditions for the density of the set P_A in the space $H[X]$ when X falls into the class of so-called Banach function spaces.

We equip $\mathbb{T}$ with the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$.

Let L^0 be the space of all measurable complex-valued functions on $\mathbb{T}$. As usual, we do not distinguish functions, which are equal almost everywhere (for the latter we use the standard abbreviation a.e.). Let L^0_+ be the subset of functions in L^0 whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ is denoted by χ_E.

Following [91, Chap. 1, Definition 1.1], a mapping $\rho : L^0_+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, for all constants $a \ge 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

$(A1) \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \rho(af) = a\rho(f), \rho(f + g) \le \rho(f) + \rho(g),$

$(A2)\ 0 \le g \le f\ \mu - \text{a.e.} \Rightarrow \rho(g) \le \rho(f) (\text{the lattice property}),$

$(A3) 0 \le f_n \uparrow f\ a.e. \Rightarrow \rho(f_n) \uparrow \rho(f) (\text{the Fatou property}),$

$(A4) m(E) < \infty \Rightarrow \rho(\chi_E) < \infty,$

$(A5) \int_E f(t) dm(t) \le C_E \rho(f)$

with the constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$.

The set X under the natural linear space operations and under this norm becomes a Banach space (see [91, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on L_+^0 by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} f(t)g(t)d\mu(t) : f \in L_+^0, \rho(f) \le 1\right\}, g \in L_+^0.$$

It is a Banach function norm itself [91, Chap. 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Kothe dual) of X. The associate space X' can be viewed a subspace of the (Banach) dual space X^*.

The distribution function mf of an a.e. finite function $f \in L^0$ is defined by

$$m_f(\lambda) := m\{t \in \mathbb{T}: |f(t)| > \lambda\}, \lambda \ge 0.$$

Two a.e. finite functions $f, g \in L^0$ are said to be equimeasurable if

$$m_f(\lambda) = m_g(\lambda) \text{ for all } \lambda \ge 0.$$

The non-increasing rearrangement of an a.e. finite function $f \in L^0$ is defined by

$$f^*(x) := \inf\{\lambda : m_f(\lambda) \le x\}, x \ge 0.$$

See [91, Chap. 2, Section 1] and [101, Chap. II, Section 2] for properties of distribution functions and non-increasing rearrangements. A Banach function space X is called rearrangement-invariant if for every pair of a.e. finite equimeasurable functions $f, g \in L^0$, one has the following property: if $f \in X$, then $g \in X$ and the equality $\|f\|_X = \|g\|_X$ holds. Lebesgue spaces $L^p, 1 \le p \le \infty$, as well as, more general Orlicz spaces, Lorentz spaces, and Marcinkiewicz spaces are classical examples of rearrangement invariant Banach function spaces (see [91, 11]). For more recent examples of rearrangement-invariant spaces, like Cesaro, Copson, and Tandori spaces, See Maligranda and Lesnik [103].

One of our motivations in the study of Harmonic Analysis in the setting of variable Lebesgue spaces [94, 96,100]. Let $\mathfrak{B}(\mathbb{T})$ be the set of all measurable functions $p: \mathbb{T} \to [1, \infty]$. For $p \in \mathfrak{B}(\mathbb{T})$, put

$$\mathbb{T}_\infty^{p(\cdot)} := \{t \in \mathbb{T}: p(t) = \infty\}.$$

For a measurable function $f: \mathbb{T} \to \mathbb{C}$, consider

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{T}\setminus\mathbb{T}_\infty^{p(\cdot)}} |f(t)|^{p(t)} dm(t) + \|f\|_{L^\infty\left(\mathbb{T}_\infty^{p(\cdot)}\right)}.$$

According to [94, Definition 2.9], the variable Lebesgue space $L^{p(\cdot)}$ is defined as the set of all measurable functions $f: \mathbb{T} \to \mathbb{C}$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. This space is a Banach function space with respect to the Luxemburg-Nakano norm given by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0: \varrho_{p(\cdot)}(f/\lambda) \le 1\}$$

(see, e.g., [94, Theorems 2.17, 2.71 and Section 2.10.3]). If $p \in \mathfrak{B}(\mathbb{T})$ is constant, then $L^{p(\cdot)}$ is nothing but the standard Lebesgue space L^p. If $\in \mathfrak{B}(\mathbb{T})$ is not constant, then $L^{p(\cdot)}$ is not rearrangement-invariant [94, Example 3.14].

Variable Lebesgue spaces are often called Nakano spaces. See Maligranda [104] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces. The associate space of $L^{p(\cdot)}$ is isomorphic to the space $L^{p'(\cdot)}$, where $p' \in \mathfrak{B}(\mathbb{T})$ is defined so that

$1/p(t) + 1/p'(t) = 1$ for a.e. $t \in \mathbb{T}$ with the usual convention $1/\infty := 0$ [96, Theorem (3.2.8).13]. For $p \in \mathfrak{B}(\mathbb{T})$, put

$$p_- := \text{ess}_{t\in\mathbb{T}} \inf p(t)\,, p_+ := \text{ess}_{t\in\mathbb{T}} \sup p(t).$$

The space variable Lebesgue space $L^{p(\cdot)}$ is separable if and only if $p_+ < \infty$ (see, e.g., [94, Theorem 2.78]).

Theorem (3.2.1)[90]: Let X be a separable rearrangement-invariant Banach function space on $\mathbb{T}$. Then the set of analytic polynomials P_A is dense in the abstract Hardy space $H[X]$. Moreover, for every $f \in H[X]$, there is a sequence of analytic polynomials $\{p_n\}$ such that $\|p_n\|_X \leq \|f\|_X$ for all $n \in \mathbb{N}$ and $p_n \to f$ in the norm of X as $n \to \infty$.

We could not find in the literature neither Theorem (3.2.1) explicitly stated nor any result on the density of P_A in abstract Hardy spaces $H[X]$ in the case when X is an arbitrary Banach function space beyond the class of rearrangement-invariant spaces. The aim is to fill in this gap.

Given $f \in L^1$, the Hardy-Littlewood maximal function is defined by

$$(Mf)(t) := \sup_{I\ni t} \frac{1}{m(I)} \int_I |f(\tau)| dm(\tau)\,, t \in \mathbb{T},$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$ containing $t \in \mathbb{T}$. The operator $f \mapsto Mf$ is called the Hardy-Littlewood maximal operator.

Theorem (3.2.2)[90]: (Main result). Suppose X is a separable Banach function space on $\mathbb{T}$. If the Hardy-Littlewood maximal operator M is bounded on the associate space X', then the set of analytic polynomials P_A is dense in the abstract Hardy space $H[X]$.

To illustrate this result in the case of variable Lebesgue spaces, we will need the following classes of variable exponents. Following [94, Definition 2.2], one says that $r: \mathbb{T} \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant $C_0 > 0$ such that

$$|r(x) - r(y)| = C_0/(-\log|x-y|)) \text{ for all } x, y \in \mathbb{T}, |x-y| < 1/2.$$

The class of all locally log-Hölder continuous functions is denoted by $LH_0(\mathbb{T})$.

If $p_+ < \infty$, then $p \in LH_0(\mathbb{T})$ if and only if $1/p \in LH_0(\mathbb{T})$. By [94, Theorem 4.7], if $p \in \mathfrak{B}(\mathbb{T})$ is such that $1 < p_-$ and $1/p \in LH_0(\mathbb{T})$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}$. This condition was initially referred to as "almost necessary" (see [94, Section 4.6.1]). However, Lerner [102] constructed an example of discontinuous variable exponent such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$.

Kapanadze and Kopaliani [97] developed further Lerner's ideas. They considered the following class of variable exponents. Recall that a function $f \in L^1$ belongs to the space BMO if

$$\|f\|_* := \sup_{I\subset\mathbb{T}} \frac{1}{m(I)} \int_I |f(t) - f_I| dm(t) < \infty,$$

where f_I is the integral average of f on the arc I and the supremum is taken over all arcs $I \subset \mathbb{T}$. For $f \in BMO$, put

$$\gamma(f, r) := \sup_{m(I)\leq r} \frac{1}{m(I)} \int_I |f(t) - f_I| dm(t).$$

Let $VMO^{1/|\log|}$ be the set of functions $f \in BMO$ such that

$$\gamma(f, r) = o(1/|\log r|) \quad as\ r \to 0.$$

Note that $VMO^{1/|\log|}$ contains discontinuous functions. We will say that $p \in \mathfrak{B}(\mathbb{T})$ belongs to the Kapanadze-Kopaliani class $\mathfrak{K}(\mathbb{T})$ if $1 < p_- \le p_+ < \infty$ and $p \in VMO^{1/|\log|}$. It is shown in [97, Theorem 2.1] that if $p \in \mathfrak{K}(\mathbb{T})$, then the Hardy-Littelwood maximal operator M is bounded on the variable Lebesgue space $L^{p(\cdot)}$.

Corollary (3.2.3)[90]: Suppose $p \in \mathfrak{B}(\mathbb{T})$. If $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$ or if $p' \in \mathfrak{K}(\mathbb{T})$, then the set of analytic polynomials P_A is dense in the abstract Hardy space $H[L^{p(\cdot)}]$ built upon the variable Lebesgue space $L^{p(\cdot)}$.

We prove that the separability of a Banach function space X is equivalent to the density of the set of trigonometric polynomials P in X and to the density of the set of all continuous functions C in X. Further, we recall a pointwise estimate of the Fejér means $f * K_n$, where K_n is the n −th Fejér kernel, by the Hardy- Littlewood maximal function Mf. We show that the norms of the operators $F_n f = f * K_n$ are uniformly bounded on a Banach function space X if X is rearrangement-invariant or if the Hardy-Littlewood maximal operator is bounded on X'. Moreover, if X is rearrangement-invariant, then $\|F_n\|_{B(X)} \le 1$ for all $n \in \mathbb{N}$. Further, we prove that under the assumptions of Theorem (3.2.1) or (3.2.2), $\|f * K_n - f\|_X \to 0$ as $n \to \infty$. It remains to observe that $f * K_n \in P_A$ if $f \in H[X]$, which will complete the proof of Theorems (3.2.1) and (3.2.2).

We start with the following elementary lemma, whose proof can be found, e.g., in [93, Chap. III, Proposition 1.6(a)]. Here and in what follows, the space of all bounded linear operators on a Banach space E will be denoted by $B(E)$.

Lemma (3.2.4)[90]: Le E be a Banach space and $\{T_n\}$ be a sequence of bounded operators on E such that

$$\sup_{n\in\mathbb{N}} \|T_n\|_{B(E)} < \infty.$$

If D is a dense subset of E and for all $x \in D$,

$$\|T_n x - x\|_E \to 0 \text{ as } n \to \infty, \tag{67}$$

then (67) holds for all $x \in E$.

A function of the form

$$q(t) = \sum_{k=-n}^{n} \alpha_k t^k, t \in \mathbb{T}, \alpha_{-n}, \dots, \alpha_n \in \mathbb{C},$$

is said to be a trigonometric (or Laurent) polynomial on $\mathbb{T}$. The set of all trigonometric polynomials is denoted by P.

Lemma (3.2.5)[90]: Let X be a Banach function space on $\mathbb{T}$. The following statements are equivalent:

(a) The set P of all trigonometric polynomials is dense in X;

(b) the space C of all continuous functions on $\mathbb{T}$ is dense in X;

(c) the Banach function space X is separable.

Proof. The proof is developed by analogy with [98, Lemma 1.3].

(a) ⇒ (b) is trivial because $P \subset C \subset X$.

(b) ⇒ (c). Since C is separable and $C \subset X$ is dense in X, we conclude that X is separable.

(c) ⇒ (a). Assume that X is separable and P is not dense in X. Then by the corollary of the Hahn-Banach theorem (see, e.g., [92, Chap. 7, Theorem 4.2]), there exists a nonzero functional $\Lambda \in X^*$ such that $\Lambda(p) = 0$ for all $p \in P$. Since X is separable, from [90, Chap. 1, Corollaries 4.3 and 5.6] it follows that the Banach dual X^* of X is canonically isometrically isomorphic to the associate space X'. Hence there exists a nonzero function $h \in X' \subset L^1$ such that

$$\int_{\mathbb{T}} p(t)h(t)dm(t) = 0 \text{ for all } p \in P.$$

Taking $p(t) = t^n$ for $n \in \mathbb{Z}$, we obtain that all Fourier coefficients of $h \in L^1$ vanish, which implies that $h = 0$ a.e. on $\mathbb{T}$ by the uniqueness theorem of the Fourier series (see, e.g., [99, Chap. I, Theorem 2.7]). This contradiction proves that P is dense in X.

Recall that L^1 is a commutative Banach algebra under the convolution multiplication defined for $f, g \in L^1$ by

$$(f * g)(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta - i\varphi}) g(e^{i\varphi}) d\varphi, e^{i\theta} \in \mathbb{T}.$$

For $n \in \mathbb{N}$, let

$$K_n(e^{i\theta}) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{i\theta k} = \frac{1}{n+1} \left(\frac{\sin \frac{n+1}{2}\theta}{\sin \frac{\theta}{2}} \right)^2, e^{i\theta} \in \mathbb{T},$$

be the n −th Fejér kernel. It is well-known that $\|K_n\|_{L^1} \le 1$. For $f \in L^1$, the n −th Fej´er mean of f is defined as the convolution $f * K_n$. Then

$$(f * K_n)(e^{i\theta}) = \sum_{k=-n}^{n} \hat{f}(k) \left(1 - \frac{|k|}{n+1}\right) e^{i\theta k}, e^{i\theta} \in \mathbb{T} \quad (68)$$

(see, e.g., [99, Chap. I]). This means that if $f \in L^1$, then $f * K_n \in P$. Moreover, if $f \in H^1 = H[L^1]$, then $f * K_n \in P_A$.

Lemma (3.2.6)[90]: For every $f \in L^1$ and $t \in \mathbb{T}$,

$$\sup_{n \in \mathbb{N}} |(f * K_n)(t)| \le \frac{\pi^2}{2} (Mf)(t). \quad (69)$$

Proof. Since $|\sin \varphi| \ge 2|\varphi|/\pi$ for $|\varphi| \le \pi/2$, we have for $\theta \in [-\pi, \pi]$,

$$\begin{aligned} K_n(e^{i\theta}) &\le \frac{\pi^2}{n+1} \sin^2 \frac{\left(\frac{n+1}{2}\theta\right)}{\theta^2} \\ &= \frac{\pi^2}{4}(n+1) \sin^2 \frac{\left(\frac{n+1}{2}\theta\right)}{\left(\frac{n+1}{2}\theta\right)^2} \\ &\le \frac{\pi^2}{4}(n+1) \min\left\{1, \left(\frac{n+1}{2}\theta\right)^{-2}\right\} \\ &\le \frac{\pi^2}{2} \frac{n+1}{1 + \left(\frac{n+1}{2}\theta\right)^2} =: \Psi_n(\theta). \end{aligned} \quad (70)$$

It is easy to see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_n(\theta)d\theta \le \frac{\pi^2}{2} \text{ for all } n \in \mathbb{N}. \tag{71}$$

From [105, Lemma 21] and estimates (70)–(71) we immediately get estimate (69).

First we consider the case of rearrangement-invariant Banach function spaces.

Lemma (3.2.7). Let X be a rearrangement-invariant Banach function space on $\mathbb{T}$. Then for each $n \in \mathbb{N}$, the operator $F_n f = f * K_n$ is bounded on X and

$$\sup_{n\in\mathbb{N}}\|F_n\|_{B(X)} \le 1.$$

Proof. By [90, Chap. 3, Lemma 6.1], for every $f \in X$ and every $n \in \mathbb{N}$,

$$\|f * K_n\|_X \le \|K_n\|_{L^1}\|f\|_X.$$

It remains to recall that $\|K_n\|_{L^1} \le 1$ for all $n \in \mathbb{N}$.

Now we will show the corresponding results for Banach function spaces such that the Hardy-Littlewood maximal operator is bounded on X'.

Theorem (3.2.8)[90]: Let X be a Banach function space on $\mathbb{T}$ such that the Hardy-Littlewood maximal operator M is bounded on its associate space X'. Then for each $n \in \mathbb{N}$, the operator $F_n f = f * K_n$ is bounded on X and

$$\sup_{n\in\mathbb{N}}\|F_n\|_{B(X)} \le \pi^2\|M\|_{X'\to X'}.$$

Proof. The idea of the proof is borrowed from the proof of [94, Theorem 5.1].

Fix $f \in X$ and $n \in \mathbb{N}$. Since $K_n \ge 0$, we have $|f * K_n| \le |f| * K_n$. Then from the Lorentz-Luxemburg theorem (see, e.g., [90, Chap. 1, Theorem 2.7]) we deduce that

$$\begin{aligned}\|f * K_n\|_X &\le \||f| * K_n\|_X = \||f| * K_n\|_{X''}\\ &= \sup\left\{\int_{\mathbb{T}}(|f| * K_n)(t)|g(t)|\,dm(t) : g \in X', \|g\|_{X'} \le 1\right\}.\end{aligned}$$

Hence there exists a function $h \in X'$ such that $h \ge 0$, $\|h\|_{X'} \le 1$, and

$$\|f * K_n\|_X \le 2\int_{\mathbb{T}}(|f| * K_n)(t)h(t)dm(t). \tag{72}$$

Taking into account that $K_n(e^{i\theta}) = K_n(e^{-i\theta})$ for all $\theta \in \mathbb{R}$, by Fubini's theorem, we get

$$\int_{\mathbb{T}}(|f| * K_n)(t)h(t)dm(t) = \int_{\mathbb{T}}(h * K_n)(t)|f(t)|dm(t).$$

From this identity and Hölder's inequality for X (see, e.g., [90, Chap. 1, Theorem 2.4]), we obtain

$$\int_{\mathbb{T}}(|f| * K_n)(t)h(t)dm(t) \le \|f\|_X\|h * K_n\|_{X'}. \tag{73}$$

Applying Lemma (3.2.6) to $h \in X' \subset L^1$, by the lattice property, we see that

$$\|h * K_n\|_{X'} \le \frac{\pi^2}{2}\|Mh\|_{X'}. \tag{74}$$

Combining estimates (72)–(74) and taking into account that M is bounded on X' and that $\|h\|_{X'} \le 1$, we arrive at

$$\|f * K_n\|_X \le \pi^2\|M\|_{X'\to X'}\|f\|_X.$$

Hence

$$\sup_{n\in\mathbb{N}}\|F_n\|_{B(X)} = \sup_{n\in\mathbb{N}}\sup_{f\in X\setminus\{0\}}\frac{\|f * K_n\|_X}{\|f\|_X} \le \pi^2\|M\|_{X'\to X'} < \infty,$$

which completes the proof.

We have the following for the proof of the main results.

Theorem (3.2.9)[90]: Suppose X is a separable Banach function space on $\mathbb{T}$. If X is rearrangement-invariant or the Hardy-Littlewood maximal operator is bounded on the associate space X', then for every $f \in X$,

$$\lim_{n\to\infty}\|f * K_n - f\|_X = 0. \tag{75}$$

Proof. It is well-known that for every $f \in C$,

$$\lim_{n\to\infty}\|f * K_n - f\|_C = 0$$

(see, e.g., [93, Chap. III, Theorem 1.1(a)] or [99, Theorem 2.11]). From the definition of the Banach function space X it follows that $C \subset X \subset L^1$,

Where both embeddings are continuous. Then, for every $f \in C$, (75) is fulfilled. From Lemma (3.2.5) we know that the set C is dense in the space X. By Lemma (3.2.7) and Theorem (3.2.8),

$$\sup_{n\in\mathbb{N}}\|F_n\|_{B(X)} < \infty,$$

where $F_n f = f * K_n$. It remains to apply Lemma (3.2.4).

This statement for rearrangement-invariant Banach function spaces is contained, e.g., in [95, p. 268]. Notice that the assumption of the separability of X is hidden there.

Now we formulate the corollary of the above theorem in the case of variable Lebesgue spaces.

Corollary (3.2.10)[90]: Suppose $p \in \mathfrak{B}(\mathbb{T})$. If $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$ or if $p' \in \mathfrak{K}(\mathbb{T})$, then for every $f \in L^p(\cdot)$,

$$\lim_{n\to\infty}\|f * K_n - f\|_{L^{p}(\cdot)} = 0.$$

For variable exponents $p \in \mathfrak{B}(\mathbb{T})$ satisfying $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$, this result was obtained by Sharapudinov [106, Section 3.1]. For $p \in \mathfrak{K}(\mathbb{T})$, the above corollary is new.

If $f \in H[X]$, then $p_n = f * K_n \in P_A$ for all $n \in \mathbb{N}$ in view of (68). By Theorem (3.2.9), $\|p_n - f\|_X \to 0$ as $n \to \infty$. Thus the set P_A is dense in in the abstract Hardy space $H[X]$ built upon X.

Moreover, if X is a rearrangement-invariant Banach function space, then from Lemma (3.2.7) it follows that $\|p_n\|_X \le \|f\|_X$ for all $n \in \mathbb{N}$.

Section (3.3): Toeplitz Operators Acting Between Hardy Type Subspaces of Different Banach Function Spaces:

For Γ be a Jordan curve, that is, a curve that homeomorphic to a circle. We suppose that Γ is rectifiable and equip it with the Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation. The Cauchy singular integral of a measurable function $f: \Gamma \to \mathbb{C}$ is defined by

$$(Sf)(t) := \lim_{\epsilon\to 0}\frac{1}{\pi i}\int_{\Gamma\setminus\Gamma(t,\varepsilon)}\frac{f(\tau)}{\tau - t}d\tau, t \in \Gamma, \tag{76}$$

where the "portion" $\Gamma(t,\varepsilon)$ is

$$\Gamma(t,\varepsilon) := \{\tau \in \Gamma: |\tau - t| < \varepsilon\}, \varepsilon > 0.$$

It is well known that $(Sf)(t)$ exists a.e. on Γ whenever f is integrable (see [118, Theorem (3.3.3)2]).

For two normed spaces X and Y, we will write $X \hookrightarrow Y$ if there is a constant $c \in (0, \infty)$ such that $\|f\|_Y \leq c\|f\|_X$ for all $f \in X, X = Y$ if X and Y coincide as sets and there are constants $c_1, c_2 \in (0, \infty)$ such that $c_1\|f\|_X \leq \|f\|_Y \leq c_2\|f\|_X$ for all $f \in X$, and $X \equiv Y$ if X and Y coincide as sets and $\|f\|_X = \|f\|_Y$ for all $f \in X$. As usual, the space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. We adopt the standard abbreviation $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$.

Let γ be a measurable subset of Γ of positive measure. The set of all measurable complex-valued functions on γ is denoted by $M(\gamma)$. Let $M^+(\gamma)$ be the subset of functions in $M(\gamma)$ whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \gamma$ is denoted by χ_E.

Following [109, Chap. 1, Definition 1.1], a mapping $\rho_\gamma: M^+(\gamma) \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in M^+(\gamma)$ with $n \in \mathbb{N}$, for all constants $a \geq 0$, and for all measurable subsets E of γ, the following properties hold:

$(A1)\rho_\gamma(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \rho_\gamma(af) = a\rho_\gamma(f), \rho(f + g) \leq \rho_\gamma(f) + \rho_\gamma(g)$,

$(A2)\ 0 \leq g \leq f\ a.e. \Rightarrow \rho_\gamma(g) \leq \rho_\gamma(f)$ (the lattice property),

$(A3)0 \leq f_n \uparrow f\ a.e. \Rightarrow \rho_\gamma(f_n) \uparrow \rho_\gamma(f)$ (the Fatou property),

$(A4)\rho_\gamma(\chi_E) < \infty$,

$(A5) \int_E f(\tau)|d\tau| \leq C_E \rho_\gamma(f)$

with the constant $C_E \in (0, \infty)$ that may depend on E and ρ_γ, but is independent of f. When functions differing only on a set of measure zero are identified, the set $X(\gamma)$ of all functions $f \in M(\gamma)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\gamma)$, the norm of f is defined by

$$\|f\|_X(\gamma) := \rho(|f|).$$

The set $X(\gamma)$ under the natural linear space operations and under this norm becomes a Banach space (see [109, Chap. 1, Theorems 1.4 and 1.6]) and

$$L^\infty(\gamma) \hookrightarrow X(\gamma) \hookrightarrow L^1(\gamma).$$

If ρ_γ is a Banach function norm, its associate norm ρ'_γ is defined on $M^+(\gamma)$ by

$$\rho'_\gamma(g) := \sup\left\{\int_\gamma f(\tau)g(\tau)|d\tau| : f \in M^+(\gamma), \rho_\gamma(f) \leq 1\right\}, g \in M^+(\gamma).$$

It is a Banach function norm itself [109, Chap. 1, Theorem (3.3.3)]. The Banach function space $X'(\gamma)$ determined by the Banach function norm ρ′ is called the associate space (Kothe dual) of $X(\gamma)$. The associate space $X'(\gamma)$ can be viewed a subspace of the dual space $X^*(\gamma)$.

Recall that, since the Lebesgue length measure $|d\tau|$ is separable (see, e.g., [120, Section 6.10]), a Banach function space $X(\gamma)$ over γ is separable if and only if its Kothe dual space $X'(\gamma)$ is isometrically isomorphic to the Banach dual space $X^*(\gamma)$ (see, e.g., [109, Chap. 1, Corollaries 4.3, 4.4]). A Banach function space $X(\gamma)$ reflexive if and only if $X(\gamma)$ and $X'(\gamma)$ are separable (see, e.g., [109, Chap. 1, Corollary 5.6]).

For Banach function spaces $X(\gamma)$ and $Y(\gamma)$, let $M\big(X(\gamma), Y(\gamma)\big)$ denote the space of point wise multipliers from $X(\gamma)$ to $Y(\gamma)$ defined by

$$M\big(X(\gamma), Y(\gamma)\big) := \{f \in M(\gamma): fg \in Y(\gamma) \text{ for all } g \in X(\gamma)\}.$$

It is a Banach function space with respect to the operator norm

$$\|f\|_{M(X(\gamma),Y(\gamma))} = \sup\{\|fg\|_{Y(\gamma)}: g \in X(\gamma), \|g\|_{X(\gamma)} \le 1\}.$$

In particular, $M\big(X(\gamma), X(\gamma)\big) \equiv L^\infty(\gamma)$. Note that it may happen that the space $M\big(X(\gamma), Y(\gamma)\big)$ contains only the zero function. For instance, if $1 \le p < q < \infty$, then $M\big(L^p(\gamma), L^q(\gamma)\big) = \{0\}$. The continuous embedding $L^\infty(\gamma) \hookrightarrow M\big(X(\gamma), Y(\gamma)\big)$ holds if and only if $X(\gamma) \hookrightarrow Y(\gamma)$. For example, if $1 \le q \le p \le \infty$, then $L^p(\gamma) \hookrightarrow L^q(\gamma)$ and $M\big(L^p(\gamma), L^q(\gamma)\big) \equiv L^r(\gamma)$, where $1/r = 1/q - 1/p$. For these and many other properties and examples, see [124,126,128,129,130].

We will write $X := X(\Gamma)$ if Γ is a rectifiable Jordan curve. If X is a reflexive Banach function space over a rectifiable Jordan curve Γ and the Cauchy singular integral operator defined by (76) is bounded on X, then in view of [121, Theorem 6.1] and the Hölder inequality for Banach function spaces (see, e.g., [109, Chap. 1, Theorem 2.4]), the curve Γ is a Carleson curve (or Ahlfors-David regular curve), that is,

$$\sup_{t\in\Gamma}\sup_{\varepsilon>0}\frac{|\Gamma(t,\varepsilon)|}{\varepsilon} < \infty.$$

Moreover, by [121, Lemma 6.4], the operators

$$P := (I + S)/2, Q := (I - S)/2$$

are bounded projections both on X and on X', the latter means that $P^2 = P$ and $Q^2 = Q$. Then we can define Hardy type subspaces PX, QX of X and PX', QX' of X'.

In what follows we will always assume that X and Y are reflexive Banach function spaces and S is bounded on both X and Y. For $a \in M(X, Y)$, define the Toeplitz operator $T(a): PX \to PY$ with symbol a by

$$T(a)f = P(af), f \in PX.$$

It is clear that $T(a) \in \mathcal{L}(PX, PY)$ and

$$\|T(a)\|_{\mathcal{L}(PX,PY)} \le \|P\|_{\mathcal{L}(Y)}\|a\|_{M(X,Y)}.$$

We note that there is a huge literature dedicated to Toeplitz operator acting between the same Hardy spaces $H^p = PL^p, 1 < p < \infty$, see, e.g., the monographs by Douglas [115], Bottcher and Silbermann [111], Gohberg, Goldberg, Kaashoek [119], Nikolski [131] for Toeplitz operators on Hardy spaces over the unit circle and the monograph by Bottcher and Karlovich [110] for Toeplitz operators on weighted Hardy spaces over Carleson curves.

We could find by Tolokonnikov [135] dedicated to Toeplitz operators acting between different Hardy spaces H^p and H^q over the unit circle. In particular, he described in [135, Theorem 4] all symbols generating bounded Toeplitz operators from H^p to H^q for $0 < p, q \le \infty$. Lesnik [125] proposed to study Toeplitz and Hankel operators between abstract Hardy spaces $H[X]$ and $H[Y]$ built upon different separable rearrangementin variant Banach function spaces X and Y over the unit circle such that $X \hookrightarrow Y$ and the space Y has nontrivial Boyd indices. Notice that the latter condition is equivalent to the boundedness of the operator S on the space Y, whence $H[Y] = PY$.

Lesnik obtained analogues of the Brown-Halmos and Nehari theorems (see [125, Theorem 4.2] and [125, Theorem 5.5], respectively), extending results of [122] for the case of a reflexive rearrangement-invariant Banach function space X (that is, $X = Y$) with nontrivial Boyd indices. He also proved [125, Theorem 6.1] that a Toeplitz operator $T(a): H[X] \to H[Y]$ is compact if and only if $a = 0$.

Inspired by Lesnik [125], we prove the following analogue of the Coburn-Simonenko theorem for Toeplitz operators $T(a): PX \to PY$ in the case when X and Y are different Banach function spaces. Notice that we do not assume that the spaces X and Y are rearrangement-invariant.

The above result was proved by Coburn [112] for the case of $X = Y = L^2$ over the unit circle and by Simonenko [134] in a more general of setting of $X = Y = L^p, 1 < p < \infty$, over so-called Lyapunov curves. See [110, Theorem 6.17], where the above theorem is proved in the case $X = Y = L^p(w)$, where $L^p(w), 1 < p < \infty$, is a Lebesgue space with a Muckenhoupt weight over a Carleson Jordan curve.

The statement of Theorem (3.3.10) has a more precise form for concrete Banach function spaces X, Y when $M(X, Y)$ can be calculated and conditions for the boundedness of S are known. Here we mention only the case of Toeplitz operators acting from the Hardy space $H^p = PL^p$ to the Hardy space $H^q = PL^q$ as the simplest example.
Corollary (3.3.1)[108]: Let $1 < q \le p < \infty$ and $1/r = 1/q - 1/p$. Suppose Γ is a Carleson Jordan curve. If $a \in L^r \backslash \{0\}$, then the Toeplitz operator $T(a) \in \mathcal{L}(H^p, H^q)$ has a trivial kernel in H^p or a dense image in H^q.

It seems that the above corollary is new even in the case of the unit circle.

We collect properties of Banach function spaces and their Hardy type subspaces proved elsewhere. We first relate the triviality of the kernel (resp. the density of the image) of a Toeplitz operator $\tilde{T}(a) \in \mathcal{L}(PX, PY)$ with the density of the range (resp. triviality of the kernel) of its companion operator $\tilde{T}(a): L(QY', QX')$ defined by $\tilde{T}(a)f = Q(af)$.

Then show that one of the operators $T(a)$ or $\tilde{T}(a)$ is injective with the aid of the Lusin-Privalov theorem and other results stated. We recall the definition of variable Lebesgue spaces $L^{p(\cdot)}$, which give a non-trivial example of Banach function spaces. Further, we describe the space $M\left(L^{p(\cdot)}, L^{r(\cdot)}\right)$ and formulate conditions for the boundedness of the operator Cauchy singular operator S on $L^{p(\cdot)}$. These results allow us to reformulate Theorem (3.3.10) for Toeplitz operators between $PL^{p(\cdot)}$ and $PL^{q(\cdot)}$ in terms of variable exponents $p, q: \Gamma \to (1, \infty)$. In particular, we immediately get Corollary (3.3.1), taking all exponents constant.

Let Γ be a rectifiable Jordan curve. It divides the plane into a bounded connected component D^+ and an unbounded connected component D^-. We provide Γ with the counter-clockwise orientation, that is, we demand that D^+ stays on the left of Γ when the curve is traced out in the positive direction. Without loss of generality we suppose that $0 \in D^+$. Put

$$L_+^1 := \left\{ f \in L^1 : \int_\Gamma f(\tau)\tau^n d\tau = 0 \text{ for } n \ge 0 \right\},$$

$$(L^1)^0_- := \left\{ f \in L^1 : \int_\Gamma f(\tau)\tau^n d\tau = 0 \text{ for } n < 0 \right\},$$
$$L^1_- := (L^1)^0_- \oplus \mathbb{C}.$$
From [132, pp. 202–206] one can extract the following result.
Lemma (3.3.2)[108]: We have $L^1_+ \cap (L^1)^0_- = \{0\}$ and $L^1_+ \cap L^1_- = \mathbb{C}$.

The proof of the following important theorem is contained in [132, p. 292] or [117, Theorem 10.3].
Theorem (3.3.3)[108]: (Lusin-Privalov). Let Γ be a rectifiable Jordan curve. If $f \in L^1_\pm$, then f vanishes either almost everywhere on Γ or almost nowhere on Γ.

We collect some well known properties of Banach function spaces and pointwise multipliers between them.
Lemma (3.3.4)[108]: ([109, Chap. 1, Proposition 2.10]). Let X, Y be Banach function spaces over a rectifiable Jordan curve Γ and let X', Y' be their associate spaces, respectively. If $X \hookrightarrow Y$, then $Y' \hookrightarrow X'$.
Lemma (3.3.5)[108]: ([124, Section 2, property (vii)]). Let X, Y be Banach function spaces over a rectifiable Jordan curve Γ and let X', Y' be their associate spaces, respectively. Then $M(X, Y) \equiv M(Y', X')$.
Lemma (3.3.6)[108]: Let X, Y be separable Banach function spaces over a rectifiable Jordan curve Γ and $a \in M(X, Y)$. Then the adjoint of the operator $aI \in \mathcal{L}(X, Y)$ of multiplication by the function a is the operator $(aI)^* = aI \in \mathcal{L}(Y', X')$.
Proof. Since X (resp., Y) is separable, its Banach dual space X^* (resp., Y^*) is isometrically isomorphic to the associate (Kothe dual) space X' (resp., Y') and
$$G(f) = \int_\Gamma f(\tau)\overline{g(\tau)}|d\tau|$$
gives the general form of a linear functional on X (resp., Y) and $\|G\|_{X^*} = \|g\|_{X'}$ (resp., $\|G\|_{Y^*} = \|g\|_{Y'}$), see, e.g, [109, Chap. 1, Corollary 4.3]. The desired statement follows immediately from the above observation and Lemma (3.3.5).

Suppose X is a reflexive Banach function space in which the Cauchy singular integral operator S is bounded. Put
$$X_+ := PX, \quad X^0_- := QX, \quad X_- := X^0_- \oplus \mathbb{C}.$$
The corresponding subspaces X'_+, $(X')^0_-$, X'_- are defined analogously.

For $f \in X \subset L^1$, consider the Cauchy type integrals
$$(C_\pm f)(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)}{\tau - z} d\tau, z \in D^\pm.$$
It is well known [132, p. 189] that the functions $(C_\pm f)(z)$ are analytic in $D^\pm$, they have nontangential boundary values $(C_\pm f)(t)$ as $z \to t$ almost everywhere on Γ.

These boundary values can be found by the Sokhotsky-Plemelj formulas
$$(C_\pm f)(t) = \frac{1}{2}f(t) \pm \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)}{\tau - t} d\tau,$$
that is,
$$(C_+ f)(t) = (Pf)(t), \ (C_- f)(t) = (Qf)(t).$$

Since the function $f \in X_+$ (respectively, $f \in X_-^0$) coincides on Γ with the boundary value of the function C_+f (respectively, C_-f) defined in D^+ (respectively, D^-), we will think of functions from X_+ (respectively, X_-^0) as of functions defined in D^+ (respectively, in D^-) by $f(z) := (C_+f)(z)$ (respectively, by $f(z) := (C_-f)(z)$).

Lemma (3.3.7)[108]: ([121, Lemma 6.9]). Let Γ be a rectifiable Jordan curve and X be a reflexive Banach function space in which the Cauchy singular integral operator S is bounded.

(a) If $f \in X_\pm$ and $g \in X'_\pm$, then $fg \in L^1_\pm$. If, in addition, $f \in X_-^0$ or $g \in (X')_-^0$, then $fg \in (L^1)_-^0$.

(b) We have

$$X_+ = L^1_+ \cap X, X_-^0 = (L^1)_-^0 \cap X, X_- = L^1_- \cap X.$$

On a rectifiable Jordan oriented curve Γ, we have

$$d\tau = e^{i\theta_\Gamma(\tau)}|d\tau|,$$

Where $\theta_\Gamma(\tau)$ is the angle made by the positively oriented real axis and the naturally oriented tangent of Γ at τ (which exists almost everywhere). Let X be a Banach function space over Γ. Define the operator $H_\Gamma: X \to X$ by

$$(H_\Gamma f)(\tau) := e^{-i\theta_\Gamma(\tau)}\overline{f(\tau)}.$$

Note that the operator H_Γ is additive but $H_\Gamma(\alpha f) = \bar{\alpha} \cdot H_\Gamma f$ for $\alpha \in \mathbb{C}$ and $f \in X$.

It is clear that H_Γ is bounded on X and $H_\Gamma^2 = I$.

Lemma (3.3.8)[108]: ([121, Lemma 6.6]). Let Γ be a rectifiable Jordan curve and X be a reflexive Banach function space in which the Cauchy singular integral operator S is bounded. Then the adjoint of $S \in \mathcal{L}(X)$ is the operator $S^* = -H_\Gamma S H_\Gamma \in \mathcal{L}(X')$ and consequently,

$$P^* = H_\Gamma Q H_\Gamma, Q^* = H_\Gamma P H_\Gamma.$$

Let X and Y be reflexive Banach function spaces over a rectifiable Jordan curve Γ. Suppose $a \in M(X,Y) \equiv M(Y',X')$ and the operator S is bounded on X and on Y. In view of Lemma (3.3.8), the operator S is also bounded on Y' and on X'. Then, along with the Toeplitz operator $T(a): X_+ \to Y_+$, we consider its companion operator $\tilde{T}(a): (Y')_-^0 \to (X')_-^0$ defined by

$$\tilde{T}(a)f = Q(af), f \in (Y')_-^0.$$

It is obvious that $\tilde{T}(a) \in \mathcal{L}((Y')_-^0, (X')_-^0)$ and

$$\left\|\tilde{T}(a)\right\|_{\mathcal{L}((Y')_-^0,(X')_-^0)} \le \|Q\|_{\mathcal{L}(X')}\|a\|_{M(X,Y)}.$$

Lemma (3.3.9)[108]: Let X and Y be reflexive Banach function spaces over a rectifiable Jordan curve. Suppose $X \hookrightarrow Y$ and the Cauchy singular integral operator S given by (76) is bounded on X and on Y. If $a \in M(X,Y)$, then the Toeplitz operator $T(a): X_+ \to Y_+$ has a trivial kernel in X_+ (resp., a dense image in Y_+) if and only if its companion operator $\tilde{T}(a): (Y')_-^0 \to (X')_-^0$ has a dense image in $(X')_-^0$ (resp., a trivial kernel in $(Y')_-^0$).

Proof. Let $\operatorname{Im} A$ and $\ker A$ denote the image and the kernel, respectively, of a bounded linear operator A acting between Banach spaces.

Since $X \hookrightarrow Y$, we have $Q \in \mathcal{L}(X,Y)$ and $PaP + Q \in \mathcal{L}(X,Y)$. The spaces X and Y decompose into the direct sums $X = X_+ \oplus X_-^0$ and $Y = Y_+ \oplus Y_-^0$. Accordingly, the operator $PaP + Q$ may be written as an operator matrix

$$\begin{pmatrix} T(a) & 0 \\ 0 & I \end{pmatrix} : \begin{pmatrix} X_+ \\ X_-^0 \end{pmatrix} \to \begin{pmatrix} Y_+ \\ Y_-^0 \end{pmatrix}.$$

Hence

$$\operatorname{Im}(PaP + Q) = \operatorname{Im} T(a) \oplus Y_-^0, \ker(PaP + Q) = \ker T(a). \qquad (77)$$

On the other hand, $Y' \hookrightarrow X'$ by Lemma (3.3.4) and $a \in M(Y', X')$ by Lemma (3.3.5). Then $P \in \mathcal{L}(Y', X')$ and $P + QaQ \in \mathcal{L}(Y', X')$. Since the spaces Y' and X' decompose into the direct sums $Y' = (Y')_+ \oplus (Y')_-^0$ and $X' = (X')_+ \oplus (X')_-^0$, the operator $P + QaQ$ may be written as an operator matrix

$$\begin{pmatrix} I & 0 \\ 0 & \tilde{T}(a) \end{pmatrix} : \begin{pmatrix} (Y')_+ \\ (Y')_-^0 \end{pmatrix} \to \begin{pmatrix} (X')_+ \\ (X')_-^0 \end{pmatrix}.$$

Therefore

$$\operatorname{Im}(P + QaQ) = (X')_+ \oplus \operatorname{Im}\tilde{T}(a), \ker(P + QaQ) = \ker \tilde{T}(a). \qquad (78)$$

Lemmas (3.3.6) and (3.3.8) yield

$$\begin{aligned}(PaP + Q)^* = P^*\bar{a}P^* + Q^* &= (H_\Gamma Q H_\Gamma)(H_\Gamma a H_\Gamma)(H_\Gamma Q H_\Gamma) + H_\Gamma P H_\Gamma \\ &= H_\Gamma(P + QaQ)H_\Gamma. \end{aligned} \qquad (79)$$

From the second identity in (77) it follows that $T(a) \in \mathcal{L}(X_+, Y_+)$ has a trivial kernel in X_+ if and only if $PaP + Q \in \mathcal{L}(X, Y)$ has a trivial kernel in X. On the other hand, from (79) and $H_\Gamma^2 = I$ we deduce that the latter fact is equivalent to the fact that $P + QaQ \in \mathcal{L}(Y', X')$ has a dense image in X' (see, e.g., [133, Section 4.12]). In turn, in view of the first identity in (78), the operator $P + QaQ$ has a dense image in X' if and only if the operator $\tilde{T}(a) \in \mathcal{L}((Y')_-^0, (X')_-^0)$ has a dense image in $(X')_-^0$.

The proof of the equivalence of the density of the image of $T(a)$ in Y_+ and the triviality of the kernel of e $\tilde{T}(a)$ in $(Y')_-^0$ is analogous.

Theorem (3.3.10)[108]: Let X and Y be reflexive Banach function spaces over a rectifiable Jordan curve. Suppose $X \hookrightarrow Y$ and the Cauchy singular integral operator S given by (76) is bounded on X and on Y. If $a \in M(X, Y)\backslash\{0\}$, then $T(a) \in \mathcal{L}(PX, PY)$ has a trivial kernel in PX or a dense image in PY.

Proof. In view of Lemma (3.3.9), it is sufficient to show that $T(a): X_+ \to Y_+$ is injective on X_+ or $\tilde{T}(a): (Y')_-^0 \to (X')_-^0$ is injective on $(Y')_-^0$.

Assume the contrary, that is, that there exist $f_+ \in X_+$ and $g_- \in (Y')_-^0$ such that $f_+ \neq 0, g_- \neq 0$, and

$$Paf_+ = 0, Qag_- = 0. \qquad (80)$$

By Lemma (3.3.7)(b), $f_+ \in X_+ \subset L_+^1$ and $g_- \in (Y')_0^- \subset L_-^1$. Since $f_+ \neq 0$ and $g_- \neq 0$, from the Lusin-Privalov Theorem (3.3.3) it follows that $f_+ \neq 0$ a.e. on Γ and $g_- \neq 0$ a.e. on Γ.

Put $f_- := af_+$ and $g_+ := ag_-$. Then from (80) it follows that $Paf_+ = Pf_- = 0$ and $Qag_- = Qg_+ = 0$. Therefore,

$$f_- = af_+ = Paf_+ + Qaf_+ = Qaf_+ \in Y_-^0,$$
$$g_+ = ag_- = Pag_- + Qag_- = Pag_- \in (X')_+.$$

Then

$$f_+g_+ = f_+(ag_-) = (f_+a)g_- = f_-g_-. \qquad (81)$$

From Lemma (3.3.7)(a) we deduce that $f_+g_+ \in L_+^1$ and $f_-g_- \in (L^1)_-^0$. Lemma (3.3.2) and identity (81) imply that $f_+g_+ = f_-g_- = f_+ag_- = 0$. Since $f_+ \neq 0$ a.e. on and $g_- \neq$

0 a.e. on Γ, we conclude that $a = 0$ a.e. on Γ, but this contradicts our hypothesis and, thus, completes the proof.

Given a rectifiable Jordan curve Γ, let $P(\Gamma)$ be the set of all measurable functions $p: \Gamma \to [1, \infty]$. For $p \in P(\Gamma)$ and a measurable subset $\gamma \subset \Gamma$, put

$$\gamma_\infty^{p(\cdot)} := \{t \in \gamma: p(t) = \infty\}.$$

For a measurable function $f: \gamma \to \mathbb{C}$, consider

$$\varrho_{p(\cdot),\gamma}(f) := \int_{\gamma \setminus \gamma_\infty^{p(\cdot)}} |f(t)|^{p(t)} |dt| + \|f\|_{L^\infty\left(\gamma_\infty^{p(\cdot)}\right)}.$$

According to [113, Definition 2.9], the variable Lebesgue space $L^{p(\cdot)}(\gamma)$ is defined as the set of all measurable functions $f: \gamma \to \mathbb{C}$ such that $\varrho_{p(\cdot),\gamma}(f/\lambda) < \infty$ for some $\lambda > 0$. This space is a Banach function space with respect to the Luxemburg-Nakano norm given by

$$\|f\|_{L^{p(\cdot)}(\gamma)} := \inf\{\lambda > 0: \varrho_{p(\cdot),\gamma}(f/\lambda) \le 1\}$$

(see, e.g., [113, Theorems 2.17, 2.71 and Section 2.10.3]). If $p \in P(\Gamma)$ is constant, then $L^{p(\cdot)}(\gamma)$ is nothing but the standard Lebesgue space $L^p(\gamma)$. Variable Lebesgue spaces are often called Nakano spaces. See Maligranda [127] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces.

The following property of the unit ball of variable Lebesgue spaces is well known (see, e.g., [113, Corollary 2.22]).

Lemma (3.3.11)[108]: Let γ be a measurable subset of a rectifiable Jordan curve Γ. If $p \in P(\Gamma)$ and f is a measurable function on γ, then the inequalities $\varrho_{p(\cdot),\gamma}(f) \le 1$ and $\|f\|_{L^{p(\cdot)}(\gamma)} \le 1$ are equivalent.

For the brevity, we will simply write $L^{p(\cdot)}$ for $L^{p(\cdot)}(\Gamma)$. For $p \in P(\Gamma)$, put

$$p_- := \operatorname{ess\,inf}_{t \in \Gamma} p(t), p_+ := \operatorname{ess\,sup}_{t \in \Gamma} p(t).$$

Lemma (3.3.12)[108]: ([113, Corollary 2.81]). Let Γ be a rectifiable Jordan curve and $p \in P(\Gamma)$.

Then $L^{p(\cdot)}$ is reflexive if and only if $1 < p_- \le p_+ < \infty$.

Embeddings of variable Lebesgue spaces are characterized as follows.

Lemma (3.3.13)[108]: ([113, Corollary 2.48]). Let Γ be a rectifiable Jordan curve. Suppose $p, q \in P(\Gamma)$. Then $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$ if and only if $q(t) \le p(t)$ for almost all $t \in \Gamma$.

We will describe the space of pointwise multipliers between variable Lebesgue spaces. The next lemma follows from [129, Section 2, Property (f) and Theorem 1] and the fact that variable Lebesgue spaces are Banach function spaces [113, Section 2.10.3].

Lemma (3.3.14)[108]: Let γ be a measurable subset of a rectifiable Jordan curve Γ and $p \in P(\Gamma)$. Then

$$M\left(L^\infty(\gamma), L^{p(\cdot)}(\gamma)\right) \equiv L^{p(\cdot)}(\gamma), M\left(L^{p(\cdot)}(\gamma), L^{p(\cdot)}(\gamma)\right) \equiv L^\infty(\gamma).$$

Now we state the following two simple statements.

Lemma (3.3.15)[108]: Let Γ be a rectifiable Jordan curve and $\gamma_1, \dots, \gamma_k$ be measurable sub-sets of Γ such that

$$\gamma_i \cap \gamma_j = \emptyset \text{ for } i, j \in \{1, \dots, k\}, \ \gamma_1 \cup \cdots \cup \gamma_k = \Gamma. \tag{82}$$

If $p \in P(\Gamma)$, then

$$L^{p(\cdot)} = L^{p(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{p(\cdot)}(\gamma_k),$$

where the norm in the direct sum $L^{p(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{p(\cdot)}(\gamma_k)$ is defined by

$$\|f\|_{L^{p(\cdot)}(76)\oplus\cdots\oplus L^{p(\cdot)}(k)} = \|f\chi_{\gamma_1}\|_{L^{p(\cdot)}(\gamma_1)} + \cdots + \|f\chi_{\gamma_k}\|_{L^{p(\cdot)}(\gamma_k)}.$$

Lemma (3.3.17)[108]: Let Γ be a rectifiable Jordan curve and $\gamma_1, \ldots, \gamma_k$ be measurable sub-sets of Γ satisfying (82). If $p, q \in P(\Gamma)$ and $q(t) \le p(t)$ for almost all $t \in \Gamma$, then

$$M\left(L^{p(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{p(\cdot)}(\gamma_k), L^{q(\cdot)}(\gamma_1) \oplus \cdots \oplus L^{q(\cdot)}(\gamma_k)\right)$$
$$= M\left(L^{p(\cdot)}(\gamma_1), L^{q(\cdot)}(\gamma_1)\right) \oplus \cdots \oplus M\left(L^{p(\cdot)}(\gamma_k), L^{q(\cdot)}(\gamma_k)\right).$$

The proofs of the above two lemmas are straightforward.

We will need the following generalized Hölder inequality.

Lemma (3.3.18)[108]: ([113, Corollary 2.28]). Let Γ be a rectifiable Jordan curve. Suppose $p, q, r \in P(\Gamma)$ are related by

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, t \in \Gamma. \quad (83)$$

Then there exists a constant $C > 0$ such that for all $f \in L^{p(\cdot)}$ and $g \in L^{r(\cdot)}$, one has $fg \in L^{q(\cdot)}$ and

$$\|fg\|_{L^{q(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}\|g\|_{L^{r(\cdot)}}.$$

The following result was obtained by Nakai [130, Example 4.1] under the additional hypothesis

$$\sup_{t\in\Gamma\setminus\Gamma_\infty^{r(\cdot)}} r(t) < \infty$$

(and in the more general setting of quasi-Banach variable Lebesgue spaces spaces over arbitrary measure spaces). Nakai also mentioned in [130, Remark 4.2] (without proof) that this hypothesis is superfluous. We provide a proof here.

Theorem (3.3.19)[108]: Let Γ be a rectifiable Jordan curve. Suppose $p, q, r \in P(\Gamma)$ are related by (83). Then $M\left(L^{p(\cdot)}, L^{q(\cdot)}\right) = L^{r(\cdot)}$.

Proof. Let $\gamma_1 := \Gamma_\infty^{p(\cdot)}, \gamma_2 := \left(\Gamma_\infty^{q(\cdot)} \cup \Gamma_\infty^{r(\cdot)}\right)\setminus\Gamma_\infty^{p(\cdot)}$, and

$$\gamma_3 := \Gamma\setminus(\gamma_1 \cup \gamma_2) = \Gamma\setminus\left(\Gamma_\infty^{p(\cdot)} \cup \Gamma_\infty^{q(\cdot)} \cup \Gamma_\infty^{r(\cdot)}\right).$$

From (83) it follows that $p(t) = \infty$ and $q(t) = r(t)$ for $t \in \gamma_1$. Then by Lemma (3.3.14),

$$M\left(L^{p(\cdot)}(\gamma_1), L^{q(\cdot)}(\gamma_1)\right) \equiv M\left(L^{\infty}(\gamma_1), L^{r(\cdot)}(\gamma_1)\right) \equiv L^{r(\cdot)}(\gamma_1). \quad (84)$$

Similarly, from (83) we also obtain $\Gamma_\infty^{q(\cdot)} \subset \Gamma_\infty^{p(\cdot)} \cap \Gamma_\infty^{r(\cdot)}$, whence $\gamma_2 = \Gamma_\infty^{r(\cdot)}\setminus\Gamma_\infty^{p(\cdot)}$.

Therefore, $p(t) = q(t) < \infty$ and $r(t) = \infty$ for $t \in \gamma_2$. Then, from Lemma (3.3.14) we get

$$M\left(L^{p(\cdot)}(\gamma_2), L^{q(\cdot)}(\gamma_2)\right) \equiv M\left(L^{p(\cdot)}(\gamma_2), L^{p(\cdot)}(\gamma_2)\right) \equiv L^{\infty}(\gamma_2) \equiv L^{r(\cdot)}(\gamma_2). \quad (85)$$

The rest of the proof is developed by analogy with the proof of [129, Theorem 4]. Let $f \in M\left(L^{p(\cdot)}(\gamma_3), L^{q(\cdot)}(\gamma_3)\right)$. The multiplication operator $T\,g = fg$ maps $L^{p(\cdot)}(\gamma_3)$ into $L^{q(\cdot)}(\gamma_3)$ and has a closed graph. Hence there exists a constant $c \in (0, \infty)$ such that

$$\|fg\|_{L^{q(\cdot)}(\gamma_3)} \le c\|g\|_{L^{p(\cdot)}(\gamma_3)} \text{ for all } g \in L^{p(\cdot)}(\gamma_3). \quad (86)$$

For $\varepsilon > 0$, put

$$f_\varepsilon(t) = \begin{cases} \dfrac{c+\varepsilon}{f(t)} \left(\dfrac{|f(t)|}{c+\varepsilon}\right)^{r(t)/q(t)} & \text{if } \ f(t) \neq 0, \\ 0, & \text{if } f(t) = 0. \end{cases} \tag{87}$$

Let us show that

$$\varrho_{p(\cdot),\gamma_3}(f_\varepsilon) \leq 1. \tag{88}$$

Assume the contrary, that is, $\varrho_{p(\cdot),\gamma_3}(f_\varepsilon) > 1$. Then from [116, Propositions A.1 and A.8] it follows that there exists a measurable set $\gamma \subset \gamma_3$ such that

$$\varrho_{p(\cdot),\gamma_3}(\chi_\gamma f_\varepsilon) = 1. \tag{89}$$

From (83) and (87) we get

$$|f_\varepsilon(t)| = \left(\frac{|f(t)|}{c+\varepsilon}\right)^{r(t)/q(t)-1} = \left(\frac{|f(t)|}{c+\varepsilon}\right)^{r(t)/p(t)}, \quad t \in \gamma. \tag{90}$$

Equality (89) and Lemma (3.3.11) imply that $\|\chi_\gamma f_\varepsilon\|_{L^{p(\cdot)}(\gamma_3)} \leq 1$. Applying (86) with $g = \chi_\gamma f_\varepsilon$, we obtain

$$\left\|\frac{\chi_\gamma f_\varepsilon f}{c}\right\|_{L^{p(\cdot)}(\gamma_3)} \leq \|\chi_\gamma f_\varepsilon\|_{L^{p(\cdot)}(\gamma_3)} \leq 1.$$

Then, in view of Lemma (3.3.11), we get

$$\varrho_{q(\cdot),\gamma_3}\left(\frac{\chi_\gamma f_\varepsilon f}{c}\right) \leq 1. \tag{91}$$

Combining (89), (87), (83), and (91), we arrive at

$$\begin{aligned} 1 = \varrho_{p(\cdot),\gamma_3}(\chi_\gamma f_\varepsilon) = \varrho_{r(\cdot),\gamma_3}\left(\frac{\chi_\gamma f}{c+\varepsilon}\right) &= \varrho_{q(\cdot),\gamma_3}\left(\frac{\chi_\gamma f_\varepsilon f}{c+\varepsilon}\right) \\ &\leq \frac{c}{c+\varepsilon}\varrho_{q(\cdot),\gamma_3}\left(\frac{\chi_\gamma f_\varepsilon f}{c}\right) \leq \frac{c}{c+\varepsilon} < 1, \end{aligned}$$

and we get a contradiction. Hence (88) is fulfilled. Applying Lemma (3.3.11) to (88), we deduce that $\|f_\varepsilon\|_{L^{q(\cdot)}(\gamma_3)} \leq 1$. Then, in view of (86), we obtain

$$\|f_\varepsilon f\|_{L^{q(\cdot)}(\gamma_3)} \leq c\|f_\varepsilon\|_{L^{p(\cdot)}(\gamma_3)} \leq c.$$

Taking into account the above inequality, equality (87) and Lemma (3.3.11), we see that

$$\varrho_{r(\cdot),\gamma_3}\left(\frac{f}{c+\varepsilon}\right) = \varrho_{q(\cdot),\gamma_3}\left(\frac{f_\varepsilon f}{c+\varepsilon}\right) \leq \varrho_{q(\cdot),\gamma_3}\left(\frac{f_\varepsilon f}{c}\right) \leq 1,$$

Whence $\|f\|_{L^{r(\cdot)}(\gamma_3)} \leq c+\varepsilon$. Letting $\varepsilon \to 0$, we obtain $\|f\|_{L^{r(\cdot)}(\gamma_3)} \leq c$. It remains to observe that the smallest constant in inequality (86) coincides with $\|f\|_{M\left(L^{p(\cdot)}(\gamma_3),L^{q(\cdot)}(\gamma_3)\right)} \leq c$. Hence

$$M\left(L^{p(\cdot)}(\gamma_3), L^{q(\cdot)}(\gamma_3)\right) \hookrightarrow L^{r(\cdot)}(\gamma_3).$$

The embedding

$$L^{r(\cdot)}(\gamma_3) \hookrightarrow M\left(L^{p(\cdot)}(\gamma_3), L^{q(\cdot)}(\gamma_3)\right)$$

follows from the generalized Hölder inequality (Lemma (3.3.18)). Thus,

$$M\left(L^{p(\cdot)}(\gamma_3), L^{q(\cdot)}(\gamma_3)\right) = L^{r(\cdot)}(\gamma_3). \tag{92}$$

Finally, from (84), (85), (92) and Lemmas (3.2.16)–(3.2.17) we obtain

$$M\left(L^{p(\cdot)}, L^{q(\cdot)}\right) = M\left(L^{p(\cdot)}(\gamma_1) \oplus L^{p(\cdot)}(\gamma_2) \oplus L^{p(\cdot)}(\gamma_3), L^{q(\cdot)}(\gamma_1) \oplus L^{q(\cdot)}(\gamma_2) \oplus L^{q(\cdot)}(\gamma_3)\right)$$
$$= M\left(L^{p(\cdot)}(\gamma_1), L^{q(\cdot)}(\gamma_1)\right) \oplus M\left(L^{p(\cdot)}(\gamma_2), L^{q(\cdot)}(\gamma_2)\right) \oplus M\left(L^{p(\cdot)}(\gamma_3), L^{q(\cdot)}(\gamma_3)\right)$$
$$= L^{r(\cdot)}(\gamma_1) \oplus L^{r(\cdot)}(\gamma_2) \oplus L^{r(\cdot)}(\gamma_3) = L^{r(\cdot)},$$

Which completes the proof.

The above proof can be extended without any change to the case of variable Lebesgue spaces over arbitrary nonatomic measure spaces. The theorem itself is also true for arbitrary measure spaces. However the proof for not necessarily nonatomic measure spaces is more complicated. It can be developed by analogy with [128].

David's theorem [114] (see also [110, Theorem 4.17]), says that the Cauchy singular integral operator S is bounded on the standard Lebesgue space $L^p, 1 < p < \infty$, over a rectifiable Jordan curve Γ if and only if Γ is a Carleson curve. To formulate the generalization of this result to the setting of variable Lebesgue spaces, we will need the following class of nice variable exponents.

Let Γ be a rectifiable Jordan curve. We say that an exponent $p \in P(\Gamma)$ is locally log-Hölder continuous (cf. [113, Definition 2.2]) if $1 < p_- \le p_+ < \infty$ and there exists a constant $C_{p(\cdot),\Gamma} \in (0, \infty)$ such that

$$|p(t) - p(\tau)| \le \frac{C_{p(\cdot),\Gamma}}{-\log|t-\tau|} \text{ for all } t, \tau \in \Gamma \text{ satisfying } |t-\tau| < 1/2.$$

The class of all locally log-Hölder continuous exponent will be denoted by $LH(\Gamma)$.

Notice that some also denote this class by $\mathbb{P}^{\log}$ (Γ), see, e.g., [123, Section 1.1.4].

Theorem (3.3.20)[108]: ([123, Theorems 2.45 and 2.49]). Let Γ be a rectifiable Jordan curve and $p \in LH(\Gamma)$. Then the Cauchy singular integral operator S is bounded on $L^{p(\cdot)}$ if and only if Γ is a Carleson curve.

Now we are in a position to give a more precise formulation of Theorem (3.3.10) in the case of Toeplitz operators acting between Hardy type subspaces $PL^{p(\cdot)}$ and $PL^{q(\cdot)}$ of variable Lebesgue spaces $L^{p(\cdot)}$ and $L^{q(\cdot)}$, respectively.

Theorem (3.3.21)[108]: Let Γ be a Carleson Jordan curve. Suppose variable exponents $p, q \in LH(\Gamma)$ and $r \in P(\Gamma)$ are related by (83). If $a \in L^{r(\cdot)} \setminus \{0\}$, then the Toeplitz operator $T(a) \in \mathcal{L}\left(PL^{p(\cdot)}, PL^{q(\cdot)}\right)$ has a trivial kernel in $PL^{p(\cdot)}$ or a dense image in $PL^{q(\cdot)}$.

Proof. We know from Lemma (3.3.12) that the spaces $L^{p(\cdot)}$ and $L^{q(\cdot)}$ are reflexive because $1 < p_-, q_-$ and $p_+, q_+ < \infty$ (in view of $p, q \in LH(\Gamma)$). Since $r \in P(\Gamma)$, we have $1 \le r(t) \le \infty$ for almost all $t \in \Gamma$. Then we deduce from (83) that $q(t) \le p(t)$ for almost all $t \in \Gamma$. Therefore, by Lemma (3.3.13), $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)}$. It follows from Theorem (3.3.20) that the Cauchy singular integral operator S is bounded on $L^{p(\cdot)}$ and $L^{q(\cdot)}$.

Now we observe that $L^{r(\cdot)} = M\left(L^{p(\cdot)}, L^{q(\cdot)}\right)$ in view of Theorem (3.3.19). It remains to apply Theorem (3.3.10).

Corollary (3.3.1) follows immediately from Theorem (3.3.21) if we take all exponents p, q, and r constant.

Chapter 4
Toeplitzness and Toeplitz Projections

We extend some of asymptotic Toeplitzness of composition operator's results but we also show that new phenomena appear in higher dimensions. We deduce an essential version of the classical Hartman–Wintner spectral inclusion theorem, give a new proof of Johnson and Parrot's theorem on the essential commutant of abelian von Neumann algebras for separable Hilbert spaces and construct short exact sequences of Toeplitz algebras. T is a Toeplitz operator (that is, $T = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}$, where M_φ is the Laurent operator on $L^2(\mathbb{T}^n)$ for some $\varphi \in L^\infty(\mathbb{T}^n)$) if and only if $T_{z_i}^* T T_{z_i} = T$ for all $i = 1, \dots, n$. We show that T is an asymptotic Toeplitz operator if and only if $T =$ Toeplitz+ compact. The case $n = 1$ is the well known results of Brown and Halmos, and Feintuch, respectively. We also present related results in the setting of vector-valued Hardy spaces over the unit disc.

Section (4.1): Composition Operators:

For $\mathbb{B}_n$ denote the unit ball and $\mathbb{S}_n$ the unit sphere in $\mathbb{C}^n$. We denote by the surface area measure on $\mathbb{S}_n$, so normalized that $\sigma(\mathbb{S}_n) = 1$. We write L^∞ for $L^\infty(\mathbb{S}_n, d\sigma)$ and L^2 for $L^2(\mathbb{S}_n, d\sigma)$. The Hardy space H^2 consists of all analytic functions h on $\mathbb{B}_n$ which satisfy

$$\|h\|^2 = \sup_{0<r<1} \int_{\mathbb{S}_n} |h(\zeta)|^2 d\sigma(\zeta) < \infty.$$

It is well known that such a function h has radial boundary limits almost everywhere. We shall still denote the limiting function by h. We then have $h(\zeta) = \lim_{r\uparrow 1} h(r\zeta)$ for a.e. $\zeta \in \mathbb{S}$ and

$$\|h\|^2 = \int_{\mathbb{S}_n} |h(\zeta)|^2 d\sigma(\zeta) = \|h\|_{L^2}^2.$$

From this we may consider H^2 as a closed subspace of L^2. We shall denote by P the orthogonal projection from L^2 onto H^2. See [146, Section 5.6] for more details about H^2 and other Hardy spaces.

We shall also need the space H^∞, which consists of bounded analytic functions on $\mathbb{B}_n$. As before, we may regard H^∞ as a closed subspace of L^∞.

For any $f \in L^\infty$, the Toeplitz operator T_f is defined by $T_f h = P(fh)$ for h in H^2. It is immediate that T_f is bounded on H^2 with $\|T_f\| \le \|f\|_\infty$.

(The equality in fact holds true but it is highly nontrivial. See [141].) We call f the symbol of T_f. The following properties are well known and can be verified easily from the definition of Toeplitz operators.

(a) $T_f^* = T_{\bar f}$ for any $f \in L^\infty$.

(b) $T_f = M_f$, the multiplication operator with symbol f, for any $f \in H^\infty$.

(c) $T_g T_f = T_{gf}$ and $T_f^* T_g = T_{\bar f g}$ for $f \in H^\infty$ and $g \in L^\infty$.

The other class of operators that we are concerned with is the class of composition operators. Let φ be an analytic mapping from $\mathbb{B}_n$ into itself. We shall call φ an analytic selfmap of $\mathbb{B}_n$. We define the composition operator C_φ by $C_\varphi h = h \circ \varphi$ for all analytic functions h on $\mathbb{B}_n$. Note that C_φ is the identity if and only if φ is the identity mapping of $\mathbb{B}_n$. In the one dimensional case, it follows from Littlewood Subordination Principle that C_φ is a bounded operator on the Hardy space H^2. In higher dimensions, C_φ may not be bounded on H^2 even when φ is a polynomial mapping. See [139, 147] for details on composition operators.

We discuss the case of one dimension, that is, $n = 1$. It is a well known theorem of Brown and Halmos [138] back in the sixties that a bounded operator T on H^2 is a Toeplitz operator if and only if

$$T_{\bar{z}} T T_z = T. \tag{1}$$

Here T_z is the Toeplitz operator with symbol $f(z) = z$ on the unit circle $\mathbb{T}$.

This operator is also known as the unilateral forward shift. There is a rich literature on the study of Toeplitz operators and see, for example, [142].

In their study of the Toeplitz algebra, Barra and Halmos [137] introduced the notion of asymptotic Toeplitz operators. An operator A on H^2 is said to be strongly asymtotically Toeplitz (''SAT'') if the sequence $\{T_{\bar{z}}^m A T_z^m\}_{m=0}^{\infty}$ converges in the strong operator topology. It is easy to verify, thanks to (1), that the limit A_∞, if exists, is a Toeplitz operator. The symbol of A_∞ is called the asymptotic symbol of A. Barra and Halmos showed that any operator in the Toeplitz algebra is SAT.

In [143], Feintuch investigated asymptotic Toeplitzness in the uniform (norm) and weak topology as well. An operator A on H^2 is uniformly asymptotically Toeplitz (''UAT'') (respectively, weakly asymptotically Toeplitz (''WAT'')) if the sequence $\{T_{\bar{z}}^m A T_z^m\}$ converges in the norm (respectively, weak) topology.
It is clear that

$$UAT \Rightarrow SAT \Rightarrow WAT$$

and the limiting operators, if exist, are the same.

The following theorem of Feintuch completely characterizes operators that are UAT. A proof can be found in [143] or [145].

Theorem (4.1.1)[136]: (Theorem 4.1 in [143]). An operator on H^2 is uniformly asymptotically Toeplitz if and only if it has the form ''Toeplitz + compact''.

Recently Nazarov and Shapiro [145] investigated the asymptotic Toeplitzness of composition operators and their adjoints. They obtained many interesting results and open problems. We list here a few of their results, which are relevant to our work.

Theorem (4.1.2)[136]: (Theorem (4.1.1) in [145]). C_φ =''Toeplitz + compact'' (or equivalently by Feintuch's Theorem, C_φ is UAT) if and only if $C_\varphi = I$ or C_φ is compact.

It is easy [145, page 7] to see that if $\omega \in \partial\mathbb{D}\backslash\{1\}$ and $\varphi(z) = \omega z$ (such a φ is called a rotation), then C_φ is not WAT. On the other hand, Nazarov and Shapiro showed that for several classes of symbols φ, the operator C_φ is WAT and the limiting operator is always zero. The following conjecture appeared in [145].

If φ is neither a rotation nor the identity map, then C_φ is WAT with asymptotic symbol zero.

We already know that the conjecture holds when C_φ is a compact operator. Nazarov and Shapiro showed that the conjecture also holds when (a) $\varphi(0) = 0$; or (b) $|\varphi| = 1$ on an open subset V of $\mathbb{T}$ and $|\varphi| < 1$ a.e. on $\mathbb{T}\backslash V$.

For the strong asymptotic Toeplitzness of composition operators, Nazarov and Shapiro proved several positive results. On the other hand, they showed that if φ is a non-trivial automorphism of the unit disk, then C_φ is not SAT.

Later, Cuckovic and Nikpour [140] proved that C_φ^* is not SAT either. We combine these results into the following theorem.

Theorem (4.1.3)[136]: Suppose φ is a non-identity automorphism of $\mathbb{D}$. Then C_φ and C_φ^* are not SAT.

A more general notion of asymptotic Toeplitzness has been investigated by Matache in [144]. An operator S on H^2 is called a (generalized) unilateral forward shift if S is an isometry and the sequence $\{S^{*m}\}$ converges to zero in the strong operator topology. An operator A is called uniformly (strongly or weakly) $S-$asymptotically Toeplitz if the sequence $\{S^{*m}AS^m\}$ has a limit in the norm (strong or weak) topology. Among other things, the results in [144] on the $S-$asymptotic Toeplitzness of composition operators generalize certain results in [145].

Motived by Nazarov and Shapiro's work discussed in the previous, we would like to study the asymptotic Toeplitz-ness of composition operators on the Hardy space H^2 over the unit sphere in higher dimensions.

To define the notion of asymptotic Toeplitzness, we need a characterization of Toeplitz operators. Such a characterization, which generalizes (1), was found by Davie and Jewell [141] back in the seventies. They showed that a bounded operator T on H^2 is a Toeplitz operator if and only if $T = \sum_{j=1}^n T_{\bar{z}_j} T T_{z_j}$.

We define a linear operator Φ on the algebra $B(H^2)$ of all bounded linear operators on H^2 by

$$\Phi(A) = \sum_{j=1}^{n} T_{\bar{z}_j} A T_{z_j}, \tag{2}$$

for any A in $B(H^2)$. It is clear that Φ is a positive map (that is, $\Phi(A) \geq 0$ whenever $A \geq 0$) and Φ is continuous in the weak operator topology of $B(H^2)$. Let S be the column operator whose components are $T_{z_1}, \ldots, T_{z_n}$.

Then S maps H^2 into the direct sum $(H^2)^n$ of n copies of H^2. In dimension $n = 1$, the operator S is the familiar forward unilateral shift. The adjoint $S^* = [T_{\bar{z}_1}, \ldots, T_{\bar{z}_n}]$ is a row operator from $(H^2)^n$ into H^2. Since

$$S^*S = T_{\bar{z}_1}T_{z_1} + \cdots + T_{\bar{z}_n}T_{z_n} = T_{\bar{z}_1 z_1 + \cdots + \bar{z}_n z_n} = I,$$

we see that S is a co-isometry. In particular, we have $\|S\| = \|S^*\| = 1$.

From the definition of Φ, we may write

$$\Phi(A) = \begin{bmatrix} T_{\bar{z}_1} \ldots T_{\bar{z}_n} \end{bmatrix} \begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A \end{bmatrix} \begin{bmatrix} T_{z_1} \\ \vdots \\ \vdots \\ T_{z_n} \end{bmatrix} = S^* \begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A \end{bmatrix} S.$$

It follows that $\|\Phi(A)\| \leq \|S^*\|\|A\|\|S\| \leq \|A\|$ for any A in $B(H^2)$. Hence Φ is a contraction. For any positive integer m, put $\Phi^m = \Phi \circ \ldots \circ \Phi$, the composition of m copies of Φ. Then we also have $\|\Phi^m(A)\| \leq \|A\|$.

The aforementioned Davie{Jewell's result shows that a bounded operator T is a Toeplitz operator on H^2 if and only if T is a fixed point of Φ, which implies that $\Phi^m(T) = T$ for all positive integers m.

We now define the notion of asymptotic Toeplitzness. An operator A on H^2 is uniformly asymptotically Toeplitz (''UAT'') (respectively, strongly asymptotically Toeplitz (''SAT'') or weakly asymptotically Toeplitz (''WAT'')) if the sequence $\{\Phi^m(A)\}$ converges in the norm topology (respectively, strong operator topology or weak operator topology). As in the one dimensional case, it is clear that

$$UAT \Longrightarrow SAT \Longrightarrow WAT$$

and the limiting operators, if exist, are the same. Let A_∞ denote the limiting operator. It follows from the continuity of Φ in the weak operator toplogy that $\Phi(A_\infty) = A_\infty$. Therefore, A_∞ is a Toeplitz operator. Write $A_\infty = T_g$ for some bounded function on $\mathbb{S}_n$. We shall call g the asymptotic symbol of A.

In the definition of the map Φ (and hence the notion of Toeplitzness), we made use of the coordinate functions $z_1, \dots, z_n$. It turns out that a unitary change of variables gives rise to the same map. More specifically, if $\{u_1, \dots, u_n\}$ is any orthonormal basis of $\mathbb{C}^n$ and we define $f_j(z) = \langle z, u_j \rangle$ for $j = 1, \dots, n$ then a direct calculation shows that

$$\Phi(A) = \sum_{j=1}^{n} T_{\bar{f}_j} A T_{f_j},$$

for every bounded linear operator A on H^2.

We devoted to the study of the Toeplitzness of composition operators in several variables. Our focus is on strong and uniform asymptotic Toeplitzness. It turns out that while some results are analogous to the one dimensional case, other results are quite different.

Let $\varphi = (\varphi_1, \dots, \varphi_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ be two analytic selfmaps of $\mathbb{B}_n$.

We also use φ and η to denote their radial limits at the boundary. We will assume that both composition operators C_φ and C_η are bounded on the Hardy space H^2. (Recall that in dimensions greater than one, composition operators may not be bounded. See [139, Section 3.5].) Suppose g is a bounded measurable function on $\mathbb{S}_n$. Using the identities $C_\varphi T_{z_j} = T_{\varphi_j} C_\varphi$ and $T_{\bar{z}_j} C_\eta^* = C_\eta^* T_{\bar{\eta}_j}$ for $j = 1, \dots, n$, we obtain

$$\Phi^m\left(C_\eta^* T_g C_\varphi\right) = \sum_{j=1}^{n} T_{\bar{z}_j} C_\eta^* T_g C_\varphi T_{z_j} = \sum_{j=1}^{n} C_\eta^* T_{\bar{\eta}_j} T_g T_{\varphi_j} C_\varphi$$

$$= C_\eta^* \left(\sum_{j=1}^{n} T_{\bar{\eta}_j g \varphi_j} \right) C_\varphi = C_\eta^* T_{g\langle\varphi,\eta\rangle} C_\varphi.$$

Here $\langle\varphi,\eta\rangle$ is the inner product of $\varphi = \langle\varphi_1, \dots, \varphi_n\rangle$ and $\eta = \langle\eta_1, \dots, \eta_n\rangle$ as vectors in $\mathbb{C}^n$. By induction, we conclude that

$$\Phi^m\left(C_\eta^* T_g C_\varphi\right) = C_\eta^* T_{g\langle\varphi,\eta\rangle^m} C_\varphi \text{ for any } m \geq 1. \qquad (3)$$

As an immediate application of the formula (3), we show that certain products of Toeplitz and composition operators on H^2 are SAT.

Proposition (4.1.4)[136]: Suppose that $|\langle\varphi,\eta\rangle| < 1$ a.e. on $\mathbb{S}_n$. Then for any bounded function g on $\mathbb{S}_n$, the operator $C_\eta^* T_g C_\varphi$ is SAT with asymptotic symbol zero.

Proof. By assumption, $\langle\varphi,\eta\rangle^m \to 0$ a.e. on $\mathbb{S}_n$ as $m \to \infty$. This, together with Lebesgue Dominated Convergence Theorem, implies that $T_{g\langle\varphi,\eta\rangle^m} \to 0$, and hence, $C_\eta^* T_{g\langle\varphi,\eta\rangle^m} C_\varphi \to 0$ in the strong operator topology. Using (3), we conclude that $\Phi^m\left(C_\eta^* T_g C_\varphi\right) \to 0$ in the strong operator topology. The conclusion of the proposition follows.

As suggested by (3), the following set is relevant to the study of the asmytotic Toeplitzness of $C_\eta^* T_g C_\varphi$:

$$\begin{aligned} E(\varphi,\eta) &= \{\zeta \in \mathbb{S}_n : \langle\varphi(\zeta),\eta(\zeta)\rangle = 1\} \\ &= \{\zeta \in \mathbb{S}_n : \varphi(\zeta) = \eta(\zeta) \text{ and } |\varphi(\zeta)| = 1\}. \end{aligned}$$

To obtain the second equality we have used the fact that $|\varphi(\zeta)| = 1$ and $|\eta(\zeta)| \le 1$ for $\zeta \in \mathbb{S}_n$. Note that $E(\varphi,\varphi)$ is the set of all $\zeta \in \mathbb{S}_n$ for which $|\varphi(\zeta)| = 1$. On the other hand, by [146, Theorem 5.5.9], if $\varphi \neq \eta$, then $E(\varphi,\eta)$ has measure zero.

Proposition (4.1.5). For any analytic selfmaps φ,η of $\mathbb{B}_n$ and any bounded function g on $\mathbb{S}_n$, we have

$$\frac{1}{m}\sum_{j=1}^{m} \Phi^j\left(C_\eta^* T_g C_\varphi\right) \to C_\eta^* T_{g\chi_{E(\varphi,\eta)}} C_\varphi \text{ in the strong operator topology}$$

as $m \to 1$.

Proof. By (3), it suffices to show that $(1/m) = \sum_{j=1}^{m} g\langle\varphi,\eta\rangle^j$ converges to $g\chi_{E(\varphi,\eta)}$ a.e. on $\mathbb{S}_n$. But this follows from the identity

$$\frac{1}{m}\sum_{j=1}^{m} g(\zeta)\langle\varphi(\zeta),\eta(\zeta)\rangle^j = \begin{cases} g(\zeta) & \text{if } \zeta \notin E(\varphi,\eta) \\ \dfrac{1}{m} g(\zeta)\left(\dfrac{1-\langle\varphi(\zeta),\eta(\zeta)\rangle^{m+1}}{1-\langle\varphi(\zeta),\eta(\zeta)\rangle}\right) & \text{if } \zeta \notin E(\varphi,\eta) \end{cases}$$

for any $\zeta \in \mathbb{S}_n$.

Proposition (4.1.5) says that any operator of the form $C_\eta^* T_g C_\varphi$ is mean strongly asymptotically Toeplitz (''MSAT'') with limit $C_\eta^* T_{g\chi_{E(\varphi,\eta)}} C_\varphi$. We now specify η to be the identity map of $\mathbb{B}_n$ and g to be the constant function 1 and obtain

Corollary (4.1.6)[136]: Let φ be a non-identity analytic selfmap of $\mathbb{B}_n$ such that C_φ is bounded on H^2. Then C_φ is MSAT with asymptotic symbol zero.

This result in the one-dimensional case was obtained by Shapiro in [148]. In fact, Shapiro considered a more general notion of MSAT. It seems possible to generalize Proposition (4.1.5) in that direction and we leave this.

Theorem (4.1.3) asserts that for φ a non-identity automorphism of the unit disk $\mathbb{D}$, the operators C_φ and C_φ^* are not SAT. In dimensions greater than one, the situation is different.

Let $A(\mathbb{B}_n)$ denote the space of functions that are analytic on the open unit ball $\mathbb{B}_n$ and continuous on the closure $\overline{\mathbb{B}}_n$. We also let $\mathrm{Lip}(\alpha)$ (for $0 < \alpha \le 1$) be the space of α −Lipschitz continuous functions on $\mathbb{B}_n$, that is, the space of all functions $f: \mathbb{B}_n \to \mathbb{C}$ such that

$$\sup\left\{\frac{|f(a)-f(b)|}{|a-b|^{\alpha}}: a,b\in\mathbb{B}_n, a\neq b\right\}<\infty.$$

We shall need the following result, see [146, p.248].

Proposition (4.1.7)[136]: Suppose $n\geq 2$. If $\frac{1}{2}<\alpha\leq 1$ and $f\in A(\mathbb{B}_n)\cap \mathrm{Lip}(\alpha)$ is not a constant function, then

$$\sigma(\{\zeta\in\mathbb{S}_n: |f(\zeta)|=\|f\|_{\infty}\})=0.$$

Before giving a proof of the theorem, we present here an immediate application. For any $n\geq 1$, a linear fractional mapping of the unit ball $\mathbb{B}_n$ has the form

$$\varphi(z)=\frac{Az+B}{\langle z,C\rangle+D},$$

where A is a linear map, B, C are vectors in $\mathbb{C}^n$ and D is a non-zero complex number. It was shown by Cowen and MacCluer that C_{φ} is always bounded on H^2 for any linear fractional selfmap φ of $\mathbb{B}_n$. We recall that when $n=1$ these operators and their adjoints are not SAT in general by Theorem (4.1.3).

In higher dimensions it follows from Theorem (4.1.9) that the opposite is true.

Corollary (4.1.8)[136]: For $n\geq 2$, both C_{φ} and C_{φ}^* are SAT with asymptotic symbol zero except in the case $\varphi(z)=\lambda z$ for some $\lambda\in\mathbb{T}$.

Theorem (4.1.9)[136]: Suppose $n\geq 2$. Let $A:\mathbb{C}^n\to\mathbb{C}^n$ be a linear operator and b be a vector in $\mathbb{C}^n$. Let f be in $A(\mathbb{B}_n)\cap\mathrm{Lip}(\alpha)$ for some $\frac{1}{2}<\alpha\leq 1$.

Suppose $\varphi(z)=f(z)(Az+b)$ is a selfmap of $\mathbb{B}_n$ and φ is not of the form $\varphi(z)=\lambda z$ with $|\lambda|=1$. Then both C_{φ} and C_{φ}^* are SAT with asymptotic symbol zero.

Proof. We claim that under the hypothesis of the theorem, the set

$$\mathcal{E}=\{\zeta\in\mathbb{S}_n: |g(\zeta)|=1\}$$

is a σ −null subset of $\mathbb{S}_n$. We may then apply Proposition (4.1.4).

There are two cases to consider.

Case 1. $A=\delta I$ for some complex number δ and $b=0$. To simplify the notation, we write $\varphi(z)=g(z)z$, where $g(z)=\delta f(z)$. Then the set $\mathcal{E}$ can be written as $\mathcal{E}=\{\zeta\in\mathbb{S}_n: |\langle\varphi(\zeta),\zeta\rangle|=1\}$.

Since φ is a selfmap of $\mathbb{B}_n$, we have $\|g\|_{\infty}\leq 1$. Now if $\|g\|_{\infty}<1$, then $\mathcal{E}=\emptyset$ so $\sigma(\mathcal{E})=0$. If $\|g\|_{\infty}=1$, then g is a non-constant function since φ is not of the form $\varphi(z)=z$ for some $|\lambda|=1$. Proposition (4.1.7) then gives $\sigma(\mathcal{E})=0$ as well.

Case 2. A is not a multiple of the identity or $b\neq 0$. Since $|\varphi(\zeta)|\leq 1$ for $\zeta\in\mathbb{S}_n$, we see that ζ belongs to $\mathcal{E}$ if and only if there is a unimodular complex number $\gamma(\zeta)$ such that $\varphi(\zeta)=\gamma(\zeta)\zeta$. This implies that $f(\zeta)\neq 0$ and

$$\big(A-\gamma(\zeta)/f(\zeta)\big)\zeta+b=0. \tag{4}$$

Equation (4) shows that $\mathcal{E}$ is contained in the intersection of $\mathbb{S}_n$ with the set

$$M=\{z\in\mathbb{C}^n: (A-\lambda)z+b=0 \text{ for some } \lambda\in\mathbb{C}\}=\bigcup_{\lambda\in\mathbb{C}}(A-\lambda)^{-1}(\{-b\}).$$

Now decompose M as the union $M=M_1\cup M_2$, where

$$M_1=\bigcup_{\lambda\in\mathbb{C}\setminus sp(A)}(A-\lambda)^{-1}(\{-b\}) \text{ and } M_2\bigcup_{\lambda\in sp(A)}(A-\lambda)^{-1}(\{-b\}).$$

We have used $sp(A)$ to denote the spectrum of A, which is just the set of eigenvalues since A is an operator on $\mathbb{C}^n$. We shall show that both sets $M_1 \backslash \mathbb{S}_n$ and $M_2 \backslash \mathbb{S}_n$ are -null sets.

For $\lambda \in \mathbb{C} \backslash sp(A)$, the equation $(A-\lambda)z + b = 0$ has a unique solution whose components are rational functions in λ by Cramer's rule. So M_1 is a rational curve parametrized by $\lambda \in \mathbb{C} \backslash sp(A)$. Since the real dimension of $\mathbb{S}_n$ is $2n-1$, which is at least 3 when $n \geq 2$, we conclude that $\sigma(M_1 \backslash \mathbb{S}_n) = 0$.

For $\lambda \in sp(A)$, the set $(A-\lambda)^{-1}(\{b\})$ is either empty or an affine subspace of complex dimension at most $n-1$ (hence, real dimension at most $2n-2$). Since M_2 is a union of infinitely many such sets and the sphere $\mathbb{S}_n$ has real dimension $2n-1$, we conclude that $\sigma(M_2 \cap \mathbb{S}_n) = 0$.

Since $\mathcal{E} \subset (M_1 \cup M_2) \cap \mathbb{S}_n$ and $\sigma(M_1 \cap \mathbb{S}_n) = \sigma(M_2 \cap \mathbb{S}_n) = 0$, we have $\sigma(\mathcal{E}) = 0$, which completes the proof of the claim.

Nazarov and Shapiro [145] showed in the one-dimensional case that if φ is an inner function which is not of the form λz for some constant λ, and $\varphi(0) = 0$, then C_φ is not SAT but C_φ^* is SAT. While we do not know what the general situation is in higher dimensions, we have obtained a partial result.

Proposition (4.1.10)[136]: Suppose f is a non-constant inner function on $\mathbb{B}_n$ and $\varphi(z) = f(z)z$ for $z \in z$ such that C_φ is bounded on H^2. Then C_φ is not
SAT but C_φ is SAT.

Proof. By formula (3), we have $\Phi^m(C_\varphi) = T_{f^m} C_\varphi$ and $\Phi^m(C'_\varphi) = C_\varphi^* T_{f^m}^*$ for all positive integers m.

It then follows that $\|\Phi^m(C_\varphi)(1)\| = \|T_{f^m} C_\varphi 1\| = \|f^m\| = 1$. Hence $\Phi^m(C_\varphi)$ does not converge to zero in the strong operator topology. Since φ is a non-identity selfmap of $\mathbb{B}_n$, Corollary (4.1.6) implies that C_φ is not SAT.

On the other hand, we claim that as $m \to \infty, T_{f^m}^*$, and hence, $\Phi^m(C_\varphi^*)$, converges to zero in the strong operator topology. This shows that C_φ^* is SAT with asymptotic symbol zero. The proof of the claim is similar to that in case of dimension one ([145, Theorem 4.2]). We provide here the details. For any $a \in \mathbb{B}_n$, there is a function $K_a \in H^2$ such that $h(a) = \langle h, K_a \rangle$ for any $h \in H^2$. Such a function is called a reproducing kernel. It is well known that $T_{f^m}^* K_a = \overline{f^m(a)} K_a$ for any integer $m \geq 1$. Since $|f(a)| < \|f\|_\infty = 1$ by the Maximum Principle, it follows that $\|T_{f^m}^* K_a\| \to 0$ as $m \to \infty$. Because the linear span of $\{K_a : a \in \mathbb{B}_n\}$ is dense in H^2 and the operator norms of $\|T_{f^m}^*\|$ are uniformly bounded by one, we conclude that $T_{f^m}^* \to 0$ in the strong operator topology.

It follows from the characterization of Toeplitz operators and the notion of Toeplitzness that any Toeplitz operator is UAT. The following lemma shows that any compact operator is also UAT. Hence, anything of the form ''Toeplitz + compact'' is UAT. This result may have appeared in the literature but for completeness, we sketch here a proof.

Lemma (4.1.11)[136]: Let K be a compact operator on H^2. Then we have

$$\lim_{m \to \infty} \|\Phi^m(K)\| = 0.$$

As a consequence, for any bounded function f, the operator $T_f + K$ is uniformly asymptotically Toeplitz with asymptotic symbol f.

Proof. Since Φ^m is a contraction for each m and any compact operator can be approximated in norm by finite-rank operators, it suffices to consider the case when K is a rank-one operator. Write $K = u \otimes v$ for some non-zero vectors $u, v \in H^2$. Here $(u \otimes v)(h) = \langle h, v\rangle u$ for $h \in H^2$. Since polynomials form a dense set in H^2, we may assume further that both u, v are polynomials.

For any multi-index α, we have $T_{\bar{z}^\alpha}(u \otimes v)T_{z^\alpha} = (T_{\bar{z}^\alpha}u) \otimes (T_{\bar{z}^\alpha}v)$. Since v is a polynomial, there exists an integer m_0 such that $T_{\bar{z}^\alpha}v = 0$ for any α with $|\alpha| > m_0$. If m is a positive integer, the definition of Φ shows that $\Phi^m(K) = \Phi^m(u \otimes v)$ is a finite sum of operators of the form $T_{\bar{z}^\alpha}(u \otimes v)T_{z^\alpha}$ with $|\alpha| = m$. This implies that $\Phi^m(K) = 0$ for all $m > m_0$. Therefore, $\lim_{m\to\infty} \|\Phi^m(K)\| = 0$.

Now for f a bounded function on $\mathbb{S}_n$, we have

$$\Phi^m\left(T_f + K\right) = \Phi^m\left(T_f\right) + \Phi^m(K) = T_f + \Phi^m(K) \to T_f$$

in the norm topology as $m \to \infty$. This shows that $T_f + K$ is UAT with asymptotic symbol f.

In dimension one, Theorem (4.1.1) shows that the converse of Lemma (4.1.11) holds. On the other hand, Theorem (4.1.1) fails when $n \geq 2$. We shall show that there exist composition operators that are UAT but cannot be written in the form ''Toeplitz + compact''.

We first show that composition operators cannot be written in the form ''Toeplitz + compact" except in trivial cases. This generalizes Theorem (4.1.2) to all dimensions.

Theorem (4.1.12)[136]: Let φ be an analytic selfmap of $\mathbb{B}_n$ such that C_φ is bounded on H^2. If C_φ can be written in the form ''Toeplitz + compact'', then either C_φ is compact or it is the identity operator.

Proof. Our proof here works also for the one-dimensional case and it is different from Nazarov-Shapiro's approach (see the proof of Theorem 1.1 in [145]). Suppose C_φ is not the identity and $C_\varphi = T_f + K$ for some compact operator K and some bounded function f. By Lemma (4.1.11), C_φ is UAT with asymptotic symbol f on the unit sphere. This then implies that C_φ is also MSAT with asymptotic symbol f. From Corollary (4.1.6) we know that C_φ, being a non-identity bounded composition operator, is MSAT with asymptotic symbol zero. Therefore $f = 0$ a.e. and hence $C_\varphi = K$. This completes the proof of the theorem.

We now provide an example which shows that the converse of Lemma (4.1.11) (and hence Theorem (4.1.1)) does not hold in higher dimensions.

Example (4.1.13)[136]: For $z = (z_1, \dots, z_n)$ in $\mathbb{B}_n$, we define

$$\varphi(z) = \left(\varphi_1(z), \dots, \varphi_n(z)\right) = (0, z_1, 0, \dots, 0).$$

Then φ is a linear operator that maps $\mathbb{B}_n$ into itself. It follows from [139, Lemma 8.1] that C_φ is bounded on H^2 and $C_\varphi^* = C_\psi$, where ψ is a linear map given by $\psi(z) = \left(\psi_1(z), \dots, \psi_n(z)\right) = (z_2, 0, \dots, 0)$.

We claim that $\Phi\left(C_\varphi\right) = 0$ For $j \neq 2$, $C_\varphi T_{z_j} = T_{\varphi_j} C_\varphi = 0$ since $\varphi_j = 0$ for such j. Also, $\left(C_{\bar{z}_2} C_\varphi\right)^* = C_\varphi^* T_{z_2} = C_\psi T_{z_2} = T_{\psi_2} C_\psi = 0$. Hence $T_{\bar{z}_2} C_\varphi = 0$.

It follows that $\Phi(C_\varphi) = T_{\bar{z}_1} C_\varphi T_{z_1} + \cdots + T_{\bar{z}_n} C_\varphi T_{z_n} = 0$, which implies $\Phi^{\mathrm{m}}(C_\varphi) = 0$ for all $m \geq 1$. Thus, C_φ is UAT with asymptotic symbol zero.

On the other hand, since $(\varphi \circ \psi)(z) = (0, z_2, 0, \ldots, 0)$, we conclude that for any non-negative integer s,

$$C_\varphi^* C_\varphi(z_2^s) = C_\psi C_\varphi(z_2^s) = C_{\varphi\circ\psi}(z_2^s) = z_2^s.$$

This shows that the restriction of C_φ on the infinite dimensional subspace spanned by $\{1, z_2, z_2^2, z_2^3, \ldots\}$ is an isometric operator. As a consequence, C_φ is not compact on H^2. Theorem (4.1.12) now implies that C_φ is not of the form ''Toeplitz + compact'' either.

Theorem (4.1.2) shows that on the Hardy space of the unit disk, a composition operator C_φ is UAT if and only if it is either a compact operator or the identity. Example (4.1.13) shows that in dimensions $n \geq 2$, there exists a non-compact, non-identity composition operator which is UAT. It turns out that there are many more such composition operators. We study uniform asymptotic Toeplitzness of composition operators induced by linear selfmaps of $\mathbb{B}_n$.

We begin with a proposition which gives a lower bound for the norm of the product $T_f C_\varphi$ when φ satisfies certain conditions. This estimate will later help us show that certain composition operators are not UAT.

Proposition (4.1.14)[136]: Let φ be an analytic selfmap of $\mathbb{B}_n$ such that C_φ is bounded. Suppose there are points $\zeta, \eta \in \mathbb{S}_n$ so that $\langle \varphi(z), \eta \rangle = \langle z, \zeta \rangle$ for a.e. $z \in \mathbb{S}_n$. Let f be a bounded function on $\mathbb{S}_n$ which is continuous at ζ.

Then we have

$$\|T_f C_\varphi\| \geq |f(\zeta)|.$$

Proof. For an integer $s \geq 1$, put $g_s(z) = (1 + \langle z, \eta \rangle)^s$ is and $h_s = C_\varphi g_s$. Then for a.e. $z \in \mathbb{S}_n$,

$$h_s(h) = g_s(\varphi(z)) = (1 + \langle \varphi(z), \eta \rangle)^s = (1 + \langle z, \zeta \rangle)^s.$$

Because of the rotation-invariance of the surface measure on $\mathbb{S}_n$, we see that $\|h_s\| = \|g_s\|$. Now, we have

$$\|T_f C_\varphi\| \geq \frac{|\langle T_f C_\varphi g_s, h_s \rangle|}{\|g_s\| \|h_s\|} = \frac{|\langle T_f h_s, h_s \rangle|}{\|g_s\| \|h_s\|} = \frac{|\langle f h_s, h_s \rangle|}{\|h_s\|^2}$$
$$= \left| \frac{\int_{\mathbb{S}_n} f(z) |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)}{\int_{\mathbb{S}_n} |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)} \right|$$

We claim that the limit as $s \to \infty$ of the quantity inside the absolute value is $f(\zeta)$. From this the conclusion of the proposition follows.

To prove the claim we consider

$$\left| \frac{\int_{\mathbb{S}_n} f(z) |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)}{\int_{\mathbb{S}_n} |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)} - f(\zeta) \right| \leq \frac{\int_{\mathbb{S}_n} |f(z) - f(\zeta)| \cdot |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)}{\int_{\mathbb{S}_n} |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)}$$
$$= \frac{\left(\int_{\mathcal{U}} + \int_{\mathbb{S}_n \setminus \mathcal{U}} \right) |f(z) - f(\zeta)| \cdot |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)}{\int_{\mathbb{S}_n} |1 + \langle z, \zeta \rangle|^{2s} d\sigma(z)}$$

$$\leq \sup_{z\in\mathcal{U}}|f(z)-f(\zeta)|+2\|f\|_\infty \frac{\int_{\mathbb{S}_n\setminus\mathcal{U}}|1+\langle z,\zeta\rangle|^{2s}d\sigma(z)}{\int_{\mathbb{S}_n}|1+\langle z,\zeta\rangle|^{2s}d\sigma(z)}. \quad (5)$$

Where $\mathcal{U}$ is any open neighborhood of ζ in $\mathbb{S}_n$. By the continuity of f at ζ, the first term in (5) can be made arbitrarily small by choosing an appropriate $\mathcal{U}$. For such a $\mathcal{U}$, we may choose another open neighborhood W of ζ with $W \subseteq \mathcal{U}$ such that

$$\sup\{|1+\langle z,\zeta\rangle|: z\in \mathbb{S}_n\setminus\mathcal{U}\} < \inf\{|1+\langle z,\zeta\rangle|: z\in W\}$$

This shows that the second term in (5) converges to 0 as $s\to\infty$. The claim then follows.

Using Proposition (4.1.14), we give a sufficient condition under which C_φ fails to be UAT.

Proposition (4.1.15)[136]: Let φ be a non-identity analytic selfmap of $\mathbb{B}_n$ such that C_φ is bounded. Suppose that φ is continuous on $\overline{\mathbb{B}}_n$ and there is a point $\zeta\in\mathbb{S}_n$ and a unimodular complex number λ so that $\langle\varphi(z),\zeta\rangle = \lambda\langle z,\zeta\rangle$ for all $z\in\mathbb{S}_n$. Then C_φ is not UAT.

Proof. Since φ is a non-identity map, Corollary (4.1.6) shows that C_φ is MSAT with asymptotic symbol zero. To prove that C_φ is not UAT, it suffices to show that C_φ is not UAT with asymptotic symbol zero.

Let $f(z)=\langle\varphi(z),z\rangle$ for $z\in\mathbb{S}_n$. By the hypothesis, the function f is continuous on $\mathbb{S}_n$ and $f(\zeta)=\langle\varphi(\zeta),\zeta\rangle=\lambda\langle\zeta,\zeta\rangle=\lambda$. For any positive integer m, formula (3) gives $\Phi^m(C_\varphi)=T_{f^m}C_\varphi$. Since φ satisfies the hypothesis of Proposition (4.1.14) with $\eta=\lambda\zeta$ and f^m is continuous at ζ, we may apply Proposition (4.1.14) to conclude that

$$\|\Phi^m(C_\varphi)\| = \|T_{f^m}C_\varphi\| \geq |f^m(\zeta)| = 1.$$

This implies that C_φ is not UAT with asymptotic symbol zero, which is what we wished to prove.

Our last result provides necessary and sufficient conditions for a class of composition operators to be UAT.

Theorem (4.1.16)[136]: Let $\varphi(z)=Az$ where $A:\mathbb{C}^n\to\mathbb{C}^n$ is a non-identity linear map with $\|A\|\leq 1$. Then C_φ is UAT if and only if all eigenvalues of A lie inside the open unit disk.

Proof. Since $\|A\|\leq 1$, all eigenvalues of A lie inside the closed unit disk.

We first show that if A has an eigenvalue λ with $|\lambda|=1$, then C_φ is not UAT. Let $\zeta\in\mathbb{S}_n$ be an eigenvector of A corresponding to λ. We claim that $A^*=\bar{\lambda}\zeta$. In fact, we have

$$\begin{aligned}|(A^*-\bar{\lambda})\zeta|^2 &= |A^*\zeta|^2 - 2\Re\langle A^*\zeta,\bar{\lambda}\zeta\rangle + |\bar{\lambda}\zeta|^2\\ &= |A^*\zeta|^2 - 2\Re\langle \zeta,\bar{\lambda}A\zeta\rangle + |\zeta|^2\\ &= |A^*\zeta|^2 - 2\Re\langle \zeta,\bar{\lambda}\lambda\zeta\rangle + |\zeta|^2\\ &= |A^*\zeta|^2 - 1 \leq 0.\end{aligned}$$

This forces $A^*\zeta,\bar{\lambda}\zeta$ as claimed. As a result, for $z\in\mathbb{S}_n$, we have

$$\langle\varphi(z),\zeta\rangle = \langle Az,\zeta\rangle = \langle z,A^*\zeta\rangle = \langle z,\bar{\lambda}\zeta\rangle = \lambda\langle z,\zeta\rangle.$$

We then apply Proposition (4.1.15) to conclude that C_φ is not UAT.

We now show that if all eigenvalues of A lie inside the open unit disk then C_φ is UAT. Put $f(z) = \langle \varphi(z), z\rangle = \langle Az, z\rangle$ for $z \in \mathbb{S}_n$. Since $|Az| \le 1$ and Az is not a unimodular multiple of z, we see that $|f(z)| < 1$ for $z \in \mathbb{S}_n$.

Since f is continuous and $\mathbb{S}_n$ is compact, we have $\|f\|_{L^\infty(\mathbb{S}_n)} < 1$. For any integer $m \ge 1$, formula (3) gives

$$\|\Phi^m(C_\varphi)\| = \|T_{f^m} C_\varphi\| \le \|T_{f^m}\| \|C_\varphi\| \le \left(\|f\|_{L^\infty(\mathbb{S}_n)}\right)^m \|C_\varphi\|.$$

Since $\|f\|_{L^\infty(\mathbb{S}_n)} < 1$, we conclude that $\lim\limits_{m\to\infty} \|\Phi^m(C_\varphi)\| = 0$. Therefore, C_φ is UAT with asymptotic symbol zero.

Section (4.2): The Essential Commutants:

A result of K. Davidson [156] from 1977, answering a question of R. Douglas, shows that the essential commutant $\mathcal{T}_a^{ec}$ of the set $\mathcal{T}_a = \{T_f; f \in H^\infty(\mathbb{T})\} \subset B(H^2(\mathbb{T}))$ of all analytic Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle is given by

$$\mathcal{T}_a^{ec} = \{T_f + K; f \in H^\infty(\mathbb{T}) + C(\mathbb{T}) \text{ and } K \in \mathcal{K}(H^2(\mathbb{T}))\},$$

Where $\mathcal{K}(H)$ denotes the set of all compact operators on a given Hilbert space H. It was observed by X. Ding and S. Sun [161] that the result of Davidson remains true on the Hardy space $H^2(\mathbb{S})$ of the unit sphere $\mathbb{S} = \partial\mathbb{B}_n$ in dimension $n > 1$ when the symbol algebra $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is replaced by the closed subalgebra $S = \{f \in L^\infty(\mathbb{S}); H_f \text{ is compact}\} \subset L^\infty(\mathbb{S})$, that is,

$$\mathcal{T}_a^{ec} = \{T_f + K; f \in S \text{ and } K \in \mathcal{K}(H^2(\mathbb{S}))\}.$$

It is well known that $H^\infty(\mathbb{S}) + C(\mathbb{S}) \subsetneq S$ is a proper subalgebra in every dimension $n > 1$ (see [158]) and that therefore the higher dimensional version of Davidson's result fails if the algebra Sis replaced by the smaller algebra $H^\infty(\mathbb{S}) + C(\mathbb{S})$.

In [160] the above results were extended to Toeplitz operators formed with respect to a quite general class of subnormal tuples on arbitrary Hilbert spaces containing, as a very particular case, Toeplitz operators on strictly pseudoconvex domains in $\mathbb{C}^n$.

Let $A \subset C(K)$ be a closed subalgebra of the Banach algebra of all $\mathbb{C}$ −valued continuous functions on a compact subset $K \subset \mathbb{C}^n$ such that A contains at least the polynomials. A subnormal tuple $T \in B(H)^n$ is called an A −isometry [163] if the spectrum of the minimal normal extension $U \in B(\widehat{H})^n$ of T is contained in the Shilov boundary ∂_A of A and if A is contained in the restriction algebra $\mathcal{R}_T$ of T. In this setting concrete T −Toeplitz operators are defined as compressions $T_f = P_H \Psi_U(f)|_H$, where $\Psi_U: L^\infty(\mu) \to B(\widehat{H})$ is the L^∞ −functional calculus of U and $f \in L^\infty(\mu)$, while abstract T −Toeplitz operators are defined as those operators $X \in B(H)$ which satisfy the Brown–Halmos condition

$$T_\theta^* X T_\theta = X$$

for all μ −inner functions θ.

By results of A. Athavale [152] and T. Itô [164] the $A(\mathbb{B}_n)$ −isometries on a given Hilbert space. Hare the spherical isometries on H, that is, the commuting tuples $T \in B(H)^n$ satisfying the identity $\sum_{1\le i\le n} T_i^* T_i = 1_H$ and the class of $A(\mathbb{D}^n)$-isometries on H is given by the commuting tuples of isometries on H. For any strictly pseudoconvex or symmetric domain $D \subset \mathbb{C}^n$, the tuple $T_z = (T_{z_1}, \dots, T_{z_n}) \in B(H^2(\sigma))^n$ on the Hardy space $H^2(\sigma)$ formed with respect to the canonical probability measure σ on the Shilov boundary of the domain algebra $A(D) = \{f \in C(\overline{D}); f|_D \in O(D)\}$ is an example of an $A(D)$ −isometry. Finally, every commuting tuple $N \in B(H)^n$ of normal operators on a Hilbert space H is a $C(\sigma(N))$ −isometry.

Under a suitable regularity condition on T, which is satisfied in all the above examples and which is needed to apply results of Aleksandrov [150] on the existence of sufficiently many μ −inner functions, it follows that the set $\mathcal{T}(T)$ of abstract T −Toeplitz operators is given by the compressions

$$\mathcal{T}(T) = P_H(U)'|_H$$

of the operators in the commutant $(U)' = W^*(U)'$ of the von Neumann algebra generated by U, while by the very definition, the concrete T −Toeplitz operators are given by the compressions of all operators in $W^*(U)$.

It follows from results of B. Prunaru [168] on families of spherical isometries that there is a completely positive unital projection $\Phi_T: B(H) \to B(H)$ onto the set $\mathcal{T}(T)$ of all abstract T −Toeplitz operators [160]. In this note we give a much more direct and straightforward construction of Toeplitz projections Φ_T. We use the properties of these projections to improve the main result of [160] on the essential commutant of analytic Toeplitz operators and to extend a number of classical results on Toeplitz operators to our general setting.

After constructing Toeplitz projections, we show that every operator S in the essential commutant of the analytic Toeplitz operators associated with an essentially normal regular A −isometry $T \in B(H)^n$ is a compact perturbation of the Toeplitz operator $\Phi_T(S)$. Thus we improve a corresponding result obtained in [160] under the additional condition that T possesses no joint eigenvalues. We obtain complete characterizations of the essential commutant of essentially normal regular A −isometries and give, as a direct application, a new proof of a classical theorem of Johnson and Parrot [165] on the essential commutant of abelian von Neumann algebras in the case of separable Hilbert spaces. We show that the Toeplitz projection associated with an arbitrary regular A −isometry annihilates the compact operators if and only if T possesses no joint eigenvalues. We conclude that the Toeplitz calculus associated with a regular A −isometry T with empty point spectrum satisfies the essential version of the Hartman–Wintner spectral inclusion theorem and that the semi-commutator ideal of Toeplitz algebras $\mathcal{T}_B$ generated by arbitrary symbol algebras Bnecessarily contains every compact operator in $\mathcal{T}_B$.

Let $T \in B(H)^n$ be a subnormal tuple on a complex Hilbert space H, that is, a commuting tuple that can be extended to a commuting tuple of normal operators on a larger Hilbert space. We denote by $U \in B(\widehat{H})^n$ the minimal normal extension of T which is unique up to unitary equivalence [155], and fix a scalar spectral measure μ for U. The measure μ is a positive regular Borel measure on the normal spectrum $\sigma_n(T) = \sigma(U)$ of T. By the spectral theorem for normal tuples there is an isomorphism of von Neumann algebras $\Psi_U: L^\infty(\mu) \to W^*(U) \subset B(\widehat{H})$ extending the polynomial calculus of U. The restriction algebra

$$\mathcal{R}_T = \{f \in L^\infty(\mu); \Psi_U(f)H \subset H\} \subset L^\infty(\mu)$$
is a weak* closed subalgebra. For $f \in L^\infty(\mu)$, we define the T −Toeplitz operator with symbol fas the compression
$$T_f = P_H \Psi_U(f)|_H.$$
Toeplitz operators of this form will be called concrete T-Toeplitz operators in the sequel.

Let $A \subset C(K)$ be a unital closed subalgebra of the Banach algebra of all $\mathbb{C}$ −valued continuous functions on a compact subset $K \subset \mathbb{C}^n$ such that A contains at least the co-ordinate functions. Then a subnormal tuple $T \in B(H)^n$ as above is called an A −isometry if $\sigma_n(T)$ is contained in the Shilov boundary ∂_A of A and $A|_{\partial_A} \subset \mathcal{R}_T$. Here the Shilov boundary $\partial_A \subset K$ is the smallest closed set such that $\|f\|_\infty, K = \|f\|_{\infty,\partial_A}$ for every $f \in A$ and we regard the scalar spectral measure μ of U as a positive measure on ∂_A via trivial extension. Since $\mathcal{R}_T \subset L^\infty(\mu)$ is weak* closed and contains A, it also contains the dual algebra
$$H_A^\infty(\mu) = \bar{A}^{w^*} \subset L^\infty(\mu).$$
The unimodular elements in $H_A^\infty(\mu)$, that is, the elements of the set
$$I_\mu = \{\theta \in H_A^\infty(\mu); |\theta| = 1 \ \mu - \text{almost everywhere on } \partial_A\}$$
will be called μ −inner functions. In [150] Aleksandrov gives a sufficient condition for $H_A^\infty(\mu)$ to contain a rich supply of μ −inner functions. The triple (A, K, μ) is called regular in the sense of Aleksandrov if, for every function $\phi \in C(K)$ with $\phi > 0$ on K, there is a sequence (ϕ_k) of functions in A with $|\phi_k| < \phi$ on K and $\lim_{k\to\infty} |\phi_k| = \phi$ μ −almost everywhere on K. It follows from the results of Aleksandrov that the regularity of the triple (A, K, μ) implies that the set $I_\mu \subset H_A^\infty(\mu)$ of μ −inner functions generates $L^\infty(\mu)$ as a von Neumann algebra, that is, $L^\infty(\mu) = W^*(I_\mu)$ (Corollary 2.5 in [159]). We call $T \in B(H)^n$ a regular A −isometry if T is an A −isometry and the triple (A, K, μ) is regular in the sense of Aleksandrov. It was observed by Aleksandrov [150] that, for every regular positive measure μ on the Shilov boundary of the domain algebra $A(D)$ of a strictly pseudoconvex or symmetric domain $D \subset \mathbb{C}^n$, the triple $(A(D), \overline{D}, \mu)$ is regular.

Let $T \in B(H)^n$ be a regular A −isometry with minimal normal extension $U \in B(\widehat{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. Since $L^1(\mu)$ is separable, its dual unit ball $B_{L^\infty(\mu)} = \{f \in L^\infty(\mu); \|f\|_{L^\infty(\mu)} \leq 1\}$ equipped with the relative weak* topology of $L^\infty(\mu) = L^1(\mu)'$ is a compact metrizable space. Hence $B_{L^\infty(\mu)}$ and its subset I_μ consisting of all μ −inner functions are separable metrizable spaces in the relative weak* topology. For any countable weak* dense subset $I \subset I_\mu$, the von Neumann algebra generated by I in $L^\infty(\mu)$ satisfies
$$W^*(I) = W^*(I_\mu) = L^\infty(\mu).$$
Let us fix any sequence $(\theta_k)_{k\geq 1}$ in I_μ with the property that
$$W^*(\{\theta_k; k \geq 1\}) = L^\infty(\mu).$$
For $r \geq 0$, the norm-closed ball $B_r = \{X \in B(\widehat{H}); X \leq r\}$ equipped with the relative topology of the weak* topology of $B(\widehat{H})$ is a compact Hausdorff space. For $X \in B(\widehat{H})$, the averages

$$\Phi_{U,k}(X) = \frac{1}{k^k} \sum_{1 \le i_1,\dots,i_k \le k} \Psi_U\left(\theta_k^{i_k} \cdot \dots \cdot \theta_1^{i_1}\right)^* X \Psi_U\left(\theta_1^{i_1} \cdot \dots \cdot \theta_k^{i_k}\right) \in B(\hat{H})$$

form a sequence $\left(\Phi_{U,k}(X)\right)_k$ in $B_{\|X\|}$. Since by Tychonoff's theorem the topological product $\prod_{X \in B(\hat{H})} B_{\|X\|}$ is compact and since convergence in the product topology is equivalent to component wise convergence, there is a subnet $\left(\Phi_{U,k_\alpha}\right)_\alpha$ of the sequence $\left(\Phi_{U,k}\right)_k$ such that the weak* limits

$$\Phi_U(X) = w^* - \lim_\alpha \Phi_{U,k_\alpha}(X) \in B(\hat{H})$$

exist simultaneously for every $X \in B(\hat{H})$. Each choice of such a subnet yields a well-defined map $\Phi_U: B(\hat{H}) \to B(\hat{H})$ with the properties that will be deduced in the sequel.

Theorem (4.2.1)[149]: The mapping

$$\Phi_U: B(\hat{H}) \to B(\hat{H}), X \longmapsto \Phi_U(X)$$

constructed above is a completely positive unital projection with

$$\operatorname{ran}(\Phi_U) = (U)'.$$

Proof. Obviously, the mappings

$$\Phi_{U,k}: B(\hat{H}) \to B(\hat{H}), X \longmapsto \Phi_{U,k}(X)$$

are completely positive and unital. Since, for each $N \in \mathbb{N}$, weak* convergence for a net in $B(\hat{H}^N)$ identified with the space $M\left(N, B(\hat{H})\right)$ of all $N \times N$ matrices over $B(\hat{H})$ is equivalent to coefficient wise weak* convergence in $B(\hat{H})$ and since the set of all positive operators on a Hilbert space is weak∗ closed, it follows that

$$\Phi_U(X) = B(\hat{H}) \to B(\hat{H}), \quad X \longmapsto w^* - \lim_\alpha \Phi_{U,k_\alpha}(X)$$

is completely positive and unital. By construction the mappings $\Phi_{U,k}$, and hence also Φ_U, act as the identity operator on the commutant $(U)' = W^*(U)'$. To complete the proof, it suffices to show that $\operatorname{ran}(\Phi_U) \subset (U)'$.

For $1 \le j \le k$ and $i = (i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k) \in \{1, \dots, k\}^{k-1}$, we use the abbreviation

$$R_{ij} = \Psi_U\left(\prod_{\substack{\nu=1 \\ \nu \ne j}}^{k} \theta_\nu^{i_\nu}\right).$$

Note that, for $X \in B(\hat{H}), k \ge 1$ and $1 \le j \le k$, the estimates

$$\begin{aligned} &\left\|\Psi_U(\overline{\theta_j})\Phi_{U,k}(X)\Psi_U(\theta_j) - \Phi_{U,k}(X)\right\| \\ &\quad \le \frac{1}{k^k}\left\|\sum_i R_{ij}^*\left(\sum_{\mu=1}^{k} \Psi_U\left(\overline{\theta_j}^{\mu+1}\right) X \Psi_U\left(\theta_j^{\mu+1}\right) - \sum_{\mu=1}^{k} \Psi_U\left(\overline{\theta_j}^{\mu}\right) X \Psi_U\left(\theta_j^{\mu}\right)\right) R_{ij}\right\| \\ &\quad \le \frac{k^{k-1}}{k^k} 2\|X\| = \frac{2\|X\|}{k} \end{aligned}$$

hold. Hence for $j \ge 1$ and $X \in B(\hat{H})$, we obtain

$$\Psi_U(\overline{\theta_j})\Phi_U(X)\Psi_U(\theta_j) = w * - \lim_\alpha \Psi_U(\overline{\theta_j})\Phi_{U,k_\alpha}(X)\Psi_U(\theta_j) = \Phi_U(X),$$

or equivalently, $\Phi_U(X)\Psi_U(\theta_j) = \Psi_U(\theta_j)\Phi_U(X)$. It follows that

$$\Phi_U(X) \in W^*(\{\Psi_U(\theta_j); j \geq 1\})' = W^*(U)' = (U)'$$

for all $X \in B(\widehat{H})$. This observation completes the proof.

A projection onto the space of all Toeplitz operators on the Hardy space of the unit circle was constructed by Arveson in [151] using a generalized limit argument. In [168] Prunaru used invariant means to construct a completely positive unital projection onto the set of Toeplitz operators associated with a commuting family of spherical isometries. In our setting, a projection onto the set of all abstract T −Toeplitz operators is obtained by compressing Φ_U to H.

For $X \in B(H)$, we denote by $\tilde{X} = X \oplus 0 \in B(\widehat{H})$ its trivial extension to $\widehat{H}$. Then for $k \geq 1$ and $X \in B(H)$, the operators

$$\Phi_{T,k}(X) = \frac{1}{k^k} \sum_{1 \leq i_1,\dots,i_k \leq k} T^*_{\theta_k^{i_k}\cdot\dots\cdot\theta_1^{i_1}} X T_{\theta_1^{i_1}\cdot\dots\cdot\theta_k^{i_k}} \in B(H)$$

are the compressions of the corresponding operators $\Phi_{U,k}(X)$, that is,

$$\Phi_{T,k}(X) = P_H \Phi_{U,k}(\tilde{X})|_H (k \geq 1, X \in B(H)).$$

As before we denote by I_μ the set of all μ-inner functions θ in $H_A^\infty(\mu)$ and write

$$\mathcal{T}(T) = \{X \in B(H); T_\theta^* X T_\theta = X \text{ for all } \theta \in I_\mu\}$$

for the set of all abstract T −Toeplitz operators on H.

Corollary (4.2.2)[149]: The mapping

$$\Phi_T: B(H) \to B(H), X \longmapsto \Phi_T(X) = w^* - \lim_\alpha \Phi_{T,k_\alpha}(X) \ = P_H \Phi_U(\tilde{X})|_H$$

is a well-defined completely positive unital projection with

$$\operatorname{ran}(\Phi_T) = \mathcal{T}(T).$$

Proof. Since the compression mapping $B(\widehat{H}) \to B(H), X \longmapsto P_H X|_H$, is weak* continuous, completely positive and unital, it follows that

$$w^* - \lim_\alpha \Phi_{T,k_\alpha}(X) \ = P_H \Phi_U(\tilde{X})|_H$$

for $X \in B(H)$ and that the map $\Phi_T: B(H) \to B(H), X \longmapsto P_H \Phi_U(\tilde{X})|_H$, is completely positive and unital. Since $\Phi_{T,k_\alpha}(X) = X$ for each abstract T −Toeplitz operator $X \in \mathcal{T}(T)$ and every $k \geq 1$, it follows that $\Phi_T(X) = X$ for $X \in \mathcal{T}(T)$. Using Theorem (4.2.1), we obtain that

$$\begin{aligned} T_\theta^* \Phi_T(X) T_\theta &= P_H \Psi_U(\theta)^* P_H \Phi_U(\tilde{X}) \Psi_U(\theta)|_H \\ &= P_H \Psi_U(\theta)^* \Phi_U(\tilde{X}) \Psi_U(\theta)|_H = P_H \Phi_U(\tilde{X})|_H = \Phi_T(X) \end{aligned}$$

for every operator $X \in B(H)$ and each μ −inner function $\theta \in I_\mu$. Hence $\operatorname{ran}(\Phi_T) = \mathcal{T}(T)$, and the proof is complete.

As a direct application of Theorem (4.2.1) and Corollary (4.2.2) we obtain a natural description of the abstract T −Toeplitz operators.

Corollary (4.2.3)[149]: Let $T \in B(H)^n$ be a regular A −isometry with minimal normal extension $U \in B(\widehat{H})^n$. Then we have

$$\mathcal{T}(T) = P_H(U)'|_H.$$

Proof. By Corollary (4.2.2) and Theorem (4.2.1) we have $\mathcal{T}(T) \subset P_H(U)'|_H$. Conversely, if $X \in (U)'$ and $\theta \in I_\mu$ is a μ −inner function, then

$$T_\theta^*(P_H X|_H)T_\theta = P_H \Psi_U(\theta)^* P_H X \Psi_U(\theta)|_H = P_H \Psi_U(\theta)^* X \Psi_U(\theta)|_H = P_H X|_H.$$

Hence also the reverse inclusion $P_H(U)'|_H \subset \mathcal{T}(T)$ holds.

Let Φ_U and Φ_T be defined as above. Then

$$\hat{\pi}: B(H) \to B(\hat{H}), X \longmapsto \Phi_U(\tilde{X})$$

defines a completely positive linear mapping with $\Phi_T(X) = P_H \hat{\pi}(X)|_H$ for all $X \in B(H)$ and $\operatorname{ran}(\hat{\pi}) \subset (U)'$. To see that equality holds here, we need some more preparations. Note that

$$I_U = \Psi_U(I_\mu) \subset W^*(U)$$

defines an abelian semigroup of unitary operators with $W^*(I_U) = W^*(U)$. The minimality of U as a normal extension of T implies that

$$\hat{H} = \bigvee(V^* H;\ V \in I_U).$$

To see this it suffices to observe that the space on the right-hand side is invariant under $W^*(I_U) = W^*(U)$.

Corollary (4.2.4)[149]: The compression mapping

$$\varrho: (U)' \to \mathcal{T}(T), X \longmapsto P_H X|_H$$

defines a completely isometric linear isomorphism with inverse given by

$$\mathcal{T}(T) \to (U)', X \longmapsto \hat{\pi}(X).$$

Proof. We know from Corollary (4.2.3) that ϱ is well-defined and surjective. As a compression mapping is completely contractive. Since

$$\langle X V^* h, W^* k\rangle = \langle \varrho(X) W h, V k\rangle$$

for all $X \in (U)', V, W \in IU$ and $h, k \in H$, the remarks preceding the corollary imply that ϱ is injective. The observation that

$$\varrho(\hat{\pi}(X)) = \Phi_T(X) = X$$

for all $X \in \mathcal{T}(T)$ shows that $\mathcal{T}(T) \to (U)', X \longmapsto \hat{\pi}(X)$, defines the inverse of the bijection $\varrho: (U)' \to \mathcal{T}(T)$. Since also $\hat{\pi}$ is completely contractive as a composition of completely contractive mappings, it follows that ϱ is completely isometric.

The restriction of $\hat{\pi}: B(H) \to B(\hat{H})$ to the C^*-algebra $C^*(\mathcal{T}(T))$ generated by all abstract T $-$Toeplitz operators is even a C^*-algebra homomorphism.

Theorem (4.2.5)[149]: The restriction

$$\pi = \hat{\pi}|_{C^*(\mathcal{T}(T))}: C^*(\mathcal{T}\ (T)) \to B(\hat{H})$$

is the minimal Stinespring dilation of the completely positive unital projection

$$C^*(\mathcal{T}(T)) \to C^*(\mathcal{T}(T)), \quad X \longmapsto \Phi_T(X).$$

For $X \in B(H)$ and $Y \in C^*(\mathcal{T}(T))$, we have

$$\hat{\pi}(XY) = \hat{\pi}(X)\hat{\pi}(Y).$$

Proof. We know that $\hat{\pi}: B(H) \to B(\hat{H})$, and hence also its restriction π, are completely positive maps. To prove that π is a homomorphism of C∗-algebras, it suffices to check its multiplicativity. Fix operators $X \in B(H)$ and $Y \in \mathcal{T}(T)$. Since $\operatorname{ran}(\hat{\pi}) \subset W^*(U)'$, it follows that

$$\langle \hat{\pi}(XY) V^* h, k\rangle = \lim_\alpha \langle V^* \Phi_{U,k_\alpha}(XY) h, k\rangle$$

for $V \in I_U$ and $h \in H, k \in \hat{H}$. Applying Corollary (4.2.4) to the operator $Y \in \mathcal{T}(T)$, we obtain the identity

$$\Psi_U(\theta)^* \widetilde{XY}\ \Psi_U(\theta) h = \Psi_U(\theta)^* X P_H \Psi_U(\theta) \hat{\pi}(Y) h$$

for $\theta \in I_\mu$ and $h \in H$. Using the definition of $\hat{\pi}(X) = \Phi_U(\tilde{X})$, we find that

$$\langle \hat{\pi}(XY)V^*h, k\rangle = \langle V^*\hat{\pi}(X)\hat{\pi}(Y)h, k\rangle = \langle \hat{\pi}(X)\hat{\pi}(Y)V^*h, k\rangle$$

for $V \in I_U$ and $h \in H, k \in \hat{H}$. By the remarks preceding Corollary (4.2.4) it follows that $\hat{\pi}(XY) = \hat{\pi}(X)\hat{\pi}(Y)$.

Inductively one obtains that

$$\hat{\pi}(X_1 \cdot \ldots \cdot X_r) = \hat{\pi}(X_1) \cdot \ldots \cdot \hat{\pi}(X_r)$$

holds for any finite number of operators $X_1, \ldots, X_r \in \mathcal{T}(T)$. Since $C^*(\mathcal{T}(T))$ is the norm-closed linear span of products of this type and since $\hat{\pi}$ is norm-continuous, the multiplicativity of $\pi = \hat{\pi}|_{C^*(\mathcal{T}(T))}$ follows.

Using the definition of $\pi(1_H) = \Phi_U(1_H \oplus 0_{H^\perp})$, one easily finds that $\pi(1_H)$ acts as the identity operator on H. Since

$$\pi(1_H)V^*h = V^*\pi(1_H)h = V^*h$$

for all $V \in I_U$ and $h \in H$, it follows that $\pi(1_H) = 1_{\hat{H}}$. As an application of Corollary (4.2.4) one obtains that $\pi(T_f) = \Psi_U(f)$ for all $f \in L^\infty(\mu)$. Hence the minimality of U implies that π is the minimal Stinespring dilation of $\Phi_T|_{C^*(\mathcal{T}(T))}$. To see that $\hat{\pi}$ possesses the additional multiplicativity property claimed in the theorem, it suffices to observe that

$$\hat{\pi}(XY_1 \cdots Y_r) = \hat{\pi}(X)\hat{\pi}(Y_1) \cdots \hat{\pi}(Y_r) = \hat{\pi}(X)\hat{\pi}(Y_1 \cdots Y_r)$$

for $X \in B(H), Y_1, \ldots, Y_r \in \mathcal{T}(T)$, and to use the norm-continuity of $\hat{\pi}$.

For $Y \in (U)'$, we define the Toeplitz operator $T_Y \in \mathcal{T}(T)$ with symbol Y as the compression $T_Y = P_H Y|_H$. In the particular case that $Y = \Psi_U(f)$ with $f \in L^\infty(\mu)$ we obtain that $T_Y = T_f$ is the Toeplitz operator with symbol f.

Corollary (4.2.6)[149] : Let $T \in B(H)^n$ be a regular A −isometry with minimal normal extension $U \in B(\hat{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. Let $\Phi_T: B(H) \to B(H)$ and $\pi: C^*(\mathcal{T}(T)) \to B(\hat{H})$ be defined as before.

(a) For $X \in \mathcal{T}(T)$, the operator $\pi(X)$ is the unique element in $(U)'$ with $X = T_{\pi(X)}$. For $Y \in (U)'$, we have $\pi(T_Y) = Y$.

(b) For $Y \in (U)'$ and $f \in L^\infty(\mu)$, we have

$$\|T_Y\| = \|Y\| \text{ and } \|T_f\| = \|f\|_{L^\infty(\mu)}.$$

(c) For $Y_{ij} \in (U)', 1 \le i \le r, 1 \le j \le s$, we have

$$\Phi_T\left(\sum_{i=1}^r \sum_{j=1}^s T_{Y_{ij}}\right) = T_{\sum_{i=1}^r \sum_{j=1}^s Y_{ij}}.$$

Proof. Part (a) and part (b) follow immediately from Corollary (4.2.4). Since by Theorem (4.2.5) the restriction $\pi = \hat{\pi}|_{C^*(\mathcal{T}(T))}$ is a C∗-algebra homomorphism, we obtain that

$$\Phi_T\left(\sum_{i=1}^r \sum_{j=1}^s T_{Y_{ij}}\right) = P_H\left(\sum_{i=1}^r \sum_{j=1}^s \pi\left(T_{Y_{ij}}\right)\right)|_H = T_{\sum_{i=1}^r \sum_{j=1}^s Y_{ij}}$$

for $Y_{ij} \in (U)'$ as in part (c).

Since the C^*-algebra $C^*(\mathcal{T}(T))$ is the norm-closure of the set of all finite sums of finite products of Toeplitz operators of the form T_Y with $Y \in (U)'$, part (c) of Corollary (4.2.6) shows in particular that the action of any Toeplitz projection $\Phi_T: B(H) \to B(H)$ defined as above is uniquely determined by T on the Toeplitz C^*-algebra $C^*(\mathcal{T}(T))$.

If $W^*(U)$ is a maximal abelian W^*-algebra, or equivalently, $W^*(U) = W^*(U)'$, then the abstract and concrete Toeplitz operators coincide, that is,

$$\mathcal{T}(T) = \{T_f; f \in L^\infty(\mu)\}.$$

This can be seen as a generalization of the classical Brown–Halmos characterization [152] of the Toeplitz operators on $H^2(\mathbb{T})$.

Let $T \in B(H)^n$ be a regular $A-$isometry with minimal normal extension $U \in B(\hat{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. Throughout the rest of this paper we denote by $\Phi_T: B(H) \to B(H)$ a Toeplitz projection defined. Recall that Φ_T is the compression

$$\Phi_T(X) = P_H \Phi_U(\tilde{X})|_H (X \in B(H))$$

of a projection $\Phi_U: B(\hat{H}) \to B(\hat{H})$ with $\operatorname{ran}(\Phi_U) = (U)'$ and that

$$\pi: C^*(\mathcal{T}(T)) :\to B(\hat{H}), X \longmapsto \Phi_U(\tilde{X})$$

is the minimal Stinespring dilation of the completely positive and unital mapping $\Phi_T|_{C^*(\mathcal{T}(T))}$. We denote by

$$\mathcal{T}_a(T) = \{T_f; f \in H_A^\infty(\mu)\} \subset B(H)$$

the weak* closed subalgebra consisting of all analytic Toeplitz operators. We calculate the essential commutant $\mathcal{T}_a(T)^{ec}$ of the set of all analytic Toeplitz operators.

Lemma (4.2.7)[149]: Suppose that $M \subset H$ is a closed reducing subspace for $\mathcal{T}(T)$. Then

$$\Phi_T(X) = \Phi_T(P_M X|_M) \oplus (P_{M^\perp} X|_{M^\perp})$$

for every operator $X \in B(H)$.

Proof. We denote by M the set of all operators $X \in B(H)$ with the property that $XM \subset M^\perp$ and $XM^\perp \subset M$. Fix an operator $X \in M$. Then $T_\theta^* X T_\theta \in M$ for all μ-inner functions $\theta \in I_\mu$ and hence also $\Phi_T(X) \in M$ (see Corollary (4.2.4)). On the other hand, the space M is reducing for the operator $\Phi_T(X) \in \mathcal{T}(T)$. Therefore $\Phi_T(X) = 0$ and the assertion follows.

Let $S \in \mathcal{T}_a(T)^{ec}$ be arbitrary. It follows from Corollary (4.2.4) that $Y_S = \hat{\pi}(S)$ is the unique operator in $(U)'$ with $\Phi_T(S) = P_H Y_S|_H$. Our aim is to show that, under suitable conditions on T, the operator Sis a compact perturbation of an abstract $T-$Toeplitz operator. Since $\Phi_T(S) \in \mathcal{T}(T)$, it suffices to show that

$$S - \Phi_T(S) \in K(H).$$

To prove this, we shall use the map

$$F: L^\infty(\mu) \to B(H), \quad f \longmapsto T_f S - P_H(Y_S \Psi_U(f))|_H.$$

Note first that $S - \Phi_T(S) = F(1)$ and that

$$F(f) = T_f S - \Phi_T(S) T_f$$

for every function $f \in H_A^\infty(\mu)$. It clearly suffices to find conditions which ensure that the whole image of F consists of compact operators. Since F is continuous linear, we only need to show that F maps the characteristic function χ_ω of each Borel set $\omega \subset \partial_A$ into $K(H)$. We begin with a very modest first step.

Lemma (4.2.8)[149]: For every point $z \in \partial_A$, the operator $F(\chi_{\{z\}})$ is compact.

Proof. We may suppose that $\mu(\{z\}) > 0$, since otherwise $F(\chi_{\{z\}}) = 0$. As shown in [159, Proposition 2.3] the regularity of T implies that $\chi_{\{z\}} \in H_A^\infty(\mu)$. Exactly as in [159], it follows that the eigenspace H_z of T associated with the joint eigenvalue z coincides with the eigenspace of U associated with z, that is,

$$\bigcap_{i=1}^{n} \ker(z_i - T_i) = \bigcap_{i=1}^{n} \ker(z_i - U_i)$$

and that $P_z = \Phi_U(\chi_{\{z\}})|_H \in B(H)$ is the orthogonal projection onto H_z. The space $H_z = P_z H$ is reducing for $\mathcal{T}(T)$, since

$$(P_H X|_H)P_z = P_H \Psi_U(\chi_{\{z\}})P_H X|_H = P_z(P_H X|_H)$$

for all $X \in (U)'$. Let $S = (S_{ij})_{i,j=1,2}$ be the matrix representation of S with respect to the decomposition $H = (H \ominus H_z) \oplus H_z$. Since $P_z = T_{\chi_{\{z\}}} \in \mathcal{T}_a(T)$, it follows that $SP_z - P_z S \in K(H)$, or equivalently, that S_{12} and S_{21} are compact. Using Lemma (4.2.7) and passing to the equivalence classes in the Calk in algebra, we find that

$$[F(\chi_{\{z\}})] = [P_z(S_{11} \oplus S_{22}) - \Phi_T(S)P_z] = [P_z(0 \oplus S_{22}) - \Phi_T(S_{11} \oplus S_{22})P_z].$$

For each μ −inner function $\theta \in I_\mu$, we have

$$(T_\theta^*(S_{11} \oplus S_{22})T_\theta)|_{H_z} = (T_\theta^* S_{22} T_\theta)|_{H_z} = S_{22}.$$

Hence the definition of Φ_T implies that $\Phi_T(S_{11} \oplus S_{22})|_{H_z} = S_{22}$. But then $P_z(0 \oplus S_{22}) - \Phi_T(S_{11} \oplus S_{22})P_z = 0$ and therefore $F(\chi_{\{z\}})$ is compact.

Let us suppose in addition that T is essentially normal. Then it follows from Lemma 3.9 (c) in [160] that all operators in the image of the map

$$F: L^\infty(\mu) \to B(H), f \longmapsto T_f S - P_H(Y_S \Psi_U(f))|_H$$

belong to the essential commutant $(T)^{ec}$ of T. Hence we can apply the following consequence of the Allan-Douglas localization principle to every operator in $\operatorname{ran}(F)$.

Proposition (4.2.9)[149]: Suppose that the regular A −isometry $T \in B(H)^n$ is essentially normal. Then for every operator $X \in (T)^{ec}$, we have

$$\|X\|_e = \sup_{z \in \partial_A} \inf\left\{\|T_f X\|_e; f \in C(\partial_A) \text{ with } f(z) = 1\right\}.$$

Proof. By Lemma 3.9 (c) and Lemma (4.2.7) in [160], the essential normality of T yields that $\mathcal{D} = (T)^{ec}$ is a C∗-algebra containing $\mathcal{T}(T) \cup K(H)$, that the C^*-algebra

$$A = \left(C^*(\{T_f; f \in C(\partial_A)\}) + K(H)\right)/K(H)$$

is contained in the center of the C^*-algebra $\mathcal{T} = D/K(H)$ and that the mapping $\tau: C(\partial_A) \to A, f \longmapsto [T_f]$, is a surjective C^*-algebra homomorphism. Hence, for each functional $\lambda \in \Delta_A$ in the character space of A, there is a unique point $z(\lambda) \in \partial_A$ with

$$\lambda([T_f]) = f(z(\lambda))(f \in C(\partial_A)).$$

For $\lambda \in \Delta_A$ and $z \in \partial_A$, let $I_\lambda \subset \mathcal{T}$ be the closed ideal generated by all elements $[T_f]$ where $f \in C(\partial_A)$ and $\lambda([T_f]) = 0$, and let $I_z \subset \mathcal{T}$ be the closed ideal generated by all elements $[T_f]$ such that $f \in C(\partial_A)$ satisfies $f(z) = 0$. Then $I_\lambda = I_{z(\lambda)}$ for all $\lambda \in \Delta_A$, and the Allan-Douglas localization principle (Theorem 7.47 in [162]) implies that

$$\|X\|_e = \sup_{\lambda \in \Delta_A} \|[X] + I_\lambda\|_{\mathcal{T}/I_\lambda} \le \sup_{z \in \partial_A} \|[X] + I_z\|_{\mathcal{T}/I_z}$$

for every $X \in (T)^{ec}$. But for $X \in (T)^{ec}$ and $f \in C(\partial_A)$ with $f(z) = 1$, the estimate

$$\|[X] + I_z\|_{\mathcal{T}/I_z} = \left\|[T_f X] + I_z\right\|_{\mathcal{T}/I_z} \le \left\|T_f X\right\|_e$$

holds. This observation completes the proof.

An application of the dominated convergence theorem (Lemma 3.4 in [160]) shows that the mapping

$$F: L^\infty(\mu) \to B(H), f \mapsto T_f S \ - P_H\big(Y_S \Psi_U(f)\big)|_H$$

is point wise boundedly SOT-continuous, that is, for every bounded sequence $(f_k)_k$ in $L^\infty(\mu)$ converging point wise μ −almost everywhere to some function $f \in L^\infty(\mu)$, it follows that $F(f) = \lim_{k\to\infty} F(f_k)$ in the strong operator topology.

Corollary (4.2.10)[149]: Suppose that the regular A −isometry $T \in B(H)^n$ is essentially normal. For a given operator $S \in T_a(T)^{ec}$, let $F: L^\infty(\mu) \to B(H)$ be defined as above. If $F\big(L^\infty(\mu)\big) \nsubseteq K(H)$, then there is a sequence $(f_k)_k$ of continuous functions $f_k \in C(\partial_A, [0,1])$ with pairwise disjoint supports such that

$$\inf_{k\ge 1} \|F(f_k)\| > 0.$$

Proof. Suppose that $F\big(L^\infty(\mu)\big) \nsubseteq K(H)$. Since every bounded measurable function can be approximated uniformly by linear combinations of characteristic functions of Borel sets, we can choose a characteristic function χ of some Borel set in ∂_A such that $\varrho = \frac{\|F(\chi)\|_e}{2} > 0$. By Proposition (4.2.9) there is a point $z \in \partial_A$ with

$$\|F(f\chi)\|_e = \left\|T_f F(\chi)\right\|_e > \varrho$$

for all $f \in C(\partial_A)$ with $f(z) = 1$. Here the first equality follows from Lemma 3.9 (c) in [160]. Let $k \ge 0$ be an integer. Suppose that $g_1, \ldots, g_k \in C(\partial_A, [0,1])$ are functions with pairwise disjoint supports such that $\left\|F(g_j\chi)\right\| > \varrho$ and $z \notin \operatorname{supp}(g_j)$ for $j = 0, \ldots, k$. Choose a function $f \in C(\partial_A, [0,1])$ with $f(z) = 1$ and $\operatorname{supp}(f) \cap \operatorname{supp}(g_j) = \emptyset$ for all $j = 0, \ldots, k$. Let $(\theta_j)_j$ be a sequence of functions in $C(\partial_A, [0,1])$ with $z \notin \operatorname{supp}(\theta_j)$ for all j such that $\theta_j(w) \to 1$ as $j \to \infty$ for every point $w \in \partial_A \backslash \{z\}$. Since F is point wise boundedly SOT-continuous, it follows that

$$F\big(\chi_{\{z\}^c} f\chi\big) = \mathrm{SOT} - \lim_{j\to\infty} F\big(\theta_j f\chi\big).$$

As an application of Lemma (4.2.8), we obtain that

$$\left\|F\big(\chi_{\{z\}^c} f\chi\big)\right\| \ \ge \|F(f\chi)\|_e > \varrho.$$

Hence there is an integer $j \ge 1$ such that $\left\|F(\theta_j f\chi)\right\| > \varrho$.

Inductively one obtains a sequence of functions $g_k \in C(\partial_A, [0,1])$ with pairwise disjoint supports and $\|F(g_k\chi)\| > \varrho$ for all j. In the inductive step, one can define $g_{k+1} = \theta_j f$ with f and θ_j as above. A standard application of Lusin's theorem (Theorem 7.4.3 and Proposition 3.1.2 in [154]) shows that there is a sequence $(h_j)_j$ in $C(\partial_A, [0,1])$ such that $h_j \to \chi$ for $j \to \infty$ μ −almost everywhere. Since

$$F(g_k\chi) = \text{SOT} - \lim_{j\to\infty} F\big(g_k h_j\big)\,(k \geq 1),$$

for each $k \geq 1$, there is an index $j_k \geq 1$ with $\|F(g_k h_{jk})\| > \varrho$. But then the resulting functions $f_k = g_k h_{jk}$ have all the required properties.

After these preparations we are able to prove the main result.

Theorem (4.2.11)[149]: Let $T \in B(H)^n$ be an essentially normal regular A −isometry. Then for every operator $S \in T_a(T)^{ec}$, we have

$$S - \Phi_T(S) \in K(H).$$

Proof. Let $F: L^\infty(\mu) \to B(H), f \longmapsto T_f S - P_H\big(Y_S \Psi_U(f)\big)|_H$, be the map considered above. Since $S - \Phi_T(S) = F(1)$, it suffices to show that $F\big(L^\infty(\mu)\big) \subset K(H)$. Let us assume that this inclusion does not hold. Then by Corollary (4.2.10) there are a positive real number $\varrho > 0$ and a sequence of functions $f_k \in C(\partial_A, [0,1])$ with pairwise disjoint supports $A_k = \text{supp}(f_k)$ such that $\|F(f_k)\| > \varrho$ for all $k \geq 1$.

Exactly as in the proof of Theorem 4.6 in [160], one can use the regularity of T to replace $(f_k)_k$ by a sequence $(g_k)_k$ of functions in A such that

$$\|g_k\|_{\infty,\partial_A} \leq 2, \|g_k\|_{\infty,\partial_A \setminus A_k} < 2^{-k}, \|F(g_k)\| > \frac{\varrho}{4}$$

for all $k \geq 1$. Recall that $F\big(g_j\big) = T_{g_j}S - \Phi_T(S)T_{g_j}$ is the weak* limit of a net consisting of operators of the form

$$T_{g_j}S - \frac{1}{k^k}\sum_{i\in\{1,\dots,k\}^k} T^*_{\theta(i)} S T_{\theta(i)} T_{g_j}$$

$$= \frac{1}{k^k} T^*_{\theta(i)} \sum_{i\in\{1,\dots,k\}^k} \Big(T_{g_j\theta(i)}S - ST_{g_j\theta(i)}\Big)$$

with suitable μ −inner functions $\theta(i) \in I_\mu$. Hence, for each $j \geq 1$, there is a function $\theta_j: \partial_A \to \mathbb{C}$ with $|\theta_j| = 1$ on ∂_A such that $\theta_j \in I_\mu$ and such that the function $h_j = g_j\theta_j \in H_A^\infty(\mu)$ satisfies

$$\|h_j\|_{\infty,\partial_A} \leq 2, \|h_j\|_{\infty,\partial_A\setminus A_j} < 2^{-k}, \left\|T_{h_j}S - ST_{h_j}\right\| > \frac{\varrho}{4}.$$

By hypothesis the commutators $K_j = \left[T_{h_j}, S\right]$ are compact. By passing to a subsequence, one can achieve that the limit

$$c = \lim_{j\to\infty}\|K_j\| \in [4\,,\infty)$$

exists. Since the sequence $\big(h_j\big)_j$ is uniformly bounded on ∂_A and converges to zero point-wise on ∂_A, it follows that the sequences $\big(K_j\big)_j$ and $\big(K_j^*\big)_j$ converge to zero strongly. A result due to Muhly and Xia (Lemma 2.1 in [166]) shows that, by passing to a subsequence again, one can achieve that the series

$$K = \sum_{j=1}^{\infty} K_j$$

converges in the strong operator topology and satisfies $\|K\|_e = c > 0$. Since each point of ∂_A belongs to at most one of the sets A_j, the partial sums of the series $\sum_{j=1}^{\infty} h_j$ are uniformly bounded on ∂_A and converge point wise to a function $h: \partial_A \to \mathbb{C}$. By the dominated convergence theorem it follows that $h \in H_A^\infty(\mu)$. Again using Lemma 3.4 from [160], one obtains that

$$T_h = \mathrm{SOT} - \sum_{j=1}^{\infty} T_{h_j}, \quad [T_h, S] = \mathrm{SOT} - \sum_{j=1}^{\infty} \left[T_{h_j}, S\right] = K.$$

But then $T_h \in T_a(T)$ would be an operator with non-compact commutator $[T_h, S] = K$. This contradiction completes the proof.

Let $T \in B(H)^n$ be a regular A −isometry with minimal normal extension $U \in B(\widehat{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. Suppose that $W^*(U) \subset B(\widehat{H})$ is a maximal abelian von Neumann algebra, that is, $W^*(U) = (U)'$. Then Corollary (4.2.3) implies that $T(T) = \{T_f; f \in L^\infty(\mu)\}$. As a consequence, we obtain a complete characterization of the essential commutant $T_a(T)^{ec}$ of the analytic Toeplitz operators in this case.

Corollary (4.2.12)[149]: Let $T \in B(H)^n$ be an essentially normal regular A −isometry with minimal normal extension $U \in B(\widehat{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. If $W^*(U) \subset B(\widehat{H})$ is a maximal abelian von Neumann algebra and $S \in B(H)$, then equivalent are:

(i) $S \in T_a(T)^{ec}$.

(ii) $S = T_f + K$ with a compact operator $K \in K(H)$ and a symbol $f \in L^\infty(\mu)$ with the property that the associated Hankel operator H_f is compact.

Proof. First, suppose that $S \in T_a(T)^{ec}$. Then $\Phi_T(S) = T_f$ with a suitable function $f \in L^\infty(\mu)$. The proof of Theorem (4.2.11) shows that the image of the bounded linear map

$$F: L^\infty(\mu) \to B(H), g \mapsto T_g S - P_H(\Psi_U(gf))|_H$$

is contained in $K(H)$. It follows that $K = F(1) = S - T_f$ is compact and the identity

$$\begin{aligned} F(\bar{f}) &= T_{\bar{f}}S - T_{|f|^2} \\ &= T_{\bar{f}}T_f - T_{|f|^2} + T_{\bar{f}}K \\ &= P_H\Psi_U(\bar{f})P_H\Psi_U(f)|_H - P_H\Psi_U(\bar{f})\Psi_U(f)|_H + T_{\bar{f}}K \\ &= -P_H\Psi_U(\bar{f})P_{H^\perp}\Psi_U(f)|_H + T_{\bar{f}}K \\ &= -H_f^* H_f + T_{\bar{f}}K \end{aligned}$$

shows that also the operator H_f is compact.

In order to prove the remaining implication, it suffices to verify that all Toeplitz operators T_f such that the corresponding Hankel operators H_f are compact essentially commute with $T_a(T)$. But this follows from the formula

$$T_f T_g - T_g T_f = T_{gf} - T_g T_f = P_H \Psi_U(g) H_f$$

which holds for all $f \in L^\infty(\mu)$ and $g \in H_A^\infty(\mu)$.

By considering Hankel operators $H_Y = (1 - P_H)Y|_H \in B(H, H^\perp)$ with symbol $Y \in (U)'$, we obtain a similar characterization of the essential commutant of the analytic Toeplitz operators in the general case.

Corollary (4.2.13)[149]: If $T \in B(H)^n$ is an essentially normal regular A −isometry with minimal normal extension $U \in B(\widehat{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$, then the following statements are equivalent:

(i) $S \in T_a(T)^{ec}$.

(ii) $S = T_Y + K$ with a compact operator $K \in K(H)$ and a symbol $Y \in (U)'$ such that the associated Hankel operator H_Y has the property that $H_{\bar{f}}^* H_Y$ is compact for every $f \in L^\infty(\mu)$.

(iii) $S = T_Y + K$ with a compact operator $K \in K(H)$ and a symbol $Y \in (U)'$ such that the associated Hankel operator H_Y has the property that $H_{\bar{f}}^* H_Y$ is compact for every $f \in H_A^\infty(\mu)$.

Proof. For arbitrary symbols $f \in L^\infty(\mu)$ and $Y \in (U)'$, an elementary calculation shows that

$$-H_{\bar{f}}^* H_Y = T_f T_Y - P_H \Psi_U(f) Y|_H.$$

Suppose that $S \in T_a(T)^{ec}$. Then Theorem (4.2.11) implies that $S = T_Y + K$ is a sum of the Toeplitz operator $T_Y = \Phi_T(S) \in B(H)$ with symbol $Y \in (U)'$ and the compact operator $K = S - \Phi_T(S) \in K(H)$. By the proof of Theorem (4.2.11) the range of the mapping

$$F: L^\infty(\mu) \to B(H), f \mapsto T_f S - P_H \Psi_U(f) Y|_H$$

is contained in $K(H)$. Consequently, $H_{\bar{f}}^* H_Y = T_f K - F(f)$ is compact for every symbol $f \in L^\infty(\mu)$. To complete the proof note that the identity

$$H_{\bar{f}}^* H_Y = P_H \Psi_U(f) Y|_H - T_f T_Y = T_Y T_f - T_f T_Y$$

holds for $f \in H_A^\infty(\mu)$ and $Y \in (U)'$.

[165], Johnson and Parrott characterized the essential commutant $\mathfrak{U}^{ec}$ of an abelian von Neumann algebra $\mathfrak{U} \subset B(H)$ as the sum $\mathfrak{U}' + K(H)$ of its commutant and the compact operators. This result has been generalized in [167] to the non-abelian case. We present an alternative proof of Johnson and Parrott's result for finitely generated abelian von Neumann algebras. To this end, let us observe that, for every compact subset $K \subset \mathbb{C}^n$, the Shilov boundary of $C(K)$ is equal to K itself and the triple $(C(K), K, \mu)$ is regular [150] for every choice of $\mu \in M^+(K)$. Consequently, every commuting tuple $N = (N_1, \dots, N_n) \in B(H)^n$ of normal operators is a regular $C(\sigma(N))$ −isometry.

Corollary (4.2.14)[149]: (Johnson–Parrott). The essential commutant of a finitely generated abelian von Neumann algebra $\mathfrak{U} \subset B(H)$ is given by

$$\mathfrak{U}^{ec} = \mathfrak{U}' + K(H).$$

Proof. Since $\mathfrak{U}$ is abelian, its generators $N_1, \dots, N_n \in B(H)$ form a commuting tuple of normal operators and hence a normal regular $C(\sigma(N))$ −isometry $N \in B(H)^n$. By Theorem (4.2.11), the inclusion $T_a(N)^{ec} \subset T(N) + K(H)$ holds. Hence it suffices to check that the analytic Toeplitz operators associated with N coincide with $\mathfrak{U} = W^*(N)$ and that the abstract N −Toeplitz operators are precisely those operators that commute with $\mathfrak{U}$. Let $\mu \in M^+(\sigma(N))$ denote the scalar spectral measure associated with N. Then $C(\sigma(N))$ is weak*-dense in $L^\infty(\mu)$, which implies that $H_{C(\sigma(N))}^\infty(\mu) = L^\infty(\mu)$ and hence

$$W^*(N) = \Psi_N(L^\infty(\mu)) = \Psi_N\left(H_{C(\sigma(N))}^\infty(\mu)\right) = T_a(N).$$

To conclude the proof, we combine the fact that $(N)' = W^*(N)' = \mathfrak{U}'$ with Corollary (4.2.4) to obtain the remaining identity $T(N) = \mathfrak{U}$.

By [157, Lemma II.2.8], the preceding result applies in particular to every abelian von Neumann algebra on a separable Hilbert space.

We characterize those regular A −isometries for which the associated Toeplitz projection Φ_T vanishes on the compact operators. By following the lines of the proof of [159, Theorem 3.3] and adapting it to the setting of regular A −isometries, we observe that a regular A −isometry $T \in B(H)^n$ has empty point spectrum if and only if

$$T(T) \cap K(H) = \{0\}.$$

Corollary (4.2.15)[149]: The Toeplitz projection Φ_T associated with a regular A −isometry $T \in B(H)^n$ vanishes on $K(H)$ if and only if $\sigma_p(T) = \emptyset$.

Proof. Recall that the Toeplitz projection acts as the identity on the Toeplitz operators. Thus, if T has an eigenvalue, we can choose a compact Toeplitz operator $X \neq 0$ satisfying $\Phi_T(X) = X \neq 0$. On the other hand, the minimal normal extension $U \in B(\widehat{H})^n$ of T is a normal regular A −isometry. Moreover, the mapping Φ_U is the corresponding Toeplitz projection. A look at Theorem (4.2.11) reveals that

$$S - \Phi_U(S) \in K(\widehat{H})$$

for every element $S \in T_a(U)^{ec}$. Now assume that $K \in K(H)$ is a compact operator. Then $\widehat{K} = K \oplus 0 \in K(\widehat{H})$ is compact and thus belongs to $T_a(U)^{ec}$. Hence the above calculation implies that $\Phi_U(K) \in K(\widehat{H}) \cap T(U)$ is a compact U −Toeplitz operator. Assuming that $\sigma_p(T) = \emptyset$, we infer that $\Phi_T(K) = P_H \Phi_U(K)|_H = 0$.

Using Corollary (4.2.15) we prove an essential spectral inclusion theorem for Toeplitz operators.

Theorem (4.2.16)[149]: Let $T \in B(H)^n$ be a regular A −isometry with minimal normal extension $U \in B(\widehat{H})^n$. Then T has empty point spectrum if and only if the spectral inclusion $\sigma(Y) \subset \sigma_e(T_Y)$ holds for every operator $Y \in (U)'$.

Proof. Suppose that $\sigma_p(T) = \emptyset$ and fix an operator $Y \in (U)'$. We first show that the left spectrum of Y is contained in the left essential spectrum of T_Y. To prove this inclusion it suffices to verify that Y is left invertible in $B(\widehat{H})$ whenever T_Y is left invertible in the Calkin algebra $C(H) = B(H)/K(H)$. Let us suppose that $X \in B(H)$ is an operator with $XT_Y - 1_H \in K(H)$. Using Corollary (4.2.4) and the proof of Theorem (4.2.5), we find that

$$\hat{\pi}(X)Y = \hat{\pi}(X)\hat{\pi}(T_Y) = \hat{\pi}(XT_Y) = 1_{\widehat{H}} + \hat{\pi}(XT_Y - 1_H).$$

Since $\sigma_p(U) = \sigma_p(T) = \emptyset$, it follows from Corollary (4.2.15) applied to U that Φ_U annihilates the compact operators. But then, using the definition of $\hat{\pi}$ (see Corollary (4.2.4)), we find that

$$\hat{\pi}(X)Y = 1_{\widehat{H}} + \Phi_U\big((XT_Y - 1_H) \oplus 0\big) = 1_{\widehat{H}}.$$

Thus we have shown that the left spectrum of Y is contained in the left essential spectrum of T_Y. Applying the same argument to $Y^* \in (U)'$, we obtain that the left spectrum of Y^* is contained in the left essential spectrum of $T_Y^* = T_{Y^*}$. By standard duality results this means precisely that the right spectrum of Y is contained in the right essential spectrum of T_Y. In total we have shown that $\sigma(Y) \subset \sigma_e(T_Y)$ for every operator $Y \in (U)'$ under the hypothesis that the point spectrum of T is empty. If x is a joint eigenvector for T, then the orthogonal projection Y of $\widehat{H}$ onto the one-dimensional subspace spanned by x belongs to the commutant $(U)'$ and T_Y is the corresponding rank-one projection on H. Then $1 \in \sigma(Y)$ while $\sigma_e(T_Y) \subset \{0\}$. Hence the essential spectral inclusion result does not hold.

Note that, for regular A −isometries with empty point spectrum, we even proved spectral inclusion theorems for the left (essential) spectra and right (essential) spectra separately. If $\sigma_p(T) = \emptyset$ and μ denotes the scalar spectral measure of the minimal normal extension $U \in B(\hat{H})^n$, then we obtain in particular that

$$\text{essran}(f) = \sigma_{L^\infty(\mu)}(f) = \sigma(\Psi_U(f)) \subset \sigma_e(T_f)$$

for every function $f \in L^\infty(\mu)$. For the particular case of Toeplitz operators on the Hardy space of the unit disc or the unit ball, this result is contained in [162] and [158].

For a given subalgebra $B \subset (U)'$, we denote by

$$\mathcal{T}_B = \overline{\text{alg}}(\{T_X;\ X \in B\}) \subset B(H)$$

the smallest norm-closed subalgebra containing all operators T_X with $X \in B$. The semi-commutator ideal $SC(\mathcal{T}_B)$ of $\mathcal{T}_B$ is defined as the norm-closed ideal in $\mathcal{T}_B$ generated by all operators $T_X T_Y - T_{XY}$ with $X, Y \in B$. Since $\mathcal{T}_B$ is the norm-closure of the set of all finite sums of finite products of operators of the form T_X with $X \in B$, a straightforward argument using part (c) of Corollary (4.2.6) shows that $\mathcal{T}_B$ is invariant under the Toeplitz projection Φ_T and that

$$SC(\mathcal{T}_B) = \ker(\Phi_{T\,|\mathcal{T}_B}) = \ker(\pi|_{\mathcal{T}_B}).$$

The last equality follows from Theorem (4.2.5) together with Corollary (4.2.4).

Corollary (4.2.17)[149]: Let $T \in B(H)^n$ be a regular A −isometry with minimal normal extension $U \in B(H)^n$. For each subalgebra $B \subset (U)'$, there is a short exact sequence

$$0 \rightarrow SC(\mathcal{T}_B) \hookrightarrow \mathcal{T}_B \overset{\pi}{\twoheadrightarrow} B \rightarrow 0$$

of Banach algebras with $\pi(T_X) = X$ for all $X \in B$. If $\sigma_p(T) = \emptyset$, then

$$\mathcal{T}_B \cap K(H) \subset SC(\mathcal{T}_B).$$

Proof. The existence of the short exact sequence follows from the remarks preceding the corollary. The last assertion is a consequence of Corollary (4.2.15).

Using part (b) and part(c) of Corollary (4.2.6) one obtains that, for every regular A −isometry $T \in B(H)^n$ and each subalgebra $B \subset (U)'$, the direct sum decomposition

$$\mathcal{T}_B = SC(\mathcal{T}_B) \oplus \{T_X; X \in \bar{B}\}$$

holds with $SC(\mathcal{T}_B) = \ker(\Phi_{T|\mathcal{T}_B})$ and $\{T_X; X \in B\} = \Phi_T(\mathcal{T}_B)$. If in addition the subalgebra $B \subset (U)'$ is self-adjoint in the sense that $X^* \in B$ whenever $X \in B$, then the sequence described in Corollary (4.2.17) is a short exact sequence of C^*-algebras.

Section (4.3): Asymptotic Toeplitz Operators on $H^2(\mathbb{D}^n)$:

Although concrete bounded linear operators on Hilbert spaces exist in great variety and can exhibit interesting properties, one of the main concerns of function theory and operator theory has generally been the study of operators which are connected with the spaces of holomorphic and integrable functions. The class of Toeplitz and analytic Toeplitz operators have turned out to be one of the most important classes of concrete operators from this point of view.

Toeplitz operators on the Hardy space (or, on the l^2 space) were first studied by O. Toeplitz (and then by P. Hartman and A. Wintner in [185]). However, a systematic study of Toeplitz operators was triggered by Brown and Halmos [173] on algebraic properties of Toeplitz operators on $H^2(\mathbb{D})$. Here $H^2(\mathbb{D})$ denote the Hardy space over the open unit disc $\mathbb{D}$ in $\mathbb{C}$. The study of Toeplitz operators on Hilbert spaces of holomorphic functions, like the Hardy space, the Bergman space and the weighted Bergman spaces, on domains in $\mathbb{C}^n$ is also one of the very active area of current research that brings together several areas of mathematics. For more information on this direction of research, see [177], [178], [179], [180], [186], [190].

Recall that a bounded linear operator T on $H^2(\mathbb{D})$ is said to be a Toeplitz operator if $T = P_{H^2(\mathbb{D})} M_\varphi|_{H^2(\mathbb{D})}$, where M_φ is the Laurent operator on $L^2(\mathbb{T})$ for some $\varphi \in L^\infty(\mathbb{T})$.

Here $P_{H^2(\mathbb{D})}$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. The well-known Brown-Halmos theorem characterizes Toeplitz operators on $H^2(\mathbb{D})$ as follows (see the matricial characterization, Theorem 6 in [173]): Let T be a bounded linear operator on $H^2(\mathbb{D})$. Then T is a Toeplitz operator if and only if

$$T_z^* \, T T_z = T.$$

One of the main results the following generalization of Brown-Halmos theorem (see Theorem (4.3.2)): A bounded linear operator T on $H^2(\mathbb{D}^n)$ is a Toeplitz operator if and only if

$$T_{z_j}^* T T_{z_j} = T,$$

for all $j = 1, \dots, n$.

The notion of Toeplitzness was extended to more general settings by Barr´ıa and Halmos [171] and Feintuch [181]. Also see Popescu [182] for Toeplitzness in the non-commutative setting.

Accordingly, following Feintuch (and Barrıa and Halmos [171]) we shall say that a bounded linear operator T on $H^2(\mathbb{D})$ is (uniformly) asymptotically Toeplitz if $\{T_z^{*m} \, T T_z^m\}_{m \geq 1}$ converges in operator norm. The following theorem due to Feintuch [181] gives a remarkable characterization of asymptotically Toeplitz operators: A bounded linear operator T on $H^2(\mathbb{D})$ is asymptotically Toeplitz if and only if $T = \text{Toeplitz} + \text{compact}$.

After the Hardy space over unit polydisc, we introduce the asymptotic Toeplitz operators in polydisc setting (see Definition (4.3.4)). In Theorem (4.3.5), we prove the following generalization of Feintuch's theorem: A bounded linear operator T on $H^2(\mathbb{D}^n)$ is asymptotically Toeplitz if and only if $\mathrm{T} = \text{Toeplitz} + \text{compact}$.

We investigate Toeplitzness and asymptotic Toeplitzness of compressions of the $n-$tuple of multiplication operators $(T_{z_1}, \dots, T_{z_n})$ to Beurling type quotient spaces of $H^2(\mathbb{D}^n)$.

More specifically, let $\theta \in H^\infty(\mathbb{D}^n)$ be an inner function, that is, $|\theta| = 1$ on the distinguished boundary $\mathbb{T}^n$ of $\mathbb{D}^n$. Set

$$Q_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n),$$

and

$$C_{z_i} = P_{Q_\theta} T_{z_i}|_{Q_\theta},$$

Where P_{Q_θ} denotes the orthogonal projection from $H^2(\mathbb{D}^n)$ onto Q_θ. A basic question is now to characterize those $T \in B(Q_\theta)$ for which

$$C_{z_i}^* T C_{z_i} = T.$$

Similarly, characterize those $T \in B(Q_\theta)$ for which

$$C_{z_i}^{*m} \, T C_{z_i}^m \to A,$$

in norm, for some $A \in B(Q_\theta)$ and for all $i = 1, \ldots, n$ (given a Hilbert space H, we denote by $B(H)$ the C^* −algebra of all bounded linear operators on H). In this general setting, to remedy the subtlety of the product domain $\mathbb{D}^n$, we modify the above condition by adding another natural condition. The main content is the following: Let $T, A \in B(Q_\theta)$. Then $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots, n$, if and only if $A = 0$. Moreover, the following are equivalent:

(i) $C_{z_i}^{*m} T C_{z_i}^m \to A$ and $C_{z_i}^{*m}(T - A)C_{z_i}^m \to 0$ in norm for all $i, j = 1, \ldots, n$;

(ii) $C_{z_i}^{*m} T C_{z_i}^m \to 0$ in norm for all $i = 1, \ldots, n$;

(iii) T is compact.

We study the above questions in the vector-valued Hardy space over the unit disc setting. To be precise, let E be a Hilbert space, and let $\Theta \in H^\infty_{B(\mathcal{E})}(\mathbb{D})$ be an inner multiplier [187]. Then the model space and the model operator are defined by $Q_\Theta = H^2_{\mathcal{E}}(\mathbb{D}) \ominus \Theta H^2_{\mathcal{E}}(\mathbb{D})$ and $S_\Theta = P_{Q_\Theta} T_z|_{Q_\Theta}$, respectively. We prove that for every $T \in B(Q_\Theta)$, the following holds: (i) $S_\Theta^* T S_\Theta = T$ if and only if $T = 0$, and (ii) $\{S_\Theta^{*m} T S_\Theta^m\}_{m \geq 1}$ converges in norm if and only if T is compact.

Let $n \geq 1$ and $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$. In the sequel, z will always denote a vector $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. The Hardy space $H^2(\mathbb{D}^n)$ over $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions f on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} := \left(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} \left| f\left(re^{i\theta_1}, \ldots, re^{i\theta_n}\right) \right|^2 d\theta \right)^{1/2} < \infty,$$

where $d\theta$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$. Let $\left(T_{z_1}, \ldots, T_{z_n}\right)$ denote the n −tuple of multiplication operators by the coordinate functions $\{z_i\}_{i=1}^n$, that is,

$$\left(T_{z_i} f\right)(w) = w_i f(\boldsymbol{w}),$$

for all $w \in \mathbb{D}^n$ and $i = 1, \ldots, n$. We will often identify $H^2(\mathbb{D}^n)$ with the n −fold Hilbert space tensor product of one variable Hardy space as $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$. In this identification, T_{z_i} can be represented as

$$I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{T_z}_{i^{th}\ place} \otimes \cdots \otimes I_{H^2(\mathbb{D})},$$

for all $i = 1, \ldots, n$. Also one can identify the Hardy space (via the radial limits of functions in $H^2(\mathbb{D}^n)$) with the closed subspace of $L^2(\mathbb{T}^n)$ in the following sense: Let $\{e_{\boldsymbol{k}}: \boldsymbol{k} \in \mathbb{Z}^n\}$ be the orthonormal basis of $L^2(\mathbb{T}^n)$, where $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $e_{\boldsymbol{k}} = e^{i\theta_1 k_1} \cdots e^{i\theta_n k_n}$. Then a function

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}} \in L^2(\mathbb{T}^n),$$

is the radial limit function of some function in $H^2(\mathbb{D}^n)$ if and only if $a_{\boldsymbol{k}} = 0$ whenever at least one of the $k_j, j = 1, \ldots, n$, is negative. In particular, the set of all monomials $\{\boldsymbol{z}^{\boldsymbol{k}}: \boldsymbol{k} \in \mathbb{Z}_+^n\}$ form an orthonormal basis for $H^2(\mathbb{D}^n)$, where $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ and $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$ (cf.[170], [189]). We use $P_{H^2(\mathbb{D}^n)}$ to denote the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$, that is,

$$P_{H^2(\mathbb{D}^n)}\left(\sum_{\boldsymbol{k}\in\mathbb{Z}^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}}\right) = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}},$$

for all $\sum_{\boldsymbol{k}\in\mathbb{Z}^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}}$ in $L^2(\mathbb{T}^n)$.

For $\varphi \in L^\infty(\mathbb{T}^n)$, the Toeplitz operator with symbol φ is the operator $T_\varphi \in B(H^2(\mathbb{D}^n))$ defined by

$$T_\varphi f = P_{H^2(\mathbb{D}^n)}(M_\varphi f)(f \in H^2(\mathbb{D}^n)),$$

Where M_φ is the Laurent operator on $L^2(\mathbb{T}^n)$ defined by $M_\varphi g = \varphi g$ for all $g \in L^2(\mathbb{T}^n)$.

Therefore

$$T_\varphi = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}.$$

For the relevant results on Toeplitz operators on $H^2(\mathbb{D}^n)$, see [172, 175, 178, 186, 188].

The following lemma will prove useful in what follows.

Lemma (4.3.1)[169]: Let H be a Hilbert space and $A \in B(H)$ be a compact operator. If R is a contraction on H, and if $R^{*m} \to 0$ in strong operator topology, then $R^{*m}A \to 0$ in norm.

Proof. This is a particular case of ([172], 1.3 (d), page 3).

In what follows, for each $\boldsymbol{k} \in \mathbb{Z}_+^n$ and $\boldsymbol{l} \in \mathbb{Z}^n$, we write $T_z^k = T_{z_1}^{k_1} \cdots T_{z_n}^{k_n}$, $M_{e^{i\theta}}^{\boldsymbol{l}} = M_{e^{i\theta_1}}^{\boldsymbol{l}_1} \cdots M_{e^{i\theta_n}}^{\boldsymbol{l}_n}$, $T_z^{*k} = T_{z_1}^{*k_1} \cdots T_{z_n}^{*k_n}$ and $M_{e^{i\theta}}^{*\boldsymbol{l}} = M_{e^{i\theta_1}}^{*\boldsymbol{l}_1} \cdots M_{e^{i\theta_n}}^{*\boldsymbol{l}_n}$.

In the following we prove a generalization of Brown and Halmos characterization [173] of Toeplitz operators on $H^2(\mathbb{D})$. This result should be compared with the algebraic characterization of Guo and Wang [184] which states that T in $B(H^2(\mathbb{D}^n))$ is a Toeplitz operator if and only if $T_\varphi^* T T_\varphi = T$ for all inner function $\varphi \in H^\infty(\mathbb{D}^n)$.

Theorem (4.3.2)[169]: Let $T \in B(H^2(\mathbb{D}^n))$. Then T is a Toeplitz operator if and only if $T_{z_j}^* T T_{z_j} = T$ for all $j = 1, \dots n$.

Proof. For each $k \in \mathbb{Z}_+$, define $k_d \in \mathbb{Z}_+^n$ by $\boldsymbol{k}_d = (k, \dots, k)$. From $T_{z_j}^* T T_{z_j} = T, j = 1, \dots n$, we obtain that

$$T_z^{*k_d} T T_z^{k_d} = T,$$

which implies that

$$\begin{aligned}\langle T e_{i+k_d}, e_{i+k_d}\rangle &= \langle T T_z^{k_d} e_i, T_z^{k_d} e_j\rangle \\ &= \langle T e_i, e_j\rangle,\end{aligned}$$

for all $k \in \mathbb{Z}_+$ and $i, j \in \mathbb{Z}_+^n$. Now for each $\boldsymbol{l}, \boldsymbol{m} \in \mathbb{Z}^n$, there exists $\boldsymbol{t} = (t_1, \dots t_n) \in \mathbb{Z}_+^n$ such that $\boldsymbol{l} + \boldsymbol{k}_d, \boldsymbol{m} + \boldsymbol{k}_d \in \mathbb{Z}_+^n$ for all $\boldsymbol{k}_d \geq \boldsymbol{t}$ (that is, $k \geq t_j$ for all $j = 1, \dots, n$). Hence setting

$$A_k = M_{e^{i\theta}}^{*k_d} T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{k_d},$$

for each $k \geq 1$, we have

$$\begin{aligned}\langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}}\rangle_{L^2(\mathbb{T}^n)} &= \langle T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{k_d} e_{\boldsymbol{l}}, M_{e^{i\theta}}^{k_d} e_{\boldsymbol{l}}\rangle_{L^2(\mathbb{T}^n)} \\ &= \langle T P_{H^2(\mathbb{D}^n)} e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d}\rangle_{L^2(\mathbb{T}^n)},\end{aligned}$$

and therefore, for all $\boldsymbol{k}_d \geq \boldsymbol{t}$, we have that

$$\begin{aligned}\langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}}\rangle_{L^2(\mathbb{T}^n)} &= \langle T e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d}\rangle_{H^2(\mathbb{D}^n)} \\ &= \langle T e_{\boldsymbol{l}+\boldsymbol{t}}, e_{\boldsymbol{m}+\boldsymbol{t}}\rangle_{H^2(\mathbb{D}^n)}.\end{aligned}$$

This implies in particular that

$$\langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}} \rangle \to \langle T e_{\boldsymbol{l}+\boldsymbol{t}}, e_{\boldsymbol{m}+\boldsymbol{t}} \rangle \; as \; k \to \infty.$$

Let the bilinear form η on the linear span of $\{e_s : \boldsymbol{s} \in \mathbb{Z}^n\}$ be defined by

$$\eta(e_{\boldsymbol{l}}, e_{\boldsymbol{m}}) = \lim_{k\to\infty} \langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}} \rangle,$$

for all $\boldsymbol{l}, \boldsymbol{m} \in \mathbb{Z}^n$. Since $\|A_k\| \leq \|T\|, k \geq 1$, it follows that η is a bounded bilinear form.

Therefore, η can be extended to a bounded bilinear form (again denoted by η) on all of $L^2(\mathbb{T}^n)$, and hence there exists a unique bounded linear operator A_∞ on $L^2(\mathbb{T}^n)$ such that

$$\eta(f, g) = \langle A_\infty f, g \rangle = \lim_{k\to\infty} \langle A_k f, g \rangle,$$

for all $f, g \in L^2(\mathbb{T}^n)$. Now let $j \in \{1, \ldots, n\}, \boldsymbol{l}, \boldsymbol{m} \in \mathbb{Z}^n$ and set

$$\epsilon_j = \left(0, \ldots, \underbrace{1}_{j^{th}\; place}, \ldots, 0\right).$$

Then for all k sufficiently large (depending on $\boldsymbol{l}, \boldsymbol{m}$ and), we have

$$\begin{aligned}\langle \left(M^{*\boldsymbol{k}_d}_{e^{i\theta}} T P_{H^2(\mathbb{D}^n)} M^{*\boldsymbol{k}_d}_{e^{i\theta}}\right) e_{\boldsymbol{l}+\epsilon_j}, e_{\boldsymbol{m}+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} &= \langle T P_{H^2(\mathbb{D}^n)} e_{\boldsymbol{l}+\boldsymbol{k}_d+\epsilon_j}, e_{\boldsymbol{m}+\boldsymbol{k}_d+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle T e_{\boldsymbol{l}+\boldsymbol{k}_d+\epsilon_j}, e_{\boldsymbol{m}+\boldsymbol{k}_d+\epsilon_j} \rangle_{H^2(\mathbb{D}^n)} \\ &= \langle T^*_{z_j} T T_{z_j} e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d} \rangle_{H^2(\mathbb{D}^n)} \\ &= \langle T e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d} \rangle_{H^2(\mathbb{D}^n)} \\ &= \langle A_{\boldsymbol{k}} e_{\boldsymbol{l}}, e_{\boldsymbol{m}} \rangle_{L^2(\mathbb{T}^n)}.\end{aligned}$$

Therefore

$$\begin{aligned}\langle A_\infty e_{\boldsymbol{l}+\epsilon_j}, e_{\boldsymbol{m}+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} &= \lim_{k\to\infty} \langle M^{*\boldsymbol{k}_d}_{e^{i\theta}} T P_{H^2(\mathbb{D}^n)} M^{*\boldsymbol{k}_d}_{e^{i\theta}} e_{\boldsymbol{l}+\epsilon_j}, e_{\boldsymbol{m}+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle A_\infty e_{\boldsymbol{l}}, e_{\boldsymbol{m}} \rangle_{L^2(\mathbb{T}^n)},\end{aligned}$$

and consequently $M^*_{e^{i\theta_j}} A_\infty M_{e^{i\theta_j}} = A_\infty$, that is, $A_\infty M_{e^{i\theta_j}} = M_{e^{i\theta_j}} A_\infty$. Hence there exists φ in $L^\infty(\mathbb{T}^n)$ such that $A_\infty = M_\varphi$ [187]. Finally, we note that for $f, g \in H^2(\mathbb{D}^n)$,

$$\begin{aligned}\langle A_\infty f, g \rangle_{L^2(\mathbb{T}^n)} &= \lim_{k\to\infty} \langle M^{*\boldsymbol{k}_d}_{e^{i\theta}} T P_{H^2(\mathbb{D}^n)} M^{*\boldsymbol{k}_d}_{e^{i\theta}} f, g \rangle_{L^2(\mathbb{T}^n)} \\ &= \lim_{k\to\infty} \langle T_z^{*\boldsymbol{k}_d} T_z^{\boldsymbol{k}_d} f, g \rangle_{H^2(\mathbb{D}^n)},\end{aligned}$$

that is,

$$\langle A_\infty f, g \rangle_{L^2(\mathbb{T}^n)} = \langle T f, g \rangle_{H^2(\mathbb{D}^n)},$$

and hence

$$\begin{aligned}\langle P_{H^2(\mathbb{D}^n)} A_\infty f, g \rangle_{H^2(\mathbb{D}^n)} &= \langle A_\infty f, g \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle T f, g \rangle_{H^2(\mathbb{D}^n)}.\end{aligned}$$

Therefore, $T = P_{H^2(\mathbb{D}^n)} A_\infty |_{H^2(\mathbb{D}^n)} = P_{H^2(\mathbb{D}^n)} M_\varphi |_{H^2(\mathbb{D}^n)}$, that is, T is a Toeplitz operator.

Conversely, let $\varphi \in L^\infty(\mathbb{T}^n)$ and $T = P_{H^2(\mathbb{D}^n)} M_\varphi |_{H^2(\mathbb{D}^n)}$ Then for $f, g \in H^2(\mathbb{D}^n)$ and $j = 1, \ldots n$, we have

$$\begin{aligned}\langle \left(T^*_{z_j} T T_{z_j}\right) f, g \rangle_{H^2(\mathbb{D}^n)} &= \langle \varphi e^{i\theta_j} f, e^{i\theta_j} g \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle \varphi f, g \rangle_{L^2(\mathbb{T}^n)},\end{aligned}$$

that is,

$$\langle\left(T_{z_j}^{*}TT_{z_j}\right)f,g\rangle_{H^2(\mathbb{D}^n)}=\langle P_{H^2(\mathbb{D}^n)}M_{\varphi}|_{H^2(\mathbb{D}^n)}f,g\rangle_{H^2(\mathbb{D}^n)},$$

and therefore $T_{z_j}^{*}TT_{z_j}=T$ for all $j = 1,\dots n$, as desired.

We now characterize compact operators on $H^2(\mathbb{D}^n)$ in terms of the multiplication operators $\{T_{z_1},\dots,T_{z_n}\}$. This characterization was proved by Feintuch [181] in the case of $n = 1$.

Theorem (4.3.3)[169]: A bounded linear map T on $H^2(\mathbb{D}^n)$ is compact if and only if $T_{z_i}^{*m}TT_{z_j}^{m}\to 0$ in norm for all $i,j \in \{1,\dots,n\}$.

Proof. Let T on $H^2(\mathbb{D}^n)$ be a bounded operator. First observe that for each $m\ge 1$, we have

$$T_z^{m}\,T_{m_z}^{*}=I_{H^2(\mathbb{D})}-P_{\mathcal{F}_m},$$

Where

$$\mathcal{F}_m=\mathbb{C}\oplus z\mathbb{C}\oplus\cdots\oplus z^{m-1}\mathbb{C},$$

is an m-dimensional subspace of $H^2(\mathbb{D})$. For each $m \ge 1$, set

$$F_m=\prod_{i=1}^{n}\left(I_{H^2(\mathbb{D}^n)}-T_{z_i}^{m}T_{z_i}^{*m}\right).$$

Then

$$F_m=\prod_{i=1}^{n}\left(I_{H^2(\mathbb{D})}\otimes\cdots\otimes\underbrace{\left(I_{H^2(\mathbb{D})}-T_z^{m}\,T_z^{*m}\right)}_{i^{th}place}\otimes\cdots\otimes I_{H^2(\mathbb{D})}\right)$$
$$=\prod_{i=1}^{n}\left(I_{H^2(\mathbb{D})}\otimes\cdots\otimes\underbrace{\left(P_{\mathcal{F}_m}\right)}_{i^{th}place}\otimes\cdots\otimes I_{H^2(\mathbb{D})}\right)$$
$$=P_{\mathcal{F}_m}\otimes\cdots\otimes P_{\mathcal{F}_m},$$

which gives that F_m is a finite rank operator and hence

$$\tilde{F}_m=TF_m+F_mT-F_mTF_m,$$

is a finite rank operator, $m\ge 1$. Moreover

$$T-\tilde{F}_m=T-(TF_m+F_mT-F_mTF_m)$$
$$=\left(I_{H^2(\mathbb{D}^n)}-F_m\right)T\left(I_{H^2(\mathbb{D}^n)}-F_m\right).$$

Finally, observe that

$$I_{H^2(\mathbb{D}^n)}-F_m=\sum_{1\le i_1<\cdots<i_l\le n}(-1)^{l+1}T_{z_{i_1}}^{m}\cdots T_{z_{i_l}}^{m}T_{z_{i_1}}^{*m}\cdots\ T_{z_{i_l}}^{*m}$$
$$=\sum_{1\le i_1<\cdots<i_l\le n}(-1)^{l+1}\left(T_{z_{i_1}}\cdots T_{z_{i_l}}\right)^{m}\left(T_{z_{i_1}}\cdots T_{z_{i_l}}\right)^{*m},$$

for all $m\ge 1$. Hence, by hypothesis and the triangle inequality we have

$$\|T-\tilde{F}_m\|=\left\|\left(I_{H^2(\mathbb{D}^n)}-F_m\right)T\left(I_{H^2(\mathbb{D}^n)}-F_m\right)\right\|\to\ 0,$$

as $m\to\infty$, that is, T is a compact operator.

The converse follows from Lemma (4.3.1). This completes the proof.

In view of the preceding theorem, it seems reasonable to define asymptotic Toeplitz operators as follows (compare this with Feintuch [181] and Barr´ıa and Halmos [171]):

Definition (4.3.4)[169]: A bounded linear operator T on $H^2(\mathbb{D}^n)$ is said to be an asymptotic Toeplitz operator if there exists $A \in B(H^2(\mathbb{D}^n))$ such that $T_{z_i}^{*m} TT_{z_i}^m \to A$ and $T_{z_i}^{*m}(T-A)T_{z_j}^m \to 0$ as $m \to \infty$ in norm, $1 \le i,j \le n$.

We close by characterizing asymptotic Toeplitz operators on $H^2(\mathbb{D}^n)$ as analogous characterization of asymptotic Toeplitz operators on $H^2(\mathbb{D})$ (see [181]).

Theorem (4.3.5)[169]: Let T be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then T is an asymptotic Toeplitz operator if and only if T is a compact perturbation of Toeplitz operator.

Proof. Let $A \in B(H^2(\mathbb{D}^n)), T_{z_i}^{*m}TT_{z_i}^m \to A$ and $T_{z_i}^{*m}(T-A)T_{z_j}^m \to 0$ in norm, as $m \to \infty$, and $1 \le i,j \le n$. Then for all $m \ge 1$,

$$\left\|A - T_{z_j}^* A T_{z_j}\right\| \le \left\|A - T_{z_j}^{*(m+1)} TT_{z_j}^{m+1}\right\| + \left\|T_{z_j}^{*(m+1)} TT_{z_j}^{m+1} - T_{z_j}^* A T_{z_j}\right\|$$
$$\le \left\|A - T_{z_j}^{*(m+1)} TT_{z_j}^{m+1}\right\| + \left\|T_{z_j}^{*m} TT_{z_j}^m - A\right\|,$$

yields $T_{z_j}^* A T_{z_j} = A$ for all $j = 1, \dots, n$. Also by Theorem (4.3.3), $T - A$ is compact on $H^2(\mathbb{D}^n)$.

The converse follows from Lemma (4.3.1) and Theorem (4.3.2). This completes the proof.

The more interesting question now is to describe bounded linear operators T on $H^2(\mathbb{D}^n)$ (in terms of Toeplitz and Hankel operators) such that $T_{z_i}^{*m} TT_{z_i}^m \to A$ and $T_{z_i}^{*m}(T-A)T_{z_j}^m \to 0$ for some $A \in B(H^2(\mathbb{D}^n))$ and as $m \to \infty, 1 \le i,j \le n$, in the weak or strong operator topology.

We extend some of the results in the case when the ambient operator is the compression of $(T_{z_1}, \dots, T_{z_n})$ to a quotient space of $H^2(\mathbb{D}^n)$, that is, a joint $(T_{z_1}^*, \dots, T_{z_n}^*)$ $-$invariant closed subspace of $H^2(\mathbb{D}^n)$. Note that a rich source of n $-$tuples of commuting contractions comes from quotient Hilbert spaces of $H^2(\mathbb{D}^n)$.

Let Q be a joint $(T_{z_1}^*, \dots, T_{z_n}^*)$ $-$invariant subspace of $H^2(\mathbb{D}^n)$. Set

$$C_{z_i} = P_Q T_{z_i}|_Q,$$

for all $i = 1, \dots, n$. Note that $Q^\perp$ is a joint invariant subspace for $(T_{z_1}, \dots, T_{z_n})$ and so

$$C_{z_i}^* = T_{z_i}^*|_Q \in B(Q).$$

In the case $n = 1, C_z$ is called a Jordan block [187]. In the several variables quotient space setting, we have the following analogue of Theorem (4.3.5).

Theorem (4.3.6)[169]: Let $T, A \in B(Q)$. Then $C_{z_i}^{*m} TC_{z_i}^m \to A$ and $C_{z_i}^{*m}(T-A)C_{z_j}^m \to 0$ in norm for all $i,j = 1, \dots, n$ if and only if $T = A + K$, where $K \in B(Q)$ is a compact operator and $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \dots n$.

Proof. We first note that, as in the proof of Theorem (4.3.5), the assumption $C_{z_i}^{*m}TC_{z_i}^m \to A$ as $m \to \infty$ implies that

$$C_{z_i}^* A C_{z_i} = A,$$

for all $i = 1, \dots n$. Now it follows from the definition of C_{z_i} that

$$C_{z_i}^{*m} = T_{z_i}^{*m}|_Q,$$

and hence

$$C_{z_i}^{*m}(T-A)C_{z_j}^m = T_{z_i}^{*m}(T-A)P_Q T_{z_j}^m|_Q,$$

for all $i, j = 1, \ldots, n$ and $m \geq 1$. By once again using the fact that

$$P_Q T_{z_j}^m P_Q = P_Q T_{z_j}^m,$$

one sees that

$$T_{z_i}^{*m}(T - A) P_Q T_{z_j}^m = T_{z_i}^{*m}(T - A) P_Q T_{z_j}^m P_Q.$$

Hence $C_{z_i}^{*m}(T - A) C_{z_j}^m \to 0$ in $B(Q)$ if and only if $T_{z_i}^{*m}(T - A) P_Q T_{z_j}^m \to 0$ in $B\big(H^2(\mathbb{D}^n)\big)$ as $m \to \infty$.

Therefore, if $C_{z_i}^{*m}(T - A) C_{z_j}^m \to 0$ as $m \to \infty$ in norm for all $i, j = 1, \ldots, n$, then $T_{z_i}^{*m}(T - A) P_Q T_{z_j}^m \to 0$ in $B\big(H^2(\mathbb{D}^n)\big)$ as $m \to \infty$, and consequently by Theorem (4.3.3), $(T - A)|_Q$ is a compact operator on $H^2(\mathbb{D}^n)$. Therefore

$$(T - A) = (T - A)|_Q,$$

is a compact operator on Q, which proves the necessary part.

Conversely, let $T - A$ be a compact operator on Q and $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots n$.

Since $C_{z_i}^{*m} \to 0$ as $m \to \infty$ in the strong operator topology, Lemma (4.3.1) implies that

$$C_{z_i}^{*m}(T - A) C_{z_j}^m \to 0,$$

as $m \to \infty$. In particular, for all $i = 1, \ldots n$

$$C_{z_i}^{*m} T C_{z_i}^m \to C_{z_i}^{*m} A C_{z_i}^m.$$

But $C_{z_i}^* A C_{z_i} = A, i = 1, \ldots n$, yields us

$$C_{z_i}^{*m} T C_{z_i}^m \to A.$$

This completes the proof.

Considering the particular case $Q_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n)$, the so called Beurling type quotient space of $H^2(\mathbb{D}^n)$, where $\theta \in H^\infty(\mathbb{D}^n)$ is an inner function, we get the following result.

Theorem (4.3.7)[169]: Let $\theta \in H^\infty(\mathbb{D}^n)$ be an inner function and $Q_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n)$ and $A \in B(Q_\theta)$. Then $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots n$, if and only if $A = 0$.

Proof. Let $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots, n$. Since

$$Q_\theta^\perp = \theta H^2(\mathbb{D}^n),$$

is a joint invariant subspace for $\big(T_{z_1}, \ldots, T_{z_n}\big)$, it follows that

$$P_{Q_\theta} T_{z_i}^*|_{Q_\theta} = T_{z_i}^* P_{Q_\theta},$$

and hence

$$\begin{aligned} A P_{Q_\theta} &= \big(C_{z_i}^* A C_{z_i}\big) P_{Q_\theta} \\ &= \big(P_{Q_\theta} T_{z_i}^*|_{Q_\theta} A P_{Q_\theta} T_{z_i}|_{Q_\theta}\big) P_{Q_\theta} \\ &= T_{z_i}^* A P_{Q_\theta} T_{z_i} P_{Q_\theta} \\ &= T_{z_i}^* A P_{Q_\theta} T_{z_i} \\ &= T_{z_i}^* \big(A P_{Q_\theta}\big) T_{z_i}. \end{aligned}$$

for all $i = 1, \ldots, n$. This and Theorem (4.3.2) implies that $A P_{Q_\theta}$ is a Toeplitz operator. Consequently, there exists $\psi \in L^\infty(\mathbb{T}^n)$ such that

$$A P_{Q_\theta} = T_\psi.$$

On the other hand, since T_θ is an analytic Toeplitz operator, it follows that

$$AP_Q T_\theta = 0.$$

Hence, using [Theorem 1, C. Gu [183]], we conclude that

$$\begin{aligned} T_{\psi\theta} &= T_\psi T_\theta \\ &= AP_{Q_\theta} T_\theta \\ &= 0. \end{aligned}$$

This completes the proof of the theorem.

Summing up the above two results and Lemma (4.3.1), we have the following generalization of Theorem 1.2 in [174].

Theorem (4.3.8)[169]: For an inner function $\theta \in H^\infty(\mathbb{D}^n)$ and bounded linear operators T and A on $Q_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n)$, the following are equivalent:

(i) $C_{z_i}^{*m} T C_{z_i}^m \to A$ and $C_{z_i}^{*m}(T - A)C_{z_j}^m \to 0$ in norm for all $i, j = 1, \ldots, n$;

(ii) $C_{z_i}^{*m}\ TCmzi \to\ 0$ in norm for all $i\ = 1, \ldots, n$;

(iii) T is compact.

For asymptotic Toeplitzness of composition operators on the Hardy space of the unit sphere in $\mathbb{C}^n$ see Nazarov and Shapiro [188], and Cuckovic and Le [176].

We characterize the compact operators on the model space $H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^p}(\mathbb{D})$, where $\Theta \in H^\infty_{B(\mathbb{C}^p)}(\mathbb{D})$ is an inner function. We note that this result for p = 1 case can be found in [174]. Moreover, our proof seems more shorter and conceptually different (for instance, compare Theorem 5.5 with Proposition 2.10 in [174]).

We begin with the definition of a Toeplitz operator with operator-valued symbol.

Definition (4.3.9)[169]: Let $\mathcal{E}$ be a Hilbert space. A bounded linear operator T on $H^2_{\mathcal{E}}(\mathbb{D})$ is said to be Toeplitz if there exists an operator-valued function Φ in $L^\infty_{B(\mathcal{E})}(\mathbb{T})$ such that $T = P_{H^2_{\mathcal{E}}(\mathbb{D})} M_\Phi|_{H^2_{\mathcal{E}}(\mathbb{D})}$.

Here let us observe, before we proceed further, the following characterization of Toeplitz operators on a vector-valued Hardy space. Since the result follows from concepts and techniques used in the proof of Theorem (4.3.2), we give a sketch of the proof.

Theorem (4.3.10)[169]: Let $\mathcal{E}$ be a Hilbert space and $T \in B\left(H^2_{\mathcal{E}}(\mathbb{D})\right)$. Then T is a Toeplitz operator if and only if $T_z^*\ TT_z = T$.

Proof. Note first that $\{e_m\eta : m \in \mathbb{Z}, \eta \in \mathcal{E}\}$ is a total set in $L^2_{\mathcal{E}}(\mathbb{D})$, where $e_m = e^{im\theta}, m \in \mathbb{Z}$. For each $k \geq\ 1$, set

$$A_k = M_{e^{i\theta}}^{*k} TPM_{e^{i\theta}}^k,$$

where $M_{e^{i\theta}}$ is the bilateral shift on $L^2_{\mathcal{E}}(\mathbb{T})$ and P is the orthogonal projection from $L^2_{\mathcal{E}}(\mathbb{T})$ onto $H^2_{\mathcal{E}}(\mathbb{D})$. If $T_z^*\ TT_z = T$ and $k \in \mathbb{Z}_+$, then

$$\langle Te_{i+k}\eta, e_{j+k}\zeta\rangle = \langle Te_i\eta, e_j\zeta\rangle,$$

for all $i, j\ \geq\ 0$. Then for each $l, m \in \mathbb{Z}$, as in the proof of Theorem (4.3.2), there exists $t\ \geq\ 0$ such that $l + k, m +\ k \geq 0$ for all $k \geq t$, and so

$$\langle A_k e_l\eta, e_m\zeta\rangle \to \langle Te_{l+t}\eta, e_{m+t}\zeta\rangle,$$

as $k \to \infty$. Then

$$(e_l\eta, e_m\zeta) \longmapsto \lim_{k\to\infty} \langle A_k e_l\eta, e_m\zeta\rangle,$$

defines a bounded bilinear form on the span of $\{e_l\eta\colon\ l \in \mathbb{Z}, \eta \in \mathcal{E}\}$.

Therefore, there exists (again, following the proof of Theorem (4.3.2)) $A_\infty \in B\left(L^2_{\mathcal{E}}(\mathbb{T})\right)$ such that

$$\langle A_\infty f, g\rangle = \lim_{k\to\infty}\langle A_k f, g\rangle,$$

for all $f, g \in L^2_{\mathcal{E}}(\mathbb{T})$. This yields

$$A_\infty M_{e^{i\theta}} = M_{e^{i\theta}} A_\infty.$$

Hence there exists a $\Phi \in L^2_{B(\mathcal{E})}(\mathbb{T})$ such that

$$A_\infty = M_\Phi,$$

and hence

$$T = P_{H^2_{\mathcal{E}}(\mathbb{D})} M_\Phi|_{H^2_{\mathcal{E}}(\mathbb{D})}.$$

The proof of the converse part proceeds verbatim as that of Theorem (4.3.2). This completes the proof of the theorem.

Following Feintuch [181] we now define an asymptotic Toeplitz operator on a vector-valued Hardy space.

Definition (4.3.11)[169]: Let $\mathcal{E}$ be a Hilbert space. A bounded linear operator T on $H^2_{\mathcal{E}}(\mathbb{D})$ is said to be an asymptotic Toeplitz operator if there exists $A \in B\left(H^2_{\mathcal{E}}(\mathbb{D})\right)$ such that $T_z^{*m}\, T T_z^m \to A$ as $m \to \infty$ in norm.

In the theorem below, we generalize the Feintuch's characterization [181] (see also Theorem F, page 195, [188]) of asymptotic Toeplitz operators on Hardy space to asymptotic Toeplitz operators on $\mathbb{C}^p$ −valued Hardy space. However, the method of proof here is adapted from the original proof by Feintuch.

Theorem (4.3.12)[169]: Let $T, A \in B\left(H^2_{\mathbb{C}^p}(\mathbb{D})\right)$. Then $T_z^{*m}\, T T_z^m \to A$ in norm if and only if A is a Toeplitz operator and $(T - A)$ is compact.

Proof. Suppose that $T_z^{*m} T T_z^m \to A$ in norm. It follows that

$$\left\|T_z^{*(m+1)}\, T T_z^{m+1} - T_z^* A T_z\right\| \le \|T_z^{*m}\, T T_z^m - A\| \to 0$$

as $m \to \infty$. This and the triangle inequality yields $A = T_z^*\, A T_z$. Now let $R_m = T_z^m\, T_z^{*m}$ and

$$Q_m = I - R_m.$$

Further, let $P_{\mathbb{C}_p}$ denote the orthogonal projection of $H^2_{\mathbb{C}_p}(\mathbb{D})$ onto the space of ($\mathbb{C}_p$-valued) constant functions. Since $T_z T_z^* = I_{H^2_{\mathbb{C}_p}(\mathbb{D})} - P_{\mathbb{C}_p}$, it follows that

$$Q_m = \sum_{k=0}^{m-1} T_z^k P_{\mathbb{C}_p} T_z^{*k} (m \ge 1).$$

Then $Q_m, m \ \ge\ 1$, is a finite rank operator, and therefore

$$F_m = (T - A)Q_m + Q_m(T - A) - Q_m(T - A)Q_m (m \ge 1),$$

is also a finite rank operator. Moreover

$$(T - A) - F_m = R_m(T - A)R_m (m \ge 1),$$

yields

$$\|(T - A) - F_m\| = \|R_m(T - A)R_m\| \le \|T_z^{*m}\, T T_z^m - A\| \to 0,$$

as $m \to \infty$. So $T - A$ is compact as desired.

The converse follows from Lemma (4.3.1). This completes the proof.

Given a Hilbert space $\mathcal{E}$ and an inner multiplier $\Theta \in H^{\infty}_{B(\mathcal{E})}(\mathbb{D})$, the model space Q_Θ and the model operator S_Θ are defined by

$$Q_\Theta = H^2_{\mathcal{E}}(\mathbb{D}) \ominus \Theta H^2_{\mathcal{E}}(\mathbb{D}),$$

and

$$S_\Theta = P_{Q_\Theta} M_z|_{Q_\Theta},$$

respectively. Model spaces (and hence model operators) represent a wide and very important class of bounded linear operators [187]. We have the following result in the model space setting.

Proposition (4.3.13)[169]: Let $\Theta \in H^{\infty}_{B(\mathcal{E})}(\mathbb{D})$ be an inner multiplier and $T \in B(Q_\Theta)$. Assume that $\Theta(e^{i\theta})$ is invertible a.e. Then $S^*_\Theta T S_\Theta = T$ if and only if $T = 0$.

Proof. The proof goes exactly along the same lines as the proof of Theorem (4.3.7). Since

$$TP_{Q_\Theta} = T^*_z\left(TP_{Q_\Theta}\right)T_z,$$

it follows from Theorem (4.3.10) that TP_Q is a Toeplitz operator. Consequently, there exists $\Psi \in L^{\infty}_{B(\mathcal{E})}(\mathbb{T})$ [187] such that

$$TP_{Q_\Theta} = T_\Psi.$$

Since T_Θ is an analytic Toeplitz operator, again as in the proof of Theorem (4.3.7), it follows that

$$T_{\Psi\Theta} = 0,$$

and hence

$$\Psi\Theta = 0.$$

Since Θ is invertible a.e., it follows that $\Psi = 0$ a.e. and hence $T = 0$. This completes the proof.

Not only is this proposition a considerable generalization of Proposition 2.10 of [174], but our proof is much simpler. The principal tool is the identity $S^*_\Theta = T^*_z|_{Q_\Theta}$.

We have the following characterization which generalizes the characterization of compact operators on Q_Θ for $p = 1$ (see the implication (i) and (iii) in Theorem 1.2 in [174]).

Theorem (4.3.14)[169]: Let $\Theta \in H^{\infty}_{B(\mathbb{C}^p)}(\mathbb{D})$ be an inner multiplier and $T \in B(Q_\Theta)$. Then T is compact if and only if $\{S^{*m}_\Theta T S^m_\Theta\}_{m\geq 1}$ converges in norm.

Proof. If T is compact on Q_Θ, then by Lemma (4.3.1), $\|S^{*m}_\Theta T S^m_\Theta\| \to 0$ as $m \to \infty$. To prove the converse, let $A \in B(Q_\Theta)$ and $S^{*m}_\Theta T S^m_\Theta \to A$, as $m \to \infty$, in norm. Then by the same argument used in the proof of Theorem (4.3.6), we have $S^*_\Theta A S_\Theta = A$. It now follows from Proposition (4.3.13) that $A = 0$ and therefore $T^{*m}_z T P_{Q_\Theta} T^m_z \to 0$ as $m \to \infty$. Now Theorem (4.3.12) implies that TP_{Q_Θ} is a compact operator on $H^2_{\mathbb{C}^p}(\mathbb{D})$. Therefore $T = TP_{Q_\Theta}$ is a compact operator on Q_Θ. This completes the proof.

Theorem (4.3.14) and Lemma (4.3.1) give us the following generalization of Theorem 1.2 in [174].

Theorem (4.3.15)[169]: Let $\Theta \in H^{\infty}_{B(\mathbb{C}^p)}(\mathbb{D})$ be an inner multiplier and $T \in B(Q_\Theta)$. Then the following are equivalent:

(i) $\{S^{*m}_\Theta T S^m_\Theta\}_{m\geq 1}$ converges in norm;
(ii) $S^{*m}_\Theta T S^m_\Theta \to 0$ in norm;
(iii) T is a compact operator.

Chapter 5
Hardy Space over the Bidisk

We show some results reflect the two variable nature of $H^2(D^2)$. We show that manifest a close tie between the operator theory in $H^2(D^2)$ and classical single operator theory. The unilateral shift of a finite multiplicity and the Bergman shift will be used as examples to illustrate some of the results. We first show the Coburn type theorem fails generally on the bidisk. But, we show that certain pluriharmonic symbols or product symbols of one variable functions induce Toeplitz operators satisfying the Coburn type theorem.

Section (5.1): Operator Theory:

Non-selfadjoint operator theory has been greatly enriched by the introduction of Hilbert spaces of analytic functions. On the one hand, analytic function theory maizes it possible to reformulate and solve many classical operator theoretical problems; on the other hand, it opens many new fields of study in which algebra, geometry and topology also play fundamental roles. A very illustrative example is the study of the *unilateral shift operator* of the *Hardy space* over the unit disk, the results of which have found many important applications. In recent years, many attempts have been made to explore a multi-variate analogue of this study. One line is the study of commuting operator tnples in which the *dilation* (cf. [203]), *joint similarity* (cf. [213]), *joint hyponormality* (cf. [202][214][215]), *joint spectrum* (cf. [199][201]) and *functional calculus* (cf. [201]) are very interesting topics. Another line is the coordinate free approach in which the language of module theory is adopted (cf. [198[200][205][206][212]). This module language emphasizes some key problems in the multlvariate operator theory from a module theoretical viewpoint and makes clear its connections with algebraic geometry and commutative algebra.

[207] and [217] start a project of building a systematic operator theory in $H^2(D^2)$. This project is based on the module language. Its ultimate goal is to make $H^2(D^2)$ into a concrete model in multi-varlable operator theory in which, on the one hand, the two variable nature of $H^2(D^2)$ has a clear operator theoretical representation; on the other hand, the transition from single operator theory to a multivariable theory becomes natural. The study of $H^2(D^2)$, and in general $H^2(D^n)$, is not new. Its function theory was laid down in [215], and some operator theoretical problems were studied, see [192], [196], [197], [208], [209], and [210]. But the operator theory in $H^2(D^2)$ is still far from being fully developed.

We devoted to a study of the evaluation operator. In the process of exhibiting elementary properties of the evaluation operator in $H^2(D^2)$, some general techniques are also developed. These techniques are used to obtain results in other topics.

We give an interpretation of characteristic operator function in $H^2(D^2)$ using evaluation operator. This interpretation is a basis for the development of some useful techniques.

Difference quotient operator is closely related to the evaluation operator. Its properties are used in many places.

Results are used to prove a spectral equivalence in our setting of $H^2(D^2)$. This proof is another important source of techniques.

Some sufficient conditions for the compactness of evaluation operators on quotient modules $H^2(D^2) \ominus M$ are studied. A necessary condition will be given.

Multiplications by coordinate functions z and w in $H^2(D^2)$ are two unilateral shift operators of infinite multiplicity. We study compressions of the two shift operators to quotient spaces. Many results will be used here. In functional model theory, compressions of the shift operators to quotient spaces serve as canonical models for a large class of operators.

From this point of view, the study, has a useful generality. We will take the Bergman shift and the unilateral shift of finite multiplicity as examples to illustrate some of the results.

We let C denote the complex plane and $C^{\,2}$ be the Cartesian product of 2 copies of C. The points of $C^{\,2}$ are thus the ordered 2-tuples (z, w). $Z_+$is the set of nonnegative integers. D^2 will be the unit bidisk in $C^{\,2}$ with distinguished boundary T^2, where D is the nnit disk and T is the unit circle. $\frac{|dz|}{2\pi}$ denotes the normalized Lebesgue measure on the unit circle T and $dm := \frac{|dz|}{2\pi}\frac{|dw|}{2\pi}$ be the product measure on the torus T^2. $H^2(D^2)$, which is equal to $H^2(D)\otimes H^2(D)$, is the Hardy space over the bidisk. No distinction will be made between $H^2(D^2)$ and $H^2(T^2)$. Bidisk algebra $A(D^2)$ is the closure of polynomials in z and w under the norm of $C(T^2)$.

$A(D^2)$ acts on $H^2(D^2)$ by pointwise multiplication of functions which turns $H^2(D^2)$ into an $A(D^2)$ module. A closed subspace M of $H^2(D^2)$ is a *submoduIe* if M is invariant under the module action, or equivalently, M is invariant under multiplications by both z and w(denoted by T_z and T_w respectively). A subspace invariunt for $T_z(or T_w)$alone will be called *z-lnvariant (or respectively w-invariant).* For any subset $X \subset H^2(D^2)$, we let *clos*$\{X\}$ denote the closure of X in $H^2(D^2)$ and

$$[X] := \text{clos}\{\text{span}\{A(D^2)X\}\}$$

denote the submodule generated by X. For example [h] is the submoduIe generated by function h.

A function $h \in H^2(D^2)$ is said to be *inner* if $|h(z,w)|$ is almost everywhere equal to 1 on t^2; and it is said to be *H-outer* if $[h] = H^2(D^2)$. It is easy to see that when h is inner, $[h] = hH^2(D^2)$.

If we denote $H^2(D^2) \ominus zH^2(D^2)$ by H_w and $H^2(D^2) \ominus wH^2(D^2)$ by H_z, then

$$H_w = \text{clos}\{\text{span}\{w^j : j \geq 0\}\}, \qquad H_z = \text{clos}\{\text{span}\{z^j : j \geq 0\}\}.$$

One sees that both H_w *and* H_z are the Hardy space over the unit disk, but they are different subspaces in $H^2(D^2)$. These two subspaces will be used in the definition of evaluation operators and some other places.

If M is a closed proper subspace of $H^2(D^2)$ and

$$p : H^2(D^2) \to M, \qquad q : H^2(D^2) \to H^2(D^2) \ominus M$$

are orthogonal projections, then we define a map

$$S : A(D^2) \to B(H^2(D^2) \ominus M)$$

by

$$S_{fg} := qfg,$$

where $f \in A(D^2)$ and $g \in H^2(D^2) \ominus M$. One sees that the operators S_z, S_w are compressions of the Toeplitz operators T_z, T_w to $H^2(D^2) \ominus M$. When M is a submodule, S is a homomorphism which turns $H^2(D^2) \ominus M$ into a quotient $A(D^2)$-module and in particular S_z commutes with S_w.

It is easy to check that a closed subspace N of $H^2(D^2) \ominus M$ is invariant for $S_z (or S_w)$ if and only if $N \oplus M$ is z-invariant(or resp. w-invariant) in $H^2(D^2)$. We will see that S_z *and* S_w have a very close tie with the evaluation operators.

Definition (5.1.1)[191]: For every $\lambda \in D$, we define *a left evaluation* operator $L(\lambda)$ from $H^2(D^2)$ to H_w *and a right evaluation* operator $R(\lambda)$ from $H^2(D^2)$ to H_z by

$$L(\lambda)f(w) = f(\lambda, w),\ \ R(\lambda)f(z) = f(z, \lambda),\ \ f \in H^2(D^2).$$

It is easy to see that $L(\lambda)$ and $R(\lambda)$ have integral representations using the Canchy integral formula from which we see that $L(\lambda)$ and $R(\lambda)$ are operator valued analytic functions in λ and

$$\|L(\lambda)\| = \|R(\lambda)\| = (1 - |\lambda|^2)^{-1/2}.$$

As manifested in [207] and [217], evaluation operators play important roles in the study of the compression operators. On the one hand, the restriction of evaluation operators to the quotient space $M \ominus zM$ is the *characteristic operator function* of S_z; on the other hand, the restriction of evaluation operators to the quotient space $H^2(D^2) \ominus M$ is in many cases compact. These two facts lead to some interesting results.

We will be mainly interested in restrictions of $L(\lambda)$ and $R(\lambda)$ to quotient spaces like $H^2(D^2) \ominus M$ and $M \ominus zM$. For simplicity, we denote these restrictions also by $L(\lambda)$ and $R(\lambda)$ in cases in which their meanings are clear from the context.

We have the following lemma.

Lemma (5.1.2)[191]: *If* $f \in H^2(D^2)$, *then* $\|L(0)f\| = \|f\|$ *if and only if* $f \in H_w$.

We mentioned that $L(\lambda)$ is an operator-valued analytic function in λ. An operator-valued analytic function $u(z)$ over D is said to be *contractive* if $\|u(z)\| \le 1$ for every $z \in D$. If $u(z)$ is contractive, then it has non-tangential limit to almost every point in $\partial D = T$. If H is a closed subspace of $H^2(D^2)$ such that $L|_H$ is contractive and

$$\|L(0)f\| < \|f\|$$

for every nonzero $f \in H$, then $L(z)$ is said to be *purely contractive on* H. Lemma (5.1.2) shows that if $L|_H$ is contractive and H contains no nonzero function which is independent of z, then $L|_H$ is purely contractive. We will need this fact.

The following lemma in [217] will be used.

Lemma (5.1.3)[191]: *If* $M \subset H^2(D^2)$ *is z-invariant, then* $R(\lambda)$ *restricted to* $M \ominus zM$ *is Hilbert-Schmidt for every* $\lambda \in D$, *and*

$$tr\big(R^*(\lambda)R(\lambda)\big) \le (1 - |\lambda|^2)^{-1}.$$

Similarly if $M \subset H^2(D^2)$ is w-invariant, then $L(\lambda)$ restricted to $M \ominus wM$ is Hilbert-Schraidt for every $\lambda \in D$. This lemma reflects the two-variable nature of $H^2(D^2)$--if M is z-invariant, then the functions in $M \ominus zM$ depends largely on variable w and hence they do not have much 'room' to vary if w variable is fixed. For example, if $M = H^2(D^2)$, then $M \ominus zM = H_w$ and $R(\lambda)|_{M \ominus zM}$ is of rank 1 for every $\lambda \in D$.

The following lemma describes the adjoints of $L(\lambda)|_{H^2(D^2)\ominus M}$ and $R(\lambda)|_{H^2(D^2)\ominus M}$. Its proof is simple.

Lemma (5.1.4)[191]: *If* M *is a subspace of* $H^2(D^2)$ *and* $L(\lambda)$, $R(\lambda)$ *are restrictions of evaluation operators to quotient* $H^2(D^2) \ominus M$, *then for every* $\phi \in H_w$, $\psi \in H_z$,

$$L^*(\lambda)\phi = q\left(1-\bar{\lambda}z\right)^{-1}\phi,\ \ R^*(\lambda)\psi = q\left(1-\bar{\lambda}w\right)^{-1}\psi.$$

We will use this lemma often later on.

For every contraction T, one can associate with it two defect operators $D_T = (1-T^*T)^{1/2}$, $D_{T^*} = (1-TT^*)^{1/2}$, and two defect spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ which are the closure of the ranges of D_T *and* D_{T^*} respectively. The operator-valued analytic function

$$\theta_T(\lambda) = [-T + \lambda D_{T^*}(1-\lambda T^*)^{-1}]|_{\mathcal{D}_T},\quad \lambda \in D \tag{1}$$

is called the *characteristic operator function* for T.

In functional model theory (cf. [214][204][216]), the defect operators and the characteristic operator functions are very useful too]s in determining the structure of contractions. In [207] it is shown that if M is a z-invariant subspaee of $H^2(D^2)$, then the evaluation operator L is *left inner* from $M \ominus zM$ to H_w and

$$M = L(z)H^2(E),$$

where $E = M \ominus zM$. By [216], there are constant unitaries U, V, W such that

$$L(\lambda)|_{M\ominus zM} = \left(U\theta_{S_z}(\lambda)V\right) \oplus W, \tag{2}$$

and by Lemma (5.1.2), $W \neq 0$ if and only if M contains nonzero functions independent of variable.

An analytic formula for $L(\lambda)|_{M\ominus zM}$, which is parallel to the formula for θ_{S_z}, can be deduced from (2) and a known result in the vector-valued Hardy space setting (cf. Theorem 5.2 in [194]). But the following treatment fits better into the context.

We first observe that in Lemma (5.1.4),

$$L^*(\lambda)\phi = q\left(1-\bar{\lambda}z\right)^{-1}\phi = q\left(1-\bar{\lambda}z\right)^{-1}q\phi = \left(1-\bar{\lambda}S_z\right)^{-1}L^*(0)\phi,$$

and therefore we have the following

Lemma (5.1.5)[191]: *For every* $f \in H^2(D^2) \ominus M, L(\lambda)f = L(0)(1-\lambda S_z^*)^{-1}f.$

Lemma (5.1.5), apart from giving analytic representations of $L(\lambda)$, displays a connection with the compression S_z. More connections will be exihibited in Section 5. In the following corollary, $D_z f = \frac{f-f(0,\cdot)}{z}$ for every $f \in H^2(D^2)$. We will make a study of it next. Here we need the fact that when M is z-invariant D_z maps $M \ominus zM$ into $H^2(D^2) \ominus M$.

Corollary (5.1.6)[191]: *On* $M \ominus zM$

$$L(\lambda) = L(0) + \lambda L(0)(1-\lambda S_z^*)^{-1}D_z. \tag{3}$$

Proof. For $f \in M \ominus zM$, if we write

$$f(z,w) = f(0,w) + zg(z,w),$$

then $g = D_z f$ which is in $H^2(D^2) \ominus M$ by a remark following the definition of difference quotient operators. It is easy to see that

$$\begin{aligned}\lambda L(\lambda)D_z f &= \lambda g(\lambda,w)\\ &= f(\lambda,w) - f(0,w)\\ &= \left(L(\lambda)-L(0)\right)f,\end{aligned}$$

and hence

$$L(\lambda)f = L(0)f + \lambda L(\lambda)D_z f.$$

By Lemma (5.1.5),

$$L(\lambda)f = L(0)f + \lambda L(0)(1-\lambda S_z^*)^{-1}D_z f.$$

This representation of the characteristic operator function for S_z is very useful since in many cases $L(0)$ and D_z are easy to compute, and this leads to a better understanding of S_z. We will see examples later. A comparison of (3) with (1) suggests that the defect operators for S_z may have a clearer expression in terms of $L(0)$ and D_z.

If M is z-invariant, then S_z on $H^2(D^2) \ominus M$ can be very general (cf. [211][216]). In fact, if S is any contraction in class $C_{.0}$, that is $\lim_{n\to\infty} (S^*)^n \to 0$ in strong topology, then there is a z-invariaalt subspace M such that S is unitarily equivalent to S_z on $H^2(D^2) \ominus M$. But if M is a submodule, S_z is much less general, and there exist $C_{.0}$ class contractions which are not unitarily equivalent to S_z for any submodule M. The following Theorem (5.1.7) shows that submodules which make S_z compact are rare. A function ϕ is said to be a factor of M if ϕ is a factor of every function in M. A submodule M is said to be *generic* if it contains no non-trivial one-variable function and has no one-variable inner factor.

Theorem (5.1.7)[191]: *If M is a generic submodule, then S_z is not compact.*

Proof. For a generic submodule, remarks preceeding Lemma (5.1.5) says that $L|_{M\ominus=M}$ differs from tile chtaracteristic operator fuuction of S_z by constant unitaries, e.g.,

$$L(\lambda)|_{M\ominus zM} = U\theta_{S_z}(\lambda)V,$$

for some unitary operators U and V. So by the formula for θ_{S_z}, if S_z is compact then $L(0)|_{M\ominus zM}$ is compact. Since $L(0)(zM) = \{0\}, L(0)$ is compact oll M. We show that this is impossible. In fact, since functions in M do not have common factor z and

$$M = \oplus_{i=0}^{\infty} w^i (M \ominus wM),$$

we can pick $a\ f \in M \ominus wM$ such that $f(0,w) \neq 0$ and $\|f\| = 1$. One checks that $\{w^j f\ : j \geq 0\}$ is an orthonormal set and

$$\|L(0)w^j f\| = \int_T \left|w^j f(0,w)\right|^2 \frac{|dw|}{2\pi} = \int_T |f(0,w)|^2 \frac{|dw|}{2\pi},$$

for every $j \geq 0$. This means that $L(0)$ can't be compact on M.

So if S_z is compact on $H^2(D^2) \ominus M$, then the submodule M must be non-generic. Some study was made for non-generic submodules in the next. A non-generic submodule is special and also simple. However it is not clear if non-generic submodules are able to produce all compact strict contractions.

Another related question is the following

Question (5.1.8)[191]: If M is a submodule with codimension $dim(H^2(D^2) \ominus M) > 1$, then can S_z be normal?

If M is a submodule of $H^2(D^2)$, then compressions S_z and S_w on $H^2(D^2) \ominus M$ are a closely related pair. In studies of operator pairs, an important problem is the jointly-invariant subspace problem. The jointly-invariant subspace problem for the pair (S_z, S_w) is very hard. Actually it is tied up with the invariant subspace problem for Hilbert space operators.

However, if either S_z or S_w is in C_o class, which means there is a non-zero r $\psi \in H^\infty(D)$ such that either $\psi(S_z) = 0$ or $\psi(S_w) = 0$, then (S_z, S_w) has a non-trivial jointly-invariant subspace. We first state a lemma. This lemma will also be used.

Lemma (5.1.9)[191]: *If ϕ is an inner function in H_z, then*

$$H^2(D^2) \ominus \phi H^2(D^2) = (H_z \ominus \phi H_z) \otimes H_w.$$

Proof. For every $f \in H_z \ominus \phi H_z$ and $g \in H_w$,

$$< fg,\ \phi z^i w^j >=< f,\ \phi z^i >< g, w^j >= 0,\ \ \forall i,j \geq 0.$$

This implies that

$$(H_z \ominus \phi H_z) \otimes H_w \subset H^2(D^2) \ominus \phi H^2(D^2).$$

Conversely, if $f(z,w) = \sum_{j=0}^{\infty} w^j f_j(z)$ is any function in $H^2(D^2) \ominus \phi H^2(D^2)$ then for each $i \geq 0$ and $k \geq 0$,

$$\begin{aligned} 0 =< f,\ \phi z^i w^k >&= \sum_{j=0}^{\infty} < w^j f_j\,, \phi z^i w^k > \\ &=< w^k f_k, \phi z^i w^k > \\ &=< f_k, \phi z^i >. \end{aligned}$$

This shows that $f_k \in H_z \ominus \phi H_z$ for every $k \geq 0$ and therefore $f \in (H_z \ominus \phi H_z) \otimes H_w$.

Theorem (5.1.10)[191]: *If $M \subset H^2(D^2)$ is a submodule of infinite codimension and either S_z or S_w is in class C_o, then S_z and S_w have a non-trivial jointly-invariant subspace.*

Proof. If S_w is in C_o, then S_w has a minimal function $\phi(w) \in H^\infty(D)$. Since

$$q\phi = q\phi(q1) = \phi(S_w)(q1) = 0, \qquad (4)$$

ϕ is in M. If $\phi = \psi F$ is the inner-outer factorization, then ψ is in M because M is a submodule. So without loss of generality we assume ϕ is inner. There are two cases.

Case 1. If ϕ can be factorized nontrivially into a product of two inner functions as

$$\phi(w) = \phi_1(w)\phi_2(w),$$

then by the minimality of ϕ and the arguments in (4) neither ϕ_1 nor ϕ_2 is in M. If we set

$$\widehat{M} = clos\{\phi_1 H^2(D^2) + M\}$$

then clearly $\widehat{M}$ is a submodule which contains M properly.

We now show that $\widehat{M} \neq H^2(D^2)$. In fact if $\widehat{M} = H^2(D^2)$, then there is a sequence $\{g_n : n \geq 0\} \subset H^2(D^2)$ and a sequence $\{h_n : n \geq 0\} \subset M$ such that

$$\lim_{n\to\infty} \phi_1 g_n + h_n = 1.$$

This implies that

$$\lim_{n\to\infty} \phi_2\phi_1 g_n + \phi_2 h_n = \phi_2.$$

But $\{\phi g_n, \phi_2 h_n : n \geq 0\}$ is a subset of M, so ϕ_2 needs to be in M which contradicts the minimality of ϕ So $\widehat{M} \ominus M$ is a non-trivial jointly invariant subspace of S_z and S_w by the remarks.

Case 2. If $\phi = \frac{w-\lambda}{1-\bar{\lambda}w}$ for some $\lambda \in D$, then $\phi H^2(D^2) = (w-\lambda)H^2(D^2) \subset M$ and by Lemma (5.1.9)

$$H^2(D^2) \ominus (w-\lambda)H^2(D^2) = \frac{1}{1-\bar{\lambda}w} H_z.$$

This in particular implies that $H^2(D^2) \ominus (w-\lambda)H^2(D^2)$ is invariant for T_z. Since M is invariant for T_z and

$$M \ominus (w-\lambda)H^2(D^2) = M \cap \big(H^2(D^2) \ominus (w-\lambda)H^2(D^2)\big),$$

$M \ominus (w-\lambda)H^2(D^2)$ is also invariant for T_z. By Beurling's theorem,

$$M \ominus (w-\lambda)H^2(D^2) = \frac{\psi(z)}{1-\bar{\lambda}w} H_z$$

for some inner function $\psi \in H_z$ and therefore

$$M = \frac{\psi(z)}{1-\bar{\lambda}w} H_z \oplus (w-\lambda)H^2(D^2).$$

If $\psi(z) = \frac{z-\mu}{1-\bar{\mu}z}$ for some $\mu \in D$, then M will be of codimension i which contradicts the assumption, so ψ must have a non-trivial inner factor, say ψ_1. If we let

$$\widehat{M} = \frac{\psi_1}{1-\bar{\lambda}w} H_z \oplus (w-\lambda)H^2(D^2),$$

then $\widehat{M} \ominus M$ is a non-trivial invariant subspace for S_z. Moreover, since $w-\lambda \in M, S_w = \lambda I$ on $H^2(D^2) \ominus M$, $\widehat{M} \ominus M$ is also invarint for S_w.

It is shown in [207] that if S_z doubly commutes with S_w,, e.g., $S_z S_w^* = S_w^* S_z$ and $S_z S_w^* = S_w S_z$, then either S_z or S_w is in class C_o. So Theorem (5.1.10) has the following

Corollary (5.1.11)[191]: *If S_z and S_w doubly commute on a quotient module $H^2(D^2) \ominus M$, then they have a non-trivial jointly-invariant subspace.*

In the study of the evaluation operator, it is necessary to make a study of another kind of operator, the difference quotient operators. For every $\lambda \in D$ we define difference, quotient operators $D_{z,\lambda}$ *and* $D_{w,\lambda}$ from $H^2(D^2)$ to itself by

$$D_{z,\lambda} f(z,w) = \frac{f(z,w) - f(\lambda,w)}{z-\lambda}, \quad D_{w,\lambda} f(z,w) = \frac{f(z,w) - f(z,\lambda)}{w-\lambda}.$$

One verifies that $D_{z,\lambda}$ and $D_{w,\lambda}$ are operator valued analytic functions in λ. The following lemma describes the adjoints of the difference quotient operators.

Lemma (5.1.12)[191]: *For every $f \in H^2(D^2)$,*

$$D_{z,\lambda}^* f = \frac{z}{1-\bar{\lambda}z} f; \quad D_{w,\lambda}^* f = \frac{w}{1-\bar{\lambda}w} f.$$

Proof. We prove the first equality. For every $f, g \in H^2(D^2)$,

$$\begin{aligned} < g, D_{z,\lambda}^* f > &= < D_{z,\lambda g}, \quad f > \\ &= < \frac{g - g(\lambda,\cdot)}{z-\lambda}, \quad f > \\ &= < g - g(\lambda,\cdot), \frac{f}{\overline{z-\lambda}} > \\ &= < g - g(\lambda,\cdot), \frac{zf}{1-\bar{\lambda}z} > = < g, \frac{zf}{1-\bar{\lambda}z} >. \end{aligned}$$

Since $\left\| \frac{1}{1-\bar{\lambda}z} \right\|_\infty = (1-|\lambda|)^{-1}$, it follows from Lemma (5.1.12) that

$$\|D_{z,\lambda}\| = \|D_{z,\lambda}^*\| = (1-|\lambda|)^{-1}, \|D_{w,\lambda}\| = \|D_{w,\lambda}^*\| = (1-|\lambda|)^{-1}.$$

The following lemma is easily checked.

Lemma (5.1.13)[191]: *For all λ and η in D,*

(a) $L(\lambda)$ *commutes with* T_w *and* $R(\lambda)$ *commutes with* T_z*;*

(b) $D_{z,\lambda}$ *commutes with* T_w *and* $D_{w,\lambda}$ *commutes with* T_z*;*

(c) $D_{z,\lambda}$ *commutes with* $R(\eta)$ *and* $D_{w,\lambda}$ *commutes with* $L(\eta)$.

Lemma (5.1.13)(a) can be used to generalize Lemma (5.1.3).

Corollary (5.1.14)[191]: *If M is z-invariant, then $R(\lambda)$ restricted to $M \ominus z^n M$ is Hiilbert-Schmidt for every $\lambda \in D$ and every integer n.*

Proof. It follows directly from Lemma (5.1.3), Lemma (5.1.13)(a) and the fact that

$$M \ominus z^n M = \oplus_{j=0}^{n-1} z^j (M \ominus zM).$$

For simplicity, we denote $D_{z,0}$ by D_z and $D_{w,0}$ by D_w. D_z and D_w are contractions.

The difference quotient operators are related to the compression operators in many ways.

One example is that when restricted to quotient modules, they are the analytic extensions of S_z^* and S_w^*. If M is z-invariant and $f \in H^2(D^2) \ominus M$, then $\bar{z}f$ is orthogonal to M and

$$\begin{aligned} S_z^* f &= q\bar{z}f \\ &= P\bar{z}f - p\bar{z}f \\ &= P\bar{z}f \\ &= \frac{f(z,w) - f(0,w)}{z} = D_z f. \end{aligned}$$

This shows that $D_z|_{H^2(D^2)\ominus M} = S_z^*$.

Another important property of D_z is that it maps space $M \ominus zM$ into $H^2(D^2) \ominus M$ when M is z-invariaalt. We state this fact as

Lemma (5.1.15)[191]: *If M is z-invariant, then $D_z(M \ominus zM) \subset H^2(D^2) \ominus M$.*

Proof. It suffices to show that $D_z h$ is orthogonal to M for every $h \in M \ominus zM$.

In fact for every $\varphi \in M$,

$$< D_z h, \varphi > = < zD_z h,\ z\varphi > = < h - h(0,\cdot),\ z\varphi > = 0.$$

By using Lemma (5.1.12) and the idea of Lemma (5.1.15), one easily checks the following

Proposition (5.1.16)[191]: *If M is z-invariant, then D_z^n maps $M \ominus z^j M$ into $H^2(D^2) \ominus M$ and*

$$D_z^n|_{M\ominus z^n M} = q\bar{z}^n|_{M\ominus z^n M} = \left(pz^n|_{H^2(D^2)\ominus M}\right)^*$$

for every natural number n.

$D_{z,\lambda}$ has an analytic expression in terms of D_z and S_z^*.

Lemma (5.1.17)[191]: *For every $g \in M \ominus zM$, $D_{z,\lambda} g = (1 - \lambda S_z^*)^{-1} D_z g$.*

Proof.

$$\begin{aligned} (1 - \lambda S_z^*) D_{z,\lambda} g &= q(1 - \lambda\bar{z}) \frac{g - g(\lambda,\cdot)}{z - \lambda} \\ &= q(1 - \lambda\bar{z}) \frac{\bar{z}\big(g - g(\lambda,\cdot)\big)}{1 - \lambda\bar{z}} \\ &= q\bar{z}g - q\bar{z}g(\lambda,\cdot) \\ &= q\bar{z}g = D_z g. \end{aligned}$$

The next corollary follows directly from Lemma (5.1.15) and Lemma (5.1.17).

Corollary (5.1.18)[191]: *If M is z-invariant, then $D_{z,\lambda}$ maps $M \ominus zM$ into $H^2(D^2) \ominus M$ for every $\lambda \in D$.*

Corollary (5.1.18) will be used.

We study the essential spectrum of S_z.

We begin by proving the first statement of the following

Theorem (5.1.19)[191]: *If M is z-invariant and $\lambda \in D$, then $S_z - \lambda$ is Fredholm on $H^2(D^2) \ominus M$ if and only if $L(\lambda)|_{M\ominus zM}$ is FredhoIm, and moreover*

$$ind(S_{z-\lambda}) = ind\big(L(\lambda)\big).$$

Proof. If A is an operator on a separable Hilbert space H, then it is well known that A is not Fredholm if and only if there is a sequence $\{x_n : n \geq 0\} \subset H$ which converges weakly to 0 and is *bounded below(e.g.,* $\|x_n\| \geq c > 0$ for some constant'e and all interger n sufficiently large) such that

$$\lim_{n\to\infty} \|Ax_n\| = 0.$$

We now assume $S_z - \lambda$ is not Fredholm and $\{f_n : n \geq 0\} \subset H^2(D^2) \ominus M$ is a sequence that converges weakly to 0 and is bounded below such that

$$\lim_{n\to\infty} \|S_{z-\lambda} f_n\| = 0. \tag{5}$$

Since

$$\begin{aligned} S_{z-\lambda} f_n &= (z-\lambda) f_n - p(z-\lambda) f_n \\ &= (z-\lambda) f_n - pz f_n, \end{aligned} \tag{6}$$

we have that

$$\|pzf_n\|^2 = \|(z-\lambda)f_n\|^2 - \|S_{z-\lambda}f_n\|^2$$

which implies that $\{pzf_n : n \geq 0\}$ is bounded below by (5). Since $pzf \in M \ominus zM$ for all $f \in H^2(D^2) \ominus M$, $\{pzf_n : n \geq 0\}$ is a sequence in $M \ominus zM$. By Proposition (5.1.16), for every $h \in M \ominus zM$,

$$< pzf_n, h > = < f_n, \ D_z h >.$$

So $\{pzf_n : n \geq 0\}$ also converges weakly to 0. Moreover, by (6)

$$L(\lambda)(S_{z-\lambda} f_n) = -L(\lambda)(pzf_n)$$

and therefore

$$\lim_{n\to\infty} \|L(\lambda)(pzf_n)\| \leq (1 - |\lambda|^2)^{-1/2} \lim_{n\to\infty} \|S_{z-\lambda} f_n\| = 0.$$

So $L(\lambda)|_{M\ominus zM}$ is not Fredholm.

Conversely, if $\{h_n : n \geq 0\} \subset M \ominus zM$ weakly converges to 0 and is bounded below such

$$\lim_{n\to\infty} \|L(\lambda) h_n\| = 0, \tag{7}$$

then first of all $\{D_{z,\lambda} h_n : n \geq 0\}$ converges wealdy to 0, and by Corollary (5.1.18) it is a sequence in $H^2(D^2) \ominus M$. Since

$$\begin{aligned} \|D_{z,\lambda} h_n\| &= \left\| \frac{h_n - L(\lambda) h_n}{z - \lambda} \right\| \\ &\geq \frac{1}{1 + |\lambda|} (\|h_n\| - \|L(\lambda) h_n\|), \end{aligned}$$

$\{D_{z,\lambda} h_n : n \geq 0\}$ is bounded below by (7). Moreover,

$$S_{z-\lambda} D_{z,\lambda} h_n = q(h_n - L(\lambda) h_n) = -q(L(\lambda) h_n)$$

and hence

$$\lim_{n\to\infty} \|S_{z-\lambda} D_{z,\lambda} h_n\| \leq \lim_{n\to\infty} \|L(\lambda) h_n\| = 0.$$

This shows that $S_{z-\lambda}$ is not Fredholm. Thus we conclude that $S_{z-\lambda}$ is Fredholm if and only if $L(\lambda)|_{M\ominus zM}$ is Fredholm.

We now prove the second statement of Theorem (5.1.19).

If $S_{z-\lambda}$ and $L(\lambda)|_{M\ominus zM}$ are Fredholm, we now show that $ind(S_{z-\lambda}) = ind\big(L(\lambda)\big)$ by proving that

$$dim\big(Ker(S_{z-\lambda})\big) = dim\Big(Ker\big(L(\lambda)\big)\Big)$$

and that

$$dim\big(CoKer(S_{z-\lambda})\big) = dim\Big(CoKer\big(L(\lambda)\big)\Big).$$

We first define a map $X :\ Ker(S_{z-\lambda}) \to Ker\big(L(\lambda)\big)$ by

$$Xf = (z-\lambda)f.$$

If $f \in Ker(S_{z-\lambda})$, then $(z-\lambda)f \in M$, and since $(z-\lambda)f$ is orthogonal to $zM, (z-\lambda)f \in M \ominus zM$. It is obvious that $(z-\lambda)f \in Ker\big(L(\lambda)\big)$. So X is well defined. It is not hard to see that X is bounded and injective. If $h \in Ker\big(L(\lambda)\big)$, then

$$D_{z,\lambda}h = \frac{h}{z-\lambda}$$

which is in $H^2(D^2) \ominus M$ by Corollary (5.1.18). Moreover $S_{z-\lambda}D_{z,\lambda}h = qh = 0$ and $XD_{z,\lambda}h = h$. This shows that X is also surjective and hence

$$dim\big(Ker(S_{z-\lambda})\big) = dim\Big(Ker\big(L(\lambda)\big)\Big).$$

To show that $dim\big(CoKer(S_{z-\lambda})\big) = dim\Big(CoKer\big(L(\lambda)\big)\Big)$, we define a map $Y :\ CoKer\big(L(\lambda)\big) \to CoKer(S_{z-\lambda})$ by

$$Yg = \frac{g(w)}{1-\bar{\lambda}z},$$

where $g \in CoKer\big(L(\lambda)\big) \subset H_w$. For every $h \in M$,

$$< Yg, h >= \int_{T^2} \frac{g(w)}{1-\bar{\lambda}z}\overline{h(z,w)}\frac{|dz|}{2\pi}\frac{|dw|}{2\pi} = \int_T g(w)\overline{\int_T \frac{h(z,w)}{1-\lambda\bar{z}}\frac{|dz|}{2\pi}}\frac{|dw|}{2\pi}$$
$$= \int_T g(w)\overline{h(\lambda,w)}\frac{|dw|}{2\pi} =< g, L(\lambda)h >.$$

Since $L(\lambda)(M \ominus zM) = L(\lambda)(M)$ (cf. [207]) and $g \in CoKer\big(L(\lambda)\big), < g, L(\lambda)h >= 0$. This shows that $Yg \in H^2(D^2) \ominus M$.

Moreover, for every $f \in H^2(D^2) \ominus M$,

$$< Yg, S_{z-\lambda}f >=< Yg, (z-\lambda)f >= \int_T g(w)\overline{\int_T \frac{(z-\lambda)f(z,w)}{1-\lambda\bar{z}}\frac{|dz|}{2\pi}}\frac{|dw|}{2\pi}$$
$$= \int_T g(w)\cdot 0\frac{|dw|}{2\pi} = 0.$$

This concludes that $Yg \in CoKer(S_{z-\lambda})$ for all $g \in CoKer\big(L(\lambda)\big)$ and hence Y is well defined. It is not hard to see that Y is bounded and injective. If ψ is any function in $CoKer(S_{z-\lambda})$, then first of all ψ is orthogonal to $(z-\lambda)M$. Moreover, for every $f \in H^2(D^2) \ominus M$,

$$0 =< \psi, S_{z-\lambda}f >=< \psi, (z-\lambda)f >.$$

This concludes that ψ is orthogonal to $(z-\lambda)M + (z-\lambda)(H^2(D^2) \ominus M)$ which is equal to $(z-\lambda)H^2(D^2)$, and hence by Lemma (5.1.9) and the fact that

$$H_z \ominus (z-\lambda)H_z = \mathrm{C}\big(1-\bar{\lambda}z\big)^{-1}$$

we have that

$$\psi(z,w) = \frac{g(w)}{1-\overline{\lambda}z}$$

for some $g \in H_w$. To show that ψ is in the range of Y, we only need to check that $g \in CoKer(L(\lambda))$. In fact, for every $h \in M$,

$$< g, L(\lambda)h \geq \int_T g(w)\overline{h(\lambda,w)}\frac{|dw|}{2\pi}$$
$$= \int_{T^2} \frac{g(w)}{1-\overline{\lambda}z}\overline{h(z,w)}\frac{|dz|}{2\pi}\frac{|dw|}{2\pi}$$
$$=< \psi, h >= 0.$$

This concludes that Y is surjective and hence

$$dimCoKer(S_{z-\lambda}) = dimCoKer(L(\lambda)).$$

We point out that based on Equality (2) and techniques in functional model theory (cf. [216]) one may give a simpler proof of Theorem (5.1.19). But the proof here fits into our setting and it contains ideas and techniques that are useful in other places.

The proof of the following corollary is similar to that of Theorem 2.6 in [207], but since it is short, we include it here.

Corollary (5.1.20)[191]: *If M is a submodule of $H^2(D^2)$ with $M \ominus (zM + wM)$ infinite dimensional, then*

$$\sigma_e(S_z) = \sigma_e(S_w) = \overline{D}.$$

Proof. If $\{g_n : n \geq 0\} \subset M \ominus (zM + wM) = (M \ominus zM) \cap (M \ominus wM)$ is an orthonormal basis, then for every $\lambda \in D$,

$$\sum_{j=0}^{\infty} \|R(\lambda)g_n\|^2 < \infty$$

by Lemma (5.1.3)**.** This in particular implies that

$$\lim_{n\to\infty} \|R(\lambda)g_n\| = 0$$

which means $\lambda \in \sigma_e(S_w)$ by Theorem (5.1.19) and the first few lines of its proof. S_w is clearly a contraction and $\sigma_e(S_w)$ is a closed set, so we have

$$\sigma_e(S_w) = \overline{D}.$$

The proof of $\sigma_e(S_z) = \overline{D}$ is similar.

Corollary (5.1.21)[191]: *If M is z-invariant and $L(\lambda)$ is compact on $H^2(D^2) \ominus M$, then $S_{z-\lambda}$ is Fredholm if and only if S_z is Fredholm, and in which case*

$$ind(S_z) = ind(S_{z-\lambda}).$$

Proof. For every $f \in M \ominus zM$, we can write

$$f(z,w) = f(0,w) + z(D_z f)(z,w),$$

and therefore,

$$L(\lambda)f = f(\lambda,w) = L(0)f + \lambda L(\lambda)D_z f.$$

So on $M \ominus zM$,

$$L(\lambda) - L(0) = \lambda L(\lambda)D_z.$$

Since D_z maps $M \ominus zM$ into $H^2(D^2) \ominus M$, if $L(\lambda)|_{H^2(D^2)\ominus M}$ compact, then $L(\lambda)|_{M\ominus zM} - L(0)|_{M\ominus zM}$ is compact, and the corollary follows from Theorem (5.1.19).

Corollary (5.1.21) suggests that the compactness of the evaluation operators on $H^2(D^2) \ominus M$ has implications on the spectral properties of the compressions. We will study some sufficient conditions under which the evaluation operators are compact on the quotient $H^2(D^2) \ominus M$.

One necessary condition will be given for singly generated submodules after a study of compression operators.

Lemma (5.1.5) implies that for any $\lambda \in D, L(\lambda)$ is compact if and only if $L(0)$ is compact. So we only need to study the compactness of $L(0)$ on $H^2(D^2) \ominus M$, and we assume except in Theorem (5.1.27), M stands for submodules in $H^2(D^2)$.

We now study some sufficient conditions for the compactness of evaluation operators on quotient $H^2(D^2) \ominus M$.

The simplest case is $M = [p]$ where p is a polynomial in H_z. The following corollary is a consequence of Lemma (5.1.9).

Corollary (5.1.22)[191]: *If $p(z)$ is a polynomial of degree N, then the right evaluation $R(\lambda)$ from $H^2(D^2) \ominus [p]$ to H_z is of at most rank N for every $\lambda \in D$.*

Proof. If $p(z) = \phi(z)F(z)$ is the inner-outer factorization of p, then $\phi(z)$ is a finite Blaschke product with at most N zeros in D and the dimension of $H_z \ominus \phi H_z$ is less than or equal to N (cf. [216]). By Lemma (5.1.9),

$$H^2(D^2) \ominus [p] = H^2(D^2) \ominus H^2(D^2) = (H_z \ominus \phi H_z) \otimes H_w$$

and hence the range of $R(\lambda)$ on $H^2(D^2) \ominus [p]$ is $H_z \ominus \phi H_z$ and therefore the rank of $R(\lambda)$ is less than or equal to N.

Since $R(0)$ maps $H^2(D^2) \ominus M$ to H_z, its adjoint $R(0)^*$ maps H_z to $H^2(D^2) \ominus M$. And it is conceivable that functions in the range of $R(0)^*$ depends largely on variable z. The following lemma reflects this phenomenon.

Lemma (5.1.23)[191]: $L(0)R^*(0) : H_z \to H_w$ is *Hilbert-Schmidt.*

Proof. For every $h \in H^2(D^2) \ominus M$ and every $f \in H_z$,

$$< R^*(0)f, h >=< f, R(0)h >=< f, h >=< qf, h >.$$

So $R^*(0)f = qf$ and hence

$$L(0)R^*(0)f = L(0)qf = L(0)(f - pf).$$

Since $pf \in M \ominus wM$ for every $f \in H_z, L(0)p|_{H_z}$ is Hilbert-Schmidt by a parallel statement of Lemma (5.1.3) for the left evaluation on $M \ominus wM$, and the corollary follows easily from the additional fact that $L(0)|_{H_z}$ is of rank 1.

Corollary (5.1.24)[191]: *If there is a bounded invertible linear map $V : H_z \to H_w$ such that for all $f \in H^2(D^2) \ominus M$*

$$L(0)f = V.R(0)f,$$

then both $L(0)$ and $R(0)$ are compact.

Proof. Since $VR(0)R^*(0) = L(0)R^*(0)$, $VR(0)R^*(0)$ is Hilbert-Schmidt by Lemma (5.1.23), and hence $R(0)R^*(0)$ is Hilbert-Schmidt since V is invertible. Therefore $R(0)$ is compact, and so is $L(0)$.

This corollary means that if functions in $H^2(D^2) \ominus M$ have certain symmetries in z and w then the evaluation operators are Hilbert-Schmidt on $H^2(D^2) \ominus M$.

Example (5.1.25)[191]: The submodule $M = [z - w]$ is mentioned in many papers. One feature of $H^2(D^2) \ominus M$ is that functions in it are symmetric in z and w, e.g.,

$f(z,w) = f(w,z)$ on D^2 for every $f \in H^2(D^2) \ominus M$. So if V is the map which sends every $g(z) \in H_z$ to $g(w)$ in H_w, then V is unitary and

$$L(0)f = VR(z)f,$$

and hence by Corollary (5.1.24) $L(0)$ and $R(0)$ are compact. We will give another proof to this fact by using a direct computation.

Another sufficient condition follows, but are need the following simple fact.

Lemma (5.1.26)[191]: *If $A : . X \to Y$ is a bounded linear map and $\|Ax\| \geq c\|x\|$ for a fixed positive constant c and every $x \in X$, then A^*A is invertibIe.*

Proof. It is easy to see that A^*A is injective and has dense range. We now show that A^*A has closed range by showing it is bounded below. In fact, for every $x \in X$ with $\|x\| = 1$,

$$\begin{aligned}\|A^*Ax\| &= \sup_{\|y\|\leq 1}|< A^*Ax, y >| \\ &\geq < A^*Ax, x > \\ &= \|Ax\|^2 \geq c^2\|x\|^2 = c^2.\end{aligned}$$

Theorem (5.1.27)[191]: *If M is z-invariant and there is an integer n such that $\|S_z^n\| < 1$ on $H^2(D^2) \ominus M$, then $R(\lambda)$ is Hilbert-Schmidt on $H^2(D^2) \ominus M$ for every $\lambda \in D$.*

Proof. Since $S_z^n f + pz^n f = z^n f$,

$$\|S_z^n f\|^2 + \|pz^n f\|^2 = \|f\|^2.$$

Therefore,

$$\|pz^n f\|^2 \geq (1 - \|S_z^n\|^2)\|f\|^2.$$

By Proposition (5.1.16) and Lemma (5.1.26), $D_z^n : M \ominus z^n M \to H^2(D^2) \ominus M$ is onto. Since for every $\lambda \in D$ $R(\lambda)$ restricted to $M \ominus z^n M$ is Hilbert-Schmidt by Corollary (5.1.14) and $R(\lambda)$ commmtes with D_z, $R(\lambda)D_z^n$ is Hilbert-Schmidt. But D_z^n is onto, so $R(\lambda)$ restricted to $H^2(D^2) \ominus M$ is Hilbert-Schmidt.

The following corollary is more concrete.

Corollary (5.1.28)[191]: *If $h(z,w) = z^n + \phi(z,w)$ for some natural number n and $\phi \in H^\infty(D^2)$ with $\|\phi\|_\infty < 1$, then $R(\lambda)$ restricted to $H^2(D^2) \ominus [h]$ is Hilbert-Schmidt for every $\lambda \in D$.*

Proof. Since $H^2(D^2) \ominus [h]$ is a quotient module,

$$S_z^n + S_\phi = S_{z^n+\phi} = S_h = 0.$$

It follows that

$$\|S_z^n\| = \|S_\phi\| \leq \|\phi\|_\infty < 1$$

and the corollary follows from Theorem (5.1.27).

The simplest case of Corollary (5.1.28) is the following

Example (5.1.29)[191]: If $M = [z - \mu w]$ for some $\mu \in D$, then $S_z = \mu S_w$ on $H^2(D^2) \ominus M$ and hence $\|S_z\| \leq |\mu| < 1$. So $R(\lambda)$ is Hilbert-Schmidt for every $\lambda \in D$. However, computation shows that $L(\lambda)$ is not compact for any $\lambda \in D$. We now give a direct proof.

If $e_0 = 1$ and

$$e_n(z,w) = c_n((\bar{\mu}z)^n + (\bar{\mu}z)^{n-1}w + \cdots + \bar{\mu}zw^{n-1} + w^n), n \geq 0,$$

where $c_n = \sqrt{\frac{1-|\mu|^2}{1-|\mu|^{2n+2}}}$, then, with some computations, one checks that $\{e_n : n \geq 0\}$ is an orthonormal basis for $H^2(D^2) \ominus [z - \mu w]$ and

$$R(0)e_n = c_n(\bar{\mu}z)^n.$$

It follows that

$$\sum_{n=0}^{\infty} \|R(0)e_n\|^2 < \infty$$

and hence $R(0)$ is ttilbert-Schmidt.

However

$$L(0)e_n = c_n w^n.$$

Since

$$\lim_{n\to\infty} c_n = \sqrt{1-|\mu|^2} > 0,$$

$L(0)$ is not compact.

The following corollary generalizes Theorem (5.1.27) in the case when $n = 1$.

Corollary (5.1.30)[191]: *If there is a $\alpha \in D$ such that $\|S_{z-\alpha}\| < 1-|\alpha|$, then $R(\lambda)$ is Hilbert-Schmidt for every , $\lambda \in D$.*

Proof. Since

$$S_{Z-\alpha}f = (z-\alpha)f - p(z-\alpha)f = (z-\alpha)f - pzf,$$

we have

$$\|S_{z-\alpha}f\|^2 + \|pzf\|^2 = \|(z-\alpha)f\|^2.$$

This implies that

$$\begin{aligned}\|pzf\|^2 &= \|(z-\alpha)f\|^2 - \|S_{z-\alpha}f\|^2 \\ &\geq (1-|\alpha|)^2\|f\|^2 - \|S_{z-\alpha}\|^2\|f\|^2.\end{aligned}$$

If $\|S_{z-\alpha}\| < 1-|\alpha|$ then $D_z^* = pz$ is bounded below and the corollary follows from the proof of Theorem (5.1.27).

This kind of submodules are subjects of many studies. It was shown in [193] that if M is unitarily equivalent to $H^2(D^2)$ as $A(D^2)$-modules, then there is a inner function, say r such that $M = \phi H^2(D^2)$. One useful corollary of this fact is that modules of this kind have simple reproducing kernels. To be precise, if ϕ is inner and $K_\eta(z)$ is the reproducing kernel of $M = \phi H^2(D^2)$, where $\eta = (\eta_1, \eta_2), z = (z_1, z_2)$ in D^2 then one verifies that If $K_\eta(z) = \frac{\phi(z)\overline{\phi}(\eta)}{(1-\overline{\eta_1}z_1)(1-\overline{\eta_2}z_2)}$, and $K_\eta^\perp(z) = \frac{1-\phi(z)\overline{\phi}(\eta)}{(1-\overline{\eta_1}z_1)(1-\overline{\eta_2}z_2)}$ is the reproducing kernel of $H^2(D^2) \ominus \phi H^2(D^2)$.

The Kilbert-Schmidtness of $L(0)$ on $H^2(D^2) \ominus \phi H^2(D^2)$ can be completely determined. We need a lemma to move on.

Lemma (5.1.31)[191]: *A bounded linear operator $T: H^2(D^2) \ominus M \to H_w$ is Hilbert-Schmidt if and only if*

$$sup_{0<r<1} \int_T \left[\iint_{T^2} \left|TK_\eta^\perp(rw)\right|^2 \frac{|d\eta_1|}{\pi}\frac{|d\eta_2|}{\pi}\right]\frac{|dw|}{2\pi} < \infty,$$

and moreover

$$trT^*T = sup_{0<r<1} \int_T \left[\iint_{T^2} \left|TK_\eta^\perp(rw)\right|^2 \frac{|d\eta_1|}{\pi}\frac{|d\eta_2|}{\pi}\right]\frac{|dw|}{2\pi}.$$

Proof. If $\{e_j | j \geq 0\}$ is an orthonormal basis for $H^2(D^2) \ominus M$, then

$$K_\eta^\perp(z) = \sum_{j=0}^{\infty} e_j(z)\overline{e_j}(\eta)$$

and

$$TK_\eta^\perp(\lambda) = \sum_{j=0}^{\infty} Te_j(\lambda)\overline{e_j}(\eta), \;\; \lambda \in D.$$

Since for every $f \in H^2(D)$ and $\lambda \in D$,

$$|f(\lambda)|^2 \leq \frac{\|f\|^2}{1-|\lambda|^2}$$

and T is Hilbert-Schmidt,

$$\sum_{j=0}^{\infty}\left|Te_j(\lambda)\right|^2 \leq \sum_{j=0}^{\infty}\frac{\left\|Te_j\right\|^2}{1-|\lambda|^2} < \infty,$$

and therefore $\overline{TK_\eta^\perp}(\lambda)$, as a function in η, is in $H^2(D^2)$, and

$$\int_{T^2}\left|\overline{TK_\eta^\perp}(\lambda)\right|^2\frac{|d\eta_1|}{\pi}\frac{|d\eta_2|}{\pi} = \sum_{j=0}^{\infty}\left|Te_j(\lambda)\right|^2.$$

Since $\left|Te_j(\lambda)\right|^2$ is subharmonic in λ for all $j \geq 0$,

$$sup_{0<r<1}\int_T\left[\int_{T^2}\left|TK_\eta^\perp(rw)\right|^2\frac{|d\eta_1|}{\pi}\frac{|d\eta_2|}{\pi}\right]\frac{|dw|}{2\pi}$$

$$= sup_{0<r<1}\int_T\sum_{j=0}^{\infty}\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi}$$

$$= \lim_{r\to 1^-}\int_T\sum_{j=0}^{\infty}\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi}$$

$$= \lim_{r\to 1^-}\sum_{j=0}^{\infty}\int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi}$$

$$= \sum_{j=0}^{\infty}\lim_{r\to 1^-}\int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi} = \sum_{j=0}^{\infty}\left\|Te_j\right\|^2.$$

The second equality from the bottom needs explanation. The inequality

$$\lim_{r\to 1^-}\sum_{j=0}^{\infty}\int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi} \geq \sum_{j=0}^{\infty}\lim_{r\to 1^-}\int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi}$$

comes from Fatou's lemma. On the other hand, by the subharmonicity

$$\lim_{r\to 1^-}\int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi} \geq \int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi}$$

for each j and all $0 < r < 1$, therefore

$$\sum_{j=0}^{\infty}\lim_{r\to 1^-}\int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi} \geq \sum_{j=0}^{\infty}\int_T\left|Te_j(rw)\right|^2\frac{|dw|}{2\pi},$$

which implies that

$$\sum_{j=0}^{\infty} \lim_{r\to 1^-} \int_T \left|Te_j(rw)\right|^2 \frac{|dw|}{2\pi} \geq \lim_{r\to 1^-} \sum_{j=0}^{\infty} \int_T \left|Te_j(rw)\right|^2 \frac{|dw|}{2\pi}.$$

Example (5.1.32)[191]: If $M = \phi H^2(D^2)$ for some inner function ϕ, then

$$K_\eta^\perp(z) = \frac{1-\phi(z)\overline{\phi}(\eta)}{(1-\overline{\eta_1}z_1)(1-\overline{\eta_2}z_2)}.$$

So

$$L(0)K_\eta^\perp = K_\eta^\perp(0,z_2) = \frac{1-\phi(0,z_2)\overline{\phi}(\eta_1,\eta_2)}{(1-\overline{\eta_2}z_2)},$$

and one checks that

$$\int_{T^2} \left|K_\eta^\perp(0,rz_2)\right|^2 \frac{|d\eta_1|}{\pi}\frac{|d\eta_2|}{\pi} = \frac{1-|\phi(0,rz_2)|^2}{1-r^2}.$$

So if $L(0)$ is Hilbert-Schmidt on $H^2(D^2) \ominus M$, then by Leman (5.1.31) and its proof,

$$\infty > \sup_{0<r<1} \int_T \int_{T^2} \left|K_\eta^\perp(0,rz_2)\right|^2 \frac{|d\eta_1|}{\pi}\frac{|d\eta_2|}{\pi}\frac{|dz_2|}{\pi} = \lim_{r\to 1^-} \int_T \frac{1-|\phi(0,rz_2)|^2}{1-r^2}\frac{|dz_2|}{\pi}$$
$$= \int_T \lim_{r\to 1^-} \frac{1-|\phi(0,rz_2)|^2}{1-r^2}\frac{|dz_2|}{\pi},$$

which is possible only if $\phi(0,z_2)$ is almost everywhere equal to 1 on T. Therefore,

$$\|\phi\| = \|L(0)\phi\| = 1,$$

and this implies that $\phi(z,w) = \phi(0,w)$.

For simplicity we denote $\phi(0,w)$ by $\phi(w)$ and continue to find out more about this $\phi(w)$. Since ϕ is inner, ϕH_w is a dosed subspace of H_w which is invariant for multiplicaton by w and it is easy to check that $H_w \ominus \phi H_w$is a subset of $H^2(D^2) \ominus M$. Since functions in $H_w \ominus \phi H_w$ are independent of z, $L(0)$ is an isometry acting on it. The compactness of $L(0)$ implies that $H_w \ominus \phi H_w$ is finite dimensional, which is possible only if ϕ is a finite Blaschke product.

On the other hand if ϕ is a finite Blaschke product with zeros $\alpha_1, \alpha_2, \dots, \alpha_n$, counting multiplicity, then

$$\phi(w) = \phi_1 \cdots \phi_n,$$

where $\phi_j = \frac{w-\alpha_j}{1-\overline{\alpha_j}w}$. Since

$$\begin{aligned} 1 - |\phi(rw)|^2 \\ &= 1-|\phi_1(rw)|^2 + |\phi_1(rw)|^2(1-|\phi_2(rw)|^2) + \cdots \\ &+ |\phi_1\cdots\phi_n(rw)|^2(1-|\phi_n(rw)|^2), \end{aligned}$$

and

$$\frac{1-\left|\phi_j(rw)\right|^2}{1-r^2} = \frac{1-\left|\alpha_j\right|^2}{\left|1-r\overline{\alpha_j}w\right|^2},$$

we have that

$$\lim_{r\to 1^-} \frac{1-|\phi(rw)|^2}{1-r^2} = \sum_{j=1}^{n} \frac{1-\left|\alpha_j\right|^2}{\left|1-\overline{\alpha_j}w\right|^2}.$$

So by Lemma (5.1.31), $L(0)$ is Hilbert-Schmidt on $H^2(D^2) \ominus M$.

It is convenient to calculate the trace for $L^*(0)L(0)$ at this point. By Lemma (5.1.31) and the argument above

$$trL^*(0)L(0) = \int_T \sum_{j=0}^{n} \frac{1-|\alpha_j|^2}{1-\overline{\alpha_j}w} \frac{|dw|}{2\pi} = n.$$

We conclude this example by

Corollary (5.1.33)[191]: If *ϕ is inner, then $L(0)$ is tfilbert-Schmidt on $H^2(D^2) \ominus \phi H^2(D^2)$ if and only if ϕ is a finite Blaschke product in w, and in which case*

$$trL^*(0)L(0) = n,$$

where n is the number of zeros of ϕ in D counting multiplicity.

When M is rank 1, e.g. $M = [h]$, a good necessary condition can be given.

We first study the relations among the compression operators, the evaluation operators and the difference quotient operators, we then show that the compactness of evaluation operators leads to interesting spectral properties of the compression operators.

Here we note that most of the results are stated for S_z, but at some places we will use the corresponding results for S_w.

The following proposition is not hard to cheek.

Proposition (5.1.34)[191]: *If M is a z-invariant subspace of $H^2(D^2)$, then for every f $f \in H^2(D^2) \ominus M$ and $g \in M \ominus zM$*

(a) $S_z^* S_z f + D_z D_z^* f = f$;

(b) $S_z S_z^* f + \left(L(0)|_{H^2(D^2)\ominus M}\right)^* \left(L(0)|_{H^2(D^2)\ominus M}\right) f = f$;

(c) $S_z D_z g + \left(L(0)|_{H^2(D^2)\ominus M}\right)^* \left(L(0)|_{M\ominus zM}\right) g = 0$;

(d) $D_z^* D_z g + \left(L(0)|_{M\ominus zM}\right)^* \left(L(0)|_{M\ominus zM}\right) g = g$.

Proof. (a).

$$\begin{aligned} S_z^* S_z f &= q\bar{z}(zf - pzf) \\ &= q(f - \bar{z}pzf) \\ &= f - q\bar{z}pzf = f - D_z D_z^* f \end{aligned}$$

by Proposition (5.1.16).

(b). Since $f \in H^2(D^2) \ominus M, \bar{z}f$ is orthogonal to M and hence

$$\begin{aligned} q\bar{z}f &= P\bar{z}f \\ &= P\bar{z}(f - L(0)f + L(0)f) = \bar{z}(f - L(0)f). \end{aligned}$$

Therefore,

$$\begin{aligned} S_z S_z^* f &= qzq\bar{z}f \\ &= qz\bar{z}(f - L(0)f) = f - qL(0)f \\ &= f - L(0)^* L(0)f. \end{aligned}$$

The last equality follows from Lemma (5.1.4) in the case $\lambda = 0$.

(c).

$$\begin{aligned} S_z D_z g &= qz\frac{g - g(0,\cdot)}{z} \\ &= qg - qL(0)g \\ &= -\left(L(0)|_{H^2(D^2)\ominus M}\right)^* L(0)g \end{aligned}$$

by Lemma (5.1.4).

(d). By Proposition (5.1.16),

$$D_z^* D_z g = pz \frac{g - g(0,\cdot)}{z} = g - pL(0)g.$$

Since g is arbitrary and for any $\phi \in H_w$

$$< p\phi,\ g >=< \phi,\ g >=< \phi,\ L(0)g >,$$

$p|_{H_w}$ is the adjoiat of $L(0)|_{M\ominus zM}$ and therefore

$$D_z^* D_z g = g - \left(L(0)|_{M\ominus zM}\right)^* \left(L(0)|_{M\ominus zM}\right)g.$$

The following corollary follows quickly from Proposition (5.1.34).

Corollary (5.1.35)[191]: *If M is z-invariant with $H^2(D^2) \ominus M$ is infinte dimensional and $L(0)$ is compact* on $H^2(D^2) \ominus M$ then $\|S_z\| = 1$.

Proof. Lemma (5.1.34)(b) shows that on $H^2(D^2) \ominus M$

$$S_z S_z^* = I - L^*(0)L(0).$$

If $H^2(D^2) \ominus M$ is infinite dimensional and $L(0)$ is compact, then the spectrum of $S_z S_z^*$ contains 1 and hence $\|S_z\|^2 = \|S_z S_z^*\| \geq 1$. But S_z is a contraction, so $\|S_z\| = 1$.

If M is a submodule such that S~ is a strict contraction, then $R(0)$ is Hilbert-Schmidt on $H^2(D^2) \ominus M$ by Theorem (5.1.27) which implies that $\|S_w\| = 1$ by Corollary (5.1.35) when $H^2(D^2) \ominus M$ is infinite diraension. We state this observation as the following

Corollary (5.1.36)[191]: *If M is a submodule such that S_z and S_w are both strict contractions then $H^2(D^2) \ominus M$ is finite dimensional.*

If $L(0)$ is compact on $H^2(D^2) \ominus M$, then a generalization of Proposition (5.1.34)(b) gives a spectral picture of S_z.

Theorem (5.1.37)[191]: *If M is z-invariant and $L(0)$ restricted on $H^2(D^2) \ominus M$ is compact, then*

$$\sigma(S_z) \cap D \subset \sigma_p(S_z) \cup \overline{\sigma_p(S_z^*)}.$$

Proof. If $\lambda \in D$ and $\varphi_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z}$ then for any $g \in H^2(D^2) \ominus M$,

$$\begin{aligned} S_{\varphi_\lambda}^* g &= q\overline{\varphi_\lambda} g \\ &= P\overline{\varphi_\lambda} g \\ &= P\overline{\varphi_\lambda}\left(g - g(\lambda,\cdot)\right) + P\overline{\varphi_\lambda} L(\lambda)g \\ &= \overline{\varphi_\lambda}\left(g - g(\lambda,\cdot)\right) + P\overline{\varphi_\lambda} L(\lambda)g. \end{aligned}$$

One checks that

$$P\overline{\varphi_\lambda} L(\lambda)g = P\overline{\varphi_\lambda} g(\lambda,\cdot) = -\bar{\lambda} g(\lambda,\cdot),$$

and hence

$$\begin{aligned} S_{\varphi_\lambda} S_{\varphi_\lambda}^* g &= q\left(g - g(\lambda,\cdot)\right) - q\bar{\lambda}\varphi_\lambda g(\lambda,\cdot) \\ &= g - q\left(1 + \bar{\lambda}\varphi_\lambda\right)L(\lambda)g \\ &= g - (1 - |\lambda|^2)q\left(1 - \bar{\lambda}z\right)^{-1} L(\lambda)g \\ &= g - (1 - |\lambda|^2)L^*(\lambda)L(\lambda)g \end{aligned}$$

by Lemma (5.1.4). So

$$S_{\varphi_\lambda} S_{\varphi_\lambda}^* = I - (1 - |\lambda|^2)L^*(\lambda)L(\lambda). \tag{8}$$

Since $L(\lambda)$ is compact, $S_{\varphi_\lambda} S_{\varphi_\lambda}^*$ is Fredholm. Moreover,

$$S_{\varphi_\lambda} = S_{(1-\bar{\lambda}z)^{-1}} S_{z-\lambda}$$

and $S_{(1-\bar{\lambda}z)^{-1}}$ is invertible, so $S_{z-\lambda}S^*_{z-\lambda}$ is Fredholm.If $S^*_{z-\lambda}$ has trivial kernel, then $S_{z-\lambda}S^*_{z-\lambda}$ *is* invertible and therefore $S_{z-\lambda}$ is onto. So if $S_{z-\lambda}$ also has trivial kernel then λ is in the resolvant set of S_z.

We give some sufficient conditions for the compactness of the evaluation operators on quotient spaces. If M is a submodule of rank 1, e.g., $M = [h]$ for some $h \in H^2(D^2)$, then behaviors of S_z and $L(0)$ on $H^2(D^2) \ominus M$ reflect properties of h. We now give a necessary condition for the compactness of $L(0)$ in terms of h. We state a lemma first.

Lemma (5.1.38)[191]: If $\{F_\lambda | \lambda \in D\}$ *is a norm continuous:family of selfadjoint Fredholm operators on a Hilbert space* H*, then* $dim(KerF_\lambda)$ *is a lower semi-continuous function in* λ.

Proof. Let $\lambda_0 \in D$ be any point and P_0 be the projection from H to $KerF_{\lambda_0}$, then one verifies that $F_{\lambda_0} + P_0$ is invertible. If λ_n is a sequence in D converging to λ_0, then $F_{\lambda_n} + P_0$ converges to $F_{\lambda_0} + P_0$ in operator norm which implies that $F_{\lambda_n} + P_0$ is invertible for all λ_n dose enough to λ_0.

Since

$$P_0|_{KerF_{\lambda_n}} = \left(F_{\lambda_n} + P_0\right)|_{KerF_{\lambda_n}},$$

Po maps $KerF_{\lambda_n}$ injectively into $KerF_{\lambda_0}$. This implies that

$$dim\left(KerF_{\lambda_n}\right) \le dim\left(KerF_{\lambda_0}\right)$$

when λ_n is sufficiently close to λ_0.

A definition is needed in order to state our next theorem. For every fixed $\lambda \in D$ and $h \in H^2(D^2)$ we let $Z_h(\lambda)$ denote the number of zeros of $h(\lambda, w)$ in D. So $Z_h(\lambda)$ is an integer-valued function in λ.

Theorem (5.1.39)[191]: If $L(0)$ *is compact on* $H^2(D^2) \ominus [h]$, *then* $Z_h(\lambda)$ *is a constant.*

Proof. *If* $\lambda \in D$ and $\phi(\lambda) = \frac{z-\lambda}{1-\bar{\lambda}z}$, then

$$S_{\phi_\lambda}S^*_{\phi_\lambda} + (1 - |\lambda|^2)L^*(\lambda)L(\lambda) = I$$

by Equality (8). If $L(0)$ is compact on $H^2(D^2) \ominus [h]$ then by Lemma (5.1.5), $L(\lambda)$ is compact on $H^2(D^2) \ominus [h]$ for every $\lambda \in D$, which implies that $S_{\phi_\lambda}S^*_{\phi_\lambda}$ is Fredholm for all $\lambda \in D$. Since $S_{\phi_\lambda} = S^{-1}_{1-\bar{\lambda}z} S_{z-\lambda} S_{z-\lambda} S^*_{z-\lambda}$ is Fredholm for all $\lambda \in D$ and it is easy to see that $\{S_{z-\lambda}S^*_{z-\lambda} : \lambda \in D\}$ is a norm-continuous family of setfadjoint operators. So by Lemma (5.1.38) $dim(KerS_{z-\lambda}S^*_{z-\lambda})$ is a finite lower semi-continuous function in λ. By the Fredholmness of $S_{z-\lambda}S^*_{z-\lambda}$, $range(S_{z-\lambda})$ is closed and its codimension is equal to $dim(KerS_{z-\lambda}S^*_{z-\lambda})$. By the relation between $L(\lambda)|_{M \ominus zM}$ and S_z (cf Equality (2)), $L(\lambda)(M \ominus zM)$ is closed in H_w, and moreover it was shown that

$$dimCoker(S_z - \lambda) = dimCoker\left(L(\lambda)|_{M \ominus zM}\right),$$

so

$$L(\lambda)(M \ominus zM) = L(\lambda)M = [h(\lambda,\cdot)],$$

and

$$\begin{aligned} dim(H_w \ominus [h(\lambda,\cdot)]) &= dimCoker\left(L(\lambda)|_{M \ominus zM}\right) \\ &= dimCoker(S_z - \lambda) = dimr(S_{z-\lambda}S^*_{z-\lambda}), \end{aligned}$$

where $[h(\lambda,\cdot)]$ is the submoduie in H_w generated by function $h(\lambda, w)$. Since

$$Z_h(\lambda) = dim\, H_w \ominus [h(\lambda,\cdot)],$$

$Z_h(\lambda)$ is a finite lower semi-continuous function in λ.

We now show that $Z_h(\lambda)$ is a constant. If we fix a $r \in (0,1)$ and let

$$N = sup\{Z_h(\lambda) : \lambda \in rD\},$$

then by the lower semi-continuity of $Z_h(\lambda)$ over D, N is finite. We define

$$E_N = \{\lambda \in rD : Z_h(\lambda) = N\}.$$

for every fixed $\lambda \in E_N$, we can choose an $\eta \in (0,1)$ such that the zeros of $h(\lambda, w)$ lie inside ηD.

By compIex function theory, if μ in rD is close enough to λ then $Z_h(\mu)$ and $Z_h(\lambda)$ have the same number of zeros in ηD which follows that $Z_h(\mu) \geq N$ and this is possible only if $Z_h(\mu) = N$ and therefore μ is in E_N. This shows that E_N is open. But by the lower semi-continuity of $Z_h(\lambda) E_N$ is also closed, so $E_N = rD$, and hence $Z_h(\lambda) = N$ for all $\lambda \in rD$. The proof is finished if we let $r \to 1^-$.

Example (5.1.40)[191]: If $h(z, w) = z - 0.5w$, then

$$Z_h(\lambda) = \begin{cases} 0, & 0.5 \leq |\lambda| < 1; \\ 1, & |\lambda| < 0.5. \end{cases}$$

This implies that $L(\lambda)$ is not compact on $H^2(D^2) \ominus [z - 0.5w]$.

The following example shows that the condition in Theorem (5.1.39) is not sufficient.

Example (5.1.41)[191]: Let $h(z, w) = \phi(z)\phi(w)$ where $\phi(w) = exp\left(i\frac{w+1}{w-1}\right)$ which is a singular inner function, and set $M = hH^2(D^2)$. It is not hard to check that $H_w \ominus \phi(w)H_w \subset H^2(D^2) \ominus M$, and since $\phi(w)$ is a singular inner function, $H_w \ominus \phi(w)H_w$ is infinite dimensional. Since $L(0)$ acts on $H_w \ominus \phi(w)H_w$ as an isometry, it is not compact.But $Z_h(\lambda)$ is constant 0.

Question (5.1.42)[191]: What is a necessary and sufficient condition for the compactness of $L(0)$ on $H^2(D^2) \ominus M$ in the case M is of rank 1?

The case when h is a polynomial seems more interesting. If p is a polynomial in $C(z, w)$ for which $L(0)$ is compact on $H^2(D^2) \ominus [p]$, and q is a factor of p, then $L(0)$ is compact on $H^2(D^2) \ominus [q]$ since $H^2(D^2) \ominus [q]$ is a subspace of $H^2(D^2) \ominus [p]$. This means that if p satisfies certain conditions which make $L(0)$ compact then so does every factor of p.

Since the evaluation operators and S_z, S_w are closely related, the compactness of $L(0)$ has an effect on their behaviors. The following theorem is an example.

Theorem (5.1.43)[191]: *If $L(0)$ is compact on $H^2(D^2) \ominus M$ and $dim(ker S_z) < \infty$, then*

(a) $D_z|_{M \ominus zM}$ *is compact*;

(b) $[S_z^*, S_w]$ *is compact*;

(c) $[S_z^*, S_z]$ *is compact*.

Proof. (a). Since

$$S_z S_z^* = 1 - L^*(0)L(0),$$

$S_z S_z^*$ is Fredholm, and hence has closed range with finite codimension. This implies that $S_z$has closed range with finite codimension. If $dim(ker S_z) < \infty$ then S_z is Fredholm. Since by Proposition (5.1.34)(c)

$$S_z D_z = -\left(L(0)|_{H^2(D^2)\ominus M}\right)^* L(0)|_{M\ominus zM}$$

which is compact under the condition, $D_z|_{M\ominus zM}$ is compact.

(b) Since for every $f \in H^2(D^2) \ominus M$ $p\bar{z}f = 0, S_z^* f = q\bar{z}f = P\bar{z}f$, where P is the orthogonal projection from $L^2(T^2)$ onto $H^2(D^2)$. Therefore

$$[S_z^*,\ S_w]f = q\bar{z}qwf - qwP\bar{z}f = q\bar{z}qwf - qw\bar{z}f$$
$$= -q\bar{z}(wf - qwf) = -q\bar{z}pwf = -D_z D_w^* f,$$

where the last equality comes from a parallel statement of Proposition (5.1.16) for D_w. Since $D_z|_{M\ominus zM}$ is compact, $[S_z^*,\ S_w]$ is compact.

(c) Since by Proposition (5.1.34)

$$[S_z^*,\ S_z] = L^*(0)L(0) - D_z D_z^*,$$

the assertions in c also follows from the fact that $D_z|_{M\ominus zM}$ is compact.

Note that if we assume in Theorem (5.1.43) that $L(0)$ is tIilbert-Schmidt on $H^2(D^2) \ominus M$ then the operators in assertions a and b are both Hilbert-Schmidt, and $[S_z^*,\ S_z]$ is trace class.

Many results obtained so far have manifested a close tie between S_z and $L(0)$. This tie is not only theorectically interesting, but also practically useful. Examples show that in some cases it is much easier to calculate $L(0)$, but in other cases it is much easier to deal with S_z. The relationship between the two makes it possible to study them even in the hard cases.

Remarks preceeding Theorem (5.1.7) say that S_z serves as a canonical model for a large class of contractions. It will be interesting to see how the results apply to some concrete examples. In the following we will take a look at the unilateral shift of a finite multiplicity and the Bergman shift. Both operators have been extensively studied. The following two examples show that we can get new results if we study these two operators of $H^2(D^2)$.

Example (5.1.44)[191]: We first look at the unilateral shift. If n is any integer and we let

$$K = H^2(D^2) \ominus w^n H^2(D^2),$$

then by Lemma (5.1.9)

$$K = H_z \oplus wH_z \oplus \cdots \oplus w^{n-1}H_z$$

which is equivalent to the vector valued Hardy space $H^2(T, C^2)$, and H_z on $K$is equivalent to $z \otimes I_n$. It is easy to see that for every $\lambda \in D$,

$$L(\lambda)(K) = span\{1, w, w^2, \dots, w^n\}$$

and hence $L(\lambda)$ is of rank n when restricted to K.

Example (5.1.45)[191]: If $M = [z - w]$, then $H^2(D^2) \ominus [z - w]$ is equivalent to $L_a^2(D)$, the Bergman space over the unit disc (cf. [215]), and S_z on $H^2(D^2) \ominus [z - w]$, whirl1 is equal to S_w, is unitarily equivalent to the Bergman shift. We see in Example (5.1.25) that evaluation operators on $H^2(D^2) \ominus M$ are compact. This fact is also obtained by the following computation. It is known that

$$e_n(z, w) = \frac{1}{\sqrt{n+1}}(z^n + z^{n-1}w + \cdots + zw^{n-1} + w^n), n \geq 0$$

is an orthonormal basis for $H^2(D^2) \ominus [z - w]$ and from which it is easy to see that the left evaluations behave similarly as the right evaluations do. We suffice to check the compactness for $R(0)$.

If $f = \sum_{j=0}^{\infty} c_j e_j$ is any function in $H^2(D^2) \ominus [z-w]$, then

$$R(0)f = \sum_{n=0}^{\infty} c_n R(0) e_n = \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n+1}} z^n.$$

If we let $U : H^2(D^2) \ominus [z-w] \to H_z$ be the operator defined by

$$U(e_n) = z^n, \quad n \geq 0;$$

and $K : H_z \to H_z$ be the map defined by

$$K(z^n) = \frac{1}{\sqrt{n+1}} z^n, \quad n \geq 0$$

then U is a unitary map and K is compact. Computations above actually yield the polar decomposition of $R(0)$:

$$R(0)f = KUf.$$

This implies that $R(\lambda) : H^2(D^2) \ominus [z-w] \to H_z$ is compact.

The observations in Example (5.1.44), (5.1.45), Corollary (5.1.21), Corollary (5.1.36) and Theorem (5.1.37) are combined to give the following

Theorem (5.1.46)[191]: *We assume that $\mathcal{H}$ is $H^2(T, C^n)$ or $L_a^2(D)$ and X is the multiplication by the coordinate function z in $\mathcal{H}$. If N is an invariant subspace of S and X is the compression of S to the quotient space $\mathcal{H} \ominus N$ then*

(a) $\mathcal{H} \ominus N$ is finite dimensional if X is a strict contraction;

(b) $\sigma(X) \cap D \subset \sigma_p(X) \cup \overline{\sigma_p(X^*)}$;

(c) if X is Fredholm, then $X - \lambda$ is Fredholm with ind $ind(X) = ind(X - \lambda)$ for all $\lambda \in D$.

Proof. If a bounded linear operator is compact, then it is compact when restricted to any closed subspace. Example (5.1.44) and (5.1.45) show that the evaluation operators are compact on $\mathcal{H}$ and hence they are compact on $\mathcal{H} \ominus N$. The theorem then follows from Corollary (5.1.21), Corollary (5.1.36) and Theorem (5.1.37).

We look at in Theorem (5.1.46)(a) in the case $\mathcal{H} = H^2(T)$.

If N is a invariant subspace of z in $H^2(T)$, then By Beurling's Theorem $N = \phi H^2(T)$ for some inner function ϕ The compression of z to $H^2(T) \ominus N$ is of class C_o and its spectrum is equal to the *spectrum* of ϕ (cf. [194][204]).If the compression is a strict contraction, then ϕ has no singular part and its zero set has no accummulation point on the unit circle which is possible only if ϕ is a finite Blaschke product and hence $H^2(T) \ominus N$ is finite dimensional.

Section (5.2): A Coburn Type Theorem:

For $\mathbb{T}$ be the boundary of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. The bidisk $\mathbb{D}^2$ and torus $\mathbb{T}^2$ are the cartesian products of 2 copies of $\mathbb{D}$ and $\mathbb{T}$ respectively. We let $L^p(\mathbb{T}^2) = L^p(\mathbb{T}^2, \sigma)$ denote the usual Lebesgue space on $\mathbb{T}^2$ where $\sigma = \sigma_2$ is the normalized Haar measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{D}^2)$ is the closure of the holomorphic polynomials in $L^2(\mathbb{T}^2)$. As is well known, we can identify a function in $H^2(\mathbb{D}^2)$ with its holomorphic extension to $\mathbb{D}^2$ via the Poisson extension. Thus, we will use the same notation for a function $f \in H^2(\mathbb{D}^2)$ and its holomorphic extension f on $\mathbb{D}^2$. Let Pdenote the orthogonal projection from $L^p(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$. For a function $u \in L^\infty(\mathbb{T}^2)$, the Toeplitz operator T_u with symbol u is defined by

$$T_u f = P(uf)$$

for functions $f \in H^2(\mathbb{D}^2)$. Then clearly T_u is a bounded linear operator on $H^2(\mathbb{D}^2)$.

On the Hardy space of the unit disk, a celebrated theorem of Coburn asserts that for a nonzero Toeplitz operator, we have either it is injective or its adjoint operator is injective. This theorem is implicit in the proof of Theorem 4.1 of [219]. See also Proposition 7.2.4 of [220]. Later, Vukotić [224] reproved the theorem by making the statement above more explicit by showing that the range of a nonzero Toeplitz operator which is not injective contains the set of all analytic polynomials.

We naturally consider the corresponding problem for Toeplitz operators acting on the Hardy space of the bidisk. First of all, we should mention that the Coburn type theorem fails generally on the bidisk. For an example, one can see

$$T_{z^2\overline{w}^2}(z) = 0 = T^*_{z^2\overline{w}^2}(z)$$

on $H^2(\mathbb{D}^2)$ where S^* denotes the adjoint operator of a bounded operator S. Thus the Toeplitz operator $T_{z^2\overline{w}^2}$ doesn't satisfy the Coburn type theorem. On the other hand, we can easily see that a Toeplitz operator with a nonzero (anti-)holomorphic symbol satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$. Also, one can check that a Toeplitz operator induced by a symbol depending only one variable satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$; see Corollary (5.2.16). In view of this observation, we naturally pose the following problem: For which symbol, does the corresponding Toeplitz operator satisfy the Coburn type theorem on $H^2(\mathbb{D}^2)$?

Motivated by examples mentioned above, we consider three classes of symbols as outlined below:

(i) symbols of the form $u = \varphi + \bar{\psi}$ where $\bar{\varphi}, \psi$ are bounded holomorphic on $\mathbb{D}^2$ and
   (a) φ is bounded by 1 and ψ is inner,
   (b) Or $\varphi = \varphi(z)$ is general and $\psi = \psi(w)$ is inner,
   (c) Or ψ is not assumed to be inner.

(ii) symbols of the form $u = f(z)g(w)$ where f, g are bounded on $\mathbb{T}$.

(iii) symbols of the form $u = \sum_{j=0}^{\infty} \overline{h_j}(z)w^j$ where h_j is holomorphic on $\mathbb{D}$.

We then provide several sufficient conditions on the symbols u for which T_u satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$. More explicitly, we first consider pluriharmonic symbols of the form $\varphi + \psi$ where $\varphi \in H^\infty(\mathbb{D}^2)$ and ψ is a non-constant inner function. Here the space $H^\infty(\mathbb{D}^2)$ denotes the space of all bounded holomorphic functions on $\mathbb{D}^2$ and we write $\|\varphi\|_\infty$ for the essential supremum norm for a function $\varphi \in L^\infty(\mathbb{D}^2)$. Also we say that a function in $H^\infty(\mathbb{D}^2)$ is called inner if the modulus of its radial limit is equal to 1 a.e. on $\mathbb{T}^2$; see [223]. For such a pluriharmonic symbol $u = \varphi + \bar{\psi}$, if $\|\varphi\|_\infty < 1$, then we first show that T_u^* is injective but T_u is not injective. Moreover we describe the kernel of T_u; see Theorem (5.2.2).

Specially, for symbols of the form $u = \varphi(z) + \bar{\psi}(w)$ where φ and ψ depend on a different single variable, we show that the boundedness condition $\|\varphi\|_\infty < 1$ can be removed. More explicitly, we characterize the injectivity of T_u and then, as an application, we show that the corresponding Toeplitz operator satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$; see Theorem (5.2.3) and Corollary (5.2.4).

We also consider general ψ other than inner functions. For such symbols $u = \varphi(z) + \bar{\psi}(w)$, if $\|\varphi\|_\infty \neq \|\psi\|_\infty$ or $\|\varphi\|_\infty = \|\psi\|_\infty = 1$ together with certain boundary conditions, then we show that T_u satisfies the Coburn type theorem; see Theorem (5.2.5) and Theorem (5.2.9). But we don't know whether a Toeplitz operator with general pluriharmonic symbol satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$.

We consider symbols which are products of two one variable functions depending on a different single variable. We consider symbols of the form $u = f(z)g(w)$ where $f, g \in L^\infty(T)$ are nonzero functions. For such a symbol, we first describe the kernel of T_u and then, as immediate consequences, obtain several kinds of symbols of Toeplitz operators satisfying the Coburn type theorem; see Theorem (5.2.13) and its corollaries.

We consider the symbols of the form $u = \sum_{j=0}^{\infty} \bar{h}_j(z)w^j$ where h_j is holomorphic on $\mathbb{D}$. We then show that the dimension of kernel of T_u can be 0 or ∞ according to choices of h_j; see Theorem (5.2.19). We were not able to characterize the injectivity for Toeplitz operators with general symbol.

We let $H^2(\mathbb{D})$ for the Hardy space of the unit disk $\mathbb{D}$ and $L^p(\mathbb{T}) = L^p(T, \sigma_1)$ denote the usual Lebesgue space on $\mathbb{T}$ where σ_1 is the normalized Lebesgue measure on $\mathbb{T}$. Also we write Q for the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. With the identification of a function in $H^2(\mathbb{D})$ with its holomorphic extension on $\mathbb{D}$, for each $z \in \mathbb{D}$, the reproducing kernel K_z for $H^2(\mathbb{D})$ is the well known Cauchy kernel given by

$$K_z(\zeta) = \frac{1}{1 - \bar{z}\zeta}, \zeta \in \mathbb{T}.$$

Thus the projection Q can be written as

$$Q_\varphi(z) = \int_{\mathbb{T}} \varphi \overline{K_z} d\sigma_1$$

for $\varphi \in L^2(\mathbb{T})$. Given $u \in L^\infty(\mathbb{T})$, the 1-dimensional Toeplitz operator t_u with symbol u is the bounded linear operator on $H^2(\mathbb{D})$ defined by

$$t_u f = Q(uf)$$

for functions $f \in H^2(\mathbb{D})$.

Recall that we can also identify a function in $H^2(\mathbb{D}^2)$ with its holomorphic extension on $\mathbb{D}^2$. With this identification, given $x = (z, w) \in \mathbb{D}^2$, the reproducing kernel R_x for $H^2(\mathbb{D}^2)$ is given by

$$R_x(y) = \frac{1}{(1 - \bar{z}\zeta)(1 - \bar{w}\eta)}, y = (\zeta, \eta) \in \mathbb{T}^2$$

and thus we can write the projection P as $P\varphi(x) = \int_{\mathbb{T}^2} \varphi \overline{R_x} d\sigma_2$ for $\varphi \in L^2(\mathbb{T}^2)$. See Chapter 3 of [223] or Chapter 9 of [225] for details and related facts. Noting $R_{(z,w)}(\zeta, \eta) = K_z(\zeta)K_w(\eta)$, we have

$$P[f(\zeta)g(\eta)](z, w) = Qf(z)Qg(w)$$

for every $f, g \in L^2(\mathbb{T})$. Since $Q\bar{f} = \overline{f(0)}$ for every $f \in H^2(\mathbb{D})$, it follows that

$$T_{z^2\bar{w}^2}(z) = P(z^3\bar{w}^2) = Q(z^3)Q(\bar{w}^2) = 0.$$

Similarly, since $T_u^* = T_{\bar{u}}$ for every $u \in L^\infty(\mathbb{T}^2)$, one sees that $T_{z^2\bar{w}^2}^*(z) = T_{z^2\bar{w}^2}(z) = 0$. This shows that both $T_{z^2\bar{w}^2}$ and its adjoint operator are not injective and hence the Coburn type theorem fails on $H^2(\mathbb{D}^2)$.

We consider several kinds of pluriharmonic symbols and then study the problem of when the corresponding Toeplitz operators are injective. Given a function $f \in H^2(\mathbb{D}^2)$, we let

$$\|f\| = \left(\sup_{0 \le r < 1} \int_{\mathbb{T}^2} |f(r\zeta, r\eta)|^2 \, d\sigma_2(\zeta, \eta) \right)^{1/2}$$

be the usual $H^2(\mathbb{D}^2)$-norm of f.

We start with the following simple and useful lemma.

Lemma (5.2.1)[218]: Let $u = \varphi + \bar{\psi}$, where ψ is an inner function on $\mathbb{D}^2$ and $\varphi \in L^\infty(\mathbb{D}^2)$ with $\|\varphi\|_\infty \le 1$. If $u \not\equiv 0$, then T_u^* is injective on $H^2(\mathbb{D}^2)$.

Proof. Suppose $T_u^* h = 0$ for some $h \in H^2(\mathbb{D}^2)$ and then $P(\bar{\varphi} h) + \psi h = 0$. Since ψ is inner, we note

$$\|h\| = \|\psi h\| = \|P(\bar{\varphi} h)\| \le \|\bar{\varphi} h\| \le \|h\|$$

and then $\|P(\bar{\varphi} h)\| = \|\bar{\varphi} h\|$. Thus $P(\bar{\varphi} h) = \bar{\varphi} h$ and

$$\bar{u} h = \bar{\varphi} h + \psi h = P(\bar{\varphi} h) + \psi h = 0.$$

Since $u \equiv 0$, we have $h = 0$ and T_u^* is injective, as desired. The proof is complete.

In addition, if $\|\varphi\|_\infty < 1$, then T_u is not injective as shown in the following. The notation $\ker L$ stands for the kernel of an operator L.

Theorem (5.2.2)[218]: Let ψ be a non-constant inner function on $\mathbb{D}^2$ and $\varphi \in H^\infty(\mathbb{D}^2)$. Assume $\|\varphi\|_\infty < 1$ and put $u = \varphi + \bar{\psi}$. Then T_u^* is injective but T_u is not injective on $H^2(\mathbb{D}^2)$. Moreover we have

$$\ker T_u = \frac{1}{1 + \varphi\psi} [H^2(\mathbb{D}^2) \ominus \psi H^2(\mathbb{D}^2)] \tag{9}$$

and $\dim \ker T_u = \infty$.

Proof. By Lemma (5.2.1), T_u^* is injective on $H^2(\mathbb{D}^2)$. To prove (9), we first note that $1/(1 + \varphi\psi) \in H^\infty(\mathbb{D}^2)$ because $\|\varphi\|_\infty < 1$. Since ψ is inner, $u = \varphi + \bar{\psi} = \bar{\psi}(\psi\varphi + 1)$ a.e. on $\mathbb{T}^2$ and hence $T_u = T_{\bar{\psi}} T_{1+\varphi\psi}$. Since $1 + \varphi\psi$ is invertible in $H^\infty(\mathbb{D}^2)$, we have

$$\ker T_u = \frac{1}{1 + \varphi\psi} \ker T_{\bar{\psi}} = \frac{1}{1 + \varphi\psi} [H^2(\mathbb{D}^2) \ominus \psi H^2(\mathbb{D}^2)]$$

because $\ker T_{\bar{\psi}} = \left(\operatorname{ran} T_\psi\right)^\perp = [H^2(\mathbb{D}^2) \ominus \psi H^2(\mathbb{D}^2)]$, so (9) holds as desired. The proof is complete.

We don't know whether condition $\|\varphi\|_\infty < 1$ in Theorem (5.2.2) is essential. But, if φ and ψ depend on a different variable each other, we will show that one of T_u and T_u^* is injective. Thus the corresponding Toeplitz operator satisfies the Coburn type theorem on $H^2(\mathbb{D}^2)$.

We let $H^2(z)$ and $H^2(w)$ be the z and w variable Hardy spaces respectively. Also, we write $H^\infty(z)$ and $H^\infty(w)$ for the spaces of all bounded holomorphic functions depending on only z and w variable respectively.

We denote by ball $H^\infty(z)$ the closed unit ball of $H^\infty(z)$. It is well known that $\varphi \in$ball $H^\infty(z)$ is an extreme point of ball $H^\infty(z)$ if and only if

$$\int_{\mathbb{T}} \log(1 - |\varphi|) \, d\sigma_1 = -\infty.$$

Moreover, if $\varphi \in$ ball $H^\infty(z)$ is not an extreme point of ball $H^\infty(z)$, then there is an outer function $\eta \in$ ball $H^\infty(z)$ satisfying $|\varphi|^2 + |\eta|^2 = 1$ a.e. on $\mathbb{T}$. See Chapter 9 of [222] for details and related facts.

Let $\psi \in H^2(w)$ be a non-constant inner function and put

$$K_\psi(w) := H^2(w) \ominus \psi(w)H^2(w).$$

Then it is known that $K_\psi(w) \neq \{0\}$ and $H^2(w) = \bigoplus_{n=0}^{\infty} K_\psi(w)\psi(w)^n$.

Moreover, we have

$$H^2(\mathbb{D}^2) = \bigoplus_{n=0}^{\infty} [H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)]\psi(w)^n$$

$$= \bigoplus_{n=0}^{\infty} \left[H^2(z) \otimes K_\psi(w)\right]\psi(w)^n.$$

Let $\{e_k : k \geq 0\}$ denote an orthonormal basis of $K_\psi(w)$. Since $\{e_k\psi^n : n, k \geq 0\}$ is an orthonormal basis of $H^2(w)$, it follows that

$$H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2) = \bigoplus_{k=0}^{\infty} e_k(w)H^2(z),$$

and

$$H^2(\mathbb{D}^2) = \bigoplus_{k=0}^{\infty} e_k(w)\psi(w)^n H^2(z). \tag{10}$$

The following result characterizes non-injective Toeplitz operators in case when such φ and ψ depend on a different variable each other.

Theorem (5.2.3)[218]: Let $\psi \in H^\infty(w)$ be a non-constant inner function and $\varphi \in H^\infty(z)$. Put $u(z,w) = \varphi(z) + \bar{\psi}(w)$. Then the following statements are equivalent.

(i) T_u is not injective on $H^2(\mathbb{D}^2)$.

(ii) $\|\varphi\|_\infty \leq 1$ and φ is not an extreme point of ball $H^\infty(z)$.

In which case, we have

$$\ker T_u = \frac{\eta(z)}{1 + \varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)]$$

and $\dim \ker T_u = \infty$, where $\eta \in \text{ball}\, H^\infty(z)$ is an outer function satisfying $|\varphi|^2 + |\eta|^2 = 1$ a.e. on $\mathbb{T}$.

Proof. First assume (i) and then $T_u f = 0$ for some nonzero function $f \in H^2(\mathbb{D}^2)$. By (10), we may write

$$f = \sum_{n,k=0}^{\infty} f_{n,k}(z)e_k(w)\psi(w)^n,\ z, w \in \mathbb{D} \tag{11}$$

where $f_{n,k}(z) \in H^2(z)$ for every $n, k \geq 0$. Since ψ is inner, we note that

$$0 = T_u f = \varphi(z)f + \sum_{n,k=0}^{\infty} P\left(f_{n,k}e_k\psi^n\bar{\psi}\right)$$

$$= \varphi(z)f + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} f_{n,k}(z)e_k(w)\,\psi(w)^{n-1}$$

$$= \sum_{n,k=0}^{\infty} \varphi(z)f_{n,k}(z)e_k(w)\psi(w)^n + \sum_{n,k=0}^{\infty} f_{n+1,k}(z)e_k(w)\psi(w)^n$$

$$= \sum_{n,k=0}^{\infty} \left[\varphi(z)f_{n,k}(z) + f_{n+1,k}(z)e_k(w)\psi(w)^n\right]$$

for all $z, w \in \mathbb{D}$. It follows that $\varphi f_{n,k} + f_{n+1,k} = 0$ for every $n, k \geq 0$ and hence, for each $k \geq 0, f_{n,k} = f_{0,k}(-\varphi)^n$ for all n. Thus

$$f = \sum_{n,k=0}^{\infty} f_{0,k}(z)\big(-\varphi(z)\big)^n e_k(w)\psi(w)^n.$$

Since $f \in H^2(\mathbb{D}^2)$ and $\{e_k\psi^n : n, k \geq 0\}$ is an orthonormal basis of $H^2(w)$, we see

$$\infty > \|f\|^2 = \sum_{n,k=0}^{\infty} \left\|f_{0,k}(-\varphi)^n\right\|^2 = \int_{\mathbb{T}} \left(\sum_{k=0}^{\infty}\left|f_{0,k}\right|^2\right)\left(\sum_{n=0}^{\infty}|\varphi|^{2n}\right) d\sigma_1.$$

Since $f \neq 0$, we have $f_{0,\ell} \neq 0$ for some $\ell \geq 0$. Thus the above shows that $|\varphi(z)| < 1$ a.e. on $\mathbb{T}$ and then $\|\varphi\|_\infty \leq 1$. Also, we have

$$\int_{\mathbb{T}} \frac{\sum_{k=0}^{\infty}\left|f_{0,k}\right|^2}{1 - |\varphi|^2}\, d\sigma_1 < \infty \tag{12}$$

and

$$f = \frac{1}{1 + \varphi(z)\psi(w)} \sum_{k=0}^{\infty} f_{0,k}(z)e_k(w). \tag{13}$$

Now, noting

$$\log\left|f_{0,\ell}\,(z)\right|^2 \leq \log \frac{\sum_{k=0}^{\infty}\left|f_{0,k}(z)\right|^2}{1 - |\varphi(z)|^2} \leq \frac{\sum_{k=0}^{\infty}\left|f_{0,k}(z)\right|^2}{1 - |\varphi(z)|^2},$$

we have by the Jensen inequality

$$-\infty < \int_{\mathbb{T}} \log\left|f_{0,\ell}\right|^2 d\sigma_1 \leq \int_{\mathbb{T}} \log \sum_{k=0}^{\infty}\left|f_{0,\ell}\right|^2 d\sigma_1 - \int_{\mathbb{T}} \log(1 - |\varphi|^2)\, d\sigma_1$$

$$\leq \int_{\mathbb{T}} \frac{\sum_{k=0}^{\infty}\left|f_{0,k}\right|^2}{1 - |\varphi|^2} d\sigma_1 < \infty.$$

We also have

$$-\infty < \int_{\mathbb{T}} \log\left|f_{0,\ell}\right|^2 d\sigma_1 \leq \int_{\mathbb{T}} \log \sum_{k=0}^{\infty}\left|f_{0,k}\right|^2 d\sigma_1$$

and

$$0 \leq -\int_{\mathbb{T}} \log(1 - |\varphi|)\, d\sigma_1 \leq \infty.$$

Then, the above inequalities imply $-\infty < \int_{\mathbb{T}} \log(1-|\varphi|)\, d\sigma_1$ and hence φ is not an extreme point of ball $H^\infty(z)$. Thus (ii) holds.

Now assume (ii) and show (i). By the remark just before this proposition, there is an outer function $\eta \in ball\ H^\infty(z)$ satisfying $|\varphi|^2 + |\eta|^2 = 1$ a.e. on $\mathbb{T}$. Thus

$$\frac{|\eta|^2}{1-|\varphi|^2} = 1 \quad \text{a.e. on } \mathbb{T}. \tag{14}$$

To prove (i), it suffices to show

$$\frac{\eta(z)}{1+\varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)] \subset \ker T_u. \tag{15}$$

To do this, we let $g \in H^2(z)$ and $h \in K_\psi(w)$. Note

$$\frac{\eta(z)g(z)h(w)}{1+\varphi(z)\psi(w)} = \sum_{n=0}^{\infty} \eta(z)g(z)h(w)\big(-\varphi(z)\big)^n \psi(w)^n$$

for all $z, w \in \mathbb{D}$. Since $h \in K_\psi(w)$, we have $h\psi^n \perp h\psi^k$ for every $n, k \geq 0$ with $n \neq k$ and then

$$\left\|\frac{\eta(z)g(z)h(w)}{1+\varphi(z)\psi(w)}\right\|^2 = \sum_{n=0}^{\infty} \left\|\eta(z)g(z)\big(-\varphi(z)\big)^n\right\|^2 \|h(w)\psi(w)^n\|^2$$

$$= \|h(w)\|^2 \int_{\mathbb{T}} \frac{|\eta(z)g(z)|^2}{1-|\varphi(z)|^2}\, d\sigma_1 = \|h(w)\|^2 \|g(z)\|^2 < \infty$$

and hence

$$\chi(z,w) := \frac{\eta(z)g(z)h(w)}{1+\varphi(z)\psi(w)} \in H^2(\mathbb{D}^2).$$

Moreover we see

$$T_u\chi = \varphi\chi + P\left(\sum_{n=0}^{\infty} \eta g h(-\varphi)^n \psi^n \bar{\psi}\right) = \varphi\chi + \sum_{n=0}^{\infty} \eta g h(-\varphi)^{n+1}\psi^n$$

$$= \sum_{n=0}^{\infty} [\varphi(-\varphi)^n + (-\varphi)^{n+1}]\eta g h \psi^n = 0$$

and thus (15) follows as desired.

To complete the proof, we need to show that the reverse inclusion of (15) holds. To prove this, let $f \in \ker T_u$ and then $T_u f = 0$. Then, using the same notation as in (11), we have from (12) and (14)

$$\infty > \int_{\mathbb{T}} \frac{\sum_{k=0}^{\infty}|f_{0,k}|^2}{1-|\varphi|^2}\, d\sigma_1 = \int_{\mathbb{T}} \frac{|\eta|^2 \sum_{k=0}^{\infty}|f_{0,k}/\eta|^2}{1-|\varphi|^2}\, d\sigma_1 = \int_{\mathbb{T}} \sum_{k=0}^{\infty} \left|\frac{f_{0,k}}{\eta}\right|^2 d\sigma_1 .$$

Thus $f_{0,k} \in \eta H^2(z)$ for every $k \geq 0$ and (13) shows

$$f \in \frac{\eta(z)}{1+\varphi(z)\psi(w)} \bigoplus_{k=0}^{\infty} e_k(w)H^2(z) = \frac{\eta(z)}{1+\varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)]$$

because $\{e_k(w)\}_{k\geq 0}$ is an orthonormal basis of $K_\psi(w)$. Thus we get

$$\ker T_u \subset \frac{\eta(z)}{1+\varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)]$$

as desired. The proof is complete.

Having Theorem (5.2.3), we have two remarks by using the same notations being there.

(a) If φ is an extreme point in ball $H^\infty(z)$, Theorem (5.2.3) shows $\ker T_u = \{0\}$. If $\|\varphi\|_\infty \le 1$ and φ is not an extreme point of ball $H^\infty(z)$, we have by Theorem (5.2.3)

$$\ker T_u = \frac{\eta(z)}{1+\varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)].$$

If $\|\varphi\|_\infty < 1$, then η is invertible in $H^\infty(z)$ and hence

$$\ker T_u = \frac{1}{1+\varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)],$$

which is already noticed in (a). Also, if $\|\varphi\|_\infty = 1$, then η is not invertible in $H^\infty(z)$, so we have

$$\ker T_u = \frac{\eta(z)}{1+\varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)]$$
$$\subsetneqq \frac{1}{1+\varphi(z)\psi(w)}[H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)].$$

(b) Given a non-constant function $\theta \in L^\infty(\mathbb{T})$, it turns out that $\ker t_\theta = p[H^2(\mathbb{D}) \ominus zqH^2(\mathbb{D})]$ for an outer function p and an inner function q on $\mathbb{T}$. Also, it has been known that the map

$$H^2(\mathbb{D}) \ominus zqH^2(\mathbb{D}) \ni f \to pf \in \ker t_\theta$$

is an isometry; see [221] for details. In view of Theorem (5.2.3) together with this result, we remark that the map

$$H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2) f \to \frac{\eta}{1+\varphi\psi} f \in \ker T_u$$

is an isometry. To see this, let $f \in H^2(\mathbb{D}^2) \ominus \psi(w)H^2(\mathbb{D}^2)$ and write

$$f(z,w) = \sum_{k=0}^{\infty} f_k(z)e_k(w)\, , f_k(z) \in H^2(z).$$

Then we note

$$\frac{\eta(z)}{1+\varphi(z)\psi(w)} f(z,w) = \sum_{k,n=0}^{\infty} \eta(z) f_k(z)\big(-\varphi(z)\big)^n e_k(w)\psi(w)^n.$$

Since $e_k\psi^n \perp e_i\psi^j$ for $(k,n) \neq (i,j)$, we have

$$\eta(z)f_k(z)\big(-\varphi(z)\big)^n e_k(w)\psi(w)^n \perp \eta(z)f_i(z)\big(-\varphi(z)\big)^j e_i(w)\psi(w)^j$$

for $(k,n) \neq (i,j)$. Hence

$$\left\|\frac{\eta}{1+\varphi\psi} f\right\|^2 = \sum_{k,n=0}^{\infty} \|\eta f_k(-\varphi)^n e_k \psi^n\|^2$$
$$= \sum_{k,n=0}^{\infty} \|\eta f_k \varphi^n\|^2 = \sum_{k=0}^{\infty} \int_{\mathbb{T}} \frac{|\eta|^2 |f_k|^2}{1-|\varphi|^2} d\sigma_1 = \|f\|^2.$$

As a simple application of Theorem (5.2.3), we have the following.

Corollary (5.2.4)[218]: Let $\psi \in H^{\infty}(w)$ be a non-constant inner function and $\varphi \in H^{\infty}(z)$. Put $u = \varphi(z) + \bar{\psi}(w)$. Then either T_u or T_u^* is injective on $H^2(\mathbb{D}^2)$.

Proof. Suppose T_u is not injective. Theorem (5.2.3) shows $\|\varphi\|_\infty \le 1$ and then $T_u^* = T_{\overline{\varphi}+\psi}$ is injective by Lemma (5.2.1). The proof is complete.

We don't know whether the condition "ψ is inner" in Corollary (5.2.4) is essential. So we have a natural question: For non-constant $u = \varphi(z) + \bar{\psi}(w)$ where $\varphi, \psi \in H^{\infty}(\mathbb{D})$, is either T_u or T_u^* injective? In the rest, we will discuss this question under certain conditions on the essential sup-norms of φ, ψ.

We let $H^{\infty}(\mathbb{D})$ denote the space of all bounded holomorphic functions on $\mathbb{D}$ and use the same notation $\|\varphi\|_\infty = \sup_{z\in\mathbb{D}}|\varphi(z)|$ for $\varphi \in H^{\infty}(\mathbb{D})$. Also, for $\varphi \in H^{\infty}(\mathbb{D})$, put

$$E(\varphi) = \{e^{i\theta} \in \mathbb{T}: |\varphi(e^{i\theta})| \ge 1\}$$

and write $|S|$ for the Lebesgue measure of a Borel set $S \subset \mathbb{T}$.

Theorem (5.2.5)[218]: Let $\varphi, \psi \in H^{\infty}(\mathbb{D})$ be non-constant and $u = \varphi(z) + \bar{\psi}(w)$. If $\|\varphi\|_\infty = \|\psi\|_\infty$, then either T_u or T_u^* is injective on $H^2(\mathbb{D}^2)$.

Proof. We may assume that $0 < \|\psi\|_\infty < 1 < \|\varphi\|_\infty$. We shall prove that T_u is injective. To prove this, suppose not. Then $T_u f = 0$ for some $f \in H^2(\mathbb{D}^2)$ with $f \neq 0$. Since

$$0 = T_u f = P(\varphi f + \bar{\psi} f) = \varphi f + T_{\psi}^* f,$$

we see $T_{\psi}^* f = -\varphi f$ and hence

$$T_{\psi}^{*2} f = T_{\psi(w)}^*(-\varphi(z) f) = -\varphi(z) T_{\psi(w)}^* f = (-\varphi)^2 f.$$

Repeating the same argument, we have $T_{\psi}^{*n} f = (-\varphi)^n f$ for each $n = 1, 2, \cdots$. Since $\|\varphi\|_\infty > 1$, we have $\sigma_2(E(\varphi) \times \mathbb{T}) > 0$ and then

$$\|T_{\psi}^{*n} f\| = \|\varphi^n f\|^2 = \int_{\mathbb{T}^2} |\varphi|^{2n} |f|^2 d\sigma_2 \ge \int_{E(\varphi)\times\mathbb{T}} |f|^2 d\sigma_2 > 0$$

for each n because f is nonzero. On the other hand, since $\|\psi\|_\infty < 1$, we have

$$\|T_{\psi}^{*n} f\| \le \|T_{\psi}^*\|^n \le \|\psi\|_\infty^n \|f\| \to 0$$

as $n \to \infty$, which is a contradiction. The proof is complete.

As an immediate consequence of Theorem (5.2.5), we have the following, because $\varphi(z) + \bar{\psi}(w) = \varphi(z) + c + \overline{\psi(w) - \bar{c}}$ for any constant c.

Corollary (5.2.6)[218]: Let $\varphi, \psi \in H^{\infty}(\mathbb{D})$ be non-constant and $u = \varphi(z) + \bar{\psi}(w)$. If $\|\varphi + c\|_\infty \neq \|\psi - \bar{c}\|_\infty$ for some $c \in \mathbb{C}$, then either T_u or T_u^* is injective.

The following lemma is quite well known. For the sake of completeness we provide a proof.

Lemma (5.2.7)[218]: Let $\varphi \in H^{\infty}(\mathbb{D})$ be a non-constant function with $\|\varphi\|_\infty \le 1$. Then $\|t_{\varphi}^{*n} f\| \to 0$ as $n \to \infty$ for every $f \in H^2(\mathbb{D})$.

Proof. For each $z \in \mathbb{D}$ and each integer $n \ge 1$, we first have

$$\|t_{\varphi}^{*n} K_z\| = |\varphi(z)|^n \|K_z\|.$$

Since $\|\varphi\|_\infty \le 1$ and φ is non-constant, we have $|\varphi(z)| < 1$. It follows that $\|t_{\varphi}^{*n} K_z\| \to 0$ as $n \to \infty$. Since the set of all linear combinations of reproducing kernels is dense $H^2(\mathbb{D})$ and $\|t_{\varphi}^{*n}\| \le 1$ for all n, we have the assertion. The proof is complete.

In case when $\|\varphi\|_\infty = \|\psi\|_\infty$, we have the following.

Lemma (5.2.8)[218]: Let $\varphi, \psi \in H^{\infty}(\mathbb{D})$ be non-constant functions satisfying that $\|\varphi\|_{\infty} = \|\psi\|_{\infty} = 1$. Put $u(z,w) = \varphi(z) + \psi(w)$. If $|E(\varphi)| > 0$, then T_u is injective on $H^2(\mathbb{D}^2)$.

Proof. Suppose there exists a nonzero $f \in H^2(\mathbb{D}^2)$ satisfying that $T_u f = 0$. By the proof of Theorem (5.2.5), we have $T^{*n}_{\psi(w)} f = \left(-\varphi(z)\right)^n f$ and hence

$$\left\|T^{*n}_{\psi} f\right\|^2 = \int_{\mathbb{T}^2} |\varphi|^{2n} |f|^2 d\sigma \geq \int_{E(\varphi)\times\mathbb{T}} |f|^2 d\sigma > 0 \quad (16)$$

for every $n \geq 1$. Writing $f(z,w) = \sum_{k=0}^{\infty} f_k(w) z^k$ where $f_k(w) \in H^2(w)$, we have

$$T^{*n}_{\psi} f(z,w) = \sum_{k=0}^{\infty} P(\bar{\psi}^n f_k)(w) z^k = \sum_{k=0}^{\infty} \left(t^{*n}_{\psi} f_k\right)(w) z^k$$

for all $z, w \in \mathbb{D}$ and hence

$$\left\|T^{*n}_{\psi} f\right\|^2 = \sum_{k=0}^{\infty} \left\|t^{*n}_{\psi} f_k\right\|^2 \leq \sum_{k=0}^{\infty} \|f_k\|^2 = \|f\|^2 < \infty.$$

By Lemma (5.2.7), applying the Lebesgue dominated convergence theorem, we have $\left\|T^{*n}_{\psi} f\right\| \to 0$ as $n \to \infty$, which contradicts to (16). The proof is complete.

As an immediate consequence, we have the following.

Theorem (5.2.9)[218]: Let $\varphi, \psi \in H^{\infty}(\mathbb{D})$ be non-constant functions satisfying that $\|\varphi\|_{\infty} = \|\psi\|_{\infty} = 1$. Put $u(z,w) = \varphi(z) + \bar{\psi}(w)$. If either $|E(\varphi)| > 0$ or $|E(\psi)| > 0$, then either T_u or T_u^* is injective on $H^2(\mathbb{D}^2)$.

In view of Theorem (5.2.9), we ask a natural question: For $u = \varphi(z) + \bar{\psi}(w)$ where $\varphi, \psi \in H^{\infty}(\mathbb{D})$, $\|\varphi\|_{\infty} = \|\psi\|_{\infty} = 1$ and $|E(\varphi)| = |E(\psi)| = 0$, is either T_u or T_u^* injective on $H^2(\mathbb{D}^2)$?

We close with another observation. Before doing this, we have the following which can be easily checked.

Lemma (5.2.10)[218]: Let $\psi = \psi(z) \in H^{\infty}(\mathbb{D})$ be a nonzero function. Then the following statements are equivalent.

(i) ψ is an outer function.

(ii) t_{ψ}^* is injective on $H^2(\mathbb{D})$.

(iii) T_{ψ}^* is injective on $H^2(\mathbb{D}^2)$.

Proposition (5.2.11)[218]: Let $\varphi, \psi \in H^{\infty}(\mathbb{D})$ be non-constant functions and $u = \varphi(z) + \bar{\psi}(w)$. If φ has a non-constant inner factor and ψ is an outer function, then T_u is injective on $H^2(\mathbb{D}^2)$.

Proof. Suppose there exists a nonzero $f \in H^2(\mathbb{D}^2)$ such that $T_u f = 0$. Write $\varphi(z) = I(z)O(z)$ for the inner-outer factorization of φ. Since

$$H^2(\mathbb{D}^2) = \bigoplus_{n=0}^{\infty} [H^2(\mathbb{D}^2) \ominus I(z)H^2(\mathbb{D}^2)] I(z)^n,$$

we may write $f = \sum_{n=\ell}^{\infty} f_n I(z)^n$, where $f_n \in H^2(\mathbb{D}^2) \ominus I(z)H^2(\mathbb{D}^2)$ for some $\ell \geq 0$ with $f_\ell \neq 0$. Then we have

$$0 = T_u f = \varphi f + T_\psi^* f = \sum_{n=\ell}^{\infty} O(z) f_n I(z)^{n+1} + \sum_{n=\ell}^{\infty} \left(T_\psi^* f_n\right) I(z)^n.$$

Since $T_\psi^* f_n \in H^2(\mathbb{D}^2) \ominus I(z) H^2(\mathbb{D}^2)$ for every $n \geq \ell$, we have $T_\psi^* f_\ell = 0$. But, since ψ is outer, we have $f_\ell = 0$ by Lemma (5.2.10). This is a contradiction and therefore T_u is injective on $H^2(\mathbb{D}^2)$. The proof is complete.

We consider symbols which are products $u = \varphi(z)\psi(w)$ of two bounded functions φ, ψ on $\mathbb{T}$.

The following lemma will be useful in our applications.

Lemma (5.2.12)[218]: Let $\varphi, \psi \in L^\infty(\mathbb{T})$ and put $u = \varphi(z)\psi(w)$. Then we have $T_u = T_{\varphi(z)} T_{\psi(w)} = T_{\psi(w)} T_{\varphi(z)}$.

Proof. Let $f, g \in H^2(\mathbb{D})$ and write $\psi(w) g(w) = g_1(w) + t_{\psi(w)} g(w)$ for some $g_1(w) \in L^2(\mathbb{T})$ with $g_1(w) \perp H^2(w)$. Then we have

$$\begin{aligned} T_u f(z) g(w) &= P\left[\varphi(z) f(z) \left(g_1(w) + t_{\psi(w)} g(w)\right)\right] \\ &= P\left[\varphi(z) f(z) t_{\psi(w)} g(w)\right] \\ &= T_{\varphi(z)} f(z) t_{\psi(w)} g(w) = T_{\varphi(z)} T_{\psi(w)} f(z) g(w). \end{aligned}$$

Similarly $T_u f(z) g(w) = T_{\psi(w)} T_{\varphi(z)} f(z) g(w)$. Since $H^2(\mathbb{D}^2) = H^2(z) \otimes H^2(w)$, we have $T_u = T_{\varphi(z)} T_{\psi(w)} = T_{\psi(w)} T_{\varphi(z)}$, as desired. The proof is complete.

We describe the kernel of a Toeplitz operator whose symbol is a product of one variable functions.

Theorem (5.2.13)[218]: Let $\varphi, \psi \in L^\infty(\mathbb{T})$ be nonzero functions and $u = \varphi(z)\psi(w)$. Then we have

$$\ker T_u = \left[\ker t_{\varphi(z)} \otimes H^2(w)\right] + \left[H^2(z) \otimes \ker t_{\psi(w)}\right].$$

Moreover if $\ker T_u \neq \{0\}$, then $\dim \ker T_u = \infty$.

Proof. By Lemma (5.2.12), we see

$$\left[\ker t_{\varphi(z)} \otimes H^2(w)\right] + \left[H^2(z) \otimes \ker t_{\psi(w)}\right] \subset \ker T_u.$$

To prove the reverse inclusion, we note that

$$\begin{aligned} H^2(\mathbb{D}^2) = &\left[\ker t_{\varphi(z)} \otimes H^2(w) + H^2(z) \otimes \ker t_{\psi(w)}\right] \\ &\oplus \left(H^2(z) \ominus \ker t_{\varphi(z)}\right) \otimes (H^2(w) \ominus \left(\ker t_{\psi(w)}\right). \end{aligned}$$

Thus, to complete the proof, it suffices to show that T_u is injective on

$$E := \left(H^2(z) \ominus \ker t_{\varphi(z)}\right) \otimes (H^2(w) \ominus \left(\ker t_{\psi(w)}\right).$$

To do this, suppose $T_u f = 0$ for some $f \in E$ and let $\{e_i(z)\}_{i \geq 0}$ be an orthonormal basis of $H^2(z) \ominus \ker t_{\varphi(z)}$. Then we may write

$$f = \sum_{i=0}^{\infty} f_i(w) e_i(z), f_i \in H^2(w) \ominus \ker t_{\psi(w)}.$$

By Lemma (5.2.12), we have

$$0 = T_u f = T_{\varphi(z)} \sum_{i=0}^{\infty} T_{\psi(w)} f_i(w) e_i(z) = T_{\varphi(z)} \sum_{i=0}^{\infty} \left(t_{\psi(w)} f_i\right)(w) e_i(z).$$

Also, since

$$\sum_{i=0}^{\infty}\left(t_{\psi(w)}f_i\right)(w)e_i(z) \in \left(H^2(z) \ominus \ker t_{\varphi(z)}\right) \otimes H^2(w),$$

we write

$$\sum_{i=0}^{\infty}\left(t_{\psi(w)}f_i\right)(w)e_i(z) = \sum_{j=0}^{\infty} g_j(z)w^j \,, g_j \in H^2(z) \ominus \ker t_{\varphi(z)}.$$

Then, we have

$$0 = T_{\varphi(z)}\sum_{j=0}^{\infty} g_j(z)w^j = \sum_{j=0}^{\infty}\left(t_{\varphi(z)}g_j\right)(z)w^j,$$

so we have $t_\varphi g_j = 0$ for every $j \geq 0$. Since $g_j \perp \ker t_\varphi$, we have $g_j = 0$ for every j. Therefore

$$\sum_{i=0}^{\infty}\left(t_{\psi(w)}f_i\right)(w)e_i(z) = 0,$$

which shows that $t_\psi f_i = 0$ for every $i \geq 0$. Also, since $f_i \perp \ker t_\psi$, we see $f_i = 0$ for every i, so $f = 0$. Thus T_u is injective on E, as desired. The proof is complete.

We remark in passing that Theorem (5.2.13) can be generalized to tensor products of bounded linear operators on Hilbert spaces by the same way as in the proof: If A and B are bounded linear operators on Hilbert spaces H and K respectively, then

$$\ker(A \otimes B) = [\ker(A) \otimes K] + [H \otimes \ker(B)].$$

As immediate consequences of Theorem (5.2.13), we have several applications.

Corollary (5.2.14)[218]: Let $\varphi, \psi \in L^\infty(\mathbb{T})$ be nonzero functions and $u = \varphi(z)\psi(w)$. Then T_u is injective on $H^2(\mathbb{D}^2)$ if and only if both t_φ and t_ψ are injective on $H^2(\mathbb{D})$.

Corollary (5.2.15)[218]: Let $\varphi, \psi \in L^\infty(\mathbb{T})$ be nonzero functions. Put $u_1 = \varphi(z)\psi(w)$ and $u_2 = \varphi(z)\bar{\psi}(w)$. Then one of $T_{u_1}, T_{u_1}^*, T_{u_2}$ and $T_{u_2}^*$ is injective on $H^2(\mathbb{D}^2)$.

Proof. By the Coburn theorem, we see that either t_φ or $t_{\bar{\varphi}}$ is injective on $H^2(\mathbb{D})$. Also either t_ψ or $t_{\bar{\psi}}$ is injective on $H^2(\mathbb{D})$. Hence there are four cases to consider and then the result follows from Corollary (5.2.14). The proof is complete.

Taking $\varphi = 1$ or $\psi = 1$ in Corollary (5.2.15), we see that Toeplitz operators with symbols which depend on only one variable satisfy the Coburn type theorem.

Corollary (5.2.16)[218]: Let $u \in L^\infty(\mathbb{T}^2)$ be a symbol depending on only one variable. Then either T_u or T_u^* is injective on $H^2(\mathbb{D}^2)$.

For symbols which are products of holomorphic and antiholomorphic functions, we have the following.

Corollary (5.2.17)[218]: Let $\varphi, \psi \in H^\infty(\mathbb{D})$ be nonzero functions and put $u = \varphi(z)\bar{\psi}(w)$. Then T_u is injective on $H^2(\mathbb{D}^2)$ if and only if ψ is an outer function.

Proof. Since $\varphi \in H^\infty(\mathbb{D})$ is nonzero, we note $t_{\varphi(z)}$ is injective. By Corollary (5.2.14), we see T_u is injective if and only if $t_{\psi(w)}^*$ is injective, which is in turn equivalent to that ψ is an outer function by Lemma (5.2.10). The proof is complete.

As an immediate consequence of Corollary (5.2.17), we have the following.

Corollary (5.2.18)[218]: Let $\varphi, \psi \in H^\infty(\mathbb{D})$ be nonzero and put $u = \varphi(z)\bar{\psi}(w)$. Then, either T_u or T_u^* is injective on $H^2(\mathbb{D}^2)$ if and only if either φ or ψ is a outer function.

We consider symbols of the form $u=\sum_{j=0}^{\infty}\bar{h}_j(z)w^j$ with certain $h_j\in H^\infty(z)$ and then characterize the kernels of the corresponding Toeplitz operators.

Theorem (5.2.19)[218]: Let $u\in L^\infty(\mathbb{T}^2)$ be a nonzero function having the following form; $u=\sum_{j=0}^{\infty}\bar{h}_j(z)w^j$ where $h_j\in H^\infty(z)$ for every $j\geq 0$ and $h_0\neq 0$. Then the following statements hold.

(i) For $f\in H^2(\mathbb{D}^2)$ with $f=\sum_{k=0}^{\infty}f_k(z)w^k$ where each $f_k\in H^2(z)$, we have $f\in \ker T_u$ if and only if $\sum_{k=0}^{\ell}t^*_{h_{\ell-k}}f_k=0$ for every $\ell\geq 0$.

(ii) If h_0 is an outer function, then T_u is injective on $H^2(\mathbb{D}^2)$.

(iii) Suppose that h_0 is a non-constant inner function and $0<r<\frac{1}{2}$. If $\|h_j\|_\infty\leq r^j$ for every $j\geq 1$, then $\dim\ker T_u=\infty$.

(iv) Suppose h_0 is a non-constant inner function and $h_1=c$ for some constant c with $c\ \geq 1$. Then T_u is injective on $H^2(\mathbb{D}^2)$.

(v) If $\{h_j\}_{j\geq 0}$ has a non-constant common inner factor, then T_u is not injective on $H^2(\mathbb{D}^2)$ and $\dim\ker T_u=\infty$.

Proof. (i) Noting

$$T_uf=P\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\bar{h}_j(z)\,f_k(z)w^{k+j}\right)=P\left(\sum_{\ell=0}^{\infty}\left(\sum_{k=0}^{\ell}\bar{h}_{\ell-k}(z)f_k(z)\right)w^\ell\right)$$
$$=\sum_{\ell=0}^{\infty}\left(\sum_{k=0}^{\ell}t^*_{h_{\ell-k}}f_k\,(z)\right)\omega^\ell,$$

we see that (i) holds.

(ii) Let $f=\sum_{k=0}^{\infty}f_k(z)w^k\in H^2(\mathbb{D}^2)$ be any nonzero function and $m\geq 0$ be the smallest integer such that $f_m\neq 0$. Since h0is outer, $t^*_{h_0}$ is injective by Lemma (5.2.10). It follows that

$$\sum_{k=0}^{m}t^*_{h_{m-k}}f_k=t^*_{h_0}f_m\neq 0.$$

By (i), $f\notin\ker T_u$ and (ii) holds.

(iii) Since h_0 is a non-constant inner function, we may take $f_0\in H^2(z)$ such that $t^*_{h_0}f_0=0$ and $\|f_0\|=1$. Inductively we define

$$f_{\ell+1}(z)=-h_0(z)\sum_{k=0}^{\ell}t^*_{h_{\ell+1-k}}f_k\,(z)+a_{\ell+1}f_0(z)\in H^2(z)$$

where $|a_{\ell+1}|\leq r^{\ell+1}$ for every $\ell\geq 0$. Then we have

$$\sum_{k=0}^{\ell}t^*_{h_{\ell+1-k}}f_k=\sum_{k=0}^{\ell}t^*_{h_{\ell+1-k}}f_k+t^*_{h_0}f_{\ell+1}=\ 0$$

for every $\ell\geq 0$. Also note

$$\|f_1\|=\left\|-\,h_0t^*_{h_1}f_0+a_1f_0\right\|\leq\|h_1\|_\infty\|f_0\|+|a_1|\|f_0\|\leq 2r.$$

By induction, we shall show that $\|f_\ell\| \le 2^\ell r^\ell$ for every $\ell \ge 1$. Suppose that $\|f_\ell\| \le 2^\ell r^\ell$ for every $1 \le \ell \le m$. Then we note

$$\begin{aligned}\|f_{m+1}\| &\le \sum_{k=0}^{m} \left\|t^*_{h_{m+1-k}} f_k\right\| + r^{m+1} \le \sum_{k=0}^{m} \|h_{m+1-k}\|_\infty \|f_k\| + r^{m+1} \\ &\le \sum_{k=0}^{m} r^{m+1-k} 2^k r^k + r^{m+1} = 2^{m+1} r^{m+1}\end{aligned}$$

and hence $\|f_\ell\| \le (2r)^\ell$ for every $\ell \ge 0$. Set $f(z,w) = \sum_{k=0}^\infty f_k(z) w^k$. By the observation above, we have

$$\|f\|^2 = \left\|\sum_{k=0}^{\infty} f_k(z) w^k\right\|^2 = \sum_{k=0}^{\infty} \|f_k(z)\|^2 \le \sum_{k=0}^{\infty} (2r)^{2k} < \infty.$$

Hence $f \in H^2(\mathbb{D}^2)$ and $f \neq 0$. Now, by (i), we have $f \in \ker T_u$ and $\ker T_u \neq \{0\}$. Moreover, the construction of f above shows that $\dim \ker T_u = \infty$.

(iv) Suppose there is a nonzero $f \in H^2(\mathbb{D}^2)$ such that $T_u f = 0$. Write $f = \sum_{k=0}^\infty f_k(z) w^k$ where each $f_k \in H^2(z)$ and let $m \ge 0$ be the smallest integer such that $f_m \neq 0$. Then, by (i)

$$\sum_{k=0}^{\ell} t^*_{h_{\ell-k}} f_k(z) = 0 \tag{17}$$

for every $\ell \ge 0$. When $\ell = m$ in (17), we have $0 = \sum_{k=0}^m t^*_{h_{m-k}} f_k = t^*_{h_0} f_m$.

Since h_0 is inner, it follows that $f_m \in H^2(z) \ominus h_0 H^2(z)$. Also, if we take $\ell = m+1$ in (17), we have

$$0 = \sum_{k=0}^{m} t^*_{h_{m+1-k}} f_k = t^*_{h_1} f_m + t^*_{h_0} f_{m+1} = c f_m + t^*_{h_0} f_{m+1}$$

and hence $f_{m+1} = -c h_0 f_m + g_{m+1}$ for some $g_{m+1} \in H^2(z) \ominus h_0 H^2(z)$. We also have

$$\begin{aligned}0 &= \sum_{k=0}^{m} t^*_{h_{m+2-k}} f_k = t^*_{h_2} f_m + t^*_{h_1} f_{m+1} + t^*_{h_0} f_{m+2} \\ &= t^*_{h_2} f_m + c f_{m+1} + t^*_{h_0} f_{m+2} = t^*_{h_2} f_m + c g_{m+1} - c^2 f_m + t^*_{h_0} f_{m+2}.\end{aligned}$$

Hence

$$f_{m+2} = -h_0\big(t^*_{h_2} f_m + c g_{m+1} - c^2 h_0 f_m\big) + g_{m+2}$$

for some $g_{m+2} \in H^2(z) \ominus h_0 H^2(z)$. Since $t^*_{h_2} f_m \in H^2(z) \ominus h_0 H^2(z)$, we have

$$\begin{aligned}f_{m+2} &= \big[g_{m+2} - h_0\big(t^*_{h_2} f_m + c g_{m+1}\big)\big] + c^2 h_0^2 f_m \\ &\in [H^2(z) \ominus h_0^2 H^2(z)] \oplus h_0^2 H^2(z).\end{aligned}$$

Repeating the same argument, we may write

$$f_{m+i} = G_i + c^i h_0^i f_m \in \big[H^2(z) \ominus h_0^i H^2(z)\big] \oplus h_0^i H^2(z)$$

for every $i \ge 1$. Hence

$$\infty > \|f\|^2 = \|f_m\|^2 + \sum_{i=1}^{\infty i} \|f_{m+i}\|^2 \ge \|f_m\|^2 + \sum_{i=1}^{\infty} c^{2i} \|f_m\|^2 = \infty$$

because $c \ge 1$, which is a contradiction. Thus $\ker T_u = \{0\}$ by (i).

(v) Let $\eta(z)$ be a non-constant common inner factor of $\{h_j\}_{j\geq 0}$. For each $j \geq 0$, we may write $h_j = \eta \tilde{h}_j$ for some $\tilde{h}_j \in H^\infty(z)$. Take a nonzero function g in $H^2(z) \ominus \eta H^2(z)$. Note $\bar{\eta} g \perp H^2(z)$ and $\bar{h}_j g \perp H^2(z)$ for each j. Hence for each nonnegative integer m, we have

$$T_u(gw^m) = P\left(\sum_{j=0}^{\infty} \bar{h}_j(z) g(z) w^{j+m}\right) = 0.$$

Therefore T_u is not injective and $[H^2(z) \ominus \eta H^2(z)] \otimes H^2(w) \subset \ker T_u$.

Hence $\dim \ker T_u = \infty$. The proof is complete.

In conjunction with Theorem (5.2.19), we finally have the following which is a consequence of Lemma (5.2.1).

Proposition (5.2.20)[218]: Let φ be a non-constant inner function on $\mathbb{D}$ and put $u = z - \varphi(z)\bar{w}$. Then T_u is injective on $H^2(\mathbb{D}^2)$.

Corollary (5.2.21)[307]: Let $u_r = \varphi_r + \overline{\psi_r}$,where ψ_r be an inner functions on $\mathbb{D}^2$ and $\varphi_r \in L^\infty(\mathbb{D}^2)$ with $\|\varphi_r\|_\infty \leq 1$. If $u_r \not\equiv 0$, then $T_{u_r}^*$ is injective on $H^2(\mathbb{D}^2)$.

Proof. Suppose $T_{u_r}^* h_r = 0$ for some $h_r \in H^2(\mathbb{D}^2)$ and then $P(\overline{\varphi_r} h_r) + \psi_r h_r = 0$. Since ψ_r is inner, we note

$$\|h_r\| = \|\psi_r h_r\| = \|P(\overline{\varphi_r} h_r)\| \leq \|\overline{\varphi_r} h_r\| \leq \|h_r\|$$

and then $\|P(\overline{\varphi_r} h_r)\| = \|\overline{\varphi_r} h_r\|$. Thus $P(\overline{\varphi_r} h_r) = \overline{\varphi_r} h_r$ and

$$\overline{u_r} h_r = \overline{\varphi_r} h_r + \psi_r h_r = P(\overline{\varphi_r} h_r) + \psi_r h_r = 0.$$

Since $u_r \equiv 0$, we have $h_r = 0$ and $T_{u_r}^*$ is injective, as desired. The proof is complete.

Corollary (5.2.22)[307]: Let ψ_r be a non-constant inner functions on $\mathbb{D}^2$ and $\varphi_r \in H^\infty(\mathbb{D}^2)$. Assume $\|\varphi_r\|_\infty < 1$ and put $u_r = \varphi_r + \overline{\psi_r}$. Then $T_{u_r}^*$ is injective but T_{u_r} is not injective on $H^2(\mathbb{D}^2)$. Moreover we have

$$\ker T_{u_r} = \frac{1}{1 + \varphi_r \psi_r} [H^2(\mathbb{D}^2) \ominus \psi_r H^2(\mathbb{D}^2)],$$

and $\dim \ker T_{u_r} = \infty$.

Proof. By Lemma (5.2.1), $T_{u_r}^*$ is injective on $H^2(\mathbb{D}^2)$. To prove (1), we first note that $1/(1 + \varphi_r \psi_r) \in H^\infty(\mathbb{D}^2)$ because $\|\varphi_r\|_\infty < 1$. Since ψ_r is inner, $u_r = \varphi_r + \overline{\psi_r} = \overline{\psi_r}(\psi_r \varphi_r + 1)$ a.e. on $\mathbb{T}^2$ and hence $T_{u_r} = T_{\overline{\psi_r}} T_{1+\varphi_r\psi_r}$. Since $1 + \varphi_r \psi_r$ is invertible in $H^\infty(\mathbb{D}^2)$, we have

$$\ker T_{u_r} = \frac{1}{1+\varphi_r\psi_r} \ker T_{\overline{\psi_r}} = \frac{1}{1+\varphi_r\psi_r} [H^2(\mathbb{D}^2) \ominus \psi_r H^2(\mathbb{D}^2)],$$

because $\ker T_{\overline{\psi_r}} = (\operatorname{ran} T_{\psi_r})^\perp = [H^2(\mathbb{D}^2) \ominus \psi_r H^2(\mathbb{D}^2)]$, so (1) holds as desired. The proof is complete.

Corollary (5.2.23)[307]: Let $\psi_r \in H^\infty(z + \varepsilon)$ be a non-constant inner functions and $\varphi_r \in H^\infty(z)$. Put $u_r(z, z + \varepsilon) = \varphi_r(z) + \overline{\psi_r}(z + \varepsilon)$. Then the following statements are equivalent.

(i) T_{u_r} is not injective on $H^2(\mathbb{D}^2)$.

(ii) $\|\varphi_r\|_\infty \leq 1$ and φ_r are not an extreme points of ball $H^\infty(z)$.

In which case, we have

$$\ker T_{u_r} = \frac{\eta_r(z)}{1+\varphi_r(z)\psi_r(z+\varepsilon)}[H^2(\mathbb{D}^2) \ominus \psi_r(z+\varepsilon)H^2(\mathbb{D}^2)]$$

and $\dim \ker T_{u_r} = \infty$, where $\eta_r \in$ ball $H^\infty(z)$ are an outer functions satisfying $|\varphi_r|^2 + |\eta_r|^2 = 1$ a.e. on $\mathbb{T}$.

Proof. First assume (i) and then $T_{u_r} f_r = 0$ for some nonzero functions $f_r \in H^2(\mathbb{D}^2)$. By (2), we may write

$$f_r = \sum_{n,k=0}^{\infty} (f_r)_{n,k}(z) e_k(z+\varepsilon)\psi_r(z+\varepsilon)^n, \qquad z, z+\varepsilon \in \mathbb{D}$$

where $(f_r)_{n,k}(z) \in H^2(z)$ for every $n, k \geq 0$. Since ψ_r are inners, we note that

$$\begin{aligned} 0 = T_{u_r} f_r &= \varphi_r(z) f_r + \sum_{n,k=0}^{\infty} P\big((f_r)_{n,k} e_k \psi_r^n \overline{\psi_r}\big) \\ &= \varphi_r(z) f_r + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (f_r)_{n,k}(z) e_k(z+\varepsilon)\, \psi_r(z+\varepsilon)^{n-1} \\ &= \sum_{n,k=0}^{\infty} \varphi_r(z)(f_r)_{n,k}(z) e_k(z+\varepsilon)\psi_r(z+\varepsilon)^n \\ &\quad + \sum_{n,k=0}^{\infty} (f_r)_{n+1,k}(z) e_k(z+\varepsilon)\psi_r(z+\varepsilon)^n \\ &= \sum_{n,k=0}^{\infty} \left[\varphi_r(z)(f_r)_{n,k}(z) + (f_r)_{n+1,k}(z) e_k(z+\varepsilon)\psi_r(z+\varepsilon)^n\right] \end{aligned}$$

for all $z, z+\varepsilon \in \mathbb{D}$. It follows that $\varphi_r (f_r)_{n,k} + (f_r)_{n+1,k} = 0$ for every $n, k \geq 0$ and hence, for each $k \geq 0$, $(f_r)_{n,k} = (f_r)_{0,k}(-\varphi_r)^n$ for all n. Thus

$$f_r = \sum_{n,k=0}^{\infty} (f_r)_{0,k}(z)\big(-\varphi_r(z)\big)^n e_k(z+\varepsilon)\psi_r(z+\varepsilon)^n.$$

Since $f_r \in H^2(\mathbb{D}^2)$ and $\{e_k \psi_r^n : n, k \geq 0\}$ is an orthonormal basis of $H^2(z+\varepsilon)$, we see

$$\infty > \|f_r\|^2 = \sum_{n,k=0}^{\infty} \left\|(f_r)_{0,k}(-\varphi_r)^n\right\|^2 = \int_{\mathbb{T}} \left(\sum_{k=0}^{\infty} \left|(f_r)_{0,k}\right|^2\right)\left(\sum_{n=0}^{\infty} |f_r|^{2n}\right) d\sigma_1.$$

Since $f_r \neq 0$, we have $(f_r)_{0,\ell} \neq 0$ for some $\ell \geq 0$. Thus the above shows that $|\varphi_r(z)| < 1$ a.e. on $\mathbb{T}$ and then $\|\varphi_r\|_\infty \leq 1$. Also, we have

$$\int_{\mathbb{T}} \frac{\sum_{k=0}^{\infty} \left|(f_r)_{0,k}\right|^2}{1 - |\varphi_r|^2}\, d\sigma_1 < \infty$$

and

$$f_r = \frac{1}{1+\varphi_r(z)\psi_r(z+\varepsilon)} \sum_{k=0}^{\infty} (f_r)_{0,k}(z) e_k(z+\varepsilon).$$

Now, noting

$$\log\left|(f_r)_{0,\ell}\,(z)\right|^2 \leq \log\frac{\sum_{k=0}^{\infty}\left|(f_r)_{0,k}(z)\right|^2}{1-|\varphi_r(z)|^2} \leq \frac{\sum_{k=0}^{\infty}\left|(f_r)_{0,k}(z)\right|^2}{1-|\varphi_r(z)|^2}.$$

We have by the Jensen inequality

$$-\infty < \int_{\mathbb{T}} \log\left|(f_r)_{0,\ell}\right|^2 d\sigma_1 \leq \int_{\mathbb{T}} \log\sum_{k=0}^{\infty}\left|(f_r)_{0,\ell}\right|^2 d\sigma_1 - \int_{\mathbb{T}} \log(1-|\varphi_r|^2)\, d\sigma_1$$

$$\leq \int_{\mathbb{T}} \frac{\sum_{k=0}^{\infty}\left|(f_r)_{0,k}\right|^2}{1-|\varphi_r|^2} d\sigma_1 < \infty.$$

We also have

$$-\infty < \int_{\mathbb{T}} \log\left|(f_r)_{0,\ell}\right|^2 d\sigma_1 \leq \int_{\mathbb{T}} \log\sum_{k=0}^{\infty}\left|(f_r)_{0,k}\right|^2 d\sigma_1.$$

and

$$0 \leq -\int_{\mathbb{T}} \log(1-|\varphi_r|)\, d\sigma_1 \leq \infty.$$

Then, the above inequalities imply $-\infty < \int_{\mathbb{T}} \log(1-|\varphi_r|)\, d\sigma_1$ and hence φ_r are not an extreme points of ball $H^\infty(z)$. Thus (ii) holds. Now assume (ii) and show (i). By the remark just before this proposition, there is an outer functions $\eta_r \in ball\ H^\infty(z)$ satisfying $|\varphi_r|^2 + |\eta_r|^2 = 1$ a.e. on $\mathbb{T}$. Thus

$$\frac{|\eta_r|^2}{1-|\varphi_r|^2} = 1 \quad \text{a.e. on } \mathbb{T}.$$

To prove (i), it suffices to show

$$\frac{\eta_r(z)}{1+\varphi_r(z)\psi_r(z+\varepsilon)}\left[H^2(\mathbb{D}^2) \ominus \psi_r(z+\varepsilon)H^2(\mathbb{D}^2)\right] \subset \ker T_{u_r}.$$

To do this, we let $g_r \in H^2(z)$ and $h_r \in K_{\psi_r}(z+\varepsilon)$. Note

$$\frac{\eta_r(z)g_r(z)h_r(z+\varepsilon)}{1+\varphi_r(z)\,\psi_r(z+\varepsilon)} = \sum_{n=0}^{\infty} \eta_r(z)g_r(z)h_r(z+\varepsilon)\left(-\varphi_r(z)\right)^n \psi_r(z+\varepsilon)^n$$

for all $z, z+\varepsilon \in \mathbb{D}$. Since $h_r \in K_{\psi_r}(z+\varepsilon)$, we have $h_r{\psi_r}^n \perp h_r{\psi_r}^k$ for every $n, k \geq 0$ with $n \neq k$ and then

$$\left\|\frac{\eta_r(z)g_r(z)h_r(z+\varepsilon)}{1+\varphi_r(z)\psi_r(z+\varepsilon)}\right\|^2$$

$$= \sum_{n=0}^{\infty}\left\|\eta_r(z)g_r(z)\left(-\varphi_r(z)\right)^n\right\|^2 \|h_r(z+\varepsilon)\ \psi_r(z+\varepsilon)^n\|^2$$

$$= \|h_r(z+\varepsilon)\|^2 \int_{\mathbb{T}} \frac{|\eta_r(z)g_r(z)|^2}{1-|\varphi_r(z)|^2}\, d\sigma_1 = \|h_r(z+\varepsilon)\|^2\|g_r(z)\|^2 < \infty$$

and hence

$$\chi(z, z+\varepsilon) := \frac{\eta(z)g_r(z)h_r(z+\varepsilon)}{1+\varphi_r(z)\,\psi_r(z+\varepsilon)} \in H^2(\mathbb{D}^2).$$

Moreover we see

$$T_{u_r}\chi = \varphi_r\chi + P\left(\sum_{n=0}^{\infty} \eta_r\, g_r h_r(-\,\varphi_r)^n \psi_r^n \overline{\psi_r}\right) = \varphi_r\chi + \sum_{n=0}^{\infty} \eta_r g_r h_r(-\,\varphi_r)^{n+1}\psi_r^n$$
$$= \sum_{n=0}^{\infty} [\varphi_r(-\varphi_r)^n + (-\varphi_r)^{n+1}]\eta_r g_r h_r \psi_r^n = 0$$
and thus (7) follows as desired.

To complete the proof, we need to show that the reverse inclusion of (7) holds. To prove this, let $f_r \in \ker T_{u_r}$ and then $T_{u_r} f_r = 0$. Then, using the same notation as in (3), we have from (4) and (6)

$$\infty > \int_{\mathbb{T}} \frac{\sum_{k=0}^{\infty} \left|(f_r)_{0,k}\right|^2}{1 - |\varphi_r|^2} d\sigma_1 = \int_{\mathbb{T}} \frac{|\eta_r|^2 \sum_{k=0}^{\infty} \left|(f_r)_{0,k}/\eta_r\right|^2}{1 - |\varphi_r|^2} d\sigma_1 = \int_{\mathbb{T}} \sum_{k=0}^{\infty} \left|\frac{(f_r)_{0,k}}{\eta_r}\right|^2 d\sigma_1 .$$

Thus $(f_r)_{0,k} \in \eta_r H^2(z)$ for every $k \geq 0$ and (5) shows

$$f_r \in \frac{\eta_r(z)}{1 + \varphi_r(z)\psi_r(z+\varepsilon)} \bigoplus_{k=0}^{\infty} e_k(z+\varepsilon) H^2(z)$$
$$= \frac{\eta_r(z)}{1 + \varphi_r(z)\psi_r(z+\varepsilon)} [H^2(\mathbb{D}^2) \ominus \psi_r(z+\varepsilon) H^2(\mathbb{D}^2)]$$

because $\{e_k(z+\varepsilon)\}_{k\geq 0}$ is an orthonormal basis of $K_{\psi_r}(z+\varepsilon)$. Thus we get

$$\ker T_{u_r} \subset \frac{\eta_r(z)}{1 + \varphi_r(z)\psi_r(z+\varepsilon)} [H^2(\mathbb{D}^2) \ominus \psi_r(z+\varepsilon) H^2(\mathbb{D}^2)]$$

as desired. The proof is complete.

Corollary (5.2.24)[307]: Let $\psi_r \in H^\infty(z+\varepsilon)$ be a non-constant inner functions and $\varphi_r \in H^\infty(z)$. Put $u_r = \varphi_r(z) + \overline{\psi_r}(z+\varepsilon)$. Then either T_{u_r} or $T_{u_r}^*$ is injective on $H^2(\mathbb{D}^2)$.

Proof. Suppose T_{u_r} is not injective. Theorem (5.2.3) shows $\|\varphi_r\|_\infty \leq 1$ and then $T_{u_r}^* = T_{\overline{\varphi_r}+\psi_r}$ is injective by Lemma (5.2.1). The proof is complete.

Corollary (5.2.25)[307]: Let $\varphi_r, \psi_r \in H^\infty(\mathbb{D})$ be non-constant and $u_r = \varphi_r(z) + \overline{\psi_r}(z+\varepsilon)$. If $\|\varphi_r\|_\infty = \|\psi_r\|_\infty$, then either T_{u_r} or $T_{u_r}^*$ is injective on $H^2(\mathbb{D}^2)$.

Proof. We may assume that $0 < \|\psi_r\|_\infty < 1 < \|\psi_r\|_\infty$. We shall prove that T_{u_r} is injective. To prove this, suppose not. Then $T_{u_r} f_r = 0$ for some $f_r \in H^2(\mathbb{D}^2)$ with $f_r \neq 0$. Since

$$0 = T_{u_r} f_r = P(\varphi_r f_r + \overline{\psi_r} f_r) = \varphi_r f_r + T_{\psi_r}^* f_r,$$

we see $T_{\psi_r}^* f_r = -\varphi_r f_r$ and hence

$$(T_{\psi_r}^*)^2 f_r = T_{\psi_r(z+\varepsilon)}^*(-\varphi_r(z) f_r) = -\varphi_r(z) T_{\psi_r(z+\varepsilon)}^* f_r = (-\varphi_r)^2 f_r.$$

Repeating the same argument, we have $(T_{\psi_r}^*)^n f_r = (-\varphi_r)^n f_r$ for each $n = 1, 2, \cdots$. Since $\|\varphi_r\|_\infty > 1$, we have $\sigma_2(E(\varphi_r) \times \mathbb{T}) > 0$ and then

$$\left\|(T_{\psi_r}^*)^n f_r\right\| = \|\varphi_r^n f_r\|^2 = \int_{\mathbb{T}^2} |\varphi_r|^{2n} |f_r|^2 d\sigma_2 \geq \int_{E(\varphi_r)\times\mathbb{T}} |f_r|^2 d\sigma_2 > 0$$

for each n because f_r is nonzero. On the other hand, since $\|\psi_r\|_\infty < 1$, we have

$$\left\|(T_{\psi_r}^*)^n f_r\right\| \le \left\|T_{\psi_r}^*\right\|^n \le \|\psi_r\|_\infty^n \|f_r\| \to 0$$
as $n \to \infty$, which is a contradiction. The proof is complete.

Corollary (5.2.26)[307]: Let $\varphi_r \in H^\infty(\mathbb{D})$ be a non-constant function with $\|\varphi_r\|_\infty \le 1$. Then $\left\|(t_{\varphi_r}^*)^n f_r\right\| \to 0$ as $n \to \infty$ for every $f_r \in H^2(\mathbb{D})$.

Proof. For each $z \in \mathbb{D}$ and each integer $n \ge 1$, we first have
$$\left\|(t_{\varphi_r}^*)^n K_z\right\| = |\varphi_r(z)|^n \|K_z\|.$$
Since $\|\varphi_r\|_\infty \le 1$ and φ_r is non-constant, we have $|\varphi_r(z)| < 1$. It follows that $\left\|(t_{\varphi_r}^*)^n K_z\right\| \to 0$ as $n \to \infty$. Since the set of all linear combinations of reproducing kernels is dense $H^2(\mathbb{D})$ and $\left\|(t_{\varphi_r}^*)^n\right\| \le 1$ for all n, we have the assertion. The proof is complete.

Corollary (5.2.27)[307]: Let $\varphi_r, \psi_r \in H^\infty(\mathbb{D})$ be non-constant functions satisfying that $\|\varphi_r\|_\infty = \|\psi_r\|_\infty = 1$. Put $u_r(z, z+\varepsilon) = \varphi_r(z) + \psi_r(z+\varepsilon)$. If $|E(\varphi_r)| > 0$, then T_{u_r} is injective on $H^2(\mathbb{D}^2)$.

Proof. Suppose there exists a nonzero $f_r \in H^2(\mathbb{D}^2)$ satisfying that $T_{u_r} f_r = 0$. By the proof of Theorem (5.2.5), we have $(T_{\psi_r}^*)_{z+\varepsilon}^n f_r = \left(-\varphi_r(z)\right)^n f_r$ and hence
$$\left\|(T_{\psi_r}^*)^n f_r\right\|^2 = \int_{\mathbb{T}^2} |\varphi_r|^{2n} |f_r|^2 d\sigma \ge \int_{E(\varphi_r)\times\mathbb{T}} |f_r|^2 d\sigma > 0$$
for every $n \ge 1$. Writing $f_r(z, z+\varepsilon) = \sum_{k=0}^\infty (f_r)_k(z+\varepsilon) z^k$ where $(f_r)_k(z+\varepsilon) \in H^2(z+\varepsilon)$, we have
$$(T_{\psi_r}^*)^n f_r(z, z+\varepsilon) = \sum_{k=0}^\infty P\left(\overline{\psi_r}^n (f_r)_k\right)(z+\varepsilon) z^k = \sum_{k=0}^\infty \left((t_{\psi_r}^*)^n (f_r)_k\right)(z+\varepsilon) z^k$$
for all $z, z+\varepsilon \in \mathbb{D}$ and hence
$$\left\|(T_{\psi_r}^*)^n f\right\|^2 = \sum_{k=0}^\infty \left\|(t_{\psi_r}^*)^n (f_r)_k\right\|^2 \le \sum_{k=0}^\infty \|(f_r)_k\|^2 = \|(f_r)\|^2 < \infty.$$
By Lemma (5.2.7), applying the Lebesgue dominated convergence theorem, we have $\left\|(T_{\psi_r}^*)^n f_r\right\| \to 0$ as $n \to \infty$, which contradicts to (8). The proof is complete.

Corollary (5.2.28)[307]: Let $\varphi_r, \psi_r \in H^\infty(\mathbb{D})$ be non-constant functions and $u_r = \varphi_r(z) + \overline{\psi_r}(z+\varepsilon)$. If φ_r has a non-constant inner factor and ψ_r are an outer functions, then T_{u_r} is injective on $H^2(\mathbb{D}^2)$

Proof. Suppose there exists a nonzero $f_r \in H^2(\mathbb{D}^2)$ such that $T_{u_r} f_r = 0$. Write $\varphi_r(z) = I(z)O(z)$ for the inner-outer factorization of φ_r. Since
$$H^2(\mathbb{D}^2) = \bigoplus_{n=0}^{\infty} [H^2(\mathbb{D}^2) \ominus I(z)H^2(\mathbb{D}^2)] I(z)^n,$$
we may write $f_r = \sum_{n=\ell}^\infty (f_r)_n I(z)^n$, where $(f_r)_n \in H^2(\mathbb{D}^2) \ominus I(z)H^2(\mathbb{D}^2)$ for some $\ell \ge 0$ with $(f_r)_\ell \ne 0$. Then we have
$$0 = T_{u_r} f_r = \varphi_r f_r + T_{\psi_r}^* f_r = \sum_{n=\ell}^\infty O(z)(f_r)_n I(z)^{n+1} + \sum_{n=\ell}^\infty \left(T_{\psi_r}^*(f_r)_n\right) I(z)^n.$$

Since $T^*_{\psi_r}(f_r)_n \in H^2(\mathbb{D}^2) \ominus I(z)H^2(\mathbb{D}^2)$ for every $n \geq \ell$, we have $T^*_{\psi_r}(f_r)_\ell = 0$. But, since ψ_r is outer, we have $(f_r)_\ell = 0$ by Lemma (5.2.10). This is a contradiction and therefore T_{u_r} is injective on $H^2(\mathbb{D}^2)$. The proof is complete.

Corollary (5.2.29)[307]: Let $\varphi_r, \psi_r \in L^\infty(\mathbb{T})$ and put $u_r = \varphi_r(z)\psi_r(z+\varepsilon)$. then we have $T_{u_r} = T_{\varphi_r(z)}T_{\psi_r(z+\varepsilon)} = T_{\psi_r(z+\varepsilon)}T_{\varphi_r(z)}$.

Proof. Let $f_r, g_r \in H^2(\mathbb{D})$ and write $\psi_r(z+\varepsilon)g_r(z+\varepsilon) = (g_r)_1(z+\varepsilon) + t_{\psi_r(z+\varepsilon)}g_r(z+\varepsilon)$ for some $(g_r)_1(z+\varepsilon) \in L^2(\mathbb{T})$ with $(g_r)_1(z+\varepsilon) \perp H^2(z+\varepsilon)$. Then we have

$$\begin{aligned} T_{u_r}f_r(z)g_r(z+\varepsilon) &= P\left[\varphi_r(z)f_r(z)\left((g_r)_1(z+\varepsilon) + t_{\psi_r(z+\varepsilon)}g_r(z+\varepsilon)\right)\right] \\ &= P\left[\varphi_r(z)f_r(z)t_{\psi_r(z+\varepsilon)}g_r(z+\varepsilon)\right] \\ &= T_{\varphi_r(z)}f_r(z)t_{\psi_r(z+\varepsilon)}g_r(z+\varepsilon) \\ &= T_{\varphi_r(z)}T_{\psi_r(z+\varepsilon)}f_r(z)g_r(z+\varepsilon). \end{aligned}$$

Similarly $T_{u_r}f_r(z)g_r(z+\varepsilon) = T_{\psi_r(z+\varepsilon)}T_{\varphi_r(z)}f_r(z)g_r(z+\varepsilon)$. Since $H^2(\mathbb{D}^2) = H^2(z) \otimes H^2(z+\varepsilon)$, we have $T_{u_r} = T_{\varphi_r(z)}T_{\psi_r(z+\varepsilon)} = T_{\psi_r(z+\varepsilon)}T_{\varphi_r(z)}$, as desired. The proof is complete.

Corollary (5.2.30)[307]: Let φ_r, $\psi_r \in L^\infty(\mathbb{T})$ be nonzero functions and $u_r = \varphi_r(z)\,\psi_r(z+\varepsilon)$. Then we have

$$\ker T_{u_r} = \left[\ker t_{\varphi_r(z)} \otimes H^2(z+\varepsilon)\right] + \left[H^2(z) \otimes \ker t_{\psi_r(z+\varepsilon)}\right].$$

Moreover if $\ker T_{u_r} \neq \{0\}$, then $\dim \ker T_{u_r} = \infty$.

Proof. By Lemma (5.2.12), we see

$$\left[\ker t_{\varphi_r(z)} \otimes H^2(z+\varepsilon)\right] + \left[H^2(z) \otimes \ker t_{\psi_r(z+\varepsilon)}\right] \subset \ker T_{u_r}.$$

To prove the reverse inclusion, we note that

$$\begin{aligned} H^2(\mathbb{D}^2) = &\left[\ker t_{\varphi_r(z)} \otimes H^2(z+\varepsilon) + H^2(z) \otimes \ker t_{\psi_r(z+\varepsilon)}\right] \\ &\oplus \left(H^2(z) \ominus \ker t_{\varphi_r(z)}\right) \otimes \left(H^2(z+\varepsilon) \ominus \left(\ker t_{\psi_r(z+\varepsilon)}\right)\right). \end{aligned}$$

Thus, to complete the proof, it suffices to show that T_{u_r} is injective on

$$E := \left(H^2(z) \ominus \ker t_{\varphi_r(z)}\right) \otimes \left(H^2(z+\varepsilon) \ominus \left(\ker t_{\psi_r(z+\varepsilon)}\right)\right).$$

To do this, suppose $T_{u_r}f_r = 0$ for some $f_r \in E$ and let $\{e_i(z)\}_{i\geq 0}$ be an orthonormal basis of $H^2(z) \ominus \ker t_{\varphi_r(z)}$. Then we may write

$$f_r = \sum_{i=0}^{\infty} (f_r)_i(z+\varepsilon)e_i(z)\,,\ (f_r)_i \in H^2(z+\varepsilon) \ominus \ker t_{\psi_r(z+\varepsilon)}.$$

By Lemma (5.2.12), we have

$$\begin{aligned} 0 = T_{u_r}f_r &= T_{\varphi_r(z)} \sum_{i=0}^{\infty} T_{\psi_r(z+\varepsilon)}(f_r)_i(z+\varepsilon)e_i(z) \\ &= T_{\varphi_r(z)} \sum_{i=0}^{\infty} \left(t_{\psi_r(z+\varepsilon)}(f_r)_i\right)(z+\varepsilon)e_i(z) \end{aligned}$$

Also, since

$$\sum_{i=0}^{\infty} \left(t_{\psi_r(z+\varepsilon)}(f_r)_i\right)(z+\varepsilon)e_i(z) \in \left(H^2(z) \ominus \ker t_{\varphi_r(z)}\right) \otimes H^2(z+\varepsilon),$$

we write

$$\sum_{i=0}^{\infty}\left(t_{\psi_r(z+\varepsilon)}(f_r)_i\right)(z+\varepsilon)e_i(z)$$
$$=\sum_{j=0}^{\infty}(g_r)_j(z)(z+\varepsilon)^j\,,(g_r)_j\in H^2(z)\ominus\ker t_{\varphi_r(z)}.$$

Then, we have

$$0=T_{\varphi_r(z)}\sum_{j=0}^{\infty}(g_r)_j(z)(z+\varepsilon)^j=\sum_{j=0}^{\infty}\left(t_{\varphi_r(z)}(g_r)_j\right)(z)(z+\varepsilon)^j,$$

so we have $t_{\varphi_r}(g_r)_j=0$ for every $j\geq 0$. Since $(g_r)_j\perp\ker t_{\varphi_r}$, we have $(g_r)_j=0$ for every j. Therefore

$$\sum_{i=0}^{\infty}\left(t_{\psi_r(z+\varepsilon)}(f_r)_i\right)(z+\varepsilon)e_i(z)=\ 0,$$

which shows that $t_{\psi_r}(f_r)_i=0$ for every $i\geq 0$. Also, since $(f_r)_i\perp\ker t_{\psi_r}$, we see $(f_r)_i=0$ for every i, so $f_r=0$. Thus T_{u_r} is injective on E, as desired. The proof is complete.

Corollary (5.2.31)[307]: Let $\varphi_r,\psi_r\in L^\infty(\mathbb{T})$ be nonzero functions. Put $(u_r)_1=\varphi_r(z)\psi_r(z+\varepsilon)$ and $(u_r)_2=\varphi_r(z)\overline{\psi_r}(z+\varepsilon)$. Then one of $T_{(u_r)_1},T^*_{(u_r)_1},T_{(u_r)_2}$ and $T^*_{(u_r)_2}$ is injective on $H^2(\mathbb{D}^2)$.

Proof. By the Coburn theorem, we see that either t_{φ_r} or $t_{\overline{\varphi_r}}$ is injective on $H^2(\mathbb{D})$. Also either t_{ψ_r} or $t_{\overline{\psi_r}}$ is injective on $H^2(\mathbb{D})$. Hence there are four cases to consider and then the result follows from Corollary (5.2.14). The proof is complete.

Corollary (5.2.32)[307]: Let $\varphi_r,\psi_r\in H^\infty(\mathbb{D})$ be nonzero functions and put $u_r=\varphi_r(z)\overline{\psi_r}(z+\varepsilon)$. Then T_{u_r} is injective on $H^2(\mathbb{D}^2)$ if and only if ψ_r are an outer functions.

Proof. Since $\varphi_r\in H^\infty(\mathbb{D})$ is nonzero, we note $t_{\varphi_r(z)}$ is injective. By Corollary (5.2.14), we see T_{u_r} is injective if and only if $t^*_{\psi_r(z+\varepsilon)}$ is injective, which is in turn equivalent to that ψ_r are an outer functions by Lemma (5.2.10). The proof is complete.

Corollary (5.2.33)[307]: Let $u_r\in L^\infty(\mathbb{T}^2)$ be a nonzero functions having the following form; $u_r=\sum_{j=0}^{\infty}\overline{(h_r)}_j(z)(z+\varepsilon)^j$ where $(h_r)_j\in H^\infty(z)$ for every $j\geq 0$ and $(h_r)_0\neq 0$. Then the following statements hold.

(i) For $f_r\in H^2(\mathbb{D}^2)$ with $f_r=\sum_{k=0}^{\infty}(f_r)_k(z)(z+\varepsilon)^k$ where each $(f_r)_k\in H^2(z)$, we have $f_r\in\ker T_{u_r}$ if and only if $\sum_{k=0}^{\ell}t^*_{(h_r)_{\ell-k}}(f_r)_k=0$ for every $\ell\geq 0$.

(ii) If $(h_r)_0$ is an outer function, then T_{u_r} is injective on $H^2(\mathbb{D}^2)$.

(iii) Suppose that $(h_r)_0$ is a non-constant inner function and $0<r<\frac{1}{2}$. If $\left\|(h_r)_j\right\|_\infty\leq r^j$ for every $j\geq 1$, then $\dim\ker T_{u_r}=\infty$.

(iv) Suppose $(h_r)_0$ is a non-constant inner function and $(h_r)_1=c$ for some constant c with $c\ \geq 1$. Then T_{u_r} is injective on $H^2(\mathbb{D}^2)$.

(v) If $\{(h_r)_j\}_{j\geq 0}$ has a non-constant common inner factor, then T_{u_r} is not injective on $H^2(\mathbb{D}^2)$ and $\dim\ker T_{u_r} = \infty$.

Proof. (i) Noting

$$T_{u_r} f_r = P\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\overline{(h_r)_j}(z)\,(f_r)_k(z)(z+\varepsilon)^{k+j}\right)$$
$$= P\left(\sum_{\ell=0}^{\infty}\left(\sum_{k=0}^{\ell}\overline{(h_r)}_{\ell-k}(z)(f_r)_k(z)\right)(z+\varepsilon)^{\ell}\right)$$
$$= \sum_{\ell=0}^{\infty}\left(\sum_{k=0}^{\ell} t^*_{(h_r)_{\ell-k}}(f_r)_k\,(z)\right)(z+\varepsilon)^{\ell},$$

we see that (i) holds.

(ii) Let $f_r = \sum_{k=0}^{\infty}(f_r)_k(z)(z+\varepsilon)^k \in H^2(\mathbb{D}^2)$ be any nonzero function and $m \geq 0$ be the smallest integer such that $(f_r)_m \neq 0$. Since $(h_r)_0$ is outer, $t^*_{(h_r)_0}$ is injective by Lemma (5.2.10). It follows that

$$\sum_{k=0}^{m} t^*_{(h_r)_{m-k}}(f_r)_k = (t^*_{h_r})_0(f_r)_m \neq 0.$$

By (i), $f_r \notin \ker T_{u_r}$ and (ii) holds.

(iii) Since $(h_r)_0$ is a non-constant inner function, we may take $(f_r)_0 \in H^2(z)$ such that $t^*_{(h_r)_0}(f_r)_0 = 0$ and $\|(f_r)_0\| = 1$. Inductively we define

$$(f_r)_{\ell+1}(z) = -(h_r)_0(z)\sum_{k=0}^{\ell}(t^*_{h_r})_{m-k}(f_r)_k\,(z) + a_{\ell+1}(f_r)_0(z) \in H^2(z)$$

where $|a_{\ell+1}| \leq r^{\ell+1}$ for every $\ell \geq 0$. Then we have

$$\sum_{k=0}^{\ell}(t^*_{h_r})_{m-k}(f_r)_k = \sum_{k=0}^{\ell}(t^*_{h_r})_0(f_r)_k + t^*_{(h_r)_0}(f_r)_{\ell+1} = 0$$

for every $\ell \geq 0$. Also note

$\|(f_r)_1\| = \left\|-(h_r)_0 t^*_{(h_r)_1}(f_r)_0 + a_1(f_r)_0\right\| \leq \|(h_r)_1\|_\infty\|(f_r)_0\| + |a_1|\|(f_r)_0\| \leq 2r_0.$

By induction, we shall show that $\|(f_r)_\ell\| \leq 2^\ell r_0^\ell$ for every $\ell \geq 1$. Suppose that $\|(f_r)_\ell\| \leq 2^\ell r_0^\ell$ for every $1 \leq \ell \leq m$. Then we note

$$\|(f_r)_{m+1}\| \leq \sum_{k=0}^{m}\left\|(t^*_{h_r})_{m-k}(f_r)_k\right\| + r^{m+1} \leq \sum_{k=0}^{m}\|(h_r)_{m+1-k}\|_\infty\,\|(f_r)_k\| + r^{m+1}$$
$$\leq \sum_{k=0}^{m} r_0^{m+1-k}2^k r_0^k + r_0^{m+1} = 2^{m+1}r_0^{m+1}$$

and hence $\|(f_r)_\ell\| \leq (2r_0)^\ell$ for every $\ell \geq 0$. Set $f_r(z, z+\varepsilon) = \sum_{k=0}^{\infty}(f_r)_k(z)(z+\varepsilon)^k$. By the observation above, we have

$$\|(f_r)\|^2 = \left\|\sum_{k=0}^{\infty}(f_r)_k(z)(z+\varepsilon)^k\right\|^2 = \sum_{k=0}^{\infty}\|(f_r)_k(z)\|^2 \le \sum_{k=0}^{\infty}(2r_0)^{2k} < \infty.$$

Hence $f_r \in H^2(\mathbb{D}^2)$ and $f_r \neq 0$. Now, by (i), we have $f_r \in \ker T_{u_r}$ and $\ker T_{u_r} \neq \{0\}$. Moreover, the construction of f_r above shows that $\dim \ker T_{u_r} = \infty$.

(iv) Suppose there is a nonzero $f_r \in H^2(\mathbb{D}^2)$ such that $T_{u_r} f_r = 0$. Write $f_r = \sum_{k=0}^{\infty}(f_r)_k(z)(z+\varepsilon)^k$ where each $(f_r)_k \in H^2(z)$ and let $m \ge 0$ be the smallest integer such that $(f_r)_m \neq 0$. Then, by (i)

$$\sum_{k=0}^{\ell} t^*_{(h_r)_{m-k}}(f_r)_k\,(z) = 0$$

for every $\ell \ge 0$. When $\ell = m$ in (9), we have $0 = \sum_{k=0}^{m} t^*_{(h_r)_{m-k}}(f_r)_k = t^*_{(h_r)_0}(f_r)_m$.

Since $(h_r)_0$ is inner, it follows that $(f_r)_m \in H^2(z) \ominus (h_r)_0 H^2(z)$. Also, if we take $\ell = m+1$ in (9), we have

$$0 = \sum_{k=0}^{m} t^*_{(h_r)_{m-k}}(f_r)_k = t^*_{(h_r)_1}(f_r)_m + t^*_{(h_r)_0}(f_r)_{m+1} = c(f_r)_m + t^*_{(h_r)_0}(f_r)_{m+1}$$

and hence $(f_r)_{m+1} = -c(h_r)_0(f_r)_m + (g_r)_{m+1}$ for some $(g_r)_{m+1} \in H^2(z) \ominus (h_r)_0 H^2(z)$. We also have

$$\begin{aligned} 0 &= \sum_{k=0}^{m} t^*_{(h_r)_{m+2-k}}(f_r)_k = t^*_{(h_r)_2}(f_r)_m + t^*_{(h_r)_1}(f_r)_{m+1} + t^*_{(h_r)_0}(f_r)_{m+2} \\ &= t^*_{(h_r)_2}(f_r)_m + c(f_r)_{m+1} + t^*_{(h_r)_0}(f_r)_{m+2} \\ &= t^*_{(h_r)_2}(f_r)_m + c(g_r)_{m+1} - c^2(f_r)_m + t^*_{(h_r)_0}(f_r)_{m+2}. \end{aligned}$$

Hence

$$(f_r)_{m+2} = -(h_r)_0\left(t^*_{(h_r)_2}(f_r)_m + c(g_r)_{m+1} - c^2(h_r)_0(f_r)_m\right) + (g_r)_{m+2}$$

for some $(g_r)_{m+2} \in H^2(z) \ominus (h_r)_0 H^2(z)$. Since $t^*_{(h_r)_2}(f_r)_m \in H^2(z) \ominus (h_r)_0 H^2(z)$, we have

$$\begin{aligned} (f_r)_{m+2} &= \left[(g_r)_{m+2} - (h_r)_0\left(t^*_{(h_r)_2}(f_r)_m + c(g_r)_{m+1}\right)\right] + c^2(h_r)_0^2(f_r)_m \\ &\in [H^2(z) \ominus (h_r)_0^2 H^2(z)] \oplus (h_r)_0^2 H^2(z). \end{aligned}$$

Repeating the same argument, we may write

$$(f_r)_{m+i} = G_i + c^i(h_r)_0^i(f_r)_m \in \left[H^2(z) \ominus (h_r)_0^i H^2(z)\right] \oplus (h_r)_0^i H^2(z)$$

for every $i \ge 1$. Hence

$$\infty > \|f_r\|^2 = \|(f_r)_m\|^2 + \sum_{i=1}^{\infty i}\|(f_r)_{m+i}\|^2 \ge \|(f_r)_m\|^2 + \sum_{i=1}^{\infty} c^{2i}\|(f_r)_m\|^2 = \infty$$

because $c \ge 1$, which is a contradiction. Thus $\ker T_{u_r} = \{0\}$ by (i).

(v) Let $\eta_r(z)$ be a non-constant common inner factor of $\{(h_r)_j\}_{j\ge 0}$. For each $j \ge 0$, we may write $(h_r)_j = \eta_r\overline{(h_r)}_j$ for some $\widetilde{(h_r)}_j \in H^\infty(z)$. Take a nonzero function g_r in $H^2(z) \ominus\ \eta_r H^2(z)$. Note $\overline{\eta_r} g_r \perp H^2(z)$ and $\overline{(h_r)}_j g_r \perp H^2(z)$ for each j. Hence for each nonnegative integer m, we have

$$T_{u_r}(g_r(z+\varepsilon)^m) = P\left(\sum_{j=0}^{\infty}\overline{(h_r)}_j(z)g_r(z)(z+\varepsilon)^{j+m}\right) = 0.$$

Therefore T_{u_r} is not injective and $[H^2(z) \ominus \eta_r H^2(z)] \otimes H^2(z+\varepsilon) \subset \ker T_{u_r}$. Hence $\dim\ker T_{u_r} = \infty$. The proof is complete.

Chapter 6
Pointwise Multipliers with Density and Brown–Halmos Theorem

We obtain characterization of pointwise multipliers between Nakano spaces. We also discuss factorization problem for Musielak–Orlicz spaces and exhibit some differences between Orlicz and Musielak–Orlicz cases. We show that there exists a separable weighted L^1 space X such that the sequence $f * F_n$ does not always converge to $f \in X$ in the norm of X. On the other hand, we prove that the set $\mathcal{P}_A$ is dense in $H[X]$ under the assumption that X is merely separable. We specify our results to the case of variable Lebesgue spaces $X = L^{p(\cdot)}$ and $Y = L^{q(\cdot)}$ and to the case of Lorentz spaces $X = Y = L^{p,q}(w), 1 < p < \infty, 1 \le q < \infty$ with Muckenhoupt weights $w \in A_p(\mathbb{T})$.

Section (6.1): Orlicz Function Spaces and Factorization:

Given two Orlicz spaces L^{φ_1} , L^{φ} over the same measure space, the space of point wise multipliers $M(L^{\varphi_1}, L^{\varphi})$ is the space of all functions x, such that $xy \in L^{\varphi}$ for each $y \in L^{\varphi_1}$, equipped with the operator norm. The problem of identifying such spaces was investigated by many, starting from Shragin [240], Ando [227], O'Neil [237] and Zabreiko–Rutickii [242], who gave a number of partial answers.

These investigations were continued in number of directions and results were presented in different forms. One of them is the following result from Maligranda–Nakaii [234], which states that if for two given Young functions φ, φ_1 there is a third one φ_2 satisfying

$$\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}, \tag{1}$$

Then

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_2}. \tag{2}$$

This result, however, neither gives any information when such a function φ_2 exists, nor says anything how to find it. Further, it was proved in [231, Cor. 6.2] that condition (1) is necessary for a wide class of φ, φ_1 functions satisfying some additional properties, but at the same time Example 7.8 from [231] ensures that in general it is not a case, i.e. there are functions φ, φ_1 such that no Young function φ_2 satisfies (1), while $M(L^{\varphi_1}, L^{\varphi}) = L^{\infty}$, which is also Orlicz space generated by the function φ_2 defined as $\varphi_2(t) = 0$ for $0 \le t \le 1$ and $\varphi_2(t) = \infty$ for $1 < t$. In particular, these functions do not satisfy (1), although (2) holds.

On the other hand, there is a natural candidate for a function φ_2 satisfying

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_2}.$$

Such a function is the following generalization of Young conjugate function (a kind of generalized Legendre transform considered also in convex analysis, for example in [241]) defined for two Orlicz functions φ, φ_1 as

$$\varphi \ominus \varphi_1(t) = \sup_{s>0}\{\varphi(st) - \varphi_1(s)\}.$$

The function $\varphi - \varphi_1$ is called to be conjugate to φ_1 with respect to φ.

Also in [231] this construction was compared with condition (1) and it happens that very often $\varphi_2 = \varphi \ominus \varphi_1$ satisfies (1) (cf. [231, Thm. 7.9]), but once again Example 7.8 from [231] shows that $\varphi_2 = \varphi \ominus \varphi_1$ need not satisfy (1). In that example, anyhow, there holds $L^{\infty} = L^{\varphi \ominus \varphi_1}$, so that $M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}$. Therefore, it is natural to expect that in general

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}, \tag{3}$$

as was already conjectured in [231]. In fact, such theorem was stated for Orlicz N functions by Maurey in [236], but his proof depends heavily on the false conjecture, that the construction $\varphi \ominus \varphi_1$ enjoys involution property, i.e. $\varphi \ominus (\varphi \ominus \varphi_1) = \varphi_1$ (see Example 7.12 in [231] for counterexample).

On the other hand, the formula (3) was already proved for Orlicz sequence spaces by Djakov and Ramanujan in [230], where they used a slightly modified construction $\varphi \ominus \varphi_1$ (the supremum is taken only over $0 < s \leq 1$). This modification appeared to be appropriate for sequence case, because then only behaviour of Young functions for small arguments is important, while cannot be used for function spaces. Anyhow, we will borrow some ideas from [230].

We show that (3) holds in full generality for Orlicz function spaces, as well over finite and infinite nonatomic measure. Then we use this result to find that $\varphi_2 = \varphi \ominus \varphi_1$ satisfies (1) if and anly if L^{φ_1} factorizes L^{φ}, which completes the discussion from [232].

Let $L^0 = L^0(\Omega, \Sigma, \mu)$ be the space of all classes of equivalence (with respect to equality μ −a.e.) of μ −measurable, real valued functions on Ω, where (Ω, Σ, μ) is a σ −finite complete measure space. A Banach space $X \subset L^0$ is called the Banach ideal space if it satisfies the so called ideal property, i.e. $x \in L^0, y \in X$ with $|x| \leq |y|$ implies $x \in X$ and $\|x\|_X \leq \|y\|_X$ (here $|x| \leq |y|$ means that $|x(t)| \leq |y(t)|$ for μ −a.e. $t \in \Omega$), and it contains a weak unit, i.e. a function $x \in X$ such that $x(t) > 0$ for μ −a.e. $t \in \Omega$. When (Ω, Σ, μ) is purely nonatomic measure space, the respective space is called Banach function space (abbreviation B.f.s.), while in case of $\mathbb{N}$ with counting measure we shall speak about Banach sequence space. A Banach ideal space X satisfies the Fatou property when given a sequence $(x_n) \subset X$, satisfying $x_n \uparrow x$ μ −a.e. and $\sup_n \|x_n\|_X < \infty$, there holds $x \in X$ and $\|x\|_X \leq \sup_n \|x_n\|_X$.

Writing $X = Y$ for two B.f.s. we mean that they are equal as set, but norms are just equivalent. Recall also that for two Banach ideal spaces X, Y over the same measure space, the inclusion $X \subset Y$ is always continuous, i.e. there is $c > 0$ such that $\|x\|_Y \leq c\|x\|_X$ for each $x \in X$.

Given two Banach ideal spaces X, Y over the same measure space (Ω, Σ, μ), the space of point wise multipliers from X to Y is defined as

$$M(X, Y) = \{y \in L^0 : xy \in Y \text{ for all } y \in X\}$$

with the natural operator norm

$$\|y\|_{M(X,Y)} = \sup_{\|x\|_X \leq 1} \|xy\|_Y.$$

When there is no risk of confusion we will just write $\|\cdot\|_M$ for the norm of $M(X, Y)$.

A space of point wise multipliers may be trivial, for example for nonatomic measure space $M(L^p, L^q) = \{0\}$ when $1 \leq p < q$, and therefore it need not be a Banach function space in the sense of above definition. Anyhow, it is a Banach space with the ideal property (see for example [235]). To provide some intuition for multipliers let us recall that $M(L^p, L^q) = L^r$ when $p > q \geq 1, 1/p + 1/r = 1/q$ and $M(X, L^1) = X$, where X is the Köthe dual of X (see [231,232,235]).

A function $\varphi : [0, \infty) \to [0, \infty]$ will be called a Young function if it is convex, non-decreasing and $\varphi(0) = 0$. We will need the following parameters

$$a_\varphi = \sup\{t \geq 0 : \varphi(t) = 0\} \text{ and } b_\varphi = \sup\{t \geq 0 : \varphi(t) < \infty\}.$$

A Young fuction φ is called Orlicz function when $b_\varphi = \infty$. For a Young function φ by φ^{-1} we understand the right-continuous inverse defined as $\varphi^{-1}(v) = \inf\{u \geq 0: \varphi(u) > v\}$ for $v \geq 0$.

Let φ be a Young function. The Orlicz space L^φ is defined as

$$L^\varphi = \{x \in L^0: I_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where the modular I_φ is given by

$$I_\varphi(x) = \int_\Omega \varphi(|x|)d\mu$$

and the Luxemburg–Nakano norm is defined as

$$\|x\|_\varphi = \inf\left\{\lambda > 0: I_\varphi\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

We point out here that the function $\varphi \equiv 0$ is excluded from the definition of Young functions, but we allow $\varphi(u) = \infty$ for each $u > 0$ and understand that in this case $L^\varphi = \{0\}$.

We will often use the following relation between norm and modular. For $x \in L^\varphi$

$$\|x\|_\varphi \leq 1 \Rightarrow I_\varphi(x) \leq \|x\|_\varphi, \quad (4)$$

(see for example [233]).

Given two Orlicz functions φ, φ_1, the conjugate function $\varphi \ominus \varphi_1$ of φ_1 with respect to φ is defined by

$$\varphi \ominus \varphi_1(u) = \sup_{0\leq s}\{\varphi(su) - \varphi_1(s)\},$$

for $u \geq 0$. Since we need to deal with Young functions, one may be confused by possibility of appearance of indefinite symbol $\infty - \infty$ in the above definition, when $b_\varphi, b_{\varphi_1} < \infty$. To avoid such a situation we understand that for Young functions φ, φ_1 the conjugate function $\varphi \ominus \varphi_1$ is defined as

$$\varphi \ominus \varphi_1(u) = \begin{cases} \sup\limits_{0\leq s<\infty}\{\varphi(su) - \varphi_1(s)\}, \text{when } b_{\varphi_1} = \infty, \\ \sup\limits_{0<s<b_{\varphi_1}}\{\varphi(su) - \varphi_1(s)\}, \text{when } b_{\varphi_1} < \infty \text{ and } \varphi_1(b_{\varphi_1}) = \infty, \\ \sup\limits_{0<s\leq b_{\varphi_1}}\{\varphi(su) - \varphi_1(s)\}, \text{when } b_{\varphi_1} < \infty \text{ and } \varphi_1(b_{\varphi_1}) < \infty. \end{cases}$$

Notice that it is just a natural generalization of conjugate function in a sense of Young, i.e. when $\varphi(u) = u$ we get the classical conjugate function φ_1^* to φ_1. Of course, functions φ, φ_1 and $\varphi \ominus \varphi_1$ satisfy the generalized Young inequality, i.e.

$$\varphi(uv) \leq \varphi \ominus \varphi_1(u) + \varphi_1(v)$$

for each $u, v \geq 0$.

We will also need the following construction.

Definition (6.1.1)[226]: For two Young functions φ, φ_1 and $0 < a < b_{\varphi_1}$ we define

$$\varphi \ominus_a \varphi_1(u) = \sup_{0\leq s\leq a}\{\varphi(su) - \varphi_1(s)\}, u \geq 0.$$

Such defined function $\varphi \ominus_a \varphi_1$ enjoys the following elementary properties.

Lemma (6.1.2)[226]: Let φ, φ_1 be two Young functions.

(i) $\varphi \ominus_a \varphi_1$ is Young function for each $0 < a < b_{\varphi_1}$.

(ii) For each $t \geq 0$ there holds

$$\lim_{a \to b_{\varphi_1}^-} \varphi \ominus_a \varphi_1(u) = \varphi \ominus_a \varphi_1(u).$$

Notice that dilations of Young functions do not change Orlicz spaces, i.e. when φ is a Young function and ψ is defined by $\psi(u) = \varphi(au)$ for some $a > 0$, then $L^{\varphi} = L^{\psi}$. It gives a reason to expect that dilating φ, φ_1 results in dilation of $\varphi \ominus \varphi_1$.

In fact, let φ, φ_1 be Young functions and put $\psi(u) = \varphi(au), \psi_1(u) = \varphi_1(bu)$. Then

$$\psi \ominus \psi_1(u) = \sup_{0<s}\big(\varphi(aus) - \varphi_1(bs)\big) = \sup_{0<s}\left(\varphi\left(\frac{aus}{b}\right) - \varphi_1(s)\right)$$
$$= \varphi \ominus \varphi_1(au/b).$$

Moreover, if $b_{\varphi} = b_{\varphi_1} < \infty$, then supremum in the definition of $\varphi \ominus \varphi_1$ is attained for each $u < 1$, i.e. for each $u < 1$ there is $0 < s < b_{\varphi_1}$ such that $\varphi \ominus \varphi_1(u) = \varphi(us) - \varphi_1(s)$. In particular, $b_{\varphi \ominus \varphi_1} = 1$.

Let us also recall that a fundamental function f_{φ} of an Orlicz space L^{φ} is defined for $0 \le t \le \mu(\Omega)$ as $f_{\varphi}(t) = \|\chi A\|_{\varphi}$, where $\mu(A) = t$. Notice that it is well defined, since $\|\chi A\|_{\varphi}$ does not depend on particular choice of measurable set $A \subset \Omega$ with $\mu(A) = t$ (in general Orlicz spaces belong to the class of the so called rearrangement invariant spaces—see for example [228] for respective definitions).

Further, it is well known that f_{φ} is given by the formula $f_{\varphi}(t) = \frac{1}{\varphi^{-1}(1/t)}$, for $0 < t < \mu(\Omega)$ and $f_{\varphi}(0) = 0$. In particular, the fundamental function of L^{φ} is right-continuous at 0 if and only if $b_{\varphi} = \infty$, or equivalently, $b_{\varphi} = \infty$ if and only if for each $\varepsilon > 0$ there is $\delta > 0$ such that if $A \in \Sigma, \mu(A) < \delta$ then $\|\chi A\|_{\varphi} < \varepsilon$.

Since now on we are interested only in Orlicz function spaces, so that the underlying measure space (Ω, Σ, μ) is understood to be purely nonatomic for all spaces below.

Lemma (6.1.3)[226]: Let φ, φ_1 be Young functions such that $b_{\varphi} < \infty$ and $b_{\varphi_1} = \infty$. Then

$$M(L^{\varphi_1}, L^{\varphi}) = \{0\}.$$

Proof. The proof follows immediately from Proposition 3.2 in [231], since under our assumptions $L^{\varphi_1} \not\subset L^{\infty}$ but $L^{\varphi} \subset L^{\infty}$.

Lemma (6.1.4)[226]: Let φ, φ_1 be Young functions and $b_{\varphi} < \infty$. Then

$$M(L^{\varphi_1}, L^{\varphi}) \subset L^{\infty}.$$

Proof. Suppose that $M(L^{\varphi_1}, L^{\varphi}) \not\subset L^{\infty}$. Then there exists $0 \le y \in M(L^{\varphi_1}, L^{\varphi})$ such that $\|y\|_M = 1$ and for each $n > 0$

$$\mu(\{t \in \Omega : y(t) \ge n\}) > 0.$$

Denote $A_n = \{t \in \Omega : y(t) \ge n\}$ for $n \in \mathbb{N}$. Then $\|n\chi_{A_n}\|_M \le 1$ and for A_{n_0} chosen in such a way that $\mu(A_{n_0}) < \infty$, it follows

$$\|y\|_M \ge \|n\chi_{A_n}\|_M \ge \frac{n}{\|\chi A_{n_0}\|_{\varphi_1}} \|\chi_{A_n}\chi_{A_{n_0}}\|_{\varphi} = \frac{n}{\|\chi_{A_{n_0}}\|_{\varphi_1}} \|\chi_{A_n}\|_{\varphi} \ge \frac{nb_{\varphi}^{-1}}{\|\chi_{A_{n_0}}\|_{\varphi_1}},$$

for each $n > n_0$. This contradiction shows that $M(L^{\varphi_1}, L^{\varphi}) \subset L^{\infty}$.

We are in a position to prove the main theorem.

Theorem (6.1.5)[226]: Let φ, φ_1 be Young functions. Then
$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}.$$
Proof. The inclusion
$$L^{\varphi \ominus \varphi_1} \subset M(L^{\varphi_1}, L^{\varphi}) \quad (5)$$
is well known (see [227,231,234] or [237]) and follows from equivalence of generalized Young inequality and inequality $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \lesssim \varphi^{-1}$. For the completeness of presentation we present the proof which employs the generalized Young inequality directly. If $\varphi \ominus \varphi_1(u) = \infty$ for each $u > 0$ then $L^{\varphi \ominus \varphi_1} = \{0\}$ and inclusion trivially holds.

Suppose $L^{\varphi \ominus \varphi_1} \neq \{0\}$, i.e. $\varphi \ominus \varphi_1(u) < \infty$ for some $u > 0$. Let $y \in L^{\varphi \ominus \varphi_1}$ and $x \in L^{\varphi_1}$ be such that
$$\|y\|_{\varphi \ominus \varphi_1} \leq \frac{1}{2} \text{ and } \|x\|_{\varphi_1} \leq \frac{1}{2}.$$
Then generalized Young inequality gives
$$I_{\varphi}(yx) \leq I_{\varphi \ominus \varphi_1}(y) + I_{\varphi_1}(x) \leq 1.$$
Consequently $yx \in L^{\varphi}$ and $\|yx\|_{\varphi} \leq 1$. Therefore, $L^{\varphi \ominus \varphi_1} \subset M(L^{\varphi_1}, L^{\varphi})$ and
$$\|y\|_M \leq 4\|y\|_{\varphi \ominus \varphi_1}.$$
To prove the second inclusion it is enough to indicate a constant $c > 0$ such that for each simple function $y \in M(L^{\varphi_1}, L^{\varphi})$ there holds
$$\|y\|_{\varphi \ominus \varphi_1} \leq c\|y\|_M. \quad (6)$$
In fact, it follows directly from the Fatou property of both $L^{\varphi \ominus \varphi_1}$ and $M(L^{\varphi_1}, L^{\varphi})$ spaces (it is elementary fact that $M(X, Y)$ has the Fatou property when Y has so). Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ and $0 \leq y_n \uparrow y$ μ −a.e., where y_n are simple functions. Then, by (6),
$$\|y_n\|_{\varphi \ominus \varphi_1} \leq c\|y_n\|_M \to c\|y\|_M$$
and so the Fatou property of $L^{\varphi \ominus \varphi_1}$ implies $y \in L^{\varphi \ominus \varphi_1}$ and $\|y\|_{\varphi \ominus \varphi_1} \leq c\|y\|_M$.

The proof of (6) will be divided into four cases, depending on finiteness of b_{φ} and b_{φ_1}.

Consider firstly the most important case $b_{\varphi} = b_{\varphi_1} = \infty$. Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ be a simple function of the form $y = \sum_k a_k \chi_{B_k}$ and such that $\|y\|_M \leq \frac{1}{2}$. We will show that for each $a > 1$
$$I_{\varphi \ominus_a \varphi_1}(y) \leq 1.$$
Let $a > 1$ be arbitrary. For each a_k there exists $b_k \geq 0$ such that
$$\varphi(a_k b_k) = \varphi \ominus_a \varphi_1(a_k) + \varphi_1(b_k).$$
This is, for $x = \sum_k b_k \chi_{B_k}$, there holds $\varphi(xy) = \varphi \ominus_a \varphi_1(x) + \varphi_1(y)$. Note that from definition of $\varphi \ominus_a \varphi_1$ we have $x(t) \leq a$ for each $t \in \Omega$. Further, since $b_{\varphi_1} = \infty$, there exists $t_a > 0$ such that $\|\chi_A\|_{\varphi_1} \leq \frac{1}{a}$ for each $A \subset \Omega$ with $\mu(A) < t_a$. Suppose $\mu(\Omega) = \infty$. Since (Ω, Σ, μ) is σ −finite and atomless, we can divide Ω into a sequence of pairwise disjoint sets (A_n) with $\mu(A_n) = t_a$ for each $n \in \mathbb{N}$ and $\Omega = \bigcup A_n$. In the case of $\mu(\Omega) < \infty$ the sequence (A_n) may be chosen finite and such that $\mu(A_n) = \delta \leq t_a$ for each $n = 1, \ldots, N$ with $\Omega = \bigcup A_n$.

In any case, for each A_n we have

$$\|yx\chi_{A_n}\|_{\varphi} \leq \|y\|_M \|x\chi_{A_n}\|_{\varphi_1} \leq \frac{a}{2}\|\chi_{A_n}\|_{\varphi_1} \leq \frac{1}{2},$$

because $\mu(A_n) \leq t_a$ and $x(t) \leq a$ for $t \in \Omega$. In consequence, using inequality $\varphi_1(x) \leq \varphi(yx)$, we have for each A_n

$$I_{\varphi_1}(x\chi_{A_n}) \leq I_{\varphi}(yx\chi_{A_n}) \leq \|yx\chi_{A_n}\|_{\varphi} \leq \frac{1}{2}. \quad (7)$$

Define now

$$x_n = \sum_{k=1}^{n} x\chi_{A_k}.$$

We claim that $I_{\varphi_1}(x_n) \leq \frac{1}{2}$ for each n. It will be shown by induction. For $n = 1$ it comes from (7). Let $n > 1$ and suppose

$$I_{\varphi_1}(x_{n-1}) \leq \frac{1}{2}.$$

It follows

$$I_{\varphi_1}(x_n) = I_{\varphi_1}(x_{n-1}) + I_{\varphi_1}(x\chi_{A_n}) \leq 1,$$

thus $\|x_n\|_{\varphi_1} \leq 1$. Moreover, inequality

$$\|yx_n\|_{\varphi} \leq \frac{1}{2}\|x_n\|_{\varphi_1} \leq \frac{1}{2}$$

together with $\varphi_1(x) \leq \varphi(yx)$ imply

$$I_{\varphi_1}(x_n) \leq I_{\varphi}(yx_n) \leq \|yx_n\|_{\varphi} \leq \frac{1}{2}.$$

It means we proved the claim and can proceed with the proof.

Clearly, $x_n \uparrow x$ μ −a.e., thus from the Fatou property of L^{φ_1} we obtain that $x \in L^{\varphi_1}$ and

$$\|x\|_{\varphi_1} \leq \sup_n \|x_n\|_{\varphi_1} \leq 1.$$

Finally, inequalities $\varphi \ominus_a \varphi_1(y) \leq \varphi(yx)$ and $\|yx\|_{\varphi} \leq \frac{1}{2}\|x\|_{\varphi_1} \leq \frac{1}{2}$ give

$$I_{\varphi\ominus_a\varphi_1}(y) \leq I_{\varphi}(yx) \leq \|yx\|_{\varphi} \leq \frac{1}{2}.$$

Applying the Fatou Lemma we obtain

$$I_{\varphi\ominus\varphi_1}(y) = \int \varphi \ominus \varphi_1(y)d\mu \leq \lim_{a\to\infty} \inf \int \varphi \ominus_a \varphi_1(y)d\mu \leq \frac{1}{2}.$$

In consequence $y \in L^{\varphi\ominus\varphi_1}$ with $\|y\|_{\varphi\ominus\varphi_1} \leq 1$. This gives also constant for inclusion, i.e.

$$\|y\|_{\varphi\ominus\varphi_1} \leq 2\|y\|_M,$$

When $y \in M(L^{\varphi_1}, L^{\varphi})$.

Let us consider the second case, this is $b_{\varphi} = \infty$ and $b_{\varphi_1} < \infty$. Without loss of generality we can assume that $b_{\varphi_1} > 1$. Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ be a simple function satisfying $\|y\|_M \leq \frac{1}{2b_{\varphi_1}}$. Notice that $b_{\varphi} = \infty$ with $b_{\varphi_1} < \infty$ imply that $b_{\varphi\ominus\varphi_1} = \infty$. Moreover, as before, there exists a simple function x such that $0 < x(t) \leq b_{\varphi_1}$ for each $t \in \Omega$ and

$$\varphi(yx) = \varphi \ominus \varphi_1(y) + \varphi_1(x)$$

As before, we can find $t_0 > 0$ such that $\mu(A) < t_0$ implies $\|\chi_A\|_{\varphi_1} \leq 1$.

Selecting the sequence (A_n) like previously, but with $\mu(A_n) \leq t_0$ for each A_n, we obtain

$$\left\|yx\chi_{A_n}\right\|_\varphi \leq \frac{b_{\varphi_1}}{2b_{\varphi_1}} \left\|\chi_{A_n}\right\|_{\varphi_1} \leq \frac{1}{2}.$$

Define further

$$x\_n = \sum_{k=1}^{n} x\chi_{A_k}.$$

Then it may be proved by the same induction as before, that $I_{\varphi_1}(x_n) \leq \frac{1}{2}$ for each n.

Following respective steps from previous case we get

$$\|y\|_{\varphi\ominus\varphi_1} \leq 2b_{\varphi_1}\|y\|_M.$$

Let now $b_\varphi, b_{\varphi_1} < \infty$. We can assume that $b_{\varphi_1} = b_\varphi = 1$.

From Lemma (6.1.4) it follows that there exists a constant $c \geq 1$ such that for each $y \in M(L^{\varphi_1}, L^\varphi)$ we have

$$\|y\|_\infty \leq c\|y\|_M.$$

Let $0 \leq y \in M(L^{\varphi_1}, L^\varphi)$ be a simple function and $\|y\|_M \leq \frac{1}{4c}$. We have $y(t) \leq \frac{1}{4} \leq \frac{b_{\varphi\ominus\varphi_1}}{2}$ for almost every $t \in \Omega$, therefore $\varphi \ominus \varphi_1\big(y(t)\big) < \infty$.

Consequently, we can choose a simple function x satisfying

$$\varphi(yx) = \varphi \ominus \varphi_1(y) + \varphi_1(x).$$

Then $x(t) \leq b_\varphi = 1$ for each $t \in \Omega$. Further, we can find $t_0 > 0$ so that inequality

$$\|\chi_A\|_{\varphi_1} \leq 2$$

is fulfilled for each A with $\mu(A) \leq t_0$, just because $\lim_{t\to 0^+} f_\varphi(t) = b_\varphi = 1$. Choosing a sequence (A_n) as in previous cases we get

$$\left\|yx\chi_{A_n}\right\|_\varphi \leq \frac{1}{4c} \left\|\chi_{A_n}\right\|_{\varphi_1} \leq \frac{1}{2}.$$

Once again we can show by induction that for each $x_n = \sum_{k=1}^{n} x\chi_{A_k}$ there holds $I_{\varphi_1}(x_n) \leq \frac{1}{2}$. Therefore $\|x_n\|_{\varphi_1} \leq 1$ and, by the Fatou property of L^{φ_1}, $\|x\|_{\varphi_1} \leq 1$. It follows

$$\|yx\|_\varphi \leq 1$$

and by inequality $\varphi \ominus \varphi_1(y) \leq \varphi(yx)$ we obtain

$$I_{\varphi\ominus\varphi_1}(y) \leq I_\varphi(yx) \leq \|yx\|_\varphi \leq 1.$$

In consequence

$$\|y\|_{\varphi\ominus\varphi_1} \leq 4c\|y\|_M.$$

Finally, there left the trivial case of $b_\varphi < \infty, b_{\varphi_1} = \infty$ to consider. However, Lemma (6.1.4) with the embedding (5) give

$$L^{\varphi\ominus\varphi_1} = M(L^{\varphi_1}, L^\varphi) = \{0\}$$

and the proof is finished.

Recall that given two B.f.s. X, Y over the same measure space, we say that X factorizes Y when

$$X \odot M(X, Y) = Y,$$

Where

$$X \odot M(X,Y) = \{z \in L^0 : z = xy \text{ for some } x \in X, y \in M(X,Y)\}.$$

The idea of such factorization goes back to Lozanovskii, who proved that each B.f.s. factorizes L^1. For more informations on factorization and its importance see [229,232] and [239] which are devoted mainly to this subject.

Also in [232] one may find a discussion on factorization of Orlicz spaces (and even more general Calderón–Lozanovskii spaces). Having in hand our representation $M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}$ we are able to complete this discussion by proving sufficient and necessary conditions for factorization in terms of respective Young functions.

We say that equivalence $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ holds for all [large] arguments when there are constants $c, C > 0$ such that

$$c\varphi^{-1}(u) \le \varphi_1^{-1}(u)\varphi_2^{-1}(u) \le C\varphi^{-1}(u)$$

for all $u \ge 0$ [for some $u_0 > 0$ and all $u > u_0$].

Theorem (6.1.8)[226]: Let φ, φ_1 be two Young functions. Then L^{φ_1} factorizes L^{φ}, i.e. $L^{\varphi_1} \odot M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi}$ if and only if

(i) equivalence $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}$ is satisfied for all arguments when $\mu(\Omega) = \infty$.

(ii) equivalence $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}$ is satisfied for large arguments when $\mu(\Omega) < \infty$,

Proof. In the light of Theorem (6.1.7)

$$L^{\varphi_1} \odot M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_1} \odot L^{\varphi \ominus \varphi_1}.$$

Therefore L^{φ_1} factorizes L^{φ} if and only if $L^{\varphi_1} \odot L^{\varphi \ominus \varphi_1} = L^{\varphi}$. The latter, however, is equivalent with $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}$ for all, or for large arguments, depending on Ω, as proved in Corollary 6 from [232].

Section (6.2): Analytic Polynomials in Abstract Hardy Spaces:

For $0 < p \le \infty$, let $L^p := L^p(\mathbb{T})$ be the Lebesgue space on the unit circle $\mathbb{T}: \{z \in \mathbb{C}: |z| = 1\}$ in the complex plane $\mathbb{C}$. For $f \in L^1$, let

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta, \quad n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of f. Let X be a Banach space continuously embedded in L^1. Following [264, p. 877], we will consider the abstract Hardy space $H[X]$ built upon the space X, which is defined by

$$H[X] := \{f \in X : \hat{f}(n) = 0 \quad \text{for all} \quad n < 0\}.$$

It is clear that if $1 \le p \le \infty$, then $H[L^p]$ is the classical Hardy space H^p.

A function of the form

$$q(t) = \sum_{k=0}^{n} \alpha_k t^k, \quad t \in \mathbb{T}, \quad \alpha_0, \dots, \alpha_n \in \mathbb{C},$$

is said to be an analytic polynomial on $\mathbb{T}$. The set of all analytic polynomials is denoted by P_A. It is well known that the set P_A is dense in H^p whenever $1 \le p < \infty$ (see, e.g., [254, Chap. III, Corollary 1.7(a)]). The density of the set P_A in the abstract Hardy spaces $H[X]$ was studied by [259] for the case when X is a so-called Banach function space.

We recall the definition of a Banach function space. We equip $\mathbb{T}$ with the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$. Let L^0 be the space of all measurable complex-valued functions on $\mathbb{T}$. As usual, we do not distinguish functions which are equal almost everywhere (for the latter we use the standard abbreviation a.e.). Let L^0_+ be the subset of functions in L^0 whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ is denoted by χ_E.

Following [252, Chap. 1, Definition 1.1], a mapping $\rho: L^0_+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, for all constants $a \geq 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

$(A1)\rho(f) = 0 \Leftrightarrow f = 0\ a.e., \rho(af) = a\rho(f), \rho(f + g) \leq \rho(f) + \rho(g),$

$(A2)0 \leq g \leq f\ a.e. \Rightarrow \rho(g) \leq \rho(f)$ (the lattice property),

$(A3)0 \leq f_n \uparrow f\ a.e. \Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

$(A4)m(E) < \infty \Rightarrow \rho(\chi E) < \infty,$

$$(A5) \int_E f(t)dm(t) \leq C_E \rho(f)$$

with a constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$.

The set X under the natural linear space operations and under this norm becomes a Banach space (see [252, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} f(t)g(t)dm(t) : f \in L^0_+, \rho(f) \leq 1\right\}, g \in L^0_+.$$

It is a Banach function norm itself [252, Chap. 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Kothe dual) of X. The associate space X' can be viewed as a subspace of the (Banach) dual space X^*.

Recall that L^1 is a commutative Banach algebra under the convolution multiplication defined for $f, g \in L^1$ by

$$(f * g)(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta - i\varphi})g(e^{i\varphi})d\varphi, \quad e^{i\theta} \in \mathbb{T}.$$

For $n \in \mathbb{N}$, let

$$F_n(e^{i\theta}) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{i\theta k} = \frac{1}{n+1} \left(\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}\right)^2, \qquad e^{i\theta} \in \mathbb{T},$$

be the n −th Fejer kernel. For $f \in L^1$, the n −th Fejer mean of f is defined as the convolution $f * F_n$.

Given $f \in L^1$, the Hardy-Littlewood maximal function is defined by

$$(Mf)(t) := \sup_{I \ni t} \frac{1}{m(I)} \int_I |f(\tau)| dm(\tau), t \in \mathbb{T};$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$ containing $t \in \mathbb{T}$. The operator $f \mapsto Mf$ is called the Hardy-Littlewood maximal operator.

Theorem (6.2.1)[251]: ([259, Theorem 3.3]). Suppose X is a separable Banach function space on $\mathbb{T}$. If the Hardy-Littlewood maximal operator is bounded on the associate space X', then for every $f \in X$,

$$\lim_{n \to \infty} \|f * F_n - f\|_X = 0. \quad (8)$$

It is well known that for $f \in L^1$ one has

$$(f * F_n)(e^{i\theta}) = \sum_{k=-n}^{n} \hat{f}(k) \left(1 - \frac{|k|}{n+1}\right) e^{i\theta k}, \ \ e^{i\theta} \in \mathbb{T}$$

(see, e.g., [260, Chap. I]). This implies that if $f \in H[X] \subset H[L^1] = H^1$, then $f * F_n \in P_A$. Combining this observation with Theorem (6.2.1), we arrive at the following.

Corollary (6.2.1)[251]:: ([8, Theorem 1.2]). Suppose X is a separable Banach function space on $\mathbb{T}$. If the Hardy-Littlewood maximal operator M is bounded on its associate space X', then the set of analytic polynomials P_A is dense in the abstract Hardy space $H[X]$ built upon the space X.

Note that if a Banach function space X is, in addition, rearrangement invariant, then the requirement of the boundedness of M on the space X'
can be omitted in Corollary (6.2.1) (see [259, Theorem 1.1] or [262, Lemma 1.3(c)]).

Lesnik [261] conjectured that the same fact should be true for arbitrary, not necessarily rearrangement-invariant, Banach function spaces.

We first observe that Theorem (6.2.1) does not hold for arbitrary separable Banach function spaces. For a function $K \in L^1$, consider the convolution operator C_K with kernel K defined by

$$C_K f = f * K, \quad f \in L^1.$$

It follows from [263, Theorem 2] that there exists a continuous function $p: \mathbb{T}[1, \infty)$ such that the sequence of the convolution operators C_{F_n} is not uniformly bounded in the variable Lebesgue space $L^{p(\cdot)}$ defined as the set of all $f \in L^0$ such that

$$\int_{\mathbb{T}} |f(t)|^{p(t)} dm(t) < \infty.$$

It is well known (see, e.g., [255, Propostion 2.12, Theorem 2.78, Section 2.10.3]) that if $p: \mathbb{T} \to [1, \infty)$ is continuous, then $L^{p(\cdot)}$ is a separable Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0: \int_{\mathbb{T}} \left|\frac{f(t)}{\lambda}\right|^{p(t)} dm(t) \leq 1\right\}$$

Since the norms of the convolution operators C_{F_n} may not be uniformly bounded on $L^{p(\cdot)}$, the standard argument, based on the uniform boundedness principle, leads us to the following.

Theorem (6.2.3)[251]:: There exist a separable Banach function space X on $\mathbb{T}$ and a function $f \in X$ such that (8) is not fullled.

We show that the separable Banach function space in Theorem (6.2.3) can be chosen as a weighted L^1 space, that is, the techniques of variable Lebesgue spaces can be omitted.

In spite of the observation made in Theorem (6.2.3) and Theorem (6.2.7), we show that the requirement of the boundedness of the Hardy-Littlewood maximal operator M on the associate space X' of a separable Banach function space X in Corollary (6.2.1) can be omitted. Thus, Lesnik's conjecture [261] is, indeed, true.

We prove that a convolution operator C_K with a nonnegative symmetric kernel $K \in L^1$ is bounded on a Banach function space X if and only if it is bounded on its associate space X'. Further, we consider a special weight $w \in L^1$ such that $w^{-1} \in L^\infty$. Then $X = L^1(w)$ is a separable Banach function space with the associate space $X' = L^\infty(w^{-1})$. We show that the sequence of convolution operators $\{C_{K_n}\}$ with nonnegative bounded symmetric kernels K_n, satisfying $\|K_n\|_{L^1} = 1$ and a natural localization property, is not uniformly bounded on $X' = L^\infty(w^{-1})$, and therefore, on its associate space $X'' = X = L^1(w)$. Applying this result to the sequence of the Fejer kernels $\{F_n\}$, we prove Theorem (6.2.7) with the aid of the uniform boundedness principle.

We recall that the separability of a Banach function space X is equivalent to $X^* = X'$. Further, we collect some facts on the identification of the Hardy spaces H^p on the unit circle and the Hardy spaces $H^p(\mathbb{D})$ of analytic functions in the unit disk $\mathbb{D}$. Finally, we give a proof of Theorem (6.2.11) based on an application of the Hahn-Banach theorem, a corollary of the Smirnov theorem and properties of the identification of H^1 with $H^1(\mathbb{D})$.

The Banach space of all bounded linear operators on a Banach space E is denoted by $B(E)$.

Lemma (6.2.4)[251]: Let X be a Banach function space on $\mathbb{T}$ and $K \in L^1$ be a nonnegative function such that $K(e^{i\theta}) = K(e^{-i\theta})$ for almost all $\theta \in [-\pi, \pi]$.

Then the convolution operator C_K is bounded on the Banach function X if and only if it is bounded on its associate space X'. In that case

$$\|C_K\|_{B(X')} = \|C_K\|_{B(X)}. \tag{9}$$

Proof. Suppose C_K is bounded on X'. Fix $f \in X\backslash\{0\}$. Since $K \geq 0$, we have $|f * K| \leq |f| * K$. According to the Lorentz-Luxemburg theorem (see, e.g., [252, Chap. 1, Theorem 2.7]), $X = X''$ with equality of the norms. Hence

$$\begin{aligned}\|f * K\|_X \leq \||f| * K\|_X = \||f| * K\|_{X''} \\ = \sup\left\{\int_{\mathbb{T}} (|f| * K)(t)|g(t)|dm(t) : g \in X', \|g\|_{X'} \leq 1\right\}.\end{aligned}$$

Then for every $\varepsilon > 0$ there exists a function $h \in X'$ such that $h \geq 0$, $\|h\|_{X'} \leq 1$, and

$$\|f * K\|_X \leq (1 + \varepsilon) \int_{\mathbb{T}} (|f| * K)(t)h(t)dm(t) \tag{10}$$

Taking into account that $K(e^{i\theta}) = K(e^{-i\theta})$ for almost all $\theta \in \mathbb{R}$, by Fubini's theorem, we get

$$\int_{\mathbb{T}} (|f| * K)(t)h(t)dm(t) = \int_{\mathbb{T}} (h * K)(t)|f(t)|dm(t).$$

From this identity, Hölder's inequality for X (see, e.g., [252, Chap. 1, Theorem 2.4]), and the boundedness of C_K on X', we obtain

$$\int_{\mathbb{T}} (|f| * K)(t)h(t)dm(t) \le \|f\|_X \|h * K\|_{X'} \le \|f\|_X \|C_K\|_{B(X')}. \quad (11)$$

It follows from (10)-(11) that

$$\|C_K\|_{B(X)} = \sup_{f \in X, f \neq 0} \frac{\|f * K\|_X}{\|f\|_X} \le (1 + \varepsilon)\|C_K\|_{B(X')}$$

for every $\varepsilon > 0$, which implies the boundedness of C_K on X and the inequality

$$\|C_K\|_{B(X)} \le \|C_K\|_{B(X')} \quad (12)$$

If C_K is bounded on X, then using the Lorentz-Luxemburg theorem and (12) with X' in place of X, we obtain that C_K is bounded on X' and

$$\|C_K\|_{B(X')} \le \|C_K\|_{B(X'')} = \|C_K\|_{B(X)} \quad (13)$$

Combining (12)-(13), we arrive at (9).

Lemma (6.2.5): Let

$$w(e^{i\theta}) := \begin{cases} \sqrt{m}, & \frac{\pi}{2m} \le |\theta| \le \frac{\pi}{2m - 1}, m \in \mathbb{N} \\ 1, & \frac{\pi}{2m + 1} < |\theta| < \frac{\pi}{2m}, m \in \mathbb{N} \end{cases} \quad (14)$$

Then the spaces

$$L^1(w) = \{f \in L^0 : fw \in L^1\}, \ \ L^\infty(w^{-1}) = \{f \in L^0 : fw^{-1} \in L^\infty\}$$

are Banach function spaces on $\mathbb{T}$ with respect to the norms

$$\|f\|_{L^1(w)} = \|fw\|_{L^1}, \qquad \|f\|_{L^\infty(w^{-1})} = \|fw^{-1}\|_{L^\infty},$$

and $\left(L^1(w)\right)' = L^\infty(w^{-1})$. Moreover, the space $L^1(w)$ is separable.

Proof. It is clear that $w^{-1} \in L^\infty$ and, since

$$\|w\|_{L^1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(e^{i\theta})d\theta$$

$$= \sum_{m=1}^{\infty} \left(\frac{1}{2m} - \frac{1}{2m + 1}\right) + \sum_{m=1}^{\infty} \sqrt{m} \left(\frac{1}{2m - 1} - \frac{1}{2m}\right) < \infty, \quad (15)$$

we also have $w \in L^1$. Then it follows from [258, Lemma 2.5] that $L^1(w)$ and $L^\infty(w^{-1})$ are Banach function spaces and $\left(L^1(w)\right)' = L^\infty(w^{-1})$. Finally, the separability of the space $L^1(w)$ follows from [258, Proposition 2.6] and [252, Chap. 1, Corollary 5.6].

Theorem (6.2.6)[251]: Let $\{K_n\}$ be a sequence of bounded functions $K_n : \mathbb{T} \to \mathbb{C}$ such that

$$K_n(e^{i\theta}) \ge 0, \ \ K_n(e^{i\theta}) = K_n(e^{-i\theta}) \ \text{a.e. on} \ [-\pi, \pi], \quad (16)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(e^{i\theta})d\theta = 1, \quad (17)$$

and

$$\lim_{n\to\infty} \sup_{\varepsilon\le|\theta|\le\pi} K_n(e^{i\theta}) = 0 \quad \text{for each } \varepsilon > 0. \tag{18}$$

If w is the weight given by (14), then the convolution operators C_{K_n} are bounded on $L^\infty(w^{-1})$ and on $L^1(w)$ for all $n \in \mathbb{N}$, however,

$$\sup_{n\in\mathbb{N}} \|C_{K_n}\|_{B(L^\infty(w^{-1}))} = \infty, \tag{19}$$

$$\sup_{n\in\mathbb{N}} \|C_{K_n}\|_{B(L^1(w))} = \infty. \tag{20}$$

Proof. By (14)-(15), $w \in L^1$ and $w^{-1} \in L^\infty$. Therefore, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\|C_{K_n} f\|_{L^1(w)} &\le \frac{1}{2\pi} \left\| \int_{-\pi}^{\pi} K_n(e^{i(\cdot-\theta)}) |f(e^{i\theta})| d\theta \right\|_{L^1(w)} \\
&\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \|K_n(e^{i(\cdot-\theta)})\|_{L^1(w)} |f(e^{i\theta})| d\theta \\
&\le \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|f\|_{L^1} \\
&= \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|w^{-1} f w\|_{L^1} \\
&\le \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|w^{-1}\|_{L^\infty} \|f\|_{L^1(w)}
\end{aligned}$$

Hence

$$\|C_{K_n}\|_{B(L^1(w))} \le \frac{1}{2\pi} \|K_n\|_{L^\infty} \|w\|_{L^1} \|w^{-1}\|_{L^\infty}, \quad n \in \mathbb{N}.$$

It follows from (16) and Lemmas (6.2.4)-(6.2.5) that the operators C_{K_n} are bounded on $L^\infty(w^{-1})$ for all $n \in \mathbb{N}$. Moreover, (19) implies (20).

Let us prove (19). Consider the sequence

$$v_m(e^{i\theta}) := \begin{cases} \sqrt{m}, \dfrac{\pi}{2m} \le \theta \le \dfrac{\pi}{2m-1}, \\ 0, \ \ \theta \in [-\pi,\pi] \setminus \left[\dfrac{\pi}{2m}, \dfrac{\pi}{2m-1}\right] \end{cases} m \in \mathbb{N}.$$

Then it follows from (14) that $\|v_m\|_{L^\infty(w^{-1})}$ for all $m \in \mathbb{N}$.

Fix $m \in \mathbb{N}$. According to (17) and the localization property (18), there exists $n(m) \in \mathbb{N}$ such that

$$\int_{-\frac{\pi}{(2m)^2}}^{0} K_n(e^{i\theta}) d\theta = \frac{1}{2} \int_{-\frac{\pi}{(2m)^2}}^{\frac{\pi}{(2m)^2}} K_n(e^{i\theta}) d\theta \ge \frac{1}{3} \ \text{ for all } n \ge n(m).$$

Since $K_n \in L^1$, for every $n \ge n(m)$, there exists $\delta_n > 0$ such that

$$\int_{-\frac{\pi}{(2m)^2}}^{-\delta_n} K_n(e^{i\theta}) d\theta \ge \frac{1}{4}$$

Therefore, for almost all $\vartheta \in \left[\frac{\pi}{2m} - \delta_n, \frac{\pi}{2m}\right]$, one gets

$$\left(C_{K_n}v_m\right)\left(e^{i\vartheta}\right)=\frac{\sqrt{m}}{2\pi}\int_{\frac{\pi}{2m}}^{\frac{\pi}{2m-1}}K_n\left(e^{i\vartheta-i\theta}\right)d\theta$$

$$\geq\frac{\sqrt{m}}{2\pi}\int_{\frac{\pi}{2m}}^{\frac{\pi}{2m}+\frac{\pi}{(2m)^2}}K_n\left(e^{i\vartheta-i\theta}\right)d\theta$$

$$=\frac{\sqrt{m}}{2\pi}\int_{\vartheta-\frac{\pi}{2m}-\frac{\pi}{(2m)^2}}^{\vartheta-\frac{\pi}{2m}}K_n\left(e^{i\eta}\right)d\eta$$

$$\geq\frac{\sqrt{m}}{2\pi}\int_{-\frac{\pi}{(2m)^2}}^{-\delta_n}K_n\left(e^{i\eta}\right)d\eta\geq\frac{\sqrt{m}}{8\pi}\qquad(21)$$

In view of (14), $w\left(e^{i\theta}\right)=1$ for all $\vartheta\in\left(\max\left\{\frac{\pi}{2m}-\delta_n,\frac{\pi}{2m+1}\right\},\frac{\pi}{2m}\right)$. Hence, it follows from (21) that

$$\left\|C_{K_n}v_m\right\|_{L^\infty(w^{-1})}\geq\frac{\sqrt{m}}{8\pi}\quad\text{for all}\quad n\geq n(m),$$

while $\|v_m\|_{L^\infty(w^{-1})}=1$. So,

$$\left\|C_{K_n}\right\|_{B\left(L^\infty(w^{-1})\right)}\geq\frac{\sqrt{m}}{8\pi}\quad\text{for all}\quad n\geq n(m)$$

Since $m\in\mathbb{N}$ is arbitrary, the latter inequality immediately implies (19).

Theorem (6.2.7)[252]: (Main result 1). There exist a nonnegative function $w\in L^1$ such that $w^{-1}\in L^\infty$ and a function f in the separable Banach function space

$$X=L^1(w)=\{f\in L^0;fw\in L^1\}$$

such that (8) is not fullled.

Proof. Let $X=L^1(w)$, where w is the weight given by (14). By Lemma (6.2.5), X is a separable Banach function space. It is well known (and not difficult to check) that the sequence $\{F_n\}$ of the Fejer kernels is a sequence of bounded functions satisfying (16)-(18). By Theorem (6.2.6), the operators C_{F_n} are bounded on X for every $n\in\mathbb{N}$.

Assume that (8) is fullled for all $f\in X$. Then, for all $f\in X$, the sequence $\{C_{F_n}f\}$ is bounded in X. Therefore, by the uniform boundedness principle, the sequence $\left\{\left\|C_{F_n}\right\|_{B(X)}\right\}$ is bounded, but this contradicts (20).

Thus, there exists a function $f\in X$ such that (8) does not hold.

Combining [252, Chap. I, Corollaries 4.3 and 5.6] and observing that the measure dm is separable (for the definition of a separable measure, see, e.g., [252,p. 27] or [257, Chap. I, Section 6.10]), we arrive at the following.

Theorem (6.2.8)[251]: Let X be a Banach function space on $\mathbb{T}$. Then X is separable if and only if its dual space X^* is isometrically isomorphic to the associate space X'.

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Recall that a function F analytic in $\mathbb{D}$ is said to belong to the Hardy space $H^p(\mathbb{D}), 0 < p \leq \infty$, if the integral mean

$$M_p(r,F) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| F\left(re^{i\theta}\right) \right|^p d\theta \right)^{1/p}, \qquad 0 < p < \infty$$

$$M_\infty(r,F) = \max_{-\pi \leq \theta \leq \pi} \left| F\left(re^{i\theta}\right) \right|,$$

remains bounded as $r \to 1$. If $F \in H^p(\mathbb{D}), 0 < p \leq \infty$, then the nontangential

$$f\left(e^{i\theta}\right) = \lim_{r \to 1-0} F\left(re^{i\theta}\right)$$

exists for almost all $\theta \in [-\pi, \pi]$ (see, e.g., [256, Theorem 2.2]) and the boundary function $f = f\left(e^{i\theta}\right)$ belongs to L^p.

The following lemma is an immediate consequence of the Smirnov theorem (see, e.g., [256, Theorem 2.11]).

Lemma (6.2.9)[251]: If $F \in H^p(\mathbb{D})$ for some $p \in (0,1)$ and its boundary function f belongs to L^1, then $F \in H^1(\mathbb{D})$.

Recall that if $f \in H^1$ then its analytic extension F into $\mathbb{D}$, given by the Poisson integral

$$F\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta - \varphi) f\left(e^{i\varphi}\right) d\varphi, \quad 0 \leq r < 1, -\pi \leq \theta \leq \pi,$$

where

$$P(r,\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \quad 0 \leq r < 1, \quad -\pi \leq \theta \leq \pi$$

is the Poisson kernel, belongs to $H^1(\mathbb{D})$ and the boundary function of F coincides with f a.e. on $\mathbb{T}$ (see, e.g., [256, Theorem 3.1]).

It is important to note that the Taylor coefficients of $F \in H^p(\mathbb{D})$ coincide with the Fourier coefficients of its boundary function $f \in L^p$. One has the following.

Theorem (6.2.10)[251]: ([256, Theorem 3.4]). Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ belong to $H^1(\mathbb{D})$ and let $\{\hat{f}(n)\}$ be the sequence of the Fourier coefficients of its boundary function $f \in L^1$. Then $\hat{f}(n) = a_n$ for all $n \geq 0$ and $\hat{f}(n) = 0$ for $n < 0$.

Theorem (6.2.11)[251]: (Main result 2). If X is a separable Banach function space on $\mathbb{T}$, then the set of analytic polynomials P_A is dense in the abstract Hardy space $H[X]$ built upon the space X.

Proof. Suppose P_A is not dense in $H[X]$. Take any function $f \in H[X]$ that does not belong to the closure of P_A with respect to the norm of X. Since X is separable, it follows from Theorem (6.2.8) that X^* is isometrically isomorphic to X'. Then, by a corollary of the Hahn-Banach theorem (see, e.g., [253, Chap. 7, Theorem 4.1]), there exists a function $g \in X' \subset L^1$ such that

$$\int_{-\pi}^{\pi} f\left(e^{i\theta}\right) g\left(e^{i\theta}\right) d\theta \neq 0 \tag{22}$$

and

$$\int_{-\pi}^{\pi} p(e^{i\theta})g(e^{i\theta})d\theta \neq 0 \quad \text{for all } p \in P_A.$$

In particular, if $p(e^{i\theta}) = e^{in\theta}$ with $n = 0,1,2, \ldots$, then

$$\hat{g}(-n) = 0 \quad \text{for all } n = 0,1,2, \ldots \tag{23}$$

Hence $g \in H[X'] \subset H^1$. For functions $f \in H[X] \subset H^1$ and $g \in H[X'] \subset H^1$, let F and G denote their analytic extensions to the unit disk $\mathbb{D}$ by means of their Poisson integrals. Then $F, G \in H^1(\mathbb{D})$. It follows from (23) and Theorem (6.2.10) that $G(0) = 0$. Since $F, G \in H^1(\mathbb{D})$, by Hölder's inequality, $FG \in H^{1/2}(\mathbb{D})$. On the other hand, since $f \in X$ and $g \in X'$, it follows from Hölder's inequality for Banach function spaces (see [252, Chap. 1, Theorem 2.4]) that $fg \in L^1$. Then it follows from Lemma (6.2.9) that $FG \in H^1(\mathbb{D})$.

Since $(FG)(0) = F(0)G(0) = 0$, applying Theorem (6.2.10) to FG, we obtain $\widehat{fg}(0) = 0$, that is,

$$\int_{-\pi}^{\pi} f(e^{i\theta})g(e^{i\theta})d\theta = 0$$

which contradicts (22).

Section (6.3): A pair of Abstract Hardy Spaces:

For $1 \leq p \leq \infty$, let $L^p := L^p(\mathbb{T})$ represent the standard Lebesgue space on the unit circle Tin the complex plane $\mathbb{C}$ with respect to the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$. For $f \in L^1$, let

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi})e^{-in\varphi}\, d\varphi \,, n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of f. For $1 \leq p \leq \infty$, the classical Hardy spaces H^p are defined by

$$H^p := \{f \in L^p : \hat{f}(n) = 0 \ \text{ for all } n < 0\}.$$

Consider the operators S and P, defined for a function $f \in L^1$ and an a.e. point $t \in \mathbb{T}$ by

$$(Sf)(t) := \frac{1}{\pi i} \text{p.v.} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t} d\tau \,, (Pf)(t) := \frac{f(t) + (Sf)(t)}{2}, \tag{24}$$

respectively, where the integral is understood in the Cauchy principal value sense. The operator S is called the Cauchy singular integral operator. It is well known that the operators P and S are bounded on L^p if $p \in (1, \infty)$ and are not bounded on L^p if $p \in \{1, \infty\}$ (see, e.g., [270, Section 4.4] or [271, Section 1.42]). Note that using the elementary equality

$$\frac{e^{i\theta}}{e^{i\theta} - e^{i\vartheta}} = \frac{1}{2}\left(1 + i \cot \frac{\vartheta - \theta}{2}\right) , \theta, \vartheta \in [-\pi, \pi],$$

one can write for $f \in L^1$ and $\vartheta \in [-\pi, \pi]$,

$$(Sf)(e^{i\vartheta}) = \frac{1}{\pi} \text{p.v.} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta} - e^{i\vartheta}} d\theta = \hat{f}(0) + i(Cf)(e^{i\vartheta}),$$

where the operator C, called the Hilbert transform, is defined for $f \in L^1$ by

$$(Cf)(e^{i\vartheta}) := \frac{1}{2\pi}\text{p.v.}\int_{-\pi}^{\pi} f(e^{i\theta})\cot\frac{\vartheta-\theta}{2}\,d\theta\,, \vartheta \in [-\pi,\pi]. \quad (25)$$

Hence the definition of Pf for $f \in L^1$ in terms of the Cauchy singular integral operator given by the second equality in (24) is equivalent to the following definition in terms of the Hilbert transform and the zeroth Fourier coefficient of f (cf. [278, p.104] and [271, Section 1.43]):

$$Pf := \frac{1}{2}(f + iCf) + \frac{1}{2}\hat{f}(0). \quad (26)$$

If $f \in L^1$ is such that $Pf \in L^1$, then

$$\widehat{Pf}(n) = \hat{f}(n) \text{ for } n \geq 0, Pf(n) = 0 \text{ for } n < 0. \quad (1.4)$$

Since we are not able to provide a precise reference to this well known fact, we will give its proof. Note that definitions (24) can be extended to more general Jordan curves in place of $\mathbb{T}$ (see, e.g., [270] and also [281,282,286]), while definitions (25) and (26) are used only in the case of the unit circle. If $1 < p < \infty$, then the operator P projects L^p onto H^p. In view of this fact, the operator P is usually called the Riesz projection.

For $a \in L^\infty$, the Toeplitz operator T_a with symbol a on $H^p, 1 < p < \infty$, is defined by

$$T_a f = P(af), f \in H^p.$$

The theory of Toeplitz operators has its origins in Otto Toeplitz [303]. Brown and Halmos [272, Theorem4] proved that an operator on H^2 is a Toeplitz operator if and only if its matrix with respect to the standard basis is a Toeplitz matrix, that is, an infinite matrix of the form $(a_{j-k})_{j,k=0}^{\infty}$ (see also [298, Part B, Theorem 4.1.4] and [300, Theorem 1.8]). An analogue of this result is true for Toeplitz operators acting on $H^p, 1 < p < \infty$ (see [271, Theorem 2.7]). Tolokonnikov [304] was the first to study Toeplitz operators acting between different Hardy spaces H^p and H^q. In particular, [304, Theorem 4] contains a description of all symbols generating bounded Toeplitz operators from H^p to H^q for $0 < p, q \leq \infty$.

Let X be a Banach function space. For the moment, we observe only that it is continuously embedded in L^1. Following [305, p.877], we consider the abstract Hardy space $H[X]$ built upon the space X, which is defined by

$$H[X] := \{f \in X : \hat{f}(n) = 0 \text{ forall } n < 0\}.$$

It is clear that if $1 \leq p \leq \infty$, then $H[L^p]$ is the classical Hardy space H^p.

Lemma (6.3.1)[265]: If the operator P defined by (24) is bounded on a Banach function space X over the unit circle $\mathbb{T}$, then its image $P(X)$ coincides with the abstract Hardy space $H[X]$ built upon X.

Since $X \subset L^1$, this lemma follows immediately from formula (27) and the uniqueness theorem for Fourier series (see, e.g., [289, Chap.1, Theorem 2.7]).

Thus, the operator P projects the Banach function space X onto the abstract Hardy space $H[X]$. We will call P the Riesz projection as in the case of the spaces L^p with $1 < p < \infty$.

The Brown–Halmos theorem was extended by the first to abstract Hardy spaces $H[X]$ built upon reflexive rearrangement-invariant Banach function spaces X with non-trivial Boyd indices [283, Theorem 4.5]. Under this assumption, the Riesz projection P

is bounded on X. Further, it was shown in [284, Theorem 1] that the Brown–Halmos theorem remains true for abstract Hardy spaces built upon arbitrarily, not necessarily rearrangement-invariant, reflexive Banach function spaces X under the assumption that the Riesz projection is bounded on X. In particular, it is true for the weighted Hardy spaces $H^p(w), 1 < p < \infty$, with Muckenhoupt weights $w \in A_p(\mathbb{T})$ [284, Corollary 9].

The space of all bounded linear operators from a Banach space E to a Banach space F is denoted by $B(E, F)$. We adopt the standard abbreviation $B(E)$ for $B(E, E)$. We will write $E = F$ if E and F coincide as sets and there are constants $c_1, c_2 \in (0, \infty)$ such that $c_1\|f\|_E \le \|f\|_F \le c_2\|f\|_E$ for all $f \in E$, and $E \equiv F$ if E and F coincide as sets and $\|f\|_E = \|f\|_F$ for all $f \in E$.

We study Toeplitz operators acting between abstract Hardy spaces $H[X]$ and $H[Y]$ built upon different Banach function spaces X and Y over the unit circle $\mathbb{T}$. We extend further the results by Leśnik [293], who additionally assumed that the Banach function spaces X and Y are rearrangement-invariant. Let L^0 be the space of all measurable complex-valued functions on $\mathbb{T}$. Following [295], let $M(X, Y)$ denote the space of point wise multipliers from X to Y defined by $M(X, Y) := \{f \in L^0 : fg \in Y \text{ for all } g \in X\}$ and equipped with the natural operator norm

$$\|f\|_{M(X,Y)} = \|M_f\|_{B(X,Y)} = \sup_{\|g\|_X \le 1} \|fg\|_Y.$$

Here M_f stands for the operator of multiplication by f defined by $(M_f g)(t) = f(t)g(t)$ for $t \in \mathbb{T}$.

In particular, $M(X, X) \equiv L^\infty$. Note that it may happen that the space $M(X, Y)$ contains only the zero function. For instance, if $1 \le p < q \le \infty$, then $M(L^p, L^q) = \{0\}$. The continuous embedding $L^\infty \subset M(X, Y)$ holds if and only if $X \subset Y$ continuously. For example, if $1 \le q \le p \le \infty$, then $L^p \subset L^q$ and $M(L^p, L^q) \equiv L^r$, where $1/r = 1/q - 1/p$. For these and many other properties and examples, we refer to [291,294,295,297].

If the Riesz projection P is bounded on the space Y, then one can define the Toeplitz operator T_a with symbol $a \in M(X, Y)$ by

$$T_a f = P(af), f \in H[X]$$

(cf. [293]). It follows from Lemma (6.3.1) that $T_a f \in H[Y]$ and, clearly,

$$\|T_a\|_{B(H[X],H[Y])} \le \|P\|_{B(Y)}\|a\|_{M(X,Y)}.$$

Let X' be the associate space of X. For $f \in X$ and $g \in X'$, put

$$\langle f, g\rangle := \int_{\mathbb{T}} f(t)\overline{g(t)}dm(t).$$

For $n \in \mathbb{Z}$ and $\tau \in \mathbb{T}$, put $\chi_n(\tau) := \tau^n$. Then the Fourier coefficients of a function $f \in L^1$ can be expressed by $\hat{f}(n) = \langle f, \chi_n\rangle$ for $n \in \mathbb{Z}$. With this notation, the main result reads as follows.

Theorem (6.3.2)[265]: (à la Brown–Halmos). Let X, Y be two Banach function spaces over the unit circle $\mathbb{T}$. Suppose that X is separable and the Riesz projection P is bounded on the space Y. If $A \in B(H[X], H[Y])$ and there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that

$$\langle A\chi_j, \chi_k\rangle = a_{k-j} \text{ for all } j, k \ge 0, \tag{28}$$

then there is a function $a \in M(X,Y)$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover,

$$\|a\|_{M(X,Y)} \leq \|T_a\|_{B(H[X],H[Y])} \leq \|P\|_{B(Y)}\|a\|_{M(X,Y)}. \quad (29)$$

Under the additional assumption that the Banach function spaces X and Yare rearrangement-invariant, this result was recently obtained by Leśnik [293, Theorem 4.2].

The above theorem and the fact that $M(X,X) \equiv L^\infty$ (see [295, Theorem 1]) immediately imply the following.

Corollary (6.3.3)[265]: Let X be a separable Banach function spaces over the unit circle $\mathbb{T}$ and let the Riesz projection P be bounded on X. If $A \in B(H[X])$ and there is a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers satisfying (28), then there exists a function $a \in L^\infty$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover,

$$\|a\|_{L^\infty} \leq \|T_a\|_{B(H[X])} \leq \|P\|_{B(X)}\|a\|_{L^\infty}.$$

Note that Corollary (6.3.3) is also new. Under the additional assumption that the Banach function space X is reflexive, it was proved by [284, Theorem 1]. On the other hand, under the additional hypothesis that X is rearrangement-invariant, it is established in [293, Corollary 4.4].

We collect preliminary facts on Banach function spaces X, including results on the density of the set of all trigonometric polynomials P in X and the density of the set of all analytic polynomials P_A in the abstract Hardy space $H[X]$ built upon X. Further, we show that if each function in the closure $(X')_b$ of all simple functions in the associate space X' has absolutely continuous norm, then the norm of any function $f \in X$ can be expressed as follows:

$$\|f\|_X = \sup\{|\langle f,p\rangle| : p \in P, \|p\|_{X'} \leq 1\}. \quad (30)$$

We conclude several facts from complex analysis on the Hilbert transform and inner functions. In particular, we recall a result by Qiu [301, Lemma 5.1] (see also [273, Theorem 7.2]) saying that, for every measurable set $E \subset \mathbb{T}$ and an arc $\gamma \subset \mathbb{T}$ of the same measure, there exists an inner function u such that $u^{-1}(\gamma)$ and E coincide almost everywhere.

We start the consequences of the boundedness of the operator P defined by (24) with a discussion of operators of weak type. It is easy to see that if the Riesz projection P is bounded on X, then the Hilbert transform C is of weak types (L^∞, X) and (L^∞, X'). Using the existence of the inner function umentioned above and properties of the Hilbert transform, we show that if C is of weak types (L^∞, X) and (L^∞, X'), then each function in the closures X_b and $(X')_b$ of the simple functions in X and X', respectively, has absolutely continuous norm. Thus, for every $f \in X$, formula (30) holds under the only assumption that $P \in B(X)$.

We present a proof of Theorem (6.3.2). Armed with the density of the set of analytic polynomials P_A in the abstract Hardy space $H[X]$ built upon a separable Banach function space X and formula (30) with Y such that $P \in B(Y)$ in place of X, we can adapt the proofs given in [271, Theorem 2.7] (for $X = Y = L^p$ with $1 < p < \infty$) and in [293, Theorem 4.2] (for the case of separable rearrangement-invariant spaces $X \subset Y$ such that Y has non-trivial Boyd indices) to our setting.

We specify the result of Theorem (6.3.2) to the case of variable Lebesgue spaces (also known as Nakano spaces) $X = L^{p(\cdot)}$ and $Y = L^{q(\cdot)}$. It is known that if $1/q(t) = 1/p(t) + 1/r(t)$ for $t \in \mathbb{T}$, then $M\big(L^{p(\cdot)}, L^{q(\cdot)}\big) = L^{r(\cdot)}$ and that the Riesz projection P is bounded on $L^{q(\cdot)}$ if the variable exponent q is sufficiently smooth and bounded away from 1 and ∞. Since the spaces $L^{p(\cdot)}$ and $L^{q(\cdot)}$ are not rearrangement-invariant, in general, the main result cannot be obtained from [293, Theorem 4.2].

We apply Corollary (6.3.3) to the case of Lorentz spaces $L^{p,q}(w), 1 < p < \infty, 1 \le q < \infty$, with Muckenhoupt weights $w \in A_p(\mathbb{T})$. Under these assumptions, $L^{p,q}(w)$ is a separable Banach function space and the Riesz projection P is bounded on $L^{p,q}(w)$. The space $L^{p,1}(w)$ is not reflexive and not rearrangement-invariant. Hence the earlier results of [284, Theorem 1] and [293, Corollary 4.4] are not applicable to the space $L^{p,1}(w)$, while Corollary (6.3.3) is.

Let L_+^0 be the subset of functions in L^0 whose values lie in $[0, \infty]$. The characteristic (indicator) function of a measurable set $E \subset \mathbb{T}$ is denoted by $\mathbb{I}_E$.

Following [266, Chap.1, Definition 1.1], a mapping $\rho: L_+^0 \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L_+^0$ with $n \in \mathbb{N}$, for all constants $a \ge 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

$(A1) \rho(f) = 0 \Leftrightarrow f = 0\ a.e., \rho(af) = a\rho(f), \rho(f + g) \le \rho(f) + \rho(g),$

$(A2)\ 0 \le g \le f\ a.e. \Rightarrow \rho(g) \le \rho(f)$ (the lattice property),

$(A3)\ 0 \le f_n \uparrow f\ a.e. \Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

$(A4) m(E) < \infty \Rightarrow \rho(\mathbb{I}_E) < \infty,$

$$(A5) \int_E f(t)dm(t) \le C_E \rho(f)$$

with a constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f. When functions differing only on a set of measure zero are identified, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$. The set X under the natural linear space operations and under this norm becomes a Banach space (see [266, Chap.1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on L_+^0 by

$$\rho'(g) := \sup\left\{\int_{\mathbb{T}} f(t)g(t)dm(t) : f \in L_+^0, \rho(f) \le 1\right\}, g \in L_+^0.$$

It is a Banach function norm itself [266, Chap.1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space (Köthe dual) of X. The associate space X' can be viewed as a subspace of the (Banach) dual space X^*.

For $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, a function of the form $\sum_{k=-n}^{n} \alpha_k \chi_k$, where $\alpha_k \in \mathbb{C}$ for all $k \in \{-n, \dots, n\}$, is called a trigonometric polynomial of order n. The set of all trigonometric polynomials is denoted by P. Further, a function of the form $\sum_{k=0}^{n} \alpha_k \chi_k$ with $\alpha_k \in \mathbb{C}$ for $k \in \{0, \dots, n\}$ is called an analytic polynomial of order n. The set of all analytic polynomials is denoted by P_A.

Following [266, Chap.1, Definition 3.1], a function f in a Banach function space X is said to have absolutely continuous norm in X if $\left\|f \mathbb{I}_{\gamma_n}\right\|_X \to 0$ for every sequence

$\{\gamma_n\}_{n\in\mathbb{N}}$ of measurable sets such that $f\mathbb{I}_{\gamma_n} \to 0$ almost everywhere as $n \to \infty$. The set of all functions of absolutely continuous norm in X is denoted by X_a. If $X_a = X$, then one says that X has absolutely continuous norm. Let S_0 be the set of all simple functions on $\mathbb{T}$. Following [266, Chap.1, Definition 3.9], let X_b denote the closure of S_0 in the norm of X.

Lemma (6.3.4)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If $X_a = X_b$, then the set of trigonometric polynomials P is dense in X_b.

Proof. The proof is analogous to the proof of [285, Lemma 2.2.1]. Assume that P is not dense in X_b. Then, by a corollary of the Hahn–Banach theorem (see, e.g., [267, Chap.7, Theorem 4.2]), there exists a nonzero functional $\Lambda \in (X_b)^*$ such that $\Lambda(p) = 0$ for all $p \in P$. It follows from [266, Chap.1, Theorems 3.10 and 4.1] that if $X_a = X_b$, then $(X_b)^* = X'$. Hence there exists a nonzero function $h \in X' \subset L^1$ such that

$$\int_{\mathbb{T}} p(t)h(t)dm(t) = 0 \quad \text{forall} \quad p \in P.$$

Taking $p(t) = t^n$ for $n \in \mathbb{Z}$, we obtain that all Fourier coefficients of $h \in L^1$ vanish, which implies that $h = 0$ a.e. on $\mathbb{T}$ by the uniqueness theorem of the Fourier series (see, e.g., [289, Chap. I, Theorem 2.7]). This contradiction proves that P is dense in X_b.

Combining the above lemma with [266, Chap.1, Corollary 5.6 and Theorem 3.11], we arrive at the following well known result.

Corollary (6.3.5)[265]: A Banach function space X over the unit circle Tis separable if and only if the set of trigonometric polynomials P is dense in X.

The analytic counterpart of the above result had a hard birth. First, observe that under the additional assumption that the Riesz projection P is bounded on X, the density of the set of analytic polynomials P_A in the abstract Hardy space $H[X]$ trivially follows from (27), Lemma (6.3.1), and Corollary (6.3.5) (see [284, Lemma 4]). Leśnik [292] conjectured that the boundedness of P is superfluous here and P_A must be dense in the abstract Hardy space $H[X]$ under the hypothesis that X is merely separable.

If X is a separable rearrangement-invariant Banach function space, then

$$\|f * F_n - f\|_X \to 0 \text{ for every } f \in X \text{ as } n \to \infty, \tag{31}$$

Where $\{F_n\}$ is the sequence of the Fejér kernels on the unit circle $\mathbb{T}$. The property in (31) implies the density of P_A in $H[X]$ (see, e.g., [293, Lemma 3.1(c)] or [285, Theorem 1.0.1]). If X is an arbitrary separable Banach function space, then (31) is true under the assumption that the Hardy–Littlewood maximal operator M is bounded on its associate space X' [285, Theorem 3.2.1], whence P_A is dense in $H[X]$ (see [285, Theorem 1.0.2]). Finally, in [287, Theorem 1.4] we constructed a separable weighted L^1 space X such that (31) does not hold. On the other hand, we proved Leśnik's conjecture.

Lemma (6.3.6)[265]: ([287, Theorem 1.5]). If X is a separable Banach function space over the unit circle $\mathbb{T}$, then the set of analytic polynomials P_A is dense in the abstract Hardy space $H[X]$ built upon the space X.

Let X be a Banach function space over the unit circle $\mathbb{T}$ and X' be its associate space. Then for every $f \in X$ and $h \in X'$, one has the following well known formulae:

$$\|f\|_X = \sup\{|\langle f, g\rangle| : g \in X', \|g\|_{X'} \leq 1\}, \tag{32}$$

$$\|f\|_X = \sup\{|\langle f, s\rangle| : g \in S_0, \|s\|_{X'} \leq 1\}, \tag{33}$$

$$\|h\|_{X'} = \sup\{|\langle h, s\rangle|: g \in S_0, \|s\|_X \le 1\}. \quad (34)$$

Equality (32) follows from [266, Chap.1, Theorem 2.7 and Lemma 2.8]. Equality (33) can be proved by a literal repetition of the proof of [288, Lemma 2.10]. Equality (34) is obtained by applying formula (33) to $h \in X'$ and recalling that $X \equiv X''$ in view of the Lorentz–Luxemburg theorem (see [266, Chap.1, Theorem 2.7]).

Lemma (6.3.7)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If $(X')_a = (X')_b$, then for every $f \in X$,

$$\|f\|_X = \sup\{|\langle f, p\rangle|: p \in P, \|p\|_{X'} \le 1\}. \quad (35)$$

Proof. Since $P \subset X'$, equality (32) immediately implies that

$$\|f\|_X \ge \sup\{|\langle f, p\rangle|: p \in P, \|p\|_{X'} \le 1\}. \quad (36)$$

Take any $g \in (X')_b$ such that $0 < \|g\|_{X'} \le 1$. Since $(X')_a = (X')_b$, it follows from Lemma (6.3.4) that there is a sequence $q_n \in P\backslash\{0\}$ such that $\|q_n - g\|_{X'} \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$, put $p_n := (\|g\|_{X'}/\|q_n\|_{X'})q_n \in P$. Then, arguing as in [284, Lemma 5], one can show that

$$|\langle f, g\rangle| = \lim_{n\to\infty} |\langle f, p_n\rangle| \le \sup_{n\in\mathbb{N}} |\langle f, p_n\rangle| \le \sup\{|\langle f, p\rangle|: p \in P, \|p\|_{X'} \le 1\}.$$

This inequality and equality (32) imply that

$$\|f\|_X \le \sup\{|\langle f, p\rangle|: p \in P, \|p\|_{X'} \le 1\}. \quad (37)$$

Combining inequalities (36) and (37), we arrive at equality (35).

Note that Leśnik proved formula (35) for arbitrary rearrangement-invariant Banach function spaces X (see [293, Lemma 3.2]). His proof relies on the interpolation theorem of Calderón (see [266, Chap.3, Theorem 2.2]), which allows one to prove that for $f \in X'$, the sequence $p_n = f * F_n \in P$ satisfies $\|p_n\|_{X'} \le \|f\|_{X'}$ for all $n \in \mathbb{N}$. In the setting of arbitrary Banach function spaces, the tools based on interpolation are not available, but one can prove (35) for translation-invariant Banach function spaces and their weighted generalizations with positive continuous weights (cf. [288, Corollary 2.13]). We show that if the Riesz projection P is bounded on a Banach function space X, then $(X')_a = (X')_b$, whence formula (35) holds.

Let $\mathbb{D}$ denote the open unit disk in the complex plane C. Recall that a function F analytic in $\mathbb{D}$ is said to belong to the Hardy space $H^p(\mathbb{D}), 0 < p \le \infty$, if

$$\|F\|_{H^p(\mathbb{D})} := \sup_{0\le r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|F\left(re^{i\theta}\right)\right|^p d\theta \right)^{1/p} < \infty, 0 < p < \infty, \|F\|_{H^\infty(\mathbb{D})}: = \sup_{z\in\mathbb{D}} |F(z)| < \infty.$$

Recall that an inner function is a function $u \in H^\infty(\mathbb{D})$ such that $\left|u\left(e^{i\theta}\right)\right| = 1$ for a.e. $\theta \in [-\pi, \pi]$.

The following important fact was observed by Nordgren (see corollary to [299, Lemma 1] and also [274, Remark 9.4.6]).

Lemma (6.3.8)[265]: If u is an inner function such that $u(0) = 0$, then u is a measure-preserving transformation from $\mathbb{T}$ onto itself.

Proof. We include a sketch of the proof for the readers' convenience. Let G be an arbitrary measurable subset of $\mathbb{T}$ and let h be the bounded harmonic function on $\mathbb{D}$ with the boundary values equal to $\mathbb{I}_G$. Then $h \circ u$ is the bounded harmonic function on $\mathbb{D}$ with the boundary values equal to $\mathbb{I}_{u^{-1}}(G)$, and

$$m(G) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{I}_G(e^{i\theta})d\theta = h(0) = h(u(0))$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{I}_{u^{-1}(G)}(e^{i\theta})d\theta = m(u^{-1}(G)),$$
which completes the proof.

The next result is one of the most important ingredients in our proof. It appeared in [301, Lemma 5.1] and [273, Theorem 7.2].

Theorem (6.3.9)[265]: If $E \subset \mathbb{T}$ is a measurable set and $\gamma \subset \mathbb{T}$ is an arc such that $m(E) = m(\gamma)$, then there exists an inner function u satisfying $u(0) = 0$ and such that the sets $u^{-1}(\gamma)$ and E are equal almost everywhere.

For $\vartheta \in [-\pi, \pi]$ and $r \in [0, 1)$, let
$$P_r(\vartheta) := \frac{1 - r^2}{1 - 2r\cos\vartheta + r^2}, Q_r(\vartheta) := \frac{2r\sin\vartheta}{1 - 2r\cos\vartheta + r^2}$$
be the Poisson kernel and the conjugate Poisson kernel, respectively.

Theorem (6.3.10)[265]: Let $1 < p < \infty$.

(a) If $f \in L^p$ is a real-valued function, then the function defined by
$$u(re^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})(P_r + iQ_r)(\vartheta - \theta)d\theta, \vartheta \in [-\pi, \pi], r \in [0, 1), \tag{38}$$
belongs to the Hardy space $H^p(\mathbb{D})$. Its nontangential boundary values $u(e^{i\vartheta})$ as $z \to e^{i\vartheta}$ exist for a.e. $\vartheta \in [-\pi, \pi]$ and
$$\operatorname{Re} u(e^{i\vartheta}) = f(e^{i\vartheta}), \operatorname{Im} u(e^{i\vartheta}) = (Cf)(e^{i\vartheta}) \text{ for } a.e.\ \vartheta \in [-\pi, \pi], \tag{39}$$
Where C is the Hilbert transform defined by (25).

(b) If $u \in H^p(\mathbb{D})$ and $\operatorname{Im} u(0) = 0$, then there is a real-valued function $f \in L^p$ such that (38) holds.

This statement is well known (see, e.g., [291, Chap. I, Section D and Chap.V, Section B.2°]).

Proof of formula (27). Since $f \in L^1$, the Cauchy integral
$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - z}d\tau, z \in \mathbb{D},$$
belongs to $H^p(\mathbb{D})$ for all $0 < p < 1$ (see, e.g., [277, Theorem 3.5]). By Privalov's theorem (see, e.g., [279, Chap. X, §3, Theorem1]), the nontangential limit of $F(z)$ as $z \to e^{i\vartheta}$ coincides with $(Pf)(e^{i\vartheta})$ for a.e. $\vartheta \in [-\pi, \pi]$. Hence, taking into account that $Pf \in L^1$, by Smirnov's theorem (see, e.g., [279, Chap. IX, §4, Theorem 4] or [277, Theorem 3.4]), $F \in H^1(\mathbb{D})$. Then (27) follows from [277, Theorem 3.4] and the formula for the Taylor coefficients of F:
$$\frac{1}{n!}F^{(n)}(0) = \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{f(\tau)}{\tau^{n+1}}d\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})e^{-in\varphi}\, d\varphi = \hat{f}(n), n \geq 0,$$
Which completes the proof.

Let X and Y be Banach function spaces over the unit circle. Following [268], we say that a linear operator $A: X \to L^0$ is of weak type (X, Y) if there exists a constant $C > 0$ such that for all $\lambda > 0$ and $f \in X$,

$$\left\|\mathbb{I}_{\{\zeta\in\mathbb{T}:|(Af)(\zeta)|>\lambda\}}\right\|_Y \le C\frac{\|f\|_X}{\lambda}. \tag{40}$$

We denote the infimum of the constants C satisfying (40) by $\|A\|_{W(Y,X)}$ and the set of all operators of weak type (X, Y) by $W(X, Y)$.

Lemma (6.3.11)[265]: Let X, Y be Banach function spaces over the unit circle $\mathbb{T}$. If $A \in B(X, Y)$, then $A \in W(X, Y)$ and $\|A\|_{W(X,Y)} \le \|A\|_{B(X,Y)}$.

Proof. For all $\lambda > 0, f \in X$ and almost all $\tau \in \mathbb{T}$, one has

$$\mathbb{I}_{\{\zeta\in\mathbb{T}:\,|(Af)(\zeta)|>\lambda\}}(\tau) \le \mathbb{I}_{\{\zeta\in\mathbb{T}:\,|(Af)(\zeta)|>\lambda\}}(\tau)\frac{|(Af)(\tau)|}{\lambda} \le \frac{|(Af)(\tau)|}{\lambda}.$$

It follows from the above inequality, the lattice property, and the boundedness of the operator A that

$$\left\|\mathbb{I}_{\{\zeta\in\mathbb{T}:\,|(Af)(\zeta)|>\lambda\}}\right\|_Y \le \left\|\frac{Af}{\lambda}\right\|_Y \le \|A\|_{B(X,Y)}\frac{\|f\|_X}{\lambda},$$

which completes the proof.

For a set $G \subset [-\pi, \pi]$, we use the following notation

$$\mathbb{I}_G^*\left(e^{i\theta}\right) := \begin{cases} 1, \theta \in G, \\ 0, \theta \in [-\pi, \pi] \setminus G. \end{cases}$$

Let $|G|$ denote the Lebesgue measure of G.

Lemma (6.3.12)[265]: For every measurable set $E \subset [-\pi, \pi]$ with $0 < |E| \le \pi/2$, there exists a measurable set $F \subset [-\pi, \pi]$ with $|F| = \pi$ such that

$$\left|(C\mathbb{I}_F^*)\left(e^{i\vartheta}\right)\right| > \frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right| \quad \text{for a.e. } \vartheta \in E. \tag{41}$$

Proof. Let $\ell := \left\{e^{i\eta} \in \mathbb{T} : \pi - |E| < \eta < \pi\right\}$. By Theorem (6.3.9), there exists an inner function V such that $V(0) = 0$ and

$$V\left(e^{i\vartheta}\right) \in \begin{cases} \ell \text{ for } a.e. \vartheta \in E, \\ \mathbb{T}\setminus\ell \text{ for a.e. } \vartheta \in [-\pi, \pi]\setminus E. \end{cases} \tag{42}$$

Consider the set

$$F := \left\{\theta \in [-\pi, \pi]: \operatorname{Im} V\left(e^{i\theta}\right) \le 0\right\}. \tag{43}$$

Since $V(0) = 0$ and V is inner, it defines a measure-preserving transformation of $\mathbb{T}$ onto itself due to Lemma (6.3.8). Therefore,

$$|F| = \left|\left\{\vartheta \in [-\pi, \pi]: \operatorname{Im} e^{i\vartheta} \le 0\right\}\right| = \pi.$$

For $\eta \in [-\pi, \pi]$ and $r \in [0, 1)$, let

$$w\left(re^{i\eta}\right) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{I}_{[-\pi,0]}^*\left(e^{i\zeta}\right)(P_r + iQ_r)\,(\eta - \zeta)d\zeta.$$

By Theorem (6.3.10), the function $w \in H^2(\mathbb{D})$ has nontangential boundary values $w\left(e^{i\eta}\right)$ as $z \to e^{i\eta}$ for a.e. $\eta \in [-\pi, \pi]$ and

$$\operatorname{Re} w\left(e^{i\eta}\right) = \mathbb{I}_{[-\pi,0]}^*\left(e^{i\eta}\right) \text{ for a.e.} \eta \in [-\pi, \pi], \tag{44}$$

$$\operatorname{Im} w\left(e^{i\eta}\right) = \left(C\mathbb{I}_{[-\pi,0]}^*\right)\left(e^{i\eta}\right) \text{ for a.e.} \eta \in [-\pi, \pi]. \tag{45}$$

It is clear that for $\eta \in (\pi - |E|, \pi)$,

$$\left(C\mathbb{I}_{[-\pi,0]}^{*}\right)\left(e^{i\eta}\right)=\frac{1}{2\pi}\int_{-\pi}^{0}\cot\frac{\eta-\zeta}{2}\,d\zeta=\frac{1}{\pi}\log\sin\frac{\eta}{2}-\frac{1}{\pi}\log\sin\frac{\eta+\pi}{2}. \tag{46}$$

Since $|E|\in(0,\pi/2]$, we have for all $\eta\in(\pi-|E|,\pi)$,

$$\log\sin\frac{\eta}{2}>\log\sin\frac{\pi}{4}=\ -\log\sqrt{2}\geq\log\sin\frac{|E|}{2}>\log\sin\frac{\eta+\pi}{2}. \tag{47}$$

It follows from (45)–(47) that for a.e. $\eta\in(\pi-|E|,\pi)$,

$$\left|\operatorname{Im} w\left(e^{i\eta}\right)\right|>\frac{1}{\pi}\left(-\log\sqrt{2}-\log\sin\frac{|E|}{2}\right)=\frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|. \tag{48}$$

Consider now the function $W=w\circ V$, which belongs to $H^2(\mathbb{D})$ (see, e.g., [277, Section 2.6]). In view of (43) and (44), we have

$$\operatorname{Re}W\left(e^{i\vartheta}\right)=\begin{cases}1 \text{ if } \operatorname{Im} V\left(e^{i\vartheta}\right)\leq 0,\\ 0 \text{ if } \operatorname{Im} V\left(e^{i\vartheta}\right)>0\end{cases}=\mathbb{I}_F^{*}\left(e^{i\vartheta}\right) \text{ for a.e. } \vartheta\in[-\pi,\pi].$$

Then, by Theorem (6.3.10),

$$\operatorname{Im} W\left(e^{i\vartheta}\right)=(C\mathbb{I}_F^{*})\left(e^{i\vartheta}\right) \text{ for a.e. } \vartheta\in[-\pi,\pi]. \tag{49}$$

If $\vartheta\in E$, then it follows from (42) that $V\left(e^{i\vartheta}\right)\in\ell$. In this case inequality (48) implies that for a.e. $\vartheta\in E$,

$$\left|\operatorname{Im} W\left(e^{i\vartheta}\right)\right|=\left|\operatorname{Im} wV\left(e^{i\vartheta}\right)\right|>\frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right.. \tag{50}$$

Combining equality (49) and inequality (50), we arrive at (41).

Lemma (6.3.13)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Hilbert transform C is of weak type (L^∞,X), then for every measurable set $E\subset[-\pi,\pi]$ with $0<|E|\leq\pi/2$, one has

$$\|\mathbb{I}_E^{*}\|_X\leq\frac{\pi\|C\|_{W(L^\infty,X)}}{\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|}. \tag{51}$$

Proof. Let

$$\lambda=\frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|.$$

By Lemma (6.3.12), there exists a measurable set $F\subset[-\pi,\pi]$ with $|F|=\pi$ such that for a.e. $\tau\in\mathbb{T}$,

$$\mathbb{I}_E^{*}(\tau)\leq\mathbb{I}_{\{\zeta\in\mathbb{T}:|(C\mathbb{I}_F^{*})(\zeta)|>\lambda\}}(\tau).$$

Therefore, by the lattice property, taking into account that $C\in W(L^\infty,X)$, we obtain

$$\|\mathbb{I}_E^{*}\|_X\leq\left\|\mathbb{I}_{\{\zeta\in\mathbb{T}:|(C\mathbb{I}_F^{*})(\zeta)|>\lambda\}}\right\|_X\leq\frac{1}{\lambda}\|C\|_{W(L^\infty,X)}\|\mathbb{I}_F^{*}\|_{L^\infty}=\frac{\pi\|C\|_{W(L^\infty,X)}}{\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|},$$

which completes the proof.

Theorem (6.3.14)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Hilbert transform C is of weak type (L^∞,X), then $X_a=X_b$.

Proof. Let $\Gamma\subset\mathbb{T}$ be a measurable set. Consider a sequence of measurable subsets $\{\gamma_n\}_{n\in\mathbb{N}}$ of $\mathbb{T}$ such that $\mathbb{I}_{\gamma_n}\to 0$ a.e. on $\mathbb{T}$. By the dominated convergence theorem,

$$m(\gamma_n) = \int_{\mathbb{T}} \mathbb{I}_{\gamma_n}(\tau)dm(\tau) \to 0 \; as \; n \to \infty.$$

Without loss of generality, one can assume that $0 < m(\gamma_n) \leq 1/4$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, there exists a measurable set $E_n \subset [-\pi, \pi]$ such that $\mathbb{I}_{\gamma_n}(\tau) = \mathbb{I}^*_{E_n}(\tau)$ for all $\tau \in \mathbb{T}$. It is clear that $|E_n| = 2\pi m(\gamma_n) \leq \pi/2$ for $n \in \mathbb{N}$. By Lemma (6.3.13), for every $n \in \mathbb{N}$,

$$\left\|\mathbb{I}_\Gamma \mathbb{I}_{\gamma_n}\right\|_X \leq \left\|\mathbb{I}_{\gamma_n}\right\|_X = \left\|\mathbb{I}^*_{E_n}\right\|_X \leq \frac{\pi\|C\|_{W(L^\infty,Y)}}{\left|\log\left(\sqrt{2}\sin\frac{|E_n|}{2}\right)\right|} = \frac{\pi\|C\|_{W(L^\infty,Y)}}{\left|\log\left(\sqrt{2}\sin\left(\pi m(\gamma_n)\right)\right)\right|}.$$

Since $m(\gamma_n) \to 0$ as $n \to \infty$, the above estimate implies that $\left\|\mathbb{I}_\Gamma \mathbb{I}_{\gamma_n}\right\|_X \to 0$ as $n \to \infty$. Thus the function $\mathbb{I}_\Gamma$ has absolutely continuous norm. By [266, Chap.1, Theorem 3.13], $X_a = X_b$.

Lemma (6.3.15)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$ and X' be its associate space. If $C \in B(X_b, X)$, then $C \in B((X')_b, X')$ and

$$\|C\|_{B((X')_b,X)} \leq \|C\|_{B(X_b,X)}. \tag{52}$$

Proof. It is well known that the operator iC is a self-adjoint operator on the space L^2 (see, e.g., [298, Section 5.7.3(a)]). Therefore, for all $s, v \in S_0 \subset L^2$, one has

$$\langle Cv, s\rangle = -\langle v, Cs\rangle. \tag{53}$$

It follows from equalities (34), (53), and Hölder's inequality (see [266, Chap.1, Theorem 2.4]) that for every $v \in S_0$,

$$\|Cv\|_{X'} = \sup\{|\langle Cv, s\rangle|: s \in S_0, \|s\|_X \leq 1\} = \sup\{|\langle v, Cs\rangle|: s \in S_0, \|s\|_X \leq 1\}$$
$$\leq \sup\{\|v\|_{X'}\|Cs\|_X: s \in S_0, \|s\|_X \leq 1\} \leq \|C\|_{B(X_b,X)}\|v\|_{X'}.$$

Since S_0 is dense in $(X')_b$, we conclude that $C \in B((X')_b, X')$ and (52) holds.

Lemma (6.3.16)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$ and X' be its associate space. If the Riesz projection P is bounded on X, then $C \in W(L^\infty, X)$ and $C \in W(L^\infty, X')$.

Proof. Since X is continuously embedded into L^1, the functional $f \longmapsto \hat{f}(0)$ is continuous on the space X. Then it follows from (26) that $P \in B(X)$ if and only if $C \in B(X)$. Since L^∞ is continuously embedded into X, one has $B(X) \subset B(L^\infty, X)$. By Lemma (6.3.11), $B(L^\infty, X) \subset W(L^\infty, X)$. These observations imply that $C \in W(L^\infty, X)$ if $P \in B(X)$. Since X_b is a Banach space isometrically embedded into X (see [266, Chap.1, Theorem 3.1]), we see that $C \in B(X) \subset B(X_b, X)$ if $P \in B(X)$. Then, by Lemma (6.3.15), $C \in B((X')_b, X')$. Taking into account that L^∞ is continuously embedded into $(X')_b$ (see, e.g., [266, Chap.1, Proposition 3.10]), we get $C \in B((X')_b, X') \subset B(L^\infty, X')$, which implies that $C \in W(L^\infty, X')$ in view of Lemma (6.3.11).

Now we are in a position to formulate the main result.

Theorem (6.3.17)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Riesz projection P is bounded on X, then $X_a = X_b$ and $(X')_a = (X')_b$.

Proof. If the Riesz projection P is bounded on a Banach function space X, then the Hilbert transform C is of weak types (L^∞, X) and (L^∞, X') in view of Lemma (6.3.16). In turn, $C \in W(L^\infty, X)$ implies that $X_a = X_b$ and $C \in W(L^\infty, X')$ implies that $(X')_a = (X')_b$ due to Theorem (6.3.14).

Combining Theorem (6.3.17) and Lemma (6.3.7), we immediately arrive at the following.

Corollary (6.3.18)[265]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Riesz projection P is bounded on X, then for every $f \in X$,

$$\|f\|_X = \sup\{|\langle f, p\rangle| : p \in P, \|p\|_{X'} \leq 1\}.$$

Lemma (6.3.19)[265]: Let X, Y be Banach functions spaces over the unit circle $\mathbb{T}$. Suppose X is separable and $A \in B(X, Y)$. If there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that

$$\langle A\chi_j, \chi_k\rangle = a_{k-j} \text{ for all } j, k \in \mathbb{Z}, \tag{54}$$

then there exists a function $a \in M(X, Y)$ such that $A = M_a$ and

$$\hat{a}(n) = a_n \text{ for all } n \in \mathbb{Z}.$$

Proof. This statement was proved in [293, Lemma 4.1] under the additional hypothesis that X and Y are rearrangement-invariant Banach function spaces. Put $a := A\chi_0 \in Y$. Then, one can show exactly as in [293] that $(af)\hat{}(j) = (Af)\hat{}(j)$ for all $j \in \mathbb{Z}$ and $f \in P$. Therefore, $Af = af$ for all $f \in P$ in view of the uniqueness theorem for Fourier series (see, e.g., [289, Chap.1, Theorem 2.7]).

Now let $f \in X$. Since the space X is separable, the set P is dense in X by Corollary (6.3.5). Then there exists a sequence $p_n \in P$ such that $p_n \to f$ in X and, whence, $Ap_n \to Af$ in X as $n \to \infty$. By [266, Chap.1, Theorem 1.4], $p_n \to f$ and $Ap_n \to af$ in measure as $n \to \infty$. Then $ap_n \to af$ in measure as $n \to \infty$ (see, e.g., [269, Corollary 2.2.6]). Hence, the sequence $Ap_n = ap_n$ converges in measure to the functions Af and af as $n \to \infty$. This implies that Af and af coincide a.e. on $\mathbb{T}$ (see, e.g., the discussion preceding [296, Theorem 2.2.3]). Thus $Af = af$ for all $f \in X$. This means that $A = M_a$ and $a \in M(X, Y)$ by the definition of $M(X, Y)$.

We present a proof of our extension of the Brown–Halmos theorem. Although it follows the scheme of the proof of [271, Theorem 2.7] with modifications that are necessary in the setting of different spaces X and Y(cf. [293, Theorem 4.2]), it uses results obtained (e.g., Theorem (6.3.17) and Corollary (6.3.18)) and in [287] (see Lemma (6.3.6) above). We provide details for the sake of completeness.

Since $P \in B(Y)$, it follows from Theorem (6.3.17) that $(Y')_a = (Y')_b$. Then, by Lemma (6.3.4), the set of trigonometric polynomials P is dense in $(Y')_b$. Therefore, $(Y')_b$ is separable. It follows from [266, Chap.1, Theorems 3.11 and 4.1] that $((Y')_b)^* = Y''$. On the other hand, by the Lorentz–Luxemburg theorem (see [266, Chap.1, Theorem 2.7]), $Y'' \equiv Y$. Thus, the Banach function space Y is canonically isometrically isomorphic to the dual space $((Y')_b)^*$ of the separable Banach space $(Y')_b$.

For $n \geq 0$, put $b_n := \chi_{-n}A\chi_n$. Then $b_n \in Y$ and

$$\begin{aligned}\|b_n\|_Y = \|A\chi_n\|_Y = \|A\chi_n\|_{H[Y]} &\leq \|A\|_{B(H[X],H[Y])}\|\chi_n\|_X \\ &= \|A\|_{B(H[X],H[Y])}\|1\|_X.\end{aligned} \tag{55}$$

Put

$$V = \left\{y \in (Y')_b : \|y\|_{Y'} < \frac{1}{\|A\|_{B(H[X],H[Y])}\|1\|_X}\right\}.$$

It follows from the Hölder inequality (see [266, Chap.1, Theorem 2.4]) and (55) that

$$|\langle b_n, y\rangle| \leq \|b_n\|_Y\|y\|_{Y'} < 1 \text{ forall } y \in V, n \geq 0.$$

Applying a corollary of the Banach–Alaoglu theorem (see, e.g., [302, Theorem 3.17]) to the neighborhood V of zero in the separable Banach space $(Y')_b$ and the sequence $\{b_n\}_{n\in\mathbb{N}} \subset Y = \left((Y')b\right)^*$, we deduce that there exists a function $b \in Y$ such that some subsequence $\{b_{n_k}\}_{k\in\mathbb{N}}$of $\{b_n\}_{n\in\mathbb{N}}$ converges to bin the weak-* topology of $((Y')_b)^*$. It follows from [266, Chap.1, Proposition 3.10] that $\chi_j \in (Y')_b$ for all $j \in \mathbb{Z}$. Hence

$$\lim_{k\to+\infty} \langle b_{n_k}, \chi_j\rangle = \langle b, \chi_j\rangle \text{ for all } j \in \mathbb{Z}. \tag{56}$$

On the other hand, we get from the definition of b_n and (28) for $n_k + j \geq 0$,

$$\langle b_{n_k}, \chi_j\rangle = \langle \chi_{-n_k} A\chi_{n_k}, \chi_j\rangle = \langle A\chi_{n_k}, \chi_{n_k+j}\rangle = a_j. \tag{57}$$

It follows from (56) and (57) that

$$\langle b, \chi_j\rangle = a_j \text{ for all } j \in \mathbb{Z}. \tag{58}$$

Now define the mapping B by

$$B: P \to Y, f \longmapsto bf. \tag{59}$$

Assume that f and g are trigonometric polynomials of order m and r, respectively. Using equalities (28) and (58) and definition (59), one can show that for $n \geq \max\{m, r\}$,

$$\langle Bf, g\rangle = \langle \chi_{-n} A(\chi_n f), g\rangle. \tag{60}$$

It is clear that for those n, one has $\chi_n f \in H[X]$. Since $A \in B(H[X], H[Y])$, we obtain

$$\begin{aligned}\|A(\chi_n f)\|_Y = \|A(\chi_n f)\|_{H[Y]} &\leq \|A\|_{B(H[X],H[Y])} \|\chi_n f\|_{H[X]} \\ &= \|A\|_{B(H[X],H[Y])} \|f\|_X.\end{aligned} \tag{61}$$

Hence, by the Hölder inequality (see [266, Chap.1, Theorem 2.4]), we deduce from (61) that

$$\begin{aligned}|\langle \chi_{-n} A(\chi_n f), g\rangle| &\leq \|\chi_{-n} A(\chi_n f)\|_Y \|g\|_{Y'} = \|A(\chi_n f)\|_Y \|g\|_{Y'} \\ &\leq \|A\|_{B(H[X],H[Y])} \|f\|_X \|g\|_{Y'}.\end{aligned} \tag{62}$$

It follows from (60) and (62) that

$$\begin{aligned}|\langle Bf, g\rangle| &\leq \limsup_{n\to\infty} |\langle \chi_{-n} A(\chi_n f), g\rangle| \\ &\leq \|A\|_{B(H[X],H[Y])} \|f\|_X \|g\|_{Y'}.\end{aligned} \tag{63}$$

Since the Riesz projection P is bounded on Y, inequality (63) and Corollary (6.3.18) imply that for every $f \in P$,

$$\|Bf\|_Y = \sup\{|\langle Bf, g\rangle| : g \in P, \|g\|_{Y'} \leq 1\} \leq \|A\|_{B(H[X],H[Y])} \|f\|_X.$$

Since X is separable, the set P is dense in X in view of Corollary (6.3.5). Hence the above inequality shows that the linear mapping defined in (59) extends to an operator $B \in B(X, Y)$ with

$$\|B\|_{B(X,Y)} \leq \|A\|_{B(H[X],H[Y])}. \tag{64}$$

We deduce from (58)–(59) that

$$\langle B\chi_j, \chi_k\rangle = \langle b\chi_j, \chi_k\rangle = \langle b, \chi_{k-j}\rangle = a_{k-j} \text{ for all } j, k \in \mathbb{Z}.$$

Then, by Lemma (6.3.19), there exists a function $a \in M(X, Y)$ such that $B = M_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover,

$$\|B\|_{B(X,Y)} = \|M_a\|_{B(X,Y)} = \|a\|_{M(X,Y)}. \tag{65}$$

It follows from the definition of the Toeplitz operator T_a that

$$\langle T_a\chi_j, \chi_k\rangle = \hat{a}(k - j) \text{ forall } j, k \geq 0.$$

Combining this identity with (28), we obtain

$$\langle T_a\chi_j, \chi_k\rangle = a_{k-j} = \langle A\chi_j, \chi_k\rangle \text{ for all } j, k \geq 0. \tag{66}$$

Since $T_a\chi_j, A\chi_j \in H[Y] \subset H^1$, it follows from (66) and the uniqueness theorem for Fourier series (see, e.g., [289, Chap.1, Theorem 2.7]) that $T_a\chi_j = A\chi_j$ for all $j \geq 0$. Therefore,

$$T_a f = Af \text{ for all } f \in P_A. \tag{67}$$

By Lemma (6.3.6), P_A is dense in $H[X]$. This observation and (67) imply that $T_a = A$ on $H[X]$ and

$$\|T_a\|_{B(H[X],H[Y])} = \|A\|_{B(H[X],H[Y])}. \tag{68}$$

Combining inequality (64) with equalities (65) and (68), we arrive at the first inequality in (29). The second inequality in (29) is obvious.

Let $B(\mathbb{T})$ be the set of all measurable functions $p: \mathbb{T} \to [1, \infty]$. For $p \in B(\mathbb{T})$, put

$$T_\infty^{p(\cdot)} := \{t \in \mathbb{T}: p(t) = \infty\}.$$

For a function $f \in L^0$, consider

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{T}\setminus\mathbb{T}_\infty^{p(\cdot)}} |f(t)|^p(t)dm(t) + \|f\|_{L^\infty\left(\mathbb{T}_\infty^{p(\cdot)}\right)}.$$

The variable Lebesgue space $L^{p(\cdot)}$ is defined (see, e.g., [275, Definition 2.9]) as the set of all measurable functions $f \in L^0$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. This space is a Banach function space with respect to the Luxemburg–Nakano norm given by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0: \varrho_{p(\cdot)}(f/\lambda) \leq 1\}$$

(see [275, Theorems 2.17, 2.71 and Section 2.10.3]). If $p \in B(\mathbb{T})$ is constant, then $L^{p(\cdot)}$ is nothing but the standard Lebesgue space L^p. Variable Lebesgue spaces are often called Nakano spaces. See Maligranda [296] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces.

For $p \in B(\mathbb{T})$, put

$$p_- := \operatorname{ess}\inf_{t\in\mathbb{T}} p(t)\,, p_+ := \operatorname{ess}\inf_{t\in\mathbb{T}} p(t).$$

It is well known that the variable Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is separable if and only if $p_+ < \infty$ and is reflexive if and only if $1 < p_-, p_+ < \infty$ (see, e.g., [275, Theorem 2.78 and Corollary 2.79]).

The following result was obtained by Nakai [297, Example 4.1]under the additional hypothesis

$$\sup_{t\in\mathbb{T}\setminus\mathbb{T}_\infty^{r(\cdot)}} r(t) < \infty$$

(and in the more general setting of quasi-Banach variable Lebesgue spaces over arbitrary measure spaces). Nakai also mentioned in [297, Remark 4.2] (without proof) that this hypothesis is superfluous. One can find its proof in the present form in [286, Theorem 4.8].

Theorem (6.3.20)[265]: Let $p, q, r \in B(\mathbb{T})$ be related by

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, t \in \mathbb{T}. \tag{69}$$

Then $M\left(L^{p(\cdot)}, L^{q(\cdot)}\right) = L^{r(\cdot)}$.

We say that an exponent $q \in B(\mathbb{T})$ is log-Hölder continuous (cf. [275, Definition 2.2]) if $1 < q_- \leq q_+ < \infty$ and there exists a constant $C_{q(\cdot)} \in (0, \infty)$ such that

$$|q(t)-q(\tau)| \leq \frac{C_{q(\cdot)}}{-\log|t-\tau|} \text{ for all } t,\tau \in \mathbb{T} \text{ satisfying } |t-\tau| < 1/2.$$

The class of all log-Hölder continuous exponent will be denoted by $LH(\mathbb{T})$. Some authors denote this class by $\mathbb{P}^{\log}(\mathbb{T})$ (see, e.g., [290, Section 1.1.4]). The following result is well known (see, e.g., [290, Section 10.1] or [284, Lemma 12]).

Theorem (6.3.21)[265]: If $q \in LH(\mathbb{T})$, then the Riesz projection P is bounded on $L^{q(\cdot)}$.

Applying Theorems (6.3.2), (6.3.20), and (6.3.21), we arrive at the following.

Theorem (6.3.22)[265]: Let $p, q, r \in B(\mathbb{T})$ be related by (69). Suppose $q \in LH(\mathbb{T})$ and $p_+ < \infty$. If a linear operator A is bounded form $H[L^{p(\cdot)}]$ to $H[L^{q(\cdot)}]$ and there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that

$$\langle A\chi_j, \chi_k\rangle = a_{k-j} \text{ for all } j,k \geq 0,$$

then there is a function $a \in L^{r(\cdot)}$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover, there exist constants $c_{p,q}, C_{p,q} \in (0,\infty)$ depending only on p and q such that

$$c_{p,q}\|a\|_{L^{r(\cdot)}} \leq \|T_a\|_{B(H[L^{p(\cdot)}],H[L^{q(\cdot)}])} \leq C_{p,q}\|P\|_{B(L^{q(\cdot)})}\|a\|_{L^{r(\cdot)}}.$$

Note that if $p, q \in LH(\mathbb{T})$ coincide, then the constants $c_{p,q}$ and $C_{p,q}$ in the above inequality are equal to one (cf. [284, Corollary 13]).

The distribution function m_f of an a.e. finite function $f \in L^0$ is given by

$$m_f(\lambda) := m\{t \in \mathbb{T}: |f(t)| > \lambda\}, \lambda \geq 0.$$

The non-increasing rearrangement of an a.e. finite function $f \in L^0$ is defined by

$$f^*(x) := \inf\{\lambda: m_f(\lambda) \leq x\}, x \in [0,1].$$

We refer to [266, Chap.2, Section 1] for properties of distribution functions and non-increasing rearrangements.

Two a.e. finite functions $f, g \in L^0$ are said to be equimeasurable if their distribution functions coincide: $m_f(\lambda) = m_g(\lambda)$ for all $\lambda \geq 0$. A Banach function space X over the unit circle $\mathbb{T}$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in L^0$, $f \in X$ implies that $g \in X$ and the equality $\|f\|_X = \|g\|_X$ holds. For a rearrangement-invariant Banach function space X, its associate space X' is also rearrangement-invariant (see [266, Chap.2, Proposition 4.2]).

Let f be an a.e. finite function in L^0. For $x \in (0,1]$, put

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(y)dy.$$

Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. The Lorentz space $L^{p,q}$ consists of all a.e. finite functions $f \in L^0$ for which the quantity

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^1 \left(x^{1/p}f^{**}(x)\right)\frac{dx}{d}\right)^{1/q}, \text{if } 1 \leq q < \infty, \\ \sup_{0<x<1}\left(x^{1/p}f^{**}(x)\right), \text{if } q = \infty, \end{cases}$$

is finite. It is well known that $L^{p,q}$ is a rearrangement-invariant Banach function space with respect to the norm $\|\cdot\|_{L^{p,q}}$ (see, e.g., [266, Chap.4, Theorem 4.6], where the case of spaces of infinite measure is considered; in the case of spaces of finite measure, the proof is the same). It follows from [266, Chap.2, Proposition 1.8 and Chap.4, Lemma 4.5] that $L^{p,p} = L^p$ (with equivalent norms).

For $q \in [1,\infty]$, put $q' = q/(q-1)$with the usual conventions $1/0 = \infty$ and $1/\infty = 0$. A function $w \in L^0_+$ is referred to as a weight if $0 < w(\tau) < \infty$ for a.e. $\tau \in \mathbb{T}$.

Let $1 < p < \infty$ and $1 \le q \le \infty$. Suppose $w: \mathbb{T} \to [0,\infty]$ is a weight such that $w \in L^{p,q}$ and $1/w \in L^{p',q'}$. The weighted Lorentz space $L^{p,q}(w)$ is defined as the set of all a.e. finite functions $f \in L^0$ such that $fw \in L^{p,q}$.

The next lemma follows directly from well known results on Lorentz spaces.

Lemma (6.3.23)[265]: Let $1 < p < \infty, 1 \le q \le \infty$ and $w: \mathbb{T} \to [0,\infty]$ be a weight such that $w \in L^{p,q}, 1/w \in L^{p',q'}$.

(a) The space $L^{p,q}(w)$ is a Banach function space with respect to the norm $\|f\|_{L^{p,q}(w)} = \|fw\|_{L^{p,q}}$ and $L^{p',q'}(1/w)$ is its associate space.

(b) If $1 < q < \infty$, then the space $L^{p,q}(w)$ is reflexive.

(c) The space $L^{p,1}(w)$ is separable and non-reflexive.

Proof.(a) In view of [266, Chap. 4, Theorem 4.7], the associate space of the Lorentz space $L^{p,q}$, up to equivalence of norms, is the Lorentz space $L^{p',q'}$. It is easy to check that $L^{p,q}(w)$ is a Banach function space and $L^{p',q'}(1/w)$ is its associate space.

(b) Note that $L^{p,q}(w) \ni f \longmapsto wf \in L^{p,q}$ is an isometric isomorphism of $L^{p,q}(w)$ and $L^{p,q}$. Hence these spaces have the same Banach space theory properties, e.g., reflexivity and separability. If $1 < p, q < \infty$, then $L^{p,q}$ is reflexive in view of [266, Chap.4, Corollary 4.8]. Then the weighted Lorentz space $L^{p,q}(w)$ is reflexive too.

(c) If $1 < p < \infty$, then $L^{p,1}$ has absolutely continuous norm and $(L^{p,1})^* = L^{p',\infty}$(see [266, Chap. 4, Corollary 4.8]). Then $L^{p,1}$ is separable in view of [266, Chap.1, Corollary 5.6]. It is known that

$$L^{p,1}(L^{p,\infty})^* = (L^{p,1})^{**}$$

(see [276, p.83]). Hence $L^{p,1}$ is non-reflexive. Therefore, $L^{p,1}(w)$ is also separable and nonrefexive.

Let $1 < p < \infty$ and w be a weight. It is well known that the Riesz projection P is bounded on the weighted Lebesgue space $L^p(w) := \{f \in L^0 : fw \in L^p\}$ if and only if the weight w satisfies the Muckenhoupt A_p −condition, that is,

$$\sup_{\gamma \subset \mathbb{T}} \left(\frac{1}{m(\gamma)} \int_\gamma w^p(\tau)dm(\tau) \right)^{\frac{1}{p}} \left(\frac{1}{m(\gamma)} \int_\gamma w^{-p'}(\tau)dm(\tau) \right)^{1/p} < \infty,$$

where the supremum is taken over all subarcs γ of the unit circle $\mathbb{T}$ (see [280]and also [270, Section 6.2], [271, Section 1.46], [298, Section 5.7.3(h)]). In this case, we will write $w \in A_p(\mathbb{T})$.

Lemma (6.3.24)[265]: Let $1 < p < \infty$ and $1 \le q \le \infty$. If $w \in A_p(\mathbb{T})$, then $w \in L^{p,q}$ and $1/w \in L^{p',q'}$.

Proof. By the stability property of Muckenhoupt weights (see, e.g., [270, Theorem 2.31]), there exists $\varepsilon > 0$ such that $w \in A_s(\mathbb{T})$ for all $s \in (p-\varepsilon, p+\varepsilon)$. Therefore, $w \in L^s$ and $1/w \in L^{s'}$ for all $s \in (p-\varepsilon, p+\varepsilon)$. In particular, if s_1, s_2 are such that $p - \varepsilon <$

$s_1 < p < s_2 < p + \varepsilon$, then $w \in L^{s_2} = L^{s_2,s_2} \subset L^{p,q}$ and $1/w \in L^{s_1'} = L^{s_1',s_1'} \subset L^{p',q'}$ in view of the embeddings of Lorentz spaces (see, e.g., [266, Chap.4, remark after Proposition 4.2]).

Lemmas (6.3.23)(a) and (6.3.24) imply that if $w \in A_p(\mathbb{T})$, then $L^{p,q}(w)$ is a Banach function space.

Theorem (6.3.25)[265]: Let $1 < p < \infty$ and $1 \le q \le \infty$. If $w \in A_p(\mathbb{T})$, then the Riesz projection P is bounded on the weighted Lorentz space $L^{p,q}(w)$.

Proof. It follows from [266, Chap.4, Theorem 4.6] and [281, Theorem 4.5] that the Cauchy singular integral operator S is bounded on $L^{p,q}(w)$. Thus, the Riesz projection P is bounded on $L^{p,q}(w)$ in view of (24).

The next theorem is an immediate consequence of Corollary (6.3.3), Lemmas (6.3.23) and (6.3.24), and Theorem (6.3.25).

Theorem (6.3.26)[265]: Let $1 < p < \infty, 1 \le q < \infty$, and $w \in A_p(\mathbb{T})$. If an operator A is bounded on the abstract Hardy space $H[L^{p,q}(w)]$ and there exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers such that

$$\langle A\chi_j, \chi_k\rangle = a_{k-j} \text{ for all } j, k \ge 0,$$

then there is a function $a \in L^\infty$ such that $A = T_a$ and $\hat{a}(n) = a_n$ for all $n \in \mathbb{Z}$. Moreover,

$$\|a\|_{L^\infty} \le \|T_a\|_{B(H[L^{p,q}(w)])} \le \|P\|_{B(L^{p,q}(w))}\|a\|_{L^\infty}.$$

For $p = q$ this result is contained in [284, Corollary 9]. For $1 < q < \infty$, this result as well follows from [284, Theorem1]. The most interesting case is when $q = 1$ because in this case the weighted Lorentz space $L^{p,1}(w)$ is separable and non-reflexive. Moreover, it is not rearrangement-invariant. Therefore [284, Theorem 1] and [293, Corollary 4.4] are not applicable, while Corollary (6.3.3) works in this case.

Corollary (6.3.27)[307]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If $X_{a^r} = X_{b^r}$, then the set of trigonometric polynomials P is dense in X_{b^r}.

Proof. The proof is analogous to the proof of [285, Lemma 2.2.1]. Assume that P is not dense in X_{b^r}. Then, by a corollary of the Hahn–Banach theorem (see, e.g., [267, Chap.7, Theorem 4.2]), there exists a nonzero functional $A_r \in (X_{b^r})^*$ such that $A_r(p) = 0$ for all $p \in P$. It follows from [266, Chap.1, Theorems 3.10 and 4.1] that if $X_{a^r} = X_{b^r}$, then $(X_{b^r})^* = X'$. Hence there exists a nonzero function $h_r \in X' \subset L^1$ such that

$$\int_{\mathbb{T}} \sum p(t)h_r(t)dm(t) = 0 \qquad \text{for all } p \in P.$$

Taking $p(t) = t^n$ for $n \in \mathbb{Z}$, we obtain that all Fourier coefficients of $h_r \in L^1$ vanish, which implies that $h_r = 0$ a.e. on $\mathbb{T}$ by the uniqueness theorem of the Fourier series (see, e.g., [289, Chap. I, Theorem 2.7]). This contradiction proves that P is dense in X_{b^r}.

Corollary (6.3.28)[307]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If $(X')_{a^r} = (X')_{b^r}$, then for every $f_r \in X$,

$$\|f_r\|_X = \sup\{|\langle f_r, p\rangle| : p \in P, \|p\|_{X'} \le 1\}.$$

Proof. Since $P \subset X'$, equality (32) immediately implies that

$$\|f_r\|_X \ge \sup\{|\langle f_r, p\rangle| : p \in P, \|p\|_{X'} \le 1\}.$$

Take any $g_r \in (X')_{b^r}$ such that $0 < \|g_r\|_{X'} \le 1$. Since $(X')_{a^r} = (X')_{b^r}$, it follows from Lemma (6.3.4) that there is a sequence $q_n \in P\backslash\{0\}$ such that $\|q_n - g_r\|_{X'} \to 0$ as

$\to \infty$. For $n \in \mathbb{N}$, put $p_n := (\|g_r\|_{X'}/\|q_n\|_{X'})q_n \in P$. Then, arguing as in [284, Lemma 5], one can show that

$$|\langle f_r, g_r \rangle| = \lim_{n\to\infty} |\langle f_r, p_n\rangle| \le \sup_{n\in\mathbb{N}} |\langle f_r, p_n\rangle| \le \sup\{|\langle f_r, 1+\varepsilon\rangle| : p \in P, \|1+\varepsilon\|_{X'} \le 1\}.$$

This inequality and equality (32) imply that

$$\|f_r\|_X \le \sup\{|\langle f_r, 1+\varepsilon\rangle| : p \in P, \|1+\varepsilon\|_{X'} \le 1\}.$$

Combining inequalities (36) and (37), we arrive at equality (35).

Corollary (6.3.29)[307]: If u_r is an inner function such that $u_r(0) = 0$, then u_r is a measure-preserving transformation from $\mathbb{T}$ onto itself.

Proof. We include a sketch of the proof for the readers' convenience. Let G_r be an arbitrary measurable subset of $\mathbb{T}$ and let h_r be the bounded harmonic function on $\mathbb{D}$ with the boundary values equal to $\mathbb{I}_{G_r}$. Then $h_r \circ u_r$ is the bounded harmonic function on $\mathbb{D}$ with the boundary values equal to $\mathbb{I}_{{u_r}^{-1}}(G_r)$, and

$$m(G_r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum \mathbb{I}_{G_r}(e^{i\theta})d\theta = \sum h_r(0) = \sum h_r(u_r(0))$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum \mathbb{I}_{{u_r}^{-1}(G_r)}(e^{i\theta})d\theta = \sum m({u_r}^{-1}(G_r)),$$

which completes the proof.

Corollary (6.3.10)[307]: Let $X, X+\varepsilon$ be Banach function spaces over the unit circle $\mathbb{T}$. If $A_r \in B(X, X+\varepsilon)$, then $A_r \in W_r(X, X+\varepsilon)$ and $\|A_r\|_{W_r(X,X+\varepsilon)} \le \|A_r\|_{B(X,X+\varepsilon)}$.

Proof. For all $\lambda > 0, f_r \in X$ and almost all $\tau \in \mathbb{T}$, one has

$$\sum \mathbb{I}_{\{\zeta\in\mathbb{T}:\, |(A_r f_r)(\zeta)|>\lambda\}}(\tau) \le \sum \mathbb{I}_{\{\zeta\in\mathbb{T}:\, |(A_r f_r)(\zeta)|>\lambda\}}(\tau)\frac{|(A_r f_r)(\tau)|}{\lambda} \le \sum \frac{|(A_r f_r)(\tau)|}{\lambda}.$$

It follows from the above inequality, the lattice property, and the boundedness of the operator A_r that

$$\left\|\sum \mathbb{I}_{\{\zeta\in\mathbb{T}:\, |(A_r f_r)(\zeta)|>\lambda\}}\right\|_{X+\varepsilon} \le \sum \left\|\frac{A_r f_r}{\lambda}\right\|_{X+\varepsilon} \le \sum \|A_r\|_{B(X,X+\varepsilon)} \sum \frac{\|f_r\|_X}{\lambda},$$

which completes the proof.

Corollary (6.3.31)[307]: For every measurable set $E \subset [-\pi, \pi]$ with $0 < |E| \le \pi/2$, there exists a measurable set $F_r \subset [-\pi, \pi]$ with $|F_r| = \pi$ such that

$$\left|\left(C_r \mathbb{I}_{F_r}^*\right)(e^{i\vartheta})\right| > \frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right| \quad \text{for a.e. } \vartheta \in E.$$

Proof. Let $\ell := \{e^{i\eta} \in \mathbb{T} : \pi - |E| < \eta < \pi\}$. By Theorem (6.3.9), there exists an inner function V_r such that $V_r(0) = 0$ and

$$V_r(e^{i\vartheta}) \in \begin{cases} \ell \text{ for } a.e.\, \vartheta \in E, \\ \mathbb{T}\setminus\ell \text{ for a.e.}\, \vartheta \in [-\pi,\pi]\setminus E. \end{cases}$$

Consider the set

$$F_r := \{\theta \in [-\pi, \pi] : \operatorname{Im} V_r(e^{i\theta}) \le 0\}.$$

Since $V_r(0) = 0$ and V_r is inner, it defines a measure-preserving transformation of $\mathbb{T}$ onto itself due to Lemma (6.3.8). Therefore,

$$|F_r| = \left|\{\vartheta \in [-\pi, \pi] : \operatorname{Im} e^{i\vartheta} \le 0\}\right| = \pi.$$

For $\eta \in [-\pi, \pi]$ and $r_0 \in [0, 1)$, let

$$w_r(r_0e^{i\eta}):=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{I}^*_{[-\pi,0]}(e^{i\zeta})\big(P_{r_0}+iQ_{r_0}\big)(\eta-\zeta)d\zeta.$$

By Theorem (6.3.10), the function $w_r\in H^2(\mathbb{D})$ has nontangential boundary values $w_r(e^{i\eta})$ as $z\to e^{i\eta}$ for a.e. $\eta\in[-\pi,\pi]$ and

$$\operatorname{Re} w_r(e^{i\eta})=\mathbb{I}^*_{[-\pi,0]}(e^{i\eta})\text{ for a.e.}\,\eta\in[-\pi,\pi],$$
$$\operatorname{Im} w_r(e^{i\eta})=\big(C_r\mathbb{I}^*_{[-\pi,0]}\big)(e^{i\eta})\text{ for a.e.}\,\eta\in[-\pi,\pi].$$

It is clear that for $\eta\in(\pi-|E|,\pi)$,

$$\big(C_r\mathbb{I}^*_{[-\pi,0]}\big)(e^{i\eta})=\frac{1}{2\pi}\int_{-\pi}^{0}\cot\frac{\eta-\zeta}{2}\,d\zeta=\frac{1}{\pi}\log\sin\frac{\eta}{2}-\frac{1}{\pi}\log\sin\frac{\eta+\pi}{2}.$$

Since $|E|\in(0,\pi/2]$, we have for all $\eta\in(\pi-|E|,\pi)$,

$$\log\sin\frac{\eta}{2}>\log\sin\frac{\pi}{4}=-\log\sqrt{2}\geq\log\sin\frac{|E|}{2}>\log\sin\frac{\eta+\pi}{2}.$$

It follows from (45)–(47) that for a.e. $\eta\in(\pi-|E|,\pi)$,

$$\left|\operatorname{Im} w_r(e^{i\eta})\right|>\frac{1}{\pi}\left(-\log\sqrt{2}-\log\sin\frac{|E|}{2}\right)=\frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|.$$

Consider now the function $W_r=w_r\circ V_r$, which belongs to $H^2(\mathbb{D})$ (see, e.g., [277, Section 2.6]). In view of (43) and (44), we have

$$\operatorname{Re}W_r(e^{i\vartheta})=\begin{cases}1\text{ if }\operatorname{Im}V_r(e^{i\vartheta})\leq 0,\\ 0\text{ if }\operatorname{Im}V_r(e^{i\vartheta})>0\end{cases}=\mathbb{I}^*_{F_r}(e^{i\vartheta})\text{ for a.e. }\vartheta\in[-\pi,\pi].$$

Then, by Theorem (6.3.10),

$$\operatorname{Im}W_r(e^{i\vartheta})=\big(C_r\mathbb{I}^*_{F_r}\big)(e^{i\vartheta})\qquad\text{for a.e.}\,\vartheta\in[-\pi,\pi].$$

If $\vartheta\in E$, then it follows from (42) that $V_r(e^{i\vartheta})\in\ell$. In this case inequality (48) implies that for a.e. $\vartheta\in E$,

$$\left|\operatorname{Im}W_r(e^{i\vartheta})\right|=\left|\operatorname{Im}w_rV_r(e^{i\vartheta})\right|>\frac{1}{\pi}|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right).$$

Combining equality (49) and inequality (50), we arrive at (41).

Corollary (6.3.32)[307]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Hilbert transform C_r is of weak type (L^∞,X), then for every measurable set $E\subset[-\pi,\pi]$ with $0<|E|\leq\pi/2$, one has

$$\|\mathbb{I}^*_E\|_X\leq\frac{\pi\|C_r\|_{W_r(L^\infty,X)}}{\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|}.$$

Proof. Let

$$\lambda=\frac{1}{\pi}\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|.$$

By Lemma (6.3.12), there exists a measurable set $F_r\subset[-\pi,\pi]$ with $|F_r|=\pi$ such that for a.e. $\tau\in\mathbb{T}$,

$$\mathbb{I}^*_E(\tau)\leq\mathbb{I}_{\{\zeta\in\mathbb{T}:|(C_r\mathbb{I}^*_{F_r})(\zeta)|>\lambda\}}(\tau).$$

Therefore, by the lattice property, taking into account that $C_r\in W_r(L^\infty,X)$, we obtain

$$\|\mathbb{I}_E^*\|_X \le \left\|\mathbb{I}_{\{\zeta\in\mathbb{T}:|(C_r\mathbb{I}_{F_r}^*)(\zeta)|>\lambda\}}\right\|_X \le \frac{1}{\lambda}\|C_r\|_{W_r(L^\infty,X)}\left\|\mathbb{I}_{F_r}^*\right\|_{L^\infty} = \frac{\pi\|C_r\|_{W_r(L^\infty,X)}}{\left|\log\left(\sqrt{2}\sin\frac{|E|}{2}\right)\right|},$$

which completes the proof.

Corollary (6.3.33)[307]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Hilbert transform C_r is of weak type (L^∞, X), then $X_{a^r} = X_{b^r}$.

Proof. Let $\Gamma \subset \mathbb{T}$ be a measurable set. Consider a sequence of measurable subsets $\{\gamma_n\}_{n\in\mathbb{N}}$ of $\mathbb{T}$ such that $\mathbb{I}_{\gamma_n} \to 0$ a.e. on $\mathbb{T}$. By the dominated convergence theorem,

$$m(\gamma_n) = \int_{\mathbb{T}} \mathbb{I}_{\gamma_n}(\tau)dm(\tau) \to 0 \; as \; n \to \infty.$$

Without loss of generality, one can assume that $0 < m(\gamma_n) \le 1/4$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, there exists a measurable set $E_n \subset [-\pi,\pi]$ such that $\mathbb{I}_{\gamma_n}(\tau) = \mathbb{I}_{E_n}^*(\tau)$ for all $\tau \in \mathbb{T}$. It is clear that $|E_n| = 2\pi m(\gamma_n) \le \pi/2$ for $n \in \mathbb{N}$. By Lemma (6.3.13), for every $n \in \mathbb{N}$,

$$\left\|\mathbb{I}_\Gamma \mathbb{I}_{\gamma_n}\right\|_X \le \left\|\mathbb{I}_{\gamma_n}\right\|_X = \left\|\mathbb{I}_{E_n}^*\right\|_X \le \frac{\pi\|C_r\|_{W_r(L^\infty,X+\varepsilon)}}{\left|\log\left(\sqrt{2}\sin\frac{|E_n|}{2}\right)\right|} = \frac{\pi\|C_r\|_{W_r(L^\infty,X+\varepsilon)}}{\left|\log\left(\sqrt{2}\sin\left(\pi m(\gamma_n)\right)\right)\right|}.$$

Since $m(\gamma_n) \to 0$ as $n \to \infty$, the above estimate implies that $\left\|\mathbb{I}_\Gamma \mathbb{I}_{\gamma_n}\right\|_X \to 0$ as $n \to \infty$. Thus the function $\mathbb{I}_\Gamma$ has absolutely continuous norm. By [266, Chap.1, Theorem 3.13], $X_{a^r} = X_{b^r}$.

Corollary (6.3.34)[307]: Let X be a Banach function space over the unit circle $\mathbb{T}$ and X' be its associate space. If $C_r \in B(X_{b^r}, X)$, then $C_r \in B((X')_{b^r}, X')$ and

$$\|C_r\|_{B((X')_{b^r},X)} \le \|C_r\|_{B(X_{b^r},X)}.$$

Proof. It is well known that the operator iC_r is a self-adjoint operator on the space L^2 (see, e.g., [298, Section 5.7.3(a)]). Therefore, for all $s, v \in S_0 \subset L^2$, one has

$$\langle C_r v, s\rangle = -\langle v, C_r s\rangle$$

It follows from equalities (34), (53), and Hölder's inequality (see [266, Chap.1, Theorem 2.4]) that for every $v \in S_0$,

$$\|C_r v\|_{X'} = \sup\{|\langle C_r v, s\rangle| : s \in S_0, \|s\|_X \le 1\} = \sup\{|\langle v, C_r s\rangle| : s \in S_0, \|s\|_X \le 1\}$$
$$\le \sup\{\|v\|_{X'}\|C_r s\|_X : s \in S_0, \|s\|_X \le 1\} \le \|C_r\|_{B(X_{b^r},X)}\|v\|_{X'}.$$

Since S_0 is dense in $(X')_{b^r}$, we conclude that $C_r \in B((X')_{b^r}, X')$ and (52) holds.

Corollary (6.3.35)[307]: Let X be a Banach function space over the unit circle $\mathbb{T}$ and X' be its associate space. If the Riesz projection P is bounded on X, then $C_r \in W_r(L^\infty, X)$ and $C_r \in W_r(L^\infty, X')$.

Proof. Since X is continuously embedded into L^1, the functional $f_r \mapsto \widehat{f_r}(0)$ is continuous on the space X. Then it follows from (26) that $P \in B(X)$ if and only if $C_r \in B(X)$. Since L^∞ is continuously embedded into X, one has $B(X) \subset B(L^\infty, X)$. By Lemma (6.3.11), $B(L^\infty, X) \subset W_r(L^\infty, X)$.These observations imply that$C_r \in W_r(L^\infty, X)$ if $P \in B(X)$. Since X_{b^r} is a Banach space isometrically embedded into X (see [266, Chap.1, Theorem 3.1]), we see that $C_r \in B(X) \subset B(X_{b^r}, X)$ if $P \in B(X)$. Then, by Lemma (6.3.15), $C_r \in B((X')_{b^r}, X')$. Taking into account that L^∞ is continuously embedded into $(X')_{b^r}$ (see, e.g., [266, Chap.1, Proposition 3.10]), we get $C_r \in$

$B((X')_{b^r}, X') \subset B(L^\infty, X')$, which implies that $C_r \in W_r(L^\infty, X')$ in view of Lemma (6.3.11).

Corollary (6.3.36)[307]: Let X be a Banach function space over the unit circle $\mathbb{T}$. If the Riesz projection P is bounded on X, then $X_{a^r} = X_{b^r}$ and $(X')_{a^r} = (X')_{b^r}$.

Proof. If the Riesz projection P is bounded on a Banach function space X, then the Hilbert transform C_r is of weak types (L^∞, X) and (L^∞, X') in view of Lemma (6.3.16). In turn, $C_r \in W_r(L^\infty, X)$ implies that $X_{a^r} = X_{b^r}$ and $C_r \in W_r(L^\infty, X')$ implies that $(X')_{a^r} = (X')_{b^r}$ due to Theorem (6.3.14).

Corollary (6.3.37)[307]: Let $X, X + \varepsilon$ be Banach functions spaces over the unit circle $\mathbb{T}$. Suppose X is separable and $A_r \in B(X, X + \varepsilon)$. If there exists a sequence $\{a_n^r\}_{n\in\mathbb{Z}}$ of complex numbers such that

$$\langle A_r \chi_j, \chi_k \rangle = a_{k-j}^r \text{ for all } j, k \in \mathbb{Z},$$

then there exists a function $a^r \in M(X, X + \varepsilon)$ such that $A_r = M_{a^r}$ and

$$\widehat{a^r}(n) = a_n^r \text{ for all } n \in \mathbb{Z}.$$

Proof. This statement was proved in [293, Lemma 4.1] under the additional hypothesis that X and $X + \varepsilon$ are rearrangement-invariant Banach function spaces. Put $a^r := A_r \chi_0 \in X + \varepsilon$. Then, one can show exactly as in [293] that $(a^r f_r)\hat{}(j) = (A_r f_r)\hat{}(j)$ for all $j \in \mathbb{Z}$ and $f_r \in P$. Therefore, $A_r f_r = a^r f_r$ for all $f_r \in P$ in view of the uniqueness theorem for Fourier series (see, e.g., [289, Chap.1, Theorem 2.7]).

Now let $f_r \in X$. Since the space X is separable, the set P is dense in X by Corollary (6.3.5). Then there exists a sequence $p_n \in P$ such that $p_n \to f_r$ in X and, whence, $A_r p_n \to A_r f_r$ in X as $n \to \infty$. By [266, Chap.1, Theorem 1.4], $p_n \to f_r$ and $A_r p_n \to a^r f_r$ in measure as $n \to \infty$. Then $a^r p_n \to a^r f_r$ in measure as $n \to \infty$ (see, e.g., [269, Corollary 2.2.6]). Hence, the sequence $A_r p_n = a^r p_n$ converges in measure to the functions $A_r f_r$ and $a^r f_r$ as $n \to \infty$. This implies that $A_r f_r$ and $a^r f_r$ coincide a.e. on $\mathbb{T}$(see, e.g., the discussion preceding [296, Theorem 2.2.3]). Thus $A_r f_r = a^r f_r$ for all $f_r \in X$. This means that $A_r = M_{a^r}$ and $a^r \in M(X, X + \varepsilon)$ by the definition of $M(X, X + \varepsilon)$.

Corollary (6.3.38)[307]: Let $0 < \varepsilon < \infty$, and $w_r: \mathbb{T} \to [0, \infty]$ be a weight such that $w_r \in L^{1+\varepsilon,1+2\varepsilon}, 1/w_r \in L^{1+\varepsilon,1+2\varepsilon}$.

(a) The space $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ is a Banach function space with respect to the norm $\|f_r\|_{L^{1+\varepsilon,1+2\varepsilon}(w_r)} = \|f_r w_r\|_{L^{1+\varepsilon,1+2\varepsilon}}$ and $L^{1+\varepsilon,1+2\varepsilon}(1/w_r)$ is its associate space.

(b) If $0 < \varepsilon < \infty$, then the space $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ is reflexive.

(c) The space $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ is separable and non-reflexive.

Proof. (a) In view of [266, Chap.4, Theorem4.7], the associate space of the Lorentz space $L^{1+\varepsilon,1+2\varepsilon}$, up to equivalence of norms, is the Lorentz space $L^{1+\varepsilon,1+2\varepsilon}$. It is easy to check that $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ is a Banach function space and $L^{1+\varepsilon,1+2\varepsilon}(1/w_r)$ is its associate space.

(b) Note that $L^{1+\varepsilon,1+2\varepsilon}(w_r) \ni f_r \mapsto w_r f_r \in L^{1+\varepsilon,1+2\varepsilon}$ is an isometric isomorphism of $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ and $L^{1+\varepsilon,1+2\varepsilon}$. Hence these spaces have the same Banach space theory properties, e.g., reflexivity and separability. If $0 < \varepsilon < \infty$, then $L^{1+\varepsilon,1+2\varepsilon}$ is reflexive in view of [266, Chap.4, Corollary 4.8]. Then the weighted Lorentz space $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ is reflexive too.

(c) If $0 < \varepsilon < \infty$, then $L^{1+\varepsilon,1}$ has absolutely continuous norm and $(L^{1+\varepsilon,1})^* = L^{1+\varepsilon,\infty}$(see [266, Chap.4, Corollary 4.8]). Then $L^{1+\varepsilon,1}$ is separable in view of [266, Chap.1, Corollary 5.6]. It is known that

$$L^{1+\varepsilon,1}(L^{1+\varepsilon,\infty})^* = (L^{1+\varepsilon,1})^{**}$$

(see [276, p.83]). Hence $L^{1+\varepsilon,1}$ is non-reflexive. Therefore, $L^{1+\varepsilon,1}(w_r)$ is also separable and nonrefexive.

Corollary (6.3.39)[307]: Let $0 < \varepsilon < \infty$. If $w_r \in (A_r)_{1+\varepsilon}(\mathbb{T})$, then $w_r \in L^{1+\varepsilon,1+2\varepsilon}$ and $1/w_r \in L^{1+\varepsilon',1+2\varepsilon'}$.

Proof. By the stability property of Muckenhoupt weights (see, e.g., [270, Theorem 2.31]), there exists $\varepsilon > 0$ such that $w_r \in (A_r)_s(\mathbb{T})$ for all $s \in (p-\varepsilon, p+\varepsilon)$. Therefore, $w_r \in L^s$ and $1/w_r \in L^{s'}$ for all $s \in (p-\varepsilon, p+\varepsilon)$. In particular, if s_1, s_2 are such that $p-\varepsilon < s_1 < p < s_2 < p+\varepsilon$, then $w_r \in L^{s_2} = L^{s_2,s_2} \subset L^{p,q}$ and $1/w_r \in L^{s_1'} = L^{s_1',s_1'} \subset L^{p',q'}$ in view of the embeddings of Lorentz spaces (see, e.g., [266, Chap.4, remark after Proposition 4.2]).

Lemmas (6.3.23)(a) and (6.3.24) imply that if $w_r \in (A_r)_{1+\varepsilon}(\mathbb{T})$, then $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ is a Banach function space.

Corollary (6.3.40)[307]: Let $1 < \varepsilon \leq \infty$. If $w_r \in (A_r)_{1+\varepsilon}(\mathbb{T})$, then the Riesz projection P is bounded on the weighted Lorentz space $L^{1+\varepsilon,1+2\varepsilon}(w_r)$.

Proof. It follows from [266, Chap.4, Theorem 4.6] and [281, Theorem 4.5] that the Cauchy singular integral operator S is bounded on $L^{1+\varepsilon,1+2\varepsilon}(w_r)$. Thus, the Riesz projection P is bounded on $L^{1+\varepsilon,1+2\varepsilon}(w_r)$ in view of (24).

List of Symbols

Symbol		Page
L^∞	: Essential Lebesgue space	1
H^2	: Hardy Space	1
mod	: module	1
$\otimes$	: Tensor product	2
$\oplus$	: Direct Sum	2
L^2	: Hilbert Space	2
inf	: infimum	3
Ker	: Kernel	4
H^p	: essential Hardy space	4
diag	: diagonal	6
L^p	: Lebesgue Integral	9
A^2	: Bergman Space	9
L^1	: Lebesgue integral on the Real line	11
Aut	: Automorphism	11
sup	: Supremum	12
H^p	: Hardy space	12
max	: Maximum	16
$\ominus$	: Direct difference	43
L_a^2	: Bergman Space	47
dim	: dimension	48
tr	: trace	56
a. e	: almost everywhere	59
G. C. D	: greatest common divisor	70
UFD	: Unique Factorization Domain	70
L^q	: Dual of Lebesgue Space	72
H^q	: Dual of Hardy Space	72
min	: minimum	80
ess	: essential	85
BMO	: Bounded Meau Oscillation	85
VMO	: Vanishing Meau Oscillation	56
Im	: Imaginary	95
SAT	: Strongly Asymptotically Toepliz	102
UAT	: Uniformly asymptotically Toepliz	102
WAT	: Weakly asymptotically Toepliz	102
MSAT	: Strongly Asymptotically Toepliz	102
Lip	: Lopschtiz	105
SP	: Spectrum	105
ec	: essential commutant	107
ran	: range	111
SOT	: strong operator topology	114

supp	: support	120
alg	: algebra	121
SC	: Semi Commutator	125
ℓ^2	: Hilbert Space of Sequences	126
clos	: closure	138
ind	: index	145
B. F. S	: Banach Function Space	174
$L^{p,q}$	: Lorentz spaces	192
Re	: Real	195

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