Sudan University of Science and Technology
College of Graduate Studies

كلية الدراسات العليا

Spectrum of Small Sets and Dubovitskiĭ–Federer Theorems with Unconditionally Saturated Banach Spaces

طيف الفئات الصغيرة و مبرهنات دوبوفيتسكي – فيدرر مع فضاءات باناخ المشبعة دون شرط

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

Acknowledgements

I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

The Mordell exponential sum estimates and sets of large trigonometric sums are presented. The exposition of Bourgain 2-source extractor and subspace of the Bourgain-Dellaen space are given. We study the generalized N-property and Morse-Sard theorem for the sharp case of Sobolev mappings and the trace theorem with Luzin N and Morse-Sard properties for the sharp case of Sobolev-Lorentz mappings. We also study Dubovitskiiĭ-Sard and Dubovitskiiĭ–Federer theorems in Sobolev spaces and the coarea formula. The operators in tight by support Banach spaces and an additive combinatorics approach relating rank to communication complexity with the structure of the spectrum of small sets and the uniform structure of the separable essential Lebesgue spaces are introduced. The Hereditarily indecomposable essential Lebesgue spaces and unconditionally saturated Banach space and that solves the scalar–plus–compact problem and property are discissed.

الخلاصة

قمنا بتقديم تقديرات جمع أسية موردیل وفئات لمجاميع حساب المثلثات الكبيرة. تم أعطاء عرض مستخرج مصدر – 2 لبورجان و الفضاء الجزئي لفضاء بورجان – ديلين. قمنا بدراسة خاصية – N المعممة و مبرهنة مورس – سارد لأجل الحالة القاطعة لرواسم سوبوليف ومبرهنة الأثر مع خصائص لوزين N ومورس – سارد للحالة القاطعة لرواسم سوبوليف – لورنتز. أيضاً درسنا مبرهنات دوبوفيتسكي – سارد و دوبوفيتسكي – فيدرر في فضاءات سوبوليف وصيغة المساحة المصاحبة. تم ادخال المؤثرات في الضيق بواسطة فضاءات باناخ الدعامة ومقارنة التوافقية الجمعية المتعلقة الرتبة الى تعقدية الاتصالات مع تشييد الطيف لفئات صغيرة والتشيد المنتظم لفضاءات لبيق الأساسية المنفصلة. تمت مناقشة فضاءات لبيق الاساسية غير القابلة للتحلل وراثياً وفضاءات باناخ المشبعة دون قيد أو شرط والتي تحل مسألة التراص – زائد – القياسية والخاصية.

Introduction

A construction of Bourgain [19] gave the first 2-source extractor to break the min-entropy rate 1/2 barrier. We write an exposition of his result, giving a high level way to view his extractor construction.

We some recent extensions of the Lusin N-property and the Sard theorem for Sobolev maps, which have been obtained in a joint work with M. Csörnyei, E. D'Aniello, and B. Kirchheim. We establish Luzin N- and Morse–Sard properties for mappings $v : \mathbb{R}^n \to \mathbb{R}^m$ of the Sobolev–Lorentz class $W^k_{p,1}$ with $k = n - m + 1$ and $p = \frac{n}{k}$ (this is the sharp case that guaranties the continuity of mappings).

It is shown that every infinite-dimensional closed subspace of the Bourgain-Delbaen space $X_{a,b}$ has a subspace isomorphic to some ℓ^p.

We show the existence of non-trivial solutions of the equation $r_1 + r_2 = r_3 + r_4$, where r_1, r_2, r_3 and r_4 belong to the set R of large Fourier coefficients of a certain subset A of $\mathbb{Z}/N\mathbb{Z}$. For a $\{0, 1\}$-valued matrix M let CC(M) denote the deterministic communication complexity of the boolean function associated with M. It is well-known since the work of Mehlhorn and Schmidt [STOC 1982] that CC(M) is bounded from above by rank(M) and from below by log rank(M) where rank(M) denotes the rank of M over the field of real numbers. Determining where in this range lies the true worst-case value of CC(M) is a fundamental open problem in communication complexity. The state of the art is $\log^{1.631} \mathrm{rank(M)} \leq \mathrm{CC(M)} \leq 0.415\, \mathrm{rank(M)}$, the lower bound is by Kushilevitz [unpublished, 1995] and the upper bound is due to Kotlov [Journal of Graph Theory, 1996]. Lovasz and Saks [FOCS 1988] conjecture ´ that CC(M) is closer to the lower bound, i.e., CC(M) ≤ $\log^{c}$(rank(M)) for some absolute constant c — this is the famous "log-rank conjecture" — but so far there has been no evidence to support it, even giving a slightly nontrivial (o(rank(M))) upper bound on the communication complexity. Our main result is that, assuming the Polynomial Freiman-Ruzsa (PFR) conjecture in additive combinatorics, there exists a universal constant c such that CC(M) ≤ c · rank(M)/ log rank(M). Although our bound is stated using the rank of M over the reals, our proof goes by studying the problem over the finite field of size 2, and there we bring to bear a number of new tools from additive combinatorics which we hope will facilitate further progress on this perplexing question. For G be a finite Abelian group and A a subset of G. The spectrum of A is the set of its large Fourier coefficients. Known combinatorial results on the structure of spectrum, such as Chang's theorem, become trivial in the regime $|A| = |G|^{\alpha}$ whenever α ≤ c, where c ≥ 1/2 is some absolute constant. On the other hand, there are statistical results, which apply only to a noticeable fraction of the elements, which give nontrivial bounds even to much smaller sets.

We show Luzin N- and Morse–Sard properties for mappings $v : \mathbb{R}^n \to \mathbb{R}^d$ of the Sobolev–Lorentz class $W^k_{p,1}, p = \frac{n}{k}$ (this is the sharp case that guarantees the continuity of mappings). The Sard theorem from 1942 requires that a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class $C^k, k > max(n - m, 0)$. In 1957 Duvovitskiĭ generalized Sard's theorem to the case of C^k mappings for all k. Namely he proved that, for almost all $y \in \mathbb{R}^m$, $H^\ell(Cf \cap f - 1\,(y)) = 0$ where $\ell = max(n - m - k + 1, 0), H^\ell$ denotes the Hausdorff measure, and Cf is the set of critical points of f. In 2001 De Pascale proved that the Sard theorem holds true for Sobolev mappings of the class $W^{k,p}_{\mathrm{loc}}\,(\mathbb{R}^n, \mathbb{R}^m), k > max(n - m, 0)$ and $p > n$. The Morse–Sard theorem requires that a mapping $v : \mathbb{R}^n \to \mathbb{R}^m$ is of class $C^k, k > max(n - m, 0)$. In 1957 Dubovitskiĭ generalized this result by proving that almost all level sets for a Ck mapping have Hs-negligible intersection with its critical set, where $s = max(n - m - k + 1, 0)$. Here the critical set, or m-critical set is defined as $Z_{v,m} = \{x \in \mathbb{R}^n : rank\,\nabla v(x) < m\}$. Another generalization was obtained independently by Dubovitskiĭ and Federer in 1966, namely for C^k mappings $v : \mathbb{R}^n \to \mathbb{R}^d$ and integers m ≤ d they proved that the set of m-critical values $v(Z_{v,m})$ is $H^{q_\circ}$ -negligible for $q_\circ = m - 1 + \frac{n-m+1}{k}$. They also established the sharpness of these results within the C^k category.

Answering the question of W. T. Gowers, we give an example of a bounded operator on a subspace of Gowers unconditional space, which is not a strictly singular perturbation of a restriction of a diagonal operator. We give an example of two non-isomorphic separable $\mathcal{L}_\infty$-spaces which are uniformly homeomorphic. This answers a question of Johnson, Lindenstrauss and Schechtman [89]. We construct a Bourgain–Delbaen $\mathcal{L}_\infty$-space X_{Kus} with structure that is strongly heterogeneous.

The Contents

Chapter 1
Mordell Exponential Sum and an Exposition

We include a proof of a generalization of Vazirani's XOR lemma that seems interesting in its own right, and an argument (due to Boaz Barak) that shows that any two source extractor with sufficiently small error must be strong.

Section (1.1): Exponential Sum Estimate Revisited

Theorem (1.1.1)[1]: Let p be prime. Given $r \in \mathbb{Z}_+$ and $\varepsilon > 0$, there is $\delta = \delta(r, \varepsilon) > 0$ satisfying the following property: If

$$f(x) = \sum_{i=1}^{r} a_i x^{k_i} \in \mathbb{Z}[x] \quad and \quad (a_i, p) = 1$$

where the exponents $1 \le k_i < p - 1$ satisfy

$$(k_i, p-1) < p^{1-\varepsilon} \quad for\ all \quad 1 \le i < r, \tag{1}$$

$$\left(k_i - k_j, p-1\right) < p^{1-\varepsilon} \quad for\ all \quad 1 < i \ne j \le r, \tag{2}$$

then there is an exponential sum estimate

$$\left|\sum_{x=1}^{p-1} e_p\big(f(x)\big)\right| < p^{1-\delta} \tag{3}$$

(denoting $e_p(y) = e^{\frac{2niy}{p}}$).

Proof. Let $1 \le k_i < p - 1 (1 \le i \le r)$ satisfy (1) and (2).
We prove that

$$\max_{(a_1,\dots,a_r,p)=1} \left|\sum_{x=1}^{p-1} e_p(a_1 x^{k_1} + \cdots + a_r x^{k_r})\right| < p^{1-\delta_r} \tag{4}$$

for some $\delta_r > 0$, by induction on r.
The case $r = 1$ appears in [5] and $r = 2$ was treated. Thus assume $r > 3$.
Let

$$H = \left\{(x^{k_1}, \dots, x^{k_r}) \middle| x \in \mathbb{F}_p^*\right\} \lhd \left(\mathbb{F}_p^*\right)^r \tag{5}$$

With

$$H = \frac{p-1}{d}, \qquad d = (k_1, \dots, k_r, p-1).$$

where δ_y is Dirac at $y \in \mathbb{F}_p^r$.
To establish (4), we may assume all $a_i \in \mathbb{F}_p^* (1 \le i \le r)$, since otherwise the problem reduces to $r - 1$ terms. Assume

$$\frac{1}{p}\left|\sum_{1}^{p-1} e_p(a_1 x^{k_1} + \cdots + a_r x^{k_r})\right| = |\hat{\mu}(a)| > p^{-\delta}. \tag{6}$$

The same argument leading to (81) (now applied on $\mathbb{F}_p^r$) implies

$$\left(\mu^{(\ell)} * \mu_-^{(\ell)}\right)(0) > p^{-\frac{r}{2} - 2\delta\ell^2}. \tag{7}$$

On the other hand, letting $\frac{r}{2} < r_1 < r$ $(r \ge 3)$, proceeding as in the binomial case, we estimate

$$\left(\mu^{(\ell)} * \mu_-^{(\ell)}\right)(0)$$
$$= (p-1)^{-2\ell}\left|\left\{(x_1,\ldots,x_{2\ell}) \in \left(\mathbb{F}_p^*\right)^{2\ell}\middle| x_1^{k_i} - x_1^{k_i} + + \cdots - x_{2\ell}^{k_i} = 0\ (1 \le i \le r)\right\}\right|$$
$$\le (p-1)^{-2\ell}\left|\left\{(x_1,\ldots,x_{2\ell}) \in \left(\mathbb{F}_p^*\right)^{2\ell}\middle| x_1^{k_i} - x_2^{k_i} + \cdots - x_{2\ell}^{k_i} = 0\ (1 \le i \le r_1)\right\}\right| \qquad (8).$$

To bound (8), express the quantity by exponential sums that may be estimated nontrivially from the induction hypothesis, since $r_1 < r$. Thus clearly

$$= (p-1)^{-2\ell} p^{-r_1} \sum_{\xi_1,\ldots,\xi_{r_1} \in \mathbb{F}_p} \left|\sum_{x=1}^{p-1} e_p\left(\xi_1 x^{k_1} + \cdots + \xi_{r_1} x^{k_{r_1}}\right)\right|^{2\ell}$$
$$< p^{r_1} + (p-1)^{-2\ell} p^{2\ell(1-\delta_{r_1})} < p^{-r_1} + 2p^{-2\ell\delta_{r_1}}. \qquad (9)$$

Taking

$$\ell = \left[\frac{r_1}{\delta_{r_1}}\right], \qquad (10)$$

(7), (9) imply

$$p^{-\frac{r}{2}-2\delta\ell^2} < 2p^{-r_1};$$

hence, from the choice of r_1

$$\delta = \frac{1}{4\ell^2} > \frac{\delta_{r_2}^2}{4r_1^2}.$$

Taking $r_1 = \left[\frac{r}{2}\right] + 1$, we proved that

$$\delta_r > \frac{\delta_{\left[\frac{r}{2}\right]+1}^2}{4r^2} \qquad (11)$$

implying Theorem (1.1.1) with

$$\delta_r > \left(\frac{\delta_2}{4r}\right)^{4r} \qquad (12)$$

where $\delta_2 = \delta_2(\varepsilon)$.

Remarks (1.1.2)[1]: (i) The result for $r = 1$ (Gauss sums) was obtained in [5]. Thus

$$\left|\sum_{x=1}^{p-1} e_p(ax^k)\right| \quad if\ a \in \mathbb{F}_p^* \quad and \quad (k, p-1) < p^{1-\varepsilon}. \qquad (13)$$

More precisely, it was shown in [5] that if $G \lhd \mathbb{F}_p^*$ and $|G| > p^{1-\varepsilon}$, then

$$\left|\sum_{x\in G} e_p(ax)\right| > |G|^{1-\delta} \qquad for\ a \in \mathbb{F}_p^*. \qquad (14)$$

See also [2] for further extensions to exponential sums of the form

$$\sum_{s=1}^{t_1} e_p(a\theta^s) \qquad (15)$$

And

$$\sum_{s,s'=1}^{t_1} e_p\left(a\theta^s + b\theta^{ss'}\right) \qquad (16)$$

where $a, \theta \in \mathbb{F}^*$ and θ is of multiplicative order $t, t \ge t_1 > p^\delta$.

The methods involved here are closely related to those used in [5] and [2] (while the results in [12] and [8] depend on Stepanov's method).
(ii) Theorem (1.1.1) improves upon the results from [7] and [8] when the exponents $\{k_i\}$ are large. Notice that the recent [7] already contains a substantial improvement over Mordell's original [13].
(iii) The role of condition (ii) above is made clear by the following example from [7] (see Example 1.1 in [7]). Let r be even and let

$$f(x)=\sum_{i=1}^{\frac{r}{2}}\left(x^{\frac{p-1}{2}+i}-x^{i}\right). \tag{17}$$

Then

$$\left|\sum_{x=1}^{p-1} e_p\big(f(x)\big)-\frac{p-1}{2}\right| \leq r\sqrt{p}. \tag{18}$$

(iv) As mentioned above, our argument follows the same pattern as in [5] and [2]. The key combinatorial ingredient in [5] is a 'sum-product' theorem for subsets A of the field $\mathbb{F}_p$ (see also [6]).

Proposition (1.1.3)[1]: Given $\varepsilon>0$, there is $\delta>0$ such that if $A\subset\mathbb{F}_p$ and

$$1<|A|<p^{1-\varepsilon}, \tag{19}$$

then

$$|A+A|+|A.A|>C|A|^{1+\delta}. \tag{20}$$

We denote here $A+A=\{x+y|x,y\in A\}$ and $A.A=\{x.y|x,y\in A\}$ for the sum and product sets (and will use the same notation if, more generally, A is a subset of a commutative ring $\mathcal{R}$).

Given $G\lhd\mathbb{F}_p^*$, consider the probability measure v on $\mathbb{F}_p$ defined by

$$v=\frac{1}{|G|}\sum_{s\in G}\delta_x. \tag{21}$$

As shown in [5], one may then derive from Proposition (1.1.3) uniform bounds on the convolution powers

$$v^{(k)}=\underbrace{v*\ldots*v}_{k-\text{fold}}$$

denoting

$$(i*\mu)(x)=\sum_{y\in\mathbb{F}_p} v(x-y)\mu(y)$$

and those bounds translate in exponential sum estimates such as (14).

It turns out that in order to establish Theorem (1.1.1) for general r, it suffices to treat the monomial ($r=1$) and the binomial case ($r=2$). Thus we are left with the problem for $r=2$. Following the scheme used for $r=1$, we need to establish a sum-product theorem for subsets A of the product $\mathbb{F}_p\times\mathbb{F}_p$. Clearly if A is a subset of the form

$$A=\{a\}\times\mathbb{F}_p, A=\mathbb{F}_p\times\{a\}\ or\ A=\big\{(x,ax)\big|x\in\mathbb{F}_p\big\},$$

one has $|A|=|A+A|=|A.A|=p$. It turns out that these are essentially the only 'exceptions' to be taken into account when reformulating Proposition (1.1.3) for $\mathbb{F}_p\times\mathbb{F}_p$.

Proposition (1.1.4)[1]: Let $A\subset\mathbb{F}_p\times\mathbb{F}_p$ satisfying for some $\varepsilon_0>0$

$$|A|>p^{\varepsilon_0}. \tag{22}$$

Assume that

$$|A + A| + |A.A| < p^{\varepsilon}|A|. \tag{23}$$

Proof. Decomposing A as $\left(A \cap \left(\mathbb{F}_p^* \times \mathbb{F}_p^*\right)\right) \cup \left(A \cap \left(\{0\} \times \mathbb{F}_p\right)\right) \cup \left(A \cap \left(\mathbb{F}_p \times \{0\}\right)\right)$, we may, in view of alternative (27), assume $\left|A \cap \left(\mathbb{F}_p^* \times \mathbb{F}_p^*\right)\right| > \frac{1}{2}|A|$ and hence $A \subset \mathbb{F}_p^* \times \mathbb{F}_p^*$. Fix $\varepsilon' > 0$ small and $k \in \mathbb{Z}_+$ (to be specified). Take ε in (49) small enough to obtain from Lemma (1.1.10) a subset $A_1 \subset A$ satisfying

$$|A_1| > p^{-\varepsilon'}|A|, \tag{24}$$

$$\left|kA_1^k\right| < p^{-\varepsilon'}|A_1|. \tag{25}$$

Next, apply Lemma (1.1.9) to the set A_1 with $\varepsilon = \varepsilon'$.

If (42) fails, say $|\pi_1(A_1)| \le p^{\varepsilon'}$, obviously for some $a \in \mathbb{F}_p$

$$\left|A \cap \left(\{a\} \times \mathbb{F}_p\right)\right| > \left|A_1 \cap \left(\{a\} \times \mathbb{F}_p\right)\right| > p^{-\varepsilon'}|A_1| > p^{-2\varepsilon'}|A|$$

and hence (51) holds.

Otherwise, either (43) or (44) holds. If (44) and assuming $k > k(\varepsilon')$ (=the integer in (44)), we get

$$p^2 = \left|kA^k\right| \overset{(25)}{<} p^{\varepsilon'}|A_1|$$

And hence

$$|A_1| > p^{2-\varepsilon^1}$$

and (26) holds.

Assume (43). Since then

$$A_1 \subset \left\{(x, ax)\middle|x \in \mathbb{F}_p\right\} \text{ for some } a \in \mathbb{F}_p^*,$$

$\left|A \cap \left\{(x, ax)\middle|x \in \mathbb{F}_p\right\}\right| \ge |A_1| \overset{(2.34)}{>} p^{-\varepsilon'}|A|$ and (28) holds.

Assuming (27) or (28), the upperbound in (29) is clear and the lower bound follows from Proposition (1.1.3).

This proves Proposition (1.1.4).

Then one of the following cases occurs:

$$|A| > p^{2-\varepsilon'}. \tag{26}$$

$$\text{There is } a \in \mathbb{F}_p \text{ such that either} \tag{27}$$

$$\left|A \cap \left(\{a\} \times \mathbb{F}_p\right)\right| > p^{-\varepsilon'}|A|$$

Or

$$\left|A \cap \left(\mathbb{F}_p \times \{a\}\right)\right| > p^{-\varepsilon'}|A|.$$

$$\text{There is } a \in \mathbb{F}_p^* \text{ such that} \tag{28}$$

$$\left|A \cap \left\{(x, ax)\middle|x \in \mathbb{F}_p\right\}\right| > p^{-\varepsilon'}|A|$$

where $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ for $\varepsilon \to 0$ with ε_0 (22) fixed.

In cases (27), (28)

$$p^{1-\varepsilon'} < |A| < p^{1+\varepsilon'} \tag{29}$$

(v) Theorem (1.1.1) has the following reformulation.

For $\theta \in \mathbb{F}_p^*$, denote by $0(\theta)$ the multiplicative order of θ in $\mathbb{F}_p^*$.

Corollary (1.1.5)[1]: Let $\mathbb{F}_1, \dots, \mathbb{F}_r \in \mathbb{F}_p^*$ satisfy for some $\varepsilon > 0$

$$0(\theta_i) > p^{\varepsilon} \quad \text{for all } i = 1, \dots, r, \tag{30}$$

$$0\left(\theta_i\theta_j^{-1}\right) > p^{\varepsilon} \quad \text{for all } 1 \le i \ne j \le r. \tag{31}$$

Then

$$\max_{a_i\in\mathbb{F}_p^*}\left|\sum_{s=1}^{p-1} e_p\left(\sum_{r=1}^{r} a_i\theta\right)\right| < p^{1-\delta} \tag{32}$$

with $\delta = \delta(\varepsilon)$.

Indeed, let ψ be a generator of $\mathbb{F}_p^*$ and write $\theta_i = \psi^{k_i}$, where thus

$$0(\theta_i) = \frac{p-1}{(p-1, k_i)}, \tag{33}$$

$$0\left(\theta_i\theta_j^{-1}\right) = \frac{p-1}{\left(p-1, k_i - k_j\right)}. \tag{34}$$

Clearly

$$\sum_{s=1}^{p-1} e_p\left(\sum_{i=1}^{r} a_i\psi^{sk_i}\right) = \sum_{s\in\mathbb{F}_p^*} e_p\left(\sum_{i=1}^{r} a_i x^{k_i}\right).$$

Since (30), (31), (33), and (34) ensure conditions (1), (2) on the exponents k_i, (32) is equivalent to (3).

The Corollary (1.1.5) remains valid for incomplete sums (the case $r = 1$ appears in [2]).

Theorem (1.1.6)[1]: Let $\varepsilon > 0$ and $\theta_i, \ldots, \theta_r \in \mathbb{F}_p^*$ satisfy (30), (31). Then for $t > p^\varepsilon$

$$\max_{a_i\in\mathbb{F}_p^*}\left|\sum_{s=1}^{t} e_p\left(\sum_{i=1}^{r} a_i\theta_i^s\right)\right| < p^{-\delta}t \tag{35}$$

Where $\delta = \delta(\varepsilon)$.

We will prove Proposition (1.1.4). We contain the proof of Theorem (1.1.1) for $f(x) = ax^k + bx^\ell$ a binomial. The general case (r arbitrary) is treated. We point out the modifications to obtain Theorem (1.1.6).

We illustrate applications to uniform distribution issues for power generators in cryptography, in the spirit of [10] and [9]. Since the module is assumed to be a product of two distinct primes (a Blum integer), we first show how to extend Theorem (1.1.6) to composite moduli which factor in distinct large primes.

We denote for $k \in \mathbb{Z}_+$

$$kA = A + A + \cdots + A \qquad (k - fold),$$
$$A^k = A.A \ldots A \qquad (k - fold)$$

where sum and product sets are defined as

$$A + B = \{x + y | x \in A, y \in B\},$$
$$A.B = \{x.y | x \in A, y \in B\}.$$

Lemma (1.1.7)[1]: (i) Let $S \subset \mathbb{F}_p, |S| > p^{\frac{3}{4}}$. Then

$$\mathbb{F}_p = 3S.S. \tag{36}$$

(ii) Let $S \in \mathbb{F}_p, |S| > p^\varepsilon$. Then

$$\mathbb{F}_p = k.S^k \quad for \quad k > k(\varepsilon). \tag{37}$$

Proof. (1) We may of course assume $S \in \mathbb{F}_p^*$. Introduce the function

$$f(x) = \frac{1}{|S|}\sum_{y\in S^{-1}} \chi_S(x, y) \tag{38}$$

satisfying $supp\ f \subset S.S$.

If $\xi \in \mathbb{F}_p$, we have

$$\hat{f}(\xi) = \sum_{x\in\mathbb{F}_p} e_p(x\xi)f(x) = \frac{1}{|S|}\sum_{y\in S}\hat{\chi}_s(y\xi)$$

and for $f \in \mathbb{F}_p^*$

$$\left|\hat{f}(\xi)\right| \leq |S|^{-\frac{1}{2}}\left(\sum_{y\in S}|\hat{\chi}_s(y\xi)|^2\right)^{\frac{1}{2}} = |S|^{-\frac{1}{2}}\left(\sum_{\eta\in\mathbb{F}_p}|\hat{\chi}_s(\eta)|^2\right)^{\frac{1}{2}} = p^{\frac{1}{2}} \quad (39)$$

Write $(F * f)(x) = \sum_{y\in\mathbb{F}_p} f(x-y)f(y)$ and

$$(f * f * f)(x) = \frac{1}{p}\sum_{\xi\in\mathbb{F}_p^*}\hat{f}(\xi)^3 e_p(x\xi).$$

Hence for all $x \in \mathbb{F}_p$

$$\left|(f * f * f)(x) - \frac{1}{p}\,|S|^3\right| \overset{(39)}{\leq} \frac{1}{\sqrt{p}}\sum\left|\hat{f}(\xi)\right|^2 = \sqrt{p}\|f\|_2^2 \overset{(2.3)}{\leq} \sqrt{p}|S|. \quad (40)$$

Since $\frac{1}{p}|S|^3 > \sqrt{p}|S|$ from assumption on S,

$$\mathbb{F}_p = supp(f * f * F) \subset 3\, supp\, f \subset 3S.S$$

proving (36).

(ii) From the sum-product theorem in $\mathbb{F}_p$ (Proposition (1.1.3)), there is $k_1 = k_1(\varepsilon)$ such that $|k_1 \cdot S^{k_1}| > p^{\frac{3}{4}}$. Here we just iterate (20) using the fact that $(A+A)\cdot(A+A) \in 4A^2$. Next apply part (i) to get $\mathbb{F}_p = S(k_1S^{k_i}(k_1S^{k_1}) \subset 3k_1^2S^{2k_1}$.

Lemma (1.1.8)[1]: If $S \subset \mathbb{F}_p \times \mathbb{F}_p$ satisfies

$$|S| > p^{1+\varepsilon},$$

Then

$$kS^k = \mathbb{F}_p \times \mathbb{F}_p \quad \text{for} \quad k \in \mathbb{Z}_+, k \geq k(\varepsilon). \quad (41)$$

Proof. Denote by $\pi_i: \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ the coordinate projections. From the assump tion, there are $a_1, a_2 \in \mathbb{F}_p$ so that

$$S_i; = \{x \in S | \pi_i(x) = a_i\}$$

Satisfies

$$|S_i| = |\pi_{3-i}(S_i)| > p^{\varepsilon} \quad (i = 1,2).$$

From Lemma (1.1.7), there is $k_1 = k_1(\varepsilon) \in \mathbb{Z}_+$ s.t.

$$\mathbb{F}_p = k_1\pi_2(S_1)^{k_1} = \pi_2\left(k_1S_1^{k_1}\right)$$

And

$$\mathbb{F}_p = k_1\pi_1(S_2)^{k_1} = \pi_1\left(k_1S_2^{k_1}\right).$$

Writing then

$$2k_1S^{k_1} \supset k_1S_1^{k_1} + k_1S_2^{k_1} = \left(\left\{k_1a_1^{k_1}\right\} \times \mathbb{F}_p\right) + \left(\mathbb{F}_p \times \left\{k_1a_2^{k_1}\right\}\right) = \mathbb{F}_p \times \mathbb{F}_p,$$

(41) follows.

Lemma (1.1.9)[1]: Let $A \subset \mathbb{F}_p^* \times \mathbb{F}_p^*$ satisfy for some $\varepsilon > 0$

$$|\pi_i(A)| > p^{\varepsilon} \quad \text{and} \quad |\pi_2(A)| > p^{\varepsilon}. \quad (42)$$

Then either

$$A \subset \left\{(x, ax)\middle| x \in \mathbb{F}_p\right\} \quad \text{for some } a \in \mathbb{F}_p^* \quad (43)$$

Or

$$kA^k = \mathbb{F}_p \times \mathbb{F}_p \quad \text{for some} \quad k = k(\varepsilon) \in \mathbb{Z}. \quad (44)$$

Proof. Applying Lemma (1.1.7) to $S = \pi_i(A) \subset \mathbb{F}_p$, we have for some $k_0 \in \mathbb{Z}_+$

$$\pi_1(k_0A^{k_0}) = k_0S^{k_0} = \mathbb{F}_p \tag{45}$$

and similarly

$$\pi_2(k_2A^{k_0}) = \mathbb{F}_p. \tag{46}$$

Clearly (45), (46) remain valid for $k > k_0$. In particular

$$|kA^k| \geq p \quad \text{for} \quad k > k_0. \tag{47}$$

Assume $k > k_0$ and $|kA^k| > p$. Then $\pi_1|_{kA^k}$ is not one-to-one and there are $z, w \in kA^k$ such that $z_1 = w_1$ and $z_2 \neq w_2$. Hence

$$2kA^{2k} - \{z - w)(kA^k) = \{(x_1, x_2 - (z_2 - w_2)y_2)| \\ x = (x_1, x_2) \in 2kA^{2k}, y = (y_1, y_2) \in kA^k\} = \mathbb{F}\_p \times \mathbb{F}\_p$$

since $\pi(2kA^{2k}) = \mathbb{F}_p = \pi_2(kA^k)$. Thus

$$\mathbb{F}_p \times \mathbb{F}_p = 2kA^{2k} - (kA^k - kA^k)(kA^k)$$

And

$$\mathbb{F}_p \times \mathbb{F}_p = k_1A^{k_1} - k_1A^{k_1} \quad \text{for} \quad k_1 = 3k^2. \tag{48}$$

Prom the Plannecke-Ruzsa sum set inequalities (see [14]) applied in the additive group $\mathbb{F}_p \times \mathbb{F}_p$ and (48)

$$p^2 = |k_1A^{k_1} - k_1A^{k_1}| \leq \left(\frac{|2k_1A^{k_1}|}{|k_1A^{k_1}|}\right)^3 |k_1A^{k_1}|$$

And hence by (47)

$$|2k_1A^{k_1}| \geq p^{\frac{4}{3}}. \tag{49}$$

We may then apply Lemma (1.1.8) to $S = 2k_1A^{k1} \subset \mathbb{F}_p \times \mathbb{F}_p$ and get $k_2 \in \mathbb{Z}_+$ s.t. $k_2A^{k_2} = \mathbb{F}_p \times \mathbb{F}_p$, hence (44).

Fix $z \in k_0A^{k_0}$ and let $P = k_0A^{k_0} - z$. Thus $0 \in P \subset P + P$ and $|P| \geq p$ by (47). If $|P + P| = |2k_0A^{k_0}| > p$, it follows from the preceding that we are in alternative (44). Assume thus $|P + P| = p = |P|$, so that $P = P + P$ and P is closed under addition. Since $\pi_1(P) = \mathbb{F}_p$ by (45), there is $c \in \mathbb{F}_p$ s.t. $(1, c) \in P$ and

$$P = \{(t, ct)|t \in \mathbb{F}_p\},$$

$$k_0A^{k_0} = \{(z_1 + t, z_2 + ct)|t \in \mathbb{F}_p\} = \{(t, ct + d)|t \in \mathbb{F}_p\} \tag{50}$$

with $d = z_2 - cz_1 \in \mathbb{F}_p$. By (46), $c \neq 0$. Assume $d \neq 0$. Writing

$$(k_0A^{k_0}).(k_0A^{k_0}) = \{(t_1t_2, t_1 + cd(t_1 + t_2) + d^2)|t_1, t_2 \in \mathbb{F}_p\},$$

it follows that

$$|k_0^2A^{2k_0}| \geq |\{(t_1t_2, t_1 + t_2)|t_1, t_2 \in \mathbb{F}_p\}| > \frac{p^2}{2} \tag{51}$$

putting us again in alternative (44).

Assume $d = 0$ in (50), i.e., $k_0A^{k_0} = \{(t, ct)|t \in \mathbb{F}_p\}$. Fix an element $w = (w_1, w_2) \in k_0A^{k_0-1}$ with $w_2 \neq 0$. Then, for all $x = (x_1, x_2) \in A, wx \in k_0A^{k_0}$ and $w_2x_2 = cw_1x_1$, implying that $A \subset \{(t, at)|t \in \mathbb{F}_p\}$ with $a = cw_1w_2^{-1}$. This is alternative (43).

Lemma (1.1.10)[1]: Let $A \subset \mathbb{F}_p^* \times \mathbb{F}_p^*$ satisfying

$$|A| > p^{\varepsilon_0}, \tag{52}$$

$$|A + A| + |A.A| \leq p^{\varepsilon}|A| \tag{53}$$

$$(\varepsilon \ll \varepsilon_0).$$

fix $k \in \mathbb{Z}_+$. There is a subset $A_1 \subset A$ such that

$$|A_1| > p^{-\delta}|A|, \tag{54}$$
$$\left|kA_1^k\right| < p^{\delta}|A_1| \tag{55}$$
where $\delta = \delta_k(\varepsilon)$ and $\delta_k(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$ for given k.
(Observe that $\left|kA^k\right|$ is nondecreasing in k).
Proof. Recall that $A \subset \mathbb{F}_p^* \times \mathbb{F}_p^*$. Write
$$|A|^2 = \sum_{x\in A}|xA| \le |A.A|^{\frac{1}{2}}\left[\sum_{x,x'\in A}|xA\cap x'A|\right]^{\frac{1}{2}}$$
and by (53)
$$\sum_{x,x'\in A}|XA\cap x'A| > p^{-\varepsilon}|A|^3. \tag{56}$$
Lemma (1.1.10) may be proven by an adjustment of the argument in [6] for subsets of $\mathbb{F}_p$ (the main point in the present is to avoid problems due to zero-divisors). We give a different argument, in particular not relying on Gowers' proof of the Balog-Szemeredi theorem.
Lemma (1.1.11)[1]: Let A_1, A_2, A_3 be finite subsets of an additive group, satisfying
$$|A_1\cap A_3| > \frac{1}{K}|A_1|, \tag{57}$$
$$|A_2\cap A_3| > \frac{1}{K}|A_2|, \tag{58}$$
$$|A_i + A_i| < \frac{1}{K}|A_i| \quad (i = 1,2,3). \tag{59}$$
Then
$$|A_1 + A_2| < K^9|A_3. \tag{60}$$
Proof. Write for $i = 1,2$
$$\chi_{A_i} \le \frac{1}{|A_i\cap A_3|}\sum_{y\in A_i-(A_i\cap A_3)}\chi_{y+(A_i\cap A_3)} \tag{61}$$
Hence
$$\chi_{A_i+A_2} \le \frac{1}{|A_1\cap A_3||A_2\cap A_3|}\sum_{\substack{y\in A_i-(A_i\cap A_3)\\ i=1,2}}\chi_{y_1+y_2+(A_1\cap A_3)+(A_2\cap A_3)}$$

and therefore
$$|A_1 + A_2| \le \frac{|A_1 - A_1||A_2 - A_2||A_3 + A_3|}{|A_1\cap A_3||A_2\cap A_3|} \le \frac{K^7|A_1||A_2||A_3|}{K^{-2}|A_1||A_2|} = K^9|A_3|$$
from (57)-(59) and the sum-difference inequalities; (see [14]).
From (56), we may specify x G A such that
$$A_1 = \left\{x \in A\middle|.\,|xA\cap \bar{x}A| > \frac{1}{2}p^{-\varepsilon}|A|\right\} \tag{62}$$
Satisfies
$$|A_1| > p^{-\varepsilon}|A|.$$
If $x_1, x_2 \in A_1$, apply Lemma (1.1.11) with $A_1 = x_1A, A_2 = x_2A, A_3 = \bar{x}A$ and $K = 2p^{\varepsilon}$. From (60)
$$|x_1A + x_2A| < p^{10\varepsilon}|A|, \tag{63}$$

Next, let $x_1, x_2, x_3, x_4 \in A_1$. Since

$$|x_1x_3A \cap x_1\bar{x}A| = |x_3A \cap \bar{x}A| > \frac{1}{2}p^{-\varepsilon}|A|,$$
$$|x_2x_4A \cap x_2\bar{x}A| > \frac{1}{2}p^{-\varepsilon}|A|,$$

we may apply Lemma (1.1.11) with $A_1 = x_1x_3A, A_2 = x_2x_4A, A_3 = x_1\bar{x}A \cup x_2\bar{x}A$ and $K = p^{10\varepsilon}$, from (63). Hence

$$|x_1x_3A + x_2x_4A| < p^{90\varepsilon}|A|. \tag{64}$$

Straightforward iteration implies that

$$|y_1A + y_2A| < p^{C\varepsilon}|A| \tag{65}$$

whenever $y_1, y_2 \in A_2^k$ and with $C = C_k$ in (65).
The same statement holds clearly also if $y_i, y_2 \in A_1^{-1}A_1^k$.
Write now

$$\chi_{A_1^k} \le \frac{1}{|A_1|}\sum_{y\in A_1^{-1}A_1^k}\chi_{yA_1} \tag{66}$$
$$\chi_{A_1^k+A_1^k} \le \frac{1}{|A_1|^2}\sum_{y_1,y_2\in A_1^{-1}A_1^k}\chi_{y_1A_1+y_2A_1}$$

and using (65)

$$\left|A_1^k + A_1^k\right| \le \frac{\left|A_1^{-1}A_1^k\right|^2}{|A_1|^2}p^{c\varepsilon}|A| < p^{c\varepsilon}\frac{\left|A^{-1}A^k\right|^2}{|A|}. \tag{67}$$

From (53) and the Plannecke-Ruzsa inequalities applied multiplicatively in the group $\mathbb{F}_p^* \times \mathbb{F}_p^*$, we have $\left|(A \cup A^{-1})^k\right| < p^{c\varepsilon}|A|$. Thus (67) gives

$$\left|A_1^k + A_1^k\right| < p^{c\varepsilon}|A| \tag{68}$$

and (65) follows from (68), applying again the sum set inequalities.

We prove Theorem (1.1.1) for $r = 2$. The case $r = 1$ was treated in [5]. First we recall a few results from combinatorics and harmonic analysis.

Lemma (1.1.12)[1]: (The Balog-Szemeredi-Gowers theorem; see [11]). Let A be a finite subset of an additive group, $|A| = N$, and assume for some $0 < \delta < \frac{1}{10}$ that

$$\{(x_1, x_2, x_3, x_4) \in A^4 | x_1 + x_2 = x_3 + x_4\}| > \delta N^3. \tag{69}$$

Then there is a subset $A_1 \subset A$ satisfying

$$|A_1| > \delta^C N \tag{70}$$

And

$$|A_1 + A_1| < \delta^{-C}|A_1| \tag{71}$$

where C is an absolute constant.
See [11].
Later on we will apply this result in the additive group $\mathbb{F}_p \times \mathbb{F}_p$ and also in the multiplicative group $\mathbb{F}_p^* \times \mathbb{F}_p^*$ (both cases may in fact be derived from the statement for subsets of $\mathbb{Z}, +$).
Next, we give an elementary fact about the Fourier transform of probability measures.

Lemma (1.1.13)[1]: Let v be a probability measure on an Abelian group G and assume $\gamma_1, \dots, \gamma_m \in T$(= dual group) such that

$$\sum_{i=1}^{m}|\hat{v}(\gamma_i)| > \delta m.$$

Then

$$\sum_{i,j=1}^{m} \left|\hat{v}\left(\gamma_i - \gamma_j\right)\right| > \delta^2 m^2.$$

Proof. Take $a \in \mathbb{C}, |a_i| = 1$, such that $a_i \hat{v}(\gamma_i) = |\hat{v}(\gamma_i)|$. Hence, identifying γ_i with the character function $G \to \{z \in \mathbb{C} \,||z| = 1\}$,

$$\delta m < \int_G \left|\sum_{i=1}^{m} a_i \gamma_i(x)\right| v(dx),$$

$$\delta^2 m^2 < \int_G \left|\sum_{i=1} a_i \gamma_i(x)\right|^2 v(dx) \leq \sum_{i,j=1}^{m} \left|\int \left(\gamma_i \bar{\gamma}_j\right)(x) v(dx)\right| = \sum_{i,j=1}^{m} \left|\hat{v}\left(\gamma_i - \gamma_j\right)\right|.$$

Returning to the exponential sum estimate, assume $1 < k_1 < k_2 < p - 1$ satisfying

$$(k_i, p-1) < p^{1-\gamma} \quad (i = 1,2) \tag{72}$$

And

$$(k_1 - k_2, p-1) < p^{1-\gamma} \tag{73}$$

For some $\gamma > 0$. Let $a_1, a_2 \in \mathbb{F}_p^*$ and assume

$$\left|\sum_{1}^{p-1} e_p(a_1 x^{k_1} + a_2 x^{k_2})\right| > P^{1-\varepsilon}. \tag{74}$$

Our purpose is to get a contradiction for $\varepsilon < \varepsilon(\gamma), \varepsilon(\gamma) > 0$ in (74).
Consider the multiplicative subgroup $H < \mathbb{F}_p^* \times \mathbb{F}_p^*$ defined by

$$H = \left\{(x^{k_1}, x^{k_2}) \middle| x \in \mathbb{F}_p^*\right\}.$$

Hence

$$|H| = \frac{p-1}{d} \quad \text{with} \quad d = (k_1, k_2, p-1). \tag{75}$$

Define the probability measures μ, μ_- on $\mathbb{F}_p \times \mathbb{F}_p$ by

$$\mu = \frac{1}{|H|} \sum_{y \in H} \delta_y,$$

$$\mu_- = \frac{1}{|H|} \sum_{y \in H} \delta_{-y}$$

where δ_y stands for the Dirac measure at y. Rephrase (74) as

$$|\hat{\mu}(a)| > p^{-\varepsilon}. \tag{76}$$

Notice that by invariance, $\hat{\mu}(\xi) = \hat{\mu}(y\xi)$ for $y \in H$.
Let $\ell \in \mathbb{Z}_+$. From (76)

$$\sum_{y \in H} |\hat{\mu}(ya)|^{2\ell} > |H| p^{-2\varepsilon\ell}. \tag{77}$$

Since $|\hat{\mu}(\xi)|^{2\ell} = \left(\mu^{(\ell)} * \mu_-^{(\ell)}\right)(\xi)$, iterated application of Lemma (1.1.13) with $v = \mu^{(\ell)} * \mu_-^{(\ell)}$ implies

$$\frac{1}{|H|^{2r}} \sum_{y_1, \dots, y_{2r} \in H} \left|\hat{\mu}\left((y_1 - y_2 + \cdots - y_{2r})a\right)\right|^{2\ell} > p^{-4\varepsilon r \ell} \tag{78}$$

assuming $\ell \in \mathbb{Z}_+$ to be a power of 2.
Hence

$$p^{-4\varepsilon r\ell} < \sum_{y\in\mathbb{F}_p^2} |\hat{\mu}(ya)|^{2\ell}\big(\mu^{(r)} * \mu_-^{(r)}\big)(y) \tag{79}$$

$$\leq \big(mu^{(r)} * \mu_-^{(r)}\big)(0) \sum_{\xi\in\mathbb{F}_p^2} |\hat{\mu}(\xi)|^{2\ell}$$

$$= p^2\big(\mu_-^{(r)}\big)(0).\big(\mu^{(\ell)} * \mu_-^{(\ell)}\big)(0). \tag{80}$$

Taking $r = \ell$, it follows that

$$\big(\mu^{(r)} * \mu_-^{(r)}\big)(0) > p^{-1-2\varepsilon r^2}. \tag{81}$$

On the other hand, there is the upperbound

$$\big(\mu^{(r)} * \mu_-^{(r)}\big)(0) = |H|^{-2r}|\{(y_1,..,y_{2r}) \in H^{2r}|y_1 - y_2 + \cdots - y_{2r} = 0\}|$$

$$= (p-1)^{-2r}\left|\left\{(x_1,\dots,x_{2r}) \in \left(\mathbb{F}_p^*\right)^{2r} \middle| \begin{matrix} x_1^{k_1} - x_2^{k_1} + \cdots - x_{2r}^{k_1} = 0 \\ x_1^{k_2} - x_2^{k_2} + \cdots - x_{2r}^{k_2} = 0 \end{matrix}\right\}\right|$$

$$= (p-1)^{-2r}\left|\left\{(x_1,\dots,x_{2r}) \in \left(\mathbb{F}_p^*\right)^{2r} \middle| x_1^{k_1} - x_2^{k_1} + \cdots - x_{2r}^{k_1} = 0\right\}\right| \tag{82}$$

to which the Gauss sum estimate applies. Write

$$(82) = (p-1)^{-2r}p^{-1}\sum_{\xi\in\mathbb{F}_p}\left|\sum_{x=1}^{p-1} e_p(\xi x^{k_1})\right|^{2r}$$

$$\leq \left(\frac{p}{p-1}\right)^{2r} p^{-1} + (p-1)^{-2r}\max_{\xi\in\mathbb{F}_p^*}\left|\sum_{x=1}^{p-1} e_p(\xi x^{k_1})\right|^{2r}.$$

In view of assumption (72), by (13), there is $\delta_0 = \delta(\gamma) > 0$ such that

$$\max_{\xi\in\mathbb{F}_p^*}\left|\sum_{x=1}^{p-1} e_p(\xi x^{k_1})\right| \leq p^{1-\delta_0} \quad (i = 1,2) \tag{83}$$

Taking $r > r_0$,

$$r_0 = \left[\frac{1}{\delta_0}\right] \tag{84}$$

and it follows that (82) $< \frac{2}{p}$

Summarizing

$$p^{-1-2\varepsilon r^2} < \left(\mu^{(r)} * \mu_r^{(r)}\right)(0) < \frac{2}{p} \quad for\ \ r \geq r_0 \tag{85}$$

Define the sets

$$\Omega_\delta = \{\xi \in \mathbb{F}_p^2 \mid |\hat{\mu}(\xi)| > p^{-\delta}\}$$

And

$$\Lambda_{r,\delta} = \left\{y \in \mathbb{F}_p^* \middle| \big(\mu^{(r)} * \mu_-^{(r)}\big)(y) > p^{-1-s}\right\}.$$

From (85) with $r = r_0$

$$\sum_{\xi} |\hat{\mu}(\xi)|^{2r_0} < p^2\frac{2}{p} = 2p. \tag{86}$$

Hence

$$|\Omega_\delta| < p^{1+2r_0\delta}. \tag{87}$$

Obviously

$$\left|\Lambda_{r,\delta}\right| < p^{1+\delta} \tag{88}$$

Apply (79) with $\ell = 1, r = r_0$. Thus

$$p^{-4\varepsilon r_0} < \sum_{y \in \mathbb{F}_p^2} |\hat{\mu}(ya)|^2 \big(\mu^{(r_0)} * \mu_-^{(r_0)}\big)(y)$$

Implying

$$\frac{1}{2} p^{-4\varepsilon r_0} < \sum_{ay \in \Omega_{3\varepsilon r_0}} \big(\mu^{(r_0)} * \mu_-^{(r_0)}\big)(y) \overset{(85)}{<} \frac{2}{p}\left|\Omega_{3\varepsilon r_0}\right|$$

And

$$|\Omega_\delta| > p^{1-5\varepsilon r_0} \quad for\ \ \delta > 3\varepsilon r_0. \tag{89}$$

Next, writing (79) with $\ell = r_0$

$$p^{-4\varepsilon r r_0} < \sum_{y \in \mathbb{F}_p^2} |\hat{\mu}(ya)|^{2r_0} \big(\mu^{(r)} * \mu_-^{(r)}\big)(y) = \sum_{y \in \Lambda_{r,\delta}} + \sum_{y \notin \Lambda_{r,\delta}}. \tag{90}$$

Since

$$\sum_{y \notin \Lambda_{r,\delta}} < p^{-1-\delta} \sum_{y \in \mathbb{F}_p^2} |\hat{\mu}(\xi)|^{2r_0} \overset{(86)}{<} 2p^{-\delta},$$

It follows from (90) that for $\delta > 4\varepsilon r r_0$

$$\left|\Lambda_{r,\delta}\right| > p^{-14\varepsilon r r_0} \quad if\ \ r \geq r_0, \delta > 4\varepsilon r r_0$$

and hence

$$\left|\Lambda_{r,\delta}\right| > p^{1-4\varepsilon r r_0} \quad if\ \ r > r_0, \delta > 4\varepsilon r r_0. \tag{91}$$

Notice also that, by (83), if $\delta < \delta_0$,

$$\Omega_\delta(\mathbb{F}_p^* \times \mathbb{F}_p^*) = \{(0,0)\};$$

hence $\Omega_\delta = \Omega_\delta^* \cup \{(0,0)\}$, denoting

$$\Omega_\delta^* = \Omega_\delta \cap (\mathbb{F}_p^* \times \mathbb{F}_p^*).$$

Put

$$\delta_1 = 5\varepsilon r_0 \tag{92}$$

and let $\xi \in \Omega_{\delta_1}^*$. Replacing in (79) a by ξ and ε by δ_1,

$$p^{-4\delta_1 r\ell} < \sum_{y \in \mathbb{F}_p^2} |\hat{\mu}(y\xi)|^{2\ell} \big(\mu^{(r)} * \mu_-^{(r)}\big)(y) \tag{93}$$

Taking $\ell = 1$ in (93),

$$\frac{1}{2} p^{-4\delta_1 r} < \sum_{y\xi \in \Omega_{2\delta_1 r}} \big(\mu^{(r)} * \mu_-^{(r)}\big)(y) \tag{94}$$

Since $\left|\Omega_{2\delta_1 r}\right| < p^{1+4\delta_1 r_0 r}$ by (87), in (94) we may further restrict the y summation to $\Lambda_{r,5\delta_1 r_0 r}$ and conclude that

$$\left|\Lambda_{r,5\delta_1 r_0 r} \cap \big(\xi^{-1}\Omega_{2\delta_1 r}\big)\right| > \frac{1}{4} p^{1-4\delta_1 r} \quad for\ r \geq r_0. \tag{95}$$

From (87), (89)

$$p^{1-\delta_1} < \left|\Omega_{\delta_1}\right| < p^{1+2r_0\delta_1}. \tag{96}$$

Inequality (95) is valid for all $\xi \in \Omega_{\delta_1}^*$. Taking $r = r_0$, (95), (96) imply

$$\sum_{\xi \in \Omega_{\delta_1}^*} \left|\big(\xi_1^{-1}\Omega_{2\delta_1 r_0}\big) \cap \Lambda_{r,5\delta_1 r_0^2}\right| > p^{2-5\delta_1 r_0}$$

and the left side is bounded by

$$\left|\Lambda_{r,5\delta_1 r_0^2}\right|^{\frac{1}{2}}\left(\sum_{\xi_1,\xi_2\in\Omega^*_{\delta_1}}\left|\left(\xi_1^{-1}\Omega^*_{2\delta_1 r_0}\right)\right|\cap\left|\left(\xi_2^{-1}\Omega^*_{2\delta_1 r_0}\right)\right|\right)^{\frac{1}{2}}$$

Therefore

$$\begin{aligned} p^{3-15\delta_1 r_0^2} &< \sum_{\xi_1,\xi_2\in\Omega^*_{\delta_1}}\left(\xi_1^{-1}\Omega^*_{2\delta_1 r_0}\right)\cap\left(\xi_2^{-1}\Omega^*_{2\delta_1 r_0}\right) \\ &< \left|\left\{(\xi_2,\xi_2,\xi_3,\xi_4)\in\left(\Omega^*_{2\delta_1 r_0}\right)^4\middle|\xi_1\xi_3=\xi_2\xi_4\right\}\right|. \end{aligned} \quad (97)$$

With e sufficiently small, we may make $\delta_1$in (92) arbitrarily small. Applying Lemma (1.1.12) to the set $\Omega^*_{2\delta_1 r_0}$ in the multiplicative group $\mathbb{F}_p^*\times\mathbb{F}_p^*$, there is a subset $\delta\subset\Omega^*_{2\delta_1 r_0}$ satisfying

$$|\Omega|>p^{1-Cr_0^2\delta_1} \quad (98)$$

And

$$|\Omega.\Omega|<p^{1+Cr_0^2\delta_1}. \quad (99)$$

We reduce Ω further to also obtain a small additive doubling set. From Lemma (1.1.13)

$$\sum_{\xi_1,\xi_2\in\Omega}|\hat{\mu}|^2(\xi_1-\xi_2)>p^{-8\delta_1 r_0}|\Omega|^2 \quad (100)$$

Implying

$$\left|\left\{(\xi_1,\xi_2)\in\Omega^2\ \middle|\xi_1-\xi_2\in\Omega_{5\delta_1 r_0}\right\}\right|>p^{3-Cr_0^2\delta_1}|\Omega|^2$$

and since $\left|\Omega_{5\delta_1 r_0}\right|<\ p^{1+10r_0^2\delta_1}$, it also holds that

$$|\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\Omega^4|\xi_1-\xi_2=\xi_3-\xi_4\}|>p^{3-Cr_0^2\delta_1}. \quad (101)$$

Now applying Lemma (1.1.12) to Ω in the additive group $\mathbb{F}_p\times\mathbb{F}_p$ gives a subset $A\subset\Omega$ such that

$$p^{1+4r_0^2\delta_1}>|A|>|\Omega|p^{-Cr_0^2\delta_1}>p^{1-Cr_0^2\delta_1}, \quad (102)$$

$$|A+A|<p^{Cr_0^2\delta_1}|A| \quad (103)$$

(here we use C to denote various numerical constants).
By (99), it also holds that

$$|A.A|<p^{Cr_0^2\delta_1}|A|. \quad (104)$$

Since A satisfies (103), (104), we may apply Proposition (1.1.4). Notice that by (92), $\varepsilon'=\varepsilon'(\delta_1)=\varepsilon'(\varepsilon)\xrightarrow{\varepsilon\to 0}0$. Either (27) or (28) holds. Assume (27), say for some $b\in\mathbb{F}_p$

$$\left|\Omega\cap\left(\{b\}\times\mathbb{F}_p\right)\right|\geq\left|A\cap\left(\{b\}\times\mathbb{F}_p\right)\right|>p^{-\varepsilon'}|A|\overset{(3.34)}{>}p^{1-2\varepsilon'}. \quad (105)$$

Applying (100) with Ω replaced by $\Omega\cap(\{b\}\times\mathbb{F}_p)$, we obtain

$$\left|\left\{(\xi_1,\xi_2)\in\mathbb{F}_p^2\middle|(0,\xi_1-\xi_2)\in\Omega_{5\delta_1 r_0}\right\}\right|>p^{2-4\varepsilon'-9\delta_1 r_0}>p^{\frac{3}{2}}$$

contradicting the fact that $\Omega_{5\delta_1 r_0}=\Omega^*_{5\delta_1 r_0}\cup\{(0,0)\}$.
Assume (28). Thus there is $c\in\mathbb{F}_p^*$ s.t. if

$$A_1=A\cap\left\{(t,ct)\middle|t\in\mathbb{F}_p\right\}, \quad (106)$$

Then

$$|A_1|>p^{-\varepsilon'}|A|>p^{1-2\varepsilon'}. \quad (107)$$

Recalling that $A_1\subset\Omega^*_{2\delta_1 r_0}$ write

$$\sum_{t=0}^{p-1} |\hat{\mu}(t, ct)|^2 \geq |A_1| p^{-4\delta_1 r_0} \overset{(3.39)}{>} p^{1-3\varepsilon'} \tag{108}$$

Where

$$\hat{\mu}(t, ct) = \frac{1}{p-1} \sum_{z=1}^{p-1} e_p\left(t(z^{k_1} + cz^{k_2})\right).$$

Hence (108) implies

$$\left\{(z, w) \in \mathbb{F}_p^* \times \mathbb{F}_p^* \middle| z^{k_1} + cz^{k_2} = w^{k_1} + cw^{k_2}\right\}| > p^{2-3\varepsilon'}. \tag{109}$$

Writing $w = v.z$, there is $v \in \mathbb{F}_p^*$ such that

$$v^{k_2} \neq 1 \tag{110}$$

and the equation

$$z^{k_2 - k_1} = \frac{1 - v^k}{c(v^{k_2} - 1)} \tag{111}$$

has at least $p^{1-3\varepsilon'}$ solutions in $z \in \mathbb{F}_p$.

To ensure (110), we used that $x^{k_2} \equiv 1$ has $(k_2, p-1) \overset{(72)}{<} p^{1-\gamma} < p^{1-3\varepsilon'}$ solutions in $\mathbb{F}_p$. By (73), (111) has at most $(k_2 - k_1, p-1) < p^{1-\gamma}$ solutions, a contradiction for ε' (hence ε in (74)) small enough.

This completes the proof of Theorem (1.1.1) in the binomial case.

(i) We comment on how δ in (iii) according to the preceding argument depends on ε in (i), (ii). For $r = 1$ (the monomial case) it was shown in [5] that we may take

$$\delta_1 > \exp(-C\varepsilon^{-C_2}). \tag{112}$$

for some constants C_1, C_2 (see [5]).

A more careful analysis of the proof of Proposition (1.1.4) and the binomial case gives a similar lower bound for S2. Therefore (11) implies

$$\delta_r > \exp\left(-C_3 r(\varepsilon^{-C_2} + \log r)\right). \tag{113}$$

(ii) Next we indicate the proof of Theorem (1.1.6). As already mentioned, the case $r = 1$ appears in [2] (these and related exponential sums have their importance in issues related to cryptography, such as the DifRe-Hellman distributions; see [2]).

We first treat the case $r = 2$. The general case is then obtained using the same strategy as described.

Also the proof of the $r = 2$ case is almost identical.

Let $\gamma > 0$ and assume

$$0(\theta_1) > p^\gamma, 0(\theta_2) > p^\gamma, 0(\theta_1\theta_2^{-1}) > p^\gamma. \tag{114}$$

Take

$$t = [p^\gamma]. \tag{115}$$

Introduce

$$H = \{(\theta_1^s, \theta_2^s) | s = 1, \dots, t\} \subset \mathbb{F}_p^* \times \mathbb{F}_p^*. \tag{116}$$

H is not a subgroup of $\mathbb{F}_p^* \times \mathbb{F}_p^*$ (but an 'approximative' subgroup in the sense of [2]).

Clearly $|H| = t$. Define μ, μ_- and assume $a \in \mathbb{F}_p^* \times \mathbb{F}_p^*$ such that

$$|\hat{\mu}(a)| > p^{-\varepsilon} \tag{117}$$

with $\varepsilon > 0$ small enough. Justifying (77) requires an additional argument, since there is no true invariance under H-multiplication. Let $t_1 = \frac{t}{10} p^{-\varepsilon}$ and write for $1 \leq s_1 \leq t_1$

$$\left|\hat{\mu}\left(\theta_1^{s_1}a_1,\theta_2^{s_1}a_2\right)-\hat{\mu}(a_1,a_2)\right|<\frac{2s_1}{t}<\frac{1}{5}p^{-\varepsilon};$$

Hence

$$\left|\hat{\mu}\left(\theta_1^{s_1}a_1,\theta_2^{s_1}a_2\right)\right|>\frac{1}{2}p^{-\varepsilon}. \tag{118}$$

Therefore

$$\sum_{y\in H}|\hat{\mu}(ya)|^{2\ell}\geq\sum_{s_1=1}^{t_1}\left|\hat{\mu}\left(\theta_1^{s_1}a_1,\theta_2^{s_1}a_2\right)\right|^{2\ell}>t_1 4^{-\ell}p^{-2\varepsilon\ell}<|H|p^{-3\varepsilon\ell} \tag{119}$$

providing (77).

Inequality (83) is substituted by the $r=1$ case of Theorem (1.1.6) (established in [2]); thus

$$\max_{\xi\in\mathbb{F}_p^*}\left|\sum_{s=1}^{t}e_p(\xi\theta_i^s)\right|<tp^{-\delta_0}\quad(r=1,2) \tag{120}$$

Where $\delta_0=\delta_0(\gamma)>0$.

We establish (85) again and continue verbatim the argument until invoking Proposition (1.1.4).

Assuming alternative (28), we obtain instead of (109) that

$$\left|\left\{s,s'=1,\ldots,t\middle|\theta_1^s+c\theta_2^s=c\theta_1^{s'}+c\theta_2^{s'}\right\}\right|>t^2p^{-3\varepsilon'} \tag{121}$$

for some $c\in\mathbb{F}_p^*$. Writing $s'=s+\bar{s}$, the equation becomes

$$(\theta_2\theta_1^{-1})^s=c^{-1}\frac{\theta_1^{\bar{s}}-1}{1-\theta_2^{\bar{s}}}. \tag{122}$$

Since $s,\bar{s}<\min\left(0(\theta_1),0(\theta_2),0(\theta_2\theta_1^{-1})\right)$ equation (122) has at most t solutions, contradicting (121).

The combinatorial methods introduced here (sum-product theorems) permit us to extend the results from [4] (in particular estimates on Gauss sums) and the results from for sparse polynomials to the case of certain composite moduli q. We assume the factorization of q involves only a bounded number of prime factors. Details will appear in [3].

If q factors as a (simple) product of a bounded number of distinct prime factors, i.e., $q=p_1\ldots p_r$, the residue ring $\mathbb{Z}_q$ identifies with $\mathbb{Z}_{p_1}\times\ldots\times\mathbb{Z}_{p_r}$ and the argument simplifies significantly. It is basically an easy variant of the methods described earlier. In view of cryptographical applications, the special case where $q=p\ell$ with p,ℓ distinct primes, $p\sim\ell$, is of particular interest (such q are called Blum integers). Our first aim is to extend the proof of Theorem (1.1.6) to such moduli. The argument extends easily to products of several (boundedly many) distinct primes involving only notational complications.

Proposition (1.1.14)[1]: Let $q=p.\ell$ with p,ℓ as above and $\theta_1,\ldots,\theta_r\in\mathbb{Z}_q^*$ where $\mathbb{Z}_q^*$ denotes the multiplicative group of $\mathbb{Z}_q$. Assume for some $S>0$

$$O_p(\theta_i)>q^5,O_\ell(\theta_i)>q^\delta\quad(1\leq i\leq r), \tag{123}$$

$$O_p\left(\theta_i\theta_j^{-1}\right)>q^\delta,(O_\ell\left(\theta_i\theta_j^{-1}\right)>q^\delta\quad(1\leq i\neq j\leq r). \tag{124}$$

Then for $J>q^\delta$

$$\max_{a_1,\ldots,a_r\neq0\left(mod\ p^\ell\right)}\left|\sum_{j=1}^{J}e_q\left(a_1\theta_1^j+\cdots+a_r\theta_r^j\right)\right|<Jq^{-\delta'}. \tag{125}$$

where $\delta'=\delta'(r,\delta)>0$.

We first specify the identification of $\mathbb{Z}_{p\ell}$ and the product $\mathbb{Z}_p \times \mathbb{Z}_\ell$. Take $\alpha \in \mathbb{Z}_p$ s.t. $\alpha\ell = 1(mod p)$ and $\beta \in \mathbb{Z}_\ell$ s.t. $\beta p = 1(mod \ell)$. Denote by $\pi_p: \mathbb{Z}_{p\ell} \to \mathbb{Z}_p, \pi_\ell: \mathbb{Z}_{p\ell} \to \mathbb{Z}_\ell$ the quotient maps. If $a \in \mathbb{Z}_{p\ell}$, clearly

$$a = \pi_p(a)\ell\alpha + \pi_\ell(a)p\beta \quad (mod p\ell) \tag{126}$$

providing a factorization of the identity on $\mathbb{Z}_{p\ell}$ as $\varphi(\pi_p \times \pi_\ell)$ where $\varphi: \mathbb{Z}_p \times \mathbb{Z}_\ell \to \mathbb{Z}_{p\ell}$ is the ring isomorphism given by $\varphi(A, B) = A\ell\alpha + Bp\beta$. Writing $\frac{a}{p^\ell} = \frac{\alpha A}{p} = \frac{\beta B}{\ell}\Big(A = \pi_p(a), B = \pi_\ell(a)\Big)$, we get for the exponential sum

$$\sum_{j \le J}\left(\sum_{s=1}^{r} a_s\theta_s^j\right) = \sum_j e_p\left(\sum_s (\alpha A_s)\theta_s^j\right) e_\ell\left(\sum_s (\beta B_s)\theta_s^j\right). \tag{127}$$

We outline the proof of Proposition (1.1.14).

In order to treat the binomial case, we also need the sum-product result in $\mathbb{Z}_p \times \mathbb{Z}_\ell, p, \ell$ distinct primes. It turns out that the situation is even simpler than for $p = \ell$.

Lemma (1.1.15)[1]: Let $S \subset \mathbb{Z}_p \times \mathbb{Z}_\ell$, where p, ℓ are distinct primes as above. Assume

$$p^\delta < |S| < (p\ell)^{1-\delta} \tag{128}$$

and ($\varepsilon > 0$ assumed small enough depending on δ)

$$|S + S| + |S.S| < |S|^{1+\varepsilon} \tag{129}$$

(addition and multiplication refer to the $\mathbb{Z}_p \times \mathbb{Z}_\ell$ (product) ring structure).

Then one of the following two alternatives holds:

$$\left|S \cap \left(\mathbb{Z}_p \times \{a\}\right)\right| > p^{-\varepsilon'}|S| \, for\ some\ a \in \mathbb{Z}_\ell, \tag{130}$$

$$|S \cap (\{a\} \times \mathbb{Z}_\ell)| > p^{-\varepsilon'}|S| \, for\ some\ a \in \mathbb{Z}_p \tag{131}$$

where $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ with $\varepsilon \to 0$.

Moreover, in case (130) (resp. (131)), $p^{1-\varepsilon'} < |S| < p^{1+\varepsilon'}$ (resp.$\ell^{1-\varepsilon'} < |S| < \ell^{1+\varepsilon'}$).

Notice that if $p \neq \ell$, we do not have to consider alternative (28) in Proposition (1.1.4).

Sketch of the proof. We follow essentially the same argument as when $p = \ell$. Assume (130), (131) do not hold. We may in particular assume $S \in \mathbb{Z}_p^* \times \mathbb{Z}_\ell^*$.

By (129), there is a subset $S_1 \subset S$ s.t. $|S_1| > p^{-C\varepsilon}|S|$ and

$$\left|kS_1^k\right| < p^{C\varepsilon}|S_1|\backslash< p^{C\varepsilon}(p\ell)^{1-\delta} < (p\ell)^{1-\delta+C\varepsilon} < (p\ell)^{1-\frac{\delta}{2}} \tag{132}$$

(here k is specified, depending on ε', and the constant C depends on k).

Denote by $\pi_p: \mathbb{Z}_p \times \mathbb{Z}_\ell \to \mathbb{Z}_p$ and $\pi_\ell: \mathbb{Z}_p \times \mathbb{Z}_\ell \to \mathbb{Z}_\ell$ the projections. If (130) fails, $\max_\alpha\left|S_1 \cap \left(\mathbb{Z}_p \times \{a\}\right)\right| < p^{-\varepsilon'}|S| < p^{-\varepsilon' + C\varepsilon}|S_1| < p^{-\frac{\varepsilon'}{2}}|S_1|$ and hence $|\pi_\ell(S_1)| > p^{\frac{\varepsilon'}{2}}$. Similarly $\left|\pi_p(S_1)\right| > p^{\frac{\varepsilon'}{2}}$.

By the sum-product theorem in prime fields and Lemma (1.1.7) we may thus (replacing S_1 by $k_0 S_1^{k_0} = S_2$ for some $k_0 \in \mathbb{Z}_+$., depending on ε') assume

$$\pi_p(S_2) = \mathbb{Z}_p \ \ and \ \ \pi_\ell(S_2) = \mathbb{Z}_\ell. \tag{133}$$

Suppose $|S_2| > p > \ell$. There are distinct elements $x_0 \neq x_1$ in S_2 s.t. $\pi_\ell(x_0) = \pi_\ell(x_1)$. Then by (133)

$$\begin{aligned} S_2^2 + (S_2 - S_2)S_2 \supset S_2^2 + (x_0 - x_1)S_2 &= S_2^2 + \left(\pi_p(x_0 - x_1)\pi_p(S_2) \times \{0\}\right) \\ &= S_2^2 + \left(\mathbb{Z}_p \times \{0\}\right) = \mathbb{Z}_p \times \mathbb{Z}_\ell \end{aligned} \tag{134}$$

and therefore

$$|2S_2^2 - S_2^2| = p\ell$$
contradicting (132).
Also, if (130), it follows from (129) and $|S| > p^\delta$ that $\left|S \cap \left(\mathbb{Z}_p \times \{a\}\right)\right| > p^{1-\varepsilon'}$; hence $p^{1-\varepsilon'} < |S| < p^{1+\varepsilon'}$.
This proves Lemma (1.1.15).
With Lemma (1.1.15) at hand, we obtain the exponential sum estimate.
Lemma (1.1.16)[1]: Let p, ℓ be as above, $p \neq \ell$. Let $\theta \in \mathbb{Z}_\ell^*, \psi \in \mathbb{Z}_\ell^*$ satisfying for some $S > 0$
$$O_p(\theta) > p^\delta,$$
$$O_\ell(\psi) > p^\delta,$$
If $J > p^\delta, a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_\ell^*$, then
$$\left|\sum_{j=0}^{J} e_p\left(a\theta^j\right)e_\ell\left(b\psi^j\right)\right| < Jp^{-\delta'} \tag{135}$$
for some $\delta' = \delta'(\delta) > 0$.
The proof is similar to the argument explained but no condition on $\frac{\theta}{\psi}$ is involved, since (28) is not an issue here.
More generally, following the argument, we get
Lemma (1.1.17)[1]: Let p, ℓ be as above, $p \neq \ell$. Let $\theta_1, \ldots, \theta_r \in \mathbb{Z}_p^*; \psi_1, \ldots, \psi_s \in \mathbb{Z}_\ell^*$ satisfy for some $\delta > 0$
$$O_p(\theta_i) > p^\delta \ (1 \le i \le r), O_p\left(\theta_i\theta_j^{-1}\right) > p^\delta \ (1 \le i \neq j \le r) \tag{136}$$
And
$$O_\ell(\psi_i) > p^\delta \ (1 \le i \le s), O_p\left(\psi_i\psi_j^{-1}\right) > p^\delta \ (1 \le i \neq j \le s) \tag{137}$$
Let $a_1, \ldots, a_r \in \mathbb{Z}_p^*$ and $b_1, \ldots, b_s \in \mathbb{Z}_\ell^*$. Let $J > p^\delta$. Then
$$\left|\sum_{j=1}^{J} e_p\left(a_1\theta_1^j + \cdots + a_r\theta_r^j\right)e_\ell\left(b_1\psi_1^j + \cdots + b_s\psi_s^j\right)\right| < Jp^{-\delta'} \tag{138}$$
With $\delta' = \delta'_{r+s}(S) > 0$.
As in the proof of Theorems (1.1.1) and (1.1.6) we proceed by induction on $r + s$. Again the case $r + s = 1$ follows from [4]. Let $r + s = 2$. There are three cases. If $r = 2$ or $s = 2$, we are in the situation $p = \ell$ discussed. If $r = s = 1$, apply Lemma (1.1.16). The case $r + s \geq 3$ is treated inductively.
From the identification of $\mathbb{Z}_{p\ell}$ and $\mathbb{Z}_p \times \mathbb{Z}_\ell$, in particular (127), Proposition (1.1.14) follows from Lemma (1.1.17).

We now discuss a few cryptographical applications related to [10], [9].

Let $q = p\ell$ with $p \neq \ell, p \sim \ell$ prime, be a Blum integer. Fix $e \in \mathbb{Z}_q^*$ and consider the sequence $\bar{u} = \{u_n\}$ defined by
$$u_{n+1} = u_n^e \text{ with initial } u_0 = \theta \in \mathbb{Z}_q^*.$$
If $e = 2, \bar{u}$ is the Blum-Blum-Shub generator.
If $\left(e, (p-1)(\ell-1)\right) = 1, \bar{u}$ is called an RSA generator.
Let $\lambda(q)$ be the smallest common multiple of $p - 1, \ell - 1$ (the Carmichael function). Denote $T = O_q(\theta)$ and $\tau = O_T(e)$. Thus $T|\lambda(q)$. Recall the result from [9] stating that almost surely in p, ℓ, θ, e we have

$$\tau \gg q^{1-\varepsilon} \tag{139}$$

for any fixed $\varepsilon > 0$.

From (139) and the results from [9], and [10] the uniform distribution of $\{u_0, \ldots, u_{r-1}\}$ $(modq)$. Using Proposition (1.1.14), we establish also the joint distribution, i.e., the uniform distribution of $(u_n, u_{n+1}, \ldots, u_{n+J-i})$ in $\mathbb{Z}_q^J$, for any fixed $J \geq 1$.

This will be an immediate consequence of the corresponding exponential sum estimate.

Proposition (1.1.18)[1]: Assume p, ℓ, θ, e satisfy (139). Then for some $\delta > 0$

$$\left|\sum_{n=0}^{\tau-1} e_q\left(a_0 u_n + a_1 u_{n+1} + \cdots + a_{n+J-1}\right)\right| < \tau q^{-\delta} \tag{140}$$

for all $(a_0, \ldots, a_{J-i}) \in \mathbb{Z}_q^J \backslash \{0\}$.

Proof. Denote $A = \{u_n | n = 0,1, \ldots, \tau - 1\} \subset G = \{\theta^j | 0 \leq j < T\} < \mathbb{Z}_q^*$ and denote by χ_A the indication function of A. Let $1 < V < \tau$ be an integer to specify and let $v = 0,1, \ldots, V$. Write

$$\sum_{n=0}^{\tau-1} e_q(\ldots) = \sum_{n=0}^{\tau-1} e_q\left(a_0 u_{n+v} + \cdots + a_{J-1} u_{n+v+J-1}\right) + 0(V)$$

$$= \frac{1}{V} \sum_{v=0}^{V-1} \sum_{n=0}^{\tau-1} e_q\left(a_0 u_n^{e^v} + a_{J-1} u_n^{e^{v+J-1}}\right) 0(V)$$

$$= \frac{1}{V} \sum_{v=0}^{V-1} \sum_{x \in G} e_q\left(a_0 x^{e^v} + \cdots + a_{j-1} x^{e^{v+J-1}}\right) \chi_A(x) + 0(V). \tag{141}$$

In order to remove the restriction $x \in A$ in the first term of (141), proceed in the usual way. Thus estimate by

$$\frac{1}{V} |A|^{\frac{1}{2}} \left(\sum_{x \backslash in G} \left| \sum_{v=0}^{V-1} e_q(\ldots) \right|^2 \right)^{\frac{1}{2}} \leq \frac{1}{V} |A|^{\frac{1}{2}} \left(V|G| + (7.4)\right)^{\frac{1}{2}}$$

Where

$$= \sum_{v_1 \neq v_2 < V} \left| \sum_{x \in G} e_q\left(a_0 x^{e^{v_1}} + \cdots + a_{J-1} x^{e^{v_1+J-1}} - a_0 x^{e^{v_1}} - \cdots - a_{J-1} x^{e^{v_2+J-1}}\right) \right|. \tag{142}$$

Rewrite the inner sum in (142) as

$$\sum_{s=0}^{T-1} e_q\left(a_0(\theta_0^s - \psi_0^s) + \cdots + a_{J-1}\left(\theta_{J-1}^s - \psi_{J-1}^s\right)\right) \tag{143}$$

Where

$$\theta_j = \theta^{e^{v_1+J}} \quad and \quad \psi_j = \theta^{e^{(v_2+J)}}. \tag{144}$$

In order to apply Proposition (1.1.14) to (143), we need to ensure that for some $\gamma > 0$

$$O_p(\theta_i), O_p(\psi_i) > p^\gamma \qquad (for\ all\ i), \tag{145}$$

$$O_p\left(\theta_i \theta_j^{-1}\right), O_p\left(\psi_i \psi_j^{-1}\right) > p^\gamma \quad (i \neq j), \tag{146}$$

$$O_p\left(\theta_i \psi_j^{-1}\right) > p^\gamma \quad (for\ all\ \ i, j) \tag{147}$$

and similarly replacing p by ℓ.

By (144), these conditions are equivalent to

$$\left(e^{v_1+j}, p-1\right) < p^{1-\gamma}, \tag{148}$$
$$\left(e^{v_2+j}, p-1\right) < p^{1-\gamma}, \tag{149}$$
$$\left(e^{v_1+i} - e^{v_1+j}, p-1\right) < p^{1-\gamma} \qquad (i \neq j), \tag{150}$$
$$\left(e^{v_1+i} - e^{v_1+j}, p-1\right) < p^{1-\gamma}, \tag{151}$$

and similarly with p replaced by ℓ.

Conditions (148), (149) are obviously satisfied since $(e, p-1) = 1 = (e, \ell - 1)$. Also (150), (151) are equivalent to

$$\left(e^{j} - 1, p-1\right) < p^{1-\gamma} \quad (0 < j \leq J) \tag{152}$$

And

$$\left(e^{v_1 - v_2 + j} 1, p-1\right) < p^{1-\gamma} \quad (|j| \leq J) \tag{153}$$

If $(e^{w} - 1, p-1) = \xi > p^{1-\gamma}, w \neq 0$, clearly

$$\#\{e^{u}(mod p-1)\} < |w|. \frac{p-1}{\xi}$$

and recalling (139)

$$q^{1-\varepsilon} < 0_T(e) < \#\{e^{u} mod (p-1)(\ell-1)\} < \left|\frac{w}{\xi}\right| q. \tag{154}$$

Therefore $|\xi| < q^{\varepsilon}|w|$ and $|w| > q^{\frac{1-\gamma}{2}-\varepsilon}$.

Take $\gamma = \frac{1}{2}$.

Thus (152) holds, since $j = w < J < q^{\frac{1}{4}-\varepsilon}$. Since

$$|v_1 - v_2 + j| \leq V + J,$$

choosing $V = \left[q^{\frac{1}{5}}\right]$ will also ensure (153) if $|v_1 - v_2| > J$.

Returning to (142), it follows from Proposition (1.1.14)

$$(7.4) < V^2 |G|^{1-\delta'} + J.V|G|$$

where $\delta' = \delta'(\gamma) = \delta'\left(\frac{1}{4}\right)$. Therefore

$$\begin{aligned} (141) &< \frac{1}{V} \tau^{\frac{1}{2}} \left(JV|G| + V^2|G|^{1-\delta'}\right)^{\frac{1}{2}} + O(V) \\ &\lesssim V^{\frac{1}{2}} (\tau T)^{\frac{1}{2}} + \tau^{\frac{1}{2}} T^{\frac{\tau-\delta'}{2}} + O(V). \\ &\lesssim q^{\frac{9}{10}} + q^{1-\frac{\delta'}{2}} < \tau q^{\varepsilon - \left(\frac{\delta'}{2} \wedge \frac{1}{10}\right)}. \end{aligned} \tag{155}$$

This proves (140).

We establish an unconditional result for the Blum-Blum-Shub gen erator. First, we choose appropriate primes p, ℓ. Fix r and let p, ℓ be distinct primes of the form

$$p = 1 + c3^{r}, \tag{156}$$
$$\ell = 1 + d3^{r}, \tag{157}$$

with $c, d \in \mathbb{Z}_+$ and

$$p \sim \ell < 3^{20r} \tag{158}$$

(which exist by Linnik's theorem).

Clearly $3^{r} | \lambda(q), q = p\ell$ and we take $\theta = u_0 \in \mathbb{Z}_q^*$ s.t.

$$O_q(\theta) = T = 3^{r} > q^{\frac{1}{20}}. \tag{159}$$

Hence

$$\tau = O_T(2) \sim T. \tag{160}$$

We verify conditions (148)-(151).

From (156)-(158)

$$\left(2^{v_1+j}, p-1\right) = \left(2^{v_1+j}, c3^r\right)\left(2^{v_1+j}, c\right) < \frac{p}{3^r} < p^{\frac{19}{20}}.$$

Condition (152) is obviously satisfied. We verify (153).

Let $(2^w - 1, p-1) = \xi > p^{1-\gamma}, w \neq 0$. Again

$$3^r \lesssim \{2^u (mod 3^r)\} < \#\{2^u (mod\, p-1)\} < |w| \frac{p-1}{\xi} < p^\gamma |w|; \qquad (161)$$

Hence

$$|w| > p^{\frac{1}{20}-\gamma}.$$

The same holds with p replaced by ℓ.

It suffices thus to choose $\gamma = \frac{1}{40}$ and $V = \left[q^{\frac{1}{50}}\right]$ in (155). We proved

Proposition (1.1.19)[1]: Take p, ℓ distinct primes as in (156)-(158) and let $u_0 = \theta$ satisfy (160). Thus the Blum-Blum-Shub generator $\{u_n\}$ satisfies (140) (for any fixed J) and hence $\bar{u}$ is jointly uniformly distributed.

Section (1.2): Bourgain 2-Source Extractor

The min-entropy of a distribution is k if

$$\max_{x \in Supp(X)} Pr[X = x] = 2^{-k}$$

We say that a function $Ext: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^m$ is a 2-source extractor for entropy k if given any 2 independent distributions (a.k.a. sources) (X, Y) with min-entropy k, $Ext(X, Y)$ is close to being uniformly random. We say that the extractor is strong if it satisfies the properties:

$$\Pr_{y \leftarrow_R X} [|Ext(X, y) - U_m| > \epsilon] < \epsilon$$

$$\Pr_{y \leftarrow_R X} [|Ext(x, Y) - U_m| > \epsilon] < \epsilon$$

with low ϵ for arbitrary independent min-entropy k distributions X,Y.

Another way to view 2-source extractors is as boolean matrices (obtained in the natural way from the truth table of the extractor) which look random in a strong sense: Every 2-source extractor for entropy k gives an $N \times N$ boolean matrix in which every $K \times K$ minor has roughly the same number of 1's and 0's, with $N = 2^n, K = 2^k$.

The probabilistic method shows that most functions are 2-source extractors requiring entropy that is just logarithmic in the total length of each of the sources, though explicit constructions of such functions are far from achieving this bound.

The question of finding explicit deterministic polynomial time computable functions that match the random construction. This question was first considered by [21], [26], [27]. The classical Lindsey Lemma gives a 2-source extractor for sources on n bits with entropy slightly greater than $n/2$. No significant progress was made in improving the entropy requirements over this, until recently. In the last few years, sparked by new results in arithmetic combinatorics [20], there were several results [16], [17], [25], [19], [24], [18] on constructing extractors for a few independent sources.

Today, the 2 source extractor that requires the lowest amount of entropy in every source is due to Bourgain [19], who showed how to get an extractor for 2 sources, when the sum of the min-entropies of both sources is large than $2n(1/2 - \alpha)$ for some universal constant α. Bourgain's construction relies on bounds coming from arithmetic combinatorics. While Bourgain's bound may not seem like a big improvement over the earlier result, it turns out to be crucial to the Ramsey graph construction of [18].

first we describe Bourgain's argument. Then we give a proof of a generalization of Vazirani's XOR lemma, that can be used to improve the output length of Bourgain's extractor. At the end we include a simple argument due to Boaz Barak that shows that any two source extractor with small enough error must be strong.

We will reserve the variable p to denote primes.

$\mathbb{F}_p$ will denote the field of size p.

$\mathbb{C}$ will denote the complex numbers.

U_m will denote the uniform distribution on m bits.

G will denote a finite abelian group.

We use the convention that $N = 2^n, M = 2^m$.

For two elements of a vector space x,y, we will use $x \cdot y$ to denote the dot product $\sum_i x_i y_i$.

For a complex number x, we will use $\bar{x}$ to represent its complex conjugate.

We state several facts without proof though all of them can be worked out easily.

Let $f: G \to \mathbb{C}$ and $g: G \to \mathbb{C}$ be two functions from a finite abelian group G to the complex numbers.

We define the inner product $\langle f, g\rangle = \left(\frac{1}{|G|}\right) \sum_{x\in G} f(x)g(x)$.

The ℓ^p norm of f is defined to be $\|f\|_{\ell^p} = (\sum_{x\in G}|f(x)|^p)^{\frac{1}{p}}$.

The L^p norm of f is defined to be $\|f\|_{L^p} = \left(\frac{\sum_{x\in G}|f(x)|^p}{|G|}\right)^{\frac{1}{p}} = |G|^{-\frac{1}{p}}\|f\|_{\ell^p}$.

The ℓ^∞ norm is defined to be $\|f\|_{\ell^\infty} = \max_x |f(x)|$.

We have the following basic relations between the norms:

Fact (1.2.1)[15]: $\|f\|_{\ell^\infty} \geq \left(\frac{1}{\sqrt{|G|}}\right)\|f\|_{\ell^2}$.

Fact (1.2.2)[15]: $\|f\|_{\ell^2} \geq \left(\frac{1}{\sqrt{|G|}}\right)\|f\|_{\ell^1}$.

Fact (1.2.3)[15]: (Triangle Inequality). $|\langle f, g\rangle| \leq \|f\|_{L^1}\|g\|_{\ell^\infty}$.

The Cauchy Schwartz inequality will play a central role in the proof.

Proposition (1.2.4)[15]: (Cauchy Schwartz). For any two functions f, g as above, $|\langle f, g\rangle| \leq \|f\|_{L^2}\|g\|_{L^2}$.

Let $\mathbb{F}$ be any field. Let $\psi: G \to \mathbb{F}^*$ be a group homomorphism. Then we call ψ a character. We call ψ non-trivial if $\psi \neq 1$. Unless we explicitly state otherwise, all characters will map into the multiplicative group of $\mathbb{C}$.

Definition (1.2.5)[15]: (Bilinear maps). We say a map $e: G \times G \to \mathbb{C}$ is bilinear if it is a homomorphism in each variable (for every ξ, both $e(\cdot,\xi)$ and $e(\xi,\cdot)$ are homomorphisms). We say that it is non-degenerate if for every $\xi, e(\xi,\cdot)$ and $e(\cdot,\xi)$ are both non-trivial. We say that it is symmetric if $e(x, y) = e(y, x)$ for every $x, y \in G$.

Let $\mathbb{Z}_r$ denote the ring $\mathbb{Z}/(r)$. It is easy to check that if we let e be the map that maps $(x, y) \longmapsto exp(2\pi xy\imath/r)$, then e is a symmetric non-degenerate bilinear map. Let $G = H_1 \oplus H_2$ be the direct sum of two finite abelian groups. Let $e_1: H_1 \times H_1 \to \mathbb{C}$ and $\epsilon_2: H_2 \times H_2 \to \mathbb{C}$ be symmetric nondegenerate bilinear maps. Then it is easy to see that the map $(x_1 \oplus y_1, x_2 \oplus y_2) \longmapsto e_1(x_1, x_2)e_2(y_1, y_2)$ is a symmetric non-degenerate bilinear map. By the fundamental theorem of finitely generated abelian groups, every finitely generated abelian group is isomorphic to a direct sum of cyclic groups. Thus the previous discussion gives that:

Fact (1.2.6)[15]: For every abelian group G, there exists a symmetric non-degenerate bilinear $e: G \times G \to \mathbb{C}$.

It can be shown that the characters of a finite abelian group G themselves form a finite abelian group $G^\wedge$ (called the dual group of G), where the group operation is pointwise multiplication. Now fix any symmetric, non-degenerate, bilinear map e. For every $x \in G$, let e_x denote the character $e(x,\cdot)$. The map $x \longmapsto e_x$ can then be shown to be an isomorphism from G to $G^\wedge$.

Fact (1.2.7) (Orthogonality)[15]: For any two characters e_x, e_y, we have that $\langle e_x, e_y\rangle = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$.

We define the fourier transform of f (with respect to the above e) to be the function $\hat{f}: G \to \mathbb{C}$ to be: $\hat{f}(\xi) = \langle f, e_\xi\rangle$. Then it is easy to check that this is a linear, invertible operation on the space of all such functions. We get that:

Fact (1.2.8) (Parseval)[15]: $\|f\|_{L^2} = \|\hat{f}\|_{\ell^2}$.

Proposition (1.2.9)[15]: $\|f\|_{\ell^1} \leq |G|^{\frac{3}{2}}\|\hat{f}\|_{\ell^\infty}$.

Proof.

$$\begin{aligned} &\|f\|_{\ell^1} \\ &\leq \sqrt{|G|}\|f\|_{l^2} \\ &= |G|\|f\|_{L^2} \\ &= |G|\|\hat{f}\|_{\ell^2} \qquad by\ Parseval(Fact\ (1.2.8)) \\ &\leq |G|^{\frac{3}{2}}\|\hat{f}\|_{\ell^\infty} \end{aligned}$$

Fact (1.2.10)[15]: (Fourier Inversion). $f(x) = |G|\hat{\hat{f}}(-x) = \sum_{\xi \in G} \hat{f}(\xi) e_\xi\ (x)$.

Fact (1.2.11)[15]: (Preservation of Inner Product). $\langle f, g\rangle = |G|\langle \hat{f}, \hat{g}\rangle$.

By the additive characters of a vector space over a finite field, we mean the characters of the additive group of the vector space. In our applications for 2-source extractors, the characters will always be additive characters of some such vector space. The following proposition is easy to check:

Proposition (1.2.12)[15]: Let $\mathbb{F}^l$ be a vector space over a finite field $\mathbb{F}$. Let ψ be any non-trivial additive character of $\mathbb{F}$. Then the map $e(x, y) = \psi(x \cdot y) = \psi(\sum_i x_i y_i)$ is symmetric, non-degenerate and bilinear.

Note that we can view every distribution on the group G as a function that maps every group element to the probability that the element shows up. Thus we will often view distributions as real valued functions in the natural way: $X(x) = Pr[X = x]$.

Fact (1.2.13)[15]: Let X be any random variable over G. Then $H_\infty(X) \geq k$ simply means that $\|X\|_{\ell^\infty} \leq 2^{-k}$ and implies that $\|X\|_{\ell^2} \leq 2^{-\frac{k}{2}}$.

Fact (1.2.14)[15]: Let X be any random variable over G, then $E_X(f(X)) = |G|\langle f, X\rangle$.

Fact (1.2.15)[15]: If X is a distribution, $\hat{X}(0) = 1/|G|$.

Let U denote the uniform distribution. Then note that $|G|U$ is simply the trivial character e_0. Thus:

Fact (1.2.16)[15]: $\widehat{U}(\xi) = \begin{cases} \frac{1}{|G|} & \xi = 0 \\ 0 & \xi \neq 0 \end{cases}$.

Let $\mathbb{F}$ be a finite field.

We will call a subset $\ell \subset \mathbb{F} \times \mathbb{F}$ a line if there exist two elements $a, b \in \mathbb{F}$ s.t. the elements of ℓ are exactly the elements of the form $(x, ax + b)$ for all $x \in \mathbb{F}$.

Let $P \subseteq \mathbb{F} \times \mathbb{F}$ be a set of points and L be a set of lines. We say that a point (x, y) has an incidence with a line ℓ if $(x, y) \in \ell$. A natural question to ask is how many incidences can we generate with just K lines and K points. Bourgain, Katz and Tao [20] proved a bound on the number of incidences for special fields when the number of lines and points is high enough. Konyagin [23] improved the bound to eliminate the need for K to be large.

Theorem (1.2.17)[15]: (Line Point Incidences). [20], [23] There exists universal constants $\beta, \alpha > 0$ such that for any prime field $\mathbb{F}_p$, if L,P are sets of K lines and K points respectively, with $K \leq p^{2-\beta_0}$, the number of incidences $I(L, P)$ is at most $O\left(K^{\frac{3}{2}-\alpha}\right)$.

An interesting thing to note is that the theorem above does not hold for pseudolines (sets with small pairwise intersections) over finite fields, though a similar theorem does hold over the reals.

When the field is of size 2^p for a prime p a weaker version of the line point incidences theorem holds.

Theorem (1.2.18)[15]: (Line Point Incidences). [20], [23] There exists a universal constant $\beta > 0$ such that for any field $\mathbb{F}_{2^p}$ of size 2^p for prime p, if L, P are sets of K lines and K points respectively with $2^{(1-\beta)p} \leq K \leq 2^{(1-\beta)p}$, the number of incidences $I(L, P)$ is at most $O\left(K^{\frac{3}{2}-\alpha}\right)$.

We describe Bourgain's construction. We start by revisiting the argument for why the hadamard matrix gives a good 2 source extractor for higher min-entropy.

We recall how to extract from two sources when the min-entropy is high. For a finite field $\mathbb{F}$, let $Had: \mathbb{F}^l \times \mathbb{F}^l \rightarrow \mathbb{F}$ be the dot product function, $Had(x, y) = x \cdot y$.

We have the following theorem.

Theorem (1.2.19)[15]: [21], [27] For every constant $\delta > 0$, there exists a polynomial time algorithm $Had: (\{0,1\}^n)^2 \rightarrow \{0,1\}^m$ s.t. if X, Y are independent $\left(n, \left(\frac{1}{2} + \delta\right) n\right)$ sources, $\mathbb{E}_Y[\|Had(X, Y) - U_m\|_{\ell^1}|] < \epsilon$ with $m = \Omega(n)$ and $\epsilon = 2^{-\Omega(n)}$.

Proof. For a convenient l, we treat both inputs as elements of $\mathbb{F}^l (so\ |\mathbb{F}|^l = N)$ and then use the dot product function as described above.

We can view the random variable X as a function $X: \mathbb{F}^l \rightarrow [0, 1]$, which for each element of F^l assigns the probability of taking on that element. We will prove the theorem by using the XOR lemma. To use the lemma, we need to bound $bias_\psi(X, Y) = \left|E\left[\psi\big(Had(X, Y)\big)\right]\right|$ for every non-trivial character ψ.

Fix such a character ψ and let $e(x, y)$ be the symmetric non-degenerate bilinear map $e(x, y) = \psi(x \cdot y)$ (Proposition (1.2.12)). Recall that e_x denotes the character $e(x,\cdot)$. Below we will use Fourier analysis according to e.

Note that

$$bias_\psi(X, Y) = \left|\sum_{y \in \mathbb{F}^l} Y(y) \sum_{x \in \mathbb{F}^l} X(x)\psi(x \cdot y)\right| \tag{162}$$

Now observe that $\sum_{x \in \mathbb{F}^l} X(x)\psi(x \cdot y) = |\mathbb{F}|^l \langle e_y, X\rangle = |\mathbb{F}|^l \overline{\widehat{X}(y)}$. Thus we get that

$$bias_\psi(X,Y) = |\mathbb{F}|\hat{}l \left| \sum_{y\in\mathbb{F}^l} Y(y)\overline{\hat{X}(y)} \right| = |\mathbb{F}|^{2l} \left| \langle Y, \hat{X} \rangle \right|$$

Using the Cauchy Schwartz inequality and the fact that $\|f\|_{\ell^2}^2 = |\mathbb{F}|^l \|f\|_{L^2}^2$ for every $f: \mathbb{F}^l \to \mathbb{C}$, we obtain the bound:

$$\begin{aligned} bias_\psi(X,Y)^2 &\leq |\mathbb{F}|^{4l} \|Y\|_{L^2}^2 \left\|\hat{X}\right\|_{L^2}^2 \\ &= |\mathbb{F}|^{2l} \|Y\|_{\ell^2}^2 \left\|\hat{X}\right\|_{\ell^2}^2 \\ &= |\mathbb{F}|^{2l} \|Y\|_{\ell^2}^2 \|X\|_{L^2}^2 \qquad by\ Parseval(Fact\ (1.2.8)) \\ &= |\mathbb{F}|^{l} \|Y\|_{\ell^2}^2 \|X\|_{\ell^2}^2 \\ &\leq 2^n 2^{-k_1} 2^{-k_2} \end{aligned}$$

Where the last inequality is obtained by Fact (1.2.13), assuming X, Y have min-entropy k_1, k_2. Thus, as long as $k_1 + k_2 > n$, the bias is less than 1.

Set l so that $N^{\frac{1}{l}} = M = |\mathbb{F}|$. By the XOR lemma Lemma (1.2.26) we get m bits which are $2^{\frac{n-k_1-k_2+m}{2}}$ close to uniform. The fact that the extractor is strong follows from Theorem (1.2.30).

One question we might ask is: is this error bound just an artifact of the proof? Does the Hadamard extractor actually perform better than this bound suggests? If $l = 1$, the answer is clearly no, since the output must have at least n bits of entropy to generate a uniformly random point of $\mathbb{F}$. If l is large the answer is still no; there exist sources X, Y with entropy exactly $n/2$ for which the above extractor does badly. For example let X be the source which picks the first half of its field elements randomly and sets the rest to 0. Let Y be the source that picks the second half of its field elements randomly and sets the rest to 0. Then each source has entropy rate exactly $1/2$, but the dot product function always outputs 0.

A key observation of Bourgain's is that the counterexample that we exhibited for the Hadamard extractor is just a pathalogical case. He shows that although the Hadamard function doesn't extract from any sources with lower entropy, there are essentially very few counterexamples for which it fails. He then demonstrates how to encode any general source in a way that ensures that it is not a counterexample for the Hadamard function. Thus his extractor is obtained by first encoding each source in some way and then applying the Hadamard function.

For instance, consider our counterexamples from the last. The counterexamples were essentially subspaces of the original space. In particular, each source was closed under addition, i.e. the entropy of the source $X + X$ obtained by taking two independent samples of X and summing them is exactly the same as the entropy of X. We will argue that when the source grows with addition (we will define exactly what we mean by this), the Hadamard extractor does not fail.

Our proof of Bourgain's theorem will be obtained in the following steps:

- First we will argue that for sources which grow with addition, the Hadamard extractor succeeds.
- Then we will show how to encode any source with sufficiently high entropy in a way that makes it grow with addition.

To show that the Hadamard extractor succeeds, we were trying to bound the bias of the output distribution of the extractor $bias_\psi(X,Y)$ Equation 1:

$$bias_\psi(X,Y) = \left|\sum_{y\in\mathbb{F}^l} Y(y) \sum_{x\in\mathbb{F}^l} X(x)\psi(x\cdot y)\right| \tag{163}$$

Now for any source X, let $X - X$ be the source that samples a point by sampling two points independently according to X and subtracting them.

Lemma (1.2.20)[15]: $bias_\psi(X,Y)^2 \le bias_\psi(X-X,Y)$

Proof.

$$bias_\psi(X,Y) = \left|\sum_{y\in\mathbb{F}^l} Y(y) \sum_{x\in\mathbb{F}^l} X(x)\psi(x\cdot y)\right|$$
$$\le \sum_{y\in\mathbb{F}^l} Y(y) \left|\sum_{x\in\mathbb{F}^l} X(x)\psi(x\cdot y)\right|$$

Then by convexity,

$$bias_\psi(X,Y)^2 = \sum_{y\in\mathbb{F}^l} Y(y) \left|\sum_{x\in\mathbb{F}^l} X(x)\psi(x\cdot y)\right|^2$$
$$= \left|\sum_{y\in\mathbb{F}^l} Y(y) \sum_{x_1,x_2\in\mathbb{F}^l} X(x_1)X(x_2)\psi(x_1\cdot y)\psi(-x_2\cdot y)\right|$$
$$= \left|\sum_{y\in\mathbb{F}^l} Y(y) \sum_{x_1,x_2\in\mathbb{F}^l} X(x_1)X(x_2)\psi\big((x_1-x_2)\cdot y\big)\right|$$

Now let X' denote the source $X - X$. Then by grouping terms, we see that the last expression is simply:

$$bias_\psi(X,Y)^2 \le \left|\sum_{y\in\mathbb{F}^l} Y(y) \sum_{x\in\mathbb{F}^l} X'(x)\psi(x\cdot y)\right| = bias(X-X,Y)$$

Notice the magic of this "squaring the sum" trick. By squaring the sum for the expectation via Cauchy Schwartz, starting with our original bound for the error of the extractor, we obtained a bound that behaves as if our original source was $X' = X - X$ instead of X! If X' has much higher entropy than X, we have made progress; we can follow the rest of the proof of Theorem (1.2.19) in the same way and obtain an error bound that is a bit worse (because we had to square the bias), but now assuming that our input source was X' instead of X.

For one thing, we see that we can easily compose this trick with itself. Applying the lemma again we obtain $bias_\psi(X,Y)^4 \le bias_\psi(X-X,Y)^2 \le bias_\psi(X-X-X+X,Y) = bias_\psi(2X-2X,Y)$.

Applying the lemma with respect to Y (by symmetry), we obtain $bias_\psi(X,Y)^8 \le bias_\psi(2X-2X,Y-Y)$.

In general, we obtain the following lemma:

Lemma (1.2.21)[15]: There exists a polynomial time computable function $Had: \mathbb{F}^l \times \mathbb{F}^l \to \{0,1\}^m$ s.t. given two independent sources X,Y taking values in $\mathbb{F}^l$ and constants c_1, c_2 with

the property that the sources $2^{c_1}X - 2^{c_1}X$ and $2^{c_2}Y - 2^{c_2}Y$ have min-entropy k_1, k_2, then $|E[\psi(Had(X,Y))]| \leq \left(|\mathbb{F}^l|2^{-(k_1+k_2)}\right)$^$(1/2^{c_1+c_2+2}$ for every non-trivial character ψ.
Note that $X - X$ has at least as high min-entropy as X, thus if it is convenient we may simply ignore the subtraction part of the hypothesis; it is sufficient to have that $2^{c_1}X, 2^{c_2}Y$ have high min-entropy to apply the above lemma.

Given Lemma (1.2.21) We find a way to encode X, Y in such a way that the resulting sources grow with addition. Then we can apply the dot product function and use the lemma to prove that our extractor works. How can we encode a source in a way that guarantees that it grows with addition? Our main weapon to do this will be bounds on the number of line point incidences (Theorem (1.2.17) or Theorem (1.2.18)). We will force the adversary to pick a distribution on lines and a distribution on points with high entropy. Then we will argue that if our encoding produces a source which does not grow with addition, the adversary must have picked a set of points and a set of lines that violates the line point incidences theorem.
We will use the following corollary of Theorem (1.2.17), which is slightly stronger than a theorem due to Zuckerman [28]. We will follow his proof closely.
Corollary (1.2.22)[15]: Let $\mathbb{F}$ and $K = 2^{(2+\alpha)k}$ be such that a line point incidences theorem holds for $\mathbb{F}$,K, with α the constant from Theorem (1.2.17). Suppose L,X are two independent sources, with min-entropy $2k, k$ with L picking an element of $\mathbb{F}^2$ and X picking an element of $\mathbb{F}$ independently. Then the distribution $(X, L(X))$ where $L(X)$ represents the evaluation of the L'th line at X is $2^{-\Omega(k)}$ -close to a source with min-entropy $(1 + \alpha/2)2k$.
Proof. Every source with min-entropy k is a convex combination of sources with min-entropy k and support of size exactly 2^k. So without loss of generality we assume that $supp(L)$ is of size 2^{2k} and that supp(X) has size 2^k.
Suppose $\left(X, L(X)\right)$ is ϵ-far from any source with min-entropy $(1 + \alpha/2)2k$ in terms of statistical distance. Then there must exist some set H of size at most $2^{\left(1+\frac{\alpha}{2}\right)2k}$ s.t. $Pr[(X, L(X)) \in H] \geq \epsilon$.
Then we have
(i) A set of points H: $2^{2k+k\alpha}$ points
(ii) A set of lines $supp(L)$: 2^{2k} lines.
Now we get an incidence whenever $(X, L(X)) \in H$. Thus the number of incidences is at least
$$Pr[(X, L(X)) \in H]|supp(L)||supp(X)| \geq \epsilon 2^{3k}$$
However, by the line point incidences theorem (Theorem (1.2.17)), the number of incidences is at most $2^{\left(\frac{3}{2}-\alpha\right)(2k+k\alpha)} = 2^{3k+3k\alpha/2-2k\alpha-k\alpha^2} < 2^{3k\left(1-\frac{\alpha}{2}\right)} = 2^{-\left(\frac{3k\alpha}{2}\right)}2^{3k}$.
These two inequalities imply that $\epsilon < 2^{-\left(\frac{3k\alpha}{2}\right)}$.
Given this corollary, we now describe several ways to encode a source so that it grows with addition. It suffices to understand any one of these encodings to complete the proof for the extractor.
Encoding 1: $x \longmapsto (x, g^x)$ We treat the input x from the source as an element of $\mathbb{F}^*$ for a field in which a version of the line point incidences theorem holds. Then we encode it into an element of $\mathbb{F}^2$ as (x, g^x) where g is a generator of the multiplicative group $\mathbb{F}^*$. Now fix an adversarially chosen source X. Consider the source $\bar{X}$ obtained by performing the above encoding.

$\bar{X}$ is a distribution on points of the form (x, g^x) where $x \neq 0$. By doing a change of variables, we think of every such point as $\left(\log_g \bar{x}, \bar{x}\right)$.
First consider the distribution of $2\bar{X}$. An element of $supp(2\bar{X})$ is of the form $\left(\log_g(\bar{x}_1\bar{x}_2), \bar{x}_1 + \bar{x}_2\right)$ for some $\bar{x}_1, \bar{x}_2$ in the support of $\bar{X}$. Notice that for each a, b with $a = \bar{x}_1\bar{x}_2$ and $b = \bar{x}_1 + \bar{x}_2$, there are at most two possible values for $(\bar{x}_1, \bar{x}_2)$, since for the solutions for $\bar{x}_1$ must satisfy some quadratic equation in a, b. This means that the min-entropy of $2\bar{X}$ is at least $2k - 1$ since the probability of getting a particular (a, b) is at most twice the probability of getting a single pair from $\bar{X}, \bar{X}$. By changing k, in the rest of this discussion we assume that the min-entropy of $2\bar{X}$ is $2k$.
Now for each $a, b \in \mathbb{F}$ with $a, b \neq 0$ define the line

$$\ell_{a,b} = \{(ax, b + x) \in \mathbb{F}^2 | x \in \mathbb{F}\} = \left\{\left(x, \frac{x}{a} + b\right) \in \mathbb{F}^2 \middle| x \in \mathbb{F}\right\}$$

Every (a, b) in our encoding then determines the line $\ell_{a,b}$. Let $L = 2\bar{X}$ be a random variable that picks a line according to $2\bar{X}$.
Every element of $supp(3\bar{X})$ is of the form $\left(\log_g(\bar{x}_1\bar{x}_2\bar{x}_3), \bar{x}_1 + \bar{x}_2 + \bar{x}_3\right)$ and determines the point $(\bar{x}_1\bar{x}_2\bar{x}_3, \bar{x}_1 + \bar{x}_2 + \bar{x}_3) \in \mathbb{F}^2$.

Now think of the distribution of $3\bar{X}$ as obtained by first sampling a line according to $2\bar{X}$ and then evaluating that line at an independent sample from $\bar{X}$ and outputting the resulting point. Then we see that we are in a position to apply Corollary (1.2.22) to get that the encoding does grow with addition.
Encoding 2: $x \longmapsto (x, x^2)$ Again we treat x as an element of the multiplicative group of a field $\mathbb{F}^*$ with charactersitic not equal to 2 in which a version of the line point incidences theorem holds. Now fix an adversarially chosen source X. Let $\bar{X}$ denote the source obtained by encoding X in the above way.

First consider the distribution of $2\bar{X}$. An element of $supp(2\bar{X})$ is of the form $(\bar{x}_1 + \bar{x}_2, \bar{x}_1^2 + \bar{x}_2^2)$ for some $\bar{x}_1, \bar{x}_2$ in the support of $\bar{X}$. Notice that for each a, b with $a = \bar{x}_1 + \bar{x}_2$ and $b = \bar{x}_1^2 + \bar{x}_2^2$, there are at most two possible values for $(\bar{x}_1, \bar{x}_2)$. This means that the min-entropy of $2\bar{X}$ is at least $2k - 1$ since the probability of getting a particular (a, b) is at most twice the probability of getting a single pair from $\bar{X}, \bar{X}$. By changing k, in the rest of this discussion we assume that the min-entropy of $2\bar{X}$ is 2k.
Now for each $a, b \in \mathbb{F}$ with $a, b \neq 0$ define the line

$$\begin{aligned} \ell_{a,b} &= \{(2ax + a^2 - b, a + x) \in \mathbb{F}^2 | x \in \mathbb{F}\} \\ &= \{(x, x/(2a) + (a^2 + b)/(2a)) \in \mathbb{F}^2 \, | x \in \mathbb{F}\} \end{aligned}$$

Every (a, b) in our encoding then determines a unique line $\ell_{a,b}$. Let $L = 2\bar{X}$ be a random variable that picks a line according to $2\bar{X}$.
Every element of $supp(3\bar{X})$ is then of the form $(\bar{x}_1 + \bar{x}_2 + \bar{x}_3, \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)$ and determines the point

$$\begin{aligned} &((\bar{x}_1 + \bar{x}_2 + \bar{x}_3)^2 - (\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2), \bar{x}_1 + \bar{x}_2 + \bar{x}_3) \\ &= (2(\bar{x}_1 + \bar{x}_2)\bar{x}_3 + (\bar{x}_1 + \bar{x}_2)^2 - (\bar{x}_1^2 + \bar{x}_2^2), (\bar{x}_1 + \bar{x}_2) + \bar{x}_3) \\ &= (2a\bar{x}_3 + a^2 - b, a + \bar{x}_3) \end{aligned}$$

Now think of the distribution of $3\bar{X}$ as obtained by first sampling a line according to $2\bar{X}$ and then evaluating that line at an independent sample from $\bar{X}$ and outputting the resulting point. Then we see that we can apply Corollary (1.2.22) to get that the encoding does grow with addition.

By picking an appropriate constant γ, we obtain the following lemma:

Lemma (1.2.23)[15]: There is a universal constant γ s.t. if X is any source that picks an element of $\mathbb{F}$ with min-entropy $\left(\frac{1}{2}-\gamma\right)\log|\mathbb{F}|$, $3\bar{X}$ is $|\mathbb{F}|^{-\Omega(1)}$-close to a source with min-entropy $\left(\frac{1}{2}+\gamma\right)\log|\mathbb{F}^2|$.

Putting together the results from the two previous and applying Lemma (1.2.25), we obtain the theorem for Bourgain's extractor.

Theorem (1.2.24)[15]: [19] There exists a univeral constant $\gamma > 0$ and a polynomial time computable function $Bou: (\{0,1\}^n)^2 \to \{0,1\}^m$ s.t. if X, Y are two independent $(n, (1/2 - \gamma)n)$ sources, $\mathbb{E}_Y[\|Bou(X,Y) - U_m\|_{\ell^1}] < \epsilon$, with $\epsilon = 2^{-\Omega(n)}$, $m = \Omega(n)$.

We will prove a generalization of Vazirani's XOR lemma.

We reserve G for a finite abelian group.

The lemma we will prove is the following:

Lemma (1.2.25) (XOR lemma for cyclic groups)[15]: For every cyclic group $G = \mathbb{Z}_N$ and every integer $M \leq N$, there is an efficiently comptable function $\sigma: \mathbb{Z}_N \to \mathbb{Z}_M = H$ with the following property: Let X be any random variable taking values in $\mathbb{Z}_N$ s.t. for every non-trivial character $\psi: \mathbb{Z}_N \to \mathbb{C}^*$, we have $|\mathbb{E}[\psi(X)]| < \epsilon$, then $\sigma(X)$ is $O\left(\log N \sqrt{M}\right) + O(M/N)$ close to the uniform distribution.

It is easy to extend this result to work for any abelian group G, though it's hard to state the result for general abelian groups in a clean way. We will discuss the proof of the above lemma and just make a few remarks about how to extend it to general abelian groups.

Before we move on to prove Lemma (1.2.25), let us first prove a special case of this lemma which is a generalization of Vazirani's XOR lemma. For the proof of this case below, we essentially follow the proof as in Goldreich's survey [22].

Lemma (1.2.26)[15]: X be a distribution on a finite abelian group G s.t. $|\mathbb{E}[\psi(X)]| \leq \epsilon$ for every non-trivial character ψ. Then X is $\sqrt{|G|}$ close to the uniform distribution: $\|X - U\|_{\ell^1} \leq \epsilon\sqrt{|G|}$.

Proof. By the hypothesis, for every non-trivial character ψ of G, $|\langle\psi, X\rangle| = \left(\frac{1}{|G|}\right)|\mathbb{E}_X[\psi(X)]| \leq \epsilon/|G|$. Then note that if $\psi \neq 1$, $|\langle\psi, X - U\rangle| = |\langle\psi, X\rangle - \langle\psi, U\rangle| = |\langle\psi, X\rangle| \leq \epsilon/|G|$. Also, since X, U are distributions, $\langle 1, X - U\rangle = \langle 1, X\rangle - \langle 1, U\rangle = 0$.

Thus we have shown that $\left\|\widehat{X - U}\right\|_{\ell^\infty} \leq \epsilon/|G|$. Proposition (1.2.9) then implies that $\|X - U\|_{\ell^1} \leq \sqrt{|G|}$.

In Lemma (1.2.26), given a bound of on the biases, the statistical distance blows up by a factor of $\sqrt{|G|}$. This is too much if ϵ is not small enough. Lemma (1.2.25) gives us the flexibility to tradeoff this blowup factor with the number of bits that we can claim are statistically close to uniform. As M is made smaller, the blowup factor is reduced, but we get "less" randomness. Our proof for the general case will work (more or less) by reducing to the case of Lemma (1.2.26).

Note that if σ is an onto homomorphism, for every non-trivial character ϕ of H, $\phi \circ \sigma$ is a non-trivial character of G. Thus the bounds on the biases of X give bounds on the biases of $\sigma(X)$ and we can reduce to the case of Lemma (1.2.26). The problem is that we cannot hope to find such a homomorphism σ for every M. For instance, if $G = \mathbb{Z}_p$ for p a large prime, G contains no non-trivial subgroup and so σ cannot be a homomorphism for $M = \lceil p/2 \rceil$. Instead, we will show that we can find a σ which approximates a homomorphism in the sense:

(i) For every non-trivial character ϕ of H, $\varphi \circ \sigma$ is approximated by a few characters of G. Formally, this is captured by bounding $\|\widehat{\phi \circ \sigma}\|_{L^1}$ (observe that if σ is a homomorphism, this quantity is $1/|G|$).

(ii) We'll ensure that $\sigma(U)$ is the close to the uniform distribution on H.

Then we will be able to use the bounds on the biases of X to give bounds on the biases of $\sigma(X) - \sigma(U)$, where U is the uniform distribution. This will allow us to apply Proposition (1.2.9) to conclude that X is a pseudorandom generator for σ, i.e. $\|\sigma(X) - \sigma(U)\|_{\ell^1}$ is small, which implies that $\sigma(X)$ is close to uniform, since $\sigma(U)$ is close to uniform.

The following lemma asserts that every -biased distribution is pseudorandom for any function σ that satisfies the first condition above.

Lemma (1.2.27)[15]: Let G, H be finite abelian groups. Let X be a distribution on G with $|\mathbb{E}_X[\psi(X)]| \leq \epsilon$ for every non-trivial character ψ of G and let U be the uniform distribution on G. Let $\sigma: G \to H$ be a function such that for every character ϕ of H, we have that

$$\|\widehat{\phi \circ \sigma}\|_{L^1} \leq \tau/|G|$$

Then $\|\sigma(X) - \sigma(U)\|_{\ell^1} < \tau\epsilon\sqrt{|H|}$.

Proof. First note that the assumption on X is equivalent to $\|\widehat{X - U}\|_{\ell^\infty} \leq \epsilon/|G|$. Let ϕ be any non-trivial character of H. Then

$|\langle \phi, \sigma(X) - \sigma(U)\rangle| = |\langle \phi, \sigma(X)\rangle - \langle \phi, \sigma(U)\rangle|$

$= \frac{|\mathbb{E}_{\sigma(X)}[\phi(\sigma(X))] - \mathbb{E}_{\sigma(U)}[\phi(\sigma(U))]|}{|H|}$ by Fact (1.2.14) applied to σ(X) and σ(U)

$= \frac{|\mathbb{E}_X[\phi(\sigma(X))] - \mathbb{E}_U[\phi(\sigma(U))]|}{|H|}$

$= \frac{|G|}{|H|}|\langle \phi \circ \sigma, X\rangle - \langle \phi \circ \sigma, U\rangle|$ by Fact (1.2.14) applied to X and U

$= \frac{|G|}{|H|}|\langle \phi \circ \sigma, X - U\rangle|$

$= \frac{|G|^2}{|H|}|\langle \widehat{\phi \circ \sigma}, \widehat{X - U}\rangle|$ by preservation of inner product (Fact (1.2.11))

$\leq \frac{|G|^2}{|H|}\|\widehat{\phi \circ \sigma}\|_{L^1}\|\widehat{X - U}\|_{\ell^\infty}$ by the triangle inequality (Fact (1.2.3))

$\leq \tau\epsilon/|H|$ since $\|\widehat{\phi \circ \sigma}\|_{L^1} \leq \tau/|G|$ and $\|\widehat{X - U}\|_{\ell^\infty} \leq \epsilon/|G|$

On the other hand, $\langle 1, \sigma(X) - \sigma(U)\rangle = 0$, since $\sigma(X)$ and $\sigma(U)$ are distributions. Thus, we have shown that $\|\widehat{\sigma(X) - \sigma(U)}\|_{\ell^\infty} \leq \tau\epsilon/|H|$, which by Proposition (1.2.9) implies that $\|\sigma(X) - \sigma(U)\|_{\ell^1} \leq \tau\epsilon\sqrt{|H|}$.

Note that when σ is the identity function (or any surjective homomorphism onto a group H), $\tau = 1$. Thus Vazirani's XOR lemma corresponds exactly to the case of σ being the identity function.

Next we show that in the special when G is a cyclic group, we can find a σ which satisfies the hypothesis of Lemma (1.2.27) with small τ.

Lemma (1.2.28)[15]: Let M, N be integers satisfying $N > M$. Let $\sigma: \mathbb{Z}_N \to \mathbb{Z}_M$ be the function $\sigma(x) = x \bmod M$. Then for every character ϕ of $\mathbb{Z}_M$, $\|\widehat{\phi \circ \sigma}\|_{L^1} \leq O(\log N)/N$

Proof. Note that if M divides N, the statement is trivial, since σ is a homomorphism. Below we show that even in the general case, this expectation is small. Define the function $\rho(x) = exp(2\pi\iota x)$. Then note that $\rho(a + b) = \rho(a)\rho(b)$.

First let ϕ be any character of $\mathbb{Z}_M$. Then $\phi(y) = \rho\left(\frac{wy}{M}\right)$ for some $w \in \mathbb{Z}_M$. Clearly, $\phi(\sigma(x)) = \rho\left(\frac{wx}{M}\right)$.

$$\|\widehat{\phi \circ \sigma}\|_{L^1} = \left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N} \left| \sum_{x \in \mathbb{Z}_N} \rho\left(\frac{tx}{N}\right) \rho\left(-\frac{wx}{M}\right) \right| = \left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N} \left| \sum_{x \in \mathbb{Z}_N} \rho\left(\frac{x(tM - wN)}{NM}\right) \right|$$

Recall that for any geometric sum $\sum_{i=0}^{N} br^i = \frac{br^N - b}{r-1}$, as long as $r \neq 1$. The inner sum in this expression is exactly such a geometric sum. Thus we get:

$$\|\widehat{\phi \circ \sigma}\|_{L^1} \leq \left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N, t \neq \frac{wN}{M}} \left| \sum_{x \in \mathbb{Z}_N} \rho\left(\frac{x(tM - wN)}{NM}\right) \right| + 1/N$$

$$= \left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N, t \neq \frac{wN}{M}} \left| \frac{\rho\left(\frac{N(tM-wN)}{NM}\right) - 1}{\rho\left(\frac{tM-wN}{NM}\right) - 1} \right| + 1/N \qquad \text{by simplifying the geometric sum}$$

$$\leq \left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N, t \neq \frac{wN}{M}} \left| \frac{2}{\rho\left(\frac{tM - wN}{NM}\right) - 1} \right| + \frac{1}{N} \qquad \textit{since} \left| \rho\left(\frac{N(tM - wN)}{NM}\right) - 1 \right| \leq 2$$

$$\leq \left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N, t \neq \frac{wN}{M}} \left| \frac{2}{\rho\left(\frac{t - (wN/M)}{N}\right) - 1} \right| + 1/N$$

Now write $wN/M = c + d$, where c is an integer, and $d \in [0, 1]$. Then, by doing a change of variable from t to $t - c$, we get that the above sum is

$$\left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N, t \neq d} \left| \frac{2}{\rho\left(\frac{t - d}{N}\right) - 1} \right| + 1/N$$

We will bound two parts of this sum separately. Let r be a constant with $0 < r < 1/4$. Now note that $\left|\rho\left(\frac{t-d}{N}\right) - 1\right| \geq \Omega(1)$ when $rN < t < (1 - r)N$, since in this situation the quantity is the distance between two points on the unit circle which have an angle of at least $2\pi r$ between them.

When t is not in this region, $\left|\rho\left(\frac{t-d}{N}\right) - 1\right| \geq \left|\sin\left(\frac{2\pi(t-d)}{N}\right)\right|$, since the sin function gives the vertical distance between the two points. This is at least $(t - d)/100N$ for r small enough, since we have that $|\sin x| > |x|$ for $-\pi/2 < x < \pi/2$. Thus, choosing r appropriately, we can bound the sum:

$$\left(\frac{1}{N^2}\right) \sum_{t \in \mathbb{Z}_N, t \neq d} \left| \frac{2}{\rho\left(\frac{t - d}{N}\right) - 1} \right| + 1/N$$

$$= \left(\frac{1}{N^2}\right) \sum_{t \neq d, t \in [rN, (1-r)N]} \left| \frac{2}{\rho\left(\frac{t - d}{N}\right) - 1} \right| + \sum_{t \neq d, t \notin [rN, (1-r)N]} \left| \frac{2}{\rho\left(\frac{t - d}{N}\right) - 1} \right| + 1/N$$

$$\leq \left(\frac{1}{N^2}\right)\left(\sum_{t\neq d, t\in[0,rN]} \frac{800N}{t-d} + \sum_{t\neq d, t\notin[rN,(1-r)N]} O(1)\right) + 1/N$$
$$\leq \left(\frac{1}{N^2}\right)\big(O(N\log N) + O(N)\big) + 1/N$$
Here the last inequality used the fact that $\sum_{i=1}^{n} 1/i = O(\log n)$. Overall this gives us a bound of $\tau \leq O\left(\frac{\log N}{N}\right)$.

On uniform input the distribution $\sigma(U)$ is quite close to uniform. Specifically, if $N = qM + r$, with q, r the quotient and remainder of N on dividing by N, we have that $\sigma(U)$ is $2r((q+1)/N - 1/M) = (2r/M)(M(q+1)/N - 1) = (2r/M)(M-r)/N = 2M/N$ close to the uniform distribution. Thus, overall we get that this σ turns any distribution which fools characters with bias at most into one that is $\epsilon \log N\sqrt{M} + O\left(\frac{M}{N}\right)$ close to uniform.

Now we discuss the situation for general abelian groups. The basic observation is that approximate homomorphisms can be combined to give a new approximate homomorphism:

Lemma (1.2.29)[15]: Let $G = G_1 \oplus G_2$ and $H = H_1 \oplus H_2$ be finite abelian groups. Let $\sigma_1: G_1 \to H_1$ and $\sigma_2: G_2 \to H_2$ be two functions that satisfy the hypotheses of Lemma (1.2.27) with constants τ_1 and τ_2 respectively. Then the function $\sigma: G \to H$ defined as $\sigma(x \oplus y) = \sigma_1(x) \oplus \sigma_2(y)$ satisfies the hypotheses of the lemma with parameters $\tau_1\tau_2$.

Given this lemma, it is clear how to get for every abelian group. Simply write the abelian group as a direct sum of cyclic groups. Then depending on how much randomness is needeed, we can compose several homomorphisms with approximate homorphisms to get a function σ that does the job.

We give an argument due to Boaz Barak showing that every 2 source extractor which has sufficiently small error is in fact strong.

Theorem (1.2.30)[15]: Let $IExt: (\{0,1\}^n)^2 \to \{0,1\}^m$ be any two source extractor for min-entropy k with error . Then $IExt$ is a strong two source extractor for min-entropy k' (strong with respect to both sources) with error $2^m\left(\epsilon + 2^{k-k'}\right)$.

Proof. Without loss of generality, we assume that X, Y have supports of size k'. Then we need to bound:
$$\sum_{y\in supp(Y)} 2^{-k'} \|IExt(X,y) - U_m\|_{\ell^1}$$
For any $z \in \{0,1\}^m$, define the set of bad y's for z
$$B_z = \{y: |Pr[IExt(X,y) = z] - 2^{-m}| \geq \epsilon\}$$
Claim (1.2.31)[15]: For every z, $|B_z| < 2^k$

Suppose not, then the flat distributions on B_z, X are two independent sources for which the extractor IExt fails. Now let $B = \cup_z B_z$. We see that $|B| < 2^k 2^m$. Thus,
$$\sum_{y\in supp(Y)} 2^{-k'} \|IExt(X,y) - U_m\|_{\ell^1}$$
$$= \sum_{y\in supp(Y)\cap B} 2^{-k'} \|IExt(X,y)q - U_m\|_{\ell^1} + \sum_{y\in supp(Y)\setminus B} 2^{-k'} \|Ext(X,y) - U_m\|_{\ell^1}$$
$$\leq 2^{-k'} 2^{k+m} + \epsilon 2^m = 2^m\left(2^{k-k'} + \epsilon\right).$$

Chapter 2
Generalized N-Property and Morse–Sard Theorem

Investigate the questions related to the uniqueness of weak solutions for the continuity equation associated to a vector field with Sobolev regularity. We show that almost all level sets are finite disjoint unions of C^1–smooth compact manifolds of dimension $n - m$.

Section (2.1): Sard Theorem for Sobolev Maps

We describe some extensions of the Lusin N-property and the Sard theorem for Sobolev maps which have been recently obtained in collaboration with M. Csornyei, E. D'Aniello, and B. Kirchheim [32], [33];

The N-property has been widely studied, mostly in connection with the area formula for Sobolev maps and other classes of weakly differentiable maps. However, the variant of this property that we are interested in arises as a key ingredient of our proof of the optimal form of Sard theorem for Sobolev maps. We consider this version of Sard theorem in the attempt—which eventually failed—to produce a counterexample to a certain uniqueness statement for the flow associated to a vector field with Sobolev regularity; this statement is in turn related to the uniqueness of weak solutions of the continuity equation (or the transport equation) associated to the same vector field.

We plan to explain the connections between these problems (Nproperty, Sard Theorem, uniqueness for the flow and for the continuity equation associated to a divergence-free vector field), and then illustrate some of our results at least in simple cases, giving when possible an outline of the proof. In writing this We tried to improve readability at the expenses of precision by omitting most technical details.

Let me finally add that similar results on the N-property and the Sard theorem for Sobolev maps have been obtained by J. Bourgain, M.V. Korobkov, and J. Kristensen [37] at about the same time as us (but with different motivations in the background).

We consider the continuity equation

$$u_t + div(bu) = 0 \tag{1}$$

where b is a vector field on $\mathbb{R}^n$ and the unknown u is a scalar function on $[0,T) \times \mathbb{R}^n$ subject to the initial condition $u(0,\cdot) = u_0$, with u_0 a given initial datum.

To understand what follows it is convenient to keep in mind the standard mechanical interpretation of (pde): consider a continuous distribution of point particles in $\mathbb{R}^n$ such that the trajectory $x = x(t)$ of each particle satisfies the ordinary differential equation

$$\dot{x} = b(x), \tag{2}$$

and let $u = u(t,x)$ be the corresponding density—that is, mass per unit volume at time t and position x. Then u satisfies (pde).

This interpretation suggests that existence and uniqueness of solutions of the Cauchy problem for (pde) are strictly related to existence and uniqueness for the Cauchy problem for (ode).

Assume for the time being that b is bounded and smooth. Under these assumptions we can construct the flow associated to (ode), namely the oneparameter family of diffeomorphisms of $\mathbb{R}^n$

$$\{\Phi_t\}_{t\geq 0}$$

defined by the fact that for every $x \in \mathbb{R}^n$ the map $t \longmapsto \Phi_t(x)$ solves the equation (ode) with initial value $\Phi_0(x) = x$.

If b is divergence-free then the flow is volume-preserving (that is, each diffeomorphism Φ_t is volume-preserving), and therefore a solution of (pde) with initial datum u_0 is

$$u(t,x) := u_0(\Phi_t^{-1}(x) \ . \tag{3}$$

It follows immediately that if u_0 is bounded then

$$\|u(t,\cdot)\|_\infty \le \|u_0\|_\infty \quad \text{for all } t. \tag{4}$$

Assume now that the vector field b is bounded, divergence-free (in the sense of distribution) but no longer smooth. We construct a solution of (pde) with initial datum u_0 as follows: let b_ε be a regularization of b by convolution (so b_ε is bounded, divergence-free, and smooth), and let u_ε be the solution of (pde) with b_ε in place of b given by formula (3); then we can use the bound (4) to pass to the limit in u_ε as $\varepsilon \to 0$, and obtain bounded function u that solves (pde) for all positive times (in the sense of distribution).
To make this argument work it is not needed that $div\ b = 0$, but it suffices that $div\ b \ge -m$ for some finite m; in this case (3) should be replaced by

$$u(t,x) := u_0\left(\Phi_t^{-1}(x)\right) \cdot \det\left(\nabla\Phi_t^{-1}(x)\right),$$

and since the derivative of $\det\left(\nabla\Phi_t(x)\right)$ with respect to the variable t agrees with $div\ b(x)$, which is larger than $-m$, then the bound (4) becomes

$$\|u(t,\cdot)\|_\infty \le e^{mt}\|u_0\|_\infty \qquad for\ all\ \ t.$$

Note that without assumptions on the divergence of b the existence of bounded solutions for all times may no longer hold, because it can happen that all particles end up in the same point and remain there; therefore after some time the particle density becomes a measure with an atom, and is no longer represented by a function (let alone a bounded function). For example, this is the case when

$$b(x) := \begin{cases} -\dfrac{x}{\sqrt{|x|}} & if\ x \ne 0, \\ 0 & if\ x = 0. \end{cases}$$

Under the only assumption that b is bounded and has bounded (or even vanishing) divergence there is in general no uniqueness for the Cauchy problem for the continuity equation (pde). However, in [41], R.J. DiPerna and P.-L. Lions proved that uniqueness holds under the additional assumption that b is (locally) of Sobolev class $W^{1,1}$, and later on L. Ambrosio [34] improved this result by showing that it suffices that b is (locally) of class BV.

Note that in both uniqueness is proved within the class of distributional solutions of (pde) that are functions for all times (actually some additional bound on the solution u is also needed, for example $\|u(t,\cdot)\|_\infty$ uniformly bounded in t for all finite time-intervals). In other words, the possibility that particles concentrate in a negligible set is excluded a priori, and not proved impossible.
It should also be noted that both results give conditions which are sufficient for uniqueness, but not necessary.

In view of the mechanical interpretation described above, one would expect that uniqueness for (pde) is related to uniqueness for (ode), and the heuristic argument should be the following: let N be the set of non-uniqueness associated to b, that is, the set of all points $z \in \mathbb{R}^n$ such that the differential equation (ode) has at least two solutions $x_z(t)$ and $\tilde{x}_z(t)$ with initial datum z. Consider now an initial distribution of particles contained in N: there are at least two possible evolutions of this distribution, one obtained by moving each particle initially located at the point z according to the trajectory $x_z(t)$, and the other one obtained by moving it according to $\tilde{x}_z(t)$. We thus expect that the densities u and $\tilde{u}$ associated to these two evolutions give different solutions of (pde) with the same initial datum.

Now, this would certainly be the case if our notion of solution included measurevalued solutions, that is, if we allowed the particle density at time t to be represented by a measure instead of a function. But since by solutions we mean functions, and sometimes even bounded functions, we quickly realize that to make the previous constructions work we need some additional assumptions.

First of all we need an initial distribution of particles with positive total mass whose density is a function and not a measure, and therefore we must assume that the non-uniqueness set N has positive measure.

Secondly, we need that at every time $t > 0$ the densities of the two distributions considered above are functions and not measures, which is obtained by assuming that the families of trajectories $\{x_z\}$ and $\{\tilde{x}_z\}$ do not "concentrate", where non-concentration (for $\{x_z\}$) means that for every set E with positive measure contained in N and every $t > 0$, the set $E_t := \{x_z(t): z \in E\}$ has positive measure. (This is the weakest notion of non-concentration: to makes sure that the solutions u and $\tilde{u}$ constructed above are bounded functions, and not just functions, one has to impose some explicit lower bound for the measure of E_t, such as $meas(E_t) \geq m\, meas(E)$ for some positive constant m.)

The argument We have just presented has been made rigorous by Ambrosio in [34] using a suitable weak notion of flow a regular Lagrangian flow associated to a vector field b on $\mathbb{R}^n$ is a family of maps $\Phi_t: \mathbb{R}^n \to \mathbb{R}^n$ parametrized by time t such that

(i) $t \longmapsto \Phi_t(x)$ solves (ode) for almost every $x \in \mathbb{R}^n$,

(ii) there exists a positive constant m such that $meas\big(\Phi_t(E)\big) \geq m\, meas(E)$ for every set E and every time t (non-concentration).

Two Lagrangian flows are said to be equivalent if they agree for almost every x and every t, and, as shown in [34], the existence of two non-equivalent regular Lagrangian flows implies non-uniqueness of bounded solutions for (pde). In particular, the uniqueness result for (pde) in [41] and [34] imply the uniqueness of regular Lagrangian flows up to equivalence.

For more details on the connection between (pde) and flows for (ode), and for a review of related uniqueness results I refer the reader to [38], [35].

The uniqueness of regular Lagrangian flows (up to equivalence) can be loosely interpreted as uniqueness for (ode) for almost every initial position. However, these two conditions are not equivalent: while the latter clearly implies the former (because of assumption (i) in the definition of regular Lagrangian flow), the converse is not true (essentially because for certain vector fields b there exist flows that satisfy condition (i) but not (ii)).

In particular, it is not know whether the uniqueness results for (pde) in [41] and [34] imply uniqueness for (ode) for almost every initial position.

We are thus led to the following question, which is still open: Is there a continuous vector field b on $\mathbb{R}^n$ with bounded divergence and of class $W^{1,p}$ for some $p \geq 1$ (that is, a vector field to which the uniqueness result in [41] applies) such that the non-uniqueness set N has positive measure?

We restrict our attention to vector fields b on $\mathbb{R}^n$ that are bounded and divergence-free. Under these assumptions there exists a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$, called potential of b, such that

$$b = (\nabla f)^{\perp} \tag{5}$$

where $v^{\perp}$ stands for the rotation of the vector v by ninety degrees counterclockwise (f exists because the rotation of b by ninety degrees clockwise is curl-free).

In [30] it is proved that the vector fields b such that there is uniqueness for the corresponding continuity equation (pde) can be characterized in terms of the critical set of the potential f.
In view of the mechanical interpretation of (pde) given at the beginning, we can rephrase the first step of this proof as follows: a particle that belongs to some level set $f^{-1}(y)$ at time 0, remains for all subsequent times in the same level set, and in the same connected component of the same level sets. This is not surprising because b is orthogonal to ∇f and therefore tangent to the level sets of f at almost every point.
It follows that solving (pde) is equivalent to solve a partial differential equation similar to (pde) on every nontrivial connected component E of a generic level set $f^{-1}(y)$ (here "nontrivial" means "containing more than one point"; "generic" means "for almost every y").
Moreover a nontrivial connected component E of a generic level set is a simple rectifiable curve (see [31]) and therefore uniqueness for (pde) reduces to uniqueness for a family of variants of the continuity equation in one space dimension. It turns out that uniqueness for these one-dimensional continuity equations is strictly related to the intersection of the connected component E and the set of critical points

$$S := \{x : \nabla f(x) = 0\}.$$

In particular, if a generic level set of f does not contains critical points (that is, if f has the Sard property) then there is uniqueness for all these onedimensional equations, and therefore also for the original two-dimensional equation (pde).

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function of class $W^{2,p}$ and with compact support, and let V be the set of all values $y \in \mathbb{R}$ such that there exists a nontrivial connected component E_y of the level set $f^{-1}(y)$ which contains one and only one critical point of f, denoted by x_y. Finally let b be the vector field with potential f, that is, the one defined by (5), and let N be the non-uniqueness set associated to b.
We claim that if the set V has positive measure then the set N has positive measure, and therefore the answer to the question is negative.
Let me argue in favour of this claim. We first recall that for almost every $y \in \mathbb{R}$ the set E_y is a rectifiable, simple, closed curve, and We observe that

(i) a particle that moves along E_y reaches x_y in finite time;
(ii) after the particle has reached the critical point x_y it can stay there for any given amount of time and then start moving again.

Statement (ii) is essentially a consequence of statement (i) (applied with reversed time) and of the fact that b vanishes in x_y. To prove statement (i), note that the time T_y taken by the particle to go all the way through the curve E_y is

$$T_y = \int_{E_y} \frac{1}{|b|} = \int_{E_y} \frac{1}{|\nabla f|} \le \int_{f^{-1}(y)} \frac{1}{|\nabla f|},$$

and therefore

$$\int_V T_y dy \le \int_{-\infty}^{+\infty} \left[\int_{f^{-1}(y)} \frac{1}{|\nabla f|} \right] dy \le \text{meas}(\text{supp}(f)) < +\infty$$

(the second inequality follows by the coarea formula and the fact that $f^{-1}(y)$ is contained in the support of f for all $y \neq 0$; the last inequality is due to the fact that the support of f is assumed to be compact, and therefore it has finite measure).
Hence T_y is finite for almost every $y \in V$, which implies statement (i). Now notice that statements (i) and (ii) together imply that for every point z contained in E_y with $y \in V$ there

are infinitely many solutions of (ode) with initial datum z, and therefore E_y is contained in the non-uniqueness set N of the vector field b. Finally, the coarea formula and the fact that V has positive measure imply that the union of all E_y with $y \in V$, and therefore also N, are sets of positive measure in the plane.

The fact that the set V in the previous construction has positive measure implies that the function f does not have the Sard property. When we started working on these problems it was only known that Sard theorem holds for functions $f: \mathbb{R}^2 \to \mathbb{R}$ of class $W^{2,p}$ with $p > 2$ but nothing was known for $p \leq 2$. So we looked for a counterexample, with the hope that it would eventually lead to a negative answer to the question raised We found out in the end that there are no counterexamples, and that Sard theorem holds for all $p \geq 1$.

Given a function $f: \mathbb{R}^n \to \mathbb{R}^m$ with $m \leq n$, the critical set of f is

$$S := \{x: rank(\nabla f(x)) < m\}$$

We say that f has the Sard property if $f(S)$ is negligible, that is, if a generic level set of f contains no critical points.

In the classical form (see [47]), Sard theorem states that if f is of class C^{n-m+1} then it has the Sard property. Note that the regularity exponent $n - m + 1$ is sharp: there exist maps of class C^{n-m} without the Sard property (see [48], [42]).

A more precise version of Sard theorem was given in [42]: given a map $f: \mathbb{R}^n \to \mathbb{R}^m$ of class C^k (without restrictions on n and m) and $h = 0, 1, \ldots$, then the set

$$S_h := \{x: rank(\nabla f(x)) \leq h\}. \tag{6}$$

is $\mathscr{H}^{h+\frac{n-h}{k}}$-negligible, where $\mathscr{H}^d$ denotes the d-dimensional Hausdorff measure. This result was later extended in [36] to maps of class $C^{k,\alpha}$.

Concerning Sobolev maps, L. De Pascale proved in [39] that continuous maps of class $W^{n-m+1,p}$ with $p > n > m$ have the Sard property. A simpler proof of this statement was later given in [43]. Note that the counterexamples mentioned before show that the differentiability exponent $n - m + 1$ is sharp. On the other hand, there are no examples showing that the bound $p > n$ on the summability exponent is optimal (and indeed it is not, as I am going to explain).

In the rest restrict for simplicity to the case $n = 2$ and $m = 1$, that is, to functions f on $\mathbb{R}^2$ to $\mathbb{R}$. (For $n = m$ Sard theorem is just a consequence of the area formula, and therefore the "interesting" cases are those with $n > m$; among these the case $n = 2$ and $m = 1$ is the simplest, and is also the one which is relevant to the construction explained.

In this case the critical set S agrees with the set S_0 of all points where the gradient ∇f vanishes, and the result by De Pascale states that a continuous function in $W^{2,p}$ with $p > 2$ has the Sard property. We give a detailed outline of the proof of this result, and then indicate how it can be extended to $W^{2,1}$.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuos function of class $W^{2,p}$ for some $p > 2$; we assume for simplicity that the singular set S_0 ha finite measure.

The starting point is the following estimate: for every ball $B = B(x, r)$ with center x and radius r there holds

$$osc(f, B) \lesssim r|\nabla f(x)| + r^2 \left(\fint_B |\nabla^2 f|^p \right)^{\frac{1}{p}}, \tag{7}$$

where $osc(f, B)$ stands for the oscillation of f over the set B (that is, the difference between the supremum and the infimum), the symbol $\lesssim$ means that the inequality holds up to some (universal) multiplicative factor, and the dashed integral stands for the average.

Since estimate (7) is scaling and translation invariant, it suffices to prove it when $B = B(0,1)$. Since $W^{2,p}$ embeds in L^∞, we can bound the oscillation of f by its $W^{2,p}$-norm (on B). Now recall that an equivalent norm on $W^{2,p}$ is given by the sum of the L^p -norm of $\nabla^2 f$ and any continuous seminorm ϕ on $W^{2,p}$ which does not vanishes on nontrivial affine functions, for example $\phi(f) := |f(0)| + |\nabla f(0)|$ (the equivalence with the usual norm of $W^{2,p}$ follows by a standard argument, see [49]). Thus

$$osc(f,B) \lesssim |f(0)| + |\nabla f(0)| + \|\nabla^2 f\|_{L^p(B)}. \tag{8}$$

Since $osc(f,B)$ is invariant under the addition of a constant to f, we can assume $f(0) = 0$ and drop the first addendum on the right-hand side of this inequality, and so we finally obtain (7).

Note that if x belongs to S_0 then $\nabla f(x) = 0$ and (7) becomes

$$osc(f,B) \lesssim r^{2-\frac{2}{p}} \left(\int_B |\nabla^2 f|^p \right)^{\frac{1}{p}}. \tag{9}$$

We now choose an open set A that contains S_0, and cover S_0 with a collections of balls $B_i = B(x_i, r_i)$ such that $x_i \in S_0$ and $B_i \subset A$. Thus the sets $f(B_i)$ cover the set $f(S_0)$, and we can use this cover to estimate the measure of $f(S0)$:

$$meas(f(S_0)) \le \sum_i meas(f(B_i)).$$

Since the measure of the set $f(B_i)$ is less than its diameter, which is $osc(f, B_i)$, using (9) we get

$$meas(f(S_0)) \lesssim \sum_i r_i^{2-\frac{2}{p}} \left(\int_{B_i} |\nabla^2 f|^p \right)^{\frac{1}{p}} \le \left(\sum_i r_i^2 \right)^{1-\frac{1}{p}} \left(\sum_i \int_{B_i} |\nabla^2 f|^p \right)^{\frac{1}{p}}$$

$$\lesssim meas(A)^{1-\frac{1}{p}} \left(\int_A |\nabla^2 f|^p \right)^{\frac{1}{p}}, \tag{10}$$

where the second inequality follows by applying Hölder inequality in the form $\sum a_i b_i \le \left(\sum a_i^q\right)^{\frac{1}{q}} \left(\sum b_i^p\right)^{\frac{1}{p}}$, and the third one holds provided that the balls B_i do not overlap too much—a property that can be obtained by the Besicovitch covering theorem.

To conclude the proof, note that we can choose the open set A so that meas(A) is arbitrarily close to $meas(S_0)$, which is finite, while $\int_A |\nabla^2 f|^p$ is arbitrarily close to $\int_{S_0} |\nabla_2 f|^p$, which is null because $\nabla f = 0$ on S_0 implies $\nabla^2 f = 0$ a.e. on S_0.

All versions of Sard theorem We mentioned so far apply to classes of maps that are differentiable at every point, and for which, consequently, the definition of critical set carries no ambiguity. However for $1 \le p \le 2$ the space $W^{2,p}(\mathbb{R}^2)$ embeds in C^0 but not in C^1, and therefore a function f in this space admits a continuous representative which in general is differentiable almost everywhere but not everywhere. Thus for such f we should consider two sets:

$$S_0 := \{x : f \text{ is differentiable at } x \text{ and } \nabla f(x) = 0\},$$
$$N := \{x : f \text{ is not differentiable at } x\}. \tag{11}$$

It turns out that Sard theorem holds in the strongest form (see [33], [37]): if f is a continuous function of class $W^{2,1}$ then $f(S_0 \cup N)$ is negligible.

The only information readily available on the set N is that it cannot be too large, and $\mathcal{H}^1(N) = 0$. Therefore we could obtain that $f(N)$ is negligible if we knew that for every set E in $\mathbb{R}^2$

$$\mathcal{H}^1(E) = 0 \Rightarrow \mathcal{H}^1\big(f(E)\big) = 0. \tag{12}$$

This is exactly a particular case of the generalized N-property.

We show how to adapt the proof to obtain that $f(S_0)$ is negligible, too. First of all, notice that this proof, as it is, does not work. The point is that we no longer have estimate (7), because for $p \leq 2$ the space $W^{2,p}$ does not embeds in C^1, and therefore the value of ∇f at a given point x is not well-defined.

The idea is to replace the term $|\nabla f(x)|$ in (7) with

$$\int_B |\nabla f| d\mu_B$$

where μ_B is a probability measure supported on B that belongs to the dual of $W^{1,1}$, in the sense that $u \longmapsto \int u \, d\mu_B$ is a well-defined bounded functional on $W^{1,1}$, and therefore $u \longmapsto \int |u| d\mu_B$ is a well-defined continuous seminorm on $W^{1,1}$ (for more details on measures in the dual of $W^{1,1}$ see [49]). Then we have the following variant of (7):

$$osc(f, B) \lesssim r \int_B |\nabla f| d\mu_B + r^2 - \!\!\!\!\!\!\int_B |\nabla^2 f|, \tag{13}$$

Let now S' be the set of all $x \in S_0$ with the following property: there exists a sequence of balls $B = B(x, r_i)$ with $r_i \to 0$ such that on each of these balls we can find a measure μ_B as above, supported on $S_0 \cap B$.

With this choice of μ_B the first integral at the right-hand side of (13) vanishes, and therefore we get once again estimate (9) (with $p = 1$). We can now repeat the rest of the proof as it is, and obtain that $f(S')$ is negligible. It remains to show that $f(S_0 \backslash S')$ is negligible. We obtain this using (12) and

$$\mathcal{H}^1(S_0 \backslash S') = 0. \tag{14}$$

To prove (14), we first need to understand when a point x belongs to S', which in turn implies understanding when the set $S_0 \cap B(x, r)$ can support a probability measure μ_B in the dual of $W^{1,1}$ and how small the dual norm of this measure can be.

So, when does a set E in $\mathbb{R}^2$ support a probability measure μ in the dual of $W^{1,1}$? Intuitively, a necessary condition should be that the set E has positive $W^{1,1}$ -capacity, or, equivalently, that $\mathcal{H}^1(E) > 0$. It turns out that a sufficient condition is that $\mathcal{H}^1_\infty(E) > 0$, where $\mathcal{H}^1_\varepsilon$ are the Hausdorff pre-measures that appear in the definition of Hausdorff measures (see [49]).

Using this sufficient condition we obtain that x belongs to S' if

$$\limsup_{r \to 0} \frac{\mathcal{H}^1_\infty\big(S_0 \cap B(x, r)\big)}{r} \geq 1/2\,, \tag{15}$$

and therefore for all $x \in S_0 \backslash S'$ the limsup in (15) is necessarily strictly smaller than 1, which implies that

$$\limsup_{r \to 0} \frac{\mathcal{H}^1_\infty(S_0 \backslash S') \cap B(x, r)}{r} < 1. \tag{16}$$

The last step of the proof consists in showing that (16) implies (14).

In [33] we prove the following Take n, k, and p so that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ embeds in C^0 (that is, $kp > n$ or $p = 1$ and $k = n$), let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map of class $W^{k,p}$, and define the sets S_0 and N as in (11). Then

$$\mathcal{H}^{\frac{n}{k}}\big(f(S_0 \cup N)\big) = 0. \tag{17}$$

This result is optimal, in the sense that

(i) the dimension n/k in (17) cannot be lowered;

(ii) if n, k, and p do not satisfy the condition above, then there are maps f on $\mathbb{R}^n$ of class $W^{k,p} \cap C^{k-1}$ for which the Hausdorff dimension of $f(S_0)$ is strictly larger than n/k, and in particular (17) fails.

To obtain the optimal statement of Sard theorem we should then prove similar estimates for the sets S_h defined in (6).

A map $f: \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq n$ has the Lusin N-property if the following implication holds for every set E contained in $\mathbb{R}^n$:

$$\mathcal{H}^n(E) = 0 \Rightarrow \mathcal{H}^n\big(f(E)\big) = 0\,.$$

This property has been widely studied in the past years, mostly in relation to the area formula. Indeed, the following statement holds: let f be a map which is differentiable (in the approximate sense) at almost every point and has the Nproperty; then the area formula holds, that is

$$\int_{y\in\mathbb{R}^m} \left[\sum_{x\in f^{-1}(y)\cap E} \varphi(x)\right] d\mathcal{H}^n(y) = \int_{x\in E} \varphi(x) J_f(x) d\mathcal{H}^n(x) \tag{18}$$

where φ is any positive Borel function on $\mathbb{R}^n$, E is any Borel subset of $\mathbb{R}^n$, and J_f is the Jacobian of f (defined at every point where f is differentiable).

The proof of this statement is elementary: since f is a.e. differentiable, it has the Lusin approximation property with Lipschitz maps, that is, there exist a sequence of Borel sets F_i and of Lipschitz maps f_i such that the sets F_i cover almost all of $\mathbb{R}^n$ and $f = f_i$ on F_i (see [42]). Using the area formula for Lipschitz maps (see [42]) we obtain that (18) holds when E is contained in the union of all F_i. It remains to show that (18) holds when E is contained in the complement of the union of all F_i. Since E is $\mathcal{H}^n$-negligible, the integral at right-hand side of (18) vanishes, and to prove that also the integral at the left-hand side vanishes it suffices to show that $f(E)$ is $\mathcal{H}^n$-negligible, which is precisely what the N-property says.

Concerning Sobolev maps, a continuous map $f: \mathbb{R}^n \to \mathbb{R}^m$ of class $W^{1,p}$ has the N-property if $p > n$ (see [45]) and this bound on the summability exponent is sharp (however, homeomorphisms of class $W^{1,n}$ also have the N-property; for this and other results on the N-property see [44]).

We focus on a generalization of the N-property that naturally arises when dealing with the Sard theorem for Sobolev maps.

Given a map f between metric spaces and positive numbers α, β, we say that f has the (α, β)-N-property if the following implication holds for every set E contained in the domain of f:

$$\mathcal{H}^\alpha(E) = 0 \Rightarrow \mathcal{H}^\beta\big(f(E)\big) = 0.$$

It follows from elementary facts that a Lipschitz map has the (α, α)-N-property for every $\alpha > 0$ and, more generally, an Hölder map with exponent γ has the $(\alpha, \alpha/\gamma) - N$-property for every $\alpha > 0$.

Concerning Sobolev maps, in [32] we prove the following: Take n, k, and p so that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ embeds in C^0 (that is, $kp > n$ or $p = 1$ and $k = n$), and let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map of class $W^{k,p}$. Then

(i) f has the (α, β)-N-property with $\beta := \frac{\alpha p}{kp+\alpha-n}$ for $\alpha < n - (k-1)p$;

(ii) f has the (α, α)-N-property for $\alpha > n - (k-1)p$.

Moreover this result is sharp, in the sense that

(iii) the value of β in (i) cannot be lowered;

(iv) if we take n, k, and p so that the Sobolev space $W^{k,p}(\mathbb{R}^n)$ does not embed in C^0, then there are continuous maps $f: \mathbb{R}^n \to \mathbb{R}^m$ of class $W^{k,p}$ that do not have the (α, β)-N-property for any $\alpha > 0$ and $\beta \leq m$; in other words, these maps take some sets of dimension arbitrarily close to 0 into sets of dimension m.

We have two different methods for proving statements (i) and (ii) above. Even though the proof can be achieved by either methods for most k, p, α, β in the range where the N-property holds, yet neither approach covers all cases (or so it seems).

Let me illustrate the first method in the case of the $(1,1)$-N-property for maps $f: \mathbb{R}^2 \to \mathbb{R}^m$ of class $W^{2,1}$. The starting point is the following estimate (the proof is essentially the same as that of estimates (7) and (13)): for every ball $B = B(x, r)$ there holds

$$osc(f, B) \lesssim r \fint_B |\nabla f| + r^2 \fint_B |\nabla^2 f|. \tag{19}$$

We now fix a set E with $\mathcal{H}^1(E) = 0$ and, given $\varepsilon > 0$, we choose a family of balls $B_i = B(x_i, r_i)$ which cover E and satisfy $\sum r_i \leq \varepsilon$. Then the sets $f(B_i)$ cover $f(E)$, and we use this cover to estimate the Hausdorff measure of $f(E)$:

$$\mathcal{H}^1(f(E)) \leq \sum_i diam(f(B_i)).$$

Since the diameter of $f(B_i)$ agrees with the oscillation of f on B_i, using (19) we obtain

$$\mathcal{H}^1(f(E)) \lesssim \sum_i \frac{1}{r_i} \int_{B_i} |\nabla f| + \sum_i \int_{B_i} |\nabla^2 f|. \tag{20}$$

We want to show that both sums at the right-hand side of (20) tends to 0 as ε tend to 0 (provided the covers $\{B_i\}$ are suitably chosen).

If the balls B_i do not overlap too much (and this can be obtained by Besicovitch covering lemma) we can estimate the second sum by the integral of $|\nabla^2 f|$ over the union A of the balls B_i, and since the area of A tends to 0 as $\varepsilon \to 0$, the same happens to the integral.

The difficult part is to handle the first sum. First of all we write it as $\int |\nabla f| d\mu$ where μ is given by the Lebesgue measure multiplied by the density

$$\rho := \sum_i \frac{1}{r_i} 1_{B_i},$$

and then we show that μ belongs to the dual of $W^{1,1}(\mathbb{R}^2)$ in the sense of [49] (the key step is to prove that $\mu(B) \lesssim r$ for every ball $B = B(x, r)$). Then the proof is concluded by a careful estimate of the norm of this measure as element of the dual of $W^{1,1}(\mathbb{R}^2)$.

Concerning the second method, let me just hint that it is related to estimates for the local Hölder exponent of Sobolev maps. The simplest version of such estimates reads as follows: if α is a real number with $0 < \alpha \leq n$ and $f: \mathbb{R}^n \to \mathbb{R}^m$ is a continuous map of class $W^{1,p}$ with $p > n$, then for $\mathcal{H}^\alpha$-almost every $x \in \mathbb{R}^n$ and every ball $B = B(x, r)$ there holds

$$osc(f,B) \lesssim r\left(⨍_B |\nabla f|^p\right)^{\frac{1}{p}} = O(r^\gamma) \quad with \;\; \gamma := \frac{p+\alpha-n}{p}. \qquad (21)$$

The inequality in (21) can be proved in the same way as estimate (7), and the equality is obtained by applying the following elementary statement with $g := |\nabla f|^p$:given a positive function g in $L^1(\mathbb{R}^n)$ and $0 < \alpha \le n$, for $\mathcal{H}^\alpha$-almost every $x \in \mathbb{R}^n$ and every ball $B = B(x,r)$ there holds

$$\int_B g = O(r^\alpha)$$

(the estimate applies in the regime $r \to 0$, and it is clearly not uniform in x).

Now, estimate (21) says more or less that we can find a sequence of sets such that the restriction of f to each of these sets is Hölder continuous of exponent γ, and the sets cover $\mathbb{R}^n$ except for a residual set which is $\mathcal{H}^\alpha$-negligible. If we neglect this residual set, we immediately obtain that f has the $\left(\alpha, \frac{\alpha}{\gamma}\right)$-N-property, and α/γ is exactly the value of β in statement (i) for $k = 1$.

Section (2.2): The Sharp Case of Sobolev Mappings

The Morse–Sard theorem is a fundamental result with many applications. In its classical form it states that the image of the set of critical points of a C^{n-m+1} smooth mapping $v : \mathbb{R}^n \to \mathbb{R}^m$ has zero Lebesgue measure in $\mathbb{R}^m$. Assuming that $n \ge m$ the set of critical points for v is $Z_v = \{x \in \mathbb{R}^n : rank\nabla v(x) < m\}$ and the conclusion is that

$$L^m(v(Z_v)) = 0. \qquad (43)$$

The theorem was proved by Morse [69] in the case $m = 1$ and subsequently by Sard [47] in the general case. It is well–known since the work of Whitney [48] that the C^{n-m+1} smoothness assumption on the mapping v cannot be weakened to C^j smoothness with j less than $n - m + 1$. While this is so Dubovitskiĭ [59] obtained results on the structure of level sets for C^j mappings v including the cases where j is smaller than $n - m + 1$ (also see [53]). An important generalization of the Morse–Sard theorem is the following result that we display as it, together with the classical result, forms the starting point for our investigations here.

Theorem (2.2.1)[50]: (Federer [61]). Let $m \in \{1, \dots, n\}, d, k \in \mathbb{N}$, and let $v : \mathbb{R}^n \to \mathbb{R}^d$ be a C^k–smooth mapping. Denote $q_\circ = m - 1 + \frac{n-m+1}{k}$. Then

$$H^{q_\circ}\left(v\left(Z_{v,m}\right)\right) = 0, \qquad (44)$$

where H^β denotes the β–dimensional Hausdorff measure and $Z_{v,m}$ denotes the set of m–critical points of v: $Z_{v,m} = \{x \in \mathbb{R}^n : rank\nabla v(x) < m\}$.

The Morse–Sard–Federer results have subsequently been generalized to mappings in more refined scales of spaces, including Hölder and Sobolev spaces. For H ölder spaces we mention ¨ in particular [36], [53], [68], [70], [78] where essentially sharp results were obtained, including examples showing that the smoothness assumption on v in Federer's theorem cannot be weakened within the scale of C^j spaces. However, it follows from [36] that the conclusion (44) remains valid for $C^{k-1,1}$ mappings v, and according to [68] it fails in general for $C^{k-1,\alpha}$ mappings whenever $\alpha < 1$. (For $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$ we say that the mapping v is of class $C^{k,\alpha}$ when v is C^k and the k–th order derivative of v is locally α–Hölder continuous.) One interpretation of ¨ these results is that for the validity of (44) one must assume existence of k derivatives of v in a suitably strong sense. At a heuristic level the general problem is then to prove analogs of the Morse–Sard–Federer results where we replace the assumption that the mapping is k times continuously differentiable by a corresponding Sobolev assumption: v has weak derivatives up to and including order k and these weak derivatives must satisfy a suitable integrability condition. The aforementioned examples show that we cannot in general reduce the degree k of differentiability. The question we wish to address here concerns the optimal local integrability condition that the k–th order weak derivative must satisfy for the validity of (44). Previous works on the Morse–Sard property of Sobolev spaces include [53], [57], [39], [43], [63], [71], [76], [77], [37], [56]. The first Morse–Sard result in the Sobolev context that we are aware of is [39]. It states that (43) holds for mappings $v \in W^k_{p,loc}(\mathbb{R}^n, \mathbb{R}^m)$ when $k \geq max(n-m+1, 2)$ and $p > n$. Note that by the Sobolev embedding theorem any mapping on $\mathbb{R}^n$ which is locally of Sobolev class W^k_p for some $p > n$ is in particular C^{k-1}, so the critical set Z_v can be defined as usual. When in the scalar case $m = 1$ we consider functions in $W^n_{p,loc}(\mathbb{R}^n)$ with $p \in [1, n]$ we are in general only assured everywhere continuity whereas the differentiability can fail at some points. Hence for such functions one must adapt the definition of critical set accordingly. We define the sets $A_v := \{x \in \mathbb{R}^n : v$ is not differentiable at $x\}$ and $Z_v := \{x \in \mathbb{R}^n \setminus A_v : \nabla v(x) = 0\}$. In these terms the results of [37], [56] imply that (43) holds with $m = 1$ for all $v \in W^n_{1,loc}(\mathbb{R}^n)$ and that also $L^1(v(A_v)) = 0$. The latter is a consequence of a more general Luzin N property with respect to one–dimensional Hausdorff content that $W^n_{1,loc}$ functions are shown to enjoy. In fact the results of [37], [56] even yield (43) with $m = 1$ and an appropriate definition of the critical set, and the Luzin N property within the more general framework of functions of bounded variation $BV_{n,loc}(\mathbb{R}^n)$.

We shall be concerned with the vectorial case $m > 1$.It is very natural to assume, that the inclusion $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ should guarantee at least the continuity of v. For values $k \in \{1, \dots, n-1\}$ it is well–known that $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is continuous for $p > \frac{n}{k}$ and could be discontinuous for $\leq \frac{n}{k}$. So the borderline case is $= p_\circ = \frac{n}{k}$. It is well–known (see for instance [62]) that really $v \in W^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$ is continuous if the derivatives of k-th order belong to the Lorentz space $L_{p_\circ,1}$, we will denote the space of such mappings by $W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$.

We prove the precise analog of the above Federer's theorem for mappings $v : \mathbb{R}^n \to \mathbb{R}^d$ locally of class $W^k_{p_\circ,1}$, $k \in \{2, \dots, n\}, m \in \{2, \dots, n\}$ (the case $k = 1$, and, consequently, $q_\circ = n$, was considered in [62], It is easy to see (using well–known results such as [58]) that such a function is (Frechet–)differentiable ´ $H^{q_\circ}$ –almost everywhere, where $q_\circ = m - 1 + \frac{n-m+1}{k}$ is the same as in above Federer's theorem. The critical set $Z_{v,m}$ is defined as the set of points x, where v is differentiable and rank$\nabla v(x) < m$. As our main result we prove that $H^{q_\circ}(v(Z_{v,m})) = 0$. In fact, the result in Theorem (2.2.18) is slightly more general and concerns mappings locally of Sobolev class $W^k_{p_\circ}$.

We also establish a related Luzin N property with respect to Hausdorff content in Theorem (2.2.14). When the mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ is of class $W^k_{p_\circ,1}$ we find for any $\varepsilon > 0$ a $\delta > 0$ such that for all subsets E of $\mathbb{R}^n$ with $H^{q_\circ}_\infty(E) < \delta$ we have $H^{q_\circ}_\infty(v(E)) < \varepsilon$. Here $H^{q_\circ}_\infty$ is the $q_\circ$–dimensional Hausdorff content. In particular, it follows that $H^{q_\circ}(v(E)) = 0$whenever $H^{q_\circ}(E) = 0$. So the image of the exceptional "bad" set, where the differential is not defined,

has zero q°–dimensional Hausdorff measure. This ties nicely with our definition of the critical set and our version of the Federer result.
Finally, using these results we prove that if $v \in W^{k}_{p^{\circ},1}(\mathbb{R}^n, \mathbb{R}^m)$ with $k = n - m + 1$ then for L^m–almost all $y \in \mathbb{R}^m$ the preimage $v^{-1}(y)$ is a finite disjoint union of C^1–smooth compact manifolds of dimension $n - m$ without boundary.
The results are in particular valid for functions v from the classical Sobolev spaces $W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ with $p > p_{\circ} = \frac{n}{k}$.
We emphasize again that the similar results were proved for $k = 1$ (i.e., $q_{\circ} = n$ for any $\in \{1, \ldots, n\}$) in [62] and for $m = 1, k = n$ in [37], [56]. We do not prove the analogs of Federer's theorem for the cases $k > n$ or $m = 1, k < n$. In fact, these cases remain open. While we have formulated all our results of euclidean spaces it is clear that the results are local and hence could, with the appropriate modifications, be formulated for Sobolev mappings between smooth Riemannian manifolds instead.
Our proofs rely on the results of [67] on advanced versions of Sobolev imbedding theorems of [51] on Choquet integrals of Hardy-Littlewood maximal functions with respect to Hausdorff content, and of [78] on the entropy estimate of near–critical values of differentiable functions. The key step in the proof of the Morse–Sard–Federer Theorem (2.2.18) is contained in Lemma (2.2.19), and it expands on a similar argument used in [56].

By an n–dimensional interval we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If I is an n–dimensional interval then we write $\ell(I)$ for its sidelength.
For a subset S of $\mathbb{R}^n$ we write $L^n(S)$ for its outer Lebesgue measure. The m–dimensional Hausdorff measure is denoted by H^m and the m–dimensional Hausdorff content by H^m_∞. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$H^m(S) = \lim_{\alpha \searrow 0} H^m_\alpha(S) = \sup_{\alpha > 0} H^m_\alpha(S),$$

where for each $0 < \alpha \leq \infty$,

$$H^m_\alpha(S) = \inf\{\sum_{i=1}^{\infty} (diam\, S_i)^m \; : diam\, S_i \leq \alpha, S \subset \bigcup_{i=1}^{\infty} S_i \; .$$

It is well known that $H^n(S) \sim H^n_\infty(S) \sim L^n(S)$ for sets $S \subset \mathbb{R}^n$.
To simplify the notation, we write $\|f\|_{L_p}$ instead of $\|f\|_{L_p(\mathbb{R}^n)}$, etc.
The Sobolev space $W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$ -valued functions $f \in L_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \leq k$ belong to $L_p(\mathbb{R}^n)$ (for detailed definitions and differentiability properties of such functions see, e.g., [60], [79], [58]). Denote by $\nabla^k f$ the vector-valued function consisting of all k-th order partial derivatives of f arranged in some fixed order. However for the case of first order derivatives $k = 1$ we shall often think of $\nabla f(x)$ as the Jacobi matrix of f at x, i.e., the $d \times n$ matrix whose r-th row is the vector of partial derivatives of the r-th coordinate function.
We use the norm

$$\|f\|_{W^k_p} = \|f\|_{L_p} + \|\nabla f\|_{L_p} + \cdots + \left\|\nabla^k f\right\|_{L_p},$$

and unless otherwise specified all norms on the spaces $\mathbb{R}^s (s \in \mathbb{N})$ will be the usual euclidean norms. We state the following result, and only remark that it is well–known and follows from the definition of Sobolev spaces. In its statement we denote by $C^\infty_c(\mathbb{R}^n)$ the space of C^∞ smooth and compactly supported functions on $\mathbb{R}^n$.

Lemma (2.2.2)[50]: Let $f \in W_p^k(\mathbb{R}^n)$. Then for any $\varepsilon > 0$ there exist functions $f_0 \in C_c^\infty(\mathbb{R}^n)$ and $f_1 \in W_p^k(\mathbb{R}^n)$ such that $f = f_0 + f_1$ and $\|f_1\|_{W_p^k} < \varepsilon$.

Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,loc}(\Omega)$, then the precise representative w^* is defined by

$$w^*(x) = \begin{cases} \lim\limits_{r \searrow 0} \fint_{B(x,r)} w(z)\, dz, & if\ the\ limit\ exists\ and\ is\ finite, \\ 0 & otherwise, \end{cases} \tag{45}$$

where the dashed integral as usual denotes the integral mean,

$$\fint_{B(x,r)} w(z) dz = \frac{1}{L^n\big(B(x,r)\big)} \int_{B(x,r)} w(z)\, dz\ ,$$

and $B(x,r) = \{y : |y - x| < r\}$ is the open ball of radius r centered at x. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

We will say that x is an L_p Lebesgue point of w (and simply a Lebesgue point when $p = 1$), if

$$\fint_{B(x,r)} |w(z) - w(x)|^p dz \to 0 \text{ as } r \searrow 0.$$

If $k < n$, then it is well-known that functions from Sobolev spaces $W_p^k(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \le p_\circ = \frac{n}{k}$ (see, e.g., [67], [79]). The Sobolev–Lorentz space $W_{p_\circ,1}^k(\mathbb{R}^n) \subset W_{p_\circ}^k(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space that for our purposes turns out to be convenient. Among other things functions that are locally in $W_{p_\circ,1}^k$ on $\mathbb{R}^n$ are in particular continuous.

Given a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, denote by $f_* : (0,\infty) \to \mathbb{R}$ its distribution function

$$f_*(s) := L^n\{x \in \mathbb{R}^n : |f(x)| > s\},$$

and by f^* the nonincreasing rearrangement of f, defined for $t > 0$ by

$$f^*(t) = inf\{s \ge 0 : f_*(s) \le t\}.$$

Since f and f^* are equimeasurable we have for every $1 \le p < \infty$,

$$\left(\int_{(\mathbb{R}^n)} |f(x)|^p dx\right)^{1/p} = \left(\int_0^{+\infty} f^*(t)^p dt\right)^{1/p}.$$

The Lorentz space $L_{p,q}(\mathbb{R}^n)$ for $1 \le p < \infty, 1 \le q < \infty$ can be defined as the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which the expresssion

$$\|f\|_{L_{p,q}} = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^{+\infty} \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} & if\ 1 \le q < \infty \\ \sup\limits_{t>0} t^{1/p} f^*(t) & if\ q = \infty \end{cases}$$

is finite. See [65], [74] or [79] for information about Lorentz spaces. However, let us remark that in view of the definition of $\|\cdot\|_{L_{p,q}}$ and the equimeasurability of f and f^* we have an identity $\|f\|_{L_p} = \|f\|_{L_{p,p}}$ so that in particular $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. Further, for a fixed exponent p and $q_1 < q_2$ we have an estimate $\|f\|_{L_{p,q_2}} \le \|f\|_{L_{p,q_1}}$, and, consequently, an embedding $L_{p,q_1}(\mathbb{R}^n) \subset L_{p,q_2}(\mathbb{R}^n)$ (see [65]). Finally we recall that $\|\cdot\|_{L_{p,q}}$ is a norm on $L_{p,q}(\mathbb{R}^n)$ for all $q \in [1,p]$ (see [65]).

Here we shall mainly be concerned with the Lorentz space $L_{p,1,}$ and in this case one may rewrite the norm as (see [65])

$$\|f\|_{p,1} = \int_0^{+\infty} [L^n(\{x \in \mathbb{R}^n : |f(x)| > t\})]^{\frac{1}{p}} \, dt. \tag{46}$$

We need the following subadditivity property of the Lorentz norm.

Lemma (2.2.3) (see, e.g., [72] or [65])[50]: Suppose that $1 \le p < \infty$ and $= \bigcup_{j\in\mathbb{N}} E_j$, where E_j are measurable and mutually disjoint subsets of $\mathbb{R}^n$. Then for all $f \in L_{p,1}$ we have

$$\sum_j \left\| f \cdot 1_{E_j} \right\|_{L_{p,1}}^p \le \|f \cdot 1_E\|_{L_{p,1}}^p ,$$

where 1_E denotes the indicator function of E.

Denote by $W_{p,1}^k(\mathbb{R}^n)$ the space of all functions $v \in W_p^k(\mathbb{R}^n)$ such that in addition the Lorentz norm $\left\|\nabla v^k\right\|_{L_{p,1}}$ is finite. For given dimensions $n, m, \in \mathbb{N}, 1 \le m \le n$, and $k \in \{1, \dots, n\}$, we denote the corresponding critical exponents by

$$p_\circ = \frac{n}{k} \text{ and } q_\circ = m - 1 + \frac{n-m+1}{k} = p_\circ + (m-1)(1-k^{-1}) . \tag{47}$$

By direct calculation, from $m \ge 1, k \ge 1$ we find

$$p_\circ \le q_\circ \le n. \tag{48}$$

Note that in the double inequality (48) we have equality in the first inequality iff $m = 1$ or $k = 1$, while in the second inequality equality holds iff $k = 1$. In particular,

$$p_\circ \le q_\circ < n \text{ for } k, m \in \{2, \dots, n\}. \tag{49}$$

For a mapping $u \in L_1(I, \mathbb{R}^d), I \subset \mathbb{R}^n$, define the polynomial $P_I[u] = P_{I,k-1}[u]$ of degree at most $k-1$ by the following rule:

$$\int_I y^\alpha (u(y) - P_I[u](y)) \, dy = 0 \tag{50}$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k - 1$.

The following well–known bound will be used on several occasions.

Lemma (2.2.4)[50]: Suppose $v \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then v is a continuous mapping and for any ndimensional interval $I \subset \mathbb{R}^n$ the estimate

$$\sup_{y\in I} |v(y) - P_I[v](y)| \le C \left\|1_I \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \tag{51}$$

holds, where C is a constant depending on n, d only. Moreover, the mapping $v_I(y) = v(y) - P_I[v](y), y \in I$, can be extended from I to the whole of $\mathbb{R}^n$ such that the extension (denoted again) $v_I \in W_{p_\circ}^k(\mathbb{R}^n, \mathbb{R}^d)$ and

$$\left\|\nabla^k v_I\right\|_{L_{p_\circ}(\mathbb{R}^n)} \le C_0 \left\|\nabla^k v\right\|_{L_{p_\circ}(I)} , \tag{52}$$

where C_0 also depends on n, d only.

Proof. By well–known estimates (see for instance [58] or [65]) we have for any Lebesgue point $y \in I$ of v,

$$|v(y) - P_I[v](y)| \le C \int_I \frac{\left|\nabla^k v(x)\right|}{|y - x|^{(n-k)}} dx \le C \left\|1_I \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \cdot \left\| \frac{(1_I)}{|y - \cdot|^{n-k}} \right\|_{L_{\frac{n}{n-k},\infty}}$$

$$\le C' \left\|1_I \cdot \nabla^k v\right\|_{L_{p_\circ,1}} .$$

From this estimate the continuity of v follows in a routine manner, and thus (51) holds. Because of coordinate invariance of estimate (52), it is sufficient to prove the assertions about extension for the case when I is a unit cube: $I=[0,1]^n$. By results of [67] for any $u\in W^{k,p_\circ}(I)$ the estimate

$$\|u\|_{W^k_{p_\circ}(I)}\le c(\|P_I[u]\|_{L_1(I)}+\left\|\nabla^k u\right\|_{L_{p_\circ}(I)}, \tag{53}$$

holds, where $c=c(n,k)$ is a constant. Taking $u(y)=v_I(y)=v(y)-P_I[v](y)$, the first term on the right hand side of (53) vanishes and so we have

$$\|u_1\|_{W^k_{p_\circ}(I)}\le\ c\left\|\nabla^k u\right\|_{L_{p_\circ}(I)}. \tag{54}$$

By the Sobolev Extension Theorem, every function $u\in W^k_{p_\circ}(I)$ on the unit cube $I=[0,1]^n$ can be extended to a function $U\in W^k_{p_\circ}(\mathbb{R}^n)$ such that the estimate $\left\|\nabla^k U\right\|_{Lp_\circ(\mathbb{R}^n)}\le c\|u\|_{W^k_{p_\circ}(I)}$ holds, see [67]). Applying this result coordinatewise to $u=v_I$ and taking into account (54), we obtain the required estimate (52).

From Lemma (2.2.4) we deduce the following oscillation estimate.

Corollary (2.2.5)[50]: Suppose $v\in W^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. Then for any n-dimensional interval $I\subset\mathbb{R}^n$ the estimate

$$\begin{aligned}diam\, v(I)&\le C\left(\frac{\|\nabla v\|_{L_1(I)}}{\ell(I)^{n-1}}+\left\|1_I\cdot\nabla^k v\right\|_{L_{p_\circ,1}}\right)\\&\le C(\frac{\|\nabla v\|_{L_q(I)}}{\ell(I)^{\frac{n}{q}-1}}+\left\|1_I\cdot\ \nabla^k v\right\|_{L_{p_\circ,1}}\end{aligned} \tag{34}$$

holds for every $q\in[1,n]$, where C depends on n,k only.

Proof. Because of coordinate invariance of estimate (34)it is sufficient to prove the estimates for the case when I is a unit cube: $I=[0,1]^n$. But for a such fixed interval I the estimate follows from (51) and from the fact that the coefficients of the polynomial $P_I[u]$ depend continuously on u with respect to L_1-norm.

We need a version of the Sobolev Embedding Theorem that gives inclusions in Lebesgue spaces with respect to suitably general positive measures. Very general and precise statements are known, but here we restrict attention to the following class of measures. For $\beta\in(0,n)$ denote by M^β the space of all nonnegative Radon measures μ on $\mathbb{R}^n$ such that

$$|||\mu|||_\beta=\sup_{I\subset\mathbb{R}^n}\ell(I)^{-\beta}\mu(I)<\infty,$$

where the supremum is taken over all n-dimensional intervals $I\subset\mathbb{R}^n$.

Theorem (2.2.6) (see [67])[50]: Let μ be a positive Radon measure on $\mathbb{R}^n$ and $p(k-1)<n, 1\le p<q<\infty$. Then for any function $v\in W^k_p(\mathbb{R}^n)$ the estimate

$$\int|\nabla v|^q\,d\mu\ \le\ C|||\mu|||_\beta\ \cdot\ \left\|\nabla^k v\right\|^q_{L_p}, \tag{35}$$

holds with $\beta=(\frac{n}{p}-k+1)q$, where C depends on n,p,q,k.

We use also the following important strong-type estimate for maximal functions.

Theorem (2.2.7) (see Theorem A, Proposition 1 and it's Corollary in [51])[50]: Let $\beta\in(0,n)$. Then for nonnegative functions $f\in C_0(\mathbb{R}^n)$ the estimates

$$\begin{aligned}\int_0^\infty H^\beta_\infty(\{x\in\mathbb{R}^n:\ Mf(x)\ge t\})\,dt\ &\le C_1\int_0^\infty H^\beta_\infty(\{x\in\mathbb{R}^n: f(x)\ge t\})\,dt\\&\le C_2\sup\{\textstyle\int f\,d\mu:\ \mu\in M^\beta,|||\mu|||_\beta\le 1\},\end{aligned}$$

hold, where the constants C_1,C_2 depend on β,n only and

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|\, dy$$

is the usual Hardy-Littlewood maximal function of f.

Applying the two foregoing theorems for $p = p_\circ = \frac{n}{k}$, $q = \beta = q_\circ = m - 1 + \frac{n-m+1}{k}$, we obtain the first key ingredient of our proof.

Corollary (2.2.8)[50]: Let $m, k \in \{2, \dots, n\}$. Then for any function $v \in W^k_{p_\circ}(\mathbb{R}^n)$ the estimates

$$\|\nabla v\|^{q_\circ}_{L_{q_\circ}(\mu)} \le C|||\mu|||_{q_\circ} \left\|\nabla^k v\right\|^{q_\circ}_{L_{p_\circ}} \quad \forall \mu \in M^{q_\circ}, \tag{36}$$

$$\int_0^\infty H^\beta_\infty(\{x \in \mathbb{R}^n : M(|\nabla v|^{q_\circ})(x) \ge t\})\, dt \le C\left\|\nabla^k v\right\|^{q_\circ}_{L_{p_\circ}}. \tag{37}$$

hold, where the exponents $p_\circ, q_\circ$ are defined by (47) and the constant C depends on n, k, m only.

For a subset A of $\mathbb{R}^m$ and $\varepsilon > 0$ the ε–entropy of A, denoted by $\mathrm{Ent}(\varepsilon, A)$, is the minimal number of balls of radius ε covering A. Further, for a linear map $L: \mathbb{R}^n \to \mathbb{R}^d$ denote by $\lambda_j(L), j = 1, \dots, d$, the lengths of the semiaxes of the ellipsoid $L(B(0,1))$ ordered by the rule $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_d$. Obviously the numbers λ_j are exactly the eigenvalues repeated according to multiplicity of the symmetric nonnegative linear map $\sqrt{LL^*} : \mathbb{R}^d \to \mathbb{R}^d$. Also for a differentiable mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ put $\lambda_j(f, x) = \lambda_j(d_x f)$, where by $d_x f$ we denote the differential of f at x. The next result is the second basic ingredient of our proof .

Theorem (2.2.9) ([78])[50]: For any polynomial $P : \mathbb{R}^n \to \mathbb{R}^d$ of degree at most k, for each ball $B \subset \mathbb{R}^n$ of radius $r > 0$, and any number $\varepsilon > 0$ the estimate

$$Ent\ (\varepsilon r, \{P(x): x \in B, \lambda_1 \le 1 + \varepsilon, \dots, \lambda_{m-1} \le 1 + \varepsilon, \lambda_m \le \varepsilon, \dots, \lambda_d \le \varepsilon\}) \le C_Y(1 + \varepsilon^{1-m}) \ ,$$

holds, where the constant C_Y depends on n, d, k only and for brevity we wrote $\lambda_j = \lambda_j(P, x)$.

The application of Corollary (2.2.8) is facilitated through the following simple estimate (see for instance Lemma 2 in [58]).

Lemma (2.2.10)[50]: Let $u \in W^1_1(\mathbb{R}^n)$. Then for any ball $B(z, r) \subset \mathbb{R}^n, B(z, r) \ni x$, the estimate

$$\left|u(x) - \int_{(B(z,r))} u(y)dy\right| \le Cr(M\nabla u)(x).$$

holds, where C depends on n only and $M\nabla u$ is the Hardy-Littlewood maximal function of $|\nabla u|$.

By use of the triangle inequality we then deduce the following oscillation estimate (cf. [55]).

Corollary (2.2.11)[50]: Let $u \in W^1_1(\mathbb{R}^n, \mathbb{R}^d)$. Then for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and for any number $\varepsilon > 0$ the estimate

$$diam(\{u(x) : x \in B, (M\nabla u)(x) \le \varepsilon\}) \le C_M \varepsilon r$$

holds, where C_M is a constant depending on n, d only.

Finally, recall the following approximation properties of Sobolev functions.

Theorem (2.2.12)[50]: (see, [79] or [54]). Let $p \in (1, \infty), k, l \in \{1, \dots, n\}, l \le k, (k - l)p < n$. Then for any $f \in W^k_p(\mathbb{R}^n)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^n$ and a function $g \in C^l(\mathbb{R}^n)$ such that

(i) each point $x \in \mathbb{R}^n \setminus U$ is a Lebesgue point for f and for $\nabla^j f, j = 1, \dots, l$;

(ii) $f \equiv g, \nabla^j f \equiv \nabla^j g$ on $\mathbb{R}^n \setminus U$ for $j = 1, \dots, l$;

(iii) $L^n(U) < \varepsilon$ if $l = k$;
(iv) $B_{k-l,p}(U) < \varepsilon$ if $l < k$, where $B_{\alpha,p}(U)$ denotes the Bessel capacity of the set U.
Since for $1 < p < \infty$ and $0 < n - \alpha p < \beta \le n$ the smallness of $B_{\alpha,p}(U)$ implies the smallness of $H^{\beta}_{\infty}(U)$ (see, e.g., [52]), we have

Corollary (2.2.13)[50]: Let $k \in \{2, \dots, n\}$ and $v \in W^k_{p_\circ}(\mathbb{R}^n)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $H^q(A_v) = 0$ for every $q \in (p_\circ, n]$ and all points of $\mathbb{R}^n \setminus A_v$ are Lebesgue points for ∇v. Further, for every $\varepsilon > 0$ and $q \in (p_\circ, n]$ there exist an open set $U \supset A_v$ and a function $g \in C^1(\mathbb{R}^n)$ such that $H^q_\infty(U) < \varepsilon$ and $v \equiv g, \nabla v \equiv \nabla g$ on $\mathbb{R}^n \setminus U$.

The main result is the following Luzin N–property with respect to Hausdorff content for $W^k_{p_\circ,1}$–mappings:

Theorem (2.2.14)[50]: Let $k \in \{2, \dots, n\}, q \in (p_\circ, n]$, and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^n$ if $H^q_\infty(E) < \delta$, then $H^q_\infty(v(E)) < \varepsilon$. In particular, $H^q(v(E)) = 0$ whenever $H^q(E) = 0$.

Proof. Fix $\varepsilon > 0$ and take $\delta = \delta(\varepsilon, v)$ from Lemma (2.2.17). Then by Corollary (2.2.5) for any regular family $\{I_\alpha\}$ of n–dimensional dyadic intervals we have if $\sum_\alpha \ell(I_\alpha)^q < \delta$, then$\sum_\alpha \left(diam\, v(I_\alpha)\right)^q < C\varepsilon$. Now we may conclude the proof of Theorem (2.2.14) by use of Lemmas (2.2.15) and (2.2.16). Indeed they allow us to find a $\delta_0 > 0$ such that if for a subset E of $\mathbb{R}^n$we have $H^q_\infty(E) < \delta_0$, then E can be covered by a regular family $\{I_\alpha\}$ of n–dimensional dyadic intervals with $\sum_\alpha \ell(I_\alpha)^q < \delta$.

For the case $d = 1, k = n$, and $q = p_\circ = 1$ the assertion of Theorem (2.2.14) was obtained in [56], and the argument given there easily adapts to cover also the cases $k = n, q = 1$, and $d > 1$. Our proof here for the remaining cases follows and expands on the ideas from [56]. We fix $k \in \{2, \dots, n\}, q \in (p_\circ, n]$, and a mapping v in $W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$ To prove Theorem (2.2.14), we need some preliminary lemmas that we turn to next. By a dyadic interval we understand an interval of the form $[\frac{k_1}{2^l}, \frac{k_1+1}{2^l}] \times \cdots \times [\frac{k_n}{2^l}, \frac{k_n+1}{2^l}]$, where k_i, l are integers. The following assertion is straightforward, and hence we omit its proof here.

Lemma (2.2.15)[50]: For any n-dimensional interval $I \subset \mathbb{R}^n$ there exist dyadic intervals $Q_1, \dots, Q_{2^n}$ such that $I \subset Q_1 \cup \cdots \cup Q_{2^n}$ and $\ell(Q_1) = \cdots = \ell(Q_{2^n}) \le 2\ell(I)$.

Let $\{I_\alpha\}_{\alpha \in A}$ be a family of n-dimensional dyadic intervals. We say that the family $\{I_\alpha\}$ is regular, if for any n-dimensional dyadic interval Q the estimate

$$\ell(Q)^q \ge \sum_{\alpha : I_\alpha \subset Q} \ell(I_\alpha)^q \tag{38}$$

holds. Since dyadic intervals are either disjoint or contained in one another, (38) implies that any regular family $\{I_\alpha\}$ must in particular consist of mutually disjoint1 intervals.

Lemma (2.2.16)[50]: (see Lemma (2.2.16) in [56]). Let $\{I_\alpha\}$ be a family of n–dimensional dyadic intervals. Then there exists a regular family $\{J_\beta\}$ of n–dimensional dyadic intervals such that $\bigcup_\alpha I_\alpha \subset \bigcup_\beta J_\beta$ and

$$\sum_\beta \ell(J_\beta)^q \le \sum_\alpha \ell(I_\alpha)^q .$$

Lemma (2.2.17)[50]: For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{I_\alpha\}$ of n–dimensional dyadic intervals we have if

$$\sum_\alpha \ell(I_\alpha)^q < \delta, \tag{39}$$

then

$$\sum_{\alpha} \left\| 1_{I_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}}^q < \varepsilon \tag{40}$$

and

$$\sum_{\alpha} \frac{1}{\ell(I_\alpha)^{n-q}} \int_{I_\alpha} |\nabla v|^q < \varepsilon\, . \tag{41}$$

Proof. Fix $\varepsilon \in (0,1)$ and let $\{I_\alpha\}$ be a regular family of n–dimensional dyadic intervals satisfying (39), where $\delta > 0$ will be specified below.
We start by checking (40). Of course, for sufficiently small δ we can achieve that $\left\| 1_{I_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}}$ is strictly less than say 1 for every α. Then in view of the inequalities $q > p_\circ$ and Lemma (2.2.3) we have

$$\sum_{\alpha} \left\| 1_{I_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}}^q \le \sum_{\alpha} \left\| 1_{I_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}}^{q_\circ} \le \left\| 1_{\cup_\alpha I_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}}^{q_\circ}$$

Using (46), we can rewrite the last estimate as

$$\sum_{\alpha} \left\| 1_{I_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}}^q \le \left(\int_0^{+\infty} \left[L^n \left(\left\{ x \in \bigcup_{\alpha} I\alpha : \left|\nabla^k v(x)\right| > t \right\} \right) \right]^{\frac{1}{p_\circ}} dt \right)^{p_\circ} . \tag{42}$$

Since

$$\int_0^{+\infty} \left[L^n \left(\left\{ x \in \mathbb{R}^n : \left|\nabla^k v(x)\right| > t \right\} \right) \right]^{\frac{1}{p_\circ}} dt < \infty$$

it follows that the integral on the right–hand side of (42) tends to zero as $L^n(\bigcup_\alpha I\alpha) \to 0$. In particular, it will be less than ε if the condition (39) is fulfilled with a sufficiently small δ. Thus (40) is established for all $\delta \in (0,\delta_1]$, where $\delta_1 = \delta_1(\varepsilon, v) > 0$.
Next we check (41). By virtue of Lemma (2.2.2), applied coordinate–wise, we can find a decomposition $v = v_0 + v_1$, where $\left\| \nabla_{v_0} \right\|_{L^\infty} \le K = K(\varepsilon, v)$ and

$$\left\| \nabla^k v_1 \right\|_{L_{p_\circ}} < \varepsilon. \tag{43}$$

Assume that $\delta \in (0, \delta_1]$ and

$$\sum_{\alpha} \ell(I_\alpha)^q < \delta < \frac{1}{K^q + 1}\, \varepsilon. \tag{44}$$

Define the measure μ by

$$\mu = \left(\sum_{\alpha} \frac{1}{\ell(I_\alpha)^{n-q}}\, 1_{I_\alpha} \right) L^n\, , \tag{45}$$

where 1_{I_α} denotes the indicator function of the set I_α.

The estimate

$$\sup_I \{ \ell(I)^{-q} \mu(I) \le 2^{n+q} \tag{46}$$

holds, where the supremum is taken over all n–dimensional intervals. Indeed, write for a dyadic interval Q

$$\mu(Q) = \sum_{\alpha: I_\alpha \subset Q} \ell(I_\alpha)^q + \sum_{\alpha: I_\alpha \not\subseteq Q} \frac{\ell(Q \cap I_\alpha)^n}{\ell(I_\alpha)^{n-q}}\, .$$

By regularity of $\{I_\alpha\}$ the first sum is bounded above by $\ell(Q)^q$. If the second sum is nonzero then there must exist an index α such that $I_\alpha \nsubseteq Q$ and I_α, Q overlap. But as the intervals $\{I_\alpha\}$ are disjoint and dyadic we must then precisely have one such interval I_α and $I_\alpha \supset Q$. But then the first sum is empty and the second sum has only the one term $\ell(Q)^n/\ell(I_\alpha)^{n-q}$, hence is at most$\ell(Q)^q$. Thus the estimate $\mu(Q) \le \ell(Q)^q$ holds for dyadic Q. The inequality (46) in the case of a general interval I follows from the above dyadic case and Lemma (2.2.15). The proof of the claim is complete.

Now return to (41). By properties (43), (35) (applied to the mapping v_1 and parameters $p = p_\circ, \beta = (\frac{n}{p_\circ} - k + 1)q = q$), we have

$$\sum_\alpha \frac{1}{\ell(I_\alpha)^{n-q}} \int_{I_\alpha} |\nabla v|^q \le \frac{K^q}{K^q + 1}\, \varepsilon + \sum_\alpha \frac{1}{\ell(I_\alpha)^{n-q}} \int_{I_\alpha} |\nabla v_1|^q$$
$$\le C'\varepsilon + \int_{I_\alpha} |\nabla v_1|^q \, d\mu \le C''\varepsilon \,.$$

Since $\varepsilon > 0$ was arbitrary, the proof of Lemma (2.2.17) is complete.

Let $k, m \in \{2, \ldots, n\}$ and $v \in W^k_{p_\circ, loc}(\Omega, \mathbb{R}^d)$, where Ω is an open subset of $\mathbb{R}^n$. Then, by Corollary (2.2.13), there exists a Borel set A_v such that $H_{q_\circ}(A_v) = 0$ and all points of the complement $\Omega \setminus A_v$ are Lebesgue points for the gradient $\nabla v(x)$. We remark that with the assumed Sobolev regularity the mapping v need not be differentiable at any point of Ω, and that $\nabla v(x)$ simply is the precise representative of the weak gradient of v. There are of course many other ways to give pointwise meaning to $\nabla v(x)$, but as these play no role in our considerations here we omit any further discussion. Denote $Z_{v,m} = \{x \in \Omega \setminus A_v : \operatorname{rank} \nabla_v(x) < m\}$.

Theorem (2.2.18)[50]: If $k, m \in \{2, \ldots, n\}, \Omega$ is an open subset of $\mathbb{R}^n$, and $v \in W^k_{p_\circ, loc}(\Omega, \mathbb{R}^d)$, then $H^{q_\circ}(v(Z_{v,m})) = 0$.

The exponents occuring in the theorem are the critical exponents that were defined in (47):

$$p_\circ = \frac{n}{k} \; and \; q_\circ = m - 1 + \frac{n-m+1}{k}.$$

We emphasize the fact that, in contrast with the Luzin N– property with respect to Hausdorff content of Theorem (2.2.14), the Morse–Sard–Federer Theorem (2.2.18) is valid within the wider context of W^k_p–Sobolev spaces (finiteness of the Lorentz norm is not required).

Before embarking on the detailed proof let us make some preliminary observations that will enable us to make some convenient additional assumptions. Namely because the result is local we can without loss in generality assume that $\Omega = \mathbb{R}^n$ and that $v \in W^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$. Indeed note that it suffices to prove that

$$H^{q_\circ}\left(v\left(Z_{v,m} \cap \Omega'\right)\right) = 0 \qquad (47)$$

for all smooth domains Ω' whose closure $\overline{\Omega'}$ is compact and contained in Ω. Now such domains Ω' are extension domains for $W^k_{p_\circ}$ and so $v|_{\Omega'}$ can be extended to $V \in W^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$ and hence proving the statement for V we deduce (47) and therefore prove the theorem.

We fix $k, m \in \{2, \ldots n\}$ and a mapping $v \in W^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$ In view of the definition of critical set adopted here we have that

$$Z_{v,m} = \bigcup_{j\in\mathbb{N}} \{x \in Z_{v,m} : |\nabla v(x)| \le j\}.$$

Consequently we only need to prove that $H^{q_\circ}(Z'_v) = 0$, where

$$Z'_v = \{x \in Z_{v,m} : |\nabla v(x)| \le 1\}.$$

The following lemma contains the main step in the proof of Theorem (2.2.18).

Lemma (2.2.19)[50]: For any n-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$H_\infty^{q_\circ}(v(Z'_v \cap I)) \le C(\|\nabla^k v\|_{L_{p_\circ(I)}}^{q_\circ} + \ell(I)^{m-1}\|\nabla^k v\|_{L_{p_\circ(I)}}^{1-m+q_\circ}) \quad (48)$$

holds, where the constant C depends on n, m, k, d only.

Proof. By virtue of (52) it suffices to prove that

$$H_\infty^{q_\circ}(v(Z'_v \cap I)) \le C(\|\nabla^k v_I\|_{L_{p_\circ(\mathbb{R}^n)}}^{q_\circ} + \ell(I)^{m-1}\|\nabla^k v_I\|_{L_{p_\circ(\mathbb{R}^n)}}^{1-m+q_\circ}) \quad (49)$$

for the mapping v_I defined in Lemma (2.2.4), where $C = C(n, m, k, d)$ is a constant. Fix an n-dimensional dyadic interval $I \subset \mathbb{R}^n$ and recall that $v_I(x) = v(x) - P_I(x)$ for all $x \in I$. Denote

$$\sigma = \|\nabla^k v_I\|_{L_{p_\circ}}^{q_\circ}, \sigma_* = \ell(I)^{m-1}\|\nabla^k v_I\|_{L_{p_\circ}}^{1-m+q_\circ},$$

and for each $j \in \mathbb{Z}$

$$E_j = \{x \in \mathbb{R}^n : (M|\nabla v_I|^{q_\circ})(x) \in (2^{j-1}, 2^j]\} \; and \; \delta_j = H_\infty^{q_\circ}(E_j).$$

Then by Corollary (2.2.8),

$$\sum_{j=-\infty}^{\infty} \delta_j 2^j \le C\sigma$$

for a constant C depending on n, m, k, d only. By construction, for each $j \in \mathbb{Z}$ there exists a family of balls $B_{ij} \subset \mathbb{R}^n$ of radii r_{ij} such that

$$E_j \subset \bigcup_{i=1}^{\infty} B_{ij} \; and \; \sum_{i=1}^{\infty} r_{ij}^{q_\circ} \le 2^{q_\circ}\delta_j .$$

Denote

$$Z_j = Z'_v \cap I \cap E_j \; and \; Z_{ij} = Z_j \cap B_{ij}.$$

By construction $Z'_v \cap I = \bigcup_j Z_j$ and $Z_j = \bigcup_i Z_{ij}$. Put

$$\varepsilon_* = \frac{1}{\ell(I)} \|\nabla^k v_I\|_{L_{p_\circ}},$$

and let j_* be the integer satisfying $\varepsilon_*^{q_\circ} \in (2^{j_*-1}, 2^{j_*}]$. Denote $Z_* = \bigcup_{j<j_*} Z_j, Z_{**} = \bigcup_{j\ge j_*} Z_j$. Then by construction

$$Z'_v \cap I = Z_* \cup Z_{**}, Z_* \subset \{x \in Z'_v \cap I : (M|\nabla v_I|^{q_\circ})(x) < \varepsilon_*^{q_\circ}\}$$

Since $\nabla P_I(x) = \nabla v(x) - \nabla v_I(x), |\nabla v_I(x)| \le 2^{j/q_\circ}, |\nabla v(x)| \le 1$, and $\lambda_\nu(v, x) = 0$ for $x \in Z_{ij}$ and $\nu \in \{m, \ldots, d\}$, we have

$$Z_{ij} \subset \{x \in B_{ij} : \lambda_1(P_I, x) \le 1 + 2^{j/q_\circ}, \ldots, \lambda_{m-1}(P_I, x) \le 1 + 2^{j/q_\circ}, \lambda_m(P_I, x) \le 2^{j/q_\circ}, \ldots, \lambda_d(P_I, x) \le 2^{j/q_\circ}\}.$$

Applying Theorem (2.2.9) and Corollary (2.2.11) to mappings P_I, v_I, respectively, with $B = B_{ij}$ and $\varepsilon = \varepsilon_j = 2^{j/q_\circ}$, we find a finite family of balls $T_s \subset \mathbb{R}^d, s = 1, \ldots, s_j$ with $s_j \le C_Y(1 + \varepsilon_j^{1-m})$, each of radius $(1 + C_M)\varepsilon_j r_{ij}$, such that

$$\bigcup_{s=1}^{s_j} T_s \supset v(Z_{ij}).$$

Therefore, for $j \geq j_*$ we have

$$H_\infty^{q_\circ}\left(v(Z_{ij})\right) \leq C_1 s_j \varepsilon_j^{q_\circ} r_{ij}^{q_\circ} = C_2\left(1+\varepsilon_j^{1-m}\right)2^j\ r_{ij}^{q_\circ}$$
$$\leq C_2(1+\varepsilon^{1-m})2^j\ r_{ij}^{q_\circ}\ , \tag{50}$$

where all the constants C_ν above depend on n, m, d only. By the same reasons, but this time applying Theorem (2.2.9) and Corollary (2.2.11) with $\varepsilon = \varepsilon_*$ and instead of the balls B_{ij} *w*e take a ball $B \supset I$ with radius $r = \sqrt{n\ell}(I)$, we have

$$H_\infty^{q_\circ}(v(Z_*)) \leq\ C_3(1\ +\varepsilon_*^{1-m}\ )\varepsilon_*^{q_\circ}\ell(I)^{q_\circ} = C_3(1+\varepsilon_*^{1-m})\sigma\ = C_3(\sigma+\sigma_*). \tag{51}$$

From (50) we get immediately

$$H_\infty^{q_\circ}(v(Z_{**})) \leq \sum_{j\geq j_*}\sum_i C_2(1+\varepsilon_*^{1-m})2^j\ r_{ij}^{q_\circ}\ \leq \sum_{j\geq j_*} C_2(1\ +\varepsilon_*^{1-m}\ )2^{j+q_\circ}\delta_j$$
$$\leq C_4(1+\varepsilon_*^{1-m})\sigma\ = C_4(\sigma+\sigma_*).$$

The last two estimates combine to give $H_\infty^{q_\circ}(v(Z_v'\cap I)) = H_\infty^{q_\circ}(v(Z_*\cup Z_{**})) \leq C(\sigma+\sigma_*)$, and hence finish the proof of the lemma.

Corollary (2.2.20)[50]: For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any subset E of $\mathbb{R}^n$ we have $H_\infty^{q_\circ}(v(Z_v'\cap E)) \leq \varepsilon$ provided $L^n(E) \leq \delta$. In particular, $H_\infty^{q_\circ}\left(v(Z_v'\cap E)\right) = 0$ whenever $L^n(E) = 0$.

Proof. Let $L^n(E) \leq \delta$, then we can find a family of disjoint n-dimensional dyadic intervals I_α such that $E \subset \bigcup_\alpha I_\alpha$ and $\sum_\alpha \ell^n(I_\alpha) < C\delta$. Of course, for sufficiently small δ the estimate $\left\|\nabla^k v\right\|_{L_{p_\circ}(I_\alpha)} < 1$ is fulfilled for every α. Then in view of $q_\circ > p_\circ$ and Lemma (2.2.3) we have

$$\textstyle\sum_\alpha \left\|\nabla^k v\right\|_{L_{p_\circ}(I_\alpha)} \leq \left\|\nabla^k v\right\|_{L_{p_\circ}(\cup\ I_\alpha)}^{p_\circ} \tag{52}$$

Analogously, by Hölder inequality and by virtue of the equalities $1-m+q_\circ = \frac{n-m+1}{k}$ and $(1-m+q_\circ)\frac{n}{n-m+1} = \frac{n}{k} = p_\circ$, we have

$$\sum_\alpha \ell(I_\alpha)^{m-1}\ \left\|\nabla^k v\right\|_{L_{p_\circ(I_\alpha)}}^{1-m+q_\circ}\ \leq \left(\sum_\alpha \ell(I_\alpha)^n\right)^{\frac{m-1}{n}}\left(\sum_\alpha\ \left\|\nabla^k v\right\|_{L_{p_\circ(I_\alpha)}}^{p_\circ}\right)^{\frac{n-m+1}{n}}$$
$$\leq \delta^{\frac{m-1}{n}}\left\|\nabla^k v\right\|_{L_{p_\circ(\cup\ I_\alpha)}}^{\frac{n-m+1}{k}}.$$

The last two estimates together with Lemma (2.2.19) allow us to conclude the required smallness of

$$\sum_\alpha H_\infty^{q_\circ}(Z_v'\cap I_a))\ \geq H_\infty^{q_\circ}(Z_v'\cap E).$$

Invoking Federer's Theorem for the smooth case $g \in C^k(\mathbb{R}^n)$, Theorem (2.2.12) (iii) (applied to the case $k = l$) implies

Corollary (2.2.21) (see, e.g., [39])[50]: There exists a set Zev of n-dimensional Lebesgue measure zero such that $H^{q_\circ}(v(Z_v'\backslash\tilde{Z}_v)) = 0$. In particular, $H^{q_\circ}(v(Z_v')) = H^{q_\circ}(v(\tilde{Z}_v))$.

From Corollaries (2.2.20) and (2.2.21) we conclude that $H^{q_\circ}\left(v(Z_v')\right) = 0$, and this ends the proof of Theorem (2.2.18).

Theorem (2.2.18) implies the following analog us of the classical Morse–Sard Theorem:

Corollary (2.2.22)[50]: Let Ω be an open subset of $\mathbb{R}^n$. If $m \in \{1, \dots, n\}$ and $v \in W^{n-m+1}_{\frac{n}{n-m+1},loc}(\Omega, \mathbb{R}^m)$, then $L^m(v(Z_{v,m})) = 0$.
This assertion follows directly from Theorem (2.2.18) for $m > 1$ and from the results of [56] for $m = 1$.

We start with the following simple technical observation.

Lemma (2.2.23)[50]: If $k \in \{2, \dots, n\}$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$, then for $H^{p_\circ}$ –almost all $x \in \mathbb{R}^n$,

$$\lim_{r\searrow 0} r^{-1} \left\| 1_{B(x,r)} \cdot \nabla^k v \right\|_{L_{p_\circ,1}} = 0$$

holds.

Proof. Fix $\varepsilon > 0$. Let $\{B_\alpha\}$ be a family of disjoint balls $B_\alpha = B(x_\alpha, r_\alpha)$ such that

$$\left\| 1_{B_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}} \geq \varepsilon r_\alpha$$

and $\sup_\alpha r_\alpha < \delta$ for some $\delta > 0$, where δ is choosen small enough to guarantee $\sup_\alpha \left\| 1_{B_\alpha} \cdot \nabla^k v \right\|_{L_{p_\circ,1}} < 1$. Then by Lemma (2.2.3) we have

$$\begin{aligned} \sum_\alpha r_\alpha^{p_\circ} &\leq \varepsilon^{-1} \sum_\alpha \left\| 1_{B_\alpha} \cdot \nabla^k v \right\|^{p_\circ}_{L_{p_\circ,1}} \\ &\leq \varepsilon^{-1} \left\| 1_{\bigcup_\alpha B_\alpha} \cdot \nabla^k v \right\|^{p_\circ}_{L_{p_\circ,1}}. \end{aligned} \tag{53}$$

Since the last term tends to 0 as $L^n(\bigcup_\alpha B_\alpha) \to 0$, and $L^n(\bigcup_\alpha B_\alpha) \leq \delta^{n-p_\circ} \sum_\alpha r_\alpha^{p_\circ}$, we get easily that $\sum_\alpha r_\alpha^{p_\circ} \to 0$ as $\delta \to 0$. Using this fact and some standard covering lemmas we arrive in a routine manner at the required assertion

$$H^{p_\circ}\{x \in \mathbb{R}^n : \lim_{r\searrow 0} sup\, r^{-1} \left\| 1_{B(x,r)} \cdot \nabla^k v \right\|_{L_{p_\circ,1}} \geq \varepsilon\} = 0.$$

From the last lemma, Corollary (2.2.13) and estimate (34) we obtain the following result that is probably well–known to specialists:

Theorem (2.2.24)[50]: Let $k \in \{2, \dots, n\}$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $H^q(A_v) = 0$ for every $q \in (p_\circ, n]$ and for any $x \in \mathbb{R}^n \setminus A_v$ the function v is differentiable (in the classical Frechet sense) at x, furthermore, the classical derivative coincides with $\nabla v(x)$, where

$$\lim_{r\searrow 0} \int_{B(x,r)} |\nabla v(z) - \nabla v(x)|\, dz = 0.$$

The case $k = 1, q = p_\circ = n$ is a classical result due to Stein [73] (see also [62]), and for $m = 1, k = n$ the result is also proved in [58].

Applying Theorems (2.2.14) and (2.2.18) in combination with the Corollary (2.2.13), we obtain

Corollary (2.2.25)[50]: Let $k, m \in \{2, \dots, n\}, v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$, and rank$\nabla v(x) \leq m$ for all $x \in \mathbb{R}^n \setminus A_v$. Then for any $\varepsilon > 0$ there exist an open set $V \subset \mathbb{R}^d$ and a mapping $g \in C_1(\mathbb{R}^n, \mathbb{R}^d)$ such that $H^{q_\circ}_\infty(V) < \varepsilon, v(A_v) \subset V$ and $|_{v^{-1}}(\mathbb{R}^d \setminus V) = g|_{v^{-1}}(\mathbb{R}^d \setminus V)$, $\nabla v|_{v^{-1}}(\mathbb{R}^d \setminus V) = \nabla g|_{v^{-1}}(\mathbb{R}^d \setminus V)$, and rank$\nabla v|_{v^{-1}}(\mathbb{R}^d \setminus V) \equiv m.$

Here A_v is the Borel set with $H^{q_\circ}(A_v) = 0$ from Theorem (2.2.24).

Theorem (2.2.26)[50]: Let $k, m \in \{2, \dots, n\}$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^m)$. Then for L^m–almost all $y \in v(\mathbb{R}^n)$ the preimage $v^{-1}(y)$ is a finite disjoint family of $(n - m)$–dimensional C^1 -smooth compact manifolds (without boundary) S_j , $j = 1, \dots, N(y)$.

Proof. The inclusion $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^m)$ and Lemma (2.2.4) easily imply the following statement (see also Remark 1.4):

(i) For any $\varepsilon > 0$ there exists $R_\varepsilon \in (0, +\infty)$ such that $|v(x)| < \varepsilon$ for all $x \in \mathbb{R}^n \setminus B(0, R_\varepsilon)$.

Fix an arbitrary $\varepsilon > 0$. Take the corresponding set $V \subset \mathbb{R}^m$ and mapping $g \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ from Corollary (2.2.25). Let $0 \neq y \in \mathbb{R}^m \setminus V$. Denote $F_v = v^{-1}(y)$, $F_g = g^{-1}(y)$. We assert the following properties of these sets.

(ii) F_v is a compact set;

(iii) $F_v \subset F_g$;

(iv) $\nabla v = \nabla g$ and rank$\nabla v =$rank$\nabla g = m$ on F_v;

(v) The function v is differentiable (in the classical sense) at each $x \in F_v$, and the classical derivative coincides with

$$\nabla v(x) = \lim_{r \searrow 0} \quad \int_{(B(x,r))} \quad \nabla v(z)\, dz.$$

Indeed, (ii) follows by continuity and from (i) since $y \neq 0$, (iii)-(iv) follow from Corollary (2.2.25), and (v) follows from the condition $v(A_v) \subset V$ of Corollary (2.2.25) (see also Theorem (2.2.24)). We require one more property of these sets:

(vi) For any $x_0 \in F_v$ there exists $r > 0$ such that $F_v \cap B(x_0, r) = F_g \cap B(x_0, r)$.

Indeed, take any point $x_0 \in F_v$ and suppose the claim (v_i) is false. Then there exists a sequence of points $F_g \setminus F_v \ni x_i \to x_0$. For $r > 0$ we put

$$H_m = \left(ker\ d_{x_0} g\right)^\perp \cap B(0,r), S_m = \left(ker\ d_{x_0} g\right)^\perp \cap \partial B(0,r),$$
$$H_m(x) = x + H_m, S_m(x) = x + S_m,$$

where $\left(ker\ d_{x_0}\ g\right)^\perp$ is the orthogonal complement of the $(n-m)$-dimensional linear subspace ker $d_{x_0}\ g$. Evidently, for sufficiently small $r > 0$ we have $H_m(x) \cap F_g = \{x\}$ for any $x \in F_g \cap B(x_0, r)$. Then by construction

$$H_m(x_i) \cap F_v = \emptyset \tag{54}$$

for sufficiently large i. Since v is differentiable (in the classical sense) at x_0 with $\nabla v(x_0) = \nabla g(x_0)$, for sufficiently small $r > 0$ we have $v(x) \neq y$ for all $x \in S_m(x_0)$, and deg$(v, H_m(x_0), y) = \pm 1$, where we denote by deg$(v, H_m(x_0), y)$ the topological degree of $v|_{H_m(x_0)}$ at y. Then for sufficiently large i we must have $v^{-1}(y) \cap S_m(x_i) = \emptyset$ and deg$(v, H_m(x_i), y) =$ deg$(v, H_m(x_0), y) = \pm 1$. But this contradicts (54) and finishes the proof of (vi).

Obviously, (ii)–(vi) imply that each connected component of the set $F_v = v^{-1}(y)$ is a compact $(n-m)$-dimensional C^1- smooth manifold (without boundary).

Chapter 3

Subspaces and HereditarIy Indecomposable L^{∞}-Space

We construct a hereditarily indecomposable Banach space with dual space isomorphic to ℓ_1. Every bounded linear operator on this space is expressible as $\lambda I + K$ with λ a scalar and K compact.

Section (3.1): The Bourgain—Delbaen Space

In 1980, Bourgain and Delbaen [85], [84] introduced some separable $\mathcal{L}^{\infty}$ spaces with surprising properties: all have the Radon—Nikodym property, and so certainly do not have subspaces isomorphic to c_o; some of them (the spaces of "Cflass X") have the Schur property; the others (C1as.s Y") have dual spaces isomorphic to ℓ^1 Despite their importance, these spaces wore not much studied subsequently, and it became habitual to remark that they were "not well-understood". There has been some renewed interest recently, partly because these spaces are interesting test-cases for questions about uniform homeomorphisms [89], [86] and smooth surjections [82], [87]. Alspach [81] has investigated their Szlenk index. An attempt to understand a bit better the subspace structure of the spaces of Class Y, that is to say, in Dourgain's notation, the spaces $X_{0,6}$ with $b < 1/2 < a < 1$ and $a + b > 1$. Bourgain and Delbaen showed that every infinite-dimensional subspace of such a space has an infinite-dimensional reflexive subspace; however, they did not characterize which reflexive spaces occur as subspaces of $X_{a,b}$; Bourgain [84] raised the question of whether $X_{0,6}$ has a subspace with no unconditional basic sequence. The main result of the present answers these questions by showing that each infinite-dimensional subspacc cf $X_{0,,,}$ has a subspacc isomorphic to P. The p in question is determined by $1/p + 1/p' = 1$ where $a^{p'} + b^{p'} = 1$.

We follow modern practice by saying that vectors $x_1, x_2, \ldots$ are successive linear combinations (or blocks) of a sequence $(\mathcal{Y}_n)$ if there are integers $m_1 \leq n_1 < m_2 \leq n_2 < m_3 \leq \ldots$ and scalars $a_1, a_2 \ldots$ such that $X_k = \sum_{j-k}^{n_k} a_j \mathcal{Y}_j$.

Closely associated with the Bourgain—Delbaen spaces are some spaces with unconditional basis, which we shall denote by $U_{a,b}$. We shall study these spaces, eventually showing that they are just er-spaces with equivalent norms. The norm $||.||_{a.b}$ is defined by a recursion similar to (but simpler than!) the one that leads to the Tsirelson space [88]. We fix real numbers a, b with $a, b < 1, a + b > 1$.

For a vector $x \in \mathbb{R}^d$, or a finitely-supported vector $x \in \mathbb{R}^{(\mathbb{N})}$, we define (recursively)

$$||x||_{a.b} = \max\left\{||x||_{\infty}, \max_{l \in \mathbb{N}} \left(a\left||x\left[[0, l]\right|\right|_{a,b} + b\left|\left|x\left[[l+1, \infty)\right|\right|_{a,b}\right)\right\}$$

That is to say that the norm $\left||x_o, x, \ldots, X_d|\right|_{a,b}$ of a vector in $\mathbb{R}^{d+1}$ is whichever is greater of

It is an elementary exercise to see that this is indeed an unambiguous definition. We then define to be the completion of R(N) with respect to this norm. It should be noted that in the definition of the space We do not need to suppose that b < 1/2 (a condition essential for the Bourgain-Delbaen construction). However, it will be convenient in all that follows to assume that b a. The symmetry of the definition of the norm II IL,, means that the main result, Theorem (3.1.2), remains true when a <b, though with a replacing b in the final estimates. The recursive calculation of norms in the space Ua,b leads naturally to the construction of a finite dyadic tree of intervals of natural numbers, and it will be useful to have a standard notation for such trees. We write For the set of all finite strings of Os and

l’s, including the empty string). In our intended application, a “O” in a string o’ will always be associated with a move to the left and a “1” with a move to the right.
We shall accordingly denote the number of 0’s and the number of l’s in a string o by and r(a-) respectively. For We write o’ < ‘r and say that u precedes r if o’ is an initial segment of r. Each element o- of E has two immediate successors, which we may denote by oO and o-l. By an admissible subtree of E we shall mean a non-empty, finite subset T of E having the property that, whenever, ail predecessors of u are also in r and, of the two immediate successors of o’, either both are in T, or else neither is. Those a’ with no successors in T form the set maxT of maximal element s of T.
A dyadic tree of intervals is a family 1(a) of non-empty intervals in N, indexed by some admissible subtree T, with the property that whenever a’ E T is non-maxima], the interval ((cr) is the disjoint union of its subintervals and I(crl), with lying to the left of We note that the intervals i(r) corresponding to form a partition of the original interval 0•
If x is a finitely supported vector in $\mathbb{R}^{(\mathbb{N})}$ and $I(\sigma)$ $(\sigma \in T)$ is any dyadic
tree of intervals, it is dear from the recursive definition of the norm that

$$||x||_{a,b} \geq \sum_{r \in mxT} a^{l(T)} b^{r(T)} ||xI(T)||_{a,b}$$

Moroovef ,for a suitably chosen tree, we have

$$||x||_{a,b} = \sum_{r \in mxT} a^{l(T)} b^{r(T)} ||xI(T)||_{\infty}$$

Notice that in the case where $||x||_{a,b} ||x||_{\infty}$ this latter equality holds for the trMaltrce $T\{O\}$.
We shall now proceed to establish the inequality $||x||_{a,b} \leq ||x||_p \leq C||x||_{a,b}$ for an arbitrary finitely-supported vector x in $\mathbb{R}^{(\mathbb{N})}$ thus showing that $||\cdot||_{a,b}$. Is equivalent to the ℓ^{P}’-norm, where $1/p + 1/p' = L = a^{p'} + b^{p'}$.
A few naïve remarks will perhaps help to clarify the calculations that follow. The inequality $||x||_{a,b} \leq ||x||_p$ Is easy to establish by induction the size of the support of a. Indeed, $||x||_{a,b}$is equal either or to $||x||_{\infty}$or to $a||x[0,k)||_{a,b} + b||xI(T)||_{a,b}$ and this latter quantity is at most

$$\left(a^{p'} + b^{p'}\right)^{1/p'} \left(||x[0,k)||_{a,b}^{p} + ||x[k,\infty)||_{a,b}^{p}\right)^{1/p} \leq \left(||x[0,k)||_{p}^{p} + ||x[k,\infty)||_{p}^{p}\right)^{1/p} = ||x||_p ,$$

by H$\ddot{o}$lder’s inequality and our inductive hypothesis. There are, of course, some vectors for which $||x||_{a,b} = ||x||_p$;they may be characterized using the condtiion for equality to occur In H$\ddot{o}$lder’s inequality Indeed, they are exactly those vectors where a norm calculation uf the kind described thorn leads to a dyadic tree of intervals with the property that the ratio $||x[0,k)||_p : ||x[0,k)||_p$is precisely $a^{p'-1} : b^{p'-1}$ for every non-maximal , and such that$||x[0,k)||_{a,b}^{p} = ||x[0,k)||_{a,b}^{p}$ for each maximal r.
If we are thinking of $||\cdot||_{a,b}$ as an approximation to $||\cdot||_p$ then, every time that we arc obliged to split an interval ocher than in the ratio $a^{p'-1} : b^{p'-1}$ with respect to the ℓ^{P} $-$in, We introduce an underestimate. The proof we give proceeds by constructing a certain dyadic tree and keeping fairly careful accounts of the accumulated underestimation- It will be

convenient to write $a = a^{P'}$ and $\beta = b^{P'}$, so that $a + \beta = 1$. As already remarked, we lose no generality in supposing that $a \geq b$.

LEMMA (3.1.1)[80]: Let $\mathcal{Y} \in \mathbb{R}^{(\mathbb{N})}$ (be a non-zero vector, with support contained in the finite interval J. Assume that i satisfies

$$||\mathcal{Y}||_{\infty}^{p} \frac{2\beta}{5} ||\mathcal{Y}||_{p}^{p}.$$

We may choose a natural number k, not an end-point of the interval J, an a natural number I (equal either to k or to $k — 1$) in such a way that

$$||\mathcal{Y}||_{p} \leq exp \frac{1}{5pp'}; \left[\frac{|\mathcal{Y}_k|^p}{||\mathcal{Y}||_p^p}\right] [a||\mathcal{Y}[0,l]||_p + b||\mathcal{Y}[k,\infty]||_p]$$

That is to say, either

$$||\mathcal{Y}||_{p} \leq exp \left[\frac{1}{5pp'} \frac{|\mathcal{Y}_k|^p}{||\mathcal{Y}||_p^p}\right] [a||\mathcal{Y}[0,k-1]||_p + b||\mathcal{Y}[k,\infty]||_p]$$

or

$$||\mathcal{Y}||_{p} \leq exp \left[\frac{1}{5pp'} \frac{|\mathcal{Y}_k|^p}{||\mathcal{Y}||_p^p}\right] [a||\mathcal{Y}[0,k]||_p + b||\mathcal{Y}[k+1,\infty]||_p]$$

Notice that in either case k is an end-point of the subinterial $J \cap [O,k\}$or $J \cap [k,\infty)$ which contains it.

Proof. It will simplify notation to suppose k that the interval J is $[1,n]$

We choose k to be the unique natural number that satisfies

$$\sum_{j=1}^{k—1} |\mathcal{Y}|^p < a||\mathcal{Y}||_p^P \leq \sum_{j=1}^{k} |\mathcal{Y}|^p$$

Our assutuption implies that $||\mathcal{Y}||_\infty^P < \beta||\mathcal{Y}||_p^P$and hence that $|\mathcal{Y}|^p < \beta||\mathcal{Y}||_p^P \leq a||\mathcal{Y}||_p^Pand\sum_{j=1}^{n-1} |\mathcal{Y}|^p = ||\mathcal{Y}||_p^P - ||\mathcal{Y}_n||_p^p - |\mathcal{Y}_n|^p > (1-\beta)||\mathcal{Y}||_p^p$. Thus k canno be either of te end points $1, n$ of the supporting interval J. By choosing l to be either $k — 1$ or k, we may arrange that

$$|\sum_{j=1}^{l} |\mathcal{Y}_j|^p - a \, ||\mathcal{Y}||_p^p \leq \frac{1}{2} |\mathcal{Y}|^p.$$

So if we write $w = \mathcal{Y}[io,ij$ and $z = \mathcal{Y}[(+l,\infty)1$ we have

$$||w||_p^p = (a + \varepsilon) \, ||\mathcal{Y}||_p^p, ||z||_p^p = (\beta - \varepsilon)||\mathcal{Y}||_p^p,$$

where $|\varepsilon| \leq \frac{1}{2}\left(|\mathcal{Y}_k|/ \, ||\mathcal{Y}||_p\right)^p$. We can now calculate as follows:

$$\begin{aligned} a||w||_p + b||z||_p &= [a(a+\varepsilon)^{1/p} + b(\beta-\varepsilon)^{1/p}||\mathcal{Y}||_p \\ &= \left[a(1+\varepsilon/a)^{1/p} + \beta(1-\varepsilon/\beta)^{1/p}\right]||\mathcal{Y}||_p. \end{aligned}$$

Of course, for small values of ,

$$a(1 + + \varepsilon a)^{1p} + \beta(1-\varepsilon\beta)^{1p} \approx \exp\left[-\frac{1}{2pp'}\left(\frac{1}{a}+\frac{1}{\beta}\right)\varepsilon^2\right]$$

and it is an elementary exercise to see that

$$a(1\ +\ +\varepsilon a)^{1p}+\beta(1-\varepsilon\beta)^{1p}>\exp\left[-\frac{1}{pp'}\left(\frac{1}{a}+\frac{1}{\beta}\right)\varepsilon^2\right]$$

whenever $|\varepsilon|<\beta/5.$ In our case, since we axe assuming that $||\mathcal{Y}||_\infty^p<\left(\frac{2\beta}{5}\right)||\mathcal{Y}||_p^p$, the quantity ε as defined above is indeed smaller than$\beta/5$.

We are thus led to the inequality

$$||\mathcal{Y}||_p\le\ exp\ \left[\frac{1}{pp'}\ \left(\frac{1}{a}+\frac{1}{\beta}\right)\varepsilon^2\right]\left[a||w||_p+b||z||_p\right]$$

$$\le\exp\left[\frac{1}{5pp'}\left|\frac{\mathcal{Y}_k}{||\mathcal{Y}||_p^p}\right|\right]\left[a||w||_p+b||z||_p\right],$$

using once again the fact that

$$|\varepsilon|\le\frac{1}{2\frac{|\mathcal{Y}_k|^p}{||\mathcal{Y}||_p^p}}\le\frac{\beta}{5}.$$

Theorem (3.1.2)[80]: Let a,b be real numbers satisfying $a,b\ <\ 1,a+\ b>\ 1$ and Let p,p' be determined by $1/p+\ l/p'\ =\ I\ =\ a^{p'}\ +\ b^{p'}.$ The norm $||\cdot||_{a,b}$ equivalent to the usual ℓ^p-norm.

Proof. As in the preceding lemma, we may suppose that $b\ <\ a$ and we retain the notation $a\ =\ a^{p'},\beta=\ b^{p'},$ We consider an arbitrary non-zero $x\in\mathbb{R}^{(\mathbb{N})}$ and give a recursive definition of an admissible tree T a dyadic tree of intervals $(I(\sigma))\sigma\in T$ and elements $i(\sigma)$ of $I(\sigma)$, which we shall use to estimate $||x||_{a,b}$. We start by taking $I(\tau)$ to be any finite interval that contains the support of x. if a string τ is already in T and $I(\tau)$ has already been defined we need to specify whether τ is going to be a maximal element of T and, if not) what the two “daughter” intervals $I(\tau_0)$ and $I(\tau_1)$ are going to be.

There will be two criteria involved in deciding if τ is maximal. First, r will be declared to be maximal if the following condition holds:

$$\|x[_{I(r)}\|_\infty\ge(2\beta/5)^{1/p}\|x[_{I(r)}\|_p.$$

If this condition does not hold, then of course LEMMA (3.1.1) is applicable to the vector $y\ =x[_{I(r)}$. We let $i(\tau)$ be the unique $i\ \in\ I(\tau)$ such that for every r in the tree. Indeed, otherwise the recursive construction would have been terminated (by criterion (B)) at a predecessor of r.

In the event that neither (A) nor (B) holds) we choose l as in LEMMA (3.1.1) and define the daughter intervals by $I(rO)\ =\ I(r)\cap[0,l],I(r1)\ =\ I(r)\cap[l\ +\ 1,\infty)$. We notice that $i(r)$ is an end-point of one or other of these intervals, and hence also of any interval $I(v)$, with $v\ \succ\ r$, which contains it.

This completes the recursive construction of $T,I(\sigma)$ and $i(\sigma)$. The set Max T of maximal elements may be partitioned as $A\cup B$, where A is the set of r for which condition (A) holds. We notice that the natural numbers $i(r)$, defined for $r\ \in\ T\setminus A$, are all distinct. Indeed, if v and r are incomparable elements of T, then $i(v)$ and $i(r)$ are elements of the disjoint intervals $I(v)$ and $I(r)$; on the other hand, if $r\prec\ v$ and $i(r)\ \in\ i(v)$ then $i(r)$ is an end-point of $I(v)$ while $i(v)$ is not.

It follows from That, whenever σ is a non-maximal element of T

$$\left|\left|x\, I(\sigma)\right|\right|_p \leq \ exp\left[\frac{1}{5pp'}\frac{\left|x_{i(\sigma)}\right|^p}{\left|\left|xI(\sigma)\right|\right|_p^p}\right]\left[a\left|\left|xI(\sigma 0)\right|\right|_p + b\left|\left|xI(\sigma 1)\right|\right|_p\right].$$

We deduce from this inequality, together with the remark we made following the introduction of criterion (B), that

$$\left|\left|x\right|\right|_p \leq \sum_{r\in mxT} a^{I(r)}b^{r(r)}\exp\left[\sum_{\sigma\prec r}\frac{1}{5pp'}\frac{x_{i(\sigma)}^p}{\left|\left|xI(\sigma)\right|\right|_p^p}\right]\left|\left|xI(r)\right|\right|_p$$

$$\leq e^{1/(pp')}\sum_{r\in mxT} a^{I(r)}b^{r(r)}c$$

$$\leq e^{1/(pp')}\left[\sum_{r\in mxT} a^{I(r)}b^{r(r)}\left(\frac{5}{2\beta}\right)^{1p}\left|\left|xI(r)\right|\right|_\infty + \sum_{r\in B} a^{I(r)}b^{r(r)}a^{I(r)}b^{r(r)}\right]$$

$$= e^{1(p/p')}[H_A + H_B],,$$

in an obvious notation. It follows from the relationship between trees and norm calculations that $H_A \leq \left(5/(2\beta)\right)^{1/p}\left|\left|x\right|\right|_{a,b}$. On the other hand, we may use Hölder's inequality and the fact that $a^{p'} + b^{p'} = 1$ to show that

$$H_B \leq \left(\sum_{r\in B}\left|\left|xI(r)\right|\right|_p^p\right)^{1/p} = \left(\sum_{j\in J}\left|x_j\right|^p\right)^{1/p},$$

where

$$J = \bigcup_{r\in B} I(r) = \left\{j \in I(\ ): \sum_{\sigma with j\in I(\sigma)}\frac{\left|x_{i(\sigma)}\right|^p}{\left|\left|xI(\sigma)\right|\right|_p^p} > 5\right\}$$

We that have

$$5H_B^p \leq \sum_{j\in J}\left|X\right|^p\sum_{\sigma with j\in I(\sigma)}\left(\frac{\left|x_{i(\sigma)}\right|^p}{\left|\left|xI(\sigma)\right|\right|_p}\right)^p$$

$$\leq \sum_{j\in J(\ \ )}\left|X_j\right|^p\sum_{\sigma with j\in I(\sigma)}\left(\frac{\left|x_{i(\sigma)}\right|^p}{\left|\left|xI(\sigma)\right|\right|_p}\right)^p$$

$$= \sum_{\sigma\in T}\left|X_{i(\sigma)}\right|^p\left|\left|xI(\sigma)\right|\right|_p^{-p}\sum_{j\in I(\sigma)}\left|x_j\right|^p = \sum_{\sigma\in T}\left|X_{i(\sigma)}\right|^p$$

and this is at most $\left|\left|x\right|\right|_p^p$since as we noted before, the $i(\sigma)$ are all distinct.

We have finally obtained the following inequalities:

$$e^{-1/(pp')}\left|\left|x\right|\right|_p \leq \ H_A \ + H_{\mathcal{B}} \leq \left(5(2/\beta)\right)^{1/P}\left|\left|x\right|\right|_{a.b} + 5^{-1/p}\left|\left|x\right|\right|_p$$

whence

$$\left(e^{-1/p} - 5^{-1/p}\right)\left|\left|x\right|\right|_p \leq \left(5(2/\beta)\right)^{1/P}\left|\left|x\right|\right|_{a.b},$$

which leads to a final estimate of the form

$$\left|\left|x\right|\right|_p \leq Cpb^{-p'/p}\left|\left|x\right|\right|_{a,b},$$

with C a cow-tant independent of p sud b (and smaller than 50).

A study of various generalizations of the spaces $U_{a,b}$ will appear [83].

We shall recall the contrction of the spaces $X_{a,b}$, using a notation consistent with the original, but differing somewhat from it. As well seeming (to the author at least!) somewhat clearer, this notation appears to be better suited to poenbie generalization. The ingredients needed in a construction of this kind are a sequence of sets $\Delta_0, \Delta_1, \ldots$ and linear mappings that we shall denote by u,,. The next paragraph sets out the properties that these sets and mappings have to satisfy.

We suppose that the sets $\Delta_0, \Delta_1, \ldots$ are disjoint and finite, end that the union $\Gamma = \bigcup_{n\in N} \Delta_{\mathrm{n}}$ is infinite. For $n \geq 0$, we write $\Gamma_n = \bigcup_{m<n} \Delta_{\mathrm{m}}$ For each $n \geq 0$, we need to have a linear operator $u_n : \ell^\infty(\Gamma_n) \to \ell^\infty(\Delta_{\mathrm{n+1}})$ and we define $i_n \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma_{n+1})$ by setting

$$(i_n f)(y) = \begin{cases} f(y) & if\ y \in \Gamma_n, \\ (u_n f)(y) & if\ y \in \Delta_{\mathrm{n+1}}. \end{cases}$$

We define $i_{m,n}$: $\ell^\infty(\Gamma_m) \to \ell^\infty(\Gamma_n)$ to be the composition $i_{n-2} o,,,,,, o\ i_m\ \ i,,\_1\ o$ and note that, for $m < n < p$ and $f \in \ell^\infty(\Gamma_m)$ we have

$$(i_{m,p} f)\Gamma_{n+1} = i_{m,p} f.$$

It follows that we may well-define a linear mapping j_m: $\ell^\infty(\Gamma_m) \to \mathbb{R}^\Gamma$ by setting

$$(j_m f) = (i_{m,n} f)(\delta)\ (\delta \in \Gamma_{\mathrm{n}}).$$

We now make the further assumption that the mapping u_m have been defined in such way that the norms of all the compositions $i_{m,n}$ are bounded by some constant λ .This tells us that the mapping $j_m$take values in $\ell^\infty(\Gamma)$ a finite-dimensional subspace $X_m = imj_m$ of $\ell^\infty(\Gamma)$ with

$$||f|| \leq ||j_m f|| \leq \lambda ||f||$$

Finally, we take X to be the closure in $\ell^\infty(\Gamma)$ of the union of the increaseing sequence of subspace X_m .Since the subspace X_m are λ-isomorohic to $\ell^\infty(\Gamma)$,the space X is a separable $\mathcal{L}_\lambda^\infty$-space, whose properties are determined (in a way that is not always straightforward to decide) by the operators u_n,The tricky part of the construction lies in finding u_n,is which are such that the norm condition on the $i_{m,n}$, satisfied.

However the u_n are defined, the space X obtained in this way has some useful structure. Each of the subspaces X_n is the range of a projection S_n, defined by $S_n x = j_n(x|_{\Gamma_n})$. If we set $P_0 = S_0$ and $P_n = S_n - S_{n-1}$ (for $n \geq 1$), then the subspaces $M_n = \operatorname{im} P_n$ form a finite-dimensional decomposition of X. We refer to the support of a vector $x \in X$, we shall be thinking in terms of this f.d.d. Thus, if $x = \sum_m z_m$ with $z_m \in M_m$, then supp(x) will mean the set of m for which $z_m \neq 0$. Similarly, we shall say that the vectors $x_1, x_2, \ldots$ are successive if there exist natural numbers $m_1 \leq n_1 < m_2 \leq n_2 < m_3 \leq \cdots$ such that supp $x_k \subseteq [m_k, n_k]$. There is a relationship between this notion of support and the more obvious one where we are thinking of the vector x as a function on Γ; namely, supp $x \cap [0, n] = \emptyset \Leftrightarrow x|_{\Gamma_n} = 0$. It is also worth noting that, since the spaces $M_n = \{j_n(x) : x \in \ell^\infty(\Gamma_n)$ and $x|_{\Gamma_{n-1}} = 0\}$ are λ-isomorphic to $\ell^\infty(\Delta_n)$ and so have uniformly bounded basis constant, the space X has a basis. Such basis vectors occur as the w_γ in below (though the fact that they form a basis is not crucial there).

We now pass to the details of the Bourgain—Delbaen construction, Let a, b be real constants with $0 < b < 1/2 < a < 1$ and $a + b > 1$. We shall show how to construct the space $X_{a,b}$ by defining (recursively) the sets Δ_n and the mappings u_n. We start by taking Δ_0 to be a set with just one element, say $\Delta_0 = \{0\}$. Now we define

$$A_{n+1} = \{n+1\} \times \bigcup_{0 \le k < n} \{k\} \times \Gamma_k \times \Gamma_n \times \{\pm 1\}.$$

So an element of Δ_n is a 5-tuple of the form

$$\delta = \{n, \xi, \eta, \pm 1\}.$$

This notation replaces the explicit enumeration that appears in [84] and [85].

It will be convenient to have names for the five coordinates of δ:

$$n = \operatorname{rank}(\delta), k = \operatorname{cut}(\delta), \xi = \operatorname{base}(\delta), \eta = \operatorname{top}(\delta), \pm 1 = \operatorname{sign}(\delta).$$

The mapping $u_n : \ell^\infty(\Gamma_n) \to \ell^\infty(\Delta_{n+1})$ defined by

$$(u_n f)(n, k, \xi, n, \pm 1) = af(\xi) \pm b(f(\eta) - \left(i_{k,n}(f|_{\Gamma_k})\right)(\eta).$$

It is shown in [84], [85] that with the above definitions, the composite mappings $i_{m,n}$ are indeed uniformly bounded with

$$\|i_{m,n}\| \le \lambda = a/(1 - 2b).$$

It is perhaps worth repeating the original argument in our modified notation.

We assume inductively that, for some n, all the mappings $i_{m,n}$ $(m \le n)$ have norm at most λ. We now consider some $f \in \ell^\infty(\Gamma_m)$ and some $\gamma = (n+1, k, \xi, \eta, \pm 1) \in \Delta_{n+1}$. By definition,

$$\begin{aligned}|(i_{m,n+1}f)(\gamma)| &= (u_n i_{m,n} f)(\gamma)\\ &\le a|(i_{m,n}f)(\xi)| + b|(i_{m,n}f)(\eta) - (i_{k,n}((i_{m,n}f)|_{\Gamma_k}))(\eta)|.\end{aligned}$$

If the cut k is greater than m, then $i_{m,n}f = i_{k,n}((i_{m,k}f) = i_{k,n}\left((i_{m,n}f)|_{\Gamma_k}\right)$ so that the second term above vanishes, leaving $|(i_{m,n+1}f)(\gamma)| \le a\|i_{m,n}f\|$, which is at most $a\lambda\|f\|$ by our inductive hypothesis. If, on the other hand, $k \le m$, it must be that $\xi \in \Gamma_k \subseteq \Gamma_m$, so that $|(i_{m,n}f)(\xi)| = |f(\xi)| \le \|f\|$. Also, $(i_{m,n}f)|_{\Gamma_k} = f|_{\Gamma_k}$ an element of $\ell^\infty(\Gamma_k)$ satisfying $f|_{\Gamma_k} \le \|f\|$. Applying our inductive hypothesis to the two mappings $i_{m,n}$ ànd $i_{k,n}$ we obtain

$$\begin{aligned}|(i_{m,n+1}f)(\gamma)| &\le a|(i_{m,n}f)(\xi)| + b|(i_{m,n}f)(\eta) - (i_{k,n}f|_{\Gamma_k}(\eta)|\\ &\le a\|f\| + 2b\lambda\|f\|\end{aligned}$$

Since $a = (1 - 2b)\lambda$, this is at most $\lambda\|f\|$, as required.

The following proposition can also be found in [84].

PROPOSITION (3.1.3)[80]: Let k, m, n be natural numbers, with $m < n$, let x be an element of X_m, and let γ be an element of Γ with $\operatorname{rank}(\gamma) = n$, $\operatorname{cut}(\gamma) = k$.

$$|x(\gamma)| \le a\|x|_{\Gamma_k}\| + b\|(I - S_k)x\| \le \|S_k x\| + b\|(I - S_k)x\|.$$

Proof. Since $x \in X_m$, x has the form $j_m f$, for some $f \in \ell^\infty(\Gamma_m)$, and so

$$\begin{aligned}x(\gamma) &= (i_{m,n}f)(\gamma) = (i_{n-1} \, o \, i_{m,n-1}f)(\gamma)\\ &= a(i_{m,n-1}f)(\xi) \pm b[(i_{m,n-1}f) - (i_{k,n-1}(f|_{\Gamma_k})](\eta)\\ &= ax(\xi) \pm b(I - S_k)x(\eta),\end{aligned}$$

where $\xi = \operatorname{base}(\gamma)$ and $\eta = \operatorname{top}(\gamma)$ as usual. The inequality is now obvious.

COROLLARY (3.1.4)[80]: For any m and any $x \in X_m$, either $\|x\| = \|x|_{\Gamma_k}\|$ or

$$\|x\| = \max_k[\|x|_{\Gamma_k}\| + b\|(I - S_k)x\|].$$

It is apparent from the construction that for a general $f \in \ell^\infty(\Gamma_m)$ we may need to go to Γ_n, with n significantly larger than m, in order to find a coordinate γ at which $j_m f$ comes close to attaining its norms However, it is worth remarking that if $f \in \ell^\infty(\Gamma_m)$ and f is zero, except on Δ_m (the "last" of the sets that make up Γ_m), then $\|i_{m,n}f\| = \|f\|$ for all n. Thus in this set-up the subspaces M_n that make up the finite-dimensional decomposition of X are actually isometric to $\ell^\infty(\Delta_n)$.

It is implicitly shown in [84] that certain sequences in $X_{a,b}$ admit lower $U_{a,b}$-estirnates (and thus, as we can now see, lower ℓ^p-estimates). These are sequences of vectors which are successive (with respect to the f.d.d. (M_n)) and which have supports sufficiently well spread out. To make this precise we choose a function $F : \mathbb{N} \to \mathbb{N}$ having the property that, for every n and every non-zero $x \in X_n$,

$$\left\| x|_{\Gamma_{F(n)}} \right\| > \frac{1}{2}\|x\|$$

This is possible by compactness of the unit ball of the finite dimensional space X_n. We shall say that a (finite or infinite) sequence (y_k) in X is F-admissible if there are integers m_k, and n_k, satisfying $m_k \le n_k, F(n_k) + k < m_{k+1}$, with $y_k \in X_{n_k}, y_k|_{\Gamma_{F(m_k)}} = 0$. In terms ofthe f.d.d. (M_n) introduced earlier, we are saying that $y_k \in \bigoplus_{m_k<n\le n_k} M_n$ for all k, or equivalently that $\operatorname{supp}(y_k) \subseteq [m_k + 1, n_k]$. Evidently, if (y_k) is admissible then so is any sequence of successive linear combinations. The following lemma is related to Lemma 3.20 of [84].

LEMMA (3.1.5)[80]: If (y_k) is an F-admissible sequemee, then, for any l,

$$\left\|\sum_{k=1}^{l} y_k\right\| > \frac{1}{6}\|(\|y_1\|,\cdot\cdot\cdot,\|y_l\|)\|_{a,b}.$$

In particular, $\|y_j\| \le 6\left\|\sum_{k=1}^{l} y_k\right\|$ for each $1 \le j \le l$.

Proof. For each k let us write p_k and q_k for the minimum and maximum, respectively, of the support of y_k. The hypothesis of F-admissibility implies that $p_{k+1} > F(q_k) + k$. We shall show that, for each subinterval $I = [j, k]$ of $[1, l]$, there exists $\gamma \in \Gamma_{F(q_k)+k-j}$ such that

$$\left|\sum_{i=j}^{k} y_i(\gamma)\right| > \frac{1}{6}\left\|\left(\|y_j\|, \|y_{j+1}\|,\cdot\cdot\cdot,\|y_k\|\right)\right\|_{a,b}.$$

We may suppose, by induction on the length of I and a possible re-indexing, that $I = [1, l]$ and that the result has already been proved for all proper subintervals of $[1, l]$.

When we come to calculate $\|(\|y_1\|,\cdot\cdot\cdot,\|y_l\|)\|_{a,b}$, there are two possibilities, the first being where this norm equals $\|y_j\|$ for some j. By the defining property of the function F, there is some $\gamma \in \Gamma_{F(q_j)}$ with

$$|y_j(\gamma)| > \frac{1}{2}\|y_j\|.$$

For $i < j, y_i(\gamma) = 0$ by F-admissibility, and so

$$\left|\sum_{i=1}^{l} y_i(\gamma)\right| = \left|\sum_{i=1}^{j} y_i(\gamma)\right|.$$

Now if this quantity is at least $\frac{1}{6}\|y_j\|$, we are home. Otherwise, is must be that

$$\left\|\sum_{i=1}^{j-1} y_i\right\| \ge \left|\sum_{i=1}^{j-1} y_i(\gamma)\right| \ge |y_j(\gamma)| - \left|\sum_{i=1}^{j} y_i(\gamma)\right| > \left(\frac{1}{2}-\frac{1}{6}\right)\|y_j\| = \frac{1}{3}\|y_j\|.$$

Now we see that there exists $\delta \in \Gamma_{F(q_{j-1})}$ such that

$$\|(\|y_1\|,\cdots,\|y_l\|)\|_{a,b} = a\|(\|y_1\|,\cdots,\|y_k\|)\|_{a,b} + b\|(\|y_{k+1}\|,\cdots,\|y_l\|)\|_{a,b}.$$

By our inductive hypothesis, there exist $\xi \in \Gamma_{F(q_k)+k-1}$ and $\eta \in \Gamma_{F(q_l)+l-k-1}$ such that

$$\left|\sum_{i=1}^{k} y_i(\xi)\right| > \frac{1}{6}\|(\|y_1\|,\cdots,\|y_k\|)\|_{a,b},$$

$$\left|\sum_{i=k+1}^{l} y_i(\eta)\right| > \frac{1}{6}\|(\|y_{k+1}\|,\cdots,\|y_l\|)\|_{a,b}.$$

If we now consider the element

$$\gamma = (F(q_l) + l - k, F(q_k) + k - 1, \xi, \eta, \pm 1)$$

of $\Gamma_{F(q_l)+l-k}$ (with an appropriate choice of sign), we see that

$$\begin{aligned}\left|\sum_{i=1}^{k} y_i(\gamma)\right| &= a\left|\sum_{i=1}^{k} y_i(\xi)\right| + b\left|\sum_{i=k+1}^{l} y_i(\eta)\right| \\ &> \frac{1}{6}a\|(\|y_1\|,\cdots,\|y_k\|)\|_{a,b} + \frac{1}{6}b\|(\|y_{k+1}\|,\cdots,\|y_l\|)\|_{a,b} \\ &= \frac{1}{6}\|(\|y_1\|,\cdots,\|y_l\|)\|_{a,b}.\end{aligned}$$

It is also shown in [84] that, for certain carefully chosen admissible sequences,
there is an upper estimate as well. This is the way in which Bourgain and Delbaen show that $X_{a,b}$ is not isomorphic to $X_{a,b'}$ if $b \neq b'$ (and then deduce the existence of a continuum of non-isomorphic separable $\mathcal{L}^\infty$- spaces). Of course we can now see that these special sequences are ℓ^p-bases.

We shall shortly show that from every admissible sequence we can form a normalized sequence of successive linear combinations which is an ℓ^p-basis. Before going on to that, however, let us note that not every normalized admissible sequence is itself an ℓ^p basis. We note that the same calculation shows that $X_{a,b}$, with $\min supp\ y_1 > m$, such that

$$\sum_{j=1}^{2^k-1} \|y_j\|^p = 1, \left\|\sum_{j=1}^{2^k-1} y_j\right\| \geq k^{1/p'}.$$

Proof. We shall prove the statement by induction on k and shall show, moreover, that the construction may be carried out in such a way that the vector $\sum_{j=1}^{2^k-1} y_j$ attains a value of at least $k^{1/p'}$ at some element ξ of Γ. The construction will use some special vectors $\omega_\gamma(\gamma \in \Gamma)$ which we shall now define. For each $\gamma \in \Gamma$ we set $n = rank\ (\gamma)$ and let e_γ be the usual unit vector in $\ell^\infty(\Gamma_n)$ define by $e_\gamma(\delta) = 1$ if $\delta = \gamma$ and 0 otherwise. We then define $w_\gamma = j_n(e_\gamma)$, noting that $\|w_\gamma\| = 1$.

We now pass to the inductive proof. For $k = 1$ there is of course no real problem, but in order to be sure about attainment of the norm, we might as well be specific, taking y_1 to be w_γ with rank (γ) sufficiently large.

Now suppose that the result is true for k. Given m there exist successive, F-admissible vectors $y'_1, \ldots, y'_{2^k-1}$, with $\min \text{supp}\ y'_1 > m$, together with an element ξ' of Γ such that

$$\sum_{j=1}^{2^k-1}\|y_j'\|^p=1,\quad \sum_{j=1}^{2^k-1} y_j'(\xi')\geq k^{1/p'}.$$

We now use our inductive hypothesis again to obtain $y_1'',\dots,y_{2^k-1}''$ and ξ'' satisfying the same conditions, and with

$$\min\operatorname{supp}\, y_1''>\max\{\operatorname{rank}(\xi'),\ F(\max\operatorname{supp}\, y_{2^k-1}'+2^k-1)\}.$$

We choose $n>\max\{\operatorname{rank}(\xi''),\ F(\max\operatorname{supp}\, y_{2^k-1}''+2^{k+1}-2)\}$ and take $\xi\in\Delta_n$ to be

$$\xi=(n,\operatorname{rank}(\xi'),\xi',\xi'',1).$$

Finally, we define $y_1,\dots,y_{2^{k+2}-1}$ by

$$y_j=\begin{cases}\dfrac{a^{p'-1}k^{1/p}}{(k+1)^{1/p}} & (1\leq j\leq 2^k-1),\\ \dfrac{b^{p'-1}k^{1/p}}{(k+1)^{1/p}}y_{j-2^k-1}'' & (2^k\leq j\leq 2^{k+1}-2),\\ (k+1)^{1/p}w_\xi & (j=2^k-1).\end{cases}$$

By construction, the sequence $y_1,\dots,y_{2^{k+1}-1}$ is F-admissible, and

$$\sum_{j=1}^{2^{k+1}-1}\|y_j\|^p=\frac{ka^{p'}}{k+1}\sum_{j=1}^{2^k-1}\|y_j'\|^p+\frac{kb^{p'}}{k+1}\sum_{j=1}^{2^k-1}\|y_j''\|^p+\frac{\|w_\xi\|}{k+1}$$
$$=\frac{ka^{p'}+kb^{p'}+1}{k+1}=1.$$

When we evaluate at ξ we obtain

$$\sum_{j=1}^{2^k-1}y_j(\xi)=a\frac{a^{p'-1}k^{1/p}}{(k+1)^{1/p}}\sum_{j=1}^{2^k-1}y_j'(\xi')$$
$$+b\frac{b^{p'-1}k^{1/p}}{(k+1)^{1/p}}\sum_{j=1}^{2^k-1}y_j''(\xi'')+\frac{1}{(k+1)^{1/p}}$$
$$\geq\frac{a^{p'}k^{1/p}}{(k+1)^{1/p}}k^{1/p'}+\frac{b^{p'}k^{1/p}}{(k+1)^{1/p}}k^{1/p'}+\frac{1}{(k+1)^{1/p}}$$
$$\geq\frac{(a^{p'}+b^{p'})k}{(k+1)^{1/p}}+\frac{1}{(k+1)^{1/p}}=(k+1)^{1/p'}.$$

COROLLARY (3.1.6)[80]: There exist normalized F-admissible sequences that are not equivalent to the usual ℓ^p-basis.

Proof. It is clear that such sequences may be constructed by normalizing and sticking together finite sequences of the kind obtained.

In view of what we have just seen it is clear that we shall have to work a bit harder in order to find ℓ^p-bases in $X_{a,b}$. We shall start with an arbitrary normalized F-admissible sequence (y_n) and then form further linear combinations. As a piece of temporary terminology, we shall say that a vector x has height h, and write $h(x)\ =\ h$, if x is a linear combination

$$x=\sum_{l=m}^{n}\alpha_l y_l$$

with $h = \max_i |\alpha_i|$. When I is a non-empty finite interval of integers, we shall write 1^* for the subinterval obtained by removing the end-points of I: thus $I^* = I\backslash\{\max I, \min I\}$.

PROPOSITION (3.1.7)[80]: There is a constant $c > 0$, depending only on a and b, with the following property: for any normalized F-admissible sequence (y_n), any sequence (x_r) of successive linear combinations, any finite interval I and any $\gamma \in \Gamma$, we have

$$c\left\|\sum_{i\in I} x_i\right\| \leq \left(\|x_{\min I}\|^p + 2\sum_{i\in I^*} 6^p\|x_i\|^p + \|x_{\max I}\|^p\right)^{1/p} \tag{1}$$

moreover, for γ with $\operatorname{rank}(\gamma) > \max \operatorname{supp} x_{\max I}$, we have

$$c\left|\sum_{i\in I} x_i(\gamma)\right| \leq \left(\|x_{\min I}\|^p + 2\sum_{i\in I^*} 6^p\|x_i\|^p + \|x_{\max I}\|^p\right)^{1/p} \\ +4\sum_{i\in I^*} h(x_i) + \frac{3}{2}h(x_{\max I}).$$

In fact, the constant c may be taken to be whichever is smaller of b and $2^{-1/p'}a$.

Proof. We proceed by induction on the length of the interval I, assuming that (1) and (2) hold for all sequences of successive linear combinations of the y_j, and all intervals shorter than I. (Of course, the case of an interval containing only one natural number is trivial.) For convenience, we shall take I to be the interval $[1, l]$; let us write x for the sum $\sum_{i=1}^{l} x_i$. We consider an arbitrary $\gamma \in \Gamma$; our aim is to show that

$$c|x(\gamma)| \leq \left(\|x_1\|^p + 2\sum_{i=2}^{l-1} 6^p\|x_i\|^p + \|x_{\max I}\|^p\right)^{1/p} \\ +4\sum_{i=2}^{l-1} h(x_i) + 3h(x_i), \tag{2}$$

with

$$c|x(\gamma)| \leq \left(\|x_1\|^p + 2\sum_{i=2}^{l-1} 6^p\|x_i\|^p + \|x_{\max I}\|^p\right)^{1/p} \\ +4\sum_{i=2}^{l-1} h(x_i) + \frac{3}{2}h(x_i), \tag{3}$$

in the special case where $\operatorname{rank}(\gamma) > \max \operatorname{supp} x_i$.

We may assume that $\operatorname{rank}(\gamma) \geq \min \operatorname{supp} x_i$. Indeed, otherwise we have $x(\gamma) = \sum_{j=1}^{i-1} x_j(\gamma)$ and our inductive hypothesis may be applied. This assumption about the rank of γ will be useful since it will allow us to apply PROPOSITION (3.1.3) to vectors like $\sum_{j=1}^{i-1} x_j$.

Let us now write $k = \operatorname{cut}(\gamma)$; we shall deal first with the two cases $k < \min \operatorname{supp} x_2$ and $k > \max \operatorname{supp} x_{l-1}$. In the first of these cases, we may estimate $|x(\gamma)|$ as follows:

$$|x(\gamma)| \leq \|x_1\| + \left|\sum_{i=2}^{i-1} x_i(\gamma)\right| + \|x_l\|$$

$$\leq \|x_1\| + \|x_l\| + a\left\|S_k\left(\sum_{i=2}^{l-1} x_i\right)\right\| + b\left\|(I - S_k)\left(\sum_{i=2}^{l-1} x_i\right)\right\| \text{ (by Prop. 1)}$$

$$= \|x_1\| + \|x_l\| + b\left\|\sum_{i=2}^{l-1} x_i\right\|.$$

Now the interval $[2, l — 1]$ is one to which our inductive hypothesis is applicable, so that we obtain

$$c|x(\gamma)| \leq c\|x_1\| + c\|x_l\| + b\left(\|x_2\|^p + 2\sum_{i=3}^{l-2} 6^p\|x_i\|^p + \|x_{i-1}\|^p\right)^{1/p}$$

$$+4b\sum_{i=3}^{l-2} h(x_i) + 3bh(x_{i-1})$$

$$\leq \left(2c^{p'} + b^{p'}\right)^{1/p'}\left(\|x_1\|^p + \|x_2\|^p + 2\sum_{i=3}^{l-2} 6^p\|x_i\|^p + \|x_{i-1}\|^p + \|x_i\|^p\right)^{1/p}$$

$$+4b\sum_{i=3}^{l-1} h(x_i),$$

by Hölder's inequality. Comparing terms and recalling that $b < 1/2$, we see that this implies inequality (2), provided that $2c^{p'} + b^{p'} \leq 1$, or equivalently $c \leq 2^{-1/p'}\left(1 - b^{p'}\right)^{1/p'} = 2^{-1/p'}a$.

The argument in the case $k >$ max supp x_{i-1} is similar:

$$c|x(\gamma)| \leq c\|x_l\| + c\left|\sum_{i=2}^{i-1} x_i(\gamma)\right|$$

$$\leq c\|x_l\| + ac\left\|S_k\left(\sum_{i=2}^{l-1} x_i\right)\right\| + bc\left\|(I - S_k)\left(\sum_{i=2}^{l-1} x_i\right)\right\| \text{ (by Prop. 1)}$$

$$= c\|x_l\| + ac\left\|\sum_{i=1}^{l-1} x_i\right\|$$

$$\leq c\|x_l\| + a\left(\|x_1\|^p + 2\sum_{i=2}^{l-2} 6^p\|x_i\|^p + \|x_{i-1}\|^p\right)^{1/p}$$

$$+4a\sum_{i=2}^{l-2} h(x_i) + 3ah(x_{i-1}),$$

$$\leq \left(c^{p'} + a^{p'}\right)^{1/p'}\left(\|x_1\|^p + 2\sum_{i=3}^{l-2} 6^p\|x_i\|^p + \|x_{i-1}\|^p + \|x_i\|^p\right)^{1/p}$$

$$+4\sum_{i=2}^{l-1} h(x_i),$$

which implies inequality (2) provided $c^{p'} + a^{p'} \leq 1$ or equivalently $c \leq b$.
From now on, we shall assume that $\min \operatorname{supp} x_2 \leq k = \operatorname{cut}(\gamma) \leq \max \operatorname{supp} x_{i-1}$. We consider next the case where $\operatorname{rank}(\gamma) > \max \operatorname{supp} x_i$ and need to establish inequality (2).
An easy case is where the cut k lies between the supports of consecutive x_i's, say max supp $x_i^* < k <$ min supp x_{i^*+1} (where $2 \leq i^* \leq l-2$ by what we have just proved). By our inductive hypothesis, we have inequality (1) for each of the intervals $[1, i^*]$ and $[i^* + 1, l]$. Moreover, PROPOSITION (3.1.3) is applicable, giving

$$c|x(\gamma)| \leq ca\|S_k x\| + cb\|(I - S_k)x\| = ac\left\|\sum_{i \leq i^*} x_i\right\| + bc\left\|\sum_{i > i^*} x_i\right\|$$

$$\leq a\left[\left(\|x_1\|^p + 2\sum_{i<i<i^*} 6^p\|x_i\|^p + \|x_{i^*}\|^p\right)^{1/p} + 4\sum_{i<i<i^*} h(x_i) + 3h(x_{i^*})\right]$$

$$+b\left[\left(\|x_{i^*+1}\|^p + 2\sum_{i^*+1<i<l} 6^p\|x_i\|^p + \|x_i\|^p\right)^{1/p} + 4\sum_{i^*+1<i<l} h(x_i) + 3h(x_i)\right]$$

$$\leq \left(\|x_1\|^p + 2\sum_{i<i<i^*} 6^p\|x_i\|^p + \|x_i\|^p\right)^{1/p} + 4\sum_{i<i<l} h(x_i) + \frac{3}{2}h(x_i),$$

by Hölder's inequality and the facts that $a < 1, b < 1/2$.
A slightly more complicated case arises if min supp $x_i \leq k \leq$ max supp x_i for some $i = i^*$, say. By what we proved earlier, it must be that $1 < i^* < l$.
We now study the fine structure of the vector x_{i^*}, recalling that

$$x_{i^*} = \sum_{n_{i^*-1}<j\leq n_{i^*}} \alpha_j y_j.$$

We may suppose that k is somewhere between min supp y_{j^*} and max supp y_{j^*}, for some j^*. We then set

$$x_{i^*}^L = \sum_{n_{i^*-1}<j<j^*} \alpha_j y_j, \qquad x_{i^*}^R = \sum_{j^*<j\leq n_{i^*}} \alpha_j y_j,$$

$$X^L = x_1 + x_2 + \ldots + x_{i^*-1} + x_{i^*}^L, \qquad X^R = x_{i^*}^R + x_{i^*+1} + \ldots + x_i.$$

By minimality of l and the fact that $1 < i^* < l$, inequality (1) is true for the vectors x^R and x^L. Hence we have

$$c\|x^L\| \leq \left(\|x_1\|^p + 2\sum_{i<i<i^*} 6^p\|x_i\|^p + \left\|x_{i^*}^L\right\|^p\right)^{1/p} + 4\sum_{i<i<i^*} h(x_i) + 3h\left(x_{i^*}^L\right)$$

$$\leq \left(\|x_1\|^p + 2\sum_{i<i<i^*} 6^p\|x_i\|^p + \left\|x_{i^*}^L\right\|^p\right)^{1/p} + 4\sum_{i<i<i^*} h(x_i) + 3h(x_{i^*}),$$

since $h(x_{i^*}^L) \leq h(x_{i^*})$ by the definition of the function h, and

$$c\|x^R\| \leq \left(\left\|x_{i^*}^R\right\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_i\|^p\right)^{1/p} + 4\sum_{i^*<i<l} h(x_i) + 3h(x_{i^*}).$$

If we now write $x^* + x^L + x^R = x - a_{j^*}y_{j^*}$, and apply PROPOSITION (3.1.3), we obtain

$$c|x(\gamma)| \leq c|x^*(\gamma)| + c\left|\alpha_{j^*}\right| \leq ac\|S_k x^*\| + bc\|(I - S_k)x^*\| + ch(x_{i^*})$$

$$= ac\|x^L\| + bc\|x^R\| + ch(x_{i^*})$$

$$\leq a\left[\left(\|x_1\|^p + 2\sum_{i<i<i^*} 6^p\|x_i\|^p + \left\|x_{i^*}^L\right\|^p\right)^{1/p} + 4\sum_{i<i<i^*} h(x_i) + 3h(x_{i^*})\right]$$

$$+b\left[\left(\left\|x_{i^*}^R\right\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\right)^{1/p} + 4\sum_{i^*<i<l} h(x_i) + 3h(x_l)\right] + ch(x_{i^*})$$

$$\leq \left(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \left\|x_{i^*}^L\right\|^p + \left\|x_{i^*}^R\right\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\right)^{1/p}$$

$$+4\sum_{i<i<i^*} h(x_i) + (3+c)h(x_{i^*}) + 4\sum_{i^*<i<l} h(x_i) + 3bh(x_l)$$

using Hölder's inequality and the values of a and b as before. LEMMA (3.1.5), applied to the admissible sequence $(x_{i^*}^L, \alpha_j y_j, x_{i^*}^R)$, implies that each of $\left\|x_{i^*}^L\right\|$ and $\left\|x_{i^*}^R\right\|$ is at most $6\|x_{i^*}\|$ so that we can finally write

$$c|x(\gamma)| \leq \left(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\right)^{1/p} + 4\sum_{1<i<l} h(x_i) + \frac{3}{2}h(x_l),$$

which is inequality (2) as required1 (Of course, we have also used the facts that $b < 1/2$ and $3 + c < 4$.)

To finish the proof, we now need to look at $|x(\gamma)|$ where $\operatorname{rank}(\gamma)$ max sup x_l and show that inequality $(1')$ holds. We do this by another induction, this time on the number $n_l - n_{l-1}$ of non-zero coefficients in the expression for the last vector x_l as a linear combination of the y_j. We set

$$x_l^* = \sum_{n_{l-1}<j<n_l} \alpha_j y_j = x_l - \alpha_{n_l} y_{n_l}, \quad x^* = \sum_{1\leq i\leq l} x_i + x_l^* = x - \alpha_{n_l} y_{n_l}.$$

Our additional inductive hypothesis is applicable to x^*, and if $\operatorname{rank}(\gamma) < \min\operatorname{supp}\ y_{n_l}$, we have $x(\gamma) = x^*(\gamma)$, giving the result immediately. If, on the other hand, $\operatorname{rank}(\gamma) \geq \min\operatorname{supp}\ y_{n_l} > \max\operatorname{supp}\ x^*$, it is inequality (2) which holds for x^*. Thus we obtain

$$c|x(\gamma)| \leq c|x^*(\gamma)| + c\left|\alpha_{n_l} y_{n_l}(\gamma)\right|$$

$$\leq \left(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l^*\|^p\right)^{1/p}$$

$$+4\sum_{1<i<l} h(x_i) + \frac{3}{2}h(x_l^*) + c\left|\alpha_{n_l}\right|$$

$$\leq \left(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\right)^{1/p} + \|x_l^* - x_l\|$$

$$+4\sum_{1<i<l} h(x_i) + \frac{3}{2}h(x_l^*) + c\left|\alpha_{n_l}\right| \quad \text{(by Minkowski's inequality)}$$

$$\leq \left(\|x_1\|^p + 2\sum_{i^*<i<l} 6^p\|x_i\|^p + \|x_l\|^p\right)^{1/p}$$

$$+4\sum_{1<i<l} h(x_i)+\frac{3}{2}h(x_l^*)+(c+1)\left|\alpha_{n_l}\right|$$
$$\leq\left(\|x_1\|^p+2\sum_{i^*<i<l}6^p\|x_i\|^p+\|x_l\|^p\right)^{1/p}+4\sum_{1<i<l} h(x_i)+3h(x_l)$$
since $c<1/2$ and $h(x_l)=\max\{\left|\alpha_{n_l}\right|,h(x_i^*)\}$. We have thus established inequality (2) as required.

Theorem (3.1.8)[80]: Let a,b be real constants satisfying $0<b<1/2<a<1, a+b>1$ and let np,p' be given by $1/p+1/p'=1=a^{p'}+b^{p'}$. Every closed infinite-dimensional subspace of $X_{a,b}$ has a subspace isomorphic to ℓ^p.

Proof. By a standard approximation argument, it is enough to consider the case of a subspace Y which is the closed linear span of a normalized F-admissible sequence (y_j). Because of the lower estimates of LEMMA (3.1.5) and Theorem (3.1.2), we may construct successive linear combinations z_i with $\|z_i\|=1$ and $h(z_i)$ very small, say $\sum_{i=1}^{\infty}h(z_i)<1$. Now, for arbitrary $l\in\mathbb{N}$ and arbitrary scalars β_i, we may apply the above proposition to the vectors $x_i=\beta_i z_i$, obtaining
$$c\left\|\sum_{i=1}^{l}\beta_i z_i\right\|\leq 12\left(\sum_{i=1}^{l}|\beta_i|^p\right)^{1/p}+4\sum_{i=1}^{l}|\beta_i|h(z_i)\leq 16\left(\sum_{i=1}^{l}|\beta_i|^p\right)^{1/p}.$$
On the other hand, from LEMMA (3.1.5) and Theorem (3.1.2) again, we get the lower estimate
$$\left\|\sum_{i=1}^{l}\beta z_i\right\|\geq\frac{1}{6}\|(\beta_1,\ldots,\beta_l)\|_{a,b}\geq d\left(\sum_{i=1}^{l}|\beta_i|^p\right)^{1/p},$$
where d is a strictly positive constant.

Section (3.2): The Scalar–Plus–Compact Problem

The question of whether there exists a Banach space X on which every bounded linear operator is a compact perturbation of a scalar multiple of the identity has become known as the "Scalar–plus–Compact Problem". It is mentioned by Lindenstrauss as Question 1 in his 1976 list of open problems in Banach space theory [114]. Lindenstrauss remarks that, by the main theorem of [101] or [115], every operator on a space of this type has a proper non-trivial invariant subspace. Related questions go further back: for instance, Thorp [123] asks whether the space of compact operators $\mathcal{K}(X;Y)$ can ever be a proper complemented subspace of $\mathcal{L}(X;Y)$. On the Gowers–Maurey space $\mathfrak{X}_{\mathrm{gm}}$ [111], every operator is a strictly singular perturbation of a scalar, and other hereditarily indecomposable (HI) spaces also have this property. Indeed it seemed for a time that $\mathfrak{X}_{\mathrm{gm}}$ might already solve the scalar–plus–compact problem. However, after Gowers[110] had shown that there is a strictly singular, non-compact operator from a subspace of $\mathfrak{X}_{\mathrm{gm}}$ to $\mathfrak{X}_{\mathrm{gm}}$, Androulakis and Schlumprecht [95] showed that such an operator can be defined on the whole of $\mathfrak{X}_{\mathrm{gm}}$. Gasparis [108] has done the same for the Argyros–Deliyanni space $\mathfrak{X}_{\mathrm{ad}}$ of [96].

We solve the scalar–plus–compact problem by combining techniques that are familiar from other HI constructions with an additional ingredient, the Bourgain– Delbaen method for constructing special ℓ_∞-spaces [85]. The initial motivation for combining these two constructions was to exhibit a hereditarily indecomposable predual of ℓ_1; such a space is, in some sense, the extreme example of a known phenomenon—that the HI property does not pass from a space to its dual [107], [100], [97]. it turned out that the additional structure was just what we needed to show that strictly singular operators are compact. It is interesting, perhaps, to note that the Schur property of ℓ_1 does not play a role in our proof and, indeed, we have no general result to say that an HI predual of ℓ_1 necessarily has the scalar–plus–compact property. We use in an essential way the specific structure of the BD construction, which embeds into our space some very explicit finite-dimensional ℓ_∞-spaces. As well as the (now) classical machinery of HI constructions—a space of Schlumprecht type, Maurey–Rosenthal coding and rapidly increasing sequences based on ℓ_1-averages—we add the possibility of splitting an arbitrary vector into pieces of comparable norm, while staying in one of these ℓ_∞^n's. This allows us to introduce two additional classes of rapidly increasing sequences, and these in turn lead to the stronger result about operators.

If A is any set, $\ell_\infty(A)$ is the space of all bounded (real-valued) functions on A, equipped with the supremum norm $\|\cdot\|_\infty$ and $\ell_1(A)$ is the space of all absolutely summable functions on A, equipped with the norm $\|x\|_1 = \sum_{a\in A}|x(a)|$. The support of a function x is the set of all a such that $x(a) \neq 0$; $c_{00}(A)$ is the space of functions of finite support. We shall write ℓ_p for the space $\ell_p(\mathbb{N})$, where $\mathbb{N}$ is the set $\{1, 2, 3, \dots\}$ of positive integers, and ℓ_p^n for $\ell_p(\{1, 2, \dots, n\})$. Even when we are dealing with these sequence spaces we shall use function notation $x(m)$, rather than subscript notation, for the m^{th} coordinate of the vector x.

When x and y are in $c_{00}(A)$ (and more generally) we shall write $\langle y, x\rangle$ for $\sum_{a\in A} x(a)y(a)$.

If we are thinking of y as a functional acting on x^* (rather than vice versa) we shall usually choose a notation involving a star, denoting y by f^*, or something of this kind. In particular, e_a and e_a^* are two notations for the same unit vector in $c_{00}(A)$ (given by $e_a(a') = \delta_{a,a'}$), to be employed depending on whether we are thinking of it as a unit vector or as the evaluation functional $x \mapsto \langle e_a^*, x\rangle = x(a)$.

We say that (finitely or infinitely many) vectors $z_1, z_2, \dots$ in c_{00} are successive, or that (z_i) is a block-sequence, if max supp $x_i <$ min supp x_{i+1} for all i. In a Banach space X we say that vectors y_j are successive linear combinations, or that (y_j) is a block sequence of a basic sequence (x_i) if there exist $0 = q_1 < q_2 < \cdots$ such that, for all $j \geq 1, y_j$ is in the linear span $[x_i : q_{j-1} < i \leq q_j]$. If we may arrange that $y_j \in [x_i : q_{j-1} < i < q_j]$ we say that (y_j) is a skipped block sequence. More generally, if X has a Schauder decomposition $X = \bigoplus_{n\in\mathbb{N}} F_n$ we say that (y_j) is a block sequence (resp. a skipped block sequence) with respect to (F_n) if there exist $0 = q_0 < q_1 < \bullet\bullet\bullet$ such that y_j is in $\bigoplus_{q_{j-1}<n\leq q_j} F_n$ (resp. $\bigoplus_{q_{j-1}<n<q_j} F_n$. A block subspace is the closed subspace generated by a block sequence.

A Banach space X is indecomposable if there do not exist infinite-dimensional closed subspaces Y and Z of X with $X = Y \oplus Z$, and is hereditarily indecomposable (HI) if every closed subspace is indecomposable. The following useful criterion, like so much else in this in this area, goes back to the original of Gowers and Maurey [111].

Proposition (3.2.1)[91]: Let X be a an infinite dimensional Banach space. Then X is HI if and only if, for every pair Y, Z of infinite-dimensional subspaces, and every $\epsilon > 0$, there exist $y \in Y$ and $z \in \mathbb{Z}$ with $\|y + z\| > 1$ and $\|y - z\| < \epsilon$. If X has a finite-dimensional decomposition $(F_n)_{n\in\mathbb{N}}$ it is enough that the above should hold for block subspaces.

We shall make use of the following well-known blocking lemma, the first part of which can be found as Lemma 1 of [116]. The proof of the second part is very similar, and, as Maurey remarks, both can be traced back to R.C. James [112].

Lemma (3.2.2)[91]: Let $n \geq 2$ be an integer, let $\epsilon \in (0,1)$ be a real number and let N be an integer that can be written as $N = n^k$ for some $k \geq 1$. Let $(x_i)_{i=1}^N$ be a sequence of vectors in the unit sphere of a Banach space X.

(i) If $\left\|\sum_{i=1}^N \pm x_i\right\| \geq (n - \epsilon)^k$ for all choices of signs ± 1, then there is a block sequence $y_1, y_2, \ldots, y_n \in [x_i : 1 \leq i \leq N]$ which is $(1 - \epsilon)^{-1}$-equivalent to the unit-vector basis of ℓ_1^n.

(ii) If $\left\|\sum_{i=1}^N \pm x_i\right\| \leq (1 + \epsilon)^k$ for all choices of signs ± 1, then there is a block sequence $y_1, y_2, \ldots, y_n \in [x_i : 1 \leq i \leq N]$ which is $(1 + \epsilon)$-equivalent to the unit-vector basis of ℓ_∞^n.

A separable Banach space X is an $L_{\infty,\lambda}$-space if there is an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite dimensional subspaces of X such that the union $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X and, for each n, F_n is λ-isomorphic to $\ell_\infty^{\dim F_n}$. It is known [113] that if a separable L_∞ space X has no subspace isomorphic to ℓ_1, then the dual space X^* is necessarily isomorphic to ℓ_1. This implies that the dual of a separable, hereditarily indecomposable L_∞-space is isomorphic to ℓ_1.

The Bourgain–Delbaen spaces $X_{a,b}$, which inspired the construction given, were the first examples of L_∞ spaces not containing c_0.

All existing HI constructions have, somewhere at the heart of them, a space of Schlumprecht type; rather than working with the original space of [122], we find it convenient to look at a different mixed Tsirelson space. We recall some notation and terminology from [99]. Let $(l_j)_j$ be a sequence of positive integers and let $(\theta_j)_j$ be a sequence of real numbers with $0 < \theta_n < 1$. We define $W\,[(A_{l_j}, \theta_j)_j]$ to be the smallest subset W of c_{00} with the following properties

(i) $\pm e_k^* \in W$ for all $k \in \mathbb{N}$;

(ii) whenever $f_1^*, f_2^*, \ldots, f_m^* \in W$ are successive vectors, $\theta_j \sum_{i\leq m} f_i^* \in W$, provided $m \leq l_j$.

We say that an element f^* of W is of Type 0 if $f^* = \pm e_k^*$ for some k and of Type I otherwise; an element of type I is said to have weight θ_j if $f^* = \theta_j \sum_{i\leq m} f_i^*$ for a suitable sequence (f_i) of successive elements of W.

The mixed Tsirelson space $T\,[(A_{l_j}, \theta_j)_j]$ is defined to be the completion of c_{00} with respect to the norm

$$\|x\| = \sup\{\langle f^*, x\rangle : f^* \in W\,[(A_{l_j}, \theta_j)_j]\}.$$

We may also characterize the norm of this space implicitly as being the smallest function $x \mapsto \|x\|$ satisfying

$$\|x\| = \max\left\{\|x\|_\infty \sup \theta_j \sum_{i=1}^{l_j} \left\|x\chi_{E_i}\right\|\right\},$$

where the supremum is taken over all j and all sequences of finite subsets $E_1 < E_2 < \cdots < E_{l_j}$. Schlumprecht's original space is the result of taking $l_j = j$ and $\theta_j = (\log_2(j + 1))^{-1}$

We shall choose to work with two sequences of natural numbers (m_j) and (n_j). We require m_j to grow quite fast, and n_j to grow even faster. The precise requirements are as follows.

Assumption (3.2.3)[91]: We assume that $\left(m_j\,,n_j\right)_{j\in\mathbb{N}}$ satisfy the following:

(i) $m_1 \ \ge\ 4$;

(ii) $m_{j+1} \ \ge\ m_j^2$;

(iii) $n_{j+1} \ \ge\ m_{j+1}^2\left(4n_j\right)^{2^{j+1}}$.

A straightforward way to achieve this is to assume that (m_j, n_j) is some subsequence of the sequence $\left(2^{2j}, 2^{2j^2+1}\right)_{j\in\mathbb{N}}$. From now on, whenever m_j and n_j appear, we shall assume we are dealing with sequences satisfying (3.2.3).

The following lemma can be found as II.9 of [99]. The proof is not affected by the small change we have made in the definition of the sequences $\left(n_j\right)_j$ and $\left(m_j\right)_j$.

Lemma (3.2.4)[91]: If $j \in \mathbb{N}$ and $f \ \in\ W\ \left[(A_{4n_j}, m_j^{-1})_j\right]$ is an element of weight m_h, then

$$\left|\langle f^*, n_{j_0}^{-1}\sum_{j=1}^{n_{j_0}} e_l\rangle\right| \le \begin{cases} 2m_h^{-1}m_{j_0}^{-1} & \text{if } i <\ j_0 \\ \ \ m_h^{-1} & \text{if } i\ \ge\ j_0. \end{cases}$$

In particular, the norm of $n_{j_0}^{-1}\sum_{j=1}^{n_{j_0}} e_l$ in $T[(A_{4n_j}, m_j^{-1})_j\ ]$ is exactly $m_{j_0}^{-1}$

If we restrict attention to $f\ \in\ W[(A_{4n_j}, m_j^{-1})_{j\ne j_0}]$ then

$$\left|\langle f^*, n_{j_0}^{-1}\sum_{j=1}^{n_{j_0}} e_l\rangle\right| \le \begin{cases} 2m_h^{-1}m_{j_0}^{-2} & \text{if } i <\ j_0 \\ \ \ m_h^{-1} & \text{if } i\ \ge\ j_0. \end{cases}$$

In particular, the norm of $n_{j_0}^{-1}\sum_{j=1}^{n_{j_0}} e_l$ in $T[(A_{4n_j}, m_j^{-1})_{j\ne j_0}]$ is at most $m_{j_0}^{-2}$.

We shall present a generalization of the Bourgain–Delbaen construction of separable L_∞-spaces. Our approach is slightly different from that of [102] and [85], but the mathematical essentials are the same. We choose to set things out in some detail partly because we believe our approach yield new insights into the original BD construction, and partly because the calculations presented here are a good introduction to the notations and methods we use later. It is perhaps worth emphasizing here that BD constructions are very different from the majority of constructions that occur in Banach space theory. Normally we start with the unit vectors in the space c_{00} and complete with respect to some (possibly exotic) norm. The only norms that occur in a BD construction are the usual norms of ℓ_∞ and ℓ_1. What we construct here are exotic vectors in ℓ_∞ whose closed linear span is the space we want.

The idea will be to introduce a particular kind of (conditional) basis for the space ℓ_1 and to study the subspace X of ℓ_∞ spanned by the biorthogonal elements. Since ℓ_1 is then in a natural way a subspace of (and in some cases the whole of) X^*, we shall be thinking of elements of ℓ_1 as functionals and, in accordance with the convention explained earlier, denote them b^*, c^* and so on. In our initial discussion we shall consider the space $\ell_1(\mathbb{N})$ (which we shall later replace with $\ell_1(\Gamma)$ with Γ a certain countable set better adapted to our needs).

Definition (3.2.5)[91]: We shall say that a basic sequence $(d_n^*)_{n\in\mathbb{N}}$ in $\ell_1(N)$ is a triangular basis if supp $d_n^* \ \subseteq\ \{1, 2, \dots, n\}$, for all n. We thus have

$$d_n^* = \sum_{m=1}^{n} a_{n,m} e_m^*,$$

where, by linear independence, we necessarily have $a_{n,n} \neq 0$. Notice that the linear span $[d_1^*, d_2^*, \dots, d_n^*]$ is the same as $[e_1^*, e_2^*, \dots, e_n^*]$, that is to say, the space $\ell_1(n)$, regarded as a subspace of $\ell_1(\mathbb{N})$ in the usual way. So, in particular, the basic sequence (d_n^*) is indeed a basis for the whole of ℓ_1. The biorthogonal sequence in ℓ_∞ will be denoted (d_n); it is a weak* basis for ℓ_∞ and a basis for its closed linear span, which will be our space X.

Proposition (3.2.6)[91]: If (d_n^*) is a triangular basis for $\ell_1(N)$, with basis constant M, then the closed linear span $X = [d_n : n \in \mathbb{N}]$ is a $L_{\infty,M}$ -space. If (d_n^*) is boundedly complete, or equivalently (d_n) is shrinking, then X^* is naturally isomorphic to $\ell_1(\mathbb{N})$ with $\|g^*\|_{X^*} \leq \|g^*\|_1 \leq M\|g^*\|_{X^*}$.

Proof. In accordance with our "star" notation, let us write P_n^* for the basis projection $\ell_1 \to \ell_1$ associated with the basis (d_n^*). Thus $P_n^*(d_m^*)$ equals d_m^* if $m \leq n$ and 0 otherwise; because $e_m^* \in \ell_1(n) = [d_1^*, \dots, d_n^*]$, we also have $P_n^* e_m = e_m$ when $m \leq n$. If we modify P_n^* by taking the codomain to be the image im $P_n^* = \ell_1(n)$, rather than the whole of ℓ_1, what we have is a quotient operator, which we shall denote q_n, of norm at most M . The dual of this quotient operator is an isomorphic embedding in : $\ell_\infty(n) \to \ell_\infty(\mathbb{N})$, also of norm at most M . If $m \leq n$ and $u \in \ell_\infty(n)$ we have

$$(i_n u)(m) = \langle e_m^*, i_n u\rangle = \langle q_n e_m^*, u\rangle = \langle e_m^*, u\rangle = u(m).$$

So in is an extension operator $\ell_\infty^n \to \ell_\infty(\mathbb{N})$ and we have

$$\|u\|_\infty \leq \|i_n u\|_\infty \leq M\|u\|_\infty$$

for all $u \in \ell_\infty^n$. In particular, the image of i_n, which is exactly $[d_1, \dots, d_n]$ is M -isomorphic to ℓ_∞^n, which implies that X is a $L_{\infty,M}$-space.

In the case where (d_n^*) is a boundedly complete basis of ℓ_1 then X^* may be identified with ℓ_1 by standard result about bases. Moreover, for $g^* \in \ell_1$, we have

$$\|g^*\|_{X^*} = \sup\{\langle g^*, x\rangle : x \in X \text{ and } \|x\|_\infty \leq 1\} \leq \|g^*\|_1.$$

On the other hand, if g^* has finite support, say supp $g^* \subseteq \{1, 2, \dots, n\}$, we can choose $u \in \ell_\infty^n$ with $\|u\| = 1$ and $\langle g^*, u\rangle = \|g^*\|_1$. The extension $x = i_n(u)$ is now in X and satisfies

$$\|x\| \leq M, \langle g^*, x\rangle = \|g^*\|.$$

We shall say that (d_n^*) is a unit-triangular basis of $\ell_1(\mathbb{N})$ if it is a triangular basis and the non-zero scalars $a_{n,n}$ are all equal to 1. We can thus write

$$d_n^* = e_n^* - c_n^*,$$

where $c_1^* = 0$ and $\operatorname{supp} c_n^* \subset \{1, 2, \dots, n-1\}$ for $n \geq 2$. The clever part of the Bourgain–Delbaen construction is to find a method of choosing the c_n^* in such a way that (d_n^*) is indeed a basic sequence. The idea is to proceed recursively assuming that, for some $n \geq 1$, we already have a unit-triangular basis $(d_m^*)_{m\leq n}$ of ℓ_1^n. The value of $P_r^* b^*$ is thus already determined when $1 \leq r \leq n$ and $b^* \in \ell_1^n$.

Definition (3.2.7)[91]: In the set-up described above, we shall say that an element c^* of $\ell_1(n)$ is a BD-functional (with respect to the triangular basis $(d_m^*)_{m=1}^n$) if there there exist real numbers $\alpha \in (0, 1]$ and $\beta \in [0, \frac{1}{2})$ such that we can express c^* in one of the following forms:

(i) αe_j^* with $1 \leq j \leq n$,

(ii) $\beta(I - P_k^*)b^*$ with $0 \leq k < n$ and $b^* \in$ ball $\ell_1(k+1, \dots, n)$,

(iii) $\alpha e_j^* + \beta(I - P_k^*)b^*$ with $1 \leq j \leq k < n$ and $b^* \in$ ball $\ell_1(k+1, \dots, n)$.

The non-negative constant β will be called the weight of the functional c^* (“weight 0” in case (0)). Note that (0) and (i) are “almost” special cases of (ii), with β (resp. α) equal to 0. In the construction presented, we do not use functionals of type (0) and the constant α in case (ii) is always equal to 1. However, it may be worth stating the following theorem in full generality.

Theorem (3.2.8) ([102], [85])[91]: Let θ be a real number with $0 < \theta < \frac{1}{2}$ and let $d_n^* = e_n^* - c_n^*$ in ℓ_1 be such that, for each $n, c_{n+1}^* \in \ell_1^n$ is a BD-functional of weight at most θ with respect to $(d_m^*)_{m=1}^n$. Then $(d_n^*)_{n\in\mathbb{N}}$ is a triangular basis of ℓ_1, with basis constant at most $M = 1/(1 - 2\theta)$.

The subspace $X = [d_n : n \in \mathbb{N}]$ of ℓ_∞ is thus a $L_{\infty,M}$-space.

Proof. Despite the disguise, this is essentially the same argument as in the original of Bourgain and Delbaen. What we need to show is that P_m^* is a bounded operator, with $\|P_m^*\| \le M$ for all m. Because we are working on the space ℓ_1 it is enough to show that $\|P_m^* e_n^*\| \le M$ for every m and n.

First, if $n \le m, P_m^* e_n^* = e_n^*$, so there is nothing to prove. Now let us assume that $\|P_k e_j^*\| \le M$ for all $k \le m$ and all $j \le n$; we then consider $P_m^* e_{n+1}^*$. We use the fact that

$$e_{n+1}^* = d_{n+1}^* + c_{n+1}^*,$$

with $c_{n+1}^* \in \ell_1^n$ a BD-functional. We shall consider a functional of type (2), which presents the most difficulty. We thus have

$$c_{n+1}^* = \alpha e_j^* + \beta(I - P_k^*)b^*,$$

where $1 \le j \le k < n$ and α, β, b^* are as in Definition (3.2.7), and $\beta \le \theta$ by our hypothesis. Now, because $n + 1 > m$ we have $P_m^* d_{n+1}^* = 0$ so

$$P_m^* e_{n+1}^* = \alpha P_m^* e_j^* + \beta(P_m^* - P_{m\wedge k}^*)b^*.$$

If $k \ge m$ the second term vanishes so that

$$\|P_m^* e_n^*\| = \alpha\|P_m^* e_j^*\| \le \|P_m^* e_j^*\|,$$

which is at most M by our inductive hypothesis.

If, on the other hand, $k < m$, we certainly have $j < m$ so that $P_m^* e_j^* = e_j^*$, leading to the estimate

$$\|P_m^* e_{n+1}^*\| \le \alpha\|e_j^*\| + \beta\|P_m^* b^*\| + \beta\|P_k^* b^*\|.$$

Now b^* is a convex combination of functionals $\pm e_l^*$ with $l \le n$, and our inductive hypothesis is applicable to all of these. We thus obtain

$$\|P_m^* e_{n+1}^*\| \le \alpha + M\beta \le 1 + 2M\beta = M,$$

by the definition of $M = 1/(1 - 2\theta)$ and the assumption that $0 \le \beta \le \theta$.

The L_∞ spaces of Bourgain and Delbaen, and those we construct are of the above type. However, the “cuts” k that occur in the definition of BD-functionals are restricted to lie in a certain subset of $\mathbb{N}$, thus naturally dividing the coordinate set $\mathbb{N}$ into successive intervals. As in [80], it will be convenient to replace the set $\mathbb{N}$ with a different countable set Γ having a structure that reflects this decomposition. This will also enable us later to use a notation in which an element $\gamma \in \Gamma$ automatically codes the BD-functional associated with it.

Theorem (3.2.9)[91]: Let $(\Delta_q)_{q\in\mathbb{N}}$ be a sequence of non-empty finite sets, with $\#\Delta_1 = 1$; write $\Gamma_q = \bigcup_{1\le p\le q} \Delta_p, \Gamma = \bigcup_{p\in\mathbb{N}} \Delta_p$. Assume that there exists $\theta < \frac{1}{2}$ and a mapping τ defined on $\Gamma\backslash\Delta_1$, assigning to each $\gamma \in \Delta_{q+1}$ a tuple of one of the forms:

(0) (α, ξ) with $0 < \alpha \le 1$ and $\in \Gamma_q$;

(1) (p, β, b^*) with $0 \le p < q, 0 < \beta \le \theta$ and $b^* \in$ ball $\ell_1(\Gamma_q \backslash \Gamma_p)$;
(2) $(\alpha, \xi, p, \beta, b^*)$ with $0 < \alpha \le 1, 1 \le p < q, \xi \in p, 0 < \beta \le \theta$ and $b^* \in$ ball $\ell_1(\Gamma_q \backslash \Gamma_p)$.
Then there exist $d_\gamma^* = e_\gamma^* - c_\gamma^* \in \ell_1(\Gamma)$ and projections $P_{(0,q]}^*$ on $\ell_1(\Gamma)$ uniquely determined by the following properties:

$$(1) P_{(0,q]}^* d_\gamma^* = \begin{cases} d_\gamma^* & if\ \gamma \in \Gamma_q \\ 0 & if\ \gamma \in \Gamma \backslash \Gamma_q \end{cases}$$

$$(2) c_\gamma^* = \begin{cases} 0 & if\ \gamma \in \Delta_1 \\ \alpha e_\xi^* & if\ \tau(\gamma) = (\alpha, \xi) \\ \beta\big(I - P_{(0,p]}^*\big) b^* & if\ \tau(\gamma) = (p, \beta, b^*) \\ \alpha e_\xi^* + \beta\big(I - P_{(0,p]}^*\big) b^* & if\ \tau(\gamma) = (\alpha, \xi, \beta, b^*)\ with\ \xi \in \Delta_p. \end{cases}$$

The family $\left(d_\gamma^*\right)_{(\gamma \in \Gamma)}$ is a basis for $\ell_1(\Gamma)$ with basis constant at most $M = (1 - 2\theta)^{-1}$. The norm of each projection $P_{(0,q]}^*$ is at most M. The biorthogonal elements d_γ generate a $L_{\infty,M}$ –subspace $X(\Gamma, \tau)$ of $\ell_\infty(\Gamma)$. For each q and each $u \in \ell_\infty(\Gamma_q)$, there is a unique $i_q(u) \in [d_\gamma : \gamma \in \Gamma_q]$ whose restriction to q is u; the extension operator $i_q : \ell_\infty(\Gamma_q) \to X(\Gamma, \tau)$ has norm at most M . The subspaces $M_n = [d_\gamma : \gamma \in \Gamma_q] = i_q\, [\ell_\infty(\Delta_q)]$ form a finite-dimensional decomposition (FDD) for X; if this FDD is shrinking then X^* is naturally isomorphic to $\ell_1(\Gamma)$.
Proof. We shall show that, with a suitable identification of Γ with $\mathbb{N}$, this theorem is just a special case of Theorem (3.2.8). Let $k_p = \#\Gamma_p$ and let $n \mapsto \gamma(n) : \mathbb{N} \to \Gamma$ be a bijection with the property that $\Delta_1 = \{\gamma(1)\}$, while, for each $q \ge 2, \Delta_q = \{\gamma(n) : k_{q-1} < n \le k_q\}$. There is a natural isometry: $J : \ell_1(\mathbb{N}) \to \ell_1(\Gamma)$ satisfying $J(e_n^*) = e_{\gamma(n)}^*$. It is straightforward to check that if $d_n^* = J^{-1}\big(d_{\gamma(n)}^*\big) = e_n^* - c_n^*$, then the hypotheses of Theorem (3.2.8) are satisfied. (The cuts k that occur in the BD-functionals c_n^* are all of the form $k = k_p$.) All the assertions in the present theorem are now immediate consequences. The projections $P_{(0,q]}^*$ whose existence is claimed here are given by $P_{(0,q]}^* = J P_{k_q}^* J^{-1}$, where P_n^* is the basis projection of Theorem (3.2.8).
When ordered as $\big(d_{\gamma(n)}\big)_{n \in \mathbb{N}}$ the vectors d_γ form a basis of their closed linear span, which is a $L_{\infty,M}$ -space. The extension operator that (by abuse of notation) we here denote by i_q is just $J i_{k_q} J^{-1}$. The assertions about the subspaces $M_q = \big[d_{\gamma(n)} : k_{q-1} < n \le k_q\big]$ follow from the fact that $(d_{\gamma(n)})$ is a basis.
We now make a few observations about the space $X = (\Gamma, \tau)$ and the functions d_γ, taking the opportunity to introduce notation that will be used in the rest. We have seen that for each $\gamma \in \Delta_{n+1}$ the functional d_γ^* has support contained in $\Gamma_n \cup \{\gamma\}$. Using biorthogonality, we see that d_γ is supported by $\{\gamma\} \cup \Gamma \backslash \Gamma_{n+1}$. It should be noted that we should not expect the support of d_γ to be finite; in fact, in all interesting cases, we have $X \cap c_0(\Gamma) = \{0\}$.
As noted above the subspaces $M_n = [d_\gamma : \gamma \in \Delta_n]$ form a finite-dimensional decomposition for X. For each interval $I \subseteq \mathbb{N}$ we define the projection $P_I : X \to \bigoplus_{n \in I} M_n$ in the natural way; this is consistent with our use of $P_{(0,n]}^*$ in Theorem (3.2.9). Most of our arguments will involve sequences of vectors that are block sequences with respect to this FDD. Since we are using the word “support” to refer to the set of γ where a given function

is non-zero, we need other terminology for the set of n such that x has a non-zero component in M_n. We define the range of x, denoted ran x, to be the smallest interval $I \subseteq N$ such that $x \in \bigoplus_{n\in I} M_n$. It is worth noting that if ran $x = (p, q]$ then we can write $x = i_q(u)$ where $u = x \upharpoonright \Gamma_q \in \ell_\infty(\Gamma_q)$ satisfies $\Gamma_p \cap \operatorname{supp} u = \emptyset$.

We now set about constructing specific BD spaces which will be modelled on mixed Tsirelson spaces, in rather the same way that the original spaces of Bourgain and Delbaen have been found to be modelled on ℓ_p. We shall adopt a notation in which elements γ of Δ_{n+1} automatically code the corresponding BD-functionals. This will allow us to write $X(\Gamma)$ rather than $X(\Gamma, \tau)$ for the resulting L_∞-space. An element γ of Δ_{n+1} will be a tuple of one of the forms:

(i) $\gamma = (n + 1, \beta, b^*)$, in which case $\tau(\gamma) = (0, \beta, b^*)$;

(ii) $\gamma = (n + 1, \xi, \beta, b^*)$ in which case $\tau(\gamma) = (1, \xi, \operatorname{rank} \xi, \beta, b^*)$.

In each case, the first co-ordinate of γ tells us what the rank of γ is, that is to say to which set Δ_{n+1} it belongs, while the remaining co-ordinates specify the corresponding BD-functional.

It will be observed that BD-functionals of Type 0 do not arise in this construction and that the p in the definition of a Type 1 functional is always 0. In the definition of a Type 2 functional that the scalar α that occurs is always 1 and p equals rank ξ. We shall make the further restriction the weight β must be of the form m_j^{-1}, where the sequences (m_j) and (n_j) satisfy Assumption (3.2.3). We shall say that the element γ has weight m_j^{-1} (sometimes dropping the -1 and referring to "weight m_j "). In the case of a Type 2 element $\gamma = (n + 1, \xi, m_j^{-1}, b^*)$ we shall insist that ξ be of the same weight m_j^{-1} as γ.

To ensure that our sets Δ_{n+1} are finite we shall admit into Δ_{n+1} only elements of weight m_j with $j \leq n + 1$. A further restriction involves a recursively defined function which we call "age". For a Type 1 element $\gamma = (n + 1, \beta, b^*)$ we define age $\gamma = 1$. For a Type 2 element $\gamma = \left(n + 1, \xi, m_j^{-1}, b^*\right)$, we define age $\gamma = 1 +$ age ξ, and further restrict the elements of Δ_{n+1} by insisting that the age of an element of weight m_j may not exceed n_j. Finally, we shall restrict the functionals b^* that occur in an element of Δ_{n+1} by requiring them to lie in some finite subset B_n of $\ell_1(\Gamma_n)$. It is convenient to fix an increasing sequence of natural numbers (N_n) and take $B_{n,p}$ to be the set of all linear combinations $b^* = \sum_{\eta\in\Gamma_n\backslash\Gamma_p} a_\eta e_\eta^*$, where $\sum_\eta |a_\eta| \leq 1$ and each is a rational number with denominator dividing $N_n!$. We may suppose the N_n are chosen in such a way that $B_{n,p}$ is a 2^{-n}-net in the unit ball of $\ell_1(\Gamma_n\backslash\Gamma_p)$. The above restrictions may be summarized as follows.

Assumption (3.2.10)[91]:

$$\Delta_{n+1} \subseteq \bigcup_{j=1}^{n} \left\{\left(n + 1, m_j^{-1}, b^*\right): b^* \in B_{n,0}\right\}$$

$$\cup \bigcup_{0<p<n} \bigcup_{j=1}^{p} \left\{\left(n + 1, \xi, m_j^{-1}, b^*\right): \xi \in \Delta_p, \text{weight } \xi = m_j^{-1}, \text{age } \xi < n_j, b^* \in B_{n,p}\right\}$$

We shall also assume that Δ_{n+1} contains a rich supply of elements of "even weight", more exactly of weight m_j with j even.

Assumption (3.2.11)[91]:

$$\Delta_{n+1} \supseteq \bigcup_{j=1}^{\lfloor (n+1)/2 \rfloor} \{(n+1, m_{2j}^{-1}, b^*) : b^* \in B_{n,0}\}$$

$$\cup \bigcup_{1 \le p < n} \bigcup_{j=1}^{p} \{(n+1, \xi, m_{2j}^{-1}, b^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_{2j}^{-1}, \text{age}\, \xi < n_{2j}, b^* \in B_{n,p}\}$$

For our main HI construction, there are additional restrictions on the elements with "odd weight" m_{2j-1}. However, there is some interest already in the space we obtain without making such restrictions. We denote this space $\mathfrak{B}_{\mathrm{mT}}$; it is an isomorphic predual of ℓ_1 that is unconditionally saturated but contains no copy of c_0 or ℓ_p. An analogous space $\mathfrak{B}_{\mathrm{T}}$, modelled on the standard Tsirelson space, rather than a mixed Tsirelson space.

Definition (3.2.12)[91]: We define $\mathfrak{B}_{\mathrm{mT}} = \mathfrak{B}_{\mathrm{mT}}[(m_j, n_j)_{j \in \mathbb{N}}]$ to be the space $X(\Gamma)$ where $\Gamma = \Gamma^{\max}$ is defined by the recursion $\Delta_1 = \{1\}$,

$$\Delta_{n+1} = \bigcup_{j=1}^{n+1} \{(n+1, m_j^{-1}, b^*) : b^* \in B_{n,0}\}$$

$$\cup \bigcup_{j=1}^{n-1} \bigcup_{j \le p < n} \{(n+1, \xi, m_j^{-1}, b^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_j^{-1}, \text{age}\, \xi < n_j, b^* \in B_{n,p}\}$$

The extra constraints that we place on "odd-weight" elements in order to obtain hereditary indecomposability will involve a coding function that will produce the analogues of the "special functionals" that occur in [111] and other HI constructions. In our case, all we need is an injective function $\sigma : \Gamma \to \mathbb{N}$ satisfying $4\sigma(\gamma) > rank\, \gamma$ for all γ. This may easily be included in our recursive construction of Γ. We then insist that a Type 1 element of odd weight must have the form

$$(n+1, m_{2j-1}^{-1}, e_\eta^*)$$

with weight $\eta = m_{4i-2} > n_{2j-1}^2$, while a Type 2 element must be

$$(n+1, \xi, m_{2j-1}^{-1}, e_\eta^*)$$

with weight $\eta = m_{4\sigma(\xi)}$.

Definition (3.2.13)[91]: We define $\mathfrak{X}_{\mathrm{K}}\left[(m_j, n_j)_{j \in \mathbb{N}}\right]$ to be the space $X(\Gamma)$ where $\Gamma = \Gamma^K$ is defined by the recursion $\Delta_1 = \{1\}$,

$$\Delta_{n+1} = \bigcup_{j=1}^{\lfloor (n+1)/2 \rfloor} \{(n+1, m_{2j}^{-1}, b^*) : b^* \in B_{n,0}\}$$

$$\cup \bigcup_{p=1}^{n} \bigcup_{j=1}^{\lfloor p/2 \rfloor} \{(n+1, \xi, m_{2j}^{-1}, b^*) : \xi \in \Delta_p, \text{weight}\, \xi = m_{2j}^{-1}, \text{age}\, \xi < n_{2j}, b^* \in B_{n,p}\}$$

$$\cup \bigcup_{j=1}^{\lfloor (n+2)/2 \rfloor} \{(n+1, \xi, m_{2j-1}^{-1}, e_\eta^*) : \eta \in \Gamma_n \text{ and weight}\, \eta = m_{4i-2} > n_{2j-1}^2\}$$

$$\cup \bigcup_{p=1}^{n} \bigcup_{j=1}^{\lfloor (p+1)/2 \rfloor} \{(n+1, \xi, m_{2j-1}^{-1}, e_\eta^*): \xi \in \Delta_p, \text{weight } \xi = m_{2j-1}^{-1}, \text{age } \xi < n_{2j-1}, \eta \in \Gamma_n \backslash \Gamma_p, \text{weight } \eta = m_{4\sigma(\xi)}\}$$

With the definition readily at hand, this is a convenient moment to record an important "tree-like" property of odd-weight elements of Γ^{K}, even though we shall not be exploiting these special elements until later on.

Lemma (3.2.14)[91]: Let γ, γ' be two elements of Γ^{K} both of weight m_{2j-1} and of ages $a \geq a'$, respectively. Let $(p_i, e_{n_i}^*, \xi_i)_{1 \leq i \leq a}$, resp. $(p_i', e_{n_i'}^*, \xi_i')_{1 \leq i \leq a'}$, be the analysis of γ, resp. γ'. There exists l with $1 \leq l \leq a'$ such that $\xi' = \xi_i$ when $i < l$, while weight $\eta_j \neq weight\, \eta_i'$ for all j when $l < i \leq a'$.

Proof. If weight $\eta_i' \neq weight\, \eta_j$ for all $i \geq 2$ and all j there is nothing to prove (we may take $l = 1$). Otherwise, let $2 \leq l \leq a$ be maximal subject to the existence of j such that $weight\, \eta_j = weight\, \eta_l'$. Now this weight is exactly $m_{4\sigma(\xi_{l-1}')}$, which means that j cannot be 1 (because the weight of η_1 has the form m_{4k-2}). Thus $\sigma(\xi_{l-1}') = \sigma(\xi_{j-1})$, which implies that $\xi_{l-1}' = \xi_{j-1}$. Since $l - 1 =$ age ξ_{l-1}' and $j - 1 =$ age ξ_{j-1}, we deduce that $j = l$. Moreover, since the elements ξ_i with $i < l - 1$ are determined by ξ_{l-1}, we have $\xi i = \xi_i'$ for $i < l$.

Although the structure of the space $X(\Gamma)$ is most easily understood in terms of the basis (d_γ) and the biorthogonal functionals d_γ^*, it is with the evaluation functionals e_γ^* that we have to deal in order to estimate norms. The recursive definition of the functionals d_γ^* can be unpicked to yield the following proposition.

Proposition (3.2.15)[91]: Assume that the set satisfies Assumption (3.2.10). Let n be a positive integer and let γ be an element of Δ_{n+1} of weight m_j and age $\leq n_j$. Then there exist natural numbers $0 = p_0 < p_1 < \cdots < p_a = n + 1$, elements $\xi_1, \ldots, \xi_a = \gamma$ of weight m_j with $\xi_r \in \Delta_{p_r}$ and functionals $b_r^* \in$ ball $\ell_1(\Gamma_{p_r-1} \backslash \Gamma_{p_r-1})$ such that

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_j^{-1} \sum_{r=1}^{a} P_{(p_r-1,\infty)}^* b_r^*$$

$$= \sum_{r=1}^{a} d_{\xi_r}^* + m_j^{-1} \sum_{r=1}^{a} P_{(p_r-1,p_r)}^* b_r^*$$

Proof. Given the assumption (3.2.10), this is an easy induction on the age a of γ. If $a = 1$ then γ has the form $(n + 1, m_j^{-1}, b^*)$ and

$$e_\gamma^* = d_\gamma^* + c_\gamma^*,$$

where c_γ^* is the Type 1 BD-functional

$$c_\gamma^* = m_j^{-1} P_{(0,\infty)}^* b^*,$$

with $b^* \in B(n, 0) \subset$ ball $\ell_1(\Gamma_n)$. Since b^* is in the image of the projection $P_{(0,n]}^*$ we have $P_{(0,n]}^* b^* = b^*$ and so

$$e_\gamma^* = d_{\xi_1}^* + m_j^{-1} P_{(p_0,\infty)}^* b_1^* = d_{\xi_1}^* + m_j^{-1} P_{(p_0,p_1)}^* b_1^*,$$

with $p_0 = 0, p_1 = n + 1, b_1^* = b^*$ and $\xi_1 = \gamma$.

If $a > 1$ then γ has the form $(n + 1, \xi, m_j^{-1}, b^*)$ and c_γ^* is the Type 2 BD-functional

$$c_\gamma^* = e_\xi^* + m_j^{-1} P_{(p,\infty)}^* b^*.$$

If we apply our inductive hypothesis to the element ξ of weight m_j, rank p and age $a-1$, we obtain the desired expression for e_γ^* .

We shall refer to the identity presented in the above proposition as the evaluation analysis of γ and shall use it repeatedly in norm estimations. The form of the second term in the evaluation analysis, involving a sum weighted by m_j^{-1}, indicates that there is going to be a connection with mixed Tsirelson spaces; the first term, involving functionals d_ξ^*, with no weight, can cause inconvenience in some of our calculations, but is an inevitable feature of the BD construction.

The data $(p_r, b_r^*, \xi_r)_{1\le r\le a}$ will be called the analysis of γ. We note that if $1 \le s \le a$ the analysis of ξ_s is just $(p_r, b_r^*, \xi_r)_{1\le r\le s}$.

We shall be dealing with a space $X = X(\Gamma)$ and shall be making the assumptions (3.2.10) and (3.2.11). Our results thus apply both to $\mathfrak{B}_{\mathrm{mT}}$ and $\mathfrak{X}_{\mathrm{K}}$.

We note that, since the weights m_j^{-1} are all at most $\frac{1}{4}$, the constant M in Theorem (3.2.9) may be taken to be 2. This leads to the following norm estimates for the extension operators i_n and for the projections P_I associated with the FDD (M_n):

$$\|i_n\| = \|P_{(0,n]}\| \le 2, \|P_{(n,\infty)}\| \le 3, \|P_{(m,n]}\| \le 4, \|d_\xi^*\| = \|P^*_{[\mathrm{rank}\,\xi,\infty)} e_\xi^*\| \le 3.$$

The assumption (3.2.11) enables to write down a kind of converse to Proposition (3.2.15) which will lead to our first norm estimate.

Proposition (3.2.16)[91]: Let j, a be positive integers with $\le n_{2j}$, let $0 = p_0 < p_1 < p_2 < \cdots < p_a$ be natural numbers with $p_1 \ge 2j$ and let b_r^* be functionals in $B(p_r - 1, p_{r-1})$ for $1 \le r \le a$. Then there are elements $\xi_r \in \Gamma_{p_r}$ such that the analysis of $\gamma = \xi_a$ is $(p_r, b_r^*, \xi_r)_{1\le r\le a}$.

Proposition (3.2.17)[91]: Let $(x_r)_{r=1}^a$ be a skipped block sequence (with respect to the FDD (M_n)) in X. If j is a positive integer such that $a \le n_{2j}$ and $2j <$ min ran x_2, then there exists an element γ of weight m_{2j} satisfying

$$\sum_{r=1}^{a} x_r(\gamma) \ge \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^{a} \|x_r\| .$$

Hence

$$\left\| \sum_{r=1}^{a} x_r \right\| \ge \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^{a} \|x_r\| .$$

Proof. Let $p_0 = 0$, and choose $p_1, p_2, \ldots, p_a$ such that ran $x_r \subseteq (p_{r-1}, p_r)$. Thus $x_r = i_{p_r-1}(u_r)$ where the element $u_r = x_r \restriction \Gamma_{p_r-1}$ has support disjoint from $\Gamma_{p_{r-1}}$. Since $\|i_n\| \le 2$ for all n we have $\|u_r\| \ge \frac{1}{2}\|x_r\|$ and so there exist $\eta_r \in \Gamma_{p_r-1} \backslash \Gamma_{p_{r-1}}$ with

$$|u_r(\eta_r)| \ge \frac{1}{2}\|x_r\|.$$

The functional $b_r^* = \pm e_{\eta_r}^*$ is certainly in $B_{p_r-1,p_{r-1}}$ and with a suitable choice of sign we may arrange that

$$\langle b_r^*, x_r\rangle = |u_r(\eta_r)| \ge \frac{1}{2}\|x_r\|.$$

By Proposition (3.2.16) there is an element γ of Δ_{p_a} whose analysis is $(p_r, b_r^*, \xi_r)_{1\le r\le a}$. We shall use the evaluation analysis to calculate

$$\sum_{s=1}^{a} x_s(\gamma) = \langle e_\gamma^*, \sum_{s=1}^{a} x_s \rangle .$$
For any r and $s, x_s \in [d_\xi : p_{s-1} < \text{rank}\, \xi < p_s]$, while rank $\xi_r = p_r$, whence
$$\langle d_{\xi_r}^* , x_s \rangle = 0 \quad \text{for all } r, s,$$
while
$$\langle P_{(p_{r-1}, p_r)}^* b_r^*, x_s \rangle = \langle b_r^*, P_{(p_{r-1}, p_r)}^* x_s \rangle = 0,$$
for all $r \neq s$. In the case $r = s$ we have
$$\langle P_{(p_{r-1}, p_r)}^* b_r^*, x_r \rangle = \langle b_r^*, P_{(p_{r-1}, p_r)}^* x_r \rangle = \langle b_r^*, x_r \rangle.$$
The evaluation analysis thus simplifies to yield
$$\sum_{r=1}^{a} x_r(\gamma) = m_{2j}^{-1} \sum_{r=1}^{a} \langle b_r^*, x^* r \rangle \geq \frac{1}{2} m_{2j}^{-1} \sum_{r=1}^{a} \|x_r\|$$
The lower estimate we have just obtained indicates that there is a close connection between our space X and mixed Tsirelson spaces of the kind considered. We can show that a normalized skipped-block sequence in X dominates the unit vector basis of $T[(A_{n_{2j}}, m_{2j}^{-1})_{j \in \mathbb{N}}]$.

We continue to work with the space $X = X(\Gamma)$, where satisfies the assumptions (3.2.10) and (3.2.11). We saw that skipped block sequences admit useful Mixed Tsirelson lower estimates. We now pass to a class of block sequences that admit upper estimates of a similar kind. The following definition is a variant of something that is familiar from other HI constructions.

Let I be an interval in $\mathbb{N}$ and let $(x_k)_{k \in I}$ be a block sequence (with respect to the FDD (M_n)). We say that (x_k) is a rapidly increasing sequence, or RIS, if there exists a constant C such that the following hold:

(i) $\|x_k\| \leq C$ for all $k \in \mathbb{N}$,

and there is an increasing sequence (j_k) such that, for all k,

(ii) $j_{k+1} >$ max ran x_k

(iii) $|x_k(\gamma)| \leq C m_i^{-1}$ whenever weight $\gamma = m_i$ and $i < j_k$

If we need to be specific about the constant, we shall refer to a sequence satisfying the above conditions as a C-RIS.

Lemma (3.2.18)[91]: Let (x_k) be a C-RIS and let (j_k) be an increasing sequence of natural numbers as in the definition. If $\gamma \in \Gamma$ and weight $\gamma = m_i$ then, for any natural number s
$$\left| \langle e_\gamma^* , P_{(s,\infty)} x_k \rangle \right| \leq \begin{cases} 5C m_i^{-1} & if\ i < j_k \\ 3C m_i^{-1} & if\ i \geq j_{k+1} \end{cases}$$
Proof. We first consider the case where $i \geq j_{k+1}$, noting that this implies that $i >$ max ran x_k by RIS condition (ii). As in Proposition (3.2.15), we may write down the evaluation analysis of γ as
$$e_\gamma^* = \sum_r d_{\xi r}^* + m_i^{-1} \sum_r d_r^* \circ P_{(p_{r-1}, \infty)},$$
where $0 = p_0 < p_1 < q_1 < p_2 < \cdots$, and b_r^* is a norm-1 element of $\ell_1(\Gamma)$, supported by $\Gamma_{p_r - 1} \backslash \Gamma_{p_{r-1}}$, while ξ_r is of rank p_r and weight m_i. Since Δ_q contains no elements of weight mi unless $q \geq i$, it must be that $p_1 \geq i$. Thus $p_1 >$ max ran x_k , from which it follows that $P_{(p_r, \infty)} \circ P_{(s,\infty)} x_k = P_{(s \vee p_r, \infty)} x_k = 0$ for all $r \geq 1$. For the same reason, we also have

$$\langle d^*_{\xi r}, P_{(s,\infty)}x_k\rangle = \langle e^*_{\xi r}, P_{(s\vee q_r,\infty)}P_{[p_r,\infty)}x_k\rangle = 0$$
for all r. We are left with
$$|\langle e^*_\gamma, P_{(s,\infty)}x_k\rangle| = m_i^{-1}|\langle d^*_{\xi r}, P_{(s,\infty)}x_k\rangle| \le m_i^{-1}\|P_{(s,\infty)}\|\|x\| \le 3Cm_i^{-1}$$
In the case where $i < j_k$, we again use the evaluation analysis, but need to be more careful about the value of s. Since we shall need this argument again, we state it as a separate lemma. Clearly the second part of the present lemma is an immediate consequence.

Lemma (3.2.19)[91]: Let i be a positive integer and suppose that $x \in X$ has the property that $\|x\| \le C$ and $|x(\xi)| \le \delta$ whenever weight $\xi = m_i$. Then for any s and any γ of weight m_i we have
$$|\langle e^*_\gamma, P_{(s,\infty)}x\rangle| \le 2\delta + 3Cm_i^{-1}.$$
Proof. As before we consider the evaluation analysis
$$e^*_\gamma = \sum_{r=1}^{a} d^*_{\xi r} + m_i^{-1}\sum_{r=1}^{a} b^*_r \circ P_{(p_{r-1},\infty)},$$
If $s \ge p_a$ then $P^*_{(s,\infty)}e^*_\gamma = 0$. If $0 < s < p_1$, by applying $P^*_{(s,\infty)}$ to each of the terms in the evaluation analysis, we see that
$$P^*_{(s,\infty)}e^*_\gamma = e^*_\gamma - m_i^{-1}P^*_{(0,s]}b^*_1,$$
which leads to
$$|\langle e^*_\gamma, P_{(s,\infty)}x_k\rangle| \le \delta + m_i^{-1}\|b^*_1\|\|P_{(p_1,s]}\|\|x_k\| \le \delta + 3Cm_i^{-1},$$
by our assumptions.

In the remaining case, there is some t with $1 \le t < a$ such that $p_t \le s$ while $p_{t+1} > s$. We may rewrite the evaluation analysis of γ as
$$e^*_\gamma = e^*_{\xi_t} + \sum_{r=t+1}^{a} d^*_{\xi r} + m_i^{-1}\sum_{r=t+1}^{a} b^*_r \circ P_{(p_{r-1},\infty)},$$
which gives us
$$P^*_{(s,\infty)}e^*_\gamma = e^*_\gamma - e^*_{\xi_t} - m_i^{-1}P^*_{(p_t,s]}b^*_{t+1}.$$
When we recall that weight ξ_t = weight γ this yields
$$|\langle e^*_\gamma, P_{(s,\infty)}x_k\rangle| \le 2\delta + 3Cm_i^{-1},$$
as above.

Proposition (3.2.20) (Basic Inequality)[91]: Let $(x_k)_{k\in I}$ be a C-RIS, let λ_k be real numbers, let s be a natural number and let γ be an element of Γ. There exist $k_0 \in I$ and and a functional $g^* \in W[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ such that:

(i) either $g^* = 0$ or weight (g^*) = weight (γ) and supp $g^* \subseteq \{k \in I : k > k_0\}$;

(ii) $|\langle e^*_\gamma, P_{(s,\infty)}\sum_{k\in I}\lambda_k x_k\rangle| \le 5C|\lambda_{k_0}| + 5C\langle g^*, \sum_k|\lambda_k|e_k\rangle$.

Moreover, if j_0 is such that
$$|\langle e^*_\xi, \sum_{k\in J}\lambda_k x_k\rangle| \le 2C\max_{k\in J}|\lambda_k|,$$
for all subintervals J of I and all $\xi \in \Gamma$ of weight m_{j_0}, then we may choose g^* to be in $W[(A_{3n_j}, m_j^{-1})_{j\ne j_0}]$.

Proof. We proceed by induction of the rank of γ, noting that if γ is of rank 1 we have $P^*_{(s,\infty)}e^*_\gamma = 0$ whenever $s \ge 1$, so that

$$\langle e_\gamma^*, P_{(s,\infty)} \sum_{k\in I} \lambda_k x_k \rangle = \begin{cases} 0 & if \quad r \geq 1 \\ \lambda_1 x_1(\gamma) & if\ r = 0. \end{cases}$$

Thus $k_0 = 1$ and $g^* = 0$ have the desired property.

Now consider an element γ of rank greater than 1, of age a and of weight mh. Taking (j_k) to be a sequence as in the definition of a RIS, we shall suppose that there is some $l \in I$ such that $j_l \leq h < j_{l+1}$. (The cases where $h < j_k$ for all $k \in I$ and where $h \geq j_{k+1}$ for all $k \in I$ are simpler.)

We split the summation over k into three parts as follows:

$$\langle e_\gamma^*, P_{(s,\infty)} \sum_{k\in I} \lambda_k x_k \rangle$$
$$= \sum_{I\ni k<l} \lambda_k \langle e_\gamma^*, P_{(s,\infty)} x_k \rangle + \langle e_\gamma^*, P_{(s,\infty)} \lambda_l x_l \rangle + \langle e_\gamma^*, P_{(s,\infty)} \sum_{I\ni k>l} \lambda_k\, x_k \rangle$$

and estimate the three terms separately.

When $k < l$ we have $h \geq j_l \geq j_{k+1}$ so that

$$\left|\langle e_\gamma^*, P_{(s,\infty)} \lambda_k x_k \rangle\right| \leq 3C\, m_h^{-1} |\lambda_k| \leq 3C\, m_{jk}^{-1} |\lambda_k|,$$

by Lemma (3.2.18). Thus

$$|\sum_{I\ni k<l} \lambda_k \langle e_\gamma^*, P_{(s,\infty)} x_k \rangle| \leq 3C \sum_{k<l} m_{jk}^{-1} |\lambda_k| \leq 3C \sum_{j=1}^{\infty} m_j^{-1} \max_{k<l} |\lambda_k| \leq C \max_{k<l} |\lambda_k|\,.$$

For the second term, we have the immediate estimate

$$\left|\langle e_\gamma^*, P_{(s,\infty)} \lambda_l x_l \rangle\right| \leq \left\|P_{(s,\infty)}\right\| |\lambda_l| \|x_l\| \leq 3C|\lambda_l|.$$

Thus putting the first two terms together we have

$$|\langle e_\gamma^*, P_{(s,\infty)} \sum_{k\leq l} \lambda_k x_k \rangle| \leq C \max_{k<l} |\lambda_k| + 3C|\lambda_l| \leq 4C\left|\lambda_{k_0}\right|, \tag{4}$$

for a suitably chosen $k_0 \leq l$.

We now have to estimate the last term

$$|\langle e_\gamma^*, \sum_{k\in I'} \lambda_k x_k' \rangle|,$$

where $I' = \{k \in I : k > l\}$ and $x_k' = P_{(s,\infty)} x_k$. We shall use the evaluation analysis of γ

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi r}^* + m_h^{-1} \sum_{r=1}^{a} b_r^* \circ P_{(p_{r-1},\infty)}.$$

Let $I_0' = \{k \in I' :$ ran x_k' contains rank ξ_r for some $r\}$ noting first that $\# I_0' \leq a$ and secondly that for $k \in I' \setminus I_0'$ the interval ran x_k' meets (p_{r-1}, p_r) for at most one value of r. If we set $I_r' = \{k \in I'$: ran x_k meets $(p_{r-1}, p_r]$ but no other $(p_{r'-1}, p_{r'})\}$ then each I_r' is a subinterval of I' and we have

$$\langle e_\gamma^*, x_k' \rangle = m_h^{-1} \langle b_r^*, P_{(p_{r-1},\infty)} x_k' \rangle = m_h^{-1} \langle b_r^*, P_{(s\vee p_{r-1},\infty)} x_k \rangle$$

if $k \in I_r'$, while

$$\langle e_\gamma^*, x_k' \rangle = 0 \quad \text{if} \quad k \in I' \setminus \bigcup_r I_r'$$

Thus $\langle e_\gamma^*, \sum_{k\in I'} \lambda_k x_k' \rangle = \langle e_\gamma^*, \sum_{k\in I_0'} \lambda_k x_k' \rangle + m_h^{-1} \sum_{r=1}^{a} \langle b_r^*, \sum_{k\in I_r'} \lambda_k x_k' \rangle$

Applying Lemma (3.2.18), we see that

$$\left|\langle e_\gamma^*, \sum_{k\in I'} \lambda_k x_k' \rangle\right| \le 5Cm_h^{-1} \sum_{k\in I_0} |\lambda_k| + m_h^{-1} \left| \sum_{r=1}^{a} \langle b_r^*, \sum_{k\in I_r'} \lambda_k x_k' \rangle \right|. \qquad (5)$$

Now, for each r, the functional b_r^* is a convex combination of functionals $\pm e_\eta^*$ with $p_{r-1} <$ rank $\eta < p_r$, so we may choose η_r to be such an η with

$$|\langle b_r^*, \sum_{k\in I_r'} \lambda_k x_k' \rangle| \le |\langle e_{\eta_r}^*, \sum_{k\in I_r'} \lambda_k x_k' \rangle|.$$

For each r, we may apply our inductive hypothesis to the element $\eta_r \in \Gamma$ and the RIS $(x_k)_{k\in I_r'}$, obtaining $k_r \in I_r'$ and $g_r^* \in W\left[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}\right]$ supported on $\{k \in I_r' : k > k_r\}$ satisfying

$$|\langle e_{\eta_r}^*, P_{(s\vee p_r,\infty)} \sum_{k\in I_r'} \lambda_k x_k \rangle| \le 5C|\lambda_{k_r}| + 5C\,\langle g_r^*, \sum_{k\in I_r'} |\lambda_k| e_k \rangle. \qquad (6)$$

We now define g^* by setting

$$g^* = m_h^{-1}\Big(\sum_{k\in I_0'} e_k^* + \sum_{r=1}^{a} \big(e_{k_r}^* + g_r^*\big)\Big).$$

This is a sum, weighted by m_h, of at most $3n_h$ functionals in $W[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$, supported by disjoint intervals, and is hence itself in $W[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$. Putting together (3.2.18), (3.2.19) and (3.2.20), we finally obtain

$$\begin{aligned}
|\langle e_\gamma^*, P_{(s,\infty)} \sum_{k\in I} \lambda_k x_k \rangle| &\le 4C|\lambda_{k_0}| + 5Cm_h^{-1} \sum_{k\in I_0'} |\lambda_k| + m_h^{-1} |\sum_{r=1}^{a} \langle b_r^*, \sum_{k\in I_r'} \lambda_k x_k' \rangle| \\
&\le 4C|\lambda_{k_0}| + 5Cm_h^{-1} \sum_{k\in I_0'} |\lambda_k| + m_h^{-1} |\sum_{r=1}^{a} \langle e_{\eta_r}^*, P_{(s,\infty)} \sum_{k\in I_r'} \lambda_k x_k \rangle| \\
&\le 4C|\lambda_{k_0}| + 5Cm_h^{-1} \left(\sum_{k\in I_0'} |\lambda_k| + \sum_{r=1}^{a} \big(|\lambda_{k_r}| + \langle g_r^*, \sum_{k\in I_r'} |\lambda_k| e_k \rangle\big) \right) \\
&\le 5C|\lambda_{k_0}| + 5C\,\langle g^*, \sum_{k\in I'} |\lambda_k| e_k \rangle.
\end{aligned}$$

If j_0 satisfies the additional condition set out in the statement of the theorem, we proceed by the same induction. The base case certainly presents no problem and if weight $\gamma = m_h$ with $h = j_0$ we have a simple way to estimate

$$\langle e_\gamma^*, P_{(s,\infty)} \sum_{k\in I} \lambda_k x_k \rangle$$

Indeed there is at most one value of k, l say, for which s is in ran x_k and $P_{(s,\infty)} x_k = 0$ for $k < l$.

If we set $J = \{k \in I : k > l\}$ we then have

$$|\langle e_\gamma^*, P_{(s,\infty)} \sum_{k\in I} \lambda_k x_k \rangle| \le |\lambda_l| \|P_{(s,\infty)}\| \|x_l\| + |e_\gamma^*(\sum_{k\in J} \lambda_k x_k)|,$$

By our usual estimate $\|P_{(s,\infty)}\| \leq 3$ and the assumed additional condition, this is at most $5C|\lambda_{k_0}|$ for some $l \leq k_0 \in I$. We can then take $g^* = 0$.

Corollary (3.2.21)[91]: Any RIS is dominated by the unit vector basis of $T[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$. More precisely, if (x_k) is a C-RIS then, for any real λ_k, we have

$$\|\sum_k \lambda_k x_k\| \leq 10C\|\sum_k \lambda_k e_k\|,$$

where the norm on the right hand side is taken in $T[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$.

As well as this domination result, we shall need the following more precise lemma.

Proposition (3.2.22)[91]: Let $(x_k)_{k=1}^{n_{j_0}}$ be a C-RIS. Then

(i) For every $\gamma \in \Gamma$ with weight $\gamma = m_h$ we have

$$|n_{j_0}^{-1}\sum_{k=1}^{j_0} x_k(\gamma)| \leq \begin{cases} 11Cm_{j_0}^{-1}m_h^{-1} & if\ \ h < j_0 \\ 5Cn_{j_0}^{-1}5Cm_h^{-1} & if\ \ h \geq j_0 \end{cases}$$

In particular,

$$|n_{j_0}^{-1}\sum_{k=1}^{j_0} x_k(\gamma)| \leq 6Cm_{j_0}^{-2},$$

if $h > j_0$ and

$$\|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k\| \leq 6Cm_{j_0}^{-1},$$

(ii) If $\lambda_k(1 \leq k \leq n_{j_0})$ are scalars with $|\lambda_k| \leq 1$ and having the property that

$$|\sum_{k\in J} \lambda_k x_k(\gamma)| \leq 2C\max_{k\in J}|\lambda_k|,$$

for every γ of weight m_{j_0} and every interval $J \subseteq \{1, 2, \ldots, n_{j_0}\}$, then

$$\|n_{j_0}^{-1}\sum_{k=1}^{j_0} \lambda_k x_k\| \leq 6Cm_{j_0}^{-2}.$$

Proof. This is a direct application of the Basic Inequality, with all the coefficients λ_k equal to $n_{j_0}^{-1}$. Indeed, for (i) there exists $g^* \in W[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ (either zero or of weight m_h) such that

$$|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)| \leq 5Cn_{j_0}^{-1} + 5Cg^*(n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} e_k),$$

Using Lemma (3.2.4) to estimate the term involving g^*, we obtain

$$|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)| \leq \begin{cases} 5Cn_{j_0}^{-1} + 10Cm_{j_0}^{-1}m_h^{-1} & if\ \ h < j_0 \\ 5Cn_{j_0}^{-1} + 5Cm_h^{-1} & if\ \ h \geq j_0 \end{cases}$$

The formulae given in (i) follow easily when we note that n_{j_0} is (much) larger than $5m_{j_0}^2$ when $j_0 \geq 2$.

If the scalars λ_k satisfy the additional condition, then the g^* whose existence is guaranteed by the Basic Inequality may be taken to be in $W[(A_{3n_j}, m_j^{-1})_{j\neq j_0}]$ so that the second part of Lemma (3.2.4) may be applied, yielding

$$|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} x_k(\gamma)| \leq \begin{cases} 5Cn_{j_0}^{-1} + 10Cm_{j_0}^{-1}m_h^{-1} & if\ \ h < j_0 \\ 5Cn_{j_0}^{-1} + 5Cm_h^{-1} & if\ \ h \geq j_0 \end{cases}$$

This leads easily to the claimed estimate for $\left\|n_{n_{j_0}}^{-1}\sum_{k=1}^{n_{j_0}}\lambda_k x_k\right\|$.

It turns out that in our space there are three useful types of RIS. One of these is based on an idea that will be familiar from other constructions, that of introducing long ℓ_1-averages. We defer our discussion of this construction until. We shall deal first with the other two types of RIS, which involve the L_∞ structure of our space, and provide the extra tool that we eventually use to solve the scalar-plus-compact problem.

We have already remarked that the support of an element of X is not of great interest — indeed the support of any nonzero element of X is an infinite set, and contains elements γ of Γ of all possible weights. There is, however, a related notion which is of much use. Recall that an element x whose range is contained in the interval $(p, q]$ can be expressed as $i_q(u)$ where $u \in \ell_\infty(\Gamma_q)$ and supp $(u) \subseteq \Gamma_q \backslash \Gamma_p$. It turns out that the support of u contains a lot of information about x. We shall refer to supp (u) as the local support. A formal (and unambiguous) definition may be formulated as follows.

Definition (3.2.23)[91]: Let x be an element of $\bigoplus_n M_n$ and let q = max ran x; thus x may be expressed as $i_q(u)$ with $u = x \restriction \Gamma_q$. The subset supp $u = \{\gamma \in \Gamma_q : x(\gamma) \neq 0\}$ is defined to be the local support of x.

The following easy lemma uses an idea that has already occurred in Lemma (3.2.18).

Lemma (3.2.24)[91]: Let $\gamma \in \Gamma$ be of weight mh and assume that weight $(\xi) \neq m_h$ for all ξ in the local support of x. Then $|x(\gamma)| \leq 3m_h^{-1}\|x\|$.

Proof. Let q = max ran x so that $x = i_q(x \restriction \Gamma_q)$ and, by hypothesis, weight $\xi \neq m_h$ whenever $\xi \in \Gamma_q$ and $x(\xi) \neq 0$. If rank $\gamma \leq q$ we thus have $x(\gamma) = 0$ and there is nothing to prove. Otherwise we consider the evaluation analysis of γ

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi r}^* + m_h^{-1}\sum_{r=1}^{a} b_r^* \circ P_{(p_{r-1},\infty)}$$

and let s be chosen maximal subject to p_s = rank $\xi_s \leq q$. (Since $\gamma = \xi_a$ such an s certainly exists.) For $r \geq s$ we have $r >$ max ran x, whence $d_{\xi_r}^*(x) = 0$ and $P_{(p_r,\infty)}x = 0$. Thus

$$x(\gamma) = \langle e_\gamma^*, x\rangle = \begin{cases} m_h^{-1}\langle b_s^*, P_{(p_{s-1},\infty)}x\rangle + \langle e_{\xi_{s-1}}^*, x\rangle & \text{if } \ s > 1 \\ m_h^{-1}\langle b_1^*, x\rangle = m_h^{-1}\langle b_1^*, P_{(p_0,\infty)}x\rangle & \text{if } \ s = 1. \end{cases}$$

Since, in the first of the above cases, we have rank $\xi_{s-1} < q$ and weight $\xi_{s-1} = m_h$, which imply $e_{\xi_{s-1}}^*(x) = 0$, we deduce that in both cases

$$|x(\gamma)| = m_h^{-1}\left|\langle b_s^*, P_{(p_{s-1},\infty)}x\rangle\right| \leq 3m_h^{-1}\|x\|.$$

We can now introduce two classes of block sequence, characterized by the weights of the elements of the local support.

Definition (3.2.25)[91]: We shall say that a block sequence $(x_k)_{k\in\mathbb{N}}$ in X has bounded local weight if there exists some j_1 such that weight $\gamma \leq m_{j_1}$ for all γ in the local support of x_k, and all values of k. We shall say that $(x_k)_{k\in\mathbb{N}}$ has rapidly increasing local weight if, for each k and each γ in the local support of x_{k+1}, we have weight $\gamma > m_{i_k}$ where $i_k = \max$ ran x_k .

Proposition (3.2.26)[91]: Let $(x_k)_{k\in\mathbb{N}}$ be a bounded block sequence. If either (x_k) has bounded local weight, or (x_k) has rapidly increasing local weight, the sequence (x_k) is a RIS.

Proof. We start with the case of rapidly increasing local weight and let m_{j_k} be the minimum weight of an element γ in the local support of x_k. By hypothesis, $j_{k+1} > \max \operatorname{supp} x_k$ so that RIS condition (2) is satisfied. Also, if $h < j_k$ and γ is of weight m_h then $|x_k(\gamma)| \le 3m_h^{-1}\|x_k\|$ by Lemma (3.2.24). So (x_k) is a C-RIS with $C = 3\sup_k\|x_k\|$.

Now let us suppose that weight $\gamma \le m_{j_1}$ for all γ in the local support of x_k and all k. For $k \ge 2$ define $j_k = 1 + \max \operatorname{supp} x_{k-1}$, thus ensuring that RIS condition (2) is satisfied. If weight $\gamma = m_h$ where $h < j_k$ there are two possibilities: if $i > j_1$ then $|x_k(\gamma)| \le 3m_i^{-1}\|x_k\|$ by Lemma (3.2.24); if $i \le j_1$ then $|x_k(\gamma)| \le \|x_k\| \le m_i^{-1}m_{j_1}\|x_k\|$. Thus (x_k) is a C-RIS, where C is the (possibly quite large) constant $m_{j_1}^{-1}\sup_k\|x_k\|$.

Proposition (3.2.27)[91]: Let Y be any Banach space and $T : X(\Gamma) \to Y$ be a bounded linear operator. If $\|T(x_k)\| \to 0$ for every RIS $(x_k)_{k\in\mathbb{N}}$ in $X(\Gamma)$ then $\|T(x_k)\| \to 0$ for every bounded block sequence sequence in $X(\Gamma)$.

Proof. It is enough to consider a bounded block sequence (x_k) and show that there is a subsequence (x_j') such that $\|T(x_j')\| \to 0$. We may write $x_k = i_{q_k}(u_k)$ with $u_k = x_k \upharpoonright \Gamma_{q_k}$ supported by $\Gamma_{q_k}\backslash\Gamma_{q_{k-1}}$. For each k and each $N \in \mathbb{N}$, we split u_k as $v_k^N + w_k^N$, where, for $\gamma \in \Gamma_{q_k}$,

$$v_k^N(\gamma) = \begin{cases} u_k(\gamma) \text{if weight}\, \gamma \le m_N \\ 0 \qquad\quad \text{otherwise} \end{cases}$$

$$w_k^N(\gamma) = \begin{cases} u_k(\gamma) \text{if weight}\, \gamma > m_N \\ 0 \qquad\quad \text{otherwise} \end{cases}$$

and set

$$y_k^N = i_{q_k}(v_k^N), z_k^N = i_{q_k}(v_k^N).$$

We notice that $\|y_k^N\| \le \frac{3}{2}\|v_k^N\| \le \frac{3}{2}\|x_k\|$, with a similar estimate for $\|z_k^N\|$, so that the sequences $(y_k^N)_k$ and $(z_k^N)_k$ are bounded. We note also that weight $\gamma \le N$ for all γ in the local support of y_k^N and weight $\gamma > N$ for all γ in the local support of z_k^N

So for each N, the sequence (y_k^N) has bounded local weight and is thus a RIS, by Proposition (3.2.26). By hypothesis, $\|T(y_k^N)\| \to 0$ for each N. Hence we can choose a sequence (k_n) tending to ∞ such that $\|T(y_{k_n}^n)\| \to 0$. If we put $n_1 = 1$ and then, recursively, set $n_{j+1} = q_{k_{n_j}}$, it is easy to see that the sequence $(z_{k_{n_j}}^{n_j})$ has rapidly increasing local weight. Thus this sequence is a RIS and we hence have $\|T(z_{k_{n_j}}^{n_j})\| \to 0$. Since $x_{k_{n_j}} = y_{k_{n_j}}^{n_j} + z_{k_{n_j}}^{n_j}$, we have found a subsequence $(x_j') = (x_{k_{n_j}})$ of (x_k) with $\|T(x_j')\| \to 0$.

The above proposition will play an important role in proving compactness of operators, but in the mean time we shall use it to give our promised proof that the dual of X is ℓ_1. There is an alternative approach using ℓ_1-averages.

Proposition (3.2.28)[91]: The dual of $X(\Gamma)$ is $\ell_1(\Gamma)$.

Proof. As we have already noted in Theorem (3.2.9) it is enough to show that the FDD (M_n) is shrinking, that is to say, that every bounded block sequence in X is weakly null. So let ϕ be an element of X^*. By the upper estimate of Proposition (3.2.22) we see that $\phi(x_k) \to 0$ for every RIS $(x_k)_{k\in\mathbb{N}}$. Now Proposition (3.2.27), applied with $T = \phi$, shows that $\phi(x_k) \to 0$ for every bounded block sequence (x_k).

We shall still only be using the assumptions (3.2.10) and (3.2.11), so that our results will apply when X is either of the spaces $\mathfrak{B}_{\mathrm{mT}}$ and $\mathfrak{X}_{\mathrm{K}}$. The special properties of the second of these spaces will come into play only from Definition (3.2.38) onwards.

Definition (3.2.29)[91]: An element x of X will be called a C-ℓ_1^n average if there exists a block sequence $(x_i)_{k=1}^n$ in X such that $x = n^{-1}\sum_{k=1}^n x_k$ and $\|x_k\| \le C$ for all k. We say that x is a normalized C-ℓ_1^n average if, in addition, $\|x\| = 1$.

A standard argument (c.f. II.22 of [99]) using the lower estimate of Lemma (3.2.17) and Lemma (3.2.2) leads to the following.

Lemma (3.2.30)[91]: Let Z be any block subspace of X. For any n and and $C > 1, Z$ contains a normalized C-ℓ_1^n average.

Proof. Write $C = (1-\epsilon)^{-1}$ and choose an integer l with $n(1-\epsilon/n)^l < 1$; next choose j sufficiently large as to ensure that $n_{2j} > \left(2m_{2j}\right)^l$; finally let k be minimal subject to

$$m_{2j} < (1-\epsilon/n)^{-k}$$

Since $\frac{1}{2}(1-\epsilon/n)^{-k} \le (1-\epsilon/n)^{-k+1} \le m_{2j}$ we have

$$n_{2j} > (2m_{2j})l \ge (1-\epsilon/n)^{-kl} > n^k .$$

If (x_i) is any normalized skipped-block sequence in Z, we can apply Lemma (3.2.17) to see that

$$\|\sum_{i=1}^n x_i\| \ge m_{2j}^{-1} n^k > (n-\epsilon)^k .$$

It now follows from Lemma (3.2.2) that there are normalized successive linear combinations $y_1, \ldots, y_n$ of (x_i) such that

$$\|\sum_{i=1}^n a_i y_i\| \ge (1-\epsilon)\sum_{i=1}^n |a_i| ,$$

for all real a_i. In particular, there is a normalized C-ℓ_1^n average.

Lemma (3.2.31)[91]: Let x be a C-$\ell_1^{n_j}$ average. For all $\gamma \in \Gamma$ we have $\left|\langle d_\gamma^*, x\rangle\right| \le 3Cn_j^{-1}$. If γ is of weight m_i with $i < j$ and $p \in \mathbb{N}$ then $|x(\gamma)| \le 2Cm_i^{-1}$.

Proof. Let $= n_j^{-1}\sum_{k=1}^{n_j} x_k$, as in the definition of a C-ℓ_1^n average. For any γ there is some k such that $\langle d_\gamma^*, x\rangle = n_j^{-1}\langle d_\gamma^*, x_k\rangle$. Thus

$$\left|\langle d_\gamma^*, x\rangle\right| \le n_j^{-1}\left\|d_\gamma^*\right\|\|x_k\| \le 3Cn_j^{-1}.$$

Let us now consider the case where weight $\gamma = m_i$, with $i < j$. From the evaluation analysis

$$e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + m_i^{-1}\sum_{r=1}^a b_r^* \circ P_{(p_{r-1},\infty)},$$

it follows that

$$|x(\gamma)| \le \sum_{r=1}^a \left|\langle d_{\xi_r}^*, x\rangle\right| + m_i^{-1}\sum_{r=1}^a \left\|P_{(p_{r-1},p_r)}x\right\|. \tag{7}$$

By what we have already observed, we have

$$\sum_{r=1}^{a}\left|\langle d_{\xi_r}^*, x\rangle\right| \le 3Can_j^{-1} \tag{8}$$

To estimate the second term in (7) we follow the argument of page 33 of [99], letting I_r (resp. J_r) be the set of k such that ran x_k is contained in (resp. meets) the interval (p_{r-1}, p_r). We have $\#J_r \le \#I_r + 2$ and $\Sigma_r \#I_r \le n_j$. Moreover, for each r, we have $P_{(p_{r-1}, p_r)}x_k = x_k$ if $k \in I_r$, while $P(p_{r-1}, p_r)x_k = 0$ if $k \notin Jr$ and

$$\left\|P_{(p_{r-1}, p_r)}x_k\right\| \le 4\|x_k\| \le 4C \quad \text{ if } \ k \in J_r \backslash I_r.$$

It follows that

$$\left\|P_{(p_{r-1}, p_r)}x_k\right\| \le n_j^{-1}(C\#I_r + 8C)) \le Cn_j^{-1}(\#I_r + 8).$$

Summing over r leads us to

$$\sum_{r\le a}\left\|P_{(p_{r-1}, p_r)}x_k\right\| \le Cn_j^{-1}\left(n_j + 8_a\right). \tag{9}$$

Combining our inequalities, and using the fact that $a \le n_i$ we obtain

$$|x(\gamma)| \le 3Can_j^{-1} + m_i^{-1}n_j^{-1}\left(Cn_j + 8Ca\right) \le Cm_i^{-1} + 5Cn_in_j^{-1} < 2Cm_i^{-1}.$$

Lemma (3.2.32)[91]: Let I be an interval in $\mathbb{N}$, et $(x_k)_{k\in I}$ be a block sequence in X and let $(j_k)_{k\ge1}$ be an increasing sequence of natural numbers. Suppose that, for each k, x_k is a C-$\ell_1^{n_{j_k}}$ -average and that $j_{k+1} >$ max ran x_k. Then (x_k) is a $2C$-RIS.

Corollary (3.2.33)[91]: Let Z be a block subspace of X, and let $C > 2$ be a real number. Then X contains a normalized C-RIS.

Definition (3.2.34)[91]: Let $C > 0$ and let $\varepsilon \in \{0,1\}$. A pair $(x,\gamma) \in X \times \Gamma$ is said to be a (C,j,ε)-exact pair if

(i) $\left|\langle d_\xi^*, x\rangle\right| \le Cm_j^{-1}$ for all $\xi \in \Gamma$;

(ii) weight $\gamma = m_j, \|x\| \le C, x(\gamma) = \varepsilon$;

(iii) for every element γ' of Γ with weight $\gamma' = m_i \ne mj$, we have

$$|x(\gamma')| \le \begin{cases} Cm_i^{-1} & \text{if } \ i < j \\ (Cm_j^{-1} & \text{if } i > j. \end{cases}$$

It will be seen that these estimates, as well as those in the definition, have much in common with those of Lemma (3.2.18). We show how we can construct $(C, 2j, 1)$-exact pairs, starting from a RIS.

Lemma (3.2.35)[91]: Let j be a positive integer and let $(x_k)_{k=1}^{n_{2j}}$ be a skipped-block C-RIS, such that min ran $x_2 \ge 2j$ and $\|x_k\| \ge 1$ for all k. Then there exists $\theta \in \mathbb{R}$, with $|\theta| \le 2$, and there exists $\gamma \in \Gamma$, such that (x,γ) is a $(22C, 2j, 1)$-exact pair, where x is the weighted sum

$$x = \theta m_{2j}n_{2j}^{-1}\sum_{k=1}^{n_{2j}} x_k \ .$$

Proof. We may apply the construction of Lemma (3.2.17) to obtain an element γ of Γ of weight m_{2j} such that

$$n_{2j}^{-1}\sum_{k=1}^{n_{2j}} x_k(\gamma) \ge \frac{1}{2}m_{2j}^{-1}.$$

For a suitably chosen $\theta \in \mathbb{R}$ with $0 < \theta \leq 2$ we have $x(\gamma) = 1$, where $x = \theta m_{2j} n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k$.
We thus have condition (ii) in the definition of an exact pair.
There is no problem establishing condition (i) since, for any ξ, there is some k satisfying $\langle d_\xi^*, x\rangle = \theta m_{2j} n_{2j}^{-1} \langle d_\xi^*, x_k\rangle$. By RIS condition (i), $\|x_k\| \leq C$ and we know that $\|d_\xi^*\| \leq 3$. Hence $|\langle d_\xi^*, x_k\rangle| \leq 6Cm_{2j}n_{2j}^{-1} < Cm_{2j}^{-1}$.
To establish condition (3), we shall use the fact that (x_k) is a C-RIS and apply Proposition (3.2.22), with $j_0 = 2j$. If weight $\gamma' = m_i$ with $i \neq 2j$, we thus have

$$|x(\gamma)| = |\theta| m_{2j} n_{2j}^{-1} \sum_{k=0}^{n_{2j}} x_k(\gamma') \leq \begin{cases} 22Cm_i^{-1} & \text{if } i < 2j \\ 10Cm_{2j}n_{2j}^{-1} + 10Cm_{2j}m_i^{-1} < 11Cm_{2j}^{1} & if\ i > 2j. \end{cases}$$

Using Lemma (3.2.33) we now immediately obtain the following.
Lemma (3.2.36)[91]: If Z is a block subspace of X then for every $j \in \mathbb{N}$ there exists a $(45, 2j, 1)$-exact pair (x, η) with $x \in Z$.
The proof of the following lemma, is very similar.
Lemma (3.2.37)[91]: Let $(x_k)_{k=1}^{n_{2j}}$ be a skipped-block C-RIS, and let $q_0 < q_1 < Q_2 < \cdots < q_{n_{2j}}$ be natural numbers such that ran $x_k \subseteq (q_{k-1}, q_k)$ for all k.Let z denote the weighted sum $x = m_{2j} n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k$. For each k let b_k^* be an element of $B_{q_{k-1}, q_{k-1}}$ with $b_k(x_k) = 0$. Then there exist $\zeta_i \in \Delta_{q_i}(1 \leq i \leq n_{2j})$ such that the element $\eta = \zeta_{n_{2j}}$ has analysis $(q_i, b_i^*, \zeta_i)_{1 \leq i \leq n_{2j}}$ and (z, η) $(12C, n_{2j}, 0)$-exact pair.
We are finally ready to make use of the special conditions governing "odd-weight" elements of Γ. We need to consider a special type of rapidly increasing sequence whose members belong to exact pairs.
Definition (3.2.38)[91]: Consider the space $\mathfrak{X}_K = X(\Gamma)$ where $\Gamma = \Gamma_K$ as defined in (3.2.13). We shall say that a sequence $(x_i)_{i \leq n_{2j_0-1}}$ is a $(C, 2j_0 - 1, \varepsilon)$-dependent sequence if there exist $0 = p_0 < p_1 < p_2 < \cdots < p_{n_{2j-1}}$, together with $\eta_i \in \Gamma_{p_i-1} \backslash \Gamma_{p_{i-1}}$ and $\xi_i \in \Delta_{p_i}$ $(1 \leq i \leq n_{2j_0-1})$ such that
(i) for each k, ran $x_k \subseteq (p_{k-1}, p_k)$;
(ii) the element $\xi = \xi_{2j_0-1}$ of $\Delta_{p_{2j_0-1}}$ has weight m_{2j_0-1} and analysis $(p_i, e_{\eta_i}^*, \xi_i)_{i=1}^{2n_{j_0-1}}$
(iii) (x_1, η_1) is a $(C, 4j_1 - 2, \varepsilon)$-exact pair;
(iv) for each $2 \leq i \leq n_{2j-1}$, (x_i, η_i) is a $(C, 4j_i, \varepsilon)$-exact pair, with ran $x_i \subseteq (p_{i-1}, p_i)$.
We notice that, because of the special odd-weight conditions in (3.2.13), we necessarily have $m_{4j_1-2} =$ weight $\eta_1 > n_{2j_0-1}^2$, and weight $\eta_{i+1} = m_{4j_{i+1}}$, where $j_{i+1} = \sigma(\xi_i)$ for $1 \leq i < n_{2j_0-1}$.
Lemma (3.2.39)[91]: $A(C, 2j_0 - 1, \varepsilon)$-dependent sequence in $\mathfrak{X}_K$ is a C-RIS.
Lemma (3.2.40)[91]: Let $(x_i)_{i \leq n_{2j_0-1}}$ be a $(C, 2j_0 - 1, 1)$-dependent sequence in $\mathfrak{X}_K$ and let J be a sub-interval of $[1, n_{2j_0-1}]$. For any $\gamma' \in \Gamma$ of weight m_{2j_0-1} we have

$$\left|\sum_{i \in I} (-1)^i x_i(\gamma')\right| \leq 4C.$$

Proof. Let ξ_i, η_i, p_i, j_i be as in the definition of a dependent sequence and let γ denote ξ_{2j_0-1}, an element of weight m_{4j_0-1}. Let $(p_i', e_{\eta_i'}^*, \xi_i')_{1 \leq i \leq a'}$ be the analysis of γ' and let the weight of ξ_i' be $m_{4j_1'-2}$ when $i = 1, m_{4j_i'}$ when $1 < i \le a'$. We note that $a' \le n_{2j_0-1}$ because γ' is of weight m_{2j_0-1}. We may thus apply the tree-like property of Lemma (3.2.14) deducing that there exists $1 \le l \le a'$ such that $(p_i', \eta_i', \xi_i') = (p_i, \eta_i, \xi_i)$ for $i < l$ while $j_k \ne j_i'$ for all $l < i \le a'$ and all $1 \le k \le n_{2j_0-1}$. Since

$$e_\gamma^* \circ P_{(0,p_{l-1}]} = e_{\xi_{l-1}}^* = e_{\xi_{l-1}}^* = e_\gamma^* \circ P_{(0,p_l)},$$

we have

$$x_k(\gamma') = x_k(\gamma) = m_{2j_0-1}^{-1} e_{\eta_k}^* \circ P_{(p_{k-1},\infty)} x_k = m_{2j_0-1}^{-1} x_k(\eta_k) = m_{2j_0-1}^{-1},$$

for $1 \le k < l$.

We may now estimate as follows

$$\begin{aligned}|\sum_{k\in I}(-1)^k x_k(\gamma')| &\le |\sum_{k\in J,k<l} m_{2j_0-1}^{-1}(-1)^k x_k(\gamma')| + |x_l(\gamma')| + |\sum_{k\in J,k>l} x_i(\gamma')| \\ &\le m_{2j_0-1}^{-1}|\sum_{k\in I,k<l}(-1)^k| + \|x_l\| + \sum_{k\in J,k<l}\sum_{i\le a'}\left|d_{\xi_i}^* m_{2j_0-1}^{-1} e_{\eta_i'}^* \circ P_{(p_{i-1}',\infty)} x_k\right| \\ &\le 1 + C + n_{2j_0-1}^2 \max_{l<k\in J, i\le a'}\left|d_{\xi_i}^* x_k + m_{2j_0-1}^{-1} e_{\eta_i'}^* \circ P_{(p_{i-1}',\infty)} x_k\right|.\end{aligned}$$

Now we know that, provided $k > l$, weight $\eta_k' \ne$ weight η_i for all i, so by the definition of an exact pair, we have

$$\begin{aligned}&\left|d_{\xi_k'}^*(x_i) + m_{2j_0-1}^{-1} P_{(p_{k-1},\infty]} x_i(\eta_k')\right| \\ &\quad\le C(\text{weight}\,\eta_i)^{-1} + 5Cm_{2j_0-1}^{-1}\max\{(\text{weight}\,\eta_k')^{-1}, (\text{weight}\,\eta_i)^{-1}\} \\ &\quad\le 2C\ \max\{(\text{weight}\,\eta_1)^{-1}, (\text{weight}\,\eta_1')^{-1}\} \\ &\quad= 2C\max\left\{m_{4j_1-2}^{-1}, m_{4j_1'-2}^{-1}\right\} \le 2Cn_{2j_0-1}^{-2},\end{aligned}$$

using the fact that m_{4j_1-2} and $m_{4j_1'-2}$ are both at least $n_{2j_0-1}^2$. We now deduce the inequality $\left|\sum_{i\in J}(-1)^i x_i(\gamma')\right| \le 4C$ as required.

Let $(x_i)_{i\le n_{2j-1}}$ be a $(C, 2_{j0} - 1, 1)$-dependent sequence in $\mathfrak{X}_{\mathrm{K}}$. Then

Lemma (3.2.41)[91]:

$$\|n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}} x_i\| \ge m_{2j_0-1}^{-1} \quad but \quad \|n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}}(-1)^i x_i\| \le 12C\, m_{2j_0-1}^{-2}.$$

Proof. Using the notation of Definition (3.2.38) is easy to show by induction on a, as in Lemma (3.2.17), that

$$\sum_{i=1}^{a} x_i(\xi_a) = m_{2j_0-1}^{-1} a,$$

whence we immediately obtain

$$\|n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}} x_i\| \ge \sum_{i=1}^{a} x_i(\xi_{2j_0-1}) \ge m_{2j_0-1}^{-1}.$$

To estimate $\|n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}}(-1)x_i\|$ we consider any $\gamma \in \Gamma$ and apply the second part of Lemma (3.2.22), with $\lambda_i = (-1)^n n_{2j_0-1}^{-1}$ and with $2j_0 - 1$ playing the role of j_0. Lemma (3.2.40) shows that the extra hypothesis of the second part of Lemma (3.2.22) is indeed satisfied, provided we replace C by $2C$. We deduce that $\|n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}}(-1)^i x_i\| \le 12Cm_{2j_0-1}^{-2}$, as claimed.

A very similar proof yields the following estimate.

Lemma (3.2.42)[91]: Let $(x_i)_{i\le n_{2j-1}}$ be a $(C, 2j_0 - 1, 0)$-dependent sequence in $\mathfrak{X}_{\mathrm{K}}$. Then

$$\|n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} x_i\| \le 4Cm_{2j_0-1}^{-2}$$

We finish the proof of one of our main theorems.

Lemma (3.2.43)[91]: Let Y and Z be block subspaces of $\mathfrak{X}_{\mathrm{K}}$. Then, for each $\epsilon > 0$, there exist $y \in Y$ and $z \in Z$ with $\|y - z\| < \epsilon\|y + z\|$.

Proof. We start by choosing j_0, j_1 with $m_{2j_0-1} > 540\epsilon^{-1}$ and $m_{4j_1-2} > n_{2j_0-1}^2$. Next we use Lemma (3.2.36) to choose a $(45, m_{4j_1-2}, 1)$-exact pair (x_1, η_1) with $x_1 \in Y$. Now, for some $p_1 >$ rank $\eta_1 \vee$ max ran x_1, we define $\xi_1 \in \Delta_{p_1}$ to be $(p_1, m_{2j_0-1}, e_{\eta_1}^*)$. We now set $j_2 = \sigma(\xi_1)$ and choose a $(45, m_{4j_2}, 1)$-exact pair (x_2, η_2) with $x_2 \in Z$ and min ran $x_2 > p_1$. We pick $p_2 >$ rank $\eta_2 \vee$ max ran x_2 and take ξ_2 to be the element $(p_2, \xi_1, m_{2j_0-1}e_{\eta_2}^*)$ of Δ_{p_2}. Notice that this tuple is indeed in Δ_{q_2+1} because we have ensured that weight $\eta_2 = m_{4\sigma(\xi_1)}$.

Continuing in this way, we obtain a $(45, 2j_0 - 1)$-dependent sequence (x_i) such that $x_i \in Y$ when i is odd and $x_i \in Z$ when i is even. We define $y = \sum_{i \text{ odd}} x_i$ and $z = \sum_{i \text{ even}} x_i$, and observe that, by Lemma (3.2.38),

$$\|y + z\| = \|\sum_{i=1}^{n_{2j_0-1}} x_i\| \ge n_{2j_0-1}m_{2j_0-1}^{-1}, \qquad \text{while}$$

$$\|y - z\| = \|\sum_{i=1}^{n_{2j_0-1}} (-1)^i x_i\| \le 12 \times 45 n_{2j_0-1}m_{2j_0-1}^{-2}.$$

Proposition (3.2.1) now yields the theorem.

Theorem (3.2.44)[91]: The space $\mathfrak{X}_{\mathrm{K}}$ is hereditarily indecomposable.

For technical reasons it will be convenient in the first few results to work with elements of $\mathfrak{X}_{\mathrm{K}}$ all of whose coordinates are rational, that is to say with elements of $\mathfrak{X}_{\mathrm{K}} \cap \mathbb{Q}^{\Gamma}$. Since (as may be readily checked) each d_ξ is in $\mathfrak{X}_{\mathrm{K}} \cap \mathbb{Q}^{\Gamma}$, as are all rational linear combinations of these, we see that $\mathfrak{X}_{\mathrm{K}} \cap \mathbb{Q}^{\Gamma}$ is dense in $\mathfrak{X}_{\mathrm{K}}$.

Lemma (3.2.45)[91]: Let $m < n$ be natural numbers and let $x \in \mathfrak{X}_{\mathrm{K}} \cap \mathbb{Q}^{\Gamma}, y \in \mathfrak{X}_{\mathrm{K}}$ be such that ran x, ran y are both contained in the interval $(m, n]$. Suppose that $\operatorname{dist}(y, \mathbb{R}x) > \delta$. Then there exists $b^* \in$ ball $\ell_1(\Gamma_n \backslash \Gamma_m)$, with rational coordinates, such that $b^*(x) = 0$ and $b^*(y) > \frac{1}{2}\delta$.

Proof. Let $u, v \in \ell_\infty(\Gamma_n \backslash \Gamma_m)$ be the restrictions of x, y respectively. Then $x = i_n u, y = i_n v$ and so, for any scalar λ, $\|y - \lambda x\| \le \|i_n\|\|v - \lambda u\|$. Hence $\operatorname{dist}(v, \mathbb{R}u) > \frac{1}{2}\delta$ and so, by the Hahn– Banach Theorem in the finite dimensional space $\ell_\infty(\Gamma_n \backslash \Gamma_m)$, there exists $a^* \in$ ball $\ell_1(\Gamma_n \backslash \Gamma_m)$ with $a^*(u) = 0$ and $a^*(v) > \frac{1}{2}\delta$. Since x has rational coordinates our vector u is in $\mathbb{Q}^{\Gamma_n \backslash \Gamma_m}$. It follows that we can approximate a^* arbitrarily well with $b^* \in \mathbb{Q}^{\Gamma_n \backslash \Gamma_m}$ retaining the condition $b^*(u) = 0$.

Lemma (3.2.46)[91]: Let T be a bounded linear operator on $\mathfrak{X}_{\mathrm{K}}$, let (x_i) be a C-RIS in $\mathfrak{X}_{\mathrm{K}} \cap \mathbb{Q}$ and assume that $\operatorname{dist}(Tx_i, \mathbb{R}x_i) > \delta > 0$ for all i. Then, for all $j, p \in \mathbb{N}$, there exist $z \in [x_i : i \in \mathbb{N}], q > p$ and $\eta \in \Delta_q$ such that

(i) (z, η) is a $(12C, 2j, 0)$-exact pair;

(ii) $(T\,z)(\eta) > \frac{7}{16}\,\delta$;

(iii) $\left\|(I \;-\; P_{(p,q)})Tz\right\| < m_{2j}^{-1}\delta$;

(iv) $\langle P^*_{(p,q]}e^*_\eta, Tz\rangle > \frac{3}{8}\delta$.

Proof. Since the sequence (Tx_i) is weakly null, we may, by taking a subsequence if necessary, assume that there exist $p \;<\; q_0 \;<\; q_1 \;<\; q_2 \;< \cdots$ such that, for all $i \;\geq\; 1$, ran $x_i \subseteq (q_{i-1}, q_i)$ and $\left\|\big(I - P_{(q_{i-1},q_i)}\big)Tx_i\right\| < \frac{1}{5}m_{2j}^{-2}\delta \leq \frac{1}{80}m_{2j}^{-1} \leq \frac{1}{1280}\delta$. It certainly follows from this that $\operatorname{dist}\big(P_{(q_{i-1},q_i)}Tx_i, \mathbb{R}x_i\big) > \frac{1279}{1280}\delta$. We may apply Lemma (3.2.45) to obtain $b_i^* \in$ ball $\ell_1(\Gamma_{q_i-1}\backslash\Gamma_{q_{i-1}})$, with rational coordinates, satisfying

$$\langle b_i^*, x_i\rangle = 0, \qquad \langle b_i^*, P_{(q_{i-1},q_i)}Tx_i\rangle > \frac{1279}{2560}\delta.$$

Taking a further subsequence if necessary, we may assume that the coordinates of b_i^* have denominators dividing $N_{q_i-1}!$, so that $b_i^* \in B_{q_{i-1},q_i-1}$, and we may also assume that $q_1 \geq 2j$.

We are thus in a position to apply Lemma (3.2.37), getting elements ξ_i of weight m_{2j} in Δ_{q_j} such that the element $\eta \;=\; \xi_{n_{2j}}$ of $\Delta_{q_{n_{2j}}}$ has evaluation analysis

$$e_\eta^* = \sum_{i=1}^{n_{2j}} d^*_{\xi_i} + m_{2j}^{-1}\sum_{i=1}^{n_{2j}} P^*_{(q_{i-1},q_i)}b_i^*.$$

and such that (x, η) is a $(12C, 2j, 0)$-exact pair, where z denotes the weighted average

$$x \;= m_{2j}m_{2j}^{-1}\sum_{i=1}^{n_{2j}} x_i.$$

We next need to estimate $(Tz)(\eta)$. For each k, we have $\left\|\big(I - P_{(q_{k-1},q_k)}\big)T\,x_k\right\| < \frac{1}{80}m_{2j}^{-1}\delta$ so that

$$\begin{aligned}(Tx_k)(\eta) \;&\geq \langle e_\eta^*, P_{(q_{k-1},q_k)}Tx_k\rangle - \frac{1}{80}m_{2j}^{-1}\delta \\ &= m_{2j}^{-1}\langle b_k^*, P_{(l_{k-1},l_k)}Tx_k\rangle - \frac{1}{80}m_{2j}^{-1}\delta > \frac{1247}{2560}m_{2j}^{-1}\delta.\end{aligned}$$

It follows that

$$(Tz)(\eta) = n_{2j}^{-1}m_{2j}\sum_{k=1}^{n_{2j}}(Tx_k)(\eta) > \frac{7}{16}\delta.$$

For inequality (3) in which we are taking $q = q_{n_{2j}}$, we note that $p < q_{k-1} < q_k \leq q$ for all k so that

$$\begin{aligned}\left\|\big(I - P_{(p,q]}\big)Tx_k\right\| &= \left\|\big(P_{(0,p]} + P_{(q,\infty)}\big)Tx_k\right\| \\ &= \left\|\big(P_{(0,p]} + P_{(q,\infty)}\big)\big(I - P_{(q_{k-1},q_k)}\big)Tx_k\right\| \\ &\leq 5\left\|\big(I - P_{(q_{k-1},q_k)}\big)Tx_k\right\| < m_{2j}^{-2}\delta,\end{aligned}$$

using our usual estimates for norms of FDD projections. The inequality for the weighted average z follows at once. Inequality (iv) follows from (ii) and (iii) thus

$$\langle P^*_{(p,q]}e^*_\eta, Tz\rangle \geq (Tz)(\eta) = \|(I - P_{(p,q]})Tz\| > \frac{7}{16}\delta - m_{2j}^{-1}\delta \geq \frac{3}{16}\delta.$$

Proposition (3.2.47)[91]: Let T be a bounded linear operator on $\mathfrak{X}_K$ and let $(x_i)_{i\in\mathbb{N}}$ be a RIS in $\mathfrak{X}_K$.

Then $\operatorname{dist}(Tx_i, \mathbb{R}x_i) \to 0$ as $i \to \infty$.

Proof. It will be enough to prove the result for a RIS in $\mathfrak{X}_K \cap \mathbb{Q}^\Gamma$. Suppose, if possible, that $\operatorname{dist}(Tx_i, \mathbb{R}x_i) > \delta > 0$ for all i. The idea is to obtain a dependent sequence in rather the same way as we did in Lemma (3.2.43), except that this time it will be a 0-dependent sequence, rather than a 1-dependent sequence.

We start by choosing j_0 such that $m_{2j_0}^{-1} > 256C\|T\|\delta^{-1}$ and j_1 such that $m_{4j_1-1} > m_{2j_0-1}^2$ Taking $p = p_0 = 0$ and $j = 2j_1 - 1$ in Lemma (3.2.46) we can find q_1 and a $(12C, 4j_1 - 2, 0)$-exact pair (z_1, η_1) with rank $\eta_1 = q_1, (Tz_1)(\eta_1) > \frac{3}{8}\delta$ and $\|(I - P_{(0,q_1]})(Tz_1)\| < m_{4j_1-2}^{-1}\delta$. Let $p_1 = q_1 + 1$ and let ξ_1 be the special Type 1 element of Δ_{p_1} given by $\xi_1 = (p_1, m_{2j_0-1}, e^*_{\eta_1})$.

Now, recursively for $2 \leq i \leq n_{2j_0-1}$, define $j_i = \sigma(\xi_{i-1})$, and use the lemma again to choose q_i and a $(12C, 4j_i, 0)$-exact pair (z_i, η_i) with rank $\eta_i = q_i$, ran $z_i \subseteq (p_{i-1}, q_i], \langle P^*_{(p_{i-1},q_i]}e^*_{\eta_i}, Tz_1\rangle > \frac{3}{8}\delta$ and $\|(I - P_{(p_i,q_i]})(Tz_i)\| < m_{4j_i}^{-1}\delta$. We now define $p_i = q_i + 1$ and let ξ_i to be the Type 2 element $(p_i, \xi_{i-1}, m_{2j_0-1}^{-1}, e^*_{\eta_i})$ of Δ_{p_i}.

It is clear that we have constructed a $(12C, 2j_0 - 1, 0)$-dependent sequence $(z_i)_{1\leq i\leq n_{2j_0-1}}$. By the estimate of Lemma (3.2.42) we have

$$\|z\| \leq 48Cm_{2j_0-1}^{-2}$$

for the average

$$z = n_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}} z_i.$$

However, let us consider the element $\gamma = \xi_{n_{2j_0-1}^{-1}}$ of $\Delta_{p_{n_{2j_0-1}}}$, which has evaluation analysis

$$e^*_\gamma = \sum_{i=1}^{n_{2j_0-1}} d^*_{\xi_i} + m_{2j_0-1}^{-1}\sum_{i=1}^{n_{2j_0-1}} P^*_{(p_{i-1},p_i)e^*_{\eta_i}}.$$

Noting that $p_k = q_q + 1$ for $k \geq 1$, and that $m_{4j_i} > m_{4j_1-2} > n_{2j_0-1}^2$, we may estimate $(Tz)(\gamma)$ as follows

$$(Tz)(\gamma) = n_{2j_0-1}^{-1}\sum_{k=1}^{n_{2j_0-1}} (Tz_k)(\gamma)$$

$$\geq n_{2j_0-1}^{-1}\sum_{k=1}^{n_{2j_0-1}} \left(\langle P^*_{(p_{k-1},p_k)}e^*_\gamma, Tz_k\rangle - \|(I - P_{(p_{k-1},q_k)})(Tx_k)\|\right)$$

$$\geq n_{2j_0-1}^{-1}\sum_{k=1}^{n_{2j_0-1}} \left(m_{2j_0-1}^{-1}\langle P^*_{(p_{k-1},p_k)}e^*_{\eta_k}, Tx_k\rangle - m_{4j_1-2}^{-1}\delta\right)$$

$$\geq \delta n_{2j_0-1}^{-1} \sum_{k=1}^{n_{2j_0-1}} \left(\frac{3}{8} m_{2j_0-1}^{-1} - 5n_{2j_0-1}^{-2}\right) > \frac{1}{4} m_{2j_0-1}^{-1} \delta.$$

So

$$\|Tz\| \geq \frac{1}{4} m_{2j_0-1}^{-1} > \frac{1}{144} C^{-1} \delta m_{2j_0-1} \|z\|$$

which is a contradiction because $\frac{1}{144} C^{-1} \delta m_{2j_0-1} > \|T\|$ by our original choice of j_0.

Theorem (3.2.48)[91]: Let T be a bounded linear operator on $\mathfrak{X}_{\mathrm{K}}$. Then there exists a scalar λ such that $T - \lambda I$ is compact.

Proof. We start by considering a normalized RIS (x_i) in $\mathfrak{X}_{\mathrm{K}}$. By Proposition (3.2.47) there exist scalars λ_i such that $\|Tx_i - \lambda_i x_i\| \to 0$. We claim that λ_i necessarily tends to some limit λ. Indeed, if not, by passing to a subsequence, we may suppose that $|\lambda_{i+1} - \lambda_i| > \delta$ for all i. Now the sequence (y_i) where $y_i = x_{2i-1} + x_{2i}$ is again a RIS, so that there exist μ_i with $\|Ty_i - \mu_i y_i\| \to 0$ by Proposition (3.2.47) again. We thus have

$$\|(\lambda_{2i} - \mu_i)x_{2i} + (\lambda_{2i-1} - \mu_i)x_{2i-1}\|$$
$$\leq \|Tx_{2i} - \lambda_{2i}x_{2i}\| + \|Tx_{2i-1} - \lambda_{2i-1}x_{2i-1}\| + \|Ty_i - \mu_i y_i\| \to 0.$$

Since the RIS (x_i) is a block sequence, there exist l_i such that $P_{(0,l_i]}y_i = x_{2i-1}$ and $P_{(l_i,\infty)}y_i = x_{2i}$. Using the assumption that the sequence (x_i) is normalized we now have

$$|\lambda_{2i-1} - \mu_i| = \|(\lambda_{2i-1} - \mu_i)x_{2i-1}\| \leq \left\|P_{(0,l_i]}\right\| \|(\lambda_{2i} - \mu_i)x_{2i} + (\lambda_{2i-1} - \mu_i)x_{2i-1}\|,$$

with a similar estimate for $|\lambda_{2i} - \mu_i|$. Each of these sequences thus tends to 0, so that $\lambda_{2i} - \lambda_{2i-1}$ also tends to 0, contrary to our assumption.

We now show that the scalar λ is the same for all rapidly increasing sequences. Indeed, if (x_i) and (x_i') are RIS with $\|Tx_i - \lambda x_i\| \to 0$ and $\|Tx_i' - \lambda' x_i\| \to 0$, we may find $i_1 < i_2 < \cdots$ such that the sequence (y_k) defined by

$$y_k = \begin{cases} x_{i_k} & \text{if } k \text{ is odd} \\ x_{i_k}' & \text{if } k \text{ is even} \end{cases}$$

is again a RIS. By the first part of the proof we must have $\lambda = \lambda'$.

We have now obtained λ such that $\|(T - \lambda I)x_i\| \to 0$ for every RIS. By Proposition (3.2.27), we deduce that $\|(T - \lambda I)x_i\| \to 0$ for every bounded block sequence in $\mathfrak{X}_{\mathrm{K}}$. This, of course, implies that $T - \lambda I$ is compact.

We devote to a proof that $\mathfrak{X}_{\mathrm{K}}$ is saturated with reflexive HI subspaces having HI duals. The proof involves reworking much of the construction of a subspace of $\mathfrak{X}_{\mathrm{K}}$ and its dual. By standard blocking arguments, it is enough to prove the following theorem.

Theorem (3.2.49)[91]: Let $L = \{l_0, l_1, l_2, \ldots\}$ be a set of natural numbers satisfying $l_{n-1} + 1 < l_n$, and for each $n \geq 1$ let F_n be a subspace of the finite-dimensional space $P_{(l_{n-1},l_n)}\mathfrak{X}_{\mathrm{K}} = \bigoplus_{l_{n-1}<k<l_n} M_k$. Then the subspace $W = \overline{\bigoplus_{n\in\mathbb{N}} F_n}$ of $\mathfrak{X}_{\mathrm{K}}$ is reflexive and has HI dual.

We note in passing the following corollary, which gives an indication of the "very conditional" nature of the basis of ℓ_1 that we have constructed. For the purposes of the statement we briefly abandon the "Γ notation" and revert to the notation of Definition (3.2.5) and Theorem (3.2.8).

Corollary (3.2.50)[91]: There exist a basis $(d_n^*)_{n\in\mathbb{N}}$ of ℓ_1 and natural numbers $k_1 < k_2 < \cdots$ with the property that the quotient $\ell_1/[d_n^* : n \in M]$ is hereditarily indecomposable whenever the subset M of $\mathbb{N}$ contains infinitely many of the intervals $(k_p, k_{p+1}]$.

The rest will be devoted to the proof of Theorem (3.2.49). We have already remarked at the end that the subspace W defined in the statement of the theorem is reflexive. The subspaces F_n form a finite-dimensional decomposition of W, the corresponding FDD projections being $Q_{(m,n]} = P_{(l_m,l_n]} \upharpoonright W = P_{(l_m,l_n)} \upharpoonright W$, when $0 \le m < n$. The dual space W^* has a dual FDD (F_n^*) and corresponding projections $Q^*_{(m,n]}$. We shall establish hereditary indecomposability of W^* via the criterion Proposition (3.2.1). We write R for the quotient mapping $\mathfrak{X}_{\mathrm{K}}^* = \ell_1 \to W^*$ and observe that if $f_n^* \in F_n^*$ for $1 \le n \le \mathbb{N}$ then the norm of $f^* = \sum_{n=1}^{N} f_n^*$ in W^* is given by

$$\left\|\sum f^*\right\|_{Y^*} = \inf\{\|g^*\| : g^* \in \mathfrak{X}_{\mathrm{K}}^* \text{ and } Rg^* = f^*\}.$$

Lemma (3.2.51)[91]: If $f^* \in \operatorname{im} Q^*_{(M,N]} = \bigoplus_{M<n\le N} F_n^* \subset W^*$ then there exists $h^* \in \mathfrak{X}_{\mathrm{K}}^* = \ell_1(\Gamma)$ with supp $h^* \subseteq \Gamma_{l_N-1} \setminus \Gamma_{l_M}$ and $\|h^*\|_1 \le 4\|f^*\|$ and $RP^*_{(l_M,l_N)}h^* = RP^*_{(l_M,\infty)}h^* = f^*$.

Proof. We extend f^* by the Hahn–Banach theorem to obtain $g^* \in \mathfrak{X}_{\mathrm{K}}^* = \ell_1(\Gamma)$ with $Rg^* = f^*$ and $\|g^*\|_{\mathfrak{X}_{\mathrm{K}}^*} = \|f^*\|_{W^*}$. We set $h^* = P_{(0,l_N)}g^* \in \ell_1(\Gamma_{l_N-1})$ and $b^* = h^*\chi_{\Gamma_{l_N-1}\setminus\Gamma_{l_N}}$, noting that

$$\|b^*\|_1 \le \|h^*\|_1 \le 2\|g^*\|_1 \le 4\|g^*\|_{\mathfrak{X}_{\mathrm{K}}^*} = 4\|f^*\|.$$

To check that $RP^*_{(l_M,l_N)}h^* = RP^*_{(l_M,\infty)}b^* = f^*$, we first note that

$$P^*_{(l_M,\infty)}b^* = P^*_{(l_M,\infty)}h^*,$$

because $P^*_{(l_M,\infty)}k^* = 0$ whenever supp $k \subseteq \Gamma_{l_M}$. Since both b^* and h^* are supported by Γ_{l_N-1} we have

$$P^*_{(l_M,l_N)}b^* = P^*_{(l_M,\infty)}P^*_{(0,l_N)}b^* = P^*_{(l_M,\infty)}b^* = P^*_{(l_M,\infty)}h^* = P^*_{(l_M,\infty)}P^*_{(0,l_N)}h^* = P^*_{(l_M,l_N)}g^*.$$

It follows that

$$R^*P^*_{(l_M,l_N)}b^* = R^*P^*_{(l_M,l_N)}g^* = g^* \circ P_{(l_M,l_N)} \upharpoonright W = g^* \circ Q_{(M,N]} = f^*.$$

Lemma (3.2.52)[91]: Let $j \ge 1, 1 \le a \le n_{2j}$ and $M \le M_0 < M_1 < M_2 < \cdots < M_a$ be natural numbers, with $2j \le M_1$. For each $i \le a$, let f_i^* be in ball $\bigoplus_{M_{i-1}<n\le M_i} F^*$ and write $f^* = \sum_{i=1}^{a} f_i^*$.

Then there exists $\gamma \in \Gamma$ with $p^*_{(0,l_M]}e^*_\gamma = 0$ and $\|4m_{2j}R(e^*_\gamma) - f^*\| \le 2^{-l_M+3}$; in particular $\|f^*\|_{Y^*} \le 5m_{2j}$.

Proof. By Lemma (3.2.51) there exist $h_i^* \in \ell_1\left(\Gamma_{l_{M_i-1}} \setminus \Gamma_{l_{M_{i-1}}}\right)$ with $\|h_i^*\|_1 \le 4$ and $R(P^*_{\left(l_{M_{i-1}},l_{M_i}\right)} = f_i^*$. Since $B_{l_{M_i-1},l_{M_{i-1}}}$ is an ϵ-net in ball $\ell_1\left(\Gamma_{l_{M_i-1}} \setminus \Gamma_{l_{M_{i-1}}}\right)$, with $\epsilon = 2^{-l_{M_i}+1} \le 2^{-l_M-2i+1}$ we can choose $b_i^* \in B_{l_{M_k},l_{M_{k-1}}}$ such $\|h_i^* - 4b_i^*\|_1 \le 2^{-l_M-2i+3}$.

Now write $p_i = l_{M_i}$ for $1 \le i \le a$ and apply the construction of Proposition (3.2.16) to obtain $\gamma \in \Delta_{p_a}$ with evaluation analysis

$$e^*_\gamma = \sum_{i=1}^{a} d^*_{\xi_i} + m^1_{2j}\sum_{i=1}^{a} P^*_{(p_{k-1},\infty)}b^*_k .$$

Since rank $\xi_i = p_i \in L$ for all i, we have $Rd^*_{\xi_i} = 0$ and so

$$\|f^* - 2m_{2j}R(e^*_\gamma)\| = \left\|\sum_{i=1}^{a}\left(f_i^* - 2RP^*_{(p_{i-1},\infty)}b_i^*\right)\right\|$$

$$\leq \sum_{i=1}^{n_{2j}} \left\| RP^*_{(p_{i-1},\infty]} h_i^* - 2RP^*_{(p_{i-1},\infty)} b_i^* \right\|$$
$$\leq 3\sum_{i=1}^{a} \|h_i^* - 2b_i^*\| \leq 3\sum_{i=1}^{\infty} 2^{-l_M-2i+2} = 2^{-l_M+2}.$$
It follows that $\|f^*\| \leq \left\|4m_{2j}R\left(e_\gamma^*\right)\right\| + 8 \leq 5m_{2j}$.

Lemma (3.2.53)[91]: Let Y be any block subspace of W^* and let n, M be positive integers. For every $C > 1$ there exists a $4C$-ℓ_1^n-average $w \in W$, with $Q_{(0,M]}w = 0$, and a functional $g^* \in$ ball Y with $Q^*_{(0,M]}g^* = 0$ and $\langle g^*, w\rangle \geq 1$.

Proof. The proof is a dualized version of Lemma (3.2.30). We suppose, without loss of generality, that $C < 2$ and choose l, j such that $C^l > n$ and $n_{2j} > \left(10n_{2j}\right)^l$; we take k minimal subject to $C^k > 5m_{2j}$ noting that
$$n_{2j} > \left(10m_{2j}\right)^l \geq (2C^{k-1})^l \geq C^{kl} > n^k.$$
Now take $(f_i^*)_{i=1}^{n^k}$ to be a normalized block sequence in $Y \cap \ker Q^*_{(0,M]}$; we may apply Lemma (3.2.52) to obtain
$$\|\sum_{i=1}^{n^k} \pm f_i^*\| \leq 5m_{2j} < C^k.$$
So by part (ii) of Lemma (3.2.2) (with $C = 1 + \epsilon$) there are successive linear combinations $g_1^*, \ldots, g_n^*$ such that $\|g_i^*\| \geq C^{-1}$ for all i, while
$$\|\sum_{i=1}^{n} \pm g_i^*\| \leq 1,$$
for all choices of sign. Since (g^*) is a block sequence in $\ker Q^*_{(0,M]}$ we can choose $M \leq N_0 < N_1 < \cdots$. such that $Q^*_{(N_{i-1},N_i]}g_i^* = g_i^*$. Now we choose, for each i an element w_i of W such that $\|w_i\| \leq C$ and $\langle g_i^*, w_i\rangle = 1$. If we set $w_i' = Q^*_{(N_{i-1},N_i]}w_i$ then we have $\|w_i'\| \leq 4C$ and $\langle g_i^*, w_i'\rangle = \langle g_i^*, w_i\rangle = 1$, while $\langle g_i^*, w_h'\rangle = 0$ when $h \neq i$. The element $w = n^{-1}\sum_{i=1}^{n} w_i'$ is thus a $4C$-ℓ_1^n average, with $Q_{(0,p]}w = 0$, and satisfies $\langle g^*, w\rangle = 1$, where $g^* = \sum_{i=1} g_i^* \in$ ball Y .

Lemma (3.2.54)[91]: Let Y be any block subspace of W^* and let N, j be positive integers. There exists a $(600, 2j, 1)$-exact pair (z, γ) with $z \in W$, $Q_{(0,N]}z = 0, P^*_{(0,l_N]}e_\gamma^* = 0$ and $\mathrm{dist}\left(Re_\gamma^*, Y\right) < 2^{-l_N}$.

Proof. By repeated applications of Lemma (3.2.53), we construct natural numbers $N \leq M_0 < M_1 < M_2 < \cdots$ and $j_1 < j_2 < \cdots$, elements $w_i = Q_{(M_{i-1},M_i]}w_i$ of W, and functionals $g_i^* = Q^*_{(M_{i-1},M_i]}g_i^* \in$ ball Y such that

(i) w_i is a 5-$\ell_1^{n_{j_i}}$-average;

(ii) $\langle g_i^*, w_i\rangle \geq 1$;

(iii) $j_{i+1} > M_i$.

It follows from Lemma (3.2.32) that (w_i) is a 10-RIS.

Writing $g^* = \sum_{i=1}^{n_{2j}} g_i^*$ and applying Lemma (3.2.52) we find γ of weight m_{2j} such that $\left\|4m_{2j}R\left(e_\gamma^*\right) - g^*\right\| \leq 2^{-N+3}$ We thus have

$$\operatorname{dist}(Re^*_\gamma, Y) \le \|Re^*_\gamma - \frac{1}{4} m_{2j}^{-1} g^*\| \le 2^{-l_N+1} m_{2j}^{-1} < 2^{-l_N},$$

and

$$4m_{2j} \sum_{i=1}^{n_{2j}} w_i(\gamma) \ge \sum_{i=1}^{n_{2j}} \langle g^*, w_i \rangle - 2^{-l_N+3} \ge n_{2j} - 16.$$

We now set $z = \theta m_{2j} n_{2j}^{-1} \sum_{i=1}^{n_{2j}} w_i$ where θ is chosen so that $z(\gamma) = 1$; by the above inequality $0 < \theta \le 4 + 128 n_{2j}^{-1} < 5$.

To estimate $\|z\|$ and $|z(\gamma')|$ when weight $\gamma' = m_h \ne m_{2j}$ we return to Lemma (3.2.22) deducing that

$$\|z\| \le 60\theta \quad \text{and} \quad |z(\gamma')| \le \begin{cases} 110\theta m_h^{-1} & \text{if } h < 2j \\ 60\theta m_{2j}^{-1} & \text{if } h > 2j. \end{cases}$$

So (z, γ) is certainly a $(600, 2j, 1)$-exact pair.

Lemma (3.2.55)[91]: Let Y_1 and Y_2 be block subspaces of W^* and let j_0 be a natural number. There exists a sequence $(x_i)_{i \le n_{2j_0-1}}$ in W, together with natural numbers $0 = p_0 < p_1 < p_2 < \cdots < p_{n_{2j_0-1}}$, and elements $\eta_i \in \Gamma_{p_i-1} \backslash \Gamma_{p_{i-1}}, \xi_i \in \Delta_{p_i} (1 \le i \le n_{2j_0-1})$, satisfying the conditions (i) to (iv) of Definition (3.2.38) with $C = 600, \varepsilon = 1$, and such that, for all $i \ge 1$, the following additional properties hold

(i) rank $\xi_i = p_i \in L$;

(ii) $P^*_{(p_{i-1}, p_i]} e^*_{\eta_i} = 0, P_{(p_{i-1}, p_i]}(x_i) = x_i$;

(iii) $\operatorname{dist}(Re^*_{\eta_i}, Y_k) < 2^{-p_{i-1}}$, where $k = 1$ for odd i and $k = 2$ for even i.

Proof. We start by choosing j_1 such that $m_{4j_1-2} > n^2_{2j_0-1}$ and then applying Lemma (3.2.54) to obtain a $(600, 4j_1 - 2, 1)$-exact pair (x_1, η_1) with $x_1 \in W$. Set $p_1 = l_{N_1}$, where N_1 is large enough to ensure that $P_{(0,p_1)} x_1 = Q_{(0,N_1]} x_1 = x_1$, rank $\eta_1 < p_1$ and $2^{p_1} > 2n_{2j_0-1}$. Let $\xi_1 = \left(p_1, m^{-1}_{2j_0-1}, \eta_1\right) \in \Delta_{p_1}$.

Continuing recursively, if for some $i < n_{2j_0-1}$, we have defined $\xi_i \in \Delta_{p_i}$, where $p_i = l_{N_i}$, we set $j_{i+1} = \sigma(\xi_i)$ and apply Lemma (3.2.54) to get a $(600, 4j_{i+1}, 1)$-exact pair (x_{i+1}, η_{i+1}) with $x_{i+1} \in W$, $Q_{(0,N_i]} x_{i+1} = P_{(0,p_i]} x_{i+1} = 0, P^*_{(0,p_i]} e^*_{\eta_{i+1}} = 0$ and $\operatorname{dist}(R^* e^*_{\eta_{i+1}}, Y_k) < 2^{-p_i}$, where k depends on the parity of $i + 1$. We now take N_{i+1} large enough, set $p_{i+1} = l_{N_{i+1}}$ and define $\xi_{i+1} = (p_{i+1}, \xi_i, m^{-1}_{2j_0-1}, \eta_{i+1}) \in \Delta_{p_{i+1}}$.

We are now ready to finish the proof of the theorem. We consider any two infinite-dimensional subspaces Y_1 and Y_2 of W^* and apply Lemma (3.2.55) obtaining a dependent sequence satisfying (i) to By property (7) we may choose, for each i, an element y_i^* of Y_k with

$$\|y_i^* - Re^*_{\eta_i}\| < 2^{-p_i}.$$

We set

$$y^* = m^{-1}_{2j_0-1} \sum_{i \text{ odd}} y_i^* \in Y_1, \quad z^* = m^{-1}_{2j_0-1} \sum_{i \text{ even}} y_i^* \in Y_2,$$

If γ is the element $\xi_{n_{2j_0}-1}$ then the evaluation analysis of γ is

$$e^*_\gamma = \sum_{i=1}^{n_{2j_0-1}} d^*_{\xi_i} + m^{-1}_{2j_0-1} \sum_{i=1}^{n_{2j_0-1}} P^*_{(p_{i-1}, p_i)} e^*_{\eta_i}$$

$$= \sum_{i=1}^{n_{2j_0-1}} d^*_{\xi_i} + m^{-1}_{2j_0-1} \sum_{i=1}^{n_{2j_0-1}} e^*_{\eta_i},$$

because $P^*_{(0,p_{i-1}]} e^*_{\eta_i} = 0$. Since rank $\xi_i = p_i \in L$ for all i we have

$$Re^*_\gamma = m^{-1}_{2j_0-1} \sum_{i=1}^{n_{2j_0-1}} Re^*_{\eta_i},$$

which leads to

$$\|y^* + z^*\| \le 1 + \|m^{-1}_{2j_0-1} \sum_{i=1}^{n_{2j_0-1}} Re^*_{\eta_i}\| = 1 + \|Re^*_\gamma\| \le 2.$$

We shall prove that $\|y^* - z^*\|$ is very large by estimating $\langle y^* - z^*, x\rangle$, where x is the average

$$x = n^{-1}_{2j_0-1} \sum_{k=1}^{n_{2j_0-1}} (-1)^k x_k,$$

about which we know from Lemma (3.2.41) that

$$\|x\| \le 7200 m^{-1}_{2j_0-1}.$$

and the definition of a 1-exact pair, we have

$$\langle e^*\eta_i, x_k\rangle = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases}$$

so that

$$\begin{aligned} \langle y^* - z^*, x\rangle &= n^{-1}_{2j_0-1} m^{-1}_{2j_0-1} \sum i,k \langle y^*_i - x_k\rangle \\ &\ge n^{-1}_{2j_0-1} m^{-1}_{2j_0-1} \sum_{i,k} (\langle e^*\eta_i, x_k\rangle - 2^{-p_i}) \\ &\ge m^{-1}_{2j_0-1}(1 - n_{2j_0-1} 2^{-p_1}) \ge \frac{1}{2} m^{-1}_{2j_0-1}, \end{aligned}$$

the last step following from our choice of p_1 with $2^{p_1} > 2n_{2j_0-1}$.
We can now deduce that

$$\|y^* - z^*\| \ge \frac{m_{2j_0-1}}{14400}.$$

We have shown that the subspaces Y_1 and Y_2 of W_* contain elements y_*, z_* with $\|y^* + z^*\| \le 2$ and $\|y^* - z^*\|$ arbitrarily large. By Proposition (3.2.1), we have established hereditary indecomposability of W^*.

If we are looking at a bounded linear operator $T : Y \to \mathfrak{X}_K$ defined only on a subspace Y of $\mathfrak{X}_K$, rather than on the whole space, then, as in other HI constructions, the arguments of the preceding can be used to show that T can be expressed as $\lambda I_Y + S$ with S strictly singular. However, as we shall now see, in this case the perturbation need not be compact.

Proposition (3.2.56)[91]: There exists a subspace Y of $\mathfrak{X}_K$ and a strictly singular, non-compact operator T from Y into $\mathfrak{X}_K$. In fact, for a suitably chosen Y, we may choose T mapping Y into itself.

Proof. By a theorem of Androulakis, Odell, Schlumprecht and Tomczak-Jaegermann [94], in order to find Y and a strictly singular, non-compact $T : Y \to \mathfrak{X}_K$, it is enough to exhibit normalized sequences (x_i) and (y_i) in $\mathfrak{X}_K$ such that (y_i) has a spreading model equivalent to the usual ℓ_1-basis, while (x_i) has a spreading model that is not equivalent to that basis. For (x_i) we may take any normalized RIS; indeed, by Proposition (3.2.20), the spreading model associated with any RIS is dominated by the unit vector basis of the Mixed Tsirelson space $\mathfrak{T}[(A_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$, and so is not equivalent to the ℓ_1-basis. For (y_i) we may take a specific sequence , setting

$$y_n = \sum_{\xi\in\Delta_n} d_\xi \,.$$

The result we need is a lemma about norms of linear combinations of these vectors.

Lemma (3.2.57)[91]: Let F be a finite set of natural numbers with $\min F \geq j$ and $\#F < 2n_{2j}$. Then, for all real scalars a_n,

$$\|\sum_{n\in F} a_n y_n\| \geq \frac{1}{4}\sum_{n\in F}|a_n|\,.$$

Proof. Without loss of generality, we may suppose that $\sum_{n\in F} a_n^+ \geq \frac{1}{2}\sum_{n\in F}|a_n|$ and we may choose $p_1, p_2, \dots, p_r$ in F, with $p_{i+1} > p_i, r \leq p_{2j}$, and

$$\sum_{i=1}^{r} a_i \geq \frac{1}{4}\sum_{n\in F}|a_n|.$$

Since $p_1 \geq \min F \geq 2j, \Delta_{n_1}$ does contain Type 1 elements of the form $\left(p_1, m_{2j}^{-1}, \pm e_{\eta_1}^*\right)$, with $\eta_1 \in \Gamma_{n_1-1}$. We take ξ_1 to be such an element, and continue recursively, for $1 \leq i < r$, taking η_{i+1} to be any element of $\Delta_{p_{i+1}}$ and ξ_{i+1} to be the Type 2 element $\left(p_{i+1}, \xi_i, m_{2j}^{-1}, \pm e_{\eta_1}^*\right)$ of $\Delta_{n_{i+1}}$. If $\gamma = \xi_r$ then the evaluation analysis of is

$$e_\gamma^* = \sum_{i=1}^{r} d_{\xi_i}^* + m_{2j}^{-1}\sum_{i=1}^{r} \pm P_{(n_{i-1},n_i)}^* e_{\eta_i}^*\,.$$

If we write $y = \sum_{n\in F} a_n y_n$, we have $\langle d_{\xi_i}^*, y\rangle = a_{n_i}$ for each i, so that

$$e_\gamma^*(y) = \sum_{i=1}^{r} a_{p_i} + m_{2j}^{-1}\sum_{i=1}^{r} \pm P_{(n_{i-1},n_i)}^* e_{\eta_i}^*\,(y).$$

We have not until now been explicit about how the signs $\pm$ were chosen, but it is now clear that this may be done in such a way that $e_\gamma^*(y) \geq \sum_{i=1}^{r} a_{p_i} \geq \frac{1}{4}\sum_{n\in F}|a_n|\,.$

It is now clear that the theorem of Androulakis et al may be applied. In order to get the refined version where T takes Y into itself, it is enough to look a little more closely at the proof given in [94]. It turns out that we may take (y_i) as above and Y to be the closed linear span $[y_i : i \in \mathbb{N}]$. It may be shown that, for any RIS (x_i), the mapping $y_i \mapsto x_i$ extends to a bounded linear operator from Y to $\mathfrak{X}_K$. Since Y, like all other infinite dimensional subspaces, contains a RIS, we may choose the x_i to lie in Y.

The original spaces $X_{a,b}$ of Bourgain and Delbaen provided, for the first time, a continuum of non-isomorphic L_∞ spaces. It has also been noted [92] that if we take Y to be Hilbert space and X to be $X_{a,b}$ with (for instance) $0 < b < \frac{1}{2} < a < 1, a^4 + b^4 = 1$, then all operators from X to Y and all operators from Y to X are compact. The constructions allow us to exhibit a continuum of spaces $X_\alpha(\alpha \in \mathfrak{c})$ such that, for all $\alpha \neq \beta, \mathcal{L}(X_\alpha, X_\beta) = \mathcal{K}(X_\alpha, X_\beta)$.

We start by taking an almost-disjoint family $(L_\alpha)_{\alpha\in\mathfrak{c}}$ of infinite subsets of $\mathbb{N}$. For each α we enumerate L_α in increasing order as l_j^α and define

$$m_j^\alpha = m_{l_j^\alpha}, n_j^\alpha = n_{l_j^\alpha},$$

where $(m_j, n_j) = (2^{2j}, 2^{2j^2+1})$ is the sequence mentioned.

Now we may take X_α to be either $\mathfrak{B}_{\mathrm{mT}}\left[(A_{n_j^\alpha}, 1/m_j^\alpha)_{j\in\mathbb{N}}\right]$ or $\mathfrak{X}_{\mathrm{K}}\left[(A_{n_j^\alpha}, 1/m_j^\alpha)_{j\in\mathbb{N}}\right]$.

Proposition (3.2.58)[91]: Assume that $\alpha \neq \beta$ and let $T : X_\alpha \to X_\beta$ be a bounded linear operator. For any RIS $(x_i)_{i\in\mathbb{N}}$ in X_α, we have $\|T(x_i)\| \to 0$ as $i \to \infty$.

Proof. Let (x_i) be a C-RIS in X_α and suppose, if possible, that $\|Tx_i\| > \delta > 0$ for all i. Since (Tx_i) is weakly null we may, by taking a subsequence, assume that (Tx_i) is a small perturbation of a skipped-block sequence in X_β. Thus, if $l = l_{2j}^\beta \in L_\beta$, we may apply Proposition (3.2.17) to conclude

$$\|n_l^{-1}\sum_{i=1}^{n_l} Tx_r\|_{X_\beta} \geq \frac{1}{4}m_{2j}^{-1}n_l^{-1}\sum_{r=1}^{n_l}\|Tx_r\| \geq \frac{1}{4}\delta m_{2j}^{-1}.$$

On the other hand, Corollary (3.2.21) tells us that

$$\|n_l^{-1}\sum_{i=1}^{n_l} x_r\|_{X_\alpha} \leq 10C\|n_l^{-1}\sum_{i=1}^{n_l} e_i\|,$$

where the norm on the right-hand side is calculated in $T[(A_{3n_j}, m_j^{-1})_{j\in L_\alpha}]$. If l is not in L_α then this norm is at most m_l^{-2} by Lemma (3.2.4), so that

$$\|n_l^{-1}\sum_{i=1}^{n_l} x_r\|_{X_\alpha} \leq 10Cm_l^{-2}.$$

By the assumed almost-disjointness of L_β and L_α we can certainly choose j such that $l_{2j}^\beta \notin L_\alpha$ and $m_l > 40\|T\|\delta^{-1}$, yielding a contradiction.

The spaces $\mathcal{L}(X)$ and $K(X)$ of bounded (respectively compact) linear operators on an infinite-dimensional Banach space X are always decomposable. (Indeed, for finite dimensional subspaces $E \subset X$ and $F \subset X^*$, the subspaces $X^* \otimes E$ and $F \otimes X$ are complemented.) So we must not hope for too much exotic structure in these spaces of operators. We shall look briefly at subspaces of $\mathcal{L}(\mathfrak{X}_{\mathrm{K}})$. Certainly, $\mathcal{L}(\mathfrak{X}_{\mathrm{K}}) = K(\mathfrak{X}_{\mathrm{K}}) \oplus \mathbb{R}I$ has HI subspaces, such as those isomorphic to $\mathfrak{X}_{\mathrm{K}}$, and subspaces isomorphic to $\mathfrak{X}_{\mathrm{K}}^* = \ell^1$. It has no subspace isomorphic to c_0 by a result of Emmanuele. (The main result of [104] shows that c_0 does not embed into $K(X_{a,b})$ and the same proof works for $\mathfrak{X}_{\mathrm{K}}$.) We shall now see that $(\mathfrak{X}_{\mathrm{K}})$ does have other subspaces with unconditional basis. It is a general fact that if (x_n) is a basic sequence in a Banach space X then the injective tensor product $\ell_1 \widehat{\otimes}_\varepsilon X$ contains a sequence equivalent to the "unconditionalization" of the basic sequence (x_n). This follows immediately from the following exact formula for the norm of a finite sum of elementary tensors in $\ell_1 \widehat{\otimes}_\varepsilon X$:

$$\|\sum_{j=1} e_j^* \otimes x_j\|_\varepsilon = \sup\|\sum_{j=1} \pm x_j\|,$$

where the supremum is over all choices of signs.

In the case of $\mathfrak{X}_{\mathrm{K}}$ the space of compact operators $K(\mathfrak{X}_{\mathrm{K}})$ is isomorphic to $\ell_1 \widehat{\otimes}_\varepsilon X$ and so contains the unconditionalization of any basic sequence in $\mathfrak{X}_{\mathrm{K}}$. An interesting special case is that of the basis (d_γ); we have chosen to prove the following proposition in a way that does not depend on the general theory of tensor products.

Proposition (3.2.59)[91]: The family $\left(e_\gamma^* \otimes d_\gamma\right)_{\gamma\in\Gamma}$ is an unconditional basis of a reflexive subspace of $K(\mathfrak{X}_{\mathrm{K}})$.

Proof. Let us write $U_\gamma = e_\gamma^* \otimes d_\gamma$ considered as the rank–1 operator

$$U_\gamma : \mathfrak{X}_{\mathrm{K}} \to \mathfrak{X}_{\mathrm{K}};\ x \longmapsto x(\gamma)d_\gamma.$$

For a finite linear combination $W = \sum_{\gamma\in\Gamma_n} w(\gamma)U\,(\gamma)$ and any $x\ \in$ ball $\mathfrak{X}_{\mathrm{K}}$ we have

$$\|W(x)\| = \|\sum_{\gamma\in\Gamma_n} (w\gamma)x(\gamma)d_\gamma\| \leq \max_{\pm}\|\sum_{\gamma\in\Gamma_n} w(\gamma)d_\gamma\|\,.$$

We shall write $|||W|||$ for the last expression on the line above. We have thus shown that $\|W\| \leq\ |||W|||$.

On the other hand, if we choose $u(\gamma) = \pm 1$ for $\gamma\ \in \Gamma_n$ in such a way as to achieve the maximum in the definition of $|||W|||$ and then set $y = i_n(u)$ we have

$$|||W||| = \|\sum_{\gamma\in\Gamma_n} u(\gamma)d_\gamma\| = \ \|W(y)\| \leq \|W\|\|i_n\| \leq\ 2\|W\|.$$

Thus the operator norm $\|\cdot\|$ and the unconditionalized norm $|||\cdot|||$ are equivalent on $\left[U_\gamma : \gamma\ \in \Gamma\right]$.

It will be convenient to work with the latter norm.

Given a linear combination $= \sum_\gamma v(\gamma)U_\gamma$, any vector $\sum_\gamma \mp v(\gamma)d_\gamma$ in $\mathfrak{X}_{\mathrm{K}}$, (whether or not the signs achieve the supremum in the definition of the unconditionalized norm), will be called a realization of W.

If the subspace $\left[U_\gamma : \gamma\ \in \Gamma\right]$ is not reflexive then by unconditionality there is a skipped block sequence equivalent to the unit vector basis of either c_0 or ℓ_1. We shall treat the case of ℓ_1.

We consider a normalized skipped block sequence with $V_i = \sum_{\gamma\in\Gamma_{p_{i-1}}\setminus\Gamma_{p_{i-1}}} v(\gamma)U_\gamma$ and suppose, if possible, that (V_i) is C-equivalent to the usual ℓ_1-basis for the norm $|||\cdot|||$. suppose that $|||V_i||| \leq C$ for all i and that

$$|||\sum_i a(i)V_i||| \geq \sum_i |a(i)|$$

for all scalars a_i. Let us note that if W is a linear combination of the form

$$W = n^{-1}\sum_{i=l+1}^{l+n} V_i,$$

then any realization $\widehat{W}$ of W is a C-ℓ_1 -average as in Definition (3.2.29)). Indeed $\widehat{W}$ is expressible as $n^{-1}\sum_{i=l+1}^{l+n}\hat{V}_i$ where the $\hat{V}_i$ are realizations of V_i, and so satisfy $\left\|\widehat{W}_i\right\| \leq |||W_i||| \leq\ C$ for all i.

We now look at Lemma (3.2.32). It should be clear that, by choosing sequences $(j_k)_{j\in\mathbb{N}}$ and $(l_k)_{j\in\mathbb{N}}$ growing sufficiently fast, we may define

$$W_k = n_{j_k}^{-1}\sum_{i=l_j+1}^{l_j+n_{j_k}} V_i\,,$$

in such a way that any realizations $\widehat{W}_k$ form a $2C$-RIS in $\mathfrak{X}_{\mathrm{K}}$. In particular

$$|||n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} W_k\,||| = \|n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} \widehat{W}_k\|$$

for suitable realizations $\widehat{W}_k$, yielding

$$|||n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} W_k\,||| \leq 12C\,m_{j_0}^{-1},$$

by Proposition (3.2.22). On the other hand,

$$|||n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} W_k\,||| = |||n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} n_{j_k}^{-1}\sum_{i=l_k+1}^{l_k+n_{n_k}} V_i\,|||$$

which is at least 1, by our assumption on (V_i).
So we have a contradiction for suitably large values of j_0.

The constructions give no clue as to whether there exists a reflexive Banach space on which all operators are scalar–plus–compact. The construction of such a space, if one exists, will need new ideas. We thus have no example of a reflexive space on which all operators have non-trivial proper invariant subspaces. It is piquant to observe that, at the other end of the spectrum, the construction of a reflexive space on which some operator has no non-trivial proper invariant subspace has also proved to be very resistant to attack. See Enflo [105], [106] and Read [119], [120] for more about the Invariant Subspace Problem, noting the more recent [121] of Read, in which a strictly singular operator is constructed which has no non-trivial proper invariant subspace.

We do not know whether an isomorphic predual of ℓ_1 which has the “few-operators” property in the scalar–plus–strictly-singular sense necessarily also has this property in the scalar–plus–compact sense. An answer to this would follow from an affirmative solution to the following more general problem.

Chapter 4
Sets and an Additive Combinatorics Approach

We discuss generalizations and applications of the results obtained. We study of the "approximate duality conjecture" which was suggested by Ben-Sasson and Zewi and studied there in connection to the PFR conjecture. We improve the bounds on approximate duality assuming the PFR conjecture. Then we use the approximate duality conjecture (with improved bounds) to get our upper bound on the communication complexity of low-rank martices. We show a theorem (due to Bourgain) goes as follows. For a noticeable fraction of pairs γ_1, γ_2 in the spectrum, $\gamma_1 + \gamma_2$ belongs to the spectrum of the same set with a smaller threshold. Here we show that this result can be made combinatorial by restricting to a large subset. We show that for any set A there exists a large subset A , such that the sumset of the spectrum of A has bounded size. Our results apply to sets of size $|A| = |G|^{\alpha}$ for any constant $\alpha > 0$, and even in some sub-constant regime

Section (4.1): Large Trigonometric Sums

For N be a positive integer. We denote by $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ the set of residues modulo N. Let $f : \mathbb{Z}_N \to \mathbb{C}$ be an arbitrary function. The Fourier transform of f is given by the formula

$$\hat{f}(r) = \sum_{n \in \mathbb{Z}_N} f(n) e(-nr), \tag{1}$$

where $e(x) = e^{-2\pi i x/N}$. The following Parseval equality holds for the Fourier coefficients of f:

$$\sum_{r \in \mathbb{Z}_N} \left|\hat{f}(r)\right|^2 = N \sum_{n \in \mathbb{Z}_N} |f(n)|^2. \tag{2}$$

Let δ and α be real numbers, $0 < \alpha \leqslant \delta \leqslant 1$, and let A be a subset of $\mathbb{Z}_N$ of cardinality δN. The symbol A will also stand for the characteristic function of this set. Consider the set $\mathcal{R}_\alpha$ of large trigonometric sums of A:

$$\mathcal{R}_\alpha = \mathcal{R}_\alpha(A) = \{r \in \mathbb{Z}_N : |\hat{A}(r)| \geqslant \alpha N\}. \tag{3}$$

For many problems of the combinatorial theory of numbers it is important to know the structure of $\mathcal{R}_\alpha$, it is important to know its properties, as will be indicated below. We only mention the fact that this problem was posed by Gowers in [125].

The elementary properties of $\mathcal{R}_\alpha$ are as follows. The definition implies that $0 \in \mathcal{R}_\alpha$ and $\mathcal{R}_\alpha = -\mathcal{R}_\alpha$, which means that $-r \in \mathcal{R}_\alpha$ if $r \in \mathcal{R}_\alpha$. Further, Parseval's equality (2) implies that $|\mathcal{R}_\alpha| \leqslant \delta/\alpha^2$. Has $\mathcal{R}_\alpha$ any other non-trivial properties?

It turns out that the answer to this question is positive.

We denote by log the logarithm to the base 2.

In 2002, M.-C. Chang proved the following theorem [126].

Theorem (4.1.1)[124]: (M.-C. Chang). Let δ and α be real numbers, $0 < \alpha \leqslant \delta \leqslant 1$, let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN and let $\mathcal{R}_\alpha$ be the set defined by (3). Then there is a set $\Lambda = \{\lambda_1, \ldots, \lambda_{|\Lambda|}\} \subseteq \mathbb{Z}_N$, $|\Lambda| \leqslant 2\left(\frac{\delta}{\alpha}\right)^2 \log(1/\delta)$, such that every element r of $\mathcal{R}_\alpha$ can be represented in the form

$$r \equiv \sum_{i=1}^{|\Lambda|} \varepsilon_i \lambda_i \pmod N, \tag{4}$$

where $\varepsilon_i \in \{-1, 0, 1\}$.

Developing the approach suggested in [127] (see also [128]), Chang applied her result to the proof of Freiman's theorem [129] on sets with small sum. Recall that $Q \subseteq \mathbb{Z}$ is called a d-dimensional arithmetic progression if

$$Q = \{n_0 + n_1\lambda_1 + \cdots + n_d\lambda_d : 0 \leqslant \lambda_i < m_i\},$$

where the m_i are positive integers and the n_i are integers.

Theorem (4.1.2)[124]: (G. A. Freiman). Let $C > 0$ be some number, let $A \subseteq \mathbb{Z}$ be an arbitrary set and let $|A + A| \leqslant C|A|$. Then one can find numbers d and K depending only on C and a d-dimensional arithmetic progression Q such that $|Q| \leqslant K|A|$ and $A \subseteq Q$.

Another application of Theorem (4.1.1) was given by B. Green in [130] (see [131], [132] and [133]). One of the main results of [130] can be stated as follows.

Theorem (4.1.3)[124]: (B. Green). Let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN. Then $A + A + A$ contains an arithmetic progression whose length is greater than or equal to

$$2^{-24}\delta^5\left(\log\left(\frac{1}{\delta}\right)\right)^{-2} N^{\delta^2/(250\log(1/\delta))}. \tag{5}$$

(See [134]), Green showed that Chang's theorem is, in a sense, exact. Let $E = \{e_1, \dots, e_{|E|}\} \subseteq \mathbb{Z}_N$ be an arbitrary set. We denote by $\mathrm{Span}(E)$ the set of all sums of the form $\sum_{i=1}^{|E|} \varepsilon_i e_i$, where $\varepsilon_i \in \{-1, 0, 1\}$.

Theorem (4.1.4)[124]: (B. Green). Let δ and α be real numbers, $\delta \leqslant 1/8, 0 < \alpha \leqslant \delta/32$.

Assume that

$$\left(\frac{1}{\delta}\right)^2 \log\frac{1}{\delta} \leqslant \frac{\log N}{\log\log N}. \tag{6}$$

Then there is an $A \subseteq \mathbb{Z}_N, |A| = [\delta N]$, such that the set $\mathcal{R}_\alpha$ defined by (3) is not contained in $\mathrm{Span}(\Lambda)$ for any Λ with $|\Lambda| \leqslant 2^{-12}\left(\frac{\delta}{\alpha}\right)^2 \log(1/\delta)$.

The structure of $\mathcal{R}_\alpha$ in the case when α is close to δ was studied in [135]–[137] (see also [138]).

We see that results on the structure of $\mathcal{R}_\alpha$ are of importance in the combinatorial theory of numbers. We prove the following theorem.

Theorem (4.1.5)[124]: Let δ and α be real numbers, $0 < \alpha \leqslant \delta$, let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN, let $k \geqslant 2$ be an even number and let $\mathcal{R}_\alpha$ be the set defined by (3). Assume $B \subseteq \mathcal{R}_\alpha \setminus \{0\}$ is an arbitrary set. Then the quantity

$$T_k(B) := |\{(r_1, \dots, r_k, r'_1, \dots, r'_k) \in B^{2k} : r_1 + \cdots + r_k = r'_1 + \cdots + r'_k\}| \tag{7}$$

is greater than or equal to

$$\frac{\delta\alpha^{2k}}{2^{4k}\delta^{2k}}|B|^{2k}. \tag{8}$$

Proof: First we prove an analogue of Lemma (4.1.6).

We claim that the assertion of Theorem (4.1.5) is non-trivial in the case when δ tends to zero as N tends to infinity (if δ does not tend to zero as $N \to \infty$, then the structure of $\mathcal{R}_\alpha$ can be arbitrary [139]–[141]). Consider the simplest case $k = 2$. Let the order of the cardinality of $\mathcal{R}_\alpha$ be equal to δ/α^2. By Theorem (4.1.5), the order of the number of solutions of the equation

$$r_1 + r_2 = r_3 + r_4, \qquad r_1, r_2, r_3, r_4 \in \mathcal{R}_\alpha \setminus \{0\}, \tag{9}$$

is greater than or equal to δ/α^4. Among these solutions there are three series of trivial solutions. In the first series $r_1 = r_3, r_2 = r_4$, in the second $r_1 = r_4, r_2 = r_3$ and, finally, in the third $r_1 = -r_2, r_3 = -r_4$. Therefore, equation (9) has at most $3|\mathcal{R}_\alpha|^2$ trivial solutions. The cardinality of $\mathcal{R}_\alpha$ does not exceed δ/α^2. Therefore, $3|\mathcal{R}_\alpha|^2$ is less than $3\delta^2/\alpha^4$. We see that this quantity is less than δ/α^4 as δ tends to zero. Thus, Theorem (4.1.5) states that equation (9) has non-trivial solutions.

Hence, $\mathcal{R}_\alpha$ has some additive structure.

The proof of Theorem (4.1.5), where we begin with a detailed consideration of the case when $k = 2$ and then prove it in the general situation.

We generalize Theorem (4.1.5) to systems of linear equations. In our proof we use properties of the Gowers norms (see [142]).

We apply the main result to some problems in the combinatorial theory of numbers. We show that M.-C. Chang's theorem can be derived from Theorem (4.1.5) and Rudin's inequality [143].

Let N be a positive integer and let $\hat{A}(r)$ be the Fourier transform of the characteristic function A. As mentioned above, the following equality holds for the Fourier coefficients of A:

$$\sum_{r\in\mathbb{Z}_N} \left|\hat{A}(r)\right|^2 = N|A|. \tag{10}$$

Are there any non-trivial relations between the Fourier coefficients $\hat{A}(r)$ other than (10)? It is obvious that the answer to this question is positive.

Consider a slightly more general situation. Let $f : \mathbb{Z}_N \to \mathbb{C}$ be an arbitrary complex function. The following inversion formula holds for the Fourier coefficients of $f(x)$:

$$f(x) = \frac{1}{N}\sum_{r\in\mathbb{Z}_N} \hat{f}(r)e(rx). \tag{11}$$

The function $f(x)$ is the characteristic function of some subset of $\mathbb{Z}_N$ if and only if

$$|f(x)|^2 = f(x) \tag{12}$$

for all x in $\mathbb{Z}_N$. Substituting (11) into (12), we obtain that

$$\frac{1}{N^2}\sum_{r',r''} \hat{f}(r')\overline{\hat{f}(r'')}e(r'x - r''x) = \frac{1}{N}\sum_{u} \hat{f}(u)e(ux). \tag{13}$$

Hence,

$$\sum_{u}\left(\frac{1}{N}\sum_{r} \hat{f}(r)\overline{\hat{f}(r-u)}\right)e(ux) = \sum_{u} \hat{f}(u)e(ux). \tag{14}$$

Since (14) holds for all $x \in \mathbb{Z}_N$, we have

$$\hat{f}(u) = \frac{1}{N}\sum_{r} \hat{f}(r)\overline{\hat{f}(r-u)}. \tag{15}$$

Hence, $f : \mathbb{Z}_N \to \mathbb{C}$ is a characteristic function if and only if equality (15) holds for its Fourier coefficients. It is clear that (15) also holds for the characteristic function $A(x)$ of the set A. Moreover, (15) contains all the relations between the Fourier coefficients of A: for example, Parseval's equality (2) can be obtained by putting $u = 0$.

We shall need the following generalization of (15). Let $f, g: \mathbb{Z}_N \to \mathbb{C}$ be arbitrary complex functions. Then

$$\frac{1}{N}\sum_{r}\hat{f}(r)\overline{\hat{g}(r-u)} = \sum_{x} f(x)\overline{g(x)}e(-xu), \tag{16}$$

and (15) obviously follows from (16).

We explain the basic idea of the proof of Theorem (4.1.5). Let $A \subseteq \mathbb{Z}_N$ be an arbitrary set, $|A| = \delta N$, and let $\mathcal{R}_\alpha$ be the set of large trigonometric sums given by (3). Consider a model situation. Assume that $|\hat{A}(r)| = \alpha N$ for all $r \in \mathcal{R}_\alpha \setminus \{0\}$ and let $\hat{A}(r) = 0$ for all $r \notin \mathcal{R}_\alpha, r \neq 0$ (the justification of such a hypothesis will be discussed below). Let $\delta \leqslant 1/4$ and let u be an arbitrary non-zero residue belonging to $\mathcal{R}_\alpha$. Then $|\hat{A}(u)| = \alpha N$. Using formula (15) and the triangle inequality, we obtain that

$$\alpha N = |\hat{A}(u)| \leqslant \frac{1}{N}\sum_{r}|\hat{A}(r)||\hat{A}(r-u)|$$

$$\leqslant \frac{1}{N}\delta N|\hat{A}(-u)| + \frac{1}{N}|\hat{A}(u)|\delta N + \frac{1}{N}\sum_{r\neq 0,u}|\hat{A}(r)||\hat{A}(r-u)|. \tag{17}$$

Hence,

$$\frac{1}{N}\sum_{r\neq 0,u}|\hat{A}(r)||\hat{A}(r-u)| \geqslant \frac{\alpha N}{2}.$$

We have $|\hat{A}(r)| = \alpha N R_\alpha(r)$ for all $r \neq 0$. Therefore,

$$\sum_{r\neq 0,u} R_\alpha(r)R_\alpha(r-u) > \frac{1}{2\alpha}. \tag{18}$$

It follows from (18) that for all $u \in R_\alpha \setminus \{0\}$ the equation $r_1 - r_2 = u$, where $r_1, r_2 \in R_\alpha \setminus \{0\}$, has at least $1/(2\alpha)$ solutions. Therefore, R_α has non-trivial additive relations.

We now proceed to the rigorous proof of Theorem (4.1.5). We shall prove it first in the case when $k = 2$ and then in the general case. Let $k = 2$ and let B be an arbitrary subset of $R_\alpha \setminus \{0\}$. We denote by $[N]$ the segment $\{1, 2, \dots, N\}$ of the positive integers.

We need the following lemma.

Lemma (4.1.6)[124]: Let δ and α' be real numbers, $0 < \alpha' \leqslant \delta$, and let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN. Assume also that

$$R'_{\alpha'} = \{r \in \mathbb{Z}_N : \alpha' N \leqslant |\hat{A}(r)| < 2\alpha' N\} \tag{19}$$

and let B' be an arbitrary subset of $R'_{\alpha'} \setminus \{0\}$. Then

$$T_2(B') \geqslant \frac{(\alpha')^4|B'|^4}{16\delta^3}.$$

Proof. Let

$$f_{B'}(x) = \frac{1}{N}\sum_{r\in B'}\hat{A}(r)e(rx).$$

Generally speaking, $f_{B'}(x)$ is a complex function. It is obvious that $\hat{f}_{B'}(r) = \hat{A}(r)B'(r)$. Consider the sum

$$\sigma = \sum_{s}\left|\sum_{r}\hat{f}_{B'}(r)\overline{\hat{A}(r-s)}\right|^2. \tag{20}$$

Using formula (16) and Parseval's equality, we obtain that

$$\sigma = N^2\sum_{s}\left|\sum_{x} f_{B'}(x)\overline{A(x)}e(-xs)\right|^2 = N^3\sum_{x}|f_{B'}(x)|^2A^2(x). \tag{21}$$

We estimate $\sum_x \left|\hat{f}_{B'}(r)\right|^2 A^2(x)$ from below using Parseval's equality and the definition of $R'_{\alpha'}$:

$$\left(\sum_x f_{B'}(x)A(x)\right)^2 = \left(\frac{1}{N}\sum_r \hat{f}_{B'}(r)\overline{\hat{A}(r)}\right)^2 = \left(\frac{1}{N}\sum_r |f_{B'}(r)|^2\right)^2$$
$$\geqslant (N(\alpha')^2|B'|)^2 = (\alpha')^4|B'|^2N^2. \qquad (22)$$

On the other hand, we have

$$\left(\sum_x f_{B'}(x)A(x)\right)^2 \leqslant \left(\sum_x |f_{B'}(x)|^2A^2(x)\right)\left(\sum_x A^2(x)\right)$$
$$= \delta N\left(\sum_x |f_{B'}(x)|^2 A^2(x)\right). \qquad (23)$$

Using inequalities (22) and (23), we obtain that

$$\sigma^2 \geqslant \frac{(\alpha')}{\delta^2}|B'|^4N^8. \qquad (24)$$

To obtain an upper bound for σ^2, we note that

$$\sigma = \sum_s \sum_{r,r'} \hat{f}_{B'}(r)\overline{\hat{f}_{B'}(r)\hat{A}(r-s)}\hat{A}(r'-s)$$
$$= \sum_u \left(\sum_r \hat{f}_{B'}(r)\overline{\hat{f}_{B'}(r-u)}\right)\overline{\left(\sum_r \hat{A}(r)\overline{\hat{A}(r-u)}\right)}, \qquad (25)$$

whence

$$\sigma^2 \leqslant \sum_u \left|\sum_r \hat{f}_{B'}(r)\overline{\hat{f}_{B'}(r-u)}\right|^2 \sum_u \left|\sum_r \hat{A}(r)\overline{\hat{A}(r-u)}\right|^2 = \sigma_1\sigma_2. \quad (26)$$

Using formula (15) and Parseval's equality, we obtain that

$$\sigma_2 = N^2\sum_u \left|\hat{A}(u)\right|^2 = \delta N^4. \qquad (27)$$

Since $\hat{f}_{B'}(r) = \hat{A}(r)B'(r)$ and $B' \subseteq R'_{\alpha'} \setminus \{0\}$, we have $|\hat{f}_{B'}(r)| \leqslant 2\alpha' B'(r)N$. Hence,

$$\sigma_1 \leqslant 16(\alpha')^4T_2(B')N^4. \qquad (28)$$

Substituting (27) and (28) into (24), we obtain that $T_2(B') \geqslant (\alpha')^4|B'|^4/(16\delta^3)$. The lemma is proved.

Let

$$B_i = \{r \in B\colon 2^{i-1}\alpha N \leqslant |\hat{A}(r)| < 2^i\alpha N\}, \qquad i \geqslant 1.$$

It is clear that $B = \coprod_{i\geqslant 1} B_i$. Applying Lemma (4.1.6) to every B_i, we obtain that $T_2(B_i) \geqslant \left(2^{i-1}\alpha\right)^4|B_i|^4/(16\delta^3),\ i \geqslant 1$. Hence,

$$T_2(B) \geqslant \sum_i T_2(B_i) \geqslant \frac{\alpha^4}{2^8\delta^3}\sum_i 2^{4i}|B_i|^4. \qquad (29)$$

We have $|B| = \sum_i |B_i|$. The Cauchy–Bunyakovsky inequality implies that

$$|B|^4 = \left(\sum_i 2^i 2^{-i}|B_i|\right)^4 \leqslant \left(\sum_i 2^{4i}|B_i|^4\right)\left(\sum_i 2^{-4i/3}\right)^3 \leqslant \sum_i 2^{4i}|B_i|^4. \qquad (30)$$

Substituting (30) into (29), we obtain the inequality

$$T_2(B) \geqslant \frac{\alpha^4}{2^8\delta^3}|B|^4. \tag{31}$$

Now consider the general case when $k > 2$.

Lemma (4.1.7)[124]: Let δ and α' be real numbers, $0 < \alpha' \leqslant \delta$, let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN and let $k \geqslant 2$ be an even number. Assume also that

$$R'_{\alpha'} = \{r \in \mathbb{Z}_N : \alpha' N \leqslant |\hat{A}(r)| < 2\alpha' N\} \tag{32}$$

and let B' be an arbitrary subset of $R'_{\alpha'} \setminus \{0\}$. Then

$$T_k(B') \geqslant \frac{\delta(\alpha')^{2k}|B'|^{2k}}{(2\delta)^{2k}}.$$

Proof. Let $f_{B'}(x)$ be the function defined by the formula

$$f_{B'}(x) = \frac{1}{N}\sum_{r\in B'} \hat{A}(r)e(rx).$$

Consider the sum

$$\sigma = \left(\sum_x f_{B'}(x)A(x)\right)^k. \tag{33}$$

Estimating σ from below as in Lemma (4.1.6), we obtain that

$$\sigma \geqslant ((\alpha')^2|B'|N)^k. \tag{34}$$

Since k is an even number, it has the form $k = 2k', k' \in \mathbb{N}$. Using Hölder's inequality, we obtain that

$$\sigma = \left(\sum_x f_{B'}(x)A(x)\right)^{2k'} \leqslant \left(\sum_x |f_{B'}(x)|^{2k'}A^2(x)\right)\left(\sum_x A(x)\right)^{k-1}$$
$$= \left(\sum_x |f_{B'}(x)|^{2k'}A^2(x)\right)(\delta N)^{k-1}. \tag{35}$$

Hence,

$$(\sigma')^2 = \left(\sum_x |f_{B'}(x)|^{2k'}A^2(x)\right)^2 \geqslant \delta^2\frac{(\alpha')^{4k}}{\delta^{2k}}|B'|^{2k}N^2. \tag{36}$$

On the other hand, the inversion formula (11) implies that

$$
\begin{aligned}
\sigma' &= \sum_x |f_{B'}(x)|^{2k'} A^2(x) \\
&= \frac{1}{N^{2k'+2}} \sum_x \sum_{r_1,\ldots,r_{k'},r_1',\ldots,r_{k'}'} \sum_{y,z} \hat{f}_{B'}(r_1)\cdots \hat{f}_{B'}(r_{k'})\overline{\hat{f}_{B'}(r_1')} \cdots \\
&\cdot \overline{\hat{f}_{B'}(r_{k'}')}\hat{A}(y)\overline{\hat{A}(z)} \times e\left(x\left(r_1+\cdots+r_{k'}-r_1'-\cdots-r_{k'}'\right)\right) e\left(x(y-z)\right) \\
&= \frac{1}{N^{2k'+2}} \sum_{u,y} \sum_{\substack{r_1,\ldots,r_{k'},r_1',\ldots,r_{k'}' \\ r_1+\cdots+r_{k'}-r_1'-\cdots-r_{k'}'-u}} \hat{f}_{B'}(r_1)\cdots \hat{f}_{B'}(r_{k'}) \times \overline{\hat{f}_{B'}(r_1')} \cdots \\
&\cdot \overline{\hat{f}_{B'}(r_{k'}')}\hat{A}(y)\overline{\hat{A}(y-u)} \\
&= \frac{1}{N^{2k'+2}} \sum_u \left(\sum_y \hat{A}(y)\overline{\hat{A}(y-u)}\right) \\
&\times \left(\sum_{\substack{r_1,\ldots,r_{k'},r_1',\ldots,r_{k'}' \\ r_1+\cdots+r_{k'}-r_1'-\cdots-r_{k'}'-u}} \hat{f}_{B'}(r_1)\cdots \hat{f}_{B'}(r_{k'})\overline{\hat{f}_{B'}(r_1')} \cdots \right. \\
&\left. \cdot \overline{\hat{f}_{B'}(r_{k'}')} \right). \qquad (37)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(\sigma')^2 &\leqslant \frac{1}{N^{4k'+2}} \sum_u \left|\sum_y \hat{A}(y)\overline{\hat{A}(y-u)}\right|^2 \\
&\times \sum_u \left| \sum_y \sum_{\substack{r_1,\ldots,r_{k'},r_1',\ldots,r_{k'}' \\ r_1+\cdots+r_{k'}-r_1'-\cdots-r_{k'}'-u}} \hat{f}_{B'}(r_1)\cdots \hat{f}_{B'}(r_{k'})\overline{\hat{f}_{B'}(r_1')} \cdots \overline{\hat{f}_{B'}(r_{k'}')} \right|^2 \\
&= \frac{1}{N^{4k'+2}} \sigma_1\sigma_2. \qquad (38)
\end{aligned}
$$

Using formula (15) and Parseval’s equality, we obtain that

$$
\sigma_1 = N^2 \sum_u \left|\hat{A}(u)\right|^2 = \delta N^4. \qquad (39)
$$

Since $B' \subseteq R_{\alpha'}' \setminus \{0\}$, we have $|\hat{f}_{B'}(r)| \leqslant 2\alpha' B'(r) N$. Hence,

$$\sigma_2 \leqslant \left((2\alpha' N)^{2k'}\right)^2 \sum_u \left| \sum_y \sum_{\substack{r_1,\dots,r_{k'},r'_1,\dots,r'_{k'} \\ r_1+\dots+r_{k'}-r'_1-\dots-r'_{k'}-u}} B'(r_1)\cdots B'(r_{k'})B'(r'_1)\cdots B'(r'_{k'}) \right|^2$$
$$= (2\alpha' N)^{2k} T_k(B'). \tag{40}$$
Using equalities (38), (39) and inequalities (36), (40), we obtain that
$$T_k(B') \geqslant \frac{\delta(\alpha')^{2k}|B'|^{2k}}{(2\delta)^{2k}}. \tag{41}$$
The lemma is proved.
Let
$$B_i = \{r \in B \colon 2^{i-1}\alpha N \leqslant |\hat{A}(r)| < 2^i \alpha N\}, \qquad i \geqslant 1.$$
It is clear that $B = \coprod_{i\geqslant 1} B_i$. Applying Lemma (4.1.7) to every B_i, we obtain that $T_k(B_i) \geqslant \delta\left(2^{i-1}\alpha\right)^{2k} |B_i|^{2k}/(2\delta)\delta^{2k}$, $i \geqslant 1$. Hence,
$$T_k(B) \geqslant \sum_i T_k(B) \geqslant \frac{\delta\alpha^{2k}}{2^{4k}\delta^{2k}} \sum_i 2^{2ki}|B_i|^{2k}. \tag{42}$$
We have $|B| = \sum_i |B_i|$. Using Hölder's inequality, we obtain that
$$|B|^{2k} = \left(\sum_i 2^i 2^{-i}|B_i|\right)^{2k} \leqslant \left(\sum_i 2^{2ki}|B_i|^{2k}\right)\left(\sum_i 2^{-2ki/(2k-1)}\right)^{2k-1}$$
$$\leqslant \sum_i 2^{2ki}|B_i|^{2k} \tag{43}$$
Substituting (43) into (42), we obtain the inequality
$$T_k(B) \geqslant \frac{\delta\alpha^{2k}}{2^{4k}\delta^{2k}}|B|^{2k}. \tag{44}$$
The theorem is proved.

Let k be a positive integer and let $d \geqslant 0$ be an integer. Let $A = (a_{ij})$ be the $2^{d+1}k \times (d+1)$ matrix whose elements a_{ij} are defined by the formula
$$a_{ij} = \begin{cases} 1 & \text{if the } (i-1)\text{st coefficient in the binary expansion of } (j-1) \\ & \text{is equal to 1 and } 1 \leqslant j \leqslant 2^d k, \\ -1 & \text{if the } (i-1)\text{st coefficient in the binary expansion of } (j-1) \\ & \text{is equal to 1 and } 2^d k < j \leqslant 2^{d+1}k, \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$
Recall that the binary expansion of a positive integer n is defined by the rule $n = \sum nl \cdot 2^{l-1}$, where $l \geqslant 1$ and $n_l \in \{0,1\}$.
For example, when $k = 2$ and $d = 2$ we have
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 \end{pmatrix}.$$
We prove the following theorem.

Let $d \geqslant 0$ be an integer and let $\{0,1\}^d = \{\omega = (\omega_1, \dots, \omega_d) : \omega_j \in \{0,1\},\ j = 1, 2, \dots, d\}$ be the ordinary d-dimensional cube. If $\omega \in \{0,1\}^d$, then $|\omega|$ is defined to be $\omega_1 + \cdots + \omega_d$. If $h = (h_1, \dots, h_d) \in \mathbb{Z}_N^d$, then $\omega \cdot h := \omega_1 h_1 + \cdots + \omega_d h_d$.

Let $\mathcal{C}$ be the operator of complex conjugation. If n is a positive integer, then $\mathcal{C}^n$ stands for the nth power of this operator. Let $\|\omega\| = \sum_{i=1}^{d} \omega_i \cdot 2^{i-1} + 1$. For every $\omega \in \{0,1\}^d$ we define a map from $\mathbb{Z}_N^{2^d}$ to $\mathbb{Z}_N$, which we denote by the same symbol ω, by the rule: if $\vec{r} \in \mathbb{Z}_N^{2^d}$, then $\omega(\vec{r})$ is the $\|\omega\|$th component of the vector $\vec{r}$.

Definition (4.1.8)[124]: Let $f : \mathbb{Z}_N \to \mathbb{C}$ be an arbitrary function. The uniform Gowers norm (or, briefly, the Gowers norm) of f is defined to be

$$\|f\|_{U^d} := \left(\frac{1}{N^{d+1}} \sum_{x \in \mathbb{Z}_N, h \in \mathbb{Z}_N^d} \prod_{\omega \in \{0,1\}^d} \mathcal{C}^{|\omega|} f(x + \omega \cdot h) \right)^{1/2^d} . \tag{46}$$

We shall need the following lemma (see [142]).

Lemma (4.1.9) (the motonicity inequality for Gowers norms)[124]: Let $f : \mathbb{Z}_N \to \mathbb{C}$ be an arbitrary function and let d be a positive integer. Then

$$\|f\|_{U^d} \leqslant \|f\|_{U^{d+1}}. \tag{47}$$

Other properties of the Gowers norms can be found in [142].

We show the following lemma.

Lemma (4.1.10)[124]: Let δ and α' be real numbers, $0 < \alpha' \leqslant \delta$, let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN, let k be a positive integer and let $d \geqslant 0$ be an integer. Assume, moreover, that

$$R'_{\alpha'} = \{r \in \mathbb{Z}_N : \alpha' N \leqslant |\hat{A}(r)| < 2\alpha' N\} \tag{48}$$

and let B' be an arbitrary subset of $R'_{\alpha'} \setminus \{0\}$. Then the number of solutions of the system (46) with $r_j \in B'$ is greater than or equal to

$$\left(\frac{\delta (\alpha')^{2k}}{2^{2k} \delta^{2k}} |B'|^{2k} \right)^{2^d} . \tag{49}$$

Proof. Let $f(x)$ be the function defined by the formula

$$f(x) = \frac{1}{N} \sum_{r \in B'} \hat{A}(r) e(rx) .$$

Using Hölder's inequality, we obtain that

$$\left| \sum_x f(x) A(x) \right|^{2k} \leqslant \left(\sum_x |f(x)|^{2k} \right) \left(\sum_x A(x) \right)^{2k-1}$$

$$= \left(\sum_x |f(x)|^{2k} \right) (\delta N)^{2k-1}. \tag{50}$$

On the other hand, using Parseval's equality and the definition of $R'_{\alpha'}$, we obtain that

$$\sum_x f(x) A(x) = \frac{1}{N} \sum_r \hat{f}(r) \overline{\hat{A}(x)} = \frac{1}{N} \sum_r \left| \hat{f}(r) \right|^2 \geqslant (\alpha')^2 |B'| N. \tag{51}$$

Consider the sum

$$\sigma = \left\| |f|^{2k} \right\|_{U^0} = \left\| |f|^{2k} \right\|_{U^1} = \frac{1}{N} \sum_{x} |f(x)|^{2k}. \tag{52}$$

It follows from (50) and (51) that

$$\sigma \succcurlyeq \frac{\delta(\alpha')^{4k}}{\delta^{2k}} |B'|^{2k}. \tag{53}$$

Using Lemma (4.1.9), we obtain that

$$\begin{aligned} \sigma^{2^d} &\preccurlyeq \frac{1}{N^{d+1}} \sum_{x \in \mathbb{Z}_N} \sum_{h \in \mathbb{Z}_N^d} \prod_{\omega \in \{0,1\}^d} |f(x + \omega \cdot h)|^{2k} \\ &= \frac{1}{N^{d+1}} \sum_{x \in \mathbb{Z}_N} \sum_{h \in \mathbb{Z}_N^d} \left| \prod_{\omega \in \{0,1\}^d} f(x + \omega \cdot h) \right|^{2k}. \end{aligned} \tag{54}$$

Using the inversion formula (11), we obtain that

$$\prod_{\omega \in \{0,1\}^d} f(x + \omega \cdot h) = \frac{1}{N^{2^d}} \sum_{\vec{r} \in \mathbb{Z}_N^{2^d}} \prod_{\omega \in \{0,1\}^d} \hat{f}(\omega(\vec{r})) e(\omega(\vec{r}))(x + \omega \cdot h). \tag{55}$$

Hence,

$$\begin{aligned} \sigma^{2^d} = \frac{1}{N^{2^{d+1}k+d+1}} &\sum_{x \in \mathbb{Z}_N} \sum_{h \in \mathbb{Z}_N^d} \sum_{r^{(1)},\dots,r^{(k)},r^{(k+1)},\dots,r^{(2k)} \in \mathbb{Z}_N^{2^d}} \\ &\times \prod_{i=1}^{k} \prod_{\omega^{(i)} \in \{0,1\}^d} \hat{f}(\omega^{(i)}(r^{(i)})) e(\omega^{(i)}\big(r^{(i)}\big)\big(x + \omega^{(i)} \cdot h\big)) \\ &\times \prod_{i=k+1}^{2k} \prod_{\omega^{(i)} \in \{0,1\}^d} \overline{\hat{f}\Big(\omega^{(i)}(r^{(i)})\Big)} e(-\omega^{(i)}\big(r^{(i)}\big)\big(x + \omega^{(i)} \\ &\cdot h\big)\big). \end{aligned} \tag{56}$$

We denote by Σ the system of equations

$$\begin{gathered} \sum_{i=1}^{k} \sum_{\omega^{(i)} \in \{0,1\}^d} \omega^{(i)}\big(r^{(i)}\big) = \sum_{i=k+1}^{2k} \sum_{\omega^{(i)} \in \{0,1\}^d} \omega^{(i)}\big(r^{(i)}\big), \\ \sum_{i=1}^{k} \sum_{\omega^{(i)} \in \{0,1\}^d, \omega_1^{(i)}=1} \omega^{(i)}\big(r^{(i)}\big) = \sum_{i=k+1}^{2k} \sum_{\omega^{(i)} \in \{0,1\}^d, \omega_1^{(i)}=1} \omega^{(i)}\big(r^{(i)}\big) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_{i=1}^{k} \sum_{\omega^{(i)} \in \{0,1\}^d, \omega_d^{(i)}=1} \omega^{(i)}\big(r^{(i)}\big) = \sum_{i=k+1}^{2k} \sum_{\omega^{(i)} \in \{0,1\}^d, \omega_d^{(i)}=1} \omega^{(i)}\big(r^{(i)}\big). \end{gathered}$$

Then

$$\sigma^{2^d}$$
$$= \frac{1}{N^{2^{d+1}k+d+1}} \sum_{r^{(1)},\dots,r^{(k)},r^{(k+1)},\dots,r^{(2k)}\in\mathbb{Z}_N^{2^d}} \prod_{i=1}^{k} \prod_{\omega^{(i)}\in\{0,1\}^d} \hat{f}(\omega^{(i)}(r^{(i)}))$$
$$\times \prod_{i'=k+1}^{2k} \prod_{\omega^{(i')}\in\{0,1\}^d} \overline{\hat{f}\left(\omega^{(i')}\left(r^{(i')}\right)\right)}$$
$$\times \sum_{x\in\mathbb{Z}_N} \sum_{h\in\mathbb{Z}_N^d} e(\omega^{(i)}\left(r^{(i)}\right)\left(x+\omega^{(i)}\cdot h\right) - \omega^{(i')}\left(r^{(i')}\right)\left(x+\omega^{(i')}\cdot h\right))$$
$$= \frac{1}{N^{2^{d+1}k}} \sum_{r^{(1)},\dots,r^{(k)},r^{(k+1)},\dots,r^{(2k)}\in\mathbb{Z}_N^{2^d}} \prod_{i=1}^{k} \prod_{\omega^{(i)}\in\{0,1\}^d} \hat{f}(\omega^{(i)}(r^{(i)}))$$
$$\times \prod_{i=k+1}^{2k} \prod_{\omega^{(i)}\in\{0,1\}^d} \overline{\hat{f}\left(\omega^{(i)}\left(r^{(i)}\right)\right)}. \tag{57}$$

The sum in (57) is taken over the $r^{(1)},\dots,r^{(k)},r^{(k+1)},\dots,r^{(2k)}$ that satisfy Σ. It is easy to verify that this system coincides with (46).

Since $\hat{f}_{B'}(r) = \hat{A}(r)B'(r)$ and $B' \subseteq R'_{\alpha'} \setminus \{0\}$, we have $|\hat{f}_{B'}(r)| \leqslant 2\alpha' B'(r)N$. Hence,

$$\sigma^{2^d} \leqslant (2^{2k}(\alpha')^{2k})^{2^d} N^{2^{d+1}k}. \tag{58}$$

Using inequalities (53), (54) and (58), we finally obtain that

$$\sum_{r^{(1)},\dots,r^{(k)},r^{(k+1)},\dots,r^{(2k)}\in\Sigma} 1 \geqslant \left(\frac{\delta(\alpha')^{4k}}{\delta^{2k}}|B'|^{2k}\right)^{2^d} \frac{1}{(2^{2k}(\alpha')^{2k})^{2^d}}$$
$$= \left(\frac{\delta(\alpha')^{2k}}{2^{2k}\delta^{2k}}|B'|^{2k}\right)^{2^d}. \tag{59}$$

The sum in (59) is taken over the $r^{(i)}, i = 1,2,\dots,2k$, whose components belong to B'. In other words, the number of solutions of the system (46) with $r_i \in B'$ is greater than or equal to

$$\left(\frac{\delta(\alpha')^{2k}}{2^{2k}\delta^{2k}}|B'|^{2k}\right)^{2^d}.$$

The lemma is proved.

Theorem (4.1.11)[124]: Let δ and α be real numbers, $0 < \alpha \leqslant \delta$, let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN, let k be a positive integer, let $d \geqslant 0$ be an integer and let R_α be the set defined by (3). Let $B \subseteq R_\alpha \setminus \{0\}$ be an arbitrary set. Consider the system of equations

$$\sum_{j=1}^{2^{d+1}k} a_{ij} r_j = 0, \qquad i = 1,2,\dots,d+1, \tag{60}$$

where the elements a_{ij} of the matrix A are defined by formula (45) and $r_j \in B$ for all j. Then the number of solutions of the system (46) is greater than or equal to

$$\left(\frac{\delta\alpha^{2k}}{2^{4k}\delta^{2k}}|B|^{2k}\right)^{2^d}. \tag{61}$$

To make it clear that Theorem (4.1.11) is a generalization of Theorem (4.1.5), it is sufficient to put the d in Theorem (4.1.11) equal to zero.

To prove Theorem (4.1.11), we need some properties of the Gowers norms (see [142]).

Proof: Let

$$B_i = \{r \in B: 2^{i-1}\alpha N \leqslant |\hat{A}(r)| < 2^i\alpha N\}, \qquad i \geqslant 1.$$

It is clear that $B = \coprod_{i\geqslant 1} B_i$.

Let E be a set. We denote by $S_{k,d}(E)$ the number of solutions of the system (46) with $r_i \in E$. Applying Lemma (4.1.10) to every B_i, we obtain that

$$S_{k,d}(B_i) \geqslant \left(\frac{\delta\left(2^{i-1}\alpha\right)^{2k}}{2^{2k}\delta^{2k}}|B_i|^{2k}\right)^{2^d},$$

where $i \geqslant 1$. Hence,

$$S_{k,d}(B) \geqslant \sum_i S_{k,d}(B_i) \geqslant \left(\frac{\delta\alpha^{2k}}{2^{2k}\delta^{2k}}\right)^{2^d} \sum_i \left(2^{2ki}|B_i|^{2k}\right)^{2^d}. \tag{62}$$

We have $|B| = \sum_i |B_i|$. Using Hölder's inequality, we obtain that

$$\begin{aligned} |B|^{2^{d+1}k} &= \left(\sum_i 2^i 2^{-i}|B_i|\right)^{2^{d+1}k} \\ &\leqslant \left(\sum_i \left(2^{2ki}|B_i|^{2k}\right)^{2^d}\right)\left(\sum_i 2^{-(2^{d+1}ki)/(2^{d+1}k-1)}\right)^{2^{d+1}k-1} \\ &\leqslant \sum_i \left(2^{2ki}|B_i|^{2k}\right)^{2^d}. \end{aligned} \tag{63}$$

Substituting (63) into (62), we obtain the desired inequality

$$S_{k,d}(B) \geqslant \left(\frac{\delta\alpha^{2k}}{2^{4k}\delta^{2k}}|B|^{2k}\right)^{2^d}. \tag{64}$$

The theorem is proved.

In the proof of Theorem (4.1.1), Chang used Rudin's theorem [143] (see also [144]) on the dissociative subsets of $\mathbb{Z}_N$. A set $\mathcal{D} = \{d_1, \dots, d_{|\mathcal{D}|}\} \subseteq \mathbb{Z}_N$ is said to be dissociative if the congruence

$$\sum_{i=1}^{|\mathcal{D}|} \varepsilon_i d_i = 0 \pmod{N}, \tag{65}$$

where $\varepsilon_i \in \{-1, 0, 1\}$, implies that all the ε_i are equal to zero.

Theorem (4.1.12) (W. Rudin)[124]: There is an absolute constant $C > 0$ such that for any dissociative set $\mathcal{D} \subseteq \mathbb{Z}_N$ and any complex numbers $a_n \in \mathbb{C}$ the inequality

$$\frac{1}{N}\sum_{x\in\mathbb{Z}_N}\left|\sum_{n\in\mathcal{D}} a_n e(nx)\right|^p \leqslant \left(C\sqrt{p}\right)^p \left(\sum_{n\in\mathcal{D}}|a_n|^2\right)^{p/2} \tag{66}$$

holds for all integers $p \geqslant 2$.

The proofs of Theorem (4.1.12) and Chang's theorem can also be found in [133], [145]. We shall use Rudin's theorem and Theorem (4.1.5) to derive an analogue of Theorem (4.1.1), which only differs from Chang's theorem in that it gives a somewhat weaker estimate for the cardinality of Λ.

Proposition (4.1.13)[124]: Let δ and α be real numbers, $0 < \alpha \preccurlyeq \delta \preccurlyeq 1$, let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN and let R_α be the set defined by (3). Then there is a set $\mathcal{D} = \{d_1, \ldots, d_{|\mathcal{D}|}\} \subseteq \mathbb{Z}_N, |\mathcal{D}| \preccurlyeq 2^8 C^2 \left(\frac{\delta}{\alpha}\right)^2 \log(1/\delta)$, such that every element r of R_α can be represented in the form

$$r \equiv \sum_{i=1}^{|\mathcal{D}|} \varepsilon_i d_i = 0 \pmod N, \tag{67}$$

where $\varepsilon_i \in \{-1, 0, 1\}$ and C is the absolute constant occurring in Rudin's inequality (66).

Proof. Let $k = 2\lceil\log(1/\delta)\rceil$ and let $\mathcal{D} \subseteq R_\alpha$ be a maximal dissociative set. Since $\mathcal{D}$ is dissociative, we have $0 \notin \mathcal{D}$. Using Theorem (4.1.5), we obtain the estimate

$$T_k(\mathcal{D}) \succcurlyeq \frac{\delta\alpha^{2k}}{2^{4k}\delta^{2k}} |\mathcal{D}|^{2k}. \tag{68}$$

On the other hand,

$$T_k(\mathcal{D}) \preccurlyeq C^{2k} 2^k k^k |\mathcal{D}|^k, \tag{69}$$

where C is the absolute constant occurring in Theorem (4.1.12). Indeed, let the a_n in (66) be equal to $\mathcal{D}(n)$ and let $p = 2k$. Then the left-hand side of (66) is $T_k(\mathcal{D})$ while the right-hand side is equal to $C^{2k} 2^k k^k |\mathcal{D}|^k$. We have $k = 2\lceil\log(1/\delta)\rceil$. Using (68) and (69), we obtain that $|\mathcal{D}| \preccurlyeq 2^8 C^2 \left(\frac{\delta}{\alpha}\right)^2 \log(1/\delta)$. Since $\mathcal{D}$ is a maximal dissociative subset of R_α, every element r of R_α can be represented in the form $r \equiv \sum_{i=1}^{|\mathcal{D}|} \varepsilon_i d_i = 0 \pmod N$, where $d_i \in \mathcal{D}$ and $\varepsilon_i \in \{-1, 0, 1\}$. Note that it is only the constant factors in the estimate $|\mathcal{D}| \preccurlyeq 2^8 C^2 \left(\frac{\delta}{\alpha}\right)^2 \log(1/\delta)$ that are different from those in the corresponding estimate in Chang's theorem. The proposition is proved.

We shall now strengthen Chang's theorem. Our method of proof has much in common with the methods used in [146]–[148].

Corollary (4.1.14)[124]: Let N be a positive integer, $(N, 6) = 1$, let δ and α be real numbers, $0 < \alpha \preccurlyeq \delta \log^{\frac{1}{2}}(1/\delta)$, and let R_α be the set defined by (3). Then there is a $\Lambda^* \subseteq \mathbb{Z}_N, |\Lambda^*| \preccurlyeq 2^{12}(\delta/\alpha)^2 \log(1/\delta)$, such that for any residue $r \in R_\alpha$ there is a set $\lambda_1^*, \ldots, \lambda_M^*$ of at most $8\log(1/\delta)$ elements of Λ^* such that $r \equiv \sum_{i=1}^{M} \varepsilon_i \lambda_i^* \pmod N$, where $\varepsilon_i \in \{-1, 0, 1\}$.

In the proof of Theorem (4.1.19) we shall use several auxiliary assertions and definitions.

Definition (4.1.15)[124]: Let k and s be positive integers. Consider a family $\Lambda(k, s)$ of subsets of $\mathbb{Z}_N$ that has the following property. If $\Lambda = \{\lambda_1, \ldots, \lambda_{|\Lambda|}\}$ belongs to $\Lambda(k, s)$, then the congruence

$$\sum_{i=1}^{|\Lambda|} \lambda_i s_i \equiv 0 \pmod N, \qquad \lambda_i \in \Lambda, \qquad s_i \in \mathbb{Z}, \qquad |s_i| \preccurlyeq s, \qquad \sum_{i=1}^{|\Lambda|} |s_i| \preccurlyeq 2k, \tag{70}$$

implies that all the s_i are equal to zero.

The definition of $\Lambda(k, 1)$ can be found in [149].

Note that for every $\Lambda \in \Lambda(k,s)$ we have $0 \notin \Lambda$ and $\Lambda \cap (-\Lambda) = \emptyset$. It is implicit in what follows that the equality of two elements of $\mathbb{Z}_N$ will always mean that they are equal modulo N. For sets belonging to $\Lambda(k,s)$, the following upper bound holds for the quantities T_k.

Assertion (4.1.16)[124]: Let k and s be positive integers, let Λ be an arbitrary set belonging to the family $\Lambda(k,s)$ and assume that $|\Lambda| \geqslant k$. Then

$$T_k(\Lambda) \leqslant 2^{3k} k^k \max\left\{1, \left(\frac{k}{|\Lambda|}\right)^k |\Lambda|^{k/s}\right\}. \tag{71}$$

Proof: Let $x \in \mathbb{Z}_N$ be an arbitrary residue and let $N_k(x)$ be the number of $(\lambda_1,\dots,\lambda_k)$ such that the λ_i belong to Λ and $\lambda_1 + \cdots + \lambda_k = x$. Then $T_k(\Lambda) = \sum_{x\in\mathbb{Z}_N} N_k^2(x)$. Let $s_1,\dots,s_l$ be positive integers such that $s_1 + \cdots + s_l = k$.

To fix ideas, we assume that $s_1,\dots,s_l$ are arranged in descending order: $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_l \geqslant 1$.

Let $E(s_1,\dots,s_l)(x) = \{(\lambda_1,\dots,\lambda_k)$: among $\lambda_1,\dots,\lambda_k$ there are precisely s_1 numbers equal to $\tilde\lambda_1$, precisely s_2 numbers equal to $\tilde\lambda_2$,.. and precisely s_l numbers equal to $\tilde\lambda_l$, so that $s_1\tilde\lambda_1 + \cdots + s_l\tilde\lambda_l = x$, and the $\tilde\lambda_i$ are all distinct$\}$. For brevity we denote $E(s_1,\dots,s_l)(x)$ by $E(\vec{s})(x)$. Recall that the numbers $s_1,\dots,s_l$ in the definition of $E(\vec{s})(x) = E(s_1,\dots,s_l)(x)$ are such that $\sum_{i=1}^{l} s_i = k$. Then

$$N_k(x) = \sum_{\vec{s}} |E(\vec{s})(x)|,$$

where the sum is taken over all vectors for which $\sum_{i=1}^{l} s_i = k$. Hence,

$$\sigma = T_k(\Lambda) = \sum_{x\in\mathbb{Z}_N}\left(\sum_{\vec{s}} |E(\vec{s})(x)|\right)^2. \tag{72}$$

Let $\vec{s} = (s_1,\dots,s_l)$ and $G = G(\vec{s}) = \{i : s_i \leqslant s\}, B = B(\vec{s}) = \{i : s_i > s\}$. Then $|G(\vec{s})| + |B(\vec{s})| = l(\vec{s}) = l$. We claim that

$$l \leqslant k - s|B|. \tag{73}$$

Indeed,

$$k = \sum_{i\in G} s_i + \sum_{i\in B} s_i' \geqslant |G| + (s+1)|B| = l + s|B|, \tag{74}$$

and (73) follows.

Example (4.1.17)[124]: Let $\log|\Lambda| > \log^2 k$ and let Λ be an arbitrary set belonging to the family $\Lambda(k,3)$. Using the inequality (71), we obtain that $T_k(\Lambda) \leqslant 2^{20k}k^k|\Lambda|^k$.

It is obvious that the order of this estimate cannot be improved, which means that $T_k(\Lambda) \geqslant \binom{|\Lambda|}{k}(k!)^2 \gg e^{-k}k^k|\Lambda|^k$ for every Λ and every positive integer k such that $\log|\Lambda| \geqslant \log^2 k$.

Lemma (4.1.18)[124]:

$$|E(\vec{s})(x)| \leqslant \frac{k!}{s_1!\cdots s_l!}|\Lambda|^{|B(\vec{s})|} \tag{75}$$

for all $\vec{s}$ with $\sum_{i=1}^{l} s_i = k$ and all $x \in \mathbb{Z}_N$.

Proof. Let $(\lambda_1,\dots,\lambda_k)$ be an arbitrary set belonging to $E(\vec{s})(x)$. Then $\sum_{i=1}^{l} s_i\tilde\lambda_i = x$, where the $\tilde\lambda_i \in \{\lambda_1,\dots,\lambda_k\}$ are distinct. Consider another element $(\lambda_1',\dots,\lambda_k')$ of $E(\vec{s})(x)$ with $\sum_{i=1}^{k}\lambda_i' = \sum_{i=1}^{l} s_i\tilde\lambda_i' = x$, where the $\tilde\lambda_i' \in \{\lambda_1',\dots,\lambda_k'\}$ are distinct. Assume that $\tilde\lambda_i = \tilde\lambda_i'$ for all $i \in B(\vec{s})$. We claim that $\tilde{\lambda}_i = \tilde{\lambda}'_i$ for all $i \in G(\vec{s})$. We have $\sum_{i=1}^{l} s_i\tilde{\lambda}_i = x = \sum_{i=1}^{l} s_i\tilde{\lambda}'_i$. Hence,

$$\sum_{i\in G} s_i\tilde{\lambda}_i = \sum_{i\in G} s_i\tilde{\lambda}'_i.$$

Moreover, $\Lambda \cap (-\Lambda) = \emptyset$. Therefore,

$$\sum_{i\in G} s_i\tilde{\lambda}_i - \sum_{i\in G} s_i\tilde{\lambda}'_i = \sum_{i\in G} s'_i\lambda_i^0 = 0,$$

where $s'_i \in \mathbb{Z}, |s'_i| \leqslant s, \sum_i |s'_i| \leqslant 2k$ and the $\lambda_i^0 \in \Lambda$ are distinct. The definition of $\Lambda(k,s)$ implies that all the s'_i are equal to zero. Hence, $\tilde{\lambda}_i = \tilde{\lambda}'_i$ for all $i \in G(\vec{s})$.
Therefore, $(\lambda'_1, \dots, \lambda'_k)$ can be obtained from $(\lambda_1, \dots, \lambda_k)$ by a permutation. By the definition of $E(\vec{s})(x)$, among $\lambda_1, \dots, \lambda_k$ there are precisely s_1 equal to $\tilde{\lambda}_1, s_2$ equal to $\tilde{\lambda}_2, \dots$ and s_l equal to $\tilde{\lambda}_l$, and $s_1\tilde{\lambda}_1 + \cdots + s_l\tilde{\lambda}_l = x$, where the $\tilde{\lambda}_i$ are all distinct.
Therefore, the number of permutations of $(\lambda_1, \dots, \lambda_k)$ is equal to $k!/(s_1! \cdots s_l!)$.
Hence, for a fixed $\tilde{\lambda}_i, i \in B$, the number of $(\lambda_1, \dots, \lambda_k)$ belonging to $E(\vec{s})(x)$ does not exceed $k!/(s_1! \cdots s_l!)$. Therefore, the cardinality of $E(\vec{s})(x)$ does not exceed $|\Lambda|^{|B(\vec{s})|}|k!/(s_1! \cdots s_l!)$. The lemma is proved.
We now return to the proof of the assertion and estimate the sum σ. Let b be a non-negative integer and let

$$\sigma_b = \sum_{x\in\mathbb{Z}_N} \left(\sum_{\vec{s}:\,|B(\vec{s})|=b} |E(\vec{s})(x)| \right)^2. \tag{76}$$

It follows from (73) that $|B(\vec{s})| \leqslant [k/s]$ for all $\vec{s}$. Combining this with the Cauchy–Bunyakovsky inequality, we obtain that $\sigma \leqslant ([(k-1)/s]+1)^2 \sum_{b=o}^{[k/s]} \sigma_b$. We now fix a b and estimate σ_b as follows. We have

$$\sigma_b \leqslant \left(\sum_{x\in\mathbb{Z}_N} \sum_{\vec{s}:\,|B(\vec{s})|=b} |E(\vec{s})(x)| \right)\left(\max_{x\in\mathbb{Z}_N} \sum_{\vec{s}:\,|B(\vec{s})|=b} |E(\vec{s})(x)| \right). \tag{77}$$

Let $P_k(\vec{s}) = k!/(s_1! \cdots s_l!)$. Then

$$\sum_{\vec{s}} P_k(\vec{s}) \leqslant \sum_{i=1}^{k} \sum_{\substack{s_1,\dots,s_l=0\\ s_1+\cdots+s_l=k}}^{k} \frac{k!}{s_1! \cdots s_l!} = \sum_{i=1}^{k} l^k \leqslant 2k^k. \tag{78}$$

Using Lemma (4.1.18), we obtain that $|E(\vec{s})(x)| \leqslant P_k(\vec{s})|\Lambda|^{|B(\vec{s})|}$. Combining this with inequality (78), we obtain that

$$\max_{x\in\mathbb{Z}_N} \sum_{\vec{s}:\,|B(\vec{s})|=b} |E(\vec{s})(x)| \leqslant 2k^k|\Lambda|^b. \tag{79}$$

Consider the sum

$$\sum_{x\in\mathbb{Z}_N} \sum_{\vec{s}:\,|B(\vec{s})|=b} |E(\vec{s})(x)|. \tag{80}$$

It follows from (73) that this sum is bounded above by the number of $(\lambda_1, \dots, \lambda_k) \in \Lambda^k$ such that at most $k - sb$ of the numbers $\lambda_1, \dots, \lambda_k$ are distinct. Therefore,

$$\sum_{x\in\mathbb{Z}_N} \sum_{\vec{s}:\,|B(\vec{s})|=b} |E(\vec{s})(x)| \leqslant \binom{|\Lambda|}{k-sb} (k-sb)^k$$

$$\leqslant \frac{|\Lambda|^{k-sb}}{(k-sb)!}(k-sb)^k \leqslant e^k k^{sb}|\Lambda|^{k-sb}. \quad (81)$$

Combining this with (79), we obtain that

$$\sigma_b \leqslant 2e^k k^k |\Lambda|^b \left(\frac{k}{|\Lambda|}\right)^{sb} |\Lambda|^k. \quad (82)$$

Hence,

$$\sigma \leqslant 2([(k-1)/s]+1)^2 e^k k^k |\Lambda|^k \sum_{b=o}^{[(k-1)/s]} \left(\frac{k^s}{|\Lambda|^{s-1}}\right)^b$$
$$= 2([(k-1)/s]+1)^2 e^k k^k |\Lambda|^k \sigma^*. \quad (83)$$

We estimate σ^* as follows. If $k^s \leqslant |\Lambda|^{s-1}$, then it is obvious that $\sigma^* \leqslant [(k-1)/s]+1$. If $k^s > |\Lambda|^{s-1}$, then$\sigma^* \leqslant ([(k-1)/s]+1)(k/|\Lambda|)^k\ |\Lambda|^{k/s}|\Lambda|^{1-1/s}/k$. In any case we have $\sigma^* \leqslant ([(k-1)/s]+1)\max\{(k/|\Lambda|)^k\ |\Lambda|^{k/s}|\Lambda|^{1-1/s}/k\}$. Therefore,

$$\sigma = T_k(\Lambda) \leqslant 2^{3k}k^k|\Lambda|^k \max\{(k/|\Lambda|)^k|\Lambda|^{k/s}\}. \quad (84)$$

The assertion is proved.

Theorem (4.1.19)[124]: Let N be a positive integer, $(N,2)=1$, let δ and α be real numbers, $0 < \alpha \leqslant \delta \leqslant 1/16$, let A be an arbitrary subset of $\mathbb{Z}_N$ of cardinality δN and let R_α be the set defined by (3). Then there is a $\Lambda^* \subseteq \mathbb{Z}_N$,

$$|\Lambda^*| \leqslant \max(2^{12}(\delta/\alpha)^2 \log(1/\delta), 2^6 \log^2(1/\delta)), \quad (85)$$

such that for any residue $r \in R_\alpha$ there is a set $\lambda_1^*, \dots, \lambda_M^*$ of at most $8\log(1/\delta)$ elements of Λ^* such that

$$r \equiv \sum_{i=1}^{M} \varepsilon_i \lambda_i^* \pmod N, \quad (86)$$

where $\varepsilon_i \in \{-1,0,1\}$.

If, moreover, N is a prime, then there is a set $\widetilde{\Lambda} \subseteq \mathbb{Z}_N$,

$$|\widetilde{\Lambda}| \leqslant 2^{12}(\delta/\alpha)^2 \log(1/\delta), 2^6 \log^2(1/\delta), \quad (87)$$

such that for every residue $r \in R_\alpha$ there is a set $\tilde{\lambda}_1, \dots, \tilde{\lambda}_M$ of at most $8\log(1/\delta)$ elements of $\widetilde{\Lambda}$ such that

$$r \equiv \sum_{i=1}^{M} \varepsilon_i \tilde{\lambda}_i \pmod N, \quad (88)$$

where $\varepsilon_i \in \{-1,0,1\}$.

Proof: Let $k = 2[\log(1/\delta)]$, let $s = 2$ and let $\Lambda = \{\lambda_1, \dots, \lambda_{|\Lambda|}\}$ be a maximal subset of $R_\alpha \setminus \{0\}$ belonging to $\Lambda(k,s)$. If $R_\alpha = \{0\}$, then the proof is obvious. If $R_\alpha \setminus \{0\}$ is non-empty, then Λ is also non-empty. Let $\Lambda^* = \left(\bigcup_{j=1}^{s} j^{-1}\Lambda\right) \cup \{0\}$. Then $|\Lambda^*| \leqslant 4|\Lambda|$ and $0 \in \Lambda^*$. We claim that for any $x \in R_\alpha \setminus \{0\}$ there is a $j \in [s]$ such that

$$xj = \sum_{i=1}^{|\Lambda|} \lambda_i s_i, \quad s_i \in \mathbb{Z}, \qquad |s_i| \leqslant s, \qquad \sum_{i=1}^{|\Lambda|} |s_i| \leqslant 2k. \quad (89)$$

Then since $j^{-1}\lambda_i \in \Lambda^*$ for all $i \in [|\Lambda|], j \in [s]$, the desired assertion will follow from (89).

Thus, let x be an arbitrary element of $R_\alpha \setminus \Lambda, x \neq 0$. Consider relations of the form $\sum_{i=1}^{|\Lambda|+1} \tilde{\lambda}_i s_i = 0$, where $\tilde{\lambda}_i \in \Lambda \coprod \{x\}$ and $s_i \in \mathbb{Z}, |s_i| \leqslant s, \sum_{i=1}^{|\Lambda|+1} |s_i| \leqslant 2k$. If all these relations are trivial, that is, if for each of them we have $s_i = 0, i \in [|\Lambda| + 1]$, then we obtain a contradiction to the maximality of Λ. Hence, there is a non-trivial relation of the form (89) such that $j, s_1, \ldots, s_{|\Lambda|}$ are not all equal to zero. We have $j \in [-s, \ldots, s]$. If $j = 0$, then we obtain a contradiction to the fact that Λ belongs to $\Lambda(k, s)$. Therefore, we can assume that $j \in [s]$. Since $2k \leqslant 8\log(1/\delta)$, we obtain that for any $x \in R_\alpha$ there is a $\{\lambda_1^*, \ldots, \lambda_M^*\} \subset \Lambda^*, M \leqslant 8\log(1/\delta)$, such that (71) holds.

We claim that $|\Lambda^*| \leqslant \max(2^{12}(\delta/\alpha)^2 \log(1/\delta), 2^6 \log^2(1/\delta))$.

If $|\Lambda| \leqslant k^2$, then $|\Lambda| \leqslant 2^4 \log^2(1/\delta)$, whence $|\Lambda^*| \leqslant 2^6 \log^2(1/\delta)$. If $|\Lambda| \leqslant k^2$, then Assertion (4.1.16) implies that $T_k(\Lambda) \leqslant 2^{3k} k^k |\Lambda|^k$. On the other hand, using Theorem (4.1.5) we obtain that $T_k(\Lambda) \geqslant \delta\alpha^{2k}|\Lambda|^{2k}/(2^{4k}\delta^{2k})$. Therefore, $|\Lambda| \leqslant 2^{10}\left(\frac{\delta}{\alpha}\right)^2 \log(1/\delta)$, whence $|\Lambda^*| \leqslant 2^{12}(\delta/\alpha)^2 \log(1/\delta)$.

In any case we have $|\Lambda^*| \leqslant \max(2^{12}(\delta/\alpha)^2 \log(1/\delta), 2^6 \log^2(1/\delta))$.

We now prove the existence of $\widetilde{\Lambda}$. Let $s = [\log\log(1/\delta)]$ and let Λ_1 be a maximal subset of $R_\alpha \setminus \{0\}$ belonging to $\Lambda(k, s), k = 2\lceil \log(1/\delta) \rceil$. Let $\widetilde{\Lambda} = \bigcup_{j=1}^{s} j^{-1}\Lambda_1$.

Then $|\widetilde{\Lambda}| \leqslant s|\Lambda_1|$. Arguments similar to those used above enable us to show that for any residue $r \in R_\alpha$ there is a set $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M\} \subset \widetilde{\Lambda}, M \leqslant 8\log(1/\delta)$, such that (73) holds.

We prove (72) as follows. If $|\Lambda_1| \leqslant k^{s/(s-1)}$, then $|\Lambda_1| \leqslant 2^{10}\log(1/\delta)$ and $|\widetilde{\Lambda}| \leqslant s|\Lambda_1| \leqslant 2^{12}\log(1/\delta)\log\log(1/\delta)$. We see that in this case (72) is proved.

Now let $|\Lambda_1| > k^{s/(s-1)}$. Using Assertion (4.1.16), we obtain that $T_k(\Lambda_1) \leqslant 2^{3k}k^k|\Lambda_1|^k$. On the other hand, Theorem (4.1.5) implies that $T_k(\Lambda_1) \geqslant \delta\alpha^{2k}|\Lambda_1|^{2k}/(2^{4k}\delta^{2k})$. Therefore, $|\Lambda_1| \leqslant 2^{10}(\delta/\alpha)^2\log(1/\delta)$, whence $|\widetilde{\Lambda}| \leqslant 2^{12}(\delta/\alpha)^2\log(1/\delta)\log\log(1/\delta)$. The theorem is proved.

We shall now apply problems in the combinatorial theory of numbers.

Let K be an arbitrary subset of $\mathbb{Z}_N$ and $\varepsilon \in (0, 1)$ any real number. Then the corresponding Bohr set is defined as

$$B(K, \varepsilon) = \left\{x \in \mathbb{Z}_N : \left\|\frac{rx}{N}\right\| < \varepsilon \quad \forall r \in K\right\},$$

where $\|\cdot\|$ denotes the integer part of a real number. Information on the properties of Bohr sets can be found in [150], where, in particular, it is proved that

$$B(K, \varepsilon) \geqslant \frac{1}{2}\varepsilon^{|K|}N. \tag{90}$$

In her proof of the quantitative version of Freiman's theorem (see [126] and [133]), Chang used the following proposition.

Proposition (4.1.20)[124]: Let N be a positive integer, $\delta \in (0, 1)$ a real number and A an arbitrary subset of $\mathbb{Z}_N$ with $|A| = \delta N$. Then $2A - 2A$ contains a Bohr set $B(K, \varepsilon)$ with $|K| \leqslant 8\delta^{-1}\log(1/\delta)$ and $\varepsilon = 1/(2^8\log(1/\delta))$.

We claim that Proposition (4.1.20) can be strengthened as follows.

To prove Proposition (4.1.21) we need the following definition.

Definition (4.1.22)[124]: Let $f, g : \mathbb{Z}_N \to \mathbb{C}$ be arbitrary functions. The convolution of f and g is defined to be the function

$$(f * g)(x) = \sum_{y \in \mathbb{Z}_N} f(y)\overline{g(y - x)}. \tag{91}$$

It is obvious that

$$(\widehat{f * g})(r) = \hat{f}(r)\overline{\hat{g}(r)}. \tag{92}$$

Proposition (4.1.21)[124]: Let N be a positive integer, $(N,2) = 1$, let $0 < \delta \leqslant 2^{-256}$ be a real number and let A be an arbitrary subset of $\mathbb{Z}_N$ with $|A| = \delta N$. Then $2A - 2A$ contains a Bohr set $B(K,\varepsilon)$ with $|K| \leqslant 2^{15}\delta^{-1}\log(1/\delta)$ and $\varepsilon = 1/(2^8\log(1/\delta))$.

Using formula (90), we obtain that the cardinality of $B(K,\varepsilon)$ in Proposition (4.1.20) is greater than or equal to $(1/2)\cdot 2^{-8\delta^{-1}(\log(1/\delta))^2}N$. The cardinality of the Bohr set in Proposition (4.1.21) is greater than or equal to $(1/2)\cdot 2^{-2^{20}\delta^{-1}\log(1/\delta)\log\log(1/\delta)}N$.

Proof: Let $\alpha = \delta^{3/2}/(2\sqrt{2})$. Applying Corollary (4.1.14) to $R_\alpha(A)$, we obtain a set $\Lambda^* \subseteq \mathbb{Z}_N, |\Lambda^*| \leqslant 2^{15}\delta^{-1}\log(1/\delta)$, such that for any residue $r \in R_\alpha$ there is a set $\lambda_1^*,\dots,\lambda_M^*$ of at most $8\log(1/\delta)$ elements of Λ^* such that (71) holds.

Let $R_\alpha^* = R_\alpha \setminus \{0\}$. Consider the Bohr set $B_1 = B(R_\alpha^*, 1/20)$. For all $x \in B_1$ and all $r \in R_\alpha^*$ we have

$$|1 - e(rx)| = 2\left|\sin\left(\frac{\pi r x}{N}\right)\right| \leqslant \frac{2\pi}{20} < \frac{1}{2}. \tag{93}$$

The expression $(A * A * A * A)(x)$ is obviously equal to the number of quadruples $(a_1,a_2,a_3,a_4) \in A^4$ such that $a_1 + a_2 - a_3 - a_4 = x$. Hence, $(A * A * A * A)(x) > 0$ if and only if $x \in 2A - 2A$. Using formulae (11) and (92), we obtain that x belongs to $2A - 2A$ if and only if $\sum_r\left|\hat{A}(r)\right|^4 e(rx) > 0$. Let $x \in B_1$. Then, using Parseval's equality (2)), we have

$$\begin{aligned}\sum_r\left|\hat{A}(r)\right|^4 e(rx) &= \sum_r\left|\hat{A}(r)\right|^4 - \sum_r\left|\hat{A}(r)\right|^4\left(10e(rx)\right)\\ &> \frac{1}{2}\sum_r\left|\hat{A}(r)\right|^4 - 2\sum_{r\notin R, r\neq 0}\left|\hat{A}(r)\right|^4 \geqslant \frac{1}{2}\delta^4N^4 - 2\max_{r\notin R, r\neq 0}\left|\hat{A}(r)\right|^2\sum_r\left|\hat{A}(r)\right|^4\\ &\geqslant \frac{1}{2}\delta^4N^4 - 2\cdot\frac{\delta^3N^2}{8}\delta N^2 = \frac{\delta^4N^4}{4} > 0. \end{aligned} \tag{94}$$

It follows from (94) that the Bohr set B_1 is contained in $2A - 2A$. Consider another Bohr set $B_2 = B(\Lambda^*, 1/(2^8\log(1/\delta))$. We claim that $B_2 \subseteq B_1$. Since for any residue $r \in R_\alpha^*$ there is a set $\lambda_1^*,\dots,\lambda_M^*$ of at most $8\log(1/\delta)$ elements of Λ^* such that (71) holds, the inequality

$$\left\|\frac{rx}{N}\right\| \leqslant \sum_{i=1}^{M}\left\|\frac{\lambda_i^* x}{N}\right\| \leqslant 8\log\left(\frac{1}{\delta}\right)\frac{1}{2^8\log(1/\delta)} < \frac{1}{20} \tag{95}$$

holds for all $x \in B_2$. Hence, every $x \in B_2$ belongs to B_1, and we have obtained a Bohr set $B_2 \subseteq 2A - 2A$ with the desired properties. The proposition is proved.

Section (4.2): Relating Rank to Communication Complexity

We present a new connection between communication complexity and additive combinatorics, showing that a well-known conjecture from additive combinatorics known as the Polynomial Freiman-Ruzsa Conjecture (PFR, in short), implies better upper bounds than currently known on the deterministic communication complexity of a boolean function in terms of the rank of its associated matrix. The results show that the PFR Conjecture implies that every boolean function has communication complexity $O(rank(M)/\log rank(M))$ where $rank(M)$ is the rank, over the reals, of the associated matrix. We view this result as interesting not only due to its being the first sublinear bound (and the first advance on this problem since 1997) but also because of its suggestion of a new connection between the two vibrant, yet seemingly unrelated, fields of communication complexity and

additive combinatorics. The analysis relies on the study of approximate duality, a concept closely related to the PFR Conjecture, which was introduced in [152].
The main technical contribution improves the bounds on approximate duality, assuming the PFR Conjecture, and it does so with simpler proof than in [152]. We view this contribution as being of independent interest because of the growing number of applications of the "approximate duality method" to theoretical computer science. These include so-far the construction of bipartite Ramsey graphs and two-source extractors [152], communication complexity (this work), and the subsequent lower bounds for matching vector locally decodable codes [153].

In the two-party communication complexity model two parties — Alice and Bob — wish to compute a function $f: X \times Y \to \{0, 1\}$ on inputs x and y where x is known only to Alice and y is known only to Bob. In order to compute the function f they must exchange bits of information between each other according to some (deterministic) protocol. The (deterministic) communication complexity of a protocol is the maximum total number of bits sent between the two parties, where the maximum is taken over all pairs of inputs x, y. We henceforth omit the adjective "deterministic" from our discourse because our results deal only with the deterministic model. The communication complexity of the function f, denoted by $CC(f)$, is the minimum communication complexity of a protocol for f.
For many applications it is convenient to associate the function $f: X \times Y \to \{0,1\}$ with the matrix $M \in \{0, 1\}X \times Y$ whose (x, y) entry equals $f(x, y)$. For a $\{0, 1\}$-valued matrix M, let $CC(M)$ denote the communication complexity of the boolean function associated with M. Let $rank(M)$ denote the rank of M over the reals. We will occasionally consider the rank of M over the two-element field $\mathbb{F}_2$ and will denote this by $rank_{\mathbb{F}_2}(M)$.
It is well-known since the work of Mehlhorn and Schmidt [154] that

$$\log rank(M) \leq CC(M) \leq rank(M) \tag{96}$$

and it is a fundamental question to find out what is the true worst-case dependency of $CC(M)$ on the rank. The famous log-rank conjecture due to Lovasz and Saks [155] postulates that the true answer is closer to the lower bound of (96).
Conjecture (4.2.1) (Log-rank)[151]: For every $\{0, 1\}$-valued $matrix\, M\, CC(M) = \log^{O(1)} rank(M)$.
Lovasz and Saks also point out that the above conjecture has several other interesting equivalent formulations. One of them, due to Nuffelen [156] and Fajtlowicz [157], is the following:
Conjecture (4.2.2)[151]: For every graph $G, \chi(\bar{G}) \leq \log^{O(1)} rank(G)$, where $\chi(\bar{G})$ is the chromatic number of the complement of G, and rank(G) is the rank of the adjacency matrix of G over the reals.

Though considerable effort has been made since 1982 in an attempt to narrow the gap between lower and upper bounds in (96), the state of the art is not far from where it was 30 years ago and currently stands at

$$\Omega\left(\log^{\log_3 6} rank(M)\right) \leq CC(M) \leq \log\left(\frac{4}{3}\right) rank(M) \tag{97}$$

where $\log_3 6 \approx 1.63 \ldots$ and $\log\left(\frac{4}{3}\right) \approx 0.41 \ldots$ The upper bound is due to Kotlov [158] and improves on the previous best bound of $CC(M) \leq rank(M)/2$ by Kotlov and Lovasz [159]. The lower bound is due to Kushilevitz (unpublished, cf. [160]) and improves on a previous bound of $\Omega\left(\log^{\log_2 3} rank(M)\right) = \Omega(\log^{1.58\ldots} rank(M))$ due to Nisan and Wigderson [160].

Our main result is stated next. It assumes a wellknown conjecture from additive combinatorics — the Polynomial Freiman-Ruzsa (PFR) conjecture — discussed.

Quoting the (current) Wikipedia definition, additive combinatorics studies "combinatorial estimates associated with the arithmetic operations of addition and subtraction". As such, it deals with a variety of problems that aim to 'quantify' the amount of additive structure in subsets of additive groups. One such a problem is that which is addressed by the Polynomial Freiman-Ruzsa conjecture (we shall encounter a different problem in additive combinatorics when we get to "approximate duality" later on).

For $A \subseteq \mathbb{F}_2^n$, let $A+A$ denote the sum-set of A

$$A+A := \{a+a' | a, a' \in A\}$$

where addition is over $\mathbb{F}_2$. It is easy to see that $|A+A| = |A|$ if and only if A is an affine subspace of $\mathbb{F}_2^n$. The question addressed by the Freiman-Ruzsa Theorem is whether the ratio of $|A+A|$ to $|A|$ also 'approximates' the closeness of A to being a subspace, or in other words, whether the fact that $A+A$ is small with respect to the size of A also implies that span (A) is small with respect to the size of A. The Freiman-Ruzsa Theorem [161] says that this is indeed the case.

Theorem (4.2.3)[151]: (Freiman-Ruzsa Theorem [161]). If $A \subseteq \mathbb{F}_2^n$ has $|A+A| \le K|A|$, then $|span\ (A)| \le K^2 2^{K^4} |A|$.

The above theorem was improved in a series of works [162]–[164], culminating in the recent work [165] which proved an upper bound on the ratio $\frac{|span(A)|}{|A|}$ of the form $2^{2k}/(2k)$. This bound can be seen to be tight (up to a multiplicative factor of 2) by letting $A = \{u_1, u_2, \ldots, u_t\}$, where $u_1, u_2, \ldots, u_t \in \mathbb{F}_2^n$ are linearly independent vectors. Then in this case we have $|A+A| \approx \frac{t}{2}|A|$, while $|span\ (A)| = 2^t$.

This example also shows that the ratio $\frac{|span(A)|}{|A|}$ must depend exponentially on K. However, it does not rule out the existence of a large subset $A' \subseteq A$ for which the ratio $\frac{|span(A')|}{|A'|}$ is just polynomial in K, and this is exactly what is suggested by the PFR Conjecture:

Conjecture (4.2.4) (Polynomial Freiman-Ruzsa (PFR))[151]: There exists an absolute constant r, such that if $A \subset \mathbb{F}_2^n$ has $|A+A| \le K|A|$, then there exists a subset $A' \subseteq A$ of size at least $K^{-r}|A|$ such that $|span\ (A')| \le |A|$.

Note that the above conjecture implies that $|span\ (A')| \le |A| \le K^r|A'|$. The PFR conjecture has many other interesting equivalent formulations, see the survey of Green [166] for some of them. It is conjectured to hold for subsets of general groups as well and not only for subsets of the group $\mathbb{F}_2^n$ but we will be interested only in the latter case. Significant progress on this conjecture has been achieved recently by Sanders [167], using new techniques developed by Croot and Sisask [168]. Sanders proved an upper bound on the ratio $\frac{|span(A')|}{|A'|}$ which is quasi-polynomial in K:

Theorem (4.2.5) (Quasi-polynomial Freiman-Ruzsa Theorem (QFR) [167])[151]: Let $A \subset \mathbb{F}_2^n$ be a set such that $|A+A| \le K|A|$. Then there exists a subset $A' \subseteq A$ of size at least $K^{-O(\log^3 K)}|A|$ such that $|span\ (A')| \le |A|$.

We mentioning several other recent applications of the PFR Conjecture to theoretical computer science. The first application, due to Samorodnitsky [169], is to the area of low-degree testing, with further results by Lovett [170] and Green and Tao [171]. The second

application is to the construction of twosource extractors due to Ben-Sasson and Zewi [152]. The latter also introduced the notion of approximate duality which plays a central role in our proof method as well. The approximate duality method has recently found another application to proving lower bounds on locally decodable matching vector codes in the subsequent work by Bhowmick, Dvir and Lovett [153]. We describe the approximate duality conjecture and our new contributions to its study.
We improv the bounds on approximate duality, assuming the PFR conjecture. The new bound lies at the heart of our proof of the Main Theorem (4.2.20). We believe that Lemma (4.2.8) and its proof are of independent interest since they improve and simplify the proof of [152], and have already found new interesting applications to the study of locally decodable codes [153].

For $, B \subseteq \mathbb{F}_2^n$, we define the duality measure of A, B in (98) as an estimate of how 'close' this pair is to being dual

$$D(A,B) := \left|E_{a\in A, b\in B}\left[(-1)^{\langle a,b\rangle_2}\right]\right|, \tag{98}$$

where $\langle a,b\rangle_2$ denotes the binary inner-product of a, b over $\mathbb{F}_2$, defined by $\langle a,b\rangle_2 = \sum_{i=1}^{n} a_i \cdot b_i$ where all arithmetic operations are in $\mathbb{F}_2$.

It can be verified that if $D(A,B) = 1$ then A is contained in an affine shift of $B^{\perp}$ which is the space dual to the linear $\mathbb{F}_2$-span of B. The question is what can be said about the structure of A, B when $D(A,B)$ is sufficiently large, but strictly smaller than 1. The following theorem from [152] says that if the duality measure is a constant very close to 1 (though strictly smaller than 1) then there exist relatively large subsets $A' \subseteq A, B' \subseteq B$, such that $D(A', B') = 1$.

Theorem (4.2.6) (Approximate duality for nearly-dual sets, [152])[151]: For every $\delta > 0$ there exists a constant $\epsilon > 0$ that depends only on δ, such that if $A, B \subseteq \mathbb{F}_2^n$ satisfy $D(A,B) \geq 1-\epsilon$, then there exist subsets $A' \subseteq A, |A'| \geq \frac{1}{4}|A|$ and $B' \subseteq B, |B'| \geq 2^{-\delta n}|B|$, such that $D(A', B') = 1$.

It is conjectured that a similar result holds also when the duality measure is relatively small, and in particular when it tends to zero as n goes to infinity. Furthermore, the following theorem from [152] gives support to this conjecture, by showing that such bounds indeed follow from the PFR conjecture.

Theorem (4.2.7) (Approximate duality assuming PFR, exponential loss [152])[151]: Assuming the PFR Conjecture (4.2.4), for every pair of constants $\alpha > \delta > 0$ there exists a constant $\zeta > 0$, depending only on α and δ, such that the following holds. If $A, B \subseteq \mathbb{F}_2^n$ satisfy $|A|, |B| > 2^{\alpha n}$ and $D(A,B) \geq 2^{-\zeta n}$, then there exist subsets $A' \subseteq A, |A'| \geq 2^{-\delta n}|A|$ and $B' \subseteq B, |B'| \geq 2^{-\delta n}|B|$ such that $D(A', B') = 1$.

Our main technical contribution is the following generalization of the above theorem.

Lemma (4.2.8) (Main technical lemma)[151]: Assuming the PFR Conjecture (4.2.4) there exists a universal integer r such that the following holds. Suppose that $A, B \subseteq \{0,1\}^n$ satisfy $D(A,B) \geq \epsilon$. Then for every $K \geq 1$ and $t = n/\log K$, there exist subsets A', B' of A, B respectively such that $D(A', B') = 1$, and

$$\frac{|A'|}{|A|} \geq \left(\left(\frac{\left(\frac{\epsilon}{2}\right)^{2^t}}{nK}\right)(4n)^{-t}\right)^r, \frac{|B'|}{|B|} \geq \left(\left(\frac{\left(\frac{\epsilon}{2}\right)^{2^t}}{nK}\right)2^{-t}\right)^r \tag{99}$$

The proof of the above lemma appears. To see that it is indeed a generalization of Theorem (4.2.7) set $K = 2^{\frac{\delta n}{3r}}, t = \frac{3r}{\delta}, \zeta = \frac{\delta}{3r \cdot 2^t} = \frac{\delta}{3r \cdot 2^{\frac{3r}{\delta}}}, \epsilon = 2^{-\zeta n}$, and note that in this case the above lemma assures the existence of $|A'| \geq 2^{-\delta n}|A|, |B'| \geq 2^{-\delta n}|B|$ such that $D(A', B') = 1$. Note that Lemma (4.2.8) actually improves on the previous Theorem (4.2.7) even in this exponential range of parameters in that its parameters do not depend on the sizes of the sets A and B as was the case in Theorem (4.2.7).

However, the main significance of Lemma (4.2.8) is that it allows one to tradeoff the loss in the sizes of A' and B' with the value of ϵ for a wider range of parameters. More specifically it allows one to achieve a loss in the sizes of A' and B' which is only sub-exponential in n by requiring ϵ be a bit larger. In particular, the following corollary of Lemma (4.2.8) will enable us to prove the new upper bound of $O(rank(M)/log\, rank(M))$ on the communication complexity of $\{0, 1\}$-valued matrices assuming the PFR conjecture.

Corollary (4.2.9) (Approximate duality assuming PFR, sub-exponential loss)[151]: Suppose that $A, B \subseteq \mathbb{F}_2^n$ satisfy $D(A, B) \geq 2^{-\sqrt{n}}$. Then assuming the PFR Conjecture (4.2.4), there exist subsets A', B' of A, B respectively such that $D(A', B') = 1$, and $|A'| \geq 2^{-\frac{cn}{\log n}}|A|, |B'| \geq 2^{-\frac{cn}{\log n}}|B|$ for some absolute constant c.

Proof: Follows from Lemma (4.2.8) by setting $K = 2^{\frac{4n}{\log n}}, t = \frac{\log n}{4}, = 2^{-\sqrt{n}}$.

Note that in Corollary (4.2.9) the ratios $|A'|/|A|, |B'|/|B|$ are bounded from below by $2^{-\frac{cn}{\log n}}$, whereas in Theorem (4.2.7) we only get a smaller bound of the form $2^{-\delta n}$ for some constant $\delta > 0$. However, this improvement comes with a requirement that the duality measure D(A, B) is larger — in the above corollary we require that it is at least $2^{-\sqrt{n}}$ while in Theorem (4.2.7) we only require it to be at least $2^{-\zeta n} \ll 2^{-\sqrt{n}}$. We note that the bound $D(A, B) \geq 2^{-\sqrt{n}}$ can be replaced by $D(A, B) \geq \exp(-n^{1-\epsilon})$ for any $\epsilon > 0$ at the price of a larger constant $c = c(\epsilon)$.

We stress that a benefit of the proof of Lemma (4.2.8) is that it simplifies the original proof of Theorem (4.2.7) in [152]. Indeed, we believe that the presentation of the proof that appears is clearer and less involved than that in [152]. Also, the fact that the parameters in Lemma (4.2.8) do not depend on the sizes of A and B allows us to deduce new equivalence between approximate duality and the PFR conjecture in the exponential range that was not previously known. We elaborate on this equivalence in the full version of [173].

First we show how our Main Theorem (4.2.20) is deduced from the improved bounds on approximate duality in Corollary (4.2.9). Then we give an overview of the proof of Lemma (4.2.8) itself.

a) From approximate duality to communication complexity upper bounds.: We follow the approach of Nisan and Wigderson from [160]. Let the size of a matrix M be the number of entries in it and if M is $\{0, 1\}$-valued let $\delta(M)$ denote its (normalized) discrepancy, defined as the absolute value of the difference between the fraction of zero-entries and one-entries in M. Informally, discrepancy measures how "unbalanced" is M, with $\delta(M) = 1$ when M is monochromatic — all entries have the same value — and $\delta(M) = 0$ when M is completely balanced.

Returning to the work of [160], they observed that to prove the log-rank conjecture it suffices to show that a $\{0, 1\}$-valued matrix M of rank r always contains a monochromatic sub-matrix of size $|M|/qpoly(r)$ where $qpoly(r) = r^{\log^{O(1)} r}$ means quasi-polynomial in

r. Additionally, they used spectral techniques (i.e., arguing about the eigenvectors and eigenvalues of M) to show that any {0, 1}-valued matrix M of rank r contains a relatively large submatrix M' — of size at least $|M|/r^{\frac{3}{2}}$ — that is somewhat biased — its discrepancy is at least $1/r^{\frac{3}{2}}$. We show, using tools from additive combinatorics, that M' in fact contains a pretty large monochromatic submatrix (though not large enough to deduce the log-rank conjecture).

We start by working over the two-element field $\mathbb{F}_2$. This seems a bit counter-intuitive because the log-rank conjecture is false over $\mathbb{F}_2$. The canonical counterexample is the inner product function $IP(x,y) = \langle x,y\rangle_2$ — It is well-known (see e.g. [174]) that $rank_{\mathbb{F}_2}(M_{IP}) = n$ while $CC(IP) = n$. However, rather than studying M over $\mathbb{F}_2$ we focus on the biased submatrix M' and things change dramatically. (As a sanity-check notice that M_{IP} does not contain large biased submatrices and this does not contradict the work of [160] because the rank of M_{IP} over the reals is $2^n - 1$.)

Thus, our starting point is a large submatrix M' that has large discrepancy. It is well-known that $rank_{\mathbb{F}_2}(M') \le rank(M') \le r$ and that this implies M' can be written as $M = A^{\top} \cdot B$ where A, B are matrices whose columns are vectors in $\mathbb{F}_2^r$. Viewing each of A, B as the set of its columns, we have in hand two sets that have a large duality measure as defined in (98), namely, $D(A,B) = \delta(M') \ge 1/r^{\frac{3}{2}}$. This is the setting in which we apply Corollary (4.2.9) and deduce that A, B contain relatively large subsets A', B' with $D(A', B') = 1$. One can now verify that the submatrix of M' whose rows and columns are indexed by A', B' respectively is indeed monochromatic, as needed. We point out that to get our bounds we need to be able to find monochromatic submatrices of M' even when M' is both small and skewed (i.e., has many more columns than rows or vice versa). Fortunately, Corollary (4.2.9) is robust enough to use in such settings.

b) Improved bounds on approximate duality assuming PFR.: We briefly sketch the proof of our Main Technical Lemma (4.2.8). We use the spectrum of a set as defined in [175]:

Definition (4.2.10) (Spectrum)[151]: For a set $B \subseteq \mathbb{F}_2^n$ and $\alpha \in [0,1]$ let the α-spectrum of B be the set

$$Spec_\alpha(B) := \{x \in \mathbb{F}_2^n \,|\, |E_{b\in B}[(-1)^{\langle x,b\rangle_2}]| \ge \alpha\}. \tag{100}$$

Notice that $A \subseteq Spec_\epsilon(B)$ implies $D(A,B) \ge \epsilon$ (cf. (98)). In the other direction, Markov's inequality can be used to deduce that $D(A,B) \ge \epsilon$ implies the existence of $A' \subseteq A$ of relatively large size — $|A'| \ge \frac{\epsilon}{2}|A|$ — such that $A' \subseteq Spec_{\frac{\epsilon}{2}}(B)$. To prove our lemma we start with $A_1 = A'$ and establish a sequence of sets

$$A_2 \subseteq A_1 + A_1, \qquad A_3 \subseteq A_2 + A_2, \ldots$$

such that $A_i \subseteq Spec_{\epsilon_i}(B)$ for all i. This holds by construction for A_1 with $\epsilon_1 = \epsilon/2$, and we show that it is maintained throughout the sequence for increasingly smaller values of ϵ_i(we shall use $\epsilon_i = \epsilon_{i-1}^2$).

Moving our problem from the field of real numbers to the two-element field $\mathbb{F}_2$ now pays off. Each A_i is of size at most 2^n so there must be an index $i \le n/\log K$ for which $|A_{i+1}| \le K|A_i|$, let t be the minimal such index. We use the PFR conjecture together with the Balog–Szemeredi–Gowers Theorem ´ II.1 from additive combinatorics to show that our assumption that $|A_{t+1}| \le K|A_t|$ implies that a large subset A_t'' of A_t has small span (over $\mathbb{F}_2$).

We now have in hand a set A_t'' which is a relatively large fraction of its span and additionally satisfies $D(A_t'', B) \ge \epsilon_t$ because by construction $A_t'' \subseteq Spec_{\epsilon_t}(B)$. We use an approximate duality claim from [152] (Lemma (4.2.12)) which applies when one of the sets is a large fraction of its span (in our case the set which is a large fraction of its span is A_t''). This claim says that A_t'' and B each contain relatively large subsets A_t', B_t' satisfying $D(A_t', B_t') = 1$. Finally, recalling A_t' is a (carefully chosen) subset of $A_{t-1} + A_{t-1}$, we argue that A_{t-1}' contains a relatively large subset A_{t-1}' that is "dual" to a large subset B_{t-1}' of B_t', where by "dual" we mean $D(A_{t-1}', B_{t-1}') = 1$ (in other words A_{t-1}' is contained in an affine shift of the space dual to $span\ (B_{t-1}')$). We continue in this manner to find pairs of "dual" subsets for $t-2, t-3, \ldots, 1$ at which point we have found a pair of "dual" subsets of A, B that have relatively large size, thereby completing the proof.

The new connection between additive combinatorics and communication complexity seems to us worthy of further study. In particular, the exciting recent advances in additive combinatorics [165], [167], [168] use a rich palette of tools that may yield further insights into problems in communication complexity. We end by briefly pointing out a few directions we find interesting.

c) Improved unconditional bounds on communication complexity: Given the recent QFR result of [167] (Theorem (4.2.5)) which comes very close to proving the PFR conjecture, it is interesting to see if it implies any unconditional improvement on communication complexity of low-rank matrices. Looking at our proof of Lemma (4.2.8), we apply the PFR conjecture to a subset A_t' of A_t which satisfies $|A_t' + A_t'| \leq K'|A_t'|$ for $K' \approx K/\epsilon^{2^t}$. For $\epsilon < \frac{1}{2}$ this gives a non-trivial bound only if $t = O(\log n)$. Since t could be as large as $n/\log K$ we are forced to choose $K = 2^{\Omega\left(\frac{n}{\log n}\right)}$ which implies in turn $K' = 2^{\Omega\left(\frac{n}{\log n}\right)}$. Thus, Sander's QFR Theorem (4.2.5) does not yield any non-trivial bounds in our case. However, for purposes of improving the unconditional upper bound of Kotlov (cf. 2) say, to $CC(M) \leq rank(M)/4$, it suffices to improve the loss in the size of A in Theorem (4.2.5) from $K^{-O(\log_3 K)}$ to $K^{-c\log K}$ for a sufficiently small constant c.

d) Improved conditional bounds: The bounds on approximate duality in can possibly be significantly improved. For all we know, the exponential loss of $2^{-O(\sqrt{n})}$ shown May be tight, and this would lead to an improved version of Corollary (4.2.9) in which the sizes of $|A'|, |B'|$ are a $2^{-O(\sqrt{n})}$ fraction of A and B respectively, instead of the $2^{-O\left(\frac{n}{\log n}\right)}$ loss we currently have. Such a result would translate directly to an upper bound on communication complexity of the form $CC(M) \leq O\left(\sqrt{rank(M)}\right)$. In order to make further progress one might want to also consider working over finite fields that are larger than 2, or over the reals. As a first step in this direction, one may wish to investigate whether there are interesting approximate duality statements over such fields.

e) Does the log-rank conjecture imply the PFR conjecture?: Alternatively, does it have any other nontrivial consequences in additive combinatorics? We believe the answer to this question is positive and make a step in this direction by showing an equivalence between approximate duality and PFR statements in the exponential range, namely, when the losses in the sizes of sets in both approximate duality and PFR is exponential in n (See [173] for an exact statement and details of the proof.)

We contain the proof of the Main Technical Lemma (4.2.8). The proof of Main Theorem (4.2.20) given Corollary (4.2.9).

We prove our Main Technical Lemma (4.2.8). We start with some additive combinatorics.

f) Additive combinatorics preliminaries: In what follows all arithmetic operations are taken over $\mathbb{F}_2$. For the proof of Lemma (4.2.8) we need two other theorems from additive combinatorics. The first is the well-known Balog–Szemeredi–Gowers Theorem of [176], [11].

Theorem (4.2.11)[151]: (Balog–Szemeredi–Gowers). There exist fixed polynomials $f(x,y), g(x,y)$ such that the following holds for every subset A of an abelian additive group. If A satisfies $\Pr_{a,a'\in A}[a+a'\in S] \geq 1/K$ for $|S| \leq C|A|$, then one can find a subset $A' \subseteq A$ such that $|A'| \geq |A|/f(K,C)$, and $|A'+A'| \leq g(K,C)|A|$.

The second is a lemma from [152] which can be seen as an approximate duality statement which applies when one of the sets has small span:

Lemma (4.2.12)[151]: (Approximate-duality for sets with small span, [152]). If $D(A,B) \geq \epsilon$, then there exist subsets $A' \subseteq A, B' \subseteq B, |A'| \geq \frac{\epsilon}{4}|A|$, $|B'| \geq \frac{\epsilon^2}{4}\frac{|A|}{|span(A)|}|B|$, such that $D(A',B') = 1$. If $A \subseteq Spec_\epsilon(B)$ then we have $|A'| \geq |A|/2$ and $|B'| \geq \epsilon^2 \frac{|A|}{|span(A)|}|B|$ in the statement above.

Recall the definition of the spectrum given in (100):

$$Spec_\alpha(B) := \left\{x \in \mathbb{F}_2^n \middle| \left|E_{b\in B}\left[(-1)^{\langle x,b\rangle_2}\right]\right| \geq \alpha\right\}.$$

Finally, for $S \subset \mathbb{F}_2^n$ and $x \in \mathbb{F}_2^n$ let $rep_S(x)$ be the number of different representations of x as an element of the form $s+s'$ where $s,s' \in S. rep_S(x)$ can also be written, up to a normalization factor, as $1_S * 1_S(x)$ where 1_S is the indicating function of the set S and $*$ denotes convolution.

g) Proof overview: We construct a decreasing sequence of constants

$$\epsilon_1 = \frac{\epsilon}{2}, \qquad \epsilon_2 = \frac{\epsilon_1^2}{2}, \qquad \epsilon_3 = \frac{\epsilon_2^2}{2}, \ldots$$

and a sequence of sets $A_1 := A \cap Spec_{\epsilon_1}(B), A_2 \subseteq (A_1 + A_1) \cap Spec_{\epsilon_2}(B), A_3 \subseteq (A_2 + A_2) \cap Spec_{\epsilon_3}(B), \ldots$

Since each of the sets in the sequence is of size at most 2^n there must be an index $i \leq n/\log K$ for which

$$|A_{i+1}| \leq K|A_i| \tag{101}$$

and let t be the minimal such index. The PFR Conjecture (4.2.4) together with the Balog–Szemeredi–Gowers Theorem (4.2.11) will be used to deduce from (101) that a large subset A_t'' of A_t has small span. Applying Lemma (4.2.12) to the sets A_t'' and B implies the existence of large subsets $A_t' \subseteq A_t$ and $B_t' \subseteq B$ such that $D(A_t', B_t') = 1$. Finally we argue inductively for $i = t-1, t-2, \ldots, 1$ that there exist large subsets $A_i' \subseteq A_i$ and $B_i' \subseteq B$ such that $D(A_i', B_i') = 1$. The desired conclusion will follow from the $i = 1$ case. To be able to "pull back" and construct a pair of large sets A_{i-1}', B_{i-1}' from the pair A_i', B_i' we make sure every element in A_i is the sum of roughly the same number of pairs in $A_{i-1} \times A_{i-1}$.

h) The sequence of sets: Let $\epsilon_1 := \epsilon/2, A_1 := A \cap Spec_{\epsilon_1}(B)$. Assuming A_{i-1}, ϵ_{i-1} have been defined set $\epsilon_i = \epsilon_{i-1}^2/2$ and let $j_i \in \{0,\ldots,n-1\}$ be an integer index which maximizes the size of

$$\left\{(a,a') \in A_{i-1} \middle| a+a' \in Spec_{\epsilon_i}(B) and\ 2^{j_i} \leq rep_{A_{i-1}}(a+a') \leq 2^{j_i+1}\right\} \tag{102}$$

and set

$$A_i := \left\{a+a' \middle| \begin{matrix} a,a' \in A_{i-1}, a+a' \in Spec_{\epsilon_i}(B)\ and \\ 2^{j_i} \leq rep_{A_{i-1}}(a+a') \leq 2^{j_{i+1}} \end{matrix}\right\} \tag{103}$$

Claim (4.2.13)[151]: For $i=1$ we have $|A_1| \geq \left(\frac{\epsilon}{2}\right)|A|$. For $i>1$ we have

$$\Pr_{a,a' \in A_{i-1}}[a+a' \in A_i] \geq \epsilon_i/n \tag{104}$$

and additionally

$$|A_i| \geq \frac{\epsilon_i}{2^{j_i+1}n}\,|A_{i-1}|^2. \tag{105}$$

Proof: The case of $i=1$ follows directly from Markov's inequality. For larger i we argue that

$$\Pr_{a,a' \in A_{i-1}}\left[a+a' \in Spec_{\epsilon_i}(B)\right] \geq \epsilon_i.$$

To see this use Cauchy-Schwarz to get

$$\mathbb{E}_{a,a' \in A_{i-1}}\left|\mathbb{E}_{b\in B}(-1)^{\langle a+a',b\rangle}\right| = \mathbb{E}_{b\in B}\left(\mathbb{E}_{a\in A_{i-1}}\left[(-1)^{\langle a,b\rangle}\right]\right)^2 \geq (\mathbb{E}^2_{a\in A_{i-1,b\in B}\left[(-1)^{\langle a,b\rangle}\right]} = \epsilon^2_{i-1}$$

and apply Markov's inequality to deduce that an ϵ_i-fraction of $(a,a') \in A_{i-1} \times A_{i-1}$ sum to an element of $Spec_{\epsilon_i}(B)$. Selecting j_i to maximize (102) yields inequality (104). Since every element $x \in A_i$ can be represented as $x = a + a'$ with $a, a' \in A_{i-1}$ in at most 2^{j_i+1} different ways we deduce (105) from (104) and complete the proof.

i) The inductive claim: Let t be the minimal index such that $|A_{t+1}| \leq K|A_t|$ and note that $t \leq n/\log K$ because all sets A_i are contained in $\mathbb{F}_2^n$. We shall prove the following claim by backward induction.

Claim (4.2.14) (Inductive claim)[151]: For $i = t, t-1, \ldots, 1$ there exist subsets

$$A_i' \subseteq A_i, \qquad B_i' \subseteq B$$

such that $D(A_i', B_i') = 1$ and A_i', B_i' are not too small:

$$|A_i'| \geq poly\left(\frac{\epsilon_{t+1}}{nK}\right)(4n)^{-(t-i)}\left(\prod_{\ell=i}^{t}\epsilon_{\ell+1}\right)|A_i|,$$

$$|B_i'| \geq poly\left(\frac{\epsilon_{t+1}}{nK}\right)2^{-(t-i)}|B|$$

We split the proof of the claim to two parts. The base case (Proposition (4.2.15)) is proved using the tools from additive combinatorics listed in the beginning. The inductive step is proved in Proposition (4.2.16) using a graph construction. Before proving Claim (4.2.14) we show how it implies Lemma (4.2.8).

Proof of Main Technical Lemma (4.2.8): Set $i=1$ in Claim (4.2.14) above. Recall that $\epsilon_{i+1} = \epsilon_i^2/2$ for all i, so

$$\epsilon_{\ell+1} = \epsilon^{2^\ell}/2^{2^\ell-1} \geq \left(\frac{\epsilon}{2}\right)^{2^\ell}.$$

Thus we have $\epsilon_{t+1} \geq \left(\frac{\epsilon}{2}\right)^{2^t}$ and $\prod_{\ell=1}^{t}\epsilon_{\ell+1} \geq \left(\frac{\epsilon}{2}\right)^{2^{t+1}}$. This gives the bounds on A', B' stated in (99).

Proposition (4.2.15) (Base case of Claim (4.2.14) ($\boldsymbol{i=t}$))[151]: There exist subsets $A_t' \subseteq A_t, B_t' \subseteq B_t$ such that $D(A_t', B_t') = 1$ and A_t', B_t' are not too small:

$$|A_t'| \geq poly\left(\frac{\epsilon_{t+1}}{nK}\right)|A_t|,$$

$$|B_t'| \geq poly\left(\frac{\epsilon_{t+1}}{nK}\right)|B|.$$

Proof:

By assumption $|A_{t+1}| \le K|A_t|$ and $\Pr_{a,a' \in A_t}[a + a' \in A_{t+1}] \ge \epsilon_{t+1}/n$ by (105). Hence we can apply the Balog–Szemeredi–Gowers Theorem (Theorem (4.2.11)) to the set A_t to obtain a subset $\tilde{A}_t \subseteq A_t$ such that

$$\left|\tilde{A}_t\right| \ge poly\left(\frac{\epsilon_{t+1}}{nK}\right)|A_t|,$$

And

$$\left|\tilde{A}_t + \tilde{A}_t\right| \le poly\left(\frac{nK}{\epsilon_{t+1}}\right)|A_t| = poly\left(\frac{nK}{\epsilon_{t+1}}\right)\left|\tilde{A}_t\right|.$$

Now we can apply the PFR Conjecture (4.2.4) to the set $\tilde{A}_t$ which gives a subset $A_t'' \subseteq \tilde{A}_t$ such that

$$|A_t''| \ge poly\left(\frac{\epsilon_{t+1}}{nK}\right)\left|\tilde{A}_t\right| = poly\left(\frac{\epsilon_{t+1}}{nK}\right)|A_t|,$$

And

$$|span\ (A_t'')| \le \left|\tilde{A}_t\right| = poly\left(\frac{nK}{\epsilon_{t+1}}\right)\ |A_t''|.$$

Recall that $A_t'' \subseteq Spec_{\epsilon_t}(B)$, and in particular $D(A_t''\ , B) \ge \epsilon_t$. Applying Lemma (4.2.12) to the sets A_t'' and B we conclude that there exist subsets $A_t' \subseteq A_t'', B' \subseteq B$ such that $D(A_t', B') = 1$, and which satisfy $|A_t'| \ge \frac{1}{2}|A_t''|$ and

$$|B_t'| \ge \epsilon_t^2 \frac{|A_t''|}{|span\ (A_t'')|}|B| = poly\left(\frac{\epsilon_{t+1}}{nK}\right)|B|.$$

This completes the proof of the base case.

Proposition (4.2.16) (Inductive step of Claim (4.2.14))[151]: For every $i = t - 1, \dots, 1$ there exist subsets $A_i' \subseteq A_i, B_i' \subseteq B$ such that $D(A_i', B_i') = 1$ and A_i', B_i' are not too small:

$$|A_i'| \ge poly\left(\frac{\epsilon_{t+1}}{nK}\right)(4n)^{-(t-i)}\left(\prod_{\ell=i}^{t}\epsilon_{\ell+1}\right)|A_i|,$$

$$|B_i'| \ge poly\left(\frac{\epsilon_{t+1}}{nK}\right)2^{-(t-i)}|B|.$$

Proof: Suppose that the claim is true for i and argue it holds for index $i - 1$. Let $G = (A_{i-1}, E)$ be the graph whose vertices are the elements in A_{i-1}, and (a, a') is an edge if $a + a' \in A_i'$. We bound the number of edges in this graph from below. Recall from (103) that every $a \in A_i'$ (where $A_i' \subseteq A_i$) satisfies $2^{j_i} \le rep_{A_{i-1}}(a) \le 2^{j_i+1}$. Using this we get

$$\begin{aligned}|E| \ge 2^{j_i} \cdot |A_i'| &\ge 2^{j_i}\left(\frac{\epsilon_{t+1}}{nK}\right)^{O(1)}\frac{|A_i|}{(4n)^{(t-i)}}\prod_{\ell=i}^{t}\epsilon_{\ell+1}\\ &\ge 2^{j_i}\left(\frac{\epsilon_{t+1}}{nK}\right)^{O(1)}\frac{|A_{i-1}|^2}{(4n)^{(t-i)}}\frac{\epsilon_i}{2^{j_i+1}n}\prod_{\ell=i}^{t}\epsilon_{\ell+1}\\ &= 2\cdot\left(\frac{\epsilon_{t+1}}{nK}\right)^{O(1)}\frac{|A_{i-1}|^2}{(4n)^{(t-(i-1))}}\prod_{\ell=i-1}^{t}\epsilon_{\ell+1}\end{aligned}$$

The first inequality follows because $rep_{A_{i-1}}(x) \ge 2^{j_i}$ for all $x \in A_i'$, the second uses the induction hypothesis and the third follows by (105).

Let $M := poly\left(\frac{\epsilon_{t+1}}{nK}\right)(4n)^{-(t-(i-1))}(\prod_{\ell=i-1}^{t}\epsilon_{\ell+1})$. Since our graph has at least $2M|A_{i-1}|^2$ edges and $|A_{i-1}|$ vertices, it has a connected component with at least $2M|A_{i-1}|$ vertices and denote by A''_{i-1} the set of vertices in it.

Choose an arbitrary element a in A''_{i-1}. Partition B'_i into two sets $B'_{i,0}$ and $B'_{i,1}$ such that all elements in $B'_{i,0}$ have inner product 0 with a, and all elements in $B'_{i,1}$ have inner product 1 with a. Let B'_{i-1} be the larger of $B'_{i,0}, B'_{i,1}$, and note that $|B'_{i-1}| \geq |B'_i|/2$. Recall that our assumption was that $D(A'_i, B'_i) = 1$. Abusing notation, let $\langle A'_i, B'_i\rangle_2$ denote the value of $\langle a', b'\rangle_2$ for some $a' \in A'_i, B'_i$ (the choice of a', b' does not matter because $D(A'_i, B'_i) = 1$). Next we consider two cases — the case where $\langle A'_i, B'_i\rangle_2 = 0$, and the case where $\langle A'_i, B'_i\rangle_2 = 1$.

In the first case we have that for every $a, a' \in A''_{i-1}$ which are neighbors in the graph, $a + a' \in A'_i$, and therefore $\langle a + a', b\rangle_2 = 0$ for every $b \in B'_{i-1}$. This implies in turn that $\langle a, b\rangle_2 = \langle a', b\rangle_2$ for all elements $a, a' \in A''_{i-1}$ which are neighbors in the graph, $b \in B'_{i-1}$. Since A''_{i-1} induces a connected component, and due to our choice of B''_{i-1}, this implies that $D(A''_{i-1}, B'_{i-1}) = 1$ so we set $A'_{i-1} = A''_{i-1}$.

In the second case we have that $\langle a + a', b\rangle_2 = 1$ for every $a, a' \in A''_{i-1}$ which are neighbors in the graph, $b \in B'_{i-1}$. In particular this implies that $\langle a, b\rangle_2 = \langle a', b\rangle_2 + 1$ for every elements $a, a' \in A''_{i-1}$ which are neighbors in the graph, $b \in B'_{i-1}$. This means that A''_{i-1} can be partitioned into two sets $A'_{i-1,0}, A'_{i-1,1}$, where the first one contains all elements in A''_{i-1} that have inner product 0 with all elements in B''_{i-1}, while the second set contains all elements in A''_{i-1} that have inner product 1 with all elements in B'_{i-1}. We set A'_{i-1} to be the larger of these two sets and get $D(A'_{i-1}, B'_{i-1}) = 1$ and $|A'_{i-1}| \geq M|A_{i-1}|$.
Concluding, in both cases we obtained subsets A'_{i-1}, B'_{i-1} of A_{i-1}, B respectively, such that $D(A'_{i-1}, B'_{i-1}) = 1$ and A'_{i-1}, B'_{i-1} are not too small:

$$\frac{|A'_{i-1}|}{|A_{i-1}|} \geq \left(\frac{\epsilon_{t+1}}{nK}\right)^{O(1)}(4n)^{-(t-(i-1))}\left(\prod_{\ell=i-1}^{t}\epsilon_{\ell+1}\right),$$

and

$$\frac{|B'_{i-1}|}{|B|} \geq \frac{1}{2}\frac{|B'_i|}{|B|} \geq \frac{1}{2}poly\left(\frac{\epsilon_{t+1}}{nK}\right)2^{-(t-i)} = poly\left(\frac{\epsilon_{t+1}}{nK}\right)2^{-(t-(i-1))}$$

This concludes the proof of the inductive claim.

We prove our main theorem, Theorem (4.2.20) given Corollary (4.2.9). The proof of the main technical lemma is deferred.

We start by repeating the necessary definitions. For a $\{0, 1\}$-valued matrix M, let $CC(M)$ denote the communication complexity of the boolean function associated with M. Let rank(M) and $rank_{\mathbb{F}_2}(M)$ denote the rank of M over the reals and over $\mathbb{F}_2$, respectively. We denote by $|M|$ the total number of entries in M, and by $|M_0|$ and $|M_1|$ the number of zero and non-zero entries of M, respectively. We say that M is monochromatic if either $|M| = |M_0|$ or $|M| = |M_1|$. Finally, we define the discrepancy $\delta(M)$ of M to be the ratio $\frac{||M_0|-|M_1||}{|M|}$. Recall the statements of Theorem (4.2.20) and Corollary (4.2.9).

Assuming the PFR conjecture (Conjecture (4.2.4)), for every $\{0, 1\}$-valued matrix M,

$$CC(M) = O\left(\frac{rank(M)}{\log rank(M)}\right).$$

Suppose that $A, B \subseteq \mathbb{F}_2^n$ satisfy $D(A, B) \geq 2^{-\sqrt{n}}$. Then assuming the PFR conjecture, there exist subsets A', B' of A, B respectively such that $D(A', B') = 1$, and $|A'| \geq 2^{-\frac{cn}{\log n}}|A|, |B'| \geq 2^{-\frac{cn}{\log n}}|B|$ for some absolute constant c.

We first prove that the above corollary is equivalent to the following one:

Lemma (4.2.17) (Main technical lemma, equivalent matrix form)[151]: Let M be a $\{0, 1\}$-valued matrix with no identical rows or columns, of rank at most r over $\mathbb{F}_2$, and of discrepancy at least $2^{-\sqrt{r}}$. Then assuming the PFR conjecture (Conjecture (4.2.4)), there exists a monochromatic submatrix M' of M of size at least $2^{-\frac{cr}{\log r}}|M|$ for some absolute constant c.

Proof: We prove only the Corollary (4.2.9) ⇒ Lemma (4.2.17) implication. The proof of the converse implication is similar. Denote the number of rows and columns of M by k, ℓ respectively. It is well known that the rank of M over a field $\mathbb{F}$ equals r if and only if M can be written as the sum of r rank one matrices over the field $\mathbb{F}$. Since $rank_{\mathbb{F}_2}(M) \leq r$ this implies in turn that there exist subsets $A, B \subseteq \mathbb{F}_2^r$, $A = \{a_1, a_2, \ldots, a_k\}, B = \{b_1, b_2, \ldots, b\}$ such that $M_{i,j} = \langle a_i, b_j \rangle_2$ for all $1 \leq i \leq k, 1 \leq j \leq \ell$. Since M has no identical rows or columns we know that $|A| = k, |B| = \ell$. Note that $D(A, B) = \delta(M) \geq 2^{-\sqrt{r}}$.

Corollary (4.2.9) now implies the existence of subsets $A' \subseteq A, B' \subseteq B, |A'| \geq 2^{-\frac{cr}{\log r}}|A|, |B'| \geq 2^{-\frac{cr}{\log r}}|B|$, such that $D(A', B') = 1$. Let M' be the submatrix of M whose rows and columns correspond to the indices in A' and B' respectively. The fact that $D(A', B') = 1$ implies that $M_{i,j} = \langle a_i, b_j \rangle_2 \equiv$ const for all $a_i \in A', b_j \in B'$. Therefore M' is a monochromatic submatrix of M of which satisfies

$$|M'| = |A'||B'| \geq 2^{-\frac{2cr}{\log r}}|A||B| = 2^{-\frac{2cr}{\log r}}|M|,$$

as required.

In order to prove Theorem (4.2.20) we follow the highlevel approach of Nisan and Wigderson [160] which was explained. They showed that in order to prove the log-rank conjecture it suffices to prove that every $\{0, 1\}$-valued matrix of low rank has a large monochromatic submatrix. We start with the following lemma.

Theorem (4.2.18)[151]: (Existence of submatrix with high discrepancy [160]). Every $\{0, 1\}$-valued matrix M has a submatrix M' of size at least $\left(rank(M)\right)^{-\frac{3}{2}}|M|$ and with $\delta(M') \geq \left(rank(M)\right)^{-\frac{3}{2}}$.

Lemma (4.2.19)[151]: (Existence of large monochromatic submatrix assuming PFR). Assuming the PFR conjecture, every $\{0, 1\}$-valued matrix M with no identical rows or columns has a monochromatic submatrix of size at least $2^{-O\left(\frac{rank(M)}{\log rank(M)}\right)}|M|$.

In order to prove the above lemma we use Lemma (4.2.17), together with the following theorem from [160], which says that every $\{0, 1\}$-valued matrix M contains a submatrix of high discrepancy:

Proof: Let $r = rank(M)$. Theorem (4.2.18) implies the existence of a submatrix M of M' with $|M'| \geq \left(rank(M)\right)^{-\frac{3}{2}}|M|$, and $\delta(M') \geq r^{-\frac{3}{2}} \gg 2^{-\sqrt{r}}$. Note also that

$$rank_{\mathbb{F}_2}(M) \leq rank(M') \leq rank(M) = r.$$

Lemma (4.2.17) then implies the existence of a monochromatic submatrix M'' of M' of size at least $2^{-\frac{cr}{\log r}}|M'|$ for some absolute constant c. So we have that M'' is a monochromatic submatrix of M which satisfies

$$|M''| \geq 2^{-\frac{cr}{\log r}}|M'| \geq 2^{-\frac{cr}{\log r}} r^{-\frac{3}{2}}|M| = 2^{-O\left(\frac{r}{\log r}\right)}|M|$$

Theorem (4.2.20) (Main)[151]: Assuming the PFR Conjecture (4.2.4), for every $\{0,1\}$-valued matrix M

$$CC(M) = O\left(\frac{rank(M)}{\log rank(M)}\right).$$

Proof:

Let M be a $\{0,1\}$-valued matrix. We will construct a deterministic protocol for M with communication complexity $O\left(\frac{rank(M)}{\log rank(M)}\right)$. We may assume w.l.o.g that M has no repeated rows or columns, otherwise we can eliminate the repeated row or column and the protocol we construct for the "compressed" matrix (with no repeated rows/columns) will also be a protocol for M.

We follow the high level approach of the proof of Theorem 2 from [160]. We will show a protocol with $2^{O\left(\frac{r}{\log r}\right)}$ leaves. This will suffice since it is wellknown that a protocol with t leaves has communication complexity at most $O(\log t)$ (cf. [174]).

Now we describe the protocol. Let Q be the largest monochromatic submatrix of M. Then Q induces a natural partition of M into 4 submatrices Q, R, S, T with R sharing the rows of Q and S sharing the columns of Q.

$$M = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$$

Let U_1 be a subset of the rows of $(Q|R)$ whose restriction to the columns of R span the rows of R. Similarly, let U_2 be a subset of the rows of $(S|T)$ whose restriction to the columns of S span the rows of S. Note that if Q is the all zeros matrix then the rows of U_1 are independent of the rows of U_2. Otherwise, if Q is the all ones matrix then the rows of U_1 are independent of all the rows of U_2 except possibly for the vector in U_2 whose restriction to the columns of S is the all ones vector (if such vector exists). Thus since Q is monochromatic we have that $rank(R) + rank(S) = |U_1| + |U_2| \leq rank(M) + 1$.

If $rank(R) \leq rank(S)$ then the row player sends a bit saying if his input belongs to the rows of Q or not. The players continue recursively with a protocol for the submatrix $(Q|R)$ or the submatrix $(S|T)$ according to the bit sent. If $rank(R) \geq rank(S)$ the roles of the row and column players are switched.

Suppose without loss of generality that $rank(R) \leq rank(S)$. Then after sending one bit we continue with either the matrix $(Q|R)$ which is of rank at most $rank(M)/2$ or with the matrix $(S|T)$ which — thanks to Lemma (4.2.19) — is of size at most $(1 - \delta)|M|$ for $\delta \geq 2^{-\frac{cr}{\log r}}$.

Let $L(m,r)$ denote the number of leaves in the protocol starting with a matrix of area at most m and rank at most r. Then we get the following recurrence relation:

$$L(m,r) \leq \begin{cases} L\left(m, \frac{r}{2}\right) + L(m(1 - \delta), r) & r > 1 \\ 1 & r = 1 \end{cases}$$

It remains to show that in the above recursion $(m,r) = 2\text{^}O\left(\frac{r}{\log r}\right)$. Applying the recurrence iteratively $1/\delta$ times to the right-most summand we get

$$L(m,r) \le \delta^{-1}L(m,r/2) + L\left(m(1 - \delta)^{\frac{1}{\delta}}, r\right) \le 2^{\frac{cr}{\log(r)}}L(m,r/2) + L(m/2,r).$$

Set $A(m,r) := 2^{-\frac{2cr}{\log r}}L(m,r)$. Then we have $A(m,r) \le A(m,r/2) + A(m/2,r)$ which together with $A(1,r), A(m,1) \le 1$ imply $A(m,r) \le \binom{\log m + \log r}{\log r}$ since we may apply the recursion iteratively at most log r times to the left term and $\log m$ times to the right term before we reach $A(1,r)$ or $A(m,1)$. This in turn implies $A(m,r) \le \binom{\log m + \log r}{\log r} \le r^{O(\log r)}$ due to the fact that $r \le m \le 2^{2r}$, since we may assume there are no identical rows or columns in the matrix M.

Concluding, we have $(m,r) \le 2^{\frac{2cr}{\log r} + O(\log^2 r)}$, which implies in turn $CC(M) = O(r/\log r)$ as claimed.

Section (4.3): The Structure of the Spectrum of Small Sets

For G be a finite Abelian group, and let A be a subset of G. For a character $\gamma \in G$, the corresponding Fourier coefficient of 1_A

$$\widehat{1}_A(\gamma) = \sum_{x \in A} \gamma(x).$$

The spectrum of A is the set of characters with large Fourier coefficients,

$$Spec_\varepsilon(A) = \{\gamma \in \hat{G} : |\widehat{1}_A(\gamma)| \ge \varepsilon|A|\}.$$

Note that the spectrum of a set is a symmetric set, that is $Spec_\varepsilon(A) = -Spec_\varepsilon(A)$, where we view G as an additive group (which is isomorphic to G). Understanding the structure of the spectrum of sets is an important topic in additive combinatorics, with several striking applications discussed below. As we illustrate, there is a gap in our knowledge between combinatorial structural results, which apply to all elements in the spectrum, and statistical structural results, which apply to most elements in the spectrum. The former results apply only to large sets, typically of the size $|A| \ge |G|^c$ for some absolute constant $c > 0$, where the latter results apply also for smaller sets. The goal is to bridge this gap.

Our interest in this problem originates from applications of it in computational complexity, where a better understanding of the structure of the spectrum of small sets can help to shed light on some of the main open problems in the area, such as constructions of two source extractors [19], [15], [152] or the log rank conjecture in communication complexity [151]. We refer to a survey by applications of additive combinatorics in theoretical computer science [179]. We focus on the core mathematical problem, and do not discuss applications further.

We assume from now on that $|A| = |G|^\alpha$ where $\alpha > 0, \varepsilon > 0$ are arbitrarily small constants, which is the regime where current techniques fail. In fact, our results extend to some range of sub-constant parameters, but only mildly. First, we review the current results on the structure of the spectrum, and their limitations.

Size bound The most basic property of the spectrum is that it cannot be too large. Parseval's identity bounds the size of the spectrum by

$$|Spec_\varepsilon(A)| \le \frac{|G|}{\varepsilon^2|A|} = \frac{|G|^{1-\alpha}}{\varepsilon^2}.$$

However, this does not reveal any information about the structure of the spectrum, except from a bound on its size.
Dimension bound A combinatorial structural result on the spectrum was obtained by Chang [126]. She discovered that the spectrum is low dimensional. For a set $\Gamma \subseteq \hat{G}$, denote its dimension as the minimal integer d, such that there exist $\gamma_1, \ldots, \gamma_d \in \hat{G}$ with the following property: any element $\gamma \in \Gamma$ can be represented as $\gamma = \sum \varepsilon_i \gamma_i$ with $\varepsilon_i \in \{-1, 0, 1\}$. With this definition, Chang's theorem asserts that

$$dim(Spec_\varepsilon(A)) \le O(\varepsilon^{-2} \, log(|G|/|A|)).$$

Chang [126] used this result to obtain improved bounds for Freiman's theorem on sets with small doubling, and Green [130] used it to find arithmetic progressions in sumsets. Moreover, Green [145] showed that the bound in Chang's theorem cannot in general be improved, at least when A is not too small. Recently, Bloom [178] obtained sharper bounds for a large subset of the spectrum. He showed that there exists a subset $\Gamma \subseteq Spec_\varepsilon(A)$ of size $|\Gamma| \ge \varepsilon \cdot |Spec_\varepsilon(A)|$ such that

$$dim(\Gamma) \le O(\varepsilon^{-1} \, log(|G|/|A|)).$$

He applied these structural results to obtain improved bounds for Roth's theorem and related problems. However, we note that in our regime of interest, where $|A| = |G|^\alpha$ with $0 < \alpha < 1$, both results become trivial if ε is a small enough constant. This is because both give a bound on the dimension of the form $O(\varepsilon^{-c}(1 - \alpha)) \cdot \log |G|$ with $c \in \{1, 2\}$. However, any set $\Gamma \subseteq \hat{G}$ trivially has dimension at most $\log |G|$. As our interest is in the regime of any arbitrarily small constant $\alpha, \varepsilon > 0$, we need to turn to a different set of techniques.
Statistical doubling Bourgain [1] showed that for many pairs of elements in the spectrum, their sum lands in a small set. Concretely,

$$\Pr_{\gamma_1, \gamma_2 \in Spec_\varepsilon(\mathcal{A})} [\gamma_1 + \gamma_2 \in Spec_{\varepsilon^2/2}(A)] \ge \varepsilon^2/2,$$

where we note that by Parseval's identity, $|Spec_{\varepsilon^2/2}(A)| \le O(|G|1^{-\alpha}/\varepsilon^4)$. He used these results to obtain improved bounds on exponential sums. Similar bounds can be obtained for linear combinations of more than two elements in the spectrum, for example as done by Shkredov [124]. If we assume that $|Spec_{\varepsilon^2/2}(A)| \le K|Spec_\varepsilon(A)|$ and apply the Balog–Szemerédi–Gowers theorem [176], [11], this implies that there exists a large subset $\Gamma \subseteq Spec_\varepsilon(A)$ such that $|\Gamma + \Gamma| \le (K/\varepsilon)^{O(106)}|\Gamma|$. However, it does not provide any bounds on the sumset of the entire spectrum, that is on $|Spec_\varepsilon(A) + Spec_\varepsilon(A)|$. In fact, we will later see an example showing that this sumset could be much large than the spectrum, whenever $\varepsilon \le 1/2$.
The motivating question for the current work is to understand whether the statistical doubling result described above, can be applied for the entire spectrum. That is, can we obtain combinatorial structural results on the sumset of the entire spectrum $Spec_\varepsilon(A) + Spec_\varepsilon(A)$.
As a first step, we ask for which $\alpha, \varepsilon > 0$ is is true that, for any set A of size $|A| = |G|^\alpha$, the sumset $Spec_\varepsilon(A) + Spec_\varepsilon(A)$ is much smaller than the entire group. There are two regimes where this is trivially true. First, when $\alpha > 1/2$, it is true since by Parseval's identity, $Spec_\varepsilon(A)$ is smaller than the square root of the group size, and hence

$$|Spec_\varepsilon(A) + Spec_\varepsilon(A)| \le |Spec_\varepsilon(A)|^2 \le \frac{|G|^2 - 2\alpha}{\varepsilon^4}.$$

Also, when $\varepsilon > 1/2$ then $Spec_\varepsilon(A) + Spec_\varepsilon(A) \subseteq Spec_{2\varepsilon-1}(A)$ (see, e.g., [175] for a proof) and hence again by Parseval's identity, the size of the sumset is bounded by

$$|Spec_\varepsilon(A) + Spec_\varepsilon(A)| \leq |Spec_\varepsilon(A)|^2 \leq \frac{|G|^{1-\alpha}}{(2\varepsilon - 1)^2}.$$

As the following example shows, the thresholds of $\alpha = 1/2, \varepsilon = 1/2$ are tight.

Example (4.3.1)[177]: Let $G = \mathbb{Z}_2^{2n}$ and $A = (\mathbb{Z}_2^n \times \{0^n\}) \cup (\{0^n\} \times \mathbb{Z}_2^n)$. Then $|A| = 2|G|^{1/2} - 1, Spec_{1/2}(A) = A$ and $A + A = G$.

So, it seems that such structural results are hopeless when $\alpha, \varepsilon < 1/2$. However, there is still hope: in the example, if we restrict to a large subset $A = \mathbb{Z}_2^n \times \{0^n\} \subseteq A$, then $Spec_{1/2}(A) = \{0^n\} \times \mathbb{Z}_2^n$ is a subgroup, and specifically the size of $Spec_{1/2}(A) + Spec_{1/2}(A)$ is bounded away from the entire group. Our first result is that this is true in general. In fact, the size of the sum set is close to the bound given by Parseval's identity, which is approximately $|G|^{1-\alpha}$.

A more refined notion of structure is that of bounded doubling. Here, we say that a set Γ has a doubling constant K if $|\Gamma + \Gamma| \leq K|\Gamma|$. Note that if $|Spec_\varepsilon(A)|$ has size close to the bound given by Parseval's identity, which is roughly $|G|^{1-\alpha}$, then Theorem (4.3.3) would show that $Spec_\varepsilon(A)$ has a small doubling constant $K = C|G|^\delta$. We conjecture that this is always the case. However, we could only show it if we are allowed to change the value of ε somewhat. We state both the theorem and the conjecture below.

Conjecture (4.3.2)[177]: Fix $0 < \delta < \alpha < 1/2$ and $0 < \varepsilon < 1/2$. Let $A \subseteq G$ of size $|A| \geq |G|\,\alpha$. Then there exists a subset $A' \subseteq A$ of size $|A'| \geq |A|/C$ such that

$$|Spec_\varepsilon(A') + Spec_\varepsilon(A')| \leq C|G|^\delta \cdot |Spec_\varepsilon(A')|,$$

where $C = C(\varepsilon, \delta)$.

We use big-O notation. For two quantities $x, \mathcal{Y}$, the expression $x = O(\mathcal{Y})$ means $x \leq c\mathcal{Y}$ for an unspecified absolute constant $c > 0$. We also use c, c', c_1, etc. to denote unspecified absolute constants, where the big-O notation may be confusing. The value of these may change between different instantiations of them. We make no effort to optimize constants. Also we use E as average operator, i.e., $\mathbb{E}_a \in A_f = \frac{1}{|A|} \sum_{a\in A} f(a)$.

We prove Theorem (4.3.3) and Theorem (4.3.13).

Theorem (4.3.3)[177]: Fix $0 < \delta < \alpha < 1/2$ and $0 < \varepsilon < 1/2$. Let $A \subseteq G$ of size $|A| \geq |G|\,\alpha$. Then there exists a subset $A \subseteq A$ of size $|A| \geq |A|/C$ such that

$$|Spec_\varepsilon(A') + Spec_\varepsilon(A)| \leq (1/\varepsilon)^{O(1/\delta)} \cdot \frac{|G|^{1+\delta}}{|A'|}$$

where $C \leq exp((1/\varepsilon)^{O(1/\delta)})$.

Proof: We begin by introducing some notation. For $A \subseteq G$ and $\Gamma \subseteq \hat{G}$, define an $|A| \times |\Gamma|$ complex matrix $M = M(A, \Gamma)$, with rows indexed by A and columns by Γ, as follows. First, denote by $\gamma(A) := \mathbb{E}_{a\in A}[\gamma(a)]$ the average value of the character γ on A. Define

$$M_{a,\gamma} := \gamma(a)\frac{\overline{\gamma(A)}}{|\gamma(A)|}.$$

With this definition, we have that for any $\Gamma \subseteq Spec_\varepsilon(A)$,

$$|1_A^T M(A,\Gamma)1_\Gamma| = \sum_{\gamma\in\Gamma} \left|\sum_{a\in A} \gamma(a)\right| \geq \varepsilon|A||\Gamma|. \quad (106)$$

We next define a notion of regularity for $M(A, \Gamma)$.

Definition (4.3.4) (Regularity for $M(A,\Gamma)$)[177]: Let $A \subseteq G, \Gamma \subseteq G$. The matrix $M = M(A,\Gamma)$ is called λ-regular if for every pair of functions $f : A \to C, g : \Gamma \to C$ such that $\langle f, 1_A \rangle = 0$ or $\langle g, 1_\Gamma \rangle = 0$ or both, it holds that
$$|f^T M g| < \lambda \|f\|_\infty \|g\|_\infty |A||\Gamma|.$$
It is conventional to use the L_2-norm in definition of regularity, however in our case, the use of L_∞-norm makes the argument more straightforward and gives better bounds. The argument informally goes as follows. We divide into two cases. First, we show if $M = M(A, Spec_\varepsilon(A))$ is λ-regular for a suitable choice of λ, then $Spec_\varepsilon(A)$ has bounded doubling. Otherwise, if M is not λ-regular, we find large subsets $A' \subseteq A, \Gamma' \subseteq Spec_\varepsilon(A)$ such that $M(A', \Gamma')$ has higher average. This allows us to revert to study $M(A', Spec_\varepsilon(A'))$ where $\varepsilon' = \varepsilon + \lambda^{O(106)}$ and iterate.
First, we analyze the case where M is regular.

Lemma (4.3.5)[177]: Fix some $0 < \varepsilon, \rho < 1$ and $\Gamma \subseteq Spec_\rho(A)$. If $M = M(A,\Gamma)$ is $\varepsilon\rho/150$-regular, then for any $\gamma \in Spec_\varepsilon(A)$, there is a subset $\Gamma_\gamma \subseteq \Gamma, |\Gamma_\gamma| \geq 0.9|\Gamma|$ such that
$$\gamma + \Gamma_\gamma \subset Spec_{\varepsilon\rho/2}(A).$$
Proof. Suppose towards contradiction that there is some $\gamma_\circ \in Spec_\varepsilon(A)$ for which the claim does not hold. That is, there exists a subset $\Gamma' \subseteq \Gamma$ of size $|\Gamma'| > 0.1|\Gamma|$ such that $\forall \gamma' \in \Gamma'$,
$$\gamma_\circ + \gamma' \notin Spec_{\varepsilon\rho/2}(A).$$
Define a pair of functions $f : A \to \mathbb{C}$ and $g : \Gamma \to \mathbb{C}$ by
$$f(a) = \gamma_\circ(a),$$
$$g(\gamma) = \frac{|\Gamma|}{|\Gamma'|} 1_{\Gamma'}(\gamma).$$
We have
$$f^T M g = \sum_{\gamma\in\Gamma} \left[\sum_{a\in A} \gamma_\circ(a)\gamma(a) \frac{\overline{\gamma(A)}}{|\gamma(A)|} \frac{|\Gamma|}{|\Gamma'|} 1_{\Gamma'}(\gamma)\right]$$
$$= \frac{|\Gamma|}{|\Gamma'|} \sum_{\gamma\in\Gamma} \frac{\overline{\gamma(A)}}{|\gamma(A)|} \sum_{a\in A} \gamma_\circ(a)\gamma(a) 1_{\Gamma'}(\gamma)$$
$$= \frac{|\Gamma|}{|\Gamma'|} \sum_{\gamma'\in\Gamma'} \frac{\overline{\gamma'(A)}}{|\gamma'(A)|} \sum_{a\in A} (\gamma_\circ + \gamma')(a).$$
By our assumption, $\forall \gamma' \in \Gamma, \gamma_\circ + \gamma' \notin Spec_{\varepsilon\rho/2}(A)$. Therefore
$$|f^T M g| \leq (\varepsilon\rho/2) \cdot |\Gamma||A|.$$
Decompose f as $f = f_1 + f_2$ with $f_1 = \mathbb{E}_{a\in A}[f(a)] \cdot 1_A$ and g as $g = g_1 + g_2$ with $g_1 = \mathbb{E}_{\gamma\in\Gamma}[g(\gamma)] \cdot 1_\Gamma = 1_\Gamma$. Then
$$f^T M g = f_1^T M g_1 + f_2^T M g_1 + f_1^T M g_2 + f_2^T M g_2. \quad (107)$$
We have that $\langle f_2, 1_A \rangle = 0, \langle g_2, 1_\Gamma \rangle = 0$ and
$$|f_1^T M g_1| = |\mathbb{E}_{a\in A} f(a) \cdot (1_A^T M 1_\Gamma)| \geq |\mathbb{E}_{a\in A}[\gamma \circ (a)]| \cdot \rho|\Gamma||A| \geq \varepsilon\rho|\Gamma||A|.$$
We show that the other terms in Equation (107) are too small to cancel out the contribution of $f_1^T M_{g_1}$. Consequently, we reach a contradiction.
In each one of the terms $f_1^T M g_2, f_2^T M g_1, f_2^T M g_2$ at least one of the functions are orthogonal to the identity function. Therefore, we can bound the size of these terms using

the $\frac{\varepsilon\rho}{150}$ -regularity assumption. We have $||f_1||_\infty \le 1, ||f_2||_\infty \le 2, ||g_1||_\infty \le 1, ||g_2||_\infty \le 10$, and hence

$$|f_2^T\, Mg_1 + f_1^T\, Mg_2 + f_2^T\, Mg_2\,| \le (20 + 10 + 20)\cdot(\varepsilon\rho/150)|A||\Gamma| = (\varepsilon\rho/3)|A||\Gamma|.$$

This implies that $|f^T Mg| \ge \frac{2}{3}\,\varepsilon\rho|A||\Gamma|$, which is a contradiction.

Next, we show how to use Lemma (4.3.5) to infer that if $M = M(A, Spec_\rho(A))$ is $\frac{\varepsilon\rho}{150}$ -regular then $|Spec_\varepsilon(A) - Spec_\varepsilon(A)|$ is small as long as $|Spec_{\varepsilon\rho}/2(A)| \approx |Spec_\rho(A)|$.

Lemma (4.3.6)[177]: If $M = M(A, Spec_\rho(A))$ is $\frac{\varepsilon\rho}{150}$ -regular, then

$$|Spec_\varepsilon(A) - Spec_\varepsilon(A)| \le 2\ \frac{\left|Spec_{\varepsilon\rho/2}(A)\right|^2}{\left|Spec_\rho(A)\right|}.$$

Proof. Fix arbitrary $\gamma_1, \gamma_2 \in Spec_\varepsilon(A)$. By Lemma (4.3.5) there exist sets $\Gamma_1, \Gamma_2 \subseteq Spec_\rho(A)$ of size $|\Gamma_1|, |\Gamma_2| \ge 0.9|Spec_\rho(A)|$ such that $\gamma_1 + \Gamma_1, \gamma_2 + \Gamma_2 \subseteq Spec_{\varepsilon\rho/2}(A)$. For any $\gamma \in \Gamma_1 \cap \Gamma_2$ we can then write

$$\gamma_1 - \gamma_2 = (\gamma_1 + \gamma) - (\gamma_2 + \gamma)$$

where $\gamma_1 + \gamma, \gamma_2 + \gamma \in Spec_{\varepsilon\rho/2}(A)$. This gives $|\Gamma_1 \cap \Gamma_2| \ge 0.8|Spec_\rho(A)|$ distinct ways to write $\gamma_1 - \gamma_2$ as the difference of a pair of elements in $Spec_{\varepsilon\rho/2}(A)$. Consequently

$$|Spec_\varepsilon(A) - Spec_\varepsilon(A)| \le \frac{\left|Spec_{\varepsilon\rho/2}(A)\right|^2}{|\Gamma_1 \cap \Gamma_2|} \le \frac{\left|Spec_{\varepsilon\rho/2}(A)\right|^2}{0.8\left|Spec_\rho(A)\right|}$$

Next, we consider the case that the matrix M is not λ-regular for $\lambda = \varepsilon\rho/150$. In the following we denote $\mathbb{E}[M] := \mathbb{E}_{a,\gamma}[M_{a,\gamma}]$.

Assuming that $M = M(A, \Gamma)$ is not λ-regular, there are functions $f : A \to \mathbb{C}$ and $g : \Gamma \to \mathbb{C}$ with $||f||_\infty = ||g||_\infty = 1$, at least one of which is orthogonal to the identity function, such that $|f^T Mg| \ge \lambda|A||\Gamma|$. As a first step towards proving Lemma (4.3.8), we approximate f, g by step functions $\tilde{f}$ and $\tilde{g}$, respectively.

Claim (4.3.7)[177]: Fix $\eta > 0$. Let $f : A \to C$ be a function with $||f||_\infty = 1$. Then there exists a function $\tilde{f}: A \to C$ such that

$$||f - \tilde{f}||_\infty \le \eta$$

with $\tilde{f} = \sum_{i=1}^{k} \alpha_i 1_{A_i}$, where $A_i \subseteq A$ are disjoint subsets and $\alpha_i \in \mathbb{C}$ with $|\alpha_i| \le 1$. Moreover, $k \le \frac{100}{\eta^2}$.

Proof. We partition A based on the phase and magnitude of f. For $r = \lceil 10/\eta \rceil$ define

$$A_{j,k} = \{a \in A : j/r < |f(a)| \le (j + 1)/r \text{ and } 2\pi k/r < \arg f(a) \le 2\pi(k + 1)/r\}.$$

We partition A to subsets $A_{j,k}$ for $j, k \in \{0, \dots, r - 1\}$. Define the step function $\tilde{f}$ as

$$\tilde{f} = \sum_{j,k=0}^{r-1} j/r \cdot e^{(2\pi i)k/r} \cdot 1A_{j,k}.$$

It is easy to verify that for all $a \in A, |f(a) - \tilde{f}(a)| \le \eta$ as claimed.

We proceed with the proof of Lemma (4.3.8).

Lemma (4.3.8)[177]: If $M = M(A, \Gamma)$ is not λ-regular, then there exist subsets $A \subseteq A, \Gamma \subseteq \Gamma$ such that

$$|\mathbb{E}\,[M(A',\Gamma')]| \geq |\mathbb{E}\,[M(A,\Gamma)]| + c\lambda^{15},$$

where $|A'| \geq c\lambda^{15}|A|, |\Gamma'| \geq c\lambda^{15}|\Gamma|$, and $c > 0$ is an absolute constant.

Proof: Let $\rho := \mathbb{E}[M']$ be the average of M, and define a matrix M by $M'_{a,\gamma} = M_{a,\gamma} - \rho$, so that $\mathbb{E}[M'] = 0$. Note that $|M'_{a,\gamma}| \leq 2$ for all $a \in A, \gamma \in \Gamma$. We may assume for simplicity that ρ is real and nonnegative, by multiplying all entries of M by an appropriate phase $e^{i\theta}$, as this does not change any of the properties at hand.

As we assume M is not λ-regular, there exist functions $f : A \to \mathbb{C}, g : \Gamma \to \mathbb{C}$ with $||f||_\infty, ||g||_\infty = 1$, one of which at least sums to zero, such that $||f^T Mg| \geq \lambda|A||\Gamma|$. Note that $f^T M' g = f^T Mg$. Let $\tilde{f}, \tilde{g}$ be their step function approximations given by Claim (4.3.7) for $\eta = \lambda/8$, where $\tilde{f} = \sum_{i=1}^{k} \alpha_i 1_{A_i}, \tilde{g} = \sum_{i=1}^{k} \beta_i 1_{\Gamma_i}$ and $k \leq \frac{100}{\eta^2}$. Moreover

$$|\tilde{f}^T M\,\tilde{g}| \geq |f^T M' g| - |(f - \tilde{f})\,TM' g| - |\tilde{f}^T M' (g - \tilde{g})| \geq \lambda/2 \cdot |A||\Gamma|.$$

That is,

$$\left|\sum_{i,j=1}^{k} \alpha_i\beta_j 1_{A_i}^T M' 1_{\Gamma_j}\right| \geq \lambda/2 \cdot |A||\Gamma|.$$

In particular, there must exist A_i, Γ_j such that

$$|1_{A_i}^T M' 1_{\Gamma_j}| \geq (\lambda/2k^2) \cdot |A||\Gamma| \geq c_1\lambda^5 \cdot |A||\Gamma|,$$

where $c_1 > 0$ is an absolute constant.

If we knew that $1_{A_i}^T M' 1_{\Gamma_j}$ is real and nonnegative, say, then we would be done by choosing $A' = A_i, \Gamma' = \Gamma_j$ as then $\mathbb{E}[M(A',\Gamma')] \geq \rho + c_1\lambda^5$. However, it may be that its real part is negative, canceling the average. To overcome this, we consider choosing $A' \in \{A_i, A_i^c\}, \Gamma' \in \{\Gamma_j, \Gamma_j^c\}$ (where $A_i^c = A \setminus A_i, \Gamma_j^c = \Gamma \setminus \Gamma_j$) and show that one of the choices satisfies the required properties. Set

$$\alpha_1 := 1_{A_i}^T M' 1_{\Gamma_j}, \alpha_2 := 1_{A_c^i}^T M' 1_{\Gamma_j}, \alpha_3 := 1_{A_i}^T M' 1_{\Gamma_j^c}, \alpha_4 := 1_{A_i^c}^T M' 1_{\Gamma_j^c}$$

and

$$\beta_1 := |A_i||\Gamma_j|, \beta_2 := |A_i^c||\Gamma_j|, \beta_3 := |A_i||\Gamma_j^c|, \beta_4 := |A_i^c||\Gamma_j^c|.$$

Fix $\delta = c\lambda^{15}$ for an absolute constant $c > 0$ to be chosen later. We will show that for some $i \in \{1,2,3,4\}$, we have $|\beta_i| \geq \delta|A||\Gamma|$ and $|\alpha_i + \rho\beta_i| \geq (\rho + \delta)\beta_i$. This implies that if we take A, Γ to be the corresponding sets, then $|A'| \geq \delta|A|, |\Gamma'| \geq \delta|\Gamma|$ and $|1_A M 1_{\Gamma'}| = |\alpha_i + \rho\beta_i| \geq (\rho + \delta)|A'||\Gamma'|$.

In order to show that, let us note that $\sum\alpha_i = 0, |\alpha_1| \geq c_1\lambda^5|A||\Gamma|, \beta_1 \geq c_1\lambda^5|A||\Gamma|$, and the β_i are real nonnegative numbers with $\sum\beta_i = |A||\Gamma|$. If for some i we have $Re(\alpha_i) \geq \delta|A||\Gamma|$ then $|\alpha_i + \rho\beta_i| \geq Re(\alpha_i + \rho\beta_i) \geq \delta|A||\Gamma| + \rho\beta_i \geq (\rho + \delta)\beta_i$ and we are done. If $Re(\alpha_i) \leq -\delta|A||\Gamma|$ then, since $\sum \alpha_i = 0$, there exists some $j \neq i$ for which $Re(\alpha_j) \geq \delta/3 \cdot |A||\Gamma|$, and we are done by the previous argument. So, we may assume that $|Re(\alpha_i)| \leq \delta|A||\Gamma|$ for all i. In particular $|Re(\alpha_1)| \leq (\delta/c_1\lambda^5)\beta_1$. Hence

$$|\alpha_1 + \rho\beta_1|^2 = |\rho\beta_1 + Re(\alpha_1)|^2 + Im(\alpha_1)^2 \geq \rho^2\beta_1^2 + |\alpha_1|^2 - 2\rho\beta_1|Re(\alpha 1)|$$
$$\geq \beta_1^2(\rho^2 + c_1^2\lambda^{10} - 2\delta/c_1\lambda^5) \geq \beta_1^2(\rho^2 + (c_1^2 - 2c/c_1)\lambda^{10}),$$

where we used our choice of $\delta = c\lambda^{15}$. If we choose $c > 0$ small enough, we conclude that also in this case, $|\alpha_1 + \rho\beta_1| \geq (\rho + \delta)\beta_1$. Note that the condition $\beta_i \geq c_1\lambda^5|A||\Gamma|$ is automatically satisfied for all i, by making sure, let's say, $|A_i| \leq |A|/2$ and $|\Gamma_j| \leq |\Gamma|/2$.

We now combine Lemma (4.3.6) and Lemma (4.3.8) in order to prove Theorem (4.3.3). The high level idea is the following. Initialize $\rho = \varepsilon, \Gamma = Spec_{\varepsilon}(A)$. If $M(A,\Gamma)$ is λ-regular for $\lambda = \varepsilon\rho/150$, and $|Spec_{\varepsilon\rho/2}(A)| \approx |\Gamma|$, then the proof follows from Lemma (4.3.6) and Parseval's identity. Otherwise, one of two cases must occur. The first case that could occur is that $M(A,\Gamma)$ is not λ −regular. Then by Lemma (4.3.8) we can replace A,Γ with A', Γ' and increase ρ by a noticeable amount. This cannot occur too many times, as $\rho \le 1$. The second case that could occur is that $|Spec_{\frac{\varepsilon\rho}{2}}(A)| \gg |\Gamma| \approx Spec_{\rho}(A)$. In such a case, we set $\rho \to \varepsilon\rho/2$ and increase the spectrum of A by a noticeable amount. As the spectrum is bounded by $|G|$, this again cannot happen too many times. Combining these steps together requires a somewhat delicate balance act.

Let $K = K(\varepsilon,\delta)$ be a parameter to be optimized later. We define a sequence of sets $A_i \subseteq A$ and parameters $\rho_i \in [0,1]$ for $i \ge 1$, where initially $A_0 = A, \rho_0 = \varepsilon$. Given A_i, ρ_i set $\lambda_i = \varepsilon\rho_i/150$ and run the following procedure:

(i) If $M(A_i, Spec_{\rho_i}(A_i))$ is λ_i-regular and $|Spec\varepsilon_{\frac{\rho_i}{2}}(A_i)| \le K|Spec_{\rho_i}(A_i)|$, then set $A^* = A_i$ and finish.

(ii) If $M(A_i, Spec_{\rho_i}(A_i))$ is not λ_i-regular then apply Lemma (4.3.8) to A_i and $Spec_{\rho_i}(A_i)$. Let $A' \subseteq A_i, \Gamma' \subseteq Spec_{\rho_i}(A_i)$ be the resulting sets such that $|A'| \ge c\lambda_i^{15}|A_i|, |\Gamma'| \ge c\lambda_i^{15}|\Gamma_i|$ and $|\mathbb{E}[M(A,\Gamma)]| \ge \rho_i + c\lambda_i^{15}$. Set $A_{i+1} = A'$ and $\rho_{i+1} = \rho_i + \left(\frac{c}{2}\right)\lambda_i^{15}$. Return to step (i).

(iii) If $|Spec\varepsilon_{\frac{\rho_i}{2}}(A_i)| > K|Spec_{\rho_i}(A_i)|$ then set $A_{i+1} = A_i$ and $\rho_{i+1} = \varepsilon\rho_i/2$. Return to step (i).

Next, we analyze this procedure. First, note that if the procedure ends with $A^* = A_i$ then by Lemma (4.3.6) and Parseval's identity we have that

$$|Spec_{\varepsilon}(A^*) - Spec_{\varepsilon}(A^*)| \le 2K\left|Spec\varepsilon_{\frac{\rho_i}{2}}(A_i)\right| \le \frac{8K|G|}{\varepsilon^2\rho_i^2}|A_i| \ . \quad (108)$$

So, we need to show that $\rho_i, |A_i|$ are never too small. Suppose that stages (ii) and (iii) occur k_1 and k_2 times, respectively. Let $\eta : \{1,\dots,k_2\} \to \{1,\dots,k_1 + k_2\}$ be the ordered indices of occurrences of stage (iii). We first bound k_1.

Claim (4.3.9)[177]: If $i < \eta(j)$ then $\rho_i \ge \left(\frac{\varepsilon}{2}\right)^j$.

Proof. The value of ρ_i increases in step (ii), and decreases in step (iii) by a factor of $\varepsilon/2$. If $i < \eta(j)$ then we applied step (iii) at most $j-1$ times, hence $\rho_i \ge (\varepsilon/2)^{j-1}\rho_0 \ge (\varepsilon/2)^j$.

Claim (4.3.10)[177]: For $\forall j \in \{1,\dots,k_2 - 1\}, |\eta(j+1) - \eta(j)| \le (1/\varepsilon)^{O(j)}$.

Proof. Consider a step i for $\eta(j) \le i \le \eta(j+1)$. We have that $\rho_{i+1} \ge \rho_i + \left(\frac{c}{2}\right)\left(\frac{\rho_i\varepsilon}{150}\right)^{15} \ge \rho_i + c'\varepsilon^{15(j+2)}$, where $c, c' > 0$ are absolute constants. As ρ_i never exceeds 1 for all i, this process cannot repeat more than $\left(\frac{1}{c}\right)\left(\frac{1}{\varepsilon}\right)^{15(j+2)}$ times. As we assume $\varepsilon <$ ½, this is bounded by $(1/\varepsilon)^{c'j}$ for a large enough $c' > 0$.

Corollary (4.3.11)[177]: $k_1 \le (1/\varepsilon)^{O(k_2)}$.

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Proof. By Claim (4.3.10), $k_1 \le \sum_{j=1}^{k_2} (1/\varepsilon)^{O(j)} \le (1/\varepsilon)^{O(k_2)}$.

We next upper bound k_2. To do so, we will show that in step (ii) we have that $Spec_{\rho_{i+1}}(A_{i+1})$ is not much smaller than $Spec_{\rho_i}(A_i)$.

Claim (4.3.12)[177]: Assume that we run step (ii) in iteration i. Then
$$|A_{i+1}| \geq c\lambda_i^{15}|A_i|$$
and
$$|Spec_{\rho_{i+1}}(A_{i+1})| \geq c\lambda_i^{30}|Spec_{\rho_i}(A_i)|,$$
where $c > 0$ is an absolute constant.

Proof. We apply in step (ii) Lemma (4.3.8) to $A_i, Spec_{\rho_i}(A_i)$. We get subsets $A_{i+1} \subseteq A_i, \Gamma' \subseteq Spec_{\rho_i}(A_i)$ such that $|A_{i+1}| \geq c\lambda_i^{15}|A_i|, |\Gamma'| \geq c\lambda_i^{15}|Spec_{\rho_i}(A_i)|$ and $\rho_{i+1} \leq |\mathbb{E}[M(A_{i+1}, \Gamma')]| - (c/2)\lambda_i^{15}$. Let $S = \Gamma \cap Spec_{\rho_{i+1}}(A_{i+1})$. Then
$$|\mathbb{E}[M(A_{i+1}, \Gamma')]| \leq \frac{|S|}{|\Gamma'|} + \left(1 - \frac{|S|}{|\Gamma'|}\right)\rho_{i+1}.$$
Hence $|Spec_{\rho_{i+1}}(A_{i+1})| \geq |S| \geq (c/2)\lambda_i^{15}|\Gamma|$ and the claim follows.

Combining Claim (4.3.10) and Claim (4.3.12), we deduce that, for any $j \in \{1, \ldots, k_2 - 1\}$, the ratio in the size of the spectrums immediately after the j-th application of step (iii), and immediately before the $j + 1$ application of step (iii), is lower bounded by
$$T_j := \frac{\left|Spec_{\rho_{\eta(j)}}(A_{\eta(j)})\right|}{\left|Spec_{\rho_{\eta(j+1)-1}}(A_{\eta(j+1)-1})\right|} \leq \prod_{i=\eta(j)}^{\eta(j+1)-2} \frac{1}{c\lambda_i^{30}} \leq \left(\frac{1}{c}\left(150 \cdot \frac{2^j}{\varepsilon^{j+1}}\right)^{30}\right)^{\eta(j+1)-\eta(j)}$$
$$\leq (1/\varepsilon)^{O(j\cdot(1/\varepsilon)^{O(j)})} \leq exp\left((1/\varepsilon)^{O(j)}\right).$$
We will choose K large enough so that $T_j \leq K^{1/2}$ for all $K < k_2$, and hence
$$|Spec_{\rho_{\eta(j+1)}}(A_{\eta(j+1)})| \geq K \cdot |Spec_{\rho_{\eta(j+1)-1}}(A_{\eta(j+1)} - 1)| \geq K^{\frac{1}{2}} \cdot |Spec_{\rho_{\eta(j)}}(A_{\eta(j)})|.$$
Fix $K = |G|^\delta$ and $C = exp((1/\varepsilon)^{O(1/\delta)})$. We may assume that $|G| \geq C$, as otherwise our bounds are trivial. Then, we must have $k_2 \leq 2/\delta$ and hence $k_1 \leq (1/\varepsilon)^{O(1/\delta)}$. We conclude that
$$\frac{|A|}{|A^*|} \leq \prod_{i=1}^{k_1+k_2} \frac{1}{c\lambda_i^{15}} \leq exp\left((1/\varepsilon)^{O(1/\delta)}\right)$$
and that plugging these estimates into Equation (108) implies that
$$|Spec_\varepsilon(A^*) - Spec_\varepsilon(A^*)| \leq (1/\varepsilon)^{O(1/\delta)} \cdot |G|\, 1 + \delta/|A^*|.$$
Since the definition of the spectrum is symmetric, $Spec_\varepsilon(A^*) = -Spec_\varepsilon(A^*)$, this implies the same bounds on $|Spec_\varepsilon(A^*) + Spec_\varepsilon(A^*)|$.

Theorem (4.3.13)[177]: Fix $0 < \delta < \alpha < 1/2$ and $0 < \varepsilon < 1/2$. Let $A \subseteq G$ of size $|A| \geq |G|^\alpha$. Then there exists a subset $A' \subseteq A$ of size $|A'| \geq |A|/C$ and $\varepsilon' \geq \varepsilon^{21/\delta}$ such that
$$|Spec_\varepsilon(A')| \geq |Spec_\varepsilon(A)|/C$$
and
$$|Spec_{\varepsilon'}(A') + Spec_{\varepsilon'}(A')| \leq C|G|^\delta \cdot |Spec_{\varepsilon'}(A')|,$$
where $C \leq exp\left((1/\varepsilon)^{O(2^{4/\delta})}\right)$.

Proof: The proof of Theorem (4.3.13) is very similar to the proof of Theorem (4.3.3), with a few small tweaks. First, we use Lemma (4.3.5) and Lemma (4.3.6) in the special case of $\rho = \varepsilon$. We restate Lemma (4.3.6) in this special case.

Lemma (4.3.14)[177]: If $M = M(A, Spec_\varepsilon(A))$ is $\frac{\varepsilon^2}{150}$ -regular, then

$$|Spec_\varepsilon(A) - Spec_\varepsilon(A)| \le 2 \frac{\left|Spec_{\varepsilon^2/2(A)}\right|^2}{|Spec_\varepsilon(A)|}.$$

We combine Lemma (4.3.14) with Lemma (4.3.8) to prove Theorem (4.3.13). The difference is in the iterative refinement process. Here, instead of setting $\lambda_i = \varepsilon\rho_i/150$, we instead set $\lambda_i = \rho_i^2/150$. To, initialize $\Gamma = Spec_\varepsilon(A)$. If $M(A,\Gamma)$ is λ-regular for $\lambda = \varepsilon^2/150$, and $|Spec_{\varepsilon^2/2}(A)| \approx |\Gamma|$, then the proof follows from Lemma (4.3.14) and Parseval's identity. Otherwise, one of the following two cases must occur. The first case that could occur is that $M(A,\Gamma)$ is not λ-regular. In this case, by Lemma (4.3.8) we can replace A, Γ with A', Γ' and increase ε by a noticeable amount. This can not occur many times as $\varepsilon \le 1$. The other case that can occur is that $|Spec_{\varepsilon^2/2}(A)| \gg |\Gamma| \approx Spec_\varepsilon(A)$. In this case, we set $\varepsilon = \varepsilon^2/2$ and increase the spectrum of A. Since the spectrum is bounded by $|G|$, this also can not occur too many times. In the following we formalize this high level argument.

Let $K = K(\varepsilon,\delta)$ be a parameter to be optimized later. Define a sequence of sets $A_i \subseteq A$ and parameters $\rho_i \in [0,1]$ for $i \ge 1$, and initialize $A_0 = A$ and $\rho_0 = \varepsilon$. Recall that δ is a parameter, chosen so that the final doubling constant is bounded by $|G|\ \delta$. Given A_i, ρ_i set $\lambda_i = \rho_i^2/150$ and run the following procedure:

(i) If $M(A_i, Spec_{\rho_i}(A_i))$ is λ_i-regular and $|Spec_{\rho_i^2/2}(A_i)| \le K|Spec_{\rho_i}(A_i)|$, then set $A^* = A_i$ and finish.

(ii) If $M(A_i, Spec_{\rho_i}(A_i))$ is not λ_i-regular then apply Lemma (4.3.8) to $A_i, Spec_{\rho_i}(A_i)$. Let $A' \subseteq A_i, \Gamma' \subseteq Spec_{\rho_i}(A_i)$ be the resulting sets such that $|A'| \ge c\lambda_i^{15}|A_i|, |\Gamma| \ge c\lambda_i^{15}|\Gamma_i|$ and $|\mathbb{E}[M(A',\Gamma')]| \ge \rho_i + c\lambda_i^{15}$. Set $A_{i+1} = A'$ and $\rho_{i+1} = \rho_i + (c/2)\lambda_i^{15}$.

(iii) If $|Spec_{\rho_i^2/2}(A_i)| > K|Spec\rho_i(A_i)|$ then set $A_{i+1} = A_i$ and $\rho_{i+1} = \rho_i^2/2$.

The analysis of this procedure is similar to the analysis of the procedure in the proof of Theorem (4.3.3). First note that if the procedure ends with $A^* = A_i$ and $\varepsilon^* = \rho_i$ then by Lemma (4.3.14) we have that

$$|Spec_{\varepsilon^*}(A^*) - Spec_{\varepsilon^*}(A^*)| \le 2K|Spec_{\varepsilon^{*2}/2}(A^*)| \le 2K^2|Spec_{\varepsilon^*}(A^*)|. \quad (4)$$

Therefore, we need to show that ε^* and $|A^*|$ are not too small. Suppose that stages (ii) and (iii) occur k_1 and k_2 times, respectively. Let $\eta : \{1,\cdots,k_2\} \to \{1,\cdots,k_1 + k_2\}$ be the ordered indices of occurrences of stage (iii). We first bound k_1.

Claim (4.3.15)[177]: If $i < \eta(j)$ then $\rho_i \ge (\varepsilon/2)^{2^j}$.

Proof. The value of ρ_i increases in step (ii), and decreases in step (iii). If $i < \eta(j)$ then we applied step (iii) at most $j - 1$ times, hence $\rho_i \ge (\varepsilon/2)^{2^j}$.

Claim (4.3.16)[177]: For $\forall j \in \{1,\ldots,k_2 - 1\}, |\eta(j+1) - \eta(j)| \le (1/\varepsilon)^{O(2^j)}$.

Proof. Consider a step i for $\eta(j) \le i \le \eta(j+1)$. We have that $\rho_i + 1 \ge \rho_i + c\left(\rho_i^2\right)^{15} \ge \rho_i + c((\varepsilon/2)^{30\cdot 2^j})$. As ρ_i never exceeds 1 for all i, this process cannot repeat more than $(1/c)(2/\varepsilon)^{30\cdot 2^j}$ times.

Corollary (4.3.17)[177]: $k_1 \le (1/\varepsilon)^{O(2^{k_2})}$.

Proof. By Claim (4.3.16), $k_1 \le \sum_{j=1}^{k_2} (1/\varepsilon)^{O(2^j)} \le (1/\varepsilon)^{O(2k_2)}$.

We next upper bound k_2. To do so, we will show that in step (ii) we have that $Spec_{\rho_{i+1}}(A_{i+1})$ is not much smaller than $Spec_{\rho_i}(A_i)$. We restate Claim (4.3.12) which was proved before.

Claim (4.3.18)[177]: Assume that we run step (ii) in iteration i. Then

$$|A_{i+1}| \geq c\lambda_i^{15} \cdot |A_i|$$

and

$$|Spec_{\rho_{i+1}}(A_{i+1})| \geq c\lambda_i^{30} \cdot |Spec_{\rho_i}(A_i)|.$$

As in the proof of Theorem (4.3.3), if we combine Claim (4.3.16) and Claim (4.3.18), then for any $j \in \{1, \dots, k_2 - 1\}$, the ratio in the size of the spectrums immediately after the j-th application of step (iii), and immediately before the $j + 1$ application of step (iii), is lower bounded by

$$T_j := \frac{\left|Spec_{\rho\eta(j)}\left(A_{\eta(j)}\right)\right|}{\left|Spec_{\rho_{\eta(j+1)-1}}\left(A_{\eta(j+1)-1}\right)\right|} \leq exp\,(1\varepsilon)^{O(2^j)} .$$

We will choose K large enough so that $T_j \leq K^{\frac{1}{2}}$ for all $j < k_2$, and hence $\left|Spec_{\rho_{\eta(j+1)}}\left(A_{\eta(j+1)}\right)\right| \geq K \cdot \left|Spec_{\rho_{\eta(j+1)-1}}\left(A_{\eta(j+1)-1}\right)\right| \geq K^{1/2} \cdot \left|Spec_{\rho_{\eta(j)}}\left(A_{\eta(j)}\right)\right|$.

Fix $K = |G|^{\delta/2}$ and $C = exp((1\varepsilon)^{O(24/\delta)})$. We may assume that $|G| \geq C$, as otherwise our bounds are trivial. Then we deduce that $k_2 \leq 4/\delta, k_1 \leq (2/\varepsilon)^{O(24/\delta)}$. We get that

$$\frac{|A|}{|A^*|} \leq \prod_{i=1}^{k_1+k_2} \frac{1}{c(\lambda i)^{15}} = exp\left((1/\varepsilon)^{O(2^{4/\delta})}\right)$$

and then by plugging these estimates into Equation (4) we conclude that

$$|Spec_{\varepsilon^*}(A^*) - Spec_{\varepsilon^*}(A^*)| \leq exp\left((1/\varepsilon)^{O(2^{4/\delta})}\right) |G|^{\delta} \cdot |Spec_{\varepsilon^*}(A^*)| .$$

Since the definition of the spectrum is symmetric, $Spec_{\varepsilon^*}(A^*) = -Spec_{\varepsilon^*}(A^*)$, this implies the same bounds on $|Spec_{\varepsilon^*}(A^*) + Spec_{\varepsilon^*}(A^*)|$.

Chapter 5
Trace Theoreaith, the Luzin N- and Morse–Sard Properties with Dubovitskiĭ-Sard Theorem and the Coarea Formula

We show a new trace theorem for Riesz potentials of Lorentz functions in a limiting case. Using these results, we find also some very natural approximation and differentiability properties for functions in $W_{p,1}^{k}$ with exceptional set of small Hausdorff content. We will show that Dubovitskii theorem can be generalized to the case of $W_{\text{loc}}^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$ mappings for all $k \in \mathbb{N}$ and $p > n$. Here we prove that Dubovitskiĭ's theorem can be generalized to the case of continuous mappings of the Sobolev–Lorentz class $W_{p,1}^{k}(\mathbb{R}^n, \mathbb{R}^d), p = \frac{n}{k}$ (this is the minimal integrability assumption that guarantees the continuity of mappings). In this situation the mappings need not to be everywhere differentiable and in order to handle the set of nondifferentiability points, we establish for such mappings an analog of the Luzin N-property with respect to lower dimensional Hausdorff content. Finally, we formulate and prove a bridge theorem that includes all the above results as particular cases. As a limiting case in this bridge theorem we also establish a new coarea type formula: if $E \subset \{x \in \mathbb{R}^n : rank\, \nabla v(x) \leq m\}$, then $\int_E J_m v(x)\, dx = \int_{\mathbb{R}^d} H^{n-m}\big(E \cap v^{-1}(y)\big) dH^m(y)$. The mapping v is $\mathbb{R}^d$-valued, with arbitrary d, and the formula is obtained without any restrictions on the image $v(\mathbb{R}^n)$ (such as m-rectifiability or σ-finiteness with respect to the m-Hausdorff measure). These last results are new also for smooth mappings, but are presented here in the general Sobolev context.

Section (5.1): The Sharp Case of Sobolev–Lorentz Mappings

We continue the study of the Luzin N- and Morse–Sard properties for the Sobolev mappings under minimal integrability assumptions initiated in [9]–[56], [50], see also [62]. It is very natural to restrict attention to continuous mappings, and so require from the considered function spaces that the inclusion $v \in W_p^k(\mathbb{R}^n, \mathbb{R}^d)$ should guarantee at least the continuity of v. For values $k \in \{1, \ldots, n-1\}$ it is well–known that $v \in W_p^k(\mathbb{R}^n, \mathbb{R}^d)$ is continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \leq \frac{n}{k}$. So the borderline case is $p = p_\circ = \frac{n}{k}$. It is well–known (see [62]) that $v \in W_{p_\circ}^k(\mathbb{R}^n, \mathbb{R}^d)$ is continuous if the derivatives of k-th order belong to the Lorentz space $L_{p_\circ,1}$, we will denote the space of such mappings by $W_{p_\circ,1}^{k}(\mathbb{R}^n, \mathbb{R}^d)$.

We prove the following Luzin N property with respect to Hausdorff content:

Theorem (5.1.1)[180]: Let $k \in \{1, \ldots, n\}, q \in [p_\circ, n]$, and $v \in W_{p_\circ,1}^{k}(\mathbb{R}^n, \mathbb{R}^d)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^n$ if $\mathcal{H}_\infty^q(E) < \delta$, then $\mathcal{H}_\infty^q\big(v(E)\big) < \varepsilon$. In particular, $\mathcal{H}^q\big(v(E)\big) = 0$ whenever $\mathcal{H}^q(E) = 0$.

Here $\mathcal{H}_\infty^q(E)$ is as usual the q–dimensional Hausdorff content:

$$\mathcal{H}_\infty^q(E) = \inf\left\{\sum_{i=1}^{\infty}(diam\, E_i)^q : E \subset \bigcup_{i=1}^{\infty} E_i\right\}.$$

Note that the case $k = 1$ was considered in [62], and the case $k > 1, q > p_\circ$ in [50], so we omit them and consider here only the remaining limiting case $q = p_\circ, k > 1$.

To study this limiting case, we need a new version of the Sobolev Embedding Theorem that gives inclusions in Lebesgue spaces with respect to suitably general positive measures. This result might also be interesting in its own right, and it is the main contribution. For $\beta \in (0, n)$ denote by M^{β} the space of all nonnegative Borel measures μ on $\mathbb{R}^n$ such that

$$|||\mu|||_{\beta} = \sup_{I \subset \mathbb{R}^n} \ell(I)^{-\beta} \mu(I) < \infty, \qquad (1)$$

where the supremum is taken over all n–dimensional cubic intervals $I \subset \mathbb{R}^n$ and $\ell(I)$ denotes side–length. Recall the following classical theorem proved by D.R. Adams [181] (see also, e.g., [67]).

Theorem (5.1.2)[180]: Let μ be a positive Borel measure on $\mathbb{R}^n$ and $\alpha > 0, 1 < p < q < \infty, \alpha p < n$. Then for any $f \in L_p(\mathbb{R}^n)$ the estimate

$$\int |I_{\alpha} f|^q d\mu \leq C |||\mu|||_{\beta} \cdot \|f\|_{L_p}^q \qquad (2)$$

holds with $\beta = (n - \alpha p) \frac{q}{p}$, where C depends on n, p, q, α only.

Here

$$I_{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y - x|^{n-\alpha}} dy$$

is the Riesz potential of order α. The above estimate (2) fails for the limiting case $q = p$. Namely, there exist functions $f \in L_p(\mathbb{R}^n)$ such that $I_{\alpha} f(x) = +\infty$ on some set of positive $(n - \alpha p)$–Hausdorff measure1 , see, e.g., [187]. We prove the following result for this limiting case $q = p$:

Theorem (5.1.3)[180]: Let μ be a positive Borel measure on $\mathbb{R}^n$ and $\alpha > 0, 1 < p < \infty, \alpha p < n$. Then for any $f \in L_{p,1}(\mathbb{R}^n)$ the estimate

$$\|I_{\alpha} f\|_{L_p(\mu)} \leq C |||\mu|||_{\beta}^{\frac{1}{p}} \cdot \|f\|_{L_{p,1}}, \qquad (3)$$

holds with $\beta = n - \alpha p$, where C depends on n, p, α only.

In view of the definition of the Lorentz spaces, it is sufficient to prove the above assertion for the simpler case when f coincides with the indicator function of some compact set:

Theorem (5.1.4)[180]: Let μ be a positive Borel measure on $\mathbb{R}^n$ and $\alpha > 0, 1 < p < \infty, \alpha p < n$. Then for any compact set $E \subset \mathbb{R}^n$ the estimate

$$\|I_{\alpha}(1_E)\|_{L_p(\mu)}^p \leq C |||\mu|||_{\beta} \ \ meas(E), \qquad (4)$$

holds with $\beta = n - \alpha p$, where 1_E is the indicator function of the set E and C depends on n, p, α only.

We emphasize that our proof of Theorem (5.1.4), and hence of Theorem (5.1.3), is self–contained, is independent of the previous proofs of this type of results, and uses only very natural and elementary arguments.

From the definition of the space $W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ of Sobolev–Lorentz mappings and the classical estimate $|\nabla v| \leq C \left| I_{k-1} \nabla_v^k \right|$, Theorem (5.1.3) implies

Theorem (5.1.5)[180]: Let μ be a positive Borel measure on $\mathbb{R}^n, k \in \{1, \ldots, n\}$. Then for any function $v \in W_{p_\circ,1}^k\left((\mathbb{R}^n)\right)$ the estimate

$$\int |\nabla v|^{p_\circ} d\mu \leq C |||\mu|||_{p_\circ} \cdot \left\|\nabla^k v\right\|_{L_{p_\circ,1}}^{p_\circ} \qquad (5)$$

holds, where C depends on n, k only.

From these results we deduce also some new differentiability and approximation properties of Sobolev–Lorentz mappings $v \in W_{p_\circ,1}^k(\mathbb{R}^n)$. Namely, for $m \leq n$ the m–order derivatives $\nabla^m v$ are well–defined $\mathcal{H}^{mp_\circ}$ -almost everywhere, a function v is m-times differentiable (in the classical Frechet–Peano sense) $\mathcal{H}^{mp_\circ}$ -almost everywhere, and, finally, it coincides with C^m-smooth function on $\mathbb{R}^n \backslash U$, where the open exceptional set U has small $H_\infty^{mp_\circ}$-Hausdorff content. Note that for mappings of the classical Sobolev space $W_{p_\circ}^k(\mathbb{R}^n)$ the corresponding exceptional set U has small Bessel capacity $\mathcal{B}_{k-m,p}(U) < \varepsilon$, and, respectively, the gradients $\nabla^m v$ are well-defined in $\mathbb{R}^n$ except for some exceptional set of zero Bessel capacity $\mathcal{B}_{k-m,p}$ (see [79] or [54]).

We discuss Morse–Sard type theorems for Sobolev–Lorentz mappings. Namely, for an open set $\Omega \subset \mathbb{R}^n$ and a mapping $v \in W_{p_\circ,1,loc}^k(\Omega, \mathbb{R}^n)$ denote $Z_{v,m} = \{x \in \Omega: v$ is differentiable at x and $rank\nabla v(x) < m\}$ (recall, that by previous results v is differentiable $\mathcal{H}^{p_\circ}$ a.e.). We state:

Theorem (5.1.6)[180]: If $k, m \in \{1, \dots, n\}, \Omega$ is an open subset of $\mathbb{R}^n$, and $v \in W_{p_\circ,1,loc}^k(\Omega, \mathbb{R}^n)$, then $\mathcal{H}^{q_\circ}\left(v(Z_{v,m})\right) = 0$.

Here

$$p_\circ = \frac{n}{k} \quad \text{and} \quad q_\circ = m - 1 + \frac{n-m+1}{k} = p_\circ + (m-1)(1-k^{-1})\,. \quad (6)$$

The theorem was proved for C^k-smooth functions by Morse [69] in 1939 for the case $k = n, m = d = q_\circ = 1$, and subsequently by Sard [47] in 1942 for $k = n - m + 1, m = d = q_\circ$. For arbitrary values $k, n, m \in \mathbb{N}$ and C^k-smooth functions the result was proved almost simultaneously by Dubovitskiı [183] in 1967 and Federer [61] in 1969.

The Morse–Sard Theorem for Sobolev spaces $W_p^k(\mathbb{R}^n, \mathbb{R}^m)$ with $p > n$ (i.e., when $W_p^k(\mathbb{R}^n) \hookrightarrow C^{k-1}(\mathbb{R}^n)$) was obtained in [39] (see also [48] for a simple proof), and for Lipschitz and Holder continuous mappings $C^{k,\lambda}$ see, e.g., in [36] and [53] respectively. See [9], [56], [50], where the above Theorem (5.1.6) was proved in the Sobolev context $W_{p_\circ}^k(\mathbb{R}^n)$ for $k, m \in \{2, \dots, n\}$. Since the case $k = 1$ $(i.e., q_\circ = n)$ can be considered folklore (see, e.g., [190]) we shall in only consider the cases $m = 1, k > 1, q_\circ = p_\circ = \frac{n}{k}$.

Let us end by noting an interesting phenomenon that occurs for functions of the Sobolev–Lorentz space $W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. On the one hand, the order of integrability of the k–th derivative, Lebesgues index $p_\circ$ and Lorentz index 1, is the minimal one on the Lorentz scale that guarantees continuity of mappings. On the other hand, these mappings a posteriori have many additional analytical regularity properties: the Luzin N–property, differentiability and approximation properties, and the Morse–Sard property (see above).

For instance, if $k = n - m + 1$, then almost all level sets of mappings $v \in W_{p_\circ,1}^k$ $(\mathbb{R}^n, \mathbb{R}^d)$ are C^1–smooth manifolds [50]. The result should be contrasted with the fact that mappings of class $W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^m)$ are continuous only and need not to be C^1–smooth in general. This property recently found some applications in mathematical fluid mechanics (see [188]).

By an n–dimensional cubic interval we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If Q is an n–dimensional cubic interval then we write $\ell(Q)$ for its sidelength.

For a subset S of $\mathbb{R}^n$ we write $\mathcal{L}^n(S)$ for its outer Lebesgue measure. The m– dimensional Hausdorff measure is denoted by $\mathcal{H}^m$ and the m–dimensional Hausdorff content by $\mathcal{H}_\infty^m$. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$\mathcal{H}^m(S) = \lim_{\alpha \searrow 0} \mathcal{H}_\alpha^m(S) = \sup_{\alpha > 0} \mathcal{H}_\alpha^m(S),$$

where for each $0 < \alpha \le \infty$,

$$\mathcal{H}_\alpha^m(S) = \inf\left\{\sum_{i=1}^{\infty}(diam\, S_i)^m : diam\, S_i \le \alpha, S \subset \bigcup_{i=1}^{\infty} S_i\right\}.$$

It is well known that $\mathcal{H}^n(S) = \mathcal{H}_\infty^n(S) \sim \mathcal{L}^n(S)$ for sets $\subset \mathbb{R}^n$.
To simplify the notation, we write $\|f\|_{L_p}$ instead of $\|f\|_{L_p(\mathbb{R}^n)}$, etc.

The Sobolev space $W_p^k(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$–valued functions $f \in L_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \le k$ belong to $L_p(\mathbb{R}^n)$ (for detailed definitions and differentiability properties of such functions see, e.g., [60], [67], [79], [58]). Denote by $\nabla^k f$ the vector-valued function consisting of all k–th order partial derivatives of f arranged in some fixed order. However, for the case of first order derivatives $k = 1$ we shall often think of $\nabla^f(x)$ as the Jacobi matrix of f at x, thus the $d \times n$ matrix whose r–th row is the vector of partial derivatives of the r–th coordinate function.
We use the norm

$$\|f\|_{W_p^k} = \|f\|_{L_p} + \|\nabla f\|_{L_p} + \cdots + \left\|\nabla^k f\right\|_{L_p},$$

and unless otherwise specified all norms on the spaces $\mathbb{R}^s (s \in \mathbb{N})$ will be the usual euclidean norms.
Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,loc}(\Omega)$, then the precise representative w^* is defined for all $x \in \Omega$ by

$$w^*(x) = \lim_{r \searrow 0} ⨍_{B(x,r)} w(z)dz, \text{if the limit exists and is finite, 0 otherwise,} \quad (7)$$

where the dashed integral as usual denotes the integral mean,

$$⨍_{B(x,r)} w(z)dz = \frac{1}{\mathcal{L}^n\big(B(x,r)\big)} \int_{B(x,r)} w(z)dz,$$

and $B(x,r) = \{y: |y - x| < r\}$ is the open ball of radius r centered at x. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.
We will say that x is an L_p–Lebesgue point of w (and simply a Lebesgue point when $p = 1$), if

$$⨍_{B(x,r)} |w(z) - w(x)|^p dz \to 0 \qquad as \quad r \searrow 0.$$

If $k < n$, then it is well-known that functions from Sobolev spaces $W_p^k(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \le p_\circ = \frac{n}{k}$ (see, e.g., [67], [79]). The Sobolev–Lorentz space $W_{p_\circ,1}^k(\mathbb{R}^n) \subset W_{p_\circ}^k(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space that for our purposes turns out to be convenient. Among other things functions that are locally in $W_{p_\circ,1}^k$ on $\mathbb{R}^n$ are in particular continuous.
Given a measurable function $f: \mathbb{R}^n \to \mathbb{R}$, denote by $f_*: (0, \infty) \to \mathbb{R}$ its distribution function

$$f_*(s) := \mathcal{L}^n\{x \in \mathbb{R}^n: |f(x)| > s\},$$

and by f^* the nonincreasing rearrangement of f, defined for $t > 0$ by

$$f^*(t) = \inf\{s \ge 0: f_*(s) \le t\}.$$

Since $|f|$ and f^* are equimeasurable, we have for every $1 \le p < \infty$,

$$\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}} = \left(\int_0^{+\infty} f^*(t)^p dt\right)^{\frac{1}{p}}.$$

The Lorentz space $L_{p,q}(\mathbb{R}^n)$ for $1 \leq p < \infty, 1 \leq q < \infty$ can be defined as the set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ for which the expresssion

$$\|f\|_{L_{p,q}} = \begin{cases} \left(\frac{q}{p}\int_0^{+\infty}\left(t^{\frac{1}{p}}f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & if \quad 1 \leq q < \infty \\ \sup_{t>0} t^{\frac{1}{p}}f^*(t) & if\ q = \infty \end{cases}$$

is finite. See [65], [74] or [79] for information about Lorentz spaces. However, let us remark that in view of the definition of $\|\cdot\|_{L_{p,q}}$ and the equimeasurability of f and f^* we have $\|f\|_{L_p} = \|f\|_{L_{p,p}}$ so that in particular $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. Further, for a fixed exponent p and $q_1 < q_2$ we have the estimate $\|f\|_{L_{p,q_2}} \leq \|f\|_{L_{p,q_1}}$, and, consequently, the embedding $L_{p,q_1}(\mathbb{R}^n) \subset L_{p,q_2}(\mathbb{R}^n)$ (see [65]). Finally we recall that $\|\cdot\|_{L_{p,q}}$ is a norm on $L_{p,q}(\mathbb{R}^n)$ for all $q \in [1,p]$ and a quasi–norm in the remaining cases $q \in (p,\infty]$ (see [65]).

Here we shall mainly be concerned with the Lorentz space $L_{p,1}$, and in this case one may rewrite the norm as (see for instance [65])

$$\|f\|_{p,1} = \int_0^{+\infty}[\mathcal{L}^n(\{x \in \mathbb{R}^n: |f(x)| > t\})]^{\frac{1}{p}}dt. \tag{8}$$

We record the following subadditivity property of the Lorentz norm for later use.

Lemma (5.1.7)[180]: (see, e.g., [72] or [65]). Suppose that $1 \leq p < \infty$ and $= \bigcup_{j\in\mathbb{N}} E_j$, where E_j are measurable and mutually disjoint subsets of $\mathbb{R}^n$. Then for all $f \in L_{p,1}$ we have

$$\sum_j \left\|f \cdot 1_{E_j}\right\|_{L_{p,1}}^p \leq \|f \cdot 1_E\|_{L_{p,1}}^p,$$

where 1_E denotes the indicator function of the set E.

Denote by $W_{p,1}^k(\mathbb{R}^n)$ the space of all functions $v \in W_p^k(\mathbb{R}^n)$ such that in addition the Lorentz norm $\left\|\nabla^k v\right\|_{L_{p,1}}$ is finite.

For a mapping $u \in L_1(Q,\mathbb{R}^d), Q \subset \mathbb{R}^n , m \in \mathbb{N}$, define the polynomial $P_{Q,m}[u]$ of degree at most m by the following rule:

$$\int_Q y^\alpha \left(u(y) - P_{Q,m}[u](y)\right) dy = 0 \tag{9}$$

for any multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$. Denote $P_Q[u] = P_{Q,k-1}[u]$.

The following well–known bound will be used on several occasions.

Lemma (5.1.8)[180]: Suppose $v \in W_{p_\circ,1}^k(\mathbb{R}^n,\mathbb{R}^d)$ with $k \in \{1,\ldots,n\}$. Then v is a continuous mapping and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate

$$\sup_{y\in Q}\left|v(y) - P_Q[v](y)\right| \leq C\left\|1_Q \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \tag{10}$$

holds, where C is a constant depending on n, d only. Moreover, the mapping $v_Q(y) = v(y) - P_Q[v](y), y \in Q$, can be extended from Q to the whole of $\mathbb{R}^n$ such that the extension (denoted again) $v_Q \in W_{p_\circ,1}^k(\mathbb{R}^n,\mathbb{R}^d)$ and

$$\left\|\nabla^k v_Q\right\|_{L_{p_\circ,1}(\mathbb{R}^n)} \leq C_0\left\|\nabla^k v\right\|_{L_{p_\circ,1}(Q)}, \tag{11}$$

where C_0 also depends on n, d only.

Proof. For continuity and the estimate (10) see [50]. Because of coordinate invariance of estimate (11), it is sufficient to prove the assertions about extension for the case when Q is a unit cube: $Q = [0,1]^n$. Put $u(y) = v_Q(y) = v(y) - P_Q[v](y)$ for $y \in Q$.
By Peetre theorem (see Theorem 6.5 in [65]) it is easy to deduce that
$$\|\nabla^m u\|_{L_{p_\circ,1}}(Q) \le C\|\nabla^k u\|_{L_{p_\circ,1}(Q)} \qquad \forall m = 0,1,\dots,k-1. \tag{12}$$
Using the standard Extension operator for Sobolev spaces (the well-known "finite-order reflection" procedure, see, e.g., [67]), function u on the unit cube $Q = [0,1]^n$ can be extended to a function $U \in W^k_{p_\circ,1}(\mathbb{R}^n)$ such that the estimate
$$\|\nabla^k U\|_{L_{p_\circ,1}(\mathbb{R}^n)} \le C' \sum_{m=0}^{k} \|\nabla^m u\|_{L_{p_\circ,1}(Q)}$$
holds. Taking into account the identity $\nabla^k u \equiv \nabla^k v$ on Q and (12), we obtain the required estimate (11).
Corollary (5.1.9)[180]: (see, e.g., [50]). Suppose $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$ with $k \in \{1,\dots,n\}$. Then v is a continuous mapping and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate
$$\begin{aligned} diam\, v(Q) &\le C\left(\frac{\|\nabla v\|_{L_1(Q)}}{\ell(Q)^{n-1}} + \|1_Q \cdot \nabla^k v\|_{L_{p_\circ,1}}\right) \\ &\le C\left(\frac{\|\nabla v\|_{L_{p_\circ}(Q)}}{\ell(Q)^{k-1}} + \|1_Q \cdot \nabla^k v\|_{L_{p_\circ,1}}\right) \end{aligned} \tag{13}$$
holds.
The above results can easily be adapted to give that $v \in C_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity (see [65]).
Analogously, from previous estimates one could deduce
Corollary (5.1.10)[180]: Suppose $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$ with $k \in \{1,\dots,n\}$. Then for all $m \in \{1,\dots,k\}$ and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate
$$\sup_{y\in Q}|v(y) - P_{Q,m-1}[v](y)| \le C\left(\frac{\|\nabla^m v\|_{L_{p_\circ}(Q)}}{\ell(Q)^{k-m}} + \|1_Q \cdot \nabla^k v\|_{L_{p_\circ,1}}\right) \tag{14}$$
holds.
Theorem (5.1.11)[180]: (Boundedness of the maximal operator, see [65]). Let $f \in L_{p,q}(\mathbb{R}^n), 1 < p < \infty, 1 \le q < \infty$. Then
$$\|\mathcal{M}_f\|_{L_{p,q}} \le C\|f\|_{L_{p,q}}.$$
Here
$$\mathcal{M}f(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|dy$$
is the usual Hardy–Littlewood maximal function of f.
Corollary (5.1.12)[180]: (Regularization in Lorentz spaces [65]). Let $f \in L_{p,q}(\mathbb{R}^n), 1 < p < \infty, 1 \le q < \infty$. Suppose that $f \in L_{p,q}(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ is a standard mollifier. Then $\psi_\delta * f \to f$ in $L_{p,q}(\mathbb{R}^n)$ as $\delta \to 0$.
Here and henceforth $C_0^\infty(\mathbb{R}^n)$ denotes the space of C^∞ smooth and compactly supported functions on $\mathbb{R}^n$.
Corollary (5.1.13)[180]: (Regularization in Sobolev–Lorentz spaces). If $f \in W^k_{p,q}(\mathbb{R}^n), 1 < p < \infty, 1 \le q < \infty$, then there exists a sequence of smooth functions $f_i \in$

$C_0^\infty(\mathbb{R}^n)$ such that $\|\nabla^m(f-f_i)\|_{L_p(\mathbb{R}^n)} \to 0$ for $m=0,1,\dots,k$, $\left\|\nabla^k(f-f_i)\right\|_{L_{p,q}(\mathbb{R}^n)} \to 0$ as $i \to \infty$.

We need also the following important Adams strong-type estimates for maximal functions.

Theorem (5.1.14)[180]: (see Theorem (5.1.2), Proposition 1 and its Corollary in [51]). Let $\beta \in (0,n)$. Then for nonnegative functions $f \in C_0(\mathbb{R}^n)$ the estimates

$$\int_0^\infty \mathcal{H}_\infty^\beta(\{x \in \mathcal{H}: \mathcal{M}f(x) \geq t\})\, dt \leq C_1 \int_0^\infty \mathcal{M}_\infty^\beta(\{x \in \mathcal{M}: f(x) \geq t\}) dt$$
$$\leq C_2 \sup\left\{\int f\, d\mu: \mu \in \mathcal{M}^\beta, |||\mu|||_\beta \leq 1\right\},$$

hold, where the constants C_1, C_2 depend on β, n only.

We need also the following classical fact (cf. with [55]).

Lemma (5.1.15)[180]: (see Lemma 2 in [58]). Let $u \in W_1^m(\mathbb{R}^n), m \leq n$. Then for any n–dimensional cubic interval $Q \subset \mathbb{R}^n, x \in Q$, and for any $j = 0,1,\dots,m-1$ the estimate

$$\left|\nabla^j u(x) - \nabla^j P_{Q,m-1}[u](x)\right| \leq C\ell(Q)^{m-j}(\mathcal{M}\nabla^m u)(x) \tag{15}$$

holds, where the constant C depends on n, m only.

Theorem (5.1.4) plays the key role among other results. Its proof splits into a number of lemmas. Fix parameters $m > 0, 1 < p < \infty, 0 < \alpha p < n$, and a positive Borel measure μ on $\mathbb{R}^n$ satisfying

$$\mu\big(B(x,r)\big) \leq r^{n-\alpha p} \tag{16}$$

for every ball $B(x,r) \subset \mathbb{R}^n$. Fix also a compact set $\subset \mathbb{R}^n$. Denote by I_E the corresponding Riesz potential $I_\alpha(1_E)$.

It is very easy to check by standard calculation that

$$0 \leq I_E(x) \leq C_0|E|_n^\alpha, \tag{17}$$

where the constant C_0 depends on n, α
only. Denote also $t_m = 2^m$ (here $m \in \mathbb{Z}$),

$$E_m = \{x \in E: I_E(x) \in [t_m, 2t_m]\},$$
$$E_m' = \{x \in E: I_E(x) \leq t_m\}, \qquad E_m'' = \left\{x \in E: I_{E(x)} > t_m\right\}.$$

We will write $f \lesssim g$, if $f \leq Cg$, where C depends on n, α, p only (really, most of the corresponding constants below up to Lemma (5.1.21) depends on n, α only).

Lemma (5.1.16)[180]: There exists a positive constant $m_0 \in \mathbb{N}$ depending on n, α only such that for any $m \in \mathbb{Z}$ and $x \in \mathbb{R}^n$ if $I_E(x) \geq t_m$, then $I_{E''_{m-m_0}}(x) \gtrsim t_m$.

Proof. The claim follows from the well-known maximum principle: $I_{E'_m}(x) \leq 2^{n-\alpha}t_m$ for every $m \in \mathbb{Z}$ (see [186]).

Lemma (5.1.17)[180]: For any $x, y \in \mathbb{R}^n$ if $I_E(y) = t$ and $|x-y| \leq (2t)^{\frac{1}{\alpha}}$ then $I_E(x) \gtrsim t$.

Proof. Let $I_E(y) = t$ and |

$$|y-x| \leq (2t)^{\frac{1}{\alpha}}. \tag{18}$$

Denote $r = |y-x|, B = B(y,r) = \{z \in \mathbb{R}^n: |z-y| < r\}$. Then by construction

$$t = I_E(y) = I_{E\cap B}(y) + I_{E\setminus B}(y). \tag{19}$$

Consider two possible situations.

(I).$I_{E\cap B}(y) \leq \frac{t}{2}$, then $I_{E\setminus B}(y) \geq \frac{t}{2}$. For any $z \in E\setminus B$ we have $|z-y| \geq r = |x-y|$, thus, $|x-z| \leq |x-y| + |z-y| \leq 2|z-y|$, consequently,

$$I_E(x) \geq I_{E\setminus B}(x) \geq 2^{n-\alpha}I_{E\setminus B}(y) \geq 2^{n-\alpha-1}t. \tag{20}$$

(II). $I_{E\cap B}(y) \geq \frac{t}{2}$. Then (17) implies $\frac{t}{2} \leq C_0|B\cap E|^{\frac{\alpha}{n}}$. Since $B\cap E \subset B(x,2r)$, by elementary estimates we have

$$I_E(x) \geq \frac{|B\cap E|}{(2r)^{n-\alpha}} \geq \frac{C' t^{\frac{n}{\alpha}}}{r^{n-\alpha}} \overset{(2.3)}{\geq} \frac{C'' t^{\frac{n}{\alpha}}}{t^{\frac{n}{\alpha}-1}} = C_2 t.$$

Denote $F_m = \{x \in \mathbb{R}^n : I_E(x) \in [t_m, 2t_m]\}$, $\mu_m = \mu(F_m), \mu_m(\cdot) = \mu \llcorner F_m$. By construction,

$$\|I_\alpha(1_E)\|^p_{L_p(\mu)} \sim \sum_{m=-\infty}^{\infty} t_m^p \mu_m.$$

So our main purpose below is to estimate tmμm. Of course, $t_m\mu_m \leq \int_{\mathbb{R}^n} I_E(x)d\mu_m(x)$. By Fubini Theorem we have

$$\begin{aligned}\int_{\mathbb{R}^n} I_E(x)d\mu_m(x) &= \int_0^\infty \rho^{-n+\alpha-1}\left[\int_{\mathbb{R}^n} |E\cap B(x,\rho)|d\mu_m(x)\right]d\rho \\ &= \int_0^\infty \rho^{-n+\alpha-1}\left[\int_E \mu_m\,[B(y,\rho)]dy\right]d\rho. \qquad (21)\end{aligned}$$

Lemma (5.1.18)[180]: The estimate

$$t_m\mu_m \lesssim \int_0^\infty \rho^{-n+\alpha-1}\left[\int_{\mathbb{R}^n} \left|E''_{m-m_0} \cap B(x,\rho)\right|d\mu_m(x)\right]d\rho \qquad (22)$$

holds, where m_0 is a constant from Lemma (5.1.16).

Proof. By Lemma (5.1.16), $I_{E''_{m-m_0}} \geq C_1 t_m$ on F_m, therefore $t_m\mu_m \leq C\int_{\mathbb{R}^n} I_{E''_{m-m_0}}(x)d\mu_m(x)$, and the last inequality implies in conjunction with Fubini's Theorem (22).

Lemma (5.1.19)[180]: There exists a constant $m_1 \in \mathbb{N}$ such that

$$t_m\mu_m \lesssim \int_{t_{m-m_1}^{\frac{1}{\alpha}}}^\infty \rho^{-n+\alpha-1}\left[\int_{\mathbb{R}^n} \left|E''_{m-m_0} \cap B(x,\rho)\right|d\mu_m(x)\right]d\rho. \qquad (23)$$

Proof. Let $m_1 \in \mathbb{N}$, its exact value will be specified below. We have $|E\cap B(x,\rho)| \leq \omega_n\rho^n$, where ω_n is a volume of a unit ball in $\mathbb{R}^n$. Thus

$$\begin{aligned}\int_0^{t_{m-m_1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1}\left[\int_{\mathbb{R}^n} |E\cap B(x,\rho)|d\mu_m(x)\right]d\rho &\leq \omega_n\mu_m \int_0^{t_{m-m_1}^{\frac{1}{\alpha}}} \rho^{\alpha-1}d\rho = \frac{\omega_n}{\alpha}\mu_m t_{m-m_1} \\ &= \frac{\omega_n}{\alpha}2^{-m_1}\mu_m t_m.\end{aligned}$$

So the target estimate (23) follows from (22) provided that $\frac{1}{\alpha}\omega_n 2^{-m_1}$ is sufficiently small.

Lemma (5.1.20)[180]: There exists a constant $i_0 \in \mathbb{N}$ such that for all $i \geq m - m_1$ the equality

$$\begin{aligned}&\int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1}\left[\int_{\mathbb{R}^n} \left|E''_{m-m_0} \cap B(x,\rho)\right|d\mu_m(x)\right]d\rho \\ &= \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1}\left[\int_{\mathbb{R}^n} \left|E_j \cap B(x,\rho)\right|d\mu_m(x)\right]d\rho \qquad (24)\end{aligned}$$

holds, where m_0, m_1 are the constants from Lemma (5.1.16), respectively.

Proof. Let $i \geq m - m_1$,

$$\rho^{\alpha} \le t_{i+1}, \tag{25}$$
and $y \in E_j \cap B(x,\rho), x \in F_m = supp\ \mu_m$. Then by definitions of these sets
$$I_E(x) \le 2t_m \tag{26}$$
and $I_E(y) \ge t_j$. Suppose $j \ge i+1$. Then (25) implies $|x-y|^{\alpha} \le t_{i+1} \le t_j$, therefore, by Lemma (5.1.17) (applying for $t = t_j$) we have $I_E(x) \ge C_2 t_j$. Thus by (26) we obtain $j \le m + m_2$ for some constant m_2 depending on α, n only.

Finally we have $j \le \max(i+1, m+m_2) \le \max(i+1, i+m_1+m_2)$ finishing the proof of the Lemma.

Lemma (5.1.21)[180]: The estimate
$$t_m \mu_m \lesssim \sum_{j=m-m_0}^{\infty} |E_j| t_{j-i_0}^{1-p} \tag{27}$$
holds for all $m \in \mathbb{Z}$, where m_0, i_0 are the constants from Lemmas 2.1, respectively.

Proof. We have
$$t_m \mu_m \sum_{i=m-m_1}^{\infty} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E''_{m-m_0} \cap B(x,\rho)| d\mu_m(x)\right] d\rho \overset{(2.9)}{\lesssim} \sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E_j \cap B(x,\rho)| d\mu_m(x)\right] d\rho$$
$$\overset{Fubini}{=} \sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{E_j} \mu_m[B(y,\rho)] dy\right] d\rho \lesssim$$
$$\sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1+(n-\alpha p)} |E_j| d\rho \lesssim$$
$$\lesssim \sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} |E_j| (t_i)^{1-p} \overset{\text{changing order of summation } i \leftrightarrow j}{\le}$$
$$\le \sum_{j=m-m_0}^{\infty} |E_j| \sum_{i=j-i_0}^{\infty} (t_i)^{1-p} \overset{\text{geometric progression}}{\lesssim} \sum_{j=m-m_0}^{\infty} |E_j| \left(t_{j-i_0}\right)^{1-p}. \tag{28}$$
Lemma (5.1.22)[180]: The estimate
$$\sum_{m=-\infty}^{\infty} t_m^p \mu_m \lesssim |E| \tag{29}$$
holds.

Proof. We have
$$\sum_{m=-\infty}^{\infty} t_m^p \mu_m \overset{(2.12)}{\lesssim} \sum_{m=-\infty}^{\infty} \sum_{j=m-m_0}^{\infty} |E_j| \left(\frac{t_m}{t_{j-i_0}}\right)^{p-1} \lesssim$$

$$\overset{\text{changing order of summation } m\leftrightarrow j}{\le} \sum_{j=-\infty}^{\infty} |E_j| \sum_{m=-\infty}^{j+m_0} \left(\frac{t_m}{t_{j-i_0}}\right)^{p-1} \lesssim$$

$$\overset{\text{geometric progression}}{\lesssim} \sum_{j=-\infty}^{\infty} |E_j| \left(\frac{t_{j+m_0}}{t_{j-i_0}}\right)^{p-1} \overset{\text{definition of } t_j}{=} \sum_{j=-\infty}^{\infty} |E_j| 2^{(m_0+i_0)(p-1)}$$

$$\lesssim |E|. \tag{30}$$

Using the established Theorem (5.1.3) and Adam's estimate from Theorem (5.1.14) with $\beta = n-(k-l)p$, we obtain the following estimates, which are key ingredients in the proof of N–property.

Corollary (5.1.23)[180]: Let $p \in (1,\infty), k, l \in \{1,\ldots,n\}, l \le k, (k-l)p < n$. Then for any function $f \in W^k_{p,1}(\mathbb{R}^n)$ the estimates

$$\left\|\nabla^l f\right\|^p_{L_p(\mu)} \le C |||\mu|||_\beta \left\|\nabla^k f\right\|^p_{L_{p,1}} \qquad \forall \mu \in \mathcal{M}^\beta, \tag{31}$$

$$\int_0^\infty \mathcal{H}^\beta_\infty\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(\left|\nabla^l f\right|^p\right)(x) \ge t\right\}\right) dt \le C \left\|\nabla^k f\right\|^p_{L_{p,1}} \tag{32}$$

hold, where $\beta = n-(k-l)p$ and the constant C depends on n, k, p only.

The main result is the following

Theorem (5.1.24)[180]: Let $p \in (1,\infty), k, l \in \{1,\ldots,n\}, l \le k, (k-l)p < n$. Then for any $f \in W^k_{p,1}(\mathbb{R}^n)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^n$ and a function $g \in C^l(\mathbb{R}^n)$ such that

(i) $\mathcal{H}^{n-(k-l)p}_\infty(U) < \varepsilon$;

(ii)each point $x \in \mathbb{R}^n \setminus U$ is an L_p-Lebesgue point for $\nabla_j f, j = 0,\ldots,l$;

(iii) $f \equiv g, \nabla_j f \equiv \nabla_j g$ on $\mathbb{R}^n \setminus U$ for $j = 1,\ldots,l$.

Note that in the analogous theorem for the case of Sobolev mappings $f \in W^k_p(\mathbb{R}^n)$ the assertion (i) should be reformulated as follows:

(i') $\mathcal{B}_{k-l,p}(U) < \varepsilon$ if $l < k$, where $\mathcal{B}_{\alpha,p}(U)$ denotes the Bessel capacity of the set U (see [79] or [54]).

Recall that for $1 < p < \infty$ and $0 < n-\alpha p < n$ the smallness of $\mathcal{H}^{n-\alpha p}_\infty(U)$ implies the smallness of $\mathcal{B}_{\alpha,p}(U)$, but that the opposite is false since $\mathcal{B}_{\alpha,p}(U) = 0$ whenever $\mathcal{H}^{n-\alpha p}(U) < \infty$. On the other hand, for $1 < p < \infty$ and $0 < n-\alpha p < \beta \le n$ the smallness of $\mathcal{B}_{\alpha,p}(U)$ implies the smallness of $\mathcal{H}^\beta_\infty(U)$ (see, e.g., [52]). So the usual assertion (i') is essentially weaker than (i).

Proof: Let the assumptions of the Theorem be fulfilled. By Theorem (5.1.11) and Corollary (5.1.13), we can choose the sequence of mappings $f_i \in C_0^\infty(\mathbb{R}^n)$ such that $\left\|\nabla^k f - \nabla^k f_i\right\|_{L_{p,1}(\mathbb{R}^n)} < 4^{-i}$. Denote $\tilde{f}_i = f - f_i$. Then by Corollary (5.1.23)

$$\mathcal{H}^{n-(k-l)p}_\infty\left(\left\{x \in \mathbb{R}^n : \mathcal{M}\left(\left|\nabla^l \tilde{f}_i\right|^p\right)(x) \ge 2^{-i}\right\}\right) < C\, 2^{-i}.$$

Then one could repeat almost word by word the proof of Theorem 3.1 in [56]. Since there are no essential differences.

We start with the following simple technical observation.

Lemma (5.1.25) (see, e.g., Lemma 4.1 in [50])[180]: If $l, k \in \{1,\ldots,n\}, l < k$, and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$, then for any $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $\mathcal{H}^{lp_\circ}_\infty(U) < \varepsilon$ and the uniform convergence

$$r^{-l}\left\|1_{B(x,r)} \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \to 0 \qquad as \;\; r \searrow 0$$

holds for $x \in \mathbb{R}^n \backslash U$.

Proof. The proof of the Lemma follows standard arguments, we reproduce it here for reader's convenience. Fix $\sigma > 0$. Let $\{B_\alpha\}$ be a family of disjoint balls $B_\alpha = B(x_\alpha, r_\alpha)$ such that

$$\left\|1_{B_\alpha} \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \geq \sigma r_\alpha^l$$

and $\sup_\alpha r\alpha < \delta$ for some $\delta > 0$, where δ is chosen small enough to guarantee that $\sup_\alpha \left\|1_{B_\alpha} \cdot \nabla^k v\right\|_{L_{p_\circ,1}} < 1$. Then We have

$$\sum_\alpha r_\alpha^{lp_\circ} \leq \sigma^{-p_\circ} \sum_\alpha \left\|1_{B_\alpha} \cdot \nabla^k v\right\|_{L_{p_\circ,1}}^{p_\circ} \leq \sigma^{-p_\circ} \left\|1_{\cup_\alpha B_\alpha} \cdot \nabla^k v\right\|_{L_{p_\circ,1}}^{p_\circ}. \quad (33)$$

Since the last term tends to 0 as $\mathcal{L}^n(\bigcup_\alpha B_\alpha) \to 0$, and $\mathcal{L}^n(\bigcup_\alpha B_\alpha) \leq c\, \delta^{n-lp_\circ} \sum_\alpha r_\alpha^{lp_\circ}$, we get easily that $\sum_\alpha r_\alpha^{lp_\circ} \to 0$ as $\delta \searrow 0$. Using this fact and some standard covering lemmas we infer in a routine manner that for a set

$$A_{\sigma,\delta} := \left\{x \in \mathbb{R}^n : \exists r \in (0, \delta] \qquad r^{-l} \left\|1_{B(x,r)} \cdot \nabla^k v\right\|_{L_{p_\circ,1}} > \sigma\right\}$$

the convergence

$$\mathcal{H}_\infty^{lp_\circ}(A_{\sigma,\delta}) \to 0 \qquad as \quad \delta \searrow 0$$

holds for any fixed $\sigma > 0$. The rest part of the proof of the lemma is straight forward.

From the last lemma (for $l = 1$), Theorem (5.1.24) (ii) and estimate (13) we obtain the following result:

Theorem (5.1.26)[180]: Let $k \in \{1, \ldots, n\}$ and $v \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $\mathcal{H}^{p_\circ}(A_v) = 0$ and for any $x \in \mathbb{R}^n \backslash A_v$ the function v is differentiable (in the classical Frechet sense) at x, furthermore, the classical derivative coincides with $\nabla v(x)$ (x is a Lebesgue point for ∇v).

The case $k = 1, p_\circ = n$ is a classical result due to Stein [73] (see also [62]), and for $k = n, p_\circ = 1$ the result is also proved in [58].

We have the following extension of Theorem (5.1.26).

Theorem (5.1.27)[180]: Let $k, l \in \{1, \ldots, n\}, l \leq k$, and $v \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $\mathcal{H}^{lp_\circ}(A_v) = 0$ and for any $x \in \mathbb{R}^n \backslash A_v$ the function v is l-times differentiable (in the classical Fr´echet–Peano sense) at x, i.e.,

$$\lim_{r \searrow 0} \sup_{y \in B(x,r) \backslash \{x\}} \frac{\left|v(y) - T_{v,l,x}(y)\right|}{|x - y|^l} = 0,$$

where $T_{v,l,x}(y)$ is the Taylor polynomial of order l for v centered at x (which is well defined $H^{lp_\circ}$-a.e. by Theorem (5.1.24)).

Proof. We consider only the case $l < n$; for $l = n$ the arguments are similar and becomes even simpler. Below we follow methods of [9] and [56]. By Theorem (5.1.24), there exists a set A_l such that $\mathcal{H}^{lp_\circ}(A_l) = 0$ and the derivatives $\nabla^j v(x)$ are well-defined for all $x \in \mathbb{R}^n \backslash A_l$ and $j = 0, 1, \ldots, l$. Further, by Lemma (5.1.25) there exists a sequence of open sets $U_i \subset \mathbb{R}^n$ such that $U_i \supset U_{i+1}, \mathcal{H}_\infty^{lp_\circ}(U_i) < 2^{-i}$ and the uniform convergence

$$r^{-l} \left\|1_{B(x,r)} \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \to 0 \quad as \quad r \searrow 0$$

holds for $x \in \mathbb{R}^n \backslash U_i$. It means that there exists a function $\omega_i : (0, +\infty) \to (0, +\infty)$ such that $\omega_i(r) \to 0$ as $r \searrow 0$ and

$$r^{-l} \left\|1_{B(x,r)} \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \leq \omega_i(r) \qquad \forall x \in \mathbb{R}^n \backslash U_i. \quad (34)$$

Take a sequence of mappings $v_i: \mathbb{R}^n \to \mathbb{R}^d$ from Corollary (5.1.13), i.e., $v_i \in C_0^\infty(\mathbb{R}^n)$ and $\|\nabla^k(v - v_i)\|_{L_{p_\circ,1}(\mathbb{R}^n)} < 4^{-i}$. Denote $\tilde{v}_i = v - v_i$ and

$$B_i = \left\{x \in \mathbb{R}^n : \mathcal{M}\left(\left|\nabla^l \tilde{v}_i\right|^{p_\circ}\right)(x) \geq 2^{-ip_\circ}\right\}, \qquad G_i = A_l \cup U_i \cup \left(\bigcup_{j=i}^\infty B_j\right).$$

Then by estimate (32) we have

$$\mathcal{H}_\infty^{lp_\circ}(B_i) \leq c^{2\,-i}, \tag{35}$$

therefore,

$$\mathcal{H}_\infty^{lp_\circ}(G_i) \leq C2^{-i}. \tag{36}$$

By construction,

$$\left|\nabla^l \tilde{v}_j(x)\right|^{p_\circ} \leq \mathcal{M}\left(\left|\nabla^l \tilde{v}_j\right|^{p_\circ}\right)(x) \leq 2^{-jp_\circ} \tag{37}$$

for all $x \in \mathbb{R}^n \backslash \mathrm{G}_i$ and all $j \geq i$. Moreover, since $v_j \in C_0^\infty(\mathbb{R}^n)$, there exists constants M_j such that $\left|\nabla^k v_j(x)\right| \leq M_j\ \forall x \in \mathbb{R}^n$, this fact and (34) implies

$$r^{-l}\left\|1_{B(x,r)} \cdot \nabla^k \tilde{v}_j\right\|_{L_{p_\circ,1}} \leq \omega_i(r) + M_j r^{n-l} \quad \forall x \in \mathbb{R}^n \backslash G_i. \tag{38}$$

We start by estimating the remainder term $\tilde{v}_j(y) - T_{\tilde{v}_j,l,x}(y)$. Fix $y \in \mathbb{R}^n, x \in \mathbb{R}^n \backslash G_{i\ ,j} \geq i$, and an n–dimensional cubic interval Q such that $x, y \in Q, |x - y| \sim \ell(Q)$. By construction and Lemma (5.1.15), for any multi–index α with $|\alpha| \leq l$ we have

$$\begin{aligned}\left|\partial^\alpha \tilde{v}_j(x) - \partial^\alpha P_{Q,l-1}[\tilde{v}_j](x)\right| &\leq C\ell(Q)^{l-|\alpha|}\left(\mathcal{M}\nabla^l \tilde{v}_j\right)(x) \\ &\leq Cr^{l-|\alpha|}2^{-j},\end{aligned} \tag{39}$$

where $r = |x - y|$. Consequently,

$$\begin{aligned}\left|\tilde{v}_j(y) - T_{l,\tilde{v}_j,x}(y)\right| &\leq \left|\tilde{v}_j(y) - P_{Q,l-1}[\tilde{v}_j](y)\right| + \left|P_{Q,l-1}[\tilde{v}_j](y) - T_{l,\tilde{v}_j,x}(y)\right| \leq \\ &\leq \left[C2^{-j}r^l + \omega_i(r)r^l + M_j r^n\right] \\ &\quad + \sum_{|\alpha| \leq l} \frac{1}{\alpha!}\left|\left(\partial^\alpha \tilde{v}_j(x) - \partial^\alpha P_{Q,l-1}[\tilde{v}_j](x)\right) \cdot (y - x)^\alpha\right| \\ &\leq \left(C_1 2^{-j} + \omega_i(r) + M_j r^{n-l}\right) r^l.\end{aligned} \tag{40}$$

Finally from the last estimate and equality $v = \tilde{v}_j + v_j$ we have

$$\begin{aligned}\left|v(y) - T_{l,v,x}(y)\right| &\leq |\tilde{v}_j(y) - T\_\left(l, \tilde{v}_j, x(y)\right| + \left|v_j(y) - T_{l,v_j,x}(y)\right| \\ &\leq \left(C_1 2^{-j} + \omega_i(r) + M_j r^{n-l}\right) r^l + \omega_{v_j}(r) r^l \\ &= \left(C_1 2^{-j} + \omega_i(r) + M_j r^{n-l} + \omega_{v_j}(r)\right) r^l,\end{aligned}$$

where $\omega_i(r) \to 0$ and $\omega_{v_j}(r) \to 0$ as $r \to 0$ (the latter holds since $v_j \in C_0^\infty(\mathbb{R}^n)$). We emphasize that the last inequality is valid for all $y \in \mathbb{R}^n, j \geq i$, and $\in \mathbb{R}^n \backslash G_i$. Therefore

$$\frac{\left|v(y) - T_{l,v,x}(y)\right|}{|x - y|^l} \to 0 \quad as \quad y \to x$$

uniformly for all $x \in \mathbb{R}^n \backslash G_i$. This means, that v is uniformly l-times differentiable (in the classical Frechet–Peano sense) at every $\in \mathbb{R}^n \backslash G_i$. Then the estimate (36) finishes the proof.

We aim to prove the assertion of Theorem (5.1.1), namely the Luzin N– property for $W_{p_\circ,1}^k$–mappings with respect to Hausdorff content $\mathcal{H}_\infty^{p_\circ}$ (i.e., when $q = p_\circ = \frac{n}{k}$). Let us for emphasis restate the result:

Recall that for the case $k = 1$ this assertion was proved in [62], and for $k = n$ it was proved in [56], so we omit these cases. Our proof here for the remaining cases follows and expands on the ideas from [56].

We fix $k \in \{2, \dots, n-1\}$, and a mapping v in $W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. To prove Theorem (5.1.31), we need some preliminary lemmas that we turn to next. Applying Corollary (5.1.23) for the case $p = p_\circ = \frac{n}{k}, l = 1$, we obtain

$$\|\nabla v\|^{p_\circ}_{L_{p_\circ}(\mu)} \le C|\|\mu\||_{p_\circ}\left\|\nabla^k v\right\|^{p_\circ}_{L_{p_\circ,1}} \quad \forall \mu \in \mathcal{M}^{p_\circ}, \tag{41}$$

where C depends on $n, p_\circ, d$ only.

By a dyadic interval we understand a cubic interval of the form $\left[\frac{k_1}{2^m}, \frac{k_1+1}{2^m}\right] \times \cdots \times \left[\frac{k_n}{2^m}, \frac{k_n+1}{2^m}\right]$, where k_i , m are integers. The following assertion is straightforward.

Lemma (5.1.28)[180]: For any n–dimensional cubic interval $J \subset \mathbb{R}^n$ there exist dyadic intervals $Q_1, \dots, Q_{2^n}$ such that $J \subset Q_1 \cup \cdots \cup Q_{2^n}$ and $\ell(Q_1) = \cdots = \ell(Q_{2^n}) \le 2\ell(J)$.

Let $\{Q_\alpha\}_{\alpha \in A}$ be a family of n-dimensional dyadic intervals. We say that the family $\{Q_\alpha\}$ is regular, if for any n-dimensional dyadic interval Q the estimate

$$\ell(Q)^{p_\circ} \ge \sum_{\alpha : Q_\alpha \subset Q} \ell(Q_\alpha)^{p_\circ} \tag{42}$$

holds. Since dyadic intervals are either nonoverlapping or contained in one another, (42) implies that any regular family $\{Q_\alpha\}$ must in particular consist of nonoverlapping intervals.

Lemma (5.1.29) (see Lemma (5.1.18) in [56])[180]: Let $\{Q_\alpha\}$ be a family of n–dimensional dyadic intervals. Then there exists a regular family $\{J_\beta\}$ of n–dimensional dyadic intervals such that $\bigcup_\alpha Q_\alpha \subset \bigcup_\beta J_\beta$ and

$$\sum_\beta \ell(J_\beta)^{p_\circ} \le \sum_\alpha \ell(Q_\alpha)^{p_\circ}.$$

Lemma (5.1.30)[180]: For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_\alpha\}$ of n–dimensional dyadic intervals we have if

$$\sum_\alpha \ell(Q_\alpha)^{p_\circ} < \delta, \tag{43}$$

then

$$\sum_\alpha \left\|1_{Q_\alpha} \cdot \nabla^k v\right\|^{p_\circ}_{L_{p_\circ,1}} < \varepsilon \tag{44}$$

and

$$\sum_\alpha \frac{1}{\ell(Q_\alpha)^{n-p_\circ}} \int_{Q_\alpha} |\nabla v|^{p_\circ} < \varepsilon. \tag{45}$$

Proof. Fix $\varepsilon \in (0,1)$ and let $\{Q_\alpha\}$ be a regular family of n–dimensional dyadic intervals satisfying (43), where $\delta > 0$ will be specified below.

Let us start by checking (44). We have

$$\sum_\alpha \left\|1_{Q_\alpha} \cdot \nabla^k v\right\|^{p_\circ}_{L_{p_\circ,1}} \le \left\|1_{\cup_\alpha Q_\alpha} \cdot \nabla^k v\right\|^{p_\circ}_{L_{p_\circ,1}}.$$

Using (8), we can rewrite the last estimate as

$$\sum_{\alpha}\left\|1_{Q_\alpha}\cdot\nabla^k v\right\|_{L_{p_\circ,1}}^{p_\circ}\leq\left(\int_0^{+\infty}\left[\mathcal{L}^n\left(\left\{x\in\bigcup_{\alpha}Q_\alpha:\left|\nabla^k v(x)\right|>t\right\}\right)\right]^{\frac{1}{p_\circ}}dt\right)^{p_\circ}. \tag{46}$$

Since

$$\int_0^{+\infty}\left[\mathcal{L}^n\left(\left\{x\in\mathbb{R}^n:\left|\nabla^k v(x)\right|>t\right\}\right)\right]^{\frac{1}{p_\circ}}dt<\infty,$$

it follows that the integral on the right–hand side of (46) tends to zero as $\mathcal{L}^n(\bigcup_\alpha Q_\alpha)$ tends to zero. In particular, it will be less than ε if the condition (43) is fulfilled with a sufficiently small δ. Thus (44) is established for all $\delta\in(0,\delta_1]$, where $\delta_1=\delta_1(\varepsilon,v)>0$. Next we check (45). By virtue of Corollary (5.1.13), applied coordinate–wise, we can find a decomposition $v=v_0+v_1$, where $\|\nabla v_0\|_{L^\infty}\leq K=K(\varepsilon,v)$ and

$$\left\|\nabla^k v_1\right\|_{L_{p_\circ,1}}<\varepsilon. \tag{47}$$

Assume that $\delta\in(0,\delta_1]$ and

$$\sum_{\alpha}\ell(Q_\alpha)^{p_\circ}<\delta<\frac{1}{K^{p_\circ+1}}\varepsilon. \tag{48}$$

Define the measure μ by

$$\mu=\left(\sum_{\alpha}\frac{1}{\ell(Q_\alpha)^{n-p_\circ}}1_{Q_\alpha}\right)\mathcal{L}^n, \tag{49}$$

where 1_{Q_α} denotes the indicator function of the set Q_α.

The estimate

$$\sup_J\{\ell(J)^{-p_\circ}\mu(J)\}\leq 2^{n+p_\circ} \tag{50}$$

holds, where the supremum is taken over all n–dimensional cubic intervals. Indeed, write for a dyadic interval Q

$$\mu(Q)=\sum_{\alpha:Q_\alpha\subset Q}\ell(Q_\alpha)^{p_\circ}+\sum_{\alpha:Q_\alpha\not\subseteq Q}\frac{\ell(Q\cap Q_\alpha)^n}{\ell(Q_\alpha)^{n-p_\circ}}.$$

By regularity of $\{Q_\alpha\}$ the first sum is bounded above by $\ell(Q)^{p_\circ}$. If the second sum is nonzero then there must exist an index α such that $Q_\alpha\not\subseteq Q$ and Q_α, Q overlap. But as the intervals $\{Q_\alpha\}$ are nonoverlapping and dyadic we must then precisely have one such interval Q_α and $Q_\alpha\supset Q$. But then the first sum is empty and the second sum has only the one term $\ell(Q)^n/\ell(Q_\alpha)^{n-p_\circ}$, hence is at most $\ell(Q)^{p_\circ}$. Thus the estimate $\mu(Q)\leq\ell(Q)^{p_\circ}$ holds for every dyadic Q. The inequality (50) in the case of a general cubic interval J follows from the above dyadic case and Lemma (5.1.28). The proof of the claim is complete.

Now return to (45). By properties (41), (47) and (48) (applied to the mapping v1), we have

$$\sum_{\alpha}\frac{1}{\ell(Q_\alpha)^{n-p_\circ}}\int_{Q_\alpha}|\nabla v|^{p_\circ}\leq\frac{2^{p_\circ-1}K^{p_\circ}}{K^{p_\circ}+1}\varepsilon+\sum_{\alpha}\frac{2^{p_\circ-1}}{\ell(Q_\alpha)^{n-p_\circ}}\int_{Q_\alpha}|\nabla v_1|^{p_\circ}$$
$$\leq C'\left(\varepsilon+\int|\nabla v_1|^{p_\circ}d\mu\right)\leq C''\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, the proof of Lemma (5.1.30) is complete.

Theorem (5.1.31)[180]: Let $k\in\{1,\dots,n\}$, and $v\in W_{p_\circ,1}^k(\mathbb{R}^n,\mathbb{R}^d)$. Then for each $\varepsilon>0$ there exists $\delta>0$ such that for any set $E\subset\mathbb{R}^n$ If $\mathcal{H}_\infty^{p_\circ}(E)<\delta$, then $\mathcal{H}_\infty^{p_\circ}\big(v(E)\big)<\varepsilon$. In particular, $\mathcal{H}^{p_\circ}\big(v(E)\big)=0$ whenever $\mathcal{H}^{p_\circ}(E)=0$.

Proof: Fix $\varepsilon > 0$ and take $\delta = \delta(\varepsilon, v)$ from Lemma (5.1.30). Then by Corollary (5.1.9) for any regular family $\{Q_\alpha\}$ of n–dimensional dyadic intervals we have if $\sum_\alpha \ell(Q_\alpha)^{p_\circ} < \delta$, then $\sum_\alpha \left(diam\, v(Q_\alpha)\right)^{p_\circ} < C\varepsilon$. Now we may conclude the proof of Theorem (5.1.31) by use of Indeed they allow us to find a $\delta_0 > 0$ such that if for a subset E of $\mathbb{R}^n$ we have $\mathcal{H}_\infty^{p_\circ}(E) < \delta_0$, then E can be covered by a regular family $\{Q_\alpha\}$ of n– dimensional dyadic intervals with $\sum_\alpha \ell(Q_\alpha)^{p_\circ} < \delta$.

Let $\mathrm{k}, \mathrm{m} \in \{1, \dots, \mathrm{n}\}$ and $\mathrm{v} \in \mathrm{W}^{\mathrm{k}}_{\mathrm{p}_\circ,1,\mathrm{loc}}(\Omega, \mathbb{R}^d)$, where Ω is an open subset of $\mathbb{R}^n$. Then, by Theorem (5.1.24) (ii), there exists a Borel set A_v such that $\mathcal{H}^{p_\circ}(A_v) = 0$ and all points of the complement $\Omega \backslash A_v$ are $L_{p_\circ}$-Lebesgue points for the gradient $\nabla v(x)$. Moreover, v is differentiable (in the classical Frechet sense) at every point of $\Omega \backslash A_v$.
Denote $Z_{v,m} = \{x \in \Omega\backslash A_v : rank \nabla v(x) < m\}$. The purpose is to prove the assertion of Theorem (5.1.6):

$$\mathcal{H}^{q_\circ}\left(v(Z_{v,m})\right) = 0. \tag{51}$$

The exponents occurring in the Theorem (5.1.2)re the critical exponents that were defined in (6):

$$p_\circ = \frac{n}{k} \quad and \quad q_\circ = m - 1 + \frac{n-m+1}{k}.$$

By an easy calculation, assumptions $n \geq m \geq 1, k \geq 1$ imply

$$p_\circ \leq q_\circ \leq n. \tag{52}$$

Note that in the double inequality (52) we have equality in the first inequality iff $m = 1$ or $k = 1$, while in the second inequality equality holds iff $k = 1$. In particular,

$$p \circ < q \circ < n \; for \; k, m \in \{2, \dots, n\}.$$

By results obtained in [9]–[56], [50] (see Theorem (5.1.6) We need only consider the case

$$m = 1, q_\circ = p_\circ = \frac{n}{k}.$$

Before embarking on the detailed proof let us make some preliminary observations that will enable us to make some convenient additional assumptions. Namely because the result is local we can without loss in generality assume that $\Omega = \mathbb{R}^n$. We fix $k \in \{2, \dots n\}$ and a mapping $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^n)$. In view of the definition of critical set we have for $m = 1$

$$Z_v = Z_{v,1} = \{x \in \mathbb{R}^n \backslash A_v : \nabla_v(x) = 0\}.$$

The following lemma provides the main step in the proof of Theorem (5.1.6). Lemma (5.1.16)2. For any n-dimensional dyadic interval $Q \subset \mathbb{R}^n$ the estimate

$$\mathcal{H}_\infty^{p_\circ}\left(v(Z_v \cap Q)\right) \leq C \left\|\nabla^k v\right\|^{p_\circ}_{L_{p_\circ,1}(Q)} \tag{53}$$

holds, where the constant C depends on n, m, k, d only.
Proof. By virtue of (11) it suffices to prove that

$$\mathcal{H}_\infty^{p_\circ}\left(v(Z_v \cap Q)\right) \leq C \left\|\nabla^k v_Q\right\|^{p_\circ}_{L_{p_\circ,1}(\mathbb{R}^n)} \tag{54}$$

for the mapping v_Q defined in Lemma (5.1.8), where $C = C(n, m, k, d)$ is a constant. To establish (54) it is possible to repeat almost verbatim the proof of Lemma 3.2 in [50]. One must observe the following minor changes: first $q_\circ = p_\circ$, and next, instead of Corollary 1.8 from [50] one must use Corollary (5.1.23) established above. Note that in the present situation the calculations simplify since for $m = 1$ many of terms from [50] disappear.

Corollary (5.1.32)[180]: For any $\varepsilon > 0$ there exists $\delta > 0$ such that for every subset E of $\mathbb{R}^n$ we have $\mathcal{H}_\infty^{p_\circ}(v(Z_v \cap E)) \le \varepsilon$ provided $\mathcal{L}^n(E) \le \delta$. In particular, $\mathcal{H}^{p_\circ}\big(v(Z_v \cap E)\big) = 0$ whenever $\mathcal{L}^n(E) = 0$.

Proof. Let $\mathcal{L}^n(E) \le \delta$, then we can find a family of nonoverlapping n-dimensional dyadic intervals Q_α such that $E \subset \cup_\alpha Q_\alpha$ and $\sum_\alpha \ell^n(Q_\alpha) < C\delta$. Of course, for sufficiently small δ the estimate $\|\nabla^k v\|_{L_{p_\circ,1}(Q_\alpha)} < 1$ is fulfilled for every α. Then in view of Lemma 1.1 we have

$$\sum_\alpha \|\nabla^k v\|^{p_\circ}_{L_{p_\circ,1}(Q\alpha)} \le \|\nabla^k v\|^{p_\circ}_{L_{p_\circ,1}(\cup Q_\alpha)} \tag{55}$$

This estimate together with Lemma (5.1.16)2 allow us to conclude the required smallness of

$$\sum_\alpha \mathcal{H}_\infty^{p_\circ}(Z_v \cap Q_\alpha) \ge \mathcal{H}_\infty^{p_\circ}(Z_v \cap E).$$

Invoking Dubovitskiı–Federer's Theorem (see commentary to the Theorem (5.1.6) for the smooth case $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$, Theorem (5.1.24) (iii) (applied to the case $l = k$) implies

Corollary (5.1.33) (see, e.g., [39])[180]: There exists a set $\tilde{Z}_v$ of n-dimensional Lebesgue measure zero such that $\mathcal{H}^{p_\circ}\left(v\big(Z_v \setminus \tilde{Z}_v\big)\right) = 0$. In particular, $\mathcal{H}^{p_\circ}\big(v(Z_v)\big) = \mathcal{H}^{p_\circ}\left(v\big(\tilde{Z}_v\big)\right)$.

We conclude that $\mathcal{H}^{p_\circ}\big(v(Z_v)\big) = 0$, and this ends the proof of Theorem (5.1.6).

Section (5.2): Sobolev Spaces

Originally proven in 1942, Arthur Sard's [47] famous theorem asserts that the set of critical values of a sufficiently regular mapping is null. We will use the following notation to represent the critical set of a given smooth map $f: \mathbb{R}^n \to \mathbb{R}^m$:

$$C_f = \{x \in \mathbb{R}^n | rank\ Df(x) < m\}.$$

We will assume that m and n are integers at least 1.

Theorem (5.2.1)[192]: (Sard). Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k. If $k > \max(n - m, 0)$, then

$$\mathcal{H}^m\left(f\big(C_f\big)\right) = 0.$$

Here and in what follows by $\mathcal{H}^k$ we denote the k-dimensional Hausdorff measure.

Several results have shown that Sard's result is optimal, see e.g. [194], [197], [198], [200], [203], [48]. In 1957 Dubovitskiı [194], extended Sard's theorem to all orders of smoothness k. See [53] for a modernized proof of ths result and some generalizations.

Theorem (5.2.2)[192]: (Dubovitskiı). Fix $n, m, k \in \mathbb{N}$. Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k. Write $\ell = \max(n - m - k + 1, 0)$. Then

$$\mathcal{H}^\ell\left(C_f \cap f^{-1}(y)\right) = 0 \quad \text{for } a.e. \;\; y \in \mathbb{R}^m.$$

This result tells us that almost every level set of a smooth mapping intersects with its critical set on an ℓ-null set. Higher regularity of the function implies a reduction in the Hausdorff dimension of the overlap between $f^{-1}(y)$ and C_f for a.e. $y \in \mathbb{R}^m$.

Notice that if $k > \max(n - m, 0)$, then $n - m - k + 1 \le 0$, and so $\mathcal{H}^\ell = \mathcal{H}^0$ is simply the counting measure on $\mathbb{R}^n$. That is, if $f: \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k and additionally $k > \max(n - m, 0)$, Dubovitskiı's theorem implies that $f^{-1}(y) \cap C_f$ is empty for almost every $y \in \mathbb{R}^m$. In other words, $\mathcal{H}^m\left(f\big(C_f\big)\right) = 0$. Thus Sard's theorem is a special case of Dubovitskiı's theorem.

Recently, many mathematicians have worked to generalize Sard's result to the class of Sobolev mappings [29], [53], [37], [56], [39], [43], [50], [191]. Specifically, in 2001 De Pascale [39] proved the following version of Sard's theorem for Sobolev mappings.

Theorem (5.2.3)[192]: Suppose $k > max(n-m, 0)$. Suppose $\Omega \subset \mathbb{R}^n$ is open. If $f \in W^{k,p}_{loc}(\Omega, \mathbb{R}^m)$ for $n < p < \infty$, then $\mathcal{H}^m\left(f(C_f)\right) = 0$.

We will use the usual notation $W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$ to indicate the Sobolev class of $L^p(\mathbb{R}^n, \mathbb{R}^m)$ mappings whose first k weak partial derivatives have finite L^p norm.

We show that also the Dubovitskiı theorem generalizes to the case of $W^{k,p}_{loc}$ mappings when $n < p < \infty$. We must be very careful when dealing with Sobolev mappings because the set $f^{-1}(y)$ depends on what representative of f we take. If $k \geq 2$, then Morrey's inequality implies that f has a representative of class $C^{k-1,1-\frac{n}{p}}$, so the critical set C_f is well defined. If $k = 1$, then D_f is only defined almost everywhere and hence the set C_f is defined up to a set of measure zero. We will say that f is precisely represented if each component fi of f satisfies

$$f_i(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f_i(y)dy$$

for all $x \in \Omega$ at which this limit exists. The Lebesgue differentiation theorem ensures that this is indeed a well defined representative of f. In what follows, we will always refer to the $C^{k-1,1-\frac{n}{p}}$ representative of f when $k \geq 2$ and a precise representation of f when $k = 1$. (Notice that the precise representative of f and the smooth representative of f are the same for $k \geq 2$.)

The main result reads as follows.

If $m > n$, then since $p > n$ we may apply Morrey's inequality combined with Hölder's inequality to show that $\mathcal{H}^n\big(f(Q)\big) < \infty$ for any cube $Q \Subset \Omega$, and so $\mathcal{H}^m\big(f(\Omega)\big) = 0$. Thus $f^{-1}(y)$ is empty for almost every $y \in \mathbb{R}^m$, and the theorem follows.

We will now discuss the details behind the argument that $\mathcal{H}^n\big(f(Q)\big) < \infty$ for any cube $Q \Subset \Omega$. Fix $\delta > 0$, and cover Q with $2^{n\nu}$ congruent dyadic cubes $\{Q_j\}_{j=1}^{2^{n\nu}}$ with pairwise disjoint interiors. According to Morrey's inequality (see Lemma (5.2.8)),

$$diam\, f(Q_j) \leq C\big(diam\, Q_j\big)^{1-\frac{n}{p}} \left(\int_{Q_j} |Df(z)|^p dz\right)^{\frac{1}{p}}$$

for every $1 \leq j \leq 2^{n\nu}$. Since $diam\, Q_j = 2^{-\nu}\, diam\, Q$, choosing ν large enough gives $\sup_j diam\, f(Q_j) < \delta$, and so we can estimate the pre-Hausdorff measure

$$\begin{aligned}\mathcal{H}^n_\delta\big(f(Q)\big) &\leq C \sum_{j=1}^{2^{n\nu}} \left(diam\, f(Q_j)\right)^n \leq C \sum_{j=1}^{2^{n\nu}} \big(diam\, Q_j\big)^{n\left(1-\frac{n}{p}\right)} \left(\int_{Q_j} |Df(z)|^p dz\right)^{\frac{n}{p}} \\ &\leq C \left(\sum_{j=1}^{2^{n\nu}} \big(diam\, Q_j\big)^n\right)^{1-\frac{n}{p}} \left(\sum_{j=1}^{2^{n\nu}} \int_{Q_j} |Df(z)|^p dz\right)^{\frac{n}{p}} \\ &\leq C\mathcal{H}^n(Q)^{1-\frac{n}{p}} \left(\int_Q |Df(z)|^p dz\right)^{\frac{n}{p}}.\end{aligned}$$

We used Hölder's inequality with exponents p/n and $p/(p-n)$ to obtain the third line. Since the right hand estimate does not depend on δ, sending $\delta \to 0^+$ yields $\mathcal{H}^n\big(f(Q)\big) < \infty$. This completes the proof of Theorem (5.2.13) when $m > n$. Hence we may assume that $m \leq n$.

We will now discuss the case $k = 1$ to avoid any confusion involving the definition of C_f. Since $m \leq n$, we may apply the following co-area formula due to Maly, Swanson, and Ziemer [203]:

Theorem (5.2.4)[192]: Suppose that $1 \leq m \leq n, \Omega \subset \mathbb{R}^n$ is open, $p > m$, and $f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^m)$ is precisely represented. Then the following holds for all measurable $E \subset \Omega$:

$$\int_E |J_m f(x)| dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(E \cap f^{-1}(y)\big) dy$$

where $|J_m f|$ is the square root of the sum of the squares of the determinants of the $m \times m$ minors of Df.

Notice that $|J_m f|$ is equals zero almost everywhere on the set $= C_f$. Therefore the above equality with $E = C_f$ reads

$$0 = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(C_f \cap f^{-1}(y)\right) dy = \int_{\mathbb{R}^m} \mathcal{H}^{\ell}\left(C_f \cap f^{-1}(y)\right) dy.$$

That is, $\mathcal{H}^{\ell}\left(C_f \cap f^{-1}(y)\right) = 0$ for a.e. $y \in \mathbb{R}^m$, and the theorem follows.

Therefore, we may assume for the remainder that $m \leq n$ and $k \geq 2$.

Most proofs of Sard-type results typically involve some form of a Morse Theorem [69] in which the critical set of a mapping is decomposed into pieces on which the function's difference quotients converge quickly. See [206] for the proof of the classical Sard theorem based on this method. A version of the Morse Theorem was also used by De Pascale [39]. However, there is another approach to the Sard theorem based on the so called Kneser-Glaeser Rough Composition theorem, and this method entirely avoids the use of the Morse theorem. We say that a mapping $f: W \subset \mathbb{R}^r \to \mathbb{R}$ of class C^k is s-flat on $A \subset W$ for $1 \leq s \leq k$ if $D^\alpha f = 0$ on A for every $1 \leq |\alpha| \leq s$.

Theorem (5.2.5) (Kneser-Glaeser Rough Composition)[192]: Fix positive integers s, k, r, n with $s < k$. Suppose $V \subset \mathbb{R}^r$ and $W \subset \mathbb{R}^n$ are open. Let $g: V \to W$ be of class C^{k-s} and $f: W \to \mathbb{R}$ be of class C^k. Suppose $A^* \subset V$ and $A \subset W$ are compact sets with

(i) $g(A^*) \subset A$ and

(ii) f is s-flat on A.

Then there is a function $F: \mathbb{R}^r \to \mathbb{R}$ of class C^k so that $F = f \circ g$ on A^* and F is s-flat on A^*. This theorem ensures that the composition of two smooth maps will have the same regularity as the second function involved in the composition provided that enough of the derivatives of this second function are zero. After a brief examination of the rule for differentiation of composite functions, such a conclusion seems very natural. Indeed, we can formally compute $D^\alpha(f \circ g)(x)$ for all $|\alpha| \leq k$ and $x \in A^*$ since any "non-existing" derivative $D^\beta g(x)$ with $|\beta| > k - s$ is multiplied by a vanishing $D^\gamma f\big(g(x)\big)$ term with $|\gamma| = |\alpha| - |\beta| < s$. Thus we can formally set $D^\gamma f\big(g(x)\big) D^\beta g(x) = 0$. However the proof of this theorem is not easy since it is based on the celebrated Whitney extension theorem. That should not be surprising after all. The existence of the extension F is proven by verification that the formal jet of derivatives of $f \circ g$ up to order k defined above satisfies the assumptions of the Whitney extension theorem.

In 1951, Kneser presented a proof of this composition result in [201]. We proved a theorem which may be obtained as an immediate corollary to the theorem of Sard, though he did so without any reference to or influence from Sard's result. The composition theorem is also discussed in a different context in a 1958 by Glaeser [196]. See [193], [202], [204]. Thom [207], quickly realized that the method of Kneser can be used to prove the Sard theorem. See also [193], [202], [205]. Recently Figalli [43] used this method to provide a simpler proof of Theorem (5.2.3). Our proof of Theorem (5.2.13) we will also be based on the KneserGlaeser result.

We will explain notation and prove some technical results related to the Morrey inequality that will be used in the proof of Theorem (5.2.13).

Consider $f: \mathbb{R}^n \to \mathbb{R}$. By $D^\alpha f$ we will denote the partial derivative of f with respect to the multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. In particular $D^{\delta_i} f = \partial f / \partial x_i$, i.e. $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$ is a multiindex with 1 on ith position. Also $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. $D^k f$ will denote the vector whose components are the derivatives $D^\alpha f, |\alpha| = k$. The classes of functions with continuous and α-Hölder continuous derivatives of order up to k will be denoted by C^k and $C^{k,\alpha}$ respectively. The integral average over a set S of positive measure will be denoted by

$$f_S = \fint_S f(x)dx = \frac{1}{|S|}\int_S f(x)dx.$$

The characteristic function of a set E will be denoted by χ_E. The k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$. In particular $\mathcal{H}^0$ is the counting measure. The Lebesgue measure in $\mathbb{R}^n$ coincides with $\mathcal{H}^n$. In addition to the Hausdorff measure notation we will also write $|S|$ for the Lebesgue measure of S. We say that a set is k-null if its kdimensional Hausdorff measure equals zero. By $\mathcal{H}^k_\delta, \delta > 0$, we denote the pre-Hausdorff measure defined by taking infimum over coverings of the set by sets of diameters less than δ so $\mathcal{H}^k(E) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E)$. Cubes in $\mathbb{R}^n$ will always have sides parallel to coordinate directions. The symbol C will be used to represent a generic constant and the actual value of C may change in a single string of estimates. By writing $C = C(n, m)$ we indicate that the constant C depends on n and m only.

We will use the following elementary result several times.

Lemma (5.2.6)[192]: Let $E \subset \mathbb{R}^n$ be a bounded measurable set and let $-\infty < a < n$. Then there is a constant $C = C(n, a)$ such that for every $x \in E$

$$\int_E \frac{dy}{|x-y|^a} \le \begin{cases} C|E|^{1-\frac{a}{n}} & if\ 0 \le a < n. \\ (diam\, E)^{-a}|E| & if\ a < 0. \end{cases}$$

Proof. The case $a < 0$ is obvious since then $|x - y|^{-a} \le (diam\, E)^{-a}$. Thus assume that $0 \le a < n$. In this case the inequality is actually true for all $x \in \mathbb{R}^n$ and not only for $x \in E$. Let $B = B(0, r), |B| = |E|$. We have

$$\int_E \frac{dy}{|x-y|^a} \le \int_B \frac{dy}{|y|^a} = C\int_0^r t^{-a}t^{n-1}dt = Cr^{n-a} = C|E|^{1-\frac{a}{n}}.$$

The following result [195] is a basic pointwise estimate for Sobolev functions.

Lemma (5.2.7)[192]: Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $S \subset D$ be a measurable set of positive measure. If $f \in W^{1,p}(D), p \ge 1$, then

$$|f(x) - f_S| \le C(n)\,\frac{|D|}{|S|}\int_D \frac{|Df(z)|}{|x-z|^{n-1}}dz\ a.e. \tag{56}$$

When $p > n$, the triangle inequality $|f(y) - f(x)| \leq |f(y) - f_D| + |f(x) - f_D|$, Hölder inequality, and Lemma (5.2.6) applied to the right hand side of (56) yield a well known

Lemma (5.2.8) (Morrey's inequality)[192]: Suppose $n < p < \infty$ and $f \in W^{1,p}(D)$, where $D \subset \mathbb{R}^n$ Is a cube or a ball. Then there is a constant $C = C(n,p)$ such that

$$|f(y) - f(x)| \leq C(diam\, D)^{1-\frac{n}{p}} \left(\int_D |Df(z)|^p dz \right)^{\frac{1}{p}} \; for\ all\ x, y \in D.$$

In particular,

$$diam\, f(D) \leq C(diam\, D)^{1-\frac{n}{p}} \left(\int_D |Df(z)|^p dz \right)^{\frac{1}{p}}.$$

Since $p > n$, the function f is continuous (Sobolev embedding) and hence the lemma does indeed hold for all $x, y \in D$.

From this lemma we can easily deduce a corresponding result for higher order derivatives. The Taylor polynomial and the averaged Taylor polynomial of f will be denoted by

$T\ k\ x\ f(y) = X\ |\alpha| \leq k\ D\ \alpha\ f(x)\ (y - x)\ \alpha\ \alpha!\ , Tk\ S\ f(y) = Z\ S\ T\ k\ x\ f(y)\ dx.$

Lemma (5.2.9)[192]: Suppose $n < p < \infty, k \geq 1$ and $f \in W^{k,p}(D)$, where $D \subset \mathbb{R}^n$ is a cube or a ball. Then there is a constant $C = C(n,k,p)$ such that

$$\left|f(y) - T_x^{k-1} f(y)\right| \leq C(diam\, D)^{k-\frac{n}{p}} \left(\int_D \left|D^k f(z)\right|^p dz \right)^{\frac{1}{p}} \; for\ all\ x, y \in D.$$

Proof. Given $y \in D$ let

$$\psi(x) := T_x^{k-1} f(y) = \sum_{|\alpha| \leq k-1} D^\alpha f(x) \frac{(y-x)^\alpha}{\alpha!} \in W^{1,p}(D).$$

Observe that $\psi(y) = f(y)$ and

$$\frac{\partial \psi}{\partial x_j}(x) = \sum_{|\alpha| = k-1} D^{\alpha + \delta_j} f(x) \frac{(y-x)^\alpha}{\alpha!},$$

where $\delta_j = (0, \dots, 1, \dots, 0)$. Indeed, after applying the Leibniz rule to $\partial\psi/\partial x_j$ the lower order terms will cancel out. Since

$$|D\psi(z)| \leq C(n,k)\left|D^k f(z)\right| |y - z|^{k-1},$$

Lemma (5.2.8) applied to ψ yields the result.

Applying the same argument to Lemma (5.2.7) leads to the following result, see [53].

Lemma (5.2.10)[192]: Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $S \subset D$ be a measurable set of positive measure. If $f \in W^{k,p}(D), p \geq 1, k \geq 1$, then there is constant $C = C(n,k)$ such that

$$\left|f(x) - T_S^{k-1} f(x)\right| \leq C \frac{|D|}{|S|} \int_D \frac{\left|D^k f(z)\right|}{|x - z|^{n-k}} dz \quad for\ a.e.\ \ x \in D. \tag{57}$$

We will improve the above estimates under the additional assumption that the derivative Df vanishes on a given subset of D. For a similar result in a different setting see [199].

Lemma (5.2.11)[192]: Let $D \subset \mathbb{R}^n$ be a cube or a ball and let $f \in W^{k,p}(D), n < p < \infty, k \geq 1$. Let

$$A = \{x \in D | Df(x) = 0\}.$$

Then for any $\varepsilon > 0$ there is $\delta = \delta(n,k,p,\varepsilon) > 0$ such that if

$$\frac{|D \backslash A|}{|D|} < \delta,$$

then

$$diam\, f(D) \le \varepsilon (diam\, D)^{k-\frac{n}{p}} \left(\int_D \left| D^k f(z) \right|^p dz \right)^{\frac{1}{p}}.$$

Proof: Although only the first order derivatives of f are equal zero in A, it easily follows that $D^\alpha f = 0$ a.e. in A for all $1 \le |\alpha| \le k$. Indeed, if a Sobolev function is constant in a set, its derivative equals zero a.e. in the set, [195], and we apply induction. Hence

$$T_A^{k-1} f(x) = fA \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Let $\varepsilon > 0$. Choose $0 < \delta < 1/2$ with $\max\left\{ \delta^{\frac{k}{n}-\frac{1}{p}}, \delta^{1-\frac{1}{p}} \right\} < \varepsilon$. Since $\delta < 1/2, |D|/|A| < 2$. Thus Lemma (5.2.10) with $S = A$ yields

$$|f(x) - fA| \le C(n) \int_{D\backslash A} \frac{\left|D^k f(z)\right|}{|x-z|^{n-k}} dz \le C(n) \left\| D^k f \right\|_{L^p(D)} \left(\int_{D\backslash A} \frac{dz}{|x-z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}}.$$

Now the result follows directly from Lemma (5.2.6). Indeed, if $k \le n$, Lemma (5.2.6) and the estimate

$$|D\backslash A| < \delta |D| \le C(n) \delta (diam\, D)^n$$

yield

$$\left(\int_{D\backslash A} \frac{dz}{|x-z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \le C(n,k,p) |D\backslash A|^{\frac{1}{n}\left(k-\frac{n}{p}\right)} \le C(n,k,p) \delta^{\frac{k}{n}-\frac{1}{p}} (diam\, D)^{k-\frac{n}{p}}.$$

If $k > n$, then we have

$$\left(\int_{D\backslash A} \frac{dz}{|x-z|^{(n-k)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \le (diam\, D)^{k-n} |D\backslash A|^{\frac{p-1}{p}} \le C(n,p) \delta^{1-\frac{1}{p}} (diam\, D)^{k-\frac{n}{p}}.$$

Hence

$$\begin{aligned} diam\, f(D) &= \sup_{x,y \in D} |f(x) - f(y)| \le 2 \sup_{x \in D} |f(x) - fA| \\ &\le C(n,k,p) \varepsilon (diam\, D)^{k-\frac{n}{p}} \left\| D^k f \right\|_{L^p(D)}. \end{aligned}$$

The proof is complete.

We will also need the following classical Besicovitch covering lemma, see e.g. [49]

Lemma (5.2.12) (Besicovitch)[192]: Let $E \subset \mathbb{R}^n$ and let $\{B_x\}_{x \in E}$ be a family of closed balls $B_x = \bar{B}(x, r_x)$ so that $\sup_{x \in E}\{r_x\} < \infty$. Then there is a countable (possibly finite) subfamily $\left\{B_{x_i}\right\}_{i=1}^{\infty}$ with the property that

$$E \subset \bigcup_{i=1}^{\infty} B_{x_i}$$

and no point of $\mathbb{R}^n$ belongs to more than $C(n)$ balls.

Theorem (5.2.13)[192]: Fix $n, m, k \in \mathbb{N}$. Suppose $\Omega \subset \mathbb{R}^n$ is open and $f \in W_{loc}^{k,p}(\Omega, \mathbb{R}^m)$ for some $n < p < \infty$. If $\ell = \max(n - m - k + 1, 0)$, then

$$\mathcal{H}^\ell \left(C_f \cap f^{-1}(y) \right) = 0 \quad \text{for } a.e. \quad y \in \mathbb{R}^m.$$

Proof: As we pointed out in Introduction we may assume that $m \leq n$ and $k \geq 2$. It is also easy to see that we can assume that $\Omega = \mathbb{R}^n$ and $f \in W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$. Indeed, it suffices to prove the claim of Theorem (5.2.13) on compact subsets of $\boldsymbol{\Omega}$ and so we may multiply f by a compactly supported smooth cut-off function to get a function in $W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$.
We will prove the result using induction with respect to n.
If $n = 1$, then $m = n = 1$. This gives $n - m - k + 1 = 1 - k \leq 0$ for any $k \in \mathbb{N}$, so $\ell = 0$. Thus the theorem is a direct consequence of Theorem (5.2.4).
We shall prove now the theorem for $n \geq 2$ assuming that it is true in dimensions less than or equal to $n - 1$.
Fix p and integers m and k satisfying $n < p < \infty, m \leq n$, and $k \geq 2$. Write $\ell = \max(n - m - k + 1, 0)$. Let $f \in W^{k,p}(\mathbb{R}^n\ \mathbb{R}^m\ )$.
We can write

$$C_f = K \cup A_1 \cup \ldots \cup A_{k-1},$$

where

$$K := \left\{x \in C_f \middle| 0 < rank\ Df(x) < m\right\}$$

And

$$A_s := \{x \in \mathbb{R}^n | D^\alpha f(x) = 0 \quad \text{for all} \quad 1 \leq |\alpha| \leq s\}$$

Note that $A_1 \supset A_2 \supset \cdots \supset A_{k-1}$ is a decreasing sequence of sets.
In the first step, we will show that $A_{k-1} \cap f^{-1}(y)$ is ℓ-null for a.e. $y \in \mathbb{R}^m$. Then we will prove the same for $(A_{s-1} \backslash A_s) \cap f^{-1}(y)$ for $s = 2, 3, \ldots, k-1$. To do this we will use the Implicit Function and Kneser-Glaeser theorems to reduce our problem to a lower dimensional one and apply the induction hypothesis. Finally, we will consider the set K and use a change of variables to show that we can reduce the dimension in the domain and in the target so that the fact that $\mathcal{H}^\ell\left(K \cap f^{-1}(y)\right) = 0$ will follow from the induction hypothesis.
Claim (5.2.14)[192]: $\mathcal{H}^\ell\left(A_{k-1} \cap f^{-1}(y)\right) = 0$ for a.e. $y \in \mathbb{R}^m$.
Proof. Suppose $x \in A_{k-1}$. Notice that $T_x^{k-1} f(y) = f(x)$ for any $y \in \mathbb{R}^n$ since $D^\alpha f(x) = 0$ for every $1 \leq |\alpha| \leq k - 1$. By Lemma (5.2.9) applied to each coordinate of $f = (f_1, \ldots, f_m)$,
we have for any cube $Q \subset \mathbb{R}^n$ containing x and any $y \in Q$,

$$|f(y) - f(x)| \leq C(diam\ Q)^{k - \frac{n}{p}} \left(\int_Q \left|D^k f(z)\right|^p dz\right)^{\frac{1}{p}}. \tag{58}$$

Hence

$$diam\ f(Q) \leq C(diam\ Q)^{k - \frac{n}{p}} \left(\int_Q \left|D^k f(z)\right|^p dz\right)^{\frac{1}{p}}. \tag{59}$$

Let $F_1 := \{x \in A_{k-1} | x \text{ is a density point of } A_{k-1}\}$ and $F_2 := A_{k-1} \backslash F_1$. We will treat the sets $F_1 \cap f^{-1}(y)$ and $F_2 \cap f^{-1}(y)$ separately.
First we will prove that $\mathcal{H}^\ell\left(F_2 \cap f^{-1}(y)\right) = 0$ for almost every $y \in \mathbb{R}^m$.
Let $0 < \varepsilon < 1$. Since $\mathcal{H}^n(F_2) = 0$, there is an open set $F_2 \subset U \subset \Omega$ such that $\mathcal{H}^n(U) < \varepsilon^{\frac{p}{p-m}}$. For any $j \geq 1$ let $\left\{Q_{ij}\right\}_{i=1}^\infty$ be a collection of closed cubes with pairwise disjoint interiors such that

$$Q_{ij} \cap F_2 \neq \emptyset, \qquad F_2 \subset \bigcup_{i=1}^{\infty} Q_{ij} \subset U, \qquad diam\, Q_{ij} < \frac{1}{j}.$$

Since $F_2 \cap Q_{ij} \neq \emptyset$, (59) yields

$$\mathcal{H}^m\left(f(Q_{ij})\right) \leq C\left(diam\, f(Q_{ij})\right)m \leq C\left(diam\, Q_{ij}\right)^{m\left(k-\frac{n}{p}\right)}\left(\int_{Q_{ij}} \left|D^k f(x)\right|^p dx\right)^{\frac{m}{p}}.$$

Case: $n - m - k + 1 \leq 0$ so $\ell = 0$.

This condition easily implies that $mk \geq n$ so we also have $\frac{mp}{p-m}\left(k - \frac{n}{p}\right) \geq n$, and by Holder's inequality,

$$\mathcal{H}^m\left(f(F_2)\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^m\left(f(Q_{ij})\right) \leq C\sum_{i=1}^{\infty}\left(diam\, Q_{ij}\right)^{m\left(k-\frac{n}{p}\right)}\left(\int_{Q_{ij}} \left|D^k f(x)\right|^p dx\right)^{\frac{m}{p}}$$

$$\leq C\left(\sum_{i=1}^{\infty}\left(diam\, Q_{ij}\right)^{\frac{pm}{p-m}\left(k-\frac{n}{p}\right)}\right)^{\frac{p-m}{p}}\left(\int_{\bigcup_{i=1}^{\infty} Q_{ij}} \left|D^k f(x)\right|^p dx\right)^{\frac{m}{p}}$$

$$\leq C\mathcal{H}^n(U)^{\frac{p-m}{p}}\left(\int_U \left|D^k f(x)\right|^p dx\right)^{\frac{m}{p}} < C\varepsilon\left\|D^k f\right\|_p \qquad (60).$$

Since $\varepsilon > 0$ can be arbitrarily small, $\mathcal{H}^m\left(f(F_2)\right) = 0$ and hence $F_2 \cap f^{-1}(y) = \emptyset$, i.e. $\mathcal{H}^\ell\left(F_2 \cap f^{-1}(y)\right) = 0$ for a.e. $y \in \mathbb{R}^m$.

Case: $\ell = n - m - k + 1 > 0$.

The sets $\left\{Q_{ij} \cap f^{-1}(y)\right\}_{i=1}^{\infty}$ form a covering of $F_2 \cap f^{-1}(y)$ by sets of diameters less than $1/j$. Since

$$diam\left(Q_{ij} \cap f^{-1}(y)\right) \leq \left(diam\, Q_{ij}\right)\chi_{f(Q_{ij})}(y)$$

the definition of the Hausdorff measure yields

$$\mathcal{H}^\ell\left(F_2 \cap f^{-1}(y)\right) \leq C \liminf_{j\to\infty} \sum_{i=1}^{\infty} diam\left(Q_{ij} \cap f^{-1}(y)\right)^\ell$$

$$\leq C \liminf_{j\to\infty} \sum_{i=1}^{\infty}\left(diam\, Q_{ij}\right)^\ell \chi_{f(Q_{ij})}(y). \qquad (61)$$

We would like to integrate both sides with respect to $y \in \mathbb{R}^m$. Note that the function on the right hand side is measurable since the sets $f(Q_{ij})$ are compact. However measurability of the function $y \longmapsto H^\ell\left(F_2 \cap f^{-1}(y)\right)$ is far from being obvious. To deal with this problem we will use the upper integral which for a non-negative function $g: X \to [0, \infty]$ defined μ-a.e. on a measure space (X, μ) is defined as follows:

$$\int_X^* g\, d\mu = \inf\left\{\int_X \phi d\mu : 0 \leq g \leq \phi \text{ and } \phi \text{ is } \mu - measurable\right\}.$$

An important property of the upper integral is that if $\int_X^* g\, d\mu = 0$, then $g = 0$ μ-a.e. Indeed, there is a sequence $\phi_i \geq g \geq 0$ such that $\int_X \phi_i\, d\mu \to 0$. That means $\phi_i \to 0$ in $L^1(\mu)$. Taking a subsequence we get $\varphi_{i_j} \to 0$ μ-a.e. which proves that $g = 0$ μ-a.e.

Applying the upper integral with respect to $y \in \mathbb{R}^m$ to both sides of (61), using Fatou's lemma, and noticing that

$$\frac{p}{p-m}\left(\ell + m\left(k - \frac{n}{p}\right)\right) \geq n$$

gives

$$\int_{\mathbb{R}^m}^{*} \mathcal{H}^{\ell}\left(F_2 \cap f^{-1}(y)\right) d\mathcal{H}^m(y) \leq C \liminf_{j\to\infty} \sum_{i=1}^{\infty} \left(diam\, Q_{ij}\right)^{\ell} H_m\left(f\left(Q_{ij}\right)\right)$$

$$\leq C \liminf_{j\to\infty} \sum_{i=1}^{\infty} \left(diam\, Q_{ij}\right)^{\ell + m\left(k-\frac{n}{p}\right)} \left(\int_{Q_{ij}} \left|D^k f(x)\right|^p dx\right)^{\frac{m}{p}} < C\varepsilon \left\|D^k f\right\|_p$$

by the same argument as in (60). Again, since $\varepsilon > 0$ can be arbitrarily small, we conclude that $\mathcal{H}^{\ell}\left(F_2 \cap f^{-1}(y)\right) = 0$ for a.e. $y \in \mathbb{R}^m$.

It remains to prove that $\mathcal{H}^{\ell}\left(F_1 \cap f^{-1}(y)\right) = 0$ for almost every $y \in \mathbb{R}^m$. The proof is similar to that in Step 1 and the arguments which are almost the same will be presented in a more sketchy form now. In Step 1 it was essential that the set F_2 had measure zero. We will compensate the lack of this property now by the estimates from.

It suffices to prove that for any cube $\tilde{Q}, \mathcal{H}^{\ell}\left(\tilde{Q} \cap F_1 \cap f^{-1}(y)\right) = 0$ for a.e. $y \in \mathbb{R}^m$. Assume that $\tilde{Q}$ is in the interior of a larger cube $\tilde{Q} \Subset Q$.

For each $x \in \tilde{Q} \cap F_1$ and $j \in \mathbb{N}$ there is $0 < r_{jx} < 1/j$ such that

$$diam\, f\left(B\left(x, r_{jx}\right)\right) \leq j^{-1} r_{jx}^{k-\frac{n}{p}} \left(\int_{B\left(x,r_{jx}\right)} \left|D^k f(z)\right|^p dz\right)^{\frac{1}{p}}.$$

We may further assume that $B\left(x, r_{jx}\right) \subset Q$.

Denote $B_{jx} = \bar{B}\left(x, r_{jx}\right)$. According to the Besicovitch Lemma (5.2.12), there is a countable subcovering $\left\{B_{jx_i}\right\}_{i=1}^{\infty}$of $\tilde{Q} \cap F_1$ so that no point of $\mathbb{R}^n$ belongs to more than $C(n)$ balls B_{jxi}.

Case: $n - m - k + 1 \leq 0$ so $\ell = 0$.

We have $\frac{pm}{p-m}\left(k - \frac{n}{p}\right) \geq n$ as before, so

$$\mathcal{H}^m\left(f\left(\tilde{Q} \cap F_1\right)\right) \leq C \sum_{i=1}^{\infty} \mathcal{H}^m\left(f\left(B_{jx_i}\right)\right) \leq Cj^{-m} \sum_{i=1}^{\infty} r_{jx_i}^{m\left(k-\frac{n}{p}\right)} \left(\int_{B_{jx_i}} \left|D^k f(z)\right|^p dz\right)^{\frac{m}{p}}$$

$$\leq Cj^{-m} \left(\sum_{i=1}^{\infty} r_{jx_i}^{n}\right)^{\frac{p-m}{p}} \left(\sum_{i=1}^{\infty} \int_{B_{jx_i}} \left|D^k f(z)\right|^p dz\right)^{\frac{m}{p}}.$$

Since the balls are contained in Q and no point belongs to more than $C(n)$ balls we conclude that

$$\mathcal{H}^m\left(f\left(\tilde{Q} \cap F_1\right)\right) \leq Cj^{-m} \mathcal{H}^n(Q)^{\frac{p-m}{p}} \left\|D^k f\right\|_p^m.$$

Since j can be arbitrarily large, $\mathcal{H}^m\left(f\left(\tilde{Q} \cap F_1\right)\right) = 0$, i.e. $\mathcal{H}^{\ell}\left(\tilde{Q} \cap F_1 \cap f^{-1}(y)\right) = 0$ for a.e. $y \in \mathbb{R}^m$.

Case: $\ell = n - m - k + 1 > 0$.

The sets $\left\{B_{jx_i} \cap f^{-1}(y)\right\}_{i=1}^{\infty}$ form a covering of $\tilde{Q} \cap F_1 \cap f^{-1}(y)$ and

$$diam\left(B_{jx_i}\cap f^{-1}(y)\right)\le Cr_{jx_i}\chi_{f\left(B_{jx_i}\right)}(y).$$

The definition of the Hausdorff measure yields

$$H^{\ell}\left(\tilde{Q}\cap F_1\cap f^{-1}(y)\right)\le C\liminf_{j\to\infty}\sum_{i=1}^{\infty} r_{jx_i}^{\ell}\chi_{f\left(B_{jx_i}\right)}(y).$$

Thus as above

$$\int_{\mathbb{R}^m}^{*}\mathcal{H}^{\ell}\left(\tilde{Q}\cap F_1\cap f^{-1}(y)\right)d\mathcal{H}^m(y)\le C\liminf_{j\to\infty}\sum_{i=1}^{\infty} r_{jx_i}^{\ell}\mathcal{H}^m\left(f\left(B_{jx_i}\right)\right)$$

$$\le C\ \liminf_{j\to\infty} j^{-m}\sum_{i=1}^{\infty} r_{jx_i}^{\ell+m\left(k-\frac{n}{p}\right)}\left(\int_{B_{jx_i}}\left|D^k f(z)\right|^p dz\right)^{\frac{m}{p}}$$

$$\le C\ \liminf_{j\to\infty} j^{-m}\mathcal{H}^n(Q)^{\frac{p-m}{p}}\left\|D^k f\right\|_p^m=0$$

since $\frac{p}{p-m}\left(\ell+m\left(k-\frac{n}{p}\right)\right)\ge n$. Therefore $\mathcal{H}^{\ell}\left(\tilde{Q}\cap F_1\cap f^{-1}(y)\right)=0$ for a.e. $y\in\mathbb{R}^m$. This completes the proof that $\mathcal{H}^{\ell}\left(F_1\cap f^{-1}(y)\right)=0$ for a.e. $y\in\mathbb{R}^m$ and hence that of Claim (5.2.14)

Claim (5.2.15)[192]: $\mathcal{H}^{\ell}\left((A_{s-1}\setminus A_s)\cap f^{-1}(y)\right)=0$ for a.e. $y\in\mathbb{R}^m, s=2,3,\ldots,k-1$.

In this step, we will use the Kneser-Glaeser composition theorem and the implicit function theorem to apply the induction hypothesis in $\mathbb{R}^{n-1}$.

Fix $s\in\{2,3,\ldots,k-1\}$ and $\bar{x}\in A_{s-1}\setminus A_s$. It suffices to show that the ℓ-Hausdorff measure of $W\cap(A_{s-1}\setminus A_s)\cap f^{-1}(y)$ is zero for some neighborhood W of $\bar{x}$ and a.e. $y\in\mathbb{R}^n$. Indeed, $A_{s-1}\setminus A_s$ can be covered by countably many such neighborhoods.

By the definitions of A_s and $A_{s-1}, D^{\gamma}f(\bar{x})=0$ for all $1\le|\gamma|\le s-1$, and $D^{\beta}f(\bar{x})\ne 0$ for some $|\beta|=s$. That is, for some $|\gamma|=s-1$ and $j\in\{1,\ldots,m\}$, $D\left(D^{\gamma}f_j\right)(\bar{x})\ne 0$ and $D^{\gamma}f_j\in W^{k-(s-1),p}\subset C^{k-s,1-\frac{n}{p}}$.

Hence, by the implicit function theorem, there is some neighborhood U of $\bar{x}$ and an open set $V\subset\mathbb{R}^{n-1}$ so that $U\cap\left\{D^{\gamma}f_j=0\right\}=g(V)$ for some $g\colon V\to\mathbb{R}^n$ of class C^{k-s}. In particular, $U\cap A_{s-1}\subset g(V)$ since $D^{\gamma}f_j=0$ on A_{s-1}.

Choose a neighborhood $W\Subset U$ of $\bar{x}$ and say $A^*:=g^{-1}(W\cap A_{s-1})$ so that A^* is compact. Since f is $s-1$ flat on the closed set A_{s-1}, f is of class C^{k-1}, g is of class $C^{(k-1)-(s-1)}$, and $g(A^*)\subset A_{s-1}$, we can apply Theorem (5.2.5) to each component of f to find a C^{k-1} function $F\colon\mathbb{R}^{n-1}\to\mathbb{R}^m$ so that, for every $x\in A^*, F(x)=(f\circ g)(x)$ and $D^{\lambda}F(x)=0$ for all $|\lambda|\le s-1$. That is, $A^*\subset C_F$. Hence

$$\mathcal{H}^{\ell}\left(A^*\cap F^{-1}(y)\right)\le\mathcal{H}^{\ell}\left(C_F\cap F^{-1}(y)\right)=0.$$

for almost every $y\in\mathbb{R}^m$. In this last equality, we invoked the induction hypothesis on $F\in C^{k-1}(\mathbb{R}^{n-1},\mathbb{R}^m)\subset W_{loc}^{k-1,p}(\mathbb{R}^{n-1},\mathbb{R}^m)$ with $\ell=\max\left((n-1)-m-(k-1)+1,0\right)$. Since g is of class C^1, it is locally Lipschitz, and so $\mathcal{H}^{\ell}\left(g\left(A^*\cap F^{-1}(y)\right)\right)=0$ for almost every $y\in\mathbb{R}^m$. Since $\cap A_{s-1}\subset g(A^*)$, we have

$$W\cap A_{s-1}\cap f^{-1}(y)\subset g\left(A^*\cap F^{-1}(y)\right)$$

for all $y\in\mathbb{R}^m$, and thus

$$\mathcal{H}^{\ell}\big(W \cap (A_{s-1} \backslash A_s) \cap f^{-1}(y)\big) \leq \mathcal{H}^{\ell}\big(W \cap A_{s-1} \cap f^{-1}(y)\big) = 0$$
for almost every $y \in \mathbb{R}^n$. The proof of the claim is complete.

Claim (5.2.16)[192]: $\mathcal{H}^{\ell}\big(K \cap f^{-1}(y)\big) = 0$ for a.e. $y \in \mathbb{R}^m$.

Proof. Write $K = \bigcup_{r=1}^{m-1} K_r$ where $K_r := \{x \in \mathbb{R}^n | rank\ Df(x) = r\}$. Fix $x_0 \in K_r$ for some $r \in \{1, \dots, m-1\}$. For the same reason as in Claim (5.2.15) it suffices to show that $\mathcal{H}^{\ell}\big((V \cap K_r) \cap f^{-1}(y)\big) = 0$ for some neighborhood V of x_0 for a.e. $y \in \mathbb{R}^m$.

Without loss of generality, assume that the submatrix $\left[\frac{\partial f_i}{\partial x_j(x_0)}\right]_{i,j=1}^{r}$ formed by the first r rows and columns of Df has rank r. Let
$$Y(x) = (f_1(x), f_2(x), \dots, f_r(x), x_{r+1}, \dots, x_n) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{62}$$
Y is of class C^{k-1} since each component of f is. Also, rank $DY\ (x_0) = n$, so by the inverse function theorem Y is a C^{k-1} diffeomorphism of some neighborhood V of x0 onto an open set $\tilde{V} \subset \mathbb{R}^n$. From now on we will assume that Y is defined in V only.

Claim (5.2.17)[192]: $Y^{-1} \in W^{k,p}_{loc}\big(\tilde{V}, \mathbb{R}^n\big)$.

Lemma (5.2.18)[192]: Let $\Omega \subset \mathbb{R}^n$ be open. If $g, h \in W^{\ell,p}_{loc}(\Omega)$, where $p > n$ and $\ell \geq 1$, then $gh \in W^{\ell,p}_{loc}(\Omega)$.

Proof. Since $g, h \in C^{\ell-1}$, it suffices to show that the classical partial derivatives $D^{\beta}(gh), |\beta| = \ell - 1$ belong to $W^{1,p}_{loc}(\Omega)$ (when $\ell = 1, \beta = 0$ so $D^{\beta}(gh) = gh$). The product rule for $C^{\ell-1}$ functions yields
$$D^{\beta}(gh) = \sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\,\delta!} D^{\gamma} g D^{\delta} h. \tag{63}$$
Each of the functions $D^{\gamma} g, D^{\delta} h$ is absolutely continuous on almost all lines parallel to coordinate axes, [60], so is their product. Thus $D^{\beta}(gh)$ is absolutely continuous on almost all lines and hence it has partial derivatives (or order 1) almost everywhere. According to a characterization of $W^{1,p}_{loc}$ by absolute continuity on lines, [60], it suffices to show that partial derivatives of $D^{\beta}(gh)$ (of order 1) belong to L^p_{loc}. This will imply that $D^{\beta}(gh) \in W^{1,p}_{loc}$ for all $\beta, |\beta| = \ell - 1$ so $gh \in W^{\ell,p}_{loc}$.

If $D^{\alpha} = D^{\delta_i} D^{\beta}$, then the product rule applied to the right hand side of (63) yields
$$D^{\alpha}(gh) = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\,\delta!} D^{\gamma} g D^{\delta} h.$$
If $|\gamma| < |\alpha| = \ell$ and $|\delta| < |\alpha| = \ell$, then the function $D^{\gamma} g D^{\delta} h$ is continuous and hence in L^p_{loc}. The remaining terms are $hD^{\alpha} g + gD^{\alpha} h$. Clearly this function also belongs to L^p_{loc} because the functions g, h are continuous and $D^{\alpha} g, D^{\alpha} h \in L^p_{loc}$. This completes the proof of the lemma.

Now we can complete the proof of Claim (5.2.17). Since Y is a diffeomorphism of class C^{k-1}, we have
$$D(Y^{-1})(y) = \big[DY\big(Y^{-1}(y)\big)\big]^{-1} \quad \text{for every} \quad y \in \tilde{V}. \tag{64}$$
It suffices to prove that $(Y^{-1}) \in W^{k-1,p}_{loc}$. It follows from (64) and a formula for the inverse matrix that
$$D(Y^{-1}) = \left(\frac{P_1(Df)}{P_2(Df)}\right) \circ Y^{-1},$$

where P_1 and P_2 and polynomials whose variables are replaced by partial derivatives of f. The polynomial $P_2(Df)$ is just $det\ DY$.
Since $Df \in W_{loc}^{k-1,p}$ and $p > n$, it follows from Lemma (5.2.18) that
$$P_1(Df), P_2(Df) \in W_{loc}^{k-1,p}.$$
Note that $P_2(Df) = det\ DY$ is continuous and different than zero. Hence
$$\frac{1}{P_2(Df)} \in W_{loc}^{k-1,p}$$
as a composition of a $W_{loc}^{k-1,p}$ function which is locally bounded away from 0 and ∞ with a smooth function $x \longmapsto x^{-1}$. Thus Lemma (5.2.18) applied one more time yields that $P_1(Df)/P_2(Df) \in W_{loc}^{k-1,p}$. Finally
$$D(Y^{-1}) = \left(\frac{P_1(Df)}{P_{2(Df)}}\right) \circ Y^{-1} \in W_{loc}^{k-1,p}$$
because composition with a diffeomorphism Y^{-1} of class C^{k-1} preserves $W_{loc}^{k-1,p}$. The proof of the claim is complete.
It follows directly from (62) that
$$f\big(Y^{-1}(x)\big) = \big(x_1, \dots, x_r, g(x)\big) \tag{65}$$
for all $x \in \tilde{V}$ and some function $g: \tilde{V} \to \mathbb{R}^{m-r}$.
Claim (5.2.19)[192]: $g \in W_{loc}^{k,p}\big(\tilde{V}, \mathbb{R}^{m-r}\big)$.
This statement is a direct consequence of the next
Lemma (5.2.20)[192]: Let $\Omega \subset \mathbb{R}^n$ be open, $p > n$ and $k \geq 1$. If $\Phi \in W_{loc}^{k,p}(\Omega, \mathbb{R}^n)$ is a diffeomorphism and $u \in W_{loc}^{k,p}\big(\Phi(\Omega)\big)$, then $u \circ \Phi \in W_{loc}^{k,p}(\Omega)$.
Proof. When $k = 1$ the result is obvious because diffeomorphisms preserve $W_{loc}^{1,p}$. Assume thus that $k \geq 2$. Since $p > n, \Phi \in C^{k-1}$ so Φ is a diffeomorphism of class C^{k-1}, but also $u \in C^{k-1} \subset C^1$ and hence the classical chain rule gives
$$D(u \circ \Phi) = \big((Du) \circ \Phi\big) \cdot D\Phi. \tag{66}$$
Since $Du \in W_{loc}^{k-1,p}$ and Φ is a diffeomorphism of class C^{k-1}, we conclude that $(Du) \circ \Phi \in W_{loc}^{k-1,p}$. Now the fact that $D\Phi \in W_{loc}^{k-1,p}$ combined with (66) and Lemma (5.2.18) yield that the right hand side of (66) belongs to $W_{loc}^{k-1,p}$ so $D(u \circ \Phi) \in W_{loc}^{k-1,p}$ and hence $\circ\ \Phi \in W_{loc}^{k,p}$. This compltes the proof of Lemma (5.2.20) and hence that of Claim (5.2.19).
Now we can complete the proof of Claim (5.2.16). Recall that we need to prove that
$$\mathcal{H}^{\ell}\big((V \cap K_r) \cap f^{-1}(y)\big) = 0 \quad for\ a.e.\ \ y \in \mathbb{R}^m \tag{67}$$
The diffeomorphism Y^{-1} is a change of variables that simplifies the structure of the mapping f because $f \circ Y^{-1}$ fixes the first r coordinates (see (65)) and hence it maps $(n-r)$-dimensional slices orthogonal to $\mathbb{R}^r$ to the corresponding $(m-r)$-dimensional slices orthogonal to $\mathbb{R}^r$. Because of this observation it is more convenient to work with $f \circ Y^{-1}$ rather than with f. Translating (67) to the case of $f \circ Y^{-1}$ it suffices to show that
$$\mathcal{H}^{\ell}\left(\left(\tilde{V} \cap Y(K_r)\right) \cap (f \circ Y^{-1})^{-1}\right)(y) = 0 \quad for\ a.e.\ \ y \in \mathbb{R}^m.$$
We used here a simple fact that the diffeomorphism Y preserves ℓ-null sets.
Observe also that
$$rank\ D(f \circ Y^{-1})(x) = r \quad for \quad x \in \tilde{V} \cap Y(K_r). \tag{68}$$

For any $\tilde{x} \in \mathbb{R}^r$ and $A \subset \mathbb{R}^n$, we will denote by $A_{\tilde{x}}$ the $(n-r)$–dimensional slice of A with the first r coordinates equal to $\tilde{x}$. That is, $A_{\tilde{x}} := \{z \in \mathbb{R}^{n-r} | (\tilde{x}, z) \in A\}$. Let $g_{\tilde{x}}: \tilde{V}_{\tilde{x}} \to \mathbb{R}^{m-r}$ be defined by $g_{\tilde{x}}(z) = g(\tilde{x}, z)$. With this notation

$$(f \circ Y^{-1})(\tilde{x}, z) = \left(\tilde{x}, g_{\tilde{x}}(z)\right)$$

and hence for $y = (\tilde{x}, w) \in \mathbb{R}^m$

$$\left(\tilde{V} \cap Y(K_r)\right) \cap (f \circ Y^{-1})^{-1}(y) = g_{\tilde{x}}^{-1}(w) \cap \left(\tilde{V} \cap Y(K_r)\right)_{\tilde{x}}.$$

The set on the left hand side is contained in an affine $(n-r)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $\mathbb{R}^r$ at $\tilde{x}$ while the set on the right hand side is contained in $\mathbb{R}^{n-r}$ but the two sets are identified through a translation by the vector $(\tilde{x}, 0) \in \mathbb{R}^n$ which identifies $\mathbb{R}^{n-r}$ with the affine subspace orthogonal to $\mathbb{R}^r$ at $\tilde{x}$.

According to the Fubini theorem it suffices to show that for almost all $\tilde{x} \in \mathbb{R}^r$ the following is true: for almost all $w \in \mathbb{R}^{m-r}$

$$\mathcal{H}^{\ell}\left(g_{\tilde{x}}^{-1}(w) \cap \left(\tilde{V} \cap Y(K_r)_{\tilde{x}}\right)\right) = 0. \tag{69}$$

As we will see this is a direct consequence of the induction hypothesis applied to the mapping $g_{\tilde{V}}: \tilde{x}_{\tilde{x}} \to \mathbb{R}^{n-r}$ defined in a set of dimension $n - r \le n - 1$. We only need to check that $g_{\tilde{x}}$ satisfies the assumptions of the induction hypothesis.

It is easy to see that for each $x = (\tilde{x}, z) \in \tilde{V}$

$$D(f \circ Y^{-1})(x) = \begin{pmatrix} d_{r\times r} & 0 \\ * & D(g_{\tilde{x}})(z) \end{pmatrix}.$$

This and (68) imply that for each $\tilde{x} \in \mathbb{R}^r$, $Dg_{\tilde{x}} = 0$ on the slice $\left(\tilde{V} \cap Y(K_r)\right)_{\tilde{x}}$. Hence the set $\left(\tilde{V} \cap Y\ (K_r)\right)_{\tilde{x}}$ is contained in the critical set of $g_{\tilde{x}}$ so

$$\mathcal{H}^{\ell}\left(g_{\tilde{x}}^{-1}(w) \cap \left(\tilde{V} \cap Y(K_r)\right)_{\tilde{x}}\right) \le \mathcal{H}^{\ell}\left(g_{\tilde{x}}^{-1}(w) \cap C_{g_{\tilde{x}}}\right). \tag{70}$$

It follows from the Fubini theorem applied to Sobolev spaces that for almost all $\tilde{x} \in \mathbb{R}^n$, $g_{\tilde{x}} \in W^{k,p}_{loc}\left(\tilde{V}_{\tilde{x}}, \mathbb{R}^{m-r}\right)$ and hence the induction hypothesis is satisfied for such mappings

$$W^{k,p}_{loc} \ni\ g_{\tilde{x}}: \tilde{V}_{\tilde{x}} \subset \mathbb{R}^{n-r} \to \mathbb{R}^{m-r}.$$

Since

$$\ell = \max(n - m - k + 1, 0) = \max\left((n - r) - (m - r) - k + 1, 0\right),$$

for almost all $w \in \mathbb{R}^{m-n}$ the expression on the right hand side of (70) equals zero and (69) follows. This completes the proof of Claim (5.2.16) and hence that of the theorem.

Section (5.3): Abridge Between Dubovitskiĭ –Federer Theorems

The Morse–Sard theorem in its classical form states that the image of the set of critical points of a C^{n-m+1} smooth mapping $v : \mathbb{R}^n \to \mathbb{R}^m$ has zero Lebesgue measure in $\mathbb{R}^m$. Assuming that $n \ge m$, the set of critical points for v is $Z_v = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla v(x) < m\}$ and the conclusion is that

$$\mathrm{L}^m(v(Z_v)) = 0. \tag{71}$$

The theorem was proved by Morse [69] in the case $m = 1$ and subsequently by Sard [47] in the general vector-valued case. The celebrated results of Whitney [48] show that the C^{n-m+1} smoothness assumption on the mapping v is sharp. However, the following result gives valuable information also for less smooth mappings.

Theorem (5.3.1)[208]: (Dubovitskiĭ 1957 [59]). Let $n, m, k \in \mathbb{N}$, and let $v : \mathbb{R}^n \to \mathbb{R}^m$ be a C^k-smooth mapping. Put $s = n - m - k + 1$. Then

$$\mathcal{H}^s(Z_v \cap v^{-1}(y)) = 0 \quad \text{for} \quad a.a.\, y \in \mathbb{R}^m, \tag{72}$$
where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure and Z_v is the set of critical points of v.
Here and in the following we interpret $\mathcal{H}^\beta$ as the counting measure when $\beta \le 0$. Thus for $k \ge n - m + 1$ we have $s \le 0$, and $\mathcal{H}^s$ in (72) becomes simply the counting measure, so the Dubovitskiĭ theorem contains the Morse–Sard Theorem (5.3.1)s particular case.
A few years later and almost simultaneously, Dubovitskiĭ [183] in 1967 and Federer [61] in 19692 published another important generalization of the Morse–Sard theorem.
Theorem (5.3.2)[208]: (Dubovitskiĭ–Federer). For $n, k, d \in \mathbb{N}$ let $m \in \{1, \ldots, min(n, d)\}$ and $v : \mathbb{R}^n \to \mathbb{R}^d$ be a C^k-smooth mapping. Put $q_\circ = m + \frac{s}{k}$. Then
$$\mathcal{H}^{q_\circ}\left(v\left(Z_{v,m}\right)\right) = 0, \tag{73}$$
where, as above, $s = n - m - k + 1$ and $Z_{v,m}$ denotes the set of m-critical points of v defined as $Z_{v,m} = \{x \in \mathbb{R}^n : \text{rank}\, \nabla v(x) < m\}$.
In view of the wide range of applicability of the above results it is a natural and compelling problem to decide to what extent they admit extensions to classes of Sobolev mappings. The first Morse–Sard result in the Sobolev context that we are aware of is due to L. De Pascale [39] (though see also [63]). It states that (71) holds for mappings v of class $W^k_{p,loc}(\mathbb{R}^n, \mathbb{R}^m)$ when $k \ge max(n - m + 1, 2)$ and $p > n$. Note that by the Sobolev embedding Theorem (5.3.1)ny mapping on $\mathbb{R}^n$ which is locally of Sobolev class W^k_p for some $p > n$ is in particular C^{k-1}, so the critical set Z_v can be defined as usual.
In [192] P. Hajłasz and S. Zimmerman proved Theorem (5.3.1) under the assumption that $v \in W^k_{p,loc}(\mathbb{R}^n, \mathbb{R}^m), p > n$, which corresponds to that used by L. De Pascale [39].
In view of the existing counter-examples to Morse–Sard type results in the classical C^k context the issue is not the value of k, — that is, how many weak derivatives are needed. Instead the question is, what are the minimal integrability assumptions on the weak derivatives for Morse–Sard type results to be valid in the Sobolev case. Of course, it is natural here to restrict attention to continuous mappings, and so to require from the considered function spaces that the inclusion $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ should guarantee at least the continuity of v (assuming always that the mappings are precisely represented). For values $k \in \{1, \ldots, n-1\}$ it is well-known that $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \le \frac{n}{k}$. So the borderline case is $= p_\circ = \frac{n}{k}$. It is well-known (see for instance [62], [50]) that $v \in W^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$ is continuous if the derivatives of k-th order belong to the Lorentz space $L_{p_\circ,1}$, we will denote the space of such mappings by $W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$.
In [180] it was shown that mappings $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$ are differentiable (in the classical Fréchet–Peano sense) at each point outside some $\mathcal{H}^{p_\circ}$ -negligible set A_v. Thus we define for integers $m \le min\{n, d\}$ the m-critical set as
$$Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : rank\, \nabla v(x) < m\}. \tag{74}$$
In previous joint of two of the authors with J. Bourgain [37], [56] and in [50], [180] this definition of critical set was used and a corresponding Dubovitskiĭ–Federer Theorem (5.3.2) was established for mappings of Sobolev class $W^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$. If, in addition, the highest derivative $\nabla^k v$ belongs to the Lorentz space $L_{p_\circ,1}$ (in particular, if $k = n$ since $L_{1,1} = L_1$), also the Luzin N-property with respect to the $p_\circ$-dimensional Hausdorff content was proven.

It implies, in particular, that the image of the set A_v of nondifferentiability points has zero measure, and consequently, C1-smoothness of almost all level sets follows. These facts found fruitful applications in fluid mechanics (see, e.g., [188]).
We prove the Dubovitskiĭ Theorem (5.3.1) for mappings of the same Sobolev– Lorentz class $W^k_{p_\circ,1}$ and with values in $\mathbb{R}^d$ for arbitrary $d \geq m$.
Theorem (5.3.3)[208]: Let $k, m \in \{1, \dots, n\}, d \geq m$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then the equality

$$\mathcal{H}^s(Z_{v,m} \cap v^{-1}(y)) = 0 \qquad for \;\; \mathcal{H}^m - a.a. y \in \mathbb{R}^d \tag{75}$$

holds, where as above $s = n - m - k + 1$ and $Z_{v,m}$ denotes the set of m-critical points of v: $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : rank\, \nabla v(x) < m\}$.
The result is new even when the mapping $v : \mathbb{R}^n \rightarrow \mathbb{R}^d$ is of class C^k since we allow here $m < d$ (compare with Theorem (5.3.1)). However, the main thrust of the result is the extension to the Sobolev–Lorentz context that we believe is essentially sharp. We also wish to emphasize that the result is in harmony with our definition of critical set (recall that $\mathcal{H}^{p_\circ}(A_v) = 0$) and the following new analog of the Luzin N-property:
Theorem (5.3.4)[208]: Let $k, m \in \{1, \dots, n\}, d \geq m$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for any set A with $\mathcal{H}^{p_\circ}(A) = 0$ we have

$$\mathcal{H}^s(A \cap v^{-1}(y)) = 0 \qquad for \;\; \mathcal{H}^m - a.a. y \in \mathbb{R}^d, \tag{76}$$

where again $s = n - m - k + 1$.
We end with remarks about the possibility to localize our results.

We extend the Dubovitskiĭ Theorem (5.3.1) to the Sobolev context (since the Federer–Dubovitskiĭ Theorem (5.3.2) had been extended before in [50], [180]). The very natural question arose. Theorem (5.3.1) asserts that $\mathcal{H}^m$-almost all preimages are small (with respect to $\mathcal{H}^s$-measure), and Theorem (5.3.2) claims that $\mathcal{H}^{q_\circ}$ -almost all preimages are empty. Could we connect these results? could we say something about $\mathcal{H}^q$-almost all preimages for other values of q, say, for $q \in [m - 1, q_\circ]$? The affirmative answer is contained in the next theorem.
Theorem (5.3.5)[208]: Let $k, m \in \{1, \dots, n\}, d \geq m$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for any $q \in (m - 1, \infty)$ the equality

$$\mathcal{H}^{\mu_q}\left(Z_{v,m} \cap v - 1(y)\right) = 0 \qquad for \;\; \mathcal{H}^q - a.a. y \in \mathbb{R}^d \tag{77}$$

holds, where

$$\mu_q := s + k(m - q), \quad s = n - m - k + 1, \tag{78}$$

and $Z_{v,m}$ again denotes the set of m-critical points of v: $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : rank\, \nabla v(x) \leq m - 1\}$.
Let us note, that the behavior of the function μ_q is very natural:
$\mu_q = 0$ for $q = q_\circ = m - 1 + \frac{n-m+1}{k}$ (Dubovitskiĭ–Federer Theorem (5.3.2))
$\mu_q < 0$ for $q > q_\circ$ [ibid.]
$\mu_q = s$ for $q = m$ (Dubovitskiĭ Theorem (5.3.1))

$$\mu_q = n - m + 1 \;\; for \; q = m - 1. \tag{79}$$

The last value cannot be improved in view of the trivial example of a linear mapping L: $\mathbb{R}^n \rightarrow \mathbb{R}^d$ of rank $m - 1$.
Thus, Theorem (5.3.5) contains all the previous theorems as particular cases and it is new even for the smooth case.

We emphasize the fact that in stating Theorem (5.3.5) we skipped the borderline case $q = m-1, \mu_q = n-m+1$. For this case we cannot assert that $\mathcal{H}^{m-1}$-almost all preimages in the m-critical set $Z_{v,m}$ have zero $\mathcal{H}^{n-m+1}$-measure as the above mentioned counterexample with a linear mapping $L: \mathbb{R}^n \to \mathbb{R}^d$ of rank $m-1$ shows. But for this borderline case we obtain instead the following analog of the classical coarea formula:

Theorem (5.3.6)[208]: Let $n, d \in \mathbb{N}, m \in \{0, \dots, min(n,d)\}$, and $v \in W^1_{n,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for any Lebesgue measurable subset E of $Z_{v,m+1} = \{x \in \mathbb{R}^n \setminus A_v : rank\, \nabla v(x) \le m\}$ we have

$$\int_E J_m v(x)\, dx = \int_{\mathbb{R}^d} \mathcal{H}^{n-m}\big(E \cap v^{-1}(y)\big) d\mathcal{H}^m(y), \qquad (80)$$

where $J_m v(x)$ denotes the m-Jacobian of v defined as the product of the m largest singular values of the matrix $\nabla v(x)$.

The proof relies crucially on the results of [219] and [214] that give criteria for the validity of the coarea formula for Lipschitz mappings between metric spaces, see also [209] and [66], [203].

Thus, to study the level sets for the borderline case $q = m-1$ in Theorem (5.3.5), one must take $m' = m-1$ instead of m in Theorem (5.3.6).

From the Coarea formula (80) it follows directly, that the set of $y \in \mathbb{R}^d$ where the integrand in the right-hand side of (80) is positive, is $\mathcal{H}^m$–σ-finite. Indeed, from Theorem (5.3.6) and [214] we obtain immediately the following more precise statement:

Corollary (5.3.7)[208]: Let $m \in \{0, \dots, min(d,n)\}$ and $v \in W^1_{n,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then the set

$$\left\{y \in \mathbb{R}^d : \mathcal{H}^{n-m}\left(Z_{v,m+1} \cap v^{-1}(y)\right) > 0\right\}$$

is $\mathcal{H}^m$-rectifiable, i.e., it is a union of a set of $\mathcal{H}^m$-measure zero and a countable family of images $g_i(S_i)$ of Lipschitz mappings $g_i : S_i \subseteq \mathbb{R}^m \to \mathbb{R}^d$. Here again $Z_{v,m+1} = \{x \in \mathbb{R}^n \setminus A_v : rank\, \nabla v(x) \le m\}$.

Again In harmony with our definition of critical set (recall that $\mathcal{H}^{p_\circ}(A_v) = 0$) because of the following analog of the Luzin N-property:

In particular,

$$\mathcal{H}^p\big(v(E)\big) = 0 \;\; whenever \;\; \mathcal{H}^p(E) = 0, p \in [p_\circ, n]. \qquad (81)$$

By a simple calculation we have for $q \in [0, q_\circ]$ that

$$\begin{aligned}\mu_q &= n - m - k + 1 + k(m-q) \\ &= (p_\circ - q)k + (m-1)(k-1) \ge max(p_\circ - q, 0). \qquad (82)\end{aligned}$$

Theorem (5.3.24) then yields

Corollary (5.3.8)[208]: Let $k, m \in \{1, \dots, n\}$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for every $q \in [0, +\infty)$ and for any set E with $\mathcal{H}^{p_\circ}(E) = 0$ we have

$$\mathcal{H}^{\mu_q}(E \cap v^{-1}(y)) = 0 \quad for\ \mathcal{H}^q\text{–}a.a.y \in \mathbb{R}^d. \qquad (83)$$

Consequently, for every $q \in [0, +\infty)$

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad for\ \mathcal{H}^q - a.a.y \in \mathbb{R}^d, \qquad (84)$$

where we recall that A_v is the set of nondifferentiability points of v (cf. with (77)).

Finally, applying the N-property (Theorem (5.3.24)) for $p = n, q = m \le n$, we obtain

Corollary (5.3.9)[208]: Let $n, d \in \mathbb{N}, m \in [0, n]$, and $v \in W^1_{n,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for any set E of zero n-Lebesgue measure $L^n(E) = 0$ the identity

$$\mathcal{H}^{n-m}\big(E \cap v^{-1}(y)\big) = 0 \quad for\ \mathcal{H}^m - a.a.y \in \mathbb{R}^d \qquad (85)$$

holds.

Thus the sets of n-Lebesgue measure zero (in particular, the set of nondifferentiability points Av) are negligible in the Coarea formula (80).
Finally, let us comment briefly on the proofs that merge ideas from the previous [56], [50], [180] and [192]. In particular, the joint [37], [56] by two with J. Bourgain contain many of the key ideas that allow us to consider nondifferentiable Sobolev mappings. For the implementation of these ideas one relies on estimates for the Hardy–Littlewood maximal function in terms of Choquet type integrals with respect to Hausdorff capacity. In order to take full advantage of the Lorentz context we exploit the recent estimates from [180] (recalled in Theorem (5.3.13) below, see also [51] for the case $p = 1$). As in [56] (and subsequently in [50]) we also crucially use Y. Yomdin's (see [78]) entropy estimates of near critical values for polynomials (recalled in Theorem (5.3.14) below).
In addition to the above mentioned there is a growing number on the topic, including [29], [210]–[53], [57], [22], [213], [68], [70], [71], [76], [191].

By an n-dimensional interval we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If Q is an n-dimensional cubic interval then we write $\ell(Q)$ for its sidelength.
For a subset S of $\mathbb{R}^n$ we write $\mathrm{L}^n(S)$ for its outer Lebesgue measure. The m-dimensional Hausdorff measure is denoted by $\mathcal{H}^m$ and the m-dimensional Hausdorff content by $\mathcal{H}^m_\infty$. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$\mathcal{H}^m(S) = \lim_{\alpha \searrow 0} \mathcal{H}^m_\alpha(S) = \sup_{\alpha > 0} \mathcal{H}^m_\alpha(S),$$

where for each $0 < \alpha \le \infty$,

$$\mathcal{H}^m_\alpha(S) = inf \left\{ \sum_{i=1}^{\infty} (diam\, S_i)^m : diam\, S_i \le \alpha, S \subset \bigcup_{i=1}^{\infty} S_i \right\}.$$

It is well known that $\mathcal{H}^n(S) = \mathcal{H}^n_\infty(S) \sim \mathrm{L}^n(S)$ for sets $S \subset \mathbb{R}^n$.
To simplify the notation, we write $\|f\|_{L_p}$ instead of $\|f\|_{L_p(\mathbb{R}^n)}$, etc.
The Sobolev space $W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$-valued functions $f \in L_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \le k$ belong to $L_p(\mathbb{R}^n)$ (for detailed definitions and differentiability properties of such functions see, e.g., [60], [218], [79], [58]). Denote by $\nabla^k f$ the vector-valued function consisting of all k-th order partial derivatives of f arranged in some fixed order. However, for the case of first order derivatives $k = 1$ we shall often think of $\nabla f(x)$ as the Jacobi matrix of f at x, thus the $d \times n$ matrix whose r-th row is the vector of partial derivatives of the r-th coordinate function.
We use the norm

$$\|f\|_{W^k_p} = \|f\|_{L^p} + \|\nabla f\|_{L^p} + \cdots + \left\|\nabla^k f\right\|_{L^p},$$

and unless otherwise specified all norms on the spaces $\mathbb{R}^s$ ($s \in \mathbb{N}$) will be the usual euclidean norms.
Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,loc}(\Omega)$, then the precise representative w^* is defined for all $x \in \Omega$ by

$$w^*(x) = \begin{cases} \lim_{r \searrow 0} \int_{B(x,r)} w(z)dz, & if\ the\ limit\ exists\ and\ is\ finite, \\ 0 & otherwise, \end{cases} \tag{86}$$

where the dashed integral as usual denotes the integral mean,

$$\fint_{B(x,r)} w(z)dz = \frac{1}{\mathrm{L}^n(B(x,r))} \int_{B(x,r)} w(z)dz,$$

and $B(x,r) = \{y : |y - x| < r\}$ is the open ball of radius r centered at x. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

We will say that x is an L_p-Lebesgue point of w (and simply a Lebesgue point when $p = 1$), if

$$\fint_{B(x,r)} |w(z) - w(x)|^p \, dz \to 0 \qquad as \qquad r \searrow 0.$$

If $k < n$, then it is well-known that functions from Sobolev spaces $W_p^k(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \le p_\circ = \frac{n}{k}$ (see, e.g., [218], [79]).

The Sobolev–Lorentz space $W_{p_\circ,1}^k(\mathbb{R}^n) \subset W_{p_\circ}^k(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space that for our purposes turns out to be convenient. Among other things functions that are locally in $W_{p_\circ,1}^k$ on $\mathbb{R}^n$ are in particular continuous.

Here we shall mainly be concerned with the Lorentz space $L_{p,1}$, and in this case one may rewrite the norm as (see [65])

$$\|f\|_{p,1} = \int_0^{+\infty} [L^n(\{x \in \mathbb{R}^n : |f(x)| > t\})]^{\frac{1}{p}} \, dt. \tag{87}$$

We record the following subadditivity property of the Lorentz norm for later use.

Lemma (5.3.10)[208]: (see, e.g., [72] or [65]). Suppose that $1 \le p < \infty$ and $= \bigcup_{j\in\mathbb{N}} E_j$, where E_j are measurable and mutually disjoint subsets of $\mathbb{R}^n$. Then for all $f \in L_{p,1}$ we have

$$\sum_j \left\|f \cdot 1_{E_j}\right\|_{L_{p,1}}^p \le \|f \cdot 1_E\|_{L_{p,1}}^p ,$$

where 1_E denotes the indicator function of the set E.

Denote by $W_{p,1}^k(\mathbb{R}^n)$ the space of all functions $v \in W_p^k(\mathbb{R}^n)$ such that in addition the Lorentz norm $\left\|\nabla^k v\right\|_{L_{p,1}}$ is finite.

For a mapping $u \in L_1(Q, \mathbb{R}^d), Q \subset \mathbb{R}^n, m \in \mathbb{N}$, define the polynomial $P_{Q,m}[u]$ of degree at most m by the following rule:

$$\int_Q y^\alpha (u(y) - P_{Q,m}[u](y)) \, dy = 0 \tag{88}$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n \le m$. Denote $P_Q[u] = P_{Q,k-1}[u]$.

The following well-known bound will be used on several occasions.

Lemma (5.3.11)[208]: (see, e.g., [180]). Suppose $v \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ with $k \in \{1, \dots, n\}$. Then v is a continuous mapping and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate

$$\sup_{y\in Q} \left|v(y) - P_Q[v](y)\right| \le C \left\|1_Q \cdot \nabla^k v\right\|_{L_{p_\circ,1}} \tag{89}$$

holds, where C is a constant depending on n, d only. Moreover, the mapping $v_Q(y) = v(y) - P_Q[v](y), y \in Q$, can be extended from Q to the whole of $\mathbb{R}^n$ such that the extension (denoted again) $v_Q \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ and

$$\left\|\nabla^k v_Q\right\|_{L_{p_\circ,1}(\mathbb{R}^n)} \le C_0 \left\|1_Q \cdot \nabla^k v\right\|_{L_{p_\circ,1}} , \tag{90}$$

where C_0 also depends on n, d only.

Corollary (5.3.12)[208]: (see, e.g., [50]). Suppose $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$ with $k \in \{1, \dots, n\}$. Then v is a continuous mapping and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimates

$$\operatorname{diam} v(Q) \le C\left(\frac{\|\nabla v\|_{L_{p_\circ}(Q)}}{\ell(Q)^{k-1}} + \left\|1_Q \cdot \nabla^k v\right\|_{L_{p_\circ,1}}\right)$$
$$\le C\left(\frac{\|\nabla v\|_{L_p(Q)}}{\ell(Q)^{\frac{n}{p}-1}} + \left\|1_Q \cdot \nabla^k v\right\|_{L_{p_\circ,1}}\right) \tag{91}$$

hold for every $p \in [p_\circ, n]$.

The above results can easily be adapted to give that $v \in C_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity (see [65]).

Let M^β be the space of all nonnegative Borel measures μ on $\mathbb{R}^n$ such that

$$\left|||\mu|||\right|_\beta = \sup_{I \subset \mathbb{R}^n} \ell(I)^{-\beta}\mu(I) < \infty, \tag{92}$$

where the supremum is taken over all n-dimensional cubic intervals $I \subset \mathbb{R}^n$ and $\ell(I)$ denotes side-length of I. We need the following important strong-type estimates for maximal functions (it was proved in [180] based on classic results of D.R. Adams [51] and some new analog of the trace theorem for Riesz potentials of Lorentz functions for the limiting case $q = p$, see Theorems 0.2–0.4 and Corollary (5.3.7) in [180]).

Theorem (5.3.13)[208]: ([180]). Let $p \in (1, \infty), k, l \in \{1, \dots, n\}, l \le k, (k - l)p < n$. Then for any function $f \in W^k_{p,1}(\mathbb{R}^n)$ the estimates

$$\left\|\nabla^l f\right\|^p_{L_p(\mu)} \le C \left|||\mu|||\right|_\beta \left\|\nabla^k f\right\|^p_{L_{p,1}} \qquad \forall \mu \in \mathrm{M}^\beta, \tag{93}$$

$$\int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : \mathcal{M}(\left|\nabla^l f\right|^p)(x) \ge t\})\, dt \le C \left\|\nabla^k f\right\|^p_{L_{p,1}} \tag{94}$$

hold, where $\beta = n - (k - l)p$, the constant C depends on n, k, p only, and

$$\mathcal{M}f(x) = \sup_{r>0} \int_{B(x,r)} |f(y)|\, dy$$

is the usual Hardy–Littlewood maximal function of f.

The result is true also for $p = 1, k > l$ and is in this case due to D.R. Adams [51].

For a subset A of $\mathbb{R}^m$ and $\varepsilon > 0$ the ε-entropy of A, denoted by $\mathrm{Ent}(\varepsilon, A)$, is the minimal number of closed balls of radius ε covering A. Further, for a linear map $L: \mathbb{R}^n \to \mathbb{R}^d$ we denote by $\lambda_j(L), j = 1, \dots, d$, its singular values arranged in decreasing order: $\lambda_1(L) \ge \lambda_2(L) \ge \dots \ge \lambda_d(L)$. Geometrically the singular values are the lengths of the semiaxes of the, possibly degenerate, ellipsoid $L(\partial B(0,1))$. We recall that the singular values of L coincide with the eigenvalues repeated according to multiplicity of the symmetric nonnegative linear map $\sqrt{LL^*} : \mathbb{R}^d \to \mathbb{R}^d$. Also for a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ we denote by $\lambda_j(L), j = 1, \dots, d$, its singular values arranged in decreasing order: $\lambda_1(L) \ge \lambda_2(L) \ge \dots \ge \lambda_d(L)$. Geometrically the singular values are the lengths of the semiaxes of the, possibly degenerate, ellipsoid $L(\partial B(0,1))$. We recall that the singular values of L coincide with the eigenvalues repeated according to multiplicity of the symmetric nonnegative linear map $\sqrt{LL^*} : \mathbb{R}^d \to \mathbb{R}^d$. Also for a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ that is approximately differentiable at $x \in \mathbb{R}^n$ put $\lambda_j(f,x) = \lambda_j(d_x f)$, where by $d_x f$ we denote the approximate differential of f at x. The next result is the second basic ingredient of our proof.

Theorem (5.3.14)[208]: ([78]). For any polynomial $P : \mathbb{R}^n \to \mathbb{R}^d$ of degree at most k, for each ball $B \subset \mathbb{R}^n$ of radius $r > 0$, and any number $\varepsilon > 0$ we have that

$$\operatorname{Ent}(\varepsilon r, \{P(x): x \in B, \lambda_1 \le 1+\varepsilon, \dots, \lambda_{m-1} \le 1+\varepsilon, \lambda_m \le \varepsilon, \dots, \lambda_d \le \varepsilon\}) \le C_Y(1 + \varepsilon^{1-m}),$$

where the constant C_Y depends on n, d, k, m only and for brevity we wrote $\lambda_j = \lambda_j(P,x)$.

The application of Theorem (5.3.13) is facilitated through the following simple estimate (see for instance Lemma 2 in [58], cf. with [55]).

Lemma (5.3.15)[208]: Let $u \in W_1^1(\mathbb{R}^n, \mathbb{R}^d)$. Then for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and for any number $\varepsilon > 0$ the estimate

$$\operatorname{diam}(\{u(x) : x \in B, (\mathcal{M}\nabla u)(x) \le \varepsilon\}) \le C_M \varepsilon r$$

holds, where C_M is a constant depending on n, d only.

We need also the following approximation result.

Theorem (5.3.16)[208]: (see Theorem (5.3.5) in [180]). Let $p \in (1,\infty), k, l \in \{1,\dots,n\}, l \le k, (k-l)p < n$. Then for any $f \in W_{p,1}^k(\mathbb{R}^n)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^n$ and a function $g \in C^l(\mathbb{R}^n)$ such that

(i) $\mathcal{H}_\infty^{n-(k-l)p}(U) < \varepsilon$;

(ii) each point $x \in \mathbb{R}^n \setminus U$ is an L_p-Lebesgue point for $\nabla^j f, j = 0,\dots,l$;

(iii) $f \equiv g, \nabla^j f \equiv \nabla^j g$ on $\mathbb{R}^n \setminus U$ for $j = 1,\dots,l$.

Note that in the analogous theorem for the case of Sobolev mappings $f \in W_p^k(\mathbb{R}^n)$ the assertion (i) should be replaced by

(i') $\mathcal{B}_{k-l,p}(U) < \varepsilon$ if $l < k$,

where $\mathcal{B}_{\alpha,p}(U)$ denotes the Bessel capacity of the set U (see [79] or [54]).

Recall that for $1 < p < \infty$ and $0 < n - \alpha p < n$ the smallness of $\mathcal{H}_\infty^{n-\alpha p}(U)$ implies the smallness of $\mathcal{B}_{\alpha,p}(U)$, but that the opposite is false since $\mathcal{B}_{\alpha,p}(U) = 0$ whenever $\mathcal{H}^{n-\alpha p}(U) < \infty$. On the other hand, for $1 < p < \infty$ and $0 < n - \alpha p < \beta \le n$ the smallness of $\mathcal{B}_{\alpha,p}(U)$ implies the smallness of $\mathcal{H}^{\beta\infty}(U)$ (see, e.g., [52]). So the usual assertion (i') is essentially weaker than (i).

We briefly recall some theorems from [50], [180] which we need. The following result is an analog of the Luzin N-property with respect to the Hausdorff content.

Theorem (5.3.17)[208]: ([50], [180]). Let $k \in \{1,\dots,n\}, q \in [p_\circ, n]$, and $v \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^n$ if $\mathcal{H}_\infty^q(E) < \delta$, then $\mathcal{H}_\infty^q(v(E)) < \varepsilon$. In particular, $\mathcal{H}^q(v(E)) = 0$ whenever $\mathcal{H}^q(E) = 0$.

The next asertion is the precise analog of the Dubovitskiĭ–Federer Theorem (5.3.2) which includes the Morse–Sard result.

Theorem (5.3.18)[208]: ([50], [180]). If $k, m \in \{1,\dots,n\}, \Omega$ is an open subset of $\mathbb{R}^n$, and $v \in W_{p_\circ,1,loc}^k(\Omega, \mathbb{R}^d)$, then $\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0$.

Recall that in our notation

$$p_\circ = \frac{n}{k}, s = n - m - k + 1, q_\circ = m + \frac{s}{k} = p_\circ + (m-1)(1-k^{-1}), \quad (95)$$

and $Z_{v,m} = \{x \in \Omega : rank\, \nabla v(x) < m\}$.

Finally, here we recall some differentiability properties of Sobolev–Lorentz functions.

Theorem (5.3.19)[208]: ([50], [180]). Let $k \in \{1,\dots,n\}$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $\mathcal{H}^{p_\circ}(A_v) = 0$ and for any $x \in \mathbb{R}^n \setminus A_v$ the function v is differentiable (in the classical Fréchet sense) at x, furthermore, the classical derivative coincides with $\nabla v(x)$ (x is an $L_{p_\circ}$ -Lebesgue point for ∇v).
Really the last assertion of the Theorem — that $\mathcal{H}^{p_\circ}$ -almost all points $x \in \mathbb{R}^n$ are the $L_{p_\circ}$ -Lebesgue points for the gradient ∇v — follows from Theorem (5.3.16) (ii).
The case $k = 1, p_\circ = n$ of the Theorem (5.3.19) is a classical result due to Stein [73] (see also [62]), and for $k = n, p_\circ = 1$ the result is due to Dorronsoro [58].
Theorem (5.3.19) admits the following generalization.
Theorem (5.3.20)[208]: ([50], [180]). Let $k,l \in \{1,\dots,n\}, l \le k$, and $v \in W^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. Then there exists a Borel set $A_{v,l} \subset \mathbb{R}^n$ such that $\mathcal{H}^{lp_\circ}(A_{v,l}) = 0$ and each point $x \in \mathbb{R}^n \setminus A_{v,l}$ is an $L_{p_\circ}$ -Lebesgue point for $\nabla^j f, j = 0,\dots,l$, moreover, the function v is l-times differentiable (in the classical Fréchet–Peano sense) at x, i.e.,

$$\lim_{r\searrow 0} \sup_{y\in B(x,r)\setminus\{x\}} \frac{|v(y) - T_{v,l,x}(y)|}{|x - y|^l} = 0,$$

where $T_{v,l,x}(y)$ is the Taylor polynomial of order l for v centered at x.
Note that the Taylor polynomial of order l for v centered at x is well defined $\mathcal{H}^{lp_\circ}$ — a.e. by Theorem (5.3.16).

We are going to prove Theorem (5.3.24) and as a consequence Theorem (5.3.4). Now fix $n \in \mathbb{N}, k \in \{1,\dots,n\}, p \in [p_\circ,n]$ and $q \in [0,p]$.

$$\mu = p - q. \tag{96}$$

Fix also a mapping $v \in W^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. For a set $E \subset \mathbb{R}^n$ define the set function

$$\Phi(E) = \inf_{E\subset\bigcup_\alpha D_\alpha} \sum_\alpha (diam\, D_\alpha)^\mu [diam\, v(D_\alpha)]^q , \tag{97}$$

where the infimum is taken over all countable families of compact sets $\{D_\alpha\}_{\alpha\in\mathbb{N}}$ such that $E \subset \bigcup_\alpha D_\alpha$. By Theorem (5.3.33), $\Phi(\cdot)$ is a countably subadditive set-function with the property

$$\Phi(E) = 0 \Rightarrow [\mathcal{H}^\mu(E \cap v^{-1}(y) = 0 \text{ for } \mathcal{H}^q - \text{almost all } y \in \mathbb{R}^d]\,. \tag{98}$$

Thus the assertion of Theorem (5.3.24) amounts to

$$\Phi(E) = 0 \quad \text{whenever } \mathcal{H}^p(E) = 0. \tag{99}$$

The proof of this follows the ideas of [50].
By a dyadic interval we understand a cubic interval of the form $\left[\frac{k_1}{2^l}, \frac{k_1+1}{2^l}\right] \times\cdots\times \frac{k_n}{2^l}, \frac{k_n+1}{2^l}]$, where k_i, l are integers. The following assertion is straightforward.
Lemma (5.3.21)[208]: For any n-dimensional cubic interval $J \subset \mathbb{R}^n$ there exist dyadic intervals $Q_1,\dots,Q_{2^n}$ such that $J \subset Q_1 \cup\cdots\cup Q_{2^n}$ and $\ell(Q_1) = \cdots = \ell(Q_{2^n}) \le 2\ell(J)$.
Let $\{Q_\alpha\}_{\alpha\in A}$ be a family of n-dimensional dyadic intervals. We say that the family $\{Q_\alpha\}$ is regular, if for any n-dimensional dyadic interval Q the estimate

$$\ell(Q)^p \ge \sum_{\alpha:Q_\alpha\subset Q} \ell(Q_\alpha)^p \tag{100}$$

holds. Since dyadic intervals are either nonoverlapping or contained in one another, (100) implies that any regular family $\{Q_\alpha\}$ must in particular consist of nonoverlapping intervals.

Lemma (5.3.22)[208]: (see Lemma 2.3 in [56]). Let $\{Q_\alpha\}$ be a family of n-dimensional dyadic intervals. Then there exists a regular family $\{J_\beta\}$ of n-dimensional dyadic intervals such that $\bigcup_\alpha \; Q_\alpha \subset \bigcup_\beta \; J_\beta$ and

$$\sum_\beta \ell(J_\beta)^p \le \sum_\alpha \ell(Q_\alpha)^p.$$

Lemma (5.3.23)[208]: (see Lemma 2.11 in [180] and Lemma 2.4 in [50]). Let $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_\alpha\}$ of n-dimensional dyadic intervals we have if

$$\sum_\alpha \ell(Q_\alpha)^p < \delta, \tag{101}$$

Then

$$\sum_\alpha \left[\left\|1_{Q_\alpha} \cdot \nabla^k v\right\|^p_{L_{p_\circ,1}} + \frac{1}{\ell(Q_\alpha)^{n-p}} \int_{Q_\alpha} |\nabla v|^p\right] < \varepsilon. \tag{102}$$

Theorem (5.3.24)[208]: Let $k \in \{1, \dots, n\}, p_\circ = n/k$ and $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for every $p \in [p_\circ, n], q \in [0, p]$ and for any set $E \subset \mathbb{R}^n$ with $\mathcal{H}^p(E) = 0$ we have

$$\mathcal{H}^{p-q}(E \cap v^{-1}(y)) = 0 \quad for\ \mathcal{H}^q - a.a.\, y \in \mathbb{R}^d. \tag{103}$$

Proof: Let $\mathcal{H}^p(E) = 0$. Take $\varepsilon > 0$ and $\delta = \delta(\varepsilon, v) < 1$ from Lemma (5.3.23). Take also the regular family $\{Q_\alpha\}$ of n-dimensional dyadic intervals such that $E \subset \bigcup_\alpha \; Q_\alpha$ and

$$\sum_\alpha \ell(Q_\alpha)^p < \delta \tag{104}$$

where the existence of such family follows directly from Then by Lemma (5.3.23) the estimate (102) holds. Denote $r_\alpha = \ell(Q_\alpha)$. By estimate (91),

$$[diam\, v(Q_\alpha)]^q \le C\left(\frac{\|\nabla v\|^q_{L_p(Q_\alpha)}}{r_\alpha^{\left(\frac{n}{p}-1\right)q}} + \left\|1_{Q_\alpha} \cdot \nabla^k v\right\|^q_{L_{p_\circ,1}}\right). \tag{105}$$

Therefore, by definition of $\Phi(E)$ (see (97)), we have

$$\begin{aligned}
\Phi(E) &\le C \sum_\alpha r_\alpha^\mu \left(\frac{\|\nabla v\|^q_{L_p(Q_\alpha)}}{r_\alpha^{\left(\frac{n}{p}-1\right)q}} + \left\|1_{Q_\alpha} \cdot \nabla^k v\right\|^q_{L_{p_\circ,1}}\right) \\
&\le c\left(\sum_\alpha r_\alpha^{\frac{\mu p}{p-q}}\right)^{\frac{p-q}{p}} \cdot \left[\sum_\alpha \left(\frac{1}{\ell(Q_\alpha)^{n-p}}\int_{Q_\alpha} |\nabla v|^p + \left\|1_{Q_\alpha} \cdot \nabla^k v\right\|^q_{L_{p_\circ,1}}\right)\right]^{\frac{q}{p}} \\
&\le c\left(\sum_\alpha r_\alpha^p\right)^{\frac{p-q}{p}} \cdot \varepsilon^{\frac{q}{p}} \\
&\le c\delta^{\frac{p-q}{p}} \cdot \varepsilon^{\frac{q}{p}}.
\end{aligned} \tag{106}$$

Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary small, (106) turns to the equality $\Phi(E) = 0$ and by (98) the required assertion is proved.

Fix integers $k, m \in \{1, \dots, n\}, d \ge m$ and a mapping $v \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then, by Theorem (5.3.19), there exists a Borel set A_v such that $\mathcal{H}^{p_\circ}(A_v) = 0$ and all points of the complement $\mathbb{R}^n \setminus A_v$ are $L_{p_\circ}$ -Lebesgue points for the weak gradient ∇v. We can arrange that v is differentiable (in the classical Fréchet sense) at every point $x \in \mathbb{R}^n \setminus Av$ with derivative $\nabla v(x)$ (so the classical derivative coincides with the precise representative of the weak gradient at x).

Denote $Z_{v,m} = \{x \in \Omega \setminus A_v : rank\, \nabla v(x) < m\}$. Fix a number

$$q \in [m-1, q_\circ).$$

$$\mu = \mu_q = n - m - k + 1 + (m-q)k. \tag{107}$$

Since $q < q_\circ = m - 1 + \frac{n-m+1}{k}$, we have $\mu > 0$.

We prove the assertion of the bridge Dubovitskiĭ–Federer Theorem (5.3.5) which is equivalent (by virtue of Theorem (5.3.33)) to

$$\Phi(Z_{v,m}) = 0 \quad \text{if} \quad q > m-1, \tag{108}$$

where for each fixed $q \in [m-1, q_\circ)$ we denoted

$$\Phi(E) = \inf_{E \subset \bigcup_\alpha D_\alpha} \sum_\alpha (diam\, D_\alpha)^\mu [diam\, v(D_\alpha)]^q . \tag{109}$$

As indicated the infimum is taken over all countable families of compact sets $\{D_\alpha\}_{\alpha \in N}$ such that $E \subset \bigcup_\alpha D_\alpha$. Note that the case $q = q_\circ, \mu_q = 0$ was considered in [50], [180].

Before embarking on the detailed proof we make some preliminary observations that allow us to make a few simplifying assumptions. In view of our definition of critical set we have that

$$Z_{v,m} = \bigcup_{j \in \mathbb{N}} \{x \in Z_{v,m} : |\nabla v(x)| \le j\}.$$

Consequently we only need to prove that $\Phi(Z'_v) = 0$ for $q \in (m-1, q_\circ)$, where

$$Z'_v = \{x \in Z_{v,m} : |\nabla v(x)| \le 1\}.$$

For convenience, below we use the notation $\|f\|_{L_{p_\circ,1}(I)}$ instead of $\|1_I \cdot f\|_{L_{p_\circ,1}}$. The following lemma contains the main step in the proof.

Lemma (5.3.25)[208]: Let $q \in [m-1, q_\circ)$. Then for any n-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$\Phi(Z'_v \cap I) \le C\left(\ell(I)^\mu \|\nabla^k v\|^q_{L_{p_\circ,1}(I)} + \ell(I)^{\mu+m-1} \|\nabla^k v\|^{q-m+1}_{L_{p_\circ,1}(I)}\right) \tag{110}$$

holds, where the constant C depends on n, m, k, d only.

Proof. By virtue of (90) it suffices to prove that

$$\Phi(Z'_v \cap I) \le C(\ell(I)^\mu \|\nabla^k v_I\|^q_{L_{p_\circ,1}(\mathbb{R}^n)} + \ell(I)^{\mu+m-1} \|\nabla^k v_I\|^{q-m+1}_{L_{p_\circ,1}(\mathbb{R}^n)} \tag{111}$$

for the mapping v_I defined in Lemma (5.3.11), where $C = C(n,m,k,d)$ is a constant.

Fix an n-dimensional dyadic interval $I \subset \mathbb{R}^n$ and recall that $v_I(x) = v(x) - P_I(x)$ for all $x \in I$. Denote

$$\sigma = \|\nabla^k v_I\|_{L_{p_\circ,1}}, \qquad r = \ell(I),$$

and for each $j \in \mathbb{Z}$

$$E_j = \{x \in I : (\mathcal{M}|\nabla v_I|^{p_\circ})(x) \in (2^{j-1}, 2^j]\} \quad and \quad \delta_j = \mathcal{H}^{p_\circ}_\infty(E_j).$$

Then by Theorem (5.3.13) (applied for the case $p = p_\circ = \frac{n}{k}, l = 1, \beta = p_\circ$),

$$\sum_{j=-\infty}^{\infty} \delta_j 2^j \le C\sigma^{p_\circ} \tag{112}$$

for a constant C depending on n, m, k, d only. By the definition of the Hausdorff measure, for each $j \in \mathbb{Z}$ there exists a family of balls $B_{ij} \subset \mathbb{R}^n$ of radii r_{ij} such that

$$E_j \subset \bigcup_{i=1}^{\infty} B_{ij} \quad \text{and} \quad \sum_{i=1}^{\infty} r_{ij}^{p_\circ} \le c\delta_j \,. \tag{113}$$

Denote

$$Z_j = Z_v' \cap E_j \qquad \text{and} \qquad Z_{ij} = Z_j \cap B_{ij} \,.$$

By construction $Z_v' \cap I = \bigcup_j Z_j$ and $Z_j = \bigcup_i Z_{ij}$. Put

$$\varepsilon_* = \frac{1}{r} \left\|\nabla^k v_I\right\|_{L_{p_\circ,1}} = \frac{\sigma}{r} \,,$$

and let j_* be the integer satisfying $\varepsilon_*^{p_\circ} \in (2^{j_*-1}, 2^{j_*}]$. Denote $Z_* = \bigcup_{j<j_*} Z_j, Z_{**} = \bigcup_{j\ge j_*} Z_j$. Than by construction

$$Z_v' \cap I = Z_* \cup Z_{**},\ Z_* \subset \{x \in Z_v' \cap I : (\mathcal{M}|\nabla v_I|^{p_\circ})(x) < \varepsilon_*^{p_\circ}\}.$$

Since $\nabla P_I(x) = \nabla v(x) - \nabla v_I(x), |\nabla v_I(x)| \le 2^{j/p_\circ}, |\nabla v(x)| \le 1,$ and $\lambda_m(v,x) = 0$ for $x \in Z_{ij}$, we have

$$Z_{ij} \subset \{x \in B_{ij} : \lambda_1(P_I,x) \le 1 + 2^{j/p_\circ}, \ldots, \lambda_{m-1}(P_I, x) \le 1 + 2^{j/p_\circ}, \lambda_m(P_I, x) \le 2^{j/p_\circ}\}.$$

Applying Theorem (5.3.14) and Lemma (5.3.15) to mappings P_I, v_I , respectively, with $B = B_{ij}$ and $\varepsilon = \varepsilon_j = 2^{j/p_\circ}$, we find a finite family of balls $T_s \subset \mathbb{R}^d, s = 1,\ldots,s_j$ with $s_j \le C_Y (1 + \varepsilon_j^{1-m})$, each of radius $(1 + C_M)\varepsilon_j r_{ij}$, such that

$$\bigcup_{s=1}^{s_j} T_s \supset v(Z_{ij}).$$

Therefore, for every $j \ge j_*$ we have

$$\begin{aligned}\Phi(Z_{ij}) &\le C_1 s_j \varepsilon_j^q r_{ij}^{q+\mu} = C_2(1 + \varepsilon_j^{1-m}) 2^{\frac{jq}{p_\circ}} r_{ij}^{q+\mu} \\ &\le C_2(1+\varepsilon_*^{1-m}) 2^{\frac{jq}{p_\circ}} r_{ij}^{q+\mu},\end{aligned} \tag{114}$$

where all the constants C_α above depend on n, m, k, d only. By the same reasons, but this time applying Theorem (5.3.14) and Lemma (5.3.15) with $\varepsilon = \varepsilon_*$ and instead of the balls B_{ij} we take a ball $B \supset I$ with radius $\sqrt{n}r$, we have

$$\begin{aligned}\Phi(Z_*) \le C_3(1 + \varepsilon_*^{1-m})\varepsilon_*^q r^{q+\mu} &\stackrel{\text{def}}{=} C_3(1 + \sigma^{1-m} r^{m-1})\sigma^q r^\mu \\ &= C_3(r^\mu \sigma^q + r^{\mu+m-1}\sigma^{q-m+1}).\end{aligned} \tag{115}$$

From (114) we get immediately

$$\Phi(Z_{**}) \le C_2(1 + \varepsilon_*^{1-m}) \sum_{j\ge j_*} \sum_i 2^{\frac{jq}{p_\circ}} r_{ij}^{q+\mu} \,. \tag{116}$$

Further estimates splits into the two possibilities.

Case I. $q \ge p_\circ$. Then

$$\Phi(Z_{**}) \le C_2(1 + \varepsilon_*^{1-m}) \left(\sum_{j\ge j_*} \sum_i 2^j r_{ij}^{(q+\mu)\frac{p_\circ}{q}} \right)^{\frac{q}{p_\circ}}$$

$$\le C_2(1 + \varepsilon_*^{1-m}) r_\mu \left(\sum_{j\ge j_*} \sum_i 2^j r_{ij}^{p_\circ} \right)^{\frac{q}{p_\circ}}$$

$$\leq\ C_4(1\ +\ \varepsilon_*^{1-m})r^{\mu}\left(\sum_{j\geq j_*}\ 2^j\delta_j\right)^{\frac{q}{p_\circ}}$$

$$\leq\ C_5(1\ +\ \varepsilon_*^{1-m})r^{\mu}\sigma^q = C_5(r^{\mu}\sigma^q + r^{\mu+m-1}\sigma^{q-m+1}) \qquad (117)$$

Case II. $q < p_\circ$. Recalling (107) we get by an elementary calculation

$$q+\mu = q+(n-m-k+1)+(m-q)k = (p_\circ - q + m - 1)(k-1) + p_\circ \geq p_\circ,$$

therefore,

$$\Phi(Z_{**}) \leq C_2(1+\varepsilon_*^{1-m})\left(\sum_{j\geq j_*}\ \sum_i\ 2^j r^{p_\circ ij}\right) r^{q+\mu-p_\circ} 2^{j_*\frac{q-p_\circ}{p_\circ}}$$

$$\leq\ C_6(1\ +\ \varepsilon_*^{1-m})\sigma^{p_\circ} r^{q+\mu-p_\circ}\left(\frac{\sigma}{r}\right)^{q-p_\circ} = C_6(1\ +\ \varepsilon_*^{1-m})\sigma^q r^{\mu}$$

$$= C_6(r^{\mu}\sigma^q + r^{\mu+m-1}\sigma^{q-m+1})\,. \qquad (118)$$

Now for both cases (I) and (II) we have by (117), (118) that $\Phi(Z_{**}) \leq C(r^{\mu}\sigma^q + r^{\mu+m-1}\sigma^{q-m+1})$, and, by virtue of the earlier estimate (115), we conclude that

$$\Phi(Z_v' \cap I)\ =\ \Phi(Z_*\ \cup\ Z_{**}) \leq \Phi(Z_*) + \Phi(Z_{**}) \leq C(r^{\mu}\sigma^q + r^{\mu+m-1}\sigma^{q-m+1})\,.$$

The lemma is proved.

Corollary (5.3.26)[208]: Let $q\ \in\ [m\ -\ 1, q_\circ)$. Then for any $\varepsilon\ >\ 0$ there exists $\delta\ >\ 0$ such that for any subset E of $\mathbb{R}^n$ we have $\Phi(Z_v' \cap E)\ \leq\ \varepsilon$ provided $\mathrm{L}^n(E)\ \leq\ \delta$. In particular, $\Phi(Z_{v,m} \cap\ E)\ =\ 0$ whenever $\mathrm{L}^n(E)\ =\ 0$.

Proof. We start by recording the following elementary identity (see (107)):

$$\frac{(\mu\ +\ m\ -\ 1)p_\circ}{p_\circ\ -\ q\ +\ m\ -\ 1} =\ n. \qquad (119)$$

Let $\mathrm{L}^n(E)\ \leq\ \delta$, then we can find a family of nonoverlapping n-dimensional dyadic intervals I_α such that $E\ \subset\ \bigcup_\alpha\ I_\alpha$ and $\sum_\alpha\ \ \ell^n(I_\alpha)\ <\ C\delta$. Of course, for sufficiently small δ the estimates

$$\left\|\nabla^k v\right\|_{L_{p_\circ,1}(I_\alpha)} < 1, \qquad \ell(I_\alpha)\ \leq\ \delta^{\frac{1}{n}} \qquad (120)$$

are fulfilled for every α. Denote

$$r_\alpha = \ell(I_\alpha), \quad \sigma_\alpha = \left\|\nabla^k v\right\|_{L_{p_\circ,1}(I_\alpha)}, \quad \sigma\ =\ \left\|\nabla^k v\right\|_{L_{p_\circ,1}}\,. \qquad (121)$$

In view of Lemma (5.3.25) we have

$$\Phi(E)\ \leq\ C\sum_\alpha\ r_\alpha^{\mu+m-1}\,\sigma_\alpha^{q-m+1} +\ C\sum_\alpha\ r_\alpha^{\mu}\sigma_\alpha^q. \qquad (122)$$

Now let us estimate the first sum. Since by our assumptions

$$q < q_\circ = m-1+\frac{n-m+1}{k} \leq m-1+p_\circ \ \ \textit{hence}\ p_\circ > q\ -\ m\ +\ 1$$

we have

$$\sum_\alpha\ r_\alpha^{\mu+m-1}\,\sigma_\alpha^{q-m+1} \overset{\textit{Hölder ineq.}}{\leq} C\left(\sum_\alpha\ \sigma_\alpha^{p_\circ}\right)^{\frac{q-m+1}{p_\circ}} \cdot \left(\sum_\alpha\ r_\alpha^{\frac{(\mu+m-1)p}{p_\circ-q+m-1}}\right)^{\frac{p_\circ-q+m-1}{p_\circ}}$$

$$\overset{(119),\,\text{Lemma (5.3.10)}}{\leq} C'\sigma^{q-m+1}\ \cdot\left(\mathrm{L}^n(E)\right)^{\frac{p_\circ-q+m-1}{p_\circ}}. \qquad (123)$$

The estimates of the second sum are again handled by consideration of two separate cases.
Case I. $q\ \geq\ p_\circ$. Then

$$\sum_{\alpha} r_{\alpha}^{\mu}\sigma_{\alpha}^{q} \overset{(120)}{\leq} \delta^{\frac{\mu}{n}} \sum_{\alpha} \sigma_{\alpha}^{p_\circ} \overset{\text{Lemma (5.3.10)}}{\leq} \sigma^{p_\circ} \cdot \delta^{\frac{\mu}{n}}. \tag{124}$$

Case II. $q < p_\circ$. Recalling (107) we get by an elementary calculation

$$\frac{\mu p_\circ}{p_\circ - q} = n \cdot \frac{n - qk + [mk - m - k + 1]}{n - qk}$$

$$= n \cdot \frac{n - qk + (m - 1)(k - 1)}{n - qk} \geq n, \tag{125}$$

Then

$$\sum_{\alpha} r_{\alpha}^{\mu}\sigma_{\alpha}^{q} \overset{\text{Hölder ineq.}}{\leq} \left(\sum_{\alpha} \sigma_{\alpha}^{p_\circ}\right)^{\frac{q}{p_\circ}} \cdot \left(\sum_{\alpha} r_{\alpha}^{\frac{\mu p_\circ}{p_\circ - q}}\right)^{\frac{p_\circ - q}{p_\circ}}$$

$$\overset{\text{Lemma (5.3.10), (125)}}{\leq} \sigma^{q}\delta^{\frac{\mu}{n}}. \tag{126}$$

Now for both cases (I) and (II) we have by (122)–(126) that $\Phi(E) \leq h(\delta)$, where the function $h(\delta)$ satisfies the condition $h(\delta) \searrow 0$ as $\delta \searrow 0$. The lemma is proved.

By Theorem (5.3.16) (iii) (applied to the case $k = l$), our mapping v coincides with a mapping $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$ off an exceptional set of small n-dimensional Lebesgue measure. This fact, together with Corollary (5.3.26) and Dubovitskiĭ Theorem (5.3.1), finishes the proof of Theorem (5.3.3) for the case $d = m$. But since Theorem (5.3.5) was not proved for C^k-smooth We have to do this step now.

Lemma (5.3.27)[208]: Let $q \in (m - 1, q_\circ)$ and $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$. Then

$$\Phi_g(Z_{g,m}) = 0, \tag{127}$$

where Φ_g is calculated by the same formula (109) with g instead of v and $Z_{g,m} = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla g(x) < m\}$.

Proof. We can assume without loss of generality that g has compact support and that $|\nabla g(x)| \leq 1$ for all $x \in \mathbb{R}^n$. We then clearly have that $g \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$, hence we can in particular apply the above results to g. The following assertion plays the key role:

$(*)$ For any n-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$\Phi\big(Z_{g,m} \cap I\big) \leq C\left(\ell(I)^{\mu}\left\|\nabla^k \bar{g}_I\right\|^{q}_{L_{p_\circ,1}(I)} + \ell(I)^{\mu+m-1}\left\|\nabla^k \bar{g}_I\right\|^{q-m+1}_{L_{p_\circ,1}(I)}\right)$$

holds, where the constant C depends on n, m, k, d only, and we denoted

$$\nabla^k \bar{g}_I\,(x) = \nabla^k g(x) - \frac{1}{\mathrm{L}^n(I)} \int_I \nabla^k g(y)\, dy.$$

The proof of $(*)$ is almost the same as that of Lemma (5.3.25), with evident modifications (we need to take the approximation polynomial $P_I(x)$ of degree k instead of $k - 1$, etc.). By elementary facts of the Lebesgue integration theory, for an arbitrary family of nonoverlapping n-dimensional dyadic intervals I_α one has

$$\sum_{\alpha} \left\|\nabla^k \bar{g}_{I_\alpha}\right\|^{p_\circ}_{L_{p_\circ,1}(I_\alpha)} \to 0 \qquad \text{as } \sup_{\alpha} \ \ell(I_\alpha) \to 0. \tag{128}$$

The proof of this estimate is really elementary since now $\nabla^k g$ is continuous and compactly supported function, and, consequently, is uniformly continuous and bounded.

From $(*)$ and (128), repeating the arguments of Corollary (5.3.26), using the assumptions on g and taking

$$\sigma_\alpha = \left\|\nabla^k \bar{g}_{I_\alpha}\right\|_{L_{p_\circ,1}(I_\alpha)}, \qquad \sigma^{p_\circ} = \sum_\alpha \sigma_\alpha^{p_\circ}$$

in definitions (121), we obtain that $\Phi_g(Z_{g,m}) < \varepsilon$ for any $\varepsilon > 0$, hence the sought conclusion (127) follows.

By Theorem (5.3.16) (iii) (applied to the case $k = l$), the investigated mapping v equals a mapping $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$ off an exceptional set of small n-dimensional Lebesgue measure. This fact together with Lemma (5.3.27) readily implies

Corollary (5.3.28)[208]: (cp. with [39]). Let $q \in (m - 1, q_\circ)$. Then there exists a set $\tilde{Z}_v$ of n-dimensional Lebesgue measure zero such that $\Phi(Z'_v \setminus \tilde{Z}_v) = 0$. In particular, $\Phi(Z'_v) = \Phi(\tilde{Z}_v)$.

From We conclude that $\Phi(Z'_v) = 0$, and this concludes the proof of Theorem (5.3.5).

Fix $v \in W^1_{n,1}(\mathbb{R}^n, \mathbb{R}^d))$. Applying Lemma (5.3.25) for $k = 1, p_\circ = n, \mu = n - m + 1$ and $q = m - 1$, and afterwards making the shift of indices $(m - 1) \to m$, we obtain the following key estimate:

Let $m \in \{0, \dots, n - 1\}$. Then for any n-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$\Phi(Z'_v \cap I) \le C\left(\ell(I)^{n-m}\left\|\nabla^k v\right\|^m_{L_{p_\circ,1}(I)} + \ell(I)^n\right) \tag{129}$$

holds, where $Z'_v = \{x \in \Omega \setminus A_v : rank\, \nabla v(x) \le m, |\nabla v(x)| \le 1\}$, the constant C depends on n, m, d only, and

$$\Phi(E) = \inf_{E \subset \bigcup_\alpha D_\alpha} \sum_\alpha (\operatorname{diam} D_\alpha)^{n-m}[\operatorname{diam} v(D_\alpha)]^m. \tag{130}$$

This implies (by the same arguments as in the proof of Corollary (5.3.26)) that for any measurable set $E \subset \mathbb{R}^n$ with $\mathrm{L}^n(E) < \infty$ the inequality

$$\Psi(Z'_v \cap E) < \infty \tag{131}$$

holds, where $\Psi(E)$ is defined as

$$\Psi(E) = \lim_{\delta \to 0} \inf_{\substack{E \subset \bigcup_\alpha D_\alpha, \\ \operatorname{diam} D_\alpha \le \delta}} \sum_\alpha (\operatorname{diam} D_\alpha)^{n-m}[\operatorname{diam} v(D_\alpha)]^m, \tag{132}$$

here the infimum is taken over all countable families of compact sets $\{D_\alpha\}_{\alpha\in\mathbb{N}}$ such that $E \subset \bigcup_\alpha D_\alpha$ and $\operatorname{diam} D_\alpha \le \delta$ for all α.

By Theorem (5.3.34), the bound (131) implies the validity of the following assertion:

$$\text{the set } \left\{y \in \mathbb{R}^d : \mathcal{H}^{n-m}\left(E \cap Z'_v \cap f^{-1}(y)\right) > 0\right\} \text{ is } \mathcal{H}^m\ \sigma-\text{finite}. \tag{133}$$

Since

$$Z_{v,m+1} = \{x \in \Omega \backslash A_v : \operatorname{rank} \nabla v(x) \le m\} = \bigcup_j \left\{x \in Z_{v,m+1} : |\nabla v(x)| \le j\right\},$$

we infer from (133) that in fact

$$\text{the set } \left\{y \in \mathbb{R}^d : \mathcal{H}^{n-m}\left(Z_{v,m+1} \cap f^{-1}(y)\right) > 0\right\} \text{ is } \mathcal{H}^m\ \sigma-\text{finite}. \tag{134}$$

Next we prove that the sets where rank $\nabla v \le m - 1$ are negligible in the coarea formula.

Lemma (5.3.29)[208]: The equality

$$\mathcal{H}^{n-m}\left(Z_{v,m} \cap v^{-1}(y)\right) = 0 \text{ for } \mathcal{H}^m-\text{almost all } y \in \mathbb{R}^d \tag{135}$$

holds, where $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \operatorname{rank} \nabla v(x) \le m - 1\}$ is the set of m-critical points.

Proof. We apply Theorem (5.3.5) with the parameters $q = m, k = 1, p_\circ = n$. Then by (77)

$$\mathcal{H}^{\mu_q}\left(Z_{v,m}\cap v^{-1}(y)\right)=0 \text{ for } \mathcal{H}^m-\text{almost all } y\in\mathbb{R}^d, \qquad (136)$$

where $\mu_q = n-m-k+1+(m-q)k = n-m$. The last identity taken together with (136) concludes the proof.

[219], [214] identified criteria for the validity of the Coarea formula for Lipschitz mappings.

Theorem (5.3.30)[208]: (see, e.g., Theorem 1.4 in [214]). Let $m\in\{0,1,\dots,n\}$, and $g\in C^1(\mathbb{R}^n,\mathbb{R}^d)$. Suppose that the set $E\subset\mathbb{R}^n$ is measurable and rank $\nabla g(x)\equiv m$ for all $x\in E$. Assume also that the set $g(E)$ is $\mathcal{H}^m$-σ-finite. Then the coarea formula

$$\int_E J_m g(x)\,dx=\int_{\mathbb{R}^d}\mathcal{H}^{n-m}\left(E\cap g^{-1}(y)\right)d\mathcal{H}^m(y) \qquad (137)$$

holds, where $J_m g(x)$ denotes the m-Jacobian of g.

(134) and (135) are in particular valid also for C^k-smooth mappings. So from Theorem (5.3.30) and properties (134)–(135) we obtain the following result which surprisingly is new even in this smooth case.

Theorem (5.3.31)[208]: Let $m\in\{0,\dots,n\}$ and $g\in C^1(\mathbb{R}^n,\mathbb{R}^d)$. Then for any measurable set $E\subset Z_{g,m+1}=\{x\in\mathbb{R}^n:\ \text{rank}\,\nabla g(x)\le m\}$ the coarea formula

$$\int_E J_m g(x)dx=\int_{\mathbb{R}^d}\mathcal{H}^{n-m}\left(E\cap g^{-1}(y)\right)d\mathcal{H}^m(y) \qquad (138)$$

holds, where $J_{g,m}(x)$ again denotes the m-Jacobian of g.

By Theorem (5.3.16) (iii) (applied to the case $k=l=1$), the investigated mapping $v\in W^1_{n,1}(\mathbb{R}^n,\mathbb{R}^d)$ coincides with a smooth mapping $g\in C^1(\mathbb{R}^n,\mathbb{R}^d)$ off a set of small n-dimensional Lebesgue measure. This fact together with Corollary (5.3.26) easily imply the required assertion of Theorem (5.3.6).

Fix numbers $n,d\in\mathbb{N},\mu\in(0,n],q\in(0,d]$, and a continuous function $f:\mathbb{R}^n\to\mathbb{R}^d$. For a set $E\subset\mathbb{R}^n$ define the set function

$$\Phi(E)=\inf_{E\subset\bigcup_\alpha D_\alpha}\sum_\alpha(\text{diam }D_\alpha)^\mu[\text{diam }v(D_\alpha)]^q\,, \qquad (139)$$

where the infimum is taken over all countable families of compact sets $\{D_\alpha\}_{\alpha\in\mathbb{N}}$ such that $E\subset\bigcup_\alpha D_\alpha$.

We devoted to the proof of following assertion:

We start by recalling the following technical fact from [211]:

Lemma (5.3.32)[208]: For any set $E\subset\mathbb{R}^n$, if $E=\bigcup_{i=1}^\infty E_i$ and $E_i\subset E_{i+1}$ for all $i\in\mathbb{N}$, then

$$\mathcal{H}^\mu_\infty(E)=\lim_{i\to\infty}\mathcal{H}^\mu_\infty(E_i). \qquad (140)$$

Theorem (5.3.33)[208]: The above defined set function $\Phi(\cdot)$ is countably subadditive and

$$\Phi(E)=0\Rightarrow\left[\mathcal{H}^\mu\left(E\cap f^{-1}(y)\right)=0 \text{ for } \mathcal{H}^q-\text{almost all } y\in\mathbb{R}^d\right]. \qquad (141)$$

Proof: The first assertion is evident. Let us prove the second one, i.e., the implication (140). Without loss of generality we can assume that f is compactly supported, and more specifically that $f^{-1}(y)$ is a compact subset of the closed unit ball $\overline{B(0,1)}$ for every $y\in\mathbb{R}^d\setminus\{0\}$.

Let $E\subset\mathbb{R}^n$ and assume that $\Phi(E)=0$. Without loss of generality we can assume that $0\notin f(E)$ and

$$E=\bigcap_{j=1}^\infty\bigcup_{i=1}^\infty D_{ij}\,,$$

where D_{ij} are compact sets in $\mathbb{R}^n$ and

$$\sum_{i=1}^{\infty} \left(\operatorname{diam} D_{ij}\right)^{\mu}\left[\operatorname{diam} f\left(D_{ij}\right)\right]^{q} \xrightarrow[j \to \infty]{} 0. \tag{142}$$

Of course, then E is a Borel set. Suppose that the assertion (140) is false, then we can assume without loss of generality that there exists a set $\mathcal{F} \subset f(E)$ such that

$$\mathcal{H}^q(F) > 0 \quad \textit{and} \quad \mathcal{H}^{\mu}_{\infty}\left(E \cap f^{-1}(y)\right) \geq \frac{5}{2} \quad \text{for all } y \in \mathcal{F}. \tag{143}$$

Unfortunately, we can not assume right now that the set $\mathcal{F}$ is Borel, so we need some careful preparations.

Denote $E_{k_j} = \bigcup_{i=1}^{k} D_{ij}$, $E_j = \bigcup_{i=1}^{\infty} D_{ij}$. In this notation $= \bigcap_{j=1}^{\infty} E_j$. Evidently, all these sets are Borel. By Lemma (5.3.32),

$$\mathcal{H}^{\mu}_{\infty}\left(E_j \cap f^{-1}(y)\right) = \lim_{k\to\infty} \mathcal{H}^{\mu}_{\infty}\left(E_{k_j} \cap f^{-1}(y)\right) \quad \text{for each } y \in f\left(E_j\right). \tag{144}$$

Denote further $F_{k_j} = f\left(E_{k_j}\right)$. Fix an arbitrary point y with the property

$$\mathcal{H}^{\mu}(E_{k_j} \cap f^{-1}(y)) \leq 1.$$

Since E_{k_j} is a compact set, the set $E_{k_j} \cap f^{-1}(y)$ is compact as well. Then it follows by elementary means that the sets $E_{k_j} \cap f^{-1}(z)$ lie in the ε-neighborhood of the set $E_{k_j} \cap f^{-1}(y)$, where $\varepsilon \searrow 0$ as $z \to y, z \in f(E_{k_j})$. Therefore, there exists $\delta = \delta(y) > 0$ such that

$$\mathcal{H}^{\mu}_{\infty}\left(E_{k_j} \cap f^{-1}(z)\right) \leq 2 \qquad \text{if } |z - y| < \delta. \tag{145}$$

Hence, there exists a relatively open set $\tilde{F}_{k_j} \subset F_{k_j}$ (i.e., $\tilde{F}_{k_j}$ is open in the induced topology of the set F_{k_j}) such that

$$\left\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}\left(E_{k_j} \cap f^{-1}(y)\right) \leq 1\right\} \subset \tilde{F}_{k_j}$$
$$\subset \left\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}\left(E_{k_j} \cap f^{-1}(y)\right) \leq 2\right\}. \tag{146}$$

Since by construction F_{k_j} is a compact set and $\tilde{F}_{k_j}$ is relatively open in F_{k_j} , we conclude that the set $\tilde{F}_{k_j}$ is Borel (this fact plays an important role here). Further, since $E_{k_j} \subset E_j$, we have for each $k \in \mathbb{N}$,

$$\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_j \cap f^{-1}(y)) \leq 1\} \subset \{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_{k_j} \cap f^{-1}(y)) \leq 1\} \subset \tilde{F}_{k_j}$$

and therefore,

$$\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_j \cap f^{-1}(y)) \leq 1\} \subset \tilde{F}_j , \tag{147}$$

where we denote $\tilde{F}_j = \bigcap_{k=1}^{\infty} \tilde{F}_{k_j}$. On other hand, (144) and the second inclusion in (146) imply $\tilde{F}_j \subset \{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_j \cap f^{-1}(y)) \leq 2\}$, so we have

$$\left\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}\left(E_j \cap f^{-1}(y)\right) \leq 1\right\} \subset \tilde{F}_j$$
$$\subset \{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_j \cap f^{-1}(y)) \leq 2\}. \tag{148}$$

Denote now $\tilde{G}_j = f(E_j) \setminus \tilde{F}_j$. Then we can rewrite (148) as

$$\left\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}\left(E_j \cap f^{-1}(y)\right) > 2\right\} \subset \tilde{G}_j$$
$$\subset \{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_j \cap f^{-1}(y)) > 1\}. \tag{149}$$

Since $\subset E_j$, we have from (143) that $F \subset \{y \in \mathbb{R}^d : \backslash sH^{\mu}_{\infty}(E_j \cap f^{-1}(y)) > 2\} \subset \tilde{G}_j$ for all $j \in \mathbb{N}$, therefore

$$\mathcal{F} \subset \tilde{G}, \tag{150}$$

where we denote $\tilde{G} = \bigcap_{j=1}^{\infty} \tilde{G}_j$. On the other hand, the second inclusion in (149) yields

$$\tilde{G} \subset \{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_j \cap f^{-1}(y)) > 1\} \tag{151}$$

for each $j \in \mathbb{N}$. Since $\tilde{G}$ is a Borel set and by (150), (143) the inequalities $\mathcal{H}^q(\tilde{G}) \geq \mathcal{H}^q(\mathcal{F}) > 0$ hold, by [212] there exists a Borel set $G \subset \tilde{G}$ and a positive constant $b \in \mathbb{R}$ such that $0 < \mathcal{H}^q(G) < \infty$ and

$$\mathcal{H}^q(G \cap B(y,r)) \leq b\, r^q \tag{152}$$

for any ball $B(y,r) = \{z \in \mathbb{R}^d : |z - y| < r\}$ with the center $y \in G$. Of course, by (151)

$$G \subset \{y \in \mathbb{R}^d : \mathcal{H}^{\mu}_{\infty}(E_j \cap f^{-1}(y)) > 1\} \tag{153}$$

for all $j \in \mathbb{N}$. For $S \subset \mathbb{R}^n$ consider the set function

$$\widetilde{\Phi}(S) = \int_G^* \mathcal{H}^{\mu}_{\infty}(S \cap f^{-1}(y)\ d\mathcal{H}^q(y), \tag{154}$$

where $\int^*$ means the upper integral. Standard facts of Lebesgue integration theory, $\widetilde{\Phi}(\cdot)$ is a countably subadditive set-function (see, e.g., [60], [192]).
From (142) and (152) it follows that

$$\sum_{i=1}^{\infty} \left(\operatorname{diam} D_{ij}\right)^{\mu}\left[\operatorname{diam} f\left(D_{ij}\right)\right]^q \geq c \sum_{i=1}^{\infty} \left(\operatorname{diam} D_{ij}\right)^{\mu} \mathcal{H}^q\left[G \cap f\left(D_{ij}\right)\right]$$

$$\geq C \sum_{i=1}^{\infty} \widetilde{\Phi}(D_{ij}) \geq C\, \widetilde{\Phi}(E_j).$$

Consequently, $\widetilde{\Phi}(E_j) \to 0$ as $j \to \infty$. On the other hand, from (153) and (154) we conclude

$$\widetilde{\Phi}(E_j) \geq \int_G^* d\mathcal{H}^q(y) = \mathcal{H}^q(G) > 0,$$

which is the desired contradiction. The proof of the Theorem (5.3.33) is finished.

Now again fix numbers $n, d \in \mathbb{N}, \mu \in (0,n], q \in (0,d]$ and a continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^d$. We define the set function by letting for a set $E \subset \mathbb{R}^n$,

$$\Psi(E) = \lim_{\delta \to 0} \inf_{\substack{E \subset \bigcup_\alpha D_\alpha, \\ \operatorname{diam} D_\alpha \leq \delta}} \sum_{\alpha} (\operatorname{diam} D_\alpha)^{\mu}[\operatorname{diam} f(D_\alpha)]^q , \tag{155}$$

where the infimum is taken over all countable families of compact sets $\{D_\alpha\}_{\alpha \in \mathbb{N}}$ such that $E \subset \bigcup_\alpha D_\alpha$ and $\operatorname{diam} D_\alpha \leq \delta$ for all α.
We devoted to the following assertion:
Theorem (5.3.34)[208]: The above defined $\Psi(\cdot)$ is a countably subadditive set-function and for any $\lambda > 0$ the estimate

$$\mathcal{H}^q\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}(E \cap f^{-1}(y) \geq \lambda\} \leq 5\frac{\Psi(E)}{\lambda} \tag{156}$$

holds.
Proof. The first assertion is evident and we focus on proving the estimate (156). Without loss of generality we can assume that $f^{-1}(y)$ is a compact subset of the closed unit ball $\overline{B(0,1)}$ for every $y \in \mathbb{R}^d \setminus \{0\}$. Let $E \subset \mathbb{R}^n$ and

$$\Psi(E) = \sigma < \infty.$$

Without loss of generality assume also that $0 \notin f(E)$ and

$$E = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} D_{ij} ,$$

where D_{ij} are compact sets in $\mathbb{R}^n$ satisfying

$$\sum_{i=1}^{\infty} \left(\operatorname{diam} D_{ij}\right)^{\mu} \left[\operatorname{diam} f\left(D_{ij}\right)\right]^{q} \overrightarrow{J \to \infty} \, \sigma, \tag{157}$$

And

$$\operatorname{diam} D_{ij} + \operatorname{diam} f\left(D_{ij}\right) \leq \frac{1}{j} . \tag{158}$$

Of course, E is a Borel set. Fix $\lambda > 0$ and take a set $\mathcal{F} \subset f(E)$ such that

$$\mathcal{H}_{\infty}^{\mu} \left(E \cap f^{-1}(y)\right) \geq \frac{5}{2} \lambda \qquad \text{for all } y \in \mathcal{F}. \tag{159}$$

Further we assume that

$$\mathcal{H}^{q}(\mathcal{F}) > 0 \tag{160}$$

since if $\mathcal{H}^{q}(\mathcal{F}) = 0$, there is nothing to prove. Denote $E_j = \bigcup_{i=1}^{\infty} D_{ij}$. Repeating almost verbatim the arguments from the proof of the previous Theorem (5.3.33), we can construct a Borel set $\tilde{G} \subset \mathbb{R}^d$ such that

$$\mathcal{F} \subset \tilde{G} \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) > \lambda\} \tag{161}$$

for each $j \in \mathbb{N}$. Since $\tilde{G}$ is a Borel set and since, by (161) and (160), the inequalities $\mathcal{H}^{q}(\tilde{G}) \geq \mathcal{H}^{q}(\mathcal{F}) > 0$ hold, we deduce by [212] the existence of a Borel set $G \subset \tilde{G}$ such that $0 < \mathcal{H}^{q}(G) < \infty$. Put

$$G_l = \left\{x \in G : \mathcal{H}^{q}\left(G \cap B(x,r)\right) \leq 2r^{q} \ \ \forall r \in (0,1/l)\right\} . \tag{162}$$

Then by construction all the sets G_l are Borel, $G_l \subset G_{l+1}$, moreover, by [60] we have

$$\mathcal{H}^{q}\left[G \setminus \left(\bigcup_{l=1}^{\infty} G_l\right)\right] = 0$$

and consequently,

$$\mathcal{H}^{q}(G) = \lim_{l \to \infty} \mathcal{H}^{q}(G_l). \tag{163}$$

For $S \subset \mathbb{R}^n$ consider the set function

$$\Psi_l(S) = \int_{G_l}^{*} \mathcal{H}_{\infty}^{\mu}\left(S \cap f^{-1}(y)\right) \, d\mathcal{H}^{q}(y), \tag{164}$$

where $\int^{*}$ means the upper integral. routine arguments of Lebesgue integration theory it follows that $\Psi(\cdot)$ is a countably subadditive set-function (see, e.g., [60], [192]).
From (157), (158) and (162) it follows for $j > l$ that

$$\sum_{i=1}^{\infty} \left(\operatorname{diam} D_{ij}\right)^{\mu} \left[\operatorname{diam} f\left(D_{ij}\right)\right]^{q} \geq \frac{1}{2} \sum_{i=1}^{\infty} \left(\operatorname{diam} D_{ij}\right)^{\mu} \mathcal{H}^{q}\left[G_l \cap f\left(D_{ij}\right)\right]$$

$$\geq \frac{1}{2} \sum_{i=1}^{\infty} \Psi_l\left(D_{ij}\right) \geq \frac{1}{2} \Psi_l\left(E_j\right). \tag{165}$$

On the other hand, the second inclusion in (161) implies

$$\Psi_l(E_j) \geq \lambda \int_{G_l}^{*} d\mathcal{H}^{q}(y) = \lambda \mathcal{H}^{q}(G_l). \tag{166}$$

From (165), (166), (157) we infer

$$\mathcal{H}^q(G_l) \le \frac{2\sigma}{\lambda}, \tag{167}$$

and therefore, by (163),

$$\mathcal{H}^q(G) \le \frac{2\sigma}{\lambda}. \tag{168}$$

Since this estimate is true for any Borel set $G \subset \tilde{G}$ with $\mathcal{H}^q(G) < \infty$, and since $\tilde{G}$ is Borel as well, we infer from [212] that

$$\mathcal{H}^q(\tilde{G}) \le \frac{2\sigma}{\lambda}. \tag{169}$$

In particular, by the inclusion $\mathcal{F} \subset \tilde{G}$, this implies

$$\mathcal{H}^q(\mathcal{F}) \le \frac{2\sigma}{\lambda}, \tag{170}$$

or in other words,

$$\mathcal{H}^q\left(y \in \mathbb{R}^d : \mathcal{H}^\mu\big(E \cap f^{-1}(y)\big) \ge \frac{5}{2}\lambda\right) \le 2\frac{\Psi(E)}{\lambda}. \tag{171}$$

The proof of Theorem (5.3.34) is complete.

Corollary (5.3.35)[260]: Let $k \in \{1, \dots, n\}, p_\circ = n/k$ and $v_{m_0} \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^{m+\epsilon})$. Then for every $\epsilon \ge 0$ and for any set $E \subset \mathbb{R}^n$ with $\mathcal{H}^{1+2\epsilon}(E) = 0$ we have

$$\sum_{m_0} \mathcal{H}^\epsilon(E \cap v_{m_0}^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^{1+\epsilon} - a.a. y \in \mathbb{R}^{m+\epsilon}. \tag{172}$$

In particular,

$$\sum_{m_0} \mathcal{H}^{p_\circ+\epsilon}\left(v_{m_0}(E)\right) = 0 \quad \text{whenever} \quad \mathcal{H}^{p_\circ+\epsilon}(E) = 0, \qquad \epsilon \ge 0. \tag{173}$$

By a simple calculation we have for $\epsilon \ge -1$ that

$$\begin{aligned} \mu_{1+\epsilon} &= n - m - k + 1 + k(m - 1 - \epsilon) \\ &= (p_\circ - 1 - \epsilon)k + (m-1)(k-1) \ge max(p_\circ - 1 - \epsilon, 0). \end{aligned} \tag{174}$$

Corollary (5.3.35) then yields

Proof. (See [208]) Let $\mathcal{H}^{p_0+\epsilon}(E) = 0$. Take $\varepsilon > 0$ and $\delta = \delta(\varepsilon, v_{m_0}) < 1$ from Lemma (5.3.23). Take also the regular family $\{Q_\alpha\}$ of n-dimensional dyadic intervals such that $E \subset \bigcup_\alpha Q_\alpha$ and

$$\sum_\alpha \ell(Q_\alpha)^{1+2\epsilon} < \delta \tag{175}$$

where the existence of such family follows directly from Lemmas (5.3.21) and (5.3.22). Then by Lemma (5.3.23) the estimate (102) holds. Denote $r_\alpha = \ell(Q_\alpha)$. By estimate (91),

$$\sum_{m_0} \left[\operatorname{diam} v_{m_0}(Q_\alpha)\right]^{1+\epsilon} \le C \sum_{m_0} \left(\frac{\left\|\nabla v_{m_0}\right\|^{1+\epsilon}_{L_{1+2\epsilon}(Q_\alpha)}}{r_\alpha^{\left(\frac{n}{1+2\epsilon}-1\right)1+\epsilon}} + \left\|1_{Q_\alpha} \cdot \nabla^k v_{m_0}\right\|^{1+\epsilon}_{L_{p_\circ,1}} \right). \tag{176}$$

Therefore, by definition of $\Phi(E)$ (see (97)), we have

$$\Phi(E) \le C \sum_\alpha \sum_{m_0} r_\alpha^\mu \left(\frac{\left\|\nabla v_{m_0}\right\|^{1+\epsilon}_{L_{1+2\epsilon}(Q_\alpha)}}{r_\alpha^{\left(\frac{n}{1+2\epsilon}-1\right)1+\epsilon}} + \left\|1_{Q_\alpha} \cdot \nabla^k v_{m_0}\right\|^{1+\epsilon}_{L_{p_\circ,1}} \right)$$

$$\overset{\text{Hölder ineq}}{\leq} c\left(\sum_{\alpha} r_{\alpha}^{\frac{\mu(1+2\epsilon)}{\epsilon}}\right)^{\frac{\epsilon}{1+2\epsilon}}$$

$$\cdot\left[\sum_{\alpha}\sum_{m_0}\left(\frac{1}{\ell(Q_\alpha)^{\epsilon}}\int_{Q_\alpha}\left|\nabla v_{m_0}\right|^{1+2\epsilon}+\left\|1_{Q_\alpha}\cdot\nabla^k v_{m_0}\right\|_{L_{p_\circ,1}}^{1+\epsilon}\right)\right]^{\frac{1+\epsilon}{1+2\epsilon}}$$

$$\overset{(5.1),(5.7)}{\leq} c\left(\sum_{\alpha} r_{\alpha}^{1+2\epsilon}\right)^{\frac{1+2\epsilon}{1+2\epsilon}}\cdot\varepsilon^{\frac{1+\epsilon}{1+2\epsilon}}$$

$$\overset{(175)}{\leq} c\delta^{\frac{\epsilon}{1+2\epsilon}}\cdot\varepsilon^{\frac{1+\epsilon}{1+2\epsilon}}. \tag{177}$$

Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary small, (177) turns to the equality $\Phi(E) = 0$ and by (5.3) the required assertion is proved.

Corollary (5.3.36)[260]: Let $1+\epsilon \in [m-1, 1+2\epsilon)$. Then for any n-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$\sum_{m_0}\Phi\left(Z'_{v_{m_0}}\cap I\right)$$

$$\leq C\sum_{m_0}\left(\ell(I)^{\mu}\left\|\nabla^k v_{m_0}\right\|_{L_{p_\circ,1}(I)}^{1+\epsilon}+\ell(I)^{\mu+m-1}\left\|\nabla^k v_{m_0}\right\|_{L_{p_\circ,1}(I)}^{2+\epsilon-m}\right) \tag{178}$$

holds, where the constant C depends on $n, m, k, m+\epsilon$ only.

Proof. By virtue of (90) it suffices to prove that

$$\sum_{m_0}\Phi\left(Z'_{v_{m_0}}\cap I\right)$$

$$\leq C\sum_{m_0}\left(\ell(I)^{\mu}\left\|\nabla^k (v_{m_0})_I\right\|_{L_{p_\circ,1}(\mathbb{R}^n)}^{1+\epsilon}+\ell(I)^{\mu+m-1}\left\|\nabla^k (v_{m_0})_I\right\|_{L_{p_\circ,1}(\mathbb{R}^n)}^{2+\epsilon-m}\right. \tag{179}$$

for the mapping $(v_{m_0})_I$ defined in Lemma (5.3.11), where $C = C(n, m, k, m+\epsilon)$ is a constant.

Fix an n-dimensional dyadic interval $I \subset \mathbb{R}^n$ and recall that $(v_{m_0})_I(x) = v_{m_0}(x) - P_I(x)$ for all $x \in I$. Denote

$$\sigma = \sum_{m_0}\left\|\nabla^k (v_{m_0})_I\right\|_{L_{p_\circ,1}}, \qquad 1+\epsilon = \ell(I),$$

and for each $j \in \mathbb{Z}$

$$E_j = \left\{x \in I : \sum_{m_0}\left(\mathcal{M}\left|\nabla (v_{m_0})_I\right|^{p_\circ}\right)(x) \in (2^{j-1}, 2^j]\right\} \quad and \quad \delta_j = \mathcal{H}_\infty^{p_\circ}(E_j).$$

Then by Theorem (5.3.13) (applied for the case $1+2\epsilon = p_\circ = \frac{n}{k}, l = 1, \beta = p_\circ$),

$$\sum_{j=-\infty}^{\infty}\delta_j 2^j \leq C\sigma^{p_\circ} \tag{180}$$

for a constant C depending on $n, m, k, m+\epsilon$ only. By the definition of the Hausdorff measure, for each $j \in \mathbb{Z}$ there exists a family of balls $B_{ij} \subset \mathbb{R}^n$ of radii r_{ij} such that

$$E_j \subset \bigcup_{i=1}^{\infty} B_{ij} \quad \text{and} \quad \sum_{i=1}^{\infty} r_{ij}^{p_\circ} \le c\delta_j \,. \tag{181}$$

Denote

$$Z_j = Z'_{v_{m_0}} \cap E_j \qquad \text{and} \qquad Z_{ij} = Z_j \cap B_{ij} \,.$$

By construction $Z'_{v_{m_0}} \cap I = \bigcup_j Z_j$ and $Z_j = \bigcup_i Z_{ij}$. Put

$$\varepsilon_* = \frac{1}{1+\epsilon} \sum_{m_0} \left\| \nabla^k (v_{m_0})_I \right\|_{L_{p_\circ,1}} = \frac{\sigma}{1+\epsilon},$$

and let j_* be the integer satisfying $\varepsilon_*^{p_\circ} \in (2^{j_*-1}, 2^{j_*}]$. Denote $Z_* = \bigcup_{j<j_*} Z_j, Z_{**} = \bigcup_{j \ge j_*} Z_j$. Than by construction

$$Z'_{v_{m_0}} \cap I = Z_* \cup Z_{**},\ \ Z_* \subset \{x \in Z'_{v_{m_0}} \cap I : \sum_{m_0} (\mathcal{M} \left|\nabla (v_{m_0})_I\right|^{p_\circ})(x) < \varepsilon_*^{p_\circ}\}.$$

Since $\nabla P_I\,(x) = \sum_{m_0} (\nabla v_{m_0}(x) - \nabla \big(v_{m_0}\big)_I(x)), \left|\sum_{m_0} \nabla \big(v_{m_0}\big)_I(x)\right| \le 2^{\frac{j}{p_\circ}}$, $\left|\sum_{m_0} \nabla v_{m_0}(x)\right| \le 1$, and $\lambda_m(v_{m_0}, x) = 0$ for $x \in Z_{ij}$, we have

$$Z_{ij} \subset \{x \in B_{ij} : \lambda_1(P_I, x) \le 1 + 2^{j/p_\circ}, \dots, \lambda_{m-1}(P_I\,, x) \le 1 + 2^{j/p_\circ}, \lambda_m(P_I\,, x) \le 2^{j/p_\circ}\}.$$

Applying Theorem (5.3.14) and Lemma (5.3.15) to mappings P_I , $(v_{m_0})_I$, respectively, with $B = B_{ij}$ and $\varepsilon = \varepsilon_j = 2^{j/p_\circ}$, we find a finite family of balls $T_s \subset \mathbb{R}^{m+\epsilon}, s = 1, \dots, s_j$ with $s_j \le C_Y\,(1 + \varepsilon_j^{1-m})$, each of radius $(1 + C_M)\varepsilon_j\, r_{ij}$, such that

$$\bigcup_{s=1}^{s_j} T_s \supset v_{m_0}(Z_{ij}).$$

Therefore, for every $j \ge j_*$ we have

$$\begin{aligned} \Phi\big(Z_{ij}\big) \le C_1 s_j \varepsilon_j^{1+\epsilon} r_{ij}^{1+\epsilon+\mu} &= C_2\big(1 + \varepsilon_j^{1-m}\big) 2^{\frac{j(1+\epsilon)}{p_\circ}} r_{ij}^{1+\epsilon+\mu} \\ &\le C_2(1 + \varepsilon_*^{1-m}) 2^{\frac{j(1+\epsilon)}{p_\circ}} r_{ij}^{1+\epsilon+\mu}, \end{aligned} \tag{182}$$

where all the constants C_α above depend on $n, m, k, m+\epsilon$ only. By the same reasons, but this time applying Theorem (5.3.14) and Lemma (5.3.15) with $\varepsilon = \varepsilon_*$ and instead of the balls B_{ij} we take a ball $B \supset I$ with radius $\sqrt{n(1+\epsilon)}$, we have

$$\begin{aligned} \Phi(Z_*) \le C_3(1 + \varepsilon_*^{1-m})\varepsilon_*^{1+\epsilon}(1+\epsilon)^{1+\epsilon+\mu} &\stackrel{\text{def}}{=} C_3(1 + \sigma^{1-m}(1+\epsilon)^{m-1})\sigma^{1+\epsilon}(1+\epsilon)^{\mu} \\ &= C_3((1+\epsilon)^{\mu}\sigma^{1+\epsilon} + (1+\epsilon)^{\mu+m-1}\sigma^{2+\epsilon-m}). \end{aligned} \tag{183}$$

From (182) we get immediately

$$\Phi(Z_{**}) \le C_2(1 + \varepsilon_*^{1-m}) \sum_{j \ge j_*} \sum_i 2^{\frac{j(1+\epsilon)}{p_\circ}} r_{ij}^{1+\epsilon+\mu} \,. \tag{184}$$

Further estimates splits into the two possibilities.

Case I. $1+\epsilon \ge p_\circ$. Then

$$\Phi(Z_{**}) \le C_2(1 + \varepsilon_*^{1-m}) \left(\sum_{j \ge j_*} \sum_i 2^j r_{ij}^{(1+\epsilon+\mu)\frac{p_\circ}{1+\epsilon}} \right)^{\frac{1+\epsilon}{p_\circ}}$$

$$\leq C_2(1+\varepsilon_*^{1-m})r_\mu\left(\sum_{j\geq j_*}\sum_i 2^j r_{ij}^{p_\circ}\right)^{\frac{1+\epsilon}{p_\circ}}$$

$$\leq C_4(1+\varepsilon_*^{1-m})(1+\epsilon)^\mu\left(\sum_{j\geq j_*}2^j\delta_j\right)^{\frac{1+\epsilon}{p_\circ}}$$

$$\leq C_5(1+\varepsilon_*^{1-m})(1+\epsilon)^\mu\sigma^{1+\epsilon} = C_5((1+\epsilon)^\mu\sigma^{1+\epsilon}+(1+\epsilon)^{\mu+m-1}\sigma^{2+\epsilon-m}) \quad (185)$$

Case II. $1+\epsilon < p_\circ$. Recalling (107) we get by an elementary calculation

$$1+\epsilon+\mu = 1+\epsilon+(n-m-k+1)+(m-1-\epsilon)k = (p_\circ-2-\epsilon+m)(k-1)+p_\circ \geq p_\circ,$$

therefore,

$$\Phi(Z_{**}) \leq C_2(1+\varepsilon_*^{1-m})\left(\sum_{j\geq j_*}\sum_i 2^j(1+\epsilon)^{p_\circ ij}\right)(1+\epsilon)^{1+\epsilon+\mu-p_\circ}2^{j_*\frac{1+\epsilon-p_\circ}{p_\circ}}$$

$$\leq C_6(1+\varepsilon_*^{1-m})\sigma^{p_\circ}(1+\epsilon)^{1+\epsilon+\mu-p_\circ}\left(\frac{\sigma}{1+\epsilon}\right)^{1+\epsilon-p_\circ} = C_6(1+\varepsilon_*^{1-m})\sigma^{1+\epsilon}(1+\epsilon)^\mu$$

$$= C_6((1+\epsilon)^\mu\sigma^{1+\epsilon}+(1+\epsilon)^{\mu+m-1}\sigma^{1+\epsilon-m+1}). \quad (186)$$

Now for both cases (I) and (II) we have by (185), (186) that $\Phi(Z_{**}) \leq C((1+\epsilon)^\mu\sigma^{1+\epsilon}+(1+\epsilon)^{\mu+m-1}\sigma^{2+\epsilon-m})$, and, by virtue of the earlier estimate (183), we conclude that

$$\Phi(Z'_{v_{m_0}}\cap I) = \Phi(Z_*\cup Z_{**}) \leq \Phi(Z_*)+\Phi(Z_{**})$$
$$\leq C((1+\epsilon)^\mu\sigma^{1+\epsilon}+(1+\epsilon)^{\mu+m-1}\sigma^{2+\epsilon-m}).$$

The lemma is proved.

Corollary (5.3.37)[260]: [208] Let $1+\epsilon \in [m-1, 1+2\epsilon)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any subset E of $\mathbb{R}^n$ we have $\sum_{m_0}\Phi(Z'_{v_{m_0}}\cap E) \leq \varepsilon$ provided $\mathrm{L}^n(E) \leq \delta$. In particular, $\Phi\left(\sum_{m_0}\left(Z_{v_{m_0},m}\cap E\right)\right) = 0$ whenever $\mathcal{L}^n(E) = 0$.

Proof. We start by recording the following elementary identity (see (107)):

$$\frac{(\mu+m-1)p_\circ}{p_\circ-2-\epsilon+m} = n. \quad (187)$$

Let $\mathrm{L}^n(E) \leq \delta$, then we can find a family of nonoverlapping n-dimensional dyadic intervals I_α such that $E \subset \bigcup_\alpha I_\alpha$ and $\sum_\alpha \ell^n(I_\alpha) < C\delta$. Of course, for sufficiently small δ the estimates

$$\left\|\sum_{m_0}\nabla^k v_{m_0}\right\|_{L_{p_\circ,1}(I_\alpha)} < 1, \qquad \ell(I_\alpha) \leq \delta^{\frac{1}{n}} \quad (188)$$

are fulfilled for every α. Denote

$$r_\alpha = \ell(I_\alpha), \quad \sigma_\alpha = \sum_{m_0}\left\|\nabla^k v_{m_0}\right\|_{L_{p_\circ,1}(I_\alpha)}, \quad \sigma = \sum_{m_0}\left\|\nabla^k v_{m_0}\right\|_{L_{p_\circ,1}}. \quad (189)$$

In view of Corollary (5.3.36) we have

$$\Phi(E) \leq C\sum_\alpha (r_\alpha^{\mu+m-1}\sigma_\alpha^{2+\epsilon-m}+r_\alpha^\mu\sigma_\alpha^{1+\epsilon}). \quad (190)$$

Now let us estimate the first sum. Since by our assumptions

$$\epsilon > 0 = m - 1 + \frac{n-m+1}{k} \leq m - 1 + p_\circ \quad hence\ p_\circ > 2 + \epsilon - m$$

we have

$$\sum_\alpha r_\alpha^{\mu+m-1}\, \sigma_\alpha^{2+\epsilon-m} \overset{Hölder\ ineq.}{\leq} C \left(\sum_\alpha \sigma_\alpha^{p_\circ}\right)^{\frac{2+\epsilon-m}{p_\circ}} \cdot \left(\sum_\alpha r_\alpha^{\frac{(\mu+m-1)p}{p_\circ-2-\epsilon+m}}\right)^{\frac{p_\circ-2-\epsilon+m}{p_\circ}}$$

$$\overset{(187),\text{Lemma }(5.3.10)}{\leq} C'\sigma^{2+\epsilon-m} \cdot \left(\mathrm{L}^n(E)\right)^{\frac{p_\circ-2-\epsilon+m}{p_\circ}}. \tag{191}$$

The estimates of the second sum are again handled by consideration of two separate cases.

Case I. $1+\epsilon \geq p_\circ$. Then

$$\sum_\alpha r_\alpha^\mu \sigma_\alpha^{1+\epsilon} \overset{(188)}{\leq} \delta^{\frac{\mu}{n}} \sum_\alpha \sigma_\alpha^{p_\circ} \overset{\text{Lemma }(5.3.10)}{\leq} \sigma^{p_\circ} \cdot \delta^{\frac{\mu}{n}}. \tag{192}$$

Case II. $1+\epsilon < p_\circ$. Recalling (5.11) we get by an elementary calculation

$$\frac{\mu p_\circ}{p_\circ - 1 - \epsilon} = n \cdot \frac{n-(1+\epsilon)k + [mk - m - k + 1]}{n-(1+\epsilon)k}$$
$$= n \cdot \frac{n-(1+\epsilon)k + (m-1)(k-1)}{n-(1+\epsilon)k} \geq n, \tag{193}$$

Then

$$\sum_\alpha r_\alpha^\mu \sigma_\alpha^{1+\epsilon} \overset{\text{Hölder ineq.}}{\leq} \left(\sum_\alpha \sigma_\alpha^{p_\circ}\right)^{\frac{1+\epsilon}{p_\circ}} \cdot \left(\sum_\alpha r_\alpha^{\frac{\mu p_\circ}{p_\circ-1-\epsilon}}\right)^{\frac{p_\circ-1-\epsilon}{p_\circ}}$$
$$\overset{\text{Lemma }(5.3.10),(193)}{\leq} \sigma^{1+\epsilon}\delta^{\frac{\mu}{n}}. \tag{194}$$

Now for both cases (I) and (II) we have by (190)–(194) that $\Phi(E) \leq h(\delta)$, where the function $h(\delta)$ satisfies the condition $h(\delta) \searrow 0$ as $\delta \searrow 0$. The lemma is proved.

Corollary (5.3.38)[260]: [208] Let $(1+\epsilon) \in (m-1, 1+2\epsilon)$ and $g_{m_0} \in C^k(\mathbb{R}^n, \mathbb{R}^{m+\epsilon})$. Then

$$\sum_{m_0} \Phi_{g_{m_0}}(Z_{g_{m_0},m}) = 0, \tag{195}$$

where $\Phi_{g_{m_0}}$ is calculated by the same formula (109) with g_{m_0} instead of v_{m_0} and $Z_{g_{m_0},m} = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla\left(\sum_{m_0} g_{m_0}(x)\right) < m\}$.

Proof. We can assume without loss of generality that g_{m_0} has compact support and that $|\sum_{m_0}\left(\nabla g_{m_0}(x)\right)| \leq 1$ for all $x \in \mathbb{R}^n$. We then clearly have that $g_{m_0} \in W^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^{m+\epsilon})$, hence we can in particular apply the above results to g_{m_0}. The following assertion plays the key role:

$(*)$ For any n-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$\sum_{m_0} \Phi\left(Z_{g_{m_0},m} \cap I\right) \leq C \sum_{m_0} \left(\ell(I)^\mu \left\|\nabla^k \overline{(g_{m_0})_I}\right\|^{1+\epsilon}_{L_{p_\circ,1}(I)} + \ell(I)^{\mu+m-1}\left\|\nabla^k\overline{(g_{m_0})_I}\right\|^{2+\epsilon-m}_{L_{p_\circ,1}(I)}\right)$$

holds, where the constant C depends on $n, m, k, m+\epsilon$ only, and we denoted

$$\sum_{m_0} \nabla^k \overline{(g_{m_0})_I}\,(x) = \sum_{m_0} \nabla^k g_{m_0}(x) - \frac{1}{\mathrm{L}^n(I)} \int_I \sum_{m_0} \nabla^k g_{m_0}(y)\, dy.$$

The proof of $(*)$ is almost the same as that of Corollary (5.3.36), with evident modifications (we need to take the approximation polynomial $P_I(x)$ of degree k instead of $k-1$, etc.). By elementary facts of the Lebesgue integration theory, for an arbitrary family of nonoverlapping n-dimensional dyadic intervals I_α one has

$$\sum_{\alpha}\sum_{m_0} \left\|\nabla^k\overline{(g_{m_0})_{I_\alpha}}\right\|^{p_\circ}_{L_{p_\circ,1}(I_\alpha)} \to 0 \qquad \text{as} \sup_{\alpha} \ \ell(I_\alpha) \to 0. \tag{196}$$

The proof of this estimate is really elementary since now $\nabla^k\left(\sum_{m_0} g_{m_0}\right)$ is continuous and compactly supported function, and, consequently, is uniformly continuous and bounded.
From $(*)$ and (196), repeating the arguments of Corollary (5.3.37), using the assumptions on g_{m_0} and taking

$$\sigma_\alpha = \sum_{m_0} \left\|\nabla^k\overline{(g_{m_0})_{I_\alpha}}\right\|_{L_{p_\circ,1}(I_\alpha)}, \qquad \sigma^{p_\circ} = \sum_{\alpha} \sigma_\alpha^{p_\circ}$$

in definitions (189), we obtain that $\sum_{m_0} \Phi_{g_{m_0}}(Z_{g_{m_0},m}) < \varepsilon$ for any $\varepsilon > 0$, hence the sought conclusion (195) follows.

Corollary (5.3.39)[260]: The equality

$$\sum_{m_0} \mathcal{H}^{n-m}\left(Z_{v_{m_0},m} \cap v_{m_0}^{-1}(y)\right) = 0 \ \text{ for } \mathcal{H}^m - \text{almost all } y \in \mathbb{R}^{m+\epsilon} \tag{197}$$

holds, where $Z_{v_{m_0},m} = \{x \in \mathbb{R}^n \setminus A_{v_{m_0}} : \operatorname{rank} \sum_{m_0} \nabla v_{m_0}(x) \le m-1\}$ is the set of m-critical points.

Proof. We apply Theorem (5.3.5) with the parameters $1+\epsilon = m$, $k=1$, $p_\circ = n$. Then by (77)

$$\sum_{m_0} \mathcal{H}^{\mu_{1+\epsilon}}\left(Z_{v_{m_0},m} \cap \ v_{m_0}^{-1}(y)\right) = 0 \ \text{ for } \mathcal{H}^m - \text{almost all } y \in \mathbb{R}^{m+\epsilon}, \tag{198}$$

where $\mu_{1+\epsilon} = n-m-k+1+(m-1-\epsilon)k = n-m$. The last identity taken together with (198) concludes the proof.

Corollary (5.3.40)[260]: The above defined set function $\Phi(\cdot)$ is countably subadditive and

$$\Phi(E) = 0 \Rightarrow \left[\sum_{m_0} \mathcal{H}^{\mu}\left(E \cap f_{m_0}^{-1}(y)\right) = 0 \ \text{ for } \mathcal{H}^{1+\epsilon} - \text{almost all } y \in \mathbb{R}^{m+\epsilon}\right]. \tag{199}$$

We start by recalling the following technical fact from [211]:

Proof. The first assertion is evident. Let us prove the second one, i.e., the implication (198). Without loss of generality we can assume that f_{m_0} is compactly supported, and more specifically that $f_{m_0}^{-1}(y)$ is a compact subset of the closed unit ball $\overline{B(0,1)}$ for every $y \in \mathbb{R}^{m+\epsilon} \setminus \{0\}$.

Let $E \subset \mathbb{R}^n$ and assume that $\Phi(E) = 0$. Without loss of generality we can assume that $0 \notin f_{m_0}(E)$ and

$$E = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} D_{ij},$$

where D_{ij} are compact sets in $\mathbb{R}^n$ and

$$\sum_{i=1}^{\infty} \sum_{m_0} \left(\operatorname{diam} D_{ij}\right)^{\mu}\left[\operatorname{diam} f_{m_0}(D_{ij})\right]^{1+\epsilon} \ , j \to \infty. \tag{200}$$

Of course, then E is a Borel set. Suppose that the assertion (199) is false, then we can assume without loss of generality that there exists a set $\mathcal{F} \subset f_{m_0}(E)$ such that

$$\mathcal{H}^{1+\epsilon}(F) > 0 \quad and \quad \sum_{m_0} \mathcal{H}_{\infty}^{\mu}\left(E \cap f_{m_0}^{-1}(y)\right) \geq \frac{5}{2} \text{ for all } y \in \mathcal{F}. \quad (201)$$

Unfortunately, we can not assume right now that the set $\mathcal{F}$ is Borel, so we need some careful preparations.

Denote $E_{k_j} = \bigcup_{i=1}^{k} D_{ij}$, $E_j = \bigcup_{i=1}^{\infty} D_{ij}$. In this notation $= \bigcap_{j=1}^{\infty} E_j$. Evidently, all these sets are Borel.

$$\sum_{m_0} \mathcal{H}_{\infty}^{\mu}\left(E_j \cap f_{m_0}^{-1}(y)\right) = \lim_{k\to\infty} \sum_{m_0} \mathcal{H}_{\infty}^{\mu}\left(E_{k_j} \cap f_{m_0}^{-1}(y)\right) \quad \text{for each } y \in f_{m_0}\left(E_j\right). (202)$$

Denote further $F_{k_j} = f_{m_0}\left(E_{k_j}\right)$. Fix an arbitrary point y with the property

$$\sum_{m_0} \mathcal{H}^{\mu}(E_{k_j} \cap f_{m_0}^{-1}(y)) \leq 1.$$

Since E_{k_j} is a compact set, the set $E_{k_j} \cap f_{m_0}^{-1}(y)$ is compact as well. Then it follows by elementary means that the sets $E_{k_j} \cap f_{m_0}^{-1}(z)$ lie in the ε-neighborhood of the set $E_{k_j} \cap f_{m_0}^{-1}(y)$, where $\varepsilon \searrow 0$ as $z \to y, z \in f_{m_0}(E_{k_j})$. Therefore, there exists $\delta = \delta(y) > 0$ such that

$$\sum_{m_0} \mathcal{H}_{\infty}^{\mu}\left(E_{k_j} \cap f_{m_0}^{-1}(z)\right) \leq 2 \qquad \text{if } |z - y| < \delta. \quad (203)$$

Hence, there exists a relatively open set $\tilde{F}_{k_j} \subset F_{k_j}$ (i.e., $\tilde{F}_{k_j}$ is open in the induced topology of the set F_{k_j}) such that

$$\left\{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_{\infty}^{\mu}\left(E_{k_j} \cap f_{m_0}^{-1}(y)\right) \leq 1\right\} \subset \tilde{F}_{k_j}$$
$$\subset \left\{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_{\infty}^{\mu}\left(E_{k_j} \cap f_{m_0}^{-1}(y)\right) \leq 2\right\}. \quad (204)$$

Since by construction F_{k_j} is a compact set and $\tilde{F}_{k_j}$ is relatively open in F_{k_j} , we conclude that the set $\tilde{F}_{k_j}$ is Borel (this fact plays an important role here). Further, since $E_{k_j} \subset E_j$, we have for each $k \in \mathbb{N}$,

$$\{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_{\infty}^{\mu}(E_j \cap f_{m_0}^{-1}(y)) \leq 1\} \subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_{\infty}^{\mu}(E_{k_j} \cap f_{m_0}^{-1}(y))$$
$$\leq 1\} \subset \tilde{F}_{k_j}$$

and therefore,

$$\{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_{\infty}^{\mu}(E_j \cap f_{m_0}^{-1}(y)) \leq 1\} \subset \tilde{F}_j , \quad (205)$$

where we denote $\tilde{F}_j = \bigcap_{k=1}^{\infty} \tilde{F}_{k_j}$. On other hand, (202) and the second inclusion in (204) imply $\tilde{F}_j \subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_{\infty}^{\mu}(E_j \cap f_{m_0}^{-1}(y)) \leq 2\}$, so we have

$$\left\{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_\infty^\mu\left(E_j \cap f_{m_0}^{-1}(y)\right) \le 1\right\} \subset \tilde{F}_j$$
$$\subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_\infty^\mu(E_j \cap f_{m_0}^{-1}(y)) \le 2\}. \tag{206}$$
Denote now $\tilde{G}_j = f_{m_0}(E_j) \setminus \tilde{F}_j$. Then we can rewrite (206) as
$$\left\{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_\infty^\mu\left(E_j \cap f_{m_0}^{-1}(y)\right) > 2\right\} \subset \tilde{G}_j$$
$$\subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_\infty^\mu(E_j \cap f_{m_0}^{-1}(y)) > 1\}. \tag{207}$$
Since $\subset\ E_j$, we have from (201) that $F \subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_\infty^\mu(E_j \cap f_{m_0}^{-1}(y)) > 2\} \subset \tilde{G}_j$ for all $j \in \mathbb{N}$, therefore
$$\mathcal{F} \subset \tilde{G}, \tag{208}$$
where we denote $\tilde{G} = \bigcap_{j=1}^{\infty} \tilde{G}_j$. On the other hand, the second inclusion in (207) yields
$$\tilde{G} \subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_\infty^\mu(E_j \cap f_{m_0}^{-1}(y)) > 1\} \tag{209}$$
for each $j \in \mathbb{N}$. Since $\tilde{G}$ is a Borel set and by (208), (201) the inequalities $\mathcal{H}^{1+\epsilon}(\tilde{G}) \ge \mathcal{H}^{1+\epsilon}(\mathcal{F}) > 0$ hold, by [212] there exists a Borel set $G \subset \tilde{G}$ and a positive constant $b \in \mathbb{R}$ such that $0 < \mathcal{H}^{1+\epsilon}(G) < \infty$ and
$$\mathcal{H}^{1+\epsilon}(G \cap B(y, 1+\epsilon)) \le b\,(1+\epsilon)^{1+\epsilon} \tag{210}$$
for any ball $B(y, 1+\epsilon) = \{z \in \mathbb{R}^{m+\epsilon} : |z-y| < 1+\epsilon\}$ with the center $y \in G$. Of course, by (209)
$$G \subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_\infty^\mu(E_j \cap f_{m_0}^{-1}(y)) > 1\} \tag{211}$$
for all $j \in \mathbb{N}$. For $S \subset \mathbb{R}^n$ consider the set function
$$\widetilde{\Phi}(S) = \int_G^* \sum_{m_0} \mathcal{H}_\infty^\mu\,(S \cap f_{m_0}^{-1}(y)\ d\mathcal{H}^{1+\epsilon}(y), \tag{212}$$
where $\int^*$ means the upper integral. Standard facts of Lebesgue integration theory, $\widetilde{\Phi}(\cdot)$ is a countably subadditive set-function (see, e.g., [60, 192]).
From (200) and (210) it follows that
$$\sum_{i=1}^{\infty}\sum_{m_0} \left(\operatorname{diam} D_{ij}\right)^\mu \left[\operatorname{diam} f_{m_0}\left(D_{ij}\right)\right]^{1+\epsilon} \ge c\sum_{i=1}^{\infty}\sum_{m_0} \left(\operatorname{diam} D_{ij}\right)^\mu \mathcal{H}^{1+\epsilon}\left[G \cap f_{m_0}\left(D_{ij}\right)\right]$$
$$\ge C\sum_{i=1}^{\infty} \widetilde{\Phi}\left(D_{ij}\right) \ge C\,\widetilde{\Phi}\left(E_j\right) \quad .$$
Consequently, $\widetilde{\Phi}(E_j) \to 0$ as $j \to \infty$. On the other hand, from (211) and (212) we conclude
$$\widetilde{\Phi}\left(E_j\right) \ge \int_G^* d\mathcal{H}^{1+\epsilon}(y) = \mathcal{H}^{1+\epsilon}(G) > 0,$$
which is the desired contradiction. The proof of the Corollary (5.3.40) is finished.

Corollary (5.3.41)[260]: The above defined $\Psi(\cdot)$ is a countably subadditive set-function and for any $\lambda > 0$ the estimate

$$\mathcal{H}^{1+\epsilon}\{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}^{\mu}\big(E \cap f_{m_0}^{-1}(y) \geq \lambda\big\} \leq 5\frac{\Psi(E)}{\lambda} \tag{213}$$

holds.

Proof. The first assertion is evident and we focus on proving the estimate (213). Without loss of generality we can assume that $f_{m_0}^{-1}(y)$ is a compact subset of the closed unit ball $\overline{B(0,1)}$ for every $y \in \mathbb{R}^{m+\epsilon} \setminus \{0\}$. Let $E \subset \mathbb{R}^n$ and

$$\Psi(E) = \sigma < \infty.$$

Without loss of generality assume also that $0 \notin f_{m_0}(E)$ and

$$E = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} D_{ij} ,$$

where D_{ij} are compact sets in $\mathbb{R}^n$ satisfying

$$\sum_{i=1}^{\infty} \sum_{m_0} \big(\operatorname{diam} D_{ij}\big)^{\mu}\big[\operatorname{diam} f_{m_0}\big(D_{ij}\big)\big]^{1+\epsilon} \xrightarrow[]{j \to \infty} \sigma, \tag{214}$$

and

$$\operatorname{diam} D_{ij} + \operatorname{diam} \sum_{m_0} f_{m_0}\big(D_{ij}\big) \leq \frac{1}{j} . \tag{215}$$

Of course, E is a Borel set. Fix $\lambda > 0$ and take a set $\mathcal{F} \subset f_{m_0}(E)$ such that

$$\sum_{m_0} \mathcal{H}_{\infty}^{\mu}\left(E \cap f_{m_0}^{-1}(y)\right) \geq \frac{5}{2}\lambda \qquad \text{for all } y \in \mathcal{F}. \tag{216}$$

Further we assume that

$$\mathcal{H}^{1+\epsilon}(\mathcal{F}) > 0, \tag{217}$$

since if $\mathcal{H}^{1+\epsilon}(\mathcal{F}) = 0$, there is nothing to prove. Denote $E_j = \bigcup_{i=1}^{\infty} D_{ij}$. Repeating almost verbatim the arguments from the proof of the previous Corollary (5.3.40), we can construct a Borel set $\tilde{G} \subset \mathbb{R}^{m+\epsilon}$ such that

$$\mathcal{F} \subset \tilde{G} \subset \{y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}_{\infty}^{\mu}(E_j \cap f_{m_0}^{-1}(y)) > \lambda\} \tag{218}$$

for each $j \in \mathbb{N}$. Since $\tilde{G}$ is a Borel set and since, by (218) and (217), the inequalities $\mathcal{H}^{1+\epsilon}(\tilde{G}) \geq \mathcal{H}^{1+\epsilon}(\mathcal{F}) > 0$ hold, we deduce by [212] the existence of a Borel set $G \subset \tilde{G}$ such that $0 < \mathcal{H}^{1+\epsilon}(G) < \infty$. Put

$$G_l = \left\{x \in G : \mathcal{H}^{1+\epsilon}\big(G \cap B(x, 1+\epsilon)\big) \leq 2(1+\epsilon)^{1+\epsilon} \quad \forall (1+\epsilon) \in (0, 1/l)\right\} . \tag{219}$$

Then by construction all the sets G_l are Borel, $G_l \subset G_{l+1}$, moreover, by [60] we have

$$\mathcal{H}^{1+\epsilon}\left[G \setminus \left(\bigcup_{l=1}^{\infty} G_l\right)\right] = 0$$

and consequently,

$$\mathcal{H}^{1+\epsilon}(G) = \lim_{l\to\infty} \mathcal{H}^{1+\epsilon}(G_l). \tag{220}$$

For $S \subset \mathbb{R}^n$ consider the set function

$$\Psi_l(S) = \int_{G_l}^{*} \sum_{m_0} \mathcal{H}_\infty^\mu \left(S \cap f_{m_0}^{-1}(y)\right) \, d\mathcal{H}^{1+\epsilon}(y), \tag{221}$$

where $\int^*$ means the upper integral. By routine arguments of Lebesgue integration theory it follows that $\Psi(\cdot)$ is a countably subadditive set-function (see, e.g., [60, 192]).
From (214), (215) and (219) it follows for $j > l$ that

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sum_{m_0} \left(\operatorname{diam} D_{ij}\right)^\mu \left[\operatorname{diam} f_{m_0}\left(D_{ij}\right)\right]^{1+\epsilon} \\
&\geq \frac{1}{2} \sum_{i=1}^{\infty} \sum_{m_0} \left(\operatorname{diam} D_{ij}\right)^\mu \mathcal{H}^{1+\epsilon}\left[G_l \cap f_{m_0}\left(D_{ij}\right)\right] \\
&\geq \frac{1}{2} \sum_{i=1}^{\infty} \Psi_l\left(D_{ij}\right) \geq \frac{1}{2} \Psi_l\left(E_j\right).
\end{aligned} \tag{222}$$

On the other hand, the second inclusion in (218) implies

$$\Psi_l(E_j) \geq \lambda \int_{G_l}^{*} d\mathcal{H}^{1+\epsilon}(y) = \lambda \mathcal{H}^{1+\epsilon}(G_l). \tag{223}$$

From (222), (223), (214) we infer

$$\mathcal{H}^{1+\epsilon}(G_l) \leq \frac{2\sigma}{\lambda}, \tag{224}$$

and therefore, by (220),

$$\mathcal{H}^{1+\epsilon}(G) \leq \frac{2\sigma}{\lambda}. \tag{225}$$

Since this estimate is true for any Borel set $G \subset \tilde{G}$ with $\mathcal{H}^{1+\epsilon}(G) < \infty$, and since $\tilde{G}$ is Borel as well, we infer from [212] that

$$\mathcal{H}^{1+\epsilon}\left(\tilde{G}\right) \leq \frac{2\sigma}{\lambda}. \tag{226}$$

In particular, by the inclusion $\mathcal{F} \subset \tilde{G}$, this implies

$$\mathcal{H}^{1+\epsilon}(\mathcal{F}) \leq \frac{2\sigma}{\lambda}, \tag{227}$$

or in other words,

$$\mathcal{H}^{1+\epsilon}\left(y \in \mathbb{R}^{m+\epsilon} : \sum_{m_0} \mathcal{H}^\mu\left(E \cap f_{m_0}^{-1}(y)\right) \geq \frac{5}{2}\lambda\right) \leq 2\frac{\Psi(E)}{\lambda}. \tag{228}$$

The proof of Corollary (5.3.41) is complete.

Chapter 6
Operators in Tight and Uniform Structure with Unconditionally Saturated Banach Space

We make some observations on operators in arbitrary tight by support Banach space, showing in particular that in such a space no two isomorphic infinitely dimensional subspaces form a direct sum. We exhibit a separable $\mathcal{L}_\infty$-space whose uniform structure determines, at least, three different linear structures. We show that any bounded operator on X_{Kus} is a compact perturbation of a multiple of the identity, whereas the space X_{Kus} is saturated with unconditional basic sequences.

Section (6.1): Support Banach Spaces

In [111], Gowers and Maurey built the first hereditarily indecomposable (HI) Banach space X_{GM}, that is, a space whose none infinitely dimensional subspace admits a non-trivial bounded projection. They also proved that any operator on a subspace of X_{GM} is a strictly singular perturbation of a multiple of the identity. Recall that an operator is strictly singular if none of its restriction to an infinitely dimensional subspace is an isomorphism onto its image. Gowers– Maurey construction opened the field of study of spaces with a small family of bounded operators. The celebrated space of Argyros and Haydon [91] provided an extreme example in the area; their space is an $\mathscr{L}_\infty$ HI space, on which any bounded operator is a compact perturbation of a multiple of the identity.

A natural question arises how small family of bounded operators on Banach spaces with an unconditional basis could be. Obviously, all diagonal operators with uniformly bounded entries are continuous on such a space, therefore the most one can expect is a hereditary 'diagonal+strictly singular' property: any bounded operator on a subspace of the space is a strictly singular perturbation of a restriction of a diagonal operator.

Among the properties to be considered are different types of tightness, studied in [224], [225], which describe the structure of the family of isomorphisms inside the space. The strongest type is tightness by support. Recall that a Banach space X with a basis is tight by support if no two disjointly supported infinitely dimensional subspaces of X are isomorphic [224]. Any tight by support basis is necessarily unconditional. The typical example of a tight by support Banach space is Gowers unconditional space X_U, the unconditional version of Gowers–Maurey space [225], [227]. It follows easily that the hereditary 'diagonal+strictly singular' property implies tightness by support. Gowers asked if the implication can be reversed [229], in particular if X_U has the hereditary 'diagonal+strictly singular' property [229]. It is known that any bounded operator on the whole space X_U is a strictly singular perturbation of a diagonal operator [230]. Adapting arguments from [222], one can prove an analogous result for any bounded operator $T: Y \to Y$, where Y is a block subspace of X_U. Gowers [228] also proved that any isomorphism between block subspaces of a tight by support Banach space is a strictly singular perturbation of a restriction of an invertible diagonal operator. We answer the questions by constructing a bounded projection on a direct sum of two block subspaces of X_U which is not a strictly singular perturbation of a restriction of a diagonal operator. The construction uses the block sequence of [231] in Schlumprecht space generating an ℓ_1-spreading model and canonical properties of Gowers unconditional space, thus can be easily adapted to other spaces of Gowers–Maurey type, and leaves the question on an example of a Banach space with the hereditary 'diagonal + strictly singular' property open. Next we reproduce the construction in an arbitrary block subspace of X_U, using the results of [226].

We also prove positive results on bounded operators on arbitrary Banach space X with a tight by support basis. We show that any bounded operator on a subspace generated by a weakly null sequence (x_n) in such a space has a restriction to a subspace generated by some subsequence $\left(x_{k_n}\right)$ of the form $S + D|_{[x_{k_n}]}$, with S strictly singular and D diagonal. If we allow restricting to a block subspace, then we can replace the diagonal operator D by a multiple of the identity, which implies that no two isomorphic infinitely dimensional subspaces of X form a direct sum.

In the case of Gowers unconditional space, one can strengthen Theorem (6.1.3), we prove that any bounded operator on a block subspace Y of X_U into X_U is of the form $+D|_Y$, with S strictly singular and D diagonal, generalizing earlier results.

Given any $E, F \subset \mathbb{N}$, we write $E < \min F$. Let X be a Banach space with a basis (e_i). Given any $G \subset \mathbb{N}$ by P_G, we denote the projection $X \to [e_i: i \in G]$. The support of a vector $x = \sum_i x_i e_i$ is the set $supp\ x = \{i \in \mathbb{N}: x_i \neq 0\}$. The support of a subspace Y is the union of supports of all elements of Y . We write $x < supp\ y$. Any sequence $(x_n) \subset X$ with $x_1 < x_2 < \cdots$ is called a block sequence, a closed subspace spanned by an infinite block sequence (x_n) is called a block subspace. Given any basic sequence (x_n) by $[x_n]$, we denote the closed vector space spanned by (x_n).

We show some positive results on bounded operators on Banach spaces which are tight by support. We recall the following definition.

Definition (6.1.1)[220]: ([224]). A basis of a Banach space is called tight by support, if no two infinitely dimensional subspaces with disjoint supports are isomorphic.

X denotes a Banach space with a tight by support basis (e_i). The main tool is provided by the following decomposition result, which uses the notion of a diagonal-free operator. We call an operator R defined on a block subspace $[x_n] \subset X$ into X diagonal-free provided $supp\ x_n \cap supp\ Rx_n = \emptyset$ for any $n \in \mathbb{N}$.

Proposition (6.1.2)[220]: Let X be a Banach space with a tight by support basis (e_i). Let $(x_n) \subset X$ be a block basis and $T: [x_n] \to X$ be a bounded operator. Then $T = D|_{[x_n]} + S + R$ for some bounded operators D, S, R with D diagonal, S strictly singular and R diagonal-free.

Moreover, if T satisfies $supp\ Tx_n \cap supp\ x_m = \emptyset$ for any $n \neq m$, then the above formula holds with $R = 0$.

Proof. Let (x_n) be a normalized block basis and $T: [x_n] \to X$ be a bounded operator with $\|T\| = C > 0$. Since X is tight by support, the operator $P \circ T$, where P is the projection on $[e_i: i \notin \cup\ supp\ (x_n)]$ is strictly singular. Thus we can assume that $\cup_n supp\ (x_n) = \mathbb{N}$.

For any $n, k \in \mathbb{N}$, put

$$A_{n,k} = \left\{i \in supp\ x_n: |x_n(i)| \leqslant \frac{1}{2^k} |T\, x_n(i)|\right\}$$

and $A_k = \cup_{n\in\mathbb{N}} A_{n,k}, k \in \mathbb{N}$.

For any $k \in \mathbb{N}$, put $T_k = P_k \circ T$, where P_k is the projection from X onto $[e_i: i \in A_k]$, and let $D_k: X \to X$ be the diagonal operator defined by $D_k(e_i) = \lambda_i e_i$, where

$$\lambda_i = \begin{cases} 0 & if\ \ i \in A_k, \\ \dfrac{Tx_n(i)}{x_n(i)} & if\ i \in supp\ x_n \backslash A_k = supp\ x_n \backslash A_{n,k}\,. \end{cases}$$

By the definition of the sets $A_{n,k}$, we have $\|D_k\| \leqslant 2^k$.

Fix $k \in \mathbb{N}$ and assume that T_k is not strictly singular. Thus T_k is an isomorphism between some infinitely dimensional subspaces $U \subset [x_n]$ and $W \subset P_k(X)$. Consider a bounded operator $R_k = (Id - P_k) \circ (T_k|_U)^{-1}: W \to [e_i: i \notin A_k]$.
As X is tight by support and supp $R_k(W) \cap supp\, W = \emptyset$, there is some infinitely dimensional subspace $V \subset W$ such that $\|R_k|_V\| \leqslant (2C)^{-1}$. As $(T_k|_U)^{-1}$ is an isomorphism, the subspace $Z = (T_k)^{-1}(V)$ is also infinitely dimensional. Take $x \in (T_k)^{-1}(V)$ and compute

$$\begin{aligned}\|T_k x\| \leqslant \|Tx\| \leqslant C\|x\| \leqslant C\|P_k x\| + C\|x - P_k x\| = C\|P_k x\| + C\|R_k(T_k x)\| \\ \leqslant C\|P_k x\| + \frac{1}{2}\|T_k x\|.\end{aligned}$$

Hence $\|T_k x\| \leqslant 2C\|P_k x\|$ for any $x \in Z$. As $Z \subset U$ also $T_k|_Z$ is an isomorphism onto its image. On the other hand, for any $x \in [x_n]$ and $i \in supp\, P_k x \subset A_k$ we have

$$|P_k x(i)| \leqslant \frac{1}{2^k}|Tx(i)| = \frac{1}{2^k}|T_k x(i)|.$$

It follows that for any $x \in [x_n]$ we have $\|P_k x\| \leqslant \left(\frac{1}{2^k}\right)\|T_k x\|$, which for sufficiently big k gives contradiction for any non-zero $x \in Z$. Therefore, for sufficiently big k the operator Tk is strictly singular.
Now we have

$$\begin{aligned}\left(D_k|_{[x_n]} + T_k - T\right)\left(\sum_n a_n x_n\right) &= \sum_n a_n \sum_{i \in supp\, x_n} \lambda_i x_n(i) e_i + \sum_n a_n P_k T x_n - \sum_n a_n T x_n \\ &= \sum_n a_n \sum_{i \in supp\, x_n \setminus A_{n,k}} T x_n(i) e_i + \sum_n a_n \sum_{i \in A_k} T x_n(i) e_i + \sum_n a_n \sum_{i \in \mathbb{N}} T x_n(i) e_i \\ &= \sum_n a_n \sum_{i \in \mathbb{N} \setminus (A_k \cup supp\, x_n)} T x_n(i) e_i.\end{aligned}$$

Therefore, the operator $R = T - D_k|_{[x_n]} - T_k$ is diagonal-free. Now let T satisfy $supp\, Tx_m \cap supp\, x_n = \emptyset$ for $n \neq m$. Then, as we assumed that $supp[x_n] = \mathbb{N}$, we have that $supp\, Tx_n \subset supp\, x_n$ for any $n \in \mathbb{N}$.
It follows that $T = D_k|_{[x_n]} + T_k$, as

$$\begin{aligned}\left(D_k|_{[x_n]} + T_k\right)\left(\sum_n a_n x_n\right) &= \sum_n a_n \sum_{i \in supp\, x_n} \lambda_i x_n(i) e_i + \sum_n a_n P_k T x_n \\ &= \sum_n a_n \sum_{i \in supp\, x_n \setminus A_{n,k}} T x_n(i) e_i + \sum_n a_n \sum_{i \in A_k} T x_n(i) e_i = \sum_n a_n T x_n.\end{aligned}$$

For the last equality, recall that $supp\, Tx_n \subset supp\, x_n$ for any $n \in \mathbb{N}$.
Proposition (6.1.2) implies immediately the following result.
Theorem (6.1.3)[220]: Let X be a Banach space with a tight by support basis. Let $T: [x_n] \to X$ be a bounded operator on a subspace spanned by a weakly null sequence $(x_n) \subset X$. Then there exists a subsequence $(x_n)_{n \in M}$ such that $T|_{[x_n:\, n \in M]} = D|_{[x_n:\, n \in M]} + S$, where $D: X \to X$ is a bounded diagonal operator and $S: [x_n: n \in M] \to X$ is a bounded strictly singular operator.
In particular, the assertion holds if (x_n) is a block sequence.
We can replace diagonal operator by a multiple of the identity, if we allow passing to a block sequence instead of subsequence.

As for any isomorphism T, any scalar α given by the above theorem is non-zero, we obtain the following corollary.

Corollary (6.1.4)[220]: Let X be a Banach space with a tight by support basis. Then for any isomorphic infinitely dimensional subspaces $Y, Z \subset X$, we have $\inf\{y - z: y \in Y, z \in Z, \|y\| = \|z\| = 1\} = 0$.

Theorem (6.1.5)[220]: Let X be a Banach space with a tight by support basis. Let $T: [x_n] \to X$ be a bounded operator on a block subspace $[x_n] \subset X$. Then there is an infinitely dimensional block subspace $W \subset [x_n]$ such that $T|_W = \alpha Id|_W + S$, for some scalar α and bounded strictly singular operator $S: [x_n] \to X$.

Proof: We can assume that the basis of X is 1-unconditional and the sequence (x_n) is normalized. Passing to a further subspace by Theorem (6.1.3), we can assume that $T|_{[x_n]} = D|_{[x_n]} + S$ with D bounded diagonal with entries (λ_n) and S compact. Let $\Lambda = \sup\limits_n |\lambda_n|$ and assume $\Lambda > 0$.

We shall prove the following claim.

Claim (6.1.6)[220]: For any $\varepsilon > 0$ in any block subspace of $[x_n]$, there are a further block subspace $[y_m]$ and a scalar α_ε with $|\alpha_\varepsilon| \leqslant \Lambda$ such that

$$(D - \alpha_\varepsilon Id)|_{[y_m]} < \varepsilon.$$

Assuming Claim (6.1.6), consider a cluster point α_0 of $(\alpha_\varepsilon)_{\varepsilon > 0}$ and pick some sequence (α_n) and descending sequence of block subspaces Y_n such that $|\alpha_n - \alpha_0| < 1/2^n$ and $(D - \alpha_n Id)|_{Y_n} < 1/2^n$. Thus $\left\|(D - \alpha_0 Id)|_{Y_n}\right\| < 1/2^{n-1}$ and on the diagonal subspace Y_0 of (Y_n) the operator $(D - \alpha_0 Id)|_{Y_0}$ is compact, which ends the proof.

Proof: Fix $\varepsilon > 0$ and consider a partition of $\{\lambda: |\lambda| \leqslant \Lambda\} = \cup_{i=1}^{d} A_i$ into pairwise disjoint subsets of diameter smaller than $\varepsilon/2$. For every n, put $I_{n,i} = \{k \in supp\, x_n: \lambda_k \in A_i\}$ and $x_{n,i} = x_n|_{I_{n,i}}$. By the unconditionality, we get $\left\|x_{n,i}\right\| \leqslant 1$. As X is tight by support, for every $i \neq j$ any restriction to a linear subspace spanned by a block sequence of $\left(x_{n,i}\right)_n$ of the operator

$$M_{i,j}: lin\{x_{n,i}: n \in \mathbb{N}\} \ni \sum_n a_n x_{n,i} \to \sum_n a_n x_{n,j} \in lin\{x_{n,j}: n \in \mathbb{N}\}$$

is either non-bounded or strictly singular. Using this observation in any block subspace of (x_n), we can find a further block sequence (y_m) satisfying for some $i_0 \leqslant d$ the following:

$$\left\|y_m|_{\cup_n I_{n,i_0}}\right\| = 1, m \in \mathbb{N} \quad \text{and} \quad \left\|y_m|_{\cup_n I_{n,i_0}}\right\| \to 0, m \to \infty \text{ for } i \neq i_0.$$

The above statement can be easily proved by induction on d. Passing to a subsequence of (y_m), we can assume that $\left\|P_{N \setminus \cup_n I_{n,i_0}}|_{[y_m]}\right\| < \varepsilon/(4\Lambda)$.

Pick any scalar $\lambda_\varepsilon \in A_{i_0}$ and compute for any vector $\sum_n b_n x_n \in [y_m]$ of norm 1:

$$\left\|\lambda_\varepsilon \sum_n b_n x_n - D\left(\sum_n b_n x_n\right)\right\|$$
$$\leqslant \left\|\sum_n b_n \sum_{k\in I_{n,i_0}} (\lambda_\varepsilon - \lambda_k) x_n(k) e_k\right\| + \sum_n b_n \sum_{k\notin I_{n,i_0}} (\lambda_\varepsilon \lambda_k) x_n(k) e_k$$
$$\leqslant \max_{k\in \cup_n I_{n,i_0}} |\lambda_k - \lambda_\varepsilon| \left\|\sum_n b_n x_n\right\| + \max_k |\lambda_k - \lambda_\varepsilon| \left\|\sum_n \sum_{i\neq i_0} b_n x_{n,i}\right\|$$
$$\leqslant \frac{\varepsilon}{2}\left\|\sum_n b_n x_n\right\| + 2\Lambda\, P_{\mathbb{N}\setminus \cup_n I_{n,i_0}}\Big|_{[y_m]} \leqslant \varepsilon,$$
which proves that $(D - \lambda_\varepsilon Id)|_{[y_m]} \leqslant \varepsilon$.

Let X be a Banach space with an unconditional basis (e_i). We shall use the following general observation concerning the form of a projection on one of the component of a direct sum formed by two block subspaces with possibly coinciding supports. Assume that we have block subspaces $Y = [y_n]$ and $Z = [z_n]$ with
(D1) $\min\{supp\ y_{n+1}, supp\ z_{n+1}\} \geqslant \max\{supp\ y_n, supp\ z_n\}, n \in \mathbb{N}$;
(D2) $\inf\{\|y - z\|: \|y\| = \|z\| = 1, y \in Y, z \in Z\} > 0$.
Consider projections $P_Y: Y + Z \ni y + z \longmapsto y \in Y, P_Z: Y + Z \ni y + z \mapsto z \in Z$. By (D2), these projections are bounded.

Lemma (6.1.7)[220]: In the situation as above, the projection P_Y is of the form $P_Y = D|_{Y+Z} + S$, with S strictly singular and $D: X \to X$ diagonal if and only if there is a partition $\mathbb{N} = F \cup G$ such that $P_G|_Y$ and $P_F|_Z$ are strictly singular. Moreover, if either of the conditions hold, the diagonal operator D can be chosen to be a projection onto a subspace spanned by a subsequence of the basis.

Proof. Assume that P_Y is of the form $P_Y = D|_{Y+Z} + S$, with S strictly singular and $D: X \to X$ diagonal with entries (λ_i). Let
$$F = \left\{i \in \mathbb{N}: |\lambda_i| > \frac{1}{2}\right\}, \qquad G = \left\{i \in \mathbb{N}: |\lambda_i| \leqslant \frac{1}{2}\right\}$$
Then for any $y = \sum_n a_n y_n \in Y$, we have $y = P_Y y = Dy + Sy$, so
$$\sum_n a_n \sum_{i\in supp\ y_n} y_n(i) e_i = \sum_n a_n \sum_{i\in supp\ y_n} \lambda_i y_n(i) e_i + S\left(\sum_n a_n y_n\right).$$
Thus
$$\sum_n a_n \sum_{i\in supp\ y_n} (1 - \lambda_i) y_n(i) e_i = S\left(\sum_n a_n y_n\right).$$
Applying the projection P_G, we get
$$\sum_n a_n \sum_{i\in supp\ y_n \cap G} (1 - \lambda_i) y_n(i) e_i = (P_G \circ S)\left(\sum_n a_n y_n\right)$$
thus by unconditionality of (e_i)

$$\left\|(P_G \circ S)\left(\sum_n a_n y_n\right)\right\| = \left\|\sum_n a_n \sum_{i \in supp\, y_n \cap G} (1-\lambda_i) y_n(i) e_i\right\|$$
$$\geqslant \frac{1}{2}\left\|\sum_n a_n \sum_{i \in supp\, y_n \cap G} y_n(i) e_i\right\| = \frac{1}{2}\left\|\sum_n a_n P_G y_n\right\| = \frac{1}{2}\left\|P_G\left(\sum_n a_n y_n\right)\right\|.$$

As S is strictly singular, also $P_G|_Y$ is strictly singular. Analogously, we prove that $P_F|_Z$ is strictly singular.

The reverse implication is straightforward. Given suitable F, G, we write $P_Y = P_F|_Y + Z + P_G \circ P_Y - P_F \circ P_Z$. By the assumption on projections P_F, P_G on corresponding subspaces, the operator $P_G \circ P_Y - P_F \circ P_Z$ is strictly singular. This reasoning proves also the 'moreover' part of the lemma.

We answer Gowers' question [229] by giving an example of an operator T on a subspace W of Gowers unconditional space X_U, which is not of the form $D|_W + S$ with D diagonal and S strictly singular. We present first the list of canonical properties of the class of spaces of Gowers–Maurey type that are needed for our construction and proceed to the proof of the main result. Next we generalize the construction to any block subspace of X_U proving that an operator which is not a strictly singular perturbation of a restriction of a diagonal operator can be built inside any infinitely dimensional subspace of X_U. However, performing the construction inside block subspaces requires more technical background concerning spaces of Gowers–Maurey type, thus we present it separately. We close with proving that even though the 'diagonal + strictly singular' property does not hold for any infinitely dimensional subspace of the space X_U, it is satisfied for block subspaces of X_U.

We recall now the definition of Schlumprecht space S, [122], and Gowers unconditional space X_U (see [227]). The spaces are defined as the completion of c_{00} under a suitable norm, defined as a limit of an increasing sequence of norms.

Let f denote the function $x \mapsto \log_2(x+1)$. The norm $\|\cdot\|_S$ of Schlumprecht space S satisfies on c_{00} the following equation:

$$\|x\|_S = \max\left\{\|x\|_\infty, \sup_n \frac{1}{f(n)} \sup\left\{\sum_{i=1}^n \|E_i x\|_S : E_1 < \cdots < E_n\right\}\right\}.$$

It is straightforward that the basis $(\tilde{e}_n)$ of S is 1-unconditional and subsymmetric, that is, equivalent to any of its infinite subsequences.

We shall sketch the definition of Gowers unconditional space X_U, referring to [227] for details, and present properties of the space we need in a list of facts given below.

The norm of X_U satisfies on c_{00} the following implicit equation:

$$\|x\| = \max\left\{\|x\|_\infty, \sup_n \frac{1}{f(n)} \sup_n\left\{\sum_{i=1}^n \|E_i x\| : E_1 < \cdots < E_n\right\}\right\},$$
$$\sup\{|x^*(x)| : x^*\ special\ functional\ of\ length\ k, k \in K\}$$

for some fixed infinite and co-infinite $K \subset \mathbb{N}$. Special functionals are described with the use of a so-called coding function σ defined on the family Q of finite sequences of vectors with rational coordinates with modulus at most 1, taking values in $\mathbb{N} \backslash K$ and satisfying certain growth condition. A special functional of length k is of the form $\left(\frac{1}{\sqrt{f(k)}}\right)\sum_{j=1}^k x_j^*$, for some

block sequence $(x_1^*,\dots,x_k^*)$ with each x_j^* of the form $x_j^* = \left(\frac{1}{f(n_j)}\right)\sum_{n=1}^{n_j} x_{j,n}^*$, where $\left(x_{j,1}^*,\dots,x_{j,n_j}^*\right)$ is a block sequence in Q of vectors with norm at most 1, and $n_{j+1} = \sigma\left(|x_1^*|,\dots,|x_j^*|\right)$ for any $j = 2,\dots,k$.
Recall that in the case of Gowers–Maurey space the coding function depends on $\left(x_1^*,\dots,x_j^*\right)$, not on $\left(|x_1^*|,\dots,|x_j^*|\right)$, which makes the space HI. In the case of Gowers space the basis is 1-unconditional, but including special functionals in the norming set forces tightness by support.
The basic tools are formed by sequences of ℓ_1-averages. A vector $x \in X_U$ is called an ℓ_1^n-average with constant $c \geqslant 1, n \in \mathbb{N}$, if $x = \left(\frac{1}{n}\right)(x_1 + \cdots + x_n)$ for some block sequence $x_1 < \cdots < x_n$ with $\|x_i\| \leqslant 1$ and $\|x\| \geqslant 1/c$. A block sequence of $\ell_1^{n_k}$-averages $(x_k)_{k=1}^N \subset X_U$ is a rapidly increasing sequence (RIS) of ℓ_1-averages, if, roughly speaking, (n_k) increases fast enough, with the length n_k of average x_k depending not only on k, but also on the support of x_{k-1}, and the length N of the sequence is small with respect to the length n_1 of the first average, with all relations described in terms of the function f.
We list now the properties of the space X_U needed in the sequel. This list indicates that the results can be easily adapted to the case of other spaces of Gowers–Maurey type.
First we recall the standard.
Fact (6.1.8)[220]: ([227]). For any $n \in \mathbb{N}$ and $c > 1$, every block subspace of X_U contains an ℓ_1^n-average with constant c.
We shall need also the following simple observation.
Fact (6.1.9)[220]: For any sequence (z_n) of ℓ_1-averages of increasing length and a common constant and any sets (D_n) in $\mathbb{N}$ with $\inf_n \left\|P_{D_n} z_n\right\| > 0$, the sequence $\left(P_{D_n} z_n\right)$ is a sequence of ℓ_1-averages of increasing length and a common constant.
We state now the canonical property of the space X_U, whose variations in different spaces of Gowers–Maurey type or Argyros–Deliyanni type are responsible for the irregular properties of the spaces, such as having a small (in different meanings) family of bounded operators.
Fact (6.1.10)[220]: (a) Fix a seminormalized block sequence $(u_n) \subset X_U$ and a seminormalized block sequence $(v_n) \subset X_U$ of ℓ_1^n-averages with a constant c, satisfying $supp\ v_n \cap\ supp\ u_m = \emptyset$, for all $n, m \in \mathbb{N}$. Then there are sequences $(w_k) \subset [u_n], (z_k) \subset [v_n]$ of the form $w_k = \sum_{n\in J_k} a_n u_n, z_k = \sum_{n\in J_k} a_n v_n$, such that $\|w_k\| = 1, k \in \mathbb{N}$ and $z_k \to 0$, as $k \to \infty$.
(b) Fix a subsequence $\left(e_{i_n}\right)$ of the basis of X_U. Then there is a normalized sequence $(w_k) \subset \left[e_{i_n}\right], w_k = \sum_{n\in J_k} a_n e_{i_n}$, such that $\sup_{(j_n)} \sum_{n\in J_k} a_n e_{j_n} \to 0$, as $k \to \infty$, where the supremum is taken over all sequences $(j_n) \subset \mathbb{N}$ with $\{j_n : n \in \mathbb{N}\} \cap \{i_n : n \in \mathbb{N}\} = \emptyset$.
The proof of (a) follows directly the lines of the proof in [227] of the fact that the space X_U satisfies assumptions of [227]. First we pass to an infinite set $N \subset \mathbb{N}$ such that any finite subsequence $\left(v_{k_1},\dots,v_{k_N}\right)$ of $(v_n)_{n\in N}$ with $k_1 > N$ forms an RIS of ℓ_1-averages. Now it suffices to take for any $k \in \mathbb{N}$ a special functional of length k of the form

$$\frac{1}{\sqrt{f(k)}}\sum_{j=1}^{k}\frac{1}{f(n_j)}\sum_{n\in C_j} u_n^*,$$

where $u_n^*(u_n) = 1, \#C_j = n_j$, $\min C_j > n_j$ and the corresponding vector

$$w_k = \frac{\sqrt{f(k)}}{k} \sum_{j=1}^{k} \frac{f(n_j)}{n_j} \sum_{n \in C_j} u_n.$$

Then for any (v_n) as above, we have

$$\frac{\sqrt{f(k)}}{k} \sum_{j=1}^{k} \frac{f(n_j)}{n_j} \sum_{n \in C_j} v_n \leqslant \varepsilon(k, c),$$

for some $\varepsilon(k, c) \to 0$ as $k \to \infty$.

In the case of subsequences of the basis, the proof is even simpler.

Recall that Facts (6.1.8) and (6.1.10) imply immediately the following theorem.

Theorem (6.1.11) ([224], [229])[220]: The unit vector basis of Gowers unconditional space X_U is tight by support.

The next fact allows us to transfer an example of a sequence needed in Theorem (6.1.14) from Schlumprecht space to Gowers unconditional space. Recall that a basic sequence (x_n) generates some subsymmetric basic sequence $(\tilde{x}_n)$ as a spreading model, if for any $(a_i)_{i=1}^k, k \in \mathbb{N}$, we have

$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \cdots \lim_{n_k \to \infty} \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| = \left\| \sum_{i=1}^{k} a_i \tilde{x}_i \right\|.$$

We say that a basic sequence generates an ℓ_1-spreading model, if it generates as a spreading model some basic sequence equivalent to the unit vector basis of ℓ_1.

Fact (6.1.12)[220]: The basis of X_U generates the basis of Schlumprecht space as a spreading model.

The proof of this fact follows the lines of the proof of [221], where the result is proved in the case of Gowers–Maurey space.

Now we are ready to prove the main result.

By Lemma (6.1.7), we obtain the following answer to Gowers' (and thus also Problem 5.12) [229].

Corollary (6.1.13)[220]: There is a bounded operator on a subspace of Gowers unconditional space X_U which is not a strictly singular perturbation of a restriction of a diagonal operator on X_U.

Theorem (6.1.14)[220]: There are block subspaces $Y = [y_n], Z = [z_n]$ in Gowers unconditional space X_U satisfying $(D_1), (D_2)$ and (D_3) for any partition $F \cup G = \mathbb{N}$ with $P_F|_Z$ strictly singular the operator $P_G|_Y$ is not strictly singular.

Proof: We shall use the seminormalized block sequence of [231] generating an ℓ_1-spreading model in Schlumprecht space. Recall that two vectors u, v have the same distribution, if for some increasing bijection $\rho: supp\ u \to supp\ v$ we have $v(\rho(i)) = u(i)$ for each $i \in supp\ u$. Let $u_j = \left(\frac{f(j)}{j}\right) \sum_{i=1}^{j} \tilde{e}_i$. Take $(\tilde{y}_n) \subset S$ to be the block sequence [231], that is, $\tilde{y}_n = \sum_{j=1}^{n} \tilde{v}_{n,j}$, where $\left(\tilde{v}_{n,j}\right)_{j=1}^{n}$ have carefully designed pairwise disjoint supports and each $\tilde{v}_{n,j}$ has the same distribution as $u_{p_j}/2$, for some fixed $p_j \nearrow \infty$. The sequence $(\tilde{y}_n)$ generates an ℓ_1-spreading model, as $\left\| \tilde{v}_{n_1,j} + \cdots + \tilde{v}_{n_p,j} \right\| \approx p/2$ for $j \gg p$ (cf. [231]).

Write $\tilde{y}_n = \sum_{j=1}^{n}\left(\frac{f(j)}{2j}\right)\sum_{i\in I_j}\tilde{e}_i\,, \#I_j = j$, for each n and consider sequence $(y_n) \subset X_U$ defined as

$$y_n = \sum_{j=1}^{n} v_{n,j} = \sum_{j=1}^{n}\frac{f(j)}{2j}\sum_{i\in K_j} e_i,$$

where the sets K_j with $\#K_j = j$ are pushed forward along the basis (e_i), so that by Fact (6.1.12) the vectors (y_n) form a seminormalized block sequence with the property $\left\|v_{n_1,j} + \cdots + v_{n_p,j}\right\| \approx p/2$ for $j \gg p$, therefore also generating an ℓ_1-spreading model.
We define the sequence (z_n) in the following way. Take a mapping $\tau{:}\cup_n \, supp\; y_n \to \mathbb{N}$ such that
(j1) $\tau|_{supp\; y_n}{:}\, supp\; y_n \to \{1,2,\ldots,\#supp\; y_n\}$ is a bijection for any $n \in \mathbb{N}$;
(j2) $\tau(r) \geqslant \tau(s)$ if and only if $y_n(r) \leqslant y_n(s)$ for any $n \in \mathbb{N}$ and $r,s \in supp\; y_n$.
Note that (j1) and (j2) imply the following property:
(j3) $\tau\big(supp\; v_{n,j}\big) = \tau\big(supp\; v_{m,j}\big)$ for any $j \leqslant n < m$.

Let

$$z_n(i) = \frac{1}{4^{\tau(i)}} y_n(i) \text{ for any } i \in supp\; y_n \text{ and } z_n(i) = 0 \text{ otherwise.}$$

In this way, we obtain two seminormalized block sequences (y_n) and (z_n) with $supp\; y_n = supp\; z_n$, thus in particular satisfying (D1).
Roughly speaking, the proof of Theorem (6.1.14) relies on the following three properties of the above sequences: for any $(i\_n) \subset \mathbb{N}$ with $i_n \in supp\, y_n$, $n \in \mathbb{N}$, the projection $P_{\{i_n:\, n\in\mathbb{N}\}}|_Y$ is strictly singular (Claim (6.1.15)), whereas $P_{\{i_n:\, n\in\mathbb{N}\}}|_Z$ is not strictly singular provided $\inf_n z_n(i_n) > 0$ (Claim (6.1.16)). Moreover, the projection on the set containing supports of almost all $\big(v_{n,j}\big)_n$ for any j restricted to Y is not strictly singular (Claim (6.1.17)).
Proof of (D2)[220]: Assume towards contradiction that $\inf\{\|y - z\| : \|y\| = \|z\| = 1, y \in Y, z \in Z\} = 0$. Thus there are some normalized block sequences $(w_k) \subset [y_n]$ and $(v_k) \subset [z_n]$ with $\|w_k - v_k\| \leqslant 1/16^k, k \in \mathbb{N}$. Thus for any $(c_k) \subset [-1,1]$, we have $\|\sum_k c_k w_k - \sum_k c_k v_k\| \leqslant \frac{1}{8}$.
Take $(c_k) \subset [-1,1]$, let $w = \sum_k c_k w_k\,, v = \sum_k c_k v_k$ and $I = \{i \in supp\, w : |w(i)| \geqslant 2|v(i)|\}$ and compute, using 1-unconditionality of the basis of X

$$\frac{1}{8} \geqslant \left\|\sum_k c_k w_k - \sum_k c_k v_k\right\| \geqslant \left\|\sum_{i\in I}\big(w(i) - v(i)\big)e_i\right\| \geqslant \frac{1}{2}\left\|\sum_{i\in I} w(i)e_i\right\|.$$

Analogously compute for $J = \{i \in supp\, w : |v(i)| \geqslant 2|w(i)|\}$.
Thus for any $w = \sum_k c_k w_k$ with norm 1 and $v = \sum_k c_k v_k$, we have

$$\left\|\sum_{i\in supp\; w:\frac{1}{2}|v(i)|<|w(i)|<\,2|v(i)|} w(i)e_i\right\| \geqslant \frac{1}{4}.$$

Let

$$w_k = \sum_{n\in I_k} a_n y_n = \sum_{(n\in I_k)b} a_n \sum_{i\in supp\; y_n} y_n(i)e_i,$$

$$v_k = \sum_{n\in I_k} d_n z_n = \sum_{n\in I_k} d_n \sum_{i\in supp\, y_n} 4^{-\tau(i)} y_n(i) e_i.$$

For any $i \in supp\, w$, we have

$\frac{1}{2}|v(i)| < |w(i)| < 2|v(i)|$ if and only if $\frac{1}{2}|d_n| < 4^{\tau(i)}|a_n| < 2|d_n|$ where $i \in supp\, y_n$.

Given $n \in \mathbb{N}$, there is at most one $i \in supp\, y_n$ satisfying $\frac{1}{2}|d_n| < 4^{\tau(i)}|a_n| < 2|d_n|$, denote it by in. Hence

$$\frac{1}{4} \leqslant \left\| \sum_{i\in supp\, w:\frac{1}{2}|v(i)|<|w(i)|<2|v(i)|} w(i)e_i \right\| = \left\| \sum_k c_k \sum_{v\in I_k} a_n y_n(i_n) e_{i_n} \right\|,$$

which implies that for any (c_k) we have

$$\left\| \sum_k c_k \sum_{n\in I_k} a_n y_n \right\| \leqslant 4 \left\| \sum_k c_k \sum_{n\in I_k} a_n y_n(i_n) e_{i_n} \right\|,$$

that is, $\left(\sum_{n\in I_k} a_n y_n\right)_k$ and $\left(\sum_{n\in I_k} a_n y_n(i_n) e_{i_n}\right)_k$ are equivalent.

On the other hand, we have the following claim, which yields a contradiction. Whereas the above reasoning holds for any (y_n) and (z_n) related by means of a suitable function τ, the next claim uses only the fact that the spreading models of the basis of X_U and of the chosen sequence (y_n) are quite different, and the basis of a variant of Schlumprecht space dominates the basis of X_U.

Claim (6.1.15)[220]: The mapping $(y_n)_n \to \left(y_n(i_n)e_{i_n}\right)_n$ extends to a strictly singular operator.

Proof. We shall prove that the mapping carrying $(y_n)_n$ to the standard basis $(\bar{e}_n)$ of some variant of Schlumprecht space, defined by the function $f(x) = \sqrt{\log_2(x+1)}$ instead of $f(x) = \log_2(x+1)$, is strictly singular. As the basis of such a variant of Schlumprecht space is subsymmetric and dominates the basis of Gowers space, it follows that the mapping $(y_n)_n \to \left(y_n(i_n)e_{i_n}\right)_n$ is strictly singular.

We apply results of [233] taking into account that the basis $(\bar{e}_n)$ of a variant of Schlumprecht space is subsymmetric. By [233], the basis $(\bar{e}_n)$ is strongly dominated by ℓ_1 (according to [233]) and by [233] satisfies for some $\delta_k \nearrow 0$ and any scalars (α_n) the following:

$$\left\| \sum_n \alpha_n \bar{e}_n \right\| \leqslant \max_k \delta_k \max_{k\leqslant n_1<n_2\cdots<n_k} \sum_{i=1}^{k} |\alpha_{n_i}|.$$

Now in order to show that the mapping $M: (y_n)_n \to (\bar{e}_n)_n$ is strictly singular, we repeat part of the proof of [233]. Take any normalized block sequence (u_m) of $(y_n), u_m = \sum_{i\in J_m} \alpha_i y_i, m \in \mathbb{N}$. Passing to a further block sequence, as X_U does not contain c_0, we can assume that $\max_{i\in J_m} |\alpha_i| \to 0$ as $m \to \infty$. Given $k_0 \in \mathbb{N}$, estimate the norm of $v_m = M(u_m)$ using the fact that (y_n) is unconditional and generates an ℓ_1-spreading model:

$$\left\| \sum_{i\in J_m} \alpha_i \bar{e}_i \right\| \leqslant \max\left\{ \max_{k=1,\dots,k_0-1} \delta_1 \sum_{k\leqslant n_1<\cdots<n_k, n_i\in J_m} |\alpha n_i|, \max_{k\geqslant k_0} \delta_{k_0} \sum_{k\leqslant n_1<\cdots<n_k, n_i\in J_m} |\alpha n_i| \right\}$$
$$\leqslant \max\left\{ \delta_1 k_0 \max_{i\in J_m} |\alpha_i|, 2\delta_{k_0} \|u_m\| \right\} \leqslant \max\left\{ \delta_1 k_0 \max_{i\in J_m} |\alpha_i|, 2\delta_{k_0} \right\}.$$

As $\delta_k \to 0$, choosing sufficiently big k_0 and m we can force the norm of v_m to be as small as needed, which proves that $v_m \to 0$ and finishes the proof of strict singularity of M and thus the proof of Claim (6.1.15).

Proof of (D3)[220]: First we introduce some notation. Given $n \in \mathbb{N}$ and $t \in \tau(supp\, y_n)$, let $i_{n,t} \in supp\, y_n$ be the unique index $i \in supp\, y_n$ with $\tau(i) = t$. Note that (j3) by the definition of $\left(v_{n,j}\right)_{n,j}$ implies the following.

(j4) for any $i \in supp\, y_n, k \in supp\, y_m$ with $\tau(i) = \tau(k)$, we have $y_n(i) = y_m(k)$.

Thus we can write $y_n = \sum_{t \in \tau(supp\, y_n)} \gamma_t e_{i_{n,t}}$ for some scalars $(\gamma_t)_{t \in \mathbb{N}} \subset [0, 1]$. Given any $t \in \mathbb{N}$, let also $N_t = \{n \in \mathbb{N}: t \in \tau(supp\, y_n)\}$.

The property (D3) follows from the next two claims. The first one is based only on properties of the subsequences of the basis described in Fact 2.3(b).

Claim (6.1.16)[220]: Take $F \subset \mathbb{N}$ with $P_F|_Z$ strictly singular. Then for any $t \in \mathbb{N}$ the set $\{i_{n,t}: n \in N_t\} \cap F$ is finite.

Proof. Assume that for some $t_0 \in \mathbb{N}$, the set $H = \{i_{n,t_0}: n \in N_{t_0}\} \cap F$ is infinite. We shall prove that the projection $P_H|_Z$ is not strictly singular, which will end the proof of the claim. Let $N = \{n \in N_{t_0}: i_{n,t_0} \in I\}$. Apply Fact 2.3(b) to the sequence $\left(e_{i_{n,t_0}}\right)_{n \in N}$ obtaining a suitable normalized sequence (w_k) with elements of the form $w_k = \sum_{n \in J_k} a_n e_{i_{n,t_0}}, k \in \mathbb{N}$. Now note that

$$\left\|\sum_{n \in J_k} a_n z_n(i_{n,t_0}) e_{i_{n,t_0}}\right\| = \left\|\sum_{n \in J_k} a_n \frac{\gamma_{t_0}}{4^{t_0}} e_{i_{n,t_0}}\right\| = \frac{\gamma_{t_0}}{4^{t_0}} \|w_k\| = \frac{\gamma_{t_0}}{4^{t_0}},$$

Whereas

$$\left\|\sum_{n \in J_k} a_n \left(z_n - z_n(i_{n,t_0}) e_{i_{n,t_0}}\right)\right\| = \sum_{t \in \mathbb{N}, t \neq t_0} \sum_{n \in J_k} a_n z_n(i_{n,t}) e_{i_{n,t}} \leqslant \sum_{t \in \mathbb{N}, t \neq t_0} \frac{\gamma_t}{4^t} \sum_{n \in J_k} a_n e_{i_{n,t}}$$
$$\leqslant \sup_{t \in \mathbb{N}} \sum_{n \in J_k} a_n e_{i_{n,t}}.$$

Since the vectors $\sum_{n \in J_k} a_n e_{i_{n,t}}$ have disjoint support with w_k for any $k \in \mathbb{N}$, the last term converges to zero as $k \to \infty$. It follows that the projection $P_H|_Z$ is not strictly singular.

The next claim seems to be a rather natural requirement.

Claim (6.1.17)[220]: Take $G \subset \mathbb{N}$ with each of the sets $\{i_{n,t}: n \in N_t\} \setminus G\ t \in \mathbb{N}$, finite. Then the projection $P_G|_Y$ is not strictly singular.

Proof. Note that by (j3) for each $j \in \mathbb{N}$, we have $supp\, v_{n,j} \subset G$ for all but finitely many integers n. Let $G' = \cup_{j \in \mathbb{N}} \cup_{n \in \mathbb{N}} supp\, v_{n,j} \cap G$. We shall prove that the projection $P_{G'}|_Y$ is not strictly singular, which will end the proof of the claim.

Recall that for $j \gg p$, we have $\left\|v_{n_1,j} + \cdots + v_{n_p,j}\right\| \approx p/2$ (see [231]). Therefore, by the assumption on G for any $s, r \in \mathbb{N}$ and $\varepsilon > 0$ we can pick $L \subset \mathbb{N}$ with $\#L = s$ and $j \in \mathbb{N}$, so that $\left(\frac{1}{\#L}\right) \sum_{n \in L} v_{n,2j}$ and $\left(\frac{1}{\#L}\right) \sum_{n \in L} v_{n,2j+1}$ are seminormalized ℓ_1^s-averages with constant 2, with $supp\, v_{n,2j} \subset G$ and $supp\, v_{n,2j} > r$ for any $n \in L$. By the definition of (y_n) (precisely since $\|v_{n,j}\| \geqslant \frac{1}{2}$), it follows that also $\left(\frac{1}{\#L}\right) \sum_{n \in L} (y_n|_{G'})$ and $\left(\frac{1}{\#L}\right) \sum_{n \in L} (y_n|_{N \setminus G'})$ are seminormalized ℓ_1^s-averages with constant 4. It follows that we can pick a successive sequence (L_s) such that the sequences (u_s) and (v_s) defined by the formula

$$u_s = \frac{1}{\#L_s}\sum_{n\in L_s}(y_n|_{G'}), \qquad v_s = \frac{1}{\#L_s}\sum_{n\in L_s}\left(y_n|_{N\setminus G'}\right), \qquad s \in \mathbb{N}$$

are seminormalized ℓ_1^s-averages with constant 4, for any $s \in \mathbb{N}$.
Now apply The sequences (u_s) and (v_s), obtaining a normalized sequence

$$\left(\sum_{s\in J_k} a_s \frac{1}{\#L_s}\sum_{n\in L_s}(y_n|_{G'})\right)_{k\in\mathbb{N}},$$

such that

$$\sum_{s\in J_k} a_s \frac{1}{\#L_s}\sum_{n\in L_s}\left(y_n|_{\mathbb{N}\setminus G'}\right) \to 0, \qquad k \to \infty.$$

This shows that the projection $P_{G'}|_Y$ is not strictly singular and ends the proof of the claim.
In order to prove (D3), take a partition $F \cup G = \mathbb{N}$ and assume that $P_F|_Z$ is strictly singular. By Claim (6.1.16), for any $t \in \mathbb{N}$ the set $\{i_{n,t}: n \in N_t\} \cap F = \{i_{n,t}: n \in N_t\}\setminus G$ is finite, whereas by Claim (6.1.17) the projection $P_G|_Y$ is not strictly singular. This ends the proof of (D3) and thus the proof of Theorem (6.1.14).
A natural question arises if one can find an operator which is not a strictly singular perturbation of a restriction of a diagonal operator inside any infinitely dimensional subspace of X_U. We shall discuss the proof of the above construction in any block subspace, with infinite RISs of special type playing the role of the basis of X_U in the previous reasoning.

The construction of an operator not of the form $D|_W + S$ in the space X_U was based on the existence of a sequence generating an ℓ_1-spreading model. As we have written above, the existence of such a sequence in Schlumprecht space was shown in [231], and the proof was based on the finite representability of c_0 in Schlumprecht space. The finite representability of c_0 in every block subspace of Schlumprecht space was later proved in [232], and recently a new proof of this property concerning a variant of Gowers–Maurey space was given in [226]. Moreover, they show that the space $c_0^n, n \in \mathbb{N}$ can be reproduced on a block sequence of a special type which also generates the basis of Schlumprecht space as a spreading model [226]. These block sequences of special type, which can be found in any block subspace, were called a special RIS (SRIS) according to their structure [226]. The proof rewritten in the case of Gowers unconditional space yields the first part of the following fact (let us note here that the technical modification of the definition of the original Gowers–Maurey space required for the main result of [226] are not needed for proving that SRIS generates as a spreading model the basis of Schlumprecht space).
Fact (6.1.18)[220]: Every block subspace of X_U contains a seminormalized SRIS (x_i) such that
- (i) (x_i) generates the basis $(\tilde{e}_i)$ of Schlumprecht space as a spreading model;
- (ii) the mapping $\bar{e}_i \to x_i, i \in \mathbb{N}$, where $(\bar{e}_i)$ is the canonical basis of a variant of Schlumprecht space defined with the use of the function $\sqrt{f}$ instead of f, extends to a bounded operator on a Schlumprecht space.

The proof of the second part of the above fact follows the lines of the proof of [232]. In the sequel, we shall use also the following simple observation: for an SRIS (x_i) any finite sequence $(x_{k_1},\dots,x_{k_N})$ with $k_1 > N$ forms an RIS of ℓ_1-averages.
Taking any sequence (x_i) as in Fact (6.1.18), we can again transfer the sequence $(\tilde{y}_n)$ of [231] generating an ℓ_1-spreading model from Schlumprecht space to $[x_i]$ by substituting the basis $(\tilde{e}_i)$ with (x_i) and repeat the construction of (z_n). Recall that $\tilde{y}_n = \sum_{j=1}^{n}(f(j)/2j)\sum_{i\in I_j}\tilde{e}_i, \#I_j = j$, for each n, and take a sequence $(y_n) \subset [x_i]$ defined as

$$y_n = \sum_{j=1}^{n} v_{n,j} = \sum_{j=1}^{n}\frac{f(j)}{2j}\sum_{i\in K_j} x_i, \qquad n \in \mathbb{N}$$

again with the sets K_j with $\#K_j = j$ pushed forward along the sequence (x_i) to guarantee that the vectors (y_n) form a seminormalized block sequence generating an ℓ_1-spreading model.
Repeat the definition of the function $\tau:\cup_n supp_{[x_i]}y_n \to \mathbb{N}$, taking into account the supports of (y_n) with respect to the basic sequence (x_i) instead of (e_i). Set

$$z_n = \sum_{j=1}^{n}\frac{f(j)}{2j}\sum_{i\in K_j}\frac{1}{4^{\tau(i)}}x_i, \qquad n \in \mathbb{N}$$

and let $Y = [y_n], Z = [z_n]$.
In order to repeat the proof of Theorem (6.1.14), we shall need the following observation, which is a more precise formulation.
Fact (6.1.19)[220]: Fix a seminormalized block sequence $(u_n) \subset X_U$. Then for any $c \geqslant 1$ and $\varepsilon > 0$, there is a normalized vector $w = \sum_{n\in J} a_n u_n$, such that $\left\|\sum_{n\in J} a_n v_n\right\| < \varepsilon$, for any RIS of ℓ_1-averages $(v_1,\dots,v_{\#J})$ with constant c and with $supp\, u_n \cap supp\, v_m = \emptyset, n, m \in \mathbb{N}$.
The proof of the property (D2) for Y,Z can be rewritten in our case since (x_i) generates $(\tilde{e}_i)$ as a spreading model by Fact (6.1.18)(a), (y_n) generates an ℓ_1-spreading model by [231], and the basis of a suitable variant of Schlumprecht space dominates (x_i) by Fact (6.1.18)(b).
The proof of the property (D3) requires more attention since considered projections can split also the supports of (x_i). However, a small modification allows us repeat the reasoning. We repeat the notation of $i_{n,t}$ for any $n, t \in \mathbb{N}$, and N_t for any $t \in \mathbb{N}$. Again for some scalars $(\gamma_t)_{t\in\mathbb{N}} \subset [0,1]$, we have

$$y_n = \sum_{t\in\tau\left(supp_{[x_i]}y_n\right)} \gamma_t x_{i_{n,t}}, \qquad z_n = \sum_{t\in\tau\left(supp_{[x_i]}y_n\right)} \frac{\gamma_t}{4^t} x_{i_{n,t}}, \qquad n \in \mathbb{N}.$$

Then we have the following version of Claim (6.1.16).
Claim (6.1.20)[220]: Take $F \subset \mathbb{N}$ with $P_F|_Z$ strictly singular. Then for every $\varepsilon > 0$ and $t \in \mathbb{N}$, the set $\{i_{n,t}: n \in N_t, \|P_F x_{i_{n,t}}\| \geqslant \varepsilon\}$ is finite.
Proof. On the contrary, assume that $\left\|P_F x_{i_{n,t_0}}\right\| \geqslant \varepsilon$ for some $\varepsilon, t \in \mathbb{N}$ and infinitely many n's. The collection of the indices i_{n,t_0} denote by H. We shall prove that the mapping $P_J|_Z$ is not strictly singular, where $J = \cup_{i\in H} supp\, x_{i_{n,t_0}} \cap F$, which will finish the proof.
Assume first that $\left((I - P_F)\, x_{i_{n,t_0}}\right)_n$ is seminormalized. Then by Fact (6.1.9) $\left((I - P_F)\, x_{i_{n,t_0}}\right)_n$ is a sequence of ℓ_1-averages of increasing length. Pick an infinite $M \subset \mathbb{N}$ so

that any N elements of the sequence $\left((I-P_F)\, x_{i_{n,t_0}}\right)_{n\in M}$ starting after Nth element form an RIS of ℓ_1- averages. Now by Fact (6.1.19) for any $k \in \mathbb{N}$ choose a vector $w_k = \sum_{n\in J_k} a_n P_F x_{i_{n,t_0}}$, such that $\sum_{n\in J_k} a_n x_{i_{n,t}} < 1/2^k$, for any $t \in \mathbb{N}, t \neq t_0$ and $\left\|\sum_{n\in J_k} a_n (I - P_F) x_{i_{n,t_0}}\right\| \leqslant 1/2^k$. It follows that

$$\left\|P_J \sum_{n\in J_k} a_n z_n\right\| = \left\|\sum_{n\in J_k} a_n\ \frac{\gamma_{t_0}}{4^{t_0}} x_{i_{n,t_0}}\right\| = \frac{\gamma_{t_0}}{4^{t_0}} \|w_k\| = \frac{\gamma_{t_0}}{4^{t_0}},$$

Whereas

$$\left\|\left(I-P_J\right) \sum_{n\in J_k} a_n z_n\right\| = \left\|\sum_{t\in\mathbb{N},t\neq t_0} \sum_{n\in J_k} a_n \frac{\gamma_t}{4^t}\ x_{i_{n,t}} + \sum_{n\in J_k} (I-P_F) \frac{\gamma_{t_0}}{4^t} x_{i_{n,t_0}}\right\|$$
$$\leqslant \sum_{t\in\mathbb{N},t\neq t_0} \frac{1}{4^t} \left\|\sum_{n\in J_k} a_n \gamma_t x_{i_{n,t}}\right\| + \frac{1}{4^{t_0}} \left\|\sum_{n\in J_k} a_n (I-P_F)\gamma_{t_0} x_{i_{n,t_0}}\right\| \leqslant \frac{1}{2^k}.$$

If $\liminf_n \left\|(I-P_F)x_{i_{n,t_0}}\right\| = 0$, then passing to a subsequence we can assume that $(I-P_F)x_{i_{n,t_0}} \leqslant 1/2^n$. It follows straightforward that in the above inequality we can again estimate $\left\|\sum_{n\in J_k} a_n (I-P_F) x_{i_{n,t_0}}\right\|$. Therefore, in both cases the above estimates prove that the projection $P_J|_Z$ is not strictly singular, which yields a contradiction.

On the other hand, we have the following version of Claim (6.1.17).

Claim (6.1.21)[220]: Take $G \subset \mathbb{N}$ with $(I-P_G)x_{i_{n,t}} \xrightarrow[n\to\infty]{} 0$ for any $t \in \mathbb{N}$. Then $P_G|_Y$ is not strictly singular.

Proof. For G, as in the claim by the definition of (y_n) and τ, we have $(I-P_G)v_{n,j} \xrightarrow[n\to\infty]{} 0$ for every j. Now we repeat the reasoning from the proof of Claim (6.1.17) defining G' in the same way and choosing successive $L_s \subset \mathbb{N}$ in such a way that the vectors

$$w_s = \frac{1}{\#L_s} \sum_{n\in L_s} v_{n,2j}\,, \qquad x_s = \frac{1}{\#L_s} \sum_{n\in L_s} v_{n,2j+1}\,, \qquad s \in \mathbb{N}$$

are seminormalized ℓ_1^s-averages with constant 4 and satisfying additionally the estimate $\left\|(I-P_G)v_{n,2j}\right\| < 1/2^n$. The last condition guarantees that $(w_s)_s$ and $(P_{G'} w_s)_s$ are equivalent, which allows for repeating the rest of the proof of Claim (6.1.17).

Now in order to obtain the property (D3) for Y and Z, take any partition $F \cup G = N$ and assume that $P_F|_Z$ is strictly singular. Then by Claim (6.1.20) for any $t \in \mathbb{N}$, we have $P_F\left(x_{i_{n,t}}\right) \xrightarrow[n\to\infty]{} 0$, which by Claim (6.1.21) implies that $P_G|_Y$ is not strictly singular. Thus we proved that subspaces $Y = [y_n]$ and $Z = [z_n]$ satisfy (D1)–(D3). As by Fact (6.1.18), such block subspaces can be found in any block subspace of X_U, by Lemma (6.1.7) and a standard perturbation argument we get the following theorem.

Theorem (6.1.22)[220]: For any infinitely dimensional subspace X of Gowers unconditional space X_U, there is an operator defined on a subspace of X which is not a strictly singular perturbation of a restriction of a diagonal operator on X_U.

We close with an observation that the ‘diagonal + strictly singular’ property holds for block subspaces of X_U. Namely, we prove the following version of [222] in the case of Gowers unconditional space, generalizing [230].

Proposition (6.1.23)[220]: Let X_U be Gowers unconditional space, Y be a block subspace of X_U. Then

(i) any bounded diagonal-free operator $T: Y \to X_U$ is strictly singular;

(ii) any bounded operator $T: Y \to X_U$ is a strictly singular perturbation of a restriction of a diagonal operator on X_U.

Proof. By Proposition (6.1.2), as X_U is tight by support, the second part follows from the first part. The proof of the first part is a variant of the proof of [222] in our setting, which uses technique of [223]. We include it for the sake of completeness.

Take a bounded operator $: Y \to X_U$, where $Y = [y_k]$ is a block subspace of X_U. Assume that T is diagonal-free, that is, $supp\, Ty_k \cap supp\, y_k = \emptyset$ for each $k \in \mathbb{N}$. We shall prove that for any sequence of (x_n) of normalized ℓ_1^n-averages Tx_n converges to zero. It follows that T is strictly singular, which ends the proof of the proposition.

Fix a block sequence $(x_n) \subset [y_k]$ of normalized ℓ_1^n-averages. Passing to subsequence, after small perturbation, we can assume that $(Tx_n + x_n)_{n\in N}$ is a block sequence. Write each x_n as $x_n = \sum_{k\in A_n} a_k y_k$. For every $B \subset \mathbb{N}$, denote by R_B the projection on $\left[e_j: j \in \cup_{i\in B} supp\, y_i\right]$.

Claim (6.1.24) (cf. [222])[220]: For any partitions $A_n = B_n \cup C_n, n \in \mathbb{N}$, we have $\lim_n R_{C_n} T R_{B_n} x_n = 0$.

Proof. Take partitions $A\_n = B_n \cup C_n, n \in \mathbb{N}$, and assume $\inf_{n\in N} R_{C_n} T R_{B_n} x_n > 0$ for some infinite $N \subset \mathbb{N}$. Then, as T is bounded, $\inf_{n\in N} R_{B_n} x_n > 0$. By Fact (6.1.9), the sequence $\left(R_{B_n} x_n\right)_{n\in N}$ is also a sequence of ℓ_1-averages of increasing length with a common constant. Apply Fact (6.1.10)(a) to the seminormalized block sequence $u_n = R_{C_n} T R_{B_n} x_n$ and $v_n = R_{B_n} x_n, n \in N$, in order to obtain sequences (z_k) and (w_k) with $z_k = \sum_{n\in J_k} b_n R_{B_n} x_n\, , w_k = \sum_{n\in J_k} b_n R_{C_n} T R_{B_n} x_n\, , w_k = 1, k \in \mathbb{N}$ and $z_k \to 0$. This contradicts the boundedness of T and ends the proof of the claim.

Let now

$$\mathcal{P}_n = \begin{cases} \left\{(B,C): B \cup C = A_n, B \cap C = \emptyset, \#B = \#C = \#\dfrac{A_n}{2}\right\} & if\ A_n\ is\ even, \\ \{(B,C): B \cup C = A_n, B \cap C = \emptyset, |\#B - \#C| = 1\} & if\ A_n\ is\ odd, \end{cases}$$

and set L_n to be the integer part of $\#A_n/2$.

Claim (6.1.25) (cf. [222])[220]: $R_{A_n} T\, x_n = \left(\frac{\lambda_n}{\#\mathcal{P}_n}\right) \sum_{(B,C)\in\mathcal{P}_n} R_B T\, R_C x_n$, where

$$\lambda_n = \begin{cases} \dfrac{2L_n(2L_n - 1)}{L_n^2} & if\ A_n\ is\ even, \\ \dfrac{2(2L_n + 1)}{L_n + 1} & if\ \ A_n\ is\ odd. \end{cases}$$

Proof. Note first that

$$R_{A_n} T x_n = R_{A_n} \sum_k a_k \left(\sum_{j\notin supp\, y_k} e_j^*\, (Ty_k) e_j \right) \text{ since } supp\, y_k \cap supp\, Ty_k = \emptyset$$

$$= \sum_{i\in A_n} \sum_{j\in supp\, y_i} \left(\sum_{k: k\neq i} a_k e_j^* (Ty_k) \right) e_j,$$

whereas for any partition (B, C) of A_n we have

$$R_B T R_C x_n = \sum_{i \in B} \sum_{j \in supp\ y_i} \left(\sum_{k \in C} a_k e_j^* (T y_k) \right) e_j.$$

Fix $i \in A_n$ and $j \in supp\ y_i$. We shall prove that

$$\sum_{k: k \neq i} a_k e_j^* \ (T\ y_k) = \frac{\lambda_n}{\# \mathcal{P}_n} \sum_{(B,C) \in \mathcal{P}_n} e_j^* (R_B T R_C x_n).$$

Indeed, by the definition of R_B, if $e_j^*(R_B T R_C\ x_n) \neq 0$, then $i \in B$. Thus for any $k \neq i$, there are as many terms $a_k e_j^*(T y_k)$ in the sum $\sum_{(B,C) \in \mathcal{P}_n} e_j^*(R_B T\ R_C x_n)$ as is the cardinality of the set $\{(B, C) \in \mathcal{P}_n : i \in B, k \in C\}$. The latter is equal to $\# \mathcal{P}_n / \lambda_n$, which ends the proof of the claim.

The following claim ends the proof of Proposition (6.1.23).

Claim (6.1.26)[220]: $\lim_n T\ x_n = 0$.

Proof. Assume $\inf_{n \in N} \|T\ x_n\| > 0$ for some infinite $N \subset \mathbb{N}$. Note that $A_n T x_n \to 0$. Indeed, by Claim (6.1.25), $A_n T x_n = \left(\frac{\lambda_n}{\# \mathcal{P}_n} \right) \sum_{(B,C) \in \mathcal{P}_n} R_B T\ R_C\ x_n$ for some $0 < \lambda_n \leqslant 4$. On the other hand, by Claim (6.1.24) we have

$$\lim_n (\sup\{\|R_C T R_B x_n\| : (B, C) partition\ of\ A_n\}) = 0.$$

Hence, after small perturbation, we can assume that $supp\ T x_n \cap supp\ x_n = \emptyset, n \in N$ with N being infinite. Apply Fact (6.1.10)(a) to $u_n = T\ x_n$ and $v_n = x_n, n \in N$, in order to obtain sequences (z_k) and (w_k) with $z_k = \sum_{n \in J_k} b_n x_n$, $w_k = \sum_{n \in J_k} b_n T x_n$, $\|w_k\| = 1, k \in \mathbb{N}$ and $z_k \to 0$, which contradicts boundedness of T.

Section (6.2): Separable $\mathcal{L}_\infty$-Spaces

Aharoni and Lindenstrauss gave in [235] an example of two non-isomorphic $\mathcal{L}_\infty$-spaces which are uniformly homeomorphic. The spaces considered in such an example were non-separable. They asked whether a similar result holds in the separable setting or not. It was asked if c_0 and $C[0, 1]$ (or c_0 and $C(\omega^\omega)$) could be uniformly homeomorphic. This last equation was answered negatively by Johnson, Lindenstrauss and Schechtman [89] who proved the following fundamental result on the uniform structure of $C(K)$-spaces:

Theorem (6.2.1)[234]: If a $C(K)$-space is uniformly homeomorphic to c_0, then it is isomorphic to c_0.

However, the uniform structure of separable $\mathcal{L}_\infty$-spaces seems to be not completely clear. Actually, the following general question about the uniform structure of Banach spaces was raised in [89]: Is every separable $\mathcal{L}_p$-space, with $1 \leq p \leq \infty$, determined by its uniform structure? A Banach space is determined by its uniform structure if it is isomorphic to every Banach space to which it is uniformly homeomorphic. We are interested only in the case $p = \infty$. In this case, there is a partial result due to Kalton. He gave an example of two non-isomorphic separable $\mathcal{L}_\infty$-spaces which are coarsely homeomorphic, see [245]. So, as far as we know, the problem to decide if every separable $\mathcal{L}_\infty$-space is determined by its uniform structure remains open. We give an example of two separable $\mathcal{L}_\infty$-spaces which are uniformly homeomorphic but not linearly isomorphic. This completes Kalton's result. The approach is based on a deep construction of Kalton in the nonlinear setting. The main idea is to combine such a construction with one of the exotic Bourgain–Pisier spaces [103]. The way to do it is to use a well-known technique of homological algebra: The push-out space. This technique has the skill to mix nicely the two quoted constructions. On one hand, the resulting push-out space inherits the nonlinear properties of Kalton's construction. On the other hand, it receives the $\mathcal{L}_\infty$-structure and the Schur property of the Bourgain–Pisier spaces. This is enough to show that separable $\mathcal{L}_\infty$-spaces are not uniquely determined by its uniform structure. However, it follows from the procedure that one of the spaces in the example is a Schur space. We go a step further giving a second example: There exist two non-isomorphic separable $\mathcal{L}_\infty$-spaces containing c_0 (and thus failing to be Schur spaces) which are uniformly homeomorphic. This construction is much more delicate. It involves the uncomplemented copy of ℓ_1 inside ℓ_1 given by Bourgain [237] and the twin brother of the push-out space: The pull-back space. The juxtaposition of these two examples gives an unexpected result: We find that there are separable $\mathcal{L}_\infty$-spaces whose uniform structure determines, at least, three different linear structures.

We contain all the necessary background to follow. We gathered in three devoted to homological algebra, linear Banach space theory and nonlinear theory respectively. We contain our first example while deal with the second and much more elaborated example. The end contains the aforementioned result about an $\mathcal{L}_\infty$-space whose uniform structure determines, at least, three different linear structures.
We recall some basic tools from homological algebra in the Banach space setting. The reader can find all the necessary details in [238]. Let V , W, X, Y , Z be Banach spaces. A short exact sequence is a diagram like

$$0 \longrightarrow Z \xrightarrow{j} Y \xrightarrow{q} X \longrightarrow 0 \qquad (1)$$

where the morphisms are linear and such that the image of each arrow is the kernel of the next one. This condition implies that Z is a subspace of Y and thanks to the Open Mapping Theorem we find that X is isomorphic to $Y/j\,(Z)$. We usually refer to Y as a twisted sum of Z and X (in that order). The twisted sum Y is trivial if $j\,(Z)$ is complemented in ; otherwise is nontrivial. In the same line, we say the sequence (1) splits or is trivial if there is a bounded linear map $R : Y \longrightarrow Z$ such that $R \circ j = Id_Z$. R receives the name of a retraction for j . In this case, it is not hard to check that Y is isomorphic to $Z \oplus X$. Of course, trivial twisted sums correspond to trivial sequences. We say $\mathcal{P}$ is a three space property if for every short exact sequence (1) where Z, X have property $\mathcal{P}$ then Y has also property $\mathcal{P}$. Let us introduce the Push-out and Pull-back constructions as in [241].

A commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{j} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle I} \\ V & \xrightarrow{J} & PO, \end{array} \qquad (2)$$

is called a push-out of

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "i"'] & Y \\ V & \end{tikzcd}$$

provided that for every commutative diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d, "\beta"] \\ V \arrow[r, "\alpha"] & W \end{tikzcd}$$

there is a unique morphism $w : PO \longrightarrow W$ so that $\alpha = wJ$, $\beta = wI$. There is a short description for the space PO in (2) called canonical push-out and defined directly as

$$PO = (Y \oplus_1 V)/\overline{D}$$

Endowed with the natural quotient norm; and with I and J the compositions of the natural mappings of Y and V into $Y \oplus_1 V$ with the quotient map from $Y \oplus_1 V$ onto PO. Given the sequence (1) and an into-isomorphism $i : Z \longrightarrow V$, we may complete (2) to produce a commutative push-out diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & & \\ 0 \arrow[r] & Z \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[r, "q"] \arrow[d, "I"'] & X \arrow[r] \arrow[d, equal] & 0 & (1) \\ 0 \arrow[r] & V \arrow[r, "J"] \arrow[d, "p"'] & PO \arrow[r, "Q"] \arrow[d, "P"'] & X \arrow[r] & 0 & (\textit{Push-out sequence}) \\ & V/i(Z) \arrow[r, equal] \arrow[d] & V/i(Z) \arrow[d] & & & \\ & 0 & 0. & & & \end{tikzcd}$$

The sequence in the second row of the diagram above is called the push-out sequence of the sequence (1) and i.

A commutative diagram

$$\begin{tikzcd} PB \arrow[r, "Q"] \arrow[d, "P"'] & V \arrow[d, "p"] \\ Y \arrow[r, "q"] & X, \end{tikzcd} \qquad (3)$$

is called a pull-back of

$$\begin{tikzcd} & V \arrow[d, "p"'] \\ Y \arrow[r, "q"] & X \end{tikzcd}$$

provided that for every commutative diagram

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\beta"'] & V \arrow[d, "p"'] \\ Y \arrow[r, "q"] & X \end{tikzcd}$$

there is a unique morphism $w : W \longrightarrow PB$ so that $\alpha = Qw, \beta = P\, w$. There is a short description for the space PB of (3) called the canonical pull-back and defined as

$$PB = (y, v) \in Y \oplus_\infty V : q(y) = p(v) .$$

Given the sequence (1) and a quotient map $p : V \longrightarrow X$, we may complete the diagram (3) to produce a commutative pull-back diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & & \\ & & \mathrm{Ker}\, p \arrow[r, phantom, "="] \arrow[d, "I"] & \mathrm{Ker}\, p \arrow[d, "i"] & & \\ 0 \arrow[r] & Z \arrow[r, "J"] \arrow[d, equal] & PB \arrow[r, "Q"] \arrow[d, "P"] & V \arrow[r] \arrow[d, "p"] & 0 & (\textit{Pull-back sequence}) \\ 0 \arrow[r] & Z \arrow[r, "j"] & Y \arrow[r, "q"] \arrow[d] & X \arrow[r] \arrow[d] & 0 & (1) \\ & & 0 & 0. & & \end{tikzcd}$$

The sequence in the first row of the diagram above is called the pull-back sequence of the sequence (1) and p.

We need to introduce the notion of a quasi-linear map and some basic facts related. This notion is necessary only to follow our proof of the technical Lemma (6.2.13). This lemma is essential to construct our second example but not the first Quasi-linear maps

The theory developed by Kalton [242] and Kalton and Peck [246] establishes that there is a correspondence between exact sequences

$$0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0,$$

and quasi-linear maps $\mathcal{F}$ from X to Z. By a quasi-linear map $\mathcal{F}$ from X to Z, we mean a homogeneous map from X with values in Z and satisfying

$$\|\mathcal{F}\,(x + \acute{x}) - \mathcal{F}\,(x) - \mathcal{F}\,(x')\|_Z \leq K\,(\|x\| + \|x'\|),$$

for some constant K and all $x, x' \in X$. Given a quasi-linear map $\mathcal{F}$ from X to Z we can construct a short exact sequence $0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0$; the space Y may be identified with $Z \oplus X$ endowed with the quasi-norm

$$\|(z, x)\| = \|z - \mathcal{F}(x)\|_Z + \|x\|_X.$$

It is usual to denote Y as $Z \oplus_{\mathcal{F}} X$ to make explicit the role of $\mathcal{F}$. The converse also holds, actually it can be proved:

Theorem (6.2.2)[234]: There is a correspondence between twisted sums $Z \oplus_{\mathcal{F}} X$ and quasi-linear maps from X to Z.

Full details can be found in [242], [246] or [238]. According to the previous theorem it is useful to write

$$0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0 \equiv \mathcal{F},$$

to make explicit the role of the correspondence between the exact sequence and $\mathcal{F}$ – the corresponding quasi-linear map provided by Theorem (6.2.2). This notation works nicely to identify quasi-linear maps in commutative diagrams. More specifically, in a pull-back diagram we may identify the quasi-linear maps as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & PB & \longrightarrow & X_1 & \longrightarrow & 0 \equiv F \circ t \quad (\textit{Pull-back sequence}) \\ & & \| & & \downarrow & & \downarrow{\scriptstyle t} & & \\ 0 & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \equiv F \qquad (1). \end{array}$$

That is, if $\mathcal{F}$ denotes the quasi-linear map representing the sequence (1), then the pull-back sequence with the operator t can be identified with $\mathcal{F} \circ t$. This fact follows from a close inspection of the proof of Theorem (6.2.2).

It is also very useful to introduce the notion of equality, that is, when two quasi-linear maps are "the same". We shall say that two quasi-linear maps $\mathcal{F}, G$ from X to Z are equivalent, and write

$$\mathcal{F} \equiv G,$$

if there is a bounded linear map $T : Z \oplus_{\mathcal{F}} X \longrightarrow Z \oplus_G X$ making commutative the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & Z \oplus_F X & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle T} & & \| & & \\ 0 & \longrightarrow & Z & \longrightarrow & Z \oplus_G X & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

If such a T exists then it must be an isomorphism by the 3-lemma (see e.g. [238]). Note that if $\mathcal{F} \equiv 0$ then $\mathcal{F}$ induces the ordinary topological direct sum $Z \oplus X$; or equivalently, the short exact sequence $0 \longrightarrow Z \longrightarrow Z \oplus_{\mathcal{F}} X \longrightarrow X \longrightarrow 0$ splits. We recall the following important theorem [246]:

Theorem (6.2.3)[234]: Let $\mathcal{F}, G$ be quasi-linear maps from X to Z. The following conditions are equivalent:

(i) $\mathcal{F} \equiv G$.

(ii) There exists a constant M and a linear (not necessarily bounded) map $A : X \longrightarrow Z$ such that

$$\|\mathcal{F}(x) - G(x) - A(x)\| \leq M\|x\|, x \in X.$$

Given two quasi-linear maps $\mathcal{F}, G$ from X to Z it is a routine to check that also $\mathcal{F} - G$ is a quasi-linear map from X to Z. Equipped with the theorem above it is trivial to check

Corollary (6.2.4)[234]:

$$\mathcal{F} \equiv G \Leftrightarrow \mathcal{F} - G \equiv 0.$$

Our notation for Banach spaces is standard, see e.g. [249]. We need to recall a few classes of operators acting between Banach spaces: strictly singular, approximable, compact and weakly compact. Recall that an operator is said to be strictly singular if it is never an isomorphism when restricted to an infinite dimensional subspace. An operator $t : X \longrightarrow X$ is said to be an approximable operator if there exists a sequence $(t_n)_{n=1}^{\infty}$ of finite rank operators such that $\|t - t_n\| \longrightarrow 0 \; as \; n \longrightarrow \infty$. Closely related to this is the concept of compact operator. We say that $T : X \longrightarrow X$ is compact (respectively weakly compact) if $T\ (B(X))$ is a relatively compact (respectively weakly compact) set. We will use the well-known fact that every compact operator $T : c_0 \longrightarrow c_0$ is approximable; see [249].

We also need to recall a few isolated properties of some Banach spaces: the Schur property, to be an $\mathcal{L}_\infty$-space and to have Pełczynski's property (V). We recall that a Banach space ´ X has the Schur property (or X is a Schur space) if weak and norm sequential convergences coincide in X. We say that Z is an $\mathcal{L}_{\infty,\lambda}$-space if and only if the following holds: For every finite dimensional subspace $\mathcal{F}$ of Z one may find a further finite dimensional subspace G of Z such that $\mathcal{F} \subseteq G$ and $d\left(G, \ell_\infty^{dimG}\right) \leq \lambda$. Then we write that Z is an $\mathcal{L}_\infty$-space if Z is an $\mathcal{L}_{\infty,\lambda}$-space for some λ. To finish, a Banach space X has Pełczy´nski's property (V) if every operator on X is either weakly compact or an isomorphism on a copy of c_0. It is well known that $C(K)$-spaces have Pełczynski's property (V) ´ [251].

We finish by quoting a couple of deep results that we will need later. The first one is due to Bourgain and Pisier and will be employed in the first example; while the second result, due to Bourgain, will be used only in our second example. In [103], Bourgain and Pisier showed that for every separable Banach space X and $\lambda > 1, X$ can be embedded into some $\mathcal{L}_{\infty,\lambda}$-space, namely $\mathcal{L}_{\infty,\lambda}(X)$, in such a way that the corresponding quotient space $\mathcal{L}_{\infty,\lambda}(X)/X$ has the Schur property. In other words,

Theorem (6.2.5)[234]: (Bourgain–Pisier). Given a separable Banach space X, there is a short exact sequence

$$0 \longrightarrow X \longrightarrow \mathcal{L}_{\infty,\lambda}(X) \longrightarrow S \longrightarrow 0 \quad (BP),$$

where $S = \mathcal{L}_{\infty,\lambda}(X)/X$ has the Schur property.

As mentioned before, our second example requires a deep result of Bourgain [237]. Roughly speaking, this result provides uncomplemented copies of ℓ_1 in ℓ_1. We may state the result as follows:

Theorem (6.2.6)[234]: (Bourgain). There exists a nontrivial exact sequence

$$0 \longrightarrow \ell_1(\ell_1^n) \longrightarrow \ell_1(\ell_1^n) \longrightarrow \ell_1(A_n) \longrightarrow 0 \quad (B).$$

Since Bourgain's proof is local and it is well known that $c_0(\ell_\infty^n)$ is isomorphic to c_0, it is not hard to see that the sequence above has a nontrivial predual. This is Corollary (6.2.4). There exists a nontrivial exact sequence

$$0 \longrightarrow c_0(\ A_n^*\ ) \xrightarrow{i} c_0 \xrightarrow{p} c_0 \longrightarrow 0 \quad (B_*).$$

In other words $(B_*) * = (B)$. The sequence (B_*) is the new ingredient for the second example. Observe that, by the definition of short exact sequence, we infer that $c_0(A_n^*)$ is a subspace of c_0. It is well known that every subspace of c_0 contains a further isomorphic copy of c_0; see e.g. [236]. Thus $c_0(A_n^*)$ contains an isomorphic copy of c_0. This last fact will be used later.

Let X, Y be Banach spaces and suppose $\phi : X \longrightarrow Y$ is any mapping. We define the modulus of continuity of ϕ by

$$\omega_\phi(t) = \sup\{\|\phi(x) - \phi(y)\| : \|x - y\| \le t\}, t > 0.$$
We say that ϕ is uniformly continuous if
$$\lim_{t \to 0} \omega_\phi(t) = 0.$$
The following notion will be frequently used through. Given a quotient map $Q : Y \longrightarrow X$, we say that Q admits or has a uniformly continuous section if there is a uniformly continuous map
$$\phi : X \longrightarrow Y$$
such that $Q \circ \phi = Id_X$. The following easy proposition shows that the existence of uniformly continuous sections provides uniformly homeomorphic spaces.

Proposition (6.2.7)[234]: Assume we have a short exact sequence
$$0 \longrightarrow Z \longrightarrow Y \xrightarrow{Q} X \longrightarrow 0,$$
where the quotient Q admits a uniformly continuous section. Then $Z \oplus X$ and Y are uniformly homeomorphic.

Proof. Let us denote by ϕ a uniformly continuous section for Q. We define $\hat{\phi} : Z \oplus X \longrightarrow Y$ by the rule
$$\hat{\phi}(z, x) = z + \phi(x)$$
with inverse
$$\hat{\phi}^{-1}(y) = (y - \phi\left(Q(y)\right), Q(y)).$$
It is a routine calculation to check that $\hat{\phi}$ gives a uniformly continuous homeomorphism.

The rest of definitions we need are taken from [243]. See [243] for further details. We define a gauge to be a function $\omega : [0, \infty) \longrightarrow [0, \infty)$ which is a continuous increasing subadditive function with $\omega(0) = 0$ and $\omega(t) \ge t$ for $0 \le t \le 1$. We say ω is strongly normalized if $\omega(t) = t$ for all $t \ge 1$ and nontrivial if $\lim_{t \to 0} \omega(t)/t = \infty$.

Let X be a Banach space and let us denote by d the natural metric $d(x, y) = \|x - y\|$. If ω is a gauge then we can form a new metric replacing d by $\omega \circ d$. We denote $Lip_\omega(X) := Lip(X, \omega \circ d)$ the space of real-valued Lipschitz functions over X for which $f(0) = 0$ under the Lipschitz norm
$$\|f\|_{Lip_\omega} = sup\left\{\frac{f(x) - f(y)}{\omega(\|x - y\|)} : x \ne y\right\}.$$
In this case we write
$$\mathcal{F}_\omega(X) := \mathcal{F}_\omega(X, \omega \circ d)$$
as the canonical predual of Lip$(X, \omega \circ d)$, that is, the closed linear span of the point evaluations $\delta_x(f) = f(x) \; in \; Lip_\omega(X)^*$. The space $\mathcal{F}_\omega(X)$ is known as the Arens–Eells space or free-Lipschitz space on the metric space $(X, \omega \circ d)$.

Let us recall a couple of key facts that are necessary through. The first one is that $\mathcal{F}_\omega(X)$ is a Schur space if ω is nontrivial [243]. As it was observed in [243], the barycentric map $\beta : \mathcal{F}_\omega(X) \to X$ is a quotient map for a strongly normalized ω. Consequently, the evaluation map $\delta : X \longrightarrow \mathcal{F}_\omega(X)$ is a uniformly continuous section for the quotient map β with modulus of continuity ω. All these can be summarized in the following deep result of Kalton which is the key for our construction:

Theorem (6.2.8) (Kalton)[234]: Fix ω a nontrivial strongly normalized gauge. Given a Banach space X, there exists a short exact sequence
$$0 \longrightarrow Ker\beta \longrightarrow \mathcal{F}_\omega(X) \xrightarrow{\beta} X \longrightarrow 0 \quad (K),$$
verifying the following conditions:

(i) $F_\omega(X)$ is a Schur space.
(ii) β admits a uniformly continuous section with modulus of continuity ω.

If we apply Proposition (6.2.7) to the exact sequence of the theorem above, we find that $Ker\beta \oplus X$ is uniformly homeomorphic to $F_\omega(X)$. If moreover, X fails the Schur property then $F_\omega(X)$ and $Ker\beta \oplus X$ are not linearly isomorphic; otherwise X would inherit the Schur property. This is Kalton's strategy to provide examples of non-isomorphic but uniformly homeomorphic Banach spaces. We use the same idea in the first counterexample.

Our first example follows trivially from the next proposition that can be regarded as an $\mathcal{L}_\infty$-analogue of [243].

Proposition (6.2.9)[234]: Let X be a separable $\mathcal{L}_\infty$-space. There is a separable $\mathcal{L}_\infty$-space with the Schur property Y and a separable $\mathcal{L}_\infty$-space Z which contains a complemented copy of X so that Y and Z are uniformly homeomorphic.

Proof. Fix ω a nontrivial strongly normalized gauge and pick a separable $\mathcal{L}_\infty$-space X. Theorem (6.2.8) provides us with a short exact sequence

$$0 \longrightarrow Ker\beta \longrightarrow F_\omega(X) \xrightarrow{\beta} X \longrightarrow 0 \quad (K),$$

where $F_\omega(X)$ is a Schur space and δ – the evaluation map – is a uniformly continuous section for β with modulus of continuity ω. Since $\mathrm{Ker}\beta$ is separable, we may isometrically embed such a kernel into a Bourgain–Pisier space, namely $\mathcal{L}_\infty(Ker\beta)$, and thus produce a short exact sequence

$$0 \longrightarrow Ker\beta \xrightarrow{i} \mathcal{L}_\infty(Ker\beta) \longrightarrow S \longrightarrow 0,$$

where S has the Schur property. We combine both sequences and obtain a push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & \mathrm{Ker}\,\beta \arrow[r] \arrow[d, "i"] & \mathcal{F}_\omega(X) \arrow[r, "\beta"] \arrow[d, "I"'] & X \arrow[r] \arrow[d, equal] & 0 & (K) \\
0 \arrow[r] & \mathcal{L}_\infty(\mathrm{Ker}\,\beta) \arrow[r] \arrow[d] & PO \arrow[r, "B"] \arrow[d] & X \arrow[r] & 0 & (\textit{Push-out sequence}) \\
 & S \arrow[r, equal] \arrow[d] & S \arrow[d] & & & \\
 & 0 & 0. & & &
\end{tikzcd} \tag{4}$$

The push-out sequence

$$0 \longrightarrow \mathcal{L}_\infty(Ker\beta) \longrightarrow PO \xrightarrow{B} X \longrightarrow 0 \quad (K_X)$$

is the key. Actually, let us check that $Y := PO$ and $Z := \mathcal{L}_\infty(Ker\beta) \oplus X$ satisfy the requirements of the proposition. Since "to be an $\mathcal{L}_\infty$-space" is a three space property [238], a quick inspection to (K_X) shows that PO is an $\mathcal{L}_\infty$-space. The Schur property is also a three space property [238]. So, we infer from the second column in the push-out diagram

$$0 \longrightarrow F_\omega(X) \longrightarrow PO \longrightarrow S \longrightarrow 0,$$

that PO is a Schur space; recall that $F_\omega(X)$ and S have the Schur property by Theorem (6.2.8) and Theorem (6.2.5) respectively. All together yields that PO is a separable $\mathcal{L}_\infty$-space with the Schur property. Next we observe that the quotient B in the push-out sequence (K_X) admits a uniformly continuous section: Since δ is a uniformly continuous section for β with modulus ω, we find that $I \circ \delta$ is a uniformly continuous section for B with modulus ω. Indeed, since the push-out diagram is commutative it must be $B \circ I = \beta$; thus $\beta \circ \delta = Id_X$ implies $B \circ I \circ \delta = Id_X$. To finish, we just need to apply Proposition (6.2.7) to the section $I \circ \delta$ of the quotient $B : PO \dashrightarrow X$; we find that $\mathcal{L}_\infty(Ker\beta) \oplus X$ is uniformly homeomorphic to PO.

If we apply the previous proposition to any separable $\mathcal{L}_\infty$-space X which fails the Schur property, we immediately obtain:

Corollary (6.2.10)[234]: There are two non-isomorphic separable $\mathcal{L}_\infty$-spaces which are uniformly homeomorphic.

Theorem (6.2.11)[234]: Every infinite dimensional Banach space X can be isometrically embedded into an $\mathcal{L}_\infty$-space Y of the same density X such that the quotient Y/X has the Radon–Nikodým and the Schur properties.

Therefore the separability assumption of Proposition (6.2.9) can be removed. The same proof works in the non-separable setting using Theorem (6.2.11) instead of Theorem (6.2.5). In particular, as in Corollary (6.2.10), this gives (new) examples of non-separable $\mathcal{L}_\infty$-spaces which are non-isomorphic but uniformly homeomorphic. Since the aim is to give counterexamples in the separable setting we have decided to introduce Proposition (6.2.9) in the separable case.

According to Kalton's approach, one of the two separable non-isomorphic $\mathcal{L}_\infty$-spaces which are uniformly homeomorphic must be always a Schur space. Let us go a step further and show that one may find separable non-isomorphic $\mathcal{L}_\infty$-spaces failing the Schur property which are uniformly homeomorphic. Let us prove the following:

To prove this proposition we need a couple of preliminary lemmas. The first one was already observed in [244]; we give an independent proof.

Lemma (6.2.12)[234]: Assume we have a commutative pull-back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & PB & \xrightarrow{Q} & X_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow t & & \\
0 & \longrightarrow & Z & \longrightarrow & Y & \xrightarrow{q} & X & \longrightarrow & 0.
\end{array}$$

If q admits a uniformly continuous section then so does Q.

Proof. Let us denote by ϕ a uniformly continuous section for q. By the very definition of the canonical pull-back space $PB = \{(y, x_1) \in Y \oplus_\infty X_1 : q(y) = t(x_1)\}$, the map $\psi : X_1 \longrightarrow PB$ given by

$$\psi(x_1) := (\phi(t(x_1)), x_1)$$

is a uniformly continuous section for Q. Recall that Q is just the map $Q(y, x_1) = x_1$.

The second lemma traces back to [240].

Lemma (6.2.13)[234]: Assume we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \xrightarrow{J} & Y & \xrightarrow{q} & X & \longrightarrow & 0 \\
 & & \| & & T\downarrow & & t\downarrow & & \\
0 & \longrightarrow & Z & \xrightarrow{j} & Y & \xrightarrow{q} & X & \longrightarrow & 0.
\end{array}$$

If t is an approximable operator then the sequence

$$0 \longrightarrow Z \xrightarrow{j} Y \xrightarrow{q} X \longrightarrow 0$$

splits.

Proof. Assume momentarily that t is a finite rank operator and form the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "J"] \arrow[d, equal] & PB \arrow[r, "Q"] \arrow[d, "T"'] & X \arrow[r] \arrow[d, "t"'] & 0 \equiv F \circ t \\ 0 \arrow[r] & Z \arrow[r, "j"] & Y \arrow[r, "q"] & X \arrow[r] & 0 \equiv F. \end{tikzcd}$$

Let us check that $F \circ t \equiv 0$. Since the rank of t is finite dimensional, one may lift t to $T^1 : X \longrightarrow Y$ such that $qT^1 = t$. For the map $T - T^1Q : PB \longrightarrow X$, we have $q(T - T^1Q) = 0$. Therefore there is a linear map $R : PB \longrightarrow Z$ such that $jR = T - T^1q$. Then

$$jRJ = (T - T^1q)\, J = T\,J = j$$

and hence $RJ = Id_Z$, i.e., R is a linear retraction for J . In other words, we have proved that $F \circ t \equiv 0$ whenever t is a finite rank operator. We perform now the general case. Let us call by F a quasi-linear map representing our exact sequence. Recall that we may identify the pull-back of F with the approximable operator t as the new quasi-linear map $F \circ t$. Since the diagram

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "j"] \arrow[d, equal] & Y \arrow[r, "q"] \arrow[d, "T"'] & X \arrow[r] \arrow[d, "t"'] & 0 \equiv F \circ t \\ 0 \arrow[r] & Z \arrow[r, "j"] & Y \arrow[r, "q"] & X \arrow[r] & 0 \equiv F, \end{tikzcd}$$

is indeed a pull-back diagram, the hypothesis of the lemma can be written as $F \circ t \equiv F$ or equivalently by Corollary (6.2.4)

$$F \circ (Id_X - t) \equiv 0. \tag{5}$$

So the claim of the lemma is that condition (5) implies $F \equiv 0$. Set $t = \lim_{n\to\infty} t_n$ with t_n finite rank operators. Then

$$0 \equiv F \circ (Id_X - t) \equiv F \circ \big(Id_X - (t - t_n)\big) - F \circ t_n.$$

Since t_n is a finite rank operator, $F \circ t_n \equiv 0$ as we already checked. Hence $0 \equiv F \circ (Id_X - (t - t_n))$.

For n large enough, $\| t - t_n \| < 1$ and we find that $Id_X - (t - t_n)$ is invertible. This last fact gives $F \equiv 0$.

Lemma (6.2.12) in [240] contains a stronger statement for compact operators. The given proof for approximable operators originates in preliminary versions of [240]. Its proof is much simpler than [240] and enough for our purposes. We are ready now to give a proof for Proposition (6.2.14).

Proposition (6.2.14)[234]: There are two separable uniformly homeomorphic $\mathcal{L}_\infty$-spaces containing an isomorphic copy of c_0 which are non-isomorphic.

Proof: We follow here an idea of [240]. Pick the push-out diagram (4) given in Proposition (6.2.9) for the particular choice $X = c_0$:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] \arrow[d] & \mathcal{F}_\omega(c_0) \arrow[r, "\beta"] \arrow[d] & c_0 \arrow[r] \arrow[d, equal] & 0 & (K) \\
0 \arrow[r] & \mathcal{L}_\infty(\operatorname{Ker}\beta) \arrow[r] \arrow[d] & PO \arrow[r, "B"] \arrow[d] & c_0 \arrow[r] & 0 & (K_{c_0}) \\
 & S \arrow[r, equal] \arrow[d] & S \arrow[d] & & & \\
 & 0 & 0. & & &
\end{tikzcd}
\tag{6}
$$

And recall that B admits a uniformly continuous section. Now, let us consider

$$0 \longrightarrow c_0(\,A_n^*) \xrightarrow{\ i\ } c_0 \xrightarrow{\ p\ } c_0 \longrightarrow 0 \quad (B_*),$$

which is the natural predual of the nontrivial exact sequence given by Bourgain in [237] (see Corollary 2 for further details on this sequence). Form the pull-back diagram of the sequences (K_{c_0}) and (B_*):

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & c_0(A_n^*) \arrow[r, equal] \arrow[d, "I"] & c_0(A_n^*) \arrow[d, "i"] & & \\
0 \arrow[r] & \mathcal{L}_\infty(\operatorname{Ker}\beta) \arrow[r] \arrow[d, equal] & PB \arrow[r, "\hat{B}"] \arrow[d, "P"] & c_0 \arrow[r] \arrow[d, "p"] & 0 & (\textit{Pull-back sequence}) \\
0 \arrow[r] & \mathcal{L}_\infty(\operatorname{Ker}\beta) \arrow[r] & PO \arrow[r, "B"] \arrow[d] & c_0 \arrow[r] \arrow[d] & 0 & (K_{c_0}), \\
 & & 0 & 0 & & \\
 & & & (B_*). & &
\end{tikzcd}
\tag{7}
$$

The pull-back sequence

$$0 \longrightarrow \mathcal{L}_\infty(Ker\beta) \longrightarrow PB \xrightarrow{\ \hat{B}\ } c_0 \longrightarrow \ 0$$

is now the key. We just need to check that $\mathcal{L}_\infty(Ker\beta) \ \oplus\ c_0$ and PB satisfy the conclusions of the proposition. The only thing immediately obvious is that $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ is an $\mathcal{L}_\infty$-space and contains an isomorphic copy of c_0. We divide the rest of the proof in four steps.

(i) PB is an $\mathcal{L}_\infty$-space. A quick look at the pull-back sequence and a three space argument (as given in Proposition (6.2.9)) confirms this.

(ii) Let us prove that PB and $\mathcal{L}_\infty(Ker\beta) \ \oplus\ c_0$ are uniformly homeomorphic. Let us recall again that the quotient B in $(Kc_0\,)$ admits a uniform section. Applying Lemma (6.2.12) to the pull-back diagram (7), one may find a uniformly continuous section for the quotient $\hat{B}\,:\ PB \ \longrightarrow\ c_0$. Thus, we are in position to apply Proposition (6.2.7) and conclude: PB and $\mathcal{L}_\infty(Ker\beta) \ \oplus\ c_0$ are uniformly homeomorphic.

(iii) Let us check now that PB contains an isomorphic copy of c_0. By the definition of short exact sequence, it is trivial to check that $c_0(A_n^*)$ is a subspace of PB and also a subspace of

c_0. Thus, since every subspace of c_0 contains an isomorphic copy of c_0 (see e.g. [236]), we find that $c_0(A_n^*)$ contains an isomorphic copy of c_0; hence also PB contains a copy of c_0.

(iv) The last but delicate point is to show that PB is not linearly isomorphic to $\mathcal{L}_\infty(Ker\beta) \oplus c_0$. We divide the argument into two claims: Claim (6.2.15) and Claim (6.2.16). For the convenience of the reader, let us rewrite the two columns in the pull-back diagram (7):

$$\begin{tikzcd}
0 \arrow[r] & c_0(A_n^*) \arrow[r, "i"] \arrow[d, equal] & c_0 \arrow[r, "p"] & c_0 \arrow[r] & 0 & (B_*) \\
0 \arrow[r] & c_0(A_n^*) \arrow[r, "I"] & PB \arrow[r, "P"] \arrow[u, "\hat{B}"] & PO \arrow[r] \arrow[u, "B"] & 0 & (1st\ column\ of\ the\ pull\text{-}back\ diagram)
\end{tikzcd}$$

Claim (6.2.15)[234]: Every operator $T : c_0(A_n^*) \longrightarrow \mathcal{L}_\infty(Ker\beta) \oplus c_0$ can be extended to c_0 through i.

Proof. First of all we need to observe that $\mathcal{L}_\infty(Ker\beta)$ is a Schur space. Consider the first column in the push-out diagram (6) (which is the Bourgain–Pisier sequence of Theorem (6.2.5)):

$$0 \longrightarrow Ker\beta \longrightarrow \mathcal{L}_\infty(Ker\beta) \longrightarrow S \longrightarrow 0.$$

Recall that by Theorem (6.2.8), $F_\omega(c_0)$ is a Schur space and thus also $Ker\beta$ is a Schur space. Since S is also a Schur space by Theorem (6.2.5), a three space argument applied to the sequence above immediately gives that $\mathcal{L}_\infty(Ker\beta)$ is a Schur space. By Pełczynski property (V) every operator $T : c_0(A_n^*) \longrightarrow \mathcal{L}_\infty(Ker\beta)$ is either weakly compact or an isomorphism on a copy of c_0 (see [251]). But Schur spaces do not contain c_0; therefore such a T must be weakly compact. Since $\mathcal{L}_\infty(Ker\beta)$ is a Schur space then T is indeed compact. Thus by Lindenstrauss' extension theorem for compact operators [247], the operator $T : c_0(A_n^*) \longrightarrow \mathcal{L}_\infty(Ker\beta)$ can be extended to c_0 through i. If the operator is of the form $T : c_0(A_n^*) \longrightarrow c_0$ then Sobczyk's theorem provides an extension to c_0. All together proves trivially Claim (6.2.15).

Claim (6.2.16)[234]: The operator $I : c_0(A_n^*) \longrightarrow PB$ cannot be extended to c_0 through i.

Proof. Assume on the contrary, $I_1 : c_0 \longrightarrow PB$ is an extension of I through i. Since $I_1 \circ i = I$, I_1 induces a morphism $\hat{I}_1$ making commutative the following diagram

$$\begin{tikzcd}
0 \arrow[r] & c_0(A_n^*) \arrow[r, "i"] \arrow[d, equal] & c_0 \arrow[r, "q"] \arrow[d, "I_1"] & c_0 \arrow[r] \arrow[d, "\hat{I}_1"] & 0 & (B_*) \\
0 \arrow[r] & c_0(A_n^*) \arrow[r, "I"] \arrow[d, equal] & PB \arrow[r, "p"] \arrow[d, "\hat{B}"] & PO \arrow[r] \arrow[d, "B"] & 0 & (1st\ column\ of\ the\ pull\text{-}back\ diagram) \\
0 \arrow[r] & c_0(A_n^*) \arrow[r, "i"] & c_0 \arrow[r, "q"] & c_0 \arrow[r] & 0 & (B_*).
\end{tikzcd}$$

Therefore we have a commutative pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & c_0(A_n^*) \arrow[r, "i"] \arrow[d, equal] & c_0 \arrow[r, "q"] \arrow[d, "\hat{B}I_1"] & c_0 \arrow[r] \arrow[d, "B\hat{I}_1"] & 0 & (B_*) \\
0 \arrow[r] & c_0(A_n^*) \arrow[r, "i"] & c_0 \arrow[r, "q"] & c_0 \arrow[r] & 0 & (B_*).
\end{tikzcd} \tag{8}$$

The idea is to show that $B\,\hat{I}_1$ is an approximable operator; therefore one may apply Lemma (6.2.13) to the diagram above and show that (B_*) splits which is absurd. Thus, let us check that $B\,\hat{I}_1$ is approximable. The first thing we observe is that B is strictly singular because PO is a Schur space. Hence $B\,\hat{I}_1$ is weakly compact by Pełczynski property (V). By Gantmacher's theorem $\left(B\,\hat{I}_1\right)^*$ is also weakly compact. But actually $\left(B\,\hat{I}_1\right)^* : \ell_1 \longrightarrow \ell_1$ is compact because ℓ_1 is a Schur space and we find $B\,\hat{I}_1$ also to be compact. Since every compact operator $T : c_0 \longrightarrow c_0$ is approximable (see [249]), we find that $B\,\hat{I}_1$ is approximable. Therefore we may apply Lemma (6.2.13) to our pull-back diagram (8) $with\, t = B \circ \hat{I}_1$ which shows us that the sequence (B_*) must split. This last statement is absurd and Claim (6.2.16) is proved.

Claim (6.2.15) and Claim (6.2.16) immediately show that PB and $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ are not linearly isomorphic: Assume they are isomorphic; then, by Claim (6.2.15), every operator $T : c_0(A_n^*) \longrightarrow PB$ can be extended to c_0 through i. But using Claim (6.2.16) for $T = I$ we reach a contradiction and part (4) is proved.

The steps (1)–(4) yield that PB is an $\mathcal{L}_\infty$-space containing c_0; PB and $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ are uniformly homeomorphic but non-isomorphic.

The proof above gives a bit more of information. To explain us, We recall that in [239] it was isolated a nontrivial subclass of $\mathcal{L}_\infty$-spaces. This class termed as Lindenstrauss–Pełczy´nski spaces is defined as those spaces X for which every X-valued operator from a subspace of c_0 admits an extension to c_0. This class contains $C(K)$-spaces by Lindenstrauss–Pełczynski's extension theorem [248]. We observe that in particular it has been proved the following:

Corollary (6.2.17)[234]: The class of Lindenstrauss–Pełczy´nski spaces is not preserved under uniform homeomorphisms.

Proof. Consider the spaces $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ and PB from Proposition (6.2.14). We are about to see that $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ (which is a Lindenstrauss–Pełczynski space) is uniformly homeomorphic to PB (which is not a Lindenstrauss–Pełczynski space): The step (ii) of the proof of Proposition (6.2.14) shows that $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ and PB are uniformly homeomorphic. Observe that Claim (6.2.15) of the previous proof works replacing $c_0(A_n^*)$ for any subspace of c_0 (see [251]); thus $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ is a Lindenstrauss–Pełczynski space. On the other hand, ´ Claim (6.2.16) asserts that PB is not a Lindenstrauss–Pełczynski space.

We summarize the information of the two examples in the next proposition: **Proposition (6.2.18)[234]:** There exist separable $\mathcal{L}_\infty$-spaces Z_1, Z_2, Z_3 verifying the following conditions:

(i) Z_1, Z_2 and Z_3 are uniformly homeomorphic.

(ii) Z_j and Z_k are not isomorphic for $j \neq k$ where $j, k = 1, 2, 3$.

Proof. Consider the spaces $Z_1 := \mathcal{L}_\infty(Ker\beta) \oplus c_0$ and $Z_2 := PO$ from the push-out diagram (6). As Z_3 pick the space PB from the pull-back diagram (7). Let us prove first part (i):

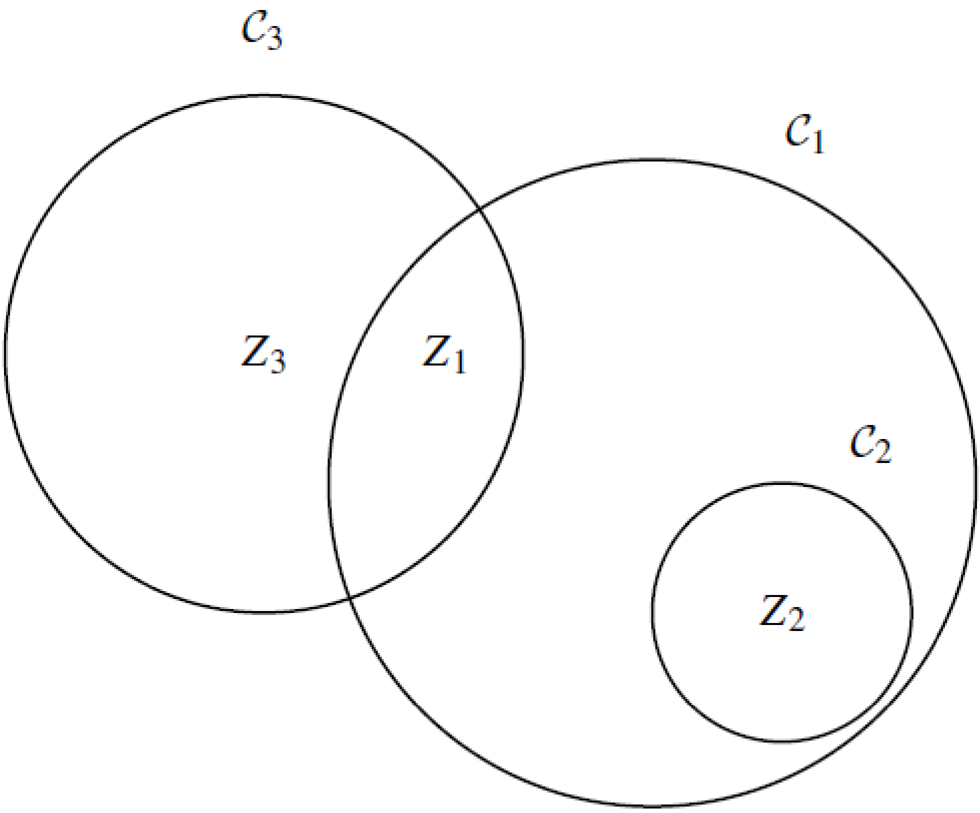
(i.1) Z_1 and Z_2 are uniformly homeomorphic: It was proved in Proposition (6.2.9) that $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ and PO are uniformly homeomorphic.

(i.2) Z_1 and Z_3 are uniformly homeomorphic: The step (2) in the proof of Proposition (6.2.14) shows that $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ and PB are uniformly homeomorphic.

(i.3) Z_2 and Z_3 are uniformly homeomorphic: By transitivity, using (i.1) and (i.2), Z_2 and Z_3 must be uniformly homeomorphic.

We pass to the proof of part (ii):
(ii.1) Z_1 and Z_2 are not isomorphic: It was proved in Proposition (6.2.9) that PO has the Schur property; hence $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ and PO are not isomorphic.
(ii.2) Z_1 and Z_3 are not isomorphic: The step (4) in the proof of Proposition (6.2.14) shows that $\mathcal{L}_\infty(Ker\beta) \oplus c_0$ and PB are not isomorphic.
(ii.3) Z_2 and Z_3 are not isomorphic: As was already quoted PO is a Schur space. But the step (iii) of Proposition (6.2.14) shows that PB contains an isomorphic copy of c_0 what makes impossible that PB and PO are linearly isomorphic.

The spaces Z_1, Z_2, Z_3 given in the proof of the previous proposition "live" in different subclasses of $\mathcal{L}_\infty$-spaces. The following picture might be useful to understand the classes involved.



C_1 = { Lindenstrauss–Pełczy´nski spaces} ,
C_2 = {$\mathcal{L}_\infty$-spaces with the Schur property},
C_3 = {$\mathcal{L}_\infty$-spaces containing c_0}

Section (6.3): The Scalar-Plus-Compact Property

In [85] a brilliant method of constructing L_∞-spaces with peculiar structure. Their method relies on a careful choice of an increasing sequence of finite dimensional subspaces $(F_n)_n$ of $\ell_\infty(\Gamma)$, with countably infinite Γ and each F_n uniformly isomorphic to $\ell_\infty^{dimF_n}$. A suitable choice of $(F_n)_n$ guarantees that the space $\overline{\cup_n F_n}$ is an L_∞-space with no unconditional basis. The Bourgain–Delbaen example contains no isomorphic copy of c_0, answering an old problem in the theory of L_∞-spaces.
Later R. Haydon [80] proved that this space is saturated with reflexive ℓ_p spaces and introduced the notation used nowadays. The Bourgain–Delbaen method was used to construct Banach spaces that solved several other long-standing conjectures on the structure of Banach spaces and showed that one may not hope for an ordinary classification of L_∞-

spaces as it happens in the $C(K)$-spaces case, see [253]–[91], [257]. See [102] and [85] for the properties of the classical Bourgain–Delbaen spaces.

In [254] a general Bourgain–Delbaen-L_∞ -space is defined and they show a remarkable fact that any separable L_∞-space is isomorphic to such a space. We recall from [254] that a BD-L_∞-space is a space $\mathcal{X} \subset \ell_\infty(\Gamma)$, with Γ countable, associated to a sequence $(\Gamma_q, i_q)_{q\in\mathbb{N}}$, where $(\Gamma_q)_q$ is an increasing sequence of finite sets with $\Gamma = \cup_{q\in\mathbb{N}} \Gamma_q$ and $(i_q)_q$ are uniformly bounded compatible extension operators $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$, i.e. $i_q(x)|_{\Gamma_q} = x$ and $i_q(x) = i_p(i_q(x)|_{\Gamma_p}))$ for any $q < p$ and $x \in \ell_\infty(\Gamma_q)$. The space $\mathcal{X} = \mathcal{X}(\Gamma_q, i_q)_q$ is defined as $\mathcal{X} = \overline{\langle d_\gamma : \gamma \in \Gamma\rangle}$, where d_γ is given by $d_\gamma = i_q(e_\gamma)$, with q chosen so that $\gamma \in \Gamma_q \setminus \Gamma_{q-1}$. An efficient method of defining particular examples of BD-L_∞-spaces as quotients of canonical BD-L_∞-spaces was given in [255]. They proved that given a BD-L_∞-space $\mathcal{X} \subset \ell_\infty(\Gamma)$ any so-called self-determined set $\Gamma' \subset \Gamma$ produces a further L_∞-space $Y = \langle d_\gamma : \gamma \in \Gamma \setminus \Gamma'\rangle$ and a BD-L_∞-space $/Y$, with the quotient map defined by the restriction of Γ to Γ' .

S.A. Argyros and R. Haydon in [91] used the Bourgain–Delbaen method in order to produce an L_∞-space $\mathcal{X}_{AH}$ which is hereditary indecomposable (HI) i.e. contains no closed infinitely dimensional subspace which is a direct sum of further two closed infinitely dimensional subspaces (in particular the space $\mathcal{X}_{AH}$ admits no unconditional basic sequence), and with dual isomorphic to ℓ_1. Moreover, using in an essential way the local unconditional structure imposed by the $\ell_\infty^{dimF_n}$ -spaces they proved that the space $\mathcal{X}_{AH}$ has the scalar-plus-compact property i.e. every bounded operator on the space is of the form $\lambda I + K$, with K compact and λ scalar.

Although it readily follows that there does not exist a Banach space with an unconditional basis and the scalar-plus-compact property, the latter property does not exclude rich unconditional structure inside the space. This is witnessed in [253], where it was shown that, among other spaces, any separable and uniformly convex Banach space embeds into an L_∞-space with the scalar plus compact property. Therefore, a naturally arising question is whether there exists a Banach space with the scalar-plus-compact property that is saturated with unconditional basic sequences.

Recall here that the first example of a space with an unconditional basis and a small family of operators is due to W.T. Gowers, who “unconditionalized” in [227] the famous Gowers–Maurey space, [111], producing a space X_G with unconditional basis that solved the hyperplane problem. Afterwards, W.T. Gowers and B. Maurey, [230], proved that any bounded operator on the space X_G is of the form $D + S$, with D diagonal and S strictly singular. Gowers asked if an analogous property holds for the operators defined on subspaces of X_G and if such property characterises a class of so-called tight by support Banach spaces, as it is in the case of complex HI spaces according to [256]. This question was answered negatively by the first two named authors [220].

An example of a space with rich unconditional structure and a small family of bounded operators of a different type was presented in [98], where they built a Banach space saturated with unconditional sequences and satisfying the following property: any bounded operator on the space is a strictly singular perturbation of a multiple of identity (recall that an operator is strictly singular provided none of its restriction to an infinitely dimensional subspace is an isomorphism onto its range). The construction used the saturated norms technique in a mixed Tsirelson space setting.

We continue the study of Banach spaces with a small family of operators by showing the existence of a Banach space with a strongly heterogeneous structure. We construct a BD-L_∞-space $\mathcal{X}_{Kus}$ with a basis satisfying the following properties:

(i) Any bounded operator $T : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ is of the form $T = \lambda Id\mathcal{X}_{Kus} + K$, with K compact and λ scalar.

(ii) The space $\mathcal{X}_{Kus}$ is saturated with unconditional basic sequences.

(iii) The dual space to $\mathcal{X}_{Kus}$ is isomorphic to ℓ_1.

The structure of the space of bounded operators $\mathcal{B}(\mathcal{X}_{Kus})$ implies that the space $\mathcal{X}_{Kus}$ is indecomposable, however, being unconditionally saturated, it admits no HI structure. The space $\mathcal{X}_{Kus}$ is thus the first example of a Banach space with the scalar-plus-compact property failing to have any HI structure. We recall that M. Tarbard in [259] constructed an indecomposable BD-L_∞-space $\mathcal{X}_\infty$, that is not HI, but the Calkin algebra $\mathcal{B}(\mathcal{X}_\infty)/K(\mathcal{X}_\infty)$ is isomorphic to ℓ_1.

In order to build $\mathcal{X}_{Kus}$ we adapt the idea of a construction of a Banach space X_{ius} of [98] to the scheme of the Argyros–Haydon construction of Bourgain–Delbaen spaces [91]. This framework allows to pass from strictly singular operators to compact ones, however, in order to profit from this key property of the Argyros–Haydon construction we need to strengthen some results of [98] in the following way: we prove that if a bounded operator on the space converges to zero on the basis, then it converges to zero on any element of a special class of basic sequences, called RIS, instead of a saturating family of RIS. In order to avoid a technical inductive construction of the space $\mathcal{X}_{Kus}$ we follow the scheme of [255], defining $\mathcal{X}_{Kus}$ as a suitable quotient of some variation of the canonical BD-L_∞-space $\mathfrak{B}_m T$ defined in [91].

The balance between unconditional saturation and the restricted form of bounded operators on the whole space in the case of X_{ius} was guaranteed by the form of so-called special functionals – the major tool in the construction of saturated norms. Any special functional in the norming set of X_{ius} is a weighted average of a sequence of functionals, where the odd parts are weighted averages of the basis. The choice of the next functional of the weighted average is determined by the previously chosen odd parts and supports of the even parts. The freedom on the side of even parts allows changing signs of parts of even functionals of the weighted average, which in turn provides saturation by unconditional sequences. On the other hand, the control over the supports of the even parts guarantees the typical property of such construction, i.e. in our case given two RIS (x_n) and (y_n) with pairwise disjoint ranges and $\epsilon > 0$ one is able to built on (y_n) an average $\sum_n a_n y_n$ of norm 1, such that $\|\sum_n a_n x_n\| < \epsilon$. This last property is crucial for proving the form of a bounded operator on a space.

The direct translation of the special functionals described above into the setting of BD-spaces is impossible, as any change of signs of a part of a norming functional changes its support. In order to overcome this obstacle we use in the definition of functionals on the space $\mathcal{X}_{Kus}$ projections on finite intervals instead of projections on right intervals of the form $[p, \infty)$ and substitute the equality of supports of even parts of special functionals by tight relation between tree-analysis of even parts (definition of special nodes). The latter notion in the setting of the Argyros–Haydon construction comes from [258] and proves to be a very efficient tool in our case.

We describe the construction of the general space we shall use, including different kinds of analyses of norming functionals. We devoted to the properties the basis, including the notion of neighbour nodes, within the general framework. We give the definition of $\mathcal{X}_{Kus}$. We study the rapidly increasing sequences (RIS) and the dependent sequences respectively. We contain the results on bounded operators on the space, whereas the proof of unconditional saturation.

We present a BD-L_∞-space $X_{\bar{\Gamma}}$, which is a minor modification of the space $\mathfrak{B}_m T$ defined in [91]. We shall define later the space $\mathcal{X}_{Kus}$ as determined by some set $\Gamma \subset \bar{\Gamma}$ following the general scheme of [255].

Pick $(m_k)_k, (n_k)_k, (l_k)_k \nearrow +\infty$ such that $m_1 = 4, n_1 = 4, l_1 = 2$ and

$$m_k m_{k-1} \leq m_1^{l_k} \left(\frac{n_{k-1}}{m_{k-1}}\right)^{l_k} \leq \frac{n_k}{m_{k-1} m_k}, k \in \mathbb{N}. \tag{9}$$

For example take $\left(2^{2^k}\right)_k, \left(2^{2^{k^2}}\right)_k, (2^k)_k$.

Following [91] we shall define recursively finite sets of nodes $\bar{\Delta}_q$ and $\bar{\Gamma}_q = \bar{\Delta}_1 \cup \cdots \cup \bar{\Delta}_q, q \in \mathbb{N}$. Along with each set $\bar{\Delta}_q$ we define functionals $\left(\bar{c}_\gamma^*\right)_{\gamma \in \bar{\Delta}_q} \subset \ell_1(\bar{\Gamma}_q)$ and further $\left(\bar{d}_\gamma^*\right)_{\gamma \in \bar{\Delta}_q} \subset \ell_1(\bar{\Gamma}_q)$ as $\bar{d}_\gamma^* = e_\gamma^* - \bar{c}_\gamma^*$. Having defined all sets $\bar{\Delta}_q, q \in \mathbb{N}$, we let $\bar{\Gamma} = \cup_q \bar{\Gamma}_q$.

We proceed now to the inductive construction. We let $\bar{\Delta}_1 = \{1\}, c_1^* = 0$ and thus $\bar{d}_1^* = e_1^*$. Assume we have defined sets $\bar{\Delta}_1, \dots, \bar{\Delta}_q$. By $\left(e_\gamma^*\right)_{\gamma \in \bar{\Gamma}_q}$ we denote the standard unit vector basis of $\ell_1(\bar{\Gamma}_q)$. We enumerate the set $\bar{\Delta}_q$ using $\{\#\bar{\Gamma}_{q-1} + 1, \dots, \#\bar{\Gamma}_q\}$ as the index set and in the set $\bar{\Gamma}_q$ we consider the corresponding enumeration. Thus we can regard the sets $\bar{\Delta}_q$ and $\bar{\Gamma}_q$ as intervals of $\mathbb{N}$. We use the notation $(\gamma_n)_n$ to refer to this enumeration. For any interval $I \subset \bar{\Gamma}_q$ let $\bar{P}_I^*$ be the projection onto $\langle \bar{d}_{\gamma_n}^* : n \in I \rangle$. For simplicity for any $n \in \mathbb{N}$ by $\bar{P}_n^*$ we denote the projection $\bar{P}_{(0,n]}^*$.

For each $q \in \mathbb{N}$ let $\mathrm{Net}_{1,q}$ be a finite symmetric ${}^1/_{4n_q^2}$ -net of $[-1,1]$ containing ± 1. We set

$B_{p,q} = \{\lambda e_\eta^* : \lambda \in \mathrm{Net}_{1,q}, \eta \in \bar{\Gamma}_q \setminus \bar{\Gamma}_p\}$,

where for $p = 0$ we let $\bar{\Gamma}_0 = \emptyset$. For simplicity we write $B_q = B_{0,q}, q \in \mathbb{N}$.

The set $\bar{\Delta}_{q+1}$ is defined to be the set of nodes

$$\begin{aligned}\bar{\Delta}_{q+1} = \bigcup_{j=1}^{q} &\{(q+1, 0, m_j, I, \epsilon, b^*) : I \text{ interval} \subset \bar{\Gamma}_q, \epsilon \in \{-1,1\}, b^* \in B_q \text{ and } \bar{P}_I^* b^* \\ &\neq 0\} \\ \cup \bigcup_{1 \leq p < q} \bigcup_{j=1} &\{(q+1, \xi, mj, I, \epsilon, b^*) : \xi \in \bar{\Delta}_p, w(\xi) = m_j^{-1}, \mathrm{age}(\xi) < n_j, \\ &\epsilon \in \{-1,1\}, b^* \in B_{p,q}, I \text{ interval} \subset \bar{\Gamma}_q \setminus \bar{\Gamma}_p, \bar{P}_I^* b^* \neq 0\}.\end{aligned}$$

For any $\gamma \in \bar{\Delta}_q$ we define $\bar{c}_\gamma^*$ as follows.

$$\bar{c}_{\gamma}^{*}=\begin{cases}\dfrac{1}{m_j}\epsilon\bar{P}_I^{*}\,b^{*} & \text{for } \gamma=\left(q+1,0,m_j,I,\epsilon,b^{*}\right)\\ e_{\xi}^{*}+\dfrac{1}{m_j}\,\epsilon\bar{P}_I^{*}\,b^{*} & \text{for } \gamma=(q+1,\xi,m_j,I,\epsilon,b^{*})\end{cases}\tag{10}$$

We let also $\bar{d}_{\gamma}^{*}=e_{\gamma}^{*}-\bar{c}_{\gamma}^{*}$.

For any $\gamma=(q+1,0,m_j,I,\epsilon,b^{*})$ we define age$(\gamma)=1$ and for $\gamma=(q+1,\xi,m_j,I,\epsilon,b^{*})$ we define age$(\gamma)=age(\xi)+1$. For any $\gamma=(q+1,0,m_j,I,\epsilon,b^{*})$ or $\gamma=(q+1,\xi,m_j,I,\epsilon,b^{*})$ we define rank$(\gamma)=q+1$ and weight $(\gamma)=m_j^{-1}$.

Adapting the reasoning of [91] we obtain the following two lemmas.

Lemma (6.3.1)[252]: $\langle\bar{d}_{\gamma_i}^{*}:i\le n\rangle=\langle e_{\gamma_i}^{*}:i\le n\rangle$ for every $n\in\mathbb{N}$.

Lemma (6.3.2)[252]: $\|\bar{P}_m^{*}\|\le\frac{m_1}{m_1-2}=2$ for every $m\in\mathbb{N}$.

The above lemma yields that $\left(\bar{d}_{\gamma_n}^{*}\right)_{n\in\mathbb{N}}$ is a triangular basis of $\ell_1(\Gamma)$ (in the sense of [91], Def. (6.3.8)). Let $\left(\bar{d}_{\gamma_n}\right)_{n\in\mathbb{N}}$ be its biorthogonal sequence. Regarding each projection $\bar{P}_n^{*}$ as an operator $\ell_1(\Gamma)\to\ell_1^n$ we consider the dual operator $\bar{\iota}_n:\ell_\infty^n\to\ell_\infty(\Gamma)$, which is an isomorphic embedding satisfying $\|\bar{\iota}_n\|\le 2$. We are ready to define the following.

Definition (6.3.3)[252]: Let $\mathcal{X}_{\bar{\Gamma}}=\langle\bar{d}_{\gamma_n}:n\in\mathbb{N}\rangle\subset\ell_\infty(\bar{\Gamma})$.

Repeating the results of [91] in our setting we obtain the following.

Theorem (6.3.4)[252]: The space $\mathcal{X}_{\bar{\Gamma}}$ is a BD-L$_\infty$-space defined by the sequence $\left(\bar{\Gamma}_q,\bar{\iota}_q\right)_q$.

For any interval $I\subset\mathbb{N}$ we denote by $\bar{P}_I$ the canonical projection $\bar{P}_I:\mathcal{X}_{\bar{\Gamma}}\to\langle\bar{d}_{\gamma_i}:i\in I\rangle$. In case $I=\{1,\dots,n\},n\in\mathbb{N}$, we write simply $\bar{P}_n$.

Given any $q\in\mathbb{N}$ we let $\bar{M}_q=\bar{\iota}_{max\bar{\Delta}_q}\,[\ell_\infty(\bar{\Delta}_q)]$. We shall consider supports and ranges of vectors, thus also block sequences, with respect both to the basis $\left(\bar{d}_{\gamma_n}\right)_{n\in\mathbb{N}}$ of $\mathcal{X}_{\bar{\Gamma}}$ and to the FDD $\left(\bar{M}_q\right)_{q\in\mathbb{N}}$ of $\mathcal{X}_{\bar{\Gamma}}$. In the first case we shall use for any $x\in\mathcal{X}_{\bar{\Gamma}}$ the notation supp x, rng x, whereas in the second we write supp$_{FDD}$ x and rng$_{FDD}$ x.

Definition (6.3.5)[252]: We say that a block sequence $(x_n)_n\subset\mathcal{X}_{\bar{\Gamma}}$ is skipped provided max rng$_{FDD}\,x_n+1<$ min rng$_{FDD}\,x_{n+1}$ for each n.

We introduce different types of analysis of a node following [91] and [258], adjusting their scheme to our situation.

The evaluation analysis of e_{γ}^{*}.

First we notice that every $\gamma\in\bar{\Gamma}$ admits a unique analysis as follows (Prop. (6.3.17) [91]). Let $(\gamma)=m_j^{-1}$. Then using backwards induction we determine a sequence of sets $\left(I_i,\epsilon_i,b_{\eta i}^{*},\xi_i\right)_{i=1}^{a}$ so that $\xi_a=\gamma,\xi_1=(q_1+1,0,m_j,I_1,\epsilon_1,b_{\eta_1}^{*})$ and $\xi_i=(q_i+1,\xi_{i-1},m_j,I_i,\epsilon_i,b_{\eta_i}^{*})$ for every $1<i\le a$, where $b_{\eta_i}^{*}=\lambda_i e_{\eta_i}^{*}$ for some $\lambda_i\in$ Net$_{1,q_i}$.

Repeating the reasoning of [91], as $e_{\xi}^{*}=\bar{d}_{\xi}^{*}+c_{\xi}^{*}$ for each $\xi\in\Gamma$, with the above notation we have

$$e_{\gamma}^{*}=\sum_{i=1}^{a}\bar{d}_{\xi_i}^{*}+m_j^{-1}\sum_{i=1}^{a}\epsilon_i\bar{P}_{I_i}^{*}b_{\eta_i}^{*}=\sum_{i=1}^{a}\bar{d}_{\xi_i}^{*}+m_j^{-1}\sum_{i=1}^{a}\epsilon_i\lambda_i\bar{P}_{I_i}^{*}\,e_{\eta_i}^{*}$$

Definition (6.3.6)[252]: Let $\gamma\in\bar{\Gamma}$. Then the sequence $\left(I_i,\epsilon_i,\lambda_i e_{\eta_i}^{*},\xi_i\right)_{i=1}^{a}$ satisfying all the above properties will be called the evaluation analysis of γ.

We define the bd-part and mt-part of e_γ^* as

$$bd(e_\gamma^*) = \sum_{i=1}^{a} \bar{d}_{\xi_i}^*, \quad mt(e_\gamma^*) = m_j^{-1} \sum_{i=1}^{a} \epsilon_i \lambda_i \bar{P}_{I_i}^* e_{\eta_i}^* .$$

The I (interval)-analysis of a functional e_γ^*.

Let $I \subset \mathbb{N}$ and $\gamma \in \Gamma$ with $\bar{P}_I^* e_\gamma^* \neq 0$. Let $w(\gamma) = m_j^{-1}, a \leq n_j$ and $\left(I_i, \epsilon_i, \lambda_i e_{\eta_i}^*, \xi_i\right)_{i=1}^{a}$ the evaluation analysis of γ. We define the I-analysis of e_γ^* as follows:

(a) If for at least one i we have $\bar{P}_{I_i \cap I}^* e_{\eta_i}^* \neq 0$, then the I-analysis of e_γ^* is of the following form

$$\left(I_i \cap I, \epsilon_i, \lambda_i e_{\eta_i}^*, \xi_i\right)_{i \in A_I},$$

where $A_I = \{i : \bar{P}_{I_i \cap I}^* e_{\eta_i}^* \neq 0\}$. In this case we say that e_γ^* is I-decomposable.

(b) If $\bar{P}_{I_i \cap I}^* e_{\eta_i}^* = 0$ for all $i = 1, \ldots, a$, then we assign no I-analysis to e_γ^* and we say that e_γ^* is I-indecomposable.

Now we introduce the tree-analysis of e_γ^* analogous to the tree-analysis of a functional in a mixed Tsirelson space (see [99]).

We denote by $(\mathcal{T}, \preccurlyeq)$ a finite tree, whose elements are finite sequences of natural numbers ordered by the initial segment partial order. Given $t \in \mathcal{T}$ denote by S_t the set of immediate successors of t.

Let $(I_t)_{t \in \mathcal{T}}$ be a tree of intervals of $\mathbb{N}$ such that $t \preccurlyeq s$ iff $I_t \supset I_s$ and t, s are incomparable iff $I_t \cap I_s = \emptyset$. For such a family $(It)_{t \in \mathcal{T}}$ and t, s incomparable we write $t < s$ iff $I_t < I_s$ (i.e. $max\, I_t < min\, I_s$).

The tree-analysis of a functional e_γ^*.

Let $\gamma \in \bar{\Gamma}$. The tree-analysis of e_γ^* is a family of the form $(I_t, \epsilon_t, \eta_t)_{t \in \mathcal{T}}$ defined inductively in the following way:

(i) $\mathcal{T}$ is a finite tree with a unique root denoted by $\emptyset$.

(ii) Set $\eta_\emptyset = \gamma, I_\emptyset = (1, max\, \Delta_{rank\, \gamma}], \epsilon_\emptyset = 1$ and let $\left(I_i, \epsilon_i, \lambda_i e_{\eta_i}^*, \xi_i\right)_{i=1}^{a}$ be the evaluation analysis of $e_{\eta_\emptyset}^*$. Set $S_\emptyset = \{(1), (2), \ldots, (a)\}$ and for every $s = (i) \in S_\emptyset, (I_s, \epsilon_s, \eta_s) = (I_i, \epsilon_i, \eta_i)$.

(iii) Assume that for $t \in \mathcal{T}$ the tuple $(I_t, \epsilon_t, \eta_t)$ is defined. Let $\left(I_i, \epsilon_i, \lambda_i e_{\eta_i}^*, \xi_i\right)_i$ be the evaluation analysis of $e_{\eta t}^*$. Consider two cases:

(a) If $e_{\eta_t}^*$ is I_t-decomposable, let $\left(I_i, \epsilon_i, \lambda_i e_{\eta_i}^*, \xi_i\right)_{i \in A_{I_t}}$ be the I_t-analysis of $e_{\eta t}^*$. Set $S_t = \{(t^\frown i) : i \in A_{I_t}\}$. For every $s = (t^\frown i) \in S_t$, let $(I_s, \epsilon_s, \eta_s) = (I_i, \epsilon_i, \eta_i)$.

(b) If $e_{\eta_t}^*$ is I_t-indecomposable, then t is a terminal node of the tree-analysis.

Definition (6.3.7)[252]: Given any $\gamma \in \Gamma$, in notation Let

$$mt - supp\, e_\gamma^* = \{\xi_t : t \in \mathcal{T}, t\ terminal\} = \{\xi_t : t \in \mathcal{T}, \bar{P}_{I_t}^* e_{\eta_t}^* = \bar{d}_{\xi_t}^*\}$$

and $\mathrm{bd} - \mathrm{supp}\, e_\gamma^* = supp\, e_\gamma^* \setminus mt - supp\, e_\gamma^*$.

We present here estimates on the averages of the basis $\left(\bar{d}_{\gamma_n}\right)_{n \in \mathbb{N}}$.

The result is crucial for the estimates in.

Definition (6.3.8)[252]: We shall call two nodes ξ_1, ξ_2 neighbours if there exists $\gamma \in \Gamma$ with $bd\left(e_\gamma^*\right) = \sum_{j=1}^{a} \bar{d}_{\zeta_j}^*$ such that $\xi_i = \zeta_{j_i}$ for some $j_1 < j_2$.

Note that from the definition it follows that for any neighbours ξ_1, ξ_2 we have $w(\xi_1) = w(\xi_2)$.

Lemma (6.3.9)[252]: Let $\left(\bar{d}_{\gamma_n}\right)_{n\in\mathbb{N}}$ be a subsequence of the basis. Then there exists infinite $M \subset N$ such that no two nodes $\gamma_n, \gamma_m, n, m \in M$, are neighbours.

The proof is based on the fact that the age is uniquely determined for each node.

Proof. If there are infinitely many nodes with different weights we are done. So assume that for all but finite nodes we have $w(\gamma_n) = m_k^{-1}$ for some fixed k.

Applying Ramsey theorem we obtain an infinite set such that either no two nodes from this set are neighbours or any two are neighbours.

In the first case we are done. Otherwise passing to a further subsequence we may assume that $\operatorname{rank}(\gamma_n) < \operatorname{rank}(\gamma_{n+1})$ for every n.

Since we have that γ_j, γ_{j+1} are neighbours it follows by a simple induction that

$$\operatorname{age}(\gamma_{j+1}) \geq \operatorname{age}(\gamma_j) + 1 \geq j + 1.$$

Take $j = n_k + 1$ and pick e_γ^* of the form

$$e_\gamma^* = \sum_{r=1}^{a} \bar{d}_{\xi_r}^* + m_k^{-1} \sum_{r=1}^{a} \epsilon_r \lambda_r e_{\eta_r}^* \bar{P}_{I_r}$$

with $\bar{d}_{\gamma_{n_k+1}}^* = \bar{d}_{\xi_r}^*$ for some r. Then $\operatorname{age}(\xi_r) \leq n_k$ which yields a contradiction and ends the proof.

In [91] it is proved that the sequence $\left(\sum_{\xi\in\Delta_n} \bar{d}_\xi\right)_{n\in\mathbb{N}}$ generates an ℓ_1-spreading model in the space $\mathcal{X}_{AH}$. We show that the norm of the vector $= n_j^{-1} \sum_{i\in F} \bar{d}_{\xi_i}$, where ξ_i's are pairwise non-neighbours, is determined by the mt-part of the nodes.

We shall use basic properties of mixed Tsirelson spaces. Recall that the mixed Tsirelson space $T[(\mathcal{A}_{n_k}, m_k^{-1})_{k\in\mathbb{N}}]$ is the completion of c_{00} with the norm defined by a norming set D, which is the smallest set in c_0 that contains the unit vectors $\{\pm e_n\}$ and satisfies for any $k \in \mathbb{N}$ the following condition: for any block sequence $f_1 < \cdots < f_d, d \leq n_k$, of elements of D the weighted average $m_k^{-1}(f_1 + \cdots + f_d)$ also belongs to D. For further details see [99].

Lemma (6.3.10)[252]: Let $= n_j^{-1} \sum_{i\in G} \bar{d}_{\xi_i}$, be such that no two ξ_i's are neighbours and $\#G \leq n_j$. Then for any $\gamma \in \Gamma$ with $w(e_\gamma^*) = m_k^{-1}$ we have the following

$$|e_\gamma^*(x)| \leq \begin{cases} \dfrac{1}{n_j} + \dfrac{2}{m_k} & if\ k \geq j \\[2mm] \dfrac{7}{m_k m_j} & if\ k < j. \end{cases}$$

In particular

$$\left\| n_j^{-1} \sum_{i=1}^{n_j} \bar{d}_{\xi_i} \right\| \leq 7 m_j^{-1}.$$

Proof. We shall construct functionals ϕ_γ in the norming set of the mixed Tsirelson space $X_{aux} = T[(\mathcal{A}_{n_k}, m_k^{-1})_{k\in\mathbb{N}}]$ such that

$$\left|e_\gamma^*(x)\right| \leq \phi_\gamma(y) + \frac{2}{m_j m_{j-1}}$$

where $y = 2 \sum_{k\in G} e_k/n_j \in c_{00}(\mathbb{N})$.
Take $\gamma \in \Gamma$ and consider its evaluation analysis $e_\gamma^* = \sum_{r=1}^{a} \bar{d}_{\beta_r}^* + m_k^{-1} \sum_{r=1}^{a} \epsilon_r \lambda_r e_{\eta_r}^* \bar{P}_{I_r}$.
Let $g_\gamma = bd(e_\gamma^*)$ and $f_\gamma = mt(e_\gamma^*)$.
We shall consider two cases.
Case 1. $w(\gamma) \le m_j^{-1}$.
Since the nodes $(\xi_i)_i$ are pairwise non-neighbours and $(\beta_i)_i$ are pairwise neighbours it follows that

$$|g_\gamma(x)| \le n_j^{-1} . \tag{11}$$

Also for every $r \le a$ using that $|e_\zeta^* (\bar{d}_\beta)| \le 2$ for all ζ, β, we get

$$\left|e_{\eta_r}^* \bar{P}_{I_r}(x)\right| \le 2\frac{\#\{i : \operatorname{rng}(d_{\xi_i}^*) \subset I_r\}}{n_j} . \tag{12}$$

It follows from (11), (12), using that $|\lambda_r| \le 1$ for every r, that

$$\left|e_\gamma^*(x)\right| \le \frac{1}{n_j} + 2m_k^{-1} \sum_{r=1}^{a} \frac{\#\{i : \operatorname{rng}(d_{\xi_i}^*) \subset I_r\}}{n_j} \le \frac{1}{n_j} + \frac{2}{m_k} . \tag{13}$$

Taking $\phi_\gamma = m_k^{-1} \sum_{n\in F} e_n^*$ where $F = \cup_{r\le a} \{n \,|\gamma_n = \xi_i, \operatorname{rng}(d_{\xi_i}^*) \subset I_r$ for some $i \in G\}$ it follows that $\#F \le n_j \le n_k$ and ϕ_γ belongs to the norming set of the mixed Tsirelson space X_{aux}.
From (13) we get

$$\left|e_\gamma^*(x)\right| \le \frac{1}{n_j} + 2m_k^{-1} \sum_{n\in F} \frac{e_n^*(e_n)}{n_j} = \frac{1}{n_j} + \phi_\gamma(y). \tag{14}$$

Case 2. $w(\gamma) = m_k^{-1} > m_j^{-1}$.
Let $(I_t, \varepsilon_t, \eta_t)_{t\in\mathcal{T}}$ be the tree-analysis of e_γ^* and $\mathcal{T}'$ be the subtree of $\mathcal{T}$ consisting of all nodes t of height at most l_j. We will describe how to define certain functionals $(\phi_t)_{t\in\mathcal{T}'}$ in the norming set of $T[(\mathcal{A}_{n_k}, m_k^{-1})_{k\in\mathbb{N}}]$ that we will use to obtain the desired estimate.
As in the previous case we get

$$|g_\gamma(x)| \le n_j^{-1} . \tag{15}$$

Using that $e_\gamma^* = g_\gamma + f_\gamma$ and $|\lambda_r| \le 1$ for every r, we get

$$|e_\gamma^*(x)| \le n_j^{-1} + |f_\gamma(x)| \le n_j^{-1} + m_k^{-1} \sum_{r=1}^{a} |e_{\eta_r}^* \bar{P}_{I_r}(x)|. \tag{16}$$

We shall split now the successors $e_{\eta_r}^*$ of e_γ^* into those with weight smaller or equal to m_j^{-1} and those with weight bigger that m_j^{-1} . For a node γ we set

$$S_{\gamma,1} = \{r \in S_\gamma : w(\eta_r) \le m_j^{-1}\} \text{ and } S_{\gamma,2} = S_\gamma \setminus S_{\gamma,1}.$$

From (16) we get

$$|e_\gamma^*(x)| \le n_j^{-1} + m_k^{-1}\left(\sum_{r\in S_{\gamma,1}} \left|e_{\eta_r}^* \bar{P}_{I_r}(x)\right| + \sum_{r\in S_{\gamma,2}} \left|e_{\eta_r}^* \bar{P}_{I_r}(x)\right|\right)$$

Using (14) for the $r \in S_{\gamma,1}$, (16) for the $r \in S_{\gamma,2}$ and that $\#S_{\gamma,1} + \#S_{\gamma,2} \le n_k, k < j$, we get

$$|e_\gamma^*(x)| \le n_j^{-1} + \frac{n_k}{m_k n_j} + \frac{1}{m_k}\left(\sum_{r\in S_{\gamma,1}} \phi_r(y) + \sum_{r\in S_{\gamma,2}} w(e_{\eta_r}) \sum_{s\in S_r} \left|e_{\eta_s}^* \bar{P}_{I_s}(x)\right|\right)$$

$$\le \frac{1}{n_j}(1 + \left(\frac{n_{j-1}}{m_{j-1}}\right) + \frac{1}{m_k}\left(\sum_{r\in S_{\gamma,1}} \phi_r(y) + \sum_{r\in S_{\gamma,2}} w(e_{\eta_r}) \sum_{s\in S_r} \left|e_{\eta_s}^* \bar{P}_{I_s}(x)\right|\right). \qquad (17)$$

Note that the functional $m_k^{-1}\left(\sum_{r\in S_{\gamma,1}} \phi_r\right)$ belongs to the norming set of the mixed Tsirelson space X_{aux} and has room for $\#S_{\gamma,2}$ more functionals.

We shall replay the above splitting for every $e_{\eta_s}^* \bar{P}_{I_s}$. To avoid complicated notation we shall set $n_s = \#S_s$ and $m_s^{-1} = w(e_{\eta_s}^*)$. From (17) using $e_{\eta_s}^* \bar{P}_{I_s}$ in the place of e_γ^* we get

$$\left|e_{\eta_s}^* \bar{P}_{I_s}(x)\right| \le \frac{1}{n_j}\left(1 + \frac{n_{j-1}}{m_{j-1}}\right)$$

$$+ m_s^{-1}\left(\sum_{t\in S_{s,1}} \phi_t(y) + \sum_{t\in S_{s,2}} m_t^{-1} \sum_{u\in S_t} \left|e_{\eta_u}^* \bar{P}_{I_u}(x)\right|\right). \qquad (18)$$

It follows that

$$\sum_{r\in S_{\gamma,2}} w(e_{\eta_r}) \sum_{s\in S_r} \left|e_{\eta_s}^* \bar{P}_{I_s}(x)\right| \le \sum_{r\in S_{\gamma,2}} m_r^{-1} \sum_{s\in S_r} \frac{1}{n_j}\left(1 + \frac{n_{j-1}}{m_{j-1}}\right) \qquad (19)$$

$$+ \sum_{r\in S_{\gamma,2}} m_r^{-1} \sum_{s\in S_r} m_s^{-1}\left(\sum_{t\in S_{\gamma,1}} \phi_t(y) + \sum_{t\in S_{\gamma,2}} m_t^{-1} \sum_{u\in S_r} \left|e_{\eta_{su}}^* \bar{P}_{I_u}(x)\right|\right)$$

$$\le n_k \frac{n_r}{m_r}\frac{1}{n_j}\left(1 + \frac{n_{j-1}}{m_{j-1}}\right) \qquad \text{since } \#S_{\gamma,2} \le n_k \text{ and } \#S_r \le n_r$$

$$+ \sum_{r\in S_{\gamma,2}} m_r^{-1} \sum_{s\in S_r} m_s^{-1}\left(\sum_{t\in S_{\gamma,1}} \phi_t(y) + \sum_{t\in S_{\gamma,2}} m_t^{-1} \sum_{u\in S_r} \left|e_{\eta_u}^* \bar{P}_{I_u}(x)\right|\right)$$

By (17) and (19), using that $\frac{n_r}{m_r}, \frac{n_k}{m_k} \le \frac{n_{j-1}}{m_{j-1}}$ we get

$$|e_\gamma^*(x)| \le \frac{1}{n_j}\left(1 + \frac{n_{j-1}}{m_{j-1}} + \left(\frac{n_{j-1}}{m_{j-1}}\right)^2 + \left(\frac{n_{j-1}}{m_{j-1}}\right)^3\right) \qquad (20)$$

$$+ \frac{1}{m_k}\left(\sum_{r\in S_{\gamma,1}} \phi_r(y) \right.$$

$$+ \sum_{r\in S_{\gamma,2}} m_r^{-1} \sum_{s\in S_r} m_s^{-1}\left(\sum_{t\in S_{s,1}} \phi_t(y)\right.$$

$$\left.\left. + \sum_{t\in S_{s,2}} m_t^{-1} \sum_{u\in S_t} \left|e_{\eta_u}^* \bar{P}_{I_u}(x)\right|\right)\right) \qquad (21)$$

Note that the functional

$$\phi_\gamma = \frac{1}{m_k} + \left(\sum_{r\in S_{\gamma,1}} \phi_r(y) \sum_{r\in S_{\gamma,2}} m_r^{-1} \sum_{s\in S_r} m_s^{-1} \sum_{t\in S_{s,1}} \phi_{t(y)} \right)$$
belongs to the norming set of the mixed Tsirelson space X_{aux} and the functional $m_s^{-1} \sum_{t\in S_{s,1}} \phi_t$ has room for $\#S_{s,2}$ more functionals.
We continue this splitting at most l_j times, see (9) for the choice of l_j , or till $S_{s,2} = \emptyset$ i.e. we do not have nodes with weight $> m_j^{-1}$.
If we stop before the l_j-th step we get that $|e_\gamma^*(x)|$ is dominated by $\phi_\gamma(y)$ plus the errors in (20), where the sum end to the l_j-th power of n_{j-1}/m_{j-1}. Since ϕ_γ belongs to the norming set of the mixed Tsirelson space X_{aux} it follows from [99], Lemma II.9, that
$$\phi_\gamma(y) \le 4m_k^{-1} m_j^{-1} .$$
If we continue the splitting l_j -times, then there exists some node with $(\gamma_t) > m_j^{-1}$. For every such node we have
$$\left(\prod_{s<t} w(e_{\gamma_s})\right)|e_{\gamma_t}^*(x)| \le \left(\frac{1}{m_1}\right)^{l_k} |e_{\gamma_t}^*(x)| \le 2m_k^{-1} m_j^{-1} \frac{\#\{i : rng(d_{\xi_i}^*) \subset I_t \cap G\}}{n_j}$$
since $m_1^{-l_j} \le (m_j m_{j-1})^{-1}$, see (9).
Summing the estimation of all those nodes we get upper estimate equal to ${}^{2\#G}/_{m_k m_j n_j} \le {}^{2}/_{m_k m_j}$.
The remaining nodes provide us with a functional in the norming set of the mixed Tsirelson space X_{aux}. By [99] its action on y is bounded by $4m_k^{-1} m_j^{-1}$.
It remains to handle the errors (20). In each case we have
$$\frac{1}{n_j}\left(1+\frac{n_{j-1}}{m_{j-1}}+\left(\frac{n_{j-1}}{m_{j-1}}\right)^2+\cdots+\left(\frac{n_{j-1}}{m_{j-1}}\right)^{l_j}\right) \le \frac{1}{n_j}\,\frac{(n_{j-1}/m_{j-1})^{l_j+1}-1}{(n_j/m_{j-1})} - 1$$
$$\le \frac{2}{m_j m_{j-1}}.$$
Summing all the above estimates we get an upper estimate $7m_k^{-1} m_j^{-1}$.
Corollary (6.3.11)[252]: Let $x = m_j n_j^{-1} \sum_{i=1}^{n_j} \bar{d}_{\xi_i}$ such that no two ξ_i's are neighbours. Let $i < j, \left(e_{\eta_p}^*\right)_{p=1}^{n_i}$ be nodes such that $w\left(e_{\eta_p}^*\right) = m_{l_p} \ne m_j$ and $m_{l_p} < m_{l_{p+1}}$ for all $p \le n_i$. Then
$$\sum_{p=1}^{n_i} |e_{\eta_p}^* \bar{P}_{I_p}(x)| \le \frac{14}{m_{p_1}} . \tag{22}$$
Proof. From Lemma (6.3.10) we get
$$\sum_{p=1}^{n_i} \left|e_{\eta_p}^* \bar{P}_{I_p}(x)\right| \le \sum_{p:l_p<j} \left|e_{\eta_p}^* \bar{P}_{I_p}(x)\right| + \sum_{p:l_p>j} |e_{\eta_p}^* \bar{P}_{I_p}(x)|$$
$$\le \sum_{p:l_p<j} \frac{7}{m_p} + \sum_{p:l_p>j} \left(\frac{1}{n_j} + \frac{2m_j}{m_p}\right)$$

$$\leq \sum_{p:l_p<j} \frac{7}{m_p} + \frac{n_i}{n_j} + \sum_{p:l_p>j} \frac{2}{m_{p-1}} \leq \frac{14}{m_{p_1}}.$$

We define the space $\mathcal{X}_{Kus}$. We shall need the following notion from [255].

Definition (6.3.12)[252]: Let $\mathcal{X}$ be a BD-L$_\infty$-subspace of $\ell_\infty(\tilde{\Gamma})$. A subset Γ of $\tilde{\Gamma}$ is called selfdetermined provided $\langle \bar{d}_\gamma^* : \gamma \in \Gamma\rangle = \langle e_\gamma^* : \gamma \in \Gamma\rangle$, where $\left(\bar{d}_\gamma^*\right)_{\gamma\in\Gamma}$ denotes the biorthogonal sequence to the basis $\left(\bar{d}_\gamma\right)_{\gamma\in\Gamma}$ and for $\gamma \in \Gamma, e_\gamma^*$ denotes the element e_γ of $\ell_1(\tilde{\Gamma})$ restricted to $\mathcal{X}$.

Now we proceed to the choice of a self-determined subset Γ of $\bar{\Gamma}$ which will determine the space $\mathcal{X}_{Kus}$. This set will consist of regular and special nodes.

We introduce first the notion which will describe the “freedom” in choosing special nodes. For any $\gamma \in \bar{\Gamma}$ we write $\operatorname{rank}(\operatorname{bd}(e_\gamma^*)) = \{\operatorname{rank} \xi_i, i \in A\}$, where $\operatorname{bd}\left(e_\gamma^*\right) = \sum_{i\in A} d_{\xi_i}^*$.

Definition (6.3.13)[252]: We say that the functionals $e_\gamma^*, e_{\tilde{\gamma}}^*, \gamma, \tilde{\gamma} \in \bar{\Gamma}$, have compatible treeanalyses if

(CT1) $e_\gamma^*, e_{\tilde{\gamma}}^*$ have tree-analyses $(I_t, \varepsilon_t, \eta_t)_{t\in\mathcal{T}}$, $(I_t, \tilde{\varepsilon}_t, \tilde{\eta}_t)_{t\in\mathcal{T}}$ respectively,

(CT2) $w(\eta_t) = w(\tilde{\eta}_t)$ for any $\in \mathcal{T}$,

(CT3) mt-supp $e_{\eta_t}^* = \text{mt} - \text{supp}\, e_{\tilde{\eta}_t}^*$ for any $\in \mathcal{T}$,

(CT4) $\operatorname{rank}(\eta_t) = \operatorname{rank}(\tilde{\eta}_t)$ for any $\in \mathcal{T}$,

(CT5) $\operatorname{rank}(\operatorname{bd}(e_{\eta_t}^*)) = \operatorname{rank}(\operatorname{bd}(e_{\tilde{\eta}_t}^*))$ for any $\in \mathcal{T}$.

For every $\gamma = (q + 1, \xi, m_k, \epsilon, I, e_\eta^*) \in \bar{\Gamma}$ and $x \in \mathcal{X}_{\bar{\Gamma}}$ we set

$$\lambda_{\gamma,x} = \begin{cases} \epsilon e_\eta^*(x) & \text{if } e_\eta^*(x) \neq 0 \\ \epsilon n_k^{-2} & \text{otherwise.} \end{cases} \tag{23}$$

Notice that in the above formula we do not use the projection P_I , which in particular yields that $|\lambda_{\gamma,x}| \leq 1$ for x with $\|x\| \leq 1$. On the other hand, for any x with $rng(x) \subset I$ we have $e_\eta^*(x) = e_\eta^* P_I\,(x)$ and we shall use the above notion in such context.

Definition (6.3.14) (The tree of the special sequences)[252]: We denote by Q the set of all finite sequences of pairs $\{(\zeta_1, \bar{x}_1), \ldots, (\zeta_k, \bar{x}_k)\}$ satisfying the following:

(i) $\zeta_i \in \bar{\Gamma}$ with $\operatorname{rank}(\zeta_i) = q_i \geq min\, rng_{FDD}\, \bar{x}_i$ for $i = 1, \ldots, k$,

(ii) $(\bar{x}_1, \ldots, \bar{x}_k)$ are vectors with rational coefficients with respect to the basis $\left(\bar{d}_\gamma\right)_{\gamma\in\bar{\Gamma}}$, successive with respect to the $FDD\, \left(\bar{M}_q\right)_q$.

We choose a one-to-one function $\sigma : Q \to \mathbb{N}$, called the coding function, so that

$$\sigma(\{(\zeta_1, \bar{x}_1), \ldots, (\zeta_k, \bar{x}_k)\}) > w(\zeta_k)^{-1}\, max\, supp_{FDD}\, \bar{x}_k\ \forall\{(\zeta_1, \bar{x}_1), \ldots, (\zeta_k, \bar{x}_k)\} \in Q. \tag{24}$$

Definition (6.3.15)[252]: A finite sequence $(\zeta_i, \bar{x}_i)_{i=1}^d \in Q$ is called a j-special sequence, $j \in \mathbb{N}$, if $d \leq n_{2j-1}$ and the following conditions are satisfied.

(i) $\zeta_1 = (q_1 + 1, 0, m_{2j-1}, I_1, \epsilon_1, e_{\eta_1}^*)$ and $\zeta_i = (q_i + 1, \zeta_{i-1}, m_{2j-1}, I_i, \epsilon_i, \lambda_i e_{\eta_i}^*)$ for every $i \leq d$,

(ii) $w(\eta_1) = m_{4l-2}^{-1} < n_{2j-1}^{-2}$ and $w(\eta_i) = m_{4\sigma((\zeta_1,\bar{x}_1),\ldots,(\zeta_{i-1},\bar{x}_{i-1}))}^{-1}$ for $i = 2, \ldots, d,$

(iii) if i is odd then $\lambda_i = 1$ and $\|\bar{x}_i\| \leq 1$,

(iv) if i is even then $\epsilon_i = 1, \eta_i$ is chosen to satisfy

$$\mathrm{mt}\left(e^*_{\eta_i}\right) = m^{-1}_{4\sigma\left(\left(\zeta_p,\bar{x}_p\right)_{p=1}^{i-1}\right)} \sum_{r=1}^{n_{4\sigma\left(\left(\zeta_p,\bar{x}_p\right)_{p=1}^{i-1}\right)}} \bar{d}^*_{\beta_r},$$

where $\left(\bar{d}_{\beta_r}\right)_r$ are pairwise non-neighbours. Moreover, we let

$$\bar{x}_i = \frac{m_{4\sigma\left((\zeta_k,\bar{x}_k)_{k=1}^{i-1}\right)}}{n_{4\sigma\left((\zeta_k,\bar{x}_k)_{k=1}^{i-1}\right)}} \sum_{r=1}^{n_{4\sigma\left((\zeta_k,\bar{x}_k)_{k=1}^{i-1}\right)}} \bar{d}_{\beta_r}$$

and $\lambda_i \in \mathrm{Net}_{1,q_i}$ is chosen to satisfy $|\lambda_i - \lambda_{\zeta_{i-1},\bar{x}_{i-1}}| < {}^{1}\!/_{4n^2_{q_{i-1}}}$.

We denote by $\mathcal{U}$ the tree of all special sequences, endowed with the natural ordering "$\sqsubseteq$" of initial segments.

Fix $\Gamma = \cup_q \Gamma_q, \Gamma_q \subset \bar{\Gamma}_q$. A j-special sequence $(\zeta_1,\bar{x}_1)$, with $\zeta_1 = (q+1, 0, m_{2j-1}, I_1, \epsilon, e^*_\eta)$ is called (Γ, j)-special if $\eta \in \Gamma_q$. A j-special sequence $(\zeta_i,\bar{x}_i)_{i=1}^{d}, d \le n_{2j-1}$, with $\zeta_i = (q_i + 1, \zeta_{i-1}, m_{2j-1}, I_i, \epsilon_i, \lambda_i e^*_{\eta_i})$ is called (Γ, j)-special if $\eta_d \in \Gamma_q \setminus \Gamma_{q_d-1}, \zeta_{d-1} \in \Gamma_{q_{d-1}+1}$ and $(\zeta_i,\bar{x}_i)_{i=1}^{d-1}$ is a $(\Gamma_{q_{d-1}}, j)$-special sequence.

Now we are ready to define inductively on $q \in \mathbb{N}$ the families of nodes $\left(\Delta_q\right)_q$ and $\left(\Gamma_q\right)_q$ satisfying $\Delta_q \subset \bar{\Delta}_q$ and $\Gamma_q = \cup_{p=1}^{q} \Delta_p$ for any $q \in \mathbb{N}$.

Set $\Gamma_1 = \Delta_1 = \bar{\Delta}_1$. Fix $q \in \mathbb{N}$ and assume we have defined all objects up to q-th level. The set of regular nodes is defined as

$$\Delta^{reg}_{q+1} = \bigcup_{j=1}^{\lfloor (q+q)/2 \rfloor} \{(q + 1, 0, m_{2j}, I, \epsilon, e^*_\eta) \in \bar{\Delta}_{q+1} : \eta \in \Gamma_q\}$$

$$\cup \bigcup_{1 \le p < q} \bigcup_{j=1}^{\lfloor (q+1)/2 \rfloor} \{(q+1, \xi, m_{2j}, I, \epsilon, e^*_\eta) \in \bar{\Delta}_{q+1} : \xi \in \Delta_p, \eta \in \Gamma_q \setminus \Gamma_p\}$$

Now we define the special nodes, i.e. the nodes compatible to the special sequences defined above (counterparts of special functionals in [98]). We start with the notion of compatibility, which is defined recursively on $\mathrm{age}(\gamma)$.

Definition (6.3.16)[252]: We say that a node $\gamma = (q + 1, 0, m_{2j-1}, I, \epsilon, e^*_\eta) \in \bar{\Delta}_{q+1}$ is compatible with a (Γ_q, j)-special sequence $(\zeta_1,\bar{x}_1)$, where $\zeta_1 = (q + 1, 0, m_{2j-1}, I, \epsilon_1, e^*_{\eta_1})$, if $\eta \in \Gamma_q$ and η, η_1 have compatible tree-analyses.

We say that a node $\gamma = (q + 1, \xi, m_{2j-1}, I, \epsilon, \lambda e^*_\eta) \in \bar{\Delta}_{q+1}$ is compatible with a (Γ_q, j)-special sequence $(\zeta_i,\bar{x}_i)_{i=1}^{age(\gamma)}$, where $\zeta_{age(\gamma)} = (q + 1, \zeta_{age(\gamma)-1}, m_{2j-1}, I, \epsilon_{age(\gamma)}, \lambda_{age(\gamma)} e^*_{\eta_{age(\gamma)}})$, provided

(i) $\eta, \xi \in \Gamma_q$,

(ii) ξ is compatible with the $(\Gamma_{rank(\xi)}, j)$-special sequence $(\zeta_i,\bar{x}_i)_{i=1}^{age(\xi)}$ (recall that $age(\gamma) = age(\xi) + 1$),

(iii) if $age(\gamma)$ is odd then $\lambda = 1 (= \lambda_{age(\gamma)})$ and $\eta, \eta_{age(\gamma)}$ have compatible tree-analyses,

(iv) if $age(\gamma)$ is even then $\epsilon = 1, \eta = \eta_{age(\gamma)}$ and $\lambda \in \mathrm{Net}_{1,q}$ is chosen to satisfy $|\lambda - \lambda_{\xi,\bar{x}_{age(\xi)}}| < {}^{1}\!/_{4n^2_{rank(\xi)}}$.

The set of special nodes is defined as

$$\Delta_{q+1}^{sp} = \bigcup_{j=1}^{\lfloor (q+1)2 \rfloor} \{\gamma = (q+1, 0, m_{2j-1}, I, \epsilon, e_\eta^*) \in \bar{\Delta}_{q+1}$$
$$: \gamma \text{ is compatible with some } (\Gamma q, j) - \text{special sequence } (\zeta_1, \bar{x}_1)\}$$
$$\cup \bigcup_{p=1}^{q} \bigcup_{j=1}^{\lfloor (q+1)/2 \rfloor} \left\{\gamma = \left(q+1, \xi, m_{2j-1}, I, \epsilon, \lambda e_\eta^*\right) \in \bar{\Delta}_{q+1}\right.$$
$$: \gamma \text{ is compatible with some } \left(\Gamma_q, j\right)$$
$$\left. - \text{special sequence } (\zeta_i, \bar{x}_i)_{i=1}^{age(\gamma)}\right\}. \qquad (25)$$

Finally we set

$$\Delta_{q+1} = \Delta_{q+1}^{reg} \cup \Delta_{q+1}^{sp} \text{ and } \Gamma_{q+1} = \Gamma_q \cup \Delta_{q+1}.$$

Obviously $\Delta_q \subset \bar{\Delta}_q$ for any $q \in \mathbb{N}$. We set $\Gamma = \cup_q \Gamma_q$. Following [255] we denote by R the restriction on $\mathcal{X}_{\bar{\Gamma}}$ of the restriction operator $\ell_\infty(\bar{\Gamma}) \to \ell_\infty(\Gamma)$ and for any $q \in \mathbb{N}$ we let $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$ be defined by $i_q(x) = R(\bar{\iota}_q(x))$ for any x. Given any $q \in \mathbb{N}$ we let $M_q = i_{\max \Gamma_q} [\ell_\infty(\Gamma_q)]$.

Proposition (6.3.17)[252]: The set Γ is a self-determined subset of $\bar{\Gamma}$, hence it defines a BD-L_∞-space $\mathcal{X}_{(\Gamma_q, i_q)_q}$.

Moreover, the restriction $R : \mathcal{X}_{\bar{\Gamma}} \to \mathcal{X}_{(\Gamma_q, i_q)_q}$ is a well-defined operator of norm at most 1 inducing the isomorphism between $\mathcal{X}_{(\Gamma_q, i_q)_q}$ and $\mathcal{X}_{\bar{\Gamma}}/Y$, where $Y = \overline{\langle d_\gamma : \gamma \in \bar{\Gamma} \setminus \Gamma \rangle}$.

Proof. According to Proposition 1.5 [255] it is enough to show that for every $\gamma \in \Delta_{q+1}$ the following holds

$$\bar{c}_\gamma^* \in \{e_\gamma^* \circ P_E : \gamma \in \Gamma_q, E \subset \mathbb{N} \cup \{0\}\}$$

This follows readily from the definition of $\bar{c}_\gamma^*$, see (10), using that $\bar{d}_\gamma^* = e_\gamma^* \circ P_{\{rank(\gamma)\}}$.
The second part of Proposition follows by Proposition 1.9 [255].

Definition (6.3.18)[252]: We let $\mathcal{X}_{Kus} = \mathcal{X}_{(\Gamma_q, i_q)_q}$.

We shall use the casual notation, $c_\gamma^*, d_\gamma^*, d_\gamma$ etc for the objects in the space X_{Kus}. We shall use also notation P_I for the projections onto $\langle d_\gamma : \gamma \in I \rangle$, notice here that we can consider I to be an interval in Γ instead of $\bar{\Gamma}$. Henceforth, by $(\gamma_n)_n$ we shall denote the enumeration of Γ instead of the one of $\bar{\Gamma}$.

Where the last term in the square brackets appears if $a \in 2\mathbb{N} + 1$, and with each $e_{\eta_{2i}}^*$ having the mt-part of the following form

$$\mathrm{mt}(e_{\eta_{2i}}^*) = w(\eta_{2i}) \sum_k P^*_{\Delta_{rank(\beta_{k,i})}} e^*_{\beta_{k,i}} = w(\eta_{2i}) \sum_k d^*_{\beta_{k,i}}.$$

Now we make some comments concerning the possible modification of the mt-part of a functional.

From now on we shall work in the space $\mathcal{X}_{Kus}$. We introduce the basic canonical tool, i.e. Rapidly Increasing Sequences and state their properties, in particular the fundamental property of Bourgain–Delbaen spaces in the Argyros–Haydon setting that allows to pass from strictly singular operators to compact ones. As the proofs of all the results stated here follows directly the reasoning of [91], we do not present them here.

Recall that skipped block sequences are defined with respect to the FDD $\left(M_q\right)_{q\in\mathbb{N}}$.

Definition (6.3.19)[252]: Let I be an interval in $\mathbb{N}$ and $(x_k)_{k\in I} \subset \mathcal{X}_{Kus}$ be a skipped block sequence. We shall say that $(x_k)_{k\in I}$ is a Rapidly Increasing Sequence with constant $C > 0$ (C-RIS) if there exists an increasing sequence $(j_k)_{n\in I} \subset \mathbb{N}$ such that
(i) $\|x_k\| \leq C$ for all $k \in I$,
(ii) $\mathrm{rng}_{FDD}\, x_k < j_{k+1}$,
(iii) $|x_k(\gamma)| \leq Cm_i^{-1}$ for all γ with $w(\gamma) = m_i^{-1}$ and $i < j_k$.
Lemma (6.3.20) (Proposition (6.3.24) [91])[252]: Let $(x_k)_{k=1}^{n_{j_0}}$ be a C-RIS and $s \in \mathbb{N}$.
a) If $\gamma \in \Gamma$ and $w(\gamma) = m_i^{-1}$ then

$$\left|e_\gamma^* P_{(s,+\infty)}\left(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} x_k\right)\right| \leq \begin{cases} 16Cm_i^{-1}m_{j_0}^{-1} & \text{if } i < j_0 \\ 5Cn_{j_0}^{-1} + 6Cm_i^{-1} & \text{if } i \geq j_0. \end{cases} \tag{26}$$

In particular for $i > j_0$ we have

$$\left|e_\gamma^*\left(\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} x_k\right)\right| \leq 10Cm_{j_0}^{-2} \tag{27}$$

and also

$$\left\|\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} x_k\right\| \leq 10Cm_{j_0}^{-1}. \tag{28}$$

b) If $\lambda_k, 1 \leq k \leq n_{j_0}$ are scalars with $|\lambda_k| \leq 1$, satisfying the property

$$\left|e_\gamma^*\left(\sum_{k\in J} \lambda_k x_k\right)\right| \leq C \max_{k\in J} |\lambda_k|$$

for every $\gamma \in \Gamma$ with $w(\gamma) = m_{j_0}^{-1}$ and every interval $J \subset \{1,\dots,n_{j_0}\}$ then we have

$$\left\|\frac{1}{n_{j_0}}\sum_{k=1}^{n_{j_0}} \lambda_k x_k\right\| \leq \frac{10C}{m_{j_0}^2}.$$

The following result is proved in a manner similar to how Lemma (6.3.10) is proved.
Corollary (6.3.21)[252]: Let $i < j \in \mathbb{N}, (x_k)_{k=1}^{n_j}$ be a C-RIS, $x = \frac{m_j}{n_j}\sum_{k=1}^{n_j} x_k$ and $\left(e_{\eta_p}^*\right)_{p=1}^{n_i}$ be nodes such that $w(e_{\eta_p}^*) = m_{l_p}^{-1}$ and $m_{l_p} \neq m_j\,, m_{l_p} < m_{l_{p+1}}$ for all $p \leq n_i$. Then for every choice of intervals $I_p, p \leq n_i$, we have

$$\sum_{p=1}^{n_i} |e_{\eta_p}^*(P_{I_p}x)| \leq 64C/m_{p_1}. \tag{29}$$

Lemma (6.3.22) (Corollary (6.3.41) [91])[252]: For every block subspace $Y \subset \mathcal{X}_{Kus}, C > 2$ and every interval $J \subset \mathbb{N}$ there exists a normalised C-RIS $(x_k)_{k\in J}$ in . Moreover, for any $\varepsilon > 0$ and $C > 2$ the sequence $(x_k)_{k\in J}$ can be chosen to satisfy $|d_\gamma^*(x_k)| < \varepsilon$ for any $k \in J$ and $\gamma \in \Gamma$.
Notice that if $x \in \bigoplus_{n=1}^q M_n$ with q minimal then there exists a unique $u \in \ell_\infty(\Gamma_q)$ such that $i_q(u) = x$. The local support of x is defined to be the set $\{\gamma \in \Gamma_q \,|u(\gamma) \neq 0\}$. Next results are again quoted from [91].

Lemma (6.3.23) (Lemma (6.3.26) [91])[252]: Let $\gamma \in \Gamma$ be of weight m_h^{-1} and assume that $w(\xi) \neq m_h^{-1}$ for all ξ in the local support of x. Then $|x(\gamma)| \leq 4m_h^{-1} \|x\|$.
We recall the two classes of block sequences, characterised by the weights of the elements of the local support.
Definition (6.3.24) (Definition (6.3.27) [91])[252]: We say that a block sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{X}_{Kus}$ has bounded local weight if there exists some j_1 such that $w(\gamma) \geq m_{j_1}^{-1}$ for all γ in the local support of x_k, and all values of k.
We say that a block sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{X}_{Kus}$ has rapidly increasing local weight if, for each k and each γ in the local support of x_{k+1}, we have $w(\gamma) < m_{i_k}^{-1}$ where $i_k = \max \operatorname{rng}_{FDD} x_k$.
Proposition (6.3.25) ([91])[252]: Let $(x_k)_{k\in\mathbb{N}} \subset \mathcal{X}_{Kus}$ be a bounded block sequence. If either (x_k) has bounded local weight, or (x_k) has rapidly increasing local weight, then the sequence (x_k) is a RIS.
Corollary (6.3.26) ([91])[252]: Let Y be any Banach space and $T : \mathcal{X}_{Kus} \to Y$ be a bounded linear operator. If $\|Tx_k\| \to 0$ for every RIS $(x_k)_k$ in $\mathcal{X}_{Kus}$ then $\|Tx_k\| \to 0$ for every bounded block sequence (x_k) in $\mathcal{X}_{Kus}$.
Corollary (6.3.27) ([91])[252]: The basis $\left(d_{\gamma_n}^*\right)_n$ is shrinking. It follows that the dual space to $\mathcal{X}_{Kus}$ is isomorphic to $\ell_1(\Gamma)$.

We introduce the classical tools in the study of spaces defined with the use of saturated norms.
Lemma (6.3.28)[252]: a) Let $j \in \mathbb{N}$ and $k \leq n_{2j}$. Let also $(x_k)_k \subset \mathcal{X}_{Kus}$ be a normalised skipped block sequence such that $\operatorname{rng}_{FDD}(x_k) = (p_{k-1}, p_k]$ for some strictly increasing (p_k) with $p_1 \geq 2j - 1$. Then there exists a node $\gamma \in \Gamma$ such that

$$e_\gamma^* = \sum_{k=1}^{n_{2j}} d_{\xi_k}^* + m_{2j}^{-1} \sum_{k=1}^{n_{2j}} \varepsilon_k e_{\eta_k}^* P_{I_k}$$

with the following properties
(i) $\operatorname{rank}(\xi_k) = p_k + 1$ for each k,
(ii) $\varepsilon_k e_{\eta_k}^* P_{I_k}(x_k) \geq {}^1/_2$ and $\eta_k \in \Gamma_{p_k} \setminus \Gamma_{p_{k-1}}$ for each k,
(iii) $e_\gamma^*\left(\sum_{k=1}^{n_{2j}} x_k\right) \geq \frac{n_{2j}}{2m_{2j}}$.

b) Let $\left(d_{\xi_i}\right)_{i=1}^{n_{2j}}$ be a finite subsequence of the basis such that $\operatorname{rank}(\xi_i) + 1 < \operatorname{rank}(\xi_{i+1})$ for every i and $\operatorname{rank}(\xi_1) \geq 2j - 1$.
Then the node

$$e_\xi^* = \sum_{i=1}^{n_{2j}} d_{\zeta_i}^* + m_{2j}^{-1} \sum_{i=1}^{n_{2j}} d_{\xi_i}^* \tag{30}$$

with $\operatorname{rank}(\zeta_i) = \operatorname{rank}(\xi_i) + 1$ is a regular node and $e_\xi^*\left(\sum_{i=1}^{n_{2j}} d_\xi\right) = \frac{n_{2j}}{m_{2j}}$.
Proof. a) (see [91], Proposition 4.8) Let $x_k = i_k(u_k)$ where $u_k \in \Gamma_{p_k} \setminus \Gamma_{p_{k-1}}$ is the restriction of x_k on Γ_{p_k} . Since

$$2\|u_k\| \geq \left\|i_{p_k}(u_k)\right\| = \|x_k\| = 1$$

we can choose $\eta_k \in \Gamma_{p_k} \setminus \Gamma_{p_{k-1}}$ such that $|e_{\eta_k}^*(u_k)| \geq {}^1/_2$. Setting $I_k = \operatorname{rng}_{FDD}(x_k) = \cup_{i=p_{k-1}+1}^{p_k} \Delta_i$, choose $\varepsilon_k \in \{-1, 1\}$ such that

$$|e_{\eta_k}^* P_{I_k}(x_k)| = \varepsilon_k e_{\eta_k}^* P_{I_k}(x_k) = \varepsilon_k e_{\eta_k}^*(u_k) \geq {}^1\!/_2. \qquad (31)$$

The nodes $\gamma_k = (p_k + 1, \gamma_{k-1}, m_{2j}, I_k, \varepsilon_k, e_{\eta_k}^*), \gamma_0 = 0, k = 1, \ldots, n_{2j}$ give the node $\gamma = \gamma_{n_{2j}}$ with the properties (i)–(iii).

b) Take the nodes $\zeta_i = (\mathrm{rank}(\xi_i) + 1, \zeta_{i-1}, m_{2j}, I_i, 1, e_{\xi_i}^*), \zeta_0 = 0$, where $I_i = \Delta_{\mathrm{rank}(\xi_i)}$.

Definition (6.3.29)[252]: Fix $j \in \mathbb{N}, C \geq 1$ with $n_{2j-1} \geq 200C$ and let $(\gamma_k, \bar{x}_k)_{k=1}^d$ be a (Γ, j)-special sequence.

A sequence $(\gamma_k, x_k)_{k=1}^d, d \leq n_{2j-1}$, with $x_k \in \mathcal{X}_{Kus}$ and $\gamma_k = (q_k + 1, \gamma_{k-1}, m_{2j-1}, I_k, 1, e_{\eta_k}^*)$ for each k, where $\gamma_0 = 0, q_1 \geq 4j_1 - 2, 2^{-q_1} \leq 1/4n_{2j-1}^2$, is called a j-dependent sequence with a constant C of length d with respect to $(\gamma_k, \bar{x}_k)_{k=1}^d$ if the following conditions are satisfied.

(i) if k is even then $x_k = R\bar{x}_k, \mathrm{rng}(x_k) = I_k$,

(ii) if k is odd then $x_k = \frac{c_k m_{l_k}}{n_{l_k}} \sum_{l=1}^{n_{l_k}} x_{k,l}$, where $(x_{k,l})_l$ is a normalised skipped block sequence which is a C-RIS of length $n_{l_k}, m_{l_k} = w(\eta_k), m_{l_1} \geq n_{2j-1}^2, \|x_k\| = {}^1\!/_2, \mathrm{rng}(x_k) \subset I_k$ and $e_{\eta_k}^*(x_k) \geq {}^1\!/_{40C}$,

(iii) $|e_\gamma^*(\bar{x}_k) - e_\gamma^*(x_k)| < {}^1\!/_{4n_{q_k}^2}$ for every $\gamma \in \Gamma$ and every k,

(iv) $(\gamma_k, x_k)_{k=1}^{d-1}$ is j-dependent of length $d - 1$ with respect to the (Γ, j)-special sequence $(\gamma_k, \bar{x}_k)_{k=1}^{d-1}$.

We say that a sequence $(\gamma_k, x_k)_{k=1}^d$ is a j-dependent sequence of length d, if it is j-dependent with respect to some (Γ, j)-special sequence.

Remark (6.3.30)[252]: Take $(x_{k,l})_l$ as in (ii) of Definition (6.3.29) with max $\mathrm{rng}_{FDD}(x_k) \geq 2l_k - 1$ for each $k \in \mathbb{N}$. Then Lemmas (6.3.20)a) and (6.3.28)a) yield that there is a node $\eta_k \in \Gamma$ such that

$$\frac{1}{2} \leq e_{\eta_k}^*\left(\frac{m_{l_k}}{n_{l_k}} \sum_{l=1}^{n_{l_k}} x_{k,l}\right) \leq \left\| \frac{m_{l_k}}{n_{l_k}} \sum_{l=1}^{n_{l_k}} x_{k,l} \right\| \leq 10C. \qquad (32)$$

Therefore c_k in Definition (6.3.29) satisfies ${}^1\!/_{20C} \leq c_k \leq 1$.

Moreover, the last condition in the property (ii) of Definition (6.3.29), i.e. $e_{\eta_k}^*(x_k) \geq {}^1\!/_{40C}$, follows from (32) using the lower bound of c_k.

Lemma (6.3.31)[252]: Let $(z_k)_k$ be a normalised block sequence in $\mathcal{X}_{Kus}$ and $(d_{\xi_n})_{n \in M}$ be a subsequence of the basis. Then for every $j \in \mathbb{N}$ there exists a j-dependent sequence of length $n_{2j-1}, (\gamma_i, x_i)_{i \leq n_{2j-1}}$, such that $x_{2i-1} \in \langle z_k : k \in \mathbb{N} \rangle$ and $x_{2i} \in \langle d_{\xi_n} : n \in M \rangle$.

Proof. Passing to a further subsequence we may assume that

$$d_{\xi_n} \text{ are pairwise non-neighbours and } \mathrm{rank}(\xi_n) + 1 < \mathrm{rank}(\xi_{n+1}). \quad (33)$$

Let j_1 be such that $m_{4j_1-2} > n_{2j-1}^2$ and choose q_1 big enough to guarantee that $4j_1 - 2 < q_1$ and $2^{-q_1} \leq {}^1\!/_{4n_{2j-1}^2}$.

Let $(x_{1,k})_{k=1}^{n_{4j_1-2}}$ be a normalised skipped block sequence of $\langle z_l : l \geq q \rangle$ which is a C-RIS. Setting

$$x_1 = \frac{c_1 m_{4j_1-2}}{n_{4j_1-2}} \sum_{k=1}^{n_{4j_1-2}} x_{1,k} \qquad \text{with } \|x_1\| = {}^1\!/_2$$

from Remark (6.3.30) we get ${}^1\!/_{20C} \le c_1 \le 2$ and that there exists a node $\eta_1 \in \Gamma$ with $w(\eta_1) = m_{4j_1-2}^{-1}$ such that

$$e_{\eta_1}^* P_{I_1}(x_1) \ge \frac{1}{40C},$$

where $I_1 = \cup \{\Delta_p : p \in \mathrm{rng}_{FDD}(x_1)\}$.

Using that R is a quotient operator of norm 1 take a block $\bar{y}_1 \in \mathcal{X}_{\bar{\Gamma}}$ such that $x_1 = R(\bar{y}_1)$ and $\|\bar{y}_1\| \le 1$. Then choose a vector $\bar{x}_1$ with rational coefficients in the unit ball of $\langle \bar{d}_\gamma : \gamma \in \bar{\Gamma}_{q_1}\rangle$ such that $\|\bar{x}_1 - \bar{y}_1\|_{\mathcal{X}_{\bar{\Gamma}}} \le {}^1\!/_{4n_{q_1}^2}$.

Note that $R(\bar{x}_1) = R(\bar{x}_1 - \bar{y}_1) + R(\bar{y}_1) = R(\bar{x}_1 - \bar{y}_1) + x_1$ and hence for every $\gamma \in \Gamma$,

$$\begin{aligned} \left|e_\gamma^*(\bar{x}_1) - e_\gamma^*(x_1)\right| &= \left|e_\gamma^* R(\bar{x}_1) - e_\gamma^* R(x_1)\right| \\ &\le \left\|e_\gamma^* \circ R\right\| \|\bar{x}_1 - \bar{y}_1\|_{\mathcal{X}_{\bar{\Gamma}}} \le {}^1\!/_{4n_{q_1}^2}. \end{aligned} \tag{34}$$

We take γ_1 to be the node

$$\gamma_1 = \left(q_1 + 1, 0, m_{2j-1}^{-1}, I_1, 1, e_{\eta_1}^*\right).$$

From the above we get that (γ_1, x_1) is a j-dependent couple of length 1 with respect to the (Γ, j)-special sequence $(\gamma_1, \bar{x}_1)$.

Set $j_2 = \sigma(\gamma_1, \bar{x}_1)$ and choose $x_2, e_{\eta_2}^*$ such that

$$x_2 = m_{4j_2} n_{4j_2}^{-1} \sum_{k \in F_2} d_{\xi_{2,k}} \in \mathcal{X}_{Kus} \quad \text{and } \mathrm{mt}(e_{\eta_2}^*) = m_{4j_2}^{-1} \sum_{k \in F_2} d_{\xi_{2,k}}^*$$

where $|F_2| = n_{4j_2}$ and $q_1 + 2 < \min \mathrm{rng}_{FDD}(x_2)$. Such a node exists by Lemma (6.3.28)(b) since $\mathrm{rank}(\xi_n) + 1 < \mathrm{rank}(\xi_{n+1})$. We also take the node

$$\gamma_2 = (q_2 + 1, \gamma_1, m_{2j-1}, I_2, 1, \lambda_2 e_{\eta_2}^*) \in \Gamma$$

where $I_2 = [p_2, q_2]$ is the range of x_2 with respect to the basis and $\lambda_2 \in \mathrm{Net}_{1,q_1}$ is chosen such that

$$|\lambda_2 - e_{\eta_1}^*(\bar{x}_1)| \le {}^1\!/_{4n_{q_1}^2}.$$

From the above equation and (34) we get

$$\left|\lambda_2 - e_{\eta_1}^*(x_1)\right| \le \frac{1}{2n_{q_1}^2} \Rightarrow \lambda_2 \ge e_{\eta_1}^*(x_1) - \frac{1}{2n_{q_1}^2} \ge \frac{c_1}{2} - \frac{1}{2n_{q_1}^2} \ge \frac{1}{45C}.$$

Pick $\bar{x}_2$ to be the corresponding average of $\left(\bar{d}_{\xi_{2,k}}\right)_{k \in F_2}$. It follows that $x_2 = R\bar{x}_2$ (recall that $d_\gamma = R\bar{d}_\gamma$ for each $\gamma \in \Gamma$) and $\bar{x}_1 < \bar{x}_2$. Then we get that $(\gamma_i, x_i)_{i=1}^2$ is j-dependent of length 2 with respect to the (Γ, j)-special sequence $(\gamma_i, \bar{x}_i)_{i=1}^2$.

Set $j_3 = \sigma(\gamma_i, \bar{x}_i)_{i=1}^2$. We continue to choose $x_3, e_{\gamma_3}^*, x_4, e_{\gamma_4}^*$ in the same way we have chosen $x_1, e_{\gamma_1}^*, x_2, e_{\gamma_2}^*$ taking care that x_1, x_2, x_3, x_4 is a skipped block sequence (with respect to the FDD) and repeat the procedure obtaining the desired dependent sequence.

Notice that for a dependent sequence $(\gamma_i, x_i)_{i \le n_{2j-1}}$ with a constant C we have $\left\|\frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}} x_i\right\| \ge {}^1\!/_{45C}$. Indeed, consider the functional $e_{\zeta_{n_{2j-1}}}^*$ determined by the nodes $(\gamma_i)_{i=1}^{n_{2j-1}}$, i.e. of the form

$$e^*_{\zeta_{n_{2j-1}}} = \sum_{i=1}^{n_{2j-1}} d^*_{\gamma_i} + m^{-1}_{2j-1} \sum_{i=1}^{n_{2j-1}/2} \left(e^*_{\eta_{2i-1}} P_{I_{2i-1}} + \lambda_{2i} e^*_{\eta_{2i}} P_{I_{2i}}\right),$$

and notice that

$$e^*_{\zeta_{n_{2j-1}}}\left(\frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}} x_i\right) \geq \frac{1}{n_{2j-1}}\left(\sum_{i=1}^{n_{2j-1}/2} e^*_{\eta_{2i-1}} P_{I_{2i-1}}(x_{2i-1}) + \lambda_{2i} e^*_{\eta_{2i}}(x_{2i})\right)$$

$$\geq \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} \left(\frac{c_{2i-1}}{2} + \frac{c_{2i-1}}{2} - \frac{1}{2n^2_{q_{2i-1}}}\right) \geq \frac{1}{45C},$$

using that $c_{2i-1} \geq {}^1/_{20C}$.

Lemma (6.3.32)[252]: Let $(\gamma_i, x_i)_{i\leq n_{2j-1}}$ be a j-dependent sequence. Then

$$\left\|\frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}} (-1)^{i+1} x_i\right\| \leq \frac{250}{m^2_{2j-1}}.$$

Proof. Let J be an interval of $\{1,\ldots,n_{2j-1}\}$ and $z = \sum_{i\in J} (-1)^{i+1} x_i$. We shall verify the assumption (b) in Lemma (6.3.20) for $j_0 = 2j - 1$.

Let $(\gamma_k, \bar{x}_k)_{k=1}^{n_{2j-1}}$ be the special sequence associated with the dependent sequence $(\gamma_k, x_k)_{k=1}^{n_{2j-1}}, \gamma_k = (q_k + 1, \gamma_{k-1}, m_{2j-1}, I_k, \epsilon_k, \lambda_k e^*_{\eta_k})$ for each k, where $\gamma_0 = 0$.

Consider a node β with evaluation analysis

$$e^*_\beta = \sum_{i=1}^{n_{2j-1}} d^*_{\xi_i} + m^{-1}_{2j-1} \sum_{i=1}^{n_{2j-1}/2} \left(\tilde{\epsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{\tilde{I}_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\tilde{\delta}_{2i}} P_{\tilde{I}_{2i}}\right)$$

which is produced from a (Γ, j)-special sequence $(\zeta_k, \bar{z}_k)_{k\leq n_{2j-1}}$. Let

$$k_0 = \min\{k \leq n_{2j-1} : (\gamma_k, \bar{x}_k) \neq (\zeta_k, \bar{z}_k)\}$$

if such a k exists. We estimate separately $|e^*_{\beta_{k_0}-1}(z)|$ and $\left|\left(e^*_\beta - e^*_{\beta_{k_0}-1}\right)(z)\right|$.

We start with $|e^*_{\beta_{k_0}-1}(z)|$. Notice that $e^*_{\beta_{k_0}-1}$, if $k_0 > 1$, has the following evaluation analysis

$$e^*_{\beta_{k_0-1}} = \sum_{i=1}^{k_0-1} d^*_{\xi_i} + m^{-1}_{2j-1} \sum_{i=1}^{\lfloor(k_0-1)/2\rfloor} (\tilde{\epsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{I_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\eta_{2i}} P_{I_{2i}})$$
$$+ [\tilde{\epsilon}_{k_0-1} e^*_{\tilde{\delta}_{k_0-1}} P_{I_{k_0-1}}],$$

where $e^*_{\tilde{\delta}_{2i-1}}$ and $e^*_{\eta_{2i-1}}$ have compatible tree-analyses and the last term in square brackets appears if $k_0 - 1$ is odd. By the definition of nodes we have $\mathrm{rank}(\xi_i) = \mathrm{rank}(\gamma_i) \in (\max \mathrm{rng}_{FDD}(x_i), \min \mathrm{rng}_{FDD}(x_{i+1}))$ for every $i < k_0$. Therefore

$$\left(\sum_{i=1}^{k_0-1} d^*_{\xi_i}\right) \sum_i (-1)^{i+1} x_i = 0. \tag{35}$$

We partition the indices $P = \{1, 2, \ldots, \lfloor(k_0 - 1)/2\rfloor\}$ into the sets $A = \{i \in P : e^*_{\tilde{\delta}_{2i-1}} P_{I_{2i-1}}(\bar{x}_{2i-1}) \neq 0\}$ and its complement B.

For every $i \in A$ from the choice of $\tilde{\lambda}_{2i}$, the fact that $\operatorname{rng}(x_{2i-1}) \subset I_{2i-1}$ and (3) of Definition (6.3.29) we have

$$\left|\tilde{\lambda}_{2i} - \tilde{\varepsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}}(\bar{x}_{2i-1})\right| \leq \frac{1}{4n^2_{2j-1}} \quad \text{and} \tag{36}$$

$$|e^*_{\tilde{\delta}_{2i-1}}(\bar{x}_{2i-1}) - e^*_{\tilde{\delta}_{2i-1}} P_{I_{2i-1}}(x_{2i-1})| = |e^*_{\tilde{\delta}_{2i-1}}(\bar{x}_{2i-1}) - e^*_{\tilde{\delta}_{2i-1}}(x_{2i-1})| \leq \frac{1}{4n^2_{2j-1}}.$$

It follows that

$$|\tilde{\varepsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{I_{2i-1}}(x_{2i-1}) + \tilde{\lambda}_{2i} e^*_{\eta_{2i}} P_{I_{2i}}(-x_{2i})|$$
$$= \left|\tilde{\varepsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{I_{2i-1}}(x_{2i-1}) - \tilde{\lambda}_{2i}\right| \leq \frac{1}{2n^2_{2j-1}} \quad \text{by (6.7).} \tag{37}$$

Similarly for every $i \in B$,

$$|\tilde{\epsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{I_{2i-1}}(x_{2i-1}) + \tilde{\lambda}_{2i} e^*_{\eta_{2i}} P_{I_{2i}}(-x_{2i})|$$
$$= \left|\tilde{\epsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{I_{2i-1}}(x_{2i-1}) - \tilde{\lambda}_{2i}\right| \leq \frac{1}{2n^2_{2j-1}}. \tag{38}$$

For an interval $J = [l, m]$ using that $\|x_{2i-1}\| = 1/2$, $\|x_{2i}\| \leq 7$ (by Lemma (6.3.10)) and inequalities (35), (38) we obtain

$$|e^*_{\beta_{k_0-1}}\left(\sum_{i \in J} (-1)^{i+1} x_i\right)| \leq 10.$$

Now we proceed to estimate $|(e^*_\beta - e^*_{\beta_{k_0-1}})(z)|$.

Observe that as x_{2l-1} is a weighted average of a normalised C-RIS of length $n_{j_{2l-1}}$ we have

$$\left|\sum_{i=k_0}^{n_{2j-1}} d^*_{\xi_i}(x_{2l-1})\right| \leq 3n_{2j-1} c_{2i-1} C \frac{m_{j_{2l-1}}}{n_{j_{2l-1}}} \leq 2m^{-2}_{j_{2l-1}} < n^{-3}_{2j-1} \tag{39}$$

The same inequality holds also for the averages of the basis i.e.

$$\left|\sum_{i=k_0}^{n_{2j-1}} d^*_{\xi_i}(x_{2l})\right| \leq n_{2j-1} \frac{m_{j_{2l}}}{n_{j_{2l}}} \leq m^{-3}_{j_{2l}} < n^{-3}_{2j-1} \quad \forall l. \tag{40}$$

We shall distinguish the cases when k_0 is odd or even. Assume first that $k_0 = 2i_0 - 1$ for some i_0.

Then for every $i < i_0$ and every $k > k_0$,

$$(\tilde{\epsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{\tilde{I}_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\tilde{\delta}_{2i}} P_{\tilde{I}_{2i}})(x_k) = 0.$$

From the injectivity of σ it follows that $w(e^*_{\tilde{\delta}_{2i-1}}), w(e^*_{\tilde{\delta}_{2i}}) \notin \left\{w\left(e^*_{\eta_{i'}}\right)\middle| i' > i_0\right\}$ for every $i > i_0$. Hence by Corollary (6.3.21), using that $|\tilde{\lambda}_{2i}| \leq 1$ and $c_k \leq 2$, we get for every odd $k > k_0$ the following

$$\left|\sum_{i \geq i_0}^{n_{2j-1}/2} \left(\tilde{\epsilon}_{2i-1} e^*_{\tilde{\delta}_{2i-1}} P_{\tilde{I}_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\tilde{\delta}_{2i}} P_{\tilde{I}_{2i}}\right)(x_k)\right|$$
$$\leq 64 c_k C\, w(\delta_1) \leq 128 C n^{-2}_{2j-1}. \tag{41}$$

Also from Corollary (6.3.11) we obtain for every even $k > k_0$ the following

$$|\sum_{i\geq i_0}^{n_{2j-1}/2} (\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P_{\tilde{I}_{2i-1}}+\tilde{\lambda}_{2i}e^*_{\tilde{\delta}_{2i}}P_{\tilde{I}_{2i}})(x_k)| \leq 14n_{2j-1}^{-2}. \quad (42)$$

For x_{k_0} we also obtain the following

$$\left|\sum_{i\geq i_0}^{n_{2j-1}/2} \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P_{\tilde{I}_{2i-1}}+\tilde{\lambda}_{2i}e^*_{\tilde{\delta}_{2i}}P_{\tilde{I}_{2i}}\right)(x_{k_0})\right| \quad (43)$$

$$\leq |e^*_{\tilde{\delta}_{k_0}}P_{\tilde{I}_{k_0}}(x_{k_0})|$$

$$+|\left(\tilde{\lambda}_{k_0+1}e^*_{\tilde{\delta}_{k_0+1}}P_{\tilde{I}_{k_0+1}}+\sum_{i\geq i_0}^{n_{2j-1}/2}\left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P_{\tilde{I}_{2i-1}}+\tilde{\lambda}_{2i}e^*_{\tilde{\delta}_{2i}}P_{\tilde{I}_{2i}}\right)\right)(x_{k_0})|$$

$$\leq 4 + 128Cn_{2j-1}^{-2},$$

using that $\|x_{k_0}\| \leq 1$ and $\|e^*_\gamma \circ P_I\| \leq \|P_I\| \leq 4$ while for the second term we get the upper bound as in (41).

The case where k_0 is even is similar, except that $|e^*_{\tilde{\delta}_{k_0}}P_{\tilde{I}_{k_0}}(x_{k_0})| \leq 7$.

Splitting J to $J_1 = J \cap [1,i_0], J_2 = J \cap (i_0, n_{2j-1})$ and considering the cases when min J_1 is odd or even we get $|\left(e^*_\beta - e^*_{\beta_{k_0-1}}\right)\left(\sum_{i\in J} (-1)^{i+1}x_i\right)| \leq 15$, using that $n_{2j+1} > 200C$.

The lemmas above imply the following.

Proposition (6.3.33)[252]: Let $M \subset \mathbb{N}$ be infinite and $(y_k)_k \subset \mathcal{X}_{Kus}$ be a normalised block sequence. Then

$$\inf\{\|x-y\| : x \in \langle d_{\gamma_n} : n \in M\rangle, y \in \langle y_k : k \in \mathbb{N}\rangle, \|x\| = \|y\| = 1\} = 0.$$

We show that the space $\mathcal{X}_{Kus}$ has the scalar-plus-compact property.

Proposition (6.3.34)[252]: Let $T : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator and $\left(d_{\gamma_n}\right)_{n\in M}$ be a subsequence of the basis. Then

$$\lim_{M\ni n\to+\infty} \operatorname{dist}(Td_{\gamma_n}, \mathbb{R}d_{\gamma_n}) = 0.$$

Proof. Assume that $\operatorname{dist}(Td_{\gamma_n}, \mathbb{R}d_{\gamma_n}) > 4\delta$ for infinitely many $n \in M$ and some $\delta > 0$. By Corollary (6.3.27) and Lemma (6.3.9) passing to a further subsequence and admitting a small perturbation we may assume that

(P1) $\left(Td_{\gamma_n}\right)_{n\in M}$ is a skipped block sequence and setting R_n to be the minimal interval containing $\operatorname{rng}(Td_{\gamma_n})$ and $\{n\}$ we have

$$\max \operatorname{rank}(R_n) + 2 < \min \operatorname{rank}(R_{n+1}),$$

(P2) no two elements of $\left(d_{\gamma_n}\right)_{n\in M}$ are neighbours.

By the assumption that $\operatorname{dist}(Td_{\gamma_n}, \mathbb{R}d_{\gamma_n}) > 4\delta$ it follows that either

$$\|P_{n-1}Td_{\gamma_n}\| \geq 2\delta \quad \text{or} \quad \|(I - P_n)Td_{\gamma_n}\| \geq 2\delta$$

(recall that P_m denotes the canonical projection onto $\langle d_{\gamma_i} : i \leq m\rangle, m \in \mathbb{N}$).

Passing to a further subsequence we may assume that one of the two alternatives holds for any $n \in \mathbb{N}$. Let

$$q_n = \begin{cases} \max \operatorname{rank}\left(P_{n-1}Td_{\gamma_n}\right) & \text{in the first case} \\ \max \operatorname{rank}\left((I-P_n)Td_{\gamma_n}\right) & \text{in the second case.} \end{cases}$$

In the first case we take $I_n = [\min \operatorname{rng}(Td_{\gamma_n}), n-1]$. Also $P_{n-1}Td_{\gamma_n} = i_{q_n}(u_n)$ where $u_n = r_{q_n}(P_{n-1}Td_{\gamma_n})$ and hence we may choose $\epsilon_n \in \{-1,1\}$ and $\eta_n \in \Gamma_{q_n} \setminus \Gamma_{max\, rank(R_{n-1})+1}$ such that

$$\epsilon_n e^*_{\eta_n} P_{I_n}(Td_{\gamma_n}) = \epsilon_n e^*_{\eta_n}(P_{n-1}Td_{\gamma_n}) = \epsilon_n e^*_{\eta_n}(u_n) \geq \delta \tag{44}$$

using that $2\delta \leq \|i_{q_n}(u_n)\| \leq 2\|u_n\|$.

In the second case we take $I_n = [n+1, \max \operatorname{rng}(Td\_\gamma_n)]$. Also since $(I - P_n)Td_{\gamma_n} = i_{q_n}(u_n)$ where $u_n = r_{q_n}((I-P_n)Td_{\gamma_n})$ we get $\epsilon_n \in \{-1,1\}, \eta_n \in \Gamma_{q_n} \setminus \Gamma_{\max \operatorname{rng} R_{n-1}+1}$ such that

$$\epsilon_n e^*_{\eta_n} P\_I_n(Td_{\gamma_n}) = \epsilon_n e^*_{\eta_n}((I-P_n)Td_{\gamma_n}) = \epsilon_n e^*_{\eta_n}(u_n) \geq \delta. \tag{45}$$

Given any $j \in \mathbb{N}$ we shall build a vector y with $\|Ty\| \geq \delta/28m_{2j-1}$ and $\|y\| \leq 420/m^2_{2j-1}$ which for sufficiently big j yields a contradiction.

Assume the first case holds. The second case will follow analogously. Notice that by (P1) for any $i \in \mathbb{N}$ and $A \subset M$ with $\#A = n_{2i}$ and $\max \operatorname{rank}(R_{\min A}) \geq 2i-1$ there is a functional e^*_ψ associated to a regular node of the form

$$e^*_\psi = \sum_{n\in A} d^*_{\xi_n} + \frac{1}{m_{2i}} \sum_{n\in A} \epsilon_n e^*_{\eta_n} P_{I_n}$$

with $\operatorname{rank}(\xi_n) = \max \operatorname{rank}(R_n) + 1$ for each $n \in A$. Let $= m_{2i}n^{-1}_{2i} \sum_{n\in A} d_{\gamma_n}$.

It follows that

$$\begin{aligned}\|Tx\| \geq e^*_\psi(Tx) &= \left(\sum_{n\in A} d^*_{\xi_n} + \frac{1}{m_{2i}} \sum_{n\in A} \epsilon_n e^*_{\eta_n} P_{I_n}\right)\left(\frac{m_{2i}}{n_{2i}} \sum_{n\in A} Td_{\gamma_n}\right) \\ &= m_{2i}n^{-1}_{2i} \sum_{n\in A} d^*_{\xi_n}(Td_{\gamma_n}) + \frac{1}{n_{2i}} \sum_{n\in A} \epsilon_n e^*_{\eta_n} P_{I_n}(Td_{\gamma_n}) \\ &= \frac{1}{n_{2i}} \sum_{n\in A} \epsilon_n e^*_{\eta_n} P_{k_{n-1}}(Td_{\gamma_n}) \geq \delta.\end{aligned}$$

Fix $j \in \mathbb{N}$ and choose inductively, as in Lemma (6.3.31), a j-dependent sequence $(\zeta_i, x_i), \zeta_i = (q_i + 1, \zeta_{i-1}, m_{2j-1}, J_i, 1, \psi_i), i = 1, \ldots, n_{2j-1}$, with $\zeta_0 = 0$, with respect to a (Γ, j)-special sequence $(\zeta_i, \bar{x}_i)$, so that it satisfies for any i the following

$$e^*_{\psi_{2i-1}} = \sum_{n\in A_i} d^*_{\xi_n} + \frac{1}{m_{j_{2i-1}}} \sum_{n\in A_i} \epsilon_n e^*_{\eta_n} P_{I_n},$$

$$x_{2i-1} = \frac{c_{2i-1}m_{j2i-1}}{n_{j_{2i-1}}} \sum_{n\in A_i} d_{\gamma_n}, \|x_{2i-1}\| = 1/2$$

with $\operatorname{rank}(\xi_n) = \max \operatorname{rank}(R_n) + 1$ for each $n \in \cup_i A_i$. Lemma (6.3.10) yields that $1/14 \leq c_{2i-1} \leq 1$. Recall that by definition each vector $\bar{x}_{2i-1}$ satisfies

$$|e^*_\gamma(\bar{x}_{2i-1}) - e^*_\gamma(x_{2i-1})| \leq 4n^{-2}_{q_{2i-1}} \quad \forall \gamma \in \Gamma.$$

For any i let $J_{2i-1} = \operatorname{rng}(e^*_{\psi_{2i-1}})$. We demand also that $\operatorname{supp} e^*_{\psi_{2i}} \cap \operatorname{supp} x_{2k-1} = \emptyset$ for any i, k, thus the even parts of the chosen special functional play no role in the estimates on the weighted averages of (x_{2i-1}). We assume also $m_{j_1}/m_{j_1+1} \leq 1/n^2_{2j-1}$.

By the previous remark we have for each i the following

$$e^*_{\psi_{2i-1}}(Tx_{2i-1}) \geq \delta/14. \tag{46}$$

Let

$$y=\frac{1}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}/2} x_{2i-1}=\frac{1}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}/2} c_{2i-1}\frac{m_{j_{2i-1}}}{n_{j_{2i-1}}}\sum_{n\in A_i} d_{\gamma_n}$$

and consider the functional associated to the special node $\zeta_{n_{2j-1}}$, i.e. of the form

$$e^*_{\zeta_{n_{2j-1}}}=\sum_{i=1}^{n_{2j-1}} d^*_{\zeta_i}+\frac{1}{m_{2j-1}}\sum_{i=1}^{n_{2j-1}2}\left(e^*_{\psi_{2i-1}}P_{J_{2i-1}}+\lambda_{2i}e^*_{\psi_{2i}}P_{J_{2i}}\right).$$

Then

$$\|Ty\|\geq e^*_{\zeta_{n_{2j-1}}}(Ty)$$

$$=\left(\sum_{i=1}^{n_{2j-1}} d^*_{\zeta_i}+\frac{1}{m_{2j-1}}\sum_{i=1}^{n_{2j-1}/2}\left(e^*_{\psi_{2i-1}}P_{J_{2i-1}}+\lambda_{2i}e^*_{\psi_{2i}}P_{J_{2i}}\right)\right)$$

$$\left(\frac{1}{n_{2j-1}}\sum_{i=1}^{n_{(2j-1)}/2} Tx_{2i-1}\right)$$

$$=\ldots$$

Notice that $J_{2i}\cap\Gamma_{\mathrm{rank}(\phi_{2i-1})}=\emptyset$, whereas by the choice of R_n and the node ϕ_{2i-1} we have $\mathrm{rng}(Tx_{2i-1})\subset\Gamma_{\mathrm{rank}(\phi_{2i-1})}$. Therefore

$$\cdots=(\sum_{i=1}^{n_{2j-1}} d^*_{\zeta_i}(\frac{1}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}/2} Tx_{2i-1})+\frac{1}{n_{2j-1}m_{2j-1}}\sum_{i=1}^{n_{2j-1}/2} e^*_{\psi_{2i-1}}P_{J_{2i-1}}(Tx_{2i-1})$$

where in the last line the first sum disappears by the choice of (q_{2i-1}), as $\mathrm{rank}(\mathrm{bd}(e^*_{\zeta_{n_{2j-1}}}))\cap\mathrm{rank}(Tx_{2i-1})=\emptyset$ for any i. Therefore we have

$$\|Ty\|\geq\frac{\delta}{28m_{2j-1}}\,. \tag{47}$$

On the other hand we estimate $\|y\|$. We shall prove that $\|y\|\leq 420/m^2_{2j-1}$ yielding for sufficiently big j a contradiction. By (P2) and Lemma (6.3.10) we get that (x_i) is 7-RIS. By Lemma (6.3.20) it is enough to estimate $|e^*_\beta(z)|$, where e^*_β is associated to a (Γ,j)-special sequence $(\delta_i,\bar{z}_i)_{i=1}^a$, and $z=\sum_{i\in J} x_{2i-1}$ for some interval $J\subset\{1,\ldots,n_{2j-1}\}$.

Let e^*_β have the following form

$$e^*_\beta=\sum_{i=1}^{a} d^*_{\tilde{\zeta}_i}+\frac{1}{m_{2j-1}}\sum_{i=1}^{\lfloor a2\rfloor}\left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\psi}_{2i-1}}P_{\tilde{J}_{2i-1}}+\tilde{\lambda}_{2i}e^*_{\tilde{\psi}_{2i}}P_{\tilde{J}_{2i}}\right)+\left[\tilde{\epsilon}_a e^*_{\psi_a}P_{\tilde{J}_a}\right]$$

with $a\leq n_{2j-1}$, where the last term appears if a is odd. Let $i_0=\max\{i\leq a:(\zeta_i,\bar{x}_i)=(\delta_i,\bar{z}_i)\}$ if such i exists. We estimate $|e^*_\beta(z)|$ assuming i_0 is well-defined. We estimate separately $|\sum_{i=1}^a d^*_{\tilde{\zeta}_i}(z)|$, $|\mathrm{mt}(e^*_{\tilde{\zeta}_{i_0}})(z)|$ and $|(\mathrm{mt}(e^*_\beta)-\mathrm{mt}(e^*_{\tilde{\zeta}_{i_0}}))(z)|$.

First notice that taking into account coordinates of z with respect to the basis (d_γ) and that $c_{2i-1}\leq 1$, we have

$$\left|\sum_{i=1}^{a} d^*_{\tilde{\zeta}_i}(z)\right|\leq n_{2j-1}\frac{m_{j_1}}{n_{j_1}}. \tag{48}$$

Now consider the tree-analysis of $e^*_{\tilde{\zeta}_{i_0}}$, recall that it is compatible with the tree-analysis of $e^*_{\zeta_{i_0}}$. Then by the definition of a special node we have

$$\mathrm{mt}\left(e^*_{\zeta_{i_0}}\right) = \begin{cases} \frac{1}{m_{2j-1}} \sum_{i=1}^{i_0/2} \left(\tilde{\varepsilon}_{2i-1} e^*_{\tilde{\psi}_{2i-1}} P_{J_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\psi_{2i}} P_{J_{2i}}\right) & \text{if } i_0 \text{ even} \\ \frac{1}{m_{2j-1}} \sum_{i=1}^{\lfloor i_0/2 \rfloor} (\tilde{\varepsilon}_{2i-1} e^*_{\tilde{\psi}_{2i-1}} P_{J_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\psi_{2i}} P_{J_{2i}}) + \tilde{\varepsilon}_{i_0} e^*_{\tilde{\psi}_{i_0}} P_{J_{i_0}} & \text{if } i_0 \text{ odd} \end{cases}$$

where for each $2i - 1 \leq i_0$ we have

$$e^*_{\tilde{\psi}_{2i-1}} = \sum_{n\in A_i} d^*_{\tilde{\xi}_n} + \frac{1}{m_{j_{2i-1}}} \sum_{n\in A_i} \tilde{\varepsilon}_n e^*_{\tilde{\eta}_n} P_{I_n} .$$

Notice that as $M \cap I_n = \emptyset$ for any n and by the choice of $e^*_{\psi_{2i}}$ and ranks of ξ_n, thus also ranks of $\tilde{\xi}_n$, we get, assuming that i_0 is even,

$$|\mathrm{mt}(e^*_{\zeta_{i_0}})(z)| = |\frac{1}{m_{2j-1}} \sum_{i=1}^{i_0 2} (\tilde{\varepsilon}_{2i-1} e^*_{\tilde{\psi}_{2i-1}} P_{J_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\psi_{2i}} P_{J_{2i}})(z)| \quad (49)$$

$$= |(\frac{1}{m_{2j-1}} \sum_{i=1}^{i_0 2} \tilde{\varepsilon}_{2i-1} \sum_{n\in A_i} d^*_{\tilde{\xi}_n})(\sum_{2i-1\in J} c_{2i-1} \frac{m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n\in A_i} d_{\gamma_n})| = 0.$$

The same holds if i_0 is odd.
Now consider $\mathrm{mt}(e^*_\beta) - \mathrm{mt}(e^*_{\zeta_{i_0}})$ assuming that $i_0 < a$. Notice that
(1) $w(\psi_s) \neq w(\tilde{\psi}_i)$ for each $s, i > i_0$ provided at least one of the indices s, i is bigger than $i_0 + 1$,
(2) $\left(\mathrm{mt}(e^*_\beta) - \mathrm{mt}\left(e^*_{\zeta_{i_0}}\right)\right)(x_{2k-1}) = 0$ for any $2k - 1 \leq i_0$.
Using Corollary (6.3.11) for the terms $\sum_{i=i_0+1}^{a} |e^*_{\tilde{\psi}_i} P_{\tilde{J}_i}(x_{2k-1})|$ and that $|e^*_{\tilde{\psi}_{i_0+1}} P_{\tilde{J}_{i_0+1}}(x_{i_0+1})| \leq 4$, it follows that

$$|(\mathrm{mt}(e^*_\beta) - \mathrm{mt}(e^*_{\zeta_{i_0}}))(z)| \leq \frac{1}{m_{2j-1}} \sum_{i=i_0+1}^{a} \sum_{2k-1=i_0+1}^{n_{2j-1}} |e^*_{\tilde{\psi}_i} P_{\tilde{J}_i}(x_{2k-1})| \quad (50)$$

$$\leq \frac{4}{m_{2j-1}} + \frac{1}{m_{2j-1}} n_{2j-1} \frac{14}{m_{j_{i_0+1}}} \leq \frac{5}{m_{2j-1}} .$$

Therefore by (48), (49), (50) and the choice of j_1 we have $|e^*_\beta(z)| \leq 6/m_{2j-1}$, thus we can apply Lemma (6.3.20) obtaining that $\|y\| \leq 60 \cdot 7/m^2_{2j-1}$. For sufficiently big j we obtain contradiction with (47) and boundedness of T.

Proposition (6.3.35)[252]: Let $T : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator. If $Td_{\gamma_n} \to 0$, then $Ty_n \to 0$ for every RIS $(y_n)_n$.

Proof. Take $T : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ with $Td_{\gamma_n} \to 0$ and suppose there are a normalised C-RIS $(y_n)_n$ and $\delta > 0$ such that $\|Ty_n\| > \delta$ for all $n \in \mathbb{N}$. Passing to a subsequence we may assume as in the proof of Proposition (6.3.34) that

$$\max \operatorname{rank} R_n + 2 < \min \operatorname{rank} R_{n+1} \text{ where } R_n = \operatorname{rng}(Ty_n) \cup \operatorname{rng}(y_n).$$

Pick $(\mu_n) \subset \{\pm 1\}$ and nodes (ψ_n) with $\mu_n e^*_{\psi_n}(Ty_n) > \delta$.

Case 1. There exist a constant $c > 0$, an infinite set $M \subset \mathbb{N}$ and nodes $(\varphi_n)_{n\in M}$ such that $|e^*_{\varphi_n}(y_n)| > c$ and $e^*_{\varphi_n}, e^*_{\psi_n}$ have compatible tree-analyses.
Pick signs $(\nu_n)_{n\in M}$ with $\nu_n e^*_{\varphi_n}(y_n) = |e^*_{\varphi_n}(y_n)| > c$ for each n. We may pass to a subsequence $(\gamma_{k_n})_n$ of $(\gamma_n)_n$ so that $\left\|Td_{\gamma_{k_n}}\right\| \leq 2^{-n}$ for all n. For a fixed $j \in \mathbb{N}, n_{2j+1} > 200C$, we pick, as in Lemma (6.3.31), a j-dependent sequence $(\zeta_i, x_i)_i$ where $\zeta_i = (q_i + 1, \zeta_{i-1}, m_{2j-1}, J_i, 1, \eta_i), i = 1, \ldots, n_{2j-1}$, with $\zeta_0 = 0$, satisfies

$$\mathrm{mt}\left(e^*_{\eta_{2i-1}}\right) = \frac{1}{m_{j2i-1}} \sum_{n\in A_{2i-1}} \nu_n e^*_{\varphi_n} P_{I_n},$$

$$x_{2i-1} = \frac{c_{2i-1} m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n\in A_{2i-1}} y_n, \quad \|x_{2i-1}\| = {}^1/_2,$$

where $I_n = [\min R_n, \max R_n]$, so that the functional associated to the special node $\zeta_{n_{2j+1}}$ with mt-part of the form

$$\mathrm{mt}\left(e^*_{\zeta_{n_{2j-1}}}\right) = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j-1}/2} \left(e^*_{\eta 2i-1} P_{J_{2i-1}} + \lambda_{2i} e^*_{\eta 2i} P_{J_{2i}}\right),$$

satisfies $J_{2i-1} \supset \mathrm{rng}(Tx_{2i-1}), J_{2i} \cap \mathrm{rng}(Tx_{2k-1}) = \emptyset$ and $\mathrm{rank}(\mathrm{bd}(e^*_{\zeta_{n_{2j+1}}})) \cap \mathrm{rank}(Tx_{2i-1}) = \emptyset$ for any i, k.
From Remark (6.3.30) we get

$${}^1/_{20C} \leq c_{2i-1} \leq 2.$$

Using gaps between sets R_n we pick nodes $(\xi_{2i-1})_{2i-1\leq n_{2j+1}}$, with

$$\mathrm{mt}\left(e^*_{\xi_{2i-1}}\right) = \frac{1}{m_{j_{2i-1}}} \sum_{n\in A_{2i-1}} \mu_n e^*_{\psi_n} P_{I_n}.$$

It follows that $e^*_{\xi_{2i-1}}(Tx_{2i-1}) > {}^\delta/_{20C}$ for each i.
Notice also that for $x_{2i} = \frac{m_{j_{2i}}}{n_{j_{2i}}} \sum_{n\in A_{2i}} d_{\gamma_n}, A_{2i} \subset \{k_n : n \in \mathbb{N}\}$, by the condition on $\left(Td_{\gamma_{k_n}}\right)$ we have $\|Tx_{2i}\| < \frac{m_{j_{2i}}}{n_{j_{2i}}} < 2^{-i}$ for each i.
Let $x = \frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} x_{2i-1}$ and $d = \frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{(2j-1)}/2} x_{2i}$. We have

$$\|Td\| \leq \frac{m_{2j-1}}{n_{2j-1}} \tag{51}$$

and by Lemma (6.3.32)

$$\|x - d\| \leq \frac{250}{m^2_{2j-1}}. \tag{52}$$

On the other hand by the choice of (φ_n) and (ψ_n) there is a well-defined special node β, associated to the same j-special sequence as $\zeta_{n_{2j+1}}$ with

$$\mathrm{mt}(e^*_\beta) = \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} e^*_{\xi_{2i-1}} P_{J_{2i-1}} + \tilde{\lambda}_{2i} e^*_{\eta_{2i}} P_{J_{2i}},$$

so that $\mathrm{rank}(\mathrm{bd}(e^*_\beta)) \cap \mathrm{rank}(Tx_{2i-1}) = \emptyset$ for any i. Thus

$$\|Tx\| \geq e^*_\beta(Tx) \geq \frac{\delta}{40C}$$

which contradicts (51) and (52) for sufficiently big j as T is bounded.

Case 2. Case 1 does not hold. Applying this assumption for $c = n_{2j-1}^{-1} m_k^{-1}, k \in \mathbb{N}$, we pick inductively an increasing sequence $(p_k) \subset \mathbb{N}$ such that for any node φ and $n > p_k$ so that $e_\varphi^*, e_{\psi_n}^*$ have compatible tree-analyses we have $|e_\varphi^*(y_n)| \leq n_{2j-1}^{-1} m_k^{-1}$. Let $M = (p_k)_k$.

Now we repeat the proof of Proposition (6.3.34), using (y_n) instead of (d_{γ_n}). For a fixed $j \in \mathbb{N}$ we pick a j-dependent sequence $(\zeta_i, x_i), \zeta_i = (q_i + 1, \zeta_{i-1}, m_{2j-1}, J_i, 1, \eta_i), i = 1, \ldots, n_{2j-1}$, with $\zeta_0 = 0$, such that for each i we have

$$\mathrm{mt}(e_{\eta_{2i-1}}^*) = \frac{1}{m_{j_{2i-1}}} \sum_{n \in A_i} \mu_n e_{\psi_n}^* P_{I_n}, x_{2i-1} = \frac{c_{2i-1} m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n \in A_i} y_n, \|x_{2i-1}\| = {}^1\!/_2,$$

with $A_i \subset M, \#A_i = n_{j_{2i-1}} J_{2i-1} = \mathrm{rng}(e_{\eta_{2i-1}}^*), J_{2i} \cap \mathrm{supp}\, x_{2k-1} = \emptyset$ for any $i, k, I_n = [\min R_n, \max R_n]$ and $\mathrm{rank}(\xi_n) = \max \mathrm{rng}\, R_n + 1$ for any n. As in the previous case, ${}^1\!/_{20C} \leq c_{2i-1} \leq 2$. Pick j_1 with $m_{j_1}/m_{j_1+1} \leq 1/n_{2j-1}^2$ and let

$$y = \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} x_{2i-1}$$

As in the proof of Proposition (6.3.34) it follows that

$$\|Ty\| \geq e_{\zeta_{n_{2j-1}}}^*(y) \geq \frac{1}{m_{2j-1} n_{2j-1}} \sum_{i=1}^{n_{2j-1}2} \frac{\delta}{2} c_{2i-1} \geq \frac{\delta}{80 C m_{2j-1}}. \quad (53)$$

We shall estimate now $\|y\|$. As before we consider a special node β which is compatible with a (Γ, j)-special sequence $(\delta_i, \bar{z}_i)_{i=1}^a, a \leq n_{2j-1}$, and estimate $|e_\beta^*(z)|$ where $z = \sum_{i \in J} x_{2i-1}$ for some interval $J \subset \{1, \ldots, n_{2j-1}\}$. Writing

$$e_\beta^* = \sum_{i=1}^{a} d_{\tilde{\zeta}_i}^* + \frac{1}{m_{2j-1}} \sum_{i=1}^{\lfloor a2 \rfloor} \left(\tilde{\varepsilon}_{2i-1} e_{\tilde{\eta}_{2i-1}}^* P_{\tilde{J}_{2i-1}} + \tilde{\lambda}_{2i} e_{\tilde{\eta}_{2i} P_{\tilde{J}_{2i}}}^* \right)$$

with $a \leq n_{2j-1}$ we pick as before $i_0 = min\{i \leq a : (\zeta_i, x_i) \neq (\delta_i, z_i)\}$ (if such i exists) and estimate separately $|\sum_{i=1}^a d_{\tilde{\zeta}_i}^*(w)|, |mt(e_{\tilde{\zeta}_{i_0}}^*)(w)|$ and $|(mt(e_\beta^*) - mt(e_{\tilde{\zeta}_{i_0}}^*))(w)|$.

Repeating the reasoning of the proof of Proposition (6.3.34), as (y_n) have norm bounded by 1 and all $\left\|d_{\tilde{\zeta}_i}^*\right\| \leq 3$, we obtain

$$|\sum_{i=1}^{a} d_{\tilde{\zeta}_i}^*(z)| \leq 3 \cdot 2 n_{2j-1} \frac{m_{j_1}}{n_{j_1}} \leq \frac{1}{m_{2j-1}}. \quad (54)$$

Using Corollary (6.3.21) and the fact that $|e_\gamma^* P_I (x_{i0+1})| \leq 4$ we obtain that

$$\left|\left(mt\left(e_\beta^*\right) - mt\left(e_{\tilde{\zeta}_{i0}}^*\right)\right)(z)\right| \leq \frac{4}{m_{2_{j-1}}} + 2 \frac{1}{m_{2_{j-1}}} n_{2_{j-1}} \frac{64C}{m_{j_{i_0+1}}} \leq \frac{5}{m_{2_{j-1}}} \quad (55)$$

using that $m_{j_1}^{-1} < n_{2_{j-1}}^2$ and $n_{2_{j+1}} > 200C$.

Now consider $e_{\tilde{\zeta}_{i_0}}^*$, recall this functional and $e_{\zeta_{i_0}}^*$ have compatible tree-analyses. Therefore

$$mt\left(e^*_{\tilde{\zeta}_{i_0}}\right) = \begin{cases} \dfrac{1}{m_{2_{j-1}}} \displaystyle\sum_{i=1}^{\frac{i_0}{2}} (\tilde{\epsilon}_{2i-1} e^*_{\tilde{\eta}_{2i-1}} P_{J_{2_{i-1}}} + \tilde{\lambda}_{2_i} e^* \_\eta_{2_i} P_{J_{2_i}}) & if\ i_0\ \ even \\ \dfrac{1}{m_{2_{j-1}}} \displaystyle\sum_{i=1}^{\frac{i_0}{2}} \left(\tilde{\varepsilon}_{2_i-1} e^*_{\tilde{\eta}_{2_i-1}} P_{J_{2_i-1}} + \tilde{\lambda}_{2_i} e^*_{\eta_{2_i}} P_{J_{2_i}}\right) + \tilde{\epsilon}_{i_0} e^*_{\tilde{\eta}_{i_0}} P_{J_{i_o}} & if\ i_0\ odd \end{cases}$$

where for each for each $2i - 1 \leq i_0$ we have

$$e^*_{\tilde{\eta}_{2i-1}} = \sum_{n \in A_i} d^*_{\tilde{\xi}_n} + \frac{1}{m_{j_{2i-1}}} \sum_{n \in A_i} \tilde{\varepsilon}_n e^*_{\varphi_n} P_{I_n} \quad .$$

By choice of the objects above we have

$$\left| mt\left(e^*_{\tilde{\zeta}_{i_0}}\right)(z)\right|$$

$$\leq \frac{1}{m_{2_{j-1}}} \left| \left(\sum_{i=1}^{\frac{n_{2_{j-1}}}{2}} \sum_{n \in A_i} d^*_{\tilde{\xi}_n} \right) \left(\sum_{2i-1 \in J} \frac{c_{2i-1} m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n \in A_i} y_n \right) \right|$$

$$+ \frac{1}{m_{2_{j-1}}} \sum_{2i-1 \in J} \frac{c_{2i-1} m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n \in A_i} \left| e^*_{\varphi_n}(y_n) \right| .$$

As for each n the nodes ψ_n, φ_n have compatible tree-analyses the last sum can be estimated by $2m^{-1}_{2j-1}$. The first sum equals 0 by the condition on ranks of ξ_n, thus also $\tilde{\xi}_n$ Therefore we have

$$\left| mt\left(e^*_{\tilde{\zeta}_{i_0}}\right)(z)\right| \leq \frac{2}{m_{2j-1}} . \tag{56}$$

As before by (54), (55), (56) we have $|e^*_\beta(z)| \leq 8/m_{2j-1}$, thus we can apply Lemma (6.3.20) obtaining that $\|y\| \leq 80C/m^2_{2j-1}$. For sufficiently big j we obtain contradiction with (53) and boundedness of T.

Theorem (6.3.36)[252]: Let $T: \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator. Then there exist a compact operator K : $\mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ and a scalar λ such that $T = \lambda Id + K$.

Proof. By Proposition (6.3.34) any $(d_{\gamma n})_{n \in N}$ has a further subsequence $(d_{\gamma_n})_{n \in M}$ such that $Td_{\gamma n} - \lambda d_{\gamma n} \to 0\ as\ M \ni n \to \infty$, for some λ. By Proposition (6.3.33) there is a universal λ so that $Td_{\gamma n} - \lambda d_{\gamma n} \to 0\ as\ n \to \infty$. Applying Proposition (6.3.35) to the operator $T - \lambda Id$ we get that $Ty_n - \lambda y_n \to 0$ for any RIS (y_n) and thus, by Proposition (6.3.26), for any bounded block sequence (y_n). It follows that the operator $T - \lambda Id$ is compact.

The above theorem implies immediately the following.

Corollary (6.3.37)[252]: The space $\mathcal{X}_{Kus}$ is indecomposable, i.e. it is not a direct sum of two its infinitely dimensional closed subspaces.

We devoted to the proof of saturation of the space $\mathcal{X}_{Kus}$ by unconditional basic sequences. We follow the idea of the proof of the corresponding fact from [98] with additional work in order to control the bd-parts of norming functionals. Below we present a construction of unconditional sequences in $\mathcal{X}_{Kus}$.

Fix a block subspace $Y \subset \mathcal{X}_{Kus}$ and pick sequences $j_k < j_{k,1} < j_{k,2} < \cdots < j_{k,n_{j_k}}, k \in N$, with (j_k) increasing, and a block sequence $(x_k)_k \subset Y$, $with\ x_k = \frac{m_{j_k}}{n_{j_k}} \sum_{i=1}^{n_{j_k}} x_{k,i}$ where for some fixed $C > 2$ and for each $k \in N$ the sequence $(x_{k,i})_i \subset Y$ is a $C-RIS$ with parameters $(j_{k,i})_i$ chosen according to Lemma (6.3.22) to satisfy $|d_\gamma^*(x_{k,i})| < 1/n_{j_k}^2\ for\ any\ i \leq n_{j_k}\ and\ \gamma \in \Gamma$. Therefore

$$|d_\gamma^*(x_k)| < C/n_{j_k}^2 \quad for\ any\ k \in N, \gamma \in \Gamma. \tag{57}$$

We fix the sequence (x_k) and the node γ with the tree-analysis $(I_t, \epsilon_t, \eta_t)_{t \in T}$ for the sequel. Recall that S_t denotes the set of immediate successors of t in the tree . We order the sets S_t with the order on $(I_s)_{s \in S_t}$ and we write $s\_$ for the immediate predecessor of s.

Definition (6.3.38)[252]: A couple of nodes (η_{s-}, η_s) is called a dependent couple with respect to γ if $s\_, s \in S_t, w(\eta_t) = m_{2j+1}^{-1}$ for some $j \in N$ and s is at the even position in the mt-part of $e_{\eta_t}^*$.

Let $\mathcal{E}_\gamma = \{s \in T : (\eta_{s\_}, \eta_s)$ is a dependent couple with respect to $\gamma\}$.

Definition (6.3.39)[252]: For $k \in \mathbb{N}$ a couple of nodes $(\eta_{s\_}, \eta_s)$ is called a dependent couple with respect to γ and $x_k\ if\ (\eta_{s\_}, \eta_s)$ is a dependent couple with respect to γ and moreover

$$\min supp(x_{k+1}) > \max supp\left(e_s^* P_{I_s}\right) \geq \min supp(x_k),$$
$$\max supp(x_{k-1}) \geq min\ supp(e_{s\_}^* P_{I_{s\_}}).$$

Let $\mathcal{F}_\gamma = \{s \in \mathcal{T} \mid (\eta_{s\_}, \eta_s)$ is a dependent couple with respect to γ and x_k for some $k\}$ and let $Q_\gamma = \sum_{s \in \mathcal{F}_\gamma} P_{I_s}$. Then we define $y_k = Q_\gamma x_k$ and $x_k' = x_k - y_k$. As our basis $(d_\gamma)_{\gamma \in \Gamma}$ is not unconditional, the projections $(Q_\gamma)_\gamma$ are not uniformly bounded. However, we have the following lemma that is proved along the lines of [98].

Lemma (6.3.40)[252]:

(i) For every $k \in \mathbb{N}$ and $t \in \mathcal{T}$ we have $|e_{\eta_t}^* P_{I_t}(y_k)| \leq 10C/m_{j_k}$.

(ii) For every $k \in \mathbb{N}$ and $t \in \mathcal{T}$ with $w(\eta_t) < m_{j_k}^{-1}$ we have $|e_{\eta_t}^* P_{I_t}(x_k')| \leq 11C/m_{j_k}$.

Proof. Concerning (i), notice first that for any $s \in \mathcal{F}_\gamma$ we have $|e_{\eta_s}^* P_{I_s}(x_k)| \leq 10C/m_{j_k}$.

Indeed, for $w(\eta_s) = m_{2j}$ for some j, we consider the following two cases. If $m_{2j}^{-1} < m_{j_k}^{-1}$ then the estimate follows by (26). If $m_{2j}^{-1} \geq m_{j_k}^{-1}$, then by the form of $e_{\eta_s}^*$ and (57) we have

$$\left|e_{\eta_s}^* P_{I_s}(x_k)\right| \leq 2n_{2j} \max_{\gamma \in \Gamma} |d_\gamma^*(x_k)| \leq 2C/n_{j_k}$$

Now, as each of the sets$\{s \in \mathcal{F}_\gamma \mid |s| = i, rng(x_k) \cap I_s \neq \emptyset\}, i \in \mathbb{N}$, has at most two elements, we have

$$\begin{aligned}\left|e_{\eta_t}^* P_{I_t}(y_k)\right| &\leq \sum_{s \in \mathcal{F}_\gamma} \left(\Pi_{t \leq u \prec s} w(\eta_u)\right) \left|e_{\eta_s}^* P_{I_s}(x_k)\right| \\ &= \sum_i \sum_{s \in \mathcal{F}_\gamma, |s|=i} \left(\Pi_{t \leq u \prec s} w(\eta_u)\right) \left|e_{\eta_s}^* P_{I_s}(x_k)\right| \leq \frac{20C}{m_{j_k}} \sum_i \frac{1}{m_1^i} = \frac{10C}{m_{j_k}}.\end{aligned}$$

Condition (ii) follows from Lemma (6.3.20) and (i).

Lemma (6.3.41). For every choice of signs (δ_k) there exists a node $\tilde{\gamma} \in \Gamma$ such that $Q_\gamma = Q_{\tilde{\gamma}}$ and $\epsilon \in \{\pm 1\}$ so that

$\left|e_\gamma^*(x_k') - \epsilon e_{\tilde{\gamma}}^*(\delta_k x_k')\right| \le \frac{6C}{m_{j_k}}$ $for\ any\ k \in \mathbb{N}$.

Proof. Define

$D = \{t \in \mathcal{T} \mid rng(x_k) \cap rng(e_t^* P_{I_t}) \neq \emptyset$ for at most one k and if $t \in S_u$ $then\ rng(x_i) \cap rng(e_u^* P_{I_u}) \neq \emptyset$ for at least two $i\}$.

Since for every branch b of $\mathcal{T}$ the set $b \cap D$ has exactly one element we can define a subtree $\mathcal{T}'$ of $\mathcal{T}$ such that D is the set of terminal nodes for $'$. Notice that $(\mathcal{T} \setminus \mathcal{T}') \cap \mathcal{F}_\gamma = \emptyset$.

If $\gamma \in D$, then we pick the unique k_0 with $rng(e_\gamma^*) \cap rng(x_{k_0}) \neq \emptyset\ (as\ I_\emptyset = [1, max\ \Delta_{rank(\gamma)}])$ and let $\tilde{\gamma} = \gamma$ and $\varepsilon = \delta_{k_0}$. Then we have the estimate in the lemma for any $k \in N$.

Assume that $\gamma \notin D$. Using backward induction on $\mathcal{T}'$ we shall define a node $\tilde{\gamma}$ with a tree-analysis $(I_t, \tilde{\epsilon}_t, \tilde{\eta}_t)_{t\in T}$ and associated scalars $\left(\tilde{\lambda}_t\right)_{t\in T}$, by modifying the nodes $(I_t, \tilde{\epsilon}_t, \eta_t)_{t\in\mathcal{T}'}$ and scalars $(\lambda_t)_{t\in\mathcal{T}'}$ starting from elements of D such that

(T1) $e_{\eta_t}^*$, $e_{\tilde{\eta}_t}^*$ have compatible tree-analyses for any $\in \mathcal{T}'$,

(T2) $F_{\tilde{\eta}_t} = F\_\eta_t$ for any $\in \mathcal{T}'$,

(T3) $\tilde{\epsilon}_t e_{\tilde{\eta}_t}^* P_{I_t}(\delta_k x_k') = \epsilon_t e_{\eta_t}^* P_{I_t}(x_k')$ for any $t \in D \setminus \mathcal{E}_\gamma$ and k, $\tilde{\lambda}_t e_{\tilde{\eta}_t}^* P_{I_t}(\delta_k x_k') = \lambda_t e_{\eta_t}^* P_{I_t}(x_k')$ for any $t \in D \cap \mathcal{E}_\gamma$ and k,

(T4) $\tilde{\epsilon}_t = \epsilon_t$ for any $t \in \mathcal{T}' \setminus D$.

We need to modify only $\epsilon_t, t \in D$, changing signs of some of them. These modifications determine changes in the rest of the tree, i.e. $\eta_u, u \in \mathcal{T}' \setminus D$ according to the rules of producing nodes and

Let $\tilde{\gamma} = \tilde{\eta}_\emptyset$. Notice that by conditions (T1)–(T2) we have $Q_{\tilde{\gamma}} = Q_\gamma$. Now we proceed to show the estimate part of the lemma. Fix $k \in N$. For any nonterminal $u \in \mathcal{T}$ let

$S_{u,k} := \{s \in S_u \mid rng(x_k) \cap rng(e_s^* P_{I_s}) = \emptyset\}$.

Let G be the set of minimal nodes u of $\mathcal{T}'$ with $u \in D$ or $(\eta_u) < m_{jk}^{-1}$. By $\mathcal{T}''$ denote the subtree of $\mathcal{T}'$ with the terminal nodes in G. We shall prove by induction starting from G that for any $u \in \mathcal{T}''$ we have

$$\left|\epsilon_u e_{\eta_u}^* P_{I_u}(x_k') - \tilde{\epsilon}_u e_{\tilde{\eta}_u}^* P_{I_u}(\delta_k x_k')\right| \le \frac{22C}{m_{jk}}. \tag{58}$$

This will end the proof as it follows by (T4) that $|\epsilon_\emptyset e_{\eta_\emptyset}^*(x_k') - \tilde{\epsilon}_\emptyset e_{\tilde{\eta}_\emptyset}^*(\delta_k x_k')| = |e_{\tilde{\gamma}}^*(x_k') - e_{\tilde{\gamma}}^*(\delta_k x_k')|$. Thus taking $\epsilon = 1$ we obtain the estimate of the lemma.

Step 1. $u \in G. If\ w(\eta_u) < m_{jk}^{-1}$ then the estimate (58) holds true by Lemma (6.3.40) (ii). If $u \in D$ then the estimate (58) holds true by (T3). **Step 2.** $u \in \mathcal{T}'' \setminus G$. In particular $(\eta_u) \ge m_{jk}^{-1}$. Obviously $S_u \subset \mathcal{T}''$.

Case 2a. $w(\eta_u) = m_{2j}^{-1}$. We estimate, using (T3) for $s \in S_{u,k} \cap D$

$$|e_{\tilde{\eta}_u}^* P_{I_u}(\delta_k x_k') - e_{\eta_u}^* P_{I_u}(x_k')|$$

$$= |\left(\sum_{s\in S_u} d_{\tilde{\xi}_s}^* + \frac{1}{m_{2j}} \sum_{s\in S_{u,k}\cap D} \tilde{\epsilon}_s e_{\tilde{\eta}_s}^* P_{I_s} + \frac{1}{m_{2j}} \sum_{s\in S_{u,k}\setminus D} \tilde{\epsilon}_s e_{\tilde{\eta}_s}^* P_{I_s}\right)(\delta_k x_k')$$

$$-\left(\sum_{s\in S_u} d^*_{\xi_s} + \frac{1}{m_{2j}} \sum_{s\in S_{u,k}\cap D} \tilde{\epsilon}_s e^*_{\eta_s} P_{I_s} + \frac{1}{m_{2j}} \sum_{s\in S_{u,k}\setminus D} \tilde{\epsilon}_s e^*_{\eta_s} P_{I_s} \right) (x'_k)|$$
$$\leq \left| \sum_{s\in S_u} d^*_{\tilde{\xi}_s} (\delta_k x'_k) \right|$$
$$+ \left| \sum_{s\in S_u} d^*_{\xi_s} (x'_k)\right| + \frac{1}{m_{2j}} \sum_{s\in S_{u,k}\setminus D} \left|\tilde{\epsilon}_s e^*_{\tilde{\eta}_s} P_{I_s} (\delta_k x'_k) - \epsilon_s e^*_{\eta_s} P_{I_s} (x'_k)\right|$$
$$\leq \ldots$$
The first two sums are estimated using (57) and $\#S_u \leq n_{2j} \leq n_{jk}$, for the third element use the inductive hypothesis and the fact that $\#(S_{u,k} \setminus D) \leq 2$, obtaining the following
$$\ldots \leq 2n_{2j} \frac{C}{n_{jk}^2} + \frac{2}{m_{2j}} \cdot \frac{22C}{m_{jk}} \leq \frac{22C}{m_{jk}}.$$
Case 2b. $w(\eta_u) = m_{2j+1}^{-1}$. Recall that by (T3) we have $\epsilon_{s_-} e^*_{\eta_{s_-}} P_{I_{s_-}} (x'_k) = \tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} I_{s_-} (\delta_k x'_k)$ for any $s \in S_u \cap \mathcal{E}_\gamma$ with $s_- \in D$ and $\lambda_s e^*_{\eta_s} P_{I_s} (x'_k) = \tilde{\lambda}_s e^*_{\tilde{\eta}_s} I_s(\delta_k x'\_k)$ for any $s \in S_u \cap \mathcal{E}_\gamma \cap D$. Moreover $\mathcal{E}_\gamma \setminus D \subset \mathcal{F}_\gamma$ thus $e^*_{\eta_s} P_{I_s} (x'_k) = 0 = e^*_{\tilde{\eta}_s} P\_I_s (\delta_k x'_k)$ for any $s \in (S_u \cap \mathcal{E}_\gamma) \setminus D$. Therefore we have
$$\left|e^*_{\tilde{\eta}_u} P_{I_u} (\delta_k x'_k) - e^*_{\eta_u} P_{I_u} (x'_k)\right|$$
$$= \left| \left(\sum_{s\in S_u} d^*_{\tilde{\xi}_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}, s\in\mathcal{E}_\gamma} \tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} P_{I_s} + \frac{1}{m_{2j+1}} \sum_{s\in S_{u,k}\cap\mathcal{E}_\gamma} \tilde{\lambda}_s e^*_{\tilde{\eta}_s} P_{I_s} \right) (\delta_k x'_k)\right.$$
$$\left. - \left(\sum_{s\in S_u} d^*_{\xi_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}, s\in\mathcal{E}_\gamma} \tilde{\epsilon}_{s_-} e^*_{\eta_{s_-}} P_{I_s} + \frac{1}{m_{2j+1}} \sum_{s\in S_{u,k}\cap\mathcal{E}_\gamma} \lambda_s e^*_{\eta_s} P_{I_s} \right) (x'_k)\right|$$
$$= \left| \left(\sum_{s\in S_u} d^*_{\tilde{\xi}_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}\setminus D, s\in\mathcal{E}_\gamma} \tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} P_{I_s} \right) (\delta_k x'_k)\right.$$
$$\left. - \left(\sum_{s\in S_u} d^*_{\xi_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}\setminus D, s\in\mathcal{E}_\gamma} \epsilon_{s_-} e^*_{\eta_{s_-}} P_{I_s} \right) (x'_k)\right|$$
$$\leq \left|\sum_{s\in S_u} d^*_{\tilde{\xi}_s} (\delta_k x'_k)\right| + \left|\sum_{s\in S_u} d^*_{\xi_s} (x'_k)\right| +$$
$$+ \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}\setminus D, s\in\mathcal{E}_\gamma} \left|\tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} P_{I_{s_-}} (\delta_k x'_k) - \epsilon_{s_-} e^*_{\eta_{s_-}} P_{I_{s_-}} (x'_k)\right|$$
$$\leq \ldots$$
Proceeding as in Case $2a$ we obtain
$$\ldots \leq 2n_{2j+1} \frac{C}{n_{jk}^2} + \frac{2}{m_{2j+1}} \cdot \frac{22C}{m_{jk}} \leq \frac{22C}{m_{jk}}.$$
Theorem (6.3.42)[252]: The space $\mathcal{X}_{Kus}$ is unconditionally saturated.

Proof. In every block subspace of $\mathcal{X}_{Kus}$ pick a sequence $(x_k)_k$ as above with $m_{j1} > 400C$. We claim that such a sequence is unconditional. To this end consider a finite sequence of scalars (a_k) with $\|\sum_k a_k x_k\| = 1$ and $(\delta_k) \subset \{\pm 1\}$. We want to estimate the norm of the vector $\sum_k \delta_k a_k x_k$. Take $\gamma \in \Gamma$ with $e_\gamma^*(\sum_k a_k x_k) \geq 3/4$. Define $Q\gamma$, (yk) and (x'_k) and consider $\tilde{\gamma}$ and ϵ provided by Lemma (6.3.41). Notice that as $Q_{\tilde{\gamma}} = Q_\gamma$, the projection $Q_{\tilde{\gamma}}$ defines also (yk) and $(x;_k)$. Estimate, applying Lemma (6.3.41) and Lemma (6.3.40) (1) both for γ and $\tilde{\gamma}$, as follows

$$\left|e_\gamma^*\left(\sum_k a_k x_k\right) - \epsilon e_{\tilde{\gamma}}^*\left(\sum_k \delta_k a_k x_k\right)\right|$$

$$\leq \left|e_\gamma^*\left(\sum_k a_k x'_k\right) - \epsilon e_{\tilde{\gamma}}^*\left(\sum_k \delta_k a_k x'_k\right)\right| + \left|e_\gamma^*\left(\sum_k a_k y_k\right)\right| + \left|e_{\tilde{\gamma}}^*\left(\sum_k \delta_k a_k y_k\right)\right|$$

$$\leq \sum_k |a_k|\left|e_\gamma^*(x'_k) - \epsilon e_\gamma^*(\delta_k x'_k)\right| + \sum_k |a_k|\left|e_\gamma^*(y_k)\right| + \sum_k |a_k|\left|e_{\tilde{\gamma}}^*(\delta_k y_k)\right|$$

$$\leq 4 \cdot 24C \sum_k m_{j_k}^{-1} \leq 200C m_{j_1}^{-1} \leq 12/$$

where in the last line we use the fact that each $|a_k|$ is dominated by twice the basic constant of the basis $(d\gamma)$. Therefore $\|\sum_k \delta_k a_k x_k\| \geq |e_{\tilde{\gamma}}^*(\sum_k \delta_k a_k x_k)| \geq 1/4$, which ends the proof.

Corollary (6.3.43)[260]: [252] Let $\left(\bar{d}^j_{\gamma_n^j}\right)_{n\in\mathbb{N}}$ be a subsequence of the basis. Then there exists infinite $M \subset N$ such that no two nodes $\gamma_n^j, \gamma_m^j, n, m \in M$, are neighbours.

The proof is based on the fact that the age is uniquely determined for each node.

Proof. If there are infinitely many nodes with different weights we are done. So assume that for all but finite nodes we have $w(\gamma_n^j) = m_k^{-1}$ for some fixed k.

Applying Ramsey theorem we obtain an infinite set such that either no two nodes from this set are neighbours or any two are neighbours.

In the first case we are done. Otherwise passing to a further subsequence we may assume that $\operatorname{rank}(\gamma_n^j) < \operatorname{rank}(\gamma_{n+1}^j)$ for every n.

Since we have that $\gamma_j^j, \gamma_{j+1}^j$ are neighbours it follows by a simple induction that

$$\operatorname{age}(\gamma_{j+1}^j) \geq \operatorname{age}(\gamma_j^j) + 1 \geq j + 1.$$

Take $j = n_k + 1$ and pick $e_{\gamma^j}^*$ of the form

$$e_{\gamma^j}^* = \sum_{r=1}^{a} \sum_j \bar{d}_{\xi_r}^{j*} + m_k^{-1} \sum_{r=1}^{a} \sum_j \epsilon_r \lambda_r^j e_{\eta_r}^* \bar{P}_{I_r}^j$$

with $\sum_j \bar{d}^{j*}_{\gamma^j_{n_k+1}} = \sum_j \bar{d}^{j*}_{\xi_r}$ for some r. Then $\operatorname{age}(\xi_r) \leq n_k$ which yields a contradiction and ends the proof.

Corollary (6.3.44)[260]: [252] Let $x^j = n_j^{-1} \sum_{i\in G} \sum_j \bar{d}^j_{\xi_i}$, be such that no two ξ_i's are neighbours and $\#G \leq n_j$. Then for any $\gamma^j \in \Gamma$ with $w(e_{\gamma^j}^*) = m_k^{-1}$ we have the following

$$|e^*_{\gamma^j}(x^j)| \leq \begin{cases} \dfrac{1}{n_{k-\epsilon}} + \dfrac{2}{m_k} & if \ \ \epsilon < 0 \\ \dfrac{7}{m_k m_{k+\epsilon}} & if \ \ \epsilon > 0. \end{cases}$$

In particular

$$\left\| n_j^{-1} \sum_{i=1}^{n_j} \sum_j \ \bar{d}^j_{\xi_i} \right\| \leq 7 \sum_j \ m_j^{-1} \, .$$

Proof. We shall construct functionals $\phi^j_{\gamma^j}$ in the norming set of the mixed Tsirelson space $X_{aux} = \ T^j[(\mathcal{A}_{n_k}, m_k^{-1})_{k\in\mathbb{N}}]$ such that

$$\sum_j \ \left|e^*_{\gamma^j}(x^j)\right| \leq \sum_j \ \left(\phi^j_{\gamma^j}(y^j) + \frac{2}{m_j m_{j-1}}\right)$$

where $y^j \ = \ 2 \ \sum_{k\in G} \ e_k/n_j \ \in \ c_{00}(\mathbb{N})$.

Take $\gamma^j \in \Gamma$ and consider its evaluation analysis $e^*_{\gamma^j} = \sum_{r=1}^{a} \sum_j \ \bar{d}^{j*}_{\beta_r} + m_k^{-1} \sum_{r=1}^{a} \sum_j \ \epsilon_r \lambda^j_r e^*_{\eta_r} \bar{P}^j_{I_r}$. Let $g^j_{\gamma^j} = \mathrm{bd}(e^*_{\gamma^j})$ and $f^j_{\gamma^j} = mt(e^*_{\gamma^j})$.

We shall consider two cases.

Case 1. $w(\gamma^j) \ \leq \ m_j^{-1}$.

Since the nodes $(\xi_i)_i$ are pairwise non-neighbours and $(\beta_i)_i$ are pairwise neighbours it follows that

$$|\sum_j \ g^j_{\gamma^j}(x^j)| \ \leq \sum_j \ n_j^{-1} \, . \tag{59}$$

Also for every $r \ \leq \ a$ using that $|\sum_j \ e^*_\zeta \ (\bar{d}^j_\beta)| \ \leq \ 2$ for all ζ, β, we get

$$\sum_j \ \left|e^*_{\eta_r} \bar{P}^j_{I_r}(x^j)\right| \leq \sum_j \ 2 \, \frac{\#\left\{i : \ \mathrm{rng}\left(d^{j*}_{\xi_i}\right) \subset I_r\right\}}{n_j} \, . \tag{60}$$

It follows from (59), (60), using that $|\lambda^j_r| \ \leq \ 1$ for every r, that

$$\left|e^*_{\gamma^j}(x^j)\right| \leq \frac{1}{n_j} + 2m_k^{-1} \sum_{r=1}^{a} \sum_j \ \frac{\#\left\{i : \ \mathrm{rng}\left(d^{j*}_{\xi_i}\right) \subset I_r\right\}}{n_j} \leq \frac{1}{\sum_j \ n_j} + \frac{2}{m_k} \, . \tag{61}$$

Taking $\phi^j_{\gamma^j} \ = \ m_k^{-1} \ \sum_{n\in F} \ e^*_n$ where $F \ = \cup_{r\leq a} \{n \ |\gamma^j_n \ = \ \xi_i, \mathrm{rng}(d^{j*}_{\xi_i}) \ \subset \ I_r$ for some $i \ \in \ G\}$ it follows that $\#F \ \leq \ n_j \ \leq \ n_k$ and $\phi^j_{\gamma^j}$ belongs to the norming set of the mixed Tsirelson space X_{aux}.

From (61) we get

$$\left|\sum_j \ e^*_{\gamma^j}(x^j)\right| \leq \sum_j \ \left(\frac{1}{n_j} + 2m_k^{-1} \sum_{n\in F} \ \frac{e^*_n(e_n)}{n_j}\right) = \sum_j \ \left(\frac{1}{n_j} + \phi^j_{\gamma^j}(y^j)\right). \tag{62}$$

Case 2. $w(\gamma^j) \ = \ m_k^{-1} \ > \ m_j^{-1}$.

Let $(I_t, \varepsilon_t, \eta_t)_{t\in\mathcal{T}}$ be the tree-analysis of $e^*_{\gamma^j}$ and $\mathcal{T}'$ be the subtree of $\mathcal{T}$ consisting of all nodes t of height at most l_j. We will describe how to define certain functionals $\left(\phi^j_t\right)_{t\in\mathcal{T}'}$ in the norming set of $T^j[\left(\mathcal{A}_{n_k}, m_k^{-1}\right)_{k\in\mathbb{N}}]$ that we will use to obtain the desired estimate.
As in the previous case we get

$$|\sum_j g^j_{\gamma^j}(x^j)| \leq \sum_j n_j^{-1}. \tag{63}$$

Using that $e^*_{\gamma^j} = g^j_{\gamma^j} + f^j_{\gamma^j}$ and $|\lambda^j_r| \leq 1$ for every r, we get

$$|\sum_j e^*_{\gamma^j}(x^j)| \leq \sum_j \left(n_j^{-1} + \left|f^j_{\gamma^j}(x^j)\right| \leq n_j^{-1} + m_k^{-1}\sum_{r=1}^{a} \left|e^*_{\eta_r}\bar{P}^j_{I_r}(x^j)\right|\right). \tag{64}$$

We shall split now the successors $e^*_{\eta_r}$ of $e^*_{\gamma^j}$ into those with weight smaller or equal to m_j^{-1} and those with weight bigger that m_j^{-1}. For a node γ^j we set

$$S_{\gamma^j,1} = \{r \in S_{\gamma^j} : w(\eta_r) \leq m_j^{-1}\} \text{ and } S_{\gamma^j,2} = S_{\gamma^j} \setminus S_{\gamma^j,1}.$$

From (64) we get

$$|\sum_j e^*_{\gamma^j}(x^j)| \leq \sum_j \left(n_j^{-1} + m_k^{-1}\left(\sum_{r\in S_{\gamma^j,1}} \left|e^*_{\eta_r}\bar{P}^j_{I_r}(x^j)\right| + \sum_{r\in S_{\gamma^j,2}} \left|e^*_{\eta_r}\bar{P}^j_{I_r}(x^j)\right|\right)\right)$$

Using (62) for the $r \in S_{\gamma^j,1}$, (64) for the $r \in S_{\gamma^j,2}$ and that $\#S_{\gamma^j,1} + \#S_{\gamma^j,2} \leq n_k, k < j$, we get

$$\begin{aligned}
|\sum_j e^*_{\gamma^j}(x^j)| & \\
&\leq n_j^{-1} + \frac{n_k}{m_k n_j} \\
&\quad + \frac{1}{m_k}\sum_j \left(\sum_{r\in S_{\gamma^j,1}} \phi^j_r(y^j) + \sum_{r\in S_{\gamma^j,2}} w(e_{\eta_r})\sum_{s\in S_r} \left|e^*_{\eta_s}\bar{P}^j_{I_s}(x^j)\right|\right) \\
&\leq \sum_j \left(\frac{1}{n_j}\left(1 + \left(\frac{n_{j-1}}{m_{j-1}}\right)\right)\right. \\
&\quad \left. + \frac{1}{m_k}\sum_j \left(\sum_{r\in S_{\gamma^j,1}} \phi^j_r(y^j) + \sum_{r\in S_{\gamma^j,2}} w(e_{\eta_r})\sum_{s\in S_r} \left|e^*_{\eta_s}\bar{P}^j_{I_s}(x^j)\right|\right)\right).
\end{aligned} \tag{65}$$

Note that the functional $m_k^{-1}\left(\sum_{r\in S_{\gamma^j,1}}\sum_j \phi^j_r\right)$ belongs to the norming set of the mixed Tsirelson space X_{aux} and has room for $\#S_{\gamma^j,2}$ more functionals.
We shall replay the above splitting for every $e^*_{\eta_s}\bar{P}^j_{I_s}$. To avoid complicated notation we shall set $n_s = \#S_s$ and $m_s^{-1} = w(e^*_{\eta_s})$. From (65) using $e^*_{\eta_s}\bar{P}^j_{I_s}$ in the place of $e^*_{\gamma^j}$ we get

$$\left|\sum_{j} e_{\eta_s}^{*}\bar{P}_{I_s}^{j}(x^j)\right|$$

$$\leq \sum_{j}\left(\frac{1}{n_j}\left(1+\frac{n_{j-1}}{m_{j-1}}\right)\right.$$

$$\left.+\,m_s^{-1}\left(\sum_{t\in S_{s,1}} \phi_t^j(y^j)+\sum_{t\in S_{s,2}} m_t^{-1}\sum_{u\in S_t} \left|e_{\eta_u}^{*}\bar{P}_{I_u}^{j}(x^j)\right|\right)\right). \quad (66)$$

It follows that

$$\sum_{r\in S_{\gamma^j,2}} w(e_{\eta_r})\sum_{s\in S_r}\sum_{j} \left|e_{\eta_s}^{*}\bar{P}_{I_s}^{j}(x^j)\right| \leq \sum_{r\in S_{\gamma^j,2}} m_r^{-1}\sum_{s\in S_r} \frac{1}{n_j}\left(1+\frac{n_{j-1}}{m_{j-1}}\right) \quad (67)$$

$$+\sum_{r\in S_{\gamma^j,2}} m_r^{-1}\sum_{s\in S_r}\sum_{j} m_s^{-1}\left(\sum_{t\in S_{\gamma^j,1}} \phi_t^j(y^j)+\sum_{t\in S_{\gamma^j,2}} m_t^{-1}\sum_{u\in S_r} \left|e_{\eta_{su}}^{*}\bar{P}_{I_u}^{j}(x^j)\right|\right)$$

$$\leq n_k\frac{n_r}{m_r}\sum_{j} \frac{1}{n_j}\left(1\,+\frac{n_{j-1}}{m_{j-1}}\right) \qquad \text{since } \#S_{\gamma^j,2}\ \leq\ n_k \text{ and } \#S_r\ \leq\ n_r$$

$$+\sum_{j}\sum_{r\in S_{\gamma^j,2}} m_r^{-1}\sum_{s\in S_r} m_s^{-1}\left(\sum_{t\in S_{\gamma^j,1}} \phi_t^j(y^j)+\sum_{t\in S_{\gamma^j,2}} m_t^{-1}\sum_{u\in S_r} \left|e_{\eta_u}^{*}\bar{P}_{I_u}^{j}(x^j)\right|\right)$$

By (65) and (67), using that $\frac{n_r}{m_r}\,,\frac{n_k}{m_k}\ \leq\frac{n_{j-1}}{m_{j-1}}$ we get

$$\left|\sum_{j} e_{\gamma^j}^{*}(x^j)\right| \leq \sum_{j} \frac{1}{n_j}\left(1+\frac{n_{j-1}}{m_{j-1}}+\left(\frac{n_{j-1}}{m_{j-1}}\right)^2+\left(\frac{n_{j-1}}{m_{j-1}}\right)^3\right) \quad (68)$$

$$+\frac{1}{m_k}\sum_{j}\left(\sum_{r\in S_{\gamma^j,1}} \phi_r^j(y^j)\right.$$

$$+\sum_{r\in S_{\gamma^j,2}} m_r^{-1}\sum_{s\in S_r} m_s^{-1}\left(\sum_{t\in S_{s,1}} \phi_t^j(y^j)\right.$$

$$\left.\left.+\sum_{t\in S_{s,2}} m_t^{-1}\sum_{u\in S_t} \left|e_{\eta_u}^{*}\bar{P}_{I_u}^{j}(x^j)\right|\right)\right) (69)$$

Note that the functional

$$\sum_{j} \phi_{\gamma^j}^{j}=\frac{1}{m_k}+\sum_{j}\left(\sum_{r\in S_{\gamma^j,1}} \phi_r^j(y^j)\sum_{r\in S_{\gamma^j,2}} m_r^{-1}\sum_{s\in S_r} m_s^{-1}\sum_{t\in S_{s,1}} \phi_{t(y^j)}^{j}\right)$$

belongs to the norming set of the mixed Tsirelson space X_{aux} and the functional $m_s^{-1} \sum_{t \in S_{s,1}} \phi_t^j$ has room for $\# S_{s,2}$ more functionals.
We continue this splitting at most l_j times, see (9) for the choice of l_j , or till $S_{s,2} = \emptyset$ i.e. we do not have nodes with weight $> m_j^{-1}$.
If we stop before the l_j-th step we get that $|e^*_{\gamma^j}(x^j)|$ is dominated by $\phi^j_{\gamma^j}(y^j)$ plus the errors in (68), where the sum end to the l_j-th power of n_{j-1}/m_{j-1}. Since $\phi^j_{\gamma^j}$ belongs to the norming set of the mixed Tsirelson space X_{aux} it follows from [99], Lemma II.9, that

$$\phi^j_{\gamma^j}(y^j) \leq 4 m_k^{-1} m_j^{-1} .$$

If we continue the splitting l_j -times, then there exists some node with $(\gamma_t^j) > m_j^{-1}$. For every such node we have

$$\sum_j \left(\prod_{s \prec t} w\left(e_{\gamma_s^j}\right)\right)\left|e^*_{\gamma_t^j}(x^j)\right| \leq \left(\frac{1}{m_1}\right)^{l_k} \sum_j \left|e^*_{\gamma_t^j}(x^j)\right|$$

$$\leq 2 \sum_j m_k^{-1} m_j^{-1} \frac{\#\left\{i : rng\left(d^{j*}_{\xi_i}\right) \subset I_t \cap G\right\}}{n_j}$$

since $m_1^{-l_j} \leq \left(m_j m_{j-1}\right)^{-1}$.
Summing the estimation of all those nodes we get upper estimate equal to ${}^{2\#G}/_{m_k m_j n_j} \leq {}^{2}/_{m_k m_j}$.
The remaining nodes provide us with a functional in the norming set of the mixed Tsirelson space X_{aux}. By [99] its action on y is bounded by $4 m_k^{-1} m_j^{-1}$.
It remains to handle the errors (68). In each case we have

$$\frac{1}{n_j}\left(1 + \frac{n_{j-1}}{m_{j-1}} + \left(\frac{n_{j-1}}{m_{j-1}}\right)^2 + \cdots + \left(\frac{n_{j-1}}{m_{j-1}}\right)^{l_j}\right) \leq \frac{1}{n_j} \frac{\left(n_{j-1}/m_{j-1}\right)^{l_j+1} - 1}{\left(n_j/m_{j-1}\right)} - 1$$

$$\leq \frac{2}{m_j m_{j-1}} .$$

Summing all the above estimates we get an upper estimate $7 m_k^{-1} m_j^{-1}$.
Corollary (6.3.45)[260]: [252] Let $x^j = m_{i+\epsilon} n_{i+\epsilon}^{-1} \sum_{i=1}^{n_{i+\epsilon}} \bar{d}^j_{\xi_i}$ such that no two ξ_i's are neighbours. Let $\epsilon > 0, \left(e^*_{\eta_{q^2+\epsilon}}\right)_{q^2+\epsilon=1}^{n_i}$ be nodes such that $w\left(e^*_{\eta_{q^2+\epsilon}}\right) = m_{l_{q^2+\epsilon}} \neq m_{i+\epsilon}$ and $m_{l_{q^2+\epsilon}} < m_{l_{q^2+\epsilon+1}}$ for all $q^2 + \epsilon \leq n_i$. Then

$$\sum_{q^2+\epsilon=1}^{n_i} \sum_j |e^*_{\eta_{q^2+\epsilon}} \bar{P}^j_{I_{q^2+\epsilon}}(x^j)| \leq \frac{14}{m_{q_1^2+\epsilon}} . \tag{70}$$

Proof. From Corollary (6.3.44) we get

$$\sum_{q^2+\epsilon=1}^{n_i}\sum_j \left|e^*_{\eta_{q^2+\epsilon}}\bar{P}^j_{I_{q^2+\epsilon}}(x^j)\right| \leq \sum_{q^2+\epsilon: l_{q^2+\epsilon}<i+\epsilon}\sum_j \left|e^*_{\eta_{q^2+\epsilon}}\bar{P}^j_{I_{q^2+\epsilon}}(x^j)\right| + \sum_{q^2+\epsilon: l_{q^2+\epsilon}>i+\epsilon}\sum_j |e^*_{\eta_{q^2+\epsilon}}\bar{P}^j_{I_{q^2+\epsilon}}(x^j)|$$

$$\leq \sum_{q^2+\epsilon: l_{q^2+\epsilon}<i+\epsilon} \frac{7}{m_{q^2+\epsilon}} + \sum_{q^2+\epsilon: l_{q^2+\epsilon}>i+\epsilon} \left(\frac{1}{n_{i+\epsilon}} + \frac{2m_{i+\epsilon}}{m_{q^2+\epsilon}}\right)$$

$$\leq \sum_{q^2+\epsilon: l_{q^2+\epsilon}<i+\epsilon} \frac{7}{m_{q^2+\epsilon}} + \frac{n_i}{n_{i+\epsilon}} + \sum_{q^2+\epsilon: l_{q^2+\epsilon}>i+\epsilon} \frac{2}{m_{q^2+\epsilon-1}} \leq \frac{14}{m_{q_1^2+\epsilon}}.$$

Corollary (6.3.46)[260]: [252] The set Γ is a self-determined subset of $\bar{\Gamma}$, hence it defines a BD-L_∞-space $\mathcal{X}_{\left(\Gamma_{q^2}, i_{q^2}\right)_{q^2}}$.

Moreover, the restriction $R: \mathcal{X}_{\bar{\Gamma}} \to \mathcal{X}_{\left(\Gamma_{q^2}, i_{q^2}\right)_{q^2}}$ is a well-defined operator of norm at most 1 inducing the isomorphism between $\mathcal{X}_{\left(\Gamma_{q^2}, i_{q^2}\right)_{q^2}}$ and $\mathcal{X}_{\bar{\Gamma}}/Y$, where $Y = \overline{\langle d^j_{\gamma^j} : \gamma^j \in \bar{\Gamma} \setminus \Gamma\rangle}$.

Proof. According to Proposition 1.5 [225] it is enough to show that for every $\gamma^j \in \Delta_{q^2+1}$ the following holds

$$\bar{c}^*_{\gamma^j} \in \{e^*_{\gamma^j} \circ P^j_E : \gamma^j \in \Gamma_{q^2}, E \subset \mathbb{N} \cup \{0\}\}$$

This follows readily from the definition of $\bar{c}^*_{\gamma^j}$, see (10), using that $\sum_j \bar{d}^{j*}_{\gamma^j} = \sum_j e^*_{\gamma^j} \circ P^j_{\{rank(\gamma^j)\}}$.

The second part of Proposition follows by Proposition 1.9 [225].

Corollary (6.3.47)[260]: [252]. a) Let $j \in \mathbb{N}$ and $k \leq n_{2j}$. Let also $(x^j_k)_k \subset \mathcal{X}_{Kus}$ be a normalised skipped block double sequence such that $\mathrm{rng}_{FDD}(x^j_k) = (q^2_{k-1}+\epsilon, q^2_k+\epsilon]$ for some strictly increasing $(q^2_k+\epsilon)$ with $q^2_1+\epsilon \geq 2j-1$. Then there exists a node $\gamma^j \in \Gamma$ such that

$$\sum_j e^*_{\gamma^j} = \sum_{k=1}^{n_{2j}}\sum_j \left(d^{j*}_{\xi_k} + m^{-1}_{2j}\sum_{k=1}^{n_{2j}} \varepsilon_k e^*_{\eta_k} P^j_{I_k}\right)$$

with the following properties

(i) $\mathrm{rank}(\xi_k) = q^2_k+\epsilon+1$ for each k,

(ii) $\varepsilon_k e^*_{\eta_k}\sum_j P^j_{I_k}(x^j_k) \geq 1/2$ and $\eta_k \in \Gamma_{q^2_k+\epsilon} \setminus \Gamma_{q^2_{k-1}+\epsilon}$ for each k,

(iii) $\sum_j e^*_{\gamma^j}\left(\sum_{k=1}^{n_{2j}} x^j_k\right) \geq \sum_j \frac{n_{2j}}{2m_{2j}}$.

b) Let $\left(d^j_{\xi_i}\right)_{i=1}^{n_{2j}}$ be a finite subsequence of the basis such that $\mathrm{rank}(\xi_i)+1 < \mathrm{rank}(\xi_{i+1})$ for every i and $\mathrm{rank}(\xi_1) \geq 2j-1$.

Then the node

$$e_\xi^* = \sum_{i=1}^{n_{2j}} \sum_j \left(d_{\zeta_i}^{j*} + m_{2j}^{-1} d_{\xi_i}^{j*}\right) \tag{71}$$

with $\text{rank}(\zeta_i) = \text{rank}(\xi_i) + 1$ is a regular node and $e_\xi^* \left(\sum_{i=1}^{n_{2j}} \sum_j d_\xi^j\right) = \sum_j \frac{n_{2j}}{m_{2j}}$.

Proof. a) (see [91]) Let $x_k^j = i_k(u_k^j)$ where $u_k^j \in \Gamma_{q_k^2+\epsilon} \setminus \Gamma_{q_{k-1}^2+\epsilon}$ is the restriction of x_k^j on $\Gamma_{q_k^2+\epsilon}$. Since

$$2\left\|\sum_j u_k^j\right\| \geq \sum_j \left\|i_{q_k^2+\epsilon}(u_k^j)\right\| = \sum_j \|x_k^j\| = 1$$

we can choose $\eta_k \in \Gamma_{q_k^2+\epsilon} \setminus \Gamma_{q_{k-1}^2+\epsilon}$ such that $|e_{\eta_k}^*(u_k^j)| \geq {}^1\!/_2$. Setting $I_k = \text{rng}_{FDD}\left(x_k^j\right) = \cup_{i=q_{k-1}^2+\epsilon+1}^{q_k^2+\epsilon} \Delta_i$, choose $\varepsilon_k \in \{-1, 1\}$ such that

$$\left|\sum_j e_{\eta_k}^* P_{I_k}^j\left(x_k^j\right)\right| = \sum_j \varepsilon_k e_{\eta_k}^* P_{I_k}^j(x_k^j) = \sum_j \varepsilon_k e_{\eta_k}^*(u_k^j) \geq {}^1\!/_2. \tag{72}$$

The nodes $\gamma_k^j = (q_k^2 + \epsilon + 1, \gamma_{k-1}^j, m_{2j}, I_k, \varepsilon_k, e_{\eta_k}^*), \gamma_0^j = 0, k = 1, \ldots, n_{2j}$ give the node $\gamma^j = \gamma_{n_{2j}}^j$ with the properties (i)–(iii).

b) Take the nodes $\zeta_i = (\text{rank}(\xi_i) + 1, \zeta_{i-1}, m_{2j}, I_i, 1, e_{\xi_i}^*), \zeta_0 = 0,$ where $I_i = \Delta_{\text{rank}(\xi_i)}$.

Corollary (6.3.48)[260]: [252]. Let $\left(z_k^j\right)_k$ be a normalised block double sequence in $\mathcal{X}_{Kus}$ and $\left(d_{\xi_n}^j\right)_{n\in M}$ be a subsequence of the basis. Then for every $j \in \mathbb{N}$ there exists a j-dependent sequence of length n_{2j-1}, $\left(\gamma_i^j, x_i^j\right)_{i \leq n_{2j-1}}$, such that $x_{2i-1}^j \in \langle z_k^j : k \in \mathbb{N}\rangle$ and $x_{2i}^j \in \langle d_{\xi_n}^j : n \in M\rangle$.

Proof. Passing to a further subsequence we may assume that

$$d_{\xi_n}^j \text{ are pairwise non}-\text{neighbours and } \text{rank}(\xi_n) + 1 < \text{rank}(\xi_{n+1}). \tag{73}$$

Let j_1 be such that $m_{4j_1-2} > n_{2j-1}^2$ and choose q_1^2 big enough to guarantee that $4j_1 - 2 < q_1^2$ and $2^{-q_1^2} \leq {}^1\!/_{4n_{2j-1}^2}$.

Let $\left(x_{1,k}^j\right)_{k=1}^{n_{4j_1-2}}$ be a normalised skipped block sequence of $\langle z_l^j : l \geq q^2\rangle$ which is a C-RIS. Setting

$$x_1^j = \frac{c_1 m_{4j_1-2}}{n_{4j_1-2}} \sum_{k=1}^{n_{4j_1-2}} \sum_j x_{1,k}^j \qquad \text{with } \sum_j \|x_1^j\| = {}^1\!/_2$$

from Remark (6.3.30) we get ${}^1\!/_{20C} \leq c_1 \leq 2$ and that there exists a node $\eta_1 \in \Gamma$ with $w(\eta_1) = m_{4j_1-2}^{-1}$ such that

$$\sum_j e_{\eta_1}^* P_{I_1}^j(x_1^j) \geq \frac{1}{40C},$$

where $I_1 = \cup \{\Delta_{q^2+\epsilon} : q^2 + \epsilon \in \text{rng}_{FDD}(x_1^j)\}$.

Using that R is a quotient operator of norm 1 take a block $\bar{y}_1^j \in \mathcal{X}_{\bar{\Gamma}}$ such that $x_1^j = R(\bar{y}_1^j)$ and $\|\bar{y}_1^j\| \leq 1$. Then choose a vector $\bar{x}_1^j$ with rational coefficients in the unit ball of $\langle \bar{d}_{\gamma^j}^j : \gamma^j \in \bar{\Gamma}_{q_1^2}\rangle$ such that $\sum_j \|\bar{x}_1^j - \bar{y}_1^j\|_{\mathcal{X}_{\bar{\Gamma}}} \leq 1/4n_{q_1^2}^2$.

Note that $\sum_j R(\bar{x}_1^j) = \sum_j R(\bar{x}_1^j - \bar{y}_1^j) + \sum_j R(\bar{y}_1^j) = \sum_j R(\bar{x}_1^j - \bar{y}_1^j) + \sum_j x_1^j$ and hence for every $\gamma^j \in \Gamma$,

$$\sum_j |e_{\gamma^j}^*(\bar{x}_1^j) - e_{\gamma^j}^*(x_1^j)| = \sum_j |e_{\gamma^j}^* R(\bar{x}_1^j) - e_{\gamma^j}^* R(x_1^j)| \leq \sum_j \left\|e_{\gamma^j}^* \circ R\right\| \|\bar{x}_1^j - \bar{y}_1^j\|_{\mathcal{X}_{\bar{\Gamma}}}$$
$$\leq 1/4n_{q_1^2}^2 . \tag{74}$$

We take γ_1^j to be the node

$$\gamma_1^j = \left(q_1^2 + 1, 0, m_{2j-1}^{-1}, I_1, 1, e_{\eta_1}^*\right).$$

From the above we get that (γ_1^j, x_1^j) is a j-dependent couple of length 1 with respect to the (Γ, j)-special sequence $(\gamma_1^j, \bar{x}_1^j)$.

Set $j_2 = \sigma(\gamma_1^j, \bar{x}_1^j)$ and choose $x_2^j, e_{\eta_2}^*$ such that

$$x_2^j = m_{4j_2}\, n_{4j_2}^{-1} \sum_{k\in F_2} \sum_j d_{\xi_{2,k}}^j \in \mathcal{X}_{Kus} \quad \text{and} \quad \mathrm{mt}(e_{\eta_2}^*) = m_{4j_2}^{-1} \sum_{k\in F_2} \sum_j d_{\xi_{2,k}}^{j*}$$

where $|F_2| = n_{4j_2}$ and $q_1^2 + 2 < \min \mathrm{rng}_{FDD}(x_2^j)$. Such a node exists by Corollary (6.3.47)(b) since $\mathrm{rank}(\xi_n) + 1 < \mathrm{rank}(\xi_{n+1})$. We also take the node

$$\gamma_2^j = (q_2^2 + 1, \gamma_1^j, m_{2j-1}, I_2, 1, \lambda_2^j e_{\eta_2}^*) \in \Gamma$$

where $I_2 = [q_2^2 + \epsilon, q_2^2]$ is the range of x_2^j with respect to the basis and $\lambda_2^j \in \mathrm{Net}_{1,q_1^2}$ is chosen such that

$$\left|\sum_j \left(\lambda_2^j - e_{\eta_1}^*(\bar{x}_1^j)\right)\right| \leq 1/4n_{q_1^2}^2 .$$

From the above equation and (74) we get

$$\left|\sum_j \left(\lambda_2^j - e_{\eta_1}^*\left(x_1^j\right)\right)\right| \leq \frac{1}{2n_{q_1^2}^2} \Rightarrow \lambda_2^j \geq \sum_j e_{\eta_1}^*\left(x_1^j\right) - \frac{1}{2n_{q_1^2}^2} \geq \frac{c_1}{2} - \frac{1}{2n_{q_1^2}^2} \geq \frac{1}{45C}.$$

Pick $\bar{x}_2^j$ to be the corresponding average of $\left(\bar{d}_{\xi_{2,k}}^j\right)_{k\in F_2}$. It follows that $x_2^j = R\bar{x}_2^j$ (recall that $d_{\gamma^j}^j = R\bar{d}_{\gamma^j}^j$ for each $\gamma^j \in \Gamma$) and $\bar{x}_1^j < \bar{x}_2^j$. Then we get that $\left(\gamma_i^j, x_i^j\right)_{i=1}^2$ is j-dependent of length 2 with respect to the (Γ, j)-special sequence $\left(\gamma_i^j, \bar{x}_i^j\right)_{i=1}^2$.

Set $j_3 = \sigma\left(\gamma_i^j, \bar{x}_i^j\right)_{i=1}^2$. We continue to choose $x_3^j, e_{\gamma_3^j}^*, x_4^j, e_{\gamma_4^j}^*$ in the same way we have chosen $x_1^j, e_{\gamma_1^j}^*, x_2^j, e_{\gamma_2^j}^*$ taking care that $x_1^j, x_2^j, x_3^j, x_4^j$ is a skipped block sequence (with respect to the FDD) and repeat the procedure obtaining the desired dependent sequence.

Notice that for a dependent sequence $(\gamma_i^j, x_i^j)_{i\le n_{2j-1}}$ with a constant C we have $\left\|\frac{m_{2j-1}}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}} x_i^j\right\| \ge {}^1/_{45C}$. Indeed, consider the functional $e^*_{\zeta_{n_{2j-1}}}$ determined by the nodes $(\gamma_i^j)_{i=1}^{n_{2j-1}}$, i.e. of the form

$$\sum_j e^*_{\zeta_{n_{2j-1}}} = \sum_{i=1}^{n_{2j-1}}\sum_j \left(d^{j*}_{\gamma_i^j} + m_{2j-1}^{-1}\sum_{i=1}^{\frac{n_{2j-1}}{2}} \left(e^*_{\eta_{2i-1}}P^j_{I_{2i-1}} + \lambda^j_{2i}e^*_{\eta_{2i}}P^j_{I_{2i}}\right)\right),$$

and notice that

$$\sum_j e^*_{\zeta_{n_{2j-1}}}\left(\frac{m_{2j-1}}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}}\sum_j x_i^j\right)$$
$$\ge \sum_j \frac{1}{n_{2j-1}}\left(\sum_{i=1}^{n_{2j-1}/2} e^*_{\eta_{2i-1}}P^j_{I_{2i-1}}(x^j_{2i-1}) + \lambda^j_{2i}e^*_{\eta_{2i}}(x^j_{2i})\right)$$
$$\ge \sum_j \frac{1}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}/2}\left(\frac{c_{2i-1}}{2} + \frac{c_{2i-1}}{2} - \frac{1}{2n^2_{q^2_{2i-1}}}\right) \ge \frac{1}{45C},$$

using that $c_{2i-1} \ge {}^1/_{20C}$.

Corollary (6.3.49)[260]: [252]. Let $(\gamma_i^j, x_i^j)_{i\le n_{2j-1}}$ be a j-dependent sequence. Then

$$\left\|\sum_j \frac{1}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}} (-1)^{i+1}x_i^j\right\| \le \sum_j \frac{250}{m^2_{2j-1}}.$$

Proof. Let J be an interval of $\{1,\dots,n_{2j-1}\}$ and $z^j = \sum_{i\in J}\sum_j (-1)^{i+1}x_i^j$. We shall verify the assumption (b) in Lemma (6.3.20) for $j_0 = 2j-1$.

Let $(\gamma_k^j, \bar{x}_k^j)_{k=1}^{n_{2j-1}}$ be the special sequence associated with the dependent sequence $(\gamma_k^j, x_k^j)_{k=1}^{n_{2j-1}}, \gamma_k^j = (q_k^2+1, \gamma_{k-1}^j, m_{2j-1}, I_k, \epsilon_k, \lambda_k^j e^*_{\eta_k})$ for each k, where $\gamma_0^j = 0$.

Consider a node β with evaluation analysis

$$e^*_\beta = \sum_{i=1}^{n_{2j-1}}\sum_j d^{j*}_{\xi_i} + m^{-1}_{2j-1}\sum_{i=1}^{n_{2j-1}/2}\sum_j \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{\tilde{I}_{2i-1}} + \tilde{\lambda}^j_{2i}e^*_{\tilde{\delta}_{2i}}P^j_{\tilde{I}_{2i}}\right)$$

which is produced from a (Γ, j)-special sequence $(\zeta_k, \bar{z}_k^j)_{k\le n_{2j-1}}$. Let

$$k_0 = \min\{k \le n_{2j-1} : (\gamma_k^j, \bar{x}_k^j) \ne (\zeta_k, \bar{z}_k^j)\}$$

if such a k exists. We estimate separately $|e^*_{\beta_{k_0}-1}(z^j)|$ and $|\left(e^*_\beta - e^*_{\beta_{k_0}-1}\right)(z^j)|$.

We start with $|e^*_{\beta_{k_0}-1}(z^j)|$. Notice that $e^*_{\beta_{k_0}-1}$, if $k_0 > 1$, has the following evaluation analysis

$$e^*_{\beta_{k_0-1}} = \sum_{i=1}^{k_0-1}\sum_j d^{j*}_{\xi_i} + m^{-1}_{2j-1}\sum_{i=1}^{\left\lfloor\frac{(k_0-1)}{2}\right\rfloor}\sum_j (\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{I_{2i-1}} + \tilde{\lambda}^j_{2i}e^*_{\eta_{2i}}P^j_{I_{2i}})$$
$$+\sum_j [\tilde{\epsilon}_{k_0-1}e^*_{\tilde{\delta}_{k_0-1}}P^j_{I_{k_0-1}}],$$
where $e^*_{\tilde{\delta}_{2i-1}}$ and $e^*_{\eta_{2i-1}}$ have compatible tree-analyses and the last term in square brackets appears if $k_0 - 1$ is odd. By the definition of nodes we have $\mathrm{rank}(\xi_i) = \mathrm{rank}(\gamma^j_i) \in (\max \mathrm{rng}_{FDD}(x^j_i), \min \mathrm{rng}_{FDD}(x^j_{i+1}))$ for every $i < k_0$. Therefore
$$\left(\sum_{i=1}^{k_0-1}\sum_j d^{j*}_{\xi_i}\right)\sum_i (-1)^{i+1}x^j_i = 0. \tag{75}$$
We partition the indices $P^j = \{1,2,\dots,\lfloor(k_0-1)/2\rfloor\}$ into the sets $A = \{i \in P^j : e^*_{\tilde{\delta}_{2i-1}}\sum_j P^j_{I_{2i-1}}(\bar{x}^j_{2i-1}) \neq 0\}$ and its complement B.

For every $i \in A$ from the choice of $\tilde{\lambda}^j_{2i}$, the fact that $\mathrm{rng}(x^j_{2i-1}) \subset I_{2i-1}$ and (3) of Definition (6.3.29) we have
$$\left|\sum_j \left(\tilde{\lambda}^j_{2i} - \tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}(\bar{x}^j_{2i-1})\right)\right| \leq \sum_j \frac{1}{4n^2_{2j-1}} \quad \text{and} \tag{76}$$
$$\left|\sum_j \left(e^*_{\tilde{\delta}_{2i-1}}(\bar{x}^j_{2i-1}) - e^*_{\tilde{\delta}_{2i-1}}P^j_{I_{2i-1}}(x^j_{2i-1})\right)\right| = \sum_j |e^*_{\tilde{\delta}_{2i-1}}(\bar{x}^j_{2i-1}) - e^*_{\tilde{\delta}_{2i-1}}(x^j_{2i-1})|$$
$$\leq \sum_j \frac{1}{4n^2_{2j-1}}.$$
It follows that
$$\left|\sum_j \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{I_{2i-1}}(x^j_{2i-1}) + \tilde{\lambda}^j_{2i}e^*_{\eta_{2i}}P^j_{I_{2i}}(-x^j_{2i})\right)\right|$$
$$= \left|\sum_j \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{I_{2i-1}}(x^j_{2i-1}) - \tilde{\lambda}^j_{2i}\right)\right| \leq \sum_j \frac{1}{2n^2_{2j-1}} \quad \text{by (6.7).} \tag{77}$$
Similarly for every $i \in B$,
$$\left|\sum_j \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{I_{2i-1}}(x^j_{2i-1}) + \tilde{\lambda}^j_{2i}e^*_{\eta_{2i}}P^j_{I_{2i}}(-x^j_{2i})\right)\right|$$
$$= \left|\sum_j \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{I_{2i-1}}(x^j_{2i-1}) - \tilde{\lambda}^j_{2i}\right)\right| \leq \sum_j \frac{1}{2n^2_{2j-1}}. \tag{78}$$
For an interval $J = [l,m]$ using that $\|x^j_{2i-1}\| = {}^1\!/_2, \|x^j_{2i}\| \leq 7$ (by Corollary (6.3.44)) and inequalities (75), (78) we obtain

$$|e^*_{\beta_{k_0-1}}\left(\sum_{i\in J}\sum_j \ (-1)^{i+1}x_i^j\right)| \leq 10.$$

Now we proceed to estimate $|(e^*_\beta \ - \ e^*_{\beta_{k_0-1}})(z^j)|$.

Observe that as x^j_{2l-1} is a weighted average of a normalised C-RIS of length $n_{j_{2l-1}}$ we have

$$\left|\sum_{i=k_0}^{n_{2j-1}}\sum_j \ d^{j*}_{\xi_i}\left(x^j_{2l-1}\right)\right| \leq 3\sum_j \ n_{2j-1}c_{2i-1}C\frac{m_{j_{2l-1}}}{n_{j_{2l-1}}} \leq \ 2m^{-2}_{j_{2l-1}} < \sum_j \ n^{-3}_{2j-1} \tag{79}$$

The same inequality holds also for the averages of the basis i.e.

$$\left|\sum_{i=k_0}^{n_{2j-1}}\sum_j \ d^{j*}_{\xi_i}\left(x^j_{2l}\right)\right| \leq \sum_j \ n_{2j-1}\frac{m_{j_{2l}}}{n_{j_{2l}}} \leq \ m^{-3}_{j_{2l}} < \sum_j \ n^{-3}_{2j-1} \quad \forall l. \tag{80}$$

We shall distinguish the cases when k_0 is odd or even. Assume first that $k_0 \ = \ 2i_0 \ - \ 1$ for some i_0.

Then for every $i \ < \ i_0$ and every $k > k_0$,

$$\sum_j \ (\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{\tilde{I}_{2i-1}} + \tilde{\lambda}^j_{2i}e^*_{\tilde{\delta}_{2i}}P^j_{\tilde{I}_{2i}})(x^j_k) = 0.$$

From the injectivity of σ it follows that $w(e^*_{\tilde{\delta}_{2i-1}}), w(e^*_{\tilde{\delta}_{2i}}) \notin \left\{w\left(e^*_{\eta_{i'}}\right)\middle| \, i' \ > i_0\right\}$ for every $i \ > \ i_0$. Hence by Corollary (6.3.21), using that $|\tilde{\lambda}^j_{2i}| \ \leq \ 1$ and $c_k \ \leq \ 2$, we get for every odd $k > k_0$ the following

$$\begin{aligned}\left|\sum_{i\geq i_0}^{n_{2j-1}/2} \ \sum_j \ \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{\tilde{I}_{2i-1}} + \tilde{\lambda}^j_{2i}e^*_{\tilde{\delta}_{2i}}P^j_{\tilde{I}_{2i}}\right)\left(x^j_k\right)\right| \\ \leq 64c_kC \ w(\delta_1) \leq 128C\sum_j \ n^{-2}_{2j-1}.\end{aligned} \tag{81}$$

Also from Corollary (6.3.45) we obtain for every even $k > k_0$ the following

$$|\sum_{i\geq i_0}^{\frac{n_{2j-1}}{2}}\sum_j \ (\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{\tilde{I}_{2i-1}} + \tilde{\lambda}^j_{2i}e^*_{\tilde{\delta}_{2i}}P^j_{\tilde{I}_{2i}})(x^j_k)| \ \leq 14\sum_j \ n^{-2}_{2j-1}. \tag{82}$$

For $x^j_{k_0}$ we also obtain the following

$$\begin{aligned}&\left|\sum_{i\geq i_0}^{n_{2j-1}/2} \ \sum_j \ \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{\tilde{I}_{2i-1}} + \tilde{\lambda}^j_{2i}e^*_{\tilde{\delta}_{2i}}P^j_{\tilde{I}_{2i}}\right)\left(x^j_{k_0}\right)\right| \\ &\leq \sum_j \ \left|e^*_{\tilde{\delta}_{k_0}}P^j_{\tilde{I}_{k_0}}\left(x^j_{k_0}\right)\right| \\ &+\sum_j \left|\left(\tilde{\lambda}^j_{k_0+1}e^*_{\tilde{\delta}_{k_0+1}}P^j_{\tilde{I}_{k_0+1}} + \sum_{i\geq i_0}^{\frac{n_{2j-1}}{2}} \ \left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\delta}_{2i-1}}P^j_{\tilde{I}_{2i-1}} + \tilde{\lambda}^j_{2i}e^*_{\tilde{\delta}_{2i}}P^j_{\tilde{I}_{2i}}\right)\right)\left(x^j_{k_0}\right)\right|\end{aligned} \tag{83}$$

$$\leq 4 + 128C \sum_{j} n_{2j-1}^{-2},$$

using that $\left\|x_{k_0}^j\right\| \leq 1$ and $\left\|\sum_j e_{\gamma^j}^* \circ P_I^j\right\| \leq \sum_j \left\|P_I^j\right\| \leq 4$ while for the second term we get the upper bound as in (81).

The case where k_0 is even is similar, except that $|\sum_j e_{\tilde{\delta}_{k_0}}^* P_{\tilde{I}_{k_0}}^j (x_{k_0}^j)| \leq 7$.

Splitting J to $J_1 = J \cap [1, i_0], J_2 = J \cap (i_0, n_{2j-1})$ and considering the cases when min J_1 is odd or even we get $|\left(e_\beta^* - e_{\beta_{k_0-1}}^*\right)\left(\sum_{i\in J}\sum_j (-1)^{i+1} x_i^j\right)| \leq 15$, using that $n_{2j+1} > 200C$.

Corollary (6.3.50)[260]: [252]. Let $T^j : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator and $\left(d_{\gamma_n^j}^j\right)_{j,n\in M}$ be a subsequence of the basis. Then

$$\lim_{M\ni n\to+\infty} \operatorname{dist}(T^j d_{\gamma_n^j}^j, \mathbb{R} d_{\gamma_n^j}^j) = 0.$$

Proof. Assume that $\operatorname{dist}(T^j d_{\gamma_n^j}^j, \mathbb{R} d_{\gamma_n^j}^j) > 4\delta$ for infinitely many $n \in M$ and some $\delta > 0$.

By Corollary (6.3.27) and Corollary (6.3.43) passing to a further subsequence and admitting a small perturbation we may assume that

(P1) $\left(T^j d_{\gamma_n^j}^j\right)_{n\in M}$ is a skipped block sequence and setting R_n to be the minimal interval containing $\operatorname{rng}(T^j d_{\gamma_n^j}^j)$ and $\{n\}$ we have

$$\max \operatorname{rank}(R_n) + 2 < \min \operatorname{rank}(R_{n+1}),$$

(P2) no two elements of $\left(d_{\gamma_n^j}^j\right)_{n\in M}$ are neighbours.

By the assumption that $\operatorname{dist}(T^j d_{\gamma_n^j}^j, \mathbb{R} d_{\gamma_n^j}^j) > 4\delta$ it follows that either

$$\sum_j \left\|P_{n-1}^j T^j d_{\gamma_n^j}^j\right\| \geq 2\delta \quad \text{or} \quad \sum_j \left\|\left(I - P_n^j\right) T^j d_{\gamma_n^j}^j\right\| \geq 2\delta$$

(recall that P_m^j denotes the canonical projection onto $\langle d_{\gamma_i^j}^j : i \leq m\rangle, m \in \mathbb{N}$).

Passing to a further subsequence we may assume that one of the two alternatives holds for any $n \in \mathbb{N}$. Let

$$q_n^2 = \begin{cases} \max \operatorname{rank}\left(P_{n-1}^j T^j d_{\gamma_n^j}^j\right) & \text{in the first case} \\ \max \operatorname{rank}\left(\left(I - P_n^j\right) T^j d_{\gamma_n^j}^j\right) & \text{in the second case.} \end{cases}$$

In the first case we take $I_n = [\min \operatorname{rng}(T^j d_{\gamma_n^j}^j), n-1]$. Also $\sum_j P_{n-1}^j T^j d_{\gamma_n^j}^j = \sum_j i_{q_n^2}(u_n^j)$ where $u_n^j = r_{q_n^2}(P_{n-1}^j T^j d_{\gamma_n^j}^j)$ and hence we may choose $\epsilon_n \in \{-1, 1\}$ and $\eta_n \in \Gamma_{q_n^2} \setminus \Gamma_{max\, rank(R_{n-1})+1}$ such that

$$\sum_j \epsilon_n e_{\eta_n}^* P_{I_n}^j (T^j d_{\gamma_n^j}^j) = \sum_j \epsilon_n e_{\eta_n}^* (P_{n-1}^j T^j d_{\gamma_n^j}^j) = \sum_j \epsilon_n e_{\eta_n}^* (u_n^j) \geq \delta \tag{84}$$

using that $2\delta \leq \sum_j \left\|i_{q_n^2}(u_n^j)\right\| \leq 2\sum_j \left\|u_n^j\right\|$.

In the second case we take $I_n = [n+1, \max \operatorname{rng}\left(T^j d^j_{\gamma^j_n}\right)]$. Also since $\sum_j \left((I - P^j_n)T^j d^j_{\gamma^j_n}\right) = \sum_j i_{q_n^2}(u^j_n)$ where $\sum_j u^j_n = r_{q_n^2}\sum_j \left((I - P^j_n)T^j d^j_{\gamma^j_n}\right)$ we get $\epsilon_n \in \{-1, 1\}, \eta_n \in \Gamma_{q_n^2} \setminus \Gamma_{\max \operatorname{rng} R_{n-1}+1}$ such that

$$\sum_j \epsilon_n e^*_{\eta_n} P^j_{I_n}(T^j d^j_{\gamma^j_n}) = \sum_j \epsilon_n e^*_{\eta_n}((I - P^j_n)T^j d^j_{\gamma^j_n}) = \sum_j \epsilon_n e^*_{\eta_n}(u^j_n) \geq \delta. \tag{85}$$

Given any $j \in \mathbb{N}$ we shall build a vector y^j with $\left\|\sum_j T^j y^j\right\| \geq \delta/28m_{2j-1}$ and $\left\|y^j\right\| \leq 420/m^2_{2j-1}$ which for sufficiently big j yields a contradiction.
Assume the first case holds. The second case will follow analogously. Notice that by (P1) for any $i \in \mathbb{N}$ and $A \subset M$ with $\#A = n_{2i}$ and $\max \operatorname{rank}(R_{\min A}) \geq 2i-1$ there is a functional e^*_ψ associated to a regular node of the form

$$e^*_\psi = \sum_{n\in A}\sum_j d^{j*}_{\xi_n} + \frac{1}{m_{2i}} \sum_{n\in A}\sum_j \epsilon_n e^*_{\eta_n} P^j_{I_n}$$

with $\operatorname{rank}(\xi_n) = \max \operatorname{rank}(R_n) + 1$ for each $n \in A$. Let $x^j = m_{2i} n^{-1}_{2i} \sum_{n\in A}\sum_j d^j_{\gamma^j_n}$.
It follows that

$$\left\|\sum_j T^j x^j\right\| \geq \sum_j e^*_\psi(T^j x^j) = \sum_j \left(\sum_{n\in A} d^{j*}_{\xi_n} + \frac{1}{m_{2i}}\sum_{n\in A} \epsilon_n e^*_{\eta_n} P^j_{I_n}\right)\left(\frac{m_{2i}}{n_{2i}}\sum_{n\in A} T^j d^j_{\gamma^j_n}\right)$$

$$= m_{2i} n^{-1}_{2i} \sum_{n\in A}\sum_j d^{j*}_{\xi_n}(T^j d^j_{\gamma^j_n}) + \frac{1}{n_{2i}}\sum_{n\in A}\sum_j \epsilon_n e^*_{\eta_n} P^j_{I_n}(T^j d^j_{\gamma^j_n})$$

$$= \frac{1}{n_{2i}}\sum_{n\in A}\sum_j \epsilon_n e^*_{\eta_n} P^j_{k_{n-1}}(T^j d^j_{\gamma^j_n}) \geq \delta.$$

Fix $j \in \mathbb{N}$ and choose inductively, as in Corollary (6.3.48), a j-dependent sequence $(\zeta_i, x^j_i), \zeta_i = (q_i^2 + 1, \zeta_{i-1}, m_{2j-1}, J_i, 1, \psi_i), i = 1, \ldots, n_{2j-1}$, with $\zeta_0 = 0$, with respect to a (Γ, j)-special sequence $(\zeta_i, \bar{x}^j_i)$, so that it satisfies for any i the following

$$e^*_{\psi_{2i-1}} = \sum_{n\in A_i}\sum_j d^{j*}_{\xi_n} + \frac{1}{m_{j_{2i-1}}}\sum_{n\in A_i}\sum_j \epsilon_n e^*_{\eta_n} P^j_{I_n},$$

$$x^j_{2i-1} = \frac{c_{2i-1} m_{j2i-1}}{n_{j_{2i-1}}}\sum_{n\in A_i}\sum_j d^j_{\gamma^j_n}, \left\|x^j_{2i-1}\right\| = 1/2$$

with $\operatorname{rank}(\xi_n) = \max \operatorname{rank}(R_n) + 1$ for each $n \in \cup_i A_i$. Corollary (6.3.44) yields that $1/14 \leq c_{2i-1} \leq 1$. Recall that by definition each vector $\bar{x}^j_{2i-1}$ satisfies

$$\left|\sum_j \left(e^*_{\gamma^j}(\bar{x}^j_{2i-1}) - e^*_{\gamma^j}(x^j_{2i-1})\right)\right| \leq 4n^{-2}_{q^2_{2i-1}} \quad \forall \gamma^j \in \Gamma.$$

For any i let $J_{2i-1} = \operatorname{rng}(e^*_{\psi_{2i-1}})$. We demand also that $\operatorname{supp} e^*_{\psi_{2i}} \cap \operatorname{supp} x^j_{2k-1} = \emptyset$ for any i, k, thus the even parts of the chosen special functional play no role in the estimates on the weighted averages of (x^j_{2i-1}). We assume also $m_{j_1}/m_{j_1+1} \leq 1/n^2_{2j-1}$.
By the previous remark we have for each i the following

$$e^*_{\psi_{2i-1}} \sum_j \left(T^j x^j_{2i-1}\right) \geq \delta/14. \tag{86}$$

Let

$$y^j = \sum_j \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} \sum_j x^j_{2i-1} = \sum_j \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} c_{2i-1} \frac{m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n \in A_i} d^j_{\gamma^j_n}$$

and consider the functional associated to the special node $\zeta_{n_{2j-1}}$, i.e. of the form

$$\sum_j e^*_{\zeta_{n_{2j-1}}} = \sum_{i=1}^{n_{2j-1}} \sum_j d^{j*}_{\zeta_i} + \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}2} \sum_j \left(e^*_{\psi_{2i-1}} P^j_{J_{2i-1}} + \lambda^j_{2i} e^*_{\psi_{2i}} P^j_{J_{2i}}\right).$$

Then

$$\left\| \sum_j T^j y^j \right\| \geq \sum_j e^*_{\zeta_{n_{2j-1}}} (T^j y^j)$$

$$= \sum_j \left(\sum_{i=1}^{n_{2j-1}} d^{j*}_{\zeta_i} \right.$$

$$\left. + \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} \left(e^*_{\psi_{2i-1}} P^j_{J_{2i-1}} + \lambda^j_{2i} e^*_{\psi_{2i}} P^j_{J_{2i}}\right) \right) \left(\frac{1}{n_{2j-1}} \sum_{i=1}^{n_{(2j-1)}/2} T^j x^j_{2i-1} \right)$$

$$= \ldots$$

Notice that $J_{2i} \cap \Gamma_{\mathrm{rank}\left(\phi^j_{2i-1}\right)} = \emptyset$, whereas by the choice of R_n and the node ϕ^j_{2i-1} we have $\mathrm{rng}(T^j x^j_{2i-1}) \subset \Gamma_{\mathrm{rank}\left(\phi^j_{2i-1}\right)}$. Therefore

$$\cdots = \sum_j \left(\left(\sum_{i=1}^{n_{2j-1}} d^{j*}_{\zeta_i} \right) \left(\frac{1}{n_{2j-1}} \sum_{i=1}^{\frac{n_{2j-1}}{2}} T^j x^j_{2i-1} \right) \right.$$

$$\left. + \frac{1}{n_{2j-1} m_{2j-1}} \sum_{i=1}^{\frac{n_{2j-1}}{2}} e^*_{\psi_{2i-1}} P^j_{J_{2i-1}} \left(T^j x^j_{2i-1}\right) \right)$$

where in the last line the first sum disappears by the choice of $\left(q^2_{2i-1}\right)$, as $\mathrm{rank}(\mathrm{bd}(e^*_{\zeta_{n_{2j-1}}})) \cap \mathrm{rank}\left(T^j x^j_{2i-1}\right) = \emptyset$ for any i. Therefore we have

$$\left\| \sum_j T^j y^j \right\| \geq \sum_j \frac{\delta}{28 m_{2j-1}}. \tag{87}$$

On the other hand we estimate $\|y^j\|$. We shall prove that $\left\|\sum_j y^j\right\| \leq \sum_j 420/m^2_{2j-1}$ yielding for sufficiently big j a contradiction. By (P2) and Corollary (6.3.44) we get that

(x_i^j) is 7-RIS. By Lemma (6.3.20) it is enough to estimate $|e_\beta^*(z^j)|$, where e_β^* is associated to a (Γ, j)-special sequence $\left(\delta_i, \bar{z}_i^j\right)_{i=1}^a$, and $z^j = \sum_{i\in J}\sum_j x_{2i-1}^j$ for some interval $J \subset \{1,\dots,n_{2j-1}\}$.

Let e_β^* have the following form

$$e_\beta^* = \sum_{i=1}^{a}\sum_j d_{\tilde{\zeta}_i}^{j*} + \sum_j \frac{1}{m_{2j-1}}\sum_{i=1}^{\lfloor a2\rfloor} \left(\tilde{\epsilon}_{2i-1} e_{\tilde{\psi}_{2i-1}}^* P_{\tilde{J}_{2i-1}}^j + \tilde{\lambda}_{2i}^j e_{\tilde{\psi}_{2i}}^* P_{\tilde{J}_{2i}}^j\right) + \sum_j \left[\tilde{\epsilon}_a e_{\psi_a}^* P_{\tilde{J}_a}^j\right]$$

with $a \le n_{2j-1}$, where the last term appears if a is odd. Let $i_0 = \max\{i \le a : (\zeta_i, \bar{x}_i^j) = (\delta_i, \bar{z}_i^j)\}$ if such i exists. We estimate $|e_\beta^*(z^j)|$ assuming i_0 is well-defined. We estimate separately $|\sum_{i=1}^a\sum_j d_{\tilde{\zeta}_i}^{j*}(z^j)|$, $|\mathrm{mt}(e_{\tilde{\zeta}_{i_0}}^*)(z^j)|$ and $|(\mathrm{mt}(e_\beta^*) - \mathrm{mt}(e_{\tilde{\zeta}_{i_0}}^*))(z^j)|$.

First notice that taking into account coordinates of z^j with respect to the basis $(d_{\gamma^j}^j)$ and that $c_{2i-1} \le 1$, we have

$$\left|\sum_{i=1}^{a}\sum_j d_{\tilde{\zeta}_i}^{j*}\left(z^j\right)\right| \le \sum_j n_{2j-1}\frac{m_{j_1}}{n_{j_1}}. \tag{88}$$

Now consider the tree-analysis of $e_{\tilde{\zeta}_{i_0}}^*$, recall that it is compatible with the tree-analysis of $e_{\zeta_{i_0}}^*$. Then by the definition of a special node we have

$$\mathrm{mt}\left(e_{\zeta_{i_0}}^*\right) = \begin{cases} \sum_j \frac{1}{m_{2j-1}} \sum_{i=1}^{i_0/2} \left(\tilde{\epsilon}_{2i-1} e_{\tilde{\psi}_{2i-1}}^* P_{J_{2i-1}}^j + \tilde{\lambda}_{2i}^j e_{\psi_{2i}}^* P_{J_{2i}}^j\right) & \text{if } i_0 \text{ even} \\ \sum_j \frac{1}{m_{2j-1}} \sum_{i=1}^{\left\lfloor\frac{i_0}{2}\right\rfloor} (\tilde{\epsilon}_{2i-1} e_{\tilde{\psi}_{2i-1}}^* P_{J_{2i-1}}^j + \tilde{\lambda}_{2i}^j e_{\psi_{2i}}^* P_{J_{2i}}^j) + \tilde{\epsilon}_{i_0} e_{\tilde{\psi}_{i_0}}^* P_{J_{i_0}}^j & \text{if } i_0 \text{ odd} \end{cases}$$

where for each $2i - 1 \le i_0$ we have

$$e_{\tilde{\psi}_{2i-1}}^* = \sum_{n\in A_i}\sum_j \left(d_{\tilde{\xi}_n}^{j*} + \frac{1}{m_{j_{2i-1}}}\sum_{n\in A_i} \tilde{\epsilon}_n e_{\tilde{\eta}_n}^* P_{I_n}^j\right).$$

Notice that as $M \cap I_n = \emptyset$ for any n and by the choice of $e_{\psi_{2i}}^*$ and ranks of ξ_n, thus also ranks of $\tilde{\xi}_n$, we get, assuming that i_0 is even,

$$\sum_j |\mathrm{mt}(e_{\zeta_{i_0}}^*)(z^j)| = \sum_j \left|\frac{1}{m_{2j-1}}\sum_{i=1}^{i_0 2}\sum_j \left(\tilde{\epsilon}_{2i-1} e_{\tilde{\psi}_{2i-1}}^* P_{J_{2i-1}}^j + \tilde{\lambda}_{2i}^j e_{\psi_{2i}}^* P_{J_{2i}}^j\right)\left(z^j\right)\right| \tag{89}$$

$$= \sum_j \left|\left(\frac{1}{m_{2j-1}}\sum_{i=1}^{i_0 2} \tilde{\epsilon}_{2i-1}\sum_{n\in A_i} d_{\tilde{\xi}_n}^{j*}\right)\left(\sum_{2i-1\in J} c_{2i-1}\frac{m_{j_{2i-1}}}{n_{j_{2i-1}}}\sum_{n\in A_i} d_{\gamma_n^j}^j\right)\right| = 0.$$

The same holds if i_0 is odd.

Now consider $\mathrm{mt}(e_\beta^*) - \mathrm{mt}(e_{\zeta_{i_0}}^*)$ assuming that $i_0 < a$. Notice that

(i) $w(\psi_s) \neq w(\tilde{\psi}_i)$ for each $s, i > i_0$ provided at least one of the indices s, i is bigger than $i_0 + 1$,

(ii) $\left(\mathrm{mt}(e^*_\beta) - \mathrm{mt}\left(e^*_{\zeta_{i_0}}\right)\right)\left(x^j_{2k-1}\right) = 0$ for any $2k - 1 \leq i_0$.

Using Corollary (6.3.45) for the terms $\sum_{i=i_0+1}^{a} \sum_j |e^*_{\tilde{\psi}_i} P^j_{\tilde{J}_i}(x^j_{2k-1})|$ and that $|e^*_{\tilde{\psi}_{i_0+1}} \sum_j P^j_{\tilde{J}_{i_0+1}}(x^j_{i_0+1})| \leq 4$, it follows that

$$\sum_j \left|\left(\mathrm{mt}(e^*_\beta) - \mathrm{mt}\left(e^*_{\zeta_{i_0}}\right)\right)(z^j)\right| \leq \sum_j \frac{1}{m_{2j-1}} \sum_{i=i_0+1}^{a} \sum_{2k-1=i_0+1}^{n_{2j-1}} |e^*_{\tilde{\psi}_i} P^j_{\tilde{J}_i}(x^j_{2k-1})| \quad (90)$$

$$\leq \sum_j \left(\frac{4}{m_{2j-1}} + \frac{1}{m_{2j-1}} n_{2j-1} \frac{14}{m_{j_{i_0+1}}}\right) \leq \sum_j \frac{5}{m_{2j-1}}.$$

Therefore by (88), (89), (90) and the choice of j_1 we have $|e^*_\beta(z^j)| \leq 6/m_{2j-1}$, thus we can apply Lemma (6.3.20) obtaining that $\|y^j\| \leq 60 \cdot 7/m^2_{2j-1}$. For sufficiently big j we obtain contradiction with (87) and boundedness of T^j.

Corollary (6.3.51)[260]: [252]. Let $T^j : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator. If $\sum_j T^j d^j_{\gamma^j_n} \to 0$, then $T^j y^j_n \to 0$ for every RIS $(y^j_n)_n$.

Proof. Take $T^j : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ with $\sum_j T^j d^j_{\gamma^j_n} \to 0$ and suppose there are a normalised C-RIS $(y^j_n)_n$ and $\delta > 0$ such that $\|\sum_j T^j y^j_n\| > \delta$ for all $n \in \mathbb{N}$. Passing to a subsequence we may assume as in the proof of Corollary (6.3.50) that

$$\max \mathrm{rank}\, R_n + 2 < \min \mathrm{rank}\, R_{n+1} \text{ where } R_n = \mathrm{rng}(T^j y^j_n) \cup \mathrm{rng}(y^j_n).$$

Pick $(\mu_n) \subset \{\pm 1\}$ and nodes (ψ_n) with $\mu_n e^*_{\psi_n}(T^j y^j_n) > \delta$.

Case 1. There exist a constant $c > 0$, an infinite set $M \subset \mathbb{N}$ and nodes $(\varphi_n)_{n\in M}$ such that $\sum_j |e^*_{\varphi_n}(y^j_n)| > c$ and $e^*_{\varphi_n}, e^*_{\psi_n}$ have compatible tree-analyses.

Pick signs $(\nu_n)_{n\in M}$ with $\sum_j \nu_n e^*_{\varphi_n}(y^j_n) = \sum_j |e^*_{\varphi_n}(y^j_n)| > c$ for each n. We may pass to a subsequence $\left(\gamma^j_{k_n}\right)_n$ of $(\gamma^j_n)_n$ so that $\left\|\sum_j T^j d^j_{\gamma^j_{k_n}}\right\| \leq 2^{-n}$ for all n. For a fixed $j \in \mathbb{N}, n_{2j+1} > 200C$, we pick, as in Corollary (6.3.48), a j-dependent sequence $(\zeta_i, x^j_i)_i$ where $\zeta_i = (q^2_i + 1, \zeta_{i-1}, m_{2j-1}, J_i, 1, \eta_i), i = 1, \ldots, n_{2j-1}$, with $\zeta_0 = 0$, satisfies

$$\mathrm{mt}(e^*_{\eta_{2i-1}}) = \frac{1}{m_{j2i-1}} \sum_{n\in A_{2i-1}} \sum_j \nu_n e^*_{\varphi_n} P^j_{I_n},$$

$$x^j_{2i-1} = \frac{c_{2i-1} m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n\in A_{2i-1}} \sum_j y^j_n, \quad \|x^j_{2i-1}\| = 1/2,$$

where $I_n = [\min R_n, \max R_n]$, so that the functional associated to the special node $\zeta_{n_{2j+1}}$ with mt-part of the form

$$\sum_j \mathrm{mt}\left(e^*_{\zeta_{n_{2j-1}}}\right) = \sum_j \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j-1}/2} \left(e^*_{\eta 2i-1} P^j_{J_{2i-1}} + \lambda^j_{2i} e^*_{\eta 2i} P^j_{J_{2i}}\right),$$

satisfies $J_{2i-1} \supset \operatorname{rng}(T^j x_{2i-1}^j), J_{2i} \cap \operatorname{rng}(T^j x_{2k-1}^j) = \emptyset$ and $\operatorname{rank}(\operatorname{bd}(e_{\zeta_{n_{2j+1}}}^*)) \cap \operatorname{rank}(T^j x_{2i-1}^j) = \emptyset$ for any i, k.
From Remark (6.3.30) we get

$$1/_{20C} \leq c_{2i-1} \leq 2.$$

Using gaps between sets R_n we pick nodes $(\xi_{2i-1})_{2i-1 \leq n_{2j+1}}$, with

$$\operatorname{mt}(e_{\xi_{2i-1}}^*) = \frac{1}{m_{j_{2i-1}}} \sum_{n \in A_{2i-1}} \sum_j \mu_n e_{\psi_n}^* P_{I_n}^j .$$

It follows that $\sum_j e_{\xi_{2i-1}}^*(T^j x_{2i-1}^j) > \delta/_{20C}$ for each i.
Notice also that for $x_{2i}^j = \frac{m_{j_{2i}}}{n_{j_{2i}}} \sum_{n \in A_{2i}} \sum_j d_{\gamma_n^j}^j, A_{2i} \subset \{k_n : n \in \mathbb{N}\}$, by the condition on $\left(T^j d_{\gamma_{k_n}^j}^j\right)$ we have $\left\|\sum_j T^j x_{2i}^j\right\| < \frac{m_{j_{2i}}}{n_{j_{2i}}} < 2^{-i}$ for each i.
Let $\sum_j x^j = \sum_j \frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} x_{2i-1}^j$ and $\sum_j d^j = \sum_j \frac{m_{2j-1}}{n_{2j-1}} \sum_{i=1}^{n_{(2j-1)}/2} x_{2i}^j$. We have

$$\left\|\sum_j T^j d^j\right\| \leq \sum_j \frac{m_{2j-1}}{n_{2j-1}} \tag{91}$$

and by Corollary (6.3.49)

$$\left\|\sum_j (x^j - d^j)\right\| \leq \sum_j \frac{250}{m_{2j-1}^2} . \tag{92}$$

On the other hand by the choice of (φ_n) and (ψ_n) there is a well-defined special node β, associated to the same j-special sequence as $\zeta_{n_{2j+1}}$ with

$$\operatorname{mt}(e_\beta^*) = \sum_j \left(\frac{1}{m_{2j-1}} \sum_{i=1}^{\frac{n_{2j-1}}{2}} e_{\xi_{2i-1}}^* P_{J_{2i-1}}^j + \tilde{\lambda}_{2i}^j e_{\eta_{2i}}^* P_{J_{2i}}^j \right),$$

so that $\operatorname{rank}(\operatorname{bd}(e_\beta^*)) \cap \operatorname{rank}(T^j x_{2i-1}^j) = \emptyset$ for any i. Thus

$$\left\|\sum_j T^j x^j\right\| \geq \sum_j e_\beta^*(T^j x^j) \geq \frac{\delta}{40C}$$

which contradicts (91) and (92) for sufficiently big j as T^j is bounded.
Case 2. Case 1 does not hold. Applying this assumption for $c = n_{2j-1}^{-1} m_k^{-1}, k \in \mathbb{N}$, we pick inductively an increasing sequence $(q_k^2 + \epsilon) \subset \mathbb{N}$ such that for any node φ and $n > q_k^2 + \epsilon$ so that $e_\varphi^*, e_{\psi_n}^*$ have compatible tree-analyses we have $|\sum_j e_\varphi^*(y_n^j)| \leq \sum_j n_{2j-1}^{-1} m_k^{-1}$. Let $M = (q_k^2 + \epsilon)_k$.
Now we repeat the proof of Corollary (6.3.50), using (y_n^j) instead of $(d_{\gamma_n^j}^j)$. For a fixed $j \in \mathbb{N}$ we pick a j-dependent sequence $(\zeta_i, x_i^j), \zeta_i = (q_i^2 + 1, \zeta_{i-1}, m_{2j-1}, J_i, 1, \eta_i), i = 1, \dots, n_{2j-1}$, with $\zeta_0 = 0$, such that for each i we have

$$\mathrm{mt}\left(e^*_{\eta_{2i-1}}\right)=\frac{1}{m_{j_{2i-1}}}\sum_{n\in A_i}\sum_j \mu_n e^*_{\psi_n}P^j_{I_n}, \qquad x^j_{2i-1}=\frac{c_{2i-1}m_{j_{2i-1}}}{n_{j_{2i-1}}}\sum_{n\in A_i}\sum_j y^j_n,$$

$$\left\|\sum_j x^j_{2i-1}\right\|={}^1\!/_2,$$

with $A_i \subset M, \#A_i = n_{j_{2i-1}} J_{2i-1} = \mathrm{rng}(e^*_{\eta_{2i-1}}), J_{2i} \cap \mathrm{supp}\, x^j_{2k-1} = \emptyset$ for any $i,k, I_n = [\min R_n, \max R_n]$ and $\mathrm{rank}(\xi_n) = \max \mathrm{rng}\, R_n + 1$ for any n. As in the previous case, ${}^1\!/_{20C} \leq c_{2i-1} \leq 2$. Pick j_1 with $m_{j_1}/m_{j_1+1} \leq 1/n^2_{2j-1}$ and let

$$y^j=\sum_j \frac{1}{n_{2j-1}}\sum_{i=1}^{n_{2j-1}/2} x^j_{2i-1}$$

As in the proof of Corollary (6.3.50) it follows that

$$\left\|\sum_j T^j y^j\right\| \geq \sum_j e^*_{\zeta_{n_{2j-1}}}\left(y^j\right) \geq \sum_j \frac{1}{m_{2j-1}n_{2j-1}}\sum_{i=1}^{n_{2j-1}2}\frac{\delta}{2}c_{2i-1} \geq \frac{\delta}{80Cm_{2j-1}}. \quad (93)$$

We shall estimate now $\|y^j\|$. As before we consider a special node β which is compatible with a (Γ, j)-special sequence $\left(\delta_i, \bar{z}^j_i\right)^a_{i=1}, a \leq n_{2j-1}$, and estimate $|e^*_\beta(z^j)|$ where $z^j = \sum_{i\in J} x^j_{2i-1}$ for some interval $J \subset \{1,\ldots,n_{2j-1}\}$. Writing

$$e^*_\beta=\sum_{i=1}^{a}\sum_j\left(d^{j*}_{\tilde{\zeta}_i}+\frac{1}{m_{2j-1}}\sum_{i=1}^{\lfloor a2\rfloor}\left(\tilde{\epsilon}_{2i-1}e^*_{\tilde{\eta}_{2i-1}}P^j_{\tilde{J}_{2i-1}}+\tilde{\lambda}^j_{2i}e^*_{\tilde{\eta}_{2i}P^j_{\tilde{J}_{2i}}}\right)\right)$$

with $a \leq n_{2j-1}$ we pick as before $i_0 = min\{i \leq a : \left(\zeta_i, x^j_i\right) \neq (\delta_i, z^j_i)\}$ (if such i exists) and estimate separately $|\sum_{i=1}^a d^{j*}_{\tilde{\zeta}_i}(w)|, |mt(e^*_{\tilde{\zeta}_{i_0}})(w)|$ and $|(mt(e^*_\beta) - mt(e^*_{\tilde{\zeta}_{i_0}}))(w)|$. Repeating the reasoning of the proof of Corollary (6.3.50), as (y^j_n) have norm bounded by 1 and all $\left\|d^{j*}_{\tilde{\zeta}_i}\right\| \leq 3$, we obtain

$$\left|\sum_{i=1}^a\sum_j d^{j*}_{\tilde{\zeta}_i}\left(z^j\right)\right| \leq \sum_j 3\cdot 2n_{2j-1}\frac{m_{j_1}}{n_{j_1}} \leq \sum_j \frac{1}{m_{2j-1}}. \quad (94)$$

Using Corollary (6.3.21) and the fact that $|\sum_j e^*_{\gamma^j} P^j_I (x^j_{i0+1})| \leq 4$ we obtain that

$$\left|\sum_j\left(mt\left(e^*_\beta\right)-mt\left(e^*_{\tilde{\zeta}_{i0}}\right)\right)\left(z^j\right)\right| \leq \sum_j\left(\frac{4}{m_{2_{j-1}}}+2\frac{1}{m_{2_{j-1}}}n_{2_{j-1}}\frac{64C}{m_{j_{i_0+1}}}\right)$$

$$\leq \sum_j \frac{5}{m_{2_{j-1}}} \quad (95)$$

using that $m^{-1}_{j_1} < n^2_{2_{j-1}}$ and $n_{2_{j+1}} > 200C$.

Now consider $e^*_{\tilde{\zeta}_{i_0}}$, recall this functional and $e^*_{\tilde{\zeta}_{i_0}}$ have compatible tree-analyses. Therefore

$$mt\left(e^*_{\tilde{\zeta}_{i_0}}\right)$$
$$=\begin{cases} \sum_j \frac{1}{m_{2_{j-1}}} \sum_{i=1}^{\frac{i_0}{2}} (\tilde{\epsilon}_{2i-1} e^*_{\tilde{\eta}_{2i-1}} P^j_{J_{2_i-1}} + \tilde{\lambda}^j_{2_i} e^*_{\eta_{2_i}} P^j_{J_{2_i}}) & if\ i_0\ even \\ \sum_j \frac{1}{m_{2_j-1}} \sum_{i=1}^{\frac{i_0}{2}} \left(\tilde{\epsilon}_{2_i-1} e^*_{\tilde{\eta}_{2_i-1}} P^j_{J_{2_i-1}} + \tilde{\lambda}^j_{2_i} e^*_{\eta_{2_i}} P^j_{J_{2_i}}\right) + \tilde{\epsilon}_{i_0} e^*_{\tilde{\eta}_{i_0}} \sum_j P^j_{J_{i_0}} & if\ i_0 \text{ odd} \end{cases}$$

where for each for each $2i - 1 \le i_0$ we have
$$e^*_{\tilde{\eta}_{2i-1}} = \sum_{n\in A_i} \sum_j d^{j*}_{\tilde{\xi}_n} + \frac{1}{m_{j_{2i-1}}} \sum_{n\in A_i} \sum_j \tilde{\epsilon}_n e^*_{\varphi_n} P^j_{I_n} \quad .$$
By choice of the objects above we have
$$\left|\sum_j mt\left(e^*_{\tilde{\zeta}_{i_0}}\right)(z^j)\right|$$
$$\le \sum_j \frac{1}{m_{2_{j-1}}} \left| \left(\sum_{i=1}^{\frac{n_{2_{j-1}}}{2}} \sum_{n\in A_i} d^{j*}_{\tilde{\xi}_n} \right) \left(\sum_{2i-1\in J} \frac{c_{2i-1} m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n\in A_i} y^j_n \right) \right|$$
$$+ \sum_j \frac{1}{m_{2_{j-1}}} \sum_{2i-1\in J} \frac{c_{2i-1} m_{j_{2i-1}}}{n_{j_{2i-1}}} \sum_{n\in A_i} \left|e^*_{\varphi_n}(y^j_n)\right| .$$
As for each n the nodes ψ_n, φ_n have compatible tree-analyses the last sum can be estimated by $2m^{-1}_{2j-1}$. The first sum equals 0 by the condition on ranks of ξ_n, thus also $\tilde{\xi}_n$ Therefore we have
$$\left|\sum_j mt\left(e^*_{\tilde{\zeta}_{i_0}}\right)(z^j)\right| \le \sum_j \frac{2}{m_{2j-1}} . \tag{96}$$
As before by (94), (95), (96) we have $|\sum_j e^*_\beta(z^j)| \le \sum_j 8/m_{2j-1}$, thus we can apply Lemma (6.3.20) obtaining that $\|\sum_j y^j\| \le \sum_j 80C/m^2_{2j-1}$. For sufficiently big j we obtain contradiction with (93) and boundedness of T^j.

Corollary (6.3.52)[260]: [252]. Let $T^j : \mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ be a bounded operator. Then there exist a compact operator K : $\mathcal{X}_{Kus} \to \mathcal{X}_{Kus}$ and a scalar λ such that $T^j = \lambda^j Id + K^j$.

Proof. By Corollary (6.3.50) any $\left(d^j_{\gamma^j n}\right)_{n\in N}$ has a further subsequence $\left(d^j_{\gamma^j_n}\right)_{n\in M}$ such that $T^j d^j_{\gamma^j n} - \lambda^j d^j_{\gamma^j n} \to 0\ as\ M \ni n \to \infty$, for some λ^j. By Proposition 6.6 there is a universal λ so that $T^j d^j_{\gamma^j n} - \lambda^j d^j_{\gamma^j n} \to 0\ as\ n \to \infty$. Applying Corollary (6.3.51) to the operator $T^j - \lambda^j Id$ we get that $T^j y^j_n - \lambda^j y^j_n \to 0$ for any RIS (y^j_n) and thus, by Proposition (6.3.26), for any bounded block sequence (y^j_n). It follows that the operator $T^j - \lambda^j Id$ is compact.

Corollary (6.3.53)[260]: [252].

(iii) For every $k \in \mathbb{N}$ and $t \in \mathcal{T}$ we have $|\sum_j e^*_{\eta_t} P^j_{I_t}(y^j_k)| \leq 10C/m_{j_k}$.

(iv) For every $k \in \mathbb{N}$ and $t \in \mathcal{T}$ with $w(\eta_t) < m^{-1}_{j_k}$ we have $|\sum_j e^*_{\eta_t} P^j_{I_t}((x^j_k)')| \leq 11C/m_{j_k}$.

Proof. Concerning (i), notice first that for any $s \in \mathcal{F}_{\gamma^j}$ we have $|\sum_j e^*_{\eta_s} P^j_{I_s}(x^j_k)| \leq 10C/m_{j_k}$.

Indeed, for $w(\eta_s) = m_{2j}$ for some j, we consider the following two cases. If $m^{-1}_{2j} < m^{-1}_{j_k}$ then the estimate follows by (26). If $m^{-1}_{2j} \geq m^{-1}_{j_k}$, then by the form of $e^*_{\eta_s}$ and (57) we have

$$\left|\sum_j e^*_{\eta_s} P^j_{I_s}\left(x^j_k\right)\right| \leq 2n_{2j} \max_{\gamma^j \in \Gamma} \sum_j |d^{j*}_{\gamma^j}(x^j_k)| \leq 2C/n_{j_k}$$

Now, as each of the sets$\{s \in \mathcal{F}_{\gamma^j} \mid |s| = i, rng(x^j_k) \cap I_s \neq \emptyset\}, i \in \mathbb{N}$, has at most two elements, we have

$$\left|\sum_j e^*_{\eta_t} P^j_{I_t}\left(y^j_k\right)\right| \leq \sum_{s \in \mathcal{F}_{\gamma^j}} \sum_j \left(\Pi_{t \preceq u \prec s}\, w(\eta_u)\right)\left|e^*_{\eta_s} P^j_{I_s}\left(x^j_k\right)\right|$$
$$= \sum_i \sum_{s \in \mathcal{F}_{\gamma^j}, |s|=i} \sum_j \left(\Pi_{t \preceq u \prec s}\, w(\eta_u)\right)\left|e^*_{\eta_s} P^j_{I_s}\left(x^j_k\right)\right| \leq \frac{20C}{m_{j_k}} \sum_i \frac{1}{m^i_1} = \frac{10C}{m_{j_k}}.$$

Condition (ii) follows from Lemma (6.3.20) and (i).

Corollary (6.3.54)[260]: [252]. For every choice of signs (δ_k) there exists a node $\tilde{\gamma}^j \in \Gamma$ such that $Q_{\gamma^j} = Q_{\tilde{\gamma}^j}$ and $\epsilon \in \{\pm 1\}$ so that

$$\left|\sum_j \left(e^*_{\gamma^j}\left(\left(x^j_k\right)'\right) - \epsilon e^*_{\tilde{\gamma}^j}\left(\delta_k\left(x^j_k\right)'\right)\right)\right| \leq \frac{6C}{m_{j_k}} \text{ for any } k \in \mathbb{N}.$$

Proof. Define

$D^j = \{t \in \mathcal{T} \mid rng(x^j_k) \cap rng(e^*_t P^j_{I_t}) \neq \emptyset$ for at most one k and if $t \in S_u$ $then\, rng(x^j_i) \cap rng(e^*_u P^j_{I_u}) \neq \emptyset$ for at least two $i\}$.

Since for every branch b of $\mathcal{T}$ the set $b \cap D^j$ has exactly one element we can define a subtree $\mathcal{T}'$ of $\mathcal{T}$ such that D^j is the set of terminal nodes for $'$. Notice that $(\mathcal{T} \setminus \mathcal{T}') \cap \mathcal{F}_{\gamma^j} = \emptyset$.

If $\gamma^j \in D^j$, then we pick the unique k_0 with $rng(e^*_{\gamma^j}) \cap rng(x^j_{k_0}) \neq \emptyset$ $(as\, I_\emptyset = [1, max\, \Delta_{rank(\gamma^j)}])$ and let $\tilde{\gamma}^j = \gamma^j$ and $\varepsilon = \delta_{k_0}$. Then we have the estimate in the lemma for any $k \in N$.

Assume that $\gamma^j \notin D^j$. Using backward induction on $\mathcal{T}'$ we shall define a node $\tilde{\gamma}^j$ with a tree-analysis $(I_t, \tilde{\epsilon}_t, \tilde{\eta}_t)_{t \in T^j}$ and associated scalars $\left(\tilde{\lambda}^j_t\right)_{t \in T^j}$, by modifying the nodes $(I_t, \tilde{\epsilon}_t, \eta_t)_{t \in \mathcal{T}'}$ and scalars $\left(\lambda^j_t\right)_{t \in \mathcal{T}'}$ starting from elements of D^j such that

(T1) $e^*_{\eta_t}, e^*_{\tilde{\eta}_t}$ have compatible tree-analyses for any $\in \mathcal{T}'$,

(T2) $F_{\tilde{\eta}_t} = F_{\eta_t}$ for any $\in \mathcal{T}'$, (T3) $\tilde{\epsilon}_t e^*_{\tilde{\eta}_t} \sum_j P^j_{I_t}\left(\delta_k\left(x^j_k\right)'\right) = \epsilon_t e^*_{\eta_t} \sum_j P^j_{I_t}\left(\left(x^j_k\right)'\right)$ for any $t \in D^j \setminus \mathcal{E}_{\gamma^j}$ and k, $\sum_j \tilde{\lambda}^j_t e^*_{\tilde{\eta}_t} P^j_{I_t}\left(\delta_k\left(x^j_k\right)'\right) = \sum_j \lambda^j_t e^*_{\eta_t} P^j_{I_t}\left(\left(x^j_k\right)'\right)$ for any $t \in D^j \cap \mathcal{E}_{\gamma^j}$ and k,

(T4) $\tilde{\epsilon}_t = \epsilon_t$ for any $t \in \mathcal{T}' \setminus D^j$.

Roughly speaking we need to modify only $\epsilon_t, t \in D^j$, changing signs of some of them. These modifications determine changes in the rest of the tree, i.e. $\eta_u, u \in \mathcal{T}' \setminus D^j$ according to the rules of producing nodes.

Step 1. Take $t \in D^j$.

Case 1a. $t \notin \mathcal{E}_{\gamma^j} \cup \bigcup_{u \in \mathcal{E}_{\gamma^j}} S_u$. We set $\tilde{\eta}_t = \eta_t$ and $\tilde{\epsilon}_t = \delta_k \epsilon_t$, if $rng(e^*_t P^j_{I_t})$ intersects $rng(x^j_k)$ for some (unique) k, otherwise $\tilde{\epsilon}_t = \delta_m \epsilon_t$ where $m = min\{i : rng\, e^*_{\eta_t} P^j_{I_t} \le rng(x^j_i)\}$.

The condition (T3) follows straitforward.

Case 1b. $t \in \mathcal{E}_{\gamma^j} \cup \bigcup_{u \in \mathcal{E}_{\gamma^j}} S_u$. In this case we set $\tilde{\eta}_t = \eta_t$ and $\tilde{\epsilon}_t = \epsilon_t (= 1)$. Moreover, for $t \in \mathcal{E}_{\gamma^j}$ we set $\tilde{\lambda}^j_t = \delta_k \lambda^j_t$. Such choice is possible since Net_{1,q^2} is symmetric. It follows that

$$|\sum_j \tilde{\lambda}^j_t - \tilde{\epsilon}_t - e^*_{\tilde{\eta}_t}\text{-}(y^{jt}_{2i-1})| = \sum_j |\delta_k \lambda^j_t - \delta_k \epsilon_t - e^*_{\eta_t} - (y^{jt}_{2i-1})|$$
$$= \sum_j \left|\lambda^j_t - \epsilon_t - e^*_{\eta_t} - \left(y^{jt}_{2i-1}\right)\right|$$

where $\left(y^{jt}_i\right)_i$ are the vectors of the suitable special sequences. In order to verify condition (T3) we consider two subcases.

(1) If $t \in \mathcal{F}_{\gamma^j}$ or $t \in S_u$ for some $u \in \mathcal{E}_{\gamma^j}$ (then $u \in \mathcal{F}_{\gamma^j}$), it follows that $rng(e^*_{\eta_t} P^j_{I_t}) \cap rng\left(x^j_k\right)' = \emptyset$ for any k by the definition of $\left(x^j_k\right)'$, thus we obtain (T3).

(2) If $t \in \mathcal{E}_{\gamma^j} \setminus \mathcal{F}_{\gamma^j}$ and $rng(e^*_{\eta_t} P^j_{I_t}) \cap rng\left(x^j_k\right)' = \emptyset$ for some k, it follows that $e^*_{t_-} \in D^j$ as well and moreover $rng(e^*_{\eta_{t_-}} P^j_{I_{t_-}})$ either intersects only $rng\, x^j_k$ or intersects no $rng\, x^j_i$. In both cases $\tilde{\epsilon}_{t_-} = \delta_k \epsilon_{t_-}$ and so $\tilde{\lambda}^j_t = \delta_k \lambda^j_t$ and (T3) holds.

Notice that in either case conditions (T1)–(T2) and (T4) are straitforwardly satisfied.

Step 2. Now we define inductively nodes in $t \in \mathcal{T}' \setminus D^j$. Take $t \in \mathcal{T}' \setminus D^j$ and assume we have defined $(\tilde{\epsilon}_s, \tilde{\eta}_s, I_s)_{s \in S_t}$ satisfying (T1)–(T4). In all cases we let $\tilde{\epsilon}_t = \epsilon_t$, thus (T4) is satisfied. Notice that $t \notin \bigcup_{u \in \mathcal{E}_{\gamma^j}} S_u$.

Case 2a. $t \in \mathcal{E}_{\gamma^j}$. In this case we set $\tilde{\eta}_t = \eta_t$. Obviously we have (T1)–(T2).

Case 2b. $t \notin \mathcal{E}_{\gamma^j}, w(\eta_t) = m^{-1}_{2j}$. Then using Remark 4.10 (1) we define $\tilde{\eta}_t$ so that

$$mt\left(e^*_{\tilde{\eta}_t}\right) = \sum_j \frac{1}{m_{2j}} \sum_{s \in S_t} \tilde{\epsilon}_s e^*_{\tilde{\eta}_s} P^j_{I_s} .$$

By definition we have (T1)–(T2).

Case 2c. $w(\eta_t) = m_{2j+1}^{-1}$, with η_t compatible with a (Γ, j)-special sequence $(\bar{x}_t^j, \bar{\eta}_t)$. Then using Remark 4.10 (2) we define a special node $\tilde{\eta}_t$ which is compatible with the same (Γ, j)-special sequence $(\bar{x}_t^j, \bar{\eta}_t)$ so that

$$mt\left(e_{\tilde{\eta}_t}^*\right) = \sum_j \frac{1}{m_{2j+1}} \sum_{s \in S_t \cap \mathcal{E}_{\gamma^j}} \left(\tilde{\epsilon}_{s_-} e_{\tilde{\eta}_{s_-}}^* P_{I_{s_-}}^j + \tilde{\lambda}_s^j e_{\tilde{\eta}_s}^* P_{I_s}^j\right) .$$

By definition we have (T1)–(T2).
Let $\tilde{\gamma}^j = \tilde{\eta}_\emptyset$. Notice that by conditions (T1)–(T2) we have $Q_{\tilde{\gamma}^j} = Q_{\gamma^j}$. Now we proceed to show the estimate part of the lemma. Fix $k \in N$. For any nonterminal $u \in \mathcal{T}$ let
$S_{u,k} := \{s \in S_u \mid rng(x_k^j) \cap rng(e_s^* P_{I_s}^j) = \emptyset\}$.
Let G be the set of minimal nodes u of $\mathcal{T}'$ with $u \in D^j$ or $(\eta_u) < m_{jk}^{-1}$. By $\mathcal{T}''$ denote the subtree of $\mathcal{T}'$ with the terminal nodes in G. We shall prove by induction starting from G that for any $u \in \mathcal{T}''$ we have

$$\left|\sum_j \left(\epsilon_u e_{\eta_u}^* P_{I_u}^j \left(\left(x_k^j\right)'\right) - \tilde{\epsilon}_u e_{\tilde{\eta}_u}^* P_{I_u}^j \left(\delta_k\left(x_k^j\right)'\right)\right)\right| \le \frac{22C}{m_{jk}} . \tag{97}$$

This will end the proof as it follows by (T4) that $\left|\sum_j \left(\epsilon_\emptyset e_{\eta_\emptyset}^* \left(\left(x_k^j\right)'\right) - \tilde{\epsilon}_\emptyset e_{\tilde{\eta}_\emptyset}^* \left(\delta_k\left(x_k^j\right)'\right)\right)\right| = \sum_j |e_{\tilde{\gamma}^j}^*((x_k^j)') - e_{\tilde{\gamma}^j}^*(\delta_k(x_k^j)')|$. Thus taking $\epsilon = 1$ we obtain the estimate of the lemma.
Step 1. $u \in G. If\ w(\eta_u) < m_{jk}^{-1}$ then the estimate (97) holds true by Corollary (6.3.53) (ii). If $u \in D^j$ then the estimate (97) holds true by (T3).
Step 2. $u \in \mathcal{T}'' \setminus G$. In particular $(\eta_u) \ge m_{jk}^{-1}$. Obviously $S_u \subset \mathcal{T}''$.
Case 2a. $w(\eta_u) = m_{2j}^{-1}$. We estimate, using (T3) for $s \in S_{u,k} \cap D^j$

$$\left|\sum_j \left(e_{\tilde{\eta}_u}^* P_{I_u}^j \left(\delta_k\left(x_k^j\right)'\right) - e_{\eta_u}^* P_{I_u}^j \left(\left(x_k^j\right)'\right)\right)\right|$$

$$= \sum_j \Bigg| \left(\sum_{s \in S_u} d_{\tilde{\xi}_s}^{j*} + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \cap D^j} \tilde{\epsilon}_s e_{\tilde{\eta}_s}^* P_{I_s}^j + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \setminus D^j} \tilde{\epsilon}_s e_{\tilde{\eta}_s}^* P_{I_s}^j \right) (\delta_k(x_k^j)')$$

$$- \sum_j \left(\sum_{s \in S_u} d_{\xi_s}^{j*} + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \cap D^j} \tilde{\epsilon}_s e_{\eta_s}^* P_{I_s}^j + \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \setminus D^j} \tilde{\epsilon}_s e_{\eta_s}^* P_{I_s}^j \right) ((x_k^j)')\Bigg|$$

$$\le \Bigg| \sum_{s \in S_u} \sum_j d_{\tilde{\xi}_s}^{j*} \left(\delta_k\left(x_k^j\right)'\right) \Bigg| + \Bigg| \sum_{s \in S_u} \sum_j d_{\xi_s}^{j*} ((x_k^j)')\Bigg|$$

$$+ \sum_j \frac{1}{m_{2j}} \sum_{s \in S_{u,k} \setminus D^j} \left|\tilde{\epsilon}_s e_{\tilde{\eta}_s}^* P_{I_s}^j \left(\delta_k\left(x_k^j\right)'\right) - \epsilon_s e_{\eta_s}^* P_{I_s}^j \left(\left(x_k^j\right)'\right)\right|$$

$$\le \dots$$

The first two sums are estimated using (8.1) and $\#S_u \le n_{2j} \le n_{jk}$, for the third element use the inductive hypothesis and the fact that $\#(S_{u,k} \setminus D^j) \le 2$, obtaining the following

$$\ldots \leq \sum_j \left(2n_{2j}\frac{C}{n_{jk}^2} + \frac{2}{m_{2j}} \cdot \frac{22C}{m_{jk}}\right) \leq \sum_j \frac{22C}{m_{jk}}.$$

Case 2b. $w(\eta_u) = m_{2j+1}^{-1}$. Recall that by (T3) we have $\epsilon_{s_-} e^*_{\eta_{s_-}} \sum_j P^j_{I_{s_-}}\left(\left(x^j_k\right)'\right) = \sum_j \tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} I_{s_-}\left(\delta_k\left(x^j_k\right)'\right)$ for any $s \in S_u \cap \mathcal{E}_{\gamma^j}$ with $s_- \in D^j$ and $\sum_j \lambda^j_s e^*_{\eta_s} P^j_{I_s}\left(\left(x^j_k\right)'\right) = \sum_j \tilde{\lambda}^j_s e^*_{\tilde{\eta}_s} I_s\left(\delta_k\left(x^j_k\right)'\right)$ for any $s \in S_u \cap \mathcal{E}_{\gamma^j} \cap D^j$. Moreover $\mathcal{E}_{\gamma^j} \setminus D^j \subset \mathcal{F}_{\gamma^j}$ thus $\sum_j e^*_{\eta_s} P^j_{I_s}\left(\left(x^j_k\right)'\right) = 0 = \sum_j e^*_{\tilde{\eta}_s} P^j_{I_s}\left(\delta_k\left(x^j_k\right)'\right)$ for any $s \in (S_u \cap \mathcal{E}_{\gamma^j}) \setminus D^j$. Therefore we have

$$\left|\sum_j \left(e^*_{\tilde{\eta}_u} P^j_{I_u}\left(\delta_k\left(x^j_k\right)'\right) - e^*_{\eta_u} P^j_{I_u}\left(\left(x^j_k\right)'\right)\right)\right|$$

$$= \sum_j \left|\left(\sum_{s\in S_u} d^{j*}_{\tilde{\xi}_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}, s\in\mathcal{E}_{\gamma^j}} \tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} P^j_{I_s} + \frac{1}{m_{2j+1}} \sum_{s\in S_{u,k}\cap\mathcal{E}_{\gamma^j}} \tilde{\lambda}^j_s e^*_{\tilde{\eta}_s} P^j_{I_s}\right)\left(\delta_k\left(x^j_k\right)'\right) - \left(\sum_{s\in S_u} d^{j*}_{\xi_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}, s\in\mathcal{E}_{\gamma^j}} \tilde{\epsilon}_{s_-} e^*_{\eta_{s_-}} P^j_{I_s} + \frac{1}{m_{2j+1}} \sum_{s\in S_{u,k}\cap\mathcal{E}_{\gamma^j}} \lambda^j_s e^*_{\eta_s} P^j_{I_s}\right)\left(\left(x^j_k\right)'\right)\right|$$

$$= \sum_j \left|\left(\sum_{s\in S_u} d^{j*}_{\tilde{\xi}_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}\setminus D^j, s\in\mathcal{E}_{\gamma^j}} \tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} P^j_{I_s}\right)\left(\delta_k\left(x^j_k\right)'\right) - \left(\sum_{s\in S_u} d^{j*}_{\xi_s} + \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}\setminus D^j, s\in\mathcal{E}_{\gamma^j}} \epsilon_{s_-} e^*_{\eta_{s_-}} P^j_{I_s}\right)\left(\left(x^j_k\right)'\right)\right|$$

$$\leq \sum_j \left|\sum_{s\in S_u} d^{j*}_{\tilde{\xi}_s}\left(\delta_k\left(x^j_k\right)'\right)\right| + \sum_j \left|\sum_{s\in S_u} d^{j*}_{\xi_s}\left(\left(x^j_k\right)'\right)\right| + \sum_j \frac{1}{m_{2j+1}} \sum_{s_-\in S_{u,k}\setminus D^j, s\in\mathcal{E}_{\gamma^j}} \left|\tilde{\epsilon}_{s_-} e^*_{\tilde{\eta}_{s_-}} P^j_{I_{s_-}}\left(\delta_k\left(x^j_k\right)'\right) - \epsilon_{s_-} e^*_{\eta_{s_-}} P^j_{I_{s_-}}\left(\left(x^j_k\right)'\right)\right|$$

$$\leq \ldots$$

Proceeding as in Case $2a$ we obtain

$$\ldots \leq \sum_j \left(2n_{2j+1}\frac{C}{n_{jk}^2} + \frac{2}{m_{2j+1}} \cdot \frac{22C}{m_{jk}}\right) \leq \sum_j \frac{22C}{m_{jk}}.$$

Corollary (6.3.55)[260]: [252]. The space $\mathcal{X}_{Kus}$ is unconditionally saturated.

Proof. In every block subspace of $\mathcal{X}_{Kus}$ pick a sequence $\left(x_k^j\right)_k$ as above with $m_{j1} > 400C$. We claim that such a sequence is unconditional. To this end consider a finite sequence of scalars (a_k) with $\left\|\sum_k \sum_j a_k x_k^j\right\| = 1$ and $(\delta_k) \subset \{\pm 1\}$. We want to estimate the norm of the vector $\sum_k \sum_j \delta_k a_k x_k^j$. Take $\gamma^j \in \Gamma$ with $\sum_j e_{\gamma^j}^*(\sum_k a_k x_k^j) \geq 3/4$. Define $Q\gamma^j$, (y_k^j) and $(\left(x_k^j\right)')$ and consider $\tilde{\gamma}^j$ and ϵ provided by Corollary (6.3.54). Notice that as $Q_{\tilde{\gamma}^j} = Q_{\gamma^j}$, the projection $Q_{\tilde{\gamma}^j}$ defines also (y_k^j) and (x_k^j). Estimate, applying Corollary (6.3.54) and Corollary (6.3.53) (i) both for γ^j and $\tilde{\gamma}^j$, as follows

$$
\begin{aligned}
&\left|\sum_j \left(e_{\gamma^j}^*\left(\sum_k a_k x_k^j\right) - \epsilon e_{\tilde{\gamma}^j}^*\left(\sum_k \delta_k a_k x_k^j\right)\right)\right| \\
&\leq \sum_j \left|e_{\gamma^j}^*\left(\sum_k a_k \left(x_k^j\right)'\right) - \epsilon e_{\tilde{\gamma}^j}^*\left(\sum_k \delta_k a_k \left(x_k^j\right)'\right)\right| + \sum_j \left|e_{\gamma^j}^*\left(\sum_k a_k y_k^j\right)\right| \\
&\quad + \sum_j \left|e_{\tilde{\gamma}^j}^*\left(\sum_k \delta_k a_k y_k^j\right)\right| \\
&\leq \sum_k \sum_j |a_k| \left|e_{\gamma^j}^*\left(\left(x_k^j\right)'\right) - \epsilon e_{\gamma^j}^*\left(\delta_k \left(x_k^j\right)'\right)\right| + \sum_k \sum_j |a_k| \left|e_{\gamma^j}^*\left(y_k^j\right)\right| \\
&\quad + \sum_k \sum_j |a_k| \left|e_{\tilde{\gamma}^j}^*\left(\delta_k y_k^j\right)\right| \\
&\leq 4 \cdot 24C \sum_k m_{j_k}^{-1} \leq 200C m_{j_1}^{-1} \leq 12/
\end{aligned}
$$

where in the last line we use the fact that each $|a_k|$ is dominated by twice the basic constant of the basis $(d^j\gamma^j)$. Therefore $\left\|\sum_k \sum_j \delta_k a_k x_k^j\right\| \geq |e_{\tilde{\gamma}^j}^*(\sum_k \sum_j \delta_k a_k x_k^j)| \geq 1/4$, which ends the proof.

List of Symbols

Symbol		Page
max:	Maximum	1
$supp$:	Support	5
mod:	modulo	16
min:	minimum	20
ext:	extractor	20
ℓ^p:	Banach space of sequences	21
L^p:	Lebesgue space	21
ℓ^∞:	Essential Banach space of sequences	21
ℓ^2:	Hilbert space of sequences	21
L^1:	Lebesgue integral in the real line	21
L^2:	Hilbert space	21
$\bigoplus$:	Orthogonal sum	21
Pr:	probability	22
Had:	Hadamard	23
del:	determinant	33
$w^{1,1}$:	Sobolev space	33
BV:	Bounded Variation	33
$meas$:	Measure	34
$w^{1,p}$:	Sobolev space	34
OSC:	Oscillation	37
L^∞:	essential Lebesgue space	37
$a.e$:	almost every where	38
sup:	Supremum	39
$w^{k,p}$:	Sobolev space	39
loc:	Local	42
$L_{po,1}$:	Lorentz space	42
$diam$:	diameter	43
$w_{po,1}^{k}$:	Lorentz space	44
inf:	Infimum	44
$L_{p,q}$:	Lorentz space	44
$f.d.d$:	Finite-dimensional decomposition	60
Bo:	Bourgain-Delbaen	70
dim:	Dimension	71
RIS:	Rapidly increasing sequence	80
dist:	Distance	93
$\bigotimes$:	Tensor product	100
PFE:	Polynomial Freiman-Ruzsa conjecture	120
CC:	Communication Complexity	121
QFR:	Quasi-polynomial Freiman-Ruzsa	122
$qpely$:	Quasi-polynomial	124
$Spec$:	Spectrum	125
Re:	Real	138
Im:	Imaginary	138
$a.a$:	Almost all	170
Ent:	Enterior	176

B_{a1p}:	Bessel capacity	176
ItI:	Hereditarily indecomposable	199
Gm:	Gowers Maurey	199
$SRIS$:	Special rapidly increasing sequence	210
ℓ_1:	Space on the real line	215
ker:	Kernel	218
Lip:	Lipschitz	220
po:	Schur space	222
rng:	Range	231
X_{aux}:	Mixed Tsirelson space	234

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