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Nonmaximal Ideals with Seminormed $*$-Subalgebra and Parabolic Algebra on L^p Spaces

المثاليات غير الأعظمية مع شبه المنتظم *- الجبري الجزئي والجبر المكافئ على فضاءات L^p

A Thesis Submitted in Fulfillment of the Requirements for the Degree of Ph.D in Mathematics

By

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Dedication

To my Family.

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I would like to thank with all sincerity Allah, and my family for their supports throughout my study. Many thanks are due to my thesis guide, Prof. Dr. Shawgy Hussein AbdAlla Sudan University of Science and Technology.

Abstract

We show the convexity properties and similarity classification with homogeneous operators on Hilbert spaces of holomorphic mappings, curves and functions. The sharp estimates of all homogeneous expansions for a class and subclass of quasi-convex mappings on the unit polydisk in the unitary space and of type B and order A with the weak version of the Bieberbach conjecture in several complex variables are given. The centers of the quasi-homogeneous polynomial differential equations of degree three and global behaviour of the period of the sum of two quasi-homogeneous vector fields are determined. We obtain the first derivative of the period function for Hamiltonian systems with homogeneous non linearities and applications. The multiplicity-free and rigidity of the flag structure with classification of homogeneous and quasi-homogeneous operators and holomorphic curves in the Cowen-Douglas class are discussed.

الخلاصة

قمنا بتوضيح الخصائص التحدبية وتصنيف التشابه مع مؤثرات التجانس على فضاءات هلبرت للرواسم الهولومورفيك و المنحنيات والدوال. قمنا باعطاء التقديرات القاطعة لكل المفكوكات المتجانسة لأجل العائلة والعائلة الجزئية للرواسم شبه-المحدبة على متعدد الأقراص في الفضاء الواحدي ومن النوع B و الرتبة A مع النسخة الضعيفة لتخمين بيبرباخ في المتغيرات المركبة المتعددة. تم تحديد المراكز للمعادلات التفاضلية كثيرة الحدود شبه – المتجانسة من الدرجة الثالثة و السلوك العالمي لفترة مجموع حقلين متجهين شبه متجانسين. تم الحصول على المشتقة الأولى لدالة الفترة لأجل أنظمة هاميلتونيان مع عدم الخطية المتجانسة والتطبيقات. قمنا بمناقشة المضاعفة الحرة والصلابة للبناء العلم مع تصنيف المؤثرات المتجانسة وشبه – المتجانسة والدوال الهولومورفيك في عائلة كوين – دوغلاس.

Introduction

Not many convex mappings on the unit ball in $\mathbb{C}^n$ for $n > 1$ are known. We introduce two families of mappings, which we believe are actually identical, that both contain the convex mappings. These families which we have named the "Quasi-Convex Mappings. We establish a sufficient condition for a quasi-convex mapping (including quasiconvex mapping of type A and quasi-convex mapping of type B) $f(x)$ defined on the unit ball in a complex Banach space.

We show, that under certain hypotheses, the planar differential equation: $\dot{x}X_1(x,y) + X_2(x,y), \dot{y}Y_1(x,y) + Y_2(x,y)$, where $(X_i, Y_i), i = 1, 2$, are quasi-homogeneous vector fields, has at most two limit cycles. We show that such systems have no isochronous centers, that the period annulus of any of its centres is either bounded or the whole plane and that the period function associated to the origin has at most one critical point.

For H be a complex separable Hilbert space. For Ω an open connected subset of C, we shall say that a map $f: \Omega \to Gr(n, H)$ is a holomorphic curve, if there exist n holomorphic H-valued functions $\gamma_1, \gamma_2, \ldots, \gamma_n$ on Ω such that $f(\lambda) = \{\gamma_1(\lambda), \ldots, \gamma_n(\lambda)\}, \forall\lambda \in \Omega$, where $Gr(n, H)$ denotes the Grassmann manifold, the set of all n-dimensional subspaces of H. We construct a large class of multiplication operators on reproducing kernel Hilbert spaces which are homogeneous with respect to the action of the Möbius group consisting of bi-holomorphic automorphisms of the unit disc $\mathbb{D}$. For every $m \in \mathbb{N}$ we have a family of operators depending on $m + 1$ positive real parameters. We construct a large class of operators in the Cowen-Douglas class Cowen-Douglas class of the unit disc D which are homogeneous with respect to the action of the group Möb – the Möbius group consisting of bi-holomorphic automorphisms of the unit disc D. The associated representation for each of these operators is multiplicity free.

The sharp estimates of all homogeneous expansions for f are established, where $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))$ is a k-fold symmetric quasi-convex mapping defined on the unit polydisk in $\mathbb{C}^n$. The sharp estimates of all homogeneous expansions for a subclass of starlike mappings on the unit ball in complex Banach spaces are first established. Meanwhile, the sharp estimates of all homogeneous expansions for the above generalized mappings on the unit polydisk in $\mathbb{C}^n$ are also obtained. The sharp estimates of all homogeneous expansions for a subclass of quasi-convex mappings of type B and order α on the unit ball in complex Banach spaces are given. The sharp estimates of all homogeneous expansions for the above generalized mappings on the unit polydisk in $\mathbb{C}^n$ are also established. In particular, the sharp estimates of all homogeneous expansions for a subclass of quasi-convex mappings (include quasi-convex mappings of type A and quasi-convex mappings of type B) in several complex variables are get accordingly.

Given a centre of a planar differential system, we extend the use of the Lie bracket to the determination of the monotonicity character of the period function. As far as we know, there are no general methods to study this function, and the use of commutators and Lie bracket was restricted to prove isochronicity. We characterize the centers of the quasi-homogeneous planar polynomial differential systems of degree three. Such systems do not admit isochronous centers. We study the global behaviour of the period function on the period annulus of degenerate centres for two families of planar polynomial vector fields. These families are the quasi-homogeneous vector fields and the vector fields given by the sum of two quasi-homogeneous Hamiltonian ones. We prove that the period function is globally decreasing, extending previous results that deal either with the Hamiltonian quasi-homogeneous case or with the general homogeneous situation.

An explicit construction of all the homogeneous holomorphic Hermitian vector bundles over the unit disc $\mathbb{D}$ is given. It is shown that every such vector bundle is a direct sum of irreducible ones. Among these irreducible homogeneous holomorphic Hermitian vector bundles over $\mathbb{D}$, the ones corresponding to operators in the Cowen–Douglas class $B_n(\mathbb{D})$ are identified. The explicit description of irreducible homogeneous operators in the Cowen–Douglas class and the localization of Hilbert modules naturally leads to the definition of a smaller class possessing a flag structure. These operators are shown to be irreducible. We study quasi-homogeneous operators, which include the homogeneous operators, in the Cowen–Douglas class. We give two separate theorems describing canonical models for these operators using techniques from complex geometry. This considerably extends the similarity and unitary classification of homogeneous operators in the Cowen–Douglas class.

The Contents

Chapter 1
Convexity Properties and the Quasi-Convex Mappings

We show that types A and B" seem to be natural generalizations of the convex mappings in the plane. It is much easier to check whether a function is in one of these classes than to check for convexity. We show that the upper and lower bounds on the growth rate of such mappings is the same as for the convex mappings. We show that sharp estimations of all homogeneous expansions for f are given, where $f(z)$ is a normalized quasi-convex mapping (including quasi-convex mapping of type A and quasi-convex mapping of type B) defined on the open unit polydisk in $\mathbb{C}^n$, and $D^m f_k(0)(z^m)$.

Section (1.1): Holomorphic Mappings in $\mathbb{C}^n$

In the complex plane analytic functions which map the unit disk onto starlike or convex domains have been extensively studied. These functions are easily characterized by simple analytic or geometric conditions and there are many well known results which help us understand their nature. In moving to higher dimensions several difficulties arise. Some are predictable, some are somewhat surprising. Imposing the condition that a mapping be convex turns out to be very restrictive and so we will introduce a larger class of mappings with properties similar to the convex mappings in the plane. We actually look at two classes, the "Quasi-Convex Mappings, Types A and B", but we suspect that they are the same. We will contain:

(a) A brief review of results in the plane with a discussion of some of the difficulties encountered in extending the results to higher dimensions.

(b) Some characterizations of convex and starlike mappings in higher dimensions.

(c) The introduction of the "Quasi-Convex" families of mappings in $\mathbb{C}^n$ along with some preliminary results.

(d) A discussion of open questions. Before going further let us define some terms which will recur.

(e) Let X be a Banach space. The ball of radius $r, B_r = \{Z \in X : \|Z\| < r\}$. If $r = 1$, we will simply use B and if $X = C$, then $B = \Delta$.

(f) A setAis convex if $z, w \in A \Rightarrow t_z + (1-t)\, w \in A$, for all $t \in [0,1]$, and a mapping is said to be convex if it maps the unit ball onto a convex domain.

(g) A set A is starlike with respect to $z_0 \in A$ if $z \in A \Rightarrow (1-t)z + tz_0 \in A$, for all $t \in [0,1]$. We will use the termstarlike to mean "starlike with respect to 0".

A mapping is said to be starlike if it maps the unit ball onto a starlike domain.

(i) $S = \{f : \Delta \to \mathbb{C}: f$ is analytic and univalent, $f(0) = 0$ and $f(0) = 1\}$.

(ii) $S^* = \{f \in S : f(\Delta)$ is starlike with respect to $0\}$.

(iii) $K = \{f \in S : f(\Delta)$ is convex$\}$.

In trying to obtain analogous results in higher dimensions we run into several problems. For example, in proving the result

$$f \in K \Leftrightarrow Re\{\, zf''(z)/f(z) + 1\} > 0,$$

we use the fact that if f is convex, then the tangent vector turns in one direction. i.e. $\frac{d}{d\theta}(arg(izf'(z))) > 0, z = re^{i\theta}$. In higher dimensions this concept has no meaning.

Similarly, the characterization,

$$f \in S^* \Leftrightarrow Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

is obtained from the observation that for f to be starlike $\frac{d}{d\theta}(\arg\ f(re^{i\theta})) > 0.$

Once again this has no meaning in higher dimensions, nor does the expression $zf'(z)/f(z)$. The analogue of the well-known equivalence, "$f \in K \Leftrightarrow zf' \in S^*$" is false in higher dimensions as we show in Examples (1.1.9) and (1.1.13).

Our intuition seems to let us down when we realize that even if we take a function $f \in K$ and form a function $F : B \subset \mathbb{C}^2 \to \mathbb{C}^2$ with $F(z,w) = (f(z), f(w))$, then F is not necessarily convex. This is demonstrated in the following example.

Example (1.1.1)[1]: Let B be the Euclidean ball in $\mathbb{C}^2$, then the mapping

$$F(z,w) = \left(\frac{z}{1-z}, \frac{w}{1-w}\right), \qquad z,w \in \mathbb{C}, |z|^2 + |w|^2 < 1,$$

is not convex even though $f(z) = z/(1-z), z \in \Delta$, is a convex function in the plane. Note that $u = f(z)$ maps the real line segment $-1 < z < 1$ onto the real halfline $u > -1/2$. A necessary condition that F is convex is that every cross-section of the image of B is convex. Consider the cross-section of $F(B), \{(u,v) \in F(B) : u,v \in R\}$. This is the image of $\Omega = \{(s,t) \in B : -1 < s < 1, -1 < t < 1\}$. This cross-section, $F(\Omega)$, is not convex. If it were, then the set $\{(u,v) : u > 0, v > 0\}$ would be contained in $F(\Omega)$. In particular the line $\{(u,u) : u > 0\} \subset F(\Omega)$. If $u = v$, then $s = t$ and $s^2 + t^2 < 1, s = t < 1/\sqrt{2}$, so that $u = v < 1/(\sqrt{2} - 1)$. We cannot get any further from the origin along this line and so it is clear that this cross-section is not convex. See Figure 1. Similar arguments show that there is no convex mapping $F(z,w) = (z/(1-z), g(w))$.

In one approach to extending these results to $\mathbb{C}^n, n \geq 2$, Suffridge [12] generalizes some of Robertson's results [10], which use the principle of subordination in the plane, to higher dimensions.

To extend these theorems to higher dimensions we first need to adapt the Schwarz Lemma accordingly. There are several ways of doing this (see Harris [5]) but the appropriate one for our purposes is as follows.

Theorem (1.1.2)[1]: Let X be a Banach space and let $B \subset X$. If $f : B \to Y$ is holomorphic, $\|f(x)\| \leq 1$ when $x \in B$ and $f(0) = 0$, then $\|f(x)\| \leq \|x\|$ for all $x \in B$.

We next need to extend the concepts of "positive real part", and "functions of positive real part". We use functionals to accomplish this. For a more complete treatment see Gurganus [4] or Suffridge [14].

fORt X be a Banach space and $x \in X, x \neq 0$, and let X^* denote the space of linear functionals from X to $\mathbb{C}$. Define

$$T(x) = \{\ell_x \in X^* : \|\ell_x\| = 1, and\ \ell_x(x) = \|x\|\} where\ \ell_x = \sup_{\|y\|=1} |\ell_x(y)|.$$

Let $H_1 = \{y \in X : Re\ \ell_x(y) = \|x\|\}$. H_1 is a supporting hyperplane for $B_{\|x\|}$ at x, because $x \in H_1$ and if $y \in H_1$, then

$$\|y\| \geq |\ell_x(y)| \geq Re\{\ell_x(y)\} = \|x\|, \qquad since\ \|\ell_x\| = 1.$$

If X has complex dimension n, then H_1 has real dimension $2n - 1$ and it is thus a hyperplane. In the infinite dimensional case, H_1 has real codimension 1.

Example (1.1.3)[1]: Let $X = \mathbb{C}^n$ with a p-norm, $1 < p < \infty$, i.e. $\|x\|_p = (\sum_{i=1}^n |x_i|^p)\ \frac{1}{p}$. Then $\ell_x \in T(x)$ is given uniquely by

$$\ell_x(w) = \frac{\sum_{\{j:xj\neq 0\}} w_j \bar{x}_j |x_j|^{p-2}}{\|x\|_p^{p-1}}.$$

Example (1.1.4)[1]: In the case of $p = 1$, $T(x)$ is the set of linear functionals of the form $\ell_x(w) = \sum_{\{j:x_j\neq 0\}} \frac{w_j \bar{x}_j}{|x_j|} + \sum_{\{j:x_j=0\}} \gamma_j w_j$ with $\gamma_j \in \mathbb{C}, |\gamma_j| \leq 1$ for all j.

And in the case $p = \infty$, we let $J = \{j : \|x\| = |x_j|\}$ and

$$\ell_x(w) = \sum_{j\in J} \frac{t_j w_j \bar{x}_j}{\|x\|} \; where\; each\; t_j \geq 0 \; and \sum_{j\in J} t_j = 1.$$

We now define three families of functions:

$N_0 = \{w : B \to X :$ w is holomorphic, $w(0) = 0$, and $Re\{\ell_x(w(x))\} \geq 0$,
$for\; all\; x \in B, x \neq 0, \ell_x \in T(x)\}$,

$N = \{w \in N\,0 : Re\{\ell_x(w(x))\} > 0,\; for\; all\; x \in B, x \neq 0,\; \ell_x \in T(x)\}$,

$M = \{w \in N : Dw(0) = I\}$.

Example (1.1.5)[1]: Let $X = \mathbb{C}, B = \Delta$, then

$$N_0 = \{w : \Delta \to \mathbb{C} : w\; is\; analytic, w(0) = 0, Re\{\tilde{z}w(z)\} \geq 0, z \in \Delta\backslash\{0\}\}.$$

However ,

$$Re\{\tilde{z}w(z)\} \geq 0 \;\Leftrightarrow\; Re\left\{\frac{|z|^2 w(z)}{z}\right\} \geq 0$$

$$\Leftrightarrow\; Re\left\{\frac{w(z)}{z}\right\} \geq 0.$$

Thus, if $w \in N_0$ either $Re\,\{w(z)/z\} \equiv 0$ or Re $\{w(z)/z\} > 0$.

We also observe that $M = \{zf : f \in \mathbb{P}\}$, where $\mathbb{P}$ is the family of functions that are analytic in the unit disk with $f(0) = 1$ and $f(\Delta)$, contained in the right half-space.

The following lemmas are Suffridge's extensions of Robertson's theorems, [14], [12].

Lemma (1.1.6)[1]: (Suffridge). Let $v : B \times I \to B$ be holomorphic in B for each $t \in I = [0,1]$ (i.e. v(·,t) is holomorphic for each fixed $t \in I$), $v(0,t) = 0$ and $v(x,0) = x$. If $\lim_{t\to 0^+} \frac{x - v(x,t)}{t} = w(x)$ exists and is holomorphic in B, then $w \in N_0$.

Lemma (1.1.7)[1]: (Suffridge). Let $f : B \to Y$ be a biholomorphic mapping of B onto an open set $f(B) \subset Y$ and let $f(0) = 0$. Assume $F : B \times I \to Y$ is holomorphic in B for each fixed $t \in I, F(x,0) = f(x), F(0,t) = 0$ and suppose $F(B,t) \subset f(B)$ for each fixed $t \in I$. Further, suppose

$$\lim_{t\to 0^+} \frac{F(x,0) - F(x,t)}{t} = G(x)$$

exists and is holomorphic. Then $G(x) = DF(x)(w(x))$ where $w \in N_0$.

From these we obtain the characterization of starlike mappings in higher dimensions. Note the similarity of this condition to that of starlike functions in the plane.

Theorem (1.1.8)[1]: The mapping $f : B \subset \mathbb{C}^n \to \mathbb{C}^n$ is starlike if and only if $f(x) = Df(x)(\omega(x))$ for some $\omega \in M$.

This result was obtained by Matsuno [7] for the Euclidean norm, and by Suffridge for the sup norm [12], and for more general norms [13].

We include two examples of mappings which are starlike.

Example (1.1.9)[1]: The function $f(Z) = (z + aw^2, w)$ where $Z = (z, w), \|Z\|^p = |z|^p + |w|^p < 1, p > 1, z, w \in \mathbb{C}$ is starlike if and only if

$$|a| \leq \left(\frac{p^2 - 1}{4}\right)^{\frac{1}{p}} \left(\frac{p+1}{p-1}\right). \qquad (1)$$

We have that f is starlike if and only if $Re\,\{\ell_Z(Df(Z)^{-1}(f(Z)))\} > 0$. Since

$$Df(Z)^{-1}(f(Z)) = \begin{bmatrix} z - aw^2 \\ w \end{bmatrix},$$

we have

$$Re\, \ell_Z\left(Df(Z)^{-1}\big(f(Z)\big)\right) = Re\,\{\|Z\|^p - aw^2\,\tilde{z}|z|^{p-2}\}\,\|Z\|^{p-1}.$$

Replacing Z by $\alpha Z, |\alpha| < 1/\|Z\|$, we apply the minimum principle for harmonic functions to see that we may assume that $\|Z\| = 1$. Thus the necessary and sufficient condition for f to be starlike is

$$Re\,\{1 - aw^2\,\tilde{z}|z|^{p-2}\} \geq 1 - |a||w|^2|z|^{p-1} = 1 - |a|(1-r^p)^{\frac{2}{p}}r^{p-1}, where\ r = |z|.$$

By elementary calculus, write $h(r) = 1 - |a|(1-r^p)^{\frac{2}{p}}r^{p-1}, h(r)$ has

$$1 - |a|\left(\frac{2}{p+1}\right)^{\frac{2}{p}}\left(\frac{p-1}{p+1}\right)\left(\frac{p-1}{p+1}\right)^{-\frac{1}{p}}$$

as its minimum value and (1) follows.

Note that this result together with Example (1.1.12) tells us that the result "$Df(Z)(Z)$ is starlike implies f is convex " does not hold for $n > 1$. If we use the 2-norm and let $f(z,w) = (z + \frac{a}{2}w^2, w)$, then $Df(z,w)(z,w) = (z + aw^2, w)$ and this is starlike for $|a| \leq 3\sqrt{3}/2$. However, f is only convex for $|a| \leq 1$.

Example (1.1.10)[1]: The mapping $f : B \subset \mathbb{C}^2 \to \mathbb{C}^2$ given by $f(z,w) = (z + azw, w)$, with $|z|^p + |w|^p < 1$ is starlike if and only if $|a| \leq 1$ for all p-norms, $1 \leq p \leq \infty$.

First assume $1 < p < \infty$, then f is starlike if and only if

$$Re\{\ell_Z(Df(Z)^{-1}(f(Z)))\} > 0$$

for $\ell_Z \in T(Z)$.

$$Re\,\left\{\ell_Z\left(Df(Z)^{-1}\big(f(Z)\big)\right)\right\} = Re\,\left\{\ell_Z\left(\frac{z}{1+aw}, w\right)\right\}.$$

If we use a p-norm and assume by the minimum principle (as before) that $\|Z\| = 1$, we find that $Re\left\{\ell_Z\left(\left(\frac{z}{1+aw}, w\right)\right)\right\} \geq Re\left\{\frac{|z|^p}{1+aw} + |w|^p\right\}$

$$= Re\left\{\frac{|z|^p + |w|^p + aw|w|^p}{1+aw}\right\}$$

$$= Re\left\{\frac{1 + aw|w|^p}{1+aw}\right\}$$

$$= Re\left\{\frac{1 + aw|w|^p + \overline{aw} + |a|^2|w|^{p+2}}{|1+aw|^2}\right\}.$$

Clearly $|a| \leq 1$ is necessary and we need to find a such that

$$Re\{1 + |a|^2|w|^{p+2} + \overline{aw} + aw|w|^p\} \geq 0.$$

It is sufficient to have

$$1 + |a|^2|w|^{p+2} - |a||w| - |a||w|^{p+1} = (1 - |a||w|^p)(1 - |a||w|^{p+1}) \geq 0$$

and hence the result readily follows. The cases $p = 1$ and $p = \infty$ are easily handled.

The last theorem might lead us to conjecture that

$$Df(x)^{-1}(D^2f(x)(x,x) + Df(x)(x)) \in M\ if\ and\ only\ if\ f\ is\ convex.$$

The mapping given in Example (1.1.1) quickly dispels this thought. It turns out that this is a necessary but not sufficient condition. We will look more extensively at this condition later on. For necessary and sufficient conditions we have the following theorem, [12].

Theorem (1.1.11)[1]: (Suffridge). Let X and Y be Banach spaces with $B \subset X$. Let $f: B \to Y$ be locally biholomorphic with $f(x) - f(y) = Df(x)(\omega(x,y))$ for $x, y \in B$.

Then f is convex if and only if $Re\{\ell_x(\omega(x,y))\} > 0$ whenever $\|y\| < \|x\|$ and $\ell_x \in T(x)$.

The condition says that $f(B)$ must be starlike with respect to each of its interior points. However, this condition, which agrees with our intuition, is difficult to apply. For a somewhat different approach, see [3]. The following examples make use of the above theorem.

Example (1.1.12)[1]: The function $f(Z) = (z + aw^2, w)$ where $Z = (z, w)$ with $\|Z\|^2 = |z|^2 + |w|^2 < 1$ and $z, w \in C$ is convex if and only if $|a| \leq 1/2$.

We will need to check when $Re\langle Df(Z)^{-1}(f(Z) - f(U), Z\rangle > 0$ where $Z = (z, w)$ and $U = (u, v)$ with $\|Z\| \geq \|U\|$.

$$\begin{aligned}
Re\langle Df(Z)^{-1}(f(Z) - f(U)), Z\rangle &= Re\,\{|z|^2 + |w|^2 - u\bar{z} - v\bar{w} - a\bar{z}(w - v)^2\} \\
&= \|Z\|^2 - Re\langle U, Z\rangle - Re\{a\tilde{z}(w - v)^2\} \\
&\geq \|Z\|^2 - Re\langle U, Z\rangle - |a||z||w - v|^2 \\
&= \|Z\|^2 - Re\langle U, Z\rangle - |a||z|(\|Z\|^2 - |z|^2 - 2Re(u\bar{z} + v\bar{w}) + 2Re\, u\bar{z} \\
&\quad + \|U\|^2 - |u|^2) \\
&= \|Z\|^2(1 - |a||z|) - Re\langle U, Z\rangle(1 - 2|a||z|) - |a||z|(\|U\|^2 - |z - u|^2) \\
&\geq \|Z\|^2\,(1 - |a||z|) - R\langle U, Z\rangle(1 - 2|a||z|) - |a||z|(\|Z\|^2 - |z - u|^2) \\
&= (\,\|Z\|^2 - Re\langle U, Z\rangle)(1 - 2|a||z|) + |a||z||z - u|^2 \\
&\geq 0\; when |a| \leq \frac{1}{2},
\end{aligned}$$

since $|Re\langle U, Z\rangle| \leq \|Z\|^2$, and $|a| \leq \frac{1}{2} \Rightarrow |a||z| \leq \frac{1}{2}$.

If $|a| > \frac{1}{2}$ we can find z such that $\bar{z}a > \frac{1}{2}, u = z, v = -w \in R$ and we obtain

$$\begin{aligned}
&Re\langle Df(Z)^{-1}(f(Z) - f(U)), Z\rangle \\
&= \|Z\|^2 - Re\langle U, Z\rangle - Re\{a\bar{z}(w - v)^2\} \\
&< \|Z\|^2 - Re\{z\bar{z} - w\bar{w}\} - \frac{1}{2}\,(2w)^2 \\
&= 0\,.
\end{aligned}$$

Example (1.1.13)[1]: The mapping $f : B \subset \mathbb{C}^2 \to \mathbb{C}^2$, with the 2-norm, given by $f(z, w) = (z + azw, w)$ is convex if and only if $|a| \leq 1/\sqrt{2}$.

It is sufficient to assume that $a > 0$. Using the above result we let $Z = (z, w)$ and $U = (u, v)$. Then

$$\begin{aligned}
\langle Df(Z)^{-1}(f(Z) - f(U)), Z\rangle &= \bar{z}(\,z - u\,)\left(\frac{1 + av}{1 + aw}\right) + \bar{w}(w - v) \\
&= \bar{z}(\,z - u\,)\left(\frac{1 - a(w - v)}{1 + aw}\right) + \bar{w}(w - v) \\
&= \langle Z - U, Z\rangle - \frac{a\bar{z}}{1 + aw}\,(z - u)(w - v).
\end{aligned}$$

So,

$$\begin{aligned}
&Re\{Df(Z)^{-1}(f(Z) - f(U)), Z\} \\
&= Re\left\{\langle Z - U, Z\rangle - \frac{a\bar{z}}{1 + aw}\,(z - u)(w - v)\right\} \\
&\geq Re\langle Z - U, Z\rangle - \frac{a|z|}{1 - a|w|}|z - u||w - v|.
\end{aligned}$$

By examining the function $\frac{ax}{1-ay}$ subject to the constraint $x^2 + y^2 = k^2$ we see that $\frac{a|z|}{1-a|w|}$ is maximized at $\frac{a\|Z\|}{1-a^2\|Z\|^2}$ when $|z| = \|Z\|\sqrt{1 - a^2\|Z\|^2}$ and $|w| = a\|Z\|^2$.

Similarly, by maximizing the product xy subject to the constraint $x^2 + y^2 = k^2$ we see that $|z - u||w - v| \le \frac{1}{2}\|Z - U\|^2$ with equality when $|z - u| = |w - v|$.

Now we have the sharp inequality

$$Re\{\langle Df(Z)^{-1}(f(Z) - f(U)), Z\rangle\} \ge Re\langle Z - U, Z\rangle - \frac{a\|Z\|}{\sqrt{1 - a^2\|Z\|^2}}\frac{1}{2}\|Z - U\|^2. \quad (2)$$

For $a\|Z\| = 1/\sqrt{2}$ this expression is positive for $\|U\| \le \|Z\|$ since

$$Re\langle Z - U, Z\rangle - \frac{a\|Z\|}{\sqrt{1 - a^2\|Z\|^2}}\frac{1}{2}\|Z - U\|^2 = Re\left\{\langle Z - U, Z\rangle - \frac{1}{2}\langle Z - U, Z - U\rangle\right\}$$
$$= \|Z\|^2 - \|U\|^2 \ge 0.$$

Since the function $\frac{x}{\sqrt{1-x^2}}$ is increasing on $[0,1)$ the inequality holds for $a\|Z\| \le 1/\sqrt{2}$.

To show that $Re\{\langle Df(Z)^{-1}(f(Z) - f(U)), Z\rangle\} < 0$ for $a\|Z\| > 1/\sqrt{2}$ we choose

$$Z = (z, w) = (k\sqrt{1 - a^2k^2}, -ak^2) \ and$$
$$U = (u, v) = (z - (z + w)cos\theta e^{i\theta}, w - (z + w)cos\theta e^{-i\theta}),$$

where $0 \le k \le 1$ and $cos\theta \ne 0$. We note that $\|Z\| = \|U\| = k$.

$$Re\left\{\langle Z - U, Z\rangle - \frac{a\bar{z}}{1 + aw}(z - u)(w - v)\right\}$$
$$= \frac{(z + w)^2\cos^2\theta}{\sqrt{1 - a^2k^2}}(\sqrt{1 - a^2k^2} - ak)$$
$$< 0, for\ ak > 1/\sqrt{2}.$$

So, on $B = \{Z : \|Z\| < 1\}$ we have $a \le 1/\sqrt{2}$ for convexity.

In the plane, $f \in K$ if and only $if\ zf' \in S^*$. This last result shows us that this is not true in higher dimensions. The mapping $Df(Z)(Z) = (z + 2azw, w)$, for $a = 1/\sqrt{2}$, is not even univalent much less starlike. To see this we note that $(z(1 + \sqrt{2}w), w) = (0, -1/\sqrt{2})$ for all $Z = (z, -1/\sqrt{2}), Z \in B$.

When we couple this with Example (1.1.9) we see that the implication does not hold in either direction.

The nature of convex mappings is strongly dependent on the norm used in the domain. Using the sup norm in $\mathbb{C}^n$ so that the unit ball is a polydisk, the only normalized convex mappings are mappings F such that Fj is a function of zj only and Fj is a convex mapping on the unit disk. On the other hand, using the 1norm, $\|Z\| = \sum_{j=1}^{n} |z_j|$, the convex maps of the unit ball are the non-singular linear mappings. In the Euclidean norm (i.e. using the 2-norm in $\mathbb{C}^n$) we have the following theorem. In view of the results stated above for the sup norm and the 1-norm, such a result cannot hold for normed linear spaces in general.

Theorem (1.1.14)[1]: Let $B = \{z \in \mathbb{C}^n : \|z\|^2 = \sum_{i=1}^{n} |z_i|^2 < 1\}$ and assume $f : B \to \mathbb{C}^n$ is holomorphic with $f(0) = 0$ and $Df(0) = 1$. Further, assume $\sum_{k=2}^{\infty} \frac{k^2}{k!}\left\|D^k f(0)\right\| \le 1$. Then $f(B)$ is convex.

Proof. Consider a function $A_k : \prod_{j=1}^{k}\mathbb{C}^n \to \mathbb{C}^n$ that is linear in each variable and symmetric. Then $A_k(z, z, \ldots, z) \equiv A_k(z^k)$ is a homogeneous polynomial of degree k and by a result of Hormander [6], we have,

$$\|A_k\| \sup_{\substack{\|z^{(j)}\|=1 \\ 1\le j\le k}} \left\|A_k(z^{(1)}, z^{(2)}, \ldots, z^{(k)})\right\| = \sup_{\|z\|=1} \|A_k(z, z, \ldots, z)\|.$$

Further, by Lemma (1.1.6) in H¨ormander's above, given $f : B \to \mathbb{C}^n$, where f is holomorphic on the unit ball B of $\mathbb{C}^n$ with k^{th} derivative at $0, D^k f(0)$. We may identify $\frac{1}{k!}\ D^k f(0)$ with A_k above. Then

$$f(z) = f(0) + \sum_{k=1}^{\infty} \frac{1}{k!}\ D^k f(0)(z^k) = f(0) + \sum_{k=1}^{\infty}\ A_k(z^k).$$

Assuming $: B \to \mathbb{C}^n$ satisfies $f(0) = 0, Df(0) = A_1 = I$ and that

$$\sum_{k=2}^{\infty}\ k^2 \|A_k\| \leq 1,$$

we proceed as follows.

First observe that

$$\sum_{k=2}^{\infty}\ k\|A_k\| \leq \frac{1}{2} \sum_{k=2}^{\infty}\ k^2 \|Ak\| \leq \frac{1}{2}$$

with equality in the first step if and only if $Ak\ \equiv 0$ when $k > 2$. We also note that

$$\left\| \sum_{k=2}^{\infty}\ kA_k(z^{k-1}, w) \right\| \leq \sum_{k=2}^{\infty}\ k\|A_k\| \|z\|^{k-1} \|w\|\ \leq \|z\|\ \|w\| \sum_{k=2}^{\infty}\ k\|A_k\|$$
$$= N\|z\|\|w\|$$

where

$$N\ = \sum_{k=2}^{\infty}\ k\|A_k\| \leq \frac{1}{2}\ and\ \|z\| \leq 1, \qquad \|w\| \leq 1.$$

Therefore, it follows that

$$\left\| \sum_{k=2}^{\infty}\ kA_k(z^{k-1}, \cdot)^p(w) \right\| \leq N^p \|z\|^p \|w\|$$

when p is a non-negative integer.

The analytic condition for f to be convex is that

$$Re\{Df(z)^{-1}(f(z) - f(w)), z\} > 0\ when\ 1 > \|z\| \geq \|w\|.$$

We have

$$Df(z)(u) = \frac{\lim\limits_{h \to 0}\ \big(f(z\ +\ hu) - f(z)\big)}{h} = \sum_{k=1}^{\infty}\ kA_k(z^{k-1}, u).$$

That is,

$$\frac{f(z\ +\ hu) - f(z)}{h} = \sum_{k=2}^{\infty} \sum_{l=1}^{k} \binom{k}{l}\ h^{l-1} A_k(z^{k-1}, u)$$
$$\to \sum_{k=1}^{\infty}\ kA_k(z^{k-1}, u)\ as\ h \to 0.$$

Therefore,

$$Df(z)^{-1} = \left[I - \big(I - Df(z)\big)\right]^{-1} \tag{3}$$
$$= \left[I\ - \sum_{k=2}^{\infty}\ - kA_k(z^{k-1}, \cdot)\right]^{-1} \tag{4}$$

$$= I + \sum_{l=1}^{\infty} (-1)^l \left(\sum_{k=2}^{\infty} kA_k(z^{k-1},\cdot)\right)^l , \quad (5)$$

because

$$\left\|\sum_{k=2}^{\infty} kA_k(z^{k-1},\cdot)\right\| \le N\|z\| \le \frac{1}{2} \quad (6)$$

for each fixed $z, \|z\| < 1$.

Also,

$$f(z) - f(w) = \sum_{k=1}^{\infty} [A_k(z^k) - A_k(w^k)] = \sum_{k=1}^{\infty} \sum_{p=1}^{k} A_k(z^{k-p}, w^{p-1}, z-w)$$

$$= z - w + \sum_{k=2}^{\infty} \sum_{p=1}^{k} A_k(z^{k-p}, w^{p-1}, z-w).$$

Therefore,

$$H(z,w) \equiv Df(z)^{-1}(f(z) - f(w))$$

$$= \left(\left(I + \sum_{l=1}^{\infty} (-1)^l \left(\sum_{k=2}^{\infty} kA_k(z^{k-1},\cdot)\right)^l\right)\right.$$

$$\left.\times \left(z - w + \sum_{p=2}^{\infty} \sum_{q=1}^{p} A_p(z^{p-q}, w^{q-1}, z-w)\right)\right)!$$

$$= z - w + \sum_{l=1}^{\infty} (-1)^l \left(\sum_{k=2}^{\infty} kA_k(z^{k-1},\cdot)\right)^l (z-w)$$

$$+ \sum_{l=1}^{\infty} (-1)^l \left(\sum_{k=2}^{\infty} kA_k(zk-1,\cdot)\right)^{l-1} \left(-\sum_{p=2}^{\infty} \sum_{q=1}^{p} A_p(z^{p-q}, w^{q-1}, z-w)\right)$$

$$= (z - w) + \sum_{l=1}^{\infty} (-1)^l \sum_{k=2}^{\infty} kA_k(z^{k-1},\cdot)^{l-1}$$

$$\left(\sum_{p=2}^{\infty} \left(\sum_{q=1}^{p} A_p(z^{p-1}, z-w) - A_p(z^{p-q}, w^{q-1}, z-w)\right)\right)$$

$$= z - w + \sum_{l=1}^{\infty} (-1)^l \left(\sum_{l=1}^{\infty} kA_k(z^{k-1},\cdot)\right)^{l-1}$$

$$\times \sum_{k=2}^{\infty} \sum_{q=2}^{k} \sum_{p=2}^{q} (A_k(z^{k-p}, w^{p-2}, (z-w)^2))$$

Now

$$\left\|\sum_{k=2}^{\infty}\sum_{q=2}^{k}\sum_{p=2}^{q} A_k(z^{k-p}, w^{p-2}, (z-w)^2)\right\|$$

$$\leq \sum_{k=2}^{\infty}\sum_{q=2}^{k}\sum_{p=2}^{q} \|A_k\|\|z-w\|^2 = \sum_{k=2}^{\infty}\sum_{q=2}^{k} (q-1)\|A_k\|\|z-w\|^2$$

$$= \sum_{k=2}^{\infty} \frac{(k-1)(k)}{2} \|A_k\|\|z-w\|^2$$

$$\leq \frac{1-N}{2}\|z-w\|^2 .$$

Now assume $\|w\| \leq \|z\| = r < 1$. Then

$$\frac{\|z-w\|^2}{2} = \frac{1}{2}\left(\|z\|^2 - 2\,Re\langle w,z\rangle + \|w\|^2\right)$$

$$\leq r^2 - Re\langle w,z\rangle > 0 \; if \; w \neq z.$$

Thus,

$$Re\langle H(z,w),z\rangle = Re\{\langle z-w,z\rangle + \langle H(z,w)-(z-w),z\rangle\}$$

while

$$\|H(z,w)-(z-w)\| \leq \sum_{l=1}^{\infty} \left\|\sum_{k=2}^{\infty} kA_k(z^{k-1},\cdot)\right\|^{l-1}$$

$$((1-N)(r^2 - Re\langle w,z\rangle)) \leq \sum_{l=1}^{\infty} (N\|z\|)^{l-1}(1-N)(r^2 - Re\langle w,z\rangle$$

$$= \frac{1-N}{1-N\|z\|}(r^2 - Re\langle w,z\rangle)$$

Since $\langle z-w,z\rangle = r^2 - \langle w,z\rangle \geq 0$ we have

$$Re\langle H(z,w),z\rangle \geq (r^2 - Re\langle w,z\rangle)\left(1 - \frac{1-N}{1-N\|z\|}\|z\|\right)$$

$$= (r^2 - Re\langle w,z\rangle)\left(\frac{1-\|z\|}{1-r\|z\|}\right) \geq 0$$

and the proof is complete.

As we have seen, the condition that a mapping be convex is somewhat restrictive and unwieldy to verify. You will recall that even the mapping $(f_1(z_1),\ldots,f_n(z_n))$ with $f_j : \Delta \to \mathbb{C}$ convex for each $j = 1,\ldots,n$, may not be convex in $\mathbb{C}^n$. This leads us to consider a set of mappings which contains the set of convex mappings for dimension two or more and has many of the "nice" properties that we would like a generalization of the convex functions in the plane to have, yet has a more readily usable definition.

The characterization

$$f \in K \; if \; and \; only \; if \; Re\left\{\frac{zf''(z)}{f'^{(z)}} + 1\right\} > 0 \tag{7}$$

is well-known and as we have mentioned comes from the fact that the curvature of the boundary of the image of any disk $|z| < r < 1$ is always positive if and only if the function is convex. A less well-known result is that (see Suffridge, [13]),

$$f \in K \text{ if and only if } Re\left\{\frac{zf'(z)}{f(z)-f(\xi)}\right\} > 0, \text{for all } z, \xi \in \Delta, |\xi| < |z|. \quad (8)$$

This characterization comes from noticing that f being convex is equivalent to f being starlike with respect to every interior point. The expression is arrived at by letting z vary on a circle of radius r and then for any fixed ξ with $|\xi| < r < 1$, the argument of the vector connecting $f(\xi)$ with $f(z)$ is an increasing function of arg(z).

If $|\xi| = r, \xi \neq z$, then from (8) we have

$$Re\left\{\frac{zf'(z)}{f(z)-f(\xi)}\right\} \geq 0, \quad (9)$$

We further note that when $|z| = |\xi|, z \neq \xi, Re\left\{\frac{z+\xi}{z-\xi}\right\} = 0$. Hence

$$Re\left\{\frac{2zf'(z)}{f(z)-f(\xi)} - \frac{z+\xi}{z-\xi}\right\} \geq 0. \quad (10)$$

We observe that the singularity at $z = \xi$ is removable and we are now working with the real part of an analytic function of z and ξ, which is thus harmonic in both z and ξ.

By fixing z and varying ξ, since we know that this function cannot attain its minimum on the interior of the disk $|\xi| < r$, the inequality is strict on the interior. Similarly, by holding ξ fixed and varying z we get the same result for z. We conclude that

$$f \in K \text{ if and only if } Re\left\{\frac{2zf'(z)}{f(z)-f(\xi)} - \frac{z+\xi}{z-\xi}\right\} \geq 0, \text{for all } z, \xi \in \Delta. \quad (11)$$

We further note that

$$\lim_{\xi \to z} Re\left\{\frac{2zf'(z)}{f(z)-f(\xi)} - \frac{z+\xi}{z-\xi}\right\} = Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\}. \quad (12)$$

When trying to generalize these ideas, we have seen in Theorem (1.1.11) that (8) does extend to higher dimensions. However, in trying to generalize the expression in (11) we find that we cannot find an appropriate second term that removes the singularity. So we need to modify the approach.

Definition (1.1.15)[1]: Let

$$\mathbb{S}_n = \{f : B \subset \mathbb{C}^n \to \mathbb{C}^n : f(0) = 0 \text{ and } Df(0) = I\},$$

and let

$$S^{2n-1} = \{U \in \mathbb{C}^n : \|U\| = 1\}.$$

represent the unit sphere in $\mathbb{C}^n$.

Consider the one-dimensional subset of B,

$$C_U = \{\alpha U : U \in S^{2n-1}, U \text{ fixed, and } \alpha \in \Delta\}.$$

On this slice of B we can mimic the expression in (11) in the following way.

Definition (1.1.16)[1]: Let $U \in \mathbb{C}^n$, with $\|U\| = 1$, and let $\ell_U \in T(U)$. For $f \in \mathbb{S}_n$ define $G_f : \Delta \times \Delta \to \hat{\mathbb{C}}$ by

$$G_f(\alpha,\beta) = \frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\left(f(\alpha U) - f(\beta U)\right)\right)} - \frac{\alpha+\beta}{\alpha-\beta}, \quad (13)$$

where $\hat{\mathbb{C}}$ is the extended plane.

We now define a family of mappings, $\mathbb{G}$, which bears some resemblance to the convex mappings in the plane. The question is, how much? The lemmas which follow the definition lead up to two theorems which assert that $\mathbb{G}$ is between the convex mappings and the starlike mappings.

Definition (1.1.17)[1]: Let

$$\mathbb{G} = \{f \in \mathbb{S}_n : Re\{G_f(\alpha,\beta)\} > 0, for\ all\ \alpha,\beta \in \Delta\ and\ any\ U \in S^{2n-1}\}.$$
We call this family of mappings the "Quasi-Convex Mappings, Type A".

Lemma (1.1.18)[1]: The mapping $G_f\,(\alpha,\beta)$ is analytic in α and β.

Proof. It suffices to show that there is a removable singularity at $\alpha = \beta$. We expand $f(\beta U)$ about αU to obtain

$$f(\beta U) = f(\alpha U) + Df(\alpha U)\big((\beta - \alpha)U\big) + \frac{1}{2} D^2 f(\alpha U)([(\beta - \alpha)U]^2) + o((\beta - \alpha)^2).$$

Therefore

$$\begin{aligned} &Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)) \\ &= -Df(\alpha U)^{-1}(Df(\alpha U)\big((\beta - \alpha)U\big) + \frac{1}{2} D^2 f(\alpha U)([(\beta - \alpha)U]^2) + o((\beta - \alpha)^2))) \\ &= (\alpha - \beta)Df(\alpha U)^{-1}(Df(\alpha U)(U) + \frac{1}{2}(\beta - \alpha)D^2 f(\alpha U)(U,U) + o((\beta - \alpha))) \\ &= (\alpha - \beta)(U + \frac{1}{2}(\beta - \alpha)Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U) + o(\beta - \alpha)). \end{aligned}$$

This gives

$$\begin{aligned} &G_f\,(\alpha,\beta) \\ &= \frac{2\alpha - (\alpha + \beta)\ell_U\,[U + \frac{1}{2}(\beta - \alpha)Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U)) + o(\beta - \alpha)]}{(\alpha - \beta)\ell_U(U + \frac{1}{2}(\beta - \alpha)Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U)) + o(\beta - \alpha))} \\ &= \frac{2\alpha - (\alpha + \beta)(1 + \frac{1}{2}(\beta - \alpha)\ell_U\,[Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U)) + o(\beta - \alpha)])}{(\alpha - \beta)\ell_U\,(U + \frac{1}{2}(\beta - \alpha)Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U)) + o(\beta - \alpha))} \\ &= \frac{(\alpha - \beta)(1 + \frac{1}{2}(\beta + \alpha)\ell_U\,[Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U)) + o(\beta - \alpha)])}{(\alpha - \beta)\ell_U\,(U + \frac{1}{2}(\beta - \alpha)Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U)) + o(\beta - \alpha))}. \end{aligned}$$

Taking limits,

$$\lim_{\beta\to\alpha} G_f\,(\alpha,\beta) \tag{14}$$

$$\begin{aligned} &= 1 + \alpha\ell_U Df(\alpha U)^{-1}(D^2 f(\alpha U)(U,U)) \\ &= \frac{1}{\alpha}\ell_U(\alpha U + Df(\alpha U)^{-1}(D^2 f(\alpha U)(\alpha U,\alpha U))) \\ &= \frac{1}{|\alpha|}\,\ell_{\alpha U}(\alpha U + Df(\alpha U)^{-1}(D^2 f(\alpha U)(\alpha U,\alpha U)))\ \text{where}\ \ell_{\alpha U} \in T\,(\alpha U) \\ &= \frac{1}{|\alpha|}\,\ell_{\alpha U}\left(Df(\alpha U)^{-1}\big(D^2 f(\alpha U)(\alpha U,\alpha U) + Df(\alpha U)(\alpha U)\big)\right), \end{aligned} \tag{15}$$

which is well defined. We conclude from (14) that G_f is indeed analytic in α and β.

The next theorem asserts a result which was really the motivation for considering the family $\mathbb{G}$.

Theorem (1.1.19)[1]: Let $f \in \mathbb{S}^n$, and assume f is convex. Then $f \in \mathbb{G}$.

Proof. Given $f \in \mathbb{S}^n$, from Theorem (1.1.11) we have that if f is convex, then $Re\{\ell_Z(Df(Z)^{-1}(f(Z) - f(V))) > 0$ where $\|V\| < \|Z\| < 1$ and $\ell_Z \in T\,(Z)$.

By considering the one-dimensional cross-section of , C_U , we have

$$f\ convex\ \Rightarrow\ Re\,\{\ell_{\alpha U}\left(Df(\alpha U)^{-1}\big(f(\alpha U)-\ f(\beta U)\big)\right)\}\ >\ 0 \text{ where } |\beta|\ <\ |\alpha|$$
$$\Rightarrow\ Re\{|\alpha|\ell_{\alpha U}\left(Df(\alpha U)^{-1}\big(f(\alpha U)-\ f(\beta U)\big)\right)\}\ >\ 0$$
since corresponding to each ℓ_U we have an $\ell_{\alpha U}(\cdot)=\frac{|\alpha|}{\alpha}\ \ell_U\ (\cdot)$ in $T\ (\alpha U)$. Thus
$$f\ convex\ \Rightarrow\ Re\ \left\{\frac{1}{\alpha}\ \ell_U\left(Df(\alpha U)^{-1}\big(f(\alpha U)-\ f(\beta U)\big)\right)\right\}>\ 0$$
$$\Rightarrow\ Re\left\{\frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\big(f(\alpha U)-\ f(\beta U)\big)\right)}\right\}>\ 0 \text{ for } |\beta|\ <\ |\alpha|.$$
As before, if we let $|\beta|\ =\ |\alpha|$ with $\beta\neq\alpha$ we have
$$Re\left\{\frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\big(f(\alpha U)-\ f(\beta U)\big)\right)}\right\}\geq 0.$$
If $|\alpha|\ =\ |\beta|\ =\ r$, then $\alpha\ =\ re^{i\theta},\beta\ =\ re^{i\varphi}$ for some $\theta,\varphi\ \in\ R$ and
$$\frac{\alpha\ +\ \beta}{\alpha\ -\ \beta}=\ -i\frac{\cos\ \frac{1}{2}(\theta\ -\ \varphi)}{\sin\ \frac{1}{2}\ (\theta\ -\ \varphi)}.$$
Hence $Re\ \left\{\frac{\alpha+\beta}{\alpha-\beta}\right\}\ =\ 0$. Thus for $|\alpha|=\ |\beta|,\alpha\neq\beta$ we have
$$Re\ \left\{\frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\big(f(\alpha U)-\ f(\beta U)\big)\right)}\right\}\ \geq\ 0$$
$$\Leftrightarrow\ Re\left\{\frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\big(f(\alpha U)-\ f(\beta U)\big)\right)}-\frac{\alpha\ +\ \beta}{\alpha\ -\ \beta}\right\}\geq\ 0.$$
That is $Re\ \{G_f(\alpha,\beta)\}\ \geq\ 0$ for $|\alpha|=\ |\beta|,\alpha\neq\beta$. As we have seen in Lemma (1.1.18) G_f is analytic in both α and β. It follows that $Re\ \{G_f(\alpha,\beta)\}$ is harmonic on $\Delta\ \times\ \Delta$. Keeping α fixed and varying β, we apply the minimum principle for harmonic functions to assert that $Re\ \{G_f(\alpha,\beta)\}$ cannot attain its minimum at an interior point, i.e. when $|\beta|\ <\ |\alpha|$. Similarly, holding β fixed and varying α, with $|\alpha|\ <\ |\beta|$, we obtain the same result for α. We conclude that on the whole polydisk $\Delta\ \times\ \Delta, Re\ \{G_f\ (\alpha,\beta)\}\ >\ 0$. Hence $f\ \in\mathbb{G}$.

Theorem (1.1.20)[1]: If $f\ \in\mathbb{G}$, then f is starlike.

Proof. If $f\ \in\mathbb{G}$, then $Re\ \{G_f\ (\alpha,\beta)\}\ >\ 0$ for all $\alpha,\beta\ \in\ \Delta$. Consider the case when $\beta\ =\ 0$. Then
$$Re\ \{G_f\ (\alpha,0)\}\ =\ Re\left\{\frac{2\alpha}{\ell_U\left(D_f(\alpha U)^{-1}\big(f(\alpha U)\big)\right)}-\ 1\right\}\ >\ 0$$
$$\Rightarrow\ Re\left\{\frac{\alpha}{\ell_U\left(D_f(\alpha U)^{-1}\big(f(\alpha U)\big)\right)}\right\}>\frac{1}{2}$$
$$\Rightarrow\ Re\left\{\frac{1}{\alpha}\ \ell_U\left(Df(\alpha U)^{-1}\big(f(\alpha U)\big)\right)\right\}>\ 0$$
$$\Rightarrow\ Re\ \frac{1}{|\alpha|}\ \{\ell_{\alpha U}(Df(\alpha U)^{-1}(f(\alpha U)))\}\ >\ 0$$

since there is a 1-1 correspondence between $T\ (\alpha U)$ and $T\ (U)$ given by $\ell_{\alpha U}(\cdot) = \frac{|\alpha|}{\alpha}\ \ell_U(\cdot)$. Thus

$$Re\ G_f(\alpha,0) > \ 0\ \Rightarrow\ Re\ \left\{\ell_{\alpha U}\left(Df(\alpha U)^{-1}\big(f(\alpha U)\big)\right)\right\} > 0$$

and this is the condition for starlikeness from (14).

The condition (8) led us to our definition of the family $\mathbb{G}$. An obvious question is, why not use the more common characterization of K, namely (7)? The analogous condition to this is

$$Re\ \left\{\ell_Z\left(Df(Z)^{-1}\big(D^2f(Z)(Z,Z) +\ Df(Z)\big)\right)\right\} > 0.$$

This leads us to define a new family of mappings, F. Naturally we will want to examine the relationship between F and $\mathbb{G}$. F, as we will see later, is defined by a local condition, whereas $\mathbb{G}$ is defined by a global condition. In the plane they are one and the same, but what about higher dimensions?

Further motivation comes from the derivation of (7). The condition

$$Re\{zf''(z)f'(z)+1\}\ >\ 0$$

for convexity in the plane is equivalent to saying that the curvature of $f(z)$ is always positive for $z = re^{it}$ with r fixed and t real. When we generalize this to the image of $C_U\ =\ \{\alpha U:\ \|U\|\ =\ 1, \alpha\ \in\ \Delta\}$, and use a 2-norm, we obtain an expression which is similar to that in the plane. That is, the condition which ensures that the curvature of $f(Ze^{it})$, where $Z\ \in\ C_U$ with Z fixed and for some U, is always positive leads us to the same condition.

Let $r(t)\ =\ f(Ze^{it})$. Then $r'(t) = iDf(Ze^{it})(Ze^{it})$ and

$$r''(t) = -(D^2f(Ze^{it})(Ze^{it}, Ze^{it}) + Df(Ze^{it})(Ze^{it})).$$

Since $r''(t) = a_T(t)T(t) + a_N(t)N(t)$ where $T(t)$ and $N(t)$ are the unit tangential and unit normal (inward) components to the curve $r(t)$. Also $a_N(t) = \kappa\|r'(t)\|^2$ where κ is the curvature and $a_N(t) = Re\langle r''(t), N(t)\rangle$.

$$N(t) = -\ \frac{\left(Df\big(Ze^{it}\big)^{-1}\right)^*\big(Ze^{it}\big)}{\|(Df(Ze^{it})^{-1})^*(Ze^{it})\|},$$

where $\left(Df\big(Ze^{it}\big)^{-1}\right)^*$ is the adjoint of the derivative. Hence

$$\kappa\left\|Df\big(Ze^{it}\big)\big(Ze^{it}\big)\right\|^2$$

$$= Re\ \langle D^2f\big(Ze^{it}\big)\big(Ze^{it}, Ze^{it}\big) + Df\big(Ze^{it}\big)\big(Ze^{it}\big), \frac{\left(Df\big(Ze^{it}\big)^{-1}\right)^*\big(Ze^{it}\big)}{\|(Df(Ze^{it})^{-1})^*(Ze^{it})\|}\rangle,$$

$$\kappa = \frac{Re\ \langle Df\big(Ze^{it}\big)^{-1}\left(D^2f\big(Ze^{it}\big)\big(Ze^{it}, Ze^{it}\big) + Df\big(Ze^{it}\big)\big(Ze^{it}\big)\right), Ze^{it}\rangle}{\|Df(Ze^{it})(Ze^{it})\|^2\|(Df(Ze^{it})^{-1})^*(Ze^{it})\|}$$

Hence for any curve $X(t)\ =\ Ze^{it}$,

$$Re\langle Df(X)^{-1}(D^2f(X)(X,X)\ +\ Df(X)(X)), X\rangle > 0$$

if and only if the curvature of $f(X(t))$ is positive.

This leads us to the following definitions.

Definition (1.1.21)[1]: Let $F\_f(Z)\ = \ell_Z(Df(Z)^{-1}(D^2f(Z)(Z,Z)\ +\ Df(Z)(Z)))$ where $\ell_Z \in T(Z)$.

Definition (1.1.22)[1]: Let $\mathbb{F} = \{f \in \mathbb{S}_n :\ Re\{F_f(Z)\} > 0$ for all $Z\ \in\ B\}$. We call this family of mappings the "Quasi-Convex Mappings, Type B".

The first relationship between $\mathbb{F}$ and $\mathbb{G}$ we prove is that $\mathbb{G}$ is a subset of $\mathbb{F}$.

Theorem (1.1.23)[1]: If $f\ \in \mathbb{G}$, then $f\ \in \mathbb{F}$.

Proof. This follows easily from (15) in Lemma (1.1.18).

As we have seen, a mapping which has a convex function of one variable in each of its coordinates is not necessarily convex. We prove here that for any absolute norm such mappings are Quasi-Convex.

Theorem (1.1.24)[1]: Let $f : B \subset \mathbb{C}^n \to \mathbb{C}^n$ be defined by $f(Z) = (f_1(z_1), \dots, f_n(z_n))$ where $Z = (z_1, \dots, z_n)$ and $f_j \in K$, for each $j = 1, 2, \dots, n$. Then $f \in \mathbb{G}$ in any absolute norm. (That is, any norm for which $|z_j| \le |w_j|$ for each j implies that $\|Z\| \le \|W\|$.)

Proof. We know that $Df(Z) = diag\{f_j'(z_j)\}_{j=1}^n$ and since $f_j'(z_j) \neq 0$ for all j, $Df(Z)$ is nonsingular. Thus $Df(Z)^{-1} = diag\left\{\frac{1}{f_j'(z_j)}\right\}_{j=1}^n$.

Let $U \in S^{2n-1}$, and $\alpha, \beta \in \Delta$ and define $W : B \times B \to \mathbb{C}^n$ by

$$W(\alpha U, \beta U) = Df(\alpha U)^{-1} \left(\frac{f(\alpha U) - f(\beta U)}{\alpha}\right).$$

Then

$$W(\alpha U, \beta U) = \left(\frac{f_j(\alpha u_j) - f_j(\beta u_j)}{\alpha f_j'(\alpha U)}\right)_{j=1}^n, \text{where } U = (u_1, \dots, u_n).$$

Let $|\alpha| = r < 1$ and let $\beta = \gamma\alpha$ with $|\gamma| < 1$. For $t \in [0, 1]$, let

$$F_j(\alpha u_j, t) = (1 - t) f_j(\alpha u_j) + t f_j(\beta u_j)$$
$$= (1 - t) f_j(\alpha u_j) + t f_j(\gamma\alpha u_j).$$

By the convexity of each of the f_j, F_j is subordinate to f_j on Δ for each $t \in [0, 1]$.

Hence $F(\alpha U, t)$ is subordinate to $f(\alpha U)$ for $\alpha U \in B$ and $t \in [0, 1]$. (The norm we have chosen guarantees this). We have,

$$F(\alpha U, 0) = \left(F_j(\alpha u_j, 0)\right)_j = \left(f_j(\alpha u_j)\right)_j = f(\alpha U).$$

We now take the following limits,

$$\lim_{t \to 0^+} \left(\frac{F(\alpha U, 0) - F(\alpha U, t)}{t}\right)$$
$$= \lim_{t \to 0^+} \left(\frac{f_j(\alpha u_j) - (1 - t) f_j(\alpha u_j) - t f_j(\gamma\alpha u_j)}{t}\right)_j$$
$$= \left(f_j(\alpha u_j) - f_j(\gamma\alpha u_j)\right)_j$$
$$= G(\alpha U), \text{say, which is holomorphic.}$$

Hence by Lemma (1.1.7), $G(\alpha U) = Df(\alpha U)(V(\alpha U))$ with $V \in N_0$

$$V(\alpha U) = Df(\alpha U)^{-1}(G(\alpha U)) = \left(\frac{f_j(\alpha u_j) - f_j(\gamma\alpha u_j)}{f_j'(\alpha U)}\right)_j = \left(\frac{f_j(\alpha u_j) - f_j(\beta u_j)}{f_j'(\alpha U)}\right)_j$$
$$= \alpha W(\alpha U, \beta U).$$

Hence $\alpha W(\alpha U, \beta U) \in N_0$ which means that

$$Re\,\{\ell_\alpha U\,(\alpha W(\alpha U, \beta U))\} > 0,$$

where $\ell_{\alpha U} \in T\,(U)$.

Since for each $\ell_{\alpha U} \in T\,(\alpha U)$ there is a corresponding $\ell_U \in T\,(U)$ related by $\ell_{\alpha U}(\cdot) = \frac{|\alpha|}{\alpha}(\cdot)$, we have $Re\,\{\ell_U(W(\alpha U, \beta U))\} > 0$. Thus

$$Re\left\{\frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\left(f(\alpha U)-\ f(\beta U)\right)\right)}\right\}>\ 0,$$

and it follows by a similar argument that

$$Re\left\{\frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\left(f(\alpha U)-\ f(\alpha U)\right)\right)}-\frac{\alpha\ +\ \beta}{\alpha\ -\ \beta}\right\}>0.$$

Hence $f\ \in \mathbb{G}$.

Theorem (1.1.25)[1]: Let B be the unit ball in $\mathbb{C}^n$ with a p-norm with $1\ \le\ p\ \le\ \infty$. Let F be a mapping $F:\ B\ \to\ \mathbb{C}^n$ with one of its coordinate maps, f_k, a function of one variable only. It is a necessary condition for $F\ \in \mathbb{F}$ that $f_k\ \in\ K$.

Proof. Without loss of generality we can assume that

$$F(Z)=(f(z),f_2(Z),\dots,f_n(Z)),\text{where } Z\ =\ (z,z_2,\dots,z_n).$$

$$DF(Z)\ =\ \begin{bmatrix} f'(z) & 0_{n-1}\\ A_{n-1} & B_{(n-1)\times(n-1)}\end{bmatrix},$$

$$DF(Z)^{-1}=\begin{bmatrix} 1/f'(z) & 0_{n-1}\\ C_{n-1} & D_{(n-1)\times(n-1)}\end{bmatrix},$$

$$D^2F(Z)(Z,Z)=\begin{bmatrix} z^2f''(z)\\ E_{n-1}\end{bmatrix}.$$

Choose $Z\ =\ (z,0,\dots,0)\ =\ (re^{i\theta},0,\dots,0)$. Then the functional $\ell_Z(U)=e^{-i\theta}u_1$ is in $T(Z)$ and

$$Re\,\ell_Z\left(DF(Z)^{-1}\left(D^2F(Z)(Z,Z)+\ DF(Z)(Z)\right)\right)$$
$$=Re\left\{e^{-i\theta}\begin{bmatrix} re^{i\theta}\\ 0_{n-1}\end{bmatrix}+e^{-i\theta}[1/f'(z)0_{n-1}]\begin{bmatrix} z^2f''(z)\\ H_{n-1}\end{bmatrix}\right\}=Re\left\{1\ +\frac{zf''(z)}{f'(z)}\right\}$$
>0 if and only if f is convex.

Corollary (1.1.26)[1]: If $f:\ B\ \to\ \mathbb{C}^n$ where $B\ \in \ell^p(n),1\ \le\ p\ \le\ \infty$ is of the form $f(Z)\ =\ (f_1(z_1),\dots,f_n(z_n))$ where for each $j\ =\ 1,\dots,n,f_j\in S$, then $f\ \in \mathbb{G}$ (and $\mathbb{F}$) if and only if $f_j\in K$ for each j.

Proof. The result follows immediately from Theorem (1.1.24) and Theorem (1.1.25).

Theorem (1.1.27)[1]: Let $f:\ B\ \to \mathbb{C}$ with $B\ \subset\ \mathbb{C}^n$ be holomorphic. Define $F:\ B\ \to\ \mathbb{C}^n$ by $F(Z)\ =\ f(Z)Z$. Further, given $U\ \in\ S^{2n-1}$, define $g_U:\ \Delta\to\mathbb{C}$ by $g_U(\alpha)\ =\ \alpha f(\alpha U)$. Then:

(i) $F\ \in\ G$ if and only if $g_U\ \in\ K$ for each $U\ \in\ S^{2n-1}$.

(ii) $F\ \in\ F$ if and only if $g_U\ \in\ K$ for each $U\ \in\ S^{2n-1}$.

Proof. Since $F(Z)\ =\ f(Z)(Z)$ where $Z^T=(z_1\dots,z_n)$ we have

$$DF(Z)\ =\ Z\nabla f(Z)^T+f(Z)I$$
$$=\ Z\nabla f^T+f\ I \text{ (for simplicity).} \tag{16}$$

It is easy to check that

$$DF(Z)^{-1}=\frac{1}{f(f\ +\ \nabla f^TZ)}\ [(f+\nabla f^TZ)I-Z\nabla f^T] \tag{17}$$

where $=\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right)^T$.

(i) From $F(Z)\ =\ f(Z)Z$, we write

$$F(\alpha U)\ -\ F(\beta U)=(g_U(\alpha)\ -\ g_U(\beta))U.$$

Also, since

$$g'_U(\alpha) = f(\alpha U) + \nabla f(\alpha U)^T(\alpha U), \tag{18}$$

we have

$$DF(\alpha U)^{-1} = \frac{\big(f(\alpha U) + \nabla f(\alpha U)^T(\alpha U)\big)I - \alpha U \nabla f(\alpha U)^T}{f(\alpha U)\big(f(\alpha U) + \nabla f(\alpha U)^T(\alpha U)\big)}$$
$$= \frac{g'_U(\alpha)I - \alpha U \nabla f(\alpha U)^T}{f(\alpha U)\big(g'_U(\alpha)\big)}.$$

So

$$DF(\alpha U)^{-1}(F(\alpha U) - F(\beta U))$$
$$= \frac{g'_U(\alpha)\big(g_U(\alpha) - g_U(\beta)\big)U - (\alpha U \nabla f(\alpha U)^T)\big(g_U(\alpha) - g_U(\beta)\big)U}{f(\alpha U) g'_U(\alpha)}$$
$$= \frac{g_U(\alpha) - g_U(\beta)}{g'_U(\alpha)}\ U.$$

It follows that

$$\ell_U(DF(\alpha U)^{-1}(F(\alpha U) - F(\beta U))) = \frac{g_U(\alpha) - g_U(\beta)}{g'_U(\alpha)} \tag{19}$$

for $\ell_U \in T(U)$.
Therefore

$$G_F(\alpha, \beta) = \frac{2\alpha g'_U(\alpha)}{g_U(\alpha) - g_U(\beta)} - \frac{\alpha + \beta}{\alpha - \beta},$$

and so $Re\,\{G_F(\alpha,\beta)\} > 0$ if and only if $g_U \in K$ by (7).
(ii) Let $\hat{f}(Z) = f(Z) + \nabla f(Z)^T Z$. Then

$$DF(Z)(Z) = Z \nabla f(Z)^T Z + f(Z) Z = \hat{f}(Z) Z.$$

Differentiating again,

$$D^2F(Z)(Z,\cdot) + DF(Z)(\cdot) = (Z \nabla \widehat{f}\,(Z)^T + \widehat{f}\,(Z) I)(\cdot)$$
$$D^2F(Z)(Z,Z) + DF(Z)(Z) = (\nabla \widehat{f}\,^T Z + \widehat{f}\,)Z.$$

So we have

$$DF(Z)^{-1}(D^2F(Z)(Z,Z) + DF(Z)(Z)) = \frac{1}{f\hat{f}}\ (\hat{f}\, I - Z\nabla f^T)(\nabla \hat{f}^T Z + \hat{f})Z$$
$$= \frac{\big(\nabla \hat{f}^T Z + \hat{f}\big)}{f\hat{f}}\ (\hat{f} - \nabla f^T Z)Z = \frac{\big(\nabla \hat{f}^T Z + \hat{f}\big)}{\hat{f}}\ Z.$$

For $\ell_Z \in T\,(Z)$ we obtain

$$\ell_Z(DF(Z)^{-1}(D^2F(Z)(Z,Z) + DF(Z)(Z))) \tag{20}$$
$$= \|Z\| \left(\frac{\nabla \hat{f}^T Z}{\hat{f}} + 1\right). \tag{21}$$

Given $U \in S^{2n-1}$ let $g_U(\alpha) = \alpha f(\alpha U)$. Then

$$g'_U(\alpha) = f(\alpha U) + \alpha \nabla f(\alpha U)^T U$$
$$= \hat{f}(\alpha U),$$
$$\alpha g''_U(\alpha U) = \nabla \hat{f}(\alpha U)^T \alpha U.$$

Therefore

$$\frac{\alpha g''_U(\alpha)}{g'_U(\alpha)} + 1 = \frac{\nabla \hat{f}(\alpha U)^T(\alpha U)}{\hat{f}(\alpha U)} + 1.$$

So from (21) with $Z = \alpha U$ we see that

$$\ell_Z\left(DF(Z)^{-1}\left(D^2F(Z)(Z,Z)+\ DF(Z)(Z)\right)\right)=|\alpha|\left(\frac{\alpha g_U''(\alpha)}{g_U'(\alpha)}+1\right)$$

We conclude that $F\ \in \mathbb{F}$ if and only if $g_U\ \in\ K$ for each $U\ \in\ S^{2n-1}$.

The following corollary involves an interesting mapping. Let us define the mapping $F:\ B\ \to\ \mathbb{C}^n$ by $F(Z)=\frac{f(\ell(Z))}{\ell(Z)}Z$, where $f\ \in\ S$ and $\ell\in T(U)$, for some $U\ \in\ \mathbb{C}^n$ with $\|U\|=1$. This mapping has the property that in the one-dimensional space $\{\alpha U:\ \alpha\ \in\ \Delta\}$ it is identical to the mapping in the plane. In any other onedimensional space described by $\{\alpha V:\ \alpha\ \in\ \Delta,\ell(V\ )=0\}$ we have $F(Z)\ =\ Z$. That is, the identity mapping. This follows from $f(z)/z$ having a removable singularity at $z\ =\ 0$. This is easily seen by the following computation.

$$F(\alpha U\ +\ \beta V\ )=\frac{f(\alpha)}{\alpha}(\alpha U\ +\ \beta V\ ).\tag{22}$$

Hence if $\beta\ =\ 0, F(\alpha U)\ =\ f(\alpha)U$ and if $=\ 0, F(\beta V\ )\ =\ \beta V$.

Corollary (1.1.28)[1]: Let $\ell\ \in\ T\ (U')$ for some $U'\ \in\ S_{2n-1}$. Define $f:\ B\ \to\mathbb{C}$ by

$$f(Z)=\frac{h\left(\ell(Z)\right)}{\ell(Z)}$$

where $h\ \in\ S$. Then $F:\ B\ \to\ \mathbb{C}^n$ given by $F(Z)\ =\ f(Z)Z$ is in $\mathbb{G}$ (or $\mathbb{F}$) if and only if $h\ \in\ K$.

Proof. Given $U\ \in\ S_{2n-1}$, we have

$$g_U(\alpha)=\alpha f(\alpha U)=\frac{h\left(\alpha\ell(U)\right)}{\ell(U)}.$$

Then

$$Re\left\{\alpha\frac{g''U(\alpha)}{g_U'(\alpha)}+1=Re\left\{\alpha\ell(U)\right)\frac{h''\left(\alpha\ell(U)\right)}{h'\left(\alpha\ell(U)\right)}+1\right\}>0 \text{ if and only if } h\ \in\ K.$$

The result follows from Theorem (1.1.27).

Example (1.1.29)[1]: The function $f(Z)=(z\ +\ aw^2,w)$ where $Z\ =\ (z,w),\|Z\|^p=|z|^p+|w|^p<1,z,w\ \in\ C$ is in $\mathbb{G}$ if and only if

$$|a|\le\frac{1}{2}\left(\frac{p^2\ -\ 1}{4}\right)^{1/p}\left(\frac{p\ +\ 1}{p\ -\ 1}\right).$$

As before,

$$Df(Z)\ =\ \begin{bmatrix}1 & 2aw\\ 0 & 1\end{bmatrix},Df(Z)^{-1}=\begin{bmatrix}1 & -2aw\\ 0 & 1\end{bmatrix}.$$

Using $f\ \in\ G$ if and only if $Re\ G_f(\alpha,\beta)\ >\ 0$ where

$$G_f(\alpha,\beta)=\frac{2\alpha}{\ell_U\left(Df(\alpha U)^{-1}\left(f(\alpha U)-\ f(\beta U)\right)\right)}-\frac{\alpha\ +\ \beta}{\alpha\ -\ \beta}$$

and $U\ =\ (z,w),\|U\|=1,|\alpha|\ <\ 1,|\beta|\ <\ 1,\alpha,\beta\ \in\ C$. It follows that

$$\begin{aligned}Df(\alpha U)^{-1}(f(\alpha U)\ -\ f(\beta U))&=\begin{bmatrix}1 & -2a\alpha w\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha z+a\alpha^2w^2-\beta z-a\beta^2w^2\\ \alpha w-\beta w\end{bmatrix}\\ &=\ (\alpha\ -\ \beta)\begin{bmatrix}z-a(\alpha-\beta)w^2\\ w\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\ell_U(Df(\alpha U)^{-1}(f(\alpha U)\ -\ f(\beta U)))&=(\alpha-\beta)(\|U\|-a(\alpha-\beta)\ell_U((w^2,0)))\\ &=(\alpha-\beta)(1-a(\alpha-\beta)\ell_U((w^2,0))).\end{aligned}$$

So

$$G_f(\alpha,\beta) = \frac{2\alpha}{(\alpha-\beta)\Big(1-a(\alpha-\beta)\ell_U\big((w^2,0)\big)\Big)} - \frac{\alpha+\beta}{\alpha-\beta}$$
$$= \frac{2\alpha-(\alpha+\beta)\Big(1-a(\alpha-\beta)\ell_U\big((w^2,0)\big)\Big)}{(\alpha-\beta)\Big(1-a(\alpha-\beta)\ell_U\big((w^2,0)\big)\Big)}$$
$$= \frac{1+a(\alpha+\beta)w^{2\bar{z}}}{1-a(\alpha-\beta)\ell_U\big((w^2,0)\big)}$$
$$= \left(1+\frac{a\ell_U\big((w^2,0)\big)}{1+a\beta\ell_U\big((w^2,0)\big)}\alpha\right) / \left(1-\frac{a\ell_U\big((w^2,0)\big)}{1+a\beta\ell_U\big((w^2,0)\big)}\alpha\right)$$
$$= \frac{1+b\alpha}{1-b\alpha}, \text{where } b = \frac{a\ell_U\big((w^2,0)\big)}{1+a\beta\ell_U\big((w^2,0)\big)}.$$

Since $\frac{1+b\alpha}{1-b\alpha} = \frac{1-|b\alpha|^2}{|1-b\alpha|^2} + \frac{2i\, Im\,\{b\alpha\}}{|1-b\alpha|^2}$ it follows that $Re\,\{G_f(\alpha,\beta) \geq 0\}$ if and only if $|b\alpha| \leq 1$. Thus we need

$$|b| = \left|\frac{a\ell_U\big((w^2,0)\big)}{1+a\beta\ell_U\big((w^2,0)\big)}\right| \leq 1.$$

Hence

$$|a||\ell_U((w^2,0))| \leq |1+a\beta\ell_U((w^2,0))|,$$

and in the worst case

$$|a||\ell_U((w^2,0))| \leq 1-|a||\ell_U((w^2,0))|.$$

That is, $2|a||\ell_U((w^2,0))| \leq 1$.

If we are using a p-norm, $1 < p < \infty, \ell_U\big((x_1,x_2)\big) = |z|^{p-2}\bar{z}x_1 + |w|^{p-2}w\bar{x}_2$. Then $\ell_U((w^2,0)) = |z|^{p-2}\bar{z}w^2$ and $|\ell_U((w^2,0))| = |z|^{p-1}(1-|z|^p)^{2/p}$. Hence $2|a||\ell_U((w^2,0))| \leq 1$ if and only if

$$|a| \leq \frac{1}{2}\left(\frac{p^2-1}{4}\right)^{1/p}\left(\frac{p+1}{p-1}\right).$$

We note that if f is in $\mathbb{G}$, then f is starlike.

Example (1.1.30)[1]: The function $f(Z) = (z+aw^2, w)$ where $Z = (z,w), \|Z\|^p = |z|^p + |w|^p < 1, z, w \in C$ is in $\mathbb{F}$ if and only if

$$|a| \leq \frac{1}{2}\left(\frac{p^2-1}{4}\right)^{1/p}\left(\frac{p+1}{p-1}\right).$$

We have that $f \in \mathbb{F}$ if and only if

$$Re\,\ell_Z(Df(Z)^{-1}(D^2f(Z)(Z,Z) + Df(Z)(Z))) > 0.$$

Therefore,

$$Df(Z)^{-1}(D^2f(Z)(Z,Z) + Df(Z)(Z)) = \begin{bmatrix}1 & -2aw\\ 0 & 1\end{bmatrix}\begin{bmatrix}2aw^2\\ 0\end{bmatrix} + \begin{bmatrix}z\\ w\end{bmatrix} = \begin{bmatrix}z+2aw^2\\ w\end{bmatrix}.$$

Hence

$$Re\left\{\ell_Z\left(Df(Z)^{-1}\big(D^2f(Z)(Z,Z) + Df(Z)(Z)\big)\right)\right\} = Re\{\|Z\| + 2a\ell_Z((w^2,0))\}$$
$$= Re\{\|Z\| + 2a\ell_Z((w^2,0))\}$$
$$\geq Re\{1 + 2a\ell_Z((w^2,0))\} \text{ (minimum principle)}$$
$$\geq Re\{1 - 2|a||\ell_Z((w^2,0))|\}.$$

This is the same condition as for the family $\mathbb{G}$ and so the same bound applies.

It should be noted that, as with $\mathbb{G}, f(z,w) = (z + aw^2, w) \in \mathbb{F} \Rightarrow f$ is starlike.
Example (1.1.31)[1]: The mapping $f(z,w) = (z + azw, w)$ with $(z,w) \in B \subset \mathbb{C}^2$ with a p-norm is in $\mathbb{G}$ if and only if

$$|a| \leq \left(\frac{2}{3}(p + 1)\right)^{1/p} \left(\frac{p + 1}{3p}\right).$$

We know that $f \in \mathbb{G}$ if and only if $Re\, G_f\,(\alpha,\beta) > 0$.

$$Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)) = (\alpha - \beta)\left[z - \frac{(\alpha-\beta)awz}{1 + aw\alpha} \atop w\right]$$

where $U = (z,w), \|U\| = 1$.
If we use a p-norm,

$$\ell_U\left((\alpha-\beta)\left(z - \frac{(\alpha-\beta)awz}{1+aw\alpha}, w\right)\right) = (\alpha - \beta)\left(1 - (\alpha-\beta)\,\frac{aw|z|^p}{1 + aw\alpha}\right).$$

Hence

$$G_f\,(\alpha,\beta) = \frac{2\alpha}{\ell_U(Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U))} - \frac{\alpha + \beta}{\alpha - \beta}$$
$$= \frac{2\alpha}{\frac{\alpha-\beta}{1+aw\alpha}(1 + aw\alpha - (\alpha - \beta)aw|z|^p)}$$
$$= \frac{1 + aw\alpha + (\alpha + \beta)aw|z|^p}{1 + aw\alpha - (\alpha - \beta)aw|z|^p}$$
$$= \frac{1 + \beta aw|z|^p + aw(1 + |z|^p)\alpha}{1 + \beta aw|z|^p - aw(1 - |z|^p)\alpha}$$
$$= \left(1 + \frac{aw\alpha|z|^p}{1 + \beta aw|z|^p + aw\alpha}\right) \Big/ \left(1 - \frac{aw\alpha|z|^p}{1 + \beta aw|z|^p + aw\alpha}\right).$$

The real part of this is positive if and only if

$$\left|\frac{aw\alpha|z|^p}{1 + \beta aw|z|^p + aw\alpha}\right| \leq 1.$$

Hence we need $|a||w||\alpha||z|^p \leq |1 + \beta aw||z|^p aw\alpha|$. The worst case is when $\alpha aw = \beta aw = -|aw|$, and we have to find a such that $|a||w||z|^p \leq 1 - |a||w||z|^p - |a||w|$. We obtain $|a||w|(3 - 2|w|^p) \leq 1$ from which we find that

$$|a| \leq \left(\frac{2}{3}(p + 1)\right)^{1/p} \left(\frac{p + 1}{3p}\right).$$

In particular we have the following values of a.
If $p = 1\ |a| \leq 8/9$,
If $p = 2\ |a| \leq 1/\sqrt{2}$,
If $p = \infty\ |a| \leq 1/3$.
Note that for $p = 2$ the values obtained for a are the same for $\mathbb{G}$ as for the convex mappings.
Example (1.1.32)[1]: The mapping $f(z,w) = (z + azw, w)$ is in $\mathbb{F}$ if and only if

$$|a| \leq \left(\frac{2}{3}(p+1)\right)^{1/p} \left(\frac{p + 1}{3p}\right).$$

This is the same result as in Example (1.1.31). This follows directly from the observation that the worst case in that example occurs when $\alpha = \beta$.

We now turn our attention to finding information on the family $\mathbb{G}$ as a whole, Theorem (1.1.35) gives us some uniform bounds, in the Euclidean norm, on $\mathbb{G}$. We first prove two lemmas.

Lemma (1.1.33)[1]: Let $f : B \to \mathbb{C}^n$ be holomorphic and univalent on B. Let $U \in \mathbb{C}^n$ with $\|U\| = 1$ and let $\alpha \in \Delta$. Then necessary conditions for $\|f(Z)\|$ to have a local maximum or minimum on $\{Z : \|Z\| = r < 1\}$ at $Z = \alpha U, |\alpha| = r$ are

$$\operatorname{Im}\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle = 0, \tag{23}$$

And

$$\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle = 0, \tag{24}$$

where $V \in \mathbb{C}^n, \|V\| = 1$ and $\langle U, V\rangle = 0$.

Proof. Let $\alpha = re^{i\theta}$ where r is fixed and θ varies.

$$\frac{d}{d\theta}(\|f(\alpha U)\|^2) = \frac{d}{d\theta}\langle f(\alpha U), f(\alpha U)\rangle$$
$$= \langle Df(\alpha U)(i\alpha U), f(\alpha U)\rangle + \langle f(\alpha U), Df(\alpha U)(i\alpha U)\rangle$$
$$= 2\, Re\langle Df(\alpha U)(i\alpha U), f(\alpha U)\rangle$$
$$= 2\, Re\,\{i\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle\}$$
$$= -2\operatorname{Im}\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle.$$

When a maximum (or minimum) of $\|f(\alpha U)\|^2$ for $|\alpha| = r$ occurs,

$$\operatorname{Im}\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle = 0.$$

That is, $\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle$ is real.

Now fix α at a point where $\|f(\alpha U)\|$ has a maximum, $|\alpha| = r$, and vary Z by letting $Z(\theta) = \alpha(U\cos\theta + \lambda V\sin\theta)$, where $\langle U, V\rangle = 0, \|V\| = 1, |\lambda| = 1$.

$$\frac{d}{d\theta}(\langle f(Z(\theta)), f(Z(\theta))\rangle)$$
$$= \langle Df(Z(\theta))(Z'(\theta)), f(Z(\theta))\rangle + \langle f(Z(\theta)), Df(Z(\theta))(Z0\,(\theta))\rangle$$
$$= 2\, Re\langle Df(Z(\theta))(Z0\,(\theta)), f(Z(\theta))\rangle.$$

We want this to have a maximum at $\theta = 0$. Hence we need

$$2\, ReDf\big(Z(\theta)\big)\big(Z'(\theta)\big), f\big(Z(\theta)\big)\Big|_{\theta=0} = 0.$$

Since $Z'(0) = \alpha(U(-\sin\theta) + \lambda V\cos\theta)|_{\theta=0}, Z'(0) = \lambda\alpha V, Z(0) = \alpha U$, this becomes

$$2\, Re\langle Df(\alpha U)(\lambda\alpha V), f(\alpha U)\rangle = 0.$$

Hence $\{\lambda\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle\} = 0$, for all $\lambda, |\lambda| = 1$. Thus it follows that $|\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle| = 0$. Consequently,

$$\langle Df(\alpha U)(\alpha V), f(\alpha U)\rangle = 0.$$

Lemma (1.1.34)[1]: Let $(r_n)_{n=1}^{\infty}$ be a monotone increasing sequence of positive numbers converging to 1. Let $f \in G$ and define $f_n(Z) = (1/r_n)f(r_nZ)$. Then

(i) $f_n \in G$, and

(ii) $f_n \to f$ uniformly on compact subsets of B.

Proof. We first note that $Df_n(Z) = Df(r_nZ)$. Hence

$$Gf_n(\alpha,\beta) = \frac{2\alpha}{\langle Df_n(\alpha U)^{-1}\big(f_n(\alpha U) - f_n(\beta U)\big), U\rangle} - \frac{\alpha+\beta}{\alpha - \beta}$$
$$= \frac{2r_n\alpha}{\langle Df(r_n\alpha U)^{-1}\big(f(r_n\alpha U) - f_n(r_n\beta U)\big), U\rangle} - \frac{r_n\alpha + r_n\beta}{r_n\alpha - r_n\beta}$$
$$= G_f(r_n\alpha, r_n\beta)$$

and $f_n \in G$.

That $f_n \to f$ uniformly on compact subsets of B follows by a standard argument.

Theorem (1.1.35)[1]: Let $f \in G$, then for all $Z \in B$, using the 2-norm,

$$\frac{\|Z\|}{1 + \|Z\|} \le \|f(Z)\| \le \frac{\|Z\|}{1 - \|Z\|}.$$

Proof. Since $Re\, G_f(\alpha,\beta) > 0$ and $G_f(0,\beta) = 1$ we can write

$$G_f(\alpha,\beta) = \frac{1 + \alpha\omega(\alpha,\beta)}{1 - \alpha\omega(\alpha,\beta)},$$

where $\alpha\omega(\alpha,\beta)$ is a Schwarz function. That is, $\alpha\omega(\alpha,\beta)$ is analytic for $\alpha,\beta \in \Delta$ and $|\alpha\omega(\alpha,\beta)| \le |\alpha|$. Thus $|\omega(\alpha,\beta)| \le 1$. We have

$$\frac{2\alpha}{\langle Df(\alpha U)^{-1}\big(f(\alpha U) - f(\beta U)\big), U\rangle} - \frac{\alpha + \beta}{\alpha - \beta} = \frac{1 + \alpha\omega(\alpha,\beta)}{1 - \alpha\omega(\alpha,\beta)}.$$

Hence

$$\frac{2\alpha}{\langle Df(\alpha U)^{-1}\big(f(\alpha U) - f(\beta U)\big), U\rangle} = \frac{1 + \alpha\omega(\alpha,\beta)}{1 - \alpha\omega(\alpha,\beta)} + \frac{\alpha + \beta}{\alpha - \beta}$$
$$= \frac{2\alpha\big(1 - \beta\omega(\alpha,\beta)\big)}{(\alpha - \beta)\big(1 - \alpha\omega(\alpha,\beta)\big)}.$$

Thus

$$\langle Df(\alpha U)^{-1}(f(\alpha U) - f(\beta U)), U\rangle = (\alpha - \beta)\frac{1 - \alpha\omega(\alpha,\beta)}{1 - \beta\omega(\alpha,\beta)}$$

And

$$Df(\alpha U)^{-1}\big(f(\alpha U) - f(\beta U)\big) = (\alpha - \beta)\frac{1 - \alpha\omega(\alpha,\beta)}{1 - \beta\omega(\alpha,\beta)}U + \sum_{j=2}^{n} d_j(\alpha,\beta)V_j$$

where $d_j(\alpha,\beta)$ is analytic in α and β, $\langle V_j, U\rangle = 0$ and $\langle V_j, V_k\rangle = 0$ for $j \ne k$. Thus

$$\big(f(\alpha U) - f(\beta U)\big) = (\alpha - \beta)\frac{1 - \alpha\omega(\alpha,\beta)}{1 - \beta\omega(\alpha,\beta)} Df(\alpha U)(U) + \sum_{j=2}^{n} d_j(\alpha,\beta) Df(\alpha U)(V_j).$$

Further, dividing by $\alpha - \beta$ and letting $\beta \to \alpha$ we have

$$Df(\alpha U)(U) = Df(\alpha U)(U) + \sum_{j=2}^{n} \lim_{\beta\to\alpha} \frac{d_j(\alpha,\beta)}{\alpha - \beta} Df(\alpha U)(V_j).$$

Therefore

$$\lim_{\beta\to\alpha} \frac{d_j(\alpha,\beta)}{\alpha - \beta} = 0 \text{ for } j = 2,\dots,n.$$

From this we conclude that $d_j(\alpha,\beta) = (\alpha - \beta)^2 c_j(\alpha,\beta)$ where $c_j(\alpha,\beta)$ is analytic in α and β. So we can write

$$f(\alpha U) - f(\beta U) = (\alpha - \beta)\frac{1 - \alpha\omega(\alpha,\beta)}{1 - \beta\omega(\alpha,\beta)} Df(\alpha U)(U) \tag{25}$$
$$+ \sum_{j=2}^{n} (\alpha - \beta)^2 c_j(\alpha,\beta) Df(\alpha U)(V_j).$$

From this we get two useful representations of $f(Z)$.
When $\beta = 0$,

$$f(\alpha U) = \alpha(1 - \alpha\omega(\alpha,0)) Df(\alpha U)(U) + \sum_{j=2}^{n} \alpha^2 c_j(\alpha,0) Df(\alpha U)(V_j), \tag{26}$$

and when $\alpha = 0$,

$$f(\beta U) = \frac{\beta}{1 - \beta\omega(0,\beta)}U - \sum_{j=2}^{n} \beta^2 c_j(0,\beta)V_j. \qquad (27)$$

Since

$$\|f(\beta U)\|^2 = \langle f(\beta U), f(\beta U)\rangle \geq \frac{|\beta|^2}{|1 - \beta\omega(0,\beta)|^2} \geq \frac{|\beta|^2}{(1 + |\beta|)^2},$$

we have the lower bound.

To obtain the upper bound requires a lot more work. We note that if $|\omega(\alpha,\beta)| = 1$, it follows that $\omega(\alpha,\beta) = e^{i\theta}$ for all $\alpha,\beta \in \Delta$ and θ is a real constant. If $\beta = |\beta|e^{-i\theta}$ then

$$\|f(\beta U)\|^2 = \frac{|\beta|^2}{(1 - |\beta|)^2} + |\beta|^4 \sum_{j=2}^{n} \left|c_j(0,\beta)\right|^2.$$

Clearly, if $c_j\left(0,|\beta|e^{-i\theta}\right) \neq 0$ for some j, then $\|f(\beta U)\| > \frac{|\beta|}{1-|\beta|}$. Our approach will implicitly show that this does not happen.

Let $(r_n)_{n=1}^{\infty}$ and fn be as in Lemma (1.1.34). Then from the lemma we know that $f_n \in \mathbb{G}$ and $f_n \to f$ uniformly on compact sets. In addition we will show that for each n, the $\omega_n(\alpha,\beta)$ associated with fn as in (25) has the property $|\omega_n(\alpha,\beta)| < 1$.

Once this is established it will suffice to show that the bound holds for mappings with $|\omega(\alpha,\beta)| < 1$.

To see that for any f_n, $|\omega_n(\alpha,\beta)| < 1$ we use

$$G_{f_n}(\alpha,\beta) = G_f(r_n\alpha, r_n\beta)$$

from Lemma (1.1.34) and

$$G_f(\alpha,\beta) = \frac{1 + \alpha\omega(\alpha,\beta)}{1 - \alpha\omega(\alpha,\beta)}.$$

From this we have $|\omega_n(\alpha,\beta)| = |r_n\omega(r_n\alpha, r_n\beta)|$. Therefore $|\omega_n(\alpha,\beta)| < 1$ since $r_n < 1$ and $|\omega(\alpha,\beta)| \leq 1$.

Now let $f \in \mathbb{G}$ have the property $|\omega(\alpha,\beta)| < 1$, for all $\alpha,\beta \in \Delta$ and let

$$T = \left\{r : \|f(Z)\| \leq \frac{\|Z\|}{1 - \|Z\|} \text{ for } \|Z\| < r\right\}.$$

We will show that T is both open and closed and conclude that $T = [0,1]$ for every f with the property that $|\omega(\alpha,\beta)| < 1$. Note that although T depends on f, for the sake of simplicity, our notation will not explicitly reflect this.

$T \neq \phi$ since $0 \in T$ (vacuously).

Next we show that there exists $\varepsilon > 0$ such that $[0,\varepsilon] \in T$.

We have seen that

$$\|f(\beta U)\|^2 = \frac{|\beta|^2}{|1-\beta\omega(0,\beta)|^2} + |\beta|^4 \sum_{j=2}^{n} \left|c_j\,(0,\beta)\right|^2.$$

Let us assume that $|\beta| \leq \frac{1}{2}$ and let $M = \max_{|\beta|\leq 1/2} \sum_{j=2}^{n} \left|c_j(0,\beta)\right|^2$. Also let $|\omega(0,\beta)| \leq \rho < 1$ for $|\beta| \leq \frac{1}{2}$. Hence

$$\|f(\beta U)\|^2 \leq \frac{|\beta|^2}{(1 - \rho|\beta|)^2} + |\beta|^4 M,$$

and we need $\|f(\beta U)\|^2 \leq \frac{|\beta|^2}{(1-|\beta|)^2}$. We will find the conditions on $|\beta|$ for $\frac{|\beta|^2}{(1-\rho|\beta|)^2} + |\beta|^4 M \leq \frac{|\beta|^2}{(1-|\beta|)^2}$ and the required inequality will follow.
Easily,

$$\frac{1}{(1-\rho|\beta|)^2} + |\beta|^2 M \leq \frac{1}{(1-|\beta|)^2}$$

if and only if

$$(1-|\beta|)^2 + |\beta|^2 M (1-|\beta|)^2 (1-\rho|\beta|)^2 \leq (1-\rho|\beta|)^2.$$

Thus, it is sufficient to obtain

$$(1-|\beta|)^2 + |\beta|^2 M \leq (1-\rho|\beta|)^2.$$

Hence we have

$$1 - 2|\beta| + |\beta|^2 + |\beta|^2 M \leq 1 - 2\rho|\beta| + \rho^2|\beta|^2,$$
$$|\beta|(1-\rho^2+M) \leq 2(1-\rho),$$
$$|\beta| \leq \frac{2(1-\rho)}{1-\rho^2+M}.$$

Since $\frac{2(1-\rho)}{1-\rho^2+M} > 0$ we can choose $\varepsilon \leq 1/2$ such that $0 < \varepsilon < \frac{2(1-\rho)}{1-\rho^2+M}$. Hence $[0,\varepsilon] \subset T$.
T is closed if r_0 is a limit point of T and $r_0 \notin T$, then there is a neighborhood, N, of r_0 such that $\|f(Z)\| > \frac{\|Z\|}{1-\|Z\|}$ for some Z with $\|Z\| \in N, \|Z\| = r_1 < r_0$. Now choose r_2 such that $r_1 < r_2 < r_0$ such that $r_2 \in T$ (r_2 exists since r_0 is a limit point). Then by the definition of $T, \|f(Z)\| \leq \frac{\|Z\|}{1-\|Z\|}$ for all Z such that $\|Z\| < r_2$. This is contradiction and so $r_0 \in T$. Hence T is closed.
To establish that T is (relatively) open we will show that if $\|f(Z)\| = \frac{\|Z\|}{1-\|Z\|}$ for some $= Z_0, \|Z_0\| = r_0 \in T$, then it would mean that $\|f(Z)\| > \frac{\|Z\|}{1-\|Z\|}$ for some $Z, \|Z\| = r < r_0$. This would contradict the definition of . So on B_{r_0} , $\|f(Z)\| < \frac{\|Z\|}{1-\|Z\|}$. Hence there is a sufficiently small $\delta > 0$ such that $r + \delta \in T$. It would follow that T is open.
Suppose $\|f(\alpha U)\| = \frac{|\alpha|}{1-|\alpha|}$ for some $\alpha, |\alpha| = r_0$. Then since $\|f(\gamma U)\| \leq \frac{|\gamma|}{1-|\gamma|}$ for all $|\gamma| = r < r_0$ (because $r \in T$), and since this is on the interior of B_{r_0} , $\|f(\alpha U)\|$ must be the maximum value of $\|f(Z)\|$ on ∂B_{r_0} . From Lemma (1.1.33) we have that at this maximum point

$$\langle Df(\alpha U)(V), f(\alpha U)\rangle = 0 \text{ where } \langle U, V\rangle = 0,$$

And

$$\langle Df(\alpha U)(\alpha U), f(\alpha U)\rangle > 0$$

by a suitable choice of coordinates.
From (26) we have

$$f(\alpha U) = \alpha(1-\alpha\omega(\alpha,0))Df(\alpha U)(U) + \sum_{j=2}^{n} \alpha^2 c_j(\alpha,0)Df(\alpha U)(V_j).$$

It follows that

$$\|f(\alpha U)\|^2 = \langle f(\alpha U), f(\alpha U)\rangle = \alpha(1-\alpha\omega(\alpha,0))\langle Df(\alpha U)(U), f(\alpha U)\rangle.$$

At this point $(1-\alpha\omega(\alpha,0)) > 0$.

By a suitable relabeling we can assume α is positive. We have

$$\frac{d}{dt}(\|f(tU)\|^2)|_{t=\alpha} = \frac{d}{d\alpha}\langle f(\alpha U), f(\alpha U)\rangle = 2\,Re\langle Df(\alpha U)(U), f(\alpha U)\rangle$$
$$= \frac{2\|f(\alpha U)\|^2}{\alpha\big(1 - \alpha\omega(\alpha,0)\big)}.$$

Hence

$$\left.\frac{\frac{d}{dt}(\|f(tU)\|^2)}{\|f(tU)\|^2}\right|_{t=\alpha} = \frac{2}{\alpha\big(1 - \alpha\omega(\alpha,0)\big)},$$

Therefore

$$\frac{d}{dt}\, log(\|f(tU)\|^2)|_{t=\alpha} = \frac{2}{\alpha\big(1 - \alpha\omega(\alpha,0)\big)}.$$

That is,

$$\frac{d}{dt}\, log(\|f(tU)\|)|_{t=\alpha} = \frac{1}{\alpha\big(1 - \alpha\omega(\alpha,0)\big)}$$
$$< \frac{1}{\alpha(1 - \alpha)} \ \text{ since } |\omega(\alpha,0)| < 1. \tag{28}$$

Therefore, for t in a sufficiently small neighborhood of α, inequality (28) holds. That is,

$$\frac{d}{dt}\, log\,\|f(tU)\| < \frac{1}{t(1 - t)}.$$

Choose $0 < \xi < \alpha$ in this neighborhood and integrate along the radial path $Z(t) = tU, t \in [\xi,\alpha]$.

$$\int_{\xi}^{\alpha} \frac{d}{dt}\, log\,\|f(tU)\| dt < \int_{\xi}^{\alpha} \frac{1}{t(1 - t)}\, dt = \left[log\,\frac{t}{1 - t}\right]_{\xi}^{\alpha},$$
$$log\frac{\|f(\alpha U)\|}{\|f(\xi U)\|} < log\left(\frac{\alpha}{1 - \alpha}\cdot\frac{1 - \xi}{\xi}\right),$$
$$\frac{\|f(\alpha U)\|}{\|f(\xi U)\|} < \frac{\alpha}{1 - \alpha}\cdot\frac{1 - \xi}{\xi}.$$

But $\|f(\alpha U)\| = \frac{\alpha}{1-\alpha}$, and so $\|f(\xi U)\| > \frac{\xi}{1-\xi}$. This contradiction shows T is open, and we conclude that $T = [0,1]$.
Thus for any $f \in \mathbb{G}$ with $|\omega(\alpha,\beta)| < 1$ the bound holds on B.
The theorem is now proved.
These bounds are sharp as the following examples will show.
Example (1.1.36)[1]: The mapping $f : B \subset \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$f(z,w) = \left(\frac{z}{1 - z}, \frac{w}{1 - z}\right)$$

attains the bounds.
The function is convex and hence in $\mathbb{G}$. We note that

$$f(z,0) = \left(\frac{z}{1 - z}, 0\right).$$

Since the mapping $\frac{z}{1-z}$ attains these bounds in the plane, then f will attain the bounds given by the theorem.
The next example shows that several mappings attain this bound.

Example (1.1.37)[1]: Any mapping of the form $\left(\frac{z}{1-z}, g(w)\right)$, where $g \in K$ is in $\mathbb{G}$ and attains the bounds.

Since the family $\mathbb{G}$ is locally uniformly bounded, it is normal and compact. Some of the implications of this are:

(i) In the Taylor expansion of $f, f(Z) = Z + \sum_{k=2}^{\infty} P_k(Z)$, the $P_k(Z)$'s, which are homogenous polynomials in $z_1, \ldots, z_n$, are uniformly bounded.

(ii) There are uniform bounds on the volume of the image of the ball of radius $R, R < 1$.

(iii) There are uniform bounds on the determinant of the Jacobian of f.

It is known that for convex maps, $\|P_k\| \leq 1$ for each k and the upper bound $\frac{\|Z\|}{1-\|Z\|}$ can be readily determined ([2]). However, in our families the mapping $(z + aw^2, w)$ can have $|a| = 3\sqrt{3}/4 \approx 1.3$ and so the bound $\|P_k\| \leq 1$ does not hold for $\mathbb{F}$ or $\mathbb{G}$.

Section (1.2): The Unit Polydisk in $\mathbb{C}^n$

In 1999, Roper and Suffridge [1] first introduced the definitions of a quasi-convex mapping of type A and a quasi-convex mapping of type B on the unit ball in a finite dimensional complex Banach space. After that, Zhang and Liu introduced the definition of another quasi-convex mapping on the unit ball in acomplex Banach space (including finite dimensional and infinite di-mensional spaces).For brevity, say the mapping is quasi-convex. They proved that the definitions of a quasi-convex mapping of type A and a quasi-convex mapping on the unit ball in a complex Banach space are the same.

With respect to the criteria for a normalized biholomorphic convex mapping, Roper and Suf-fridge [1] provided a sufficient condition for a normalized biholomorphic convex mapping on the open Euclidean unit ball in $\mathbb{C}^n$. In 2003, Zhu gave a concise proof of the above result.

On the other hand, at present, only a few works treat the estimation of homogeneous expansions for subclasses of biholomorphic mappings in the case of several complex variables. See [15], [18], [19]. These estimations still arouse great interest. The reason is that the estimation of all homogeneous expansions for star like mappings on the open unit poly disk D^n in C^n is analogous with the famous Bieberbach conjecture in the case of one complex variable.

Conjecture (1.2.1)[14]: (See [15], [17].) If $f : D^n \to \mathbb{C}^n$ is a normalized biholomorphic starlike mapping, where D^n is the open unit polydisk in $\mathbb{C}^n$, then

$$\frac{\|D^m f(0)(z^m)\|}{m} \leq m\|z\|^m, z \in D^n, m = 2,3,\ldots.$$

Until now, only the case of $m = 2$ (see [15]) was proved. For the estimations of all homogeneous expansions for normalized biholomorphic convex mappings on the unit ball in a complex Banach space, the analogous results as in the case of one complex variable are not difficult to get. However, with respect to the estimation of homogeneous expansion for quasi-convex mappings of type A and quasi-convex mappings of type B on the Euclidean unit ball B^n in $\mathbb{C}^n$, Roper and Suffridge [1] provided a counterexample to point out that the above similar conjecture does not hold for m=2, hence we mainly investigate the estimation of homogeneous expansion for quasi-convex mappings (including quasi-convex mappings of type A and quasi-convex mappings of type B) on D^n in $\mathbb{C}^n$.

For X be a complex Banach space with norm $\|.\|$, X^* be the dual space of X, B be the open unit ball in X, D denote the Euclidean open unit disk in $\mathbb{C}, D^n$ represent the open unitpolydiskin C^n. Let ∂D^n be the boundary of D^n, $\partial_0 D^n$ be the distinguished boundary of D^n.

Let the symbol $'$ mean transpose. For each $x \in X\backslash\{0\}$ we define $T(x) = \{T_x \in X^*: \|T_x\| = 1, T_x(x) = \|x\|\}$. From the Hahn–Banach theorem, $T(x)$ is nonempty. Let $H(B)$ be the set of all holomorphic mappings from B into X. It is well known that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n),$$

for all y in some neighborhood of $x \in B$, where $D^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x) \underbrace{(y-x, \dots, y-x)}_{n}.$$

Moreover, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^{n} X$ into X.

A holomorphic mapping $f: B \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f: B \to X$ is a holomorphic mapping, then f is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where I stands for the identity operator from X into X.

We say that a normalized biholomorphic mapping $f: B \to X$ is starlike if $f(B)$ is a starlike domain with respect to the origin. Also, a normalized biholomorphic mapping $f: B \to X$ is said to be convex if $f(B)$ is a convex domain.

Definition (1.2.2)[14]: (See [16].) Suppose $f: B \to X$ is a normalized locally biholomorphic mapping, denote

$$G_f(\alpha,\beta) = \frac{2\alpha}{T_u\left[\left(Df(\alpha u)\right)^{-1}\left(f(\alpha u) - f(\beta u)\right)\right]} - \frac{\alpha+\beta}{\alpha-\beta}.$$

If

$$\Re e\, G_f(\alpha,\beta) \geq 0, \qquad u \in \partial B, \alpha, \beta \in D,$$

then f is said to be quasi-convex of type A.

It is known that a quasi-convex mapping of type A is biholomorphic on B.

Definition (1.2.3)[14]: (See [16].) Suppose $f: B \to X$ is a normalized locally biholomorphic mapping. If

$$\Re e\{T_x[Df(x)^{-1}(D^2 f(x)(x^2) + Df(x)x)]\} \geq 0, \qquad x \in B,$$

then f is said to be quasi-convex of type B.

At present, we do not still know whether a quasi-convex mapping of type B is biholomorphic on B or not.

When $X = \mathbb{C}^n$, Definitions (1.2.2) and (1.2.3) introduced by Roper and Suffridge [1].

Definition (1.2.4)[14]: Suppose $f: B \to X$ is a normalized locally biholomorphic mapping. If

$$\Re e\{T_x[Df(x)^{-1}(f(x) - f(\xi x))]\} \geq 0, \qquad x \in B, \xi \in \overline{D},$$

then f is said to be quasi-convex.

It is known that a quasi-convex mapping is biholomorphic on B.

When $X = \mathbb{C}$, Definitions (1.2.2)–(1.2.4) are the same, that is, a quasi-convex function is equivalent to a normalized biholomorphic convex function in one complex variable.

Let $S(B)$ denote the set of all normalized biholomorphic mappings on B. Let $K(B)$ be the set of all normalized biholomorphic convex mappings on B. Let $Q_A(B)$ (respectively

$Q_B(B)$) denote the set of all quasi-convex mappings of type A (respectively type B) on B and $Q(B)$ be the set of all quasi-convex mappings on B.

We establish a sufficient condition for quasi-convex mappings (including quasi-convex mappings of type A and quasi-convex mappings of type B) on the unit ball B in a complex Banach space, furthermore, we will obtain sharp estimations of all homogeneous expansions for quasi-convex mappings (including quasi-convex mappings of type A and quasi-convex mappings of type B) defined on the open unit poly disk D^n in $\mathbb{C}^n$, which satisfy a certain condition.

In order to prove the main theorem, we need to establish some lemmas.

Lemma (1.2.5)[14]: (See [16].) $K(B) \subset Q(B) = Q_A(B) \subset Q_B(B)$. In some concrete complex Banach spaces, we even have $K(B)\ Q(B)$.

Lemma (1.2.6)[14]: (See [1].) Suppose B is the unit ball in $\mathbb{C}^n$, $f \in S(B)$. Then $F(z) = \frac{f(T_u(z))}{T_u(z)}\ z \in\ Q_A(B)\ (or\ Q_B(B))$ if and only if $f \in K(D)$, where$\|u\|$=1.

It is not difficult to prove the following

Lemma (1.2.7)[14]: Suppose f is a normalized locally biholomorphic mappingon D^n.Then $f \in Q(D^n)$ if and only if

$$\Re e \frac{g_j(\xi,z)}{z_j} \geq 0, \qquad z = (z_1,\dots,z_n) \in D^n,$$

where $g(\xi,z) = (g_1(\xi,z),\dots,g_n(\xi,z)) = \left(Df(z)\right)^{-1}(f(z) - f(\xi z))\ , z \in D^n, \xi \in \overline{D}$ is a column vector in $\mathbb{C}^n$, j satisfies $|z_j| = \|z\| = \max\ 1 \leq k \leq n\{|z_k|\}$.

Theorem (1.2.8)[14]: If $f \in H(B), f(0) = 0, Df(0) = I$, and $\sum_{m=2}^{\infty} \frac{m^2\|D^m f(0)\|}{m!} \leq 1$, where

$$\|D^m f(0)\| = \sup_{\|x^{(k)}\|=1, 1\leq k\leq m} \left\|D^k f(0)(x^{(1)}, x^{(2)},\dots,x^{(m)})\right\|,$$

then $f \in Q(B)$, furthermore, $f \in Q_A(B)$ and $f \in Q_B(B)$.

Proof. Since $f \in H(B)$ and $f(x) = x + \sum_{m=2}^{\infty} \frac{D^m f(0)(x^m)}{m!}, x \in B$, then

$$Df(x) = I + \sum_{m=2}^{\infty} \frac{mD^m f(0)(x^{m-1},.)}{m!},. \tag{29}$$

Also since $\sum_{m=2}^{\infty} \frac{m^2\|D^m f(0)\|}{m!} \leq 1$, then from (29), we obtain

$$\|Df(x) - I\| \leq \sum_{m=2}^{\infty} \frac{m\|D^m f(0)\|\ \|x\|^{m-1}}{m!} \leq \frac{1}{2}\sum_{m=2}^{\infty} \frac{m^2\|D^m f(0)\|\ \|x\|}{m!} \leq \frac{\|x\|}{\|2\|} < 1.$$

According, $Df(x) = I - (I - Df(x))$ exists a bounded inverse operator $\left(Df(x)\right)^{-1}$, and

$$\|Df(x)^{-1}\| \leq \frac{1}{1 - \|I - Df(x)\|} \leq \frac{1}{1 - \sum_{m=2}^{\infty} \frac{m\|D^m f(0)\|\|x\|^{m-1}}{m!}}. \tag{30}$$

On the other hand,

$$\|f(x) - f(\xi x) - (1-\xi)Df(x)x\| = \left\|\sum_{m=2}^{\infty} \frac{[(1-\xi^m) - m(1-\xi)]D^m f(0)(x^m)}{m!}\right\|$$

$$\leq \sum_{m=2}^{\infty} \frac{|(1-\xi m) - m(1-\xi)|\|D^m f(0)(x^m)\|}{m!}$$

$$\leq \sum_{m=2}^{\infty} \frac{|1-\xi||1-m+\xi+\cdots+\xi^{m-1}|\|D^m f(0)\|\|x\|^m}{m!}$$
$$\leq \sum_{m=2}^{\infty} \frac{|1-\xi|^2(1+2+\cdots+m-1)\|D^m f(0)\|\|x\|^m}{m!}$$
$$= \sum_{m=2}^{\infty} \frac{|1-\xi|^2 \frac{m(m-1)}{2} \|D^m f(0)\|\|x\|^m}{m!}$$
$$\leq \|x\| \frac{|1-\xi|^2}{2}\left(1-\sum_{m=2}^{\infty} \frac{m\|D^m f(0)\|\|x\|^{m-1}}{m!}\right). \qquad (31)$$

So, according to (30) and (31), $\forall \xi \in \bar{D}$, we obtain

$$\Re e\left\{T_x\left[\left(Df(x)\right)^{-1}(f(x)-f(\xi x))\right]\right\}$$
$$= \Re e\left\{T_x\left[\left(Df(x)\right)^{-1}(f(x)-f(\xi x)-(1-\xi)Df(x)x+(1-\xi)Df(x)x)\right]\right\}$$
$$\geq \|x\|\Re e(1-\xi)-\left|T_x\left[\left(Df(x)\right)^{-1}(f(x)-f(\xi x)-(1-\xi)Df(x)x)\right]\right|$$
$$\geq \|x\|\Re e(1-\xi)-\frac{\|x\|\frac{|1-\xi|^2}{2}\left(1-\sum_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-1}}{m!}\right)}{1-\sum_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-1}}{m!}}$$
$$= \|x\|\frac{1-|\xi|^2}{2} \geq 0.$$

By Definition (1.2.4), we obtain that $f \in Q(B)$. From Lemma (1.2.5), we deduce that $f \in QA(B)$ and $Q_B(B)$. This completes the proof.

Theorem (1.2.8) tells us that $f(x) = x + \sum_{m=2}^{\infty} \frac{D^m f(0)(x^m)}{m!} \in Q(B)$ ($Q_A(B)$ and $Q_B(B)$) if $\frac{\|D^m f(0)\|}{m!}$ $(m = 2,3,\ldots)$is small enough.

When $X = \mathbb{C}, B = D$, Theorem (1.2.8) is the same as the corresponding result of normalized biholomorphic convex function.

When $X = \mathbb{C}^n, B = D^n$, from Theorem (1.2.8), we immediately obtain the following corollary.

Corollary (1.2.9)[14]: If $f \in H(D^n), f(0) = 0, Df(0) = I$, and $\sum_{m=2}^{\infty} \frac{m^2\|D^m f(0)\|}{m!} \leq 1$, where $\|D^m f(0)\| = \sup_{\|z^{(k)}\|=1, 1\leq k\leq m} \left\|D^m f(0)(z^{(1)}, z^{(2)}, \ldots, z^{(m)})\right\|$, then $f \in Q(D^n)$, furthermore, $f \in Q_A(D^n)$ and $Q_B(D^n)$.

Example (1.2.10)[14]: If $\sum_{k=2}^{n} |a_{km}| \leq \frac{1}{m(m-1)}$, $m = 2,3,\ldots$, then

$$f(z) = \left(z_1 + \sum_{k=2}^{n} a_{km} z_k^m, z_2, \ldots, z_n\right)' \in Q(D^n)\ (Q_A(D^n)\ and\ Q_B(D^n)), m = 2,3,\ldots.$$

Proof. Obviously, f is a normalized biholomorphic mapping on D^n. Straight forward computation shows that

$$\left(Df(z)\right)^{-1}(f(z)-f(\xi z))$$
$$= \left((1-\xi)z_1 + 1 - \xi^m - m(1-\xi)\sum_{k=2}^{n} a_{km} z_k^m, (1-\xi)z_2, \ldots, (1-\xi)z_n\right)'.$$

If there exists $j(\ 2 \leq j \leq n)$ which satisfies $|z_j| \geq |z_1|$, then

$$\Re e\frac{g_j(\xi,z)}{z_j} = \Re e(1-\xi) \geq 0, \tag{32}$$

where $g(z) = (g_1(\xi,z),\ldots,g_j(\xi,z),\ldots,g_n(\xi,z)) = \left(Df(z)\right)^{-1}(f(z) - f(\xi z))\,,\xi \in \overline{D}$; if $|z_k| < |z_1|, k = 2,\ldots,n$, then

$$\begin{aligned}
\Re e\frac{g_1(\xi,z)}{z_1} &= \Re e\left[(1-\xi) + \frac{\sum_{k=2}^{n}\ a_{km}z_k^m}{z_1}\ (1-\xi^m - m(1-\xi))\right] \\
&\Re e(1-\xi) - \left(\sum_{k=2}^{n}\ |a_{km}|\right)|1-\xi|1-m+\xi\ +\xi^2+\cdots+\xi^{m-1} \\
&\geq \Re e(1-\xi) - \left(\sum_{k=2}^{n}\ |a_{km}|\right)\frac{m(m-1)}{2}\ |1-\xi|^2 \\
&\geq \Re e(1-\xi) - \frac{1}{2}|1-\xi|^2 = \Re e(1-\xi) - \frac{1}{2}(1-2\Re e\xi + |\xi|^2) \\
&= \frac{1}{2}(1-|\xi|^2) \geq 0,
\end{aligned} \tag{33}$$

where $g(\xi,z) = (g_1\ (\xi,z),\ldots,g_j(\xi,z),\ldots,g_n(\xi,z)) = (Df(z)) - 1(f(z) - f(\xi z))\,,\xi \in D$. Sofrom Lemma (1.2.7), (32) and (33), we conclude that $f(z) = (z_1 + \sum_{k=2}^{n}\ a_{km}z_k^m\,, z_2,\ldots,z_n)' \in Q(D^n), m = 2,3,\ldots$. Also from Lemma (1.2.5), we have that $f \in Q_A(Dn)$ and $Q_B(D^n), m = 2,3,\ldots$. This completes the proof.

Obviously, each $z_1 + \sum_{k=2}^{n}\ a_{km}z_k^m\ (m = 2,3,\ldots)$ cannot be written as the form of $z_1 g_1(z)$, where $g_1(z) \in H(D^n)$, Example (1.2.10).

By Lemmas (1.2.5) and (1.2.6), we immediately deduce the following example.

Example (1.2.11)[14]: If $f(z) = \left(\frac{z_1}{1-z_1}\,,\frac{z_2}{1-z_1}\,,\ldots,\frac{z_n}{1-z_1}\right)$,then $f(z) \in Q(D^n)$ ($Q_A(D^n)$ and $Q_B(D^n)$).

In order to get the main theorem, it is necessary to establish the following lemma.

Lemma (1.2.12)[14]: If $f(z)$ is a normalized locally biholomorphic mapping on D^n, and $g(z) = \left(Df(z)\right)^{-1}(D^2f(z)(z^2) + Df(z)z) \in H(D^n)$, then

$$\frac{D^2f(0)(z^2)}{2!} = \frac{1}{2}\cdot\frac{D^2g(0)(z^2)}{2!},$$

$$m(m-1)\frac{D^m f(0)(z^m)}{m!} = \frac{D^m g(0)(z^m)}{m!} + \frac{2D^2f(0)z,\frac{D^{m-1}\ g(0)(z^{m-1})}{(m-1)!}}{2!} + \cdots + \frac{(m-1)D^{m-1}f(0)z^{m-2},\frac{D^2g(0)(z^2)}{2!})}{(m-1)!},$$

$$z \in D^n, m = 3,4,\ldots.$$

Proof. Since $g(z) = \left(Df(z)\right)^{-1}(D^2f(z)(z^2) + Df(z)z) \in H(D^n)$, then $D(Df(z)z)z = D^2f(z)(z^2) + Df(z)z = Df(z)g(z)$.

Therefore,

$$z+\frac{4}{2!}D^2f(0)(z^2)+\cdots+\frac{m^2}{m!}D^mf(0)(z^m)+\cdots$$
$$=\left(I\ +\frac{2}{2!}D^2f(0)(z,.)+\cdots+\frac{m}{m!}D^mf(0)(z^{m-1}),.+\cdots\right)$$
$$\cdot\left(Dg(0)z+\frac{D^2g(0)(z^2)}{2!}\ +\cdots+\frac{D^mg(0)(z^m)}{m!}\ +\cdots\right).$$

Comparing with the homogeneous expansion of two sides of the above equality, we have $Dg(0)z=z,\ \frac{4}{2!}D^2f(0)(z^2)=\frac{D^2g(0)(z2)}{2!}\ +\frac{2}{2!}D^2f(0)(z,Dg(0)z),$

$$\frac{m^2}{m!}D^mf(0)(z^m)=\frac{D^mg(0)(z^m)}{m!}\ +\frac{2D^2f(0)\left(z,\frac{D^{m-1}g(0)(z^{m-1})}{(m-1)!}\right)}{2!}$$
$$+\cdots+\frac{(m-1)D^{m-1}f(0)\left(z^{m-2},\frac{D^2g(0)(z^2)}{2!}\right)}{(m-1)!}\ +\frac{m}{m!}\ D^mf(0)(z^m),$$
$$m\ =3,4,\ldots.$$

This implies that

$$\frac{D^2f(0)(z^2)}{2!}\ =\frac{1}{2}\cdot\frac{D^2g(0)(z^2)}{2!},$$
$$m(m-1)\frac{D^mf(0)(z^m)}{m!}\ =\frac{D^mg(0)(z^m)}{m!}\ +\frac{2D^2f(0)\left(z,\frac{D^{m-1}g(0)(z^{m-1})}{(m-1)!}\ \right)}{2!}$$
$$+\cdots+\ \frac{(m-1)D^{m-1}f(0)\left(z^{m-2},\frac{D^2g(0)(z^2)}{2!}\right)}{(m-1)!},$$
$$z\in D^n,m=3,4,\ldots.$$

This completes the proof.

Lemma (1.2.13)[14]: (See [19].) Suppose $g(z)=\left(g_1(z),g_2(z),\ldots,g_n(z)\right)'\in H(D^n),g(0)=\ 0,Dg(0)=I.If\ \Re e\frac{g_j(z)}{z_j}\ \geq 0,z\in D^n,$

where $|z_j|=\|z\|=\ \max\limits_{1\leq k\leq n}\{|z_k|\}$, then

$$\frac{\|D^mg(0)(z^m)\|}{m!}\ \leq 2\|z\|^m,z\ \in D^n,m=2,3,....$$

It is easy to prove the following

Lemma (1.2.14)[14]: Suppose f is a normalized locally biholomorphic mapping on D^n. Then $f\ \in\ Q_B(D^n)$ if and only if

$$\Re e\frac{gj(z)}{z_j}\geq 0,\qquad z\ =(z_1,\ldots,z_n)'\in D^n,$$

where $g(z)=(g_1(z),\ldots,g_n(z))'=\left(Df(z)\right)^{-1}(D^2f(z)(z^2)+Df(z)z)$ is a column vector in $\mathbb{C}^n, j$ satisfies $|z_j|=\|z\|=\ \max\limits_{1\leq k\leq n}\{|z_k|\}$.

Now we can prove the following theorem.

Theorem (1.2.15)[14]: If $f\ \in Q_B(D^n)\ (Q_A(D^n)$ or $Q(D^n))$, and $D^mf_k(0)(z^m)=z_k(\sum_{l=1}^n\ a_{klm}z_l^{m-1}),\ z\in D^n,k=1,2,\ldots,n\ ,m=2,3,\ldots$, where $a_{klm}=\frac{\partial^mf_k(0)}{\partial_{zk}\partial z_l^{m-1}}\ ,k,l=1,2,\ldots,n\ ,m=2,3,\ldots$, then

$$\frac{\|D^m f(0)(z^m)\|}{m!} \le \|z\|^m, z \in D^n, m = 2,3,\ldots.$$

The above estimations are sharp.

Proof. Suppose $f \in Q_B(D^n), \forall z \in D^n \setminus \{0\}$, denote $z_0 = \frac{z}{\|z\|}$. According to Lemmas (1.2.12) (the case of m=2), (1.2.13) (the case of m=2) and (1.2.14), we obtain $\frac{\|D^2 f(0)(z^2)\|}{2!} \le \|z\|^2$

That is, the desired result holds.

Assume that

$$\frac{\|D^s f(0)(z^s)\|}{s!} \le \|z\|^s, \qquad z \in D^n, s = 2,3,\ldots,m. \tag{34}$$

From (34), we have

$$\|D^s f(0)\ (z_0^s)\| \le s!. \tag{35}$$

Also since $D^s f_k(0)(z^s) = z_k(\sum_{l=1}^n a_{kls} z_l^{s-1})$, $z \in D^n, k = 1,2,\ldots,n, s = 2,3,\ldots,m$, where $a_{kls} = \partial^s f_k(0)/\partial z_k \partial z_l^{s-1}, k, l = 1,2,\ldots,n, s = 2,3,\ldots,m$, therefore, by (35), we obtain

$$\left|\frac{z_j}{\|z\|}\left(\sum_{l=1}^n a_{jls}\left(\frac{z_l}{\|z\|}\right)^{s-1}\right)\right| = \left|\sum_{l=1}^n a_{jls}\left(\frac{z_l}{\|z\|}\right)^{s-1}\right| \le s!,$$

where $|zj| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$. Especially, when $z_l = e^{-\frac{i\alpha_l}{s-1}}\|z\|$, where $a_l = \arg\ a_{jls}, l = 1,2,\ldots,n$, it yields that

$$\sum_{l=1}^n |aj_{ls}| \le s!. \tag{36}$$

Denote $w = \frac{D^{m-s+2} g(0)(z^{m-s+2})}{(m-s+2)!}$, $s = 2,3,\ldots,m$, $\forall \lambda \in \overline{D}$, we obtain

$$D^s f_j(0)\Bigg(\underbrace{\frac{z+\lambda w}{2}, \frac{z+\lambda w}{2}, \ldots, \frac{z+\lambda w}{2}}_{s}\Bigg)$$
$$= \frac{D^s f_j(0)(z^s)}{2^s} + \frac{s D^s f_j(0)(z^{s-1}, w)}{2^s}\lambda + \cdots + \frac{D^s f_j(0)(w^s)}{2^s}\lambda^s, \tag{37}$$

where j satisfies $|zj| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$. Note that $D^s f_k(0)(z^s) = z_k(\sum_{l=1}^n a_{kls} z_l^{s-1}), z \in D^n, k = 1,2,\ldots,n, s = 2,3,\ldots,m$, hence

$$D^s f_j(0)\Bigg(\underbrace{\frac{z+\lambda w}{2}, \frac{z+\lambda w}{2}, \ldots, \frac{z+\lambda w}{2}}_{s}\Bigg) = \frac{z_j + \lambda w_j}{2^s}\sum_{l=1}^n a_{jls}(z_l + \lambda w_l)^{s-1}$$
$$= \frac{z_j}{2^s}\left(\sum_{l=1}^n a_j l_s z_l^{s-1}\right) + \frac{(s-1)z_j\left(\sum_{l=1}^n a_j l_s z_l^{s-2} w_l\right) + w_j\left(\sum_{l=1}^n a_j l_s z_l^{s-1}\right)}{2^s}\lambda$$
$$+ \cdots + \frac{w_j\left(\sum_{l=1}^n a_{jls} w_l^{s-1}\right)}{2^s}\lambda^s. \tag{38}$$

Comparing the coefficient of the right sides of (37) and (38) with respect to λ, we have

$$D^s f_j(0)(z^{s-1}, w) = \frac{1}{s}\left[(s-1)z_j\left(\sum_{l=1}^n a_{jls} z_l^{s-2} w_l\right) + w_j\left(\sum_{l=1}^n a_{jls} z_l^{s-1}\right)\right]. \tag{39}$$

From Lemma (1.2.13), (36) and (39), we obtain

$$\left|D^s f_j(0)\left(z_0^{s-1}, \frac{D^{m-s+2}g(0)(z_0^{m-s+2})}{(m-s+2)!}\right)\right|$$

$$= \frac{1}{s}\ (s-1)\frac{z_j}{\|z\|}\left(\sum_{l=1}^{n}\ a_{jls}\left(\frac{z_l}{\|z\|}\right)^{s-2}\frac{D^{m-s+2}g_l(0)(z_0^{m-s+2})}{(m-s+2)!}\right)$$

$$+\ \frac{D^{m-s+2}g_l(0)(z_0^{m-s+2})}{(m-s+2)!}\ \left(\sum_{l=1}^{n}\ a_{jls}\left(\frac{z_l}{\|z\|}\right)^{s-1}\right)$$

$$\leq \frac{1}{s}\left[(s-1)\left|\sum_{l=1}^{n}\ a_{jls}\left(\frac{z_l}{\|z\|}\right)^{s-2}\frac{D^{m-s+2}g_l(0)(z_0^{m-s+2})}{(m-s+2)!}\right|\right.$$

$$\left.+\frac{D^{m-s+2}g_l(0)(z_0^{m-s+2})}{(m-s+2)!}\left|\sum_{l=1}^{n}\ a_{jls}\left(\frac{z_l}{\|z\|}\right)^{s-1}\right|\right]$$

$$\leq \frac{1}{s}\left[(s-1)\sum_{l=1}^{n}\ a_{jls}\left(\frac{|z_l|}{\|z\|}\right)^{s-2}\ \frac{|D^{m-s+2}g_l(0)(z_0^{m-s+2})|}{(m-s+2)!}\right.$$

$$\left.+\frac{|D^{m-s+2}g_l(0)(z_0^{m-s+2})|}{(m-s+2)!}\sum_{l=1}^{n}\ a_{jls}\left(\frac{|z_l|}{\|z\|}\right)^{s-1}\right] \leq \frac{1}{s}[2(s-1)s! + 2\cdot s!]$$

$$= 2\cdot s!.$$

That is,

$$\left|D^s f_j(0)\left(z_0^{s-1},\qquad \frac{D^{m-s+2}g\,(0)(z_0^{m-s+2})}{(m-s+2)!}\right)\right| \leq\ 2\cdot s!,\quad z_0 \in \partial D^n. \tag{40}$$

Especially, when $z_0 \in \partial_0 D^n$, by (40), it yields that

$$\left|D^s f_k(0)\left(z_0^{s-1},\qquad \frac{D^{m-s+2}g\,(0)(z_0^{m-s+2})}{(m-s+2)!}\right)\right| \leq\ 2\cdot s!\,,\quad k\ = 1,2,\ldots,n. \tag{41}$$

In view of $D^s f_k(0)(z^{s-1},\ \frac{D^{m-s+2}g\,(0)(z_0^{m-s+2})}{(m-s+2)!}) \in H(\overline{D^n}), k = 1,2,\ldots,n$, by the maximum modulus theorem of holomorphic functions on the unit polydisk and (41), we obtain

$$\left|D^s f_k(0)\left(z_0^{s-1}, \frac{D^{m-s+2}g\,(0)(z_0^{m-s+2})}{(m-s+2)!}\right)\right| \leq 2\cdot s!, z_0 \in \partial D^n,\ \ k = 1,2,\ldots,n.$$

We conclude that

$$\left|D^s f(0)\left(z_0^{s-1}, \frac{D^{m-s+2}g\,(0)(z_0^{m-s+2})}{(m-s+2)!}\right)\right| \leq 2\cdot s!,$$

That is,

$$\left\|D^s f(0)\left(z^{s-1}, \frac{D^{m-s+2}g\,(0)(z_0^{m-s+2})}{(m-s+2)!}\right)\right\| \leq 2\cdot s!\,\|z\|^{m+1}, z \in D^n, s = 2,3,\ldots,m. \quad (42)$$

From Lemma (1.2.13) and (42), we have

$$\frac{(m+1)m\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}$$
$$\leq \frac{\|D^{m+1}g(0)(z^{m+1})\|}{(m+1)!} + \frac{2\left\|D^2 f(0)\left(z\,,\frac{D^m g(0)(z^m)}{m!}\right)\right\|}{2!}$$
$$+\cdots+\frac{(m-1)\left\|D^{m-1}f(0)\left(z^{m-2},\frac{D^3 g(0)(z^3)}{3!}\right)\right\|}{(m-1)!}+\frac{m\left\|D^m f(0)\left(z^{m-1},\frac{D^2 g(0)(z^2)}{2!}\right)\right\|}{m!}$$
$$\leq 2\|z\|^{m+1}+2\cdot 2\|z\|^{m+1}+\cdots+(m-1)\cdot 2\|z\|^{m+1}+m\cdot 2\|z\|^{m+1}$$
$$=(m+1)m\|z\|^{m+1}.$$
That is,
$$\frac{\|D^{m+1}f(0)(z^{m+1})\|}{(m+1)!}\ \|z\|^{m+1}, \qquad z\in D^n.$$
Therefore, the desired result holds. By Lemma (1.2.5), the desired result for $f\in Q_A(D^n)$ or $Q(D^n)$ also holds. This completes the proof.

According to Example (1.2.11), it is not difficult to verify
$$f(z)=\left(\frac{z_1}{1-z_1},\frac{z_2}{1-z_1},\ldots,\frac{z_n}{1-z_1}\right), z\in D^n,$$
satisfies the condition of Theorem (1.2.15). Taking $z=(r,0,\ldots,0)'$ $(0\leq r<1)$, then
$$\frac{\|D^m f(0)(z^m)\|}{m!}=r^m, \quad m=2,3,\ldots.$$
Hence the estimations of Theorem (1.2.15) are sharp.

From Lemmas (1.2.12) and (1.2.13), it is not difficult to deduce the following

According to Theorem (1.2.15), we can prove the following corollaries.

Corollary (1.2.16)[14]: If $f\in Q_B(D^n)$ $(Q_A(D^n)$ or $Q(D^n))$, and $D^m f_k(0)(z^m)=z_k(\sum_{l=1}^n a_{klm}z_l^{m-1})$, $z\in D^n, k=1,2,\ldots,n, m=2,3,\ldots$, where $a_{klm}=\frac{\partial^m f_k(0)}{\partial z_k\partial z_l^{m-1}}$, $k,l=1,2,\ldots,n, m=2,3,\ldots$, then
$$\|f(z)\|\leq\frac{\|z\|}{1-\|z\|}, \qquad z\in D^n.$$
The above estimation is sharp.

Proof. From Theorem (1.2.15), we obtain
$$\frac{\|D^m f(0)(z^m)\|}{m!}\|z\|^m, \qquad z\in D^n, m=2,3,\ldots.$$
Also,
$$f(z)=z+\sum_{m=2}^{\infty}\frac{D^m f(0)(z^m)}{m!},$$
hence,
$$\|f(z)\|\leq \|z\|+\sum_{m=2}^{\infty}\frac{\|D^m f(0)(z^m)\|}{m!}\sum_{m=1}^{\infty}\|z\|^m=\frac{\|z\|}{1-\|z\|}.$$
That is, $\|f(z)\|\leq\frac{\|z\|}{1-\|z\|}$, $z\in D^n$.

This completes the proof.

Corollary (1.2.17)[14]: If $f \in Q_B(D^n)$ $(Q_A(D^n)$ or $Q(D^n))$, and $D^m f_k(0)(z^m) = z_k(\sum_{l=1}^{n} a_{klm} z_l^{m-1})$, $z \in D^n, k = 1,2,\dots,n, m = 2,3,\dots,$ where $a_{klm} = \frac{\partial^m f_k(0)}{\partial z_k \partial z_l^{m-1}}$, $k, l = 1,2,\dots,n, m = 2,3,\dots,$
then

$$\|Df(z)z\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \qquad z \in D^n.$$

The above estimation is sharp.

Proof. From Theorem (1.2.15), we obtain

$$\frac{\|D^m f(0)(z^m)\|}{m!} \|z\|^m, \qquad z \in D^n, m = 2,3,\ldots.$$

Also,

$$Df(z)z = z + \sum_{m=2}^{\infty} \frac{mD^m f(0)(z^m)}{m!},$$

hence,

$$\|Df(z)z\| \leq \|z\| + \sum_{m=2}^{\infty} \frac{\|D^m f(0)(z^m)\|}{m!} \sum_{m=1}^{\infty} m\|z\|^m = \frac{\|z\|}{(1-\|z\|)^2}.$$

That is,

$$\|Df(z)z\| \leq \frac{\|z\|}{(1-\|z\|)^2}, \qquad z \in D^n.$$

This completes the proof.

According to Example (1.2.11), it is not difficult to verify

$$f(z) = \left(\frac{z_1}{1-z_1}, \frac{z_2}{1-z_1}, \ldots, \frac{z_n}{1-z_1}\right)', z \in D^n,$$

satisfies the condition of Corollaries (1.2.16) and (1.2.17). Taking $z = (r, 0,\dots,0)'$ $(0 \leq r < 1)$, then

$$\|f(z)\| = \frac{r}{1-r}, \qquad \|Df(z)z\| = \frac{r}{(1-r)^2}.$$

Hence the estimations of Corollaries (1.2.16) and (1.2.17) are sharp. Corollaries (1.2.16) and (1.2.17) show that the upper bounds of growth theorem and distortion theorem for a normalized quasi-convex mapping (including quasi-convex mapping of type A and quasi-convex mapping of type B) $f(z)$ hold, where $f(z)$ satisfies

$$D^m f_k(0)(z^m) = z_k(\sum_{l=1}^{n} a_{klm} z_l^{m-1}), \quad z \in D^n, k = 1,2,\dots,n, m = 2,3,\dots,$$

where

$$a_{klm} = \frac{\partial^m f_k(0)}{\partial z_k \partial z_l^{m-1}}, k, l = 1,2,\dots,n, m = 2,3,\dots,$$

Chapter 2
Differential Equations and the Period Function

We show that the main tools used in the proof are the generalized polar coordinates, introduced by Lyapunov to study the stability of degenerate critical points, and the analysis of the derivatives of the Poincaré return map. The results generalize those obtained for polynomial systems with homogeneous non-linearitie. We deal with Hamiltonian systems with homogeneous nonlinearities.

Section (2.1): The Sum of Two Quasi-Homogeneous Vector Fields

Given $p, q, s \in \mathbb{N}$, we will say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is (p,q) −quasi-homogeneous of degree s if $f(\lambda^p x, \lambda^q y) = \lambda^s f(x,y)$ for $\lambda \in \mathbb{R}$, (see [21]). A vector field $X = (P,Q) \colon \mathbb{R}^2 \to \mathbb{R}^2$ is called (p,q) −quasihomogeneous of degree r if P and Q are (p,q) −quasi-homogeneousfunctions of degrees $p + r - 1$ and $q + r - 1$ respectively, see [22].

Observe that the above definition is the natural one for the following reasons:

(i) When $p = q = 1$, it coincides with the usual definition of homogeneous vector field of degree r.

(ii) The differential equation $\frac{dy}{dx} = \frac{Q}{P}$, associated with X, is invariant by the change of variables $\bar{x} = \lambda^p x, \bar{y} = \lambda^q y$.

(iii) Homogeneous vector fields can be integrated using polar coordinates whereas (p,q) −quasi-homogeneous vector fields can be integrated using the (p,q) −polar coordinates. These generalized polar coordinates were introduced by Lyapunov in his study of the stability of degenerate critical points, see [34]. We consider a small modification of these coordinates and their main properties.

The (p,q) −polar coordinates have also been applied recently to study properties of planar differential equations, see [24], [28].

We study differential equations of type:

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \big(P(x,y), Q(x,y)\big) = X(x,y) = X_n(x,y) + X_m(x,y), \qquad (1)$$

where $m > n$, and X_u is a $(p,q) - q$uasi-homogeneous vector field of degree $u - p - q + 2pq$, for $u \in \{n, m\}$.

Note that when $p = q = 1, X = X_n + X_m$ is the sum of two homogeneous vector fields with n and m degrees of homogeneity respectively and includes quadratic differential equations$(p = q = n = 1, m = 2)$ and polynomial systems with homogeneous nonlinearities $(p = q = n = 1)$, see [26], [27], [29], [30], [32].

In the (p,q) −polar coordinates, $(x,y) = \big(\rho^p\, Cs(\varphi), \rho^q\, Sn(\varphi)\big)$, defined, and with a new time variable s, given by $\frac{dt}{ds} = \rho^{p+q-2pq}$, the differential equation (1) becomes

$$\dot{\rho} = \frac{d\rho}{ds} = \rho^{p+q+1-4pq}[x^{2q-1}P(x,y) + y^{2p-1}Q(x,y)],$$

$$\dot{\varphi} = \frac{d\varphi}{ds} = \rho{-}^{2pq}[pxQ(x,y) - qyP(x,y)].$$

Using (1) we obtain

$$\begin{aligned} \dot{\rho} &= \bar{a}_n(\varphi)\rho^n + \bar{a}_m(\varphi)\rho^m, \\ \dot{\varphi} &= b_n(\varphi)\rho^{n-1} + b_m(\varphi)\rho^{m-1}, \end{aligned} \qquad (2)$$

where

$$\begin{pmatrix} \bar{a}_u(\varphi) \\ b_u(\varphi) \end{pmatrix} = \begin{pmatrix} Cs^{2q-1}(\varphi) & Sn^{2p-1}(\varphi) \\ -q\,Sn(\varphi) & pCs(\varphi) \end{pmatrix} \begin{pmatrix} P_u\left(Cs(\varphi), Sn(\varphi)\right) \\ Q_u\left(Cs(\varphi), Sn(\varphi)\right) \end{pmatrix},$$

$u \in \{n, m\}$(and $\mathrm{Sn}(\varphi)$ and $\mathrm{Cs}(\varphi)$ are also defined.

Finally taking the new coordinates r and φ and a new time variable v, given by $r = \rho^{m-n}, \varphi = \varphi, \frac{dv}{ds} = \rho^{\mathrm{n}-1}$, the differential equation (2) writes as

$$\dot{r} = \frac{dr}{dv} = a_n(\varphi)r + a_m(\varphi)r^2, \tag{3}$$

$$\dot{\varphi} = \frac{d\varphi}{dv} = b_n(\varphi) + b_m(\varphi)r,$$

where $a_u(\varphi) = \dot{a}_{\mathrm{u}}(\varphi) \cdot (m - n)$ for u $\in \{n, m\}$.

For the values (r, φ) for which $b_n(\varphi) + b_m(\varphi)r \neq 0$, equation (3) can be transformed into a new equation as follows

$$\frac{dr}{d\varphi} = \mathrm{S}(r, \varphi) = \frac{a_n(\varphi)r + a_m(\varphi)r^2}{b_n(\varphi) + b_m(\varphi)r}. \tag{4}$$

Most properties that we will prove for system (1) will be studied in coordinates r, φ in which this system can be written as (3) or (4). We will define the functions:

$$F(\varphi) = a_n(\varphi)b_m(\varphi) - a_m(\varphi)b_n(\varphi), \text{and } A(\varphi) = b_m(\varphi)F(\varphi). \tag{5}$$

Note that the function $\mathrm{b_m}(\varphi)$ controls the infinite critical points of (1) in the (p, q) Poincare´ compactification. The functions $\mathrm{F}(\varphi)$ and $\mathrm{b_m}(\varphi)$ control the finite critical points of (1). On the other hand, $b_n(\varphi)$ gives information about the origin: if $b_n(\varphi) \neq 0, (0, 0)$ is a critical point of center or focus type, while if $\mathrm{b_n}(\varphi)$ vanishes, (0, 0) can be the ,αorω −limit set for some trajectory of system (3). As the following results show, hypothesizing on $A, F,$ or bn we can establish the number of limit cycles in (1).

The main results are listed in the following theorems. A more detailed account of these results and related ones, such as cases $b_m(\varphi) \equiv 0, F(\varphi) \equiv 0$.

Note that Theorem (2.1.19) gives new information only if $A(\varphi)$ changes sign. Theorems (2.1.15), (2.1.17) and (2.1.19) generalize several results obtained for differential equations with homogeneous non-linearities to systems of type (1) (see again [27], [29], [30], [32]). We would like to point out that most of the proofs that we present differ from the proofs that appear. In the main, use the transformation of equation (3) into an Abel differential equation (see [25], [33]) whereas our different proofs are based directly on the expression (3), although the ideas used are similar.

We contain some results on the location of the critical points and limit cycles of system (1). We give the proofs of Theorems (2.1.15), (2.1.17) and (2.1.19) with more detailed information about the number of limit cycles. There we also consider some examples. Finally, there are three appendices. The first two of them have already been mentioned. The third one discusses how to verify the existence of n and m in such a way that a differential equation can be written in form (1).

We study the situation of the finite critical points and periodic orbits of system (1). Here we will use the generalized tangent function $Tn(\varphi) = Sn^p(\varphi)Cs^q(\varphi)$ and its inverse Arc$Tn(x)$. Let $T = T(p, q)$ be the period of the functions $Sn(\varphi)$ and $Cs(\varphi)$.

Let $C_{\varphi 0}$ be the half-curve of points of $\mathbb{R}^2 - \{(0, 0)\}$ that has the generalized polar angle of its points equal to φ_0 in the (r, φ) coordinates considered. Note that

$$C_{\varphi 0} \cup C_{\varphi 0} + \left\{ (x, y) \in \mathbb{R}^2 : ArcTn\,; \left(\frac{y^p}{x^q}\right) = \varphi_0 \right\}$$

$$\left\{ (x,y) \in \mathbb{R}^2 : \; ArcTn\,; \left(\frac{y^p}{x^q}\right) = \varphi_0 + \frac{T}{2} \right\}.$$

We have the following result

Lemma (2.1.1)[20]: (a) If $b_n(\varphi_1) \cdot b_m(\varphi_1) = 0$ and $b_n(\varphi_1) + b_m(\varphi_1) \neq 0$ or if $F(\varphi_1) \neq 0$ then system (3) has no critical points on C_{φ_1}.

(b) If $b_n(\varphi_1) = b_m(\varphi_1) = 0$ then C_{φ_1} is an invariant curve for (3).

(c) If $F(\varphi_1) = 0$ and $b_n(\varphi_1) and\ b_m(\varphi_1) = 0$, then system (3) has exactly one finite critical point on C_{φ_1}.

(d) If $F(\varphi_1) = 0$ and $b_n(\varphi_1) and\ b_m(\varphi_1) < 0$, then system (3) has no finite critical points on C_{φ_1}.

Proof. (a) In the first case, $b_u(\varphi_{1_1}) = 0$ for some $u \in \{n, m\}$, and then $b_n(\varphi_1) + b_m(\varphi_1) r \neq 0$ for all $r \neq 0$. To prove that if $F(\varphi_1) \neq 0$, (3) has no critical points on $C\varphi_1$, note that if (r_1, φ_1) is a critical point different from the origin then $a_n(\varphi_1) + a_m(\varphi_1) r_1 = b_n(\varphi_1) + b_m(\varphi_1) r_1 = 0$, and $F(\varphi_1) = 0$.

(b) It is obvious from expression (3).

(c) If we take $r_1 = -\frac{b_n(\varphi_1)}{b_m(\varphi_1)}$, then (r_1, φ_1) is a critical point of system (3).

(d) This case follows from (c) because if (r_1, φ_1) is a critical point, then $r_1 \geq 0$.

Let K be the subset of points of $\mathbb{R}^2$ on which the angular component of the vector field (3), $\dot{\varphi}$ vanishes. In the following lemma we study the geometry of K, when it has no curves like (e) of Figure 1. We exclude this case because, as we will see in Proposition (2.1.3)(i), the presence of such curves forces the non existence of periodic orbits. This lemma improves Lemma 2.2 of [27].

Lemma (2.1.2)[20]: Let X be the vector field associated with system (3). Then

(a) K is the graph of the function $r = \frac{-b_n(\varphi)}{b_m(\varphi)}$.

(b) At point $a = (x_0, y_0) \in K, X(a)$ is tangent to the half-curve C_φ where $\varphi =$ArcTn$(\frac{y_0^p}{x_0^q})$.

(c) If K has no curves of type (e) given in Figure 1, then K is either the finite union of curves given by sectors of type (a), (b), (c), (d) and (f) of Figure 1, or K is one of the curves which delimit the sets shown in Figure 2.

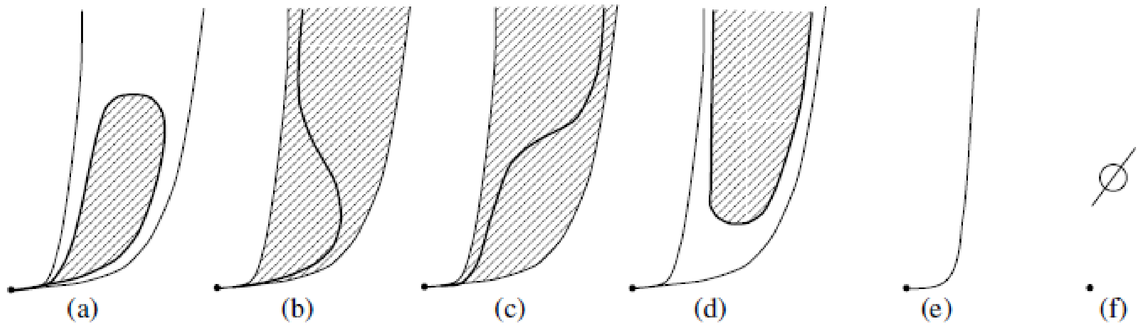
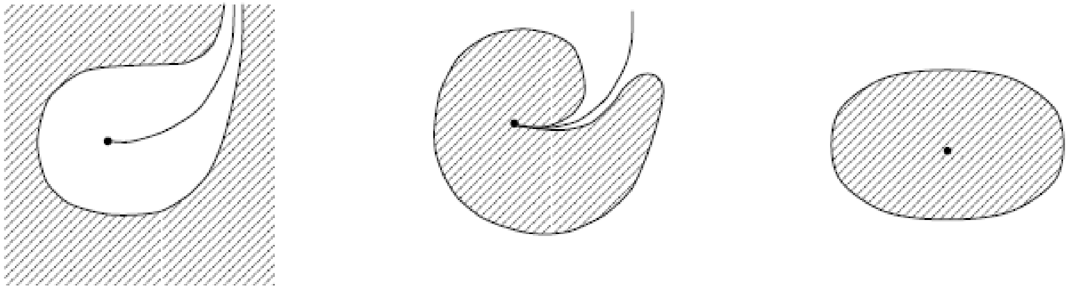


Figure (1)[20]:

The subset K can be a finite union of the curves given by these sectors. The shadowed regions in cases (b) and (c) are either positively or negatively invariant by the flow of (3). In cases (a) and (d) the same happens when one of the hypotheses assumed in Proposition (2.1.3)(iii) is satisfied.



(h) (i) (j)

Figure (2)[20]:
The subset K can be one of the curves which delimit the shadowed regions. These shadowed regions are either positively or negatively invariant by the flow of (3), when one of the hypotheses assumed in Proposition (2.1.3)(iii) is satisfied.

Proof. Parts (a) and (b) follow from direct calculations.

(c) When there are φ_1 and φ_2, not equal, and with $b_n(\varphi)b_m(\varphi) < 0$ for all φ in (φ_1, φ_2) we have (a) if $b_n(\varphi_1) = b_n(\varphi_2) = 0$; (b) if $b_n(\varphi_1) = b_m(\varphi_2) = 0$; (c) if $b_n(\varphi_2) = b_m(\varphi_1) = 0$; (d) if $b_m(\varphi_1) = b_m(\varphi_2) = 0$. When there is only one φ_1 such that $b_n(\varphi_1)\, n\, 0$, $b_m(\varphi_1) \neq 0$ and $b_n(\varphi)b_m(\varphi) > 0$, for all $\varphi \neq \varphi_1$, then we have case (f). When for all φ in some interval (φ_1, φ_2) we have $b_n(\varphi)b_m(\varphi) > 0$, then K has no points in thisregion and we are in case (f). When there exists φ_1 such that $b_n(\varphi_1) = b_m(\varphi_1) = 0$, then $\varphi = \varphi_1$ is invariant by the flow of system (3), and we get case (e).

When there is only one φ_1 such that $b_n(\varphi_1) \neq 0$ and $b_m(\varphi 1) = 0$ with $b_n(\varphi)b_m(\varphi) < 0$ for all $\varphi \neq \varphi_1$, we are in case (h). When there is only one φ_1 such that $b_n(\varphi_1) = 0$ and $b_m(\varphi_1) \neq 0$ with $b_n(\varphi)b_m(\varphi) < 0$ for all $\varphi \neq \varphi_1$, then the form of K is that given in (i) of Figure 2. When for all φ we have $b_n(\varphi)b_m(\varphi) < 0$, then we obtain case (j).

The following proposition gives information about the periodic orbits of system (3) that surround the origin.

Proposition (2.1.3)[20]: (i) Assume that K has associated some sector of type (b), (c) or (e) of Figure 1, then equation (3) has no periodic orbits surrounding the origin.

(ii) Assume that γ is a periodic orbit of (3) surrounding the origin, then $\gamma \cap K = \emptyset$

(iii) Assume that one of the functions $F(\varphi)$ or $A(\varphi)$ or $A(\varphi)b_n(\varphi)$, associated with the differential equation (3), does not change sign. If γ is a periodic orbit of (3), then s surrounds the origin. Furthermore, assume that K has associated no sectors of type (a) of Figure 1 orthat the curve$K$ is not like the curves given in (i) or(j) of Figure 2, then the origin is the only critical point surrounded by s; otherwise s can surround other critical points.

Proof. (i) Let γ be a periodic orbit of system (3), then s cannot cross those sectors, given by K, because of the sign of $b_n(\varphi) + b_m(\varphi)r$ in (3) in cases (b) and (c) (note that
the shadowed regions in those sectors are either positively or negatively invariant by the flow of system (3)), or because $\varphi = \varphi_1$ is an invariant curve of system (3) in case (e).

(ii) Assume that $\gamma \cap K \neq \emptyset$. Then y crosses K transversally because, otherwise, this contact point will be a critical point of system (3). Hence, γ must cross sectors (a) or (d) or subsets (h) or (i) or (j) in two points, R and S, because γ surrounds the origin. In essence, we will have the situation given in Figure 3, where we mark the direction of rotation of the flow of the vector field (3), by means of small arrows. We also take into account that K separates the regions where the directions of rotation are opposed. So by the uniqueness of the solutions we have a contradiction and, therefore, y cannotsurround the origin. (iii) From the Index Theory, y has to surround a critical point. This point belongs to the set K. Note that on $K, \dot{r}(r,w) = \frac{F(\varphi)r(\varphi)b_m}{\varphi} = -\frac{b_n(\varphi)A(\varphi)}{b_m^3}(\varphi) = \frac{A(\varphi)r(\varphi)}{b_m^2}(\varphi)$, so $\dot{r}$ does not change sign on the connected components of K. Hence γ must surround the origin. If, in addition, K has associated no sectors of type (a) or (i) or (j) of Figures 1 and 2, the origin will be the unique critical point that γ surrounds because the shadowed regions of Figures 1 and 2 are invariant under the flow of the system (3). In the other cases γ can surround critical points different from the origin. The examples: $(a)\ \dot{r} = r(10 - r)\cos^2 \varphi, \dot{\varphi} = 5 - (1 + \sin^2 \varphi)r$; $(b)\dot{r} = r(10 - r)\sin^2 \varphi, \dot{\varphi} = 5\cos^2 \varphi - r$, illustrate this situation, see Figure 4.

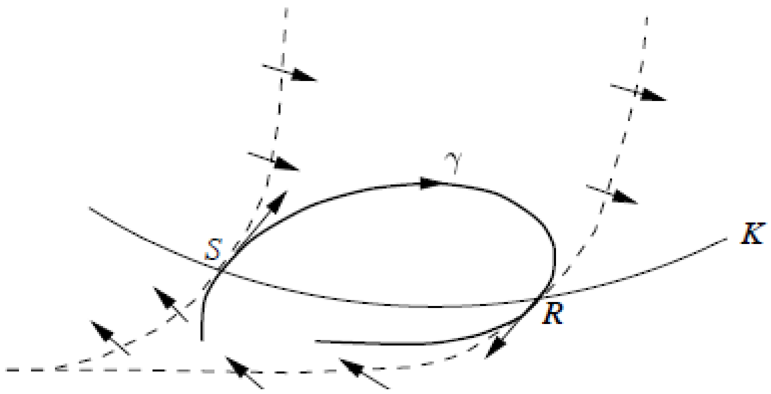


Figure (3)[20]: Standard situation that occurs when $\gamma \cap K \neq \emptyset$, and y surrounds the origin.

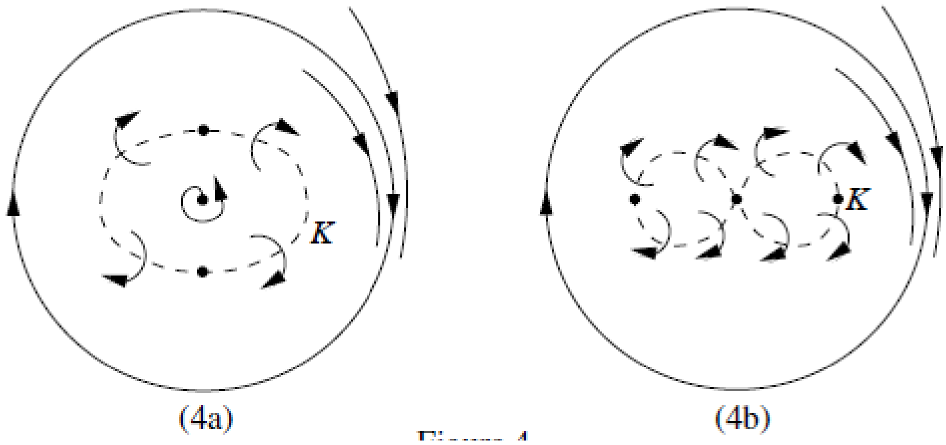


Figure (4)[20]: Limit cycles for system (3) surrounding several critical points.

Corollary (2.1.4)[20]: Periodic orbits of differential equation (3) surrounding the origin can be studied as solutions of (4) satisfying $r(\varphi_1) = r(\varphi_1 + T)$ for any φ_1.

Proof. Follows from (ii) of Proposition (2.1.3).

Given a subset C_ψ, we define $D_\psi \subset C_\psi$ as the subset of points of C_ψ for which the Poincare´ return map, h, is defined, i.e., the set of points, $\alpha \in C_\psi$ for which $h(a) := (T, a)$ is defined and belongs to C_ψ, where $\psi\ (\varphi, a)$ is the solution of (4) such that $\psi(0, a) = a$. Note that D_ψ is always an open subset of C_ψ.

Proposition (2.1.5)[20]: Assume that either the function $F(\varphi)$ or $A(\varphi)$ or $A(\varphi)b_n(\varphi)$, associated with the equation (3) does not change sign and K is not a simple closed curve (case j of Figure 2). Then there is a ψ, such that

(i) All the periodic orbits of (3) belong to the closest connected component to the origin of D_ψ.

(ii) If $b_n(\varphi)$ does not vanish, $0 \in \overline{D_\psi}$.

Proof. (i) If equation (3) has no periodic orbits, there is nothing to be proved. So, from Proposition (2.1.3)(i), cases (b), (c) and (e) will not be considered. We can assume that there is a ψ such that C_ψ is a half curve without contact. Assume, now, that onC_ψ, D_ψ has, at least,

two connected components D_1 and D_2 and equation (3) has a periodic orbit … on D_2 (D_1 is closer to the origin than D_2). From Proposition (2.1.3)(ii)–(iii), we have $\gamma \cap K =$ and an orbit $\tilde{\gamma}$ through a point in D_1 must, always, surround sectors like (a) of Figure 1, if K has associated some of them. Hence, if we take a point q on $C_\psi \backslash D_\psi$ between D_1 and D_2, its -limit or -limit set must be non-empty. This is impossible because between $\tilde{\gamma}$and γ there are no critical points. Thus, all periodic orbits of (3) cut D_1, and (i) follows (see Figure 5).
(ii) The proof follows from (i) taking into account that when $b_n(\varphi)$ does not vanish the origin behaves like a periodic orbit.

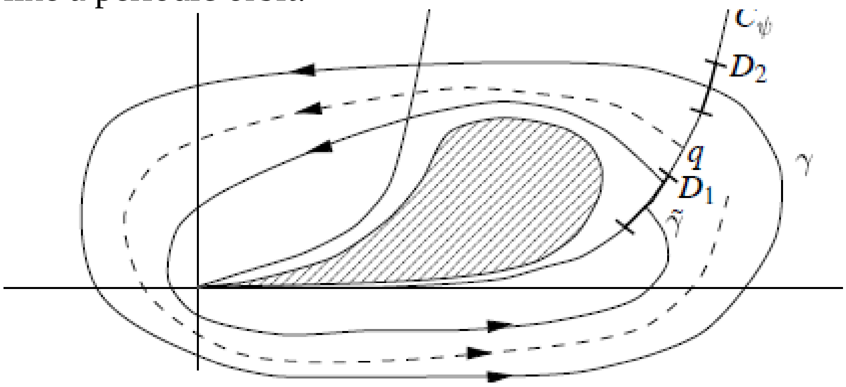


Figure (5)[20]: C_ψ with two different connected components.

First, we will give some preliminary results.

Proposition (2.1.6)[20]: (See [35]). Let $h(x)$ be the return map associatedwith the differential equation $dr/d\varphi = S(r,\varphi)$, then

$$(i) h'(x) = \exp \int_0^T \frac{\partial S}{\partial r}\,(r(\varphi,x),\varphi)d\varphi,$$

$$(ii) h''(x) = h'(x)\left[\int_0^T \frac{\partial^2 S}{\partial r^2}\,(r(\varphi,x),\varphi)\exp\left\{\int_0^\varphi \frac{\partial S}{\partial r}\,(r(s,x),s)ds\right\}d\varphi,\right]$$

$$(iii) h'''(x) = h'(x)\left[\frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2 + \int_0^T \frac{\partial^3 S}{\partial r^3}\,(r(\varphi,x),\varphi)\, exp\left\{2\int_0^\varphi \frac{\partial S}{\partial r}\,(r(s,x),s)\,ds\right\}d\varphi\right],$$

where $r(\varphi,x)$ denotes the solution of the differential equation such that $r(0,x) = x$. Direct calculations give the following lemma,

Lemma (2.1.7)[20]: For equation (4) we have:

$$(i) S(r,\varphi) = \frac{a_m(\varphi)}{b_m(\varphi)}r + \frac{F(\varphi)}{b_m^2(\varphi)} - \frac{F(\varphi)b_n(\varphi)}{b_m^2(\varphi)(b_n(\varphi)+b_m(\varphi)r)},$$

$$\frac{(ii)\partial S}{\partial r}\,(r,\varphi) = \frac{a_m(\varphi)}{b_m(\varphi)} + \frac{F(\varphi)b_n(\varphi)}{b_m(\varphi)(b_n(\varphi)+b_m(\varphi)r)^2},$$

$$(iii)\frac{\partial^2 S}{\partial r^2}\,(r,\varphi) = -\frac{2F(\varphi)b_n(\varphi)}{(b_n(\varphi)+b_m(\varphi)r)^3},$$

$$(iv)\frac{\partial^3 S}{\partial r^3}\,(r,\varphi) = \frac{6A(\varphi)b_n(\varphi)}{(b_n(\varphi)+b_m(\varphi)r)^4}.$$

When the return map is defined we obtain the next result,

Lemma (2.1.8)[20]: The first derivative of the return map associated to a periodic orbit, $r(\varphi)$, of equation (4) is

$$(i)\exp\left\{\int_0^T \frac{a_n(\varphi)}{b_n(\varphi)}\,d\varphi\right\}, if\ r\ \equiv\ 0,$$

$$(ii)\exp\left\{-\int_0^T \frac{F(\varphi)r(\varphi)}{\left(b_n(\varphi)+b_m(\varphi)r(\varphi)\right)^2}\,d\varphi\right\}, if\ r\ \not\equiv\ 0.$$

Proof. (i) follows from the expression obtained for the function $S(r,\varphi)=\frac{dr}{d\varphi}$ in Lemma (2.1.7)(ii), and from Proposition (2.1.6)(i).
To prove (ii), note that from equation (4),

$$0=\int_0^T \frac{r'(\varphi)}{r(\varphi)}\,d\varphi=\int_0^T \frac{a_n(\varphi)+a_m(\varphi)r}{b_n(\varphi)\ +\ b_m(\varphi)r}d\varphi$$
$$=\int_0^T \frac{(a_n(\varphi)+\ a_m(\varphi)r)\big(\,b_n(\varphi)+\ b_m(\varphi)r\,\big)}{(\,b_n(\varphi)+\ b_m(\varphi)r)^2}\,d\varphi,$$

and, from this last expression, we have that

$$\int_0^T \frac{a_n(\varphi)b_n(\varphi)+a_m(\varphi)b_m(\varphi)r^2}{(\,b_n(\varphi)+\ b_m(\varphi)r\,)^2}\,d\varphi=\int_0^T \frac{-r\big(a_n(\varphi)b_m(\varphi)+b_n(\varphi)a_m(\varphi)\big)}{(\,b_n(\varphi)+\ b_m(\varphi)r\,)^2}\,d\varphi.$$

Hence, using this equality, (i) of Proposition (2.1.6) and (ii) of Lemma (2.1.7), (ii) holds.
The calculations made in the following lemma are inspired by [36] and are straightforward.
Lemma (2.1.9)[20]: Let $r_1(\varphi)>\ r_2(\varphi)>r_3(\varphi)$ be three positive solutions of (4). If

$$\mathcal{H}\ (\varphi):=\ \frac{S(r_1,\varphi)-S(r_2,\varphi)}{r_1(\varphi)-\ r_2(\varphi)}-\frac{S(r_1,\varphi)-S(r_3,\varphi)}{r_1(\varphi)\cdot\ r_3(\varphi)}-\frac{S(r_2,\varphi)}{r_2(\varphi)}+\frac{S(r_3,\varphi)}{r_3(\varphi)}\ , \qquad (6)$$

where $S(r_i\ ,\varphi)$, for $i=\ 1,2,3$ is defined in (4), then we have

$$\mathcal{H}\ (\varphi)=\frac{A(\varphi)r_1(\varphi)\big(r_2(\varphi)-\ r_3(\varphi)\big)}{(b_n(\varphi)+\ b_m(\varphi)r_1)(b_n(\varphi)+\ b_m(\varphi)r_2)(b_n(\varphi)+\ b_m(\varphi)r_3\,)}\ . (7)$$

The next lemma follows from direct computations and is based on the change of variables made in [25].
Lemma (2.1.10)[20]: If $b_n(\varphi)$ does not vanish, the transformation $\mathcal{T}\ (r,\varphi)=\ (\rho,\varphi)$, where

$$\rho=\frac{r}{b_n(\varphi)\ +\ b_m(\varphi)r},$$

is a diffeomorphism between $\mathbb{R}^2\backslash\ K$ and its image. Furthermore, the differential equation (4) is transformed into the following Abel differential equation:

$$\frac{d\rho}{d\varphi}\ \alpha(\varphi)\rho^3\ +\ \beta(\varphi)\rho^2\ +\ \gamma(\varphi)\rho, \qquad (8)$$

where

$$\alpha(\varphi)=\frac{b_m(\varphi)}{b_n(\varphi)}\ [a_n(\varphi)b_m(\varphi)-a_m(\varphi)b_n(\varphi)]=\frac{F(\varphi)b_m(\varphi)}{b_n(\varphi)}=\frac{A(\varphi)}{b_n(\varphi)},$$
$$\beta(\varphi)=\frac{1}{b_n(\varphi)}\ [b_n(\varphi)a_m(\varphi)-\ 2a_n(\varphi)b_m(\varphi)]+\ \frac{b_n'\,(\varphi)b_m(\varphi)-\ b_n(\varphi)b'{}_m(\varphi)}{b_n(\varphi)}$$

$$\gamma(\varphi) = \frac{a_n(\varphi) - b_n'(\varphi)}{b_n(\varphi)},$$

Following [27], equation (8) can be written in a different way asthe next lemma shows. **Lemma (2.1.11)[20]:** Equation (8) is equivalent to

$$(9)\ \frac{d(\rho^{-1} - b_m(\varphi))}{d\varphi} = \left(\rho^{-1} b_m(\varphi)\right)\left(\frac{F(\varphi)}{b_n(\varphi)}\rho - \frac{a_n(\varphi)}{b_n(\varphi)} + \frac{b_n'(\varphi)}{b_n(\varphi)}\right).$$

Lemma (2.1.12)[20]: It is not restrictive, when the function $A(\varphi)b_n(\varphi)$ does not change sign, to consider $A(\varphi)b_n(\varphi) \geq 0$ for every φ.

Proof. By using the following change of variables, $(r, \varphi) \to (r, T - \varphi)$, the lemma follows.

Proposition (2.1.13)[20]: Assume that the function $A(\varphi)b_n(\varphi)$ does not change sign. Then the third derivative of the Poincare´ return map, h, of (4) is positive.

Proof. Using Lemma (2.1.12), if $A(\varphi)b_n(\varphi)$ does not change sign, one can assume that $A(\varphi)b_n(\varphi) \geq 0$. Since, for Lemma (2.1.7)(iv), $\frac{\partial^3 S}{\partial r^3}(r, \varphi) = \frac{6A(\varphi)b_n(\varphi)}{(b_n(\varphi)+b_m(\varphi)r)^4} \geq 0$, it follows from Proposition (2.1.6) that $h'''(x) > 0$ for all x for which h is defined.

In a similar way asin equation (4), we can define a Poincare´ return map $\tilde{h}$ for equation (8) between $\varphi = 0$ and $\varphi = T$. For this map $\tilde{h}$ we have the following result which has already been proved, see for instance [29].

Proposition (2.1.14)[20]: Assume that the function $b_n(\varphi)$ does not vanish and $A(\varphi)b_n(\varphi)$ does not change sign, then the third derivative of the Poincare´ return map, $\tilde{h}$, of (8) is positive.

Proof. Since $\frac{\partial^3}{\partial\rho^3}(\alpha(\varphi)\rho^3 + \beta(\varphi)\rho^2 + \gamma(\varphi)\rho) = 6\alpha(\varphi) = 6\frac{A(\varphi)}{b_n(\varphi)}$, does not change sign, the proof follows in the same way as the proof of Proposition (2.1.13).

First we will prove Theorems (2.1.15), (2.1.17) and (2.1.19) only when $F(\varphi), A(\varphi)$ or $b_n(\varphi)$ are not identically zero. The case in which one of the three functions identically vanishes is easier and is studied at the end.

Theorem (2.1.15)[20]: Given system (1), assume that the function $F(\varphi)$, defined in (5), does not change sign. Thus, this system has, at most, one limit cycle and, when it exists, it is hyperbolic, and surrounds the origin.

Furthermore, there are examples of (1), with the above hypotheses, and with one limit cycle.

Proof. From Proposition (2.1.3)(iii), any periodic orbit of (3) surrounds the origin. As explained, we will divide the proof into two cases:

Case I. K is not a simple closed curve.

From Proposition (2.1.5), there exists a ψ such that all periodic orbits are in the connected component D_ψ of C_ψ. Take a periodic orbit γ of (3). From Lemma (2.1.8), since F does not change sign, it is a hyperbolic stable (resp. unstable) limit cycle if $F(\varphi)$ is greater than or equal to (resp. less than or equal to) zero. Hence γ is unique.

CASE II. K is a simple closed curve.

Periodic orbits of (3) can cut different connected components of C_ψ. Of course, the proof of case a) shows that, in case b), our system has, at most, two limit cycles, one turning clockwise and another one turning counterclockwise but, as we will see, they can not coexist.

From Proposition (2.1.3), periodic orbits of (3) surround the origin, furthermore and since $b_n(\varphi)$ does not vanish, we can study the periodic orbits of (3) as $T-$ periodic solutions of

(9). Let $r(\varphi)$ be a periodic orbit of (3). It gives a T −periodic solution of (9), $\rho(\varphi)$. From Lemma (2.1.11), we have that:

$$\frac{d}{d\varphi} = \ln\ \left(\rho^{-1}\,(\varphi) - b_m(\varphi)\right)\ = \left(\frac{F(\varphi)}{b_n(\varphi)}\rho(\varphi) - \frac{a_n(\varphi)}{b_n(\varphi)}\ + \frac{b_n'\varphi}{b_n(\varphi)}\right),$$

and since $\rho(\varphi)$ is T −periodic,

$$0 = \int_0^T \frac{F(\varphi)}{b_n(\varphi)}\,\rho(\varphi)\,d\varphi\ +\ k, \qquad (10)$$

where $k = -\int_0^T\ a_n(\varphi)/b_n(\varphi)\ \mathrm{d}\varphi$. Observe that if $r_1(\varphi)$ and $r_2(\varphi)$ are two periodic orbits of (3), they induce two T −periodic solutions of (9), $\rho_1(\varphi)$ and $\rho_2(\varphi)$. We can assume that $\rho_1(\varphi) > \rho_2(\varphi)$. But since $F(\varphi)/b_n(\varphi)$ does not change sign,

$$\int_0^T \frac{F(\varphi)}{b_n(\varphi)}\,\rho_1(\varphi)d\varphi\ \neq \int_0^T \frac{F(\varphi)}{b_n(\varphi)}\,\rho_2(\varphi)\,d\varphi,$$

and this contradicts (10). Hence (3) has, at most, one periodic orbit. Using Lemma (2.1.8), it is hyperbolic.

Corollary (2.1.16)[20]: Given the differential equation (1), assume that $F(\varphi)\ \not\equiv\ 0$, does not change sign and that $b_n(\varphi)$ does not vanish. Set $c =\ \int_0^T \frac{a_n(\varphi)}{b_n(\varphi)}\,d\varphi$. Then

a) If K is not a simple closed curve, the unique limit cycle for system (1) only exists when $\operatorname{sign}(F)\ \cdot\ c >\ 0$.

b) If K is a simple closed curve, it divides $\mathbb{R}^2$ in two connected components, one bounded K_b and one unbounded K_u. Thus, if the limit cycle exists in system (I), it is in K_b (resp. K_u) if $\operatorname{sign}(F)\ \cdot\ c$ is plus (resp. minus).

Proof. Follows easily from Lemma (2.1.8) and Theorem (2.1.15).

Theorem (2.1.17)[20]: Given system (1), assume that the function $A(\varphi)$, defined in (5), does not change sign. Thus, this system has, at most, two limit cycles and, when they exist, they surround the origin. Furthermore, if $b_n(\varphi)$ does not vanish, the sum of the multiplicities of the limit cycles is, at most, two.

Moreover, there are examples of (1), with the above hypothesis, with two, one or no limit cycles.

Proof. In our hypotheses and from Proposition (2.1.3), all periodic orbits of system (1) surround the origin and do not cut K. Assume thatsystem (1) hasthree limit cycles $r_1(\varphi) > r_2(\varphi) >\ r_3(\varphi)$. From Corollary (2.1.4), $r_i(\varphi), i =\ 1, 2, 3,$ can be considered as positive solutions of equation (4). Since from Lemma (2.1.9), $A(\varphi)$ does not change sign, we have that $\mathcal{H}\ (\varphi)$ does not change sign and is a continuous function. But, on the other hand, we have that:

$$0 =\ log\ \left\{\frac{\left(r_1(\varphi) -\ r_2(\varphi)\right)r_3(\varphi)}{\left(r_1(\varphi) -\ r_3(\varphi)\right)r_2\ (\varphi)}\right\}\Big|_0^T = \int_0^T\ \ \mathcal{H}\ (\varphi)\,d\varphi,$$

and this contradicts the continuity of $\mathcal{H}\ (\varphi)$. Hence system (1) has, at most, two limit cycles. Now we have to prove that, when $b_n(\varphi)$ does not vanish, the sum of the multiplicities of the limit cycles is, at most, two. In this case, when K is not a simple closed curve, from Proposition (2.1.5) and Corollary (2.1.4), all periodic orbits of (4), included the origin, belong to the same connected component of D_ψ. Furthermore, the third derivative of the Poincare´ return map of (4), h, is positive. Whence we conclude, from Rolle's Theorem, that

$h(x) = x$ has, at most, two simple solutions besides the origin. Therefore the theorem follows. When K is a simple closed curve $b_m(\varphi)$ does not vanish. Hence, $F(\varphi) = A(\varphi)/b_m(\varphi)$ neither changes sign. Therefore from Theorem (2.1.15), system (1) has, at most, one hyperbolic limit cycle and again the theorem follows.

In the case where $A(\varphi) \not\equiv 0$ does not change sign and $b_n(\varphi) \neq 0$, for all φ (this is the case where the local phase portrait of the origin of system (1) is of focus or center type), we obtain a more precise distribution of limit cycles, as we can see in the next theorem. This theorem is based on [29].

Theorem (2.1.18)[20]: Assume that in system(1), $A(\varphi) \not\equiv 0$ does not change sign, $b_n(\varphi) \neq 0$, for all φ, and K is not a simple closed curve. Then Table I shows the distribution of limit cycles when $A(\varphi)b_n(\varphi) \geq 0$, according to the different values of c and d. (The case $A(\varphi)b_n(\varphi) \leq 0$ has associated the table obtained reversing the inequalities for c and d, in accordance with Lemma (2.1.12)).

	$c < 0$	$c = 0$			$c > 0$		
		$d < 0$	$d = 0$	$d > 0$	$d < 0$	$d = 0$	$d > 0$
(I)	1	1	0	0	2	0	0
(II)	1	2	3	2	1	1	1

Table (I)[20]: Maximum number of limit cycles of equation (1) when $A(\varphi)b_n(\varphi) \geq 0$. Here

$$c = \int_0^T \frac{a_n(\varphi)}{b_n(\varphi)}\, d\varphi, \qquad d = \int_0^T \frac{-2F(\varphi)}{b_n^2(\varphi)} exp\left(\int_0^{\varphi} \frac{a_n(s)}{b_n(s)}\, ds\right) d\varphi.$$

(I) maximum number of limit cycles, taking into account their multiplicity.
(II) multiplicity of the solution $r \equiv 0$.

Proof. Using Corollary (2.1.4), to study the limit cycles of (1), it is sufficient to consider equation (4). From the hypotheses, we have that set K is not like the curve in (j) of Figure 2. Therefore, from Proposition (2.1.5), there exists some ψ such that all periodic orbits of (4) cut a connected subset of D_ψ, I and, furthermore, $0 \in \bar{I}$.

If we define $H(x) = h(x) - x$, where $h(x)$ is the Poincare´ return map associated with (4) and with , we have the following properties for H:

(i) $H'''(x) > 0$, for all $x \in I$ (Proposition (2.1.6)(iii) and Lemma (2.1.7)(iv))

(ii) $H'(0) = e^c - 1$, and $H''(0) = e^c d$ (Proposition (2.1.6) and Lemma (2.1.8)(i))

Note that $x = 0$ corresponds to solution $r \equiv 0$, and the fixed points of H correspond with the periodic orbits of (4). Therefore, using (i) and (ii) and arguing as in the proof of [29], we obtain Table I.

Theorem (2.1.19)[20]: Given system (1), assume that the function $A(\varphi)b_n(\varphi)$ does not change sign. Thus, for this system, if there are limit cycles, they surround the origin and the sum of their multiplicities is, at most, three.

Proof. From Proposition (2.1.3)(iii), if (1) has some limit cycle, it surrounds the origin

CASE I. K is not a simple closed curve.

From Proposition (2.1.5) we have that all periodic orbits cut the same connected component D_ψ of C_ψFrom Proposition (2.1.13), the third derivative of the return map, h, when it is defined, is positive. Therefore, by Rolle's Theorem, the sum of the multiplicities of the limit cycles is, at most, three.

CASE II. K is a simple closed curve.

Since in this case $b_n(\varphi)$ does not vanish, the result is that $A(\varphi)$ does not change sign. So, in fact, there is, at most, one limit cycle.
Fixed φ set $S(r) = S(r, \varphi)$. Remember that given $r_i \in \mathbb{R}, i = 1, \ldots, n$, we can define inductively the divided differences of S, as:

$$S[r_i, r_{i+1}, \ldots, r_{i+j+1}] = \frac{S[r_{i+1}, \ldots, r_{i+j+1}] - S[r_i, \ldots, r_{i+j}]}{r_{i+j+1} - r_i},$$

where $S[r_i] = S(r_i)$, see [31]. It turns out that S[ ] is a symmetric function of its variables. As usual, we call it $S_{i,\ldots,i+j+1}$ for short. Then, with this notation, and using $S(0) = 0$,

$$\mathcal{H}(\varphi) = S_{1,2} - S_{1,3} - S_{2,0} + S_{3,0} = (S_{2,1,3} - S_{2,0,3})(r_2 - r_3) = S_{0,1,2,3}(r_2 - r_3)r_1.$$

At the same time, it is well known that $S_{0,1,2,3,\ldots,n} = \frac{S(n)(\xi)}{n!}$, where $\xi \in \langle r_0, r_1, \ldots, r_n \rangle$. Therefore, we have that

$$\mathcal{H}(\varphi) = \frac{1}{3!} \, \mathrm{r}_1(\varphi)\mathrm{r}_2(\varphi) - \mathrm{r}_3(\varphi))\partial 3 \, \mathrm{S}(\mathrm{r},) \ \partial\text{^}3 \ \ \mathrm{r} \ \ 3 \, \mathrm{r}(\varphi, \mathrm{r}_1(\varphi), \mathrm{r}_2(\varphi), \mathrm{r}_3(\varphi)) .$$

When $b_n(\varphi) \equiv 0$ or $A(\varphi) \equiv 0$, it is possible to have more precise information about the limit cycles. And we go on to deal with this below.
When $b_n(\varphi) \equiv 0$ and $b_m(\varphi) \not\equiv 0$ (in the case $b_n(\varphi) \equiv b_m(\varphi) \equiv 0$, system (3) has the solution $\varphi =$ constant, for all), or $A(\varphi) \equiv 0$, we can integrate system (3). Hence, in these cases, we can know exactly the trajectories of all closed solutions. Their initial conditions are given in the following lemma.

Lemma (2.1.20)[20]: In system (3) we assume $b_n(\varphi) \equiv 0, d_1 = \int_0^T \frac{a_m(\varphi)}{b_m(\varphi)} \, d\varphi, and\ d_2 =$ - $\int_0^T \frac{a_n(\varphi)}{b_m(\varphi)}$ exp$(-\int_0^{\varphi} \frac{a_m(s)}{b_m(s)} \, ds) \, d\varphi$.Thus, the following hold.
(i) If $d_1 = d_2 = 0$, all trajectories of (3), in a neighbourhood of $r \equiv 0$, are closed.
(ii) If $|d_1| + |d_2| \neq 0$, system (3) has at most two closed solutions. Furthermore, these solutions are the ones with initial conditions

$$\left.\begin{matrix} r(0) = 0 \\ \varphi(0) = 0 \end{matrix}\right\} , and \ \left.\begin{matrix} r(0) = \dfrac{d_2 e^{d_1}}{1\varphi e^{d_1}} \\ \varphi(0) = 0 \end{matrix}\right\}.$$

Proof. The proof follows by direct calculations.

Proposition (2.1.21)[20]: In system (3), assuming $A(\varphi) \equiv 0$,
(i) If $F(\varphi) \equiv 0$ and $b_m(\varphi) \not\equiv 0$, then system (3) has no limit cycles. Moreover, if $c = \int_0^T \frac{a_n(\varphi)}{b_n(\varphi)} \, d\varphi = 0$, then the origin is a center for system (3). (ii) If $F(\varphi) \not\equiv 0$ and $b_m(\varphi) \equiv 0$, then system (3) has, at most, one limit cycle. Moreover, if $d = \int_0^T \frac{-F(\varphi)}{b_n^2(\varphi)} \, exp\,(\int_0^{\varphi} \frac{a_n(s)}{b_n(s)} \, ds) \, d\varphi$, and c is the value given in (i), the following holds:
(a) If $d = c = 0$, all trajectories of (3), in a neighbourhood of $r \equiv 0$, are closed.
(b) If $|\, c \,| + |\, d \,| \neq 0$, system (3) has, at most, two closed solutions with initial conditions

$$\left.\begin{matrix} r(0) = 0 \\ \varphi(0) = 0 \end{matrix}\right\} , and \ \left.\begin{matrix} r(0) = \dfrac{(1 - e^c)}{d} \\ \varphi(0) = 0 \end{matrix}\right\}.$$

(iii) Assume $F(\varphi) \equiv b_m(\varphi) \equiv 0$. If $b_n(\varphi) \equiv 0$, then all straight lines through the origin are invariant and if $b_n(\varphi) \not\equiv 0$ and $a_m(\varphi) \equiv 0$, then the origin is a center.

Proof. If $F(\varphi) \equiv 0$ and assuming $b_m(\varphi) \not\equiv 0$, we have that system (3) is equivalent to $\frac{dr}{d\varphi} = \frac{a_m(\varphi)}{b_m(\varphi)} r$. With the condition $F(\varphi) \equiv 0$ and integrating this equation we obtain the solutions

$$r(\varphi) = r(0) \cdot exp\left(\int_0^{\varphi} \frac{a_n(s)}{b_n(s)} \, ds \right).$$

Then (i) follows.

If $\mathrm{b_m}(\varphi) \equiv$, system (3) becomes

$$\dot{r} = a_n(\varphi) r + a_m(\varphi) r^2 ,$$
$$\dot{\varphi} = b_n(\varphi),$$

or, equivalently, $\frac{dr}{d\varphi} = \frac{a_n(\varphi)}{b_n(\varphi)} r + \frac{a_m(\varphi)}{b_n(\varphi)} r^2$, and the solutions, $r(\varphi)$, of this Riccati equation are

$$r(\varphi) = \frac{exp\left(\int_0^{\varphi} \frac{a_n(s)}{b_n(s)} \, ds\right)}{-\int_0^{\varphi} \frac{a_m(s)}{b_n(s)} \; exp\left(\int_0^{s} \frac{a_n(\tau)}{b_n(\tau)} \, d\tau\right) ds + r^{-1} \; (0)} .$$

From this expression, (ii) follows. The proof of (iii) is trivial.

The natural generalization of the example given in [30]

$$\dot{x} = apx + \gamma y^{p/q} - \left(a\rho^{2pq} + \gamma x^{2q-1} y^{p/q}\right)(px + y^{2p-1} \rho^{p+q-2pq}),$$
$$\dot{y} = aqy - (a\rho^{2pq} + \gamma x^{2q-1} y^{p/q})(qy - x^{2q-1} \rho^{(p+q-2pq)}),$$

where $\rho = \sqrt[2pq]{px^{2q} + qy^{2p}}$, is a system of type (1). For some values of a, γ, p, and q, this system has one or two limit cycles and it is in accordance with the hypotheses of Theorem (2.1.15) and (2.1.17). Therefore, it shows that the results of Theorem (2.1.15) and (2.1.17) cannot be improved. We also present some different examplesfor which the above theorems apply. We stress that they have not homogeneous nonlinearities. Consider

$$(\dot{x}, \dot{y}) = \left(-y^{2p-1} + P_m(x,y), x^{2q-1} + Q_m(x,y)\right),$$

where P_m and Q_{m} are (p,q) −quasihomogeneous polynomials of degrees $m + 2pq - (q + 1)$ and $m + 2pq - (p + 1)$ respectively. For these systems

$$F(x,y) = -\left(x^{2q-1} P_m(x,y) + y^{2p-1} Q_m(x,y)\right),$$

and

$$A(x,y) = F(x,y)\left(px Q_m(x,y) - qy P_m(x,y)\right).$$

For instance, for system

$$(\dot{x}, \dot{y}) = (-y + ax^3 + bxy, x^3 + cx^4 + dx^2 y),$$

with $(b + c)2 - 4ad < 0$, we get $F(x,y) = ax^6 + (b + c)x^4 y + dx^2 y^2$ and taking $y = \lambda x^2$, we can prove that F does not change sign. On the other hand, consider

$$(\dot{x}, \dot{y}) = (-y + ax^5 + bx^2 y, x^5 + cx^7 + dx^4 y + 3bxy^2),$$

where $b = b(a,c,d) = -\frac{a(3a-d)^3 + c^2 (3a-d)^2 + dc^2 (3a-d)}{3c^3 + c(3aFd)^2}$. For this system we have $F(x,y) = x\,(cx^3 + (d - 3a)y)\,(ay^2 + \beta x^3 y + \gamma x^6)$, and $A(x,y) = x^6\,(cx^3 + (d - 3a)y)^2 (ay^2 + \beta x^3 y + \gamma x^6)$, where α, β and γ are real values depending on a, c and d. If we assume that $\Delta = \Delta(a,c,d) = (d + 3b\lambda)^2 - 12b(3b\lambda^2 + d\lambda + b + c) < 0$, where $\lambda = \frac{c}{3a-d}$, then it can be proved that $\alpha y^2 + \beta x^3 y + \gamma x^6$ does not change sign. So, Theorem (2.1.17) can be applied to the above system under condition $\Delta < 0$. We observe that this last condition is not empty because, for instance, $\Delta(a, 3a - d, d) = d^2 -$

$6ad - 3a^2$. In fact, when $a_n(\varphi) \equiv 0$ and $b_n(\varphi) \equiv 1$, Theorem (2.1.17) can be improved by using Propositions (2.1.3) and (2.1.14), because in this case $\rho = 0$ is a periodic orbit of multiplicity two of system (8), and then system (3) has at most one limit cycle. So the above example has at most one limit cycle.
Before ending we give, for some family of systems of type (1), a compact expression of functions F and A in complex coordinates $(z = x + iy)$. Consider system

$$\dot{x} = \lambda x - y + P_m(x,y),$$
$$\dot{y} = x + \lambda y + Q_m(x,y),$$

where P_m and Q_m are real homogeneous polynomials of degree m on x and y. It also writes as $\dot{z} = (i + \lambda)z + H_m(z,\bar{z})$, where $H_m(z,\bar{z})$ is a complex homogeneous polynomial of degree m on z and $\bar{z}$. In this setting, the functions F and A, that appear in (5), are

$$F = (1 - m)Re\big((1 + \lambda i)H_m(z,\bar{z})\bar{z}\big)$$

and

$$A = (1 - m)Re\left((1 + \lambda i)H_m(z,\overline{z})\bar{z}\right) Im\left(H_m(z,\bar{z})\bar{z}\right),$$

evaluated at $z = e^{i\varphi}, \overline{z} = e^{-i\varphi}$.

Following Lyapunov [34], we introduce the (p,q) −trigonometric functions$z(\varphi) = Sn(\varphi)$ and$w(\varphi) = Cs(\varphi)$, asthe solutions of the Cauchy problem:

$$\dot{z} = -w^{2p-1},$$
$$\dot{w} = z^{2q-1}, \tag{11}$$
$$z(0) = \sqrt[2q]{\frac{1}{p}}, \quad w(0) = 0,$$

where p and q, are positive integers. Observe that we do not explicitly put the dependence of $Sn(\varphi)$ and$\mathrm{Cs}(\varphi)$ with respect to p and q. Also note that for $p = q = 1$, $\mathrm{Sn}(\varphi) = \sin(\varphi)$ and $Cs(\varphi) = cos(\varphi)$. Therefore, it is natural to say that the argument of the functions $Sn(\varphi)$ and $Cs(\varphi)$ is an angle.
We define $Tn(\varphi), Ctn(\varphi), Sec(\varphi), Csc(\varphi)$, by

$$Tn(\varphi) = \frac{Sn^p\varphi}{Cs^q(\varphi)}, Ctn(\varphi) = \frac{Cs^q(\varphi)}{Sn^p(\varphi)},$$
$$Sec(\varphi) = \frac{1}{Cs^q(\varphi)}, \text{and } Csc(\varphi) = \frac{1}{Sn^p(\varphi)}.$$

From these definitions, direct calculations give the following lemma.
Lemma (2.1.22)[20]: The functions defined above satisfy the following properties
(i)$pCs^{2q}(\varphi) + q\,Sn^{2p}(\varphi) = 1$,
(ii)$p + q\,Tn^2(\varphi) = Sec^2(\varphi)$,
(iii)$pCtn^2(\varphi) + q = Csc^2(\varphi)$,
(iv)$\frac{dSn(\varphi)}{d\varphi} = Cs^{2q-1}(\varphi)$,
(v)$\frac{dCs(\varphi)}{d\varphi} = Sn^{2p-1}(\varphi)$,
(vi)$\frac{dTn(\varphi)}{d\varphi} = \frac{Sn^{p-1}\varphi}{Cs^{q+1}}(\varphi)$,
(vii)$\frac{dCsc(\varphi)}{d\varphi} = -p\frac{Cs^{2q-1}(\varphi)}{Sn^{p+1}(\varphi)}$,
(viii)$\frac{dSec(\varphi)}{d\varphi} = q\frac{Sn^{2p-1}(\varphi)}{Cs^{q+1}(\varphi)}$,
(ix)$\frac{dCtn(\varphi)}{d\varphi} = -\frac{Cs^{q-1}(\varphi)}{Sn^{p+1}(\varphi)}$.

Lemma (2.1.23)[20]: $Sn(\varphi)$ and $Cs(\varphi)$ are T −periodic functions (whose period is T) and T is given by

$$T = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2p}} \int_0^1 (1-t)^{\frac{(1-2p)}{2p}} t^{\frac{(1-2p)}{2p}} dt = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2q}} \frac{\Gamma\left(\frac{1}{2p}\right) \cdot \Gamma\left(\frac{1}{2q}\right)}{\Gamma(\frac{1}{2p} + \frac{1}{2q})}.$$

Proof. Since $f(z,w) = qw^{2p} + pz^{2q}$, is a first integral for system (11), there exists $T > 0$ such that $Sn(\varphi)$ and $Cs(\varphi)$ are T −periodic functions.
From Lemma (2.1.22):

$$d\frac{Sn(\varphi)}{d\varphi} = \sqrt[2q]{\left(\frac{1-qSn^{2p}\varphi}{p}\right)^{2q-1}},$$

so

$$\frac{\frac{dSn(\varphi)}{d\varphi}}{\sqrt[2q]{\left(\frac{1-qSn^{2p}(\varphi)}{p}\right)^{2q-1}}} = 1, or \frac{d}{d\varphi}\left(\int_0^{Sn(\varphi)} \frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1-qx^{2p})^{2q-1}}} dx\right) = 1,$$

hence

$$\int_0^{Sn(\varphi)} \frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1-qx^{2p})^{2q-1}}} dx = \varphi + k,$$

where $k = 0$, because $Sn(0) = 0$, (from the initial conditions of the Cauchy problem (11)). Otherwise, φ is the parameter of derivation in (11), so the period T is given by:

$$T = 4\int_0^{Sn(\frac{T}{4})} \frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1- qx^{2p})^{2q-1}}} dx = 4\int_0^{\sqrt[2p]{\frac{1}{q}}} \frac{\sqrt[2q]{p^{2q-1}}}{\sqrt[2q]{(1- qx^{2p})^{2q-1}}} dx,$$

where we have used Lemma (2.1.22)(i). Integrating this last expression we obtain the desired result.
More properties of $Sn(\varphi)$ and $Cs(\varphi)$, are listed in the next lemma.
Lemma (2.1.24)[20]: Functions $Sn(\varphi)$ and $Cs(\varphi)$, satisfy the following relations:
(i)$Cs(-\varphi) = Cs(\varphi)$,
(ii)$Sn(-\varphi) = -Sn(\varphi)$,
(iii)$Cs\left(\frac{T}{2} - \varphi\right) = -Cs(\varphi)$,
(iv)$Sn\left(\frac{T}{2} - \varphi\right) = Sn(\varphi)$,
(v)$Cs\left(\frac{T}{2} + \varphi\right) = -Cs(\varphi)$,
(vi)$Sn\left(\frac{T}{2} + \varphi\right) = -Sn(\varphi)$.
Proof. The relations are obtained from the invariance of system (11) under the transformations: $(z,w,t) \to (z,n-,-t), (z,w,t) \to (-z,w,-t)$ and $(z,w,t) \to (-z,-w,t)$.

Given a point $(x,y) \neq (0,0) \in \mathbb{R}^2$, we can associate the positive real number $\sqrt[2pq]{px^{2q} + qy^{2p}}$, with it. Hence, $\varphi \in \mathbb{R}/[0,T]$ and r give the so-called (p,q) −polar coordinates of $\mathbb{R}^2$. In other words,

$$x = r^p\, Cs(\varphi), \qquad y = r^q\, Sn(\varphi).$$

Using these coordinates and a new time variable, given by $\frac{dt}{ds} = r^{p+q-2pq}$, the system

$$\dot{x} = P(x,y), \dot{y} = Q(x,y),$$

is transformed into

$$\dot{r} = r^{p+q+1-4pq}\,[x^{2q-1}\,\dot{x} + y^{2p-1}\,\dot{y}],$$
$$\dot{\varphi} = r^{-2pq}[p\dot{y}x - qy\dot{x}].$$

In order to study the behaviour of the orbits in a neighbourhood of infinity we follow a generalization of the approach to the usual Poincare´ compactification, [37], explained in [23].

Let $X = (P,Q)$ be a polynomial vector field of usual degree $n \geq 1$. Set $M = \{(i,j) \in \{0,1,\ldots,n\}^2 \,|\, 0 \leq i + j \leq n\}$, and

$$P(x,y) = \sum_{(i,j)\in M} a_{ij}x^i\,y^j\,, Q(x,y) = \sum_{(i,j)\in M} b_{ij}x^i\,y^j\,.$$

Fixed $p,q \in +\mathbb{N}\,, p \geq q,$ we define the following subset of $Z, A = \{ip + jq + 1 - p \,|\, (i,j) \in M\} \cup \{ip + jq + 1 - q \,|\, (i,j) \in M\}$. Observe that the smallest element of A is $1 - p$ and the biggest one is $np + 1 - q$. Given that $k \in Z$ and $r \in \{p,q\}$, consider the subset of $M, L_k^r = \{(i,j) \in M | ip + jq + 1 - r = k\}$. Define the vector field:

$$X_k = \left(P_k(\varphi), Q_k\,(\varphi)\right) = \left(\sum_{(i,j)\in L_k^p} a_{ij}x^i\,y^j\,, \sum_{(i,j)\in L_k^q} b_{ij}x^i\,y^j\right).$$

It is clear that X_k is a (p,q) −homogeneous function of degree k. Thus $X = \sum_{k\in A} X_k$, is the decomposition of X in $(p,q) - q$uasi-homogeneous vector fields.

The expression of $(\dot{x},\dot{y}) = X(x,y)$ in the (p,q) −polar coordinates is:

$$\dot{r} = r^{p+q-2pq}\sum_{k\in A} f_k(\varphi)r^{k+1}\,,$$
$$\dot{\varphi} = r^{p+q-2pq}\sum_{k\in A} g_k\,(\varphi)r^k\,,$$

Where

$$f_k\,(\varphi) = Cs^{2q-1}\,(\varphi)P_k\left(Cs(\varphi), Sn\,(\varphi)\right) + Sn^{2p-1}Q_k\left(Cs(\varphi), Sn(\varphi)\right), and$$
$$g_k\,(\varphi) = pCs(\varphi)Q_k\left(Cs(\varphi), Sn(\varphi)\right) - q\,Sn(\varphi)P_k\left(Cs(\varphi), Sn(\varphi)\right)\,.$$

Putting $\rho = r^{-1}$, and replacing the old time t by a new one t_1, given by the relation $\frac{dt_1}{dt} = r^{(n+1-2q)p+1}$, we get

$$\dot{\rho} = \sum_{k\in A} f_k\,(\varphi)\rho^{np+2-q-k}\,,$$
$$\dot{\varphi} = \sum_{k\in A} g_k\,(\varphi)\rho^{np+1-q-k}\,.$$

This last expression gives the (p,q) −Poincare´ compactification of the vector field X. Observe that $\rho = 0$ (the equator) is invariant and the infinite critical points of X are the points with $\rho = 0$ and φ satisfying $g_{np+1-q}(\varphi) = 0$. Finally, we would like to point out that when $p = q = 1$, this procedure gives the usual Poincare´ compactification.

We characterize vector fields defined by the sum of two quasi-homogeneous vector fields. This method is based on the Newton diagram, see for instance [22]. Given a polynomial vector field $X = (P,Q)$, where

$$P(x,y)=\sum_{i+j=0}^{n} a_{ij}x^i y^j ,Q(x,y)=\sum_{i+j=0}^{n} b_{ij}x^i y^j ,$$

we define its support, S_X, as the following subset of $\mathbb{R}^2$:

$$S_X=\{(i+1,j)|b_{ij}\neq 0\}\cup\{(i,j+1)|a_{ij}\neq 0\}.$$

The next lemma follows from direct computations.

Lemma (2.1.25)[20]: Let X be a polynomial vector field, and let p and q be natural numbers with $(p,q)=1$.

Then, X is given by the sum of two (p,q) −quasi-homogeneous vector fields of degrees $k_1+1-(p+q)$ and $k_2+1-(p+q)$ respectively, if and only if there are two straight lines, $l_i=\{(x,y)\in\mathbb{R}^2|px+qy=k_i\}$ for $i=1,2$, such that $S_X\subset l_1\cup l_2$. Furthermore, $S_{X_i}\subset l_i$, for $i=1,2$.

Observe that, from the above lemma, in order to know if X is given by the sum of two (p,q) −quasi-homogeneous vector fields, for some p and q, it is sufficient to plot its support SX in $\mathbb{R}^2$ and to check if it is contained in the union of two parallel straight lines.

Example (2.1.26)[20]: The vector field

$$X=(y^8+x^3y^6+x^6y^4+y^{11}+x^3y^9+x^6y^7,x^8y^3+x^{11}y+x^8y^6+x^{11}y^4+x^{14}y^2+x^{17}), \tag{12}$$

can be decomposed as the sum of two (2, 3)-quasi-homogeneous vector fields of degrees 23 and 32. See Figure 6

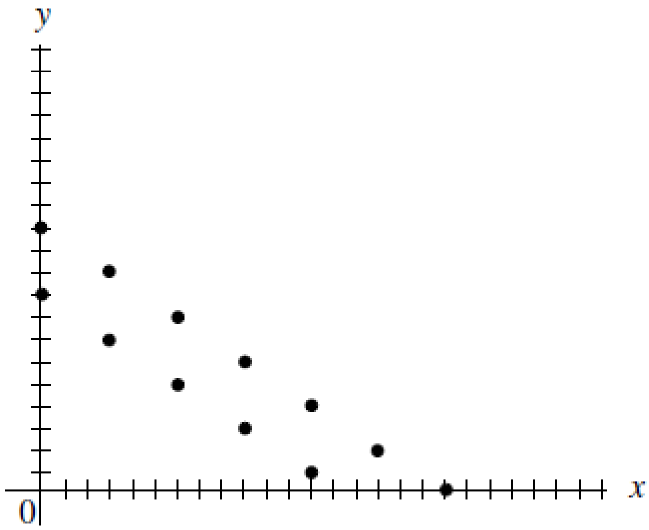


Figure (6)[20]: Support of the vector field (12).

Section (2.2): Hamiltonian Systems with Homogeneous Nonlinearities

We deal with Hamiltonian systems of the form

$$\begin{cases}\dot{x}=-H_y(x,y),\\ \dot{y}=H_x(x,y),\end{cases} \tag{13}$$

where $H(x,y)=(x^2+y^2)/2+H_{n+1}(x,y)$, and H_{n+1} is a non zero homogeneous polynomial of degree $n+1,n\geq 2$. The solutions of system (13) are contained in the level curves $\{H(x,y)=h,h\in\mathbb{R}\}$. Furthermore, the origin is a centre. For any centre p of a planar differential system, the largest neighbourhood of p which is entirely covered by periodic orbits is called the period annulus of p. The function which associates to any closed curve its period is called the period function. When the period function is constant, the centre is called the isochronous centre. We are interested in obtaining the global description of the period function $T(h)$ defined in the origin's period annulus.

It has been proved by several that the origin of (13) cannot be an isochronous centre: For $n = 2$ and 3 this fact was observed by Loud [54] and Pleshkan [56], respectively. In the general case, Christopher and Devlin [44] used geometrical and dynamical methods, and Schuman [57] used Birkhoff's normal form. Another natural approach is the computation of the period constants (see [43] for definitions). Using this last approach we obtain the same result (see Corollary (2.2.10) of the Appendix). One advantage of this method is that it also provides information about the behaviour in a neighbourhood of the origin of the period function, giving lower bounds for the number of critical points of this function (critical periods) associated with the origin's period annulus. Our estrategy for the study of $T(h)$ consists of using the knowledge of the period constants, the knowledge of some properties of the phase portrait of (13) and a criterion to decide when a function has at most one critical point (see Theorem (2.2.3)).

To enunciate the main result we must introduce the following notation: system (13) can be written in complex coordinates as

$$\dot{z} = iz + F_n(z, \bar{z}), \text{with } z \in \mathbb{C},$$

$$F_n(z, \bar{z}) = \sum_{k+l=n} f_{kl}\, z^k z^{-l}, \text{and } Re(\partial F_n(z, \bar{z}) \partial z) \equiv 0.$$

Theorem (2.2.7) (a) was obtained for $n = 2$ by Li Ji-Bin [53], Coppel and Gavrilov [49].

Notice that Theorem (2.2.7) (a) cannot be applied to other centres different from the origin because the structure of (13) is broken under translations (except for $n = 2$). Statement (b) of the above theorem shows that the period function is more complicated for these centers.

A similar difference could exist with other problems. The most relevant is that of isochronicity. From Theorem (2.2.7), it is obvious that systems of type (13) cannot have isochronous centres at the origin. In fact, this result is already known; see [44] and [57]. But, since the structure of (13) is broken under translations, what can be said about the isochronicity of the other centres different from the origin? Are there isochronous centres inside the family of Hamiltonian systems with homogeneous nonlinearities? As far as we know, there was no answer to this question. We prove that:

Our proof of Theorems (2.2.7) and (2.2.8) uses some knowledge of the phase portrait of (13). In particular, we need to study the structure of the hyperbolic sectors at infinity in Poincare's compactification of (13). According to the definitions used in [45], given an infinite critical point q and a hyperbolic sector $\mathcal{H}$ associated to q, we say that $\mathcal{H}$ is degenerate if its two separatrices are contained in the equator of the Poincare's disk. Otherwise, we say that $\mathcal{H}$ is non-degenerate. The control of this kind of points is important for knowing the type of boundary of the period annulus, and for solving Conti's problem for system (13); see [47]. We prove the following result.

We give the proof of Theorem (2.2.2) and we devoted to proving Theorems (2.2.7) and (2.2.8).

We compute the first Lyapunov and period constants for the origin of a system with homogeneous nonlinearities (not necessarily Hamiltonian). They play a key role in the proof of Theorem (2.2.7), but we prefer to show the computations apart, as a technical result. Furthermore, the way of computing these constants and their final expressions help to improve a known result of Conti (see [48]) about the characterization of the centres at the origin of (13) with constant angular speed, see also [55]. While Conti gave an integral characterization of those systems, we provide an explicit expression.

First of all we need a preliminary result that can be also found in [45]. We include the proof here for the sake of completeness and because it is simpler than that of [45].
Let q be an infinite critical point of any planar polynomial Hamiltonian vector field in the Poincare's compactification. We will say that $\mathcal{H}$ does not contain straight lines if given any finite straight line l which passes through q (in Poincare's compactification) there exists compact set K large enough so that $l \cap (\mathbb{R}^2\backslash K)$ is not contained in the interior of $\mathcal{H}$.
Lemma (2.2.1)[39]: Let q be an infinite critical point of a Hamiltonian system with a hyperbolic sector $\mathcal{H}$. Then either $\mathcal{H}$ is degenerate or it does not contain straight lines. Moreover, in this case, the Hamiltonian takes the same value on both separatrices, which are finite.

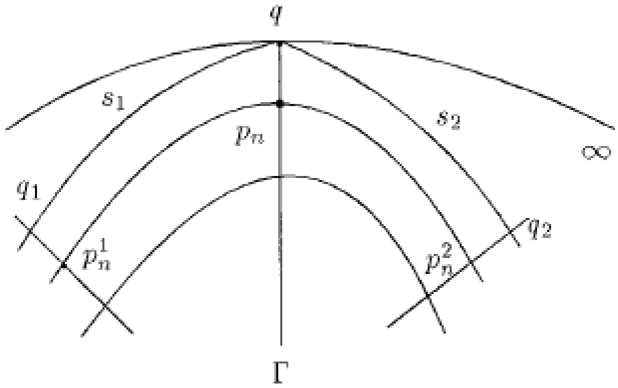


Fig. (1)[39]: Construction used in the proof of Lemma (2.2.1).
Proof. Let s_1 and s_2 be the two separatrices of $\mathcal{H}$. First we will prove that if s_1 is not included in the equator of the Poincaredisk, then s_2 is not contained either. Set $x \in s_2$ and $\{p_n\}_n$ a sequence of points in the interior of $\mathcal{H}$ such that $\lim_{n\to+\infty} p_n = x$. Since $\mathcal{H}$ is a hyperbolic sector, there exists a sequence $\{p'_n\}_n$ in the interior of $\mathcal{H}$ such that $H(p_n) = H(p'_n)$ and moreover, $\lim_{n\to+\infty} p'_n = x' \in s_1$. Thus
$$H(x') = \lim_{n\to+\infty} H(p'_n) = \lim_{n\to+\infty} H(p_n).$$
Hence, we have that $\lim_{z\to x} H(z)$ exists for all $x \in s_2$ when z is in the interior of $\mathcal{H}$. Since $\mathcal{H}$ is a polynomial, s_2 cannot lie on the equator of the Poincare disk, and we are done.
When $\mathcal{H}$ is non-degenerate we can assume, then, that $\mathcal{H}$ has two finite separatrices, s_1 and s_2. From the above equality these separatrices have the same value of the energy (h). First we will prove that if $\Gamma \subset \mathcal{H}$ is any path going to q, we have that (see Fig. 1)
$$\lim_{p\to q, p\in\Gamma} H(p) = h.$$
Let $\{p_n\}_n$ be a sequence of points in the interior of $\mathcal{H}$ satisfying $\lim_{n\to+\infty} p_n = q$. Since $\mathcal{H}$ is a hyperbolic sector, there exist sequences of points $\{p_n^i\}_n$, for $i = 1, 2$, such that $\lim_{n\to+\infty} p_n^i = qi \in s_i$ and $H(p_n^i) = H(p_n)$, for $i = 1, 2$. Then
$$\lim_{n\to+\infty} (p_n^i) = \lim_{n\to+\infty} H(p_n) = H(q_i) = h,$$
and so
$$\lim_{p\to q} H(p) = h.$$
Suppose now that Γ is a straight line. Without loss of generality, we can suppose that this straight line is $x = 0$. From the above argument, if we set
$$H(x, y) = H_0(x) + yH_1(x) + y^2H_2(x) ... + y^{n+1}H_{n+1}(x),$$

then $\lim_{y\to+\infty} H(0,y) = h$. However, this is possible if and only if $H_0(0) = h$, and $H_j(0) = 0$ for all $j = 1,\ldots,n+1$; that is, $H(x,y)\big|_{x=0} \equiv h$ and so $x = 0$ is formed by solutions, which contradicts the fact that 1 is included in $\mathcal{H}$.

We will introduce polar cordinates in order to prove Theorem (2.2.2). The Hamiltonian function is now written as

$$H(r,\theta) = \frac{r^2}{2} + g(\theta)r^{n+1},$$

where $g(\theta)$ is a trigonometric polynomial of degree $n+1$, and system (13) becomes

$$\begin{cases} \dot{r} = -g'(\theta)r^n, \\ \dot{\theta} = 1 + (n+1)g(\theta)r^{n-1}, \end{cases} \tag{14}$$

defined on the cylinder $C = \{(r,\theta): r \in \mathbb{R}^+, \theta \in [0,2\pi]\}$. Observe that the critical points of (14) are (r_*,θ_*) such that $g(\theta_*) < 0$ and $g'(\theta_*) = 0$, and $r_* = \big(-1/\big((n+1)g(\theta_*)\big)\big)^{1/(n-1)}$.

Theorem (2.2.2)[39]: The following statements hold for systems of type (13).

(i) If q is a an infinite critical point in Poincare's compactification having a hyperbolic sector at the infinity $\mathcal{H}$, then $\mathcal{H}$ is degenerate.

(ii) The origin of (13) either is a global center or has a bounded period annulus. Furthermore, the origin is a global centre of (13) if and only if $g(\theta) = H_{n+1}(cos\ \theta, sin\ \theta) \geq 0$, and this can only occur when n is odd.

(iii) A centre p of (13) different from the origin has a bounded period annulus. For $n = 2$, statements (ii) and (iii) of the above Theorem (2.2.2)an be deduced from [41].

Proof. (i) Suppose that q is an infinite critical point of system (13) having a nondegenerate hyperbolic sector $\hbar$. From Lemma (2.2.1), we know that both separatrices must be finite. Without loss of generality, we can suppose that q is determined by the direction $x = 0$ and again, by Lemma (2.2.1), that the separatrices lien on right side of $x = 0$.

Assume that the separatrices of $\hbar$ have energy level h. Then, the energy equation written in polar coordinates is $r^2/2 + g(\theta)r^{n+1} = h$, and so we have that

$$g(\theta) = \frac{2h - r^2}{2r^{n+1}}. \tag{15}$$

We set $F_{h,n}(r) := (2h - r^2)/2r^{n+1}$. If the situation described above were possible for any fixed $\theta \in (\pi/2 - \varepsilon, \pi/2)$, there would be two arbitrarily large

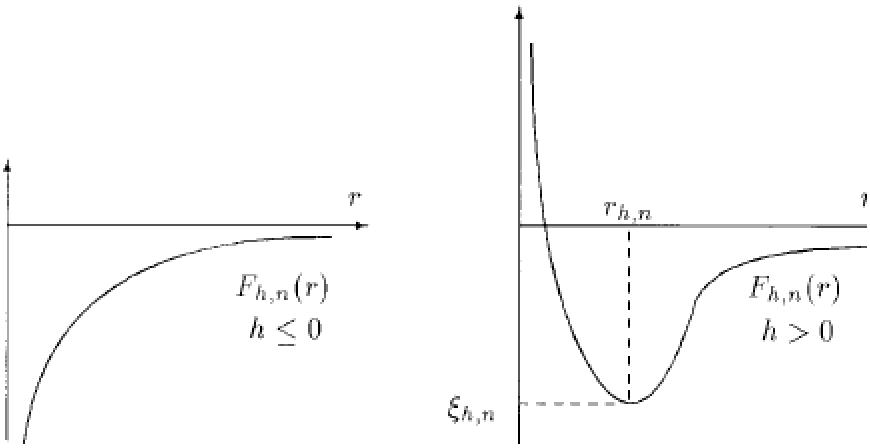


Fig. (2)[39]: Graph of $F_{h,n}(r)$ for $h \leq 0$ (left) and for $h > 0$ (right).

pre-images of $F_{h,n}(r)$ satisfying (15), but this contradicts the behaviour of $F_{h,n}(r)$, for any value of h (see Fig. 2).
(ii-iii) Suppose that p is a centre whose period annulus, N_p, is unbounded but not global. Under this assumption, there must exist a hyperbolic sector at infinity with at least one separatrix contained in ∂N_p. This implies the existence of a non-degenerate hyperbolic sector at infinity, in contradiction to statement (i).
Therefore, ∂N_p either is bounded (moreover, by the analyticity of (13), ∂N_p cannot be a periodic orbit and it contains at least one critical point) or is the empty set. In the latter case, p is the unique critical point and it is a global centre (in fact, p is the origin).
To end the proof we will characterize global centers. From Eq. (14), we see that any critical point (r_*, θ_*) different from the origin must satisfy $g(\theta_*) < 0$ and $g'(\theta_*) = 0$. Thus, it is clear that the origin is the unique critical point if and only if $g(\theta) \geq 0$ for all $\theta \in [0,2\pi)$, and from part (i) this implies that it is a global center. Finally, notice that if n is even, then $g(\theta)$ is a trigonometric polynomial of odd degree and so $g(\theta) = -g(\theta + \pi)$. Consequently, the property $g(\theta) \geq 0$, for all $\theta \in [0,2\pi)$, can only hold when n is odd.

In order to prove Theorems (2.2.7) and (2.2.8), we need the following preliminary results.

Theorem (2.2.3)[39]: An analytic function $f: I = (i^-, i^+) \subset \mathbb{R} \to \mathbb{R}$ has at most one non-degenerate critical point if and only if there exists an analytic function $\varphi: I \to \mathbb{R}$ such that, for all $x \in I$,
$$f''(x) + .(x)f'(x) \neq 0. \tag{16}$$
Proof. Suppose that there exists an analytic function $\varphi: I \to \mathbb{R}$ such that Eq. (16) holds. Let ψ be a primitive of φ. Consider $h: J = (j^-, j^+) \to I$,a solution of the differential equation $h' = exp(\psi(h))$, defined in its maximal interval of definition. Observe that since $h' > 0$ and it is defined in its maximal interval of definition, then $\lim_{x \to j\pm} h(x) = i^{\pm}$. Sohis h a diffeomorphism.

Since $h' \neq 0$ and h is bijective, f has at most one non-degenerate critical point if and only if $f \circ h$ does so. In order to see this last property it suffices to see that $(f \circ h)'' \neq 0$. We prove this as follows:
$$\begin{aligned}(f \circ h)''(x) &= \big(f'(h(x))h'(x)\big)' = f''\big(h(x)\big)\big(h'(x)\big)^2 + f'\big(h(x)\big)h''(x) \\ &= f''\big(h(x)\big)e^{2\psi(h(x))} + f'\big(h(x)\big)e^{2\psi(h(x))}\psi'(h(x)) \\ &= e^{2\psi(h(x))}\,(f''(h(x)) + \varphi(h(x))f'(h(x))) \neq 0.\end{aligned}$$
Let us now prove the converse.
Suppose that f has no critical points. Then, it suffices to choose $.\varphi(x) = (f'(x) - f''(x)/f'(x)$.
If f has a non-degenerate critical point, we can assume, without loss of generality, that it is $x = 0$ and that $f(0) = f'(0) = 0$ and $f''(0) = A > 0$. Hence
$$f(x) = Ax^2 + O(x^3).$$
We choose
$$\varphi(x) = \frac{(f')^2 - 2f''f}{2ff'}(x).$$
Clearly, since f is an analytic function for all $x \neq 0$, $.\varphi$is analytic. We must prove that it is also analytic on $x = 0$. An easy computation shows that $\lim_{x \to 0} .\varphi(x)$ is finite. So φ is analytic on I.

Since $x = 0$ is the unique finite critical point of $f, f(x) \neq 0$ and $f'(x) \neq 0$ for all $x \neq 0$. Hence, we have that, as we wanted to prove, $(f''+.f')(x) = (f')^2/(2f) \neq 0$, for all $x \neq 0$. On the other hand, it is easy to see that $\lim_{x\to 0}(f''+.\varphi f')(x) = 2A \neq 0$.

We will use this last result to prove that the period function associated with the origin's period annulus has at most one critical period. Before proving this fact, we will see that in any Hamiltonian system the set of all periodic orbits, Γ, can be parameterized by the energy in any period annulus W. Consider in W the following total ordering:
Given $\gamma_1, \gamma_2 \in \Gamma$ we say that $\gamma_1 < \gamma_2$ if and only if Int $\gamma_1 \subset$Int γ_2 , where Int γ_i denotes the bounded domain of $\mathbb{R}$ surrounded by γ_i.

Now we endow Γ with the order topology. Clearly, the Hamiltonian function H over Γ is continuous with respect this topology and applies Γ in some interval $I = (0, b)$ of the real line ($b \in \mathbb{R} \cup [+\infty]$). To see that this map is orderpreserving it suffices to show that it is one to one. To prove this, suppose that $H(\gamma_1) = H(\gamma_2)$ with $\gamma_1 < \gamma_2$ and consider the map H restricted to the compact annulus $K = \overline{Int\ \gamma_2 Int\ \gamma_1}$. This map attaches a maximum and a minimum in K. Since $\partial K = \gamma_1 \cup \gamma_2$ and $H\big|_{\partial K}$ is constant, either $H\big|_K$ is constant or $H\big|_K$ has a local extremum in its interior. In both cases we can ensure the existence of $x \in K \subset W$ with $(\{\nabla H)_x = 0$, that is, a critical point of the Hamiltonian vector field in the interior of W, which is a contradiction. So the map H over Γ is order-preserving (in fact it is an order-preserving homeomorphism).

Hence, it seems natural to consider the period function over I instead of the original period function which is defined over the period annulus W, because we can use standard techniques of differential analysis to study the properties of the period function. Therefore, in the sequel we will talk about the period function $T(h)$ which gives the period of the closed orbit with energy $H = h$. From Eq. (14), $T(h)$ can be computed as

$$T(h) = \int_0^{2\pi} \frac{d\theta}{1 + (n+1)g(\theta)r(\theta,h)^{n-1}} \tag{17}$$

(for short, in the following we denote $r := r(\theta, h)$) while $\dot{\theta}\,\big|_{H=h} = 1 + (n+1)g(\theta)r(\theta,h)^{n-1}$ does not vanish. This condition is verified in a deleted neighbourhood of the origin because $\lim_{r\to 0} \dot{\theta} = 1$. The following lemma asserts that this condition holds in the whole period annulus of the origin W. This result is well known (see [27], [20] or [46]), but we include here, for the sake of completeness, a different proof.

Lemma (2.2.4)[39]: The period annulus associated with the origin of (13), W, has no points (r, θ) on which $\theta == 1 + (n+1)g(\theta)r^{n-1} = 0$.

Proof. First we prove that there are no points in $x \in W$ for which $\dot{\theta}\,(x) = \ddot{\theta}(x) = 0$. Consider $\dot{\theta}\,(x) = 1 + (n+1)g(\theta)r^{n-1}$. Then, $\ddot{\theta}\,(x) = (n^2-1)g(\theta)r^{n-2}\,\dot{r} + (n+1)g'(\theta)r^{n-1}\dot{\theta}$. Hence, $\dot{\theta}(x) = \ddot{\theta}\,(x) = 0$ implies that $\dot{\theta} = \dot{r} = 0$ and, as a consequence, x is a critical point different from the origin, which contradicts the fact that $x \epsilon W$.
Set $I = [0, a)$, the image of W by H (remember that H is a homeomorphism between the set of periodic orbits Γ and I). For each $h \in I$ denote by γ_h the closed curve of $H = h$ contained in W. Define the map $L: I \to \mathbb{R}$ by

$$L(h) = min\ |1 + (n+1)g(\theta)r^{n-1}|_{\gamma_h}.$$

This function is clearly well defined and continuous. If $L^{-1}(0) = \emptyset$ there is nothing to prove. Suppose that $L^{-1}(0) \neq \emptyset$. Then $L^{-1}(0)$ is a closed set which does not contain 0 because $L(0) = 1$. Let h_0 be the infimum of $L^{-1}(0)$. Then the orbit y_{h_0} is the first orbit (in the

ordering of T) such that there exists $x \in y_{h_0}$ with $\dot{\theta}(x) = 0$. Set $.\varphi_y(t) = (r_y(t), \theta_y(t))$ be the solution of (14) with initial condition y. Since $\ddot{\theta}\ (x) \neq 0$, the function $\theta_x(t)$ has a local extremum at 0. This implies that, for $\varepsilon => 0$ small enough, the function $\theta_y(t)$ also has a local extremum for $y \in y_{h_0-\varepsilon} =$. Therefore there exists $z \in y_{h_0-\varepsilon} =$ with $\dot{\theta}\ (z) = 0$ and hence $L(h_0 - \varepsilon) = 0$. This last equality is in contradiction to the fact that h_0 =inf $L^{-1}(0)$.
From the above result and the energy equation $r^2/2 + g(\theta)r^{n+1} = h$, it follows that

$$\frac{dh}{dr} = r(1 + (n+1)g(\theta)r^{n-1}) > 0 \qquad (18)$$

in the whole period annulus. Furthermore, any fixed periodic orbit in the origin's period annulus has positive energy. Finally, observe that the above results imply that $T(h)$ is an analytic function.

Lemma (2.2.5)[39]: The period function associated to the period annulus of the origin of (13) satisfies

$$T(h) = \frac{d}{dh}\int_0^{2\pi} \frac{r^2}{2}\, d\theta.$$

Proof. Let γ denote a closed orbit of energy h corresponding to a solution $r(\theta, h)$ of (14). From the expression (17), using (18), we have

$$T(h) = \int_0^{2\pi} \frac{d\theta}{1 + (n+1)g(\theta)r(\theta,h)^{n-1}}$$

$$= \frac{d}{dh}\int_0^{2\pi} \frac{r^2}{2}\, d\theta.$$

Theorem (2.2.6)[39]: The period function associated with the period annulus of the origin of (13) has at most one critical period.

Proof. As we have seen in Lemma (2.2.5), $T(h) = (d/dh)\ \int_0^{2\pi}\ (r^2/2)d\theta$. So Eq. (16) can be written as

$$T''(h) + \varphi(h)T'(h) = \frac{1}{2}\int_0^{2\pi} \frac{d^3}{dh^3}(r^2) + \varphi(h)\frac{d^2}{dh^2}(r^2)\, d\theta \neq 0. \qquad (19)$$

We set $M(r, \theta) = 1 + (n+1)g(\theta)r^{n-1}$ (we call it M, for the sake of brevity). Taking into account Eq. (18), we have that the middle part of expression (19) can be written as

$$\frac{1}{2}\int_0^{2\pi} \frac{-2(n^2-1)(n-3)g(\theta)r^{n-5}M + 6(n^2-1)^2 g^2(\theta)r^{2n-6}}{M^5}$$

$$+\varphi(h)\frac{-2(n^2-1)g(\theta)r^{n-3}}{M^3}d\theta. \qquad (20)$$

We choose $.\varphi(h) = [(n-3)/2]1/h$, defined in $I = (0, a)$, for some $a\epsilon\mathbb{R}^+ \cup \{+\infty\}$ (notice that the fact that the energy in the period annulus takes only positive values plays an important role here). Tedious computation transforms the expression (20) into

$$2\int_0^{2\pi} \frac{(n+1)n(n-1)^2}{hM^5}\ g(\theta)^2 r^{2n-4}\left(1 + \frac{n+3}{4n}\ (n+1)g(\theta)r^{n-1}\right) + d\theta.$$

Note that $0 < (n+3)/(4n) < 1$. Then, since by Lemma (2.2.4) in the whole origin's period annulus $1 + (n+1)g(\theta)r^{n-1} > 0$ holds, we have that

$$1 + \frac{n+3}{4n}(n+1)g(\theta)r^{n-1} > 0,$$

and then

$$\int_0^{2\pi} \frac{(n+1)n(n-1)^2}{hM^5} g(\theta)^2 r^{2n-4}\left(1 + \frac{n+3}{4n}(n+1)g(\theta)r^{n-1}\right) d\theta > 0.$$

Since $T(h)$ is analytic, the theorem follows by applying Theorem (2.2.3). K Now we are able to prove Theorem (2.2.7).

Theorem (2.2.7)[39]: (a) Let $T(h)$ be the period function associated to the origin's period annulus of system (13). $T(h)$ satisfies one of the following properties:

(i) It is monotonic decreasing.

(ii) It is monotonic increasing and it tends to infinity when the periodic orbit tends to the boundary of the period annulus.

(iii) It has a unique nondegenerate critical period (a minimum) and it tends to infinity when the periodic orbit tends to the boundary of the period annulus.

Furthermore,

(i) It is monotonic decreasing if and only if n is odd and $g(\theta) = H_{n+1}(\cos\theta, \sin\theta) \geq 0$,

(ii) It is monotonic increasing if and only if

(I) either n is even,

(II) or n is odd, and $Im(f_{(n+1)/2\,,(n-1)/2}) \leq 0$.

(iii) It has a unique nondegenerate critical period if and only if n is odd, $Im(f_{(n+1)/2,(n-1)/2}) > 0$, and there exists $\theta \in [0,2\pi)$such that $g(\theta) < 0$.

(b) There are systems of type (13) having a critical point of center type (different from the origin) for which the period function has at least two critical periods.

Proof. (a) From Eq. (17), and taking into account (18), we have that

$$\frac{dT(h)}{dh} = -(n+1)(n-1)\int_0^{2\pi} \frac{g(\theta)r^{n-3}d\theta}{(1+(n+1)g(\theta)r^{n-1})^3} \tag{21}$$

To prove statement (i), we recall that $1 + (n+1)g(\theta)r^{n-1} \neq 0$ in the whole origin's period annulus. Hence, if $g(\theta) = H_{n+1}(\cos\theta, \sin\theta) \geq 0$, from (21) we directly obtain that $dT/dh(h) < 0$. Conversely, suppose that $T(h)$ is monotonic decreasing. This impliesusing Theorem (2.2.2) (ii)that the origin is a global centre (otherwise, the boundary of the origin's period annulus would have a critical point and $T(h)$ would tend increasingly to infinity) and, again by Theorem (2.2.2) (ii), if the origin is a global centre then $g(\theta) \geq 0$.

Suppose now that $g(\theta)$ takes negative values. By Theorem (2.2.2) (ii), we also know that the period annulus of the origin is bounded and contains some critical point. This fact implies that the period function tends to infinity as the closed orbits tend to this boundary. If instead of parameterizing the closed curves of the period annulus W by the Hamiltonian energy we use the point of intersection of any closed curve of W with the OX^+ $-$axis we get another period function called $t(r)$. Observe that this can be done in the whole W, because in this set $1 + (n+1)g(0)r^{n-1} > 0$, and $t(r) = T(r^2/2 + g(\theta)r^{n+1})$. Hence

$$T'(h) = \frac{1}{r(1+(n+1)g(0)r^{n-1})}t'(r), \tag{22}$$

where $h = r^2/2 + g(0)r^{n+1}$.

From the above expression we get that the main preliminary result we have obtained, Theorem (2.2.6), is still valid for $t(r)$.

To prove statements (ii) and (iii), we use the results. From Proposition (2.2.9) of the Appendix, we know that

$$b_1 = \begin{cases} 0, & if\ n\ is\ even, \\ -2\pi\ Im\big(f_{(n+1)/2,(n-1)/2}\big), & if\ n\ is\ odd. \end{cases}$$

Moreover, in the proof of Corollary (2.2.10), we deduce that $b_2 > 0$. We distinguish then two cases, depending on the value of the first period-Abel constant:

(ii)$b_1 \geq 0$.

Depending on whether b_1 vanishes or not, the period function in polar coordinates may be written (see Corollary (2.2.11)) as

$$t(r) = 2\pi + b_2\, r^{2n-2} + O(r^{2n-1}), \text{with } b_2 > 0,$$

or

$$t(r) = 2\pi + b_1\, r^{n-1} + O(r^{n}), \text{with } b_1 > 0.$$

In both cases, in a neighbourhood $(0, \delta)$, the period function $t(r)$ is monotonically increasing. Thus, Theorem (2.2.2) (ii) and Theorem (2.2.6) ensure that $t(r)$ is monotonic increasing in its domain and tends to infinity near the boundary of the origin's period annulus, and so does $T(h)$.

(iii) $b_1 < 0$.

Thus, the period function in polar coordinates may be written as

$$t(r) = 2\pi + b_1 r^{n-1} + O(r^{n}), \text{with } b_1 < 0.$$

Therefore, in a neighbourhood $(0,\ \delta)$, the period function $T(r)$ is monotonicaly decreasing. As in the statement (ii), we recall Theorem (2.2.2) (ii) and Theorem (2.2.6). In the current case, they imply that $T(h)$ reaches a unique minimum and then it tends to infinity as the closed orbits tend to boundary of the period annulus.

(b) Consider the system

$$\begin{cases} \dot{x} = -y - ex^4 - 2dx^3\, y + 3x^2y^2 + y^4, \\ \dot{y} = x + 5cx^4 + 4ex^3y + 3dx^2\, y^2 - 2xy^3. \end{cases} \tag{23}$$

It has a centre at the point (0, 1). By a translation to the origin and the linear change of time $dt/d\tau = -1\sqrt{3}$, it is transformed in the following quartic system:

$$\begin{cases} \dot{x} = -y - 3x^2 - 2y^2 + 2\sqrt{3}dx^3 - 6x^2y - \frac{4}{3}y^3 \\ \qquad +3ex^4 + 2\sqrt{3}dx^3\, y - 3x^2y^2 - \frac{1}{3}\, y^4, \\ \dot{y} = x - 3\sqrt{3}dx^2 + 6xy - 12ex^3 6\sqrt{3}dx^2y \\ +6xy^2 - 15\sqrt{3}cx^4 - 12ex^3y - 3\sqrt{3}dx^2y^2 + 2xy^3. \end{cases} \tag{24}$$

The first two period constants (we call them p_2 and p_4) are known for general systems, see for instance [52]. Straightforward computations give that

$$p_2 = \frac{129}{4} + \frac{135}{4}d^2 + \frac{9}{2}e.$$

$$p_4 = \frac{832{,}883}{2{,}304} + \frac{945}{8}cd - \frac{25{,}095}{128}d^2 - \frac{152{,}685}{256}d^4.$$

Since p_2 and p_4 are independent and can take any real value, standard arguments imply that there are values of the parameters for which the period function associated with the period annulus of (0, 1) has at least two critical points in a neighbourhood of the critical point.

Theorem (2.2.8)[39]: System (13) has no isochronous centres.

Proof. Let p be a centre of system (13) and N_p its period annulus. From Theorem (2.2.2) (ii), we know that either N_p is bounded and its boundary contains a critical pointand then it cannot be an isochronous centreor p is a global centre. The last case is possible if and only if n is odd and $g(\theta) = H_{n+1}(cos\,\theta, sin\,\theta) \geq 0$. From Theorem (2.2.7) (i), the period function $T(h)$ defined in the origin's period annulus is globally monotonic decreasing, and so it cannot be an isochronous centre.

Consider

$$\dot{z} = iz + F_n(z, \bar{z}), \text{with } z \in \mathbb{C}, \tag{25}$$

where $F_n(z, \bar{z})$ is a homogeneous polynomial of degree n. We will usually write $F_n(z, \bar{z}) = \sum_{k+l=n} f_{k,l} z^k \bar{z}^l$, where $f_{k,l} \in \mathbb{C}$. For the sake of simplicity, we define, for a fixed n:

$$g_l = \begin{cases} f_{(n+l+1)/2,(n\&l\&1)/2} & if\ l \in \Omega_n, \\ 0 & if\ l \notin \Omega_n, \end{cases} \tag{26}$$

where $\Omega_n = \{l \in \mathbb{Z}: (n+l) \text{ is odd and } -(n+1) \leq l \leq n-1\}$.

Our interest is mainly focused on computing the so-called Lyapunov and period constants for system (25). To this end, we perform the following changes of variables:

If we first introduce the usual polar coordinates by setting $R^2 = z\bar{z}$and $\theta =$arctan $(Im\,z/Rez)$, and then apply the change $r = R^{n-1}/(1 + Im(S_n(\theta))R^{n-1})$ (suggested in [42]), system (25) may be written:

$$\begin{cases} \dot{r} = \dfrac{A_2(\theta)r^2 + A_3(\theta)r^3}{1 - Im(S_n(\theta))r}, \\ \dot{\theta} = \dfrac{1}{1 - Im(S_n(\theta))r}, \end{cases} \tag{27}$$

where$S_n(\theta)$is a trigonometric polynomial defined by $S_n(\theta) = e^{-i\theta} F_n(e^{i\theta}, e^{-i\theta})$; thus,

$A_2(\theta) = Re((n-1)S_n(\theta) + iS_n'(\theta))$ and $A_3(\theta) = [(n-1)/2]\ Re(iS_n^2(\theta))$.

By eliminating the time, we reach the Abel equation:

$$\frac{dr}{d\theta} = A_2(\theta)r^2 + A_3(\theta)r^3. \tag{28}$$

Following [40], for this differential equation, consider the solution $r(\theta, \rho)$ that takes the value ρ when $\theta = 0$. Therefore,

$$r(\theta, \rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + \ldots, \text{with } u_k(0) = 0 \ for\ k \geq 2. \tag{29}$$

Let $P(\rho) = r(2\pi, \rho)$ be the return map between $\mathbb{R} \times [0]$ and $\mathbb{R} \times \{2\pi\}$. We will say that system (28) has a centre at $r = 0$ if and only if $u_k(2\pi) = 0$, for all $k \geq 2$. On the other hand, it has a focus if it exists some k such that $u_k(2\pi) \neq 0$. When, for system (25), $u_j(2\pi) = 0$ for $j = 1, \ldots, m-1$, we will say that its LyapunovAbel constant of order m is $a_m = u_m(2\pi)$.

Substituting (29) in (28) we easily get the following relations, which suggest a recurrent way to find the LyapunovAbel constants a_j:

$$\begin{gathered} u_2' = A_2, \\ u_3' = A_3 + 2A_2u_2, \\ u_4' = A_2u_2^2 + 2A_2u_3 + 3A_3u_2, \ldots. \end{gathered} \tag{18}$$

Once we know that the origin of (25) is a centre, there is a simple way to give the conditions for it to be an isochronous centre. We observe that we cannot use the Abel equation (28), since this equation does not take into account the time variable. The idea we will use is suggested in [50]: if we take the second equation of (27) and we integrate the time, we obtain

$$\bar{t}(\rho) = \int_0^{2\pi} 1 - Im\big(S_n(\theta)\big) r(\theta,\rho) d\theta = 2\pi - \int_0^{2\pi} Im(S_n(\theta)) r(\theta,\rho) d\theta, \quad (31)$$

where $r(\theta,\rho)$ is given above.

The system (25) has an isochronous centre at the origin if it is a centre and, furthermore,

$$\int_0^{2\pi} Im(S_n(\theta)) r(\theta,\rho) d\theta = \int_0^{2\pi} Im\big(S_n(\theta)\big) \left(\sum_{j\geq 1} u_j(\theta)\rho^j \right) d\theta$$

$$= \sum_{j\geq 1} \left(\int_0^{2\pi} Im\big(S_n(\theta)\big) u_j(\theta) d\theta \right) \rho^j = 0.$$

Hence, the conditions to have an isochronous centre are

$$b_j := -\int_0^{2\pi} Im\big(S_n(\theta)\big) u_j(\theta) d\theta = 0, for\ j \geq 1. \quad (32)$$

The numbers b_j will be called periodAbel constants.

In the main result we give some of the first LyapunovAbel and periodAbel constants for all systems of type (25) in terms of the coefficients of the equation and valid for all $n \in \mathbb{N}$. The above approach has been already used in [51] to give integral expressions for the Lyapunov and period constants. As we will see in the applications, our result allows us to establish general properties for systems of type (25) of any degree; see for instance Corollary (2.2.10) and Proposition (2.2.12).

The statement basically follows by integrating the recurrences given in (30). We set first some useful notation for the integration steps:

Given a trigonometric polynomial $p(\theta) = \sum_{k\in K} p_k e^{ik\theta} + p_0$ with K a finite subset of $\mathbb{Z}\backslash\{0\}$, we define

$$\tilde{p}\,(\theta) = \int_0^{\theta} p(\xi) d\xi = \sum_{k\in K} \left[\frac{p_k}{ik}\, e^{ik\theta} + p_0\theta\right]_0^{\theta} = \sum_{k\in K} \frac{p_k}{ik}\,(e^{ik\theta} - 1) + p_0\theta,$$

$$\hat{p}\,(\theta) = \sum_{k\in K} \frac{pk}{ik}\, e^{ik\theta} + p_0\theta,$$

and $\{p\}^{\sim} = \tilde{p}\,, \{p\}^{\wedge} = \hat{p}$. In general, we can write $\tilde{p}(\theta\,) = \hat{p}(\theta) - \hat{p}\,(0)$.

The difference between both primitives of $p(\theta)$ is that $\tilde{p}$ contains an ``extra'' constant term, while $\hat{p}(\theta)$ is the primitive of $p(\theta)$ which has no constant terms. This fact will be crucial for the fluency of our computations.

Observe also that

$$\tilde{p}(\theta) = \left\{ \sum_{k\neq 0} ikp_k e^{ik\theta} \right\}^{\sim} = \sum_{k\neq 0} p_k\big(e^{ik\theta} - 1\big) = p(\theta) - p(0),$$

$$(21)$$

$$\tilde{p}(\theta) = \left\{ \sum_{k\neq 0} ikp_k e^{ik\theta} \right\}^{\wedge} = \sum_{k\neq 0} p_k e^{ik\theta} = p(\theta) - p_0.$$

The last one, then, becomes a trigonometric polynomial without constant terms.

Proposition (2.2.9)[39]: The following assertions are true for systems of type (25), with $F_n(z,\bar{z})$ homogeneous of degree n:

(a) The first three LyapunovAbel constants are

$$a_2 = 2\pi(n-1)Re(g_0),$$

$$a_3 = (1-n)\pi \sum Im(g_l g_{-l}),$$

$$a_4 = \frac{\pi(1-n)}{2} Re\left(\sum_{l,k,l+k\neq 0} \frac{g_l g_k}{l+k}((n-1+l+k)g_{-(l+k)} + (n-1-l-k)\bar{g}_{(l+k)}\right).$$

(b) The first two periodAbel constants are

$$b_1 = -2\pi Im(g_0),$$

$$b_2 = -\pi\left(\sum_{l\neq 0} \frac{n-l-1}{l} g_l \bar{g}_l + 2\sum_{l>0} g_l g_{-l}\right).$$

Proof. (i) To integrate (30) we compute the expressions of $S_n(\theta), A_2(\theta)$, and $A_3(\theta)$ in terms of the coefficients given in (26):

$$S_n(\theta) = \sum_l g_l e^{il\theta},$$

$$A_2(\theta) = Re\sum_l (n-1-l)g_l e^{il\theta}, \tag{34}$$

$$A_3(\theta) = -\frac{n-1}{2} Im \sum_{l,k} g_l g_k e^{i(l+k)\theta}.$$

By using (30) and the above expressions we have that

$$u_2'(\theta) = A_2(\theta) = Re\sum_l (n-1-l)g_l e^{il\theta}.$$

This implies that

$$u_2(\theta) = \tilde{A}_2(\theta) = Re\sum_l \int_0^\theta (n-1-l)g_l e^{il\theta} d\theta$$

$$= Re\left[(n-1)g_0\theta + \sum_{l\neq 0} \frac{(n-1-l)}{il} g_l e^{il\theta}\right]_0^\theta$$

.

Thus, $a_2 = u_2(2\pi) = 2\pi(n-1)Reg_0$.

To compute the subsequent a_i, we will assume that $a_2 = 0$ and so $Reg_0 = 0$ (this assumption may also be read as $u_2(2\pi) = 0$, $\tilde{A}_2(2\pi) = 0$ or $\widehat{A_2}(2\pi) = \widehat{A_2}(0)$). Of course, we also must re-consider the functions $A_2, \tilde{A}_2 = u_2$, and $\widehat{A_2}$ under this restriction. As a consequence, $\widehat{A_2}(\theta)$ becomes a trigonometric polynomial without constant terms.

The second equality of (30) gives that

$$u_3(\theta) = \{A_3 + 2A_2u_2\}^{\sim}(\theta) = \tilde{A}_3(\theta) + 2\{A_2u_2\}^{\sim}(\theta) = \tilde{A}_3(\theta) + \left\{\left(\tilde{A}_2^{\,2}\right)'\right\}^{\sim}(\theta)$$

$$= \tilde{A}_3(\theta) + \tilde{A}_2^{\,2}(\theta) - \tilde{A}_2^{\,2}(0).$$

Then, imposing that $a_2 = 0$,

$$a_3 = u_3(2\pi) =$$

$$\tilde{A}_3(2\pi)$$

$$= \frac{1-n}{2} Im \left(\sum_{l+k=0} g_l g_k \xi + \sum_{l+k\neq 0} \frac{g_l g_k}{i(l+k)} e^{i(l+k)\xi} + \right)_0^{2\pi}$$
$$= \pi(1-n) Im \sum_{l+k=0} g_l\, g_k$$
$$= \pi(1-n) Im \sum_{l} g_l g_{-l}.$$
Again from (30), and using that $u_2 = \widetilde{A}_2$, we get that
$$u_4(\theta) = \{A_2 u_2^2 + 2A_2 u_3 + 3A_3 u_2\}^{\sim}\,(\theta) = \left\{A_2 \widetilde{A}_2^{\,2} + 2A_2 u_3 + 3A_3 u_2\right\}^{\sim}(\theta)$$
$$= \frac{1}{3}\,\{(\widetilde{A}_2^{\,3})' + 2[(\widetilde{A}_2\ \widetilde{A}_3)']^{\sim} + \{A_3 \widetilde{A}_2\}^{\sim}.$$
To compute a_4 we must assume that $\widetilde{A}_2\,(2\pi) = \widetilde{A}_3(2\pi) = 0$. Thus,
$$a_4 = u_4(2\pi) = \{A_3 \widetilde{A}_2\}^{\sim}(2\pi).$$
Moreover, there exists some constant C such that $\left[A_3 \widetilde{A}_2\right]^{\sim}(2\pi) = \left\{A_3 \widehat{A_2}\right\}^{\sim}(2\pi) + C\widetilde{A}_3\,(2\pi)$, and so
$$a_4 = [A_3 \widehat{A_2}\,]^{\sim}\,(2\pi).$$
This simple trick clarifies the forthcoming computations,
$$A_3 \widehat{A_2} = \left(\frac{1-n}{2} Im \sum_{l+k\neq 0} g_l g_k e^{i(l+k)\theta} \right) \left(Re \sum_{j\neq 0} \frac{n-j-1}{ij} g_j e^{ij\theta} \right)$$
$$= \frac{1-n}{4}\, Im \sum_{\Delta} g_l g_k e^{i(l+k)\theta} \frac{n-j-1}{ij} (g_j e^{ij\theta} - \overline{g_j} e^{-ij\theta}),$$
where $\Delta = \{(j,l,k): l+k \neq 0, j \neq 0\}$; and so,
$$a_4 = \frac{1-n}{4} Im \sum_{\Delta} \int_0^{2\pi} g_l g_k\, e^{i(l+k)\theta} \frac{n-j-1}{ij} \left(g_j e^{ij\theta} - \overline{g_j}\, e^{-ij\theta}\right) d\theta,$$
$$\frac{1-n}{4}\, Im \left[\sum_{j+k+l=0} \frac{n-j-1}{ij} g_l g_k g_j\, \theta - \sum_{j+k+l=0} \frac{n-j-1}{ij}\, g_l g_k \overline{g_j} \theta + \sum_{s\neq 0} \psi_s e^{is\theta} \right]_0^{2\pi}$$
$$= \frac{\pi(1-n)}{2} Im \sum_{l,k,l+k\neq 0} \frac{g_l g_k}{i(l+k)}\ (-g_{-(l+k)}(n+l+k-1)$$
$$-\overline{g_{l+k}}(n-l-k-1))$$
$$= \frac{\pi(1-n)}{2} Re \sum_{l,k,l+k\neq 0} \frac{g_l g_k}{l+k} (g_{-(l+k)}(n+l+k-1) + \overline{g_{l+k}}(n-l-k-1)),$$
as we wanted to prove.

(ii) Referring to the period constants, since $u_1(\theta) \equiv 1$, we immediately obtian the expression for b_1:
$$b_1 = \int_0^{2\pi} Im\, S_n(\theta) d\theta = -2\pi Im g_0.$$
On the other hand, from (32), and assuming that $a_i = 0$ for all i and $b_1 = 0$, we see that

$$b_2 = \int_0^{2\pi} Im\, S_n(\theta)\tilde{A}_2\,(\theta) = \int_0^{2\pi} Im\, S_n(\theta)\widehat{A_2}\,(\theta)$$
$$= -\int_0^{2\pi} \left(Im \sum_{l\neq 0} g_l e^{il\theta} \right) + \left(Re \sum_{j\neq 0} \frac{n-j-1}{ij} g_j e^{ij\theta} \right)$$
$$= -\frac{1}{2} Im \int_0^{2\pi} \sum_{j,l\neq 0} \frac{n-j-1}{ij} g_j e^{il\theta}\left(g_l e^{il\theta} - \overline{g_l}\, e^{-il\theta}\right)$$
$$= \frac{1}{2} Re \int_0^{2\pi} \sum_{j,l\neq 0} \frac{n-j-1}{j} g_j\left(g_l e^{i(j+l)\theta} - \overline{g_l}\, e^{-i(j-l)\theta}\right)$$
$$= \pi Re \sum_{l\neq 0} \frac{1}{l}\left((n+l-1)g_l g_{-l} + (n-l-1)g_l\overline{g_l}\right).$$
By using that $(n+l-1)/l - (n-l-1)/l = 2$ and that $a_3 = 0$, we get that the real part of the above expression can be removed and then
$$b_2 = -\pi \sum_{l\neq 0} \frac{1}{l}\left((n+l-1)g_l g_{-l} + (n-l-1)g_l\overline{g_l}\right)$$
$$= -\pi\left(2\sum_{l>0} g_l g_{-l} + \sum_{l\neq 0} \frac{n-l-1}{l} g_l\overline{g_l} + \right), \qquad (35)$$
which gives an expression for b_2.
As a consequence of Proposition (2.2.9), we can state the following results.
Corollary (2.2.10)[39]: Suppose that system (25) is Hamiltonian. Then the origin cannot be an isochronous centre.
Proof. We will prove that for such systems the second period−Abel constant is always positive, and hence that the origin cannot be an isochronous centre.
In the case of Hamiltonian systems we have that $Re(\partial F/\partial z) \equiv 0$ and so we get the following characterization:
$$(n+l+1)\overline{g_l} + (n-l+1\,)g_{-l} = 0.$$
By substituting the relation given by (35), we get
$$b_2 = -\pi \sum_{l\neq 0} \frac{g_l\overline{g_l}}{l}\left(\frac{-(n+l-1)(n+l+1)}{n-l+1} + (n-l-1)\right) +$$
$$= -\pi \sum_{l\neq 0} \frac{g_l\overline{g_l}}{l}\frac{-4nl}{n-l+1} = \pi \sum_{l\neq 0} \frac{4n g_l\overline{g_l}}{n-l+1} > 0.$$
Corollary (2.2.11)[39]: Assume that system (25) has a center at the origin. For r small enough let $t(r)$ denote the period function of the solution of (25) which starts at the point $z = r + 0i$. Let b_1 and b_2 be given by Proposition (2.2.9). Then the following hold:
$(i) if\ b_1 \neq 0\ then\ t(r) = 2\pi + b_1 r^{n-1} + O(r^n),$
$(ii) if\ b_1 = 0\ and\ b_2 \neq 0\ then\ t(r) = 2\pi + b_2 r^{2n-2} + O(r^{2n-1}).$
Proof. Consider $b_1 \neq 0$. By the definition of b_1, see (32), it turns out that
$$\bar{t}(\rho) = 2\pi + b_1\rho + O(\rho^2),$$
where $\bar{t}(p)$ is given in (31). From the change used to get (27), we have that

$$t(r) = \bar{t}\left(\frac{r^{n-1}}{1 + Im\big(S_n(0)\big)r^{n-1}}\right).$$

Hence the proof follows by direct substitution. The case $b_1 = 0$ and $b_2 \neq 0$ can be proved in a similar way.

The expression of the Lyapunov−Abel constants in the way given in Proposition (2.2.9) is also a good language in which prove and write more explicitly a result of Conti, see [48], which gives necessary conditions for the origin of a system of type (25) satisfying

$$\frac{d\theta}{dt} = 1$$

to be a centre. When this centre exists, it is obvious that it is an isochronous one.

In real variables, these systems admit the general form:

$$\begin{cases} \dot{x} \ = y + x \displaystyle\sum_{k=0}^{n} \ c_{n-k,k} x^{n-k} y^k \\ \dot{y} = x + y \displaystyle\sum_{k=0}^{n} \ c_{n-k,k} x^{n-k} y^k. \end{cases} \tag{36}$$

The above system expressed in complex coordinates turns out to be:

$$\dot{z} \ = iz + F_{n+1}(z, \bar{z}\,), \tag{37}$$

with

$$F_{n+1}(z, \bar{z}\,) = \frac{1}{2^n} \sum_{k=0}^{n} \ c_{n-k,k} \, z(z + \bar{z}\,)^{n-k} \, (\, z - \ \bar{z}\,)^k \, (-i)^k.$$

Expanding the binomials, we finally obtain that

$$F_{n+1}(z, \bar{z}\,) = \sum_{l+m=n+1} \ f_{l,m} z^l \bar{z}^m,$$

where

$$f_{l,m} = \frac{1}{2^n} \sum_{\Delta} \ (-1)^{j_2} \, (-i)^k \binom{n-k}{j_1} \binom{k}{j_2} \, c_{n-k,k,}$$

$$n = l + m - 1, and$$

$$\Delta = \{(k, j_1, j_2) \colon 0 \leq k \leq n, 0 \leq j_1 \leq n - k, 0 \leq j_2 \leq k \,, j_1 + j_2 = m\}.$$

Proposition (2.2.12)[39]: (i) A system of type (36) (which in complex coordinates is written as (37)) has a center at the origin if and only if its first Lyapunov−Abel constant a_2 is zero.
(ii)

$$a_2 = \begin{cases} 0 & if\ n\ is\ odd, \\ \dfrac{2\pi n}{2^n} : \displaystyle\sum_{\Delta'} \ (-1)^{j_2} \, (-i)^k \binom{n-k}{j_1} \binom{k}{j_2} \, c_{n-k,}k & if\ n\ is\ even, \end{cases} \tag{38}$$

where

$$\Delta' = \{(k, j_1, j_2) \colon 0 \leq k \leq n, 0 \leq j_1 \leq n - k, 0 \leq j_2 \leq k \,, j_1 + j_2 = n/2\}.$$

Conditions for several n obtained applying (38) are

$n = 2 \ c_{0\,,2} + c_{2,0} = 0,$

$n = 4 \ 3c_{0,4} + c_{2,2} + 3c_{4,0} = 0,$

$n = 6 \ 5c_{0,6} + c_{2,4} + c_{4,2} + 5c_{6,0} = 0,$

$n = 14 \ 429c_{0,14} + 33c_{2,12} + 9c_{4,10} + 5c_{6,8} + 5c_{8,6} + 9c_{10,4} + 33c_{12,2} \ + 429c14,0$
$= 0,$

$$n = 20\ 46{,}189c_{0,20} + 2{,}431c_{2,18} + 429c_{4,16} + 143c_{6,14} + 77c_{8,12} + 63c1_{0,10} + 77c_{12,8} + 143c_{14,6} + 429c_{16,4} + 2{,}431c_{18,2} + 46{,}189c_{20,0} = 0.$$

Proof. (i) The necessity is obvious. To prove the sufficiency, suppose that Re $g_0 = 0$. By using (34) this last equality is equivalent to

$$Re\int_0^{2\pi} S_{n+1}(v)d\theta = 0.$$

Then, integrating system (37) in polar coordinates, we will obtain that all the orbits are closed, and so that the origin is a centre. This is done in the following. From $r^2 = z\bar{z}$ and (37), it follows that

$$r\dot{r} = Re\big(\bar{z}F_n + 1(z,\bar{z})\big) = Re\left(re^{-i\theta}F_{n+1}\left(re^{i\theta}, re^{-i\theta}\right)\right),$$

$\frac{\dot{r}}{r^{n+1}} = Re(e^{-i\theta}F^{n+1}(e^{i\theta}, e^{-i\theta})) = Re(S_{n+1}(\theta))$, and, finally,

$$-\frac{1}{nr^n} = Re\int_0^{2\pi} S_{n+1}(\theta)d\theta = 0.$$

Finally we will prove (ii). In our notation, this constant is written as $a_2 = 2\pi nReg_0$ (see Proposition (2.2.9)), where g_l are defined as in (26). As we have pointed out before, if there is a center in this sytem it is isochronous. So the first period−Abel constant b_1 is always zero. Therefore (see Proposition (2.2.9)), Re $g_0 = g_0$.
From (26) we obtain that $g_0 = 0$ if n is odd, and that

$$g_0 = f_{(n+2)/2,n/2} = \frac{1}{2^n}\sum_{\Delta'} (-1)^{j_2}\ (-i)^k \binom{n-k}{j_1}\binom{k}{j_2} c_{n-k,k,}$$

where

$$\Delta' = \left\{(k, j_1, j_2) : 0 \le k \le n, 0 \le j1 \le n-k, 0 \le j_2 \le k\ , j_1 + j_2 = \frac{n}{2}\right\} = ,$$

if n is even, as we wanted to prove.

Chapter 3
Similarity Classification and Homogeneous Operators Multiplicity

We give a similarity classification of some holomorphic curves by using the K-group of its commutant algebra as an invariant. We show that the kernel function is calculated explicitly. It is proved that each of these operators is bounded, lies in the Cowen - Douglas class of $\mathbb{D}$ and is irreducible. These operators are shown to be mutually pairwise unitarily inequivalent. Here we give a different independent construction of all homogeneous operators in the Cowen-Douglas class with multiplicity free associated representation and verify that they are exactly the examples constructed previously.

Section (3.1): Holomorphic Curves

Let $\mathcal{H}$ be a complex separable Hilbert space and $Gr(n,\mathcal{H})$ denote the n-dimensional Grassmann manifold, the set of all n-dimensional subspaces of $\mathcal{H}$. If $\dim \mathcal{H} < +\infty, Gr(n,\mathcal{H})$ is a complex manifold. For Ω an open connected subset of C, we shall say that a map $f: \Omega \to Gr(n,\mathcal{H})$ is a holomorphic curve, if there exist n holomorphic $\mathcal{H}$-valued functions $\gamma_1, \gamma_2, \ldots, \gamma_n$ on Ω such that $f(\lambda) = \bigvee\{\gamma_1(\lambda), \ldots, \gamma_n(\lambda)\}$ for $\lambda \in \Omega$. If $f: \Omega \to Gr(n,\mathcal{H})$ is a holomorphic curve, then a natural n-dimensional hermitian holomorphic vector bundle E_f is induced over Ω, that is,

$$E_f = \{(x,\lambda) \in \mathcal{H} \times \Omega | x \in f(\lambda)\} \text{ and } \pi: E_f \to \Omega, where\, \pi(x,\lambda) = \lambda.$$

Given two holomorphic curves f and $\tilde{f}: \Omega \to Gr(n,\mathcal{H})$, we have two vector bundles E_f and $E_{\tilde{f}}$ over Ω. If there exists a unitary operator U on $\mathcal{H}$ such that $\tilde{f} = Uf$, then f and $\tilde{f}$ are said to be congruent and E_f and $E_{\tilde{f}}$ are equivalent. Let $\mathcal{H}$ be a complex separable Hilbert space and $L(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$, and f and $\tilde{f}$ are said to be similar equivalent if there exists an invertible operator $X \in L(\mathcal{H})$ such that $\tilde{f} = Xf$, and $E_{\tilde{f}}$ is similar equivalent to E_f [61].

In 1978, M.J. Cowen and R.G. Douglas gave the unitary classification of holomorphic curves in [61]. They introduced a class of geometry operators which are called Cowen–Douglas operators by using the concept of complex bundles.

Let Ω be a bounded connected open set; Cowen–Douglas operator of index n denoted by $B_n(\Omega)$ is the set of operators $T \in \mathcal{L}(\mathcal{H})$ satisfying:

(a) $\Omega \subset \sigma(T) = \{z \in C;\ T - z \text{ is not invertible}\}$;

(b) $ran(T - z) := \{(T - z)x;\ x \in \mathcal{H}\} = \mathcal{H}$ for z in Ω;

(c) $\bigvee_{z\in\Omega} \ker(T - z) = \mathcal{H}$; and

(d) $dim\, ker(T - z) = n$ for z in Ω.

By the definition, we can easily find a holomorphic frame $\big(e_1(z), e_2(z), \ldots, e_n(z)\big)$ such that

$$Ker(z - T) = \bigvee_{k=1}^{n} \{e_k(z), z \in \Omega\}, \qquad \forall T \in B_n(\Omega).$$

It is obvious that each Cowen–Douglas operator induces a holomorphic curve.

When $n = 1$, M.J. Cowen and R.G. Douglas define a curvature function and show that the curvature function is the unitary invariant of operators in $B_1(\Omega)$.

A natural question is posed by M.J. Cowen and R.G. Douglas in [61]. What is the similarity invariants of holomorphic curves? It is obvious that the curvature function defined by M.J. Cowen and R.G. Douglas is not the similarity invariant of holomorphic curves.

We have to find new terms to characterize the similarity invariants of holomorphic curves. Fortunately, we notice a series of great works of G. Elliott [63], [64], [65], [66], [67], G. Gong [68], [69], and Dadarlat [62] about classification of C*-algebra by using of K-theory. These works stimulated us to apply the K-theory to the exploration of similarity invariants of holomorphic curves.

We introduce the commutant of holomorphic curves first, and then we shall show that the K_0-group of the commutant of the holomorphic curve is a complete similarity invariant of the holomorphic curve.

We will introduce some basic properties of holomorphic curves. We will prove our main theorem using the properties of the holomorphic curves and K-theory and complete the similarity classification of 1-dimensional curves. We complete the similarity of some n-dimensional curves in the same way.

Let $f: \Omega \to Gr(n, \mathcal{H})$ be a holomorphic curve. If there exists no invertible operator $X \in L(\mathcal{H})$ such that $Xf: \Omega \to Gr(n, \mathcal{H})$ can be written as the orthogonal direct sum of two holomorphic curves, then we shall say that f is an indecomposable curve.

Example (3.1.1)[58]: Every 1-dimensional holomorphic curve is an indecomposable curve. In fact, if $f: \Omega \to Gr(1, \mathcal{H})$ for $\lambda \in \Omega$ is a 1-dimensional indecomposable holomorphic curve, then there exist an invertible operator X and holomorphic curves f_1, f_2 such that $Xf(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$. Since $\dim f_1(\lambda) \geqslant 1, \dim f_2(\lambda) \geqslant 1$ for $\lambda \in \Omega$, then this is a contradiction as f is a 1-dimensional holomorphic curve.

Definition (3.1.2)[58]: For a holomorphic curve $f: \Omega \to Gr(n, \mathcal{H})$, we use $\mathcal{A}'(f)$ to denote the commutant of f which is the set $\{A \in \mathcal{L}(\mathcal{H}) \mid Af\ (\lambda) \subseteq f(\lambda), \forall \lambda \in \Omega\}$. We can see it is a unital Banach algebra and $rad\ \mathcal{A}'(f)$ denotes the Jocaboson radical of $\mathcal{A}'(f)$.

Theorem (3.1.3)[58]: A holomorphic curve $f: \Omega \to Gr(n, \mathcal{H})$ is indecomposable if and only if there exist no nontrivial idempotents in $\mathcal{A}'(f)$.

Proof. ($\Rightarrow$) If $P \in \mathcal{A}'(f)$ is a nontrivial idempotent, then $f(\lambda) = Pf(\lambda) + (I - P)f(\lambda)$ for $\lambda \in \Omega$ and there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $XP\mathcal{H} = P\mathcal{H}$ and $X(I - P)\mathcal{H} = (P\mathcal{H})^\perp$. So we can see that $Xf(\lambda) = XPf\ (\lambda) \oplus X(I - P)f\ (\lambda)$. This contradicts the indecomposition of f.

($\Leftarrow$) If $Xf(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$, then $f(\lambda) = X^{-1}f_1(\lambda) + X^{-1}f_2(\lambda)$. Note that X^{-1} is invertible and $f_1(\lambda)$ and $f_2(\lambda)$ are orthogonal; we can suppose that

$$f_1(\lambda) = \bigvee\{e_1(\lambda), \ldots, e_m(\lambda)\}, \qquad f_2(\lambda) = \bigvee\{e_{m+1}(\lambda), \ldots, e_n(\lambda)\}.$$

Since $\langle e_i(\lambda), e_j(\lambda)\rangle = 0$ for $i \neq j$, $1 \leqslant i \leqslant m, m + 1 \leqslant j \leqslant n$, if $|\lambda - \lambda_0|$ is near to zero enough, we have $\langle e_i(\lambda), e_j(\lambda_0)\rangle = 0$ for $i \neq j$. By the property of the analytic function, we have $\bigvee_{\lambda \in \Omega} X^{-1}f_1(\lambda) \dotplus \bigvee_{\lambda \in \Omega} X^{-1}f_2(\lambda) = \mathcal{H}$, where $\dotplus$ denotes the algebra direct sum. For $x \in \mathcal{H}, x = x_1 + x_2, x_1 \in \bigvee_{\lambda \in \Omega} X^{-1}f_1(\lambda), x_2 \in \bigvee_{\lambda \in \Omega} X^{-1}f_2(\lambda)$. Let $P\, x = x_1$, then $P \in \mathcal{A}'(f)$ is a nontrivial idempotent.

Theorem (3.1.4)[58]: Let $f: \Omega \to Gr(n, \mathcal{H})$ be a holomorphic curve, $P \in \mathcal{A}'(f)$ is an idempotent, then $Pf: \Omega \to Gr(m, P\mathcal{H})$ is still a holomorphic curve, where $m = \dim P\big(f(\lambda)\big)$ for $\lambda \in \Omega$ and P is minimal if and only if Pf is indecomposable.

Proof. Let $f(\lambda) = \bigvee\{\gamma_1(\lambda), \ldots, \gamma_n(\lambda)\}, \lambda \in \Omega$ and $Pf(\lambda) = \bigvee\{P\gamma_1(\lambda), \ldots, P\gamma_n(\lambda)\}$. Since $P\gamma_i(\lambda) \subseteq f(\lambda)$ for $i = 1, 2, \ldots, n$, then there exists Ω_0 such that $\{\gamma_i'(\lambda)\}_{i=1}^m$ for $\lambda \in \Omega_0$ to be the frames of E_{Pf} and satisfy $Pf(\lambda) = \bigvee\{\gamma_1'\ (\lambda), \ldots, \gamma_m'(\lambda)\}$, where $m = \dim Pf(\lambda)$ for $\lambda \in \Omega$ and $\Omega - \Omega_0$ is a finite set. So we assume that $m = \dim P\big(f(\lambda)\big)$ for $\lambda \in \Omega$ and $Pf: \Omega \to Gr(m, P\mathcal{H})$ is still a holomorphic curve.

And if there exists a $P' \in \mathcal{A}'(f), P'\mathcal{H} \subseteq P'\mathcal{H}$, then $Pf(\lambda) = PP'f(\lambda) + P\,(I - P')f(\lambda)$. If Pf is decomposable, then there is an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $XPf(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$ for $\lambda \in \Omega$. Similarly to the above proof, it is easy to see that P is not a minimal idempotent of $\mathcal{A}'(f)$.

Theorem (3.1.5)[58]: Let $f: \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve, then the following are equivalent:

(i) there exist m minimal idempotents $P_1, P_2, \dots, P_m \in \mathcal{A}'(f)(m \leqslant n)$ such that $P_iP_j = 0$ and $\sum_{i=1}^{m} P_i = I_{\mathcal{H}}$ (identity operator on $\mathcal{H}$);

(ii) there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that Xf can be written as an orthogonal direct sum of m indecomposable curves.

Proof. (i) $\Rightarrow$ (ii). Since $P_i \in \mathcal{A}'(f), P_iP_j = 0, f(\lambda) = P_1f(\lambda) + P_2f(\lambda) + \cdots + P_mf(\lambda)$ for $\lambda \in \Omega$ and $\mathcal{H} = P_1\mathcal{H} + P_2\mathcal{H} + \cdots + P_m\mathcal{H}$, then we can find an invertible operator $X \in \mathcal{L}(\mathcal{H})$ to satisfy $XP_i\mathcal{H}_i = \mathcal{H}_i'$ and $\mathcal{H} = \mathcal{H}_1' \oplus \mathcal{H}_2' \oplus \cdots \oplus H_m'$. So $Xf(\lambda) = XP_1f(\lambda) \oplus XP_2f(\lambda) \cdots XP_mf\,(\lambda)$ and $XP_if(\lambda) \in \mathcal{H}_i'$ for $\lambda \in \Omega$.

(ii) $\Rightarrow$ (i). If (ii) is satisfied, then by the proof of Theorem (3.1.3), there exist $\{f_i'(\lambda)\}_{i=1}^{m}$ indecomposable which satisfy $f(\lambda) = f_1'(\lambda) + \cdots + f_m'(\lambda)$ for $\lambda \in \Omega$ and

$$\bigvee_{\lambda \in \Omega} f_i'(\lambda) \cap \bigvee_{\lambda \in \Omega} f_j'(\lambda) = \emptyset, \qquad i \neq j.$$

So there exists $P_i: \mathcal{H} \to \bigvee_{\lambda \in \Omega} f_i'(\lambda)$ such that $P_i \in \mathcal{A}'(f), P_i^2 = P_i, P_iP_j = 0$ for $i \neq j$ and $\sum_{i=1}^{m} P_i = I_{\mathcal{H}}$.

Definition (3.1.6)[58]: If $f: \Omega \to Gr(n, \mathcal{H})$ satisfies any condition of Theorem (3.1.5), then we say f has a finite decomposition.

Let $\{P_1, P_2, \dots, P_m\}$ and $\{Q_1, Q_2, \dots, Q_n\}$ are arbitrary two decompositions of f and if following are satisfied:

(i) $m = n$;

(ii) there exists an invertible operator $X \in \mathcal{A}'(f)$ and a permutation $\Pi \in S_n$ such that $XQ_{\Pi(i)}X^{-1} = P_i$ for $1 \leqslant i \leqslant n$,

then we say f has a unique decomposition up to similarity.

Proposition (3.1.7)[58]: Let $\Omega_0 \subseteq \Omega$ be a bounded connected open set of Ω; if $f: \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve which satisfies $\bigvee_{\lambda \in \Omega} f(\lambda) = \mathcal{H}$, then $\bigvee_{\lambda \in \Omega_0} f(\lambda) = \mathcal{H}$.

Proof. We only need to prove that if each element x in $\mathcal{H}$ satisfies $\langle x, \bigvee_{\lambda \in \Omega_0} f(\lambda) \rangle = 0$, then $x = 0$.

Let $\{e_1(\lambda), e_2(\lambda), \dots, e_n(\lambda)\}$ for $\lambda \in \Omega$ be a holomorphic frame of Ef. Note that $\langle e_i(\lambda), x \rangle$ for $i = 1, 2, \dots, n$ is analytic on Ω. Hence $\langle e_i(\lambda), x \rangle = 0$ for $\lambda \in \Omega_0$ implies that $\langle e_i(\lambda), x \rangle = 0, \lambda \in \Omega$. This shows that $x = 0$.

Proposition (3.1.8)[58]: For a holomorphic curve $f: \Omega \to Gr(n, \mathcal{H}), f(\lambda) = \bigvee\{e_1(\lambda), \dots, e_n(\lambda)\}$ for $\lambda \in \Omega$. Let $f^{(k)}(\lambda)$ denote the set $\bigvee\left\{e_1^{(k)}(\lambda), \dots, e_n^{(k)}(\lambda)\right\}$ for $k = 1, 2, \dots$ and $\lambda \in \Omega$, where $e_i^{(k)}(\lambda)$ denotes the k derivate of $e_i(\lambda)$ at λ. If $\bigvee_{\lambda \in \Omega} f(\lambda) = \mathcal{H}$, then $\bigvee_{k=1}^{\infty} f^{(k)}(\lambda_0) = \mathcal{H}$, where $\lambda_0 \in \Omega$.

Proof. Since $f(\lambda) = \bigvee\{e_1(\lambda), \dots, e_n(\lambda)\}, \lambda \in \Omega$, then there exists a neighborhood Δ_0 of λ_0 such that

$$e_i(\lambda) = \sum_{k=0}^{\infty} \frac{e_i^{(k)}(\lambda_0)}{k!}(\lambda - \lambda_0) \in \bigvee_{k=1}^{\infty} f^{(k)}(\lambda_0), \qquad \forall \lambda \in \Delta_0, i = 1, 2, \dots, n.$$

By the proof of Proposition (3.1.7), it is not difficult to prove that

$$\bigvee_{\lambda\in\Delta_0} f(\lambda) = \bigvee_{\lambda\in\Delta_0}\{e_1(\lambda),\dots,e_n(\lambda)\} = \mathcal{H} \subseteq \bigvee_{k=1}^{\infty} f^{(k)}(\lambda_0).$$

So we get $\bigvee_{k=1}^{\infty} f^{(k)}(\lambda_0) = \mathcal{H}$.

M.J. Cowen and R.G. Douglas in [61] give a character of the commutant of Cowen–Douglas operators. In the following, we shall imitate their proof to describe the commutant of holomorphic curve f.

Proposition (3.1.9)[58]: Let $f: \Omega \to Gr(n,\mathcal{H})$ and $\bigvee_{\lambda\in\Omega} f(\lambda) = \mathcal{H}$, then the map $\Gamma_f: \mathcal{A}'(f) \to \mathcal{H}^{\infty}_{\mathcal{L}(E_f)}(\Omega)$ is a contractive monomorphism, where $\Gamma_f(X)(\lambda) = X|_{f(\lambda)}, X \in \mathcal{A}'(f)$ and $\mathcal{H}^{\infty}_{\mathcal{L}(E_f)}(\Omega)$ denotes the collection of bounded bundle endomorphisms on E_f.

Proof. If $X \in \mathcal{A}'(f)$, then $Xf(\lambda) \subseteq f(\lambda)$. If $e(\lambda)$ is a holomorphic cross-section of E_f or $e(\lambda) \in f(\lambda)$ then so is $Xe(\lambda)$. Let

$$\Gamma_f X(x,\lambda) = (X(x),\lambda), \qquad X \in \mathcal{A}'(f), \qquad x \in f(\lambda).$$

Since $\|(\Gamma_f X)(\lambda)\| = \|X|_{f(\lambda)}\| \leqslant \|X\|$, then $\Gamma_f(X) \in \mathcal{H}^{\infty}_{\mathcal{L}(E_f)}(\Omega)$, and Γ_f is contractive. By $\bigvee_{\lambda\in\Omega} f(\lambda) = \mathcal{H}, \Gamma_f$ is one-to-one.

Now suppose Φ is an element of $\mathcal{H}^{\infty}_{\mathcal{L}(E_f)}(\Omega)$ for which there exists a bounded operator $X \in \mathcal{A}'(f)$ such that

$$\Gamma_f(X)(\lambda) = X|_{f(\lambda)}, \qquad X \in \mathcal{A}'(f).$$

If $\{e_1(\lambda), e_2(\lambda),\dots,e_n(\lambda)\}$ is a holomorphic frame of E_f, then by differentiating we obtain

$$Xe_i'(\lambda) = (\Phi(\lambda)e_i(\lambda))' = \Phi(\lambda)e_i'(\lambda) + \Phi'(\lambda)e_i(\lambda),$$
$$Xe_i''(\lambda) = (\Phi(\lambda)e_{i(\lambda)})'' = \Phi(\lambda)e_i''(\lambda) + 2\Phi'(\lambda)e_i'(\lambda) + \Phi''(\lambda)e_i(\lambda),$$
$$\vdots$$
$$Xe_i^{(N)}(\lambda) = (\Phi(\lambda)e_i(\lambda))^{(N)} = \Phi(\lambda)e_i^{(N)}(\lambda) + \cdots + \Phi^{(N)}(\lambda)e_i(\lambda).$$

In other words the block matrix for $X|_{f^{(k)}(\lambda)}$ relative to the basis $\left\{e_i^{(j)}(\lambda)\right\}_{i=1}^{n}{}_{j=1}^{N}$ is

$$\begin{bmatrix} \Phi(\lambda) & \Phi'(\lambda) & \Phi''(\lambda) & \cdots & \Phi^{(N)}(\lambda) \\ 0 & \Phi(\lambda) & 2\Phi'(\lambda) & \cdots & N\Phi^{(N-1)}(\lambda) \\ 0 & 0 & \Phi(\lambda) & \cdots & \frac{N(N-1)}{2}\Phi^{(N-2)}(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Phi(\lambda) \end{bmatrix}.$$

For $\Phi \in \mathcal{H}^{\infty}_{\mathcal{L}(E_f)}(\Omega)(\Omega)$, the following are equivalent:

(i) $\Phi = \Gamma_f(X)$ for some X in $\mathcal{A}'(f)$;

(ii) $\sup\{\|\Phi_N(\lambda)\|: \lambda \in \Omega, N = 0,1,2,\dots\} = C_1 < \infty$; and

(iii) $\sup\{\|\Phi_N(\lambda_0)\|: N = 0,1,2,\dots\} = C_2 < \infty$ for some λ_0 in Ω.

Moreover, if these conditions hold, then $\|X\| = C_1 = C_2$ and we can easily get the following lemma.

Lemma (3.1.10)[58]: Let $f: \Omega \to Gr(n,\mathcal{H})$ be a holomorphic curve, $\Omega_0 \subseteq \Omega$ be a bounded connected open set of Ω. If we set $f': \Omega_0 \to Gr(n,\mathcal{H})$ and $f'(\lambda) = f(\lambda)$ for $\lambda \in \Omega_0$, then $\mathcal{A}'(f) = \mathcal{A}'(f')$.

Proof. It is obvious that $\mathcal{A}'(f) \subseteq \mathcal{A}'(f')$. We only need to prove that $\mathcal{A}'(f') \subseteq \mathcal{A}'(f)$. Let $\lambda_0 \in \Omega_0$ and $\{e_i(\lambda)\}_{i=1}^{n}$ be the frame of E_f. If λ is near to λ_0, we can get that

$$e_i(\lambda) = \sum_{k=1}^{\infty} \frac{e_i^{(k)}(\lambda_0)}{k!}(\lambda - \lambda_0).$$

By the above proof, we can assume $X\big(e_i(\lambda)\big) = \Phi(\lambda)e_i(\lambda), \forall\lambda \in \overline{\Omega_0}$, where $X \in \mathcal{A}'(f')$ and $\Phi(\lambda) \in \mathcal{H}^{\infty}_{\mathcal{L}(E_f)}(\Omega)$. By the above result, we obtain

$$Xe_i(\lambda) = X\left(\sum_{k=1}^{\infty} \frac{e_i^{(k)}(\lambda_0)}{k!}(\lambda - \lambda_0)^k\right) = \left(\sum_{k=1}^{\infty} \frac{\Phi(k)(\lambda_0)}{k!}(\lambda - \lambda_0)^k\right)e_i(\lambda),$$

where $\lambda_0 \in \partial\Omega_0$ and λ is contained by some neighborhood of λ_0. So we can see that $X \in \mathcal{A}'(f), \forall X \in \mathcal{A}'(f')$.

In order to understand our work well, we will introduce some notations of K_0-theory of a Banach algebra.

Let $\mathcal{A}$ be a unital Banach algebra, and e and f be idempotents in $\mathcal{A}$; we call e and f algebraically equivalent, if there exist $x, y \in \mathcal{A}$ with $xy = e, yx = f$ (denoted by $e \sim_a f$). The algebraic equivalence class containing p is denoted by [p]. We call e and f similar if there exists an invertible operator z in A such that $zez^{-1} = f$ (denoted by $e \sim f$). Obviously, $e \sim_a f$ and $e \sim f$ are equivalence relations.

For a holomorphic curve, $f: \Omega \to Gr(n, \mathcal{H})$, let $M_k\big(\mathcal{A}'(f)\big)$ be the collection of $k \times k$ matrices with entries from $\mathcal{A}'(f)$.

$$M_\infty\big(\mathcal{A}'(f)\big) = \bigcup_{k=1}^{\infty} M_k\big(\mathcal{A}'(f)\big).$$

Let $Proj\big(\mathcal{A}'(f)\big)$ be the set of algebraic equivalence classes of idempotents in $\mathcal{A}'(f)$ and $\bigvee\big(\mathcal{A}'(f)\big) = Proj\left(M_\infty\big(\mathcal{A}'(f)\big)\right)$. Note that $\bigvee\left(M_n\big(\mathcal{A}'(f)\big)\right)$ is isomorphic to $\bigvee\big(\mathcal{A}'(f)\big)$ (denoted by $\bigvee\left(M_n\big(\mathcal{A}'(f)\big)\right) \cong \bigvee\big(\mathcal{A}'(f)\big)$). If p, q are idempotents in $Proj\big(\mathcal{A}'(f)\big)$, then $p \sim_{st} q$ if and only if $p \oplus r \sim_a q \oplus r$ for some idempotent r in $Proj\big(\mathcal{A}'(f)\big)$. The relation $\sim_{st}$ is called stable equivalence. $K_0\big(\mathcal{A}'(f)\big)$ is the Grothendieck group of $\bigvee\big(\mathcal{A}'(f)\big)$ (cf. [60], [71]).

A pair (G, G^+) is called an ordered Abelian group, if G is an Abelian group, G^+ is a subset of G, and

(i) $G^+ + G^+ \subseteq G^+$;
(ii) $G^+ \cap (-G^+) = \{0\}$;
(iii) $G^+ - G^+ = G$.

Define a relation '$\leqslant$' on G by $x \leqslant y$, if $y - x$ belongs to G^+.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and α be a homomorphism from $\mathcal{A}$ into $\mathcal{B}$. Then the map $\alpha_n: M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is given by

$$\alpha_n \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}_{n\times n} = \begin{bmatrix} \alpha(a_{11}) & \cdots & \alpha(a_{1n}) \\ \vdots & \ddots & \vdots \\ \alpha(a_{n1}) & \cdots & \alpha(a_{nn}) \end{bmatrix}_{n\times n}$$

and there is a homomorphism α_* induced by α from $K_0(\mathcal{A})$ into $K_0(\mathcal{B})$.

Let $GL_n\big(\mathcal{A}'(f)\big)$ denote the invertible elements of $M_n\big(\mathcal{A}'(f)\big)$, and $GL_n\big(\mathcal{A}'(f)\big)_0$ denote the connected components of the identity. Since the group is locally path-connected, i.e. the group of the path components of the identity, it is an open subgroup. We embed $GL_n\big(\mathcal{A}'(f)\big)$ into $GL_{n+1}\big(\mathcal{A}'(f)\big)$ by $x \to diag(x, 1)$. Let $GL_\infty\big(\mathcal{A}'(f)\big) =$

$\lim\limits_{\rightarrow} GL_n(\mathcal{A}'(f))$ (the inductive limit). $GL_\infty(\mathcal{A}'(f))$ is a topological group with the inductive limit topology. The embedding of $GL_n(\mathcal{A}'(f))$ into $GL_{n+1}(\mathcal{A}'(f))$ maps $GL_n(\mathcal{A}'(f))_0$ into $GL_{n+1}(\mathcal{A}'(f))_0$ and $GL_\infty(\mathcal{A}'(f))_0 = \lim\limits_{\rightarrow} GL_n(\mathcal{A}'(f))_0$.

Let

$$K_1(\mathcal{A}'(f)) = GL_\infty(\mathcal{A}'(f))/GL_\infty(\mathcal{A}'(f))_0 = \lim_{\rightarrow}[GL_n(\mathcal{A}'(f))/GL_n(\mathcal{A}'(f))_0].$$

The suspension of $\mathcal{A}'(f)$, denoted by $S\mathcal{A}'(f)$, is the set $\{f: R \to \mathcal{A}'(f) \mid f$ is continuous and $\lim\limits_{x\to\infty} \|f(x)\| = 0\}$. With pointwise operations and the sup norm, $S\mathcal{A}'(f)$ is a Banach algebra. Then $K_1(\mathcal{A}'(f))$ is naturally isomorphic to $K_0(S\mathcal{A}'(f))$, i.e. there is an isomorphism $\theta_{\mathcal{A}'(f)} : K_1(\mathcal{A}'(f)) \to K_0(S\mathcal{A}'(f))$ such that whenever $\phi: \mathcal{A}'(f) \to \mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd} K_1(\mathcal{A}'(f)) \arrow[r, "\phi_*"] \arrow[d, "\theta_{\mathcal{A}'(f)}"'] & K_1(\mathcal{B}) \arrow[d, "\theta_{\mathcal{B}}\downarrow"'] \\ K_0(S\mathcal{A}'(f)) \arrow[r] & K_0(S\mathcal{B}). \end{tikzcd}$$

(In the language of category theory, θ gives an invertible natural transformation from K_1 to K_0S, where $\mathcal{B}$ is another unital Banach algebra.)

The main theorems are the following:

In order to prove the theorem, we will introduce the following notations and results.

Lemma (3.1.11)[58]: Let $e, f: \Omega \to Gr(n, \mathcal{H})$ be two holomorphic curves and $\Phi: \mathcal{A}'(e) \cong \mathcal{A}'(f)$, then $\{P_i\}_{i=1}^n$ is a unit finite decomposition of e if and only if $\{\Phi(P_i)\}_{i=1}^n$ is a unit decomposition of f. In particular, if $e \sim f$ then $\mathcal{A}'(e) \cong \mathcal{A}'(f)$.

Proof. Since Φ is an isomorphism satisfying $0 = \Phi(P_iP_j) = \Phi(P_i)\Phi(P_j)$ for $i \neq j$ and $\sum_{i=1}^n \Phi(P_i) = I$, then we need only to prove that $\Phi(P_i)f$ is indecomposable. Otherwise, there exist two nonzero idempotents Q_1 and Q_2 in $\mathcal{A}'(f)$, so that $Q_2Q_1 = Q_1Q_2 = 0$ and $\Phi(P_i) = Q_1 + Q_2$. Note that $\Phi^{-1}(Q_1), \Phi^{-1}(Q_2)$ are two nonzero idempotents in $\mathcal{A}'(e)$ and $P_i = \Phi^{-1}(Q_1) + \Phi^{-1}(Q_2)$. This is a contradictions as P_ie is indecomposable.

If e is similar to f, then there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ satisfying $Xe(\lambda) = f(\lambda)$ for $\lambda \in \Omega$. Define a mapping $\Phi: \Phi(T) = XTX^{-1}$ for $T \in \mathcal{A}'(e)$. It is clear that Φ is an isomorphism from $\mathcal{A}'(e)$ to $\mathcal{A}'(f)$.

Lemma (3.1.12)[58]: Let $e: \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve and $P_1, P_2 \in \mathcal{A}'(e)$ are idempotent operators. If $P_1 \sim (\mathcal{A}'(e))P_2$, then $P_1e \sim P_2e$.

Proof. Since $P_1 \sim (\mathcal{A}'(e))P_2$, then there exists an invertible operator $X \in GL(\mathcal{A}'(e))$ such that $XP_1X^{-1} = P_2$ and $XP_1 = P_2X, XP_1e(\lambda) = P_2Xe(\lambda) = P_2XX^{-1}e(\lambda)$. Note that $X^{-1} \in \mathcal{A}'(e)$; we obtain $P_2e(\lambda) = P_2XX^{-1}e(\lambda) = XP_1e(\lambda)$ for $\lambda \in \Omega$. That is $P_1e \sim P_2e$.

Lemma (3.1.13)[58]: Let $e: \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve and $\bigvee_{\lambda\in\Omega} e(\lambda) = \mathcal{H}, \{P_i\}_{i=1}^n, \{Q_i\}_{i=1}^n$ are two unit finite decompositions of e. If there exists an operator $X_i \in GL(\mathcal{L}(P_i\mathcal{H}, Q_i\mathcal{H}))$ such that

$$X_iP_ie(\lambda) = Q_ie(\lambda), \qquad \forall\lambda \in \Omega, i = 1, 2, \ldots, n,$$

then $X = X_1 \dotplus X_2 \dotplus \cdots \dotplus X_n \in GL(A'(e))$.

Proof. Since $Xe(\lambda) = (X_1 \dotplus X_2 \dotplus \cdots \dotplus X_n)(P_1e_1(\lambda) \dotplus P_2e_2(\lambda) \dotplus \cdots \dotplus P_ne_n(\lambda))$, and $X_i \in L(P_i\mathcal{H}, Q_i\mathcal{H})$, then

$$Xe(\lambda) = (X_1P_1e(\lambda) \dotplus X_2P_2e(\lambda) \dotplus \cdots \dotplus X_nP_ne(\lambda) = Q_1e(\lambda) \dotplus Q_2e(\lambda) \dotplus \cdots \dotplus Q_ne(\lambda)$$

and $X_i^{-1}Q_ie(\lambda) = P_ie(\lambda)$ for $\lambda \in \Omega$. So $X \in \mathcal{A}'(e)$ and $X \in GL(\mathcal{A}'(e))$.

Lemma (3.1.14)[58]: Suppose that $\{P_1,\dots,P_m,P_{m+1},\dots,P_n\}$ and $\{Q_1,\dots,Q_m,Q_{m+1},\dots,Q_n\}$ are two sets of idempotent operators in $\mathcal{A}'(e)$, where $e:\Omega\to Gr(n,\mathcal{H})$ is a holomorphic curve. If there exist $X,Y\in GL(\mathcal{A}'(e))$ and a permutation $\Pi\in S_n$ satisfying

(i) $XP_iX^{-1}=Q_i, 1\leqslant i\leqslant m$;

(ii)$Y\,P_iY^{-1}=Q_{\Pi(i)}, 1\leqslant i\leqslant n$,

then for $Q_r, m<r'\leqslant n$, there exist $P_{r'}, m<r\leqslant n$, and Z_r, a finite product of Y and X, such that $Z_rQ_rZ_r^{-1}=P_{r'}$. Moreover, $\{P_{r'}\}_{r=m+1}^{n}$ is exactly a rearrangement of $\{P_{r'}\}_{r'=m+1}^{n}$.

Proof. Given $Q_r, m<r\leqslant n$, it follows from (ii) that there exists $P_{j_1}, 1\leqslant j_1\leqslant n$, such that $YQ_rY^{-1}=P_{j_1}$. If $m<j_1\leqslant n$, then we set $Z_r=Y$ and $P_{r'}=P_{j_1}$. If $1\leqslant j_1\leqslant m$, then by (ii) there exists an operator $Q_{j_1}, j_1=r$, such that

$$XYQ_rY^{-1}X^{-1}=Q_{j_1}.$$

By (ii), $YQ_{j_1}Y^{-1}=P_{j_2}$ for some j_2. If $m<j_2\leqslant n$, then we set

$$Z_r=YXY,\qquad P_{r'}=P_{j_2}.$$

If $1\leqslant j_2\leqslant m$, it is obvious that $j_1\neq j_2$. Otherwise,

$$Q_{j_1}=Y^{-1}P_{j_2}Y=Y^{-1}P_{j_1}Y=Q_r,$$

which contradicts $m<r\leqslant n$. Using (ii) again, we can find P_{j_3} such that

$$YQ_{j_2}Y^{-1}=P_{j_3}.$$

Similarly, $j_3\notin\{j_1,j_2\}$.If $m<j_3\leqslant n$, then we set $Z_r=YXYXY, P_{r'}=P_{j_3}$. Or we can continue the procedure above. Since n is finite, after $s\leqslant m+1$ steps we will force $P_s\in\{P_{m+1},\dots,P_n\}$. Set $P_{r'}=P_{j_s}, Z_r=YXY\cdots XY$,where X appears S times. Then $Z_rQ_rZ_r^{-1}=P_{j_s}$. We claim that if $r_1\neq r_2$, where $r_1,r_2\in\{m+1,\dots,n\}$, then $j_{s_1}\neq j_{s_2}$. Otherwise, there exist $Z_{r_1}=YXY\cdots YXY$ (X appears s_1 times) and $Z_{r_2}=YXY\cdots YXY$ (X appears s_2 times) such that

$$Z_{r_1}Q_{r_1}Z_{r_1}^{-1}=Z_{r_2}Q_{r_2}Z_{r_2}^{-1}.$$

Without loss of generality, we may assume that $s_1\geqslant s_2$.If $s_1>s_2$, then

$$Z_{r_2}^{-1}Z_{r_1}Q_{r_1}Z_{r_1}^{-1}Z_{r_2}=Q_{r_2}\in\{Q_{m+1},\dots,Q_n\}.$$

Note that $Z_{r_2}^{-1}Z_{r_1}=XY\cdots XY$ (X appears $j_{s_1}-j_{s_2}$ times). Set

$$R=YXY\cdots XY,$$

where X appears $j_{s_1}-j_{s_2}-1$ times. By the procedure of the choice, we have $RQ_{r_1}R^{-1}\in\{P_1,P_2,\dots,P_m\}$. Thus

$$XRQ_{r_1}R^{-1}X^{-1}\in\{Q_1,Q_2,\dots,Q_m\}.$$

But $XRQ_{r_1}R^{-1}X^{-1}=Z_{r_2}^{-1}Z_{r_1}Q_{r_1}Z_{r_1}^{-1}Z_{r_2}=Q_{r_2}\in\{Q_{m+1},\dots,Q_n\}$. This is a contradiction. Thus $s_1=s_2$. But if $s_1=s_2$, we can easily prove that $Q_{r_1}=Q_{r_2}$, which is also a contradiction. This completes the proof of our claim and the lemma.

By the similar argument of Lemma (3.1.14), we can prove the following result.

Lemma (3.1.15)[58]: Let

$$e:\Omega\to Gr(n,\mathcal{H}),\left\{P_1,P_2,\dots,P_{m_1},\dots,P_{m_{k-1}-1},\dots,P_{m_k},P_{m_{k+1}},\dots,P_n\right\}$$

and

$$\left\{Q_1,Q_2,\dots,Q_{m_1},\dots,Q_{m_{k-1}-1},\dots,Q_{m_k},Q_{m_{k+1}},\dots,Q_n\right\}$$

be two sets of idempotent operators of $\mathcal{A}'(e)$. If there exist $X_1,X_2,\dots,X_k,Y\in GL\left(\mathcal{A}'(e)\right)$ and a permutation $\Pi\in S_n$ satisfying

$$X_i P_j X_i^{-1} = Q_j, \qquad m_i + 1 \leqslant j \leqslant m_{i+1}, i = 0,1,\ldots,k-1, m_0 = 0,$$
and
$$Y^{-1} P_j Y = Q_{\Pi(i)}, 1 \leqslant i \leqslant n,$$
then for each $r, m_k < r < n$,there exists Z_r, a finite product of $\{Y, X_1, \ldots, X_k\}$, so that $\{Z_r Q_r Z_r^{-1}\}_{r=m_k+1}^{n}$ is exactly a rearrangement of $\{P_r\}_{r=m_k+1}^{n}$.

Lemma (3.1.16)[58]: Suppose that $\{P_1, \ldots, P_m, P_{m+1}, \ldots, P_n\}$ and $\{Q_1, \ldots, Q_{m+1}, \ldots, Q_n\}$ are two unit decompositions of e. If the following properties are satisfied:

(i) for each P_i, there exists an $X_i \in GL(P_i\mathcal{H}, Q_i\mathcal{H})$ satisfying $X_i P_i e(\lambda) = Q_i e(\lambda)$ for $\lambda \in \Omega, 1 \leqslant i \leqslant m$;

(ii)there exists $Y \in GL(\mathcal{A}'(e))$ and a permutation $\Pi \in S_n$ satisfying $Y^{-1} P_i Y = Q_{\Pi(i)}$;

then given $Q_r, r \in \{m+1, \ldots, n\}$, there exist $r' \in \{m+1, \ldots, n\}$ and $Z_r \in GL(Q_r\mathcal{H}, P_{r'}\mathcal{H})$ satisfying $Z_r Q_r e(\lambda) = P_{r'} e(\lambda)$ for $\lambda \in \Omega$. Furthermore, if $r_1 = r_2$, then $r_1 = r_2$.

Proof. Given $r \in \{m+1, \ldots, n\}$, by (ii) of the lemma, there exists an operator $P_{j_1} \in \{P_i\}_{i=1}^{n}$ such that $Y Q_r Y^{-1} = P_{j_1}$. If $m < j_1 \leqslant n$, set $Z_r = Y|_{Q_r\mathcal{H}}$. Otherwise, it follows from $(Y Q_r Y^{-1}) e(\lambda) = P_{j_1} e(\lambda)$ and (i) that $X_{j_1} P_{j_1} e(\lambda) = Q_{j_1} e(\lambda)$. Using condition (ii) again, we can find $j_2 \in \{1,2,\ldots,n\}$ such that $Y Q_{j_1} Y^{-1} = Q_{j_2}$. Clearly, $j_1 \neq j_2$. If $j_2 \in \{m+1, \ldots, n\}$, set $Z_r = Y|_{Q_j\mathcal{H}}\, X_{j_1} Y|_{Q_r\mathcal{H}}, P_{r'} = P_{j_2}$. Thus $Z_r Q_r e(\lambda) = P_{r'} e(\lambda)$. Otherwise, by the similar arguments used in the proof of Lemma (3.1.14), after finite steps, we can find $P_{r'} \in \{P_k\}_{k=m+1}^{n}$ such that $Z_r Q_r e(\lambda) = P_{r'} e(\lambda)$. Similarly, we can prove that $r_1' \neq r_2'$ if $r_1 \neq r_2$.

Lemma (3.1.17)[58]: Let $e: \Omega \to Gr(n, \mathcal{H})$ and suppose e has a unique finite decomposition up to similarity, then for an arbitrary idempotent P in $\mathcal{A}'(e)$, P e has a unique finite decomposition up to similarity.

Proof. Since e has a unique finite decomposition up to similarity, P e has a finite decomposition and all the cardinalities must be the same.

Let $\{P_i\}_{i=1}^{m}$ and $\{Q_i\}_{i=1}^{m}$ be two unit decompositions of P e and $\{P_i\}_{i=m+1}^{n}$ be a unit decomposition of $(I-P)e$, then $\{\{P_i\}_{i=1}^{m}, \{P_i\}_{i=m+1}^{n}\}$ and $\{\{Q_i\}_{i=1}^{m}, \{P_i\}_{i=m+1}^{n}\}$ are two unit decompositions of e. By the uniqueness, we can find a $Y \in GL(\mathcal{A}'(e))$ such that
$$\{Y P_i Y^{-1}\} = \{Q_1, \ldots, Q_m, P_{m+1}, \ldots, P_n\}.$$
By Lemma (3.1.16), we can find $Z_i \in GL(\mathcal{L}(Q_i\mathcal{H}, P_i\mathcal{H}))$ and a permutation $\Pi \in S_n$ satisfying
$$Z_i Q_i e(\lambda) = P_{\Pi(i)} e(\lambda), \qquad 1 \leqslant i \leqslant m.$$
Set $Z = Z_1 \dotplus Z_2 \dotplus \cdots \dotplus Z_n, Z_k = I|_{P_k\mathcal{H}}, m+1 \leqslant k$. By Lemma (3.1.13), $Z \in GL(\mathcal{A}'(e))$ and $PZ \in GL(\mathcal{A}'(Pe))$. Note that $(Z|P\mathcal{H}) Q_i (Z|P\mathcal{H})^{-1} = P_{\Pi(i)}$ for $1 \leqslant i \leqslant m$.

Lemma (3.1.18)[58]: Let $e: \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve. If e has a unique finite decomposition up to similarity, P,Q in $\mathcal{A}'(e)$ are two idempotents, then the following are equivalent:

(i) $P \sim (\mathcal{A}'(e)) Q$;

(ii)$Pe \sim Qe$.

Proof. (i) ⇒ (ii). This a straightforward consequence of Lemma (3.1.13).

(ii) ⇒ (i). By Lemma (3.1.17), $Pe, Qe, (I-P)e, (I-Q)e$ all have a unique finite decomposition up to similarity. Since $Pe \sim Qe$, $\exists X \in GL(\mathcal{L}(P\mathcal{H}, Q\mathcal{H}))$ such that
$$XPe(\lambda) = Qe(\lambda), \qquad \forall \lambda \in \Omega.$$

If $\{P_i\}_{i=1}^{m}$ is a unit decomposition of P e, then $\{XP_iX^{-1}\}_{i=1}^{m}$ is a unit decomposition of Qe. Note that

$$\begin{aligned}Qe(\lambda) = XPe(\lambda) &= X(P_1e(\lambda)\dot{+}P_2e(\lambda)\dot{+}\cdots\dot{+}P_me(\lambda)) \\ &= X(P_1P_1e(\lambda)\dot{+}P_2P_2e(\lambda)\dot{+}\cdots\dot{+}P_mP_me(\lambda)) \\ &= XP_1X^{-1}Q_1e(\lambda)\dot{+}XP_2X^{-1}Q_2e(\lambda)\dot{+}\cdots\dot{+}XP_mX^{-1}Q_me(\lambda).\end{aligned}$$

Let $\{P_i\}_{i=m+1}^{n}$ and $\{Q_i\}_{i=m+1}^{n}$ be an arbitrary decomposition of $(I-P)e$ and $(I-Q)e$, then $\{P_i\}_{i=1}^{n}$ and $\{\{XP_iX^{-1}\}_{i=1}^{m}, \{Q_k\}_{k=m+1}^{n}\}$ are two unit decompositions of e. By Lemma (3.1.16), for $r \in \{m+1,\ldots,n\}$, we can find $P_{r'}, r' \in \{m+1,\ldots,n\}$, and $Z_r \in GL\big(\mathcal{L}(Q_r\mathcal{H}, P_{r'}\mathcal{H})\big)$ satisfying

$$Z_rQ_re(\lambda) = P_{r'}e(\lambda), \qquad \forall\lambda \in \Omega.$$

$r_1' = r_2'$ if and only if $r_1 = r_2$. Set $Z = Z_1\dot{+}Z_2 \dot{+}\cdots\dot{+}Z_n \in GL\big(\mathcal{A}'(e)\big)$, then

$$ZP = QZ, \qquad Z \in GL\big(\mathcal{A}'(e)\big).$$

Note that $ZPZ^{-1} = Q$; we can deduce that $P \sim \big(\mathcal{A}'(e)\big)Q$, by using Lemma (3.1.13).

Lemma (3.1.19)[58]: Let $e{:}\,\Omega \to Gr(n,\mathcal{H})$ be a holomorphic curve and P,Q be idempotents in $\mathcal{A}'(e)$. If P e is not similar to Qe, then for each natural number $n, P \oplus 0_{\mathcal{H}^{(n)}}$ is not similar to $Q \oplus 0_{\mathcal{H}^{(n)}}$.

Proof. If not, there exist $n \in N$ and $X \in GL\left(\mathcal{A}'\left(e^{(n+1)}\right)\right)$ satisfying

$$X\big(P \oplus 0_{\mathcal{H}^{(n)}}\big)X^{-1} = \big(Q \oplus 0_{\mathcal{H}^{(n)}}\big).$$

According to Lemma (3.1.12)

$$\big(P \oplus 0_{\mathcal{H}^{(n)}}\big)e^{(n+1)} \sim (Q \oplus 0_{\mathcal{H}^{(n)}}e^{(n+1)}.$$

Note that $(P \oplus 0_{\mathcal{H}^{(n)}}e^{(n+1)} \sim Pe$ and $(Q \oplus 0_{\mathcal{H}^{(n)}}e^{(n+1)} \sim Qe$. Thus $Pe \sim Qe$, that contradicts $Pe \sim Qe$.

Lemma (3.1.20)[58]: Let $e{:}\,\Omega \to Gr(n,\mathcal{H})$ is a holomorphic curve and $\bigvee_{\lambda\in\Omega} e(\lambda) = \mathcal{H}$, then the following are equivalent:

(i) $e \sim \bigoplus_{i=1}^{k} (P_ie)^{(n_i)}, \mathcal{H} = \bigoplus_{i=1}^{k} \mathcal{H}_i^{(m_i)}, P_i{:}\,\mathcal{H} \to \mathcal{H}_i, P_i^2 = P_i$ for $k, n_i < \infty$, P_ie is indecomposable, $P_ie \sim P_je$ for $i \neq j$, and $e^{(l)}$ have a finite unique decomposition up to similarity for $l \in N$.

(ii) $\bigvee\big(\mathcal{A}'(e)\big) \cong N^{(k)}$ under the isomorphism h that sends [I] to $(n_1, n_2, \ldots, n_k)$, i.e., $h([I]) = n_1e_1 + n_2e_2 + \cdots + n_ke_k$, where I is the unit of $\mathcal{A}'(e)$ and $0 \neq n_i \in N$ for $i = 1,2,\ldots,k, \{e_i\}_{i=1}^{k}$ are the generators of $N^{(k)}$.

Proof. (i) $\Rightarrow$ (ii). Let E in $\mathcal{A}'\big(e^{(n)}\big)$ be an idempotent, then $Ee^{(n)}$ and $(I-E)e^{(n)}$ have finite decompositions.

If $\{Q_1,\ldots,Q_a\}$ is a decomposition of $Ee^{(n)}$ and $\{Q_{a+1},\ldots,Q_b\}$ is a decomposition of $(I-E)e^{(n)}$, then $\{Q_1,\ldots,Q_b\}$ is a decomposition of $e^{(n)}$. Since we also have a decomposition of $e^{(n)}$ using nni copies of each of projections P_i, the uniqueness implies that there is $X \in GL\left(\mathcal{A}'\left(e^{(n)}\right)\right)$ such that $XQ_jX^{-1} = P_i$. Since $E = Q_1 + Q_2 + \cdots + Q_a$, there are integers $m_i, 0 \leqslant m_i \leqslant nn_i, XEX^{-1} = \sum_{i=1}^{k} P_i^{(m_i)}$. Define a map $h{:}\bigvee\big(\mathcal{A}'(e)\big) \to N^{(k)}$ by

$$h([E]) = (m_1, m_2, \ldots, m_k).$$

To see that h is well defined, we observe that if $[E] = [F]$, then

$$F \sim E \sim \sum_{i=1}^{k} P_i^{(m_i)}.$$

If F can be similar at most to one projection of the form $\sum_{i=1}^{k} P_i^{(m_i)}$, it follows that if $h([F]) = h([E])$, then $F \sim E$, so h is one-to-one. For any k-tuple $(m_1, m_2, \dots, m_k)$ of nonnegative integers, we can find n so that $m_i \leqslant n n_i$ for all i and then h sends $\sum_{i=1}^{k} P_i^{(m_i)}$ to $(m_1, \dots, m_k)$. This shows that h is onto. Thus, $\vee(\mathcal{A}'(e)) \cong N^k$ and by our construction, $h([I]) = (n_1, \dots, n_k)$.

(ii)⇒(i). Suppose $\vee(\mathcal{A}'(e)) \cong N^{(k)}$ and h is the isomorphism, then there exist a natural number r and $Q_1, \dots, Q_k, k$ idempotents of $\mathcal{A}'(e^{(r)})$, satisfying $h([Q_i]) = e_i, 1 \leqslant i \leqslant k$. Since $\vee\left(\mathcal{A}'(e^{(n)})\right) \cong \vee(\mathcal{A}'(e))$, we need only to prove that e has a unique finite decomposition up to similarity. At first, we will prove the following:

(a) For an arbitrary idempotent $P \in \mathcal{A}'(e)$, if P e is indecomposable, then there exists i, $1 \leqslant i \leqslant k$, satisfying $h([P]) = e_i$.

Let $h([P]) = \sum_{i=1}^{k} \lambda_i e_i = \sum_{i=1}^{k} \lambda_i h([Q_i])$ for $\lambda_i \in N, \omega = r \sum_{i=1}^{k} \lambda_i$, then we can find natural number $n > \omega$ satisfying

$$P \oplus 0_{\mathcal{H}^{(n-1)}} \sim \left(\mathcal{A}'(e^{(n)})\right) \sum_{i=1}^{k} Q_i^{(\lambda_i)} \oplus 0_{\mathcal{H}^{(n-1)}}$$

and

$$(P \oplus 0_{\mathcal{H}^{(n-1)}}) e^{(n)} \sim \left(\sum_{i=1}^{k} Q_i^{(\lambda_i)} \oplus 0_{\mathcal{H}^{(n-1)}}\right) e^{(n)}.$$

So $Pe \sim \sum_{i=1}^{k} Q_i^{(\lambda_i)} e^{(\omega)}$. Note that P e is indecomposable, but the right-hand side of this similarity is indecomposable only if one λ_i is 1 and the rest are zeros. Thus, there exists $i, 1 \leqslant i \leqslant k, h([P]) = e_i$.

(b) For arbitrary idempotents P and Q in $\mathcal{A}'(e^{(n)})$, if $h([P]) = h([Q])$, then $Pe \sim Qe$.

Let $\{P_1, \dots, P_m\}$ be a unit decomposition of e and $h([P_i]) = \sum_{i=1}^{k} \lambda_{ij} e_j$, where $\lambda_{ij} \in N$, then

$$h([I\,]) = h\left(\left[\sum_{i=1}^{m} P_i\right]\right) = \sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_{ij} e_j.$$

Note that $h([I]) = \sum_{i=1}^{k} n_i e_i$, so that $\sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_{ij} = \sum_{i=1}^{k} n_i$, so $m \leqslant \sum_{i=1}^{k} n_i$. This shows that e has a finite decomposition.

Furthermore, let $\{P_1, \dots, P_t\}$ be a unit decomposition of e, then

$$h\left(\sum_{i=1}^{t} [P_i]\right) = h([I\,]) = \sum_{i=1}^{k} n_i e_i.$$

By (a), $t = \sum_{i=1}^{k} n_i$ and for each $i, 1 \leqslant i \leqslant k$, there exist $P_{i_1}, \dots, P_{i_{n_i}} \in \{P_1, \dots, P_t\}$ satisfying $h([P_{i_1}]) = \cdots = h\left(\left[P_{i_{n_i}}\right]\right) = e_i$. By (b), $P_{i_j} e \sim P_{i_k} e, \forall 1 \leqslant j, k \leqslant n_i$.

$$e \sim \sum_{i=1}^{k} P_i^{(n_i)} e.$$

Suppose $\{P_1', \dots, P_s'\}$ is another unit decomposition of e, then in the same way we know $r = \sum_{i=1}^{k} n_i$ and for each $i, 1 \leqslant i \leqslant k$, there exist n_i idempotents in $\{P_1', \dots, P_s'\}$ and h sends each of them to e_i. By (b) again, if $h([P_i]) = h([P_j]), 1 \leqslant i, j \leqslant \sum_{i=j}^{k} n_i$, then $P_i e \sim P_j e$. By Lemma (3.1.13), e has a unique finite decomposition up to similarity.

This completes the proof of Lemma (3.1.20).
Let $\mathcal{H} = l^2, (\alpha_0, \alpha_1, \ldots) \in l^2, (\alpha_0, \alpha_1, \ldots) \in l^2$. Define $T_z^*(\alpha_0, \alpha_1, \ldots) = (\alpha_1, \alpha_2, \ldots)$, and T_z is the adjoint of T_z^*.

Theorem (3.1.21)[58]: Let $S \in \mathcal{L}(\mathcal{H})$ be a pure isometry operator, then S is unitary equivalent to $\bigoplus_{k=1}^{l} T_z$, where $l = \dim ker\, S^*$.

Let $\mathcal{H} = l^2, f(\lambda) = \{(1, \lambda, \lambda^2, \ldots)\}$, then $f: D \to Gr(1, \mathcal{H})$ is a holomorphic curve. By the Theorem (3.1.21) we can get

Lemma (3.1.22)[58]: For $P \in \mathcal{A}'\left(f^{(n)}\right)$ an idempotent, if $m = \dim Pf^{(n)}(\lambda)$ for $\lambda \in D$, then there exists a unitary operator U such that

(i) $U\left(P\mathcal{H}^{(n)}\right) = \mathcal{H}^{(m)} \oplus 0^{(n-m)}$;

(ii)let $V = U|_{P\mathcal{H}^{(n)}}$, then $VPf^{(n)}(\lambda) = f^{(m)}(\lambda), \forall \lambda \in D$.

Proof. Note that $T_z^*(1, \lambda, \lambda^2, \ldots) = \lambda(1, \lambda, \lambda^2, \ldots), |\lambda| < 1$. So $Ker(T_z^* - \lambda) = f(\lambda)$ and $Ker(T_z^{*(n)} - \lambda) = f^{(n)}(\lambda)$.

Note that T_z, the adjoint of T_z^*, is a pure isometry operator. Let $P \in \mathcal{A}'\left(T_z^{(n)}\right)$ is an idempotent and $S = T_z^{(n)}|_{ran\, P'}$ and $m = \dim ker\, S^*$.

At first, we shall prove that S is unitary equivalent to $T_z^{(m)}$. By the Theorem (3.1.21), S is unitarily equivalent to $T_z^{(m)}$. Thus there is a unitary operator $V: P'\mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ such that $V\, SV^* = T_z^{(m)}$. Note that if $m < n, \mathcal{H}^{(n)} \oplus P'\mathcal{H}^{(m)}$ is infinite dimensional. Therefore, there exists a unitary operator $W: \mathcal{H}^{(n)} \ominus P'\mathcal{H}^{(n)} \to \mathcal{H}^{(n-m)}$. Let $= V \oplus W, V = U|_{ran\, P'}$, then $U\left(P'\mathcal{H}^{(n)}\right) = \mathcal{H}^{(m)} \oplus 0^{(n-m)}, V\, SV^* = T_z^{(m)}$.

Let $Q = (I_{\mathcal{H}^{(n)}-P}^*$, then $Q \in \mathcal{A}'\left(T_z^{(n)}\right)$ is an idempotent and

$$\left(Q(T_z)^{(n)}Q\right)^* = \left(I_{\mathcal{H}^{(n)}} - P\right)(T_z^*)^{(n)}\left(I_{\mathcal{H}}^{(n)} - P\right).$$

Thus $\left(T_z^{(n)}|_{ranQ}\right)^* = (T_z^*)^{(n)}|_{ran\left(I_{\mathcal{H}^{(n)}-P}\right)}$.

Since $P \in \mathcal{A}'\left(T_z^{*(n)}\right), \dim Ker\left(T_z^{(n)}|_{ranQ}\right)^* = \dim Ker((T_z^*)^{(n)}|_{ran\left(I_{\mathcal{H}^{(n)}-P}\right)} = n - m$.

From the above proof, we know that there exists a unitary operator U_1 such that

$$U_1\left(Q\mathcal{H}^{(n)}\right) = \mathcal{H}^{(n-m)} \oplus \mathcal{H}^{(m)} \text{ and } U_1\left(Q(T_z)^{(n)}Q\right)U_1^* = \begin{bmatrix} T_z^{(n-m)} & * \\ 0 & 0 \end{bmatrix}\begin{matrix} \mathcal{H}^{(n-m)} \\ \mathcal{H}^{(m)} \end{matrix}.$$

Thus

$$U_1 P\, U_1^* = \begin{bmatrix} 0 & 0 \\ * & I_{\mathcal{H}^{(m)}} \end{bmatrix}\begin{matrix} \mathcal{H}^{(n-m)} \\ \mathcal{H}^{(m)} \end{matrix} \quad and \quad U_1 P\, T_z^{*(n)} P\, U_1^* = \begin{bmatrix} 0 & 0 \\ * & T_z^{*(m)} \end{bmatrix}\begin{matrix} \mathcal{H}^{(n-m)} \\ \mathcal{H}^{(m)} \end{matrix}.$$

Define $U_2: \mathcal{H}^{(n)} = \mathcal{H}^{(n-m)} \oplus \mathcal{H}^{(m)} \xrightarrow{onto} \mathcal{H}^{(m)} \oplus \mathcal{H}^{(n-m)}$ by $U_2(x \oplus y) = (y \oplus x)$ for $x \in \mathcal{H}^{(n-m)}$ and $y \in \mathcal{H}^{(m)}$. Then U_2 is a unitary operator. Let $U = U_2 U_1$, then U satisfies:

(i) $U\left(P\mathcal{H}^{(n)}\right) = \mathcal{H}^{(m)} \oplus 0^{(n-m)}, i.e., UPU^* = \begin{bmatrix} I_{\mathcal{H}^{(m)}} & * \\ 0 & 0 \end{bmatrix}\begin{matrix} \mathcal{H}^{(m)} \\ \mathcal{H}^{(n-m)} \end{matrix}$;

(ii)let $V = (U|_{ranP}), V\left(T_z^{*(n)}{}_{ranP}\right)V^* = T_z^{*(m)}$, then $VPf^{(n)}(\lambda) = f^{(m)}(\lambda)$ for $\lambda \in D$.

By Lemma (3.1.20), we obtain that $\vee\left(\mathcal{A}'\left(f0^{(n)}\right)\right) \cong N, K_0\left(\mathcal{A}'\left(f^{(n)}\right)\right) \cong Z$.

Lemma (3.1.23)[58]: Let f is the holomorphic curve described before Lemma (3.1.20), then $\mathcal{A}'\left(f^{(n)}\right) \cong M_n(\mathcal{H}^\infty)$.

Proof. It is obvious.

Theorem (3.1.24)[58]: (See [61].) Let Λ be an open connected subset of C^k, and f and $\tilde{f}$ be holomorphic maps from Λ to $Gr(n,\mathcal{H})$ such that $\bigvee_{\lambda\in\Lambda} f(\lambda) = \bigvee_{\lambda\in\Lambda} \tilde{f}(\lambda) = \mathcal{H}$. Then f and $\tilde{f}$ are congruent if and only if E_f and $E_{\tilde{f}}$ are locally equivalent hermitian holomorphic vector bundles over Λ.

Lemma (3.1.25)[58]: Let $\mathcal{H} = l^2, e: \Omega \to Gr(1,\mathcal{H}), \bigvee_{\lambda\in\Omega} e(\lambda) =$ and $P \in \mathcal{A}'\left(e^{(n)}\right)$ is an arbitrary idempotent. If $\dim Pe^{(n)}(\lambda) = m$, for $\lambda \in \Omega$, then $P\, e^{(n)} \sim e^{(m)}$.

Proof. Without loss of generality, we can assume that $D \subseteq \Omega$. Then we can find $\mathcal{H}$-valued holomorphic functions $v(\lambda)$ and $e(\lambda)$ on D to be the frames of E_e and E_f respectively. Set

$$v_k(\lambda) = (0,\dots,0,v(\lambda),0,\dots,0),$$
$$e_k(\lambda) = (0,\dots,0,e(\lambda),0,\dots,0), \qquad k = 1,2,\dots,n, \lambda \in D.$$

Set $P(\lambda) = P|_{e^{(n)}(\lambda)}$, then $P(\lambda) = \left(P_{ij}(\lambda)\right)_{n\times n} \in M_n(\mathcal{H}^\infty)$ is an idempotent. By Lemma (3.1.23), $P(\lambda) \in \mathcal{A}'\left(f^{(n)}\right)$, so there is $Q \in \mathcal{A}'\left(f^{(n)}\right)$ such that

$$Q|_{f^{(n)}(\lambda)} = P(\lambda).$$

Since $\dim Pe^{(n)}(\lambda) = \dim Qf^{(n)}(\lambda) = m, \forall \lambda \in D$, by Lemma (3.1.22), there exists a unitary operator U such that $U\left(Q\mathcal{H}^{(n)}\right) = \mathcal{H}^{(m)} \oplus 0^{(n-m)}$ and if let $V = U|_{ranQ}$, then $VQ(\lambda) = f^{(m)}(\lambda)$.

Since $U^*\left(\mathcal{H}^{(m)} \oplus 0^{(n-m)}\right) = V^*\left(\mathcal{H}^{(m)} \oplus 0^{(n-m)}\right) = Q\mathcal{H}^{(n)}$, then $U^*e_i(\lambda) \in Q(\lambda) \subseteq f^{(n)}(\lambda), 1 \leqslant i \leqslant m$. So

$$U^*e_i(\lambda) = \lambda_{i1}e_1(\lambda) + \lambda_{i2}e_2(\lambda) + \cdots + \lambda_{in}e_n(\lambda), \qquad 1 \leqslant i \leqslant m, \lambda_{ij} \in C.$$

Since $\langle e_i(\lambda), e_j(\lambda)\rangle = \delta_{ij}\,\langle e(\lambda), e(\lambda)\rangle, 1 \leqslant i,j \leqslant n$ and U^* is unitary, we have

$$\lambda_{i1}(\lambda)\overline{\lambda_{j1}(\lambda)} + \cdots + \lambda_{in}(\lambda)\overline{\lambda_{jn}(\lambda)} = \delta_{ij}, 1 \leqslant i \leqslant m, \lambda \in D.$$

Since $UQ(\lambda)U^*e_i(\lambda) = UP\,(\lambda)U^*e_i(\lambda) = I_{\mathcal{H}^{(m)}}\left(e_i(\lambda)\right) = e_i(\lambda)$ for $1 \leqslant i \leqslant m, \lambda \in D$, then

$$P(\lambda)U^*e_i(\lambda) = U^*e_i(\lambda), \qquad 1 \leqslant i \leqslant m, \qquad \lambda \in D.$$

That means

$$\left(P_{ij}(\lambda)\right)_{n\times n}\left(\lambda_{i1}(\lambda),\dots,\lambda_{in}(\lambda)\right) = \left(\lambda_{i1}(\lambda),\dots,\lambda_{in}(\lambda)\right). \tag{1}$$

Let $\omega_i(\lambda) = \lambda_{i1}v_1(\lambda) + \lambda_{i2}v_2(\lambda) + \cdots + \lambda_{in}v_n(\lambda)$ for $1 \leqslant i \leqslant m$. Since

$$\langle v_i(\lambda), v_j(\lambda)\rangle = \delta_{ij}\langle v(\lambda), v(\lambda)\rangle$$

then $\langle \omega_i(\lambda), \omega_j(\lambda)\rangle = \delta_{ij}\langle v(\lambda), v(\lambda)\rangle$ for $1 \leqslant i \neq j \leqslant m$. From (1) we can see that $P(\lambda)\omega_i(\lambda) = \omega_i(\lambda)$, then $\omega_i(\lambda) \in Pe^{(n)}(\lambda)$ and $\left(\omega_1(\lambda),\dots,\omega_n(\lambda)\right)$ forms a holomorphic frame of $E_{Pe^{(n)}}$. Define $U(\lambda): e^{(m)}(\lambda) \to Pe^{(n)}(\lambda)$ as follows:

$$U(\lambda)v_i(\lambda) = \omega_i(\lambda), \qquad 1 \leqslant i \leqslant m,$$

and note that

$$\langle U(\lambda)v_i(\lambda), U(\lambda)v_j(\lambda)\rangle = \langle v_i(\lambda), v_j(\lambda)\rangle = \delta_{ij}\langle v(\lambda), v(\lambda)\rangle, \qquad 1 \leqslant i,j \leqslant m.$$

Since $U(\lambda)$ is a holomorphic isometric bundle map and $\bigvee_{\lambda\in D} Pe^{(n)}(\lambda) = P\mathcal{H}^{(n)}$, by the Theorem (3.1.24), we have $Pe^{(n)} \sim e^{(m)}$.

Lemma (3.1.26)[58]: Let $e: \Omega \to Gr(1,\mathcal{H})$ is a holomorphic curve and $\bigvee_{\lambda\in\Omega} e(\lambda) = \mathcal{H}$, then $\bigvee\left(\mathcal{A}'(e)\right) \cong N, K_0\left(\mathcal{A}'(e)\right) \cong Z$.

Proof. By Lemmas (3.1.25) and (3.1.20), we can prove Lemma (3.1.26) immediately.

Theorem (3.1.27)[58]: Let $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ be Banach algebras and

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2,$$

then

$$\bigvee(\mathcal{A}) \simeq \bigvee(\mathcal{A}_1) \oplus \bigvee(\mathcal{A}_2), \qquad K_0(\mathcal{A}) \simeq K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2),$$
$$\bigvee\left(M_n(\mathcal{A})\right) \simeq \bigvee(\mathcal{A}) \quad \text{and} \quad K_0\left(M_n(\mathcal{A})\right) \simeq K_0(\mathcal{A}),$$
where "$\simeq$" means isomorphism.

A (finite or infinite) sequence of Banach algebras and homomorphisms
$$\cdots\to \mathcal{A}_n \xrightarrow{\varphi_n} \mathcal{A}_{n+1} \xrightarrow{\varphi_{n+1}} \mathcal{A}_{n+2} \to\cdots$$
is said to be exact if $Im(\varphi_n) = Ker(\varphi_{n+1})$ for all n. An exact sequence of the form:
$$0 \to I \xrightarrow{\varphi} \mathcal{A} \xrightarrow{\psi} \mathcal{B} \to 0$$
is called short exact.

We also need to characterize the commutant of $\mathcal{A}'(f \oplus g)$, where $f, g: \Omega \to Gr(n, \mathcal{H})$ are holomorphic curves. In fact, let $P \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, then
$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} f(\lambda) \\ g(\lambda) \end{bmatrix} = \begin{bmatrix} P_{11}f(\lambda) + P_{12}g(\lambda) \\ P_{21}f(\lambda) + P_{22}g(\lambda) \end{bmatrix}, \qquad \forall \lambda \in \Omega.$$
Let $\ker \tau_{f,g} \triangleq \{Q \in \mathcal{L}(\mathcal{H}) | Qf(\lambda) \subseteq g(\lambda), \forall \lambda \in \Omega\}$. Then
$$\mathcal{A}'(f \oplus g) = \begin{bmatrix} \mathcal{A}'(f) & \ker \tau_{f,g} \\ \ker \tau_{g,f} & \mathcal{A}'(g) \end{bmatrix}.$$

Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{J}$ be its ideal, then we have the following standard exact sequence:
$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{J} \to 0$$
and the following exact cyclic sequence:
$$\begin{tikzcd} K_0(\mathcal{J}) \arrow[r, "i_*"] & K_0(\mathcal{A}) \arrow[r, "\pi_*"] & K_0(\mathcal{A}/\mathcal{J}) \arrow[d, "\partial"] \\ K_1(\mathcal{A}/\mathcal{J}) \arrow[u, "\partial"] & K_1(\mathcal{A}) \arrow[l] & K_1(\mathcal{J}). \arrow[l] \end{tikzcd}$$

Lemma (3.1.28)[58]: Let $f_1, f_2: \Omega \to Gr(1, \mathcal{H})$ be indecomposable holomorphic curves. Assume that $f_1 \sim f_2$ and $F = f_1 \oplus f_2$. Then there exists $\mathcal{J} \in \mathrm{M}\left(\mathcal{A}'(F)\right)$ and $\mathcal{J}$ is of the following form:
$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{11} & \ker \tau_{f_1,f_2} \\ \ker \tau_{f_2,f_1} & \mathcal{A}'(f_2) \end{bmatrix},$$
where $\mathcal{J}_{11} \in \mathrm{M}\left(\mathcal{A}'(f_1)\right)$.

Proof. Let $\mathcal{J}_{11}$ be a maximal ideal in $\mathcal{A}'(f_1)$, then we can prove that
$$\mathcal{J} = \begin{bmatrix} J_{11} & \ker \tau_{f_1,f_2} \\ \ker \tau_{f_2,f_1} & \mathcal{A}'(f_2) \end{bmatrix}$$
is a maximal ideal of $\mathcal{A}'(f_1 \oplus f_2)$. It is easy to see that $\mathcal{J}$ is a proper ideal of $\mathcal{A}'(f_1 \oplus f_2)$ and if proper ideal $\mathcal{J} \in \mathcal{A}'(f_1 \oplus f_2)$ satisfies $\mathcal{J} \subset \mathcal{J}'$, then $\mathcal{J}'$ must be of the form:
$$\mathcal{J}' = \begin{bmatrix} \mathcal{J}'_{11} & \ker \tau_{f_1,f_2} \\ \ker \tau_{f_2,f_1} & \mathcal{A}'(f_2) \end{bmatrix}$$
where $\mathcal{J}'_{11}$ is a proper ideal of $\mathcal{A}'(f_1)$ and $\mathcal{J}_{11} \subset \mathcal{J}'_{11}$. This is a contradiction, since $\mathcal{J}_{11}$ is maximal.

Theorem (3.1.29)[58]: Let f, $g: \Omega \to Gr(1, \mathcal{H})$ be two holomorphic curves, then $f \sim g$ if and only if $K_0(\mathcal{A}'(f \oplus g)) \cong Z$.

Proof. By Lemma (3.1.26),
$$\bigvee\left(\mathcal{A}'(f)\right) \cong \bigvee\left(\mathcal{A}'(g)\right) \cong N \quad \text{and} \quad K_0\left(\mathcal{A}'(f)\right) \cong K_0\left(\mathcal{A}'(g)\right) \cong Z.$$

Suppose $f \sim g$, then $(f \oplus g) \sim f^{(2)}$. So
$$\bigvee(\mathcal{A}'(f \oplus g)) \cong \bigvee\left(\mathcal{A}'(f^{(2)})\right) \cong \bigvee\left(M_2(\mathcal{A}'(f))\right) \cong N \text{ and } K_0(\mathcal{A}'(f \oplus g) \cong Z.$$
In order to prove the "if" part, we shall introduce the following notations and results. Proof of the "if" part. We need only to show that if $f \nsim g$, then $K_0(\mathcal{A}'(f \oplus g)) \neq Z$. Otherwise, we assume that $K_0(\mathcal{A}'(f \oplus g) \cong Z$. Since $f \nsim g$, there exists a maximal ideal $\mathcal{J}$ in $\mathcal{A}'(f \oplus g)$ such that $\mathcal{A}'(f \oplus g)/\mathcal{J} \cong C$, where
$$\mathcal{J} = \begin{bmatrix} \mathcal{J}' & \ker \tau_{f,g} \\ \ker \tau_{g,f} & \mathcal{A}'(g) \end{bmatrix}$$
and $\mathcal{J}'$ is a maximal ideal of $\mathcal{A}'(f)$. Since $\mathcal{J}\dot{+}1 \cong \mathcal{A}'(f \oplus g)$ is stable finite, we know that $K_0(\mathcal{J}) \neq 0$. By $\mathcal{A}'(f \oplus g)/\mathcal{J} \cong C$, we also have the following separating exact sequence:
$$0 \to \mathcal{J} \overset{l}{\to} \mathcal{A}'(f \oplus g) \underset{\overset{\leftrightarrows}{\lambda}}{\pi} \mathcal{A}'(f \oplus g)/\mathcal{J} \to 0.$$
By Proposition 8.3.6 in [60], we get the exact sequence:
$$0 \to K_0(\mathcal{J}) \overset{l_*}{\to} K_0(\mathcal{A}'(f \oplus g) \overset{\pi_*}{\to} K_0(\mathcal{A}/\mathcal{J}) \to 0.$$
Note that $K_0\big(\mathcal{A}'(f \oplus g)\big) \cong K_0\left(\frac{\mathcal{A}}{J}\right) \cong Z$, therefore $K_0(\mathcal{J}) = 0$. This contradicts $K_0(\mathcal{J}) \neq 0$.

We always assume that $\vee_{\lambda \in \Omega} f(\lambda) = \mathcal{H}$ and if $\forall P \in \mathcal{A}'(f)$ is an idempotent, then $\sigma\big(P(\lambda)\big) = \sigma\big(P|_{f(\lambda)}\big)$ for $\lambda \in \Omega$ is connected, for each holomorphic curve $f: \Omega \to Gr(n, \mathcal{H})$, where $\sigma\big(P(\lambda)\big)$ denotes the spectrum of $P(\lambda)$.

Example (3.1.30)[58]: Let $T \in B_1(\Omega)$, then for each natural number n $(1 \leqslant n \leqslant \infty)$, we define
$$A = \begin{bmatrix} T & I & \cdots & 0 \\ 0 & T & \ddots & 0 \\ 0 & \cdots & \ddots & I \\ 0 & \cdots & 0 & T \end{bmatrix}_{n \times n}.$$
Let $f(\lambda) = Ker(A - \lambda)$ for $\lambda \in \Omega$, then $f: \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve and $\vee_{\lambda \in \Omega} f(\lambda) = \mathcal{H}$, and if $\forall P \in \mathcal{A}'(f)$ is an idempotent, then $\sigma(P(\lambda)) = \sigma\left(P|_{f(\lambda)}\right)$ is connected.

Without loss of generality, we assume that $n = 2$, then $A = \begin{bmatrix} T & I \\ 0 & T \end{bmatrix}$. We can prove that
$$\begin{aligned} Ker(T-\lambda) \oplus Ker(T-\lambda) &\subseteq \{(x,y) | y \in Ker(T-\lambda), (T-\lambda)x = y\} = Ker(A-\lambda) \\ &\subseteq Ker(T-\lambda)^{(2)} \oplus Ker(T-\lambda). \end{aligned} \quad (2)$$
For $\forall P \in \mathcal{A}'(f), x \in Ker(A-\lambda), (A-\lambda)P\,x = P\,(A-\lambda)x = 0$. Since $\vee_{\lambda \in \Omega} Ker(A - \lambda) = \mathcal{H}$, then we can assume that $y = \sum_{\alpha \in I} x_\alpha \in \mathcal{H}, x_\alpha \in Ker(A - \lambda_\alpha)$ for $\lambda_\alpha \in \Omega$. So
$$(A-\lambda_\alpha)Px_\alpha = P(A-\lambda_\alpha)x_\alpha = 0, \qquad i.e., \qquad (AP - P\,A)x_\alpha = 0$$
and $(AP - P\,A)y = (AP - P\,A)\sum_{\alpha \in I} x_\alpha = 0$ for $y \in \mathcal{H}$.

Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, then
$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} T & I \\ 0 & T \end{bmatrix} = \begin{bmatrix} p_{11}T & p_{11} + p_{12}T \\ p_{21}T & p_{21} + p_{22}T \end{bmatrix},$$
$$\begin{bmatrix} T & I \\ 0 & T \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} Tp_{11} + p_{21} & Tp_{12} + p_{22} \\ Tp_{21} & Tp_{22} \end{bmatrix}.$$
So $p_{11} = p_{22} \in \mathcal{A}'(T), p_{12} \in \mathcal{A}'(T), p_{21} = 0$. Since P is an idempotent, then p_{11}, p_{22} are both idempotents.

By the above proof and Lemmas 1.22 and 1.23 in [61], we know $\sigma\big(P(\lambda)\big)$ is a singleton for $\lambda \in \Omega$.

Lemma (3.1.31)[58]: Let $f: \Omega \to Gr(n, \mathcal{H})$ is a holomorphic curve, then $\mathcal{A}'(f)/rad\mathcal{A}'(f)$ is commutative.

Proof. Since σ $(P(\lambda))$ is connected, let A and B be in $\mathcal{A}'(f)$, then we have

$$\sigma\big((AB - BA)(\lambda)\big) = \sigma\big(A(\lambda)B(\lambda) - B(\lambda)A(\lambda)\big) = \{0\}, \qquad \lambda \in \Omega.$$

Hence there is a positive integer $m \leqslant n$ such that

$$\big(A(\lambda)B(\lambda) - B(\lambda)A(\lambda)\big)^{(m)} = 0, \qquad \lambda \in \Omega.$$

Since $\bigvee_{\lambda\in\Omega} f(\lambda) = \mathcal{H}$ for $\lambda \in \Omega$, then $(AB - BA)^{(m)} = 0$. So $\mathcal{A}'(f)/rad\,\mathcal{A}'(f)$ is commutative [59].

Lemma (3.1.32)[58]: Let $f: \Omega \to Gr(n, \mathcal{H})$ and $\{P_1, P_2, \ldots, P_m\}$ is a unit decomposition of $f^{(l)}$, then $m = l$ and $\dim P_i f^{(l)}(\lambda) = n, i = 1, 2, \ldots, m$ for $\lambda \in \Omega$.

Proof. At first, we show that $m \leqslant l$. By Lemma (3.1.31), $\mathcal{A}'(f)/rad\mathcal{A}'(f)$ is commutative. By the Gelfand Theorem, there exists a continuous natural homomorphism φ from $\mathcal{A}'(f)$ into $C\left(M\big(\mathcal{A}'(f)\big)\right)$, where $M\big(\mathcal{A}'(f)\big)$ denotes the maximal ideal space of $\mathcal{A}'(f)$. So φ can induce a continuous homomorphism ψ from $\mathcal{A}'\big(f^{(l)}\big)$ into $M_l\left(M\big(\mathcal{A}'(f)\big)\right)$ defined by

$$\psi(S)(J) = \Big(\psi\big(S_{ij}\big)(\mathcal{J})\Big)_{l\times l}, \qquad \forall S = \big(S_{ij}\big)_{l\times l} \in \mathcal{A}'\big(f^{(l)}\big)\ and\ \mathcal{J} \in \mathrm{M}\left(\mathcal{A}'(f)\right).$$

Set $P_k = \big(P_{ij}\big)_{l\times l}$ for $k = 1, 2, \ldots, m$. Then $\psi(P_k)(\mathcal{J}) = \Big(\varphi\big(P_{ij}^k\big)(\mathcal{J})\Big)_{l\times l}$. Set

$$tr(\psi(P_k)(\mathcal{J}) = \sum_{i=1}^{l} \psi\big(P_{ii}^k\big)(\mathcal{J}).$$

Then $tr(.)$ defines a continuous function on $\mathrm{M}\big(\mathcal{A}'(f)\big)$. Since $A'(f)/rad\mathcal{A}'(f)$ is commutative and f is indecomposable, $\mathrm{M}\big(\mathcal{A}'(f)\big)$ is connected, by Proposition (3.1.7)7 of [70]. Since $\psi(P_k)(\mathcal{J})$ is an idempotent, $tr(\psi(P_k)(\mathcal{J})) \equiv n_k \geqslant 1$. Note that $\sum_{k=1}^m P_k = I$ and $P_k P_{k'} = \delta_{kk'} P_k$; we have $\sum_{k=1}^m tr\big(\psi(P_k)(\mathcal{J})\big) = l$. Hence

$$\sum_{k=1}^{m} tr\big(\psi(P_k)\big)(\mathcal{J}) = \sum_{k=1}^{m} n_k = l.$$

So $m \leqslant l$.

Now we show that $\dim P_i f^{(l)}(\lambda) = n$. Otherwise, we may assume that

$$\dim P_i f^{(l+1)}(\lambda) = k \quad and \quad k < n.$$

Let $S = f \oplus P_i f^{(l+1)}$. We can find an $\mathcal{J}_1 \in \mathrm{M}\big(\mathcal{A}'(S)\big)$ such that

$$\mathcal{A}'(S)/\mathcal{J}_1 \cong C,$$

and

$$\mathcal{A}'(\frac{f^{(l+1)}}{\mathcal{J}} \cong M_{l+1}(C) \ \text{ for } \ \mathcal{J} \in \mathrm{M}\left(\mathcal{A}'\big(f^{(l+1)}\big)\right).$$

Note that

$$f^{(l+1)} \sim f \oplus P_1 f \oplus \cdots \oplus P_m f \quad \text{and} \quad m \leqslant l;$$

we can find $\mathcal{J}$_2 $\in \mathrm{M}\left(\mathcal{A}'\big(f^{(l+1)}\big)\right)$ such that

$$\mathcal{A}'\big(f^{(l+1)}\big)/\mathcal{J}_2 \cong M_d(C) \quad \text{and} \quad d < l + 1.$$

This contradicts $\mathcal{A}'\left(f^{(l+1)}\right)/\mathcal{J} \cong M_{l+1}(C)$. Similarly, we can show that it is impossible for $k \geqslant n$. So $k = n$ and $m = l$. We complete the proof of Lemma (3.1.32).
Similarly to the proof of Lemma (3.1.23), we have

Lemma (3.1.33)[58]: Let $\{P_k(\lambda)\}_{k=1}^{m}$ be a family of holomorphic idempotent elements in $M_n\left(\mathcal{H}^\infty(D)\right)$ such that $\sum_{k=1}^{m} P_k(\lambda) = I_n$, and $P_i(\lambda)P_j(\lambda) = \delta_{ij}P_j(\lambda)$ for $1 \leqslant i,j \leqslant m$ and $\lambda \in \Omega$. Then there exists a holomorphic invertible element $X(\lambda) \in M_n\left(\mathcal{H}^\infty(\Omega)\right)$ such that

$$X^{-1}(\lambda)P_j(\lambda)X(\lambda) = I_{C^{k_j}} \oplus 0_{n-k}$$

and $X(\lambda)|_{ranP_j(\lambda)}$ is a holomorphic isometric bundle map from $ran\, P_j(\lambda)$ onto $f^{(k)}(\lambda)$, where $f(\lambda) = \vee\{(1,\lambda,\lambda^2,\dots)\}$ for $j = 1,2,\dots,m$ and $k_j = rank\, P_j(\lambda)$.

Lemma (3.1.34)[58]: Let $f\colon \Omega \to Gr(n,\mathcal{H}), F = f^{(n)}$ and P is an idempotent operator in $\mathcal{A}'(F)$ satisfying P F is indecomposable. Then P F is similar to f.

Proof. Without loss of generality, we may assume that $\mathcal{H} = l^2, e(\lambda) = \vee\{(1,\lambda,\lambda^2,\dots)\}, \Omega = D$. We will prove Lemma (3.1.34) only in the case $n = 2$. Now, $F = f \oplus f$. Note that P is an idempotent in $\mathcal{A}'(F)$; we can find an idempotent P_1 in $\mathcal{A}'(F)$ and B in $rad\, A'(F)$ such that $P(\lambda) = P_1(\lambda) + B(\lambda)$, where

$$P_1(\lambda) = \begin{bmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{bmatrix} \begin{matrix} f(\lambda) \\ f(\lambda) \end{matrix}, \qquad B(\lambda) = \begin{bmatrix} B_{11}(\lambda) & B_{12}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) \end{bmatrix} \begin{matrix} f(\lambda) \\ f(\lambda) \end{matrix},$$

where scalar function $f_{ij}(\lambda) \in \mathcal{H}^\infty(D)$ and $B_{ij}\,(\lambda) \in rad\, \mathcal{A}'(f)$. Set $G = -I_{\mathcal{H}^{(2)}} + (2P_1 + B)$. Since $B \in rad\, \mathcal{A}'(F)$, G is an invertible operator in $\mathcal{A}'(F)$ and $PG = GP_1$. This shows $G^{-1}PG = P_1 \in \mathcal{A}'(F)$. Without loss of generality, we now assume that $P = P_1$. That is

$$P(\lambda) = \begin{bmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{bmatrix} \begin{matrix} f(\lambda) \\ f(\lambda) \end{matrix}.$$

Also set

$$P'(\lambda) = \begin{bmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{bmatrix} \begin{matrix} e(\lambda) \\ e(\lambda) \end{matrix}.$$

Since P F is indecomposable and by Lemma (3.1.32), we can show that $\dim PF(\lambda) = n$ and $tr\left(P'(\lambda)\right) = 1$ for each $\lambda \in D$. By Lemma (3.1.33), we can find a holomorphic invertible element $X(\lambda) \in M_2\left(\mathcal{H}^\infty(D)\right)$ such that

$$X(\lambda)P'(\lambda)X^{-1}(\lambda) = \begin{bmatrix} I_C & 0 \\ 0 & 0 \end{bmatrix},$$

$$X(\lambda)\left(I - P'(\lambda)\right)X^{-1}(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & I_C \end{bmatrix}.$$

$X(\lambda)|_{ran\, P'(\lambda)}$ and $X(\lambda)|_{ran(I-P)'(\lambda)}$ are holomorphic isometric bundle maps from $ran\, P'(\lambda)$ and $ran\left(I - P'(\lambda)\right)$ onto $e(\lambda)$, respectively.
Set

$$X(\lambda) = \begin{bmatrix} u_{11}(\lambda) & u_{12}(\lambda) \\ u_{21}(\lambda) & u_{22}(\lambda) \end{bmatrix},$$

and

$$\hat{X}(\lambda) = \begin{bmatrix} u_{11}(\lambda)I_{f(\lambda)} & u_{12}(\lambda)I_{f(\lambda)} \\ u_{21}(\lambda)I_{f(\lambda)} & u_{22}(\lambda)I_{f(\lambda)} \end{bmatrix}.$$

Then

$$\hat{X}(\lambda)P(\lambda)\hat{X}^{-1}(\lambda) = \begin{bmatrix} I_{f(\lambda)} & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $\hat{X}(\lambda)f^{(2)}(\lambda) = f^{(2)}(\lambda)$. Now we claim that $\hat{G}(\lambda) = \hat{X}(\lambda)\big|_{ran\,P(\lambda)}$ is a holomorphic isometric bundle map from $ran\,P(\lambda)$ onto $f(\lambda)$.

Note that $G(\lambda) = X(\lambda)|_{ran\,P'(\lambda)}$ is a holomorphic isometric bundle map from $ran\,P'(\lambda)$ onto $e(\lambda)$. Let $t_1(\lambda) = e(\lambda) \oplus 0$ and $t_2(\lambda) = 0 \oplus e(\lambda)$. Then $\big(t_1(\lambda), t_2(\lambda)\big)$ is a holomorphic frame of $e^{(2)}(\lambda)$. Let $l(\lambda)$ be a holomorphic frame of $E_{P'e^{(2)}}$. Then

$$l(\lambda) = \alpha(\lambda)t_1(\lambda) + \beta(\lambda)t_2(\lambda),$$

where $\alpha(\lambda)$ and $\beta(\lambda)$ are analytic functions on D.

Since $G(\lambda)$ is a holomorphic isometry, we can find a holomorphic function $C(\lambda)$ on D such that $G(\lambda)l(\lambda) = C(\lambda)e(\lambda)$ and

$$\|l(\lambda)\|^2 = (|\alpha(\lambda)|^2 + |\beta(\lambda)|^2)\|e(\lambda)\|^2 = |C(\lambda)|^2\|e(\lambda)\|^2 = \|G(\lambda)l(\lambda)\|^2, \lambda \in D.$$

Let $\big(S_1(\lambda), \ldots, S_n(\lambda)\big)$ be a holomorphic frame of $E_{f(\lambda)}, v_j(\lambda) = S_j(\lambda) \oplus 0$ and $u_j(\lambda) = 0 \oplus S_j(\lambda)$ for $j = 1, 2, \ldots, m$. Then $\big(v_1(\lambda), \ldots, v_n(\lambda), u_1(\lambda), \ldots, u_n(\lambda)\big)$ is a holomorphic frame EF.

Set $f_j(\lambda) = \alpha(\lambda)v_j(\lambda) + \beta(\lambda)u_j(\lambda)$ for $j = 1, 2, \ldots, n$. Then $\big(f_1(\lambda), \ldots, f_n(\lambda)\big)$ is a holomorphic frame of E_{PF} and set $\hat{G}(\lambda)f_j(\lambda) = C(\lambda)v_j(\lambda)$.

Let $k_1(\lambda), \ldots, k_n(\lambda)$ be analytic functions on D and

$$\begin{aligned} g(\lambda) &= k_1(\lambda)f_1(\lambda) + \cdots + k_n(\lambda)f_n(\lambda) \\ &= k_1(\lambda)\big(\alpha(\lambda)v_1(\lambda) + \beta(\lambda)u_1(\lambda)\big) + \cdots + k_n(\lambda)\big(\alpha(\lambda)v_n(\lambda) + \beta(\lambda)u_n(\lambda)\big). \end{aligned}$$

Then

$$\hat{G}(\lambda)g(\lambda) = C(\lambda)\big(k_1(\lambda)v_1(\lambda) + \cdots + k_2(\lambda)v_2(\lambda) + \cdots + k_n(\lambda)v_n(\lambda)\big) = g'(\lambda).$$

Note that $\langle v_i(\lambda), v_j(\lambda)\rangle = \langle u_i(\lambda), u_j(\lambda)\rangle = \langle S_i(\lambda), S_j(\lambda)\rangle,\ \lambda \in D$. So

$$\begin{aligned} \langle g(\lambda), g(\lambda)\rangle &= \sum_{i=1}^{n}|k_i(\lambda)|^2(|\alpha(\lambda)|^2 + |\beta(\lambda)|^2)\|S_i(\lambda)\|^2 \\ &\quad + \sum_{i,j=1}^{n} k_i(\lambda)\overline{k_j(\lambda)}(|\alpha(\lambda)|^2 + |\beta(\lambda)|^2)\langle S_i, S_j\rangle, \end{aligned}$$

also

$$\langle g'(\lambda), g'(\lambda)\rangle = \sum_{i=1}^{n}|k_i(\lambda)|^2|C(\lambda)|^2\|S_i(\lambda)\|^2 + \sum_{i,j=1}^{n} k_i(\lambda)\overline{k_j(\lambda)}|C(\lambda)|^2\langle S_i, S_j\rangle.$$

This shows that $\|\hat{G}(\lambda)g(\lambda)\| = \|g(\lambda)\|$, and then our claim is verified.

Similarly, we can deduce that $\hat{X}\big|_{ran(I-P(\lambda))}$ is a holomorphic isometric bundle map from $ran\big(I - P(\lambda)\big)$ onto $f(\lambda)$. By the Theorem (3.1.24), we can find two isometric operators $U_1 \in \mathcal{L}\big(P\mathcal{H}^{(2)}, \mathcal{H} \oplus 0\big)$ and $U_2 \in \mathcal{L}\Big((I - P)\mathcal{H}^{(2)}, 0 \oplus \mathcal{H}\Big)$ such that $X = U_1 + U_2 \in \mathcal{A}'(F)$ and $XPX^{-1} = I_{\mathcal{H}} \oplus 0$. So $PF \sim (I_{\mathcal{H}} \oplus 0)F \sim f$. This completes the proof of the lemma.

Using Lemmas (3.1.20) and (3.1.34), we can immediately obtain the following:

Lemma (3.1.35)[58]: Let $e: \Omega \to Gr(n, \mathcal{H})$ and $E = e^{(n)}$. Then E has a unique decomposition up to similarity and

$$\bigvee\big(\mathcal{A}'(e)\big) \cong N, \qquad K_0\big(\mathcal{A}'(e)\big) \cong Z.$$

Using Lemmas (3.1.20) and (3.1.35), we have the following result similar to the proof of Theorem (3.1.29).

Theorem (3.1.36)[58]: Let f_1 and $f_2 : \Omega \to Gr(n, \mathcal{H})$ be two indecomposable holomorphic curves satisfying $\bigvee_{\lambda \in \Omega} f_i(\lambda) = \mathcal{H}$ and for $P \in \mathcal{A}'(f_i)$ an idempotent, $\sigma(P(\lambda)) = \sigma(P|_{f_i(\lambda)})$ is connected for $\lambda \in \Omega, i = 1, 2$. Then $f_1 \sim f_2$ if and only if

$$K_0(\mathcal{A}'(f_1 \oplus f_2) \cong Z.$$

Section (3.2): Hilbert Spaces of Holomorphic Functions

A homogeneous operator on a Hilbert space $\mathcal{H}$ is a bounded operator T whose spectrum is contained in the closure of the unit disc $\mathbb{D}$ in $\mathbb{C}$ and is such that $g(T)$ is unitarily equivalent to T for all linear fractional transformations g which map $\mathbb{D}$ to $\mathbb{D}$. This class of operators has been studied in [76], [78], [75], [83], [77], [82], [73], [80]. It is known that every homogeneous operator is a block shift, that is, $\mathcal{H}$ is the orthogonal direct sum of subspaces V_n, indexed by all integers, all non-negative integers or all non-positive integers, such that $T(V_n) \subseteq V_{n+1}$ for each n.

The case where $\dim V_n = 1$ for each n is completely known, the corresponding operators have been classified in [77]. The classification in the case where $\dim V_n \leq 2$ and T belongs to the Cowen - Douglas class of $\mathbb{D}$ is complete and the operators are explicitly described in [83]. Beyond this there are only some results of a general nature, and not too many examples are known (cf. [76]).

We construct a large family of examples. For every natural number m we construct a family depending on $m + 1$ parameters. Each one of the examples is realized as the multiplication operator on a reproducing kernel space of vector-valued holomorphic functions. All of these reproducing kernel Hilbert spaces admit a direct sum decomposition $\bigoplus_{n \geq 0} V_n$ with $\dim V_n = n + 1$ if $0 \leq n < m$ and $\dim V_n = m + 1$ for $n \geq m$. The reproducing kernels are described explicitly. All our examples are irreducible operators and their adjoints belong to the Cowen-Douglas class.

We have chosen a presentation as elementary as possible, based on explicit computations. This seemed to be appropriate here since our goal was a complete explicit description of the examples. On the other hand, it does not explain the deeper background of the results. To remedy this situation we have added a final which discusses a more conceptual approach to the examples.

The more conceptual approach will play a leading role in the sequel, where a description of all homogeneous Cowen-Douglas operators will be given albeit in a less explicit way than our present examples.

The results are also the subject of a short note presented to the Comptes Rendus de l'Acad´emie des Sciences, Paris [81].

We denote by $\mathbb{D}$ the open unit disc in C and by G the group of Mobius transformations $z \longmapsto \frac{az+b}{\bar{b}z+\bar{a}}, |a|^2 - |b|^2 = 1$. Let G_0 be the group $(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$. So, $G = G_0 / \{\pm I\}$. We denote by $\tilde{G}$, the universal covering group of G.

All Hilbert spaces $\mathcal{H}$ considered will be spaces of holomorphic functions $f : \mathbb{D} \to V$ taking their values in a finite dimensional Hilbert space V and possessing a reproducing kernel K. A reproducing kernel is a function $K : \mathbb{D} \times \mathbb{D} \to Hom(V, V)$ holomorphic in the first variable and anti-holomorphic in the second, such that $K_\omega \zeta$ defined by $(K_\omega \zeta)(z) := K(z, \omega)\zeta$ is in $\mathcal{H}$ for each $\in \mathbb{D}, \zeta \in V$, and

$$\langle f, K_\omega \zeta \rangle_{\mathcal{H}} = \langle f(\omega), \zeta \rangle_V \tag{3}$$

for all $f \in \mathcal{H}$.

As is well known, if $\{e_n\}_{n=0}^{\infty}$ is any orthonormal basis of $\mathcal{H}$, then we have

$$K(z,\omega)=\sum_{n=0}^{\infty}e_n(z)e_n(\omega)^* \tag{4}$$

with the sum converging pointwise. Here we interpret a formal product $\xi\eta^*$ for $\xi,\eta\in V$ as the transformation $\zeta\longmapsto\langle\zeta,\eta\rangle\xi$; when $V=\mathbb{C}^k, k\in\mathbb{N}$, and its elements are written as column vectors, $\xi\eta^*$ is just the usual matrix product.

We will be concerned with multiplier representations of $\tilde{G}$ on the Hilbert space $\mathcal{H}$. A multiplier is a continuous function $J:\tilde{G}\times\mathbb{D}\to Hom(V,V)$, holomorphic on $\mathbb{D}$, such that

$$J(gh,z)=J(h,z)J(g,hz) \tag{5}$$

for all $g,h\in\tilde{G}$ and $z\in\mathbb{D}$. For $g\in\tilde{G}$, we define $U(g)$ on on $Hol(\mathbb{D},V)$ by

$$(U(g)f)(z)=J(g^{-1},z)f\big(g^{-1}(z)\big). \tag{6}$$

It is easy to see that the multiplier identity (5) is equivalent to $U(gh)=U(g)U(h)$.

Suppose that the action $g\longmapsto U(g), g\in\tilde{G}$, defined in (6) preserves $\mathcal{H}$ and is unitary on it, then we say that U is a unitary multiplier representation of $\tilde{G}$.

Also, if the reproducing kernel K transforms according to the rule

$$J(g,z)K\big(g(z),g(\omega)\big)J(g,\omega)^*=K(z,\omega) \tag{7}$$

for all $g\in\tilde{G}; z,\omega\in\mathbb{D}$, then we say that K is quasi-invariant.

Proposition (3.2.1)[72]: Suppose $\mathcal{H}$ has a reproducing kernel K. Then U defined by (6) is a unitary representation if and only if K is quasi-invariant.

Proof. Assume that K is quasi-invariant. We have to show that the linear transformation U defined in (6) is unitary. We note, writing $\tilde{\omega}=g^{-1}(\omega')$ and $\tilde{\omega}'=g^{-1}(\omega')$,

$$\begin{aligned}\langle U(g^{-1})K(\cdot,\omega)\xi,U(g^{-1})K(\cdot,\omega')\eta\rangle &=\langle J(g,\cdot)K(g(\cdot),\omega)\xi,J(g,\cdot)K(g(\cdot),\omega')\eta\rangle\\ &=\langle K(\cdot,\tilde{\omega})J(g,\tilde{\omega})^{*-1}\xi,K(\cdot,\tilde{\omega}')J(g,\tilde{\omega}')^{*-1}\eta\rangle\\ &=\langle K(\tilde{\omega}',\tilde{\omega})J(g,\tilde{\omega})^{*-1}\xi,J(g,\tilde{\omega}')^{*-1}\eta\rangle=\langle J(g,\tilde{\omega}')^{-1}K(\tilde{\omega}',\tilde{\omega})J(g,\tilde{\omega})^{*-1}\xi,\eta\rangle\\ &=\langle K(\omega',\omega)\xi,\eta\rangle\end{aligned}$$

and it follows that $U(g^{-1})$ is isometric.

On the other hand, if U of (6) is unitary then the reproducing kernel K of the Hilbert space $\mathcal{H}$ satisfies the transformation rule (7). A reproducing kernel K has the expansion (4). It follows from the uniqueness of the reproducing kernel that the expansion is independent of the choice of the orthonormal basis. Consequently, we also have $K(z,\omega)=\sum_{\ell=0}\big(U_{g^{-1}}e_\ell\big)(z)\big(U_{g^{-1}}e_\ell\big)(\omega)^*$ which verifies the equation (7).

When we are in the situation of the Proposition and if we can prove that the operator M defined by $(Mf)(z)=zf(z)$ is bounded on $\mathcal{H}$, then M is a homogeneous operator. This is wellknown and trivial: Clearly, $(g(M)f)(z)=g(z)f(z)$ and hence $(MU(g^{-1})f)(z)=zJ(g,z)f(g(z))=J(g,z)g^{-1}\big(g(z)\big)f\big(g(z)\big)=\big(U(g^{-1})\big(g^{-1}(M)\big)f\big)(z)$, for all $g\in\tilde{G}, f\in\mathcal{H}, z\in\mathbb{D}$. If, in addition, $\dim ker(M-\omega I)^*=n$ and the operator $(M-\omega I)^*$ is bounded below, on the orthogonal complement of its kernel, for every $\omega\in\mathbb{D}$ then M^* is in the Cowen-Douglas class (see [61]) $B_n(\mathbb{D})$.

In the case of reproducing kernel Hilbert spaces of scalar functions (i.e. when $\dim V=1$) the unitary multiplier representations of $\tilde{G}$ are well-known. We describe them here because they will be used. They are the elements of the holomorphic discrete series depending on one real parameter $\lambda>0$. They act on the Hilbert space $A^{(\lambda)}(\mathbb{D})$ characterized by its reproducing kernel $B^\lambda(z,\omega)=(1-z\bar{\omega})^{-2\lambda}$. Here $B(z,\omega)=(1-z\bar{\omega})^{-2}$ is the reproducing kernel of the Bergman space $A^2(\mathbb{D})$, the Hilbert space of square integrable (with respect to normalized area measure) holomorphic functions on the unit disc $\mathbb{D}$.

For $g \in \tilde{G}, g'(z)^\lambda$ is a real analytic function on the simply connected set $\tilde{G} \times \mathbb{D}$, holomorphic in z. Also $g'(z)^\lambda \neq 0$ since g is one-one and holomorphic. Given any $\lambda \in \mathbb{C}$, taking the principal branch of the power function when g is near the identity, we can uniquely define $g'(z)^\lambda$ as a real analytic function on $\tilde{G} \times \mathbb{D}$ which is holomorphic on $\mathbb{D}$ for all fixed $g \in \tilde{G}$. The multiplier $j_\lambda(g,z) = g'(z)^\lambda$ defines on $A^{(\lambda)}(\mathbb{D})$ the unitary representation D_λ^+ by the formula (6), that is,

$$D_\lambda^+(g^{-1})(f) = (g')^{2\lambda}(f \circ g), \qquad f \in A^{(\lambda)}(\mathbb{D}), g \in \tilde{G}. \tag{8}$$

An orthonormal basis of the space is given by $\left\{\sqrt{\frac{(2\lambda)_n}{n!}} z^n\right\}_{n\geq 0}$, where $(x)_n = x(x+1)\ldots(x+n-1)$ is the Pochhammer symbol. The operator M is bounded on the Hilbert space $A^{(\lambda)}(\mathbb{D})$. It is easily seen to be in the Cowen-Douglas class $B_1(\mathbb{D})$.

Let $Hol(\mathbb{D}, \mathbb{C}^k)$ denote the vector space of all holomorphic functions on $\mathbb{D}$ taking values in $C^k, k \in \mathbb{N}$. Let λ be a real number and m be a positive integer satisfying $2\lambda - m > 0$. For brevity, we will write $2\lambda_j = 2\lambda - m + 2j$.

For each $j, 0 \leq j \leq m$, define the operator $\Gamma_j: A^{(\lambda_j)}(\mathbb{D}) \to Hol(\mathbb{D}, \mathbb{C}^{m+1})$ by the formula

$$\left(\Gamma_j f\right)_\ell = \begin{cases} \binom{\ell}{j} \dfrac{1}{\left(2\lambda_j\right)_{\ell-j}} f^{(\ell-j)} & if\ \ell \geq j \\ 0 & if\ \ell < j, \end{cases}$$

for $f \in A^{(\lambda_j)}(\mathbb{D}), 0 \leq \ell \leq m$. Here $\left(\Gamma_j f\right)_\ell$ denotes the ℓth component of the function $\Gamma_j f$ and $f^{(\ell-j)}$ denotes the $(\ell - j)$th derivative of the holomorphic function f.

We denote the image of Γ_j by $A^{(\lambda_j)}(\mathbb{D})$ and transfer to it the inner product of $A^{(\lambda_j)}(\mathbb{D})$, that is, we set $\langle \Gamma_j f, \Gamma_j g\rangle = \langle f, g\rangle$, for $f, g \in A^{(\lambda_j)}(\mathbb{D})$. The Hilbert space $A^{(\lambda_j)}(\mathbb{D})$ is a reproducing kernel space because the point evaluations $f \longmapsto \left(\Gamma_j f\right)(\omega)$ are continuous for each $\omega \in \mathbb{D}$. Let $B^{(\lambda_j)}$ denote the reproducing kernel for the Hilbert space $A^{(\lambda_j)}(\mathbb{D})$.

The algebraic sum of the linear spaces $A^{(\lambda_j)}(\mathbb{D}), 0 \leq j \leq m$ is direct. This is easily seen. If $\sum_{j=0}^m \Gamma_j f_j = 0, f_j \in A^{(\lambda_j)}(\mathbb{D})$, then $f_0 = (\Gamma_0 f_0)_0 = 0$ since $\left(\Gamma_j f_j\right)_0 = 0$ for $j > 0$. Similarly, $f_1 = (\Gamma_1 f_1)_1 = 0$ since $\left(\Gamma_j f_j\right)_1 = 0$ for $j > 1$. Continuing in thpositive numbersis fashion, we see that $f_m = 0$. It follows that we can choose m positive numbers, $\mu_j, 1 \leq j \leq m$, set $\mu_0 = 1$, write $\mu = (\mu_0, \mu_1, \ldots, \mu_m)$, and define an inner product on the direct sum of the $A^{(\lambda_j)}(\mathbb{D})$ by setting

$$\langle \sum_{j=o}^m \Gamma_j f_j, \sum_{j=o}^m \Gamma_j g_j\rangle = \sum_{j=0}^m \mu_j^2 \langle f_j, g_j\rangle, \qquad f_j, g_j \in A^{(\lambda_j)}(\mathbb{D}). \tag{9}$$

We obtain a Hilbert space in this manner which we denote by $A^{(\lambda,\mu)}(\mathbb{D})$. It has the reproducing kernel $B^{(\lambda,\mu)} = \sum_{j=0}^m \mu_j^2 B^{(\lambda_j)}$.

The direct sum of the discrete series representations $D_{\lambda_j}^+$ on $\bigoplus_{j=0}^m A^{(\lambda_j)}$ can be transferred to $A^{(\lambda_j)}(\mathbb{D})$ by the map $\Gamma = \bigoplus_{j=0}^m \mu_j \Gamma_j$. It is a unitary representation of the group $\tilde{G}$ which we call U. Its irreducible subspaces are the $A^{(\lambda_j)}(\mathbb{D})$.

We will show that U is a multiplier representation. For each $A^{(\lambda_j)}(\mathbb{D})$ separately this is fairly obvious by checking the effect of Γ_j. The important point is that the multiplier is the same on each $A^{(\lambda_j)}(\mathbb{D})$.

We need a relation between $g''(z)$ and $g'(z)$. The elements of G_0 are the matrices $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, $|a|^2-|b|^2=1$, acting on $\mathbb{D}$ by fractional linear transformations. The inequalities

$$|a-1|<1/2, |b|<1/2 \tag{10}$$

determine a simply connected neighborhood U_0 of e in G_0. Under the natural projections, it is diffeomorphic with a neighborhood U of e in G and with a neighborhood $\widetilde{U}$ of e in $\tilde{G}$. So, we may use a, b satisfying (10) to parametrize $\widetilde{U}$. For $g\in\widetilde{U}, z\in\mathbb{D}$ we have $g'(z)=\left(\bar{b}z+\bar{a}\right)^{-2}$ and $g''(z)=-2\bar{b}\left(\bar{b}z+\bar{a}\right)^{-3}$, which gives a relation

$$g''(z)=-2cg'(z)^{3/2}, \tag{11}$$

where $c=c_g$ depends on g real analytically and is independent of z; the meaning of $g'(z)^{\frac{3}{2}}$ is as defined earlier. Since both sides are real analytic, (11) remains true on all of $\tilde{G}\times\mathbb{D}$.

Definition (3.2.2)[72]: Let $J:\tilde{G}\times\mathbb{D}\to\mathbb{C}^{m+1\times m+1}$ be the function given by the formula

$$J(g,z)_{p,\ell}=\begin{cases}\binom{p}{\ell}(-c)^{p-\ell}(g')^{\lambda-\frac{m}{2}+\frac{p+\ell}{2}}(z) & if\ p\geq \ell \\ 0 & if\ p<\ell,\end{cases} \tag{12}$$

for $g\in\tilde{G}$. Here c is the constant depending on g as in (11)

The following Lemma is used for showing that U is a multiplier representation.

Lemma (3.2.3)[72]: For any $g\in\tilde{G}$, we have the formula

$$\left((g')^{\ell}(f\circ g)\right)^{(k)}=\sum_{i=0}^{k}\binom{k}{i}(2\ell+i)_{k-i}(-c)^{k-i}(g')^{\ell+\frac{k+i}{2}}\left(f^{(i)}\circ g\right).$$

Proof. The proof is by induction, using the formula (11). For $k=0$, the formula is an identity. Assume the formula to be valid for some k. Then

$$\begin{aligned}
&\left((g')^{\ell}(f\circ g)\right)^{(k+1)} \\
&\quad=\sum_{i=0}^{k}\binom{k}{i}(2\ell+i)_{k-i}(-c)^{k-i}\left\{\left(\ell+\frac{k+i}{2}\right)(g')^{\ell+\frac{k+i}{2}-1}g''\left(f^{(i)}\circ g\right)\right. \\
&\quad\quad\left.+(g')^{\ell+\frac{k+i}{2}}\left(f^{(i+1)}\circ g\right)g'\right\} \\
&=\sum_{i=0}^{k}\binom{k}{i}(2\ell+i)_{k-i}(-c)^{k-i}\left\{(2\ell+k+i)(-c)(g')^{\ell+\frac{k+i+1}{2}}\left(f^{(i)}\circ g\right)\right. \\
&\quad\quad\left.+(g')^{\ell+\frac{k+i+2}{2}}\left(f^{(i+1)}\circ g\right)\right\} \\
&=\sum_{i=0}^{k}\binom{k}{i}(2\ell+i)_{k-i}(2\ell+k+i)(-c)^{k+1-i}(g')^{\ell+\frac{k+i+1}{2}}\left(f^{(i)}\circ g\right) \\
&\quad+\sum_{i=1}^{k+1}\binom{k}{i-1}(2\ell+i-1)_{k+1-i}(-c)^{k+1-i}(g')^{\ell+\frac{k+i+1}{2}}\left(f^{(i)}\circ g\right).
\end{aligned}$$

Now, we observe that

$$\binom{k}{i}(2\ell+i)_{k-i}(2\ell+k+i)+\binom{k}{i-1}(2\ell+i-1)_{k+1-i}$$
$$=(2\ell+i)_{k-i}\left\{\binom{k}{i}(2\ell+k+i)+\binom{k}{i-1}(2\ell+i-1)\right\}$$
$$=(2\ell+i)_{k-i}\left\{\binom{k}{i}+\binom{k}{i-1}(2\ell+k)+i\binom{k}{i}+(i-1+k)\binom{k}{i-1}\right\}$$
$$=(2\ell+i)_{k+1-i}\binom{k+1}{i}.$$

Thus $((g')^{\ell}(f\circ g)^{(k+1)}=(2\ell+i)_{k+1-i}\binom{k+1}{i}(-c)^{k+1-i}(g')^{\ell+\frac{k+i+1}{2}}$ completing the induction step.

We can now prove the main theorem.

Theorem (3.2.4)[72]: The image of $\bigoplus_0^m D_{\lambda_j}^+$ under Γ is a multiplier representation with the multiplier given by $J(g,z)$ as in (12).

Proof. It will be enough to show

$$\Gamma_j\left(D_{\lambda_j(g^{-1})f}^+\right)=J(g,\cdot)(\Gamma_j f)\circ g)$$

for each $j, 0\le j\le m$. We compute the p'th component on both sides.

For $p<j$, both sides are zero by definition of Γ_j and knowing that $J(g,z)_{p,\ell}=0$ for $\ell>p$.

For $p\ge j$, we have using the Lemma,

$$\left(\left(\Gamma_j D_{\lambda_j}^+(\varphi^{-1})f\right)\right)_p=\binom{p}{j}\frac{1}{(2\lambda_j)_{p-j}}\left((g')^{\lambda_j}f\circ g\right)^{p-j}$$
$$=\binom{p}{j}\frac{1}{(2\lambda_j)_{p-j}}\sum_{i=0}^{p-j}\binom{p-j}{i}(2\lambda_j+i)_{p-j-i}(-c)^{p-j-i}(g')^{\lambda_j+\frac{p-j+i}{2}}\left(f^{(i)}\circ g\right)$$
$$=\binom{p}{j}\frac{1}{(2\lambda_j)_{p-j}}\sum_{\ell=j}^{p}\binom{p-j}{\ell-j}(2\lambda_j+\ell-j)_{p-\ell}(-c)^{p-\ell}(g')^{\lambda_j-j+\frac{p+\ell}{2}}\left(f^{(\ell-j)}\circ g\right)$$
$$=\sum_{\ell=j}^{m}\frac{p!}{j!\,(\ell-j)!\,(p-\ell)!}\frac{1}{(2\lambda_j)_{\ell-j}}(-c)^{p-\ell}(g')^{\lambda_j-j+\frac{p+\ell}{2}}\left(f^{(\ell-j)}\circ g\right)$$
$$=\sum_{\ell=0}^{m}J(\varphi,\cdot)_{p,\ell}\left((\Gamma_j f)\circ g\right)_{\ell}.$$

The vectors $e_n^j(z):=\Gamma_j\left(\sqrt{\frac{q(2\lambda_j)_n}{n!}}z_n\right)$ clearly form an orthonormal basis in the Hilbert space $A^{(\lambda_j)}(\mathbb{D})$. We have, by definition of Γ_j,

$$\left(e_n^j(z)\right)_\ell=\begin{cases}0 & \ell<j \text{ or } \ell>n+j\\ \binom{\ell}{j}\dfrac{\sqrt{n!}}{(n-\ell+j)!}\dfrac{\sqrt{(2\lambda_j)_n}}{(2\lambda_j)_{\ell-j}}z^{n-\ell+j} & \ell\ge j \text{ and } \ell\le n+j.\end{cases}\tag{13}$$

We compute the reproducing kernel $B^{(\lambda_j)}$ for the Hilbert space $A^{(\lambda_j)}(\mathbb{D})$. We have

$$B^{(\lambda_j)}(z,\omega)=\sum_{n=0}^{\infty}\left((\Gamma_j e_n^j)(z)\right)\left((\Gamma_j e_n^j)(\omega)\right)^*=\left(\Gamma_j\sum_{n=0}^{\infty}e_n^j(z)\right)\left(\Gamma_j\sum_{n=0}^{\infty}e_n^j(\omega)\right)^*$$
$$=\Gamma_j^{(z)}\Gamma^{\bar{\omega}}j\;B\;\lambda j\;(4.2)(z,\omega),\tag{14}$$

since the series converges uniformly on compact subsets. Explicitly,

$$B^{(\lambda_j)}(z,\omega)_{p,\ell}=\begin{cases}\binom{\ell}{j}\binom{p}{j}\frac{1}{(2\lambda_j)_{\ell-j}}\frac{1}{(2\lambda_j)_{p-j}}\,\partial^{(p-j)}\bar{\partial}^{(\ell-j)}B^{\lambda_j}(z,\omega)\ if\ \ell,p\geq j\\ 0 \qquad\qquad\qquad\qquad \text{otherwise.}\end{cases} \tag{15}$$

In particular, it follows that $B^{(\lambda_j)}(0,0)$ is diagonal, and

$$B^{(\lambda_j)}(0,0)_{\ell,\ell}=\begin{cases}0 & if\ \ell<j\\ \binom{\ell}{j}^2\frac{(\ell-j)!}{(2\lambda_j)_{\ell-j}} & if\ \ell\geq j.\end{cases} \tag{16}$$

Then

$$B^{(\lambda,\mu)}(0,0)_{\ell,\ell}=\sum_{j=0}^{m}B^{(\lambda_j)}(0,0)\mu_j^2. \tag{17}$$

A more general formula for $B^{(\lambda,\mu)}(z,\omega)$ can be easily obtained using (7). For $z\in\mathbb{D}$, we set $p_z=\frac{1}{\sqrt{(1-|z|^2)}}\begin{pmatrix}1 & z\\ \bar{z} & 1\end{pmatrix}\in SU(1,1)$. We also write p_z for the corresponding element of $\tilde{G}$ such that p_z depends contnuously on $z\in\mathbb{D}$ and $p_0=e$. Then $p_z(0)=z$; $p_z^{-1}=p_{-z}$. By Theorem (3.2.4), formula (7) holds for $B^{\lambda,\mu}$ and gives

$$J_{p-z}(z)B^{\lambda,\mu}(0,0)J_{p-z}(z)^*=B^{\lambda,\mu}(z,z). \tag{18}$$

We have $p'_{-z}(\zeta)=\frac{1-|z|^2}{(1-\bar{z}\zeta)^2}$; $p'_{-z}(z)=(1-|z|^2)^{-1}$. The $-c$ of (11) corresponding to p_{-z} is $\frac{\bar{z}}{1-|z|^2}$. So (12) gives

$$J_{p-z}(z)_{p,\ell}=\begin{cases}(1-|z|^2)^{-\lambda-\frac{m}{2}}\binom{p}{\ell}\bar{z}^{p-\ell}(1-|z|^2)^{m-p} & p\geq\ell\\ 0 & p<\ell,\end{cases}$$

which can be written in matrix form as

$$J_{p-z}(z)=(1-|z|^2)^{-\lambda-\frac{m}{2}}D(|z|^2)exp(\bar{z}S_m), \tag{19}$$

where $D(|z|^2)_{p,\ell}=(1-|z|^2)^{m-\ell}\delta_{p,\ell}$ is diagonal and S_m is the forward shift on $\mathbb{C}^{m+1}$ with weight sequence $\{1,\dots,m\}$, that is, $(S_m)_{\ell,p}=\ell\delta_{p+1,\ell}$, $0\leq p,\ell\leq m$. Substituting (19) into (7) and polarizing we obtain

$$B^{(\lambda,\mu)}(z,\omega)=(1-z\bar{\omega})^{-2\lambda-m}D(z\bar{\omega})\exp(\bar{\omega}S_m)\,B^{(\lambda,\mu)}(0,0)\exp(zS_m^*)\,D(z\bar{\omega}). \tag{20}$$

In general, let $\mathcal{H}$ be a Hilbert space consisting of holomorphic functions on the open unit disc $\mathbb{D}$ with values in $\mathbb{C}^{m+1}$. Assume that $\mathcal{H}$ possesses a reproducing kernel $K:\mathbb{D}\times\mathbb{D}\to\mathbb{C}^{(m+1)\times(m+1)}$. The set of vectors $\mathcal{H}_0=\{K_\omega\xi:\omega\in\mathbb{D},\xi\in\mathbb{C}^{m+1}\}$ span the Hilbert space $\mathcal{H}$. On the dense set of vectors $\mathcal{H}_0$, we define a map T by the formula $TK_\omega\xi=\bar{\omega}K_\omega\xi$ for $\omega\in\mathbb{D}$. The following Lemma gives a criterion for boundedness of the operator T.

Lemma (3.2.5)[72]: The densely defined operator T is bounded if and only if for some positive constant c and for all $n\in\mathbb{N}$

$$\sum_{i,j=1}^{n}\langle(c-\omega_j\bar{\omega}_i)K(\omega_j,\omega_i)x_i\,,x_j\rangle\geq 0$$

for $x_1,\dots,x_n\in\mathbb{C}^{m+1}$ and $\omega_1,\dots,\omega_n\in\mathbb{D}$. If the map $T:\mathcal{H}_0\to\mathcal{H}_0\subseteq\mathcal{H}$ is bounded then it is the adjoint of the multiplication operator on $\mathcal{H}$.

The proof is well-known and easy in the scalar case. We omit the obvious modifications required in the general case.

It is known and easy to verify that for every $\epsilon > 0$, the multiplication operator $M^{(\epsilon)}$, defined by $\left(M^{(\epsilon)}f\right)(z) = zf(z)$, is bounded on $A^{(\epsilon)}$. Consequently, the kernel Bϵ satisfies the positivity condition of the Lemma above for $\epsilon > 0$. Fix $m \in \mathbb{N}$. Consider the reproducing kernel $B^{(\lambda,\mu)}$. We recall that $B^{(\lambda,\mu)}$ is a positive definite kernel on the unit disc $\mathbb{D}$ if and only if $\lambda > m/2$.

Theorem (3.2.6)[72]: The multiplication operator $M^{(\lambda,\mu)}$ on the Hilbert space $A^{(\lambda,\mu)}$ is bounded for all $\lambda > m/2$.

Proof. Let ϵ be a positive real number such that $\lambda - \epsilon > m/2$. Let us find μ' with $\mu'_j > 0, 0 \le j \le m$, such that

$$B^{(\lambda,\mu)}(z,\omega) = (1 - z\overline{\omega})^{-2\epsilon} B^{(\lambda-\epsilon,\mu')}(z,\omega). \tag{21}$$

Since the multiplication operator is bounded on the Hilbert space whose reproducing kernel is $(1 - z\overline{\omega})^{-2\epsilon}$ for every $\epsilon > 0$, it follows that we can find $r > 0$ such that $(r - z\overline{\omega})(1 - z\overline{\omega})^{-2\epsilon}$ is positive definite. Assuming the existence of μ' as above, we conclude that $(r - z\overline{\omega})B^{(\lambda,\mu)}(z,\omega)$ is positive definite finishing the proof. To find such a μ', it is enough to prove $B^{(\lambda,\mu)}(0,0) = B^{(\lambda-\epsilon,\mu')}(0,0)$, because then (18) and (19) (or (20)) immediately imply (21).

By (17), writing $L(\lambda)_{\ell_j} = B^{(\lambda_j)}(0,0)_{\ell\ell}$, the question becomes whether we can find positive numbers μ'_j satisfying the equations

$$\sum_j L(\lambda)_{\ell_j}\mu_j^2 = \sum_j L(\lambda - \epsilon)_{\ell_j}{\mu'_j}^2. \tag{22}$$

By (16) each $L(\lambda)_{\ell_j}$ is continuous in λ,; also $L(\lambda)_{\ell_j} = 0$ for $\ell < j$, and $L(\lambda)_{00} = 1$. It follows that for small $\epsilon > 0$, the system (22) has solutions satisfying ${\mu'_0}^2 = 1, {\mu'_j}^2 > 0$ $(1 \le j \le m)$.

Next we compute the matrix of M with respect to the orthonormal basis $\{\mu_j e_n^j(z) : n \ge 0;\ 0 \le j \le m\}$. Let $\mathcal{H}(n)$ be the linear span of the vectors $\left\{e_{n-j}^j(z) : 0 \le j \le \min(m,n)\right\}$. It is clear that M maps the space $\mathcal{H}(n)$ into $\mathcal{H}(n+1)$. (The subspace $\mathcal{H}(n)$ of $A^{(\lambda,\mu)}(\mathbb{D})$ is a “K-type” of the representation U.) We therefore have

$$M\mu_j e_{n-j}^j = \sum_{k=0}^{m} M(n)_{k,j}\mu_k e_{n+1-k}^k.$$

Let $E(n)$ be the matrix, determined by (13), such that $\left(e_{n-j}^j(z)\right)_\ell = E(n)_{\ell,j} z^{n-\ell}, n \ge j, 0 \le j \le m$. In this notation,

$$E(n)_{\ell,j\mu_j} = \sum_{k=0}^{m} M(n)_{k,j} E(n+1)_{\ell,k\mu_k}.$$

In matrix form, this means

$$E(n)D(\mu) = E(n+1)D(\mu)M(n), \text{which gives}$$
$$M(n) = D(\mu)^{-1}E(n+1)^{-1}E(n)D(\mu),$$

where $D(\mu)$ is the diagonal matrix with $D(\mu)_{\ell,\ell} = \mu_\ell$. (These are the blocks of M regarded as a “block shift” with respect to the orthogonal decomposition of $A^{(\lambda,\mu)}(\mathbb{D}) = \bigoplus_{n=0}^{\infty} \mathcal{H}(n)$.) To get information about $M(n)$, we note that, as $n \to \infty$, Stirling’s formula gives, for any fixed $b \in \mathbb{R}$,

$$\Gamma(n+b)\sim\sqrt{2\pi}(n+b)^{n+b-\frac{1}{2}}e^{-(n+b)}\sim\sqrt{2\pi}n^{n+b-\frac{1}{2}}\left(1+\frac{b}{n}\right)^{n}\ e^{-(n+b)}\sim e^{b}n^{n+b-\frac{1}{2}}.$$

Applying this we immediately get, by (13),

$$E(n)_{\ell,j}\sim n^{\ell}n^{\lambda-\frac{m}{2}-\frac{1}{2}}E_{\ell,j},$$

where E is the matrix with entries

$$E_{\ell,j}=\begin{cases}\binom{\ell}{j}\dfrac{\sqrt{\Gamma(2\lambda-m+2j)}}{\Gamma(2\lambda-m+\ell+j)} & \ell\geq j\\ 0 & \ell<j\end{cases}$$

independent of n. Using the diagonal matrix $d(n)$ with $d(n)_{\ell,\ell}=n^{\ell}$, we can write

$$E(n)\sim n^{\lambda-\frac{m}{2}-\frac{1}{2}}d(n)E.$$

It follows that

$$\begin{aligned}M(n)&=D(\mu)^{-1}E(n+1)^{-1}E(n)D(\mu)\\&\sim\left(\frac{n}{n+1}\right)^{\lambda-\frac{m}{2}-\frac{1}{2}}D(\mu)^{-1}E^{-1}d(n+1)^{-1}d(n)ED(\mu).\end{aligned}$$

Since $\frac{n}{n+1}=1+O\left(\frac{1}{n}\right)$, this implies

$$M(n)=I+O\left(\frac{1}{n}\right),$$

where I is the identity matrix of order $m+1$ and $O\left(\frac{1}{n}\right)$ stands for a $(m+1)\times(m+1)$ matrix each of whose entries is $O\left(\frac{1}{n}\right)$.

We denote by U_{+} the operator on $A^{(\lambda,\mu)}(\mathbb{D})$ defined by $U_{+}e_{n-j}^{j}=e_{n+1-j}^{j}(0\leq j\leq \min(m,n),n-j\geq 0)$.

Theorem (3.2.7)[72]: The operator M on $A^{(\lambda,\mu)}(\mathbb{D})$ is the sum of U_{+} and of an operator in the HilbertSchmidt class. In particular, M is bounded and its adjoint belongs to the Cowen-Douglas class.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two reproducing kernel Hilbert spaces consisting of holomorphic functions on $\mathbb{D}$ taking values in $\mathbb{C}^{m+1}$. Suppose that the multiplication operator M on these two Hilbert spaces are bounded. Furthermore, assume that the standard set of $m+1$ orthonormal vectors $\varepsilon_0,\ldots,\varepsilon_m$ in $\mathbb{C}^{m+1}$, thought of as constant functions on $\mathbb{D}$, are in both $\mathbb{D}_1$ and $\mathbb{D}_2$. Since $(\sum_{i=0}^{m}p_i(M)\varepsilon_i)(z)=\sum_{i=0}^{m}p_i(z)\varepsilon_i$ for polynomials p_i with scalar coefficients, it follows that the polynomials $p(z)=\sum_{i=0}^{m}p_i(z)\varepsilon_i$ belong to these Hilbert spaces. We assume that the polynomials p are dense in both of these Hilbert spaces. Suppose that there is a bounded operator $X:\mathcal{H}_1\to\mathcal{H}_2$ satisfying $MX=XM$. Then

$$\begin{aligned}(Xp)(z)&=\left(X\sum_{i=0}^{m}p_i\varepsilon_i\right)(z)=\left(X\sum_{i=0}^{m}p_i(M)\varepsilon_i\right)(z)=\left(\sum_{i=0}^{m}p_i(M)X\varepsilon_i\right)(z)\\&=\left(\sum_{i=0}^{m}p_i(M)X\varepsilon_i\right)(z).\end{aligned}$$

Now, if we let $(X\varepsilon_i)(z)=\sum_{j=0}^{m}x_i^{j}(z)\varepsilon_j$, then $(Xp)(z)=\Phi_X(z)p(z)$, where $\Phi_X(z)=\left(\left(x_i^{j}(z)\right)\right)_{j,i=0}^{m}$. Since the polynomials p are dense, it follows that $(Xf)(z)=\Phi_X(z)f(z)$ for all $f\in\mathcal{H}_1$.

We calculate the adjoint of the intertwining operator X. We have

$$\langle XK_1(\cdot,\omega)\xi, K_2(\cdot,u)\eta\rangle = \langle \Phi_X(\cdot)K_1(\cdot,\omega)\xi, K_2(\cdot,u)\eta\rangle = \langle \Phi_X(u)K_1(u,\omega)\xi,\eta\rangle$$
$$= \langle K_1(u,\omega)\xi, \overline{\Phi_X(u)}^{tr}\eta\rangle = \langle K_1(\cdot,\omega)\xi, K_1(\cdot,u)\overline{\Phi_X(u)}^{tr}\eta\rangle$$

for all $\xi,\eta \in \mathbb{C}^{m+1}$, that is,

$$X^*K_2(\cdot,u)\eta = K_1(\cdot,u)\overline{\Phi_X(u)}^{tr}\eta, \tag{23}$$

for all $\eta \in \mathbb{C}^{m+1}$ and $u \in \mathbb{D}$. Hence the intertwining operator X is unitary if and only if there exists an invertible holomorphic function $\Phi_X: \mathbb{D}_0 \to \mathbb{C}^{(m+1)\times(m+1)}$ for some open subset $\mathbb{D}_0$ of $\mathbb{D}$ satisfying

$$K_2(z,\omega) = \Phi_X(z)K_1(z,\omega)\overline{\Phi_X(\omega)}^{tr}. \tag{24}$$

Let $\mathcal{H}$ be a Hilbert space consisting of $\mathbb{C}^n$-valued holomorphic functions on $\mathbb{D}$. Assume that $\mathcal{H}$ has a reproducing kernel, say K. Let Φ be a $n\times n$ invertible matrix valued holomorphic function on $\mathbb{D}$ which is invertible. For $f \in \mathcal{H}$, consider the map $X: f \longmapsto \tilde{f}$, where $\tilde{f}(z) = \Phi(z)f(z)$. Let $\widetilde{H} = \{\tilde{f}: f \in \mathcal{H}\}$. The requirement that the map X is unitary, prescribes a Hilbert space structure for the function space $\widetilde{\mathcal{H}}$. The reproducing kernel for $\widetilde{\mathcal{H}}$ is clearly

$$\widetilde{K}(z,\omega) = \Phi(z)K(z,w)\Phi(\omega)^*. \tag{25}$$

It is easy to verify that XMX^* is the multiplication operator $M: \tilde{f} \longmapsto z\tilde{f}$ on the Hilbert space $\widetilde{\mathcal{H}}$. Suppose we have a unitary representation U given by a multiplier J acting on $\mathcal{H}$ according to (7). Transplanting this action to $\widetilde{\mathcal{H}}$ under the isometry X, it becomes

$$\left(\widetilde{U}_{g-1}\tilde{f}\right)(z) = \tilde{J}(g,z)\tilde{f}(g\cdot z),$$

where the new multiplier $\tilde{J}$ is given in terms of the original multiplier J by

$$\tilde{J}(g,z) = \Phi(z)J(g,z)\Phi(g\ \cdot\ z)^{-1}. \tag{26}$$

Now $\widetilde{K}$ transforms according to (7), with the aid of $\tilde{J}$.

Lemma (3.2.8)[72]: Suppose that the operator M acting on the Hilbert space $\mathcal{H}$ with reproducing kernel K is bounded, the constant vectors $\varepsilon_0,\dots,\varepsilon_m$ are in $\mathcal{H}$, and that the polynomials p are dense in $\mathcal{H}$. If there exists a (self adjoint) projection X commuting with the operator M then

$$\Phi_X(z)K(z,\omega) = K(z,\omega)\overline{\Phi_X(\omega)}^{tr}$$

for some holomorphic function $\Phi_X: \mathbb{D} \to \mathbb{C}^{(m+1)\times(m+1)}$ with $\Phi_X^2 = \Phi_X$.

Proof. We have already seen that any such operator X is multiplication by a holomorphic function Φ_X. To complete the proof, note that

$$\Phi_X(\cdot)K(\cdot,\omega)\xi = XK(\cdot,\omega)\xi = X^*K(\cdot,\omega)\xi = K(\cdot,\omega)\overline{\Phi_X(\omega)}^{tr}\xi$$

for all $\xi \in \mathbb{C}^{m+1}$.

From the Lemma, putting $\omega = 0$, we see that $\Phi_X(z) = K(z,0)\overline{\Phi(0)}^{tr}K(z,0)^{-1}$ for any self adjoint intertwining operator X. Furthermore, $X_0 := \Phi_X(0)$ is an ordinary projection on $\mathbb{C}^{m+1}$, if $K(0,0) = I$. The multiplication operator on the two Hilbert spaces $\mathcal{H}$ with reproducing kernel K and $\mathcal{H}_0$ with reproducing kernel $K_0(z,\omega) = K(0,0)^{-\frac{1}{2}}K(z,\omega)K(0,0)^{-\frac{1}{2}}$ are unitarily equivalent via the unitary map $f \longmapsto K(0,0)^{-\frac{1}{2}}f$. The reproducing kernel K_0 has the additional property that $K_0(0,0) = I$. Therefore, we conclude that M is reducible if and only if there exists a projection X_0 on $\mathbb{C}^{m+1}$ satisfying

$$X_0K_0(z,0)^{-1}K_0(z,\omega)K_0(0,\omega)^{-1} = K_0(z,0)^{-1}K_0(z,\omega)K_0(0,\omega)^{-1}X_0. \tag{27}$$

This is the same as requiring the existence of a projection X_0 which commutes with all the coefficients in the power series expansion of the function $\widehat{K_0}(z,\omega) := K_0(z,0)^{-1}K_0(z,\omega)K_0(0,\omega)^{-1}$ around 0. We also point out that $\widehat{K_0}$ is the normalized kernel in the sense of [79] and is characterized by the property $\widehat{K_0}(z,0) \equiv 1$.

We set $B := B^{(\lambda,\mu)}(0,0)$ and $S := S_m$.

Lemma (3.2.9)[72]: The operator $M := M^{(\lambda,\mu)}$ on the Hilbert space $A^{(\lambda,\mu)}$ is irreducible if and only if there is no projection X_0 on $\mathbb{C}^{m+1}$ commuting with all the coefficients in the power series expansion of the function

$(1 - z\overline{\omega})^{-2\lambda - m} B^{\frac{1}{2}} \exp(-zS^*)\, B^{-1} D(z\overline{\omega}) \exp(\overline{\omega} S)\, B \exp(zS^*)\, D(z\overline{\omega}) B^{-1} \exp(-\overline{\omega} S)\, B^{\frac{1}{2}}$, around 0.

Proof. From (20), we have $B_0^{(\lambda,\mu)}(z,0) = B^{\frac{1}{2}} \exp(zS^*)\, B^{-\frac{1}{2}}$, where $B_0^{(\lambda,\mu)} := B^{-\frac{1}{2}} B^{(\lambda,\mu)} B^{-\frac{1}{2}}$. To complete the proof, using (20), we merely verify that

$$\widehat{B_0}(z,\omega) = \left(B_0^{(\lambda,\mu)}(z,0)\right)^{-1} B_0^{(\lambda,\mu)}(z,\omega) \left(B_0^{(\lambda,\mu)}(0,\omega)\right)^{-1}$$
$$= (1 - z\widehat{\omega})^{-2\lambda - m} B^{\frac{1}{2}} \exp(-zS^*)\, B^{-1} D(z\widehat{\omega}) \exp(\widehat{\omega} S)\, B \exp(zS^*)\, D(z\widehat{\omega}) B^{-1} \exp(-\widehat{\omega} S)\, B^{\frac{1}{2}}.$$

Let D_s denote the coefficient of $(-1)^s z^s \overline{\omega}^s$ in the expansion of $D(z\overline{\omega})$ and $\widetilde{D}_s = B^{-1} D_s$. (The choice of D_s ensures that the diagonal sequence in $\widetilde{D}_s$ is positive.)

Lemma (3.2.10)[72]: If $\left(S^{*i} \widetilde{D}_s S^p B S^{*q} \widetilde{D}_t S^j\right)_{k,n} \neq 0$ for some choice of i, j, s, t, p, q in $\{0,1,\ldots,m\}$ then

$$0 \le s \le m - k - i, \qquad 0 \le t \le m - n - j;$$
$$0 \le p \le k + i, \qquad 0 \le q \le n + j;$$

and $k + i - p = n + j - q$.

Proof. Recall that

$$S: \begin{cases} e_\ell \longmapsto (\ell+1) e_{\ell+1} & if\ 0 \le \ell \le m-1 \\ e_m \longmapsto 0 & \text{otherwise.} \end{cases}$$

So

$$S^p: \begin{cases} e_\ell \longmapsto \iota_{e_{\ell+p}} & if\ 0 \le i \le m-1 \\ e_m \longmapsto 0 & \ell > m-p \end{cases},$$

where $\iota = (\ell+1)\ell \ \cdots (\ell - p)$. Also,

$$\widetilde{D}_s: \begin{cases} e_\ell \longmapsto c_{e_\ell} & if\ 0 \le \ell \le m-s \\ e_m \longmapsto 0 & \ell \ge m-s+1, \end{cases}$$

where c is a non-zero constant depending on ℓ, s. Therefore

$$Q := S^{*i} \widetilde{D}_s S^p: \begin{cases} e_\ell \longmapsto c' e_{\ell+p-i} & if\ 0 \le i \le m-p-s\ and\ \ell + p - i \ge 0 \\ e_m \longmapsto 0 & otherwise \end{cases}$$

for some non-zero constant c'. Hence the full condition for $Q_{k,\ell} \neq 0$ is

$$i - p \le \ell \le m - p - s, k = \ell + p - i. \tag{28}$$

Let $:= S^* q \widetilde{D}_t S^j$. By what we have just proved, it follows that $R_{\ell,n} \neq 0$ if and only if

$$q - j \le n \le m - j - t, n = \ell - j + q. \tag{29}$$

The conditions (28) and (29) simplify as follows:

$$0 \le \ell = k + i - p = n + j - q = \ell \le m, k + i \le m - s\ and\ n + j \le m - t. \tag{30}$$

Let $a(\ell)$ denote the coefficient of $z^{m+\ell+1} \overline{\omega}^{m+\ell}$ in the polynomial A with

$$A(z,\omega) = \exp(-zS^*)\, B^{-1} D(z\overline{\omega}) \exp(\overline{\omega} S)\, B \exp(zS^*)\, D(z\overline{\omega}) B^{-1} \exp(-\overline{\omega} S)$$
$$= \sum (-1)^i \frac{S^{*i}}{i!} z^i (-1)^s \widetilde{D}_s z^s \overline{\omega}^s \ \frac{S^p}{p!} \overline{\omega}^p B \frac{S^{*q}}{q!} z^q (-1)^t \widetilde{D}_t z^t \overline{\omega}^t (-1)^j \frac{S^j}{j!} \overline{\omega}^j,$$

where the sum is over $0 \le i, j, p, q, s, t \le m$.

Lemma (3.2.11)[72]: For $0 \le \ell \le m - 1$,

$$a(\ell)_{k,n} = \begin{cases} not\ zero & if\ k = m - \ell - 1\ and\ n = m - \ell \\ zero & if\ k - n\ 6 =\ 1\ or\ k > m - \ell - 1 \end{cases}.$$

Proof. Clearly, $A(z,w) = \sum A_{ijpqst} z^{i+s+q+t} \overline{\omega}^{s+p+t+j}$, where the sum is over $0 \le i,j,p,q,s,t \le m$. Therefore, $(\ell) = \sum cS^{*i}\widetilde{D}_s S^p BS^{*q}\widetilde{D}_t S^j$, where the sum is over all i,j,p,q,s,t such that $s+t+i+q = m+\ell+1$ and $s+t+p+j = m+\ell$; $c = \frac{(-1)^{i+j+s+t}}{i!j!p!q!}$.

It follows from the preceding Lemma that if $a(\ell)_{k,n} \ne 0$, then $i - j + q - p = n - k$. However, for the terms occuring in the sum, we now have $i - j + q - p = (s+t+i+q) - (s+t+p+j) = 1$. Thus if $a(\ell)_{k,n} \ne 0$ then $n - k = 1$.

Furthermore, if $a(\ell)_{k,n} \ne 0$, then we also have $m + \ell + 1 = (s+t+i+q)$. Hence $m + \ell + 1 - (s+t+i) = q \le n + j$ from the last inequality of the preceding Lemma, that is, $s+t+i+j \ge m+\ell+1-n$. This along with $s+t+i+j \le 2m-k-n$, which is obtained by adding the first two inequalities of the preceding Lemma, gives $k \le m - \ell - 1$ The proof of the second part of the Lemma is now complete.

If $k = m - \ell - 1$ and $n = m - \ell$, for the terms occuring in the sum for $a(\ell)$, we have $s+t+i+j = 2\ell+1$. It follows that $a(\ell)_{m-\ell-1,m-\ell}$ is a sum of negative numbers. This proves the first part of the Lemma.

Theorem (3.2.12)[72]: The multiplication operator $M := M^{\lambda,\mu}$ on the Hilbert space $A^{(\lambda,\mu)}$ is irreducible.

Proof. Suppose there exists a non-trivial projection P commuting with $\widehat{B_0}(z,\omega)$ for all $z,\omega \in \mathbb{D}$. Then by Lemma (3.2.9) such a projection must commute with $B^{\frac{1}{2}}A(z,\omega)B^{\frac{1}{2}}$ for all $z,\omega \in \mathbb{D}$. However, Lemma (3.2.11) shows that there is no non-trivial projection commuting with the set of matrices $\left\{\frac{B^{\frac{1}{2}}a(\ell)B1}{2} : 0 \le \ell \le m-1\right\}$. This completes the proof.

Let $pr: E_T \to \mathbb{D}$ be the holomorphic vector bundle corresponding to an operator $T \in B_k(\mathbb{D})$. The operator T is homogeneous if and only if for any $g \in G$, there exists an automorphism $\hat{g}$ of the bundle E_T covering g, that is, the diagram $\begin{array}{ccc} E_T & \xrightarrow{\hat{g}} & E_T \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} \\ \mathbb{D} & \xrightarrow{g} & \mathbb{D} \end{array}$ is commutative.

Theorem (3.2.13)[72]: If T is a homogeneous operator in $B_k(\mathbb{D})$ then the the universal covering group $\tilde{G}$ of G acts on E_T by automorphisms.

Proof. Let $\hat{G}$ be the group of automorphisms of E_T. This is a Lie group. Let $p: \hat{G} \to G$ be the natural homomorphism. Let $N = ker\ p$, the automorphisms fixing all the points of $\mathbb{D}$. Then $\frac{\hat{G}}{N} \simeq G$, and for the corresponding Lie algebras, we have $\hat{g}/n \simeq g$. Since g is semisimple, by the Levi decomposition, there is a subalgebra $\hat{g}_0 \subseteq \hat{g}$ such that $\hat{g} = \hat{g}_0 + n$, where the sum is a vector space direct sum. Let $\hat{G}_0$ be the corresponding analytic subgroup.

There is a neigbourhood U of $e \in \hat{G}_0$ such that $p_{|U}$ is a homeomorphism onto a neighbourhood $p(U)$ of $e \in G$. But then $p(\hat{g}U) = p(\hat{g})p(U)$. So, p is a homeomorphism of a neighbourhood of any point $\hat{g} \in \hat{G}_0$ to a neighbourhood of $p(\hat{g})$ in G. It follows that the image of p is an open subgroup and so must equal G. Therefore, $\hat{G}_0$ is a covering group of G.

Now, $\hat{G}_0$ acts on E_T by automorphisms and projects to G. The universal cover $\hat{G}$ now also acts on E_T see [76].

Theorem (3.2.14)[72]: For every $m \geq 1$, the operators $M^{(\lambda,\mu)}, \lambda > \frac{m}{2}$; $\mu_1, \dots, \mu_m > 0$ are mutually unitarily inequivalent.

Proof. Suppose $M^{(\lambda,\mu)}$ and $M^{(\lambda',\mu')}$ are unitarily equivalent. Then the corresponding Hermitian holomorphic bundles are isomorphic [61]. Hence the multipliers J and J' giving the $\tilde{G}$ action on $\mathbb{A}^{(\lambda,\mu)}$ and $\mathbb{A}^{(\lambda',\mu')}$ are equivalent in the sense that there exists a invertible matrix function $\phi(z)$, holomorphic in z, such that

$$\Phi(z)J(g,z)\Phi(gz)^{-1} = J'(g,z)$$

on $\tilde{G} \times \mathbb{D}$ which is nothing but (26). Setting here $g = p_{-z}$, (19) gives

$$\Phi(z) = (1 - |z|^2)^{\lambda-\lambda'} D(|z|^2) \exp(-\bar{z}S_m)\, D(|z|^2)F(0) \exp(\bar{z}S_m)\, D(|z|^2)^{-1}.$$

The right hand side is real anlytic in $z, \bar{z}$ on $\mathbb{D}$. Since Φ is holomorphic, $\Phi(z) = \Phi(0)$ identically. Looking at the Taylor expansion, we obtain

$$S_m\Phi(0) = \Phi(0)S_m.$$

This implies that $\Phi(0) = p(S_m)$, a polynomial in S_m. (Note that S_m is conjugate to S, the unweighted shift with entries $S_{\ell p} = \delta_{p+1\,\ell}$, which is its Jordan canonical form. For S the corresponding property is easy to see.) We write

$$D^{1,1} = \left.\frac{\partial^2}{\partial z \partial z}\right|_0 D(|z|^2) = -\begin{pmatrix} m & & & & \\ & m-1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix},$$

and for the Taylor coefficient of $z\bar{z} = |z|^2$ we obtain

$$(\lambda - \lambda')\Phi(0) + D^{1,1}\Phi(0) - \Phi(0)D^{1,1} = 0.$$

Consider the diagonal of this matrix equality. All diagonal elements of $\Phi(0) = p(s_m)$ are the same number $x \neq 0$ (since p(Sm) is triangular and invertible). Hence $\lambda - \lambda' = 0$. Now, since the diagonal entries of $D^{1,1}$ are all different, $\Phi(0)$ must be diagonal. So, $\Phi(0) = xI_{m+1}$. Also, $\Phi(0)$ intertwines the operators $M^{(\lambda,\mu)}$ and $M^{(\lambda',\mu')}$, hence $\Phi(0)B^{(\lambda,\mu)}(z,\omega)\Phi(0)^* = B^{(\lambda',\mu')}(z,\omega)$ as in (24). Using this with $z = \omega = 0$ and using (16), (17) we get $|x|^2\mu_j^2 = {\mu'_j}^2$ for all j. Suince $\mu_0 = 1 = \mu'_0$, it follows that $|x|^2 = 1$ and $\mu_j = \mu'_j$ for $1 \leq j \leq m$ see [73], [74].

Section (3.3): Free Homogeneous Operators in the Cowen-Douglas Class

The homogeneous operators form a class of bounded operators T on a Hilbert space H. The operator T is said to be homogeneous if its spectrum is contained in the closed unit disc and for every Mobius transformation g the operator $g(T)$, defined via the usual holomorphic functional calculus, is unitarily equivalent to T. To every homogeneous irreducible operator T there corresponds an associated unitary representation π of the universal covering group $\tilde{G}$ of the Mobius group G:

$$\pi(g)^*T\pi(\hat{g}) = (p\hat{g})(T), \qquad \hat{g} \in \hat{G},$$

where $p: \tilde{G} \to G$ is the natural homomorphism. In [72] (see also [86]), it was shown that each homogeneous operator T, not necessarily irreducible, in $B^{m+1}(\mathbb{D})$ admits an associated representation. The representations of $\tilde{G}$ are quite well-known, but we are still far from a complete description of the homogeneous operators. In [72], the following theorem was proved.

Theorem (3.3.1)[84]: For any positive real number $\lambda > m/2, m \in \mathbb{N}$ and an $(m+1)$-tuple of positive reals $\mu = (\mu_0, \mu_1, \dots, \mu_m)$ with $\mu_0 = 1$, there exists a reproducing kernel $K^{(\lambda,\mu)}$ on the unit disc such that the adjoint of the multiplication operator $M^{(\lambda,\mu)}$ on the corresponding Hilbert space $A^{(\lambda,\mu)}(\mathbb{D})$ is homogeneous. The operators $\left(M^{(\lambda,\mu)}\right)^*$ are in the Cowen-Douglas class $B_{m+1}(\mathbb{D})$, irreducible and mutually inequivalent.

[72], presented the operators $M^{(\lambda,\mu)}$ in as elementary a way as possible, but this presentation hides the natural ways in which these operators can be found to begin with. Here we will describe another independent construction of the operators $M^{(\lambda,\mu)}$. We will also give an exposition of some of the fundamental background material. Finally, we will prove that if T is an irreducible homogeneous operator in $B_{m+1}(\mathbb{D})$ whose associated representation is multiplicity free then, up to equivalence, T is the adjoint of of the multiplication operator $M^{(\lambda,\mu)}$ for some $\lambda > m/2$ and $\mu \geq 0$.

Although, we intend to discuss homogeneous operators in the Cowen-Douglas class $B_n(\mathbb{D})$, the material below is presented in somewhat greater generality. Here we discuss commuting tuples of operators in the Cowen-Douglas class $B_n(\mathcal{D})$ for some bounded open connected set $\mathcal{D} \subseteq C^m$. The unitary equivalence class of a commuting tuple in $B_n(\mathcal{D})$ is in one to one correspondence with a certain class of holomorphic Hermitian vector bundles $(hHvb)$ on $\mathcal{D}$ [61]. These are distinguished by the property, among others, that the Hermitian structure on the fibre at $w \in \mathcal{D}$ is induced by a reproducing kernel K. It is shown in [61] that the corresponding operator can be realized as the adjoint of the commuting tuple multiplication operator M on the Hilbert space $\mathcal{H}$ of holomorphic functions with reproducing kernel K.

Start with a Hilbert space $\mathcal{H}$ of $\mathbb{C}^n$- valued holomorphic functions on a bounded open connected set $\mathcal{D} \subseteq C^m$. Assume that the Hilbert space $\mathcal{H}$ contains the set of vector valued polynomials and that these form a dense subset in $\mathcal{H}$. We also assume that there is a reproducing kernel K for $\mathcal{H}$. We use the notation $K_w(z) := K(z,w)$.

Recall that a positive definite kernel $K: D \times D \to \mathbb{C}^{n\times n}$ on $\mathcal{D}$ defines an inner product on the linear span of $\{K_w(\cdot)\xi : w \in \mathcal{D}, \xi \in \mathbb{C}^n\} \subseteq Hol(\mathcal{D}, \mathbb{C}^n)$ by the rule

$$\langle K_w(\cdot)\xi, K_u(\cdot)\eta\rangle = \langle K_w(u)\xi, \eta\rangle, \qquad \xi, \eta \in \mathbb{C}^n.$$

(On the right hand side $\langle , \rangle$ denotes the inner product of $\mathbb{C}^n$. We denote by $\varepsilon_1, \ldots, \varepsilon_n$ the natural basis of $\mathbb{C}^n$.) The completion of this subspace is then a Hilbert space $\mathcal{H}$ of holomorphic functions on $\mathcal{D}$ (cf. [85]) in which the set of vectors $\{K_w : w \in \mathcal{D}\}$ is dense. The kernel K has the reproducing property, that is,

$$\langle f, K_w\xi\rangle = \langle f(w), \xi\rangle, f \in \mathcal{H}, w \in \mathcal{D}, \xi \in \mathbb{C}^m.$$

Now, for $1 \leq i \leq m$, we have

$$M_i^* K_w \xi = \bar{w}_i K_w \xi, w \in \mathcal{D}, \text{where } (M_i f)(z) = z_i f(z), f \in \mathcal{H}$$

and $\{K_w \varepsilon_i\}_{i=1}^n$ is a basis for $\cap_{i=1}^m ker(M_i - w_i)^*, w \in \mathcal{D}$.

The joint kernel of the commuting m - tuple $M^* = (M_1^*, \ldots, M_m^*)$, which we assume to be bounded, then has dimension n. The map $\sigma_i : w \mapsto K_{\bar{w}} \bar{\varepsilon}_i, w \in \bar{\mathcal{D}}, 1 \leq i \leq n$, provides a trivialization of the corresponding bundle E of Cowen - Douglas (cf. [61]). Here $\bar{\mathcal{D}} := \{z \in \mathbb{C}^m | \bar{z} \in D\}$).

On the other hand, suppose we start with an abstract Hilbert space $\mathcal{H}$ and a m-tuple of commuting operators $T = (T_1, \ldots, T_m)$ in the Cowen - Douglas class $B_n(\mathcal{D})$. Then we have a holomorphic Hermitian vector bundle E over $\mathcal{D}$ with the fibre $E_w = \cap_{i=1}^n ker(T_i - w_i)$ at $w \in \mathcal{D}$. Following [61], one associates to this a reproducing kernel Hilbert space $\widehat{H}$ consisting of holomorphic functions on $\bar{D}$ as follows. Take a holomorphic trivialization $\sigma_i : \mathcal{D} \to \mathcal{H}$ with $\sigma_i(w), 1 \leq i \leq n$, spanning E_w. For $f \in \mathcal{H}$, define $\hat{f}_j(w) := \langle f, \sigma_{j(\bar{w})}\rangle_{\mathcal{H}}, w \in \bar{\mathcal{D}}$. Set $\langle \hat{f}, \hat{g}\rangle_{\widehat{H}} := \langle f, g\rangle_H$. The function $K_w \varepsilon_j := \widehat{\sigma_j(\bar{w})}$ then serves as the reproducing kernel for the Hilbert space $\widehat{H}$. Note that

$$\langle K_w(z)\varepsilon_j, \varepsilon_i\rangle_{\mathbb{C}^n} = \langle K_w\varepsilon_j, K_z\varepsilon_i\rangle_{\widehat{\mathcal{H}}} = \langle\widehat{\sigma_j(\bar{w})}, \widehat{\sigma_i(\bar{z})}\rangle_{\widehat{\mathcal{H}}} = \langle\sigma_j(\bar{w}), \sigma_i(\bar{z})\rangle_{\mathcal{H}}, z, w \in \bar{\mathcal{D}}.$$

If one applies this construction to the case where $\mathcal{H}$ is a Hilbert space of holomorphic functions on $\mathcal{D}$, possesses a reproducing kernel, say K, and the operator M^* is in $B_n(\bar{\mathcal{D}})$ then using the trivialization $\sigma_i(w) = K_{\bar{w}}\varepsilon_i, w \in \bar{\mathcal{D}}$ for the bundle E defined on $\bar{\mathcal{D}}$, the reproducing kernel for $\widehat{\mathcal{H}}$ is

$$\langle K_w(z)\varepsilon_j, \varepsilon_i\rangle_{\mathbb{C}^n} = \langle K_w\varepsilon_j, K_z\varepsilon_j\rangle_{\mathcal{H}} = \langle\sigma_j(\bar{w}), \sigma_i(\bar{z})\rangle_{\mathcal{H}} = \langle K_w\varepsilon_j, K_z\varepsilon_i\rangle_{\widehat{\mathcal{H}}}, z, w \in \mathcal{D}.$$

Thus $\mathcal{H} = \widehat{\mathcal{H}}$.

Let G be a Lie group acting transitively on the domain $\subseteq \mathbb{C}^{\mathrm{d}}$. Let $GL(n,\mathbb{C})$ denote the set of non-singular $n \times n$ matrices over the complex field $\mathbb{C}$. We start with a multiplier J, that is, a smooth family of holomorphic maps $J_g: \mathcal{D} \to \mathbb{C}^{n\times n}$ satisfying the cocycle relation

$$J_{gh}(z) = J_h(z)J_g(h \cdot z), \qquad \text{for all } g, h \in G, z \in \mathcal{D}, \qquad (31)$$

Let $Hol(\mathcal{D}, \mathbb{C}^n)$ be the linear space consisting of all holomorphic functions on $\mathcal{D}$ taking values in $\mathbb{C}^n$. We then obtain a natural (left) action U of the group G on $Hol(\mathcal{D}, \mathbb{C}^n)$:

$$(U_g f)(z) = J_{g^{-1}}(z)f(g^{-1} \cdot z), \qquad f \in Hol(\mathcal{D}, \mathbb{C}^n), z \in \mathcal{D}. \qquad (32)$$

Let $\mathbb{K} \subseteq G$ be the compact subgroup which is the stabilizer of 0. For h, k in $\mathbb{K}$, we have $J_{kh}(0) = J_h(0)J_k(0)$ so that $k \longmapsto J_k(0)^{-1}$ is a representation of $\mathbb{K}$ on $\mathbb{C}^n$.

As in [72], we say that if a reproducing kernel K transforms according to the rule

$$J(g,z)K\big(g(z), g(\omega)\big)J(g,\omega)^* = K(z,\omega) \qquad (33)$$

for all $g \in \tilde{G}; z, \omega \in \mathbb{D}$, then K is quasi-invariant.

Proposition (3.3.2)[84]: ([72], Proposition 2.1). Suppose $\mathcal{H}$ has a reproducing kernel K. Then U defined by (32) is a unitary representation if and only if K is quasi-invariant.

Let g_z be an element of G which maps 0 to z, that is $g_z \cdot 0 = z$.

For quasi-invariant K we have

$$K(g_z \cdot 0, g_z \cdot 0) = \left(J_{g_z}(0)\right)^{-1} K(0,0)\left(J_{g_z}(0)^*\right)^{-1}, \qquad (34)$$

which shows that $K(z,z)$ is uniquely determined by $K(0,0)$. For each z in $\mathcal{D}$, the positive definite matrix $K(z,z)$ gives the Hermitian structure of our vector bundle.

Given any positive definite matrix $K(0,0)$ such that

$$J_k(0)^{-1}K(0,0) = K(0,0)J_k(0)^* \text{ for all } k \in \mathbb{K}, \qquad (35)$$

that is, the inner product $\langle K(0,0) \cdot | \cdot\rangle$ is invariant under $J_k(0)$, (34) defines a Hermitian structure on the homogeneous vector bundle determined by $J_g(z)$. In fact, $K(z,z)$, for any $z \in \mathcal{D}$ is well defined, because if g'_z is another element of G such that $g'_z \cdot 0 = z$ then $g'_z = g_z k$ for some $k \in \mathbb{K}$. Hence

$$\begin{aligned}
K(g'_z \cdot 0, g'_z \cdot 0) = K(g_z k \cdot 0, g_z k \cdot 0) &= \left(J_{g_z}k(0)\right)^{-1} K(0,0)\left(J_{g_z}k(0)^*\right)^{-1} \\
&= \left(J_k(0)J_{g_z}(k \cdot 0)\right)^{-1} K(0,0)\left(J_{g_z}(k \cdot 0)^* J_k(0)^*\right)^{-1} \\
&= \left(J_{g_z}(0)\right)^{-1} \left(J_k(0)\right)^{-1} K(0,0)(J_k(0)^*)^{-1}\left(J_{g_z}(0)^*\right)^{-1} \\
&= \left(J_{g_z}(0)\right)^{-1} K(0,0)\left(J_{g_z}(0)^*\right)^{-1} = K(g_z \cdot 0, g_z \cdot 0)
\end{aligned}$$

This gives a good overview of all the Hermitian structures of a homogeneous holomorphic vector bundle. But not all such bundles arise from a reproducing kernel. Starting with a positive matrix satisfying (35), (34) gives us $K(z,z)$, but there is no guarantee (and is false in general) that $K(z,z)$ extends to a positive definite kernel on $\mathcal{D} \times \mathcal{D}$. It is, however, true that if there is such an extension then it is uniquely determined by $K(z,z)$ (because $K(z,w)$ is holomorphic in z and antiholomorphic in w).

This leaves us with the following possible strategy for finding the homogeneous operators in the Cowen - Douglas class. Find all multipliers, (i.e., holomorphic homogeneous vector bundles (hhvb)) such that there exists $K(0,0)$ satisfying (35) and consider all such $K(0,0)$. Then determine which of the $K(z,z)$ obtained form (34) extends to a positive definite kernel on $\mathcal{D} \times \mathcal{D}$. Then check if the multiplication operator is well-defined and bounded on the corresponding Hilbert space.

For $\mathcal{H}$ be a Hilbert space consisting of $\mathbb{C}^n$-valued holomorphic functions on some domain $\mathcal{D}$ possessing a reproducing kernel K. The corresponding holomorphic Hermitian vector bundle defined on $\mathcal{D}$ have many different realizations. The connection between two of these is given by a $n \times n$ invertible matrix valued holomorphic function φ on $\mathcal{D}$. For $f \in \mathcal{H}$, consider the map $\Gamma_\varphi : f \longmapsto \tilde{f}$, where $\tilde{f}(z) = \varphi(z) f(z)$. Let $\widetilde{\mathcal{H}} = \{\tilde{f} : f \in \mathcal{H}\}$. The requirement that the map Γ_φ is unitary, prescribes a Hilbert space structure for the function space $\widetilde{\mathcal{H}}$. The reproducing kernel for $\widetilde{\mathcal{H}}$ is easily calculated

$$\widetilde{K}(z,w) = \varphi(z) K(z,w) \varphi(w)^*. \tag{36}$$

It is also easy to verify that $\Gamma_\varphi M \Gamma_\varphi^*$ is the multiplication operator $M : \tilde{f} \longmapsto z\tilde{f}$ on the Hilbert space $\widetilde{\mathcal{H}}$. Suppose we have a unitary representation U given by a multiplier J acting on $\mathcal{H}$ according to (32). Transplanting this action to $\widetilde{\mathcal{H}}$ under the isometry Γ_φ, it becomes

$$\left(\widetilde{U}_{g^{-1}} \tilde{f}\right)(z) = \tilde{J}_g(z) \tilde{f}(g \cdot z),$$

where the new multiplier $\tilde{J}$ is given in terms of the original multiplier J by

$$\tilde{J}_g(z) = \varphi(z) J_g(z) \varphi(g \cdot z)^{-1}.$$

Now $\widetilde{K}$ transforms according to (33), with the aid of $\tilde{J}$. If we want, we can now ensure that, by passing from $\mathcal{H}$ to an appropriate $\widetilde{\mathcal{H}}, \widetilde{K}(z,0) \equiv 1$. We merely have to set $(z) = K(0,0)^{\frac{1}{2}} K(z,0)^{-1}$. Thus the reproducing kernel $\widetilde{K}$ is almost unique. The only freedom left is to multiply $\varphi(z)$ by a constant unitary $n \times n$ matrix. Once the kernel is normalized, we have

$$J_k(z) = J_k(0), \qquad z \in \mathbb{D}, k \in \mathbb{D}$$

In fact,

$$I = K(z,0) = J_k(z) K(k \cdot z, 0) J_k(0)^* = J_k(z) J_k(0)^{-1}$$

and the statement follows. Therefore, once the kernel K is normalized, we have

$$(U_{k^{-1}} f)(z) = J_k(0) f(k \cdot z), \qquad k \in \mathbb{K}.$$

Given a multiplier J, there is always the following method for constructing a Hilbert space with a quasi-invariant Kernel K transforming according to (34). We look for a functional Hilbert space possessing this property among the weighted L^2 spaces of holomorphic functions on $\mathcal{D}$. The norm on such a space is

$$\|f\|^2 = \int_{\mathcal{D}} f(z)^* Q(z) f(z) dV(z) \tag{37}$$

with some positive matrix valued function $Q(z)$. Clearly, this Hilbert space possesses a reproducing kernel K. The condition that $U_{g^{-1}}$ in (32) is unitary is

$$\int_{\mathcal{D}} f(g \cdot z)^* J^* g\ (z) Q(z) J_g(z) f(g \cdot z) dV(z) = \int_{\mathcal{D}} f(w)^* Q(w) f(w) dV(w)$$

$$= \int_{\mathcal{D}} f(g \cdot z)^* Q(g \cdot z) f(g \cdot z) \left| \frac{\partial (g \cdot z)}{\partial (z)} \right|^2 dV(z),$$

that is,

$$Q(g\cdot z)=J_g(z)^*Q(z)J_g(z)\left|\frac{\partial(g\cdot z)}{\partial(z)}\right|^{-2}, \tag{38}$$

which is equation (33) with $J_g(z)$ replaced by $\frac{\partial(g\cdot z)}{\partial(z)}Jg(z)^{*-1}$.

Given the multiplier $Jg(z), Q(z)$ is again determined by $Q=Q(0)$, and (just as in the case of $K(0,0)=A$) it must be a positive matrix commuting with all $J_k(0), k\in\mathbb{K}$. (It is assumed that each $J_k(0)$ is unitary).

In this way, we can construct many examples of homogeneous operators in $B_n(\mathcal{D})$ but not all.

Even, not all the the homogeneous operators in $B_1(\mathbb{D})$ come from this construction. There is a homogeneous operator in the class $B1\,(\mathbb{D})$ corresponding to the multiplier $J(g,z)=\left(g'(z)\right)^{\lambda}, \lambda\in\mathbb{R}$ exactly when $\lambda>0$. The reproducing kernel is $(z,w)=(1-z\bar{w})^{-2\lambda}$. But such an operator arises from the construction outlined above only if $\lambda\geq 1/2$.

The homogeneous operators constructed in the manner described above are of interest since they happen to be exactly the subnormal homogeneous operators in this class (cf. [74]).

In the case of $B_n(\mathbb{D})$, it is shown in [72] that the bundle corresponding to a homogeneous Cowen-Douglas operator admits an action of the covering group $\tilde{G}$ of the group $G=Mob$ via unitary bundle maps. This suggests the strategy of first finding all the homogeneous holomorphic Hermitian vector bundles (a problem easily solved by known methods) and then determining which of these correspond to an operator in the Cowen-Douglas class.

We use the method of holomorphic induction. For this, first we describe some basic facts and fix our notation. We follow the notation of [88] which we will use as a reference.

The Lie algebra $\mathfrak{g}$ of $\tilde{G}$ is spanned by $X_1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, X_0=\frac{1}{2}\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ and $Y=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$.

The subalgebra $\mathfrak{k}$ corresponding to $\tilde{\mathbb{K}}$ is spanned by X_0. In the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$, we mostly use the complex basis h, x, y given by

$$h=-iX_0=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$
$$x=X_1+iY=\begin{pmatrix}0&1\\0&0\end{pmatrix}$$
$$y=X_1-iY=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

We write $G^{\mathbb{C}}$ for the (simply connected group) $SL(2,\mathbb{C})$. Let $G_0=SU(1,1)$ be the subgroup corresponding to $\mathfrak{g}$. The group GC has the closed subgroups $\mathbb{K}^{\mathbb{C}}=\left\{\begin{pmatrix}z&0\\0&\frac{1}{z}\end{pmatrix}: z\in\mathbb{C}, z\neq 0\right\}, P^+=\left\{\begin{pmatrix}1&z\\0&1\end{pmatrix}: z\in\mathbb{C}\right\}, P^-=\left\{\begin{pmatrix}1&0\\z&1\end{pmatrix}: z\in\mathbb{C}\right\}$; the corresponding Lie algebras $\mathfrak{k}^{\mathbb{C}}=\left\{\begin{pmatrix}c&0\\0&-c\end{pmatrix}: c\in\mathbb{C}\right\}, \mathfrak{p}^+=\left\{\begin{pmatrix}0&c\\0&0\end{pmatrix}: c\in\mathbb{C}\right\}, \mathfrak{p}^-=\left\{\begin{pmatrix}0&0\\c&0\end{pmatrix}: c\in\mathbb{C}\right\}$ are spanned by h, x and y, respectively. The product $\mathbb{K}^{\mathbb{C}}P^-=\left\{\begin{pmatrix}a&0\\b&\frac{1}{a}\end{pmatrix}: 0\neq a\in\mathbb{C}, b\in\mathbb{C}\right\}$ is a closed subgroup to be denoted T; its Lie algebra is $t=\mathbb{C}h+\mathbb{C}y$. The product set $P^+\mathbb{K}^{\mathbb{C}}P^-=P^+T$ is dense open in $G^{\mathbb{C}}$, contains G, and the product decomposition of each of its elements is unique. ($G^{\mathbb{C}}/T$ is the Riemann sphere, $g\tilde{\mathbb{K}}\to gT, (g\in G)$ is the natural embedding of $\mathbb{D}$ into it.)

According to holomorphic induction [87] the isomorphism classes of homogeneous holomorphic vector bundles are in one to one correspondence with equivalence classes of linear representations ϱ of the pair $(\mathfrak{t}, \widetilde{\mathbb{K}})$. Since $\widetilde{\mathbb{K}}$ is connected, here this means just the representations of $\mathfrak{t}$. Such a representation is completely determined by the two linear transformations $\varrho(h)$ and $\varrho(y)$ which satisfy the bracket relation of h and y, that is,

$$[\varrho(h), \varrho(y)] = -\varrho(y). \tag{39}$$

The $\tilde{G}$-invariant Hermitian structures on the homogeneous holomorphic vector bundle (making it into a homogeneous holomorphic Hermitian vector bundle), if they exist, are given by $\varrho(\widetilde{\mathbb{K}})$-invariant inner products on the representation space. An inner product is $\varrho(\widetilde{\mathbb{K}})$- invariant if and only if $\varrho(h)$ is diagonal with real diagonal elements in an appropriate basis.

We will be interested only in bundles with a Hermitian structure. So, we will assume without restricting generality, that the representation space of ϱ is $\mathbb{C}^d$ and that $\varrho(h)$ is a real diagonal matrix.

We will be interested only in irreducible homogeneous holomorphic Hermitian vector bundles, this corresponds to ϱ not being the orthogonal direct sum of non-trivial representations. Suppose we have such a ϱ; we write V_α for the eigenspace of $\varrho(h)$ with eigenvalue α. Let $-\eta$ be the largest eigenvalue of $\varrho(h)$ and m be the largest integer such that $-\eta, -(\eta + 1), \ldots, -(\eta + m)$ are all eigenvalues. From (39) we have $\varrho(y)V_\alpha \subseteq V_{\alpha-1}$; this and orthogonality of the eigenspaces imply that $V = \bigoplus_{j=0}^{m} V_{-(\eta+j)}$ and its orthocomplement are invariant under ϱ. So, V is the whole space, and have proved that the eigenvalues of $\varrho(h)$ are $-\eta, \ldots, -(\eta + m)$.

From this it is clear that ϱ can be written as the tensor product of the one dimensional representation σ given by $\sigma(h) = -\eta, \sigma(y) = 0$, and the representation ϱ^0 given by $\varrho^0(h) = \varrho(h) + \eta I, \varrho^0(y) = \varrho(y)$. Correspondingly, the bundle for ϱ is the tensor product of a line bundle L_η and the bundle corresponding to ϱ^0.

The representation ϱ^0 has the great advantage that it lifts to a holomorphic representation of the group T. It follows that the homogeneous holomorphic vector bundle it determines for $\mathbb{D}, \tilde{G}$, can be obtained as the restriction to $\mathbb{D}$ of the homogeneous holomorphic vector bundle over $G^{\mathbb{C}}/T$ obtained by ordinary induction in the complex analytic category. So, (as a convenient choice) take the local holomorphic cross section $z \longmapsto s(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ of $G^{\mathbb{C}}/T$ over $\mathbb{D}$. In the trivialization given by $s(z)$, the multiplier then appears for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^{\mathbb{C}}$ as

$$\begin{aligned} J_g^0(z) = \varrho^0\big(s(z)^{-1} g^{-1} s(g \cdot z)\big) = \varrho^0 \begin{pmatrix} cz + d & 0 \\ -c & (cz + d)^{-1} \end{pmatrix} \\ = \varrho^0 \left(\exp\left(-\frac{c}{cz + d} y\right)\right) \varrho^0(\exp(2\, log(cz + d)h)) \end{aligned} \tag{40}$$

The last two equalities are simple computations.

For the line bundle L_η, the multiplier is $g'(z)^\eta$ (we write $g'(z) = \frac{\partial g}{\partial z}(z)$) . Consequently, the multiplier corresponding to the original ϱ is

$$J_g(z) = \big(g'(z)\big)^\eta J_g^0(z). \tag{41}$$

We now assume that we have a homogeneous holomorphic vector bundle induced by ϱ and that it has a reproducing kernel. Then we derive conditions about the action of $\tilde{G}$ that follow from this hypothesis. We will show that these conditions are sufficient: they lead directly to the construction of all homogeneous operators the Cowen-Douglas class with multiplicity free representations.

Under our hypothesis there is a Hilbert space structure in which the action of $\tilde{G}$ given by (34) is unitary. We will study this representation through its $\mathbb{K}$ - types (i.e., its restriction to $\tilde{\mathbb{K}}$). We first compute the infinitesimal representat.

For $X \in \mathfrak{g}$, and holomorphic f, we have

$$(U_X f)(z) := \left(\frac{d}{dt}\right)_{|t=0} \left(U_{\exp(tX)} f(z)\right)$$

$$= \left(\frac{d}{dt}\right)_{|t=0} \left\{\left(\frac{\partial(\exp(-tX)\cdot z)}{\partial z}\right)^{\eta} J^0_{\exp(-tX)}(z) f(exp(-tX)\cdot z)\right\}. \qquad (42)$$

There is a local action of $G^{\mathbb{C}}$, so this formula remains meaningful also for $X \in \mathfrak{g}^{\mathbb{C}}$. There are three factors to differentiate. For the last one, $\left(\frac{d}{dt}\right)_{|t=0} f(\exp(-tX)\cdot z) = -(Xz)f'(z)$, and we see that $\exp(tx)\cdot z = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\cdot z = z + t$ gives $x\cdot z = 1$; by similar computations, $y\cdot z = -z^2, h\cdot z = z$. For the first factor, we interchange the differentiations and get $-\eta\frac{\partial}{\partial z}(X\cdot z)$, i.e., $0, 2\eta z, -\eta$, respectively for x, y and h.

To differentiate the factor in the middle, we use its expression (40). First for $X = y$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \varrho^0(exp(-t(tz+1)^{-1}y) = \left.\frac{d}{dt}\right|_{t=0} (\exp\left(-t(tz+1)^{-1}\varrho^0(y)\right) = -\varrho^0 \quad (43)$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \varrho^0(\exp(2\log(tz+1)\,h)) = \left.\frac{d}{dt}\right|_{t=0} exp\left(2\,log(tz+1)\varrho^0(h)\right) = 2z\varrho^0 \quad (44)$$

¿From these, following the conventions of [88] in defining H,E,F, it follows that

$$(F\,f)(z) := \left(U_{-y}f\right)(z) = \left.\frac{d}{dt}\right|_{t=0} J_{\exp(ty)}(z) f(\exp(ty)\cdot z)$$

$$= \left(-2\eta z I + 2z\varrho^0(h) - \varrho^0(y)\right)f(z) - z^2 f'. \qquad (45)$$

Similar, simpler computations give, for $g = \exp(tx) = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$

$$(Ef)(z) := (U_x f)(z) = -f'(z). \qquad (46)$$

Finally, for $= \exp(th) = \begin{pmatrix} e^{\frac{t}{2}} & 0\\ 0 & e^{-\frac{t}{2}}\end{pmatrix}$, we have

$$J_{\exp(th)}(z) = \varrho\begin{pmatrix} e^{-\frac{t}{2}} & 0\\ 0 & e^{\frac{t}{2}}\end{pmatrix} = \exp(-t)\,\varrho^0(h).$$

Hence it is not hard to verify that

$$(Hf)(z) := (U_h f)(z) = \left(-\,\eta I + \varrho^0(h)\right)f(z) - zf'(z). \qquad (47)$$

Under our hypothesis, we have a reproducing kernel and U is unitary. From our computations above, we can determine how U decomposes into irreducibles. The infinitesimal representation of U acts on the vector valued polynomials; a good basis for this space is $\{\varepsilon_j z^n : n \geq 0\}$; ε_j is the jth natural basis vector in $\mathbb{C}^d$. We have $H\left(\varepsilon_j z^n\right) = -(\eta + j + n)\left(\varepsilon_j z^n\right)$, so the lowest $\mathbb{K}$ - types of the irreducible summands are spanned by the ε_j. This space also, the kernel of E. So, U is direct sum of discrete series representations ($U^{\eta+j}$, in the notation of [88]), each one appearing as many times as $-(\eta + j)$ appears on the diagonal of $\varrho(h)$.

In order to be able to use the computations of [72] without confusion, we introduce the parameter $= \eta + \frac{m}{2}$.
From the last remark, it is clear that if U is multiplicity-free then $\varrho(h)$ is an $(m+1)\times(m+1)$ matrix with eigenvalues $-\lambda+\frac{m}{2}, -\lambda+\frac{m}{2}-1, \dots, -\lambda-\frac{m}{2}$. As $(h)\varepsilon_j = -\left(\lambda - \frac{m}{2}+j\right)\varepsilon_j$, (39) shows that

$$\varrho(h)\big(\varrho(y)\varepsilon_j\big) = -\left(\lambda+\frac{m}{2}+j+1\right)\varrho(y)\varepsilon_j \text{ ,that is, } \varrho(y)\varepsilon_j = const\ \varepsilon_{j+1}.$$

So, $\varrho(y)$ is a lower triangular matrix (with non-zero entries, otherwise we have a reducible bundle). The homogeneous holomorphic vector bundle determines $\varrho(y)$ only up to a conjugacy by a matrix commuting with $\varrho(h)$, that is, a diagonal matrix. So, we can choose the realization of our bundle by applying an appropriate conjugation such that $\varrho(y) = S_m$, the triangular matrix whose $(j, j-1)$ element is j for $1 \le j \le m$.
By standard representation theory of $SL(2,\mathbb{R})$, the vectors $(-F)^n\varepsilon_j$ are orthogonal and the irreducible subspaces $\mathcal{H}^{(j)}$ for U are span$\{(-F)^n\varepsilon_j : n \ge 0\}$ for $0 \le j \le m$. There is also precise information about the norms.
Using this, we can construct an orthonormal basis for our representation space.
For any $n \ge 0$, we let $u_n^j(z) = (-F)^n\varepsilon_j$.
To proceed further, we need to find the vectors $u_n^j(z)$ explicitly. This is facilitated by the following Lemma.
Lemma (3.3.3)[84]: Let u be a vector with $u_\ell(z) = u_\ell z^{n-\ell}, 0 \le \ell \le m$ and $n \ge 0$. We then have

$$(-Fu)_\ell(z) = (2\lambda - m + \ell + n)u\ell z^{n+1-\ell} + \ell u_{\ell-1}z^{n+1-\ell}, 0 \le \ell \le m.$$

Proof. We recall (45) that $-(Ff)(z) = 2\lambda z f(z) + S_m f(z) - 2zD_m f(z) + z^2 f'(z)$ for $f \in \mathcal{H}(n)$, where $D_m = -\varrho^0(h)$ is the diagonal operator with diagonal $\left\{-\frac{m}{2}, -\frac{m}{2}+1, \dots, \frac{m}{2}\right\}$ and S_m is the forward weighted shift with weights $1, 2, \dots, m$. Therefore we have

$$(-Fu)\ell(z) = (2\lambda u_\ell + \ell u_{\ell-1} - (m-2\ell)u_\ell + (n-\ell)u_\ell)z^{n+1-\ell}$$

completing the proof.
Lemma (3.3.4)[84]: For $0 \le j \le m$ and $0 \le \ell \le m$, we have

$$u_{n,\ell}^j(z) = \begin{cases} 0 & if\ 0 \le \ell \le j-1 \\ \binom{n}{k}(j+1)_k(2\lambda - m + 2j + k)_{n-k}z^{n-k} & if\ j \le \ell \le m, k = \ell - j, \end{cases}$$

where $u_{n,\ell}^j(z)$ is the scalar valued function at the position ℓ of the $\mathbb{C}^{m+1}$- valued function $u_n^j(z) := (-F)^n\varepsilon_j$.
Proof. The proof is by induction on n. The vectors u_n^j are in $\mathcal{H}(n)$ for $0 \le j \le m$. For a fixed but arbitrary positive integer $j, 0 \le j \le m$, we see that $u_{n,\ell}^{j0}(z)$ is 0 if $n < \ell - j$. We have to verify that $\left(-F\, u_n^j\right)(z) = u_{n+1}^j(z)$. From the previous Lemma, we have

$$\left(-F\, u_n^j\right)_\ell(z) = (2\lambda - m + \ell + n + j)u_{n,\ell}^j z^{n+j+1-\ell} + \ell u_{n,\ell-1}^j z^{n+j+1-\ell}$$

where $\left(-F\, u_n^j\right)_\ell(z)$ is the scalar function at the position ℓ of the $\mathbb{C}^{m+1}$- valued function $\left(-F\, u_n^j\right)(z)$. To complete the proof, we note (using $k = \ell - j$) that

$$\left(-F\, u_n^j\right)_{j+k}(z)$$

$$= (\binom{n}{k} (j+1)_k (2\lambda - m + 2j + k)_{n-k} (2\lambda - m + 2j + k + n)$$
$$+ \left(\binom{n}{k-1} (j+1)_k (2\lambda - m + 2j + k - 1)_{n-k}\right) z^{n+1-k}$$
$$= (j+1)_k (2\lambda - m + 2j + k)_{n-k} (\binom{n}{k} \ (2\lambda - m + 2j + k + n)$$
$$+ \binom{n}{k-1} \ (2\lambda - m + 2j + k - 1) z^{\,n+1-k})$$
$$=(j+1)_k (2\lambda - m + 2j + k)_{(n-k)} ((\binom{n}{k} + \binom{n}{k-1} \ (2\lambda - m + 2j + k - 1) + (n + 1) \binom{n}{k}) z^{n+1-k}$$
$$= (j+1)_k (2\lambda - m + 2j + k)_{(n-k)} \left(\binom{n+1}{k} \ (2\lambda - m + 2j + k - 1)\right.$$
$$\left. + \binom{n+1}{k} \ (n - k + 1) + \right) z^{n+1-k}$$
$$= (j+1)_k (2\lambda - m + 2j + k)_{n-k} \left(\binom{n+1}{k} \ (2\lambda - m + 2j + n)\right) z^{\,n+1-k}$$
$$= (j+1)_k \left(\binom{n+1}{k} \ (2\lambda - m + 2j + k)_{n+1-k}\right) z^{n+1-k} = \ u_{n+1,j+k}^j(z)$$

for a fixed but arbitrary j, $0 \le j \le m$ and k, $0 \le k \le m-j$. This completes the proof. On H(j), we have the representation U^{λ_j} acting $(0 \le j \le m)$, where $\lambda_j = \lambda - \frac{m}{2} + j$. Its lowest $\mathbb{K}$ -type is spanned by $\varepsilon_j (= u_0^j)$ and $\varepsilon_j = \lambda_j \varepsilon_j$. By [88] we have $\left\|(-F)^k \varepsilon_j\right\|^2 = \left\|\sigma_k^j (-F)^{k-1} \varepsilon_j\right\|^2$ with

$$\sigma_k^j = \left(2\lambda_j + k - 1\right) k$$

for all $k \ge 1$. (Here we used that the constant q in [88] equals $\lambda_j \ (1 - \lambda_j)$ by [88].) We write

$$\sigma_n^j = \prod_{k=1}^{n} \sigma_k^j$$

which can be written in a compact form

$$\sigma_n^j = \left(\left(2\lambda_j\right)_n (1)_n\right), \qquad (48)$$

where $(x)_n = (x+1) \cdots (x+n-1)$. We stipulate that the binomial co-efficient $\binom{n}{k}$ as well as $(x)_{n-k}$ are both zero if $n < k$.

The positivity of the normalizing constants $\left(\sigma_{n-j}^j\right)^{\frac{1}{2}} (n \ge j)$ is equivalent to the existence of an inner product for which the set of vectors e_{n-j}^j defined by the formula:

$$e_{n-j}^j = \left(\sigma_n^j\right)^{-\frac{1}{2}} u_{n-j}^j(z), \qquad n \ge j, 0 \le j \le m$$

forms an orthonormal set. Of course, the positivity condition is fulfilled if and only if $2\lambda > m$.

In this way, for fixed j, each e_{n-j}^j has the same norm for all $n \ge j$. Hence the only possible choice for an orthonormal system is $\left\{\mu_j e_{n-j}^j : n \ge j\right\}$ for some positive $(0 \le j \le m)$.

However, we may choose the norm of the first vector, that is, the vector $e_0^j, 0 \le j \le m$, arbitrarily. Therefore, all the possible choices for an orthonormal set are

$$\mu_j e_{n-j}^j(z) = \frac{\mu_j}{\sqrt{(2\lambda - m + 2j)_{n-j}}\sqrt{(1)_{n-j}}} u_{n-j}^j(z),$$

$n \ge j, 0 \le j \le m,$ and $\mu_j, 0 \le j \le m$ are $m+1$ arbitrary positive numbers.

Let us fix a positive real number λ and $m \in \mathbb{N}$ satisfying $2\lambda > m$. Let $\mathcal{H}^{(\lambda,\mu)}$ denote the closed linear span of the vectors $\{\mu_j e_{n-j}^j : 0 \le j \le m, n \ge j\}$. Then the Hilbert space $\mathcal{H}^{(\lambda,\mu)}$ is the representation space for U defined in (32). Since the vectors $u_n^j \perp u_p^k$ as long as $j \ne k$, it follows that the Hilbert space $\mathcal{H}^{(\lambda,\mu)}$ is the orthogonal direct sum $\bigoplus_{j=0}^m 1/\mu_j \mathcal{H}^{(j)}$. We proceed to compute the reproducing kernel by using the orthonormal system $\{\mu_j e_{n-j}^j : n \ge j\}, 0 \le j \le m$. We point out that for $0 \le \ell \le m$, the entry $e_{n-j}^{\ell,j} z^{n-j}$ at the position ℓ of the vector $e_{n-j}^j\,(z)$ is 0 for $n < \ell$. Consequently, e_{n-j}^j is the zero vector unless $n \ge j$. The set of vectors $\{\mu_j e_{n-j}^j : 0 \le j \le m, n \ge j\}$ is orthonormal in the Hilbert space $\mathcal{H}^{(\lambda,\mu)}$. We note that

$$e_{n-j}^j(z) = \left(\left(e_{n-j}^{\ell,j} z^{n-k}\right)\right)_{\ell=0}^m,$$

$$\left(e_{n-j}^j(z)\right)_\ell = \begin{cases} 0, & 0 \le \ell \le j-1 \\ \sqrt{\dfrac{(2\lambda + 2j - m + k)_{n-j-k}}{(1)_{n-j-k}}}\sqrt{\dfrac{(n-j-k+1)_k}{(2\lambda+2j-m)_k}}\dfrac{(j+1)_k}{(1)_k} z^{n-k}, j \le \ell \le m, k = \ell - j. \end{cases} \tag{50}$$

We have under the hypothesis that we have a reproducing kernel Hilbert space on which the representation U is unitary, explicitly determined an orthonormal basis for this space. Now we are able to answer the question of whether this space really exists. For this it is enough to show that $\sum e_n(z)\overline{e_n(w)}^{tr}$ converges pointwise, the sum then represents the reproducing kernel for this Hilbert space. We will sum the series explicitly, and will verify that it gives exactly the kernels constructed in [72]. This will complete the program by proving that the examples of [72] give all the homogeneous operators in the Cowen-Douglas class whose associated representation is multiplicity free.

To compute the kernel function, it is convenient to set, for any $n \ge 0$,

$$\begin{aligned} G(\mu, n, z) &= \begin{pmatrix} \mu_0 e_n^{0,0} z^n & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \mu_0 e_n^{j,0} z^{n-j} & \cdots & \mu_j e_{n-j}^{j,j} z^{n-j} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \mu_0 e_n^{m,0} z^{n-m} & \cdots & \mu_j e_{n-j}^{m,j} z^{n-m} & \cdots & \mu_m e_{n-m}^{m,m} z^{n-m} \end{pmatrix} \\ &= \begin{pmatrix} z^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z^{n-m} \end{pmatrix} \begin{pmatrix} e_n^{0,0} & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ e_n^{j,0} & \cdots & e_{n-j}^{j,j} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ e_n^{m,0} & \cdots & e_{n-j}^{m,j} & \cdots & e_{n-m}^{m,m} \end{pmatrix} \begin{pmatrix} \mu_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_m \end{pmatrix} \\ &= D_n(z) G(n) D(\mu) \end{aligned} \tag{51}$$

where $D_n(z), D(\mu)$ are the two diagonal matrices and $G(n) = \left(e^{\ell,j}_{n-j}\right)^m_{\ell,j=0}$ with $e^{\ell,j}_{n-j} = 0$ if $\ell < j$ or if $n < \ell$. The nonzero entries of the lower triangular matrix $G(n)$, using (50), are

$$\begin{aligned} G_{j+k,j}(n) &= \frac{\binom{n-j}{k}(j+1)_k(2\lambda - m + 2j + k)_{n-j-k}}{\sqrt{(2\lambda - m + 2j)_{n-j}}\sqrt{(1)_{n-j}}} \\ &= \frac{\sqrt{(2\lambda - m + 2j + k)_{n-j-k}}}{\sqrt{(2\lambda - m + 2j)_k}} \frac{(n\ - p\, j - k + 1)_k}{\sqrt{(1)_{n-j}}} \frac{(j+1)_k}{(1)_k} \\ &= \frac{\sqrt{(2\lambda - m + 2j + k)_{n-j-k}}}{\sqrt{(2\lambda - m + 2j)_k}} \sqrt{\frac{(n - j - k + 1)_k}{(1)_{n-j-k}} \frac{(j+1)_k}{(1)_k}} \end{aligned} \tag{52}$$

for $0 \leq k \leq m - j$.

Now, we are ready to compute the reproducing kernel K_j for the Hilbert space $\mathcal{H}^{(j)} = span\{e^j_{n-j}: n \geq j\}, 0 \leq j \leq m$. Recall that $K(z,w) = \sum_{n=0}^{\infty} e_n(z)e_n(w)^*$ for any orthonormal basis $e_n, n \geq 0$. This ensures that K is a positive definite kernel. For our computations, we will use the particular orthonormal basis e^j_{n-j} as described in (49). Since there are j zeros at the top of each of these basis vectors, it follows that (ℓ, p) will be 0 if either $\ell < j$ or $p < j$. We will compute $\left(K_j(z,w)\right)$, at (ℓ, p) for $j \leq \ell, p \leq m$. For ℓ, p as above, we have

$$\left(K_j(z,w)\right)_{\ell,p} = \sum_{n \geq \max(\ell,p)}^{\infty} e^j_{n-j,\ell}(z)\overline{e^j_{n-j,p}(w)} = \sum_{n \geq \max(\ell,p)}^{\infty} G_{\ell,j}(n)G_{p,j}(n)z^{n-\ell}\overline{w}^{n-p}.$$

We first simplify the coefficient $G_{\ell,j}(n)G_{p,j}(n)$ of $z^{n-\ell}\overline{w}^{n-p}$. The values of $G_{\ell,j}(n)$ are given in (52). Therefore, we have

$$\begin{aligned} &G_{\ell,j}(n)G_{p,j}(n) \\ &\quad= \left(\frac{\left(2\lambda_j + \ell - j\right)_{n-\ell}}{\left(2\lambda_j\right)_{\ell-j}} \frac{(n - \ell + 1)_{\ell-j}}{(1)_{n-\ell}} \frac{\left(2\lambda_j + p - j\right)_{n-p}}{\left(2\lambda_j\right)_{p-j}} \frac{(n - \ell + 1)_{\ell-j}}{(1)_{n-p}}\right)^{\frac{1}{2}} \\ &\quad\times \frac{(j+1)_{\ell-j}}{(1)_{\ell-j}} \frac{(j+1)_{p-j}}{(1)_{p-j}} \\ &\quad= \frac{\left(2\lambda_j + p - j\right)_{n-p}(n - \ell + 1)_{\ell-j}}{\left(2\lambda_j\right)_{\ell-j}(1)_{n-p}} \left(\frac{\left(2\lambda_j + \ell - j\right)_{p-\ell}(n - p + 1)_{p-\ell}}{\left(2\lambda_j + \ell - j\right)_{p-\ell}(n - p + 1)_{p-\ell}}\right)^{\frac{1}{2}} \\ &\quad\times \frac{(j+1)_{\ell-j}}{(1)_{\ell-j}} \frac{(j+1)_{p-j}}{(1)_{p-j}} \\ &= \frac{\left(2\lambda_j\right)_{p-j}\left(2\lambda_j + p - j\right)_{n-p}(n - \ell + 1)_{\ell-j}(n - p + 1)_{p-j}}{\left(2\lambda_j\right)_{p-j}\left(2\lambda_j\right)_{\ell-j}(1)_{n-p}(n - p + 1)_{p-j}} \frac{(j+1)_{\ell-j}}{(1)_{\ell-j}} \frac{(j+1)_{p-j}}{(1)_{p-j}} \\ &\quad= \frac{\left(2\lambda_j\right)_{n-j}(n - \ell + 1)_{\ell-j}(n - p + 1)_{p-j}}{\left(2\lambda_j\right)_{p-j}\left(2\lambda_j\right)_{\ell-j}(1)_{n-j}} \frac{(j+1)_{\ell-j}}{(1)_{\ell-j}} \frac{(j+1)_{p-j}}{(1)_{p-j}}. \end{aligned}$$

Theorem (3.3.5)[84]: Given an arbitrary set $\mu_0, \ldots, \mu_m$ of positive numbers, and $2\lambda > m$, we have

$$K^{(\lambda,\mu)}(z,w) = \sum_{j=0}^{m} \mu_j^2 K_j(z,w) = B^{(\lambda,\mu)}(z,w).$$

As a result, the two Hilbert spaces $\mathcal{H}^{(\lambda,\mu)}$ and $A^{(\lambda,\mu)}$ of [72] are equal.

Proof. We now compare the co-efficients $\left(K_j(z,w)\right)_{\ell,p}$ with that of a known Kernel. Let $B^{\lambda_j}(z,w) = (1 - z\bar{w})^{-2\lambda_j}$, where $B(z,w) = (1 - z\bar{w})^{-2}$ is the Bergman kernel on the unit disc. We let ∂ and $\bar{\partial}$ denote differentiation with respect to z and $\bar{w}$ respectively. Put

$$\tilde{B}^{(\lambda_j)}(z,w) = \left(\partial^{\ell-j}\bar{\partial}^{p-j}(1 - z\bar{w})^{-2\lambda_j}\right)_{j\le\ell,p\le m}.$$

We expand the entry at the position (ℓ,p) of $\tilde{B}^{(\lambda_j)}(z,w)$ to see that

$$\begin{aligned}\left(\tilde{B}^{(\lambda_j)}(z,w)\right)_{\ell,p} &= \sum_{\nu\ge max(\ell-j,p-j)} \frac{(2\lambda_j)_\nu}{(1)_\nu}(\nu - \ell + j + 1)_{\ell-j}(\nu + j - p \\ &\quad + 1)_{p-j} z^{\nu-(\ell-j)}\bar{w}^{\nu-(p-j)} \\ &= \sum_{n\ge max(\ell,p)} \frac{(2\lambda_j)_{n-j}}{(1)_{n-j}}(n - \ell + 1)_{\ell-j}(n - p + 1)_{p-j} z^{n-\ell}\bar{w}^{n-p},\end{aligned}$$

where we have set $n = m + j$. Comparing these coefficients with that of $G_{\ell,j}(n)G_{p,j}(n)$, we find that

$$K_j(z,w) = D_j\tilde{B}^{(\lambda_j)}(z,w)D_j, \tag{53}$$

where D_j is a diagonal matrix with $\frac{1}{(2\lambda_j)_{\ell-j}}\frac{(j+1)_{\ell-j}}{(1)_{\ell-j}}$ at the (ℓ,ℓ) position with $j \le \ell \le m$. Hence $K_j(z,w) = B^{(\lambda_j)}(z,w)$ which was defined in the equation ([72]).

Clearly, we can add up the kernels K_j to obtain the kernel $K^{(\lambda,\mu)}$ for the Hilbert space $\mathcal{H}^{(\lambda,\mu)} = \frac{\oplus_{j=0}^{m} 1}{\mu_j}\mathcal{H}^{(j)}$. Hence the proof of the theorem is complete.

Corollary (3.3.6)[84]: The irreducible homogeneous operators in the Cowen - Douglas class whose associated representation is multiplicity free are exactly the adjoints of $M^{(\lambda,\mu)}$ constructed in [72].

Proof. In our discussion up to here we proved that the Hilbert space $\mathcal{H}^{(\lambda,\mu)}$ corresponding to a homogeneous operator in the Cowen - Douglas class has a reproducing kernel given by $K^{(\lambda,\mu)} = \sum_0^m \mu_j^2 K_j$, $2\lambda > 1, \mu_1, \ldots, \mu_m > 0$. It follows from the Theorem that the kernels obtained this way are the same as (are equivalent to) the kernels constructed in [72]. These operators were shown to be irreducible [72].

We now consider the action of the multiplication operator $M^{(\lambda,\mu)}$ on the Hilbert space $\mathcal{H}^{(\lambda,\mu)}$. Let $\mathcal{H}(n)$ be the linear span of the vectors

$$\{e_n^0(z), \ldots, e_{n-j}^j(z), \ldots, e_{n-m}^m(z)\},$$

where as before, for $0 \le \ell \le m, e_{n-\ell}^j(z)$ is zero if $n - \ell < 0$. Clearly, $\mathcal{H}^{(\lambda,\mu)} = \bigoplus_{n=0}^{\infty} \mathcal{H}(n)$. We have

$$zG(n,z) = D_n(z)G(n)D(\mu) = D_{n+1}(z)G(n)D(\mu)$$
$$= D_{n+1}(z)G(n+1)D(\mu)(D(\mu)^{-1}G(n+1)^{-1}G(n)D(\mu).$$

If we let $W(n) = D(\mu)^{-1}G(n+1)^{-1}G(n)D(\mu)$,then we see that $ze^{j}_{n-j}(z) = G(\mu,n+1,z)W_j(n)$, where $W_j(n)$ is the jth column of the matrix $W(n)$. It follows that the operator $M^{(\lambda,\mu)}$ defines a block shift W on the representation space $\mathcal{H}^{(\lambda,\mu)}$. The block shift W is defined by the requirement that $W: \mathcal{H}(n) \to \mathcal{H}(n+1)$ and $W_{|\mathcal{H}(n)} = W_n^{tr}$.

Here, we have a construction of the representation space $\mathcal{H}^{(\lambda,\mu)}$ along with the matrix representation of the operator $M^{(\lambda,\mu)}$ which is independent of the corresponding results from [72].

Example (3.3.7)[84]: Recall that $G(\mu,n,z) = D_n(z)G(n)D(\mu)$. Once we determine the matrix $G(n)$ explicitly, we can calculate both the block weighted shift and the kernel function.

We discuss these calculations in the particular case of $m = 1$. First, it is easily seen that

$$G(n) = \begin{pmatrix} \left(\frac{(2\lambda-1)_n}{(1)_n}\right)^{\frac{1}{2}} & 0 \\ \left(\frac{n}{2\lambda-1}\right)^{\frac{1}{2}}\left(\frac{(2\lambda)_{n-1}}{(1)_{n-1}}\right)^{\frac{1}{2}} & \left(\frac{(2\lambda+1)_{n-1}}{(1)_{n-1}}\right)^{\frac{1}{2}} \end{pmatrix}. \quad (54)$$

The block W_n of the weighted shift W is

$$W_n = \begin{pmatrix} \left(\frac{n+1}{2\lambda+n-1}\right)^{\frac{1}{2}} & 0 \\ -\frac{1}{\mu_1}\left(\frac{2\lambda}{2\lambda-1}\right)^{\frac{1}{2}}\left(\frac{1}{(2\lambda+n-1)(2\lambda+n)}\right)^{\frac{1}{2}} & \left(\frac{n}{2\lambda}+n\right)^{\frac{1}{2}} \end{pmatrix}. \quad (55)$$

Finally, the reproducing kernel $K^{(\lambda,\mu)}$ with $m = 1$ is easily calculated:

$$K^{(\lambda,\mu)}(z,w) = \begin{pmatrix} \frac{1}{(1-\bar{w}z)^{2\lambda-1}} & z(1-\bar{w}z)^{2\lambda} \\ \frac{\bar{w}}{(1-\bar{w}z)^{2\lambda}} & \frac{1}{2\lambda-1}\frac{1+(2\lambda-1)\bar{w}z}{(1-\bar{w}z)^{2\lambda+1}} \end{pmatrix}$$
$$+ \mu_1^2\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{(1-\bar{w}z)^{2\lambda+1}} \end{pmatrix}. \quad (56)$$

One might continue the explicit calculations, as above, in the particular case of $m = 2$ as well. We begin with the matrix

$$G(n) = \begin{pmatrix} \left(\frac{(2\lambda-2)_n}{(1)_n}\right)^{\frac{1}{2}} & 0 & 0 \\ \left(\frac{n}{2\lambda-2}\right)^{\frac{1}{2}}\left(\frac{(2\lambda-1)_{n-1}}{(1)_{n-1}}\right)^{\frac{1}{2}} & \left(\frac{(2\lambda)_{n-1}}{(1)_{n-1}}\right)^{\frac{1}{2}} & 0 \\ \left(\frac{n(n-1)}{(2\lambda-2)(2\lambda-1)}\right)^{\frac{1}{2}}\left(\frac{(2\lambda)_{n-2}}{(1)_{n-2}}\right)^{\frac{1}{2}} & 2\left(\frac{n-1}{2\lambda}\right)^{\frac{1}{2}}\left(\frac{(2\lambda+1)_{n-2}}{(1)_{n-2}}\right)^{\frac{1}{2}} & \left(\frac{(2\lambda+2)_{n-2}}{(1)_{n-2}}\right)^{\frac{1}{2}} \end{pmatrix}. (58)$$

The block W_n of the weighted shift W, in this case, is

$$\begin{pmatrix} \left(\frac{n+1}{2\lambda+n-2}\right)^{\frac{1}{2}} & 0 & 0 \\ \frac{-1}{\mu_1}\left(\frac{2\lambda-1}{2\lambda-2}\right)^{\frac{1}{2}}\left(\frac{1}{(2\lambda+n-1)(2\lambda+n-2)}\right)^{\frac{1}{2}} & \left(\frac{n}{2\lambda+n-1}\right)^{\frac{1}{2}} & 0 \\ \frac{-2}{\mu_2}\left(\frac{2\lambda+1}{(2\lambda-2)_3}\right)^{\frac{1}{2}}\left(\frac{n}{(2\lambda+n-2)_3}\right)^{\frac{1}{2}} & \frac{(-2)\mu_1}{\mu_2}\left(\frac{2\lambda+1}{2\lambda}\right)^{\frac{1}{2}}\left(\frac{1}{(2\lambda+n-1)(2\lambda+n)}\right)^{\frac{1))}{2}} & \left(\frac{n-1}{2\lambda+n}\right)^{\frac{1}{2}} \end{pmatrix}.$$

Finally, the reproducing kernel $K^{(\lambda,\mu)}$ with $m = 2$ has the form:

$$K^{(\lambda,\mu)}(z,w) =$$

$$\begin{pmatrix} \frac{1}{(1-\bar{w}z)^{2\lambda-2}} & \frac{z}{(1-\bar{w}z)^{2\lambda-1}} & \frac{z^2}{(1-\bar{w}z)^{2\lambda}} \\ \frac{\bar{w}}{(1-\bar{w}z)^{2\lambda-1}} & \frac{1+(2\lambda-2)\bar{w}z}{(2\lambda-2)(1-\bar{w}z)^{2\lambda}} & \frac{z(2+(2\lambda-2)\bar{w}z)}{(2\lambda-2)(1-\bar{w}z)^{2\lambda+1}} \\ \frac{\bar{w}^2}{(1-\bar{w}z)^{2\lambda}} & \frac{\bar{w}(2+(2\lambda-2)\bar{w}z)}{(2\lambda-2)(1-\bar{w}z)^{2\lambda+1}} & \frac{2+4(2\lambda-1)\bar{w}z+(2\lambda-1)(2\lambda-2)z^{2\bar{w}^2}}{(2\lambda-1)(2\lambda-2)(1-\bar{w}z)^{2\lambda+2}} \end{pmatrix}$$

$$+\mu_1^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{(1-\bar{w}z)^{2\lambda}} & 2\frac{z}{(1-\bar{w}z)^{2\lambda+1}} \\ 0 & 2\frac{\bar{w}}{(1-\bar{w}z)^{2\lambda+1}} & 2\frac{2}{2\lambda}\frac{1+2\lambda\bar{w}z}{(1-\bar{w}z)^{2\lambda+2}} \end{pmatrix} + \mu_2^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(1-\bar{w}z)^{2\lambda+2}} \end{pmatrix}. \quad (59)$$

Chapter 4
Sharp Estimates of all Homogeneous Expansions and a Proof of a Weak Version

We establish the sharp upper bounds of growth theorem and distortion theorem for a k-fold symmetric quasi-convex mapping. These results show that in the case of quasi-convex mappings, Bieberbach conjecture in several complex variables is partly proved, and many known results are generalized. The results show that a weak version of the Bieberbach conjecture in several complex variables is proved, and the obtained conclusions reduce to the classical results in one complex variable. The results state that a weak version of the Bieberbach conjecture for quasi-convex mappings of type B and order α in several complex variables is proved, and the derived conclusions are the generalization of the classical results in one complex variable.

Section (4.1): A Class of Quasi-convex Mappings on the Unit Polydisk in $\mathbb{C}^n$*

In the case of one complex variable, the following Bieberbach conjecture (i.e., de Branges theorem) is well-known.

Theorem (4.1.1)[89]: (see [15]) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a biholomorphic function on the unit disk U, then

$$|a_n| \leqslant n, \qquad n = 2, 3, \dots .$$

However, in the case of several complex variables, Cartan [90] pointed out that the above theorem does not hold. So people mainly investigated the case of the subclasses of biholomorphic mappings Bieberbach conjecture in several complex variables. In 1992, Zhang, Dong and Wang [91] first established the sharp estimates of all homogeneous expansions for normalized biholomorphic convex mappings on the unit ball in a complex Banach space with a brief proof. But with respect to the estimates of all homogeneous expansions for normalized biholomorphic starlike mappings, quasi-convex mappings of type A and quasi-convex mappings of type B on the Euclidean unit ball B^n in $\mathbb{C}^n$, Roper and Suffridge [1] stated that the corresponding Bieberbach conjecture does not hold about the second homogeneous expansions with concrete counterexamples. Taking into account the above reason, people chiefly show interest in studying the estimates of homogeneous expansions for the subclasses of biholomorphic mappings on the unit polydisk U^n in $\mathbb{C}^n$. In 1999, Gong [15] posed the following conjecture.

Conjecture (4.1.2)[89]: If $f: U^n \to \mathbb{C}^n$ is a normalized biholomorphic starlike mapping on the unit polydisk U^n in $\mathbb{C}^n$, then

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant m\|z^m\|, \qquad z \in U^n, \qquad m = 2, 3, \cdots .$$

It is obvious that the above conjecture is quite similar to the famous Bieberbach conjecture in one complex variable. Recently, Liu [14] investigated the estimates of all homogeneous expansions for a class of quasi-convex mappings (including quasi-convex mappings of type A and quasiconvex mappings of type B), Liu and Liu [92] established the estimates of all homogeneous expansions for a class of k-fold symmetric quasi-convex mappings of type B and order α.

On the other hand, at present, the sharp growth, covering and distortion theorem for quasiconvex mappings of type B on U^n is not given, the sharp distortion theorem for quasi-convex mappings (including quasi-convex mappings of type A) on U^n is not given as well (see [93]).

In view of the additional condition for the above mappings (see [14], [92]) are somewhat special, a natural question arises to how to weaken the additional condition in order to obtain the generalization of known results.

For X be a complex Banach space with norm $\|\cdot\|, X^*$ be the dual space of X and let $T(x) = \{T_x \in X^*: \|T_x\| = 1, T_x(x) = x\}$. Let E be the unit ball in X, let ∂E be the boundary of E and let $\bar{E}$ be the closure of E. Let U stand for the Euclidean unit disk in $\mathbb{C}$, let U^n be the unit polydisk in $\mathbb{C}^n$, and let $\partial_0 U^n$ denote the characteristic boundary (i.e., the boundary on which the maxium modulus of the holomorphic function can be attained) of U^n. Let the symbol $'$ mean transpose. Let $\mathbb{N}$ be the set of all positive integers.

Definition (4.1.3)[89]: (see [16]) Suppose that $f: E \to X$ is a normalized locally biholomorphic mapping. Denote

$$G_f(\alpha, \beta) = \frac{2\alpha}{T_u\left[\left(Df(\alpha u)\right)^{-1}\left(f(\alpha u) - f(\beta u)\right)\right]} - \frac{\alpha + \beta}{\alpha - \beta}.$$

If

$$Re\, G_f(\alpha, \beta) \geqslant 0, \qquad u \in \partial E, \qquad \alpha, \beta \in U,$$

then f is said to be a quasi-convex mapping of type A on E.

Definition (4.1.4)[89]: (see [16]) Suppose that $f: E \to X$ is a normalized locally biholomorphic mapping. If

$$Re\left\{T_x\left[\left(Df(x)\right)^{-1}\left(D^2 f(x)(x^2) + Df(x)x\right)\right]\right\} \geqslant 0, \qquad x \in E,$$

then f is said to be a quasi-convex mapping of type B on E.

When $X = \mathbb{C}^n$, Definitions (4.1.3) and (4.1.4) were originally introduced by Roper and Suffridge [1].

Definition (4.1.5)[89]: (see [94]) Suppose that $f: E \to X$ is a normalized locally biholomorphic mapping. If

$$Re\ \left\{T_x\left[\left(Df(x)\right)^{-1}\left(f(x) - f(\xi x)\right)\right]\right\} \geqslant 0, \qquad x \in E, \xi \in \bar{U},$$

then f is said to be a quasi-convex mapping on E.

When $X = \mathbb{C}$, Definitions (4.1.3)– (4.1.4) are the same; this implies that a quasi-convex function is equivalent to a normalized biholomorphic convex function in one complex variable.

Definition (4.1.6)[89]: (see [95]) Suppose $f \in H(E)$. It is said that f is k-fold symmetric if $e^{-\frac{2\pi i}{k}} f\left(e^{\frac{2\pi i}{k}} x\right) = f(x)$ for all $x \in E$, where $k \in \mathbb{N}$ and $i = \sqrt{-1}$.

Definition (4.1.7)[89]: (see [96]) Suppose that Ω is a domain (connected open set) in X which contains 0. It is said that $x = 0$ is a zero of order k of $f(x)$ if $f(0) = 0, \dots, D^{k-1} f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$.

Definition (4.1.8)[89]: (see [92]) Suppose that $\alpha \in [0, 1)$ and $f: E \to X$ is a normalized locally biholomorphic mapping. If

$$Re\ \left\{T_x\left[\left(Df(x)\right)^{-1}\left(D^2 f(x)(x^2) + Df(x)x\right)\right]\right\} \geqslant \alpha\|x\|, \qquad x \in E,$$

then f is said to be quasi-convex of type B and order α on E.

Let $K(E)$ denote the set of all normalized biholomorphic convex mappings on E. Let $Q_A(E)$ (resp. $Q_B(E)$) be the set of all quasi-convex mappings of type A (resp. type B) on E and let $Q(E)$ be the set of all quasi-convex mappings on E.

We shall first give the following lemmas. It is easy to prove the following results.

Lemma (4.1.9)[89]: Suppose that f is a normalized locally biholomorphic mapping on U^n. Then $f \in Q_B(U^n)$ if and only if

$$Re\frac{g_j(z)}{z_j} \geqslant 0, \qquad z = (z_1, \dots, z_n)' \in U^n,$$

where $g(z) = \left(g_1(z), \dots, g_n(z)\right)' = \left(Df(z)\right)^{-1}(D^2f(z)(z^2) + Df(z)z)$ is a column vector in $\mathbb{C}^n$, j satisfies $|z_j| = \|z\| = \max_{1\leqslant k\leqslant n}\{|z_k|\}$.

Lemma (4.1.10)[89]: (see [97]) Suppose $g(z) = \left(g_1(z), g_2(z), \dots, g_n(z)\right)' \in H(U^n), g(0) = 0, Dg(0) = I$. If $Re\frac{g_j(z)}{z_j} \geqslant 0$ $(z \in U^n)$, where $|z_j| = \|z\| = \max_{1\leqslant k\leqslant n}\{|z_k|\}$, then

$$\frac{\|D^m g(0)(z^m)\|}{m!} \leqslant 2\|z\|^m, \qquad z \in U^n, \qquad m = 2, 3, \dots.$$

Lemma (4.1.11)[89]: (see [14]) If $f(z)$ is a normalized locally biholomorphic mapping on U^n, and $g(z) = \left(Df(z)\right)^{-1}(D^2f(z)(z^2) + Df(z)z) \in H(U^n)$, then

$$\frac{D^2f(0)(z^2)}{2!} = \frac{1}{2}\cdot\frac{D^2g(0)(z^2)}{2!},$$

$$m(m-1)\frac{D^mf(0)(z^m)}{m!}$$
$$= \frac{D^mg(0)(z^m)}{m!} + \frac{2D^2f(0)\left(z, \frac{D^{m-1}g(0)(z^{m-1})}{(m-1)!}\right)}{2!} + \cdots$$
$$\cdot + \frac{(m-1)D^{m-1}f(0)\left(z^{m-2}, \frac{D^2g(0)(z^2)}{2!}\right)}{(m-1)!}, z \in U^n, m = 3, 4, \cdots.$$

Lemma (4.1.12)[89]: (see [16]) $K(E) \subset Q(E) = Q_A(E) \subset Q_B(E)$. In some concrete complex Banach spaces, we even have $K(E) \leqslant Q(E)$.

Lemma (4.1.13)[89]: Suppose $f(z) \in H(U^n)$, and

$$\frac{D^mf_p(0)(z^m)}{m!} = \sum_{l_1,l_2,\dots,l_m=1}^{n} a_{pl_1l_2\cdots l_m}z_{l_1}z_{l_2}\cdots z_{l_m}, \qquad p = 1, 2, \cdots, n,$$

where $a_{pl_1l_2\dots l_m} = \frac{1}{m!}\frac{\partial^m f_p(0)}{\partial z_{l_1}\partial z_{l_2}\dots\partial z_{l_m}}, l_1, l_2, \dots, l_m = 1, 2, \dots, n, m = 2, 3, \dots$. Then

$$\frac{1}{m!}D^mf_p(0)(z^{m-1}, w)$$
$$= \frac{1}{m}\left(\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{pl_1l_2\dots l_m}w_{l_1}z_{l_2}\dots z_{l_m} + \sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\,l_1l_2\dots l_m}z_{l_1}w_{l_2}z_{l_3}\dots z_{l_m}\right.$$
$$\left. + \cdots + \sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\,l_1l_2\dots l_m}z_{l_1}z_{l_2}\dots z_{l_{m-1}}w_{l_m}\right), z \in U^n, p = 1, 2, \dots, n, m$$
$$= 2, 3, \dots,$$

where $w = (w_1, w_2, \dots, w_n)' \in \mathbb{C}^n$ which satisfies $\|w\| = \max_{1\leqslant p\leqslant n}\{|w_p|\} < 2$.

Proof: $\forall\lambda \in \mathbb{C}$ with $|\lambda| \leqslant \frac{1}{2}$, by a straightforward computation, it yields that

$$D^m f_p(0)\underbrace{\left(\frac{z+\lambda w}{2},\frac{z+\lambda w}{2},\dots,\frac{z+\lambda w}{2}\right)}_{m}$$
$$=\frac{D^m f_p(0)(z^m)}{2^m}+\frac{mD^m f_p(0)(z^{m-1},w)}{2^m}\lambda+\cdots$$
$$+\frac{D^m f_p(0)(w^m)}{2^m}\lambda^m. \qquad (1)$$

Note that $\frac{D^m f_p(0)(z^m)}{m!}=\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\dots l_m} z_{l_1} z_{l_2}\dots z_{l_m}$. Therefore,

$$D^m f_p(0)\underbrace{\left(\frac{z+\lambda w}{2},\frac{z+\lambda w}{2},\dots,\frac{z+\lambda w}{2}\right)}_{m} m$$
$$=\frac{m!}{2m}\left(\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\dots l_m}(z_{l_1}+\lambda w_{l_1})(z_{l_2}+\lambda w_{l_2})\dots(z_{l_m}+\lambda w_{l_m})\right)$$
$$=\frac{m!}{2m}\left(\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\dots l_m} z_{l_1} z_{l_2}\dots z_{l_m}\right)$$
$$+\frac{m!}{2m}\left(\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\dots l_m} w_{l_1} z_{l_2}\dots z_{l_m}\right.$$
$$+\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\dots l_m} z_{l_1} w_{l_2} z_{l_3}\dots z_{l_m}+\cdots$$
$$\left.+\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\dots l_m} z_{l_1} z_{l_2}\dots z_{l_{m-1}} w_{l_m}\right)\lambda$$
$$+\cdots+\frac{m!}{2m}\left(\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\dots l_m} w_{l_1} w_{l_2}\dots w_{l_m}\right)\lambda^m. \qquad (2)$$

Comparing with the coefficient of the right-hand sides of (1) and (2) with respect to λ, we obtain

$$\frac{1}{m!}D^m f_p(0)(z^{m-1},w)$$
$$=\frac{1}{m}\left(\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\cdots l_m} w_{l_1} z_{l_2}\dots z_{l_m}+\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\cdots l_m} z_{l_1} w_{l_2} z_{l_3}\cdots z_{l_m}\right.$$
$$\left.+\cdots+\sum_{l_1,l_2,\dots,l_m=1}^{n} a_{p\ l_1 l_2\cdots l_m} z_{l_1} z_{l_2}\cdots z_{l_{m-1}} w_{l_m}\right), z\in U^n, p=1,2,\cdots,n, m$$
$$=2,3,\cdots,$$

where $w=(w_1,w_2,\cdots,w_n)'\in\mathbb{C}^n$ which satisfies $w=\max_{1\leqslant p\leqslant n}\{|w_p|\}<2$. This completes the proof.

Lemma (4.1.14)[89]: Let

$$\left\| \begin{pmatrix} \sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{1\,l_1l_2\cdots l_m}| e^{i\frac{\theta_{1l_1}+\theta_{1l_2}+\cdots+\theta_{1l_m}}{m}} z_{l_1}z_{l_2}\ldots z_{l_m} \\ \sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{2\,l_1l_2\cdots l_m}| e^{i\frac{\theta_{2l_1}+\theta_{2l_2}+\cdots+\theta_{2l_m}}{m}} z_{l_1}z_{l_2}\ldots z_{l_m} \\ \vdots \\ \sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{n\,l_1l_2\cdots l_m}| e^{i\frac{\theta_{nl_1}+\theta_{nl_2}+\cdots+\theta_{nl_m}}{m}} z_{l_1}z_{l_2}\ldots z_{l_m} \end{pmatrix} \right\| \leqslant C_m\|z\|^m,$$

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in U^n,$$

where $m = 2, 3,\cdots$, each $a_{pl_1l_2\ldots l_m}(p, l_1, l_2,\cdots, l_m = 1, 2,\cdots, n)$ is a complex number which is independent of $z_p(p = 1, 2,\cdots, n), i = \sqrt{-1}$, each $\theta_{pl_q} \in (-\pi, \pi](q = 1, 2,\cdots, m;\ p, l_1, l_2,\cdots, l_m = 1, 2,\cdots, n)$ which is independent of $z_p(p = 1, 2,\cdots, n), \|z\| = \max_{1\leqslant p\leqslant n}\{|z_p|\}$, and each $C_m(m = 2, 3,\cdots)$ is a nonnegative real constant which is only dependent on m. Then

$$A_m = \max_{1\leqslant p\leqslant n}\left\{\sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{p\,l_1l_2\cdots l_m}|\right\} \leqslant C_m, m = 2, 3,\cdots.$$

Proof: $\forall z \in U^n\backslash\{0\}$, according to the hypothesis of Lemma (4.1.14), we have

$$\left|\sum_{l_1,l_2,\cdots,l_m=1}^{n} |a_{p\,l_1l_2\cdots l_m}| e^{i\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_m}}{m}} \frac{z_{l_1}}{\|z\|}\frac{z_{l_2}}{\|z\|}\cdots\frac{z_{l_m}}{\|z\|}\right| \leqslant C_m, \qquad p = 1, 2,\cdots, n.$$

In particular, taking $z_{l_q} = e^{-i\frac{\theta_{pl_q}}{m}}\|z\|, q = 1, 2,\cdots, m$, we conclude that

$$\sum_{l_1,l_2,\cdots,l_m=1}^{n} |a_{pl_1l_2\ldots l_m}| \leqslant C_m, \qquad p = 1, 2, \ldots, n.$$

That is

$$A_m = \max_{1\leqslant p\leqslant n}\left\{\sum_{l_1,l_2,\cdots,l_m=1}^{n} |a_{p\,l_1l_2\cdots l_m}|\right\} \leqslant C_m, \qquad m = 2, 3,\cdots.$$

This completes the proof.

Now we shall prove the following theorem.

Theorem (4.1.15)[89]: If $f \in Q_B(U^n)(Q_A(U^n)$ or $Q(U^n))$, and

$$\frac{D^sf_p(0)(z^s)}{s!} = \sum_{l_1,l_2,\cdots,l_s=1}^{n} |a_{p\,l_1l_2\cdots l_s}| e^{i\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_s}}{s}} z_{l_1}z_{l_2}\ldots z_{l_s}, p = 1, 2,\cdots, n,$$

where $|a_{p\,l_1l_2\ldots l_s}| e^{i\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_s}}{s}} = \frac{1}{s!}\frac{\partial^s f_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_s}}, i = \sqrt{-1}, \theta_{pl_q} \in (-\pi, \pi](q = 1, 2,\cdots, s), l_1, l_2,\cdots, l_s = 1, 2,\cdots, n, s = 2, 3,\cdots, m - 1$,then

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant \frac{2}{m(m-1)}\left(1+\sum_{s=2}^{m-1} sA_s\right)\|z\|^m, \qquad z \in U^n, m = 3,4,\cdots,$$

where $A_s = \max\limits_{1\leqslant p\leqslant n}\left\{\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{p\, l_1 l_2\cdots l_s}\right|\right\}$, $s = 2,3,\cdots,m-1$.

Proof: Assume $f \in Q_B(U^n), \forall\, z \in U^n\backslash\{0\}$. Denote $z_0 = \frac{z}{\|z\|}$. Let $g(z) = (Df(z)) - 1 \cdot (D^2 f(z)(z^2) + Df(z)z), w = \frac{D^{m-s+1}g(0)(z^{m-s+1})}{(m-s+1)!}, s = 2,3,\cdots,m-1$, and j satisfies $|z_j| = \|z\| = \max\limits_{1\leqslant k\leqslant n}\{|z_k|\}$. In view of the hypothesis of Theorem (4.1.15), Lemmas (4.1.9), (4.1.10) and (4.1.13), we conclude that

$$\left|\frac{1}{s!}D^s f_j(0)\left(z_0^{s-1}, \frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\right)\right|$$

$$= \frac{1}{s}\left|\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{D^{m-s+1}g_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_2}}{\|z\|}\cdots \cdot \frac{z_{l_s}}{\|z\|}\right.$$

$$+ \sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|}\frac{D^{m-s+1}g_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_3}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|} + \cdots$$

$$\left.+ \sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|}\cdots\frac{z_{l_{s-1}}}{\|z\|}\frac{D^{m-s+1}g_{l_s}(0)(z_0^{m-s+1})}{(m-s+1)!}\right|$$

$$\leqslant \frac{1}{s}\left(\left|\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{D^{m-s+1}g_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_2}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|}\right|\right.$$

$$+ \left|\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|}\frac{D^{m-s+1}g_{l_2}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_3}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|}\right| + \cdots$$

$$\left.+ \left|\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|}\cdots\frac{z_{l_{s-1}}}{\|z\|}\frac{D^{m-s+1}g_{l_s}(0)(z_0^{m-s+1})}{(m-s+1)!}\right|\right)$$

$$\leqslant \frac{1}{s}\left(\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right|\frac{D^{m-s+1}g_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{z_{l_2}}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|}\right.$$

$$+\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right|\frac{|z_{l_1}|}{\|z\|}\frac{D^{m-s+1}g_{l_2}(0)(z_0^{m-s+1})}{(m-s+1)!}\frac{|z_{l_3}|}{\|z\|}\cdots\frac{z_{l_s}}{\|z\|}+\cdots$$

$$\left.+\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{jl_1l_2\cdots l_s}\right|\frac{|z_{l_1}|}{\|z\|}\cdots\frac{|z_{l_{s-1}}|}{\|z\|}\frac{D^{m-s+1}g_{l_s}(0)(z_0^{m-s+1})}{(m-s+1)!}\right)$$

$$\leqslant \frac{1}{s}\underbrace{(2A_s+2A_s+\cdots+2A_s)}_{s}=2A_s.$$

This implies that

$$\frac{1}{s!}\left|D^s f_j(0)\left(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\right)\right|\leqslant 2A_s,\qquad z_0\in\partial U^n. \quad (3)$$

In particular, when $z_0\in\partial_0 U^n$, by (3), we deduce that

$$\frac{1}{s!}\left|D^s f_p(0)\left(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\right)\right|\leqslant 2A_s,\qquad p=1,2,\cdots,n. \quad (4)$$

Taking into account

$$D^s f_p(0)(z^{s-1},\frac{D^{m-s+1}g(0)(z^{m-s+1})}{(m-s+1)!}\in H(\overline{U^n}),\qquad p=1,2,\cdots,n,$$

by the maximum modulus theorem of holomorphic functions on the unit polydisk and (4), it yields that

$$\frac{1}{s!}\left|D^s f_p(0)\left(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\right)\right|\leqslant 2A_s, z_0\in\partial U^n, p=1,2,\cdots,n.$$

We have

$$\frac{1}{s!}\left\|D^s f(0)\left(z_0^{s-1},\frac{D^{m-s+1}g(0)(z_0^{m-s+1})}{(m-s+1)!}\right)\right\|\leqslant 2A_s.$$

That is,

$$\frac{1}{s!}\left\|D^s f(0)\left(z^{s-1},\frac{D^{m-s+1}g(0)(z^{m-s+1})}{(m-s+1)!}\right)\right\|\leqslant 2A_s\|z\|^m, z\in U^n, s=2,3,\cdots,m-1. \quad (5)$$

By Lemma (4.1.11) and (5), we obtain

$$\frac{m(m-1)\|D^m f(0)(z^m)\|}{m!}$$

$$\leqslant\frac{\|D^m g(0)(z^m)\|}{m!}+\frac{2\left\|D^2 f(0)\left(z,\frac{D^{m-1}g(0)(z^{m-1})}{(m-1)!}\right)\right\|}{2!}+\cdots$$

$$+\frac{(m-1)D^{m-1}f(0)\left(z^{m-2},\frac{D^2 g(0)(z^2)}{2!}\right)}{(m-1)!}\leqslant 2\left(1+\sum_{s=2}^{m-1}sA_s\right)\|z\|^m.$$

This implies that

$$\frac{\|D^m f(0)(z^m)\|}{m!}\leqslant\frac{2}{m(m-1)}\left(1+\sum_{s=2}^{m-1}sA_s\right)\|z\|^m, z\in U^n, m=3,4,\cdots,$$

where $A_s = \max\limits_{1\leqslant p\leqslant n}\left\{\sum_{l_1,l_2,\cdots,l_s=1}^{n}\left|a_{p\,l_1l_2\ldots l_s}\right|\right\}, s = 2,3,\cdots,m-1$. Consequently, the desired result holds. By Lemma (4.1.12), the desired result for $f \in Q_A(U^n)$ or $Q(U^n)$ also holds.
This completes the proof.
Corollary (4.1.16)[89]: Suppose $k \in \mathbb{N}$. If $f \in Q_B(U^n)\big(Q_A(U^n)\ or\ Q(U^n)\big)$, and $z = 0$ is a zero of order $k+1$ of $f(z) - z$, then

$$\frac{\left\|D^{k+1}f(0)(z^{k+1})\right\|}{(k+1)!} \leqslant \frac{2}{(k+1)k}\|z\|^{k+1}, \qquad z \in U^n.$$

The above estimate is sharp.
Proof When $k = 1$, in view of the hypothesis of Corollary (4.1.16) and Lemma (4.1.11) (the case of $m = 2$), the result follows. When $k \geqslant 2, m = k+1$, according to the hypothesis of Corollary (4.1.16), it is known that $A_s = 0, s = 2,3,\cdots,k$. From Theorem (4.1.15), we deduce that

$$\frac{\left\|D^{k+1}f(0)(z^{k+1})\right\|}{(k+1)!} \leqslant \frac{2}{(k+1)k}\|z\|^{k+1}, \qquad z \in U^n.$$

This completes the proof.
It is not difficult to verify that

$$f(z) = (\int_0^{z_1}\frac{dt}{(1-t^k)^{\frac{2}{k}}}, \frac{z_2}{z_1}\int_0^{z_1}\frac{dt}{(1-t^k)^{\frac{2}{k}}}, \ldots, \frac{z_n}{z_1}\int_0^{z_1}\frac{dt}{(1-t^k)^{\frac{2}{k}}}, z \in U^n$$

satisfies the hypothesis of Corollary (4.1.16). Taking $z = (r,0,\cdots,0)'(0 \leqslant r < 1)$, we have

$$\frac{\left\|D^{k+1}f(0)(z^{k+1})\right\|}{(k+1)!} = \frac{2}{(k+1)k}r^{k+1}.$$

Hence, the estimate of Corollary (4.1.16) is sharp.
Corollary (4.1.17)[89]: Suppose $k \in \mathbb{N}$. If f is a k-fold symmetric quasi-convex mapping of type B (quasi-convex mapping of type A or quasi-convex mapping) defined on U^n, and

$$\frac{D^{tk+1}f_p(0)(z^{tk+1})}{(tk+1)!} = \sum_{l_1,l_2,\cdots,l_{tk+1}=1}^{n}\left|a_{p\,l_1l_2\ldots l_{tk+1}}\right|e^{i\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_{tk+1}}}{tk+1}}z_{l_1}z_{l_2}\ldots z_{l_{tk+1}}, p = 1,2,\cdots,n,$$

where $\left|a_{p\,l_1l_2\cdots l_{tk+1}}\right|e^{i\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_{tk+1}}}{tk+1}} = \frac{1}{(tk+1)!}\frac{\partial^{tk+1}f_p(0)}{\partial z_{l_1}\partial z_{l_2}\ldots\partial z_{l_{tk+1}}}, \theta_{pl_q} \in (-\pi,\pi]\ (q = 1,2,\cdots,tk+1), l_1,l_2,\cdots,l_{tk+1} = 1,2,\cdots,n, t = 1,2,\cdots$, then

$$\frac{\left\|D^{tk+1}f(0)(z^{tk+1})\right\|}{(tk+1)!} \leqslant \frac{\prod_{r=1}^{t}\big((r-1)k+2\big)}{(tk+1)\cdot t!\,k^t}\|z\|^{tk+1}, \quad z \in U^n, t = 1,2,\cdots. \tag{6}$$

The above estimates are sharp.
Proof. It is known that $z = 0$ is a zero of order $k+1 (k \in \mathbb{N})$ of $f(z) - z$ if f is a k-fold symmetric normalized holomorphic mapping $f(z)$ $(f(z) \not\equiv z)$ defined on U^n. In view of the hypothesis of Corollaries (4.1.17) and (4.1.16), we conclude that

$$\frac{\left\|D^{k+1}f(0)(z^{k+1})\right\|}{(k+1)!} \leqslant \frac{2}{(k+1)k}\|z\|^{k+1}, \qquad z \in U^n.$$

That is, (6) holds for $t = 1$. Assume now that (6) holds for $t = 1,2,\cdots,j$ for some integer $j \geqslant 2$. This implies

$$\frac{D^{tk+1}f(0)(z^{tk+1})}{(tk+1)!} \leqslant \frac{\prod_{r=1}^{t}\big((r-1)k+2\big)}{(tk+1)\cdot t!\,k^t}\|z\|^{tk+1}, z \in U^n, t = 1,2,\ldots,j. \quad (7)$$

In view of (7), we take

$$C_{tk+1} = \frac{\prod_{r=1}^{t}\big((r-1)k+2\big)}{(tk+1)\cdot t!\,k^t}, \qquad t = 1,2,\cdots,j.$$

Notice that $A_m = 0, 2 \leqslant m \neq tk+1$ $(t = 1,2,\cdots)$ from the hypothesis of Corollary (4.1.17). Again according to Lemma (4.1.14) and Theorem (4.1.15), we deduce that

$$\begin{aligned}
&\frac{\left\|D^{(j+1)k+1}f(0)\big(z^{(j+1)k+1}\big)\right\|}{[(j+1)k+1]!}\\
&\leqslant \frac{2}{[(j+1)k+1](j+1)k}\left[1+\sum_{t=1}^{j}(tk+1)A_{tk+1}\right]\|z\|^{(j+1)k+1}\\
&\leqslant \frac{2}{[(j+1)k+1](j+1)k}\left[1 + \sum_{t=1}^{j}(tk+1)\frac{\prod_{r=1}^{t}\big((r-1)k+2\big)}{(tk+1)\cdot t!\,k^t}\right]\|z\|^{(j+1)k+1}\\
&= \frac{2}{[(j+1)k+1](j+1)k}\left[1+\sum_{t=1}^{j}\frac{\prod_{r=1}^{t}\big((r-1)k+2\big)}{t!\,k^t}\right]\|z\|^{(j+1)k+1}\\
&= \frac{2}{[(j+1)k+1](j+1)k}\cdot\frac{\prod_{r=2}^{j+1}\big((r-1)k+2\big)}{t!\,k^t}\|z\|^{(j+1)k+1}\\
&= \frac{\prod_{r=1}^{j+1}\big((r-1)k+2\big)}{((j+1)k+1)\cdot(j+1)!\,k^{j+1}}\|z\|^{(j+1)k+1}.
\end{aligned}$$

That is, (6) holds for $t = j+1$. This completes the proof.

It is easy to verify that

$$f(z) = \left(\int_0^{z_1}\frac{dt}{(1-t^k)^{\frac{2}{k}}}, \frac{z_2}{z_1}\int_0^{z_1}\frac{dt}{(1-t^k)^{\frac{2}{k}}}, \cdots, \frac{z_n}{z_1}\int_0^{z_1}\frac{dt}{(1-t^k)^{\frac{2}{k}}}\right)', z \in U^n$$

satisfies the hypothesis of Corollary (4.1.17). Taking $z = (r,0,\cdots,0)'(0 \leqslant r < 1)$, we have

$$\frac{D^{tk+1}f(0)(z^{tk+1})}{(tk+1)!} \leqslant \frac{\prod_{r=1}^{t}\big((r-1)k+2\big)}{(tk+1)\cdot t!\,k^t}r^{tk+1}, t = 1,2,\cdots.$$

Hence, the estimate of Corollary (4.1.17) is sharp.

When $l_1 = p, l_2 = \cdots = l_{tk+1} = l$ $(l = 1,2,\cdots,n)$, we notice that

$$arg\ a_{pp\underbrace{l\cdots l}_{tk}} = \frac{\theta_{pp} + tk\theta_{p_l}}{tk+1}$$

for $a_{pp\underbrace{l\cdots l}_{tk}} \neq 0$. It is obvious that Corollary (4.1.17) (the case of $f \in Q\_B(U^n)$) is the corresponding result of [92] (the case of $\alpha = 0$), and the methods of their proofs are different.

Setting $k = 1$ in Corollary (4.1.17), we can deduce the following result.

Corollary (4.1.18)[89]: If $f \in Q_B(U^n)(Q_A(U^n)$ or $Q(U^n))$, and

$$\frac{D^m f_p(0)(z^m)}{m!} = \sum_{l_1,l_2,\cdots,l_m=1}^{n} \left|a_{p\,l_1 l_2 \dots l_m}\right| e^{i\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_m}}{m}} z_{l_1} z_{l_2} \dots z_{l_m}, p = 1,2,\cdots,n,$$

where

$$\left|a_{p\,l_1 l_2 \dots l_m}\right| e^{i\frac{\theta_{p\,l_1}+\theta_{pl_2}+\cdots+\theta_{pl_m}}{m}} = \frac{1}{m!}\frac{\partial^m f_p(0)}{\partial z_{l_1}\partial z_{l_2} \dots \partial z_{l_m}}, i = \sqrt{-1}, \theta_{pl_q}$$
$$\in (-\pi,\pi](q = 1,2,\cdots,m), l_1, l_2,\cdots, l_m = 1,2,\cdots,n, m = 2,3,\cdots,$$

then

$$\frac{D^m f(0)(z^m)}{m!} \leqslant \|z\|^m, \qquad z \in U^n, m = 2,3,\cdots.$$

The above estimates are sharp.

Proof. Take $k = 1$ in Corollary (4.1.17). Denote $m = t + 1$. It follows the result. This completes the proof. The example which shows that the estimates of Corollary (4.1.18) are sharp is the same as the example in [14].

When $l_1 = p, l_2 = \cdots = l_m = l\ (l = 1,2,\cdots,n)$, note that $arg\ a_{pp\underbrace{l\cdots l}_{m-1}} = \frac{\theta_{pp}+(m-1)\theta_{p_l}}{m}$ for $a\_(pp\ \underbrace{l\cdots l}_{m-1} \neq 0$. It is clear that Corollary (4.1.18) is Theorem (4.1.19) of [14].

Corollary (4.1.19)[89]: With the same assumptions of Corollary (4.1.17), we have

$$f(z) \leqslant \int_0^{\|z\|} \frac{dt}{(1 - t^k)^{\frac{2}{k}}}, z \in U^n.$$

The above estimate is sharp.

Proof. By Corollary (4.1.17), applying a method similar to that in [14], Corollary (4.1.19) can be proved.

Corollary (4.1.20)[89]: With the same assumptions of Corollary (4.1.17), then we have

$$\|Df(z)z\| \leqslant \frac{\|z\|}{(1 - \|z\|^k)^{\frac{2}{k}}}, \qquad z \in U^n.$$

The above estimate is sharp.

Proof. According to Corollary (4.1.17), with an analogous method in [14], Corollary (4.1.20) can be proved.

It is easy to verify that

$$f(z) = \left(\int_0^{z_1} \frac{dt}{(1 - t^k)^{\frac{2}{k}}}, \frac{z_2}{z_1}\int_0^{z_1} \frac{dt}{(1 - t^k)^{\frac{2}{k}}}, \cdots, \frac{z_n}{z_1}\int_0^{z_1} \frac{dt}{(1 - t^k)^{\frac{2}{k}}}\right)', z \in U^n$$

satisfies the condition of Corollaries (4.1.19) and (4.1.20).

Taking $z = (r, 0,\cdots,0)'(0 \leqslant r < 1)$, we have

$$f(z) = \int_0^{r} \frac{dt}{(1 - t^k)^{\frac{2}{k}}} \quad \text{and} \quad \|Df(z)z\| = \frac{r}{(1 - r^k)^{\frac{2}{k}}}.$$

Therefore the estimates of Corollaries (4.1.19) and (4.1.20) are both sharp.

When $k = 1, l_1 = p, l_2 = \cdots = l_m = l\ (l = 1,2,\cdots,n)$, in [14], respectively.

Section (4.2): Bieberbach Conjecture in Several Complex Variables

In one complex variable, the following theorem is classical and well-known.

Theorem (4.2.1)[98]: (See [99]). If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a normalized biholomorphic function on the unit disk U in $\mathbb{C}$, then

$$|a_n| \leq n, n = 2,3,\ldots,$$

for biholomorphic starlike mappings via their geometric properties. Naturally, people attempt to study the estimates of all homogeneous expansions for biholomorphic starlike mappings from their analytic properties. In 1999, Roper and Suffridge [1] pointed out that the estimate of the second homogeneous expansion for biholomorphic starlike mappings is invalid by providing a counterexample. Owing to this reason, subsequently Gong [15] proposed the following conjecture.

Conjecture (4.2.2)[98]: If $f : U^n \to \mathbb{C}^n$ is a normalized biholomorphic starlike mapping, where U^n is the open unit polydisk in $\mathbb{C}^n$, then

$$\frac{\|D^m f(0)(z_m)\|}{m!} \leq m \|z\|^m, z \in U^n, m = 2,3,\ldots$$

In fact, the above conjecture is the Bieberbach conjecture in several complex variables due to the facts that the Bieberbach conjecture for biholomorphic mappings in several complex variables does not hold and the properties of biholomorphic starlike functions are the most similar to biholomorphic functions among the subclasses of biholomorphic functions. Up to now, only the estimates of the second and third homogeneous expansions for biholomorphic starlike mappings were in essence discussed. It is shown that the difficulties of the Bieberbach conjecture in several complex variables is not less than the Bieberbach conjecture in one complex variable. The related results may consult refs [100], [14], [101], [103].

For X denote a complex Banach space with the norm $\|\cdot\|$, X^* be the dual space of X, B be the open unit ball in X, and U be the Euclidean open unit disk in $\mathbb{C}$. Also, let U^n be the open unit polydisk in $\mathbb{C}^n$, and let $\mathbb{N}$ be the set of all positive integers. We denote by ∂U^n the boundary of U^n, and$\partial_0 U^n$ the distinguished boundary of U^n. Let the symbol$'$mean transpose. For each $x \in X\backslash\{0\}$, we define

$$T(x) = \{Tx \in X^* : \|T_x\| = 1, Tx(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

For $H(B)$ be the set of all holomorphic mappings from B into X. We know that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y - x)^n),$$

for all y in some neighborhood of $x \in B$, where$D^n f(x)$ is then-th-Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x) \underbrace{(y-x,\ldots,y-x)}_{n}.$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\Pi_{j=1}^{n} X$ into X. We say that a holomorphic mapping $f: B \to X$ is biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fr´echet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, then we say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X.

We say that a normalized biholomorphic mapping $f : B \to X$ is a starlike mapping if $f(B)$ is a starlike domain with respect to the origin.

Suppose that $\Omega \in \mathbb{C}^n$ is a bounded circular domain. The first Fr´echet derivative and the m $(m \geq 2)$th Fr´echet derivative of a mapping $f \in H(\Omega)$ at point $z \in \Omega$ are written by $Df(z), D^m f(z)(a^{m-1},\cdot)$, respectively. The matrix representations are

$$Df(z) = \left(\frac{\partial fp(z)}{\partial z_k}\right)_{1\le p,k\le n,} \quad D^m f(z)(a^{m-1},\cdot)$$

$$= \left(\sum_{l_1,l_2,\dots,l_{m-1}=1}^{n} \frac{\partial^m f_p(z)}{\partial z_k \partial z l_1 \cdots \partial z l_{m-1}} \; a l_1 \cdots a l_{m-1}\right)_{1\le p,k\le n},$$

where $f(z) = \left(f_1(z), f_2(z), \dots, f_n(z)\right)'$, $a = (a_1, a_2, \dots, a_n)' \in \mathbb{C}^n$.

We now recall some definitions as follows.

Definition (4.2.3)[98]: (See [95]). Let $f \in H(B)$. It is said that f is $k-$fold symmetric if $e^{\frac{-2\pi i}{k}} f(e^{\frac{2\pi i}{k}} x) = f(x)$for all $x \in B$, where $k \in \mathbb{N}$ and $i = \sqrt{-1}$.

Definition (4.2.4)[98]: (See [96]). Suppose that Ω is a domain (connected open set) in X which contains 0. It is said that $x = 0$ is a zero of orderk of $f(x)$ if $f(0) = 0, \dots, D^{k-1} f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$. The definitions both reduce to the case $X = C$. According to Definitions (4.2.3) and (4.2.4), it is shown that $x = 0$ is a zero of order $k+1$($k \in \mathbb{N}$) of $f(x) - x$ if f is a $k-$fold symmetric normalized holomorphic mapping ($f(x) \not\equiv x$) defined on B. However, the converse is fail.

We denote by $S^*(B)$ the set of all normalized biholomorphic starlike mappings on B. We will establish the sharp estimates of all homogeneous expansions for a subclass of starlike mappings on the unit ball in complex Banach spaces, and the sharp estimates of all homogeneous expansions for the above generalized mappings on the unit polydisk in $\mathbb{C}^n$. It is shown that the Bieberbach conjecture in several complex variables is proved under the restricted conditions, and the derived conclusions reduce to the classical results in one complex variable.

In order to obtain the desired results, we need to provide the following lemmas.

Lemma (4.2.5)[98]: Let $f, g : B \to \mathbb{C} \in H(B), f(0) = g(0) = 1$,

$$f\left(e^{\frac{2\pi i}{k}} x\right) = f(x), g\left(e^{\frac{2\pi i}{k}} x\right) = g(x) (\, k \in \mathbb{N}),$$

where $i = \sqrt{-1}. If f(x) + Df(x)x = g(x)f(x)$, then

$$\frac{kD^k f(0)(x^k)}{k!} = \frac{D^k g(0)(x^k)}{k!}, \frac{skD^{sk} f(0)(x^{sk})}{(s^k)!}$$

$$= \frac{D^{sk} g(0)(x^{sk})}{(s^k)!} + \frac{D^{(s-1)}\, kg(0)\left(x^{(s-1)k}\right)}{\left((s-1)k\right)!} \cdot \frac{D^k f(0)(x^k)}{k!} + \cdots$$

$$\cdot + \frac{D^k g(0)(x^k)}{k!} \cdot \frac{D^{(s-1)k} f(0)\left(x^{((s-1))k}\right)}{\left((s-1)k\right)!}, x \in B, s = 2,3,\dots$$

Proof. According to the hypothesis of Lemma (4.2.5), it yields that

$$1+\frac{(k\ +\ 1)D^{k}f(0)(x^{k})}{k!}+\frac{(2k\ +\ 1)D^{2k}f(0)(x^{2k})}{(2k\ )!}+\cdots+\frac{(sk\ +\ 1)D^{sk}f(0)(x^{sk})}{(sk)!}+\cdots$$
$$=\left(1+\frac{D^{k}g(0)(x^{k})}{k!}+\frac{D^{2k}g(0)(x^{2k})}{(2k)!}+\cdots+\frac{D^{sk}g(0)(z^{sk})}{(sk)!}+\cdots\right)$$
$$\times\left(1+\frac{D^{k}f(0)(x^{k})}{k!}+\frac{D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{D^{sk}f(0)(x^{sk})}{(sk)!}+\cdots\right).$$
A direct computation shows that
$$1+\frac{(k\ +\ 1)D^{k}f(0)(x^{k})}{k!}+\frac{(2k\ +\ 1)D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{(sk\ +\ 1)D^{sk}f(0)(x^{sk})}{(sk)!}+\cdots$$
$$=1+\frac{D^{k}f(0)(x^{k})}{k!}+\frac{D^{k}g(0)(x^{k})}{k!}+\frac{D^{2k}f(0)(x^{2k})}{(2k)!}+\frac{D^{k}g(0)(x^{k})}{k!}$$
$$\cdot\frac{D^{k}f(0)(x^{k})}{k!}+\frac{D^{2k}g(0)(x^{2k})}{(2k)!}+\cdots+\frac{D^{sk}f(0)(x^{sk})}{(sk)!}+\frac{D^{k}g(0)(x^{k})}{k!}$$
$$\cdot\frac{D^{(s-1)}kf(0)\left(x^{(s-1)}k\right)}{\left((s-1)k\right)!}+\cdots+\frac{D^{(s-1)k}g(0)\left(x^{(s-1)k}\right)}{\left((s-1)k\right)!}\cdot\frac{D^{k}f(0)(x^{k})}{k!}$$
$$+\frac{D^{sk}g(0)(x^{sk})}{(sk)!}+\cdots.$$
Compare the homogeneous expansions of both sides in the above equality. The result follows, as desired. This completes the proof.

Lemma (4.2.6)[98]: Suppose that $g:\ B\ \to\mathbb{C}\ \in H(B), g(0)\ =\ 1.$ If $Re(g(x))\ >\ 0, x\in B$, then
$$\frac{|D^{m-1}g(0)(x^{m-1})|}{(m-1)!}\ \le 2\|x\|^{m-1}, x\ \in B, m\ =2\ ,3,\ldots$$
Proof. Fix $x\ \in\ B\setminus\{\,0\}$, and denote $x\,0\ =\frac{x}{\|x\|}$. We define $p(\xi)=g(\xi x0), \xi\ \in\ U$. Then $p\ \in\ H(U), p(0)\ =\ 1, Re(p(\xi))\ =\ Re(g(\xi x0))\ >\ 0, \xi\in U$, and
$$1+\sum_{m=2}^{\infty}\ b_{m-1}\xi^{m-1}\ =1+\sum_{m=2}^{\infty}\ D^{m-1}\frac{g(0)(x_0^{m-1}\ )}{(m-1)!}\xi^{m-1}\ .$$
Comparing the coefficients of both sides in the above equality, we have
$$\frac{D^{m-1}g(0)(x_0^{m-1}\ )}{(m-1)!}=\ b_{m-1}, m=2\ ,3,\ldots$$
Note that $|b_{m-1}|\le\ 2, m=2\ ,3,\ldots$ (see [17]). Consequently, the desired result follows . This completes the proof.

Lemma (4.2.7)[98]: Let $f,g:\ B\ \to\mathbb{C}\ \in\ H(B), f(0)\ =\ g(0)\ =\ 1,$ and $x\ =0$ be a zero of order $k\ +1of\ xf(x)-x\ (resp. xg(x)-x)$. If $f(x)+Df(x)x\ =\ g(x)f(x)$, then
$$\frac{(m-1)D^{m-1}f(0)(x^{m-1})}{(m-1)!}\ =\frac{D^{m-1}g(0)(x^{m-1})}{(m-1)!}\ , m$$
$$=\ k\ +1, k+2,\ldots,2k,\frac{(m-1)D^{m-1}f(0)(x^{m-1})}{(m-1)!}$$
$$=\frac{D^{m-1}g(0)(x^{m-1})}{(m-1)!}\ +\frac{D^{k}g(0)(x^{k})}{k!}\cdot\frac{D^{k}f(0)(x^{k})}{k!}\ , m=2\ k\ +1\ .$$
Proof. In view of the hypothesis of Lemma (4.2.7), it is shown that

$$1+\frac{(k+1)D^k f(0)(x^k)}{k!}+\frac{(k+2)D^{k+1}f(0)(x^{k+1})}{(k+1)!}+\cdots+\frac{mD^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots$$
$$=\left(1+\frac{D^k g(0)(x^k)}{k!}+\frac{D^{k+1}g(0)(x^{k+1})}{(k+1)!}+\cdots+\frac{D^{m-1}g(0)(x^{m-1})}{(m-1)!}+\cdots\right)$$
$$\times\left(1+\frac{D^k f(0)(x^k)}{k!}+\frac{D^{k+1}f(0)(x^{k+1})}{(k+1)!}+\cdots+D^{m-1}\frac{f(0)(x^{m-1})}{(m-1)!}+\cdots\right).$$
A simple calculation shows that
$$1+\frac{(k+1)D^k f(0)(x^k)}{k!}+\frac{(k+2)D^{k+1}f(0)(x^{k+1})}{(k+1)!}+\cdots+\frac{mD^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots$$
$$=1+\frac{D^k f(0)(x^k)}{k!}+\frac{D^k g(0)(x^k)}{k!}+\frac{D^{k+1}f(0)(x^{k+1})}{(k+1)!}+\frac{D^{k+1}g(0)(x^{k+1})}{(k+1)!}$$
$$+\cdots+\frac{D^{2k-1}f(0)(x^{2k-1})}{(2k-1)!}+\frac{D^{2k-1}g(0)(x^{2k-1})}{(2k-1)!}+\frac{D^{2k}f(0)(x^{2k})}{(2k)!}$$
$$+\frac{D^k f(0)(x^k)}{k!}\cdot\frac{D^k g(0)(x^k)}{k!}+\frac{D^{2k}g(0)(x^{2k})}{(2k)!}+\cdots.$$
Comparing the homogeneous expansions of both sides in the above equality, we obtain the desired results. This completes the proof.

We now present the main theorems

Theorem (4.2.8)[98]: Let $f: B \to \mathbb{C} \in H(B), F(x)=xf(x) \in S^*(B)$, and F be a k $(k \in \mathbb{N})$ $-$fold symmetric mapping on B. Then
$$\frac{D^{sk+1}F(0)(x^{sk+1})}{(sk+1)!} \le \frac{\Pi_{r=1}^{s}\big((r-1)k+2\big)}{s!\,k^s}\|x\|^{sk+1}, x \in B, s=1,2,\ldots,$$
and the above estimates are sharp.

Proof. We write $W(x)=\big(DF(x)\big)^{-1}F(x)$. Note that $f(x) \neq 0$ if $F(x)=xf(x) \in S^*(B)$ (see [17]). A simple calculation shows that
$$\big(DF(x)\big)^{-1}F(x)=\frac{xf(x)}{f(x)+Df(x)x}, x \in B$$
and
$$Re\left[T_x\left(\big(DF(x)\big)^{-1}F(x)\right)\right]>0 \Leftrightarrow Re\left(1+\frac{Df(x)x}{f(x)}\right)>0, x \in B\backslash\{0\}.$$
In view of $F(x)=xf(x) \in S^*(B)$, then it is shown that
$$Re\frac{\|x\|}{T_x\big(W(x)\big)}=Re\left(1+\frac{Df(x)x}{f(x)}\right)>0, x \in B\backslash\{0\} \quad (8)$$
(see [102]). Letting $g(x)=1+Df(x)x\,f(x), x \in B$, then $g: B \to \mathbb{C} \in H(B), g(0)=f(0)=1, Re\,g(x)>0, x \in B$, and
$$f(x)+Df(x)x=g(x)f(x) \quad (9)$$
from (8). Also since $F(x)=xf(x)$ is a k $(k \in \mathbb{N})$ $-$fold symmetric mapping, we know that $f(e^{\frac{2\pi i}{k}}x)=f(x)$ and $g(e^{\frac{2\pi i}{k}}x)=g(x)$, where $i=\sqrt{-1}$. By inductive method, we now prove that
$$\frac{\left|D^{sk}f(0)(x^{sk})\right|}{(sk)!} \le \frac{\Pi_{r=1}^{s}\big((r-1)k+2\big)}{s!\,k^s}\|x\|^{sk}, x \in B, s=1,2,\ldots \quad (10)$$
When $s=1$, (10) holds from (9), Lemmas (4.2.5) and (4.2.6) (the case $m=k+1$). Assume that

$$\frac{\left|D^{sk}f(0)(x^{sk})\right|}{(sk)!} \le \frac{\Pi_{r=1}^{s}\big((r-1)k + 2\big)}{s!\,k^s}\|x\|^{sk}\ x \in B, s = 1,2,\ldots,q. \tag{11}$$

We need only to prove that (10) holds for $s = q + 1$. To see this, taking into account Lemmas (4.2.5), (4.2.6) and (11), it is shown that

$$\frac{(q+1)k\left|D^{(q+1)k}f(0)\big(x^{(q+1)k}\big)\right|}{\big((q+1)k\big)!} =$$

$$\frac{D^{(q+1)k}g(0)\big(x^{(q+1)k}\big)}{\big((q+1)k\big)!} + \frac{D^{q}kg(0)(x^{q}k)}{(qk)!}\cdot\frac{D^{k}f(0)(x^{k})}{k!} + \cdots + \frac{D^{k}g(0)(x^{k})}{k!}$$

$$\cdot\frac{D^{q}kf(0)(x^{q}k)}{(qk)!}\frac{\left|D^{(q+1)k}g(0)\big(x^{(q+1)k}\big)\right|}{\big((q+1)k\big)!} + \frac{|D^{q}kg(0)(x^{q}k)|}{(qk)!}$$

$$\cdot\frac{\left|D^{k}f(0)(x^{k})\right|}{k!} + \cdots + \frac{\left|D^{k}g(0)(x^{k})\right|}{k!}\cdot\frac{|D^{q}kf(0)(^{q}k)|}{(qk)!}$$

$$\le 2\|x\|^{(q+1)^k} + \frac{2\cdot 2}{k}\|x\|^{(q+1)^k} + \cdots + 2$$

$$\cdot\frac{\Pi_{(r=1)}^{q}\big((r-1)k + 2\big)}{q!\,k^q}\|x\|(q+1)^k$$

$$= \frac{\Pi_{r=1}^{q+1}\big((r-1)k + 2\big)}{q!\,k^q}\|x\|(q+1)^k.$$

This implies that

$$\frac{\left|D^{(q+1)k}f(0)\big(x^{(q+1)k}\big)\right|}{\big((q+1)k\big)!} \le \frac{\Pi_{r=1}^{q+1}\big((r-1)k + 2\big)}{(q+1)!\,k^{q+1}}\|x\|^{(q+1)k}, x \in B.$$

On the other hand, we see that

$$\frac{D^{sk+1}F(0)(x^{sk+1})}{(sk+1)!} = x\frac{D^{sk}f(0)(x^{sk})}{(sk)!}, x \in B, s = 1,2,\ldots \tag{12}$$

if $F(x) = xf(x)$. Hence according to (10) and (12), we derived the desired result. It is not difficult to verify that

$$F(x) = \frac{x}{\Big(1 - \big(Tu(x)\big)^k\Big)^{\frac{2}{k}}}, x \in B$$

satisfies the condition of Theorem (4.2.8), where= 1. We set $x = ru$ $(0 \le r < 1)$, a direct computation shows that

$$\frac{D^{sk+1}F(0)(x^{sk+1})}{(sk+1)!} = \frac{\Pi_{r=1}^{s}\big((r-1)k + 2\big)}{s!\,k^s} r^{sk+1}, s = 1,2,\ldots$$

Then it is shown that the estimates of Theorem (4.2.8) are sharp. This completes the proof. When $k = 1$, we immediately obtain the following corollary.

Corollary (4.2.9)[98]: Let $f : B \to \mathbb{C} \in H(B), F(x) = xf(x) \in S^*(B)$. Then

$$\frac{\|D^{m}F(0)(x^{m})\|}{m!} \le m\|x\|^{m}, x \in B, m = 2,3,\ldots,$$

and the above estimates are sharp.

Theorem (4.2.10)[98]: Let $f : B \to \mathbb{C} \in H(B), F(x) = xf(x) \in S^*(B)$, and $x = 0$ be a zero of order $k + 1 of\ F(x) - x$. Then

$$\frac{\|D^mF(0)(x^m)\|}{m!} \le \begin{cases} \dfrac{2}{m-1}\|x\|^m, m = k+1, k+2, \dots, 2k, \\ \qquad \dfrac{2(k+2)}{(m-1)k}\|x\|^m, m = 2k+1 \end{cases}$$

for $x \in B$. The above estimates are sharp for $m = k+1$and $m = 2k+1$.

Proof. According to the conditions of Theorem (4.2.10), Lemmas (4.2.6) and (4.2.7), we deduce that

$$\frac{|D^{m-1}f(0)(x^{m-1})|}{(m-1)!} \le \frac{2}{m-1}\|x\|^{m-1}, x \in B, m = k+1, k+2, \dots, 2k,$$

and

$$\frac{(m-1)|D^{m-1}f(0)(x^{m-1})|}{(m-1)!}$$
$$= \left|\frac{D^{m-1}g(0)(x^{m-1})}{(m-1)!} + \frac{D^kg(0)(x^k)}{k!} \cdot \frac{D^kf(0)(x^k)}{k!}\right| \le \frac{|D^{m-1}g(0)(x^{m-1})|}{(m-1)!} + \left|\frac{D^kg(0)(x^k)}{k!} \cdot \frac{D^kf(0)(x^k)}{k!}\right|$$
$$\le 2\|x\|^{2k} + \frac{2^2}{k}\|x\|^{2k} = \frac{2(k+2)}{k}\|x\|^{2k}, x \in B, m = 2k+1.$$

The result follows, as desired. The example which shows the sharpness of Theorem (4.2.10) is the same as Theorem (4.2.8). This completes the proof.

Theorems (4.2.8), (4.2.10) and Corollary (4.2.9) show that almost every homogeneous expansion for $F(x)$ lies in a ball exactly if the image set of $F(x) = xf(x)$ is a starlike domain with respect to the origin.

Let each m_l $(l = 1, 2, \dots, n)$ be a non-negative integer, $N = m_1 + m_2 + \cdots + m_n \in \mathbb{N}$, and $m_l = 0$ mean the corresponding components in Z and $F(Z)$ are omitted. We denote by U^{ml} $(resp. U^N)$ the unit polydisk of C^{m_l} $(resp. \mathbb{C}^N)$.

Theorem (4.2.11)[98]: Let $fl : U^{m_l} \to \mathbb{C} \in H(U^{m_l}), l = 1, 2, \dots, n$,

$$F(Z) = (Z_1f_1(Z_1), Z_2f_2(Z_2), \dots, Z_nf_n(Z_n))' \in S^*(U^N),$$

and $F(Z)$ be a k $(k \in \mathbb{N})$ $-$fold symmetric mapping on U^N. Then

$$\frac{\|D^{sk+1}F(0)(Z^{sk+1})\|}{(s^{k+1})!} \le \frac{\Pi_{r=1}^{s}\big((r-1)k+2\big)}{s!\,k^s}\|Z\|^{sk+1},$$
$$Z = (Z_1, Z_2, \dots, Z_n) \in U^N, s = 1, 2, \dots, \tag{13}$$

and the above estimates are sharp.

Proof. Let $F(Z) = (F_1(Z_1), F_2(Z_2), \dots, F_n(Z_n))'$. According to the hypothesis of Theorem (4.2.11), for any $Z = (Z_1, Z_2, \dots, Z_n)' \in U^N$, it yields that

$$\big(DF(Z)\big)^{-1}F(Z)(\big(DF_1(Z_1)\big)^{-1}F_1(Z_1), \big(DF_2(Z_2)\big)^{-1}F_2(Z_2), \dots, \big(DF_n(Z_n)\big)^{-1}F_n(Z_n))'$$

by a simple computation. Note that

$$\big(DF(Z)\big)^{-1}F(Z) = \Big(0, \dots, \big(DF_l(Z_l)\big)^{-1}F_l(Z_l), \dots, 0\Big)\, if\ Z = (0, \dots, Z_l, \dots, 0)' \in U^N, l = 1, 2, \dots, n.$$

We set

$$W(Z) = (W_1, W_2, \dots, W_n)' = (W_{11}, \dots, W_{1m1}, W_{21}, \dots, W_{2m2}, \dots, W_{n1}, \dots, W_{nmn})' = \big(DF(Z)\big)^{-1}F(Z).$$

Then it yields that

$$F \in S^*(U^N) \Leftrightarrow F_l \in S^*(U^{m_l}), l = 1, 2, \dots, n$$

from the criterion of starlike mappings on U^n (see [12]). Also it is easy to see that each F_l $(l = 1,2,\ldots,n)$ is a k $(k \in \mathbb{N})$ −fold symmetric mapping on U^{m_l} if F is a k $(k \in \mathbb{N})$ −fold symmetric mapping on U^N. Taking into account the facts $D^m F(0)(Z_m) = \max_{1\le l\le n} \{D^m F_l(0)(Z_l^m)\}$ and $Z = \max_{1\le l\le n}\{Z_l\}$, here $\|Z_l\|_{m_l}$ $(resp. \|Z\|_N)$ is briefly written as$\|Z_l\|$ $(resp. \|Z\|)$, it is shown that (13) holds from Theorem (4.2.8) (the case of $X = \mathbb{C}^n, B = U^n$). It is easy to verify that

$$F(Z) = \left(\frac{Z_1}{\left(1 - Z_{11}^k\right)^{\frac{2}{k}}}, \frac{Z_2}{\left(1 - Z_{21}^k\right)^{\frac{2}{k}}}, \ldots, \frac{Z_n}{\left(1 - Z_{n1}^k\right)^{\frac{2}{k}}}\right)', Z = (Z_1, Z_2, \ldots, Z_n) \in U^N$$

satisfies the condition of Theorem (4.2.8), where $Z_l = (Z_{l1}, Z_{l2}, \ldots, Z_{lm_l})' \in U^{ml}, l = 1,2,\ldots,n$. Taking $Z_l = (r, 0, \ldots, 0)$ $(0\ r < 1), l = 1,2,\ldots,n$, we easily see that

$$\frac{\left\|D^{sk+1}F(0)(Z^{sk+1})\right\|}{(s^{k+1})!} = \frac{\Pi_{r=1}^{s}(r-1)k + 2}{s!\,k^s}\, r^{sk+1}, s = 1,2,\ldots$$

It is shown that the estimates of Theorem (4.2.11) are sharp. This completes the proof.

Taking $k = 1$ in Theorem (4.2.11), we readily obtain the following corollary.

Corollary (4.2.12)[98]: Let $fl : U^{m_l} \to \mathbb{C} \in H(U^{m_l}), l = 1,2,\ldots,n,$

$$F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \ldots, Z_n f_n(Z_n))' \in S^*(U^N).$$

Then

$$\frac{\|D^m F(0)(Z_m)\|}{m!} \le m\,\|Z\|^m, Z = (Z_1, Z_2, \ldots, Z_n)' \in U^N s, m = 2,3,\ldots,$$

and the above estimates are sharp.

Theorem (4.2.13)[98]: Let $fl : U^{ml} \to \mathbb{C} \in H(U^{ml}), l = 1,2,\ldots,n,$

$$F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \ldots, Z_n f_n(Z_n))' \in S^*(U^N),$$

and $Z = 0$be a zero of order $k + 1 (k \in \mathbb{N})$ of $F(Z) - Z$. Then

$$\frac{\|D^m F(0)(Z^m)\|}{m!} \le \begin{cases} \dfrac{2}{m-1}\|Z\|^m, m = k+1, k+2, \ldots, 2k, \\ \dfrac{2(k+2)}{(m-1)k}\|Z\|^m, m = 2k+1 \end{cases}$$

for $Z \in U^N$. The above estimates are sharp for $m = k + 1 and\ m = 2k + 1$.

Proof. Similar to that in the proof of Theorem (4.2.11), we derive the desired result from Theorem (4.2.10) (the case of $X = \mathbb{C}^n, B = U^n$). The sharpness of Theorem (4.2.13) is similar to that in Theorem (4.2.11). This completes the proof.

Theorems (4.2.11), (4.2.13) and Corollary (4.2.12) state that almost every homogeneous expansion for $F(Z)$ lies in a polydisk exactly if the image set of $F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \ldots, Z_n f_n(Z_n))$ is a starlike domain with respect to the origin.

Theorem (4.2.14)[98]: Suppose that $F(z) = (F_1(z), F_2(z), \ldots, F_n(z))' \in H(U^n)$, and$F(z)$ is a k $(k \in \mathbb{N})$ −fold symmetric mapping on $U^n. If Re \frac{DF_j(z)z}{F_j(z)} > 0, z \in U^n \setminus \{0\}$, wherej satisfies the condition $|z_j| = \|z\| = \max_{1\le l\le n}\{|zl|\}$, then

$$\frac{\left\|D^{sk+1}F(0)(z^{sk+1})\right\|}{(sk+1)!} \le \frac{\Pi_{r=1}^{s}\left((r-1)k + 2\right)}{s!\,k^s}\|z\|^{sk+1}, z \in U^n, s = 1,2,\ldots,$$

and the above estimates are sharp.

Proof. Fix $z \in U^n \setminus\{0\}$, and denote $z_0 = \frac{z}{\|z\|}$. We define

$$h_j(\xi) = \frac{\|z\|}{z_j} F_j(\xi z_0), \xi \in U, \tag{14}$$

where j satisfies the condition $|z_j| = \|z\| = \max_{1 \le l \le n}\{|z_l|\}$. A direct calculation shows that

$$\frac{h_j(\xi)\xi}{h_j(\xi)} = \frac{DF_j(\xi z_0)\xi z_0}{F_j(\xi z_0)}, \xi \in U \backslash \{0\}.$$

Hence, we conclude that

$$Re\left(\frac{h_j'(\xi)\xi}{h_j(\xi)}\right) > 0, \xi \in U \backslash \{0\}$$

from $Re \frac{DF_j(z)z}{F_j(z)} > 0, z \in U^n \backslash \{0\}$. This implies that $h_j \in S^*(U)$, and h_j is a k −fold symmetric function.

On the other hand, it is not difficult to see that

$$\xi + \sum_{m=2}^{\infty} a_m \xi^m = \xi + \frac{\|z\|}{z_j} \sum_{m=2}^{\infty} \frac{D^m F_j(0)(z_0^m)}{m!} \xi^m$$

from (14). Compare the coefficients of both sides in the above equality. We obtain

$$\frac{\|z\|}{z_j} \frac{D^m F_j(0)(z_0^m)}{m!} = a_m, m = 2, 3, \ldots$$

Therefore, we deduce that

$$\frac{\left|D^{sk+1} F_j(0)\left(z_0^{sk+1}\right)\right|}{(s^{k+1})!} \le \frac{\prod_{r=1}^{s}\left((r-1)k + 2\right)}{s!\, k^s}, z_0 \in \partial U^n$$

from Theorem (4.2.8) (the case of $X = \mathbb{C}, B = U$). When $z_0 \in \partial_0 U^n$, we have

$$\frac{\left|D^{sk+1} F_l(0)\left(z_0^{sk+1}\right)\right|}{(s^{k+1})!} \le \frac{\prod_{r=1}^{s} (r-1)k + 2}{s!\, k^s}, l = 1, 2, \ldots, n.$$

Also since $D^{sk+1}F_l(0)(z^{sk+1})$ is a holomorphic function on $\overline{U^n}$, applying the maximum modulus theorem of holomorphic functions on the unit polydisk, we obtain

$$\frac{\left|D^{sk+1} F_l(0)\left(z_0^{sk+1}\right)\right|}{(s^{k+1})!} \le \frac{\prod_{r=1}^{s} (r-1)k + 2}{s!\, k^s}, z_0 \in \partial U^n, l = 1, 2, \ldots, n,$$

i.e.,

$$\frac{\left|D^{sk+1} F_l(0)\left(z_0^{sk+1}\right)\right|}{(s^{k+1})!} \le \frac{\prod_{r=1}^{s} (r-1)k + 2}{s!\, k^s} \|z\|^{sk+1}, z \in U^n, l = 1, 2, \ldots, n.$$

Hence,

$$\frac{\left|D^{sk+1} F_l(0)\left(z_0^{sk+1}\right)\right|}{(s^{k+1})!} \frac{\prod_{r=1}^{s} (r-1)k + 2}{s!\, k^s} \|z\|^{sk+1}, z \in U^n.$$

It is easy to check that

$$F(z) = \left(\& \frac{z_1}{\left(1 - z_1^k\right)^{\frac{2}{k}}}, \frac{z_2}{\left(1 - z_2^k\right)^{\frac{2}{k}}}, \ldots, \frac{z_n}{\left(1 - z_n^k\right)^{\frac{2}{k}}} \right), z = (z_1, z_2, \ldots, z_n)' \in U^n$$

satisfies the condition of Theorem (4.2.14). We set $z = (r, 0, \ldots, 0)'$ $(0 \le r < 1)$, a simple calculation shows that

$$\frac{\left\|D^{sk+1} F(0)(x^{sk+1})\right\|}{(sk + 1)!} = \frac{\prod_{r=1}^{s} ((r-1))k + 2}{s!\, k^s} r^{sk+1}, s = 1, 2, \ldots$$

Then it is shown that the estimates of Theorem (4.2.14) are sharp. This completes the proof.

When $k = 1$, we directly have the following corollary.

Corollary (4.2.15)[98]: Suppose that $F(z) = \left(F_1(z), F_2(z), \ldots, F_n(z)\right)' \in H(U^n)$. If $Re\frac{DF_j(z)z}{Fj(z)} > 0, z \in U^n \setminus\{0\}$, where j satisfies the condition $|z_j| = \|z\| = \max_{1\le l\le n}\{|z_l|\}$, then

$$\frac{\|D^m F(0)(z^m)\|}{m!} \le m\|z\|^m, z \in U^n, m = 2, 3, \ldots,$$

and the above estimates are sharp.

Theorem (4.2.16)[98]: Suppose that $F(z) = \left(F_1(z), F_2(z), \ldots, F_n(z)\right)' \in H(U^n)$, and $z = 0$ is a zero of order $k + 1 (k \in \mathbb{N})$ of $F(z) - z$. If $Re\frac{DF_j(z)z}{F_j(z)} > 0, z \in U^n \setminus\{0\}$, wherej satisfies the condition $|z_j| = \|z\| = max\, 1 \le l \le n\{|z_l|\}$, then

$$\frac{\|D^m F(0)(z^m)\|}{m!} \le \begin{cases} \frac{2}{m-1}\|z\|^m, m = k+1, k+2, \ldots, 2k, \\ \frac{2(k+2)}{(m-1)k}\|z\|^m, m = 2k+1 \end{cases}$$

for $z \in U^n$. The above estimates are sharp for $m = k + 1$ and $m = 2k + 1$.

Proof. With the analogous arguments as in the proof of Theorem (4.2.14), it follows the desired result from Theorem (4.2.10) (the case of $X = \mathbb{C}, B = U$). The sharpness of Theorem (4.2.16) is the same as Theorem (4.2.14). This completes the proof.

Theorems (4.2.14), (4.2.16) and Corollary (4.2.15) show that almost every homogeneous expansion for $F(z) = \left(F_1(z), F_2(z), \ldots, F_n(z)\right)'$ lies in a polydisk exactly if the image set of $\frac{DF_j(z)z}{F_j(z)}$ is the right half plane of the complex plane $\mathbb{C}$.

Section (4.3): A Subclass of Quasi-Convex Mappings of Type $\mathbb{B}$ and Order α in Several Complex Variables

In geometric function theory of one complex variable, people show great interest in the following classical theorem.

Theorem (4.3.1)[104]: (see [105]) If $f(z) = z + \sum_{n=2}^{\infty} a_n$ is a normalized biholomorphic convex function of order α on the unit disk U in $\mathbb{C}$, then

$$|a_n| \le \frac{1}{n!}\prod_{k=2}^{n}(k - 2\alpha), n = 2, 3, \ldots,$$

and the above estimates are sharp.

We are naturally to ask whether the corresponding result in several complex variables holds or not? We shall in part provide an affirmative answer.

Concerning the sharp estimates of all homogeneous expansions for a subclass of quasi-convex mappings (include quasi-convex mappings of type $\mathbb{A}$ and quasi-convex mappings of type $\mathbb{B}$) in several complex variables, it was shown that the above result in general is invalid (see [1]). However, on a special domain, such as the unit polydisk in $\mathbb{C}^n$, Liu [14], Liu and Liu [89] obtained the sharp estimates of all homogeneous expansions for quasi-convex mappings (include quasiconvex mappings of type $\mathbb{A}$ and quasi-convex mappings of type $\mathbb{B}$) under different restricted conditions respectively. On the other hand, Liu and Liu [92] derived the sharp estimates of all homogeneous expansions for a subclass of quasi-convex mappings of type $\mathbb{B}$ and order α (include quasi-convex mappings, quasi-convex mappings of type $\mathbb{A}$ and quasi-convex mappings of type $\mathbb{B}$). We mention that the family of quasi-convex mappings of type $\mathbb{B}$ and order α is a significant family of holomorphic mappings in several complex variables, and the Bieberbach conjecture in several complex variables (i.e., the sharp estimates of all homogeneous expansions for biholomorphic starlike mappings on the unit polydisk in $\mathbb{C}^n$ hold) (see [15], [17], [101]) is a very significant and extremal difficult problem. Owing to this reason, the sharp estimates of all homogeneous expansions for quasi-convex mappings of type $\mathbb{B}$ and order α seem to be a meaningful problem as well.

For X denote a complex Banach space with the norm $\|.\|$, let X^* denote the dual space of X, let B be the open unit ball in X, and let U be the Euclidean open unit disk in $\mathbb{C}$. We also denote by U^n the open unit polydisk in $\mathbb{C}^n$, Bn the Euclidean unit ball in $\mathbb{C}^n$ and $\mathbb{N}^*$ the set of all positive integers. Let ∂U^n denote the boundary of U^n, $(\partial U)^n$ be the distinguished boundary of U^n. Let the symbol $'$ mean transpose. For each $x \in X\backslash\{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $H(B)$ be the set of all holomorphic mappings from B into X. We know that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n)$$

for all y in some neighborhood of $x \in B$, where $D^n f(x)$ is the nth-Frechet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x) \underbrace{(y-x, \ldots, y-x)}_{n}.$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^{n} X$ into X.

We say that a holomorphic mapping $f: B \to X$ is biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Frechet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f: B \to X$ is a holomorphic mapping, then we say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X.

We say that a normalized biholomorphic mapping $f: B \to X$ is a starlike mapping if $f(B)$ is a starlike domain with respect to the origin.

Suppose that $\Omega \in \mathbb{C}^n$ is a bounded circular domain. The first Frechet derivative and the $m(m \geqslant 2)$-th Frechet derivative of a mapping $f \in H(\Omega)$ at point $z \in \Omega$ are written by $Df(z), D^m f(z)$, respectively.

Definition (4.3.2)[104]: (see [92]) Suppose that $\alpha \in [0,1)$ and $f: B \to X$ is a normalized locally biholomorphic mapping. If

$$\Re e\left\{T_x\left[\left(Df(x)\right)^{-1}(D^2 f(x)(x^2) + Df(x)x)\right]\right\} \geq \alpha\|x\|, \qquad x \in B,$$

then f is said to be quasi-convex of type $\mathbb{B}$ and order α.

Let $Q_B^\alpha(B)$ be the set of all quasi-convex mapping of type $\mathbb{B}$ and order α on B.

Definition (4.3.3)[104]: (see [94]) Suppose that $f: B \to X$ is a normalized locally biholomorphic mapping, and denote

$$G_f(\alpha,\beta) = \frac{2\alpha}{T_u\left[\left(Df(\alpha u)\right)^{-1}\left(f(\alpha u) - f(\beta u)\right)\right]} - \frac{\alpha+\beta}{\alpha-\beta}.$$

If

$$ReG_f(\alpha,\beta) \geqslant 0, \qquad u \in \partial B, \alpha,\beta \in U,$$

then f is said to be a quasi-convex mapping of type $\mathbb{A}$ on B.

We denote by $Q_A(B)$ the set of all quasi-convex mapping of type $\mathbb{A}$ on B.

Definition (4.3.4)[104]: (see [16]) Suppose that $f: B \to X$ is a normalized locally biholomorphic mapping. If

$$Re\left\{T_x\left[\left(Df(x)\right)^{-1}\left(D^2 f(x)(x^2) + Df(x)x\right)\right]\right\} \geqslant 0, \qquad x \in B,$$

then f is said to be a quasi-convex mapping of type $\mathbb{B}$ on B.

We refer to the set $Q_{\mathbb{B}}(B)$ as the set of all quasi-convex mapping of type $\mathbb{B}$ on B.

When $X = \mathbb{C}^n$, Definitions (4.3.2) and (4.3.3) are the same definitions which were introduced by Roper and Suffridge [1].

Definition (4.3.5)[104]: (see [94]) Suppose that $f: B \to X$ is a normalized locally biholomorphic mapping. If

$$Re\left\{T_x\left[\left(Df(x)\right)^{-1}\left(f(x) - f(\xi x)\right)\right]\right\} \geqslant 0, \qquad x \in B, \xi \in \bar{U},$$

then f is said to be a quasi-convex mapping on B.

Let $Q(B)$ be the set of all quasi-convex mapping of type $\mathbb{B}$ on B. Gong [16] proved the inclusion relation

$$Q_A(B) = Q(B) \subset Q_{\mathbb{B}}(B).$$

Indeed, Definitions (4.3.3), (4.3.4) and (4.3.5) reduce to the criteria of biholomorphic convex functions in one complex variable.

Definition (4.3.6)[104]: (see [95]) Let $f \in H(B)$. It is said that f is k-fold symmetric if

$$\exp(-(2\pi i)/k))f(e^{\frac{2\pi i}{k}}x) = f(x) \quad \text{for all} \quad x \in B,$$

where $k \in \mathbb{N}^*$ and $i = \sqrt{-1}$.

Definition (4.3.7)[104]: (see [96]) Suppose that Ω is a domain (connected open set) in X which contains 0. It is said that $x = 0$ is a zero of order k of $f(x)$ if $f(0) = 0, \dots, D^{k-1}f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}^*$.

According to Definitions (4.3.5) and (4.3.6), it is easily shown that $x = 0$ is a zero of order $k+1$ $(k \in \mathbb{N})$ of $f(x) - x$ if f is a k-fold symmetric normalized holomorphic mapping $f(x)(f(x) \not\equiv x)$ defined on B. However, the converse is fail.

Let $Q_{\mathbb{A},k+1}(B)$ (resp. $Q_{\mathbb{B},k+1}(B), Q_{k+1}(B)$) be the subset of $Q_{\mathbb{A}}(B)$ (resp. $Q_{\mathbb{B}}(B), Q(B)$) of mappings f such that $z = 0$ is a zero of order $k+1$ of $f(z) - z$.

In order to prove the desired results, we need to provide some lemmas as follows.

Lemma (4.3.8)[104]: Let $\alpha \in [0,1), f, p: B \to \mathbb{C} \in H(B), f(0) = p(0) = 1, f\left(\frac{e^{2\pi i}}{k}x\right) = f(x), p\left(\frac{e^{2\pi i}}{k}x\right) = p(x) (k \in \mathbb{N}^*)$, and $f(x) + 3Df(x)x + D^2 f(x)(x^2) = (f(x) + Df(x)x)\big(\alpha + (1-\alpha)p(x)\big)$. Then

$$\frac{k(k+1)D^k f(0)(x^k)}{k!} = \frac{(1-\alpha)D^k p(0)(x^k)}{k!},$$

$$\frac{sk(sk+1)D^{sk} f(0)(x^{sk})}{(sk)!} = \frac{(1-\alpha)D^{sk}p(0)(x^{sk})}{(sk)!} + \frac{(1-\alpha)D^{(s-1)k}p(0)\left(x^{(s-1)k}\right)}{\big((s-1)k\big)!} \cdot \frac{(k+1)D^k f(0)(x^k)}{k!} + \dots + \frac{(1-\alpha)D^k p(0)(x^k)}{k!} \cdot \frac{\big((s-1)k+1\big)D^{(s-1)k} f(0)\left(x^{(s-1)k}\right)}{\big((s-1)k\big)!}, x \in B, s = 2, 3, \dots.$$

Proof In view of the hypothesis of Lemma (4.3.8), we have

$$1+\frac{(k+1)^2D^kf(0)(x^k)}{k!}+\frac{(2k+1)^2D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{(sk+1)^2D^{sk}f(0)(x^{sk})}{(sk)!}+\cdots$$
$$=\left(1+\frac{(k+1)\,D^kf(0)(x^k)}{k!}+\frac{(2k+1)\,D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{(sk+1)\,D^{sk}f(0)(x^{sk})}{(sk)!}+\cdots\right)$$
$$\cdot\left(1+\frac{(1-\alpha)\,D^kf(0)(x^k)}{k!}+\frac{(1-\alpha)\,D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{(1-\alpha)\,D^{sk}f(0)(x^{sk})}{(sk)!}+\cdots\right)$$

A simple calculation shows that
$$1+\frac{(k+1)^2D^kf(0)(x^k)}{k!}+\frac{(2k+1)^2D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{(sk+1)^2D^{sk}f(0)(x^{sk})}{(sk)!}+\cdots$$
$$=1+\frac{(k+1)D^kf(0)(x^k)}{k!}+\frac{(1-\alpha)D^kp(0)(x^k)}{k!}+\frac{(2k+1)D^{2k}f(0)(x^{2k})}{(2k)!}+\frac{(1-\alpha)D^kp(0)(x^k)}{k!}\cdot\frac{(k+1)D^kf(0)(x^k)}{k!}+\frac{(1-\alpha)D^{2k}p(0)(x^{2k})}{(2k)!}+\cdots+\frac{(sk+1)D^{sk}f(0)(x^{sk})}{(sk)!}+\frac{(1-\alpha)D^kp(0)(x^k)}{k!}\cdot\frac{\big((s-1)k+1\big)D^{(s-1)k}f(0)\big(x^{(s-1)k}\big)}{\big((s-1)k\big)!}+\cdots+\frac{(1-\alpha)D^{(s-1)k}p(0)\big(x^{(s-1)k}\big)}{\big((s-1)k\big)!}\cdot\frac{(k+1)D^kf(0)(x^k)}{k!}+\frac{(1-\alpha)D^{sk}p(0)(x^{sk})}{(sk)!}+\cdots.$$

Compare the homogeneous expansions of the two sides in the above equality. We derived the desired result.

Lemma (4.3.9)[104]: Let $\alpha\in[0,1), f,p\colon B\to\mathbb{C}\in H(B), f(0)=p(0)=1$. If $x=0$ is a zero of order $k+1(k\in\mathbb{N}^*)$ of $xf(x)-x$ (resp. $xp(x)-(x)$), and $f(x)+3Df(x)x+D^2f(x)(x^2)=(f(x)+Df(x)x)\big(\alpha+(1-\alpha)p(x)\big)$, then for any $x\in B$,
$$\frac{m(m-1)D^{m-1}f(0)(x^{m-1})}{(m-1)!}=\begin{cases}\dfrac{(1-\alpha)D^{m-1}p(0)(x^{m-1})}{(m-1)!}, & m=k+1,\ldots,2k,\\[2ex] \dfrac{(1-\alpha)D^{m-1}p(0)(x^{m-1})}{(m-1)!}+\dfrac{(1-\alpha)D^kp(0)(x^k)}{k!}\cdot\dfrac{(k+1)D^kf(0)(x^k)}{k!}, & m=2k+1.\end{cases}$$

Proof. According to the conditions of Lemma (4.3.9), we obtain

$$1+\frac{(k+1)^2D^kf(0)(x^k)}{k!}+\frac{(2k+1)^2D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{m^2D^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots$$
$$=\left(1+\frac{(k+1)\,D^kf(0)(x^k)}{k!}+\frac{(2k+1)\,D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots\right.$$
$$\left.+\frac{mD^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots\right)$$
$$\cdot\left(1+\frac{(1-\alpha)D^kf(0)(x^k)}{k!}+\frac{(1-\alpha)D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots\right.$$
$$\left.+\frac{(1-\alpha)D^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots\right)$$

A direct computation shows that

$$1+\frac{(k+1)^2D^kf(0)(x^k)}{k!}+\frac{(2k+1)^2D^{2k}f(0)(x^{2k})}{(2k)!}+\cdots+\frac{m^2D^{m-1}f(0)(x^{m-1})}{(m-1)!}+\cdots$$
$$=1+\frac{(k+1)\,D^kf(0)(x^k)}{k!}+\frac{(1-\alpha)D^kf(0)(x^{2k})}{(2k)!}+\frac{(1-\alpha)D^kp(0)(x^k)}{k!}$$
$$\cdot\frac{(k+1)D^kf(0)(x^k)}{k!}+\frac{(1-\alpha)D^{2k}p(0)(x^{2k})}{(2k)!}+\cdots+\frac{mD^{m-1}f(0)(x^{m-1})}{(m-1)!}$$
$$+\cdots+\frac{(1-\alpha)D^{m-1}p(0)(x^{m-1})}{(m-1)!}+\cdots.$$

Compare the homogeneous expansions of the two sides in the above equality. It follows the desired result.

We now begin to establish the desired results.

Theorem (4.3.10)[104]: Let $\alpha\in[0,1), f: B\to\mathbb{C}\in H(B), f(x)+Df(x)x\neq 0, x\in B, F(x)=xf(x)\in Q_B^\alpha(B)$, and F is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping on B. Then

$$\frac{\left\|D^{sk+1}F(0)(x^{sk+1})\right\|}{(sk+1)!}\leq\frac{\prod_{r=1}^{s}\big((r-1)k+2-2\alpha\big)}{(sk+1)s!\,k^s}\|x\|^{sk+1}, x\in B, s=1,2,\ldots$$

and the above estimates are sharp.

Proof Let $W(x)=\big(DF(x)\big)^{-1}D(DF(x)x)x$. A straightforward computation shows that

$$\big(DF(x)\big)^{-1}D(DF(x)x)x=\frac{\big(f(x)+3Df(x)x+D^2f(x)(x^2)\big)x}{f(x)+Df(x)x}, x\in B.$$

Since $F(x)=xf(x)\in Q_{\mathbb{B}}^\alpha(B)$, then according to Definition (4.3.2), we see that

$$Re\left(\frac{f(x)+3Df(x)x+D^2f(x)(x^2)-\alpha(f(x)+Df(x)x)}{(1-\alpha)(f(x)+Df(x)x)}\right)$$
$$=Re\left(\frac{T_x\big(W(x)\big)-\alpha\|x\|}{1-\alpha\|x\|}\right)>0,\qquad x\in B\backslash\{0\}. \tag{15}$$

Letting

$$p(x)=\begin{cases}\dfrac{f(x)+3Df(x)x+D^2f(x)(x^2)-\alpha(f(x)+Df(x)x)}{(1-\alpha)(f(x)+Df(x)x)}, & x\in B\{0\};\\ 1, & x=0,\end{cases} \tag{16}$$

then $p: B\to\mathbb{C}\in H(B), p(0)=f(0)=1$,

$$f(x)+3Df(x)x+D^2f(x)(x^2)=(f(x)+Df(x)x)\big(\alpha+(1-\alpha)p(x)\big).$$

Also since $F(x)=xf(x)$ is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping, then $f\left(e^{\frac{2\pi i}{k}}x\right)=f(x)$ and $p\left(e^{\frac{2\pi i}{k}}x\right)=p(x)$. We now deduce that

$$\frac{\left|D^{sk}f(0)(x^{sk})\right|}{(sk)!}\leq\frac{\prod_{r=1}^{s}\big((r-1)k+2-2\alpha\big)}{(sk+1)s!\,k^s}\|x\|^{sk},x\in B,s=1,2,\dots \quad (17)$$

hold by inductive method. When $s=1$, (17) holds from Lemma (4.3.8) and [98] (the case $m=k+1$). We assume that

$$\frac{\left|D^{sk}f(0)(x^{sk})\right|}{(sk)!}\leq\frac{\prod_{r=1}^{s}\big((r-1)k+2-2\alpha\big)}{(sk+1)s!\,k^s}\|x\|^{sk},x\in B,s=1,2,\dots,q. \quad (18)$$

It suffices to prove that (17) holds for $s=q+1$. For this purpose, by applying Lemma (4.3.8), (18) and [98], we know that

$$\frac{(q+1)k\big((q+1)k+1\big)\left|D^{(q+1)k}f(0)\big(x^{(q+1)k}\big)\right|}{\big((q+1)k\big)!}$$
$$=\Bigg|\frac{(1-\alpha)D^{(q+1)k}p(0)\big(x^{(q+1)k}\big)}{\big((q+1)k\big)!}+\frac{(1-\alpha)D^{qk}p(0)(x^{qk})}{(qk)!}\cdot\frac{(k+1)D^{k}f(0)(x^{k})}{k!}+\cdots$$
$$+\frac{(1-\alpha)D^{k}p(0)(x^{k})}{k!}\cdot\frac{(qk+1)D^{qk}f(0)(x^{qk})}{(qk)!}$$
$$\leq\frac{(1-\alpha)\left|D^{(q+1)k}p(0)\big(x^{(q+1)k}\big)\right|}{\big((q+1)k\big)!}+\frac{(1-\alpha)\left|D^{qk}p(0)(x^{qk})\right|}{(qk)!}$$
$$\cdot\frac{(k+1)\left|D^{k}f(0)(x^{k})\right|}{k!}+\cdots+\frac{(1-\alpha)\left|D^{k}p(0)(x^{k})\right|}{k!}$$
$$\cdot\frac{(qk+1)\left|D^{qk}f(0)(x^{qk})\right|}{(qk)!}$$
$$\leq 2(1-\alpha)\|x\|^{(q+1)k}+(k+1)\cdot\frac{2(1-\alpha)\cdot2(1-\alpha)}{k(k+1)}\|x\|^{(q+1)k}+\cdots+(qk$$
$$+1)\cdot\frac{\big(2(1-\alpha)\cdot\prod_{r=1}^{q}\big((r-1)k+2-2\alpha\big)}{(qk+1)q!\,k^q}\|x\|^{(q+1)k}$$
$$=\frac{\prod_{r=1}^{q+1}\big((r-1)k+2-2\alpha\big)}{q!\,k^q}\|x\|^{(q+1)k}.$$

That is

$$\frac{\left|D^{(q+1)k}f(0)\big(x^{(q+1)k}\big)\right|}{\big((q+1)k\big)!}\leq\frac{\prod_{r=1}^{q+1}\big((r-1)k+2-2\alpha\big)}{\big((q+1)k+1\big)(q+1)!\,k^{q+1}}\|x\|^{(q+1)k},x\in B.$$

Note that

$$\frac{D^{sk+1}F(0)(x^{sk+1})}{(sk+1)!}=x\frac{D^{sk}f(0)(x^{sk})}{(sk)!},\qquad x\in B,s=1,2,\dots,\quad (19)$$

when $F(x)=xf(x)$. Therefore in view of (17) and (19), it follows the result, as desired.
It is easy to check that

$$F(x)=\frac{x}{T_u(x)}\int_0^{T_u(x)}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}},\qquad x\in B$$

satisfies the condition of Theorem (4.3.10), where $\|u\|=1$. Taking $x=ru(0\leq r<1)$, it yields that

$$\frac{\left\|D^{sk+1}F(0)(z^{sk+1})\right\|}{(sk+1)!}=\frac{\prod_{r=1}^{s}\big((r-1)k+2-2\alpha\big)}{(sk+1)s!\,k^{s}}r^{sk+1},\qquad s=1,2,\dots.$$

We see that the estimates of Theorem (4.3.10) are sharp.

Put $\alpha=0$ in Theorem (4.3.10). Then we obtain the following corollary immediately.

Corollary (4.3.11)[104]: Let $f: B\to\mathbb{C}\in H(B), f(x)+Df(x)x\neq 0, x\in B, F(x)=xf(x)\in Q_{\mathbb{B}}(B)$, and F is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping on B. Then

$kDsk+1F(0)(x\ sk+1)k\ (sk+1)!\leq Qs\ r$

$=1\ ((r-1)k+2)\ (sk+1)s!\ k\ s\ kxk\ sk+1, x\in B, s=1,2,\dots,$

and the above estimates are sharp.

Note that $f(x)+Df(x)x\neq 0, x\in B$ due to the growth theorem of $F(x)=xf(x)\in Q_{\mathbb{A}}(B)$ (or $Q(B)$) and

$$Re\left[T_x\left(\left(DF(x)\right)^{-1}F(x)\right)\right]=Re\left(\frac{\|x\|f(x)}{f(x)+Df(x)x}\right)>0\Leftrightarrow Re\left(1+\frac{Df(x)x}{f(x)}\right)>0$$

from $S^*\ (B)\subset Q_{\mathbb{A}}(B)=Q(B)$ and $Q(B)=Q_{\mathbb{A}}(B)\subset Q_{\mathbb{B}}(B)$ (see [16]). We readily get the following corollary from Corollary (4.3.11).

Corollary (4.3.12)[104]: Let $f: B\to\mathbb{C}\in H(B), F(x)=xf(x)\in Q_{\mathbb{A}}(B)$ (resp. $Q(B)$), and F is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping on B. Then

$$\frac{\left\|D^{sk+1}F(0)(x^{sk+1})\right\|}{(sk+1)!}\leq\frac{\prod_{r=1}^{s}\big((r-1)k+2-2\alpha\big)}{(sk+1)s!\,k^{s}}\|x\|^{sk+1}, x\in B, s=1,2,\dots$$

and the above estimates are sharp.

By making use of Theorem (4.3.10), the Taylor expansion of $F(x)=xf(x)$ and the triangle inequality of the norm in complex Banach spaces, we deduce the following two corollaries immediately.

Corollary (4.3.13)[104]: Let $\alpha\in[0,1), f: B\to\mathbb{C}\in H(B), f(x)+Df(x)x\neq 0, x\in B, F(x)=xf(x)\in Q_B^{\alpha}(B)$, and F is a $k(k\in\mathbb{C}^*)$-fold symmetric mapping. Then

$$\|F(x)\|\leq\int_0^{\|x\|}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}},\qquad x\in B,$$

and the above estimate is sharp.

The example of the sharpness of Corollary (4.3.11) is similar to that in Theorem (4.3.10), we need only to mention that

$$\|F(x)\|=\frac{\|x\|}{|T_u(x)|}\left|\int_0^{T_u(x)}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}\right|=\int_0^{r}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}$$

holds for $x=ru(0\leq r<1)$.

Corollary (4.3.14)[104]: Let $\alpha\in[0,1), f: B\to\mathbb{C}\in H(B), f(x)+Df(x)x\neq 0, x\in B, F(x)=xf(x)\in Q_B^{\alpha}(B)$, and $F(x)$ is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping, where B is the unit ball of a complex Hilbert space X. Then

$$\|DF(x)\xi\|\leq\|\xi\|/(1-\|x\|^k)^{\frac{2-2\alpha}{k}},\qquad x\in B,\xi\in X$$

and the above estimate is sharp.

Proof According to Corollary (4.3.11), triangle inequalities with respect to the norm in complex Banach spaces and the fact

$$\sup_{\|x\|=\|\xi\|=1}\frac{\|D^mF(0)(x^{m-1},\xi)\|}{m!}=\sup_{\|x\|=1}\frac{\|D^mF(0)(x^m)\|}{m!}$$

(see [6]), then it follows the result, as desired. Considering

$$F(x)=\frac{x}{\langle x,e\rangle}\int_0^{\langle x,e\rangle}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}},\qquad x\in B,$$

where $\|e\|=1$, then F satisfies the conditions of Corollary (4.3.14). It is shown that

$$DF(x)\xi=\frac{\xi}{\langle x,e\rangle}\int_0^{\langle x,e\rangle}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}+\frac{\langle\xi,e\rangle x}{\langle x,e\rangle(1-(\langle x,e\rangle)^k)^{\frac{2-2\alpha}{k}}}$$

$$-\frac{\int_0^{\langle x,e\rangle}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}\langle\xi,e\rangle x}{\langle x,e\rangle^2},\qquad x\in B,\xi\in X$$

by a direct calculation. We set $x=re,\xi=Re(0\le r<1,R\ge 0)$. Then

$$\|DF(x)\xi\|=R/(1-r^k)^{\frac{2-2\alpha}{k}}.$$

We see that the estimate of Corollary (4.3.14) is sharp.

Taking $\alpha=0$ in Corollaries (4.3.11) and (4.3.12), we directly obtain the corollaries as follows.

Corollary (4.3.15)[104]: Let $f:B\to\mathbb{C}\in H(B),f(x)+Df(x)x\neq 0,x\in B,F(x)=xf(x)\in Q_{\mathbb{B}}(B)$, and F is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping. Then

$$\|F(x)\|\le\int_0^{\|x\|}\frac{dt}{(1-t^k)^{\frac{2}{k}}},\qquad x\in B,$$

and the above estimate is sharp.

Corollary (4.3.16)[104]: Let $f:B\to\mathbb{C}\in H(B),f(x)+Df(x)x\neq 0,x\in B,F(x)=xf(x)\in Q_{\mathbb{B}}(B)$, and $F(x)$ is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping, where B is the unit ball of a complex Hilbert space X. Then

$$\|DF(x)\xi\|\le\frac{\|\xi\|}{(1-\|x\|^k)^{\frac{2}{k}}},\qquad x\in B,\xi\in X$$

and the above estimate is sharp.

With the analogous explanation of Corollary (4.3.14), we get the following corollary from Corollary (4.3.16).

Corollary (4.3.17)[104]: Let $f:B\to\mathbb{C}\in H(B),F(x)=xf(x)\in Q(B)\big(or\ Q_{\mathbb{A}}(B)\big)$, and $F(x)$ is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping, where B is the unit ball of a complex Hilbert space X. Then

$$\|DF(x)\xi\|\le\frac{\|\xi\|}{(1-\|x\|^k)^{\frac{2}{k}}},\qquad x\in B,\xi\in X$$

and the above estimate is sharp [1], [94].

Theorem (4.3.18)[104]: Let $\in[0,1),f:B\to\mathbb{C}\in H(B),f(x)+Df(x)x\neq 0,x\in B,F(x)=xf(x)\in Q^{\alpha}_{B,k+1}(B)$. Then

$$\frac{\|D^mF(0)(x^m)\|}{m!}\le\begin{cases}\dfrac{2-2\alpha}{m(m-1)}\|x\|^m, & m=k+1,k+2,\dots,2k;\\ \dfrac{(2-2\alpha)(k+2-2\alpha)}{m(m-1)k}\|x\|^m, & m=2k+1\end{cases}$$

for $x\in B$. The above estimates are sharp for $m=k+1$ and $m=2k+1$.

Proof In view of the hypothesis of Theorem (4.3.18), Lemma (4.3.9) and [98], it yields that

$$\frac{|D^m f(0)(x^{m-1})|}{(m-1)!} \le \frac{2-2\alpha}{m(m-1)}\|x\|^{m-1}, \qquad x \in B, m = k+1, k+2, \dots, 2k$$

and

$$\frac{m(m-1)|D^{m-1} f(0)(x^{m-1})|}{(m-1)!}$$
$$= \left|\frac{(1-\alpha)D^{m-1}p(0)(z^{m-1})}{(m-1)!} + \frac{(1-\alpha)D^k p(0)(z^k)}{k!} \cdot \frac{(k+1)D^k f(0)(z^k)}{k!}\right|$$
$$\le \frac{(1-\alpha)|D^{m-1}p(0)(x^{m-1})|}{(m-1)!} + \left|\frac{(1-\alpha)D^k p(0)(x^k)}{k!} \cdot \frac{(k+1)D^k f(0)(x^k)}{k!}\right|$$
$$\le 2(1-\alpha)\|x\|^{2k} + \frac{4(1-\alpha)^2}{k}\|x\|^{2k} = \frac{(2-2\alpha)(k+2-2\alpha)}{k}\|x\|^{2k},$$
$$x \in B, m = 2k+1.$$

Noticing that

$$\frac{D^m F(0)(x^m)}{m!} = x\frac{D^{m-1} f(0)(x^{m-1})}{(m-1)!}, \qquad x \in B, s = 1, 2, \dots$$

if $F(x) = xf(x)$. Then we derive the desired result. The example which shows the sharpness of Theorem (4.3.18) is similar to that in Theorem (4.3.10).

Letting $\alpha = 0$, it is easy to obtain the corollary as follow.

Corollary (4.3.19)[104]: Let $f: B \to \mathbb{C} \in H(B), f(x) + Df(x)x \neq 0, x \in B, F(x) = xf(x) \in Q_{\mathbb{B},k+1}(B)$. Then

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \begin{cases} \dfrac{2}{m(m-1)}\|x\|^m, & m = k+1, k+2, \dots, 2k; \\ \dfrac{2(k+2)}{m(m-1)k}\|x\|^m, & m = 2k+1 \end{cases}$$

for $x \in B$.The above estimates are sharp for $m = k+1$ and $m = 2k+1$.

Similar to that in the explanation of Corollary (4.3.14), we drive the following corollary from Corollary (4.3.19).

Corollary (4.3.20)[104]: Let $f: B \to \mathbb{C} \in H(B), F(x) = xf(x) \in Q_{k+1}(B) \left(or\ Q_{\mathbb{A},k+1}(B)\right)$. Then

$$\frac{\|D^m F(0)(x^m)\|}{m!} \le \begin{cases} \dfrac{2}{m(m-1)}\|x\|^m, & m = k+1, k+2, \dots, 2k; \\ \dfrac{2(k+2)}{m(m-1)k}\|x\|^m, & m = 2k+1 \end{cases}$$

for $x \in B$. The above estimates are sharp for $m = k+1$ and $m = 2k+1$.

Let each m_j be a non-negative integer, $N = m_1 + m_2 + \dots + m_n \in \mathbb{N}^*$, and $m_j = 0$ implies that the corresponding components in Z and $F(Z)$ are omitted. U^{m_l} (resp. U^N) is denoted by the unit polydisk of $\mathbb{C}^{m_l}(l = 1, 2, \dots, n)$ (resp. $\mathbb{C}^N$).

It is necessary to establish the following lemmas in order to get the desired results.

Lemma (4.3.21)[104]: (see [92]) Suppose that $\alpha \in [0, 1)$, and f is a normalized locally biholomorphic mapping on U^n. Then $f \in QK_B^\alpha(U^n)$ if and only if

$$Re\frac{g_j(z)}{z_j} \ge \alpha, z = (z_1, \dots, z_n)' \in U^n,$$

where $g(z) = \left(g_1(z), \dots, g_n(z)\right)' = \left(Df(z)\right)^{-1}(D^2f(z)(z^2) + Df(z)z), z \in U^n$ is a column vector in $\mathbb{C}^n$ and j satisfies $\left|z_j\right| = \|z\| = \max_{1\le k\le n}\{|z_k|\}$.

Theorem (4.3.22)[104]: Let $\alpha \in [0,1), f_l: U^{m_l} \to \mathbb{C} \in H(U^{m_l})$,

$$f_l(Z_l) + Df_l(Z_l)Z_l \neq 0, Z_l \in U^{m_l}, l = 1,2,\dots,n, F(Z) = \left(F_1(Z_1), F_2(Z_2), \dots, F_n(Z_n)\right)'$$
$$= \left(Z_1f_1(Z_1), Z_2f_2(Z_2), \dots, Z_nf_n(Z_n)\right)' \in Q_{\mathbb{B}}^{\alpha}(U^N),$$

and F is a $k(k \in \mathbb{N}^*)$-fold symmetric mapping. Then

$$\frac{D^{sk+1}F(0)(Z^{sk+1})}{(sk+1)!} \le \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha}{(sk+1)s!\,k^s}\|Z\|^{sk+1},$$
$$Z = (Z_1, Z_2, \dots, Z_n)' \in U^N, \qquad s = 1,2,\dots,$$

and the above estimates are sharp.

Proof In view of the condition of Theorem (4.3.22), for any $Z = (Z_1, Z_2, \dots, Z_n)' \in U^N$, it is shown that

$$\left(DF(Z)\right)^{-1}D(DF(Z)Z)Z$$
$$= \left(\left(DF_1(Z_1)\right)^{-1}D(DF(Z_1)Z_1)Z_1, \dots, \left(DF_n(Z_n)\right)^{-1}D(DF(Z_n)Z_n)Z_n\right)'$$

by a direct calculation. We pay attention to that

$$\left(DF(Z)\right)^{-1}D(DF(Z)Z)Z = \left(0, \dots, \left(DF_l(Z_l)\right)^{-1}D(DF(Z_l)Z_l)Z_l, \dots, 0\right)'$$

if $Z = (0, \dots, Z_l, \dots, 0)' \in U^N, l = 1,2,\dots,n$. Let

$$G(Z) = (G_1, G_2, \dots, G_n)' = \left(G_{11}, \dots, G_{1m_1}, G_{21}, \dots, G_{2m_2}, \dots, G_{n1}, \dots, G_{nm_n}\right)'$$
$$= \left(DF(Z)\right)^{-1}D(DF(Z)Z)Z.$$

Then we know that

$$F \in Q_{\mathbb{B}}^{\alpha}(U^N) \Leftrightarrow F_l \in Q_{\mathbb{B}}^{\alpha}(U^{m_l}), \qquad l = 1,2,\dots,n$$

from Lemma (4.3.21). Noticing that

$$\|D^mF(0)(Z^m)\| = \max_{1\le l\le n}\{\|D^mF_l(0)(Z_l^m)\|\}, \qquad \|Z\| = \max_{1\le l\le n}\{\|Z_l\|\},$$

here $\|Z_l\|_{m_l}$ (resp. $\|Z\|_N$) is briefly denoted by $\|Z_l\|$ (resp. $\|Z\|$), it follows the desired result.
For any $Z = (Z_1, Z_2, \dots, Z_n)' \in U^N$, it is not difficult to check that

$$F(Z) = \left(\frac{Z_1}{Z_{11}}\int_0^{Z_{11}}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}, \frac{Z_2}{Z_{21}}\int_0^{Z_{21}}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}, \dots, \frac{Z_n}{Z_{n1}}\int_0^{Z_{n1}}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}\right)'$$

satisfies the condition of Theorem (4.3.22), where $Z_l = \left(Z_{l1}, Z_{l2}, \dots, Z_{lm_l}\right)' \in U^{m_l}, l = 1,2,\dots,n$.
We set $Z_l = (R, 0, \dots, 0)'(0 \le R < 1), l = 1,2,\dots,n$. It is easy to obtain

$$\frac{\left\|D^{sk+1}F(0)(Z^{sk+1})\right\|}{(sk+1)!} = \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha}{(sk+1)s!\,k^s}R^{sk+1}, s = 1,2,\dots$$

Hence the estimates of Theorem (4.3.22) are sharp.
We set $\alpha = 0$ in Theorem (4.3.22). Then we easily get the following corollary.

Corollary (4.3.23)[104]: Let $f_l: U^{m_l} \to \mathbb{C} \in H(U^{m_l})$,

$$f_l(Z_l) + Df_l(Z_l)Z_l \neq 0, Z_l \in U^{m_l}, l = 1,2,\dots,n, F(Z) = \left(F_1(Z_1), F_{2(Z_2)}, \dots, F_n(Z_n)\right)'$$
$$= \left(Z_1f_1(Z_1), Z_2f_2(Z_2), \dots, Z_nf_n(Z_n)\right)' \in Q_{\mathbb{B}}(U^N),$$

and F is a $k(k \in \mathbb{N}^*)$-fold symmetric mapping. Then

$$\frac{\left\|D^{sk+1}F(0)(Z^{sk+1})\right\|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^{s}}\|Z\|^{sk+1}, Z=(Z_1,Z_2,\ldots,Z_n)' \in U^N, s = 1,2,\ldots,$$

and the above estimates are sharp.

With the similar interpretation of Corollary (4.3.14), it is apparent to obtain the corollary as follow.

Corollary (4.3.24)[104]: Let $f_l: U^{m_l} \to C \in H(U^{m_l}), l=1,2,\ldots,n,$

$$F(Z) = \big(F_1(Z_1),F_2(Z_2),\ldots,F_n(Z_n)\big)' = \big(Z_1f_1(Z_1),Z_2f_2(Z_2),\ldots,Z_nf_n(Z_n)\big)' \in Q(U^N)\big(Q_{\mathbb{A}}(U^N)\big),$$

and F is a $k(k \in \mathbb{N}^*)$- fold symmetric mapping. Then

$$\frac{\left\|D^{sk+1}F(0)(Z^{sk+1})\right\|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^{s}}\|Z\|^{sk+1}, Z=(Z_1,Z_2,\ldots,Z_n)' \in U^N, s = 1,2,\ldots,$$

and the above estimates are sharp.

Theorem (4.3.25)[104]: Let $\alpha \in [0,1), f_l: U^{m_l} \to \mathbb{C} \in H(U^{m_l}),$

$$f_l(Z_l) + Df_l(Z_l)Z_l \neq 0, Z_l \in U^{m_l}, l=1,2,\ldots,n, F(Z) = \big(Z_1f_1(Z_1),Z_2f_2(Z_2),\ldots,Z_nf_n(Z_n)\big)' \in Q^{\alpha}_{\mathbb{B},k+1}(U^N).$$

Then

$$\frac{\|D^mF(0)(Z^m)\|}{m!} \leq \begin{cases} \dfrac{2-2\alpha}{m(m-1)}\|Z\|^m, & m=k+1,k+2,\ldots,2k; \\ \dfrac{(2-2\alpha)(k+2-2\alpha)}{m(m-1)k}\|Z\|^m, & m=2k+1 \end{cases}$$

for $Z=(Z_1,Z_2,\ldots,Z_n)' \in U^N$.The above estimates are sharp for $m=k+1$ and $m=2k+1$.

Proof. With the analogous arguments as in the proof of Theorem (4.3.18), it follows the desired result.

Put $\alpha = 0$ in Theorem (4.3.25). Then we readily obtain the following corollary.

Corollary (4.3.26)[104]: Let $f_l: U^{m_l} \to \mathbb{C} \in H(U^{m_l}), f_l(Z_l) + Df_l(Z_l)Z_l \neq 0, Z_l \in U^{m_l}, l=1,2,\ldots,n, F(Z) = \big(Z_1f_1(Z_1),Z_2f_2(Z_2),\ldots,Z_nf_n(Z_n)\big)' \in Q_{\mathbb{B},k+1}(U^N)$. Then

$$\frac{\|D^mF(0)(Z^m)\|}{m!} \leq \begin{cases} \dfrac{2}{m(m-1)}\|Z\|^m, & m=k+1,k+2,\ldots,2k; \\ \dfrac{2(k+2)}{m(m-1)k}\|Z\|^m, & m=2k+1 \end{cases}$$

for $Z=(Z_1,Z_2,\ldots,Z_n)' \in U^N$. The above estimates are sharp for $m=k+1$ and $m=2k+1$.

Similar to that in the interpretation of Corollary (4.3.12), we easily obtain the corollary as follow.

Corollary (4.3.27)[104]: Let $f_l: U^{m_l} \to \mathbb{C} \in H(U^{m_l}), l=1,2,\ldots,n, F(Z) = \big(Z_1f_1(Z_1),Z_2f_2(Z_2),\ldots,Z_nf_n(Z_n)\big)' \in Q_{k+1}(U^N)\left(Q_{\mathbb{A},k+1}(U^N)\right)$. Then

$$\frac{\|D^mF(0)(Z^m)\|}{m!} \leq \begin{cases} \dfrac{2}{m(m-1)}\|Z\|^m, & m=k+1,k+2,\ldots,2k; \\ \dfrac{2(k+2)}{m(m-1)k}\|Z\|^m, & m=2k+1 \end{cases}$$

for $Z=(Z_1,Z_2,\dots,Z_n)'\in U^N$. The above estimates are sharp for $m=k+1$ and $m=2k+1$.

Theorem (4.3.28)[104]: Suppose that $\alpha\in[0,1), F(z)=\left(F_1(z),F_2(z),\dots,F_n(z)\right)'\in H(U^n)$, and $F(z)$ is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping on U^n. If $Re\left(1+\frac{D^2F_j(z)(z^2)}{DF_j(z)z}\right)>\alpha, z\in U^n\backslash\{0\}$, where j satisfies the condition $|z_j|=\|z\|=\max_{1\le l\le n}\{|z_l|\}$, then

$$\frac{\left\|D^{sk+1}F(0)(Z^{sk+1})\right\|}{sk+1!}\le\frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s}\|z\|^{sk+1},$$
$$z=(z_1,z_2,\dots,z_n)'\in U^n,\qquad s=1,2,\dots,$$

and the above estimates are sharp.

Proof. Fix $z\in U^n\backslash\{0\}$. We write $z_0=z/\|z\|$.Let

$$h_j(\xi)=\frac{\|z\|}{z_j}F_j(\xi z_0),\qquad \xi\in U, \tag{20}$$

where j satisfies the condition $|z_j|=\|z\|=\max_{1\le l\le n}\{|z_l|\}$. It yields that

$$1+\frac{h_j''(\xi)\xi}{h_j'(\xi)}=1+\frac{D^2F_j(\xi z_0)(\xi z_0,\xi z_0)}{DF_j(\xi z_0)\xi z_0},\qquad \xi\in U\backslash\{0\}$$

by a simple calculation. Therefore, we have

$$Re(1+h_j''(\xi)\xi/h_j'(\xi)>\alpha,\qquad \xi\in U\backslash\{0\}$$

if $Re(1+\frac{D^2F_j(z)(z^2)}{DF_j(z)z}>\alpha, z\in U^n\backslash\{0\}$. That is, $h_j\in K_\alpha(U)$ and h_j is a k-fold symmetric function.

It is also easy to know that

$$\xi+\sum_{m=2}^{\infty}b_m\xi^m=\xi+\frac{\|z\|}{z_j}\sum_{m=2}^{\infty}\frac{D^mF_j(0)(z_0^m)}{m!}\xi^m$$

from (20). Comparing the coefficients of the two sides in the above equality, it is shown that

$$\frac{\|z\|}{z_j}\frac{D^mF_j(0)(z_0^m)}{m!}=b_m,\qquad m=2,3,\dots.$$

Hence, by Theorem (4.3.10)(the case $X=\mathbb{C}, B=U$), we conclude that

$$\frac{\left|D^{sk+1}F_j(0)\left(z_0^{sk+1}\right)\right|}{(sk+1)!}\le\frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s},\qquad z_0\in\partial U^n.$$

When $z_0\in(\partial U)^n$, it yields that

$$\frac{\left|D^{sk+1}F_l(0)\left(z_0^{sk+1}\right)\right|}{(sk+1)!}\le\frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s}, l=1,2,\dots,n\,.$$

Also in view of $D^{sk+1}F_l(0)(z^{sk+1})$ is a holomorphic function on $\overline{U^n}$, we have

$$\frac{\left|D^{sk+1}F_l(0)\left(z_0^{sk+1}\right)\right|}{(sk+1)!}\le\frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s}, z_0\in\partial U^n, l=1,2,\dots,n$$

by the maximum modulus theorem of holomorphic functions on the unit polydisk. This implies that

$$\frac{\left|D^{sk+1}F_l(0)(z^{sk+1})\right|}{(sk+1)!}\le\frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s}\|z\|^{sk+1}, z\in U^n, l=1,2,\dots,n\,.$$

Therefore,

$$\frac{\left\|D^{sk+1}F(0)(z^{sk+1})\right\|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s}\|z\|^{sk+1}, z \in U^n.$$

It is not difficult to verify that

$$F(z) = \left(\int_0^{z_1}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}, \frac{z_2}{z_1\int_0^{z_2}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}}, \dots, \frac{z_n}{z_1}\int_0^{z_n}\frac{dt}{(1-t^k)^{\frac{2-2\alpha}{k}}}\right)'$$

satisfies the condition of Theorem (4.3.28). Put $z = (r, 0, \dots, 0)'(0 \leq r < 1)$, we see that

$$\frac{\left\|D^{sk+1}F(0)(x^{sk+1})\right\|}{(sk+1)!} = \frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s}r^{sk+1}, s = 1, 2, \dots$$

by a direct computation. Then we know that the sharpness for the estimates of Theorem (4.3.28).

Taking $\alpha = 0$ in Theorem (4.3.28), we get the following corollary immedatel.

Corollary (4.3.29)[104]: Suppose that $F(z) = \left(F_1(z), F_2(z), \dots, F_n(z)\right)' \in H(U^n)$, and $F(z)$ is a $k(k \in N^*)$-fold symmetric mapping on U^n. If $Re\left(1 + \frac{D^2F_j(z)(z^2)}{DF_j(z)z}\right) > 0, z \in U^n\backslash\{0\}$, where j satisfies the condition $\left|z_j\right| = \|z\| = \max_{1\leq l\leq n}\{|z_l|\}$, then

$$\frac{\left\|D^{sk+1}F(0)(z^{sk+1})\right\|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha}{(sk+1)s!\,k^s}\|z\|^{sk+1}, z = (z_1, z_2, \dots, z_n)' \in U^n, s = 1, 2, \dots,$$

and the above estimates are sharp.

Theorem (4.3.30)[104]: Suppose that $\alpha \in [0, 1), F(z) = \left(F_1(z), F_2(z), \dots, F_n(z)\right)' \in H(U^n)$, and $z = 0$ is a zero of order $k+1$ of $F(z) - z$. If $Re\left(1 + \frac{D^2F_j(z)(z^2)}{DF_j(z)z}\right) > \alpha, z \in U^n\backslash\{0\}$, where j satisfies the condition $\left|z_j\right| = \|z\| = \max_{1\leq l\leq n}\{|z_l|\}$, then

$$\frac{\|D^mF(0)(z^m)\|}{m!} \leq \begin{cases} \frac{2-2\alpha}{m(m-1)}\|z\|^m, & m = k+1, k+2, \dots, 2k; \\ \frac{(2-2\alpha)(k+2-2\alpha)}{m(m-1)k}\|z\|^m, & m = 2k+1 \end{cases}$$

for $z = (z_1, z_2, \dots, z_n)' \in U^n$. The above estimates are sharp for $m = k+1$ and $m = 2k+1$. We set $\alpha = 0$ in Theorem (4.3.30). Then it is obvious to obtain the corollary as follow.

Corollary (4.3.31)[104]: Suppose that $F(z) = \left(F_1(z), F_2(z), \dots, F_n(z)\right)' \in H(U^n)$, and $z = 0$ is a zero of order $k+1$ of $F(z) - z$. If $Re\left(1 + \frac{D^2F_j(z)(z^2)}{DF_j(z)z}\right) > 0, z \in U^n\backslash\{0\}$, where j satisfies the condition $\left|z_j\right| = \|z\| = \max_{1\leq l\leq n}\{|z_l|\}$, then

$$\frac{\|D^mF(0)(z^m)\|}{m!} \leq \begin{cases} \frac{2}{m(m-1)}\|z\|^m, & m = k+1, k+2, \dots, 2k; \\ \frac{2(k+2)}{m(m-1)k}\|z\|^m, & m = 2k+1 \end{cases}$$

for $z = (z_1, z_2, \dots, z_n)' \in U^n$. The above estimates are sharp for $m = k+1$ and $m = 2k+1$.

Corollary (4.3.32)[209]: Let $\alpha_r \in [0,1), f_r, p_r: B \to \mathbb{C} \in H(B), f_r(0) = p_r(0) = 1, f_r\left(\frac{e^{2\pi i}}{k}x\right) = f_r(x), p_r\left(\frac{e^{2\pi i}}{k}x\right) = p_r(x)(k \in \mathbb{N}^*)$, and $f_r(x) + 3Df_r(x)x + D^2 f_r(x)(x^2) = (f_r(x) + Df_r(x)x)\big(\alpha_r + (1-\alpha_r)p_r(x)\big)$. Then

$$\frac{k(k+1)D^k f_r(0)(x^k)}{k!} = \frac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!},$$

$$\frac{sk(sk+1)D^{sk} f_r(0)(x^{sk})}{(sk)!} = \frac{(1-\alpha_r)D^{sk} p_r(0)(x^{sk})}{(sk)!} + \frac{(1-\alpha_r)D^{(s-1)k} p_r(0)\big(x^{(s-1)k}\big)}{\big((s-1)k\big)!} \cdot \frac{(k+1)D^k f_r(0)(x^k)}{k!} + \cdots + \frac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!} \cdot \frac{\big((s-1)k+1\big)D^{(s-1)k} f_r(0)\big(x^{(s-1)k}\big)}{\big((s-1)k\big)!}, x \in B, s = 2, 3, \ldots.$$

Proof In view of the hypothesis of Corollary (4.3.31), we have

$$1 + \frac{(k+1)^2 D^k f_r(0)(x^k)}{k!} + \frac{(2k+1)^2 D^{2k} f_r(0)(x^{2k})}{(2k)!} + \cdots + \frac{(sk+1)^2 D^{sk} f_r(0)(x^{sk})}{(sk)!} + \cdots$$
$$= \left(1 + \frac{(k+1)\, D^k f_r(0)(x^k)}{k!} + \frac{(2k+1)\, D^{2k} f_r(0)(x^{2k})}{(2k)!} + \cdots + \frac{(sk+1)\, D^{sk} f_r(0)(x^{sk})}{(sk)!} + \cdots\right)$$
$$\cdot \left(1 + \frac{(1-\alpha_r)\, D^k f_r(0)(x^k)}{k!} + \frac{(1-\alpha_r)\, D^{2k} f_r(0)(x^{2k})}{(2k)!} + \cdots + \frac{(1-\alpha_r)\, D^{sk} f_r(0)(x^{sk})}{(sk)!} + \cdots\right)$$

A simple calculation shows that

$$1 + \frac{(k+1)^2 D^k f_r(0)(x^k)}{k!} + \frac{(2k+1)^2 D^{2k} f_r(0)(x^{2k})}{(2k)!} + \cdots + \frac{(sk+1)^2 D^{sk} f_r(0)(x^{sk})}{(sk)!} + \cdots$$
$$= 1 + \frac{(k+1)D^k f_r(0)(x^k)}{k!} + \frac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!} + \frac{(2k+1)D^{2k} f_r(0)(x^{2k})}{(2k)!} + \frac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!} \cdot \frac{(k+1)D^k f_r(0)(x^k)}{k!} + \frac{(1-\alpha_r)D^{2k} p_r(0)(x^{2k})}{(2k)!} + \cdots + \frac{(sk+1)D^{sk} f_r(0)(x^{sk})}{(sk)!} + \frac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!} \cdot \frac{\big((s-1)k+1\big)D^{(s-1)k} f_r(0)\big(x^{(s-1)k}\big)}{\big((s-1)k\big)!} + \cdots$$

$$+\frac{(1-\alpha_r)D^{(s-1)k}p_r(0)\left(x^{(s-1)k}\right)}{\left((s-1)k\right)!}\cdot\frac{(k+1)D^k f_r(0)(x^k)}{k!}+\frac{(1-\alpha_r)D^{sk}p_r(0)(x^{sk})}{(sk)!}+\cdots.$$

Compare the homogeneous expansions of the two sides in the above equality. We derived the desired result.

Corollary (4.3.33)[209]: Let $\alpha_r\in[0,1), f_r, p_r: B\to\mathbb{C}\in H(B), f_r(0)=p_r(0)=1$. If $x=0$ is a zero of order $k+1(k\in\mathbb{N}^*)$ of $xf_r(x)-x$ (resp. $xp_r(x)-(x)$), and $f_r(x)+3Df_r(x)x+D^2f_r(x)(x^2)=(f_r(x)+Df_r(x)x)\left(\alpha_r+(1-\alpha_r)p_r(x)\right)$, then for any $x\in B$,

$$\frac{m(m-1)D^{m-1}f_r(0)(x^{m-1})}{(m-1)!}=$$
$$\begin{cases}\dfrac{(1-\alpha_r)D^{m-1}p_r(0)(x^{m-1})}{(m-1)!}, & m=k+1,\ldots,2k,\\ \dfrac{(1-\alpha_r)D^{m-1}p_r(0)(x^{m-1})}{(m-1)!}+\dfrac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!}\cdot\dfrac{(k+1)D^k f_r(0)(x^k)}{k!}, & m=2k+1.\end{cases}$$

Proof According to the conditions of Corollary (4.3.33), we obtain

$$\begin{aligned}1&+\frac{(k+1)^2D^k f_r(0)(x^k)}{k!}+\frac{(2k+1)^2D^{2k}f_r(0)(x^{2k})}{(2k)!}+\cdots+\frac{m^2D^{m-1}f_r(0)(x^{m-1})}{(m-1)!}+\cdots\\&=\left(1+\frac{(k+1)\,D^k f_r(0)(x^k)}{k!}+\frac{(2k+1)\,D^{2k}f_r(0)(x^{2k})}{(2k)!}+\cdots\right.\\&\left.+\frac{mD^{m-1}f_r(0)(x^{m-1})}{(m-1)!}+\cdots\right)\\&\cdot\left(1+\frac{(1-\alpha_r)D^k f_r(0)(x^k)}{k!}+\frac{(1-\alpha_r)D^{2k}f_r(0)(x^{2k})}{(2k)!}+\cdots\right.\\&\left.+\frac{(1-\alpha_r)D^{m-1}f_r(0)(x^{m-1})}{(m-1)!}+\cdots\right)\end{aligned}$$

A direct computation shows that

$$\begin{aligned}1&+\frac{(k+1)^2D^k f_r(0)(x^k)}{k!}+\frac{(2k+1)^2D^{2k}f_r(0)(x^{2k})}{(2k)!}+\cdots+\frac{m^2D^{m-1}f_r(0)(x^{m-1})}{(m-1)!}+\cdots\\&=1+\frac{(k+1)\,D^k f_r(0)(x^k)}{k!}+\frac{(1-\alpha_r)D^k f_r(0)(x^{2k})}{(2k)!}\\&+\frac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!}\cdot\frac{(k+1)D^k f_r(0)(x^k)}{k!}+\frac{(1-\alpha_r)D^{2k}p_r(0)(x^{2k})}{(2k)!}\\&+\cdots+\frac{mD^{m-1}f_r(0)(x^{m-1})}{(m-1)!}+\cdots+\frac{(1-\alpha_r)D^{m-1}p_r(0)(x^{m-1})}{(m-1)!}+\cdots.\end{aligned}$$

Compare the homogeneous expansions of the two sides in the above equality. It follows the desired result.

Corollary (4.3.34)[209]: Let $\alpha_r\in[0,1), f_r: B\to\mathbb{C}\in H(B), f_r(x)+Df_r(x)x\neq 0, x\in B, F_r(x)=xf_r(x)\in Q_B^{\alpha_r}(B)$, and f_r is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping on B. Then

$$\frac{\left\| D^{sk+1}F_r(0)(x^{sk+1}) \right\|}{(sk+1)!} \le \frac{\prod_{r=1}^{s}\left((r-1)k+2-2\alpha_r\right)}{(sk+1)s!\,k^s}\|x\|^{sk+1}, x \in B, s = 1, 2, \ldots$$

and the above estimates are sharp.

Proof Let $W_r(x) = \left(DF_r(x)\right)^{-1}D(DF_r(x)x)x$. A straightforward computation shows that

$$\left(DF_r(x)\right)^{-1}D(DF_r(x)x)x = \frac{\left(f_r(x) + 3Df_r(x)x + D^2f_r(x)(x^2)\right)x}{f_r(x) + Df_r(x)x}, x \in B.$$

Since $F_r(x) = xf_r(x) \in Q_{\mathbb{B}}^{\alpha_r}(B)$, then according to Definition 1.1, we see that

$$Re\left(\frac{f_r(x) + 3Df_r(x)x + D^2f_r(x)(x^2) - \alpha_r(f_r(x) + Df_r(x)x)}{(1-\alpha_r)(f_r(x) + Df_r(x)x)}\right)$$

$$= Re\left(\frac{T_x\left(W_r(x)\right) - \alpha_r\|x\|}{1-\alpha_r\|x\|}\right) > 0, \qquad x \in B\backslash\{0\}. \tag{21}$$

Letting

$$p_r(x) = \begin{cases} \dfrac{f_r(x) + 3Df_r(x)x + D^2f_r(x)(x^2) - \alpha_r(f_r(x) + Df_r(x)x)}{(1-\alpha_r)(f_r(x) + Df_r(x)x)}, & x \in B\{0\}; \\ 1, & x = 0, \end{cases} \tag{22}$$

then $p_r\colon B \to \mathbb{C} \in H(B), p_r(0) = f_r(0) = 1$,

$$f_r(x) + 3Df_r(x)x + D^2f_r(x)(x^2) = (f_r(x) + Df_r(x)x)\left(\alpha_r + (1-\alpha_r)p_r(x)\right).$$

Also since $F_r(x) = xf_r(x)$ is a $k(k \in \mathbb{N}^*)$-fold symmetric mapping, then $f_r\left(e^{\frac{2\pi i}{k}}x\right) = f_r(x)$ and $p_r\left(e^{\frac{2\pi i}{k}}x\right) = p_r(x)$. We now deduce that

$$\frac{\left|D^{sk}f_r(0)(x^{sk})\right|}{(sk)!} \le \frac{\prod_{r=1}^{s}\left((r-1)k+2-2\alpha_r\right)}{(sk+1)s!\,k^s}\|x\|^{sk}, x \in B, s = 1, 2, \ldots \tag{23}$$

hold by inductive method. When $s = 1$, (23) holds from Corollary (4.3.32) and [98] (the case $m = k+1$). We assume that

$$\frac{\left|D^{sk}f_r(0)(x^{sk})\right|}{(sk)!} \le \frac{\prod_{r=1}^{s}\left((r-1)k+2-2\alpha_r\right)}{(sk+1)s!\,k^s}\|x\|^{sk}, x \in B, s = 1, 2, \ldots, q. \tag{24}$$

It suffices to prove that (23) holds for $s = q+1$. For this purpose, by applying Corollary (4.3.32), (24) and [98], we know that

$$\frac{(q+1)k\left((q+1)k+1\right)\left|D^{(q+1)k}f_r(0)\left(x^{(q+1)k}\right)\right|}{\left((q+1)k\right)!}$$

$$= |\frac{(1-\alpha_r)D^{(q+1)k}p_r(0)\left(x^{(q+1)k}\right)}{\left((q+1)k\right)!}+\frac{(1-\alpha_r)D^{qk}p_r(0)(x^{qk})}{(qk)!}\cdot\frac{(k+1)D^kf_r(0)(x^k)}{k!}$$

$$+\cdots+\frac{(1-\alpha_r)D^kp_r(0)(x^k)}{k!}\cdot\frac{(qk+1)D^{qk}f_r(0)(x^{qk})}{(qk)!}$$

$$\le\frac{(1-\alpha_r)\left|D^{(q+1)k}p_r(0)\left(x^{(q+1)k}\right)\right|}{\left((q+1)k\right)!}+\frac{(1-\alpha_r)\left|D^{qk}p_r(0)(x^{qk})\right|}{(qk)!}$$

$$\cdot\frac{(k+1)\left|D^kf_r(0)(x^k)\right|}{k!}+\cdots+\frac{(1-\alpha_r)\left|D^kp_r(0)(x^k)\right|}{k!}$$

$$\cdot\frac{(qk+1)\left|D^{qk}f_r(0)(x^{qk})\right|}{(qk)!}$$

$$\le 2(1-\alpha_r)\|x\|^{(q+1)k}+(k+1)\cdot\frac{2(1-\alpha_r)\cdot 2(1-\alpha_r)}{k(k+1)}\|x\|^{(q+1)k}+\cdots$$

$$+(qk+1)\cdot\frac{\left(2(1-\alpha_r)\cdot\prod_{r=1}^{q}\left((r-1)k+2-2\alpha_r\right)\right.}{(qk+1)q!\,k^q}\|x\|^{(q+1)k}$$

$$=\frac{\prod_{r=1}^{q+1}\left((r-1)k+2-2\alpha_r\right)}{q!\,k^q}\|x\|^{(q+1)k}.$$

That is

$$\frac{\left|D^{(q+1)k}f_r(0)\left(x^{(q+1)k}\right)\right|}{\left((q+1)k\right)!}\le\frac{\prod_{r=1}^{q+1}\left((r-1)k+2-2\alpha_r\right)}{\left((q+1)k+1\right)(q+1)!\,k^{q+1}}\|x\|^{(q+1)k}, x\in B.$$

Note that

$$\frac{D^{sk+1}F_r(0)(x^{sk+1})}{(sk+1)!}=x\frac{D^{sk}f_r(0)(x^{sk})}{(sk)!},\qquad x\in B, s=1,2,\dots,\quad (25)$$

when $F_r(x)=xf_r(x)$. Therefore in view of (23) and (25), it follows the result, as desired.

It is easy to check that

$$F_r(x)=\frac{x}{T_u(x)}\int_0^{T_u(x)}\frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}},\qquad x\in B$$

satisfies the condition of Corollary (4.3.34), where $\|u\|=1$. Taking $x=ru(0\le r<1)$, it yields that

$$\frac{\left\|D^{sk+1}F_r(0)(z^{sk+1})\right\|}{(sk+1)!}=\frac{\prod_{r=1}^{s}\left((r-1)k+2-2\alpha_r\right)}{(sk+1)s!\,k^s}r^{sk+1},\qquad s=1,2,\dots.$$

We see that the estimates of Corollary (4.3.34) are sharp.

Corollary (4.3.35)[209]: Let $\alpha_r\in[0,1), f_r: B\to\mathbb{C}\in H(B), f_r(x)+Df_r(x)x\ne 0, x\in B, F_r(x)=xf_r(x)\in Q_B^{\alpha_r}(B)$, and $F_r(x)$ is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping, where B is the unit ball of a complex Hilbert space X. Then

$$\|DF_r(x)\xi\|\le\|\xi\|/(1-\|x\|^k)^{\frac{2-2\alpha_r}{k}},\qquad x\in B,\xi\in X$$

and the above estimate is sharp.

Proof According to Corollary (4.3.11), triangle inequalities with respect to the norm in complex Banach spaces and the fact

$$\sup_{\|x\|=\|\xi\|=1} \frac{\|D^m F_r(0)(x^{m-1},\xi)\|}{m!} = \sup_{\|x\|=1} \frac{\|D^m F_r(0)(x^m)\|}{m!}$$

(see [6]), then it follows the result, as desired. Considering

$$F_r(x) = \frac{x}{\langle x,e\rangle}\int_0^{\langle x,e\rangle} \frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}}, \qquad x \in B,$$

where $\|e\| = 1$, then F_r satisfies the conditions of Corollary (4.3.35). It is shown that

$$DF_r(x)\xi = \frac{\xi}{\langle x,e\rangle}\int_0^{\langle x,e\rangle} \frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}} + \frac{\langle \xi,e\rangle x}{\langle x,e\rangle(1-(\langle x,e\rangle)^k)^{\frac{2-2\alpha_r}{k}}} - \frac{\int_0^{\langle x,e\rangle} \frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}}\langle \xi,e\rangle x}{\langle x,e\rangle^2}, \qquad x \in B, \xi \in X$$

by a direct calculation. We set $x = re, \xi = Re(0 \le r < 1, R \ge 0)$. Then

$$\|DF_r(x)\xi\| = R/(1-r^k)^{\frac{2-2\alpha_r}{k}}.$$

We see that the estimate of Corollary (4.3.35) is sharp.

Corollary (4.3.36)[209]: Let $\alpha_r \in [0,1), f_r: B \to \mathbb{C} \in H(B), f_r(x) + Df_r(x)x \neq 0, x \in B, F_r(x) = xf_r(x) \in Q_{B,k+1}^{\alpha_r}(B)$. Then

$$\frac{\|D^m F_r(0)(x^m)\|}{m!} \le \begin{cases} \frac{2-2\alpha_r}{m(m-1)}\|x\|^m, & m = k+1, k+2, \dots, 2k; \\ \frac{(2-2\alpha_r)(k+2-2\alpha_r)}{m(m-1)k}\|x\|^m, & m = 2k+1 \end{cases}$$

for $x \in B$. The above estimates are sharp for $m = k+1$ and $m = 2k+1$.

Proof In view of the hypothesis of Corollary (4.3.36), Corollary (4.3.33) and [98], it yields that

$$\frac{|D^m f_r(0)(x^{m-1})|}{(m-1)!} \le \frac{2-2\alpha_r}{m(m-1)}\|x\|^{m-1}, \qquad x \in B, m = k+1, k+2, \dots, 2k$$

and

$$\frac{m(m-1)|D^{m-1} f_r(0)(x^{m-1})|}{(m-1)!} = \left| \frac{(1-\alpha_r)D^{m-1}p_r(0)(z^{m-1})}{(m-1)!} + \frac{(1-\alpha_r)D^k p_r(0)(z^k)}{k!} \cdot \frac{(k+1)D^k f_r(0)(z^k)}{k!} \right|$$

$$\leq \frac{(1-\alpha_r)|D^{m-1}p_r(0)(x^{m-1})|}{(m-1)!} + \left|\frac{(1-\alpha_r)D^k p_r(0)(x^k)}{k!} \cdot \frac{(k+1)D^k f_r(0)(x^k)}{k!}\right|$$
$$\leq 2(1-\alpha_r)\|x\|^{2k} + \frac{4(1-\alpha_r)^2}{k}\|x\|^{2k} = \frac{(2-2\alpha_r)(k+2-2\alpha_r)}{k}\|x\|^{2k},$$
$$x \in B, m = 2k+1.$$

Noticing that

$$\frac{D^m F_r(0)(x^m)}{m!} = x\frac{D^{m-1}f_r(0)(x^{m-1})}{(m-1)!}, \qquad x \in B, s = 1, 2, \ldots$$

if $F_r(x) = xf_r(x)$. Then we derive the desired result. The example which shows the sharpness of Corollary (4.3.36) is similar to that in Corollary (4.3.34).

Corollary (4.3.37)[209]: Let $\alpha_r \in [0,1), (f_r)_l: U^{m_l} \to \mathbb{C} \in H(U^{m_l})$, $(f_r)_l(Z_l) + D(f_r)_l(Z_l)Z_l \neq 0$, $Z_l \in U^{m_l}$, $l = 1, 2, \ldots, n$, $F_r(Z) = \big((F_r)_1(Z_1), (F_r)_2(Z_2), \ldots, (F_r)_n(Z_n)\big)' = \big(Z_1(f_r)_1(Z_1), Z_2(f_r)_2(Z_2), \ldots, Z_n(f_r)_n(Z_n)\big)' \in Q_{\mathbb{B}}^{\alpha_r}(U^N)$, and F_r is a $k(k \in \mathbb{N}^*)$-fold symmetric mapping. Then

$$\frac{D^{sk+1}F_r(0)(Z^{sk+1})}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha_r}{(sk+1)s!\,k^s}\|Z\|^{sk+1},$$
$$Z = (Z_1, Z_2, \ldots, Z_n)' \in U^N, \qquad s = 1, 2, \ldots,$$

and the above estimates are sharp.

Proof In view of the condition of Corollary (4.3.37), for any $Z = (Z_1, Z_2, \ldots, Z_n)' \in U^N$, it is shown that

$$\big(DF_r(Z)\big)^{-1}D(DF_r(Z)Z)Z$$
$$= \Big(\big(D(F_r)_1(Z_1)\big)^{-1}D(DF_r(Z_1)Z_1)Z_1, \ldots, \big(D(F_r)_n(Z_n)\big)^{-1}D(DF_r(Z_n)Z_n)Z_n\Big)'$$

by a direct calculation. We pay attention to that

$$\big(DF_r(Z)\big)^{-1}D(DF_r(Z)Z)Z = \Big(0, \ldots, \big(D(F_r)_l(Z_l)\big)^{-1}D(DF_r(Z_l)Z_l)Z_l, \ldots, 0\Big)'$$

if $Z = (0, \ldots, Z_l, \ldots, 0)' \in U^N, l = 1, 2, \ldots, n$. Let

$$G(Z) = (G_1, G_2, \ldots, G_n)' = \big(G_{11}, \ldots, G_{1m_1}, G_{21}, \ldots, G_{2m_2}, \ldots, G_{n1}, \ldots, G_{nm_n}\big)'$$
$$= \big(DF_r(Z)\big)^{-1}D(DF_r(Z)Z)Z.$$

Then we know that

$$F_r \in Q_{\mathbb{B}}^{\alpha_r}(U^N) \Leftrightarrow (F_r)_l \in Q_{\mathbb{B}}^{\alpha_r}(U^{m_l}), \qquad l = 1, 2, \ldots, n$$

from Lemma (4.3.21). Noticing that

$$\|D^m F_r(0)(Z^m)\| = \max_{1\leq l\leq n}\{\|D^m(F_r)_l(0)(Z_l^m)\|\}, \qquad \|Z\| = \max_{1\leq l\leq n}\{\|Z_l\|\},$$

here $\|Z_l\|_{m_l}$ (resp. $\|Z\|_N$) is briefly denoted by $\|Z_l\|$ (resp. $\|Z\|$), it follows the desired result.

For any $Z = (Z_1, Z_2, \ldots, Z_n)' \in U^N$, it is not difficult to check that

$$F_r(Z)=\left(\frac{Z_1}{Z_{11}}\int_0^{Z_{11}}\frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}},\frac{Z_2}{Z_{21}}\int_0^{Z_{21}}\frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}},\dots,\frac{Z_n}{Z_{n1}}\int_0^{Z_{n1}}\frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}}\right)'$$

satisfies the condition of Corollary (4.3.37), where $Z_l=\left(Z_{l1},Z_{l2},\dots,Z_{lm_l}\right)'\in U^{m_l},l=1,2,\dots,n$.

We set $Z_l=(R,0,\dots,0)'(0\le R<1),l=1,2,\dots,n$. It is easy to obtain

$$\frac{\left\|D^{sk+1}F_r(0)(Z^{sk+1})\right\|}{(sk+1)!}=\frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha_r}{(sk+1)s!\,k^s}R^{sk+1},s=1,2,\dots$$

Hence the estimates of Corollary (4.3.36) are sharp.

Corollary (4.3.38)[209]: Let $\alpha_r\in[0,1),(f_r)_l:U^{m_l}\to\mathbb{C}\in H(U^{m_l}),(f_r)_l(Z_l)+D(f_r)_l(Z_l)Z_l\ne 0,Z_l\in U^{m_l},l=1,2,\dots,n,F_r(Z)=\left(Z_1(f_r)_1(Z_1),Z_2(f_r)_2(Z_2),\dots,Z_n(f_r)_n(Z_n)\right)'\in Q_{\mathbb{B},k+1}^{\alpha_r}(U^N)$. Then

$$\frac{\|D^mF_r(0)(Z^m)\|}{m!}\le\begin{cases}\dfrac{2-2\alpha_r}{m(m-1)}\|Z\|^m, & m=k+1,k+2,\dots,2k;\\ \dfrac{(2-2\alpha_r)(k+2-2\alpha_r)}{m(m-1)k}\|Z\|^m, & m=2k+1\end{cases}$$

for $Z=(Z_1,Z_2,\dots,Z_n)'\in U^N$.The above estimates are sharp for $m=k+1$ and $m=2k+1$.

Proof With the analogous arguments as in the proof of Corollary (4.3.35), it follows the desired result.

Corollary (4.3.39)[209]: Suppose that $\alpha_r\in[0,1),F_r(z)=\left((F_r)_1(z),(F_r)_2(z),\dots,(F_r)_n(z)\right)'\in H(U^n)$, and $F_r(z)$ is a $k(k\in\mathbb{N}^*)$-fold symmetric mapping on U^n. If $Re\left(1+\frac{D^2(F_r)_j(z)(z^2)}{D(F_r)_j(z)z}\right)>\alpha_r,z\in U^n\backslash\{0\}$, where j satisfies the condition $|z_j|=\|z\|=\max\limits_{1\le l\le n}\{|z_l|\}$, then

$$\frac{\left\|D^{sk+1}F_r(0)(Z^{sk+1})\right\|}{(sk+1)!}\le\frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha_r}{(sk+1)s!\,k^s}\|z\|^{sk+1},$$
$$z=(z_1,z_2,\dots,z_n)'\in U^n,\qquad s=1,2,\dots,$$

and the above estimates are sharp.

Proof Fix $z\in U^n\backslash\{0\}$. We write $z_0=z/\|z\|$.Let

$$h_j(\xi)=\frac{\|z\|}{z_j}(F_r)_j(\xi z_0),\qquad \xi\in U, \tag{26}$$

where j satisfies the condition $|z_j|=\|z\|=\max\limits_{1\le l\le n}\{|z_l|\}$. It yields that

$$1+\frac{h_j''(\xi)\xi}{h_j'(\xi)}=1+\frac{D^2(F_r)_j(\xi z_0)(\xi z_0,\xi z_0)}{D(F_r)_j(\xi z_0)\xi z_0},\qquad \xi\in U\backslash\{0\}$$

by a simple calculation. Therefore, we have

$$Re(1 + h_j''(\xi)\xi/h_j'(\xi) > \alpha_r, \qquad \xi \in U\backslash\{0\}$$

if $Re(1 + \frac{D^2(F_r)_j(z)(z^2)}{D(F_r)_j(z)z} > \alpha_r, z \in U^n\backslash\{0\}$. That is, $h_j \in K_{\alpha_r}(U)$ and h_j is a k-fold symmetric function.

It is also easy to know that

$$\xi + \sum_{m=2}^{\infty} b_m \xi^m = \xi + \frac{\|z\|}{z_j} \sum_{m=2}^{\infty} \frac{D^m(F_r)_j(0)(z_0^m)}{m!} \xi^m$$

from (26). Comparing the coefficients of the two sides in the above equality, it is shown that

$$\frac{\|z\|}{z_j} \frac{D^m(F_r)_j(0)(z_0^m)}{m!} = b_m, \qquad m = 2, 3, ...$$

Hence, by Corollary (4.3.33)(the case $X = \mathbb{C}, B = U$), we conclude that

$$\frac{\left|D^{sk+1}(F_r)_j(0)\left(z_0^{sk+1}\right)\right|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha_r}{(sk+1)s!\,k^s}, \qquad z_0 \in \partial U^n.$$

When $z_0 \in (\partial U)^n$, it yields that

$$\frac{\left|D^{sk+1}(F_r)_l(0)\left(z_0^{sk+1}\right)\right|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha_r}{(sk+1)s!\,k^s}, l = 1, 2, \dots, n\,.$$

Also in view of $D^{sk+1}(F_r)_l(0)(z^{sk+1})$ is a holomorphic function on $\overline{U^n}$, we have

$$\frac{\left|D^{sk+1}(F_r)_l(0)\left(z_0^{sk+1}\right)\right|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha_r}{(sk+1)s!\,k^s}, z_0 \in \partial U^n, l = 1, 2, \dots, n$$

by the maximum modulus theorem of holomorphic functions on the unit polydisk. This implies that

$$\frac{\left|D^{sk+1}(F_r)_l(0)(z^{sk+1})\right|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha_r}{(sk+1)s!\,k^s} \|z\|^{sk+1}, z \in U^n, l = 1, 2, \dots, n\,.$$

Therefore,

$$\frac{\left\|D^{sk+1}F_r(0)(z^{sk+1})\right\|}{(sk+1)!} \leq \frac{\prod_{r=1}^{s}(r-1)k + 2 - 2\alpha_r}{(sk+1)s!\,k^s} \|z\|^{sk+1}, z \in U^n.$$

It is not difficult to verify that

$$F_r(z) = \left(\int_0^{z_1} \frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}}, \frac{z_2}{z_1 \int_0^{z_2} \frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}}}, \dots, \frac{z_n}{z_1} \int_0^{z_n} \frac{dt}{(1-t^k)^{\frac{2-2\alpha_r}{k}}} \right)'$$

satisfies the condition of Corollary (4.3.38). Put $z = (r, 0, \dots, 0)'(0 \leq r < 1)$, we see that

$$\frac{\left\| D^{sk+1}F_r(0)(x^{sk+1}) \right\|}{(sk+1)!} = \frac{\prod_{r=1}^{s}(r-1)k+2-2\alpha_r}{(sk+1)s!\,k^s} r^{sk+1}, s = 1, 2, \ldots$$

by a direct computation. Then we know that the sharpness for the estimates of Corollary (4.3.38).

Chapter 5
First Derivative and Centers with Global Behaviour

We give several examples and a special method which simplifies the computations when a first integral is known. At most one limit cycle can bifurcate from the periodic orbits of a center of a cubic homogeneous polynomial system using the averaging theory of first order. We show after adding some more additional hypotheses, we show that the period function of the origin is either decreasing or has at most one critical period and that both possibilities may happen. This result also extends some previous results that deal with the situation where both vector fields are homogeneous and the origin is a non-degenerate centre.

Section (5.1): The Period Function with Applications

In the latest years, there have been many developments concerning the problem of centres for systems of ordinary differential equations on the plane. By one side, improvements have been done in the direction of solving the centre-focus problem (see [119] or [126]); however, the problem is far to be solved. By the other side, questions about either the kind of period annulus or the shape of the period function of a centre have also been tackled (the period annulus, $\mathcal{P}$ from now on, is the greatest neighbourhood of the centre filled of periodic orbits; given a transversal of the period annulus, the time function defined on it is called the period function).

A first question is to decide whether the centre is isochronous or not. A recent survey on this problem is given in [111]. We remark that in the works of Sabatini and Villarini (see [129], [131]) they settled the strong relationship between Lie brackets and isochronicity. This idea has been used fruitfully by many. In [121] we have also found a full description of the link between commutators and isochronicity. A second question is that of controlling the number of critical points of the period function. This question has been treated for special families of vector fields by (Chicone–Dumortier [115], [116], for some polynomial systems; Chow and Wang [117], and Gavrilov [122] for potential systems; Coppel and Gavrilov [49], Collins [46], and Gasull et al. [39], for Hamiltonian centres with homogeneous non-linearities; Rothe [128], for some Hamiltonian families; Freire et al. [120], for perturbation of isochronous centres, etc.) They mainly focus on seeking for conditions of monotonicity of the period function and seldom examples of more than one critical period are found. Maybe one of the most relevant approach to give general tools for proving the monotonicity of the period function is due to Chicone (see [113]) who gave an expression for the first derivative of the period function as a dynamical interpretation of a result of Diliberto.

Inspired in the geometrical ideas involved in the Lie bracket, we give a method to prove that some centres have either an increasing or a decreasing period function. This method is based on a formula for computing the derivative of the period function, which is obtained from the knowledge of the set of normalizers of the centre. See the definitions and more detailed comments after the statement of the following theorem, which is the key point.

As we have already explained, the aim of Theorem (5.1.2) is to give a tool to study the shape of the period function, that is, features like its monotonicity, its number of critical periods or knowing when it is constant (isochronicity problem). To be useful we need to be able to compute $\mu,$ and control its integral. The existence of U and m satisfying $[X\ U] = \mu X;$ for sufficiently regular vector fields X with a non-degenerate centre at p is already known, see [107]. Note also that our expression of T' given in (2), and based on the knowledge of $U,$ is simpler that the one obtained in [113].

Part 2 of Theorem (5.1.2) tries to give a procedure to compute m and U when an integrating factor for X is known. It can be seen as a reciprocal of the following well known result of S. Lie: Assume that a vector field $U = (R, S)$ such that $[X, U] = \mu X$ is known. If ψ is a first integral of X or a constant—usually ψ is taken to be 1—then a solution f of the system

$$\begin{cases} f_x P + f_y Q = 0, \\ f_x R + f_y S = \psi, \end{cases} \tag{1}$$

exists and it is also a first integral of X, (see [108]). Our result is an extension of a previous one of S. Lie, see Theorem 2.48 in [125] or Proposition 1.1 in [133], which just covers the case $\emptyset' = 0$.

Observe that another interpretation of part 2 of Theorem (5.1.2) is the following: if for a given Hamiltonian vector field $\nabla H^{\perp} >= (-H_y, H_x)$ we are able to find an U such that $[\nabla H^{\perp}, U] = \mu_H \nabla H^{\perp}$, then if we consider $\mu = \mu H \ (\nabla V \cdot U)/V$ it is satisfied that $[X, U] = \mu X$, where $X = V \nabla H^{\perp}$.

We also want to comment that it is very easy to find a formal solution $U = (R, S)$ of $RH_x + SH_y = H$, when $div\ X \neq 0$, It suffices to take $U = (R, S) = (V_y, V_x)/div\ X$, Nevertheless, in most cases U is a not well defined vector field in a neighbourhood of p and it is not useful. The freedom to choose $\emptyset$ is a key point of the method proposed to obtain a well-defined U in $\mathcal{P}$,

The first part is devoted to prove Theorem (5.1.2). In the second part we apply it to prove the monotonicity of the period function for several families of planar systems. Hence, once an U and a μ are obtained we are interested to prove that integral (2) has constant sign. In the systems that we study it sometimes happens that the μ that we have makes difficult these computations. A second step of our way of approach is try to get a more suitable μ.

From a geometrical point of view, the vector field U is the infinitesimal generator of the Lie group of symmetries of X. As usual in Lie theory, we call the set of infinitesimal generators the normalizer of X, while the set of commuting vector fields is called the centralizer, see [132]. Accordingly, our work can be seen as giving the same dynamical interpretation for normalizers than Sabatini's and Villarini's results do for centralizers. Moreover, the set of normalizers of a given vector field X has the nice structure that we show in the following proposition.

Proposition (5.1.3) gives a practical tool. For proving monotonicity one has to figure out in each case whether it is better to compute the value of $\int \ \mu$ as Theorem (5.1.2) suggests, or to find a new element of the normalizer whose corresponding m is more suitable. Note that, in general, $\int \ \mu \neq \int \ \mu^*$ on the same periodicorbit of the period annulus because of the different parameterization given by the first integral ψ, However, the sign is preserved and so are the deductions on the qualitative behaviour of the period function.

We can summarize our approach to study the monotonicity of the period function in a method which, as far as we know, is a new one:

A method in three steps for proving the monotonicity of the period function:

(i) Try to compute U and μ defined in all the period annulus of p and satisfying $[X, U] = \mu X$. If X admits an integrating factor, use part 2 of Theorem (5.1.2).

(ii) Try to control the sign of the integral of μ which appears in (2). If you do not succeed then pass to the next steep.

(iii) Use Proposition (5.1.3) to get a more suitable μ, Go again to step (ii).

The most interesting examples to which we have been able to apply our method. Some of the results that we get were already known but, even in these cases, we want to stress how our method enables to shorten the proofs.

We study Hamiltonian systems of type $H(x,y) = F(x) + G(y)$ and give some applications to physical problems. We go through a miscellanea of examples: Lotka–Volterra centre, quadratic systems, Lie´nard systems and polynomial Hamiltonian systems with homogeneous nonlinearities. Maybe the clearest application of our method is given in Proposition (5.1.16), where we prove that the period function of a family of quadratic systems is decreasing.

We end this introduction by noticing that from part 1 of Theorem (5.1.2) it can be deduced the following result on isochronicity:

Corollary (5.1.1)[106]: Consider a $\mathcal{C}^1$ vector field X having a centre at a point p and period annulus $P \subset \mathbb{R}^2$. Let U be a vector field $U \in \mathcal{C}^1$ $(\mathcal{P})$, transversal to X in $\mathcal{P}\{p\}$, and such that $[X, U] = \mu X$ for some smooth scalar function $\mu : \mathcal{P} \to \mathbb{R}$. Let $\gamma = \{(x(t), y(t)), t \in= [0, T_\gamma]\}$ be any periodic orbit of X in $\mathcal{P}$.

Then, if there is a neighbourhood of p such that for any g contained in it,

$$\int_0^{T_\gamma} \mu(x(t), y(t))\, dt = 0,$$

the centre is isochronous.

In [121] the converse of the above corollary is also proved and some applications of it are given.

Theorem (5.1.2)[106]: Consider a $\mathcal{C}^1$ vector field X having a centre at a point p with period annulus $\mathcal{P}$, The following statements hold:

(i) Let U be a vector field, $U \in \mathcal{C}^1$ ðPÞ; transversal to X in $P\backslash\{p\}$, and such that $[X, U] = \mu X$ on $\mathcal{P}$, for some $\mathcal{C}^1$ function $\mu : \mathcal{P} \subset \mathbb{R}^2 \to \mathbb{R}$: Denote by $\psi = \psi(s)$ a trajectory of U such that $\psi(s_0\psi \in \mathcal{P}$, Then,

$$T'(s_0) = \int_0^{T(s_0)} \mu(x(t), y(t))dt, \tag{2}$$

where $(x(t)y(t))$ is the orbit of X such that $(x(0)y(0)) = \psi(s_0)$ and $T(s)$ the period of the orbit of X passing through $\psi(s)$.

(ii) Assume that

(a) the vector field $X = (P, Q)$ admits an integrating factor $V(x,y)^{-1}$ in $\mathcal{P}$; that is, there exist $V(x,y)$ and $H(x,y)$ such that $X = (P,Q) = V(-H_y, H_x)$ in $\mathcal{P}$;

(b) there exist scalar functions R and S such that $RH_x + SH_y = \emptyset(H)$, for some smooth scalar function $\emptyset$.

Then, by taking the vector field $U = (R, S)$, it satisfies $[X, U] = \mu X$, with

$$\mu(x,y) = div\, U - \frac{\nabla V \cdot U}{V} \emptyset'(H) \tag{3}$$

Proof. Part 1: Let $\gamma(t)$ be a periodicorbit of period T of X, and $p = \gamma(0) = \gamma(T)$. Take a transversal $\sum$ given by

$$g : (-\varepsilon, \varepsilon) \to \sum ,$$

being $g(s)$ a solution of $x' = U(x)$ such that $g(0) = p$; that is, $g'(s) = U(g(s))$.

Consider as well the return map of X defined on $\sum$:

$$\pi : \sum_0 \subset \sum \to \sum .$$

If we call $\varphi(t,x)$ the flow defined by X, then

$$\pi(g)(s)) = \varphi(T + t(s), g(s))$$

Moreover, observe that in case that $\gamma(t)$ is a closed orbit of the interior of a period annulus, $T + \tau(s)$ is the period of the closed orbit passing through $g(s)$. In this notation, it is easy to see that the monodromy matrix of the variational equation of the return map in the basis $\{X(p), U(p)\}$ is

$$\begin{pmatrix} 1 & -\tau'(0) \\ 0 & 1 \end{pmatrix}.$$

A key point of our proof is to note that the hypothesis $[X, U] = \mu X$ implies that

$$Y(t) := U\big(\gamma(t)\big) - \left\{ \int_0^t \mu\big(\gamma(u)\big)\, du \right\} X(\gamma(t)),$$

is a solution of the variational equation, since

$$\frac{d}{dt} Y(t) = DU(\gamma(t))X(\gamma(t\gamma\, \mu\big(\gamma(t)\big)X\big(\gamma(t)\big) - \int_0^t \mu\big(\gamma(u)\big)du\, DX\big(g(t)\big)X\big(\gamma(t)\big)$$

$$= DX\big(\gamma(t)\big)U\big(\gamma(t)\big) + \mu\big(\gamma(t)\big)X\big(\gamma(t)\big)u\big(\gamma(t)\big)X\big(\gamma(t)\big)$$

$$- \int_0^t \mu\big(\gamma(u)\big)du\, DX\big(\gamma(t)\big)X\big(\gamma(t)\big) = DX(\gamma(t))Y(t)$$

Finally, by observing that $Y(0) = U(p)$ and $Y(T) = U(p) - \int_0^T \mu(\gamma(t))\, dtX(p)$ we get that

$$\begin{pmatrix} \int_0^T \mu\big(\gamma(t)\big)dt \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\tau'(0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so,

$$\tau'(0) = \int_0^T \mu\big(\gamma(t)\big)dt,$$

as we wanted to prove.

Part 2: Let V^{-1} be an integrating factor of X, that is, $X = V(-H_y, H_x)$ for some Hamiltonian function H. Let us take the vector field $U = (R, S)$ satisfying $H_x R + H_y S = \emptyset(H)$. Then, straightforward computations give

$$[X, U] = \begin{pmatrix} R_x\ R_y \\ S_x\ S_y \end{pmatrix} \begin{pmatrix} -VH_y \\ +VH_x \end{pmatrix} - V \begin{pmatrix} -H_{yx} H_{yy} \\ H_{xx}\ H_{xy} \end{pmatrix} \begin{pmatrix} R \\ S \end{pmatrix}$$

$$- \begin{pmatrix} -V_x H_y\ V_y H_y \\ V_x H_x\ V_y H_x \end{pmatrix} \begin{pmatrix} R \\ S \end{pmatrix}$$

$$= \begin{pmatrix} -(R_x + S_y)VH_y + (RV_x + SV_y)Hy + V(R_yH_x + RH_{yx} + SH_{yy} + S_yH_y) \\ (R_x + S_y)VH_x - (RV_x + SV_y + H_x - V(R_xH_x + RH_{xx} + SHxy + S_xH_y) \end{pmatrix}$$
$$= \begin{pmatrix} -div(U)VH_y + (\nabla V \cdot U)H_y + V\frac{\eth}{\eth y}\emptyset'(H) \\ div(U)VH_x + (\nabla V \cdot U)H_x + V\frac{\eth}{\eth y}\emptyset'(H) \end{pmatrix}$$
$$= \left(div\, U - \frac{\nabla V \cdot U}{V}\ \emptyset'(H)\right)\ X,$$
and thus the desired result.

Proposition (5.1.3)[106]: Consider the set of normalizers of X,
$$\mathcal{N}\ (X) = \{U: [X, U] = \mu X\ for\ some\ \mu\},$$
and take $U \in \mathcal{N}\ (X)$ that satisfies $[X, U] = \mu X$. Then, if $U^* \in \mathcal{N}\ (X)$, it can be written as $U = \psi U + gX$, where ψ is either a first integral of X or a non-zero constant and g is any C^1 function. Moreover, $[X, U^*] = \mu^* X$, with $\mu^* = (\psi\mu + \nabla g^t \cdot X)$.

Proof. First of all, we observe that
$$[X, U] = [X, \psi U + gX] = \psi[X, U] + (\nabla\psi^t X)U + g[X, X] + (\nabla g^t X)X$$
$$= (\psi\mu + \nabla g^t X)X, \tag{4}$$
where in the last step we use that $[X, U] = \mu X$ and that ψ is either a first integral of X or a non-zero constant.

Last assertion tells us that any U^* of the prescribed type is a normalizer of X, and also gives the formula for μ, The property that any normalizer can be written in this way follows from the fact that U and X form a basis of $\mathbb{R}^2$ just because they are transversal. Then, there exist f and g such that $U = fU + gX$. Equality (4) with $\psi = f$ and $U^* \in \mathcal{N}\ (X)$ forces $\nabla f^t X = 0$, which implies that f is either a first integral of X or a non-zero constant (if it was zero, U would not be transversal to X), as we wanted to prove.

We present some results and comments related with Theorem (5.1.2) and Proposition (5.1.3). The first one is about a method given in [110] to compute m and U when the first integral of X is a polynomial.

The first one dealing with the general properties (finding normalizers and adapting part 2 of Theorem (5.1.2) to the specific family), the second one containing some examples and applications to physical problems and the third one with the routine computations. This family has been also studied in [118], [128], [130].

We start with some notation and the technicalities to look for normalizers of the vector field induced by $H(x, y) = F(x) + G(y)$.

Define the numbers $x_L = \max\{x < 0 : F'(x) = 0\}$ $x_R = \min\{x > 0 : F'(x) = \}$ $y_L = \max\{y < 0 : G'(y) = 0\}, y_R = \min\{y > 0 : G'(y) = 0\}$If some of these sets is empty, then the corresponding number is $\pm\infty(-for\ L;\ +for\ R)$. Denote also by $\mathcal{R}$ the rectangle $\mathcal{R} = (x_L, x_R) \times (y_L, y_R) \subset \mathbb{R}^2$.

Lemma (5.1.4)[106]: Let F and G two real analytic functions at 0, such that $F(0) = G(0) = 0$ and they have a non-degenerate minimum at 0. Then,

(i) Let X be the vector field given by
$$\begin{cases} \dot{x} = G'(y), \\ \dot{y} = F'(x), \end{cases} \tag{5}$$
and U the vector field

$$U = \begin{cases} \dot{x} = \dfrac{F(x)}{F'(x)}, \\ \dot{y} = \dfrac{G(y)}{G'(y)}, \end{cases}$$

then U is well-defined in $\mathcal{R}$ and satisfies $[X, U] = \mu X$, where

$$\mu(x, y) = \; div\, U - 1 = \frac{d}{dx}\left(\frac{F(x)}{F'(x)}\right) + \left(\frac{d}{dy}\frac{G(y)}{G'(y)}\right) - \; 1,$$

(ii) The origin of (5) is a centre, which period annulus is contained in $\mathcal{R}$, and the associated period function T satisfies:

$$T'(s) = \int_0^{T(s)} \mu\big(x\,(t), y(t)\big)\, dt,$$

where s refers to the parameterization of the orbits of U.

Proof. The vector field U is well-defined in $\mathcal{R}$ since F and G are analytical with a non-degenerate minimum at 0. Furthermore, the non-degeneracy of functions F and G guarantees the presence of a centre. Notice that the orbits of the period annulus of the origin cannot intersect the lines that form the boundary of $\mathcal{R}$.

Straightforward computations from part 2 of Theorem (5.1.2) with $V(x, y) \equiv \; 1$ and $\emptyset(x) = \; x$ lead to the desired result.

Observe that for the function m given in Lemma (5.1.4), we can equally separate the contribution of F and G in the expression and extract some useful sufficient conditions for monotonicity avoiding integration of m: According to this goal, given
a function F and following the previously quote, we define

$$\nu_F\,(x) = \; F'(x)^2 - 2F(x)F''(x),$$

$$\varphi F(x) = \left(\frac{F(x)}{F'(x)^2}\right) = \; \nu_F\,(x)/F'(x)^3.$$

This notation suggests to consider the following subclasses of $\mathcal{C}^2$ real functions of one variable:

Definition (5.1.5)[106]: Let $J \in \mathcal{C}^2\,(O, \mathbb{R})$ for some $\Omega \subseteq \mathbb{R}$: We say that J is

(i) of class $\mathcal{I}$ if either $\nu_J \geq 0$ or φ_J is increasing in $\Omega\,\big(\nu_J \not\equiv 0\big)$,

(ii) of class $\mathcal{N}$ if either $\nu_J \; \equiv \; 0$ or φ_J is constant in Ω,

(iii) of class $\mathcal{D}$ if either $\nu_J \; \leq 0$ or φ_J is decreasing in $\Omega\,\big(\nu_J \not\equiv 0\big)$.

We also say that a pair of functions $\{l_1, l_2\}$ form a $\mathcal{L}_1$–$\mathcal{L}_2$ pair if l_1 is of class $\mathcal{L}_1$ and $\mathcal{L}_2$ is of class $\mathcal{L}_2$, where $\mathcal{L}_j$ stands for $\mathcal{I}, \mathcal{N}$ or $\mathcal{D}$.

Since the initial value problem $\nu_F\,(x) = \; 0$ with $F(0) = F'(0\,) = \; 0$ has the only solution $F(x) = \; kx^2, k \in \mathbb{R}$; class $\mathcal{N}$ becomes quite artificial. We keep it as a class only for aestheticpurposes.

On the other hand, under the hypotheses of Lemma (5.1.4), the periodicorbits of the period annulus of the origin are contained in $\mathcal{R}$, Notice also that in $\mathcal{R}$, the horizontal and vertical isoclines are, respectively, the axes $x = \; 0$ and $y = \; 0$.

Definition (5.1.6)[106]: Let γ be a periodicorbit of the period annulus of the origin of system (5). We denote by $(x_M, 0), (0, y_M), (x_m, 0)$ and $(0, y_m)$ the intersections of γ with the axes, see Fig. 1.

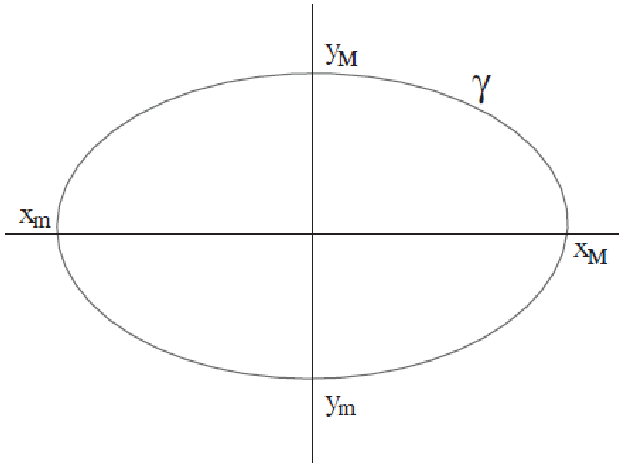


Fig. (1)[106]: Definitions of x_m, x_M, y_m and y_M.

Next proposition gives sense to these definitions and shows that the functions φ and ν are suitable to find simpler ways to prove monotonicity.

We remark that the possibilities $\mathcal{I}-\mathcal{D}$ and $\mathcal{D}-\mathcal{I}$ are not reflected in Proposition (5.1.7).

In principle, these situations could lead either to isochronous centres, or periodincreasing, or period-decreasing or even more complicated behaviours, see the $\mathcal{I}-\mathcal{D}$ family of systems explored in Fig. 2 (the fact that these systems are of type $\mathcal{I}-\mathcal{D}$ is proved in Proposition (5.1.15)). Of course, it is also possible that the functions F and G that define system (5), are not of any of the classes considered in Definition (5.1.5).

Proposition (5.1.7)[106]: (see also [118] for the second part). Consider the Hamiltonian system (5) generated by $H(x,y) = F(x) + G(y)$, with F and G two real analytic functions at 0, such that $F(0) = G(0) = 0$ and they have a non-degenerate minimum at 0. Then, the following hold:

(i) The function m of Lemma (5.1.4) is defined in the rectangle $\mathcal{R}$ and can be written as

$$\mu(x,y) = \nu_F(x)\frac{1}{2F'(x)^2} + \nu_G(y)\frac{1}{2G'(y)^2}. \tag{6}$$

(ii) By using the notation introduced in Definition (5.1.6) and in Fig. 1, the derivative of the period function in the period annulus of the origin can be written as

$$\int_0^{T(s)} \mu\big(x(t),y(t)\big)dt$$
$$= \frac{1}{2}\int_{y_m}^{y_M} \left[\varphi_F\big(x_+(y)\big) - \varphi_F\big(x(y)\big)\right] dy$$
$$+ \frac{1}{2}\int_{x_m}^{x_M} \left[\varphi_G\big(y_+(x)\big) - \varphi_G\big(y(x)\big)\right] dx \tag{7}$$

(iii) The centre at the origin

(a) is isochronous if $\{F,G\}$ form $a\ \mathcal{N}-\mathcal{N}$ pair.

(b) has an increasing period function if $\{F,G\}$ form one of the following pairs: $\mathcal{I}-\mathcal{I}, \mathcal{N}-\mathcal{I}, \mathcal{I}-\mathcal{N}$.

(c) has a decreasing period function if $\{F,G\}$ form one of the following pairs: $\mathcal{D}-\mathcal{D}, \mathcal{N}-\mathcal{D}, \mathcal{D}-\mathcal{N}$.

Proof. Part 1 of the proposition follows from the equalities $\mu(x,y) = \left(\frac{F(x)}{F'(x)}\right)' - \frac{1}{2} + \left(\frac{G(y)}{G'(y)}\right)' - \frac{1}{2} = -\frac{1}{2}\frac{F(x)F''(x)}{F'(x)^2} + \frac{1}{2}\frac{G(y)G''(y\text{Þ})}{G'(y)^2} = \nu_F \frac{(x)1}{2F'(x)^2} + \frac{\nu_G(y)1}{2G'(y)^2}$

Note also that $\mu(x,y) = \frac{1}{2}\left(\varphi_F\,(x)F'(x) + \varphi_G(y)G'(y)\right)$.

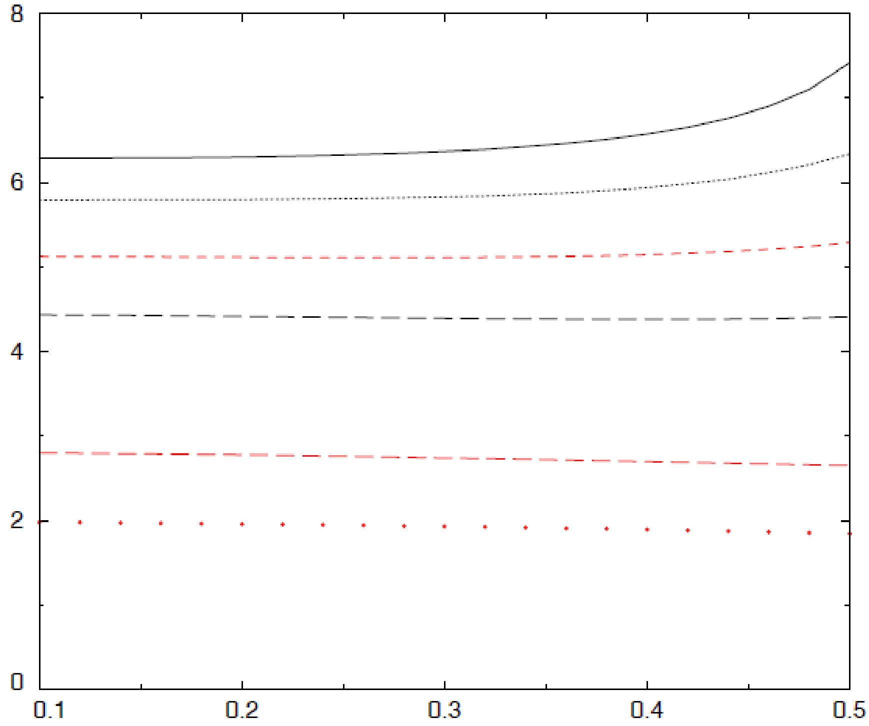


Fig. (2)[106]: Numerical computations of the period function associated to $H(x,,y) = (x^2/2 + x^3/3) + y^2/2 + y^4/4$, for different values of $k, k = 1, 1{,}17525, 1{,}5, 2, 5, 10$ from above to below. While for $k = 1$ the period is increasing, from $k \approx 1.17525$ to some value it presents a minimum (so inappreciable in the scale of the figure that the centre seems to be isochronous) and it is decreasing for larger values of k like $k = 5, 10$.

To prove part 2, take a periodicorbit γ of (5), for some value h of the Hamiltonian. Call x_m, x_M, y_m and y_M the intersections of γ with the axes, as shown in Fig. 1. For each y; call $x_-(y)$ and $x_+(y)$ its two pre-images and, similarly, define $y_-(x)$ and $y_+(x)$. Then, using the hypotheses on F and G and the differential equations themselves, we obtain,

$$\int_0^{T(s)} \mu(x(t),y(t))dt = \int_0^{T(s)} \left[\frac{1}{2}(\varphi_G(x)F'(x)+\varphi_G(y)G'(x))\right]_{x=x(t),y=y\,(t)} dt$$
$$= \int_{y_m}^{y_M} \left(\frac{1}{2}\varphi_F(x)\right)_{x=x_+(y)} dy - \int_{y_m}^{y_M} \left(\frac{1}{2}\varphi_F(x)\right)_{x=x_+(y)} dy$$
$$+ \int_{y_m}^{y_M} \left(\frac{1}{2}\varphi_F(y)\right)_{y=y_+(x)} dx - \int_{x_m}^{x_M} \left(\frac{1}{2}\varphi_F(y)\right)_{y=y_-(x)} dx$$
$$= \frac{1}{2}\int_{y_m}^{y_M} \left[\varphi_F(x_-(y)) - \varphi_F(x_-y)\right) dy$$
$$+ \frac{1}{2}\int_{x_m}^{x_M} [\varphi_G(y+(x)) - \varphi_G(y(x))]\, dx.$$

The statements of part 3 mainly follow from the fact that the two variables play separate roles. Let us suppose, for instance, that $F(x$Þ is of class I; both if $\nu_F \geq 0$ and if φ_F is increasing, the term $\frac{1}{2}\int_{y_m}^{y_M} (\varphi_F(x_+(y)) - \varphi_F(x_-y)))\, dy$ will be strictly positive. Similar reasonings apply for $G(y)$ and for the other two different classes of functions, $\mathcal{N}$ and $\mathcal{D}$.

Bearing in mind the definitions of classes $\mathcal{I}, \mathcal{N}$ and $\mathcal{D}$, in the next result we group all the functions that we will need from now on along so that the remaining results will not need detailed proofs. The list does not pretend to be exhaustive and tries to show the strength and clearness of the method.

Theorem (5.1.8)[106]: The 54 parametric families of Hamiltonian systems associated either to $H(x,y) = cI_i(x) + kI_j(y)$, for $i = 1, \ldots, 9, i \leq j \leq 9$; or to $H(x,y) = cI_i(x) + ky^2, for\ i = 1, \ldots, 9$ and $c > 0, k > 0$, have increasing period function in the period annulus of the origin.

The 5 parametric families of Hamiltonian systems associated either to $H(x,y) = cD_i(x) + kD_j(y)$, for $i = 1, \ldots, 2, i \leq j \leq 2$, or to $H(x,y) = cD_i(x) + ky^2$, for $i = 1, \ldots, 2$, and $c > 0, k > 0$, have decreasing period function in the period annulus of the origin.

Proof. The theorem follows directly from Propositions 10(iii) and 11. Only a nuance in the case of function D_1 must be underlined: note that when $n > 1$ the centre is degenerate, which breaks the first condition of Lemma (5.1.4). However, both the transversal vector field U, and the function m are well-defined and the proofs and conclusions are still valid. Observe that in this case—as in any degenerate centre— the period function tends to infinity when the periodic orbits tend to the critical point.

The general family of Hamiltonian systems (5) treated has connections with many physical problems and other well-known examples. Among the 59 cases presented in Theorem (5.1.8), we would like to stress how our method works for the non-forced pendulum, some applications to celestial mechanics and to relativistic mechanics, the Lotka–Volterra model and a number of potential systems. First of all, using function I_6, we get:

Example (5.1.9)[106]: The non-forced pendulum, the Hamiltonian system with

$$H(x, y) = \frac{y^2}{2} - \cos x + 1,$$

has increasing period.

A less trivial potential Hamiltonian arises when using function D_1:

Example (5.1.10)[106]: The potential Hamiltonian systems with $H(x, y) = \frac{y^2}{2} + a_m \frac{x^{2m}}{2m} + an \frac{x^{2n}}{2n}$, with $a_m \geq 0, a_n > 0$ and $m > n \geq 1$ have decreasing periods.

The features of $I_7(z) = D_2(z) = (p + qz)^a - p^a$ provide two interesting applications. When $a = \frac{1}{2}$; the resulting Hamiltonian is used in relativistic mechanics, where the problem of finding constant period oscillators (isochronous centres) has some interest, see [124] and the references therein. The authors find numerical approximations of a function V such that the Hamiltonian $H(x, y) = V(x) + K(y)$ where $K(y) = \sqrt{m^2 + y^2/c^2} - m$, is isochronous. We think also that a nice way to find isochronous centres would be looking for V such that ν_V compensates ν_K : Here we give an example of decreasance of the period function.

Example (5.1.11)[106]: The period function associated to the centre of the Hamiltonian system given by $H(x, y) = \frac{1}{2} x^2 + \sqrt{m^2 + y^2/c^2} - m$ is decreasing.

The function $I_7(z)$ when $a = \frac{1}{2}$ leads to a Hamiltonian used in celestial mechanics to study the Sitnikov motion problem, see [109].

Example (5.1.12)[106]: The period function associated to the centre of the Hamiltonian system given by $H(x, y) = \frac{1}{2} y^2 - \frac{1}{\sqrt{x^2 + \frac{1}{4}}} + 2$ is increasing.

Remark (5.1.13)[106]: Theorem (5.1.8) covers many of the examples of Chow and Wang, see [117], where they study, for potential Hamiltonian systems, not only the first derivative of the period function but also give an expression for the second

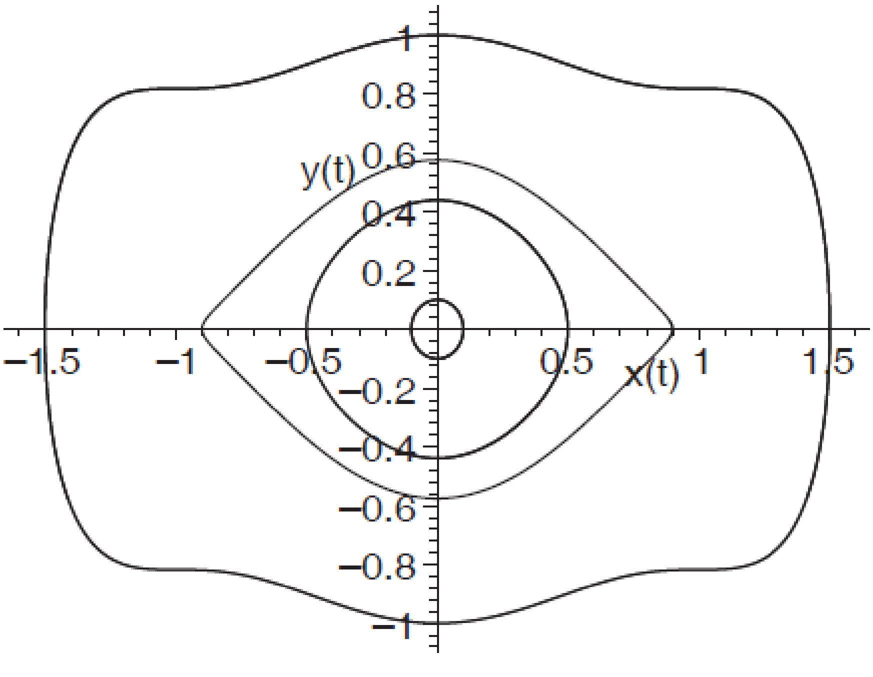


Fig. (3)[106]: The phase portrait of the Hamiltonian system derived from $H(x, y) = \frac{z^6}{6} - \frac{z^4}{2} + \frac{z^2}{2} + \frac{y^2}{2}$.

derivative. In the current context, potential Hamiltonian systems are equivalent to $G\eth(y) = y^2/2$. In particular, taking $F(x) = I_1(x)$, $I_2(x)$, $I_3(x)$, $I_4(x)$ and $I_5(x)$ we obtain the increasing periods showed in [117] in Examples 1, 2, 3.a, 3.b and 5, respectively; and taking $F(x) = D_1(x)$ with $\{a_m = 0, n = 2\}$ and $\{m = 4, n = 2, a_m = a_n = 1\}$ we obtain the decreasing periods given in [117] in Examples 3.d and 3.c $(b = 0)$, These are all the examples where they succeed to prove monotonicity.

The case when $F(x) = I_5(x) = \frac{z^6}{6}\frac{z^4}{2} + \frac{z^2}{2}$ and $G(y) = \frac{y^2}{2}$ deserves some attention. The vector field has exactly three critical points: the centre at the origin and two cusps at $(\pm 1, 0)$. All the orbits of the vector field are closed, except for the two heteroclinics that link the two cusps, see Fig. 3. We have proved that the period function of the origin's period annulus is increasing; moreover, it must go to infinity as it approaches to those heteroclinics. Outside the heteroclinics, the normalizer U (see Lemma (5.1.4)) is no longer transversal to the vector field and so, we cannot deduce that the period function is increasing. Indeed, there are strong numerical evidences that it is decreasing as the orbits go to infinity.

The increasance of periods for the Lotka–Volterra predator–prey system is one of the most known results related to periods in planar ODEs. It was first stated by Hsu [123], but some gap was found in the proof. Afterwards, it has been proved [127], [130], [134].

Example (5.1.14)[106]: The centre of the classical Lotka–Volterra predator–prey system,

$$\begin{cases} \dot{x} = x(a - \beta y), \\ \dot{y} = y(gx - m), \end{cases} \tag{8}$$

has an increasing period function.

Here, we give a short proof of such a fact. By means of a change of variables $u = \log((\gamma x)/m)$ $v = log((\beta y)/a)$, the Lotka–Volterra system can be transformed into a Hamiltonian system of type $H(u, v) = F(u) + G(v)$ with $F(u) = a(e^u - u - 1)$ and $G(v) = m(e^v - v - 1)$, Then, Theorem (5.1.8) with function I_1 gives the result.

However, an advantage of our method is that we do not need to do any transformation and we can apply it directly to the original system.

We devote the rest to prove all the cases listed in Proposition (5.1.15). The proof is quite technical and straightforward.

Proposition (5.1.15)[106]: (i) The following functions are of class $\mathcal{I}$ in Ω:

(a) $I_1(z) = e^z - z - 1, \Omega = \mathbb{R}$.

(b) $I_2(z) = z^3/3 + z^2/2, \Omega = (-5/2, +\infty)$.

(c) $I_3(z) = z^2\left(\frac{z^2}{4} + \frac{a+1}{3} z + \frac{a}{2}\right)$ $, with\ 0 < a < 1\ and\ \Omega = (-a, 1)$.

(d) $I_4(z) = z^2\left(\frac{z^2}{4} + \frac{a+1}{3} z + \frac{a}{2}\right)$ $with\ 0 < a \le 1\ and\ \Omega = (-a + \infty)$.

(e) $I_5(z) = \frac{z^6}{6}\frac{z^4}{2} + \frac{z^2}{2}, \Omega = \mathbb{R}\backslash\{-1, 1\}$.

(f) $I_6(z) = 1 - cos\, z, \Omega = \mathbb{R}$:

(g) $I_7(z) = (p + qz)^a - p^a$ with p, q positive real numbers and $a \notin [0, 1)\Omega = \mathbb{R}$.

(h) $I_8(z) = \frac{z^2}{1+z^2}, +z^2 = \mathbb{R}$,

(i) $I_9(z) = z$ arctan $z - \frac{1}{2}\ln(1 + z^2)\Omega = \mathbb{R}$.

(ii) The following functions are of class $\mathcal{D}$ in Ω:

(a) $D_1(z) = a_m \frac{z^2 m}{2m} + a_n \frac{z^2 n}{2n}, with\ a_n > 0, a_m \ge 0\ and\ m > n \ge 1, \Omega = \mathbb{R}$:

(b) $D_2(z) = (p + qz)^a - p^a$ with p, q positive real numbers and $a \in (0, 1)\Omega = \mathbb{R}$:

Proof. To avoid cumbersome notations, in the whole proof we drop the subscripts for φ and ν.

1. Functions of class $\mathcal{I}$.

(a) For $I_1(z) = e^z - z - 1, \frac{d}{dz}\varphi(z) = \frac{e^z}{(e^z-1)^4}(e^{2z} - 4e^z + 4ze^z + 2z + 5)$.

The function $e^{2z} - 4e^z + 4ze^z + 2z + 5$ is always a negative function and so φ increasing.

(b) For $I_2(z) = z^3/3 + z^2/2, \frac{d}{dz}\varphi(z) = \frac{1}{3}\frac{2z+5}{(z+1)^4}$.

(c) Consider $I_3(z) = -z^2\left(\frac{z^2}{4} + \frac{a-1}{3}z - \frac{a}{2}\right)$, with $0 < a < 1$ and $a < z < 1$. Elementary computations give:

$$\frac{d}{dz}\varphi(z) = \frac{1}{6}\frac{P(z,a)}{(z+a)^4(z-1)^4},$$

where $P(z,a) = (-10 + 4z)a^3 + (11 - 42z + 16z^2)a^2 + (24z^3 - 10 - 68z^2 + 42z)a - 4z - 24z^3 + 16z^2 + 9z^4$. The proof is finished in (d) together with that of I_4.

(d) For $I_4(z) = z^2\left(\frac{z^2}{4} + \frac{a+1}{3}z + \frac{a}{2}\right)$, with $0 < a \le 1$ and $-a < z < +\infty$, we obtain in a similar way:

$$\frac{d}{dz}\varphi(z) = \frac{1}{6}\frac{P(-z,-a)}{(z+a)^4(z+1)^4},$$

so that to prove that both I_3 and I_4 are of class $\mathcal{I}$, we need

$P(z,a) \le 0$for all $(z,a) \in R_1 := (-a,1) \times (0,1)$,

$P(z,a) \ge 0$ for all $(z,a) \in R_2 := (-\infty,a) \times (-1,0)$,

From standard computations it is easy to see that:

(i) The restriction of $P(z,a)$ to $ðR_1$ is negative except at $(z,a) = (0,0)$ and $(z,a) = (-1,1)$, where it is zero.

(ii) The restriction of $P(z,a)$ to $ðR_2$ is positive except at $(z,a) = (0,0)$, where it is zero.

(iii) $\frac{ð}{ða}P(ðz,a)$ never vanishes in $R_1 \cup R_2$.

Then, the result follows.

(e)For$I_5(z) = \frac{z^6}{6} - \frac{z^4}{2} + \frac{z^2}{2}, \frac{d}{dz}\varphi(z) = \frac{1}{3}\frac{10z^6-39z^4+60z^2+9}{(z^2-1)^6}$.

The polynomial $10w^3 - 39w^2 + 60w + 9$ has two non-real roots and one real negative, so $10z^6 - 39z^4 + 60z^2 + 9$ has no real roots and φ' turns out to be positive everywhere it is defined.

(f) For $I_6(z) = 1 - \cos z, n(z) = (1 - \cos z)^2 \ge 0$.

(g) Since $I_7(z) = (p + qz)^a - p^a$ and $D_2(z)$ are the same function we are going to give the proof together.

For the sake of simplicity, we write M instead of I_7 or D_2, We first compute $\nu = \nu_M$

$$\nu(z) = M'(z)^2 - 2M(z)M''(z) = 4a\frac{p}{q}p^{2a}w^{a-2}h(w).$$

where $w = (1 + pz^2/q) \ge 1$ and

$$h(w) = (a-1)w^{(a+1)} + (2-a)w^a + (1-2a)w + (2a-2).$$

This expression tells us that all the cases behave as $p = q = 1$, that is, $M(z) = (1 + z^2)^a - 1$; because it reduces the study of the sign of ν_M to that of $h(w)$ Some elementary calculus gives the following properties: $h(1) = h'(1) = 0$ and $h''(w) = w^{a-2}\big(a(a^2-1)w + a(a-1)(2-a)\big)$. Then, if h'' does not change sign, the function

h also keeps the same sign. We can easily see that: when $a > 1, h''(w) \geq 0 \Leftrightarrow w > (a-\ 2) = (a+\ 1)$;
when $0 < a < 1, h''(w) \leq 0 \Leftrightarrow w > (a\ -2)/(a\ +\ 1)$;
when $a < 0, h''(w) \geq 0 \Leftrightarrow w < (a-\ 2)/(a+\ 1)$:
For the function $(a-2)/(a+\ 1)$, it is straightforward to see that the last three inequalities on w are true and so, $h(w) \geq 0$ for all w if $a \notin (0,1)$ and $h(w) \leq 0$ for all w if $a \in (0,1)$, The first and the third give the statement referred to function I_7 while the second one leads to that of D_2:
(h) For $I_8(z) = \frac{z^2}{1+z^2}, v(z) =\ 12\frac{z^4}{(1+z^2)^4} \geq 0$.
(i) For $I_9(z) =\ z \arctan z - \frac{1}{2}\ln(1+z^2), v(z) = \frac{\arctan^2\ (z)(1-z)^2 + \ln(1+z^2)}{1+z^2} \geq 0$.
(ii) Functions of class $\mathcal{D}$.
(a) Consider $D_1(z) =\ a_m\frac{z^{2m}}{2m} +\ a_n\frac{z^{2n}}{2n}$, with $a_n \neq 0, a_m/an \geq 0$ and $m > n \geq 1$.
Denoting $w =\ w(z) =\ z^{2(m-n)}$ it turns out that

$$v(z) =\ z^{4n-2}\ (Aw^2 +\ Bw +\ C),$$

with $A := a_m^2\left(1+\frac{1}{m}\right), B := \frac{a_m a_n}{m_n}\ (m - 2m^2 +\ n +\ 2mn - 2n^2)$ and $C :=\ a_n^2\left(1+\frac{1}{n}\right)$. For $w =\ 0$ the value of the second degree polynomial is
$a_n^2(1+\frac{1}{n}) \leq 0$. Now, if we prove that it does not have positive solutions, we are done. So, we impose $\frac{B}{2A} \geq 0$ and $AC \geq 0$. The last inequality always holds since $AC =\ a_m^2 a_n^2(-1+\frac{1}{m})(-1+\frac{1}{n}) \geq 0$. On the other hand,

$$\frac{B}{2A} = \frac{a_m}{2n(1-m)a_n}\ (m - 2m^2 +\ n +\ 2mn - 2n^2).$$

Since $m-\ 2m^2 +\ n +\ 2mn - 2n^2 =\ m(1-n) +\ n(1-m) -\ 2(m-n)^2 < 0, B/(2A) \geq 0$ reduces then to $a_m/a_n \geq 0$, which is true by hypothesis. Finally, although it is not necessary for D_1 being of class $\mathcal{D}$, we need to assume that an is positive so that the origin is a centre.
(b) For $D_2(z)$ see the proof of $I_7(z)$.

We devoted to a new result about a family of quadratic systems with a decreasing period function. A bigger family of quadratic systems including the next one was treated in [110]. Despite they obtain a general expression for $\mu(x,y)$; it is too difficult to handle for our purposes. We have considered the following case, which is also a Loud's system, see [54] and also [114].

Proposition (5.1.16)[106]: The quadratic system

$$\begin{cases}\dot{x} =\ -y +\ 2Dx^2\ Dy^2, \\ \quad\dot{y} =\ x +\ Dxy,\end{cases} \tag{9}$$

has a decreasing period function.

Proof. First of all, notice that the change of variables $\tilde{x} =\ Dx, \tilde{y} =\ Dy$ eliminates the parameter D in (9) and so we can consider only the case $D =\ 1$:

$$\begin{cases}\dot{x} =\ -y +\ 2x^2 - y^2, \\ \quad\dot{y} =\ x +\ xy,\end{cases} \tag{10}$$

A first integral for (10) is

$$H(x,y) = \frac{1}{2}\frac{x^2}{(1+\ y)^4} - \frac{1}{6}\frac{(1+\ 3y)}{(1+\ y)^3} + \frac{1}{6},$$

which associated integrating factor is $1/V$, where $V = (1+\ y)^5$.
It is not difficult to see that the periodic orbits γ_h corresponding to the period annulus of the origin are included in the sets $\{H =\ h, 0 < h < 1/6\}$ (they are one of the two connected components of the level sets). When $h \to 1/6$; the periodic orbits approach to the curve $x^2 =\ y^2 +\ 4/3y +\ 1/3$.
Note that for H of the special form $H(x,y) =\ A(y) +\ B(y)x^2$ it is easy to prove that

$$U = \left(\left(1 - \frac{A(y)}{A'(y)}\frac{B'(y)}{B(y)}\right)\frac{x}{2}\ , \frac{A(y)}{A'(y)}\right)$$

is a normalizer of $\nabla H^{\perp}$ see also [121]. We have that the same U is also a normalizer for any system of the form $V(x,y)\nabla H^{\perp}$, By performing these computations in our case we can take $U = (R,S)$ where

$$R(x,y) =\ x\left(\frac{1}{2} + \frac{1}{3}\ y(3+\ y)\right)\ and\ S(x,y) = \frac{1}{6}\ y(3+\ y)(1+\ y).$$

Furthermore, $RH_x +\ SH_y =\ H$. By using part 2 of Theorem (5.1.2) we have that $a\,\mu$ associated to system (10) is

$$\mu = \left(1 + \frac{7}{3}\ y + \frac{5}{6}\ y^2\ \right) - \left(\frac{5}{6}\ y(3+\ y)\right) -\ 1 = \frac{1}{6}\ y$$

and hence

$$T'(s) = \int_0^{T(s)} \mu\big(x(t),y(t)\big)dt = \frac{1}{6}\int_0^{T(s)} y(t)dt.$$

Fixed y and s, the first integral H tells us that there exists only a pair of values of $x, -\,x_-(y) =\ x_+(y) > 0$, such $(x <\ y) \in \gamma_s$ For a fixed s, define y_m and y_M the two intersection of gs with the y $-$axis, see also Fig. 1.
At this point the integration could be cumbersome and perhaps not possible. We make use, now, of Proposition (5.1.3). It turns out that taking $g(x,y) = \frac{x}{6(1+y)}$, and defining

$$\mu^*\ (x,y) =\ \mu(y) + \nabla g^t \cdot\ X = \frac{1}{6}\frac{x^2}{1+\ y}$$

we can compute T' as

$$T'(w) = \frac{1}{6}\left(\int_{y_m}^{y_M} \frac{x_+(y)}{(1+\ y)^2}\ dy - \int_{y_m}^{y_M} \frac{x_-(y)}{(1+\ y)^2}dy\right) = \frac{1}{3}\left(\int_{y_m}^{y_M} \frac{x_+(y)}{(1+\ y)^2}dy\ \right),$$

because of the symmetry on x, Clearly, the argument of the last integral is always positive and so, we can assert that the period is decreasing.

We give another proof of the monotonicity of the period function for the Lotka–Volterra system

$$\begin{cases} \dot{x} =\ x(a - \beta y) = xyH_y(x,y), \\ \dot{y} =\ y(\gamma x -\ m) =\ xyH_x(x,y) \end{cases} \tag{11}$$

which works directly on (11), without changing variables. Here $H(x,y) =\ F(x) +\ G(y)$ where

$$F(x) =\ \gamma x -\ m\ \left(ln\ \left(\frac{\gamma x}{m}\right) +\ 1\right) \text{and } G(y) =\ \beta y -\ a\left(\ln\left(\frac{\beta y}{a}\right) +\ 1\right)$$

$U = \left(\frac{F(x)}{F'(x)}, \frac{G(y)}{G'(y)}\right)$ is a normalizer of (11) and m is

$$\mu(x,y) = 1 - \frac{\gamma x F(x)}{(\gamma x - m)^2} - \frac{\beta y G(y)}{(\beta y - a)^2};$$

Now, we want to see that $T'(s) > 0$, where s is the parameter of some orbit of U, because this parameter increases forward from the critical point.

We perform the integration in the following way:

$$\int_0^{T(s)} \mu(x(t), y(t))dt$$

$$= \int_{y_m}^{y_M} \left[\frac{1}{y(\gamma x - m)}\left(\frac{1}{2} - \frac{\gamma x F(x)}{(\gamma x - m)^2}\right)\right]_{x_-(y)}^{x_+(y)} dy$$

$$+ \int_{x_m}^{x_M} \left[\frac{1}{x(a - \beta y)}\left(\frac{1}{2} - \frac{\beta y G(y)}{(a - \beta y)^2}\right)\right]_{y_+(x)}^{y_-(x)} dx.$$

Then, making the change of variables $u = \ln(\gamma x/m)\,, v = \ln(\beta y/a)$ in both integrals we obtain

$$\int_0^{T(s)} \mu(x(t), y(t))dt$$

$$= \frac{1}{m}\int_{v_m}^{v_M} \left[\frac{1 + 2ue^u - e^{2u}}{2(e^u - 1)^3}\right]_{u-(v)}^{u+(v)} dv$$

$$+ \frac{1}{a}\int_{v_m}^{v_M} \left[\frac{1 + 2ve^v - e^{2v}}{2(1 - e^v)^3}\right]_{v_+(u)}^{v_-(u)} du.$$

If we denote

$$H(\xi) = \frac{1 + 2\xi e^{\xi} - e^{2\xi}}{2(e^{\xi} - 1)^3}$$

it turns out that

$$H'(\xi) = \frac{1}{2}\frac{e^{\xi}\left(-4e^{\xi} + 5 + 4\xi e^{\xi} + 2\xi\, e^{2\xi}\right)}{(e^{\xi} - 1)^4}.$$

The function in parenthesis in the numerator is negative (it is the same function that the one that appears in the proof of Proposition (5.1.15), function I1). Then, $H(\xi)$ is an increasing function. This fact and the preceding computations clearly imply the result.

We prove that a subfamily of Lie´nard systems—which includes the quadraticone with $A(x) = x^2/2$ studied in [112]—has an increasing period function in the period annulus of the origin.

Proposition (5.1.17)[106]: The family of Lie´nard equations

$$\begin{cases} \dot{x} = -y + A(x), \\ \dot{y} = A'(x) \end{cases} \tag{12}$$

with A an smooth function satisfying $A(0) = A'(0) = 0$; has a centre at the origin. Furthermore, if $A(x) = kI_i(x)$, for some $i = 1, \dots, 9$ and $k > 0$ where I_i are the functions which appear in Proposition (5.1.15) or $A(x) = kx^2$, then the period function of (12) is increasing in the period annulus of the origin.

Proof. By using the change of variables $(u, v) = (x, y)\, A(x))$ we get the new system
$$\begin{cases} \dot{u} = -v, \\ \dot{v} = A'(u)(1 + v), \end{cases}$$
Applying the new change $(z, w) = (u, \log(1 + v))$ we arrive to
$$\begin{cases} \dot{z} = 1\, e^{w}, \\ \dot{w} = A'(z), \end{cases}$$
which is of the form of the systems for which Theorem (5.1.8) applies. By applying it with the Hamiltonians $I_1(w) + kI_j(z)$, with $j = 1, \ldots .,9$; or with the Hamiltonian $I_1(w) + k_z^2$ the result follows.

To finish, we give an overview to one of the families where the period function is better understood: that of Hamiltonian systems obtained from
$$H(x, y) = \frac{1}{2}\,(x^2 + y^2) + H_{n+1}(x, y)$$
where H_{n+1} is a homogeneous polynomial of degree $n + 1$, It has been shown, see [39], that the period function of the centre at the origin is always increasing when n is even and has at most one critical period when n is odd. Here, we will see how formula (9) for the derivative of the period function used in [39] can be obtained using our method.
To achieve this goal it is convenient to express the system in polar coordinates. We remark that the Lie bracket does not depend on the chosen variables. Therefore, the vector field is
$$\begin{cases} \dot{r} = r^n g'(\theta), \\ \dot{\theta} = 1 + (n + 1) r^{n-1} g(\theta), \end{cases}$$
while the Hamiltonian writes now as $H(r, \theta) = \frac{1}{2}\, r^2\, r^{n+1} g(\theta)$
Aiming to use part 2 of Theorem (5.1.2), we search for $R = R(r, \theta)$ and $S = S(r, \theta)$ such that $RH_x + SH_y = H$: We observe first that $R_1 = r/2$ and $S_1 = (1 - \frac{n}{2}) g(\theta)/g'(\theta)$ satisfy $R_1 H_r + S_1 H_\theta = H$: Then, using that $H_r = H_x \cos\theta + H_y \sin\theta$ and $H_\theta - rH_x \sin\theta + rH_y \cos\theta$, it turns out that
$$R = R_1 \cos\theta - S_1 r \sin\theta,$$
$$S = R_1 \sin\theta + S_1 r \cos\theta,$$
satisfy the required relation. Hence, from our main theorem, we know that $\mu(\theta) = \frac{1}{2}\frac{g'(\theta)^2\,(1-n) + g''(\theta) g(\theta)(n-1)}{g'(\theta)^2} = \frac{1-n}{2}\frac{d}{d\theta}\left(\frac{g(\theta)}{g'(\theta)}\right)$.
Integrating by parts it becomes (except for a positive constant) the same formula used in [114].

Section (5.2): Quasi-Homogeneous Polynomial Differential Equations of Degree Three

Poincaré in [159] was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a center is a singular point surrounded by a neighborhood filled of closed orbits with the unique exception of the singular point. Since then the center–focus problem, i.e. the problem to distinguish when a singular point is either a focus or a center is one of the hardest problems in the qualitative theory of planar differential systems, see [136]. We deal mainly with the characterization of the center problem for the class of quasi-homogeneous polynomial differential systems of degree 3. In the literature we found classifications of polynomial differential systems having a center. For the quadratic systems see Dulac [143], Kapteyn [147], [148], Bautin [138] among others. In [162] Schlomiuk, Guckenheimer and Rand gave a brief history of the center problem for quadratic systems.

There are many partial results about centers for polynomial differential systems of degree greater than two. Some of them are for instance, the classification by Malkin [153] and Vulpe and Sibirskii [164] about the centers for cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree three. Note that for polynomial differential systems of the form linear with homogeneous nonlinearities of degree $k > 3$ the centers are not classified. However, there are some results for $k = 4, 5$, see for instance the works by Chavarriga and Giné [140], [141]. It seems difficult for the moment to obtain a complete classification of the centers for the class of all polynomial differential systems of degree 3. Actually, there are some subclasses of cubic systems well studied like the ones of Rousseau and Schlomiuk [160] and the ones of Żoładek [165], [166]. Some centers for arbitrary degree polynomial differential systems have been studied in [152].

We denote by $\mathbb{R}[x, y]$ the ring of all polynomials in the variables x and y and coefficients in the real numbers $\mathbb{R}$. We consider polynomial differential systems of the form

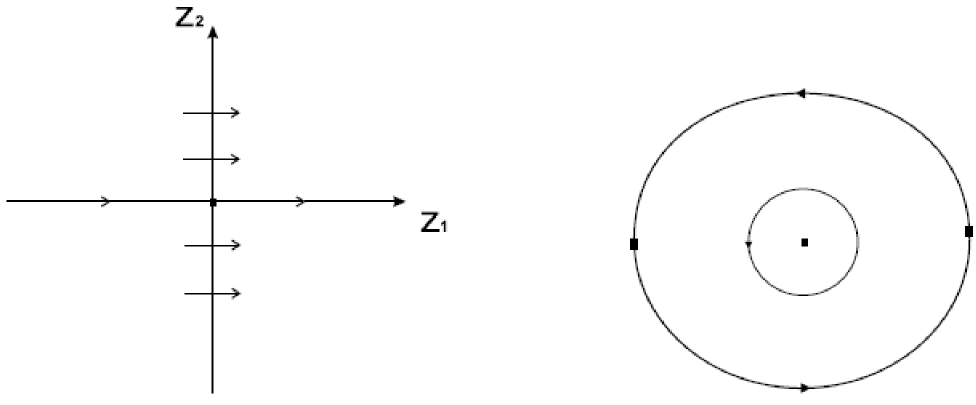
$$\dot{x} = P(x, y), \dot{y} = Q(x, y), \qquad (13)$$

with $P, Q \in \mathbb{R}[x, y]$ and its corresponding vector field $X = (P, Q)$. Here the dot denotes derivative with respect to the time t (independent variable). The degree of the differential polynomial system (13) is the maximum of the degrees of the polynomials P and Q.

System (13) is a quasi-homogeneous polynomial differential system if there exist natural numbers s_1, s_2, d such that for an arbitrary non-negative real number α it holds

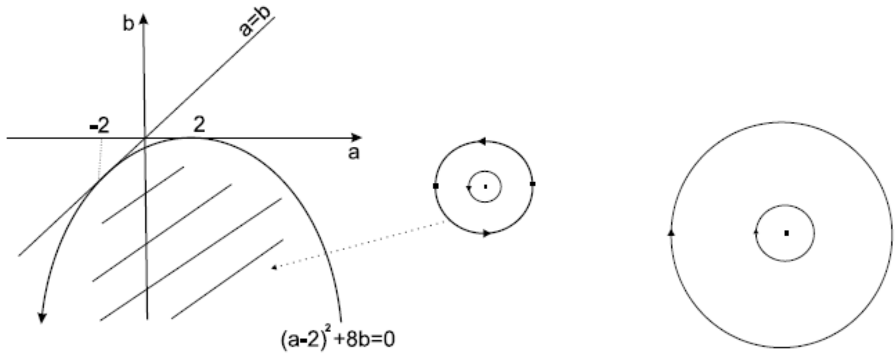
$$P\,(\alpha^{s_1}\, x, \alpha^{s_2}\, y) = \alpha^{s_1+d-1} P(x, y), Q\,(\alpha^{s_1}\, x, \alpha^{s_2}\, y) = \alpha s^{2+d-1} Q(x, y). \qquad (14)$$

The natural numbers s_1 and s_2 are the weight exponents of system (13) and d is the weight degree with respect to the weight exponents s_1 and s_2. When $s_1 = s_2 = s$ we obtain the classical homogeneous polynomial differential system of degree $s + d - 1$.



(a) $(z_1, z_2) = (0, 0)$ on the $(U_1, F_1.)$ (b) On the Poincaré disk.

Fig. (1)[135]: (a) The local phase portrait at the origin in the local chart U1. (b) Phase portrait of a cubic quasi-homogeneous non-homogeneous system (18) in the Poincaré disk. This system has a global center.



(a) cubic quasi–homogeneous (b) cubic homogeneous

Fig. (2)[135]: (a) The parameter space (a, b) and the phase portrait of cubic quasi-homogeneous systems (18). (b) Cubic homogeneous systems (20) having a center, see also [142].

It is well known that all quasi-homogeneous vector fields are integrable with a Liouvillian first integral [145], [146], [150].

From Theorem (5.2.8) of [151] we have that there are only two families of cubic polynomial differential homogeneous systems with a center.

In the next result we characterize all the centers of quasi-homogeneous polynomial differential systems.

The proof of Theorem (5.2.7).

In addition to the classification of centers, another classical problem in the qualitative theory of planar differential systems is the study of their limit cycles. Recall that a limit cycle of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system. Thus in what follows we study, using the averaging theory of first order, the limit cycles which bifurcate from the periodic orbits of the centers (19) and (20) of Theorem (5.2.7) when these centers are perturbed inside the class of all cubic polynomial differential systems.

For proving Theorem (5.2.7) we should need the following result.

Proposition (5.2.1)[135]: A quasi-homogeneous non-homogeneous cubic polynomial differential systems (13) with P and Q coprime and $s_1 > s_2$ after a rescaling of the variables can be written as one of the following systems.

(a)$\dot{x} = y\,(ax + by^2)$, $\dot{y} = x + y^2$, with $a \neq b, or\ \dot{x} = y\,(ax \pm y^2)\,, y' = x$, and both with minimal weight vector $(2, 1, 2)$.

(b)$\dot{x} = x^2 + y^3, \dot{y} = axy$, with $a \neq 0$ and minimal weight vector $(3, 2, 4)$.

(c) $\dot{x} = y^3, \dot{y} = x^2$, and minimal weight vector $(4, 3, 6)$.

(d)$\dot{x} = x\,(x + ay^2), \dot{y} = y(bx + y^2)$, with $(a, b) \neq (1, 1)$, and minimal weight vector $(2, 1, 3)$.

(e)$\dot{x} = axy^2, \dot{y} = \pm x^2 + y^2$, with $a \neq 0$ and minimal weight vector $(3, 2, 5)$.

(f)$\dot{x} = axy^2, \dot{y} = x + y^3$, with $a \neq 0$ and minimal weight vector $(3, 1, 3)$.

(g) $\dot{x} = ax + y^3, \dot{y} = y, or\ \dot{x} = ax, \dot{y} = y$ with $a \neq 0$, and minimal weight vector $(3, 1, 1)$.

Proof. See [146].

A singular point is nilpotent if both eigenvalues of its linear part are zero but its linear part is not identically zero. Andreev [137] was the first in characterizing the local phase portraits of the nilpotent singular points. In what follows we summarize the results of the local phase portraits of the nilpotent singular points, for more details see Theorem 3.5 of [144].

Theorem (5.2.2)[135]: Let (0, 0) be an isolated singular point of the vector field X given by

$$\dot{x} = y + A(x,y), \dot{y} = B(x,y),$$

where A and B are analytic in a neighborhood of the point (0, 0) starting with terms of second degree.

Let $y = f(x)$ be the solution of the equation $y + A(x,y) = 0$ in a neighborhood of the point (0, 0), and consider $F(x) = B(x, f(x))$ and $G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x))$. Then the origin can be a focus or a center if and only if one of the following statements holds:

(a) If $G(x) \equiv 0$ and $F(x) = ax^m + o(x^m)$ for $m \in \mathbb{N}$ with $m \geq 1, m$ odd and $a < 0$, then the origin of X is a center or a focus.

(b) If $F(x) = \alpha x^m + o(x^m)$ with $\alpha < 0, m \in \mathbb{N}, m \geq 2, m$ odd, and $G(x) = \beta x^n + o(x^n)$ with $\beta \neq 0, n \in \mathbb{N}, n \geq 1$ and if either $m < 2n + 1$, or $m = 2n + 1$ and $\beta^2 + 4\alpha(n + 1) < 0$, then the origin of X is a center or a focus.

The following result characterizes the isochronous centers.

Theorem (5.2.3)[135]: A center of an analytic system is isochronous if and only if there exists an analytic change of coordinates of the form $u = x + o(x,y)$ and $v = y + o(x,y)$ changing the system to the linear isochronous system

$$\dot{u} = -kv, \dot{v} = ku,$$

where k is a real constant.

For a proof of Theorem (5.2.3), see [154].

Assume that the origin is an isochronous center for system (13). Then Theorem (5.2.3) guarantees that there exists an analytic change of coordinates $u = x + o(x,y)$ and $v = y + o(x,y)$ such that $\dot{u} = -kv, \dot{v} = ku$. Then since $\ddot{u} + u = 0$, and $\ddot{v} + v = 0$, and doing a rescaling we can take $k = 1$.

In order to plot the global phase portrait of the polynomial vector field (13) of degree m we should be able to control the orbits that come or escape at infinity. For this reason we consider the so called Poincaré compactification of the polynomial vector field X . Consider $\mathbb{R}^2$ as the plane in $\mathbb{R}^3$ defined by $(y_1, y_2, y_3) = (x_1, x_2, 1)$. We also consider the Poincaré sphere $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 + y_2 + y_3 = 1\}$ (see also [158]) and we denote by $T(0,0,1)\mathbb{S}^2$ the tangent space to $\mathbb{S}^2$ at the point (0, 0, 1). The Poincaré compactified vector field $p(X)$ of X is an analytic vector field induced on $\mathbb{S}^2$ in the following way: We consider the central projection $f : T(0,0,1) : \mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of X , one in the northern hemisphere $\{y \in \mathbb{S}^2 : y_3 > 0\}$ and the other in the southern hemisphere. We denote by $\tilde{X}$ the vector field $Df \circ X$ defined on $\mathbb{S}^2$ except on its equator. We notice that the points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator of $\mathbb{S}^2, \mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ and so we identify $\mathbb{S}^1$ to be the infinity of $\mathbb{R}^2$.

Now we would like to extend the induced vector field $\tilde{X}$ from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to $\mathbb{S}^2$. It is possible that $\tilde{X}$ does not stay bounded as we get close to $\mathbb{S}^1$. However, it turns out that if we multiply $\tilde{X}$ by the factor y_3^{m-1} , namely, if we consider the vector field $y_3^{m-1}\tilde{X}$ the extension is possible in the whole $\mathbb{S}^2$.

Note that on $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of X and knowing the behavior of $p(X)$ around $\mathbb{S}^1$, we know the behavior of X at infinity. The Poincaré disk D^2 is the projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longrightarrow (y_1, y_2)$. Moreover, $\mathbb{S}^1$ is invariant under the flow of $p(X)$.

We also say that two polynomial vector fields X and $\backslash Y$ on $\mathbb{R}^2$ are topologically equivalent if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$. The homeomorphism should preserve or reverse simultaneously the sense of all orbits of the two compactified vector fields $p(X)$ and $p(\mathcal{Y})$.

Since $\mathbb{S}^2$ is a differentiable manifold we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$ and the diffeomorphisms $Fi : V_i \dashrightarrow \mathbb{R}^2$ and $G_i : V_i \dashrightarrow \mathbb{R}^2$ are the inverses of the central projections from the planes tangent at the points (1, 0, 0), (−1, 0, 0), (0, 1, 0), (0, −1, 0), (0, 0, 1) and (0, 0, −1) respectively. Now we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$. Then we obtain the following expressions of the compactified vector field $p(X)$ of X (for more details we refer to Chapter V of [144] and the references therein)

$$z_2^n\,\Delta(z)\left(Q\left(\frac{1}{z_2},\frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2},\frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_1},\frac{z_1}{z_2}\right)\right)\ in\ U_1,$$

$$z_2^n\,\Delta(z)\left(P\left(\frac{z1}{z2},\frac{1}{z2}\right) - z_1 Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2},\frac{1}{z_2}\right)\right)\ in\ U_2,$$

$$\Delta(z)\big(P(z_1, z_2), Q(z_1, z_2)\big)\ in\ U3,$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{1}{2(n-1)}}$. Note that in the two sets U_i and V_i the expressions of the vector field $p(X)$ are the same and they only differ by the multiplicative factor $(-1)^{n-1}$. In these coordinates $z_2 = 0$ always denotes the points of $\mathbb{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(X)$ and so we obtain a polynomial vector field in each local chart.

Let $p(X)$ be the Poincaré compactification of $\mathbb{S}^2$ of a polynomial vector field X in $\mathbb{R}^2$. In what follows we consider the definition of parallel flows given by Markus [155] and Neumann in [156]. Let $\emptyset$ be a C^ω local flow on the two dimensional manifold $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$. The flow $(M, \emptyset)$ is C^k parallel if it is C^ω −equivalent to one of the following ones:
$strip$: $(\mathbb{R}^2, \emptyset)$ with the flow $\emptyset$ defined by $\dot{x} = 1, \dot{y} = 0$;
$annular$: $(\mathbb{R}^2 \setminus \{0\}, \emptyset)$ with the flow $\emptyset$ defined (in polar coordinates) by $\dot{r} = 0, \dot{\theta} = 1$;
$spiral$: $(\mathbb{R}^2 \setminus \{0\}, \emptyset)$ with the flow $\emptyset$ defined by $\dot{r} = 0, \dot{\theta} = 1$.

It is known that the separatrices of the vector field $p(X)$ in the Poincaré disk D are

(i) all the orbits of $p(X)$ which are in the boundary $\mathbb{S}^1$ of the Poincaré disk (recall that $\mathbb{S}^1$ is the infinity of $\mathbb{R}^2$);
(ii) all the finite singular points of $p(X)$;
(iii) all the limit cycles of $p(X)$; and
(iv) all the separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(X)$.

We denote by Σ the union of all separatrices of the flow $(D, \emptyset)$ defined by the compactified vector field $p(X)$ in the Poincaré disk D. Then Σ is a closed invariant subset of D. Every connected component of $D \setminus \Sigma$, with the restricted flow, is called a canonical region of $\emptyset$.

For a proof of the following result see [149] and [156].

Theorem (5.2.4)[135]: Let $\emptyset$ be a C^ω flow in the Poincaré disk with finitely many separatrices, and let Σ be the union of all its separatrices. Then the flow restricted to every canonical region is C^ω parallel.

The separatrix configuration Σ_c of a flow $(D, \emptyset)$ is the union of all the separatrices Σ of the flow together with an orbit belonging to each canonical region. The separatrix configuration Σ_c of the flow $(D, \emptyset)$ is said to be topologically equivalent to the separatrix configuration $\tilde{\Sigma}_c$ of the flow $(D, \widetilde{\emptyset})$ if there exists a homeomorphism from D to D which transforms orbits of Σ_c into orbits of $\tilde{\Sigma}_c$, and orbits of Σ into orbits of $\tilde{\Sigma}$.

Theorem (5.2.5)[135]: Let $(D, \emptyset)$ and $(D, \widetilde{\emptyset})$ be two compactified Poincaré flows with finitely many separatrices coming from two polynomial vector fields (13). Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.

For a proof of Theorem (5.2.5) see [155], [156], [157].

From Theorem (5.2.5) it follows that in order to classify the phase portraits in the Poincaré disk of a planar polynomial differential system having finitely many finite and infinite separatrices, it is enough to describe their separatrix configuration.

We consider the system

$$x'(t) = F_0(t, x), \tag{15}$$

with $F_0 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ a C^2 function, T −periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. We assume that system (15) has a submanifold of periodic solutions.

Let ε be sufficiently small and we consider a perturbation of system (15) of the form

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \tag{16}$$

with $F_1 : \mathbb{R} \times \omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions, T −periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. Averaging theory deals with the problem of the bifurcation of T −periodic solutions of system (16), see also for more information on the averaging theory [161], [163].

Let $x(t, z)$ be the periodic solution of the unperturbed system (15) satisfying the initial condition $x(0, z) = z$. Now we consider the linearization of system (15) along the solution $x(t, z)$, namely

$$y' = D_x F_0\,(t, x(t, z))\, y,$$

and let $M_z(t)$ be a fundamental matrix of this linear system satisfying that $M(0)$ is the identity matrix.

For a proof of the following theorem see [139].

Theorem (5.2.6)[135]: (Perturbations of an isochronous set). We assume that there exists an open bounded set V with $Cl(V) \subset \Omega$ such that for each $z \in Cl(V)$, the solution $x(t, z)$ is T −periodic, then we consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^n$

$$\mathcal{F}(z) = \int_0^T \mathcal{M}_z^{-1}\,(t, z) F_1\,(t, x(t, z))\, dt. \tag{17}$$

If there exist $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\big((d\mathcal{F}/dz)(a)\big) \neq 0$, then there exists a T −periodic solution $\emptyset(t, \varepsilon)$ of system (16) such that $\emptyset(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

Theorem (5.2.7)[135]: The following two statements hold.

(a) The unique cubic quasi-homogeneous non-homogeneous polynomial differential system (13) with P and Q coprime and $s_1 > s_2$ having a center after a rescaling of the variables can be written as

$$\dot{x} = y\,(ax + by^2)\,, \dot{y} = x + y^2, \tag{18}$$

with $(a - 2)^2 + 8b < 0$. For all a and b satisfying $(a - 2)^2 + 8b < 0$ the phase portrait in the Poincaré disk of system (18) is topologically equivalent to the one given in

Fig. 1(b). Moreover, its parameter space (a, b) is described in Fig. 2(a). Additionally, these centers are not isochronous.

(b) The unique cubic homogeneous polynomial differential systems having a center after a linear transformation and a rescaling of independent variable can be written in one of the following four forms:

$$\dot{x} = -3\alpha\mu x^2 y - \alpha y^3 + P_3, \dot{y} = \alpha x^3 + 3\alpha\mu x y^2 + Q_3, \qquad (19)$$

$where\ \alpha = \pm 1, \mu > -1/3$ and $\mu \neq 1/3$;

$$\dot{x} = -\alpha x^2 y - \alpha y^3 + P_3, \dot{y} = \alpha x^3 + \alpha x y^2 + Q_3, \qquad (20)$$

$with\ \alpha = \pm 1$. Here $P_3 = p_1 x^3 + p_2 x^2 y - p_1 x y^2$ and $Q_3 = p_1 x^2 y + p_2 x y^2 - p_1 y^3$. The phase portraits in the Poincaré disk of systems (19) and (20) are topologically equivalent to the ones of Fig. 2(b). Moreover, these centers are not isochronous.

Proof. All quasi-homogeneous non-homogeneous cubic polynomial differential systems are given by Proposition (5.2.1). Note that all those systems have the origin as the unique singular point.

Now we consider the first system of statement (a) of Proposition (5.2.1). This system admits the real first integral

$$(by^4 + (a - 2)xy^2 - 2x)^2(\Delta x - 2by^2 - ax + 2x)^{\frac{3a+6+\Delta}{\Delta-a-2}}(2by^2 + ax - 2x + \Delta x),$$

with $(a - 2)^2 + 8b \geq 0$ and $\Delta = \sqrt{(a - 2)^2 + 8b}$. Note that the real invariant curve $2by^2 + ax - 2x + \Delta x = 0$ passes through the origin. Hence, the origin is not a center.

Now we consider the case where $(a - 2)^2 + 8b < 0$. Under the change of coordinates $x \to Y, y \to X$ and after renaming (X, Y) by (x, y) we obtain

$$\dot{x} = y + x^2, \dot{y} = x(ay + bx^2). \qquad (21)$$

Now we apply Theorem (5.2.2) to system (21). We have $A(x, y) = x^2$ and $B(x, y) = x(ay + bx^2)$. We have $F(x) = B(x, -x^2) = (b - a)x^3$ and $G(x) = (a + 2)x$. Since $a \neq b$ we have that $F \not\equiv 0$. Following the notation of Theorem (5.2.2) we have $m = 3, \alpha = b - a, n = 1$ and $\beta = a + 2$.

For $a \neq -2$ we have that $G(x) \equiv 0$ and $b < -2$. So $\alpha < 0$ and by Theorem (5.2.2)(a) the origin is a focus or a center. System (21) has the real first integral

$$H = \left(y - \left(-1 + \frac{1}{2}\sqrt{2(2 + b)}\right)x^2\right)\left(y - \left(-1 - \frac{1}{2}\sqrt{2(2 + b)}\right)x^2\right),$$

well defined at the origin and consequently the origin is a center.

For $a \neq -2$ we have $G(x) \not\equiv 0$. In order that the origin of system (21) can be a focus or a center, from Theorem (5.2.2)(b), we need that $\alpha = b - a < 0$ and $(a - 2)^2 + 8b < 0$. We notice that system (21) under these assumptions admits the real first integral

$$H(x, y) = , \frac{(16y^2 + 16x^2y - 8x^2ay + 8x^4c^2 + 4x^4 - 4x^4a + x^4a^2)^{2c}}{\left(e^{(\sqrt{2}(2+a)\arctan)\left(\frac{\sqrt{2}}{4}-\frac{4y-2x^2+x^2a}{x^2c}\right)}\right.}$$

with $c = \sqrt{-2((a - 2)^2 + 8b)/4}$. Since this first integral is defined at the origin, the origin is a center.

The second family of systems of statement (a) of Proposition (5.2.1) admits the real invariant curves $\sqrt{a^2 + 8x} \pm 2y^2 \pm ax = 0$ which pass through the origin. So these systems have no centers.

Easy computations show that systems (b), (c), (d), (e), (f) and (g) have real invariant curves passing through the origin. Therefore these systems have no centers.

In short, the quasi-homogeneous non-homogeneous cubic polynomial differential systems having a center are the system (18) satisfying either $a = -2$ and $b < -2$, or $a \neq -2, b - a < 0$ and $(a - 2)^2 + 8b < 0$. An easy computation (see Fig. 2) shows that these conditions for existence of the center in system (18) reduce to the unique condition $(a - 2)^2 + 8b < 0$.

Now we shall study the phase portrait in the Poincaré disk D and the parameter space of system (18). So, we study the infinite singular points of system (18). On the local chart U_1 we obtain

$$\begin{aligned} \dot{z}_1 &= z_2^2 + (1 - a)z_1^2 z_2 - bz_1^4, \\ \dot{z}_2 &= -z_1 z_2 (az_2 + bz_1^2). \end{aligned} \tag{22}$$

Since $8b + (a - 2)^2 < 0$ we have that $(z_1, z_2) = (0, 0)$ is the only infinite singular point in U_1 and it is linearly zero. In order to classify this infinite singular point we use the standard blow-up techniques, see for instance [144]. Then we obtain that the local phase portrait at the origin (0, 0) of system (22) is topologically equivalent to the one described in Fig. 1(a). Additionally, note that in the chart (U_2, F_2) there are no infinite singular points. Hence, in the Poincaré disk the origin and $\mathbb{S}^1$ are the only separatrices. If we remove the origin and $\mathbb{S}^1$, then we have only one canonical region homomorphic to $\mathbb{R}^2 \setminus \{0\}$ and the flow is locally annular. According to Theorem (5.2.4) we obtain that the center is globally defined in $\mathbb{R}^2 \setminus \{0\}$. Hence, the phase portrait of the differential system (18) is topologically equivalent to the one of Fig. 1(b).

The parameter space and phase portrait of system (18) is given in Fig. 2(a). Now we will study the isochronicity of the center of system (18). System (18) written in the polar coordinates is

$$\begin{aligned} \dot{r} &= P_1(\theta)r + P_2(\theta)r^2 + P_3(\theta)r^3, \\ \dot{\theta} &= Q_0(\theta) + Q_1(\theta)r + Q_2(\theta)r^2, \end{aligned}$$

with

$$P_1 = \cos\theta \sin\theta, P_2 = (\sin^2\theta + a\cos^2\theta)\sin\theta, P_3 = b\cos\theta\sin^3\theta, Q_0 = \cos^2\theta, Q_1 = -(a - 1)\sin^2\theta\cos\theta, Q_2 = -b\sin^4\theta.$$

Consider the analytic function $H(r, \theta) = \sum_{n=1}^{\infty} H_n(\theta)r^n$ where $H_n(\theta)$ are trigonometric polynomials of degree n. If the condition

$$\ddot{H} + H = 0,$$

is satisfied then in the new variables $(H, -\dot{H})$, system (18) could be transformed into the form

$$\dot{u} = -v, \dot{v} = u.$$

So system (18) could have an isochronous center at the origin.

If we expand $\ddot{H} + H = 0$ in power series of r we obtain a recursive system of differential equation. The coefficient of rn for $n = 1, 2, \ldots$ in this expansion is the differential equation of the form

$$\cos^4\theta H''_n(\theta) + 2(n - 1)\sin\theta\cos\theta H'_n(\theta) + n\cos^2\theta H_n(\theta)(n - 1) - (n - 2)\cos^2\theta + H_n(\theta) = 0,$$

and its general solution for $n = 1$ is

$$H_1(\theta) = \cos\theta \left(C_1 \sin\left(\frac{\sin\theta}{\cos\theta}\right) + C_2\cos\left(\frac{\sin\theta}{\cos\theta}\right)\right).$$

For $n = 2, 3, \ldots$ we have

$$H_n(\theta) = (\cos 2\theta + 1)^{\frac{n}{2}} \left(C_1 \sin\left(\frac{\sin 2\theta}{\cos 2\theta + 1}\right) + C_2\cos\left(\frac{\sin 2\theta}{\cos 2\theta + 1}\right)\right).$$

Since these solutions $H_n(\theta)$ must be polynomials of trigonometric functions we have that $H_n \equiv 0$ for all n. Hence we do not have an isochronous center and the proof of Theorem (5.2.7)(a) is completed.

Now we are going to prove Theorem (5.2.7)(b). The usual forms given in (20) for the cubic homogeneous polynomial differential systems having a center were obtained in Proposition 1 and Theorem (5.2.8) of [151]. The phase portraits were classified in [142]. See also Fig. 2(b).

In order to study the isochronicity of systems (19) and (20) we can repeat the same mechanism used in the proof of statement (a). In polar coordinates system (20) takes the form

$$\dot{r} = P_3(\theta)r^3, \qquad \dot{\theta} = \alpha r^2,$$

where $P_3 = p_1(\cos^2\theta - \sin^2\theta) + p_2 \sin\theta \cos\theta$.

We can see that

$$H_1(\theta) = H_2(\theta) = H_3(\theta) = H_4(\theta) = 0,$$

and for $n \geq 5$ we have that

$$H_n(\theta) = -(\alpha^2 H''_{n-4} + 2(n-3)\alpha P_3 H'_{n-4} + (n-4)H_{n-4}(\alpha P'_3 + (n-2)P_3^2)).$$

Clearly each $H_n \equiv 0$, for all n, so system of (20) is not an isochronous center. System (19) can be written in polar coordinates as

$$\dot{r} = P_3(\theta)r^3, \dot{\theta} = \alpha Q_2(\theta) r^2,$$

where

$$P_3 = p_1 \cos^2\theta - \sin^2\theta + p_2 \sin\theta\cos\theta,$$
$$Q_2 = \cos^4\theta + 6\mu\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Again we obtain

$$H_1(\theta) = H_2(\theta) = H_3(\theta) = H_4(\theta) = 0,$$

and for $n \geq 5$ we have

$$H_n(\theta) = -(\alpha^2 Q_2^2 H''_{n-4} + (\alpha^2 Q'^2 Q_2 + 2(n-3)\alpha Q_2 P_3)H'_{n-4} + (n-4)H_{n-4}(\alpha P'_3 Q_2 + (n-2)P_3^2)).$$

Clearly each $H_n \equiv 0$, for all n and therefore system (19) is not an isochronous center. This completes the proof of Theorem (5.2.7).

Theorem (5.2.8)[135]: Consider the cubic homogeneous systems (19) and (20) and their perturbation inside the class of all cubic polynomial differential systems. Then, for $|\varepsilon| \neq 0$ sufficiently small one limit cycle can bifurcate from the continuum of the periodic orbits of the center of systems (19) and (20) using averaging theory of first order.

Proof. System (20) in polar coordinates can be written into the form

$$\dot{r} = r_3(p_1\cos^2\theta + p_2 \sin\theta\cos\theta - p_1\sin^2\theta,)\ \dot{\theta} = \alpha r^2,$$

or equivalently

$$\frac{dr}{d\theta} = \frac{r}{\alpha}(p_1\cos^2\theta + \sin\theta p_2 \cos\theta - p_1\sin^2\theta),$$

its solution satisfying the initial condition $r(0) = r_0$ is

$$\tilde{r}(\theta, r_0) = r_0 \exp((p_2 + 2p_1 \sin(2\theta) - p_2\cos(2\theta)/(4\alpha)).$$

Now the fundamental matrix of the linearized equation evaluated on a closed orbit is

$$Mr_0(\theta) = M(\theta) = \exp((p_2 + 2p_1\sin(2\theta) - p_2\cos(2\theta)/(4\alpha)),$$

and satisfies the condition $M(0) = 1$.

Now we perturb system (20) inside the class of all cubic polynomial differential systems and we have

$$\dot{x} = p_1 x^3 + (p_2 - \alpha)x^2 y - p_1 x y^2 - \alpha y^3 + \varepsilon\left(\sum_{0\le i+j\le 3} a_{ij} x^i y^j\right),$$
$$\dot{y} = \alpha x^3 + p_1 x^2 y + (p_2 + \alpha)x y^2 - p_1 y^3 + \varepsilon\left(\sum_{0\le i+j\le 3} b_{ij} x^i y^j\right).$$
The corresponding differential equation in polar coordinates becomes
$$\frac{dr}{d\theta} = F_0(\theta, r) + \varepsilon F_1(\theta, r) + O(\epsilon^2),$$
with
$$F_0(\theta, r) = \frac{r}{\alpha}\,(p_1\,(2\cos^2\theta - 1) + p_2 \sin\theta\cos\theta),$$
$$F_1(\theta, r) = \frac{1}{\alpha r^3}\,(B_4 r^4 + B_3 r^3 + B_2 r^2 + B_1 r),$$
where
$$B_4 = \frac{1}{\alpha}\,(B46\cos^6\theta + B_{45}\sin\theta\cos^5\theta + B_{44}\cos^4\theta + B_{43}\sin\theta\cos^3\theta$$
$$+ B_{42}\cos^2\theta + B_{41}\sin\theta\cos\theta + B_{40}),$$
$$B3 = -\frac{1}{\alpha}\,(B_{35}\cos^5\theta + B_{34}\sin\theta\cos^4\theta + B_{33}\cos^3\theta + B_{32}\sin\theta\cos^2\theta$$
$$+ B_{31}\theta\cos\theta + B_{30}\sin\theta),$$
$$B_2 = -\frac{1}{\alpha}\,(B_{24}\cos^4\theta + B_{23}\sin\theta\cos^3\theta + B_{22}\cos^2\theta + B_{21}\sin\theta\cos\theta + B_{20}),$$
$$B_1 = -\frac{1}{\alpha}\,(B_{13}\cos^3\theta + B_{12}\, sin\,\theta\cos^2\theta + B_{10}\, sin\,\theta),$$
with
$$B_{46} = 2p_1a_{03} + 2p_1b_{12} - 2p_1a_{21} - 2p_1b_{30} + p_2a_{12} - p_2a_{30} + p_2b_{21} - p_2b_{03},$$
$$B_{45} = -2p_1a_{12} + 2p_1a_{30} + p_2a_{03} - p_2b_{30} - 2p_1b_{21} - p_2a_{21} + p_2b_{12}$$
$$+ 2p_1b_{03},$$
$$B_{44} = -5p_1a_{03} + 3p_1a_{21} - 3p_1b_{12} - b_{21}\alpha - a_{12}\alpha + a_{30}\alpha + p_2a_{30} - p_2b_{21}$$
$$+ p_1b_{30} + b_{03}\alpha + 2p_2b_{03} - 2p_2a_{12},$$
$$B_{43} = -p_2b_{12} + 3p_1a_{12} - a_{03}\alpha + p_2a_{21} + b_{30}\alpha - 3p_1b_{03} - p_1a_{30} - b_{12}\alpha$$
$$+ p_1b_{21} + a_{21}\alpha - 2p_2a_{03},$$
$$B_{42} = 4p_1a_{03} + p_1b_{12} - p_2b_{03} + b_{21}\alpha + a_{12}\alpha - 2b_{03}\alpha - p_1a_{21} + p_2a_{12},$$
$$B_{41} = -p_1a_{12} + p_2a_{03} + b_{12}\alpha + a_{03}\alpha + p_1b_{03},$$
$$B_{40} = b_{03}\alpha - p_1a_{03},$$
$$B_{35} = 2p_1b_{02} - 2p_1a_{11} - 2p_1b_{20} + p_2a_{02} + p_2b_{11} - p_2a_{20},$$
$$B_{34} = -2p_1b_{11} - p_2b_{20} + 2p_1a_{20} - p_2a_{11} - 2a_{02}p_1 + p_2b_{02},$$
$$B_{33} = -a_{02}\alpha + p_1b_{20} + a20\alpha + p2a20 - 2p2a02 - p2b11 - 3p1b02$$
$$+ 3p1a11 - b11\alpha,$$
$$B_{32} = -b_{02}\alpha + p_2a_{11} - p_1a_{20} + b_{20}\alpha + p_1b_{11} + 3a_{02}p_1 + a_{11}\alpha - p_2b_{02},$$
$$B_{31} = -p_1a_{11} + p_2a_{02} + b_{11}\alpha + a_{02}\alpha + p_1b_{02},$$
$$B_{30} = -a_{02}p_1 + b_{02}\alpha,$$
$$B_{24} = -2a_{01}p_1 - 2p_1b_{10} + p_2b_{01} - p_2a_{10},$$
$$B_{23} = -2p_1b_{01} - p_2b_{10} - p_2a_{01} + 2p_1a_{10},$$
$$B_{22} = 3a_{01}p_1 + a_{10}\alpha + \alpha b_{01} + p_2a_{10} + p_1b_{10} - p_2b_{01},$$
$$B_{21} = p_1b_{01} - p_1a_{10} + b_{10}\alpha + a_{01}\alpha + p_2a_{01},$$

$B_{20} = -a_{01}p_1 + \alpha b_{01},$
$B_{13} = -2p_1 b_{00} - p_2 a_{00},$
$B_{12} = 2a_{00}p_1 - p_2 b_{00},$
$B_{11} = p_1 b_{00} + p_2 a_{00} + \alpha a_{00},$
$B_{10} = b_{00}\alpha - a_{00}p_1.$

Note that

$$\mathcal{F}(r_0) = \int_0^{2\pi} M^{-1}(\theta)F_1(\theta, \tilde{r}(\theta, r_0))d\theta$$
$$= \frac{1}{r_0} A_0 I_0 + \frac{2}{r_0^2} \left(A_1 I_1 + A_2 I_2 + A_3 I_3 + A_4 I_4 + \pi C_1 + \frac{3\pi}{4} C_2\right)$$
$$+ 2r_0\pi(\alpha b_{03} - p_1 a_{03}) + \frac{5\pi}{8} C_3,$$

where we have

$$I_0 = \int_0^{2\pi} E\, d\theta, I_1 = \int_0^{2\pi} E \cos\theta \sin\theta\, d\theta,$$
$$I_2 = \int_0^{2\pi} E \cos^2 \theta\, d\theta, I_3 = \int_0^{2\pi} E \cos^3\theta\, sin\,\theta\, d\theta,$$
$$I_4 = \int_0^{2\pi} E \cos^4 \theta\, d\theta, \quad E = exp\left(\frac{sin\theta(2p_1\, cos(\theta) + p_2\, sin\,\theta)}{\alpha}\right),$$

and

$$A_0 = -a_{01}p_1 + \alpha b_{01},$$
$$A_1 = -\frac{1}{2}\left((a_{10} - b_{01})p_1 - p_2 a_{01} - \alpha\left(a_{01} + b_{1,0}\right)\right) r_0,$$
$$A_2 = \left(\left(\frac{3}{2} a_{01} + \frac{1}{2} b_{10}\right) p_1 + \frac{1}{2}(p_2 + \alpha)(a_{10} - b_{01})\right) r_0,$$
$$A3 = \left((a_{10} - b_{01})p_1 - \frac{1}{2} p_2(a_{01} + b_{10})\right) r_0,$$
$$A_4 = \left((-a_{01} - b_{10})p_1 - \frac{1}{2} p_2(a_{10} - b_{01})\right) r_0,$$
$$C_1 = (\left(2_{a_{03}} + \frac{1}{2} b_{12} - \frac{1}{2} a_{21}\right) p_1 + \left(\frac{1}{2} a_{12} - \frac{1}{2} b_{03}\right) p_2$$
$$- \left(-\frac{1}{2} a_{12} + b_{03} - \frac{1}{2} b_{21})\ \alpha\right)\ r_0^3,$$
$$C_2 = \left(\left(\frac{3}{2} a_{21} + \frac{1}{2} b_{30} - 5/2 a_{03} - 3/2 b_{12}\right) p_1\right.$$
$$+ \left(\frac{1}{2} a_{30} - a_{1,2} - \frac{1}{2} b_{21} + b_{03}\right) p_2$$
$$\left. + \frac{1}{2}\alpha(b_{03} + a_{30} - a_{12} - b_{21})\right) r_0^3,$$

$$C_3 = 2\left((b_{12} + a_{03} - a_{21} - b_{30})p_1 - \frac{1}{2} p_2(b_{03} + a_{30} - a_{12} - b_{21})\right) r_0.$$

In short, the function F(r) of Theorem (5.2.6) is of the form

$$\mathcal{F}(r) = \frac{\alpha r^2 + \beta}{r},$$

so it has at most one real positive root given by $r = \sqrt{-\beta/\alpha}$. Moreover, we have that $\mathcal{F}\,(\sqrt{-\beta/\alpha}\,) = 2\alpha$. So by Theorem (5.2.6) if $-\beta/\alpha > 0$ then there is one limit cycle bifurcating from a periodic orbit of the center of system (20). This completes the proof of Theorem (5.2.8) for system (20).

The rest of the proof of Theorem (5.2.8) for system (19) is completely analogous to the one done for system (20), only the computations change, and we do not repeat it here.

We give an example satisfying the result of Theorem (5.2.8) for system (20). We consider the system

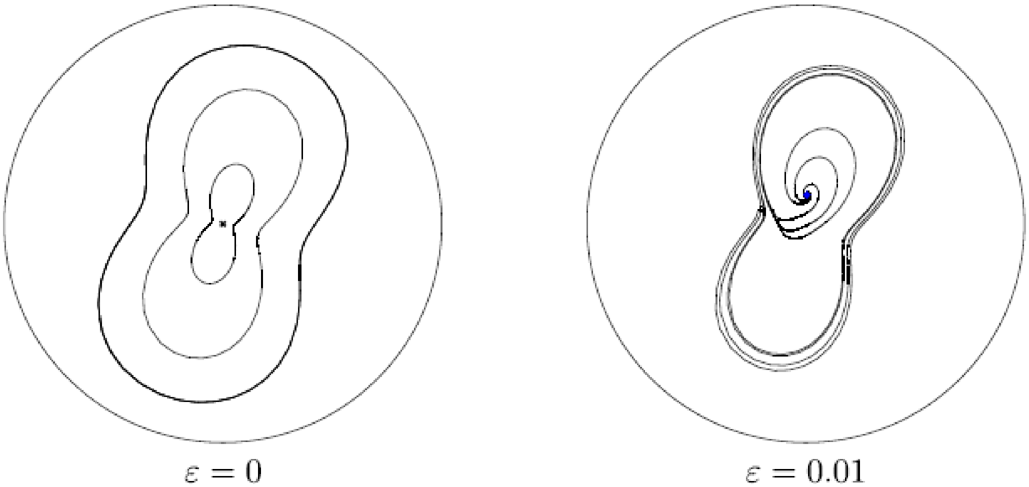


Fig. (3)[135]: Phase portrait of system (23) in the Poincaré disk.

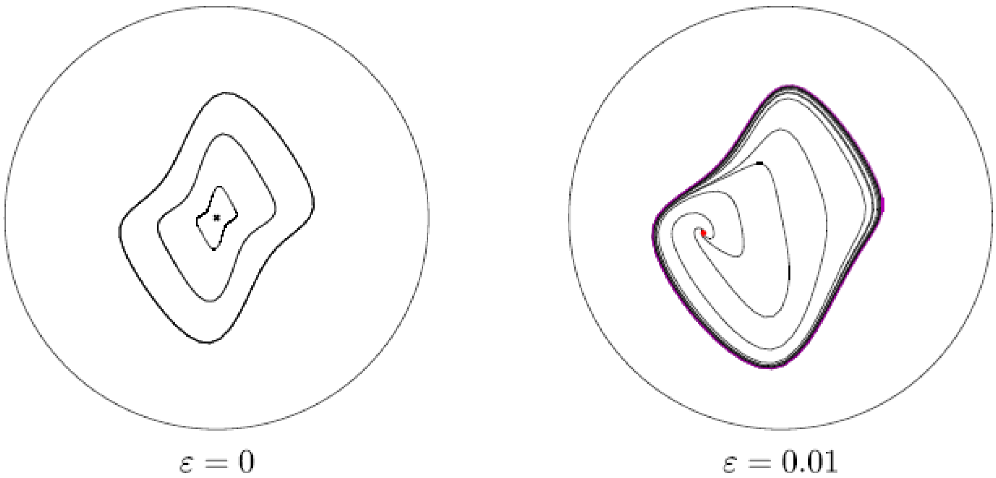


Fig. (4)[135]: Phase portrait of system (24) in the Poincaré disk.

$$\dot{x} = x^3 + 2x^2y - xy^2 - y3, \dot{y} = x^3 + x^2y + 4xy^2 - y^3,$$

and its perturbation

$$\dot{x} = x^3 + 2x^2y - xy^2 - y^3 + \epsilon(\,\varepsilon\, 4y^3 + 3xy^2 + 3x^2y + 5x^3 + 3y^2 + 3xy + 3x^2 - y - x + 2\,),$$
$$\dot{y} = x^3 + x^2y + 4xy^2 - y^3 + \varepsilon\,(-3y^3 + xy^2 + x^2y + x^3 + 5y^2 + xy + x^2 + y + 2x + 1). \qquad (23)$$

Then

$$\mathcal{F}(r_0) = \frac{11.78097245r_0^2 - 4.108168642}{r_0}$$

and $\mathcal{F}(r) = 0$ gives $r = 0.5905185728$. So according to Theorem (5.2.6) at most one limit cycle can be bifurcated from the origin, see also Fig. 3.

Example (5.2.9)[135]: Now we give an example satisfying the result of Theorem (5.2.8) for system (19). For $\varepsilon = 0$ the origin of the system

$$\dot{x} = x^3 - 6x^2y - xy^2 - y^3 + \epsilon(2y^3 + 3xy^2 + 3x^2y - 5x^3 + 3y^2 + 10xy + 3x^2 - y - x - 20),$$
$$\dot{y} = x^3 + x^2y + 12xy^2 - y^3 + -\epsilon(3y^3 + xy^2 - 10x^2y + x^3 + 5y^2 + xy + 1/5x^2 + y + 2x + 100), \quad (24)$$

is a center and for ε = 0.01 one limit cycle is produced, see Fig. 4.

Section (5.3): The Period Function of the Sum of Two Quasi-Homogeneous Vector Fields

A planar polynomial vector field $X(x,y) = (P(x,y), Q(x,y))$is called (p,q)quasi-homogeneous of quasidegree n if there exist $p,q,n \in \mathbb{N}$ such that

$$P(\lambda^p x, \lambda^q y) = \lambda^{n+p-1}P(x,y), Q(\lambda^p x, \lambda^q y) = \lambda^{n+q-1}Q(x,y),$$

for all $\lambda \in \mathbb{R}$. It is not restrictive to take p and q coprime. The numbers p and q are usually called weights. It is well known that its associated differential equation

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

can be integrated by writing it in the so called generalized polar coordinates, see [34]. Notice that homogeneous vector fields of degree n are quasi-homogeneous of quasi-degree n and weights (1, 1). Moreover, in this case the generalized polar coordinates are the usual polar ones.

We are concerned with vector fields having a degenerate critical point at the origin of centre type, and being either quasi-homogeneous or given by the sum of two quasi-homogeneous ones sharing the same weights (p,q). In the latter case, additionally we will assume that the vector field is Hamiltonian. We write the vector field in both situations as $X(x,y) = X_n(x,y) + X_m(x,y)$, with associated differential equation

$$\begin{cases} \dot{x} = P_n(x,y) + P_m(x,y), \\ \dot{y} = Q_n(x,y) + Q_m(x,y), \end{cases} \quad 1 < n < m, \quad (25)$$

where each $X_j = (P_j, Q_j), j \in \{n,m\}$, is (p,q) quasi-homogeneous of quasi-degree j. We assume that $X_n(x,y) \not\equiv 0$ but we admit that $X_m(x,y) \equiv 0$.

We want to know the global behaviour of the period function on the period annulus of the origin when we assume that the differential equation associated to X has a degenerate centre at this point. Recall that a centre is a critical point that has a punctured neighbourhood full of periodic orbits. The largest of such neighbourhoods is called the period annulus of the centre. When the eigenvalues of DX at the centre are not purely imaginary, then the centre is called degenerate. This is our situation because $n > 1$. The function that associates to any closed curved of the period annulus its period is called the period function of the centre. It is well known that the period function tends to infinity when the orbits in a period annulus approach either to a degenerate centre or to a polycycle with some finite critical point, see for instance [44].

In general, given a system with a centre, we will write $T(x,y)$ to denote the period of the orbit passing through the point (x,y). When the system is Hamiltonian, it is sometimes more convenient to parameterize the periodic orbits by their energy h and write $T(h)$ to denote their corresponding periods. The critical periods are the zeroes of the derivative of the period function once the continuum of periodic orbits is parameterized by a smooth one-parameter function. This parameter can be the energy in the Hamiltonian situation, or anyone describing a transversal to the orbits. It is not difficult to prove that the number of critical periods does not depend neither on the transversal, nor on its parametrization. When a zero of the derivative of the period function is simple we will say that the system has a simple critical period.

Some motivations to know properties of the period function come from its role in the study of several differential equations. For instance, it appears in mathematical models in physics or ecology, see [176], [106], [181], [134]. It is important in the study of the bifurcations from a continuum of periodic orbits giving rise to isolated ones, see [173], in the description of the dynamics of some discrete dynamical systems, see [170], [174], [175] or for counting the solutions of some boundary value problems, see [112], [172].

The period function for homogeneous vector fields (both Hamiltonian and non-Hamiltonian) was characterized in [46], while the quasi-homogeneous Hamiltonian were studied in [182]. Our main result for the quasi-homogeneous case, i.e. $X_m(x,y) \equiv 0$, completely characterizes the period function in the general case, extending their results.

Recall that a critical point is called monodromic if there are no orbits tending or leaving the point with a given direction. For analytic vector fields, monodromic points are either centre or focus, and the problem of distinguishing between both options is called the centre-focus problem. The solution of the centre-focus problem for quasi-homogeneous vector fields is relatively easy. As we will see in the proof of the previous theorem, in order to have a centre at the origin we only need to guarantee that the origin is monodromic and moreover that some definite integral, that can be obtained from the expression in quasi-homogeneous polar coordinates, is zero, see (38).

In the particular case that the system considered in Theorem (5.3.8) is also Hamiltonian the above result can be rewritten as follows, recovering the result in [182].

As we will see, the constants T_j , $j = 1, 2, 3$ are closely related and all them can be expressed in terms of two iterated integrals, see (39). Moreover in some cases they can be explicitly computed. For instance, consider the (1, 2) quasi-homogeneous Hamiltonian system, of quasi-degree 2,

$$\begin{cases} \dot{x} = -y + bx^2, \\ \dot{y} = x^3 - 2bxy, \end{cases}$$

with Hamiltonian $H(x,y) = y^2/2 - bx^2y + x^4/4$. When $b^2 < 1/2$, it has a centre at the origin and its period function, for $\xi > 0$, is (see Example *(5.3.11)*)

$$T(\xi,0) = \frac{T_1}{\xi} = \frac{2\pi^{3/2}}{\sqrt[4]{1 - 2b^2}\ \Gamma^2\,(3/4)}\frac{1}{\xi},$$

where Γ is the Gamma function. Equivalently, for $\eta > 0$,

$$T(0,\eta) = \frac{T_1}{\sqrt{\eta}} \ \textit{and}\ T(h) = \frac{T_1}{\sqrt{2}}\frac{1}{\sqrt[4]{h}}.$$

For general systems (25) the centre-focus problem is still an open question. Moreover, even for quadratic systems with a reversible centre, the global behaviour of the period function is not fully understood, see for instance [180]. Therefore to ensure that the origin is a centre and to start with the most tractable case, we will restrict our attention to the Hamiltonian subcase. Notice that for system (25) the condition of being a Hamiltonian vector field implies the existence of two (p, q) quasi-homogeneous functions $H_n(x, y)$ and $H_m(x, y)$, with respective quasi-degrees $n + p + q - 1$ and $m + p + q - 1$, such that

$$H_k(\lambda^p x, \lambda^q y) = \lambda^{k+p+q-1} H_k(x, y),$$

$$\frac{\partial H_k(x, y)}{\partial x} = Q_k(x, y), \frac{\partial H_k(x, y)}{\partial y} = -P_k(x, y), \qquad k = n, m,$$

and the Hamiltonian is $H(x, y) = H_n(x, y) + H_m(x, y)$.

We obtain the following results, where recall that a centre is called global if its associated basin of attraction is the whole plane

We remark that previous result strongly relies on two facts. The first one is that the associated vector field is given by the sum of two (p, q) quasi-homogeneous ones, while the second fact is that $n > 1$. As an example of the necessity of both hypotheses, consider for instance the globally linearizable isochronous system associated to the Hamiltonian $H(x, y) = x^2 + (y + x^2)^2$, for which all orbits have period π. The corresponding Hamiltonian vector field can be considered as the sum of three homogeneous ones, which violates the first required assumption. On the other hand, the same vector field can be also considered as the sum of two (1, 2) quasi-homogeneous ones of quasi-degrees 0 and 2, respectively. In this case the second assumption fails.

For the particular case when X_n and X_m are both homogeneous vector fields $((p, q) = (1, 1))$, we obtain the following result*:*

The above corollary extends the results obtained in [46], [177], [39] for the case of Hamiltonian vector fields of the form $X_n + X_m$, with $n = 1 < m$, where it is also proved that the period function on the period annulus of the origin has at most one critical period. When $n \geq 2$ (indeed n has to be odd to have a centre at the origin) the same result holds when $m \geq 2n - 1$. Our attempts to cover the remaining cases have not succeeded. For instance, by applying our result we know the behaviour of the period function on the period annulus of the origin for all Hamiltonian systems of the form $X_3 + X_m, m \geq 5$, and the only open case is $m = 4$.

We give some preliminary results and we introduce the generalized polar coordinates*;* We deal with quasi-homogeneous vector fields, not necessarily Hamiltonian, and is devoted to prove Theorem (5.3.8). Finally, the proofs of Theorems (5.3.12) and (5.3.13) and Corollary (5.3.14) about Hamiltonian vector fields of the form $X_n + X_m$ are given.

We start recalling the generalized polar coordinates and the generalized trigonometric functions. They were introduced by Lyapunov in his study of the stability of degenerate critical points, see [34]. These new functions are defined as the unique solution of the initial value problem

$$\begin{cases} \dot{x} = -y^{2p-1}, \\ \dot{y} = x^{2q-1}, \end{cases} \tag{26}$$

with the initial conditions $x(0) = \sqrt[2]{1/p}, y(0) - 0$. We will denote them by $x(\theta) = Cs(\theta), y(\theta) = Sn(\theta)$. When $p = q = 1$ we recover the usual trigonometric functions. The generalized trigonometric functions satisfy the equality $p\, Cs^{2q}(\theta) + q\, Sn^{2p}(\theta) = 1$ and they are periodic, with period

$$\Omega = \Omega_{p,q} = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2p}} \frac{\Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2q}\right)}{\Gamma\left(\frac{1}{2p} + \frac{1}{2q}\right)}, \tag{27}$$

where Γ is the Gamma function. Associated to them we can introduce the quasi-homogeneous polar coordinates by the change

$$x = r^p \, Cs(\theta), \qquad y = r^q \, Sn(\theta). \tag{28}$$

With these coordinates, it holds that $px^{2q} + qy^{2p} = r^{2pq}$.

In general, a system

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \tag{29}$$

by doing the change to generalized polar coordinates, is transformed into

$$\begin{cases} \dot{r} = r^{1-2pq} \left[x^{2q-1}P(x,y) + y^{2p-1}Q(x,y)\right], \\ \quad \dot{\theta} = r^{-p-q}[pxQ(x\, y) - qyP(x,y)], \end{cases} \tag{30}$$

where x and y have to be substituted using (28). In the particular case that the vector field $X = (P,Q)$ is (p,q) quasi-homogeneous of quasi-degree n, we obtain

$$\begin{cases} \dot{r} = r^n \left[Cs^{2q-1}(\theta)P\big(Cs(\theta),Sn(\theta)\big) + Sn^{2p-1}(\theta)Q\big(Cs(\theta),Sn(\theta)\big)\right], \\ \dot{\theta} = r^{n-1} \left[p\, Cs(\theta)Q\big(Cs(\theta),Sn(\theta)\big) - q\, Sn(\theta)P\big(Cs(\theta),Sn(\theta)\big)\right]. \end{cases} \tag{31}$$

Moreover, system (26), which has quasi-degree $n = 2pq - p - q + 1$, is transformed into

$$\begin{cases} \qquad \dot{r} = 0, \\ \dot{\theta} = r^{n-1} = r^{2pq-p-q}. \end{cases}$$

Notice that each polynomial vector field can be decomposed in different ways according to some chosen (p,q) weights. For instance, the vector field (y,x^2) decomposes as $(y,0) + (0,x^2)$ in its homogeneous components or as itself when one takes (2, 3) quasi-homogeneous ones. Therefore, given X, and a couple of weights (p,q), we have a unique decomposition

$$X(x,y) = \sum_{j=n}^{m} X_j\,(x,y),$$

where $n = n(p,q) \leq m = m(p,q)$ and each $X_j\,(x,y) = X_j\,(x,y,p,q)$ is (p,q) quasi-homogeneous of quasidegree j. Observe that in general many X_j are identically zero.

Associated to a given (p,q) decomposition and motivated by the expressions of $\dot{\theta}$ in (30) and (31) we define the (p,q) characteristic quasi-directions at the origin of the vector field $X = (P,Q)$, as the curves $\lambda \to (\lambda^p \overline{x},\ \lambda^q \bar{y})$, where $(\overline{x},\overline{y}) \neq (0,0)$ is a real zero of the quasi-homogeneous polynomial

$$F^0_{p,q}(x,y) := pxQ_{n(p,q)}(x,y) - qyP_{n(p,q)}(x,y). \tag{32}$$

Smilarly, we define the (p,q) characteristic quasi-directions at infinity of X as the curves $\lambda \to (\lambda^p \bar{x},\ \lambda^q \bar{y})$, where $(\bar{x},\bar{y}) \neq (0,0)$ is a real zero of the quasi-homogeneous polynomial

$$F^\infty_{p,q}(x,y) := pxQ_{m(p,q)}(x,y) - qyP_{m(p,q)}(x,y). \tag{33}$$

Notice that, as a result of the quasi-homogeneity of $F^0_{p,q}$ and $F^\infty_{p,q}$, the control of the zeroes and signs of these functions is a one-variable problem. For instance, (u,v) gives a characteristic quasi-direction at the origin if either $F^0_{p,q}(0,1) = 0$ or $F^0_{p,q}(1,w) = 0$, where $w = v\,u^{-q/p}$. Using this fact it makes sense to talk about the multiplicity of the characteristic quasi-directions as the multiplicity of the one-variable associated functions.

Observe that if X is a (p,q) quasi-homogeneous vector field, then any (p,q) characteristic quasi-direction is invariant by X.

Based on the ideas of [168], [169] it is not difficult to prove the following result, wherein the definition for infinity to be monodromic is essentially the same as for the origin. This result states some folklore results that appear in many places only when $(p, q) = (1, 1)$.

Proposition (5.3.1)[167]: Consider a polynomial vector filed X. The following holds*:*

(i) If the origin is a critical point and some orbit tends to it asymptotically to some curve $\lambda \to (\lambda^p \bar{x},\ \lambda^q \bar{y})$, then this direction has to be a (p, q) characteristic quasi-direction, that is a zero of $F^0_{p,q}$.

(ii) (ii) If the origin is a monodromic critical point then given any pair of weights p, q, either the point has not (p, q) characteristic quasi-directions at the origin or all its (p, q) characteristic quasi-directions at the origin have even multiplicity. Moreover when X is (p, q) quasi-homogeneous a necessary and sufficient condition to be monodromic is that the point has not (p, q) characteristic quasi-directions.

(iii) (iii) If the infinity is monodromic, given any pair of weights p, q, then either it has not (p, q) characteristic quasi-directions at infinity or all its (p, q) characteristic quasi-directions at infinity have even multiplicity.

Following with the vector field $X = (y, x^2)$ considered at the beginning, we get
$$F^0_{1,1}(x, y) = y^2, F^0_{2,3}(x, y) = 2x^3 - 3y^2.$$
On one hand, since $F^0_{1,1}$ has a double characteristic direction, the above proposition taking $(p, q) = (1, 1)$ does not allow to conclude whether there is an orbit tending to the origin in positive or negative time. On the other hand taking $(p, q) = (2, 3)$ we can conclude that the origin is a non-monodromic point because $F^0_{2,3}(1, w)$ has a simple root.

Next result gives a well known extension of Euler Theorem to smooth (p, q) quasi-homogeneous functions. We will need this extension to prove Lemma (5.3.3) on non-vanishing of quasi-characteristic polynomials of Hamiltonian systems.

Lemma (5.3.2)[167]: Let $F : \mathbb{R}^2 \to \mathbb{R}$ be $a\ C^1$, (p, q) quasi-homogeneous function of quasi-degree k, i.e. such that for all $\lambda \in \mathbb{R}$,
$$F(\lambda^p x, \lambda^q y) = \lambda^k F(x, y). \tag{34}$$
Then
$$px \frac{\partial F(x, y)}{\partial x} + qy \frac{\partial F(x, y)}{\partial y} = k\, F(x, y).$$
Proof. Derivating with respect to λ the equality (34) we get that
$$\frac{\partial F(\lambda^p x, \lambda^q y)}{\partial x}\, p\lambda^{p-1} x + \frac{\partial F(\lambda^p x, \lambda^q y)}{\partial y}\, q\lambda^{q-1} y = k\lambda^{k-1} F(x, y).$$
The result follows substituting $\lambda = 1$ in the above expression.

Lemma (5.3.3)[167]: The quasi-characteristic polynomial at the origin or at infinity of a Hamiltonian system can not be identically null.

Proof. Consider a Hamiltonian function written in its (p, q) quasi-homogeneous components $H(x, y) = H_n(x, y) + \ldots + H_m(x, y)$, with $H_n, H_m \equiv 0$, and its associated system
$$\begin{cases} x^{\cdot} = -\dfrac{\partial H(x, y)}{\partial y} = P(x, y) = P_n(x, y) + P_{n+1}(x, y) + \ldots + P_m(x, y), \\ y^{\cdot} = \dfrac{\partial H(x, y)}{\partial x} = Q(x, y) = Q_n(x, y) + Q_{n+1}(x, y) + \ldots + Q_m(x, y). \end{cases}$$
Its quasi-characteristic polynomial at the origin is

$$F^0_{p,q}(x,y) = -qyP_n(x,y) + pxQ_n(x,y) = qy\frac{\partial H_n(x,y)}{\partial y} + px\frac{\partial H_n(x,y)}{\partial x} =$$
$$= (n + p + q - 1)H_n(x,y),$$
where in the last equality we have used Lemma (5.3.2). Then, since $H_n(x,y) \not\equiv 0, F^0_{p,q}$ can not be null.
The case of infinity is completely analogous but substituting H_n by H_m.
Next proposition can be proved as in [178]. The results at infinity can be inferred from the ones at the origin by doing the change $r = 1/\rho$. We use the following notation:
$$f(x) \sim g(x)\ at\ x = x_0 \in \mathbb{R} \cup \{\infty\},$$
if $\lim_{x\to x_0} f(x)/g(x) = k \neq 0.$

Proposition (5.3.4)[167]: Given $1 \leq n < m$, consider a vector field $X = X_n + X_{n+1} + \cdots + X_m$, with $X_n \equiv 0, X_m \equiv 0$ and each X_i being a (p, q) quasi-homogeneous polynomial of quasi-degree i.
(i) If the origin is a centre and has not (p, q) characteristic quasi-directions then for $\xi > 0, T(\xi,0) \sim \xi^{\frac{1-n}{p}}$ at $\xi = 0$.
(ii) If the infinity has a neighbourhood full of periodic orbits and has not (p,q) characteristic quasi-directions then $T(\xi,0) \sim \xi^{\frac{1-m}{p}}\ at\ \xi = \infty$.
The following proposition extends the results of item (ii) of [45], that deals with polynomial Hamiltonian systems with Hamiltonian $H(x,y) = (x^2 + y^2)/2 + H_m(x,y)$, with H_m homogeneous, to Hamiltonian system of the form (25) with $p = q = 1$. Its proof is similar to the one and we omit it. It will be one of the key points for proving Corollary (5.3.14).

Proposition (5.3.5)[167]: Consider a Hamiltonian system of the form (25) with $p = q = 1$ and a centre at the origin. Then either it has a global centre or its period annulus is bounded.
In order to prove that the bound for the number of critical periods is one, a way is to compute the second derivative of the period function and verify that it does not change sign. Next result gives an alternative for this computation that, moreover, has the freedom of choosing a function ϕ.

Theorem (5.3.6)[167]: ([39]). Let I be a real open interval. An analytic function $f : I \to \mathbb{R}$ has at most one simple critical point if and only if there exists an analytic function $\phi : I \to \mathbb{R}$ such that for all $x \in I$
$$f''x) + \phi(x)f'(x) = 0.$$
Consider a vector field $X = X_n + \cdots + X_m, n \geq 1$, decomposed as sum of homogeneous components X_j of respective degrees j. It is well known that if the origin is monodromic, then n must be odd. This can be seen, for instance by using item (ii) of Proposition (5.3.1), because either it has not characteristic directions or all its characteristic directions have to have even multiplicity. Hence, the polynomial that gives these directions must have even degree. Thus $n + 1$ has to be even. Next result extends this property to (p,q) quasi-homogeneous vector fields.

Lemma (5.3.7)[167]: Consider the (p,q) quasi-homogeneous system of quasi-degree
$$\begin{cases} \dot{x} = P_n(x,y), \\ \dot{y} = Q_n(x,y). \end{cases} \qquad (35)$$
If it has a monodromic point at the origin, then $n = 2kpq - p - q + 1$ for some $k \in \mathbb{N}$.

Proof. In order to be monodromic at the origin the function $P_n(x,y)$ must satisfy that $P_n(0,y) = 0$ and $Q_n(x,0) \neq 0$. Otherwise it would have an invariant line through it. Thus,

$$P_n(0,y) = a_1 y^{k_1}, Q_n(x,0) = a_2 x^{k_2}, with\, a_1 a_2 \neq 0,$$

for some natural numbers $k_1, k_2 \geq 1$.

Moreover, since the vector field is (p,q) quasi-homogeneous of quasi-degree n, it holds that:

$$P_n(0,\lambda^q y) = a_1 \lambda^{k_1 q} y^{k_1} = \lambda^{n+p-1} P_n(0,y) = a_1 \lambda^{n+p-1} y^{k_1}, Q_n(\lambda^p x, 0) = a_2 \lambda^{k_2 p} x^{k_2} = \lambda^{n+q-1} Q_n(x,0) = a_2 \lambda^{n+q-1} x^{k_2}.$$

Consequently $n + p - 1 = k_1 q$ and $n + q - 1 = k_2 p$. From these equalities $(k_1 + 1)q = (k_2 + 1)p$. But p and q are coprime numbers, hence $k_1 + 1 = K_p$ and $k_2 + 1 = K_q$ for some $K \in \mathbb{N}$. By substituting one gets

$$n - 1 = k_1 q - p = (Kp - 1)q - p = Kpq - p - q.$$

It remains to be proved that K is even. By item (ii) of Proposition (5.3.1), since the origin is monodromic, it can not have (p,q) characteristic quasi-directions at the origin. Consequently, if we consider

$$F_{p,q}^0(x,y) = pxQ_n(x,y) - qyP_n(x,y),$$

it happens that $F_{p,q}^0(1,y) = pQ_n(1,y) - qyP_n(1,y)$ has no real roots. The term of higher degree of the previous expression is y^{k1+1} and hence, $k_1 + 1 = Kp$ must be even. Doing the same reasoning but now with $F_{p,q}^0(x,1)$ one gets that $k_2 + 1 = Kq$ must also be even. But p, q can not be both even at the same time, as they are coprime. Consequently, $K = 2k$.

Theorem (5.3.8)[167]: Consider a (p,q) quasi-homogeneous vector field of quasi-degree n, that is (25) with $X_m = 0$, with a degenerate centre at the origin. Then its associated period function is monotonic decreasing. Moreover it can be written as

$$T(\xi,0) = T_1\, \xi^{\frac{1-n}{p}}, or\, T(0,\eta) = T_2\, \eta^{\frac{1-n}{q}},$$

for $\xi, \eta \in \mathbb{R}^+$, and some non-zero constants T_1 and T_2.

Proof. By using the quasi-homogeneous polar coordinates we can write system (25) as

$$\begin{cases} \dot{r} = r^n A(\theta), \\ \dot{\theta} = r^{n-1} B(\theta), \end{cases} \tag{36}$$

where

$$A(\theta) = Cs^{2q-1}(\theta) P(Cs(\theta), Sn(\theta)) + Sn^{2p-1}(\theta) Q(Cs(\theta), Sn(\theta)), B(\theta) = p\, Cs(\theta) Q(Cs(\theta), Sn(\theta)) - qSn(\theta) P(Cs(\theta), Sn(\theta)),$$

see system (31). From the above expressions it is clear that the monodromy condition in this situation is: the function $B(\theta)$ does not vanish. Then, clearly the origin has not (p,q) characteristic quasi-directions. Under this monodromy assumption we can write the above system as

$$\frac{dr}{d\theta} = \frac{A(\theta)}{B(\theta)} r,$$

which can be easily integrated, giving

$$r(\theta; r_0) = r_0 \exp\left(\int_0^\theta \frac{A(\psi)}{B(\psi)}\, d\psi \right), \tag{37}$$

where $r_0 > 0$ denotes the initial condition at $\theta = 0$. Hence, the centre condition $r(\Omega_{p,q}; r_0) = r_0$ writes as

$$\int_0^{\Omega_{p,q}} \frac{A(\psi)}{B(\psi)}\, d\psi = 0. \tag{38}$$

Moreover, from the second equation of (36) and (37) it holds that

$$\tilde{T}\,(r_0) = \int_0^{\Omega_{p,q}} \frac{d\theta}{B(\theta) r^{n-1(\theta;\, r_0)}}$$

$$= \left(\int_0^{\Omega_{p,q}} \frac{1}{B(\theta)} \exp\left[(1 - n) \int_0^{\theta} \frac{A(\psi)}{B(\psi)} d\psi \right] d\theta \right) \frac{1}{r_0^{n-1}}, \tag{39}$$

where $\tilde{T}(r_0)$ denotes the period of the orbit passing through the point with generalized polar coordinates $r = r_0$ and $\theta = 0$, that is the point $(x, y) = (r_0^p \sqrt[2q]{1/p, 0}\,)$. Hence

$$T(\xi, 0) = T_1\, \xi^{\frac{1-n}{p}},$$

for some constant $T_1 \neq 0$, as we wanted to prove. If the initial condition of the periodic orbit is taken to be $(0, \eta), \eta > 0$, then similarly we get that $T(0, \eta) = T_2\, \eta^{\frac{1-n}{q}}$.

Corollary (5.3.9)[167]: Under the hypotheses of Theorem (5.3.8), if moreover the system is Hamiltonian, with $H(0, 0) = 0$ and closed ovals $H(x, y) = h \geq 0$, then the period function parameterized by the energy level h is

$$T(h) = T_3\, h^{\frac{1-n}{n+p+q^{-1}}},$$

for some non-zero constant T_3.

Proof. If the quasi-homogeneous vector field $X = (P_n, Q_n)$ is Hamiltonian, then its Hamiltonian function, satisfying $H(0, 0) = 0$, can be obtained as

$$H(x, y) = \int_0^x Q_n(u, y)\, du + R(y) = \frac{a_2\, x^{k_2} + 1}{k_2 + 1} + yS(x, y),$$

for some polynomial functions R and S, with $R(0) = 0$, where we keep the same notation as in the proof of Lemma (5.3.7). Then, using that $k_2 + 1 = 2kq$, see again Lemma (5.3.7), the energy level of the solution passing through the point $(\xi, 0)$, called h, satisfies $h = H(\xi, 0) = \frac{a_2}{2kq} \xi 2kq$. Applying now Theorem (5.3.8) we get that

$$T(h) = T_3\, h^{\frac{1-n}{2kpq}} = T_3\, h^{\frac{1-n}{n+p+q-1,}}$$

because $2kpq = n + p + q - 1$.

We end with a couple of examples.

Example (5.3.10)[167]: Consider the classical (p, q) quasi-homogeneous system:

$$\begin{cases} \dot{x} = -y^{2p-1}, \\ \dot{y} = x2^{q-1} \end{cases}$$

It has quasi-degree $n = 2pq - p - q + 1$ and it is Hamiltonian, with $H(x, y) = \frac{x^{2q}}{2q} + \frac{y^{2p}}{2p}$. Recall that in the generalized polar coordinates the previous system writes as $r^{\cdot} =$

$0, \theta^{\cdot} = r^{n-1}$. Since $H\big(r^p\, Cs(\theta), r^q\, Sn(\theta)\big) = \frac{r^{2pq}}{2pq}$ it holds that the orbit γh with energy $h > 0$ $is\, r = (2pqh)\frac{1}{2pq}$.

By the proof of Theorem (5.3.8), the period function of γh can be explicitly computed as

$$T(h) = \int_0^{\Omega_{p,q}} \frac{1}{r^{n-1}}\, d\theta = T_3\, h^{\frac{1-n}{2pq}}, with\, T_3 = (2pq)\frac{1-n}{2pq}\, \Omega p, q.$$

Example (5.3.11)[167]: Let us consider next (1, 2) quasi-homogeneous systems of quasi-degree 2:

$$\begin{cases} \dot{x} = -y + bx^2, \\ \dot{y} = x^3 + axy, \end{cases} \tag{40}$$

with $(a - 2b)^2 < 8$. As it is proved in [135] the previous system is the only cubic quasi-homogeneous (and non-homogeneous) system having a centre at the origin (after a rescaling of the variables, if necessary). Notice that the condition $(a-2b)^2 < 8$ is, precisely, the condition of non-existence of characteristic quasi-directions, because this function

$$\begin{aligned} pxtQ\,(x,y) - qyP(x,y) &= x(x^3 + axy) - 2y(-y + bx^2) \\ &= x^4 + (a - 2b)x^2y + 2y^2, \end{aligned}$$

does not vanish at $(x,y) = (0,0)$ if and only if $(a - 2b)^2 - 8 < 0$. Moreover, the origin is a centre because it is invariant by the change of variables and time $(x,y,t) \to (-x,y,-t)$, and so it is reversible.

When $b = 0$, system (40) is the one studied in [171], where an explicit expression for the period function is given. When a = −2b the previous system is Hamiltonian with $H(x,y) = y^2/2 - bx^2y + x^4/4$.

We will compute the period function in the general case, getting a closed expression when the system is Hamiltonian.

Following Theorem (5.3.8) and its proof, $T(\xi,0) = T_1\xi^{\frac{1-n}{p}} = T_1/\xi$, with

$$T_1 = T_1(a,b) = \int_0^{\Omega_{1,2}} exp\frac{\left(-\int_0^{\theta} \frac{Cs(\phi)\big(b\, Cs^4(\phi) + a\, Sn^2(\phi)\big)}{1 + (a-2b)Cs^2(\phi)Sn(\phi)}\, d\phi\right)}{1 + (a - 2b)Cs^2(\phi)\, Sn(\phi)}\, d\theta.$$

When $b = 0$ the formula given in [171] is recovered.

In the Hamiltonian case, $a = -2b$, the integral in the numerator of the expression of T_1 can be computed explicitly in the following way:

$$\frac{\int_0^{\theta} Cs(\phi)\big(b\, Cs^4(\phi) - 2b\, Sn^2(\phi)\big)}{1 - 4b\, Cs^2(\phi)\, Sn(\phi)}\, d\phi = \frac{-1}{4}\, ln\,\big(1 - 4b\, Cs^2(\theta)Sn(\theta)\big),$$

where we have used that $\dot{Cs}(\theta) = -\dot{Sn}(\theta), Sn(\theta) = Cs^3(\theta)$.

Substituting now in the expression of T_1 one gets:

$$T_1 = T_1(b) = \int_0^{\Omega_{!,2}} \frac{d\theta}{\big(1 - 4b\, Cs^2(\theta)Sn(\theta)\big)^{\frac{3}{4}}}.$$

Now, by using the change of variables $x = Sn(\theta)/Cs^2(\theta)$, we can write

$$T_1(b) = \int_{-\infty}^{\infty} \frac{2\,dx}{(1 - 4bx + 2x^2)^{\frac{3}{4}}} = \frac{2\sqrt{2}}{\sqrt{\sqrt[4]{1 - 2b^2}}} \int_0^{\infty} \frac{dx}{(1 + x^2)^{\frac{3}{4}}}$$
$$= \frac{4}{\sqrt[4]{1 - 2b^2}} F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) = \frac{2\pi^{\frac{3}{2}}}{\sqrt[4]{1 - 2b^2}\,\Gamma^2(3/4)} = \frac{\Omega_{1,2}}{\sqrt[4]{1 - 2b^2}},$$
where F is the elliptic integral of the first kind. See [179], for instance. We observe that $F(\frac{\pi}{2}, k) = K(k)$, where K is a complete elliptic integral of the first kind.

Theorem (5.3.12)[167]: Consider a Hamiltonian system of the form (25) with a global centre at the origin. Then its period function is monotone decreasing to zero.

Proof. First of all, we transform system (25) into generalized polar coordinates. Using (30) we get
$$\begin{cases} \dot{r} = R(r,\theta) = a_n(\theta)r^n + a_m(\theta)r^m, \\ \acute{\theta} = \Phi(r,\theta) = (n + p + q - 1)H_n(\theta)r^{n-1} + (m + p + q - 1)H_m(\theta)r^{m-1}, \end{cases} \quad (41)$$
where $H_k(\theta) = H_k(Cs\,\theta, Sn\,\theta), k = n, m,$ and $a_n(\theta)$ and $a_m(\theta)$ are Ω −periodic functions. Notice that we have used Euler Theorem for (p,q) quasi-homogeneous functions, see Lemma (5.3.2).

From the results of [20] we know that the periodic orbits of the system (41) that surround the origin never cut the curve $\Phi(r,\theta) = 0$. Moreover, the sign of $\Phi(r,\theta)$ in a neighbourhood of the origin is given by the sign of $H_n(\theta)$ that we will assume without loss of generality that is positive. Another important fact is that, as the period annulus is global and by item (iii) of Proposition (5.3.1), the function $F_{p,q}^{\infty}(x,y)$ does not change sign. Then, the same holds for
$$H_m(\theta) = \frac{1}{m + p + q - 1} F_{p,q}^{\infty}(Cs\,\theta, Sn\,\theta),$$
where we have used Lemma (5.3.3). In fact $H_m(\theta)$ has to have the same sign as $H_n(\theta)$. Otherwise, the direction of rotation will be opposite at the origin and at infinity, what would imply that the orbits of the global centre would cut $\Phi(r,\theta) = 0$.

Let us prove that the period function tends to zero as it approaches to infinity. If $H_m(\theta) > 0$ this is simply a consequence of item (ii) in Proposition (5.3.4). The proof in the case $H_m(\theta) \geq 0$ is more delicate. By using the second equation of (41) we get that
$$T(h) = \int_0^{\Omega} \frac{d\theta}{\Phi\,(r,\theta)} = \int_0^{\Omega} \frac{d\theta}{\Phi(\,r(\theta,h),\theta)}, \quad (42)$$
where $r = r(\theta,h)$ denotes the solution of the implicit closed curve given by
$$h = r^{n+p+q-1}H_n(\theta) + r^{m+p+q-1}H_m(\theta). \quad (43)$$
Notice also that for each fixed $\theta = \theta^* \in \Omega$,
$$H_n(\theta^*) \geq 0, H_m(\theta^*) \geq 0 \text{ and } H_n^2(\theta^*) + H_m^2(\theta^*) > 0. \quad (44)$$
The last inequality holds because otherwise the ray $\theta = \theta^*$ would be invariant and this is not possible because the origin is a centre.

Let us prove first that there exists $\tilde{h} > 0$ such that for $h \geq \tilde{h}$ and all $\theta \in [0,\Omega]$,
$$\frac{1}{\Phi\,(\,r(\theta,h),\theta)} \leq 1. \quad (45)$$
Recall that the origin is a global centre. Additionally, the fact that for each $\theta = \theta^* \in [0,\Omega]$fixed, equation (43) is polynomial in r with three monomials and (44) holds, implies that for each given θ^*

$$\lim_{h\to\infty} r(\theta^*, h) = \infty. \tag{4.6}$$

Moreover, using again (44) we have that the function $h \to r(\theta^*, h)$ is increasing and

$$\lim_{h\to\infty} \Phi\,(\,r(\theta^*, h), \theta^*) = \infty. \tag{47}$$

Therefore, given $\theta = \theta^*$ there exists $h(\theta^*)$ such that $\Phi\left(r\big(\theta^*, h(\theta^*)\big), \theta^*\right) \geq 2$. By continuity, there exists an open neighbourhood of θ^*, say U_{θ^*} , such that

$$\Phi\left(r\big(\theta, h(\theta^*)\big), \theta\right) \geq 1, for\ all\ \theta \in \mathcal{U}_{\theta^*}.$$

By using the monotonicity of $h \to r(\theta, h)$ and of $h \to \Phi(r(\theta, h), \theta)$ it holds that

$$\Phi\,(\,r(\theta, h), \theta) \geq 1, for\ all\ \theta \in \mathcal{U}_{\theta^*}\ and\ all\ h \geq h(\theta^*).$$

By compactness of $[0, \Omega]$ we can cover it by finitely many $U_{\theta j}$ $, j = 1, \ldots, k,$ in such a way that for $h \geq \tilde{h} := max\left(h(\theta_1), h(\theta_2), \ldots, h(\theta_k)\right)$ it holds that

$$\Phi\,(\,r(\theta, h), \theta) \geq 1, for\ \theta \in [0, \Omega]\ and\ h \geq \tilde{h}.$$

Then (45) follows. Moreover, by (47)

$$\lim_{h\to\infty} \frac{1}{\Phi\left(r\big(\dot{\theta}, h\big), \dot{\theta}\right)} = 0. \tag{48}$$

Since inequality (45) holds we can use the dominated convergence theorem to compute $\lim\limits_{h\to\infty} T(h)$. Therefore

$$\lim_{h\to\infty} T(h) = \lim_{h\to\infty} \int_0^{\Omega} \frac{d\theta}{\Phi\,(\,r(\theta, h), \theta)} = \int_0^{\Omega} \lim_{h\to\infty} \frac{d\theta}{\Phi\,(\,r(\theta, h), \theta)} = 0,$$

as we wanted to prove.

Recall also that from the results of [44], as the origin is a degenerate centre, its period function goes to infinity as it approaches to it.

We claim that the period function of the centre at the origin of system (25) has at most one simple critical period. If the claim holds, as the behaviour of the function is the one proved above (begins at zero being infinity and tends to zero at infinity) the period function will have no simple critical periods and it will be monotone decreasing. Hence, Theorem (5.3.12) will be proved.

To prove the claim, our approach is based on Theorem (5.3.6) and uses similar ideas that the ones of [39]. We have to compute $T''(h) + \phi(h)T'(h)$ for a suitable ϕ and prove that this expression does not change sign. By using (42) and

$$\frac{dh}{dr} = r^{p+q-1}\Phi(r, \theta) = r^{p+q-1}\frac{d\theta}{dt},$$

we get that

$$T(h) = \frac{d}{dh}\int_0^{\Omega} \frac{r^{p+q}}{p+q}\, d\theta, and\ T'(h) = \frac{d^2}{dh^2}\int_0^{\Omega} \frac{r^{p+q}}{(p+q)}\, d\theta.$$

Developing latter expression one gets:

$$T'(h) = -\int_0^{\Omega} \frac{1}{\Phi^3(r, \theta)}\ (n + p + q - 1)(n - 1)r^{n-p-q-1}H_n(\theta)$$
$$+ + (m + p + q - 1)(m - 1)r^{m-p-q-1}H_m(\theta)\ d\theta,$$

where $\Phi(r, \theta) > 0$ on all the period annulus. Recall again that in all the expressions $r = r(\theta, h)$ denotes the implicit closed curve given by $h = r^{n+p+q-1}H_n(\theta) + r^{m+p+q-1}H_m(\theta)$.

Similarly we compute the second derivative of the period function. In order to apply Theorem (5.3.6) we consider $\phi(h) = k/h$, where k is a constant value that will be fixed according each one of the two cases in which we split the proof of this theorem. So, after several computations, we get that

$$T''(h) + \phi(h)T'(h) = \int_0^{\Omega} \frac{1}{\Phi^5(r,\theta)hr^5} \left(c_1 H_n^2(\theta)H_m(\theta)r^{2n+m} + + c_2 H_n(\theta)H_m^2(\theta)r^{n+2m} + c_3 H_n^3(\theta)r^{3n} + c_4 H_m^3(\theta)r^{3m}\right)d\theta, \quad (49)$$

where $c_j = c_j(m,n,p,q,k), j = 1,2,3,4$. Their expressions are large and for the sake of shortness we omit the explicit expressions of three of them. As an example

$$c_3 = (1-n)(n+p+q-1)^2(k(n+p+q-1) - 2n - p - q + 2).$$

The proof of the theorem will be divided into two cases: the first one when $n < m < 2n - 1$ and the second case the opposite, $m \geq 2n - 1$. We begin with the first one: $n < m < 2n - 1$. In this case, in the expression (49) we choose a k such that $c_3 = 0$, that is

$$k = \frac{2n + p + q - 2}{n + p + q - 1}.$$

Hence, the parenthesis of the integrand of the previous expression (49) becomes:

$$P(h,\theta) = c_1 H_n^2(\theta)H_m(\theta)r^{2n+m} + c_2 H_n(\theta)H_m^2(\theta)r^{n+2m} + c_4 H_m^3(\theta)r^{3m},$$

with

$$c_1 = (2n - m - 1)(m - n)(m - n + p + q)(n + p + q - 1) > 0,$$

$$c_2 = (m-n)(m+p+q-1)\big((m-n)(2m-n-1) + 2(n-1)(p+q)\big) > 0,$$

$$c_4 = \frac{(m-1)(m-n)(p+q)(m+p+q-1)^2}{n+p+q-1} > 0.$$

Consequently, $T''(h) + \phi(h)T''(h) > 0$ and according to Theorem (5.3.6) the period function T has at most one critical period and, if it exists, it is simple.

Now we proceed with the second case $m \geq 2n - 1$. In this situation we choose k in such a way that $c_1 = 0$ in the expression (49). It can be seen that the parenthesis of the integrand of (49) becomes:

$$P(h,\theta) = c_2 H_n(\theta)H_m^2(\theta)r^{n+2m} + c_3 H_n^3(\theta)r^{3n} + c_4 H_m^3(\theta)r^{3m}, \quad (50)$$

where

$$c_2 = \frac{(m-n)(m+p+q-1)}{m+2n-3}(4(m-n)^3 + 2(m-n)^2(4(n-1) + p + q) + 3(m-n)(n-1)(p+q) + 3(n-1)^2(p+q) > 0, \quad (4.11)$$

$$c_3 = \frac{(n-1)(m-2n+1)(m-n)(n+p+q-1)^2(m-n+p+q)}{(m+2n-3)(m+p+q-1)} \geq 0,$$

$$c_4 = \frac{(m-1)(m-n)(m+p+q-1)^2}{(m+2n-3)(n+p+q-1)}(m - 2n + 1)(m-n) + + 2(m-1)(p+q) > 0.$$

Again $T''(h) + \phi(h)T'(h) > 0$ and according to Theorem (5.3.6) the period function has at most one critical period and, if it exists, it is simple. Then the claim is proved.

Theorem (5.3.13)[167]: Consider a Hamiltonian system of the form (25) with a centre at the origin. For

$$m \geq 2n - 1 \; and \; p + q \leq \frac{(m - n)(3m^2 + 2mn - 4n^2 - 8m + 6n + 1)}{(m - 2n + 1)(n - 1)}$$

the period function of the origin has at most one critical period and, when it exists it is simple.

Proof. The proof starts with the same computations and notations that the one of the second case of previous theorem, $m \geq 2n - 1$. Hence we have to prove that the function $P(h, \theta)$ given in (50) and with the constants $c_j > 0$ given in (51) is positive, where recall that we are assuming without loss of generality that $H_n(\theta) > 0$. The main difference is that the period annulus is not necessarily global. Hence the function $H_m(\theta)$ can change sign along it and we do not still know if the sign of $P(h, \theta)$ is constant. For the values of θ such that $H_m(\theta) \geq 0$ there is nothing to be proved because $P(h, \theta)$ is a sum of nonnegative quantities. Consider a value of θ such that $H_m(\theta) < 0$. We rewrite the function $P(h, \theta)$ in the following way:

$$\begin{aligned} P(h,\theta) = {} & c_3 H_n^3(\theta)^{r3n} + \frac{c_2}{n + p + q - 1} H_m^2(\theta) r^{2m+1} \times \\ & \times \left((n + p + q - 1) H_n(\theta) r^{n-1} \right. \\ & \left. + \frac{c_4(n + p + q - 1)}{c_2} H_m(\theta) r^{m-1} \right). \end{aligned}$$

We claim now that $\frac{c_4(n+p+q-1)}{c_2} \leq m + p + q - 1$. If that is true, it holds that

$$\frac{c_4(n + p + q - 1)}{c_2} H_m(\theta) \geq (m + p + q - 1) H_m(\theta).$$

Thus

$$\begin{aligned} & (n + p + q - 1) H_n(\theta) r^{n-1} + \frac{c_4(n + p + q - 1)}{c_2} H_m(\theta) r^{m-1} \\ & \quad \geq (n + p + q - 1) H_n(\theta) r^{n-1} + (m + p + q - 1) H_m(\theta) r^{m-1} \\ & \quad = \Phi(r, \theta) > 0. \end{aligned}$$

Then $P(h, \theta)$ will be also positive on the whole period annulus. Applying Theorem (5.3.6) to the period function with the ℓ given, we know that it will have at most one (simple) critical period. We prove now the claim. The previous inequality is equivalent to $c := (m + p + q - 1)c_2 - (n + p + q - 1)c_4 \geq 0$. This function c can be written in the following way:

$$\begin{aligned} c = {} & \frac{(m - n)(m + p + q - 1)^2}{m + 2n - 3} \big((m - n)(3m^2 + 2mn - 8m - - 4n^2 + 6n \\ & + 1) - (n - 1)(m - 2n + 1)(p + q) \big). \end{aligned}$$

It is a straightforward computation proving that $c \geq 0$ is equivalent to

$$p + q \leq \frac{(m - n)(3m^2 + 2mn - 4n^2 - 8m + 6n + 1)}{(n - 1)(m - 2n + 1)},$$

which is precisely one of the hypotheses of the theorem. Then the result follows.

Corollary (5.3.14)[167]: Consider a Hamiltonian system of the form (25) with $p = q = 1$. *If* $m \geq 2n - 1$ then the period function of the origin of system (25) has at most one critical period and, if it exists, it is simple. More specifically,

(i) if m is even, it has exactly one critical period.

(ii) if m is odd, it can have none or one critical period. Moreover both possibilities may occur.

Proof. The homogeneous case can be recovered from the quasi-homogeneous one by setting $p = q = 1$ in Theorem (5.3.13). Then it is enough with proving that

$$2 \leq \frac{(m - n)(3m^2 + 2mn - 4n^2 - 8m + 6n + 1)}{(n - 1)(m - 2n + 1)}$$

The previous inequality is equivalent to the chain of inequalities,

$$(m - n)(3m^2 + 2mn - 4n^2 - 8m + 6n + 1) - 2(n - 1)(m - 2 + 1) \geq 0, 3(m - n)^3 + 8(m - n)^2(n - 1) + (m - n)(n - 3)(n - 1) + 2(n - 1)^2 \geq 0,$$

and this last inequality is obviously true for $n \geq 3$. It remains the case $n = 2$, but it follows by a straightforward computation.

We prove the second part of the corollary. We first study the case m even. As system (25) is Hamiltonian, then the quasi-characteristic polynomial at the infinity, $F_{1,1,}^{\infty}$ can not be identically null, as it has been proved in Lemma (5.3.3). Then, as the degree of the characteristic polynomial at infinity is odd, it must have an orbit tending to infinity in positive or negative time. Consequently, the period annulus P of the origin can not be global. Then, by Proposition (5.3.5) the period annulus of the origin must be bounded. Therefore, there must exist another critical point in the exterior boundary ∂P of P. As a consequence, since the period function tends to infinity when it approaches to the origin and also to ∂P (see [44]), we know that the period function must have, at least, one critical period. But we have just proved that it has at most one critical period. Hence, if m is even the period function has exactly one simple critical period.

We study the case in which $m = 2\ell - 1$ is odd. We have to prove that there exist Hamiltonian systems with a centre at the origin having one simple critical period, and systems with a centre at the origin having zero simple critical periods. Consider the following Hamiltonian $H(x, y) = (x^2 + y^2)k + a(x^2 + y^2)^{\ell}$, with $1 < k < \ell, a \neq 0$, and the differential system associated to it:

$$\begin{cases} \dot{x} = -2y(k(x^2 + y^2)^{k-1} + a(x^2 + y^2)^{\ell-1}), \\ \dot{y} = 2x(k(x^2 + y^2)^{k-1} + a(x^2 + y2)^{\ell-1}). \end{cases} \tag{52}$$

In polar coordinates it writes as

$$\begin{cases} \dot{r} = 0, \\ \dot{\theta} = 2(kr^{2k-2} + a\, r^{2\ell-2}). \end{cases} \tag{53}$$

Observe that previous system has a continuum of critical points when $a < 0$,andthus the period annulus is bounded, while the period annulus is global in the opposite case. Therefore when $a > 0$ the period function is monotone decreasing and when $a < 0$ it has exactly one (simple) critical period. Indeed, in this particular example, where the periodic orbits are circles, the period function parameterized by the radius, $\tilde{T}(r)$, can be explicitly given, because

$$\tilde{T}(r) = \int_0^{2\pi} \frac{d\theta}{2(k^{2k-2} + a\, r^{2\ell-2})} = \frac{\pi}{kr^{2k-2} + a\, r^{2\ell-2}}.$$

Hence the decreasing behaviour of T when $a > 0$ and the existence of exactly one critical period when $a < 0$ is clear. Moreover, when $a < 0$, the critical period corresponds to $r = r_0$ with $\tilde{T}(r_0) = 0$. Then r_0 is the positive solution of $k(k - 1) + a\ell(\ell - 1)r^{2(\ell-k)} = 0$.

Chapter 6
A Classification and Rigidity of the Flag Structure

We show that the classification of homogeneous operators in $B_n(\mathbb{D})$ is completed using an explicit realization of these operators. We also show how the homogeneous operators in $B_n(\mathbb{D})$ split into similarity classes. It is also shown that the flag structure is rigid, that is, the unitary equivalence class of the operator and the flag structure determine each other. A complete set of unitary invariants, which are somewhat more tractable than those of an arbitrary operator in the Cowen–Douglas class, is obtained. In a significant generalization of the properties of the homogeneous operators, we show that quasi-homogeneous operators are irreducible and determine which of them are strongly irreducible. Applications include the equality of the topological and algebraic K-group of a quasi-homogeneous operator and an affirmative answer to a well-known question of Halmos.

Section (6.1): Homogeneous Operators in the Cowen–Douglas Class

An operator T is said to be homogeneous if its spectrum is contained in the closed unit disc and for every Möbius transformation g of the unit disc $\mathbb{D}$, the operator $g(T)$, defined via the usual holomorphic functional calculus, is unitarily equivalent to T. To every homogeneous irreducible operator T there corresponds (cf. [77]) an associated projective unitary representation U of the Möbius group G_0:

$$U_g^* \, T \, U_g = g(T), g \in G_0.$$

The projective unitary representations of G_0 lift to unitary representations of the universal cover $\tilde{G}_0$ which are quite well known. We can choose (cf. [77]) U_g such that $k \longmapsto U_k$ is a representation of the rotation group. If

$$\mathcal{H}(n) = \{x \in \mathcal{H}: U_{k\theta}\, x = e^{in\theta 0}\, x\},$$

where $k_\theta(z) = e^{i\theta} z$, then $T : \mathcal{H}(n) \to \mathcal{H}(n+1)$ is a block shift. A complete classification of these for $\dim\mathcal{H}(n) \leq 1$ was obtained in [77] using the representation theory of $\tilde{G}_0$. First examples for $\dim\mathcal{H}(n) = 2$ appeared in [83]. Recently [72], [84], an m-parameter family of examples with $\dim\mathcal{H}(n) = m$ was constructed. We will use the ideas of [72], [84] to obtain a complete classification of the homogeneous operators in the Cowen–Douglas class. Finally, we describe the similarity classes within the homogeneous Cowen–Douglas operators. As a consequence, we obtain an affirmative answer to a question of Halmos (cf. [187]) for this class of operators. We also include a somewhat new conceptual presentation of the Cowen–Douglas theory and a brief description of the method of holomorphic induction, which will be our main tool. The essentially self contained and can be read without the knowledge of [72] and [84]. The results were announced in [186] except for Theorem (6.1.10).

For M be a complex manifold and suppose $\pi : E \to M$ is a complex vector bundle. We write, as usual, $E_z = \pi^{-1}(z)$. For a trivialization, $\varphi : E \to M \times \mathbb{C}^n$, we write $\varphi(v) = (z, \varphi_z(v))$ for $v \in E_z$ with $\varphi_z : E_z \to \mathbb{C}^n$ $l$inear. (All we are going to say here would be valid using local trivializations, but in this article we will always work with global trivializations.)

We write E_z^* for the complex anti-linear dual of $E_z, z \in M$, and we write $[u, v]$ for $u(v), u \in E_z^*, v \in E_z$. We consider $\mathbb{C}^n$ to be equipped with its natural inner product and identify it with its own anti-linear dual ($so\ \xi \in \mathbb{C}^n$ is identified with the anti-linear map $\eta \langle \xi, \eta \rangle \to \mathbb{C}^n$). Then $\varphi_z^* : \mathbb{C}^n \to E_z^*$ is well defined. We set $\psi_z = \varphi_z^{*-1}$ and $\psi(u) =$

$(z, \psi_z(u))$ for $u \in E_z^*$. This makes E^* a complex vector bundle with trivialization ψ. We call φ and ψ, the associated trivializations of E and E^*. If E is a holomorphic vector bundle then E^* is an anti-holomorphic vector bundle (meaning that for any two trivializations, ψ_α and ψ_β , the transition functions $z \longmapsto (\psi_\alpha)_z \circ (\psi_\beta)_z^{-1}$ are anti-holomorphic) and vice-versa.

If E has a Hermitian structure, we automatically equip E^* with the dual structure (giving the dual norm of E_z to E_z^* for all $z \in M$).

By an automorphism of $\pi : E \to M$, we mean a diffeomorphism $\hat{g} : E \to E$ such that $\pi \circ \hat{g} = g \circ \pi$ for some automorphism g of M. We write g_z for the restriction of $\hat{g}$ to E_z. The automorphism $\hat{g}$ also acts on f of E, by $(\hat{g}^* f)(z) = g_z^{-1} f(gz)$. When G is the group of automorphisms of E (acting on the left, as usual) we have a representation U of G given by $U_{\hat{g}} f = (\hat{g}^{-1})^* f$, that is,

$$(U_{\hat{g}} f)(z) = g_z f(g^{-1} z) .$$

Given an automorphism g of E, there is a corresponding automorphism of E^*, where the place of g_z is taken by g_z^{*-1} . This also remains true in the category of Hermitian bundles. It follows that a group G of automorphisms of E also acts as a group of automorphisms of E^*. If E is homogeneous, that is, the action of G is transitive on M, then so is E^*, and vice-versa.

We describe, essentially following [184], how the usual formalism of reproducing kernels can be adapted to vector bundles. Suppose $\mathcal{H}$ is a Hilbert space whose elements of a vector bundle $E \to M$ and suppose the maps $ev_z : \mathcal{H} \to E_z$ are continuous for all $z \in M$. Then setting $K_z = ev_z^*$, we have

$$[u, f(z)] = [u, ev_z(f)] = \langle K_z u, f \rangle_{\mathcal{H}}, u \in E_z^* , f \in \mathcal{H}. \qquad (1)$$

For all $w \in M$, the $K_w u$ is in $\mathcal{H}$ and is linear in u. So, we can write $K_w(z)u = ev_z(K_w u) = ev_z ev_w^*(u)$. We also write $K(z,w) = K_w(z) = ev_z ev_w^*$ which is a linear map $E_w^* \to E_z$, and is called the reproducing kernel of $\mathcal{H}$, (1) is the reproducing property. Clearly, $K(w,z) = K(z,w)^*$. We have the positivity $\sum_{j,k} [u_k, K(z_k, z_j) u_j] \geq 0$ for any $z_1, \ldots, z_p$ in M and $u_1, \ldots, u_p \in E_z^*$ which is nothing but the inequality

$$\sum_{j,k} \left((ev_{zk})^* u_k, (ev_{zj})^* u_j\right)_{\mathcal{H}} \geq 0.$$

Conversely, a K with these properties is always the reproducing kernel of a Hilbert space of E (cf. [184]).

Suppose we have a vector bundle E and a Hilbert space $\mathcal{H}$ of E with reproducing kernel K; suppose $\hat{g}$ is an automorphism of E. Then $\hat{g}$ acts of E by $(g^* f)(z) = g_z^{-1} f(g_z)$. By the density of linear combinations of the form Kw_u, the condition for this action to preserve $\mathcal{H}$ and act on it isometrically is

$$\langle g^*(K_w u), K_z \acute{u} \rangle_{\mathcal{H}} = \langle K_w u, (g^{-1})^* (K_z \acute{u}) \rangle_{\mathcal{H}}$$

for all $z, w;\ u, u$. Evaluating both sides using (1), this amounts to

$$K(gz, gw) = g_z K(z,w)_{g_w}^* , \text{for all } z, w \in M.$$

The following remarks will be important for us. Suppose each ev_z is non-singular, that is, its range is the whole of E_z. (This is so in the important case where $\mathcal{H}$ is dense in the space of E in the topology of uniform convergence on compact sets.) Then $K_z = ev_z^*$ is an embedding of E_z^* into $\mathcal{H}$. Postulating that this embedding is an isometry we obtain a canonical Hermitian structure on E^*. Using (1) we can write for the norm on E^*,

$$\|u\|_z^2 = \|K_z u\|_{\mathcal{H}}^2 = [u, K(z,z)u], u \in E_z^*.$$

The vector bundle E has the dual Hermitian structure, for $v \in E_z$ we have

$$\|v\|_z^2 = [K(z,z)^{-1}v, v].$$

In fact this statement amounts to

$$|[u,v]|^2 \leq [K(z,z)^{-1}v, v][u, K(z,z)u].$$

for all u, v with equality reached for some u, v. Since $K(z, w)$ is bijective by hypothesis, any $v \in E_z$ can be written as $v = K(z,z)u$ with $\acute{u} \in E_z^*$ and the inequality to be proved is equivalent to

$$|[u, K(z,z)\acute{u}]|^2 \leq [\acute{u}, K(z,z)\acute{u}]\,[u, K(z,z)u].$$

But this is just the Cauchy–Schwarz inequality.

When E is a holomorphic vector bundle, $K(z,w)$ depends on z holomorphically and on w anti-holomorphically. Hence $K(z,w)$ is completely determined by $K(z,z)$. It follows that $K(z,w)$ is completely determined by the canonical Hermitian structure of E (or E^*). In the last paragraphs, we had a Hilbert space $\mathcal{H}$ of E and (under the assumption that each evz is surjective) we associated to it a family of embeddings of E_z^*, the fibres of E^*, into $\mathcal{H}$. This procedure can be reversed which is of importance for what follows. Suppose now that E is a vector bundle and the fibres E_z^* of E^* form a smooth family of subspaces of some Hilbert space H which together span H, that is, E^* is an anti-holomorphic sub-bundle of the trivial bundle $M \times H$. We write $\iota_z : E_z^* \to H$ for the (identity) embeddings. We define, $\tilde{f}(z) = \iota_z^* f$ for $f \in H, z \in M$. Then $\tilde{f}$ of E and $\text{evz}(\tilde{f}) = \iota_z^* f$. If we denote by $\mathcal{H}$ the Hilbert space of all $\tilde{f}, f \in H$, with norm $\|\tilde{f}\| = \|f\|$, each evz is continuous, so we have a reproducing kernel Hilbert space. The reproducing kernel is $K_z u = \iota_z u$.

We modify the definition of the class of operators introduced in [61] in an inessential way. A conceptual presentation in which the role of the dual of the bundle constructed in [61] is apparent follows. Given a domain $\Omega \subseteq \mathbb{C}$, we say the bounded operator T on the Hilbert space H is in $B_n(\Omega)$ if $\bar{z}$ is an eigenvalue of T, the range of the operator $T - \bar{z}$ is closed, and the corresponding eigenspaces F_z are of constant dimension n for $z \in \Omega$. It is proved in [61] that the spaces F_z span an anti-holomorphic Hermitian vector bundle $F \subseteq \Omega \times H$. (In [61] the eigenvalues are $z \in \Omega$ and so F is a holomorphic vector bundle; it is more convenient for us to change this.) We write, for $z \in \Omega, \iota_z : F_z \to H$ for the identity embedding. Now, $E = F^*$ is a holomorphic vector bundle, this will be the primary object for us. The bundle F is identified with E^*, in what follows we refer to it as E^*. We are now in the situation discussed.

To the elements f of H there correspond $\tilde{f}$ of E (defined by $\tilde{f}(z) = \iota_z^* f$) and form a Hilbert space $\mathcal{H}$ isomorphic with H and having a reproducing kernel $K_z u = \widetilde{\iota_z u}$.

Under this isomorphism, the operator on $\mathcal{H}$ corresponding to T is M^*, where M is the multiplication operator $(M\tilde{f})(z) = z\tilde{f}(z)$. In fact (cf. [61]) for any $u \in E_z^*$,

$$\begin{aligned}\left[u, \widehat{T^*} f(z)\right] &= \langle \iota_z u, T^* f \rangle_H = \langle T \iota_z u, f \rangle_H = \bar{z}\langle \iota_z u, f \rangle_H = \left[u, z\tilde{f}(z)\right] \\ &= \left[u, M\tilde{f}(z)\right].\end{aligned}$$

Finally, we describe how the preceding material appears when the vector bundle is trivialized. We always use associated trivializations φ, ψ of E and E^*. As explained in the beginning, this means that $\psi_z = \varphi_z^{*-1}$, that is, $[u,v] = \langle \psi_z u, \varphi_z v \rangle_{\mathbb{C}^n}$ for $u \in E_z^*$ and $v \in E_z$. We will consider here only the case where E is a holomorphic vector bundle. When g is an automorphism of E, in the trivialization $g_z : E_z \to E_{g_z}$ becomes $\varphi_{gz} \circ g_z \circ \varphi_z^{-1}$, which we write as the matrix $J_g(z)^{-1}$. When g is followed by another automorphism h, the relation $\left(h_g\right)_z = h_{g_z} \circ g_z$ becomes the multiplier identity

$$J_{hg}(z) = J_g(z)J_h(gz). \tag{2}$$

For the induced automorphism of E^*, the place of $J_g(z)$ is taken by $J_g(z)^{*-1}$.

E (resp. E^*) in the trivialization become the holomorphic (resp. antiholomorphic) functions $\hat{f}(z) = \varphi_z(f(z))$ $(resp. \psi_z(f(z)))$. The action g^*f of an automorphism g becomes $(g^*\hat{f})(z) = J_g(z)\hat{f}(g_z)$. If G is a group of automorphisms of E, the representation $U$of G described becomes the "multiplier representation

$$\left(\hat{U}\hat{f}\right)(z) = J_{g^{-1}}(z)\hat{f}(g^{-1}z). \tag{3}$$

A Hermitian structure on E becomes a family of inner products on $\mathbb{C}^n$, parametrized by $z \in M$. One can always write

$$\|\xi\|^2_{E_z} = \langle H(z)\xi, \xi\rangle_{\mathbb{C}^n}$$

with a positive definite matrix $H(z), z \in M$. The structure is invariant under a bundle automorphism $\hat{g}$ if and only if $H(g_z)J_g(z)^{-1}\xi, J_g(z) - 1\xi\ \mathbb{C}^n = H(z)\xi, \xi\ \mathbb{C}^n$, that is,

$$H(g\,z) = J_g(z)^*H(z)J_g(z).$$

The dual Hermitian structure of E^* is given by $\|\xi\|^2_{E^*} = \langle H(z)^{-1}\xi, \xi\rangle_{\mathbb{C}^n}$.

The Hilbert space $\mathcal{H}$ of E becomes a space $\widehat{\mathcal{H}}$ of holomorphic functions from M to $\mathbb{C}^n$. The reproducing kernel becomes $\widehat{K}(z,w) = \varphi_z \circ \widehat{K}(z,w) \circ \psi_w^{-1}$, a matrix valued function,holomorphic in z and anti-holomorphic in w. The reproducing property appears as

$$\langle \hat{f}(z), \xi\rangle_{\mathbb{C}^n} = \langle \hat{f}, \widehat{K}_z\xi\rangle_{\widehat{\mathcal{H}}},$$

the positivity as

$$\sum_{j,k} \langle \widehat{K}\left(z_j, z_k\right)\xi_k, \xi_j\rangle_{\mathbb{C}^n} \geq 0,$$

and the isometry of the $G-$action as

$$J_g(z)\widehat{K}(gz, gw)J_g(w)^* = \widehat{K}(z,w).$$

The canonical Hermitian structure of E is then given by $H(z) = K(z,z)^{-1}$.

We briefly recall some known facts of representation theory. Let G, H be real (or, complex) Lie groups and $H \subseteq G$ be closed. Given a representation ϱ of H on a complex finite dimensional vector space V, let $\mathcal{F}(G,V)^H$ denote the linear space of C^∞ (or holomorphic) functions $F : G \to V$ satisfying

$$F(gh) = \varrho(h)^{-1}F(g), g \in G, h \in H.$$

The induced representation (cf. [87]) $U := Ind_H^G(\varrho)$ acts on the linear space $\mathcal{F}(G,V)^H$ by left translation: $(\mathbb{U}_{g1}f)(g_2) = f(g_1^{-1}\ g_2)$.

From the linear representations (ϱ, V) of H, one obtains all the $G-$homogeneous vector bundles over $M = G/H$ as $G \times_H V$, which is $(G \times V)/\sim$, where

$$(gh, v) \sim (g, \varrho(h)v), h, g \in G, v \in V.$$

The map $(g,v) \longmapsto gH$ is clearly constant on the equivalence class $[(g,v)]$ and hence defines a map $\pi : G \times_H V \to M$. An action $\hat{g}, g \in G$, of the group G is now defined on $G \times_H V$ by setting $\hat{g}([[(g,v)]]) = [(g\,g, v)]$. This definition is independent of the choice of the representatives chosen. Thus $G \times_H V$ is a homogeneous vector bundle on M. There is a representation U of G of $G \times_H V$, where $(U(g)s)(x) = \hat{g}(s(g^{-1} \cdot x))$. The lift to G of the vector bundle $G \times_H V$ is $\tilde{s} : G \to V$ with $\hat{s}(g) := g\hat{g}^{-1}s(gH)$. These again form the space $\mathcal{F}(G,V)^H$ which shows that U is just another realization of the representation $\mathbb{U}$.

When M is a manifold with a group G acting on it transitively, we use the usual identification $M = G/H$, where H is the stabilizer in G of a chosen fixed point $o \in M$. The map $q : g \mapsto g \cdot o$ is the quotient map from G to M. Suppose that there exists a global cross-section $p : M \mapsto G$, that is, a map with $q \circ p = id_{|M}$. Then p gives a trivialization of the bundle $E = G \times_H V$. The trivializing map φ is given for $v \in E_z$ by $\varphi(v) = (z, p(z)^{-1}v)$, that is, $\varphi_z = p(z)^{-1}$. (This φ actually maps to $M \times E_0$, but E_0 with H acting on it by the bundle action can be identified with $(, V).$) The action of G on E_z becomes $J_g(z)^{-1} = \varphi_{gz} \circ g_z \circ \varphi_z^{-1}$ which is now the group product $p(gz)^{-1}gp(z)$ (preserving the fibre E_0) followed by the identification of E_0 with $V = \mathbb{C}^n$, that is,

$$J_g(z) = \varrho(p(z)^{-1}g^{-1}p\ (g(z)))\,, z \in M, g \in G. \tag{4}$$

The representation U appears now as the multiplier representation with multiplier (4).
Even though not needed, we point out that given any $J : G \times M \to GL_n(\mathbb{C})$ satisfying the cocycle condition (2), the map $(U_g f)(z) = J^{-1}\ \ g\,(z) f\,(g^{-1} \cdot z)$ defines a multiplier representation of the group G. Also, it defines a representation : $h \longmapsto J_{h^{-1}}\ (0)$ of the group H on the vector space V . The representation induced by this is equivalent to U. In fact, the multiplier corresponding to the cross section p and the representation $\varrho$is

$$\varrho(p(z)^{-1}g^{-1}p(g \cdot z)) = J_{p(g\cdot z)^{-1}gp(z)}(0) = J_{p(z)}(0)J_{p(g\cdot z)^{-1}g}\,(\,p(z)\cdot\,0)$$
$$= J_{p(z)}(0)J_{g(z)}J_{p(g\cdot z)^{-1}}\,(g \cdot z) = J_{p(z)}(0)J_g(z)J_{p(g\cdot 0)}(0)^{-1},$$

which is equivalent to the multiplier J .

We remark that the inducing construction always gives a multiplier such that $J_g(z) \in \varrho(H)$ for all g, z. Not all multipliers possess this additional property. However, given any multiplier J , we can always find another multiplier $\acute{J}$ equivalent to J such that $\acute{J}_g(z) \in \varrho(H)$, where $\varrho(h) = J_{h-1}\,(0)$. This is achieved by taking any section p and setting

$$\acute{J}_g(z) = J_{p(z)}(0)J_{g(z)}J_{p(g\cdot z)}(0)^{-1}.$$

Holomorphic induced representation is a refinement of the induced representation in the case of real groups G, H such that G/H has a $G-$invariant complex structure. The complex structure determines a subalgebra b of $g^{\mathbb{C}}$, namely the isotropy algebra in the local action of $g^{\mathbb{C}}$ on G/H. The holomorphic induced representation is the restriction of the induced representation to a subspace of $\mathcal{F}(G, V\,)^H$, defined by the differential equations $XF = -\varrho(X)F$ for all $X \in b$, where now is a representation of the pair (H, b). It is an important fact that every $G-$homogeneous holomorphic vector bundle arises by holomorphic induction from a simultaneous finite dimensional representation of H and b (cf. [87]). We will use this fact to determine all the holomorphic vector bundles which are homogeneous under the universal cover of the Möbius group.

We explicitly construct all the irreducible homogeneous holomorphic Hermitian vector bundles over the unit disc $\mathbb{D}$. Every homogeneous holomorphic Hermitian vector bundle on $\mathbb{D}$ is then obtained as a direct sum of the irreducible ones (Corollary (6.1.5)). We determine which ones of these irreducible homogeneous holomorphic Hermitian vector bundles over $\mathbb{D}$ correspond to operators in the Cowen–Douglas class $B_n(\mathbb{D})$.

Let G_0 be the Möbius group – the group of bi-holomorphic automorphisms of the unit disc $\mathbb{D}, G = SU(1,1)$ and $\mathbb{K} \subseteq G$ be the rotation group. Let $\tilde{G}$ be the universal covering group of G (and also that of the group G_0). The group G acts on the unit disc $\mathbb{D}$ according to the rule

$$g : z \mapsto (az + b)(\bar{b}z + \bar{a})^{-1}, g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G, z \in \mathbb{D}.$$

The group $\tilde{G}$ also acts on $\mathbb{D}$ (by $g \cdot z = q(g) \cdot z$, where $q : \tilde{G} \to G$ is the covering map), we denote the stabilizer of 0 in it by $\widetilde{\mathbb{K}}$. So $\mathbb{D} \cong G/\mathbb{K} \cong \tilde{G}/\widetilde{\mathbb{K}}$. The complexification $G^{\mathbb{C}}$ of the group G is the (simply connected) group $SL(2,\mathbb{C})$.

We use the notation of [72], [84], which is the notation used in [88]. The Lie algebra g of the group G is spanned by $X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $Y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The subalgebra $\mathbb{k}$ corresponding to $\mathbb{k}$ is spanned by X_0. In the complexified Lie algebra $g^{\mathbb{C}}$, we mostly use the complex basis h, x, y given by

$$h = -iX_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$x = X_1 + iY = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$y = X_1 - iY = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The subgroup of $G^{\mathbb{C}}$ corresponding to g is G. The group $G^{\mathbb{C}}$ has the closed subgroups

$$\mathbb{K}^{\mathbb{C}} = \left\{\begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} : z \in C, z \neq 0\right\}, P^+ = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\right\}, P^- = \left\{\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} : z \in \mathbb{C}\right\};$$

the corresponding Lie algebras $\mathbb{k}^{\mathbb{C}} = \{\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} : c \in \mathbb{C}\}, p^+ = \{\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} : c \in \mathbb{C}\}, p^- = \{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in \mathbb{C}\}$ are spanned by h, x and y, respectively. The product $\mathbb{K}^{\mathbb{C}}P^- = \{\begin{pmatrix} a & 0 \\ b & \frac{1}{a} \end{pmatrix} : 0 \neq a \in \mathbb{C}, b \in \mathbb{C}\}$ is a closed subgroup to be also denoted B; its Lie algebra is $b = \mathbb{C}h + \mathbb{C}y$. The product set $P^+\mathbb{K}^{\mathbb{C}}P^- = P^+B$ is dense open in $G^{\mathbb{C}}$, contains G, and the product decomposition of each of its elements is unique. ($G^{\mathbb{C}}/B$ is the Riemann sphere, $g\mathbb{K} \to gB, (g \in G)$ is the natural embedding of $\mathbb{D} \cong G/\mathbb{K}$ into it.) Linear representations (ϱ, V) of the algebra $b \subseteq g\mathbb{C} = sl(2,\mathbb{C})$, that is, pairs of linear transformations $\varrho(h), \varrho(y)$ satisfying

$$[\varrho(h), \varrho(y)] = -\varrho(y) \tag{5}$$

are automatically representations of $\mathbb{K}$ as well. Therefore they give, by holomorphic induction, all the homogeneous holomorphic vector bundles.

A homogeneous holomorphic vector bundle that admits a $\tilde{G}$ −invariant Hermitian structure will be called Hermitizable. Since the vector bundles corresponding to operators in the Cowen– Douglas class are of this type, we only consider these bundles. The $\tilde{G}$ −invariant Hermitian structures on the homogeneous holomorphic vector bundle (making it into a homogeneous holomorphic Hermitian vector bundle), if they exist, are given by $(\widetilde{K})$ −invariant inner products on the representation space V . A $(\widetilde{K})$ −invariant inner product exists if and only if $\varrho(h)$ is diagonal with real diagonal elements in an appropriate basis. So, we will assume without restricting generality, that the representation space of is $\mathbb{C}^d$ and that $\varrho(h)$ is a real diagonal matrix.

Furthermore, we will be interested only in irreducible homogeneous holomorphic Hermitian vector bundles, this corresponds to not being the orthogonal direct sum of non-trivial representations.

Let V_λ be the eigenspace of $\varrho(h)$ with eigenvalue λ. We say that a Hermitizable homogeneous holomorphic vector bundle is elementary if the eigenvalues of $\varrho(h)$ form an uninterrupted string: $-\eta, -(\eta + 1), \ldots, -(\eta + m)$. Every irreducible homogeneous holomorphic Hermitian vector bundle is elementary. In fact, let $-\eta$ be the largest eigenvalue of $\varrho(h)$ and m be the largest integer such that $-\eta, -(\eta + 1), \ldots, -(\eta + m)$ are all eigenvalues. From (5) we have $\varrho(y)V_\lambda \subseteq V_{\lambda-1}$; this and orthogonality of the eigenspaces imply that $V = \bigoplus_{j=0}^{m} V_{-(\eta+j)}$ and its orthocomplement are invariant under ϱ. So, V is the whole space $\mathbb{C}^d$, and we have proved that the bundle is elementary. We can write $V_{(\eta+j)} = \mathbb{C}^{d_j}$ and hence $(\varrho, \mathbb{C}^d)$ is described by the two matrices:

$$\varrho(h) = \begin{pmatrix} -\eta I_0 & & \\ & \ddots & \\ & & -(\eta + m)I_m \end{pmatrix},$$

where I_j is the identity matrix on $\mathbb{C}^{d_j}$ and

$$Y := \varrho(y) = \begin{pmatrix} 0 & & & \\ Y_1 & 0 & & \\ & Y_2 & 0 & \\ & & \ddots & \ddots \\ & & Y_m & 0 \end{pmatrix}$$

for some choice of matrices $Y_1, \ldots, Y_m$ that represent the lineartransformations $Y_j : \mathbb{C}^{d_{j-1}} \to \mathbb{C}^{d_j}$. Let $E^{(\eta,Y)}$ denote the holomorphic bundle induced by this representation.

It is clear that ϱ can be written as the tensor product of the one dimensional representation σ given by $\sigma(h) = -\eta, \sigma(y) = 0,$ and the representation 0 given by $\varrho^0(h) = \varrho(h) + \eta I, \varrho^0(y) = (y)$. Correspondingly, the bundle $E^{(\eta,Y)}$ for is the tensor product of a line bundle L_η and the bundle corresponding to ϱ^0, that is, $E^{(\eta,Y)} = L_\eta \otimes E^{(0,Y)}$.

For $g \in \tilde{G}, \acute{g}(z)$ (we write $\acute{g}(z) = \frac{\partial g}{\partial z}(z)$) is a real analytic function on the simply connected set $\tilde{G} \times \mathbb{D}$, holomorphic in z. Also $\acute{g}(z) \neq 0$ since g is one-one and holomorphic. Given any $\lambda \in \mathbb{R}$, taking the principal branch of the power function when g is near the identity, we can uniquely define $\acute{g}(z)^\lambda$ as a real analytic function on $\tilde{G} \times D$ which is holomorphic on $\mathbb{D}$ for all fixed $g \in \tilde{G}$.

For the line bundle L_η, the multiplier is $\acute{g}(z)^\eta$. Consequently, the multiplier corresponding to the original is

$$J_g(z) = (\acute{g}(z)^\eta J_g^0(z), \tag{6}$$

where J^0 is the multiplier obtained from ϱ^0.

The advantage of ϱ^0 is that it is also a representation of G (not only of $\tilde{G}$) and extends to a representation of $G^{\mathbb{C}}$. The (ordinary) induced representation (in the holomorphic category) $Ind_T^G(\varrho)$operates on functions $F : G^{\mathbb{C}} \to V$ such that $F(gt) = \varrho^0(t)^{-1}F(g)$ $(g \in G^{\mathbb{C}}, t \in T)$. The restrictions of these functions F to G then give exactly the functions $\Phi : G \to V$ which satisfy $\Phi(gk) = \varrho^0(k)^{-1}\Phi(g)$ $(g \in G^{\mathbb{C}}, t \in T)$ and $(X\Phi)(g) = -\varrho^0(X)\Phi(g)$ $(g \in G, X \in b)$, that is, the space of the representation holomorphically induced by ϱ^0. Taking a holomorphic local cross section p of $G^{\mathbb{C}}$ defined on $\mathbb{D}$, the functions $f(z) = F(p(z))$ give a trivialization of $E^{(0,Y)}$.

We use the local cross section $p : \mathbb{D} \to G^{\mathbb{C}}, z \to p(z) := \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$. Apply (4) to compute the corresponding multiplier $J_g^0(z)$. For g = $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, we have

$$J_g^0(z) = \varrho^0\left(\begin{pmatrix} 1 - g \cdot z \\ 0\ 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)^{-1} = \varrho^0\begin{pmatrix} cz + d & 0 \\ -c & (cz + d)^{-1} \end{pmatrix}$$

$$= \varrho^0\begin{pmatrix} (cz + d)^{\frac{1}{2}} & 0 \\ 0 & (cz + d)^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} (cz + d)^{\frac{1}{2}} & 0 \\ 0 & (cz + d)^{-\frac{1}{2}} \end{pmatrix}$$

$$= \varrho^0\left(exp\left(2\,log(cz + d)^{\frac{1}{2}}\,h\right)\right)\varrho^0\,(exp(-cy))\varrho^0\left(exp\left(2\,log(cz + d)^{\frac{1}{2}}\,h\right)\right) = D_g(z)exp(-cY)D_g(z), \tag{7}$$

where $D_g(z)$ is the block diagonal matrix with

$$D_g(z)_{jj} = (cz + d)^{-\frac{j}{2}} I_{d_j}, 0 \leq j \leq m.$$

Computing the matrix entries of the exponential using (6), we obtain for $g \in \tilde{G}, z \in \mathbb{D}$,

$$(J_g^{(\eta,Y)}(z))_{p\ell} := (\acute{g}(z))^{\eta} J_g^0(z)$$
$$= \begin{cases} \dfrac{1}{(p -)!}(-cg)^{p-\ell}\acute{g}(z)^{\frac{\eta + p +}{2}} Y_p \cdots Y_{\ell+1}\ if\ p \leq \ell, \\ 0\ if\ p < \ell.. \end{cases} \tag{8}$$

In this formula cg for $g \in \tilde{G}$ is to be understood as the (2, 1) entry of the matrix $q(g)$ in G, where q is the covering map. We note here, for later use, that there is also another way to interpret c_g for $g \in \tilde{G}$. Taking a small neighborhood $\widetilde{U}$ of the identity in $\tilde{G}$ such that the projection is a diffeomorphism onto a neighborhood U of the identity in G, by computing in U, we find that

$$g''(z) = -2c_g\acute{g}(z)^{3/2} \tag{9}$$

holds with c_g an analytic function of g on U, independent of z. This is then true for $g \in \widetilde{U}$ and by analytic continuation for all $g \in \tilde{G}\,.So\ Eq.(9)$ can serve as a definition for c_g.

Proposition (6.1.1)[183]: All elementary Hermitizable homogeneous holomorphic vector bundles are of the form $E^{(\eta,Y)}$ with $\eta \in \mathbb{R}$ and Y as before. The bundles $E^{(\eta,Y)}$ and $E^{(\eta,Y)}$ are isomorphic if and only if $\eta = \acute{\eta}$ and $\acute{Y} = AYA^{-1}$ with a block diagonal matrix A.

Proof. The induced bundles are isomorphic if and only if the inducing representations $\varrho, \acute{\varrho}$, are linearly equivalent, that is, $\acute{\varrho} = A\varrho A^{-1}$ for some A. Since we have normalized the representations by fixing the matrix $\varrho(h)$, the equivalence must be given by an A which commutes with $\varrho(h)$, that is, by a block diagonal A.

Thus $E^{(\eta,\{Y\})} = L_\eta \otimes E^{(\{Y\})}$ parametrizes the equivalence classes of elementary Hermitizable homogeneous holomorphic vector bundles. Here, we have let $\{Y\}$ denote the conjugacy class of Y under conjugation by a block diagonal matrix A.

We proceed to discuss homogeneous holomorphic Hermitian vector bundles. From here on we will always use the trivialization we just described. We will not always make a careful distinction between sections of $E^{(\eta,Y)}$ and the functions from $\mathbb{D}$ to $\mathbb{C}^d$ on which G acts by the multiplier $J_g^{(\eta,Y)}(z)$. A Hermitian structure appears in the trivialization as a family of quadratic forms $H(z)\xi, \xi$, which because of the homogeneity is determined by a single positive definite block-diagonal matrix $H = H(0)$. We denote by $(E^{(\eta,Y)}, H)$ the bundle $E^{(\eta,Y)}$ equipped with the Hermitian structure determined by H.

Proposition (6.1.2)[183]: The Hermitian vector bundles $(E^{(\eta,Y)}, H)$ and $(E^{(\acute{\eta},Y')}, H')$ are isomorphic if and only if $\eta = \acute{\eta}$, $Y' = AYA^{-1}$ and $H' = A^{*-1}HA$ with a block diagonal matrix A.

Proof. The trivialization obtained by starting with $\varrho'(resp. = A\varrho A^{-1})$ are related as $f'(z) = Af(z)$. Now, $H'(z)$ gives the same metric as $H(z)$ if and only if $\langle H'(z)f'(z), f'(z)\rangle = \langle H(z)f(z), f(z)\rangle$. From this, the statement follows.

For any H, clearly there is an A such that $A^{*-1}HA = I$. This means that every elementary homogeneous holomorphic Hermitian vector bundle is isomorphic to one of the form $(E^{(\eta,Y)}, I)$. Two vector bundles of this form are equivalent if and only if $Y' = AYA^{-1}$ with A such that $A^{*-1} IA^{-1} = I$, that is, with a block-diagonal unitary A. We denote by $[Y]$ the equivalence class of Y under conjugation by block-diagonal unitaries and write $E^{(\eta,[Y])}$ for the equivalence class of $\left(E^{(\eta,Y),I}\right)$, omitting the I. We now have the first half of the following Proposition.

Proposition (6.1.3)[183]: For $\eta \in \mathbb{R}$, $[Y]$ a block unitary conjugacy class of matrices Y, the vector bundles $E^{(\eta,[Y])}$ form a parametrization of the elementary homogeneous holomorphic Hermitian vector bundles. The Hermitian vector bundle $E^{(\eta,[Y])}$ is irreducible if and only if Y cannot be split into orthogonal direct $sum\, Y' \oplus Y'' \, with\, Y', Y''$ having the same block diagonal form as Y.

Proof. The last statement follows since the irreducibility of $E^{(\eta,[Y])}$ is the same as the possibility of splittingϱ. into an orthogonal direct sum of two sub-representations.

Proposition (6.1.3), with a different proof, also appears in [86].

The following theorem is important because its hypothesis is exactly what we know about the vector bundle corresponding to a homogeneous operator in the Cowen–Douglas class $B_n(\mathbb{D})$. It was stated in [72] but proved without the uniqueness statement. Here we give a complete proof.

Theorem (6.1.4)[183]: Let E be a Hermitian holomorphic vector bundle over $\mathbb{D}$ and suppose that for all $g \in G$, there exists an automorphism of E whose action on $\mathbb{D}$ coincides with g. Then the full automorphism group of E is reductive and $\hat{G}$ acts on E by automorphisms in a unique way.

Proof. Let $\hat{G}$ denote the connected component of the automorphism group of E. It is a Lie group because it is the connected component of the isometry group of E under the Riemannian metric defined for vectors tangent to the fibres by the Hermitian structure and for vectors horizontal with respect to the Hermitian connection by the $G-$invariant metric of $\mathbb{D}$.

Let $N \subseteq \hat{G}$be the subgroup of elements acting on $\mathbb{D}$ as the identity map. The subgroup N is normal, and the projection $\pi : \hat{G} \to G$ is a homomorphism with kernel N. Let $\mathbb{K}$ be the stabilizer of 0 in G and let $\hat{k} = \pi^{-1}(K)$. The group $\hat{k}$ contains N and is compact because it is the stabilizer of the origin in the fiber over 0.

Let $\hat{g}, g, k, n, \hat{k}$ be the Lie algebras of the groups defined above, and let $g = k + p$ be the Cartan decomposition. Since Kˆ is compact, we can choose an $Ad(\widehat{\mathbb{k}})-$invariant complement $\hat{p}$ to $\hat{k}$ in $\hat{g}$. Now, $\pi *$ maps $\hat{k}$ onto k with kernel n. By counting dimension, it follows that $\pi *$ maps $\hat{p}$to p bijectively.

We set $\hat{k}_0 = [\hat{p},\hat{p}]$. Then $\pi * (\hat{k}_0) = [\pi_*\hat{p},\pi_*\hat{p}] = k$, therefore $\hat{k}_0 \subseteq \pi_*^{-1}\ (k) = \hat{k}$. It follows that $[\hat{k}_0,\hat{p}] \subseteq \hat{p}$ and by the Jacobi identity, $\hat{g}_0 = \hat{k}_0 + \hat{p}$ is a subalgebra. Similarly, $[n,\hat{p}] \subseteq \hat{p}$ since $n \subseteq \hat{k}$. But n is an ideal, so $[n,p] = 0$, and by the Jacobi identity $[n,\hat{g}_0] = 0$. Finally, $\hat{g} = n \oplus \hat{g}_0$ and g is reductive. The analytic subgroup $\hat{G}_0 \subseteq \hat{G}$ corresponding to $\hat{g}_0$ is a covering group of G and therefore it acts on E by automorphisms. It is the unique subgroup of $\hat{G}$ with this property because $\hat{g}_0$, being the maximal semisimple ideal in the reductive algebra $\hat{g}$, is uniquely determined. The action of $\hat{G}_0$ now lifts to a unique action of $\tilde{G}$.

Theorem (6.1.4) implies that every homogeneous operator in the Cowen–Douglas class $B_n(\mathbb{D})$ has an associated representation irrespective of whether it is irreducible or not. The following corollary has also appeared in [86].

Corollary (6.1.5)[183]: If a Hermitian holomorphic vector bundle E is homogeneous and is reducible ($E = E_1 \oplus E_2$) as a Hermitian holomorphic vector bundle then it is reducible as a homogeneous Hermitian holomorphic vector bundle, that is, E_1 and E_2 are also homogeneous.

Proof. We consider the automorphisms exp th of E, where

$$h = \begin{cases} iI\ on\ E_1, \\ -iI\ on\ E_2. \end{cases}$$

Then h is in n since exp th ($t \in \mathbb{R}$) preserves fibres. So, h commutes with $\hat{g}_0$. E_1, E_2 are characterized as eigensections of h corresponding to different eigenvalues. Thus $\hat{G}_0$, and its universal covering $\tilde{G}$ preserve the eigensections of h.

We determine which ones of the elementary homogeneous holomorphic Hermitian vector bundles have their Hermitian structure coming from a reproducing kernel. In other words, which are the homogeneous holomorphic vector bundles that have a $\tilde{G}$ −invariant reproducing kernel $K(z,w)$. When there is a reproducing kernel K, it gives a canonical Hermitian structure by setting $H = K(0,0)^{-1}$. Let $p_z = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \in G$, so $p_z \cdot 0 = z$. Writing $J_z^{(\eta,Y)}$ for $J_{pz}^{(\eta,Y)}\ (z)$, we have

$$K(z,z) = J_z^{(\eta,Y)}\, K(0,0) J_z^{(\eta,Y)^*}\ . \tag{10}$$

So, the question amounts to enumeration of all the possibilities for $K(0,0)$. 3.1. An intertwining map

For $\lambda > 0$, let $\mathbb{A}^{(\lambda)}$ be the Hilbert space of holomorphic functions on the unit disc with reproducing kernel $(1 - z\bar{w})^{-2\lambda}$. It corresponds to the homogeneous line bundle L_λ. The group $\tilde{G}$ acts on it unitarily with the multiplier $g'(z)^\lambda$. This action is the Discrete series representation D_g^λ .

Let $\mathbb{C}^d = \bigoplus_{j=0}^{d_j} \mathbb{C}^{d_j}$. We think of functions $f : \mathbb{D} \to \mathbb{C}^d$ as having components $f_j : \mathbb{D} \to \mathbb{C}^{dj}$. Let $A^{(\eta)} = \otimes_{j=0}^{m} A^{(\eta+j)} \otimes \mathbb{C}^{dj}$. For $\eta > 0$, Y as before and $f_j \in \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{dj}$, define

$$\left(\Gamma^{(\eta,Y)} f_j\right)_\ell = \begin{cases} \dfrac{1}{(\ell-j)!} \dfrac{1}{(2\eta+2j)_{\ell-j}}\ Y_\ell \cdots Y_{j+1} f_j^{(\ell-j)}\ if\ \ell \geq j, \\ 0\ if\ \ell < j. \end{cases} \tag{11}$$

So, $\Gamma^{(\eta,Y)}$ maps Hol($\mathbb{D}, \mathbb{C}^d$) into itself. Let N be an invertible $d \times d$ block diagonal matrix with blocks N_j , $0 \leq j \leq m, d = d_0 + \cdots + d_m$. We will assume throughout that $N_0 = I_{d_0}$. This is only a normalizing condition. We can normalize further by assuming that each block diagonal matrix with $d_j \times d_j$ blocks N_j is positive definite but this is not important. We can think of N as changing the natural inner product of each $\mathbb{C}^{dj}$ to $\langle N_j u, N_j v\rangle_{\mathbb{C}^{dj}}$. We let $\Gamma_N^{(\eta,Y)} = \Gamma^{(\eta,Y)} \circ N$ and $\mathcal{H}_N^{(\eta,Y)}$ denote the image of $\Gamma_N^{(\eta,Y)}$ in the space of holomorphic functions $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^d)$.

Theorem (6.1.6)[183]: The map $\Gamma_N^{(\eta,Y)}$ is a $\tilde{G}$ −equivariant isomorphism of $A^{(\eta)}$ onto the Hilbert space $\mathcal{H}_N^{(\eta,Y)}$ on which the $\tilde{G}$ action is unitary via the multiplier $J_g^{(\eta,Y)}(z)$. It has a reproducing kernel $K_N^{(\eta,Y)}(z,w)$ such that

$$(K_N^{(\eta,Y)}(0,0))_{\ell\ell} = \sum_{j=0}^{\ell} \frac{1}{(\ell - j)!} \frac{1}{(2\eta + 2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} N_j N^* j\, Y_{j+1}^* \cdots Y_\ell^* .$$

Proof. The injectvity of the map $\Gamma_N^{(\eta,Y)}$ is clear from its definition. It is also apparent that the image $\mathcal{H}_N^{(\eta,Y)}$ is the algebraic direct sum of the summands $\mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{dj}$ of $A^{(\eta)}$. We define a norm on $\mathcal{H}_N^{(\eta,Y)}$ by stipulating that $\Gamma_N^{(\eta,Y)}$ is a Hilbert space isometry. This gives us the Hilbert space $\mathcal{H}_N^{(\eta,Y)}$ and the unitary action U_g of $\tilde{G}$ on it. We have to show that it is the multiplier action given by $J_g^{(\eta,Y)}(z)$. For this, we must verify that

$$\Gamma_N^{(\eta,Y)} \circ \left(\oplus\, d_j D_{g-1}^{(\eta+j)} \right) = U_{g^{-1}} \circ \Gamma_N^{(\eta,Y)} . \qquad (12)$$

Since N obviously intertwines $\oplus\, d_j D^{(\eta+j)}$ with itself, it suffices to prove (12) for $\Gamma^{(\eta,Y)}$ in place of $\Gamma_N^{(\eta,Y)} = \Gamma^{(\eta,Y)} \circ N$. Furthermore, it is enough to verify this relation for each $f \in \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{dj}$, that is, to show

$$\Gamma^{(\eta,Y)}\left(\left((g')\right)^{\eta+j} (f \circ g) \right) = J_g\left((\Gamma^{(\eta,Y)} f) \circ g \right), f \in \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{dj}, 0 \le j \le m.$$

We will show that the th components on both sides are equal. First, if $\ell < j$ then both sides are 0. Second if $\ell \geq j$, on the one hand, using Lemma 3.1 of [72] which is easily proved by induction starting from Eq. (9) and says that

$$\left((g')^\ell (f \circ g)\right)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} \left((2\ell + i)_{k-i} (-c)^{k-i} (g')^{\ell + \frac{k+i}{2}} f^{(i)} \circ g \right) \quad (13)$$

for any $g \in \tilde{G}$, we have

$$\begin{aligned}
\Gamma^{(\eta,Y)}\left((g')^{\eta+j} (f \circ g) \right) &= \left(\frac{1}{(\ell - j)!} \frac{1}{(2\eta + 2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} \right) \left((g')^{\eta+j} (f \circ g) \right)^{\ell-j} \\
&= \left(\frac{1}{(\ell - j)!} \frac{1}{(2\eta + 2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} \right) \\
&\quad \times \left(\sum_{i=0}^{\ell-j} \binom{\ell - j}{i} (2\eta + 2j + i)_{\ell-j-i} (-c)^{\ell-j-i} (g')^{\eta+j+\frac{\ell-j+i}{2}} \left(f^{(i)}\right) \circ g \right) \\
&= Y_\ell \cdots Y_{j+1} \sum_{i=0}^{\ell-j} \frac{1}{(\ell - j - i)!\, i!} \frac{1}{(2\eta + 2j) i} (-c)^{\ell-j-i} (g')^{\eta+j+\frac{\ell-j+i}{2}} \left(f^{(i)}\right) \circ g.
\end{aligned}$$

On the other hand,

$$\sum_{p=j}^{m} (J_g)_{\ell p} \left(\left(\Gamma^{(\eta,Y)} f\right)_p \circ g \right)$$

$$= \left(\sum_{p=j}^{\ell} (-c)^{\ell-p} \frac{1}{(\ell - p)!} (g')^{\eta + \frac{p+\ell}{2}} Y_\ell \cdots Y_{p+1} \frac{1}{(p - j)!} \right)$$

$$\times \left(\frac{1}{(2\eta + 2j)_{p-j}} Y_p \cdots Y_{j+1} f^{(p-j)} \circ g \right)$$

$$= \sum_{p=j}^{\ell} \frac{1}{(\ell - p)!} \frac{1}{(\ell - p)!} \frac{1}{(2\eta + 2j)_{p-j}} (-c)^{\ell-p} (g')^{\eta + \frac{p+\ell}{2}} Y_\ell \cdots Y_{j+1} f^{(p-j)} \circ g$$

$$= \sum_{i=0}^{\ell-j} \frac{1}{(\ell - i - j)!} \frac{1}{i!} \frac{1}{(2\eta + 2j)_i} (-c)^{\ell-j-i} (g')^{\eta + \frac{p+\ell}{2}} Y_\ell \cdots Y_{j+1} f^{(p-j)} \circ g.$$

This completes the verification of (12). Finally, we observe that $\mathcal{H}_N^{(\eta,Y)}$ has a reproducing kernel $K_N^{(\eta,Y)}(z,w)$ because it is the image of $A^{(\eta)}$ under an isomorphism given by a holomorphic differential operator, so point evaluations remain continuous. Then $K_N^{(\eta,Y)}(z,w)$ is obtained by applying $\Gamma_N^{(\eta,Y)}$ to the reproducing kernel of $A^{(\eta)}$ once as a function of z and once as a function of w. This computation is easily carried out and gives the formula for $K_N^{(\eta,Y)}$ (0, 0).

Writing $H := H_N^{(\eta,Y)} = K_N^{(\eta,Y)}(0,0))^{-1}$, the Hilbert space $\mathcal{H}_N^{(\eta,Y)}$ is a space of sections of the homogeneous holomorphic Hermitian vector bundle $(E^{(\eta,Y)}, H)$ in our (canonical) trivialization.

Theorem (6.1.7)[183]: The construction with $\Gamma_N^{(\eta,Y)}$ gives all elementary homogeneous holomorphic Hermitian vector bundles which possess a reproducing kernel, namely, those of the form

$$(E^{(\eta,Y)}, \quad (K_N^{(\eta,Y)}(0,0))^{-1}),$$

where $\eta > 0, Y$ are arbitrary and $K_N^{(\eta,Y)}$ (0, 0) is of the form given in Theorem (6.1.6). The vector bundles $(E^{(\eta,Y)}, (K_N^{(\eta,Y)}(0,0))^{-1})$ and $(E^{(\eta,Y)}, K_N^{(\eta,Y)}(0,0))^{-1})$ are equivalent if and only if $\eta = \acute{\eta}, Y' = AYA^{-1}$ and $N' N^* = ANN^*A^*$ for some invertible block diagonal matrix A of size $d \times d$.

Proof. The existence of a reproducing kernel implies that the vector bundle is Hermitizable. Such a bundle is of the form $(E^{(\eta,Y)}, H)$ by Propositions (6.1.1) and (6.1.2). When it has a reproducing kernel, then in our canonical trivialization this is a matrix valued function $K(z,w)$, and we have $H = K(0,0)^{-1}$. The $\tilde{G}$ action U which is now unitary, is given by the multiplier $J_g^{(\eta,Y)}(z)$.

$Eq.$ (8) shows that the action of $\widetilde{\mathbb{K}}$ is diagonalized by the polynomial vectors: If $v_j \in \mathbb{C}^{d_j}$ and $f(z) = z^n v_j$, then for k_θ such that $k_\theta(z) = e^{i\theta} z$, we have $U_{k\theta} f = e^{i\theta(\eta+j+k)} f$. It follows that U is a direct sum of the Discrete series representations $D^{(\eta+j)}, 0 \le j \le m$. In particular, it follows that $\eta > 0$.

The map $\Gamma^{(\eta,Y)}$ (and $\Gamma_N^{(\eta,Y)}$ for any block diagonal N) intertwines the representations U and $\oplus_{j=0}^{m} d_j D^{(\eta+j)}$, both of which are unitary. By Schur's Lemma it follows that N can be chosen such that $\Gamma^{(\eta,Y)} \circ N$ is unitary. This proves that the bundle $E^{(\eta,Y)}$ corresponds to the Hilbert space $\mathcal{H}_N^{(\eta,Y)}$.

The statement about equivalence follows from the analogous statement in Proposition (6.1.2).

One way to prove this is to use the “Inverse propagation theorem” of T. Kobayashi [185]. If the action of $\tilde{G}$ is unitary, then so is the $\widetilde{K}$ action on the fibres, and we are back in the situation of Theorem (6.1.7).

Here we sketch a more direct proof which also shows what the non-Hermitizable homogeneous holomorphic vector bundles are like.

A general E is still gotten from two matrices $Z = \varrho(h), Y = (y)$ such that $[Z, Y] = -Y$ by holomorphic induction. The inclusion $YV_\lambda \subseteq V_{\lambda-1}$ still holds for the generalized eigenspaces of Z. Using some easy identities for $g'(z)$, we can then verify that

$$J_g(z) = \exp\left(\frac{1}{2}\, log\big(g'(z)\big)'\ Y\ exp(-log\, g'(z)Z)\right),$$

which, in the case where Z is diagonal, is just another way to write (11), is a multiplier. Writing U_g for the action of $\tilde{G}$ on $Hol(D, V)$ given by $J_g(z)$, we compute

$$(U_{(exp\, tihf\,)}(z) = exp(itZ)f\left(e^{-it} z\right) \quad . \tag{14}$$

Hence $(U_h f)(0) = Zf(0)$ and by a similar computation $(U_y f)(0) = Yf(0)$. This shows that $J_g(z)$ gives a trivialization of our E. It also shows that U_k, k in $\widetilde{K}$ maps the spaces $\mathcal{M}_p$ of monomials of degree p to $\mathcal{M}_p$ for all $p \geq 0$. Hence the $\widetilde{K}$-finite vectors are exactly the (V − valued) polynomials.

Now if U is unitary with respect to some inner product, then it is a sum of irreducible representations. The $\widetilde{K}$ −types of these (i.e. the eigenspaces of U_h) are known to be one dimensional and together they span the space of $\widetilde{K}$ −finite vectors. By (14), U_h maps any $z^p v \in \mathcal{M}_p$ to $z^p(Zv - pv)$. It follows that Z must be diagonalizable, otherwise the eigenfunctions of U_h could not span $\mathcal{M}_p$.

The description of the homogeneous holomorphic Hermitian vector bundles given in Theorem (6.1.7) can be made more explicit. We now proceed to determine, in terms of the parametrization $E^{(\eta,[Y])}$ of elementary homogeneous holomorphic Hermitian vector bundles as in Proposition (6.1.3), exactly which ones of these have their Hermitian structure come from a reproducing kernel.

Theorem (6.1.8)[183]: The Hermitian structure of $E^{(\eta,[Y])}$ comes from a ($\tilde{G}$ −invariant)reproducing kernel if and only if $\eta > 0$ and

$$I - Y_j\left(\sum_{k=0}^{j-1} \frac{(-1)^{j+k}}{(j-k)!\,(2\eta + j + k - 1)_{j-k}}\, Y_{j-1} \cdots Y_{k+1}Y_{k+1}^* \cdots Y_{j-1}^* \text{-}\right) Y_j^* > 0$$

for $j = 1, 2, \dots, m$.

Proof. We have a description of all the vector bundles with reproducing kernel in Theorem (6.1.7). To see how this description appears in the parametrization $E^{(\eta,Y)}$, we have to answer the question: For what $\eta, [Y]$, is it possible to find a block-diagonal N such that $K_N^{(\eta,Y)}(0,0) = I$. Writing this more explicitly, we have the system of equations

$$I_\ell - \sum_{j=0}^{\ell} \frac{1}{(\ell - j)!} \frac{1}{(2\eta + j + k - 1)_{j-k}} Y_\ell \cdots \quad Y_{j+1} N_j N_j^*\, Y_{j+1}^* \cdots Y_\ell^* = 0, \tag{15}$$

$\ell = 1, \dots, m$ and the question is if the solution $N_j N_j^*$, $j = 1, \dots, m$ consists of positive definite matrices.

We claim that the solution of (15) is given by

$$N_jN_j^* = \sum_{k=0}^{j} \frac{(-1)^{j+k}}{(j - k)!\,(2\eta + j + k - 1)_{j-k}} Y_j \cdots Y_{k+1}Y_{k+1}^* \cdots Y_j^*, \tag{16}$$

for $j = 1,\ldots,m$.
In fact, substituting the expression for $N_jN_j^*$ from (16) into (15), we have

$$I_\ell - \sum_{j=0}^{\ell} \sum_{k=0}^{j} \frac{1}{(\ell - j)!\,(2\eta + 2j)_{\ell-j}} \frac{(-1)^{j+k}}{(j - k)!\,(2\eta + j + k - 1)_{j-k}} Y(k) = 0,$$

where $Y(k) = Y_\ell \cdots Y_{k+1}Y_{k+1}^* \cdots Y_\ell^*$. The coefficient of $Y(k)$, from the above, is seen to be

$$\frac{1}{(\ell - k)!^2} \sum_{j=k}^{\ell} (-1)^{j+k} \binom{\ell - k}{j - k} (2\eta + 2j - 1)B(2\eta + k + j - 1, \ell - k + 1),$$

where $B(x,y) = \frac{\Gamma(x)\Gamma y}{\Gamma(x+y)}$ is the usual Beta function. Using the binomial formula and the integral representation: $B(x,y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\, dt$, it simplifies to

$$\begin{aligned}
&\frac{1}{(\ell - k)!^2} \int_0^1 \{ (2\eta + 2k - 1)t^{2\eta+2k-2}(1 - t)^{2(\ell-k)} \\
&\quad - 2(\ell - k)t^{2\eta+2k-1}(1 - t)^{2(\ell-k)-1}\}\, dt \\
&\quad = \frac{1}{(\ell - k)!^2} \int_0^1 \{ t^{2\eta+2k-2}(1 - t)^{2(\ell-k)-1}((2\eta + 2k - 1) \\
&\quad - (2\eta + 2\ell - 1)t)\}\, dt \\
&\quad = \frac{1}{(\ell - k)!^2} \big(xB(x,y) - (x + y)B(x + 1,y)\big),
\end{aligned}$$

where $x = 2\eta + 2k - 1$ and $y = 2\ell - 2k$, which is zero except when $k = 0 = \ell$. This verifies the claim.
The right-hand side of Eq. (16) is exactly the expression given in the statement of the theorem, so its positivity is the condition we were seeking.

When Y is given, we may ask what are the values of η for which the positivity condition of the theorem holds. It obviously holds when η is large. We can also see that there exists a number $\eta_Y > 0$ such that it holds if and only if $\eta > \eta_Y$. For this we only have to see that if $E^{(\eta,Y)}$ has a reproducing kernel for some $\eta > 0$, then so does $E^{(\eta+\varepsilon,Y)}$ for all $\varepsilon > 0$. Now $E^{(\eta+\varepsilon,Y)} = L_\varepsilon \otimes E^{(\eta,Y)}$ which shows that the product $K(z,w) = (1 - z\bar{w})^{-2\varepsilon}K_I^{(\eta,Y)}(z,w)$ is a reproducing kernel for $E^{(\eta+\varepsilon,Y)}$, and $K(0,0) = I$ still holds.
When $m = 1$, the condition of the Theorem (6.1.8) reduces to

$$I - \frac{1}{\eta} Y_1Y_1^* > 0.$$

In this case, we have $\eta_Y = \frac{1}{2}\| Y_1Y_1^*\|$ in terms of the usual matrix norm.

The following theorem together with Theorems (6.1.6) and (6.1.7), and Corollary (6.1.5) gives a complete classification of homogeneous operators in the Cowen–Douglas class.

Theorem (6.1.9)[183]: All the homogeneous holomorphic Hermitian vector bundles with a reproducing kernel correspond to homogeneous operators in the Cowen–Douglas class. The irreducible ones are the adjoint of the multiplication operator M on the space $\mathcal{H}_I^{(\eta,Y)}$ for some $\eta > 0$ and irreducible Y . The block matrix Y is determined up to conjugacy by block diagonal unitaries.

Proof. First we note that by Theorems (6.1.7) and (6.1.8) every homogeneous holomorphic Hermitian vector bundle can be written in the form $(E^{(\eta,Y)}, I)$ with $\eta > 0$. The Hilbert space $\mathcal{H}_I^{(\eta,Y)}$ is a subspace of the (trivialized) holomorphic sections of $(E^{(\eta,Y)}, I)$ which is the image under the map $\Gamma_N^{(\eta,Y)}$ of $A^{(\eta)}$. We have to show only that the operator M^* on $\mathcal{H}_I^{(\eta,Y)}$ belongs to the Cowen–Douglas class. For this we compute the matrix of M in an appropriate orthonormal basis. Each of the Hilbert spaces $\mathbb{A}^{(\eta+j)}$ $(0 \le j \le m)$ has a natural orthonormal basis

$$\left\{ e_j^n(z) := \sqrt{\frac{(2\eta + 2j)_n}{n!}}\, z^n : n \ge 0 \right\}.$$

Hence $\mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$ has the basis $e_j^n\, \varepsilon_q^{(j)}$, where $\{\varepsilon_q^{(j)} : 1 \le q \le d_j\}$ is the natural basis of $\mathbb{C}^{d_j}$. The Hilbert space $A^{(\eta)}$ then has the orthonormal basis $e_j^n\, \varepsilon_{jq}$ with $\varepsilon_{jq} := \varepsilon_j \otimes \varepsilon_q^{(j)}$, where $\{\varepsilon_j : 0 \le j \le m\}$ is the natural basis for $\mathbb{C}^{m+1}$. Each $e_j^n\, \varepsilon_{jq}$ is a function on $\mathbb{D}$ taking values in $\mathbb{C}^d$; its part in $\mathbb{C}^{dj}$ is $\varepsilon_j \otimes \varepsilon_q^{(j)}$, and its part in every other $\mathbb{C}^{dk}$ $(k \ne j)$ is 0. Defining

$$e_{jq}^n := \Gamma^{(\eta,Y)}\left(e_j^n\, \varepsilon_{jq}\right), \tag{17}$$

we have an orthonormal basis for $\mathcal{H}^{(\eta,Y)}$.

We identify the "K –types" in $\mathcal{H}^{(\eta,Y)}$, that is, the subspaces on which the representation U restricted to $\widetilde{K}$ acts by scalars. For $k_\theta \in \widetilde{K}$ given by $k_\theta\ (z) = e_z^{i\theta}$, we have $D_{k_\theta}^{(\eta+j)}\, e_j^n = e^{-i\theta(\eta+j+n)e_j^n}$ on $\mathbb{A}^{(\eta+j)}$. By the intertwining property of $\Gamma^{(\eta,Y)}$, the basis elements of $\mathcal{H}^{(\eta,Y)}$ then satisfy $U_{k_\theta}\, e_{jq}^n = e^{-i\theta(\eta+j+n)} e_{jq}^n$. It follows that the subspace

$$\mathcal{H}^{(\eta,Y)}(n) := \left\{ f \in \mathcal{H}^{(\eta,Y)} : U_{k\theta}\, f = e^{-i\theta(\eta+n)} f \right\}$$

is spanned by the basis elements $\{e_{jq}^{n-j} : 1 \le q \le d_j\, , 0 \le j \le min(m,n)\}$ and $\mathcal{H}^{(\eta,Y)}$ equals the direct sum $\bigoplus_{n \ge 0} \mathcal{H}^{(\eta,Y)}(n)$.

Clearly, the operator M maps each $\mathcal{H}^{(\eta,Y)}(n)$ to $\mathcal{H}^{(\eta,Y)}(n+1)$. We write $M(n)$ for the matrix of the restriction of M to $\mathcal{H}^{(\eta,Y)}(n)$, that is,

$$M e_{jq}^{n-j} = \sum_{\ell,r} M(n)_{(\ell r)(jq)} e_{\ell r}^{n+1-\ell}. \tag{18}$$

We write $e_{(jq),(st)}^{n-j}(z)$ for the (s,t) –component $(0 \le s \le min(m,n), 1 \le\ t\, d_s)$ of the function e_{jq}^{n-j} taking values in $\mathbb{C}^d$. This can be regarded as a matrix of monomials in z. The coefficients of these monomials form a numerical matrix which we denote by $E(n)$.

Applying the operator M, which is multiplication by z, to the monomials does not change their coefficients. Therefore, Eq. (18) amounts to the matrix equality

$$E(n) = E(n+1)M(n). \tag{19}$$

We use (17) to compute $E(n)$ explicitly. The part in $\mathbb{C}^{d_j}$ of the vector valued function $e_j^{n-j}\ \varepsilon_{jq}$ is $e_j^{n-j}\ \varepsilon_q^{(j)}$, its part in $\mathbb{C}^{d_k}$ with $k \neq j$ is 0. So (11) gives, for the part of e_{jq}^{n-j} $(0 \leq j \leq m)$ in $\mathbb{C}^d$,

$$\left(e_{jq}^{n-j}(z)\right)_\ell = \begin{cases} c(\eta,\ell,j,n) z^{n-\ell}\left(Y_\ell \cdots Y_{j+1}\right)\varepsilon_q^{(j)}\, z^{n-\ell} \ if\ \ell \geq j, \\ 0\ if\ \ell < j \end{cases} \tag{20}$$

where the constant $c(\eta,\ell,j,n)$ is the coefficient of $z^{n-\ell}$ in

$$\frac{1}{(\ell-j)!}\frac{1}{(2\eta+2j)_{\ell-j}}\left(\frac{d}{dz}\right)^{\ell-j} e_j^{n-j}(z)$$
$$= \frac{1}{(\ell-j)!}\frac{1}{(2\eta+2j)_{\ell-j}}\sqrt{\frac{(2\eta+2j)_n}{n!}}\left(\frac{d}{dz}\right)^{\ell-j} z^{n-j}.$$

We can regard $E(n)$ as a block matrix with blocks $E(n)_{j\ell}$ of size $d_j \times d_\ell$. The (q,r) entry of $E(n)_{j\ell}$ being $E(n)_{(jq)(\ell r)}$ defined above. Then Eq. (20) says that

$$E(n)_{j\ell} = \begin{cases} c(\eta,\ell,j,n) Y_\ell \cdots Y_{j+1}\ if\ \ell \geq j, \\ 0\ if\ \ell < j \end{cases}$$

Now, we consider the behavior of $c(\eta,\ell,j,n)$ for large n. First, since

$$\sqrt{\frac{(2\eta+2j)_{n-j}}{(n-j)!}}\left(\frac{d}{dz}\right)^{\ell-j} z^{n-j} = \frac{\sqrt{(n-j)!\,(2\eta+2j)_{n-j}}}{(n-\ell)!} z^{n-\ell},$$

it follows that

$$c(\eta,\ell,j,n) = \frac{1}{(2\eta+2j)_{\ell-j}(\ell-j)!}\ \frac{\sqrt{\Gamma(n-j+1)\Gamma(2\eta+j+n)}}{\sqrt{\Gamma(2\eta+2j)}\Gamma(n-\ell+1)}.$$

From Stirling's formula, we obtain

$$c(\eta,\ell,j,n) \sim \frac{1}{\sqrt{\Gamma(2\eta+2j)(2\eta+2j)_{\ell-j}(\ell-j)!}}\ \frac{\sqrt{\left(e^{-n}n^{n-j+\frac{1}{2}}\right)\left(e^{-n}n^{n+2\eta+j-\frac{1}{2}}\right)}}{e^{-n}n^{n-\ell}+\frac{1}{2}}$$
$$\sim \frac{\sqrt{\Gamma(2\eta+2j)}}{\Gamma\left((\ell-j+1)\right)\Gamma(2\eta+2j+\ell)} n^{\eta-\frac{1}{2}+\ell}.$$

If we define the block matrix E by

$$E_{\ell j} = \begin{cases} \dfrac{\sqrt{\Gamma(2\eta+2j)}}{\Gamma\left((\ell-j+1)\right)\Gamma(2\eta+2j+\ell)} Y_\ell \cdots Y_{j+1}\ if\ \ell \geq j, \\ 0\ if\ \ell < j \end{cases}$$

and the diagonal block matrix $D(n)$ by $D(n)_{\ell\ell} = n^\ell I_{d_\ell}$ then we can write our result as

$$E(n) \sim n^{\eta-\frac{1}{2}} D(n)E.$$

From (19), for large n, it follows that

$$M(n) = E(n+1)^{-1}E(n)$$
$$\sim \left(\frac{n}{n+1}\right)^{\eta-\frac{1}{2}} E^{-1}D(n+1)^{-1}D(n)E$$
$$\sim I + O(1/n). \tag{21}$$

Therefore, the operator M which is a "weighted block shift" is the direct sum of an ordinary (unweighted) block shift and a Hilbert–Schmidt operator. Hence M is bounded and standard results from Fredholm theory ensure that the adjoint operator M^* is in the Cowen–Douglas class B_d ($\mathbb{D}$).

The similarity classes of the homogeneous Cowen–Douglas operators are easily described in terms of the parameter η and the multiplicities $d_0, \dots, d_m$. For a somewhat smaller class of operators, the similarity classes were described in [188].

Theorem (6.1.10)[183]: The multiplication operator M on $\mathcal{H}_I^{(\eta,Y)}$ and on $\mathcal{H}_I^{(\eta\prime,Y\prime)}$ are similar if and only if the blocks in Y and Y' are of the same size and $\eta = \eta'$.

Proof. To prove the forward direction, first we show that $\Gamma^{(\eta,Y)}$ maps $A^{(\eta)}$ onto itself, that is, $A^{(\eta)} = \mathcal{H}_I^{(\eta,Y)}$ as linear spaces. The derivative $\frac{d}{dz} : \mathbb{A}^\alpha \to \mathbb{A}^{(\alpha+1)}$ defines a surjective bounded linear operator for any $\alpha > 0$. For any $f \in A^{(\eta)}$,

$$\left(\Gamma^{(\eta,Y)} f\right)_\ell = \sum_{j=0}^{\ell} \left(\Gamma^{(\eta,Y)} f_j\right)_\ell$$

and (11) shows that each term of the sum is in $d_\ell \mathbb{A}^{(\eta+\ell)}$. On the other hand, given $g = (g_1, \dots, g_m) \in A^{(\eta)}$, we find $f \in A^{(\eta)}$ satisfying $\Gamma^{(\eta,Y)} f = g$. The functions $f_0, \dots, f_d$ are determined recursively. Suppose, we have already determined f_j , $j < \ell$. Then from the definition of the map $\Gamma^{(\eta,Y)}$, we see that taking

$$f_\ell = g_\ell - \sum_{j=0}^{\ell-1} \left(\Gamma^{(\eta,Y)} fj\right)_\ell$$

we have the required f.

Clearly, $M : A^{(\eta)} \to A^{(\eta)}$ is similar to $M : H_I^{(\eta,Y)} \to \mathcal{H}_I^{(\eta,Y)}$ via the map $f \longmapsto f$, which is bounded and invertible by the closed graph theorem.

For the proof in the other direction, let $K(n) \subseteq A^{(\eta)} = \bigoplus_{j=0}^{m} d_j \mathbb{A}^{(\eta+j)}$ be the linear span of the vectors $\{e_{jq}^n : 0 \le j \le m, 1 \le q \le d_j\}$. The multiplication operator M on $A^{(\eta)}$ maps $K(n)$ into $K(n+1)$. If M_n is the matrix representing $M_{|K(n)} : K(n) \to K(n+1)$ then M is a block shift with blocks $\{M_n : n \ge 0\}$, which are diagonal matrices of size $d \times d$. Let M' be the multiplication operator on $A^{(\eta')} = \bigoplus_{j=0}^{m} d_j \mathbb{A}^{(\eta+j)}$ with a similar block decomposition. Assume without loss of generality that $\eta' > \eta$. Suppose $L : A^{(\eta)} \to A^{(\eta')}$ is a bounded and invertible linear map consisting of $d \times d$ blocks with $LM = M'L$. Then $d = d_0 + \dots + d_m = \operatorname{codim}(\operatorname{ran} M) = \operatorname{codim}(\operatorname{ran} M') = d'_0 + \dots + d'_{m'}$.

It then follows that $L_{0k} = 0$ for all $k \ge 1$ and consequently L_{00} is non-singular. We also have $L_{nn} M_{n-1} = M'_{n-1} L_{n-1n-1}$ from which it follows that

$$L_{nn} = M'_{n-1} \cdots M'_0 L_{00} M_0^{-1} \cdots M_{n-1}^{-1} = F'_n L_{00} F_n^{-1},$$

where $F_n = M_0 \cdots M_{n-1}$ and $F' = M'_{n-1} \cdots M'_0$ are diagonal matrices. The diagonal elements are

$$(F_n)_{kk} = \sqrt{\frac{\left(2\eta + 2j(k)\right)_n}{n}} \left(respectively \ (F'_n)_{\ell\ell} = \sqrt{\frac{\left(2\eta' + 2j'(\ell)\right)_n}{n}} \right),$$

where $j(k) = j$ if $d_0 + \dots + d_j - 1 < k \le d_0 + \dots + d_j$ and $j'(\ell)$ is defined analogously. By Stirling's formula, we have

$$(L_{nn})_{\ell k} = (F'_n)_{\ell\ell} (L_{00})_{\ell k} (F_n^{-1})_{kk} \sim n^{\eta' - \eta + j(\ell) - j(k)} (L_{00})_{\ell k}.$$

Since L_{00} is non-singular, for any k with $j(k) = 0$, there is an ℓ such that $(L_{00})_{\ell k} = 0$. Now, unless $\eta = \eta'$, we have $(L_{nn})_{\ell k} \to \infty$ contradicting the boundedness of L. Therefore, we have $\eta = \eta'$ and $(L_{nn})_{\ell k} \sim n^{j} (\ell)^{-j(k)}(L_{00})_{\ell k}$. Take all those k for which $j(k) = 0$. For each of these, we can find a different ℓ_k such that $(L_{00})_{\ell k}\, k \neq 0$. (The columns of the non-singular matrix L_{00} with these indices are linearly independent and therefore cannot have only zeros in more than $d - k$ slots.) Again, unless $j'(\ell_k) = 0$, we have $(L_{nn})_{\ell k}\, k \to \infty$. This shows that $d'_0 \geq d_0$. Similarly, $d'_j \geq d_j$, $1 \leq j \leq m$. From the equality $\sum_{j=0}^{m'} d'_j = \sum_{j=0}^{m'} d_j$, it follows that $m' = m$ and $d'_j = d_j$ for $j = 1, \ldots, m$.

The following corollary, the proof of which is evident, implies that polynomially bounded homogeneous operators in the Cowen–Douglas class are similar to contractions.

Corollary (6.1.11)[183]: A homogeneous operator in the Cowen–Douglas class is either similar to a contraction or it is not power bounded.

We discuss how some formerly known examples fit into the present framework.

This case was already studied in [72]. Here each Y_j is a number, non-zero in the irreducible case. The unitaries implementing the equivalence are diagonal, and clearly the conjugacy class $[Y]$ under these has exactly one representative with $y_j > 0, 1 \leq j \leq m$. The positive $m + 1 -$ tuples satisfying the condition given in Theorem (6.1.8) give a parametrization of homogeneous Cowen–Douglas operators. For each one, $K(0,0) = I$ and $J_g^{(\eta,Y)}$ is given by the formula (8).

Another good parametrization is possible with the aid of Theorem (6.1.6). All possible $Y - s$ are now conjugate under diagonal unitaries A, so we may fix an arbitrary $Y^{(0)}$ (for example, $y_j = 1$ for all j, or, as in [72], $y_j = j$ for all j). Take any positive diagonal matrix N with diagonal elements $1 = \mu_0, \mu_1, \ldots, \mu_m$. By Proposition (6.1.2), $Y^{(0)}, N$ and $Y^{(0)}, N'$ give isomorphic vector bundles if and only if A is diagonal and hence $N = N'$. It follows that the positive numbers $\eta, \mu_1, \ldots, \mu_m$ give a parametrization of the homogeneous operators in the Cowen–Douglas class $B_{m+1}(\mathbb{D})$. Here $J^{(\eta,Y^{(0)})}\, g$ depends only on η and $K_N^{(\eta,Y^{(0)})}(0,0)$ is given by the formula in Theorem (6.1.6). This is the parametrization used in [72].

In the case $m = 1$, for any d_0 and d_1, the class $[Y]$ always contains a member for which Y is diagonal. So, the corresponding bundle is reducible unless $d_0 = d_1 = 1$. When $m = 2$, it is easy to see that $d_0 = 2$ or $d_2 = 2$ gives only reducible bundles. So, the first non-trivial case occurs (apart from the case $d_0 = d_1 = d_2 = 1$, which has been dealt with previously) when $d_0 = d_2 = 1, d_1 = 2$.

For this case, again there are two natural parametrizations. Conjugating Y with a blockdiagonal unitary having blocks u_0, U_1, u_2 changes Y_1, Y_2 into $U_1 Y_1 u_0^{-1}, u_2 Y_2 U_1^{-1}$. Now, U_1 can be chosen so that $Y_1 = \binom{a}{0}$. Then u_0, u_2 and a scalar factor in front of U_1 can be found with $a \geq 0$ and $Y_2 = (b\ c)$ with $b, c \geq 0$. We have irreducibility if and only if $a, b, c \neq 0$ and no two such triples give equivalent $Y - s$. So, we have a parametrization of the irreducible $E^{(\eta,Y)}$ by four arbitrary non-zero parameters. There is a reproducing kernel (and hence an operator in $B_4(\mathbb{D})$) if and only if the right-hand side of Eq. (16) is positive; in terms of the parameters, this is

$$a^2 < 2\eta,$$

$$b^2 < \frac{2\eta + 2}{1 - \frac{a^2}{2(2\eta+1)}},$$
$$c^2 < 2\eta + 2.$$

The positive quadruple (η, a, b, c) subject to this condition parametrizes the homogeneous operators in $B_4(\mathbb{D})$. In each case, $K(0,0) = I$ and J_g can be expressed in terms of the parameters using (8).

The other parametrization of the $(d_0, d_1, d_2) = (1, 2, 1)$ case is found using Theorem (6.1.6). Simple arguments show that Y can always be conjugated by a block diagonal A so that $Y_1 = (\,1\ 0)$ and $Y_2 = (1\ 0)$ or $(0\ 1)$. When $Y_2 = (0\ 1)$, the bundle will be reducible for any choice of Hermitian structure. So, we can fix $Y^{(0)}$ with $Y_1 = \binom{1}{0}$ and $Y_2 = (1\ 0)$. The block diagonal A that conjugates this $Y^{(0)}$ to itself is a diagonal matrix with (p, p, q, p) on the diagonal. If N is any positive diagonal $\operatorname{diag}(n_0, N_1, n_2)$ with $n_0 = 1, N_1 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \gamma \end{pmatrix}$ and $n_2 \geq 0$, then we can ensure $n_1 = 1 = n_2$ and $\alpha, \beta, \gamma > 0$ after conjugating by an A. Thus the homogeneous bundles with reproducing kernel (and hence the homogeneous operators in $B_4(\mathbb{D})$) of type (1, 2, 1) are now parametrized by four positive numbers $(\eta, \alpha, \beta, \gamma)$ subject to the condition $\beta^2 < \alpha\gamma$.

By a different construction, a large subset of these examples already occurs in [189].

Section (6.2): A Class of Cowen–Douglas Operators

In [61], Cowen and Douglas initiated the study of the following important class of operators.

Definition (6.2.1)[190]: For a connected open subset Ω of $\mathbb{C}$ and a positive integer n, let $B_n(\Omega) = \{T \in \mathcal{L}(\mathcal{H}) \mid \Omega \subset \sigma(T),$
$$ran\,(T - w) = \mathcal{H}\ for\ w \in \Omega,$$
$$\bigvee_{w \in \Omega} ker(T - w) = \mathcal{H},$$
$$dim\,ker(T - w) = n\ for\ w \in \Omega\},$$
where $\mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$ and $\sigma(T)$ is the spectrum of the operator T.

It is shown in [61] that if T is in $B_n(\Omega)$, then it is possible to choose n eigenvectors in $ker(T - w)$, which are holomorphic as functions of $w \in \Omega$. Thus $w \mapsto ker(T - w)$ defines a rank n holomorphic Hermitian vector bundle E_T over Ω. It therefore follows that the holomorphic Hermitian vector bundle E_T is the sub-bundle of the trivial holomorphic Hermitian bundle $\Omega \times \mathcal{H}$ defined by
$$E_T = \{(w, x) \in \Omega \times \mathcal{H} : x \in ker(T - w)\}$$
with the natural projection map $\pi : E_T \to \Omega, \pi(w, x) = w$ (cf. [61]). Here is one of the main results from [61].

Theorem (6.2.2)[190]: The operators T and $\tilde{T}$ in $B_n(\Omega)$ are unitarily equivalent if and only if the corresponding holomorphic Hermitian vector bundles E_T and $E_{\tilde{T}}$ are equivalent.

They also find a set of complete invariants for this equivalence consisting of the curvature of E_T and its covariant derivatives. Unfortunately, these invariants are not easy to compute except when the rank of the bundle is 1. In this case, the curvature
$$\mathcal{K}(w)dw \wedge d\bar{w} = -\frac{\partial^2 log\|\gamma(w)\|^2}{\partial w \partial \bar{w}}\, dw \wedge d\bar{w}$$

of the line bundle E_T , defined with respect to a non-zero holomorphic section γ of E_T , is a complete unitary invariant of the operator T. The definition of the curvature, in this case, is independent of the choice of the non-vanishing section γ: If γ_0 is another holomorphic (non-vanishing) section of E, then $\gamma_0 = \phi\gamma$ for some holomorphic function ϕ on an open subset Ω_0 of Ω, consequently the harmonicity of $log\ |\phi|$ completes the verification. However, if the rank of the vector bundle is strictly greater than 1, then only the eigenvalues of the curvature are independent of the choice of the holomorphic frame. This limits the use of the curvature and its covariant derivative if the rank of the bundle is not 1. It is difficult to determine, in general, when an operator $T \in B_n(\Omega)$ is irreducible, again except in the case $n = 1$. In this case, the rank of the vector bundle is 1 and therefore it is irreducible and so is the operator T.

We ask: For what class of holomorphic Hermitian vector bundles, defined on a bounded open connected set $\Omega \subseteq \mathbb{C}$, of rank n, the curvature remains a complete invariant. Refining the proof of [61], one may infer that the curvature is a complete invariant for the class consisting of the n-fold direct sum of line bundles. Examples were given in [198] to show that the class of the curvature alone does not determine the class of the vector bundle except in the case of a line bundle. The splitting of a holomorphic Hermitian vector bundle into a direct sum is determined by the vanishing of the second fundamental form (see [196]). We isolate those irreducible holomorphic Hermitian vector bundles, namely, the ones possessing a flag structure, for which the curvature together with the second fundamental form is a complete set of invariants. Among these, we study in detail the ones that correspond to irreducible operators in the Cowen–Douglas class $B_2(\Omega)$. All irreducible homogeneous operators in $B_2(\mathbb{D})$ are in this class. We obtain, using the methods developed, a description of all these operators. This classification was given earlier by D. Wilkins [83] using a sophisticated mix of Riemannian geometry and operator theory. We also investigate the case of $n > 2$, where together with the curvature and the second fundamental form, we find a set of $\frac{n(n-1)}{2} + 1$ invariants, which are easy to compute. Finally, we show that these are a complete set of unitary invariants.

We discuss this new class of operators in $B_2(\Omega)$ separately and then provide the details for the case of $n > 2$. One important reason for separating out the case of $n = 2$ is that the proofs that appear in this case are often necessary to begin an inductive proof in the case of an arbitrary $n \in \mathbb{N}$.

We construct similarity invariants for the operators in this new class (See [194]).

Definition (6.2.3)[190]: If T is an operator in $B_2(\Omega)$, then there exists a pair of operators T_0 and T_1 in $B_1(\Omega)$ and a bounded operator S such that $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$. This is Theorem 1.49 of [70]. We show, the other way round, that two operators T_0 and T_1 from $B_1(\Omega)$ combine with the aid of an arbitrary bounded linear operator S to produce an operator in $B_2(\Omega)$.

Proposition (6.2.4)[190]: Let T be a bounded linear operator of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$. Suppose that the two operators T_0, T_1 are in $B_1(\Omega)$. Then the operator T is in $B_2(\Omega)$.

Proof. Suppose T_0 and T_1 are defined on the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Elementary considerations from index theory of Fredholm operators show that the operator T is Fredholm and $ind(T) = ind(T_0) + ind(T_1)$ (cf. [191]). Therefore, to complete the proof that T is in $B_2(\Omega)$, all we have to do is prove that the vectors in the kernel $\ker(T - w), w \in \Omega$, span the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.

Let γ_0 and t_1 be non-vanishing holomorphic sections for the two line bundles E_0 and E_1 corresponding to the operators T_0 and T_1, respectively. For each $w \in \Omega$, the operator $T_0 - w$ is surjective. Therefore we can find a vector $\alpha(w)$ in $\mathcal{H}_0$ such that $(T_0 - w)\alpha(w) = -S(t_1(w)), w \in \Omega$. Setting $a(w) = \alpha(w) + t_1(w)$, we see that

$$(T - w)a(w) = 0 = (T - w)\gamma_0(w).$$

Thus $\{\gamma_0(w), a(w)\} \subseteq ker\,(T - w)$ for w in Ω. If x is any vector orthogonal to $ker(T - w), w \in \Omega$, then in particular it is orthogonal to the vectors $\gamma_0(w)$ and $a(w), w \in \Omega$, forcing it to be the zero vector.

We impose one additional condition on these operators, namely, $T_0S = ST_1$ and assume that the operator S is non-zero. With this seemingly innocuous hypothesis, we show that (i) it is irreducible, (ii) and that any intertwining unitary operator between two of these operators must be diagonal and (iii) the curvature of E_{T_0} together with the second fundamental form of the inclusion $E_{T_0} \subseteq E_T$ forms a complete set of unitary invariants for the operator T. It is therefore natural to isolate this class of operators.

Definition (6.2.5)[190]: We let $\mathcal{F}B_2(\Omega)$ denote the set of all bounded linear operators T of the form $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, where the two operators T_0, T_1 are assumed to be in the Cowen–Douglas class $B_1(\Omega)$ and the operator S is assumed to be a non-zero intertwiner between them, that is, $T_0S = T_1S$.

Specifically, if the operator $T_i, i = 0, 1$, is defined on the separable complex Hilbert space $\mathcal{H}_i$, then S is assumed to be a non-zero bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$ such that $T_0S = T_1S$. The operator T is defined on the Hilbert space $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$.

Each of the operators in $\mathcal{F}B_2(\Omega)$ is also in the Cowen–Douglas class $B_2(\Omega)$ by virtue of Proposition (6.2.4). Thus $\mathcal{F}B_2(\Omega) \subseteq B_2(\Omega)$.

Although, in the definition of the class $\mathcal{F}B_2(\Omega)$ given above, we have only assumed that S is non-zero, its range must be dense as is shown below.

Proposition (6.2.6)[190]: Suppose T_0 and T_1 are two operators in $B_1(\Omega)$, and S is a bounded operator intertwining T_0 and T_1, that is, $T_0S = ST_1$. Then S is non-zero if and only if range of S is dense if and only if S^* is injective.

Proof. Let γ be a holomorphic frame of E_{T_1} . Assume that S is a non-zero operator. The intertwining relationship $T_0S = ST_1$ implies that $S \circ \gamma$ is a section of E_{T_0} . Clearly, there exists an open set Ω_0 contained in Ω such that $S \circ \gamma$ is not zero on Ω_0, otherwise S has to be zero. Since $S(\gamma)$ is a holomorphic frame of E_{T_0} on Ω_0, it follows that the closure of the linear span of the vectors $\{S(\gamma(w)) : w \in \Omega_0\}$ must equal $\mathcal{H}_0$. Hence the range of the operator S is dense.

The following Proposition provides several equivalent characterizations of operators in the class $\mathcal{F}B_2(\Omega)$.

Proposition (6.2.7)[190]: Suppose T is a bounded linear operator on a Hilbert space $\mathcal{H}$, which is in $B_2(\Omega)$. Then the following conditions are equivalent.

(i) There exist an orthogonal decomposition $\mathcal{H}_0 \oplus \mathcal{H}_1$ of $\mathcal{H}$ and operators $T_0 : \mathcal{H}_0 \to \mathcal{H}_0, T_1 : \mathcal{H}_1 \to \mathcal{H}_1$, and $S : \mathcal{H}_1 \to \mathcal{H}_0$ such that $= \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, where $T_0, T_1 \in B_1(\Omega)$ and $T_0S = ST_1$, that is, $T \in \mathcal{F}B_2(\Omega)$.

(ii) There exists a holomorphic frame $\{\gamma_0, \gamma_1\}$ of E_T such that $\frac{\partial}{\partial w} \|\gamma_0(w)\|^2 = \langle\gamma_1(w), \gamma_0(w)\rangle$.

(iii) There exists a holomorphic frame $\{\gamma_0, \gamma_1\}$ of E_T such that $\gamma_0(w)$ and $\frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$ are orthogonal for all w in Ω.

Proof. (i) $\Longrightarrow$ (ii): Pick any two non-vanishing holomorphic sections t_0 and t_1 for the line bundles E_{T_0} and E_{T_1} respectively. Then

$$\begin{aligned}(T - w)t_1(w) &= (T_1 - w)t_1(w) + S(t_1(w)) \\ &= S(t_1(w)).\end{aligned}$$

Since $T_0 S = S T_1$, it induces a bundle map from E_{T_1} to E_{T_0} , so $S(t_1(w)) = \psi(w)t_0(w)$ for some holomorphic function ψ defined on Ω. Thus $(T - w)t_1(w) = \psi(w)t_0(w)$. Setting $\gamma_0(w) := \psi(w)t_0(w)$ and $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$, we see that $\{\gamma_0(w), \gamma_1(w)\} \subset ker\,(T - w)$. Now assume that

$$\alpha_0\gamma_0(w) + \alpha_1\gamma_1(w) = 0 \qquad (22)$$

for a pair of complex numbers α_0 and α_1. Then

$$0 = \langle \alpha_0\gamma_0(w) + \alpha_1\gamma_1(w), t_1(w)\rangle = \alpha_1\langle \gamma_1(w), t_1(w)\rangle = -\alpha_1\|t_1(w)\|^2. \quad (23)$$

From equations (22) and (23), it follows that $\alpha_0 = \alpha_1 = 0$. Thus $\{\gamma_0, \gamma_1\}$ is a holomorphic frame of E_T . Since $\langle t_1(w), \gamma_0(w)\rangle = 0$, we see that

$$\frac{\partial}{\partial w}\|\gamma_0(w)\|^2 = \langle \gamma_1(w), \gamma_0(w)\rangle.$$

(ii) $\Longleftrightarrow$ (iii): This equivalence is evident from the definition.

(iii) $\Longrightarrow$ (i): Set $t_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be the closed linear span of $\{\gamma_0(w) : w \in \Omega\}$ and $\{t_1(w) : w \in \Omega\}$, respectively. Set $T_0 = T|_{\mathcal{H}_0}$, $T_1 = P_{\mathcal{H}_1} T|_{\mathcal{H}_1}$ and $= P_{\mathcal{H}_0} T|_{\mathcal{H}_1}$.

We see that the closed linear span of the vectors $\{\gamma_0(w), t_1(w) : w \in \Omega\}$ is $\mathcal{H}$: Suppose x in $\mathcal{H}$ is orthogonal to this set of vectors. Then clearly, $x \perp \gamma_0(w)$ and $x \perp t_1(w)$ for all w in Ω. Or, equivalently $x \perp \gamma_0(w)$ and $x \perp \gamma_1(w)$ for all w in Ω. Therefore x must be the 0 vector. Next, we show that the two operators T_0 and T_1 are in $B_1(\Omega)$.

Clearly, $(T_1 - w)$ is onto. Thus index $(T_1 - w) = dim\,ker\,(T_1 - w)$ and $2 = index\,(T - w) = index\,(T_0 - w) + index\,(T_1 - w)$. It follows that $dim\,ker(T_1 - w) = 1$ or 2.

Suppose $dim\,ker\,(T_1 - w) = 2$ and $\{s_1(w), s_2(w)\}$ be a holomorphic choice of linearly independent vectors in $ker\,(T_1 - w)$. Then we can find holomorphic functions ϕ_1, ϕ_2 defined on Ω such that $S(s_1(w)) = \phi_1(w)\gamma_0(w)$ and $S\big(s_2(w)\big) = \phi_2(w)\gamma_0(w)$. Setting

$$\tilde{\gamma}_0(w) := \gamma_0(w),$$

$$\tilde{\gamma}_1(w) := \frac{\partial}{\partial w}(\phi_1(w)\gamma_0(w)) - s_1(w) \text{ and}$$

$$\tilde{\gamma}_2(w) := \frac{\partial}{\partial w}(\phi_2(w)\gamma_0(w)) - s_2(w),$$

we see that $(T - w)(\tilde{\gamma}_i(w)) = 0$ for $0 \leq i \leq 2$. If $\sum_{i=0}^{2} \alpha_i\tilde{\gamma}_i(w) = 0, \alpha_i \in \mathbb{C}$, then

$$\alpha_0\gamma_0(w) + \frac{\partial}{\partial w}\Big(\big(\alpha_1\phi_1(w) + \alpha_2\phi_2(w)\big)\gamma_0(w)\Big) + \alpha_1 s_1(w) + \alpha_2 s_2(w) = 0.$$

It follows that $\alpha_1 s_1(w) + \alpha_2 s_2(w) = 0$ since $\mathcal{H}_0$ is orthogonal to $\mathcal{H}_1$. Hence $\alpha_1 = \alpha_2 = 0$ implying $\alpha_0 = 0$. Thus we have $dim\,ker(T - w) \geq 3$. This contradiction proves that $dim\,ker(T_0 - w) = 1$ and hence T_1 is in $B_1(\Omega)$.

To show that T_0 is in $B_1(\Omega)$, pick any $x \in \mathcal{H}_0$, and note that there exists $z \in \mathcal{H}$ such that $(T - w)z = x$ since $T - w$ is onto. Let $z_{\mathcal{H}_1}$ and $z_{\mathcal{H}_0}$ be the projections of z to the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. We have $[(T_0 - w)z_{\mathcal{H}_0} + S(z_{\mathcal{H}_1})] + (T_1 - w)z_{\mathcal{H}_1} = x$. Therefore $(T_1 - w)z_{\mathcal{H}_1} = 0$ and $(T_0 - w)z_{\mathcal{H}_0} + S(z_{\mathcal{H}_1}) = x$. Since $dim\, ker\, (T_1 - w) = 1$, so $z_{\mathcal{H}_1} = c_1 t_1(w)$, it follows that

$$\begin{aligned} x &= (T_0 - w)z_{\mathcal{H}_0} + S(z_{\mathcal{H}_1}) = (T_0 - w)z_{\mathcal{H}_0} + S(c_1 t_1(w)) \\ &= (T_0 - w)z_{\mathcal{H}_0} + c_1\gamma_0(w) = (T_0 - w)z_{\mathcal{H}_0} + (T_0 - w)(c_1 \frac{\partial}{\partial w}\gamma_0(w)) \\ &= ((T_0 - w)(z_{\mathcal{H}_0} + c_1 \frac{\partial}{\partial w}\gamma_0(w)). \end{aligned}$$

Thus $T_0 - w$ is onto. We have $2 = dim\, ker\, (T - w) = dim\, ker\, (T_0 - w) + dim\, ker\, (T_1 - w)$. Hence $dim\, ker\, (T_0 - w) = 1$ and we see that T_0 is in $B_1(\Omega)$. Finally, since $S(t_1(w)) = \gamma_0(w)$, it follows that $T_0 S = S T_1$.

An operator $T \in \mathcal{F}B_2(\Omega)$ is also in $B_2(\Omega)$, therefore as is well-known (cf. [61], [79]), it can be realized as the adjoint of a multiplication operator on some reproducing kernel Hilbert space of holomorphic $\mathbb{C}^2$-valued functions. These functions are defined on $\Omega^* := \{w : \bar{w} \in \Omega\}$. An explicit description for operators in $\mathcal{F}B_2(\Omega)$ follows.

Let E_T be the holomorphic Hermitian vector bundle over Ω corresponding to the operator T. Since T is in $\mathcal{F}B_2(\Omega)$, we may find a holomorphic frame $\gamma = \{\gamma_0, \gamma_1\}$ such that $\gamma_0(w)$ and $\frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$ are orthogonal for all w in Ω. Define $\Gamma : \mathcal{H} \to \mathcal{O}(\Omega^*, \mathbb{C}^2)$ as follows:

$$\Gamma(x)(z) = (\langle x, \gamma_0(\bar{z}), \langle x, \gamma_1(\bar{z})\rangle\rangle)^{tr} \qquad z \in \Omega^*, x \in \mathcal{H},$$

where $\mathcal{O}(\Omega^*, \mathbb{C}^2)$ is the space of holomorphic functions defined on Ω^* which take values in $\mathbb{C}^2$. Here $(\cdot\, ,\cdot)^{tr}$ denotes the transpose of the vector $(\cdot\, ,\cdot)$.

The map Γ is injective and therefore transplanting the inner product from $\mathcal{H}$ on the range of Γ, we make it unitary from $\mathcal{H}$ onto $\mathcal{H}_\Gamma := ran\, \Gamma$. Define K_Γ to be the function on $\Omega^* \times \Omega^*$ taking values in the 2×2 matrices $\mathcal{M}_2(\mathbb{C})$:

$$K_\Gamma(z, w) = \left(\left(\langle \gamma_j(\bar{w}), \gamma_i(\bar{z})\rangle\right)\right)_{i,j=0}^{1}$$

$$= \begin{pmatrix} \gamma_0(\bar{w}), \gamma_0(\bar{z}) & \frac{\partial}{\partial \bar{w}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle \\ \frac{\partial}{\partial \bar{z}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \frac{\partial^2}{\partial z \partial \bar{w}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle + \langle t_1(\bar{w}), t_1(\bar{z})\rangle \end{pmatrix}$$

$$= \begin{pmatrix} K_0(z, w) & \frac{\partial}{\partial \bar{w}} K_0(z, w) \\ \frac{\partial}{\partial \bar{w}} K_0(z, w) & \frac{\partial^2}{\partial z \partial \bar{w}} K_0(z, w) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K_1(z, w) \end{pmatrix} \tag{24}$$

where $t_1(\bar{w}) = \frac{\partial}{\partial \bar{w}}\gamma_0(\bar{w}) - \gamma_1(\bar{w}), K_0(z, w) = \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle$ and $K_1(z, w) = \langle t_1(\bar{w}), t_1(\bar{z})\rangle$ for z, w in Ω^*. Set $(K_\Gamma)_w(\cdot) = K_\Gamma(\cdot, w)$. It is then easily verified that K_Γ has the following properties:

(i) The reproducing property: $\langle \Gamma(x)(\cdot), (K_\Gamma)_w(\cdot)\eta\rangle_{ran\, \Gamma} = \langle \Gamma(x)(w), \eta\rangle_{\mathbb{C}^2}, x \in \mathcal{H}, \eta \in \mathbb{C}^2, w \in \Omega^*$.

(ii) The unitary operator Γ intertwines the operators T defined on $\mathcal{H}$ and M^* defined on $\mathcal{H}_\Gamma$, namely, $\Gamma T^* = Mz\Gamma$.

(iii) Each w in Ω is an eigenvalue with eigenvector $(K_\Gamma)_{\bar{w}}(\cdot)\eta, \eta \in \mathbb{C}^2$, for the operator $M^* = \Gamma T \Gamma^*$.

Once we represent an operator T from $\mathcal{F}B_2(\Omega)$ in this form, the possibilities for the change of frame are limited. The admissible ones are described in the following lemma.

Lemma (6.2.8)[190]: Let T be an operator in $\mathcal{F}B_2(\Omega)$. Suppose $\{\gamma_0, \gamma_1\}, \{\tilde{\gamma}_0, \tilde{\gamma}_1\}$ are two frames of the vector bundle E_T such that $\gamma_0(w) \perp (\frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w))$ and $\tilde{\gamma}_0(w) \perp (\frac{\partial}{\partial w}\tilde{\gamma}_0(w) - \tilde{\gamma}_1(w))$ for all $w \in \Omega$. If $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ is any change of frame between $\{\gamma_0, \gamma_1\}$ and $\{\tilde{\gamma}_0, \tilde{\gamma}_1\}$, that is,

$$\{\tilde{\gamma}_0, \tilde{\gamma}_1\} = \{\gamma_0, \gamma_1\} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix},$$

then $\phi_{21} = 0, \phi_{11} = \phi_{22}$ and $\phi_{12} = \phi_{11}'$.

Proof. Define the unitary map Γ, as above, using the holomorphic frame $\gamma = \{\gamma_0, \gamma_1\}$. The operator T is then unitarily equivalent to the adjoint of the multiplication operator on the Hilbert space $\mathcal{H}_\Gamma$ possessing a reproducing kernel K_Γ of the form (24). Let e_1 and e_2 be the standard unit vectors in $\mathbb{C}^2$. Clearly, $(K_\Gamma)_w(\cdot)e_1$ and $(K_\Gamma)_w(\cdot)e_2$ are two linearly independent eigenvectors of M^* with eigenvalue $\bar{w}$.

Similarly, corresponding to the holomorphic frame $\tilde{\gamma} = \{\tilde{\gamma}_0, \tilde{\gamma}_1\}$, the operator T is unitarily equivalent to the adjoint of multiplication operator on the Hilbert space $\mathcal{H}_{\tilde{\Gamma}}$.

The reproducing kernel $K_{\tilde{\Gamma}}$ is again of the form (24) except that K_0 and K_1 must be replaced by $\widetilde{K}_0$ and $\widetilde{K}_1$, respectively.

For $i = 0, 1$, set $s_i(w) := (K_\Gamma)(w)e_i$, and $\tilde{s}_i(w) := (K_{\tilde{\Gamma}})(w)e_i$. Let $\phi(w) := \begin{pmatrix} \phi_{00}(w) & \phi_{01}(w) \\ \phi_{10}(w) & \phi_{11}(w) \end{pmatrix}$ be the holomorphic function, taking values in 2×2 matrices, such that

$$(\tilde{s}_0(w), \tilde{s}_1(w)) = \big(s_0(w), s_1(w)\big)\phi(w).$$

This implies that

$$\tilde{s}_0(w) = \phi_{00}(w)s_0(w) + \phi_{10}(w)s_1(w) \tag{25}$$

And

$$\tilde{s}_1(w) = \phi_{01}(w)s_0(w) + \phi_{11}(w)s_1(w). \tag{26}$$

From Equation (25), equating the first and the second coordinates separately, we have

$$\big(\widetilde{K}_0\big)w(\cdot) = \phi_{00}(w)(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial}{\partial \bar{w}}(K_0)_w(\cdot) \tag{27}$$

And

$$\frac{\partial}{\partial z}\big(\widetilde{K}_0\big)_w(\cdot) = \phi_{00}(w)\frac{\partial}{\partial z}(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial^2}{\partial z \partial \bar{w}}(K_0)_w(\cdot)$$
$$+\phi_{10}(w)(K_1)_w(\cdot). \tag{28}$$

From these two equations, we get

$$\phi_{00}(w)\frac{\partial}{\partial z}(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial^2}{\partial z \partial \bar{w}}(K_0)_w(\cdot) =$$
$$\phi_{00}(w)\frac{\partial}{\partial z}(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial^2}{\partial z \partial \bar{w}}(K_0)_w(\cdot) + \phi_{10}(w)(K_1)_w(\cdot),$$

which implies that $\phi_{10} = 0$. Finally, from Equation (26), we have

$$\frac{\partial}{\partial \bar{w}}\big(\widetilde{K}_0\big)_w(\cdot) = \phi_{01}(w)(K_0)_w(\cdot) + \phi_{11}(w)\frac{\partial}{\partial \bar{w}}(K_0)_w(\cdot). \tag{29}$$

The Equations (26) and (29) together give

$$\phi_{01} = \phi_{00}' \quad \text{and} \quad \phi_{00} = \phi_{11}$$

completing the proof.

A very important consequence of this Lemma is that the decomposition of the operators in the class $\mathcal{FB}_2(\Omega)$ is unique in the sense described in the following proposition.

Proposition (6.2.9)[190]: Let $T, \tilde{T} \in \mathcal{FB}_2(\Omega)$ be two operators of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1$, respectively. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1$ be a unitary operator such that

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

then $U_{12} = U_{21} = 0$.

Proof. Let $\{\gamma_0, \gamma_1\}$ and $\{\tilde{\gamma}_0, \tilde{\gamma}_1\}$ be holomorphic frames of E_T and $E_{\tilde{T}}$ respectively with the property that $\gamma_0 \perp (\frac{\partial}{\partial w}\gamma_0 - \gamma_1)$ and $\tilde{\gamma}_0 \perp (\frac{\partial}{\partial w}\tilde{\gamma}_0 - \tilde{\gamma}_1)$. Set $t_1 := (\frac{\partial}{\partial w}\gamma_0 - \gamma_1)$ and $\tilde{t}_1 := (\frac{\partial}{\partial w}\tilde{\gamma}_0 - \tilde{\gamma}_1)$. Since U intertwines T and $\tilde{T}$, it follows that $\{U\gamma_0, U\gamma_1\}$ is a second holomorphic frame of $E_{\tilde{T}}$ with the property $U\gamma_0 \perp (\frac{\partial}{\partial w}(U\gamma_0) - U\gamma_1) = U(t_1)$. By Lemma (6.2.8), we have that

$$U(\gamma_0) = \phi\tilde{\gamma}_0 \tag{30}$$

And

$$U(\gamma_1) = \phi'\tilde{\gamma}_0 + \phi\tilde{\gamma}_1. \tag{31}$$

From equations (30) and (31), we get

$$U(t_1) = \phi\tilde{t}_1. \tag{32}$$

From equations (30) and (32), it follows that U maps $\mathcal{H}_0$ to $\mathcal{H}_0$ and $\mathcal{H}_1$ to $\mathcal{H}_1$. Thus U is a block diagonal from $\mathcal{H}_0 \oplus \mathcal{H}_1$ onto $\widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1$.

Corollary (6.2.10)[190]: For $i = 0, 1$, let T_i be any two operators in $B_1(\Omega)$. Let S and $\tilde{S}$ be bounded linear operators such that $T_0 S = S T_1$ and $T_0\tilde{S} = \tilde{S}T_1$. If $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} T_0 & \tilde{S} \\ 0 & T_1 \end{pmatrix}$, then T is unitarily equivalent to $\tilde{T}$ if and only if $\tilde{S} = e^{i\theta}S$ for some real number θ.

Proof. Suppose that $UT = \tilde{T}U$ for some unitary operator U. We have just shown that such an operator U must be diagonal, say $= \begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix}$. Hence we have

$$U_{11}T_0 = T_0U_{11}, U_{22}T_1 = T_1U_{22}, U_{11}S = \tilde{S}U_{22}. \tag{33}$$

Since U_{11} is unitary, the first of the equations (33) implies that

$$U_{11} \in \{T_0, T_0^*\} := \{W \in \mathcal{L}(\mathcal{H}_0) : WT_0 = T_0W \text{ and } WT_0^* = T_0^*W\}.$$

Since T_0 is an irreducible operator, we conclude that $U_{11} = e^{i\theta_1}I_{\mathcal{H}_0}$ for some $\theta_1 \in \mathbb{R}$. Similarly, $U_{22} = e^{i\theta_2}I_{\mathcal{H}_1}$ for some $\theta_2 \in \mathbb{R}$. Hence the third equation in (33) implies that $\tilde{S} = e^{i(\theta_1 - \theta_2)}S$.

Conversely suppose that $\tilde{S} = e^{i\theta}S$ for some real number θ. Then evidently the operator $U := \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right)I_{\mathcal{H}_0} & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right)I_{\mathcal{H}_1} \end{pmatrix}$ is unitary on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $UT = \tilde{T}U$.

Corollary (6.2.11)[190]: For $i = 0, 1$, let T_i be two operators in $B_1(\Omega)$. Let S be a non-zero bounded linear operator such that $T_0S = ST_1$. If $T_\mu = \begin{pmatrix} T_0 & \mu S \\ 0 & T_1 \end{pmatrix}$ and $T_{\tilde{\mu}} = \begin{pmatrix} T_0 & \tilde{\mu} S \\ 0 & T_1 \end{pmatrix}, \mu, \tilde{\mu} > 0$, then T_μ is unitarily equivalent to $T_{\tilde{\mu}}$ if and only if $\mu = \tilde{\mu}$.

The following theorem lists a complete set of unitary invariants for operators in $\mathcal{F}B_2(\Omega)$.

Theorem (6.2.12)[190]: Suppose that $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ are any two operators in $\mathcal{F}B_2(\Omega)$. Then the operators T and $\tilde{T}$ are unitarily equivalent if and only if $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}$ (or, $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$) and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$, where t_1 and $\tilde{t}_1$ are non-vanishing holomorphic sections for the vector bundles E_{T_1} and $E_{\tilde{T}_1}$, respectively.

Proof. On a small open subset of Ω, we can assume that $S(t_1)$ and $\tilde{S}(\tilde{t}_1)$ are holomorphic frames of the bundle E_{T_0} and $E_{\tilde{T}_0}$, respectively. First suppose that $\bar{\partial}\partial log\|S(t_1)\|^2 = \bar{\partial}\partial\, log \left\|\tilde{S}(t_1)\right\|^2$ and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$. Then we claim that T and $\tilde{T}$ are unitarily equivalent. The equality of the curvatures, namely, $\bar{\partial}\partial\, log\, \|S(t_1)\|^2 = \bar{\partial}\partial\, log \left\|\tilde{S}(\tilde{t}_1)\right\|^2$ implies that $\|S(t_1)\|^2 = |\phi|^2\left\|\tilde{S}(\tilde{t}_1)\right\|^2$ for some non-vanishing holomorphic function ϕ on Ω. It may be that we have to shrink, without loss of generality, to a smaller open set Ω_0. The second of our assumptions gives $\|t_1\|^2 = |\varphi|^2\|\tilde{t}_1\|^2$. Let $\gamma_0(w) := S(t_1(w))$ and $\tilde{\gamma}_0(w) := \tilde{S}\big(\tilde{t}_1(w)\big)$; $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$ and $\tilde{\gamma}_1(w) := \frac{\partial}{\partial w}\tilde{\gamma}_0(w) - \tilde{t}_1(w)$. It follows that $\{\gamma_0, \gamma_1\}$ and $\{\tilde{\gamma}_0, \tilde{\gamma}_1\}$ are holomorphic frames of E_T and $E_{\tilde{T}}$, respectively. Define the map $\Phi : E_T \to E_{\tilde{T}}$ as follows:

(i) $\Phi(\gamma_0(w)) = \phi(w)\tilde{\gamma}_0(w)$,

(ii) $\Phi(\gamma_1(w)) = \phi'(w)\tilde{\gamma}_0(w) + \phi(w)\tilde{\gamma}_1(w)$.

Clearly, Φ is holomorphic. Note that

$$\begin{aligned}\langle\Phi(\gamma_0(w)), \Phi(\gamma_1(w))\rangle &= \langle\phi(w)\tilde{\gamma}_0(w), \phi'(w)\tilde{\gamma}_0(w) + \phi(w)\tilde{\gamma}_1(w)\rangle \\ &= \langle\phi(w)\tilde{\gamma}_0(w), \phi'(w)\tilde{\gamma}_0(w) + \phi(w)(\frac{\partial}{\partial w}\tilde{\gamma}_0(w) - \tilde{t}_1(w))\rangle \\ &= \langle\phi(w)\tilde{\gamma}_0(w), \frac{\partial}{\partial w}(\phi(w)\tilde{\gamma}_0(w)) - \phi(w)\tilde{t}_1(w)\rangle = \frac{\partial}{\partial \bar{w}}\|\phi(w)\tilde{\gamma}_0(w)\|^2 \\ &= \frac{\partial}{\partial \bar{w}}\|\gamma_0(w)\|^2\end{aligned}$$

and

$$\langle\gamma_0(w), \gamma_1(w)\rangle = \langle\gamma_0(w), \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)\rangle = \frac{\partial}{\partial \bar{w}}\|\gamma_0(w)\|^2.$$

Hence we have $\langle\Phi(\gamma_0(w)), \Phi(\gamma_1(w))\rangle = \langle\gamma_0(w), \gamma_1(w)\rangle$. Similarly, $\|\Phi(\gamma_0(w))\| = \|\gamma_0(w)\|$ and $\|\Phi(\gamma_1)\| = \|\gamma_1\|$. Thus E_T and $E_{\tilde{T}}$ are equivalent as holomorphic Hermitian vector bundles. Hence T and $\tilde{T}$ are unitarily equivalent by Theorem (6.2.2) of Cowen and Douglas.

Conversely, suppose T and $\tilde{T}$ are unitarily equivalent. Let $U : \mathcal{H} \to \widetilde{\mathcal{H}}$ be the unitary map such that $= \tilde{T}U$. By Proposition (6.2.9), U takes the form $\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$ for some pair of unitary operators U_1 and U_2. Hence we have $U_1(S(t_1)) = \phi_1(\tilde{S}(\tilde{t}_1))$ and $U_2t_1 = \phi_2\tilde{t}_1$.

The intertwining relation $U_1 S = \tilde{S} U_2$ implies that $\phi_1 = \phi_2$. Thus $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$ and

$$\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|U_1(S(t_1))\|^2}{\|U_2(t_1)\|^2} = \frac{\|\phi_1 \tilde{S}(\tilde{t}_1)\|^2}{\|\phi_2 \tilde{t}_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}.$$

This verification completes the proof.

We relate the invariants of Theorem (6.2.12) to the second fundamental form of the inclusion $E_0 \subseteq E$. The computation of the second fundamental form is given below following [80]. Here, E_0 is the line bundle corresponding to the operator T_0 and E is the vector bundle of rank 2 corresponding to the operator T in $\mathcal{F}B_2(\Omega)$. Let $\{\gamma_0, \gamma_1\}$ be a holomorphic frame for E such that γ_0 and $t_1 := \partial\gamma_0 - \gamma_1$ are orthogonal. One obtains an orthonormal frame, say, $\{e_0, e_1\}$, from the holomorphic frame $\{\gamma_0, \gamma_1\}$ by the usual Gram–Schmidt process: Set $h = \langle \gamma_0, \gamma_0 \rangle$, and observe that

$$e_1 = h^{-1/2}\gamma_0,$$

$$e_2 = \frac{\gamma_1 - \dfrac{\gamma_0 \langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2}}{\left(\|\gamma_1\|^2 - \dfrac{|\langle \gamma_1, \gamma_0 \rangle|^2}{\|\gamma_0\|^2} \right)^{1/2}}$$

are orthogonal. The canonical hermitian connection D for the vector bundle E_T is given, in terms of e_1 and e_2 by the formula:

$$\begin{aligned} D\, e_1 = D^{1,0} e_1 + D^{0,1} e_1 &= \alpha_{11} e_1 + \alpha_{21} e_2 + \bar{\partial} e_1 = \left(\alpha_{11} - \bar{\partial}(\log h) \right) e_1 + \alpha_{21} e_2 \\ &= \theta_{11} e_1 + \theta_{21} e_2, \end{aligned}$$

where α_{11}, α_{21} are $(1,0)$ forms to be determined. Similarly, we have

$$D\, e_2 = D^{1,0} e_2 + D^{0,1} e_2 = \alpha_{12} e_1 + \alpha_{22} e_2 + \bar{\partial} e_2$$

$$= \left(\alpha_{12} - h^{1/2} \frac{\bar{\partial}(h^{-1} \langle \gamma_1, \gamma_0 \rangle)}{\left(\|\gamma_1\|^2 - \dfrac{|\langle \gamma_1, \gamma_0 \rangle|^2}{\|\gamma_0\|^2} \right)^{1/2}} \right) e_1 + \left(\alpha_{22} - \frac{1}{2} \frac{\bar{\partial}\left(\|\gamma_1\|^2 - \dfrac{\langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2} \right)}{\|\gamma_1\|^2 - \dfrac{\langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2}} \right) e_2$$

$$= \theta_{12} e_1 + \theta_{22} e_2,$$

where α_{12}, α_{22} are $(1,0)$ forms to be determined. Since we are working with an orthonormal frame, the compatibility of the connection with the Hermitian metric gives

$$\begin{aligned} \langle D\, e_i, e_j \rangle + \langle e_i, D e_j \rangle &= \theta_{ji} + \bar{\theta}_{ij} \\ &= 0 \; for \; 1 \leq i,j \leq 2. \end{aligned}$$

$For\ 1 \leq i,j \leq 2$,equating (1, 0) and (0, 1) forms separately to zero in the equation $\theta_{ij} + \bar{\theta}_{ji} = 0$, we obtain $\alpha_{11} = \partial(\log h), \alpha_{12} = 0, \alpha_{21} = h^{1/2} \frac{\bar{\partial}(h^{-1}\langle \gamma_1, \gamma_0 \rangle)}{\left(\|\gamma_1\|^2 - \frac{|\langle \gamma_1, \gamma_0 \rangle|^2}{\|\gamma_0\|^2} \right)^{1/2}}$ and $\alpha_{22} = \frac{1}{2} \frac{\bar{\partial}\left(\|\gamma_1\|^2 - \frac{\langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2} \right)}{\|\gamma_1\|^2 - \frac{\langle \gamma_1, \gamma_0 \rangle}{\|\gamma_0\|^2}}$. Hence the second fundamental form for the inclusion $E_0 \subset E$ is given by the formula:

$$\theta_{12} = -h^{12} \frac{\bar{\partial}(h^{-1} \langle \gamma_1, \gamma_0 \rangle)}{\left(\|\gamma_1\|^2 - \dfrac{|\langle \gamma_1, \gamma_0 \rangle|^2}{\|\gamma_0\|^2} \right)^{1/2}} = -\frac{\dfrac{\partial^2}{\partial z \partial \bar{z}} log h d\bar{z}}{\left(\dfrac{\|t_1\|^2}{\|\gamma_0\|^2} + \dfrac{\partial^2}{\partial z \partial \bar{z}} log\ h \right)^{1/2}},$$

where $t_1 = \gamma_0' - \gamma_1$. If $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ is an operator in $\mathcal{F}B_2(\Omega)$ and t_1 is a non-vanishing holomorphic section of the vector bundle E_1 corresponding to the operator T_1, then we may assume, without loss of generality, that $S(t_1)$ is a holomorphic frame of E_0. The second fundamental form θ_{12} of the inclusion $E_0 \subseteq E$, in this case, is therefore equal to

$$-\frac{\frac{\partial^2}{\partial z \partial \bar{z}} log\|S(t_1)\|^2 d\bar{z}}{\left(\frac{\|t_1\|^2}{\|S(t_1)\|^2} + \frac{\partial^2}{\partial z \partial \bar{z}}\ log\ \|S(t_1)\|^2\right)^{1/2}} .$$

It follows from Theorem (6.2.12) that the second fundamental form of the inclusion $E_0 \subseteq E$ and the curvature of E_1 form a complete set of invariants for the operator T. We restate Theorem (6.2.12) using the second fundamental form θ_{12}.

Theorem (6.2.13)[190]: Suppose that $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ are any two operators in $\mathcal{F}B_2(\Omega)$. Then the operators T and $\tilde{T}$ are unitarily equivalent if and only if $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}(or\ \mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0})$ and $\theta_{12} = \tilde{\theta}_{12}$.

We use the machinery developed here to list the unitary equivalence classes of homogeneous operators in $B_n(\mathbb{D}), n = 2$. For $n = 1$ this was done in [197] and in [83] for $n = 2$. The classification of homogeneous operators in $B_n(\mathbb{D})$ was given in [183] for an arbitrary n. The proofs of [83] and [183] use tools from Differential geometry and the representation theory of Lie groups respectively. While the description below is very close to the spirit of [197].

Definition (6.2.14)[190]: An operator T is said to be homogeneous if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T.

Proposition (6.2.15)[190]: ([197]). An operator T in $\mathcal{B}_1(\mathbb{D})$ is homogeneous if and only if

$$\mathcal{K}_T\,(w) = -\lambda(1 - |w|^2)^{-2}$$

for some positive real number λ.

Proposition (6.2.16)[190]: Let T be an operator in $\mathcal{F}B_2(\mathbb{D})$ and let t_1 be a non-vanishing holomorphic section of the bundle E_1 corresponding to the operator T_1. For any φ in Möb, set $t_{1,\varphi} = t_1 o \varphi^{-1}$. The operator T is homogeneous if and only if T_0, T_1 are homogeneous and $\frac{\|S(t_{1,\varphi})\|^2}{\|t_{1,\varphi}\|^2} = |(\varphi^{-1})'|^2 \frac{\|S(t_1)\|^2}{\|t_1\|^2}$ for all φ in Möb.

Proof. Using the intertwining property in the class $\mathcal{F}B_2(D)$, we see that

$$\varphi(T) = \begin{pmatrix} \varphi(T_0) & S\varphi'(T_1) \\ 0 & \varphi(T_1) \end{pmatrix}.$$

Suppose that T is homogeneous, that is, T is unitarily equivalent to $\varphi(T)$ for φ in Möb. From Theorem (6.2.12), it follows that T_0 is unitarily equivalent to $\varphi(T_0), T_1$ is unitarily equivalent to $\varphi(T_1)$ and

$$\frac{\|S\varphi'(T_1)(t_1, \varphi(w))\|^2}{\|t_{1,\varphi}(w)\|^2} = \frac{\|S(t_1(w))\|^2}{\|t_1(w)\|^2} . \tag{34}$$

Now, we have

$$\frac{\left\|S\,\varphi'(T_1)\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}=\frac{\left\|S\varphi'\left(\varphi^{-1}(w)\right)\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}$$
$$=\frac{\left|\varphi'\left(\varphi^{-1}(w)\right)\right|^2\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}$$
$$=\frac{|(\varphi^{-1})'(w)|^{-2}\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}\,. \tag{35}$$

From equations (34) and (35), it follows that

$$\frac{\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}=|(\varphi^{-1})'(w)|^2\;\frac{\left\|S\left(t_1(w)\right)\right\|^2}{\|t_1(w)\|^2}\,. \tag{36}$$

Conversely suppose that T_0, T_1 are homogeneous operators and

$$\frac{\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}=|(\varphi^{-1})'\,(w)|^2\;\frac{\left\|S\left(t_1(w)\right)\right\|^2}{\|t_1(w)\|^2}$$

for all φ in Möb. From equations (35), (36) and Theorem (6.2.12), it follows that T is a homogeneous operator.

Corollary (6.2.17)[190]: An operator T in $\mathcal{F}B_2(\mathbb{D})$ is homogeneous if and only if

(i) T_0 and T_1 are homogeneous operators;

(ii) $\mathcal{K}_{T_1}(w)=\mathcal{K}_{T_0}(w)+\mathcal{K}_{B^*}(w), w\in\mathbb{D}$, where B is the forward Bergman shift;

(iii) $S(t_1(w))=\alpha\gamma_0(w)$ for some positive real number α and $\|t_1(w)\|^2=\frac{1}{(1-|w|^2)^{\lambda+2}}$, $\|\gamma_0(w)\|^2=\frac{1}{(1-|w|^2)^{\lambda}}$.

Proof. Suppose T is a homogeneous operator. Then Proposition (6.2.16) shows that T_0 and T_1 are homogeneous operators. We may therefore find non-vanishing holomorphic sections γ_0 and t_1 of E_0 and E_1, respectively, such that $\|\gamma_0(w)\|^2=(1-|w|^2)^{-\lambda}$ and $\|t_1(w)\|^2=(1-|w|^2)^{-\mu}$ for some positive real λ and μ. For φ in Möb, set $\gamma_{0,\varphi}=\gamma_0\circ\varphi^{-1}$ and $t_{1,\varphi}=t_1\circ\varphi^{-1}$. Clearly, $\left\|\gamma_{0,\varphi}(w)\right\|^2=|(\varphi^{-1})'(w)|^{-\lambda}\|\gamma_0(w)\|^2$ and $\left\|t_{1,\varphi}(w)\right\|^2=|(\varphi^{-1})'(w)|^{-\mu}\|t_1(w)\|^2$. Let $S(t_1(w))=\psi(w)\gamma_0(w)$ for some holomorphic function ψ on $\mathbb{D}$. We have $S(t_{1,\varphi}(w))=S(t_1(\varphi^{-1}(w)))=\psi\left(\varphi^{-1}(w)\right)\gamma_0(\varphi^{-1}(w))=\psi\left(\varphi^{-1}(w)\right)\gamma_{0,\varphi}(w)$ and

$$\frac{\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}=|(\varphi^{-1})'(w)|^2\;\frac{\left\|S\left(t_1(w)\right)\right\|^2}{\|t_1(w)\|^2}\,. \tag{37}$$

Combining these we see that

$$\frac{\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}=\left|\psi\left(\varphi^{-1}(w)\right)\right|^2\;\frac{\left\|\left(\gamma_{0,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2}$$
$$=\left|\psi\left(\varphi^{-1}(w)\right)\right|^2|(\varphi^{-1})'(w)|^{\mu-\lambda}\;\;\frac{\left\|\left(\gamma_0(w)\right)\right\|^2}{\|t_1(w)\|^2}\,. \tag{38}$$

From the equations (37) and (38), we get

$$|\psi(w)|^2|(\varphi^{-1})'(w)|^{\lambda+2-\mu}=|\psi(\varphi^{-1}w)|^2. \tag{39}$$

Pick $\varphi = \varphi_u$, where $\varphi_u(w) = \frac{w-u}{1-\bar{u}w}$ and put $w = 0$ in the equation (39). Then

$$|\psi(0)|^2(1 - |u|^2)^{\lambda+2-\mu} = |\psi(u)|^2. \quad (40)$$

If $\psi(0) = 0$ then equation (40) implies that $\psi(u) = 0$ for all $u \in \mathbb{D}$, which makes $S = 0$ leading to a contradiction. Thus $\psi(0) \neq 0$. Taking log and differentiating both sides of the equation (40), we see that

$$(\lambda + 2 - \mu) \frac{\partial^2}{\partial u \partial \bar{u}} log(1 - |u|^2) = 0.$$

Hence we conclude that $\mu = \lambda + 2$. Putting $\mu = \lambda + 2$ in the equation (40) we find that ψ must be a constant function. Hence there is a constant α such that $S(t_1(w)) = \alpha\gamma_0(w)$ for all $w \in \Omega$. Finally,

$$\begin{aligned}\mathcal{K}_{T_1}(w) &= \bar{\partial}\partial \, log\|t_1(w)\|^2 = \bar{\partial}\partial \, log(1 - |w|^2)^{-\mu} = \bar{\partial}\partial \, log(1 - |w|^2)^{-\lambda-2} \\ &= \bar{\partial}\partial log(1 - |w|^2)^{-\lambda} + \bar{\partial}\partial \, log(1 - |w|^2)^{-2} \\ &= \bar{\partial}\partial log \, \|\gamma_0(w)\|^2 + \bar{\partial}\partial \, log(1 - |w|^2)^{-2} = \mathcal{K}_{T_0}(w) + \mathcal{K}_{B^*}(w).\end{aligned}$$

Conversely, suppose that conditions (i), (ii) and (iii) are met. We need to show that T is a homogeneous operator. Condition (ii) is equivalent to $\mu = \lambda + 2$. By Proposition (6.2.16), it is sufficient to show that

$$\frac{\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2} = |(\varphi^{-1})'(w)|^2 \, \frac{\left\|S\left(t_1(w)\right)\right\|^2}{\|t_1(w)\|^2}.$$

However, we have

$$\begin{aligned}\frac{\left\|S\left(t_{1,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2} &= |\alpha|^2 \, \frac{\left\|\left(\gamma_{0,\varphi}(w)\right)\right\|^2}{\left\|t_{1,\varphi}(w)\right\|^2} = |\alpha|^2|(\varphi^{-1})'(w)|^{\mu-\lambda} \, \frac{\left\|\left(\gamma_0(w)\right)\right\|^2}{\|t_1(w)\|^2} \\ &= |\alpha|^2|(\varphi^{-1})'(w)|^2 \, \frac{\left\|\left(\gamma_0(w)\right)\right\|^2}{\|t_1(w)\|^2} = |(\varphi^{-1})'(w)|^2 \, \frac{\left\|S\left(t_1(w)\right)\right\|^2}{\|t_1(w)\|^2}.\end{aligned}$$

We show that an operator T in $\mathcal{F}B_2(\Omega)$ is irreducible. Furthermore, if the intertwining operator S is invertible, then T is strongly irreducible. (Recall that an operator T is said to be strongly irreducible if the commutant $\{T\}'$ of the operator T contains no idempotent operator.) We also provide a more direct proof of Proposition (6.2.9), which easily generalizes to the case of an arbitrary n.

Definition (6.2.18)[190]: Let T_1 and T_2 be any two bounded linear operators on the Hilbert space $\mathcal{H}$. Define $\sigma_{T_1,T_2} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ to be the operator

$$\sigma_{T_1,T_2}(X) = T_1X - XT_2, \quad X \in \mathcal{L}(\mathcal{H}).$$

Let $\sigma_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be the operator $\sigma_{T,T}$.

An operator T defined on a Hilbert space $\mathcal{H}$ is said to be quasi-nilpotent if $\lim\limits_{n\to\infty} \|T^n\|^{1/n} = 0$.

Lemma (6.2.19)[190]: Suppose T is in $B_1(\Omega)$ and X is a quasi-nilpotent operator such that $TX = XT$. Then $X = 0$.

Proof. Let γ be a non-vanishing holomorphic section for E_T . Since $TX = XT$, we see that $X(\gamma)$ is also a holomorphic section of E_T . Hence $X(\gamma(w)) = \phi(w)\gamma(w)$ for some holomorphic function ϕ defined on Ω. Clearly, $X^n(\gamma(w)) = \phi(w)^n\gamma(w)$. Now, we have

$$\begin{aligned}|\phi(w)|^n\|\gamma(w)\| &= \|\phi(w)^n\gamma(w)\| \\ &= \|X^n(\gamma(w))\| \\ &\leq \|X^n\|\|\gamma(w)\|.\end{aligned}$$

Thus, for $n \in \mathbb{N}$ and $w \in \Omega$, we have $|\phi(w)| \leq \|X^n\|^{1/n}$ implying $\phi(w) = 0, w \in \Omega$. Hence $X = 0$.

The following theorem from [195] is the key to an alternative proof of the Proposition (6.2.9) and its generalization in the following.

Theorem (6.2.20)[190]: Let P, T be two bounded linear operators. If $\in ran\, \sigma_T \cap ker\, \sigma_T$, then P is a quasi-nilpotent.

A second proof of Proposition (6.2.9). Suppose T is unitarily equivalent to $\tilde{T}$ via the unitary U, namely, $UT = TU$. Then

$$U_{21}S + U_{22}T_1 = \tilde{T}_1 U_{22} \tag{41}$$

$$U_{21}T_0 = \tilde{T}_1 U_{21}. \tag{42}$$

Equivalently, we also have $TU^* = U^*\tilde{T}$, which gives an additional relationship:

$$T_1 U_{12}^* = U_{12}^* \tilde{T}_0. \tag{43}$$

Using these equations, we compute

$$\begin{aligned} U_{21}SU_{12}^*\tilde{S} &= \left(\tilde{T}_1 U_{22} - U_{22}T_1\right)U_{12}^*\tilde{S} = \tilde{T}_1 U_{22} U_{12}^* \tilde{S} - U_{22}T_1 U_{12}^*\tilde{S} \\ &= \tilde{T}_1 U_{22}U_{12}^*\tilde{S} - U_{22}U_{12}^*\tilde{T}_0\tilde{S} = \tilde{T}_1 U_{22}U_{12}^*\tilde{S} - U_{22}U_{12}^*\tilde{S}\tilde{T}_1 = \sigma_{\tilde{T}_1}(U_{22}U_{12}^*\tilde{S}), \end{aligned}$$

and

$$\left(U_{21}SU_{12}^*\tilde{S}\right)\tilde{T}_1 = U_{12}SU_{12}^*\tilde{T}_0\tilde{S} = U_{21}ST_1U_{12}^*\tilde{S} = U_{21}T_0SU_{12}^*\tilde{S} = \tilde{T}_1(U_{12}S\tilde{U}_{12}^*\tilde{S}).$$

Thus $U_{21}SU_{12}^*\tilde{S} \in ran\, \sigma_{\tilde{T}_1} \cap ker\, \sigma_{\tilde{T}_1}$. From Lemma (6.2.19) and Theorem (6.2.20), it follows that

$$U_{21}SU_{12}^*\tilde{S} = 0.$$

Since $\tilde{S}$ has dense range, we have $U_{21}SU_{12}^* = 0$. Let us consider the two possibilities for U_{12}^*, namely, either $U_{12}^* = 0$ or $U_{12}^* \neq 0$. If $U_{12}^* \neq 0$, then from equation (43), U_{12}^* must have dense range. Since S also has dense range, we have $U_{21} = 0$. To complete the proof, we consider two cases.

Case 1: Suppose $U_{21} = 0$. In this case, we have to prove that $U_{12} = 0$. From $U^*U = I$, we get $U_{11}^*U_{11} = I$ and $U_{12}^*U_{11} = 0$. From $= \tilde{T}U$, we get $U_{11}T_0 = \tilde{T}_0 U_{11}$, so U_{11} has dense rang. Since U_{11} is an isometry and has dense range, it follows that U_{11} is onto. Hence U_{11} is unitary. Since U_{11} is unitary and $U_{12}^*U_{11} = 0$, it follows that $U_{12} = 0$.

Case 2: Suppose $U_{12} = 0$. In this case, we have to prove that $U_{21} = 0$. We have $U_{11}U_{11}^* = I$ and $U_{21}U_{11}^* = 0$. The intertwining relation $TU^* = U^*\tilde{T}$ gives $T_0U_{11}^* = U_{11}^*\tilde{T}_0$.

So U_{11}^* has dense range. Since U_{11}^* is an isometry and it has dense range, we must conclude that U_{11}^* is onto. Hence U_{11} is unitary and we have $U_{21}U_{11}^* = 0$ forcing U_{21} to be the 0 operator.

Proposition (6.2.21)[190]: Any operator T in $\mathcal{F}B_2(\Omega)$ is irreducible. Also, if $= \begin{pmatrix} T_0 & I \\ 0 & T_0 \end{pmatrix}$, then it is strongly irreducible.

Proof. Let $P = \left(P_{ij}\right)_{2\times 2}$ be a projection in the commutant $\{T\}'$ of the operator T, that is,

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

This equality implies that $P_{11}T_0 = T_0P_{11} + SP_{21}, P_{11}S + P_{12}T_1 = T_0P_{12} + SP_{22}, P_{21}T_0 = T_1P_{21}$ and $P_{21}S + P_{22}T_1 = T_1P_{22}$. Now

$$(P_{21}S)T_1 = P_{21}(ST_1) = P_{21}(T_0S) = (P_{21}T_0)S = T_1(P_{21}S).$$

Thus $P_{21}S \in ker\, \sigma_{T_1}$. Also note that

$$P_{21}S = T_1P_{22} - P_{22}T_1 = \sigma_{T_1}(P_{22}).$$

Hence $P_{21}S \in ran\,\sigma_{T_1} \cap ker\,\sigma_{T_1}$. Thus from Lemma (6.2.19) and Theorem (6.2.20), it follows that $P_{21}S = 0$. The operator P_{21} must be 0 since S has dense range.
To prove the first statement, we may assume that the operator P is self-adjoint and conclude P_{12} is 0 as well. Since both the operators T_0 and T_1 are irreducible and the projection P is diagonal, it follows that T must be irreducible.

For the proof of the second statement, note that if P is an idempotent of the form $\begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$, both P_{11} and P_{22} must be idempotents. By our hypothesis, P_{11} and P_{22} must also commute with T_0, which is strongly irreducible, hence $P_{11} = 0$ or I and $P_{22} = 0$ or I. By using Theorem (6.2.20), we see that if $P = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}$ or $P = \begin{pmatrix} 0 & P_{12} \\ 0 & I \end{pmatrix}$, then P does not commute with $\begin{pmatrix} T_0 & I \\ 0 & T_0 \end{pmatrix}$. Thus $P = \begin{pmatrix} I & P_{12} \\ 0 & I \end{pmatrix}$ or $P = \begin{pmatrix} 0 & P_{12} \\ 0 & 0 \end{pmatrix}$. Now, using the equation $P^2 = P$, we conclude that P_{12} must be zero. Thus $P = I$ or $P = 0$.
We now give a sufficient condition for an operator T in $\mathcal{F}B_2(\Omega)$ to be strongly irreducible.

Proposition (6.2.22)[190]: Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ be an operator in $\mathcal{F}B_2(\Omega)$. If the operator S is invertible, then the operator T is strongly irreducible.

Proof. By our hypothesis, the operator $X = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ is invertible. Now
$$XTX^{-1} = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}^{-1} = \begin{pmatrix} T_0 & I \\ 0 & ST_1S^{-1} \end{pmatrix} = \begin{pmatrix} T_0 & I \\ 0 & T_0 \end{pmatrix}.$$
Thus T is similar to a strongly irreducible operator and consequently it is strongly irreducible.
We conclude with a characterization of strong irreducibility in $\mathcal{F}B_2(\Omega)$.

Proposition (6.2.23)[190]: An operator $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ in $\mathcal{F}B_2(\Omega)$ is strongly irreducible if and only if $S \notin ran\,\sigma_{T_0,T_1}$.

Proof. Let P be an idempotent in the commutant $\{T\}'$ of the operator T. The proof of the Proposition (6.2.21) shows that P must be upper triangular: $\begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$. The commutation relation $PT = TP$ gives us $P_{11}T_0 = T_0P_{11}, P_{22}T_1 = T_1P_{22}$ and
$$P_{11}S - SP_{22} = T_0P_{12} - P_{12}T_1. \tag{44}$$
Since $P_{i+1i+1} \in \{T_i\}'$ for $0 \le i \le 1$, it follows that P_{ii} can be either I or 0. If either $P_{11} = I$ and $P_{22} = 0$ or $P_{11} = 0$ and $P_{22} = I$, then S is in ran σ_{T_0,T_1} contradicting our assumption. Thus P is of the form $\begin{pmatrix} I & P_{12} \\ 0 & I \end{pmatrix}$ or $\begin{pmatrix} 0 & P_{12} \\ 0 & 0 \end{pmatrix}$. Since P is an idempotent operator, we must have $P_{12} = 0$. Hence T is strongly irreducible.
Assume that the operator S is in ran σ_{T_0,T_1}. In this case, we show that T cannot be strongly irreducible completing the proof. Since $\in ran\,\sigma_{T_0,T_1}$, we can find an operator P_{12} such that
$$S = \sigma_{T_0,T_1}(P_{12}) = T_0P_{12} - P_{12}T_1. \tag{45}$$
The operator $P = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}$ is an idempotent operator. We have
$$\begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} T_0 & S + P_{12}T_1 \\ 0 & 0 \end{pmatrix} \tag{46}$$
And
$$\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_0 & T_0P_{12} \\ 0 & 0 \end{pmatrix}. \tag{47}$$

From these equations, we have $PT = TP$ proving that the operator T is not strongly irreducible.

We begin by describing, what one may think of as, a natural generalization of the class $\mathcal{F}B_2(\Omega)$ to operators in $B_n(\Omega)$ for an arbitrary $n \in \mathbb{N}$.

Definition (6.2.24)[190]: We let $\mathcal{F}B_n(\Omega)$ be the set of all bounded linear operators T defined on some complex separable Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus\cdots\oplus \mathcal{H}_{n-1}$, which are of the form

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix}$$

where the operator $T_i : \mathcal{H}_i \to \mathcal{H}_i$, defined on the complex separable Hilbert space $\mathcal{H}_i, 0 \le i \le n-1$, is assumed to be in $B_1(\Omega)$ and $S_{i,i+1} : \mathcal{H}_{i+1} \to \mathcal{H}_i$, is assumed to be a non-zero intertwining operator, namely, $T_i S_{i,i+1} = S_{i,i+1} T_{i+1}, 0 \le i \le n-2$.

Even without mandating the intertwining condition, the set of operators described above belongs to the Cowen–Douglas class $B_n(\Omega)$. An inductive proof presents no difficulty starting with the base case of $n = 2$, which was proved. Therefore, in particular, $\mathcal{F}B_n(\Omega) \subseteq B_n(\Omega)$. We begin with a preparatory Lemma for proving the rigidity theorem.

Lemma (6.2.25)[190]: Let X be an invertible operator that intertwines two operators in $\mathcal{F}B_n(\Omega)$. Set $Y = X^{-1}$. If $X = \left(\left(X_{i,j}\right)\right)_{n\times n}, Y = \left(\left(Y_{i,j}\right)\right)_{n\times n}$ are the block decompositions of the two operators X and , then $X_{n-1,j} = 0, 0 \le j \le n-2$, and $Y_{n-1,j} = 0, 0 \le j \le n-2$.

Proof. Consider the three possibilities:

(i) $X_{n-1,j} = 0, 0 \le j \le n-2$, but $Y_{n-1,j} \neq 0$ for some $0 \le j \le n-2$.

(ii) $Y_{n-1,j} = 0, 0 \le j \le n-2, X_{n-1,j} \neq 0$ for some $0 \le j \le n-2$.

(iii) $X_{n-1,j} \neq 0$ for some $0 \le j \le n-2$ and $Y_{n-1,k} \neq 0$ for some $0 \le k \le n-2$.

In each of these cases, we arrive at a contradiction proving the Lemma.

Case 1: Choose l to be the smallest index such that $Y_{n-1,l} \neq 0$, that is, $Y_{n-1,i} = 0$ for $0 \le i \le l-1$ but $Y_{n-1,l} \neq 0$. For this index l, the intertwining relation $TY = Y\tilde{T}$ implies $T_{n-1}Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l$. Since $Y_{n-1,l} \neq 0$, it follows from Proposition (6.2.6) that $Y_{n-1,l}$ has dense range. From $XY = I$, we get $X_{n-1,n-1}Y_{n-1,l} = 0$ and $X_{n-1,n-1}Y_{n-1,n-1} = I$. Since $Y_{n-1,l}$ has dense range and $X_{n-1,n-1}Y_{n-1,l} = 0$, we conclude that $X_{n-1,n-1} = 0$. This contradicts the identity: $X_{n-1,n-1}Y_{n-1,n-1} = I$.

Case 2: The contradiction in this case is arrived at exactly in the same manner as in the first case after interchanging the roles of X and .

Case 3: Pick j, l to be the smallest indices such that $X_{n-1,j} \neq 0$ and $Y_{n-1,l} \neq 0$. We have that $= \tilde{T}X$. Consequently,

$$X_{n-1,j}T_j = \tilde{T}_{n-1}X_{n-1,j}, X_{n-1,j}S_{j,j+1} + X_{n-1,j+1}T_{j+1} = \tilde{T}_{n-1}X_{n-1,j+1}. \quad (48)$$

Since $T_k S_{k,k+1} = S_{k,k+1}T_{k+1}$ for $k = 0, 1, 2, \cdots, n-1$, multiplying the second equation in (48) by $S_{j+1,j+2} \cdots S_{n-2,n-1}$, and replacing $T_{j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}$ with $S_{j+1,j+2} \cdots S_{n-2,n-1}T_{n-1}$, we have

$$\begin{aligned} X_{n-1,j}S_{j,j+1} \cdots S_{n-2,n-1} + X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}T_{n-1} \\ = \tilde{T}_{n-1}X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}. \end{aligned} \quad (49)$$

We also have $TY = Y\tilde{T}$, which gives us

$$T_{n-1}Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l. \tag{50}$$

Now, multiply both sides of the equation (49) by $Y_{n-1,l}$, using the commutation $T_{n-1}Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l$, then again multiplying both sides of the resulting equation by $S_{l,l+1} \cdot \cdot\cdot S_{n-2,n-1}$ and finally using the commutation relations $\tilde{T}_k\tilde{S}_{k,k+1} = \tilde{S}_{k,k+1}\tilde{T}_{k+1}, 0 \le k \le n-1$, we have

$$\begin{aligned} &X_{n-1,j}S_{j,j+1}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1} \\ &+ X_{n-1,j+1}S_{j,j+1}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1}\tilde{T}_{n-1} \\ &= \tilde{T}_{n-1}X_{n-1,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1}. \end{aligned} \tag{51}$$

Therefore, we see that

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{\square,l+1}\cdots \tilde{S}_{n-2,n-1}$$

is in the range of the operator $\sigma_{\tilde{T}_{n-1}}$. Indeed it is also in the kernel of $\tilde{\sigma}_{\tilde{T}_{n-1}}$, as is evident from the following string of equalities:

$$\begin{aligned} &X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1}\tilde{T}_{n-1} \\ &= X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{T}_l\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1} \\ &= X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}T_{n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1} \\ &= X_{n-1,j}T_jS_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1} \\ &= \tilde{T}_{n-1}X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1}. \end{aligned}$$

Thus

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1} \in ker\,\sigma_{\tilde{T}_{n-1}} \cap ran\,\sigma_{\tilde{T}_{n-1}}.$$

Consequently, using Lemma (6.2.19) and Theorem (6.2.20), we conclude that

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots \tilde{S}_{n-2,n-1} = 0.$$

By hypothesis, all the operators $S_{k,k+1}, \tilde{S}_{k,k+1}, k = 0, 1, \cdots, n-2$ have dense range. Since $Y_{n-1,l} \ne 0$, then equation (50) and Proposition (6.2.6) ensure that $Y_{n-1,l}$ has dense range. Hence $X_{n-1,j} = 0$. This contradicts the assumption $X_{n-1,j} \ne 0$.

The following proposition is the first step in the proof of the rigidity theorem.

Proposition (6.2.26)[190]: If X is an invertible operator intertwining two operators T and $\tilde{T}$ from $\mathcal{F}B_n(\Omega)$, then X and X^{-1} are upper triangular.

Proof. The proof is by induction on n. The validity of the case $n = 2$, is immediate from Lemma (6.2.25). Let us write the two operators $T, \tilde{T}$ in the form of 2×2 block matrix:

$$T = \begin{pmatrix} T_{n-1\times n-1} & T_{n-1\times 1} \\ 0 & T_{n-1,n-1} \end{pmatrix}, \tilde{T} = \begin{pmatrix} \tilde{T}_{n-1\times n-1} & \tilde{T}_{n-1\times 1} \\ 0 & \tilde{T}_{n-1,n-1} \end{pmatrix}.$$

Using Lemma (6.2.25), the operators X, Y can be written in the form of 2×2 block matrix:

$$X = \begin{pmatrix} X_{n-1\times n-1} & X_{n-1\times 1} \\ 0 & X_{n-1,n-1} \end{pmatrix}, Y = \begin{pmatrix} Y_{n-1\times n-1} & Y_{n-1\times 1} \\ 0 & Y_{n-1,n-1} \end{pmatrix}$$

without loss of generality. Here $X_{n-1\times n-1}$ and $Y_{n-1\times n-1}$ are the operators $\left((X_{i,j})\right)_{i,j=0}^{n-2}$ and $\left((Y_{i,j})\right)_{i,j=0}^{n-2}$ respectively and

$$T_{n-1\times n-1} = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-2} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-3} & S_{n-3,n-2} \\ 0 & \cdots & \cdots & 0 & T_{n-2} \end{pmatrix}, \tilde{T}_{n-1\times n-1}$$

$$= \begin{pmatrix} \tilde{T}_0 & \tilde{S}_{0,1} & \tilde{S}_{0,2} & \cdots & \tilde{S}_{0,n-2} \\ 0 & \tilde{T}_1 & \tilde{S}_{1,2} & \cdots & \tilde{S}_{1,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{T}_{n-3} & \tilde{S}_{n-3,n-2} \\ 0 & \cdots & \cdots & 0 & \tilde{T}_{n-2} \end{pmatrix}.$$

From the relations $XT = \tilde{T}X$, $TY = Y\tilde{T}$ and $XY = YX = I$, we get

$$X_{n-1\times n-1}T_{n-1\times n-1} = \tilde{T}_{n-1\times n-1}X_{n-1\times n-1}, T_{n-1\times n-1}Y_{n-1\times n-1} = Y_{n-1\times n-1}\tilde{T}_{n-1\times n-1}$$

and

$$X_{n-1\times n-1}Y_{n-1\times n-1} = Y_{n-1\times n-1}X_{n-1\times n-1} = I.$$

Now, to complete the proof by induction, we assume that any invertible operator X intertwining two operators $T, \tilde{T}$ in $\mathcal{F}B_k(\Omega)$ is upper triangular along with its inverse for all $k < n$. Thus the induction hypothesis guarantees that $X_{n-1\times n-1}$ and $Y_{n-1\times n-1}$ must be upper triangular completing the proof.

Employing these techniques, we show that any operator X, not necessarily invertible, in the commutant of $T \in \mathcal{F}B_n(\Omega)$, must be upper triangular.

Proposition (6.2.27)[190]: Suppose T is in $\mathcal{F}B_n(\Omega)$ and X is a bounded linear operator in the commutant of T. Then X is upper triangular.

Proof. The proof is by induction n. To begin the induction, for $n = 2$, following the method of the proof in Proposition (6.2.21), we see that an operator commute with an operator in $\mathcal{F}B_2(\Omega)$ must be upper triangular. Now, assume that any operator commute with an operator in $\mathcal{F}B_k(\Omega)$ is upper triangular for all $k < n$.

Step 1: We claim that $X_{n-1,i} = 0$ for $0 \leq i \leq n-2$. Suppose on contrary this is not true. Then let $l, 0 \leq l \leq n-2$, be the smallest index such that $X_{n-1,l} \neq 0$. For this index l, the commuting relation $XT = TX$ implies that

$$X_{n-1,l}T_l = T_{n-1}X_{n-1,l} \text{ and } \sum_{k=0}^{l} X_{n-1,k}S_{k,l+1} + X_{n-1,i+1}T_{l+1} = T_{n-1}X_{n-1,l+1}. \tag{52}$$

From equation (52), we have

$$X_{n-1,l}S_{l,l+1}S_{1,2}\ldots S_{n-2,n-1} \in ker\,\sigma_{T_{n-1}},$$
$$X_{n-1,l}S_{l,l+1}S_{1,2}\ldots S_{n-2,n-1} = \sigma_{T_{n-1}}\left(X_{n-1,l+1}S_{l+1,l+2}, \ldots S_{n-2,n-1}\right).$$

Therefore $X_{n-1,l}S_{l,l+1}S_{l+1,l+2}\ldots S_{n-2,n-1}$ is in ran $\sigma_{T_{n-1}} \cap ker\,\sigma_{T_{n-1}}$ .Combining Proposition (6.2.6) with Lemma (6.2.19) and Theorem (6.2.20), we conclude that $X_{n-1,l} \neq 0$. This contradicts the assumption $X_{n-1,l} \neq 0$.

Step 2: Write

$$X = \begin{pmatrix} X_{n-1\times n-1} & X_{n-1\times 1} \\ 0 & X_{n-1,n-1} \end{pmatrix}$$

And

$$T = \begin{pmatrix} T_{n-1\times n-1} & T_{n-1\times 1} \\ 0 & T_{n-1,n-1} \end{pmatrix},$$

where meaning of $X_{n-1\times n-1}$ and $T_{n-1\times n-1}$ are same as in Proposition (6.2.26). It follows from the commuting relation $XT = TX$ that

$$X_{n-1\times n-1}T_{n-1\times n-1} = T_{n-1\times n-1}X_{n-1\times n-1}.$$

Now, the induction hypothesis guarantees that $X_{n-1\times n-1}$ must be upper triangular completing the proof.

Finally, we prove a rigidity theorem for the operators in $\mathcal{F}B_n(\Omega)$. In other words, we show that any intertwining unitary between two operators in the class $\mathcal{F}B_n(\Omega)$ must be diagonal. We refer to this phenomenon as "rigidity."

Theorem (6.2.28)[190]: (Rigidity). Any two operators T and $\tilde{T}$ in $\mathcal{F}B_n(\Omega)$ are unitarily equivalent if and only if there exist unitary operators $U_i, 0 \le i \le n-1$, such that $U_iT_i = \tilde{T}_i\square_i$ and $U_iS_{i,j} = \tilde{S}_{i,j}U_j$, $i < j$.

Proof. Clearly, it is enough to prove the necessary part of this statement. Let U be a unitary operator such that $= \tilde{T}U$. By Proposition (6.2.26), both U and $U^* = U^{-1}$ must be upper triangular, that is,

(a) $U = \left(\left(U_{ij}\right)\right)_{i,j=1}^{n}$, $U_{ij} = 0$ whenever $i > j$;

(b) $U^* = \left(\left(U_{ji}^*\right)\right)_{i,j=1}^{n}$, $U_{j,i}^* = 0$ whenever $i > j$.

It follows that the operator U must be diagonal.

We use the rigidity theorem just proved to extract a complete set of unitary invariants for operators in the class $\mathcal{F}B_n(\Omega)$.

Theorem (6.2.29)[190]: Suppose T is an operator in $\mathcal{F}B_n(\Omega)$ and t_{n-1} is a non-vanishing holomorphic section of $E_{T_{n-1}}$. Then

(i) the curvature $\mathcal{K}_{T_{n-1}}$,

(ii) $\frac{\|t_{i-1}\|}{\|t_i\|}$, where $t_{i-1} = S_{i-1,i}(t_i), 1 \le i \le n-1$,

(iii) $\frac{\langle S_{i,j}(t_j),t_i\rangle}{\|t_i\|^2}$, for $0 \le i < j \le n-2$ with $j - i \ge 2$

are a complete set of unitary invariants for the operator T.

Proof. Suppose $T, \tilde{T}$ are in $\mathcal{F}B_n(\Omega)$ and that there is a unitary U such that $= T\widetilde{U}$. Such an intertwining unitary must be diagonal, that is, $U = U_0 \oplus\cdots\oplus U_{n-1}$, for some choice of n unitary operators $U_0, \ldots, U_{n-1}$.

Since $U_iT_i = \tilde{T}_iU_i, 0 \le i \le n-1$, and $U_iS_{i,i+1} = \tilde{S}_{i,i+1}U_{i+1}, 0 \le i \le n-2$, we have

$$U_i(t_i(w)) = \phi(w)\tilde{t}_i(w), 0 \le i \le n-1, \tag{53}$$

where ϕ is some non-zero holomorphic function. Thus

$$\mathcal{K}_{T_{n-1}} = \mathcal{K}_{\tilde{T}_{n-1}} \quad \text{and} \quad \frac{\|t_{i-1}\|}{\|\tilde{t}_{i-1}\|} = \frac{\|t_i\|}{\|\tilde{t}_i\|}, 1 \le i \le n-1.$$

For $0 \le i < j \le n-2$ with $j - i \ge 2$ and $w \in \Omega$, we have

$$\frac{\langle S_{i,j}\left(t_j(w)\right), t_i(w)\rangle}{\|t_i(w)\|^2} = \frac{\langle U_i\left(S_{i,j}\left(t_j(w)\right)\right), U_i\left(t_i(w)\right)\rangle}{\left\|U_i\left(t_i(w)\right)\right\|^2} = \frac{\langle \tilde{S}_{i,j}\left(U_j\left(t_j(w)\right)\right), U_i\left(t_i(w)\right)\rangle}{\left\|U_i\left(t_i(w)\right)\right\|^2}$$

$$= \frac{\langle \tilde{S}_{i,j}\left(\phi(w)\tilde{t}_j(w)\right), \phi(w)\tilde{t}_i(w)\rangle}{\|\phi(w)\tilde{t}_i(w)\|^2} = \frac{\langle \tilde{S}_{i,j}\left(\tilde{t}_j(w)\right), \tilde{t}_i(w)\rangle}{\|\tilde{t}_i(w)\|^2}.$$

Conversely assume that T and $\tilde{T}$ are operators in $\mathcal{F}B_n(\Omega)$ for which these invariants are the same. Equality of the two curvature $\mathcal{K}_{T_{n-1}} = \mathcal{K}_{\tilde{T}_{n-1}}$ together with the equality of the second fundamental forms $\frac{\|t_{i-1}\|}{\|\tilde{t}_{i-1}\|} = \frac{\|t_i\|}{\|\tilde{t}_i\|}, 1 \le i \le n-1$ implies that there exists a non-zero holomorphic function ϕ defined on Ω (if necessary, one may choose a domain $\Omega_0 \subseteq \Omega$ such that ϕ is non-zero on Ω_0) such that

$$\|t_i(w)\| = |\phi(w)|\|\tilde{t}_i(w)\|, \quad 0 \le i \le n-1.$$

For $0 \le i \le n-1$, define $U_i : \mathcal{H}_i \to \widetilde{\mathcal{H}}_i$ by the formula

$$U_i\big(t_i(w)\big) = \phi(w)\tilde{t}_i(w), \quad w \in \Omega,$$

and extend to the linear span of these vectors. For $0 \le i \le n-1$,

$$\|U_i(t_i(w))\| = \|\phi(w)\tilde{t}_i(w)\| = |\phi(w)|\|\tilde{t}_i(w)\| = \|t_i(w)\|.$$

Thus U_i extend to an isometry from $\mathcal{H}_i$ to $\widetilde{\mathcal{H}}_i$. Since U_i is isometric and $U_iT_i = \tilde{T}_iU_i$, it follows, using Proposition (6.2.6), that each U_i is unitary. It is easy to see that $U_iS_{i,i+1} = \tilde{S}_{i,i+1}U_{i+1}$ for $0 \le i \le n-2$ also. For $0 \le i < j \le n-2$ with $j - i \ge 2$ and $w \in \Omega$,

$$\begin{aligned}\langle U_i(S_{i,j}(t_j(w))), U_i(t_i(w))\rangle &= \langle S_{i,j}(t_j(w)), t_i(w)\rangle = \frac{\|t_i(w)\|^2}{\|\tilde{t}_i(w)\|^2}\langle \tilde{S}_{i,j}(\tilde{t}_j(w)), \tilde{t}_i(w)\rangle \\ &= |\phi(w)|^2\langle \tilde{S}_{i,j}(\tilde{t}_j(w)), \tilde{t}_i(w)\rangle = \langle \phi(w)\tilde{S}_{i,j}(\tilde{t}_j(w)), \phi(w)\tilde{t}_i(w)\rangle \\ &= \langle \tilde{S}_{i,j}(\phi(w)\tilde{t}_j(w)), \phi(w)\tilde{t}_i(w)\rangle = \langle \tilde{S}_{i,j}(U_j(t_j(w))), U_i(t_i(w))\rangle.\end{aligned}$$

Polarizing the real analytic functions

$$\langle U_i(S_{i,j}(t_j(w))), U_i(t_i(w))\rangle \text{ and } \langle \tilde{S}_{i,j}(U_j(t_j(w))), U_i(t_i(w))\rangle$$

to functions which are holomorphic in the first and anti-holomorphic in the second variable, we obtain the equality:

$$\langle U_i(S_{i,j}(t_j(z))), U_i(t_i(w))\rangle = \langle \tilde{S}_{i,j}(U_j(t_j(z))), U_i(t_i(w))\rangle, z, w \in \Omega.$$

Hence for w in Ω and $0 \le i < j \le n-2$ with $j - i \ge 2$, we have

$$U_i(S_{i,j}(t_j(w))) = \tilde{S}_{i,j}(U_j(t_j(w)))$$

which implies that

$$U_iS_{i,j} = \tilde{S}_{i,j}U_j\,.$$

Now, setting $U = U_0 \oplus \cdots \oplus U_{n-1}$, we see that U is unitary and $UT = \tilde{T}U$ completing the proof.

Proposition (6.2.30)[190]: If an operator T is in $\mathcal{F}B_n(\Omega)$, then it is irreducible.

Proof. Let P be a projection in the commutant $\{T\}'$ of the operator T. The operator P must therefore be upper triangular by Proposition (6.2.27). It is also a Hermitian idempotent and therefore must be diagonal with projections $P_{ii}, 0 \le i \le n-1$, on the diagonal. We are assuming that $PT = TP$, which gives

$$P_{ii}S_{i,i+1} = S_{i,i+1}P_{i+1i+1}, 0 \le i \le n-2.$$

None of the operators $S_{i,i+1}, 0 \le i \le n-2$, are zero by hypothesis. It follows that $P_{ii} = 0$, if and only if $P_{i+1\,i+1} = 0$. Thus, for any projections $P_{ii} \in \{T_i\}'$, we have only two possibilities:

$$P_{00} = P_{11} = P_{22} = \cdots = P_{n-1n-1} = I, \text{or } P_{00} = P_{11} = P_{22} = \cdots = P_{n-1n-1} = 0.$$

Hence T is irreducible.

The localization of a module at a point of the spectrum is obtained by tensoring with the one dimensional module of evaluation at that point. The localization technique has played a prominent role in the structure theory of modules. More recently, they have found their way into the study of Hilbert modules (cf. [193]). An initial attempt was made in [192]

to see if higher order localizations would be of some use in obtaining invariants for quotient Hilbert modules. Here we give an explicit description of the module tensor products over the polynomial ring in one variable.
There are several different ways in which one may define the action of the polynomial ring on $\mathbb{C}^k$. The following lemma singles out the possibilities for the module action which evaluates a function at w along with a finite number of its derivatives, say $k-1$, at w.
Let f be a polynomial in one variable. Set

$$\mathcal{J}_\mu(f)(z)=\begin{pmatrix} \mu_{1,1}f(z) & 0 & \cdots & \\ \mu_{2,1}\,\partial z\, f(z) & \mu_{2,2}f(z) & \cdots & 0 \\ & & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k,1}\,\partial_{k-1}\,\partial z_{k-1}\, f(z) & \mu_{k-1,1}\,\dfrac{\partial^{k-2}}{\partial z^{k-2}}f(z) & \vdots & \mu_{k,k}f(z) \end{pmatrix}$$

where $\mu=\left(\left(\mu_{i,j}\right)\right)$ is a lower triangular matrix of complex numbers with $\mu_{i,i}=1, 1\le i\le k$.
Lemma (6.2.31)[190]: The following are equivalent.
(i) $\mathcal{J}_\mu(fg)=\mathcal{J}_\mu(f)\mathcal{J}_\mu(g)$.
(ii) $(p+1-j-l)\mu_{p+1-j,l}=\mu_{p+1-j,l+1}\mu_{l+1,l}, 1\le l\le p-2, 1\le j<p-l+1$.
(iii) $\mu_{p,l}\mu_{l,i}=\binom{p-i}{l-i}\mu_{p,i}, 1\le p,l,i\le k, i\le l\le p$.
Proof. All the implications of the Lemma are easy to verify except for one, which we verify here. For $1\le i,j\le k$ and $i\le j$, note that

$$\begin{aligned}\left(\mathcal{J}_\mu(f)(z)\mathcal{J}_\mu(g)(z)\right)_{i,j} &=\sum_{l=0}^{i-j}\mu_{i,j+l}\mu_{j+l,j}\left(\frac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\right)\left(\frac{\partial^l}{\partial z^l}g(z)\right)\\ &=\sum_{l=0}^{i-j}\binom{i-j}{i-j-l}\mu_{i,j}\left(\frac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\right)\left(\frac{\partial^l}{\partial z^l}g(z)\right)\\ &=\mu_{i,j}\sum_{l=0}^{i-j}\binom{i-j}{i-j-l}\left(\frac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\right)\left(\frac{\partial^l}{\partial z^l}g(z)\right)\\ &=\mu_{i,j}\frac{\partial^{i-j}}{\partial z^{i-j}}(fg)(z)\\ &=\left(\mathcal{J}_\mu(fg)(z)\right)_{i,j}.\end{aligned}$$

For $i>j$,

$$\left(\mathcal{J}_\mu(f)(z)\mathcal{J}_\mu(g)(z)\right)_{i,j}=\left(\mathcal{J}_\mu(fg)(z)\right)_{i,j}=0.$$

Hence we have

$$\mathcal{J}_\mu(fg)=\mathcal{J}_\mu(f)\mathcal{J}_\mu(g).$$

For x in $\mathbb{C}^k$, and f in the polynomial ring $P[z]$, define the module action as follows:

$$f\cdot x=\mathcal{J}_\mu(f)(w)x.$$

Suppose $T_0:\mathcal{M}\to\mathcal{M}$ is an operator in $B_1(\Omega)$. Assume that the operator T has been realized as the adjoint of a multiplication operator acting on a Hilbert space of functions possessing a reproducing kernel K. Then the polynomial ring acts on the Hilbert space $\mathcal{M}$ naturally by point-wise multiplication making it a module. We construct a module of k-jets by setting

$$J\mathcal{M} = \left\{\sum_{l=0}^{k-1} \frac{\partial^i}{\partial \square^i} h \otimes \epsilon_{i+1} : h \in \mathcal{M}\right\},$$
where $\epsilon_{i+1}, 0 \le i \le k-1$, are the standard basis vectors in $\mathbb{C}^k$. There is a natural module action on $J\mathcal{M}$, namely,
$$\left(f, \sum_{l=0}^{k-1} \frac{\partial^i}{\partial z^i} h\right) \mapsto \mathcal{J}(f)\left(\sum_{l=0}^{k-1} \frac{\partial^i}{\partial z^i} h \otimes \epsilon_{i+1}\right), f \in P[z], h \in \mathcal{M},$$
Where
$$\mathcal{J}(f)_{i,j} = \begin{cases} \binom{i-1}{j-1} \partial^{i-j} f & \text{if } i \ge j, \\ 0 & \text{otherwise.} \end{cases}$$
The module tensor product $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$ is easily identified with the quotient module $\mathcal{N}^\perp$, where $\mathcal{N} \subseteq \mathcal{M}$ is the sub-module spanned by the vectors
$$\left\{\sum_{l=1}^{k} \left(J_f \cdot h_l \otimes \epsilon_l - h_l \otimes \left(\mathcal{J}_\mu(f)\right)(w) \cdot \epsilon_l\right): h_l \in J\mathcal{M}, \epsilon_l \in \mathbb{C}^k, f \in P[z]\right\}.$$
Following the proof of the Lemma (6.2.32) in [192], we can prove:

Lemma (6.2.32)[190]: The module tensor product $J\mathcal{M} \otimes_{P[z]} \mathbb{C}_w^k$ is spanned by the vector $e_p(w)$ in $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$, where
$$e_p(w) = \sum_{l=1}^{p} b_{p,l} JK(\cdot, w)\epsilon_{p-l+1} \otimes \epsilon_l, 1 \le p \le k$$
and for a fixed p,
$$b_{p,l} = \frac{\mu_{p-j+1,l}}{\binom{p-l}{j-1}} b_{p,p-j+1}, l + j < p + 1.$$
The set of vectors $\{e_p(w) : w \in \Omega^*, 1 \le p \le k\}$ defines a natural holomorphic frame for a vector bundle, say $J_{loc}(\mathcal{E})$. This vector bundle also inherits a Hermitian structure from that of $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$, which furthermore defines a positive definite kernel on $\Omega \times \Omega$:
$$\begin{aligned} J_{loc}K(z,w) &= \left(\left(\langle e_p(w), e_q(z)\rangle\right)\right) \\ &= \sum_{l=1}^{k} D(l) J_{k-l+1} K(z,w) D(l), \end{aligned}$$
where $J_r K(z,w) = \begin{pmatrix} 0_{k-r \times k-r} & 0_{k-r \times r} \\ 0_{r \times k-r} & \tilde{J}_r K(z,w) \end{pmatrix}$ and $D(l)$ is diagonal. Moreover, $D(l)_{m,m} = b_{m+l-1,l}$ and
$$\tilde{J}_r K(z,w) = \begin{pmatrix} K(z,w) & \frac{\partial}{\partial \widetilde{w}} K(z,w) & \cdots & \frac{\partial^{r-1}}{\partial \widetilde{w}^{r-1}} K(z,w) \\ \frac{\partial}{\partial z} K(z,w) & \partial^2/\partial z \partial \widetilde{w} K(z,w) & \cdots & \frac{\partial^r}{\partial z \partial \widetilde{w}^{r-1}} K(z,w) \\ \vdots & \vdots & & \cdots \\ \frac{\partial^{r-1}}{\partial z^{r-1}} & \frac{\partial^r}{\partial z^{r-1} \partial \widetilde{w}} K(z,w) & \ddots \cdots & \frac{\partial^{2r-2}}{\partial z \partial \widetilde{w}^{r-1}} K(z,w) \end{pmatrix}.$$
The two Hilbert spaces $\mathcal{M}$ and $\mathcal{M} \otimes \mathbb{C}^k$ may be identified via the map J_{k-l+1}, which is given by the formula

$$J_{k-l+1}(h) = \sum_{p=0}^{k-l} b_{p+l-1,l} \frac{\partial^p}{\partial z^p} h \otimes \epsilon_{p+l}.$$

Since J_{k-l+1} is injective, we may choose an inner product on $J_{p-l+1}\mathcal{M}$ making it unitary.

Proposition (6.2.33)[190]: [192] The Hilbert module $J_{loc}(\mathcal{M})$ admits a direct sum decomposition of the form $\bigoplus_{l=1}^{k} J_{k-l+1}\mathcal{M}$, and the corresponding reproducing kernel is the sum

$$\sum_{l=1}^{k} D(l)J_{k-l+1}K(z,w)D(l).$$

Let γ_0 be a non-vanishing holomorphic section for the line bundle E corresponding to the operator T_0. Put $b_{1,1}t_0(w) = \gamma_0(w)$ and for $1 \leq l \leq k-1$, let

(i) $t_l(w) := \sum_{i=0}^{k-l-1} b_{l+1+i,l+1} \frac{\partial^i}{\partial z^i} K(\cdot,w) \otimes \epsilon_{l+1+i}$,

(ii) $\gamma_l(w) = \sum_{i=1}^{l+1} b_{l+1,i} \frac{\partial^{l+1-i}}{\partial \bar{w}^{l+1-i}} t_{i-1}(w)$.

Now, $\{\gamma_0, \gamma_1, \cdots, \gamma_{k-1}\}$ are eigenvectors of the operator $M_z^* - \bar{w}$ acting on the Hilbert space $\mathcal{M}_{loc}$.

Since $(M_z^* - \bar{w})\gamma_1(w) = 0$, it follows that $(M_z^* - \bar{w})t_1(w) = -\frac{b_{2,1}}{b_{2,2}} t_0(w)$, which is equivalent to $(M_z^* - \bar{w})t_1(w) = -\mu_{2,1}t_0(w)$.

Suppose $(M_z^* - \bar{w})t_l(w) = -\mu_{l+1,l}t_{l-1}(w)$ for $1 \leq l \leq r$. Again, since $(M_z^* - \bar{w})\gamma_{r+1}(w) = 0$, it follows that

$$(M_z^* - \bar{w})t_{r+1}(w) = \frac{1}{b_{r+2,r+2}} \{(-(r+1)b_{r+2,1}\bar{\partial}^r t_0(w))$$
$$-\sum_{i=2}^{r+1} b_{r+2,i}(-\mu_{i,i-1}\bar{\partial}^{r+2-i}t_{i-2}(w) + (r+2-i)\bar{\partial}^{r+1-i}t_{i-1}(w))\}$$
$$= \frac{1}{b_{r+2,r+2}}\left\{\sum_{i=1}^{r} \left(-(r+2-i)b_{r+2,i} + b_{r+2,i+1}\mu_{i+1,i}\right)\bar{\partial}^{r+1-i}t_{i-1}(w) - b_{r+2,r+1}t_r(w)\right\}$$
$$= \frac{b_{r+2,r+1}}{b_{r+2,r+2}} t_r(w) = \mu_{r+2,r+1}t_r(w).$$

Let $\Gamma := J_k \oplus J_{k-1} \oplus \ldots \oplus J_1$, be the unitary from $\widetilde{\mathcal{M}} := \mathcal{M}_0 \oplus \cdots \mathcal{M}_{k-1}$ to $\mathcal{M}_{loc}$, where each of the summands $\mathcal{M}_0, \ldots, \mathcal{M}_{k-1}$ is equal to $\mathcal{M}$. Let $K_l(\cdot,w) := J_{k-l}^* t_l(w) = K(\cdot,w), 0 \leq l \leq k-1$. Now, we describe the operator $T := \Gamma^* M^* \Gamma$, where M is the multiplication operator on $\mathcal{M}_{loc}$. For $1 \leq l \leq k-1$, set $T_l := P_{\mathcal{M}_l} T|_{\mathcal{M}_l}$ and note that

$$T(K_l(\cdot,w)) = (\Gamma^* M^* \Gamma)K_l(\cdot,w) = \Gamma^* M_z^* t_l(w) = \Gamma^*(\bar{w}t_l(w) + \mu_{l+1,l}t_{l-1}(w))$$
$$= \bar{w}K_l(\cdot,w) + \mu_{l+1,l}K_{l-1}(\cdot,w).$$

Now,

$$T_l\big(K_l(\cdot,w)\big) = P_{\mathcal{M}_l}T|_{\mathcal{M}_l}(K_l(\cdot,w)) = P_{\mathcal{M}_l}T(K_l(\cdot,w))$$
$$= P_{\mathcal{M}_l}(\bar{w}K_l(\cdot,w) + \mu_{l+1,l}K_{l-1}(\cdot,w)) = \bar{w}K_l(\cdot,w).$$

Let $S_{l-1,l} : \mathcal{M}_l \to \mathcal{M}_{l-1}$ be the bounded linear operator defined by the rule $S_{l-1,l}(K_l(\cdot,w)) := \mu_{l+1,l}K_{l-1}(\cdot,w), 1 \leq l \leq k-1$. Since $\mathcal{M}_l = \mathcal{M}_{l-1} = \mathcal{M}$, it follows that $S_{l-1,l} = \mu_{l+1,l}I$. Hence the operator T has the form:

$$T = \begin{pmatrix} T_0 & \mu_{2,1}I & 0 & \cdots & 0 & 0 \\ 0 & T_0 & \mu_{3,2}I & \cdots & 0 & 0 \\ 0 & 0 & T_0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mu_{k-1,k-2}I & 0 \\ 0 & 0 & 0 & \cdots & T_0 & \mu_{k,k-1}I \\ 0 & 0 & 0 & \cdots & 0 & T_0 \end{pmatrix}.$$

Thus T is in $\mathcal{F}B_k(\Omega)$ and defines, up to unitary equivalence via the unitary Γ, the module action in $\mathcal{M}_{loc}$. In consequence, setting $\mathbb{C}_w^k[\mu]$ to be the Hilbert module with the module action induced by $\mathcal{J}_\mu(f)(w)$, we have the following theorem as a direct application of Theorem (6.2.29).

Theorem (6.2.34)[190]: The Hilbert modules corresponding to the localizations $J\mathcal{M} \otimes_{P[z]} \mathbb{C}_w^k[\mu_i], i = 1,2$, are in $\mathcal{F}B_k(\Omega)$ and they are isomorphic if and only if $\mu_1 = \mu_2$.

We attempt to relate the frame of the holomorphic vector bundle E_T ,T in $\mathcal{F}B_n(\Omega)$, to that of the direct sum of the line bundles $E_{T_0} \oplus \cdots \oplus T_{n-1}$.

Let $t = \{t_0, t_1, \ldots, t_{n-1}\}$ be a set of non-vanishing holomorphic sections for the line bundles $E_{T_0}, \ldots, E_{T_{n-1}}$, respectively. Suppose that a suitable linear combination of these non-vanishing sections $t_i, i = 0, \ldots, n-1$, and their derivatives produces a holomorphic frame $\gamma := \{\gamma_0, \ldots, \gamma_{n-1}\}$ for the vector bundle E_T , that is,

$$\gamma_i = t_0^{(i)} + \mu_{1,i} t_1^{(i-1)} + \cdots + \mu_{i-1,i} t_{i-1}^{(1)} + t_i$$

for some choice of non-zero constants $\mu_{1,i}, \ldots, \mu_{i-1,i}, 0 \le i \le k-1$. The existence of such an orthogonal frame is not guaranteed except when $n = 2$. Assuming that it exists, the relationship between these vector bundles can be very mysterious as shown below. This justifies, to some extent, the choice of the smaller class of operators in the next section.

If $\tilde{t}$ is another set of non-vanishing sections for the line bundles $E_{T_1}, \ldots, E_{T_{n-1}}$, then the linear combination of these with exactly the same constants μ_{ij} is a second holomorphic frame, say $\tilde{\gamma}$ of the vector bundle E_T . Let Φ_k be a change of frame between the two sets of non-vanishing orthogonal frames t and $\tilde{t}$, and Ψ_k be a change of frame between γ and $\tilde{\gamma}$. We now describe the relationship between Φ_k and Ψ_k explicitly:

(i) $\Phi_k(i,j) := \phi_{i,j} = \psi_{i,j} := \Psi_k(i,j) = 0, i > j$, that is, Φ_k and Ψ_k are upper-triangular.

(ii) For $0 \le i \le k-1$, we have $\phi_{i,i} = \psi_{i,i} = \phi_{0,0}$, and for $i < k-1$, we have

$$\psi_{i,k-1} = C_{k-1}^i \phi_{0,0}^{(k-1-i)} + \cdots + C_{k-1-j}^i \mu_{j,k-1} \phi_{0,j}^{(k-1-j-i)} + \cdots + \mu_{k-1-i,k-1} \phi_{0,k-1-i},$$

where C_r^n stands for the binomial coefficient $\binom{n}{r}$.

(iii) In particular, for $1 \le i \le k-1$, if we choose $\phi_{0,i}$, then $\psi_{i,k-1} = C_{k-1}^i \phi_{0,0}^{(k-1-i)}$. In this case, we have (a)

$$\Psi_k = \begin{pmatrix} \psi & \psi^1 & \psi^{(2)} & \cdots & \psi^{(k-2)} & \psi^{(k-1)} \\ & \psi & 2\psi^{(1)} & \cdots & C_{k-2}^1 \psi^{(k-3)} & C_{k-1}^1 \psi^{(k-2)} \\ & & \psi & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \psi & C_{k-1}^{\square-2} \psi^{(1)} \\ & & & & & \psi \end{pmatrix};$$

(b) and there are $\frac{(k-2)(k-1)}{2}$ equations in $\frac{(k-1)k}{2}$ variables, namely, $\mu_{i\,j}, 1 \le i < j, j \le k-1$. Thus these coefficients are determined as soon as we make an arbitrary choice of the coefficients $\mu_{1,k-1}, \dots, \mu_{k-2,k-1}$.

We prove the statements (i) and (ii) by induction on k. These statements are valid for $k = 2$ as was noted. To prove their validity for an arbitrary $k \in \mathbb{N}$, assume them to be valid for $k-1$. Let Φ_k^i and Ψ_k^i denote the ith row of Φ and Ψ, respectively.

Suppose that $(\tilde{t}_0, \tilde{t}_1, \cdots, \tilde{t}_k) = (t_0, t_1, \cdots, t_k)\Phi_k$ and $(\tilde{\gamma}_0, \tilde{\gamma}_1, \cdots, \tilde{\gamma}_k) = (\gamma_0, \gamma_1, \cdots, \gamma_k)\Psi_k$. Then we have

$$\tilde{t}_j = (t_0, t_1, \cdots, t_{k-1})\Phi_{k-1}^j + t_k \psi_{k,j}, j < k.$$

For any $i < k$, we have

$$\tilde{\gamma}_i = (\gamma_0, \gamma_1, \cdots, \gamma_{k-1})\Psi_{k-1}^i + \gamma_k \psi_{k,i}$$
$$= (\gamma_0, \gamma_1, \cdots, \gamma_{k-1})\Psi_{k-1}^i + \left(t_0^{(k)} + \mu_{1,k} t_1^{(k-1)} + \cdots + \mu_{i,k} t_i^{(k-i)} + \cdots + t_k\right)\psi_{k,i}$$

And

$$\tilde{\gamma}_i = \tilde{t}_0^{(i)} + \mu_{1,i}\tilde{t}_1^{(i-1)} + \cdots + \mu_{i-1,i}\tilde{t}_{i-1}^{(1)} + \tilde{t}_{i,i} < k.$$

From these equations, it follows that

$$(\gamma_0, \gamma_1, \cdots, \gamma_{k-1})\Psi_{k-1}^i + \left(t_0^{(k)} + \mu_{1,k} t_1^{(k-1)} + \cdots + \mu_{i,k} t_i^{(k-i)} + \cdots + t_k\right)\psi_{k,i}$$
$$= \tilde{t}_0^{(i)} + \mu_{1,i}\tilde{t}_1^{(i-1)} + \cdots + \mu_{i-1,i}\tilde{t}_{i-1}^{(1)} + \tilde{t}_i.$$

We note that $\mu_{i,k}\psi_{k,i} t_i^{(k-i)}$ appears only once in this equation to conclude $\psi_{k,i} = 0, i < k$. Comparing the coefficients of t_i on both sides of the equation, we also conclude that $\psi_{k,i} = \phi_{k,i}, i < k$ completing the induction step for the first statement of our claim.

Our assumption that $(\tilde{t}_0, \tilde{t}_1, \cdots, \tilde{t}_k) = (t_0, t_1, \cdots, t_k)\Phi_k$ and $(\tilde{\gamma}_0, \tilde{\gamma}_1, \cdots, \tilde{\gamma}_k) = (\gamma_0, \gamma_1, \cdots, \gamma_k)\Psi_k$ gives

$$\sum_{i=0}^{k} \left(t_0^i + \mu_{1,i} t_1^{(i-1)} + \cdots + \mu_{i-1,i} t_{i-1}^{(1)} + t_i\right)\psi_{i,k} = \sum_{i=0}^{k} \mu_{i,k}\left(t_0\phi_{0,i} + \cdots + t_i\phi_{0,0}\right)^{(k-i)}, i < k.$$

A comparison of the coefficients of $t_0^{(i)}$ leads to

$$\psi_{i,k} = C_k^i \phi_{0,0}^{(k-i)} + \cdots + C_{k-j}^i \mu_{j,k}\phi_{0,j}^{(k-j-i)} + \cdots + \mu_{k-i,k}\phi_{0,k-i}, i < k$$

completing the proof of the second statement. For the third statement, from the equations

$$\sum_{i=0}^{k-1} \left(t_0^i + \mu_{1,i} t_1^{(i-1)} + \cdots + \mu_{i-1,i} t_{i-1}^{(1)} + t_i\right)\psi_{i,k-1}$$
$$= \sum_{i=0}^{k-1} \mu_{i,k-1}\left(t_0\phi_{0,i} + \cdots + t_i\phi_{0,0}\right)^{(k-1-i)}, i < k-1,$$

setting $\phi_{0,i} = 0$, and comparing the coefficients of $t_i, i > 0$, we have that $\phi_{i,k-1} = c_{i,k-1}\phi_{0,0}^{(k-1-i)}$ for some $c_{i,k-1} \in \mathbb{C}$. Putting this back in the equation given above, we obtain $\frac{(k-2)(k-1)}{2}$ equations involving $\frac{(k-1)k}{2}$ coefficients. This completes the proof of the third statement.

Corollary (6.2.35)[209]: Let T_{s-2}^2 be a bounded linear operator of the form $\begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}$. Suppose that the two operators T_{s-1}^2, T_s^2 are in $B_s(\Omega)$. Then the operator T_{s-2}^2 is in $B_{s+1}(\Omega)$.

Proof. Suppose T_{s-1}^2 and T_s^2 are defined on the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Elementary considerations from index theory of Fredholm operators show that the operator T_{s-2}^2 is Fredholm and $ind(T_{s-2}^2) = ind(T_{s-1}^2) + ind(T_s^2)$ (cf. [191]). Therefore, to complete the proof that T_{s-2}^2 is in $B_{s+1}(\Omega)$, all we have to do is prove that the vectors in the kernel $\ker(T_{s-2}^2 - w_{s-2}^2), w_{s-2}^2 \in \Omega$, span the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.
Let γ_{s-1} and t_s be non-vanishing holomorphic sections for the two line bundles E_0 and E_1 corresponding to the operators T_{s-1}^2 and T_s^2, respectively. For each $w_{s-2}^2 \in \Omega$, the operator $T_{s-1}^2 - w_{s-2}^2$ is surjective. Therefore we can find a vector $\alpha_{s-2}(w_{s-2}^2)$ in $\mathcal{H}_0$ such that $(T_{s-1}^2 - w_{s-2}^2)\alpha_{s-2}(w_{s-2}^2) = -S_{s-2}(t_s(w_{s-2}^2)), w_{s-2}^2 \in \Omega.$ Setting $a(w_{s-2}^2) = \alpha_{s-2}(w_{s-2}^2) + t_s(w_{s-2}^2)$, we see that
$$(T_{s-2}^2 - w_{s-2}^2)a(w_{s-2}^2) = 0 = (T_{s-2}^2 - w_{s-2}^2)\gamma_{s-1}(w_{s-2}^2).$$
Thus $\{\gamma_{s-1}(w_{s-2}^2), a(w_{s-2}^2)\} \subseteq \ker(T_{s-2}^2 - w_{s-2}^2)$ for w_{s-2}^2 in Ω. If x is any vector orthogonal to $\ker(T_{s-2}^2 - w_{s-2}^2), w_{s-2}^2 \in \Omega$, then in particular it is orthogonal to the vectors $\gamma_{s-1}(w_{s-2}^2)$ and $a(w_{s-2}^2), w_{s-2}^2 \in \Omega$, forcing it to be the zero vector.
Corollary (6.2.36)[209]: Suppose T_{s-1}^2 and T_s^2 are two operators in $B_s(\Omega)$, and S_{s-2} is a bounded operator intertwining T_{s-1}^2 and T_s^2, that is, $T_{s-1}^2 S_{s-2} = S_{s-2}T_s^2$. Then S_{s-2} is non-zero if and only if range of S_{s-2} is dense if and only if S_{s-2}^* isinjective.
Proof. Let γ_{s-2} be a holomorphic frame of $E_{T_s^2}$. Assume that S_{s-2} is a non-zero operator. The intertwining relationship $T_{s-1}^2 S_{s-2} = S_{s-2}T_s^2$ implies that $S_{s-2} \circ \gamma_{s-2}$ is a section of $E_{T_{s-1}^2}$. Clearly, there exists an open set Ω_0 contained in Ω such that $S_{s-2} \circ \gamma_{s-2}$ is not zero on Ω_0, otherwise S has to be zero. Since $S_{s-2}(\gamma_{s-2})$ is a holomorphic frame of $E_{T_{s-1}^2}$ on Ω_0, it follows that the closure of the linear span of the vectors $\{S_{s-2}(\gamma_{s-2}(w_{s-2}^2)) : w_{s-2}^2 \in \Omega_0\}$ must equal $\mathcal{H}_0$. Hence the range of the operator S_{s-2} is dense.
Corollary (6.2.37)[209]: Suppose T_{s-2}^2 is a bounded linear operator on a Hilbert space $\mathcal{H}$, which is in $B_{s+1}(\Omega)$. Then the following conditions are equivalent.
(i) There exist an orthogonal decomposition $\mathcal{H}_0 \oplus \mathcal{H}_1$ of $\mathcal{H}$ and operators $T_{s-1}^2 : \mathcal{H}_0 \to \mathcal{H}_0, T_s^2 : \mathcal{H}_1 \to \mathcal{H}_1$, and $S_{s-2} : \mathcal{H}_1 \to \mathcal{H}_0$ such that $T_{s-2}^2 = \begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}$, where $T_{s-1}^2, T_s^2 \in B_s(\Omega)$ and $T_{s-1}^2 S_{s-2} = S_{s-2}T_s^2$, that is, $T_{s-2}^2 \in \mathcal{F}B_{s+1}(\Omega)$.
(ii) There exists a holomorphic frame $\{\gamma_{s-1}, \gamma_s\}$ of $E_{T_{s-2}^2}$ such that $\frac{\partial}{\partial w_{s-2}^2} \|\gamma_{s-1}(w_{s-2}^2)\|^2 = \langle \gamma_s(w_{s-2}^2), \gamma_{s-1}(w_{s-2}^2)\rangle$.
(iii) There exists a holomorphic frame $\{\gamma_{s-1}, \gamma_s\}$ of $E_{T_{s-2}^2}$ such that $\gamma_{s-1}(w_{s-2}^2)$ and $\frac{\partial}{\partial w_{s-2}^2} \gamma_{s-1}(w_{s-2}^2) - \gamma_s(w_{s-2}^2)$ are orthogonal for all w_{s-2}^2 in Ω.
Proof. (i) =⇒ (ii): Pick any two non-vanishing holomorphic sections t_{s-1} and t_s for the line bundles $E_{T_{s-1}^2}$ and $E_{T_s^2}$ respectively. Then
$$(T_{s-2}^2 - w_{s-2}^2)t_s(w_{s-2}^2) = (T_s^2 - w_{s-2}^2)t_s(w_{s-2}^2) + S_{s-2}(t_s(w_{s-2}^2))$$
$$= S_{s-2}(t_s(w_{s-2}^2)).$$
Since $T_{s-1}^2 S_{s-2} = S_{s-2}T_s^2$, it induces a bundle map from $E_{T_s^2}$ to $E_{T_{s-1}^2}$, so $S_{s-2}(t_s(w_{s-2}^2)) = \psi(w_{s-2}^2)t_{s-1}(w_{s-2}^2)$ for some holomorphic function ψ defined on Ω. Thus $(T_{s-2}^2 - w_{s-2}^2)t_s(w_{s-2}^2) = \psi(w_{s-2}^2)t_{s-1}(w_{s-2}^2)$. Setting $\gamma_{s-1}(w_{s-2}^2) := \psi(w_{s-2}^2)t_{s-1}(w_{s-2}^2)$ and $\gamma_s(w_{s-2}^2) := \frac{\partial}{\partial w_{s-2}^2} \gamma_{s-1}(w_{s-2}^2) - t_s(w_{s-2}^2)$, we see that $\{\gamma_{s-1}(w_{s-2}^2), \gamma_s(w_{s-2}^2)\} \subset \ker(T_{s-2}^2 - w_{s-2}^2)$. Now assume that

$$\alpha_{s-1}\gamma_{s-1}(w_{s-2}^2) + \alpha_s\gamma_s(w_{s-2}^2) = 0 \tag{54}$$

for a pair of complex numbers α_{s-1} and α_s. Then

$$\begin{aligned} 0 &= \langle \alpha_{s-1}\gamma_{s-1}(w_{s-2}^2) + \alpha_s\gamma_s(w_{s-2}^2), t_s(w_{s-2}^2)\rangle \\ &= \alpha_s\langle\gamma_s(w_{s-2}^2), t_s(w_{s-2}^2)\rangle = -\alpha_s\|t_s(w_{s-2}^2)\|^2. \end{aligned} \tag{55}$$

From equations (54) and (55), it follows that $\alpha_{s-1} = \alpha_s = 0$. Thus $\{\gamma_{s-1}, \gamma_s\}$ is a holomorphic frame of $E_{T_{s-2}^2}$. Since $\langle t_s(w_{s-2}^2), \gamma_{s-1}(w_{s-2}^2)\rangle = 0$, we see that

$$\frac{\partial}{\partial w_{s-2}^2}\|\gamma_{s-1}(w_{s-2}^2)\|^2 = \langle\gamma_s(w_{s-2}^2), \gamma_{s-1}(w_{s-2}^2)\rangle.$$

(ii) $\Longleftrightarrow$ (iii): This equivalence is evident from the definition.

(iii) $\Longrightarrow$ (i): Set $t_s(w_{s-2}^2) := \frac{\partial}{\partial w_{s-2}^2}\gamma_{s-1}(w_{s-2}^2) - \gamma_s(w_{s-2}^2)$. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be the closed linear span of $\{\gamma_{s-1}(w_{s-2}^2) : w_{s-2}^2 \in \Omega\}$ and $\{t_s(w_{s-2}^2) : w_{s-2}^2 \in \Omega\}$, respectively. Set $T_{s-1}^2 = T_{s-2}^2|_{\mathcal{H}_0}$, $T_s^2 = (P_{s-2})_{\mathcal{H}_1} T_{s-2}^2|_{\mathcal{H}_1}$ and $S_{s-2} = (P_{s-2})_{\mathcal{H}_0} T_{s-2}^2|_{\mathcal{H}_1}$.

We see that the closed linear span of the vectors $\{\gamma_{s-1}(w_{s-2}^2), t_s(w_{s-2}^2) : w_{s-2}^2 \in \Omega\}$ is $\mathcal{H}$: Suppose x in $\mathcal{H}$ is orthogonal to this set of vectors. Then clearly, $x \perp \gamma_{s-1}(w_{s-2}^2)$ and $x \perp t_s(w_{s-2}^2)$ for all w_{s-2}^2 in Ω. Or, equivalently $x \perp \gamma_{s-1}(w_{s-2}^2)$ and $x \perp \gamma_s(w_{s-2}^2)$ for all w_{s-2}^2 in Ω. Therefore x must be the 0 vector. Next, we show that the two operators T_{s-1}^2 and T_s^2 are in $B_s(\Omega)$.

Clearly, $(T_s^2 - w_{s-2}^2)$ is onto. Thus index $(T_s^2 - w_{s-2}^2) = \dim\ker(T_s^2 - w_{s-2}^2)$ and $2 = \text{index}\,(T_{s-2}^2 - w_{s-2}^2) = \text{index}\,(T_{s-1}^2 - w_{s-2}^2) + \text{index}\,(T_s^2 - w_{s-2}^2)$. It follows that $\dim\ker(T_s^2 - w_{s-2}^2) = 1$ or 2.

Suppose $\dim\ker(T_s^2 - w_{s-2}^2) = 2$ and $\{s_s(w_{s-2}^2), s_2(w_{s-2}^2)\}$ be a holomorphic choice of linearly independent vectors in $\ker(T_s^2 - w_{s-2}^2)$. Then we can find holomorphic functions ϕ_s, ϕ_{s+1} defined on Ω such that $S_{s-2}(s_s(w_{s-2}^2)) = \phi_s(w_{s-2}^2)\gamma_{s-1}(w_{s-2}^2)$ and $S_{s-2}\big(s_{s+1}(w_{s-2}^2)\big) = \phi_{s+1}(w_{s-2}^2)\gamma_{s-1}(w^2)$. Setting

$$\tilde{\gamma}_{s-1}(w_{s-2}^2) := \gamma_{s-1}(w_{s-2}^2),$$

$$\tilde{\gamma}_s(w_{s-2}^2) := \frac{\partial}{\partial w_{s-2}^2}\left(\phi_s(w_{s-2}^2)\gamma_{s-1}(w_{s-2}^2)\right) - s_s(w_{s-2}^2)$$

and

$$\tilde{\gamma}_{s+1}(w_{s-2}^2) := \frac{\partial}{\partial w_{s-2}^2}\left(\phi_{s+1}(w_{s-2}^2)\gamma_{s-1}(w_{s-2}^2)\right) - s_{s+1}(w_{s-2}^2),$$

we see that $(T_{s-2}^2 - w_{s-2}^2)(\tilde{\gamma}_i(w_{s-2}^2)) = 0$ for $s-1 \le i \le s+1$. If $\sum_{i=s-1}^{s+1} \alpha_i\tilde{\gamma}_i(w_{s-2}^2) = 0, \alpha_i \in \mathbb{C}$, then

$$\begin{aligned} \alpha_{s-1}\gamma_{s-1}(w_{s-2}^2) + \frac{\partial}{\partial w_{s-2}^2}\Big(\big(\alpha_s\phi_s(w_{s-2}^2) + \alpha_{s+1}\phi_{s+1}(w_{s-2}^2)\big)\gamma_{s-1}(w_{s-2}^2)\Big) + \alpha_s s_s(w_{s-2}^2) \\ + \alpha_{s+1}s_{s+1}(w_{s-2}^2) = 0. \end{aligned}$$

It follows that $\alpha_s s_s(w_{s-2}^2) + \alpha_{s+1}s_{s+1}(w_{s-2}^2) = 0$ since $\mathcal{H}_0$ is orthogonal to $\mathcal{H}_1$. Hence $\alpha_s = \alpha_{s+1} = 0$ implying $\alpha_{s-1} = 0$. Thus we have $\dim\ker(T_{s-2}^2 - w_{s-2}^2) \ge 3$. This contradiction proves that $\dim\ker(T_{s-1}^2 - w_{s-2}^2) = 1$ and hence T_s^2 is in $B_s(\Omega)$.

To show that T_{s-1}^2 is in $B_s(\Omega)$, pick any $x \in \mathcal{H}_0$, and note that there exists $z \in \mathcal{H}$ such that $(T_{s-2}^2 - w_{s-2}^2)z = x$ since $T_{s-2}^2 - w_{s-2}^2$ is onto. Let $z_{\mathcal{H}_1}$ and $z_{\mathcal{H}_0}$ be the projections of z to the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. We have $[(T_{s-1}^2 - w_{s-2}^2)z_{\mathcal{H}_0} + S_{s-2}(z_{\mathcal{H}_1})] + (T_s^2 - w_{s-2}^2)z_{\mathcal{H}_1} = x$. Therefore $(T_s^2 - w_{s-2}^2)z_{\mathcal{H}_1} = 0$ and $(T_{s-1}^2 -$

$w_{s-2}^2)z_{\mathcal{H}_0} + S_{s-2}(z_{\mathcal{H}_1}) = x$. Since $\dim\ker(T_s^2 - w_{s-2}^2) = 1$, so $z_{\mathcal{H}_1} = c_s t_s(w_{s-2}^2)$, it follows that

$$\begin{aligned} x &= (T_{s-1}^2 - w_{s-2}^2)z_{\mathcal{H}_0} + S_{s-2}(z_{\mathcal{H}_1}) \\ &= (T_{s-1}^2 - w_{s-2}^2)z_{\mathcal{H}_0} + S_{s-2}(c_s t_s(w_{s-2}^2)) \\ &= (T_{s-1}^2 - w_{s-2}^2)z_{\mathcal{H}_0} + c_s\gamma_{s-1}(w_{s-2}^2) \\ &= (T_{s-1}^2 - w_{s-2}^2)z_{\mathcal{H}_0} + (T_{s-1}^2 - w_{s-2}^2)(c_s \frac{\partial}{\partial w_{s-2}^2}\gamma_{s-1}(w_{s-2}^2)) \\ &= ((T_{s-1}^2 - w_{s-2}^2)(z_{\mathcal{H}_0} + c_s\frac{\partial}{\partial w_{s-2}^2}\gamma_{s-1}(w_{s-2}^2)). \end{aligned}$$

Thus $T_{s-1}^2 - w_{s-2}^2$ is onto. We have $2 = \dim\ker(T_{s-2}^2 - w_{s-2}^2) = \dim\ker(T_{s-1}^2 - w_{s-2}^2) + \dim\ker(T_s^2 - w_{s-2}^2)$. Hence $\dim\ker(T_{s-1}^2 - w_{s-2}^2) = 1$ and we see that T_{s-1}^2 is in $B_s(\Omega)$.

Finally, since $S_{s-2}(t_s(w_{s-2}^2)) = \gamma_{s-1}(w_{s-2}^2)$, it follows that $T_{s-1}^2 S_{s-2} = S_{s-2}T_s^2$.

Corollary (6.2.38)[209]: Let T_{s-2}^2 be an operator in $\mathcal{F}B_{s+1}(\Omega)$. Suppose $\{\gamma_{s-1},\gamma_s\},\{\tilde{\gamma}_{s-1},\tilde{\gamma}_s\}$ are two frames of the vector bundle $E_{T_{s-2}^2}$ such that $\gamma_{s-1}(w_{s-2}^2) \perp (\frac{\partial}{\partial w_{s-2}^2}\gamma_{s-1}(w_{s-2}^2) - \gamma_s(w_{s-2}^2))$ and $\tilde{\gamma}_{s-1}(w_{s-2}^2) \perp (\frac{\partial}{\partial w_{s-2}^2}\tilde{\gamma}_{s-1}(w_{s-2}^2) - \tilde{\gamma}_s(w_{s-2}^2))$ for all $w_{s-2}^2 \in \Omega$. If $\phi_{s-2} = \begin{pmatrix} \phi_{ss} & \phi_{s\,s+1} \\ \phi_{s+1\,s} & \phi_{s+1\,s+1} \end{pmatrix}$ is any change of frame between $\{\gamma_{s-1},\gamma_s\}$ and $\{\tilde{\gamma}_{s-1},\tilde{\gamma}_s\}$, that is,

$$\{\tilde{\gamma}_{s-1},\tilde{\gamma}_s\} = \{\gamma_{s-1},\gamma_s\}\begin{pmatrix} \phi_{ss} & \phi_{s\,s+1} \\ \phi_{s+1\,s} & \phi_{s+1\,s+1} \end{pmatrix},$$

then $\phi_{s+1\,s} = 0, \phi_{ss} = \phi_{s+1\,s+1}$ and $\phi_{s\,s+1} = \phi_{ss}'$.

Proof. Define the unitary map Γ, as above, using the holomorphic frame $\gamma_{s-2} = \{\gamma_{s-1},\gamma_s\}$. The operator T_{s-2}^2 is then unitarily equivalent to the adjoint of the multiplication operator on the Hilbert space $\mathcal{H}_\Gamma$ possessing a reproducing kernel $(K_{s-2})_\Gamma$ of the form (24). Let e_s and e_{s+1} be the standard unit vectors in $\mathbb{C}^2$. Clearly, $((K_{s-2})_\Gamma)_{w_{s-2}^2}(\cdot)e_s$ and $((K_{s-2})_\Gamma)_{w_{s-2}^2}(\cdot)e_{s+1}$ are two linearly independent eigenvectors of M^* with eigenvalue $\bar{w}_{s-2}^2$.

Similarly, corresponding to the holomorphic frame $\tilde{\gamma}_{s-2} = \{\tilde{\gamma}_{s-1},\tilde{\gamma}_s\}$, the square operator T_{s-2}^2 is unitarily equivalent to the adjoint of multiplication operator on the Hilbert space $\mathcal{H}_{\tilde{\Gamma}}$.

The reproducing kernel $(K_{s-2})_{\tilde{\Gamma}}$ is again of the form (24) except that K_{s-1} and K_s must be replaced by $\tilde{K}_{s-1}$ and $\tilde{K}_s$, respectively.

For $i = s-1, s$, set $s_i(w_{s-2}^2) := ((K_{s-2})_\Gamma)(w_{s-2}^2)e_i$, and $\tilde{s}_i(w_{s-2}^2) := ((K_{s-2})_{\tilde{\Gamma}})(w_{s-2}^2)e_i$. Let $\phi_{s-2}(w_{s-2}^2) := \begin{pmatrix} \phi_{s-1\,s-1}(w_{s-2}^2) & \phi_{s-1\,s}(w_{s-2}^2) \\ \phi_{s\,s-1}(w_{s-2}^2) & \phi_{ss}(w_{s-2}^2) \end{pmatrix}$ be the holomorphic function, taking values in 2×2 matrices, such that

$$(\tilde{s}_{s-1}(w_{s-2}^2), \tilde{s}_s(w_{s-2}^2)) = \big(s_{s-1}(w_{s-2}^2), s_s(w_{s-2}^2)\big)\phi_{s-2}(w_{s-2}^2).$$

This implies that

$$\tilde{s}_{s-1}(w_{s-2}^2) = \phi_{s-1\,s-1}(w_{s-2}^2)s_{s-1}(w_{s-2}^2) + \phi_{s\,s-1}(w_{s-2}^2)s_s(w_{s-2}^2) \tag{56}$$

and

$$\tilde{s}_s(w_{s-2}^2) = \phi_{s-1\,s}(w_{s-2}^2)s_{s-1}(w_{s-2}^2) + \phi_{ss}(w_{s-2}^2)s_s(w_{s-2}^2). \tag{57}$$

From Equation (56), equating the first and the second coordinates separately, we have

$$\left(\widetilde{K}_{s-1}\right)w_{s-2}^2(\cdot)$$
$$= \phi_{s-1\,s-1}(w_{s-2}^2)(K_{s-1})_{w_{s-2}^2}(\cdot) + \phi_{s\,s-1}(w_{s-2}^2)\frac{\partial}{\partial \overline{w}_{s-2}^2}(K_{s-1})_{w_{s-2}^2}(\cdot) \quad (58)$$
and
$$\frac{\partial}{\partial z}\left(\widetilde{K}_{s-1}\right)_{w_{s-2}^2}(\cdot)$$
$$= \phi_{s-1\,s-1}(w_{s-2}^2)\frac{\partial}{\partial z}(K_{s-1})_{w_{s-2}^2}(\cdot) + \phi_{s\,s-1}(w_{s-2}^2)\frac{\partial^2}{\partial z\partial \overline{w}_{s-2}^2}(K_{s-1})_{w_{s-2}^2}(\cdot)$$
$$+\phi_{s\,s-1}(w_{s-2}^2)(K_s)_{w_{s-2}^2}(\cdot). \quad (59)$$
From these two equations, we get
$$\phi_{s-1\,s-1}(w_{s-2}^2)\ \frac{\partial}{\partial z}(K_{s-1})_{w_{s-2}^2}(\cdot) + \phi_{s\,s-1}(w_{s-2}^2)\frac{\partial^2}{\partial z\partial \overline{w}_{s-2}^2}(K_{s-1})_{w_{s-2}^2}(\cdot)\ =$$
$$\phi_{s-1\,s-1}(w_{s-2}^2)\frac{\partial}{\partial z}\ (K_{s-1})_{w_{s-2}^2}(\cdot) + \phi_{s\,s-1}(w_{s-2}^2)\frac{\partial^2}{\partial z\partial \overline{w}_{s-2}^2}(K_{s-1})_{w_{s-2}^2}(\cdot)$$
$$+\ \phi_{s\,s-1}(w_{s-2}^2)(K_s)_{w_{s-2}^2}(\cdot),$$
which implies that $\phi_{s\,s-1}\ =\ 0$. Finally, from Equation (57), we have
$$\frac{\partial}{\partial \overline{w}_{s-2}^2}\left(\widetilde{K}_{s-1}\right)_{w_{s-2}^2}(\cdot)$$
$$= \phi_{s-1\,s}(w_{s-2}^2)(K_{s-1})_{w_{s-2}^2}(\cdot) + \phi_{ss}(w_{s-2}^2)\frac{\partial}{\partial \overline{w}_{s-2}^2}(K_{s-1})_{w_{s-2}^2}(\cdot). \quad (60)$$
The Equations (57) and (60) together give
$$\phi_{s-1\,s} = \phi'_{s-1\,s-1}\ \text{ and }\ \phi_{s-1\,s-1} = \phi_{ss}$$
completing the proof.

Corollary (6.2.39)[209]: Let $T_{s-2}^2, \tilde{T}_{s-2}^2 \in \mathcal{F}B_{s+1}(\Omega)$ be two operators of the form $\begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}$ and $\begin{pmatrix} \tilde{T}_{s-1}^2 & \tilde{S}_{s-2} \\ 0 & \tilde{T}_s^2 \end{pmatrix}$ with respect to the decomposition $\mathcal{H}\ =\ \mathcal{H}_0 \oplus\ \mathcal{H}_s$ and $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1$, respectively. Let $U_{s-2} = \begin{pmatrix} U_{ss} & U_{s\,s+1} \\ U_{s+1\,s} & U_{s+1\,s+1} \end{pmatrix}:\ \mathcal{H}_0\ \oplus\ \mathcal{H}_1\ \to \widetilde{\mathcal{H}}_0\ \oplus \widetilde{\mathcal{H}}_1$ be a unitary operator such that
$$\begin{pmatrix} U_{ss} & U_{s\,s+1} \\ U_{s+1\,s} & U_{s+1\,s+1} \end{pmatrix}\begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix} = \begin{pmatrix} \tilde{T}_{s-1}^2 & \tilde{S}_{s-2} \\ 0 & \tilde{T}_s^2 \end{pmatrix}\begin{pmatrix} U_{ss} & U_{s\,s+1} \\ U_{s+1\,s} & U_{s+1\,s+1} \end{pmatrix},$$
then $U_{s\,s+1}\ =\ U_{s+1\,s}\ =\ 0$.

Proof. Let $\{\gamma_{s-1}, \gamma_s\}$ and $\{\tilde{\gamma}_{s-1}, \tilde{\gamma}_s\}$ be holomorphic frames of $E_{T_{s-2}^2}$ and $E_{\tilde{T}_{s-2}^2}$ respectively with the property that $\gamma_{s-1}\ \perp\ (\frac{\partial}{\partial w_{s-2}^2}\ \gamma_{s-1}\ -\ \gamma_s)$ and $\tilde{\gamma}_{s-1} \perp (\frac{\partial}{\partial w_{s-2}^2}\ \tilde{\gamma}_{s-1}\ -\ \tilde{\gamma}_s)$. Set $t_s := (\frac{\partial}{\partial w_{s-2}^2}\ \gamma_{s-1}\ -\ \gamma_s)$ and $\tilde{t}_s := (\frac{\partial}{\partial w_{s-2}^2}\ \tilde{\gamma}_{s-1}\ -\ \tilde{\gamma}_s)$. Since U_{s-2} intertwines T_{s-2}^2 and $\tilde{T}_{s-2}^2$, it follows that $\{U_{s-2}\gamma_{s-1}\ , U_{s-2}\gamma_s\}$ is a second holomorphic frame of $E_{\tilde{T}_{s-2}^2}$ with the property $U_{s-2}\gamma_{s-1}\ \perp\ (\frac{\partial}{\partial w_{s-2}^2}\ (U_{s-2}\gamma_{s-1})\ -\ U_{s-2}\gamma_s)\ =\ U_{s-2}(t_s)$. By Corollary (6.2.38), we have that
$$U_{s-2}(\gamma_{s-1})\ = \phi_{s-2}\tilde{\gamma}_{s-1} \quad (61)$$
And
$$U_{s-2}(\gamma_s)\ = \phi'_{s-2}\tilde{\gamma}_{s-1}\ + \phi_{s-2}\tilde{\gamma}_s. \quad (62)$$
From equations (61) and (62), we get

$$U_{s-2}(t_s) = \phi_{s-2}\tilde{t}_s. \tag{63}$$

From equations (61) and (63), it follows that U_{s-2} maps $\mathcal{H}_0$ to $\mathcal{H}_0$ and $\mathcal{H}_1$ to $\mathcal{H}_1$. Thus U_{s-2} is a block diagonal from $\mathcal{H}_0 \oplus \mathcal{H}_1$ onto $\widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1$.

Corollary (6.2.40)[209]: For $i = s-1, s$, let T_i^2 be any two operators in $B_s(\Omega)$. Let S_{s-2} and $\tilde{S}_{s-2}$ be bounded linear operators such that $T_{s-1}^2 S_{s-2} = S_{s-2}T_s^2$ and $T_{s-1}^2\tilde{S}_{s-2} = \tilde{S}_{s-2}T_s^2$. If $T^2 = \begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}$ and $\tilde{T}_{s-2}^2 = \begin{pmatrix} T_{s-1}^2 & \tilde{S}_{s-2} \\ 0 & T_s^2 \end{pmatrix}$, then T_{s-2}^2 is unitarily equivalent to $\tilde{T}_{s-2}^2$ if and only if $\tilde{S}_{s-2} = e_{s-2}^{i\theta_{s-2}} S_{s-2}$ for some real number θ_{s-2}.

Proof. Suppose that $U_{s-2}T_{s-2}^2 = \tilde{T}_{s-2}^2 U_{s-2}$ for some unitary operator U_{s-2}. We have just shown that such an operator U_{s-2} must be diagonal, say $U_{s-2} = \begin{pmatrix} U_{ss} & 0 \\ 0 & U_{s+1\,s+1} \end{pmatrix}$. Hence we have

$$U_{ss}T_{s-1}^2 = T_{s-1}^2 U_{ss}, U_{s+1\,s+1}T_s^2 = T_s^2 U_{s+1\,s+1}, U_{ss}S_{s-2} = \tilde{S}_{s-2}U_{s+1\,s+1}. \tag{64}$$

Since U_{ss} is unitary, the first of the equations (64) implies that

$$U_{ss} \in \{T_{s-1}^2, T_{s-1}^{2*}\} := \{W \in \mathcal{L}(\mathcal{H}_0) : WT_{s-1}^2 = T_{s-1}^2 W \text{ and } WT_{s-1}^{2*} = T_{s-1}^{2*}W\}.$$

Since T_{s-1}^2 is an irreducible operator, we conclude that $U_{ss} = e_{s-2}^{i\theta_s} I_{\mathcal{H}_0}$ for some $\theta_s \in \mathbb{R}$. Similarly, $U_{s+1\,s+1} = e_{s-2}^{i\theta_{s+1}} I_{\mathcal{H}_1}$ for some $\theta_{s+1} \in \mathbb{R}$. Hence the third equation in (64) implies that $\tilde{S}_{s-2} = e_{s-2}^{i(\theta_s - \theta_{s+1})} S_{s-2}$.

Conversely suppose that $\tilde{S}_{s-2} = e_{s-2}^{i\theta_{s-2}} S_{s-2}$ for some real number θ_{s-2}. Then evidently the operator $U_{s-2} := \begin{pmatrix} \exp\left(i\frac{\theta_{s-2}}{2}\right) I_{\mathcal{H}_0} & 0 \\ 0 & \exp\left(-i\frac{\theta_{s-2}}{2}\right) I_{\mathcal{H}_1} \end{pmatrix}$ is unitary on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $U_{s-2}T_{s-2}^2 = \tilde{T}_{s-2}^2 U_{s-2}$.

Corollary (6.2.41)[209]: Suppose that $T_{s-2}^2 = \begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}$ and $\tilde{T}_{s-2}^2 = \begin{pmatrix} \tilde{T}_{s-1}^2 & \tilde{S}_{s-2} \\ 0 & \tilde{T}_s^2 \end{pmatrix}$ are any two operators in $\mathcal{F}B_{s+1}(\Omega)$. Then the operators T_{s-2}^2 and $\tilde{T}_{s-2}^2$ are unitarily equivalent if and only if $\mathcal{K}_{T_s^2} = \mathcal{K}_{\tilde{T}_s^2}$ (or, $\mathcal{K}_{T_{s-1}^2} = \mathcal{K}_{\tilde{T}_{s-1}^2}$) and $\frac{\|S_{s-2}(t_s)\|^2}{\|t_s\|^2} = \frac{\|\tilde{S}_{s-2}(\tilde{t}_s)\|^2}{\|\tilde{t}_s\|^2}$, where t_s and $\tilde{t}_s$ are non-vanishing holomorphic sections for the vector bundles $E_{T_s^2}$ and $E_{\tilde{T}_s^2}$, respectively.

Proof. On a small open subset of Ω, we can assume that $S_{s-2}(t_s)$ and $\tilde{S}_{s-2}(\tilde{t}_s)$ are holomorphic frames of the bundle $E_{T_{s-1}^2}$ and $E_{\tilde{T}_{s-1}^2}$, respectively. First suppose that $\bar{\partial}\partial log\|S_{s-2}(t_s)\|^2 = \bar{\partial}\partial\, log\left\|\tilde{S}_{s-2}(t_s)\right\|^2$ and $\frac{\|S_{s-2}(t_s)\|^2}{\|t_s\|^2} = \frac{\|\tilde{S}_{s-2}(\tilde{t}_s)\|^2}{\|\tilde{t}_s\|^2}$. Then we claim that T_{s-2}^2 and $\tilde{T}_{s-2}^2$ are unitarily equivalent. The equality of the curvatures, namely, $\bar{\partial}\partial\, log\,\|S_{s-2}(t_s)\|^2 = \bar{\partial}\partial\, log\left\|\tilde{S}_{s-2}(\tilde{t}_s)\right\|^2$ implies that $\|S_{s-2}(t_s)\|^2 = |\phi_{s-2}|^2\left\|\tilde{S}_{s-2}(\tilde{t}_s)\right\|^2$ for some non-vanishing holomorphic function ϕ_{s-2} on Ω. It may be that we have to shrink, without loss of generality, to a smaller open set Ω_0. The second of our assumptions gives $\|t_s\|^2 = |\phi_{s-2}|^2\|\tilde{t}_s\|^2$. Let $\gamma_{s-1}(w_{s-2}^2) := S_{s-2}(t_s(w_{s-2}^2))$ and $\tilde{\gamma}_{s-1}(w_{s-2}^2) := \tilde{S}_{s-2}(\tilde{t}_s(w_{s-2}^2))$; $\gamma_s(w_{s-2}^2) := \frac{\partial}{\partial w_{s-2}^2}\gamma_{s-1}(w_{s-2}^2) - t_s(w_{s-2}^2)$ and $\tilde{\gamma}_s(w_{s-2}^2) := \frac{\partial}{\partial w_{s-2}^2}\tilde{\gamma}_{s-1}(w_{s-2}^2) - \tilde{t}_s(w_{s-2}^2)$. It follows that $\{\gamma_{s-1}, \gamma_s\}$ and $\{\tilde{\gamma}_{s-1}, \tilde{\gamma}_s\}$ are

holomorphic frames of $E_{T_{s-2}^2}$ and $E_{\tilde{T}_{s-2}^2}$, respectively. Define the map $\Phi : E_{T_{s-2}^2} \to E_{\tilde{T}_{s-2}^2}$ as follows:
(i) $\Phi(\gamma_{s-1}(w_{s-2}^2)) = \phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2)$,
(ii) $\Phi(\gamma_s(w_{s-2}^2)) = \phi'_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2) + \phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_s(w_{s-2}^2)$.
Clearly, Φ is holomorphic. Note that

$$\begin{aligned}
&\langle \Phi(\gamma_{s-1}(w_{s-2}^2)), \Phi(\gamma_s(w_{s-2}^2))\rangle \\
&= \langle \phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2), \phi'_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2) + \phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_s(w_{s-2}^2)\rangle \\
&= \langle \phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2), \phi'_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2) + \phi_{s-2}(w_{s-2}^2)(\frac{\partial}{\partial w_{s-2}^2}\tilde{\gamma}_{s-1}(w_{s-2}^2) \\
&\quad - \tilde{t}_s(w_{s-2}^2))\rangle \\
&= \langle \phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2), \frac{\partial}{\partial w_{s-2}^2}(\phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2)) - \phi_{s-2}(w_{s-2}^2)\tilde{t}_s(w_{s-2}^2)\rangle \\
&= \frac{\partial}{\partial \overline{w}_{s-2}^2}\|\phi_{s-2}(w_{s-2}^2)\tilde{\gamma}_{s-1}(w_{s-2}^2)\|^2 \\
&= \frac{\partial}{\partial \overline{w}_{s-2}^2}\|\gamma_{s-1}(w_{s-2}^2)\|^2
\end{aligned}$$

and

$$\begin{aligned}
\langle \gamma_{s-1}(w_{s-2}^2), \gamma_s(w_{s-2}^2)\rangle &= \langle \gamma_{s-1}(w_{s-2}^2), \frac{\partial}{\partial w_{s-2}^2}\gamma_{s-1}(w_{s-2}^2) - t_s(w_{s-2}^2)\rangle \\
&= \frac{\partial}{\partial \overline{w}_{s-2}^2}\|\gamma_{s-1}(w_{s-2}^2)\|^2.
\end{aligned}$$

Hence we have $\langle \Phi(\gamma_{s-1}(w_{s-2}^2)), \Phi(\gamma_s(w_{s-2}^2))\rangle = \langle \gamma_{s-1}(w_{s-2}^2), \gamma_s(w_{s-2}^2)\rangle$. Similarly, $\|\Phi(\gamma_{s-1}(w_{s-2}^2))\| = \|\gamma_{s-1}(w_{s-2}^2)\|$ and $\|\Phi(\gamma_s)\| = \|\gamma_s\|$. Thus $E_{T_{s-2}^2}$ and $E_{\tilde{T}_{s-2}^2}$ are equivalent as holomorphic Hermitian vector bundles. Hence T_{s-2}^2 and $\tilde{T}_{s-2}^2$ are unitarily equivalent by Theorem (6.2.2) of Cowen and Douglas.
Conversely, suppose T_{s-2}^2 and $\tilde{T}_{s-2}^2$ are unitarily equivalent. Let $U_{s-2} : \mathcal{H} \to \widetilde{\mathcal{H}}$ be the unitary map such that $T_{s-2}^2 = \tilde{T}_{s-2}^2 U_{s-2}$. By Corollary (6.2.39), U_{s-2} takes the form $\begin{pmatrix} U_s & 0 \\ 0 & U_{s+1}\end{pmatrix}$ for some pair of unitary operators U_s and U_{s+1}. Hence we have $U_s(S_{s-2}(t_s)) = \phi_s(\tilde{S}_{s-2}(\tilde{t}_s))$ and $U_{s+1}t_s = \phi_{s+1}\tilde{t}_s$.
The intertwining relation $U_sS_{s-2} = \tilde{S}_{s-2}U_{s+1}$ implies that $\phi_s = \phi_{s+1}$. Thus $\mathcal{K}_{T_{s-1}^2} = \mathcal{K}_{\tilde{T}_{s-1}^2}$ and

$$\frac{\|S_{s-2}(t_s)\|^2}{\|t_s\|^2} = \frac{\|U_s(S_{s-2}(t_s))\|^2}{\|U_{s+1}(t_s)\|^2} = \frac{\|\phi_s\tilde{S}_{s-2}(\tilde{t}_s)\|^2}{\|\phi_{s+1}\tilde{t}_s\|^2} = \frac{\|\tilde{S}_{s-2}(\tilde{t}_s)\|^2}{\|\tilde{t}_s\|^2}.$$

This verification completes the proof.
Corollary (6.2.42)[209]: Let T_{s-2}^2 be an operator in $\mathcal{F}B_{s+1}(\mathbb{D})$ and let t_s be a non-vanishing holomorphic section of the bundle E_1 corresponding to the operator T_s^2. For any φ in Möb, set $t_{s,\varphi} = t_s \circ \varphi^{-1}$. The operator T_{s-2}^2 is homogeneous if and only if T_{s-1}^2, T_s^2 are homogeneous and $\frac{\|S_{s-2}(t_{s,\varphi})\|^2}{\|t_{s,\varphi}\|^2} = |(\varphi^{-1})'|^2 \frac{\|S_{s-2}(t_s)\|^2}{\|t_s\|^2}$ for all φ in Möb.
Proof. Using the intertwining property in the class $\mathcal{F}B_{s+1}(D)$, we see that

$$\varphi(T_{s-2}^2) = \begin{pmatrix} \varphi(T_{s-1}^2) & S_{s-2}\varphi'(T_s^2) \\ 0 & \varphi(T_s^2)\end{pmatrix}.$$

Suppose that T_{s-2}^2 is homogeneous, that is, T_{s-2}^2 is unitarily equivalent to $\varphi(T_{s-2}^2)$ for φ in Möb. From Corollary (6.2.41), it follows that T_{s-1}^2 is unitarily equivalent to $\varphi(T_{s-1}^2), T_s^2$ is unitarily equivalent to $\varphi(T_s^2)$ and

$$\frac{\left\|S_{s-2}\varphi'(T_s^2)\big(t_s,\varphi(w_{s-2}^2)\big)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}=\frac{\left\|S_{s-2}\big(t_s(w_{s-2}^2)\big)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}\,. \tag{65}$$

Now, we have

$$\begin{aligned}\frac{\left\|S_{s-2}\;\varphi'(T_s^2)\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}&=\frac{\left\|S_{s-2}\varphi'\big(\varphi^{-1}(w_{s-2}^2)\big)\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}\\ &=\frac{\left|\varphi'\big(\varphi^{-1}(w_{s-2}^2)\big)\right|^2\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}\\ &=\frac{|(\varphi^{-1})'(w_{s-2}^2)|^{-2}\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}\,.\end{aligned} \tag{66}$$

From equations (65) and (66), it follows that

$$\frac{\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}=|(\varphi^{-1})'(w_{s-2}^2)|^2\;\frac{\left\|S_{s-2}\big(t_s(w_{s-2}^2)\big)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}\,. \tag{67}$$

Conversely suppose that T_{s-1}^2, T_s^2 are homogeneous operators and

$$\frac{\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}=|(\varphi^{-1})'\;(w_{s-2}^2)|^2\;\frac{\left\|S_{s-2}\big(t_s(w_{s-2}^2)\big)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}$$

for all φ in Möb. From equations (66), (67) and Corollary (6.2.41), it follows that T_{s-2}^2 is a homogeneous operator. We have (see [190]).

Corollary (6.2.43)[209]: An operator T_{s-2}^2 in $\mathcal{F}B_{s+1}(\mathbb{D})$ is homogeneous if and only if

(i) T_{s-1}^2 and T_s^2 are homogeneous operators;

(ii) $\mathcal{K}_{T_s^2}(w_{s-2}^2)=\mathcal{K}_{T_{s-1}^2}(w_{s-2}^2)+\mathcal{K}_{B_{s-2}^*}(w_{s-2}^2), w_{s-2}^2\in\mathbb{D}$, where B_{s-2} is the forward Bergman shift;

(iii) $S_{s-2}(t_s(w_{s-2}^2))=\alpha_{s-2}\gamma_{s-1}(w_{s-2}^2)$ for some positive real number α_{s-2} and $\|t_s(w_{s-2}^2)\|^2=\frac{1}{\left(1-|w_{s-2}^2|^2\right)^{\lambda^2+2}}\,, \|\gamma_{s-1}(w_{s-2}^2)\|^2=\frac{1}{\left(1-|w_{s-2}^2|^2\right)^{\lambda^2}}\,.$

Proof. Suppose T_{s-2}^2 is a homogeneous operator. Then Corollary (6.2.42) shows that T_{s-1}^2 and T_s^2 are homogeneous operators. We may therefore find non-vanishing holomorphic sections γ_{s-1} and t_s of E_0 and E_1, respectively, such that $\|\gamma_{s-1}(w_{s-2}^2)\|^2=(1-|w_{s-2}^2|^2)^{-\lambda^2}$ and $\|t_s(w_{s-2}^2)\|^2=(1-|w_{s-2}^2|^2)^{-\mu^2}$ for some positive real λ^2 and μ^2. For φ in Möb, set $\gamma_{s-1,\varphi}=\gamma_{s-1}\circ\varphi^{-1}$ and $t_{s,\varphi}=t_s\circ\varphi^{-1}$. Clearly, $\left\|\gamma_{s-1,\varphi}(w_{s-2}^2)\right\|^2=|(\varphi^{-1})'(w_{s-2}^2)|^{-\lambda^2}\|\gamma_{s-1}(w_{s-2}^2)\|^2$ and $\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2=|(\varphi^{-1})'(w_{s-2}^2)|^{-\mu^2}\|t_s(w_{s-2}^2)\|^2$. Let $S_{s-2}(t_s(w_{s-2}^2))=\psi(w_{s-2}^2)\gamma_{s-1}(w_{s-2}^2)$ for some holomorphic function ψ on $\mathbb{D}$. We have $S_{s-2}(t_{s,\varphi}(w_{s-2}^2))=S_{s-2}(t_s(\varphi^{-1}(w_{s-2}^2)))=\psi\big(\varphi^{-1}(w_{s-2}^2)\big)\gamma_{s-1}(\varphi^{-1}(w_{s-2}^2))=\psi\big(\varphi^{-1}(w_{s-2}^2)\big)\gamma_{s-1,\varphi}(w_{s-2}^2)$ and

$$\frac{\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2} = |(\varphi^{-1})'(w_{s-2}^2)|^2 \frac{\left\|S_{s-2}\left(t_s(w_{s-2}^2)\right)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}. \tag{68}$$

Combining these we see that

$$\frac{\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2} = \left|\psi\left(\varphi^{-1}(w_{s-2}^2)\right)\right|^2 \frac{\left\|\left(\gamma_{s-1,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}$$

$$= \left|\psi\left(\varphi^{-1}(w_{s-2}^2)\right)\right|^2 |(\varphi^{-1})'(w_{s-2}^2)|^{\mu^2-\lambda^2} \frac{\left\|\left(\gamma_{s-1}(w_{s-2}^2)\right)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}. \tag{69}$$

From the equations (68) and (69), we get

$$|\psi(w_{s-2}^2)|^2 |(\varphi^{-1})'(w_{s-2}^2)|^{\lambda^2+2-\mu^2} = |\psi(\varphi^{-1}w_{s-2}^2)|^2. \tag{70}$$

Pick $\varphi = \varphi_u$, where $\varphi_u(w_{s-2}^2) = \frac{w_{s-2}^2-u}{1-\bar{u}w_{s-2}^2}$ and put $w_{s-2}^2 = 0$ in the equation (70). Then

$$|\psi(0)|^2(1 - |u|^2)^{\lambda^2+2-\mu^2} = |\psi(u)|^2. \tag{71}$$

If $\psi(0) = 0$ then equation (71) implies that $\psi(u) = 0$ for all $u \in \mathbb{D}$, which makes $S_{s-2} = 0$ leading to a contradiction. Thus $\psi(0) \neq 0$. Taking log and differentiating both sides of the equation (71), we see that

$$(\lambda^2 + 2 - \mu^2) \frac{\partial^2}{\partial u \partial \bar{u}} log(1 - |u|^2) = 0.$$

Hence we conclude that $\mu^2 = \lambda^2 + 2$. Putting $\mu^2 = \lambda^2 + 2$ in the equation (71) we find that ψ must be a constant function. Hence there is a constant α_{s-2} such that $S_{s-2}(t_s(w_{s-2}^2)) = \alpha_{s-2}\gamma_{s-1}(w_{s-2}^2)$ for all $w_{s-2}^2 \in \Omega$. Finally,

$$\mathcal{K}_{T_s^2}(w_{s-2}^2) = \bar{\partial}\partial\, log\|t_s(w_{s-2}^2)\|^2 = \bar{\partial}\partial\, log(1 - |w_{s-2}^2|^2)^{-\mu^2}$$
$$= \bar{\partial}\partial\, log(1 - |w_{s-2}^2|^2)^{-\lambda^2-2}$$
$$= \bar{\partial}\partial log(1 - |w_{s-2}^2|^2)^{-\lambda^2} + \bar{\partial}\partial\, log(1 - |w_{s-2}^2|^2)^{-2}$$
$$= \bar{\partial}\partial log\, \|\gamma_{s-1}(w_{s-2}^2)\|^2 + \bar{\partial}\partial\, log(1 - |w_{s-2}^2|^2)^{-2}$$
$$= \mathcal{K}_{T_{s-1}^2}(w_{s-2}^2) + \mathcal{K}_{B_{s-2}^*}(w_{s-2}^2).$$

Conversely, suppose that conditions (i), (ii) and (iii) are met. We need to show that T_{s-2}^2 is a homogeneous operator. Condition (ii) is equivalent to $\mu^2 = \lambda^2 + 2$. By Corollary (6.2.42), it is sufficient to show that

$$\frac{\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2} = |(\varphi^{-1})'(w_{s-2}^2)|^2 \frac{\left\|S_{s-2}\left(t_s(w_{s-2}^2)\right)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}.$$

However, we have

$$\frac{\left\|S_{s-2}\left(t_{s,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2} = |\alpha_{s-2}|^2 \frac{\left\|\left(\gamma_{s-1,\varphi}(w_{s-2}^2)\right)\right\|^2}{\left\|t_{s,\varphi}(w_{s-2}^2)\right\|^2}$$

$$= |\alpha_{s-2}|^2 |(\varphi^{-1})'(w_{s-2}^2)|^{\mu^2-\lambda^2} \frac{\left\|\left(\gamma_{s-1}(w_{s-2}^2)\right)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}$$

$$= |\alpha_{s-2}|^2 |(\varphi^{-1})'(w_{s-2}^2)|^2 \frac{\left\|\left(\gamma_{s-1}(w_{s-2}^2)\right)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}$$

$$= |(\varphi^{-1})'(w_{s-2}^2)|^2 \frac{\left\|S_{s-2}\left(t_s(w_{s-2}^2)\right)\right\|^2}{\|t_s(w_{s-2}^2)\|^2}.$$

Corollary (6.2.44)[209]: Suppose T_{s-2}^2 is in $B_s(\Omega)$ and X is a quasi-nilpotent operator such that $T_{s-2}^2 X = X T_{s-2}^2$. Then $X = 0$.

Proof. Let γ_{s-2} be a non-vanishing holomorphic section for $E_{T_{s-2}^2}$. Since $T_{s-2}^2 X = X T_{s-2}^2$, we see that $X(\gamma_{s-2})$ is also a holomorphic section of $E_{T_{s-2}^2}$. Hence $X(\gamma_{s-2}(w_{s-2}^2)) = \phi_{s-2}(w_{s-2}^2)\gamma_{s-2}(w_{s-2}^2)$ for some holomorphic function ϕ_{s-2} defined on Ω. Clearly, $X^n(\gamma_{s-2}(w_{s-2}^2)) = \phi_{s-2}(w_{s-2}^2)^n \gamma_{s-2}(w_{s-2}^2)$. Now, we have

$$|\phi_{s-2}(w_{s-2}^2)|^n \|\gamma_{s-2}(w_{s-2}^2)\| = \|\phi_{s-2}(w_{s-2}^2)^n \gamma_{s-2}(w_{s-2}^2)\| = \|X^n(\gamma_{s-2}(w_{s-2}^2))\| \leq \|X^n\|\|\gamma_{s-2}(w_{s-2}^2)\|.$$

Thus, for $n \in \mathbb{N}$ and $w_{s-2}^2 \in \Omega$, we have $|\phi_{s-2}(w_{s-2}^2)| \leq \|X^n\|^{1/n}$ implying $\phi_{s-2}(w_{s-2}^2) = 0, w_{s-2}^2 \in \Omega$. Hence $X = 0$.

Corollary (6.2.45)[209]: Any operator T_{s-2}^2 in $\mathcal{F}B_{s+1}(\Omega)$ is irreducible. Also, if $T_{s-2}^2 = \begin{pmatrix} T_{s-1}^2 & I \\ 0 & T_{s-1}^2 \end{pmatrix}$, then it is strongly irreducible.

Proof. Let $P_{s-2} = \left(P_{s+i-1\, s+j-1}\right)_{2\times 2}$ be a projection in the commutant $\{T_{s-2}^2\}'$ of the operator T_{s-2}^2, that is,

$$\begin{pmatrix} P_{ss} & P_{ss+1} \\ P_{s+1\, s} & P_{s+1\, s+1} \end{pmatrix} \begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix} = \begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix} \begin{pmatrix} P_{ss} & P_{ss+1} \\ P_{s+1\, s} & P_{s+1\, s+1} \end{pmatrix}.$$

This equality implies that $P_{ss}T_{s-1}^2 = T_{s-1}^2 P_{ss} + S_{s-2}P_{s+1\, s}, P_{ss}S_{s-2} + P_{s\, s+1}T_s^2 = T_{s-1}^2 P_{s\, s+1} + S_{s-2}P_{s+1\, s+1}, P_{s+1\, s}T_{s-1}^2 = T_s^2 P_{s+1\, s}$ and $P_{s+1\, s}S_{s-2} + P_{s+1\, s+1}T_s^2 = T_s^2 P_{s+1\, s+1}$. Now

$$(P_{s+1\, s}S_{s-2})T_s^2 = P_{s+1\, s}(S_{s-2}T_s^2) = P_{s+1\, s}(T_{s-1}^2 S_{s-2}) = (P_{s+1\, s}T_{s-1}^2)S_{s-2} = T_s^2(P_{s+1\, s}S_{s-2}).$$

Thus $P_{s+1\, s}S_{s-2} \in \ker \sigma_{T_s^2}$. Also note that

$$P_{s+1\, s}S_{s-2} = T_s^2 P_{s+1\, s+1} - P_{s+1\, s+1}T_s^2 = \sigma_{T_s^2}(P_{s+1\, s+1}).$$

Hence $P_{s+1\, s}S_{s-2} \in ran\, \sigma_{T_s^2} \cap \ker \sigma_{T_s^2}$. Thus from Corollary (6.2.44) and Theorem (6.2.20), it follows that $P_{s+1\, s}S_{s-2} = 0$. The operator $P_{s+1\, s}$ must be 0 since S_{s-2} has dense range.

To prove the first statement, we may assume that the operator P_{s-2} is self-adjoint and conclude $P_{s\, s+1}$ is 0 as well. Since both the operators T_{s-1}^2 and T_s^2 are irreducible and the projection P_{s-2} is diagonal, it follows that T_{s-2}^2 must be irreducible.

For the proof of the second statement, note that if P_{s-2} is an idempotent of the form $\begin{pmatrix} P_{ss} & P_{s\, s+1} \\ 0 & P_{s+1\, s+1} \end{pmatrix}$, both P_{ss} and $P_{s+1\, s+1}$ must be idempotents. By our hypothesis, P_{ss} and $P_{s+1\, s+1}$ must also commute with T_{s-1}^2, which is strongly irreducible, hence $P_{ss} = 0$ or I and $P_{s+1\, s+1} = 0$ or I. By using Theorem (6.2.20), we see that if $P_{s-2} = \begin{pmatrix} I & P_{s\, s+1} \\ 0 & 0 \end{pmatrix}$ or $P_{s-2} = \begin{pmatrix} 0 & P_{s\, s+1} \\ 0 & I \end{pmatrix}$, then P_{s-2} does not commute with $\begin{pmatrix} T_{s-1}^2 & I \\ 0 & T_{s-1}^2 \end{pmatrix}$. Thus $P_{s-2} = \begin{pmatrix} I & P_{s\, s+1} \\ 0 & I \end{pmatrix}$ or $P_{s-2} = \begin{pmatrix} 0 & P_{s\, s+1} \\ 0 & 0 \end{pmatrix}$. Now, using the equation $P_{s-2}^2 = P_{s-2}$, we conclude that $P_{s\, s+1}$ must be zero. Thus $P_{s-2} = I$ or $P_{s-2} = 0$.

We now give a sufficient condition for a square operator T_{s-2}^2 in $\mathcal{F}B_{s+1}(\Omega)$ to be strongly irreducible (see [190]).

Corollary (6.2.46)[209]: Let $T_{s-2}^2 = \begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}$ be an operator in $\mathcal{F}B_{s+1}(\Omega)$. If the operator S_{s-2} is invertible, then the operator T_{s-2}^2 is strongly irreducible.

Proof. By our hypothesis, the operator $X = \begin{pmatrix} I & 0 \\ 0 & S_{s-2} \end{pmatrix}$ is invertible. Now

$$XT_{s-2}^2X^{-1} = \begin{pmatrix} I & 0 \\ 0 & S_{s-2} \end{pmatrix}\begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & S_{s-2} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} T_{s-1}^2 & I \\ 0 & S_{s-2}T_s^2S_{s-2}^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} T_{s-1}^2 & I \\ 0 & T_{s-1}^2 \end{pmatrix}.$$

Thus T_{s-2}^2 is similar to a strongly irreducible operator and consequently it is strongly irreducible.

We conclude with a characterization of strong irreducibility in $\mathcal{F}B_{s+1}(\Omega)$ (see [190]).

Corollary (6.2.47)[209]: An operator $T_{s-2}^2 = \begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}$ in $\mathcal{F}B_{s+1}(\Omega)$ is strongly irreducible if and only if $S_{s-2} \notin ran\, \sigma_{T_{s-1}^2, T_s^2}$.

Proof. Let P_{s-2} be an idempotent in the commutant $\{T_{s-2}^2\}'$ of the operator T_{s-2}^2. The proof of the Corollary (6.2.45) shows that P_{s-2} must be upper triangular: $\begin{pmatrix} P_{ss} & P_{s\,s+1} \\ 0 & P_{s+1\,s+1} \end{pmatrix}$. The commutation relation $P_{s-2}T_{s-2}^2 = T_{s-2}^2P_{s-2}$ gives us $P_{ss}T_{s-1}^2 = T_{s-1}^2P_{ss}, P_{s+1\,s+1}T_s^2 = T_s^2P_{s+1\,s+1}$ and

$$P_{ss}S_{s-2} - S_{s-2}P_{s+1\,s+1} = T_{s-1}^2P_{s\,s+1} - P_{s\,s+1}T_s^2. \tag{72}$$

Since $P_{i+1i+1} \in \{T_i^2\}'$ for $s-1 \le i \le s$, it follows that P_{ii} can be either I or 0. If either $P_{ss} = I$ and $P_{s+1\,s+1} = 0$ or $P_{ss} = 0$ and $P_{s+1\,s+1} = I$, then S_{s-2} is in ran $\sigma_{T_{s-1}^2, T_s^2}$ contradicting our assumption. Thus P_{s-2} is of the form $\begin{pmatrix} I & P_{s\,s+1} \\ 0 & I \end{pmatrix}$ or $\begin{pmatrix} 0 & P_{s\,s+1} \\ 0 & 0 \end{pmatrix}$. Since P_{s-2} is an idempotent operator, we must have $P_{s\,s+1} = 0$. Hence T_{s-2}^2 is strongly irreducible.

Assume that the operator S_{s-2} is in ran $\sigma_{T_{s-1}^2, T_s^2}$. In this case, we show that T_{s-2}^2 cannot be strongly irreducible completing the proof. Since $S_{s-2} \in ran\, \sigma_{T_{s-1}^2, T_s^2}$, we can find an operator $P_{s\,s+1}$ such that

$$S_{s-2} = \sigma_{T_{s-1}^2, T_s^2}(P_{s\,s+1}) = T_{s-1}^2P_{s\,s+1} - P_{s\,s+1}T_s^2. \tag{73}$$

The operator $P_{s-2} = \begin{pmatrix} I & P_{s\,s+1} \\ 0 & 0 \end{pmatrix}$ is an idempotent operator. We have

$$\begin{pmatrix} I & P_{s\,s+1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ s-1 & T_s^2 \end{pmatrix} = \begin{pmatrix} T_{s-1}^2 & S_{s-2} + P_{s\,s+1}T_s^2 \\ 0 & 0 \end{pmatrix} \tag{74}$$

and

$$\begin{pmatrix} T_{s-1}^2 & S_{s-2} \\ 0 & T_s^2 \end{pmatrix}\begin{pmatrix} I & P_{s\,s+1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{s-1}^2 & T_{s-1}^2P_{s\,s+1} \\ 0 & 0 \end{pmatrix}. \tag{75}$$

From these equations, we have $P_{s-2}T_{s-2}^2 = T_{s-2}^2P_{s-2}$ proving that the operator T_{s-2}^2 is not strongly irreducible.

Corollary (6.2.48)[209]: Let X be an invertible operator that intertwines two operators in $\mathcal{F}B_{s+n-1}(\Omega)$. Set $Y = X^{-1}$. If $X = \left(\left(X_{i,j}\right)\right)_{n\times n}, Y = \left(\left(Y_{i,j}\right)\right)_{n\times n}$ are the block decompositions of the two operators X and Y, then $X_{n-1,j} = 0, s-1 \leq j \leq s+n-3$, and $Y_{n-1,j} = 0, s-1 \leq j \leq s+n-3$.

Proof. Consider the three possibilities:

(a) $X_{n-1,j} = 0, s-1 \leq j \leq s+n-3$, but $Y_{n-1,j} \neq 0$ for some $s-1 \leq j \leq s+n-3$.

(b) $Y_{n-1,j} = 0, s-1 \leq j \leq s+n-3, X_{n-1,j} \neq 0$ for some $s-1 \leq j \leq s+n-3$.

(c) $X_{n-1,j} \neq 0$ for some $s-1 \leq j \leq s+n-3$ and $Y_{n-1,k} \neq 0$ for some $s-1 \leq k \leq s+n-3$.

In each of these cases, we arrive at a contradiction proving the Lemma.

Case 1: Choose l to be the smallest index such that $Y_{n-1,l} \neq 0$, that is, $Y_{n-1,i} = 0$ for $s-1 \leq i \leq s+l-2$ but $Y_{n-1,l} \neq 0$. For this index l, the intertwining relation $T_{s-2}^2 Y = Y\tilde{T}_{s-2}^2$ implies $T_{s+n-2}^2 Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l^2$. Since $Y_{n-1,l} \neq 0$, it follows from Corollary (6.2.36) that $Y_{n-1,l}$ has dense range. From $XY = I$, we get $X_{n-1,n-1}Y_{n-1,l} = 0$ and $X_{n-1,n-1}Y_{n-1,n-1} = I$. Since $Y_{n-1,l}$ has dense range and $X_{n-1,n-1}Y_{n-1,l} = 0$, we conclude that $X_{n-1,n-1} = 0$. This contradicts the identity: $X_{n-1,n-1}Y_{n-1,n-1} = I$.

Case 2: The contradiction in this case is arrived at exactly in the same manner as in the first case after interchanging the roles of X and Y.

Case 3: Pick j, l to be the smallest indices such that $X_{n-1,j} \neq 0$ and $Y_{n-1,l} \neq 0$. We have that $T_{s-2}^2 = \tilde{T}_{s-2}^2 X$. Consequently,

$$X_{n-1,j}T_j^2 = \tilde{T}_{n-1}^2 X_{n-1,j}, X_{n-1,j}S_{j,j+1} + X_{n-1,j+1}T_{j+1}^2 = \tilde{T}_{n-1}^2 X_{n-1,j+1}. \quad (76)$$

Since $T_k^2 S_{k,k+1} = S_{k,k+1}T_{k+1}^2$ for $k = s, s, s+1, \cdots, s+n-2$, multiplying the second equation in (76) by $S_{j+1,j+2} \cdots S_{n-2,n-1}$, and replacing $T_{j+1}^2 S_{j+1,j+2} \cdots S_{n-2,n-1}$ with $S_{j+1,j+2} \cdots S_{n-2,n-1}T_{n-1}^2$, we have

$$\begin{aligned} X_{n-1,j}S_{j,j+1} \cdots S_{n-2,n-1} + X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}T_{n-1}^2 \\ = \tilde{T}_{n-1}^2 X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}. \end{aligned} \quad (77)$$

We also have $T_{s-2}^2 Y = Y\tilde{T}_{s-2}^2$, which gives us

$$T_{n-1}^2 Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l^2. \quad (78)$$

Now, multiply both sides of the equation (77) by $Y_{n-1,l}$, using the commutation $T_{n-1}^2 Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l^2$, then again multiplying both sides of the resulting equation by $S_{l,l+1} \cdots S_{n-2,n-1}$ and finally using the commutation relations $\tilde{T}_k^2\tilde{S}_{k,k+1} = \tilde{S}_{k,k+1}\tilde{T}_{k+1}^2, s-1 \leq k \leq s+n-2$, we have

$$\begin{aligned} X_{n-1,j}S_{j,j+1} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} \\ + X_{n-1,j+1}S_{j,j+1} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}\tilde{T}_{n-1}^2 \\ = \tilde{T}_{n-1}^2 X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}. \end{aligned} \quad (79)$$

Therefore, we see that

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}$$

is in the range of the operator $\sigma_{\tilde{T}_{n-1}^2}$. Indeed it is also in the kernel of $\tilde{\sigma}_{\tilde{T}_{n-1}^2}$, as is evident from the following string of equalities:

$$\begin{aligned} X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}\tilde{T}_{n-1}^2 \\ = X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{T}_l^2\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} \end{aligned}$$

$$= X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}T_{n-1}^2 Y_{n-1,l}\tilde{S}_{l,l+1}\cdots\tilde{S}_{n-2,n-1}$$
$$= X_{n-1,j}T_j^2 S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots\tilde{S}_{n-2,n-1}$$
$$= \tilde{T}_{n-1}^2 X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots\tilde{S}_{n-2,n-1}.$$

Thus

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots\tilde{S}_{n-2,n-1} \in \ker\sigma_{\tilde{T}_{n-1}^2}\cap ran\,\sigma_{\tilde{T}_{n-1}^2}.$$

Consequently, using Corollary (6.2.44) and Theorem 2.19, we conclude that

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2}\cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1}\cdots\tilde{S}_{n-2,n-1} = 0.$$

By hypothesis, all the operators $S_{k,k+1},\tilde{S}_{k,k+1}, k = s-1, s,\cdots, s+n-3$ have dense range. Since $Y_{n-1,l}\neq 0$, then equation (78) and Corollary (6.2.36) ensure that $Y_{n-1,l}$ has dense range. Hence $X_{n-1,j} = 0$. This contradicts the assumption $X_{n-1,j}\neq 0$.

The following proposition is the first step in the proof of the rigidity theorem (see [190]).

Corollary (6.2.49)[209]: If X is an invertible operator intertwining two operators T_{s-2}^2 and $\tilde{T}_{s-2}^2$ from $\mathcal{F}B_{s+n-1}(\Omega)$, then X and X^{-1} are upper triangular.

Proof. The proof is by induction on n. The validity of the case $n = s+1$, is immediate from Corollary (6.2.48). Let us write the two operators T_{s-2}^2, $\tilde{T}_{s-2}^2$ in the form of 2×2 block matrix:

$$T_{s-2}^2 = \begin{pmatrix} T_{n-1\times n-1}^2 & T_{n-1\times s}^2 \\ 0 & T_{n-1,n-1}^2\end{pmatrix},\qquad \tilde{T}_{s-2}^2 = \begin{pmatrix} \tilde{T}_{n-1\times n-1}^2 & \tilde{T}_{n-1\times s}^2 \\ 0 & \tilde{T}_{n-1,n-1}^2\end{pmatrix}.$$

Using Corollary (6.2.48), the operators X, Y can be written in the form of 2×2 block matrix:

$$X = \begin{pmatrix} X_{n-1\times n-1} & X_{n-1\times 1} \\ 0 & X_{n-1,n-1}\end{pmatrix},\qquad Y = \begin{pmatrix} Y_{n-1\times n-1} & Y_{n-1\times 1} \\ 0 & Y_{n-1,n-1}\end{pmatrix}$$

without loss of generality. Here $X_{n-1\times n-1}$ and $Y_{n-1\times n-1}$ are the operators $\left(\left(X_{i,j}\right)\right)_{i,j=s-1}^{s+n-3}$ and $\left(\left(Y_{i,j}\right)\right)_{i,j=s-1}^{s+n-3}$ respectively and

$$T_{n-1\times n-1}^2 = \begin{pmatrix} T_{s-1}^2 & S_{s-1,s} & S_{s-1,s+1} & \cdots & S_{s-1,n-2} \\ 0 & T_s^2 & S_{s,s+1} & \cdots & S_{s,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-3}^2 & S_{n-3,n-2} \\ 0 & \cdots & \cdots & 0 & T_{n-2}^2\end{pmatrix},$$

$$\tilde{T}_{n-1\times n-1}^2 = \begin{pmatrix} \tilde{T}_{s-1}^2 & \tilde{S}_{s-1,s} & \tilde{S}_{s-1,s+1} & \cdots & \tilde{S}_{s-1,n-2} \\ 0 & \tilde{T}_s^2 & \tilde{S}_{s,s+1} & \cdots & \tilde{S}_{s,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{T}_{n-3}^2 & \tilde{S}_{n-3,n-2} \\ 0 & \cdots & \cdots & 0 & \tilde{T}_{n-2}^2\end{pmatrix}.$$

From the relations $XT_{s-2}^2 = \tilde{T}_{s-2}^2 X$, $T_{s-2}^2 Y = Y\tilde{T}_{s-2}^2$ and $XY = YX = I$, we get $X_{n-1\times n-1}T_{n-1\times n-1}^2 = \tilde{T}_{n-1\times n-1}^2 X_{n-1\times n-1}, T_{n-1\times n-1}^2 Y_{n-1\times n-1} = Y_{n-1\times n-1}\tilde{T}_{n-1\times n-1}^2$ and

$$X_{n-1\times n-1}Y_{n-1\times n-1} = Y_{n-1\times n-1}X_{n-1\times n-1} = I.$$

Now, to complete the proof by induction, we assume that any invertible operator X intertwining two operators $T_{s-2}^2,\tilde{T}_{s-2}^2$ in $\mathcal{F}B_{s+k-1}(\Omega)$ is upper triangular along with its inverse for all $k < n$. Thus the induction hypothesis guarantees that $X_{n-1\times n-1}$ and $Y_{n-1\times n-1}$ must be upper triangular completing the proof.

Corollary (6.2.50)[209]: Suppose T_{s-2}^2 is in $\mathcal{F}B_{s+n-1}(\Omega)$ and X is a bounded linear operator in the commutant of T_{s-2}^2. Then X is upper triangular.

Proof. The proof is by induction n. To begin the induction, for $n = 2$, following the method of the proof in Corollary (6.2.45), we see that an operator commute with an operator in $\mathcal{F}B_{s+1}(\Omega)$ must be upper triangular. Now, assume that any operator commute with an operator in $\mathcal{F}B_{s+k-1}(\Omega)$ is upper triangular for all $k < n$.

Step 1: We claim that $X_{n-1,i} = 0$ for $s-1 \leq i \leq s+n-3$. Suppose on contrary this is not true. Then let $l, s \leq l \leq s+n-3$, be the smallest index such that $X_{n-1,l} \neq 0$. For this index l, the commuting relation $XT_{s-2}^2 = T_{s-2}^2 X$ implies that

$$X_{n-1,l}T_l^2 = T_{n-1}^2 X_{n-1,l} \text{ and } \sum_{k=s}^{l} X_{n-1,k}S_{k,l+1} + X_{n-1,i+1}T_{l+1}^2 = T_{n-1}^2 X_{n-1,l+1}. \tag{80}$$

From equation (80), we have

$$X_{n-1,l}S_{l,l+1}S_{1,2}\ldots S_{n-2,n-1} \in \ker \sigma_{T_{n-1}^2},$$
$$X_{n-1,l}S_{l,l+1}S_{1,2}\ldots S_{n-2,n-1} = \sigma_{T_{n-1}^2}\left(X_{n-1,l+1}S_{l+1,l+2},\ldots S_{n-2,n-1}\right).$$

Therefore $X_{n-1,l}S_{l,l+1}S_{l+1,l+2}\ldots S_{n-2,n-1}$ is in ran $\sigma_{T_{n-1}^2} \cap \ker \sigma_{T_{n-1}^2}$ .Combining Corollary (6.2.36) with Corollary (6.2.44) and Theorem (6.2.20), we conclude that $X_{n-1,l} \neq 0$. This contradicts the assumption $X_{n-1,l} \neq 0$.

Step 2: Write

$$X = \begin{pmatrix} X_{n-1\times n-1} & X_{n-1\times 1} \\ 0 & X_{n-1,n-1} \end{pmatrix}$$

And

$$T_{s-2}^2 = \begin{pmatrix} T_{n-1\times n-1}^2 & T_{n-1\times 1}^2 \\ 0 & T_{n-1,n-1}^2 \end{pmatrix},$$

where meaning of $X_{n-1\times n-1}$ and $T_{n-1\times n-1}^2$ are same as in Corollary (6.2.49). It follows from the commuting relation $XT_{s-2}^2 = T_{s-2}^2 X$ that

$$X_{n-1\times n-1}T_{n-1\times n-1}^2 = T_{n-1\times n-1}^2 X_{n-1\times n-1}.$$

Now, the induction hypothesis guarantees that $X_{n-1\times n-1}$ must be upper triangular completing the proof.

Corollary (6.2.51)[209]: (Rigidity). Any two operators T_{s-2}^2 and $\tilde{T}_{s-2}^2$ in $\mathcal{F}B_{s+n-1}(\Omega)$ are unitarily equivalent if and only if there exist unitary operators $U_i, s-1 \leq i \leq s+n-2$, such that $U_i T_i^2 = \tilde{T}_i^2 U_i$ and $U_i S_{i,j} = \tilde{S}_{i,j} U_j$, $i < j$.

Proof. Clearly, it is enough to prove the necessary part of this statement. Let U be a unitary operator such that $T_{s-2}^2 = \tilde{T}_{s-2}^2 U_{s-2}$. By Corollary (6.2.49), both U_{s-2} and $U_{s-2}^* = U_{s-2}^{-1}$ must be upper triangular, that is,

(a) $U_{s-2} = \left(\left(U_{ij}\right)\right)_{i,j=1}^{n}, U_{ij} = 0$ whenever $i > j$;

(b) $U_{s-2}^* = \left(\left(U_{ji}^*\right)\right)_{i,j=1}^{n}, U_{j,i}^* = 0$ whenever $i > j$.

It follows that the operator U_{s-2} must be diagonal.

Corollary (6.2.52)[209]: Suppose T_{s-2}^2 is an operator in $\mathcal{F}B_{s+n-1}(\Omega)$ and t_{s+n-2} is a non-vanishing holomorphic section of $E_{T_{n-1}^2}$. Then

(i) the curvature $\mathcal{K}_{T_{n-1}^2}$,

(ii) $\frac{\|t_{i-1}\|}{\|t_i\|}$, where $t_{i-1} = S_{i-1,i}(t_i), s \le i \le s+n-2$,

(iii) $\frac{\langle S_{i,j}(t_j), t_i \rangle}{\|t_i\|^2}$, for $s-1 \le\ i\ <\ j\ \le s+n\ -3$ with $j\ -\ i\ \ge s+1$

are a complete set of unitary invariants for the operator T^2_{s-2}.

Proof. Suppose $T^2_{s-2}, \tilde{T}^2_{s-2}$ are in $\mathcal{F}B_{s+n-1}(\Omega)$ and that there is a unitary U_{s-2} such that $T^2_{s-2}\ =\ T^2_{s-2}\widetilde{U}_{s-2}$. Such an intertwining unitary must be diagonal, that is, $U_{s-2}\ =\ U_{s-1}\ \oplus \cdots \oplus\ U_{s+n-2}$, for some choice of n unitary operators $U_{s-1}, \ldots, U_{s+n-2}$.

Since $U_i T_i^2\ =\ \tilde{T}_i^2 U_i, s-1 \le\ i\ \le s+n\ -2$, and $U_i S_{i,i+1} = \tilde{S}_{i,i+1} U_{i+1}, s-1 \le\ i\ \le s+n\ -3$, we have

$$U_i\big(t_i(w^2_{s-2})\big) = \phi_{s-2}(w^2_{s-2})\tilde{t}_i(w^2_{s-2}), s-1 \le i \le s+n\ -3, \tag{81}$$

where ϕ_{s-2} is some non-zero holomorphic function. Thus

$$\mathcal{K}_{T^2_{n-1}} = \mathcal{K}_{\tilde{T}^2_{n-1}} \quad \text{and} \quad \frac{\|t_{i-1}\|}{\|\tilde{t}_{i-1}\|} = \frac{\|t_i\|}{\|\tilde{t}_i\|}, s \le\ i\ \le s+n\ -2.$$

For $s-1 \le i\ <\ j\ \le s+n\ -3$ with $j\ -\ i\ \ge\ s+1$ and $w^2_{s-2}\ \in\ \Omega$, we have

$$\frac{\langle S_{i,j}\left(t_j(w^2_{s-2})\right), t_i(w^2_{s-2})\rangle}{\|t_i(w^2_{s-2})\|^2} = \frac{\langle U_i\left(S_{i,j}\left(t_j(w^2_{s-2})\right)\right), U_i\big(t_i(w^2_{s-2})\big)\rangle}{\left\|U_i\big(t_i(w^2_{s-2})\big)\right\|^2}$$

$$= \frac{\langle \tilde{S}_{i,j}\left(U_j\left(t_j(w^2_{s-2})\right)\right), U_i\big(t_i(w^2_{s-2})\big)\rangle}{\left\|U_i\big(t_i(w^2_{s-2})\big)\right\|^2}$$

$$= \frac{\langle \tilde{S}_{i,j}\left(\phi_{s-2}(w^2_{s-2})\tilde{t}_j(w^2_{s-2})\right), \phi_{s-2}(w^2_{s-2})\tilde{t}_i(w^2_{s-2})\rangle}{\|\phi_{s-2}(w^2_{s-2})\tilde{t}_i(w^2_{s-2})\|^2} = \frac{\langle \tilde{S}_{i,j}\left(\tilde{t}_j(w^2_{s-2})\right), \tilde{t}_i(w^2_{s-2})\rangle}{\|\tilde{t}_i(w^2_{s-2})\|^2}.$$

Conversely assume that T^2_{s-2} and $\tilde{T}^2_{s-2}$ are operators in $\mathcal{F}B_{s+n-1}(\Omega)$ for which these invariants are the same. Equality of the two curvature $\mathcal{K}_{T^2_{n-1}} = \mathcal{K}_{\tilde{T}^2_{n-1}}$ together with the equality of the second fundamental forms $\frac{\|t_{i-1}\|}{\|\tilde{t}_{i-1}\|} = \frac{\|t_i\|}{\|\tilde{t}_i\|}, s \le\ i\ \le s+n\ -2$ implies that there exists a non-zero holomorphic function ϕ_{s-2} defined on Ω (if necessary, one may choose a domain $\Omega_0\ \subseteq\ \Omega$ such that ϕ_{s-2} is non-zero on Ω_0) such that

$$\|t_i(w^2_{s-2})\| = |\phi_{s-2}(w^2_{s-2})|\|\tilde{t}_i(w^2_{s-2})\|, \quad s-1\ \le\ i\ \le s+n\ -2.$$

For $s-1\ \le\ i\ \le s+n\ -2$, define $U_i\ :\ \mathcal{H}_i\ \to \widetilde{\mathcal{H}}_i$ by the formula

$$U_i\big(t_i(w^2_{s-2})\big) = \phi_{s-2}(w^2_{s-2})\tilde{t}_i(w^2_{s-2}), \quad w^2_{s-2}\ \in\ \Omega,$$

and extend to the linear span of these vectors. For $s-1\ \le\ i\ \le s+n\ -2$,

$$\|U_i(t_i(w^2_{s-2}))\|\ =\ \|\phi_{s-2}(w^2_{s-2})\tilde{t}_i(w^2_{s-2})\| =\ |\phi_{s-2}(w^2_{s-2})|\|\tilde{t}_i(w^2_{s-2})\| =\ \|t_i(w^2_{s-2})\|.$$

Thus U_i extend to an isometry from $\mathcal{H}_i$ to $\widetilde{\mathcal{H}}_i$. Since U_i is isometric and $U_i T_i^2\ =\ \tilde{T}_i^2 U_i$, it follows, using Corollary (6.2.36), that each U_i is unitary. It is easy to see that $U_i S_{i,i+1} = \tilde{S}_{i,i+1} U_{i+1}$ for $s-1 \le i \le s+n-3$ also. For $s-1 \le i < j \le s+n\ -3$ with $j\ -\ i\ \ge s+1$ and $w^2_{s-2}\ \in\ \Omega$,

$$\langle U_i(S_{i,j}(t_j(w^2_{s-2}))), U_i(t_i(w^2_{s-2}))\rangle\ =\ \langle S_{i,j}(t_j(w^2_{s-2})), t_i(w^2_{s-2})\rangle$$

$$= \frac{\|t_i(w^2_{s-2})\|^2}{\|\tilde{t}_i(w^2_{s-2})\|^2}\ \langle \tilde{S}_{i,j}(\tilde{t}_j(w^2_{s-2})), \tilde{t}_i(w^2_{s-2})\rangle$$

$$=\ |\phi_{s-2}(w^2_{s-2})|^2 \langle \tilde{S}_{i,j}(\tilde{t}_j(w^2_{s-2})), \tilde{t}_i(w^2_{s-2})\rangle$$

$$= \langle \phi_{s-2}(w^2_{s-2})\tilde{S}_{i,j}(\tilde{t}_j(w^2_{s-2})), \phi_{s-2}(w^2_{s-2})\tilde{t}_i(w^2_{s-2})\rangle$$

$$=\ \langle \tilde{S}_{i,j}(\phi_{s-2}(w^2_{s-2})\tilde{t}_j(w^2_{s-2})), \phi_{s-2}(w^2_{s-2})\tilde{t}_i(w^2_{s-2})\rangle$$

$$= \langle \tilde{S}_{i,j}(U_j(t_j(w_{s-2}^2))), U_i(t_i(w_{s-2}^2))\rangle.$$

Polarizing the real analytic functions $\langle U_i(S_{i,j}(t_j(w_{s-2}^2))), U_i(t_i(w_{s-2}^2))\rangle$ and $\langle \tilde{S}_{i,j}(U_j(t_j(w_{s-2}^2))), U_i(t_i(w_{s-2}^2))\rangle$ to functions which are holomorphic in the first and anti-holomorphic in the second variable, we obtain the equality:

$$\langle U_i(S_{i,j}(t_j(z))), U_i(t_i(w_{s-2}^2))\rangle = \langle \tilde{S}_{i,j}(U_j(t_j(z))), U_i(t_i(w_{s-2}^2))\rangle, z, w_{s-2}^2 \in \Omega.$$

Hence for w in Ω and $s-1 \le i < j \le s+n-3$ with $j - i \ge s+1$, we have

$$U_i(S_{i,j}(t_j(w_{s-2}^2))) = \tilde{S}_{i,j}(U_j(t_j(w_{s-2}^2)))$$

which implies that

$$U_i S_{i,j} = \tilde{S}_{i,j} U_j \,.$$

Now, setting $U_{s-2} = U_{s-1} \oplus \cdots \oplus U_{s+n-2}$, we see that U_{s-2} is unitary and $U_{s-2}T_{s-2}^2 = \tilde{T}_{s-2}^2 U_{s-2}$ completing the proof.

Corollary (6.2.53)[209]: If an operator T_{s-2}^2 is in $\mathcal{F}B_{s+n-1}(\Omega)$, then it is irreducible.

Proof. Let P_{s-2} be a projection in the commutant $\{T_{s-2}^2\}'$ of the operator T_{s-2}^2. The operator P_{s-2} must therefore be upper triangular by Corollary (6.2.50). It is also a Hermitian idempotent and therefore must be diagonal with projections $P_{ii}, s-1 \le i \le s+n-2$, on the diagonal. We are assuming that $P_{s-2}T_{s-2}^2 = T_{s-2}^2 P_{s-2}$, which gives

$$P_{ii}S_{i,i+1} = S_{i,i+1}P_{i+1i+1}, s-1 \le i \le s+n-3.$$

None of the operators $S_{i,i+1}, s-1 \le i \le s+n-3$, are zero by hypothesis. It follows that $P_{ii} = 0$, if and only if $P_{i+1\,i+1} = 0$. Thus, for any projections $P_{ii} \in \{T_i^2\}'$, we have only two possibilities:

$$P_{s-1s-1} = P_{ss} = P_{s+1\,s+1} = \cdots = P_{n-1n-1} = I, \text{or } P_{s-1s-1} = P_{ss} = P_{s+1\,s+1} = \cdots = P_{s+n-2\,s+n-2} = 0.$$

Hence T_{s-2}^2 is irreducible.

Corollary (6.2.54)[209]: The following are equivalent.

(i) $\mathcal{J}_\mu(fg) = \mathcal{J}_\mu(f)\mathcal{J}_\mu(g)$.

(ii) $(p + 1 - j - l)\mu_{p+1-j,l} = \mu_{p+1-j,l+1}\mu_{l+1,l}, 1 \le l \le p-2, 1 \le j < p-l+1$.

(iii) $\mu_{p,l}\mu_{l,i} = \binom{p-i}{l-i} \mu_{p,i}, 1 \le p,l,i \le k, i \le l \le p$.

Proof. All the implications of the corollary are easy to verify except for one, which we verify here. For $1 \le i,j \le k$ and $i \le j$, note that

$$\left(\mathcal{J}_\mu(f)(z)\mathcal{J}_\mu(g)(z)\right)_{i,j} = \sum_{l=s-1}^{i-j} \mu_{i,j+l}\mu_{j+l,j}\left(\frac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\right)\left(\frac{\partial^l}{\partial z^l}\,g(z)\right)$$
$$= \sum_{l=s-1}^{i-j} \binom{i-j}{i-j-l}\mu_{i,j}\left(\frac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\right)\left(\frac{\partial^l}{\partial z^l}g(z)\right)$$
$$= \mu_{i,j}\sum_{l=s-1}^{i-j} \binom{i-j}{i-j-l}\left(\frac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\right)\left(\frac{\partial^l}{\partial z^l}\,g(z)\right) = \mu_{i,j}\,\frac{\partial^{i-j}}{\partial z^{i-j}}(fg)(z)$$
$$= \left(\mathcal{J}_\mu(fg)(z)\right)_{i,j}.$$

For $i > j$,

$$\left(\mathcal{J}_\mu(f)(z)\mathcal{J}_\mu(g)(z)\right)_{i,j} = \left(\mathcal{J}_\mu(fg)(z)\right)_{i,j} = 0.$$

Hence we have

$$\mathcal{J}_\mu(fg) = \mathcal{J}_\mu(f)\mathcal{J}_\mu(g).$$

For x in $\mathbb{C}^k$, and f in the polynomial ring $P_{s-2}[z]$, define the module action as follows:

$$f \cdot x = \mathcal{J}_\mu(f)(w_{s-2}^2)x.$$

Suppose $T_{s-1}^2 : \mathcal{M} \to \mathcal{M}$ is an operator in $B_s(\Omega)$. Assume that the operator T_{s-2}^2 has been realized as the adjoint of a multiplication operator acting on a Hilbert space of functions possessing a reproducing kernel K_{s-2}. Then the polynomial ring acts on the Hilbert space $\mathcal{M}$ naturally by point-wise multiplication making it a module. We construct a module of k-jets by setting

$$J\mathcal{M} = \left\{ \sum_{l=s-1}^{k-1} \frac{\partial^i}{\partial z^i} h \otimes \epsilon_{i+1} : h \in \mathcal{M} \right\},$$

where $\epsilon_{i+1}, s-1 \le i \le s+k-2$, are the standard basis vectors in $\mathbb{C}^k$. There is a natural module action on $J\mathcal{M}$, namely,

$$\left(f, \sum_{l=s-1}^{k-1} \frac{\partial^i}{\partial z^i} h \right) \mapsto \mathcal{J}(f) \left(\sum_{l=s-1}^{k-1} \frac{\partial^i}{\partial z^i} h \otimes \epsilon_{i+1} \right), f \in P_{s-2}[z], h \in \mathcal{M},$$

Where

$$\mathcal{J}(f)_{i,j} = \begin{cases} \binom{i-1}{j-1} \partial^{i-j} f & \text{if } i \ge j, \\ 0 & \text{otherwise.} \end{cases}$$

The module tensor product $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_{w_{s-2}^2}^k$ is easily identified with the quotient module $\mathcal{N}^\perp$, where $\mathcal{N} \subseteq \mathcal{M}$ is the sub-module spanned by the vectors

$$\left\{ \sum_{l=1}^{k} \left(J_f \cdot h_l \otimes \epsilon_l - h_l \otimes \left(\mathcal{J}_\mu(f) \right) (w_{s-2}^2) \cdot \epsilon_l \right) : h_l \in J\mathcal{M}, \epsilon_l \in \mathbb{C}^k, f \in P_{s-2}[z] \right\}.$$

Section (6.3): Quasi-Homogeneous Holomorphic Curves and Operators in the Cowen–Douglas Class

For a plane domain Ω, in [61], Cowen and Douglas introduced an important class of operators $B_n(\Omega)$. It was shown by them that for operators T in $B_n(\Omega)$, the local geometry of the corresponding vector bundle E_T of rank n (curvature tensor and its higher derivatives) yields a complete set of unitary invariants for the operator T. But a tractable set of unitary (or similarity) invariants has not been found yet. The analysis of holomorphic Hermitian vector bundles in case $n > 1$ is much more complicated, see [198].

In [194], [190], a class $\mathcal{F}B_n(\Omega)$ of operators in the Cowen–Douglas class possessing a flag structure was isolated. A complete set of unitary invariants for this class of operators were listed. Recently, Jiang and Ji have introduced methods from K −theory to classify flags of holomorphic curves in the Grassmannian in order to reduce the questions involving operators in $B_n(\Omega)$ to the case of $n = 1$ (cf. [203], [58]). On the other hand, the classification of homogeneous holomorphic Hermitian vector bundles over the unit disc has been completed recently (cf. [183]) using tools from representation theory of semi-simple Lie groups. Although not complete, a similar classification over an arbitrary bounded symmetric domain is currently under way [205], [206].

The methods of K-theory developed in [203], [58] together with the methods of [190] makes it possible to study a much larger class of "quasi-homogeneous" operators, where the techniques from representation theory are no longer available. These methods, applied to the class of "quasi-homogeneous" operators leads to a unitary classification. In addition the bundle maps describing the triangular decomposition of Jiang and Ji have an explicit realization in terms of the inherent harmonic analysis. A model for these operators is

described explicitly, which shows, among other things, that the well-known Halmos problem for the class of "quasi-homogeneous" operators has an affirmative answer.
Prompted by these results, one might imagine that the multi-variate case (replacing the planar domain Ω by the unit ball or a bounded symmetric domain) may also be accessible to these new techniques.

For $\mathcal{H}$ be a complex separable Hilbert space and let $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For an open connected subset Ω of the complex plane $\mathbb{C}$, and $n \in \mathbb{N}$, Cowen and Douglas introduced the class of operators $B_n(\Omega)$ in [61]. An operator T acting on a Hilbert space $\mathcal{H}$ belongs to this class if each $w \in \Omega$, is an eigenvalue of the operator T of constant multiplicity n, these eigenvectors span the Hilbert space $\mathcal{H}$ and the operator $T - w, w \in \Omega$, is surjective. They showed that for an operator T in $B_n(\Omega)$, there exists a holomorphic choice of n linearly independent eigenvectors, that is, the map $w \to ker(T - w)$ is holomorphic. Thus $\pi : E_T \to \Omega$, where

$$E_T = \{ker(T - w) : w \in \Omega, \pi(ker(T - w)) = w\}$$

defines a Hermitian holomorphic vector bundle on Ω.

The Grassmannian $Gr(n, \mathcal{H})$, is the set of all $n-$dimensional subspaces of the Hilbert space $\mathcal{H}$. A map $t : \Omega \to Gr(n, \mathcal{H})$ is said to be a holomorphic curve, if there exist n (point-wise linearly independent) holomorphic functions $\gamma_1, \gamma_2, \cdots, \gamma_n$ on Ω taking values in a Hilbert space $\mathcal{H}$ such that $t(w) = \vee\{\gamma_1(w), \cdots, \gamma_n(w)\}, w \in \Omega$. Any holomorphic curve $t : \Omega \to Gr(n, \mathcal{H})$ gives rise to a $n-$dimensional Hermitian holomorphic vector bundle E_t over Ω, namely,

$$E_t = \{(x, w) \in \mathcal{H} \times \Omega \mid x \in t(w)\} \text{ and } \pi : E_t \to \Omega, \text{where } \pi(x, w) = w.$$

Given two holomorphic curves $t, \tilde{t} : \Omega \to Gr(n, \mathcal{H})$, if there exists a unitary operator U on $\mathcal{H}$ such that $\tilde{t} = Ut$, that is, the restriction $U(w) := U|_{E_t}(w)$ of the unitary operator U to the fiber $E_t(w)$ of E at w maps it to the fiber of $E_{\tilde{t}}(w)$, then t and $\tilde{t}$ are said to be congruent. If t and $\tilde{t}$ are congruent, then clearly the vector bundles E_t and $E_{\tilde{t}}$ are equivalent via the holomorphic bundle map induced by the unitary operator U.

Furthermore, t and $\tilde{t}$ are said to be similar if there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $t_{\tilde{t}} = Xt$, that is, $X(w) := X|_{E_t}(w)$ is an isomorphism except that $X(w)$ is no longer an isometry. In this case, we say that the vector bundles E_t and $E_{\tilde{t}}$ are similar.

An operator T in the class $B_n(\Omega)$ determines a non-constant holomorphic curve $t : \Omega \to Gr(n, \mathcal{H})$, namely, $t(w) = ker(T - w), w \in \Omega$. However, if t is a holomorphic curve, setting $Tt(w) = wt(w)$, defines a linear transformation on a dense subspace of the Hilbert space H. In general, we have to impose additional conditions to ensure that the operator T is bounded. Assuming that t defines a bounded linear operator T, unitary and similarity invariants for the operator T are then obtained from those of the vector bundle E_t.

The motivation for this work comes from three very different directions. The attempt is to describe a canonical model and obtain invariants for operators in the Cowen– Douglas class with respect to equivalence via conjugation under a unitary or invertible linear transformation. These questions have been successfully addressed using ideas from $K-$theory and representation theory of Lie groups. First, the detailed study of the Cowen–Douglas class of operators, reported in [70] provides a basic structure theorem for these operators: if T is an operator in the Cowen–Douglas class $B_n(\Omega)$, then there exists operators $T_0, T_1, \ldots, T_{n-1}$ in $B_1(\Omega)$ such that

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix}. \quad (82)$$

A slight paraphrasing of it clearly implies that if $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ is a holomorphic frame for the vector bundle E_t, and $\mathcal{H} = \bigvee\{\gamma_i(w), w \in \Omega, 0 \le i \le n-1\}$, then there exists non-vanishing holomorphic curves $t_i : \Omega \to Gr(1, \mathcal{H}_i), 0 \le i \le n-1$, such that

$$\gamma_j = \emptyset_{0,j}(t_0) + \cdots + \emptyset_{i,j}(t_i) + \cdots + \emptyset_{j-1,j}(t_{j-1}) + t_j, \ 0 \le j \le n-1, \quad (83)$$

where $\emptyset_{i,j}$ are certain holomorphic bundle maps. One would expect these bundle maps to reflect the properties of the operator T. However the tenuous relationship between the operator T and the bundle maps $\emptyset_{i,j}$ becomes a little more transparent only after we impose a natural set of constraints.

Secondly, to a large extent, these constraints were anticipated in [190]. A class of operators $\mathcal{F}B_n(\Omega)$ in $B_n(\Omega)$ possessing, what we called, a flag structure were isolated. A operator $T \in B_n(\Omega)$ in (6.3.1) belongs to $\mathcal{F}B_n(\Omega)$ if and only if $T_i S_{i,i+1} = S_{i,i+1} T_{i+1}, i \le n-1$. The flag structure was shown to be rigid. It was then shown that the complex geometric invariants like the curvature and the second fundamental form of the vector bundle E_T are unitary invariants of the operator T. Indeed, a complete set of unitary invariants were found.

Lemma (6.3.1)[199]: ([190]) If X is an invertible operator intertwining any two operators T and $\tilde{T}$in $\mathcal{F}B_n(\Omega)$, then the operators X and X^{-1} are upper triangular relative to the $n \times n$ block decomposition of the operators T and $\tilde{T}$.

It is evident that if the intertwining operator X is assumed to be unitary, then X must be diagonal. This observation is the key to finding a set of tractable complete unitary invariants for the operators in the class $\mathcal{F}B_n(\Omega)$, see [190].

Finally, recall that an operators T in $B_n(\mathbb{D})$ is said to be homogeneous if the unitary orbit of T under the action of the Möbius group is itself, that is, $\varphi(T)$ is unitarily equivalent to T for φ in some open neighbourhood of the identity in the Möbius group (cf. [77]). A canonical element $T^{(\lambda,\mu)}$ in each unitary equivalence class of the homogeneous operators in $B_n(\mathbb{D})$ was constructed in [183]. It was then shown that two operators $T^{(\lambda,\mu)}$ and $T^{(\lambda',\mu')}$ are similar if and only if $\lambda = \lambda'$. In particular choosing $\mu = 0$, one verifies that a homogeneous operator in $B_n(\mathbb{D})$ is similar to the n −fold direct sum $T_0 \oplus \cdots \oplus T_n$, where T_i is the adjoint of the multiplication operator $M^{(\lambda_i)}$ acting on the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$ determined by the positive definite kernel $\frac{1}{(1-z\bar{w})^{\lambda_i}}$ defined the unit disc $\mathbb{D}, 0 \le i \le n-1, \lambda_i > 0$.

We study a class of operators, to be called quasi-homogeneous, for which we can prove results very similar to those for the homogeneous operators building on the techniques developed in [190]. This class of operators, as one may expect, contains the homogeneous operators and is characterized by the requirement that all the bundle maps of (83) take their values in a certain (full) jet bundle $\mathcal{J}_i(t)$ of the holomorphic curve t.

For a detailed account of the jet bundles, see [208].

Definition (6.3.2)[199]: If t is a holomorphic curve in the Grassmannian of rank 1, that is, $t : \Omega \rightarrow Gr(1, \mathcal{H})$. Let $\gamma(w)$ be a non-vanishing holomorphic section for the line bundle E_t.
derivatives $\gamma^{(j)}, j \in \mathbb{N}$, taking values again in the Hilbert space $\mathcal{H}$ are holomorphic. (It can be shown that they are linearly independent.) The jet bundle $\mathcal{J}_n E_t(\gamma)$ is defined by the holomorphic frame $\{\gamma^{(0)}(:= \gamma), \gamma^{(1)}, \cdots, \gamma^{(n)}\}$. The jet bundle $\mathcal{J}_n E_t(\gamma)$ has a natural Hermitian structure obtained by taking the inner product of $\gamma^{(i)}(w)$ and $\gamma^{(j)}(w)$ in the Hilbert space $\mathcal{H}$.

In the following definition we assume, implicitly, that the bundle map $\emptyset_{i,j}$ of (83) are from the holomorphic line bundles E_i to a jet bundle $\mathcal{J}_j E_i$, where for brevity of notation and when there is no possibility of confusion, we will let E_i denote the vector bundle induced by the holomorphic curve $t_i, 0 \leq i \leq n - 1$.

Definition (6.3.3)[199]: ($\mathcal{J}$ −holomorphic curve). Let t be a holomorphic curve in the Grassmannian $Gr(n, \mathcal{H})$ of a complex separable Hilbert space $\mathcal{H}$ and $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ be a holomorphic frame for t. We say that t admits an atomic decomposition if there exists holomorphic curves $t_i : \Omega \rightarrow Gr(1, \mathcal{H}_i)$, to be called the atoms of t, corresponding to operators $T_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$ in $B_1(\Omega)$ and complex numbers $\mu_{i,j} \in \mathbb{C}, 0 \leq j \leq i \leq n - 1$, such that $\mathcal{H} = \mathcal{H}_0\, \mathcal{H}_{n-1}$ and

$$
\begin{aligned}
\gamma_0 &= \mu_{0,0} t_0 \\
\gamma_1 &= \mu_{0,1} t_0^{(1)} + \mu_{1,1} t_1 \\
\gamma_2 &= \mu_{0,2} t_0^{(2)} + \mu_{1,2} t_1^{1} + \mu_{2,2} t_2 \\
&\vdots \quad \vdots \\
\gamma_j &= \mu_0, j\, t_0^{(j)} + \cdots + \mu_{i,j}\, t_i^{(j-i)} + \cdots + \mu_{j,j}\, t_j \\
&\vdots \quad \vdots \\
\gamma_{n-1} &= \mu_{0,n-1} t_0^{(n-1)} + \cdots + \mu_{i,n-1} t_i^{(n-1-i)} + \cdots + \mu_{n-1,n-1} t_{n-1}.
\end{aligned}
$$

If t admits an atomic decomposition, we call it a $\mathcal{J}$ −holomorphic curve.

Fix i in $\{0, \ldots, n - 1\}$. We say that the holomorphic curve ti is homogeneous if for $w \in \mathbb{D}, \mathbb{C}[t_i(w)] = ker(T_i - w)$ for some homogeneous operator T_i in $B_1(\mathbb{D})$. We realize, up to unitary equivalence, such a homogeneous operator T_i in $B_1(\mathbb{D})$ as the adjoint of the multiplication operator $M^{(\lambda_i)}$ on the weighted Bergman spaces $A^{(\lambda_i)}(\mathbb{D})$. Thus for a fixed $w \in \mathbb{D}$, there exists a canonical (holomorphic) choice of eigenvectors $t_i(w)$, namely, $(1 - z\bar{w})^{-\lambda_i}$.

Definition (6.3.4)[199]: (Quasi-homogeneous curve). We say that a J −holomorphic curve t is quasi-homogeneous if each of the atoms ti is homogeneous, $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ and the difference $\lambda_{i+1} - \lambda_i, 0 \leq i \leq n - 2$, is a fixed positive real number $\Lambda(t)$, which is called the valency of t.

We say that the J −holomorphic curve t defines a bounded linear operator if the linear span of $\{\gamma_i(w) : 0 \leq i \leq n - 1\}, w \in \Omega$, is dense in $\mathcal{H}$ and the linear map defined by the rule $T(\gamma_i(w)) = w\gamma_i(w), 0 \leq i \leq n - 1$, extends to a bounded operator on the Hilbert space $\mathcal{H}$.

We determine conditions on the scalars $\mu_{i,j}$ and the valency $\Lambda(t)$, which ensure that the quasi-holomorphic curve t defines bounded operator T, see Proposition (6.3.7).

We make the standing assumption that these conditions for boundedness are fulfilled. We shall use the terms quasi-homogeneous holomorphic curve t, quasi-homogeneous operator T and quasi-homogeneous holomorphic vector bundle E_t (or, even E_T) interchangeably.

If T is a quasi-homogeneous operator then it belong to the class $\mathcal{F}B_n(\mathbb{D})$ introduced in [194], [190], see Theorem (6.3.8). All quasi-homogeneous operators are therefore irreducible. All the quasi-homogeneous operators that are strongly irreducible are identified in Theorem (6.3.16). Theorem (6.3.19) gives a canonical model for a quasi-homogeneous operator in the equivalence class under conjugation by an invertible transformation.

As an application of our results, in Theorem (6.3.26), we show that the (topological) K_0 group and the (algebraic) K_0 group of a quasi-homogeneous operator are equal. In the context of the usual K_0 and K_0 groups, this is a consequence of the well-known theorem of R.G. Swan. As a second application, we obtain an affirmative answer for the Halmos question on similarity of an operator admitting the closed unit disc as a spectral set to a contraction.

A quasi-homogeneous vector bundle E_t is indeed homogeneous if $\Lambda(t) = 2$ and the constants $\mu_{i,j}$ are certain explicit functions of λ as we point out at the end of the following. However, a quasi-homogeneous vector bundle need not be homogeneous as the following example shows.

Example (6.3.5)[199]: Let S be the adjoint of the multiplication operator on arbitrary weighted Bergmann space $A(\lambda)$ $(\mathbb{D})$ and let T be the operator

$$T = \begin{pmatrix} S\ \mu_1\ I\ 0 & \cdots & 0 \\ 0\ S\ \mu_2\ I & \cdots & 0 \\ \vdots\ \ddots\ \ddots & \ddots & \vdots \\ 0\ \cdots\ 0\ S\ \mu_n\ I \\ 0\ \cdots\cdots\ 0\ S \end{pmatrix} , \mu_i \in C,$$

defined on the $n + 1$ fold direct sum $\bigoplus \mathbb{A}(\lambda)$ $(\mathbb{D})$. Then T is in $\mathcal{F}B_{n+1}(\mathbb{D})$ and therefore belongs to $B_{n+1}(\mathbb{D})$ and the corresponding holomorphic curve $t(w) = ker(T - w), w \in \mathbb{D},$ is quasi homogeneous with $\Lambda(t) = 0$. In fact, in this Example, if we replace S with an arbitrary operator, say $R,$ from $B_1(\mathbb{D}),$ then the resulting operator T while no longer quasi-homogeneous, remains a member of $\mathcal{F}B_{n+1}(\mathbb{D})$. Indeed, it has already appeared, via module tensor products, in [190].

The class of quasi-homogeneous operators, contrary to what might appear to be a rather small class of operators, contains apart from the homogeneous operators, many other operators. Indeed, in rank 2, for instance, it is parametrized by the multiplier algebra of two homogeneous operators. In the definition of the quasi-homogeneous operators given above, if we let the atoms occur with some multiplicity rather than being multiplicity-free, it will make it even larger. This would cause additional complications, which we are not able to resolve at this time. In another direction, we need not assume that the atoms themselves are homogeneous. Most of our results would appear to go through if we merely assume that the kernel function $K^{(\lambda)}(w, w) \sim \frac{1}{(1-|w|^2)^\lambda}, |w| < 1$. Deep results about such functions were obtained by Hardy and Littlewood (cf. [202]) and have already appeared in the context of similarity, see [200].

An operator T in the Cowen and Douglas class $B_n(\Omega)$ is determined, modulo unitary equivalence, by the curvature (of the vector bundle E_T) together with a finite number of its partial derivatives. However, if the rank n of this vector bundle is $> 1,$ then the computation of the curvature and its derivatives is somewhat impractical. Here we show that if the operator is quasi-homogeneous, it is enough to restrict ourselves to the computation of the curvature of the atoms and a $n - 1$ second fundamental forms of pair-wise neighbouring vector bundles. We first recall, following [61], [79], that an operator T in $B_n(\Omega)$ may be realized as the adjoint of a multiplication operator on a Hilbert space of holomorphic functions on $\Omega^* := \{w : \bar{w} \in \Omega\}$ possessing a reproducing kernel.

For an operator T in the Cowen–Douglas class $B_n(\Omega)$, acting on a Hilbert space $\mathcal{H}$, there is a holomorphic frame $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ and atoms $t_0, \ldots, t_{n-1}$, for which we have

$$\gamma_i = \mu_{0,i} t_0^{(i)} + \cdots + \mu_{j,i} t_i^{(i-j)} + \cdots + \mu_{i,i} t_i, \mu_{j,i} \in \mathbb{C}.$$

At this point, assuming that the operator is quasi-homogeneous makes the atoms $T_0, T_1, \ldots, T_{n-1}$ homogeneous. Conjugating with a diagonal unitary, if necessary, we assume without loss of generality that t_i is the holomorphic curve defined by $t_i(w) := (1 - \bar{w}z)^{-\lambda_i}, \lambda_i = \lambda_{0+i}\,\Lambda(t), 0 \le i \le n-1, \lambda_0 > 0,$
in the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. We assume without loss of generality that $\mu_{i,i} = 1, 0 \le i \le n - 1$.

Let t be a quasi-homogeneous holomorphic curve in $Gr(n, \mathcal{H})$. Assume that it defines a bounded linear operator T on the Hilbert space $\mathcal{H}$. An appeal to the decomposition (82) provides, what we would now call an atomic decomposition for the operator T. This decomposition has several additional properties arising out of our assumption of quasi-homogeneity.

Proposition (6.3.6)[199]: Let t be a $\mathcal{J}$ −holomorphic curve with atoms $\{t_0, \ldots, t_{n-1}\}$ and let $\{\gamma_0, \ldots, \gamma_{n-1}\}$ be a holomorphic frame for the vector bundle E_t. Let $\mathcal{H}$ be the closed linear span of the set of vectors $\{\gamma_0(w), \ldots, \gamma_{n-1}(w) : w \in \Omega\}$ and $\mathcal{H}_i$ be the closed linear span of the set of vectors $\{t_i(w), w \in \Omega\}, 0 \le i \le n-1$. We have
(i) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{n-1}$;
(ii) There exists an operator T, defined on a dense subset of vectors in $\mathcal{H}$, which is upper triangular with respect to the direct sum decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$:

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & 0 & \ldots & 0 & T_{n-1} \end{pmatrix},$$

where $S_{i,j}(t_j(w)) = m_{i,j}\, t_i^{(j-i-1)}(w), T_i(t_i(w))) = w\, t_i(w), \quad w \in \Omega, i, j = 0, 1, \cdots, n-1$, for some choice of complex constants $m_{i,j}$ depending on the $\mu_{i,j}$. (iii) The constants mi,j and $\mu_{i,j}$ determine each other.

For convenience of notation, in the proof below, we set $S_{i,i} := T_i, 0 \le i \le n-1$, in the proof. We will adopt this practice often and call $T_0, T_1, \ldots, T_{n-1}$, the atoms of T. Also, $S_{i,i+1}(t_{i+1}) = \mu_{i,i+1} t_i$, with the assumption that $\mu_{i,i} = 1, 0 \le i \le n-2$.

Proof. Note that $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ is a frame for E_t and the atoms $t_i, 0 \le i \le n-1$ are pairwise orthogonal. From Definition (6.3.3), the first statement of the Proposition is included in the definition of a holomorphic quasi-homogeneous curve.

For $0 \le i \le j \le n-1$, let $S_{i,j} : \mathcal{H}_j \to \mathcal{H}$ be the linear transformation induced by bundle maps $s_{i,j} : E_{t_j} \to J_{j-i} - 1E_{t_i}$, namely,

$$\sum_{i \le j} s_{i,j}\left(\gamma_k(w)\right) = w\gamma_k(w), \qquad w \in \Omega.$$

It follows that

$$\left(s_{k,k}-w\right)\left(\mu_{k,k}t_k(w)\right)=0,\left(s_{k-1,k-1}-w\right)\left(\mu_{k-1,k}t_{k-1}^{(1)}(w)\right)+s_{k-1,k}\left(\mu_{k,k}t_k(w)\right)$$
$$=0. \tag{84}$$

Thus $s_{k,k}$ induces an operator $S_{k,k}$ with $\ker(S_{k,k}-w)=\mathbb{C}[t_k(w)]$ and $s_{k-1,k}$ is a bundle map from $E_{t_k}(w)(:=\mathbb{C}[t_k(w)])$ to $E_{t_{k-1}}(w)(:=\mathbb{C}[t_{k-1}(w)])$.

For any $i\le j\le n-1, s_{i,j}$ is a bundle map from E_{t_j} to $\mathcal{J}_{j-i-1}E_{t_i}$ and there exists $m_{i,j}\in\mathbb{C}$ such that $S_{i,j}(t_j(w))=m_{i,j}\,t_i^{(j-i-1)}(w), w\in\Omega$.

Since $(s_{0,0}-w)\gamma_1(w)=(s_{0,0}-w)(\mu_{0,1}t_0^{(1)}(w))+s_{0,1}(\mu_{1,1}t_1(w))=0$, we have

$$s_{0,1}(t_1(w))=m_{0,1}t_0(w),$$

where $m_{0,1}=-\frac{\mu_{0,1}}{\mu_{1,1}}$. Thus we have

$$s_{0,2}\big(t_2(w)\big)=-\frac{2\mu_{0,2}+\mu_{1,2}m_{0,1}}{\mu_{2,2}}t_0^{(1)}(w)=m_{0,2}t_0^{(1)}(w).$$

Now assume that for any fixed k and some $k<j\le n-1$. there exits $m_{k,i}\in\mathbb{C}$ such that

$$s_{k,i}(t_i(w))=m_{k,i}t_k^{(i-k-1)}(w),\qquad i<j.$$

Then from equation (84), we have

$$(s_{k,k}-w)(\mu_{k,j}\,t_k^{(j-k)})(w)+s_{k,k+1}(\mu_{k+1,j}\,t_{k+1}^{(j-k-1)}(w))+\cdots+s_{k,j}(\mu_{j,j}\,t_j(w))$$
$$=0$$

and from the induction hypothesis, we may rewrite this as

$$\mu_{k,j}(j-k)t_k^{(j-k-1)}(w)+\mu_{k+1,j}m_{k,k+1}t_k^{(j-k-1)}(w)+\cdots+\mu_{j,j}\;s_{k,j}(t_j(w))$$
$$=0.$$

Thus

$$s_{k,j}(t_j(w))=m_{k,j}\,t_k^{(j-k-1)}(w),$$

or, equivalently

$$m_{k,j}=-\frac{\mu_{k,j}(j-k)+\sum_{l=1}^{j-k-1}\mu_{k+l,j}m_{k,k+l}}{\mu j,j} \tag{85}$$

completing the proof of the second statement of the Proposition.

Claim: For any operator T in $B_n(\Omega)$ with atomic decomposition exactly as in the second statement of the lemma, there exists $\mu_{i,j}$ satisfying the conditions in Definition (6.3.3), that is, there exists a holomorphic frame for E_T, which is a linear combination of the non-vanishing holomorphic sections of E_{t_i} and a certain number of jets.

Indeed, the proof of the second part of the Proposition already verifies this Claim for $n\le 2$. To prove the Claim by induction, let us assume that it is valid for $k\le n-2$. Note that the operator $\left(\left(S_{i,j}\right)\right)_{i,j\le n-2}$ is in $B_{n-1}(\Omega)$. By the induction hypothesis, we can find $m_{i,j}, i,j\le n-2$ verifying Claim 2 for any operator $\left(\left(S_{i,j}\right)\right)_{i,j\le n-2}$. If we consider the operator

$$\begin{pmatrix}T_{n-2} & S_{n-2,n-1}\\ 0 & T_{n-1}\end{pmatrix},$$

then we have that $S_{n-2,n-1}(t_{n-1})=m_{n-2,n-1}t_{n-2}$. Now, setting $\mu_{n-2,n-1}=-m_{n-2,n-1}$, we can define all the coefficients $\mu_{n-k,n-1}, 2\le k\le n$ recursively. In fact, if we consider

$$\begin{pmatrix} T_{n-k} & S_{n-k,n-k+1} & S_{n-k,n-k+2} & \cdots & S_{n-k,n-1} \\ & T_{n-k+1} & S_{n-k+1,n-k+2} & \cdots & S_{n-k+1,n-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & 0 & & & \\ & & & T_{n-2} & S_{n-2,n-1} \\ & & & & T_{n-1} \end{pmatrix},$$

where $2 \leq k \leq n$, and set

$$\mu_{n-k,n-1} = -\frac{\sum_{i=1}^{k-2} m_{n-k,n-k+i}\mu_{n-k+i,n-1} + m_{n-k,n-1}}{k-1},$$

then $\mu_{n-k,n-1}$ is defined involving only the coefficients $\mu_{n-k+i,n-1}$ which exist by the induction hypothesis. Thus coefficients $\mu_{i,j}$ depends only on the $m_{i,j}$, $i,j \leq n-1$. By a direct computation, $\gamma_k = \mu_{0,k}t_0^{(k)} + \mu_{1,k}t_1^{(k-1)} + \cdots + \mu_{k,k}t_k, 0 \leq k < n-1$ together defines a frame for E_T . This completes the proof of the Claim and the third statement of the lemma.

Having shown that a holomorphic quasi-homogeneous curve t defines a linear transformation on a dense subset of $\mathcal{H}_t$, we determine when it extends to a bounded linear operator on all of $\mathcal{H}_t$. We make the following conventions here which will be in force throughout.

The positive definite kernel $K^{(\lambda)}(z,w)$ is the function $(1-\bar{w}z)^{-\lambda}$ defined on $\mathbb{D} \times \mathbb{D}$ and is the reproducing kernel for the weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{D})$. The coefficient $a_n(\lambda)$ of $\bar{w}^n z^n$ in the power series expansion for $K^{(\lambda)}$ (in powers of $z\bar{w}$) is of the form $a_n(\lambda) \sim n^{\lambda-1}$ using Stirling's formula: $a(\lambda) = \frac{\Gamma(\lambda+n)}{\Gamma(n)} \sim n^{\lambda}$. The set of vectors $e_n^{(\lambda)} := \sqrt{a_n(\lambda)}z^n, n \geq 0$, is an orthonormal basis in $\mathbb{A}^{(\lambda)}(\mathbb{D})$. The action of the multiplication operator on $\mathbb{A}^{(\lambda)}(\mathbb{D})$ is easily determined:

$$M(e_n^{(\lambda)}) \sim \left(\frac{n}{n+1}\right)^{\frac{\lambda-1}{2}} e_{n+1}^{(\lambda)}.$$

Often, one sets $w_n^{(\lambda)} := \sqrt{\frac{a_n(\lambda)}{a_{n+1}(\lambda)}}$ and says that M is a weighted shift with weights $w_n^{(\lambda)}$ since $M(e_n^{(\lambda)}) = w_n^{(\lambda)} e_{n+1}^{(\lambda)}$. The other way round, $\prod^n{}_{i=0} w_i^{(\lambda)} = \sqrt{\frac{a_n(\lambda)}{a_{n+1}(\lambda)}} \sim (n+1)^{\frac{1-\lambda}{2}}$.

The adjoint of this operator is then given by the formula:

$$M^*(e_n^{(\lambda)}) = w_{n-1}^{(\lambda)} e_{n-1}^{(\lambda)} \sim \left(\frac{n-1}{n}\right)^{\frac{\lambda-1}{2}} e_{n-1}^{(\lambda)}.$$

The following Proposition shows that if the valency $\Lambda(t)$ is less than 2, then every possible linear combination of the atoms and their jets need not define a bounded linear transformation. However, from the proof of this Proposition, we infer that no such obstruction can occur if $\Lambda(t) \geq 2$.

Proposition (6.3.7)[199]: Fix a natural number $n \geq 2$. Let t be a quasi-homogeneous holomorphic curve with atoms $t_i, i = 0,1,\ldots,n-1$. For $0 \leq i,j \leq n-1$, let $s_{i,j}(t_j(w)) = m_{i,j}\, t_i^{(j-i-1)}(w)$ be the bundle map from E_{t_j} to $\mathcal{J}_{j-i-1}E_{t_i}$ and $S_{i,j}$:

$\mathcal{H}_j \to \mathcal{H}_i$ be the densely defined linear transformation induced by the maps $s_{i,j}$. The linear transformation of the form

$$T = \begin{pmatrix} T_0 \, S_{0,1} \, S_{0,2} \, \cdots \, S_{0,n-1} \\ 0 \, T_1 \, S_{1,2} \, \cdots \, S_{1,n-1} \\ \vdots \ddots \ \ddots \ \ddots \ddots \ \vdots \\ 0 \ldots 0 \, T_{n-2} \, S_{n-2,n-1} \\ 0 \quad 0 \ldots 0 \, T_{n-1} \end{pmatrix}$$

is densely defined on the Hilbert space $\mathbb{A}^{(\lambda_0)}(\mathbb{D}) \oplus \cdots \oplus \mathbb{A}^{(\lambda_{n-1})}(\mathbb{D})$. Suppose that $\Lambda(t) < 2$.

(i) If $\Lambda(t) \in [1 + \frac{n-3}{n-1}, 2), n \geq 2$, then T is bounded.

(ii) If $\Lambda(t) \in [1 + \frac{n-k-4}{n-k-2}, 1 + \frac{n-k-3}{n-k-1})$, the operator T is bounded only if we set $m_{i,j} = 0$ whenever $j - i \geq n - k - 2, n - 1 > k \geq 0, n \geq 4$, that is, T must be of the form

$$T = \begin{pmatrix} S_{0,0} \, S_{0,1} \, \cdots \, S_{0,n-k-2} \, 0 \, \cdots \, 0 \, 0 \\ S_{1,1} \, S_{1,2} \, \cdots \, S_{1,n-k-1} \, 0 \, \cdots \, 0 \\ \ddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \ \vdots \\ \ddots \ddots \ddots \ddots \qquad 0 \\ 0 \, S_{k+1,k+1} \, S_{k+1,k+2} \, \cdots \, S_{k+1,n-1} \\ \ddots \quad \ddots \quad \vdots \\ S_{n-2,n-2} \, S_{n-2,n-1} \\ S_{n-1,n-1} \end{pmatrix}$$

(iii) If $\Lambda(t) \in (0, 1)$, then the densely defined linear transformation T is bounded only if we set $m_{i,j} = 0, i < j + 1, i = 0, 1, \cdots, n - 2, n \geq 3$.

Proof. For $i = 0, 1, \cdots, n - 1$, the operators $S_{i,i}$ are homogeneous by definition. Thus the operator $S_{i,i}$, as we have said before, is realized as the adjoint of the multiplication operator on the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. The reproducing kernel $K^{(\lambda_i)}(z, w)$ for this Hilbert space is of the form $\frac{1}{(1-z\bar{w})^{\lambda_i}}$. Consequently,

$$\ker \left(S_{i,i} - w\right)^* = \mathbb{C}[t_i(\bar{w})] = \mathbb{C}[K^{(\lambda_i)}(z, w)], w \in \mathbb{D}.$$

Claim : If $\lambda_j - \lambda_i > 2(j - i) - 2, j > i = 0, 1, 2, \cdots, n - 2$, then each $s_{i,j}$ induces a non-zero linear bounded operator $S_{i,j}$.

Without loss of generality, we set $s_{i,j}(t_j) = m_{i,j} \, t_i^{(j-i-1)}, m_{i,j} \in \mathbb{C}, i, j = 0, 1, \cdots, n - 1$ and

$$t_i(w) = \frac{1}{(1 - zw)^{\lambda_i}}, t_j(w) = \frac{1}{(1 - zw)^{\lambda_i}}.$$

Then the linear transformation $S_{i,j} : \mathcal{H}_j \to \mathcal{H}_i$ induced by $s_{i,j}$ is densely defined by the rule

$$S_{i,j}(t_j) = m_{i,j} \, t_i^{(j-i-1)}, i, j = 0, 1, \cdots, n - 1.$$

We have that

$$||S_{i,j}|| = |m_{i,j}| \max_{\ell} \{\frac{\sqrt{\prod_{l=0}^{\ell-(j-i)} w_l(\lambda_j)}}{\sqrt{\prod_{l=0}^{\ell-1} w_l(\lambda_j)}} \ell(\ell - 1) \cdots (\ell - (j - i) + 2)\}.$$

By a direct computation,

$$\frac{\sqrt{\Pi_{l=0}^{\ell-(j-i-1)} \, w_l(\lambda_j)}}{\sqrt{\Pi_{l=0}^{\ell-1} \, w_l(\lambda_j)}} \ell(\ell-1)\cdots(\ell-(j-i)+2) \sim \left(\frac{1}{\ell^{\frac{\lambda_j-\lambda_i}{2}-(j-i-1)}} \right).$$

It follows that each $S_{i,j}$ is a non-zero bounded linear operator if and only if

$$\frac{\lambda_j - \lambda_i}{2} \geq j - i - 1, \text{that } is, \lambda_j - \lambda_i \geq 2(j-i) - 2.$$

If $\Lambda(t) \geq 1 + \frac{n-3}{n-1}$, then

$$\lambda_{n-1} - \lambda_0 = (n-1)\Lambda(t) \geq 2(n-2).$$

By the argument given above, we obtain $S_{0,n-1}$ is non-zero and bounded. If $\Lambda(t) < 1 + \frac{n-3}{n-1}$, then we might deduce that $m_{0,n-1} = 0$ or $\mu_{0,n-1} = 0$, i.e. $S_{0,n} = 0$. Thus the proof of the first statement is complete.

For the general case, if $\Lambda(t) \in [1 + \frac{n-k-4}{n-k-2}, 1 + \frac{n-k-3}{n-k-1}), k \geq 0$, then we have

$$(n-k-1)\Lambda(t) < 2(n-k-1) - 2.$$

On the other hand, if $j - i \geq n - k - 1$, then we obtain $\lambda_j - \lambda_i \leq 2(j-i) - 2$. By the argument above, we have $S_{i,j} = 0, j - i \geq n - k - 1$, and S has the following matrix form:

$$T = \begin{pmatrix} S_{0,0} & \cdots & S_{0,n-k-2} & 0 & \cdots & 0 \\ & S_{1,1} & \cdots & S_{1,n-k-1} & 0 & \cdots & 0 \\ & & \ddots & & \ddots & \ddots & \vdots \\ & & & \ddots & & \ddots & 0 \\ 0 & & & S_{k+1,k+1} & \cdots & & S_{k+1,n-1} \\ & & & & \ddots & & \vdots \\ & & & & & \ddots & \vdots \\ & & & & & & S_{n-1,n-1} \end{pmatrix} \tag{86}$$

This completes the proof of the second statement. In particular, if $0 \leq \Lambda(t) < 1$ and $j - i \geq 2$, then we have $\lambda_j - \lambda_i \leq 2(j-i) - 2$, which implies

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & 0 & & \cdots & 0 \\ 0 & & S_{1,1} & S_{1,2} & \cdots & 0 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ 0 & & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & & 0 & S_{n-1,n-1} \end{pmatrix}, \Lambda(t) \in [0,1).$$

This completes the proof of the third statement.

Having disposed off the question of boundedness of a quasi-homogeneous operator, we show that all quasi-homogeneous operators are in the class $F\mathcal{B}_n(\mathbb{D})$.

Theorem (6.3.8)[199]: Suppose T is a quasi-homogeneous operator and $\left((S_{i,j})\right)_{n\times n}$ is its atomic decomposition. Then we have

$$S_{i,i} \, S_{i,i+1} = S_{i,i+1} S_{i+1,i+1}, i = 0, 1, \cdots, n-2,$$

or equivalently, T is in $\mathcal{F}B_n(\mathbb{D})$.

Proof. We have found constants $m_{i,j} \in \mathbb{C}$ such that

$$S_{i,j}(t_j) = m_{i,j} \, t_i^{(j-i-1)}, i < j = 0, 1, \ldots, n-1$$

in the second statement of Proposition (6.3.6). Since $(S_{i,i} - w)(t_i(w)) = 0, w \in \Omega$, it follows that

$$S_{i,i}S_{i,i+1}(t_{i+1}(w)) = S_{i,i+1}S_{i+1,i+1}(t_{i+1}(w)).$$

We have $\mathcal{H}_i = Span_{w\in\Omega}\{t_i(w)\}, i = 0,1 \cdots, n-1$, therefore

$$S_{i,i}S_{i,i+1} = S_{i,i+1}S_{i+1,i+1}, i = 0,1,\cdots, n-2.$$

In [80], an explicit formula for the second fundamental form of a holomorphic Hermitian line bundle in its first order jet bundle of rank 2 was given. The second fundamental form, in a slightly different guise, was shown to be a unitary invariant for the class of operators $\tilde{\mathcal{F}}B_n(\Omega)$ in [190]. We give the computation of the second fundamental form here, yet again, keeping track of certain constants which appear in the description of the quasi-homogeneous operators. We compute the second fundamental form of the inclusion E_0 in E, where $\{\gamma_0,\gamma_1\}$ is a frame for E with atoms t_0 and t_1. The line bundle defined by the atom t_0 is E_0. By necessity, we have

$$\gamma_0 = t_0 \, \gamma_1 = \mu_{01}t_0' + t_1$$

with $t_0 \perp t_1$. As in [80], [190], setting $h = \langle \gamma_0,\gamma_0 \rangle$, the second fundamental form $\theta_{0,1}$ is seen to be of the form

$$\theta_{0,1} = -h^{1/2} \frac{\bar{\partial}(h^{-1}\langle \gamma_1,\gamma_0\rangle)}{\left(\|\gamma_1\|^2 - \frac{|\langle \gamma_1,\gamma_0\rangle|^2}{\|\gamma_0\|^2}\right)^{1/2}}.$$

It is important, for what follows, to express $\theta_{0,1}$ in terms of the atoms t_0 and t_1 giving the formula

$$\theta_{0,1} = \frac{\mu_{0,1}\mathcal{K}_0}{\left(\frac{\|t_1\|^2}{\|t_0\|^2} - |\mu_{0,1}|^2\mathcal{K}_0\right)^{1/2}}, \tag{87}$$

where $\mathcal{K}_0$ is the curvature of the line bundle E_{t0} given by the formula $-\partial\partial\, log$ $\|t_0\|^2$. The following lemma shows the key role of the second fundamental form in determining the unitary equivalence class of a quasi-homogeneous holomorphic curve.

Lemma (6.3.9)[199]: Suppose that t and $\tilde{t}$ are quasi-holomorphic curves with the same atoms t_0, t_1. Then the following statements are equivalent.

(i) The two curves t and $\tilde{t}$ are unitarily equivalent;

(ii) The second fundamental forms $\theta_{0,1}$ and $\tilde{\theta}_{0,1}$ are equal;

(iii) The two constants $\mu_{0,1}$ and $\tilde{\mu}_{0,1}$ are equal.

Proof. The equivalence of the first two statements was proved in [190]. The equality of $\theta_{0,1}$ and $\tilde{\theta}_{0,1}$ is clearly equivalent to

$$\tilde{\mu}_{0,1}\left(\frac{\|t_1\|^2}{\|t_0\|^2} + |\mu_{0,1}|^2\bar{\partial}\partial\, log\, \|t_0\|^2\right)^{1/2} = \mu_{0,1}\left(\frac{\|t_1\|^2}{\|t_0\|^2} + |\tilde{\mu}_{0,1}|^2\bar{\partial}\partial\, log\, \|t_0\|^2\right)^{1/2}.$$

From this equality, we infer that $arg(\mu_{0,1}) = arg(\tilde{\mu}_{0,1})$.

Given that we have assumed, without loss of generality, $\|t_0\|^2 = (1-|w|^2)^{-\lambda_0}$ and $\|t_1\|^2 = (1-|w|^2)^{-\lambda_1}$, squaring both sides and then taking the difference of the equality displayed above, we find that

$$\bar{\partial}\partial\, log\, \|t_0\|^2 = \lambda_0(1-|w|^2)^{-2},$$

which can be equal to $\frac{\|t_1\|^2}{\|t_0\|^2}$ if and only if $\lambda_1 - \lambda_0 = 2$. Thus except when $\Lambda(t) = 2$, we must have $\mu_{0,1}^2 - \tilde{\mu}_{0,1}^2 = 0$. Clearly, $\tilde{\mu}_{0,1} = -\mu_{0,1}$ is not an admissible solution. So, we must have $\tilde{\mu}_{0,1} = \mu_{0,1}$. In case $\lambda_1 - \lambda_0 = 2$, if we assume $\tilde{\mu}_{0,1} \neq \mu_{0,1}$, then we must have

$$\left(\frac{1 + \lambda_0 |\tilde{\mu}_{0,1}|^2}{1 + \lambda_0 |\mu_{0,1}|^2}\right)^{\frac{1}{2}} = \frac{|\tilde{\mu}_{0,1}|}{|\mu_{0,1}|},$$

from which it follows that $|\tilde{\mu}_{0,1}| = |\mu_{0,1}|$. The arguments of these complex numbers being equal, they must be actually equal.

When we consider the inclusion of the line bundle E_{t_i} in the vector bundle $E_{\left\{t_i, \frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j\right\}}$ of rank 2, the situation is slightly different. This is the vector bundle which corresponds to the 2×2 operator block $T_{i,j} := \begin{pmatrix} S_{i,i} & S_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$.

Clearly, $\left\{t_i, -\frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j\right\}$ is the frame for $E_{T_{i,j}}$. By the formulae above, setting temporarily $\gamma_0 = t_i, \gamma_1 = -\frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j$, we have that

(i) $h_i = ||\gamma_0||^2 = ||t_i||^2, h_j = ||t_j||^2$;

(ii) $||\gamma_1||^2 = \left|\frac{m_{i,j}}{j-i}\right|^2 \partial^{j-i} \bar{\partial}^{j-i} ||t_i||^2 + ||t_j||^2 = \left|\frac{m_{i,j}}{j-i}\right|^2 \partial^{j-i} \bar{\partial}^{j-i} h_i + h_j$;

(iii) $< \gamma_1, \gamma_0 >= -\frac{m_{i,j}}{(j-i)} \partial^{j-i} ||t_i||^2 = -\frac{m_{i,j}}{j-i} \partial^{j-i} h_i$;

(iv) $|< \gamma_1, \gamma_0 >|^2 = \left|\frac{m_{i,j}}{j-i}\right|^2 \partial^{j-i} h_i \bar{\partial}^{j-i} h_i$.

The second fundamental form $\theta_{i,j}$ for the inclusion $E_{t_i} \subseteq E_{\left\{t_i, \frac{m_{i,j}}{j-i} t_i^{(j-i)} + t_j\right\}}$ is given by the formula

$$\theta_{i,j} = \frac{\frac{m_{i,j}}{j-i} \bar{\partial}\left(h_i^{-1} \partial^{j-i} h_i\right)}{\left(\frac{h_j}{h_i} + \left|\frac{m_{i,j}}{j-i}\right|^2 \left(\frac{h_i \partial^{j-i} \bar{\partial}^{j-i} h_i - \partial^{j-i} h_i \bar{\partial}^{j-i} h_i}{h_i^2}\right)\right)^{\frac{1}{2}}}. \tag{88}$$

Lemma (6.3.10)[199]: Let $T_{i,j} := \begin{pmatrix} S_{i,i} & S_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$ and $\tilde{T}_{i,j} := \begin{pmatrix} S_{i,i} & \tilde{S}_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$ with $\tilde{S}_{i,j}(t_j) = \widetilde{m}_{i,j}\, i, j\, t_i^{(j-i-1)}$.

The second fundamental forms $\theta_{i,j}$ and $\tilde{\theta}_{i,j}$ of the operators $T_{i,j}$ and $\tilde{T}_{i,j}$ are equal, that is, $\theta_{i,j} = \tilde{\theta}_{i,j}$ if and only if $m_{i,j} = \theta$.

Proof. Without loss of generality, we will give the proof only for the case $i = 0, j = k, j \neq 1$. In this case, $\theta_{0,k} = \tilde{\theta}_{0,k}$ is equivalent to the equality:

$$\frac{\left(\frac{hk}{h0} + \left|\frac{m0,k}{k}\right|^2 \left(\frac{h_0 \partial^k \bar{\partial}^k h_0 - \partial^k h_0 \bar{\partial}^k h_0}{h_0^2}\right)\right)^{\frac{1}{2}}}{\left(\frac{h_k}{h_0} + \left|\bar{m}_0, \frac{k}{k}\right|^2 \left(\frac{h_0 \partial^k \bar{\partial}^k h_0 - \partial^k h_0 \bar{\partial}^k h_0}{h_0^2}\right)\right)^{\frac{1}{2}}} = \frac{m_{0,k}}{\widetilde{m}_{0,k}}$$

For simplicity, let g_0 denote $\left(\frac{h_0 \partial^k \bar{\partial}^k h_0 - \partial^k h_0 \bar{\partial}^k h_0}{h_0^2}\right)$ and let $\widetilde{m}$, mdenote $\frac{m_{0,k}}{k}, \frac{\bar{m}_{0,k}}{k}$ respectively. Then the equation given above may be rewritten as

$$\frac{\left(\frac{h_k}{h_0} + |m|^2 g_0\right)^{\frac{1}{2}}}{\left(\frac{h_k}{h_0} + |\tilde{m}|^2 \ g_0\right)^{\frac{1}{2}}} = \frac{m}{m}$$

From this equality, we infer that $arg(m) = arg(\tilde{m})$. Now, squaring both sides and then taking the difference, we have

$$\frac{h_k}{h_0}(\tilde{m}^2 - m^2) - \tilde{m}^2 m^2 g_0(\bar{m}^2 - \bar{\tilde{m}}^2) = 0.$$

Having assumed, without loss of generality, $h_0 = (1 - |w|^2)^{-\lambda_0}$ and $h_k = (1 - |w|^2)^{-\lambda_1}$, we find that g_0 is a polynomial of degree > 1 in $(1 - |w|^2)^{-1}$. Thus g_0 can be equal to hk h_0 if and only if $\lambda_1 - \lambda_0 = 2$. Therefore, except when $\Lambda(t) = 2$, we must have $m^2 - \tilde{m}^2 = 0$. Clearly, $m = -\tilde{m}$
is not an admissible solution. So, we must have $m = \tilde{m}$. Hence $m_{0,k} = \tilde{m}_{0,k}$.

Recall that a positive definite kernel $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$ is said to be normalized at $w_0 \in \Omega, if\ K(z, w_0) = I, z \in \Omega$. An operator T in $B_n(\Omega)$ may be realized, up to unitary equivalence, as the adjoint of a multiplication operator on a Hilbert space possessing a normalized reproducing kernel (cf. [79]). Realized in this form, the operator is determined completely modulo multiplication by a constant unitary operator acting on $\mathbb{C}^n$. As one might expect, finding the normalized kernel if $n > 1$ is not easy. The theorem below illustrates a rigidity phenomenon in the spirit of what was proved by Curto and Salinas for operators in $B_n(\mathbb{D})$. For quasi-homogeneous operators, the atoms are homogeneous operators in $B_1(\mathbb{D})$. These are assumed to be realized in normal form. Consequently, if T is a quasi-homogeneous operator, a set of $n - 1$ fundamental forms determine the operator T completely, that is, two of them are unitarily equivalent if and only if they are equal assuming they have the same set second fundamental forms.

Theorem (6.3.11)[199]: Suppose that t and $\tilde{t}$ are unitarily equivalent. Then if the second fundamental forms are the same, that is, $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}, 0 \le i \le n - 2$, then $t = \tilde{t}$.

Proof. If necessary, conjugating by a diagonal unitary, without loss of generality, we may assume that the atoms of the operators T and $\tilde{T}$ are the same. By Lemma (6.3.1), if there exists a unitary operator U such that $TU = U\tilde{T}$, then U must be diagonal with unitaries $U_0, U_1, \ldots U_{n-1}$ on its diagonal. Then we have

$$U_i S_{i,j} = \tilde{S}_{i,j} U_j, i,j = 0,1,\ldots,n - 1.$$

In particular, U_i commutes with the fixed set of atoms T_i, which are irreducible, therefore there exists $\beta_i \in [0, 2\pi]$ such that

$$U_i = e^{\iota\beta i} I_{\mathcal{H}_i}, i = 0,1,\cdots,n - 1.$$

Then on the one hand, we have

$$U_i S_{i,i+1}(t_{i+1}) = U_i(-\mu_{i,i+1} t_i) = -\mu_{i,i+1} e^{\iota\beta i}\, t_i$$

and on the other hand, we have

$$\tilde{S}_{i,i+1} Ui + 1(t_{i+1}) = S_{i,i+1}(e^{\iota\beta i+1}\, t_{i+1}) = -\tilde{\mu}_{i,i+1} e^{\iota\beta i+1}\, t_i.$$

Consequently,

$$-\mu_{i,i+1} e^{\iota\beta i} = -\tilde{\mu}_{i,i+1} e^{\iota\beta i+1}, 0 \le i \le n - 2.$$

The assumption that the second fundamental forms are the same for the two operators T and $\tilde{T}$ implies that $\mu_{i,i+1} = \tilde{\mu}_{i,i+1}$. Therefore, we have $\beta_i = \beta_{i+1} := \beta, i = 0,1,\ldots,n-2$. Since

$$U_i S_{i,j} = \tilde{S}_{i,j} U_j, i,j = 0,1,\dots,n-1,$$
we have
$$U_i S_{i,j}(t_j) = e^{\iota\beta} m_{i,j} t_i^{(j-i-1)} = e^{\iota\beta} \widetilde{m}_{i,j} t_i^{(j-i-1)} = \tilde{S}_{i,j} U_j(t_j).$$
Then $m_{i,j} = \widetilde{m}_{i,j}, i,j = 0,1,\dots,n-1$. It follows that $S_{i,j} = \tilde{S}_{i,j}$ and $t = \tilde{t}$.

Remark (6.3.12)[199]: It is natural to ask which of the quasi-homogeneous operators are homogeneous. A comparison with the homogeneous operators given in [84] shows that a quasi-homogeneous operator is homogeneous if and only if
$$\mu_{i,j} = \frac{\Gamma_{i,j}(\lambda)\mu_i}{\mu_j}, \Gamma_{i,j}(\lambda) = \binom{i}{j} \frac{1}{(2\lambda_j)_{i-j}}, \lambda_j = \lambda - \frac{m}{2} + j, \quad (89)$$
for some choice of positive constants $\mu_0 (:= 1), \mu_1, \dots, \mu_{n-1}$. Here $(\alpha) := \alpha(\alpha+1)\cdots(\alpha+\ell-1)$ is the Pochhammer symbol. Clearly, if two homogeneous operators with (λ,μ) and $(\tilde{\lambda}, \tilde{\mu})$ were unitarily equivalent, then λ must equal $\tilde{\lambda}$. Since it is easy to see that $\mu_{i,i+1} = \tilde{\mu}_{i,i+1}$ if and only if $\mu_i = \tilde{\mu}_{i+1}$, we conclude that two of these homogeneous operators are unitarily equivalent if and only if they are equal recovering previous results of [84].

The main focus is on the question of reducibility and strong irreducibility of a quasi-homogeneous operator. We recall that an operator T is said to be strongly irreducible if there is no idempotent in its commutant, or equivalently, there does not exist an invertible operator L for which LTL^{-1} is reducible. The (multiplicity-free) homogeneous operators in the Cowen–Douglas class of rank n are irreducible (cf. [84]). However, they were shown (cf. [183]) to be similar to the $n-$fold direct sum of their atoms making them strongly reducible. It is this phenomenon that we investigate here for quasihomogeneous operators. Along the way, we determine when two quasi-homogeneous operators are similar. Our investigations show that there is dichotomy which depends on whether or not the valency $\Lambda(t)$ is less than 2 or greater or equal to 2. In what follows, we will say that a holomorphic curve $t : \mathbb{D} \to Gr(n,\mathcal{H})$ is strongly irreducible if there is no invertible operator X on the Hilbert space $\mathcal{H}$ for which Xt splits into orthogonal direct sum of two holomorphic curves, say t_1 and t_2, in $Gr(n1,\mathcal{H})$ and $Gr(n_2,\mathcal{H}), n_1 + n_2 = n$, respectively.

Suppose $t : \mathbb{D} \to Gr(n,\mathcal{H})$ is a quasi-homogeneous holomorphic curve with atoms $t_0, t_1, \dots, t_{n-1}$. Then t is strongly reducible, $t \sim t_0 \oplus t_1 \cdots \oplus t_{n-1}$, if $\Lambda(t) \geq 2$ and strongly irreducible otherwise. The dichotomy involving the valency $\Lambda(t)$ is also clear from the main theorem on similarity Theorem (6.3.19) of quasi-homogeneous holomorphic curves.

The atoms of a quasi-homogeneous operator are homogeneous operators in $B_1(\mathbb{D})$ by definition. Therefore, they are uniquely determined not only up to unitary equivalence but up to similarity as well. Now, pick any two quasi-homogeneous operators. They possess an atomic decomposition by virtue of Proposition (6.3.6). Any invertible operator intertwining these two quasi-homogeneous operators is necessarily upper triangular:

Lemma (6.3.13)[199]: Let t and $\tilde{t}$ be two quasi-homogeneous holomorphic curves with atomic decomposition $\{t_i : i = 0,1,\dots,n-1\}$ and $\{\tilde{t}_i : i = 0,1,\dots,n-1\}$, respectively. If they are quasi-similar via the intertwining operators X and , that is, $Xt = \tilde{t}$ and $Y\tilde{t} = t$, then for $i \leq n-1$, we have
$$X\left(\bigvee \{t_0(w), t_1(w), \cdots, t_i(w) : w \in \mathbb{D}\}\right) \subseteq \bigvee \{\tilde{t}_0(w), \tilde{t}_1(w), \cdots, \tilde{t}_i(w) : w \in \mathbb{D}\},$$

$$Y\left(\bigvee\{\tilde{t}_0(w),\tilde{t}_1(w),\cdots,\tilde{t}_i(w): w\in\mathbb{D}\}\right)\subseteq\bigvee\{t_0(w),t_1(w),\cdots,t_i(w): w\in\mathbb{D}\}$$

This is easily proved by modifying the proof [190] slightly. Hence if two quasi-homogeneous operators are similar, then each of the atoms for one must be similar to the other. Consequently, to determine equivalence of quasi-homogeneous operators T under an invertible linear transformation, we may assume (as before) without loss of generality that the atoms are fixed with the weight λ_0 and the valency $\Lambda(t)$. Clearly, the valency $\Lambda(t)$ is both an unitary as well as a similarity invariant of the quasi-homogeneous curve t.

Note that if we let R be the $n\times n$ diagonal matrix with $\left(\prod_{\ell=0}^{i}\mu_{\ell,\ell+1}\right)\left(\prod_{\ell=0}^{i}\tilde{\mu}_{\ell,\ell+1}\right)^{-1}$ on its diagonal and set $\tilde{t}=R\,t\,R^{-1}$, then $\tilde{S}_{i,i+1}(t_{i+1})=\tilde{\mu}_{i,i+1}, 0\le i\le n-2$. Thus up to similarity, we may assume that the constants $\mu_{i,i+1}$ and $\tilde{\mu}_{i,i+1}$ are the same. Or equivalently (see Lemma (6.3.9)), we may assume that the choice of the second fundamental forms $\theta_{i,i+1}, 0\le i\le n-2$, does not change the similarity class of a quasi-homogeneous holomorphic curve. Therefore the condition in the second statement of the theorem given below is not a restriction on the similarity class of the holomorphic curves t and $\tilde{t}$.

The following lemma is the key to determining when a bundle map that intertwines two quasi-homogeneous holomorphic vector bundles extends to an invertible bounded operator. It reveals the intrinsic structure of the intertwiners between two quasi-homogeneous bundles. Recall that if A and B are two operators in $\mathcal{L}(\mathcal{H}_i), i=1,2$ respectively, then the Rosenblum operator $\sigma_{A,B}$ is defined to be the operator $\sigma_{A,B}(X)=AX-XB, X\in\mathcal{L}(\mathcal{H}_2,\mathcal{H}_1)$. If $A=B$, then we set $\sigma_A:=\sigma_{A,B}$.

Lemma (6.3.14)[199]: Let E_t be a quasi-homogeneous vector bundle and $s_{i,j}, i,j=0,1,\cdots,n-1$, be the induced bundle maps and $S_{i,j}:\mathcal{H}_j\to\mathcal{H}_i$ be the operators induced by these bundle maps. The following conditions are equivalent:

(i) $S_{r,s}\in ran\,\sigma_{S_{r,r},S_{s,s}}, 0\le r<s<n-1$

(ii) $\Lambda(t)\ge 2$.

Proof. Let T_r and T_s be the operators induced by $s_{i,i}, i=r,s$ respectively as in Proposition (6.3.6). These are then necessarily the operators $M^{(\lambda_r)^*}$ and $M^{(\lambda_s)^*}$ acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_r)}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_s)}(\mathbb{D})$, respectively. Set $k=s-r-1$.

The kernel of the operator $(T_i-w), w\in\mathbb{D}$, is spanned by the vector $t_i(w):=(1-z\bar{w})^{-\lambda_i}, i=r,s$. For $j=r$ or $j=s$, the set of vectors $e^{(\lambda_j)}:=\sqrt{a\ell(\lambda_j)}z^\ell, \ell\ge 0$ is an orthonormal basis in $\mathbb{A}^{(\lambda_j)}(\mathbb{D})$, where $a_\ell(\lambda_j)$ is the coefficient of $\bar{w}^l z^l$ in the power series expansion for $(1-z\bar{w})^{-\lambda_j}$. The matrix representation for the operator $S_{r,s}:\mathbb{A}^{(\lambda_s)}(\mathbb{D})\to\mathbb{A}^{(\lambda_r)}(\mathbb{D})$ with respect to this orthonormal basis is obtained from the computation:

$$S_{r,s}\left(e^{(\lambda_s)}\right)=\frac{(\ell+k)!}{\ell!}\sqrt{\frac{a_\ell+k(\lambda r)}{a_\ell(\lambda_s)}}\,e_{\ell+k}^{(\lambda_r)}, k=s-r-1.$$

Thus $S_{r,s}$ is a forward shift of multiplicity k. We claim that if $\Lambda(t)\ge 2$, then we can find a forward shift X of multiplicity $k+1$, namely, $X(e^{(\lambda_s)})=x_\ell e_{\ell+k+1}^{(\lambda_r)}$ which has the required intertwining property. Thus evaluating the equation $S_{r,r}X - XS_{s,s} = S_{r,s}$ on the vectors $e_\ell^{(\lambda_s)}, \ell \geq 0$, we obtain

$$\frac{(\ell+k)!}{\ell!} \frac{\prod_{i=0}^{\ell-1} w_i^{(\lambda_s)}}{\prod_{i=0}^{\ell+k-1} w_i^{(\lambda_r)}} e_{\ell+k}^{(\lambda_r)} = (x_\ell w_{\ell+k}^{(\lambda_r)} - (x_{\ell-1} w_{\ell-1}^{(\lambda_s)} e_{\ell+k}^{(\lambda_r)}. \quad (90)$$

From this we find x recursively:

$$w_k^{(\lambda_r)} x_0 = k! \frac{\sqrt{a_k(\lambda_r)}}{\sqrt{a_0(\lambda_s)}}$$

and for $\ell \geq 1$,

$$x_\ell = \frac{\sqrt{a_{k+\ell}(\lambda_r)}}{\sqrt{a_\ell(\lambda_s)}} \sum_{i=1}^{k} (\ell)_i \sim \left(\ell \frac{2k+2-\lambda_s-\lambda_r}{2}\right),$$

where $(\ell)_k := \ell(\ell+1)\cdots(\ell+k-1) = \frac{\Gamma(+k)}{\Gamma(k)}$ is the Pochhammer symbol as before. Here, using the Stirling approximation for the Γ function, we infer that $\sum_{i=1}^{k} (\ell)i \sim \ell^{k+1}$. If $\Lambda(t) \geq 2$, then $\lambda_{r+1} - \lambda_r \geq 2, \lambda_{r+2} - \lambda_{r+1} \geq 2, \cdots, \lambda_s - \lambda_{s-1} \geq 2$. Consequently, $\lambda_s - \lambda_r \geq 2k + 2$ making the operator X bounded.

It follows that if $\Lambda(t) \geq 2$, then the shift X of multiplicity n that we have constructed is bounded and has the desired intertwining property.

To show that there is no such intertwining operator if $\Lambda(t) < 2$, assume to the contrary the existence of such an operator. To arrive at a contradiction, suppose

$$X\left(e^{(\lambda_s)}\right) = \sum_{i=0}^{\infty} x_{i,\ell} e_i^{(\lambda_r)}, X = ((x_{i,\ell})).$$

Then

$$\left(S_{r,r}X - XS_{s,s}\right) e_\ell^{(\lambda_s)} = \sum_{i=0}^{\infty} (x_{i+1,\ell+}1 w_i^{(\lambda_r)} - x_{i,\ell} w_{\ell-1}^{(\lambda_s)} e_i^{(\lambda_r)}.$$

In particular, we have

$$(x+k+1,+1\, w_{\ell+k}^{(\lambda_r)} - x+k, w_{\ell-1}^{\lambda_s})\, (e_{l+k}^{(\lambda_r)}) = S_{r,s} e_\ell^{(\lambda_s)}).$$

Repeating the proof given above, we conclude that $x_{l+k,l} \to \infty, l \to \infty$. This means X is unbounded which is the desired contradiction.

Lemma (6.3.15)[199]: Let t be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \leq i \leq n-1$. Let $T := (\left(S_{i,j}\right))$ be the atomic decomposition of the operator T representing t as in Proposition (6.3.6).

(i) If $\Lambda(t) \in [1 + \frac{n-3}{n-1}, 1 + \frac{n-2}{n})$, then for any $1 \leq r < 1 - n - 3\, n - 1$, we have

$$S_{0,r} S_{r,r+1} \cdots S_{n-2,n-1} \in ran\, \sigma S_{0,0}, S_{n-1,n-1}.$$

(ii) Suppose that $\Lambda(t) \geq 2$. Then there exists a bounded linear operator $X \in L(\mathcal{H}_{n-1}, \mathcal{H}_{n-2})$ such that

$$S_{n-2,n-2}X - XS_{n-1,n-1} = S_{n-2,n-1}$$

and

$$S_{n-3,n-2}X \in ran\sigma S_{n-3,n-3}, S_{n-1,n-1}.$$

Proof. We only prove that $S_{0,n-2}S_{n-2,n-1}$ is in $ran\sigma_{S_{0,0},S_{n-1,n-1}}$. Clearly, as can be seen from the proof we present below, the proof in all the other cases are exactly the same.

Let T_0, T_{n-2} and T_{n-1} be the operators induced by $s_{0,0}, s_{n-2,n-2}$ and s_{n-1} as in Proposition (6.3.6). These are then necessarily the operators $M^{(\lambda_0)^*}, M^{(\lambda_{n-2})^*}$ and $M^{(\lambda_{n-1})^*}$ acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D}), \mathbb{A}^{(\lambda_{n-2})}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_{n-1})}(\mathbb{D})$, respectively.
As in the proof of Lemma (6.3.14), equations (90), we have that

$$S_{0,n-2}\left(e_\ell^{(\lambda_{n-2})}\right) = \frac{(\ell+n-3)!}{\ell!}\sqrt{\frac{a_{\ell+n-3}(\lambda_0)}{a_\ell(\lambda_{n-2})}}\, e_{\ell+n-3}^{(\lambda_0)},$$

$$S_{n-2,n-1}(e_\ell^{(\lambda_{n-1})}) = \frac{\sqrt{a_\ell(\lambda_{n-2})}}{\sqrt{a_\ell(\lambda_{n-1})}}\, e_\ell^{(\lambda_{n-2})}$$

and

$$S_{0,n-2}S_{n-2,n-1}\left(e_\ell^{(\lambda_{n-1})}\right) = \frac{(\ell+n-3)!}{\ell!}\sqrt{\frac{a_{\ell+n-3}(\lambda_0)}{a_\ell(\lambda_{n-2})}}\, e_{\ell+n-3}^{(\lambda_0)}.$$

Thus $S_{0,n-2}S_{n-2,n-1}$ is a forward shift of multiplicity $n-3$. We claim that if $\Lambda(t) \geq 1 + \frac{n-3}{n-1}$, then we can find a forward shift X of multiplicity $n-2$, namely, $X(e_\ell^{(\lambda_{n-1})}) = x_\ell e_{\ell+n-2}^{(\lambda_0)}$ which has the required intertwining property. Thus evaluating the equation $S_{0,0}X - XS_{n-1,n-1} = S_{0,n-1}$ on the vectors e (λ_{n-1}), $\ell \geq 0$, we obtain

$$w_{n-3}^{(\lambda_0)}x_0 = (n-3)!\,\frac{\sqrt{a_{n-3}(\lambda_0)}}{\sqrt{a_\ell(\lambda_{(n-1)})}}$$

and for $\ell \geq 1$, we have that

$$w_{l+n-3}^{(\lambda_0)}x_\ell - x_{\ell-1}w_l^{(\lambda_{n-1})} = \frac{(\ell+n-3)!}{\ell!}\frac{\sqrt{a_{\ell+n-3}(\lambda_0)}}{\sqrt{a_\ell(\lambda_{(n-1)})}}.$$

It follows that

$$x_\ell = \frac{\sqrt{a_{\ell+n-3}(\lambda_0)}}{\sqrt{a_\ell(\lambda_{(n-1)})}}\sum_{i=1}^{n-3}(\ell)_i \sim \left(\ell^{\frac{\lambda_0-\lambda_{n-1}+2n-4}{2}}\right).$$

Note that when $\Lambda(t) > 1 + \frac{n-3}{n-1}$, we obtain

$$\lambda_{n-1} - \lambda_0 = (n-1)\Lambda(t) > (n-1)\frac{2n-4}{n-1} = 2n-4$$

making X bounded. This completes the proof of the first statement.
For the proof of the second statement, note that by virtue of Lemma (6.3.14), we have $S_{n-2,n-1} \in Ran\sigma S_{n-2,n-1}$. So there exists a bounded operator X such that

$$S_{n-2,n-2}X - XS_{n-1,n-1} = S_{n-2,n-1}.$$

Repeating the proof for the first part, we conclude

$$S_{n-3,n-2}X \in ran\,\sigma S_{n-3,n-3}, S_{n-1,n-1}.$$

We now show that a quasi-homogeneous holomorphic curve t is strongly irreducible or strongly reducible according as $\Lambda(t)$ is less than 2 or greater equal to 2. We recall that homogeneous operators (in this case, $\Lambda(t) = 2$) were shown to be irreducible but strongly reducible in [183]

Theorem (6.3.16)[199]: Fix a quasi-homogeneous holomorphic curve t with atoms t_i and let $T = ((S_{i,j}))$ be its atomic decomposition.

(i) If $\Lambda(t) \geq 2$, then T is strongly reducible, indeed T is similar to the direct sum of its atoms, namely, $\bigoplus_{i=0}^{n-1} T_i$ and

(ii) if $\Lambda(t) < 2$, then T is strongly irreducible.

Proof. If $\Lambda(t) \geq 2$, then we claim that the operator T is similar to $T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$. When $n = 2$, Let $T = \begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}$. By Lemma (6.3.14), there exists $X_{0,1}$ such that

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = S_{0,1}.$$

Set $Y_{0,1} = \begin{pmatrix} I & X_{0,1} \\ 0 & I \end{pmatrix}$, then we have that

$$Y_{0,1}TY_{0,1}^{-1} = \begin{pmatrix} S_{0,0} & 0 \\ 0 & S_{1,1} \end{pmatrix}$$

Notice that $Y_{0,1}$ is invertible, we have that $T \sim S_{0,0} \oplus S_{1,1}$.

In this case, using Lemma (6.3.14), we find an invertible bounded linear operator $X_{0,n-1}$ such that

$$S_{0,0}X_{0,n-1} - X_{0,n-1}S_{n-1,n-1} = S_{0,n-1}.$$

For any $i < j$, applying Lemma (6.3.14) to the operators

$$\begin{pmatrix} S_{i,i} & S_{i,i+1} & S_{i,i+2} & \cdots & S_{i,j} \\ 0 & S_{i+1,i+1} & S_{i+1,i+2} & \cdots & S_{i+1,j} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{j-1,j}-1 & S_{j-1,j} \\ 0 & 0 & \cdots & 0 & S_{j,j} \end{pmatrix},$$

we find an invertible bounded linear operator $X_{i,j}$ such that $S_{i,i}X_{i,j} - X_{i,j}S_{j,j} = S_{i,j}$. Set $Y_{n-2,n-1} := \begin{pmatrix} I^{(n-2)} & 0 \\ 0 & I\, X_{n-2,n-1} \\ 0 & I \end{pmatrix}$ and note that $Y_{n-2,n-1}^{-1} = \begin{pmatrix} I^{(n-2)} & 0 \\ 0 & I\, -X_{n-2,n-1} \\ 0 & I \end{pmatrix}$. Now, we have

$$\begin{pmatrix} I^{(n-2)} & 0 \\ 0 & I\, X_{n-2,n-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1} \\ & 0 & \ddots & \ddots\ddots & \vdots \\ 0 \ldots & & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & 0 & \ldots & 0 & S_{n-1,n-1} \end{pmatrix} \begin{pmatrix} I^{(n-2)} & 0 \\ 0 & I\, X_{n-2,n-1} \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} - S_{0,n-2}X_{n-2,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & S_{n-3,n-1} - S_{n-3,n-2}X_{n-2,n-1} \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}.$$

By Lemma (6.3.15), we have

$$S_{n-3,n-2}X_{n-2,n-1} \in ran\, \sigma S_{n-1,n-1}, S_{n-3,n-3} .$$

Therefore, there exists an invertible bounded linear operator X such that

$$S_{n-3,n-3}\tilde{X} - \tilde{X}S_{n-1,n-1} = S_{n-3,n-1} - S_{n-3,n-2}X_{n-2,n-1}.$$

Let $X_{n-3,n-1} := \tilde{X}$ and $Y_{n-3,n-1} = \begin{pmatrix} I^{(n-3)} & & 0 \\ & & I\; 0\; X_{n-3,n-2} \\ 0 & 0\; I & 0 \\ & 0\; 0 & I \end{pmatrix}$. Now, we have

$$Y_{n-3,n-1}\begin{pmatrix} S_{0,0}\ S_{0,1}\ S_{0,2}\ \cdots\ S_{0,n-1}-S_{0,n-2}X_{n-2,n-1} \\ 0\ \ddots\ddots\ddots\ddots\vdots \\ \vdots\ddots\ S_{n-3,n-3}\ S_{n-3,n-2}\ S_{n-3,n-1}-S_{n-3,n-2}X_{n-2,n-1} \\ 0\ \ldots\ 0\ S_{n-2,n-2}\ 0 \\ 0\ \ldots\ \ldots\ 0\ S_{n-1,n-1} \end{pmatrix} Y^{-1}_{n-3,n-1}$$

$$= \cdot \begin{pmatrix} S_{0,0}\ S_{0,1}\ S_{0,2}\ \cdots\ S_{0,n-1}-S_{0,n-2}X_{n-2,n-1}-S_{0,n-3}X_{n-3,n-1} \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots\ S_{n-3,n-3} & \cdots & & 0 \\ 0 & \cdots & 0 & S_{n-2,n-2} & & 0 \\ 0 & \cdots & \cdots & 0 & & S_{n-1,n-1} \end{pmatrix}.$$

Continuing in this manner, we clearly have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots\ddots & S_{n-3,n-3} & S_{n-3,n-2} & & S_{n-3,n-1} \\ 0\ldots 0 & S_{n-2,n-2} & & & S_{n-2,n-1} \\ 0\ldots & & \ldots & 0 & S_{n-1,n-1} \end{pmatrix} \sim \begin{pmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,n-2} & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots\ S_{n-3,n-3} & S_{n-3,n-2} & 0 \\ 0 & \ldots 0 & S_{n-2,n-2} & & 0 \\ 0 & \ldots & \ldots & 0 & S_{n-1,n-1} \end{pmatrix}.$$

This completes the proof of the induction step. We have therefore proved the first statement.

To prove the second statement, assuming that $\Lambda(t) < 2$, we must show that E_t is strongly irreducible. First, we prove that E_t is irreducible. By Lemma (6.3.13), any projection $P = \left((P_{i,j})\right)_{n\times n}$ in $\mathcal{A}'(E_t)$ is diagonal. Thus

$$P_{i,i}^2 = P_{i,i} \in \mathcal{A}'(E_{t_i}).$$

It follows zthat for any $0 \le i \le n-1, P_{i,i} = 0$ or $P_{i,i} = I$. Since $PT = TP$, we have

$$P_{i,i}S_{i,i+1} = S_{i,i+1}P_{i+1,i+1}.$$

Therefore

$$P_{i,i} = P_{j,j}, i,j = 0,1,\cdots,n-1.$$

Consequently, $P = 0$ or $P = I$ and E_t is irreducible.

We first prove that E_t is also strongly irreducible for $n = 2$. By Lemma (6.3.14), we have

$$S_{0,1} \notin ran\,\sigma_{S_{0,0}S_{1,1}}.$$

In [190], it follows that E_t is strongly irreducible.

To complete the proof of the second statement by induction, suppose that it is valid for any $n \le k-1$. For $n = k$, let $P \in \mathcal{A}'(E_t)$ be an idempotent operator. By Lemma (6.3.13), P has the following form:

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots & P_{0,k} \\ 0 & P_{1,1} & P_{1,2} & \cdots & P_{1,k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & P_{k-1,k-1} & P_{k-1,k} \\ 0\ldots & \ldots & & 0 & P_{k,k} \end{pmatrix},$$

and $P\left((S_{i,j})\right)_{k\times k} = \left((S_{i,j})\right)_{k\times k}P$. It follows that

$$((P_{i,j}))\left((S_{i,j})\right) = \left((S_{i,j})\right)((P_{i,j})), 0 \le i,j \le k-1, ((P_{i,j}))\left((S_{i,j})\right)$$
$$= \left((S_{i,j})\right)((P_{i,j})), 1 \le i,j \le k.$$

Both $\left((P_{ij})\right)_{i,j=0}^{k-1}$ and $\left((P_{i,j})\right)_{i,j=1}^{k}$ are idempotents. Since $\Lambda(t) < 2$, we have

$$Sr,s \notin ran\,\sigma_{S_{r,r},S_{s,s}}, r,s \le n.$$

By the induction hypothesis, we have

$$P_{i,j} = 0, i \neq j \leq k - 1,$$

and

$$P_{0,0} = P_{1,1} = \cdots = P_{k,k} = 0, or\ P_{0,0} = P_{1,1} = \cdots = P_{k,k} = I.$$

Thus P has the following form:

$$P = \begin{pmatrix} I\ 0\ 0\ \cdots\ P_{0,k} \\ 0\ \ I\ \ 0\ \cdots\ \ \ 0 \\ \vdots\ddots\ \ \ddots\ \ \ddots\ \ \vdots \\ 0\ \cdots\ 0\ I\ 0 \\ 0\ 0 \ldots 0\ I \end{pmatrix}\ or\ P = \begin{pmatrix} 00\ 0\ \cdots\ P_{0,k} \\ 0\ \ 0\ 0\ \cdots\ 0 \\ \vdots\ \ddots\ \ddots\ \ \ddots\ \ \vdots \\ \vdots\ \ 0\ \cdots\ 0\ 0 \\ 0\ \cdots\cdots\cdots\ 0 \end{pmatrix}$$

Since P is an idempotent, it follows that $P_{0,k} = 0$.

By Lemma (6.3.13), an intertwining operator between two quasi-homogeneous operators with respect to any atomic decomposition must be upper triangular. Thus any operator X in the commutant of such an operator, say T, must also be upper-triangular. In particular, $X_{i,i}$ belongs to the commutant of $S_{i,i}, 0 \leq i \leq n-1$. Since $S_{i,i}$ is a homogeneous operator in $B_1(\mathbb{D})$, it follows that the commutant of $S_{i,i}$ is isomorphic to $\mathcal{H}^\infty(\mathbb{D})$, the space of bounded analytic functions on the unit disc $\mathbb{D}$. Consequently, for any $\emptyset \in \mathcal{H}^\infty(\mathbb{D})$, the operator $\emptyset(S_{i,i})$ is in the commutant A $(S_{i,i})$. In the following lemma, we give a description of the commutant of T. We will construct an operator X in the commutant of T, where the diagonal elements are induced by the same holomorphic function $\emptyset \in \mathcal{H}^\infty(\mathbb{D})$, that is, $\emptyset(S_{i,i}) = X_{i,i}$.

Lemma (6.3.17)[199]: Let t be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \leq i \leq 1$. Let $T = \left(\left(S_{i,j}\right)\right)_{i,j\leq 1}$ be its atomic decomposition. Suppose that $X = \left(\left(X_{i,j}\right)\right)_{i,j\leq 1}$ is in $\mathcal{A}'(T)$. Then there exists $\emptyset \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i} = \emptyset(S_{i,i}), i = 0, 1$ and we also have that

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1} = 0.$$

In particular, $X_{0,1}$ can be chosen as zero.

Proof. Set $X = \left(\left(X_{i,j}\right)\right)_{i,j\leq 1} \in \mathcal{A}'(T)$, we have the following equation

$$\begin{pmatrix} S_{0,0}\ S_{0,1} \\ 0\ S_{1,1} \end{pmatrix}\begin{pmatrix} X_{0,0}\ \ X_{0,1} \\ X_{1,0}\ X_{1,1} \end{pmatrix} = \begin{pmatrix} X_{0,0}\ X_{0,1} \\ X_{1,0}\ X_{1,1} \end{pmatrix}\begin{pmatrix} S_{0,0}\ S_{0,1} \\ 0\ S_{1,1} \end{pmatrix}.$$

By Lemma (6.3.1), we have $X_{1,0} = 0$. Then

$$S_{0,0}X_{0,1} + S_{0,1}X_{1,1} = X_{0,0}S_{0,1} + X_{0,1}S_{1,1},$$

and

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1}.$$

Note that there exist holomorphic functions $\emptyset_{0,0}$ and $\emptyset_{1,1}$ such that

$$X_{0,0}(t_0) = \emptyset_{0,0}t_0, X_{1,1}(t_1) = \emptyset_{1,1}t_1,$$

and by the definition of $S_{0,1}$, there exist constant function $\emptyset_{0,1}$ such that

$$S_{0,1}(t_1) = \emptyset_{0,1}t_0.$$

Then

$$X_{0,0}S_{0,1}(t_1) - S_{0,1}X_{1,1}(t_1) = \left(\emptyset_{0,0}\emptyset_{0,1} - \emptyset_{1,1}\emptyset_{0,1}\right)t_0,$$

and $X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$ also intertwines $S_{0,0}$ and $S_{1,1}$. Taking $X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$ the place of $S_{0,1}$ and using the proof of Lemma (6.3.14), we might deduce that

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1} = 0, \emptyset_{0,0} = \emptyset_{1,1}.$$

Thus we can choose $X_{0,1} = 0$ and there exists a holomorphic function $\emptyset = \emptyset_{0,0} = \emptyset_{(1,1)} \in \mathcal{H}^\infty(\mathbb{D})$ such that $X = \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix}$ where $X_{i,i} = \emptyset(S_{i,i})$ satisfies that

$$\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}\begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix}\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}.$$

Lemma (6.3.18)[199]: Let t be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \le i \le n-1$. Let $T = ((S_{i,j}))$ be its atomic decomposition. Let $\emptyset \in \mathcal{H}^\infty(\mathbb{D})$ be a holomorphic function. If $\Lambda(t) < 2$, then there exists a bounded linear operator $X \in \mathcal{A}'(T)$ such that $X_{i,i} = \emptyset(S_{i,i}), i = 0, 1, \cdots, n-1$.

Proof. Firstly, by Lemma (6.3.17), the lemma is true for the case of $n = 2$. For $n = 3$, let

$$X = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \in \mathcal{A}'(E_t).$$ Then we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix}\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix}\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix}\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} =$$

and it follows that

(i) $S_{0,0}X_{0,1} + S_{0,1}X_{1,1} = X_{0,0}S_{0,1} + X_{0,1}S_{1,1}$, that is, $S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$;

(ii) $S_{1,1}X_{1,2} + S_{1,2}X_{2,2} = X_{1,1}S_{1,2} + X_{1,2}S_{2,2}$, that is, $S_{1,1}X_{1,2} - X_{1,2}S_{2,2} = X_{1,1}S_{1,2} - S_{1,2}X_{2,2}$.

By Lemma (6.3.17), we may choose, without loss of generality, $X_{0,1} = 0$ and $X_{1,2} = 0$. And there exists $\emptyset \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i} = \emptyset(S_{i,i}), i = 0, 1, 2$. It is therefore enough to find an operator $X_{0,2}$ satisfying

$$S_{0,0}X_{0,2} - X_{0,2}S_{2,2} = X_{0,0}S_{0,2} - S_{0,2}X_{2,2}.$$

Clearly, we have

$$\begin{aligned}(X_{0,0}S_{0,2} - S_{0,2}X_{2,2})(t_2(w)) &= X_{0,0}(m_{0,2}t_0^{(1)}(w)) - S_{0,2}(\emptyset(w)t_2(w)) \\ &= m_{0,2}\big(\emptyset(w)t_0(w)\big)^{(1)} - m_{0,2}\emptyset(w)t_0^{(1)}(w) \\ &= m_{0,2}\emptyset^{(1)}(w)t_0(w).\end{aligned}$$

We therefore set $X_{0,2}$ be the operator: $X_{0,2}\big(t_2(w)\big) = m_{0,2}\emptyset^{(1)}(w)t_0^{(1)}(w)$.

To complete the proof by induction, we assume that we have the validity of the conclusion for $n = k-1$. Thus we assume the existence of a bounded linear operator $X = \big((X_{i,j})\big)$ such that $\big((S_{i,j})\big)\ \big((X_{i,j})\big)\ \big((X_{i,j})\big)\ \big((S_{i,j})\big)$ where $X_{i,i} = \emptyset(S_{i,i})$ and $X_{i,i+1} = 0$.

And there exists $l_{i,j}^r$ such that $X_{i,j}\left(t_j\right) = \sum_{r=1}^{j-i-1} l_{i,j}^r \emptyset^{(j-k)} t_i^{(k)}$. To complete the inductive step, we only need to find the operator $X_{0,k}$ satisfying the following equation:

$$S_{0,0}X_{0,k} - X_{0,k}S_{k,k} = X_{0,0}S_{0,k} - S_{0,k}X_{k,k} + \Big(\sum_{i=2}^{k-1} X_{0,i}S_{i,k} - \sum_{i=2}^{k-2} S_{0,i}X_{i,k}\Big) \quad (91)$$

Note that the induction hypothesis ensures the existence of constants $c_{0,k}^s$ (depending on $m_{i,j}$) such that

$$(X_{0,0}S_{0,k} - S_{0,k}X_{k,k} + \sum_{i=2}^{k-1} X_{0,i}S_{i,k} - \sum_{i=1}^{k-2} S_{0,i}X_{i,k})(t_k) = \sum_{s=2}^{k-1} c_{0,k}^s \emptyset(s) t_0^{(k-s-1)} .$$

Now, suppose that $X_{0,k}(t_k) = \sum_{s=1}^{k-1} l_{0,k}^s \emptyset^{(s)} t_0^{(k-s)}$, where the constants $l_{0,k}^s$ are to be found. Then we must have

$$(S_{0,0}X_{0,k} - X_{0,k}S_{k,k})(t_k(w)) = \sum_{s=1}^{k-1} c_{0,k}^s \emptyset^{(s)} t_0^{(k-s)} (w)$$

It follows that if we choose $l_{0,k}^s = \frac{c_{0,k}^s}{k-s}$, then $X_{0,k}$ with this choice of the constants validates equation (91). This completes the induction step.

In particular, when $\mu_{i,j}$ are all chosen to be 1, then $m_{i,j} = -1$, that is, $S_{i,j}(t_j) = -t_j^{(j-i-1)}$. In this case, $X_{0,k}(t_0) = -\sum_{s=1}^{k-1} \emptyset(s) t_0^{(k-s)}$. Now, if $m_{i,j} = -1, i,j = 0,1,\cdots,n-1$, then by a similar argument, we have

$$X_{i,j}(t_j) = -\sum_{s=1}^{j-i-1} \emptyset^{(s)} t_i^{(j-i-s)}, i,j = 0,1,\cdots,n-1. \tag{92}$$

Theorem (6.3.19)[199]: Suppose t and $\tilde{t}$ are quasi-homogeneous holomorphic curves.

(i) If $\Lambda(t) \geq 2$, then t is similar to the $n-$fold direct sum of the atoms $t_0 \oplus t_1 \oplus\cdots\oplus t_{n-1}$.

(ii) If $\Lambda(t) = \Lambda(\tilde{t}) < 2$ and $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}, i = 0,1,\cdots,n-2$, then t and $\tilde{t}$ are similar if and only if they are equal.

Proof. First, if "$\Lambda(t) \geq 2$", then the first conclusion of the theorem follows from Theorem (6.3.16). So, it remains for us to verify the second statement of the theorem, where $\Lambda(t) < 2$.

Let T and $\tilde{T}$ be the operators representing t and $\tilde{t}$ respectively. Recall from Proposition (6.3.6) that $S_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}, \tilde{S}_{i,j}(t_j) = \tilde{m}_{i,j} t_i^{(j-i-1)}$. Up to similarity, we can assume that $m_{i,i+1} = \tilde{m}_{i,i+1}$. Then T and $\tilde{T}$ have the following atomic decomposition:

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & 0 & \ldots & 0 & S_{n-1,n-1} \end{pmatrix} and$$

$$\tilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} & \cdots & c_{0,n-1}S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & c_{1,n-1}S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & S_{n-2,n-2} & c_{n-2,n-1}S_{n-2,n-1} \\ 0 & 0 & \ldots & 0 & S_{n-1,n-1} \end{pmatrix}$$

where $c_{i,j} = \frac{\tilde{m}_{i,j}}{m_{i,j}}$. Now it is enough to prove the Claim stated below.

Claim: If $T \sim \tilde{T}$, then $c_{i,j} = 1, i,j = 0,1,\cdots,n$.

Consider the following possibilities:

(i)$\Lambda(t) \in [0,1)$

(ii)$n = 3, \Lambda(t) \in [1,2)$; $n > 3, \Lambda(t) \in \left[1,\frac{4}{3}\right)$

(iii)$n = 4, \Lambda(t) \in \left[\frac{4}{3}, 2\right); n > 4, \Lambda(t) \in \left[\frac{4}{3}, \frac{3}{2}\right)$
(iv) $n = 5, \Lambda(t) \in \left[\frac{3}{2}, 2\right); n > 5, \Lambda(t) \in \left[\frac{3}{2}, \frac{8}{5}\right)$
The method of the proof below combined with Lemma (6.3.18) and equation (92) completes the proof in the remaining cases.
In what follows, without loss of generality, we will always choose $m_{i,j} = -1, i, j = 0, 1, \cdots, n-1$.
Case (i): By Proposition (6.3.7), we have

$$T = \tilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & 0 & \cdots & 0 \\ & S_{1,1} & S_{1,2} & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & & & S_{n-2,n-2} & S_{n-1,n} \\ & & & & S_{n-1,n-1} \end{pmatrix},$$

In this case, we clearly have $K_{t_i} = K_{s_i}$ and $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}, i = 0, 1, \cdots, n-1$.
Case (ii): By Proposition (6.3.7), we have

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & 0 & 0 \\ & S_{1,1} & S_{1,2} & S_{1,3} & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & S_{n-2,n-2} & S_{n-1,n} \\ & & & & & S_{n-1,n-1} \end{pmatrix},$$

and

$$\tilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} & 0 & \cdots & 0 \\ & S_{1,1} & S_{1,2} & c_{1,3}S_{1,3} & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & S_{n-2,n-2} & S_{n-2,n-1} & c_{n-2,n}S_{n-2,n} & \\ & 0 & & S_{n-1,n-1} & S_{n-1,n} & \\ & & & & S_{n,n} & \end{pmatrix}.$$

In this case, by Proposition (6.3.7), we first assume that n = 3. By Lemma (6.3.1), we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \quad (93)$$

By Lemma (6.3.17) and Lemma (6.3.14), without loss of generality, $X_{0,1}$ and $X_{1,2}$ may be chosen to be zero. Therefore we have the equalities:

$$S_{i,i+1}X_{i+1,i+1} = X_{i,i}S_{i,i+1}, i = 0, 1, and$$
$$S_{0,0}X_{0,2} + S_{0,2}X_{2,2} = c_{0,2}X_{0,0}S_{0,2} + X_{0,2}S_{2,2}.$$

Note that $\mathcal{A}'\left(S_{i,i}\right) \cong \mathcal{H}^{\infty}(\mathbb{D})$, by Lemma (6.3.17), we can find a holomorphic function $\emptyset \in \mathcal{H}^{\infty}(\mathbb{D})$ such that $X_{i,i}\, t_i = \emptyset t_i$. Since $X_{i,i}$ is invertible, $\emptyset(S_{i,i})$ is also invertible. Note that

$$\begin{aligned}(c_{0,2}X_{0,0}S_{0,2} - S_{0,2}X_{2,2})(t_2) &= c_{0,2}X_{0,0}(-t_0^{(1)}) - S_{0,2}(\emptyset t_2) \\ &= \left(c_{0,2-1}\right)S_{0,2\emptyset}(S_{2,2})(t_2) - c_{0,2}S_{0,1}S_{1,2\emptyset^{(1)}}(S_{2,2})(t_2). \end{aligned} \quad (94)$$

By Lemma (6.3.15), we have $c_{0,2}S_{0,1}S_{1,2}\emptyset^{(1)}(S_{2,2}) \in ran\sigma_{S_{0,0},S_{2,2}}$. From (94), it follows that

$$\left(c_{0,2-1}\right)S_{0,2}\emptyset(S_{2,2}) \in ran\, \sigma_{S_{0,0},S_{2,2}}.$$

By Lemma (6.3.7), $S_{0,2} \notin ran\, \sigma_{S_{0,0},S_{2,2}}$. Since $\emptyset(S_{2,2})$ is invertible and $\emptyset\varphi(S_{2,2}) \in \mathcal{A}'(S_{2,2})$, we have

$$S_{0,2}\emptyset(S_{2,2}) \notin ran\, \sigma_{S_{0,0},S_{2,2}}$$

it follows from Theorem (6.3.14). This shows that $c_{0,2} = 1$.

In the following, we will prove the general case. Now suppose that we have proved Claim 1 for $n = k - 1$. Pick $X = \begin{pmatrix} X_{0,0}\; 0 \;\cdots\; X_{0,k} \\ 0\; X_{1,1} \;\cdots\; X_{1,k} \\ \cdots\cdots\cdots\cdots \\ 0\; 0 \;\cdots\; X_{k,k} \end{pmatrix}$ such that $X\tilde{T} = TX$. Then it follows that

$$X_0((\tilde{S}_{i,j})_{i,j=0}^{k-1}) = \left((S_{i,j})_{i,j=0}^{k-1}\right)X_0, X_1((\tilde{S}_{i,j})_{i,j=1}^{k}) = \left((S_{i,j})_{i,j=1}^{k}\right)X_1,$$

where

$$X_0 = \begin{pmatrix} X_{0,0} & 0 & \cdots & X_{0,k-1} \\ 0 & X_{1,1} & \cdots & X_{1,k-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{k-1,k-1} \end{pmatrix}, X_1 = \begin{pmatrix} X_{1,1} & 0 & \cdots & X_{1,k} \\ 0 & X_{2,2} & \cdots & X_{2,k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{k,k} \end{pmatrix}.$$

Since X is invertible, X_0 and X_1 are both invertible. By the induction hypothesis $c_{i,i+2} = 1, i = 0, 1, \cdots, n - 3$.

Case (iii) and Case (iv): By Proposition (6.3.7), $\tilde{T} = ((\tilde{S}_{i,j}))$, $\tilde{S}_{i,j} = 0, j - i \geq 4$ and $\tilde{T} = ((\tilde{S}_{i,j}))$, $\tilde{S}_{i,j} = 0, j - i \geq 5$. Following the proof given above, by Proposition (6.3.7), we only need to consider the case of $n = 4$ and $n = 5$. For case 3, we only consider n = 4 and the other cases would follow by induction. In this case, we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} \\ 0 & S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & 0 & S_{2,2} & S_{2,3} \\ 0 & 0 & 0 & S_{3,3} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} \\ 0 & X_{1,1} & 0 & X_{1,3} \\ 0 & 0 & X_{2,2} & 0 \\ 0 & 0 & 0 & X_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} X_{0,0}\; 0\; X_{0,2} & X_{0,3} \\ 0\; X_{1,1}\; 0 & X_{1,3} \\ 0\; 0\; X_{2,2} & 0 \\ 0\; 0\; 0 & X_{3,3} \end{pmatrix} \begin{pmatrix} S_{0,0}\; S_{0,1}\; S_{0,2} & c_{0,3}S_{0,3} \\ 0\; S_{1,1}\; S_{1,2} & S_{1,3} \\ 0\; 0\; S_{2,2} & S_{2,3} \\ 0\;\; 0\;\; 0 & S_{3,3} \end{pmatrix} \quad = \quad .$$

It follows that $\begin{pmatrix} X_{0,0}\; 0\; X_{0,2} \\ 0\; X_{1,1}\; 0 \\ 0\; 0\; X_{2,2} \end{pmatrix}$ commutes with $\begin{pmatrix} S_{0,0}\; S_{0,1}\; S_{0,2} \\ 0\; S_{1,1}\; S_{1,2} \\ 0\; 0\; S_{2,2} \end{pmatrix}$ and $\begin{pmatrix} X_{1,1}\; 0\; X_{1,3} \\ 0\; X_{2,2}\; 0 \\ 0\; 0\; X_{3,3} \end{pmatrix}$ commutes with $\begin{pmatrix} S_{1,1}\; S_{1,2}\; S_{1,3} \\ 0\; S_{2,2}\; S_{2,3} \\ 0\; 0\; S_{3,3} \end{pmatrix}$. By equation (92), we see that $X_{0,2}$ and $X_{1,3}$ are equal to $S_{0,2\emptyset^{(1)}}(S_{2,2})$ and $S_{1,3\emptyset^{(1)}}(S_{3,3})$. Note that

$$S_{0,0}X_{0,3} + S_{0,1}X_{1,3} + S_{0,3}X_{3,3} = c_{0,3}X_{0,0}S_{0,3} + X_{0,2}S_{2,3} + X_{0,3}S_{3,3}. \quad (95)$$

Then

$$X_{0,2}S_{2,3} - S_{0,1}X_{1,3} = S_{0,2}\ \emptyset^{(1)}(S_{2,2})S_{2,3} - S_{0,1}S_{1,3}\ \emptyset^{(1)}(S_{3,3})$$
$$= (S_{0,2}S_{2,3} - S_{0,1}S_{1,3})\emptyset^{(1)}(S_{3,3}) = 0$$

So we only need to consider

$$S_{0,0}X_{0,3} - X_{0,3}S_{3,3} = c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3}.$$

Since

$\left(c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3}\right)(t_3) = \left(1 - c_{0,3}\right)\emptyset t_0^{(2)} - 2c_{0,3}\ \emptyset^{(1)}t_0^{(1)} - c_{0,3}\ \emptyset^{(2)}t_0,$
we obtain

$$\begin{aligned} c_{0,3}X_{0,0}S_{0,3} &- S_{0,3}X_{3,3} \\ &= \left(c_{0,3-1}\right)S_{0,3}\,\emptyset\left(S_{3,3}\right) + 2c_{0,3}S_{0,1}S_{1,3}\,\emptyset^{(1)}\left(S_{3,3}\right) \\ &+ c_{0,3}S_{0,1}S_{1,2}S_{2,3}\emptyset^{(2)}(S_{3,3}). \end{aligned}$$

By Lemma (6.3.15) and equation (95), we have

$$2c_{0,3}S_{0,1}S_{1,3}\emptyset^{(1)}\left(S_{3,3}\right) + c_{0,3}S_{0,1}S_{1,2}S_{2,3}\emptyset^{(2)}(S_{3,3}) \in Ran\sigma_{S_{0,0},S_{3,3}}.$$

Since $\emptyset(S_{3,3})$ is invertible, we deduce that

$$\left(c_{0,3} - 1\right)S_{0,3} \in ran\,\sigma_{S_{0,0},S_{3,3}}.$$

Note that $S_{0,3} \notin ran\,\sigma_{S_{0,0},S_{3,3}}$, we have $c_{0,3} = 1$. For case 4 with $n = 5$, we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} & S_{0,4} \\ & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ & & S_{2,2} & S_{2,3} & S_{2,4} \\ & 0 & & S_{3,3} & S_{3,4} \\ & & & & S_{4,4} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} & X_{0,4} \\ & X_{1,1} & 0 & X_{1,3} & X_{1,4} \\ & & X_{2,2} & 0 & X_{2,4} \\ & 0 & & X_{3,3} & 0 \\ & & & & X_{4,4} \end{pmatrix}$$

$$= \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} & X_{0,4} \\ & X_{1,1} & 0 & X_{1,3} & X_{1,4} \\ & & X_{2,2} & 0 & X_{2,4} \\ & 0 & & X_{3,3} & 0 \\ & & & & X_{4,4} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S0{,}3 & c_{0,4}S_{0,4} \\ & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ & & S_{2,2} & S_{2,3} & S_{2,4} \\ & 0 & & S_{3,3} & S_{3,4} \\ & & & & S_{4,4} \end{pmatrix}$$

Therefore $\left(\left(X_{ij}\right)\right)_{4\times4}$ commutes with $\left(\left(S_{i,j}\right)\right)_{4\times4}$ for $i,j = 0,1,2,3$ and $\left(\left(X_{ij}\right)\right)_{4\times4}$ commutes with $\left(\left(S_{i,j}\right)\right)_{4\times4}$ for $i,j = 1,2,3,4$. Then from Lemma (6.3.18), we find that $X_{i,j}\,,(i,j) \neq (0,4)$. We also have

$$\begin{aligned} S_{0,0}X_{0,4} &- X_{0,4}S_{4,4} \\ &= (c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4}) + (X_{0,2}S_{2,4} + X_{0,3}S_{3,4}) - (S_{0,1}X_{1,4} \\ &+ S_{0,2}X_{2,4}). \end{aligned} \tag{96}$$

By Lemma (6.3.18), we have

$$X_{0,2}S_{2,4} - S_{0,2}X_{2,4} = S_{0,2}\emptyset^{(1)}\left(S_{2,2}\right)S_{2,4} - S_{0,2}S_{2,4}\emptyset^{(1)}\left(S_{4,4}\right) = S_{0,2}S_{2,3}S_{3,4}\emptyset^{(2)}(S_{4,4}).$$

Lemma (6.3.18) together with the equation (92) gives

$$X_{0,3} = S_{0,2}S_{2,3}\emptyset^{(2)}\left(S_{3,3}\right) + S_{0,3}\emptyset^{(1)}\left(S_{3,3}\right), X_{1,4} = S_{1,3}S_{3,4}\emptyset^{(2)}\left(S_{4,4}\right) + S_{1,4}\emptyset^{(1)}(S_{4,4}).$$

Note that $S_{0,2}S_{2,3} = S_{0,1}S_{1,3}$ and $S_{0,3}S_{3,4} = S_{0,1}S_{1,4}$, we also have

$$\begin{aligned} X_{0,3}S_{3,4} &- S_{0,1}X_{1,4} \\ &= \left(S_{0,2}S_{2,3}\emptyset^{(2)}\left(S_{3,3}\right) + S_{0,3}\emptyset^{(1)}\left(S_{3,3}\right)\right)S_{3,4} - S_{0,1}\left(S_{1,3}S_{93,4}\right)\emptyset^{(2)}\left(S_{4,4}\right) \\ &+ S_{1,4}\emptyset^{(1)}(S_{4,4})) = 0. \end{aligned}$$

Since

$$\begin{aligned} \left(c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4}\right)(t_4) &= c_{0,4}X_{0,0}S_{0,4}(t_4) - S_{0,4}(\emptyset t_4) \\ &= \left(1 - c_{0,4}\right)\emptyset t_0^{(3)} - 3c_{0,4}\emptyset^{(2)}t_0^{(1)} - 3c_{0,4}\emptyset^{(1)}t_0^{(2)} - c_{0,4}\emptyset^{(3)}t_0, \end{aligned}$$

we also have

$$\begin{aligned} c_{0,4}X_{0,0}S_{0,4} &- S_{0,4}X_{4,4} \\ &= \left(c_{0,4} - 1\right)S_{0,4}\emptyset\left(S_{4,4}\right) + 3c_{0,4}S_{0,1}S_{1,3}\emptyset^{(1)}\left(S_{3,3}\right) \\ &+ 3c_{0,4}S_{0,1}S_{1,2}S_{2,3}\emptyset^{(2)}\left(S_{3,3}\right) + c_{0,4}S_{0,1}S_{1,2}S_{2,3}\emptyset^{(3)}(S_{3,3}). \end{aligned}$$

Combining Lemma (6.3.15) with the equation (96), we obtain

$$3c_{0,4}S_{0,1}S_{1,3}\emptyset^{(1)}(S_{3,3}) + 3c_{0,4}S_{0,1}S_{1,2}S_{2,3}\emptyset^{(2)}(S_{3,3}) + c_{0,4}S_{0,1}S_{1,2}S_{2,3}\emptyset^{(3)}(S_{3,3})$$
$$\in ran\,\sigma S_{0,0}, S_{4,4}\,, S_{0,2}S_{2,3}S_{3,4}\emptyset^{(2)}(S_{4,4}) \in ran\sigma S_{0,0}, S_{4,4}\,.$$

Then it follows that

$$(c_{0,4} - 1)S_{0,4}\emptyset(S_{4,4}) \in ran\,\sigma S_{0,0}, S_{4,4}\,.$$

Note that $\emptyset(S_{4,4})$ is invertible, therefore

$$(c_{0,4} - 1)S_{0,4} \in ran\,\sigma_{S_{0,0},S_{4,4}}\,.$$

Since $S_{0,4} \notin ran\,\sigma S_{0,0}, S_{4,4}$, it follows that $c_{0,4} = 1$.

We give two different applications of our results. First of these shows that the topological and algebraic K −groups defined must coincide. Second, we show that the Halmos' question on similarity has an affirmative answer for quasi-homogeneous operators.

We begin with some preliminaries on $K-$ groups. Let $t : \Omega \to Gr(n,\mathcal{H})$ be a holomorphic curve. Recall that the commutant $\mathcal{A}'\,(E_t)$ of such a holomorphic curve t is defined to be

$$\mathcal{A}'\,(E_t) = \{A \in \mathcal{L}(\mathcal{H}) : A\,t(w) \subseteq t(w), w \in \Omega\}.$$

Definition (6.3.20)[199]: For a holomorphic curve $t : \Omega \to Gr(n,\mathcal{H})$, the Jocaboson radical Rad $\mathcal{A}'\,(E_t)$ of $\mathcal{A}'\,(E_t)$ is defined to be

$$\{S \in \mathcal{A}'\,(E_t) | \sigma_{\mathcal{A}'}(E_t)(SA) = 0, A \in \mathcal{A}'\,(E_t)\},$$

where $\sigma_{\mathcal{A}'}(E_t)(SA)$ denotes the spectrum of SA in the algebra $\mathcal{A}'\,(E_t)$.

The discussion below follows closely in [58] of the first two authors.

Definition (6.3.21)[199]: A holomorphic curve $t : \Omega \to Gr(n,\mathcal{H})$ is said to have a finite decomposition if it meets one of the equivalent conditions given in [58].

Suppose $\{P_1, P_2, \cdots, P_m\}$ and $\{Q_1, Q_2, \cdots, Q_n\}$ are two distinct decompositions of t.

If $m = n$, there exists a permutation $\Pi \in S_n$ such that $XQ_{\Pi(i)}X^{-1} = P_i$ for some invertible operator X in $\mathcal{A}'(E_t), 1 \le i \le n$, then we say that t (or E_t) has a unique decomposition up to similarity.

For a holomorphic curve, $f : \Omega \to Gr(n,\mathcal{H})$, let $M_k(\mathcal{A}'\,(E_t))$ be the collection of $k \times k$ matrices with entries from $\mathcal{A}'\,(E_t)$. Let

$$M_\infty(\mathcal{A}'\,(E_t)) = \bigcup_{k=1}^{\infty} M_k(\mathcal{A}'\,(E_t)),$$

and Proj$(M_k(\mathcal{A}'\,(E_t)))$ be the algebraic equivalence classes of idempotents in $M_\infty(\mathcal{A}'\,(E_t))$. If p, q are idempotents in Proj$(\mathcal{A}'\,(E_t))$, then say that $p \sim_{st} q$ if $p \oplus r \sim_a q \oplus r$ for some idempotent r in Proj $(\mathcal{A}'\,(E_t))$. The relation $\sim_{st}$ is known as stable equivalence.

Let X be a compact Hausdorff space, and $\xi = (E, \pi, X)$ be a (topological) vector bundle. A well-known theorem due to $R.G.$ Swan says that a vector bundle $\xi = (E, \pi, X)$ is a direct summand of the trivial bundle, that is,

$$\xi \oplus \eta \sim= (X \times \mathbb{C}^n, \pi, X)$$

for some vector bundle $\eta = (F, \rho, X)$.

None of what we have said so far applies to holomorphic vector bundles over an open subset of $\mathbb{C}$ since they are already trivial by Graut's theorem. However, the study of holomorphic vector bundles over an open subset of $\mathbb{C}$ is central to operator theory. In the context of operator theory, as shown in the foundational of Cowen and Douglas [61], the vector bundles of interest are equipped with a Hermitian structure inherited from a fixed inner product of some Hilbert space $\mathcal{H}$. This makes it possible to ask questions about their equivalence under a unitary or an invertible linear transformation of $\mathcal{H}$. In [61], questions regarding unitary equivalence were dealt with quite successfully while equivalence under an invertible linear transformation remains somewhat of a mystery to date. However, we can ask if the uniqueness of the summand, which was a consequence of Swan's theorem, remains valid of Cowen–Douglas operators.

We will need the following lemma.

Lemma (6.3.22)[199]: Let E_t be a quasi-homogeneous bundle. Then $\mathcal{A}'(E_t)/Rad(\mathcal{A}'(E_t))$ is commutative.

Proof. Let
$$S = \{Y : \sigma(Y) = 0, Y \in \mathcal{A}'(E_t)\}.$$
Claim 1: S is an ideal of the algebra $\mathcal{A}'(E_t)$.

By Lemma (6.3.13), Y is upper-triangular if $Y \in S$. Since the spectrum $\sigma(Y)$ of Y is $\{0\}$, the operator Y must be of the form
$$Y = \begin{pmatrix} 0 & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,n-1} \\ 0 & 0 & Y_{1,2} & \cdots & Y_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & Y_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$
and it follows that each quasi-nilpotent element in the commutant of the holomorphic curve t of rank one is zero. Using Lemma (6.3.13) again, each element $X \in \mathcal{A}'(E_t)$ is upper-triangular. Thus $\sigma(XY) = \sigma(YX) = 0$. This completes the proof of Claim 1 and $S = Rad(\mathcal{A}'(E_t))$.

Claim 2: $\mathcal{A}'(E_t)/Rad(\mathcal{A}'(E_t))$ is commutative.

Note that if $X \in \mathcal{A}'(E_t)$ is (block) nilpotent, then $X \in S. A$ simple computation shows that $\mathcal{A}'(E_t)/Rad(\mathcal{A}'(E_t))$ is commutative.

For any holomorphic curve t, we let t n denote the n −fold direct sum of t. For any two natural numbers n and m, let E_r and E_s be the sub-bundles of E_{t^n} and E_{t^m}, respectively. If $m > n$, then both E_r and E_s can be regarded as a sub-bundle of E_{t^m}.

Two holomorphic Hermitian vector bundles E_r and E_s are said to be similar if there exist an invertible operator $X \in \mathcal{A}'(E_{t^n})$ such that $XE_r = E_s$. Analogous to the definition of $Vect(X)$, we let $Vect^0(E_t)$ be the set of equivalence classes $\overline{E_s}$ of the sub-bundles E_s of $E_{t^n}, n = 1, 2, \cdots$. An addition on $Vect^0(E_t)$ is defined as follows, namely,
$$\overline{E_r} + \overline{E_s} = \overline{E_r \oplus E_s},$$
where E_r and E_s are both sub-bundles of E_t. Now, the group $K_0(E_t)$ is the Grothendieck group of $(Vect^0(E_t), +)$. We have the following theorem.

Lemma (6.3.23)[199]: Let E_t be a quasi-homogeneous bundle. Then
$$Vect(\mathcal{A}'(E_t)) \cong Vect(\mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t)).$$
Proof. Note that $M_n(\mathcal{A}'(E_t)) \cong \mathcal{A}'(\oplus^n E_t)$. Let $p \in M_n(\mathcal{A}'(E_t))$ be an idempotent. Define a map $\sigma : Vect(\mathcal{A}'(E_t)) \to Vect(\mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t))$ as the following:
$$\sigma[P] = [\pi(P)],$$
where $\pi : \mathcal{A}'(E_t) \to Vect(\mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t))$.

Claim σ is well defined and it is an isomorphism.

If $[p] = [q]$, where $p \in M_n(\mathcal{A}'(E_t))$ and $q \in M_m(\mathcal{A}'(E_t))$ are both idempotents, then there exists $k \geq max\{m, n\}$ and an invertible element $u \in M_k(\mathcal{A}'(E_t))$ such that
$$u\left(p \oplus 0^{(k-n)}\right)u^{-1} = q \oplus 0^{(k-m)}.$$
Thus we have

$$\pi(u)\pi\big(p \oplus 0^{(k-n)}\big)\pi(u)^{-1} = \pi\big(u\big(p \oplus 0^{(k-n)}\big)u^{-1}\big) = \pi(q \oplus 0^{(k-m)}).$$

That means $[\pi(p)] = [\pi(q)]$, and σ is well defined.

Now, we would prove that σ is injective. In fact, if $p \in M_n(\mathcal{A}'(E_t))$ and q $\in M_m(\mathcal{A}'(E_t))$ are idempotents with

$$\sigma[p] = [\pi(p)] = [\pi(q)] = \sigma[q],$$

then we can find $k \geq max\{m,n\}$ and an invertible element $\pi(u) \in M_k(\mathcal{A}'(E_t))/Rad(M_k(\mathcal{A}'(E_t)))$ such that

$$\pi(u)\left(\pi\big(p \oplus 0^{(k-n)}\big)\right)\pi(u)^{-1} = \pi(q \oplus 0^{(k-m)}).$$

Since $\pi(u)$ is invertible, there exists $\pi(s) \in Rad(M_k(\mathcal{A}'(E_t)))$ such that $\pi(u)^{-1} = \pi(s)$. Then we have

$$us = I - R_1, su = I - R_2,$$

where $R_1, R_2 \in Rad(M_k(\mathcal{A}'(E_t)))$. Since $\sigma(R_1) = \sigma(R_2) = \{0\}$, then us, su are both invertible. Therefore, u is invertible and thus

$$\pi\big(u\big(p \oplus 0^{(k-n)}\big)u^{-1}\big) = \pi(u)\left(\pi\big(p \oplus 0^{(k-n)}\big)\right)\pi(u)^{-1} = \pi(q \oplus 0^{(k-m)}).$$

Thus

$$u\big(p \oplus 0^{(k-n)}\big)u^{-1} = q \oplus 0^{(k-m)} + R$$

for some $R \in Rad(M_k(\mathcal{A}'(E_t)))$. Let $W_1 = 2(q \oplus 0^{(k-m)}) - I$. Since $\sigma(Q \oplus 0^{(k-m)}) \subseteq \{0,1\}$, then W_1 is invertible. Since we have $R \in Rad(M_k(\mathcal{A}'(E_t)))$ and $W_1^{-1} \in M_k(\mathcal{A}'(E_t))$, then $RW_1^{-1} \in Rad(M_k(\mathcal{A}'(E_t)))$, so $I + RW_1^{-1}$ is invertible. Set

$$W = 2\big(q \oplus 0^{(k-m)}\big) - I + R = W1 + R = (I + RW_1^{-1})W_1,$$

and W is invertible. Since $p \oplus 0^{(k-n)}$ is an idempotent, it follows that $u\big(p \oplus 0^{(k-n)}\big)u^{-1}$ and hence $(q \oplus 0^{(k-m)}) + R$ is an idempotent as well. Thus

$$\big(q \oplus 0^{(k-m)}\big)^2 + (q \oplus 0^{(k-m)})R + R(q \oplus 0^{(k-m)}) + R^2$$
$$= (q \oplus 0^{(k-m)}) + R.$$

Similarly, $q \oplus 0^{(k-m)}$ is an idempotent, therefore

$$(q \oplus 0^{(k-m)})R + R(q \oplus 0^{(k-m)}) + R^2 = R.$$

So we have

$$\begin{aligned} &W((q \oplus 0^{(k-m)} + R) \\ &= (q \oplus 0^{(k-m)}) + R(q \oplus 0^{(k-m)}) + 2(q \oplus 0^{(k-m)})R - R + R^2 \\ &= (q \oplus 0^{(k-m)}) + (q \oplus 0^{(k-m)})R \\ &= (q \oplus 0^{(k-m)})W \end{aligned}$$

and

$$u\big(p \oplus 0^{(k-n)}\big)u^{-1} = \big(q \oplus 0^{(k-m)}\big) + R = W\big(q \oplus 0^{(k-m)}\big)W^{-1}.$$

It follows that $p \sim_a q$, and σ is injective. Finally, we show that σ is surjective. For each $[\pi(p)] \in Vect\left(\frac{\mathcal{A}'(E_t)}{Rad\mathcal{A}'(E_t)}\right)$ with $\pi(p) \in \frac{M_n\left(\mathcal{A}'(E_t)\right)}{Rad\left(M_n(\mathcal{A}'(E_t))\right)}$, $p \in M_n\big(\mathcal{A}'(E_t)\big)$ and $\pi^2(p) = \pi(p)$, we have

$$p^2 - p = R_0, R_0 \in Rad(M_n(\mathcal{A}'(E_t))).$$

Note that $p = B + R$, where $B \in M_n(\mathcal{A}'(E_t))$ is a block-diagonal matrix over $\mathbb{C}$ and R is in $Rad(M_n(\mathcal{A}'(E_t)))$. Then $\pi(p) = \pi(B)$ and

$$R_0 = p^2 - p = (B + R)^2 - (B + R) = B^2 - B + (BR + RB + R^2 - R).$$

Since $Rad(M_n(\mathcal{A}'(E_t)))$ is an ideal of $M_n(\mathcal{A}'(E_t))$, then we have

$$B^2 - B \in Rad(M_n(\mathcal{A}'(E_t))).$$

Since B is a block-diagonal matrix, then we have B is also an idempotent. Then we have

$$\sigma([B]) = [\pi(p)].$$

That means σ is also a surjective. And we also can see that σ is homomorphism. Then σ is an isomorphism and

$$Vect(\mathcal{A}'(E_t)) \sim= Vect(\mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t)).$$

Proposition (6.3.24)[199]: Let E_t and $E_{\tilde{t}}$ be two quasi-homogeneous bundles with matchable bundles $\{E_{t_i}\}_{i=0}^{n-1}$ and $\{E_{s_i}\}_{i=0}^{n-1}$ respectively. If $\Lambda(t) < 2$, then E_t and $E_{\tilde{t}}$ are similarity equivalent if and only if

$$K_0(\mathcal{A}'(E_t \oplus E_{\tilde{t}})) \cong \mathbb{Z}.$$

If $\Lambda(t) \geq 2$, then E_t and E_t are similarity equivalent if and only if

$$K_0(\mathcal{A}'(E_t \oplus E_{\tilde{t}})) \cong \mathbb{Z}^n.$$

Proof. Suppose that $\Lambda(t) < 2$. Let

$$T = \begin{pmatrix} S_{0,0}\ S_{0,1}\ S_{0,2} & \cdots & S_{0,n-1} \\ S_{1,1}\ S_{1,2} & \cdots & S_{1,n-1} \\ \ddots & \ddots & \vdots \\ & S_{n-1,n-1} & S_{n-1,n} \\ & & S_{n,n} \end{pmatrix} and\ X$$

$$= \begin{pmatrix} X_{0,0}\ X_{0,1}\ X_{0,2} & \cdots & X_{0,n-1} \\ X_{1,1}\ X_{1,2} & \cdots & X_{1,n-1} \\ \ddots & \ddots & \ddots \\ & & X_{n-1,n} \\ & X_{n-1,n-1} & X_{n,n} \end{pmatrix}.$$

Claim 1: If $XT = TX$, then we have $X_{i,i} = \emptyset(S_{i,i})$ for any i, where $\emptyset \in \mathcal{H}^\infty(\mathbb{D})$. In fact, for any $i = 0,1,\cdots,n-1$, we have

$$S_{i,i}X_{i,i+1} + S_{i,i+1}X_{i+1,i+1} = X_{i,i}S_{i,i+1} + X_{i,i+1}S_{i+1,i+1},$$

and

$$S_{i,i}X_{i,i+1} - X_{i,i+1}S_{i+1,i+1} = X_{i,i}S_{i,i+1} - S_{i,i+1}X_{i+1,i+1} = 0.$$

Since $X_{i,i} \in \mathcal{A}'(E_{t_i})$ and each E_{t_i} induces a Hilbert functional space $\mathcal{H}_i$ with reproducing kernel $\frac{1}{(1-z\bar{w})^{\lambda_i}}$, then we have $\mathcal{A}'(E_{t_i}) \cong \mathcal{H}^\infty(\mathbb{D})$. Then there exists $\emptyset_{i,i} \in \mathcal{H}^\infty(\mathbb{D})$ such that

$$X_{i,i} = \emptyset_{i,i}(S_{i,i}), i = 0,1,\cdots,n-1.$$

Thus we have

$$\emptyset_{i,i}(S_{i,i})S_{i,i+1} - S_{i,i+1}\emptyset_{i+1,i+1}(S_{i+1,i+1}) = 0.$$

Since $S_{i,i}S_{i,i+1} = S_{i,i+1}S_{i+1,i+1}$, then

$$S_{i,i+1}(\emptyset_{i,i} - \emptyset_{i+1,i+1})(S_{i+1,i+1}) = 0.$$

Note that $S_{i,i+1}$ has a dense range, then we can set

$$\emptyset_{i,i} = \emptyset, i = 0,1,\cdots,n-1.$$

Claim 2: $\mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t) \cong \mathcal{H}^\infty(\mathbb{D})$.

Recall that $Rad\mathcal{A}'(E_t) = \{S \in \mathcal{A}'(E_t) | \sigma_{\mathcal{A}'(E_t)}(SS') = 0, S \in \mathcal{A}'(E_t)\}$. Any $X \in \mathcal{A}'(E_t)$ is upper triangular by Lemma (6.3.13) and $\mathcal{A}'(E_t)/Rad\ \mathcal{A}'(E_t)$ is commutative by Lemma (6.3.22). Therefore if Y is in Rad $\mathcal{A}'(E_t)$, then we have

$$Y = \begin{pmatrix} 0 & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,n-1} \\ & 0 & Y_{1,2} & \cdots & Y_{1,n-1} \\ & & & & \ddots \\ 0 & & \ddots & \ddots & Y_{n-1,n} \\ & & & 0 & 0 \end{pmatrix}.$$

Define a map $\Gamma : \mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t) \rightarrow \mathcal{H}^\infty(\mathbb{D})$ by the rule:

$$\Gamma([X]) = \emptyset, where\ X = \left((X_{i,j})\right)_{n\times n}, X_{i,i} = \emptyset(S_{i,i}).$$

Obviously, Γ is well defined and if $\Gamma([X]) = 0$, then $\emptyset = 0$. Then $X_{i,i} = 0$, it follows that $X \in Rad\mathcal{A}'(E_t)$ and $[X] = 0$. So Γ is injective.

For any $\emptyset \in \mathcal{H}^\infty(\mathbb{D})$, set $X_{i,i} \neq \emptyset(S_{i,i}), i = 0, 1, 2, \cdots, n-1$. By Lemma (6.3.18), we can construct the operators $X_{i,j}, j \neq i$ such that $X := \left((X_{i,j})\right)_{n\times n} \in \mathcal{A}'(E_t)$. That means Γ is surjective. Then Γ is an isomorphism and

$$\mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t) \cong \mathcal{H}^\infty(\mathbb{D}).$$

By [58] and ([204]), we have

$$Vect(\mathcal{A}'(E_t)) \cong \mathbb{N}, K_0(\mathcal{A}'(E_t)) \cong \mathbb{Z}.$$

By [58], we have E_t has a unique finite decomposition up to similarity. Similarly, $E_{\tilde{t}}$ also has a unique finite decomposition up to similarity.

If $E_t \sim E_{\tilde{t}}$, then $(t \oplus \tilde{t}) \sim t^{(2)}$. So we have

$$Vect(\mathcal{A}'(t \oplus \tilde{t})) \cong Vect(\mathcal{A}'(t^{(2)})) \cong Vect M_2(\mathcal{A}'(t))) \cong \mathbb{N}$$

And

$$K_0(\mathcal{A}'(t \oplus \tilde{t})) \cong \mathbb{Z}.$$

On the other hand, note that t and $\tilde{t}$ are both strongly irreducible. If $K_0(\mathcal{A}'(t \oplus \tilde{t})) \cong \mathbb{Z}$ and $Vect(\mathcal{A}'(t \oplus \tilde{t})) \cong \mathbb{N}$, then by [58], we have $t \sim \tilde{t}$, otherwise we will have

$$Vect\left(\mathcal{A}'(t \oplus \tilde{t})\right) \cong \mathbb{N}^2.$$

This is a contradiction.

If $\Lambda(t) \geq 2$, then by Theorem (6.3.16), we have

$$E_t \sim E_{t0} \oplus E_{t1} \oplus \cdots \oplus E_{t_{n-1}}\ and\ E_{\tilde{t}} \sim E_{\tilde{t}_0} \oplus E_{\tilde{t}_1} \oplus \cdots \oplus E\tilde{t}_{n-1}.$$

By [203] and [204], we have that $E_t \sim E_{\tilde{t}}$ if and only if

$$E_t \oplus E_{\tilde{t}} \sim E_{t_0}^{(2)} \oplus E_{t_1}^{(2)} \oplus \cdots \oplus E_{t_{n-1}}^{(2)},$$

and in this case, it follows from [203] that

$$K_0(\mathcal{A}'(E_t \oplus E_{\tilde{t}})) \sim= \mathbb{Z}^n.$$

If E_t is not similar to $E_{\tilde{t}}$, then there exists $\{i_0, i_1 \cdots i_{n-1}\} = \{1, 2, \cdots, n-1\}$ and $\{j_0, j_1, \cdots, j_{n-1}\} = \{1, 2 \cdots, n-1\}$ such that $E_{t_{i_m}} \sim E_{\bar{t}_{jm}}, m = 0, 1, \cdots, l, l \leq n-2$.

Note that none of the holomorphic curves $Et_{i_{l+1}}, Et_{i_{l+2}}, \cdots, E_{t_{i_{n-1}}}$ is similar to any of the $E_{\bar{t}_0}, E_{\bar{t}_1}, \ldots, E_{\bar{t}_{n-1}}$. Thus, we have

$$E_t \oplus E_{\bar{t}} \sim E_{\bar{t}_1} \oplus \cdots \oplus E_{\bar{t}_{j0}}^{(2)} \oplus E_{\bar{t}_{j0+1}} \oplus \cdots \oplus E_{\bar{t}_{jl}}^{(2)} \oplus \cdots \oplus E_{\bar{t}_{n-1}} \oplus E_{t_{i_{l+1}}} \oplus \cdots \oplus E_{t_{i_{n-1}}}.$$

By [203], we have that

$$K_0\left(\mathcal{A}'(E_t \oplus E_{\tilde{t}})\right) \sim= \mathbb{Z}^{2n-l-1}.$$

Since $2n - l - 1 > n$, the proof is complete.

Theorem (6.3.25)[199]: For any quasi-homogeneous holomorphic curve t with atoms $t_i, 0 \leq i \leq n - 1$, we have that

(i) E_t has no non-trivial sub-bundle induced by a non-trivial idempotent of $\mathcal{A}'(E_t)$ whenever $\Lambda(t) < 2$, and

(ii) if $\Lambda(t) \geq 2$, then for any direct summand E_r of E_t, there exists a unique subbundle E_s, up to equivalence under an invertible map, such that $E_r \oplus E_s$ is similar to E_t.

Proof. When $\Lambda(t) < 2$, by Theorem (6.3.16), we have E_t is strongly irreducible. So there exists no non-trivial idempotent in $\mathcal{A}'(E_t)$, which is the same as saying that the vector bundle E_t has no non-trivial sub-bundle could be induced by a non-trivial idempotent of the commutant of E_t.

When $\Lambda(t) \geq 2$, by Theorem (6.3.16), we have
$$E_t \sim E_{t_0} \oplus E_{t_1} \oplus \cdots \oplus E_{t_{n-1}}.$$
Since $\mathcal{A}'(E_{t_i}) \sim= \mathcal{H}^\infty(\mathbb{D})$, we have
$$\mathcal{A}'(E_t) \sim= \mathcal{H}^\infty(\mathbb{D})^{(n)},$$
and by [58],
$$Vect(\mathcal{A}'(E_t)) \cong \mathbb{N}^{(n)}, K_0(\mathcal{A}'(E_t)) \cong \mathbb{Z}^{(n)}.$$
Then by [58], we have E_t has a unique finite decomposition up to similarity. Then for any non-trivial reducible sub-bundle of E_t denoted by E_r, with
$$\mathcal{H}_r = Span_{w\in\Omega}\{E_r(w)\}.$$
Let P_r be the projection from $\mathcal{H}$ to $\mathcal{H}_r$. Then
$$E_t \sim E_r \oplus (E_t \ominus E_r) = P_r E_t \oplus (I - P_r)E_t.$$
Let
$$P_{t_i} : \mathcal{H} \to \mathcal{H}_i := Span_{\lambda\in\Omega}\{E_{t_i}(w)\}, i = 0, 1, \cdots, n - 1$$
be projections in $\mathcal{A}'(E_r)$. Then there exists an invertible operator X such that $E_r = X(\bigoplus_{i=0}^{s} E_{tk_i})$. Suppose that
$$\bigoplus_{i=0}^{n-1} E_{t_i} = (\bigoplus_{i=0}^{s} E_{t_{k_i}}) \oplus (\bigoplus_{i=0}^{n-s} E_{t_{l_i}}).$$
Set $E_s = X(\bigoplus_{i=0}^{n-s} E_{t_{l_i}})$, then we have
$$E_r \oplus E_s \sim E_t.$$
If there exists another bundle E_s such that
$$E_r \oplus E_s \sim E_t.$$
Since E_r has a unique finite decomposition up to similarity, then we have
$$E_{s'} \sim \bigoplus_{i=0}^{n-s} E_{t_{l_i}} \sim E_s.$$

Theorem (6.3.26)[199]: $K^0(E_t) \cong K_0(\mathcal{A}'(E_t))$.

Proof. Let $P \in P_n(\mathcal{A}'(E_t)) = P(\mathcal{A}'(E_{t^n}))$ be an idempotent. Then we have PEtn be a sub-bundle of E_{t^n}. Define map
$$\Gamma : V(\mathcal{A}'(E_t))) \to V^0 \quad (E_t)$$
with $\Gamma([p]_0) = \overline{PE_{t^n}}$.

First, we prove that Γ is well defined. In fact, for any $P \sim Q \in [P]_0$, there exists positive integer n such that $P, Q \in \mathcal{A}'(E_{t^n})$. Since $Q = XPX^{-1}, X \in \mathcal{A}'(E_{t^n})$, then we have
$$QE_{t^n} = XPX^{-1}E_{t^n} \sim PX^{-1}E_{t^n}.$$
And note that $X, X^{-1} \in \mathcal{A}'(E_{t^n})$, then we have
$$X^{-1}t^n(w) = t^n(w), \text{for any } w \in \Omega.$$

Thus

$$QE_{t^n} \sim PXE_{t^n},$$

and $\overline{QE_{t^n}} = \overline{PE_{t^n}}$. So Γ is well defined.

Second, we prove that Γ is surjective. Suppose that E_r is a sub-bundle of E_{t^n} with dimension K, where n is positive integer. Suppose that

$$\mathcal{H}_r := \bigvee_{w\in\Omega} \{\gamma_1(w), \gamma_2(w), \cdots, \gamma_K(w)\},$$

where $K \in \mathbb{N}$ and P_r is the projection from $\mathcal{H}$ to $\mathcal{H}_r$, then we have $P_r \in \mathcal{A}'(E_{t^n})$ and

$$P_rE_{t^n} \sim E_r.$$

Then it follows that Γ is surjective.

Finally, we prove that Γ is also injective. Let $P, Q \in \mathcal{A}'(E_{t^n})$. Suppose that there exists an invertible operator $X \in \mathcal{A}'(E_{t^n})$ such that

$$XPE_{t^n} = QE_{t^n}.$$

Let $\{p_1, p_2, \cdots, p_m\}$ be a decomposition of P. Then $\{X_{p_1}X^{-1}, X_{p_2}X^{-1}, \cdots, X_{p_m}X^{-1}\}$ be a decomposition of Q. In fact, we have

$$\begin{aligned}&X_{p_1}X^{-1}QE_{t^n} + X_{p_2}X^{-1}QE_{t^n} + \cdots + X_{p_m}X^{-1}QE_{t^n}\\ &\qquad = X_{p_1}E_{t^n} + X_{p_2}E_{t^n} + \cdots + X_{p_m}E_{t^n} = XPE_{t^n} = QE_{t^n}.\end{aligned}$$

Suppose that $\{p_{m+1}, p_{m+2}, \cdots, p_N\}$ and $\{q_{m+1}, q_{m+2}, \cdots, q_N\}$ be the decompositions of $(I - P)E_{t^n}$ and $(I - Q)E_{t^n}$ respectively. Then we have

$$\{p_1, p_2, \cdots, p_N\} \ and \ \{Xp_1X^{-1}, Xp_2X^{-1}, \cdots, Xp_mX^{-1}, q_{m+1}, q_{m+2}, \cdots, q_N\}$$

are two different decompositions of E_{t^n} . By the uniqueness of decomposition of E_{t^n} , there exists an invertible bounded linear operator $Y \in A'(E_{t^n})$ such that $\{Y^{-1}p_iY\}$ is a rearrangement of

$$\{Xp_1X^{-1}, Xp_2X^{-1}, \cdots, Xp_mX^{-1}, q_{m+1}, q_{m+2}, \cdots, q_N\}.$$

By [58]) (or [204]), for any $v \in \{m+1, m+2, \cdots, N\}$, we can find Z_v in $GL(L(q_v\mathcal{H}, p_v\mathcal{H}))$ and $p_{v\prime}$, $v' \in \{m+1, \cdots, N\}$ such that

$$Z_vq_vE_{t^n} = p_vE_{t^n}, \text{and } v_1' = v_2', \text{when } v_1 = v_2.$$

Note that

$$Z := \sum_{k=1}^{m} Z_k + \sum_{v=m+1}^{N} Z_v \in GL\mathcal{A}'(E_{t^n}),$$

and

$$ZPZ^{-1} = Q.$$

It follows that Γ is injective. Since Γ is also a homomorphism, then we have

$$Vect^0(E_t) \cong Vect(\mathcal{A}'(E_{t^n})), K^0(E_t) \cong K_0(\mathcal{A}'(E_t)).$$

The well-known question of Halmos asks if $\varrho : \mathbb{C}[z] \to \mathcal{L}(\mathcal{H})$ is a continuous (for $p \in \mathbb{C}[z]$, the norm $p = \sup_{z\in\mathbb{D}} |p(z)|$) algebra homomorphism induced by an operator S, that is, $\varrho(p) = p(S)$, then does there exist an invertible linear operator L and a contraction T on the Hilbert space $\mathcal{H}$ so that $S = LTL^{-1}$. After the question was raised in [201], an affirmative answer for several classes of operators were given.

A counter example was found by Pisier in 1996 (cf. [207]). It was pointed out in Korányi [183] that the Halmos' question has an affirmative answer for homogeneous operators in the Cowen–Douglas class $B_n(\mathbb{D})$. Thus it is natural to ask if the Halmos' question has an affirmative answer for quasi-homogeneous operators. If $\Lambda(t) \geq 2$, the answer is evidently "yes":

In this case, the quasi-homogeneous operator T is similar to the $n -$ fold direct sum of the homogeneous operators T_i (adjoint of the multiplication operator) acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_i)}(\mathbb{D}), i = 0,1,\dots,n-1$. Now, if $\lambda_0 \geq 1$, this direct sum is contractive and we are done. If $\lambda_0 < 1$, then T_0 is not even power bounded and therefore neither is the operator T. So, there is nothing to prove when $\lambda_0 < 1$.

If $\Lambda(t) < 2$, then the operator T is strongly irreducible. Therefore, we can't answer the Halmos' question purely in terms of the atoms of the operator T. Never the less, the answer is affirmative even in this case. To show this, we first prove the following useful lemma.

For $i = 1,2$, let $\mathcal{H}_i$ be a Hilbert space of holomorphic function on $\mathbb{D}$ possessing a reproducing kernel, say K_i, and T_i be the adjoint of the multiplication operator on $\mathcal{H}_i$. Assume that $\mathcal{H}_0 \subseteq \mathcal{H}_1$ and let $\iota : \mathcal{H}_0 \to \mathcal{H}_1$ be the inclusion map. Then the adjoint ι^* of the inclusion map has the property $\iota^*(K_1(\cdot,w)) = K_0(\cdot,w), w \in \mathbb{D}$.

Lemma (6.3.27)[199]: Assume that $K_i(z,w) = \frac{1}{(1-z\bar{w})^{\lambda_{i,i}}} = 0,1$. Suppose that $S : \mathcal{H}_0 \to \mathcal{H}_1$ is a bounded linear operator with the intertwining property $T_0 S = S T_1$. Then there exists a holomorphic function $\emptyset$ such that $S = \emptyset(T_0)\iota^*$.

Proof. The operators $T_{i,i} = 0,1$ are in $B_1(\mathbb{D})$. If $S : \mathcal{H}_0 \to \mathcal{H}_1$ is a bounded linear operator and $T_0 S = S T_1$, then there exists a holomorphic function ψ such that $S^* = M_\psi$. This is easily proved as in [72]. Let $\emptyset$ be the holomorphic function defined on the unit disc by the formula $\emptyset(w) = \psi(w), w \in \mathbb{D}$. For any $f \in \mathcal{H}_0$, we have that

$$f(z), \emptyset(T_0)\iota^*\big(K_1(z,w)\big) = \langle f(z), \emptyset(\bar{w})K_0(z,w)\rangle = \overline{\emptyset(\bar{w})}\langle f(z), K_0(z,w)\rangle$$
$$= \langle f(z), M_\psi^*\big(K_1(z,w)\big)\rangle = \langle f(z), S(K_1(z,w))\rangle .$$

Consequently, $S = \emptyset(T_0)\iota^*$.

Lemma (6.3.28)[199]: Suppose that t is a quasi-homogeneous holomorphic curve. Assume that $\Lambda(t) < 2$ and $\lambda_0 \geq 1$. Then the operator T is not power bounded.

Proof. The top 2×2 block in the atomic decomposition of the quasi-homogeneous operator T is of the form $\begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$. As always, we assume that the operators T_0 and T_1 are the adjoints of the multiplication operator on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_1)}(\mathbb{D})$, respectively. The operator $S_{0,1}$ has the intertwining property $T_0 S_{0,1} = S_{0,1} T_1$.

Let ι denote the inclusion map from $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ to $\mathbb{A}^{(\lambda_1)}(\mathbb{D})$. Then $\iota^*(t_1)(w) = t_0(w), w \in \mathbb{D}$, and the operator $S_{0,1}$ must be of the form $\emptyset(T_0)\iota^*$ for some holomorphic function $\emptyset$ on the unit disc $\mathbb{D}$, as we have shown in Lemma (6.3.27). Indeed, $S_{0,1}(t_1(w)) = \emptyset(w)t_1(w) = \emptyset(T_0)\iota^*(t_1(w))$.

Without loss of generality, we assume that $\emptyset(w) = \sum_{i=0}^{\infty} \emptyset_i w_i$ and $\emptyset_0 \neq 0$. For $j = 0,1$, the set of vectors $e_\ell^{(\lambda_j)} := \sqrt{a_\ell(\lambda_j)}\, z^\ell, \ell \geq 0$, is an orthonormal basis in $\mathbb{A}^{(\lambda_j)}(\mathbb{D})$. Then we have that

$$T_0^{n-1}\left(e_\ell^{(\lambda_0)}\right) = \Pi_{i=\ell-n+1}^{\ell-1} w_i(\lambda_0) e_{\ell-n+1}(\lambda_0), S_{0,1}\left(e_\ell^{(\lambda_1)}\right) = \emptyset_0 \frac{\Pi_{i=0}^{\ell-1} w_i(\lambda_1) e}{\Pi_{i=0}^{\ell-1} w_i(\lambda_0)} e_\ell^{(\lambda_0)}.$$

Consequently,

$$nT_0^{n-1} S_{0,1}\left(e_\ell\big((\lambda_1)\big)\right) = n\emptyset_0 \frac{\Pi_{i=0}^{\ell-1} wi(\lambda_1)}{\Pi_{i=0}^{\ell-n} w_i(\lambda_0)} e_{\ell-n+1}(\lambda_0).$$

It is then easily deduced that $||nT_0^{n-1} S_{0,1}|| \to \infty$ as $n \to \infty$.

Let $T|_{2\times 2}$ denote the top 2×2 block $\begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$ in the operator T. Since $T^n_{|2\times 2} = \begin{pmatrix} T_0^n & nT_0^{n-1} S_{0,1} \\ 0 & T_1^n \end{pmatrix}$, and $||T^n_{|2\times 2}|| \geq ||nT_0^{n-1} S_{0,1}||$, it follows that $||T^n_{|2\times 2}|| \to \infty$ as n $\to \infty$. Clearly, $||T^n|| \geq ||T^n_{|2\times 2}||$ completing the proof.

Since a quasi-homogeneous operator for which $\lambda_0 < 1$ can't be power bounded, the lemma we have just proved shows that if T is quasi-homogeneous and $\Lambda(t) < 2$, then the operator T is not power bounded. Therefore we have proved the following theorem answering the Halmos' question in the affirmative.

Theorem (6.3.29)[199]: If a quasi-homogeneous operator T has the property $\|p(T)\|_{op} \leq K\|p\|_{\infty,\mathbb{D}}, p \in \mathbb{C}[z]$, then it must be similar to a contraction.

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