Sudan University of Science and Technology

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Complex Symmetric Generators and Dixmier Property with Tracial States of C^*-algebras

مولدات التماثل المركبة وخاصية ديكسمير مع الحالات الأثرية الي جبريات $C^* -$

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Dedication

To My Family

Acknowledgments

First I would to thank the Almighty God. For helping me to achieve this work, and I would like to express deep appreciation and gratitude to my supervisor Prof. Dr. Shawgy Hussein AbdAlla of Sudan University of Science and Technology for all his guidance and inspiration. Thanks are due to my family for all their help and support.

Abstract

We study the $C^* -$ algebras $of\ labelled\ graphs$ II simplicity results with a finite and an infinite projection and of labelled spaces and their diagonal C*-subalgebras. The group actions and the structure of Gauge-invariant ideals of labelled graph with nuclear dimension of C^*-algebras are dealt with. The applications and approximations of the complex symmetric operators and generators of $C^* -$ algebras are shown. The dixmier approximations theorem with property and symmetric amenability with tracial states for the $C^* -$ algebras are discussed.

الخلاصة

قمنا بدراسة جبريات-C^* لنتائج البساطة II للرسوم البيانية المسمي مع المسقط المنتهي واللانهائي وللفضاءات المسمي والجبريات الجزئية-C^* القطرية لها. تعاملنا مع اجراءات الزمرة وبناء مقياس - المثاليات اللامتغيرة للرسم البياني المسمي مع البعد النووي لجبريات-C^*. قمنا بتوضيح التطبيقات والتقريبات لمؤثرات ومولدات موثرات التماثل المركبة لجبريات-C^*. تمت مناقشة مبرهنة تقريب ديكسمير مع الخاصية وقابلية التماثل مع حالات التتبع لاجل جبريات-C^*.

Introduction

We show simplicity and pure infiniteness results for a certain class of labelled graph C^*-algebras. We show, by example, that this class of unital labelled graph C^*-algebras is strictly larger than the class of unital graph C^*-algebras.We introduce the notion of the action of a group on a labelled graph and the quotient object, also a labelled graph.

We study a few classes of Hilbert space operators whose matrix representations are complex symmetric with respect to a preferred orthonormal basis. The existence of this additional symmetry has notable implications and, in particular, it explains from a unifying point of view some classical results. A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is called complex symmetric if $T = CT^*C$, where C is a conjugation (an isometric, antilinear involution of $\mathcal{H}$). We show that $T = CJ|T|$, where J is an auxiliary conjugation commuting with $|T| = \sqrt{T^*T}$.

We give a complete and computationally simple description of the certain sets of any self-adjoint element of a general von Neumann algebra $\mathcal{R}$**.** We answer the following question raised by Cuntz that can asimple C^*-algebras contain both a finite and an infinite projection.

We consider the gauge-invariant ideal structure of a C^* $-$algebra $C^*(E, \mathcal{L}, B)$ associated to a set-finite, receiver set-finite and weakly left-resolving labelled space $(E, \mathcal{L}, B)$, where $\mathcal{L}$ is a labelling map assigning an alphabet to each edge of the directed graph E with no sinks. It is obtained that if an accommodating set B is closed under relative complements, there is a one-to-one correspondence between the set of all hereditary saturated subsets of B and the gauge-invariant ideals of $C^*(E, \mathcal{L}, B)$. Motivated by Exel's inverse semigroup approach to combinatorial C*-algebras. We construct a representation of the C*-algebra of a labelled space, inspired by how one might cut or glue labelled paths together, that proves that non-zero elements in the inverse semigroup correspond to non-zero elements in the C*-algebra.

An operator Ton a complex Hilbert space $\mathcal{H}$is called a complex symmetric operator if there exists a conjugate-linear, isometric involution $C: \mathcal{H} \to \mathcal{H}$ so that $CTC = T^*$. We study the approximation of complex symmetric operators. By virtue of an intensive analysis of compact operators in singly generated C^*-algebras, we obtain a complete characterization of norm limits of complex symmetric operators and provide a classification of complex symmetric operators up to approximate unitary equivalence.

We give necessary and sufficient conditions for an essentially normal operator T to have its C^*-algebra $C^*(T)$ generated by a complex symmetric operator.

We introduce the nuclear dimension of a C^*-algebra; this is a noncommutative version of topological covering dimension based on a modification of the earlier concept of decomposition rank. Our notion behaves well with respect to inductive limits, tensor products, hereditary subalgebras (hence ideals), quotients, and even extensions. It can be computed for many examples; in particular, it is finite for all UCT Kirchberg algebras. In fact, all classes of nuclear C^*-algebras which have so far been successfully classified consist of examples with finite nuclear dimension, and it turns out that finite nuclear dimension implies many properties relevant for the classification program.

We study some general properties of tracial C^*-algebras. In the first part, we consider Dixmier type approximation theorem and characterize symmetric amenability for C^*-algebras. In thesecond part, we consider continuous bundles of tracial von Neumann algebras and classify some of them. It is shown that a unital C^*-algebra A has the Dixmier property if and only if it is weakly central and satisfies certain tracial conditions. This generalises the Haagerup–Zsidó theorem for simple C^*-algebras. We also study a uniform version of the Dixmier property, as satisfied for example by von Neumann algebras and the reduced C^*-algebras of powers groups, but not by all C^*-algebras with the Dixmier property, and we obtain necessary and sufficient conditions for a simple unital C^*-algebra with unique tracial state to have this uniform property.

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Chapter 1
C^*-Algebras of Labelled Graphs

We define a skew product labelled graph and use it to prove a version of the Gross-Tucker theorem for labelled graphs. We then apply these results to the C^*-algebra associated to a labelled graph and provide some applications in non-Abelian duality.

Section (1.1): Simplicity Results

The first is to continue the development of the C^*-algebras of labelled graphs begun in [3] and the second is to provide a tractable example which illustrates why they are worthy of further study.

A labelled graph is a directed graph E in which the edges have been labelled by symbols coming from a countable alphabet. By considering the sequences of labels carried by the bi-infinite paths in E one obtains a shift space X; the labelled graph is then called a presentation of X. A directed graph is a (trivial) example of a labelled graph, and the shift space it presents is a shift of finite type (see [13]). In [3] we showed how to associate a C^*-algebra to a labelled space, which consists of a labelled graph together with a certain collection of subsets of vertices. By making suitable choices of the labelled spaces it was shown in [3, Proposition 5.1, Theorem 6.3] that the class of labelled graph C^*-algebras includes graph C^*-algebras, the ultragraph C^*-algebras of [21], [22] and the C^*-algebras of shift spaces in the sense of [14], [4].

We shall work almost exclusively with the labelled spaces which arise in connection with shift spaces. We shall be interested in identifying key properties of the labelled spaces which allow us to prove results about the simplicity and pure infiniteness of the associated C^*-algebra (see Theorem (1.1.20) and Theorem (1.1.24)).

The examples of labelled spaces that we have considered have turned out to have C^*-algebras isomorphic to the C^*-algebra of the underlying directed graph (see [3, Theorem 6.6]). We tum our attention to the question of whether the class of C^*-algebras of labelled spaces that we are considering is strictly larger than the class of graph C^*-algebras. We give presentations of the Dyck shifts D_N and show that their associated C^*-algebras cannot be unital graph C^*-algebras. We present a labelled graph which presents an irreducible non-sofic shift, whose C^*-algebra is simple and purely infinite.

The C^*-algebras associated to shift spaces (see [7], [6], [14], [16], [5], [4], [2], [3] for example). A drawback to some of the approaches is that the canonical C^*-algebra associated to an irreducible shift space is often not simple (see [3, Remark 6.10]). We believe that an equally valid way to study the C^*-algebra associated to a shift space is to study the C^*-algebras of the various labelled graphs which present it. This belief is founded on the observation that the labelled graph $(E_1, \mathcal{L}_1)$ of Examples (1.1.20) (i) is a presentation of an irreducible sofic shift (called the even shift) whose C^*-algebra is simple (see [3, Remark 6.10]) whereas the C^*-algebra associated to the even shift in [4] is not simple.

The work of Matsumoto on symbolic matrix systems and their associated λ-graph systems gives us an important method for studying shift spaces using labelled graphs (see [18], [15], [16], [17] amongst others). However, we feel that there is an extra facility afforded by the approach. Whilst λ-graph systems are indeed labelled graphs, they are quite complicated. This makes them difficult to visualise; for instance the labelled graphs in Examples (1.1.20) (i) give rise to the same C^*-algebras as the ones for the symbolic matrix systems described on [16, p. 297]. Furthermore we believe that our presentations of the Dyck shifts in give us a more tractable way of studying them. Of equal importance is the fact that the labelled spaces are ideally suited to handle shift spaces over countably infinite alphabets.

We give an important embellishment to the treatment of labelled spaces in [3] by identifying the basic objects in a labelled space, which we call the generalised vertices. In Proposition (1.1.4) we establish concrete connections between the work and that of Matsumoto by showing how to associate a symbolic matrix system to a labelled graph.

We recall the definition of the C^*-algebra of a labelled space from [3]. In Proposition (1.1.7) we give a new description of the canonical spanning set for a labelled graph C^*-algebra in terms of generalised vertices. Then in Proposition (1.1.8) we use this new description to show the relationship between the C^*-algebra of a labelled graph and the λ-graph C^*-algebra of the associated symbolic matrix system.

We give a description of the $A\Gamma$ core of a labelled graph C^*- algebra before moving on to prove the Cuntz-Krieger uniqueness theorem. The central hypothesis to the Cuntz-Krieger uniqueness theorem for labelled graphs is the notion of disagreeability, which replaces the aperiodicity hypothesis in the corresponding theorem for directed graphs (see [1, Theorem 3.1]).

We give the simplicity and pure infiniteness results for labelled graph C^*-algebras. To prove the simplicity result (Theorem (1.1.20)) we need a notion of cofinality appropriate for labelled graphs. The notion of cofinality for labelled graphs is much more subtle than that for directed graphs as many different infinite paths in the underlying directed graph can have the same labels. To prove the pure infiniteness result (Theorem (1.1.24)) we need to examine how periodic paths arise in labelled graphs. The situtation is much more complicated than for directed graphs since periodic points in the shift space associated to a labelled graph need not arise from a loop in the underlying directed graph.

We provide two new examples of labelled graphs to which our main results apply. We provide a labelled graph presentation of the Dyck shifts D_N. In Proposition (1.1.26) show that these presentations give rise to simple purely infinite labelled graph C^*-algebras. We give a formula for the K-theory of our labelled graph C^*-algebras which demonstrates that the C^*-algebras we associate to Dyck shifts cannot be isomorphic to graph C^*-algebras. We provide a presentation of an interesting new irreducible non-sofic shift whose labelled graph C^*-algebra is simple and purely infinite.

A directed graph E consists of a quadruple (E^0, E^1, r, s) where E^0 and E^1 are (not necessarily countable) sets of vertices and edges respectively and $r, s : E^1 \to E^0$ are maps giving the direction of each edge. A path $\lambda = e_1 \dots e_n$ is a sequence of edges $e_i \in E^1$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$, we define $s(\lambda) = s(e_1)$ and $r(\lambda) = r(e_n)$. The collection of paths of length n in E is denoted E^n and the collection of all finite paths in E by E^*, so that $E^* = \bigcup_{n\geq 0} E^n$.

A loop in E is a path which begins and ends at the same vertex, that is $\lambda \in E^*$ with $s(\lambda) = r(\lambda)$. We say that E is *row-finite* if every vertex emits finitely many edges. The graph E is called *transitive* if given any pair of vertices, $v \in E^0$ there is a path $\lambda \in E^*$ with $s(\lambda) = u$ and $r(\lambda) = v$. We denote the collection of all infinite paths in E by E^∞.

We will assume that our directed graphs E are *essential*: all vertices emit and receive edges (i.e., E has no sinks or sources).

A labelled graph $(E,\mathcal{L})$ over a countable alphabet $\mathcal{A}$ consists of a directed graph E together with a labelling map $\mathcal{L} : E^1 \to \mathcal{A}$. Without loss of generality we may assume that the map $\mathcal{L}$ is onto.

Let $\mathcal{A}^*$ be the collection of all *words* in the symbols of $\mathcal{A}$. The map $\mathcal{L}$ extends naturally to a map $\mathcal{L} : E^n \to \mathcal{A}^*$, where $n \geq 1$: for $\lambda = e_1 \ldots e_n \in E^n$ put $\mathcal{L}(\lambda) = \mathcal{L}(e_1) \ldots \mathcal{L}(e_n)$; in this case the path $\lambda \in E^n$ is said to be a representative of the labelled path $\mathcal{L}(e_1) \ldots \mathcal{L}(e_n)$. Let $\mathcal{L}(E^n)$ denote the collection of all labelled paths in $(E,\mathcal{L})$ of length n where we write $|\alpha| = n$ if $\alpha \in \mathcal{L}(E^n)$. The set $\mathcal{L}^*(E) = \bigcup_{n\geq 1} \mathcal{L}(E^n)$ is the collection of all labelled paths in the labelled graph $(E,\mathcal{L})$. We may similarly extend $\mathcal{L}$ to E^∞.

The labelled graph $(E,\mathcal{L})$ is *left-resolving* if for all $v \in E^0$ the map $\mathcal{L} : r^{-1}(v) \to \mathcal{A}$ is injective. The left-resolving condition ensures that for all $v \in E^0$ the labels $\{\mathcal{L}(e) : r(e) = v\}$ of all incoming edges to v are all different. For α in $\mathcal{L}^*(E)$ we put

$$s_{\mathcal{L}}(\alpha) = \{s(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\} \text{ and } r_{\mathcal{L}}(\alpha) = \{r(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\},$$

so that $r_{\mathcal{L}}, s_{\mathcal{L}} : \mathcal{L}^*(E) \to 2^{E^0}$. We shall drop the subscript on $r_{\mathcal{L}}$ and $s_{\mathcal{L}}$ if the context in which it is being used is clear.

Let $(E,\mathcal{L})$ be a labelled graph. For $A \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ the relative range of α with respect to A is defined to be

$$r_{\mathcal{L}}(A,\alpha) = \{r(\lambda) : \lambda \in E^*, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A\}.$$

A collection $\mathcal{R} \subseteq 2^{E^0}$ of subsets of E^0 is said to be closed under relative ranges for $(E,\mathcal{L})$ if for all $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$ we have $(A,\alpha) \in \mathcal{B}$. If $\mathcal{R}$ is closed under relative ranges for $(E,\mathcal{L})$, contains $r(\alpha)$ for $all\ \alpha \in \mathcal{L}^*(E)$ and is also closed under finite intersections and unions, then we say that $\mathcal{R}$ is accommodating for $(E,\mathcal{L})$.

Let $\mathcal{G}^{0,-}$ denote the smallest subset of 2^{E^0} which is accommodating for $(E,\mathcal{L})$. Since $\mathcal{G}^{0,-}$ is generated by a countable family of subsets of E^0, under countable operations, it follows that $\mathcal{G}^{0,-}$ is countable, even though E^0 itself may be uncountable. Of course, 2^{E^0} is the largest accommodating collection of subsets for $(E,\mathcal{L})$.

A labelled space consists of a triple $(E,\mathcal{L},\mathcal{B})$, where $(E,\mathcal{L})$ is a labelled graph and $\mathcal{B}$ is accommodating for $(E,\mathcal{L})$.

A labelled space $(E,\mathcal{L},\mathcal{B})$ is *weakly left-resolving* if for every, $B \in \mathcal{B}$ and every $\alpha \in \mathcal{L}^*(E)$ we have $(A,\alpha) \cap r(B,\alpha) = r(A \cap B,\alpha)$.

For $\ell \geq 1$ and $A \subseteq E^0$ let $E^\ell A = \{\lambda \in E^\ell : r(\lambda) \in A\}$ and $AE^\ell = \{\lambda \in E^\ell : s(\lambda) \in A\}$. The labelled space $(E,\mathcal{L},\mathcal{R})$ is *receiver set-finite* if for all $A \in \mathcal{B}$ and all $\ell \geq 1$ the set $\mathcal{L}(E^\ell A)$ $:= \{\mathcal{L}(\lambda) : \lambda \in E^\ell A\}$ is finite. In particular, the labelled space $(E,\mathcal{L},\mathcal{R})$ is receiver set-finite if each $A \in \mathcal{R}$ receives only finitely labelled paths of length ℓ (even though A may receive infinitely many paths of each length ℓ). More generally, for $\ell \geq 1$ and $A \subseteq E^0$ let

$$\mathcal{L}(E^{\leq \ell}) = \bigcup_{j=1}^{\ell} \mathcal{L}(E^j) \text{ and } \mathcal{L}(E^{\leq \ell}A) = \bigcup_{j=1}^{\ell} \mathcal{L}(E^j A) .$$

We say that the labelled space $(E,\mathcal{L},\mathcal{B})$ is *set-finite* if for all $A \in \mathcal{B}$ the set $\mathcal{L}(AE^1) := \{\mathcal{L}(\lambda) : \lambda \in AE^1\}$ is finite. One may similarly define $\mathcal{L}(AE^n)$ (note that $\mathcal{L}(AE^n)$ was denoted L_A^n in [3]).

We shall focus exclusively on the (minimal) accommodating labelled space $(E,\mathcal{L},\mathcal{G}^{0,-})$ associated to a labelled graph $(E,\mathcal{L})$. We do this in order to relate our work to that of Matsumoto (see [14], [15], [16], [17], [18]).

We will assume that $(E,\mathcal{L},\mathcal{G}^{0,-})$ is receiver set-finite, set-finite and weakly leftresolving. For $v \in E^0$ and $\ell \geq 1$ let

$$\Lambda_\ell(v) = \{\lambda \in \mathcal{L}(E^{\leq \ell}) :\ v \in r(\lambda)\} = \mathcal{L}(E^{\leq \ell} v)\ .$$

The relation $\sim_\ell$ on E^0 is defined by $v \sim_\ell w$ if and only if $\Lambda_\ell(v) = \Lambda_\ell(w)$; hence $v \sim_\ell w$ if v and w receive exactly the same labelled paths of length at most ℓ. Evidently $\sim_\ell$ is an equivalence relation and we use $[v]_\ell$ to denote the equivalence class of $v \in E^0$. We call the $[v]_\ell$ generalised vertices as they play the same role in labelled spaces as vertices in a directed graph.

Set $\Omega_\ell = E^0 / \sim_\ell$ and Ω be the disjoint union of the Ω_ℓ for $\ell \geq 1$. If the alphabet $\mathcal{A}$ is finite, then Ω_ℓ is finite. If there is $L \geq 1$ such that $\Omega_\ell = \Omega_L$ for all $\ell \geq L$, then the underlying shift $X_{E,\mathcal{L}}$ is a sofic shift (see [4], [13]). Conversely, if X is a sofic shift then every presentation $(E, \mathcal{L})$ of the shift X has this property (see [13, Exercise (3.2.6)]).

For $\ell \geq 1$ let $\mathcal{G}_\ell^{0,-} \subseteq \mathcal{G}^{0,-}$ be the smallest subset of 2^{E^0} which contains $r(\lambda)$ for all $\lambda \in \mathcal{L}(E^{\leq \ell})$ and is closed under finite intersections and unions. Evidently $\mathcal{G}_\ell^{0,-} \subseteq \mathcal{G}_{\ell+1}^{0,-}$. We have $\mathcal{G}^{0,-} = \bigcup_{\ell=1}^{\infty} \mathcal{G}_\ell^{0,-}$ For $v \in E^0$ and $\ell \geq 1$, the equivalence class $[v]_\ell$ does not necessarily belong to $\mathcal{G}_\ell^0$ however, as we shall see in Proposition (1.2.2) (i), $[v]_\ell$ may be expressed as a difference of elements of $\mathcal{G}_\ell^{0,-}$ First we need the following technical lemma.

Lemma (1.1.1)[443]: Let $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be a labelled space, $v \in E^0$ and $\ell \geq 1$.

(i) The set $\Lambda_\ell(v)$ is finite and $X_\ell(v) := \bigcap_{\lambda \in \Lambda_\ell(v)} r(\lambda) \in \mathcal{G}_\ell^{0,-}$ Moreover $[v]_\ell \subseteq X_\ell(v)$.

(ii) The set of labelles $Y_\ell(v) := \bigcup_{w \in X_\ell(v)} \Lambda_\ell(w) \backslash \Lambda_\ell(v)$ is finite, and $r(Y_\ell(v)) \in \mathcal{G}_\ell^{0,-}$

Proof. For the first statement let $A \in \mathcal{G}^{0,-}$ be such that $v \in A$. Since $(E, \mathcal{L}, \mathcal{G}^{0,-})$ is receiver set-finite $\mathcal{L}(E^j v) \subseteq \mathcal{L}(E^j A)$ is finite for all $j \geq 1$ and hence $\Lambda_\ell(v) = \bigcup_{j=1}^{\ell} \mathcal{L}(E^j v)$ is finite for all $\ell \geq 1$. It now follows that $X_\ell(v)$ is a finite intersection of elements of $\mathcal{G}_\ell^{0,-}$ and hence $X_\ell(v) \in \mathcal{G}_\ell^{0,-}$ Since $X_\ell(v)$ is the set of vertices which receive at least the same labelled paths as v up to length ℓ we certainly have $[v]_\ell \subseteq X_\ell(v)$.

For the second statement observe that $Y_\ell(v) = \mathcal{L}(E^{\leq \ell} X_\ell(v)) \backslash \Lambda_\ell(v)$. Since $(E, \mathcal{L}, \mathcal{G}^{0,-})$ is receiver set-finite and $X_\ell(v) \in \mathcal{G}^{0,-}$ the sets $\mathcal{L}(E^{\leq \ell} X_\ell(v))$ and $Y_\ell(v)$ must be finite. Note that $r(Y_\ell(v)) = \bigcup_{\mu \in Y_\ell(v)} r(\mu)$ belongs to $\mathcal{G}_\ell^{0,-}$ as it is a finite union of elements of $\mathcal{G}_\ell^{0,-}$ The set $Y_\ell(v)$ denotes the additional labelled paths of length at most ℓ received by those vertices which receive at least the same labelled paths as v up to length ℓ.

Proposition (1.1.2)[443]: Let $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be a labelled space, $v \in E^0$ and $\ell \geq 1$.

(i) We have $[v]_\ell = X_\ell(v) \backslash r(Y_\ell(v))$.

(ii) For every set $A \in \mathcal{G}_\ell^{0,-}$ we can find vertices $v_1, \ldots, v_m \in A$ such that $A = \bigcup_{i=1}^{m} [v_i]_\ell$.

(iii) There are $w_1, \ldots, w_n \in [v]_\ell$ such that $[v]_\ell = \bigcup_{i=1}^{n} [w_i]_{\ell+1}$.

Proof. For the first statement observe that $[v]_\ell$ consists of those vertices which receive exactly the labelled paths from $\Lambda_\ell(v)$ whereas other vertices in $X_\ell(v)$ may receive more labelled paths. Hence, to form $[v]_\ell$ we remove those vertices from $X_\ell(v)$ which receive different labelled paths of length ℓ from v -these are precisely the vertices in $(Y_\ell(v))$.

Any $A \in \mathcal{G}_\ell^{0,-}$ can be written as a finite union of elements of the form $B_k = \bigcap_{i=1}^{n} r(\beta_i)$ where $\beta_i \in \mathcal{L}(E^{\leq \ell})$. If $v_1 \in B_k$ then $[v_1]_\ell \subseteq B_k$ as v_1 , and hence every vertex in $[v_1]_\ell$, must receive $\beta_1, \ldots, \beta_n$ and so lie in B_k. If $B_k \neq [v_1]_\ell$, there is $v_2 \in B_k$ with $\Lambda_\ell(v_1) \neq$

$\Lambda_\ell(v_2)$. Again we have $[v_2]_\ell \subseteq B_k$. Since $(E, \mathcal{L}, \mathcal{G}^0)$ is receiver set-finite $B_k \in \mathcal{G}_\ell^{0,-}$ receives only finitely many different labelled paths oflength at most ℓ. Hence there are vertices $\{v_i : 1 \leq i \leq m\}$ in B_k such that $B_k = \bigcup_{i=1}^m [v_i]_\ell$ and our result is established.

For the final statement we observe that since $\mathcal{G}_\ell^{0,-} \subseteq \mathcal{G}_{\ell+1}^{0,-}$ the first statement shows that $[v]_\ell$ may be written as a difference $A \backslash B$ of elements of $\mathcal{G}_{\ell+1}^{0,-}$. The result then follows by applying the second statement to $A, B \in \mathcal{G}_{\ell+1}^{0,-}$ and noting that the $[w_i]_{\ell+1}$'s are disjoint.

Let $(E, \mathcal{L})$ be a labelled graph. The subshift X_E is defined by $X_E = \{x \in (E^1)^Z : s(x_{i+1}) = r(x_i)$ for all $\in Z\}$. The subshift $(X_{E,\mathcal{L}}, \sigma)$ is defined by

$$X_{E,\mathcal{L}} = \{y \in \mathcal{A}^Z : \text{there exists } x \in X_E \text{ such that } y_i = \mathcal{L}(x_i) \text{ for all } \in Z\},$$

where σ is the shift map $\sigma(y)_i = y_{i+1}$ for $i \in Z$. The labelled graph $(E, \mathcal{L})$ is said to be a presentation of the shift space $X_{E,\mathcal{L}}$ with language $\mathcal{L}^*(E)$.

We are primarily interested in one-sided shift spaces, namely

$$X_{E,\mathcal{L}}^+ = \{y \in \mathcal{A}^N : \text{there exists } x \in E^\infty \text{ such that } y_i = \mathcal{L}(x_i) \text{ for all } \in N\}$$

and we restrict the shift map to $X_{E,\mathcal{L}}^+$. For an infinite labelled path $x \in X_{E,\mathcal{L}}^+$ we define $s_\mathcal{L}(x)$ to be the set of all $v \in E^0$ for which there is an infinite path $\hat{x} \in E^\infty$ with $s \cap x = v$ and $\mathcal{L} \cap x = x$. The infinite path $\hat{x}$ is said to be a representative of x.

An infinite labelled path $x \in X_{E,\mathcal{L}}^+$ is *periodic* if $\sigma^n x = x$ for some $n \geq 1$. A path which is not periodic is called aperiodic.

Example (1.1.3)[443]: If E is a directed graph then we may consider it as a labelled graph when endowed with the trivial labelling $\mathcal{L}_t$. In this case $\mathcal{G}^{0,-}$ consists of all finite subsets of E^0 (see [3, Examples 4.3(i)]) and $[v]_\ell = \{v\}$ for all $\ell \geq 1$. We shall identify $\mathcal{L}_t^*(E)$ with E^* and $X_{E,\mathcal{L}_t}^+$ with E^∞.

Essential symbolic matrix systems are defined in [16, §2]. To a left-resolving labelled graph $(E, \mathcal{L})$ over a finite alphabet we associate matrices $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell \geq 1}$ as follows: For $\ell \geq 1$, write $\Omega_\ell = \{[v_i]_\ell : i = 1, \ldots, m(\ell)\}$, then $I(E)_{\ell,\ell+1}$ is a $m(\ell) \times m(\ell+1)$ matrix with entries 0, 1 determined by

$$I(E)_{\ell,\ell+1}([v_i]_\ell, [w_j]_{\ell+1}) = \begin{cases} 1 & if\ [w_j]_{\ell+1}\ subseteq [v_i]_\ell \\ 0 & otherwise. \end{cases} \quad (1)$$

The symbolic matrix $M(E)_{\ell,\ell+1}$ is the same size as $I(E)_{\ell,\ell+1}$ with entries determined as follows: For $v \in E^0$ let $\langle v \}_\ell$ denote the collection of labelled paths of length exactly ℓ which arrive at v. Since $(E, \mathcal{L})$ is left-resolving we may partition the set of labelled paths of length $\ell + 1$ arriving at w to write $\langle w \}_{\ell+1}$ as the disjoint union

$$\langle w \}_{\ell+1} = \bigcup_{e \in r^{-1}(w)} \langle s(e) \rangle_\ell \mathcal{L}(e),$$

where $\langle s(e) \rangle_\ell \mathcal{L}(e)$ denotes the set of labelled paths of length $\ell + 1$ formed by the juxtaposition of the symbol $\mathcal{L}(e)$ at the end of each labelled path in $\langle s(e) \rangle_\ell$. Since all vertices in $[v_i]_\ell$ and $[w_j]_{\ell+1}$ receive the same labelled paths of length ℓ and $\ell + 1$ respectively we may unambiguously define

$$M(E)_{\ell,\ell+1}([v_i]_\ell, [w_j]_{\ell+1}) = \sum_{e \in s^{-1}(v_i) \cap r^{-1}(w_j)} \mathcal{L}(e) \quad (2)$$

where the right hand-side is treated as a formal sum.

Proposition (1.1.4)[443]: *Let* $(E,\mathcal{L})$ be a left-resolving labelled graph over a finite alphabet. Then the matrices $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$ defined above form an essential symbolic matrix system.

Proof. If suffices to check that the matrices $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$ satisfy the conditions on [16, p.290]: Since E is essential it is straightforward to check from the definition of $I(E)_{\ell,\ell+1}$ and $M(E)_{\ell,\ell+1}$ that conditions (1), (2), (2-a), (2-b), (3), (5-i) and (5-ii) are satisfied. It remains to check that for $\ell \geq 1$ we have

$$M(E)_{\ell,\ell+1} I(E)_{\ell+1,\ell+2} = I(E)_{\ell,\ell+1} M(E)_{\ell+1,\ell+2}.$$

For $\ell \geq 1$ we form the entry $M(E)_{\ell,\ell+1} I(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+1})$ as follows: For each $[v_j]_{\ell+1}$ which receives an edge from $[u_i]_\ell$, the entry is the formal sum of the labels received by the unique $[v_j]_{\ell+1}$ of which $[w_k]_{\ell+2}$ is a subset. In which case

$$M(E)_{\ell,\ell+1} I(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+2}) = \sum_{e\in s^{-1}(u_i)\cap r^{-1}(w_k)} \mathcal{L}(e).$$

On the other hand, to form the entry $I(E)_{\ell,\ell+1} M(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+2})$ we take each $[v_j]_{\ell+1}$ which is a subset of $[u_i]_\ell$ and then formally sum the labels of the edges to $[w_k]_{\ell+1}$. In which case

$$I(E)_{\ell,\ell+1} M(E)_{\ell+1,\ell+2}([u_i]_\ell, [w_k]_{\ell+2}) = \sum_{[v_j]_{\ell+1}\subseteq [u_i]_\ell} \sum_{e\in s^{-1}(v_j)\cap r^{-1}(w_k)} \mathcal{L}(e)$$

$$= \sum_{e\in s^{-1}(u_i)\cap r^{-1}(w_k)} \mathcal{L}(e).$$

Hence for $\ell \geq 1$ we have $M(E)_{\ell,\ell+1} I(E)_{\ell+1,\ell+2} = I(E)_{\ell,\ell+1} M(E)_{\ell+1,\ell+2}$ as required.

We recall from [3] the definition of the C^*-algebra associated to the labelled space $(E,\mathcal{L},\mathcal{G}^{0,-})$.

Definition (1.1.5)[443]: Let $(E,\mathcal{L},\mathcal{G}^{0,-})$ be a labelled space. A representation of $(E,\mathcal{L},\mathcal{G}^{0,-})$ consists of projections $\{p_A : A \in \mathcal{G}^0\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ with the properties that

(i) If $A, B \in \mathcal{G}^{0,-}$ then $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, where $p\emptyset = 0$.

(ii) If $a \in \mathcal{A}$ and $A \in \mathcal{G}^{0,-}$ then $p_A s_a = s_a p_{r(A,a)}$.

(iii) If $a, b \in \mathcal{A}$ then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$.

(iv) For $A \in \mathcal{G}^{0,-}$ we have

$$p_A = \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^*. \tag{3}$$

Definition (1.1.6)[443]: Let $(E,\mathcal{L},\mathcal{G}^{0,-})$ be a labelled space, then $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$ is the universal C^*-algebra generated by a representation of $(E,\mathcal{L},\mathcal{G}^{0,-})$.

The universal property of $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$ allows us to define a strongly continuous action V of T on $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$ called the *gauge action* (see [3, Section 5 As in [20, Proposition 3.2] we denote by Φ the conditional expectation of $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$ onto the fixed point algebra $C^*(E,\mathcal{L},\mathcal{G}^{0,-})^\gamma$. If $(E,\mathcal{L},\mathcal{G}^{0,-})$ is a labelled space then by [3, Lemma 4.4] we have

$$C^*(E,\mathcal{L},\mathcal{G}^{0,-}) = \overline{span}\{s_\alpha p_A s_\beta^* : \alpha,\beta \in \mathcal{L}^*(E), A \in \mathcal{G}^{0,-}$$

Indeed, we can write down a more informative spanning set for $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$.

Proposition (1.1.7)[443]: Let $(E,\mathcal{L},\mathcal{G}^{0,-})$ be a labelled space. Then

$$C^*(E,\mathcal{L},\mathcal{G}^{0,-}) = \overline{span}\{s_\alpha p_{[v]_\ell} s_\beta^* : \alpha,\beta \in \mathcal{L}^*(E), [v]_\ell \in \Omega_\ell\}$$

Where

$$p_{[v]_\ell} := p_{X_\ell(v)} - p_{r(Y_\ell(v))p_{X_\ell(v)}} = \sum_{a\in\mathcal{L}([v]_\ell E^1)} s_a p_{r([v]_\ell,a)} s_a^*. \quad (4)$$

Proof. The first assertion holds from repeated applications of Proposition (1.1.2). Applying (3) of Definition (1.1.1) we have

$$p_{[v]_\ell} = p_{X_l}(v) - pX_\ell(v) \cap r\big(Y_\ell(v)\big)$$
$$= \sum_{a\in\mathcal{L}(X_\ell(v)E^1)} s_a p_{r(X_\ell(v),a)} s_O^* - \sum_{b\in\mathcal{L}(X_\ell(v)\cap r(Y_\ell(v))E^1)} s_b p_{r(X_\ell(v)\cap r(Y_\ell(v)),b)} s_b^*.$$

In order to eliminate double counting of labels that are emitted by both $X_\ell(v)$ and $r(Y_\ell(v))$ we need to split $\mathcal{L}(X_\ell(v)E^1)$ into two disjoint parts (the labels that come only out of $X_\ell(v)$ and those that come out of both $X_\ell(v)$ and $r(Y_\ell(v))$) to obtain

$$p_{[v]_\ell} = \sum_{a\in\mathcal{L}(X_\ell(v)E^1)\setminus\mathcal{L}(X_\ell(v)\setminus r(Yp(v))E^1)} s_a p_{r(X_\ell(v),a)} s_O^*$$
$$+ \sum_{b\in\mathcal{L}(X_\ell(v)\cap r(Y_\ell(v))E^1)} s_b\big(p_{r(X_\ell(v),b)} - p_{r(X_\ell(v)\cap r(Y_\ell(v)),b)}\big)s_b^*.$$

We may replace $X_\ell(v)$ in the first sum by $[v]_\ell$ as the labels a are emitted only by the vertices in $[v]_\ell$ and not by the vertices in $X_\ell(v) \cap r(Y_\ell(v))$. In the second sum the labels b are emitted by both $[v]_\ell$ and $X_\ell(v) \cap r(Y_\ell(v))$, but we subtract the projections corresponding to the copies emitted by $X_\ell(v) \cap r(Y_\ell(v))$ and so we have equation (4) as required.

Recall from Proposition (1.1.4) that to a labelled graph $(E,\mathcal{L})$ over a finite al- phabet $\mathcal{A}$ we may associate an essential symbolic matrix system $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$. By [16, Proposition 2.1] there is a unique (up to isomorphism) λ-graph system $\mathfrak{L}_{E,\mathcal{L}}$ associated to $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$. By [17, Theorem 3.6] one may associate a C^*-algebra $\theta_{\mathfrak{L}_{E,\mathcal{L}}}$ to the λ-graph system $\mathfrak{L}_{E,\mathcal{L}}$ which is the universal C^*-algebra generated by partial isometries $\{t_a : a \in \mathcal{A}\}$ and projections $\{E_i^\ell : i = 1,\ldots,m(\ell)\}$ satisfying relations

$$\sum_{a\in\mathcal{A}} t_a t_a^* = 1 \quad (5)$$

$$\sum_{i=1}^{m(\ell)} E_i^\ell = 1,\ \ E_i^\ell = \sum_{j=1}^{m(\ell+1)} I(E)_{\ell,\ell+1}(i,j)E_j^{\ell+1} \quad for\ i = 1,\ldots,m(\ell) \quad (6)$$

$$t_a t_a^* E_i^\ell = E_i^\ell t_a t_a^* \quad for\ \ a \in \mathcal{A}\ and\ i = 1,\ldots,m(\ell) \quad (7)$$

$$t_a^* E_i^\ell t_a = \sum_{j=1}^{m(\ell+1} A_{\ell,\ell+1}(i,a,j)E_j^{\ell+1}\ for\ a \in \mathcal{A}\ and\ i = 1,\ldots,m(\ell) \quad (8)$$

where $A_{\ell,\ell+1}(i,a,j) = 1$ if a occurs in the formal sum $M(E)_{\ell,\ell+1}([v_i]_\ell, [v_j]_{\ell+1})$ and is 0 otherwise.

Proposition (1.1.8)[443]: Let $(E,\mathcal{L})$ be a left-resolving labelled graph overa finite alphabet. Then we have $C^*(E,\mathcal{L},\mathcal{G}^{0,-}) \cong \theta_{\mathfrak{L}_{E\mathcal{L}}}$ *where* $\mathfrak{L}_{E,\mathcal{L}}$ *is the* λ-graph system associated to the symbolic matrix system $(M(E)_{\ell,\ell+1}, I(E)_{\ell,\ell+1})_{\ell\geq 1}$.

Proof. By Proposition (1.1.7) the elements $\{s_a : a \in \mathcal{A}\}$ and $\{p_{[v_j]_\ell} : i = 1,\ldots,m(\ell)\}$ form a generating set for $C^*(E, \mathcal{L}, \mathcal{G}^0)$ Let $T_a = s_a$ and $F_i^\ell = p_{[v_j]_\ell}$ then $\{T_a, F_i^\ell\}$ satisfy relations (5)-(8) above. Hence by the universal property of $\theta_{\mathfrak{L}_{E\mathcal{L}}}$ there is a map $\pi_{T,F} : \theta_{\mathfrak{L}_{E,\mathcal{L}}} \to C^*(E,\mathcal{L},\mathcal{G}^0)$ characterised by $\pi_{T,F}(t_a) = T_a$ and $\pi_{T,F}(E_i^\ell) = F_i^\ell$.

Let $\{t_a : a \in \mathcal{A}\}$ and $\{E_i^\ell : i = 1,\ldots,m(\ell)\}$ be generators for $\theta_{\mathfrak{L}_{E\mathcal{L}}}$. For $A \in \mathcal{G}_\ell^{0,-}$ and $a \in \mathcal{A}$ let $P_A = \sum_{i:[v]_\ell \subseteq A} E_i^\ell$ and $S_a = t_a$. One checks that $\{S_a, P_A\}$ is a representation of the labelled space $(E, \mathcal{L}, \mathcal{G}^0)$ By universality of $C^*(E, \mathcal{L}, \mathcal{G}^0)$ there is a map $\pi_{S,P} : C^*(E,\mathcal{L},\mathcal{G}^{0,-}) \to \theta_{\mathfrak{L}_{E\mathcal{L}}}$ characterised by $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_A) = P_A$. In particular, we have $\pi_{S,P}(p_{[v_j]_\ell}) = P_{[v_j]_\ell}$ for all $i \in 1,\ldots,m(\ell)$. The result follows since $\pi_{T,F}$ and $\pi_{S,P}$ are inverses of one another.

We perform a detailed analysis of the AF core of $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$.

Definition (1.1.9)[443]: For $1 \le k \le \ell$ let

$$\mathcal{F}^k(\ell) = \overline{span}\{s_\alpha p_A s_\beta^* : \alpha,\beta \in \mathcal{L}(E^k), A \in \mathcal{G}_\ell^{0,-}\}.$$

For $\ell \ge 1$ and $[v]_\ell \in \Omega_\ell$ we have $p_{[v]_\ell} \in \mathcal{F}^k(\ell)$ as $X_\ell(v)$, $r(Y_\ell(v)) \in \mathcal{G}_\ell^{0,-}$ by Lemma (1.1.1)(ii).

Definition (1.1.10)[443]: For $1 \le k \le \ell$ and $[v]_\ell \in \Omega_\ell$ let

$$\mathcal{F}^k([v]_\ell) = \overline{span}\{s_\alpha p_{[v]_\ell} s_\beta^* : \alpha,\beta \in \mathcal{L}(E^k)\}.$$

Proposition (1.1.11)[443]: *For $1 \le k \le \ell$ we have*

(i) $\mathcal{F}^k(\ell) \cong \bigoplus_{[v]_\ell} \mathcal{F}^k([v]_\ell)$, *where each $\mathcal{F}^k([v]_\ell)$ is a finite-dimensional matrix algebra.*

(ii) *For each $v \in E^0$* there are $w_1,\ldots,w_n \in [v]_\ell$ *such that* $\mathcal{F}^k([v]_\ell) = \bigoplus_{i=1}^n \mathcal{F}^k([w_i]_{\ell+1})$. *Hence $\mathcal{F}^k(\ell) \subseteq \mathcal{F}^k(\ell+1)$.*

(iii) There is an embedding of $\mathcal{F}^k(\ell)$ *into* $\mathcal{F}^{k+1}(\ell+1)$.

Proof. For the first statement of (i), applying Proposition (1.1.2)(ii) shows that every element $s_\alpha p_A s_\beta^* \in \mathcal{F}^k(\ell)$ can be written as a finite sum of elements of the form $s_\alpha p_{[v]_\ell} s_\beta^* \in \mathcal{F}^k([v]_\ell)$. The result follows as the summands in the decomposition are mutually orthogonal since $|\alpha| = |\beta| = k$ and the equivalence classes $[v]_\ell$ are disjoint. For the second statement of (i) note that since $[v]_\ell$ can be written as the difference of two elements of $\mathcal{G}^{0,-}$ it receives only finitely many different labelled paths of length k and hence the set $\{s_\alpha p_{[v]_\ell} s_\beta^* : |\alpha| = |\beta| = k\}$ is finite. It is straight forward to show that the elements $s_\alpha p_{[v]_\ell} s_\beta^*$ form a system of matrix units in $\mathcal{F}^k([v]_\ell)$ and the result follows.

Part (ii) follows by Proposition (1.1.2) (iii). Part (iii) follows from Definition (1.1.5) (iv).

Theorem (1.1.12)[443]: Let $(E,\mathcal{L},\mathcal{G}^{0,-})$ *be a labelled space, then* $\mathcal{F} = \overline{\bigcup_{k,\ell} \mathcal{F}^k(\ell)}$ is an AF algebra with $\mathcal{F} \cong C^*(E,\mathcal{L},\mathcal{G}^{0,-})^\gamma$.

Proof. The first statement follows from Proposition (1.1.11). The second statement follows by an argument similar to that of [1, Lemma 2.2].

Recall from [11, §3] that the directed graph E satisfies condition (L) if every loop has an exit; that is if $\lambda \in E^n$ is a loop, then there is some $1 \le i \le n$ such that the vertex $r(\lambda_i)$ emits more than one edge. Condition (L) is the key hypothesis for the Cuntz-Krieger uniqueness theorem for directed graphs (see [11, Theorem 3.7], [1, Theorem 3.1]). Since periodic paths in E^∞ arise from loops in E, condition (L) guarantees that there are lots of paths in E^∞ which are aperiodic.

We seek an analogue for condition (L) of labelled graphs which will allow us to prove a Cuntz-Krieger uniqueness theorem for labelled graph C^*-algebras. The correct analogue for condition (L) must ensure the existence of aperiodic paths in $X_{E,\mathcal{L}}^+$. The two key difficulties to overcome of labelled graphs are that we must accommodate the generalised vertices $[v]_\ell$ in a labelled graph and deal with the fact that a periodic path $x \in X^{+}_{E,\mathcal{L}}$ need not arise from a loop in E.

The following definition is inspired by [20, Lemma 3.7].

Definition (1.1.13)[443]: Let $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be a labelled space, $[v]_\ell \in \Omega_\ell$ and $\alpha \in \mathcal{L}^*(E)$ be such that $|\alpha| > 1$ and $s(\alpha) \cap [v]_\ell \neq \emptyset$. We say that α is *agreeable* for $[v]_\ell$ if there are α', $\beta,\ V \in \mathcal{L}^*(E)$ with $|\beta| = |\gamma| \leq \ell$ and $\alpha = \beta\alpha' = \alpha'\gamma$. Otherwise we say that α is disagreeable for $[v]_\ell$.

We say that $[v]_\ell$ is *disagreeable* if there is an $N > 0$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}^*(E)$ with $|\alpha| \geq n$ that is disagreeable for $[v]_\ell$.

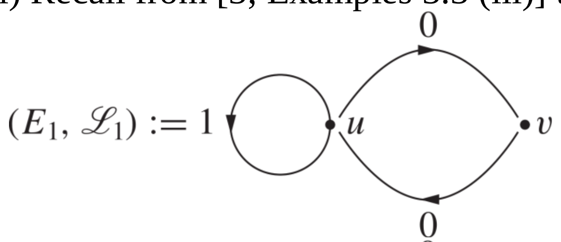
The labelled space $(E, \mathcal{L}, \mathcal{G}^{0,-})$ is disagreeable if for every $v \in E^0$ there is an $L_v > 0$ such that $[v]_\ell$ is disagreeable for all $\ell > L_v$.

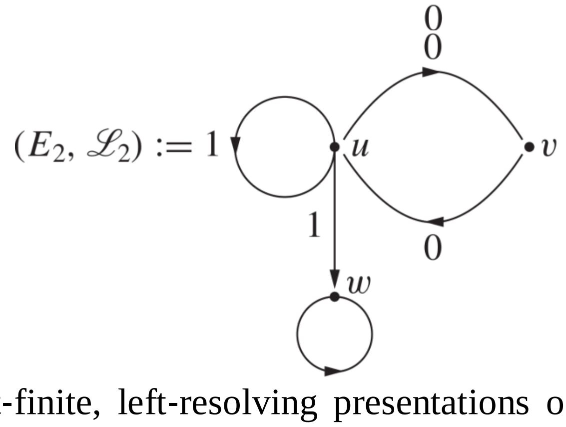
The following Lemma shows that the notion of disagreeability reduces to condition (L) for directed graphs and so is the appropriate condition for us to use in our Cuntz-Krieger uniqueness theorem and simplicity results.

Lemma (1.1.14)[443]: The directed graph E satisfies condition (L) if and only if the labelled space $(E, \mathcal{L}, \mathcal{G}^{0,-})$ is disagreeable.

Proof. Suppose that E satisfies condition (L). Observe that for all $\ell \geq 1$ and all $v \in E^0$, $[v]_\ell = \{v\}$. We show that every $v \in E^0$ is disagreeable. Let $L_v = 1$, $N = 1$, fix $n > N$ and $\ell > L_v$. If v does not lie on a loop, then any path α with $|\alpha| \geq n$ is disagreeable for $[v]_\ell = \{v\}$. If v does lie on a loop $\alpha = \alpha_1 \dots \alpha_m$, without loss of generality we may assume that $s(\alpha) = v$. Since E satisfies condition (L) there is a path β with $s(\beta) = v$ and $\beta_{|\beta|} \notin \{\alpha_1, \dots, \alpha_m\}$. The path $\alpha^n\beta$ has length $\geq n$ and is disagreeable for $[v]_\ell$. Suppose E does not satisfy condition (L). Then there is a $v \in E^0$ and a simple loop α with $s(\alpha) = v$ that has no exit. Let $N > 0$. Then there is an n such that $|\alpha^n| > N$. Suppose $n \geq 2$. We claim that $\lambda = \alpha^n$ is agreeable for every $\ell > |\alpha|$. Set $\beta = V = \alpha$ and $\lambda' = \alpha^{n-1}$. Since $\lambda = \beta\lambda' = \lambda'\gamma$ where $|\beta| = |\gamma| \leq \ell$ it follows that $[v]_\ell = \{v\}$ is agreeable for ℓ. Since α^n is the only path of length $n|\alpha|$ emitted by v, it follows that v is not disagreeable. Thus the labelled space $(E, \mathcal{L}, \mathcal{G}^{0,-})$ is not disagreeable.

Examples (1.1.15)[443]: (i) Recall from [3, Examples 3.3 (iii)] the labelled graphs





are set-finite, receiver set-finite, left-resolving presentations of the even shift. Consider $(E_1, \mathcal{L}_1)$. We claim that $(E_1 , \mathcal{L}_1 , \mathcal{G}_1^{0,-})$ is disagreeable. Now for all $\ell \geq 1$ we have $[u]_\ell =$

$\{u\}$. Let $L_u = 1$ and $N = 3$. Then for $n > N$ the labelled path $\alpha_n = 11^n 0$ satisfies $|\alpha_n| = n + 2 \geq N$ and α_n is disagreeable for $[u]_\ell$ as its first and last symbols disagree. Also for all $\ell \geq 1$ we have $[v]_\ell = \{v\}$. If we let $N = 4$ and $L_v = 1$, then for each $n > N$ the path $\alpha_n = 0^{2n+1}1$ satisfies $|\alpha_n| = 2n + 2 \geq n$ and α_n is disagreeable for $[v]_\ell$ as its first and last symbols disagree. Thus the labelled space $(E_1, \mathcal{L}_1, \mathcal{G}_1^{0,-})$ is disagreeable and our claim is established.

Consider $(E_2, \mathcal{L}_2)$. We claim that $[w]_\ell$ is agreeable for all $\ell \geq 2$. Now $[w]_\ell = \{w\}$ for all $\ell \geq 2$, and any labelled path α satisfying $s(\alpha) \cap [w]_\ell \neq \emptyset$ must have the form $\alpha = 0^n$ for some n. But $\alpha = 0^n$ is agreeable for $[w]_\ell$ for all $\ell \geq 2$ whenever $\geq \ell + 1$: set $\alpha' = 0^{n-\ell}$, $\beta = V = 0$. Thus $(E_2, \mathcal{L}_2, \mathcal{G}_2^{0,-})$ is not disagreeable.

(ii) Let G be a group with a finite set of generators $S = \{g_1, \ldots, g_m\}$, such that $g_i \neq g_j$ for $i \neq j$. The (right) *Cayley graph of G with respect to S* is the essential row-finite directed graph $E_{G,S}$ where $E^0_{G,S} = G$, $E^1_{G,S} = G \times S$ with range and source maps given by $r\,(h, g_i) = hg_i$ and $s(h, g_i) = h$ for $i = 1\,, \ldots, m$. The map $\mathcal{L}_{G,S}(h, g_i) = g_i$ gives us a set-finite, receiver set-finite, labelled graph $(E_{G,S}, \mathcal{L}_{G,S})$. Since G is cancelative it follows that $(E_{G,S}, \mathcal{L}_{G,S})$ is leftresolving. As each vertex in $E_{G,S}$ receives the same labelled paths it follows that $[g]_\ell = G$ for all $g \in G$ and $\ell \geq 1$ and so $\mathcal{G}^0_{G',,S} = \{\emptyset, G\}$. Each $g \in G$ emits the same m^ℓ labelled paths of length ℓ. So if $m = |S| > 1$, it follows that for all $[g]_\ell = G$ there is a disagreeable labelled path of length $n > 1$ beginning at $[g]_\ell = G$. Hence $(E_{G,S}, \mathcal{L}_{G,S}, \mathcal{G}^{0,-}_{G',S})$ is disagreeable.

Theorem (1.1.16)[443]: *Let $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be a disagreeable labelled space. If $\{T_\alpha, Q_A\}$ and $\{S_\alpha, P_A\}$ are two representations of $(E,\ \mathcal{L},\ \mathcal{G}^0)$ in which all the projections p_A , P_A are nonzero, then there is an isomorphism ϕ of $C^*(T_\alpha, Q_A)$ onto $C^*\ (S_\alpha, P_A)$ such that $\phi(T_\alpha) = S_\alpha$ and $\phi(Q_A) = P_A$.*

To prove this theorem we show that the representations $\pi_{T,Q}$ and $\pi_{S,P}$ of $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})$ are faithful. The required isomorphism will then be $\phi = \pi_{S,P} \circ \pi^{-1}_{T,Q}$. The usual approach is to invoke symmetry and prove that

(i) $\pi_{S,P}$ is faithful on $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})^\gamma$ and

(ii) $\|\pi_{S,P}(\Phi(a))\| \leq \|\pi_{S,P}(a)\|$ for all $a \in C^*(E, \mathcal{L}, \mathcal{G}^{0,-})$

Part (i) is proved in [3, Theorem 5.3]. To prove (ii) we must do a little more work than is needed for graph C^*-algebras because of the more complicated structure of $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})^\gamma$.**Proof.** Every element of $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})$ may be approximated by elements of the form

$$a = \sum_{(\alpha,[w]_\ell,\beta)\in F} c_{\alpha,[w]_\ell,\beta}\, s_\alpha p_{[w]_\ell} s^*_\beta$$

where F is finite, and so it is enough to prove (b) for such elements a.

Let $k = \max\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$. By Proposition (1.2.7) we may suppose (changing F if necessary), that every $(\alpha, [w]_\ell, \beta) \in F$ is such that $\min\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\} = k$. Let $k = \max\{|\alpha|\ ,\ |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$ and $L = \max\{L_w : (\alpha, [w]_\ell, \beta) \in F\}$. By Proposition (1.1.2)(iii) we may suppose (again changing F if necessary, but not M or k) that $\ell \geq \max\{L, M - k\}$.

Since $|\alpha| = |\beta|$ implies that $|\alpha| = k$ we have

$$\Phi(a) = \sum_{(\alpha,[w]_\ell,\beta)\in F,|\alpha|=|\beta|} c_{\alpha,[w]_\ell,\beta}\, s_\alpha p_{[w]_l} s^*_\beta \in \mathcal{F}^k(\ell)$$

where Φ is the conditional expectation of $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$ onto $C^*(E\ ,\ \mathcal{L},\ \mathcal{G}^{0,-})^{\gamma}$. By Proposition (1.1.11) (i) $\mathcal{F}^k(\ell)$ decomposes as the C^*-algebraic direct sum $\bigoplus_{[w]_\ell}\mathcal{F}^k([w]_\ell)$, so does its image under $\pi_{S,P}$, and there is a $[v]_\ell \in \Omega_\ell$ such that $\|\pi_{S,P}(\Phi(a))\|$ is attained on $\mathcal{F}^k([v]_\ell)$. Let $F_{[v]_\ell}$ denote the elements of F of the form $(\alpha,[v]_\ell,\beta)$, then we have

$$\left\|\pi_{S,P}\big(\Phi(a)\big)\right\| = \left\|\sum_{(\alpha,[v]_\ell,\beta)\in F_{[v]_\ell},|\alpha|=|\beta|} c_{\alpha,[v]_\ell,\beta}S_\alpha P_{[v]_\ell}S_\beta^*\right\|.$$

We write

$$b_v = \sum_{(\alpha,[v]_\ell,\beta)\in F_{[v]_\ell},|\alpha|=|\beta|} c_{\alpha,[v]_\ell,\beta}S_\alpha P_{[v]_\ell}S_\beta^*$$

and let $G = \{\alpha\ :\ \text{either}\ (\alpha,[v]_\ell,\beta)\in F_{[v]_\ell}\ \text{or}\ (\beta,[v]_\ell,\alpha)\in F_{[v]_\ell}\ \text{with}\ |\alpha| = |\beta|\}$. Then $span\{S_\alpha P_{[v]_\ell}S_\beta^*\ :\ \alpha,\beta\ \in\ G\}$ is a finite dimensional matrix algebra containing b_v.

Since $\ell > L$, $[v]_\ell$ is disagreeable. Hence there is an $n > M$ and a λ with $|\lambda| \geq n$ and $[v]_\ell \cap s(\lambda) \neq \emptyset$ which has no factorisation $\lambda = \lambda'\lambda'' = \lambda''\gamma$ for $\lambda', V \in \mathcal{L} \leq (M-k)(E)$ (as $M - k \leq \ell$). We claim that

$$Q = \sum_{v\in G} S_{v\lambda}\,P_{r([v]_\ell\lambda,)}S_{v\lambda}^*$$

is such that

$$\left\|Q\pi_{S,P}\big(\Phi(a)\big)Q\right\| = \left\|\pi_{S,P}\big(\Phi(a)\big)\right\|, \tag{9}$$

and

$$QS_\alpha P_{[v]_\ell}S_\beta^* Q = 0\ when\ (\alpha,[v]_\ell,\beta)\in F\ and\ |\alpha|\neq|\beta|. \tag{10}$$

The formula for Q can be made sense of by a calculation similar. A routine calculation verifies (9).

Now suppose that $(\alpha,[v]_\ell,\beta)\in F$ satisfies $|\alpha|\neq|\beta|$. Either α or β has length k, say $|\alpha| = k$. As before, $S_{v\lambda}^* S_\alpha$ is non-zero if and only if $v = \alpha$. Thus

$$\begin{aligned} QS_\alpha P_{[v]_\ell}S_\beta^* Q &= \sum_{v\in G} S_{\alpha\lambda}P_{r([v]_{\ell,\lambda})}S_{\alpha\lambda}^* S_\alpha P_{[v]_\ell}S_\beta^* S_{v\lambda}P_{r([v]_{\ell,\lambda})}S_{v\lambda}^* \\ &= \sum_{v\in G} S_{\alpha\lambda}P_{r([v]_{\ell,\lambda})}S_{\beta\lambda}^* S_{v\lambda}P_{r([v]_{\ell,\lambda})}S_{v\lambda}^*. \end{aligned}$$

For $P_{r([v]_{\ell,\lambda})}S_{\beta\lambda}^* S_{v\lambda}P_{r([v]_{\ell,\lambda})}$ to be non-zero $\beta\lambda$ must extend $v\lambda$, which implies that $\beta\lambda = v\lambda\gamma$ for some V. But then we have $\beta = v\lambda'$ for some initial segment λ' of λ as $|\beta| > |v|$. Hence $\lambda = \lambda'\lambda''$ which then implies that $\lambda = \lambda''\gamma$ as

$$\beta\lambda = v\lambda'\lambda'' = v\lambda\gamma = v\lambda'\lambda''\gamma$$

and that $|\lambda'| = |\gamma|$. Since $|\beta| \leq M\ and\ |v| = k$ it follows that $|\lambda'| \leq M - k \leq \ell$. Thus λ is agreeable for $[v]_\ell$, a contradiction. Thus $QS_\alpha P_{[v]_\ell}S_\beta^* Q = 0$, and we have verified (10).

The rest of the proof is now standard (see, for example, [20, p. 31])

Recall from [10, Corollary 6.8] that a directed graph E is *cofinal* if for all $x \in E^\infty$ and $v \in E^0$ there is a path $\lambda \in E^*$ and $N \geq 1$ such that $s(\lambda) = v$ and $r(\lambda) = r(x_N)$. Along with condition (L), cofinality is the key hypothesis in the simplicity results for directed graphs (see [10, Corollary 6.8], [1, Proposition 5.1]).

We seek an analogue for cofinality of labelled graphs which will allow us to prove a simplicity theorem for labelled graph C^*-algebras. The two key difficulties to overcome of labelled graphs are that we must accommodate the generalised vertices $[v]_\ell$ in a labelled graph and the fact that there may be many representatives of a given infinite labelled path $x \in X^+_{E,\mathcal{L}}$.

Definition (1.1.17)[443]: Let $(E, \mathcal{L}, \mathcal{G}^{0,-})^\gamma$ be a labelled space and $\ell \geq 1$. We say that $(E, \mathcal{L}, \mathcal{G}^{0,-})^\gamma$ is *ℓ-cofinal* if for all $x \in X^+_{E,\mathcal{L}}$, $[v]_\ell \in \Omega_\ell$, and $w \in s(x)$ there is an $R(w) \geq \ell$, an $N \geq 1$ and a finite number of labelled paths $\lambda_1, \ldots, \lambda_m$ such that for all $d \geq R(w)$ we have $\bigcup_{i=1}^m r([v]_\ell, \lambda_i) \subseteq r([w]_d, x1. . . x_N)$.

We say that $(E, \mathcal{L}, \mathcal{G}^{0,-})^\gamma$ is *cofinal* if there is an $L > 0$ such that $(E, \mathcal{L}, \mathcal{G}^{0,-})^\gamma$ is *ℓ-cofinal* for all $\ell > L$.

Examples (1.1.18)[443]: (i) Recall from Example (1.1.3) that a directed graph E may be considered to be a labelled graph with the trivial labelling $\mathcal{L}_t$. Let E be a cofinal directed graph and fix $v \in E^0$, $x \in E^\infty$. Since $w = s(x)$ is the only vertex with $r(w, x_1. . . x_n) \neq \emptyset$ for all n, we may put $R(w) = 1$ and invoke cofinality of E to get the required N and λ so that $((E, \mathcal{L}_t, \mathcal{G}^{0,-})^\gamma$ is cofinal with $L = 1$. Thus the definition of cofinality for labelled graphs reduces to the usual definition of cofinality for directed graphs.

(ii) The labelled space $(E_2, \mathcal{L}_2, \mathcal{G}_2^0)$ of Example (1.1.15)(i) is not ℓ-cofinal for $\ell \geq 2$, and so not cofinal. To see this, observe that $[w]_\ell = \{w\}$ for $\ell \geq 2$ and there is no labelled path joining w to the infinite path $(100)^\infty$.

The following result will allow us to prove cofinality for many interesting examples.

Lemma (1.1.19)[443]: *Let* $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be a labelled space. If E is row-finite, transitive and $\mathcal{G}^{0,-}$ contains $\{v\}$ *for all* $v \in E^0$ *then* $(E, \mathcal{L}, \mathcal{G}^{0,-})$ *is* cofinal with $L = 1$.

Proof. Let $w \in E^0$. Since $\{w\} \in \mathcal{G}^{0,-}$ there must be an $R(w) \geq 1$ such that $[w]_d = \{w\}$ for all $\geq R(w)$.

Let $\ell \geq 1$ and choose $[v]_\ell \in \Omega_\ell$. Let $w \in E^0$, and choose $R(w)$ as in the first paragraph. Let $x \in X^+_{E,\mathcal{L}}$ be such that $w \in s(x)$. Let $N \geq 1$. Then as E is row-finite there are only finitely many paths $\mu_1, \ldots, \mu_m$ in E with $s(\mu_i) = w$ and $\mathcal{L}(\mu_i) = x_1 \ldots x_N$. By transitivity of E there are paths $\lambda_1, \ldots, \lambda_m \in E^*$ with $s(\lambda_i) = v$ and $r(\lambda_i) = r(\mu_i)$. Then

$$\bigcup_{i=1}^m r([v]_\ell, \mathcal{L}(\lambda_i)) \supseteq r([w]_d, x_1 \ldots x_N)$$

as required. Thus $(E, \mathcal{L}, \mathcal{G}^{0,-})$ is cofinal with $L = 1$.

Theorem (1.1.20)[443]: *Let* $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be cofinal and disagreeable. Then $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})$ is simple.

Proof. Since every ideal in a C^*-algebra is the kernel of a representation, it suffices to prove that every non-zero representation $\pi_{S,P}$ of $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})$ is faithful. Suppose $\pi_{S,P}$ is a non-zero representation of $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})$ If we have $P_{[v]_\ell} = 0$ for all $v \in E^0$ and $\ell \geq 1$ then $\pi_{S,P}$ $= 0$. Thus there is a $w \in E^0$ and a $d \geq 1$ with $P_{[w]_d} \neq 0$. Fix $[v]_\ell \in \Omega_\ell$. We aim to prove that $P_{[v]_\ell} \neq 0$. Since $[w]_d$ is the disjoint union of finitely many equivalence classes $[w_i]_k$ whenever $k \geq d$, for each k there is an i such that $P_{[w_i]_k} \neq 0$. So without loss of generality, for a given $[v]_\ell \in \Omega_\ell$, we may assume that $d \geq R(w)$.

Since $(E, \mathcal{L}, \mathcal{G}^{0,-})$ is set-finite we apply (4) of Proposition (1.1.7) to obtain

$$P_{[w]_d} = \sum_{x_1 \in \mathcal{L}([w]_d E^1)} S_{x_1} P_{r([w]_d, x_1)} S^*_{x_1}.$$

Since the left-hand side is nonzero it follows that $S_{x_1}P_{r([w]_d,x_1)}S_{x_1}^* \neq 0$ for some $x_1 \in \mathcal{L}([w]_d E^1)$ which implies that $P_{r([w]_d,x_1)} \neq 0$. Arguing as in the proof of Proposition (1.1.7) we have

$$P_{r([w]_d,x_1)} = \sum_{x_2 \in \mathcal{L}(r([w]_d,x_1)E^1)} S_{x_2}P_{r(r([w]_d,x_1),x_2)}S_{x_2}^*$$

and so we may deduce that there is an x_2 with $P_{r(r([w]_d,x_1),x_2)} = P_{r([w]_d,x_1x_2)} \neq 0$. Continuing in this way we produce $x = x_1\, x_2 . . . \in X_{E,\mathcal{L}}^+$ such that $P_{r([w]_d,x_1\dots x_n)} \neq 0$ for all $n \geq 1$.

Let $\ell \geq 1$ and $[v]_\ell \in \Omega_\ell$. Since $d > R(w)$, by cofinality, there are finitely many labelled paths λ_1 , . . . , λ_m and an $N \geq 1$ such that $\bigcup_{i=1}^m r\ ([v]_\ell, \lambda_i) \supseteq r\ ([w]_d\ , \mathrm{x}_1 . . . x_N)$. Since $P_{r([w]_d,x_1,\dots,x_N)} \neq 0$ we must have $P_{r([v]_l,\lambda_j)} \neq 0$ for some $i \in \{1, \dots , m\}$. Since $r\ ([v]_\ell, \lambda_i) \subseteq r(\lambda_i)$ it then follows that $P_{r(\lambda_i)} \neq 0$ and hence $S_{\lambda_i} \neq 0$. Since $P_{[v]_\ell} = \sum_{\lambda\in\mathcal{L}} S_\lambda P_{r([v]_\ell,\lambda)}S_\lambda^*$ it then follows that $P_{[v]_\ell} \neq 0$ as required.

Thus all the projections $P_{[v]_\ell}$ are non-zero and Theorem (1.1.20) implies that $\pi_{S,P}$ is faithful, completing our proof.

Examples (1.1.21)[443]: (i) The labelled space $(E_1, \mathcal{L}_1, \mathcal{G}_1^{0,-})$ shown to be disagreeable in Examples (1.1.15)(i) is cofinal with $L = 1$. This follows by Lemma (1.1.19) (i) since E_1 is row-finite, transitive and $\{v\} \in \mathcal{G}_1^{0,-}$ for all $v \in E_1^0$. Hence $C^*(E_1\ , \mathcal{L}_1\ , \mathcal{G}_1^0\ )$ is simple by Theorem (1.1.20).

(ii) The labelled space $(E_{G,S}, \mathcal{L}_{G,S}, \mathcal{G}_{G',S}^{0,-})$ of Examples (1.1.18) (ii) is cofinal with $L = 1$. To see this recall that $[g]_\ell = E_{G,S}^0 = G$ for all $\ell \geq 1$. Fix $[g]_\ell \in \Omega_\ell$ and $x \in X_{E_{GS},\mathcal{L}_{GS}}^+$. For $h \in G$, $d \geq R(h) = 1$ and $n = 1$ we have $r\ ([h]_d, x_1) = G$. Let λ_1 be any element of S, then $r([g]_\ell, \lambda_1) = G = r\ ([h]_d, x_1)$. Hence $C^*(E_{G,S}, \mathcal{L}_{G,S}, \mathcal{G}_{G',S}^{0,-})$ is simple by Theorem (1.1.20).

We now tum our attention to the question of pure infiniteness for simple labelled graph C^*-algebras. For graph C^*-algebras the key hypotheses are condition (L) and every vertex connects to a loop (see [11, Theorem 3.9], [1, Proposition 5.4]). As we already have an analogue of condition (L), we must now seek to find a suitable replacement for the requirement that every vertex connects to a loop in the context of labelled graphs. Again, there are two difficulties to overcome: we must accommodate the generalised vertices $[v]_\ell$ in a labelled graph and find the correct analogue of a loop.

Definitions (1.1.22)[443]: The labelled path α is *repeatable* if $\alpha^n \in \mathcal{L}^*(E)$ for all $n \geq 1$. We say that every vertex connects to a repeatable labelled pat*h* if for every $[v]_m \in \Omega_m$ there is a $w \in E^0$, $L(w) \geq 1$ and labelled paths $\alpha, \delta \in \mathcal{L}^*(E)$ with $w \in r([v]_m, \delta\alpha)$ such that $[w]_\ell \subseteq r([w]_\ell, \alpha)$ for all $\ell \geq L(w)$

The pure infiniteness result requires the following lemma whose proof follows along similar lines to that of [1, Lemma 5.4].

Lemma (1.1.23)[443]: Let $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be a labelled space, $v \in E^0$ *and* $\ell \geq 1$. *Let* t be a positive element of $\mathcal{F}^k([v]_\ell)$. Then there is a projection r in th*e* C^*-subalgebra of $\mathcal{F}^k([v]_\ell)$ generated by t such that $rtr = \|t\|r$.

Theorem (1.1.24)[443]: Let $(E, \mathcal{L}, \mathcal{G}^{0,-})$ be cofinal and disagreeable. If every vertex connects to a repeatable labelled path then $C^*(E, \mathcal{L}, \mathcal{G}^{0,-})$ is simple and purely infinite.

Proof. We know that $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$ is simple by Theorem (1.1.20). We show that every hereditary subalgebra A of $C^*(E,\mathcal{L},\mathcal{G}^{0,-})$ contains an infinite projection; indeed we shall produce one which is dominated by a fixed positive element $a \in A$ with $\|\Phi(a)\| = 1$.
By Proposition (1.1.7) we may choose a positive element $\in span\{s_\alpha p_{[v]_\ell} s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E)$, $[v]_\ell \in \Omega_\ell\}$ such that $\|a - b\| < \frac{1}{4}$. Suppose $b = \sum_{c_{\alpha,[w]_\ell,\beta \in F}} s_\alpha p_{[w]_\ell} s_\beta^*$ where F is a finite subset of $\mathcal{L}^*(E) \times \Omega \times \mathcal{L}^*(E)$. The element $b_0 := \Phi(b)$ is positive and satisfies $\|b_0\| \geq \frac{3}{4}$. Let $k = \max\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$. By repeatedly applying (4) we may suppose(changing F ifnecessary) that $\min\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\} = k$. Let $M = \max\{|\alpha|, |\beta| : (\alpha, [w]_\ell, \beta) \in F\}$, $L_F = \max\{L_w : (\alpha, [w]_\ell, \beta) \in F\}$ and let L be the smallest number such that $(E, \mathcal{L}, \mathcal{G}^0)$ is ℓ-cofinal for all $\ell \geq L$. Then from Proposition (1.1.2) we may assume that $b_0 \in \bigoplus_{w:(\alpha,[w]_\ell,\beta)\in F} \mathcal{F}^k([w]_m)$ for some $\geq \max\{L, L_F, M\}$. In fact, $\|b_0\|$ must be attained in some sum and $\mathcal{F}^k([v]_m)$. Let b_1 be the component of b_0 in $\mathcal{F}^k([v]_m)$, and note that $b_1 \geq 0$ and $\|b_1\| = \|b_0\|$. By Lemma (1.1.23) there is a projection $r \in C^*(b_1) \subseteq \mathcal{F}^k([v]_m)$ such that $r b_1 r = \|b_1\| r$. Since b_1 is a finite sum of $s_\alpha p_{[v]_m} s_\beta^*$ we can write r as a sum $\sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^*$ over all pairs of paths in

$$S = \{\alpha \in \mathcal{L}(E^k): either\ (\alpha, [w]_\ell, \beta) \in F\ or\ (\beta, [w]_\ell, \alpha) \in F\ and\ [w]_\ell \subseteq r(\alpha)\}.$$

As $m \geq L_v$, $[v]_m$ is disagreeable and there is an $n > M$ and a $\lambda \in \mathcal{L}^*(E)$ with $|\lambda| \geq n$ which is disagreeable for $[v]_m$. Since $m \geq M \geq M - k$ as well we may employ the same argument as in the proof ofTheorem (1.1.16) to produce a projection $Q := \sum_{\gamma \in S} s_{\gamma\lambda} p_{r([v]_m,\lambda)} s_{\gamma\lambda}^*$ such that $Q s_\alpha p_{[v]_m} s_\beta^* Q = 0$ unless

$|\alpha| = |\beta| = k$ and $[v]_m \subseteq r(\alpha) \cap r(\beta)$. Since $r \in C^*(b_1)$ we have

$$r = \sum c_{\alpha\beta} s_\alpha p_{[v]_m} s_\beta^* = \sum c_{\alpha\beta} s_\alpha (s_\lambda p_{r([v]_m,\lambda)} s_\lambda^* + (p_{[v]_m} - s_\lambda p_{r([v]_m,\lambda)} s_\lambda^*)) s_\beta^* \geq Q$$

so that

$$QbQ = Qb_0Q = Qrb_1rQ = \|b_1\| rQ = \|b_0\| Q \geq \frac{3}{4} Q.$$

Since $\|a - b\| \leq \frac{1}{4}$ we have $QaQ \geq QbQ - \frac{1}{4} Q \geq \frac{1}{2} Q$ and so QaQ is invertible in $QC^*(E, \mathcal{L}, \xi^{0,-})Q$. Let c denote its inverse and put $v = c^{1/2} Q a^{1/2}$. Then $v^* = c^{1/2} QaQ c^{1/2} = Q$, and $v^* v = a^{1/2} QcQ a^{1/2} \leq \|c\| a$ and so $v^* v$ belongs to the hereditary subalgebra A. To finish, we must show that $v^* v$ is an infinite projection.

We wish to find a labelled path β with $r([v]_m, \beta) \neq \emptyset$ whose initial segment is λ and whose terminal segment is a repeatable labelled path. We choose $x \in r([v]_m, \lambda)$. Then $[x]_{m+|\lambda|} \subseteq r([v]_m, \lambda)$ and by hypothesis $[x]_{m+|\lambda|}$ connects to a repeatable path: That is, there is a $w \in E^0$, $L(w) \geq 1$ and paths $\alpha, \delta \in \mathcal{L}^*(E)$ such that $w \in r([x]_{m+|\lambda|}, \delta\alpha)$, and $[w]_n \subseteq r([w]_n, \alpha)$ for all $n \geq L(w)$. The required path is $\beta = \lambda\delta\alpha$. Let $N = \max\{L_w, L(w)\}$. We claim that $p_{[w]_n}$ is an infinite projection for all $n \geq N$. As $n \geq L(w)$, that we have $r([w]_n, \alpha^i) \neq \emptyset$, for $i \geq 1$. Moreover, as $n \geq L_w$ we know that $[w]_n$ is disagreeable. Hence there must be a labelled path V with $[w]_n \cap s(\gamma) \neq \emptyset$ and $i \geq 1$ with $|\gamma| = |\alpha^i|$, and $V \neq \alpha^i$. We compute

$$p_{[w]_n} \leq s_{\alpha^i} p_{r([w]_n,\alpha^i)} s_{\alpha^j}^* < s_{\alpha^i} p_{r([w]_n,\alpha^j)} s_{\alpha^j}^* + s_\gamma p_{r([w]_n,\gamma)} s_\gamma^* \leq p_{[w]_n}$$

and our claim is established.

We now demonstrate the existence of an infinite subprojection of Q. If μ is any labelled path with $|\mu| = k \leq M \leq m$ and $r(\mu) \cap s(\lambda) \cap [v]_m \neq \emptyset$ then for $n \geq N$ such that $[w]_n \subseteq$

$r([v]_m, \lambda\delta\alpha)$ (note that such an n exists as $[w]_n \subseteq r([v]_m, \lambda\delta\alpha)$ for all sufficiently large n) we have

$$p_{[w]_n} = p_{[w]_n} s^*_{\mu\lambda\delta\alpha} s_{\mu\lambda\delta\alpha} \sim s_{\mu\lambda\delta\alpha} p_{[w]_n} s^*_{\mu\lambda\delta\alpha} \leq s_{\mu\lambda} p_{r([v]_m,\lambda)} s^*_{\mu\lambda}.$$

Because the projection $s_{\mu\lambda} p_{r([v]_m,\lambda)} s^*_{\mu\lambda}$ is a minimal projection in the matrix algebra $span\{s_{\mu\lambda} p_{r([v]_m,\lambda)} s^*_{\nu\lambda} : \mu, \nu \in S\}$, it is equivalent to a subprojection of Q. It follows that Q is infinite, and, since $Q = vv^* \sim v^*v$ this completes the proof.

Examples (1.1.25)[443]: (i) In the labelled space $(E_1, \mathcal{L}_1, \mathcal{G}_1^{0,-})$ of Examples (1.1.15) (i) every vertex connects to the repeatable path 0. Since $(E_1, \mathcal{L}_1, \mathcal{G}_1^{0,-})$ is cofinal and disagreeable, $C^*(E_1, \mathcal{L}_1, \mathcal{G}_1^{0,-})$ is simple and purely infinite by Theorem (1.1.24).

(ii) Suppose that, for a group G, the set S contains (not necessarily distinct) elements $g_1, \dots, g_n$ such that $g_1 \dots g_n = 1_G$. Then every vertex in the labelled graph $(E_{G,S}, \mathcal{L}_{G,S})$ of Examples (1.1.15) (ii) connects to the repeatable labelled path $g_1 \dots g_n$. If in addition we have $|S| > 1$, then by Examples (1.1.15) (ii) and Examples (1.1.21) (ii) $(E_{G,S}, \mathcal{L}_{G,S}, \mathcal{G}^{0,-}_{G',,S})$ is cofinal and disagreeable and so C^* $(E_{G,S}, \mathcal{L}_{G,S}, \mathcal{G}^{0,-}_{G',,S})$ is simple and purely infinite by Theorem (1.1.24).

We associate a labelled graph to a Dyck shift in such a way that the resulting labelled space C^*-algebra is simple and purely infinite.

First we recall the definition of the Dyck shift (see [19], [18]). Let $N \geq 1$ be a fixed positive integer. The Dyck shift D_N has alphabet $\mathcal{A} = \{\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N\}$ where the symbols α_i correspond to opening brackets of type i and the symbols β_i are their respective closing brackets. We say that a word $\gamma_1 \dots \gamma_n \in \mathcal{A}^*$ is admissible if $\gamma_1 \dots \gamma_n$ does not contain any substring $\alpha_i\beta_j$ with $\neq j$. Thus the language of the Dyck shift consists of all strings of properly matched brackets of types $\alpha_1, \dots \alpha_N$.

The following algorithm gives a labelled graph presentation of a Dyck shift.

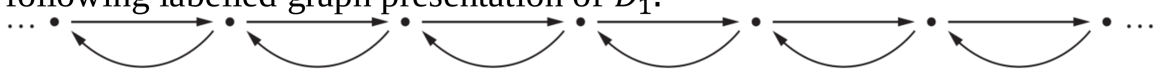
i) Fix $N \geq 1$ and an alphabet $\{\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N\}$.

ii) Draw an unrooted, infinite, directed tree in which every vertex receives one edge and emits N edges (i.e. an N- ary tree). Label the N branches from each node, working from left to right, by $\alpha_1, \dots, \alpha_N$.

iii) For each $i \in \{1, \dots, N\}$ and each edge e labelled α_i , draw an edge from $r(e)$ to $s(e)$ with label β_i.

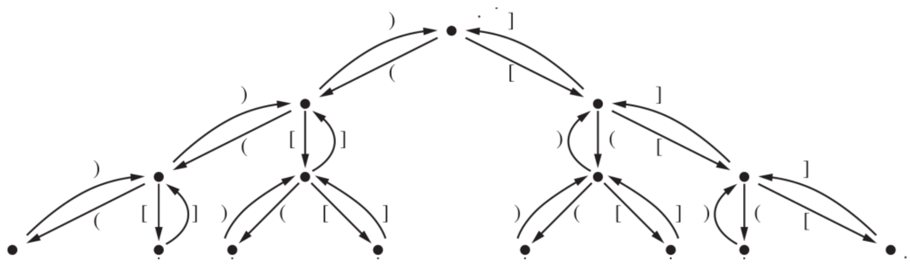
The resulting labelled graph $(E_N, \mathcal{L}_N)$ is a left-resolving labelled graph which presents the Dyck shift D_N.

Examples (1.1.26)[443]: (a) Let $N = 1$ and $\mathcal{A} = \{(,)\}$. The above algorithm gives the following labelled graph presentation of D_1.



The above labelled graph is not the optimal presentation of D_1 , as D_1 has no constraints and so is the full shift on the symbols (and).

(b) Let $N = 2$ and let $\mathcal{A} = \{(, [,),]\}$. The above algorithm gives the following labelled graph presentation of D_2.



We thank W. Krieger for pointing out that the above labelled graph is an asynchronizing Shannon graph of the Dyck shift (see [8]).

Proposition (1.1.27)[443]: Let $N \geq 1$ *and* $\mathcal{A} = \{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\}$. Then $C^*\left(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-}\right)$ is simple and purely infinite.

Proof. Let $\Lambda_\ell = \{\lambda_1^\ell, \ldots, \lambda_{N^\ell}^\ell\}$ be the labelled paths of length ℓ which consist of only α_i's (opening braces), and let $\overline{u^{\mapsto}}\ell = \{\mu_1^\ell, \ldots, \mu_{N^\ell}^\ell\}$ be the labelled paths of length ℓ which consist of only β_i's (closing braces), organised in such a way that for all i the word $\lambda_i^\ell \mu_i^\ell$ belongs to the language of D_N. Since every vertex $v \in E_N^0$ receives one opening brace and N closing braces, it follows that v receives a unique $\lambda_i^\ell \in \Lambda_\ell$ one sees that $\Omega_\ell = \{[v_i^\ell]_\ell : i = 1, \ldots, N^\ell\}$ where v_i^ℓ is some vertex in $r(\lambda_i^\ell)$. Moreover, every vertex $v \in E_N^0$ emits exactly one closing brace (the closing version of the one it receives) and N opening braces, so every v which receives λ_i^ℓ also emits μ_i^ℓ.

For $1 \leq i, j \leq N^\ell$ let $\mu_{ij}^\ell = \mu_i^\ell \lambda_j^\ell$ then $s(\mu_{ij}^\ell) = [v_i^\ell]_\ell$ as the only vertices which emit μ_i^ℓ are those which receive λ_i^ℓ. We have $r(\mu_{ij}^\ell) = r(\lambda_j^\ell) = [v_j^\ell]_\ell$ since every vertex in E_N^0 (emits the labelled path λ_i^ℓ and hence) receives a labelled path μ_i^ℓ which originates from a vertex in $[v_i^\ell]_\ell$, that it $r([v_i^\ell]_\ell, \mu_i^\ell) = E_N^0$.

Fix $\ell \geq 1$, $[v]_\ell \in \Omega_\ell$ and $\in X_{E_N, \mathcal{L}_N}^+$. Without loss of generality suppose that $[v]_\ell = [v_1^\ell]_\ell$. Then by definition of the μ_{ij}^ℓ we have

$$\bigcup_{j=1}^{N^\ell} r\left([v_1^\ell]_\ell, \mu_{1j}^\ell\right) = E_N^0$$

and hence the labelled space $\left(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-}\right)$ is cofinal with $L = 1$.

We now show that $(E_N , \mathcal{L}_N , \mathcal{G}_N^{0,-})$ is disagreeable. For $n \geq 1$, every vertex v emits the labelled path $\alpha_1^n \beta_1$, which is disagreeable for $[v]_\ell$. Hence $[v]_\ell$ is disagreeable for all $\ell \geq 1$. It follows that $C^*\left(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-}\right)$ is simple by Theorem (1.2.20).

Since every vertex $v \in E_N^0$ emits the repeatable labelled path $\alpha_1 \beta_1$ it follows that every generalised vertex in $\left(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-}\right)$ connects to a repeatable labelled path. Thus $C^*\left(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-}\right)$ is purely infinite by Theorem (1.1.23).

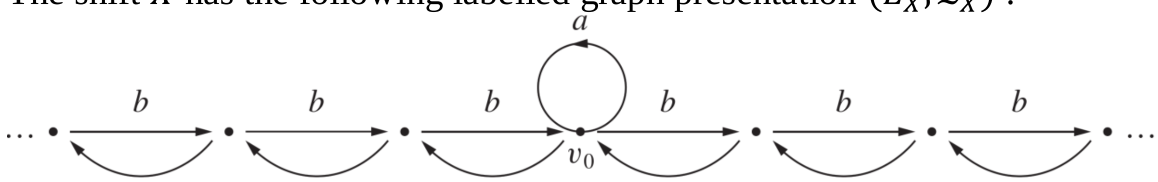
where $C(\Omega, Z)$ denotes the abelian group of all Z-valued continuous functions on the Cantor set Ω. Since the K-theory of $C^*(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-})$ is not finitely generated it follows that $C^*(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-})$ cannot be isomorphic to a unital graph algebra (indeed $C^*(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-})$ is not semiprojective).

Note that the essential symbolic matrix system $(M(E_N)_{\ell,\ell+1}, I(E_N)_{\ell,\ell+1})_{\ell \geq 1}$ associated to $(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-})$ is not the same as the one described in [9, Proposition 2.1]. In [9] the λ-graphs associated to symbolic matrix systems are "upward directed" whereas in [16] they are "downward directed This results from the change of time direction mentioned on [9, p.

81]. Hence to form the appropriate "upward directed" versions for $(E_N, \mathcal{L}_N, \mathcal{G}_N^{0,-})$ it would seem natural to reverse the arrows in E_N.

Consider the shift space X over the alphabet $\mathcal{A} = \{a, b, c\}$ whose language consists of all words in $\{a, b, c\}$ such that the numbers of b's and c's occurring between any pair of consecutive a's are equal.

Note that the shift X is not sofic: suppose otherwise. Then there is a finite labelled graph $(E_X, \mathcal{L}_X)$ with $|E_X^0| = n$ which presents X. Let α be a path in $(E_X, \mathcal{L}_X)$ which presents $ab^{2n}c^{2n}a$. Then since the number of c's in $\mathcal{L}_X(\alpha)$ is greater than n, α must contain a cycle τ such that $\mathcal{L}_X(\tau) = c^m$ for some $m \le n$. Write $= \alpha'\tau\alpha''$. Then $\beta = \alpha'\tau^2\alpha''$ is a path in E_X^* which presents the forbidden word $ab^{2n}c^{2n+m}a$.
The shift X has the following labelled graph presentation $(E_X, \mathcal{L}_X)$:



Since the graph E_X is transitive, it is straightforward to check from the above presentation that X is irreducible.

Since each vertex in E_X to the right (resp. left) of v_0 receives a unique labelled path of the form ab^n (resp. ac^n) it follows that $\{v\} \in \mathcal{G}^{0,-}$ for all $v \in E_X^0$. Since E_X is row-finite it follows that $(E_X, \mathcal{L}_X, \mathcal{G}_X^{0,-})$ is cofinal by Lemma (1.1.7).
For $n \ge 1$ every $v \in E_X^0$ emits the labelled path $b^n c$, which is disagreeable for $[v]_\ell$. Hence $[v]_\ell$ is disagreeable for all $\ell \ge 1$ and so $C^*(E_X, \mathcal{L}_X, \mathcal{G}_X^{0,-})$ is simple by Theorem (1.1.20).

Every $v \in E_X^0$ emits the repeatable path bc and since E_X is transitive, it follows that every generalised vertex connects to a repeatable path. Thus $C^*(E_X, \mathcal{L}_X, \mathcal{G}_X^{0,-})$ is purely infinite by Theorem (1.1.23).

Section (1.2): Group Actions

A labelled graph $(E, \mathcal{L})$ is a directed graph $E = (E^0, E^1, r, s)$ together with a function $\mathcal{L} : E^1 \to \mathcal{A}$ where $\mathcal{A}$ is called the alphabet. Labelled graphs are a model for studying symbolic dynamical systems; the labelled path space is a shift space whose properties may be inferred from the labelled graph presentation (cf. [34]). Labelled graph algebras were introduced in [24], [25], their theory has been developed in [23], [29], [30] and has found applications in mirror quantum spheres in [38].

We introduce the notion of a group action on a labelled graph and study the crossed products formed by the induced action on the associated C^*-algebra. Before we do this, we update the definition of the C^*-algebra associated to a labelled graph. In order to circumvent a technical error, we add a new condition to ensure that the resulting C^*-algebra satisfies a version of the gauge-invariant uniqueness theorem. Since a directed graph is a labelled graph where $\mathcal{L}$ is injective, we will be generalizing a suite of results for directed graphs and their C^*-algebras (see [27], [33], [31]). This is not as straightforward as it may seem since two distinct edges may carry the same label, so new techniques will be needed to prove our results.

An action of a group G on a labelled graph $(E, \mathcal{L})$ is an action of G on E together with a compatible action of G on $\mathcal{A}$ so that we may sensibly define the quotient object $(E/G, \mathcal{L}/G)$ as a labelled graph. In [28], Gross and Tucker introduce the notion of a skew product graph

$E \times_c G$ formed from a map $c: E^1 \to G$ and show that G acts freely on $E \times_c G$ with quotient E. The Gross-Tucker theorem [28, Theorem 2.1.2] takes a free action of G on E and recovers (up to equivariant isomorphism) the original graph and action from the quotient graph E/G. We describe a skew product construction for labelled graphs and prove a version of the Gross- Tucker theorem for free actions on labelled graphs (Theorem (1.2.25)). Since a group action on a labelled graph is a pair of compatible actions, a new approach is needed: In Definition (1.2.11), we define a skew product labelled graph $(E \times_c G, \mathcal{L}_d)$ to be a skew-product graph $E \times_c G$ together with a labelling $\mathcal{L}_d : (E \times_c G)^1 \to \mathcal{A} \times G$ which is defined using a new function $d: E^1 \to G$. The purpose of the new function d is to accommodate the possibility that two edges carry the same label. We discuss the importance of d. We then turn our attention to applications of the results on labelled graph actions to the C^*-algebras, $C^*(E, \mathcal{L})$ we have associated to labelled graphs. A function $c: E^1 \to G$ on a directed graph gives rise to a coaction δ of G on $C^*(E)$ such that $C^*(E) \times \delta G \cong C^*(E \times_c G)$ (cf. [31]). In Proposition (1.2.27), we show that a skew product labelled graph $(E \times_c G, \mathcal{L}_d)$ gives rise to a coaction δ of G on $C^*(E, \mathcal{L})$ provided that $c: E^1 \to G$ is consistent with the labelling map $\mathcal{L} : E^1 \to \mathcal{A}$. Then in Theorem (1.3.32) we show that $C^*(E, \mathcal{L}) \times \delta G \cong C^*(E \times_c G, \mathcal{L}_1)$ where 1: $E^1 \to G$ is given by $1(e) = 1_G$ for all $e \in E^1$. Since this isomorphism is equivariant for the dual action of G on $C^*(E, \mathcal{L}) \times \delta G$ and the action of G on $C^*(E \times_c G, \mathcal{L}_1)$ induced by left translation of G on $(E \times_c G, \mathcal{L}_1)$, Takai duality then gives us

$$C^*(E \times_c G, \mathcal{L}_1) \times_{\tau,r} G \cong C^*(E, \mathcal{L}) \otimes \mathcal{K}(\ell^2(G))$$

in Corollary (1.2.32). Indeed if d is consistent with the labelling map $\mathcal{L} : E^1 \to \mathcal{A}$, then $C^*(E \times_c G, \mathcal{L}_d)$ is equivariantly isomorphic to $C^*(E \times_c G, \mathcal{L}_1)$ (see Proposition (1.2.29)).
For a directed graph E, a function $c: E^1 \to \mathbb{Z}$ given by $c(e) = 1$ for all $e \in E^1$ gives rise to a skew product graph $E \times_c G$ whose C^*-algebra which is strongly Morita equivalent to the fixed point algebra $C^*(E)^\gamma$ for the gauge action. In the case of labelled graphs, if , $d: E^1 \to \mathbb{Z}$ are given by $c(e) = 1$, $d(e) = 0$ for all $e \in E^1$, then $C^*(E \times_c G, \mathcal{L}_d)$ is strongly Morita equivalent to $C^*(E, \mathcal{L})^\gamma$ (see Theorem (1.2.34)).
An action α of G on a directed graph E induces an action of G on $C^*(E)$, moreover if the action is free, then using the Gross-Tucker theorem we have

$$C^*(E) \times_{\alpha,r} G \cong C^* \left(\frac{E}{G}\right) \otimes \mathcal{K}\big(\ell^2(G)\big) \tag{11}$$

by [33, Corollary 3.10]. We show that an action of G on $(E, \mathcal{L})$ induces an action of G on $C^*(E, \mathcal{L})$. If we wish to use the Gross-Tucker theorem for labelled graphs to prove the labelled graph analog (1), we need to know when the maps , $d: (E/G)^1 \to G$ provided by Theorem (1.2.24) are consistent with the quotient labelling $\mathcal{L}/G$. The answer to this question is provided by Theorem (1.2.37: It happens precisely when the action α has a fundamental domain. Hence, if the free action of G on $(E, \mathcal{L})$ has a fundamental domain, then in Corollary (1.2.38). we show that

$$C^*(E, \mathcal{L}) \times_{\alpha,r} G \cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G)) .$$

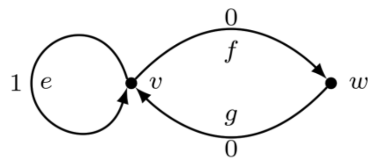
We begin with a collection of definitions, which are taken from [24]. A directed graph $E = (E^0, E^1, r, s)$ consists of a vertex set E^0, an edge set E^1, and range and source maps r, s: $E^1 \to E^0$. That E is row-finite and essential, that is

$$r^{-1}(v) \neq \emptyset \text{ and } 1 \leq \# s^{-1}(v) < \infty$$

for all $v \in E^0$. We let E^n denote the set of paths of length n and set $E^+ = \bigcup_{n \geq 1} E^n$.

Definition (1.2.1)[444]: A labelled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E together with a labelling map $\mathcal{L} : E^1 \to \mathcal{A}$.
We may assume that $\mathcal{L} : E^1 \to \mathcal{A}$ is surjective. Let $\mathcal{A}^*$ be the collection of all *words* in the symbols of $\mathcal{A}$. For $n \geq 1$, the map $\mathcal{L}$ extends naturally to a map $\mathcal{L} : E^n \to \mathcal{A}^*$: for $\lambda = \lambda_1 \cdots \lambda_n \in E^n$ we set $\mathcal{L}(\lambda) = \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n)$ and we say that λ is a representative of the labelled path $\mathcal{L}(\lambda)$. Let $\mathcal{L}(E^n)$ denote the collection of all labelled paths in $(E, \mathcal{L})$ of length n. Then $\mathcal{L}^+(E) = \bigcup_{n\geq1} \mathcal{L}\,(E^n)$ denotes the collection of all labelled paths in $(E, \mathcal{L})$, that is all words in the alphabet $\mathcal{A}$ which may be represented by paths in E.
Examples (1.2.2)[444]:
(a) Every directed graph E gives rise to a labelled graph $(E, \mathcal{L}_\tau)$ over the alphabet E^1 where $\mathcal{L}_\tau : E^1 \to E^1$ is the identity map.
(b) The directed graph E whose edges e, f, g have been labelled using the alphabet {0, 1} as shown below is an example of a labelled graph

$(E, \mathcal{L}) :=$



Let $(E, \mathcal{L})$ be a labelled graph. Then for $\beta \in \mathcal{L}^+(E)$ we set

$$r(\beta) = \{r(\lambda): \mathcal{L}(\lambda) = \beta\}, s(\beta) = \{s(\lambda): \mathcal{L}(\lambda) = \beta\}.$$

For $A \subseteq E^0$ and $\beta \in \mathcal{L}^+(E)$, the *relative range of* β *with respect to* A is

$$r(A, \beta) = \{r(\lambda): \lambda \in E^+, \mathcal{L}(\lambda) = \beta, s(\lambda) \in A\}.$$

The labelled graph $(E, \mathcal{L})$ is *left-resolving*, if for all $v \in E^0$ the map $\mathcal{L}$ restricted to $r^{-1}(v)$ is injective. The labelled graph $(E, \mathcal{L})$ is *weakly left-resolving* if for all A, $B \subseteq E^0$ and $\beta \in \mathcal{L}^+(E)$ we have

$$r(A \cap B, \beta) = r(A, \beta) \cap r(B, \beta).$$

If $(E, \mathcal{L})$ is left-resolving, then it is weakly left-resolving. Examples (1.2.2)(a) and (b) are examples of left-resolving labelled graphs.

A collection $B \subseteq 2^{E^0}$ of subsets of E^0 is closed under relative ranges for $(E, \mathcal{L})$ if for all $A \in B$ and $\beta \in \mathcal{L}^+(E)$ we have $r(A, \beta) \in B$. If B is closed under relative ranges for $(E, \mathcal{L})$, contains $r(\beta)$ for all $\beta \in \mathcal{L}^+(E)$ and unions, then B is *accommodating* for $(E, \mathcal{L})$ and the triple $(E, \mathcal{L}, B)$ is called a *labelled space*. Let $\mathcal{E}^{0,-}$ be the smallest accommodating collection of subsets of E^0 for $(E, \mathcal{L})$.
Definition (1.2.3)[444]: For $A \subseteq E^0$ and $n \geq 1$, let $\mathcal{L}_A^n := \{\beta \in \mathcal{L}(E^n) : A \cap s(\beta) \neq \emptyset\}$ denote those labelled paths of length n whose source intersects A nontrivially.
Though E is row finite it is possible for $\mathcal{L}_A^1$ to be infinite; for example if $\mathcal{L}$ is trivial, then $\mathcal{L}_{E^0}^1 = E^1$, which is infinite if E^1 is infinite. A labelled space $(E, \mathcal{L}, B)$ is *set-finite* if $\mathcal{L}_A^1$ is finite for all $A \in B$. The following definition is given in [24].
Definition (1.2.4)[444]: A representation of a weakly left-resolving, set-finite labelled space $(E, \mathcal{L}, B)$ consists of projections $\{p_A: A \in B\}$ and partial isometries $\{s_a: a \in \mathcal{A}\}$ such that
(i) If $A, B \in B$, then $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, where $p_\emptyset = 0$.
(ii) If $a \in \mathcal{A}$ and $A \in B$, then $p_A s_a = s_a p_{r(A,a)}$.
(iii) If a, $b \in \mathcal{A}$, then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$.
(iv) For $A \in B$, we have

$$p_A = \sum_{a \in \mathcal{L}_A^1} s_a \, p_{r(A,a)} s_a^*.$$

$C^*(E, \mathcal{L}, B)$ is the universal C^*-algebra generated by a representation of $(E, \mathcal{L}, B)$. Let $\gamma : \mathbb{T} \to Aut C^*(E, \mathcal{L}, B)$ be the gauge action determined by

$$\gamma_z p_A = p_A, \gamma_z s_a = z s_a \text{ for } A \in B, a \in \mathcal{A}.$$

The problem in [24, Lemma 5.2(ii)] arises because, under the hypotheses on a labelled space used in [24], it is possible to have $A \supsetneq B \in B$ with $p_A = p_B$ in $C^*(E, \mathcal{L}, B)$. To rectify this problem, we must assume that B is closed under relative complements; that is if A, $B \in B$ are such that $A \supsetneq B$, then $A \backslash B \in B$. If B is closed under relative complements, then we also recover the formula in [25, Remark 3.5].

Before stating the Gauge Invariant Uniqueness theorem, we give a corrected version of [24, Lemma 5.2] using the new hypothesis.

Lemma (1.2.4)[444]: Let $(E, \mathcal{L}, B)$ be a weakly left-resolving, set-finite labelled space where B is closed under relative complements and $\{s_a, p_A\}$ be a representation $(E, \mathcal{L}, B)$. *Let* $Y = \{s_{\alpha_i} p_{A_i} s_{\beta_i}^* : i = 1, \ldots, N\}$ be a set of partial isometries in $C^*(E, \mathcal{L}, B)$ which is closed under multiplication and taking adjoints. If q is a minimal projection in $C^*(Y)$, then either

(i) $q = s_{\alpha_i} p_{A_i} s_{\alpha_i}^*$ *for some* $1 \le i \le N$

(ii) $q = s_{\alpha_i} p_{A_i} s_{\alpha_i}^* - q'$ *where* $q' = \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s_{\alpha_{k(l)}}^*$ *and* $1 \le i \le N$; *more- over there is a nonzero* $r = s_{\alpha_i \beta} p_{r(A_i, \beta)} s_{\alpha_i \beta}^* \in C^*(E, \mathcal{L}, B)$ *such that* $q' r = 0$ *and* $q \ge r$.

Proof. By projection in $C^*(Y)$ may be written as

$$\sum_{j=1}^{n} s_{\alpha_{i(j)}} p_{A_{i(j)}} s_{\alpha_{i(j)}}^* - \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s_{\alpha_{k(l)}}^*,$$

where the projections in each sum are mutually orthogonal and for each l there is a unique j such that $s_{\alpha_{i(j)}} p_{A_{i(j)}} s_{\alpha_{i(j)}}^* \ge s_{\alpha_{k(l)}} p_{A_{k(l)}} s_{\alpha_{k(l)}}^*$.

If $q = \sum_{j=1}^{n} s_{\alpha_{i(j)}} p_{A_{i(j)}} s_{\alpha_{i(j)}}^* - \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s_{\alpha_{k(l)}}^*$ is a minimal projection in $C^*(Y)$, then we must have $n = 1$. If $m = 0$, then $q = s_{\alpha_i} p_{A_i} s_{\alpha_i}^*$ for some $1 \le i \le N$. If $m \ne 0$, then

$$q = s_{\alpha_i} p_{A_i} s_{\alpha_i}^* - \sum_{\ell=1}^{m} s_{\alpha_{k(\ell)}} p_{A_{k(\ell)}} s_{\alpha_{k(\ell)}}^*,$$

where A_i, $A_{k(\ell)} \in B$ for $1 \le \ell \le m$. If we apply Definition (1.2.4) (iv), we may write

$$q = \sum_{j=1}^{n} s_{\alpha_i \beta_j} p_{r(A_i, \beta_j) \beta_j} s_{\alpha_i}^* - \sum_{h=1}^{t} \sum_{\ell=1}^{m} s_{\alpha_{k(\ell)} \kappa_h} p_{r(A_{k(\ell)}, \kappa_h)} s_{\alpha_{k(\ell)} \kappa_h}^*,$$

where all $\alpha_i \beta_j$ and $\alpha_{k(\ell)} \kappa_h$ have the same length. Since q is a nonzero projection there is $1 \le j \le n$ and $H_j \subseteq \{1, \ldots, t\} \times \{1, \ldots, m\}$ such that $\alpha_i \beta_j = \alpha_{k(\ell)} \kappa_h$ for all $(h, \ell) \in H_j$ and

$$Y_j := \bigcup_{(h,\ell) \in H_j} r\big(A_{k(\ell)}, \kappa_h\big) \subsetneq r(A_i, \beta_j).$$

Since B is closed under finite unions we have $Y_j \in B$. Then for this j define $X_j = r(A_i, \beta_j) \backslash Y_j \ne \emptyset$, then $X_j \in B$ since B is closed under relative complements. Hence, the projection $r = s_{\alpha_i\beta_j} p_{X_j} s^*_{\alpha_i\beta_j}$ is nonzero and $q \geq r$ since $X_j \subset r(A_i, \beta_j)$. If we set $q' = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q$, then since $X_j \cap Y_j = \emptyset$ we have $q'r = 0$ as required.

Theorem (1.2.5)[444]: Let $(E, \mathcal{L}, B)$ be a weakly left-resolving, set-finite labelled space where B is closed under relative complements and $\{S_a, P_A\}$ be a representation $(E, \mathcal{L}, B)$ on Hilbert space. Take $\pi_{S,P}$ to be the representation of $C^*(E, \mathcal{L}, B)$ satisfying $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_A) = P_A$. Suppose that $P_A \neq 0$ *for all* $\emptyset \neq A \in B$ and that there is a strongly continuous action γ' *of* $\mathbb{T}$ *on* $C^*(\{S_a, P_A\})$ such that for all $z \in \mathbb{T}$, $\gamma'_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$. Then $\pi_{S,P}$ is faithful.

Proof. The proof is the same as given in [24, Theorem 5.3], using Lemma (1.3.5) instead of [24, Lemma 5.2].

Definition (1.2.6)[444]: Let $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labelled graph, then we define $\mathcal{E}(r, \mathcal{L})$ to be the smallest accommodating collection of subsets of E^0 which is closed under relative complements.

This remark motivates the following definition.

Definition (1.2.7)[444]: Let $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labelled graph. A Cuntz-Krieger $(E, \mathcal{L})$-family consists of commuting projections $\{p_{r(\beta)} : \beta \in \mathcal{L}^+(E)\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ with the properties that:

(CKla) For all $\beta, \omega \in \mathcal{L}^+(E)$, $p_{r(\beta)} p_{r(\omega)} = 0$ if and only if $r(\beta) \cap r(\omega) = \emptyset$.

(CKlb) For all $\beta, \omega, \kappa \in \mathcal{L}^+(E)$, if $(\beta) \cap r(\omega) = r(\kappa)$, then $p_{r(\beta)} p_{r(\omega)} = p_{r(\kappa)}$, if $r(\beta) \cup r(\omega) = r(\kappa)$, then $p_{r(\beta)} + p_{r(\omega)} - p_{r(\beta)} p_{r(\omega)} = p_{r(\kappa)}$ and if $r(\beta) \to\supset r(\omega)$, then $p_{r(\beta)} - p_{r(\omega)} \neq 0$.

(CK2) If $a \in \mathcal{A}$ and $\beta \in \mathcal{L}^+(E)$, then $p_{r(\beta)} s_a = s_a p_{r(\beta a)}$.

(CK3) If $a, b \in \mathcal{A}$, then $s^*_a s_a = p_{r(a)}$ and $s^*_a s_b = 0$ unless $a = b$.

(CK4) For $\beta \in \mathcal{L}^+(E)$, if $\mathcal{L}^1_{r(\beta)}$ is finite and nonempty, then we have

$$p_{r(\beta)} = \sum_{a \in \mathcal{L}^1_{r(\beta)}} s_a p_{r(\beta a)} s^*_a. \tag{12}$$

Let $C^*(E, \mathcal{L})$ be the universal C^*-algebra generated by a Cuntz-Krieger $(E, \mathcal{L})$-family.

Let $\gamma' : \mathbb{T} \to Aut C^*(E, \mathcal{L})$ be the gauge action determined by

$$\gamma'_z p_{r(\beta)} = p_{r(\beta)}, \gamma'_z s_a = z s_a \text{ for } \beta \in \mathcal{L}^+(E) , a \in \mathcal{A}.$$

Theorem (1.2.8)[444]: *Let* $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labelled graph. Then $C^*(E, \mathcal{L})$ is isomorphic to $C^*(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L})$; moreover

$$C^*(E, \mathcal{L}) = \overline{span}\{s_\alpha p_A s^*_\beta : \alpha, \beta \in \mathcal{L}^+(E) , A \in \mathcal{E}(r, \mathcal{L})$$

Proof. Let $\{s_a, p_{r(\beta)}\}$ be a universal Cuntz-Krieger $(E, \mathcal{L})$-family and $\{t_a, q_A\}$ be a universal representation of the labelled space $(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$ For $a \in \mathcal{A}$, set $T_a = s_a$.

By (CKla), we may define $Q_\emptyset = 0$. For $\alpha, \beta \in \mathcal{L}^+(E)$, we may define $Q_{r(\alpha) \cap r(\beta)} = Q_{r(\alpha)} Q_{r(\beta)}$ and $Q_{r(\alpha \cup r(\beta)} = Q_{r(\alpha)} + Q_{r(\beta)} - Q_{r(\alpha) \cap r(\beta)}$ in $C^*(E, \mathcal{L})$. If $(\alpha) \supsetneq r(\beta)$, then we may define $Q_{r(\alpha) \setminus r(\beta)} = Q_{r(\alpha)} - Q_{r(\beta)} \neq 0$ in $C^*(E, \mathcal{L})$. By using the inclusion/exclusion law we may define Q_A in $C^*(E, \mathcal{L})$ for all $A \in \mathcal{E}(r, \mathcal{L})$.

It is a routine calculation to show that $\{T_a, Q_A\}$ is a representation of the labelled space $(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$ in $C^*(E, \mathcal{L})$. By the universal property of $C^*(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L}))$ there exists a homomorphism $\Phi: C^*(E, \mathcal{L}, \mathcal{E}(r, \mathcal{L})) \to C^*(E, \mathcal{L})$ such that $\Phi(t_a) = T_a$ and $\Phi(q_A) = Q_A$. It is straightforward to see that $\gamma'_z \circ \Phi = \Phi \circ \gamma_z$ for $z \in \mathbb{T}$. The first statement then follows by Theorem (1.2.6), and the final statement follows by applying Φ to an arbitrary element of $C^*(E,\mathcal{L},\mathcal{E}(r,\mathcal{L}))$ (see [24, Lemma 4.4]).

We begin by defining what a labelled graph morphism is and use the definition to define a labelled graph automorphism. Then in Theorem (1.2.10). we show that a labelled graph automorphism of $(E,\mathcal{L})$ induces an automorphism of $C^*(E,\mathcal{L})$.

Definition (1.2.9)[444]: Let $(E,\mathcal{L})$ and (F,M) be labelled graphs over alphabets $\mathcal{A}_E$ and $\mathcal{A}_F$ respectively. A labelled graph morphism is a triple $\phi:=(\phi^0,\phi^1,\phi^{A_E}) : (E,\mathcal{L})\to(F,M)$ such that

(i) for all $e\in E^1$, we have $\phi^0(r(e))=r(\phi^1(e))$ and $\phi^0(s(e))=s(\phi^1(e))$;

(ii) $\phi^{A_E} : \mathcal{A}_E\to\mathcal{A}_F$ is a map such that $M\circ\phi^1=\phi^{A_E}\circ\mathcal{L}$.

If the maps ϕ^0, ϕ^1, ϕ^{A_E} are bijective, then the triple $\phi:=(\phi^0,\phi^1,\phi^{A_E})$ is called a labelled graph isomorphism. In the case that $F=E$, $\mathcal{A}_E=\mathcal{A}_F$ and $\mathcal{L}=M$, we call (ϕ^0,ϕ^1,ϕ^A) a labelled graph automorphism.

For a labelled graph morphism $=(\phi^0,\phi^1,\phi^{A_E})$, we shall omit the super- scripts on ϕ when the context in which it is being used is clear.

The set $Aut(E,\mathcal{L}) :=\{\phi\colon \phi$ is a labelled graph automorphism of $(E,\mathcal{L})\}$ forms a group under composition. The following result follows easily from the universal definition of $C^*(E,\mathcal{L})$.

Theorem (1.2.10)[444]: *Let* ϕ be an automorphism of a weakly left-resolving, set- finite labelled graph $(E,\mathcal{L})$ *and* $\{s_a,p_{r(\beta)}\}$ be a universal Cuntz-Krieger $(E,\mathcal{L})$-family. The maps $s_a\mapsto s_{\phi(a)}$ *and* $p_{r(\beta)}\mapsto p_{\phi(r(\beta))}$ induce an automorphism of $C^*(E,\mathcal{L})$.

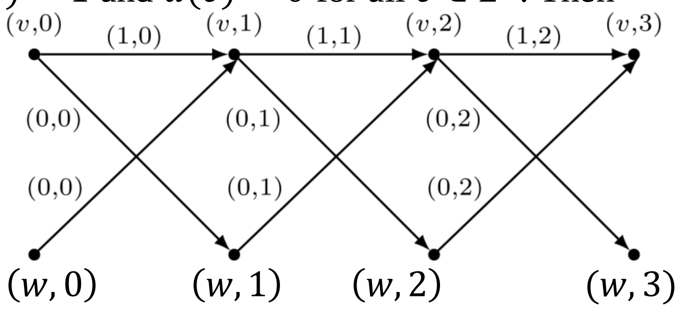
We shall define a skew product labelled graph and define what it means for a group to act on a labelled graph.

Definition (1.2.11)[444]: Let $(E,\mathcal{L})$ be a labelled graph and let c, $d\colon E^1\to G$ be functions. The *skew product labelled graph* $(E\times_c G,\mathcal{L}_d)$ over alphabet $\mathcal{A}\times G$ consists of the skew product graph $(E^0\times G,E^1\times G,r_c,s_c)$ where

$$r_c(e,g)=(r(e),gc(e))\,,s_c(e,g)=(s(e),g)$$

together with the labelling $\mathcal{L}_d : (E\times_c G)^1\to\mathcal{A}\times G$ given by $\mathcal{L}_d(e,g):=(\mathcal{L}(e),gd(e))$ .Since the labels received by $(v,g)\in(E\times_c G)^0$ are in one-to-one correspondence with the labels received by $v\in E^0$ it follows that if $(E,\mathcal{L})$ is left- resolving, then so is $(E\times_c G,\mathcal{L}_d)$

.**Examples (1.2.12)[444]:** For the labelled graph $(E,\mathcal{L})$ of Examples (1.2.2) (b) let c, d: $E^1\to\mathbb{Z}$ be given by $c(e)=1$ and $d(e)=0$ for all $e\in E^1$. Then



For $\mu\in E^*$ the map $(\mu,g)\mapsto\mu_g$ identifies $E^*\times G$ with $(E\times_c G)^*$ Then for $(\mu,g)\in E^*\times G$ we have

$$s(\mu,g)=(s(\mu),g) \text{ and } r(\mu,g)=(r(\mu),gc(\mu)) \qquad (13).$$

Let $(E,\mathcal{L})$ be a labelled graph over the alphabet $\mathcal{A}$. A *labelled graph action of G on* $(E,\mathcal{L})$ is a triple $((E,\mathcal{L}),G,\phi)$ where $\phi\colon G\to Aut(E,\mathcal{L})$ is a group homomorphism. In particular, for all $e\in E^1$ and $g\in G$ we have

$$\mathcal{L}(\phi_g(e)) = \phi_g(\mathcal{L}(e)) \quad (14) .$$

If we ignore the label maps, a labelled graph action $((E,\mathcal{L}),G,\phi)$ restricts to a graph action of G on E; we denote this restricted action by (E,G,ϕ) . The labelled graph action $((E,\mathcal{L}),G,\alpha)$ is *free* if $\phi_g(v) = v$ for some $v \in E^0$, then $g = 1_G$ and if $\phi_g(a) = a$ some $a \in \mathcal{A}$, then $g = 1_G$.

The following lemma shows that skew product labelled graphs provide a rich source of examples of free labelled graph actions.

Lemma (1.2.13)[444]: *Let* $(E,\mathcal{L})$ be a labelled graph, $c,d: E^1 \to G$ be functions and $(E \times_c G, \mathcal{L}_d)$ be the associated skew product labelled graph. Then

(i) *For* $(x,h) \in (E \times_c G)^i$, $(a,h) \in \mathcal{A} \times G$, $g \in G$ *and* $i = 0,1$ *let* $\tau_g^i(x,h) = (x,gh)$ *and* $\tau_g^A(a,h) = (a,gh)$. *Then* $\tau_g = (\tau_g^0, \tau_g^1, \tau_g^A)$ is a labelled graph automorphism.

(ii) *The map* $\tau = (\tau^0,\tau^1,\tau^A) : G \to Aut(E \times_c G, \mathcal{L}_d)$ *defined by* $g \mapsto \tau_g$ is a homomorphism.

(iii) The triple $((E \times_c G, \mathcal{L}_d), G, \tau)$ is a free labelled graph action.

Definition (1.2.14)[444]: The map $\tau = (\tau^0,\tau^1,\tau^A) : G \to Aut(E \times_c G, \mathcal{L}_d)$ as given in Lemma (1.2.13) (ii) is called the left labelled graph translation map, and the action $((E \times_c G, \mathcal{L}_d), G, \tau)$ *the left labelled graph translation action*.

Two labelled graph actions $((E,\mathcal{L}),G,\phi)$ and $((F,\mathcal{M}),G,\psi)$ are isomorphic if there is a labelled graph isomorphism : $(E,\mathcal{L}) \to (F,\mathcal{M})$ which is *equivariant* in the sense that $\varphi \circ \phi_g = \psi_g \circ \varphi$ for all $g \in G$.

Theorem (1.2.15)[444]: *Let* $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labelled graph, and $((E,\mathcal{L}),G,\alpha)$ be a labelled graph action. Let $\{s_a, p_{r(\beta)}\}$ be a universal Cuntz-Krieger $(E,\mathcal{L})$-family. Then for h ∈ G the maps

$$\alpha_h s_a = s_{\alpha_h a} \text{ and } \alpha_h p_{r(\beta)} = p_{\alpha_h r(\beta)}$$

determine an action of G *on* $C^*(E,\mathcal{L})$. *If* $((E,\mathcal{L}),G,\phi)$ *and* $((F,M),G,\psi)$ *are isomorphic then* $C^*(E,\mathcal{L}) \times_\phi G \cong C^*(F,M) \times_\psi G$.

Proof. Follows by a straight forward application of Theorem (1.2.3) and the universal property of crossed products.

We prove a version of the Gross-Tucker theorem for labelled graphs. For directed graphs, the Gross-Tucker theorem says, that up to equivariant isomorphism, every free action α of a group G on a directed graph E is a left translation automorphism τ on a skew product graph $(E/G) \times_c G$ built from the quotient graph E/G.

We to prove a similar result for labelled graphs. The new ingredient is the map $d: E^1 \to G$ found in the definition of a skew product labelled graph for labelled graphs. Before giving the main result, Theorem (1.2.24).

Definition (1.2.16)[444]: Let $((E,\mathcal{L}),G,\alpha)$ be a labelled graph action. For $i = 0$, 1 and $x \in E^i$ let $Gx := \{\alpha_g^i(x) : g \in G\}$ and $(E/G)^i = \{Gx: x \in E^i\}$. For $a \in \mathcal{A}$ let

$$Ga = \{\alpha_g^A(a): g \in G\} \text{ and } \mathcal{A}/G = \{Ga: a \in \mathcal{A}\}.$$

The proof of the following lemma is straightforward.

Lemma (1.2.17)[444]: *Let* $((E,\mathcal{L}),G,\alpha)$ be a labelled graph action. The maps , $s: (E/G)^1 \to (E/G)^0$ *given by*

$$r(Ge) = Gr(e) \text{ and } s(Ge) = Gs(e) \text{ for } Ge \in (E/G)^1 \quad (15)$$

and the map $\mathcal{L}/G$: $(E/G)^1 \to \mathcal{A}/G$ *given by* $(\mathcal{L}/G)(Ge) = G\mathcal{L}(e)$ are well defined. Consequently, $(E/G, \mathcal{L}/G)$ is a labelled graph over the alphabet $\mathcal{A}/G$. *The map* $q = (q^0, q^1, q^A) : (E, \mathcal{L}) \to (E/G, \mathcal{L}/G)$ *given by* $q^i(x) = Gx$ *for* $i = 0, 1, x \in E^i$ *and* $q^A(a) = Ga$ *for* $a \in \mathcal{A}$ is a surjective labelled graph morphism.

Definition (1.2.18)[444]: Let $((E, \mathcal{L}), G, \alpha)$ be a labelled graph action. The quotient labelled graph $(E/G, \mathcal{L}/G)$ is the labelled graph described in Lemma (1.2.18), the map q: $(E, \mathcal{L}) \to (E/G, \mathcal{L}/G)$ is the quotient labelled map.

The following Proposition is an analog of [28, Theorem 2.2.1] whose proof is routine.

Proposition (1.2.19)[444]: Let $(E, \mathcal{L})$ be a labelled graph, c, d: $E^1 \to G$ be functions and $(E \times_c G, \mathcal{L}_d)$ be the associated skew product labelled graph. Let $((E \times_c G, \mathcal{L}_d), G, \tau)$ be the left labelled graph translation action. Then

$$((E \times_c G)/G, \mathcal{L}_d/G) \cong (E, \mathcal{L}) .$$

Examples (1.2.20)[444]: Recall the labelled graphs $(E, \mathcal{L})$ and $(E \times_c \mathbb{Z}, \mathcal{L}_d)$ from Example (1.2.12). For the left labelled graph translation action $((E \times_c \mathbb{Z}, \mathcal{L}_d), \mathbb{Z}, \tau)$, we have $((E \times_c \mathbb{Z})/\mathbb{Z}, \mathcal{L}_d/\mathbb{Z}) \cong (E, \mathcal{L})$ by Proposition (1.2.20).

The Gross-Tucker theorem is a converse to Proposition (1.2.20). It states that if we have a free action of a group on a labelled graph, then we can recover the original graph from the quotient via a skew product. Recall the following definition for directed graphs.

Definition (1.2.21)[444]: Let F, E be directed graphs. A surjective graph morphism p: $F \to E$ has the unique path lifting property if given $u \in F^0$ and $e \in E^1$ with $s(e) = p^0(u)$ there is a unique edge $f \in F^1$ with $s(f) = u$ and $p^1(f) = e$.

Definition (1.2.22)[444]: Let $((E, \mathcal{L}), G, \alpha)$ be a labelled graph action and $q = (q^0, q^1, q^A)$: $(E, \mathcal{L}) \to (E/G, \mathcal{L}/G)$ be the quotient labelled map. A *section* for q^i is a map $\eta^i : (E/G)^i \to E^i$ for $i = 0,1$ such that $q^i \circ \eta^i = id_{(E/G)^i}$. A *section* for q^A is η^A: $\mathcal{A}/G \to \mathcal{A}$ such that $q^A \circ \eta^A = id_{A/G}$.

Lemma (1.2.23)[444]: *Let* (E, G, α) be a graph action and $q = (q^0, q^1) : E \to E/G$ be the quotient map. Given a section η^0 *for* q^0 there is a unique section η¹ for q¹ such that

$$s(\eta^1(Ge)) = \eta^0(s(Ge)) \text{ for all } e \in E^1 \quad (16)$$

Proof. By the quotient map q: $E \to E/G$ has the unique path lifting property. Hence if we fix $Gv \in (E/G)^0$, then for each $Ge \in (E/G)^1$ with $s(Ge) = Gv$ there is a unique $f \in E^1$ with $q^1(f) = Ge = Gf$ and $s(f) = \eta^0(Gv)$. Put $\eta^1(Ge) = f$, then $\eta^1 : (E/G)^1 \to E^1$ is well defined and the source map on $(E/G)^1$ is well defined. Since $q^1(\eta^1(Ge)) = q^1(f) = Ge$ it follows that η^1 is a section satisfying (16). Uniqueness of η^1 follows from the unique path lifting property of q.

The following is a version of the Gross-Tucker theorem (cf. [28, Theorem 2.2.2]) for labelled graphs.

Theorem (1.2.24)[444]: *Let* $((E, \mathcal{L}), G, \alpha)$ be a free labelled graph action. Let η^0, η^A *be* sections for q^0, q^A respectively. There are functions , d: $(E/G)^1 \to G$ such that $((E, \mathcal{L}), G, \alpha)$ is isomorphic to $((E/G \times_c G, (\mathcal{L}/G)_d), G, \tau)$.

Proof. Fix a section η^0: $(E/G)^0 \to E^0$ for q^0. By Lemma (1.2.24), there is a section η^1 for qlsatisfying (16). For $Ge \in (E/G)^1$ set $f = \eta^1(Ge)$, then

$$q^0\left(r(\eta^1(Ge))\right) = q^0(r(f)) = Gr(f) = r(Gf) = r(Ge) = q^0\left(\eta^0(r(Ge))\right).$$

As (E, G, α) is free, there is a unique $h \in G$ such that $\alpha_h^0\eta^0(r(Ge)) = r(\eta^1(Ge))$ and we may set $c(Ge) = h$. Define ϕ: $E/G \times_c G \to E$ by

$$\phi_c^0(Gv,g) = \alpha_g^0\eta^0(Gv) \quad and \quad \phi_c^1(Ge,g) = \alpha_g^1\eta^1(Ge)$$

for $(Gv,\ g) \in (E/G \times_c G)^0$ and $(Ge,\ g) \in (E/G \times_c G)^1$. One checks that ϕ_c : $(E/G \times_c G) \to E$ is an isomorphism of directed graphs.

We claim that ϕ_c is equivariant. Notice that for all $(Gv,\, h) \in (E/G \times_c G)^0$ and $g \in G$ we have

$$\phi_c^0\left(\tau_g^0(Gv,h)\right) = \phi_c^0(Gv,gh) = \alpha_{gh}^0\eta^0(Gv) = \alpha_g^0\alpha_h^0\eta^0(Gv) = \alpha_g^0\phi_c^0(Gv,h)$$

and so $\phi_c^0 0 \tau_g^0 = \alpha_g^0 0 \phi_c^0$ for all $g \in G$. The argument for ϕ_c^1 is similar and the claim follows.

We now construct an equivariant bijection $\phi_d^{A/G\times G}$: $\mathcal{A}/G \times G \to \mathcal{A}$ which satisfies condition (b) of Definition (1.2.10). Fix a section η^A: $\mathcal{A}/G \to \mathcal{A}$ for q^A.

We now define a map $d\colon (E/G)^1 \to G$. Fix $Ge \in (E/G)^1$ and set $f = \eta^1(Ge)$ so that $q^1(f) = Ge$. Since

$$q^A\eta^A\ (\mathcal{L}/G(Ge)) = q^A\eta^A(G\mathcal{L}(f)) = q^A\mathcal{L}\eta^1(Ge)$$

and the graph action $((E,\mathcal{L}),G,\alpha)$ is free, there is a unique $k \in G$ such that $\alpha_k^A\eta^A$ $((\mathcal{L}/G)(Ge)) = \mathcal{L}(\eta^1(Ge))$ and we may define $d(Ge) = k$. The function $d\colon (E/G)^1 \to G$ described in this way is such that $d(Ge)$ is the unique element of G with the property that

$$\alpha_{d(Ge)}^A\eta^A\big((\mathcal{L}/G)(Ge)\big) = \mathcal{L}\big(\eta^1(Ge)\big). \qquad (17)$$

For each $(Ga,g) \in \mathcal{A}/G \times G$ we define $\phi_d^{A/G\times G}$: $\mathcal{A}/G \times G \to \mathcal{A}$ by $\phi_d^{A/G\times G}(Ga,g) = \alpha_g^A\eta^A(Ga)$. We claim that $\phi_d^{A/G\times G}$ satisfies $\phi_d^{A/G\times G\circ}(\mathcal{L}/G)_d = \mathcal{L} \circ \phi_c^1$: By (17) for all $(Ge,$ $h) \in (E/G \times_c G)^1$ we have

$$\begin{aligned}\phi_d^{A/G\times G\circ}(\mathcal{L}/G)_d(Ge,\,h) &= \alpha_h^A\alpha_{d(Ge)}^A\eta_A(\mathcal{L}/G(Ge))\\ &= \mathcal{L}\ (\alpha_h^1\eta^1(Ge)) = \mathcal{L} \circ \phi_c^1(Ge,h)\end{aligned}$$

as required.

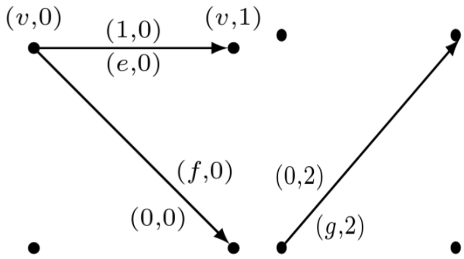
It is straightforward to see that $\phi_d^{A/G\times G}$ is bijective. To see that $\phi_d^{A/G\times G}$ is equivariant notice that we have

$$\begin{aligned}\phi_d^{A/G\times G}\ (\tau_g^{A/G\times G}(Ge,\,h)) = \phi_d^{A/G\times G}(Ge,\,gh) &= \alpha_g^A\alpha_h^A\eta^A\ (Ge)\\ &= \alpha_g^A\phi_d^{A/G\times G}\ (Ge,h)\end{aligned}$$

for all $(Ge,h) \in (E/G \times G)^1$ and $g \in G$. Thus $\phi_{c,d} = (\phi_c^0, \phi_c^1, \phi_d^{A/G\times G})$ is the required labelled graph isomorphism.

Example (1.2.25)[444]: Recall from Example (1.2.21) the labelled graph $(E \times_c \mathbb{Z}, \mathcal{L}_d)$ has a free action of $\mathbb{Z}$ such that the quotient labelled graph is $(E,\mathcal{L})$. We use this example to illustrate the point made:

Suppose we choose a section η^0: $E^0 \to (E \times_c \mathbb{Z})^0$ such that $\eta^0(v) = (v,0)$ and $\eta^0(w) = (w,2)$, then the section η^1 : $E^1 \to (E \times_c \mathbb{Z})^1$ as defined in Lemma (1.3.24) is given by $\eta^1(e) = (e,0)$, $\eta^1(f) = (f,0)$, and $\eta^1(g) = (g,2)$ whose image in $(E \times_c \mathbb{Z}, \mathcal{L}_d)$ is as shown below.



$$(w,0)\ \ (w,1)(v,2)\ (v,3)(w,2)\ (w,3)$$
Note that $(e)=1$, $c(f)=-1$, and $c(g)=3$.
Observe that $f,g\in E^1$ are such that $\mathcal{L}(f)=\mathcal{L}(g)=0$ however,
$$\mathcal{L}(\eta^1(f))=\mathcal{L}(f,0)=(0,0)\neq(0,2)=\mathcal{L}(g,2)=\mathcal{L}(\eta^1(g))\ .$$
The function d accounts for this difference. By Equation (17), we have $(g)=2$, since $\alpha_2^A(0,0)=(0,2)$, whereas $d(f)=0$. Observe that $d(g)\neq d(f)$ even though
$$\mathcal{L}(g)=\mathcal{L}(f)\ .$$
In [31] it is shown that a function $c\colon E^1\to G$ induces a coaction δ of G on the graph algebra $C^*(E)$ such that $C^*(E)\times\delta G\cong C^*(E\times_c G)$. One should expect, therefore, that the functions , $d\colon E^1\to G$ would induce a coaction δ of G on $C^*(E,\mathcal{L})$ such that $C^*(E,\mathcal{L})\times\delta G\cong C^*(E\times_c G,\mathcal{L}_d)$. However in order to obtain such a result we must assume that both functions c, d are label consistent (see Definition (1.2.27) below). For further information about coactions of discrete groups see [37], amongst others.

Definition (1.2.26)[444]: Let $(E,\mathcal{L})$ be a labelled graph over alphabet $\mathcal{A}$. A function $c\colon E^1\to G$ is label consistent if there is a function $C\colon\mathcal{A}\to G$ such that $c=C\circ\mathcal{L}$.

For any labelled graph $(E,\mathcal{L})$ the function $1\colon E^1\to G$ given by $1\ (e)=1_G$ for all $e\in E^1$ is label consistent. First, we show that if c is label consistent then there is a coaction of G on $(E,\mathcal{L})$.

Proposition (1.2.27)[444]: *Let* $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labelled graph, G be a discrete group, and $c\colon E^1\to G$ be a label consistent function. Then there is a maximal normal coaction: $C^*(E,\mathcal{L})\to C^*(E,\mathcal{L})\otimes C^*(G)$ such that
$$\delta(s_a)=s_a\otimes u_{C(a)}\ \textit{and}\ \delta\big(p_{r(\beta)}\big)=p_{r(\beta)}\otimes u_{1_G},\qquad(18)$$
where $\{s_a,p_{r(\beta)}\}$ *is a universal Cuntz-Krieger* $(E,\mathcal{L})$*-family and* $\{u_g\colon g\in G\}$ *are the canonical generators of* $C^*(G)$.

Proof. The first part of the result follows by the same argument given in [31, Lemma 3.2]. That the coaction δ is normal and maximal follows by essentially the same arguments as the ones given in [27, Lemma 3.3] and [35, Theorem 7.1 (v)].

The next result shows that if d is label consistent then we may as well assume that $d=1$.

Proposition (1.2.28)[444]: *Let* $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labelled graph and $c\colon E^1\to G$ *a function. If* d_1, $d_2\colon E^1\to G$ are label consistent functions, then $((E\ \times_c G,\mathcal{L}_{d_1}),G,\tau)\cong((E\ \times_c G,\mathcal{L}_{d_2}),G,\tau)$ where τ is the left translation action. Hence, if $d\colon E^1\to G$ is a label consistent function then there is an isomorphism from $C^*(E\times_c G,\mathcal{L}_d)$ *to* $C^*(E\times_c G,\mathcal{L}_1)$ which is equivariant for the G-action induced by τ.

Proof. For the first statement, let $\phi^i:(E\times_c G)^i\to(E\times_c G)^i$ be the identity map for $i=0$, 1 and define $\phi^{A\times G}:\mathcal{A}\times G\to\mathcal{A}\times G$ by
$$\phi^{A\times G}(a,g)=(a,gD_1^{-1}(a)D_2(a))\ .$$
For $(e,g)\in(E\times_c G)^1$, after a short calculation we have
$$\emptyset^{A\times c_{\mathcal{L}_{d_1}(e,g)=}}(\mathcal{L}(e),d_2(e))=\mathcal{L}_{d_2}(e,g)\ .$$
It is then straightforward to check that $\phi=(\phi^0,\phi^1,\phi^{A\times G})$ is a labelled graph isomorphism. Since for all $h\in G$ we have
$$\tau_h(\phi^{A\times G}(a,g))=(a,hgD_1^{-1}(a)D_2(a))=\phi^{A\times G}(\tau_h(a,g))$$
it follows that $((E\times_c G,\mathcal{L}_{d_1}),G,\tau)\cong((E\times_c G,\mathcal{L}_{d_2}),G,\tau)$.
The final statement follows from Theorem (1.2.16).

Next, we shall show that if $d = 1$ then there is a natural identification $\mathcal{L}_1^+(E \times_c G)$, the labelled path space of $(E \times_c G, \mathcal{L}_1)$ with $\mathcal{L}^+(E) \times G$.

Lemma (1.2.29)[444]: *Let* $(E, \mathcal{L})$ be a labelled graph and $c\colon E^1 \to G$ label consistent. For $\mu \in E^+$ and $g \in G$ the map

$$\mathcal{L}_1(\mu, g) \mapsto (\mathcal{L}(\mu), g)$$

establishes a bijection from $\mathcal{L}_1^+(E \times_c G)$ *to* $\mathcal{L}^+(E) \times G$.

Proof. It follows that for $n \geq 1$ every path in $(E \times_c G)^n$ has the form $(\mu, g) = (\mu_1, g)(\mu_2, gc(\mu_1)) \cdots (\mu_n, gc(\mu'))$, for some $\mu \in E^n$ and $g \in G$. Then by definition we have

$$\mathcal{L}_1(\mu, g) = (\mathcal{L}(\mu_1), g)\big(\mathcal{L}(\mu_2), gc(\mu_1)\big) \cdots \big(\mathcal{L}(\mu_n), gc(\mu')\big) \quad (19)$$

If we define the right-hand side of (19) to be $(\mathcal{L}(\mu), g)$ the result follows.

The following lemma indicates the behavior of the range map under the identification of $\mathcal{L}_1^+(E \times_c G)$ with $\mathcal{L}^+(E) \times G$.

Lemma (1.2.30)[444]: Let $(E, \mathcal{L})$ be a labelled graph and $c\colon E^1 \to G$ be a label consistent function. Let $a \in \mathcal{A}$, $\beta \in \mathcal{L}^+(E)$, *and* $g \in G$. Then under the identification of $\mathcal{L}^+(E) \times G$ with $\mathcal{L}_1^+(E \times_c G)$ *we have* $r(\beta, g) = (r(\beta), gC(\beta)) \in \mathcal{E}(r, \mathcal{L}) \times G$.

Proof. Observe that for $(\beta, g) \in \mathcal{L}^+(E) \times G$, we have

$$\begin{aligned} r(\beta, g) &= \{r(\mu, g)\colon (\mu, g) \in E^* \times G, \mathcal{L}(\mu) = \beta\} \\ &= \big\{\big(r(\mu), gC(\beta)\big)\colon \mathcal{L}(\mu) = \beta\big\} \quad (20) \end{aligned}$$

since the function $c\colon E^1 \to G$ is label consistent. Hence, we may identify $r(\beta, g)$ with $(r(\beta), gC(\beta)) \in \mathcal{E}(r, \mathcal{L}) \times G$.

With the above identifications in mind, we turn our attention to the main resul. By Theorem (1.2.16) the left labelled graph translation action $((E \times_c G, \mathcal{L}_1), G, \tau)$ defined in Definition (1.2.15) induces an action $\tau\colon G \to \operatorname{Aut} C^*(E \times_c G, \mathcal{L}_1)$. When we identify $\mathcal{L}_1^+(E \times_c G)$ with $\mathcal{L}^+(E) \times G$ this action may be described on the generators of $C^*(E \times_c G, \mathcal{L}_1)$ as follows: For $h, g \in G$, $a \in \mathcal{A}$, and $\beta \in \mathcal{L}^+(E)$ we have

$$\tau_h(s_{(a,g)}) = s_{(a,hg)} \text{ and } \tau_h(p_{(r(\beta)}, g)) = p_{(r(\beta),hg)}. \quad (21)$$

The method of proof for the next result closely follows that of [31, Theorem 2.4], however we give some of the details as they rely heavily on the identification we made in Lemma (1.2.31).

Theorem (1.2.31)[444]: *Let* $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labelled graph. Suppose that G is a discrete group, $c\colon E^1 \to G$ is a label consistent function, and δ is the coaction from Proposition (1.2.28). *Let* $j_{C^*(E,\mathcal{L})}$, j_G denote the canonical covariant homomorphisms of $C^*(E, \mathcal{L})$ and $C^*(G)$ *into* $M(C^*(E, \mathcal{L}) \times \delta G)$ and $\{s_{(a,g)}, p_{(r(\beta)}, g)\}$ be the canonical generating set of $C^*(E \times_c G, \mathcal{L}_1)$. Then the map $\phi\colon C^*(E \times_c G, \mathcal{L}_1) \to C^*(E, \mathcal{L}) \times \delta G$ given by

$$\begin{aligned} \phi(s_{(a,g)}) &=_{jC^*(E,\mathcal{L})} (s_a)_{jG}(\chi_{C(a)^{-1}}) , \\ \phi(p_{(r(\beta),g)}) &= j_{C^*(E,\mathcal{L})}(p_{r(\beta)}) j_G(\chi_{g-1}) \end{aligned}$$

is an isomorphism.

proof. For each $g \in G$, let $C^*(E, \mathcal{L})_g = \{b \in C^*(E, \mathcal{L}) : \delta(b) = b \otimes u_g\}$ denote the corresponding spectral subspace; we write b_g to denote a generic element of $C^*(E, \mathcal{L})_g$. Then $C^*(E, \mathcal{L}) \times \delta G$ is densely spanned by the set $\{(b_g, h) : b_g \in C^*(E, \mathcal{L})_g \text{ and } g, h \in G\}$, and the algebraic operations are given on this set by

$$(b_g, x)(b_h, y) = (b_g b_h, y) \text{ if } y = h^{-1}x \text{ (and 0 if not), and}$$

$$(b_g, x)^* = (b_g^*, gx) .$$

If $(j_{C^*(E,\mathcal{L})}, j_G)$ denotes the canonical covariant homomorphism of $C^*(E,\mathcal{L})$ into the multiplier algebra of $C^*(E,\mathcal{L}) \times \delta G$, then (b_g, x) is by definition $(j_{c^*}(b_g) j_G(\chi_{\{x\}}))$.

Using Lemma (1.2.31), we may show that for $(a, g) \in \mathcal{A} \times G$, $\beta \in \mathcal{L}^+(E)$ and $g \in G$

$$t_{(a,g)} = (s_a, C(a)^{-1} g^{-1}) \text{ and } q_{(r(\beta),g)} = (p_{r(\beta)}, g^{-1})$$

is a Cuntz-Krieger $(E \times_c G, \mathcal{L}_1)$-family in $C^*(E,\mathcal{L}) \times \delta G$.

By universality of C^* $(E \times_c G, \mathcal{L}_1)$ there is a homomorphism $\pi_{t,q}$ from C^* $(E \times_c G, \mathcal{L}_1)$ to $C^*(E,\mathcal{L}) \times \delta\, G$ such that $\pi_{t,q}(s_{(a,g)}) = t_{(a,g)}$ and $\pi_{t,q}(p_{(r(\beta),g)}) = q_{(r(\beta),g)}$ which we may show is injective using the argument from [31, Theorem 2.4] and Theorem (1.2.6).

We show that $\pi_{t,q}$ is surjective. That $C^*(E,\mathcal{L}) \times \delta G$ is generated by (s_a, g) and $(p_{r(\beta)}, h)$. Since $\pi_{t,q}(-1 = t_{(a,g-1} - 1 = (s_a, C(a)^{-1} C(a) g)$, and $\pi_{t,q}(p_{(r(\beta),h^{-1})}) = (p_{r(\beta)}, h)$ we see that $\pi_{t,q}$ is surjective. Hence, $\pi_{t,q}$ is the desired isomorphism.

We need to check that $\pi_{t,q}$ is equivariant for the G actions, that is $\pi_{t,q} \circ \tau_g = \widehat{\delta_g} \circ \pi_{t,q}$ for all $g \in G$. It is enough to check on generators: Notice that for all $s_{(a,h)} \in C^*(E \times_c G, \mathcal{L}_1)$

$$\pi_{t,q} \circ \tau_g(s_{(a,h)}) = \pi_{t,q}(s_{(a,gh)}) = (s_a, C(a)^{-1} h^{-1} g^{-1})$$
$$= \hat{\delta}(s_a, C(a)^{-1} h^{-1}) = \widehat{\delta_g} \circ \pi_{t,q}(s_{(a,h)})$$

and similarly $\pi_{t,q} 0 \tau_g(p_{(r(\beta),h)}) = \widehat{\delta_g} \circ \pi_{t,q}(p_{(r(\beta),h)})$ for $p_{(r(\beta),h)} \in C^*(E \times_c G, \mathcal{L}_1)$.

We claim that $\pi_{t,q}$ is equivariant for the $\mathbb{T}$ actions, that is $\pi_{t,q} \circ \gamma_z = (\gamma_z \times G) \circ \pi_{t,q}$ for all $z \in \mathbb{T}$. It is enough to check this on generators: Notice that for all $s_{(a,h)} \in C^*(E \times_c G, \mathcal{L}_1)$ and $z \in \mathbb{T}$ we have

$$\pi_{t,q} \circ \gamma_z(s_{(a,h)}) = \pi_{t,q}(z s_{(a,h)}) = (z s_a, C(a)^{-1} h^{-1}) = (\gamma_z \times G)(s_a, C(a)^{-1} h^{-1})$$
$$= (\gamma_z \times \delta G) \circ \pi_{t,q}(s_{(a,h)}) .$$

Similarly, $\pi_{t,q} \circ \gamma_z\big(p_{(r(\beta),h)}\big) = (\gamma_z \times G) \circ \pi_{t,q}\big(p_{(r(\beta),h)}\big)$ for all $p_{(r(\beta),h)} \in C^*(E \times_c G, \mathcal{L}_1)$.

Corollary (1.2.32)[444]: Let $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labelled graph. Suppose that G is a discrete group, c: $E^1 \to G$ be a label consistent function, and τ the induced action of G *on* $C^*(E \times_c G, \mathcal{L}_1)$. *Then*

$$C^*(E \times_c G, \mathcal{L}_1) \times_{\tau,r} G \cong C^*(E,\mathcal{L}) \otimes \mathcal{K}(\ell^2(G)) .$$

Proof Since the isomorphism of C^* $(E \times_c G, \mathcal{L}_1)$ with $C^*(E,\mathcal{L}) \times \delta G$ is equivariant for the G-actions , $\hat{\delta}$, respectively, it follows that

$$C^*(E \times_c G, \mathcal{L}_1) \times_{\tau,r} G \cong C^*(E,\mathcal{L}) \times \delta G \times_{\hat{\delta},r} G.$$

Following the argument in [31, Corollary 2.5], Katayama's duality theorem [32] gives us that $C^*(E,\mathcal{L}) \times \delta G \times_{\hat{\delta},r} G$ is isomorphic to $C^*(E,\mathcal{L}) \otimes \mathcal{K}(\ell^2(G))$, as required.

In order to provide a version of Corollary (1.2.33) for group actions, we must first characterise when the functions c, d in the Gross-Tucker Theorem (1.2.25) are label consistent maps.

Recall from [37, p. 209] that a coaction δ of a discrete group G on a C^*- algebra A is saturated if for each $s \in G$ we have $\overline{A_s A_s^*} = A^\delta$ where A_s is the spectral subspace $A_s = \{b \in A :\ \delta(b) = b \otimes u_s\}$ and A^δ is the fixed point algebra for δ

$$A^\delta := \{b \in A :\ \delta(a) = a \otimes u_{1_G}\}.$$

Lemma (1.2.33)[444]: *Let* $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labelled graph and c: $E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. Then the coaction δ of $\mathbb{Z}$ *on* $C^*(E,\mathcal{L})$ induced by c is saturated.

Proof. The coaction δ of $\mathbb{Z}$ on $C^*(E,\mathcal{L})$ defined in Proposition (1.2.28) is such that the fixed point algebra $C^*(E,\mathcal{L})^\delta$ is precisely the fixed point algebra $C^*(E,\mathcal{L})^\gamma$ for the canonical gauge action of $\mathbb{T}$ on $C^*(E,\mathcal{L})$ by the Fourier transform (cf. [26, Corollary 4.9]). By an argument similar to that in [36, Section 2], we have

$$C^*(E,\mathcal{L})^\gamma = \overline{span}\{s_\alpha p_A s_\beta^*: \alpha, \beta \in \mathcal{L}^n(E)\ , A \in \mathcal{E}(r, \mathcal{L})$$

Since E has no sinks it follows by a similar argument to that in [36, Lemma 4. 1.1] that $C^*(E,\mathcal{L})$ is saturated.

Theorem (1.2.34)[444]: *Let* $(E,\mathcal{L})$ be a weakly left-resolving, set-finite labelled graph. Then $C^*(E,\mathcal{L})^\gamma$ is strongly Morita equivalent to $C^*(E \times_c \mathbb{Z}, \mathcal{L}_1)$ *where* c: $E^1 \to \mathbb{Z}$ *is* given by $c(e) = 1$ *for all* $e \in E^1$

Proof. Since c is label consistent it follows by Theorem (1.2.32) that

$$C^*(E \times_c \mathbb{Z}, \mathcal{L}_1) \cong C^*(E,\mathcal{L}) \times \delta\mathbb{Z}.$$

By Lemma (1.2.34), the coaction is δ is saturated and since $C^*(E,\mathcal{L})^\delta \cong C^*(E,\mathcal{L})^\gamma$ the result follows.

We examine conditions on the free labelled graph action $((E,\mathcal{L}), G, \alpha)$ which ensure that the functions c, d from Theorem (1.2.25) are label consistent.

Recall that a fundamental domain for a graph action (E, G, α) is a subset T of E^0 such that for every $v \in E^0$ there exists $g \in G$ and a unique $w \in T$ such that $v = \alpha_g^0 w$. Every free graph action has a fundamental domain.

Definition (1.2.35)[444]: Let $((E,\mathcal{L}), G, \alpha)$ be a free labelled graph action. A fundamental domain for $((E,\mathcal{L}), G, \alpha)$ is a fundamental domain $T \subseteq E^0$ for the restricted graph action such that for every $e, f \in E^1$ we have

(a) if (e) , $r(f) \in T$ and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$ and

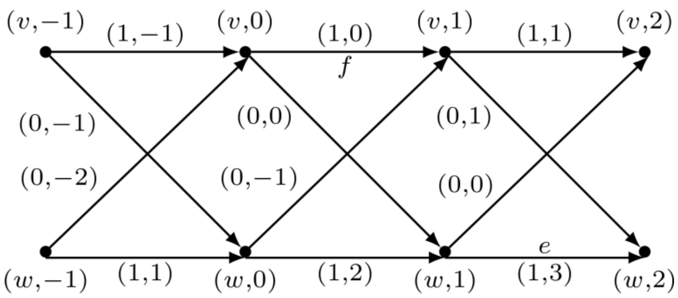
(b) if (e) , $s(f) \in T$ and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$.

We see that not every free action of a group on a labelled graph has a fundamental domain.

Examples (1.2.36)[444]:

(i) Consider the following labelled graph

$(E,\mathcal{L}) :=$



The group $\mathbb{Z}$ acts freely on $(E,\mathcal{L})$ by addition in the second coordinate of the vertices, edges and labels as indicated in the picture above; call this action α. Let $T = \{(v, 0)\ , (w, 1)\}$ then T is a fundamental domain for the restricted graph action $(E, \mathbb{Z}, \alpha)$. However when considering the labelled graph action $((E,\mathcal{L}), \mathbb{Z}, \alpha)$ the set T does not satisfy Definition (1.2.36)(b). Consider the edges e, f as shown above with $\mathcal{L}(e) = (1,3)$ and $\mathcal{L}(f) = (1,0)$ respectively. We have $s(e) = (w, 1) \in T$ and $s(f) = (v, 0) \in T$ and $\mathbb{Z}\mathcal{L}(e) = \mathbb{Z}\mathcal{L}(f) = \{(1, n) : n \in \mathbb{Z}\}$, however $\mathcal{L}(e) = (1,3) \neq (1,0) = \mathcal{L}(f)$. Indeed any fundamental domain for the restricted action $(E, \mathbb{Z}, \alpha)$ will also fail Definition (1.3.36)(b).

(ii) Let , d: $E^1 \to G$ be label consistent functions and $\left((E \times_c G, \mathcal{L}_d), G, \tau\right)$ be the associated left labelled graph translation action. Then one checks that $T = \{(v, 1_G): v \in E^0\}$ is a fundamental domain for $((E \times_c G, \mathcal{L}_d), G, \tau)$.

The following result shows that when we add the fundamental domain hypothesis to the free labelled graph action, the functions , d: $(E/G)^1 \to G$ in the labelled graph version of the Gross-Tucker theorem (Theorem (1.3.25)) may be chosen to be label consistent.

Theorem(1.2.37)[444]: *Let* $((E, \mathcal{L}), G, \alpha)$ be a free labelled graph action with a fundamental domain. Then there are label consistent functions, d: $(E/G)^1 \to G$ *such that*

$$((E, \mathcal{L}), G, \alpha) \cong ((E/G) \times_c G, (\mathcal{L}/G)_d) , G, \tau).$$

Proof. Let T be a fundamental domain for $((E, \mathcal{L}), G, \alpha)$. For every $Gv \in (E/G)^0$ there exists a unique $w \in T$ such that $Gw = Gv$. Hence, if we define $\eta^0(Gv) = w$, then η^0 : $(E/G)^0 \to T$ is a section for q^0. Then we may define η^1, c, d, and η^A as in Theorem (1.2.25). It suffices to show that c and d are label consistent. To see that d is label consistent suppose $Ge, Gf \in (E/G)^1$ are such that $(\mathcal{L}/G)\ (Ge) = (\mathcal{L}/G)\ (Gf) = Ga \in \mathcal{A}/G$. Let $= \eta^A(Ga) \in \mathcal{A}$, $d(Ge) = k \in G$, and $d(Gf) = l \in G$. Then by the definition of d we have

$$\mathcal{L}(\eta^1(Ge)) = \alpha_k^A \eta^A (\mathcal{L}/G)(Ge) = \alpha_k^A b \qquad (22)$$

$$\mathcal{L}\left(\eta^1(Gf)\right) = \alpha_l^A \eta^A (\mathcal{L}/G)(Gf) = \alpha_l^A b. \qquad (23)$$

This implies that $G\mathcal{L}(\eta^1(Ge)) = Ga = G\mathcal{L}(\eta^1(Gf))$ and so $\mathcal{L}(\eta^1(Ge)) = \mathcal{L}(\eta^1(Gf))$ since $s(\eta^1(Ge))$, $(\eta^1(Gf)) \in T$. From Equations (22) and (23) we have $\alpha_k^A b = \alpha_l^A b$ and so $k = l$ since the G action on $\mathcal{A}$ is free. Therefore, d is label consistent.

To see that c is label consistent suppose that $Ge, Gf \in (E/G)^1$ are such that $(\mathcal{L}/G)(Ge) = (\mathcal{L}/G)(Gf) = Ga \in \mathcal{A}/G$, say. Let $b = \eta^A(Ga) \in \mathcal{A}$, $c_\eta(Ge) = k \in G$, and $c(Gf) = l \in G$. Then by the definition of c we have

$$r(\eta^1(Ge)) = \alpha_k^0 \eta^0\left(r(Ge)\right), \qquad (24)$$

$$r\left(\eta^1(Gf)\right) = \alpha_l^0 \eta^0\left(r(Gf)\right). \qquad (25)$$

Then if we let $e = \alpha_{-k}^1\ (\eta^1(Ge))$ and $f = \alpha_{-l}^1(\eta^1(Gf))$ we have e, $f \in E^1$ with $(e) = \eta^0(r(Ge))$, $r(f) = \eta^0(r(Gf)) \in T$ and $G\mathcal{L}(e) = G\mathcal{L}(f)$. Since T is a fundamental domain, we have $\mathcal{L}(e) = \mathcal{L}(f)$ and hence $\alpha_{-k}^A(\mathcal{L}\ (\eta^1(Ge))) = \mathcal{L}(e) = \mathcal{L}(f) = \alpha_{-l}^A(\mathcal{L}$ $(\eta^1(Gf)))$. Since $\mathcal{L}(\eta^1(Ge)) = \mathcal{L}(\eta^1\ (Gf))$ we can conclude that $k = l$ as in the previous paragraph. Therefore, c is label consistent and our result is established.

Corollary (1.2.38)[444]: Let $(E, \mathcal{L})$ be a weakly left-resolving, set-finite labelled graph. Suppose that $((E, \mathcal{L}), G, \alpha)$ is a free labelled graph action which admits a fundamental domain. Then

$$C^*(E, \mathcal{L}) \times_{\alpha,r} G \cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G)) .$$

Proof. By Theorem (1.2.38), there are label consistent functions, d: $E^1/G \to G$ such that

$$((E, \mathcal{L}), G, \alpha\ E/G \times_c G, (\mathcal{L}/G)_d), G, \tau) ,$$

so we have

$$C^*(E, \mathcal{L}) \times_{\alpha,r} G \cong C^*(E/G \times_c G, (\mathcal{L}/G)_d)\ \times_{\tau,r} G.$$

By Proposition (1.2.29) and Corollary (1.2.33), we have

$$C^*(E/G \times_c G, (\mathcal{L}/G)_d)\ \times_{\tau,r} G \cong C^*(E/G \times_c G, (\mathcal{L}/G)_1)\ \times_{\tau,r} G$$
$$\cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G))$$

which gives the desired result.

Chapter 2
Complex Symmetric Operators

We explore applications of this symmetry to Jordan canonical models, self- adjoint extensions of symmetric operators, rank-one unitary perturbations of the compressed shift, Darlington synthesis and matrix-valued inner functions, and free bounded analytic interpolation in the disk. We consider numerous examples, including the Poincaré-Neumann singular integral (bounded) operator and the Jordan model operator (compressed shift). The decomposition $T = CJ|T|$ also extends to the class of unbounded C-selfadjoint operators, originally introduced by Glazman. It provides a method for estimating the norms of the resolvents of certain unbounded operators

Section (2.1): Application in Complex Symmetric Operators

The simultaneous diagonalization and spectral analysis of two Hermitian forms goes back to the origins of Hilbert space theory and, in particular, to the spectral theorem for self-adjoint operators. Even today the language of forms is often used when dealing with unbounded operators (see [69, 81]). The similar theory for a Hermitian and a nondefinite sesquilinear form was motivated by the Hamiltonian mechanics of strings or continuous media models; from a mathematical point of view this theory leads to Hilbert spaces with a complex linear J-involution and the associated theory of J unitary and J-contractive operators (see [57,69,79]) . Less studied, but not less important, is the simultaneous analysis of a pair consisting of a Hermitian form and a bilinear form; this framework has appeared quite early in function theory ([47,86,89]) , functional analysis ([75]) , and elasticity theory ([55]). Some of the main results in this direction were estimates derived from variational principles for eigenvalues of symmetric matrices (such as Grunsky's or Friedrichs' inequalities).

We motivated by the observation that all scalar (Jordan) models in operator theory are complex symmetric with respect to a well-chosen orthonormal basis; cf. [46,77]. Put into a pair of a Hermitian and a bilinear form, this remark reveals an extra symmetry of these model operators, shared rather surprisingly by quite a few other basic classes of operators such as normal, Hankel, compressed Toeplitz, and some Volterra operators. It is no accident that exactly this symmetry appears in one of Siegel's matrix realizations of Cartan domains [87].

We consider a complex Hilbert space $\mathcal{H}$ and an antilinear, isometric involution C on it. A bounded linear operator T is called C-symmetric if $CT = T^*C$. This is equivalent to the symmetry of T with respect to the bilinear form $[f, g] = \langle f, Cg\rangle$. It is easy to show that there exists an orthonormal basis $(e_i)_{i\in I}$ of $\mathcal{H}$ which is left invariant by C: $Ce_i = e_i$. With respect to the basis $(e_i)_{i\in I}$, C-symmetry is simply complex symmetry of the associated matrix. Already at this general level the symmetry $CT = T^*C$ has strong effects on the spectral picture of T; for instance, the generalized eigenspaces $Ker(T-\lambda)^p$ and $Ker(T^* - \overline{\lambda})^p$ are antilinearly isometrically isomorphic via C. Thus a Fredholm C-symmetric operator has zero index.

The examples of C-symmetric operators are numerous and quite diverse. Besides the expected normal operators, certain Volterra and Toeplitz operators are C-symmetric. For example, consider a finite Toeplitz matrix with complex entries:

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_{-1} & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_{-n} & a_{-n+1} & \cdots & a_0 \end{bmatrix}$$

The symmetry with respect to the second diagonal leaves this matrix invariant, and this is exactly the C-symmetry noticed and exploited a long time ago by Schur and Takagi [89]. At the level of functional models, the symmetry

$$f \mapsto Cf := \overline{fz}\varphi$$

maps the standard model space $\mathcal{H}_\varphi = H^2 \ominus \varphi H^2$ onto itself and makes the compression of the unilateral shift a C-symmetric operator. Above H^2 is the Hardy space of the disk and φ is a nonconstant inner function.

The applications of C-symmetry we discuss can be grouped into the following categories: extension and dilation results, rank one perturbations of Jordan operators, matrix-valued inner functions and free interpolation theory in the disk.

The first three of these subjects are interconnected by a simple matrix completion observation. Namely, every C-symmetric operator admits C-symmetric extensions and dilations. At the level of real symmetric operators and their self-adjoint extensions this remark goes back to von Neumann [75], and also explicitly appears in the computations of M.G. Krein [66]. The same phenomenon is present in Clark's unitary perturbations of Jordan operators, or in the study of real Volterra operators pursued by the Ukrainian school, [65] and also [45,62,63,74]. It was this last group of researchers who investigated for the first time C-symmetries of various linear systems appearing in mathematical physics or engineering. At the abstract level, we observe that every C-symmetric contraction has a C-symmetric Sz.-Nagy unitary dilation.

We also examine the canonical model spaces $\mathcal{H}_\varphi$ and the compressed Toeplitz operators carried by them from the viewpoint of C-symmetry. In particular, we show how to use Clark's theory [49] to produce complex symmetric matrix realizations for Jordan operators. Also C-symmetry turns out to be fundamental in understanding the structure of inner 2×2 matrix-valued functions in the disk. This subject is related to Darlington's synthesis problem in systems theory, and our approach offers a concrete parametrization of all solutions to the scalar Darlington problem.

Is the classical free interpolation problem in the unit disk. For an interpolating Blaschke product φ and the associated involution C on the model space $\mathcal{H}_\varphi$, we show the identity between a Fourier type *orthogonal* decomposition with respect to the bilinear form $[\cdot,\cdot] := \langle \cdot, C\, \cdot \rangle$:

$$f = \sum_{n=1}^{\infty} \frac{[f, e_n]}{[e_n, e_n]} e_n$$

for f in $\mathcal{H}_\varphi$ and the standard division and interpolation results. The novelty in the above representation formula is the orthogonality of its terms with respect to the new bilinear form.

The last contains a couple of simple examples of quotients of Hilbert spaces of analytic functions defined on domains of $\mathbb{C}^n$. They illustrate the possible complications arising from the generalization of the complex symmetry of Jordan operators to several complex variables.

Let $\mathcal{H}$ denote a separable Hilbert space and C an isometric antilinear involution of $\mathcal{H}$. By isometric we mean that $\langle f,g\rangle = \langle Cg, Cf\rangle$ for all f, g belonging to $\mathcal{H}$.
A typical example of a symmetry C as above is the complex conjugation of functions belonging to a Sobolev space of a domain in $\mathbb{R}^n$. Another example is the term by term complex conjugation

$$C(z_0, z_1, z_2, \dots) = (\overline{z_0}, \overline{z_1}, \overline{z_2}, \dots)$$

of a vector in $l^2(\mathbb{N})$. As proved below, this example is typical.
Lemma (2.1.1)[445]: If C is an isometric antilinear involution on the Hilbert space $\mathcal{H}$, then there exists an orthonormal basis e_n such that $e_n = Ce_n$ for all n. Each h in $\mathcal{H}$ can be written uniquely in the form $h_1 + ih_2$ where $h_1 = Ch_1$ and $h_2 = Ch_2$. *Moreover*, $\|h\|^2 = \|h_1\|^2 + \|h_2\|^2$
Proof. Let e_n be an orthonormal basis for the *real* Hilbert subspace $(I + C)\mathcal{H}$ of $\mathcal{H}$. Hence every vector in $(I + C)\mathcal{H}$ is of the form $\sum_{n=0}^{\infty} a_n\, e_n$ where a_n is a square-summable sequence of real numbers. Noting the decomposition

$$h = \frac{1}{2}(I + C)h + i\frac{1}{2i}(I - C)h = \frac{1}{2}(I + C)h + i\frac{1}{2}(I + C)(-ih) \qquad (1)$$

we see that every h in $\mathcal{H}$ lies in the complex linear span of $(I + C)\mathcal{H}$ and hence e_n is an orthonormal basis for $\mathcal{H}$. The remainder of the proposition follows immediately from (1) and a straightforward computation using the isometric property of C.
As a consequence of the preceding proposition, we will sometimes refer to C as a *conjugation operator*. Although the existence of a *self-conjugate basis* or C*-real basis* is guaranteed by Lemma (2.1.1), it is sometimes difficult to explicitly describe one.
Example (2.1.2)[445]: Consider the typical non-trivial invariant subspace for the backward shift operator on the classical Hardy space H^2. It is well known (see [48] for example) that the proper, nontrivial invariant subspaces for the backward shift operator are precisely the subspaces

$$\mathcal{H}_\varphi := H^2 \ominus \varphi H^2 \qquad (2)$$

where φ is a nonconstant inner function. Since

$$\{\overline{fz}\varphi, \overline{zh}\} = \{\varphi h, f\} = 0$$

and

$$\{\overline{fz}\varphi, \varphi h\} = \{\overline{zh}, f\} = 0$$

for each f in $\mathcal{H}_\varphi$ and h in H^2, we see that

$$Cf := \overline{fz}\varphi \qquad (3)$$

defines a conjugation operator on $\mathcal{H}_\varphi$. In particular, we see that $\overline{fz}\varphi$, despite its appearance, is the boundary function for an H^2 function.

Even at this basic level, C-symmetry is a powerful concept. The decomposition (1) yields an explicit function-theoretic characterization of $\mathcal{H}_\varphi$ [59] and hence of functions which are pseudocontinuable of bounded type (see [52, 84]). By Lemma (2.1.1), it suffices to classify self-conjugate functions. Suppose that ζ is a point on $\partial\mathbb{D}$ such that φ has a nontangential limiting value at ζ of unit modulus and c is a unimodular constant satisfying $c^2 = \overline{\zeta}\varphi(\zeta)$. By (3), a self-conjugate function f satisfies $f = \overline{fz}\varphi$ a.e on $\partial\mathbb{D}$ and hence $f(z) = cr(z)K_\zeta(z)$ where

$$K_\zeta(z) = \frac{1 - \overline{\varphi(\zeta)}\varphi(z)}{1 - \overline{\zeta}z}$$

and $r(z)$ is a function in the Smirnov class N^+ whose boundary values are real $a.e.$ on $\partial\mathbb{D}$. Such functions are described explicitly in [58, 61,72].

We remark that some of this can be generalized to the de Branges-Rovnyak setting, although we do not pursue that course in detail here. If b is an extreme point of the unit ball of H^∞ (that is, if $\log(1 - |b(e^{it})|)$ is not integrable [53, Thm. 7.9]) and μ_b is the measure on $\partial\mathbb{D}$ whose Poisson integral is the real part of $(1+b)/(1-b)$, then one can define a conjugation operator on the associated de Branges-Rovnyak space $H(b)$ that naturally corresponds to complex conjugation in $L^2(\mu_b)$ [73, Sect. 9].

Example (2.1.3)[445]: Consider a bounded, positive continuous weight ρ on the interval $[-1, 1]$, symmetric with respect to the midpoint of the interval: $\rho(t) = \rho(-t)$ for t in $[0,1]$. Let P_n be the associated orthogonal polynomials, normalized by the conditions

$$\int_{-1}^{1} P_n(t)^2\rho(t)dt = 1, \lim_{x\to\infty} P_n(x)/x^n = 1.$$

Due to their uniqueness, these polynomials have real coefficients and satisfy

$$P_n(-t) = (-1)^n P_n(t)$$

for all t. Thus,

$$e_n(t) = i^n P_n(t)$$

for $n \geq 0$ is a C-real basis for $L^2([-1,1], \rho dt)$ with respect to the symmetry $Cf(t) := \overline{f(-t)}$.

Let us assume now that $\mathcal{H}$ is a reproducing kernel Hilbert space (of scalar-valued functions) on a space X. If $\mathcal{H}$ is endowed with an isometric conjugation operator C, then

$$Cf(w) = \{Cf, K_w\} = \{CK_w, f\}$$

for f in $\mathcal{H}$. Therefore the *conjugate kernel* $Q_w := CK_w$ reproduces the values of $Cf(w)$ via the formula $(w) = \{Q_w, f\}$. This is to be expected, since $f \mapsto Cf(w)$ is a bounded antilinear functional on $\mathcal{H}$.

While the reproducing kernel is antisymmetric ($K_w(z) = \overline{K_z(w)}$ for all z, w in) , the conjugate kernel Q_w is symmetric in z and w:

$$Q_w(z) = \{CK_w, K_z\} = \{CK_z, K_w\} = Q_z(w).$$

Indeed, if e_n is a C-real basis, then

$$Q_w(z) = \sum_{n=1}^{\infty} e_n(z)e_n(w)$$

for all z, w in X. This follows from the well-known formula

$$K_w(z) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}$$

which holds for any orthonormal basis e_n.

Example (2.1.4)[445]: Let us return to the subspace $\mathcal{H}_\varphi$ and the conjugation operator C of Example (2.1.2) The reproducing kernel of $\mathcal{H}_\varphi$ is

$$K_w(z) = \frac{1 - \overline{\varphi(w)}\varphi(z)}{1 - \overline{w}z} \tag{4}$$

where z, w belong to the unit disk $\mathbb{D}$. The corresponding conjugate kernel is

$$Q_w(z) = \frac{\varphi(z) - \varphi(w)}{z - w}. \tag{5}$$

We will refer to these two functions often in the following pages.

Each conjugation operator C is equivalent, via the Riesz representation theorem, to a symmetric bilinear form

$$[f,g]:=\langle f,Cg\rangle \tag{6}$$

defined for f, g in $\mathcal{H}$. This form is nondegenerate and isometric, in the sense that

$$\sup|[f,g]| = \|f\|$$

for all f in $\mathcal{H}$. Conversely, if a nondegenerate, bilinear, symmetric, and isometric form $[\cdot,\cdot]$ is given, then there exists an isometric antilinear operator C on $\mathcal{H}$ satisfying (6). Since $\|Cf\| = \|f\|$ we infer that

$$\langle f,f\rangle = \langle Cf,Cf\rangle = [Cf,f] = [f,Cf] = \langle f,C^2f\rangle,$$

hence $C^2 = I$ and C is a conjugation operator.

The main object of study is a linear (usually bounded) operator T acting on a separable, complex Hilbert space $\mathcal{H}$ and satisfying

$$CT = T^*C,$$

Where C is a conjugation operator on $\mathcal{H}$. We say then that T is *C-symmetric* and refer to $(\mathcal{H},T,C)$ as a *C-symmetric triple*.

For a fixed C, we consider the set

$$C^o := \{T \in B(\mathcal{H}) : CT = T^*C\} \tag{7}$$

of all C-symmetric operators. Clearly, C^o is a $*$-closed linear manifold in $B(\mathcal{H})$ containing the identity. It is a small exercise to check that C^o is closed in the norm, weak operator, and strong operator topologies and that the adjoint is continuous on C^o with respect to all three topologies.

The next proposition contains a few remarks based on the definition of C- symmetry.

Proposition (2.1.5)[445]: *Let $(\mathcal{H},T,C)$ be a C-symmetric triple. Then*:

(i) *T is left invertible if and only if T is right invertible. If T^{-1} exists, then T^{-1} is also C-symmetric.*

(ii) *$KerT$ is trivial if and only if* Ran T *is dense in $\mathcal{H}$.*

(iii) *If T is Fredholm, then* ind $T = 0$.

(iv) *$p(T)$ is C-symmetric for any polynomial $p(z)$.*

(v) *For each λ and $n \geq 0$, the map C establishes an antilinear isometric isomorphism between $Ker(T-\lambda I)^n$ and $Ker(T^*-\overline{\lambda}I)^n$.*

The preceding proposition has several immediate spectral consequences. The last statement implies that the point spectra of T and T^* correspond under complex conjugation. Since C is isometric, the same correspondence holds for the approximate point spectra of T and T^*, as well as other spectral structures.

We first examine a few examples of C-symmetric matrices. We will later examine more sophisticated examples of C-symmetric operators and then the present finiterank examples will be instructive.

Example (2.1.6)[445]: One of the simplest, and perhaps most important, families of C-symmetric operators are the finite Jordan blocks. Let λ be a complex number and consider the Jordan block $J_n(\lambda)$ of order n corresponding to λ. In other words,

$$J_n(\lambda) := \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix}$$

If C_n denotes the isometric antilinear operator

$$C_n(z_1, z_2, \dots, z_n) := (\overline{z_n}, \dots, \overline{z_2}, \overline{z_1}) \tag{8}$$

on $\mathbb{C}^n$, then one readily computes that $(\mathbb{C}^n, J_n(\lambda), C_n)$ is a C_n-symmetric triple for any λ. In particular, the operators $J_n(\lambda)$ for $\lambda \in \mathbb{C}$ are simultaneously C_n- symmetric. Since a direct sum of finite rank Jordan blocks is clearly C-symmetric, any operator on a finite dimensional space is similar to a C-symmetric operator (see [56,68]).

The proper notion of equivalence for C-symmetric operators, or more appropriately C-symmetric triples, is unitary equivalence. Given a C-symmetric triple $(\mathcal{H}_1, T_1, C_1)$ and a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$, we obtain a new C-symmetric triple $(\mathcal{H}_2, T_2, C_2)$ where $T_2 = UT_1U^*$ and $C_2 = UC_1U^*$ Indeed, since $C_1T_1 = T_1^*C_1$, we see that

$$C_2T_2 = (UC_1U^*)(UT_1U^*) = (UT_1^*U^*)(UC_1U^*) = T_2^*C_2.$$

We say that two triples $(\mathcal{H}_1, T_1, C_1)$ and $(\mathcal{H}_2, T_2, C_2)$ are *equivalent* if there exists a unitary operator $U: \mathcal{H}_1 \to \mathcal{H}_2$ such that $T_2 = UT_1U^*$ and $C_2 = UT_1U^*$ This is clearly an equivalence relation.

Example (2.1.7)[445]: For any complex number a the matrix

$$T = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$$

defines a C-symmetric operator on $\mathbb{C}^2$. By performing a unitary change of coordinates, we may assume that a is real. Since C must map the one-dimensional eigenspaces of T corresponding to the eigenvalues 0 and 1 onto the corresponding eigenspaces of T^*, one can readily verify that $(\mathbb{C}^2, T, C)$ is a C-symmetric triple where

$$C\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{1+a^2}} & \dfrac{a}{\sqrt{1+a^2}} \\ \dfrac{a}{\sqrt{1+a^2}} & -\dfrac{1}{\sqrt{1+a^2}} \end{pmatrix}\begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}.$$

Example (2.1.8)[445]: All 2×2 complex matrices define C-symmetric operators on $\mathbb{C}^2$, with a proper choice of the C-symmetry. By unitary equivalence, it suffices to consider upper triangular 2×2 matrices. Since $T - \lambda I$ is C-symmetric if and only if T is, we need only appeal to Example **(2.1.7)** to draw the desired conclusion.

Example (2.1.9)[445]: The preceding example indicates that we must look to 3×3 matrices to find the simplest operators that are not C-symmetric. The matrix

$$T = \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}$$

is C-symmetric if and only if $|a| = |b|$. If $|a| = |b|$, then T is unitarily equivalent to

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 1 \end{pmatrix}$$

which is C-symmetric with respect to $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$.

Now suppose that $|a| \neq |b|$ and observe that T has eigenvalues 0, 1, 1 but does not have two linearly independent eigenvectors corresponding to the eigenvalue 1. To see that T is not C-symmetric, note that

$$|\langle e_0, e_1 \rangle| \neq |\langle f_1, f_0 \rangle|$$

whenever e_0, e_1 and f_0, f_1 are unit eigenvectors (corresponding to the eigenvalues 0 and 1, respectively) for T and τ^*, respectively. Take, for instance,

$$e_0 = \begin{pmatrix} -\frac{a}{s} \\ \frac{1}{s} \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 \\ \frac{1}{t} \\ -\frac{\bar{b}}{t} \end{pmatrix}, f_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $s = \sqrt{1 + |a|^2}$ and $t = \sqrt{1 + |b|^2}$.

The preceding example shows that not all finite rank operators are C-symmetric. A geometric explanation lies in the fact that the angles between the eigenspaces of a C-symmetric operator T must coincide (via C) with the complex conjugates of the corresponding angles between the eigenspaces of T^* This does not occur for general finite rank T.

We can characterize C-symmetric operators in terms of certain matrix representations. Let $(\mathcal{H}, T, C)$ be a C-symmetric triple and let e_n be the orthonormal basis for $\mathcal{H}$ provided by Lemma (2.1.1) with respect to the basis e_n, the matrix associated to T is complex symmetric: $\{ Te_n, e_m\} = \{Te_m, e_n\}$ for all n, m. Indeed, this follows from a straightforward computation based on the equation $CT = T^*C$ and the isometric property of C:

$$\langle Te_n, e_m\rangle = \langle Ce_m, CTe_n\rangle = \langle e_m, T^*Ce_n\rangle = \langle Te_m, Ce_n\rangle = \langle Te_m, e_n\rangle.$$

Thus we have proved the following proposition.

Proposition (2.1.10)[445]: *Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. The following conditions are equivalent*:

(i) T *is* C-symmetric for an isometric antilinear involution C.

(ii) There exists an isometric, symmetric bilinear form $[f, g]$ *on* $\mathcal{H}$ with respect to which T is symmetric.

(iii) There exists an orthonormal basis of $\mathcal{H}$ with respect to which T has a symmetric matrix representation.

Before proceeding to our next example, we briefly remark that the set C^o defined by (7) is not closed under multiplication except in the trivial case where $\mathcal{H}$ and C are simultaneously unitarily equivalent to $\mathbb{C}$ and complex conjugation, respectively. Indeed, it is easy to find complex symmetric 2×2 matrices whose product is not complex symmetric.

Example (2.1.11)[445]: Hankel operators are C-symmetric operators since every Hankel matrix is complex symmetric. For instance, the *Carleman operator*

$$(\Gamma f)(x) = \int_0^\infty \frac{f(y)}{x + y} dy$$

On $L^2(0, \infty)$ is C-symmetric since it can be represented as a Hankel matrix with respect to a certain orthonormal basis [75, p. 55].

Example (2.1.12)[445]: As a simple example, consider the Jordan block $J := J_3(\lambda)$ of order 3 acting on $\mathbb{C}^3$. That is,

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

The vectors $e_1 = \frac{1}{\sqrt{2}}(1,0,1), e_2 = \frac{1}{\sqrt{2}}(i, 0, -i)$, and $e_3 = (0,1,0)$ are orthonormal and self-conjugate with respect to the symmetry

$$C(z_1, z_2, z_3) := (\overline{z_3}, \overline{z_2}, \overline{z_1}).$$

The matrix for J with respect to the basis $\{e_1, e_2, e_3\}$ is the matrix

$$\begin{pmatrix} \lambda & 0 & \frac{1}{\sqrt{2}} \\ 0 & \lambda & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & \lambda \end{pmatrix}$$

which is complex symmetric, as expected. Similar results hold of course for Jordan blocks of higher order.

Example (2.1.13)[445]: A Toeplitz matrix of *finite* order n defines a C-symmetric operator on $\mathbb{C}^n$. Indeed we have $C_nT = T^*C_n$ where C_n denotes the involution (8) on $\mathbb{C}^n$. Toeplitz operators on H^2 are in general not C-symmetric, although their compressions to coinvariant subspaces for the unilateral shift are .

One of the oldest and most important results about complex symmetric matrices is the following theorem (originating in the work of Takagi [89] and reproved in different contexts at least by Schur, Hua, Siegel and Jacobson; see the comments in [68]) . The infinite dimensional proof below is a simple adaptation of Siegel's proof, [87, Lemma 1].

Theorem (2.1.14)[445]: (Takagi Factorization). *Let* $T = T^t$ be a symmetric matrix representation of a C-symmetric operator. There exists a unitary matrix U and a normal and symmetric matrix N (with respect to the same basis), such that

$$T = UNU^t$$

Proof. Note that $T\overline{T} = TT^*$ is a self-adjoint matrix, therefore there exists a unitary U and a real symmetric matrix S such that $T\overline{T} = USU^*$ Then note that $\overline{T}T = \overline{U}SU^t$ and that the matrix $N = U^*T\overline{U}$ is normal ($NN^* = N^*N = S$) and symmetric ($N = N^t$) . Thus $T = UNU^t$ as stated. In the case of finite matrices, N can be further diagonalized by a real orthogonal matrix O: $N = ODO^t$; see [68].

In a similar spirit, we have the following theorem:

Theorem (2.1.15)[445]: If T is a C-symmetric operator, then the antilinear operator CT commutes with the spectral measure of T^*T. In other words, if $E = E_{T^*T}$ denotes the spectral measure of T^*T, *then*

$$CTE(\sigma) = E(\sigma)CT$$

for every Borel subset σ *of* $[0, \infty)$.

Proof. CT commutes with T^*T since $(CT)^2 = T^*T$. Thus CT commutes with $p(T^*T)$ for any polynomial $p(x)$ with real coefficients and hence with each (σ).

Equivalently, one can also say that the antilinear operator TC commutes with the spectral measure of TT^*.

is Siegel's correspondent of the unit ball, and

$$H(C) = \{T \in C^o : \ \operatorname{Im} T > 0\}$$

of the upper half-plane; see [87] or [71].

The connection between the homogeneous complex structure of $H(C)$ and similar matrix realizations of symmetric domains and operator theory was long ago established and exploited by Potapov, Krein, Livsic and their followers; see for instance [65,66,79]. Within our framework we mention only that a self-adjoint C-symmetric operator A (bounded or not) has a resolvent $R(z) = (A - z)^{-1}$ defined in the upper half-plane, and with values in (C) :

$$C(A - z)^{-1} = (A - z)^{-1}C,$$

and

$$\frac{1}{2i}[(A-\overline{z})^{-1}-(A-z)^{-1}] = (A-\overline{z})^{-1}\frac{\overline{z}-z}{2i}(A-z)^{-1} > 0, \qquad \operatorname{Im} z > 0.$$

The homogeneous structure of $H(C)$ can lead, as in the cases studied by the above, to canonical representations of such resolvent functions. We do not follow this direction here. Is to provide a series of (quite distinct) examples of C-symmetric operators.

The building blocks (that is orthogonal summands) of any normal operator are the multiplication operators M_z on a Lebesgue space $L^2(\mu)$ of a planar, positive Borel measure μ with compact support. It is clear that complex conjugation $Cf = \overline{f}$ is isometric and that

$$CM_z = M_z^* C.$$

Subnormal operators are not in general C-symmetric, due to the fact that they tend to have nonzero Fredholm index on some part of their spectrum (see [90]). For instance, the unilateral shift represented as the multiplication operator M_z on the Hardy space H^2 of the disk cannot be C-symmetric, as was already clear from Proposition (2.1.5) The same conclusion obviously applies to the Bergman shift operator M_z with respect to any bounded planar domain.

Recall that $u \otimes v$ denotes the rank- one operator $(u \otimes v)f := \{f, v\}u$ and that any rank-one operator on $\mathcal{H}$ has such a representation.

Lemma (2.1.16)[445]: The operator $T = u \otimes v$ satisfies $CT = T^*C$ if and only if T is a constant multiple of $u \otimes Cu$.

Proof. Indeed, it is easy to see that $C(u \otimes v) = (Cu \otimes Cv)C$ since

$$C\{f, v\}u = \{v, f\}Cu = \{Cf, Cv\}Cu$$

for all $f, u, v \in \mathcal{H}$. Now $(u \otimes v)^* = v \otimes u$ and hence

$$C(u \otimes v) = (u \otimes v)^*C$$

if and only if $v \otimes u = Cu \otimes Cv$.

Passing now to compact operators, it is easy to construct C-symmetric ones. For instance, if u_n is a sequence of unit vectors in $\mathcal{H}$ and a_n is an absolutely summable sequence of scalars, then the operator

$$T = \sum_{n=1}^{\infty} a_n\,(u_n \otimes Cu_n)$$

is bounded and satisfies $CT = T^*C$. Under certain circumstances, we can use Theorem (2.1.15) to obtain a similar decomposition of a compact C-symmetric operator. Consider the following example.

Example (2.1.17)[445]: If T is a compact C-symmetric operator such that TT^* is injective and has simple spectrum, then we may write

$$TT^* = \sum_{n=1}^{\infty} c_n\,(u_n \otimes u_n),$$

where the c_n are distinct positive constants tending to 0 and the vectors u_n form an orthonormal basis of the underlying Hilbert space. By Theorem (2.1.15), the one-dimensional eigenspaces of TT^* are fixed by the antilinear operator TC and hence there exist complex constants a_n such that

$$TCu_n = a_n u_n, n \geq 1.$$

Since $Cu_n = TT^*u_n = c_n u_n$, we see that $|a_n|^2 = c_n$. This yields the decomposition

$$T = \sum_{n=1}^{\infty} a_n \, (u_n \otimes Cu_n)$$

of the operator T. Convergence is assured by the orthonormality of the vectors u_n and by the fact that the coefficients a_n tend to 0.

We leave it to make the appropriate modifications in the case where TT^* does not have simple spectrum.

It is worth mentioning at this point Hamburger's example of a compact operator K, with a complete system of root vectors, such that K^* does not have a complete system of root vectors [67]. Thus, K can not be similar to a C-symmetric operator.

Consider the simplest Volterra operator

$$Vf(x) = \int_0^x f\,(t)dt$$

on $L^2[0,1]$. The involution $Cf(t) \;:=\; \overline{f(1-t)}$ is a conjugation operator and a straightforward computation shows that V is C-symmetric.

We can treat more general Volterra type operators. Let $\mathcal{L}$ be an auxiliary Hilbert space with an antilinear, isometric involution J, and let A: $[0,1] \to B(\mathcal{L})$ be an essentially bounded, measurable, operator-valued function, with values in J^o:

$$A(t)J = JA(t)^*, t \in [0,1], a.e.$$

On the vector-valued Lebesgue space $\mathcal{H} = L^2[0,1]\widehat{\otimes}\mathcal{L}$ we define the involution

$$(Cf)(t) = J(f(1-t\; f \in \mathcal{H}, t \in \; [0,1].$$

A straightforward computation shows that the Volterra type operator

$$Vf(t) = \int_0^t A\,(t-s)f(s)ds, f \in \mathcal{H},$$

satisfies the C-symmetry relation $CV = V^*C$.

To give a numerical example, on $L^2[0, 1]$, we consider the Abel-Liouville potentials:

$$(J_\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (\,t-s)^{\alpha-1} f(s)ds, \alpha > 0.$$

They are simultaneously C-symmetric with respect to the symmetry $Cf(t) = \overline{f(1-t)}$. Volterra operators V of real type (that is, satisfying $CV = VC$) were extensively studied by the Ukrainian school; see [65]. It is interesting to note that the canonical models for these real operators involve only the pointwise involution $(Cf)(t) = \overline{f(t)}$, but not the argument inversion $(t \mapsto 1-t)$ we had above.

We maintain the notation of Example (2.1.2) and freely identity functions in H^2 with their boundary values on the unit circle. In particular, recall the definitions (2) and (3) of the Hilbert space $\mathcal{H}_\varphi$ and conjugation operator C, respectively.

For a nonconstant function u belonging to L^∞, the Toeplitz operator with symbol u is the operator on H^2 given by

$$T_u f := P(uf)$$

where P denotes the orthogonal projection from L^2 onto H^2. It is well known that $T_u^* = T_{\overline{u}}$ for each u in L^∞. For a nonconstant inner function φ, the compression of T_u to $\mathcal{H}_\varphi$ is the operator

$$T_u := P_\varphi T_u P_\varphi$$

where P_φ denotes the orthogonal projection from H^2 onto $\mathcal{H}_\varphi$. These operators are simultaneously C-symmetric.

Proposition (2.1.18)[445]: If φ is a nonconstant inner function, then $(\mathcal{H}_\varphi, T_u, C)$ *is a* C-symmetric triple for each u belonging to L^∞.

Proof. If f and g belong to $\mathcal{H}_\varphi$, then

$$\begin{aligned}\langle CT_u f, g\rangle &= \langle Cg, T_u f\rangle = \langle Cg, P_\varphi T_u P_\varphi f\rangle \\ &= \langle P_\varphi Cg, T_u f\rangle = \langle Cg, P(uf)\rangle \\ &= \langle PCg, uf\rangle = \langle Cg, uf\rangle \\ &= \langle \overline{gz}\varphi, uf\rangle = \langle \overline{fz}\varphi, ug\rangle \\ &= \langle Cf, ug\rangle = \langle PP_\varphi Cf, ug\rangle \\ &= \langle P_\varphi Cf, T_u g\rangle = \langle Cf, P_\varphi T_u P_\varphi g\rangle \\ &= \langle Cf, T_u g\rangle = \langle T_u^* Cf, g\rangle.\end{aligned}$$

Hence $CT_u = T_u^* C$ as desired.

The compression of the unilateral shift to $\mathcal{H}_\varphi$ (known as a standard model operator or *Jordan operator*) is C-symmetric (see [73, Lemma 9.2].

If h belongs to H^∞, then several well-known classical results (see, [46, 77]) follow from the fact that the compression T_h of the multiplication operator T_h (on H^2) is C-symmetric. The correspondence between portions of the spectra of a C-symmetric operator and its adjoint discussed after Proposition (2.1.5) apply immediately to the operators T_h and T_h^*. This is in stark contrast to the (uncompressed) operators T_h and T_h^* on H^2. In general, the spectra of T_h and T_h^* are structurally quite different. For example, if h is nonconstant, then T_h has empty point spectrum whereas the point spectrum of T_h^* contains $\overline{h(\mathbb{D})}$.

Compared to the dilation and extension theory in spaces with an indefinite metric (see, [40]), the analogous results for C-symmetric operators are much simpler. We consider only two illustrative situations.

Let $S: \mathcal{D} \to \mathcal{H}$ be a densely defined, closed graph symmetric operator. Recall von-Neumann's criterion for the existence of a self-adjoint extension of S: If there exists an antilinear involution $C : \mathcal{D} \to \mathcal{D}$ such that $CS = SC$, then the defect numbers of S are equal, hence at least one self-adjoint extension of S exists; see [75 and [81, 83].

The special case of an isometric involution C, actually considered by von Neumann ([75],p.101) is interesting for us, because among all self-adjoint extensions, only part of them turn out to be C-symmetric. These operators, and extensions, were called *real* by von Neumann.

Proposition (2.1.19)[445]: *Let* S: $\mathcal{D} \to \mathcal{H}$ be a closed graph, densely defined symmetric operator and assume that there exists an antilinear, isometric involution $C: \mathcal{H} \to \mathcal{H}$ *mapping* $\mathcal{D}$ *into itself and satisfying the symmetry relation* $SC = CS$. *Then the* C*-symmetric self-adjoint extensions* A $(i.e. A = CAC)$ *of* S *are parametrized by all isometric maps* $V: Ker(S^* - i) \to Ker(S^* + i)$ *satisfying* $V^*C = CV$.

Proof. Indeed, the involution C maps the defect space $Ker(S^* - i)$ into $(S^* + i)$. Let A be a C-symmetric self-adjoint extension of S corresponding to the isometry $V: Ker\ (S^* - i) \to Ker(S^* + i)$; see for instance [81]. Then the graph of A consists of Graph $(S) \oplus \{(f, Vf)$: $f \in Ker(S^* - i$ and $Af = if$, $AVf = -iVf$. Since the domain of A is invariant under C we infer (CVf, Cf) =(g, Vg) for some $g \in Ker(S^* - i)$. In other words, $V^*C = CV$. □

The case of defect indices (1, 1) is simple, for any self-adjoint extension of S is C-symmetric, due to the observation $C(e^{it}) = e^{-it}C$ for any real parameter t. For higher defect indices, however, not all self-adjoint extensions are C-symmetric.

We investigated the change acquired in the spectrum of an unbounded C-symmetric operator S (i.e. $S \subset CS^*C$) when completed to one of its C-self-adjoint extensions $\tilde{S}$ (i.e. $S \subset \tilde{S}$ and $\tilde{S} = C\tilde{S}C$). Interestingly enough, these observations apply to SturmLiouville operators of the form $-u'' + q(x)u, u \in L^2(-\infty, \infty)$, where $q(x)$ is a nonreal potential [62, Sect. 23-34].

We turn now to a C-symmetric contractive operator $T \in B(\mathcal{H})$. The defect spaces of T are

$$\mathcal{D}_+ = \text{Ran } (I - T^*T)^{1/2}, \mathcal{D}_- = \text{Ran } (I - TT^*)^{1/2},$$

where Ran A denotes the norm closure of the range of the operator A. If $CT = T^*C$, then $CT^*T = TT^*C$ and hence

$$C(I - T^*T)^{1/2} = (I - TT^*)^{1/2}C.$$

In particular, this shows that

$$C: \mathcal{D}_+ \to \mathcal{D}_-$$

is an isometric antilinear map.

Thus a C-symmetric contraction must have equal dimensional defect spaces. The Sz.-Nagy minimal unitary dilation U of T can be constructed as an infinite matrix (see [16]) as recalled below.

Let

$$\mathcal{K} = \cdots \oplus \mathcal{D}_- \oplus \mathcal{D}_- \oplus \mathcal{H} \oplus \mathcal{D}_+ \oplus \mathcal{D}_+ \ldots$$

be a direct sum Hilbert space with $\mathcal{H}$ on the 0-th position (marked below in bold face characters). Let T: $\mathcal{K} \to \mathcal{K}$ be the operator explicitly defined by

$U(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (\ldots, x_{-3}, x_{-2}, (I - TT^*)^{1/2}x_{-1} + Tx_0, -T^*x_{-1} + (I - T^*T)^{1/2}x_0, x_1, . .$

It is easy to prove that U is a unitary operator which dilates T, in the sense that $\{U^n x_0, x_0\} = \{T^n x_0, x_0\}, n \in \mathbb{N}$, where x_0 is a vector supported by the 0-th position.

We define the isometric antilinear involution $\tilde{C} : \mathcal{K} \to \mathcal{K}$ by the formula

$$\tilde{C}(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (. . . , Cx_2, Cx_1, Cx_0, Cx_{-1}, Cx_{-2}, . .)$$

A straightforward computation shows that $UCU = C$ and hence U is C-symmetric. In conclusion we have proved the following result.

Theorem (2.1.20)[445]: Let T *be a* C-symmetric contraction. The map C extends to an antilinear, isometric involution $\tilde{C}$ on the space of the unitary dilation U of T, such that

$$\tilde{C}U = U^*\tilde{C}.$$

As an almost tautological example we consider the following typical analysis of a self-adjoint extension of defect indices (1,1).

Example (2.1.21)[445]: Let s_0, s_1, . . . be an indeterminate moment sequence of a probability measure on the line. Let $\mathcal{H}$ be the completion of the space of polynomials $\mathbb{C}[x]$ in the norm given by the associated positive definite Hankel matrix

$$\| \sum_{k=0}^{n} c_k \, x^k \|^2 = \sum_{k,l=0}^{n} s_{k+l} \, c_k \overline{c_l}.$$

Let $P_k(x)$ be the associated orthogonal polynomials, normalized by the condition $\|P_k\| = 1$ and the leading term of each P_k is positive. In this way $\mathcal{H}$ can be identified with $l^2(\mathbb{N})$.

These orthogonal polynomials have real coefficients, hence they are invariant under the involution $(Cq)(x) = \overline{q(x)}$ for x in $\mathbb{R}$.

The unbounded operator of multiplication by the variable x can be represented by a Jacobi matrix J, formally symmetric in the norm of $\mathcal{H}$. By considering J on its maximal domain of definition $\mathcal{D}$ we obtain a closed graph, symmetric operator with defect indices (1, 1) (due to the fact the problem is indeterminate). For all details see [39,66].

Obviously, the C-symmetry relation $CJ = J^*C = JC$ holds on $\mathcal{D}$. The self- adjoint extensions of J are parametrized by a complex number α of modulus one and can explicitly be given as follows. Fix an arbitrary nonreal complex number λ and consider the vector $\Pi_\lambda = (P_0(\lambda), P_1(\lambda), \dots) \in Ker(J^* - \lambda)$. This vector belongs to $l^2(\mathbb{N})$ by the indeterminate nature of the moment sequence. Define $P_\lambda = \Pi_\lambda / \|\Pi_\lambda\|$. Then the rank-one operator $P_\lambda \otimes CP_\lambda = P_\lambda \rangle CP_\lambda$ satisfies

$$[P_\lambda \otimes CP_\lambda](P_0(\lambda), P_1(\lambda), \dots) = C(P_0(\lambda), P_1(\lambda), \dots) = (P_0(\lambda), P_1(\lambda), \dots)$$
$$= (P_0(\overline{\lambda}), P_1(\overline{\lambda}), \dots),$$

and thus it maps the defect space $Ker(J^* - \lambda)$ isometrically onto $Ker(J^* - \overline{\lambda})$.

Thus all self-adjoint extensions of the Jacobi matrix J can be described, on the enlarged domain of definition $\mathcal{D} + \mathbb{C}\Pi_\lambda$, as

$$S_\alpha = J + \alpha(P_\lambda \otimes CP_\lambda) \tag{9}$$

for $|\alpha| = 1$. A direct computation, or the proposition above, shows that the family S_α is simultaneously C-symmetric.

The compressions of the unilateral shift onto its coinvariant subspaces $\mathcal{H}_\varphi$ from the viewpoint of C-symmetry. Maintaining the conventions and notation of Examples (2.1.2) and (2.1.4), we show here that the rank-one unitary perturbations of the compressed unilateral shift considered by Clark [49] are jointly C-symmetric with respect to the symmetry (3). Indeed, we consider a slight generalization at little extra expense.

For λ in the unit disk define

$$b_\lambda(z) := \frac{z - \lambda}{1 - \overline{\lambda} z} \tag{10}$$

and consider the operator

$$S_\lambda := P_\varphi T_{b_\lambda} P_\varphi$$

on $\mathcal{H}_\varphi$. Hence S_λ is simply the compression to $\mathcal{H}_\varphi$ of the multiplication operator (on H^2) with symbol b_λ. The case $\lambda = 0$ corresponds to the compression of the unilateral shift. Proposition (2.1.18) tells us that the operators S_λ are jointly C-symmetric. The following lemma is a generalization of Clark's initial observation and is phrased in terms of the conjugation operator C. Recall the formulas (4) and (5) for the functions K_λ and Q_λ described in Example (2.1.4).

Lemma (2.1.22)[445]: *For each λ in D*, the following statements hold:

(a) $S_\lambda f = b_\lambda f$ if and only if f is orthogonal to Q_λ.

(b) $S_\lambda^* f = f / b_\lambda$ if and only if f is orthogonal to K_λ.

Proof. Clearly, $S_\lambda^* f = f / b_\lambda$ if and only if f / b_λ belongs to H^2. This happens if and only if $f(\lambda) = 0$, or equivalently, if and only if $\{f, K_\lambda\} = 0$. By the preceding, $S_\lambda^* Cf = (Cf)/b_\lambda$ if and only if $\{Cf, K_\lambda\} = 0$, or equivalently, if and only if f is orthogonal to $Q_\lambda = CK_\lambda$. Since $CS_\lambda = S_\lambda^* C$, this implies that $S_\lambda f = C[Cf / b_\lambda] = b_\lambda f$ if and only if f is orthogonal to Q_λ.

At this point it is convenient to introduce the normalized kernel functions k_λ and q_λ defined by

$$k_\lambda := \frac{K_\lambda}{\|K_\lambda\|}, q_\lambda := \frac{Q_\lambda}{\|Q_\lambda\|}. \tag{11}$$

For each α of unit modulus, the operator

$$U_{\lambda,\alpha} := S_\lambda[I - (q_\lambda \otimes q_\lambda)] + \alpha(k_\lambda \otimes q_\lambda) \tag{12}$$

is unitary by the preceding proposition. Moreover, it is a rank-one perturbation of S_λ since

$$U_{\lambda,\alpha} = S_\lambda + \big(\alpha + \varphi(\lambda)\big)(k_\lambda \otimes Ck_\lambda) \tag{13}$$

as a straightforward computation shows. The proof that the $U_{\lambda,\alpha}$ are the only rank- one unitary perturbations of S_λ is a straightforward generalization of the original proof [49].

The interest in these perturbations stems from the fact that they are jointly C-symmetric. The following proposition follows immediately from Proposition (2.1.18) and Lemma (2.1.16).

Proposition (2.1.23)[445]: If φ is a nonconstant inner function, then the operators $U_{\lambda,\alpha}$ are jointly C-symmetric. That is, $(\mathcal{H}_\varphi, U_{\lambda,\alpha}, \mathrm{C})$ *is a* C-symmetric triple for each $|\lambda| < 1$ and each $|\alpha| = 1$.

By (13) we have

$$U_{\lambda,\alpha}f = S_\lambda f + \big(\alpha + \varphi(\lambda)\big)\overline{Cf(\lambda)}\frac{1 - |\lambda|^2}{1 - |\varphi(\lambda)|^2}K_\lambda \tag{14}$$

for each f in $\mathcal{H}_\varphi$. Thus the antilinear operator C plays a hidden role in the structure of the $U_{\lambda,\alpha}$, so the rank-one perturbing operator involves the twisted point evaluation $f \mapsto \overline{Cf(\lambda)}$. Under certain circumstances we can explicitly furnish a self-conjugate orthonormal basis e_n for $\mathcal{H}_\varphi$. Although the existence of such a basis is guaranteed by Lemma (2.1.1) we are interested here in producing them for the purpose of computing the corresponding matrix representation of the compressed shift operator $S := S_0$.

Let α be a unimodular constant and consider the unitary operator $U_\alpha := U_{0,\alpha}$. It was shown in [49, Lemma 3.1] that a complex number ζ is an eigenvalue of U_α if and only if φ has a finite angular derivative $\varphi'(\zeta)$ at ζ (note that $|\zeta| = 1$ since U_α is unitary) and $\varphi(\zeta) = \beta$ where the unimodular constant β is defined by

$$\beta := \frac{\alpha + \varphi(0)}{1 + \overline{\varphi(0)}\alpha}. \tag{15}$$

The corresponding unit eigenvector will be k_ζ, where the definition (11) of k_ζ is extended to include unimodular ζ in the obvious way. Although the finiteness of the angular derivative is not explicit in [49], it is easily seen to be equivalent to the condition above (see [82, p. 367] or [41]).

Let ζ_n denote an enumeration of the (at most countably many) eigenvalues of U_α. Clark showed that if the operator U_α has pure point spectrum, then the corresponding eigenvectors $k_n := k_{\zeta_n}$ of U_α form an orthonormal basis for $\mathcal{H}_\varphi$. This occurs [49, Theorem 7.1] if the set of points on the unit circle at which φ does not have a finite angular derivative is countable. For example, the eigenvectors of U_α span $\mathcal{H}_\varphi$ if φ is a Blaschke product whose zeros cluster only on a countable set or if φ is a singular inner function such that the closure of the support of the associated singular measure is countable. In such cases the modulus $|\varphi'(\zeta_n)|$ of the angular derivative of φ at ζ_n is finite and equals $\|K_n\|^2$ where $K_n := K_{\zeta_n}$.

Suppose now that φ is an inner function and $|\alpha| = 1$ such that the corresponding operator U_α has pure point spectrum $\{\zeta_1, \zeta_2, \dots\}$ and fix t_n for $n = 0, 1, 2, \dots$ such that $e^{it_0} = \beta$ and $\zeta_n = e^{it_n}$ for each $n \geq 1$. The preceding discussion tells us that the functions k_n form an orthonormal basis for $\mathcal{H}_\varphi$. Multiplying the k_n by suitable unimodular constants yields a self-conjugate orthonormal basis e_n defined by

$$e_n := e^{\frac{i}{2}(t_0 - t_n)} k_n$$

with respect to which each f in $\mathcal{H}_\varphi$ enjoys the expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{e^{\frac{i}{2}(t_n - t_0)}}{\sqrt{|\varphi'(\zeta_n)|}} f(\zeta_n) e_n(z).$$

Whence we obtain the inner product formula

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \frac{f(\zeta_n)\overline{g(\zeta_n)}}{|\varphi'(\zeta_n)|}$$

for f, g in $\mathcal{H}_\varphi$. Since C is simply the complex conjugation with respect to the basis e_n, we see that

$$\Phi f := \left(\frac{e^{\frac{i}{2}(t_n - t_0)}}{\sqrt{|\varphi'(\zeta_n)|}} f(\zeta_n)\right)_{n=1}^{\infty}$$

is an isometric isomorphism of $\mathcal{H}_\varphi$ onto a certain weighted l^2 space such that

$$\Phi(Cf) = \overline{\Phi f}, f \in \mathcal{H}_\varphi.$$

If the sequence $w_n/\sqrt{|\varphi'(\zeta_n)|}$ is square-summable, then there exists a function f in $\mathcal{H}_\varphi$ whose nontangential limiting values at the points ζ_n interpolate the values w_n.

We now explicitly compute the matrix representation of the compressed shift operator S with respect to the basis e_n. In particular, we will see that the matrix $(\{Se_n, e_m\rangle)_{n,m=1}^{\infty}$ is complex symmetric (as expected) and, moreover, that the entries are related to the eigenvalues ζ_n in a simple way. By (14) we have

$$Se_n = \zeta_n e_n - \overline{e_n(0)} \frac{\alpha + \varphi(0)}{1 - |\varphi(0)|^2} K_0$$

where we used the fact that e_n are self-conjugate eigenvectors for U_α. Thus

$$\begin{aligned}
\{Se_n, e_m\} &= \zeta_n \langle e_n, e_m \rangle + \overline{e_n(0)} \frac{\alpha + \varphi(0)}{1 - |\varphi(0)|^2} \langle K_0, e_m \rangle = \zeta_n \delta_{nm} + \overline{e_n(0) e_m(0)} \frac{\alpha + \varphi(0)}{1 - |\varphi(0)|^2} \\
&= \zeta_n \delta_{nm} + \frac{e^{\frac{i}{2}(t_n - t_0) + (t_m - t_0)}}{||K_n|| ||K_m||} (1 - \beta\overline{\varphi(0)})^2 \frac{\alpha + \varphi(0)}{(1 - |\varphi(0)|^2)} \\
&= \zeta_n \delta_{nm} + \frac{\zeta^{\frac{1}{n^2}} \zeta^{\frac{1}{m^2}}}{\left|\varphi'(\zeta_n)|^{\frac{1}{2}}\right| \varphi'(\zeta_m)|^{\frac{1}{2}}} \frac{(1 - \beta\overline{\varphi(0)})^2 (\alpha + \varphi(0))}{(1 - |\varphi(0)|^2)}. \qquad (16)
\end{aligned}$$

Here δ_{nm} denotes the Kronecker δ-function and the square roots of ζ_n and ζ_m are defined in the obvious way. From this calculation we observe that $\{Se_n, e_m\rangle = \{Se_m, e_n\}$ for all n, m and hence the matrix for S with respect to the basis e_n is complex symmetric. The matrix representations of the unitary operator U_α and the perturbing operator are evident in (16). Summing up, we have proved the following result.

Theorem (2.1.24)[445]: Let φ be a nonconstant inner function (on the unit disk) and let S be the standard Jordan operator (the compressed shift) on the model space $\mathcal{H}_\varphi := H^2 \ominus \varphi H^2$ If the rank-one unitary perturbation U_α of S has pure point spectrum $\{\zeta_n;\ n \geq 1\}$, then there exists an orthonormal basis e_n of eigenvectors for U_α such that

(i) $Ce_n = e_n$ *for all* $n \geq 1$ *where* $Cf := \overline{fz}\varphi$.

(ii) The matrix of S with respect to the basis e_n is complex symmetric and has the form (16).

In the case $\varphi(0) = 0$ our computation reduces to

$$\{Se_n, e_m\rangle = \zeta_n \delta_{nm} + \alpha \frac{\zeta^{\frac{1}{n^2}} \zeta^{\frac{1}{m^2}}}{|\varphi'(\zeta_n)|^{\frac{1}{2}} |\varphi'(\zeta_m)|^{\frac{1}{2}}}.$$

We relate the C-symmetry $Cf = \overline{fz}\varphi$ on the model space $\mathcal{H}_\varphi := H^2 \ominus \varphi H^2$ to the inner-outer factorization of functions f in $\mathcal{H}_\varphi$. In terms of boundary functions $\mathcal{H}_\varphi$ we have

$$\mathcal{H}_\varphi = H^2 \cap \varphi \overline{zH^2}$$

and hence the following easy lemma.

Lemma (2.1.25)[445]: Two functions f *and* g *in* H^2 satisfy $= \overline{fz}\varphi$ *a.e. on* ∂D *if and only if* f *and* g *belong to* $\mathcal{H}_\varphi$ *and* $Cf = g$.

Suppose now that f belongs to $\mathcal{H}_\varphi$. Since the functions f and Cf have the same modulus $a.e.$ on $\partial\mathbb{D}$, they share the same outer factor, say F. We may therefore write $f = I_f F$ and $Cf = I_{Cf} F$ where I_f and I_{Cf} denote the inner factors of f and Cf, respectively.

$$I_f I_{Cf} F = \overline{Fz}\varphi$$

$a.e.$ on ∂D. This shows that F belongs to $\mathcal{H}_\varphi$ and satisfies $CF = I_f I_{Cf} F$. More-over, the inner function $I_f I_{Cf}$ depends only upon F and φ and not on the particular pair of conjugate functions f, Cf with common outer factor F. We call the inner function $I_f I_{Cf}$ the associated inner function of F (with respect to φ) and denote it $\mathcal{I}_F$. The functions $f = I_f F$ in $\mathcal{H}_\varphi$ with outer factor F are precisely those functions whose inner factors I_f divide $\mathcal{I}_F$. This yields the following lemma.

Lemma (2.1.26)[445]: For any outer function F *in* $\mathcal{H}_\varphi$ there exists a unique inner *function* $\mathcal{I}_F$ *such that* $\mathcal{I}_F F = \overline{F}\overline{z}\varphi$ *a.e. on* ∂D. *If I is an inner function, then if belongs to* $\mathcal{H}_\varphi$ *if and only if I divides* $\mathcal{I}_F$.

Example (2.1.27)[445]: Fix a nonconstant inner function φ and consider the kernel functions K_λ and $Q_\lambda = CK_\lambda$ defined by (4) and (5). The associated inner function for K_λ is the inner factor

$$\mathcal{I}_{K_\lambda} = \frac{b_{\varphi(\lambda)}(\varphi(z))}{b_\lambda(z)}$$

of Q_λ. Here $b_{\varphi(\lambda)}$ and b_λ are Möbius transformations defined by (10).

To sum up, a function f belonging to $\mathcal{H}_\varphi$ possesses the representations

(i) $f = I_f F$ where F is outer, I_f is the inner factor of f, and $I_f | \mathcal{I}_F$.

(ii) $f = f_1 + i f_2$ where $Cf_k = f_k$ for $k = 1, 2$.

In light of the fact that $|f|^2 = |f_1|^2 + |f_2|^2$ $a.e.$ on $\partial\mathbb{D}$, it is not difficult to pass from one representation to the other.

We consider below the structure of the N-dimensional model space $\mathcal{H}_\varphi$ corresponding to a finite Blaschke product

$$\varphi(z) = \prod_{n=1}^{N} \frac{z - \lambda_n}{1 - \overline{\lambda_n} z}$$

with N (not necessarily distinct) zeroes λ_n. In particular, we make extensive use of the bilinear form $[\cdot, \cdot]$ arising from the conjugation operator $Cf = \overline{fz}\varphi$ on $\mathcal{H}_\varphi$. Select any w in the closed unit disk such that the equation $\varphi(z) = w$ has N distinct solutions $z_1, \ldots, z_N$. The N functions $Q_n(z) := Q_{z_n}(z)$ defined by (5) are pairwise orthogonal with respect to the bilinear form $[\cdot, \cdot]$:

$$[Q_n, Q_m] = \begin{cases} \varphi'(z_n), & n = m, \\ 0, & n \neq m, \end{cases} \tag{17}$$

and are linearly independent since $\varphi'(z_n) \neq 0$ for all n. Therefore, the Q_n form a basis for $\mathcal{H}_\varphi$.

For any f in $\mathcal{H}_\varphi$ the well-known interpolation formula

$$f(z) = \sum_{n=1}^{N} \frac{f(z_n)}{\varphi(z_n)} Q_n(z)$$

follows immediately from (17). The space $\mathcal{H}_\varphi$ is essentially a weighted version of $\mathbb{C}^n$ twisted by C:

$$\langle f, g \rangle = \sum_{n=1}^{N} \frac{f(z_n) C g(z_n)}{\varphi(z_n)}.$$

With respect to the bilinear form $[\cdot,\cdot]$ we have

$$[f, g] = \sum_{n=1}^{N} \frac{f(z_n) g(z_n)}{\varphi'(z_n)}. \tag{18}$$

We can make these computations more explicit. Each function f belonging to $\mathcal{H}_\varphi$ is of the form $f = F/R$ where F is a polynomial of degree $\leq N - 1$ and

$$R(z) = \left(1 - \overline{\lambda_1} z\right) \cdots \left(1 - \overline{\lambda_N} z\right). \tag{19}$$

The conjugation operator on $\mathcal{H}_\varphi$ is given by the formula

$$C(F/R) = F^{\neq}/R. \tag{20}$$

where the polynomial $F^{\neq}$ is defined by

$$F^{\neq}(z) = z^{N-1} \overline{F(1/\overline{z})}.$$

Observe that $\varphi = P/R$ where $P = z^N \overline{R(1/\overline{z})}$ and then choose any w in the closed unit disk such that the equation

$$P(z) - wR(z) = 0$$

has N distinct solutions $z_1, \ldots, z_N$. In other words, select w so that the level set $\varphi^{-1}(\{w\})$ contains N distinct points.

Letting $f = F/R$ and $g = G/R$ denote arbitrary functions in $\mathcal{H}_\varphi$ we have by (18),

$$[f, g] = \sum_{n=1}^{N} \frac{F(z_n) G(z_n)}{\Delta(z_n)},$$

$$\langle f, g\rangle = \sum_{n=1}^{N} \frac{F(z_n)G^{\neq}(z_n)}{\Delta\,(z_n)}$$
where the polynomial Δ is defined by
$$\Delta = RP' - PR'.$$
Note that in the above formulas, the products $[f, g]$ and $\{f, g\}$ are intrinsic, while the right-hand sides depend on the chosen fibre of the function φ.

This is again a classical and well charted territory which we touch upon only briefly. In this framework C-symmetry is a unifying concept and transparent formalism.

Let λ_n be a sequence of distinct points in the unit disk and define
$$\varphi(z) = \prod_{n=1}^{\infty} \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1 - \overline{\lambda_n} z}.$$
Since the λ_n are distinct, it follows that $\varphi'(\lambda_n) \neq 0$.

Consider the unit vectors e_n defined by
$$e_n = \frac{Q_n}{\|Q_n\|} = (1 - |\lambda_n|^2)^{\frac{1}{2}} Q_n$$
where $Q_n := Q_{\lambda_n}$ is defined by (5). The e_n are orthogonal with respect to the bilinear form $[\cdot,\cdot]$ introduced since
$$[e_n, e_m] = \begin{cases} (1 - |\lambda_n|^2)\varphi'(\lambda_n)\,, & n = m, \\ 0, & n \neq m. \end{cases}$$
We are led therefore to the formal Fourier-type expansion
$$f = \sum_{n=1}^{\infty} \frac{[f, e_n]}{[e_n, e_n]} e_n \tag{21}$$
whose convergent behavior (for all f in $\mathcal{H}_\varphi$) is naturally linked to the uniform boundedness from below of $|[e_n, e_n]|$. This is Carleson's famous interpolation theorem.

Recall that a sequence λ_n in the unit disk is called uniformly separated if there exists a $\delta > 0$ such that the Carleson condition
$$\prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{1 - \overline{\lambda_k} \lambda_n} \right| \geq \delta \tag{22}$$
holds for every n. Thus, according to our computations, this is equivalent to asserting that
$$|[e_n, e_n]| \geq \delta$$
for all n. In other words, the unit vectors e_n are not "asymptotically isotropic" with respect to the bilinear form $[\cdot,\cdot]$.

Carleson's interpolation theorem for H^2 asserts that the operator
$$Tf := f(\lambda_n)\sqrt{1 - |\lambda_n|^2}$$
maps $\mathcal{H}_\varphi$ onto l^2 if and only if the sequence λ_n is uniformly separated. Since $Tf = [f, e_n]$ for all n, the interpolation theorem implies the following result.

Theorem (2.1.28)[445]: *Let* λ_n be a sequence of distinct points in the unit disk, let φ be the associated Blaschke product, and let e_n be the normalized evaluation elements of the model space $\mathcal{H}_\varphi := H^2 \ominus \varphi H^2$:
$$[f, e_n] = f(\lambda_n)\sqrt{1 - |\lambda_n|^2}.$$
The $[\cdot,\cdot]$*-orthogonal series*

$$f = \sum_{n=1}^{\infty} \frac{[f, e_n]}{[e_n, e_n]} e_n$$

converges for every $f \in \mathcal{H}_\varphi$ *if and only if there exists a positive constant* δ *satisfying*: $|[e_n, e_n]| \geq \delta$ *for* $n \geq 1$.

This can be stated in terms of the theory of Riesz bases. The functions e_n form a Riesz basis for $\mathcal{H}_\varphi$ if and only if the sequence λ_n is uniformly separated. From this point of view,

$$e'_n := \frac{Ce_n}{[e_n, e_n]}$$

is a biorthogonal sequence to e_n.

If the λ_n satisfy the Carleson condition, then each f in $\mathcal{H}_\varphi$ is given by the interpolation formula

$$f(z) = \sum_{n=1}^{\infty} \frac{f(\lambda_n)}{\varphi'(\lambda_n)} Q_n(z) \tag{23}$$

which converges in norm. This also gives the orthogonal projection from H^2 onto $\mathcal{H}_\varphi$ of any function interpolating the values $f(\lambda_n)$ at the nodes λ_n.

As in the finite dimensional case, the inner product on $\mathcal{H}_\varphi$ has a simple representation in terms of the conjugation operator C. In light of the interpolation formula (23) we have

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \frac{f(\lambda_n) Cg(\lambda_n)}{\varphi'(\lambda_n)} \tag{24}$$

for any f, g in $\mathcal{H}_\varphi$. The antilinearity in the second argument of the inner product on $\mathcal{H}_\varphi$ is clearly reflected by the presence of the C operator in the preceding formula. In some sense, there is an asymmetry in (24) that is unnecessary. We can easily remedy this by considering the bilinear form $[\cdot,\cdot]$, with respect to which $\mathcal{H}_\varphi$ is simply a weighted sequence space:

$$[f, g] = \sum_{n=1}^{\infty} \frac{f(\lambda_n) g(\lambda_n)}{\varphi'(\lambda_n)}.$$

Although the bilinear form is not positive definite, we still have

$$c_1 \sum_{n=1}^{\infty} |[f, e_n]|^2 \leq \|f\|^2 \leq c_2 \sum_{n=1}^{\infty} |[f, e_n]|^2$$

for some constants c_1 and c_2 since the e_n form a Riesz basis for $\mathcal{H}_\varphi$.

We consider a basic matrix extension problem arising in electrical network theory from the viewpoint of C-symmetry. We consider the scalar-valued Darlington synthesis problem: Given a function $a(z)$ belonging to H^2, do there exist functions b, c, and d also belonging to H^2 such that the matrix

$$U = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \tag{25}$$

is unitary $a.e.$ on the unit circle $\partial\mathbb{D}$? In other words, when can we extend the 1×1 matrix (a) to a 2×2 *inner* matrix?

If a matrix U of the form (25) is unitary $a.e.$ on $\partial\mathbb{D}$, then its determinant $\det U$ is an inner function, say φ. It turns out that the entries of U (including a itself) belong to $\mathcal{H}_{z\varphi}$, the backward shift invariant subspace of H^2 generated by φ. The following theorem from [59, 60] gives the exact relationship between a and $\det U$.

Theorem (2.1.29)[445]: *If φ is* a nonconstant inner function, then U is unitary a. e on $\partial\mathbb{D}$ *and* $\det U = \varphi$ if and only if:

(i) a, b, c, d *belong to* $\mathcal{H}_{z\varphi} := H^2 \ominus z\varphi H^2$

(ii) $Ca = d$ *and* $Cb = c$.

(iii) $|a|^2 + |b|^2 = 1$ *a.e. on* $\partial\mathbb{D}$.

Here C denotes the conjugation operator $Cf = \overline{f}\varphi$ on the model space $\mathcal{H}_{z\varphi}$ (see Examples (2.1.2),(2.1.4)). From the viewpoint of C-symmetry, 2×2 matrix inner functions resemble quaternions of unit modulus, for

$$U = \begin{pmatrix} a & -b \\ Cb & Ca \end{pmatrix} \tag{26}$$

where $|a|^2 + |b|^2 = 1$ $a.e.$ on $\partial\mathbb{D}$. The connection between matrix inner functions and C-symmetry is not surprising. Indeed, the connection between the Darlington synthesis problem and the backward shift operator (via pseudocontinuations [84]) was noted by several authors [42, 50, 51].

Note that $\|a\|_\infty \leq 1$ is necessary for the scalar-valued Darlington synthesis problem with data $a(z)$ to have a solution. The following theorem (also from [59, 60]) is the key to our approach.

Theorem (2.1.30)[445]: *If the function $a(z)$ belongs to $\mathcal{H}_{z\varphi}$ for some nonconstant inner function φ and $\|a\|_\infty \leq 1$, then there exists a function $b(z)$ in $\mathcal{H}_{z\varphi}$ such that*

$$|a|^2 + |b|^2 = 1 \ a.e. \ on \ \partial\mathbb{D}.$$

Returning to Theorem (2.1.29), we may write $a = I_aF$, $Ca = I_{Ca}F$, $b = I_bG$, and $Cb = I_{Cb}G$ where I_a, I_{Ca}, I_b, I_{Cb} are inner functions and F and G are outer. With this notation we have

$$\begin{aligned} \varphi &= aCa + bCb & (27)\\ &= \mathcal{I}_F F^2 + \mathcal{I}_G G^2, & (28) \end{aligned}$$

where $\mathcal{I}_F$ and $\mathcal{I}_G$ denote the associated inner functions of F and G, respectively.

Suppose that U is a solution to the scalar-valued Darlington synthesis problem with data (z) . By Theorem (2.1.29), $\det U = \varphi$ is inner and

$$U = \begin{pmatrix} a & -b \\ Cb & Ca \end{pmatrix}$$

where Ca and Cb are the conjugates of a and b in $\mathcal{H}_{z\varphi}$. Observe that if I_1 and I_2 are any inner functions, then

$$U' = \begin{pmatrix} a & -I_1 b \\ I_2 Cb & I_1 I_2 Ca \end{pmatrix} \tag{29}$$

is another solution and $\det U$ divides $\det U'$.

We say that a solution U is *primitive* if the inner function $\varphi = \det U$ is the minimal inner function such that $\det U$ divides $\det U'$ for any other solution U'. This is equivalent to requiring that φ is the minimal inner function such that a belongs to $\mathcal{H}_{z\varphi}$. Note also that every primitive solution shares the same determinant, up to a unimodular constant factor. We call such a φ a *minimal determinant* for the problem (with data $a(z)$). Recall that Arov [43,44] considered a related concept in his classification of minimal D-representations in the operator-valued case (which clearly covers the scalar case). Our techniques in the scalar case, however, are completely different, since we have available the concept of determinants and C- symmetry. The following easy proposition is from [60].

Proposition (2.1.31)[445]: *Fix a minimal determinant φ corresponding to the data* (z) *. If U' is any solution, then U' can be obtained via* (29) *from a primitive solution U with* $\det U = \varphi$.

A complete collection of primitive solutions sharing the same minimal determinant is called a *primitive solution set*. Fix a minimal determinant φ to our problem. We wish now to describe all solutions U with determinant φ. By condition (iii) of Theorem (2.1.29), we may identify each solution with the *inner* factor of the upper-left corner (z) . This inner factor must be a divisor of $\mathcal{J}_G$ (which is determined by (28)) and hence there is a bijective correspondence between a primitive solution set and the inner divisors of $\mathcal{J}_G$.

Example (2.1.32)[445]: *If* $\mathcal{J}_G$ is constant, then each primitive solution set consists of precisely one solution and all possible solutions can be constructed via (29) from a single primitive solution. This is the case for the data $a = \frac{1}{2}(1 + \varphi)$ where φ is an inner function. The minimal determinant is φ and the corresponding b is given by $b = \frac{1}{2i}(1 - \varphi)$.

Example (2.1.33)[445]: If $\mathcal{J}_G$ is the square of an inner function, then *symmetric* primitive solutions exist. By a symmetric solution, we mean here a solution U such that $U = U^t$ where U^t denotes the transpose of U. Observe that if $\mathcal{J}_G = I^2$ where I is an inner function, then the function $b = IG$ belongs to $\mathcal{H}_{z\varphi}$ and $Cb = b$. Using (29) with $I_1 = -i$ and $I_2 = i$ gives the symmetric solution

$$\begin{pmatrix} a & ib \\ ib & Ca \end{pmatrix}.$$

We sketch now an approach (see [22]) to the Darlington problem for rational data based on C-symmetry. Given a rational function $a(z)$ (not a finite Blaschke product) in H^∞ satisfying $\|a\|_\infty \leq 1$, we may write $a = P/R$ where $P(z)$ is a polynomial relatively prime to a polynomial $R(z)$ of the form (19). We consider only the case $\deg P \leq \deg R$ here, the other case is similar.

The data $a(z)$ belongs to $\mathcal{H}_{z\varphi}$ where φ denotes the finite Blaschke product

$$\varphi(z) = \prod_{k=1}^{N} \frac{z - \lambda_k}{1 - \overline{\lambda_k} z}.$$

We will easily verify that the finite Blaschke product φ is the minimal determinant corresponding to $a(z)$ and that the C operator on $\mathcal{H}_{z\varphi}$ assumes the form $C(F/R) = F^{\neq}/R$ where $\# = z^N\overline{F(1/\overline{z})}$. In particular, we have $\varphi = R^{\neq}/R$ since 1 and φ are conjugate functions in $\mathcal{H}_{z\varphi}$.

By Theorem (2.1.29) and (27) we seek solutions U of the form

$$U = \begin{pmatrix} a & -b \\ Cb & Ca \end{pmatrix}$$

where $\varphi = aCa + bCb$. Let us write $b = I_b G$ and $Cb = I_{Cb} G$ where I_b and I_{Cb} are inner and G denotes the common outer factor of b and Cb. (27) and (28) imply that

$$\mathcal{J}_G G^2 = I_b I_{Cb} G^2 = bCb = \frac{R^{\neq}R - P^{\neq}P}{R^2} \qquad (30)$$

where $\mathcal{J}_G$ denotes the associated inner function for G.

Since G belongs to $\mathcal{H}_{z\varphi}$, we have $G = S/R$ where $S(z)$ is a polynomial of degree $\leq n$. Since $G(z)$ and $R(z)$ are outer, the polynomial $S(z)$ is also outer and thus (30) reduces to

$$\mathcal{J}_G S^2 = R^{\#}R - P^{\#}P$$

where $\mathcal{J}_G$ is a finite Blaschke product (possibly constant). On $\partial\mathbb{D}$ we have

$$\frac{R^{\neq}R - P^{\neq}P}{R^2} = \varphi(1-|a|^2)$$

and hence the roots of $R^{\neq}R - P^{\neq}P$ which lie on $\partial\mathbb{D}$ are exactly the points at which $|a| = 1$. Since the zeroes of $R^{\neq}R - P^{\neq}P$ occur in pairs symmetric with respect to $\partial\mathbb{D}$, the number of zeros of $\mathcal{I}_G$ (counted according to multiplicity) depends on the degree of $R^{\neq}R - P^{\neq}P$ and the number of times (according to multiplicity) that the data function $a(z)$ assumes its maximum possible modulus of one on $\partial\mathbb{D}$. The number of solutions in a primitive solution set, therefore, depends on how many times the data $a(z)$ assumes extreme values. Since the Schur-Cohn algorithm [32] can detect the number of zeroes of a polynomial inside the disk, on its boundary, and outside, we can in principle find the number of solutions in a primitive solution set without explicitly finding the roots of polynomials.

We may factor $R^{\neq}R - P^{\neq}P$ into inner and outer factors without necessarily knowing its zeroes, obtaining S^2 and hence S. This yields the (possibly identical) solutions

$$\begin{pmatrix} P/R & -S/R \\ S^{\neq}/R & P^{\neq}/R \end{pmatrix} \text{ and } \begin{pmatrix} P/R & -S^{\neq}/R \\ S/R & P^{\neq}/R \end{pmatrix}$$

to our problem.

Since $G = S/R$ is an outer function in $\mathcal{H}_{z\varphi}$, we have

$$\hat{G} = \mathcal{I}_G G = \frac{S^{\neq}}{R}$$

and therefore the desired inner function $\mathcal{I}_G$ is given by $\mathcal{I}_G = S^{\neq}/S$. Since S is outer, the zeroes of $\mathcal{I}_G$ are precisely the zeros of $S^{\neq}$ lying in the open unit disk. Once these zeroes have been found, we can complete our primitive solution set since these solutions can be identified with the functions

$$b(z) = I_b G = I_{b\frac{S}{R}}$$

where I_b is an inner divisor of $\mathcal{I}_G$. This yields the following procedure.

Suppose that we are given a rational function (not a finite Blaschke product) $a(z)$ satisfying $\|a\|_\infty \leq 1$.

(i) *Write $a(z) = P(z)/R(z)$ where $R(z)$ has constant term 1 and $P(z)$ is relatively prime to (z) . Let the degrees of P and R be denoted m and n, respectively.*

(ii) *If $m \leq n$, then form the polynomial $R^{\neq}R - P^{\neq}P$ (of degree at most $2n$) using the definition $F^{\neq}(z) = z^n\overline{F(1/\overline{z})}$ for polynomials $F(z)$ of degree $\leq n$.*

(a) *The outer factor of $R^{\neq}R - P^{\neq}P$ is a polynomial S^2 of degree $\leq 2n$. The matrices*

$$\begin{pmatrix} P/R & -S/R \\ S^{\neq}/R & P^{\neq}/R \end{pmatrix} \textit{ and } \begin{pmatrix} P/R & -S^{\neq}/R \\ S/R & P^{\neq}/R \end{pmatrix}$$

are primitive solutions with determinant $\varphi = R^{\neq}/R$.

(b) Find the roots of the polynomial

$$S' := \frac{S^{\neq}}{gcd(S, S\#)}$$

(*of degree $N \leq n$*). These zeroes all lie inside the unit disk.

(c) *For each subset $\{\omega_1, \ldots, \omega_k\}$ of the roots of S' such that $k \leq L\frac{N}{2}\rfloor$,*

$$T(z)=S(z)\prod_{j=1}^{k}\frac{z-\omega_j}{1-\overline{\omega_j}z}$$
is a polynomial of degree $N-k$ yielding the primitive solutions
$$(\tau\#/RP/R\ P^{\neq}/R-T/R)\ \textit{and}\ \begin{pmatrix} P/R & -T^{\neq}/R \\ T/R & P^{\neq}/R \end{pmatrix}.$$
This yields a complete set of primitive solutions with determinant φ.

(iii) *If $m>n$, then form the polynomial $R^{\neq}R-P^{\neq}P$ (of degree at most $2m$) using the definition $F^{\neq}(z)=z^m\overline{F(1/\overline{z})}$ for polynomials $F(z)$ of degree $\le m$. Proceed as in the previous case.*

We now briefly consider invariant subspaces of C-symmetric operators. In particular, we are primarily interested in subspaces that are simultaneously invariant under a C-symmetric operator T and the underlying involution C.

Proposition (2.1.34)[445]: *Let $(\mathcal{H},T,C)$ denote a C-symmetric triple.*

(i) *$\mathcal{M}$ is C-invariant if and only if $\mathcal{M}^{\perp}$* is *C-invariant.*

(ii) *If $\mathcal{M}$ is a subspace of $\mathcal{H}$ that is invariant under C and T, then $\mathcal{M}$ reduces T.*

(iii) *$\mathcal{M}$ reduces T if and only if $C\mathcal{M}$ reduces T.*

(iv) *If $\mathcal{M}$ is a C invariant subspace of $\mathcal{H}$ and P denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, then the compression $A=PTP$ of T to $\mathcal{M}$ satisfies $CA=A^*C$.*

Example (2.1.35)[445]: Consider the C-symmetric triple $(\mathbb{C}^n, J_n(\lambda), C)$ of Example (2.1.2) There are no proper, nontrivial subspaces of $\mathbb{C}^n$ that are simultaneously invariant for both the Jordan block $J:=J_n(\lambda)$ and the involution C. If $\mathcal{M}$ is a nontrivial subspace of $\mathbb{C}^n$ which is J-invariant, then it must contain the vector $(1,0,\dots,0)$. However, $C(1,0,\dots,0)=(0,\dots,0,1)$ and inductively one can see that if $\mathcal{M}$ is also C-invariant, then $\mathcal{M}$ must be all of $\mathbb{C}^n$.

Example (2.1.36)[445]: Consider the C-symmetric triple $(L^2[0,1],V,C)$ It is well known that the only invariant subspaces for the Volterra integration operator are the subspaces $\chi_{[0,\,a)}L^2[0,\,1]$ where $a\in[0,\,1]$ and $\chi_{[0,a)}$ denotes the characteristic function of the interval $[0,a)$. It is clear that there are no proper, nontrivial V- invariant subspaces of $L^2[0,\,1]$ that are also C-invariant.

Example (2.1.37)[445]: We return to the notation of Example (2.1.2) There are no proper nontrivial subspaces of $\mathcal{H}_\varphi$ which are invariant under both C and the backward shift operator B. Restricted to $\mathcal{H}_\varphi$, B is simply the compression to $\mathcal{H}_\varphi$ of the Toeplitz operator $T_{\overline{z}}$ and hence, by Proposition (2.1.18) B is C-symmetric. Suppose that $\mathcal{M}$ is a subspace of $\mathcal{H}_\varphi$ that is invariant for both B and C. Without loss of generality, there exists a function f in $\mathcal{M}$ with a nonconstant outer factor, say F. The function $g:=F+CF$ belongs to $\mathcal{M}$ and satisfies $Cg=g$. Since F and CF share the same outer factor, namely F, the function g itself is outer. However, a self- conjugate outer function in $\mathcal{H}_\varphi$ must generate $\mathcal{H}_\varphi$ by a proposition in [21] (which is a simple restatement of [14, Theorem 3.1.5] in terms of conjugation operators).

Despite these examples, there are many C-symmetric triples $(\mathcal{H},T,C)$ such that $\mathcal{H}$ has subspaces that are invariant for both T and C. If the matrix representation for T with respect to the basis e_n furnished by Lemma (2.1.1) has a diagonal block, then T clearly has a subspace that is simultaneously invariant for T and C. Finally, we note in passing that

Theorem (2.1.15) immediately implies that the antilinear operators CT and TC always admit nontrivial invariant subspaces.

With the analogue of Jordan operators in several complex variables and, in particular, the question whether they can still be C- symmetric. The following observation can produce many examples of product C- symmetric operators.

Lemma (2.1.38)[445]: Let $(\mathcal{H}_1, T_1, C_1)$ and $(\mathcal{H}_2, T_2, C_2)$ *be* C-symmetric triples. Then the hilbertian tensor product $(\mathcal{H}_1 \otimes \mathcal{H}_2, T_1 \otimes T_2, C_1 \otimes C_2)$ is also a C-symmetric triple.

However, the general picture on an analytic model space on the polydisk is more involved. To start with such a product example, let

$$\Phi(z_1, z_2, \dots, z_n) = \varphi_1(z_1)\varphi_2(z_2)\dots\varphi_n(z_n)$$

be a product of inner functions in the respective variables. The function Φ is inner in the polydisk and a standard algebraic argument shows that

$$\mathcal{K}_1 = H^2(D^n) \ominus \sum_{k=1}^{n} \varphi_k H^2(D^n) \cong \otimes_{k=1}^{n}(H^2 \ominus \varphi_k H^2) .$$

The associated product conjugation on this space is

$$Cf(z) = \varphi_1(z_1)\dots\varphi_n(z_n)\overline{[z_1\dots z_n f(z)]},$$

as a direct computation can also verify that Cf is jointly analytic and orthogonal to $\sum_{k=1}^{n} \varphi_k H^2(Dn)$. Let $f \in C(\overline{D^n})$ be a continuous function and let T_f be the Toeplitz operator with symbol f compressed to the space $\mathcal{K}_1$. Since the function f is approximable in the uniform norm by real analytic monomials, the above lemma implies $CT_f = T_f^* C$.

We consider the model space $\mathcal{K}_2 = H^2(D^2) \ominus z_1 z_2 H^2(D2)$. The orthogonal decomposition

$$H^2(D^2) \ominus z_1 z_2 H^2(D^2) \cong H^2(D_1) \oplus z_2 H^2(D2),$$

holds, where the subscripts indicate the corresponding variable. Thus the compressed Toeplitz operator T_{z_1} is unitarily equivalent to the standard unilateral shift T_z and therefore has nontrivial Fredholm index. Consequently, the model space $\mathcal{K}_2$ cannot carry an involution C with respect to which the respective Jordan operators are C-symmetric.

One may ask what properties distinguish the quotient analytic modules $\mathcal{K}_1$ and $\mathcal{K}_2$ so that the compressed Toeplitz operators are C-symmetric on one, but not on the other.

Section (2.2): Advanced Application in Complex Symmetric Operators

In his consideration of the classical Carathéodory-Feje'r problem in function theory, Takagi [115] observed the relevance of the antilinear eigenvalue problem $Tx = \lambda\bar{x}$, where T is an n × n symmetric complex matrix and x denotes complex conjugation of a vector x in $\mathbb{C}^n$. He noted that this equation implies that $T^*Tx = |\lambda|^2 x$ and hence that $|\lambda|$ is an eigenvalue of $|T| = \sqrt{T^*T}$. This observation has many consequences, for example a formula for $\|T\|$ which does not explicitly involve the computation of $|T|$:

$$\|T\| = sup\{\sigma \geq 0 : (\exists x \in \mathbb{C}^n)((x \neq 0) \wedge (Tx = \sigma\bar{x}))\}.$$

We consider Takagi's antilinear eigenproblem in a much more general setting.

We now pass to a separable complex Hilbert space $\mathcal{H}$ which carries a conjugation C. Specifically, C is an antilinear operator $\mathcal{H}$ which is involutive ($C^2 = I$) and isometric, meaning that (x, y) = (Cy, Cx) holds for all x, y in $\mathcal{H}$. A bounded operator T: $\mathcal{H} \to \mathcal{H}$ is called C-symmetric if $T = CT^*C$ and complex symmetric if it is C-symmetric with respect to some conjugate-on C.

In particular, an n× n matrix T is symmetric if and only if $T = CT^*C$ where C denotes the standard conjugation $C(z_1, z_2, \dots, z_n) = (\overline{z_1}, \overline{z_2}, \dots, \overline{z_n})$ on $\mathbb{C}^n$. Thus complex symmetric operators generalize the notion of complex symmetric matrices. In fact, T is C-symmetric if and only if it has a symmetric matrix representation with respect to an orthonormal basis whose elements are fixed by C.

The class of complex symmetric operators is surprisingly large. It includes all normal operators, Hankel operators, compressed Toeplitz operators (including finite Toeplitz matrices and the compressed shift), and many standard integral operators such as the Volterra operator [98]. Somewhat confusingly, the unbounded analogues of C-symmetric operators are sometimes referred to as J-selfadjoint, although neither concept should be confused with the notion of J-selfadjointness arising in the theory of Krein spaces (where J is a linear involution).

Analyze the structure of complex symmetric operators beyondTakagi's decomposition. We prove, for example, that a bounded C-symmetric operator T factors as T = CJ $|T|$,where J is an auxiliary conjugation which commutes with $|T|$. This can be viewed as a generalization of a theorem of Godic and Lucenko which states that every unitary operator U on $\mathcal{H}$ decomposes as the product $\mathrm{U} = \mathrm{CJ}$ of two conjugations [102]. We use the decomposition $\mathrm{T} = \mathrm{CJ}|T|$ to attack Takagi's antilinear eigenvalue problem in a more general setting.

Glazman pioneered the study of unbounded complex symmetric operators [100, 101] and proved that a parallel to von Neumann's theory of selfadjoint extensions of a symmetric operator exists. Specifically, one says that a closed-graph, denselydefined, unbounded operator T is C-symmetric if $\mathrm{T} \subset CT^{*}\mathrm{C}$ and C-selfadjoint if $\mathrm{T} = CT^{*}\mathrm{C}$. In concrete applications, C is typically of the form $|Cf|(\mathrm{x}) = \overline{f(x)}$ or $[Cf](x) = \overline{f(-x)}$ on an appropriate L^2 space. Since Glazman's time, his fundamental ideas have been applied to several classes of differential operators (see [93, 106, 110]). Moreover, the complex scaling technique, a standard tool in the theory of Schrödinger operators, naturally leads to the consideration of C-selfadjoint operators [99].

We show that every unbounded C-selfadjoint operator T with zero in its resolvent admits a decomposition of the form $\mathrm{T} = \mathrm{CJ}\,|T|$, where $|T|$ is positive and selfadjoint (in the usual sense) and J is a conjugation strongly commuting with $|T|$ This establishes a direct connection between C-selfadjoint and operators and leads to a new method of estimating the norm of C-selfadjoint operators with compact resolvent.

If T is an unbounded C-selfadjoint operator which has compact resolvent at zero, then there exists an orthonormal basis u_{n} of $\mathcal{H}$ consisting of solutions to the antilinear eigenvalue problem $Tf = \sigma Cf\ (for\ \sigma \geq 0)$. Moreover, we have the formula

$$\|T\| = \sup\{\sigma \geq 0 : (\exists f \in \mathcal{H})\big((f \neq 0) \wedge (Tf = \sigma Cf)\big)\}.$$

On the other hand, the linear eigenvalue problem $Tf = \lambda f\ (for\ \lambda\ in\ C)$ for the same operator does not in general produce an orthonormal system of eigenfunctions, nor a complete system of them (see [103, 105]). Several applications of this approach, dealing with Schrödinger operators with spectral gaps and the scaled Hamiltonians appearing in the problem of resonances, can be found in [99].

We deal with the abstract structure of complex symmetric operators and briefly explores several basic examples. We discuss Jordan model operators (compressed shifts) and their rank-one unitary perturbations. We devoted to applications to unbounded operators.

We first review a beautiful, yet little-known, result of Godiˇc and Lucenko (Theorem (2.2.1)) on the structure of unitary operators before proving a broad generalization (Theorem (2.2.7)) of their theorem to the class of all complex symmetric operators. The remainder is devoted to various examples and applications. It is well-known that any planar rotation can be obtained as the product of two reflections. The following theorem of Godič and Lucenko [102] generalizes this simple geometric notion and provides an interesting perspective on the structure of unitary operators:

Theorem (2.2.1)[446]: (Godic-Lucenko). If U is a unitary operator on a Hilbert space $\mathcal{H}$, then there exist conjugations C and J on $\mathcal{H}$ such that $U = CJ$.

This theorem is remarkable, for it states that all unitary operators (on a fixed Hilbert space $\mathcal{H}$) can be constructed using essentially the same antilinear operator. Indeed, any conjugation on $\mathcal{H}$ can be represented as entry-by-entry complex conjugation with respect to a certain orthonormal basis (i.e. can be represented as the canonical conjugation on an appropriate l2-space). In this sense, the conjugations $C\ and\ J$ in Theorem (2.2.1) are essentially identical objects. Thus the fine structure of unitary operators arises entirely in how two copies of the same object are put together. The converse of Theorem (2.2.1) is also true:

Lemma (2.2.2)[446]: If $C\ and\ J$ are conjugations on a Hilbert space $\mathcal{H}$, then $U = CJ$ is a unitary operator. Moreover, U is both C-symmetric and J-symmetric.

Proof. If $U = CJ$, then (by the isometric property of $C\ and\ J$) it follows that $\langle f, U^*g\rangle = \langle Uf, g\rangle = \langle CJf, g\rangle = \langle Cg, Jf\rangle = \langle f, JCg\rangle$ for all f, g in $\mathcal{H}$. Thus U^*= JC from which $CU = U^*C$ and $JU = U^*J$ both follow.

Example (2.2.3)[446]: Let $U: \mathbb{C}^n \to \mathbb{C}^n$ be a unitary operator with n (necessarily unimodular) eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$ and corresponding orthonormal eigenvectors $e_1, e_2, \ldots, e_n$. If C and J are defined by setting $C_{ek} = \xi_k e_k$ and $Je_k = e_k$ for $k = 1, 2, \ldots, n$ and extending antilinearly to all of $\mathbb{C}^n$, then clearly $U = CJ$. By introducing offsetting unimodular parameters in the definitions of $C\ and\ J$, one sees that the Godic-Lucenko decomposition of U is not unique.

Example(2.2.4)[446]: If U denote the unitary operator $[Uf](e^{i\theta}) = e^{i\theta} f(e^{i\theta}) on\ L^2(\partial\mathbb{D}, \mu), then\ U = CJ\ where$

$$[Cf](e^{i\theta}) = e^{\frac{1}{2}\theta}\overline{f(e^{\iota\theta})}, [Jf](e^{i\theta}) = e^{-\frac{1}{2}\theta}\overline{f(e^{\iota\theta})}$$

for all f in $L^2(\partial\mathbb{D}, \mu)$. Clearly, the proof of Theorem (2.2.1) follows from the spectral theorem and this simple example.

Example (2.2.5)[446]: Let $\mathcal{H} = L^2(\mathbb{R}, dx)$ and let

$$[Ff](\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi} f(x)dx$$

denote the Fourier transform of a function f in $L^2(\mathbb{R})$. Complex conjugation

$[Jf](x) = \overline{f(x)}$ satisfies $J\mathcal{F}^*$=$J\mathcal{F}$, whence $\mathcal{F}$ is a J-symmetric unitary operator. Thus $C = \mathcal{F} J$ is another conjugation operator on $L^2(\mathbb{R})$. The Fourier transform is thus the product of two simple conjugations: C is a complex conjugation in the frequency domain and J is a complex conjugation in the state space domain.

Example (2.2.6)[446]: Let $\mathcal{H} = L^2(\mathbb{R}, dx)$ and let

$$[J\, f](x) = p.v.\frac{1}{\pi i}\int_R \frac{f(y)}{y-x}$$

denote the (self-adjoint) Hilbert transform of a function f in $L^2(\mathbb{R})$. One can verify that H is C-symmetric with respect to the conjugation $[Cf](x) = \overline{f\,(-x)}$ on $L^2(\mathbb{R})$ and that the conjugation J is given by

$$[J\, f](x) = p.v.\frac{-1}{\pi i}\int_R \frac{\overline{f(y)}}{y+x}$$

Surprisingly, Theorem (2.2.1) has a natural generalization to the entire class of complex symmetric operators. We discuss this result below.

Recall that the polar decomposition $T = \overline{U|T}|$ of an operator T expresses T uniquely as the product of a positive operator $|T| = \sqrt{T^*T}$ and a partial isometry U that satisfies ker $T = ker\ U = ker\ |T|$ and that maps the initial space (ker $|T|$) $^{\perp}$ onto the final space cl($ran\ T$), the closure of the range of T .

If T is a C-symmetric operator, then it turns out that the partial isometry U is also C-symmetric (for the same C). Furthermore, U can be written as the product $U = CJ$ of the original conjugation C and a partial conjugation J which commutes with $|T|$. In the case where T is unitary, this decomposition reduces to the Godic- Lucenko decomposition for unitary operators.

We say that an antilinear operator J is a partial conjugation if J restricts to a conjugation on (ker $J)^{\perp}$ (having values in the same space). In particular, the linear operator J^2 is the orthogonal projection onto the closed subspace ran J= (ker $J)^{\perp}$. Note that a partial conjugation J can always be extended to a conjugation $J\,\bar{J}$ on the entire space $\mathcal{H}$ by forming the internal direct sum $\bar{J} = J \oplus \grave{J}$ where $\grave{J}$ is any partial conjugation with support ker J.

Theorem (2.2.7)[446]: If $T = U|T|$ is the polar decomposition of a C-symmetric operator T, then $T = CJ|T|$ where J is a partial conjugation, supported on $cJ(ran|T|)$, which $|T| = \sqrt{T^*T}$ commutes with. In particular, the partial isometry U is C-symmetric and factors as $U = CJ$.

Proof. Write the polar decomposition $T = U\,|T|$ of T and note that

$$T = CT^*C = C\,|T|U^*C = C(U^*U\,)|T|U^*C = (CU^*C)(CU\ |T|U^*C) \qquad (31)$$

since U^*U is the orthogonal projection onto cl(ran $|T|$). Setting $W = CU^*C$, it follows that $W^* = CUC$ and hence $WW^*W = W$
since $U^*UU^* = U^*$. Thus W is a partial isometry. Since $A\ CU\,T\,U^*C$ is clearly positive, if we can show that ker A = ker $W\,ker\,T$, then the uniqueness of the factors in the polar decomposition of T will allow us to conclude that $W = U$ and $A = |T|$.

Since U and U^* have cl(ran$|T|$) as their initial and final spaces, respectively, it follows that ker $W = ker\,A = ker\,U^*C.$ We claim that $ker\,T = ker\,U^*C.$ Clearly $ker\,U^*\,C \subseteq ker\,T$ by (31). Conversely, if $Tf = 0$, then (31) implies that $|T|U^*Cf = 0$. Since the final space of U^* is c(ran $|T|$), we must have U^* Cf = 0 and hence $ker\ T = ker\ U^*\ C$. This proves that $U = W\ and\ |T| = A$.

The equality $U = CU^*\ C$ shows that U is C-symmetric. Writing $J = CU = U^*\ C$, we see that $J^2 = (U^*\ C)(CU\ ) = U^*\ U$, the orthogonal projection onto cl(ran |T |). Since $CU\ |T|U^*C = |T|$, it follows that $J|T||J = |T|$ and hence $J|T\ | = |T|J$.

From $J = CU$, it follows that ker $J = ker\ U = ker|T| = (cl(ran\ |T|))^{\perp}$. Since $J = U^*C$, it follows that ran $J = ran\ U^* = cl(ran\ |T|)$. Finally, J is clearly isometric on cl(ran $|T|$) since CU is isometric there. Thus J is a partial conjugation supported on cl(ran $|T|$) which commutes with $|T|$.This concludes the proof. Theorem (2.2.7) provides a simple scheme for constructing complex symmetric operators. Fix a conjugation C, then select a positive bounded operator A and a conjugation J commuting with it. Many such J exist, for they can be obtained from the spectral representation of A as a multiplication operator on a direct sum of Lebesgue spaces. It is easy to verify that the $T = CJA$ is C-symmetric and satisfies $|T| = A$. Finally, we remark that given two conjugations C and Cj, the map $T \mapsto \grave{C}CT$ establishes a bijection between the class of C-symmetric and Cj-symmetric operators.

Using Theorem (2.2.7), we can also obtain several strong statements about complex symmetric operators. For instance, it turns out that the partial isometry in the polar decomposition of T can always be extended to a unitary operator:

Corollary (2.2.8)[446]: If T is a C-symmetric operator, then T = W$|T|$ where W is a C-symmetric unitary operator.

Proof. If $cl(ran\ |T\ |) = \mathcal{H}$, then J is a conjugation on all of $\mathcal{H}$ and $U = CJ$ is already a C-symmetric unitary operator. Otherwise, write $T = CJ|T\ |$ and extend J to a conjugation $\bar{J}$ on all of $\mathcal{H}$ using the remarks preceding Theorem (2.2.7). By Lemma (2.2.2), the operator $W \Rightarrow C\bar{J}$ is C-symmetric and unitary.

Corollary (2.2.9)[446]: If T is a complex symmetric operator, then T is invertible if and only if its modulus $|T| = (T^*T\ )^{1/2}$ is invertible.

Proof. This follows immediately from the preceding corollary.

Corollary (2.2.10)[446]: If T is a complex symmetric operator, then $T^*\ T$ and TT^* are unitarily equivalent.

Proof. If T is C-symmetric, then write $T = CJ\ |T|$ where, without loss of generality, we assume that J is a conjugation on all of $\mathcal{H}$. Since J commutes with $|T|$, it also commutes with $|T|^2 = T^*$T . Therefore $CJ(T^*T\ ) = CT^*TJ = TCTJ = (TT^*)CJ$. By Lemma (2.2.2), CJ is unitary and thus T^*T and TT^* are unitarily equivalent.

The unitary equivalence of T^*T and TT^* is necessary, but not sufficient to imply the existence of a conjugation C with respect to which T is C-symmetric. Indeed, if T is any operator on a finite dimensional Hilbert space, then $\sqrt{T^*T}$ and $\sqrt{TT^*}$ are unitarily equivalent. Nevertheless, there exist operators on $\mathbb{C}^3$ which fail to be C-symmetric for any choice of a conjugation C (see [98]).

In the infinite-dimensional setting, it is easily seen that the preceding three corollaries are not true without the assumption that T is complex symmetric. Indeed, the unilateral shift provides immediate counterexamples to all three such assertions. The unilateral shift forms the basis of the following example:

Example (2.2.11)[446]: Let S denote the unilateral shift $S(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$ on $\mathcal{H} = \iota^2(\mathbb{N})$ both S and its adjoint S^* $(a_0, a_1 \ldots) = (a_1, a_2, \ldots)$ commute with the canonical conjugation $C(a_0, a_1, \ldots) = (\overline{a_0}, \overline{a_1}, \ldots)$ on $\mathcal{H}$. The operator $T = S^* \oplus S$ on $\mathcal{H} \oplus \mathcal{H}$ is C-symmetric with respect to the conjugation

$$C = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$, and a computation shows that

$$\underbrace{\begin{pmatrix} S^* & 0 \\ 0 & S \end{pmatrix}}_{T} = \underbrace{\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 0 & CS \\ CS^* & 0 \end{pmatrix}}_{\mathcal{J}} \underbrace{\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}}_{|T|}$$

where P denotes the orthogonal projection $P\,(a_0, a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$. In particular, the partial isometry $u = C\,\mathcal{J}$ in the polar decomposition of $\mathcal{T}$ is simply $\mathcal{T}$ itself. It is easy to check that $\mathcal{J}$ is a partial conjugation supported on ran |T| commuting with $|T| = P \oplus I$. In fact,

$$\mathcal{J}\begin{pmatrix} a_0, a_1, \ldots. \\ b_0, b_1, \ldots. \end{pmatrix} = \begin{pmatrix} 0, \bar{b}_0, \bar{b}_1, \ldots. \\ \bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots. \end{pmatrix}$$

From here it is easy to see how to extend $\mathcal{J}$ to a conjugation $\bar{\mathcal{J}}$ on all of $\mathcal{H} \oplus \mathcal{H}$:

$$\bar{\mathcal{J}}\begin{pmatrix} a_0, a_1, \ldots. \\ b_0, b_1, \ldots. \end{pmatrix} = \begin{pmatrix} 0, \bar{b}_0, \bar{b}_1, \ldots. \\ \bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots. \end{pmatrix}$$

Moreover, the operator $W = C\bar{\mathcal{J}}$ from Corollary (2.2.8) is clearly unitary:

$$W\begin{pmatrix} a_0, a_1, \ldots. \\ b_0, b_1, \ldots. \end{pmatrix} = \begin{pmatrix} a_0, b_0, b_1, \ldots. \\ a_1, a_2, a_3, \ldots. \end{pmatrix}$$

Using the decomposition $T = CJ\,|T|$ of Theorem (2.2.7), one can prove many results about compact C-symmetric operators. For instance, the following theorem shows that they have special singular-value (or Schmidt) decompositions. Without loss of generality, we consider the case $\dim \mathcal{H} = \infty$.

Theorem (2.2.12)[446]: *Every compact C-symmetric operator T is of the for*

$$\mathrm{T} = \sum_{n=0}^{\infty} \sigma n (C_{e_n} \otimes e_n) \tag{32}$$

where the e_n are certain orthonormal eigenvectors of $|T| = \sqrt{T^*T}$ and the σ_n are the nonzero eigenvalues of $|T|$, repeated according to multiplicity.

Proof. Since T is compact, the mutually orthogonal eigenspaces ε_n of $|T|$ corresponding to the *distinct* nonzero eigenvalues λ_n are finite dimensional, say of dimension d_n. Let $0 \le n < N$, where N is finite if T is of finite rank, or set

$N = \infty$ otherwise. By Theorem (2.2.7), we may write $CT = J|T|$ where J is a partial conjugation supported on cl(ran $|T|$) commuting with $|T|$. In particular, J restricts to a conjugation on each spectral subspace ε_n of $|T|$ and hence (see [98, Lemma (2.2.2)] or [91, p.94]) there exists an orthonormal basis $u_{n1}, u_{n2}, \dots, u_{nd_n}$ for E_n which is fixed by J. In other words, we have $CTu_{nk} = \lambda_n u_{nk}$ for $k = 1, 2, \dots, d_n$ which shows that the operator

$$T - \sum_{n=0}^{N-1} \lambda_n \sum_{k=1}^{dn} (Cu_{nk} \otimes u_{nk}) \tag{33}$$

vanishes on cl(ran T) = (ker $|T|$)$^{\perp}$. Since ker T = ker $|T|$, it follows that (33) vanishes identically. Convergence is guaranteed since the u_{nk} are orthonormal and λ_n tends to 0. The desired representation (32) follows upon a suitable relabeling of terms.

Corollary (2.2.13)[446]: *If T is a compact C-symmetric operator, then*

$$\|T\| = \sup\{\sigma \geq 0: (\exists f)((f \neq 0)\wedge(Tf = \sigma Cf))\}.$$

A famous theorem of Adamyan, Arov, and Krein (AAK) states that if T is compact Hankel operator, then its singular values $\sigma_0, \sigma_1, \dots$, repeated according to multiplicity, are given by

$$\sigma_n = \inf_{\substack{rank\, T' = n \\ T' \text{Hankel}}} \|T - T'\|$$

An analogous theorem holds for the class of C-symmetric operators:

Theorem (2.2.14)[446]: (C-symmetric AAK). *If T is a compact C-symmetric operator with singular values $\sigma_0, \sigma_1, \dots$, repeated according to multiplicity, then*

$$\sigma_n = \inf_{\substack{rank\, T' = n \\ T'\ C-symmetric}} \|T - T'\|$$

Proof . Write $T = CJ|T|$ by Theorem (2.2.7), and using the method of proof of Theorem (2.2.12), write $T = \sum_{k=0}^{\infty} \sigma_k e_k \otimes e_k$ where $J_{e_k} = e_k$ for all k. Let $A = 0$ and $A = \sum_{k=0}^{\infty} \sigma_k (e_e \otimes e_k)$ for n$\geq$ 1 and note that $T' = CJA_n$ satisfies

$$\|T - T'\| = \|CJ|T| - CJA_n\| = \|CJ|T| - A_n\| = \||T| - A_n\| = \sigma_n$$

The operator T' has rank n and (since J commutes with A_n) is C-symmetric by the comments following Theorem (2.2.7).

Among the simplest examples of compact complex symmetric operators are certain integral operators. If (X, μ) is a σ-finite measure space (with μ real-valued), then a set function Φ is called a measure-preserving symmetry of X if $\mu \circ \emptyset = \mu$ and $\emptyset^2 = I$. With a slight abuse of notation, each measure-preserving symmetry Φ provides a conjugation on $L^2(X, \mu)$ via the formula $[Cf] = \overline{f(\emptyset(x))}$. The proof of the following lemma is straightforward.

Lemma (2.2.15)[446]: *A bounded integral operator of the form*

$$[Tf] = \int_{x} K(x,y)f(y)d\mu(y)$$

on $L^2(X,\mu)$ *is C-symmetric with respect to [C f](x) =* $\overline{f\ (\Phi(x))}$ *if and only if the kernel satisfies* $K(\emptyset x, \emptyset y) = k(y,x)\ for\ all\ \ x,y \in X.$

The Volterra operator illustrates many of the concepts developed above. More- over, it demonstrates how the *C*-symmetry of an integral operator is related to functional equations satisfied by its kernel and the measure theoretic symmetries the underlying measure space. It also illustrates the special singular value decomposition (Theorem (2.2.12)) of a compact complex symmetric operator and its relationship to the double Fourier expansion of the integral kernel. A more traditional analysis of the Volterra operator can be found in [104, Problem 188].

Example (2.2.16)[446]: Consider the Volterra integration operator

$$[Tf](x) = \int_0^x f(t)dt$$

On $L^2[0,1]$,which is *C*-symmetric with respect to $[Cf](x) = \overline{f(1-x)}$(see [98]). Indeed, Lemma (2.2.15) says that we can read this directly from the functional equation $K(x,y) = k(1-y, 1-x)$, satisfied by the integral kernel, the characteristic function of the triangle $\{(x, y) : 0 \leq y \leq x \leq 1\}$.

Since ker *T* is obviously trivial, by Theorem (2.2.7) we may write $T = CJ|T|$ where J is a conjugation on $L^2[0,1]$ which commutes with $|T|$ and its spectral projections. Since each spectral subspace of $|T|$ has an orthonormal basis fixed by J, to diagonalize $|T|$ we consider the antilinear equation $|T|f = \sigma Jf$, where $\sigma \geq 0$. In light of the decomposition $T = CJ|T|$, this is equivalent to $Tf = \sigma Cf$:

$$\int_0^y f(y)dy = \sigma\overline{f(1-x)} \qquad (34)$$

The preceding equation yields the boundary condition $f(1) = 0$. Dierentiation of (34) yields $f(x) = -\sigma\overline{f'(1-x)}$ and hence (after back-substitution)

$$\int_0^x f(y)dy \quad = -\sigma^2 \grave{f}'(x) \qquad (35)$$

giving the second boundary condition $f'(0) = 0$. Dierentiation of (35) provides the second order boundary value problem

$$f'' + \frac{1}{\sigma^2} f = 0, f(1) = 0, f'(0) = 0.$$

Solving the boundary value problem yields $\sigma_n = [(n+\frac{1}{2})\pi]^{-1}$ of $|T|$ and the associated normalized eigenfunctions $\sqrt{2}\,cos\left(n + \frac{1}{2}\right)\pi x$ ($where$ $n \geq 0$). To satisfy (34), we multiply these eigenfunctions by suitable unimodular constants, obtaining the unit eigenfunctions $e_n(x) = i^n\sqrt{2}\cos(n + \frac{1}{2})\pi x$ of $|T|$, all of which are fixed by the J conjugation.

Applying C, we obtain $[Ce_n](x) = (-i)^n\sqrt{2}\sin(n + \frac{1}{2})\pi x$ and hence (by theorem (2.2.12)) The singular value decomposition

$$T = \sum_{n=0}^{\infty} \frac{2}{(n+\frac{1}{2})} \left[\sin\left(\pi\left(n+\frac{1}{2}\right)x\right) \otimes cos\left(\pi\left(n+\frac{1}{2}\right)x\right)\right] \qquad (36)$$

of the Volterra operator. From (36), we immediately read the numerical quantities $\| T \| = 2/\pi$ and tr $T^*T = 1/2$. Writing (36) explicitly, we find that

$$[Tf](x) = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{2}{\left(n + \frac{1}{2}\right)\pi} [\sin(\pi(n + {}^1)x)\cos(\pi(n + \frac{1}{2})x)\right] f(y)dy$$

the term in brackets being a double Fourier expansion of the Volterra kernel The next example is slightly more involved, dealing with the classical two dimensional version of the double layer potential, written in complex coordina

In the potential theory of a simply connected planar domain Ω with piecewise smooth boundary, the operator

$$T_\Omega f(z) = \frac{1}{\pi} \int_\Omega \frac{f(w)dA(w)}{(\bar{w} - \bar{z})^2}$$

defined for f in $L^2(\Omega, dA)$, plays a significant role ([114]). Here dA stands for area measure and the integral is taken as a Cauchy principal value. If $[Cf](z) = \overline{f\ (z)}$ denotes complex conjugation of a function f of $L^2(\Omega)$ (we henceforth suppress the *dA*), then clearly T_Ω is a *C*-symmetric operator.

The case $\Omega = \mathbb{C}$ is particularly important. Some simple manipulations with single and double layer potentials carried by $\partial\Omega$ (or any closed curve) reveal that $C\,T_{\mathbb{C}}$ is a conjugation on $L^2(\mathbb{C})$ [114]. Returning to our formalism, we infer that $T_{\mathbb{C}}^* T_{\mathbb{C}} = CT_{\mathbb{C}}CT_{\mathbb{C}} = I$. In other words, TC is a C-symmetric unitary operator. We may therefore write $T_{\mathbb{C}} = CJ$, where the conjugation

$$[Jf](z) = \frac{1}{\pi} \int_\Omega \frac{\overline{f(w)}dA(w)}{(\bar{w} - \bar{z})^2}$$

on $L^2(\mathbb{C})$ is called by Schiffer the *Hilbert transform* of ff .
In general, T_Ω is a compression of $T_{\mathbb{C}}$ to the subspace $L^2(\Omega)$ of $L^2(\mathbb{C})$. Indeed, if P_Ω denotes the orthogonal projection from of $L^2(\mathbb{C})$. onto $L^2(\Omega)$:

$$p_\Omega f(z) = \begin{cases} f(z), & z \in \Omega \\ 0, & z \in \mathbb{C}/\Omega \end{cases}$$

then $T_\Omega = P_\Omega T_{\mathbb{C}} P_\Omega$ (with a slight abuse of notation). Moreover, the commutativity of C and P_Ω implies that T_Ω is a C-symmetric operator.

$Let\ L^2(\Omega)\ denote\ the\ Bergman\ space\ of\ \Omega, the\ subspace\ of\ all$ holomorphic functions in $L^2(\Omega)$, and let P denote the orthogonal projection of $L^2(\Omega)$,onto $L^2(\Omega)$,otherwise known as the *Bergman projection*. A short computation shows that $\acute{P} = CPC$ is the orthogonal projection onto the subspace $CL^2(\Omega)$ which consists of all *anti* -analytic functions in $L^2(\Omega)$.

It turns out that the operator $T = \acute{P}\,T_\Omega P$, which one can regard as an operator from $L_a^2(\Omega)$ to $CL_a^2(\Omega)$, is C-symmetric:

$$CT = C(\acute{P}\,T_\Omega P) = C(CPC)T_\Omega P = PCT_\Omega P = PT_\Omega^*CP = (PT_\Omega^*\acute{P})C = T^*C.$$

Using the C-symmetry of $\acute{P}\,T_\Omega P$, we obtain the following Hilbert variant of a series of observations due to Bergman and Schiffer:

Theorem (2.2.17)[446]: If Ω is a bounded planar domain with C^2 boundary, then there exists an orthonormal basis $(u_n)_{n=0}^{\infty}$ of the Bergman space $L_a^2(\Omega)$ and a sequence $(\sigma_n)_{n=0}^{\infty}$ of positive numbers such that:

$$\frac{1}{\pi}\int_{\Omega} \frac{u_n(w)dA(w)}{(\bar{w}-\bar{z})^2} = \sigma_n\overline{u_n(z)} \tag{37}$$

for all z in Ω.

Proof. The operator T_Ω is compact (see [92], specifically the analysis of the L-kernel) and hence so is $T = \acute{P}\, T_\Omega P$. Since T is supported on $L_a^2(\Omega)$, the result follows from Theorem (2.2.12).

The values σ_n for which (37) is solvable are known as the *Fredholm eigenvalues* of Ω, and the associated eigenfunctions u_n (canonically attached by (37) to any bounded planar domain) are remarkable in many respects. For instance they simultaneously diagonalize the Bergman kernel K_Ω and the L-kernel $L(\Omega)$ of the domain:

$$K_\Omega(z,w) = -\frac{2}{\pi}\frac{\partial 2G(z,w)}{\partial z\partial w} = \sum_{n=0}^{\infty} u_n(z)\overline{u_n(w)}\,, z, w \in \Omega,$$

and

$$L_\Omega(z,w) = -\frac{2}{\pi}\frac{\partial 2G(z,w)}{\partial z\partial w} = \sum_{n=0}^{\infty} \sigma_n u_n(z)u_n(w), z, w \in \Omega,$$

where $G(z,w)$ is the Green function of Ω (see [92, 114]).

As an extended example, we briefly discuss the decompositions $T = CJ\,|T|$(of Theorem (2.2.7)) for the standard Jordan model operators and consider the Godič-Lucenko decompositions (Theorem (2.2.1)) of their rank-one unitary perturbations. Complete details, including all computations, can be found in [97].

We work here in the Hardy space H^2 of the unit disk $\mathbb{D}$, and we freely identify functions in H^2 with their nontangential boundary values which exist a.e. on the unit circle $\partial\,\mathbb{D}$. Most of the following preliminary material can be found in [95, 107] or the more specialized [108].

The interest lies in the so-called model spaces $H^2 \ominus \varphi H^2$, where φ denotes a nonconstant inner function. There is a natural interplay between function theory and operator theory on the spaces $H^2 \ominus \varphi H^2$ for they are examples of reproducing kernel Hilbert spaces. Indeed, it is not hard to derive from the standard properties of the *Szegö kernel* $e\lambda(z) = (1 - \overline{\lambda}_z)^{-1}$ and the definition of $H^2 \ominus \varphi H^2$ that the formula $f(\lambda) = \langle f, K_\lambda\rangle$ holds for every f in $H^2 \ominus \varphi H^2$. Here K_λ denotes the *reproducing kernel*

$$K_\lambda(z) = \frac{1-\emptyset(\lambda)\emptyset(z)}{1-\overline{\lambda_z}} \tag{38}$$

For $H^2 \ominus \varphi H^2$

Recall that Toeplitz *operator* with *symbol* u in L^∞ $(\partial\,\mathbb{D})$ is the operator $T_u : H^2 \longrightarrow H^2$ defined by $T_u f = P\,(uf\,)$ where P denotes the orthogonal projection from L^2 onto H^2. Also recall that the adjoint of a Toeplitz operator is given by the simple formula $T_{\bar{u}}^* = T_u$.

A compressed Toeplitz operator is an operator of the form $P_\varphi T_u P_\varphi$ where T_u is *a* standard Toeplitz operator and P_φ *Toeplitz* operator and P_φ denotes the orthogonal projection from H^2 onto $H^2 \ominus \varphi H^2$. With a slight abuse of notation, we will regard compressed Toeplitz operators as operators acting on the space $H^2 \ominus \varphi H^2$, rather than H^2 itself. It turns out that compressed Toeplitz operators are complex symmetric operators with respect to the conjugation

$$[Cf](z) = fz\varphi \tag{39}$$

on $H^2 \ominus \varphi H^2$ [97, 98]:

If φ *is* a nonconstant inner function and u belongs to $L^\infty(\partial\mathbb{D})$, then the compressed Toeplitz operator $P_\varphi T_u P_\varphi$ is *C-symmetric* with respect to the conjugation (39) on $H^2 \ominus \varphi H^2$.

Although $\overline{fz\varphi}$ does not at first appear to be the boundary function of an analytic function, let alone one in $H^2 \ominus \varphi H^2$, it is not hard to verify. Indeed, it succes to check that both $(Cf, \overline{zh})$ and $(Cf, \varphi\, h)$ vanish whenever f belongs to $H^2 \ominus \varphi H^2$.and h belongs to H^2.

We obtain the refined polar decomposition for the compressed shift (or Jordan operator) guaranteed by Theorem (2.2.7) In fact, we are able to consider a slight generalization of the Jordan model operator with little additional effort.

In our computations, we will make frequent use of disk automorphisms and we adopt the following notation. For each w in D, we let b_w denote the function

$$b_w(z) = \frac{z - w}{1 - \overline{w}z}. \tag{40}$$

We also require the reproducing kernels K_w (38) and their conjugates under (39) :

$$[CK_w](z) = \frac{\varphi(z) - \varphi(w)}{z - w}. \tag{41}$$

Furthermore, we frequently refer to the normalized kernel functions $k_w = K_w/\|K_w\|$ For each λ in $\mathbb{D}$, we consider the compression

$$S_\lambda f = P_\varphi(\lambda_\lambda f) \tag{42}$$

of the analytic Toeplitz operator T_{b_λ} to $\mathcal{H}_\varphi$. The operators S_λ are simple generalizations of the compressed shift $S_0 f = P_\varphi(zf)$. We also remark that $S_\lambda^* f = P_\varphi(\overline{b_\lambda} f)$ and that the operators S_λ are C-symmetric with respect to (39).

We explicitly describe the factorization $S_\lambda = CJ_\lambda|S_\lambda|$ of these operators. We first require several computational lemmas, the first of which generalizes [94, Lem. 2.1]. Detailed proofs can be found in [97].

Lemma (2.2.18)[446]: $S_\lambda^* f = f/b_\lambda$ *if and only if* f *is orthogonal to* k_λ. $S_\lambda f = b_\lambda f$ *if and only if* f *is orthogonal to* Ck_λ.

To find the modulus $|S_\lambda|$ of S_λ, we need only describe the positive operator $S_\lambda^* S_\lambda$. By Lemma (2.2.12) it follows that if f is orthogonal to Ck_λ, then $S_\lambda^* S_\lambda f = S_\lambda^*(b_\lambda f) = f$. Hence $|S_\lambda|$ restricts to the identity operator on the orthocomplement of the one dimensional subspace spanned by the function Ck_λ. This tells us, for example, that $|S_\lambda|$ maps the function Ck_λ onto a nonnegative constant multiple of itself. In fact:

Lemma (2.2.19)[446]: $S_\lambda Ck_\lambda = -\varphi(\lambda)k_\lambda$ *and hence* $|S_\lambda|Ck_\lambda = |\varphi(\lambda)|Ck_\lambda$.

Summing up, the modulus $|S_\lambda|$ of S_λ is given by:

$$|S_\lambda| = [I - (Ck_\lambda \otimes Ck_\lambda)] + |\varphi(\lambda)|(Ck_\lambda \otimes Ck_\lambda). \tag{43}$$

In light of (43) and Lemma (2.2.18) we assume that $\varphi(\lambda) \neq 0$ since otherwise the polar decomposition of S_λ is already evident. Indeed, if $\varphi(\lambda) = 0$, then $kerS_\lambda$ equals the one-dimensional subspace spanned by Ck_λ and the operator S_λ acts isometrically (multiplication by b_λ) on the and the operator acts isometrically (multiplication by) on the orthocomplement of this subspace.
We may write $S_\lambda = CJ_\lambda|S_\lambda|$ where J_λ is a partial conjugation supported on cl(ran $|S_\lambda|$) $= H^2 \ominus \varphi H^2$ which commutes with $|S_\lambda|$. In particular, we see that the assumption that $\varphi(\lambda) \neq 0$ implies that J_λ is a conjugation on all of $H^2 \ominus \varphi H^2$. To find J_λ, we write

$$J_\lambda|S_\lambda| = CS_\lambda \tag{44}$$

and compute the action of J_λ on the spectral subspaces of $|S_\lambda|$.

If f is orthogonal to Ck_λ, then $|S_\lambda|f = f$ by (43) and hence $J_\lambda f = CS_\lambda f = C(b_\lambda f)$ by (44) and Lemma (2.2.18) Since $\varphi(\lambda) \neq 0$ we have

$$|\varphi(\lambda)|J(Ck_\lambda) = J|S_\lambda|(Ck_\lambda) = C(S_\lambda Ck_\lambda) = -\overline{\varphi(\lambda)}Ck_\lambda,$$

the two equalities following from (44) and Lemma (2.2.19) respectively. Putting these calculations together, we have the following explicit formula for J_λ:

$$J_\lambda f = \begin{cases} C(b_\lambda f), & f \perp Ck_\lambda, \\ \overline{\alpha}Ck_\lambda, & f = Ck_\lambda \end{cases} \tag{45}$$

where $\alpha = -\varphi(\lambda)/|\varphi(\lambda)|$.

We can now compute the partial isometry $U_\lambda = CJ_\lambda$ in the polar decomposition of S_λ using (45). By our assumption that $\varphi(\lambda) \neq 0$, U_λ is actually unitary, since C and J_λ are both conjugations on $H^2 \ominus \varphi H^2$. Applying C to (45) yields

$$U_\lambda f = \begin{cases} b_\lambda f, & f \perp Ck_\lambda, \\ \alpha k_\lambda, & f = Ck_\lambda, \end{cases}$$

and hence (using Lemma (2.2.18) U_λ is given by the formula

$$U_\lambda = S_\lambda[I - (Ck_\lambda \otimes Ck_\lambda)] + \alpha(k_\lambda \otimes Ck_\lambda). \tag{46}$$

We can see directly that U_λ is C-symmetric, for a short computation shows that U_λ is a rank-one *C-symmetric* unitary perturbation of S_λ:

$$U_\lambda = S_\lambda + \big(\alpha + \varphi(\lambda)\big)(k_\lambda \otimes Ck_\lambda). \tag{47}$$

We summarize our results in the following theorem:

Theorem (2.2.20)[446]: *Let φ be a nonconstant inner function and let λ be a point in $\mathbb{D}$ such that $\varphi(\lambda) \neq 0$. The polar decomposition of the compressed Toeplitz operator $S_\lambda f = P_\varphi(b_\lambda f)$ is given by $S_\lambda = U_\lambda|S_\lambda|$ where U_λ is the C-symmetric unitary operator* (47) *and $|S_\lambda|$ is given by* (43). *Moreover, $U_\lambda = CJ_\lambda$ where the conjugation J_λ is given by* (45).

The operator U_λ defined by (47) is *not* the only rank-one C-symmetric unitary perturbation of S_λ. Indeed, for any unimodular constant α, the operator

$$U_{\lambda,\alpha} = S_\lambda + \big(\alpha + \varphi(\lambda)\big)(k_\lambda \otimes Ck_\lambda) \tag{48}$$

is C-symmetric and unitary, regardless of whether the inner function φ vanishes at λ. This can be seen by expressing $U_{\lambda,\alpha}$ in a form analogous to (46) and applying the lemmas of the preceding.

We refer to operators of the form (48) as *generalized Aleksandrov-Clark operators* due to their similarity to the operators considered by Clark in [94] and later by A. B.

Aleksandrov and others (see (49) for background). That each $U_{\lambda,\alpha}$ has the Godič-Lucenko decomposition

$$U_{\lambda,\alpha} = CJ_{\lambda,\alpha}$$

where the conjugation $J_{\lambda,\alpha}$ is given by

$$J_{\lambda,\alpha}f = \begin{cases} C(b_\lambda f), & f \perp Ck_\lambda, \\ \overline{\alpha}Ck_\lambda, & f = Ck_\lambda, \end{cases} \tag{49}$$

the parameter α now being allowed to vary over the unit circle. This decomposition makes it easy to compute the eigenvalues and eigenvectors (if any) of each $U_{\lambda,\alpha}$. A function f is an eigenvector of $U_{\lambda,\alpha}$ corresponding to the (necessarily unimod- ular) eigenvalue ξ if and only if

$$J_{\lambda,\alpha}f = \overline{\xi}Cf. \tag{50}$$

In light of the explicit formula (49) for $J_{\lambda,\alpha}$, we take the orthogonal decomposition of f with respect to the one-dimensional subspace spanned by Ck_λ. After possibly multiplying by a constant, we may assume that f is of the form $f = g + CK_\lambda$ where g is orthogonal to CK_λ. Substituting this into (50) we deduce that

$$J_\lambda(g + CK_\lambda) = \overline{\xi}(Cg + K_\lambda)$$

By (49), this can be written

$$C(b_\lambda g) + \overline{\alpha}CK_\lambda = \overline{\xi}Cg + \overline{\xi}K_\lambda.$$

Applying C to the equation and solving for g gives us

$$g = \frac{\xi CK_\lambda - \alpha K_\lambda}{b_\lambda - \xi}.$$

Using the explicit formulas (38) and [101] for K_λ and Ck_λ we find (see [97]) that f is a constant multiple of the function

$$f_\xi(z) := \frac{1 - b_{-\varphi(\lambda)}(\alpha)\varphi(z)}{1 - \overline{b_{-\lambda}(\xi)}z} \tag{51}$$

where b_w denotes the generic disk automorphism (40). Conversely, we see that if ξ is a unimodular constant such that f_ξ belongs to H^2, then f_ξ is an eigenvector of $U_{\lambda,\alpha}$ corresponding to the eigenvalue ξ. Moreover, the computation above shows that the eigenspaces of $U_{\lambda,\alpha}$ are one-dimensional.

A necessary condition for a function of the form (51) to belong to H^2 is that φ have the nontangential limiting value $b_{-\varphi(\lambda)}(\alpha)$ at the point $b_{-\lambda}(\xi)$. In other words, the condition

$$\varphi\left(\frac{\xi + \lambda}{1 + \overline{\lambda}\xi}\right) = \frac{\alpha + \varphi(\lambda)}{1 + \overline{\varphi(\lambda)}\alpha} \tag{52}$$

is necessary for f_ξ to be an eigenvector of $U_{\lambda,\alpha}$ corresponding to the eigenvalue ξ. In general, this condition is not sufficient and we must examine the angular derivative (most easily via the local Dirichlet integral (52) of φ at the point $b_{-\lambda}(\xi)$. We do not wish to pursue the function theoretic details here and simply remark that [112 generalizes , Thm. 3.2].

The following lemma shows that we may select a unit vector, fixed by C, from each of the (necessarily one-dimensional) eigenspaces of $U_{\lambda,\alpha}$:

Lemma (2.2.21)[446]: *If T is a normal C-symmetric operator, then the eigenspaces of T are fixed by C.*

Proof. By normality, $Tf = \lambda f$ implies that $T^*f = \overline{\lambda}f$. Applying C to the preceding gives $T(Cf) = \lambda(Cf)$ and thus the eigenspaces of T are invariant under C.

In summary, if λ and α are values (in $\mathbb{D}$ and on $\partial\mathbb{D}$, respectively) such that the operator $U_{\lambda,\alpha}$ has a pure point spectrum, then we can construct an orthonormal basis of $H^2 \ominus \varphi H^2$ consisting of self-conjugate vectors. In particular, the matrix representation of any C-symmetric operator with respect to such a basis will be symmetric. Conditions which ensure that $U_{\lambda,\alpha}$ has a pure point spectrum can be obtained by suitably generalizing several theorems in [94].

Let $T: \mathcal{D}(T) \to \mathcal{H}$ be a closed graph, densely defined linear operator acting on a complex Hilbert space $\mathcal{H}$ and let C be a conjugation on $\mathcal{H}$. Such an operator is called C-*symmetric* if $T \subset CT^*C$ or, equivalently, if

$$\langle CTf, g\rangle = \langle CTg, f\rangle \qquad (53)$$

for all f, g in $\mathcal{D}(T)$. We say that an operator T is C-*selfadjoint* if $T = CT^*C$ (in particular, a bounded C-symmetric operator is C-selfadjoint). Unbounded C- selfadjoint operators are sometimes called J-*selfadjoint*, although this should not be confused with the notion of J-selfadjointness in the theory of Krein spaces.

In contrast to the classical extension theory of von Neumann, it turns out that a C-symmetric operator always has a C-selfadjoint extension [100,101] (see also [96,110] . Indeed, the maximal *antilinear* symmetric operators S (in the sense that $\{Sf, g\} = \{Sg, f\}$ for all f, g in $\mathcal{D}(S)$) produce C-selfadjoint operators CS. Because of this, we use the term *complex symmetric operator* freely in both the bounded and unbounded situations when we are not explicit about the conjugation C. Much of this theory was developed by Glazman, whose early book [101] remains unsurpassed for its depth and elegance.

In concrete applications, C is typically derived from complex conjugation on an appropriate L^2 space over a domain in $\mathbb{R}^n$ and T is a particular non-selfadjoint differential operator. For instance, the articles [96,110] contain a careful analysis and parametrization of boundary conditions for Sturm-Liouville type operators with complex potentials which define C-selfadjoint operators. Such operators also arise in studies related to Dirac-type operators The complex scaling technique, a standard tool in the theory of Schrödinger operators, also leads to the consideration of C-selfadjoint operators [99] and the related class of C-*unitary* operators [113].

A useful criterion for C-selfadjointness can be deduced from the equality

$$\mathcal{D}(CT^*C) = \mathcal{D}(T) \oplus \{f \in \mathcal{D}(T^*CT^*C) : \ T^*CT^*Cf + f = 0\}$$

(see[110]). A different criterion goes back to Zhikhar [116]: if the C-symmetric operator T satisfies $\mathcal{H} = (T - zI)\mathcal{D}(T)$ for some complex number z, then T is C-selfadjoint. The resolvent set of T consists of exactly the points z fulfilling the latter condition. We denote the inverse to the right by $(T - zI)^{-1}$ and note that it is a bounded linear operator defined on all of $\mathcal{H}$.

Unlike their selfadjoint counter parts, unbounded C-selfadjoint operators do not, in general, possess a spectral resolution and fine functional calculus. When an unbounded C-selfadjoint operator has a compact resolvent, a canonically associated antilinear eigenvalue problem always has a complete set of mutually orthogonal eigenfunctions.

Theorem (2.2.22)[446]: *If* T: $\mathcal{D} \to \mathcal{H}$ *is an unbounded C-selfadjoint operator with compact resolvent $(T - zI)^{-1}$ for some complex number z, then there exists an orthonormal basis $u_0, u_1, \ldots$ of $\mathcal{H}$ consisting of solutions of the antilinear eigenvalue problem*:

$$(T - zI)u_n = \sigma_n C u_n \qquad (54)$$

where $\sigma_0, \sigma_1, \ldots$ is an increasing sequence of positive numbers tending to ∞.

Proof. For f, g in $\mathcal{D}(T)$ we have $\{C(T - zI)f, g\rangle = \{C(T - zI)g, f\rangle$. Let S denote the compact operator $(T - zI)^{-1}$ and let $f = Sx$ and $g = Sy$. Since $\{Cx, Sy\} = \{Cy, Sx\}$ for all x, y, S is a compact C-symmetric operator. By Theorem (2.2.12) there exists an orthonormal basis u_n of $\mathcal{H}$ such that $SCu_n = \sigma_n^{-1} u_n$ for all n, where σ_n^{-1} is a decreasing sequence of positive numbers tending to zero. Since each u_n belongs to ran $A = \mathcal{D}(T)$, we apply $T - zI$ to both sides of the preceding equation and the desired result follows.

We note several useful corollaries of the preceding theorem:

Corollary (2.2.23)[446]: *If $T: \mathcal{D} \to \mathcal{H}$ is an unbounded C-selfadjoint operator with compact resolvent at $z = 0$, then a vector $f = \sum_{n=0}^{\infty} a_n u_n$ in $\mathcal{H}$ belongs to $\mathcal{D}(T)$ if and only if $\sum_{n=0}^{\infty} \sigma_n^2 |a_n|^2 < \infty$.*

Corollary (2.2.24)[446]: *Under the conditions of Theorem* (2.2.22)

$$\|(T - zI)^{-1}\| = \frac{1}{\sigma_0}. \quad (55)$$

In the spirit of Theorem (2.2.22) we have the following C-selfadjointness criterion:

Theorem (2.2.25)[446]: *Let* T: $\mathcal{D}(T) \to \mathcal{H}$ *be a closed, densely defined, C-symmetric operator. If there exists a complete system of vectors u_n in $\mathcal{D}(T)$ and an increasing positive sequence σ_n tending to infinity satisfying $Tu_n = \sigma_n C u_n$ for all n, then T is C-selfadjoint.*

Proof. Since T is C-symmetric, $\sigma_j\{u_j, u_k\rangle = \{CTuj, u_k\rangle = \{CTu_k, u_j\rangle = \sigma_k\{u_k, u_j\rangle$ and hence $u_j \perp u_k$ whenever $\sigma_j \neq \sigma_k$. In the case of higher multiplicities, say $\sigma_n = \sigma_{n+1} = \cdots = \sigma_{n+p}$, we may assume that the vectors $u_n, \ldots, u_{n+p}$ are mutually orthogonal. Indeed, if these vectors were not orthogonal, we could simply replace them with an orthonormal basis for the *real* vector space generated by $u_n, \ldots, u_{n+p}$. We can therefore assume that u_n, and hence Cu_n, form orthonormal bases of $\mathcal{H}$. Let $f = \sum_{j=0}^{\infty} a_j Cu_j$ represent an arbitrary vector in $\mathcal{H}$. For each finite n, the vector $f_n = \sum_{j=0}^{n} a_j \sigma_j^{-1} u_j$ belongs to $\mathcal{D}(T)$ by Corollary (2.2.23) and satisfies $Tf_n = \sum_{j=0}^{n} a_j Cu_j$. Since the graph of T is closed, it is not hard to see that $T: \mathcal{D}(T) \to \mathcal{H}$ is surjective. According to the criterion of [116], T is C-selfadjoint.

We can generalize the refined polar decomposition $T = CJ|T|$ of Theorem (2.2.7) to the case of unbounded C-selfadjoint operators, modulo several minor modifications.

Theorem (2.2.26)[446]: *If T is a C-selfadjoint operator with zero in its resolvent, then $T = CJ|T|$ where $|T|$ is a positive operator (in the von Neumann sense) satisfying $\mathcal{D}(|T|) = \mathcal{D}(T)$ and J is a conjugation on $\mathcal{H}$ which strongly commutes with $|T|$.* Conversely, any operator of the form described above is C-selfadjoint.

Proof. If T is a C-selfadjoint operator with zero in its resolvent, then $T: \mathcal{D}(T) \to \mathcal{H}$ is surjective and we let $R^* : \mathcal{H} \to \mathcal{H}$ denote the bounded right inverse (the resolvent at 0) of T. Since $\mathcal{H} = \mathcal{D}(T)$, we use the fact that $TR = I$ and (53) to conclude that $\{Cf, Rg\rangle = \{CTRf, Rg\rangle = \{CTRg, Rf\rangle = \{Cg, Rf\rangle$ for all f, g in $\mathcal{H}$. This implies that $\{R^*Cf, g\rangle = \{CRf, g\rangle$ for all f, g in $\mathcal{H}$ and hence R is a bounded C-symmetric operator. In particular, R^* is a bounded C-symmetric operator that is injective and has dense range.

Let $R^* = CJ|R^*|$ be the decomposition of R^* guaranteed by Theorem (2.2.7) where without loss of generality we assume J is a conjugation on all of $\mathcal{H}$ which commutes with $|R^*|$. Taking the adjoint of this equation and substituting it into the equation $TR = I$, we see that $T|R^*|JC = I$ and hence $T|R^*| = CJ$. We read from here that $|R^*|\mathcal{H} = \mathcal{D}(T)$ and hence the unbounded positive operator $|R^*|^{-1}$ has the same domain as T. This implies that $JCT|R^*| = I$, or equivalently, $JCT = |R^*|^{-1}$ as unbounded operators. This yields the decomposition $T = CJ|T|$ where the positive self-adjoint operator $|T|$ is defined to be $|R^*|^{-1}$.

Regarding the terminology of Theorem (2.2.26) we say that J *strongly commutes* with $|T|$ if J commutes with the spectral measure of $|T|$. Equivalently, we could say that J commutes with the *bounded* selfadjoint operator $|T|^{-1}$. Also observe that the operator $U = CJ$ in Theorem (2.2.26) is a unitary C-symmetric operator.

Although we do not pursue this direction further, we remark that Theorem (2.2.26) can be used to characterize the C-selfadjoint extensions of an arbitrary C-symmetric operator.

Example (2.2.27)[446]: Using the techniques above, we briefly discuss a simple example of a first order differential operator with a *nonselfadjoint* two point boundary condition. More sophisticated examples and applications to quantum systems are explored in [99]. Given the following example.

Let $q(x)$ be a real valued, continuous, even function on [−1, 1] and let α be a nonzero complex number satisfying $|\alpha| < 1$. For a small parameter $\varepsilon > 0$, we define the operator

$$[T_\alpha f](x) = -if'(x) + \varepsilon q(x)f(x) \qquad (56)$$

with domain

$$\mathcal{D}(T_\alpha) = \{f \in L^2[-1,1] : f' \in L^2[-1,1], f(1) = \alpha f(-1)\}.$$

Clearly T_α is a closed operator and $\mathcal{D}(T_\alpha)$ is dense in $L^2[-1,1]$.

If C denotes the conjugation $[Cu]\ (x) = \overline{u(-x)}$ on $L^2[-1,1]$, then it follows that that *nonselfadjoint* operator T_α satisfies $T_\alpha = CT_{1/\overline{\alpha}}C$. A short computation shows that $T_\alpha^* = T_{1/\overline{\alpha}}$ and hence T_α is a C-selfadjoint operator.

In the case $\varepsilon = 0$, we have $T_\alpha f = -if'$ and we can explicitly compute the resolvent R_α of T_α at $z = 0$:

$$[R_\alpha^{-1} f](x) = i\int_{-1}^{x} f(t)dt + \frac{i}{\alpha - 1}\int_{-1}^{1} f(t)dt,$$

for f in $L^2[-1,1]$. In particular, $\mathcal{D}(T_\alpha) = R_\alpha^{-1}L^2[-1,1]$ and $T_\alpha R_\alpha = I$. According to Theorem (2.2.22) the antilinear problem

$$-if'(x) = \sigma\overline{f(-x)}, f \in \mathcal{D}(T_\alpha) \qquad (57)$$

admits nontrivial solutions for certain positive σ_n tending to ∞. Moreover, the solutions u_0, u_1, . . . can be chosen to form a complete orthonormal system in $L^2[-1,1]$. Taking another derivative in (57) and using back-substitution (see also Example [96], we find that the u_n are solutions to $f'' + \sigma_n^2 f = 0$, and thus

$$u_n(x) = a_n e^{i\sigma_n x} + b_n e^{-i\sigma_n x}$$

for certain constants a_n and b_n. The boundary condition $u_n(1) = \alpha u_n(-1)$ shows that $a_n b_n \neq 0$ for all .

Returning to the original first order antilinear equation (57), we see that

$$\sigma_n a_n e^{i\sigma_n x} - \sigma_n b_n e^{-i\sigma_n x} = \sigma_n \overline{a_n} e^{i\sigma_n x} + \sigma_n \overline{b_n} e^{-i\sigma_n x},$$

whence $a_n = \overline{a_n}$ and $b_n = -\overline{b_n}$. Multiplying u_n by a suitable real constant, we obtain the (nonnormalized) eigenfunctions

$$U_n(x) = e^{i\sigma_n x} + i\gamma_n e^{-i\sigma_n x},$$

where γ_n belongs to $\mathbb{R}\backslash\{0\}$. Moreover, the boundary condition $U_n(1) = \alpha U_n(-1)$ yields the equation

$$e^{i\sigma_n} + i\gamma_n e^{-i\sigma_n} = \alpha[e^{-i\sigma_n} + i\gamma_n e^{i\sigma_n}],$$

which implies that

$$e^{2i\sigma_n} = \frac{\alpha - i\gamma_n}{1 - i\gamma_n \alpha}$$

The image of the real line under the linear fractional transformation

$$G(z) = \frac{\alpha - iz}{1 - iz\alpha}$$

is either a circle or a line which intersects the unit circle at exactly two points since $|G(0)| = |\alpha| < 1$ and $|G(\infty)| = |1/\alpha| > 1$. In fact, the solutions γ_0 and γ_1 to $|G(z)| = 1$ can be given in closed form:

$$\frac{2 \text{ Im } \alpha \pm \sqrt{1 - 2 \text{ Re } \alpha^2 + |\alpha|^2}.}{1 - |\alpha|^2}$$

We may assume, after a possible relabeling, that the principal arguments σ_0 and σ_1 satisfying $e^{2i\sigma_0} = G(\gamma_0)$ and $e^{2i\sigma_1} = G(\gamma_1)$ satisfy $0 < \sigma_0 < \sigma_1 < \pi$.

Retracing our steps, we have:

$$\sigma_{2n} = \sigma_0 + n\pi,$$
$$\sigma_{2n+1} = \sigma_1 + n\pi$$

for $n \geq 0$. The associated (nonnormalized) eigenfunctions are:

$$U_{2n}(x) = e^{i(\sigma_0 + n\pi)x} + i\gamma_0 e^{-i(\sigma_0 + n\pi)x},$$
$$U_{2n+1}(x) = e^{i(\sigma_1 + n\pi)x} + i\gamma_1 e^{-i(\sigma_1 + n\pi)x}$$

Using Corollary (2.2.24) we obtain the norm of the resolvent at $z = 0$:

$$\|R_\alpha\| = \frac{1}{\sigma_0}.$$

A familiar argument in perturbation theory shows that for $\varepsilon\|q\|_\infty < \|R_\alpha\|$ the original operator (56) still has $z = 0$ in its resolvent, and that a similar antilinear spectral picture holds. For instance, an estimate of $\|T_\alpha^{-1}\|$ is easily within reach.

Chapter 3
A Simple C^*-Algebras

We present the Dixmier approximation theorem and study the simple C^*-algebra with its characterizations of a finite and infinite projection.

Section (3.1): The Dixmier Approximation Theorem

Suppose that $\mathcal{R}$ is a von Neumann algebra, acting on a Hilbert space $\mathcal{K}$, with centre $\mathcal{C}$ and unitary group $\mathcal{U}$. For each A in $\mathcal{R}$, let $u_{\mathcal{R}}(A)$ be the subset $\{UAU^*: U \in \mathcal{U}\}$ of $\mathcal{R}$, and write $co_{\mathcal{R}}(A)$ for the convex hull of $u_{\mathcal{R}}(A)$. The norm closure $co_{\mathcal{R}}(A)^{=}$ of $co_{\mathcal{R}}(A)$, and the weak-operator closure $co_{\mathcal{R}}(A)^{-}$, are again subsets of $\mathcal{R}$, since $\mathcal{R}$ is closed in both the norm and weak-operator topologies.

The Dixmier approximation theorem [118] asserts that $co_{\mathcal{R}}(A)^{=}$ meets $\mathcal{C}$. This theorem has many applications; as a first example, we mention its use in relating the structure of norm-closed two-sided ideals in $\mathcal{R}$ to the much simpler structure of norm-closed ideals in $\mathcal{C}$. For some of the applications, it is necessary to have more detailed information about the nature of the intersection $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$. The single most important result, on this topic, concerns finite von Neumann algebras; when $\mathcal{R}$ is finite, $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ consists of just one point, for each A in $\mathcal{R}$. This deep result is more or less equivalent to the existence of the centre-valued trace $\tau: \mathcal{R} \to \mathcal{C}$ on $\mathcal{R}$; the unique element of $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ is $\tau(A)$. For all this material, we see [119, Chapitre 3, §§5,8]. In the case of a countably decomposable Type III von Neumann algebra, $co_{\mathcal{R}}(A)^{=}$ contains a non-zero element of $\mathcal{C}$ whenever A is a non-zero element of $\mathcal{R}$ [457, Lemma 1]. For a countably decomposable Type III factor, this had been proved earlier, and $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ had been described completely for self-adjoint elements A of $\mathcal{R}$ [460, p. 133, Lemma 15(b), p. 136, Corollary 16]; the methods used in [460] extend routinely to the case of Type III algebras. These results on the Type III case have been used in a variety of ways in the study of derivations of operator algebras [121,122,458].

There is some interest in studying also the intersection $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$. For a finite von Neumann algebra $\mathcal{R}$, it follows easily, from weak-operator continuity on bounded sets of the centre-valued trace $\tau: \mathcal{R} \to \mathcal{C}$, that $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ consists of the single point $\tau(A)$. In the case of an infinite factor $\mathcal{R}$, the set $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ has been described [117, Theorem 3] in terms of the numerical range of A (when $\mathcal{R}$ is Type III) or the essential numerical range (when $\mathcal{R}$ is Type I_∞ or II_∞). This approach has been generalized by Halpern [120, Theorems 4.12, 4.16], who introduces certain concepts of 'essential central range', and uses them to describe both $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ and $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ for an arbitrary element A of a properly infinite von Neumann algebra $\mathcal{R}$. As a consequence of these results, he shows [120, Corollaries 4.14, 4.17] that $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ is weak-operator closed, and coincides with $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ when $\mathcal{R}$ is countably decomposable.

We give a complete and computationally simple description of the sets $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ and $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ where A is any self-adjoint element of a general von Neumann algebra $\mathcal{R}$. Both sets are 'closed intervals' in the partially ordered Banach space $\mathcal{C}_h$ of self-adjoint elements of $\mathcal{C}$; that is, there exist elements $C_1^n, C_2^n, C_1^w, C_2^w$, of $\mathcal{C}_h$ (the superscripts n and w stand for 'norm' and 'weak-operator') such that

$$co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} = \{C \in \mathcal{C}_h: C_1^n \le C \le C_2^n\},$$
$$co_{\mathcal{R}}(A)^{-} \cap \mathcal{C} = \{C \in \mathcal{C}_h: C_1^w \le C \le C_2^w\}.$$

We determine the 'endpoints', $C_1^n, C_2^n, C_1^w, C_2^w$, by specifying their spectral resolutions in terms of the spectral resolution of the self-adjoint operator A. The fact that $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ is a closed interval is elementary; the fact that $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ is a closed interval is less obvious, but is nevertheless a simple consequence of [120, Corollary 4.14]. Both these results are included, along with other preliminary material. The main thrust of the argument is concerned with the determination of the endpoints of the intervals; almost as a byproduct, it turns out that the results just described can all be established without appeal to [120].

We have already noted that the two sets $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ and $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ coincide when $\mathcal{R}$ is countably decomposable. In fact, this property (whether required for all A in $\mathcal{R}$, or just for self-adjoint elements) is characteristic of a slightly larger class of von Neumann algebras, described in Theorem (3.1.8).

The results are specialized to the factor case; in this situation, most of the results are known, or only marginally original. We also discuss some rather fragmented results about the intersection $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$, where $u_{\mathcal{R}}(A)^{-}$ denotes the weak-operator closure of $u_{\mathcal{R}}(A)$.

We denote by $B(\mathcal{K})$ the von Neumann algebra of all bounded linear operators acting on the Hilbert space $\mathcal{K}$. When $\mathcal{X} \subseteq \mathcal{K}$ and $\mathcal{Y} \subseteq (\mathcal{K})$, we write $[\mathcal{X}]$ for the norm-closed subspace of $\mathcal{K}$ generated by $\mathcal{X}, co\ \mathcal{Y}$ for the convex hull of $\mathcal{Y}, \mathcal{Y}^{=}$ for the norm closure of $\mathcal{Y}, \mathcal{Y}^{-}$ for the weak-operator closure of $\mathcal{Y}$, and $\mathcal{Y}'$ for the commutant of $\mathcal{Y}$.

Suppose that $\mathcal{R}$ is a von Neumann algebra with centre $\mathcal{Y}$. We write $\mathcal{R}^{+}$ for the positive cone in $\mathcal{R}$, and $(\mathcal{R})_r$ for the closed ball in $\mathcal{R}$ with centre 0 and radius r. We adopt the standard notations concerning the comparison of projections in $\mathcal{R}$ (note, however, that, in contrast with [119], we use the symbol $\precsim$ for 'weaker than or equivalent to', and reserve $\prec$ for 'weaker than and not equivalent to'). The central carrier of a projection E in $\mathcal{R}$ is denoted by C_E. The comparison theorem, applied to the projections E and I, implies that there is a projection P in $\mathcal{C}$ such that $PE \sim P$ and $P_1E \prec P_1$ whenever P_1 is a non-zero central subprojection of $I-P$. Also, there is a projection Q in $\mathcal{C}$ such that the projection $(I-Q)E$ in $\mathcal{R}$ is finite, and Q_1E is infinite for every non-zero central subprojection Q_1 of Q. Of course, E is infinite if and only if $Q \neq 0$, and is properly infinite if and only if $Q = C_E \neq 0$. The sets $u_{\mathcal{R}}(A), co_{\mathcal{R}}(A), co_{\mathcal{R}}(A)^{=}$, and $co_{\mathcal{R}}(A)^{-}$, associated with an element A of $\mathcal{R}$, have already been defined. Clearly,

$$u_{\mathcal{R}}(A) \subseteq co_{\mathcal{R}}(A) \subseteq co_{\mathcal{R}}(A)^{=} \subseteq co_{\mathcal{R}}(A)^{-} \subseteq (\mathcal{R})_{\|A\|};$$

so all these sets are bounded, and $co_{\mathcal{R}}(A)^{-}$ is weak-operator compact. If A is self-adjoint, or positive, the same is true of each element of $co_{\mathcal{R}}(A)^{-}$. If A lies in a norm-closed two-sided ideal $\mathcal{J}$ in $\mathcal{R}$, then $co_{\mathcal{R}}(A)^{=} \subseteq \mathcal{J}$. If P_1 and P_2 are projections with sum I in the centre $\mathcal{C}$ of $\mathcal{R}$, then

$$co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} = \{C_1 + C_2 : C_j \in co_{\mathcal{R}P_j}(AP_j)^{=} \cap \mathcal{C}P_j, (j = 1,2)\},$$

and there is an analogous result for $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$. By taking for P_1 the largest projection in $\mathcal{C}$ that is finite in $\mathcal{R}$, most questions concerning $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ and $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ can be treated by considering separately the two cases in which $\mathcal{R}$ is either finite, or properly infinite.

Proposition (3.1.1)[447]: Suppose that $\mathcal{A}$ is an abelian von Neumann algebra, $\mathcal{A}_h$ is the real Banach space consisting of all self-adjoint elements of $\mathcal{A}$, and the usual lattice operations in $\mathcal{A}_h$ are denoted by $\wedge, \vee$. Let $\mathcal{K}$ be a norm-closed convex subset of $\mathcal{A}_h$ with the following property: if $K_1, \ldots, K_n \in \mathcal{K}$, and $P_1, \ldots, P_n$ are projections in $\mathcal{A}$ with sum I, then $K_1P_1 + \ldots + K_nP_n \in \mathcal{K}$. Then

(i) if $C_1, C_2 \in \mathcal{K}$, then $C_1 \wedge C_2, C_1 \vee C_2 \in \mathcal{K}$;

(ii) if $C_1, C_2 \in \mathcal{K}, C \in \mathcal{A}_h$, and $C_1 \leq C \leq C_2$, then $C \in \mathcal{K}$;

(iii) if $\mathcal{K}$ is weak-operator compact, there exist C_1 and C_2 in $\mathcal{A}_h$ such that

$$\mathcal{K} = \{C\mathcal{A}_h : C_1 \leq C \leq C_2\}.$$

Proof, (i) Let $\{E_\lambda\}$ be the spectral resolution of the self-adjoint operator $C_1 - C_2$ in $\mathcal{A}$. Then $E_\lambda \in \mathcal{A}$ for all real λ; in particular, $P_1, P_2 \in \mathcal{A}$ (and $P_1 + P_2 = I$), where P_1 is E_0 and P_2 is $I - E_0$. Since

$$C_1 \wedge C_2 = C_1P_1 + C_2P_2, \qquad C_1 \vee C_2 = C_2P_1 + C_1P_2,$$

it now follows from the stated property of $\mathcal{K}$ that $C_1 \wedge C_2, C_1 \vee C_2 \in \mathcal{K}$.

(ii) There is a $*$-isomorphism from $\mathcal{A}$ onto the C^*-algebra $C(X)$ of all complex-valued continuous functions on an extremely disconnected compact Hausdorff space X (see, for example, [459, p. 310, Theorem 5.2.1]). We adopt the convention that an element of $\mathcal{A}$ is denoted by a capital letter and its representing function in $C(X)$ is denoted by the corresponding small letter. We write $\mathcal{K}_0$ for the norm-closed convex subset $\{k: K \in \mathcal{K}\}$ of $C(X)$. We have to prove that $c \in \mathcal{K}_0$; to this end, it suffices to show that for each $\varepsilon(> 0)$, there is an element c_0 of $\mathcal{K}_0$ such that $\|c - c_0\| \le \varepsilon$.

Given any x in X, we have $c_1(x) \le c(x) \le c_2(x)$. For a suitably chosen convex combination k_x of c_1 and $c_2, k_x \in \mathcal{K}_0$ and $k_x(x) = c(x)$. The open set

$$\{y \in X: |c(y) - k_x(y)| < \varepsilon\}$$

has closure a clopen set V_x such that $x \in V_x$ and

$$|c(y) - k_x(y)| \le \varepsilon \qquad (y \in V_x).$$

The clopen covering $\{V_x: x \in X)$ of X has a finite subcovering $\{V_{x(1)}, \dots, V_{x(n)}\}$. For $j = 1, \dots, n$, write k_j for $k_{x(j)}$, let W_j be the clopen set

$$V_{x(j)} / \bigcup_{r=1}^{j-1} V_{x(r)},$$

and let p_j be the characteristic function of W_j. Then X is the disjoint union of $W_1, \dots, W_n$, and $k_1, \dots, k_n \in \mathcal{K}_0$; so $P_1, \dots, P_n$ are projections in $\mathcal{A}$ with sum I, and $K_1, \dots, K_n \in \mathcal{K}$. Thus $K_1P_1 + \cdots + K_nP_n \in \mathcal{K}$, and $k_1p_1 + \cdots + k_np_n$ is an element C_0 of $\mathcal{K}_0$. For each y in $X, y \in W_j (\subseteq V_{x(j)}))$ for some j in $\{1, \dots, n\}$, and

$$|c(y) - c_0(y)| = |c(y) - k_j(y)| = |c(y) - k_{x(j)}(y)| \le \varepsilon.$$

Thus $\|c - c_0\| \le \varepsilon$.

(iii) Suppose that $\mathcal{K}$ is weak-operator compact (and, hence, bounded), and let $\le$ denote the usual partial order relation on $\mathcal{A}_h$. With this ordering, and indexed by itself, $\mathcal{K}$ is a bounded increasing net in $\mathcal{A}_h$ and is therefore strong-operator (hence, also, weak-operator) convergent to an element C_2 of $\mathcal{A}_h$ which is the least upper bound of $\mathcal{K}$ in $\mathcal{A}_h$. Since the set $\mathcal{K}$ is weak-operator closed, $C_2 \in \mathcal{K}$. Similarly, by considering $\mathcal{K}$ with the ordering $\ge$, it follows that $\mathcal{K}$ has a greatest lower bound C_1 in $\mathcal{A}_h$, and $C_1 \in \mathcal{K}$. Since $C_1, C_2 \in \mathcal{K}$ and $C_1 \le C \le C_2$ for each C in $\mathcal{K}$, it now follows from (ii) that $\mathcal{K} = \{C \in \mathcal{A}_h: C_1 \le C \le C_2\}$.

Suppose that $\mathcal{R}$ is a von Neumann algebra with centre $\mathcal{C}$ and unitary group $\mathcal{U}$. We denote by $\mathcal{D}$ the set of all mappings $\alpha: \mathcal{R} \to \mathcal{R}$ that can be defined by an equation of the form

$$\alpha(A) = \sum_{j=1}^{n} a_j U_j A U_j^* \qquad (A \in \mathcal{R}),$$

where $U_1, \dots, U_n \in \mathcal{U}$ and $a_1, \dots, a_n$ are positive real numbers with sum 1. Thus, when $A \in \mathcal{R}$, we have

$$co_{\mathcal{R}}(A) = \{\alpha(A): \alpha \in \mathcal{D}\}.$$

A straightforward calculation shows that the composite mapping $\alpha \circ \beta$ lies in $\mathcal{D}$ whenever $\alpha, \beta \in \mathcal{D}$. Each α in $\mathcal{D}$ is a positive (hence, hermitian) norm-decreasing linear mapping, and is continuous with respect to the weak-operator or strong-operator topology (as well as the norm topology).

From the properties just noted, it follows that each of the sets $co_{\mathcal{R}}(A), co_{\mathcal{R}}(A)^{=}$, and $co_{\mathcal{R}}(A)^{-}$ (where $A \in \mathcal{R}$) is invariant under all the mappings in $\mathcal{D}$. Moreover,

$$co_{\mathcal{R}}(B) \subseteq co_{\mathcal{R}}(A) \qquad if\ B \in co_{\mathcal{R}}(A),$$
$$co_{\mathcal{R}}(B)^{=} \subseteq co_{\mathcal{R}}(A)^{=} \qquad if\ B \in co_{\mathcal{R}}(A)^{=},$$
$$co_{\mathcal{R}}(B)^{-} \subseteq co_{\mathcal{R}}(A)^{-} \qquad if\ B \in co_{\mathcal{R}}(A)^{-}.$$

Finally, we observe that $\alpha(AC) = \alpha(A)C$ when $\alpha \in \mathcal{D}, A \in \mathcal{R}$, and $C \in \mathcal{C}$, and that

$$\mathcal{C} = \{A \in \mathcal{R} : \alpha(A) = A\ for\ each\ \alpha\ in\ \mathcal{D}\}.$$

Lemma (3.1.2)[447]: Suppose that $\mathcal{R}$ is a von Neumann algebra with centre $\mathcal{C}$ and $P_1, \ldots, P_n$ are projections in $\mathcal{C}$ with sum I.

(i) If $\alpha_1, \ldots, \alpha_n \in \mathcal{D}$, there is an element α of $\mathcal{D}$ such that

$$\alpha(A) = \alpha_1(A)P_1 + \cdots + \alpha_n(A)P_n \qquad (A \in \mathcal{R}).$$

(ii) If $A \in \mathcal{R}$ and $B_1, \ldots, B_n \in co_{\mathcal{R}}(A)^{=}$, then

$$B_1P_1 + \ldots + B_nP_n \in co_{\mathcal{R}}(A)^{=}.$$

(iii) If $A \in \mathcal{R}$ and $B_1, \ldots, B_n \in co_{\mathcal{R}}(A)^{-}$, then

$$B_1P_1 + \cdots + B_nP_n \in co_{\mathcal{R}}(A)^{-}.$$

(iv) If $A \in \mathcal{R}$ and $C_1, \ldots, C_n \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$, then

$$C_1P_1 + \ldots + C_nP_n \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}.$$

(v) If $A \in \mathcal{R}$ and $C_1, \ldots, C_n \in co(A)^{-} \cap \mathcal{C}$, then

$$C_1P_1 + \ldots + C_nP_n \in co(A)^{-} \cap \mathcal{C}.$$

Proof, (i) We show first that, if P is a projection in $\mathcal{C}$ and $\beta \in \mathcal{D}$, there is an β^P element in $\mathcal{D}$ such that

$$\beta^P(A) = \beta(A)P + A(I - P) \qquad (A \in \mathcal{R}).$$

To this end, we may suppose that

$$\beta(A) = \sum_{j=1}^{k} a_j U_j A U_j^* \qquad (A \in \mathcal{R}),$$

where $U_1, \ldots, U_k \in \mathcal{U}$ and $a_1, \ldots, a_k$ are positive real numbers with sum 1. It suffices to define

$$\beta^P(A) = \sum_{j=1}^{k} a_j V_j A V_j^* \qquad (A \in \mathcal{R}),$$

where, for $j = 1, \ldots, k, V_j$ is the unitary operator $PU_j + I - P$.

From the preceding paragraph, we can define $\beta_1, \ldots, \beta_n$ in $\mathcal{D}$ by

$$\beta_j(A) = \alpha_j(A)P_j + A(I - P_j) \qquad (A \in \mathcal{R}).$$

For $j = 1, \ldots, n$, the subspace $\mathcal{R}P_j$ of $\mathcal{R}$ is invariant under each of the mappings $\beta_1, \ldots, \beta_n$; moreover, $\beta_j|\mathcal{R}P_j = \alpha_j|\mathcal{R}P_j$, and $\beta_k|\mathcal{R}P_j$ is the identity mapping on $\mathcal{R}P_j$ when $k \neq j$. With α the element $\beta_1, \beta_2, \ldots, \beta_n$, of $\mathcal{D}, \alpha|\mathcal{R}P_j = \alpha_j|\ \mathcal{R}P_j (j = 1, \ldots, n)$. Thus

$$\alpha(A) = \sum_{j=1}^{n} \alpha(AP_j) = \sum_{j=1}^{n} \alpha_j(AP_j) = \sum_{j=1}^{n} \alpha_j(A)P_j\,.$$

(ii),(iii) In the argument that follows, topological terms can be interpreted as relating to the norm topology on *m* (to prove (ii)) or the weak-operator topology (to prove (iii)).

Let N be a convex neighbourhood of 0 in $\mathcal{R}$. For each $j = 1, \ldots, n,\ B_j$ lies in the closure of the set $co_{\mathcal{R}}(A)\ (= \{\alpha(A) : \alpha \in \mathcal{D}\})$, and the mapping $R \to RP_j : \mathcal{R} \to \mathcal{R}$ is continuous. Hence there exists α_j in $\mathcal{D}$ such that $[\alpha_j(A) - B_j]P_j \in n^{-1}N$. From the convexity of N, it follows that

$$\sum_{j=1}^{n}\left[\alpha_j(A) - B_j\right] P_j \in N;$$

that is, $\alpha(A) - (B_1P_1 + \ldots + B_nP_n) \in N$, where α (in $\mathcal{D}$) is defined as in (i). Hence $B_1P_1 + \ldots + B_nP_n$ lies in the closure of $co_{\mathcal{R}}(A)$.

(iv),(v) These assertions are immediate consequences of (ii) and (iii), respectively.

Corollary (3.1.3)[447]: Suppose that $\mathcal{R}$ is a von Neumann algebra with centre $\mathcal{C}$, that $A = A^* \in \mathcal{R}$, and that $\mathcal{K}$ is either $co_{\mathcal{R}}(A)^= \cap \mathcal{C}$ or $co_{\mathcal{R}}(A)^- \cap \mathcal{C}$.

(i) If $C_1, C_3 \in \mathcal{K}, C = C^* \in \mathcal{C}$, and $C_1 < C < C_2$, then $C \in \mathcal{K}$.

(ii) There exist self-adjoint elements C_1 and C_2 of $\mathcal{C}$ such that

$$\mathcal{K} = \{C \in \mathcal{C}: C = C^*, C \leq C \leq C_2\}.$$

Proof, (i) Both $co_{\mathcal{R}}(A)^= \cap \mathcal{C}$ and $co_{\mathcal{R}}(A)^- \cap \mathcal{C}$ are norm-closed convex subsets of $\mathcal{C}_h$ (the partially ordered Banach space consisting of all self-adjoint elements of the abelian von Neumann algebra $\mathcal{C}$). By Lemma (3.1.2) iv),(v), they both have the additional property required of the norm-closed convex set $\mathcal{K}$ occurring in Proposition (3.1.1). The stated result now follows from Proposition (3.1.1) (ii).

(ii) The set $co_{\mathcal{R}}(A)^= \cap \mathcal{C}$ is weak-operator closed (by [120, Corollary 4.14]), and the same is true of $co_{\mathcal{R}}(A)^- \cap \mathcal{C}$, so the stated result follows from Proposition (3.1.1) (iii). In obtaining our main results, we appeal to part (i) of Corollary (3.1.3), but not to part (ii). Accordingly, our proof of these main results does not depend on [120] (and is, in fact, quite elementary). It is for this reason that Corollary (3.1.3) has been set out in two parts, even though the first is subsumed in the second.

Suppose that $\mathcal{R}$ is a von Neumann algebra with centre $\mathcal{C}$. We begin with two lemmas that provide information about the sets $co_{\mathcal{R}}(E)^= \cap \mathcal{C}$ and $co_{\mathcal{R}}(E)^- \cap \mathcal{C}$, respectively, when E is a projection in $\mathcal{R}$. With the aid of these lemmas, we then give a complete description of $co_{\mathcal{R}}(E)^= \cap \mathcal{C}$ and $co_{\mathcal{R}}(E)^- \cap \mathcal{C}$, where A is any self-adjoint element of $\mathcal{R}$. Finally, we characterize those von Neumann algebras $\mathcal{R}$ with the property that $co_{\mathcal{R}}(E)^= \cap \mathcal{C}$ coincides with $co_{\mathcal{R}}(E)^- \cap \mathcal{C}$ for each (self-adjoint) A in $\mathcal{R}$. It suffices to consider the case in which $\mathcal{R}$ is properly infinite.

Lemma (3.1.4)[447]: Suppose that $\mathcal{R}$ is a properly infinite von Neumann algebra with centre $\mathcal{C}$, that E is a projection in $\mathcal{R}$, and that P is a projection in $\mathcal{C}$.

(i) If $PE \sim P$, then $P \in co_{\mathcal{R}}(PE)^=$.

(ii) If $QE \prec Q$ whenever Q is a projection in $\mathcal{C}$ and $0 < Q \leq P$, then $co_{\mathcal{R}}(PE)^= \cap \mathcal{C} = \{0\}$.

Proof, (i) We may assume that $P \neq 0$. From this, P is a properly infinite projection, and so is PE since $PE \sim P$. It follows that, for each positive integer n, there exist projections $F_0, E_1, \ldots, E_n$ in $\mathcal{R}$ such that

$$P \sim PE = F_0 + E_1 + \ldots + E_n \sim F_0 \sim E_1 \sim \ldots \sim E_n.$$

Let $E_0 = F_0 + (P - PE)$. Then

$$PE = (p - E_0) + F_0 \geq P - E_0.$$

Also, $P \sim F_0 \leq E_0 \leq P$, whence $E_0 \sim P$, and

$$P = E_0 + E_1 + \ldots + E_n \sim E_0 \sim E_1 \sim \ldots \sim E_n.$$

For each $j = 1, \ldots, n$ there is a partial isometry V_j in $\mathcal{R}$, with E_0 and E_j as initial and final projection, respectively. Define unitary operators $U_0, U_1, \ldots, U_n$ in $\mathcal{R}$, and α in $\mathcal{D}$, by

$$U_0 = I, \quad U_j = V_j + V_j^* + I - E_0 - E_j \quad (j = 1, \ldots, n),$$

$$\alpha(A)=\frac{1}{n+1}\sum_{j=0}^{n}U_jAU_j^* \qquad (A\in\mathcal{R}).$$

Then

$$P=\alpha(P)\geq\alpha(PE)\geq\alpha(P-E_0)=P-\alpha(E_0)$$
$$=P-\frac{1}{n+1}\sum_{j=0}^{n}U_jE_0U_j^*$$
$$=P-\frac{1}{n+1}\sum_{j=0}^{n}E_j=\left(1-\frac{1}{n+1}\right)P,$$

and

$$0\leq P-\alpha(PE)\leq(n+1)^{-1}P.$$

Thus $\|P-\alpha(PE)\|\leq(n+1)^{-1}$. Since we can find such an element α in $\mathcal{D}$, corresponding to any choice of the positive integer n, it follows that $P\in co_{\mathcal{R}}(PE)^{=}$.

(ii) Under the conditions stated in (ii), we have $Q(PE)\prec Q$ for every non-zero projection Q in $\mathcal{C}$. From [459, Corollary 2.2], PE lies in $J(\{0\})$, the largest norm-closed two-sided ideal in $\mathcal{R}$ that has intersection $\{0\}$ with $\mathcal{C}$. Accordingly,

$$co_{\mathcal{R}}(PE)^{=}\subseteq j(\{0\}),\qquad co_{\mathcal{R}}(PE)^{=}\cap\mathcal{C}\subseteq J(\{0\})\cap\mathcal{C}=\{0\},$$

and thus $co_{\mathcal{R}}(PE)^{=}\cap\mathcal{C}=\{0\}$.

Lemma (3.1.5)[447]: Suppose that $\mathcal{R}$ is a properly infinite von Neumann algebra with centre $\mathcal{C}$, and that E is a projection in $\mathcal{R}$.

(i) If E is properly infinite, $C_E\in co_{\mathcal{R}}(E)^{-}$.

(ii) If E is finite, $co_{\mathcal{R}}(E)^{-}\cap\mathcal{C}=\{0\}$.

Proof, (i) Suppose that E is properly infinite, and let $\mathcal{E}$ be the set of all projections F in $\mathcal{R}$ such that $F\sim E\leq F$. When $F\in\mathcal{E}, F$ is a properly infinite projection in $\mathcal{R}$ (since $F\sim E$); so E is a projection, equivalent to the unit F, in a properly infinite von Neumann algebra $F\mathcal{R}F$. From Lemma (3.1.4) (i), $F\in co_{F\mathcal{R}F}(E)^{=}$. Now each unitary element V of $F\mathcal{R}F$ extends to a unitary element $U(=V+I-F)$ of $\mathcal{R}$, and $VEV^*=UEU^*$. It follows that

$$F\in co_{F\mathcal{R}F}(E)^{=}\subseteq co_{\mathcal{R}}(E)^{=}\subseteq co_{\mathcal{R}}(E)^{-}.$$

Thus $\mathcal{E}\subseteq co_{\mathcal{R}}(E)^{-}$.

It now suffices to show that $C_E\in\mathcal{E}^{-}$. We prove this by showing that, given any finite set $\{x_1,\ldots,x_n\}$ of vectors, there is an element F of $\mathcal{E}$ such that $Fx_j=C_EX_j$ $(j=1,\ldots,n)$. To this end, let G be the projection whose range is the closure of the linear subspace

$$\{A_1'C_Ex_1+\ldots+A_n'C_Ex_n: A_1',\ldots,A_n'\in\mathcal{R}'\}.$$

Since the range of G is generated by the action of $\mathcal{R}'$ on a finite set of vectors, G is a countably decomposable projection in $\mathcal{R}$. Also, $G\leq C_E$, and the range of G contains $C_Ex_1,\ldots,C_Ex_n$. Thus $E\vee G$ is a projection F_0 in $\mathcal{R}$, and

$$E\leq F_0\leq C_E,\qquad F_0x_j=C_Ex_j\quad(j=1,\ldots,n).$$

By [113, p. 299, Corollaire 5], $G\precsim E$. Also, since the projection E in $\mathcal{R}$ is properly infinite, we have $E=E_1+E_2\sim E_1\sim E_2$ for suitable projections E_1 and E_2 in $\mathcal{R}$. Since

$$F_0-E=E\vee G-E\sim G-E\wedge G\leq G\precsim E\sim E_1$$

and

$$F_0=(F_0-E)+E\precsim E_1+E_2=E\leq F_0,$$

it follows that $F_0\sim E\leq F_0$. This completes the proof of (i), with F_0 the required element F of $\mathcal{E}$.

(ii) Suppose that E is a finite projection in $\mathcal{R}$, and $0 \neq C \in co_{\mathcal{R}}(E)^{-} \cap \mathcal{C}$; we shall in due course obtain a contradiction.

Since $0 \leq E \leq C_E$, we have $0 \leq C \leq C_E$. From this, and since $0 \neq C \in \mathcal{C}$, there is a non-zero projection P_1 in $\mathcal{C}$ and a positive real number c such that

$$cP_1 \leq C, \qquad P_1 \leq C_E.$$

Since P_1 is the central carrier of the finite projection P_1E in $\mathcal{R}$, the von Neumann algebra $\mathcal{R}P_1$ is semi-finite. By [119, p. 99, Proposition 9] there is a faithful normal semifinite trace τ on $(\mathcal{R}P_1)^{+}$. There is an element F of $(\mathcal{R}P_1)^{+}$ such that $0 < F \leq P_1E$ and $\tau(F) < \infty$; upon replacing F by an appropriately chosen spectral projection of F, we may assume that F is a projection. Let $\{F_1, \dots, F_k\}$ be a (necessarily finite) maximal orthogonal family of subprojections of P_1E in $\mathcal{R}$, each equivalent to F. With F_0 defined as $P_1E - \sum_{j=1}^{k} F_j$ it follows from the maximality assumption that $F \not\precsim F_0$. By the comparison theorem, there is a projection P in $\mathcal{C}$ such that $0 < P \leq P_1$ and $PF_0 \precsim PF$. Thus

$$T(PF_0) \leq \tau(PF) = \tau(PF_j) \leq \tau(F) < \infty \qquad (j = 1, \dots, n),$$

and

$$\tau(PE) = \sum_{j=0}^{n} \tau(PF_j) < \infty.$$

Let $\mathcal{K}$ be the set $\{A \in \mathcal{R}^{+} : \tau(PA) \leq \tau(PE)\}$. Then $\mathcal{K}$ is convex, contains E, and is invariant under the mapping $A \to UAU^{*} : \mathcal{R} \to \mathcal{R}$, for each unitary operator U in $\mathcal{R}$. Moreover, $\mathcal{K}$ is weak-operator closed; for there is a family $\{x_a : a \in \mathbb{A}\}$ of vectors such that

$$\tau(A) = \sum_{a \in A} \langle Ax_a, x_a \rangle \qquad (A \in \mathcal{R}^{+})$$

[119, p. 85, Corollaire], and $\mathcal{K}$ is the intersection of the weak-operator closed sets

$$\{A \in \mathcal{R}^{+} : \sum_{a \in F} \langle PAx_a, x_a \rangle \leq \tau(PE)\}.$$

associated with arbitrary finite subsets $\mathbb{F}$ of $\mathbb{A}$. From the properties of $\mathcal{K}$ just noted, it follows that $co_{\mathcal{R}}(E)^{-} \subseteq \mathcal{K}$; in particular, $C \in \mathcal{K}$. Accordingly,

$$\tau(P) = \tau(PP_1) \leq c^{-1}\tau(PC) \leq c^{-1}\tau(PE) < \infty,$$

and this gives a contradiction since P is an infinite projection in $\mathcal{R}$ and τ is a faithful trace on $\mathcal{R}^{+}$. It follows that $co_{\mathcal{R}}(E)^{-} \cap \mathcal{C} = \{0\}$.

Theorem (3.1.6)[447]: Suppose that $\mathcal{R}$ is a prbperly infinite von Neumann algebra with centre $\mathcal{C}$, that $A = A^{*} \in \mathcal{R}$, and that $\{E_\lambda\}$ is the spectral resolution of A. For each real number λ, define projections F_λ^n and G_λ^n in $\mathcal{C}$ by

$$F_\lambda^n = \bigwedge_{\mu > \lambda} P_\mu^n, \qquad G_\lambda^n = \bigwedge_{\mu > \lambda} (I - Q_\mu^n),$$

Where P_λ^n and Q_λ^n are the largest projections in $\mathcal{C}$ such that $P_\lambda^n E_\lambda \sim P_\lambda^n$ and $Q_\lambda^n(I - E_\lambda) \sim Q_\lambda^n$. Then $\{F_\lambda^n\}$ and $\{G_\lambda^n\}$ are the spectral resolutions of two self-adjoint elements of $\mathcal{C}$, C_{min}^n and C_{max}^n, respectively; moreover, $C_{min}^m \leq C_{max}^n$, and

$$co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} = \{C \in \mathcal{C} : C = C^{*}, C_{min}^n \leq C \leq C_{max}^n\}.$$

Proof. The superscript 'n' (for 'norm'), occurring on the symbols $F_\lambda^n, G_\lambda^n, P_\lambda^n, Q_\lambda^n, C_{min}^n$, and C_{max}^n, has been introduced only to distinguish these operators from similar ones that arise in the description (see Theorem (3.1.7)) of $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$; it will be omitted, throughout the proof of the present theorem.

When $\lambda \leq \mu$, we have

$$P_\lambda \sim P_\lambda E_\lambda \leq P_\lambda E_\mu \leq P_\lambda,$$

$$\mathcal{Q}_\mu \sim \mathcal{Q}_\mu(I - E_\mu) \le \mathcal{Q}_\mu(I - E_\lambda) \le \mathcal{Q}_\mu,$$
and thus $P_\lambda E_\mu \sim P_\lambda$ and $\mathcal{Q}_\mu(I - E_\lambda) \sim \mathcal{Q}_\mu$. It now follows, from the maximality property in the definitions of P_μ and $\mathcal{Q}_\lambda$, that $P_\lambda \le P_\mu$ and $\mathcal{Q}_\mu \le \mathcal{Q}_\lambda$; so
$$P_\lambda \le P_\mu, \qquad I - \mathcal{Q}_\lambda \le I - \mathcal{Q}_\mu \text{ when } \lambda \le \mu. \tag{1}$$
When $\lambda < -\|A\|, E_\lambda = 0$; so $P_\lambda = 0$ and $\mathcal{Q}_\lambda = I$. When $\lambda \ge \|A\|, E_\lambda = I$; so $P_\lambda = I$ and $\mathcal{Q}_\lambda = 0$. Thus
$$P_\lambda = I - \mathcal{Q}_\lambda = 0 \quad (\lambda < -\|A\|), \tag{2}$$
$$p_\lambda = I - \mathcal{Q}_\lambda = I \qquad (\lambda > \|A\|). \tag{3}$$
From (1), (2), and (3), the families $\{P_\lambda\}$ and $\{I - \mathcal{Q}_\lambda\}$ of projections in $\mathcal{q}$ have all the properties, except strong-operator continuity on the right, that characterize the spectral resolutions of bounded self-adjoint operators. Upon replacing P_λ and $I - \mathcal{Q}_\lambda$ by their (strong-operator) limits on the right, F_λ and G_λ, respectively, we obtain the spectral resolutions $\{F_\lambda\}$ and $\{G_\lambda\}$ of two self-adjoint elements of $\mathcal{q}$,
$$C_{min} = \int_{\mathbb{R}} \lambda dF_\lambda, \qquad C_{max} = \int_{\mathbb{R}} \lambda dG_\lambda. \tag{4}$$
From Lemma (3.1.4) (i), together with the defining property of P_λ and $\mathcal{Q}_\lambda$,
$$P_\lambda \in co_{\mathcal{R}}(P_\lambda E_\lambda)^=, \quad \mathcal{Q}_\lambda \in co_{\mathcal{R}}(\mathcal{Q}_\lambda - \mathcal{Q}_\lambda E_\lambda)^=. \tag{5}$$
Since $\alpha(P_\lambda - P_\lambda E_\lambda) = P_\lambda - \alpha(P_\lambda E_\lambda)$ for each α in $\mathcal{D}$, it follows easily that
$$P_\lambda \in co_{\mathcal{R}}(P_\lambda - P_\lambda E_\lambda)^= = \{P_\lambda - B : B\ co_{\mathcal{R}}(P_\lambda E_\lambda)^=\}.$$
From this and the corresponding result with $\mathcal{Q}_\lambda$ in place of P_λ, together with (5), we obtain
$$0 \in co_{\mathcal{R}}(P_\lambda - P_\lambda E_\lambda)^=, 0 \in co_{\mathcal{R}}(\mathcal{Q}_\lambda E_\lambda). \tag{6}$$
From the maximality property in the definitions of P_λ and $\mathcal{Q}_\lambda$, we have $PE_\lambda \prec P$ and $\mathcal{Q}(I - E_\lambda) \prec \mathcal{Q}$ whenever P and $\mathcal{Q}$ are projections in $\mathcal{q}$ such that $0 < P \le I - P_\lambda$ and $0 < \mathcal{Q} \le I - \mathcal{Q}_\lambda$. This, together with Lemma (3.1.4) (ii), implies that
$$co_{\mathcal{R}}((I - P_\lambda)E_\lambda)^= \cap \mathcal{q} = \{0\}, \ \ co_{\mathcal{R}}((I - \mathcal{Q}_\lambda)(I - E_\lambda))^= \cap \mathcal{q} = \{0\}. \tag{7}$$
Suppose that $C \in co_{\mathcal{R}}(A)^= \cap \mathcal{q}$. Given any positive real number ε, we can choose α_0 in $\mathcal{D}$ so that $\|C - \alpha_0(A)\| < \varepsilon$. For each real number λ,
$$A \ge \lambda(I - E_\lambda) - \|A\|\, E_\lambda = \lambda I - (\|A\| + \lambda)E_\lambda,$$
and hence $\alpha_0(A) \ge \lambda I - (\|A\| + \lambda)\alpha_0(E_\lambda)$. It follows that
$$C \ge \alpha_0(A) - \varepsilon I \ge (\lambda - \varepsilon)I - (\|A\| + \lambda)\alpha_0(E_\lambda),$$
and that
$$C(I - P_\lambda) \ge (\lambda - \varepsilon)(I - P_\lambda) - (\|A\| + \lambda)\alpha_0\big((I - P_\lambda)E_\lambda\big).$$
Upon applying an arbitrary element β of $\mathcal{D}$ to this last inequality, we obtain
$$C(I - P_\lambda) \ge (\lambda - \varepsilon)(I - P_\lambda) - (\|A\| + \lambda)\beta\alpha_0\big((I - P_\lambda)E_\lambda\big). \tag{8}$$
Since
$$\phi \ne co_{\mathcal{R}}(\alpha_0((I - P_\lambda)E_\lambda))^= \cap \mathcal{q} \subseteq co_{\mathcal{R}}((I - P_\lambda)E_\lambda)^= \cap \mathcal{q},$$
it follows from (7) that $co_{\mathcal{R}}(\alpha_0((I - P_\lambda)E_\lambda))^= \cap \mathcal{q} = \{0\}$. Hence the last term on the right-hand side of (8) can be made arbitrarily small in norm by appropriate choice of β in $\mathcal{D}$. Thus $C(I - P_\lambda) \ge (\lambda - \varepsilon)(I - P_\lambda)$; since this has been proved for every positive ε,
$$C(I - P_\lambda) \ge \lambda(I - P_\lambda) \qquad (\lambda \in \mathbb{R}).$$
Since $F_\mu \ge F_\lambda \ge P_\lambda$ when $\mu > \lambda$, we obtain
$$C\big(F_\mu - F_\lambda\big) \ge \lambda\big(F_\mu - F_\lambda\big) \quad (\lambda, \mu \in \mathbb{R}, \lambda < \mu)$$
upon multiplying the previous inequality by $F_\mu - F_\lambda$. It follows that, for any real numbers $\lambda_0, \lambda_1, \ldots, \lambda_m$ such that $\lambda_0 < -\|A\|, \lambda_m \ge \|A\|$, and $\lambda_0 < \lambda_1 < \ldots < \lambda_m$,

$$C = \sum_{j=1}^{m} C(F_{\lambda_j} - F_{\lambda_{j-1}}) \geq \sum_{j=1}^{m} \lambda_{j-1}\left(F_{\lambda_j} - F_{\lambda_{j-1}}\right).$$

Hence $C > \int dF_\lambda$; so we have shown that

$$C \leq C_{min} \qquad (C \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{q}). \tag{9}$$

An argument very similar to the one set out in the preceding paragraph shows that

$$C \leq C_{max} \qquad (C \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{q}). \tag{10}$$

Given C in $co_{\mathcal{R}}(A)^{=} \cap \mathcal{q}$ and $\varepsilon(> 0)$, we again choose α_0 in $\mathcal{D}$ so that $\|C - \alpha_0(A)\| < \varepsilon$. From the inequalities

$$A \leq vE_v + \|A\|(I - E_v) \leq vI + (\|A\| - v)(I - E_v),$$

we deduce (corresponding to (8)) that

$$C(I - Q_v) \leq (v + \varepsilon)\,(I - Q_v) + (\|A\| - v)\beta\alpha_0((I - Q_v)(I - E_v))$$

for all v in $\mathbb{R}$ and β in $\mathcal{D}$. With v fixed for the time being, it follows from (7) that the second term on the right-hand side of the last inequality can be made arbitrarily small in norm by an appropriate choice of β in $\mathcal{D}$. Thus

$$C(I - Q_v) \leq (v + \varepsilon)(I - Q_v).$$

When first $\varepsilon \downarrow 0$ and then $v \downarrow \mu$, we obtain $CG_\mu \leq \mu G_\mu$, and this leads easily to a proof of (10).

So far, we have constructed the elements C_{min} and C_{max} of $\mathcal{q}$ from their spectral resolutions, and proved that

$$C_{min} \leq C \leq C_{max}$$

for each C in $co_{\mathcal{R}}(A)^{=} \cap \mathcal{q}$. In order to complete the proof of the theorem, we have to show that $co_{\mathcal{R}}(A)^{=} \cap \mathcal{q}$ contains each self-adjoint C in $\mathcal{q}$ such that $C_{min} \leq C \leq C_{max}$. To this end, it suffices (Corollary (3.1.3) (i)) to prove that C_{min} and C_{max} lie in $co_{\mathcal{R}}(A)^{=} \cap \mathcal{q}$.

Given real numbers $\mu, \varepsilon(> 0)$, choose v so that $\mu < v < \mu + \frac{1}{2}\varepsilon$. We have already noted that

$$A \leq vI + (\|A\| - v)(I - E_v),$$

so

$$P_vA \leq vP_v + (\|A\| - v)(P_v - P_vE_v). \tag{11}$$

From (6), we can choose α_0 in $\mathcal{D}$ so that

$$\|(\|A\| - v)\alpha_0(P_v - P_vE_v)\| < \frac{1}{2}\varepsilon.$$

It now follows from (11) that

$$\begin{aligned}\alpha_0(P_vA) &\leq vP_v + (\|A\| - v)\alpha_0(P_v - P_vE_v)\\ &\leq \left(\mu + \frac{1}{2}\varepsilon\right)P_v + (\|A\| - v)\alpha_0(P_v - P_vE_v)P_v\\ &\leq (\mu + \varepsilon)P_v.\end{aligned}$$

Since $\mu < v$, we have $F_\mu < P_v$; so, upon multiplying the last inequality by F_μ, we obtain

$$\alpha_0(F_\mu A) \leq (\mu + \varepsilon)F_\mu. \tag{12}$$

Accordingly, we have shown that if $\mu \in \mathbb{R}$ and $\varepsilon > 0$, there exists α_0 in $\mathcal{D}$ such that (12) is satisfied.

Let $\lambda_0, \ldots, \lambda_m$ be real numbers such that

$$\lambda_0 < -\|A\|, \quad \lambda_m \geq \|A\|, \quad \lambda_0 < \lambda_1 < \ldots < \lambda_m,$$

And $\lambda_j - \lambda_{j-1} < \varepsilon (j = 1, \ldots, m)$. From the preceding paragraph, we can choose $\alpha_1, \ldots, \alpha_m$ in $\mathcal{D}$ so that

$$\alpha_j\left(F_{\lambda_j}A\right) \leq \left(\lambda_j + \varepsilon\right)F_{\lambda_j} \quad (j = 1, \ldots, m).$$

Upon multiplying this inequality by $I - F_{\lambda_{j-1}}$, and then summing the resulting inequality over $j = 1, \dots, m$, we obtain

$$\sum_{j=1}^{m} \alpha_j \left(\left(F_{\lambda_j} - F_{\lambda_{j-1}} \right) A \right) \leq \sum_{j=1}^{m} (\lambda_j + \varepsilon) \left(F_{\lambda_j} - F_{\lambda_{j-1}} \right)$$

$$= \sum_{j=1}^{m} \lambda_j \left(F_{\lambda_j} - F_{\lambda_{j-1}} \right) + \varepsilon I$$

$$\leq C_{min} + 2\varepsilon I.$$

Since the projections $F_{\lambda_j} - F_{\lambda_{j-1}} (j = 1, \dots, m)$ lie in $\mathcal{C}$ and have sum I, it follows from Lemma (3.1.2) (i) that the operator on the left-hand end of the last chain of inequalities has the form $\alpha(A)$, for some α in $\mathcal{D}$. Since $\alpha(A) \leq C_{min} + 2\varepsilon I$, it follows that $\beta\alpha(A) \leq C_{min} + 2\varepsilon I$ for each β in $\mathcal{D}$, and hence that $B \leq C_{min} + 2\varepsilon I$ for each B in $co_{\mathcal{R}}(\alpha(A))^{=}$. In particular, we can take for B an element C_0 of $co_{\mathcal{R}}(\alpha(A))^{=} \cap \mathcal{C}$; then, $C_0 \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}, C_0 \leq C_{min} + 2\varepsilon I$, and $C_{min} \leq C_0$ by (9). These inequalities entail

$$\|C_{min} - C_0\| \leq 2\varepsilon.$$

Since C_{min} can be approximated arbitrarily closely in norm by elements of the normclosed set $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$, it follows that $C_{min} \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$.

A similar argument shows that $C_{max} \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ (and so completes the proof of the theorem). We have already noted that

$$A \geq \lambda I - (\|A\| + \lambda) E_\lambda,$$

and from this

$$\mathcal{Q}_\lambda A \geq \lambda \mathcal{Q}_\lambda - (\|A\| + \lambda) \mathcal{Q}_\lambda E_\lambda.$$

From (6), we can choose α_0 in $\mathcal{D}$ so that

$$\|(\|A\| + \lambda)\alpha_0(\mathcal{Q}_\lambda E_\lambda)\| < \varepsilon.$$

It then follows that $\alpha_0(\mathcal{Q}_\lambda A) \geq (\lambda - \varepsilon)\mathcal{Q}_\lambda$. Upon multiplying this last inequality by $I - G_\lambda (\leq \mathcal{Q}_\lambda$, because $I - \mathcal{Q}_\lambda \leq G_\lambda$ from the definition of G_λ), we obtain

$$\alpha_0((I - G_\lambda)A) \geq (\lambda - \varepsilon)(I - G_\lambda). \tag{13}$$

Accordingly, we have shown that if $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, there exists α_0 in $\mathcal{D}$ such that (13) is satisfied. An argument similar to that of the preceding paragraph now completes the proof that $C_{max} \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$.

Theorem (3.1.7)[447]: Suppose that $\mathcal{R}$ is a properly infinite von Neumann algebra with centre $\mathcal{C}, A = A^* \in \mathcal{R}$, and $\{E_\lambda\}$ is the spectral resolution of A. For each real number λ, define projections F_λ^w and G_λ^w in $\mathcal{C}$ by

$$F_\lambda^w = \bigwedge_{\mu > \lambda} P_\mu^w, \qquad G_\lambda^w = \bigwedge_{\mu > \lambda} (I - \mathcal{Q}_\mu^w),$$

Where $I - P_\lambda^w$ and $I - \mathcal{Q}_\lambda^w$ are the largest projections in $\mathcal{C}$ such that the projections $(I - P_\lambda^w)E_\lambda$ and $(I - \mathcal{Q}_\lambda^w)(I - E_\lambda)$ in $\mathcal{R}$ are finite. Then $\{F_\lambda^w\}$ and $\{G_\lambda^w\}$ are the spectral resolutions of two self-adjoint elements of $\mathcal{C}, C_{min}^w$ and C_{max}^w, respectively; moreover, $C_{min}^w \leq C_{max}^w$, and

$$co_{\mathcal{R}}(A)^{-} \cap \mathcal{C} = \{C \in \mathcal{C} : C = C^*, C_{min}^w \leq C \leq C_{max}^w\}.$$

Proof. In its broad structure, the proof is similar to (but a little simpler than) that of Theorem (3.1.6) Throughout the argument, we shall omit the superscript 'w' (for 'weakoperator') occurring on the symbols $F_\lambda^w, G_\lambda^w, P_\lambda^w, \mathcal{Q}_\lambda^w, C_{min}^w$, and C_{max}^w.

When $\lambda \leq \mu$, we have

$$\left(I - P_\mu\right)E_\lambda \leq \left(I - P_\mu\right)E_\mu \quad and \quad (I - \mathcal{Q}_\lambda)\left(I - E_\mu\right) \leq (I - \mathcal{Q}_\lambda)(I - E_\lambda),$$

and the projections on the right-hand sides of these two inequalities are finite in $\mathcal{R}$; so $(I - P_\mu)E_\lambda$ and $(I - \mathcal{Q}_\lambda)(I - E_\mu)$ are finite projections in $\mathcal{R}$. It now follows, from the maximality property in the definitions of $I - P_\lambda$ and $I - \mathcal{Q}_\mu$, that $I - P_\mu \le I - P_\lambda$ and $I - \mathcal{Q}_\lambda \le I - \mathcal{Q}_\mu$. Thus

$$P_\lambda \le P_\mu, \quad I - \mathcal{Q}_\lambda \le I - \mathcal{Q}_\mu, \text{ when } \quad \lambda \le \mu \tag{14}$$

When $\lambda < -\|A\|, E_\lambda$is 0; so $I - P_\lambda$ is I and (since $\mathcal{R}$ is properly infinite) $I - \mathcal{Q}_\lambda$ is 0. When $\lambda \ge \|A\|$,E_λ is I; so $I - P_\lambda$ is 0 (because $\mathcal{R}$ is properly infinite) and $I - \mathcal{Q}_\lambda$ is I. Thus

$$P_\lambda = I - \mathcal{Q}_\lambda = 0 \qquad (\lambda < -\|A\|\ ), \tag{15}$$

$$p_\lambda = I - \mathcal{Q}_\lambda = I \quad (\lambda \ge \|X\|). \tag{16}$$

The argument used at the corresponding stage of the proof of Theorem (3.1.6) now shows that $\{F_\lambda\}$ and $\{G_\lambda\}$ are the spectral resolutions of two self-adjoint elements of $\mathcal{q}$,

$$C_{min} = \int_{\mathbb{R}} \lambda dF_\lambda, \quad C_{max} = \int_{\mathbb{R}} \lambda dG_\lambda. \tag{17}$$

It is apparent from the defining properties of P_λ and $\mathcal{Q}_\lambda$ that the projections $(I - P_\lambda)E_\lambda, (I - \mathcal{Q}_\lambda)(I - E_\lambda)$ in $\mathcal{R}$ are finite, but that $P_\lambda E_\lambda$ (unless $P_\lambda = 0$) and $\mathcal{Q}_\lambda(I - E_\lambda)$ (unless $\mathcal{Q}_\lambda = 0$) are properly infinite, and have central carriers P_λ and $\mathcal{Q}_\lambda$, respectively. From Lemma (3.1.5) (i),

$$P_\lambda \in co_{\mathcal{R}}(P_\lambda, E_\lambda)^-, \quad \mathcal{Q}_\lambda \in co_{\mathcal{R}}(\mathcal{Q}_\lambda - \mathcal{Q}_\lambda E_\lambda)^-; \tag{18}$$

so

$$0 \in co_{\mathcal{R}}(P_\lambda - P_\lambda E_\lambda)^-, \quad 0 \in co_{\mathcal{R}}(\mathcal{Q}_\lambda E_\lambda)^-. \tag{19}$$

From Lemma (3.1.5) (ii),

$$co_{\mathcal{R}}((I - P_\lambda)E_\lambda)^= \cap \mathcal{q} = \{0\}, co_{\mathcal{R}}\big((I - \mathcal{Q}_\lambda)(I - E_\lambda)\big)^- \cap \mathcal{q} = \{0\}. \tag{20}$$

Suppose that $C \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}$. There is a net $\{\alpha_j\}$ in $\mathcal{D}$ such that $\alpha_j(A) \to C$ (in the weak-operator topology). For each real λ we have $A \ge \lambda I - (\|A\| + \lambda)E_\lambda$, whence

$$A(I - P_\lambda) \ge \lambda(I - P_\lambda) - (\|A\| + \lambda)(I - P_\lambda)E_\lambda$$

and

$$\alpha_j(A)(I - P_\lambda) \ge \lambda(I - P_\lambda) - (\|A\| + \lambda)\alpha_j\big((I - P_\lambda)E_\lambda\big). \tag{21}$$

Since $co_{\mathcal{R}}((I - P_\lambda)E_\lambda)^-$ is weak-operator compact, we may assume (upon replacing $\{\alpha_j\}$ by a suitable subnet) that $\{\alpha_j((I - P_\lambda)E_\lambda)\}$ is weak-operator convergent to an element B of $co_{\mathcal{R}}((I - P_\lambda)E_\lambda)^-$. Upon taking limits in (21), we obtain

$$C(I - P_\lambda) \ge \lambda(I - P_\lambda) - (\|A\| + \lambda)B,$$

and thus

$$C(I - P_\lambda) \ge \lambda(I - P_\lambda) - (\|A\| + \lambda)\beta(B)(\beta \in \mathcal{D}). \tag{22}$$

Since $B \in co_{\mathcal{R}}((I - P_\lambda)E_\lambda)^-$, it results from (20) that

$$\phi \ne co_{\mathcal{R}}(B)^- \cap \mathcal{q} \subseteq co_{\mathcal{R}}((I - P_\lambda)E_\lambda)^- \cap \mathcal{q} = \{0\};$$

so there is a net $\{\beta_k\}$ in $\mathcal{D}$ such that $\beta_k(B) \to 0$. Upon writing β_k for β in (22), and taking limits over k, we obtain

$$C(I - P_\lambda) \ge \lambda(I - P_\lambda).$$

We can now deduce that

$$C \ge C_{min}(C \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}). \tag{23}$$

An argument very similar to the one set out in the preceding paragraph shows that

$$C \ge C_{max}(C \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}). \tag{24}$$

Corresponding to (21) and (22), we have the inequalities

$$\alpha_j(A)(I - \mathcal{Q}_v) \le v(I - \mathcal{Q}_v) + (\|A\| - v)\alpha_j(I - \mathcal{Q}_v)(I - E_v)),$$

$$C(I - \mathcal{Q}_v) \le v(I - \mathcal{Q}) + (\|A\| - v)\beta(B)\ (\beta \in \mathcal{D}),$$

where B is some element of $co_{\mathcal{R}}((I-Q_v)(I-E_v))^-$. It follows from (20) that $co_{\mathcal{R}}(B)^- \cap \mathcal{q} = \{0\}$; so by taking limits in the last inequality, as β runs through an appropriate net in $\mathcal{D}$, we have

$$C(I-Q_v) \leq v(I-Q_v).$$

When $v \downarrow \mu$, we obtain $CG_\mu \leq \mu G_\mu$, and this leads easily to a proof of (24).

So far, we have constructed the elements C_{min} and C_{max} of $\mathcal{q}$ from their spectral resolutions, and proved that

$$C_{min} \leq C \leq C_{max} \quad (C \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}).$$

It remains to show that $co_{\mathcal{R}}(A) \cap \mathcal{q}$ contains each self-adjoint C in $\mathcal{q}$ such that $C_{min} \leq C \leq C_{max}$. From Corollary (3.1.3) (i), it suffices to prove that C_{min} and C_{max} lie in $co_{\mathcal{R}}(A)^- \cap \mathcal{q}$.

For each real v, it follows from (25) (in the proof of Theorem (2.1.7)) that

$$\alpha(A)P_v \leq vP_v + (\|A\| - v)\alpha(P_v - P_vE_v) \quad (\alpha \in \mathcal{D}). \tag{25}$$

By (19), there is a net $\{\alpha_j\}$ in $\mathcal{D}$ such that $\alpha_j(P_v - P_vE_v) \to 0$ in the weak-operator topology. Since $co_{\mathcal{R}}(A)^-$ is compact, we may assume also (upon replacing $\{\alpha_j\}$ by an appropriate subnet) that $\{\alpha_j(A)\}$ is weak-operator convergent to an element B_v of $co_{\mathcal{R}}(A)^-$. By-writing α_j in place of α in (25), and taking limits over j, we obtain $B_vP_v \leq vP_v$.

Given λ in $\mathbb{R}$, we note that $F_\lambda \leq P_v$ and thus $B_vF_\lambda \leq vF_\lambda$, whenever $v > \lambda$. The family $\{B_v : v > \lambda\}$ is a net in $co_{\mathcal{R}}(A)^-$ (with the indices v directed downward), and so has a subnet convergent to an element A_λ of $co_{\mathcal{R}}(A)^-$. By taking limits over this subnet, in the relation $B_vF_\lambda \leq vF_\lambda$, we obtain $A_\lambda F_\lambda \leq \lambda F_\lambda$.

Let $\lambda_0, \ldots, \lambda_m$ be real numbers such that

$$\lambda_0 < -\|A\|, \qquad \lambda_m \geq \|A\|, \qquad \lambda_0 < \lambda_1 < \ldots < \lambda_m,$$

and $\lambda_j - \lambda_{j-1} < \varepsilon$ $(j = 1, \ldots, m)$. From the preceding paragraph, we can choose $A_1, \ldots, A_m$ in $co_{\mathcal{R}}(A)^-$ so that $A_jF_{\lambda_j} \leq \lambda_jF_{\lambda_j}$. Multiplication by $I - F_{\lambda_{j-1}}$, followed by summation over $j = 1, \ldots, m$, gives

$$\sum_{j=1}^{m} A_j(F_{\lambda_j} - F_{\lambda_{j-1}}) \leq \sum_{j=1}^{m} \lambda_j(F_{\lambda_j} - F_{\lambda_{j-1}}) \leq C_{min} + \varepsilon I.$$

From Lemma (3.1.2) (iii), the operator on the left-hand side of this chain of inequalities is an element A_0 of $co_{\mathcal{R}}(A)^-$. Choose C_0 in $co_{\mathcal{R}}(A_0)^{-1} \cap \mathcal{q}$. Then $C_0 \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}, C_{min} \leq C_0$ by (23), and $C_0 \leq C_{min} + \varepsilon I$ since $A_0 < C_{min} + \varepsilon I$; so $\|C_{min} - C_0\| < \varepsilon$. Since C_{min} can be approximated, arbitrarily closely in norm, by elements of the (weak-operator, hence norm-) closed set $co_{\mathcal{R}}(A)^- \cap \mathcal{q}$, it now follows that $C_{min} \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}$.

A similar (but slightly simpler) argument shows that $C_{max} \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}$. In place of (25) we have the inequality

$$\alpha(A)Q_\lambda \geq \lambda Q_\lambda - (\|A\| + \lambda)\alpha(Q_\lambda E_\lambda) \quad (\alpha \in \mathcal{D}). \tag{26}$$

From this, together with (19), we can deduce that $A_\lambda Q_\lambda \geq \lambda Q_\lambda$ for some A_λ in $co_{\mathcal{R}}(A)^-$. Since $I - G_\lambda \leq Q_\lambda$ (because $I - Q_\lambda \leq G_\lambda$ from the definition of G_λ), it now follows that

$$A_\lambda(I - G_\lambda) \geq \lambda(I - G_\lambda).$$

An argument similar to that of the preceding paragraph now completes the proof that

$$C_{max} \in co_{\mathcal{R}}(A)^- \cap \mathcal{q}.$$

For this purpose, let P be the largest projection in $\mathcal{q}$ such that the von Neumann algebra $\mathcal{R}P$ is finite, and let τ be the centre-valued trace on $\mathcal{R}P$. It has already been noted that the sets $co_{\mathcal{R}P}(AP)^= \cap \mathcal{q}P$ and $co_{\mathcal{R}P}(AP)^- \cap \mathcal{q}P$ both consist of the single point $\tau(AP)$; for example, this is a consequence of the norm continuity, and strong operator continuity on bounded sets, of τ. If $P \neq I$, the von Neumann algebra $\mathcal{R}(I-P)$ is properly infinite, and so the sets

$$co_{\mathcal{R}(I-P)}(A(I-P))^{=}, \quad co_{\mathcal{R}(I-P)}(A(I-P))^{-},$$

intersect $\ell(I-P)$ in closed intervals, $[C^n_{min}, C^n_{max}]$ and $[C^w_{min}, C^w_{max}]$, respectively, the endpoints being determined as in Theorems (3.1.7) and (3.1.8). It now suffices to define

$$C^n_1 = \tau(AP) + C^n_{min}, \quad C^n_2 = \tau(AP) + C^n_{max},$$
$$C^n_1 = \tau(AP) + C^n_{min}, \quad C^n_2 = \tau(AP) + C^n_{max}.$$

The inequalities $C^w_1 \leq C^n_1, C^n_2 \leq C^w_2$ follow from the inclusion

$$co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} \subseteq co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}.$$

by Lemma (3.1.4) (ii) (with $P = I$) and Lemma (3.1.5) (i). Thus

$$co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} \neq co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}.$$

Theorems (3.1.8)[447]: If $\mathcal{R}$ is a von Neumann algebra with centre $\mathcal{C}$, the following four conditions are equivalent:

(i) each projection P in $\mathcal{C}$ that is countably decomposable relative to $\mathcal{C}$ is countably decomposable relative to $\mathcal{R}$;

(ii) $E \sim C_E$ for each properly infinite projection E in $\mathcal{R}$;

(iii) $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} = co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ for each self-adjoint element A of $\mathcal{R}$;

(iv) $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} = co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ for each A in $\mathcal{R}$.

Proof. We prove the equivalence of the first three conditions without appeal to **[120],** and then make use of [120, Corollaries 4.14 and 4.17] in showing that these three conditions are equivalent to (iv). Upon expressing $\mathcal{R}$ as the direct sum of a finite von Neumann algebra and a properly infinite von Neumann algebra, it suffices to consider the two summands separately. In the case of a finite von Neumann algebra, Condition (i) is satisfied by [119, p. 99, Proposition 9 (ii)], Condition (ii) is satisfied vacuously, and Conditions (iii) and (iv) are satisfied because $co_{\mathcal{R}}(A)^{=}$ and $co_{\mathcal{R}}(A)^{-}$ both meet $\mathcal{C}$ at just one point, the (centrevalued) trace of A. We therefore assume henceforth that $\mathcal{R}$ is properly infinite.

Suppose that Condition (i) is satisfied. If E is a properly infinite projection in $\mathcal{R}$, let $\{P_j\}$ be an orthogonal family of non-zero projections in $\mathcal{C}$, each cyclic in $\mathcal{C}$, with sum C_E. For each index J, P_jE is a properly infinite projection in $\mathcal{R}$, with central carrier P_j that is countably decomposable (in $\mathcal{C}$, and hence, by (i), in $\mathcal{R}$). From [119, p. 299, Corollaire 5], $P_jE \sim P_j$; so

$$E = C_E E = \sum P_j E \sim \sum P_j = C_E.$$

Hence (i) implies (ii).

We prove next that (ii) implies (i). To this end, suppose that Condition (ii) is satisfied, and that P is a projection in $\mathcal{C}$ and countably decomposable relative to $\mathcal{C}$. We have to show that P is countably decomposable relative to $\mathcal{R}$. From [119, p. 19, Corollaire], the abelian von Neumann algebra $\mathcal{C}P$ has a separating vector x; so P has range $[\mathcal{C}'x]$, and $P = C_E$, where E is the cyclic projection in $\mathcal{R}$ with range $[\mathcal{R}'x]$. By the comparison theorem, applied to the projections $E(\leq P)$ and P, there is a projection Q in $\mathcal{C}$ such that $Q \leq P, QE \sim Q$, and $P_1E \prec P_1$ whenever P_1 is a projection in $\mathcal{C}$ and $0 < P_1 \leq P - Q$. From this last condition, $(P-Q)E$ is finite; for otherwise, P_1E is properly infinite for some non-zero central subprojection P_1 of $P - Q$, and $P_1E \sim C_{P,E} = P_1$ by (ii), contradicting the final assertion of the preceding sentence. Since $QE(\leq E)$ is cyclic (and, hence, countably decomposable) in $\mathcal{R}$, the same is true of $Q(\sim QE)$.

In order to complete the proof that (ii) implies (i), it now remains to show that $P - Q$ is countably decomposable relative to $\mathcal{R}$. Since $P - Q$ is the central carrier of the finite projection $(P-Q)E$, the von Neumann algebra $\mathcal{R}(P-Q)$ is semi-finite (as well as properly infinite). It follows (for example, as a consequence of [119, p. 218, Corollaire 2] that there

is an orthogonal family $\{Q_j\}$ of projections in $\mathcal{C}$ with sum $P-Q$, each Q_j being itself the sum of a (necessarily infinite) orthogonal family of projections in $\mathcal{R}$ each equivalent to the finite projection Q_jE. Since $P-Q$ is countably decomposable relative to $\mathcal{C}$, the family $\{Q_j\}$ is countable, and it suffices to prove that each $Q_j$is countably decomposable in $\mathcal{R}$. Now Q_j has the form $\sum F_k$, where $\{F_k\}$ is an infinite orthogonal family of projections in $\mathcal{R}$ each equivalent to Q_jE. For each k, F_k is countably decomposable in $\mathcal{R}$ (since $F_k \precsim E$). If G is the sum of a countably infinite subfamily of $\{F_k\}$, then G is a properly infinite countably decomposable projection in
$\mathcal{R}$, and

$$C_G = C_{F_k} = C_{Q_jE} = Q_jC_E = Q_j.$$

By Condition (ii), $Q_j \sim G$, and therefore Q_j is countably decomposable in $\mathcal{R}$. This completes the proof that (ii) implies (i).

We prove next that (ii) implies (iii). Suppose that $\mathcal{R}$ (still assumed properly infinite) satisfies Condition (ii). Let $\{E_\lambda\}$ be the spectral resolution of a self-adjoint element A of $\mathcal{R}$, and define central projections $P_\lambda^n, Q_\lambda^n, P_\lambda^w, Q_\lambda^w$ as in Theorems (3.1.6) and (3.1.7). Since $P_\lambda^n E_\lambda \sim P_\lambda^n$ and P_λ^n is properly infinite, the projection $P_\lambda^n E_\lambda$ in $\mathcal{R}$ is properly infinite and has central carrier P_λ^n. At the same time, $(I-P_\lambda^w)E_\lambda$ is finite. Thus, $P_\lambda^n(I-P_\lambda^w)=0$, and $P_\lambda^n \le P_\lambda^w$. On the other hand, the definition of P_λ^w implies that $P_\lambda^w E_\lambda$ is properly infinite and has central carrier P_λ^w; so $P_\lambda^w E_\lambda \sim P_\lambda^w$, by (ii). Since P_λ^n is the largest central projection such that $P_\lambda^n E_\lambda \sim P_\lambda^n$, we have $P_\lambda^w \le P_\lambda^n$.

We have now proved that $P_\lambda^n = P_\lambda^w$, and a similar argument shows that $Q_\lambda^n = Q_\lambda^w$. It follows (with the notation introduced in Theorems (3.1.6) and (3.1.7)) that $F_\lambda^n = F_\lambda^w, G_\lambda^n, = G_\lambda^w, C_{min}^n = C_{min}^w$, and $C_{max}^n = C_{max}^w$; so

$$co_{\mathcal{R}}(A)^= \cap \mathcal{C} = [C_{min}^n, C_{max}^n] = [C_{min}^w, C_{max}^w] = co_{\mathcal{R}}(A)^- \cap \mathcal{C}.$$

This completes the proof that (ii) implies (iii).

Now, suppose that (iii) is satisfied, and let E be a properly infinite projection in $\mathcal{R}$. From Lemma (3.1.5) (i), $C_E \in co_{\mathcal{R}}(E)^- \cap \mathcal{C}$; so by (iii),

$$C_E \in co_{\mathcal{R}}(E)^= \cap \mathcal{C}. \tag{27}$$

We have to show that $E \sim C_E$. If this is not so, it follows from the comparison theorem that there is a projection P in $\mathcal{C}$ such that $0 < P \le C_E$, and such that $QE \prec Q$ whenever Q is a projection in $\mathcal{C}$ and $0 < Q \le P$. From (27),

$$0 \ne P \in co_{\mathcal{R}}(PE)^= \cap \mathcal{C},$$

and this contradicts the conclusion of Lemma (3.1.4) (ii). Hence $E \sim C_E$ (and (iii) implies (ii)).

So far, we have proved the equivalence of Conditions (i), (ii), and (iii). It is apparent that (iv) implies (iii); so it now suffices to show that (i) implies (iv). Suppose, then, that (i) is satisfied. Given A in $\mathcal{R}$ and C in $co_{\mathcal{R}}(A)^- \cap \mathcal{C}$, we have to show that $C \in co_{\mathcal{R}}(A)^= \cap \mathcal{C}$. Let $\{P_j\}$ be an orthogonal family of cyclic projections in $\mathcal{C}$, with sum I. The finite subsums of $\sum P_j$, form an increasing net $\{Q_k\}$ of projections in $\mathcal{C}$, strong-operator convergent to I, in which every Q_k is countably decomposable (in $\mathcal{C}$, and hence, by (i), in $\mathcal{R}$). For each index k,

$$CQ_k \in co_{\mathcal{R}Q_k}(AQ_k)^- \cap \mathcal{C}Q_k,$$

and since $\mathcal{R}Q_k$is countably decomposable, it follows from [120, Corollary 4.17] that

$$CQ_k \in co_{\mathcal{R}Q_k}(AQ_k)^= \cap \mathcal{C}Q_k.$$

If C_0 is any element of $co_{\mathcal{R}}(A)^= \cap \mathcal{C}$, it now follows that

$$C + (C_0 - C)(I - Q_k) = CQ_k + C_0(I - Q_k) \in co_{\mathcal{R}}(A)^= \cap \mathcal{C}.$$

Upon taking limits as $Q_k \uparrow I$, and using the fact [120, Corollary 4.14] that $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ is weak-operator closed, we deduce that $C \in co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$. Thus (i) implies (iv). A von Neumann algebra that satisfies the first (and, hence, all four) of the conditions set out in Theorem (3.1.8) is said to be countably decomposable over its centre (see [119, p. 138, Exercise 5]).

Suppose that $\mathcal{R}$ is a factor, and write $\mathcal{C}$ for the centre $\{cI: c \in \mathbb{C}\}$ of $\mathcal{R}$. If A is a selfadjoint element of $\mathcal{R}$, each of the sets $co_{\mathcal{R}}(A)^{=} \cap \mathcal{C}$ and $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ is a bounded closed convex subset of the one-dimensional real Banach space $\mathcal{C}_h (= \{rI: r \in \mathbb{R}\})$; so there are bounded closed intervals $[a_n, b_n]$ and $[a_w, b_w]$ such that

$$co_{\mathcal{R}}(A)^{=} \cap \mathcal{C} = \{rI: r \in [a_n, b_n]\}, \tag{28}$$
$$co_{\mathcal{R}}(A)^{-} \cap \mathcal{C} = \{rI: r \in [a_w, b_w]\}. \tag{29}$$

In the notation,

$$C_1^n = a_n I, \qquad C_2^n = b_n I, \qquad C_1^w = a_w I, \qquad C_2^w = b_w I.$$

In the factor case, is to provide a method of determining a_n, b_n, a_w, b_w.
If $\mathcal{R}$ is a finite factor, $a_n = b_n = a_w = b_w = \tau(A)$, where τ is the unique tracial state of $\mathcal{R}$. Suppose next that $\mathcal{R}$ is an infinite factor, and that $\{E_\lambda\}$ is the spectral resolution of the self-adjoint element A of $\mathcal{R}$. Since 0 and I are the only projections in $\mathcal{C}$, the definitions of P_λ^n and Q_λ^n, as set out in Theorem (3.1.6), can be reformulated as follows:

$$P_\lambda^n = \begin{cases} 0 & if E_\lambda \prec I, \\ I & if\ E_\lambda \sim I, \end{cases} \qquad Q_\lambda^n = \begin{cases} 0 & if I - E_\lambda \prec I, \\ I & if\ I - E_\lambda \sim I. \end{cases}$$

From this, $C_{min}^n = a_n I$ and $C_{max}^n = b_n I$, where

$$a_n = \sup\{\lambda \in \mathbb{R}: E_\lambda \prec I\}, \quad b_n = \inf\{\lambda \in \mathbb{R}: I - E_\lambda \prec I\}. \tag{30}$$

In a similar way, it follows from Theorem (3.1.7) that $C_{min}^w = a_w I$ and $C_{max}^w = b_w I$, where

$$a_w = \sup\{\lambda \in \mathbb{R}: E_\lambda\ is\ finite\}, \quad b_w = \inf\{\lambda \in \mathbb{R}: I - E_\lambda\ is\ finite\}. \tag{31}$$

If $\mathcal{R}$ is a countably decomposable Type III factor, each non-zero projection E in $\mathcal{R}$ is infinite, and satisfies $E \sim I$ by [119, p. 299, Corollaire]. In this case, it follows from (30) and (31) that

$$a_n = a_w = \sup\{\lambda \in \mathbb{R}: E_\lambda = 0\},$$
$$b_n = b_w = \inf\{\lambda \in \mathbb{R}: E_\lambda = I\}.$$

Thus $[a_n, b_n]$ and $(a_w, b_w]$ both coincide with the smallest interval containing the spectrum, $sp(A)$, of A [461, p. 136, Corollary 16].
Now suppose that $\mathcal{R}$ is a factor of Type III but is not countably decomposable. Since each non-zero projection in $\mathcal{R}$ is infinite, the reasoning of the preceding paragraph still applies to show that $[a_w, b_w]$ is the smallest interval that contains $sp(A)$. Since $\mathcal{R}$ is not countably decomposable, it has (non-zero) proper norm-closed two-sided ideals (for example, the set of all operators in $\mathcal{R}$ having countably decomposable range projections), and among these ideals is a greatest one, $\mathcal{J}$ [119, p. 256, Corollaire 3]. Moreover, a projection E in $\mathcal{R}$ lies in $\mathcal{J}$ if and only if $E \prec I$ (see, for example, [460, Corollary 2.2]). Thus (30) can be rewritten in the form

$$a_n = \sup\{\lambda \in \mathbb{R}: \varphi(E_\lambda) = 0\}, \qquad b_n = \inf\{\lambda \in \mathbb{R}: \varphi(E_\lambda) = I\},$$

where $\varphi: \mathcal{R} \to \mathcal{R}/\mathcal{J}$ is the quotient mapping. From this, it is not difficult to deduce that $[a_n, b_n]$ is the smallest interval that contains $sp((\varphi(A))$. It should be noted that the results described in this paragraph (and the next one) relate to [117, Lemma 8] and (conceptually) to [120, Theorems 4.12, 4.16].

Finally, suppose that $\mathcal{R}$ is a factor of Type I_∞ or II_∞. Among the (non-zero) proper norm-closed two-sided ideals in $\mathcal{R}$, there is a smallest $\mathcal{J}_1$ and a largest $\mathcal{J}_2$. The ideal $\mathcal{J}_1$ is the norm-closure of the set of all operators in $\mathcal{R}$ with finite range projections. A projection

E lies in $\mathcal{J}_1$ if and only if E is finite, and in $\mathcal{J}_2$ if and only if $E \prec I$. Moreover, $\mathcal{J}_1 = \mathcal{J}_2$ if and only if $\mathcal{R}$ is countably decomposable. Let $\varphi_j: \mathcal{R} \to \mathcal{R}/\mathcal{J}_j$ be the quotient mapping, for $j = 1,2$. By the type of reasoning used in the previous paragraph, we can show that $[a_n, b_n]$ is the smallest interval containing $sp(\varphi_2(A))$, and that $[a_w, b_w]$ is the smallest interval containing $sP(\varphi_1(A))$.

Given an element A of a von Neumann algebra $\mathcal{R}$, we have studied the norm-closed convex hull $co_{\mathcal{R}}(A)^{=}$ and the weak-operator closed convex hull $co_{\mathcal{R}}(A)^{-}$ of the set $\{UAU^*: U \in \mathcal{U}\}$ $(= u_{\mathcal{R}}(A))$, where $\mathcal{U}$ is the unitary group of $\mathcal{R}$. We now consider, briefly, the weak-operator closure $u_{\mathcal{R}}(A)^{-}$ of $u_{\mathcal{R}}(A)$. It is shown in [462, Corollary 2] that, if $\mathcal{R}$ is a properly infinite von Neumann algebra with centre $\mathcal{C}$, then $u_{\mathcal{R}}(A)^{-}$ meets $\mathcal{C}$, and the weak-operator closed convex hull of $u_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ is $co_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$. The fact that $u_{\mathcal{R}}(A)^{-}$ meets $\mathcal{C}$ can be deduced also, in the case of an infinite factor, from [117, Lemma 4]. It is of interest to ask [461, Problem] whether $u_{\mathcal{R}}(A)^{-} \cap \mathcal{C}$ is already convex. Another fragment of information concerning $u_{\mathcal{R}}(A)^{-}$, this time in the case of certain factors $\mathcal{R}$ of Type II$_1$ is given in Proposition (3.1.9) below.

Suppose that G is a discrete group with unit element e, in which each element other than e has an infinite conjugacy class. For each g in G, we denote by L_g the unitary operator acting on the Hilbert space $l_2(G)$, defined by

$$(L_g x)(h) = x(g^{-1}h) \ (x \in l_2(G), h \in G).$$

The weak-operator closed linear span of the set $\{L_g: g \in G)$ is a factor $\mathcal{Y}_G$ of Type II$_1$ acting on $l_2(G)$. If x_g (in $l_2(G)$) is defined by

$$x_g(g) = 1, \qquad x_g(h) = 0 \quad (h \in G, h \neq g),$$

then $\{x_g: g \in G\}$ is an orthonormal basis of $l_2(G)$ and each x_g is a separating and generating trace vector for $\mathcal{Y}_G$. Each element A in $\mathcal{Y}_G$ is the operator of left convolution by an element $w(= Ax_e)$ of $l_2(G)$; that is,

$$(Ax)(h) = (w * x)(h) = \sum_{g \in G} w(hg^{-1})x(g) \ (x \in l_2(G), h \in G).$$

For these facts, see [119, p. 282].

The following result was observed during the collaborative investigation that led to the joint [475].

Proposition (3.1.9)[447]: There is a net $\{g_j\}$ of elements of G such that

$$\lim_j L_{g_j}^* A L_{g_j} = \tau(A)I$$

(in the weak-operator topology) for each A in $\mathcal{Y}_G$, where τ is the unique tracial state of $\mathcal{Y}_G$. If the group G is countable, the net $\{g_j\}$ can be a sequence.

Proof. The inner automorphisms of G, restricted to the set $G \backslash \{e\}$, form a group Π of permutations of $G \backslash \{e\}$, and each element of $G \backslash \{e\}$ has an infinite orbit under the action of Π. Given any finite subset S of $G \backslash \{e\}$, it follows from [456, Corollary 3.2] that $S \cap g_s S {g_s}^{-1} = \emptyset$, for some element g_s of G. When the finite subsets S of $G \backslash \{e\}$ are directed by inclusion, $\{g_s\}$ becomes a net of elements of G.

Suppose that $A \in \mathcal{Y}_G$, and let $w = Ax_e$. Given g, h, k in G, note that

$$\langle L_k^* A L_k x_g, x_h \rangle = \langle A L_k x_g, L_k x_h \rangle = \langle A x_{kg}, x_{kh} \rangle$$
$$= (A x_{kq})(kh) = (w * x_{kg})(kh),$$

and hence that

$$\langle L_k^* A L_k x_g, x_h \rangle = w(khg^{-1}k^{-1}) \qquad (g, h, k \in G). \qquad (32)$$

Suppose that $g \neq h$. Given $E(> 0)$, let S_ε be the finite subset

$$\{hg^{-1}\} \cup \{g_1 \in G\backslash\{e\}: |w(g_1)| \geq \varepsilon\}$$
of $G\backslash\{e\}$. If S is a finite subset of $G\backslash\{e\}$ and contains S_ε, then
$$g_s hg^{-1}g_s{}^{-1} \in g_s S g_s^{-1} \subseteq G\backslash(\{e\} \cup S) \subseteq G\backslash(\{e\} \cup S_\varepsilon),$$
and thus $|w(g_s hg^{-1}g_s{}^{-1})| < \varepsilon$; so, by (32),
$$\left|\langle L_{g_s}^* A L_{g_s} x_g, x_h\rangle\right| < \varepsilon \qquad (S \supseteq S_\varepsilon).$$
This shows that
$$\lim_s \langle L_{g_s}^* A L_{g_s} x_g, x_h\rangle = 0 \quad (g, h \in G, g \neq h). \tag{33}$$
At the same time, it follows from (32) that
$$\langle L_{g_s}^* A L_{g_s} x_g, x_g\rangle = w(e) = \tau(A),$$
and hence that
$$\lim_s \langle L_{g_s}^* A L_{g_s} x_g, x_g\rangle = \tau(A). \tag{34}$$
Since the net $\{L_{g_s}^* A L_{g_s}\}$ in $\mathcal{Y}_G$ is bounded, and $\{x_g: g \in G\}$ is an orthonormal basis of the Hilbert space on which $\mathcal{Y}_G$ acts, it follows from (33) and (34) that this net is weak-operator convergent to $\tau(A)I$.

If the group G is countable, then $G\backslash\{e\}$ is the union of an increasing sequence $\{S_1, S_2, S_3, \dots\}$ of finite sets. Let $\{g_1, g_2, g_3, \dots\}$ be the sequence in G defined by $g_j = g_{S_j}$. Then $\{g_j\}$ is a (cofinal) subnet of $\{g_S\}$; so $\{L_{g_j}^* A L_{g_j}\}$ is a subnet of $\{L_{g_s}^* A L_{g_j}\}$, and is therefore weak-operator convergent to $\tau(A)I$.

Section (3.2): A Finite and an Infinite Projection

The first interesting class of simple C^*-algebras (not counting the simple von Neumann algebras) were the UHF-algebras, also called Glimm algebras, constructed by Glimm in 1959 [144]. Several other classes of simple C^*-algebras were found over the following 25 years including the (simple) AF-algebras, the irrational rotation C^*-algebras, the free group C^*-algebras $C_{red}^*(F_n)$ (and other reduced group C^*-algebras), the Cuntz algebras O_n and the Cuntz-Krieger algebras O_A, C^*-algebras arising from minimal dynamical systems and from foliations, and certain inductive limit C^*-algebras, among many other examples. Parallel with the appearance of these examples of simple C^*-algebras it was asked if there is a classification for simple C^*-algebras similar to the classification of von Neumann factors into types. Inspired by work of Dixmier in the 1960's, Cuntz studied this and related questions about the structure of simple C^*-algebras in [136], [139] and [137].

A von Neumann algebra is simple precisely when it is either a factor of type I_n for $n < \infty$ (in which case it is isomorphic to $M_n(C)$), a factor of type II_1, or a separable factor of type III. This leads to the question if (non-type I) simple C^*-algebras can be divided into two subclasses, one that resembles type II_1 factors and another that resembles type III factors. A II_1 factor is an infinite-dimensional factor in which all projections are finite (in the sense of Murray-von Neumann's comparison theory for projections), and II_1 factors have a unique trace. A factor is of type III if all its non-zero projections are infinite, and type III factors admit no traces. Cuntz asked in [139] if each simple C^*-algebra similarly must have the property that its (non-zero) projections either all are finite or all are infinite. Or can a simple C^*-algebra contain both a (non-zero) finite and an infinite projection? We answer the latter question in the affirmative. In other words, we exhibit a simple (non-type I) C^*-algebra that neither corresponds to a type II_1 or to a type III factor.

It was shown in the early 1980's that simple C^*-algebras, in contrast to von Neumann factors, can fail to have non-trivial projections. Blackadar [127] and Connes [134] found examples of unital, simple C^*-algebras with no projections other than 0 and 1 $-$before it was shown that C_{red}^* (F_2) is a simple unital C^*-algebra with no non-trivial projections.

Simple C^*-algebras can fail to have projections in a more severe way: Blackadar found in [126] an example of a stably projectionless simple C^*-algebra. (A C^*-algebra A is stably projectionless if 0 is the only projection in $A \otimes k$.) Blackadar and Cuntz proved in [130] that every stably projectionless simple C^*-algebra is finite in the sense of admitting a (densely defined) quasitrace. (Every quasitrace on an exact C^*-algebra extends to a trace as shown by Haagerup [145] (and Kirchberg [148]).) These results lead to the dichotomy for a simple C^*-algebra A: Either A admits a (densely defined) quasitrace (in which case A is stably finite), or A is stably infinite, i.e., $A \otimes K$ contains an infinite projection.

Cuntz defined in [138] a simple C^*-algebra to be purely infinite if all its non-zero hereditary sub-C^*-algebras contain an infinite projection. Cuntz showed in [135] that his algebras $O_n, 2 \leq n \leq \infty$, are simple and purely infinite. The separable, nuclear, simple, purely infinite C^*-algebras are classified up to isomorphism by K- or KK-theory by the spectacular theorem of Kirchberg [149], [150] and Phillips [157]. This result has made it an important question to decide which simple C^*-algebras are purely infinite. We show here that not all stably infinite simple C^*-algebras A are purely infinite.

Villadsen [163] was the first to show that the K_0-group of a simple C^*-algebra need not be weakly unperforated; Villadsen [162] also showed that a unital, finite, simple C^*-algebra can have stable rank different from one thus answering in the negative two long-standing open questions for simple C^*-algebras.

If B is a unital, simple C^*-algebra with an infinite and a non-zero finite projection, then its semigroup of Murray−von Neumann equivalence classes of projections must fail to be weakly unperforated (see Question (3.2.35)). It is therefore no surprise that Villadsen's ideas play a crucial role in this article. The article is also a continuation of the work by the author in [159] and [160] where it is shown that one can find a C^*-algebra A such that $M_2(A)$ is stable but A is not stable; and, related to this, one can find a (non-simple) unital C^*-algebra B such that B is finite and $M_2(B)$ is properly infinite. We show here (Theorem (3.2.16)) that one can make this example simple by passing to a suitable inductive limit.

In ∮ 6 (added March 2002) an example is given of a crossed product C^*-algebra $D \rtimes_\alpha Z$, where D is an inductive limit of type I C^*-algebras, such that $D \rtimes_\alpha Z$ is simple and contains an infinite and a non-zero finite projection. This new example is nuclear and separable. It shows that simple C^*-algebras with this rather pathological behavior can arise from a quite natural setting. It shows that Elliott's classification conjecture (in its present formulation) does not hold (cf. Corollary (3.2.36)); and it also serves as an example of a separable nuclear simple C^*-algebra that is tensorially prime (cf. Corollary (3.2.33)).

A projection p in a C^*-algebra A is called infinite if it is equivalent (in the sense of Murray and von Neumann) to a proper subprojection of itself; and p is said to be finite otherwise. If p is non-zero and if there are mutually orthogonal subprojections p_1 and p_2 of p such that $p \sim p_1 \sim p_2$, then p is properly infinite. A unital C^*-algebra is said to be properly infinite if its unit is a properly infinite projection.

If p and q are projections in A, then let $p \oplus q$ denote the projection $\mathrm{diag}(p, q)$ in $M_2(A)$. Two projections $p \in M_n(A)$ and $q \in M_m(A)$ can be compared as follows: Write $p \sim q$ if there exists v in $M_{m,n}(A)$ such that $v^*v = p$ and $vv^* = q$, and write $p \precsim q$ if p is equivalent (in this sense) to a subprojection of q.

Where some well-known properties of properly infinite projections are recorded, O_∞ denotes the Cuntz algebra generated by infinitely many isometries with pairwise orthogonal range projections, and $\mathcal{E}_2$ is the Cuntz-Toeplitz algebra generated by two isometries with orthogonal range projections [135].

Proposition (3.2.1)[448]: The following five conditions are equivalent for every non-zero projection p in a C^*-algebra A:

(i) p is properly infinite;

(ii) $p\oplus p \precsim p$;

(iii) there is a unital $*$-homomorphism $\mathcal{E}_2 \longrightarrow pAp$;

(iv) there is a unital $*$-homomorphism $\mathcal{O}_\infty \to pAp$;

(v) for every closed two-sided ideal I in A, either $p \in I$ or $p + I$ is infinite in A/I.

The equivalences between (i), (ii) and (iii) are trivial. The equivalence between (iii) and (iv) follows from the fact that there are unital embeddings $\mathcal{E}_2 \to \mathcal{O}_\infty$and $\mathcal{O}_\infty \to \mathcal{E}_2$. The equivalence between (i) and (v) is proved in [151, Corollary 3.15]; a result that extends Cuntz' important observation from [136] that every infinite projection in a simple C^*-algebra is properly infinite.

We shall use the following two well-known results about properly infinite projections.

Lemma (3.2.2)[448]: Let p and q be projections in a C^*-algebra A. Suppose that p is properly infinite. Then $q \precsim p$ if and only if q belongs to the closed two-sided ideal in A generated by p.

Proof. If $q \precsim p$, then, by definition, $q \sim q_0 \leq p$ for some projection q_0 in A. This entails that q belongs to the ideal generated by p. Conversely, if q belongs to the ideal generated by p, then $q \precsim \oplus_{j=1}^{n} p$ for some n (cf. [162, Exercise 4.8]), and $\oplus_{j=1}^{n} P \precsim P$ if p is properly infinite by iterated applications of Proposition (3.2.1) (ii).

Proposition (3.2.3)[448]: Let B be the inductive limit of a sequence $B_1 \to B_2 \to B_3 \to \cdots$ of unital C^*-algebras with unital connecting maps. Then B is properly infinite if and only if B_n is properly infinite for all n larger than some n_0.

Proof. If B_n is properly infinite for some n, then there are unital $*$-homomorphisms $\mathcal{E}_2 \to B_n \to B$, and hence B is properly infinite. Conversely, if B is properly infinite, then there is a unital $*$-homomorphism $\mathcal{E}_2 \to B$. The C^*-algebra $\mathcal{E}_2$is semiprojective, as shown by Blackadar in [128]. By semiprojectivity (see again [128]), the unital $*$-homomorphism $\mathcal{E}_2 \to B$ lifts to a unital $*$-homomorphism $\mathcal{E}_2 \to \prod_{n=n_0}^{\infty} B_n$ for some n_0. This shows that B_n is properly infinite for all $n \geq n_0$.

We consider here complex vector bundles over the sphere S^2 and over finite products of spheres, $(S^2)^n$.

For each $k \leq n$, let $\pi_k: (S^2)^n \to S^2$ denote the kth coordinate mapping, and let $\varrho_{m,n}: (S^2)^m \to (S^2)^n$ be given by

$$\varrho_{m,n}(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_n), (x_1, x_2, \ldots, x_m) \in (S^2)^m, \quad (35)$$

when $m \geq n$.

Whenever $f: X \to Y$ is a continuous map and ξ is a k-dimensional complex vector bundle over Y, let $f^*(\xi)$ denote the vector bundle over X induced by f. Let $e(\xi) \in H^{2k}(Y, Z)$ denote the Euler class of ξ. Denote also by f^* the induced map $H^*(Y, Z) \to H^*(X, Z)$. By functoriality of the Euler class we have $f^*(e(\xi)) = e(f^*(\xi))$.

For any vector bundle ξ over $(S^2)^n$ and for every $m \geq n$ we have a vector bundle $\xi' = \varrho_{m,n}^*(\xi)$ over $(S^2)^m$. It follows from the Kunneth Theorem (see [155, Theorem A6]) that the map

$$\varrho_{m,n}^*: H^*((S^2)^n, Z) \to H^*((S^2)^m, Z)$$

is injective; so if $e(\xi)$ is non-zero, then so is $e(\xi')$. The main concern with vector bundles will be whether or not they have non-zero Euler class, and from that point of view it does not matter if we replace the base space $(S^2)^n$ with $(S^2)^m$ for some $m \geq n$.

We remind of some properties of the Euler class for complex vector bundles $\xi_1, \xi_2, \dots, \xi_n$ over a base space X. First of all we have the product formula (see [155, Property 9.6]):

$$e(\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n): e(\xi_1).\, e(\xi_2) \cdot \ldots \cdot e(\xi_n). \quad (36)$$

Let θ denote the trivial complex line bundle over X. The Euler class of θ is zero; and so it follows from the product formula that $e(\xi) = 0$ whenever ξ is a complex vector bundle that dominates θ in the sense that $\xi \cong \theta \oplus \eta$ for some complex vector bundle η.

Combining the formula

$$ch(\xi) = 1 + e(\xi) + \frac{1}{2} e(\xi)^2 + \frac{1}{6} e(\xi)^3 + \dots,$$

that relates the Chern character and the Euler class of a complex line bundle ξ(see 155, Problem 16-B]), with the fact that the Chern character is multiplicative, yields the formula

$$e(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n) = e(\xi_1) + e(\xi_2) + \ldots + e(\xi_n), \quad (37)$$

that holds for all complex line bundles $\xi_1, \dots, \xi_n$ over X.

Let ζ be a complex line bundle over S^2 such that its Euler class $e(\zeta)$, which is an element in $H^2(S^2, Z)$, is non-zero. (Any such line bundle will do, but we may take to be the Hopf bundle over S^2.*)* For each natural number n and for each lion-empty, finite subset $I = \{n_1, n_2, \dots, n_k\}$ of N define complex line bundles ζ_n and ζ_1 over $(S^2)^m$ (for all $m \geq n$ and $m \geq \max\{n_1, \dots, n_k\}$, respectively) by

$$\zeta_n = \pi_n^*(\zeta), \quad \zeta_I = \zeta_{n_1} \otimes \zeta_{n_2} \otimes \cdots \otimes \zeta_{n_k}, \quad (38)$$

where, as above, $\pi_n: (S^2)^m \to S^2$ is the nth coordinate map. The Euler classes (in $H^2((S^2)^m, Z)$) of these line bundles are by functoriality and equation (3.3) given by

$$e(\zeta_n) = \pi_n^*(e(\zeta)), \quad (39)$$

$$e(\zeta_I) = \pi_{n_1}^*(e(\zeta)) + \pi_{n_2}^*(e(\zeta)) + \ldots + \pi_{n_k}^*(e(\zeta)). \quad (40)$$

Lemma (3.2.4)[448]: For each n and for each $m \geq n$ there is a complex line bundle η_n over $(S^2)^m$ such that $\zeta_n \oplus \zeta_n \cong \theta \oplus \eta_n$.

Proof. Since

$$dim(\zeta \oplus \zeta) = 2 > 1 \geq \frac{1}{2}(dim(S^2) - 1),$$

it follows from [146, 9.1.2] that there is a complex vector bundle η over S^2 of dimension $dim(\eta) = 2 - 1 = 1$ such that $\zeta \oplus \zeta \cong \theta \oplus \eta$. We conclude that

$$\zeta_n \oplus \zeta_n = \pi_n^*(\zeta \oplus \zeta) \cong \pi_n^*(\theta \oplus \eta) = \theta \oplus \pi_n^*(\eta).$$

Proposition (3.2.5)[448]: Let $I_1, I_2, \dots, I_m$ be non-empty, finite subsets of N. The following three conditions are equivalent:

(i) $e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \ldots \oplus \zeta_{I_m}) \neq 0$;

(ii) for all subsets F of $\{1,2, \dots, m\}$ we have $\left|\bigcup_{j \in p} I_j\right| \geq |F|$;

(iii) There exists a matching $t_1 \in I_1, t_2 \in I_2, \dots, t_m \in I_m$ (i.e., the elements $t_1, \dots, t_m$ are pairwise distinct).

Proof. Choose N large enough so that each ζ_{I_j} is a vector bundle over $(S^2)^N$.

(ii)⟺(iii) is the Marriage Theorem (see any textbook on combinatorics).

(i)⇒(ii). Assume that $\left|\bigcup_{j \in F} I_j\right| < |F|$ for some (necessarily non-empty) subset $F = \{j_1, j_2, \dots, j_k\}$ of $\{1,2 \dots, m\}$, and write

$$J \stackrel{\text{def}}{=} \bigcup_{j \in F} I_j = \{n_1, n_2, \dots, n_l\}.$$

Let $\varrho: (S^2)^N \to (S^2)^l$ be given by $\varrho(x) = (\pi_{n_1}(x), \pi_{n_2}(x), \dots, \pi_{n_l}(x))$. Then

$$\xi \stackrel{\text{def}}{=} \zeta_{I_{j_1}} \oplus \zeta_{I_{j_2}} \oplus \cdots \oplus \zeta_{I_{j_k}} = \varrho^*(\eta)$$

for some k-dimensional vector bundle η over $(S^2)^l$. Now, $e(\eta)$ belongs to $H^{2k}((S^2)^l, Z)$, and $H^{2k}((S^2)^l, Z) = 0$ because $2k > 2l$. Hence $e(\xi) = \varrho^*(e(\eta)) = 0$, so by the product formula (2) we get

$$e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \ldots \oplus \zeta_{I_m}) = e(\xi). \prod_{j \notin F} e\left(\zeta_{I_j}\right) = 0.$$

(iii)⇒(i). Put

$$x_j = \pi_j^*\big(e(\zeta)\big) \in H^2((S^2)^N, Z), \qquad j = 1, 2, \ldots, N.$$

The element

$$z = x_1 \cdot x_2 \cdot \ldots x_N \in H^{2N}((S^2)^N, Z)$$

is non-zero by the Kunneth Theorem [135, Theorem A6]. Using that $x_i^2 = 0$ and that $x_i x_j = x_j x_i$ for all i, j it follows that if $i_1, i_2, \cdots, N$ belong to $\{1, 2, \ldots, N\}$, then

$$x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_N} = \begin{cases} z, & if\ i_1, \ldots, i_N\ are\ distinct, \\ 0 & ,otherwise. \end{cases} \tag{41}$$

Now, by (36) and (40),

$$e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \ldots \oplus \zeta_{I_m}) = e\big(\zeta_{I_1}\big) \cdot e\big(\zeta_{I_2}\big) \cdot \ldots \cdot e\big(\zeta_{I_m}\big)$$

$$= \left(\sum_{i \in I_1} \pi_i^*\big(e(\zeta)\big)\right) \cdot \left(\sum_{i \in I_2} \pi_i^*\big(e(\zeta)\big)\right) \cdot \ldots \cdot \left(\sum_{i \in I_m} \pi_i^*\big(e(\zeta)\big)\right)$$

$$= \left(\sum_{i \in I_1} x_i\right) \cdot \left(\sum_{i \in I_2} x_i\right) \cdot \ldots \cdot \left(\sum_{i \in I_m} x_i\right)$$

$$= \sum_{(i_1, \ldots, i_m) \in I_1 \times \ldots \times I_m} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_m}.$$

Assume that (iii) holds, and write

$$\{1, 2, \ldots, N\} \backslash \{t_1, t_2, \ldots, t_m\} = \{s_1, s_2, \ldots, S_{N-m}\}.$$

Let k denote the number of permutations σ on $\{1,2,\ldots,m\}$ such that $t_{\sigma(j)} \in I_j$ for $j = 1, 2, \ldots, m$. The identity permutation has this property, so $k \geq 1$. The formula for $e(\zeta_{I_1} \oplus \ldots \oplus \zeta_{I_m})$ above and equation (41) yield

$$e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \ldots \oplus \zeta_{I_m}) \cdot x_{S_1} \cdot x_{S_2} \cdot \ldots \cdot x_{S_{N-m}} = kz \neq 0.$$

It follows that $e(\zeta_{I_1} \oplus \ldots \oplus \zeta_{I_m}) \neq 0$ as desired.

There is a well-known one-to-one correspondence between isomorphism classes of complex vector bundles over a compact Hausdorff space X and Murray−von Neumann equivalence classes of projections in matrix algebras over $C(X)$ (and in $C(X) \otimes K$). The vector bundle corresponding to a projection p in $M_n(C(X)) = C(X, M_n(C))$ is

$$\xi_p = \{(x, v): x \in X, v \in p(x)(C^n)\}$$

(equipped with the topology given from the natural inclusion $\xi_p \subseteq X \times C^n$), so that the fibre $(\xi_p)_x$ over $x \in X$ is the range of the projection $p(x)$. If p and q are two projections in $C(X) \otimes K$, then $\xi_p \cong \xi_q$ if and only if $p \sim q$. It follows from Swan's theorem, which to each complex vector bundle ξ gives a complex vector η bundle η such that $\xi \oplus \eta$ is isomorphic to the trivial n-dimensional complex vector bundle over X for some n, that every complex vector bundle is isomorphic to ξ_p for some projection p in $M_n(C(X))$ for some n.

View each matrix algebra $M_n(C)$ as a sub-C^*-algebra of K via the embeddings

$$C \hookrightarrow M_2(C) \hookrightarrow M_3(C) \hookrightarrow \ldots \hookrightarrow K,$$

where $M_n(C)$ is mapped into the upper left corner of $M_{n+1}(C)$. Identify $C(X,K)$ with $C(X)\otimes K$ and identify $C(X, M_n(C))$ with $C(X)\otimes M_n(C)$.

We picked a non-trivial complex line bundle ζ over S^2 (which could be the Hopf bundle). This line bundle ζ corresponds to a projection p in some matrix algebra over $C(S^2)$, and, as is well known, such a projection p can be found in $M_2(C(S^2)) = C(S^2, M_2)$. (The projection $p \in M_2(S^2, M_2)$ corresponding to the Hopf bundle is in operator algebra texts often referred to as the Bott projection.) Put

$$Z = \prod_{n=1}^{\infty} S^2.$$

Let $\pi_n: Z \to S^2$ be the nth coordinate map, and let $\varrho_{\infty,n}: Z \to (S^2)^n$ be given by

$$\varrho_{\infty,n}(x_1, x_2, x_3, \dots) = (x_1, x_2, \dots, x_n), \quad (x_1, x_2, x_3, \dots) \in Z$$

With $\hat{\varrho}_n: C((S^2)^n) \to C((S^2)^{n+1})$ being the $*$-homomorphism induced by the map $\varrho_n = \varrho_{n+1,n}$ defined in (35) we obtain that $C(Z)$ is the inductive limit

$$c(S^2) \xrightarrow{\hat{\varrho}_1} C((S^2)^2) \xrightarrow{\hat{\varrho}_2} C((S^2)^{\ 3}) \xrightarrow{\hat{\varrho}_3} \dots \longrightarrow C(z)$$

with inductive limit maps $\hat{\varrho}_{\infty,n}: C((S^2)^n) \to C(Z)$.

For n in N and for each non-empty finite subset $I = \{n_1, n_2, \dots, n_k\}$ of N, let P_n and P_I be the projections in $C(Z)\otimes K = C(Z,K)$ given by

$$P_n(x) = p(x_n), \qquad (42)$$

$$P_I(x) = p(x_{n_1})\otimes p(x_{n_2})\otimes \dots \otimes p(x_{n_k})$$
$$= p_{n_1}(x)\otimes p_{n_2}(x)\otimes \dots \otimes p_{n_k}(x), \qquad (43)$$

for all $x = (x_1, x_2, \dots) \in Z$ (identifying M_2 and $M_2\otimes M_2\otimes \dots \otimes M_2$, respectively, with sub-C^*-algebras of K).

We shall now make use of the multiplier algebra, $M(C(Z)\otimes K)$, of $C(Z)\otimes K = C(Z,K)$. We can identify this multiplier algebra with the set of all bounded functions $f: Z \to B(H)$ for which f and f^* are continuous, when $B(H)$, the bounded operators on the Hilbert space H on which K acts, is given the strong operator topology.

It is convenient to have a convention for adding finitely or infinitely many projections in $M(C(Z)\otimes K)$ or more generally in $M(A)$, where A is any stable C^*-algebra a convention that extends the notion of forming direct sums of projections discussed .

Assuming that A is a stable C^*-algebra, so that $A = A_0\otimes K$ for some C^*-algebra A_0, then we can take a sequence $\{T_j\}_{j=1}^{\infty}$ of isometries in $C\otimes B(H) \subseteq M(A_0\otimes K) = M(A)$ such that $1 = \sum_{j=1}^{\infty} T_j T_j^*$ in the strict topology. (Notice that 1 is a properly infinite projection in $M(A)$.) For any sequence $q_1, q_2, \dots$ of projections in A and for any sequence $Q_1, Q_2, \dots$ of projections in $M(A)$, define

$$q_1\oplus q_2\oplus \dots \oplus q_n = \sum_{j=1}^{n} T_j q_j T_j^* \in A, \qquad (44)$$

$$\oplus_{j=1}^{\infty} q_j = \sum_{j=1}^{\infty} T_j q_j T_j^* \in M(A), \qquad (45)$$

$$Q_1\oplus Q_2\oplus \dots \oplus Q_n = \sum_{j=1}^{n} T_j Q_j T_j^* \in M(A), \qquad (46)$$

$$\oplus_{j=1}^{\infty} Q_j = \sum_{j=1}^{\infty} T_j Q_j T_j^* \in M(A). \qquad (47)$$

Observe that $q_j' = T_j q_j T_j^* \sim q_j$, that the projections $q_1', q_2', \dots$ are mutually orthogonal, and that the sum $\sum_{j=1}^{\infty} q_j'$ is strictly convergent. The projections in (44)-(47) are, up to unitary equivalence in $M(A)$, independent of the choice of isometries $\{T_j\}_{j=1}^{\infty}$. Indeed, if $\{R_j\}_{j=1}^{\infty}$ is another sequence of isometries in $M(A)$ with $1 = R_j R_j^*$ then $U = \sum_{j=1}^{\infty} R_j T_j^*$ is a unitary element in $M(A)$ and

$$\sum_{j=1}^{\infty} R_j X_j R_j^* = U\left(\sum_{j=1}^{\infty} T_j X_j T_j^*\right) U^*$$
for any bounded sequence $\{X_j\}_{j=1}^{\infty}$ in $M(A)$. It follows in particular that

$$\bigoplus_{j=1}^{\infty} q_j \sim \bigoplus_{j=1}^{\infty} q_{\sigma(j)} \quad (48)$$

for every permutation σ on N.

The correspondence between projections and vector bundles is given by the mapping $p \longmapsto \xi_p$ defined at the beginning of this section. By identifying the projections $p_n, p_I, p_{I_1}, \cdots, p_{I_k}$ with projections in $C((S^2)^N) \otimes K$, where N is any integer large enough to ensure that these projections belong to the image of

$$\hat{\varrho}_{\infty,N} \otimes id_K : C((S^2)^N) \otimes K \to C(Z) \otimes K,$$

we can take the base space to be $(S^2)^N$.

Lemma (3.2.6)[448]: Let ζ_n and ζ_I be the complex line bundles defined in (38).

(i) The vector bundle ζ_n corresponds to p_n for each n in N.

(ii) The vector bundle ζ_I corresponds to p_I for each non-empty finite subset I of N.

(iii) The vector bundle $\zeta_{I_1} \oplus \zeta_{I_2} \oplus \ldots \oplus \zeta_{I_k}$ corresponds to $p_{I_1} \oplus p_{I_2} \oplus \cdots \oplus p_{I_k}$ whenever $I_1, \ldots, I_k$ are non-empty finite subsets of N.

Proof. (i) Since p corresponds to $\zeta, p_n = p \circ \pi_n$ corresponds to $\zeta_n = \pi_n^*(\zeta)$, where $\pi_n : (S^2)^N \to S^2$ is the nth coordinate map.

(ii) Write $I = \{n_1, n_2, \ldots, n_k\}$. We shall here view p_n, as a projection in $C((S^2)^N, M_2)$ and p_l as a projection in $C((S^2)^N, M_2 \otimes \cdots \otimes M_2)$ By (i), ζ_n is the complex line bundle over $(S^2)^N$ whose fibre over $x \in (S^2)^N$ is equal to $p_n,(x)(C^2)$. The fibre of tile complex line bundle $\zeta_I = \zeta_{n_1} \otimes \cdots \otimes \zeta_{n_k}$ over $x \in (S^2)^N$ is by definition

$$\begin{aligned}(\zeta_I)_x &= (\zeta_{n_1})_x \otimes (\zeta_{n_2})_x \otimes \cdots \otimes (\zeta_{n_1})_x \\ &= p_{n_1}(x)(C^2) \otimes p_{n_2}(x)(C^2) \otimes \cdots \otimes p_{n_k}(x)(C^2) \\ &= p_I(x)(C^2 \otimes C^2 \otimes \cdots \otimes C^2).\end{aligned}$$

This shows that ζ_I corresponds to P_I.

(iii) This follows from (ii) and additivity of the map $p \longmapsto \xi_p$.

The next three lemmas are formulated for an arbitrary stable C^*-algebra A and its multiplier algebra $M(A)$, but they shall primarily be used in the case where $A = C(Z) \otimes K$.

The lemma below is a trivial, but much used, generalization of (48):

Lemma (3.2.7)[448]: Let A be a stable C^*-algebra, and let $q_1, q_2, \ldots$ and $r_1, r_2, \ldots$ be two sequences of projections in A. Assume that there is a permutation σ on N such that $q_j \precsim r_{\sigma(j)}$ and $q_j \sim r_{\sigma(j)}$, respectively, in A for all j in N. Then $\bigoplus_{j=1}^{\infty} q_j \precsim \bigoplus_{j=1}^{\infty} r_j$ and $\bigoplus_{j=1}^{\infty} q_j \sim \bigoplus_{j=1}^{\infty} r_j$, respectively, in $M(A)$.

An element in a C^*-algebra A is said to be full in A if it is not contained in any proper closed two-sided ideal of A.

Lemma (3.2.8)[448]: Let A be a stable C^*-algebra. The following three conditions are equivalent for all projections Q in $M(A)$:

(i) $Q \sim I$;

(ii) Q is properly infinite and full in $M(A)$;

(iii) $I \precsim Q$.

Proof. (i)⟹(iii) is trivial. Assume that $I \precsim Q$. Then Q is full in $M(A)$ (the closed two-sided ideal in $M(A)$ generated by Q contains 1 and hence all of $M(A)$). It was noted above (44) that 1 is properly infinite in $M(A)$, and so $Q \oplus Q \leq 1 \oplus 1 \precsim 1 \precsim Q$ whence Q is properly infinite; cf. Proposition (3.2.1). This proves (iii)⟹(ii). Assume finally that Q is properly infinite and full in $M(A)$. Since $K_0(M(A)) = 0$ (see [129, Proposition 12.2.1]) the two projections Q and 1 represent the same element in $K_0(M(A))$; and since these two projections both are properly infinite and full they must be Murray-von Neumann equivalent (see [138, w or [162, Exercise 4.9 (iii)]), i.e., $Q \sim I$.

Lemma (3.2.9)[448]: Let A be a stable C^*-algebra and let $q, q_1, q_2, \dots$ be projections in A. If $q \precsim \bigoplus_{j=1}^{\infty} q_j$ in $M(A)$, then $q \precsim q_1 \oplus q_2 \oplus \cdots \oplus q_k$ in A for some k.

Proof. We have $\bigoplus_{j=1}^{\infty} q_j = \sum_{j=1}^{\infty} q_j' \, (= Q)$ for some strictly summable sequence of mutually orthogonal projections $q_1', q_2', \dots$ in A with $q_j' \sim q_j$. By the assumption that $q \precsim Q$ there is a partial isometry v in $M(A)$ such that $vv^* = q$ and $v^*v \le Q$. As $v = qv, v$ belongs to A, and by the strict convergence of the sum $Q = \sum_{j=1}^{\infty} q_j$ there is k such that

$$\left\| v - v \sum_{j=1}^{k} q_j' \right\| < \frac{1}{2}.$$

Put $x = v \sum_{j=1}^{k} q_j'$. Then $xx^* \le q, x^*x \le q_1' + \dots + q_k'$ and $\|xx^* - q\| < 1$. This shows that xx^* is invertible in qAq with inverse $(xx^*)^{-1}$. Put $u = (xx^*)^{-1/2} x$. Then $uu^* = q$ and $u^*u \le q_1' + \cdots + q_k'$, whence $q \precsim q_1 \oplus \dots \oplus q_k$.

Let g be a constant 1-dimensional projection in $C(Z, K) = C(Z) \otimes K$ (that corresponds to the trivial complex line bundle θ over X).

Proposition (3.2.10)[448]: Let $I_1, I_2, \dots$ be a sequence of non-empty, finite subsets of N. Put

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in M(C(Z) \otimes K).$$

(i) If $\left| \bigcup_{j \in F} I_j \right| \ge |F|$ for all finite subsets F of N, then $g \not\precsim Q$ and Q is not properly infinite.

(ii) $g \precsim P_n \oplus P_n$ for every natural number n.

(iii) If infinitely many of the sets $I_1, I_2, \dots$ are singletons, then $Q \oplus Q$ is properly infinite and $Q \oplus Q \sim 1$ in $M(C(Z) \otimes K)$.

Proof. (i) We show first that $g \not\precsim Q$ in $M(C(Z) \otimes K)$ Indeed, assume to the contrary that $g \precsim Q$. Then

$$g \precsim P_{I_1} \oplus P_{I_2} \oplus \dots \oplus P_{I_k} \tag{49}$$

in $C(Z) \oplus K$ for some k by Lemma (3.2.9). As noted earlier, $C(Z) \otimes K$ is an inductive limit

$$C(S^2) \otimes K \xrightarrow{\hat{\varrho}_1 \otimes id_K} C((S^2)^2) \otimes K \otimes \xrightarrow{\hat{\varrho}_2 \otimes id_K} C((S^2)^3) \otimes K \longrightarrow \dots \longrightarrow C(Z) \otimes K.$$

Take N such that all projections appearing in (49) belong to the image of

$$\hat{\varrho}_{\infty,n} \otimes id_K : C((S^2)^n) \otimes K \longrightarrow C(Z) \otimes K$$

whenever $n \ge N$. Use a standard inductive limit argument to see that (49) holds relatively to $C((S^2)^n) \otimes K$ for some large enough $n \ge N$. In the language of vector bundles over $(S^2)^n$, (49) and Lemma (3.2.6) imply that

$$\theta \oplus \eta \cong \zeta_{I_1} \oplus \zeta_{I_2} \oplus \dots \oplus \zeta_{I_k} \tag{50}$$

for some vector bundle η over $(S^2)^n$. Now, (50) and (36) imply that $e(\zeta_{I_1} \oplus \dots \oplus \zeta_{I_k}) = 0$, in contradiction with Proposition (3.2.5) and the assumption on the sets I_j.

The projection P_{I_1} is a full element in $C(Z) \otimes K$ and $p_{I_1} \le Q$. Hence g belongs to the ideal generated by Q. It now follows from Lemma (3.2.2) and from the fact that $g \not\precsim Q$ that Q cannot be properly infinite.

(ii) follows from Lemma (3.2.4) and Lemma (3.2.6).

(iii) The unit 1 of $M(C(Z)\otimes K)$ can be written as a strictly convergent sum $1=\sum_{j=1}^{\infty} g_j$, where $g_j \sim g$ for all j. Let Γ denote the infinite subset of N consisting of those j for which I_j is a singleton. By Lemma (3.2.7) and (ii) we get

$$1\sim \oplus_{j=1}^{\infty} g \precsim \oplus_{j\in\Gamma}\left(p_{I_j}\oplus p_{I_j}\right) \precsim \oplus_{j=1}^{\infty}(p_{I_j}\oplus p_{I_j}) \sim Q\oplus Q.$$

Lemma (3.2.8) now tells us that $Q\oplus Q$ is properly infinite and that $Q\oplus Q\sim 1$.

We construct here a simple, unital C^*-algebra that contains a finite and an infinite projection; thus proving one of our main results: Theorem (3.2.16) below.

Let again Z denote the infinite product space $\prod_{j=1}^{\infty} S^2$ Set $A=C(Z)\otimes K=C(Z,K)$; recall that $M(A)$ denotes the multiplier algebra of A and that it can be identified with the set of bounded $*$-strongly continuous functions $f: Z\to B(H)$.

Choose an injective function $v: Z\times N\to N$. Choose points $c_{j,i}\in S^2$ for all $j,i\in N$ with $j\geq i$ such that

$$\overline{\{(c_{j,1},c_{j,2},\dots,c_{j,n}) | j\geq n\}} = S^2\times S^2\times\ldots\times S^2 \qquad (51)$$

for every natural number n. Set

$$I_j=\{v(j,1),v(j,2),\dots,v(j,j)\} \qquad (52)$$

for $j\in N$.

Define $*$-homomorphisms $\varphi_j: A\to A$ for all integers j as follows. For $j\leq 0$, set

$$\varphi_j(f)(x)=f\left(x_{v(j,1)},x_{v(j,2)},x_{v(j,3)},\dots\right), f\in A, x=(x_1,x_2,\dots)\in Z. \quad (53)$$

Let P_n and P_I be the projections in $A=C(Z,K)$ defined in (42) and (43). Choose an isomorphism $\tau: K\otimes K\to K$ For f in A, $x=(x_1,x_2,\dots)$ in Z and $j\geq I$ define

$$\varphi_j(f)(X): \tau\left(f\left(e_{j,1},\dots,C_{j,j},x_{v(j,j+1)},x_{v(j,j+2)},\dots\right)\otimes p_{I_j}(x)\right). \quad (54)$$

Choose a sequence $\{S_j\}_{j=-\infty}^{\infty}$ of isometries in $M(A)$ such that $\sum_{j=-\infty}^{\infty} S_jS_j^*=1$ with the sum being strictly convergent. Define a $*$-homomorphism $\psi: A\to M(A)$ by

$$\psi(f)=\sum_{j=-\infty}^{\infty} S_j\varphi_j(f)S_j^*, \quad f\in A. \qquad (55)$$

Lemma (3.2.11)[448]: Let $\{e_n\}_{n=1}^{\infty}$ be an increasing approximate unit for A. Then $\{\psi(e_n)\}_{n=1}^{\infty}$ converges strictly to a projection $F\in M(A)$, and F is equivalent to the identity 1 in $M(A)$.

Proof. If $\psi(e_n)$ converges strictly to $F\in M(A)$ for some approximate unit $\{e_n\}$ for A, then this conclusion will hold for all approximate units for A. We can therefore take $\{e_n\}_{n=1}^{\infty}$ to be the approximate unit given by $e_n(x)=\hat{e}_n$ where $\{\hat{e}_n\}_{n=1}^{\infty}$ is an increasing approximate unit for K.

We show first that $\{\varphi_j(e_n)\}_{n=1}^{\infty}$ converges strictly to a projection F_j in $M(A)$ for each $j\in Z$. Indeed, since $\varphi_j(e_n)=e_n$ when $j\leq 0$ it follows that $\varphi_j(e_n)\to 1$ strictly; and so $F_j=I$ when $j\leq 0$. Consider next the case $j\geq I$. Here we have $\varphi_j(e_n)(x)=\tau(\hat{e}_n\otimes P_{I_j}(x))$.

Extend $\tau: K\otimes K\to K$ to a strongly continuous unital $*$-homomorphism $\bar{\tau}: B(H\otimes H)\to B(H)$ and define F_j in $M(A)$ by $F_j(x)=\bar{\tau}(1\otimes p_{I_j}(x))$ for $x\in Z$. Then F_j is a projection and $\{\varphi_j(e_n)\}_{n=1}^{\infty}$ converges strictly to F_j.

Now,

$$\psi(e_n)=\sum_{j=-\infty}^{\infty} S_j\varphi_j(e_n)S_j^* \xrightarrow[n\to\infty]{strictly} \sum_{j=-\infty}^{\infty} S_jF_jS_j^* \stackrel{\text{def}}{=} F\in M(A).$$

As $I=F_0\sim S_0F_0S_0^*\leq F$ it follows from Lemma (3.2.8) that $F\sim 1$ in $M(A)$.

Take an isometry T in $M(A)$ with $TT^*=F$ (where F is as in Lemma (3.2.11)). Define

$$\varphi(f)=T^*\psi(f)T=\sum_{j=-\infty}^{\infty} T^*S_j\varphi_j(f)S_j^*T, \quad f\in A. \quad (56)$$

Then $\varphi: A \to M(A)$ is a $*$-homomorphism that maps an approximate unit for A into a sequence in $M(A)$ that converges strictly to the identity in $M(A)$ (by Lemma (3.2.11) and the choice of T). It follows from [154, Proposition 2.5] that φ extends to a unital *-homomorphism $\bar{\varphi}: M(A) \to M(A)$.

We collect below some properties of the $*$-homomorphisms φ and $\bar{\varphi}$. A subset of a C^*-algebra A is called full in A if it is not contained in any proper closed two-sided ideal in A.

Proposition (3.2.12)[448]: Let p_1 be the projection in A defined in (42), and let g be a constant 1-dimensional projection in $A = C(Z, K)$.

(i) $\varphi(g) \sim 1$ in $M(A)$, and $\varphi(f)$ is full in $M(A)$ for every full element f in A.

(ii) If f is a non-zero element in $.M(A)$, then $\bar{\varphi}(f)$ does not belong to A, and $A\bar{\varphi}(f)$ is full in A.

(iii) If f is a non-zero element in $M(A)$, then $A\bar{\varphi}^k(f)$ is full in A for every $k \in N$.

(iv) None of the projections $\bar{\varphi}^k(p_1), k \in N$, are properly infinite in $M(A)$.

It follows immediately from (ii) that $\bar{\varphi}$ and φ are injective, $\bar{\varphi}(M(A)) \cap A = \{0\}$ and $\varphi(A) \cap A = \{0\}$.

Is divided into a few lemmas, the first of which (included for emphasis) is standard and follows from the fact that any closed two-sided ideal in $C(Z, K)$ is equal to $C_0(U, K)$ for some open subset U of Z.

Lemma (3.2.13)[448]: Let f be an element in $A = C(Z, K)$. Then f is full in A if and only if $f(x) \neq 0$ for all $x \in Z$.

Proof. Observe first that $\varphi_j(g) = g$ for every $j \leq 0$. Accordingly,

$$1 \sim \bigoplus_{j=-\infty}^{0} g \sim \sum_{j=-\infty}^{0} T^* S_j \varphi_j(g) S_j^* T \leq \varphi(g) \ in\ M(A).$$

This and Lemma (3.2.8) imply that $\varphi(g) \sim 1$ and that $\varphi(g)$ is full in $M(A)$. If f is any full element in A, then the closed two-sided ideal generated by $\varphi(f)$ contains $\varphi(g)$ and therefore all of $M(A)$. This proves the second claim in (i).

Proof. Take a non-zero element f in $M(A)$. There is an element a in A such that $af \neq 0$. The two claims in (ii) will clearly follow if we can show that $\bar{\varphi}(af) \notin A$ and that $A\bar{\varphi}(af)$ is full in A, and we can therefore, upon replacing f by af, assume that f is a non-zero element in $A = C(Z, K)$.

There are $\delta > 0, r \in N$ and non-empty open subsets $U_1, \ldots, U_r$ of S^2 such that

$$x \in U_1 \times U_2 \times \ldots \times U_r \times S^2 \times S^2 \times \ldots \Longrightarrow \|f(x)\| \geq \delta. \quad (57)$$

Use (44) to find an infinite set A of integers $j \geq r$ such that

$$(C_{j,1}, C_{j,2}, \ldots, C_{j,r}) \in U_1 \times U_2 \times \ldots \times U_r \text{ for all } j \in A. \quad (58)$$

It follows from Lemma (3.2.13), (56), (57) and (58) that $\|\varphi_j(f)\| \geq \delta$ and $\varphi_j(f)$ is full in A for every j in the infinite set Λ. This entails that $\varphi(f) = \sum_{j=-\infty}^{\infty} T^* S_j \varphi_j(f) S_j^* T$ does not belong to A. (A strictly convergent sum $\sum_{j=-\infty}^{\infty} a_j$ of pairwise orthogonal elements from A belongs to A if and only if $\lim_{j \to \pm\infty} \|a_j\| = 0.$) The closed two-sided ideal in A generated by $A\varphi(f)$ contains the full element $\varphi_j(f) = S_j^* T \varphi(f) T^* S_j$ and therefore all of A (for each− and hence at least one $-j$ in A).

Proof. This follows from injectivity of $\bar{\varphi}$.

We proceed to prove Proposition (3.2.12) (iv).

Lemma (3.2.14)[448]: Let J be a finite subset of N, and j an integer. Then $\varphi_j(p_{.J}) \sim p_{cr_j}(J)$’ where

$$\alpha_j(J) = \begin{cases} v(j,J), & j \le 0, \\ v(j,J\backslash\{12,\dots,j\}) \cup I_j, & j \ge 1. \end{cases} \tag{59}$$

We have in particular that $v(j,J) \subseteq \alpha_j(J)$ for all finite subsets J of N and for all $j \in Z$.

Proof. Write $J = \{t_1, t_2, \dots, t_k\}$, where $t_1 < t_2 < \dots < t_k$. We consider first the case where $j \le 0$. Then

$$\begin{aligned} \varphi_j(p_j)(x) &= p_J(x_{v(j,1)}, x_{v(j,2)}, x_{v(j,3)}, \dots) \\ &= p(x_{v(j,t_1)}) \otimes p(x_{v(j,t_2)}) \otimes \cdots \oplus p(x_{v(j,t_k)}) \\ &= p_{v(j,t_1)}(x) \otimes p_{v(j,t_2)}(x) \otimes \cdots \otimes p_{v(j,t_k)}(x) = p_{v(j,J)}(x), \end{aligned}$$

as desired.

Suppose next that $j \ge 1$, and put $q(x) = p_j(c_{j,1}, \dots, c_{j,j}, x_{v(j,j+1)}, x_{v(j,j+2)}, \dots)$. Then $\varphi_j(p_j)(x) = \tau(q(x) \otimes p_{I_j}(x))$ Suppose that $1 \le j < t_k$ and let m be such that $t_{m-1} \le j < t_m$ (with the convention $t_0 = 0$). Then

$$\begin{aligned} q(x) &= p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_{m-1}}) \otimes p(x_{v(j,t_m)}) \otimes \dots \otimes p(x_{v(j,t_k)}) \\ &= p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_{m-1}}) \otimes p_{v(j,t_m)}(x) \otimes \dots \otimes p_{v(j,t_k)}(x) \\ &= p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_{m-1}}) \otimes P_{v(j,J\backslash\{1,2\,\dots\dots j\})}(x). \end{aligned}$$

Thus $q \sim p_{v(j,J\backslash\{1,2\,\dots\dots j\})}$, which shows that $\varphi_j(p_J)$ is equivalent to the projection defined by

$$x \longmapsto \tau\left(p_{v(j,J\{1,2\,\dots\dots j\})}(x) \otimes p_{I_j}(x)\right),$$

and this projection is equivalent to $p_{v(j,J\backslash\{1,2\,\dots\dots j\}) \cup I_j}$. If $j \ge t_k$, then $J\backslash\{1, 2\, \dots, j\} = \phi$ and $q(x) = p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_k})$, i.e., q is a constant projection. In this case, $\varphi_j(p_J) \sim p_{I_j}$, thus affirming the first claim of the lemma.

The last claim follows from the definition of the sets I_j in (51).

Lemma (3.2.15)[448]: Let $J_1, J_2, \dots$ be finite subsets of N. Put $Q = \bigoplus_{i=1}^{\infty} p_{J_1} \in M(A)$. Then

$$\bar{\varphi}(Q) \sim \bigoplus_{i=1}^{\infty} \bigoplus_{j=-\infty}^{\infty} p_{\alpha_j(J_i)},$$

where α_j is as defined in (58). Moreover, if $|U_{i\in F} J_i| \ge |F|$ for all finite subsets F of N then $\left|U_{(j,i)\in G} \alpha_j(J_i)\right| \ge |G|$ for all finite subsets G of $Z \times N$.

Proof. By (11), $Q = T_i p_{j_i} T_i^*$; and because $\bar{\varphi}$ is strictly continuous we get

$$\bar{\varphi}(Q) = \sum_{i=1}^{\infty} \bar{\varphi}(T_i) \varphi(p_{J_i}) \bar{\varphi}(T_i)^* \sim \bigoplus_{i=1}^{\infty} \varphi(p_{J_i}) \sim \bigoplus_{i=1}^{\infty} \bigoplus_{j=-\infty}^{\infty} \varphi_j(p_{J_i}) \sim \bigoplus_{i=1}^{\infty} \bigoplus_{j=-\infty}^{\infty} p_{\alpha_j(J_i)},$$

where the first equivalence is proved below (10)−(13), and the last equivalence follows from Lemma (3.2.14).

By the Marriage Theorem we can find natural numbers $t_i \in J_i$ such that $\{t_i\}_{i\in N}$ are mutually distinct. Set $s_{j,i} = v(j, t_i)$. Then $s_{j,i}$ belongs to $\alpha_j(J_i)$ by Lemma (3.2.14), and $\{s_{j,i}\}_{(j,i)\in Z\times N}$ are mutually distinct because v is injective and the t_i's are mutually distinct. This proves the second claim of the lemma.

Proof of Proposition (3.2.11) (iv). Put $Q_0 = p_1$ and put $Q_n = \bar{\varphi}^n(Q_0)$. We must show that none of the projections $Q_n, n \ge 0$, are properly infinite. It is clear that Q_0 is finite, and hence not properly infinite.

Use Lemmas (3.2.14) and (3.2.15) to see that

$$Q_1 = \sum_{j=-\infty}^{\infty} T^* S_j \varphi_j(p_1) S_j^* T \sim \bigoplus_{j=-\infty}^{\infty} \varphi_j\,(p_1) \sim \bigoplus_{j=-\infty}^{0} p_{v(j,1)} \oplus \bigoplus_{j=1}^{\infty} p_{I_j} = \bigoplus_{j=-\infty}^{\infty} p_{J_i},$$

where $J_j = \{v(j,1)\}$ for $j \leq 0$ and $J_j = I_j$ for $j \geq 1$. It is easily seen that the sequence of sets $\{J_j\}_{j=-\infty}^{\infty}$ satisfies the condition $\left|U_{j\in F} J_j\right| \geq |F|$ for all finite subsets F of Z. Hence Q_1 is not properly infinite by Proposition (3.2.10) (i).
The claim that Q_n is not properly infinite for all n follows by induction using Lemma (3.2.15) and Proposition (3.2.10) (i).
Theorem (3.2.16)[448]: Consider the inductive limit B of the sequence

$$M(C(Z)\oplus K) \xrightarrow{\bar{\varphi}} M(C(Z)\otimes K) \xrightarrow{\bar{\varphi}} M(C(Z)\oplus K) \xrightarrow{\bar{\varphi}} \ldots \longrightarrow B.$$

Then B has the following properties:

(i) B is unital and simple.
(ii) The unit of B is infinite.
(iii) B contains a non-zero finite projection.
(iv) $K_0(B) = 0$ and $K_1(B) = 0$.

Proof. (i) B is unital being the inductive limit of a sequence of unital C^*-algebras with unital connecting maps.

Write again A for $C(Z)\otimes K$, and let $\bar{\varphi}_{\infty,n}: M(A) \to B$ be the inductive limit map from the nth copy of $M(A)$ into B. Let L be a non-zero closed two sided ideal in B, and set

$$L_n = \bar{\varphi}_{\infty,n}^{-1}(L) \lhd M(A).$$

Then L_n is non-zero for some n. Since A is an essential ideal in $M(A)$, also $A \cap L_n$ is non-zero.
Take a non-zero element e in $A \cap L_n$. Then $\bar{\varphi}(e)$ belongs to L_{n+1}, hence $A\bar{\varphi}(e) \subseteq L_{n+1}$, and so it follows from Proposition (3.212) (ii) that $A \subseteq L_{n+1}$. Take now a full element f in $A \subseteq L_{n+1}$. Then $\bar{\varphi}(f)$ belongs to L_{n+2}. It follows from Proposition (3.2.12) (i) that $\bar{\varphi}(f)$ is full in $M(A)$ and therefore $L_{n+2} = M(A)$. Hence $L = B$, and this shows that B is simple.
(ii) This is clear because the unit of $M(A)$ is infinite.
(iii) As in the proof of Proposition (3.2.12) (iv), set $Q_0 = p_1$ and $Q_n = \bar{\varphi}^n(Q_0)$ for $n \geq 1$. Put $Q = \bar{\varphi}_{\infty,0}(Q_0) \in B$. It is shown in Proposition (3.2.12) (ii) that $\bar{\varphi}$ is injective, which implies that $\bar{\varphi}_{\infty,0}$ is injective, and hence Q is non-zero. We show next that Q is finite.
Assume that Q were infinite. Then Q is properly infinite by Cuntz' result (see Proposition (3.2.1)) because B is simple. Applying Proposition (3.2.3) to the sequence

$$Q_0M(A)Q_0 \xrightarrow{\lambda_0} Q_1M(A)Q_1 \xrightarrow{\lambda_1} Q_2M(A)Q_2 \longrightarrow \ldots \longrightarrow QBQ,$$

with the unital connecting maps $\lambda_j = \bar{\varphi}|_{Q_jM(A)Q_j}$, we obtain that Q_n is properly infinite for all sufficiently large n. But this contradicts Proposition (3.2.12) (iv).
(iv) This follows from the fact that the multiplier algebra of a stable C^*-algebra has trivial K-theory (see [129, Proposition 12.2.1]).

It follows from Proposition (3.2.10) (ii) and Proposition (3.2.14) (i) that the finite projection Q in B (found in part (iii) above) satisfies

$$Q \oplus Q \sim \bar{\varphi}_{\infty,0}(Q_0 \oplus Q_0) = \bar{\varphi}_{\infty,0}(p_1 \oplus p_1) \gtrsim \bar{\varphi}_{\infty,0}(g) = \bar{\varphi}_{\infty,1}(\varphi(g)) \sim 1,$$

whence $Q \oplus Q \sim 1$ by Lemma (3.2.8). In other words, the corner C^*-algebra QBQ is unital, finite and simple, and $M_2(QBQ) \cong B$ is infinite.
The C^*-algebra B from Theorem (3.2.16) is not separable and not exact. To see the latter, note that $B(H)$, the bounded operators on a separable, infinite-dimensional Hilbert space H, can be embedded into $M(A) = M(C(Z)\oplus K)$ and hence into B. As $B(H)$ is non-exact (see Wassermann [163, 2.5.4]) it follows from Kirchberg's result that exactness passes to sub-C^*-algebras (see [163, 2.5.2]) that B is non-exact. We use the lemma below from [125] to construct a non-exact separable example.

Lemma (3.2.17) [448]: Let B be a simple C^*-algebra and let X be a countable subset of B. It follows that B has a separable, simple sub-C^*-algebra B_0 that contains X.

Corollary (3.2.18)[448]: There exists a unital, separable, non-exact, simple C^*-algebra B_0 such that B_0 contains an infinite and a non-zero finite projection.

Proof. Let s be a non-unitary isometry in B and let q be a non-zero finite projection in B. The universal C^*-algebra, $C^*(F_2)$, generated by two unitaries is separable and non-exact (see Wassermann [163, Corollary 3.7]). It admits an embedding into $M(C(Z)\oplus K)$ and hence into B. Let $u, v \in B$ be the images of the two (canonical) unitary generators in $C^*(F_2)$. Use Lemma (3.2.17) to find a separable, simple, and unital C^*-algebra B_0 that contains $\{u, v, s, q\}$.

Then B_0 is infinite because it contains the non-unitary isometry s; and it contains the finite projection q. Finally, B_0 is non-exact because it contains the non-exact sub-C^*-algebra $C^*(u, v) \cong C^*(F_2)$.

We show here that an elaboration of the construction yields a nuclear and separable example of a simple C^*-algebra with a finite and an infinite projection.

The construction requires that we make a specific choice for the injective map $v: Z \times N \longrightarrow N$.

Let $\{\Lambda_r\}_{r=0}^{\infty}$ be a partition of the set N such that $\Lambda_0 = \{1\}$ and such that Λ_r is infinite for each $r \geq 1$. For each $r \geq 1$ choose an injective map $\gamma_r: Z \times \Lambda_{r-1} \to \Lambda_r$ and define $v: Z \times N \to N$ by

$$v(j,t) = \gamma_r(j,t), \quad r \in N, t \in \Lambda_{r-1}, j \in Z. \tag{60}$$

Observe that

$$t \in \Lambda_r \Leftrightarrow v(j,t) \in \Lambda_{r+1}, j \in Z. \tag{61}$$

To see that v is injective assume that $v(j,t) = v(i,s)$. Then $v(j,t) = v(i,s) \in \Lambda_r$ for some $r \geq 1$. Therefore both s and t belong to Λ_{r-1}. Now, $\gamma_r(j,t) = v(j,t) = v(i,s) = \gamma_r(i,s)$, which entails that $(j,t) = (i,s)$ by injectivity of γ_r.

Let α_j be as defined in Lemma (3.2.14) (with respect to the new choice of v). Let $\Gamma_0 \subseteq P(N)$ be the family containing the one set $\{1\}$, and set

$$\Gamma_{n+1} = \{ \alpha_j(I) | I \in \Gamma_n, j \in Z\} \subseteq P(N)$$

for $n \geq 0$. Set $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$. Observe that each $I \in F$ is a finite subset of N.

Put $Q_0 = p_1 \in A$ (cf. (42)) and put $Q_n = \bar{\varphi}^n(Q_0) \in M(A)$ (where $\bar{\varphi}$ is the endomorphism on $M(A)$ defined above Proposition (3.2,12)). It then follows by induction from Lemma (3.2.15) that

$$Q_n \sim \oplus_{I\in\Gamma_n} p_I, \quad n \geq 0, \tag{62}$$

when $p_I \in A$ is as defined in (42).

Lemma (3.2.19)[448]: There is an injective function $t: \Gamma \to N$ such that $t(I) \in I$ for all $I \in \Gamma$. It follows in particular that

$$\left| \bigcup_{I\in F} I \right| \geq |F|$$

for all finite subsets F of Γ.

Proof. Define t recursively on each Γ_n as follows. For $n = 0$ we set $t(\{1\}) = 1$. Assume that t has been defined on F_{n-1} for some $n \geq 1$. Then define t on Γ_n by $t(\alpha_j(I)) = v(j, t(I))$ for $I \in \Gamma_{n-1}$ and $j \in Z$. It follows from Lemma (3.2.4) that

$$t(I) \in I \Longrightarrow t\left(\alpha_j(I)\right) \in \alpha_j(I), \qquad I \in \Gamma, j \in Z.$$

It therefore follows by induction that $t(I) \in I$ for all $I \in F$.

We show next that $t(I) \in \Lambda_n$ if $I \in \Gamma_n$. This is clear for $n = 0$. Let $n \geq 1$ and let $I \in \Gamma_n$ be given. Then $I = \alpha_j(I')$ for some $I' \in \Gamma_{n-1}$ and some $j \in Z$. It follows that $t(I) = t(\alpha_j(I')) = v(j, t(I'))$. Hence $t(I) \in \Lambda_n$ if $t(I') \in \Lambda_{n-1}$, cf. (60). Now the claim follows by induction on n.

We proceed to show that t is injective. If $I, J \in \Gamma$ are such that $t(I) = t(J)$, then $t(I) = t(J) \in \Lambda_n$ for some n, whence I, J both belong to Γ_n. It therefore suffices to show that $t|_{\Gamma_n}$ is injective for each n. We prove this by induction on n. It is trivial that $t|_{\Gamma_0}$ is injective. Assume that $t|_{\Gamma_{n-1}}$, is injective for some $n \geq 1$. Let $I, J \in \Gamma_n$ be such that $t(I) = t(J)$. Then $I = \alpha_i(I')$ and $J = \alpha_j(J')$ for some $i, j \in Z$ and some $I', J' \in \Gamma_{n-1}$, and

$$v(i, t(I')) = t(\alpha_i(I')) = t(I) = t(J) = t(\alpha_j(J')) = v(j, t(J')).$$

Since v is injective we deduce that $i = j$ and $t(I') = t(J')$. By injectivity of $t|_{\Gamma_{n-1}}$ we obtain $I' = J'$, and this proves that $I = J$. It has now been shown that $t|_{\Gamma_n}$is injective, and the induction step is complete.

Let $g \in A = C(Z, K)$ be a constant 1-dimensional projection, and let Q_n be as defined above (62).

Lemma (3.2.20)[448]: For each natural number m we have

$$g \not\precsim Q_0 \oplus Q_1 \oplus \cdots \oplus Q_m \ in\ M(A).$$

Proof. From (28) (and Lemma (3.2.7)) we deduce that

$$Q_0 \oplus Q_1 \oplus \cdots \oplus Q_m \sim \oplus_{I \in \Gamma_0 \cup \ldots \cup \Gamma_n} p_I.$$

The claim of the lemma now follows from Proposition (3.2.9) (i) together with Lemma (3.2.19).

As in Theorem (3.2.16) consider the inductive limit

$$M(A) \xrightarrow{\bar\varphi} M(A) \xrightarrow{\bar\varphi} M(A) \xrightarrow{\bar\varphi} \ldots \longrightarrow B, \qquad (63)$$

where $A = C(Z) \otimes K$ Let $\mu_{\infty,n}: M(A) \to B$ be the inductive limit map (from the nth copy of $M(A)$) for $n \geq 0$, and let $\mu_{m,n}: M(A) \to M(A)$ be the connecting map from the nth copy of $M(A)$ to the mth copy of $M(A)$ for $n < m$, i.e., $\mu_{m,n} = \bar\varphi^{m-n}$. The endomorphism $\bar\varphi$ on $M(A)$ extends to an automorphism α on B that satisfies $\alpha(\mu_{\infty,n}(x)) = \mu_{\infty,n}(\bar\varphi(x))$ for $x \in M(A)$ and all $n \in N$. (The inverse of α is on the dense subset $\bigcup_{n=0}^{\infty} \mu_{\infty,n}(M(A))$ of B given by $\alpha^{-1}(\mu_{\infty,n}(x)) = \mu_{\infty,n+1}(x).)$

Put $A_0 = \mu_{\infty,0}(A) \subseteq B$, put $A_n = \alpha^n(A_0) \subseteq B$ for all $n \in Z$, and put

$$D_n = C^*(A_{-n}, A_{-n+1}, \ldots, A_0, \ldots, A_{n-1}, A_n), \ \ D = \overline{\bigcup_{n=1}^{\infty} D_n}. \qquad (64)$$

It is shown in Lemma (3.2.24) below that each D_n is a type I C^*-algebra, and so the C^*-algebra D is an inductive limit of type I algebras. In particular, D is nuclear and belongs to the UCT class N. Moreover, D is α-invariant (by construction). Observe that $A_{m-n} = \mu_{\infty,n}(\bar\varphi^m(A))$ for all non-negative integers m and n.

Put $Q = \mu_{\infty,n}(p_1)(= \mu_{\infty,n}(Q_n))$ in $D \subseteq B$, and, as above, let $g \in A = C(Z, K)$ be a constant 1-dimensional projection.

Lemma (3.2.21)[448]: The following two relations hold in D and in B:

(i) $\mu_{\infty,0}(g) \precsim Q \oplus Q$;

(ii) $\mu_{\infty,0}(g) \not\precsim \oplus_{j=-N}^{N} \alpha^j(Q)$ for all natural numbers N.

Proof. (i) follows immediately from Proposition (3.2.10) (ii).

(ii) Assume, to reach a contradiction, that $\mu_{\infty,0}(g) \precsim \sum_{j=-N}^{N} \alpha^j(Q)$ in B (or in D) for some $N \in N$. For $j \geq -N$ we have

$$\alpha^j(Q) = \alpha^j(\mu_{\infty,0}(Q_0)) = \alpha^j(\mu_{\infty,N}(\bar\varphi^N(Q_0))) = \mu_{\infty,N}(\bar\varphi^{N+j}(Q_0)).$$

The relation $\mu_{\infty,0}(g) \precsim \sum_{j=-N}^{N} \alpha^j(Q)$ can therefore be rewritten as

$$\mu_{\infty,N}(\bar{\varphi}^N(g)) \lesssim \bigoplus_{j=0}^{2N} \mu_{\infty,N}\left(\bar{\varphi}^j(Q_0)\right) \ \ in \ \ B.$$

By a standard property of inductive limits this entails that

$$\mu_{M,N}(\bar{\varphi}^N(g)) \lesssim \bigoplus_{j=0}^{2N} \mu_{M,N}\left(\bar{\varphi}^j(Q_0)\right) \ \ in \ \ M(A)$$

for some $M \geq N$, or, equivalently,

$$\bar{\varphi}^M(g) \lesssim \bigoplus_{j=0}^{2N} \bar{\varphi}^{j+M-N}(Q_0) = \bigoplus_{j=M-N}^{N+M} \bar{\varphi}^j(Q_0) = \bigoplus_{j=M_N}^{N+M} Q_j \lesssim \bigoplus_{j=0}^{N+M} Q_j \ in \ M(A)$$

Use now that $g \lesssim \bar{\varphi}^M(g)$ (which holds because $\varphi_j(g) = g$ for $j \leq 0$, cf. (62)) to conclude that $g \lesssim \bigoplus_{j=0}^{N+M} Q_j$ in $M(A)$, in contradiction with Lemma (3.2.20).

Let C be an arbitrary unital C^*-algebra and let γ be an automorphism on C.

Let K denote the compact operators on $l^2(Z)$ and let $\{e_{i,j}\}_{i,j\in Z}$ be a set of matrix units for K Define a unital injective $*$-homomorphism $\psi: C \to M(C\otimes K)$ and a unitary $U \in M(C\otimes K)$ by

$$\psi(c) = \sum_{n\in Z} \gamma^n(c)\otimes e_{n,n} \quad U = \sum_{n\in Z} I\otimes e_{n,n+1}, \ \ c \in C,$$

(the sums converge strictly in $M(C\otimes K)$) It is easily seen that

$$U\psi(c)U^* = \psi(\gamma(c)), \ \ c \in C,$$

so that ψ extends to a representation $\tilde{\psi}: C \ltimes_\gamma Z \longrightarrow M(C\otimes K)$ The following standard argument shows that the representation $\tilde{\psi}$ is faithful.

Put $V_t = \sum_{n\in Z} I\otimes t^{-n} e_{n,n} \in M(C\otimes K)$ for $t \in T$, and check that V_t is a unitary element that satisfies $V_t\psi(c)V_t^* = \psi(c)$ and $V_tUV_t^* = tU$ for all $t \in T$. Let $E: C \ltimes_\gamma Z \to C$ be the canonical faithful conditional expectation, and define $F: Im(\tilde{\psi}) \to Im(\tilde{\psi})$ by $F(x) = \int_T V_t x V_t^* dt$. Then $F((\tilde{\psi}(x)) = \psi(E(x))$ for all $x \in C \ltimes_\gamma Z$. Now, if $\tilde{\psi}(x) = 0$ for some positive element x in $C \ltimes_\gamma Z$, then $\psi(E(x)) = F(\tilde{\psi}(x)) = 0$, whence $E(x) = 0$ (by injectivity of ψ and $x = 0$ (because E is faithful).

Lemma (3.2.22)[448]: Let C be a unital C^*-algebra and let γ be an automorphism on C. Suppose that p and q are projections in C such that

(i) $p \lesssim \bigoplus_{j=1}^m q$ in C for some natural number m, and

(ii) $p \not\lesssim \bigoplus_{j=-N}^N \gamma^j(q)$ for all natural numbers N.

Then q is not properly infinite in $C \rtimes_\gamma Z$.

Proof. It suffices to show that $\psi(q)$ is not properly infinite in $M(C\otimes K)$. Assume, to reach a contradiction, that $\psi(q)$ is properly infinite in $M(C\otimes K)$.Then $\bigoplus_{j=1}^m \psi(q) \lesssim \psi(q)$ by Proposition (3.2.1). As $q\otimes e_{0,0} \leq \psi(q)$ we can use (i) to obtain

$$p\otimes e_{0,0} \lesssim \bigoplus_{j=1}^m q\otimes e_{0,0} \leq \bigoplus_{j=1}^m \psi(q) \lesssim \psi(q) = \sum_{j=-\infty}^{\infty} \gamma^j(q)\otimes e_{j,j}$$

in $M(C\otimes K)$ By Lemma (3.2.9) this entails that

$$p\otimes e_{0,0} \lesssim \sum_{j=-\infty}^{\infty} \gamma^j(q)\otimes e_{j,j} \ \ in \ \ C\otimes K$$

for some $N \in N$, or, equivalently, that $p \precsim \bigoplus_{j=-N}^{N} \gamma^j(q)$ in C, in contradiction with assumption (ii).

Returning now to our specific C^*-algebra B from (3.2.22), Lemmas (3.2.21) and Lemma (3.2.22) imply:

Lemma (3.2.23)[448]: The projection $Q = \mu_{\infty,0}(p_1)$ is not properly infinite in $B \ltimes_\alpha Z$.

Lemma (3.2.24)[448]: The C^*-algebra $D_n = C^*(A_{-n}, A_{-n+1}, \dots, A_0, \dots, A_n)$ is of type I for each $n \in N$.

Proof. Note first that

$$A_n A_m \subseteq A_{\min\{n,m\}}, \quad n, m \in Z. \tag{65}$$

Indeed, we can assume without loss of generality that $n \leq m$, and then deduce

$$A_n A_m = \alpha^n(\mu_{\infty,0}(A\bar{\varphi}^{m-n}(A))) \subseteq \alpha^n(\mu_{\infty,0}(A)) = A_n.$$

Since $A \cap \bar{\varphi}^{m-n}(A) = \{0\}$ when $n < m$, cf. Proposition (3.2.12) (ii), it follows also that

$$A_n \cap A_m = \{0\}, \quad n \neq m. \tag{66}$$

Use (31) to see that the C^*-algebra $D_{m,n}$ generated by $A_m, A_{m+1}, \dots, A_n$, for $m \leq n$, is equal to

$$D_{m,n} = A_m + A_{m+1} + \dots + A_{n-1} + A_n. \tag{67}$$

(To see that the right-hand side of (67) is norm closed, use successively the fact that if E is a C^*-algebra, I is a closed two-sided ideal in E, and F is a sub-C^*-algebra of E, then $I + F$ is a sub C^*-algebra of E.) It follows from (65), (66) and (67) that we have a decomposition series

$$0 \lhd A_{-n} \lhd D_{-n,-n+1} \lhd D_{-n,-n+2} \lhd \dots \lhd D_{-n,n-1} \lhd D_{-n,n} = D_n$$

for D_n and that each successive quotient is isomorphic to $A = C(Z) \otimes K$ This proves that D_n is a type I C^*-algebra.

Lemma (3.2.25)[448]: The crossed product C^*-algebra $D \rtimes_\alpha Z$ contains an infinite projection and a non-zero projection which is not properly infinite. The C^*-algebra D has no nontrivial α^n-invariant closed two-sided ideal for any non-zero integer n.

Proof. The projection $Q = \mu_{\infty,0}(p_1)$ belongs to $A_0 = \mu_{\infty,0}(A) \subseteq D$, and it is non-zero because $\mu_{\infty,0}$ is injective (which again is because $\bar{\varphi}$ is injective). We have $D \subseteq B$ and hence

$$Q \in D \rtimes_\alpha Z \subseteq B \rtimes_\alpha Z.$$

Since Q is not properly infinite in $B \rtimes_\alpha Z$ (by Lemma (3.2.25)) it follows that Q is not properly infinite in $D \rtimes_\alpha Z$.

Put $p = \mu_{\infty,0}(g) \in A_0 \subseteq D$, where g is a constant 1-dimensional projection in $A = C(Z, K)$. We have

$$g = \varphi_0(g) \sim S_0\varphi_0(g)S_0^* < \sum_{j=-\infty}^{\infty} S_j\varphi_j(g)S_j^* = \bar{\varphi}(g),$$

cf. (19). Hence $P = \mu_{\infty,0}(g)$ is equivalent to a proper subprojection of $\mu_{\infty,0}(\bar{\varphi}(g))$. As $\mu_{\infty,0}(\bar{\varphi}(g)) = \alpha(\mu_{\infty,0}(g)) \sim P$ in $D \rtimes_\alpha Z$ we conclude that P is an infinite projection in $D \rtimes_\alpha Z$.

Suppose that n is a non-zero integer (that we can take to be positive) and that I is a non-zero closed two-sided α^n-invariant ideal in D. Then $I \cap D_{kn}$ is non-zero for some natural number k, cf. (64). As I is α^n-invariant, $I \cap \alpha^{kn}(D_{kn})$ is non-zero, and

$$\alpha^{kn}(D_{kn}) = C^*(A_0, A_1, \dots, A_{2kn}) = \mu_{\infty,0}(C^*(A, \bar{\varphi}(A), \dots, \bar{\varphi}^{2kn}(A))).$$

Because $A_0 = \mu_{\infty,0}(A)$ is an essential ideal in $\alpha^{kn}(D_{kn})$ it follows that $I \cap A_0$ is non-zero. Take a non-zero element f in $I \cap A_0$, and write $f = \mu_{\infty,0}(f_0)$ for some non-zero element f_0 in A. Use Proposition (3.2.12) (iii) to conclude that

$$A_{-m}f = \mu_{\infty,m}(A\bar{\varphi}^m(f_0))$$
is full in $\mu_{\infty,m}(A) = A_{-m}$, and hence that $A_{-m} \subseteq I$, for every natural number m. Since I is α^n-invariant, $A_{-m+rn} = \alpha^{rn}(A_{-m}) \subseteq I$ for all $m \in N$ and all $r \in Z$. This shows that $A_m \subseteq I$ for all m, which finally entails that $I = D$.
We remind the notion of properly outer automorphism introduced by Elliott in [141]:
Definition (3.2.26)[448]: An automorphism γ on a C^*-algebra E is called properly outer if for every non-zero γ-invariant closed two-sided ideal I of E and for every unitary u in $M(I)$ one has $\|\gamma|_I - Adu\| = 2$ (the norm is the operator norm).

In [156, Theorem 6.6] eleven conditions on an automorphism γ that all are equivalent to γ being properly outer. We shall use the following sufficient (but not necessary) condition for being properly outer: If E has no non-trivial γ-invariant ideals and if $\gamma(p) \nsim p$ for some projection p in E, then γ is properly outer. To see this, note first that $p \sim upu^* = (Adu)(p)$ for every unitary u in $M(E)$ (the equivalence holds relatively to E). We therefore have $\gamma(p) \nsim (Adu)(p)$, whence $\|\gamma(p) - (Adu)(p)\| = 1$. This shows that $\|\gamma - Adu\| \geq 1$ for all unitaries u in $M(E)$, whence γ is properly outer (by (ii)⟺(iii) of [156, Theorem 6.6]).

(One can argue along another line by taking an approximate unit $\{e_\lambda\}$ for E, such that $e_\lambda \geq p$ for all λ, and set $x_\lambda = 2p - e_\lambda$. Then x_λ is a contraction in E for all λ, and one can check that $\lim_{\lambda\to\infty} \|\gamma(x_\lambda) - (Adu)(x_\lambda)\| = 2$, thus showing directly that $\|\gamma - Ad\,u\| = 2$ for all unitaries u in $M(E)$ whenever $\gamma(p) \nsim p$ for some projection p in E.)
More generally, γ is properly outer if for each non-zero γ-invariant ideal I of E there is a projection p in I such that $\gamma(p) \nsim p$.
Lemma (3.2.27) [448]: The automorphism α^n on D is property outer for all non-zero integers n.
Proof. We know from Lemma (3.2.25) that D has no α^n-invariant ideals (when $n \neq 0$), so the lemma will follow from the claim (verified below) that $\alpha^n(Q) \nsim Q$ for all $n \neq 0$ (where Q is as in Lemma (3.2.21)).
Assume, to reach a contradiction, that $\alpha^n(Q) \sim Q$ for some non-zero integer n (that we can take to be positive). Then, by Lemma (3.2.21) (i),
$$\mu_{\infty,0}(g) \lesssim Q \oplus Q \sim Q \oplus \alpha^n(Q) \lesssim \oplus_{j=0}^{n} \alpha^j(Q) in\ D,$$
in contradiction with Lemma (3.2.21) (ii).
We now have to prove the main result:
Theorem (3.2.28)[448]: There is a separable C^*-algebra D and an automorphism α on D such that
(i) D is an inductive limit of type I C^*-algebras;
(ii) $D \rtimes_\alpha Z$ is simple and contains an infinite and a non-zero finite projection;
(iii) $D \rtimes_\alpha Z$ is nuclear and belongs to the UCT class N.
Proof. Let D be the C^*-algebra and let α the automorphism on D defined in (and above) (30). Since D is the union of an increasing sequence of sub- C^*-algebras D_n (cf. (64)) and each D_n is of type I (by Lemma (3.2.24)), we conclude that D is an inductive limit of type I C^*-algebras, and hence that the crossed product $D \rtimes_\alpha Z$ is nuclear, separable and belongs to the UCT class N.

Since D has no non-trivial α-invariant ideals (by Lemma (3.2.25)) and α^n is properly outer for all $n \neq 0$ (by Lemma (3.2.27)), it follows from Olesen and Pedersen [156, Theorem 7.2] (a result that extends results from Elliott [141] and Kishimoto [153]) that $D \rtimes_\alpha Z$ is simple. By simplicity of $D \rtimes_\alpha Z$, the (non-zero) projection Q, which in Lemma (3.2.25) is proved to be not properly infinite, must be finite in $D \rtimes_\alpha Z$, cf. Proposition

(3.2.1). The existence of an infinite projection in $D \rtimes_\alpha Z$ follows from Lemma (3.2.25), and this completes the proof.

We begin by listing some corollaries to Theorems (3.2.16) and (3.2.28).

Corollary (3.2.29)[448]: There is a nuclear, unital, separable, infinite, simple C^*-algebra A in the UCT class N such that A is not purely infinite.

Proof. Take the C^*-algebra $D \rtimes_\alpha Z$ from Theorem (3.2.28), and take a properly infinite projection p and a non-zero finite projection q in that C^*-algebra. Then $q \sim q_0 \leq p$ for some projection q_0 in $D \rtimes_\alpha Z$ by Lemma (3.2.2). Hence $A = p(D \rtimes_\alpha Z)p$ is infinite; and A is not purely infinite because it contains the non-zero finite projection q_0.

Corollary (3.2.30)[448]: There is a nuclear, unital, separable, finite, simple C^*-algebra A that is not stably finite, and hence does not admit a tracial state (nor a non-zero quasitrace).

Proof. Take the C^*-algebra $E = D \rtimes_\alpha Z$ from Theorem (3.2.28) and a non-zero finite projection q in E. Put $A = qEq$. Then A is finite, simple and unital. Since $A \otimes K \cong E \otimes K$ we conclude that $A \otimes K$ (and hence $M_n(A)$ for some large enough n) contains an infinite projection, so A is not stably finite.

Every simple, infinite C^*-algebra is properly infinite, so $M_n(A)$ is properly infinite. No properly infinite C^*-algebra can admit a non-zero trace (or a quasitrace), so $M_n(A)$, and hence A, do not admit a tracial state (nor a non-zero quasitrace).

A C^*-algebra A is said to have the cancellation property if the implication

$$p \oplus r \sim q \oplus r \implies p \sim q \qquad (68)$$

holds for all projections p, q, r in $A \otimes K$ It is known that all C^*-algebras of stable rank one have the cancellation property and that no infinite C^*-algebra has the cancellation property. There is no example of a stably finite, simple C^*-algebra which is known not to have the cancellation property (but Villadsen's C^*-algebras from [163] are candidates). A C^*-algebra A is said to have the weak cancellation property if (68) holds for those projections p, q, r in $A \otimes K$ where p and q generate the same ideal of A.

Corollary (3.2.31)[448]: There is a nuclear, unital, separable, simple C^*-algebra A that does not have the weak cancellation property.

Proof. And take a non-zero finite projection q in A.

Since A is properly infinite, we can find isometries s_1, s_2 in A with orthogonal range projections; cf. Proposition (3.2.1). Put $p = s_1 q s_1^*(1 - S_1 S_1^*)$. Then p is infinite because $s_2 s_2^* \leq p$, and so $p \nsim q$ (because q is finite). On the other hand, q and p generate the same ideal of A −namely A itself−and

$$p \oplus 1 = (s_1 q s_1^* + (1 - s_1 s_1^*)) \oplus 1 \sim s_1 q s_1^* \oplus (1 - s_1 s_1^*) \oplus s_1 s_1^* \sim q \oplus 1.$$

It was shown in [152, Theorem (5.3.18)] that the following implications hold for any separable C^*-algebra A and for any free filter ω on N:

A is purely infinite $\implies$ A is weakly purely infinite

$\Leftrightarrow$ A_ω is traceless

$\implies$ A is traceless,

and the first three properties are equivalent for all simple C^*-algebras A. (A C^*-algebra is here said to be traceless if no algebraic ideal in A admits a non-zero quasitrace. See [152] for the definition of being weakly purely infinite.) It was not known in [152] if the reverse of the third implication holds (for simple or for non-simple C^*-algebras), but we can now answer this in the negative:

Corollary (3.2.32)[448]: Let ω be any free filter on N. There is a nuclear, unital, separable, simple C^*-algebra A which is traceless, but where $l^\infty(A)$ and A_ω admit non-zero quasitraces defined on some (possibly non-dense) algebraic ideal.

Proof. Then A is algebraically simple and A admits no (everywhere defined) non-zero quasitrace. Hence A is traceless in the sense of [152].
Because A is simple and not purely infinite, A_ω cannot be traceless. Since A_ω is a quotient of $l^\infty(A)$, the latter C^*-algebra cannot be traceless either.

Kirchberg has shown in [150] (see also [161, Theorem 4.1.10]) that every exact simple C^*-algebra which is tensorially non-prime (i.e., is isomorphic to a tensor product $D_1 \otimes D_2$, where D_1 and D_2 both are simple non-type I C^*-algebras) is either stably finite or purely infinite. Liming Ge has proved in [143] that the II_1-factor $\mathcal{L}(F_2)$ is (tensorially) prime (in the sense of von Neumann algebras), and it follows easily from this result that the C^*-algebra $C^*_{red}(F_2)$ is tensorially prime. We can now exhibit a simple, nuclear C^*-algebra that is tensorially prime:

Corollary (3.2.33)[448]: The C^*-algebra $D \rtimes_\alpha Z$ from Theorem (3.2.28) is simple, separable, nuclear and tensorially prime, and so is $p(D \rtimes_\alpha Z)p$ for every non-zero projection p in $D \rtimes_\alpha Z$.

Proof. The C^*-algebra $D \rtimes_\alpha Z$ is simple, separable, nuclear; cf. Theorem (3.2.28). It is not stably finite because it contains an infinite projection, and it is not purely infinite because it contains a non-zero finite projection. The (unital) C^*-algebra $p(D \rtimes_\alpha Z)p$ is stably isomorphic to $D \rtimes_\alpha Z$ and is hence also simple, separable, nuclear, and neither stably finite nor purely infinite. It therefore follows from Kirchberg's theorem (quoted above) that these C^*-algebras must be tensorially prime.

Villadsen's C^*-algebras from [162] and [163] are, besides being simple and nuclear, probably also tensorially prime Jiang and Su have in [147] found a non-type I, unital, simple C^*-algebra Z for which $A \cong A \otimes Z$ is known to hold for a large class of well-behaved simple C^*-algebras A, such as for example the irrational rotation C^*-algebras and more generally all C^*-algebras that are covered by a classification theorem (cf. [142] or [161]). Such C^*-algebras A are therefore not tensorially prime.

The real rank of the C^*-algebras found in Theorems (3.2.16) and (3.2.28) have not been determined, but we guess that they have real $rank \geq 1$. That leaves open the following question:

Question (3.2.34)[448]: Does there exist a (separable) unital, simple C^*-algebra A such that A contains an infinite and a non-zero finite projection, and such that:

(i) A is of real rank zero?

(ii) A is both nuclear and of real rank zero?

It appears to be difficult (if not impossible) to construct simple C^*-algebras of real rank zero that exhibit bad comparison properties; below.
George Elliott suggested the following:

Question (3.2.35)[448]: Does there exist a (separable), (nuclear), unital, simple C^*-algebra A such that all non-zero projections in A are infinite but A is not purely infinite?

If Question (3.2.35) has affirmative answer, and A is a unital, simple C^*-algebra whose non-zero projections are infinite and A is not purely infinite, then the real rank of A cannot be zero. Indeed, a simple C^*-algebra is purely infinite if and only if it has real rank zero and all its non-zero projections are infinite.
for every natural number n. But $[1] \not\leq [e]$ because e is finite and 1 is infinite.

This shows that if A is a simple C^*-algebra with a finite and an infinite projection, then the semigroup $D(A)$ of Murray−von Neumann equivalence classes of projections in $A \otimes K$ is not weakly unperforated.

(An ordered abelian semigroup $(S, +, \leq)$ is said to be weakly unperforated if

$$ng < nh \Longrightarrow g \leq h, \text{ for all } g, h \in S \text{ and all } n \in N.$$
The order structure on $D(A)$ is the algebraic order given by $g \leq h$ if and only if $h = g + f$ for some f in $D(A)$.)

In [163] that $K_0(A)$, and also the semigroup $D(A)$, of a simple, stably finite C^*-algebra A can fail to be weakly unperforated. The present is a natural continuation of Villadsen's work to the stably infinite case.

For $(S, +)$ be an abelian semigroup with a zero-element 0. An element $g \in S$ is called infinite if $g + x = g$ for some non-zero $x \in S$, and g is called finite otherwise. The sets of finite and infinite elements in S are denoted by S_{fin} and S_{inf}, respectively. One has $S = S_{fin} \coprod S_{inf}$ and $S + S_{inf} \subseteq S_{inf}$, but the sum of two finite elements can be infinite.

It is standard and easy to see that the finite and infinite elements in the semigroup $D(A)$ are given by
$$D_{fin}(A) = \{[f]: f \text{ is a finite projection in } A \otimes K\},$$
$$D_{inf}(A) = \{[f]: f \text{ is an infinite projection in } A \otimes K\}.$$
If A is a simple C^*-algebra that contains an infinite projection, then the Grothendieck *map* $\gamma: D(A) \to K_0(A)$ restricts to an isomorphism $D_{inf}(A) \to K_0(A)$ as shown by Cuntz in [138, ∮ 1]. We can therefore identify $D_{inf}(A)$ with $K_0(A)$, in which case we can write
$$D(A) = D_{fin}(A) \coprod K_0(A).$$
Note that [0] belongs to $D_{fin}(A)$, and that $D_{fin}(A) = \{[0]\}$ if and only if all non-zero projections in $A \otimes K$ are infinite. One can therefore detect the existence of non-zero finite elements in $A \otimes K$ from the semigroup $D(A)$; and $K_0(A)$ contains all information about $D(A)$ if and only if all non-zero projections in $A \otimes K$ are infinite.

When A is simple and contains both infinite and non-zero finite projections, then $D_{fin}(A)$ can be very complicated and large. One can show that $D_{fin}(B)$ is uncountable, when B is as in Theorem (3.2.16). We have no description of $D(A)$, when $A = D \rtimes_\alpha Z$ from Theorem (3.2.28).

That if A is simple and if g is a non-zero element in $D_{fin}(A)$, then $ng \in D_{inf}(A)$ for some $n \in N$. In other words, $D_{inf}(A)$ eventually absorbs all non-zero elements in $D(A)$.

The example found in Theorem (3.2.28) provides a counterexample to Elliott's classification conjecture (see for example [20]) as it is formulated (by the author) in [161, ∮ 2.2]. The conjecture asserts that
$$\left(K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A), r_A: T(A) \to S\big(K_0(A)\big)\right) \tag{69}$$
is a complete invariant for unital, separable, nuclear, simple C^*-algebras. If A is stably infinite (i.e., if $A \otimes K$ contains an infinite projection), then $K_0(A)^+ = K_0(A)$ and $T(A) = \emptyset$. The Elliott invariant for unital, simple, stably infinite C^*-algebras therefore degenerates to the triple $(K_0(A), [1_A]_0, K_1(A))$. (We say that $(K_0(A), [1_A]_0, K_1(A)) \cong (G_0, g_0, G_1)$ if there are group isomorphisms $\alpha_0: K_0(A) \to G_0$ and $\alpha_1: K_1(A) \to G_1$ such that $\alpha_0([1_A]_0) = g_0$.)

Corollary (3.2.36)[448]: There are two non-isomorphic nuclear, unital, separable, simple, stably infinite C^*-algebras A and B (both in the UCT class N) such that
$$(K_0(A), [1_A]_0, K_1(A)) \cong (K_0(B), [1_B]_0, K_1(B)).$$
Proof. Take the C^*-algebra A from Corollary (3.1.29). It follows from [158, Theorem 3.6] that there is a nuclear, unital, separable, simple, purely infinite C^*-algebra B in the UCT class N such that
$$(K_0(A), [1_A]_0, K_1(A)) \cong (K_0(B), [1_B]_0, K_1(B)).$$
Since B is purely infinite and A is not purely infinite, we have $A \ncong B$.

One can amend the Elliott invariant by replacing the triple $(K_0(A), K_0(A)^+, [1_A]_0)$ (for a unital C^*-algebra A) with the pair $(D(A), [1_A])$, above, where $D(A)$ carries the structure of a semigroup. In the unital, stably infinite case, the amended invariant will then become $(D(A), [1_A], K_1(A))$. (Since $K_0(A)$ is the Grothendieck group of $D(A)$, and $K_0(A)^+$ and $[1_A]_0$ are the images of $D(A)$ and $[1_A]$, respectively, under the Grothendieck map $\gamma: D(A) \to K_0(A)$, one can recover $(K_0(A), K_0(A)^+, [1_A]_0)$ from $(D(A), [1_A]):)$

The invariant $(D(A), [1_A])$ can detect if A has a non-zero finite projection, cf. The triples $(D(A), [1_A], K_1(A))$ and $(D(B), [1_B], K_1(B))$ are therefore nonisomorphic, when A and B are as in Corollary (3.1.36). We have no example to show that $(D(A), [1_A], K_1(A))$ is not a complete invariant for nuclear, unital, simple, separable, stably infinite C^*-algebras. On the other hand, there is no evidence to suggest that $(D(A), [1_A], K_1(A))$ indeed is a complete invariant for this class of C^*-algebras.

The Elliott conjecture can also be amended by restricting the class of C^*-algebras that are to be classified. One possibility is to consider only those unital, separable, nuclear, simple C^*-algebras A for which $A \cong A \otimes Z$ where Z is the Jiang-Su algebra (see the comment below Corollary (3.2.33)). It seems plausible that the Elliott invariant (35) actually is a complete invariant for this class of C^*-algebras; and one could hope that the condition $A \cong A \otimes Z$ has an alternative intrinsic equivalent formulation, for example in terms of the existence of sufficiently many central sequences.
with non-zero index map $\delta: K_1(C(S^3)) \to K_0(K)$. Then A is finite and $M_2(A)$ is infinite.

The proof uses that any isometry or co-isometry s in A (or in a matrix algebra over A) is mapped to a unitary element u in (a matrix algebra over) $C(S^3)$; and every unitary u in $M_n(C(S^3))$ lifts to an isometry or a co-isometry s in $M_n(A)$. Moreover, the isometry or co-isometry s is non-unitary if and only if the unitary element u has non-zero index. The unitary group of $C(S^3)$ is connected, so all unitaries here have zero index. Hence A contains no non-unitary isometry, so A is finite. By construction of the extension, the generator of $K_1(C(S^3))$, which is a unitary element in $M_2(C(S^3))$, has non-zero index, and so it lifts to a non-unitary isometry or co-isometry in $M_2(A)$, whence $M_2(A)$ is infinite.

The C^*-algebra $M_2(A)$ is not properly infinite since the quotient, $M_2(A)/M_2(E) \cong M_2(C(S^3))$, is finite.

An example of a unital, finite, (non-simple) C^*-algebra A such that $M_2(A)$ is properly infinite was found in [160].
is an inductive limit with unital connecting maps, and that B is a simple C^*-algebra such that B is finite and $M_2(B)$ is infinite. Then $M_2(B)$ is properly infinite, and it follows from Proposition (3.2.3) that B_n is finite and $M_2(B_n)$ is properly infinite for all sufficiently large n. It is therefore not possible to construct an example of a simple C^*-algebra, which is finite, but not stably finite, by taking an inductive limit of C^*-algebras arising as in the example described.

Such that $\varphi_1(e)$ is a finite projection in $M_2(B)$ whenever e is a 1-dimensional projection in $M_2(C)$.

The existence of B (already obtained in the non-simple case in [160]) shows that the image of e in the universal unital free product C^*-algebra $M_2(C) * \mathcal{O}_\infty$ is not properly infinite.

It is tempting to turn this around and seek a simple C^*-algebra A with a finite and an infinite projection by defining A to be a suitable free product of $M_2(C)$ and $\mathcal{O}_\infty$. However, the universal unital free product $M_2(C) * \mathcal{O}_\infty$ is not simple. The reduced free product C^*-algebra

$$(A, \varrho) = (M_2(C), \varrho_1) * (\mathcal{O}_\infty, \varrho_2),$$
with respect to faithful states ϱ_1 and ϱ_2, is simple (at least for many choices of the states ϱ_1 and ϱ_2, see for example [124]) and properly infinite, but no non-zero projection e in $M_2(C)$ is finite in A. The Cuntz algebra $\mathcal{O}_\infty$ contains a sequence of non-zero mutually orthogonal projections, and it therefore contains a projection f with $\varrho_2(f) < \varrho_1(e)$. Now, e and f are free with respect to the state ϱ and $\varrho(f) < \varrho(e)$. This implies that $f \precsim e$ (see [123]), and therefore e must be infinite.

In [140] that reduced free product C^*-algebras often have weakly unperforated K_0-groups, which is another reason why this class of C^*-algebras is unlikely to provide an example of a simple C^*-algebra with finite and infinite projections.

We conclude by remarking that ring theorists for a long time have known about finite simple rings that are not stably finite:

An example of a unital, simple ring which is weakly finite but not weakly 2-finite was constructed by $P.M.$ Cohn as follows:

Take natural numbers $2 \leq m < n$ and consider the universal ring $V_{m,n}$ generated by $2mn$ elements $\{x_{ij}\}$ and $\{y_j\}, i = 1, \ldots, m$ and $j = 1, \ldots, n$, satisfying the relations $XY = I_m$ and $YX = I_n$, where $X = (x_{ij}) \in M_{m,n}(R), Y = (y_{ij}) \in M_{n,m}(R)$, and I_m and I_n are the units of the matrix rings $M_m(R)$ and $M_n(R)$. The rings $M_m(V_{m,n})$ and $M_n(Y_{m,n})$ are isomorphic, and $M_n(V_{m,n})$ is not weakly finite. Therefore $M_m(V_{m,n})$ is not weakly finite. In other words, $V_{m,n}$ is not weakly m-finite.

It is shown by Cohn in [133, Theorem 2.11.1] that $V_{m,n}$ is a so-called $(m-1) - fir$, and hence a $1 - fir$; and a ring is a $1 - fir$ if and only if it is an integral domain (i.e., if it has no non-zero zero-divisors). Cohn proved in [132] that every integral domain embeds into a simple integral domain. In particular, $V_{m,n}$ is a subring of a simple integral domain $R_{m,n}$ whenever $2 \leq m < n$. Now, $R_{m,n}$ is weakly finite (an integral domain has no idempotents other than 0 and 1, and must hence be weakly finite), and $R_{m,n}$ is not weakly m-finite (because it contains $V_{m,n}$).

This example cannot in any obvious way be carried over to C^*-algebras, first of all because no C^*-algebra other than C is an integral domain.

Chapter 4
Labelled Graph C^*-algebras of Labelled Spaces

We introduce a quotient labelled space $(E,\mathcal{L},[B]_R)$ arising from an equivalence relation $\sim_R$ on B and show the existence of the C^* −algebra $C^*(E,\mathcal{L},[B]_R)$ generated by a universal representation of $(E,\mathcal{L},[B]_R)$. We give necessary and sufficient conditions for simplicity of certain labelled graph C^* −algebras. We also show that the spectrum of it diagonal C*-subalgebra is homeomorphic to the tight spectrum of the inverse semigroup associated with the labelled space.

Section (4.1): The Structure of Gauge-Invariant Ideals

In [166], Bates and Pask introduced a class of C^* −algebras associated to labelled graphs. Their motivation was to simultaneously generalize ultragraph C^* −algebras [179,180] and the shift space C^* −algebras [169,177]. A labelled graph C^* −algebra $C^*(E,\mathcal{L},B)$ is the universal C^* −algebra generated by a family of partial isometries s_a indexed by labels a and projections p_A indexed by vertex subsets A in an accommodating set B satisfying certain conditions. By definition $C^*(E,\mathcal{L},B)$ depends on the choice of an accommodating set B as well as a labelled graph $(E,\mathcal{L})$, where $\mathcal{L}$ is a labelling map assigning a label to each edge of E. An accommodating set B is a collection of vertex subsets ($B\subset 2^{E^0}$) containing the ranges of all labelled paths which is closed under finite unions, finite intersections, and relative ranges. Among accommodating sets of a labelled graph $(E,\mathcal{L})$, the smallest one $\varepsilon^{0,-}$ was mainly dealt with in [167] under the assumptions that $(E,\mathcal{L})$ is essential (E has no sinks and no sources), set-finite and receiver set-finite (every $A\in\varepsilon^{0,-}$)emits and receives only finitely many labelled edges). Some conditions on $(E,\mathcal{L},\varepsilon^{0,-})$ were investigated to explore the simplicity of $C^*(E,\mathcal{L},\varepsilon^{0,-})$ in [167]. Since the accommodating set $\varepsilon^{0,-}$is not closed under relative complements in general, it may not contain generalized vertices $[v\,]l$ despite the fact that these generalized vertices were used effectively in [167] as the canonical spanning set of labelled graph C^* −algebras $C^*(E,\mathcal{L},\varepsilon^{0,-})$. We consider an alternative of $\varepsilon^{0,-}$ in [171], that is, the smallest accommodating set $\overline{\varepsilon}$ which is closed under relative complements (or equivalently, the smallest accommodating set containing all generalized vertices). It was then proven that if $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple, $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal [171, Theorem 3.8] and if in addition $\{v\}\in\overline{\mathcal{E}}$ for every vertex $v\in E^0$, the labelled space $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable [171 , Theorem 3.14]. Furthermore, a slight modification of the proof of Theorem 6.4 in [167], shows that if $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal and disagreeable, the C^* −algebra $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple [171, Theorem 3.16].Even when $\mathcal{E}^{0,-}\neq\overline{\mathcal{E}}$ if both $(E,\mathcal{L},\mathcal{E}^{0,-})$ and $(E,\mathcal{L},\overline{\mathcal{E}})$ are weakly left-resolving, $C^*(E,\mathcal{L},\mathcal{E}^{0,-})\cong C^*(E,\mathcal{L},\overline{\mathcal{E}})$ (Corollary (4.1.18)).

By the universal property, $C^*(E,\mathcal{L},B)$ admits the gauge action of the unit circle. As for the gauge-invariant ideal structure of graph C^* −algebras, it is known [165] that the set of gaugeinvariant ideals I of a row-finite graph C^* −algebra $C^*(E)=C^*(s_e,p_v)$ is in bijective correspondence with the set of hereditary saturated vertex subsets H in such a way that I is the ideal generated by the projections $p_v,\ v\in H$. By an ideal we always mean a closed two-sided one. A more general description on the gauge-invariant ideal structure of an arbitrary graph C^* −algebra is obtained in [164]. Also, for the class of ultragraph C^* −algebras [179] which contains all graph algebras (see [174,175,164,165] among others) and Exel Laca algebras [170], the structure of gauge-invariant ideals was described via a one-to-one correspondence with the set of admissible pairs of the ultragraph [173] using the results known for the C^* −algebras of topological graphs and topological quivers [172,178].

We analyze the structure of gauge-invariant ideals of a labelled graph C^* −algebra $C^*(E,\mathcal{L},B)$ when E has no sinks and $(E,\mathcal{L},B)$ is a set−finite, receiver set-finite and weakly left−resolving labelled space such that B is closed under relative complements. One might expect that a one-to-one correspondence like the correspondence mentioned above for graph C^* −algebras could be easily established by similar arguments used in the proofs for graph C^* −algebras as done in [165]. But an essential difficulty lies in the fact that the quotient algebra $C^*(E,\mathcal{L},B)/I$ by a gauge-invariant ideal I is not known to be realized as a labelled graph C^* −algebra. So we introduce a notion of quotient labelled space $(E,\mathcal{L},[B]_R)$ which is similar to a labelled space but with the equivalence classes $[B]_R$ of an equivalence relation $\sim_R$ on B in place of B in the labelled space $(E,\mathcal{L},B)$. Then in Theorem (4.1.14) we associate a universal C^* −algebra $C^*(E,\mathcal{L},[B]_R)$ to a quotient labelled space and prove that every C^* −algebra $C^*(E,\mathcal{L},[B]_R)$ of a quotient labelled space is isomorphic to a quotient algebra $C^*(E,\mathcal{L},B)/I$ by a gauge-invariant ideal I of $C^*(E,\mathcal{L},B)$ in Corollary (4.1.8) which follows from the gauge-invariant uniqueness theorem (Theorem (4.1.17)) for the C^* −algebras of quotient labelled spaces. It is obtained that if I is a gauge-invariant ideal of $C^*(E,\mathcal{L},B)$, the quotient algebra $C^*(E,\mathcal{L},B)/I$ is isomorphic to a C^* −algebra $C^*(E,\mathcal{L},[B]_R)$ associated to certain quotient labelled space. We then apply these isomorphism relations to obtain the main result (Theorem 4.1.21) that there exists a one-toone correspondence between the set of hereditary saturated subsets H (which we shall define) of B and the set of gauge-invariant ideals I_H of $C^*(E,\mathcal{L},B)$.

Returning to the labelled spaces $(E,\mathcal{L},\overline{\mathcal{E}})$ and the simplicity of $C^*(E,\mathcal{L},\overline{\mathcal{E}})$, we consider a labelled graph $(E,\mathcal{L})$ such that for each $v \in E^0$, a generalized vertex $[v]l$ is a finite set for some l. For the merged labelled graph $(F,\mathcal{L}_F)$ (Definition (4.1.22)) of $(E,\mathcal{L})$, we show that $\overline{\mathcal{F}}$ has the property that every set of single vertex belongs to $\overline{\mathcal{F}}$ and $C^*(E,\mathcal{L},\overline{\mathcal{E}}) \cong C^*(F,\mathcal{L}_F,\overline{\mathcal{F}})$ (Theorem (4.1.30)). It is shown that $(F,\mathcal{L}_F,\overline{\mathcal{F}})$ is strongly cofinal (respectively, disagreeable) if and only if $(F,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal (respectively, disagreeable) (Theorem (4.1.31)). This then proves that if $(F,\mathcal{L},\overline{\mathcal{E}})$is a labelled space such that for each $v \in E^0$, a generalized vertex $[v]l$ is finite for some l, then $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ is simple if and only if $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal and disagreeable (Corollary (4.1.32)).

We use the notational conventions of [174] for graphs and graph C^* −algebras and of [167] for labelled spaces and their C^* −algebras. *A directed graph* is a quadruple $E = (E^0,E^1,r,s)$ consisting of a countable set of vertices E^0, a countable set of edges E^1,and the range, source maps $r_E,s_E : E^1 \to E^0$ (we often write r and s for r_E and s_E, respectively). By E^n we denote the set of all finite paths $\lambda = \lambda_1 \cdots \lambda_n$ of *length* n ($|\lambda| = n$) ($\lambda_i \in E^1, r(\lambda_i) = s(\lambda_{i+1})$,$(1 \le i \le n-1)$ and use the notation $E^{\le n} := \bigcup_{i=1}^{n} E^i$ and $E^{\ge n} := \bigcup_{i=1}^{\infty} E^i s$ naturally extend to $E^{\ge n}$. If a sequence of edges $\lambda_i \in E^1 (i \ge 1)$ *satisfies* $r(\lambda_i) = s(\lambda_{i+1})$, one obtains an infinite path $\lambda_1\lambda_2\lambda_3 \cdots$ with the source $s(\lambda_1\lambda_2\lambda_3 \cdots) := s(\lambda_1)$. E^∞ denotes the set of all infinite paths.

A labelled graph $(E,\mathcal{L})$ over a countable alphabet A consists of a directed graph E and a *labelling map* $L : E^1 \to A$. We assume that $\mathcal{L}$ is onto. Let A^* and A^∞be the sets of all finite sequences (of length greater than or equal to 1) and infinite sequences, respectively. Then

$\mathcal{L}(\lambda) := \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n) \in A^* if\ \lambda = \lambda_1 \cdots \lambda_n \in E^n, and\ \mathcal{L}(\delta) := \mathcal{L}(\delta_1)\mathcal{L}(\delta_2) \cdots \in \mathcal{L}(E^\infty) \subset A^\infty\ if\ \delta = \delta_1\delta_2 \cdots \in E^\infty$.

We use the notation $\mathcal{L}^*(E) \coloneqq \mathcal{L}(E^{\geq 1})$.The range $r(\alpha)$and source r(α)of a 1abelleb path $\alpha \in \mathcal{L}^*(E)$ are subsets of E^0 defined by

$$r(\alpha) = \{r(\lambda): \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha\},$$
$$s(\alpha) = \{s(\lambda): \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha\}.$$

The relative range of $\alpha \in \mathcal{L}^*(E)$ *with respect to* $A \subset 2^{E^0}$ is defined to be

$$r(A,\alpha) = \{r(\lambda): \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha\ , s(\lambda) \in A\}.$$

If $B \subset 2^{E^0}$ is a collection of subsets of E^0 such that $r(A,\alpha) \in B$ whenever $A \in B$ and $\alpha \in \mathcal{L}^*(E)$, B is said to be closed under relative ranges for $(E,\mathcal{L})$. We call B an accommodating set for $(E,\mathcal{L})$ if it is closed under relative ranges, finite intersections and unions and contains r(α) for all $\alpha \in \mathcal{L}^*(E)$. If B is accommodating for $(E,\mathcal{L})$, the triple $(E,\mathcal{L},B)$ is called a labelled space. A labelled space $(E,\mathcal{L},B)$ is weakly left-resolving if

$$r(A,\alpha) \cap r(B,\alpha) = r(A \cap B,\alpha)$$

for all $A, B \in B$ and $\alpha \in \mathcal{L}^*(E)$.

For $A, B \in 2^{E^0}$and $n \geq 1$, let

$$AE^n = \{\lambda \in E^n: s(\lambda) \in A\} \quad E^nB = \{\lambda \in E^n: r(\lambda) \in B\},$$

and $AE^nB = AE^n \cap E^nB$.. We write E^nv for $E^n\{v\}$ and vE^n for $\{v\}E^n$, and will use the notation $AE^{\geq k}$and vE^∞ which should have obvious meaning. A labelled space $(E,\mathcal{L},B)$ is said to be set-finite (receiver set-finite, respectively) if for every $A \in B$ the set $\mathcal{L}(AE^1)$ ($\mathcal{L}(E^1A)$, respectively) is finite.

We assume that E has no sinks, that is $|s^{-1}(v)| > 0$ for all $v \in E^0$.

Definition (4.1.1)[449]: (See [166, Definition 4.1].) Let $(E,\mathcal{L},B)$ be a weakly left-resolving labelled space. A representation of $(E,\mathcal{L},B)$ consists of projections $\{p_A : A \in B\}$ and partial isometries $\{s_a : a \in A\}$ such that for $A, B \in B$ and $a, b \in A$,

(i) $P_\emptyset = 0,\ PAPB = PA \cap B, and\ pA \cup B = p_A + p_B - p_{A\cap B},$

(ii) $_{PA}s_a = s_a \Pr(A,a),$

(iii) $s_a^* s_a = {}_{pr(a)}$and $s_a^* s_a = 0$ unless $a = b,$

(iv) for $A \in B,$ if $\mathcal{L}(AE^1)$ is finite and nonempty, then

$$P_A = \sum_{a \in \mathcal{L}(AE^1)} s_a P_{r(A,a)} s_a^*.$$

Remark (4.1.2)[449]: It is known [166, Theorem 4.5] that if $(E,\mathcal{L},B)$ is a weakly left-resolving labelled space, there exists a C^* −algebra $C^*(E,\mathcal{L},B)$ generated by a universal representation $\{s_a, P_A\}$ of $(E,\mathcal{L},B)$. In this case, we simply write $C^*(E,\mathcal{L},B) = C^*(s_a, P_A)$ and call , *labelled graph C^* −algebra* of a labelled space$(E,\mathcal{L},B)$Furthermore, $s_a \neq 0$ and $p_A \neq 0$ for $a \in A$ and $A \in B, A \neq \emptyset$. Note also that$s_\alpha p_A s_\beta^* \neq 0$ if and only if $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$. If we assume that $(E,\mathcal{L},B)$ is set-finite, by [166, Lemma 4.4] and Definition (4.1.1)(iv) it follows that

$$P_A = \textstyle\sum_{\sigma \in \mathcal{L}(AE^n)} s_\sigma P_{r(A,\sigma)} s_\sigma^*. \quad \text{for } A \in B,\ n \geq 1 \tag{1}$$

and

$$C^*(E,\mathcal{L},B) = \overline{\operatorname{span}}\{s_\alpha p_A s_\beta^*: \alpha, \beta \in \mathcal{L}^*(E), A \in B\}. \tag{2}$$

By the universal property of $C^*(E,\mathcal{L},B)C^*(s_a, p_A)$, there exists a strongly continuous action $\gamma : \mathbb{T} \to \operatorname{Aut}(C^*(E,\mathcal{L},B))$, called the gauge action, such that $\gamma_z(s_a) = zs_a$ and $\gamma_z(p_A) = p_A$. We have the following gauge-invariant uniqueness theorem for labelled graph C^* −algebras $C^*(E,\mathcal{L},B)$

Theorem (4.1.3)[449]: (See [166, Theorem 5.3].) Let $(E,\mathcal{L},B)$ be a weakly left-resolving labelled space and let $\{S_a, P_A\}$ be a representation of $(E,\mathcal{L},B)$ on Hilbert space. Take $\pi_{S,P}$ to be the representation of $C^*(E,\mathcal{L},B)$ satisfying $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_A) = P_A$. Suppose that each P_A is nonzero whenever $A \neq \emptyset$, and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_a, P_A)$ such that for all $z \in \mathbb{T}$, $\beta_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$. Then $\pi_{S,P}$ is faithful.

For $v, w \in E^0$, we write$v \sim_l w$ $\mathcal{L}(E^{\leq l}w)$ as in [167]. Then $\sim_l$ is an equivalence relation on E^0. The equivalence class$[v]l$ of v is called a generalized vertex. Let $\Omega_l(E) := E^0/\sim_l$. For $k > l$ and $v \in E^0$, $[v]_k \subset [v]_l$ is obvious and $[v]_l = \cup_{i=1}^m [v_i]_{l+1}$for some vertices $v_1, \dots, v_m \in [v]_l$ [171, Proposition 2.4].

From now on we assume that our labelled space $(E, \mathcal{L}, B)$ is set-finite and receiver set-finite for any accommodating set B.

Let $\mathcal{E}^{0,-}$ be the smallest accommodating set for $(E, \mathcal{L})$. Then $\mathcal{E}^{0,-}$ consists of the sets of the form $\cup_{k=1}^m \cap_{i=1}^n r\left(\beta_{i,k}\right)\beta_{i,k} \in \mathcal{L}^*(E)$, as mentioned in [171, Remark 2.1], and is contained

in every accommodating set B for $(E, \mathcal{L})$. Let $\overline{\mathcal{E}}$ be the smallest one among the accommodating

sets B for $(E, \mathcal{L})$ such that $A \setminus B \in B$ whenever $A, B \in B$. Then $\overline{\mathcal{E}}$ contains all generalized vertices $[v]_l$ since every $[v]_l$ is the relative complement of sets in $\mathcal{E}^{0,-}$. More precisely, $[v]_l = X_l(v) \setminus r(Y_l(v))$, where $X_l(v) := \cap_{\alpha \in \mathcal{L}(E^{\leq l}v)} r(\alpha)$ and $Y_l(v) \cup_{w \in Y_l(v)} \mathcal{L}(E^{\leq l} w) \setminus \mathcal{L}(E^{\leq l} v)$ [167, Proposition 2.4]. If $\overline{\mathcal{E}}$ is weakly left-resolving then

$$\overline{\mathcal{E}} = \left\{\cup_{i=1}^n [v_i]_{l_i} : v_i \in E^0, l_i \geq 1, n \geq 1\right\} \tag{3}$$

(see [171, Proposition 3.4]).

Let B_1 and B_2 be two accommodating sets for $(E, \mathcal{L})$ such that $B_1 \subset B_2$. If $C^*(E, \mathcal{L}, B_1) = C^*(t_a, q_A)$ and $C^*(E, \mathcal{L}, B_2) = C^*(s_a, p_A)$, since $\{s_a, p_A : a \in A, A \in B_1\}$ is a representation of $(E, \mathcal{L}, B_1)$, by the universal property of $C^*(E, \mathcal{L}, B_1)$ there exists a $*$ $-$homomorphism $\iota : C^*(E, \mathcal{L}, B_1) \to C^*(E, \mathcal{L}, B_2)$, such that $\iota(t_a) = s_a$ and $q_A = P_A$ for $a \in A$, $A \in B_1$. Let α and β be the gauge actions of $\mathbb{T}$ on , $C^*(E, \mathcal{L}, B_1)$ and , $C^*(E, \mathcal{L}, B_2)$, respectively. Then $\iota \circ \alpha_z = \beta_z \circ \iota$ for $z \in \mathbb{T}$ and $\iota(q_A) = p_A \neq 0$ for $A \in B_1$, hence by Theorem (4.1.3).

Proposition (4.1.4)[449]: Let $B_1 \subset B_2$be two accommodating sets for a labelled graph $(E, \mathcal{L})$ such that $(E, \mathcal{L}, B_i)$ is weakly left-resolving for $i = 1,2$. If $C^*(E, \mathcal{L}, B_1) = C^*(t_a, q_A)$ and $C^*(E, \mathcal{L}, B_2) = C^*(s_a, p_B)$, the homomorphism $\iota : C^*(E, \mathcal{L}, B_1) \to C^*(E, \mathcal{L}, B_2)$ such that $\iota(t_a) = s_a$ and $\iota(q_A) = p_A$ is injective.

Corollary(4.1.5)[449]: Let $(E, \mathcal{L}, \mathcal{E}^{0,-})$ and $\left(E, \mathcal{L}, \overline{\mathcal{E}}\right)$ be weakly left-resolving $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ labelled spaces. Then

Proof. Let $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) = C^*(s_a, p_A)$ and $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(t_a, q_B)$, $a \in A$, $A \in \mathcal{E}^{0,-}$, $B \in \overline{\mathcal{E}}$. Then the map $\iota : C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \to C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ such that $\iota(s_a) = t_a$ and $\iota(p_A) = q_A$, $A \in \mathcal{E}^{0,-}$, is an isomorphism by Proposition (4.1.4). For any $[v]_l \in \overline{\mathcal{E}}$, there are two sets $A, B \in \mathcal{E}^{0,-}$such that $[v]_l = A \setminus B$. Since $A = (A \setminus B) \cup (A \cap B)$ and $A \setminus B, A \cap B \in \overline{\mathcal{E}}$, we have $q_A = q_A \setminus B + q_{A \cap B}$ and so

$$q_{[v]_l} = q_{A \setminus B} = q_A - q_{A \cap B} = \iota\left(C^*(E, \mathcal{L}, \mathcal{E}^{0,-})\right).$$

Hence ι is surjective by (3).

We assume that every labelled space $(E, \mathcal{L}, B)$ is weakly left-resolving and B is closed under relative complements.

Definition (4.1.6)[449]: Let $(E, \mathcal{L}, B)$ be a labelled space and $\sim_R$ an equivalence relation on B. Denote the equivalence class of $A \in B$ by $[A]$ (or $[A]_R$ in case we need to specify the relation. $\sim_R$) and let

$$A_R := \{a \in A : [r(a)] \neq [\emptyset]\}.$$

If the following operations $\cup,\cap$, and $\backslash$,

$$[A]\cup[B]:=[A\ \cup\ B],\quad [A]\cap[B]:=[A\ \cap\ B],\quad [A]\setminus[B]:=[A\ \setminus\ B]$$

are well-defined on the equivalence classes $[B]_R:=\{[A]: A\ \in B\}$, and if the relative range,

$$r([A],\alpha):=[r(A,\alpha)],$$

Is Awell-defined for $[A]\in[B]_R, \alpha\in C^*(E)\cap(A_R)^*$ so that $r([A],\alpha)=[\emptyset]$ for all $\alpha\in C^*(E)\cap(A_R)^*$ implies $[A]=[\emptyset]$ we call a triple $(E,\mathcal{L},[B]_R)$a quotient labelled space of $(E,\mathcal{L},[B]_R)$([A],α) = $[\emptyset]\alpha\ \in(E,L*L(E,B)\ \cap)$.

Note weakly that left-resolving $r(*([\emptyset]\alpha)\cap\ A=R[\emptyset]$. If and r $([A],\alpha)[A\cap r([B]=,\alpha)[\emptyset]$ =whenever $([A]\cap[BA],\alpha)=r([A]\cap[B],\alpha)$ holds for all $[A],[B]\in[B]_R)$ is and $\alpha\,\mathcal{L}^*(E)\cap(A_R)^*$.

A labelled space itself is a quotient labelled space with the relation of equality and $A_R=A$. For a nontrivial and important example of quotient labelled spaces, see Proposition (4.1.10) below.

In a similar way to Definition (4.1.1), we define a representation of a quotient labelled space as follows.

Definition (4.1.7)[449]: Let $(E,:\mathcal{L},[B]_R)$ be a weakly left-resolving quotient labelled space of labelled space $(E,\mathcal{L},B)$. A representation of $(E,:\mathcal{L},[B]_R)$ consists of projections $\{p_{[A]}:[A]\ \in\ [B]_R\}$ and partial isometries $\{s_a: a\ \in\ A_R\}$ subject to the relations:

(i) $p_{[\emptyset]}=\ 0,\ p_{[A]}p_{[B]}\ =\ p_{[A]\cap[B]}$, and $p_{[A]\cup[B]}=p_{[A]}+p_{[B]}-\ p_{[A]\cap[B]}$,

(ii) $p_{[A]}s_a=s_a\,p_{r([A],a)}$,

(iii) $s_a^*s_a=\ p_{[r(a)]}$ and $s_a^*s_b=0$ unless $a\ =\ b$,

(iv) for $[A]\in[B]_R$, if $\mathcal{L}([A]E^1)\cap A_R$is nonempty, then

$$p_{[A]}=\sum_{a\in\mathcal{L}([A]E^1)\cap A_R} s_a\,p_{r([A],a)s_a^*}.$$

The sum $p_{[A]}=\sum_{a\in\mathcal{L}([A]E^1)\cap A_R}s_a\,p_{r([A],a)s_a^*}$ of Definition (4.1.8) (iv) exists since $s_a\,p_{r([A],a)s_a^*=0}$ for all but finitely many $a\in\mathcal{L}([A]E^1)\cap A_R$. In fact, if $[A]=[A']$ and $a\in\mathcal{L}(AE^1)\backslash\mathcal{L}\left(A'^{E^1}\right)$, then from $r([A],a)=r([A'],a)$ we must have $[r(A,a)]=[r(A',a)]=[\emptyset]$ and hence $p_{r([A],a)}=0$.Thus $s_a\,p_{r([A],a)}s_a^*\neq0$ is possible only when $a\in\bigcap_{[A']=[A]}\mathcal{L}(A'E^1)$, but the set $\bigcap_{[A']=[A]}\mathcal{L}(A'E^1)$ is finite since we assume that $(E,\mathcal{L},B)$ is set-finite.

Definition (4.1.8)[449]: Let H be a subset of an accommodating set B. H is said to be hereditary if H satisfies the following:

(i) $r(A,\alpha)\in H$ for all $A\ \in H,\alpha\in\mathcal{L}^*(E)$,

(ii) $A\cup B\in H$ For all $A,B\in H$,

(iii) if $A\ \in\ H$ and $B\ \in\ B$ with $B\ \subset\ A$, then $B\ \in\ H$.

and $B\ \in\ B$ since $A\setminus B\ \subset\ A\ \in\ H$ and $A\backslash B\ \in\ B$. A hereditary subset H of B is called saturated if for any $A\ \in\ B$, $\{r(A,a): a\ \in\ A\}\subset\ H$ implies that $A\ \in\ H$. We write $\overline{H}$ for the smallest hereditary saturated set containing H.

Lemma (4.1.9)[449]: Let I be a nonzero ideal in $C^*(E,\mathcal{L},B)\ =C^*(s_a,p_A)$. Then the set

$$H_I:=\{A\ \in\ B\ :\ p_A\ \in\ I\}$$

is hereditary and saturated. If I is gauge-invariant, $H_I\neq\{\emptyset\}$.

Proof. To show that H_I is hereditary, let $A\in H_I$.Then $p_As_a=s_a\,p_{r(A,a)}=s_a^*s_a\,p_{r(A,a)}\in I$ and $r(A,a)\in H_I$ for all $a\in A$. A1so if $A,B\in H_I, P_{A\cup B}=P_A+p_B-p_Ap_{B}$a) is in I, that is $A\cup B\in H_I$. If $A\in H_I$ and $B\in B$, with $B\subset A$, then $p_B=p_{A\cap B}=p_Ap_B\in I$ and $B\in H_I$.

Now let $A \in B$ and $r(A,a) \in H_I$ for All $a \in A$. Then the projection $p_A = \sum_{a \in \mathcal{L}(AE^1)} s_a \times p_{r(A,a)} s_a^*$ belongs to I, that is, $A \in H_I$ for all a ∈ A. and the hereditary set H_I is saturated

Finally, suppose that I is a gauge-invariant ideal of $C^*(E,\mathcal{L},B)$. such that $H_I = \{\emptyset\}$.If π : $C^*(E,\mathcal{L},B)$ $C^*(E,\mathcal{L},B)/I$ is the quotient $*-$homomorphism $\{\pi(p_A),\pi(s_a)\}$ is a representation $C^*(E,\mathcal{L},B)$ such that the projections $\pi(p_A)$ are nonzero for $A \neq \emptyset$ and the action γ'on $C^*(E,\mathcal{L},B)/I$ induced By the gauge actionγ on $C^*(E,\mathcal{L},B)$ satisfies $\gamma' \circ \pi = \pi \circ \gamma$ (note that $\gamma_z(I) \subset I$).by Theorem (4.1.3) π is faithful, which is a contradiction.

Proposition (4.1.10)[449]: Let I be a nonzero gauge-invariant ideal of $\mathrm{C}^*(\mathrm{E},\mathcal{L},\mathrm{B})$. Then the relation

$\mathrm{A} \sim_{\mathrm{I}} \mathrm{B} \Longleftrightarrow \mathrm{A} \cup \mathrm{W} = \mathrm{B} \cup \mathrm{W}$ for some $\mathrm{W} \in \mathrm{H}_{\mathrm{I}}$

defines an equivalence relation $\sim_{\mathrm{I}}$ on B such that $(\mathrm{E},\mathcal{L},[\mathrm{B}]_{\mathrm{I}})$ is a weakly left-resolving quotient labelled space of $(\mathrm{E},\mathcal{L},\mathrm{B})$.

Proof. Clearly $\sim_I$ is reflexive and symmetric. It is transitive since

$A \sim_I B$ and $B \sim_I C$

$\Rightarrow$ $A \cup W = B \cup W$ and $B \cup V = C \cup V$ for some $W,V \in H_I$

$\Rightarrow A \cup (W \cup V) = C \cup (W \cup V)$

$\Rightarrow A \sim_I C$ because $W \cup V \in H_I$.

To see that we have well-defined operations ∪, ∩ and on $[B]_I$, let $[A] = [A']$ and $[B] = [B']$ .Choose $W,V \in H_I$ such that $A \cup W = A' \cup W$ and $B \cup V = B' \cup V$. Then

$$(A \cup B) \cup (W \cup V) = (A' \cup B') \cup (W \cup V),$$
$$\begin{aligned}(A \cap B) \cup (W \cup V) &= (A \cup (W \cup V)) \cap (B \cup (W \cup V)) \\ &= (A' \cup (W \cup V)) \cap (B' \cup (W \cup V)) \\ &= (A' \cap B') \cup (W \cup V),\end{aligned}$$
$$(A \setminus B) \cup (W \cup V) = (A' \setminus B') \cup (W \cup V).$$

Thus $[A] \cup [B] = [A'] \cup [B']$, $[A] \cap [B] = [A'] \cap [B']$, and $[A] \backslash [B] = [A'] \backslash [B']$.

We claim that $[r(A,\alpha)] = [r(A',\alpha)]$ for $[A']$ and $\alpha \in \mathcal{L}^*(E) \cap A_I^*$ where $A_I = \{a \in A: [r(a)] \neq [\emptyset] = \{a \in A: p_{r(a)} \notin I\}$. Let $A \cup W = A' \cup W$ for $W \in H_I$.Then $r(A,\alpha) \cup r(W,\alpha) = r(A \cup W,\alpha) = r(A' \cup W,\alpha) = r(A',\alpha) \cup r(W,\alpha)$.Since $r(W,\alpha) \in H_I$ we have $[r(A,\alpha) = [\emptyset]$ and see that the relative ranges $r([A],\alpha)$ are well-defined.

If $r([A],\alpha) = [\emptyset]$ for all $\alpha \in \mathcal{L}^*(E) \cap A_I^*$, then $r(A,a) \in H_I$ for all $a \in A_I$. Since $r(A,a) \in H_I$ for all $a \notin A_I$ and H_I is saturated, $A \in H_I$, that is, $[A] = [\emptyset]$ follows.

Finally, $[B]_I$ is weakly left-resolving since $r([A],\alpha) \cap r([B],\alpha) = [r([A],\alpha)] \cap [r([B],\alpha)] = [r(A,\alpha) \cap r(\cap B,\alpha)] = [r(A \cap B,\alpha)] = r([A \cap B],\alpha) = r([A] \cap [B],\alpha)$.

Lemma (4.1.11)[449]: Let H be a hereditary subset of B. and Then the ideal I_{H} of $\mathrm{C}^*(\mathrm{E},\mathcal{L},\mathrm{B})$ generated by the projections $\{\mathrm{p}_{\mathrm{A}}: \mathrm{A} \in \mathrm{H}\}$ is gauge-invariant and

$$I_H = I_{\overline{H}} = \overline{span}\{s_\alpha p_A s_\beta; \alpha,\beta \in \mathcal{L}^*(E), A \in \overline{H}\}.$$

Proof. By $H_{I_H} = \{A \in \bar{\varepsilon} : P_A \in I_H\}$ is a hereditary saturated subset of B, and $H \subset H_{I_H}$. It is easy to see that $J := \overline{span}\{s_\alpha p_A s_\beta; \alpha,\beta \in \mathcal{L}^*(E), A \in \overline{H}\}$ is a gauge-invariant ideal of $\mathcal{L}^*(E,\mathcal{L},B)$ such that $J \subset I_H$. But J contains the generators $\{p_A: A \in \overline{H}\}$ of $I_{\overline{H}}$ by Definition (4.1.8) (iv). Hence $I_{\overline{H}} \subset J$.

Let $(E,\mathcal{L},[\mathcal{B}]_R)$ be a quotient labelled space. In a similar way as in [166], we set $[\mathcal{B}]_R^* = (E,\mathcal{L},(\mathcal{A}_R)^*) \cup [\mathcal{B}]_R$ and extend r,s to $[\mathcal{B}]_R$ be$r([A]) = [A]$ and $s[A] = [A]$ for $[A] \in [\mathcal{B}]_R$ Also put $s_{[A]} = p_{[A]}$ so that s_β is defined for all $\beta \in [\mathcal{B}]_R^*$. The following lemma can be proved by the same arguments in [166, Lemma 4.4].

Lemma (4.1.12)[449]: Let $(E,\mathcal{L},[\mathcal{B}]_R)$be a weakly left-resolving quotient labelled space and $\{s_a, p_{[A]}\}$ a representation of $(E,\mathcal{L},[\mathcal{B}]_R)$. Then any nonzero products of s_a , $p_{[A]}$, and

s_β^* can be written as a finite linear combination of elements of the form $s_\alpha p_{[A]} s_\beta^*$ for some $A \in [\mathcal{B}]_R$and $\alpha, \beta \in [\mathcal{B}]_R^*$ with $[A] \subset [r(\alpha) \cap r(\beta)] \neq [\emptyset]$. Moreover we have the following:

$$(s_\alpha p_{[A]} s_\beta^*)(s_\gamma p_{[B]} s_\delta^*) = \begin{cases} s_{\alpha\gamma'} p_{r([A],\gamma') \cap [B] s_\delta^*}, & if \gamma = \beta\gamma' \\ s_\alpha p_{[A]} \cap r([B]\beta') s_{\delta\beta'}^*, & if \beta = \beta\beta' \\ s_\alpha p_{[A]} \cap [B] s_{\delta\beta'}^*, & if \beta = \beta\gamma, \\ 0, & otherwise. \end{cases}$$

Theorem (4.1.13)[449]: Let (E, L, B) be a labelled space and I be a nonzero gauge-invariant ideal of $C^*(E, L, B)$. Then there exists a C^* −algebra $C^*(E, L, [B]_I)$ generated by a universal representation $\{ta, p[A]\}$ of $(E, L, [B]_I)$. Furthermore $p[A] \neq 0$ for $[A] \neq [\emptyset]$ and $t_a \neq 0$ for $\in A_I$ ..

Proof. The existence of the C^* −algebra $C^*(E, \mathcal{L}, [B]_I)$ with the desired universal property can be shown by the same argument in the first part of the proof of [166, Theorem 4.5], and here we show the second assertion of our theorem. If $C^*(E, \mathcal{L}, B) = C^*(s_a, p_A)$, it is easy to see that $\{s_a + I, p_A + I : a \in A_I, [A] \in [B]_I\}$ is a representation of $C^*(E, \mathcal{L}, [B]_I)$, hence there is a $*$ −homomorphism $\psi: C^*(E, \mathcal{L}, [B]_I) \to C^*(E, \mathcal{L}, B)/I$ such that

$$\psi(t_a) = s_a + I, \qquad \psi(p_{[A]}) = p_A + I.$$

If $\psi(p_{[A]}) = p_A + I = I$, then $p_A \in I$ and it follows that $A \in A_I$, that is, $[A] = [\emptyset]$. Thus if $[A] \neq [\emptyset] then\ \psi(p_{[A]}) \neq I$, and so $p[A] \neq 0$. If $\psi(t_a) = s_a + I = I$, then $s_a^* s_a + I = p_r(a) + I = I$, and so $[r(a)] = [\emptyset]$, that is, $a \notin A_I$. Thus if $a \in A_I$, then $\psi(t_a) \neq I$, hence $t_a \neq 0$.

The following theorem together with Theorem (4.1.18) shows that the C^* −algebra $C^*(E, \mathcal{L}, [B]_R)$ of a weakly left-resolving quotient labelled space $(E, \mathcal{L}, [B]_R)$ is always isomorphic to a C^* −algebra $C^*(E, \mathcal{L}, [B]_I)$ for some gauge invariant ideal I of $C^*(E, \mathcal{L}, B)$ (Corollary 4.1.19).

Theorem (4.1.14)[449]: Let $(E, \mathcal{L}, [B]_R)$ be a weakly left-resolving quotient labelled space of $(E, \mathcal{L}, B)$. Then there exists a C^* −algebra $C^*(E, \mathcal{L}, [B]_R)$ generated by a universal. Representation Moreover the ideal $\{t_b, q_{[A]}\}$ *of* $(E, \mathcal{L}, [B]_R)$ such that $q_{[A]} \neq 0$ for $[A] \neq [\emptyset]$ and $t_b \neq 0$ for $b \in A_R$. Moreover the ideal I of $C^*(E, \mathcal{L}, B) = C^*(s_a, p_A)$ generated by the projections $p_A, [A] = [\emptyset]$, is gauge-invariant and there exists a surjective $*$ −homomorphism

$$\phi: C^*(E, \mathcal{L}, |B|_R) \to C^*(E, \mathcal{L}, B)/I$$

such that $\phi(t_b) = s_b + I$ *and* $\phi(q_{[A]}) = p_A + I$.

Proof. One can show the existence of $C^*(E, \mathcal{L}, [\mathcal{B}]_R)$ with the, universal, property as usual. $[A]) \in [\mathcal{B}]_R$. Let $C^*(E, \mathcal{L}, \mathcal{B}) = C^*(s_a, p_A), a \in \mathcal{A},\ A \in B, and\ let\ C^*(t_b q_{[A]}), b \in A_R$, $[A] \in [\mathcal{B}]_R$. It is almost obvious that the ideal I generated by the projections p_A,$[A] = [\emptyset]$, is gauge-invariant.

Now we show that $\mathcal{A}_R = \mathcal{A}_1$ (recall $\mathcal{A}_R = \{a \in \mathcal{A}: [r(a)] \neq [\emptyset]\}$ and $\mathcal{A}_I \coloneqq \{a \in \mathcal{A}: [r(a)]_I \neq [\emptyset]_I\} = \{a \in \mathcal{A}: P_{r(a)} \notin I\}$)

$A_I \subset A_R$ follows from the fact that $P_{r(a)} \in I$ whenever $[r(a)][\ \emptyset]$] by definition of I To prove the reverse inclusion, we first show that

$$p_A \notin I \qquad \text{when } [A] \neq [\emptyset] \qquad (4)$$

then, since $a \in A_R$ if and only if $[r(a)] \neq [\emptyset]$, by (4) $a \in A_R$ implies $P_{r(a)} \notin I$, thus $a \in A_I$. To prove (4), we suppose $[A] \neq [\emptyset]$ and $p_A \in I$ It is not hard to see from (2) that the span of elements of the form $s_\alpha pBs_\beta^*$, $[B\ ]\ =\ [\emptyset]$, is dense in I. So we can find $c_i\ \in$ ℂ, $\alpha_i,\beta_i\ \in \mathcal{L}^*(E)$, and $B_i \in \mathcal{B}$ with $[B_i] = [\emptyset]$ and $B_i \subset r(\alpha_i) \cap r(\alpha_i)$ for $i = 1, \cdots, n$ such that

$$1 > \left\| p_A - \sum_{i=1}^{N} c_i\, s_{\alpha_i} pB_i s_{\beta_i}^* \right\|.$$

Using (1) we may assume that the lengths $|\alpha_i|$,$1 \le i \le n$ are all equal to, say l, and write p_A as a finite sum

$$p_A = \sum_{|r|=\iota} s_\gamma\, p_{r(a,\gamma)} s_\gamma^*.$$

Since $[A] \neq [\emptyset]$ there exists a γ_0 such that $|\gamma_0| = \iota$ and $[r(A,\gamma_0)] \neq [\emptyset]$ Then $r(A,\gamma_0) \bigcup_{i=1}^{n} B_i \neq \emptyset$ and we have the following contradiction,

$$\begin{aligned}
1 &> \left\| p_A - \sum_{i=1}^{n}\ c_i s_{\alpha_i} pB_i s_{\beta_i}^* \right\| \\
&= \left\| \sum_{|r|=\iota} s_\gamma\, p_{r(a,\gamma)} s_\gamma^* - \sum_{i=1}^{n}\ c_i s_{\alpha_i} pB_i s_{\beta_i}^* \right\| \\
&\geq \left\| s_{\gamma_0}^* \left(\sum_{|r|=\iota} s_\gamma\, p_{r(a,\gamma)} s_\gamma^* \right) s_{\gamma_0}^* - s_{\gamma_0}^* \left(\sum_{i=1}^{n}\ c_i s_{\alpha_i} pB_i s_{\beta_i}^* \right) s_{\gamma_0}^* \right\| \\
&= \left\| p_{r(A,\gamma_0)} (\textstyle\bigcup_{i\in\Lambda} B_i) \big(p_{r(A,\gamma_0)} - \sum_{i\in\Lambda} c_i pB_i s_{\beta_i}^*\, s_{\gamma_0} \big) \right\| \\
&\geq \left\| p_{r(A,\gamma_0)} \setminus (\bigcup_{i\in\Lambda} B_i) \left(p_{r(A,\gamma_0)} - \sum_{i\in\Lambda} c_i pB_i s_{\beta_i}^*\, s_{\gamma_0} \right) \right\| \\
&= \left\| p_{r(A,\gamma_0)} \setminus (\textstyle\bigcup_{i\in\Lambda} B_i) \right\| \\
&= 1.
\end{aligned}$$

Defined Set $T_a := s_a + I$ and $Q_{[A]} :=\ p_A +\ I$ for $a\ \in A_R(= A_I)$ and $[A] \in [B]_R$. Here $Q[A]$ is well defined since $p_A - p_B \in I$ whenever $[A] = [B]$. In fact, since $[A] = [B]$ implies $[A\backslash B] = [A]\backslash[B] =\ [\emptyset] = [B\backslash A]$, we have $p_{A\backslash B}, p_{B\backslash A} \in I$, hence $p_A - p_B = (p_{A\backslash B} + p_{A\cap B}) - (p_{B\backslash A} +\ p_{A\cap B})\ =\ p_{A\backslash B}\ - p_{B\backslash A} \in I$. Note that $Q[\emptyset]\ =\ p_\emptyset + I = I$ and $Q_{[A]}Q_{[B]}\ =\ (p_A\ +\ I)(p_B\ +\ I)\ =\ p_A p_B\ +\ I\ =\ p_{A\cap B}\ +\ I\ =\ Q_{[A\cap B]}\ =\ Q_{[A]\cap[B]}$. Similarly, $Q_{[A]\cap[B]}\ =\ Q_{[A]}\ +\ Q_{[B]}\ -\ Q_{[A]\cap[B]}$ Also (iii), (iv) of Definition (4.1.8) can be easily shown to hold. Thus $\{\, T_a, Q_{[A]}\, \}$ is a representation of $(E, \mathcal{L}, [B\ ]_R)$, and by the universal property *there exists a* $*-$*homomorphism*

$$\phi: C^*(E, \mathcal{L}, [B\ ]_R) \longrightarrow C^*(E, \mathcal{L}, [B\ ]\ )/I$$

such that $\phi(t_a)\ =\ T_a = s_a + I$ and $\varphi(q_{[A]})\ =\ Q_{[A]}\ =\ p_A + I$ for $a\ \in\ A_R$ and $[A]\ \in [B]_R$. Since$C^*(E, \mathcal{L}, B)/I$ is generated by

$$\{s_a + I, p_A + I\ :\ a\ \in\ A_I\ , [A]\ \neq [\emptyset]\}$$

and $A_I = A_R$, it follows that ρ is surjective.

If $[A]\ \neq [\emptyset]$, by (4) $p_A/\!\in I$, hence $\phi(q_{[A]})\ =\ p_A +\ I\ \neq I$. Thus $q_{[A]}\ \neq 0$. If $b\ \in A_R$,namely $[r(b)]\ \neq [\emptyset]$, then $\phi(t_b^* t_b)\ =\ s_b^* s_b + I\, p_{r(b)} + I\ \neq\ I$ again by (4). Hence $t_b \neq\ 0\ in\ C^*(E, \mathcal{L}, [B]_R)$.

Definition (4.1.15)[449]: We call the C^*-algebra $C^*(E, L, [B]_R)$ of Theorem (4.1.14) the *quotient labelled graph* C^* *−algebra.*

By the universal property, it follows that every quotient labelled graph C^* −algebra $C^*(E, L, [B]_R) = C^*(s_a, p_{[A]})$ admits the gauge action γ of $\mathbb{T}$ such that

$$\gamma_z(s_a) = zs_a \qquad and \qquad \gamma_z(p_{[A]}) = p_{[A]}$$

for $a \in A_R$ and $[A] \in [B]_R$.

The gauge-invariant uniqueness theorem for quotient labelled graph C^* −algebras can be proved by the same arguments used in the proof of [166, Theorem 5.3] for the C^* −algebras of labelled spaces. But we give a sketch of the proof with some minor corrections to the proof of [166, Lemma 5.2] and [166, Theorem 5.3]

Lemma (4.1.16)[449]: Let $(\mathrm{E}, \mathcal{L}, [\mathrm{B}]_\mathrm{R})$ be a weakly left-resolving quotient labelled space of a labelled space $(\mathrm{E}, \mathcal{L}, \mathrm{B}), \{\mathrm{s_a}, \mathrm{p}_{[\mathrm{A}]}\}$ a representation of $(\mathrm{E}, \mathrm{L}, [\mathrm{B}]_\mathrm{R})$, and $\mathrm{Y} = \{s_{\alpha_i} p_{[A_i]} s^*_{\beta_i} : i = 1, \dots, N\}$ be a set of partial isometries in $C^*(E, \mathcal{L}, [B]_R)$ which is closed under multiplication and taking adjoints. Then any minimal projection of $C^*(Y)$ that is either

(i) $q = s_{\alpha_i} p_{[A_i]} s^*_{\alpha_i}$ for some $1 \geq i \geq N$;

(ii) $q = s_{\alpha_i} p_{[A_i]} s^*_{\alpha_i} - q'$,where $q' = \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{[A_{k(l)}]} s^*_{\alpha_{k(l)}}$ and $1 \leq i \leq N$; moreover there is nonzero $r = s_{\alpha_i}\beta p_{[r(A_i,\beta)]} s^*_{\alpha_i\beta} \in C^*(E, \mathcal{L}, [B]_R)$ such that $q'r = 0$ and $q \geq r$.

.**Proof.** A minimal projection of the finite dimensional C^* −algebra $C^*(Y)$ is unitarily equivalent to a projection of the form

$$\sum_{j=1}^{n} s_{\alpha_{i(j)}} p_{[A_{i(j)}]} s^*_{\alpha_{i(j)}} - \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{[A_{k(l)}]} s^*_{\alpha_{k(l)}}$$

where the projections in each sum are mutually orthogonal and for each l there is a unique j such that$s_{\alpha_{i(j)}} p_{[A_{i(j)}]} s^*_{\alpha_{i(j)}} \geq s_{\alpha_{k(l)}} p_{[A_{k(l)}]} s^*_{\alpha_{k(l)}}$. Then the same argument of the proof of [166, Lemma 5.2] proves the assertion.

Theorem (4.1.17)[449]: *Let* $(E, \mathcal{L}, [B]_R)$ *be a weakly left-resolving quotient labelled space* and $C^*(E, \mathcal{L}, [B]_R) = C^*\left(S_a, p_{[A]}\right)$. *Let* $\{S_a, p_{[A]}\}$ be a representation of $(E, \mathcal{L}, [B]_R)$ such that each $p_{[A]} \neq 0$ whenever $[A] \neq [\emptyset]$ and $S_a \neq 0$ whenever $[r(a)] \neq [\emptyset]$. If $\pi_{S,P}: C^*(E, \mathcal{L}, [B]_R) \to C^*(S_a, P_{[A]})$ is the homomorphism satisfying

$$\pi_{S,P}(sa) = s_a, \pi_{S,P}(p_{[A]}) = P_{[A]}$$

and if there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_a, P_{[A]})$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$, then $\pi_{S,P}$ is faithful

.**Proof.** It is standard (for example, see the proof of [166, Theorem 5.3]) to show that the fixed point algebra $C^*(E, \mathcal{L}, [B]_R)^\gamma$ is equal to

$$\overline{\operatorname{span}}\{s_\alpha p_{[A]} s^*_\beta : \alpha, \beta \in \mathcal{L}^*(E) \cap (A_R)^*, |\alpha| = |\beta|, and\ [A] \in [B]_R\}.$$

Note that $C^*(E, \mathcal{L}, [B]_R)^\gamma$is an AF algebra. In fact, if Y is a finite subset of $C^*(E, \mathcal{L}, [B]_R)^\gamma$,each element $y \in Y$ can be approximated by linear combinations of $s_\alpha p_{[A]} s^*_\beta$ with $|\alpha| = |\beta|$, hence we may assume that $Y = \{s_{\alpha_i} p_{[A_i]} s^*_{\beta_i} : |\alpha_i| = |\beta_i|, i = 1, \dots, N\}$. Using Lemma (4.1.13), we may also assume that Y is closed under multiplication and taking adjoints, so that $C^*(Y) = span(Y)$ is finite dimensional. Thus $C^*(E, \mathcal{L}, [B]_R)^\gamma$ is an AF algebra. Now we show that $\pi := \pi_{S,P}$ is faithful on $C^*(E, \mathcal{L}, [B]_R)^\gamma$.. Let $\{Y_n : n \geq 1\}$ be an increasing family of finite subsets of $C^*(E, \mathcal{L}, [B]_R)^\gamma$ which are closed under multiplication and taking adjoints such that $C^*(E, \mathcal{L}, [B]_R)^\gamma = \bigcup C^*(Y_n)$. Suppose π is not faithful on $C^*(Y_n)$ for some $Y_n = \{s_{\alpha_i} p_{[A_i]} s^*_{\beta_i} : i = 1, \dots, N(n)\}$. Since $C^*(Y_n)$is finite dimensional, the kernel of $\pi|_{C^*(Y_n)}$ has a minimal projection. By Lemma (4.1.17), each

minimal projection in the kernel of $\pi|_{C^*(Y_n)}$ is unitarily equivalent to a projection which is either
$q = s_{\alpha_i}p_{[A_i]}s^*_{\beta_i}(1 \le i \le N(n))$ or $q = s_{\alpha_i}p_{[A_i]}s^*_{\beta_i} - q', q' = \sum_{k=1}^{m} s_{\alpha_{i(k)}}p_{[A_{i(k)}]}s^*_{\beta_{i(k)}}$ $(1 \le i \le N(n))$. As in the proof of Theorem (4.1.3)[166, Theorem 5.3], one obtains $\pi(q) = 0$ in either case. Then $\pi(u_q u^*) = 0$ for any unitary $u \in C^*(Y_n)$, namely π maps every minimal projection to a nonzero element and hence is faithful on $C^*(E, \mathcal{L}, [B]_R)^\gamma$.

Therefore we conclude that π is faithful by [168, Lemma 2.2] since the following holds: For $a \in C^*(E, \mathcal{L}, [B]_R)^\gamma$,

$$\left\|\uparrow \pi\left(\int_{\mathbb{T}} \gamma_z(a)dz\right)\right\| \le \int_{\mathbb{T}} \|\pi(\gamma_z(a))\|dz = \int_{\mathbb{T}} \|\beta_z(\pi(a))\|dz = \|\pi(a)\|.$$

Corollary (4.1.18)[449]: Let $(E, \mathcal{L}, [B]_R)$ be a weakly left-resolving quotient labelled space of $(E, \mathcal{L}, B)$ and let $C^*(E, \mathcal{L}, [B]_R) = C^*(t_b, q_{[A]})$. If $C^*(E, \mathcal{L}, B) = C^*(s_a, p_A)$ and I is the ideal generated by the projections $p_A, [A] = [\emptyset]$, there is a $*$ −isomorphism

$$\phi: C^*(E, \mathcal{L}, [B]_R)^\gamma \to C^*(E, \mathcal{L}, B)/I$$

such that $\phi(t_b) = s_b + I$, $\phi(q_{[A]}) = P_A + I$ for $b \in A_R (= A_I)$, $[A] \in [B]_R$.

Proof. By Theorem (4.1.15) ,we have a surjective $*$ −homomorphism φ with the desired properties except injectivity. But it is injective by the gauge-invariant uniqueness theorem sinc $\phi(q_{[A]}) = P_A + I$ is nonzero for $[A] \in [B]_R, [A] \neq [\emptyset]$ *by* (4), and $\beta_z \circ \varphi = \varphi \circ \gamma_z$ *for* $z \in \mathbb{T}$, all the gauge action on $C^*(E, L, B)/I$ induced by the gauge action on $C^*(E, \mathcal{L}, B)$ and γ is the gauge action on $C^*(E, \mathcal{L}, [B]_R)^\gamma$.

Recall that for a labelled space $(E, \mathcal{L}, B)$ and a hereditary saturated subset H of B, I_H denotes the ideal of $C^*(E, \mathcal{L}, B)$ generated by the projections $p_A, A \in H$ (see Lemma (4.1.12)).

Lemma (4.1.19)[449]: Let $(E, \mathcal{L}, B)$ be a labelled space. Then the map $H \longmapsto I_H$ is an inclusion preserving injection from the set of nonempty hereditary saturated subsets of B into the set of nonzero gauge-invariant ideals of $C^*(E, \mathcal{L}, B)$.

Proof. Clearly the map is inclusion preserving. For injectivity, we show that the composition of $H \longmapsto I_H$ and $I \longmapsto H_I$ is the identity on the set of hereditary saturated subsets of B, that is, we show that $H_{I_H} = H$. From the easy fact that $I_{H_J} \subset J$ holds for any ideal J, we see with $J = I_H$ that $I_{(H_{I_H})} \subset I_H$, which then shows $H_{I_H} \subset H$. Since $H \subset H_{I_H}$ is rather obvious, we have $H_{I_H} = H$.

Theorem (4.1.20)[449]: Let $(E, \mathcal{L}, B)$ be a labelled space. Then every nonzero gauge-invariant ideal I of $C^*(E, \mathcal{L}, B)$ is of the form $I = I_H$ for the hereditary saturated subset $H = \{A \in B: p_A \in I\}$ of B, and there exists an isomorphism of $C^*(E, \mathcal{L}, [B]_I)$ onto the quotient algebra)/I. Moreover the map $H \longmapsto I_H$ gives an inclusion preserving bijection between the nonempty hereditary saturated subsets of B and the nonzero gauge-invariant ideals of $C^*(E, \mathcal{L}, B)$.

Proof. Let I be a nonzero gauge-invariant ideal of $C^*(E, \mathcal{L}, B) = C^*(s_a, p_v)$ and $\sim I$ the equivalence relation on B defined by
$A \sim_I B \iff A \cup W = B \cup W$ for some $W \in H$,
where $H := H_I = \{A \in B : p_A \in I\}$ is the hereditary saturated subset of B Lemma (4.1.10). Then $(E, \mathcal{L}, [B]_I)$ is a weakly left-resolving quotient labelled space by Proposition (4.1.11) and we see from the proof of Theorem (4.1.14) that there exists a surjective $*$ −homomorphism

$$\psi : C^*(E, \mathcal{L}, [B]_I) \to C^*(E, \mathcal{L}, B)/I$$

such that. $\psi(t_b) = s_b + I, \psi(q_{[A]}) = p_A + I \; for \; b \in A_I, [A] \in [B]_I$. Moreover $p_A + I \neq I$ and $s_b + I \neq I$. By applying the gauge-invariant uniqueness theorem (Theorem 4.1.18), we see that is an isomorphism. On the other hand, the ideal $I_H (\subset I) \; of \; C^*(E, \mathcal{L}, B)$ generated by the projections $p_A \in I$ is gauge-invariant and $A_I = A_{I_H}$ since

$$[A] = [\emptyset] \quad \Leftrightarrow \quad p_A \in I_H \quad \Leftrightarrow \quad p_A \in I.$$

By Corollary (4.1.19) with $\sim_I$ in place of $\sim_R$, we have a $* -$ isomorphism

$$\phi : C^*(E, \mathcal{L}, [B]_I) \to C^*(E, \mathcal{L}, B)/I_H$$

such that, $\phi(q_{[A]}) = p_A + I_H, \phi(t_b) = s_b + I_H, \; for \; b \in A_I \; and \; [A] \in [B]_I$.where $C^*(E, \mathcal{L}, [B]_I) = C^*(t_b, q_{[A]})$. Then the composition of ϕ^{-1} and ψ,

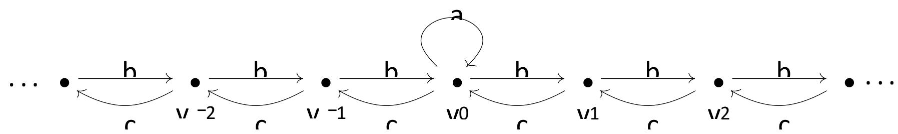
$$\psi \circ \phi^{-1} : C^*(E, \mathcal{L}, B)/I_H \to C^*(E, \mathcal{L}, B)/I$$

is a $* -$isomorphism such that

$$\psi \circ \phi^{-1}(p_A + I_H) = p_A + I, \psi \circ \phi^{-1}(s_a + I_H) = s_a + I,$$

which shows $I = I_H$. Finally Lemma (4.1.20) completes the proof.

Example (4.1.21)[449]: (See [167, Example 7.2].) If (E, L) is the following labelled graph



then $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E, \mathcal{L}, \bar{\mathcal{E}})$ by Corollary (4.1.8) while

$$\mathcal{E}^{0,-} = \{E^0\} \cup \{A \subset E^0 : A \text{is finite,}\},$$
$$\overline{\mathcal{E}} = \mathcal{E}^{0,-} \cup \{A \subset E^0 : E^0 \setminus A \text{ is finite,}\}.$$

Theorem (6.4) of [167] states that if $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is cofinal and disagreeable, then $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$ is simple, but there was a mistake in the (first paragraph of the) proof and it turns out that

if $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple then $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal ([171, Theorem 3.8], (see also Remark 3.15 of [171]). Thus $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not simple since $(E, \mathcal{L}, \overline{\mathcal{E}})$ is not strongly cofinal (see where we will consider the simplicity of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$). Let I be the gauge-invariant ideal of corresponding to the hereditary saturated set .

$H = \{A \subset E^0 : A$ is finite$\}$. Then $[\overline{\mathcal{E}}]_I = \{[E^0], [\emptyset]\}$ and $A_I = \{b, c\}$. Let $C^*(E, L, [\overline{\mathcal{E}}]_I) = C^*(p_{[E^0]}, s_b, s_c)$. Since $s_b^* s_b = p_{[r(b)]} = p_{[E^0]} = s_c^* s_c, s_b^* s_c = 0, p_{[E^0]} s_b = s_b p_{r([E^0], b)} = s_b p_{[E^0]}$, and similarly $p_{[E^0]} s_c = s_c p_{[E^0]}, C^*(E, \mathcal{L}, [\overline{\mathcal{E}}]_I)$ is the universal $C^* -$algebra generated by two isometries with orthogonal ranges with the unit $p_{[E^0]}$. Therefore $C^*(E, \mathcal{L}, [\overline{\mathcal{E}}]_I)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$ and by Theorem (4.1.21), we have $C^*(E, \mathcal{L}, \overline{\mathcal{E}})/I \cong \mathcal{O}_2$. (For the ideal I, see [171, Remark 3.7].)

We consider labelled spaces $(E, \mathcal{L}, \overline{\mathcal{E}})$ such that for every $v \in E^0, [v]_l$ is finite for some $l \geq 1$ and using their merged labelled graphs we provide an equivalent condition for the $C^* -$algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ to be simple.

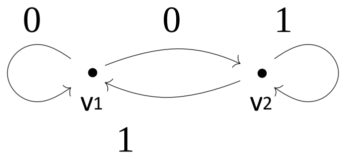
Definition (4.1.22)[449]: Let $(E, \mathcal{L}_E, \overline{\mathcal{E}})$ be a labelled space. Then $v \sim w$ if and only if $[v]_l = [w]_l$ for all $l \geq 1$ defines an equivalence relation $\sim$ on E^0. Let $[v]_\infty$ denote the equivalence class $\{w \in E^0 : w \sim v\}$ of v and let

$$F^0 := E^0/\sim = \{[v]_\infty : v \in E^0\}.$$

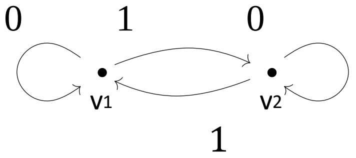
If $\lambda \in E^1$ is an edge such that $s(\lambda) \in [v]_\infty, r(\lambda) \in [w]_\infty$, then draw an edge e_λ from $[v]_\infty$ to $[w]_\infty$ and label e_λ with $\mathcal{L}_F(e_\lambda) := \mathcal{L}_E(\lambda). If\ \lambda_1, \lambda_2 \in E^1$ are edges with $s(\lambda_i) \in [v]_\infty, r(\lambda_i) \in [w]_\infty, i = 1, 2$, and $\mathcal{L}_F(\lambda_1) = \mathcal{L}_e(\lambda_2)$, we identify e_{λ_1} with e_{λ_2}. Then $F = (F^0, F^1 := \{e_\lambda : \lambda \in E^1\})$ is a graph with the range, source maps given by $r(e_\lambda) := [r(\lambda)]_\infty$, $s(e_\lambda) := [s(\lambda)]_\infty$, respectively. We call $(F, \mathcal{L}_F)$ the merged labelled graph of $(E, \mathcal{L}_E)$ (cf. [179]).

Example (4.1.23)[449]: Consider the following labelled graphs:

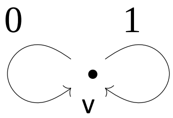
$(E_1, \mathcal{L}_1)$



$(E_2, \mathcal{L}_2)$



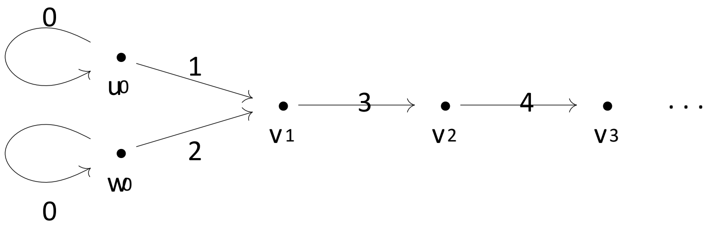
$(F, \mathcal{L}_F)$



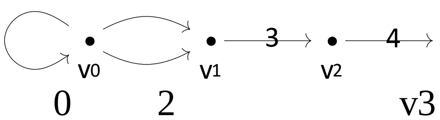
Note that $\overline{\mathcal{E}_i} = \{\emptyset, \{v_1, v_2\}\}$ and $\{v_j\} / \in \overline{\mathcal{E}_j}$ for $i, j = 1, 2$ while $\{v\} \in \overline{\mathcal{F}}$ for $v \in F^0$. $(F, \mathcal{L}_F)$ is the merged labelled graph of $(E_i, \mathcal{L}_i)$ with $v = [v_1]_\infty = [v_2]_\infty$. The $C^* -$algebras $C^*(E_i, \mathcal{L}_i, \overline{\mathcal{E}_i}), i = 1, 2$, and $C^*(F, \mathcal{L}_F, \overline{\mathcal{F}})$ are all isomorphic to the Cuntz algebra $\mathcal{O}_2$.

Example (4.1.24)[449]: The labelled graph $(F, \mathcal{L}_F)$ shown below is the merged labelled graph of $(E, \mathcal{L}_E)$ with $v_0 = [u_0]_\infty = [w_0]_\infty$, and obviously $C^*(E, \mathcal{L}_E, \overline{\mathcal{E}})$ is isomorphic to $C^*\big(F, \mathcal{L}_F, \overline{\mathcal{F}}\big)$.

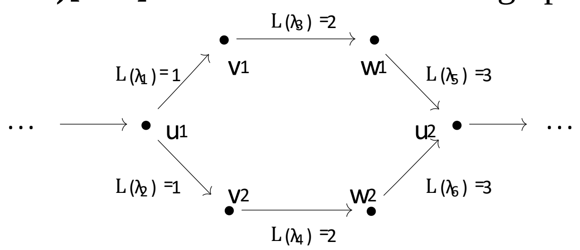
$(E, \mathcal{L}_E)$



$(E, \mathcal{L}_F)$



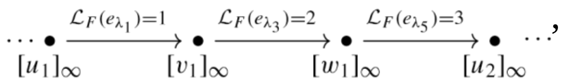
Example (4.1.25)[449]: Consider the labelled graph $(E, \mathcal{L} := \mathcal{L}_E)$.



$(E, \mathcal{L})$

The merged labelled graph $(F, \mathcal{L}_F)$ of $(E, \mathcal{L})$ is as follows.

$(F, \mathcal{L}_F)$



where $e_{\lambda_1} = e_{\lambda_2}, e_{\lambda_3} = e_{\lambda_4}, e_{\lambda_5} = e_{\lambda_6}, [v_i]_\infty = \{v_1, v_2\}$, and $[w_i]_\infty = \{w_1, w_2\}, i = 1,2$.

Definition (4.1.26)[449]: be Let $(F,\mathcal{L}_F)$ the merged labelled graph of $(E,\mathcal{L}_E)$. For $A \subset E^0, B \subset F^0$, we define $[A]_\infty \subset F^0, \hat{B} \subset E^0$ by

$$[A]_\infty := \{[v]_\infty : v \in A\}, \quad \hat{B} := \{v : [v]_\infty \in B\}.$$

Note that $[A_1 \cap A_2]_\infty \subset [A_1]_\infty \cap [A_2]_\infty$ and $[A_1 \cup A_2]_\infty = [A_1]_\infty \cup [A_2]_\infty$ hold whenever $A_1, A_2 \subset E^0$. For $A \subset E^0$ and $B \subset F^0$, it is easy to see that

$$A \subset \widehat{[A]_\infty} \quad \text{and} \quad B = [\hat{B}]_\infty. \tag{5}$$

Lemma (4.1.27)[449]: *Let $(E,\mathcal{L}_E,\overline{\mathcal{E}})$ be a labelled space such that $[v]_\infty \in \overline{\mathcal{E}}$ for all $v \in E^0$ and let $(F,\mathcal{L}_F)$ be the merged labelled graph of $(E,\mathcal{L}_E)$. Then*

$$\mathcal{L}_E([u]_\infty E^k [v]_\infty) \tag{6}$$

for all $k \geq 1$ and $u, v \in E^0$. Moreover we have the following:

(i) $r(\alpha) = \widehat{r_F(\alpha)}$ and $[r(\alpha)]_\infty = r_F(\alpha)$ for $\alpha \in \mathcal{L}^*(E)$.
(ii) $s(\alpha) \subset \widehat{s_F(\alpha)}$ and $[s(\alpha)]_\infty = s_F(\alpha)$ for $\alpha \in \mathcal{L}^*(E)$.
(iii) $[[v]_l]_\infty = [[v]_\infty]_l$ for $v \in E^0, l \geq 1$.
(iv) $[A \cap B]_\infty = [A]_\infty \cap [B]_\infty$ for $A, B \in \overline{\mathcal{E}}$.
(v) $A = \widehat{[A]_\infty}$ and $[A]_\infty \in \overline{\mathcal{F}}$ for $A \in \overline{\mathcal{E}}$.

Proof. For simplicity of notation, we write $\mathcal{L}$ for $\mathcal{L}_E$ omitting the subscript E. Note that each $[u]_\infty \in F^0$ is also a subset of E^0 so that an expression like $[u]_\infty E^k$ has obvious meaning. Since $\mathcal{L}([u]_\infty E^k v) \subset \mathcal{L}_F([u]_\infty F^k [u]_\infty)$ is clear, we only need to show the reverse inclusion for (6) when $k \geq 1$.

Let $k = 1$. If $e_\lambda \in [u]_\infty F^1 [v]_\infty$ and $L(e_\lambda) = \alpha, \lambda \in E^1$ is an edge such that $s(\lambda) \in [u]_\infty, r(\lambda) \in [v]_\infty$ and $\mathcal{L}(\lambda) = \alpha$. Since $[v]_\infty = [r(\lambda)]_\infty$, there exists an edge $\lambda' \in E^1$ with $r(\lambda') = v$ and $\mathcal{L}(\lambda') = \alpha$. We claim that $[s(\lambda')]_\infty = [s(\lambda)]_\infty$. Since $[s(\lambda)]_\infty \in \mathcal{E}$ (ii), $r(\lambda) \in r([s(\lambda)]_\infty, \alpha) \in \overline{\mathcal{E}}$ hence $[r(\lambda)]_\infty \subset r([s(\lambda)]_\infty, \alpha)$. Similarly, $v = r(\lambda') \in r([s(\lambda)]_\infty, \alpha) \in \overline{\mathcal{E}}$ implies that $[v]_\infty \subset r([s(\lambda')]_\infty, \alpha)$. Suppose $[s(\lambda)]\infty \neq [s(\lambda')]_\infty$. Then $[s(\lambda)]_\infty \cap [s(\lambda')]_\infty = \emptyset$ (ii) since $(E,\mathcal{L},\overline{\mathcal{E}})$ is weakly left-resolving, $[v]_\infty = [r(\lambda)]_\infty \subset r([s(\lambda)]_\infty, \alpha) \cap r([s(\lambda')]_\infty, \alpha) = r([s(\lambda)]_\infty \cap [s(\lambda')]_\infty, \alpha) = \emptyset$, a contradiction. Thus $[s(\lambda)]_\infty = [s(\lambda')]_\infty$, namely $s(\lambda') \in [u]_\infty$, and we have

$$\mathcal{L}([u]_\infty E^1 v) = \mathcal{L}_F([u]_\infty E^1 [v]_\infty) \tag{7}$$

Now let $k = 2$ and $e_{\lambda_1} e_{\lambda_2} \in F^2$ be a path with $[u]_\infty := s_F(e_{\lambda_1})$, $[w]_\infty := r_F(e_{\lambda_1}) = s_F(e_{\lambda_2})$, $[v]_\infty := r_F(e_{\lambda_2})$, and $\mathcal{L}_F(e_{\lambda_1} e_{\lambda_2}) = \alpha_1 \alpha_2$. Then by (7) $\lambda_1', \lambda_2' \in E^1$, there exist

$$s(\lambda_2') \in [w]_\infty, \quad r(\lambda_2') = v, \quad \mathcal{L}(\lambda_2') = \alpha_2,$$
$$s(\lambda_1') \in [u]_\infty, \quad r(\lambda_1') = s(\lambda_2'), \quad \mathcal{L}(\lambda_1') = \alpha_1.$$

Then $\lambda = \lambda_1' \lambda_2' \in [u]_\infty E^2 v$ is a path with $e_{\lambda_1'} e_{\lambda_2'} = e_{\lambda_1} e_{\lambda_1}$.Thus $\mathcal{L}([u]_\infty E^2 v) = \mathcal{L}_F([u]_\infty E^2 [v]_\infty)$. For $k \geq 3$, one can repeat the process inductively. Moreover (6) implies that

$$\mathcal{L}(E^k v) = \mathcal{L}_F(F^k [v]_\infty), \quad k \geq 1. \tag{8}$$

(i) To show $r(\alpha) = \widehat{r_F(\alpha)}$ for $\alpha \in \mathcal{L}^*(E)$, let $v \in r(\alpha)$. Then there exists $\lambda \in E^{\geq 1}$ such that r $(\lambda) = v$ and $\mathcal{L}(\lambda) = \alpha$. The edge $e_\lambda \in E^{\geq 1}$ has the range vertex $r(e_\lambda) = [v]_\infty$ and the label $\mathcal{L}_F(e_\lambda) = \alpha$. Hence $[v]_\infty \in r_F(\alpha)$, namely $v \in \widehat{r_F(\alpha)}$. Conversely, if $v \in \widehat{r_F(\alpha)}$, that is, $[v]_\infty \in r_F(\alpha)$ by (8), there is a path $\lambda \in E^{\geq 1}$ with $\mathcal{L}(\lambda) = \alpha, r(\lambda) = v$. Hence $v \in r(\alpha)$. Also, by (5) wehave $[r(\alpha)]_\infty = [\widehat{r(\alpha)}]_\infty = r_F(\alpha)$.

(ii) Since $s(\alpha) \subset \widehat{r_F(\alpha)}$ is clear, we have $[s(\alpha)]_\infty \subset [\widehat{s_F(\alpha)}]_\infty = s_F(\alpha)$ by (5). Also (6) shows that $[s(\alpha)]_\infty \supset s_F(\alpha)$.

(ii) The equality $[[v]_l]_\infty = [[v]_\infty]_l$ follows from

$$\begin{aligned} w \in [v]_l &\Leftrightarrow [w]_l = [v]_l \\ &\Leftrightarrow \mathcal{L}(E^{\leq l}w) = \mathcal{L}(E^{\leq l}w) \\ &\Leftrightarrow \mathcal{L}_F(E^{\leq l}[w]_\infty) = \mathcal{L}_F(E^{\leq l}[v]_\infty)(by(8)) \\ &\Leftrightarrow [w]_\infty \in [[v]_\infty]_l. \end{aligned}$$

(iv) It suffices to show that $[A]_\infty \cap [B]_\infty \subset [A \cap B]_\infty$. Let $[v_1]_\infty = [v_2]_\infty \in [A]_\infty \cap [B]_\infty$ for some $v_1 \in A, v_2 \in B$. Since $A, B \in \overline{\mathcal{E}}$, there exists $l \geq 1$ such that $[v_1]_\infty \subset [v_1]_l \subset A$ and $[v_2]_\infty \subset [v_2]_l \subset B$ Hence $[v_1]_l = [v_2]_l$ and so $v_1 \in A \cap B$ and $[v_1]_\infty \in [A \cap B]_\infty$ (v) $A \subset \widehat{[A]_\infty}$ is clear. If $v \in \widehat{[A]_\infty}, [v]_\infty \in [A]_\infty$ and so $[v]_\infty = [w]_\infty$ for some $w \in A$.

Writing $A = \cup_j [w_j]_l \in \overline{\mathcal{E}}$, we have $w \in [w_j]_l$ for some j, then $v \in [w_j]_l \subset A$ because $\sim_w \sim_l w_j$. By (iii) we also have

$$[A]_\infty = \left[\bigcup_j [w_j]_l\right]_\infty = \bigcup_j [[w_j]_l]_\infty = \bigcup_j [[w_j]_\infty]_l \in \overline{\mathcal{F}}.$$

Proposition (4.1.28)[449]: *Let $(E, L, \overline{\mathcal{E}})$ be a labelled space such that $[v]_\infty \in \overline{\mathcal{E}}$ for all $v \in E^0$. Then the map $A \longmapsto [A]_\infty : \overline{\mathcal{E}} \to \overline{\mathcal{F}}$ is bijective and*

$$[r(A, \alpha)]_\infty = r_F([A]_\infty, \alpha)$$

holds for $A \in \overline{\mathcal{E}}, \alpha \in \mathcal{L}^*(E)$.

Proof. To show that the map is surjective, let $B = \cup_j [[v_j]_\infty]_l$. Then by Lemma (4.1.27) (iii),

$$\hat{B} = \left\{v : [v]_\infty \in \bigcup_j \left[[v_j]_\infty\right]_l\right\} = \left\{v : [v]_\infty \in \bigcup_j \left[[v_j]_l\right]_\infty\right\} = \bigcup_j [v_j]_l \in \overline{\mathcal{E}}$$

and $B = [\hat{B}]_\infty$ by (5). For injectivity, let $[A_1]_\infty = [A_2]_\infty, A_1, A_1 \in \overline{\mathcal{E}}$. Then by Lemma (4.1.27)(v), $A_1 = \widehat{[A_1]_\infty} = \widehat{[A_2]_\infty} = A_2$.

Now we show that $[r(A, \alpha)]_\infty = r_F([A]_\infty, \alpha)$ for $A \in \overline{\mathcal{E}}, \alpha \in \mathcal{L}^*(E)$. Clearly $[r(A, \alpha)]_\infty \subset r_F([A]_\infty, \alpha)$ holds. If $[v]_\infty \in r_F([A]_\infty, \alpha)$, there exists a path $e_{\lambda_1} \cdots e_{\lambda_n} \in F^n$ with rF $(e_{\lambda_1} \cdots e_{\lambda_n}) = [v]_\infty$. Let $[u]_\infty := s_F(e_{\lambda_1} \cdots e_{\lambda_n}), [u]_\infty \in [A]_\infty$, and $\alpha = \mathcal{L}_F(e_{\lambda_1} \cdots e_{\lambda_n})$. We may assume that $u \in A$ since $[u]_\infty = [u']_\infty$ for some $u' \in A$. By (6), we can find a path $\lambda \in E^n$ with $r(\lambda) = v, s(\lambda) \in [u]_\infty (\subset A)$, and $\mathcal{L}(\lambda) = \alpha$. Thus $v \in r([u]_\infty, \alpha) \subset r(A, \alpha)(\overline{\mathcal{E}})$. Then $[v]_\infty \subset r(A, \alpha)$ and we conclude that $[v]_\infty \subset r(A, \alpha)$ and we conclude that $[v]_\infty \in [r(A, \alpha)]_\infty$.

Clearly the merged labelled space $(F, \mathcal{L}, \overline{\mathcal{F}})$ has no sinks since we assume that $(E, \mathcal{L}, \overline{\mathcal{E}})$ has no sinks. Besides, $(F, \mathcal{L}, \overline{\mathcal{F}})$ has the following properties.

Proposition (4.1.29)[449]: Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a labelled space such that $[v]_\infty \in \overline{\mathcal{E}}$ for all $v \in E^0$. Then the merged labelled space $(F, \mathcal{L}, \overline{\mathcal{F}})$ is set-finite and receiver set-finite, respectively if and only if $(E, \mathcal{L}, \overline{\mathcal{E}})$ is set-finite and receiver set-finite, respectively. Moreover $(F, \mathcal{L}, \overline{\mathcal{F}})$ is weakly leftresolving whenever $(F, \mathcal{L}, \overline{\mathcal{F}})$ is weakly left−resolving.

Proof. By Lemma (4.1.27), we know that $A \longmapsto [A]_\infty : \overline{\mathcal{E}} \to \overline{\mathcal{F}}$ forms a bijection. From the following

$$\mathcal{L}(E^l A) = \bigcup_{v\in A} \mathcal{L}(E^l A) = \bigcup_{v\in A} \mathcal{L}_F(F^l[v]_\infty)$$
$$\bigcup_{[v]_\infty\in[A]_\infty} \mathcal{L}_F(F^l[v]_\infty)$$
$$\mathcal{L}_F(F^l[v]_\infty)$$
we have that $(E,\mathcal{L},\overline{\mathcal{E}})$ is receiver set-finite if and only if $(E,\mathcal{L},\overline{\mathcal{E}})$ is receiver set-finite. Since $[r(A,\alpha)]_\infty = r_F([A]_\infty,\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$ (Proposition 4.1.30)
$$\begin{aligned}\mathcal{L}_F([A]_\infty F^1) &= \{a \in A: r_F([A]_\infty,a) \neq \emptyset\}\\ &= \{a \in A: [r(A,a)]_\infty \neq \emptyset\}\\ &= \mathcal{L}(AE^1),\end{aligned}$$
which proves the equivalence of set-finiteness of $(E,\mathcal{L},\overline{\mathcal{E}})$ and $\left(F,\mathcal{L},\overline{\mathcal{F}}\right)$.

Since $(E,\mathcal{L},\overline{\mathcal{E}})$ is weakly left-resolving, by Lemma (4.1.27) (iv) and Proposition (4.1.30), we have
$$\begin{aligned} & r(A,\alpha)\cap r(B,\alpha) = r(A\cap B,\alpha)\\ \Leftrightarrow\ & [r(A,\alpha)\cap r(B,\alpha)]_\infty = [r(A\cap B,\alpha)]_\infty\\ \Leftrightarrow\ & r_F([A]_\infty,\alpha)\cap r_F([B]_\infty,\alpha) = r_F([A]_\infty\cap[B]_\infty,\alpha).\end{aligned}$$
Thus $(F,\mathcal{L},\overline{\mathcal{F}})$ is weakly left$-$resolving.

Theorem (4.1.30)[449]: Let $(E,\mathcal{L},\overline{\mathcal{E}})$ be a labelled space such that $[v]_\infty \in \overline{\mathcal{E}}$ for all $v \in E^0$, and let $(F,\mathcal{L})$ the merged labelled graph of $(E,\mathcal{L})$. Then $\{[v]_\infty\} \in \overline{\mathcal{F}}$ for every vertex $[v]_\infty \in F^0$ and
$$C^*\left(E,\mathcal{L},\overline{\mathcal{E}}\right) \cong C^*\left(E,\mathcal{L},\overline{\mathcal{F}}\right).$$
Proof. For $v \in E^0$, with $A := [v]_\infty \in \overline{\mathcal{E}}$, we have $[A]_\infty \in \overline{\mathcal{F}}$ by Proposition (4.1.28). But $[A]_\infty = \{[v]_\infty\}$.

Let $C^*(E,\mathcal{L},\overline{\mathcal{E}}) = C^*(p_A,s_a)$ and $C^*(F,\mathcal{L}_F,\overline{\mathcal{F}}) = C^*(q_{[A]_\infty},t_a)$. Then $\{P_A := q_{[A]_\infty}: A \in \overline{\mathcal{E}}\} \cup \{S_a := t_a: a \in A\}$ is a representation of $\left(E,\mathcal{L},\overline{\mathcal{E}}\right)$:

(i) If $A,B \in \overline{\mathcal{E}}$, then $P_AP_B = q_{[A]_\infty}q_{[B]_\infty} = q_{[A]_\infty\cap[B]_\infty} = q_{[A\cap B]_\infty} = P_{A\cap B}$ *and* $P_{A\cup B} = q_{[A\cup B]_\infty} = q_{[A]_\infty\cup[B]_\infty} = q_{[A]_\infty+q[B]_\infty} - q_{[A\cap B]_\infty} = P_A + P_B - P_{A\cap B}$, where $P_\emptyset = q_\emptyset = 0$.

(ii) If $A \in \overline{\mathcal{E}}$ and $a \in A$, then $P_AS_a = q_{[A]_\infty}t_a = t_aq_{r_F([A]_\infty,a)} = t_aq_{[r(A,a)]_\infty} = S_aP_{r(A,a)}$.

(iii) If $a,b \in A, S_a^*S_a = t_a^*t_a = q_{r_F(a)} = q_{[r(a)]_\infty} = P_{r(a)}$ and $S_a^*S_b = t_a^*t_b = 0$ unless $a = b$.

(iv) For $A \in \overline{\mathcal{E}}$,
$$\begin{aligned}P_A = q_{[A]_\infty} &= \sum_{a\in\mathcal{L}_F([A]_\infty F^1)} t_a\, q_{r_F([A]_\infty,a)}t_a^*\\ &= \sum_{a\in\mathcal{L}(AE^1)} t_a\, q_{[r(A,a)]_\infty}t_a^*\\ &= \sum_{a\in\mathcal{L}(AE^1)} S_a\, P_{r(A,a)}S_a^*.\end{aligned}$$

Thus there exists a surjective $*$-homomorphism $\Phi : C^*(E,\mathcal{L},\overline{\mathcal{E}}) \to C^*(F,\mathcal{L},\overline{\mathcal{F}})$ such that $\Phi(p_A) = q_{[A]_\infty}$ and $\Phi(s_a) = t_a$ for $A \in \overline{\mathcal{E}}, a\ A$. By Theorem (4.1.3), Φ is an isomorphism.

Recall [126,130] that a labelled space $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable if for each $[v]_l$, there exists an $N \geq 1$ such that for all $n \geq N$ there is a labelled path $\alpha \in \mathcal{L}([v]_l E^{\geq n}$ that is not agreeable, that is, not of the form $\alpha = \beta\alpha' = \alpha'\gamma$ for some $\alpha',\beta,\gamma \in \mathcal{L}(E^{\geq n})$ with $|\beta| = |\gamma| \leq l$. Also $(E,\mathcal{L},B)$ is strongly cofinal [171] if for all $x \in \mathcal{L}(E^\infty), w \in s(x)$, and $[v]_l \in \Omega_l(E)$, there are $N \geq 1$ and a finite number of labelled paths $\lambda_1,\ldots,\lambda_m$ such that

$$r([w]_1, x_1 \cdots x_N) \subset \bigcup_{i=1}^{m} r([w]_1, \lambda_i).$$

Theorem (4.1.31)[449]: Let $(\mathrm{E},\mathcal{L},\overline{\mathcal{E}})$ be a labelled space such that if $\mathrm{v} \in \mathrm{E}^0, [\mathrm{v}]_\mathrm{l}$ is finite for some $\mathrm{l} \geq 1$, and let $(\mathrm{F},\mathcal{L}_\mathrm{F})$ be the merged labelled graph of $(\mathrm{E},\mathcal{L})$. Then we have the following:

Proof. First note that $[v]_\infty (\in \overline{\mathcal{E}})$ is a finite set for each $v \in E^0$.

(i) Suppose $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal and let $x = x_1 x_2 \cdots \in \mathcal{L}_F(F^\infty), [u_0]_\infty \in s_F(x)$ and $[[v]_\infty]_l \in \Omega_l(F)$. Fix $[u_i]_\infty \in r_F(x_i)$ for each i. Then $x_1 \cdots x_i \in \mathcal{L}([u_0]_\infty F^i\ [u_i]_\infty)$ for $i \geq 1$. Since $\mathcal{L}([u_0]_\infty E^i u_i) = \mathcal{L}_F([u_0]_\infty F^i\ [u_0]_\infty)\ (by\ (6)), x_1 \cdots x_1([u_0]_\infty E^i u_i)$ for all $i \geq 1$. Then the finite set $[u_0]_\infty$ must have a vertex $u_0' \in [u_0]_\infty$such that $x_1 \cdots x_i \in \mathcal{L}(u_0 E^i u_i)$ for infinitely many $i's$, which means that $x \in \mathcal{L}(u_0'\ E^\infty)$. Since $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal, there exists an $N \geq 1$ and a finite number of labelled paths $\lambda_1,\ldots,\lambda_m \in \mathcal{L}(E^{\geq 1})$ such that $r([u_0']_1, x_1 \cdots x_N) \subset \bigcup_{j=1}^{m} r([v]_l,\lambda_j)$ Then $[r([u_0']_1, x_1 \cdots x_N)]_\infty \subset [\bigcup_{j=1}^{m} r([v]_l, \lambda j\ )]_\infty$,that is,

$$r_F([[u_0']_1]_\infty, x_1 \cdots x_N) \subset \bigcup_{j=1}^{m} r_F\big([[v]_l]_\infty, \lambda_j\big) = \bigcup_{j=1}^{m} r_F\big([[v]_\infty]_l, \lambda_j\big),$$

and we see that $(F,\mathcal{L}_F,\overline{\mathcal{F}})$ is strongly cofinal.

Conversely, assuming that $(F,\mathcal{L}_F,\overline{\mathcal{F}})$ is strongly cofinal, if $x \in \mathcal{L}(E^\infty)$ is an infinite labelled path with $u \in s(x)$ and $[v]_l \in \Omega_l(E)$, clearly $x \in \mathcal{L}_F([u]_\infty F^\infty)$ and so there exist an $N \geq 1$ and a finite number of labelled paths $\lambda_1,\cdots,\lambda_m \in \mathcal{L}_F(F^{\geq 1})$ such that

$$r_F([[u]_\infty]_1, x_1 \cdots x_N) \subset \bigcup_{j=1}^{m} r_F\big([[v]_\infty]_l, \lambda_j\big).$$

Hence, we have $[r([u]_1, x_1 \cdots x_N)]_\infty \subset [\bigcup_{j=1}^{m} r([v]_l, \lambda_j)]_\infty$ by Proposition (4.1.26). Then Lemma (4.1.27) (v) shows that $r([u]_1, x_1 \cdots x_N) \subset \bigcup_{j=1}^{m} r([v]_l, \lambda_j)$ and so $(E,\mathcal{L},\overline{\mathcal{E}})$ is strongly cofinal.

(ii) Note that for each $v \in E^0$, there is an l_v with $[v]_{l_v} = [v]_k$ for all $k \geq l_v$. Thus by [171, Proposition 3.9] we see that $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable if and only if $[v]_k$ is disagreeable for $k \geq l_v$ Now let $v \in E^0$ and $l := l_v$. Then $[v]_l$ is the union of finitely many equivalence classes $[v]_\infty$ of $v \in [v]_l$. From (6), we have

$$\begin{aligned}\mathcal{L}([v]_l E^n) &= \bigcup_{v' \in [v]_l} \mathcal{L}([v']_\infty E^n) \\ &= \bigcup_{v' \in [v]_l, w \in E^0} \mathcal{L}([v']_\infty E^n w) \\ &= \bigcup_{[v']_\infty \in [[v]_l]_\infty, [w]_\infty \in F^0} \mathcal{L}_F([v']_\infty E^n [w]_\infty) \\ &= \bigcup_{[v']_\infty \in [[v]_\infty]_l} \mathcal{L}_F([v']_\infty E^n) \\ &= \mathcal{L}_F([[v]_\infty]_l E^n),\end{aligned}$$

which shows the assertion.

It is known that if $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple, $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal [171, Theorem 3.8] and if, in addition, $\{v\} \in \overline{\mathcal{E}}$ for all $v \in E^0$, $(E, \mathcal{L}, \overline{\mathcal{E}})$ is disagreeable [171, Theorem 3.14]. Also if $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal and disagreeable, $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple by [171, Theorem 3.16]. Therefore by Theorem (4.1.31) and Theorem (4.1.31) we have the following corollary.

Corollary (4.1.32)[449]: Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a set-finite, receiver set-finite, and weakly left-resolving labelled space such that for each $v \in E^0$, $[v]_l$ is finite for some $l \geq 1$.. Then $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is simple if and only if $(E, \mathcal{L}, \overline{\mathcal{E}})$ is strongly cofinal and disagreeable.

Section (4.2): The Diagonal C^*-Subalgebras

Many C^*-algebras are defined in terms of partial isometries and relations. The motivation for these relations often comes from a kind of combinatorial object such as a graph [190] or a 0-1 matrix [186]. A set of partial isometries in a C^* −algebra is generally not closed under multiplication; however, in the examples cited above, one can find an inverse semigroup of partial isometries in the C^*-algebra. In [167], Exel found a topological space on which this inverse semigroup acts, the *tight spectrum*, and by constructing the C^*-algebra of the groupoid of germs of this action, we arrive at the original C^*-algebra.

Among the combinatorial objects mentioned above are *labelled spaces*, which were introduced by Bates and Pask in [182]. The C^*-algebras associated with labelled spaces generalize several others such as graph algebras, ultragraph algebras and Carlsen–Matsumoto algebras [182]. These C^*-algebras were further studied in [181,183,184,188,189]. It is interesting to notice that the definition given in [182] was later changed in [181] because certain projections in the C^*-algebra could turn out to be zero when it was desired for them not to be.

In [185], applied Exel's construction [187] to an inverse semigroup defined from a labelled space with multiplication inspired by the relations defining the C^*-algebra of the labelled space. The tight spectrum was characterized in Theorem 6.7 of [185]. In the particular case of a labelled space defined from a graph as in [182], found [185, Proposition 6.9] that the tight spectrum is homeomorphic to the boundary path space of the underlying graph (studied by Webster in [192]).

Webster shows that the boundary path space of a graph is the spectrum of a certain commutative C^*-subalgebra of the graph C^*-algebra called the *diagonal* C^*-subalgebra [192]. There is a natural generalization of the diagonal C^*-subalgebra in the case of labelled spaces. By comparing it with the case of graphs, one would expect the spectrum of this new diagonal C^*-subalgebra to be the tight spectrum found by [185]. We show that this is actually the case.

We present the definition of the C^* −algebra associated with a labelled space and some of its properties. A rather technical one that prepares for the building of a representation of our new version of the C^*-algebra. The representation is constructed, we show that non-zero elements in the inverse semigroup correspond to non-zero elements in the C^*-algebra. The diagonal C^*-subalgebra is homeomorphic to the tight spectrum of the inverse semigroup.

We give an example to show that the new definition gives a non-trivial quotient of the C^* −algebra defined in the preprint of [181].We begin by techniques developed in [185] .

A *(directed) graph* $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ consists of non-empty sets $\mathcal{E}^0$ (of *vertices*), $\mathcal{E}^1$ (of *edges*), and *range* and *source* functions $r, s : \mathcal{E}^1 \to \mathcal{E}^0$.

We say that $v \in \mathcal{E}^0$ is a *source* if $r^{-1}(v) = \emptyset$, a *sink* if $s^{-1}(v) = \emptyset$ and an *infinite emitter* if $s^{-1}(v)$ is an infinite set. Also, v is *singular* if it is either a sink or an infinite emitter, and regular otherwise.

A *path of length* n on a graph $\mathcal{E}$ is a sequence $\lambda = \lambda_1\lambda_2 \cdots \lambda_n$ of edges such that $r(\lambda_i) = s(\lambda_{i+1})$ for all $i = 1, \cdots, n-1$. We write $|\lambda| = n$ for the length of λ and regard vertices as paths of length 0. $\mathcal{E}^n$ stands for the set of all paths of length n and $\mathcal{E}^* = \bigcup_{n\geq 0} \mathcal{E}^n$. We extend the range and source maps to $\mathcal{E}^*$ by defining $s(\lambda) = s(\lambda_1)$ and $r(\lambda) = r(\lambda_1)$ if $n \geq 2$ and $s(v) = v = r(v)$ for $n = 0$. Similarly, we define a *path of infinite length* (or an *infinite path*) as an infinite sequence $\lambda = \lambda_1\lambda_2 \cdots$ of edges such that $r(\lambda_i) = s(\lambda_{i+1})$ for all $i \geq 1$; for such a path, we write $|\lambda| = \infty$. The set of all infinite paths will be denoted by $\mathcal{E}^\infty$.

A *labelled graph* consists of a graph $\mathcal{E}$ together with a surjective *labelling map* $\mathcal{L}: \mathcal{E}^1 \to A$, where A is a fixed non-empty set, called an *alphabet*. Elements of A are called letters. The set of all finite words over A, together with the *empty word* ω, is denoted by A^*,and A^∞ stands for the set of all infinite words over A.

A labelled graph is said to be *left-resolving* if for each $v \in \mathcal{E}^0$ the restriction of $\mathcal{L}$ to $r^{-1}(v)$ is injective.

The labelling map $\mathcal{L}$ can be used to produce labelling maps $\mathcal{L}: \mathcal{E}^n \to A^*$ for all $n \geq 1$, via $\mathcal{L}(\lambda) = \mathcal{L}(\lambda_1) \ldots \mathcal{L}(\lambda_n)$; similarly, it also gives rise to a map $\mathcal{L}: \mathcal{E}^n \to A^\infty$ in the obvious fashion. Using these maps, the elements of $\mathcal{L}^n = \mathcal{L}(\mathcal{E}^n)$ are the *labelled paths of length* n and the elements of $\mathcal{L}^\infty = \mathcal{L}(\mathcal{E}^\infty)$ are the *labelled paths of infinite length*. We set $\mathcal{L}^{\geq 1} = \bigcup_{n\geq 1} \mathcal{L}^n$, $\mathcal{L}^* = \{\omega\} \cup \mathcal{L}^{\geq 1}$, and $\mathcal{L}^{\leq\infty} = \mathcal{L}^* = \cup \mathcal{L}^\infty$.

For a subset A of $\mathcal{E}^0$, let

$$\mathcal{L}(A\mathcal{E}^1) = \{\mathcal{L}(e) \mid e \in \mathcal{E}^1 \text{ and } s(e) \in A\}. \qquad (1)$$

If α is a labelled path, a path λ on the graph such that $\mathcal{L}(\lambda) = \alpha$ is called a representative of α. The *length of* α, denoted by $|\alpha|$, is the length of any one of its representatives. By definition, ω is also a labelled path, with $|\omega| = 0$. If $1 \leq i \leq j \leq |\alpha|$, let $\alpha_{i,j} = \alpha_i\alpha_{i+1} \cdots \alpha_j$ if $j < \infty$ and $\alpha_{i,j} = \alpha_i\alpha_{i+1} \cdots$ if $j = \infty$. If $j < i$ set $\alpha_{i,j} = \omega$.

We say that a labelled path α is a *beginning of* a labelled path β if $\beta = \alpha\beta'$ for some labelled path β'; also, α and β are *comparable* if either one is a beginning of the other.

For $\alpha \in \mathcal{L}^*$ and $A \in \mathcal{P}(\mathcal{E}^0)$ (where $\mathcal{P}(\mathcal{E}^0)$ denotes the power set of $\mathcal{E}^0$), the *relative range of α with respect to A*, denoted by $r(A,\alpha)$, is the set

$$r(A,\alpha) = \{r(\lambda) | \lambda \in \mathcal{E}^*, \mathcal{L}(\lambda) = \alpha, \quad s(\lambda) \in A\}$$

if $\alpha \in \mathcal{L}^{\geq 1}$ and $r(A,\omega) = A$ if $\alpha = \omega$. The *range of* α, denoted by $r(\alpha)$, is the set

$$r(\alpha) = r(\mathcal{E}^0,\alpha).$$

For $\alpha \in \mathcal{L}^{\geq 1}$ we also define the *source of* α as the set

$$s(\alpha) = \{s(\lambda) \in \mathcal{E}^0 | \mathcal{L}(\lambda) = \alpha\}.$$

It follows that $r(\omega) = \mathcal{E}^0$ and, if $\alpha \in \mathcal{L}^{\geq 1}$, then $r(\alpha) = \{r(\lambda) \in \mathcal{E}^0 \mid \mathcal{L}(\lambda) = \alpha\}$. The definitions above give maps $s : \mathcal{L}^{\geq 1} \to \mathcal{P}(\mathcal{E}^0)$ and $r : \mathcal{L}^* \to \mathcal{P}(\mathcal{E}^0)$. Also, if $\alpha,\beta \in \mathcal{L}^*$ are such that $\alpha\beta \in \mathcal{L}^*$then $r(r(A,\alpha),\beta) = r(A,\alpha\beta)$. Furthermore, for $A,B \in \mathcal{P}(\mathcal{E}^0)$ and $\alpha \in \mathcal{L}^*$, it holds that $r(A \cup B,\alpha) = r(A,\alpha) \cup r(B,\alpha)$. Finally, observe that for $A \in \mathcal{P}(\mathcal{E}^0)$ one has $\mathcal{L}(A\mathcal{E}^0) = \{a\, A \mid r(A,a) = \emptyset\}$.

A *labelled space* is a triple $(\mathcal{E},\mathcal{L},B)$ where $(\mathcal{E},\mathcal{L})$ is a labelled graph and B is a family of subsets of $\mathcal{E}^0$ that is *accommodating* for $(\mathcal{E},\mathcal{L})$; that is, B is closed under finite intersections and finite unions, contains all $r(\alpha)$ for $\alpha \in \mathcal{L}^{\geq 1}$, and is *closed under relative ranges*, that is, $r(A,\alpha) \in B$ for all $A \in B$ and all $\alpha \in \mathcal{L}^*$.

We say that a labelled space $(\mathcal{E},\mathcal{L},B)$ is *weakly left-resolving* if for all $A,B \in B$ and all $\alpha \in \mathcal{L}^{\geq 1}$ we have $r(A \cap B,\alpha) = r(A,\alpha) \cap r(B,\alpha)$.

For a given $\alpha \in \mathcal{L}^*$,let

$$B_\alpha = B \cap \mathcal{P}\big(r(\alpha)\big).$$

If B is closed under relative complements, then the set B_α is a Boolean algebra for each $\alpha \in \mathcal{L}^{\geq 1}$, and $B_w = B$ is a generalized Boolean algebra as in [191]. By Stone duality there is a topological space associated with each B_α with $\alpha \in \mathcal{L}^*$which we denote by X_α,consisting of the set of ultrafilters in B_α.

By a *filter* in a partially ordered set P with least element 0 we mean a subset ξ of P such that

(*i*) $0 \notin \xi$;

(*ii*) if $x \in \xi$ and $x \leq y$, then $y \in \xi$;

(*iii*) if $x,y \in \xi$, there exists $z \in \xi$ such that $z \leq x$ and $z \leq y$.

If P is a (meet) semilattice, condition (iii) may be replaced by $x \wedge y \in \xi$ if $x,y \in \xi$.

An *ultrafilter* is a filter which is not properly contained in any filter.

For a given $x \in P$, define

$$\uparrow x = \{y \in P \mid x \leq y\} \quad , \quad \downarrow x = \{y \in P \mid y \leq x\},$$

and for subsets X,Y of P define

$$\uparrow X = \bigcup_{x\in X} \uparrow x = \{y \in P \mid x \leq y \text{ for some } x \in X\},$$

and $\uparrow_Y X = Y \cap\uparrow X$; the sets $\uparrow_Y x, \downarrow_Y x, \downarrow X$ and $\downarrow_Y X$ are defined analogously.

From now on, E denotes a semilattice with 0.

Proposition (4.2.1)[450]: ([187], Lemma 12.3). Let E be a semilattice with 0. *A filter ξ in* E is an ultrafilter if and only if

$$\{y \in E \mid y \wedge x \neq 0\ \forall x \in \xi\} \subseteq \xi.$$

A *character* of E is a non-zero function ϕ from E to the Boolean algebra $\{0,1\}$ such that
(a) $\phi(0) = 0$;

(b) $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$, for all $x, y \in E$.
The set of all characters of E is denoted by $\hat{E}_0$ and we endow $\hat{E}_0$ with the topology of pointwise convergence.

Given $x \in E$, a set $Z \subseteq \downarrow x$ is a *cover* for x if for all non-zero $y \in \downarrow x$, there exists $z \in Z$ such that $z \wedge y \neq 0$.

A character ϕ of E is *tight* if for every $x \in E$ and every finite cover Z for x, we have

$$\bigvee_{z \in Z} \phi(z) = \phi(x).$$

The set of all tight characters of E is denoted by $\hat{E}_{tight}$, and called the *tight spectrum* of E. We can associate each filter ξ in a semilattice E with a character ϕ_ξ of E given by

$$\phi_\xi(x) = \begin{cases} 1, & \text{if } x \in \xi, \\ 0, & \text{otherwise.} \end{cases}$$

Conversely, when ϕ is a character, $\xi_\phi = \{x \in E \mid \phi(x) = 1\}$ is a filter in E. There is thus a bijection between $\hat{E}_0$ and the set of filters in E. We denote by $\hat{E}_\infty$the set of all characters ϕ of E such that ξ_ϕ is an ultrafilter, and a filter ξ in E is said to be *tight* if ϕ_ξ is a tight character.

It is a fact that every ultrafilter is a tight filter ([187], Proposition 12.7), and that $\hat{E}_{tight}$ is the closure of $\hat{E}_\infty$ in $\hat{E}_0$ ([187], Theorem 12.9).

For a given labelled space $(\mathcal{E}, \mathcal{L}, B)$ *that is weakly left-resolving*, consider the set

$$S = \{(\alpha, A, \beta) \mid \alpha, \beta \in \mathcal{L}^* \text{ and } A \in B_\alpha \cap B_\beta \text{ with } A \neq \emptyset\} \cup \{0\}.$$

A binary operation on S is defined as follows: $s \cdot 0 = 0 \cdot s = 0$ for all $s \in S$ and, given $s = (\alpha, A, \beta)$ and $t = (\gamma, B, \delta)$ in S,

$$s.t = \begin{cases} (\alpha\gamma', r(A, \gamma') \cap B, \delta), & \text{if } \gamma = \beta\gamma' \text{ and } r(A, \gamma') \cap B \neq \emptyset, \\ (\alpha, A \cap r(B, \beta'), \delta\beta'), & \text{if } \beta = \gamma\beta' \text{ and } A \cap r(B, \beta') \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

If for a given $s = (\alpha, A, \beta) \in S$ we define $s^* = (\beta, A, \alpha)$, then the set S, endowed with the operation above, is an inverse semigroup with zero element 0 ([185], Proposition 3.4), whose set of idempotents is

$$E(S) = \{(\alpha, A, \alpha) \mid \alpha \in \mathcal{L}^* \text{ and } A \in B_\alpha\} \cup \{0\}.$$

The natural order in the semilattice $E(S)$ is described in the next proposition.

Proposition (4.2.2)[450]: ([185] Proposition 4.1*). Let* $p = (\alpha, A, \alpha)$ *and* $q = (\beta, B, \beta)$ be non-zero elements in $E(S)$.

Then $p \leq q$ *if and only if* $\alpha = \beta\alpha'$ *and* $A \subseteq r(B, \alpha')$.

Filters in $E(S)$ are classified in two types: a filter ξ in $E(S)$ is of *finite type* if there exists a word $\alpha \in \mathcal{L}^*$ such that $(\alpha, A, \alpha) \in \xi$ for some $A \in B_\alpha$ and α has the largest length among all β such that $(\beta, B, \beta) \in \xi$ for some $B \in B_\beta$. If ξ is not of finite type, then we say it is of *infinite type*.

Let ξ be a filter in $E(S)$ and $p = (\alpha, A, \alpha)$ and $q = (\beta, B, \beta)$ in ξ. Since ξ is a filter, then $pq \neq 0$; hence, α and β are comparable. This says the words of any two elements in a filter are comparable; in particular, for a filter of finite type there is a unique word with largest length associated with it as above.

There is a bijective correspondence between the set of filters of finite type in $E(S)$ and the set of pairs (α, F) where $\alpha \in \mathcal{L}^*$ and F is a filter in B_α, given by the following two results.

Proposition (4.2.3)[450]: ([185] Proposition 4.3). *Let $\alpha \in \mathcal{L}^*$ and F be a filter in B_α. Then*

$$\xi = \bigcup_{A\in F} \uparrow(\alpha, A, \alpha)$$
$$= \{(\alpha_{1,i}, A, \alpha_{1,i}) \in E(S) \mid 0 \leq i \leq |\alpha| \text{ and } r(A, \alpha_{i+1,|\alpha|}) \in F\}$$

Is a filter of finite type in $E(S)$, with largest word α.

Proposition (4.2.4)[450]: ([185] Proposition 4.4). *Let ξ be a filter in of finite type in $E(S)$ with largest word α, and set*

$$F = \{A \in B \mid (\alpha, A, \alpha) \in \xi\}.$$

Then F is a filter in B_α and

$$\xi = \bigcup_{A\in F} \uparrow(\alpha, A, \alpha).$$

The situation for filters of infinite type is a bit more involved, for there is no word with largest length in this case: let $\alpha \in \mathcal{L}^\infty$ and $\{F_n\}_{n\geq 0}$ be a family such that F_n is a filter in $B_{\alpha_{1,n}}$ for every $n > 0$ and F_0 is either a filter in B or $F_0 = \emptyset$. The family $\{F_n\}_{n\geq 0}$ is said to be *admissible for α* if

$$F_n \subseteq \{A \in B_{\alpha_{1,n}} \mid r(A, \alpha_{\mathrm{n+1}}) \in F_{n+1}\}$$

for all $n \geq 0$, and is said to be *complete for α* if

$$F_n = \{A \in B_{\alpha_{1,n}} \mid r(A, \alpha_{\mathrm{n+1}}) \in F_{n+1}\}$$

for all $n \geq 0$.

Note that any filter F_n in a complete family completely determines all filters that come before it in the sequence. In fact, it can be shown that $\{F_n\}_{n\geq 0}$ is complete for α if and only if

$$F_n = \{A \in B_{\alpha_{1,n}} \mid r(A, \alpha_{\mathrm{n+1,m}}) \in F_m\}$$

for all $n \geq 0$ and all $m > n$ ([185], Lemma 4.6).

The following two results give a bijective correspondence between the set of filters of infinite type in $E(S)$ and the set of pairs $(\alpha, \{F_n\}_{n\geq 0})$, where $\alpha \in \mathcal{L}^\infty$ and $\{F_n\}_{n\geq 0}$ is a complete family for α.

Proposition (4.2.5)[450]: ([185], Proposition 4.7). *Let $\alpha \in \mathcal{L}^\infty$, $\{F_n\}_{n\geq 0}$ be an admissible family for α and define*

$$\xi = \bigcup_{n=0}^{\infty} \bigcup_{A\in F_n} \uparrow(\alpha_{1,n}, A, \alpha_{1,n}).$$

Then ξ is a filter in $E(S)$.

Proposition (4.2.6)[450]: ([185], Proposition 4.8). *Let ξ be a filter of infinite type in $E(S)$. Then there exists $\alpha \in \mathcal{L}^\infty$ such that every $p \in \xi$ can be written as $p = (\alpha_{1,n}, A, \alpha_{1,n})$ for some $n \geq 0$ and some $A \in B_{1,n}$. Moreover, if we define for each $n \geq 0$,*

$$F_n = \{A \in B \mid (\alpha_{1,n}, A, \alpha_{1,n}) \in \xi\},$$

then $\{F_n\}_{n\geq 0}$ is a complete family for α.

Admissible families can be completed, in the following sense: if $\alpha \in \mathcal{L}^*$ and $\{F_n\}_{n\geq 0}$ is an admissible family for α, then there is a complete family for α, say $\{\bar{F}_n\}_{n\geq 0}$, such that $F_n \subseteq \bar{F}_n$, for all $n \geq 0$ and both $(\alpha, \{F_n\}_{n\geq 0})$ and $(\alpha, \{\bar{F}_n\}_{n\geq 0})$ generate the same filter in $E(S)$ ([185], Proposition 4.11).

One can also talk of admissible and complete families for a labelled path α of finite length: the definition is the same, only with a finite family $\{F_n\}_{0\le n\le|\alpha|}$ satisfying the above-mentioned conditions. Note then that $F_{|\alpha|}$ determines completely all other filters in the family.

Proposition (4.2.7)[450]: ([185], Proposition 4.12). Let ξ be a filter of finite type in E(S) and (α, F) be its associated pair. For each $n \in \{0,1,\ldots,|\alpha|\}$, define

$$F_n = \{A \in B \mid (\alpha_{1,n}, A, \alpha_{1,n}) \in \xi\}.$$

Then $F_{|\alpha|} = F$ *and* $\{F_n\}_{0\le n\le|\alpha|}$ *is a complete family for* α.

We can thus bring about a common description for filters of finite and infinite type, summarized in the following theorem.

Theorem (4.2.8)[450]: ([185], Theorem 4.13). Let $(E,\mathcal{L},B)$ be a labelled space which is weakly left-resolving, and let S be its associated inverse semigroup. Then there is a bijective correspondence between filters in $E(S)$ and pairs $(\alpha,\{F_n\}_{0\le n\le|\alpha|})$, where $\alpha \in \mathcal{L}^{\le\infty}$ and $\{F_n\}_{0\le n\le|\alpha|}$ is a complete family for α (understanding that $0\le n\le|\alpha|$ means $0 \le n < \infty$ when $\alpha \in \mathcal{L}^{\infty}$).

We may occasionally denote a filter ξ in $E(S)$ with associated labelled path $\alpha \in \mathcal{L}^{\le\infty}$ by ξ^α to stress the word α; in addition, the filters in the complete family associated with ξ^α will be denoted by ξ_n^α ($or \xi_n$ when there is no risk of confusion about the associated word). Specifically,

$$\xi_n^\alpha = \{A \in B \mid (\alpha_{1,n}, A, \alpha_{1,n}) \in \xi^\alpha\}$$

and the family $\{\xi_n^\alpha\}_{0\le n\le|\alpha|}$ satisfies

$$\xi_n^\alpha = \{A \in B_{\alpha_{1,n}} \mid r(A, \alpha_{n+1+m}) \in \xi_m^\alpha\}$$

for all $0 \le n < m \le |\alpha|$.

In what follows, F denotes the set of all filters in $E(S)$ and F_α stands for the subset of F of those filters whose associated word is $\alpha \in \mathcal{L}^{\le\infty}$.

The following is a complete description of the ultrafilters in $E(S)$.

Theorem (4.2.9)[450]: ([185], Theorem 5.10). Let $(E,\mathcal{L},B)$ be a labelled space which is weakly left-resolving, and let S be its associated inverse semigroup. Then the ultrafilters in $E(S)$ are:

i. The filters of finite type ξ^α such that $\xi_{|\alpha|}$ is an ultrafilter in B_α and for each $b \in A$ there exists $A \in \xi_{|\alpha|}$ such that $r(A,b) = \emptyset$.

ii. The filters of infinite type ξ^α such that the family $\{\xi_n\}_{n\ge0}$ is maximal among all complete families for α (that is, if $\{\xi_n\}_{n\ge0}$ is a complete family for α such that $\xi_n \subseteq \mathcal{F}_n$ for all $n \ge 0$, then $\xi_n = F_n$ for all $n \ge 0$).

Suppose in addition that the accommodating family B is closed under relative complements. Then (ii) can be replaced with (ii) The filters of infinite type ξ^α such that ξ_n is an ultrafilter for every $n > 0$ and ξ_0 is either an ultrafilter or the empty set.

Let T be the set of all tight filters in $E(S)$, endowed with the topology induced from the topology of pointwise convergence of characters, via the bijection between tight characters and tight filters given at the end . Note then that T is (homeomorphic to) the tight spectrum of $E(S)$, and will be treated as such.

For each $\alpha \in \mathcal{L}^*$ recall from the end that

$$X_\alpha = \{F \subseteq B_\alpha \mid F \textit{ is an ultrafilter in } B_\alpha\}.$$

Also, define

$$X_{\propto}^{sink} = \{\mathcal{F} \in X_\alpha \parallel \forall b \in A, \exists A \in \mathcal{F} \ such \ \ that \ r(a,b) = \emptyset\}$$

Suppose the accommodating family B to be closed under relative complements. In this case, for every $\alpha, \beta \in \mathcal{L}^{\geq 1}$ such that $\alpha\beta \in \mathcal{L}^{\geq 1}$, the relative range map $r(\cdot, \beta) : B\alpha \longrightarrow B_{\alpha\beta}$ is a morphism of Boolean algebras and, therefore, we have its dual morphism

$$f_{\alpha[\beta]} : X_{\alpha\beta} \longrightarrow X_\alpha$$

given by $f_{\alpha[\beta]}(F) = \{A \in B_\alpha \mid r(A,\beta) \in F\}$. When $\alpha = \omega$, if $F \in B_\beta$ then $\{A \in B \mid r(A,\beta) \in f\}$ either an ultrafilter in $B = B_\omega$ or the empty set, and we can therefore consider $f_{\omega[\beta]} : X_\beta \longrightarrow X_\omega \cup \{\emptyset\}$.

Employing this new notation, if ξ^α is a filter in $E(S)$ and $0 \leq m < n \leq |\alpha|$, then $\xi_m = f_{\alpha_{1,m}}[\alpha_{m+1,n}](\xi_n)$.

If we endow the sets X_α with the topology given by the convergence of filters stated at the end (this is the pointwise convergence of characters), it is clear that the functions $f_{\alpha[\beta]}$ are continuous. Furthermore, it is easy to see that $f_{\alpha[\beta y]} = f_{\alpha[\beta]} \circ f_{\alpha\beta[\gamma]}$.

Under the hypothesis of B being closed under relative complements, the only tight filters of infinite type in $E(S)$ are the ultrafilters of infinite type ([185], Corollary 6.2). The next result classifies the tight filters of finite type.

Proposition (4.2.10)[450]: ([185], Proposition 6.4). Suppose the accommodating family B to be closed under relative complements and let $\xi^\propto$ be a filter of finite type. Then $\xi^\propto$ is a tight filter if and only if $\xi_{|\alpha|}$ is an ultrafilter and at least one of the following conditions hold:

(a) There is a net $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq X_\propto^{sink}$ converging to $\xi_{|\alpha|}$.

(b) There is a net $\{(t_\lambda, F_\lambda)\}_{\lambda \in \Lambda}$, where t_λ is a letter in A and $F_\lambda \in X_{\alpha t \lambda}$ for each $\lambda \in \Lambda$, such that $\{f_{\alpha[t\lambda]}(F_\lambda)\}_{\lambda \in \Lambda}$ converges to $\xi_{|\alpha|}$ and $\{t_\lambda\}_\lambda$ converges to infinity (that is, for every $b \in A$ there is $\lambda_b \in \Lambda$ with $t_\lambda \neq b$ for all $\lambda \geq \lambda_b$).

There is a more algebraic description for the tight filters in $E(S)$, given by the following theorem.

Theorem (4.2.11)[450]: ([185], Theorem 6.7). Let $(\varepsilon, \mathcal{L}, \mathcal{B})$ be a labelled space which is weakly left-resolving and whose accommodating family $\mathcal{B}$ is closed under relative complements, and let S be its associated inverse semigroup.

Then the tight filters in E(S) are:

(i) The ultrafilters of infinite type.

(ii) The filters of finite type ξ^α such that $\xi_{|\alpha|}$ is an ultrafilter in B_α and for each $A \in \xi_{|\alpha|}$ at least one of the following conditions hold:

(a) $\mathcal{L}(A\varepsilon^1)$ is infinite.

(b) There exists $B \in B_\alpha$ such that $\emptyset \neq B \subseteq A \cap \varepsilon^0_{sink}$.

Finally, the next result shows the relation between the tight spectrum T and the boundary path space of a directed graph (see also [185], Example 6.8).

Proposition (4.2.12)[450]: ([185], Proposition 6.9). Let $(\mathcal{E}, \mathcal{L})$ be a left-resolving labelled graph such that $\mathcal{E}^0$ is a finite set and let $B = P(\mathcal{E}^0)$. Then the tight spectrum T of the labelled space $(\varepsilon, \mathcal{L}, \mathcal{B})$ is homeomorphic to the boundary path space ∂E of the graph $\mathcal{E}$.

We present the C^*-algebra associated with a labelled space, following [181].

Definition (4.2.13)[450]: ([181], Definition 2.1). Let $(\varepsilon, \mathcal{L}, \mathcal{B})$be a weakly left-resolving labelled space. The C^**-algebra associated with* $(\varepsilon, \mathcal{L}, \mathcal{B})$, denoted by $C^*(\varepsilon, \mathcal{L}, \mathcal{B})$, is the universal C^*-algebra generated by projections $\{p_A \mid A \in B\}$ and partial isometries $\{s_a \mid a \in A\}$ subject to the relations

(i) $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ *and* $p\emptyset = 0$, for every $A, B \in B$;

(ii) $p_A s_a = s_a p_{r(A,a)}$, for every $A \in B$ and $a \in A$;

(iii) $s_a^* s_a = p_r(a)$ and $s_b^* s_a = 0$ if $b \neq a$, for every $a, b \in A$;

(iv) For every $A \in B$ for which $0 < \#\mathcal{L}(A\varepsilon^1) < \infty$ and there does not exist $B \in B$ such that $\emptyset \neq B \subseteq A \cap \varepsilon^0_{sink}$,

$$p_A = \sum_{\propto \in \mathcal{L}(A\mathcal{E}^1)} s_a p_{r(A,a)} s_a^*$$

In [181, Remark 2.4], Bates, Pask and Carlsen observed that this definition would lead to zero vertex projections for sinks. Therefore, they proposed a new definition modifying item (iv). In a preprint version of [181], they proposed

When we begun, the published version of [181] containing Definition (4.2.13) was not yet available and, indeed, one of the goals was to point out that Definition (4.2.13) of the C^*-algebra associated with a labelled space is more adequate than the previously given definitions. Item (iv) looks like the relation $n \sum_{i=1}^{n} s_i s_i^* = 1$ in the Cuntz algebra $\mathcal{O}_n$. To classify a definition as adequate or inadequate is sometimes difficult what would the criteria be? In [187], Exel applied the theory of tight filters associated with inverse semigroups to show that several C^*-algebras obtained from combinatorial objects created since the Cuntz algebras are, in fact, C^*-algebras of inverse semigroups acting on their tight spectra. We believe this establishes a good criterion to choose the adequate definition for the C^*-algebra of a labelled space, and we expected Definition (4.2.13) to be the adequate one, due to item (ii)(b) of Theorem (4.2.11). After the first preprint, Carlsen pointed out to us that our expected definition was indeed equivalent to the one present in the published version of [181]. We present an example that shows Definition (4.2.13) is different from that in the preprint version of [181].

Now, we develop some basic properties of $C^*(\varepsilon, \mathcal{L}, \mathcal{B})$; most of these were already discussed by Bates and Pask in [182].

Let (E,L,B) be a weakly left-resolving labelled space and consider its C^*-algebra $C^*(\varepsilon, \mathcal{L}, \mathcal{B})$For each word $\alpha = a_1 a_2 \cdots a_n$, define $s_\alpha = s_{a1} s_{a2} \cdots s_{a_n}$; we also set $s_\omega = 1$, where ω is the empty word.

Proposition (4.2.14)[450]: *The following properties are valid in* $C^*(\varepsilon, \mathcal{L}, \mathcal{B})$.

(i) If $\alpha \in C^*$, then $s_\alpha = 0$.

(ii) $p_A s_\alpha = s_a p_{r(A,a)}$, for every $A \in B$ and $\alpha \in C^*$.

(iii) $s_\alpha^* s_\alpha = p_{r(\alpha)}$ and $s_\beta^* s_\alpha = 0$ and α are not comparable, for every $\alpha, \beta \in \mathcal{L}^{\geq 1}$.

(iv) For every $\alpha \in \mathcal{L}^{\geq 1} s_\alpha$ is a partial isometry.

(v) Let $\alpha, \beta \in C^*$ and $A \in B$. If $s_\propto p_A s_\beta^* \neq 0$, then $A \cap r(\alpha) \cap r(\beta) = \emptyset$ and $s_\propto p_A s_\beta^* = s_\propto p_{A\cap r(\propto)\cap r(\beta)} s_\beta^*$.

(vi) Let $\alpha, \beta, \gamma, \delta \in \mathcal{L}^*$, $A \in B_\alpha \cap B_\beta$ and $B \in B_\gamma \cap B_\delta$.

Then

$$(s_\propto p_A s_\beta^*)(s_\gamma p_B s_\delta^*) = \begin{cases} s_{\alpha\gamma'} p_{r(A,\acute{\gamma})\cap B} s_{\delta,}^* & if \gamma = \beta\grave{\gamma} \\ s_{\alpha'} p_{A\cap r(B,\acute{\beta})} s_{\delta\hat{\beta}}^* & if \beta = \gamma\hat{\beta} \\ 0, & otherwise \end{cases}$$

In particular , $s_\propto p_A s_\beta^*$ *is a partial isometry.*

(vii) Every non-zero finite product of terms of types s_a, p_B *and* s_b^* *can be written as* $s_\propto p_A s_\beta^*$*, where* $A \in B_\alpha \cap B_\beta$*.*

(viii) $C^*(\varepsilon, \mathcal{L}, \mathcal{B}) = \overline{span}\{s_\propto p_A s_\beta^* | \alpha, \beta \in \mathcal{L}^* \text{ and } B_\alpha \cap B_\beta\}$*.*

(ix) The elements of the form $s_\propto p_A s_\alpha^*$*, where* $\alpha \in \mathcal{L}^*$ *,are commuting projections. Furthermore,*

$$(s_\propto p_A s_\alpha^*)(s_\beta p_B s_\beta^*) = \begin{cases} s_\beta p_{r(A,\beta)} s_{\beta,}^* & if \beta = \alpha\grave{\beta} \\ s_{\alpha'} p_{A\cap r(B,\grave{\alpha})} s_\alpha^* & if \alpha = \gamma\grave{\alpha} \\ 0, & otherwise \end{cases}$$

Proof.

(i) If $\alpha = a_1 a_2 \cdots a_n \notin \mathcal{L}^*$, then $r(\cdots r(r(a_1), a_2) \cdots, a_n) = \emptyset$. Therefore, by using that s_{a1} is a partial isometry and items (i), (ii) and (iii) of Definition (4.2.13), we have

$s_\alpha = s_{a_1} s_{a_2} \cdots s_{a_n} = s_{a_1} p_r(a_1) s_{a_2} \cdots s_{a_n} = s_{a_1} s_{a_2} \cdots s_{a_n} p_r(\cdots r(r(a_1), a_2) \cdots, a_n) = 0.$

(ii) This is clear from item (ii) of Definition (4.2.13).

(iii) The first equality follows by induction using that $s_a^* b^s ab = s_b^* s_a^* s_a s_b = s_b^* p_{r(a)} s_b = s_b^* s_b p_{r(r(a),b)} = p_{r(b)} p_{r(r(a),b)} = p_{r(r(a),b)} = p_{r(ab).}$ To see the second one, suppose $\alpha, \beta \in \mathcal{L}^{\geq 1}$ are not comparable, that is, there are $\grave{\alpha}, \grave{\beta} \in \mathcal{L}^*$ and $a, b \in A$ with $a \neq b$ such that $\alpha = \gamma a \grave{\alpha}$ and $\beta = \gamma b \acute{\beta}$. Therefore,

$$s_\beta^* s_\alpha = (s_\gamma s_b s_{\acute{\beta}})^* (s_\gamma s_a s_{\acute{\alpha}}) = s_{\acute{\beta}}^* s_b^* s_\gamma^* s_\gamma s_a s_{\acute{\alpha}} = s_{\acute{\beta}} s_b^* p_{r(\gamma)} s_a s_{\acute{\alpha}}$$
$$= s_{\acute{\beta}}^* s_b^* s_a p_{r(r(\gamma),a)} s_{\acute{\alpha}} = 0$$

since $s_b^* s_a = 0$

(iv) It follows by the previous item, since $s_\alpha^* s_\alpha$ is a projection.

(v) Since $s_\propto p_A s_\beta^* = s_\propto p_{r(\alpha)} p_A p_{r(\beta)} s_\beta^* = s_\propto p_{r(\alpha)\cap A\cap r(\beta)} s_\beta^*$, then $r(\alpha) \cap A \cap r(\beta) \neq \emptyset$ if $s_\propto p_A s_\beta^* \neq 0$.

(vi) If β and γ are not comparable, then $(s_\propto p_A s_\beta^*)(s_\gamma p_A s_\delta^*) = 0$, by item (iii). Now, suppose $\gamma = \beta\acute{\gamma}$ (the other case is similar and both cases coincide if $\beta = \gamma$). Then

$$(s_\propto p_A s_\beta^*)(s_\gamma p_B s_\delta^*) = s_\alpha p_A s_\beta^* s_\beta s_{\acute{\gamma}} p_B s_\delta^* = s_\alpha p_A p_{r(\beta)} s_{\acute{\gamma}} p_B s_\delta^* = s_\alpha p_A s_{\acute{\gamma}} p_B s_\delta^*$$
$$= s_\alpha s_{\acute{\gamma}} p_{r(A,\acute{\gamma})} p_B s_{\alpha\acute{\gamma}} p_{r(A,\acute{\gamma})\cap B} s_\delta^* .$$

Applying this product rule, we see that $s_\alpha p_A s_\beta^*$ is a partial isometry, since

$$(s_\alpha p_A s_\beta^*)(s_\alpha p_A s_\beta^*)^* (s_\alpha p_A s_\beta^*) = (s_\propto p_A s_\beta^*)(s_\beta p_A s_\alpha^*)(s_\alpha p_A s_\beta^*) = (s_\alpha p_A s_\alpha^*)(s_\alpha p_A s_\beta^*)$$
$$= s_\alpha p_A s_\beta^*.$$

Consider a non-zero product as in the statement. If we have a s_b^*on the left of a s_a, we must have $a = b$, since the product is non-zero. In this case, we can replace $s_a^* s_a$ by $p_{r(a)}$. If we have p_B on the left of a s_a, we can replace $p_B s_a$ by $s_a p_{r(B,a)}$. Similarly, we can replace

$s_b^* p_B$ by $s_a p_{r(B,a)}$. Applying these replacements whenever possible, we end up with a product like

$$s_{a_1} s_{a_2} \cdots s_{a_n} p_{B_1} p_{B_2} \cdots p_{B_m} s_{b_k}^* s_{b_{k-1}}^* \cdots s_{b_1}^*.$$

Taking $\alpha = a_1 a_2 \cdots a_n$, $A = B_1 \cap \cdots \cap B_m$ and $\beta = b_1 b_2 \cdots b_k$, the product reduces to $s_\alpha p_A s_\beta^*$ and, by item (v), we can suppose $A \in B_\alpha \cap B_\beta$.

(vii) Immediate from the previous item.

(viii) Applying item (vi) to the product $(s_\alpha p_A s_\alpha^*)(s_\beta p_A s_\beta^*)$, we obtain

$$(s_\alpha p_A s_\alpha^*)(s_\beta p_A s_\beta^*), \begin{cases} s_\beta p_{r(A,\beta')\cap B s_\beta^*}, & if\ \beta = \alpha\beta', \\ s_\alpha p_{r(A,\alpha')\cap B s_\alpha^*}, & if\ \alpha = \beta\alpha', \\ 0, & \text{otherwise.} \end{cases}$$

Interchanging α and A with β and B, it is clear that $s_\alpha p_A s_\alpha^*$ commutes with $s_\beta p_A s_\beta^*$. Finally, taking $\beta = \alpha$ and $B = A$, we see that $s_\alpha p_A s_\alpha^*$ is a projection.

Consider the subset $R = \{s_\alpha p_A s_\beta^*\} \alpha, \beta \in \mathcal{L}^*$ and $A \in \beta_\alpha \cap \beta_\beta\}\ of\ C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$. Properties (v) to $(viii)$ from the previous proposition say that R is a semigroup whose linear span is dense in $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$. The inverse semigroup was defined based on R, but they might not be isomorphic: two idempotents of S may give the same element in the C^*-algebra, as in Example (4.2.42) (there, triples of the form $(a^n, \mathcal{E}^0, a^n)$ are all different for $n \geq 1$, but $s_{a^n} p_{\mathcal{E}^0} s_{a^n}^* = 1$ for all $n \geq 1$).

Definition (4.2.15)[450]: Let $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. The diagonal C^*-algebra associated with $(\mathcal{E}, \mathcal{L}, \mathcal{B})$ denoted by $\Delta(\mathcal{E}, \mathcal{L}, \mathcal{B})$, is the C^*-subalgebra of $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$ generated by the elements $s_\alpha p_A s_\alpha^*$, that is,

$$\Delta(\mathcal{E}, \mathcal{L}, \mathcal{B}) = C^*(\{s_\alpha p_A s_\alpha^* | \alpha \in \mathcal{L}^*\ and\ A \in \mathcal{B}_\alpha\}).$$

By item (ix) of Proposition (4.2.12), $\Delta(\mathcal{E}, \mathcal{L}, \mathcal{B})$ is an abelian C^*-algebra generated by commuting projections and

$$\Delta(\mathcal{E}, \mathcal{L}, \mathcal{B}) = \overline{span}\{s_\alpha p_A s_\alpha^*\} | \alpha \in \mathcal{L}^*\ and\ A \in \mathcal{B}_\alpha\}.$$

We define functions that are going to be used later on to construct a representation of $C^*(\mathcal{E}, \mathcal{L}, \mathcal{B})$. These functions generalize two operations that can easily be done with paths on a graph $\mathcal{E}$: gluing paths, that is, given μ and ν paths on $\mathcal{E}$ such that $r(\mu) = s(\nu)$, it is easy to see that $\mu\nu$ is a new path on $\mathcal{E}$; and cutting paths, that is, given a path $\mu\nu$ on $\mathcal{E}$ then ν is also a path on the graph.

In the labelled spaces, we have an extra layer of complexity because filters in $E(S)$ are described not only by a labelled path but also by a complete family of filters associated with it, as in Theorem (4.2.8). When we cut or glue labelled paths, the Boolean algebras where the filters lie change because they depend on the labelled path. We also note that, since we are only interested in tight filters in $E(S)$, the families considered below will consist only of ultrafilters.

Let us begin with the problem of describing new filters by gluing labelled paths. Consider composable labelled paths $\alpha \in \mathcal{L}^{\geq 1}\ and\ \beta \in \mathcal{L}^*$, and an ultrafilter $\mathcal{F} \in X_\beta$; a simple way to produce a subset $\mathcal{J}$ of $\mathcal{B}_{\alpha\beta}$ from $\mathcal{F}$ is by cutting the elements of $\mathcal{F}$ by $r(\alpha\beta)$, that is,

$$\mathcal{J} = \{C \cap r(\alpha\beta) \mid C \in \mathcal{F}\}.$$

It may be the case, however, that $C \cap r(\alpha\beta) = \emptyset$ for some $C \in \mathcal{F}$. Since $(\alpha\beta) \in \mathcal{B}_\beta$, by Proposition(4.2.1) and using the fact that $\mathcal{F}$ is an ultrafilter it can be seen that the intersections $C \cap r(\alpha\beta)$ are non-empty for all $C \in \mathcal{F}$ if and only if $r(\alpha\beta) \in \mathcal{F}$. The following simple result is useful for the present argument, and a proof is included for convenience.

Lemma (4.2.16)[450]: Given E a meet semilattice with $0, y \in$ E and $\mathcal{F}$ an ultrafilter in E, consider

$$\mathcal{J} = \{x \wedge y \mid x \in \mathcal{F}\}.$$

Then $\mathcal{J}$ is an ultrafilter in $\downarrow y$ if and only if $y \in \mathcal{F}$.

Proof. Clearly $\mathcal{J} \subseteq \downarrow y$ and $\mathcal{J}$ is closed by finite meets. Proposition (4.2.1) ensures $y \in \mathcal{F}$ if and only if $x \wedge y \neq 0$ for all $x \in \mathcal{F}$, and so if $y \notin \mathcal{F}$ one sees already that $\mathcal{J}$ is not a filter.

Suppose then that $y \in \mathcal{F}$. Given $x \in F$ and $z \in \downarrow y$ such that $x \wedge y \leq \mathcal{Z}$, since $\mathcal{F}$ is a filter it follows that $x \wedge y \in \mathcal{F}$ and thus $\mathcal{Z} \in \mathcal{F}$, whence $\mathcal{Z} = \mathcal{Z} \wedge y \in \mathcal{J}$, showing $\mathcal{J}$ is a filter.

To show $\mathcal{J}$ is an ultrafilter, let $\mathcal{Z} \in \downarrow y$ be such that $\mathcal{Z} \wedge u \neq 0$ for all $u \in \mathcal{J}$. Given that $\mathcal{Z} = y \wedge \mathcal{Z}$ one has

$$x \wedge \mathcal{Z} = x \wedge (y \wedge \mathcal{Z}) = (x \wedge y) \wedge \mathcal{Z} \neq 0$$

for all $x \in \mathcal{F}$, hence $\mathcal{Z} \in \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter; but then $\mathcal{Z} = \mathcal{Z} \wedge y \in \mathcal{J}$, and the result follows.

Lemma (4.2.17)[450]: *Let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^*$ be such that $\alpha\beta \in \mathcal{L}^{\geq 1}$, let $\mathcal{F} \in X_\beta$, and consider*

$$\mathcal{J} = \{C \cap r(\alpha\beta) \mid C \in \mathcal{F}\}.$$

Then $\mathcal{J} \in X_{\alpha\beta}$ if and only if $r(\alpha\beta) \in \mathcal{F}$.

Proof. Follows immediately from Lemma (4.2.1), with $E = \mathcal{B}_\beta$ and $y = r(\alpha\beta)$ (note that $\downarrow r(\alpha\beta) = \mathcal{B}_{\alpha\beta}$).

For labelled paths $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^*$ such that $\alpha\beta \in \mathcal{L}^{\geq 1}$, denote by $X(\alpha)\beta$ the set of ultrafilters in B_β that give rise, via cut down by $r(\alpha\beta)$, to ultrafilters in $B\alpha\beta$. More precisely, using Lemma (4.2.17),

$$X(\alpha)\beta = \{\mathcal{F} \in X_\beta \mid r(\alpha\beta) \in \mathcal{F}\}.$$

There is thus a map $g(\alpha)\beta : X(\alpha)\beta \to X_{\alpha\beta}$, that associates to each ultrafilter $\mathcal{F} \in X_{(\alpha)\beta,}$ the ultrafilter in $B_{\alpha\beta}$ given by

$$g_{(\alpha)\beta}(F) = \{C \cap r(\alpha\beta) \mid C \in \mathcal{F}\}.$$

Also, for $\alpha = \omega$ define $X_{(\omega)\beta} = X\beta$ and let $g_{(\omega)\beta}$ denote the identity function on X_β.

The following lemmas describe properties of these sets and maps, and how they behave with respect to the maps $f_{\alpha[\beta]} : X_{\alpha\beta} \to X_\alpha$ between ultrafilter spaces.

Lemma (4.2.18)[450]: *Let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta, \gamma \in \mathcal{L}^*$ with $\alpha\beta\gamma \in \mathcal{L}^{\geq 1}$ be given. Then*

(i) $f_{\beta[\gamma]}(X_{(\alpha)\beta\gamma}) \subseteq X_{(\alpha)\beta}$;

(ii) ${f^{-1}}_{\beta[\gamma]}(X_{(\alpha)\beta\gamma}) \subseteq X_{(\alpha)\beta\gamma}$;

Proof. To prove (i), given $\mathcal{F}' \in X_{(\alpha)\beta\gamma,}$ one has $r(\alpha\beta\gamma) \in \mathcal{F}'$ and also

$$f_{\beta[r]}(\mathcal{F}') = \left\{C \in \mathcal{B}_\beta \middle| r(C, \gamma) \in \mathcal{F}'\right\}.$$

Since $r(\alpha\beta) \in \mathcal{B}_\beta$ and $r(r(\alpha\beta), \gamma) = r(\alpha\beta\gamma) \in \mathcal{F}'$, it follows that $(\alpha\beta) \in f_{\beta[r]}(\mathcal{F}')$) and therefore $f_{\beta[r]}(\mathcal{F}') \in X_{(\alpha)\beta}$.

As for (ii), if $\mathcal{F}' \in {f^{-1}}_{\beta[\gamma]}(X_{(\alpha)\beta})$ then $f_{\beta[r]}(\mathcal{F}') \in X_{(\alpha)\beta,}$ and thus $r(\alpha\beta) \in \left\{C \in \mathcal{B}_\beta \middle| r(C, \gamma) \in \mathcal{F}'\right\}$. This gives

$$r(\alpha\beta\gamma) = r(r(\alpha\beta),\gamma) \in \mathcal{F}',$$

whence $\mathcal{F}' \in X_{(\alpha)\beta\gamma}$.

Lemma (4.2.19)[450]: *Let* $\alpha \in \mathcal{L}^{\geq 1}$ *and* $\beta,\gamma \in \mathcal{L}^*$ *with* $\alpha\beta\gamma \in \mathcal{L}^{\geq 1}$ *be given. Then,*

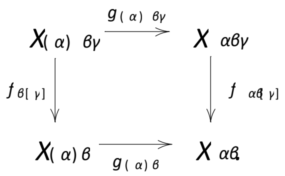
(i) $X_{(\alpha\beta)\gamma} \subseteq X_{(\beta)\gamma}$;

(ii) $g_{(\beta)\gamma}(X_{(\alpha\beta)\gamma}) \subseteq X_{(\alpha)\beta\gamma}$;

(iii) $If\ \mathcal{F} \in X_{(\alpha\beta)\gamma}$, *then*

$$g_{(\alpha\beta)\gamma}(\mathcal{F}) = g_{(\alpha)\beta\gamma} o\, g_{(\beta)\gamma}(\mathcal{F}).$$

(iv) Suppose the labelled space $(\mathcal{E},\mathcal{L},\mathcal{B})$ is weakly left-resolving. Then, the following diagram is commutative:



Proof. If $\mathcal{F} \in X_{(\alpha\beta)\gamma}$ then $r(\alpha\beta\gamma) \in \mathcal{F}$ and, since $\mathcal{F}$ is a filter in $B\gamma$, the fact that $r(\alpha\beta\gamma) \subseteq r(\beta\gamma)$ gives $r(\beta\gamma) \in \mathcal{F}$, whence $\mathcal{F} \in X_{(\beta)\gamma}$; this proves (i). Also,

$$r(\alpha\beta\gamma) = r(\alpha\beta\gamma) \cap r(\beta\gamma) \in \{C \cap r(\beta\gamma) \mid C \in \mathcal{F}\} = g_{(\beta)\gamma}(\mathcal{F}),$$

From where (ii) follows. It is clear that

$$\begin{aligned} g_{(\alpha)\beta\gamma}\, o\, g_{(\beta)\gamma}(\mathcal{F}) &= g_{(\alpha)\beta\gamma}(\{ C \cap r(\beta\gamma) \mid C \in \mathcal{F}\}) \\ &= \{(C \cap r(\beta\gamma)) \cap r(\alpha\beta\gamma) \mid C \in \mathcal{F}\} \\ &= \{C \cap r(\alpha\beta\gamma) \mid C \in \mathcal{F}\} = g_{(\alpha\beta)\gamma}(\mathcal{F}), \end{aligned}$$

which is (iii). As for (iv), Lemma (4.2.17) ensures $f_{\beta[\gamma]}$ maps $X_{(\alpha)\beta\gamma}$ into $X_{(\alpha)\beta}$. If $\mathcal{F} \in X_{(\alpha)\beta\gamma}$, then

$$(g_{(\alpha)\beta}\, o\, f_{\beta[\gamma]})(\mathcal{F}) = \{C \cap r(\alpha\beta) \mid C \in \mathcal{B}_\beta \text{ and } r(C,\gamma) \in \mathcal{F}\}$$

and

$$(f_{\alpha\beta[\gamma]}\, o\, g_{(\alpha)\beta\gamma})(\mathcal{F}) = \{D \in \mathcal{B}_{\alpha\beta} \mid r(D,\gamma) \in g_{(\alpha)\beta\gamma}(\mathcal{F})\}.$$

Given $C \in \mathcal{B}_\beta$ such that $r(C,\gamma) \in \mathcal{F}$, note that $C \cap r(\alpha\beta) \in \mathcal{B}_{\alpha\beta}$ and that

$$r(C \cap r(\alpha\beta),\gamma) = r(C,\gamma) \cap r(\alpha\beta\gamma) \in g_{(\alpha)\beta\gamma}(\mathcal{F}),$$

using the weakly left-resolving hypothesis; therefore

$$(g_{(\alpha)\beta}\, o\, f_{\beta[\gamma]})(\mathcal{F}) \subseteq (f_{\alpha\beta[\gamma]}\, o\, g_{(\alpha)\beta\gamma})(\mathcal{F}),$$

And since both terms are ultrafilters, equality follows.

Lemma (4.2.20)[450]: *Let* $\alpha \in \mathcal{L}^{\geq 1}$ *and* $\beta \in \mathcal{L}^*$ *be such that* $\alpha\beta \in \mathcal{L}^{\geq 1}$ *. Then,.*

(i) $g_{(\alpha)\beta}(X_{(\alpha)\beta} \to X_\beta^{sink}) \subseteq X_{\alpha\beta}^{sink}$

(ii) $g_{(\alpha)\beta} : X_{(\alpha)\beta} \to X_{\alpha\beta}$ *is continuous.*

(iii) $X_{(\alpha)\beta}$ *is an open subset of* X_β.

Proof. Given $\mathcal{F} \in X_{(\alpha)\beta} \cap X_\beta^{sink}$, for each $a \in \mathcal{A}$ there is a $D \in \mathcal{F}$ with $(D,a) = \emptyset$; consequently,

$D \cap r(\alpha\beta) \in g_{(\alpha)\beta}(\mathcal{F})$ is such that $r(D \cap r(\alpha\beta),a) = \emptyset$, hence $g_{(\alpha)\beta}(\mathcal{F}) \in X_{\alpha\beta}^{sink}$, proving (i).

To prove (ii), consider a net $\{\mathcal{F}_\lambda\}_\lambda \subseteq X_{(\alpha)\beta}$ that converges to $\mathcal{F} \in X_{(\alpha)\beta}$. For an arbitrary $D \in g_{(\alpha)\beta}(\mathcal{F})$ say $D = C \cap r(\alpha\beta)$ for some $C \in \mathcal{F}$, the convergence above ensures there is λ_0 such that $\lambda \geq \lambda_0$ implies $C \in \mathcal{F}_\lambda$, and therefore $D = C \cap r(\alpha\beta) \in g_{(\alpha)\beta}(\mathcal{F}_\lambda)$.

If on the other hand $D \in B_{\alpha\beta \backslash g(\alpha)\beta}(\mathcal{F})$, using that $g_{(\alpha)\beta}(\mathcal{F})$ is an ultrafilter there must be an element of it that does not intersect D, that is, there is $C \in \mathcal{F}$ such that $D \cap (C \cap r(\alpha\beta)) = \emptyset$, by Proposition (4.2.1). Since $B_{\alpha\beta} \subseteq \mathcal{B}_\beta$ one has $D \in \mathcal{B}_\beta$ and from $D = D \cap r(\alpha\beta)$ it can be concluded that $D \cap C = (D \cap r(\alpha\beta) \cap C = \emptyset$, and thus $D \notin \mathcal{F}$, again by Proposition (4.2.1) and using that $\mathcal{F}$ is an ultrafilter. That means that there must be an index λ such that $\lambda \geq \lambda'$ implies $D \notin \mathcal{F}_{\lambda,}$ ensuring the existence of an element $C_\lambda \in \mathcal{F}_\lambda$ with $D \cap C_\lambda = \emptyset$ for each such λ; the element $C_\lambda \cap r(\alpha\beta) \in g_{(\alpha)\beta}(\mathcal{F}_\lambda)$ satisfies

$$D \cap (C_\lambda \cap r(\alpha\beta)) = \emptyset,$$

from where it can be established that $D \notin g_{(\alpha)\beta}(\mathcal{F}_\lambda)$ for all $\lambda \geq \lambda'$. This concludes the proof that $g_{(\alpha)\beta}(\mathcal{F}_\lambda)$ converges to $g_{(\alpha)\beta}(\mathcal{F})$ and that $g_{(\alpha)\beta}$ is therefore a continuous map.

As for (iii), suppose $\mathcal{F} \in X_{(\alpha)\beta}$ and $\{\mathcal{F}_\lambda\}_\lambda \subseteq X_\beta$ is a net that converges to $\mathcal{F}$. Since $r(\alpha\beta) \in \mathcal{F}$, the pointwise convergence says there is an index λ_0 such that $\lambda \geq \lambda_0$ implies $r(\alpha\beta) \in \mathcal{F}_{\lambda,}$ and thus $\mathcal{F}_\lambda \in X_{(\alpha)\beta}$, whence $X_{(\alpha)\beta}$ is open.

Let us proceed with the problem of describing new filters by cutting labelled paths.

Consider composable labelled paths $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^*$, and an ultrafilter $\mathcal{F} \in X_{\alpha\beta}$. Note that $\mathcal{F} \subseteq \mathcal{B}_{\alpha\beta} \subseteq \mathcal{B}_\beta$, but $\mathcal{F}$ may not be a filter in $\mathcal{B}_\beta$, for $\mathcal{B}_\beta$ may contain elements above agiven element of $\mathcal{F}$ that are not in $\mathcal{B}_{\alpha\beta}$. If we add to $\mathcal{F}$ these elements, however, the resulting set is an ultrafilter in $\mathcal{B}_\beta$, as the following result shows.

Proposition (4.2.21)[450]: Let $\alpha \in \mathcal{L}^{\geq 1}$and $\beta \in \mathcal{L}^*$ be such that $\alpha\beta \in \mathcal{L}^{\geq 1}$, and $\mathcal{F} \in X_{\alpha\beta}$. Then $\uparrow_{\mathcal{B}\beta}\mathcal{F} \in X_{(\alpha)\beta}$.

Proof. Suppose $C \in \mathcal{B}_\beta$ satisfies $C \cap B = \emptyset$ for all $B \in \uparrow \mathcal{B}_\beta\mathcal{F}$; in particular, $C \cap B \neq \emptyset$ for all $B \in \mathcal{F}$. Note that $r(\alpha\beta) \in \mathcal{F}$ (since $F \in X_{\alpha\beta}$), so for any given $B \in \mathcal{F}$ one has $B \cap r(\alpha\beta) = B$, which in turn gives, for the element $(C \cap r(\alpha\beta)) \in \downarrow r(\alpha\beta) = B_{\alpha\beta}$,

$$(C \cap r(\alpha\beta)) \cap B = C \cap (r(\alpha\beta) \cap B) = C \cap B \neq \emptyset.$$

F is an ultrafilter, so $C \cap r(\alpha\beta) \in F$ and thus $C \in \uparrow B_\beta(C \cap r(\alpha\beta)) \subseteq \uparrow B_\beta F$. Proposition (4.2.1) now ensures that $\uparrow B_\beta F \in X_\beta$, and clearly $r(\alpha\beta) \in \uparrow B_\beta F$, proving $\uparrow B_\beta F \in X_{(\alpha)\beta}$ as desired.

Proposition (4.2.21) gives rise to a function $h_{[\alpha]\beta} : X_{\alpha\beta} \to X_{(\alpha)\beta}$ that associates to each ultrafilter $F \in X_{\alpha\beta}$, the ultrafilter in B_β given by

$$h_{[\alpha]\beta}(F) = \uparrow B_\beta F.$$

The following lemmas describe some of the properties of the maps.

Lemma (4.2.22)[450]: Let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta, \gamma \in \mathcal{L}^*$ with $\alpha_{\beta\gamma} \in \mathcal{L}^{\geq 1}$ be given. Then,

(i) $h_{[\beta]\gamma} \circ h_{[\alpha]\beta\gamma} = h_{[\alpha\beta]\gamma}$.

(ii) *The following diagram is commutative:*

$$\begin{array}{ccc} X_{\alpha\beta\gamma} & \xrightarrow{h_{[\alpha]\beta\gamma}} & X_{(\alpha)\beta\gamma} \\ \downarrow{\scriptstyle f_{\alpha\beta[\gamma]}} & & \downarrow{\scriptstyle f_{\beta[\gamma]}} \\ X_{\alpha\beta} & \xrightarrow[h_{[\alpha]\beta}]{} & X_{(\alpha)\beta}. \end{array}$$

Proof. For a given $F \in X_{\alpha\beta\gamma}$, since $F \subseteq\uparrow_{B_{\beta\gamma}} F$ one obtains

$$h_{[\alpha\beta]\gamma}(F) = \uparrow_{B_\gamma} F \subseteq\uparrow_{B_\gamma} (\uparrow_{B_{\beta\gamma}} F) = h_{[\beta]\gamma} \circ h_{[\alpha]\beta\gamma}(F),$$

which implies these are equal since they are both ultrafilters in B_γ. This proves (i). As for (ii), first observe that *Lemma (4.2.18).*(i) says the arrow on the right of the diagram makes sense. If $F \in X_{\alpha\beta\gamma}$ then

$$h_{[\alpha]\beta} \circ f_{\alpha\beta[\gamma]}(F) = \{D \in B_\beta \,|\, \exists C \in B_{\alpha\beta}\; with\; r(C,\gamma) \in F\; and\; C \subseteq D\}.$$

For a given $D \in h_{[\alpha]\beta} \circ f_{\alpha\beta[\gamma]}(F)$, if $C \in B_{\alpha\beta}$ is as above then $r(C,\gamma) \subseteq r(D,\gamma) \in B_{\beta\gamma}$, so that

$$r(D,\gamma) \in\uparrow_{B_{\beta\gamma}} r(C,\gamma) \subseteq\uparrow_{B_{\beta\gamma}} F = h_{[\alpha]\beta\gamma}(F),$$

and therefore $D \in f_{\beta[\gamma]} \circ h_{[\alpha]\beta\gamma}(F)$. The result now follows since both $f_{\beta[\gamma]} \circ h_{[\alpha]\beta\gamma}(F)$ and $h_{[\alpha]\beta} \circ f_{\alpha\beta[\gamma]}(F)$ are ultrafilters in B_β

Lemma (4.2.23)[450]: *Let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^*$ be such that $\alpha\beta \in \mathcal{L}^{\geq 1}$. Then,*

(i) The functions $h_{[\alpha]\beta} : X_{\alpha\beta} \to X_{(\alpha)\beta}$ and $g_{(\alpha)\beta} : X_{(\alpha)\beta} \to X_{\alpha\beta}$ are mutual inverses.

(ii) $h_{[\alpha]\beta}(X_{\alpha\beta}^{sink}) \subseteq X_\beta^{sink}$.

(iii) $h_{[\alpha]\beta} : X_{\alpha\beta} \longrightarrow X_{(\alpha)\beta}$ is continuous.

Proof. To prove (i), if $F \in X_{(\alpha)\beta}$ then $h_{[\alpha]\beta} \circ g_{(\alpha)\beta}(F)$ is an ultrafilter in B_β; also, for each $C \in F$ one has $C \in\uparrow_{B_\beta} \cap\, r(\alpha\beta) \subseteq h_{[\alpha]\beta} \circ g_{(\alpha)\beta}(F)$, which shows that $F \subseteq h_{[\alpha]\beta} \circ g_{(\alpha)\beta}(F)$, hence they are equal. On the other hand if $F \in X_{\alpha\beta}$, then

$$g_{(\alpha)\beta} \circ h_{[\alpha]\beta}(F) = \{D \cap r(\alpha\beta) \,|\, D \in B_\beta\; such\; that\; \exists C \in F\; with\; C \subseteq D\}.$$

Given $C \in \mathcal{F}$, from $B_{\alpha\beta} \subseteq \mathcal{B}_\beta$ it can be seen that $C = C \cap r(\alpha\beta)$ and thus $C \in g_{(\alpha)\beta}\; o\; h_{[\alpha]\beta}(\mathcal{F})$, establishing $\mathcal{F} \subseteq g_{(\alpha)\beta}\; o\; h_{[\alpha]\beta}(\mathcal{F})$ and again equality follows.

Now, let us prove (ii): suppose $\mathcal{F} \in X_{\alpha\beta}^{sink}$; then, for each $a \in \mathcal{A}$ there exists $D \in \mathcal{F}$ such that $r(D,a) = \emptyset$. Since $D \in \mathcal{F} \subseteq\uparrow_{\mathcal{B}\beta} \mathcal{F} = h_{[\alpha]\beta}(\mathcal{F})$ it can be concluded that for each $a \in \mathcal{A}$ there exists $D \in h_{[\alpha]\beta}(\mathcal{F})$ such that $r(D,a) = \emptyset$, which means $h_{[\alpha]\beta}(\mathcal{F}) \in X_\beta^{sink}$ as desired.

As for (iii), consider a net $\{\mathcal{F}_\lambda\}_\lambda \subseteq X_{\alpha\beta}$ that converges to $\mathcal{F} \in X_{\alpha\beta}$, and let $D \in \mathcal{B}_\beta$ be arbitrary. If $D \in h_{[\alpha]\beta}(\mathcal{F})$ then there exists $C \in \mathcal{F}$ such that $C \subseteq D$. The convergence of the $\mathcal{F}_\lambda$ say then that there is an index λ_0 such that $\lambda \geq \lambda_0$ implies $C \in \mathcal{F}_\lambda$, and hence
$D \in\uparrow_{\mathcal{B}\beta} C \subseteq\uparrow_{\mathcal{B}\beta} \mathcal{F}_\lambda = h_{[\alpha]\beta}(\mathcal{F}_\lambda)$.
On the other hand, if $D \in \mathcal{B}_{\beta\setminus h[\alpha]\beta}(\mathcal{F})$ then there exists $C \in h_{[\alpha]\beta}(\mathcal{F})$ such that $C \cap D = \emptyset$, by Proposition (4.2.1). What has been just shown above says there exists λ_0 such that $\lambda \geq \lambda_0$ implies $C \in h_{[\alpha]\beta}(\mathcal{F}_\lambda)$,.and again Proposition (4.2.1) can be used to ensure $\lambda \geq \lambda_0$implies $D \in h_{\beta\setminus h[\alpha]\beta}(\mathcal{F}_\lambda)$; it then follows that the net $\{h_{[\alpha]\beta}(\mathcal{F}_\lambda)\}_\lambda$ converges to $h_{[\alpha]\beta}(\mathcal{F})$ hence $h_{[\alpha]\beta}$ is continuous.

The functions above can be used to produce tools for working with filters in $E(S)$. Let us begin with a function for gluing labelled paths to a filter ξ in $E(S)$. Given labelled paths

$\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^{\leq\infty}$ such that $\alpha\beta \in \mathcal{L}^{\leq\infty}$, the problem is to construct a complete family of ultrafilters for $\alpha\beta$ starting from a complete family of

ultrafilters for β. Roughly, one begins with the complete family for β, say $\{\mathcal{F}_n\}_n$ with $0 \leq n \leq |\beta|$, cuts each ultrafilter $\mathcal{F}_n$ by the range of $\alpha\beta_{1,n}$, and adds new ultrafilters at the beginning of the family, one for each $\alpha_{1,i,}$ $0 \leq i \leq |\alpha|$, to obtain a complete family $\{\mathcal{J}_i\}_i$ for $\alpha\beta$ (remembering that, if the resulting family is to be complete, we have no real choice as to which filters to add at the beginning of the family).

We consider first the case $\beta = \omega$. In this case the complete family $\{\mathcal{F}_n\}_n$ for ω consists of a single filter $\mathcal{F}_0 \subseteq B$. Define

$$\mathcal{I}_{|\alpha|} = \{C \cap r(\alpha) \mid C \in \mathcal{F}_0\} = g_{(\alpha)\omega}(\mathcal{F}_0)$$

and, for $0 \leq i < |\alpha|$, set

$$\mathcal{J}_i = \{D \in \mathcal{B}_{\alpha 1,i} \mid r(D, \alpha_{i+1,|\alpha|}) \in \mathcal{I}_{|\alpha|}\} = f_{\alpha_{1,i}[\alpha_{i+1,|\alpha|}]}(\mathcal{I}_{|\alpha|}|).$$

Under a suitable condition on the filter $\mathcal{F}_0$ (Proposition (4.1.1) (i) below), the family $\{\mathcal{J}_i\}_i$ obtained is a complete family for $\alpha = \alpha\omega$.

Now, for the case where $\beta \neq \omega$: the ultrafilter $\mathcal{F}_1 \in X_{\beta_1}$ is translated $|\alpha|$ units to create the ultrafilter

$\mathcal{I}_{|\alpha|+1} \in X_{\alpha\beta 1,}$ given by

$$\mathcal{I}_{|\alpha|+1} = \{C \cap r(\alpha\beta_1) \mid C \in \mathcal{F}_1\} = g_{(\alpha)\beta 1}(\mathcal{F}_1).$$

More generally, for $1 \leq n \leq |\beta|$ $(or\ n < |\beta|$ if β is infinite) one defines

$$\mathcal{I}_{|\alpha|+1} = \{C \cap r(\alpha\beta_{1,n}) \mid C \in \mathcal{F}_n\} = g_{(\alpha)\beta_{1,n}}(\mathcal{F}_n).$$

that is, $\mathcal{F}_{n+1,} \in X_{(\alpha)\beta_{1,n+1}}$.

Also, for $0 \leq i \leq |\alpha|$, define

$$\mathcal{J}_i = f_{\alpha_{1,i}[\alpha_{i+1,|\alpha|}\,\beta_1]}(\mathcal{J}_{|\alpha|+1})$$
$$= \{D \in B_{\alpha_{1,i}} \mid r(D, \alpha_{i+1,|\alpha|}\beta_1) \in \mathcal{J}_{|\alpha|+1}\}.$$

Proposition (4.2.24)[450]: *Suppose the labelled space $(\mathcal{E}, \mathcal{L}, B)$ is weakly left-resolving, and let $\alpha \in \mathcal{L}^{\geq 1}$.*

(i) If $\mathcal{F}_0 \in X_{(\alpha)\omega}$, then $\{\mathcal{J}_i\}_i$, $0 \leq i \leq |\alpha|$ as above is a complete family of ultrafilters for α.

(ii) If $\beta \in \mathcal{L}^{\geq 1}\backslash\{\omega\}$ is such that $\alpha\beta \in \mathcal{L}^{\geq 1}$ and $\{\mathcal{F}_n\}_n$ is a complete family of ultrafilters for β such that $\mathcal{F}_1 \in X_{(\alpha)\beta_1}$, then $\{\mathcal{J}_i\}_i$ as above is a complete family of ultrafilters for $\alpha\beta$.

Proof. The proofs of both items are similar. We prove (ii): for $0 \leq i < |\alpha|$,

$$\mathcal{J}_i = f_{\alpha_1 i[\alpha_i+1,|\alpha|\beta_1]}(\mathcal{J}_{|\alpha|+1})$$
$$= f_{\alpha_1,i[\alpha_i+1]} \circ f_{\alpha_1,i+1[\alpha_i+2,|\alpha|\beta_1]}(\mathcal{J}_{|\alpha|+1}) = f_{\alpha_1,i[\alpha_i+1]}(\mathcal{J}_{i+1}),$$

and additionally $\mathcal{J}_{|\alpha|} = f_{\alpha[\beta_1]}(\mathcal{J}_{|\alpha|+1})$ by definition. On the other hand for $i > |\alpha|$, say $i = |\alpha| + n$ for $n \geq 1$, the definitions above coupled with Lemma (4.2.19).(iv) and the fact the family $\{\mathcal{F}_n\}_n$ is complete for β give

$$\mathcal{J}_i = g_{(\alpha)\beta_{1,n}}(\mathcal{F}_n) = g_{(\alpha)\beta_{1,n}}\big(f_{\beta_{1,n}[\beta_{n+1}]}(\mathcal{J}_{i+1})\big),$$
$$= f_{\alpha\beta_{1,n}[\beta_{n+1}]} \circ g_{(\alpha)\beta_{1+1n}}(\mathcal{F}_{n+1}) = f_{\alpha)\beta_{1,n}}\big(f_{\alpha\beta_{1,n}[\beta_{n+1}]}(\mathcal{J}_{i+1}\big),$$

and thus the family $\{\mathcal{J}_i\}_i$ is complete for $\alpha\beta$, as claimed.

Continuing the discussion above, let us now concentrate on the tight filters in $E(S)$: for $\beta \in \mathcal{L}^{\leq\infty}$ let $T_{(\alpha)\beta}$ denote the subset of T_β given by

$$T_{(\alpha)\beta} = \{\xi \in T_\beta \mid \xi_0 \in X_{(\alpha)\omega}\}.$$

Observe that, for $\beta = \omega, \xi_0 \in X_{(\alpha)\omega}$ is equivalent to $\xi_1 \in X_{(\alpha)\beta_1}$ and $r(\alpha) \in \xi_0$ is equivalent to $r(\alpha\beta_1) \in \xi_1$.

For a given tight filter $\xi \in T_{(\alpha)\beta}$, Proposition (4.2.24) says the complete family $\{\xi_n\}_n$ for β gives rise to a complete family $\{\mathcal{J}_i\}_i$ of ultrafilters for $\alpha\beta$, and therefore to a filter $\eta \in F_{\alpha\beta}$ associated with this family.

The purpose of the next theorem is to show this resulting filter is also tight.

Theorem (4.2.25)[450]: *Suppose the labelled space $(\mathcal{E}, \mathcal{L}, B)$ is weakly weakly left-resolving, that B is closed under relative complements, and let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^{\geq 1}$ be such that $\alpha\ \beta \in \mathcal{L}^{\leq \infty}$ If $\xi \in T_{(\alpha)\beta}$ and $\{\mathcal{J}_i\}_i$ is the complete family for $\alpha\beta$ constructed as above from $\{\xi_n\}_n$, then the filter $\eta \in F_{\alpha\beta}$ associated with $\{\mathcal{J}_i\}_i$ is tight.*

Proof. Suppose first that $\beta \in \mathcal{L}^{\infty}$. In this case $\{\xi_n\}_n$is a complete family of ultrafilters for β (by Theorems (4.2.11).(i) and (4.2.12).(ii)'), and $\xi_1 \in X_{(\alpha)\beta_1}$ since $\xi \in T_{(\alpha)\beta}$ Proposition (4.2.24) then says the family $\{\mathcal{J}_i\}_i$ constructed from $\{\xi_n\}_n$ is a complete family of ultrafilters for $\alpha\beta$, and thus the filter $\eta \in F_{\alpha\beta}$ associated with this family is tight, again by Theorems (4.2.11).(i) and (4.2.12).(ii)'.

Next, suppose $\beta \in \mathcal{L}^*$; from Proposition (4.2.10), ξ may be of one of two kinds of tight filters, and we consider each in turn: for the first kind, there exists a net $\{\mathcal{F}_\lambda\}_\lambda$ that converges to $\xi|\ \beta| \in X_{(\alpha)\beta}$. The set $X_{(\alpha)\beta}$ is open in X_β by Lemma (4.2.18).(iii), so there is an index λ_0 such that $\lambda \geq \lambda_0$ implies $\mathcal{F}_\lambda \in X_{(\alpha)\beta}$. Now $\{g_{(\alpha)\beta}(\mathcal{F}_\lambda)\}_{\lambda \geq \lambda_0}$ is a net in $X_{\alpha\beta}^{sink}$ due to Lemma (4.2.20).(i), and it converges to $\mathcal{J}_{|\alpha\beta|} = g_{(\alpha)\beta}(\xi_{|\beta|})$, by the continuity of $g_{(\alpha)\beta}$ established with Lemma (4.2.20).(ii), whence η is tight.

For the second kind, there is a net $\{(t_\lambda, \mathcal{F}_\lambda)\}_\lambda$ with $t_\lambda \in A$ and $\mathcal{F}_\lambda \in X_{\beta t_\lambda}$ for all λ such that $\{t_\lambda\}_\lambda$ converges to infinity in A and $\{f_{\beta[t_\lambda]}(\mathcal{F}_\lambda)\}_\lambda$ converges to $\xi_{|\beta|}$ in X_β. Again there is an index λ_0 such that $\lambda \geq \lambda_0$ implies $\{f_{\beta[t_\lambda]}(\mathcal{F}_\lambda) \in X_{(\alpha)\beta}$, and for these λ one must then have $\mathcal{F}_\lambda \in X_{(\alpha)\beta t_\lambda}$, by Lemma (4.2.18). The net $\{g_{(\alpha)\beta t_\lambda}(\mathcal{F}_\lambda)\}_{\lambda \geq \lambda_0}$satisfies, as given by Lemma (4.2.19).(iv),

$$f_{\alpha\beta[t_\lambda]}(g_{(\alpha)\beta t_\lambda}(\mathcal{F}_\lambda)) = g_{(\alpha)\beta}(f_{\alpha\beta[t_\lambda]}(\mathcal{F}_\lambda)),$$

and this converges to $g_{(\alpha)\beta}(\xi_{|\beta|}) = \mathcal{J}_{|\alpha\beta|}$, again showing that η is tight, for $\{t_\lambda\}_{\lambda_0}$still converges to infinity
in A.

Under the hypotheses of Theorem (4.2.25) above, it is then possible to define a function

$$G_{(\alpha)\beta} : T_{(\alpha)\beta} \longrightarrow T_{\alpha\beta}$$

taking a tight filter $\xi \in T_{(\alpha)\beta}$ to the tight filter $\eta \in T_{\alpha\beta}$ given by the theorem. Also, for $\alpha = \omega$ define $T_{(\omega)\beta} = T_\beta$ and let $G_{(\omega)\beta}$ be the identity function on T_β.

Lemma (4.2.26)[450]: *Suppose the labelled space $(\mathcal{E}, \mathcal{L}, B)$ is weakly left-resolving, that B is closed under relative complements, and let $\alpha, \beta \in \mathcal{L}^{\geq 1}$ and $\gamma \in \mathcal{L}^{\leq \infty}$ with $\alpha\beta_\gamma \in \mathcal{L}^{\leq \infty}$be given. Then,*

(i) $T_{(\alpha\beta)\gamma} \subseteq T_{(\beta)\gamma}$;

(ii) $G_{(\alpha\beta)\gamma} = G_{(\alpha)\beta\gamma}\, G_{(\beta)\gamma}$.

Proof. For (i), Lemma (4.2.19) (i) states $X_{(\alpha\beta)\gamma_1} \subseteq X_{(\beta)\gamma_1}$, at once giving $T_{(\alpha\beta)\gamma} \subseteq T_{(\beta)\gamma}$. As for (ii), consider an arbitrary $\xi \in T_{(\alpha\beta)\gamma}$, and let $\eta = G_{(\alpha\beta)\gamma}(\xi), \sigma = G_{(\beta)\gamma}(\xi)$ and $\rho = G_{(\alpha)\beta\gamma}(\sigma)$; η and ρ are both tight filters in $B_{\alpha\beta\gamma}$. For any given $n \geq 1$, note that

$$\eta_{|\alpha\beta|+n} = g_{(\alpha\beta)\gamma_1,n}(\xi_n) = g_{(\alpha)\beta\gamma_1,n} \circ g_{(\beta)\gamma_1,n}(\xi_n)$$
$$= g_{(\alpha)\beta\gamma_1,n}(\sigma_{|\beta|+n}) = \rho_{|\alpha|+|\beta|+n} = \rho_{|\alpha\beta|+n},$$

and from this one sees that η and ρ are associated with the same complete family of ultrafilters (remember, each ultrafilter in a complete family determines uniquely the ones preceding it). This means $\eta = \rho$, and (ii) follows.

Next is a function for removing labelled paths from a filter $\xi \in E(S)$. The problem now is to construct, from a complete family of ultrafilters for $\alpha\beta$, a new complete family of ultrafilters for β. This is achieved with the following result.

Lemma (4.2.27)[450]: *Let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^{\leq\infty}$ be such that $\alpha\beta \in \mathcal{L}^{\leq\infty}$ If $\{\mathcal{F}_n\}_n$ is a complete family of ultrafilters for $\alpha\beta$, then*

$$\{h_{[\alpha]\beta_1,n}(\mathcal{F}_{|\alpha|+n})\}_n$$

is a complete family of ultrafilters for β.

Proof. If $\beta = \omega$ then $h_{[\alpha]\omega}(\mathcal{F}_{|\alpha|}) \in X_{(\alpha)\omega} \subseteq X_\omega$, so there is nothing to do. Suppose then $\beta \neq \omega$; for any given $n \geq 1$, from Lemma (4.2.22).(ii) one has

$$h_{[\alpha]\beta_1,n}(\mathcal{F}_{|\alpha|+n}) = h_{[\alpha]\beta_1,n}(f_{\alpha\beta_1,n[\beta_{n+1}]}(\mathcal{F}_{|\alpha|+n+1}))$$
$$= f_{\beta_1,n[\beta_{n+1}} \circ h_{[\alpha]\beta_1,n+1}(\mathcal{F}_{|\alpha|+n+1}),$$

which is precisely the desired completeness.

Theorem (4.2.28)[450]: Suppose that the labelled space $(\mathcal{E}, \mathcal{L}, \mathrm{B})$ is weakly left-resolving, that B is closed under relative complements, and let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^{\leq\infty}$ be such that $\alpha\beta \in \mathcal{L}^{\leq\infty}$. If $\xi \in T_{\alpha\beta}$, then the filter $\eta \in F_\beta$ associated with the complete family $\{h_{[\alpha]\beta_1,n}(\xi_{n+|\alpha|})\}_n$ for β is a tight filter.

Proof. Again the proof is split into cases, using Theorem (4.2.11) and Proposition (4.2.10). Begin with the case $\beta \in \mathcal{L}^\infty$, so that ξ is an ultrafilter of infinite type; Lemma (4.2.27) says the family $\{h_{[\alpha]\beta_1,n}(\xi_{n+|\alpha|})\}_n$ of ultrafilters is indeed complete for β, hence the filter $\eta \in F_\beta$ associated with it is an ultrafilter and thus tight.

Next, consideris $\beta \in \mathcal{L}^*$ continuous,and and preserves suppose theresinks existsby a net $\{\mathcal{F}_\lambda\}_\lambda \subseteq X_{\alpha\beta}^{sink}$ Lemma (4.2.22), meaning that that h converges to $\mathcal{F}_{|\alpha\beta|} \in X_{\alpha\beta}$.The Map $h_{[\alpha]\beta}$ that converges to $h_{[\alpha]\beta}(\xi_{|\alpha\beta|})$, implying in turn that η is tight.

Finally, suppose that $\beta \in \mathcal{L}^*$ and that there exists a net $\{(t_\lambda, \mathcal{F}_\lambda)\}_\lambda$ with $t_\lambda \in A$ and $\mathcal{F}_\lambda \in X_{\alpha\beta t_\lambda}$ for all λ such that $\{t_\lambda\}_\lambda$ converges to infinity in A and $\{f_{\alpha\beta[t_\lambda]}(\mathcal{F}_\lambda)\}_\lambda$ converges to $\xi_{|\alpha\beta|}$ in $X_{\alpha\beta}$. The commutativity of the diagram in Lemma (4.2.22).(ii) ensures that

$$h_{[\alpha]\beta}(f_{\alpha\beta[t_\lambda]}(\mathcal{F}_\lambda)) = f_{\beta[t_\lambda]}(h_{[\alpha]\beta t_\lambda}(\mathcal{F}_\lambda)),$$

and from here it can be seen, using the continuity of $h_{[\alpha]\beta}$, that the net $\{(t_\lambda, h_{[\alpha]\beta t_\lambda}(\mathcal{F}_\lambda))\}_\lambda$ is such that $\{t_\lambda\}_\lambda$ converges to infinity in A and the net $\{f_{\beta[t_\lambda]}(h_{[\alpha]\beta t_\lambda}(\mathcal{F}_\lambda))\}_\lambda$ in X_β converges to $h_{[\alpha]\beta}(\xi_{|\alpha\beta|})$, whence η is tight.

Under the hypotheses of Theorem (4.2.28) one can therefore define a function$H_{[\alpha]\beta}$: $T_{\alpha\beta} \longrightarrow T_{(\alpha)\beta}$ by $H_{[\alpha]\beta}(\xi^{\alpha\beta}) = \eta^\beta$ where, for all n with $0 \leq n \leq |\beta|$,

$$\eta_n^\beta = h_{[\alpha]\beta_1,n}(\xi_{n+|\alpha|}) \in X_{(\alpha)\beta_1,n}.$$

For $\alpha = \omega$ define $H_{[\omega]\beta}$ to be the identity function over T_β.

Lemma (4.2.29)[450]: Suppose the labelled space $(\mathcal{E},\mathcal{L},\beta)$ is weakly left-resolving, that β is closed under relative complements, and let $\alpha,\beta \in \mathcal{L}^{\geq 1}$ and $\gamma \in \mathcal{L}^{\leq\infty}$ with $\alpha\beta\gamma \in \mathcal{L}^{\leq\infty}$ be given. Then

$$H_{[\beta]\gamma} \circ H_{[\alpha]\beta\gamma} = H_{[\alpha\beta]\gamma}.$$

Proof. Immediate from Lemma (4.2.22)

Theorem (4.2.30)[450]: Suppose the labelled space $(\mathcal{E},\mathcal{L},\beta)$ is weakly left-resolving, that B is closed under relative complements, and let $\alpha \in \mathcal{L}^{\geq 1}$ and $\beta \in \mathcal{L}^{\leq\infty}$ be such that $\alpha\beta \in \mathcal{L}^{\leq\infty}$. Then $\mathrm{H}_{[\alpha]\beta} \circ \mathrm{G}_{[\alpha]\beta}$ and $\mathrm{G}_{(\alpha\beta)} \circ \mathrm{H}_{[\alpha]\beta}$ are the identity maps over $\mathrm{T}_{(\alpha)\beta}$ and $\mathrm{T}_{\alpha\beta}$, respectively.

Proof. We consider the case $\beta \neq \omega$, as the case $\beta = \omega$ is analogous. Let $\xi \in T_{(\alpha)\beta}$ be arbitrary, and denote $\sigma^{\alpha\beta} = G_{(\alpha)\beta}(\xi), \eta^\beta = H_{[\alpha]\beta}(\sigma)$. For any given integer n with $1 \leq n \leq |\beta|$ (note that ξ_0 may be empty),

$$\eta_n = h_{[\alpha]\beta_1,n}(\sigma_{n+|\alpha|}) = h_{[\alpha]\beta_1,n}(g_{(\alpha)\beta_1,n}(\xi_n)) = \xi_n,$$

as a consequence of Lemma (4.2.23).(i), giving immediately $\eta = \xi$, from where it can be concluded that $H_{[\alpha]\beta} \circ G_{[\alpha]\beta}$ is the identity map over $T_{(\alpha)\beta}$.

On the other hand, begin with an arbitrary $\xi \in T_{\alpha\beta}$ and let $\eta^\beta = H_{[\alpha]\beta}(\xi), \sigma^{\alpha\beta} = G_{[\alpha]\beta}(\eta)$. If n is an integer with $1 \leq n \leq |\beta|$ then

$$\sigma_{n+|\alpha|} = g_{(\alpha)\beta_1,n}(\eta_n) = g_{(\alpha)\beta_1,n}(h_{[\alpha]\beta_1,n}(\xi_{n+|\alpha|})) = \xi_{n+|\alpha|},$$

again from Lemma (4.2.23).(i), giving $\sigma_{n+|\alpha|} = \xi_{n+|\alpha|}$ and thus $\sigma = \xi$ (for $\sigma_{1+|\alpha|} = \xi_{1+|\alpha|}$ implies $\sigma_i = \xi_i$ for all $i \leq |\alpha|$), that is, $G_{(\alpha\beta)} \circ H_{[\alpha]\beta}$ is the identity map over $T_{\alpha\beta}$.

Let $(\mathcal{E},\mathcal{L},B)$ be a weakly left-resolving labelled space such that B is closed under relative complements, and let S be its inverse semigroup. The goal is to build a representation of $C^*(\mathcal{E},\mathcal{L},B)$ that shows that an element $s_\alpha p_A s_\beta^* \in C^*(\mathcal{E},\mathcal{L},B)$is non-zero whenever $(\alpha,A,\beta) \in S\setminus\{0\}$.

To achieve this goal, we make use of the functions G and H defined.

Let $\mathcal{H} = \ell^2(T)$. We make an abuse of notation in that ξ represents both an element of T and of $\mathcal{H}$. For each $A \in B$, define

$$\mathcal{H}_A = \overline{\operatorname{span}}\{\xi \in T|(\omega,A,\omega) \in \xi\} = \overline{\operatorname{span}}\{\xi \in T|A \in \xi_0\}.$$

Also, for $a \in A$, define

$$\mathcal{H}_a = \overline{\operatorname{span}}\{\xi^\alpha \in T|\alpha_1 = a\}.$$

For a given $A \in \mathcal{B}$, let $P_A \in B(\mathcal{H})$ be the orthogonal projection onto $\mathcal{H}_A$; additionally, for $a \in \mathcal{A}$, we can use Theorem (4.2.30) to define a partial isometry $S_a \in \mathcal{B}(\mathcal{H})$ with initial space $\mathcal{H}_{r(a)}$ and final space $\mathcal{H}_{a,}$ given by $S_a\,(\xi^\beta) = G_{(a)\beta}(\xi)$, for all $\xi = \xi^\beta \in H_{r(a)}$. It follows that S_a^* is given by $S_a^*(\xi^{a\beta}) = H_{[a]\beta,}(\xi)$ for all $\xi = \xi^{a\beta} \in \mathcal{H}_a$. If $= \alpha_1 \cdots \alpha_n \in \mathcal{L}^{\geq 1}, set\ S_\alpha = S_{\alpha_1} \cdots S_{\alpha_n}$.

Proposition (4.2.31)[450]: The family $\{\mathrm{P_A},\mathrm{S_a}\}$ satisfies the relations defining $\mathrm{C}^*(\mathcal{E},\mathcal{L},\mathcal{B})$ in Definition (4.2.13).

Proof. Note that $\mathcal{H}_\emptyset = \{0\}$ since $(\omega,\emptyset,\omega) \notin \xi$ and therefore $(\omega,\emptyset,\omega) \notin \xi$ for all $\xi \in \mathrm{T}$. It follows that

$$P_\emptyset = 0.$$

Now, for $A,B \in \mathcal{B}$ and $\xi \in T$

$$P_A P_B(\xi) = [A \in \xi_0 \wedge B \in \xi_0]_\xi = [A \cap B \in \xi_0]_\xi = P_{A\cap B}(\xi)$$

where [ ] represents the boolean function that returns 1 if the argument is true and 0 otherwise; and the second equality follows from $\xi_0$being a filter. Also, using that ξ_0 is an ultrafilter and therefore a prime filter, we have

$$\begin{aligned} P_{A\cup B}(\xi) &= [A \cup B \in \xi_0]_\xi \\ &= ([A \in \xi_0] + [B \in \xi_0] - [A \cap B \in \xi_0])\,\xi \\ &= (P_A + P_B - P_{A\cap B})(\xi). \end{aligned}$$

Let $a, b \in \mathcal{A}$. It is easy to see that $S_a^* S_a = P_{r(a)}$ and $S_a^* S_b = 0 \; if \; a \neq b$. We check that

$$P_A S_a = S_a P_{r(A,a)}.$$

On one hand

$$\begin{aligned} P_A S_a(\xi^\beta) &= P_A([r(a) \in \xi_0] G_{(a)\beta}(\xi^\beta)) \\ &= [A \in G_{(a)\beta}(\xi^\beta)_0 \wedge r(a) \in \xi_0] G_{(a)\beta}(\xi^\beta). \end{aligned}$$

On the other hand

$$S_a P_{r(A,a)}(\xi^\beta) = S_a([r(A,a) \in \xi_0]\xi^\beta) = [r(A,a) \in \xi_0] G_{(a)\beta}(\xi^\beta)$$

where the last equality makes sense since $r(A,a) \subseteq r(a)$. If $r(a) \notin \xi_0$ then $r(A, a) \notin \xi_0$, so that both expressions above are zero. Suppose then that $r(a) \in \xi_0$. We have that $(G_{(a)\beta}(\xi^\beta))_1 = g_{(a)\omega}(\xi_0)$; hence,

$A \in (G_{(a)\beta}(\xi^\beta))_0 \Leftrightarrow r(A,a) \in (G_{(a)\beta}(\xi^\beta))_1 \Leftrightarrow \exists C \in \xi_0 : r(A,a) = C \cap r(a)$, so that

$$[r(A,a) \in \xi_0] = [A \in G_{(a)\beta}(\xi^\beta)_0 \wedge r(a) \in \xi_0]$$

and $P_A S_a = S_a P_{r(A,a)}$.

For the last relation, let $A \in B$ be such that $0 < \#L(A\mathcal{E}^1) < \infty$, and such that there is no $C \in \mathcal{B}$ such that $\emptyset \neq C \subseteq A \cap \mathcal{E}^0_{sink}$. We need to verify that

$$\begin{aligned} P_A &= \sum_{a\in\mathcal{L}(A\mathcal{E}^1)} S_a P_{r(A,a)} S_a^* \\ &= P_A \textstyle\sum_{a\in\mathcal{L}(A\mathcal{E}^1)} S_a S_a^*. \end{aligned}$$

Since $\mathcal{H}_a$ and $\mathcal{H}_b$ are orthogonal if $a = b$, the rightmost sum above is a sum of orthogonal projections

and therefore a projection itself. It is then sufficient to show that $H_A \subseteq \bigoplus_{\mathcal{L}(A\mathcal{E}^1)} H_a$. Suppose that $\xi^\beta \in H_A$, that is, $\xi^\beta \in T$ is such that $(\omega, A, \omega) \in \xi$. That $\beta = \omega$follows from the above condition on A and Theorem (4.2.11). Now, $(\omega, A, \omega) \in \xi$ is equivalent to $A \in \xi_0$, and in this case $\beta_1 \in \mathcal{L}(A\mathcal{E}^1)$ because $r(A, \beta_1) \in \xi_1$ and ξ_1 is a filter. Therefore, $\mathcal{E}^\beta \in \bigoplus_{\mathcal{L}(A\mathcal{E}^1)} H_a$.

Proposition (4.2.32)[450]: *If* $(\alpha, A, \beta) \in S \setminus\{0\}, then\; S_\alpha P_A S_\beta^* \neq 0$.

Proof. Observe that if $\alpha \in \mathcal{L}^{\geq 1}$ then for $\xi \in T$

$$S_\alpha(\xi^\beta) = S_{\alpha_1} \cdots S_{\alpha_\alpha}(\xi^\beta) = [r(\alpha) \in \xi_0] G_{(\alpha)\beta}(\xi)$$

and

$$S_\alpha^* = H_{(\alpha)\beta}(\xi).$$

The above equalities are also true for $\alpha = \omega$ if we define $S_\omega = Id_{\mathcal{H}}$.

Let $(\alpha, A, \beta) \in S$ be given so that $\emptyset \neq A \subseteq r(\alpha) \cap r(\beta)$.

Let us verify that $S_\alpha P_A S_\beta^* = 0$.Since $A \neq 0$, the set $\tilde{\eta} = \{(\omega, C, \omega) \mid C \in \uparrow_B A\}$ is a filter in $E(S)$ and so it is contained in a ultrafilter η^γ in $E(S)$, which is also element T. From $A \subseteq r(\beta)$, it follows that $\xi^{\beta\gamma} = G_{(\beta)\gamma}(\eta^\gamma)$ is well defined. Then $\eta = H_{[\beta]\gamma}(\xi)$ and $A \in H_{[\beta]\gamma}(\xi)_0$. It follows that

$$\begin{aligned} S_\alpha P_A S_\beta^*(\xi) &= S_\alpha P_A (H_{[\beta]\gamma}(\xi) \\ &= S_\alpha(\eta) \\ &= G_{(\alpha)\gamma}(\eta) \neq 0 \end{aligned}$$

where the third equality follows from $A \subseteq r(\alpha)$ so that $G_{(\alpha)\gamma}(\eta)$ is a well defined element of T.

Theorem (4.2.33)[450]: *Let $(\mathcal{E}, \mathcal{L}, B)$ be a weakly left-resolving labelled space whose accommodating family B is closed under relative complements. There exists a representation of $C^*(\mathcal{E}, \mathcal{L}, B)$ such that the image of $S_\alpha P_A S_\beta^*$ is not zero for all $(\alpha, A, \beta) \in S \setminus \{0\}$.*

Let $(\mathcal{E}, \mathcal{L}, B)$ be a weakly left-resolving labelled space such that B is closed under relative complements, and let $\Delta := \Delta(\mathcal{E}, \mathcal{L}, B)$ be the diagonal C^* −subalgebra of $C^*(\mathcal{E}, \mathcal{L}, B)$, as in Definitions (4.2.17) and (4.2.13). The spectrum of this C^* −subalgebra, $\hat{\Delta}$, is homeomorphic to T.

Proposition (4.2.34)[450]: For each , $\varphi \in \hat{\Delta}$, the set

$$\xi = \{(\alpha, A, \alpha) \in E(S) | \varphi(s_\alpha p_A s_\alpha^*) = 1\}$$

is a tight filter. In particular, the map $\Phi: \hat{\Delta} \to T$ *given by*

$$\Phi(\varphi) = \{(\alpha, A, \alpha) \in E(S) | \varphi(s_\alpha p_A s_\alpha^*) = 1\}$$

is well defined.

Proof. Observe that $(\alpha, A, \alpha) \leq (\beta, B, \beta)$ in $E(S)$ if and only if $s_\alpha p_A s_\alpha^* \leq s_\beta p_B s_\beta^*$ in $C^*(\mathcal{E}, \mathcal{L}, B)$.Using that φ is a $*$_homomorphism, it then follows that ξ is a filter in $E(S)$. We have to prove that ξ is tight. Let α be the labelled path associated with ξ.

First, let us consider the case that ξ is of infinite type. By Theorem (4.2.9), it is sufficient to show that ξ_n is an ultrafilter in the Boolean algebra $B_{\alpha_{1,n}}$ for each $n > 0$. In order to establish this, observe that

$(\alpha_{1,n}, r(\alpha_{1,n}), \alpha_{1,n}) \in \xi$, so that $\varphi(s\alpha_{1,n} p r_{(\alpha_{1,n})} s_{\alpha_{1,n}}) = 1$. If $A \in B_{\alpha_{1,n}}$, then $pr_{(\alpha_{1,n})} = p_A + pr_{(\alpha_{1,n})\setminus A}$. It follows that

$$1 = \varphi(s_{\alpha_{1,n}} p_{r(\alpha_{1,n})} s_{\alpha_{1,n}}) = \varphi(s\alpha_{1,n} p_{r(\alpha_{1,n})} s_{\alpha_{1,n}}) + \varphi(s_{\alpha_{1,n}} p_{r(\alpha_{1,n})\setminus A} s_{\alpha_{1,n}})$$

and hence $\varphi\left(s_{\alpha_{1,n}} p_{r(\alpha_{1,n})\setminus A} s_{\alpha_{1,n}}\right) = 1$ or $\varphi\left(s_{\alpha_{1,n}} p_A s_{\alpha_{1,n}}\right) = 1$. That means that $A \in \xi_n$ or $p_{r(\alpha_{1,n})} \setminus A \in \xi_n$, that is, ξ_n is an ultrafilter.

For the case that ξ is of finite type, we use (ii) of Theorem (4.2.11). If $|\alpha| > 0$, the same argument as above shows that $\xi_{|\alpha|}$ is an ultrafilter. If $|\alpha| = 0$, suppose by contradiction that ξ_0 is not an ultrafilter. Then there exists $C \in B \setminus \xi_0$such that $C \cap A \neq \emptyset$ for all $A \in \xi_0$. For a fixed $A \in \xi_0$.

$$1 = \varphi_{(p_A)} = \varphi_{(p_A)\setminus C} + p_{A\cap C}) = \varphi(\varphi_{p_A \setminus C}) + \varphi(p_{A\cap C}) = \varphi(\varphi_{p_A\setminus C}),$$

where the last equality holds because $A \cap C \subseteq C \notin \xi_0$. Hence $A \setminus C \in \xi_0$, but $(A \setminus C) \cap C = \emptyset$ which is contradiction. So, in all cases, $\xi_{|\alpha|}$ is an ultrafilter . Assume now that ξ is not tight; by (ii) of Theorem (4.2.11) there exists $A \in \xi_{|\alpha|}$ such that $\mathcal{L}(A\mathcal{E}^1)$ is finite and there is no $B \in B_\alpha$ with $\emptyset \neq B \subseteq A \cap \mathcal{E}^0_{sink}$. In particular $A \cap \mathcal{E}^0_{sink} = \emptyset$ so that $\# \mathcal{L}(A\mathcal{E}^1) > 0$ and

$$p_A = \sum_{b\in\mathcal{L}(A\mathcal{E}^1)} s_b p_{r(A,b)} s_b^*$$

holds. Since $(\alpha, A, \alpha) \in \xi$,

$$\varphi(s_\alpha p_A s_\alpha^*) = \sum_{b\in\mathcal{L}(A\mathcal{E}^1)} s_b p_{r(A,b)} s \varphi\left(s_\alpha s_b p_{r(A,b)} s_b^* s_\alpha^*\right) ,$$

which implies that $(\alpha b, r(A,b), \alpha b) \in \xi$ for some $b \in \mathcal{L}(A\mathcal{E}^1)$; but this contradicts the fact that α is the word associated to ξ. Therefore, ξ is tight.

To construct the inverse of Φ from the above proposition, we have to show that if $\xi \in T$ then there exists an element $\varphi \in \hat{\Delta}$ such that

$$\varphi(s_\alpha p_A s_\alpha^*) = [(\alpha, A, \alpha) \in \xi].$$

We would like to simply extend the above expression linearly to span$\{s_\alpha p_A s_\alpha^* | \alpha \in \mathcal{L}^*$ and $A \in B_\alpha\}$ but, in doing so, care must be taken to ensure the result is indeed a well-defined linear map. In order to show that this can be done, and that the resulting map extends to an element of $\hat{\Delta}$,we control the norm of a finite linear combination on elements of the form $s_\alpha p_A s_\alpha^*$ by rewriting the sum as a finite linear combination of orthogonal projections. We follow some of the ideas of [192].

Lemma (4.2.35)[450]: Let $F \subseteq E(S) \setminus \{0\}$ be a finite set such that for all $u, v \in F, uv = 0, u \leq v$ or $v \leq u$. For each $u = (\alpha, A, \alpha) \in F$, define

$$q_u^F = q_{(\alpha,A,\alpha)}^F := s_\alpha p_A s_\alpha^* \prod_{\substack{(\beta,B,\beta)\in F \\ (\beta,B,\beta)<(\alpha,A,\alpha)}} \left(s_\alpha p_A s_\alpha^* - s_\beta p_B s_\beta^*\right).$$

Then, for all $u, v \in F$ with $u \neq v$, the projections q_u^F and q_v^F are mutually orthogonal projections in span$\{s_\beta p_B s_\beta^* (\beta, B, \beta) \in F\}$. Also for $(\alpha, A, \alpha) \in F$

$$s_\alpha p_A s_\alpha^* = \sum_{\substack{u\in F \\ u\leq(\alpha,A,\alpha)}} q_u^F. \tag{2}$$

Proof. For all $u \in F, q_u^F$ is a product of commuting projections and therefore a projection in span$\{s_\beta p_B s_\beta^*\ (\beta, B, \beta) \in F\}$. Let $u = (\alpha, A, \alpha)$ and $v = (\beta, B, \beta)$ be elements of F such that $u \neq v$. If $uv = 0$, then $s_\alpha p_A s_\alpha^* s_\beta p_B s_\beta^* = 0$, so that $q_u^F q_v^F = 0$. If $u < v$, then $(s_\beta p_B s_\beta^* - s_\alpha p_A s_\alpha^*)$ is a factor of q_v^F. Since $s_\alpha p_A s_\alpha^*$ is a factor of q_u^F and $s_\alpha p_A s_\alpha^* s_\beta p_B s_\beta^* = s_\alpha p_A s_\alpha^*$,we have that $q_u^F q_v^F = 0$. The case $v < u$ is analogous.

To prove (2), we use induction on $\#F$. The result is immediate if $\#F = 1$. Let $n > 1$ and suppose that the result is true for all F with $\#F < n$. Let $F \subseteq E(S)$ be as in the hypothesis of the lemma, and with $\#F = n$. Chose a minimal element (γ, C, γ) in F and define $G = F \setminus \{(\gamma, C, \gamma)\}$. Since (γ, C, γ) is minimal in F then

$$\sum_{\substack{u\in F \\ u\leq(\gamma,C,\gamma)}} q_u^F = q^F_{(\gamma,C,\gamma)=s_\gamma PCs_\gamma^*},$$

that is, (2) holds for (γ, C, γ).

Observe that, for a given $(\alpha, A, \alpha) \in G$,

$$q^F_{(\alpha,A,\alpha)} = \begin{Bmatrix} q^G_{(\alpha,A,\alpha)}, & if\ (\alpha,A,\alpha)(\gamma,C,\gamma) = 0; \\ q^G_{(\alpha,A,\alpha)} - q^G_{(\alpha,A,\alpha)^{S_\gamma P_C S^*_\gamma}}, & if\ (\gamma,C,\gamma) \le (\alpha,A,\alpha). \end{Bmatrix}$$

Indeed, the above equality is trivially true if $(\alpha,A,\alpha)(\gamma,C,\gamma) = 0$ and, in the case $(\gamma,C,\gamma) \le (\alpha,A,\alpha)$, it holds since

$$q^F_{(\alpha,A,\alpha)} = S_\gamma P_C S^*_\gamma \qquad \prod_{\substack{(\beta,B,\beta)\in F \\ (\beta,B,\beta)<(\alpha,A,\alpha)}} (S_\alpha P_A S^*_\alpha - S_\beta P_A S^*_\beta)$$

$$S_\alpha P_A S^*_\alpha\ (S_\alpha P_A S^*_\alpha - S_\alpha P_A S^*_\alpha) \quad \prod_{\substack{(\beta,B,\beta)\in G \\ (\beta,B,\beta)<(\alpha,A,\alpha)}} (S_\alpha P_A S^*_\alpha - S_\beta P_A S^*_\beta)$$

$$= q^G_{(\alpha,A,\alpha)} - q^G_{(\alpha,A,\alpha)^{S_\alpha P_A S^*_\alpha}}$$

Now, if $(\alpha,A,\alpha)(\gamma,C,\gamma) = 0$, then

$$\sum_{\substack{u\in F \\ u\le(\alpha,A,\alpha)}} q^F_u = \sum_{\substack{u\in G \\ u\le(\alpha,A,\alpha)}} q^G_u = {}^{S_\alpha P_A S^*_\alpha},$$

where the last equality follows from the induction hypothesis. If $(\gamma,C,\gamma) \le (\alpha,A,\alpha)$, then

$$\sum_{\substack{u\in F \\ u\le(\alpha,A,\alpha)}} q^F_u = s_\gamma p_C s^*_\gamma + \sum_{\substack{u\in G \\ u\le(\alpha,A,\alpha)}} q^F_u$$

$$= s_\gamma p_C s^*_\gamma + \sum_{\substack{u\in G \\ u\le(\alpha,A,\alpha)}} \left(q^G_u - q^G_u s_\gamma p_C s^*_\gamma\right)$$

$$= s_\gamma p_C s^*_\gamma + s_\alpha p_A s^*_\alpha - s_\alpha p_A s^*_\alpha s_\gamma p_C s^*_\gamma = s_\alpha p_A s^*_\alpha,$$

where the second equality follows since $q^F_u = q^G_u - q^G_u s_\gamma p_C s^*_\gamma$ even when $u \cdot (\gamma,C,\gamma) = 0$ and the third equality follows from the induction hypothesis. Thus, (2) holds.

Lemma (4.2.36)[450]: For all finite $F \subseteq E(S)\setminus\{0\}$, there exists $F' \subseteq E(S)\setminus\{0\}$ such that F' satisfies the hypothesis of Lemma (4.2.35) and the following conditions*:*

(i) for all $(\alpha,A,\alpha) \in F$, there exist $(\alpha, A_1,\alpha),\dots,(\alpha, A_n,\alpha) \in F'$ such that A is the union of $A_1,\dots, A_n$;

(ii) the labelled paths that appear in elements of F' are the same as those that appear in elements of F;

(iii) if $(\alpha,A,\alpha),(\alpha,B,\alpha) \in F'$ and $A \ne B$ then $A \cap B = \emptyset$.

Proof. For each finite $F \subseteq E(S)\setminus\{0\}$, define $m = \max\{|\alpha| \mid (\alpha,A,\alpha) \in F\}$.
We prove the lemma by induction on m. If $m = 0$, then

$$F = \{(\omega,B_1,\omega),\dots,(\omega,B_l,\omega)\}.$$

Define

$$I = \left\{\bigcap_{i\in I_1} B_i \setminus \bigcap_{i\in I_2} B_j \,|I_1 \cup I_2 = \{1,\dots,l\}, I_1 \cap I_2 = \emptyset\right\}\setminus\{\emptyset\}$$

and $F = \{(\omega,B,\omega) \mid B \in I\}$. Clearly, F' satisfies the conditions in the statement.
For $m > 0$, suppose that the result is true for all finite $G \subseteq E(S)$ with $max\{|\alpha| \mid (\alpha,A,\alpha) \in G\} < m$. Let us write $F = G_1 \cup G_2$ where $G_1 = \{(\alpha,A,\alpha) \in F \mid |\alpha| < m\}$ and $G_2 = \{(\alpha,A,\alpha) \in F \mid |\alpha| = m\}$. By the induction hypothesis there exists G'_1 associated to G_1 as in the statement. Denote by L_2 the set of all $\alpha \in \mathcal{L}^*$ such that $(\alpha,A,\alpha) \in$

G_2 for some $A \in B$. Fix $\alpha \in L_2$, define $J_\alpha = \{r(A,\alpha'')|(\alpha',A,\alpha') \in G'_1 \cup G_2$ and $\alpha'\alpha'' = \alpha\}$
and consider I_α constructed from J_α as I was constructed from $\{B_1,\dots,B_l\}$ in the case $m = 0$. Finally, define $F_2' = \cup_{\alpha \in L_2} \{(\alpha,B,\alpha) \in E(S) \mid B \in I_\alpha\}$ and $F' = G_1' \cup F_2'$, observing that $0 \notin F'$.

Let $u = (\alpha,A,\alpha)$ and $v = (\beta,B,\beta)$ be elements of F'. If u and v are both elements of either G_1' or both elements of F_2' then, by the definitions of G_1' and F_2', u and v are such that $uv = 0, u \le v$ or $v \le u$. Suppose, then, that $u \in F_2'$ and $v \in G_1'$ so that $|\beta| < |\alpha|$. If α and β are not comparable then $uv = 0$. Otherwise, $\alpha = \beta\alpha'$ and, by the construction of I_α, $A \subseteq r(B,\alpha')$ or $A \cap r(B,\alpha') = \emptyset$ (observe that $r(B,\alpha') \in J_\alpha$ In this case $u \le v$ or $uv = 0$.

The other conditions from the statement are easily verified.

Lemma (4.2.37)[450]: *Let $\xi \in T$ and a finite $F \subseteq E(S)\backslash\{0\}$ be such that $\xi \cap F \neq \emptyset$ and for all $u,v \in F, uv = 0, u \le v$ or $v \le u$. Let also $w = min(\xi \cap F)$. Then there exists a non-zero $z \in E(S)$ such that $z \le w$ and $zu = 0$ for all $u \in F$ with $u < w$.*

Proof. The result is trivial if there is no $u \in F$ with $u < w$ (take $z = w$). Suppose, then, that there exists at least one such u. Let α be the word associated to ξ. Since $w \in \xi, w = (\alpha_{1,l},A,\alpha_{1,l})$ for some $l \ge 0$ and $A \in B$. We consider the cases given by Theorem (4.2.11).

Case (i): ξ^α is of infinite type. Let n be the greatest of all lengths of labelled paths β that appears in an element $v = (\beta,B,\beta) \in F$ with $v < w$ and observe that $\iota \le n$. For an element $u \in F$ with $u < w$ of the form $u = (\alpha_{1,m},B,\alpha_{1,m})$, we have that $u \notin \xi$ and hence $r(B,\alpha_{m+1,n}) \notin \xi_n$ since $\{\xi_n\}_n$ is a complete family. Since ξ_n is an ultrafilter, there exists $C_u \in \xi_n$such that $C_u \cap r(B,\alpha_{m+1,n}) = \emptyset$. Let $C = \cap_{u\in F, u<w} C_u \in \xi_n$and define $z = (\alpha_{1,n}, C \cap r(A,\alpha_{l+1,n}),\alpha_{1,n})$, which is non-zero because $C \cap r(A,\alpha_{l+1,n}) \in \xi_n$. Then $z \le w$ and it is easily verified that $zu = 0$ for all $u \in F$ with $u < w$.

Case (ii): ξ^α is of finite type. Using a similar argument as the previous case we find $C \in \xi_{|\alpha|}$ such that for all $u \in F$ with $u < w$ of the form $= (\alpha_{1,m},B,\alpha_{1,m})$, we have that $C \cap r(B,\alpha_{m+1,|\alpha|}) = \emptyset$. Define $D = C \cap r(A,\alpha_{l+1,|\alpha|})1 \in \xi_{|\alpha|}$.

Case (ii)(a): $\mathcal{L}(D\mathcal{E}^1)$ is infinite. Choose $b \in \mathcal{L}(D\mathcal{E}^1)$ be a letter that is different from $\beta_{|\alpha|+1}$ for all labelled paths β such that $|\beta| \ge |\alpha| + 1$ and that it appears in an element $v = (\beta,B,\beta) \in F$. Define $z = (\alpha b, r(D,b),\alpha b)$. By construction this z satisfies all of the conditions in the statement.

Case (ii)(b): there exists $G \in B_\alpha$ such that $\emptyset \neq G \subseteq D \cap \mathcal{E}^0_{sink}$. Define $z = (\alpha,G,\alpha)$ so that z is non-zero and $z \le w$. Let $u = (\beta,B,\beta) \in F$ be such that $u < w$. If β is not comparable with α then it is immediate that $zu = 0$. If β is a beginning of α then $zu = 0$ by the construction of z. Finally, if $\beta = \alpha\gamma$ for some $\gamma \in \mathcal{L}^{\ge 1}$, i.e. $\gamma \neq \omega$, then $zu = 0$ because $r(G,\gamma) = \emptyset$.

Lemma (4.2.38)[450]: Let $\xi \in T$ be given and let $F \subseteq E(S)\backslash\{0\}$ be a finite set. For each $u \in F$, let $\lambda_u \in \mathbb{C}\backslash\{0\}$.
Then

$$\left|\sum_{u\in F\cap\xi}\lambda_u\right| \le \left\|\sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}s_\alpha p_A s_\alpha^*\right\|$$

Proof. The result is trivial if $F \cap \xi = \emptyset$, so suppose $F \cap \xi \neq \emptyset$. Let F' be the set constructed from F as in Lemma (4.2.35). Observe that $F' \cap \xi \neq \emptyset$; indeed, for a given $(\alpha, A, \alpha) \in F \cap \xi$ there exist $(\alpha, B_1, \alpha), \ldots, (\alpha, B_m, \alpha) \in F'$ that A is the disjoint union of $B_1, \ldots, B_m$. Since $A \in \xi_{|\alpha|}$ and $\xi_{|\alpha|}$ is a prime filter, there exists $i_0 \in \{1, \ldots, m\}$ such that $B_{i_0} \in \xi_{|\alpha|}$and therefore $(\alpha, B_{i_0}, \alpha) \in F' \cap \xi$. If follows that $F' \cap \xi \neq \emptyset$, as claimed.

For each $(\gamma, C, \gamma) \in F'$ and $(\alpha, A, \alpha) \in F$ with $(\alpha, A, \alpha) \ge (\gamma, C, \gamma)$, define

$$\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)} = \#\{(\alpha, B, \alpha) \in F' | (\alpha, A, \alpha) \ge (\alpha, B, \alpha) \ge (\gamma, C, \gamma)\}.$$

Let us prove that $\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)} \le 1$: let $(\alpha, B, \alpha), (\alpha, B', \alpha) \in F'$ be such that $(\alpha, A, \alpha) \ge (\alpha, B, \alpha) \ge (\gamma, C, \gamma)$ and $(\alpha, A, \alpha) \ge (\alpha, B', \alpha) \ge (\gamma, C, \gamma)$. Then, in the semilattice $E(S)$, $(\alpha, B, \alpha)(\alpha, B', \alpha) = (\alpha, B \cap B', \alpha) \ge (\gamma, C, \gamma) \neq 0$ and hence $B \cap B' \neq \emptyset$. Therefore, by property (iii) of F' in Lemma (4.2.35), we must have $B = B'$. It follows that $\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)} \le 1$.

Let $w = \min(F' \cap \xi)$. if $(\alpha, B_{i_0}, \alpha)$ is as in the first paragraph of the proof, then $w \le (\alpha, B_{i_0}, \alpha) \le (\alpha, A, \alpha)$ and so $\eta_{w,(\alpha,A,\alpha)} \ge 1$, hence $\eta_{w,(\alpha,A,\alpha)} = 1$.

Therefore,

$$\begin{aligned}
\sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}s_\alpha p_A s_\alpha^* &\overset{(1)}{=} \sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}s_\alpha\left(\sum_{\substack{(\alpha,B,\alpha)\in F'\\ B\subseteq A}}p_B\right)s_\alpha^* \\
&= \sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}\sum_{\substack{(\alpha,B,\alpha)\in F'\\ B\subseteq A}}s_\alpha p_B s_\alpha^* \\
&\overset{(2)}{=} \sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}\sum_{\substack{(\alpha,B,\alpha)\in F'\\ B\subseteq A}}\left(\sum_{\substack{(\gamma,C,\gamma)\in F'\\ (\gamma,C,\gamma)\le(\alpha,B,\alpha)}}q_{(\gamma,C,\gamma)}^{F'}\right) \\
&\overset{(3)}{=} \sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}\sum_{\substack{(\gamma,C,\gamma)\in F'\\ (\gamma,C,\gamma)\le(\alpha,B,\alpha)}}\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)}q_{(\gamma,C,\gamma)}^{F'} \\
&\overset{(4)}{=} \sum_{(\gamma,C,\gamma)\in F'}\left(\sum_{\substack{(\alpha,B,\alpha)\in F\\ (\alpha,B,\alpha)\le(\gamma,C,\gamma)}}\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)}\lambda_{(\alpha,A,\alpha)}\right)q_{(\gamma,C,\gamma)}^{F'}
\end{aligned}$$

The equalities above are justified as follows: (1) is a consequence of Lemma (4.2.35), (2) follows from Lemma (4.2.35), (3) is due to the definition of $\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)}$, and (4) is true since the sums on both sides are over all pairs $(\alpha, A, \alpha) \in F, (\gamma, C, \gamma) \in F'$ such that $(\gamma, C, \gamma) \le (\alpha, A, \alpha)$.

Hence

$$\left\|\sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}s_\alpha p_A s_\alpha^*\right\| = \left\|\sum_{(\gamma,C,\gamma)\in F'}\left(\sum_{\substack{(\alpha,B,\alpha)\in F\\(\alpha,B,\alpha)\le(\gamma,C,\gamma)}}\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)}\lambda_{(\alpha,A,\alpha)}\right)q^{F'}_{(\gamma,C,\gamma)}\right\|$$

$$\overset{(5)}{=}\max_{\substack{(\gamma,C,\gamma)\in F'\\ q^{F'}_{(\gamma,C,\gamma)}\neq 0}}\left|\sum_{\substack{(\alpha,B,\alpha)\in F\\(\alpha,B,\alpha)\ge(\gamma,C,\gamma)}}\eta_{(\gamma,C,\gamma),(\alpha,A,\alpha)}\lambda_{(\alpha,A,\alpha)}\right|$$

$$\overset{(6)}{=}\left|\sum_{\substack{(\alpha,B,\alpha)\in F\\(\alpha,B,\alpha)\ge w}}\eta_{w,(\alpha,A,\alpha)}\lambda_{(\alpha,A,\alpha)}\right|$$

$$\overset{(7)}{=}\left|\sum_{(\alpha,B\alpha)\in F\cap\xi}\eta_{w,(\alpha,A,\alpha)}\lambda_{(\alpha,A,\alpha)}\right|$$

$$\overset{(8)}{=}\left|\sum_{(\alpha,B\alpha)\in F\cap\xi}\lambda_{(\alpha,A,\alpha)}\right|,$$

where (5) is due to the fact that the projections $q^{F'}_{(\gamma,C,\gamma)}$ are pairwise orthogonal (Lemma (4.2.35)), (6), is true since ξ is a filter and (8) is a consequence of $\eta_{w,(\alpha,A,\alpha)} = 1$.

Proposition (4.2.39)[450]: *For each $\xi \in T$ there is a unique $\varphi \in \widehat{\Delta}$ such that $\varphi(s_\alpha p_A s_\alpha^*) = [(\alpha,A,\alpha) \in \xi]$ for all $(\alpha,A,\alpha) \in E(S)$.*

Proof. An element $x \in \operatorname{span}\{s_\alpha p_A s_\alpha^* \,|(\alpha,A,\alpha) \in E(S)\}$ can be written as

$$x = \sum_{(\alpha,A,\alpha)\in F}\lambda_{(\alpha,A,\alpha)}s_\alpha p_A s_\alpha^*$$

for some finite $F \subseteq E(S)$. By Lemma (4.2.39),

$$\varphi(x) = \sum_{(\alpha,A,\alpha)\in F\cap\xi}\lambda_{(\alpha,A,\alpha)}$$

gives a well defined continuous linear map from span $\{s_\alpha p_A s_\alpha^* \,|(\alpha,A,\alpha) \in E(S)\}$ into $\mathbb{C}$. It is easily verified that φ preservers products so that it extends to an element $\varphi \in \widehat{\Delta}$ that satisfies the equality from the statement. The uniqueness is immediate.

Theorem (4.2.40)[450]: *Let $(E,\mathcal{L},B)$ be a weakly left-resolving labelled space such that B is closed under relative complements. Then, there exists a homeomorphism between T and $\widehat{\Delta}$.*

Proof. Putting together Propositions (4.2.34) and (4.2.39), we have a bijection $\Phi\colon \widehat{\Delta} \to T$ given by

$$\Phi(\varphi) = \{(\alpha,A,\alpha) \in E(S) | \varphi(s_\alpha p_A s_\alpha^*) = 1\}.$$

Since the topologies on $\widehat{\Delta}$ and T are both given by pointwise convergence, it follows that Φ is continuous. A standard $\varepsilon/3$ argument shows that Φ^{-1} is continuous.

We present an example of a labelled space whose C^*−algebra given by Definition (4.2.13) is a non-trivial quotient of the C^*−algebra considered in a preprint of [181] (see Remark 4.2.14). To see this, we construct a representation of the C^*−algebra based on this alternative definition such that the images of the generating projections and partial isometries do not satisfy the additional relations given in Definition (4.2.13).

From now on, fix a labelled space $(E, \mathcal{L}, B)$ whose labelled graph is left-resolving. Let Y be an infinite set such that $\#Y \geq max\{\#\varepsilon^0, \#\varepsilon^1\}$. For each $e \in \varepsilon^1$, let E_e be a copy of Y and, for each $v \in \varepsilon^0 \backslash \varepsilon^0_{sink}$, let D_v be also a copy of Y, so that all copies of Y are pairwise disjoint. For each $v \in \varepsilon^0 \backslash \varepsilon^0_{sink}$, define[1] $D_v = \sqcup_{e \in s^{-1}(v)} E_e$. It is easy to see that the $D_v$vare pairwise disjoint. Now, for each $e \in \varepsilon^1$, choose a bijection $h_e: D_{r(e)} \to {E_e}^2$.For each letter $a \in A$, the union $\cup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)}$is disjoint, since the labelled graph is left-resolving. Thus, we can define a bijection $h_e{:}\sqcup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)} \to \sqcup_{e \in \mathcal{L}^{-1}(a)} E_e$ by gluing the functions h_e, where $\mathcal{L}(e) = a$. Finally, set $X = \sqcup_{v \in \varepsilon^0} D_v$.

Consider the Hilbert space $\ell^2(X)e$ let $\{\delta_x\}_{x \in X}$ be its canonical basis.

For each letter $a \in A$, define

$$S_a: \ell^2(X) \to \ell^2(X)$$
$$\delta_x \longmapsto [x \in \sqcup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)}]\delta_{h_a(x)}.$$

recalling that [ ] represents the boolean function that returns 1 if the argument is true and 0 otherwise. In this way, S_a is a partial isometry with $\ell^2(\sqcup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)})$ as its initial space and $\ell^2(\sqcup_{e \in \mathcal{L}^{-1}(a)} E_e)$ as its final space.

For each $A \subseteq \varepsilon^0$, define

$$P_A: \ell^2(X) \to \ell^2(X)$$
$$\delta_x \longmapsto [x \in \sqcup_{v \in A} D_v]\delta_x.$$

Thus, P_A is the projection onto $\ell^2(\sqcup_{v \in A} D_v)$.

Proposition (4.2.41)[450]: *The operators* $\{S_a\}_{a \in A}$ *and* $\{P_A\}_{A \in B}$ satisfy the conditions of Definition (4.2.13), replacing item (iv) with

(iv) For every $A \in B$ such that $0 < \#\mathcal{L}(A\varepsilon^1) < \infty$ *and* $A \cap \varepsilon^0_{sink} = \emptyset$,

$$p_A = \sum_{a \in \mathcal{L}(A\varepsilon^1)} s_\alpha p_{r(A,a)} s_\alpha^*$$

Proof. We already know that S_a is a partial isometry and P_A is a projection. We only show items (ii) and (iv), since (i) and (iii) are trivial.

Observe that

$$P_A S_a(\delta_x) = [x \in \sqcup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)}][h_a(x) \in \sqcup_{v \in A} D_v]\delta_{h_a(x)}$$

and

$$S_a P_{r(A,a)}(\delta_x) = [x \in \sqcup_{v \in r(A,a)} D_v][x \in \sqcup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)}]\delta_{h_a(x)}$$

Thus, to see (ii), it suffices to show that $[h_a(x) \in \sqcup_{v \in A} D_v] = [x \in \sqcup_{v \in r(A,a)} D_v]$ whenever $\left[x \in \sqcup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)}\right] = 1$. Indeed, if $x \in \sqcup_{v \in r(A,a)} D_v$, then there exists $e \in \mathcal{L}^{-1}(a)$with $s(e) \in A$ such that $x \in D_{r(e)}$. Therefore, $h_a(x) \in E_e \subseteq D_{s(e)} \subseteq \sqcup_{v \in A} D_{r(e)}$. On the other hand, since $x \in \sqcup_{e \in \mathcal{L}^{-1}(a)} D_{r(e)}$, then there exists $e \in \mathcal{L}^{-1}(a)$ such that $x \in D_{r(e)}$and, hence, $h_a(x) \in E_e$. Thus, if $h_a(x) \in \sqcup_{v \in A} D_v$, then $E_e \in \sqcup_{v \in A} D_v$, which says that $s(e) \in A$. In other words, $r(e) \in r(A, a)$, showing that $x \in D_{r(e)} \subseteq \sqcup_{v \in r(A,a)} D_v$.

Let $A \in B$. be such that $\mathcal{L}(A\varepsilon^1)$ is finite and $\cap\, \varepsilon^0_{sink} = \emptyset$. Now that (ii) has been established, proving

(iv) is equivalent to showing that $s_a s_a^*$ and $s_b s_b^*$ are orthogonal projections for $a, b \in \mathcal{L}(A\varepsilon^1)$ with $a \neq b$, and

$$P_A \leq \sum_{a\in\mathcal{L}(A\varepsilon^1)} s_a s_a^*.$$

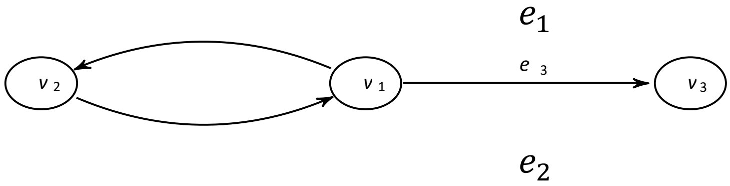
By item (iii), it is clear that $s_a s_a^*$ and $s_b s_b^*$ are orthogonal if $a \neq b$. The operator P_A is the projection onto $\ell^2(\sqcup_{v\in A} D_v)$and the operator $\sum_{a\in\mathcal{L}(A\varepsilon^1)} s_a s_a^*$ is the projection onto $\ell^2(\sqcup_{a\in\mathcal{L}(A\varepsilon^1)}\sqcup_{e\in\mathcal{L}^{-1}(a)} E_e)$. Since $A \cap \varepsilon_{sink}^0 = \emptyset$, then $\sqcup_{v\in A} D_v = \sqcup_{v\in A}\sqcup_{e\in s^{-1}(v)} E_e$ and it is clearly contained in $\sqcup_{a\in\mathcal{L}(A\varepsilon^1)}\sqcup_{e\in\mathcal{L}^{-1}(a)} E_e$.
Therefore,

$$P_A \leq \sum_{a\in\mathcal{L}(A\varepsilon^1)} s_a s_a^*.$$

This Proposition ensures that there exists a homomorphism from $C^*(E,\mathcal{L},B)$ (with the alternative item (iv)) to $B(\ell^2(X))$ which sends s_a to S_a and p_A to P_A. To simplify the writing, we refer to Definition (4.2.13) with the alternative item (iv) as the *alternative* definition.

Now, we are ready to see the promised example.

Example (4.2.42)[450]: Consider the graph below.



For each arrow, assign the label a and consider the family

$$B = \{\emptyset,\{v_1\},\{v_2,v_3\},\{v_1,v_2,v_3\}\},$$

which is easily seen to be accommodating and closed under relative complements.
The C^* −algebra of this labelled space (under any of the two given definitions) is generated by four elements: s_a, $p_{\{v_1\}}$, $p_{\{v_2,v_3\}}$ and $p_{\{v_1,v_2,v_3\}}$. Clearly, using (i), (ii) and (iii), we see that

$$p_{\{v_1\}} + p_{\{v_2,v_3\}} = p_{\{v_1,v_2,v_3\}} = s_a^* s_a = 1.$$

Observe that $A = \{v_1,v_2,v_3\}$ does not satisfy $A \cap \varepsilon_{sink}^0 = \emptyset$, and there is no $B \in \mathcal{B}$ such that $\emptyset \neq B \subseteq A \cap \varepsilon_{sink}^0$. This means there is a relation in Definition (4.2.13) that does not appear in the alternative definition.

Now, suppose we are under Definition (4.2.13). Since $\{v_1,v_2,v_3\}$satisfies the conditions of item (iv), we conclude that $s_a s_a^* = 1$. Furthermore, $\{v_1\} = r(\{v_1,v_2\}, a)$, $\{v_2,v_1\} = r(\{v_1\},a)$ and $\{v_1,v_2,v_3\} = r(\{v_1,v_2,v_3\},a)$, which says that every set in $\mathcal{B}$ is the relative range (by a) of another set in $\mathcal{B}$. Thus, for every$A \in \mathcal{B}$, $s_a p_A = p_B s_a$ for some $B \in \mathcal{B}$. Therefore, every element of the form $s_\alpha p_A s_\alpha^*$ is equal to $p_B$for some $B \in \mathcal{B}$, from where we conclude that $p_{\{v_1\}}$and $p_{\{v_2,v_3\}}$ generate $\Delta(\varepsilon,\mathcal{L},\mathcal{B})$. Hence, the spectrum of $\Delta(\varepsilon,\mathcal{L},\mathcal{B})$is a set with two points.

This labelled space to obtain $s_a s_a^* = p_{\{v_1,v_2\}} \neq p_{\{v_1,v_2,v_3\}} = 1$. We see that $p_{\{v_1\}}, p_{\{v_2,v_3\}}, p_{\{v_1,v_2,v_3\}}$ and $s_a s_a^*$ are all different and non-zero. Therefore, in this case, the spectrum of $\Delta(\varepsilon,\mathcal{L},\mathcal{B})$ possesses more than two points. This shows that $C^*(\varepsilon,\mathcal{L},\mathcal{B})$ given by Definition (4.2.13) is a non-trivial quotient of the C^* −algebra given by the alternative definition. Furthermore, we conclude that the diagonal C^* −algebras are different and, hence, the spectrum of the diagonal C^* −algebra of the alternative definition is not homeomorphic to the tight spectrum of the inverse semigroup associated with the labelled space.

Chapter 5
Approximation and C^*-Algebras

We give a general solution to the norm closure problem for complex symmetric operators. As an application, we provide a concrete description of partial isometries which are norm limits of complex symmetric operators. Also it is completely determined when $C^*(T)$ is $*$-isomorphic to a C^*-algebra singly generated by complex symmetric operators. These both depend only on the singular part of T.

Section (5.1): Approximation of Complex Symmetric Operators

[219], which provides a C^*-algebra approach to complex symmetric operators. We shall develop further some C^*-algebra techniques to solve in a general sense the norm closure problem for complex symmetric operators. The approach employs some classical results from the representation theory of C^*-algebras.

We let $\mathcal{H}$ denote a separable, infinite dimensional complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. We always denote by $B(\mathcal{H})$ the collection of bounded linear operators on $\mathcal{H}$, and by $K(\mathcal{H})$ the ideal of compact operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, we let $C^*(A)$ denote the C^*-algebra generated by A and the identity operator.

Definition (5.1.1)[451]: A map C on $\mathcal{H}$ is called a conjugation if C is conjugate-linear, $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$.

Definition (5.1.2)[451]: An operator $T \in B(\mathcal{H})$ is called a complex symmetric operator (CSO) if there is a conjugation C on $\mathcal{H}$ such that $CTC = T^*$. We denote by $S(\mathcal{H})$ the set of all $CSOs$ on $\mathcal{H}$.

Note that an operator $T \in B(\mathcal{H})$ is complex symmetric if and only if there exists an orthonormal basis (ONB, for short) $\{e_n\}$ of $\mathcal{H}$ such that $\langle Te_i, e_j\rangle = \langle Te_j, e_i\rangle$ for all i, j, that is, T admits a symmetric matrix representation with respect to $\{e_n\}$ (see [208, Lemma 1]). Thus the notion of CSO can be viewed as a generalization of symmetric matrix in the context of Hilbert space.

The general study of $CSOs$ was initiated by Garcia and Putinar [208,209], and has many motivations in function theory, matrix analysis and other areas. Many significant results concerning $CSOs$ have been obtained (see [197,211,213,214,218,230,232]).
In particular, it is worth mentioning that $CSOs$ are closely related to the study of truncated Toeplitz operators, which was initiated in Sarason's seminal [223] and has led to rapid progress in function-theoretic operator theory [196,198,199,215,216,225,226].
The reader is referred to [208,209] for more about $CSOs$ and its connections to other subjects.
We will concentrate on the following norm closure problem.

Problem (5.1.3)[451]: Characterize the norm closure of the set $S(\mathcal{H})$.

There are several motivations for us to study Problem (5.1.3). Firstly, although much attention has been paid to $CSOs$, the internal structure of $CSOs$ is still not well understood. In particular, Garcia posed many concrete questions concerning $CSOs$ (see [210–212,217]). A basic problem is to give a characterization, in "simple terms", of when an operator is complex symmetric. In a real sense such a characterization is very far from existing even in finite dimensional spaces. So people naturally restrict attention to special classes of operators. In this aspect, partial isometries, weighted shifts and some other operators are studied [197,211,230,332]. Another alternative is to consider the approximation of $CSOs$, that is, to characterize which operators are the norm limits of $CSOs$. Maybe the answer is relatively easy to state. This may help to achieve a meaningful classification. In fact, Problem (5.1.3) has inspired many interesting results [213,214,219,220,230–232]. A classification of $CSOs$ up to approximate unitary equivalence. Recall that two operators $A, B \in B(\mathcal{H})$ are approximately unitarily equivalent if there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of unitary operators such that $U_n A - BU_n \to 0$ as $n \to \infty$ (see [204, page 57]).

The second motivation lies in connections between $CSOs$ and anti-automorphisms of singly generated C^*-algebras. Recall that an anti-automorphism of a C^*-algebra A is a vector space isomorphism $\varphi: A \to A$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in A$. Anti-automorphisms play an important role in the study of the real structure of C^*-algebras [195,222,223,227,228]. It is not necessary that each C^*-algebra possesses an anti-automorphism on it [200,201]. So a basic problem is to determine when a C^*-algebra possesses an anti-automorphism on it.

Definition (5.1.4)[451]: We say that an operator $T \in B(H)$ is g-normal if it satisfies

$$\|p(T^*, T)\| = \|\tilde{p}(T, T^*)\|$$

for any polynomial $p(z_1, z_2)$ in two free variables. Here $\tilde{p}(z_1, z_2)$ is obtained from $p(z_1, z_2)$ by conjugating each coefficient.

The notion of g-normal operator was suggested in [217]. It is proved in [219] that (1) an operator $T \in B(\mathcal{H})$ is g-normal if and only if there exists an antiautomorphism φ of $C^*(T)$ such that $\emptyset(T) = T$, and (2) each operator in $\overline{S(\mathcal{H})}$ is g-normal. Moreover each C^*-algebra generated by an operator in $\overline{S(\mathcal{H})}$ possesses a real structure. This suggests a C^*-algebra approach to $CSOs$ and its norm closure problem.

Another motivation of our study stems from some recent interest in the study of $S(\mathcal{H})$ itself as a subset of $B(\mathcal{H})$. In [210], Garcia showed that the set $S(\mathcal{H})$ is invariant under the Aluthge transform, an important transformation which originally arose in the study of hyponormal operators. In [211], Garcia and Wogen showed that $S(\mathcal{H})$ is not closed in the strong operator topology (SOT). In [213], Garcia and Poore proved that the sot closure of $S(\mathcal{H})$ is $B(\mathcal{H})$. As for the norm topology, things become very complicated.

The norm closure problem for $CSOs$ was posed and first studied by Garcia and Wogen [211]. In particular, they asked whether or not the set $S(\mathcal{H})$ is norm closed. Zhu et al. [231] answered this question negatively by proving that the Kakutani shift is not complex symmetric but belongs to $\overline{S(\mathcal{H})}$. The proof there depends on a construction of finite-dimensional truncated weighted shifts. Almost immediately, using the unilateral shift and its adjoint, Garcia and Poore [214] constructed another completely different operator in $\overline{S(\mathcal{H})} \backslash S(\mathcal{H})$.

Generalizing the Kakutani shift, Garcia and Poore [213] constructed some special weighted shifts, the so-called approximately Kakutani shifts. A unilateral weighted shift $T \in B(\mathcal{H})$ with positive weights $\{\alpha_k\}_{k=1}^{\infty}$ is said to be approximately Kakutani if for each $n \geq 1$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $0 < \alpha_N < \varepsilon$ and

$$1 \leq k \leq n \Longrightarrow |\alpha_k - \alpha_{N-k}| < \varepsilon.$$

It was proved that approximately Kakutani shifts are norm limits of $CSOs$ [213, Theorem 10]. Moreover they conjectured the converse also holds [213, Conjecture 1]. Guo et al.

provided a C^*-algebra approach to the norm closure problem for $CSOs$, and gave a positive answer to the conjecture [219, Theorem 2.4]. In fact, more results were obtained there.

As observed in many significant results in operator theory, there is a subtle interplay between compact perturbation and approximation. In fact, in a large number of interesting cases, the norm closure of a subset $\mathcal{E}$ of $B(\mathcal{H})$ is contained in the set of all compact perturbations of operators in $\mathcal{E}$. For example, an operator T is a norm limit of triangular operators if and only if T is a compact perturbation of triangular operators if and only if there exist triangular operators $\{T_n\}_{n=1}^{\infty}$ such that $T_n \to T$ and $T_n - T$ is compact for each $n \geq 1$ (see [221, Theorem 6.4]). This motivates the following definition.

Definition (5.1.5)[451]: Let $\mathcal{E}$ be a subset of $B(\mathcal{H})$. The compact closure of $\mathcal{E}$, denoted by $\bar{\mathcal{E}}^c$, is defined to be the set of all operators $A \in B(\mathcal{H})$ satisfying: for any $\varepsilon > 0$, there exists $K \in K(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $A + K \in \mathcal{E}$.

It is clear that $\mathcal{E} \subset \bar{\mathcal{E}}^c \subset \bar{\mathcal{E}}$ and $\bar{\mathcal{E}}^c \subset [E + K(\mathcal{H})]$. Thus $\mathcal{E}^c$ can be viewed as the set of all small compact perturbations of operators in $\mathcal{E}$.

Definition (5.1.6)[451]: Let $T \in B(\mathcal{H})$. An operator $A \in B(\mathcal{H})$ is called a transpose of T, if $A = CT^*C$ for some conjugation C on H.

The notion "transpose" for operators is in fact a generalization of that for matrices.

In general, an operator has more than one transpose [233, Example 2.2]. However, as indicated in [219], any two transposes of an operator are unitarily equivalent. We often write T^t to denote a transpose of T. In general, there is no ambiguity especially when we write $T \cong T^t$ or $T \cong_a T^t$. Here and in what follows, the notation $\cong$ denotes unitary equivalence, and $\cong_a$ denotes approximate unitary equivalence.

Guo, Ji and Zhu obtained the following theorem which characterizes irreducible unilateral weighted shifts in $\overline{S(\mathcal{H})}$.

Theorem (5.1.7)[451]: [219, Theorem 2.4] Let $T \in B(\mathcal{H})$ be a unilateral weighted shift with positive weights. Then the following are equivalent:

(i) $T \in \overline{S(\mathcal{H})}$;
(ii) $T \in \overline{S(\mathcal{H})}^c$;
(iii) $\exists\, A \in S(\mathcal{H})$ such that $A \cong_a T$;
(iv) $T \cong_a T^*$;
(v) T is g-normal;
(vi) T is approximately Kakutani.

Furthermore, Guo, Ji and Zhu gave a description of those operators T in $\overline{S(\mathcal{H})}$ satisfying

$$C^*(T) \cap K(\mathcal{H}) = \{0\}.$$

Theorem (5.1.8)[451]: Let $T \in B(\mathcal{H})$ and assume that $C^*(T) \cap K(\mathcal{H}) = \{0\}$. Then the following are equivalent:

(i) $T \in \overline{S(\mathcal{H})}$;
(ii) $T \in \overline{S(\mathcal{H})}^c$;
(iii) $\exists\, A \in S(\mathcal{H})$ such that $A \cong_a T$;
(iv) T is g-normal;
(v) $T \cong_a T^t$.

All these results mentioned above suggest that the structure of the set $\overline{S(\mathcal{H})}$ may admit some special form, and it needs and deserves much more study. On the other hand, these results suggest a C^*-algebra approach to $CSOs$. By virtue of an intensive analysis of compact

operators in singly generated C^*-algebras, we employ the representation theory of C^*-algebras to give a complete description of operators in $\overline{S(\mathcal{H})}$.
As an application of Theorem (5.1.30), we shall give a concrete description of partial isometries which are norm limits of $CSOs$. In [211], Garcia and Wogen proved that a partial isometry T is complex symmetric if and only if the compression of T to its initial space is complex symmetric. We shall prove the following theorem, which can be viewed as an analogue of their result in the setting of approximation.

Definition (5.1.9)[451]: Let $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$. We write $A \trianglelefteq B$ if there is a $*$-homomorphism ρ of $C^*(B)$ into $C^*(A)$ such that $\rho(B) = A$; if, in addition, ρ annihilates $C^*(B) \cap K(\mathcal{H}_2)$, then we write $A \lhd B$.

It is easy to see that $A \trianglelefteq B$ if and only if $\|p(A^*, A)\| \leq \|p(B^*, B)\|$ for any polynomial $p(z_1, z_2)$ in two free variables z_1, z_2.

Lemma (5.1.10)[451]: Let $T \in B(\mathcal{H})$ and $T = A \oplus B$, where $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$. Assume that $A \lhd B$. Then

$$C^*(T) \cap K(\mathcal{H}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} : K \in C^*(B) \cap K(\mathcal{H}_2) \right\};$$

in particular, if $C^*(T)$ contains an operator of the form $Y_1 \oplus Y_2$, where $Y_1 \in B(\mathcal{H}_1)$ and $Y_2 \in K(\mathcal{H}_2)$, then $Y_1 = 0$.

Proof. Assume that $A \trianglelefteq B$. Then, by definition, there is a $*$-homomorphism ρ of $C^*(B)$ into $C^*(A)$ such that $\rho(B) = A$. For a polynomial $p(\cdot,\cdot)$ in two free variables, note that

$$p(T^*, T) = \begin{bmatrix} p(A^*, A) & 0 \\ 0 & p(B^*, B) \end{bmatrix} = \begin{bmatrix} \rho(p(B^*, B)) & 0 \\ 0 & p(B^*, B) \end{bmatrix}.$$

It follows immediately that

$$C^*(T) = \left\{ \begin{bmatrix} \rho(X) & 0 \\ 0 & X \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} : X \in C^*(B) \right\}.$$

In view of this, the result follows readily.

The following lemma is clear.

Lemma (5.1.11)[451]: Let A, B and C be three Hilbert space operators satisfying $C \lhd B$. Then (1) $A \trianglelefteq B$ if and only if $A \trianglelefteq C \oplus B$, (2) $A \lhd B$ if and only if $A \lhd C \oplus B$.

Lemma (5.1.12)[451]: [205, Corollary 5.41] If A is a C^*-subalgebra of $B(\mathcal{H})$ which contains $K(\mathcal{H})$ and ρ is an irreducible representation of A on some Hilbert space $\mathcal{H}_\rho$ such that $\rho|_{K(\mathcal{H})}$ is not zero, then there exists unitary $U: \mathcal{H} \to \mathcal{H}_\rho$ such that $\rho(X) = UXU^*$ for $X \in A$.

Lemma (5.1.13)[451]: [204, Corollary II.5.5] Suppose ρ is a non-degenerate representation of a separable C^*-subalgebra A of $B(\mathcal{H})$ into $B(\mathcal{H}_\rho)$ such that $\rho(A \cap K(\mathcal{H})) = \{0\}$. Then $id \cong_a id \oplus \rho$, where id is the identity representation of A.

Let $T \in B(\mathcal{H})$. If σ is a nonempty clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \backslash \sigma] \cap \overline{\Omega} = \emptyset$. We let $E(\sigma; T)$ denote the Riesz idempotent of T corresponding to σ [221, page 2], that is,

$$E(\sigma; T) = \frac{1}{2\pi i} \int_\Gamma (\lambda - T)^{-1} d\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. If T is self-adjoint, then it is obvious that $E(\sigma; T)$ is an orthogonal projection.

Lemma (5.1.14)[451]: Let $T = A \oplus C \oplus B$, where $A \in B(\mathcal{H}_1), B \in B(\mathcal{H}_2)$ and $C \in B(H_3)$.

(i) If $A \ntrianglelefteq B$, then $C^*(T\,)$ contains an operator Z of the form

$$Z = \begin{bmatrix} X & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3, \\ \mathcal{H}_2 \end{matrix}$$

where $X \neq 0$ and each omitted entry is 0.

(ii) If $A \trianglelefteq B$ but $A \ntriangleleft B$, then $C^*(T)$ contains an operator Z of the form

$$Z = \begin{bmatrix} X & & \\ & Y & \\ & & K \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3, \\ \mathcal{H}_2 \end{matrix}$$

where $X \neq 0$ and $K \in K(\mathcal{H}_2)$.

Proof. (i) If $A \ntrianglelefteq B$, then, by definitions, there exists a polynomial $p(\cdot,\cdot)$ in two free variables such that $\|p(A^*, A)\| > \|p(B^*, B)\|$. Denote $D = |p(A^*, A)|, E = |p(C^*, C)|$ and $F = |p(B^*, B)|$. Then it follows that

$$|p(T^*, T)| = \begin{bmatrix} D & & \\ & E & \\ & & F \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_2 \end{matrix}$$

is a positive operator in $C^*(T\,)$ with $\|D\| > \|F\|$. Set $\delta = \frac{\|D\|+\|F\|}{2}$ and define

$$h(t) = \begin{cases} 0, & 0 \leq t \leq \delta, \\ t - \delta & , t > \delta. \end{cases}$$

Thus h is continuous on $[0, +\infty)$ and one can see that $h(D) \neq 0$ and $h(F) = 0$.
Set $X = h(D), Y = h(E)$ and $Z = h(|p(T^*, T)|)$. Then $Z = X \oplus Y \oplus 0 \in C^*(T\,)$ and $X \neq 0$.
(ii) Since $A \trianglelefteq B$, there is a $*$-homomorphism of $C^*(B)$ onto $C^*(A)$ such that $\rho(B) = A$. It is easy to see that $\rho(p(B^*, B)) = p(A^*, A)$ for any polynomial $p(\cdot,\cdot)$ in two free variables. By the hypothesis, there exists $D \in C^*(B) \cap K(\mathcal{H}_2)$ such that $\rho(D) \neq 0$. So $D \neq 0$. Obviously, we can choose polynomials $\{p_n(\cdot,\cdot)\}$ in two free variables such that $p_n(B^*, B) \to D$. Thus

$$p_n(A^*, A) = \rho(p_n(B^*, B)) \to \rho(D).$$

Hence $|p_n(B^*, B)| \to |D|$ and $|p_n(A^*, A)| \to \rho(|D|)$. Note that $|\rho(D)| = \rho(|D|)$.
Since $|D|$ is compact, there exists $\delta < \frac{\|\rho(|D|)\|}{2}$ such that $\delta \notin \sigma(|D|)$. Noting that $\|\rho(|D|)\| \leq \|D\|$, we have $\sigma(|D|) = \sigma_1 \cup \sigma_2$, where $\sigma_1 \subset (-1, \delta)$ and $\emptyset \neq \sigma_2 \subset (\delta, \|D\| + 1)$. Moreover, the Riesz idempotent of $|D|$ corresponding to σ_2, denoted by $E(\sigma_2; |D|)$, is of finite rank and $E(\sigma_2; |D|) \neq 0$.
By the upper semi-continuity of spectra in approximation, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\sigma(|p_n(B^*, B)|) = \sigma_1' \cup \sigma_2'$ with $\sigma_1' \subset (-1, \delta)$ and $\sigma_2 \subset (\delta, \|D\| + 1)$; moreover, $rank\ E(\sigma_2';\ |p_n(B^*, B)|) = rank\ E(\sigma_2; |D|) < \infty$ (see [221, Corollary 1.6]). Also it can be required that $\|p_n(A^*, A)\| > \delta$ for any $n > N$.
Define

$$h(t) = \begin{cases} 0, & 0 \leq t \leq \delta, \\ t - \delta, & t > \delta. \end{cases}$$

Then h is nonnegative and continuous on $[0, +\infty)$. Now fix an $n > N$. Set

$X = h(|p_n(A^*, A)|), Y = h(|p_n(C^*, C)|)$ and $K = h(|p_n(B^*, B)|)$. It is evident that $X \neq 0, K \in K(H_2)$ and $X \oplus Y \oplus K \in C^*(T)$. This completes the proof.
For convenience, we write $0_{\mathcal{H}}$ to denote the sub-algebra $\{0\}$ of $B(\mathcal{H})$.
Lemma (5.1.15)[451]: Let $T = A \oplus C \oplus B$, where $A \in B(\mathcal{H}_1), B \in B(\mathcal{H}_2)$ and $C \in B(\mathcal{H}_3)$. If $K(\mathcal{H}_1) \oplus 0_{H_3} \subset C^*(A \oplus C)$ and $C^*(T)$ contains an operator Z of the form

$$Z = \begin{bmatrix} X & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3, \\ \mathcal{H}_2 \end{matrix}$$

where $X \neq 0$, then $K(\mathcal{H}_1) \oplus 0_{H_3} \oplus 0_{\mathcal{H}_2} \subset C^*(T)$.
Proof. Arbitrarily choose two unit vectors $e, f \in \mathcal{H}_1$. It suffices to prove that $C^*(T)$ contains the operator

$$Z = \begin{bmatrix} f \otimes e & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3. \\ \mathcal{H}_2 \end{matrix}$$

For convenience we denote $D = A \oplus C$. Since $X \neq 0$, there exist nonzero vectors $e_0, f_0 \in H_1$ such that $Xe_0 = f_0$. On the other hand, noting that $K(\mathcal{H}_1) \oplus 0_{H_1} \subset C^*(D)$, we can choose polynomials $\{p_n(\cdot,\cdot)\}$ and $\{q_n(\cdot,\cdot)\}$ in two free variables such that

$$p_n(D^*, D) \longrightarrow \begin{bmatrix} f \otimes \frac{f_0}{\|f_0\|^2} & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3 \end{matrix}$$

and

$$q_n(D^*, D) \longrightarrow \begin{bmatrix} e_0 \otimes e & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3 \end{matrix}.$$

It follows that

$$p_n(T^*, T) Z q_n(T^*, T) \longrightarrow \begin{bmatrix} f \otimes e & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_3, \\ \mathcal{H}_2 \end{matrix}$$

which completes the proof.
Lemma (5.1.16)[451]: Let $T \in B(\mathcal{H})$. Assume that $T = \bigoplus_{i=1}^n T_i$, where $T_i \in B(\mathcal{H}_i)$ and $K(\mathcal{H}_i) \subset C^*(T_i)$ for each $1 \leq i \leq n$ and $T_1 \not\trianglelefteq T_j$ whenever $j \neq 1$. If there exists $K = \bigoplus_{i=1}^n K_i \in C^*(T) \cap K(\mathcal{H})$ with $K_1 \neq 0$, then

$$K(\mathcal{H}_1) \oplus 0_{\mathcal{H}_2} \oplus \cdots \oplus 0_{\mathcal{H}_n} \subset C^*(T).$$

Proof. We shall proceed by induction on n. When $n = 1$, the result is clear.
Now assume that the result is true when $n \leq k$.We shall prove that the result holds when $n = k + 1$.
Since $T_1 \not\trianglelefteq T_n$, by Lemma (5.1.15)(i), $C^*(T)$ contains an operator $X = (\bigoplus_{i=1}^{n-1} X_i) \oplus 0$, where $X_i \in B(\mathcal{H}_i)$ for $1 \leq i \leq n-1$ and $X_1 \neq 0$. Since $\|X_1\| \cdot \|K_1\| \neq 0$, there exist nonzero vectors $e_i, f_i (i = 1, 2)$ such that $X_1 e_1 = e_2$ and $K_1 f_1 = f_2$.
Noting that $f_1 \otimes e_2 \in C^*(T_1) \cap K(\mathcal{H}_1)$, there exists a sequence $\{p_n(\cdot,\cdot)\}_{n=1}^\infty$ of polynomials in two free variables such that $p_n(T_1^*, T_1) \to f_1 \otimes e_2$; hence we have $K_1 p_n(T_1^*, T_1) X_1 \to K_1 (f_1 \otimes e_2) X_1 \neq 0$. Then some $n_0 \in \mathbb{N}$ exists such that $K_1 p_{n_0}(T_1^*, T_1) X_1 \neq 0$. Set $C_i = K_i p_{n_0}(T_i^*, T_i) X_i$ for each $1 \leq i \leq n-1$. Then $C := K p_{n_0}(T^*, T) X = (\bigoplus_{i=1}^{n-1} C_i) \oplus 0$ is a compact operator in $C^*(T)$ with $C_1 \neq 0$.

In particular, $\bigoplus_{i=1}^{n-1} C_i$ is a compact operator in $C^*(\bigoplus_{i=1}^{n-1} T_i)$ with $C_1 \neq 0$. By the induction hypothesis, we have

$$K(\mathcal{H}_1) \oplus 0_{\mathcal{H}_2} \oplus \cdots \oplus 0_{\mathcal{H}_{n-1}} \subset C^*\left(\bigoplus_{i=1}^{n-1} T_i\right).$$

Since we have proved that $C^*(T)$ contains $C_1 \oplus (\bigoplus_{n=1}^{n-1} C_i) \oplus 0$ with $C_1 \neq 0$, the desired result follows immediately from Lemma (5.1.15).

Given a set Γ, we write card Γ for the cardinality of Γ. For $T \in B(\mathcal{H})$ and a cardinal n with $1 \leq n \leq \aleph_0$, we let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$ and let $T^{(n)}$ denote the direct sum of n copies of T, acting on $\mathcal{H}^{(n)}$ (see [202, Definition 6.3]). For convenience, $\mathcal{H}^{(\aleph_0)}$ and $T^{(\aleph_0)}$ are denoted by $\mathcal{H}^{(\infty)}$ and $T^{(\infty)}$.

Lemma (5.1.17)[451]: Let $T \in B(\mathcal{H})$ and $T = \bigoplus_{i\in\Lambda} T_i^{(n_i)}$, where $T_i \in B(\mathcal{H}_i)$ for each $i \in \Lambda$. If there exists nonzero $K \in C^*(T) \cap K(\mathcal{H})$ with $= \bigoplus_{i\in\Lambda} K_i^{(n_i)}$, then $C^*(T)$ contains nonzero $C \in C^*(T) \cap K(\mathcal{H})$ with the form $C = \bigoplus_{i\in\Lambda} C_i^{(n_i)}$ satisfying $card\{i \in \Lambda: C_i \neq 0\} < \infty$.

Proof. Without loss of generality, we may directly assume that K is positive. Set $\delta = \frac{\|K\|}{2}$ and define

$$h(t) = \begin{cases} 0, & 0 \leq t < \delta, \\ t - \delta, & \delta \leq t \leq \|K\|. \end{cases}$$

Then h is a nonnegative, continuous function on $[0, \|K\|]$. Set $C = h(K)$ and $C_i = h(K_i)$ for each $i \in \Lambda$. Then $= \bigoplus_{i\in\Lambda} C_i^{(n_i)}$. It remains to show that C satisfies all requirements.

Noting that K is compact and $h(0) = 0$, we have $C \in C^*(T) \cap K(\mathcal{H})$. Since K is compact and $K \neq 0$, it immediately follows that $0 < card\{i \in \Lambda: \|K_i\| > \delta\} < \infty$.

For each , note that $C_i = h(K_i) \neq 0$ if and only if $\|K_i\| > \delta$. Thus one can deduce that $\{i \in \Lambda: C_i \neq 0\}$ is finite. This completes the proof.

Corollary (5.1.18)[451]: Let $T \in B(\mathcal{H})$ with $T = \bigoplus_{i\in\Lambda} T_i$, where $T_i \in B(\mathcal{H}_i)$ is irreducible for each $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \neq j$. If $C^*(T) \cap K(\mathcal{H}) \neq \{0\}$, then there exists $i_0 \in \Lambda$ such that

$$K(\mathcal{H}_{i0}) \oplus 0_{H_{i_0}^{\perp}} \subset C^*(T).$$

Proof. By the hypothesis, we can choose a nonzero $K \in C^*(T) \cap K(\mathcal{H})$. Since $T = \bigoplus_{i\in\Lambda} T_i$, K can be written as $K = \bigoplus_{i\in\Lambda} K_i$, where $K_i \in B(\mathcal{H}_i)$ for $i \in \Lambda$. By Lemma (5.1.17), we may assume that $\Lambda_0 := \{i \in \Lambda : K_i \neq 0\}$ is a finite set.

Now fix an $i_0 \in \Lambda_0$. Set $A = T_{i_0}, C = \bigoplus_{i\in\Lambda_0\setminus\{i_0\}} T_i$ and $= \bigoplus_{i\in\Lambda\setminus\Lambda_0} T_i$. Then $T = A \oplus C \oplus B$. Denote $\mathcal{H}_A = \mathcal{H}_{i_0}$, $\mathcal{H}_C = \bigoplus_{i\in\Lambda_0\setminus\{i_0\}} \mathcal{H}_i$ and $\mathcal{H}_B = \bigoplus_{i\in\Lambda\setminus\Lambda_0} \mathcal{H}_i$.

Claim $T_{i_0} \ntrianglelefteq T_i$ for any $i \in \Lambda_0$ with $i \neq i_0$.

In fact, if not, then there exists $j \in \Lambda_0$ with $j \neq i_0$ such that $T_{i_0} \neq T_j$. So there exists a $*$-homomorphism ρ of $C^*(T_j)$ onto $C^*(T_{i_0})$ such that $(T_j) = T_{i_0}$. Then ρ is an irreducible representation of $C^*(T_j)$. Noting that $K_j \in C^*(T_j)$ and T_j is irreducible, we have $K(\mathcal{H}_j) \subset C^*(T_j)$. On the other hand, since $K_{i_0} \oplus K_j \in C^*(T_{i_0} \oplus T_j)$, one can see that $K_{i_0} = \rho(K_j)$. It follows that $K(\mathcal{H}_j) \nsubseteq ker\ \rho$. Then, by Lemma (5.1.12), ρ is unitarily implemented, which implies that $T_{i_0} \cong T_j$, a contradiction. This proves the claim.

Set $S = \bigoplus_{i\in\Lambda_0} T_i$. Then $S = A \oplus C$ and, by the hypothesis, $\bigoplus_{i\in\Lambda_0} K_i$ is a compact operator in $C^*(S)$ with $K_1 \neq 0$. Since $T_{i_0} \not\trianglelefteq T_i$ for any $i \in \Lambda_0$ with $i \neq i_0$, it follows from Lemma (5.1.16) that

$$K(\mathcal{H}_{i_0}) \oplus 0_{\mathcal{H}_C} \subset C^*(S).$$

Note that K is an operator in $C^*(T)$ which can be written as $K = X \oplus Y \oplus 0$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_C \oplus \mathcal{H}_B$, where $X \neq 0$. Hence the desired result follows immediately from Lemma (5.1.15).

Corollary (5.1.19)[451]: Let $T \in B(\mathcal{H})$ with $T = \bigoplus_{i\in\Lambda} T_i^{(n_i)}$, where $T_i \in B(\mathcal{H}_i)$ is irreducible for each $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \neq j$. If $C^*(T) \cap K(\mathcal{H}) \neq \{0\}$, then there exists $i \in \Lambda$ such that

$$\left\{\begin{bmatrix} X^{(n_i)} & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_i^{(n_i)} \\ \mathcal{H} \ominus \mathcal{H}_i^{(n_i)} \end{matrix} : X \in K(\mathcal{H}_i)\right\} \subset C^*(T) \cap K(\mathcal{H}).$$

Lemma (5.1.20)[451]: Let $T \in B(\mathcal{H})$ with $T = A \oplus B$, where $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$. Assume that $\{C_n\}_{n=1}^{\infty}$ is a sequence of conjugations on $\mathcal{H}$ such that $C_nTC_n \to T^*$. If P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1$ and $(I - P)C_nPx \to 0$ for any $x \in \mathcal{H}$, then A is g-normal.

Proof. For each $n \geq 1$, we may assume that

$$C_n = \begin{bmatrix} C_{1,1}^n & C_{1,2}^n \\ C_{2,1}^n & C_{2,2}^n \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

By the hypothesis, we have $C_{2,1}^n \to 0$ for any $x \in \mathcal{H}_1$.

Now fix a polynomial $p(\cdot,\cdot)$ in two free variables. Assume that

$$p(T^*, T) = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} \text{ and } \tilde{p}(T, T^*) = \begin{bmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Thus $X = p(A^*, A)$ and $\tilde{X} = \tilde{p}(A, A^*)$. For each n, a matrix multiplication shows that

$$C_np(T^*, T)C_n = \begin{bmatrix} C_{1,1}^n X C_{1,1}^n + C_{1,2}^n Y C_{2,1}^n & * \\ * & * \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

Since $C_n\, p(T^*, T)C_n \to \tilde{p}(T, T^*)$, it follows that

$$C_{1,1}^n X C_{1,1}^n + C_{1,2}^n Y C_{2,1}^n \longrightarrow \tilde{X} \; as \; n \longrightarrow \infty.$$

For $x \in \mathcal{H}_1$, noting that $\left\|C_{1,2}^n Y C_{2,1}^n x\right\| \leq \left\|Y C_{2,1}^n x\right\| \longrightarrow 0$, we have $C_{1,1}^n X C_{1,1}^n x \longrightarrow \tilde{X}x$. Thus

$$\|\tilde{x}\| = \lim_n \left\|C_{1,1}^n X C_{1,1}^n x\right\| \leq \lim_n sup \left\|X C_{1,1}^n x\right\| \leq \|X\| \cdot \|x\|.$$

Thus we deduce that $\left\|\tilde{X}\right\| \leq \|X\|$, that is, $\|\tilde{p}(A, A^*)\| \leq \|p(A^*, A)\|$. By symmetry, we obtain $\|\tilde{p}(A, A^*)\| = \|p(A^*, A)\|$, which implies that A is g-normal.

Lemma (5.1.21)[451]: [287, Proposition 2.4] If $T \in B(\mathcal{H})$, then T admits the decomposition $T = T_0 \oplus (\bigoplus_{i\in\Gamma} T_i)$, where $T_0 \in B(\mathcal{H}_0)$ is completely reducible and $T_i \in B(\mathcal{H}_i)$ is irreducible for all $i \in \Gamma$.

Recall that an operator $T \in B(\mathcal{H})$ is called completely reducible if T has no nonzero minimal reducing subspace. Following Arveson [194], we let $\sum_{i\in\Lambda} A_i$ denote the direct sum of a family $\{A_i\}_{i\in\Lambda}$ of C^*-algebras. Given a C^*-algebra A of operators and n with $1 \leq n \leq \aleph_0$, we denote by $A^{(n)}$ the C^*-algebra $\{A^{(n)} \colon A \in A\}$.

Lemma (5.1.22)[451]: Let $T \in B(\mathcal{H})$. If $C^*(T) \cap K(\mathcal{H}) \neq \{0\}$, then T is unitarily equivalent to an operator $A = T_0 \oplus (\bigoplus_{i\in\Lambda} T_i^{(n_i)})$, where $T_0 \in B(\mathcal{H}_0)$, each $T_i \in B(\mathcal{H}_i)$ is irreducible with $K(\mathcal{H}_i) \subset C^*(T_i)$ for $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \neq j$; moreover, $C^*(A) \cap K(\widehat{\mathcal{H}}) = 0_{\mathcal{H}_0} \oplus \sum_{i\in\Lambda} K(\mathcal{H}_i)^{(n_i)}$, where $\widehat{\mathcal{H}} = \mathcal{H}_0 \oplus (\bigoplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)})$.

Proof. The proof is omitted since it is a minor modification of [219, Corollary 5.5].

The following result is a consequence of Voiculescu's theorem [229]. (see [202, Theorem 42.1] for a proof).

Lemma (5.1.23)[451]: Each operator in $B(\mathcal{H})$ is approximately unitarily equivalent to a direct sum of irreducible operators.

Corollary (5.1.24)[451]: Let $T \in B(\mathcal{H})$. If $C^*(T) \cap K(\mathcal{H}) = \{0\}$ and $T \in \overline{S(\mathcal{H})}$, then T is approximately unitarily equivalent to a direct sum of operators of the form $A \oplus A^t$, where A is irreducible.

Proof Since $C^*(T) \cap K(\mathcal{H}) = \{0\}$, it follows from [202, Proposition 42.9] that $T \cong_a T \oplus T$ .By Theorem (5.1.8), $T \in \overline{S(\mathcal{H})}$ implies that $T \cong_a T^t$ .Hence we obtain $T \cong_a T \oplus T^t$. On the other hand, by Lemma (5.1.23), there exists a family $\{A_i\}_{i\in\Lambda}$ of irreducible operators such that $T \cong_a \bigoplus_{i\in\Lambda} A_i$. Therefore we obtain

$$T \cong_a \bigoplus_{i\in\Lambda} (A_i \oplus A_i^t),$$

which completes the proof.

We first give some auxiliary results.

Lemma (5.1.25)[451]: Let $P \in B(\mathcal{H})$ be a finite-rank projection and $\{C_n\}_{n=1}^{\infty}$ be a sequence of conjugations on $\mathcal{H}$ so that $\{C_n P C_n\}_{n=1}^{\infty}$ converges to an operator $Q \in B(\mathcal{H})$. Then Q is a projection on $\mathcal{H}$ with $rank\ P = rank\ Q$ and there exists a subsequence $\{n_j\}_{n=1}^{\infty}$ of $\mathbb{N}$ such that for each $x \in ran\ P$ the sequence $\{C_{n_j} x\}_{j=1}^{\infty}$ converges to a vector in $ran\ Q$.

Proof. Since P is a finite-rank projection, we may assume that $rank\ P = m$ and $P = \sum_{i=1}^{m} e_i \otimes e_i$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal subset of $\mathcal{H}$. First, it is evident that Q is a projection. By the hypothesis, $lim_n\ C_n P C_n = Q$; by the lower semi-continuity of the rank in approximation (see [221, Proposition 1.12]), it follows that

$$rank\ P = \lim_n inf\ rank\ C_n P C_n \geq rank\ Q.$$

On the other hand, since $C_n^2 = I$ for any n, we have $lim_n\ C_n Q C_n = P$; by the lower semi-continuity of the rank in approximation again, we have $rank\ Q \geq rank\ P$. So we obtain

$$rank\ Q = rank\ P.$$

Now fix a k with $1 \leq k \leq m$. By the hypothesis,

$$\|(I - Q)C_n e_k\| = C_n\, Pe_k - QC_n e_k \to 0\ as\ n \to \infty. \tag{1}$$

It follows that $\|QC_n e_k\| \to 1$. Since dim ran $Q < \infty$ and $sup_n \|QC_n e_k\| \leq 1$, there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that $\{QC_{n_j} e_k\}_{j=1}^{\infty}$ converges to a unit vector x_k in ran Q. In view of (1), it follows that $C_{n_j} Pe_k = C_{n_j} e_k \to x_k$ as $j \to \infty$.

In view of the above discussion, applying the diagonal process, we can find a subsequence, denoted by $\{n_j\}_{j=1}^{\infty}$ again, such that for each $1 \leq k \leq m$ we have $C_{n_j} e_k \to x_k$ as $j \to \infty$. Since $\{e_k\}_{k=1}^{m}$ is an ONB of ran P, it can be seen that $\{C_{n_j} x\}_{j=1}^{\infty}$ converges to a vector in ran Q for each $x \in ran\ P$.

Definition (5.1.26)[451]: Let $T \in B(\mathcal{H})$. Denote by $\mathcal{H}_e$ the closed linear span of the following set

$$\{Kx: K \in C^*(T) \cap K(\mathcal{H})\ and\ x \in \mathcal{H}\}$$
and set $\mathcal{H}_r = \mathcal{H} \ominus \mathcal{H}_e$. It is easy to see that $\mathcal{H}_e$ and $\mathcal{H}_r$ both reduce T. Denote $T_e = T\mid_{\mathcal{H}_e}$ and $T_r = T\mid_{\mathcal{H}_r}$.

The proof of the following lemma follows readily from Lemma (5.1.22).

Lemma (5.1.27)[451]: Let $T \in B(\mathcal{H})$. Then $C^*(T_e) \cap K(\mathcal{H}_e)$ is non-degenerate and
$$C^*(T) \cap K(\mathcal{H}) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}\begin{matrix}\mathcal{H}_r \\ \mathcal{H}_e\end{matrix}: K \in C^*(T_e) \cap K(\mathcal{H}_e)\right\}.$$

Theorem (5.1.28)[451]: If $T \in \overline{S(\mathcal{H})}$, then T_e is a CSO.

Proof. We may directly assume that
$$T = T_0 \oplus \left(\oplus_{i\in\Lambda} T_i^{(n_i)}\right),$$
where $T_0 \in B(\mathcal{H}_0)$ and each $T_i \in B(\mathcal{H}_i)$ is irreducible for $i \in \Lambda$. Moreover, we assume that
$$C^*(T) \cap K(\mathcal{H}) = 0_{H_0} \oplus \textstyle\sum_{i\in\Lambda} K(\mathcal{H}_i)^{(n_i)}. \quad (2)$$
It is obvious that $n_i < \infty$ for all $i \in \Lambda$. Note that $T_e = \oplus_{i\in\Lambda} T_i^{(n_i)}$ and $H_e = \oplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}$.

Since $T \in \overline{S(H)}$, it follows from [213, Lemma 6] that there is a sequence $\{C_n\}_{n=1}^{\infty}$ of conjugations on $\mathcal{H}$ such that $C_nTC_n \to T^*$. Then it is easy to check that $C_n\, p(T^*,T)C_n \to \tilde{p}(T,T^*)$ for each polynomial $p(\cdot,\cdot)$ in two free variables. So $lim_n\, C_nXC_n$ exists for each $X \in C^*(T)$. Define $\varphi(X) = lim_n\, C_n\, X^*C_n$ for $X \in C^*(T)$. Then φ is an anti-automorphism of $C^*(T)$ and $\varphi^{-1} = \varphi$.

In view of (2), we can choose a sequence $\{P_n\}_{n=1}^{\infty}$ of finite-rank projections in $C^*(T) \cap K(\mathcal{H})$ with $P_m\, P_l = 0$ whenever $m \neq l$ such that $\oplus_{n=1}^{\infty} ran\, P_n = \mathcal{H}_e$. For each k, denote $Q_k = \phi(P_k)$, that is, $Q_k = lim_n\, C_nP_kC_n$. By Lemma (5.1.25), each Q_k is a projection in $C^*(T)$ with $Q_k = rank\, P_k$, and there is a subsequence $\{n_j(k)\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that for each $x \in ran\, P_k$ the sequence $\{C_{n_j(k)}x\}_{j=1}^{\infty}$ converges to a vector in $ran\, Q_k$. Applying the diagonal process, we can choose a subsequence $\{n_j\}$ of $\mathbb{N}$ such that for each $x \in \cup_{k=1}^{\infty} ran\, P_k$ the sequence $\{C_{n_j}x\}_{j=1}^{\infty}$ converges to a vector in $\cup_{k=1}^{\infty} ran\, Q_k$.

Noting that $Q_k \in C^*(T) \cap K(\mathcal{H})$ and $ran\, Q_k \subset \mathcal{H}_e$, we have found a subsequence $\{n_j\}$ of $\mathbb{N}$ such that $lim_j\, C_{n_j}x \in \mathcal{H}_e$ for each $x \in \cup_{k=1}^{\infty} ran\, P_k$. Since each C_{n_j} is isometric, one can easily see that $lim_j\, C_{n_j}x$ exists for each $x \in \oplus_{k=1}^{\infty} ran\, P_k = \mathcal{H}_e$ and $lim_j\, C_{n_j}x \in \mathcal{H}_e$. For $x \in \mathcal{H}_e$, define $Ex = lim_j\, C_{n_j}x$. Then, by the discussion above, the map $E: \mathcal{H}_e \to \mathcal{H}_e$ is well defined. Since each C_{n_j} is a conjugation, it is obvious that E is isometric and conjugate-linear.We claim that E is indeed a conjugation on $\mathcal{H}_e$. For fixed $x \in \mathcal{H}_e$, it suffices to check that $E^2x = x$. In fact,
$$\|E^2x - x\| = \lim_j \left\|C_{n_j}Ex - x\right\| = \lim_j \left\|C_{n_j}(Ex - C_{n_j}x)\right\|$$
$$= \lim_j \left\|Ex - C_{n_j}x\right\| = 0.$$
Thus E is a conjugation.

Now it remains to check that $ET_e = T_e^*E$. Given a vector $y \in \mathcal{H}_e$, we have
$$ET_ey = \lim_j C_{n_j}T_ey = \lim_j C_{n_j}Ty$$
$$= \lim_j T^*C_{n_j}y = T^*(\lim_j C_{n_j}y)$$
$$= T^*Ey = T_e^*Ey.$$

This shows that $ET_e = T_e^* E$. Therefore T_e is a CSO.

Corollary (5.1.29)[451]: Let $T \in B(\mathcal{H})$. If $T = T_e$, then $T \in \overline{S(\mathcal{H})}$ if and only if $T \in S(\mathcal{H})$.

Now we can give a short proof of the following result which was first proved in [219, Theorem 2.8].

Corollary (5.1.30)[451]: Let $T \in B(\mathcal{H})$ be essentially normal. Then $T \in \overline{S(\mathcal{H})}$ if and only if $T \in S(\mathcal{H})$.

Proof. It suffices to prove the necessity. Note that $T = T_r \oplus T_e$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_r \oplus \mathcal{H}_e$. Since $T \in \overline{S(\mathcal{H})}$, it follows from Theorem (5.1.28) that T_e is a CSO. Note that $T^*T - TT^* = (T_r^* T_r - T_r T_r^*) \oplus (T_e^* T_e - T_e T_e^*)$ is compact. Thus $ran\,(T^*T - TT^*) \subset \mathcal{H}_e$, which implies that $T_r^* T_r - T_r T_r^* = 0$. Then T_r is normal; furthermore, $T = T_r \oplus T_e$ is a CSO.

Let S denote the unilateral shift on $\mathcal{H}$. Recall that a Foguel operator is an operator on $\mathcal{H} \oplus \mathcal{H}$ of the form

$$R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix},$$

where $T \in B(\mathcal{H})$. More generally, we refer to an operator of the form

$$R_{T,n} = \begin{bmatrix} (S^*)^n & T \\ 0 & S^n \end{bmatrix},$$

as a Foguel operator of order n.

Corollary (5.1.31)[451]: Let $R_{T,n}$ be a Foguel operator of order n, where $T \in B(\mathcal{H})$ and $n \in \mathbb{N}$. Then $R_{T,n}$ is a norm limit of $CSOs$ if and only if $R_{T,n}$ is a CSO.

Proof. By Corollary (5.1.29), it suffices to prove that $(R_{T,n})_e = R_{T,n}$, that is, $\mathcal{H}^{(2)} = (H^{(2)})e$.

For convenience, we write

$$R_{T,n} = \begin{bmatrix} (S^*)^n & T \\ 0 & S^n \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Fix an $m \in \mathbb{N}$. Denote $A = R_{T,n}^m$. Since $R_{T,n}$ is a Fredholm operator, it follows that A and A^*A are both Fredholm and $ker\ A = ker\ A^*A$. Since $dim\ ker\ A^*A < \infty$, one can see that $P_{ker}\ A = P_{ker}\ A^*A \in C^*(A^*A) \subset C^*(R_{T,n})$.

Then $P_{kerA} \in [C^*(R_{T,n}) \cap K(\mathcal{H}^{(2)})]$. Noting that $\vee_{m\geq 1}\ ker\ R_{T,n}^m = \mathcal{H}_1$, we obtain $\mathcal{H}_1 \subset (\mathcal{H}^{(2)})_e$.

Applying the above argument to $R_{T,n}^*$, one can prove that $\mathcal{H}_2 \subset \mathcal{H}_e^{(2)}$. Thus $\mathcal{H}^{(2)} = (\mathcal{H}^{(2)})_e$. This completes the proof.

Lemma (5.1.32)[451]: Let $T \in B(\mathcal{H})$. If T can be written as a direct sum of some irreducible operators, then there exists no nonzero reducing subspace M of T such that $T\mid_M$ is completely reducible.

Proof. Since T can be written as a direct sum of some irreducible operators, it follows from [206, Theorem 3.1] that the commutant algebra $\{T, T^*\}'$ is $*$-isomorphic to $\sum_{i\in\Lambda} M_{n_i}(\mathbb{C})$, where $1 \leq n_i \leq \infty$ for each $i \in \Lambda$. Thus each nonzero projection $P \in \{T, T^*\}'$ admits a nonzero minimal subprojection. Then each nonzero reducing reducing subspace M of T contains a nonzero minimal reducing subspace of T. This completes the proof.

Proposition (5.1.33)[451]: If $T \in S(\mathcal{H})$ and $T = T_e$, then T can be written as a direct sum of irreducible $CSOs$ and operators of the form $A \oplus A^t$, where A is irreducible and not a CSO.

Proof. By [220, Theorem 1.6], T can be written as a direct sum of completely reducible $CSOs$, irreducible $CSOs$ and operators of the form $A \oplus A^t$, where A is irreducible and not a CSO. In view of Lemma (5.1.32), it suffices to prove that T is a direct sum of irreducible operators. Since $T = T_e$, by Lemma (5.1.22), T can be written as a direct sum of irreducible operators. This completes the proof.

Lemma (5.1.34)[451]: [203, page 793] If $T \in B(\mathcal{H})$, then $T \oplus T^t$ is complex symmetric.

Now we can give the Proof of Theorem (5.1.35).

Theorem (5.1.35)[451]: Let $T \in B(\mathcal{H})$. Then the following are equivalent:

(i) $T \in \overline{S(\mathcal{H})}$;

(ii) $T \in \overline{S(\mathcal{H})^c}$;

(iii) $\exists\, R \in S(\mathcal{H})$ such that $T \cong_a R$;

(iv) T is approximately unitarily equivalent to an operator which can be written as a direct sum of irreducible $CSOs$ and operators of the form $A \oplus A^t$, where A is irreducible and not a CSO.

Proof of Theorem (5.1.35) The implication "(iv)⟹(iii)" follows from Lemma (5.1.34). By definitions, the implications "(iii)⟹(ii)⟹(i)" are obvious. It suffices to prove "(i) ⟹ (iv)".

"(i)⟹(iv)" $T \in \overline{S(\mathcal{H})}$ implies that any operator approximately unitarily equivalent to T lies in $\overline{S(\mathcal{H})}$. Thus, in view of Lemma (5.1.23), we may directly assume that T is a direct sum of irreducible operators.

Note that $T = T_r \oplus T_e$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_r \oplus \mathcal{H}_e$. If T_e or T_r is absent, then, by Corollary (5.1.24) and Proposition (5.1.33), the result is clear. So we may assume that neither T_e nor T_r is absent. By Theorem (5.1.28), T_e is a CSO.

By Lemma (5.1.21), we may also assume that $T_r = T_0 \oplus \left(\bigoplus_{i\in\Lambda} T_i^{(n_i)}\right)$, where T_0 is completely reducible, each $T_i \in B(\mathcal{H}_i)$ is irreducible for $i \in \Lambda$ and $T_i \ncong T_j$ whenever $i \neq j$. Since T is a direct sum of irreducible operators, it follows from Lemma (5.1.32) that T_0 is absent. Thus $\mathcal{H}_r = \bigoplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}$. Hence we have

$$T = T_r \oplus T_e = \left(\bigoplus_{i\in\Lambda} T_i^{(n_i)}\right) \oplus T_e$$

with respect to the decomposition $\mathcal{H} = \left(\bigoplus_{i\in\Lambda} \mathcal{H}_i^{(n_i)}\right) \oplus \mathcal{H}_e$.

Denote $\Lambda_2 = \{i \in \Lambda : T_i \lhd T_e\}$ and $\Lambda_1 = \Lambda \backslash \Lambda_2$. Set

$$A = \bigoplus_{i\in\Lambda_1} T_i^{(n_i)} \text{ and } B = \bigoplus_{i\in\Lambda_2} T_i^{(n_i)}.$$

Denote $\mathcal{H}_A = \bigoplus_{i\in\Lambda_1} \mathcal{H}_i^{(n_i)}$ and $\mathcal{H}_B = \bigoplus_{i\in\Lambda_2} \mathcal{H}_i^{(n_i)}$. Then $A \in B(\mathcal{H}_A)$ and $B \in B(\mathcal{H}_B)$. Moreover $T_r = A \oplus B$ and $T = A \oplus B \oplus T_e$.

We give the rest of the proof by proving three claims.

Claim (5.1.36)[451]: $B \oplus T_e \cong_a T_e$.

Since $T_j \lhd T_e$ for all $j \in \Lambda_2$ and $B = \bigoplus_{i\in\Lambda_2} T_j^{(n_j)}$, it follows that $B \lhd T_e$ and there exists a unital $*$-homomorphism ρ of $C^*(T_e)$ into $C^*(B)$ such that $\rho(T_e) = B$ and ρ annihilates $C^*(T_e) \cap K(\mathcal{H}_e)$. Then, by Lemma (5.1.15), we obtain $id \cong_a id \oplus \rho$, where id is the identity representation of $C^*(T_e)$. It follows that $T_e \cong_a T_e \oplus B$.

Claim (5.1.37)[451]: $C^*(A) \cap K(\mathcal{H}_A) = \{0\}$.

For a proof by contradiction, we assume that $C^*(A) \cap K(\mathcal{H}_A) \neq \{0\}$. In view of Corollary (5.1.21), this implies that there exists $j \in \Lambda_1$ such that

$$\left\{\begin{bmatrix} X^{(n_j)} & \\ & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)} \end{matrix} : X \in K(\mathcal{H}_j)\right\} \subset C^*(A).$$

Since $T_j \ntriangleleft T_e$, by Lemma (5.1.13), we have $T_j \ntriangleleft B \oplus T_e$. Now there are two possible cases.

Case $T_j \ntrianglelefteq B \oplus T_e$.

In this case, it follows from Lemma (5.1.16) (i) that there exists $Z \in C^*(T\ )$ with

$$Z = \begin{bmatrix} X^{(n_j)} & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_B \oplus \mathcal{H}_e \end{matrix}$$

and $X \neq 0$. By Lemma (5.1.17), it follows that $K(\mathcal{H}_j)^{(n_j)} \oplus 0 \oplus 0 \subset C^*(T) \cap K(\mathcal{H})$ and $\mathcal{H}_j^{(n_j)} \subset \mathcal{H}_e$, which is absurd.

Case $T_j \trianglelefteq B \oplus T_e$ and $T_j \ntriangleleft B \oplus T_e$.

In this case, it follows from Lemma (5.1.16) (ii) that there exists $Z \in C^*(T)$ with

$$Z = \begin{bmatrix} X^{(n_j)} & & \\ & Y & \\ & & K \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)}, \\ \mathcal{H}_B \oplus \mathcal{H}_e \end{matrix}$$

where $X \neq 0$ and K is compact acting on $\mathcal{H}_B \oplus \mathcal{H}_e$. Since $B \triangleleft T_e$, it follows from Lemma (5.1.12) that K has the form

$$K = \begin{bmatrix} 0 & \\ & \bar{K} \end{bmatrix} \begin{matrix} \mathcal{H}_B \\ \mathcal{H}_e \end{matrix},$$

where $\bar{K} \in C^*(T_e) \cap K(\mathcal{H}_e)$. Then, by Lemma (5.1.29), $C^*(T)$ contains the element

$$\begin{bmatrix} 0 & & \\ & 0 & \\ & & \bar{K} \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \\ \mathcal{H}_e \end{matrix} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & K \end{bmatrix} \begin{matrix} \mathcal{H}_j^{(n_j)} \\ \mathcal{H}_A \ominus \mathcal{H}_j^{(n_j)}. \\ \mathcal{H}_B \oplus \mathcal{H}_e \end{matrix}$$

Therefore we deduce that $C^*(T)$ contains

$$\begin{bmatrix} X^{(n_j)} & & \\ & Y & \\ & & 0 \end{bmatrix}.$$

One can prove that $K(\mathcal{H}_j^{(n_j)}) \oplus 0 \oplus 0 \in C^*(T) \cap K(\mathcal{H})$, a contradiction.

Denote $D = A \oplus T_e$ and $\mathcal{H}_D = \mathcal{H}_A \oplus \mathcal{H}_e$. Then $D \in B(\mathcal{H}_D)$. By Claim (5.1.38), $C^*(A) \cap K(\mathcal{H}_A) = \{0\}$. One can see that $D_e = T_e$. Denote by P_e the orthogonal projection of $\mathcal{H}_D$ onto $\mathcal{H}_e$. By Claim (5.1.37), we have $T = A \oplus B \oplus T_e \cong_a A \oplus T_e = D$.

So $T \in \overline{S(\mathcal{H})}$ implies that D a norm limit of $CSOs$. Hence we can choose a sequence $\{C_n\}$ of conjugations on $\mathcal{H}_D$ such that $C_n D C_n \to D^*$. For $R \in C^*(D)$, define $\varphi(R) = lim_n\, C_n\, R^* C_n$. Then φ is an anti-automorphism of $C^*(D)$ and $\varphi^{-1} = \varphi$. By Lemma

(5.1.27), $rank\ \varphi(X) = rank\ X$ for all $X \in C^*(D)$. So $\varphi(C^*(D) \cap K(\mathcal{H}_D)) = C^*(D) \cap K(\mathcal{H}_D)$.

Since $\mathcal{H}_A = \bigoplus_{i\in\Lambda_1} \mathcal{H}_i^{(n_i)}$, by Lemma (5.1.22), it suffices to prove for each $i \in \Lambda_1$ and each $x \in \mathcal{H}_i^{(n_i)}$ that the sequence $\{P_e C_n x\}$ converges to 0.

Now fix an $i_0 \in \Lambda_1$. For convenience we may directly assume that $n_{i_0} = 1$. The proof for $n_{i_0} > 1$ follows easily. Arbitrarily choose a vector $f \in \mathcal{H}_{i_0}$. It suffices to prove that $P_e C_n f \to 0$.

Since $T_{i_0} \ntrianglelefteq T_e$, using a similar argument as in the proof of Claim (5.1.38), one can check that $C^*(D)$ contains an operator Z of the form

$$Z = \begin{bmatrix} X & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_{i_0} \\ \mathcal{H}_A \ominus \mathcal{H}_{i_0}, \\ \mathcal{H}_e \end{matrix}$$

where $X \neq 0$. Then there exist nonzero vectors $g_0, f_0 \in \mathcal{H}_{i_0}$ such that $X g_0 = f_0$. Note that T_{i_0} is irreducible. By the well-known Double Commutant Theorem, we can find a sequence $\{p_k(\cdot,\cdot)\}$ of polynomials in two free variables such that

$$p_k(T_{i_0}^*, T_{i_0}) \xrightarrow{SOT} \frac{f \otimes f_0}{\|f_0\|^2}\ as\ k \to \infty. \qquad (3)$$

Set $\tilde{Z} = \varphi(Z)$. For any $K \in C^*(D) \cap K(\mathcal{H}_D)$, since $K(I - P_e) = 0$, it follows that $K\,Z = 0$ and $\tilde{Z}\,\varphi(K) = \varphi(K\,Z) = 0$. Noting that $\varphi(C^*(D) \cap K(\mathcal{H}_D)) = C^*(D) \cap K(\mathcal{H}_D)$, we obtain $\tilde{Z} P_e = 0$. Thus $\tilde{Z}$ admits the following matrix representation

$$Z = \begin{bmatrix} \tilde{X} & & \\ & \tilde{Y} & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_{i_0} \\ \mathcal{H}_A \ominus \mathcal{H}_{i_0}. \\ \mathcal{H}_e \end{matrix}$$

For any k, we have

$$\begin{aligned} \lim_n C_n\, p_k(D^*, D) Z C_n &= \lim_n (C_n\, p_k(D^*, D) C_n) \cdot (C_n Z C_n) \\ &= \tilde{p}_k(D, D^*)\varphi(Z^*) \\ &= \tilde{p}_k(D, D^*)\varphi(Z)^* \\ &= \tilde{p}_k(D, D^*)(\tilde{Z})^*. \end{aligned}$$

For each k, noting that $P_e \tilde{p}_k(D, D^*)(\tilde{Z})^* = 0$, we obtain

$$\|P_e C_n\, p_k(D^*, D) Z\| = \|P_e C_n\, p_k(D^*, D) Z C_n\| \longrightarrow 0$$

as $n \to \infty$. It follows that

$$\begin{aligned} \lim_n sup \|P_e C_n\, f\| &= \lim_n sup \|P_e C_n f - P_e C_n\, p_k(D^*, D) Z g_0\| \\ &\leq \|f - p_k(D^*, D)\, f_0\| \\ &= \left\| f - p_k(T_{i_0}^*, T_{i_0})\, f_0 \right\| \end{aligned}$$

for any $k \geq 1$. In view of (3), one can deduce that $\|P_e C_n\, f\| \to 0$.

It follows from Theorem (5.1.8) that A is a norm limit of $CSOs$. By Corollary (5.1.26), A is approximately unitarily equivalent to an operator which can be written as a direct sum of irreducible $CSOs$ and operators of the form $R \oplus R^t$, where R is irreducible and not a CSO. Note that T_e is a CSO and $C^*(T_e) \cap K(\mathcal{H}_e)$ is non-degenerate. In view of Proposition (5.1.35) , T_e can be written as a direct sum of irreducible $CSOs$ and operators of the form $R \oplus R^t$, where R is irreducible and not a CSO. Noting that

$$T = A \oplus B \oplus T_e \cong_a A \oplus T_e,$$

we conclude the proof.

We first give some auxiliary results.

Proposition (5.1.38)[451]: Let $T \in B(\mathcal{H})$ have the form

$$T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $A \in B(\mathcal{H}_1)$ satisfies $C^*(A) \cap K(\mathcal{H}_1) = \{0\}$ and $\overline{ran\, A + ran\, A^*} = \mathcal{H}_1$. Then $T \in \overline{S(\mathcal{H})}$ if and only if A is a norm limit of $CSOs$.

Proof. It is obvious that we need only prove the necessity. Since $C^*(A) \cap K(\mathcal{H}_1) = \{0\}$, by Theorem (5.1.8), it suffices to prove that A is g-normal.

Since $T \in \overline{S(\mathcal{H})}$, we can choose conjugations $\{C_n\}_{n=1}^{\infty}$ on $\mathcal{H}$ such that $C_n T C_n \to T^*$. For each $n \geq 1$, assume that

$$C_n = \begin{bmatrix} C_{1,1}^n & C_{1,2}^n \\ C_{2,1}^n & C_{2,2}^n \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}.$$

By Lemma (5.1.22), we need only verify for each $x \in \mathcal{H}_1$ that $C_{2,1}^n x \to 0$.

A direct matrix calculation shows that

$$C_n T - T^* C_n = \begin{bmatrix} * & * \\ C_{2,1}^n A & * \end{bmatrix} \text{ and } C_n T^* - T C_n = \begin{bmatrix} * & * \\ C_{2,1}^n A^* & * \end{bmatrix}.$$

Since $C_n T - T^* C_n \to 0$ and $C_n T^* - T C_n \to 0$, we deduce that $C_{2,1}^n A x \to 0$ and $C_{2,1}^n A^* x \to 0$ for each $x \in \mathcal{H}_1$. Noting that $\overline{ran\, A + ran\, A^*} = \mathcal{H}_1$ and $sup_n \|C_{2,1}^n\| \leq 1$, one can see that $C_{2,1}^n\, y \to 0$ for each $y \in \mathcal{H}_1$. This completes the proof.

Lemma (5.1.39)[451]: Let $T \in B(\mathcal{H})$ have the form

$$T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $A \in B(\mathcal{H}_1)$ and

$$\overline{ran\, A + ran\, A^*} = \mathcal{H}_1.$$

If $C^*(T) \cap K(\mathcal{H}) = \{0\}$, then $C^*(A) \cap K(\mathcal{H}_1) = \{0\}$.

Proof. For a proof by contradiction, we assume that $C^*(A) \cap K(\mathcal{H}_1) \neq \{0\}$. Then, by Lemma (5.1.24), we may directly assume that

$$A = \begin{bmatrix} A_1^{(n)} & 0 \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{H}_{1,1}^{(n)} \\ \mathcal{H}_{1,2} \end{matrix},$$

where $\mathcal{H}_1 = \mathcal{H}_{1,1}^{(n)} \oplus \mathcal{H}_{1,2}$, $A_i \in B(\mathcal{H}_{1,i}) (i = 1, 2), n \in \mathbb{N}$ and

$$K(\mathcal{H}_{1,1})^{(n)} \oplus 0_{H_{1,2}} \subset C^*(A).$$

For convenience, we may directly assume that $n = 1$.

Case (5.1.40)[451]: $\|p(A_1^*, A_1)\| \leq |p(0,0)|$ for any polynomial $p(\cdot,\cdot)$ in two free variables. In this case, it follows readily that $\|A_1^* A_1\| \leq 0$. So $A_1 = 0$, contradicting the fact that $\overline{ran\, A + ran\, A^*} = \mathcal{H}_1$.

Case (5.1.41)[451]: There exists a polynomial $p(\cdot,\cdot)$ in two free variables such that

$$\|p(A_1^*, A_1)\| > |p(0,0)|.$$

In this case, by Lemma (5.1.16) (i), $C^*(T)$ contains an operator of the form

$$\begin{bmatrix} X & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_{1,1} \\ \mathcal{H}_{1,2}, \\ \mathcal{H}_2 \end{matrix}$$

where $X \neq 0$. Note that $K(\mathcal{H}_{1,1}) \oplus 0_{H_{1,2}} \subset C^*(A)$. Then it follows from Lemma (5.1.17) that $K(\mathcal{H}_{1,1}) \oplus 0_{\mathcal{H}_{1,2}} \oplus 0_{H_2} \subset C^*(T)$, a contradiction. Therefore we conclude the proof.

Lemma (5.1.42)[451]: Let $T \in B(\mathcal{H})$ have the form

$$T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \end{matrix},$$

where $\mathcal{H}_A \oplus \mathcal{H}_B = \mathcal{H}, A \in B(\mathcal{H}_A)$ and $B \in B(\mathcal{H}_B)$. Assume that $C^*(B) \cap K(\mathcal{H}_B)$ is non-degenerate. Denote $M = \overline{ran\ B + ran\ B^*}$. If $A \lhd B$, then $A \lhd (B|_M)$.

Proof. It is obvious that M reduces B. Denote $B_1 = B|_M$. Then T can be written as

$$T = \begin{bmatrix} A & & \\ & 0 & \\ & & B_1 \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \ominus M \\ M \end{matrix}.$$

It suffices to prove $A \lhd B_1$.

Since $C^*(B) \cap K(\mathcal{H}_B)$ is non-degenerate, we can see that $\dim \mathcal{H}_B \ominus M < \infty$. We claim that $A \trianglelefteq B_1$. In fact, if not, then there exists $p(\cdot,\cdot)$ such that $\|p(A^*, A)\| > \|p(B_1^*, B_1)\|$. Since

$$\|p(A^*, A)\| \leq \|p(B^*, B)\| = \max\{|p(0,0)|, \|p(B_1^*, B_1)\|\},$$

we obtain $\|p(B_1^*, B_1)\| < \|p(A^*, A)\| \leq |p(0,0)|$. Let δ be a positive number satisfying $\|p(B_1^*, B_1)\| < \delta < \|p(A^*, A)\|$. Define

$$f(t) = \begin{cases} 0, & 0 \leq t \leq \delta, \\ t - \delta, & t > \delta. \end{cases}$$

Then $f(|p(T^*, T)|) \in C^*(T)$ has the form of

$$\begin{bmatrix} X & & \\ & Y & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \ominus M, \\ M \end{matrix}$$

where $X \neq 0$ and $Y \neq 0$. Noting that $\dim \mathcal{H}_B \ominus M < \infty$, Y is a nonzero compact operator and hence $Y \oplus 0 \in C^*(B) \cap K(\mathcal{H}_B)$. Since $A \lhd B$, it follows from Lemma (5.1.12) that $X = 0$, a contradiction. Thus we have proved that $A \trianglelefteq B_1$.

It remains to prove $A \lhd B_1$. In fact, if not, then, by Lemma (5.1.16), $C^*(T)$ contains an operator of the form

$$\begin{bmatrix} X_1 & & \\ & Y_1 & \\ & & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B \ominus M, \\ M \end{matrix}$$

where $X_1 \neq 0$ and $K_1 \in K(M)$. Since $\dim \mathcal{H}_B \ominus M < \infty, Y_1 \oplus K_1 \in C^*(B)$ is compact on $\mathcal{H}_B$. Noting that $A \lhd B$, it follows from Lemma (5.1.12) that $X_1 = 0$, a contradiction. This completes the proof.

The following result extends [211, Lemma 1] in the sense of approximation.

Theorem (5.1.43)[451]: Let $T \in B(\mathcal{H})$ have the form

$$T = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $R \in B(\mathcal{H}_1)$. Then $T \in \overline{S(\mathcal{H})}$ if and only if R is a norm limit of $CSOs$.

Proof. It is obvious that we need only prove the necessity. Assume that $T \in \overline{S(\mathcal{H})}$. By Lemma (5.1.25), we may assume that R is a direct sum of irreducible operators. Then, under this hypothesis, T is also a direct sum of irreducible operators. Without loss of generality, we may also assume that $\overline{ran\ R + ran\ R^*} = \mathcal{H}_1$. It can be seen from the proof of "(i)⟹(iv)" in Theorem (5.1.9) that T admits the following matrix representation

$$T = \begin{bmatrix} A & & \\ & B & \\ & & T_e \end{bmatrix} \begin{matrix} \mathcal{H}_A \\ \mathcal{H}_B, \\ \mathcal{H}_e \end{matrix}$$

where $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B \oplus \mathcal{H}_e$ and

(a) A is a norm limit of $CSOs$ and $C^*(A) \cap K(\mathcal{H}_A) = \{0\}$,

(b) T_e is a $CSO, T_e \cong_a B \oplus T_e, B \lhd T_e$ and $C^*(T_e) \cap K(\mathcal{H}_e)$ is non-degenerate,

(c) if M_1, M_2, M_3 are nonzero minimal reducing subspaces of A, B and T_e respectively, then any two of $A|_{M_1}$,$B|_{M_2}$ and $T_e|_{M_3}$ are not unitarily univalent.

By condition (c), one can deduce that exactly one of the following three holds: $\mathcal{H}_2 \subset \mathcal{H}_A$, $\mathcal{H}_2 \subset \mathcal{H}_B$ and $\mathcal{H} \subset \mathcal{H}_e$. So the rest of the proof is divided into three cases.

Case (5.1.44)[451]: $\mathcal{H}_2 \subset \mathcal{H}_A$. In this case, we can write

$$A = \begin{bmatrix} 0 & \\ & A_1 \end{bmatrix} \begin{matrix} \mathcal{H}_2 \\ \mathcal{H}_A \ominus \mathcal{H}_2 \end{matrix}.$$

Since $\overline{ran\, R + ran\, R^*} = \mathcal{H}_1$, one can see $\overline{ran\, A_1 + ran\, A_1^*} = \mathcal{H}_A \ominus \mathcal{H}_2$. By Lemma (5.1.40), $C^*(A_1)$ contains no nonzero compact operator. Since A is a norm limit of $CSOs$, it follows from Proposition (5.1.39) that A_1 is a norm limit of $CSOs$. Note that $R = A_1 \oplus B \oplus T_e \cong_a A_1 \oplus T_e$. Thus we deduce that R is a norm limit of $CSOs$.

Case (5.1.45)[451]: $\mathcal{H} \subset \mathcal{H}_B$. In this case, we can write

$$B = \begin{bmatrix} 0 & \\ & B_1 \end{bmatrix} \begin{matrix} \mathcal{H}_2 \\ \mathcal{H}_B \ominus \mathcal{H}_2 \end{matrix}.$$

Since $B \lhd T_e$, one can see $B_1 \lhd T_e$. Then, by Lemma (5.1.15), we obtain $B_1 \oplus T_e \cong_a T_e$. Noting that $R = A \oplus B_1 \oplus T_e \cong_a A \oplus T_e$, we can deduce that R is a norm limit of $CSOs$.

Case (5.1.46)[451]: $\mathcal{H}_2 \subset \mathcal{H}_e$. In this case, we can write

$$T_e = \begin{bmatrix} 0 & \\ & T_1 \end{bmatrix} \begin{matrix} \mathcal{H}_2 \\ \mathcal{H}_e \ominus \mathcal{H}_2 \end{matrix}.$$

Since T_e is a CSO, it follows from [211, Lemma 1] that T_1 is a CSO. Noting that $\overline{ran\, R + ran\, R^*} = \mathcal{H}_1$, one can see $\overline{ran\, T_1 + ran\, T_1^*} = \mathcal{H}_e \ominus \mathcal{H}_2$.

Since $C^*(T_e) \cap K(\mathcal{H}_e)$ is non-degenerate and $B \lhd T_e$, it follows from Lemma (5.1.43) that $B \lhd T_1$. By Lemma (5.1.15) ,we obtain $B \oplus T_1 \cong_a T_1$. Thus $R = A \oplus B \oplus T_1 \cong_a A \oplus T_1$. Noting that A is a limit of $CSOs$ and T_1 is a CSO, we deduce that R is a norm limit of $CSOs$. Thus we conclude the proof.

Lemma (5.1.47)[451]: Let T, R be two partial isometries on $\mathcal{H}$ and $T \cong_a R$. Denote by A_1 the compression of T to $(ker\, T)^\perp$, and by A_2 the compression of R to $(ker\, R)^\perp$. Then $A_1 \cong_a A_2$.

Proof. We first assume that

$$T = \begin{bmatrix} A_1 & 0 \\ B_1 & 0 \end{bmatrix} \begin{matrix} (kerT)^\perp \\ ker\, T \end{matrix} \quad \text{and} \quad R = \begin{bmatrix} A_2 & 0 \\ B_2 & 0 \end{bmatrix} \begin{matrix} (kerR)^\perp \\ ker\, R \end{matrix}.$$

Since T, R are two partial isometries, it follows that $A_1^* A_1 + B_1^* B_1 = I_1$ and $A_2^* A_2 + B_2^* B_2 = I_2$, where I_1 is the identity operator on $(ker\, T)^\perp$ and I_2 is the identity operator on $(ker\, R)^\perp$. Noting that $T \cong_a R$, we can choose unitary operators $\{U_n\}_{n=1}^\infty$ on $\mathcal{H}$ such that $TU_n - U_n R \to 0$ as $n \to \infty$. Without loss of generality, we assume that

$$U_n = \begin{bmatrix} U_{1,1}^n & U_{1,2}^n \\ U_{2,1}^n & U_{2,2}^n \end{bmatrix},$$

where

$$U_{1,1}^{n} \in B((ker\ R)^{\perp}, (ker\ T)^{\perp}), U_{1,2}^{n} \in B(ker\ R, (ker\ T\ )^{\perp})$$

and

$$U_{2,1}^{n} \in B((ker\ R)^{\perp}, ker\ T), U_{2,2}^{n} \in B(ker\ R, ker\ T).$$

A matrix computation shows that

$$TU_n - U_nR\ = \begin{bmatrix} * & A_1U_{1,2}^{n} \\ * & B_1U_{1,2}^{n} \end{bmatrix}.$$

Thus $A_1U_{1,2}^{n} \to 0$ and $B_1U_{1,2}^{n} \to 0$ as $n \to \infty$. Noting that

$$U_{1,2}^{n} = (B_1^*\, B_1 + A_1^*\, A_1)U_{1,2}^{n} = B_1^*\, B_1U_{1,2}^{n} + A_1^*\, A_1U_{1,2}^{n},$$

we obtain $U_{1,2}^{n} \to 0$ as $n \to \infty$.

Note that

$$RU_n^* - U_n^*\, T = \begin{bmatrix} * & A_2\left(U_{2,1}^{n}\right)^* \\ * & B_2\left(U_{2,1}^{n}\right)^* \end{bmatrix} \to 0$$

as $n \to \infty$. Using a similar argument as above, one can prove that $\|U_{2,1}^{n}\| = \|(U_{2,1}^{n})^*\| \to 0$ as $n \to \infty$. Since $U_n^*U_n = I = U_nU_n^*$, we have

$$(U_{1,1}^{n})^*U_{1,1}^{n} + (U_{2,1}^{n})^*U_{2,1}^{n} = I_2 \text{ and } U_{1,1}^{n}(U_{1,1}^{n})^* +\ U_{1,2}^{n}(U_{1,2}^{n})^*\ = I_1.$$

It follows readily that $U_{1,1}^{n}$ is invertible for n large enough and $(U_{1,1}^{n})^*U_{1,1}^{n} \to I_2$. Hence $|U_{1,1}^{n}| \to I_2$ as $n \to \infty$.

For each $n \geq 1$, assume that $U_{1,1}^{n} = V_{1,1}^{n}|U_{1,1}^{n}|$ is the polar decomposition of $U_{1,1}^{n}$, where $V_{1,1}^{n}: (ker\ R)^{\perp} \to (ker\ T)^{\perp}$ is a partial isometry. Then, by the discussion above, $V_{1,1}^{n}$ is invertible and hence unitary for n large enough. Moreover, since $|U_{1,1}^{n}| \to I_2$, we deduce that $\|V_{1,1}^{n} - U_{1,1}^{n}\| \to 0$ as $n \to \infty$.

Since $TU_n - U_n\, R \to 0$, a direct calculation shows that

$$A_1U_{1,1}^{n} - U_{1,1}^{n}A_2 -\ U_{1,2}^{n}B_2 \longrightarrow 0.$$

Noting that $V_{1,1}^{n} - U_{1,1}^{n} \to 0$ and $U_{1,2}^{n} \to 0$ as $n \to \infty$, we deduce that $A_1V_{1,1}^{n} - V_{1,1}^{n}A_2 \to 0$, that is, $A_1 \cong_a A_2$.

Now we can give the Proof of Theorem (5.1.10).

Theorem (5.1.48)[451]: Let $T \in B(\mathcal{H})$ be a partial isometry. Denote by A the compression of T to its initial space. Then $T \in \overline{S(\mathcal{H})}$ if and only if A is a norm limit of $CSOs$.

We let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ denote the set of complex numbers, the set of real numbers and the set of positive integers respectively. Given $A \in B(\mathcal{H})$, we let $\sigma(A)$ and $\sigma_e(A)$ denote the spectrum and the essential spectrum of A respectively. Denote by $ker\ A$ and ran A the kernel of A and the range of A respectively. As usual, given two representations ρ_1 and ρ_2 of a C^*-algebra, we write $\rho_1 \cong \rho_2 (\rho_1 \cong_a \rho_2)$ to denote that ρ_1 and ρ_2 are unitarily equivalent (approximately unitarily equivalent, respectively).

Proof of Theorem (5.1.48) "⟹" Since $T \in \overline{S(\mathcal{H})}$, it follows from Theorem (5.1.9) that there exists $F \in S(\mathcal{H})$ such that $T \cong_a F$. It is easy to check that F is also a partial isometry. Denote by A_1 the compression of F to $(ker\ F)^{\perp}$. By [211, Theorem 2], $F \in S(\mathcal{H})$ implies that A_1 is a CSO. Noting that $T \cong_a F$, it follows from Lemma (5.1.49) that $A \cong_a A_1$. Thus A is a norm limit of $CSOs$.

"⟸" Since A is a norm limit of $CSOs$, there exists a CSO A_1 on $(ker\ T)^{\perp}$ such that $A \cong_a A_1$. We assume that

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (ker\ T)^{\perp} \\ ker\ T \end{matrix}.$$

Since T is a partial isometry, it follows that $A^*A + B^*\ B = I_1$ and hence $|B| = \sqrt{I_1 -\ A^*A}$, where I_1 is the identity operator on $(ker\ T)^{\perp}$. The rest of the proof is divided into three cases.

Case (5.1.49)[451]: $dimker\ B = dim\ ker\ B^*$.

Assume that $B = V|B|$ is the polar decomposition of B, where $V: (ker\ T)^{\perp} \to ker\ T$ is a partial isometry. Since $dim\ ker\ B = dim\ ker\ B^*$, we have dimker $V = dim\ ker\ V^*$. Then V can be extend to a unitary operator $U: (ker\ T)^{\perp} \to ker\ T$. Then $U^*V|B| = |B|$. Define $W: (ker\ T)^{\perp} \oplus (ker\ T)^{\perp} \longrightarrow H$ as $W : (x, y) \longrightarrow x + Uy$.

Thus W is a unitary operator. A direct matrical calculation shows that

$$W^*TW = \begin{bmatrix} A & 0 \\ |B| & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ \sqrt{I_1 - A^*A} & 0 \end{bmatrix} \begin{matrix} (ker\ T)^{\perp} \\ (ker\ T)^{\perp} \end{matrix}.$$

Noting that $A \cong_a A_1$, it is easy to check that

$$W^*TW \cong_a \begin{bmatrix} A_1 & 0 \\ \sqrt{I_1 - A_1^*A_1} & 0 \end{bmatrix} \triangleq L.$$

It is clear that

$$L^*L = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (ker\ T)^{\perp} \\ (ker\ T)^{\perp} \end{matrix}.$$

Thus L is a partial isometry and the compression of L to its initial space is A_1. Since A_1 is a CSO, it follows from [211, Theorem 2] that L is complex symmetric, and hence $T \in \overline{S(H)}$.

Case (5.1.50)[451]: $dimker\ B < dim\ ker\ B^*$.

Since $ker\ B^* = ker\ T \ominus \overline{ran\ B}$, there exists a subspace M of $ker\ T$ such that $ran\ B \subset M$ and $dim\ M \ominus \overline{ran\ B} = dim\ ker\ B$. Then T admits the following matrix representation

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T \end{matrix} = \left[\begin{array}{c|cc} A & 0 & 0 \\ \hline B_1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{matrix} (\ker T)^{\perp} \\ M \\ (\ker T) \ominus M \end{matrix},$$

where $B_1 \in B((ker\ T)^{\perp}, M)$. Note that $ker\ B =\ ker\ B_1$ and $ran\ B = ran\ B_1$. We have

$$dim\ ker\ B_1 = dim\ ker\ B = dim\ M \ominus \overline{ran\ B}$$
$$= dim\ M \ominus \overline{ran\ B_1} = dim\ ker\ B_1^*.$$

Set

$$F = \begin{bmatrix} A & 0 \\ B_1 & 0 \end{bmatrix} \begin{matrix} (ker\ T)^{\perp} \\ M \end{matrix}.$$

One can check that F is a partial isometry and $(ker\ F)^{\perp} = (ker\ T)^{\perp}$. Noting that $A \cong_a A_1$ and $dim\ ker\ B_1 = dim\ ker\ B_1^*$, it can be seen from the argument in Case (5.1.49) that F is a norm limit of $CSOs$. Then it follows readily that $T \in \overline{S(H)}$.

Case (5.1.51)[451]: $dimker\ B > dim\ ker\ B^*$.

In this case, we can choose a Hilbert space M such that $dim\ M + dim\ ker\ B^* = dim\ ker\ B$. Set

$$R = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ M \end{matrix} = \left[\begin{array}{c|cc} A & 0 & 0 \\ \hline B & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{matrix} (\ker T)^{\perp} \\ \ker T \\ M \end{matrix} \triangleq \begin{bmatrix} A & 0 \\ B_2 & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T \oplus M \end{matrix}.$$

Noting that $ker\ B_2 = ker\ B, ran\ B_2\ = ran\ B$ and $ker\ B_2^* = ker\ B^* \oplus M$, we obtain

$$dim\ ker\ B_2^* = dim\ ker\ B^* + dim\ M = dim\ ker\ B = dim\ ker\ B_2.$$

Obviously, R is still a partial isometry and A is the compression of R to its initial space. Since $A \cong_a A_1$ and $dim\ ker\ B_2 = dim\ ker\ B_2^*$, it can be seen from the proof in Case (5.1.49) that R is a norm limit of $CSOs$. By Theorem (5.1.44), it follows that $T \in \overline{S(H)}$. This completes the proof.

Let $T \in B(H)$. Assume that $T = U|T|$ be the polar decomposition of T. Recall that the Aluthge transform of T is defined to be the operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (see [193]).

Corollary (5.1.52)[451]: Let $T \in B(H)$ be a partial isometry. Then $T \in \overline{S(H)}$ if and only if the Aluthge transform of T is a norm limit of $CSOs$.

Proof : We first assume that

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (ker\ T)^{\perp} \\ ker\ T \end{matrix}.$$

Then

$$T^*T = \begin{bmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where I_1 is the unit operator on $ker\ T$. Since T is a partial isometry, the Aluthge transform of T is

$$|T|^{\frac{1}{2}}\, T\, |T|^{\frac{1}{2}} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (ker\ T)^{\perp} \\ ker\ T \end{matrix}.$$

In view of Theorems (5.1.44) and (5.1.10), the desired result follows readily.

Section (5.2): Complex Symmetric Generators

$\mathcal{H}(\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{K}\ etc.)$ will always denote a complex separable Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. We let $\mathfrak{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathfrak{B}(\mathcal{H})$, we let $C^*(T)$ denote the C^*-algebra generated by T and the identity I. If $\mathcal{A}$ is a C^*-subalgebra of $\mathfrak{B}(\mathcal{H})$ and $\mathcal{A} = C^*(T)$for some $T \in \mathfrak{B}(\mathcal{H})$, then T is called a generator of $\mathcal{A}$.

We are interested in C^*-algebras which are singly generated by complex symmetric operators.

Definition (5.2.1)[452]: A map C on $\mathcal{H}$ is called a conjugation if C is conjugate-linear, $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$.

Definition (5.2.2)[452]: An operator $T \in \mathfrak{B}(\mathcal{H})$ is called a complex symmetric operator (CSO for short) if there exists a conjugation C on $\mathcal{H}$ so that $CTC = T^*$.

$CSOs$ can be viewed as a generalization of symmetric matrices to the case of operators on Hilbert spaces. The general study of $CSOs$ was initiated by Garcia, Putinar and Wogen in [245,246,248,249]. $CSOs$ have many motivations in function theory, matrix analysis and other areas. In particular, $CSOs$ are closely related to the study of truncated Toeplitz operators [247], which was initiated in Sarason's seminal [259]. Recently, some interesting results concerning $CSOs$ have been obtained (see [237,244,250–252,256,263,264]).

Since $CSOs$ have certain nice structural properties, it is natural to explore the algebraic aspects of the theory of $CSOs$. Recently certain connections between $CSOs$ and C^*-algebras generated by them are established, and a C^*-algebraic approach has been developed to answer a number of open questions concerning $CSOs$ (see [251,252,263]). This proves to be very fruitful.

In [260], where many von Neumann algebras and C^*-algebras prove to have a single complex symmetric generator. We shall concentrate on those C^*-algebras singly generated by essentially normal operators, which have been the subject of much interest since the seminal [236].

First we are interested in the following question.

Question (5.2.3)[452]: When does an essentially normal operator T have $C^*(T)$generated by a complex symmetric operator?

There exist operators T lying outside the class of $CSOs$ such that $C^*(T)$ admits a complex symmetric generator (see Examples (5.2.11) and (5.2.42)). Hence the above question is natural and worth answering.

We give a complete answer to Question (5.2.3)(see Theorem (5.2.14)). We give a decomposition of such operators. Our result shows that whether or not $C^*(T)$ has a complex symmetric generator depends heavily on the spectral picture of the restrictions of T to its minimal reducing subspaces. The proof of our result depends on some approximation results, which are developed using tools from BDF theory, Voiculescu's theorem and noncommutative approximation theory of operators [254].

Two $*$-isomorphic C^*-algebras have the same algebraic properties. The following question arises naturally.

Question (5.2.4)[452]: When is $C^*(T)$ $*$-isomorphic to a C^*-algebra singly generated by $CSOs$?

When T is essentially normal, we give an answer to the above question (see Theorem (5.2.44)). In order to answer Question (5.2.4), we need to introduce an algebraic analogue of $CSOs$.

Given a polynomial $p(z_1, z_2)$ in two free variables z_1, z_2, we let $\tilde{p}(z_1, z_2)$ denote the polynomial obtained from $p(z_1, z_2)$ by conjugating each coefficient.

Definition (5.2.5)[452]: An operator $A \in \mathfrak{B}(\mathcal{H})$ is said to be g-normal if it satisfies $\|p(A^*, A)\| = \|\tilde{p}(A, A^*)\|$ for any polynomial $p(\cdot,\cdot)$ in two free variables.

The above concept was inspired by Garica, Lutz and Timotin [243], and posed by Guo, Ji and the author [251]. It was proved that an operator A is g-normal if and only if there is an anti-automorphism φ of $C^*(A)$ such that $\varphi(A) = A$ (see [251, Lem. 1.7]). G-normal operators, containing all $CSOs$, play an important role in solving the norm closure problem for $CSOs$ (see [251,263]). Obviously, g-normal elements in a C^*-algebra can be defined in the same manner as in Definition (5.2.5).

That an operator is g-normal if and only if it is algebraically equivalent to a CSO (see Theorem (5.2.45)). Thus the notion of g-normal operator is a suitable algebraic analogue of $CSOs$. Recall that two operators A, B are algebraically equivalent (write $A \approx B$) if there is a $*$-isomorphism of $C^*(A)$ onto $C^*(B)$ which carries A into B

We shall solve Question (5.2.3) in the irreducible case. We shall prove some approximation results and give necessary and sufficient conditions for an essentially normal operator to have a complex symmetric generator for its C^*-algebra. We study the algebraical equivalence of certain special operators and give a complete answer to Question (5.2.4) in the essentially normal case (see Theorem (5.2.44)).

For convenience, we write $A \in (cs)$ to denote that $C^*(A)$ admits a complex symmetric generator.

We shall use the BDF Theorem to derive a necessary spectral condition for an essentially normal operator T to satisfy $T \in (cs)$(see Lemma (5.2.7)), and then prove that the spectral condition is also sufficient when T is irreducible (see Theorem (5.2.8)).

$\mathbb{C}$ and $\mathbb{N}$ denote respectively the set of complex numbers and the set of natural numbers. In the following, unless otherwise stated, $\mathcal{H}$ is always assumed to be a complex separable infinite-dimensional Hilbert space. We let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators in $\mathfrak{B}(\mathcal{H})$.

Let $T \in \mathfrak{B}(\mathcal{H})$. We denote by $\sigma(T)$ the spectrum of T. Denote by $kerT$ and $ranT$ the kernel of T and the range of T respectively. T is called a semi-Fredholm operator, if $ranT$ is closed and either dimker$T$ or dimker T^* is finite; in this case, $indT := dimkerT - dimkerT^*$ is called the index of T. In particular, if $-\infty < indT < \infty$, then T is called a Fredholm operator. The Wolf spectrum of T and the essential spectrum of T are defined respectively as

$$\sigma_{lre}(T) \coloneqq \{\lambda \in \mathbb{C}: T - \lambda \ is \ not \ semi - Fredholm\}$$

and

$$\sigma_e(T) := \{\lambda \in \mathbb{C}: T - \lambda \ is \ not \ Fredholm\}$$

The spectral picture of an operator T, denoted by $\Lambda(T)$, consists of the Wolf spectrum and the values of the index function off the Wolf spectrum. So two operators A, B have the same spectral picture if and only if $\sigma_{lre}(A) = \sigma_{lre}(B)$ and $ind(A - \lambda) = ind(B - \lambda)$ for $\lambda \notin \sigma_{lre}(A)$.

Recall that an operator T is essentially normal if $T^*T - TT^*$ is compact. It is well known that $\sigma_{lre}(T) = \sigma_e(T)$ when T is essentially normal. The classical BDF Theorem classifies essentially normal operators up to unitary equivalence modulo compacts.

Theorem (5.2.6)[452]: Let $A, B \in \mathfrak{B}(\mathcal{H})$ be essentially normal. Then there exists $K \in \mathcal{K}(\mathcal{H})$ such that $A \cong B + K$ if and only if $\Lambda(A) = \Lambda(B)$.

Here and in what follows, $\cong$ denotes unitary equivalence.

Following Berg and Davidson [235], we say that an operator T is almost normal if $T = N + K$ for some normal N and some compact K. Then almost normal operators are always essentially normal. By Theorem (5.2.1), an essentially normal operator A is almost normal if and only if $ind(A - \lambda) = 0$ for all $\lambda \notin \sigma_e(A)$. By the continuity of the index function, one can see that the class of almost normal operators on $\mathcal{H}$ is norm closed.

Lemma (5.2.7)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If $C^*(T)$ admits a complex symmetric generator, then T is almost normal.

Proof. Assume that $A \in \mathfrak{B}(\mathcal{H})$ is complex symmetric and $C^*(T) = C^*(A)$. Then there is a conjugation C on $\mathcal{H}$ such that $CAC = A^*$. Then for each $\lambda \notin \sigma_{lre}(A)$ one can check that

$$ind\ (A - \lambda) = ind\ C(A - \lambda)C = ind(A - \lambda)^* = -ind\ (A - \lambda).$$

So $ind(A - \lambda) = 0$ for $\lambda \notin \sigma_{lre}(A)$. On the other hand, since T is essentially normal and $A \in C^*(T)$, it follows that A is essentially normal. By the BDF Theorem, A has the form "normal plus compact". Since $T \in C^*(A)$, T is also of the form "normal plus compact".

Theorem (5.2.8)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T is irreducible, then $T \in (cs)$ if and only if T is almost normal.

The proof of the preceding result depends on a key approximation result.

Proposition (5.2.9)[452]: Given a normal operator $T \in \mathfrak{B}(\mathcal{H}))$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is an irreducible CSO.

Proof. By the Weyl–von Neumann Theorem, we may directly assume that T is a diagonal operator with respect to some ONB $\{e_n\}_{n=1}^{\infty}$of $\mathcal{H}$. Assume that $\{\lambda_n\}_{n=1}^{\infty}$are the eigenvalues of T satisfying $Te_n = \lambda_n e_n$ for $n \geq 1$. For each $n \geq 1$, denote $a_n = Re\lambda_n$ and $b_n = Im\lambda_n$. Up to a small compact perturbation, we may assume that $a_i \neq a_j$ for $i \neq j$. Set

$$A = \sum_{i=1}^{\infty} a_i e_i \otimes e_i, \quad B = \sum_{i=1}^{\infty} b_i e_i \otimes e_i.$$

Then $T = A + iB$. For $i, j \geq 1$, set $d_{i,j} = \frac{\varepsilon}{2^{i+j}}$. Define a compact operator $K_1 \in \mathcal{K}(\mathcal{H})$ by

$$K_1 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \cdots \\ d_{2,1} & d_{2,2} & d_{2,3} & \cdots \\ d_{3,1} & d_{3,2} & d_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix}.$$

It is obvious that $K_1 \in \mathcal{K}(\mathcal{H})$ is self-adjoint and $\|K_1\| < 2\left(\sum_{n=1}^{\infty} \frac{\varepsilon}{2^{1+n}}\right) = \varepsilon$. Set $K = iK_1$. Then it remains to check that $T + K$ is an irreducible CSO.

Note that $T + K = A + iB_1$, where $B_1 = B + K_1$. Then A, B_1 are both self-adjoint. Assume that $P \in \mathfrak{B}(\mathcal{H})$ is an orthogonal projection commuting with $T + K$. It follows that

$$PA = AP \text{ and } PB_1 = B_1P.$$

Since $A = \sum_{i=1}^{\infty} a_i e_i \otimes e_i$ and $a_i \neq a_j$ whenever $i \neq j$, it follows from $AP = PA$ that $P^2 = \sum_{i=1}^{\infty} \mu_i e_i \otimes e_i$, where $\mu_i = 0$ or $\mu_i = 1$ for each $i \geq 1$. On the other hand, for $i, j \geq 1$ with $i \neq j$, we have

$$\langle PB_1 e_j, e_i \rangle = \langle B_1 e_j, Pe_i \rangle = \langle B_1 e_j, \mu_i e_i \rangle = \mu_i \langle Be_j, e_i \rangle + \mu_i \langle K_1 e_j, e_i \rangle = \mu_i d_{i,j} = \frac{\mu_i}{2^{i+j}}$$

and

$$\langle B_1 Pe_j, e_i \rangle = \langle Pe_j, B_1 e_i \rangle = \mu_j \langle e_j, B_1 e_i \rangle = \mu_j \langle Be_j, e_i \rangle + \mu_j \langle K_1 e_j, e_i \rangle = \mu_j d_{i,j} = \frac{\mu_j}{2^{i+j}}.$$

Since $PB_1 = B_1P$, it follows that $\mu_i = \mu_j$. Then either $P = 0$ or P Is the identity operator on $\mathcal{H}$, which implies that $T + K$ is irreducible.

Now it remains to show that $T + K$ is a CSO. In fact, if C is the conjugation on $\mathcal{H}$ defined by $Ce_i = e_i$ for $i \geq 1$, then one can check that $C(A + K)C = (A + K)^*$. Since each of the operators A, B, K_1 admits a complex symmetric matrix representation with respect to the same ONB $\{e_n\}$, one can also see that $T + K = A + i(B + K_1)$ is complex symmetric.

We remark that the proof of Proposition (5.2.9) is inspired by the proof of Lemma (5.2.31) (see [254, Lem. 4.33]or [258]).

Proof. The necessity follows from Lemma (5.2.7).

"$\Leftarrow$". Since T is almost normal, there exist a normal operator N and $K \in \mathcal{K}(\mathcal{H})$ so that $T = N + K$. By Proposition (5.2.9), we can find compact K_0 such that $R := N + K_0$ is an irreducible CSO. Since T, R are both irreducible and essentially normal, we have $\mathcal{K}(\mathcal{H}) \subset C^*(T) \cap C^*(R)$. It follows that $T - R = K - 0 \in C^*(T) \cap C^*(R)$. Thus $C^*(T) = C^*(R)$. This completes the proof.

In general, the condition of irreducibility in Theorem (5.2.8) can not be canceled. That is, the spectral condition "$ind(T - \lambda) = 0, \forall \lambda \notin \sigma_e(T)$" is necessary and not sufficient for $T \in (cs)$. Before giving an example, we first introduce a useful result.

Recall that an operator A is said to be abnormal if A has no nonzero reducing subspace $\mathcal{M}$ such that $A|_{\mathcal{M}}$ is normal. If an irreducible operator is not normal, then it is abnormal. Each Hilbert space operator T admits the unique decomposition

$$T = T_{nor} \oplus T_{abnor},$$

where T_{nor} is normal and T_{abnor} is abnormal. The operators T_{nor} and T_{abnor} are called the normal part and the abnormal part of T respectively. see [254, p. 116].

Lemma (5.2.10)[452]: ([251, Lem. 3.2]). An operator T is complex symmetric if and only if T_{abnor} is complex symmetric.

Example (5.2.11)[452]: Let $S \in \mathfrak{B}(\mathcal{H}_1)$ be the unilateral shift of multiplicity one and $N \in \mathfrak{B}(\mathcal{H}_2)$ be a normal operator with $\sigma(N) = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\}$. Denote $T = N \oplus S$. Then T is essentially normal. Note that $\sigma_{lre}(N) = \sigma(N) \supset \sigma(S)$. Thus $\sigma(T) = \sigma_{lre}(T)$ and $ind(T - \lambda) = 0$ for $\lambda \notin \sigma_{lre}(T)$. It follows from Theorem (5.2.1)that T is almost normal.

Now we shall show that $C^*(T)$ does not have a complex symmetric generator. For a proof by contradiction, we assume that $A \in \mathfrak{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is a complex symmetric generator of $C^*(T)$. Obviously, A can be written as $A = A_1 \oplus A_2$, where $A_i \in \mathfrak{B}(\mathcal{H}_i),\ i = 1, 2$. So $C^*(A_1) = C^*(N)$ and $C^*(A_2) = C^*(S)$. It follows immediately that A_1 is normal, A_2 is irreducible and not normal. So A_2 is abnormal. Hence $A_2 = A_{nor}$ and $A_2 = A_{abnor}$. Since A is complex symmetric, it follows from Lemma (5.2.10) that 2 is complex symmetric. Thus $C^*(S)$ has a complex symmetric generator A_2^2. By Lemma (5.2.7), S is almost normal. This is a contradiction, since S is Fredholm and $ind\ S = -1$.

We shall characterize when an essentially normal operator has a complex symmetric generator for its C^*-algebra. To state our main result, we need several extra definitions.

Definition (5.2.12)[452]: ([251, Def. 1.8]). Let $T \in \mathfrak{B}(\mathcal{H})$. An operator $A \in \mathfrak{B}(\mathcal{H})$ is called a transpose of T if $A = CT^*C$ for some conjugation C on $\mathcal{H}$.

The concept "transpose" of an operator is in fact a generalization of that for matrices. By definition, an operator $T \in \mathfrak{B}(\mathcal{H})$ is complex symmetric if and only if T is a transpose of itself. In general, an operator has more than one transpose [262, Ex. 2.2]. However, one can check that any two transposes of an operator are unitarily equivalent ([251]). We often write T^t to denote a transpose of T. In general, there is no ambiguity especially when we write $T \cong T^t$. It is easy to check that $\sigma(T) = \sigma(T^t)$, $\sigma_{lre}(T) = \sigma_{lre}(T^t)$ and $ind(T - \lambda) = -ind(T^t - \lambda)$ for $\lambda \notin \sigma_{lre}(T)$.

If $\mathcal{M}$ is a nonzero reducing subspace of $T \in \mathfrak{B}(\mathcal{H})$ and $T|_{\mathcal{M}}$ is irreducible, then $\mathcal{M}$ is called a minimal reducing subspace (m.r.s. for short) of λ. Given an essentially normal operator $T \in \mathfrak{B}(\mathcal{H})$, define

$$\mathcal{H}_s = \bigvee\{\mathcal{M} \subset \mathcal{H}: \mathcal{M} \text{ is am.r.s.of } T \text{ and } T|_{\mathcal{M}} \text{ is notalmostnormal}\},$$

where $\vee$ denotes closed linear span. It is obvious that $\mathcal{H}_s$ is either absent or a nonzero reducing subspace of T. Denote by T_s the restriction of T to $\mathcal{H}_s$. We call T_s the singular part of T.

We say that two operators A, B are disjoint if there exist no nonzero reducing subspace $\mathcal{M}_1$of A and nonzero reducing subspace $\mathcal{M}_2$ of B such that $A|_{\mathcal{M}_1} \cong B|_{\mathcal{M}_2}$.

Definition (5.2.13)[452]: An essentially normal operator T is called type C, if $T = T_s$ and T is unitarily equivalent to an operator of the form $A \oplus B$, where (a) $A, B \in \mathfrak{B}(\mathcal{H})$ are disjoint, (b) $C^*(A) \cap \mathcal{K}(H) = C^*(B) \cap \mathcal{K}(\mathcal{H})$, and (c) there exists compact $K \in C^*(A)$ such that $A + K$ is a transpose of B and $C^*(A+K) \cap \mathcal{K}(\mathcal{H}) = C^*(A) \cap \mathcal{K}(\mathcal{H})$.

One can check that if an essentially normal operator T is of type C, then T is almost normal. In fact, by the discussion right after Definition (5.2.12), we have

$$\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B) = \sigma_e(A)$$

and

$$ind\ (T-\lambda) = ind(A-\lambda) + ind(B-\lambda) = 0 \text{ for all } \lambda \notin \sigma_e(T^2).$$

By Theorem (5.2.6), T is almost normal.

Theorem (5.2.14)[452]: If $T \in \mathfrak{B}(\mathcal{H})$ is essentially normal, then $T \in (cs)$ if and only if T_s is either absent or of type C.

By Theorem (5.2.14), whether or not an essentially normal operator T has a complex symmetric generator for its C^*-algebra depends only on the behavior of T_s.

We give a concrete description of the essentially normal operators of type C. We first make some preparation.

Let $\{\mathcal{A}_i\}_{i\in\Gamma}$ be a family of C^*-algebras. We denote by $\prod_{i\in\Gamma} \mathcal{A}_i$ the direct product of $\{\mathcal{A}_i\}_{i\in\Gamma}$, and by $\bigoplus_{i\in\Gamma} \mathcal{A}_i$ the direct sum of $\{\mathcal{A}_i\}_{i\in\Gamma}$.

Let $A \in \mathfrak{B}(\mathcal{H})$. We let $W^*(A)$ denote the von Neumann algebra generated by A. By the von Neu-mann Double Commutant Theorem, we have $W^*(A) = C^*(A)''$. Here and in what follows, $\mathcal{A}'$ denotes the commutant algebra of $\mathcal{A}$. See [240, Thm. 3.1] for a proof of the following result.

Lemma (5.2.15)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ and assume that $T = \bigoplus_{i\in\Gamma} T_i$, where $T_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_i \ncong T_j$ whenever $i \neq j$. Then

$$C^*(T)' = \prod_{i\in\Gamma} \mathbb{C}I_i, \qquad W^*(T) = \prod_{i\in\Gamma} \mathfrak{B}(\mathcal{H}_i),$$

where I_i is the identity operator on $\mathcal{H}_i$ and $\mathbb{C}I_i = \{\lambda I_i : \lambda \in \mathbb{C}\}$ for $i \in \Gamma$.

For convenience, we let $0_{\mathcal{H}}$ denote the subalgebra $\{0\}$ of $\mathfrak{B}(\mathcal{H})$. Given $e, f \in \mathcal{H}$, the operator $e \otimes f$ is defined as $(e \otimes f)(x) = \langle x, f\rangle e$ for $x \in \mathcal{H}$.

Corollary (5.2.16)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal and $T = N \oplus (\bigoplus_{i=1}^{\infty} T_i)$, where

- (i) $N \in \mathfrak{B}(\mathcal{H}_0)$ is normal,
- (ii) $T_i \in B(\mathcal{H}_i)$ is irreducible and not normal for $i \geq 1$, and
- (iii) $T_i \ncong T_j$ whenever $i \neq j$.

Then $0_{\mathcal{H}_0} \oplus (\bigoplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)) \subset C^*(T)$. Moreover, if N is absent, then

$$C^*(T) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i).$$

Proof. For any fixed $i \geq 1$ and fixed $e, f \in \mathcal{H}_i$, it suffices to prove that $f \otimes e \in C^*(T)$. Set $K = T^*T - TT^*$. By the hypothesis, we may assume $K = 0 \oplus (\bigoplus_{j}^{\infty} K_j)$, where $K_j \in \mathcal{K}(\mathcal{H}_j)$ for $j \geq 1$. It is obvious that $K_j \neq 0$ for all $j \geq 1$ since T_j is not normal. There exist nonzero $e_1, f_1 \in \mathcal{H}_i$ such that $K_i e_1 = f_1$. We may assume that $\|f_1\| = 1$.

Set $A = \bigoplus_{i=1}^{\infty} T_j$. Since each T_j is irreducible and $T_{j_1} \ncong T_{j_2}$ for $j_1 \neq j_2$, it follows from Lemma (5.2.15) that each operator commuting with both A and A^* has the form $\bigoplus_{j=1}^{\infty} \lambda_j I_j$, where I_j is the identity operator on $\mathcal{H}_j$. Moreover, we have

$$W^*(A^2) = \prod_{j=1}^{\infty} \mathfrak{B}(\mathcal{H}_j)\,.$$

So $f \otimes e \in W^*(A)$ and, by the von Neumann Double Commutant Theorem, we have $f \otimes f_1, e_1 \otimes e, f \otimes e \in \overline{C^*(A)}^{SOT}$. Here SOT denotes the strong operator topology. Using the Kaplansky Density Theorem [238, Thm.I.7.3, Rem.I.7.4], we can choose polynomials $\{p_N(\cdot,\cdot)\}$ and $\{q_N(\cdot,\cdot)\}$ in two free variables so that

$$p_n(A^*, A) \xrightarrow{SOT} f \otimes f_1, \qquad q_n(A^*, A) \xrightarrow{SOT} e_1 \otimes e.$$

Since $\bigoplus_{j=1}^{\infty} K_j$ is compact, we obtain

$$p_n(A^*, A)\left(\bigoplus_{j=1}^{\infty} K_j\right) q_n(A^*, A) \xrightarrow{\|\cdot\|} f \otimes e.$$

Moreover, we obtain

$$p_n(T^*, T) K q_n(T^*, T) = \begin{bmatrix} 0 & 0 \\ 0 & p_n(A^*, A)(\bigoplus K_i) q_n(A^*, A) \end{bmatrix} \xrightarrow{\|\cdot\|} \begin{bmatrix} 0 & 0 \\ 0 & f \otimes e \end{bmatrix},$$

which completes the proof.

Recall that an operator is said to be completely reducible if it does not admit any minimal reducing subspace [242].

Lemma (5.2.17)[452]: If an essentially normal operator T is completely reducible, then T^2 is normal.

Proof. Assume that $T \in \mathfrak{B}(\mathcal{H})$. Since T is completely reducible, by [242, Lem. 2.5], we have $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Noting that T is essentially normal, we obtain $T^*T - TT^* \in C^*(T) \cap \mathcal{K}(\mathcal{H})$. Thus $T^*T - TT^* = 0$.

If d is a cardinal number and $\mathcal{H}$ is a Hilbert space, let $\mathcal{H}^{(d)}$denote the direct sum of $\mathcal{H}$ with itself d times. If $A \in \mathfrak{B}(\mathcal{H}), A^{(d)}$ is the direct sum of A^2 with itself d times.

Lemma (5.2.18)[452]: ([242, Prop. 2.4]). Each operator $T \in \mathfrak{B}(\mathcal{H})$ is unitarily equivalent to an operator of the form

$$T_0 \oplus \left(\bigoplus_{i \in \Gamma} T_i^{(n_i)}\right),$$

where T_0 is completely reducible, each T_i is irreducible and $T_i \ncong T_j$ for $i, j \in \Gamma$ with $i \neq j$.

Lemma (5.2.19)[452]: [252, Prop. 2.3]. Let $T \in \mathfrak{B}(\mathcal{H})$ and $T = T_0 \oplus \left(\bigoplus_{i \in \Gamma} T_i^{(n_i)}\right)$, where T_0 is completely reducible, T_i is irreducible and $1 \leq n_i \leq \infty$ for $i \in \Gamma$; moreover, $T_i \ncong T_j$ whenever $i, j \in \Gamma$ and $i \neq j$. Then each reducing subspace $\mathcal{M}$ of T has the form of $\mathcal{M}_0 \oplus (\bigoplus_{i \in \Gamma} \mathcal{M}_i)$, where $\mathcal{M}_0$ is a reducing subspace of T_0 and M_i is a reducing subspace of $T_i^{(n_i)}$ for $i \in \Gamma$.

Lemma (5.2.20)[452]: ([252, Lem. 2.6]). Let $T = A^{(n)}$, where $A \in \mathfrak{B}(\mathcal{H})$ is irreducible and $1 \leq n \leq \infty$. If $\mathcal{M}$ is a nonzero reducing subspace of T, then $T|_{\mathcal{M}} \cong A$ if and only if $T|_{\mathcal{M}}$ is irreducible.

Lemma (5.2.21)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. Then T_{abnor} is unitarily equivalent to an operator of the form

$$\bigoplus_{i \in \Gamma} T_i^{(n_i)},$$

where each T_i is irreducible, not normal and $T_i \ncong T_j$ for $i, j \in \Gamma$ with $i \neq j$. Moreover, T_s is the restriction of T_{abnor} to a reducing subspace and

$$T_s \cong \bigoplus_{i \in \Gamma_0} T_i^{(n_i)},$$

where $\Gamma_0 = \{i \in \Gamma: T_i$ is not almost normal$\}$.

Proof. By Lemma (5.2.18), T_{abnor} is unitarily equivalent to an operator of the form

$$T_0 \oplus \left(\oplus_{i\in\Gamma} T_i^{(n_i)}\right),$$

where $T_0 \in \mathfrak{B}(\mathcal{H}_0)$ is completely reducible, each $T_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible and $T_i \ncong T_j$ for $i, j \in \Gamma$ with $i \neq j$. Note that T_i is abnormal for $i \in \Gamma$. Since T_0 is completely reducible and essentially normal, it follows from Lemma (5.2.17) that T_0 is normal. Note that T_{abnor} is abnormal; so T_0 is absent. Then $T_{abnor} \cong \oplus_{i\in\Gamma} T_i^{(n_i)}$. For convenience we directly assume that $T_{abnor} = \oplus_{i\in\Gamma} T_i^{(n_i)}$. Thus

$$T = T_{nor} \oplus \left(\oplus_{i\in\Gamma} T_i^{(n_i)}\right).$$

By definition, it is obvious that $\oplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)} \subset \mathcal{H}_s$. On the other hand, if $\mathcal{M}$ is a m.r.s. of T and $T|_{\mathcal{M}}$ is not almost normal, then, by Lemmas (5.2.19) and (5.2.20), there exists $i_0 \in \Gamma$ such that $\mathcal{M} \subset \mathcal{H}_{i_0}^{(n_{i_0})}$ and $T|_{\mathcal{M}} \cong T_{i_0}$. So T_{i_0} is not almost normal and $\mathcal{H}_{i_0}^{(n_{i_0})} \subset \oplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Thus $\mathcal{M} \subset \oplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Furthermore we obtain $\mathcal{H}_s \subset \oplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Therefore $\mathcal{H}_s = \oplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$.

Corollary (5.2.22)[452]: If $T \in \mathfrak{B}(\mathcal{H})$ is essentially normal, then $T = T_s$ if and only if T is the direct sum of a family of essentially normal operators which are irreducible and not almost normal.

Proposition (5.2.23)[452]: An essentially normal operator T is of type C if and only if T is unitarily equivalent to an operator of the form

$$\oplus_{1\le i<\upsilon} (A_i \oplus B_i)^{(n_i)}, \qquad 1 \le n_i < \infty,$$

where (i) $\upsilon \in \mathbb{N}$ or $\upsilon = \infty, \{A_i, B_i: 1 \le i < \upsilon\}$ are irreducible and no two of them are unitarily equivalent, (ii) A_i is not almost normal and there exists compact K_i such that $A_i + K_i$ is a transpose of B_i for each i, and (iii) $\|K_i\| \to 0$ if $\upsilon = \infty$.

Proof."⟸". Assume that $A_i, B_i \in \mathfrak{B}(\mathcal{H}_i)$ for $1 \le i < \upsilon$. Denote $\mathcal{H} = \oplus_{1\le i<\upsilon} \mathcal{H}_i^{(n_i)}$ and

$$A = \oplus_{1\le i<\upsilon} A_i^{(n_i)}, \qquad B = \oplus_{1\le i<\upsilon} B_i^{(n_i)}.$$

Then $A, B \in \mathfrak{B}(\mathcal{H})$ are essentially normal and $T \cong A \oplus B$. For convenience we directly assume that $T = A \oplus B$ and $\upsilon = \infty$.

Since $\{A_i, B_i: 1 \le i < \upsilon\}$ are irreducible, not normal and no two of them are unitarily equivalent, it follows from Corollary (5.2.16) that

$$C^*(A) \cap \mathcal{K}(\mathcal{H}) = \oplus_{1\le i<\upsilon} \mathcal{K}(\mathcal{H}_i)^{(n_i)} = C^*(B) \cap \mathcal{K}(\mathcal{H}). \tag{4}$$

Moreover, if $\mathcal{M}$ is a m.r.s. of T, then, by Lemmas (5.2.19) and (5.2.20), there exists unique i_0 with $1 \le i_0 < \upsilon$ such that exactly one of the following holds

$$T|_{\mathcal{M}} \cong A_{i_0}, \qquad T|_{\mathcal{M}} \cong B_{i_0}.$$

It follows that A, B are disjoint; moreover, $T|_{\mathcal{M}}$ is not almost normal. Thus, by Corollary (5.2.22), $T = T_s$.

By statement (ii), for each $1 \le i < \upsilon$, we can find a conjugation C_i on $\mathcal{H}_i$ so that

$$A_i + K_i = C_i B_i^* C_i.$$

Set
$$K = \oplus_{1\le i<\upsilon} K_i^{(n_i)}, C = \oplus_{1\le i<\upsilon} C_i^{(n_i)}.$$

Then C is a conjugation on $\mathcal{H}$ and, by (4), $K \in C^*(A) \cap \mathcal{K}(\mathcal{H})$, since $\|K_j\| \to 0$; moreover,

$$CB^*C = A + K.$$

On the other hand, since $\{B_i\}$ are irreducible, not normal and no two of them are unitarily equivalent, so are $\{A_i + K_i\}$. It follows from Corollary (5.2.16) that

$$C^*(A+K) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{1\le i<v} \mathcal{K}(\mathcal{H}_i)^{(n_i)} = C^*(A) \cap \mathcal{K}(\mathcal{H}).$$

"$\Longrightarrow$". Now assume that $T = T_s$ and $T = A \oplus B$, where $A, B \in \mathfrak{B}(\mathcal{H})$ satisfy conditions (a), (b) and (c) in Definition (5.2.13). Since $T = T_s$, it follows that $A = A_s$. Then, by Corollary (5.2.22), we may assume that

$$A = \bigoplus_{i\in\Gamma} A_i^{(n_i)}, 1 \le n_i < \infty,$$

where each $A_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible, not almost normal and $A_i \ncong A_j$ whenever $i \ne j$. By Corollary (5.2.16), we have

$$C^*(A) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{i\in\Gamma} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$

Then K can be written as

$$K = \bigoplus_{i\in\Gamma} K_i^{(n_i)},$$

where $K_i \in \mathcal{K}(\mathcal{H}_i)$ for $i \in \Gamma$, and $\|K_i\| \to 0$ if Γ is infinite. Since $C^*(A) \cap \mathcal{K}(\mathcal{H}) = C^*(A) \cap \mathcal{K}(\mathcal{H})$ is an ideal of $C^*(B)$, B can be written as

$$B = \bigoplus_{i\in\Gamma} E_i^{(n_i)},$$

moreover, this means that $\mathcal{K}(\mathcal{H}_i) \subset C^*(E_i)$, E_i is irreducible and $E_i \ncong E_j$ whenever $i \ne j$. Since A, B are disjoint, we deduce that no two of $\{A_i,\ E_i : i \in \Gamma\}$ are unitarily equivalent.

That $A + K = \bigoplus_{i\in\Gamma} (A_i + K_i)^{(n_i)}$ and $C^*(A+K) \cap \mathcal{K}(\mathcal{H}) = C^*(A) \cap \mathcal{K}(\mathcal{H})$. As we have done to B, we can also deduce that $\{A_i + K_i\}$ are irreducible and no two of them are unitarily equivalent.

By the hypothesis, $A + K$ is a transpose of B Thus $\bigoplus_{i\in\Gamma} (A_i + K_i)^{(n_i)}$ and $\bigoplus_{i\in\Gamma} (E_i^t)^{(n_i)}$ are unitarily equivalent, and their m.r.s.'s correspond one to one. Then, by Lemmas (5.2.19) and (5.2.20), there exists a bijective map $\tau: \Gamma \to \Gamma$ such that $A_i + K_i \cong E_{\tau(i)}^t$ and $n_i = n_{\tau(i)}$ for all $i \in \Gamma$. For each $i \in \Gamma$, set $B_i = E_{\tau(i)}$. Then, up to unitary equivalence, $A_i + K_i$ is a transpose of B_i for each $i \in \Gamma$.

Lemma (5.2.24)[452]: Let $\mathcal{H} = \bigoplus_{i\in\Gamma} \mathcal{H}_i$ and $A \in \mathfrak{B}(\mathcal{H})$ with $A = \bigoplus_{i\in\Gamma} A_i$, where $A_i \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. If $B \in \mathfrak{B}(\mathcal{H})$ and $C^*(A) = C^*(B)$, then there exist $B_i \in \mathfrak{B}(\mathcal{H}_i), i \in \Gamma$, such that $B = \bigoplus_{i\in\Gamma} B_i$ and

(i) for any subset Γ_0 of $\Gamma, C^*(\bigoplus_{i\in\Gamma_0} A_i) = C^*(\bigoplus_{i\in\Gamma_0} B_i)$,

(ii) for each $i \in \Gamma$, the reducing subspaces of A_i coincide with that of B_i,

(iii) for each $i \in \Gamma, A_i$ is irreducible if and only if B_i is irreducible,

(iv) for any $i, j \in \Gamma, A_i \cong A_j$ if and only if $B_i \cong B_j$.

Proof. Since $C^*(A) = C^*(B)$, it is clear that B^2 has the form $B = \bigoplus_{i\in\Gamma} B_i$, where $B_i \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Statement (i) is also clear.

(ii) By (i), we have $C^*(A_i) = C^*(B_i)$. Thus $C^*(A_i)' = C^*(B_i)'$ and the assertion holds.

(iii) This follows immediately from (ii).

(iv) We directly assume $i \ne j$. By (i), we have $C^*(A_i \oplus A_j) = C^*(B_i \oplus B_j)$. If $A_i \cong A_j$, then there exists unitary operator $U: \mathcal{H}_j \to \mathcal{H}_i$ such that $A_j = U^* A_i U$. Then, for any polynomial $p(\cdot,\cdot)$ in two free variables, we have $p(A_j^*, A_j) = U^* p(A_i^*, A_i) U$. It follows immediately that each operator in $C^*(A_i \oplus A_j)$ has the form $X \oplus U^* X U$, where $X \in C^*(A_i)$.

Sine $B_i \oplus B_j \in C^*(A_i \oplus A_j)$, we obtain $B_j = U^* B_i U$, that is, $B_i \cong B_j$. Thus $A_i \cong A_j$ implies $B_i \cong B_j$. Likewise, one can see the converse.

Lemma (5.2.25)[452]: Let $T, R \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If $C^*(T) = C^*(R)$, then
(i) T_s is absent if and only if R_s is absent, and
(ii) $C^*(T_s) = C^*(R_s)$.

Proof. In view of Lemma (5.2.21), we may assume that

$$T = T_{nor} \oplus \left(\oplus_{i\in\Gamma} T_i^{(n_i)}\right), 1 \le n_i < \infty,$$

where $T_{nor} \in \mathfrak{B}(\mathcal{H}_0)$, $T_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible and not normal for $i \in \Gamma$; moreover, $T_i \ncong T_j$ whenever $i \neq j$. Since $C^*(T) = C^*(R), R$ can be written as

$$R = R_0 \oplus \left(\oplus_{i\in\Gamma} R_i^{(n_i)}\right),$$

where $R_0 \in \mathfrak{B}(\mathcal{H}_0)$ and $R_i \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Thus $C^*(R_0) = C^*(T_{nor})$ and $C^*(R_i) = C^*(T_i)$ for $i \in \Gamma$. Then R_0 is normal; moreover, by Lemma (5.2.24), each R_i is irreducible, not normal and $R_i \ncong R_j$whenever $i \neq j$. For each $i \in \Gamma$, we note that R_i is almost normal if and only if T_i is almost normal.

Denote $\Gamma_0 = \{i \in \Gamma : T_i \text{is notalmostnormal}\}$. Then $\Gamma_0 = \{i \in \Gamma : R_i \text{ is notalmostnormal}\}$. Thus, by Lemma (5.2.21),

$$T_s = \oplus_{i\in\Gamma_0} T_i^{(n_i)}, \qquad R_s = \oplus_{i\in\Gamma_0} R_i^{(n_i)}.$$

From $C^*(T) = C^*(R)$, we deduce that $C^*(T_s) = C^*(R_s)$. This completes the proof.

Lemma (5.2.26)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal and $T = \oplus_{i=1}^{\infty} A_i$, where $A_i \in \mathfrak{B}(\mathcal{H}_i)$ for $i \ge 1$. Assume that $B_i \in \mathfrak{B}(\mathcal{H}_i)$ is a transpose of A_i for $i \ge 1$. If $p(z_1, z_2)$ is a polynomial in two free variables, then there exists $\oplus_{i=1}^{\infty} K_i \in \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$ such that $p(B_i^*, B_i) + K_i$ is a transpose of $p(A_i^*, A_i)$ for $i \ge 1$.

Proof. By the hypothesis, there exist conjugations $\{C_i\}_{i=1}^{\infty}$ such that $B_i = C_i A_i^* C_i, i \ge 1$. Set $E_i = A_i^* A_i - A_i A_i^*$ for $i \ge 1$. Since T is essentially normal, we have $T^*T - TT^* = \oplus_{i=1}^{\infty} E_i \in \mathcal{K}(\mathcal{H})$. So $E_i \in \mathcal{K}(\mathcal{H}_i)$ for $i \ge 1$ and $\| E_i \| \to 0$.

For convenience, we assume that $p(z_1, z_2) = z_1 z_2 z_1$. The proof in general case is similar. Compute to see that

$$\begin{aligned} C_i p(A_i^*, A_i)^* C_i &= C_i A_i A_i^* A_i C_i = B_i^* B_i B_i^* \\ &= B_i^*(B_i B_i^*) B_i^* - B_i^*(B_i^* B_i) B_i^* + B_i^*(B_i^* B_i) B_i^* \\ &= B_i^*(B_i B_i^* - B_i^* B_i) B_i^* + p(B_i^*, B_i) \\ &= B_i^*(C_i E_i C_i) B_i^* + p(B_i^*, B_i). \end{aligned}$$

Set $K_i = B_i^*(C_i E_i C_i) B_i^*$. So K_i is compact and $p(B_i^*, B_i) + K_i$ is a transpose of $p(A_i^*, A_i)$; moreover, we have

$$\|K_i\| \le \|B_i\| \cdot \|E_i\| = \|A_i\| \cdot \| E_i\| \le \|T\| \cdot \| E_i\| \to 0.$$

Hence $\oplus_{i=1}^{\infty} K_i \in \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$. This completes the proof.

Proposition (5.2.27)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$be essentially normal and

$$T = \oplus_{j=1}^{\infty} (A_j \oplus B_j),$$

where $A_j, B_j \in \mathfrak{B}(\mathcal{H}_j)$ and B_j is a transpose of A_j for $j \ge 1$. Then each operator $R \in C^*(T)$ can be written as $R = \oplus_{i=1}^{\infty} (F_j \oplus G_j)$, where $G_j \in \mathfrak{B}(\mathcal{H}_j)$ is a compact perturbation of some transpose F_j^t of F_j^2 and $\left\|G_j - F_j^t\right\| \to 0$.

Proof. Since B_j is a transpose of A_j, there exists a conjugation C_j such that $B_j = C_j A_j^* C_j$. Assume that $\{p_n\}_{n=1}^{\infty}$ are polynomials in two free variables and $p_n(T^*, T) \to R$. Note that

$\bigoplus_{j=1}^{\infty} A_j$ is essentially normal. Then, by Lemma (5.2.26), for each $n \geq 1$, there exist compact operators $\{K_{j,n}\}_{j\geq 1}$ such that

$$p_n(B_j^*, B_j) + K_{j,n} = C_j p_n(A_j^*, A_j)^* C_j$$

and $\|K_{j,n}\| \to 0$ as $j \to \infty$. Then $\bigoplus_{j=1}^{\infty} K_{j,n}$ is compact for each $n \geq 1$.

Note that $p_n(T^*, T) \to R$ as $n \to \infty$ and

$$p_n(T^*, T) = \bigoplus_{j=1}^{\infty} \left(p_n(A_j^*, A_j) \oplus p_n(B_j^*, B_j)\right), \qquad n \geq 1.$$

Then $\bigoplus_{j=1}^{\infty} p_n(A_j^*, A_j)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} F_j$ and $\bigoplus_{j=1}^{\infty} p_n(B_j^*, B_j)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} G_j$ as $n \to \infty$. Then

$$\bigoplus_{j=1}^{\infty} C_j p_n(A_j^*, A_j)^* C_j \longrightarrow \bigoplus_{j=1}^{\infty} C_j F_j^* C_j.$$

So, as $n \to \infty$, we have

$$\bigoplus_{j=1}^{\infty} K_{j,n} = \bigoplus_{j=1}^{\infty} \left(C_j p_n(A_j^*, A_j)^* C_j - p_n(B_j^*, B_j)\right) \longrightarrow \bigoplus_{j=1}^{\infty} \left(C_j F_j^* C_j - G_j\right).$$

For each $n \geq 1$, note that $\bigoplus_{j=1}^{*} K_{j,n}$ is compact. Thus their norm limit $\bigoplus_{j=1}^{\infty} \left(C_j F_j^* C_j - G_j\right)$ is also compact. Hence $C_j F_j^* C_j - G_j$ is compact for each j and $\|C_j F_j^* C_j - G_j\| \to 0$ as $j \to \infty$. Note that $R = lim_n p_n(T^*, T) = \bigoplus_{j=1}^{\infty} (F_j \oplus G_j)$. This completes the proof.

Now we are going to give the proof for the necessity of Theorem (5.2.14).

Proof for the necessity of Theorem (5.2.14) Assume that $R \in \mathfrak{B}(\mathcal{H})$ is complex symmetric and $C^*(T) = C^*(R)$. Also we assume that T_s is not absent. Then, by Lemma (5.2.25), R_s is not absent. Since T is essentially normal, so is R. By Lemma (5.2.10), R_{abnor} is complex symmetric. By [251, Thm. 2.8], R_{abnor} is a direct sum of irreducible $CSOs$ and operators with form of $Z \oplus Z^t$, where Z is irreducible and not complex symmetric. Note that each essentially normal CSO is almost normal. Then, up to unitary equivalence, we may assume that

$$R = N \oplus \left(\bigoplus_{i\in\Gamma_1} R_i^{(m_i)}\right) \oplus \left(\bigoplus_{j\in\Gamma_2} (A_j \oplus B_j)^{(n_j)}\right), \tag{5}$$

where

(i) $N = R_{nor}$ is normal, $\{R_i, A_j, B_j : i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(ii) each R_i is almost normal and not normal;

(iii) A_j is not almost normal and B_j is a transpose of A_j for $j \in \Gamma_2$.

Note that each of $\{R_i, A_j, B_j : i \in \Gamma_1, j \in \Gamma_2\}$ is abnormal. Since R is essentially normal, it follows that $1 \leq m_i,\ n_j < \infty$ for all i, j.

We assume that $N \in \mathfrak{B}(\mathcal{H}_0)$, $R_i \in \mathfrak{B}(\mathcal{H}_{1,i})$ and $A_j, B_j \in \mathfrak{B}(\mathcal{H}_{2,j})$ for $i \in \Gamma_1$ and $j \in \Gamma_2$. Hence

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\bigoplus_{i\in\Gamma_1} \mathcal{H}_{1,i}^{(m_i)}\right) \oplus \left(\bigoplus_{j\in\Gamma_2} (\mathcal{H}_{2,j} \oplus \mathcal{H}_{2,j})^{(n_j)}\right). \tag{6}$$

Since $C^*(T) = C^*(R)$, in view of Lemma (5.2.24), T can be written as

$$T = D \oplus \left(\bigoplus_{i\in\Gamma_1} E_i^{(m_i)}\right) \oplus \left(\bigoplus_{j\in\Gamma_2} (F_j \oplus G_j)^{(n_j)}\right) \tag{7}$$

with respect to the decomposition (6); moreover, by statements (i)–(ii), we have

(v) D is normal, $\{E_i, F_j, G_j : i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(vi) each E_i is almost normal and not normal for $i \in \Gamma_1$;

(vii) $F_j,\ G_j$ are essentially normal and not almost normal for $j \in \Gamma_2$.

By Lemma (5.2.21), we have $T_s = \bigoplus_{j\in\Gamma_2} (F_j \oplus G_j)^{(n_j)}$. On the other hand, note that

$$\bigoplus_{j\in\Gamma_2} (F_j \oplus G_j) \in C^*\left(\bigoplus_{j\in\Gamma_2} (A_j \oplus B_j)\right).$$

It follows from Proposition (5.2.27) that G_j is a compact perturbation of a transpose F_j^t of F_j for $j \in \Gamma_2$, and $\left\|G_j - F_j^t\right\| \to 0$ if Γ_2 is infinite. By Proposition (5.2.23), T_s is of type C. This proves the necessity.

To give the proof for the sufficiency of Theorem (5.2.14), we need to prove several approximation results.

Lemma (5.2.28)[452]: ([255, Lem.3.2.6]). Let $T \in \mathfrak{B}(\mathcal{H})$ and suppose that $\emptyset \neq \Delta \subset \sigma_{lre}(K)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that

$$T + K = \begin{bmatrix} N & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where N is a diagonal normal operator of uniformly infinite multiplicity, $\sigma(N) = \sigma_{lre}(N) = \bar{\Delta}, \sigma(T) = \sigma(A)$ and $\Lambda(T) = \Lambda(A)$.

Corollary (5.2.29)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ and suppose that $\lambda \in \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that

$$T + K = \begin{bmatrix} \lambda & * \\ 0 & A^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^{\perp} \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector and $A \in B(\{e\}^{\perp})$ satisfies $\sigma(T) = \sigma(A)$.

Proof. By Lemma (5.2.28), there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that

$$T + K = \begin{bmatrix} \lambda I_1 & * \\ 0 & A_0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}, dim\mathcal{H}_1 = \infty, I_1$ is the identity operator on $\mathcal{H}_1$and $A_0 \in \mathfrak{B}(\mathcal{H}_2)$ satisfies $\sigma(A_0) = \sigma(T)$. Choose a unit vector $e \in \mathcal{H}_1$. Then $T + K$ can be written as

$$T + K = \begin{bmatrix} \lambda & 0 & E \\ 0 & \lambda I_2 & F \\ 0 & 0 & A_0 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \mathcal{H}_1 \ominus \mathbb{C}e, \\ \mathcal{H}_2 \end{matrix}$$

where I_2 is the identity operator on $\mathcal{H}_1 \ominus \mathbb{C}e$. Set

$$A = \begin{bmatrix} \lambda I_2 & F \\ 0 & A_0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \ominus \mathbb{C}e \\ \mathcal{H}_2 \end{matrix}.$$

Since $\lambda \in \sigma(T) = \sigma(A_0)$, it follows that $\sigma(A) = \sigma(T)$. Noting that

$$T + K = \begin{bmatrix} \lambda & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \mathcal{H} \ominus \mathbb{C}e \end{matrix},$$

we conclude the proof.

Given a subset Δ of $\mathbb{C}$, we write isoΔ for the set of all isolated points of Δ. For $\lambda \in C$ and $\varepsilon > 0$, denote $B(\lambda, \varepsilon) = \{z \in \mathbb{C}: |z - \lambda| < \varepsilon\}$.

Lemma (5.2.30)[452]: Let $A, B \in \mathfrak{B}(\mathcal{H})$. Assume that $\lambda \in iso\ \sigma(A)$ and $\lambda \notin \sigma(B)$. Then there exists $\delta > 0$ such that

$$\text{“}E, F \in \mathfrak{B}(\mathcal{H}), \quad \|E\| < \delta, \|F\| < \delta\text{”} \Longrightarrow \text{“}\sigma(A + E) \neq \sigma(B + F)\text{”}.$$

Proof. Since $\lambda \in iso\ \sigma(A)$ and $\lambda \notin \sigma(B)$, there exists $\varepsilon > 0$ such that $B(\lambda, \varepsilon)^- \cap \sigma(A) = \{\lambda\}$ and $B(\lambda, \varepsilon)^- \cap \sigma(B) = \emptyset$. Then, by the upper semi-continuity of spectrum (see [254, Thm. 1.1]), there exists $\delta > 0$ such that

(i) $B(\lambda, \varepsilon)^- \cap \sigma(A + E) \neq \emptyset$ for any $E \in \mathfrak{B}(\mathcal{H})$ with $\|E\| < \delta$, and

(ii) $B(\lambda, \varepsilon)^- \cap \sigma(B + F) = \emptyset$ for any $F \in \mathfrak{B}(\mathcal{H})$ with $\|F\| < \delta$.

Hence we conclude the proof.

In the preceding lemma, A, B can be operators acting on different separable Hilbert spaces.

Lemma (5.2.31)[452]: ([253]). Given $T \in \mathfrak{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K$ is irreducible.

Lemma (5.2.32)[452]: Let $\{A_i\}_{i=1}^n$ be operators on separate Hilbert spaces with pairwise distinct spectra. Then, given $B \in \mathfrak{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $A_{n+1} := B + K$ is irreducible, and $\{\sigma(A_i)\}_{i=1}^{n+1}$ are pairwise distinct.

Proof. Choose a point λ_0 in $\partial\sigma(B) \cap \sigma_{lre}(B)$. By Corollary (5.2.29), there exists compact K_0 with $\|K_0\| < \frac{\varepsilon}{2}$ such that

$$B + K_0 = \begin{bmatrix} \lambda_0 & E \\ 0 & B_0 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector and $\sigma(B_0) = \sigma(B)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ outside $\sigma(B^2)$ such that $sup_{1 \le i \le n+1} |\lambda_i - \lambda_0| < \frac{\varepsilon}{4}$. For each $1 \le i \le n+1$, set

$$B_i = \begin{bmatrix} \lambda_i & E \\ 0 & B_0 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix}.$$

Then $\|B + K_0 - B_i\| < \frac{\varepsilon}{4}$, $\in iso\, \sigma(B_i)$ and $\lambda_j \notin \sigma(B_i)$ whenever $i \neq j$. By Lemma (5.2.31), there exist compact operators F_i with $\|F_i\| < \frac{\varepsilon}{4}$ such that each $B_i + F_i$ is irreducible; moreover, by Lemma (5.2.30), we may also assume that $\{\sigma(B_i + F_i)\}_{i=1}^{n+1}$ are pairwise distinct. Then there exists some $i_0, 1 \le i_0 \le n+1$, such that $\sigma(B_{i_0} + F_{i_0}) \neq \sigma(A_j)$ for $1 \le j \le n$. Set $K = F_{i_0} + B_{i_0} - B$ and $A_{n+1} = B + K$. Then $A_{n+1} = B_{i_0} + F_{i_0}$ is irreducible. Noting that $K = F_{i_0} + B_{i_0} - (B + K_0) + K_0$ is compact,

$$\|K\| \le \left\|F_{i_0}\right\| + \left\|B_{i_0} - (B + K_0)\right\| + \|K_0\| < \varepsilon$$

and $\{\sigma(A_i)\}_{i=1}^{n+1}$ are pairwise distinct, we complete the proof.

In view of Lemma (5.2.32), the following corollary is clear.

Corollary (5.2.33)[452]: Given a sequence $\{A_i\}_{i=1}^\infty$ of operators and $\varepsilon > 0$, there exist compact operators $\{K_i\}_{i=1}^\infty$ with

$$sup_i \|K_i\| < \varepsilon \text{ and } lim_i \|K_i\| = 0$$

such that each $A_i + K_i$ is irreducible for $i \ge 1$ and $\{\sigma(A_i + K_i)\}_{i=1}^\infty$ are pairwise distinct.

Lemma (5.2.34)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be normal. Then, given $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist irreducible $CSOs$ $T_1, T_2, \cdots, T_n \in \mathfrak{B}(\mathcal{H})$ with pairwise distinct spectra such that $T_i - T \in \mathcal{K}(\mathcal{H})$ and $\|T_i - T\| < \varepsilon$ for all $1 \le i \le n$.

Proof. Choose a point λ in $\partial\sigma(T) \cap \sigma_{lre}(T)$. By the classical Weyl–von Neumann Theorem, there exists compact K with $\|K\| < \frac{\varepsilon}{2}$ such that

$$T + K = \begin{bmatrix} \lambda & 0 \\ 0 & N \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector, N is normal and $\sigma(N) = \sigma(T)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1, \lambda_2, \cdots, \lambda_n$ outside $\sigma(T)$ such that $sup_{1 \le i \le n} |\lambda_i - \lambda_0| < \frac{\varepsilon}{4}$. For each $1 \le i \le n$, set

$$A_t = \begin{bmatrix} \lambda_i & 0 \\ 0 & N \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix}.$$

Then $\|T+K-A_i\|<\frac{\varepsilon}{4}$, $\lambda_i \in \operatorname{iso}\sigma(A_i)$ and $\lambda_j \notin \sigma(A_i)$ whenever $i \neq j$. By Proposition (5.2.15), there exist compact operators F_i with $\|F_i\|<\frac{\varepsilon}{4}$ such that each A_i+F_i is irreducible and complex symmetric; moreover, by Lemma (5.2.30), it can be required that $\{\sigma(A_i+F_i)\}_{i=1}^{n}$ are pairwise distinct. Set $T_i=A_i+F_i$ for $1 \le i \le n$. Then $\{T_i: 1 \le i \le n\}$ satisfy all requirements.

Corollary (5.2.35)[452]: Let $\{T_i\}_{i=1}^{\infty}$ be normal operators on separable Hilbert spaces. Then, given $\varepsilon>0$, there exist compact operators $\{K_i\}_{i=1}^{\infty}$ with

$$sup_i \|K_i\|<\varepsilon, \qquad \lim_i \|K_i\| = 0$$

such that

(i) T_i+K_i is complex symmetric and irreducible for $i \geq 1$, and

(ii) $\sigma(T_i+K_i) \neq \sigma(T_j+K_j)$ whenever $i \neq j$.

Proof. For convenience, we assume that $T_i \in \mathfrak{B}(\mathcal{H}_i)$ for $i \geq 1$. We shall construct $\{K_i\}_{i=1}^{\infty}$ by induction.

By Proposition (5.2.15), we can choose $K_1 \in \mathcal{K}(\mathcal{H}_1)$ with $\|K_1\|<\varepsilon$ such that T_1+K_1 is irreducible and complex symmetric.

Now assume that we have chosen compact operators $K_i \in \mathcal{K}(\mathcal{H}_i), 1 \le i \le n$, satisfying that (a) $\|K_i\|<\varepsilon/i$ for $1 \le i \le n$, (b) T_i+K_i is complex symmetric and irreducible for $1 \le i \le n$, and (c) $\sigma(T_i+K_i) \neq \sigma(T_j+K_j)$ whenever $1 \le i \neq j \le n$. We are going to choose $K_{n+1} \in K(H_{n+1})$ with $\|K_{n+1}\|<\varepsilon/(n+1)$ such that $T_{n+1}+K_{n+1}$ is irreducible and complex symmetric; moreover, $\sigma(T_i+K_i) \neq \sigma(T_{n+1}+K_{n+1})$ for $1 \le i \le n$.

By Lemma (5.2.34), we can find $F_1, F_2, \cdots, F_{n+1} \in \mathcal{K}(\mathcal{H}_{n+1})$ with $\|F_i\|<\varepsilon/(n+1)$ such that $T_{n+1}+F_i$ is irreducible and complex symmetric for $1 \le i \le n+1$; moreover, $\sigma(T_{n+1}+F_i) \neq \sigma(T_{n+1}+F_j)$ whenever $i \neq j$. So some $i_0, 1 \le i_0 \le n+1$, exists such that $\sigma(T_{n+1}+F_{i_0}) \neq \sigma(T_j+K_j)$ for all $1 \le j \le n$. Set $K_{n+1}=F_{i_0}$. Then K_{n+1} satisfies all requirements. By induction, this completes the proof.

In [257], Huaxin Lin solved the problem that an approximate normal matrix is close to a normal matrix in the affirmative. As an application, he proved a conjecture of Berg [234], which implies the following result.

Lemma (5.2.37)[452]: ([257, Thm. 4.4]). Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of almost normal operators. Assume that $sup_n\|T_n\|<\infty$ and $\|T_n^*T_n-T_nT_n^*\| \to 0$ as $n \to \infty$. Then there exists a sequence $\{N_n\}_{n=1}^{\infty}$of normal operators such that T_n-N_n is compact for $n \geq 1$ and $\|T_n-N_n\| \to 0$.

Proof for the sufficiency of Theorem (5.2.14): By the hypothesis, Lemma (5.2.21) and Proposition (5.2.23), we may assume that

$$T=N \oplus \left(\oplus_{i \in \Gamma_1} T_i^{(n_i)}\right) \oplus \left(\oplus_{i \in \Gamma_2}(A_j \oplus B_j)^{(n_i)}\right),$$

where

(i) N is normal, $\{T_i, A_j, B_j: i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(ii) T_i is almost normal and not normal for $i \in \Gamma_1$;

(iii) A_j is not almost normal and there exists a compact operator K_j such that B_j+K_j is a transpose of A_j for $j \in \Gamma_2$;

(iv) $1 \le n_i,\ n_j < \infty$ for all $i \in \Gamma_1$ and $j \in \Gamma_2$, and $\|K_j\| \to 0$ if Γ_2 is infinite.

Assume that $N \in \mathfrak{B}(\mathcal{H}_0)$, $T_i \in \mathfrak{B}(\mathcal{H}_{1,i})$ for $i \in \Gamma_1$ and $A_j, B_j \in \mathfrak{B}(\mathcal{H}_{2,j})$ for $j \in \Gamma_2$. For convenience, we may directly assume that Γ_1, Γ_2 are countable and $n_i = 1$ for all $i \in \Gamma_1 \cup \Gamma_2$. The proof for the general case is similar. Then

$$T = N \oplus (\oplus_{i=1}^{\infty} T_i) \oplus \left(\oplus_{j=1}^{\infty} (A_j \oplus B_j)\right)$$

and

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\oplus_{i=1}^{\infty} \mathcal{H}_{1,i}\right) \oplus \left(\oplus_{j=1}^{\infty} (\mathcal{H}_{2,j} \oplus \mathcal{H}_{2,j})\right). \tag{8}$$

The rest of the proof is divided into three steps.

Step 1.Compact perturbations of the operators $\{T_i:\ i \ge 1\}$.

Since T is essentially normal, it follows that $T^*T - TT^* \in \mathcal{K}(\mathcal{H})$ and hence $T_i^*T_i - T_iT_i^* \in \mathcal{K}(\mathcal{H}_{1,i})$ and $\|T_i^*T_i - T_iT_i^*\| \to 0$. By Lemma (5.2.36), we can choose $D_{1,i} \in \mathcal{K}(\mathcal{H}_{1,i}), i \ge 1$, so that $\|D_{1,i}\| \to 0$ and $N_i := T_i + D_{1,i}$ is normal for all $i \ge 1$. By Corollary (5.2.35), there are compact operators $D_{2,1} \in K(H_{1,i}) (i \ge 1)$ with $\|D_{2,i}\| \to 0$ such that $S_i \coloneqq N_i + D_{2,i}$ is irreducible, complex symmetric and $S_i \not\cong S_j$ whenever $i \ne j$.

Set $D_i = D_{1,i} + D_{2,i}$ for $i \ge 1$. Then $S_i = T_i + D_i$, $D_i \in \mathcal{K}(\mathcal{H}_{1,i})$ and $\| D_i \| \to 0$. From statement (ii), each T_i acts on a space of dimension ≥ 2. Thus S_i is almost normal and not normal.

Step 2.Compact perturbations of the operators $\{A_j, B_j: j \ge 1\}$.

For each $j \ge 1$, by the hypothesis, there exists a conjugation C_j on $\mathcal{H}_{2,j}$ such that $C_jA_j^*C_j = B_j + K_j$. Note that $\|K_j\| \to 0$.

Since each A_j is irreducible, it follows from Corollary (5.2.33) that we can find compact operators $\{E_j\}_{j=1}^{\infty}$ with $\|E_j\| \to 0$ such that $R_j := A_j + E_j$ is irreducible for all $j \ge 1$ and $\{\sigma(R_j)\}_{j=1}^{\infty}$ are pairwise distinct.

For each $j \ge 1$, set $G_j = K_j + C_jE_j^*C_j$. Then $G_j \in \mathcal{K}(\mathcal{H}_{2,j})$ and $\|G_j\| \to 0$. On the other hand, note that

$$C_jR_j^*C_j = C_jA_j^*C_j + C_jE_j^*\, C_j = B_j + K_j + C_jE_j^*C_j = B_j + G_j.$$

Step 3.Construction and verification.

Set

$$R = N \oplus (\oplus_{i=1}^{\infty} S_i) \oplus \left(\oplus_{j=1}^{\infty} (R_j \oplus C_jR_j^*C_j)\right).$$

By [252, Thm. 1.6] or [251, Lem. 3.6], R is complex symmetric. Define $K \in \mathfrak{B}(\mathcal{H})$ with respect to the decomposition (8) as

$$K = 0 \oplus (\oplus_{i=1}^{\infty} D_i) \oplus \left(\oplus_{j=1}^{\infty} (E_j \oplus G_j)\right). \tag{9}$$

Then K is compact and one can check that $R = T + K$. Now it remains to prove $C^*(T) = C^*(R)$. Clearly, we need only prove $K \in C^*(T) \cap C^*(R)$.

In view of (9), it suffices to prove that

$$0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i=1}^{\infty} \mathcal{K}\,(\mathcal{H}_{1,i})\right) \oplus \left(\bigoplus_{j=1}^{\infty} (\mathcal{K}(\mathcal{H}_{2,j}) \oplus \mathcal{K}(\mathcal{H}_{2,j})\right) \subset C^*(T) \cap C^*(R).$$

By statements (i)–(iii), it follows from Corollary (5.2.16) that

$$0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i=1}^{\infty} \mathcal{K}\ (\mathcal{H}_{1,i}) \right) \oplus \left(\bigoplus_{j=1}^{\infty} (\mathcal{K}(\mathcal{H}_{2,j}) \oplus \mathcal{K}(\mathcal{H}_{2,j}) \right) \subset\ C^*(T).$$

Since $\{S_i, R_i, C_i R_i^* C_i : i \geq 1\}$ are irreducible and not normal, by Corollary (5.2.16), it suffices to prove that no two of them are unitarily equivalent.

Noting that $\sigma(C_i R_i^* C_i) = \sigma(R_i) \neq \sigma(R_j) = \sigma(C_j R_j^* C_j)$ whenever $i \neq j$, we deduce that $R_i \ncong R_j$, $R_i \ncong C_j R_j^* C_j$ and $C_i R_i^* C_i \ncong C_j R_j^* C_j$ whenever $i \neq j$. On the other hand, note that R_j is a compact perturbation of A_j and A_j is not almost normal for $j \geq 1$. Then, for each $j \geq 1$, we can choose $\lambda \in C$ such that $R_j - \lambda$ is Fredholm and $ind(R_j - \lambda) \neq 0$. So

$$\begin{aligned} ind\ (R_j - \lambda) &= -ind\ (R_j - \lambda)^* \\ &= -ind\ C_j (R_j - \lambda)^* C_j \\ &= -ind\ (C_j R_j^* C_j - \lambda), \end{aligned}$$

which implies that $R_j \ncong C_j R_j^* C_j$.

By the preceding argument, $S_i \ncong S_j$ whenever $i \neq j$. Since each of $\{S_i : i \geq 1\}$ is almost normal, we have $S_i \ncong C_j R_j^* C_j$ and $S_i \ncong R_j$ for all $i, j \geq 1$. Hence we deduce that no two of $\{S_i, R_i, C_i R_i^*\ C_i:\ i \leq 1\}$ are unitarily equivalent. This completes the proof.

Corollary (5.2.37)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If the restriction of T to its every reducing subspace is almost normal, then $T \in (cs)$.

Corollary (5.2.38)[452]: Each compact operator has a complex symmetric generator for its C^*-algebra.

Proof. Assume that $T \in \mathfrak{B}(\mathcal{H})$ is compact. Then the restrictions of T to its minimal reducing subspaces are all compact and hence almost normal. Hence the result follows readily from Corollary (5.2.37).

Corollary (5.2.39)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T_s is not absent, then the following are equivalent:

(i) $T \in (cs)$.

(ii) $T_{abnor} \in (cs)$.

(iii) $T_s \in (cs)$.

Proof. Note that $(T_s)_s = T_s = (T_{abnor})_s$. Then the result follows readily from Theorem (5.2.14).

Corollary (5.2.40)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal and assume that $T = N \oplus A^{(n)}$, where $1 \leq n < \infty$, N is normal, A is abnormal and irreducible. Then $T \in (cs)$ if and only if A is almost normal.

Proof. If A is almost normal, then T_s is absent. By Theorem (5.2.14), we have $T \in (cs)$. If A is not almost normal, then $T_s = A^{(n)}$ is not almost normal. So T_s is not of type C. By Theorem (5.2.14), we have $T \notin (cs)$.

Using the above corollary, one can deduce immediately that the operator T in Example (5.2.11) does not have a complex symmetric generator for its C^*-algebra.

Corollary (5.2.41)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal and $T = A^{(m)} \oplus B^{(n)}$, where A, B are irreducible, not normal and $A \ncong B$. Then $T \in (cs)$ if and only if exactly one of the following holds:

(i) Both A and B are almost normal;

(ii) neither $A\ nor\ B$ is almost normal, $m = n$ and $\Lambda(A^t) = \Lambda(B)$.

Proof. Since T is essentially normal, it follows immediately that $1 \leq m, n < \infty$.

"⟸". If (i) holds, then T_s is absent. By Theorem (5.2.14), we have $T \in (cs)$. If (ii) holds, then $T = T_s$; moreover, by the BDF Theorem, $\Lambda(A^t) = \Lambda(B^2)$ implies that B is a compact perturbation of A^t. So, by Proposition (5.2.23), T is of type C. The conclusion follows immediately from Theorem (5.2.14).

"⟹". We assume that $T \in (cs)$and (i) does not hold. It suffices to prove that (ii) holds. For convenience we assume that $A \in \mathfrak{B}(\mathcal{H}_1)$ and $B \in \mathfrak{B}(\mathcal{H}_2)$.

We claim that neither A nor B is almost normal. For a proof by contradiction, without loss of generality, we assume that A is almost normal. Then, by the hypothesis, B is not almost normal. So $T_s = B^{(n)}$ is not almost normal. Then T_s is not of type C and $T \notin (cs)$, a contradiction. This proves the claim, which means that $T = T_s$.

Since $T \in (cs)$, it follows that T is of type C. Noting that $A \not\cong B$, by the definition, it follows that $m = n$ and there exists compact K such that $A + K$ is unitarily equivalent to a compact perturbation of B^t. So $\Lambda(A) = \Lambda(B^t)$ and, equivalently, $\Lambda(A^t) = \Lambda(B)$.

Here we give another example of essentially normal operator which lies outside the class of $CSOs$ and has a complex symmetric generator for its C^*-algebra.

Example (5.2.42)[452]: Let $\{e_i\}_{i=1}^{\infty}$be an ONB of $\mathcal{H}$. Define $A, B \in \mathfrak{B}(\mathcal{H})$ as

$$Ae_i = \begin{cases} \dfrac{e_2}{2}, & i = 1 \\ e_{i+1}, & i \geq 2. \end{cases}, Be_i = e_{i+1}, \forall i \geq 1.$$

It is easy to verify that A, B are both essentially normal and irreducible; moreover, A is a compact perturbation of B. Note that A, B are Fredholm operators and $\operatorname{ind} A = -1 = \operatorname{ind} B$. So neither A nor B is complex symmetric. Set $T = A \oplus B^*$. It is obvious that $T = T_s$. Define a conjugation C on $\mathcal{H}$ as $C: \sum_{i=1}^{\infty} \alpha_i e_i \mapsto \sum_{i=1}^{\infty} \alpha_i e_i$. It is easy to check that $CB^*C = B^*$, so B^* is a transpose of B and, equivalently, B is a transpose of B^*. Then A is a compact perturbation of a transpose of B^*. Then T is of type C. By Theorem (5.2.14), we have $T \in (cs)$. In view of [263, Thm. 4.1]or [252, Thm. 1.6], T is not complex symmetric.

By Example (5.2.12), a compact perturbation of $CSOs$ need not have its C^*-algebra generated by a CSO. It is natural to ask if $T \in \mathfrak{B}(\mathcal{H})$ and there exists compact $K \in C^*(T)$ such that $T + K$ is complex symmetric, then does it follow that $C^*(T)$ can be generated by a CSO? No. Here is a counterexample.

Example (5.2.43)[452]: Let S be the unilateral shift of multiplicity one on $\mathcal{H}$. By Lemma (5.2.31), there exists compact K on $\mathcal{H} \oplus \mathcal{H}$ such that $A := (S \oplus 2I) + K$ is irreducible.

Set $T = A \oplus S^*$. Note that A, S^* are irreducible, essentially normal and neither A nor S^* is almost normal. So $T_s = T$. Since $\sigma_e(A^t) = \sigma_e(A) \neq \sigma_e(S^*)$, we deduce that $A \not\cong S^*$ and A^t is not unitarily equivalent to a compact perturbation of S^*. So T is not of type C. By Theorem (5.2.14), $C^*(T)$ does not admit a complex symmetric generator.

We write $\mathcal{H}_1$ and $\mathcal{H}_2$ for the underlying subspace of A and S^* respectively. By Corollary (5.2.16), we have

$$\mathcal{K}(\mathcal{H}_1) \oplus \mathcal{K}(\mathcal{H}_2) \subset C^*(T).$$

So $K_0 = (-K) \oplus 0$ is a compact operator in $C^*(T)$, and $T + K_0 = S \oplus 2I \oplus S^*$. Since S^* is a transpose of S, it follows from [264, Thm. 4.1] that $T + K_0$ is complex symmetric.

For convenience, we write $T \in (wcs)$ to denote that $C^*(T)$ is $*$-isomorphic to some C^*-algebra singly generated by $CSOs$.

The proof of our result depends on some results concerning algebraical equivalence of operators. Multiplicity-free operators are introduced and studied. A^t, we shall give a concrete form of those essentially normal operators T satisfying $T \in (wcs)$ (see Corollary (5.2.59)).

Note that $A \approx B$ if and only if $\|p(A^*, A)\| = \|p(B^*, B)\|$ for all polynomials $p(z_1, z_2)$ in two free variables. It is obvious that g-normal operators are invariant under algebraical equivalence.

Two operators A, B are approximately unitarily equivalent (write $A \cong_a B$) if there is a sequence of unitary operators U_n such that $lim_n U_n A U_n^* = B$. It is obvious that approximate unitary equivalence implies algebraical equivalence.

Theorem (5.2.44)[452]: For $T \in \mathfrak{B}(\mathcal{H})$, the following are equivalent:

(i) there is a faithful representation ρ of $C^*(T)$ such that $\rho(T)$ is complex symmetric;

(ii) T is g-normal;

(iii) T is algebraically equivalent to a CSO.

Proof. "(i)⟹(ii)". Assume that ρ is a faithful representation of $C^*(T)$ on $\mathcal{H}_\rho$ with $A = \rho(T)$ being complex symmetric. Then, for any polynomial $p(z_1, z_2)$ in two free variables, we have $\rho(p(T^*, T)) = p(A^*, A)$ and $\rho(\tilde{p}(T, T^*)) = \tilde{p}(A, A^*)$. Since ρ is faithful, we have

$$\|p(T^*, T)\| = \|p(A^*, A)\|, \qquad \|\tilde{p}(T, T^*)\| = \|\tilde{p}(A, A^*)\|.$$

Since each CSO is g-normal, it follows that

$$\|p(T^*, T)\| = \|p(A^*, A)\| = \|\tilde{p}(A, A^*)\| = \|\tilde{p}(T, T^*)\|.$$

So T is g-normal.

"(ii)⟹(iii)". Denote $R = T^{(\infty)}$. Then R is still g-normal and $R \approx T$; moreover, $C^*(R)$ contains no nonzero compact operator. By [251, Thm. 2.1], R is approximately unitarily equivalent to some complex symmetric operator X. Then $T^2 \approx X$.

"(iii)⟹(i)". By definition, the implication is obvious.

An operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be multiplicity-free if $T|_{\mathcal{M}} \ncong T|_{\mathcal{N}}$ for any distinct minimal reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ of T.

Lemma (5.2.45)[452]: Each operator is algebraically equivalent to a multiplicity-free operator.

Proof. Let $T \in \mathfrak{B}(\mathcal{H})$. By Lemma (5.2.18), we may assume that

$$T = T_0 \oplus \left(\oplus_{i \in \Gamma} T_i^{(n_i)}\right),$$

where T_0 is completely reducible, $T_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \ncong T_{i_2}$ whenever $i_1 \neq i_2$.

Set $R = T_0 \oplus (\oplus_{i \in \Gamma} T_i)$. Then it is obvious that $\|p(T^*, T)\| = \|p(R^*, R)\|$ for any polynomial $p(z_1, z_2)$ in two free variables. So $T \approx R$. It remains to prove that R is multiplicity-free.

By Lemma (5.2.19), $\{\mathcal{H}_i : i \in \Gamma\}$ are all minimal reducing subspaces of R. For $i_1, i_2 \in \Gamma$ with $i_1 \neq i_2$, we have $R|_{\mathcal{H}_{i_1}} = T_{i_1} \ncong T_{i_2} = R|_{\mathcal{H}_{i_2}}$. This completes the proof.

Recall that two representations ρ_1 and ρ_2 of a separable C^*-algebra $\mathcal{A}$ are approximately unitarily equivalent (write $\rho_1 \cong_a \rho_2$) if there is a sequence of unitary operators U_n such that

$$\rho_1(A) = lim_n U_n^* \rho_2(A) U_n \text{ for all } A \in \mathcal{A}.$$

The following result can be viewed as a consequence of Voiculescu's Theorem [261].

Lemma (5.2.46)[452]: ([238, Thm. II.5.8]). Let $\mathcal{A}$ be a separable C^*-algebra, and let ρ_1 and ρ_2 be non-degenerate representations of A on separable Hilbert spaces. Then the following are equivalent:

(i) $\rho_1 \cong_a \rho_2$;

(ii) $rank\rho_1(X) = rank\rho_2(X)$ for all $X \in \mathcal{A}$.

Lemma (5.2.47)[452]: ([239, Thm. 5.40]). If φ is a $*$-homomorphism of $\mathcal{K}(\mathcal{H})$ into $\mathfrak{B}(\mathrm{K})$, then there exists a unique direct sum of $\mathcal{K} = \mathcal{K}_0 \oplus (\bigoplus_{\alpha\in\Gamma} \mathcal{K}_\alpha)$ such that each K_α reduces $\varphi(\mathcal{K}(\mathcal{H})), \varphi(T)|_{K_0} = 0$ for $T \in \mathcal{K}(\mathcal{H})$, and there exists a unitary operator U_α from $\mathcal{H}$ onto K_α for $\alpha \in \Gamma$ such that $\varphi(T)|_{K_\alpha} = U_\alpha T U_\alpha^*$ for $T \in \mathcal{K}(\mathcal{H})$.

Theorem (5.2.48)[452]: Let $T, R \in \mathfrak{B}(\mathcal{H})$ be multiplicity-free. Then $T \approx R$ if and only if $T \cong_a R$.

Proof. The sufficiency is obvious.

"$\Longrightarrow$". We let $\varphi: C^*(T) \longrightarrow C^*(R)$ denote the $*$-isomorphism carrying T into R. It suffices to prove that

$$rank\ X = rank\ \varphi(X), \quad \forall X \in C^*(T) \cap \mathcal{K}(\mathcal{H}) \tag{10}$$

and

$$rank\ \varphi^{-1}(Y) = rank\ Y, \quad \forall Y \in C^*(R) \cap \mathcal{K}(\mathcal{H}). \tag{11}$$

In fact, if these equalities hold, then $rank\ \varphi(X) = rankX$ for all $X \in C^*(T)$. By Lemma (5.2.47), this implies $\varphi \cong_a$ id, where id$(\cdot)$ denotes the identity representation of $C^*(T)$. So $R = \varphi(T) \cong_a id(T) = T$.

Denote $\mathcal{A} = C^*(T) \cap \mathcal{K}(\mathcal{H})$. By [238, Thm.I.10.8] we may assume that

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\bigoplus_{i\in\Gamma} \mathcal{H}_i^{(k_i)} \right), \qquad \mathcal{A} = 0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i\in\Gamma} \mathcal{K}(\mathcal{H}_i)^{(k_i)} \right),$$

where the dimensions of $\mathcal{H}_0$ and $\mathcal{H}_i (i \in \Gamma)$ may be finite or $\aleph_0$, and $1 \le k_i < \infty$ for $i \in \Gamma$. Since $\mathcal{A}$ is an ideal of $C^*(T)$, T can be written as

$$T = D_0 D_0 \oplus \left(\bigoplus_{i\in\Gamma} D_i^{(k_i)} \right),$$

where $D_0 \in \mathfrak{B}(\mathcal{H}_0)$ and $D_i \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Then $\mathcal{K}(\mathcal{H}_i) \subset C^*(D_i)$ for each $i \in \Gamma$. Hence each D_i is irreducible. Noting that T is multiplicity-free, we have $k_i = 1$ for all $i \in \Gamma$. Then each compact operator in $C^*(T)$ has the form $0 \oplus (\bigoplus_{i\in\Gamma} X_i)$, where $X_i \in \mathcal{K}(\mathcal{H}_i)$. For $i \in \Gamma$, denote by P_i the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_i$.

Claim (5.2.49)[452]: For each $i \in \Gamma$, there exist unique subspace $\mathcal{K}_i$ of $\mathcal{H}$ and a unitary operator U_i: $\mathcal{K}_i \to \mathcal{H}_i$ such that

$$\varphi(P_i K P_i) = 0 \oplus U_i^*\ K U_i, \qquad \forall K \in \mathcal{K}(\mathcal{H}_i).$$

Now fix an $i \in \Gamma$. Define $\varphi_i: \mathcal{K}(\mathcal{H}_i) \longrightarrow \mathfrak{B}(\mathcal{H})$ as

$$\varphi_i(F) = \varphi(P_i F P_i), \qquad \forall F \in \mathcal{K}(\mathcal{H}_i).$$

Then φ_i is an isometric $*$-homomorphism. By Lemma (5.2.48), there exists a unique direct sum of $\mathcal{H} = k_0 \oplus (\bigoplus_{\alpha\in\Upsilon} \mathcal{K}_\alpha)$ with respect to which

$$\varphi_i(K) = 0 \oplus \left(\bigoplus_{\alpha\in\Upsilon} U_\alpha^* K U_\alpha \right), \qquad \forall K \in \mathcal{K}(\mathcal{H}_i),$$

where $U_\alpha: \mathcal{K}_\alpha \to \mathcal{H}_i$ is unitary for each $\alpha \in \Upsilon$. To prove Claim (5.2.50), it suffices to prove that $card\Upsilon = 1$. Here "card" denotes cardinality. For a proof by contradiction, we assume that $card\, \Upsilon > 1$.

Note that $\mathcal{J} := \{P_i K P_i : K \in \mathcal{K}(\mathcal{H}_i)\}$ is an ideal of $C^*(T)$ and φ is an $*$-isomorphism. Then $\varphi(\mathcal{J}) = \varphi_i(\mathcal{K}(\mathcal{H}_i))$ is an ideal of $C^*(R)$. One can directly check that R can be written as

$$R = X_0 \oplus \left(\bigoplus_{\alpha \in \Upsilon} X_\alpha \right)$$

With respect to the decomposition $\mathcal{H} = \mathcal{K}_0 \oplus (\bigoplus_{\alpha \in \Upsilon} \mathcal{K}_\alpha)$. Then $\mathcal{K}(\mathcal{K}_\alpha) \subset C^*(X_\alpha)$ and X_α is irreducible for each $\alpha \in \Upsilon$.

Since $card\, \Upsilon > 1$, we can find distinct $\alpha_1, \alpha_2 \in \Upsilon$. Since $\varphi_i(\mathcal{K}(\mathcal{H}_i))$ is an ideal of $C^*(R)$, for any $F \in \mathcal{K}(\mathcal{H}_i)$, we have $\varphi_i(F)R \in \varphi_i(\mathcal{K}(\mathcal{H}_i))$. So there exists unique $G \in \mathcal{K}(\mathcal{H}_i)$ such that $\varphi_i(F)R = \varphi_i(G)$, that is,

$$0 \oplus \left(\bigoplus_{\alpha \in \Upsilon} U_\alpha^* F U_\alpha U_\alpha \right) = 0 \oplus \left(\bigoplus_{\alpha \in \Upsilon} U_\alpha^* G U_\alpha \right).$$

It follows that $U_{\alpha_1}^* F U_{\alpha_1} X_{\alpha_1} = U_{\alpha_1}^* G U_{\alpha_1}$ and $U_{\alpha_2}^* F U_{\alpha_2} X_{\alpha_2} = U_{\alpha_2}^* G U_{\alpha_2}$. So

$$F U_{\alpha_1} X_{\alpha_1} U_{\alpha_1}^* = F U_{\alpha_2} X_{\alpha_2} U_{\alpha_2}^*.$$

Since $F \in \mathcal{K}(\mathcal{H}_i)$ is arbitrary, one can see that $U_{\alpha_1} X_{\alpha_1} U_{\alpha_1}^* = U_{\alpha_2} X_{\alpha_2} U_{\alpha_2}^*$. Then $X_{\alpha_1} \cong X_{\alpha_2}$, contradicting the fact that R is multiplicity-free. This proves Claim (5.2.50).

Claim (5.2.50)[452]: $\{K_i : i \in \Gamma\}$ are pairwise orthogonal.

For i_1, i_2 with $i_1 \neq i_2$, if $K_1 \in \mathcal{K}(\mathcal{H}_{i_1})$ and $K_2 \in \mathcal{K}(\mathcal{H}_{i_2})$, then

$$\varphi(P_{i_1} K_1 P_{i_1}) \varphi(P_{i_2} K_2 P_{i_2}) = \varphi(P_{i_1} K_1 P_{i_1} P_{i_2} K_2 P_{i_2}) = 0.$$

Since $K_1 \in \mathcal{K}(\mathcal{H}_{i_2})$ and $K_2 \in \mathcal{K}(\mathcal{H}_{i_2})$ are arbitrary, one can deduce that K_{i_1} is orthogonal to K_{i_2}.

Now we can conclude the proof by verifying that (10) and (11) hold.

Let $K \in C^*(T) \cap \mathcal{K}(\mathcal{H})$. Then, by our hypothesis, K can be written as

$$K = 0 \oplus \left(\bigoplus_{i \in \Gamma} K_i \right),$$

where $K_i \in \mathcal{K}(\mathcal{H}_i)$. It is obvious that $\|K_i\| \to 0$ if Γ is infinite. By Claims (5.2.50) and (5.2.51), we have

$$\varphi(K^2) = \varphi \sum_{i \in \Gamma} P_i K_i P_i = \sum_{i \in \Gamma} \varphi(P_i K_i P_i) = 0 \oplus \left(\oplus_{i \in \Gamma} U_i^* K_i U_i \right).$$

It follows immediately that rank $\varphi(K) = \sum_{i \in \Gamma}$ rank $K_i =$ rank K. This proves (10). By the symmetry, one can also deduce that (11) holds.

Lemma (5.2.51)[452]: ([254, Prop. 4.27]). Let $T, R \in \mathfrak{B}(\mathcal{H})$ and assume that T is essentially normal. If $T \cong_a R$, then $T_{abnor} \cong R_{abnor}$.

Corollary (5.2.52)[452]: Let $t, R \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If $T \approx R$, then $T_{abnor} \approx R_{abnor}$ and $T_s \approx R_s$.

Proof. By Lemma (5.2.50), we may assume that

$$t = T_0 \oplus \left(\oplus_{i \in \Gamma} T_i^{(m_i)} \right),$$

where $T_0 \in \mathfrak{B}(\mathcal{H}_0)$ is completely reducible, $T_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \ncong T_{i_2}$ whenever $i_1 \neq i_2$. Likewise, we assume that

$$R = R_0 \oplus \left(\oplus_{j\in\Upsilon} R_j^{(m_i)}\right),$$

where $R_0 \in \mathfrak{B}(\mathcal{K}_0)$ is completely reducible, $R_j \in \mathfrak{B}(\mathcal{K}_j)$ is irreducible for $j \in \Upsilon$ and $R_{j_1} \ncong R_{j_2}$whenever $j_1 \neq j_2$. Noting that T_0, R_0 are essentially normal, it follows from Lemma (5.2.49) that T_0, R_0 are normal.

Denote

$$\Gamma_1 = \{i \in \Gamma: T_i \text{ is not normal}\}, \qquad \Gamma_2 = \{i \in \Gamma: T_i \text{ is not almost normal}\}.$$

Then $\Gamma_2 \subset \Gamma_1$ and

$$T_{abnor} = \oplus_{i\in\Gamma_1} T_i^{(m_i)}, T_s = (T_{abnor})_s = \oplus_{i\in\Gamma_2} T_i^{(m_i)}.$$

Denote

$$\Upsilon_1 = \{j \in \Upsilon: R_j \text{ is not normal}\}, \qquad \Upsilon_2 = \{j \in \Upsilon: R_j \text{ is not almost normal}\}.$$

Then $\Upsilon_2 \subset \Upsilon_1$ and

$$R_{abnor} = \oplus_{j\in\Upsilon_1} R_j^{(n_j)}, \qquad R_s = (R_{abnor})_s = \oplus_{j\in\Upsilon_2} R_j^{(n_j)}.$$

Set

$$A = T_0 \oplus (\oplus_{i\in\Gamma} T_i), \qquad B = R_0 \oplus \left(\oplus_{j\in\Upsilon} R_j\right).$$

From the proof of Lemma (5.2.46), one can see that A, B are both multiplicity-free, $T \approx A$ and $R \approx B$. Since $T \approx R$, we obtain $A \approx B$. By Theorem (5.2.49), we have $A \cong_a B$. Note that A, B are both essentially normal. In view of Lemma (5.2.53), it follows that $A_{abnor} \cong B_{abnor}$. Hence $(A_{abnor})_s \cong (B_{abnor})_s$.

Note that

$$A_{abnor} = \oplus_{i\in\Gamma_1} T_i, \qquad A_s = (A_{abnor})_s = \oplus_{i\in\Gamma_2} T_i,$$

and

$$B_{abnor} = \oplus_{j\in\Upsilon_1} R_j, \qquad B_s = (B_{abnor})_s = \oplus_{j\in\Upsilon_2} R_j.$$

We obtain

$$\oplus_{i\in\Gamma_1} T_i \cong \oplus_{j\in\Upsilon_1} R_j, \qquad \oplus_{i\in\Gamma_2} T_i \cong \oplus_{j\in\Upsilon_2} R_j.$$

This implies that

$$\oplus_{i\in\Gamma_1} T_i^{(m_i)} \approx \oplus_{j\in\Upsilon_1} R_j^{(n_j)}, \qquad \oplus_{i\in\Gamma_2} T_i^{(m_i)} \approx \oplus_{j\in\Upsilon_2} R_j^{(n_j)}.$$

Thus we obtain $T_{abnor} \approx R_{abnor}$ and $T_s \approx R_s$.

Lemma (5.2.53)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be multiplicity-free. Then each generator of $C^*(T)$ is multiplicity-free.

Proof. By Lemma (5.2.46), we may assume that

$$T = T_0 \oplus (\oplus_{i\in\Gamma} T_i),$$

where $T_0 \in \mathfrak{B}(\mathcal{H}_0)$ is completely reducible, $T_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \ncong T_{i_2}$ whenever $i_1 \neq i_2$. Note that $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$.

Assume that $R \in \mathfrak{B}(\mathcal{H})$ and $C^*(T) = C^*(R)$. Then R can be written as $R = R_0 \oplus (\oplus_{i\in\Gamma} R_i)$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$. By Lemma (5.2.24), R_0 is completely reducible and R_i is irreducible for $i \in \Gamma$; moreover, $R_i \ncong R_j$ for $i, j \in \Gamma$ with $i \neq j$. In view of the proof of Lemma (5.2.46), R is multiplicity-free.

An operator is said to be UET if $T \ncong T^t$. In view of the BDF Theorem, if an essentially normal operator T is UET, then T is almost normal.

Lemma (5.2.54)[452]: ([251, Thm. 5.1]). Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. Then T is g-normal if and only if it is unitarily equivalent to a direct sum of (some of the summands may be absent)

(i) normal operators,

(ii) irreducible UET operators, and

(iii) operators with the form of $A^{(m)} \oplus (A^t)^{(n)}$, where A is irreducible, not UET and $1 \leq m, n < \infty$.

Lemma (5.2.55)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T is multiplicity-free and g-normal, then $T \in (cs)$.

Proof. Since T is essentially normal and g-normal, by Lemma (5.2.55), we may assume that

$$T = N \oplus \left(\oplus_{i \in \Gamma} T_i^{(l_i)}\right) \oplus \left(\oplus_{j \in \Upsilon} A_j^{(m_j)} \oplus B_j^{(n_j)}\right),$$

where $N = T_{nor}$ is normal, $\{T_i, A_j, B_j : i \in \Gamma, j \in \Upsilon\}$ are abnormal, irreducible and no two of them are unitarily equivalent; moreover, each T_i is UET and A_j is a transpose of B_j for $j \in \Upsilon$. So $\Lambda(A_j) = \Lambda(B_j^t)$ for $j \in \Upsilon$. It follows that A_j is almost normal if and only if B_j^t (or, equivalently, B_j) is almost normal. On the other hand, since T is multiplicity-free, we deduce that $l_i = m_j = n_j = 1$ for all $i \in \Gamma$ and all $j \in \Upsilon$.

Denote $\Upsilon_0 = \{j \in \Upsilon : A_j \text{ is not almost normal}\}$. Note that T_i is almost normal for $i \in \Gamma$. It follows that

$$T_s = \oplus_{j \in \Upsilon_0} \left(A_j \oplus B_j\right).$$

By Proposition (5.2.23), T_s is of type C. In view of Theorem (5.2.14), we have $T \in (cs)$.

Theorem (5.2.56)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. Then $T \in (wcs)$ if and only if there exists an essentially normal operator $R \in (cs)$ such that $T \approx R$.

Proof . The sufficiency is obvious.

"⟹". Assume that

$$T = T_0 \oplus \left(\oplus_{i \in \Gamma} T_i^{(n_i)}\right),$$

where T_0 is completely reducible, $T_i \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1} \ncong T_{i_2}$ whenever $i_i \neq i_2$. Set $A = T_0 \oplus (\oplus_{i \in \Gamma} T_i)$. Then $A \approx T$ is essentially normal and, by Lemma (5.2.46), A is multiplicity-free.

Assume that $S \in \mathfrak{B}(\mathcal{K})$ is complex symmetric and $C^*(S)$ is $*$-isomorphic to $C^*(T)$. By Lemma (5.2.46), S is algebraically equivalent to some multiplicity-free operator B. By Theorem (5.2.45), B is g-normal.

Since $C^*(S)$ is $*$-isomorphic to $C^*(T), A \approx T$ and $B \approx S$, we can find a $*$-isomorphism $\varphi : C^*(A) \to C^*(B)$. Denote $R = \varphi(A)$. Then $A \approx R$ and $C^*(A) = C^*(R)$. Noting that B is multiplicity-free, it follows from Lemma (5.2.55) that R is also multiplicity-free. By Theorem (5.2.49), we obtain $A \cong_a R$. Since A is essentially normal, so is R. This combining $C^*(B) = C^*(R)$ implies that B is also essentially normal. Since B is multiplicity-free and g-normal, it follows from Lemma (5.2.57) that $C^*(B^2) = C^*(R)$ admits a complex symmetric generator, that is, $A \in (cs)$. Noting that $T \approx A$ and $A \cong_a R$, we obtain $T \approx R$.

Corollary (5.2.57)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T_s is not absent, then the following are equivalent:

(i) $T \in (wcs)$;

(ii) $T_{abnor} \in (wcs)$;

(iii) $T_s \in (wcs)$;

(iv) T_s is algebraically equivalent to an essentially normal operator of type C.

Proof."(i)⟹(ii)". By Theorem (5.2.44), $T \in (wcs)$ implies that there exists an essentially normal operator $A \in \mathfrak{B}(\mathcal{H})$ such that $A \in (cs)$ and $T \approx A$. By Corollary (5.2.54), we have $T_{abnor} \approx A_{abnor}$, and it follows from Corollary (5.2.39) that $A_{abnor} \in (cs)$. Using Theorem (5.2.44), we obtain $T_{abnor} \in (wcs)$.

"(ii)⟹(iii)". By Theorem (5.2.44), $T_{abnor} \in (wcs)$ implies that there exists an essentially normal operator $A \in \mathfrak{B}(\mathcal{H})$ such that $A \in (cs)$ and $T_{abnor} \approx A$. By Corollary (5.2.54), we have $T_s = (T_{abnor})_s \approx A_s$, and it follows from Corollary (5.2.39) that $A_s \in (cs)$. Using Theorem (5.2.44), we obtain $T_s \in (wcs)$.

"(iii)⟹(iv)". By Theorem (5.2.44), $T_s \in (wcs)$ implies that there exists an essentially normal operator $A^2 \in \mathfrak{B}(\mathcal{H})$ such that $A^2 \in (cs)$ and $T_s \approx A$. Then, by Corollary (5.2.53), $T_s = (T_s)_s \approx A_s$. By Theorem (5.2.14), $A \in (cs)$ implies that A_s is of type C. This proves the implication "(iii) ⟹(iv)".

"(iv) ⟹(i)". Assume that $A \in \mathfrak{B}(\mathcal{H})$ is an essentially normal operator of type C and $T_s \approx A$. Denote by B the restriction of T to $\mathcal{H} \ominus \mathcal{H}_s$. Then the restriction of B to its each nonzero reducing subspace is almost normal. It follows that $T = T_s \oplus B \approx A \oplus B$. Noting that $(A \oplus B)_s = A_s = A$ is of type C, by Theorem (5.2.14), we have $A \oplus B \in (cs)$. By Theorem (5.2.44), we conclude that $T \in (wcs)$.

Lemma (5.2.58)[452]: Let $A, B \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If A, B are abnormal, then $A \approx B$ if and only if $A^{(\infty)} \cong B^{(\infty)}$.

Corollary (5.2.59)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. Then $T \in (wcs)$ if and only if T_s is either absent or unitarily equivalent to an essentially normal operator of the form

$$\bigoplus_{1\le i<v} \left(A_i^{(m_i)} \oplus B_i^{(n_i)}\right), \qquad 1 \le m_i, n_i < \infty,$$

where $\{A_i, B_i : 1 \le i < v\}$ are essentially normal operators satisfying the conditions (i), (ii) and (iii) in Proposition (5.2.23).

Proof. Obviously, we need only consider the case that T_s is not absent. By Lemma (5.2.46) and Proposition (5.2.23), each essentially normal operator of type C is algebraically equivalent to a multiplicity-free operator of the form

$$R = \oplus_{1\le i<v}(A_i \oplus B_i), \tag{12}$$

where $\{A_i, B_i : 1 \le i < v\}$ satisfy the conditions (i), (ii) and (iii) in Proposition (5.2.23). Then, by Corollary (5.2.58), an essentially normal operator T satisfies $T \in (wcs)$ if and only if T_s is algebraically equivalent to an operator R of the form (12). Noting that both T_s and R are abnormal, in view of Lemma (5.2.59), the latter is equivalent to

$$T_s^{(\infty)} \cong \oplus_{1\le i<v} \left(A_i^{(\infty)} \oplus B_i^{(\infty)}\right). \tag{13}$$

By Lemmas (5.2.19) and (5.2.20), the condition (13) holds if and only if there exist $m_i, n_i, 1 \le i < v$, such that

$$T_s \cong \bigoplus_{1\le i<v} \left(A_i^{(m_i)} \oplus B_i^{(n_i)}\right).$$

For each i, note that both $A_i^* A_i - A_i A_i^*$ and $B_i^* B_i - B_i B_i^*$ are nonzero compact operators. Since T_s is essentially normal, if such m_i, n_i exist, then it is necessary that $m_i, n_i < \infty$ for each i.

Corollary (5.2.60)[452]: Let $T \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T is irreducible, then the following are equivalent:

(i) $T \in (cs)$;

(ii) $T \in (wcs)$;

(iii) T is almost normal.

Proof. The implication "(i) ⟹(ii)" is trivial, and the equivalence "(i)⟺(iii)" follows from Theorem (5.2.8).

"(ii) ⟹(iii)". If T^2 is not almost normal, then $T^2 = T_s$ and T_s is not absent. By Corollary (5.2.59), T_s is reducible, a contradiction. This ends the proof.

Corollary (5.2.61)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If $C^*(T^2)$ admits a complex symmetric generator, then T^2 is almost normal.

Proof. Assume that $A^2 \in \mathfrak{B}(\mathcal{H})$ is complex symmetric and $C^*(T^2) = C^*(A^2)$. Then there is a conjugation C on $\mathcal{H}$ such that $CA^2C = (A^*)^2$. Then for each $\lambda^2 \notin \sigma_{lre}(A^2)$ one can check that

$$ind\ (A^2 - \lambda^2) = ind\ C(A^2 - \lambda^2)C = ind(A^2 - \lambda^2)^* = -ind\ (A^2 - \lambda^2).$$

So $ind(A^2 - \lambda^2) = 0$ for $\lambda^2 \notin \sigma_{lre}(A^2)$. On the other hand, since T^2 is essentially normal and $A^2 \in C^*(T^2)$, it follows that A^2 is essentially normal. By the BDF Theorem, A^2 has the form "normal plus compact". Since $T^2 \in C^*(A^2)$, T^2 is also of the form "normal plus compact".

Corollary (5.2.62)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T^2 is irreducible, then $T^2 \in (cs)$ if and only if T^2 is almost normal.

The proof of the preceding result depends on a key approximation result.

Proof. The necessity follows from Lemma (5.2.7).

"⟸". Since T^2 is almost normal, there exist a square normal operator N^2 and $K^2 \in \mathcal{K}(\mathcal{H})$ so that $T^2 = N^2 + K^2$. By Proposition (5.2.9), we can find compact K_0^2 such that $R^2 := N^2 + K_0^2$ is an irreducible CSO. Since T^2, R^2 are both irreducible and essentially normal, we have $\mathcal{K}(\mathcal{H}) \subset C^*(T^2) \cap C^*(R^2)$. It follows that $T^2 - R^2 = K^2 - K_0^2 \in C^*(T^2) \cap C^*(R^2)$. Thus $C^*(T^2) = C^*(R^2)$. This completes the proof.

In general, the condition of irreducibility in Theorem (5.2.8) can not be canceled. That is, the spectral condition "$ind(T^2 - \lambda^2) = 0, \forall \lambda^2 \notin \sigma_e(T^2)$" is necessary and not sufficient for $T^2 \in (cs)$. Before giving an example, we first introduce a useful result.

Recall that an operator A^2 is said to be abnormal if A^2 has no nonzero reducing subspace $\mathcal{M}$ such that $A^2|_{\mathcal{M}}$ is normal. If an irreducible operator is not normal, then it is abnormal. Each Hilbert space operator T^2 admits the unique decomposition

$$T^2 = T^2_{nor} \oplus T^2_{abnor},$$

Where T^2_{nor} is normal and T^2_{abnor} is abnormal. The operators T^2_{nor} and T^2_{abnor} are called the normal part and the abnormal part of T^2 respectively. The reader is referred to [254, p. 116] for more details.

Corollary (5.2.63)[462]: Given a normal operator $T^2 \in \mathfrak{B}(\mathcal{H}))$ and $\varepsilon > 0$, there exists $K^2 \in \mathcal{K}(\mathcal{H})$ with $\|K^2\| < \varepsilon$ such that $T^2 + K^2$ is an irreducible $CSSO$.

Proof. By the Weyl–von Neumann Theorem, we may directly assume that T^2 is a diagonal operator with respect to some ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$. Assume that $\{\lambda_n^2\}_{n=1}^{\infty}$ are the eigenvalues of T^2 satisfying $T^2 e_n = \lambda_n^2 e_n$ for $n \geq 1$. For each $n \geq 1$, denote $a_n^2 = Re\lambda_n^2$ and $b_n^2 = Im\lambda_n^2$. Up to a small compact perturbation, we may assume that $a_i^2 \neq a_j^2$ for $i \neq j$. Set

$$A^2=\sum_{i=1}^{\infty} a_i^2\, e_i \otimes e_i, \qquad B^2=\sum_{i=1}^{\infty} b_i^2\, e_i \otimes e_i.$$

Then $T^2=A^2+iB^2$. For $i,j\geq 1$, set $d_{i,j}^2=\frac{\varepsilon}{2^{i+j}}$. Define a compact operator $K_1^2\in\mathcal{K}(\mathcal{H})$ by

$$K_1^2=\begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \cdots \\ d_{2,1} & d_{2,2} & d_{2,3} & \cdots \\ d_{3,1} & d_{3,2} & d_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}^2 \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix}.$$

It is obvious that $K_1^2\in\mathcal{K}(\mathcal{H})$ is self-adjoint and $\|K_1^2\|<2\left(\sum_{n=1}^{\infty}\frac{\varepsilon}{2^{1+n}}\right)=\varepsilon$. Set $K^2=iK_1^2$. Then it remains to check that T^2+K^2 is an irreducible $CSSO$.

Note that $T^2+K^2=A^2+iB_1^2$, where $B_1^2=B^2+K_1^2$. Then A^2,B_1^2 are both self-adjoint. Assume that $P^2\in\mathfrak{B}(\mathcal{H})$ is an orthogonal projection commuting with T^2+K^2. It follows that $P^2A^2=A^2P^2$ and $P^2B_1^2=B_1^2P^2$.

Since $A^2=\sum_{i=1}^{\infty}a_i^2 e_i\otimes e_i$ and $a_i^2\neq a_j^2$ whenever $i\neq j$, it follows from $A^2P^2=P^2A^2$ that $P^2=\sum_{i=1}^{\infty}\mu_i e_i\otimes e_i$, where $\mu_i=0$ or $\mu_i=1$ for each $i\geq 1$. On the other hand, for $i,j\geq 1$ with $i\neq j$, we have

$$\langle P^2B_1^2e_j,e_i\rangle=\langle B_1^2e_j,P^2e_i\rangle=\langle B_1^2\, e_j,\mu_ie_i\rangle=\mu_i\langle B^2e_j,e_i\rangle+\mu_i\langle K_1^2e_j,e_i\rangle=\mu_i d_{i,j}^2 =\frac{\mu_i}{2^{i+j}}$$

and

$$\langle B_1^2P^2e_j,e_i\rangle=\langle P^2e_j,B_1^2e_i\rangle=\mu_j\langle e_j,B_1^2e_i\rangle=\mu_j\langle B^2e_j,e_i\rangle+\mu_j\langle K_1^2e_j,e_i\rangle=\mu_j\, d_{i,j}^2 =\frac{\mu_j}{2^{i+j}}.$$

Since $P^2B_1^2=B_1^2P^2$, it follows that $\mu_i=\mu_j$. Then either $P^2=0$ or P^2 is the identity operator on $\mathcal{H}$, which implies that T^2+K^2 is irreducible.

Now it remains to show that T^2+K^2 is a $CSSO$. In fact, if C is the conjugation on $\mathcal{H}$ defined by $Ce_i=e_i$ for $i\geq 1$, then one can check that $C(A^2+K^2)C=(A^2+K^2)^*$. Since each of the operators A^2,B^2,K_1^2 admits a complex symmetric matrix representation with respect to the same ONB $\{e_n\}$, one can also see that $T^2+K^2=A^2+i(B^2+K_1^2)$ is complex symmetric.

Corollary (5.2.64)[462]: If $T^2\in\mathfrak{B}(\mathcal{H})$ is essentially normal, then $T^2\in(cs)$ if and only if T_s^2 is either absent or of type C.

By Theorem (5.2.14), whether or not an essentially square normal operator T^2 has a complex symmetric square generator for its C^*-algebra depends only on the behavior of T_s^2.

We give a concrete description of the essentially square normal operators of type C. We first make some preparation.

Let $\{\mathcal{A}_i\}_{i\in\Gamma}$ be a family of C^*-algebras. We denote by $\prod_{i\in\Gamma}\mathcal{A}_i$ the direct product of $\{\mathcal{A}_i\}_{i\in\Gamma}$, and by $\bigoplus_{i\in\Gamma}\mathcal{A}_i$ the direct sum of $\{\mathcal{A}_i\}_{i\in\Gamma}$.

Let $A^2\in\mathfrak{B}(\mathcal{H})$. We let $W^*(A^2)$ denote the von Neumann algebra generated by A^2. By the von Neumann Double Commutant Theorem, we have $W^*(A^2)=C^*(A'')^2$. Here and in what follows, $\mathcal{A}'$ denotes the commutant algebra of $\mathcal{A}$.

We referred to [240, Thm. 3.1] for a proof of the following result.

Proof for the Necessity of Theorem Assume that $R^2 \in \mathfrak{B}(\mathcal{H})$ is complex symmetric and $C^*(T^2) = C^*(R^2)$. Also we assume that T_s^2 is not absent. Then, by Lemma (5.2.25), R_s^2 is not absent. Since T^2 is essentially normal, so is R^2. By Lemma (5.2.10), R_{abnor}^2 is complex symmetric. By [251, Thm. 2.8], R_{abnor}^2 is a direct sum of irreducible $CSSO$s and operators with form of $Z \oplus Z^t$, where Z is irreducible and not complex symmetric. Note that each essentially normal $CSSO$ is almost normal. Then, up to unitary equivalence, we may assume that

$$R^2 = N^2 \oplus \left(\oplus_{i\in\Gamma_1} R_i^{2(m_i)}\right) \oplus \left(\oplus_{j\in\Gamma_2} (A_j^2 \oplus B_j^2)^{(n_j)}\right), \tag{14}$$

where

(i) $N^2 = R_{nor}^2$ is normal, $\{R_i^2, A_j^2, B_j^2 : i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(ii) each R_i^2 is almost normal and not normal;

(iii) A_j^2 is not almost normal and B_j^2 is a transpose of A_j^2 for $j \in \Gamma_2$.

Note that each of $\{R_i^2, A_j^2, B_j^2 : i \in \Gamma_1, j \in \Gamma_2\}$ is abnormal. Since R^2 is essentially normal, it follows that $1 \le m_i,\ n_j < \infty$ for all i, j.

We assume that $N^2 \in \mathfrak{B}(\mathcal{H}_0)$, $R_i^2 \in \mathfrak{B}(\mathcal{H}_{1,i})$ and A_j^2, $B_j^2 \in \mathfrak{B}(\mathcal{H}_{2,j})$ for $i \in \Gamma_1$ and $j \in \Gamma_2$. Hence

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\oplus_{i\in\Gamma_1} \mathcal{H}_{1,i}^{(m_i)}\right) \oplus \left(\oplus_{j\in\Gamma_2} (\mathcal{H}_{2,j} \oplus \mathcal{H}_{2,j})^{(n_j)}\right). \tag{15}$$

Since $C^*(T^2) = C^*(R^2)$, in view of Lemma (5.2.24) , T^2 can be written as

$$T^2 = D^2 \oplus \left(\oplus_{i\in\Gamma_1} E_i^{2(m_i)}\right) \oplus \left(\oplus_{j\in\Gamma_2} (F_j^2 \oplus G_j^2)^{(n_j)}\right) \tag{16}$$

with respect to the decomposition (16); moreover, by statements (i)–(ii), we have

(iv) D^2 is normal, $\{E_i^2, F_j^2, G_j^2 : i \in \Gamma_1,\ j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(v) each E_i^2 is almost normal and not normal for $i \in \Gamma_1$;

(vi) F_j^2, G_j^2 are essentially normal and not almost normal for $j \in \Gamma_2$.

By Lemma (5.2.21), we have $T_s^2 = \oplus_{j\in\Gamma_2} (F_j^2 \oplus G_j^2)^{(n_j)}$. On the other hand, note that

$$\oplus_{j\in\Gamma_2} (F_j^2 \oplus G_j^2) \in C^*\left(\oplus_{j\in\Gamma_2} (A_j^2 \oplus B_j^2)\right).$$

It follows from Proposition (5.2.27) that G_j^2 is a compact perturbation of a transpose $(F_j^2)^t$ of F_j^2 for $j \in \Gamma_2$, and $\left\|G_j^2 - (F_j^2)^t\right\| \to 0$ if Γ_2 is infinite. By Proposition (5.2.23), T_s^2 is of type C. This proves the necessity.

To give the proof for the sufficiency of Theorem (5.2.14), we need to prove several approximation results.

Corollary (5.2.65)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal and $T^2 = N^2 \oplus (\oplus_{i=1}^{\infty} T_i^2)$, where

(i) $N^2 \in \mathfrak{B}(\mathcal{H}_0)$ is normal,

(ii) $T_i^2 \in B(H_i)$ is irreducible and not normal for $i \ge 1$, and

(iii) $T_i^2 \ncong T_j^2$ whenever $i \ne j$.

Then $0_{\mathcal{H}_0} \oplus (\oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)) \subset C^*(T^2)$. Moreover, if N^2 is absent, then

$$C^*(T^2) \cap \mathcal{K}(\mathcal{H}) = \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i).$$

Proof. For any fixed $i \geq 1$ and fixed $e,\ f \in \mathcal{H}_i$, it suffices to prove that $f \otimes e \in C^*(T^2)$. Set $K^2 = (T^*)^2\, T^2 - T^2(T^*)^2$. By the hypothesis, we may assume $K^2 = 0 \oplus (\bigoplus_j^\infty K_j^2)$, where $K_j^2 \in \mathcal{K}(\mathcal{H}_j)$ for $j \geq 1$. It is obvious that $K_j^2 \neq 0$ for all $j \geq 1$ since T_j^2 is not normal. There exist nonzero $e_1,\ f_1 \in \mathcal{H}_i$ such that $K_i^2 e_1 = f_1$. We may assume that $\|f_1\| = 1$.

Set $A^2 = \bigoplus_{i=1}^\infty T_j^2$. Since each T_j^2 is irreducible and $T_{j_1}^2 \ncong T_{j_1}^2$ for $j_1 \neq j_2$, it follows from Lemma (5.2.15) that each operator commuting with both A^2 and $(A^*)^2$ has the form $\bigoplus_{j=1}^\infty \lambda_j^2 I_j$, where I_j is the identity operator on $\mathcal{H}_j$. Moreover, we have

$$W^*(A^2) = \prod_{j=1}^{\infty} \mathfrak{B}(\mathcal{H}_j)\,.$$

So $f \otimes e \in W^*(A^2)$ and, by the von Neumann Double Commutant Theorem, we have $f \otimes f_1, e_1 \otimes e, f \otimes e \in \overline{C^*(A^2)}^{SOT}$. Here SOT denotes the strong operator topology. Using the Kaplansky Density Theorem ([238, Thm.I.7.3, Rem.I.7.4]), we can choose polynomials $\{p_N(\cdot,\cdot)\}$ and $\{q_N(\cdot,\cdot)\}$ in two free variables so that

$$p_n((A^*)^2, A^2) \xrightarrow{SOT} f \otimes f_1, \qquad q_n((A^*)^2, A^2) \xrightarrow{SOT} e_1 \otimes e.$$

Since $\bigoplus_{j=1}^\infty K_j^2$ is compact, we obtain

$$p_n((A^*)^2, A^2)\big(\bigoplus_{j=1}^\infty K_j^2\big) q_n((A^*)^2, A^2) \xrightarrow{\|\cdot\|} f \otimes e.$$

Moreover, we obtain

$$p_n((T^*)^2, T^2) K^2 q_n((T^*)^2, T^2) = \begin{bmatrix} 0 & 0 \\ 0 & p_n((A^*)^2, A^2)\big(\bigoplus K_i^2\big) q_n((A^*)^2, A^2) \end{bmatrix} \xrightarrow{\|\cdot\|} \begin{bmatrix} 0 & 0 \\ 0 & f \otimes e \end{bmatrix},$$

which completes the proof.

Recall that an operator is said to be completely reducible if it does not admit any minimal reducing subspace ([242]).

Corollary (5.2.66)[462]: If an essentially square normal operator T^2 is completely reducible, then T^2 is normal.

Proof. Assume that $T^2 \in \mathfrak{B}(\mathcal{H})$. Since T^2 is completely reducible, by [242, Lem. 2.5], we have $C^*(T^2) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Noting that T^2 is essentially normal, we obtain $(T^*)^2\, T^2 - T^2(T^*)^2 \in C^*(T^2) \cap \mathcal{K}(\mathcal{H})$. Thus $(T^*)^2\, T^2 - T^2(T^*)^2 = 0$.

If d is a cardinal number and $\mathcal{H}$ is a Hilbert space, let $\mathcal{H}^{(d)}$ denote the direct sum of $\mathcal{H}$ with itself d times. If $A^2 \in \mathfrak{B}(\mathcal{H})$, $A^{2(d)}$ is the direct sum of A^2 with itself d times.

Corollary (5.2.67)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially square normal. Then T_{abnor}^2 is unitarily equivalent to an operator of the form

$$\bigoplus_{i\in\Gamma} T_i^{2(n_i)},$$

where each T_i^2 is irreducible, not normal and $T_i^2 \ncong T_j^2$ for $i, j \in \Gamma$ with $i \neq j$. Moreover, T_s^2 is the restriction of T_{abnor}^2 to a reducing subspace and

$$T_s^2 \cong \bigoplus_{i\in\Gamma_0} T_i^{2(n_i)},$$

where $\Gamma_0 = \{i \in \Gamma : T_i^2 \text{ is not almost normal}\}$.

Proof. By Lemma (5.2.18), T_{abnor}^2 is unitarily equivalent to an operator of the form

$$T_0^2 \oplus \left(\bigoplus_{i\in\Gamma} T_i^{2(n_i)}\right),$$

where $T_0^2 \in \mathfrak{B}(\mathcal{H}_0)$ is completely reducible, each $T_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible and $T_i^2 \ncong T_j^2$ for $i, j \in \Gamma$ with $i \neq j$. Note that T_i^2 is abnormal for $i \in \Gamma$. Since T_0^2 is completely reducible and essentially normal, it follows from Lemma (5.2.17) that T_0^2 is normal. Note that T_{abnor}^2 is abnormal ; so T_0^2 is absent. Then $T_{abnor}^2 \cong \bigoplus_{i\in\Gamma} T_i^{2(n_i)}$. For convenience we directly assume that $T_{abnor}^2 = \bigoplus_{i\in\Gamma} T_i^{2(n_i)}$. Thus

$$T^2 = T_{nor}^2 \oplus \left(\bigoplus_{i\in\Gamma} T_i^{2(n_i)}\right).$$

By definition, it is obvious that $\bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)} \subset \mathcal{H}_s$. On the other hand, if $\mathcal{M}$ is a m.r.s. of T^2 and $T^2|_{\mathcal{M}}$ is not almost normal, then, by Lemmas (5.2.19) and (5.2.20), there exists $i_0 \in \Gamma$ such that $\mathcal{M} \subset \mathcal{H}_{i_0}^{(n_{i_0})}$ and $T^2|_{\mathcal{M}} \cong T_{i_0}^2$. So $T_{i_0}^2$ is not almost normal and $\mathcal{H}_{i_0}^{(n_{i_0})} \subset \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Thus $\mathcal{M} \subset \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Furthermore we obtain $\mathcal{H}_s \subset \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$. Therefore $\mathcal{H}_s = \bigoplus_{i\in\Gamma_0} \mathcal{H}_i^{(n_i)}$.

Corollary (5.2.68)[462]: An essentially square normal operator T^2 is of type C if and only if T^2 is unitarily equivalent to an operator of the form

$$\bigoplus_{1\le i<\upsilon} (A_i^2 \oplus B_i^2)^{(n_i)}, \qquad 1 \le n_i < \infty,$$

where (i) $\upsilon \in \mathbb{N}$ or $\upsilon = \infty$, $\{A_i^2, B_i^2 : 1 \le i < \upsilon\}$ are irreducible and no two of them are unitarily equivalent,(ii) A_i^2 is not almost normal and there exists compact K_i^2 such that $A_i^2 + K_i^2$ is a transpose of B_i^2 for each i, and (iii) $\|K_i^2\| \to 0$ if $\upsilon = \infty$.

Proof."⟸". Assume that $A_i^2, B_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ for $1 \le i < \upsilon$. Denote $\mathcal{H} = \bigoplus_{1\le i<\upsilon} \mathcal{H}_i^{(n_i)}$ and

$$A^2 = \bigoplus_{1\le i<\upsilon} A_i^{2(n_i)}, \quad B^2 = \bigoplus_{1\le i<\upsilon} B_i^{2(n_i)}.$$

Then $A^2, B^2 \in \mathfrak{B}(\mathcal{H})$ are essentially normal and $T^2 \cong A^2 \oplus B^2$. For convenience we directly assume that $T^2 = A^2 \oplus B^2$ and $\upsilon = \infty$.

Since $\{A_i^2, B_i^2 : 1 \le i < \upsilon\}$ are irreducible, not normal and no two of them are unitarily equivalent, it follows from Corollary (5.2.19) that

$$C^*(A^2) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{1\le i<\upsilon} \mathcal{K}(\mathcal{H}_i)^{(n_i)} = C^*(B^2) \cap \mathcal{K}(\mathcal{H}). \qquad (17)$$

Moreover, if $\mathcal{M}$ is a m.r.s. of T^2, then, by Lemmas (5.2.19) and (5.2.20), there exists unique i_0 with $1 \le i_0 < \upsilon$ such that exactly one of the following holds

$$T^2|_{\mathcal{M}} \cong (A_{i_0})^2, \qquad T^2|_{\mathcal{M}} \cong (B_{i_0})^2.$$

It follows that A^2, B^2 are disjoint; moreover, $T^2|_{\mathcal{M}}$ is not almost normal. Thus, by Corollary (5.2.22), $T^2 = T_s^2$.

By statement (ii), for each $1 \le i < \upsilon$, we can find a conjugation C_i on $\mathcal{H}_i$ so that $A_i^2 + K_i^2 = C_i(B_i^*)^2 C_i$. Set

$$K^2 = \bigoplus_{1\le i<\upsilon} K_i^{2(n_i)}, \qquad C = \bigoplus_{1\le i<\upsilon} C_i^{(n_i)}.$$

Then C is a conjugation on $\mathcal{H}$ and, by (14), $K^2 \in C^*(A^2) \cap \mathcal{K}(\mathcal{H})$, since $\|K_j^2\| \to 0$; moreover, $C(B^*)^2 C = A^2 + K^2$.

On the other hand, since $\{B_i^2\}$ are irreducible, not normal and no two of them are unitarily equivalent, so are $\{A_i^2 + K_i^2\}$. It follows from Corollary (5.2.16) that

$$C^*(A^2 + K^2) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{1\le i<\upsilon} \mathcal{K}(\mathcal{H}_i)^{(n_i)} = C^*(A^2) \cap \mathcal{K}(\mathcal{H}).$$

"⟹". Now assume that $T^2 = T_s^2$ and $T^2 = A^2 \oplus B^2$, where $A^2, B^2 \in \mathfrak{B}(\mathcal{H})$ satisfy conditions (a), (b) and (c) in Definition (5.2.13). Since $T^2 = T_s^2$, it follows that $A^2 = A_s^2$. Then, by Corollary (5.2.22), we may assume that

$$A^2 = \bigoplus_{i\in\Gamma} A_i^{2(n_i)}, 1 \le n_i < \infty,$$

where each $A_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible, not almost normal and $A_i^2 \ncong A_j^2$ whenever $i \neq j$. By Corollary (5.2.16), we have

$$C^*(A^2) \cap \mathcal{K}(\mathcal{H}) = \bigoplus_{i\in\Gamma} \mathcal{K}(\mathcal{H}_i)^{(n_i)}.$$

Then K^2 can be written as

$$K^2 = \bigoplus_{i\in\Gamma} K_i^{2(n_i)},$$

where $K_i^2 \in \mathcal{K}(\mathcal{H}_i)$ for $i \in \Gamma$, and $\|K_i^2\| \to 0$ if Γ is infinite. Since $C^*(B^2) \cap \mathcal{K}(\mathcal{H}) = C^*(A^2) \cap \mathcal{K}(\mathcal{H})$ is an ideal of $C^*(B^2)$, B^2 can be written as

$$B^2 = \bigoplus_{i\in\Gamma} E_i^{2(n_i)},$$

moreover, this means that $\mathcal{K}(\mathcal{H}_i) \subset C^*(E_i^2)$, E_i^2 is irreducible and $E_i^2 \ncong E_j^2$ whenever $i \neq j$. Since A^2, B^2 are disjoint, we deduce that no two of $\{A_i^2, E_i^2 : i \in \Gamma\}$ are unitarily equivalent.

that $A^2 + K^2 = \bigoplus_{i\in\Gamma} (A_i^2 + K_i^2)^{(n_i)}$ and $C^*(A^2 + K^2) \cap \mathcal{K}(\mathcal{H}) = C^*(A^2) \cap \mathcal{K}(\mathcal{H})$. As we have done to B^2, we can also deduce that $\{A_i^2 + K_i^2\}$ are irreducible and no two of them are unitarily equivalent.

By the hypothesis, $A^2 + K^2$ is a transpose of B^2 Thus $\bigoplus_{i\in\Gamma} (A_i^2 + K_i^2)^{(n_i)}$ and $\bigoplus_{i\in\Gamma} ((E_i^2)^t)^{(n_i)}$ are unitarily equivalent, and their m.r.s.'s correspond one to one. Then, by Lemmas (5.2.19) and (5.2.20), there exists a bijective map $\tau: \Gamma \to \Gamma$ such that $A_i^2 + K_i^2 \cong (E_{\tau(i)}^2)^t$ and $n_i = n_{\tau(i)}$ for all $i \in \Gamma$. For each $i \in \Gamma$, set $B_i^2 = E_{\tau(i)}^2$. Then, up to unitary equivalence, $A_i^2 + K_i^2$ is a transpose of B_i^2 for each $i \in \Gamma$.

Corollary (5.2.69)[462]: Let $\mathcal{H} = \bigoplus_{i\in\Gamma} \mathcal{H}_i$ and $A^2 \in \mathfrak{B}(\mathcal{H})$ with $A^2 = \bigoplus_{i\in\Gamma} A_i^2$, where $A_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. If $B^2 \in \mathfrak{B}(\mathcal{H})$ and $C^*(A^2) = C^*(B^2)$, then there exist $B_i^2 \in \mathfrak{B}(\mathcal{H}_i)$, $i \in \Gamma$, such that $B^2 = \bigoplus_{i\in\Gamma} B_i^2$ and

(i) for any subset Γ_0 of Γ, $C^*\left(\bigoplus_{i\in\Gamma_0} A_i^2\right) = C^*(\bigoplus_{i\in\Gamma_0} B_i^2)$,

(ii) for each $i \in \Gamma$, the reducing subspaces of A_i^2 coincide with that of B_i^2 ,

(iii) for each $i \in \Gamma$, A_i^2 is irreducible if and only if B_i^2 is irreducible,

(iv) for any $i, j \in \Gamma$, $A_i^2 \cong A_j^2$ if and only if $B_i^2 \cong B_j^2$.

Proof. Since $C^*(A^2) = C^*(B^2)$, it is clear that B^2 has the form $B^2 = \bigoplus_{i\in\Gamma} B_i^2$, where $B_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Statement (i) is also clear.

(ii) By (i), we have $C^*(A_i^2) = C^*(B_i^2)$. Thus $C^*(A_i')^2 = C^*(B_i')^2$ and the assertion holds.

(iii) This follows immediately from (ii).

(iv) We directly assume $i \neq j$. By (i), we have $C^*(A_i^2 \oplus A_j^2) = C^*(B_i^2 \oplus B_j^2)$. If $A_i^2 \cong A_j^2$, then there exists unitary operator $U: \mathcal{H}_j \to \mathcal{H}_i$ such that $A_j^2 = U^* A_i^2 U$. Then, for any polynomial $p(\cdot,\cdot)$ in two free variables, we have $p((A_j^*)^2, A_j^2) = U^* p((A_i^*)^2, A_i^2)U$. It follows immediately that each operator in $C^*(A_i^2 \oplus A_j^2)$ has the form $X \oplus U^*XU$, where

$X \in C^*(A_i^2)$. Sine $B_i^2 \oplus B_j^2 \in C^*(A_i^2 \oplus A_j^2)$, we obtain $B_i^2 = U^* B_i^2\, U$, that is, $B_i^2 \cong B_j^2$. Thus $A_i^2 \cong A_j^2$ implies $B_i^2 \cong B_j^2$. Likewise, one can see the converse.

Corollary (5.2.70)[462]: Let $T^2, R^2 \in \mathfrak{B}(\mathcal{H})$ be essentially square normal. If $C^*(T^2) = C^*(R^2)$, then

(i) T_s^2 is absent if and only if R_s^2 is absent, and

(ii) $C^*(T_s^2) = C^*(R_s^2)$.

Proof. In view of Lemma (5.2.21), we may assume that

$$T^2 = T_{nor}^2 \oplus \left(\oplus_{i\in\Gamma} T_i^{2(n_i)}\right), 1 \le n_i < \infty,$$

Where $T_{nor}^2 \in \mathfrak{B}(\mathcal{H}_0)$, $T_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible and not normal for $i \in \Gamma$; moreover, $T_i^2 \ncong T_j^2$ whenever $i \neq j$. Since $C^*(T^2) = C^*(R^2)$, R^2 can be written as

$$R^2 = R_0^2 \oplus \left(\oplus_{i\in\Gamma} R_i^{2(n_i)}\right),$$

where $R_0^2 \in \mathfrak{B}(\mathcal{H}_0)$ and $R_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Thus $C^*(R_0^2) = C^*(T_{nor}^2)$ and $C^*(R_i^2) = C^*(T_i^2)$ for $i \in \Gamma$. Then R_0^2 is normal; moreover, by Lemma (5.2.24), each R_i^2 is irreducible, not normal and $R_i^2 \ncong R_j^2$ whenever $i \neq j$. For each $i \in \Gamma$, we note that R_i^2 is almost normal if and only if T_i^2 is almost normal.

Denote $\Gamma_0 = \{i \in \Gamma : T_i^2$ is not almost normal$\}$. Then $\Gamma_0 = \{i \in \Gamma : R_i^2$ is not almost normal$\}$. Thus, by Lemma (5.2.21),

$$T_s^2 = \oplus_{i\in\Gamma_0} T_i^{2(n_i)}, \qquad R_s^2 = \oplus_{i\in\Gamma_0} R_i^{2(n_i)}.$$

From $C^*(T^2) = C^*(R^2)$, we deduce that $C^*(T_s^2) = C^*(R_s^2)$. This completes the p

Corollary (5.2.71)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal and $T^2 = \oplus_{i=1}^{\infty} A_i^2$, where $A_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ for $i \ge 1$. Assume that $B_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ is a transpose of A_i^2 for $i \ge 1$. If $p(z_1, z_2)$ is a polynomial in two free variables, then there exists $\oplus_{i=1}^{\infty} K_i^2 \in \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$ such that $p((B_i^*)^2, B_i^2) + K_i^2$ is a transpose of $p((A_i^*)^2, A_i^2)$ for $i \ge 1$.

Proof. By the hypothesis, there exist conjugations $\{C_i\}_{i=1}^{\infty}$ such that $B_i^2 = C_i (A_i^*)^2 C_i$, $i \ge 1$. Set $E_i^2 = (A_i^*)^2 A_i^2 - (A_i^*)^2$ for $i \ge 1$. Since T^2 is essentially normal, we have $(T^*)^2 T^2 - T^2 (T^*)^2 = \oplus_{i=1}^{\infty} E_i^2 \in \mathcal{K}(\mathcal{H})$. So $E_i^2 \in \mathcal{K}(\mathcal{H}_i)$ for $i \ge 1$ and $\|E_i^2\| \to 0$.

For convenience, we assume that $p(z_1, z_2) = z_1^2 z_2 z_1$. The proof in general case is similar. Compute to see that

$$C_i p((A_i^*)^2, A_i^2)^* C_i = C_i\, A_i^2 (A_i^*)^2 A_i^2 C_i = (B_i^*)^2 B_i^2\, (B_i^*)^2$$
$$= (B_i^*)^2 \left(B_i^2 (B_i^*)^2\right)(B_i^*)^2 - (B_i^*)^2((B_i^*)^2 B_i^2)(B_i^*)^2 + (B_i^*)^2((B_i^*)^2 B_i^2)(B_i^*)^2$$
$$= (B_i^*)^2 ({B_i}^2 (B_i^*)^2 - (B_i^*)^2 B_i^2)(B_i^*)^2 + p({B_i^*}^2, B_i^2)$$
$$= (B_i^*)^2 (C_i E_i^2 C_i)(B_i^*)^2 + p((B_i^*)^2, B_i^2).$$

Set $K_i^2 = (B_i^*)^2 (C_i E_i^2 C_i)(B_i^*)^2$. So K_i^2 is compact and $p((B_i^*)^2, B_i^2) + K_i^2$ is a transpose of $p((A_i^*)^2, A_i^2)$; moreover, we have

$$\|K_i^2\| \le \|B_i^2\|^2 \cdot \|E_i^2\| = \|A_i^2\|^2 \cdot \|E_i^2\| \le \|T^2\|^2 \cdot \|E_i^2\| \to 0.$$

Hence $\oplus_{i=1}^{\infty} K_i^2 \in \oplus_{i=1}^{\infty} \mathcal{K}(\mathcal{H}_i)$. This completes the proof.

Corollary (5.2.72)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$be essentially normal and

$$T^2 = \oplus_{j=1}^{\infty} \left(A_j^2 \oplus B_j^2\right),$$

where A_j^2, $B_j^2 \in \mathfrak{B}(\mathcal{H}_j)$ and B_j^2 is a transpose of A_j^2 for $j \geq 1$. Then each operator $R^2 \in C^*(T^2)$ can be written as $R^2 = \bigoplus_{i=1}^{\infty} (F_j^2 \oplus G_j^2)$, where $G_j^2 \in \mathfrak{B}(\mathcal{H}_j)$ is a compact perturbation of some transpose $(F_j^2)^t$ of F_j^2 and $\|G_j^2 - (F_j^2)^t\| \to 0$.

Proof. Since B_j^2 is a transpose of A_j^2, there exists a conjugation C_j such that $B_j^2 = C_j(A_j^*)^2 C_j$. Assume that $\{p_n\}_{n=1}^{\infty}$ are polynomials in two free variables and $p_n((T^*)^2, T^2) \to R^2$. Note that $\bigoplus_{j=1}^{\infty} A_j^2$ is essentially normal. Then, by Lemma (5.2.26), for each $n \geq 1$, there exist compact operators $\{K_{j,n}^2\}_{j\geq 1}$ such that

$$p_n((B_j^*)^2,\ B_j^2) + K_{j,n}^2 = C_j p_n((A_j^*)^2,\ A_j^2)^* C_j$$

and $\|K_{j,n}^2\| \to 0$ as $j \to \infty$. Then $\bigoplus_{j=1}^{\infty} K_{j,n}^2$ is compact for each $n \geq 1$.

Note that $p_n((T^*)^2, T^2) \to R^2$ as $n \to \infty$ and

$$p_n((T^*)^2, T^2) = \bigoplus_{j=1}^{\infty} \left(p_n\left((A_j^*)^2,\ A_j^2\right) \oplus\ p_n\left((B_j^*)^2,\ B_j^2\right)\right), \qquad n \geq\ 1.$$

Then $\bigoplus_{j=1}^{\infty} p_n((A_j^*)^2,\ A_j^2)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} F_j^2$ and $\bigoplus_{j=1}^{\infty} p_n((B_j^*)^2,\ B_j^2)$ converges to an operator of the form $\bigoplus_{j=1}^{\infty} G_j^2$ as $n \to \infty$. Then

$$\bigoplus_{j=1}^{\infty} C_j p_n((A_j^*)^2,\ A_j^2)^* C_j \longrightarrow \bigoplus_{j=1}^{\infty} C_j (F_j^*)^2 C_j.$$

So, as $n \to \infty$, we have

$$\bigoplus_{j=1}^{\infty} K_{j,n}^2 = \bigoplus_{j=1}^{\infty} \left(C_j p_n((A_j^*)^2,\ A_j^2)^* C_j -\ p_n((B_j^*)^2, B_j^2)\right) \longrightarrow \bigoplus_{j=1}^{\infty} \left(C_j (F_j^*)^2 C_j - G_j^2\right).$$

For each $n \geq 1$, note that $\bigoplus_{j=1}^{*} K_{j,n}^2$ is compact. Thus their norm limit $\bigoplus_{j=1}^{\infty} \left(C_j (F_j^*)^2 C_j - G_j^2\right)$ is also compact. Hence $C_j (F_j^*)^2 C_j - G_j^2$ is compact for each j and $\left\|C_j (F_j^*)^2 C_j - G_j^2\right\| \to 0$ as $j \to \infty$. Note that $R^2 = lim_n p_n((T^*)^2, T^2) = \bigoplus_{j=1}^{\infty} (F_j^2 \oplus G_j^2)$. This completes the proof.

Corollary (5.2.73)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ and suppose that $\lambda^2 \in \sigma_{lre}(T^2)$. Then, given $\varepsilon > 0$, there exists a compact operator K^2 with $\|K^2\| < \varepsilon$ such that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & * \\ 0 & A^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^{\perp} \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector and $A^2 \in B^2(\{e\}^{\perp})$ satisfies $\sigma(T^2) = \sigma(A^2)$.

Proof. By Lemma (5.2.28), there exists $K^2 \in \mathcal{K}(\mathcal{H})$ with $\|K^2\| < \varepsilon$ such that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 I_1 & * \\ 0 & A_0^2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$, $dim\mathcal{H}_1 = \infty$, I_1 is the identity operator on $\mathcal{H}_1$ and $A_0^2 \in \mathfrak{B}(\mathcal{H}_2)$ satisfies $\sigma(A_0^2) = \sigma(T^2)$. Choose a unit vector $e \in \mathcal{H}_1$. Then $T^2 + K^2$ can be written as

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & 0 & E^2 \\ 0 & \lambda^2 I_2 & F^2 \\ 0 & 0 & A_0^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \mathcal{H}_1 \ominus \mathbb{C}e, \\ \mathcal{H}_2 \end{matrix}$$

Where I_2 is the identity operator on $\mathcal{H}_1 \ominus \mathbb{C}e$. Set

$$A^2 = \begin{bmatrix} \lambda^2 I_2 & F^2 \\ 0 & A_0^2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \ominus \mathbb{C}e \\ \mathcal{H}_2 \end{matrix}.$$

Since $\lambda^2 \in \sigma(T^2) = \sigma(A_0^2)$, it follows that $\sigma(A^2) = \sigma(T^2)$. Noting that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & * \\ 0 & A^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \mathcal{H} \ominus \mathbb{C}e \end{matrix},$$

we conclude the proof.

Corollary (5.2.74)[462]: Let $A^2, B^2 \in \mathfrak{B}(\mathcal{H})$. Assume that $\lambda^2 \in iso\ \sigma(A^2)$ and $\lambda^2 \notin \sigma(B^2)$. Then there exists $\delta > 0$ such that

$$\text{“}E^2, F^2 \in \mathfrak{B}(\mathcal{H}), \|E^2\| < \delta, \|F^2\| < \delta\text{”} \Longrightarrow \text{“}\sigma(A^2 + E^2) \neq \sigma(B^2 + F^2)\text{”}.$$

Proof. Since $\lambda^2 \in iso\, \sigma(A^2)$ and $\lambda^2 \notin \sigma(B^2)$, there exists $\varepsilon > 0$ such that $B^2(\lambda^2, \varepsilon)^- \cap \sigma(A^2) = \{\lambda^2\}$ and $B^2(\lambda^2, \varepsilon)^- \cap \sigma(B^2) = \emptyset$. Then, by the upper semi-continuity of spectrum (see [254, Thm. 1.1]), there exists $\delta > 0$ such that

(iii) $B^2(\lambda^2, \varepsilon)^- \cap \sigma(A^2 + E^2) \neq \emptyset$ for any $E^2 \in \mathfrak{B}(\mathcal{H})$ with $\|E^2\| < \delta$, and

(iv) $B^2(\lambda^2, \varepsilon)^- \cap \sigma(B^2 + F^2) = \emptyset$ for any $F^2 \in \mathfrak{B}(\mathcal{H})$ with $\|F^2\| < \delta$.

Hence we conclude the proof.

Corollary (5.2.75)[462]: Let $\{A_i^2\}_{i=1}^n$ be operators on separate Hilbert spaces with pairwise distinct spectra. Then, given $B^2 \in \mathfrak{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K^2 \in \mathcal{K}(\mathcal{H})$ with $\|K^2\| < \varepsilon$ such that $A_{n+1}^2 := B^2 + K^2$ is irreducible, and $\{\sigma(A_i^2)\}_{i=1}^{n+1}$ are pairwise distinct.

Proof. Choose a point λ_0^2 in $\partial\sigma(B^2) \cap \sigma_{lre}(B^2)$. By Corollary (5.2.73), there exists compact K_0^2 with $\|K_0^2\| < \frac{\varepsilon}{2}$ such that

$$B^2 + K_0^2 = \begin{bmatrix} \lambda_0^2 & E^2 \\ 0 & B_0^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

Where $e \in \mathcal{H}$ is a unit vector and $\sigma(B_0^2) = \sigma(B^2)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1^2, \lambda_2^2, \cdots, \lambda_{n+1}^2$ outside $\sigma(B^2)$ such that $sup_{1\le i\le n+1}|\lambda_i^2 - \lambda_0^2| < \frac{\varepsilon}{4}$. For each $1 \le i \le n+1$, set

$$B_i^2 = \begin{bmatrix} \lambda_i^2 & E^2 \\ 0 & B_0^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix}.$$

Then $\left\|B^2 + K_0^2 - B_i^2\right\| < \frac{\varepsilon}{4}$, $\lambda_i^2 \in iso\ \sigma(B_i^2)$ and $\lambda_j^2 \notin \sigma(B_i^2)$ whenever $i \neq j$. By Lemma (5.2.31), there exist compact operators F_i^2 with $\left\|F_i^2\right\| < \frac{\varepsilon}{4}$ such that each $B_i^2 + F_i^2$ is irreducible; moreover, by Lemma (5.2.30), we may also assume that $\{\sigma(B_i^2 + F_i^2)\}_{i=1}^{n+1}$ are pairwise distinct. Then there exists some i_0, $1 \le i_0 \le n+1$, such that $\sigma(B_{i_0}^2 + F_{i_0}^2) \neq \sigma(A_j^2)$ for $1 \le j \le n$. Set $K^2 = (F_{i_0})^2 + (B_{i_0})^2 - B^2$ and $A_{n+1}^2 = B^2 + K^2$. Then $A_{n+1}^2 = (B_{i_0})^2 + (F_{i_0})^2$ is irreducible. Noting that $K^2 = (F_{i_0})^2 + (B_{i_0})^2 - (B^2 K_0^2) + K_0^2$ is compact,

$$\|K^2\| \le \left\|(F_{i_0})^2\right\| + \left\|(B_{i_0})^2 - (B^2 + (K_{i_0})^2)\right\| + \|K_0^2\| < \varepsilon$$

and $\{\sigma(A_i^2)\}_{i=1}^{n+1}$ are pairwise distinct, we complete the proof.

Corollary (5.2.76)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be normal. Then, given $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist irreducible $CSOs$ $T_1^2, T_2^2, \cdots, T_n^2 \in \mathfrak{B}(\mathcal{H})$ with pairwise distinct spectra such that $T_1^2 - T^2 \in \mathcal{K}(\mathcal{H})$ and $\left\|T_i^2 - T^2\right\| < \varepsilon$ for all $1 \le i \le n$.

Proof. Choose a point λ^2 in $\partial\sigma(T^2) \cap \sigma_{lre}(T^2)$. By the classical Weyl–von Neumann Theorem, there exists compact K^2 with $\|K^2\| < \frac{\varepsilon}{2}$ such that

$$T^2 + K^2 = \begin{bmatrix} \lambda^2 & 0 \\ 0 & N^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^\perp \end{matrix},$$

where $e \in \mathcal{H}$ is a unit vector, N^2 is normal and $\sigma(N^2) = \sigma(T^2)$.

For given $\varepsilon > 0$, we can choose pairwise distinct points $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$ outside $\sigma(T^2)$ such that $sup_{1\le i\le n}|\lambda_i^2 - \lambda_0^2| < \frac{\varepsilon}{4}$. For each $1 \le i \le n$, set

$$A_t = \begin{bmatrix} \lambda_i^2 & 0 \\ 0 & N^2 \end{bmatrix} \begin{matrix} \mathbb{C}e \\ \{e\}^{\perp} \end{matrix}.$$

Then $\|T^2 + K^2 - A_i^2\| < \frac{\varepsilon}{4}$, $\lambda_i^2 \in \text{iso}\, \sigma(A_i^2)$ and $\lambda_j \notin \sigma(A_i^2)$ whenever $i \ne j$. By Lemma (5.2.31), there exist compact operators F_i^2 with $\|F_i^2\| < \frac{\varepsilon}{4}$ such that each $A_i^2 + F_i^2$ is irreducible and complex symmetric; moreover, by Lemma (5.2.30), it can be required that $\{\sigma(A_i^2 + F_i^2)\}_{i=1}^n$ are pairwise distinct. Set $T_i^2 = A_i^2 + F_i^2$ for $1 \le i \le n$. Then $\{T_i^2 : 1 \le i \le n\}$ satisfy all requirements.

Corollary (5.2.77)[462]: Let $\{T_i^2\}_{i=1}^{\infty}$ be normal operators on separable Hilbert spaces. Then, given $\varepsilon > 0$, there exist compact operators $\{K_i^2\}_{i=1}^{\infty}$with

$$sup_i \|K_i^2\| < \varepsilon, \qquad \lim_i \|K_i^2\| = 0$$

such that

(i) $T_i^2 + K_i^2$ is complex symmetric and irreducible for $i \ge 1$, and

(ii) $\sigma(T_i^2 + K_i^2) \ne \sigma(T_j^2 + K_j^2)$ whenever $i \ne j$.

Proof. For convenience, we assume that $T_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ for $i \ge 1$. We shall construct $\{K^2\}_{i=1}^{\infty}$ by induction.

By Proposition (5.2.9), we can choose $K_1^2 \in \mathcal{K}(\mathcal{H}_1)$ with $\| K_1^2\| < \varepsilon$ such that $T_1^2 + K_1^2$ is irreducible and complex symmetric.

Now assume that we have chosen compact operators $K_i^2 \in \mathcal{K}(\mathcal{H}_i), 1 \le i \le n$, satisfying that (a) $\|K_i^2\| < \varepsilon/i$ for $1 \le i \le n$, (b) $T_i^2 + K_i^2$is complex symmetric and irreducible for $1 \le i \le n$, and (c) $\sigma(T_i^2 + K_i^2) \ne \sigma(T_j^2 + K_j^2)$ whenever $1 \le i \ne j \le n$. We are going to choose $K_{n+1}^2 \in K^2(H_{n+1})$ with $\|K_{n+1}^2\| < \varepsilon/(n+1)$ such that $T_{n+1}^2 + K_{n+1}^2$ is irreducible and complex symmetric; moreover, $\sigma(T_i^2 + K_i^2) \ne \sigma(T_{n+1}^2 + K_{n+1}^2)$ for $1 \le i \le n$.

By Lemma (5.2.34), we can find $F_1^2, F_2^2, \cdots, F_{n+1}^2 \in \mathcal{K}(\mathcal{H}_{n+1})$ with $\|F_i^2\| < \varepsilon/(n+1)$ such that $T_{n+1}^2 + F_i^2$ is irreducible and complex symmetric for $1 \le i \le n+1$; moreover, $\sigma(T_{n+1}^2 + F_i^2) \ne \sigma(T_{n+1}^2 + F_j^2)$ whenever $i \ne j$. So some i_0, $1 \le i_0 \le n+1$, exists such that $\sigma(T_{n+1}^2 + (F_{i_0})^2) \ne \sigma(T_j^2 + K_j^2)$ for all $1 \le j \le n$. Set $K_{n+1}^2 = (F_{i_0})^2$. Then K_{n+1}^2 satisfies all requirements. By induction, this completes the proof.

Corollary (5.2.78)[462]: ([257, Thm. 4.4]) Let $\{T_n^2\}_{n=1}^{\infty}$ be a sequence of almost normal operators. Assume that $sup_n\|T_n^2\| < \infty$ and $\|(T_n^*)^2\, T_n^2 - T_n^2\, (T_n^*)^2\| \to 0$ as $n \to \infty$. Then there exists a sequence $\{N_n^2\}_{n=1}^{\infty}$ of normal operators such that $T_n^2 - N_n^2$ is compact for $n \ge 1$ and $\|T_n^2 - N_n^2\| \to 0$.

Proof. for the sufficiency of Theorem (5.2.14) By the hypothesis, Lemma (5.2.21) and Proposition (5.2.23), we may assume that

$$T^2 = N^2 \oplus \left(\oplus_{i\in\Gamma_1} T_i^{2(n_i)}\right) \oplus \left(\oplus_{i\in\Gamma_2} (A_j^2 \oplus B_j^2)^{(n_i)}\right),$$

where

(i) N^2 is normal, $\{{T_i}^2, A_j^2, B_j^2 : i \in \Gamma_1, j \in \Gamma_2\}$ are irreducible operators and no two of them are unitarily equivalent;

(ii) T_j^2 is almost normal and not normal for $i \in \Gamma_1$;

(iii) A_j^2 is not almost normal and there exists a compact operator K_j^2 such that $B_j^2 + K_j^2$ is a transpose of A_j^2 for $j \in \Gamma_2$;

(iv) $1 \le n_i,\ n_j < \infty$ for all $i \in \Gamma_1$ and $j \in \Gamma_2$, and $\|K_j{}^2\| \to 0$ if Γ_2 is infinite.

Assume that $N^2 \in \mathfrak{B}(\mathcal{H}_0),\ T_i^2 \in \mathfrak{B}(\mathcal{H}_{1,i})$ for $i \in \Gamma_1$ and $A_j^2,\ B_j^2 \in \mathfrak{B}(\mathcal{H}_{2,j})$ for $j \in \Gamma_2$.

For convenience, we may directly assume that $\Gamma_1,\ \Gamma_2$ are countable and $n_i = 1$ for all $i \in \Gamma_1 \cup \Gamma_2$. The proof for the general case is similar. Then

$$T^2 = N^2 \oplus \left(\oplus_{i=1}^{\infty} T_i^2\right) \oplus \left(\oplus_{j=1}^{\infty} A_j^2 \oplus B_j^2)\right)$$

and

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\oplus_{i=1}^{\infty} \mathcal{H}_{1,i}\right) \oplus \left(\oplus_{j=1}^{\infty} (\mathcal{H}_{2,j} \oplus \mathcal{H}_{2,j})\right). \quad (18)$$

The rest of the proof is divided into three steps.

Step 1.Compact perturbations of the operators $\{T_i^2 : i \ge 1\}$.

Since T^2 is essentially normal, it follows that $(T^*)^2\, T^2 - T^2 (T^*)^2 \in \mathcal{K}(\mathcal{H})$ and hence $(T_i^*)^2\, T_i^2 - T_i^2 (T_i^*)^2 \in \mathcal{K}(\mathcal{H}_{1,i})$ and $\|(T_i^*)^2\, T_i^2 - T_i^2 (T_i^*)^2\| \to 0$. By Lemma (5.2.36), we can choose $D_{1,i}^2 \in \mathcal{K}(\mathcal{H}_{1,i}),\ i \ge 1$, so that $\|D_{1,i}^2\| \to 0$ and $N_i^2 := T_i^2 + D_{1,i}^2$ is normal for all $i \ge 1$. By Corollary (5.2.35), there are compact operators $D_{2,1}^2 \in \mathrm{K}^2(H_{1,i}) (i \ge 1)$ with $\|D_{2,i}^2\| \to 0$ such that $S_i{}^2 \coloneqq N_i^2 + D_{2,i}^2$ is irreducible, complex symmetric and $S_i^2 \ncong S_j^2$ whenever $i \ne j$.

Set $D_i^2 = D_{1,i}^2 + D_{2,i}^2$ for $i \ge 1$. Then $S_i^2 = T_i{}^2 + D_i^2,\ D_i^2 \in \mathcal{K}(\mathcal{H}_{1,i})$ and $\|D_i^2\| \to 0$. From statement (ii), each T_i^2 acts on a space of dimension ≥ 2. Thus S_i^2 is almost normal and not normal.

Step 2.Compact perturbations of the operators $\{A_j^2,\ B_j^2 : j \ge 1\}$.

For each $j \ge 1$, by the hypothesis, there exists a conjugation C_j on $\mathcal{H}_{2,j}$ such that $C_j (A_j^*)^2\, C_j = B_j^2 + K_j^2$. Note that $\|K_j^2\| \to 0$.

Since each A_j^2 is irreducible, it follows from Corollary (5.2.33) that we can find compact operators $\{E_j^2\}_{j=1}^{\infty}$ with $\|E_j^2\| \to 0$ such that $R_j^2 := A_j^2 + E_j^2$ is irreducible for all $j \ge 1$ and $\{\sigma R_j^2\}_{j=1}^{\infty}$ are pairwise distinct.

For each $j \ge 1$, set $G_j^2 = K_j^2 + C_j (E_j^*)^2\, C_j$. Then $G_j^2 \in \mathcal{K}(\mathcal{H}_{2,j})$ and $\|G_j^2\| \to 0$. On the other hand, note that

$$C_j (R_j^*)^2\, C_j = C_j (A_j^*)^2\, C_j + C_j (E_j^*)^2\, C_j = B_j{}^2 + K_j{}^2 + C_j (E_j^*)^2\, C_j = B_j^2 + G_j^2.$$

Step 3.Construction and verification.

Set

$$R^2 = N^2 \oplus \left(\oplus_{i=1}^{\infty} S_i^2\right) \oplus \left(\oplus_{j=1}^{\infty} (R_j^2 \oplus C_j (R_j^*)^2 C_j)\right).$$

By [252, Thm. 1.6] or [251, Lem. 3.6], R^2 is complex symmetric. Define $K^2 \in \mathfrak{B}(\mathcal{H})$ with respect to the decomposition (18) as

$$K^2 = 0 \oplus \left(\oplus_{i=1}^{\infty} D_i^2\right) \oplus \left(\oplus_{j=1}^{\infty} \left(E_j^2 \oplus G_j^2\right)\right). \quad (19)$$

Then K^2 is compact and one can check that $R^2 = T^2 + K^2$. Now it remains to prove $C^*(T^2) = C^*(R^2)$. Clearly, we need only prove $K^2 \in C^*(T^2) \cap C^*(R^2)$.

In view of (19), it suffices to prove that

$$0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i=1}^{\infty} \mathcal{K}\ (\mathcal{H}_{1,i}) \right) \oplus \left(\bigoplus_{j=1}^{\infty} (\mathcal{K}(\mathcal{H}_{2,j}) \oplus \mathcal{K}(\mathcal{H}_{2,j}) \right) \subset C^*(T^2) \cap C^*(R^2).$$

By statements (i)–(iii), it follows from Corollary (5.2.16) that

$$0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i=1}^{\infty} \mathcal{K}\ \left(\mathcal{H}_{1,i}\right) \right) \oplus \left(\bigoplus_{j=1}^{\infty} (\mathcal{K}(\mathcal{H}_{2,j}) \oplus \mathcal{K}(\mathcal{H}_{2,j}) \right) \subset C^*(T^2).$$

Since $\{S_i^2, R_i^2, C_i(R_i^*)^2\, C_i: i \geq 1\}$ are irreducible and not normal, by Corollary (5.2.16), it suffices to prove that no two of them are unitarily equivalent.

Noting that $\sigma(C_i(R_i^*)^2\, C_i) = \sigma(R_i^2) \neq \sigma(R_j^2) = \sigma(C_j(R_j^*)^2\, C_j)$ whenever $i \neq j$, we deduce that $R_i^2 \ncong R_j^2$, ${R_i}^2 \ncong C_j(R_j^*)^2\, C_j$ and $C_i(R_i^*)^2\, C_i \ncong C_j(R_j^*)^2\, C_j$ whenever $i \neq j$. On the other hand, note that R_j^2 is a compact perturbation of A_j^2 and A_j^2 is not almost normal for $j \geq 1$. Then, for each $j \geq 1$, we can choose $\lambda^2 \in C$ such that $R_j^2 - \lambda^2$ is Fredholm and $ind(R_j^2 - \lambda^2) \neq 0$. So

$$\begin{aligned} ind\ (R_j^2 - \lambda^2) &= -ind\ (R_j^2 - \lambda^2)^* \\ &= -ind\ C_j(R_j^2 - \lambda^2)^*\, C_j \\ &= -ind\ (C_j(R_j^*)^2\, C_j - \lambda^2), \end{aligned}$$

which implies that $R_j^2 \ncong C_j(R_j^*)^2\, C_j$.

By the preceding argument, $S_i^2 \ncong S_j^2$ whenever $i \neq j$. Since each of $\{S_i^2: i \geq 1\}$ is almost normal, we have $S_i^2 \ncong C_j(R_j^*)^2\, C_j$ and $S_i^2 \ncong R_j^2$ for all $i, j \geq 1$. Hence we deduce that no two of $\{S_i^2,\ R_i^2, C_i(R_i^*)^2\, C_i : i \leq 1\}$ are unitarily equivalent. This completes the proof.

Corollary (5.2.79)[462]: Each compact operator has a complex symmetric generator for its C^*-algebra.

Proof. Assume that $T^2 \in \mathfrak{B}(\mathcal{H})$ is compact. Then the restrictions of T^2 to its minimal reducing subspaces are all compact and hence almost normal. Hence the result follows readily from Corollary (5.2.37).

Corollary (5.2.80)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T_s^2 is not absent, then the following are equivalent:

(i) $T^2 \in (cs)$.

(ii) $T_{abnor}^2 \in (cs)$.

(iii) $T_s^2 \in (cs)$.

Proof. Note that $(T_s^2)_s = T_s^2 = (T_{abnor}^2)_s$. Then the result follows readily from Theorem (5.2.14).

Corollary (5.2.81)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal and assume that $T^2 = N^2 \oplus A^{2(n)}$, where $1 \leq n < \infty$, N is normal, A^2 is abnormal and irreducible. Then $T^2 \in (cs)$ if and only if A^2 is almost normal.

Proof. If A^2 is almost normal, then T_s^2 is absent. By Theorem (5.2.14), we have $T^2 \in (cs)$. If A^2 is not almost normal, then $T_s^2 = (A^2)^{(n)}$ is not almost normal. So T_s^2 is not of type C. By Theorem (5.2.14), we have $T^2 \notin (cs)$.

Corollary (5.2.82)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal and $T^2 = A^{2(m)} \oplus B^{2(n)}$, where A^2, B^2 are irreducible, not normal and $A^2 \ncong B^2$. Then $T^2 \in (cs)$ if and only if exactly one of the following holds:

(i) Both A^2 and B^2 are almost normal;

(ii) Neither $A^2\ nor\ B^2$ is almost normal, $m = n$ and $\Lambda(A^2)^t = \Lambda(B^2)$.

Proof. Since T^2 is essentially normal, it follows immediately that $1 \le m,\ n < \infty$.

"⟸". If (i) holds, then T_s^2 is absent. By Theorem (5.2.14), we have $T^2 \in (cs)$. If (ii) holds, then $T^2 = T_s^2$; moreover, by the BDF Theorem, $\Lambda(A^2)^t = \Lambda(B^2)$ implies that B^2 is a compact perturbation of $(A^2)^t$. So, by Proposition (5.2.23), T^2 is of type C. The conclusion follows immediately from Theorem (5.2.14).

"⟹". We assume that $T^2 \in (cs)$and (i) does not hold. It suffices to prove that (ii) holds. For convenience we assume that $A^2 \in \mathfrak{B}(\mathcal{H}_1)$ and $B^2 \in \mathfrak{B}(\mathcal{H}_2)$.

We claim that neither A^2 nor B^2 is almost normal. For a proof by contradiction, without loss of generality, we assume that A^2 is almost normal. Then, by the hypothesis, B^2 is not almost normal. So $T_s^2 = B^{2(n)}$ is not almost normal. Then T_s^2 is not of type C and $T^2 \notin (cs)$, a contradiction. This proves the claim, which means that $T^2 = T_s^2$.

Since $T^2 \in (cs)$, it follows that T^2 is of type C. Noting that $A^2 \ncong B^2$, by the definition, it follows that $m = n$ and there exists compact K^2 such that $A^2 + K^2$ is unitarily

$$\Lambda(A^2)^t = \Lambda(B^2).$$

Corollary (5.2.83)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially square normal. Then $T^2 \in (wcs)$ if and only if there exists an essentially square normal operator $R^2 \in (cs)$ such that $T^2 \approx R^2$.

Proof The sufficiency is obvious.

"⟹". Assume that

$$T^2 = T_0^2 \oplus \left(\oplus_{i \in \Gamma} T_i^{2(n_i)} \right),$$

where T_0^2 is completely reducible, $T_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $(T_{i_1})^2 \ncong \left(T_{i_2}\right)^2$ whenever $i_1 \neq i_2$. Set $A^2 = T_0^2 \oplus (\oplus_{i\in\Gamma} T_i^2)$. Then $A^2 \approx T^2$ is essentially normal and, by Lemma (5.2.46) , A^2 is multiplicity-free.

Assume that $S^2 \in \mathfrak{B}(\mathcal{K})$ is complex symmetric and $C^*(S^2)$ is $*$-isomorphic to $C^*(T^2)$. By Lemma (5.2.46), S^2 is algebraically equivalent to some multiplicity-free operator B^2. By Theorem (5.2.45), B^2 is g-normal.

Since $C^*(S^2)$ is $*$-isomorphic to $C^*(T^2)$, $A^2 \approx T^2$ and $B^2 \approx S^2$, we can find a $*$-isomorphism $\varphi : C^*(A^2) \to C^*(B^2)$. Denote $R^2 = \varphi(A^2)$. Then $A^2 \approx R^2$ and $C^*(B^2) = C^*(R^2)$. Noting that B^2 is multiplicity-free, it follows from Lemma (5.2.22) that R^2 is also multiplicity-free. By Theorem (5.2.45), we obtain $A^2 \cong_a R^2$. Since A^2 is essentially normal, so is R^2. This combining $C^*(B^2) = C^*(R^2)$ implies that B^2 is also essentially normal. Since B^2 is multiplicity-free and g-normal, it follows from Lemma (5.2.56) that $C^*(B^2) = C^*(R^2)$ admits a complex symmetric square generator, that is, $R^2 \in (cs)$. Noting that $T^2 \approx A^2$ and $A^2 \cong_a R^2$, we obtain $T^2 \approx R^2$.

Corollary (5.2.84)[462]: For $T^2 \in \mathfrak{B}(\mathcal{H})$, the following are equivalent:

(i) there is a faithful representation ρ of $C^*(T^2)$ such that $\rho(T^2)$ is complex symmetric;

(ii) T^2 is g-normal;

(iii) T^2 is algebraically equivalent to a $CSSO$.

Proof."(i)⟹(ii)". Assume that ρ is a faithful representation of $C^*(T^2)$ on $\mathcal{H}_\rho$ with $A^2 = \rho(T^2)$ being complex symmetric. Then, for any polynomial $p(z_1, z_2)$ in two free variables, we have $\rho(p((T^*)^2, T^2)) = p((A^*)^2, A^2)$ and $\rho(\tilde{p}(T^2, (T^*)^2)) = \tilde{p}(A^2, (A^*)^2)$. Since ρ is faithful, we have

$$\|p((T^*)^2, T^2)\| = \|p((A^*)^2, A^2)\|, \qquad \|\tilde{p}(T^2, (T^*)^2)\| = \|\tilde{p}(A^2, (A^*)^2)\|.$$

Since each $CSSO$ is g-normal, it follows that

$$\|p(T^*)^2, T^2)\| = \|p((A^*)^2, A^2)\| = \|\tilde{p}(A^2, (A^*)^2)\| = \|\tilde{p}(T^2, (T^*)^2)\|.$$

So T^2 is g-normal.

"(ii)⟹(iii)". Denote $R^2 = T^{2(\infty)}$. Then R^2 is still g-normal and $R^2 \approx T^2$; moreover, $C^*(R^2)$ contains no nonzero compact operator. By [251, Thm. 2.1], R^2 is approximately unitarily equivalent to some complex symmetric square operator X. Then $T^2 \approx X$.

"(iii)⟹(i)". By definition, the implication is obvious.

An operator $T^2 \in \mathfrak{B}(\mathcal{H})$ is said to be multiplicity-free if $T^2|_{\mathcal{M}} \not\cong T^2|_{\mathcal{N}}$ for any distinct minimal reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ of T^2.

Corollary (5.2.85)[462]: Each operator is algebraically equivalent to a multiplicity-free operator.

Proof. Let $T^2 \in \mathfrak{B}(\mathcal{H})$. By Lemma (5.2.18), we may assume that

$$T^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^{2(n_i)}\right),$$

where T_0^2 is completely reducible, $T_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1}^2 \not\cong T_{i_2}^2$ whenever $i_1 \neq i_2$.

Set $R^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^2\right)$. Then it is obvious that $\|p((T^*)^2, T^2)\| = \|p((R^*)^2, R^2)\|$ for any polynomial $p(z_1, z_2)$ in two free variables. So $T^2 \approx R^2$. It remains to prove that R^2 is multiplicity-free.

By Lemma (5.2.19), $\{\mathcal{H}_i : i \in \Gamma\}$ are all minimal reducing subspaces of R^2. For $i_1, i_2 \in \Gamma$ with $i_1 \neq i_2$, we have $R^2|_{\mathcal{H}_{i_1}} = T_{i_1}^2 \not\cong T_{i_2}^2 = R^2|_{\mathcal{H}_{i_2}}$. This completes the proof.

Corollary (5.2.86)[462]: Let $T^2, R^2 \in \mathfrak{B}(\mathcal{H})$ be multiplicity-free. Then $T^2 \approx R^2$ if and only if $T^2 \cong_a R^2$.

Proof. The sufficiency is obvious.

"⟹". We let $\varphi: C^*(T^2) \longrightarrow C^*(R^2)$ denote the $*$-isomorphism carrying T^2 into R^2. It suffices to prove that

$$rank\ X = rank\ \varphi(X), \quad \forall X \in C^*(T^2) \cap \mathcal{K}(\mathcal{H}) \tag{20}$$

and

$$rank\ \varphi^{-1}(Y) = rank\ Y, \quad \forall Y \in C^*(R^2) \cap \mathcal{K}(\mathcal{H}) \tag{21}$$

In fact, if these equalities hold, then $rank\ \varphi(X) = rankX$ for all $X \in C^*(T^2)$. By Lemma (5.2.48), this implies $\varphi \cong_a$ id, where id$(\cdot)$ denotes the identity representation of $C^*(T^2)$. So $R^2 = \varphi(T^2) \cong_a id(T^2) = T^2$.

Denote $\mathcal{A} = C^*(T^2) \cap \mathcal{K}(\mathcal{H})$. By [238, Thm.I.10.8] we may assume that

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\bigoplus_{i\in\Gamma} \mathcal{H}_i^{(k_i)}\right), \qquad \mathcal{A} = 0_{\mathcal{H}_0} \oplus \left(\bigoplus_{i\in\Gamma} \mathcal{K}(\mathcal{H}_i)^{(k_i)}\right),$$

where the dimensions of $\mathcal{H}_0$ and $\mathcal{H}_i (i \in \Gamma)$ may be finite or $\aleph_0$, and $1 \le k_i < \infty$ for $i \in \Gamma$. Since $\mathcal{A}$ is an ideal of $C^*(T^2)$, T^2 can be written as

$$T^2 = D_0^2 \oplus \left(\bigoplus_{i\in\Gamma} D_i^{2(k_i)} \right),$$

where $D_0^2 \in \mathfrak{B}(\mathcal{H}_0)$ and $D_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ for $i \in \Gamma$. Then $\mathcal{K}(\mathcal{H}_i) \subset C^*(D_i^2)$ for each $i \in \Gamma$. Hence each D_i^2 is irreducible. Noting that T^2 is multiplicity-free, we have $k_i = 1$ for all $i \in \Gamma$. Then each compact operator in $C^*(T^2)$ has the form $0 \oplus (\oplus_{i\in\Gamma} X_i)$, where $X_i \in \mathcal{K}(\mathcal{H}_i)$. For $i \in \Gamma$, denote by $(P_i)^2$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_i$.

Corollary (5.2.87)[462]: Let $T^2, R^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If $T^2 \approx R^2$, then $T^2_{abnor} \approx R^2_{abnor}$ and $T_s^2 \approx R_s^2$.

Proof. By Lemma (5.2.18), we may assume that

$$T^2 = T_0^2 \oplus \left(\oplus_{i\in\Gamma} T_i^{2(m_i)}\right),$$

where $T_0^2 \in \mathfrak{B}(\mathcal{H}_0)$ is completely reducible, $T_0^2 \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1}^2 \ncong T_{i_2}^2$ whenever $i_1 \neq i_2$. Likewise, we assume that

$$R^2 = R_0^2 \oplus \left(\oplus_{j\in\Upsilon} R_j^{2(m_i)}\right),$$

where $R_0^2 \in \mathfrak{B}(\mathcal{K}_0)$ is completely reducible, $R_j^2 \in \mathfrak{B}(\mathcal{K}_j)$ is irreducible for $j \in \Upsilon$ and $R_{i_1}^2 \ncong R_{i_2}^2$ whenever $j_1 \neq j_2$. Noting that T_0^2, R_0^2 are essentially square normal, it follows from Lemma (5.2.17) that R_0^2, R_0^2 are normal.

Denote

$$\Gamma_1 = \{i \in \Gamma: T_i^2 \text{ is not normal}\}, \qquad \Gamma_2 = \{i \in \Gamma: T_i^2 \text{ is not almost normal}\}.$$

Then $\Gamma_2 \subset \Gamma_1$ and

$$T^2_{abnor} = \oplus_{i\in\Gamma_1} T_i^{2(m_i)}, \qquad T_S^2 = (T^2_{abnor})_s = \oplus_{i\in\Gamma_2} T_i^{2(m_i)}.$$

Denote

$$\Upsilon_1 = \{j \in \Upsilon: R_j^2 \text{ is not normal}\}, \qquad \Upsilon_2 = \{j \in \Upsilon: R_j^2 \text{ is not almost normal}\}.$$

Then $\Upsilon_2 \subset \Upsilon_1$ and

$$R^2_{abnor} = \oplus_{j\in\Upsilon_1} R_j^{2(n_j)}, \qquad R_S^2 = (R^2_{abnor})_s = \oplus_{j\in\Upsilon_2} R_j^{2(n_j)}.$$

Set

$$A^2 = T_i^2 \oplus \left(\oplus_{i\in\Gamma} T_i^2\right), \qquad B^2 = R_0^2 \oplus \left(\oplus_{j\in\Upsilon} R_j^2\right).$$

From the proof of Lemma (5.2.46),, one can see that A^2, B^2 are both multiplicity-free, $T^2 \approx A^2$ and $R^2 \approx B^2$. Since $T^2 \approx R^2$, we obtain $A^2 \approx B^2$. By Theorem (5.2.49), we have $A^2 \cong_a B^2$. Note that A^2, B^2 are both essentially normal. In view of Lemma (5.2.52) , it follows that $A^2_{abnor} \cong B^2_{abnor}$. Hence $(A^2_{abnor})_s \cong (B^2_{abnor})_s$.

Note that

$$A^2_{abnor} = \oplus_{i\in\Gamma_1} T_i^2, \qquad A_S^2 = (A^2_{abnor})_s = \oplus_{i\in\Gamma_2} T_j^2,$$

and

$$B^2_{abnor} = \oplus_{j\in\Upsilon_1} R_j^2, \qquad A_S^2 = (B^2_{abnor})_s = \oplus_{j\in\Upsilon_2} R_j^2.$$

We obtain

$$\oplus_{i\in\Gamma_1} T_i^2 \cong \oplus_{j\in\Upsilon_1} R_j^2, \qquad \oplus_{i\in\Gamma_2} T_i^2 \cong \oplus_{j\in\Upsilon_2} R_j^2.$$

This implies that

$$\oplus_{i\in\Gamma_1} T_i^{2(m_i)} \approx \oplus_{j\in\Upsilon_1} R_j^{2(n_j)}, \qquad \oplus_{i\in\Gamma_2} T_i^{2(m_i)} \approx \oplus_{j\in\Upsilon_2} R_j^{2(n_j)}.$$

Thus we obtain $T^2_{abnor} \approx R^2_{abnor}$ and $T_s^2 \approx R_s^2$.

Corollary (5.2.88)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be multiplicity-free. Then each generator of $C^*(T^2)$ is multiplicity-free.

Proof. By Lemma (5.2.46), we may assume that

$$T_0^2 \oplus (\oplus_{i\in\Gamma} T_i^2),$$

where $T_0^2 \in \mathfrak{B}(\mathcal{H}_0)$ is completely reducible, $T_i^2 \in \mathfrak{B}(\mathcal{H}_i)$ is irreducible for $i \in \Gamma$ and $T_{i_1}^2 \ncong T_{i_2}^2$ whenever $i_1 \neq i_2$. Note that $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$.

Assume that $R^2 \in \mathfrak{B}(\mathcal{H})$ and $C^*(T^2) = C^*(R^2)$. Then R^2 can be written as $R^2 = R_0^2 \oplus \left(\oplus_{i\in\Gamma} R_i^2\right)$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus (\oplus_{i\in\Gamma} \mathcal{H}_i)$. By Lemma (5.2.24), R_0^2 is completely reducible and R_i^2 is irreducible for $i \in \Gamma$; moreover, $R_i^2 \ncong R_j^2$ for $i, j \in \Gamma$ with $i \neq j$. In view of the proof of Lemma (5.2.46), R^2 is multiplicity-free.

Corollary (5.2.89)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T^2 is multiplicity-free and g-normal, then $T^2 \in (cs)$.

Proof. Since T^2 is essentially normal and g-normal, by Lemma (5.2.55), we may assume that

$$T^2 = N^2 \oplus \left(\oplus_{i\in\Gamma}\ T_i^{2(l_i)}\right) \oplus \left(\oplus_{j\in\Upsilon} A_j^{2(m_j)} \oplus B_j^{2(n_j)}\right),$$

where $N^2 = T_{nor}^2$ is normal, $\{T_i^2,\ A_j^2,\ B_j^2: i \in \Gamma, j \in \Upsilon\}$ are abnormal, irreducible and no two of them are unitarily equivalent; moreover, each T_i^2 is UET and A_j^2 is a transpose of B_j^2 for $j \in \Upsilon$. So $\Lambda(A_j^2) = \Lambda((B_j^2)^t)$ for $j \in \Upsilon$. It follows that A_j^2 is almost normal if and only if $(B_j^2)^t$ (or, equivalently, B_j^2) is almost normal. On the other hand, since T^2 is multiplicity-free, we deduce that $l_i = m_j = n_j = 1$ for all $i \in \Gamma$ and all $j \in \Upsilon$.

Denote $\Upsilon_0 = \{j \in \Upsilon:\ A_j^2$ is not almost normal$\}$. Note that T_i^2 is almost normal for $i \in \Gamma$. It follows that

$$T_s^2 = \oplus_{j\in\Upsilon_0} \left(A_j^2 \oplus\ B_j^2\right).$$

By Proposition (5.2.23), T_s^2 is of type C. In view of Theorem (5.3.14), we have $T^2 \in (cs)$.

Corollary (5.2.90)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T_s^2 is not absent, then the following are equivalent:

(i) $T^2 \in (wcs)$;

(ii) $T_{abnor}^2 \in (wcs)$;

(iii) $T_s^2 \in (wcs)$;

(iv) T_s^2 is algebraically equivalent to an essentially square normal operator of type C.

Proof."(i)⟹(ii)". By Theorem (5.2.44), $T^2 \in (wcs)$ implies that there exists an essentially square normal operator $A^2 \in \mathfrak{B}(\mathcal{H})$ such that $A^2 \in (cs)$ and $T^2 \approx A^2$. By Corollary (5.2.53),, we have $T_{abnor}^2 \approx A_{abnor}^2$, and it follows from Corollary (5.2.39), that $A_{abnor}^2 \in (cs)$. Using Theorem (5.2.44), we obtain $T_{abnor}^2 \in (wcs)$.

"(ii)⟹(iii)". By Theorem (5.2.44),, $T_{abnor}^2 \in (wcs)$ implies that there exists an essentially square normal operator $A^2 \in \mathfrak{B}(\mathcal{H})$ such that $A^2 \in (cs)$ and $T_{abnor}^2 \approx A^2$. By Corollary (5.2.85), we have $T_s^2 = (T_{abnor}^2)_s \approx A_s^2$, and it follows from Corollary (5.2.44), that $A_s^2 \in (cs)$. Using Theorem (5.2.44),, we obtain $T_s^2 \in (wcs)$.

"(iii)⟹(iv)". By Theorem (5.2.44),, $T_s^2 \in (wcs)$ implies that there exists an essentially square normal operator $A^2 \in \mathfrak{B}(\mathcal{H})$ such that $A^2 \in (cs)$ and $T_s^2 \approx A^2$. Then, by

Corollary (5.2.53),, $T_s^2 = (T_s^2)_s \approx A_s^2$. By Theorem (5.2.14),, $A^2 \in (cs)$ implies that A_s^2 is of type C. This proves the implication "(iii) ⟹(iv)".
"(iv) ⟹(i)". Assume that $A^2 \in \mathfrak{B}(\mathcal{H})$ is an essentially square normal operator of type C and $T_s^2 \approx A^2$. Denote by B^2 the restriction of T^2 to $\mathcal{H} \ominus \mathcal{H}_s$. Then the restriction of B^2 to its each nonzero reducing subspace is almost normal. It follows that $T^2 = T_s^2 \oplus B^2 \approx A^2 \oplus B^2$. Noting that $(A^2 \oplus B^2)_s = A_s^2 = A^2$ is of type C, by Theorem (5.2.14),, we have $A^2 \oplus B^2 \in (cs)$. By Theorem (5.2.44),, we conclude that $T^2 \in (wcs)$.

Corollary (5.2.91)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. Then $T^2 \in (wcs)$ if and only if T_s^2 is either absent or unitarily equivalent to an essentially square normal operator of the form

$$\bigoplus_{1 \le i < v} \left(A_i^{2(m_i)} \oplus B_i^{2(n_i)} \right), \qquad 1 \le m_i,\, n_i < \infty,$$

where $\{ A_i^2, B_i^2 : 1 \le i < v\}$ are essentially square normal operators satisfying the conditions (i), (ii) and (iii) in Proposition (5.2.23),.

Proof. Obviously, we need only consider the case that T_s^2 is not absent. By Lemma (5.2.46), and Proposition (5.2.23),, each essentially square normal operator of type C is algebraically equivalent to a multiplicity-free operator of the form

$$R^2 = \oplus_{1 \le i < v} \left(A_i^2 \oplus B_i^2 \right), \tag{22}$$

where $\{ A_i^2, B_i^2 : 1 \le i < v\}$ satisfy the conditions (i), (ii) and (iii) in Proposition (5.2.23),. Then, by Corollary (5.2.57),, an essentially square normal operator T^2 satisfies $T^2 \in (wcs)$ if and only if T_s^2 is algebraically equivalent to an operator R^2 of the form (22). Noting that both T_s^2 and R^2 are abnormal, in view of Lemma (5.2.58),, the latter is equivalent to

$$T_s^{2(\infty)} \cong \oplus_{1 \le i < v} \left((A_i^2)^{(\infty)} \oplus (B_i^2)^{(\infty)} \right). \tag{23}$$

By Lemmas (5.2.19), and (5.2.20),, the condition (23) holds if and only if there exist m_i, n_i, $1 \le i < v$, such that

$$T_s^2 \cong \bigoplus_{1 \le i < v} \left((A_i^2)^{(m_i)} \oplus (B_i^2)^{(m_i)} \right).$$

For each i, note that both $(A_i^*)^2 A_i^2 - A_i^2 (A_i^*)^2$ and $(B_i^*)^2 B_i^2 - B_i^2 (B_i^*)^2$ are nonzero compact operators. Since T_s^2 is essentially normal, if such m_i, n_i exist, then it is necessary that $m_i,\ n_i < \infty$ for each i.

Corollary (5.2.92)[462]: Let $T^2 \in \mathfrak{B}(\mathcal{H})$ be essentially normal. If T^2 is irreducible, then the following are equivalent:

(i) $T^2 \in (cs)$;

(ii) $T^2 \in (wcs)$;

(iii) T^2 is almost normal.

Proof. The implication "(i) ⟹(ii)" is trivial, and the equivalence "(i)⇔(iii)" follows from Theorem (5.2.8).
"(ii) ⟹(iii)". If T^2 is not almost normal, then $T^2 = T_s^2$ and T_s^2 is not absent. By Corollary (5.2.59),, T_s^2 is reducible, a contradiction. This ends the proof.

Chapter 6
C^*-Algebras with Dixmier Approximation and Property

We show that the concept is linked to coarse geometry, since for a discrete metric space of bounded geometry the nuclear dimension of the associated uniform Roe algebra is dominated by the asymptotic dimension of the underlying space. We consider Dixmier type approximation theorem and characterize symmetric amenability for C^*-algebras. We consider continuous bundles of tracial von Neumann algebras and classify some of them. We give further examples of C^*-algebras with the uniform Dixmier property, namely all C^*-algebras with the Dixmier property and finite radius of comparison-by-traces. We determine the distance between two Dixmier sets, in an arbitrary unital C^*-algebra, by a formula involving tracial data and algebraic numerical ranges.

Section (6.1): The Nuclear Dimension

Recent developments in noncommutative topology suggest that dimension type conditions play a crucial role for the understanding of noncommutative spaces and their applications, cf. [271], [300], [303], [260] and [270]. While in the commutative case the various definitions of covering dimension tend to coincide (at least for sufficiently well-behaved spaces), their generalizations to the noncommutative situation yield vastly different notions, such as stable rank, real rank, or decomposition rank (cf. [285], [268], [280]), each of which has turned out to be highly useful and interesting in its own right. The known applications, e.g. to the classification of nuclear C^*-algebras, are all limited to somewhat special situations – although, it should be possible to handle many of these in a unified manner. There are also notions which have not yet been generalized to the noncommutative setting, such as Gromov's asymptotic dimension (and the latter should clearly be accessible from a noncommutative point of view, as it has already been shown to be closely related to the coarse Baum–Connes conjecture).

We will propose a notion of noncommutative covering dimension which on the one hand is flexible enough to cover large classes of (nuclear) C^*-algebras, and which on the other hand is intimately related to many other regularity properties of noncommutative spaces. The concept is linked to the classification program for nuclear C^*-algebras, as well as to the theory of dynamical systems and to coarse geometry. We hope that it will contribute to a deeper understanding of the interplay between these fields, but also shed new light on the role of dimension type conditions in other areas of noncommutative geometry.

The nuclear dimension is seemingly only a small variation of the decomposition rank, a notion introduced by Kirchberg and [280] (this in turn was based on earlier concepts introduced in [294] and [295]). The decomposition rank models the dimension type condition in terms of a decomposition property of noncommutative partitions of unity. Nuclear dimension is defined in a similar manner, only now we add a little more flexibility to the partitions of unity under consideration. The outcome is a notion of integer valued covering dimension for nuclear C^*-algebras, which still coincides with covering dimension of the spectrum in the commutative case, and which still has nice permanence properties. But now, the added flexibility in the choice of the partitions of unity makes the theory accessible to much larger classes of C^*-algebras.

The decompostion rank has turned out to be extremely useful for the classification of stably finite, separable, simple, nuclear C^*-algebras. In fact, all classes of such C^*-algebras which by now have been classified consist of ones with finite decomposition rank – and it seems well possible that separable simple C^*-algebras with finite decomposition rank are entirely classifiable by their K-theory data. An important step in this direction was achieved in [298], where it was shown that, for separable, simple, unital C^*-algebras, finite decomposition rank implies Z-stability, i.e., all such C^*-algebras absorb the Jiang–Su algebra Z tensorially. (The Jiang– Su algebra was introduced in [275]; see [289] for alternative characterizations.) The decomposition rank can take finite values only for quasidiagonal C^*-algebras, so its use beyond the stably finite case of the classification program will be limited. On the other hand, Kirchberg and Phillips have very successfully classified purely infinite simple C^*-algebras. Although in their initial approach, topological dimension type conditions do not show up explicitly, these nontheless have turned out to be important both in the simple and the nonsimple case, cf. [278], [266]. We will show that the C^*-algebras covered by Kirchberg–Phillips classification all have finite nuclear dimension, so that our theory covers large parts of the classification program, both in the stably finite and in the purely infinite case. We make progress on a unified approach to the classification problem for nuclear C^*-algebras, i.e., an approach that does not require genuinely different methods in the finite and the infinite case.

We have already mentioned that, in the simple and unital case, finite decomposition rank implies Z-stability. Using results of Kirchberg, we will be able to derive an infinite version of this statement, namely, that a separable simple C^*-algebra with finite nuclear dimension and no nontrivial trace is purely infinite, hence absorbs the Cuntz algebra $\mathcal{O}_\infty$. We do not know whether simplicity and finite nuclear dimension will imply Z-stability in general; however, there are promising results pointing in this direction, see [284] (where the corona factorization property is confirmed for simple, unital C^*-algebras with finite nuclear dimension) ; cf. also Conjecture (6.1.38), below.

A natural touchstone for any kind of invariant for C^*-algebras will be its behavior with respect to standard constructions, such as direct sums, limits, tensor products, quotients, ideals, or hereditary subalgebras. Decomposition rank and nuclear dimension behave equally well in this respect. There is, however, one exception: since finite decomposition rank implies finiteness, the Toeplitz extension shows that finite decomposition rank does not pass from quotients and ideals to extensions in general-a problem circumvented by the additional flexibility of nuclear dimension.

The situation for crossed products is more subtle. At this point, we only have partial results about the topological dimension of crossed products; for example, it is known that the transformation group C^*-algebra of a minimal diffeomorphism on a compact smooth manifold has finite decomposition rank – and the proof is extremely technical, cf. [282] and [296]. In [293], show that the transformation group C^*-algebra of a minimal homeomorphism on an infinite, compact, finite dimensional, metrizable space has finite nuclear dimension and this time, the proof is much simpler and more conceptual. (Even more, the methods introduced are an important step towards completing the classification of C^*-algebras associated to uniquely ergodic, minimal homeomorphisms on infinite, compact, finite dimensional, metrizable spaces, as achieved in [293]; see also [291].)

Another natural situation to consider is when a group satisfies certain geometric dimension type conditions. Here, we face a genuine problem since the (full or reduced) group C^*-algebra will in general not be nuclear. However, one might as well look at the so-called uniform Roe algebra; it then turns out that if a discrete group (with word length metric) has finite asymptotic dimension in the sense of Gromov, then its uniform Roe algebra has finite nuclear dimension. This statement can be generalized to discrete metric spaces of bounded geometry. At this point it is an open question how much information about the underlying space the Roe algebra actually contains. It will be interesting to approach this question, i.e., analyze what finite nuclear dimension of the Roe algebra means for the underlying space. The problem is particularly relevant since Yu (in [303]) has shown that a group with finite asymptotic dimension satisfies the coarse Baum–Connes conjecture. By now, we know that the latter also holds in more general situations, so one might ask whether finite nuclear dimension of the Roe algebra is a strong enough regularity property to ensure the coarse Baum–Connes conjecture of the underlying group.

We recall some facts about order zero maps and completely positive approximations of nuclear C^*-algebras. We introduce our nuclear dimension, compare it to the decomposition rank and derive its permanence properties with respect to inductive limits, quotients, ideals, extensions and hereditary subalgebras.The special structure of completely positive approximations realizing nuclear dimension; namely, we show that the outgoing maps can always be chosen to be approximately order zero. We compare nuclear dimension to Kirchberg's covering number. These observations together with a result of Kirchberg are to obtain a dichotomy result on sufficiently noncommutative C^*-algebras with finite nuclear dimension: they either have a nontrivial trace or are purely infinite. We collect a number of examples both with finite and with infinite nuclear dimension. We show that Kirchberg algebras satisfying the Universal Coefficient Theorem have finite nuclear dimension, and that, for a discrete countable metric space of bounded geometry, the nuclear dimension of the associated uniform Roe algebra is dominated by the asymptotic dimension of the space. We close with a number of open problems and possible future developments.

We recall some facts about order zero maps. These are $c.p.$ maps preserving orthogonality; they are particularly well-behaved, and will serve as building blocks of our noncommutative partitions of unity, similar as in [294], [295] and [280].

Definition (6.1.1)[453]: Let A and B be C^*-algebras, and $\varphi: A \to B$ a $c.p.$ map. We say φ has order zero, if, for $a, b \in A_+$,

$$a \perp b \Rightarrow \varphi(a) \perp \varphi(b).$$

The following structure theorem for order zero maps was derived in [301] (based on results from [302], and generalizing [297, 1.2], which only covers the case of finite-dimensional domains).

Theorem (6.1.2)[453]: Let A and B be C^*-algebras and $\varphi: A \to B\ a\ c.p.$ order zero map. Let $C := C^*(\varphi(A)) \subset B$, then there is a positive element $h \in M(C) \cap C'$ with $\|h\| = \|\varphi\|$ and a $*$-homomorphism

$$\pi_\varphi: A \to M(C) \cap \{h\}' \subset B^{**}$$

such that

$$\pi_\varphi(a)h = \varphi(a)\ for\ a \in A.$$

If A is unital, then $h = \varphi(1_A) \in C$.

We call π_φ the canonical supporting $*$-homomorphism of φ.
We shall have use for the following easy consequence of Theorem (6.1.2), cf. [301].
Corollary (6.1.3)[453]: Let A, B be C^*-algebras and $\psi: A \to B$ $a\ c.p.c.$ order zero map. If τ is a positive tracial functional on B, then $\tau \circ \psi$ is a positive tracial functional on A.
By [295, 1.2.3], order zero maps with finite-dimensional domains can be described in terms of generators and relations which are weakly stable in the sense of [283]. The following is a straightforward reformulation of [280, Proposition 2.5].
Proposition (6.1.4)[453]: Let F be a finite dimensional C^*-algebra. For any $\eta > 0$ there is $\delta > 0$ such that the following holds: If A is a C^*-algebra and $\phi: F \to A$ $a\ c.p.c.$ order zero map, and if $d \in A^+$ is a positive contraction in the unitization of A satisfying $\|[d, \varphi(x)]\| \leq \delta\|x\|$ for all $x \in F$, then there is $a\ c.p.c.$ order zero map $\hat{\varphi}: F \to A$ such that $\left\|\hat{\varphi}(x) - d^{\frac{1}{2}}\varphi(x)d^{\frac{1}{2}}\right\| \leq \eta\|x\|$ for all $x \in F$.
Below we define our notion of noncommutative dimension, compare it to other concepts such as topological covering dimension or decomposition rank, and derive its most important permanence properties.
Definition (6.1.5)[453]: A C^*-algebra A has nuclear dimension at most n, if there exists a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ such that the F_λ are finite-dimensional C^*-algebras, and such that $\psi_\lambda: A \to F_\lambda$ and $\varphi_\lambda: F_\lambda \to A$ are completely positive maps satisfying

(i) $\psi_\lambda \circ \varphi_\lambda(a) \to a$ uniformly on finite subsets of A;

(ii) $\|\psi_\lambda\| \leq 1$;

(iii) for each λ, F_λ decomposes into $n + 1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(n)}$ such that $\varphi_\lambda|_{F_\lambda^{(i)}}$ is $a\ c.p.c.$ order zero map for $i = 0, 1, \ldots, n$.

We write $\dim_{nuc} A \leq n$ in this case and refer to the maps φ_λ as piecewise contractive n-decomposable $c.p.$ maps, and to the triples $(F_\lambda, \psi_\lambda, \varphi_\lambda)$ as piecewise contractive n-decomposable $c.p.$ approximations.
The following permanence properties are derived just as for the completely positive rank or for the decomposition rank, cf. [294] and [280]. Note that there is no need to specify the tensor product in (6.1.6)(ii), since the values can be finite only for nuclear C^*-algebras.
Proposition (6.1.6)[453]: Let A, B, C, D and E be C^*-algebras; suppose $C = \lim_\to C_i$ is an inductive limit of C^*-algebras and D is a quotient of E. Then,

(i) $\dim_{nuc}(A \oplus B) = \max(\dim_{nuc} A, \dim_{nuc} B)$

(ii) $\dim_{nuc}(A \otimes B) \leq (\dim_{nuc} A + 1)(\dim_{nuc} B + 1) - 1$; if B is an AF algebra, then $\dim_{nuc}(A \otimes B) \leq \dim_{nuc} A$

(iii) $\dim_{nuc} C \leq \lim\inf(\dim_{nuc} C_i)$

(iv) $\dim_{nuc} D \leq \dim_{nuc} E$.

Just as the decomposition rank, nuclear dimension agrees with covering dimension of the spectrum in the separable commutative case. In the nonseparable case, nuclear dimension and decomposition rank still coincide, and they agree with the respective definition of covering dimension. The only reason why we distinguish between the separable and the nonseparable case is that the various characterizations of dimension tend to disagree for spaces which are not second countable.
Proposition (6.1.7)[453]: Let X be a locally compact Hausdorff space. Then,
$$\dim_{nuc} C_c(X) = drC_0(X).$$

In particular, if X is second countable, we have

$$\dim_{unc} C_0(X) = drC_0(X) = \dim X.$$

Proof. We have $\dim_{nuc} C_0(X) \le drC_0(X)$. For the reverse estimate, let us assume that $\dim_{nuc} C_0(X) = n < \infty$. Suppose $\mathcal{F} \subset C_0(X)$ is a finite subset of positive normalized elements, and that $\varepsilon > 0$ is given. We may assume that the elements of $\mathcal{F}$ have compact support and that there is a positive normalized function $h \in C_0(X)$ such that $ha = a$ for all $a \in \mathcal{F}$.

Choose a piecewise contractive n-decomposable $c.p.$ approximation $(F = F^{(0)} \oplus \ldots \oplus F^{(n)}, \psi, \varphi)$ for $\mathcal{F} \cup \{h\}$ within $\varepsilon/2$. Since φ has order zero on each matrix block of F, we see from [294, Remark 2.16 (ii)] that F is commutative. By cutting down F to the hereditary subalgebra generated by $\psi(h)$, we may assume that $\psi(h)$ is invertible in F. Define $c.p.$ maps

$$\hat{\psi} : C_0(X) \to F \; and \; \hat{\varphi}: F \to C_0(X)$$

by

$$\hat{\psi}(f) := \psi(h)^{-\frac{1}{2}} \psi(hf) \psi(h)^{-\frac{1}{2}} \, for \; f \in C_0(X)$$

and

$$\hat{\varphi}(x) := \left(1 - \frac{\varepsilon}{2}\right) \cdot \varphi(\psi(h)^{\frac{1}{2}} \, x \psi(h)^{\frac{1}{2}}) \quad for \; x \in F.$$

It is clear that $\hat{\psi}$ is contractive, and that $\hat{\varphi}$ is n-decomposable with respect to $F = F^{(0)} \oplus \ldots \oplus F^{(n)}$. Moreover,

$$\begin{aligned} \hat{\varphi}(1_F) &= \hat{\varphi} \, \hat{\psi}(h) \\ &= \left(1 - \frac{\varepsilon}{2}\right) \cdot \phi\psi(h) \\ &\le \left(1 - \frac{\varepsilon}{2}\right)\left(1 + \frac{\varepsilon}{2}\right) \cdot h \\ &\le 1, \end{aligned}$$

whence $\hat{\varphi}$ is contractive. Finally, we have

$$\begin{aligned} \left\|\hat{\varphi}\hat{\psi}(f) - f\right\| &\le \left\|\hat{\varphi} \, \hat{\psi}(f) - \varphi\psi(f)\right\| + \|\phi\psi(f) - f\| \\ &< \left\|\left(1 - \frac{\varepsilon}{2}\right) \cdot \varphi\psi(hf) - \varphi\psi(f)\right\| + \frac{\varepsilon}{2} \\ &\le \varepsilon \end{aligned}$$

for $f \in \mathcal{F}$, so $(F, \hat{\psi}, \hat{\varphi})$ is an n-decomposable $c.p.c.$ approximation for $\mathcal{F}$ within ε. Therefore, $drC_0(X) \le n$.

The statement about the second countable case is [280, Proposition 3.3].

We do not wish to impose any separability restrictions on our definition of nuclear dimension. However, in many situations one can non the less restrict to the separable case, using the following observation. We next show that, just like for decomposition rank, finite nuclear dimension passes to hereditary subalgebras. Combined with Brown's Theorem, this result shows that nuclear dimension is a stable invariant, cf. Corollary (6.1.10) below.

Proposition(6.1.8)[453]: $\dim_{nuc} B \le \dim_{nuc} A$ when $B \subseteq A$ is a hereditary C^*-subalgebra.

Proof. We may assume $n := \dim_{nuc} A$ to be finite, for otherwise there is nothing to show. Let $b_1, \ldots, b_m \in B_+$ be normalized elements and let $\varepsilon > 0$ be given. We have to find a piecewise contractive n-decomposable $c.p.$ approximation (of B) for $\{b_1, \ldots, b_m\}$ within ε.

Using an idempotent approximate unit, by slightly perturbing the b_j we may (as in [280, Remark 3.2(ii)]) assume that there are positive normalized elements $h_0, h_1 \in B_+$ such that

$$h_0 h_1 = h_1 \quad and \quad h_1 b_j = b_j$$

for $j = 1, \dots, m$.
Set

$$\eta := \min\left\{\frac{\varepsilon^8}{13(n+1)}, \frac{1}{2^{16}}\right\}$$

and choose a piecewise contractive n-decomposable $c.p.$ approximation

$$(F = F^{(0)} \oplus \dots F^{(n)}, \psi, \varphi)$$

(of A) for $\{h_0, h_1, b_1, \dots, b_m\}$ within η.
Define a projection $p \in F$ by

$$p := g_{\eta^{\frac{1}{2}}}(\psi(h_1)),$$

where $g_{\eta^{\frac{1}{2}}}$ is given by

$$g_{\eta^{\frac{1}{2}}}(t) := \begin{cases} 0 \ for \ t < \eta^{\frac{1}{2}}, \\ 1 \ for \ t \ \geq \ \eta^{\frac{1}{2}}. \end{cases}$$

Set

$$\hat{F} := pFp, \hat{F}^{(i)} := pF^{(i)}p \ and \ p^{(i)} := 1_{F^{(i)}}p$$

for $i \in \{0, \dots, n\}$ and define $a\ c.p.c.$ map

$$\hat{\psi}: B \to \hat{F}$$

by

$$\hat{\psi}(b) := p\psi(b)p, b \in B.$$

For $i \in \{0, \dots, n\}$ we have

$$\begin{aligned} \left\|\varphi^{(i)}(p^{(i)})(1-h_0)\right\| &= \left\|(1-h_0)\varphi^{(i)}(p^{(i)})^2(1-h_0)\right\|^{\frac{1}{2}} \\ &\leq \left\|(1-h_0)\varphi^{(i)}(p^{(i)})(1-h_0)\right\|^{\frac{1}{2}} \\ &\leq \|(1-h_0)\varphi(p)(1-h_0)\|^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\eta^{\frac{1}{2}}}\|(1-h_0)\varphi\psi(h_1)(1-h_0)\|\right)^{\frac{1}{2}} \\ &\leq \left(\frac{\eta}{\eta^{\frac{1}{2}}}\right)^{\frac{1}{2}} \\ &= \eta^{\frac{1}{4}}\left(\leq \frac{1}{16}\right). \end{aligned}$$

Now by [280, Lemma 3.6] (applied to $\varphi^{(i)}|_{\hat{F}^{(i)}}$ in place of φ and h_0 in place of h) there are $c.p.c.$ order zero maps

$$\hat{\varphi}^{(i)}: \hat{F}^{(i)} \to \overline{h_0 A h_0} \subset B$$

such that

$$\left\|\hat{\varphi}^{(i)}(x)-\varphi^{(i)}(x)\right\| \leq 8\eta^{\frac{1}{8}}\|x\|$$
for all $0 \leq x \in \hat{F}^{(i)}$ and $i \in \{0,\ldots,n\}$. Set
$$\hat{\varphi} := \sum_{i=0}^{n} \hat{\varphi}^{(i)} : \hat{F} \to B.$$
The map $\hat{\varphi}$ is a sum of $n+1$ $c.p.c.$ order zero maps by construction, and we have
$$\left\|\hat{\varphi}\,\hat{\psi}(b_j)-\varphi\hat{\psi}(b_j)\right\| \leq 8(n+1)\eta^{\frac{1}{8}}, j=1,\ldots,m. \quad (1)$$
To check that $\hat{\varphi}\hat{\psi}(b_j)$ is close to b_j, note first that
$$\begin{aligned}\left\|\varphi((1_F-p)\psi(b_j))\right\| &\leq \left\|\varphi((1_F-p)\psi(b_j))\varphi(\psi(b_j)(1_F-p))\right\|^{\frac{1}{2}}\\ &\leq \left\|\varphi((1_F-p)\psi(b_j)^2(1_F-p))\right\|^{\frac{1}{2}}\\ &\leq \left\|\varphi\big((1_F-p)\psi(h_1)(1_F-p)\big)\right\|^{\frac{1}{2}}\\ &\leq \left((n+1)\eta^{\frac{1}{2}}\right)^{\frac{1}{2}}\\ &\leq (n+1)\eta^{\frac{1}{4}}\end{aligned}$$
for $j=1,\ldots,m$, which in particular implies that
$$\left\|\varphi([p,\psi(b_j)])\right\| \leq 2(n+1)\eta^{\frac{1}{4}}.$$
We now obtain
$$\begin{aligned}\left\|\varphi\psi(b_j)-\varphi\hat{\psi}(b_j)\right\| &\leq \left\|\varphi\big(\psi(b_j)-p\psi(b_j)+\psi(b_j)p-p\psi(b_j)p\big)\right\|+2(n+1)\eta^{\frac{1}{4}}\\ &\leq 4(n+1)\eta^{\frac{1}{4}} \quad (2)\end{aligned}$$
for $j=1,\ldots,m$, whence
$$\begin{aligned}\left\|\hat{\varphi}\hat{\psi}(b_j)-b_j\right\| &\leq \left\|\hat{\varphi}\hat{\psi}(b_j)-\varphi\,\hat{\psi}(b_j)\right\|+\left\|\varphi\hat{\psi}(b_j)-\varphi\psi(b_j)\right\|+\left\|\varphi\psi(b_j)-b_j\right\|\\ &\overset{(1),(2)}{<} 8(n+1)\eta^{\frac{1}{8}}+4(n+1)\eta^{\frac{1}{4}}+\eta\\ &< \varepsilon.\end{aligned}$$
Therefore, the approximation $(\hat{F},\hat{\psi},\hat{\varphi})$ is as desired.

Proposition (6.1.9)[453]: Let B be a C^*-algebra. For any countable subset $S \subset B$ there is a separable C^*-subalgebra $C \subset B$ such that $S \subset C$ and $\dim_{nuc} C \leq \dim_{nuc} B$.

Proof. Let $\dim_{nuc} B = n < \infty$. Set $S_0 := S$ and choose
$$(F_{0,\lambda},\psi_{0,\lambda},\varphi_{0,\lambda})_{\lambda\in\mathbb{N}},$$
a system of piecewise contractive n-decomposable $c.p.$ approximations (of B) for S_0. If $S_k \subset B$ and
$$(F_{k,\lambda},\psi_{k,\lambda},\varphi_{k,\lambda})_{\lambda\in\mathbb{N}}$$
have been constructed, choose a countable dense subset
$$S_{k+1} \subset C^*\left(\bigcup_{l\leq k,\lambda\in\mathbb{N}} \varphi_{l,\lambda}(F_{l,\lambda}) \cup S_k\right) \subset B$$
and choose
$$(F_{k+1,\lambda},\psi_{k+1,\lambda},\varphi_{k+1,\lambda})_{\lambda\in\mathbb{N}},$$

a system of piecewise contractive, n-decomposable $c.p.$ approximations (of B) for S_{k+1}. Continue inductively and define

$$C := \overline{\bigcup_{k\in\mathbb{N}} S_k}\,;$$

it is straightforward to check that C has the right properties, and that a system of piecewise contractive, n-decomposable $c.p.$ approximations of C is given by

$$(F_{k,\lambda}, \psi_{k,\lambda}, \varphi_{k,\lambda})_{k,\lambda\in\mathbb{N}}.$$

Corollary (6.1.10)[453]: Let A be a C^*-algebra.

(i) For any $r \in \mathbb{N}$ we have $\dim_{nuc} A = \dim_{nuc}(M_r \otimes A) = \dim_{nuc}(K \otimes A)$.

(ii) If $B \subset A$ is a full hereditary C^*-subalgebra, then $\dim_{nuc} B = \dim_{nuc} A$.

Proof. (i) We have $\dim_{nuc} A \leq \dim_{nuc}(M_r \otimes A) \leq \dim_{nuc}(K \otimes A)$ by Proposition (6.1.8) and $\dim_{nuc}(K \otimes A) \leq \dim_{nuc} A$ by Proposition (6.1.6).

(ii) We have $n := \dim_{nuc} B \leq \dim_{nuc} A$ by Proposition (6.1.8),, so it remains to show that $\dim_{nuc} A \leq \dim_{nuc} B$.

Given $a_1, \ldots, a_m \in A_+$, by there is a separable C^*-subalgebra $D \subset A$ such that $\{a_1, \ldots, a_m\} \subset D$, such that $C := D \cap B$ is full in D and such that $\dim_{nuc} C \leq \dim_{nuc} B$.

Now by Brown's Theorem [267, Theorem 2.8], we have $K \otimes C \cong K \otimes D$, hence $\dim_{nuc} D = \dim_{nuc} C (\leq \dim_{nuc} B)$ by part (i) of the corollary. We may thus find arbitrarily close piecewise contractive n-decomposable $c.p.$ approximations for $a_1, \ldots, a_m$.

We are now ready to describe the first significant difference between decomposition rank and nuclear dimension. We already know that both theories behave well with respect to quotients and ideals; it has been observed in [280] that finite decomposition rank passes to quasidiagonal extensions, and that one cannot expect a general statement. The additional flexibility in the definition of nuclear dimension, however, ensures that finite nuclear dimension indeed passes to arbitrary extensions. So we obtain a noncommutative version of the sum theorem for covering dimension, cf. [274, III.2.B)]. This behavior will also make large new classes of C^*-algebras accessible to our theory, cf.

Proposition (6.1.11)[453]: Let $0 \to J \to E \to A \to 0$ be an exact sequence of C^*-algebras. Then,

$$\max\{\dim_{nuc} A, \dim_{nuc} J\} \leq \dim_{nuc} E \leq \dim_{nuc} A + \dim_{nuc} J + 1.$$

Proof. The first inequality follows.

For the second inequality, we may assume that both $m := \dim_{nuc} J$ and $n := \dim_{nuc} A$ are finite, for otherwise there is nothing to show. Let positive and normalized elements $e_1, \ldots, e_k \in E$ and $\varepsilon > 0$ be given.

Choose a piecewise contractive n-decomposable $c.p.$ approximation

$$(F_A = F_A^{(0)} \oplus \ldots \oplus F_A^{(n)}, \psi_A, \varphi_A)$$

(of A) for $\{\pi(e_1), \ldots, \pi(e_k)\}$ within $\frac{\varepsilon}{5}$. By [295, Proposition 1.2.4] (essentially using that cones over finite-dimensional C^*-algebras are projective), each $\varphi_A^{(j)}$ lifts to a $c.p.c.$ order zero map

$$\bar{\varphi}_A^{(j)} : F_A^{(j)} \to E,$$

so that

$$\bar{\varphi}_A := \sum_{j=0}^{n} \bar{\varphi}_A^{(j)}$$

will be a piecewise contractive n-decomposable $c.p.$ lift of φ_A.

From [295, 1.2.3], we know that the relations defining order zero maps are weakly stable; this in particular implies that there is $\delta > 0$ such that the assertion of Proposition (6.1.4) holds for each $F_A^{(j)}$ in place of F and $\frac{\varepsilon}{5(n+1)}$ in place of η.

Using a quasicentral approximate unit for J relative to E, it is straightforward to find a positive normalized element $h \in J$ such that the following hold:

(a) $\left\| [(1-h), \bar{\varphi}_A^{(j)}(x)] \right\| \leq \delta \|x\|$ for $x \in F_A^{(j)}, j = 0, \dots, n$

(b) $\left\| h^{\frac{1}{2}} e_l h^{\frac{1}{2}} + (1-h)^{\frac{1}{2}} e_l (1-h)^{\frac{1}{2}} - e_l \right\| < \frac{\varepsilon}{5}$ for $l = 1, \dots, k$

(c) $\left\| (1-h)^{\frac{1}{2}} (\bar{\varphi}_A \psi_A \pi(e_l) - e_l)(1-h)^{\frac{1}{2}} \right\| < \frac{2\varepsilon}{5}$ for $l = 1, \dots, k$.

(To obtain (c), we use that

$$\|\pi(\bar{\varphi}_A \psi_A \pi(e_l) - e_l)\| = \|\varphi_A \psi_A \pi(e_l) - e_l\| < \frac{\varepsilon}{5},$$

whence $\bar{\varphi}_A \psi_A \pi(e_l) - e_l$ is at most $\frac{\varepsilon}{5}$ away from J.)

Now by (a) and Proposition (6.1.4) there are $c.p.c.$ order zero maps

$$\hat{\varphi}_A^{(j)} : F_A^{(j)} \to E$$

such that

$$\left\| \hat{\varphi}_A^{(j)}(x) - (1-h)^{\frac{1}{2}} \bar{\varphi}_A^{(j)}(x)(1-h)^{\frac{1}{2}} \right\| \leq \frac{\varepsilon}{5(n+1)} \|x\|$$

for $x \in F_A^{(j)}, j = 0, \dots, n$; set

$$\hat{\varphi}_A := \sum_{j=0}^{n} \hat{\varphi}_A^{(j)},$$

then

$$\left\| \hat{\varphi}_A(x) - (1-h)^{\frac{1}{2}} \bar{\varphi}_A(x)(1-h)^{\frac{1}{2}} \right\| \leq \frac{\varepsilon}{5} \|x\| \; for \; x \in F_A.$$

Next, choose a piecewise contractive m-decomposable $c.p.$ approximation

$$(F_J = F_J^{(0)} \oplus \dots \oplus F_J^{(m)}, \psi_J, \varphi_J)$$

(of J) for $\{h^{\frac{1}{2}} e_l h^{\frac{1}{2}} | l = 1, \dots, k\}$ within $\frac{\varepsilon}{5}$.

Set

$$F := F_J \oplus F_A, \psi(\,.\,) := \psi_J(h^{\frac{1}{2}} . h^{\frac{1}{2}}) \oplus \psi_A \pi(\,.\,) \; and \; \varphi := \varphi_J + \hat{\varphi}_A,$$

then ψ is $c.p.c.$ and φ is piecewise contractive $c.p.$; φ is $(m+n+1)$-decomposable with respect to $F = \bigoplus_{j=0}^{m+n+1} F^{(j)}$, where

$$F^{(j)} := \begin{cases} F_J^{(j)} & for \; j = 0, \dots, m \\ F_A^{(j-m-1)} & for \; j = m+1, \dots, m+n+1. \end{cases}$$

It remains to be checked that (F, ψ, φ) indeed approximates the e_l within ε, i.e.,

$$\begin{aligned}\|\varphi\psi(e_l)-e_l\| &\overset{(b)}{<} \left\|\varphi_J\psi_J(h^{\frac{1}{2}}\,e_l h^{\frac{1}{2}})-h^{\frac{1}{2}}\,e_l h^{\frac{1}{2}}\right\| \\ &\quad+\left\|\hat{\varphi}_A\psi_A\pi(e_l)-(1-h)^{\frac{1}{2}}\,e_l(1-h)^{\frac{1}{2}}\right\|+\frac{\varepsilon}{5} \\ &\le \frac{\varepsilon}{5}+\left\|(1-h)^{\frac{1}{2}}(\bar{\varphi}_A\psi_A\pi(e_l)-e_l)(1-h)^{\frac{1}{2}}\right\|+\frac{\varepsilon}{5}+\frac{\varepsilon}{5} \\ &\overset{(c)}{<} \varepsilon.\end{aligned}$$

Corollary (6.1.12)[453]: Let A be a separable continuous trace C^*-algebra. Then,

$$\dim_{nuc} A = drA = \dim \hat{A}.$$

Proof. The proof follows that of [280, Corollary 3.10] almost verbatim.

However, following the lines of [280, Proposition 3.11], one can even show that the nuclear dimension of a C^*-algebra agrees with that of its smallest unitization. In the separable commutative case, the respective statement also holds for the maximal compactification. One cannot quite expect a noncommutative generalization of the latter result to our context, since multiplier algebras in general are not nuclear.

In [280] it was shown that C^*-algebras with finite decomposition rank are quasidiagonal. The reason was that the n-decomposable $c.p.c.$ approximations may always be chosen so that the maps $\psi_\lambda: A \to F_\lambda$ are almost multiplicative, cf. [280, Proposition 5.1]. We prove an analogous result for nuclear dimension and piecewise contractive n-decomposable $c.p.$ approximations, saying that the latter may always be chosen to be almost orthogonality preserving. We first need a simple technical observation.

Proposition (6.1.13)[453]: Let A be a C^*-algebra, and let $0 \le a \le b$ and $0 \le a' \le b'$ be positive elements of norm at most one. Then, $\|aa'\|^2 \le \|bb'\|$.

Proof. We simply estimate

$$\begin{aligned}\|bb'\| &\ge \left\|b^{\frac{1}{2}}b'bb'b^{\frac{1}{2}}\right\| \\ &= \left\|b^{\frac{1}{2}}b'b^{\frac{1}{2}}\right\|^2 \\ &\ge \left\|b^{\frac{1}{2}}a'b^{\frac{1}{2}}\right\|^2 \\ &= \left\|(a')^{\frac{1}{2}}b(a')^{\frac{1}{2}}\right\|^2 \\ &\ge \left\|(a')^{\frac{1}{2}}a(a')^{\frac{1}{2}}\right\|^2 \\ &\ge \|a'aa'\|^2 \\ &\ge \|a'a^2a'\|^2 \\ &= \|aa'\|^2.\end{aligned}$$

Proposition (6.1.14)[453]: Let A be a C^*-algebra with $\dim_{nuc} A = n < \infty$. Then, there is a system $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ of almost contractive n-decomposable $c.p.$ approximations such that the map

$$\bar{\psi}: A \to \prod\nolimits_{\Lambda} F_\lambda \,/\bigoplus\nolimits_{\Lambda} F_\lambda$$

induced by the ψ_λ has order zero.

Proof. Let us first assume A to be separable. In this case, it will suffice to show the following: For any $0 < \varepsilon < \frac{1}{(n+2)^4}$ and any finite subset $\mathcal{F} \subset A$ of positive normalized elements, there is a piecewise contractive n-decomposable $c.p.$ approximation (F, ψ, φ) of A such that

$$\|\varphi\psi(b) - b\| < \varepsilon^{\frac{1}{16}} \; for \quad b \in \mathcal{F}$$

and

$$\|\psi(c)\psi(c')\| < \varepsilon^{\frac{1}{16}}$$

whenever $c, c' \in \mathcal{F}$ satisfy $\|cc'\| < \varepsilon$.

So, let ε and $\mathcal{F}$ as above be given. Choose a piecewise contractive n-decomposable $c.p.$ approximation $(\tilde{F}, \tilde{\psi}, \tilde{\varphi})$ of A such that

$$\left\|\tilde{\varphi}\tilde{\psi}(b) - b\right\| < \varepsilon \; for \; b \in \mathcal{F}.$$

Write $\tilde{F} = M_{r_1} \oplus \ldots \oplus M_{r_s}$ and denote the respective components of $\tilde{\varphi}$ and $\tilde{\psi}$ by $\tilde{\varphi}_j$ and $\tilde{\psi}_j$, respectively. Define

$$I := \{j \in \{1, \ldots, s\} |$$
$$\left\|\tilde{\psi}_j(c)\tilde{\psi}_j(c')\right\| \geq \varepsilon^{-\frac{1}{8}} \left\|\tilde{\varphi}\tilde{\psi}(c)\tilde{\varphi}\tilde{\psi}(c')\right\|^{\frac{1}{4}}$$
$$\text{for some } c, c' \in \mathcal{F} \text{ with } \|cc'\| < \varepsilon\}.$$

Let

$$\pi_j : M_{r_j} \to A''$$

denote the canonical supporting $*$-homomorphism for $\tilde{\varphi}_j$ (cf. (6.1.2)), so that we have

$$\tilde{\varphi}_j(x) = \tilde{\varphi}_j(1_{M_{r_j}})\pi_j(x) \; for \; all \; x \in M_{r_j}.$$

We estimate that

$$\begin{aligned}
&\left\|\tilde{\varphi}\,\tilde{\psi}(b)\tilde{\varphi}\,\tilde{\psi}(b')\right\| \\
&\overset{(6.1.13)}{\geq} \left\|\tilde{\varphi}_j\tilde{\psi}_j(b)\tilde{\varphi}_j\tilde{\psi}_j(b')\right\|^2 \\
&= \left\|\tilde{\varphi}_j\left(1_{M_{r_j}}\right)^2 \pi_j(\tilde{\psi}_j(b)\tilde{\psi}_j(b'))\right\|^2 \\
&\geq \Big\|\pi_j(\tilde{\psi}_j(b')\,\tilde{\psi}_j(b)) \\
&\quad \tilde{\varphi}_j\left(1_{M_{r_j}}\right)^2 \pi_j(\tilde{\psi}_j(b)\tilde{\psi}_j(b'))\Big\|^2 \\
&= \left\|\tilde{\varphi}_j(\tilde{\psi}_j(b)\tilde{\psi}_j(b'))\right\|^4 \\
&= \left\|\tilde{\varphi}_j\left(1_{M_{r_j}}\right)\right\|^4 \left\|\tilde{\psi}_j(b)\tilde{\psi}_j(b')\right\|^4
\end{aligned}$$

for all $j \in \{1, \ldots, s\}$ and normalized $b, b' \in A$.

It follows that for each $j \in I$ there are $c, c' \in \mathcal{F}$ such that $\|cc'\| < \varepsilon$ and

$$\begin{aligned}
&\left\|\tilde{\varphi}\tilde{\psi}(c)\tilde{\varphi}\,\tilde{\psi}(c')\right\| \\
&\geq \left\|\tilde{\varphi}_j\left(1_{M_{r_j}}\right)\right\|^4 \varepsilon^{-\frac{1}{2}} \left\|\tilde{\varphi}\tilde{\psi}(c)\tilde{\varphi}\tilde{\psi}(c')\right\|,
\end{aligned}$$

whence

$$\left\|\tilde{\varphi}_j\left(1_{M_{r_j}}\right)\right\| \leq \varepsilon^{\frac{1}{8}}$$

and

$$\left\| \sum_{j\in I} \tilde{\varphi}_j\left(1_{M_{r_j}}\right)\right\| \leq (n+1)\varepsilon^{\frac{1}{8}}.$$

Set

$$F := \bigoplus_{j\in\{1,\dots,s\}\setminus I} M_{r_j}$$

and denote the respective components of $\tilde{\varphi}$ and $\tilde{\psi}$ by φ and ψ, respectively. Then, we have

$$\begin{aligned}\|b-\varphi\psi(b)\| &\leq \left\|b-\tilde{\varphi}\,\tilde{\psi}(b)\right\| - \left\|\tilde{\varphi}\tilde{\psi}(b)-\phi\psi(b)\right\| \\ &\leq \varepsilon + \left\|\sum_{j\in I}\tilde{\varphi}_j\left(1_{M_{r_j}}\right)\right\| \\ &\leq \varepsilon + (n+1)\varepsilon^{\frac{1}{8}} \\ &< \varepsilon^{\frac{1}{16}}\end{aligned}$$

for $b \in \mathcal{F}$.

Moreover, if $c, c' \in \mathcal{F}$ satisfy $\|cc'\| < \varepsilon$, then by the definition of ψ and I, we have

$$\begin{aligned}\|\psi(c)\psi(c')\|^4 &= \max_{j\notin I}\left\|\tilde{\psi}_j(c)\tilde{\psi}_j(c')\right\|^4 \\ &< \varepsilon^{-\frac{1}{2}}\left\|\tilde{\varphi}\tilde{\psi}(c)\tilde{\varphi}\tilde{\psi}(c')\right\| \\ &< \varepsilon^{-\frac{1}{2}}(\|cc'\|+2\varepsilon) \\ &\leq 3\varepsilon^{\frac{1}{2}} \\ &< \varepsilon^{\frac{1}{4}},\end{aligned}$$

so

$$\|\psi(c)\psi(c')\| < \varepsilon^{\frac{1}{16}},$$

as desired.

Now if A is not necessarily separable, then the set

$$\Gamma := \{B \mid B \subset A \text{ is a separable } C^*\text{-subalgebra with } \dim_{nuc} B \leq \dim_{nuc} A\}$$

is directed with the order given by inclusion. Equip

$$\Lambda := \Gamma \times \mathbb{N}$$

with the alphabetical order, then Λ is directed as well. Use the first part of the proof to obtain an almost order zero, piecewise contractive, n-decomposable system of $c.p.$ approximations

$$(F_{B,v}, \psi_{B,v}, \varphi_{B,v})_{v\in\mathbb{N}}$$

for each $B \in \Gamma$. Using Proposition (6.1.9), it is straightforward to check that this yields an almost order zero, piecewise contractive, n-decomposable system of $c.p.$ approximations

$$(F_{B,v}, \psi_{B,v}, \varphi_{B,v})_{(B,v)\in\Lambda}$$

for A as desired.

We shall call $(F_\lambda, \psi_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ as in Proposition (6.1.14) a system of almost order zero, piecewise contractive, n-decomposable $c.p.$ approximations.

The next result says that, if A is sufficiently noncommutative, then so may be chosen the piecewise contractive, n-decomposable $c.p.$ approximations. Where we derive a dichotomy result for C^*-algebras with finite nuclear dimension.

Proposition (6.1.15)[453]: Let A be a separable C^*-algebra with $\dim_{nuc} A \leq n < \infty$, and let $k \in \mathbb{N}$ be given. Suppose that A has no irreducible representation of rank strictly less than k.
Then, there is a system $(E_\nu, \varrho_\nu, \sigma_\nu)_{\nu\in\mathbb{N}}$ of almost order zero, piecewise contractive, n-decomposable $c.p.$ approximations of A such that the irreducible representations of each E_ν have rank at least k.
Proof. Choose a system

$$(\bar{E}_\nu, \bar{\varrho}_\nu, \bar{\sigma}_\nu)_{\nu\in\mathbb{N}}$$

of almost order zero, piecewise contractive, n-decomposable $c.p.$ approximations of A. For each ν, write

$$\bar{E}_\nu = E_\nu \oplus \bar{E}_\nu,$$

where $\check{E}_\nu$ consists precisely of those matrix blocks of $\bar{E}_\nu$with rank at most $k-1$. Let $\varrho_\nu, \check{\varrho}_\nu, \sigma_\nu$ and $\check{\sigma}_\nu$ denote the respective components of $\bar{\varrho}_\nu$ and $\bar{\sigma}_\nu$.
Let $h \in A$ be a normalized strictly positive element, and set

$$\mu := \lim_{\nu\in\mathbb{N}} \sup \|\check{\varrho}_\nu(h)\| = \|\check{\varrho}_\nu(h)\|,$$

where

$$\check{\varrho}: A \to \check{E}_\nu / \oplus \check{E}_\nu$$

is the $c.p.c.$ order zero map induced by the $\check{\varrho}_\nu$. Using a free ultrafilter on N and the fact that $\prod \check{E}_\nu$ is $(k-1)$-subhomogeneous, it is straightforward to construct an irreducible representation

$$\pi: \prod \check{E}_\nu / \oplus \check{E}_\nu \to M_l$$

for some $l \leq k-1$ such that

$$\|\pi\check{\varrho}(h)\| = \mu.$$

Since π is a $*$-homomorphism, $\pi\check{\varrho}$ again is a $c.p.c.$ order zero map, so by Theorem (6.1.2) there are a $*$-homomorphism

$$\sigma: A \to M_l$$

and $0 \leq d \leq 1_l \in M_l$ such that

$$d\sigma(a) = \sigma(a)d = \pi\check{\varrho}(a)$$

for any $a \in A$. But by our assumption on A, σ has to be zero, whence

$$\|\check{\varrho}(h)\| = \mu = \|\pi\check{\varrho}(h)\| = 0.$$

Using that $\check{\varrho}$ is a positive map and that h is a strictly positive element, it is straightforward to conclude that $\check{\varrho} = 0$. It follows that $(E_\nu, \varrho_\nu, \sigma_\nu)_{\nu\in\mathbb{N}}$ is a system of $c.p.$ approximations with the right properties.

In [278, Definition 3.1], Kirchberg introduced a new integer valued invariant for a unital C^*-algebra. This covering number is closely related to both decomposition rank and nuclear dimension. It does not directly generalize topological covering dimension though, since it measures how many order zero maps one needs to cover a noncommutative space, as opposed to approximating it. We recall the definition and some facts from [278], and then compare the covering number to nuclear dimension.
Definition (6.1.16)[453]: Let A be a unital C^*-algebra and $n \in \mathbb{N}$. A has covering number at most $n, covA \leq n$, if the following holds:
For any $k \in \mathbb{N}$, there are a finite-dimensional C^*-algebra $F, d^{(1)}, \dots, d^{(n)} \in A$ and a $c.p.$ map $\varphi: F \to A$ such that

(i) F has no irreducible representation of rank less than k.

(ii) φ is $(n-1)$-decomposable with respect to $F = F^{(1)} \oplus \ldots \oplus F^{(n)}$

(iii) $1_A = (d^{(j)})^* \varphi^{(j)}(1_{F^{(j)}})d^{(j)}$.

We recall some more facts and notation from [278].

If A is a C^*-algebra and $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ a free ultrafilter, we denote by A_ω the ultrapower C^*-algebra

$$A_\omega := \ell_\infty(A)/c_\omega(A);$$

we will often consider A as a subalgebra of A_ω via the canonical embedding as constant sequences. We denote the two-sided annihilator of A in $A_\omega \cap A'$ by $Ann(A)$, i.e.,

$$Ann(A) := \{b \in A_\omega | \; bA = Ab = \{0\}\}.$$

Then, $Ann(A)$ is a closed ideal in $A_\omega \cap A'$; if A is σ-unital, then $A_\omega \cap A'/Ann(A)$ is a unital C^*-algebra, cf. [278, Proposition 1.9].

We shall see below that $covA \leq \dim_{nuc} A + 1$ for any sufficiently noncommutative unital C^*-algebra. However, the results of [278] show that the covering number of the quotient $A_\omega \cap A'/Ann(A)$ often is much more relevant than that of A. The next result relates the nuclear dimension of A to the covering number of $A_\omega \cap A'/Ann(A)$. This will be particularly. It will also play a key role in [284], where Ng show that finite nuclear dimension implies the corona factorization property, at least for sufficiently noncommutative unital C^*-algebras.

Proposition (6.1.17)[453]: Let A be a separable C^*-algebra with $\dim_{nuc} A \leq n < \infty$, and suppose that no hereditary C^*-subalgebra of A has a finite-dimensional irreducible representation. Then,

$$cov(A_\omega \cap A'/Ann(A)) \leq (n+1)^2.$$

Proof. By [278, Proposition 1.9], $A_\omega \cap A'/Ann(A)$ is unital. Lift the unit 1 to a positive normalized element $e \in A_\infty \cap A'$; e may be represented by an approximate unit $(e_\lambda)_{\lambda\in\mathbb{N}}$ of A.

By Proposition (6.1.14) there is a system

$$(F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(n)}, \psi_\lambda, \varphi_\lambda)_{\lambda\in\mathbb{N}}$$

of almost order zero, piecewise contractive, n-decomposable $c.p.$ approximations for A. By passing to a subsequence of the approximations, and by rescaling, if necessary, we may assume that

$$\|\varphi_\lambda\psi_\lambda(e_\lambda)\| \leq 1 \; \forall \; \lambda \in \mathbb{N},$$

that

$$\varphi_\lambda\psi_\lambda\left(e_\lambda^{\frac{1}{2}} a e_\lambda^{\frac{1}{2}}\right) \to a \; \forall \; a \in A \tag{3}$$

and that

$$\|\varphi_\lambda\psi_\lambda(e_\lambda) - e_\lambda\| \to 0. \tag{4}$$

Define $c.p.c.$ maps

$$\tilde{\psi}_\lambda : A^+ \to F_\lambda$$

by

$$\tilde{\psi}_\lambda(\,.\,) := \psi_\lambda\left(e_\lambda^{\frac{1}{2}} \,.\, e_\lambda^{\frac{1}{2}}\right).$$

For each λ, we define

$$\hat{\psi}_\lambda(\,.\,) := \psi_\lambda(e_\lambda)^{-\frac{1}{2}}\,\tilde{\psi}_\lambda(\,.\,)\psi_\lambda(e_\lambda)^{-\frac{1}{2}},$$

where the inverses are taken in the hereditary subalgebras $\tilde{F}_\lambda$ generated by the $\psi_\lambda(e_\lambda)$, and

$$\hat{\varphi}_\lambda(\,.\,) := \varphi\left(\psi_\lambda(e_\lambda)^{\frac{1}{2}}\,.\,\psi_\lambda(e_\lambda)^{\frac{1}{2}}\right),$$

then

$$\hat{\varphi}_\lambda\hat{\psi}_\lambda = \varphi_\lambda\tilde{\psi}_\lambda;$$

moreover, the

$$\hat{\psi}_\lambda: A^+ \to \tilde{F}_\lambda$$

are unital $c.p.$ and the

$$\hat{\varphi}_\lambda: \tilde{F}_\lambda \to A$$

are $c.p.c.$ maps.

From [280, Lemma 3.5], we see that for $i \in \{0, \dots, n\}$ and $\lambda \in \mathbb{N}$ and any projection $p_\lambda \in \tilde{F}_\lambda$,

$$\left\|\hat{\varphi}_\lambda(p_\lambda\hat{\psi}_\lambda(a)) - \hat{\varphi}_\lambda(p_\lambda)\hat{\varphi}_\lambda\,\hat{\psi}_\lambda(a)\right\| \le 3\cdot\max\{\left\|\hat{\varphi}_\lambda\hat{\psi}_\lambda(a) - a\right\|, \left\|\hat{\varphi}_\lambda\hat{\psi}_\lambda(a^2) - a^2\right\|\},$$

from which follows that

$$\left\|\varphi_\lambda^{(i)}\tilde{\psi}_\lambda^{(i)}(a) - \varphi_\lambda^{(i)}\tilde{\psi}_\lambda^{(i)}(1_A)\varphi_\lambda\tilde{\psi}_\lambda(a)\right\| \xrightarrow{\lambda\to\infty} 0 \qquad (5)$$

for any $a \in A$.

Let $\tilde{F}_{\lambda,l}^{(i)}, l \in \{1, \dots, r_\lambda^{(i)}\}$ denote the matrix blocks of $\tilde{F}_\lambda^{(i)}$, and denote the components of $\varphi_\lambda^{(i)}$ and $\tilde{\psi}_\lambda^{(i)}$ by $\varphi_{\lambda,l}^{(i)}$ and $\tilde{\psi}_{\lambda,l}^{(i)}$ accordingly.

By Proposition (6.1.15) and the hypotheses on A, for each $\lambda \in \mathbb{N}, i \in \{0, \dots, n\}$ and $l \in \{1, \dots, r_\lambda^{(i)}\}$ there is

$$\left(E_{\lambda,l,v}^{(i)} = E_{\lambda,l,v}^{(i,0)} \oplus \dots \oplus E_{\lambda,l,v}^{(i,n)}, \varrho_{\lambda,l,v}^{(i)}, \sigma_{\lambda,l,v}^{(i)}\right)_{v\in\mathbb{N}}, \qquad (6)$$

an almost order zero, piecewise contractive, n-decomposable system of $c.p.$ approximations of her $(\varphi_{\lambda,l}^{(i)}(e_{11})) \subset A$ with the additional property that the matrix blocks of each $E_{\lambda,l,v}^{(i)}$ have rank at least k. A moment's thought shows that there is a unital $*$-homomorphism

$$\theta_{\lambda,l,v}^{(i)}: M_k \oplus M_{k+1} \to E_{\lambda,l,v}^{(i)}$$

for any i, λ, l, ν.

Let

$$\bar{\sigma}_{\lambda,l,v}^{(i)}: E_{\lambda,l,v}^{(i)} \to M_{r_{\lambda,l}^{(i)}} \otimes her(\varphi_{\lambda,l}^{(i)}(e_{11})) \cong her\left(\varphi_{\lambda,l}^{(i)}\left(1_{M_{r_{\lambda,l}^{(i)}}}\right)\right) \subset A$$

be the amplification of $\sigma_{\lambda,l,v}^{(i)}$, using the canonical supporting $*$-homomorphism $\pi_{\lambda,l}^{(i)}$ of $\varphi_{\lambda,l}^{(i)}$, i.e.,

$$\bar{\sigma}_{\lambda,l,v}^{(i)}(e) := \sum_{s=1}^{r_{\lambda,l}^{(i)}} \pi_{\lambda,l}^{(i)}(e_{s1})\sigma_{\lambda,l,v}^{(i)}(e)\pi_{\lambda,l}^{(i)}(e_{1s}) \;\; for\ e \in E_{\lambda,l,v}^{(i)}.$$

Note that

$$\left[\bar{\sigma}_{\lambda,l,v}^{(i)}\left(E_{\lambda,l,v}^{(i)}\right), \varphi_{\lambda,l}^{(i)}\left(M_{r_{\lambda,l}^{(i)}}\right)\right] = 0 \qquad (7)$$

and that $\bar{\sigma}^{(i)}_{\lambda,l,v}$ is decomposable into a sum of $n+1$ $c.p.c.$ order zero maps $\bar{\sigma}^{(i,j)}_{\lambda,l,v}$ with respect to $E^{(i)}_{\lambda,l,v} = \bigoplus_{j=0}^{n} E^{(i,j)}_{\lambda,l,v}$.

Let us fix a finite subset $\mathcal{F} \subset A$ of positive normalized elements and $\varepsilon > 0$. By (5), (4) and (3), we can find $\lambda_0 \in \mathbb{N}$ such that, for all $\bar{\lambda} \geq \lambda_0$,

$$\left\| \varphi_{\bar{\lambda}} \, \tilde{\psi}_{\bar{\lambda}}(a) \varphi^{(i)}_{\lambda} \left(1_{F^{(i)}_{\lambda}}\right) - \varphi^{(i)}_{\bar{\lambda}} \tilde{\psi}^{(i)}_{\bar{\lambda}}(a) \right\| < \frac{\varepsilon}{4},$$

$$\| \varphi_{\bar{\lambda}} \psi_{\bar{\lambda}}(e_{\bar{\lambda}}) - e_{\bar{\lambda}} \| < \frac{\varepsilon}{2}$$

and such that

$$\left\| \varphi_{\bar{\lambda}} \, \tilde{\psi}_{\bar{\lambda}}(a) - a \right\| < \frac{\varepsilon}{4}$$

for $a \in \mathcal{F}$. Choose some

$$0 < \zeta < \frac{1}{8(n+1)} \varepsilon.$$

Fix some $\bar{\lambda} \geq \lambda_0$. By the choice of the approximations in (6), there is $\bar{v} \in \mathbb{N}$ such that

$$\sigma^{(i)}_{\bar{\lambda},l,\bar{v}}, \varrho^{(i)}_{\bar{\lambda},l,\bar{v}}(g_{\zeta,2\zeta}(\varphi^{(i)}_{\bar{\lambda},l}(e_{11}))) - g_{\zeta,2\zeta}(\varphi^{(i)}_{\bar{\lambda},l}(e_{11})) < \zeta$$

for each $i \in \{0, \ldots, n\}$ and $l \in \{1, \ldots, r^{(i)}_{\bar{\lambda}}\}$. Here, we define $g_{\zeta,2\zeta} \in C([0,1])$ by

$$g_{\zeta,2\zeta}(t) := \begin{cases} 0 & for\ 0 \leq t \leq \zeta, \\ 1 & for\ t \geq 2\zeta, \\ linear & else. \end{cases}$$

We then have

$$\sum_{l} \bar{\sigma}^{(i)}_{\bar{\lambda},l,\bar{v}} \left(1_{E^{(i)}_{\bar{\lambda},l,\bar{v}}}\right) \geq g_{\zeta,2\zeta}\left(\varphi^{(i)}_{\bar{\lambda}}\left(1_{F^{(i)}_{\bar{\lambda}}}\right)\right) - \zeta.$$

For $i, j \in \{0, \ldots, n\}$ define

$$E^{(i,j)} := \bigoplus_{l} E^{(i,j)}_{\bar{\lambda},l,\bar{v}}$$

and

$$\sigma^{(i,j)} := \bigoplus_{l} \bar{\sigma}^{(i,j)}_{\bar{\lambda},l,\bar{v}};$$

note that

$$\sigma^{(i,j)} \colon E^{(i,j)} \to A$$

is a $c.p.c.$ order zero map. Let $\theta^{(i,j)}$ denote the respective component of $\bigoplus_l \theta^{(i)}_{\bar{\lambda},l,\bar{v}}$.
Define

$$\bar{\Phi} \colon E^{(i,j)} \to A$$

by

$$\bar{\Phi}^{(i,j)}(x) := \sigma^{(i,j)}(x) \varphi^{(i)}_{\bar{\lambda}} \tilde{\psi}^{(i)}_{\bar{\lambda}}(1_{A^+})\ for\ x \in E^{(i,j)}.$$

Note that by (7), $\bar{\Phi}^{(i,j)}$ is a $c.p.c.$ order zero map, whence

$$\Phi^{(i,j)} := \bar{\Phi}^{(i,j)} \circ \theta^{(i,j)}$$

also is a $c.p.c.$ order zero map. We have

$$\sum_{i,j=0}^{n} \Phi^{(i,j)}(1_{M_k \oplus M_{k+1}})$$

$$
\begin{aligned}
&= \sum_{i,j=0}^{n} \Phi^{(i,j)}(1_{E^{(i,j)}}) \\
&= \sum_{i,j}\sum_{l} \bar{\sigma}^{(i,j)}_{\bar{\lambda},l,\bar{v}}(1_{E^{(i,j)}_{\bar{\lambda},l,\bar{v}}})\varphi^{(i)}_{\bar{\lambda}}\bar{\psi}^{(i)}_{\bar{\lambda}}(1_{A^+}) \\
&= \sum_{i}\sum_{l} \bar{\sigma}^{(i)}_{\bar{\lambda},l,\bar{v}}(1_{E^{(i)}_{\bar{\lambda},l,\bar{v}}})\varphi^{(i)}_{\bar{\lambda}}\bar{\psi}^{(i)}_{\bar{\lambda}}(1_{A^+}) \\
&\geq \sum_{i}(g_{\zeta,2\zeta}(\varphi^{(i)}_{\bar{\lambda}}(1_{F^{(i)}_{\bar{\lambda}}}))\varphi^{(i)}_{\bar{\lambda}}\bar{\psi}^{(i)}_{\bar{\lambda}}(1_{A^+}-\zeta) \\
&\geq \sum_{i}(\varphi^{(i)}_{\bar{\lambda}}\bar{\psi}^{(i)}_{\bar{\lambda}}(1_{A^+})-2\zeta) \\
&\geq \varphi_{\bar{\lambda}}\bar{\psi}_{\bar{\lambda}}(1_{A^+})-(n+1)2\zeta \\
&= \varphi_{\bar{\lambda}}\bar{\psi}_{\bar{\lambda}}(e_{\bar{\lambda}})-(n+1)2\zeta \\
&\geq e_{\bar{\lambda}}-\frac{\varepsilon}{2}-(n+1)2\zeta \\
&\geq e_{\bar{\lambda}}-\varepsilon.
\end{aligned}
$$

Furthermore, we estimate for $a \in \mathcal{F}$ and $x \in E^{(i,j)}$ that

$$
\begin{aligned}
&\left\|[\bar{\Phi}^{(i,j)}(x),a]\right\| \\
&\quad\leq \left\|\left[\bar{\Phi}^{(i,j)}(x),\varphi_{\bar{\lambda}}\tilde{\psi}_{\bar{\lambda}}(a)\right]\right\| + 2\frac{\varepsilon}{4}\|x\| \\
&\quad= \left\|\sigma^{(i,j)}(x)\varphi^{(i)}_{\bar{\lambda}}(1_{F^{(i)}_{\bar{\lambda}}})\varphi_{\bar{\lambda}}\bar{\psi}_{\bar{\lambda}}(a)-\varphi_{\bar{\lambda}}\bar{\psi}_{\bar{\lambda}}(a)\varphi^{(i)}_{\bar{\lambda}}(1_{F^{(i)}_{\bar{\lambda}}})\sigma^{(i,j)}(x)\right\| + \frac{\varepsilon}{2}\|x\| \\
&\quad\leq \left\|\sigma^{(i,j)}(x)\varphi^{(i)}_{\bar{\lambda}}\tilde{\psi}^{(i)}_{\bar{\lambda}}(a)-\varphi^{(i)}_{\bar{\lambda}}\tilde{\psi}_{\bar{\lambda}}(a)\sigma^{(i,j)}(x)\right\| + \frac{\varepsilon}{2}\|x\| \\
&\quad\overset{(7)}{=} \varepsilon\|x\|,
\end{aligned}
$$

from which follows that

$$\left\|[\Phi^{(i,j)}(y),a]\right\| \leq \varepsilon\|y\| \quad for\ y \in M_k \oplus M_{k+1}.$$

Since $\mathcal{F}$ and $\varepsilon > 0$ were arbitrary, and since the construction above works for any $\bar{\lambda} \geq \lambda_0$, it is now straightforward to construct $c.p.c.$ order zero maps

$$\widetilde{\Phi}^{(i,j)}: M_k \oplus M_{k+1} \to A_\infty \cap A'$$

for $i,j = 0,\dots,n$, satisfying

$$\widetilde{\Phi}^{(i,j)}(1_{M_k \otimes M_{k+1}}) \geq e.$$

The $\widetilde{\Phi}^{(i,j)}$ drop to $c.p.c.$ order zero maps

$$\widehat{\Phi}^{(i,j)}: M_k \oplus M_{k+1} \to A_\omega \cap A'/Ann(A)$$

satisfying

$$\sum_{i,j}\widehat{\Phi}^{(i,j)}(1_{M_k \otimes M_{k+1}}) \geq 1.$$

It follows that

$$cov(A_\omega \cap A'/Ann(A)) \leq (n+1)^2.$$

Combining the idea of [278, Proposition 3.5] with the use of Proposition (6.1.15) as in the preceding proof, one can also show the following generalization of [278, Proposition 3.5].

Proposition (6.1.18)[453]: Let A be a separable unital C^*-algebra with $\dim_{nuc} A \leq n < \infty$, and suppose that A has no finite-dimensional irreducible representation. Then,

$$cov(A) \leq n + 1.$$

We will combine Proposition (6.1.15) above with [278, Proposition 3.7] to prove a dichotomy result for C^*-algebras with finite nuclear dimension: They either have nontrivial quasitraces, or they are weakly purely infinite. This statement becomes particularly satisfactory in the simple case. We first need some background results on lower semicontinuous ($l.s.c.$) traces.

Proposition (6.1.19)[453]: Let A be a separable C^*-algebra and $J \lhd A$ a closed ideal. Suppose τ is a densely defined $l.s.c.$ trace on J. Then, τ extends to a (not necessarily densely defined) $l.s.c.$ trace on A.

Proof. Choose an increasing approximate unit $(e_v)_{v\in\mathbb{N}}$ for J. Using that τ is densely defined, a standard modification shows that we may even assume that $\tau(e_v^{\frac{1}{2}}) < \infty$ for all $v \in \mathbb{N}$. Since τ is a trace and the e_v are increasing, for any $a \in A_+$ we obtain an increasing sequence of positive numbers

$$\left(\tau\left(e_v^{\frac{1}{2}} a e_v^{\frac{1}{2}}\right)\right)_v$$

(these are all finite since they are dominated by the numbers $\tau(e_v^{\frac{1}{2}})\|a\|$). We may thus define

$$\bar{\tau}(a) := \lim_v \tau\left(e_v^{\frac{1}{2}}\, a\, e_v^{\frac{1}{2}}\right) \; for\ a \in A_+.$$

Then, $\bar{\tau}$ is a well-defined map from A_+ to $[0,\infty]$. It is clear by lower semicontinuity that $\bar{\tau}$ extends τ, that it is $l.s.c.$ and that

$$\bar{\tau}(s \cdot a + t \cdot b) = s \cdot \bar{\tau}\,(a) + t \cdot\, \bar{\tau}(b)$$

if $\bar{\tau}(a), \bar{\tau}(b) < \infty$ and $s, t \in \mathbb{R}_+$. It remains to check that

$$\bar{\tau}\,(x^*x) = \bar{\tau}\,(xx^*)$$

for all $x \in A$. To this end, note that for $x \in A, \mu \in \mathbb{N}$ and $\varepsilon > 0$, we may choose v_0 so large that

$$\left\|e_\mu^{\frac{1}{4}} x^*(1 - e_v) x e_\mu^{\frac{1}{4}}\right\| < \frac{\varepsilon}{\tau\left(e_\mu^{\frac{1}{2}}\right)}$$

for any $v \geq v_0$ (this is where we use that J is an ideal in A). We then estimate

$$\begin{aligned}\tau\left(e_\mu^{\frac{1}{2}} x^* x e_\mu^{\frac{1}{2}}\right) &= \tau\left(e_\mu^{\frac{1}{2}} x^* e_v x e_\mu^{\frac{1}{2}}\right) + \tau\left(e_\mu^{\frac{1}{4}} e_\mu^{\frac{1}{4}} x^*(1 - e_v) x e_\mu^{\frac{1}{4}} e_\mu^{\frac{1}{4}}\right) \\ &\leq \tau(e_\mu^{\frac{1}{2}} x^* e_v x e_\mu^{\frac{1}{2}}) + \frac{\varepsilon}{\tau\left(e_\mu^{\frac{1}{2}}\right)} \cdot \tau\left(e_\mu^{\frac{1}{2}}\right) = \tau\left(e_v^{\frac{1}{2}} x e_\mu x^* e_v^{\frac{1}{2}}\right) + \varepsilon \\ &\leq \tau\left(e_v^{\frac{1}{2}}\, x x^* e_v^{\frac{1}{2}}\right) + \varepsilon \leq \bar{\tau}(xx^*) + \varepsilon.\end{aligned}$$

Since μ and ε were arbitrary, it follows that $\bar{\tau}(x^*x) \leq \bar{\tau}(xx^*)$; since the argument is symmetric in x and x^*, we see that $\bar{\tau}(x^*x) = \bar{\tau}(xx^*)$, as desired.

Corollary (6.1.20)[453]: Let A be a separable C^*-algebra and $B \subset A$ a hereditary C^*-subalgebra. If τ is a bounded nontrivial trace on B, then there is a (possibly unbounded) nontrivial $l.s.c.$ trace τ' on A.

Proof. Let $J \lhd A$ be the (closed) ideal generated by B. By Brown's Theorem ([267, Theorem 2.8]), $B \otimes K \cong J \otimes K$, since B is full in J.

Let Tr denote the standard $l.s.c.$ trace on K, then $\tau \otimes Tr$ yields a densely defined nontrivial $l.s.c.$ trace on $J \otimes K$. Let $\bar{\tau}$ denote the restriction to J; it is straightforward to check that $\bar{\tau}$ again is densely defined, $l.s.c.$ and nontrivial. By Proposition (6.1.19), $\bar{\tau}$ extends to a $l.s.c.$ trace τ' on A; since $\bar{\tau}$ is nontrivial, so is τ'.

Proposition (6.1.21)[453]: Let A be a separable C^*-algebra and suppose A has no nontrivial $l.s.c.$ trace.

Then, no hereditary C^*-subalgebra of A has a finite-dimensional irreducible representation.

Proof. If $B \subset A$ was a hereditary C^*-subalgebra with a finited-dimensional irreducible representation, then B also had a (necessarily nontrivial) tracial state. By Corollary (6.1.20), this would yield a nontrivial $l.s.c.$ trace on A, a contradiction.

Theorem (6.1.22)[453]: Let A be a separable C^*-algebra with $\dim_{nuc} A \le n < \infty$.

If A has no nontrivial $l.s.c.$ 2-quasitrace, then A is weakly purely infinite.

In particular, if A is simple, it is either strongly purely infinite, hence absorbs the Cuntz algebra $\mathcal{O}_\infty$, or it is stably finite with at least one densely defined trace.

Proof. Suppose A has no nontrivial $l.s.c.$ 2-quasitrace. By Proposition (6.1.21), no hereditary C^*-subalgebra of A has a finite-dimensional irreducible representation. By Proposition (6.1.17) this yields

$$cov(A_\omega \cap A'/Ann(A)) \le (n+1)^2 < \infty.$$

By [278, Proposition 3.7], this implies that A is weakly purely infinite.

For the second statement, suppose A is simple but not purely infinite. Then, A is not weakly purely infinite by [279], so A admits a nontrivial $l.s.c.$ 2-quasitrace.

Therefore, A contains a nonzero hereditary C^*-subalgebra B with a bounded 2-quasitrace, which is a trace by [273] or [277] since B is nuclear. But then $B \otimes K$ has a densely defined trace τ. By Brown's Theorem, A is a hereditary subalgebra of $B \otimes K$, and it is straightforward to check that τ restricts to a (nonzero) densely defined trace on A.

For the statement that a simple purely infinite C^*-algebra absorbs $\mathcal{O}_\infty$ see [276] or [288, Theorem 7.2.6].

Example (6.1.23)[453]: We have already seen that decomposition rank dominates nuclear dimension, and that the two theories agree in the zero-dimensional and in the commutative case, and for continuous trace C^*-algebras. This makes most examples of [277]. In particular, for irrational rotation algebras A_θ, we have

$$\dim_{nuc} A_\theta = \begin{cases} 1 \; if \; \theta \; is \; irrational \\ 2 \;\; if \; \theta \; is \; rational. \end{cases}$$

Example (6.1.24)[453]: In [293] it will be shown that, if α is a minimal homeomorphism of an infinite, compact, finite-dimensional, metrizable space X, then

$$\dim_{nuc}(C(X) \rtimes_\alpha \mathbb{Z}) \le 2 \dim X + 1.$$

Examples suggest that this is not the best possible estimate in general (see above), and that the nuclear dimension of the crossed product should be bounded by $\max\{1, \dim X\}$, at least

in the minimal case. However, for applications it often only matters whether or not the topological dimension is finite.
For the decomposition rank, the latter estimate, i.e.,

$$dr(C(X) \rtimes_\alpha \mathbb{Z}) \leq \max\{1, \dim X\},$$

is known in special cases, e.g. when the action α is a minimal diffeomorphism on a compact smooth manifold X. The known proofs of such results, however, require the full strength of the classification theory for stably finite nuclear C^*-algebras. The result of [293] has the advantage that its proof is much simpler, and more conceptual. In particular, it does not factor through classification theorems of any kind.

Example (6.1.25)[453]: Being an extension of $C(S^1)$ by the compacts, the Toeplitz algebra T has nuclear dimension at most 2 by Proposition (6.1.11). As of this moment, we do not know whether the precise value is 1 or 2 (it is not 0, since T is not AF). Since the Toeplitz algebra is infinite, hence not quasidiagonal, its decomposition rank is infinite. This in particular shows that decomposition rank and nuclear dimension do not agree.

Example (6.1.26)[453]: In [287], Rørdam constructed a simple, separable, unital, and nuclear C^*-algebra containing a finite and an infinite projection. This example does not have a nontrivial (quasi-)trace, nor is it purely infinite, so by Theorem (6.1.22) it has infinite nuclear dimension.

Below we will establish that classifiable simple purely infinite C^*-algebras have finite nuclear dimension. It suffices to prove this for classical Cuntz algebras and then use inductive limit representations of classifiable algebras.

Fix $n \in \mathbb{N}, n \geq 2$ and Cuntz-Toeplitz algebra $\mathcal{T}_n$ is the universal C^*-algebra generated by n isometries $T_1, \ldots, T_n$ subject to the relations $T_i^* T_j = \delta_{ij} 1$, whereas the Cuntz algebra On is the universal C^*-algebra generated by n isometries $S_1, \ldots, S_n$ subject to the relations $S_i^* S_j = \delta_{ij} 1$ and $\sum_{i=1}^n S_i S_i^* = 1$.

Let $I = \{1, \ldots, n\}$ and $W_n = \bigcup_{k=0}^\infty I^k$ be the set of multi-indices or words in the alphabet I. For $\mu = i_1 \ldots i_k \in W_n$ let $|\mu| = k$ be the length of the word μ and define $S_\mu = S_{i_1} \ldots S_{i_k}$, similarly $T_\mu = T_{i_1} \ldots T_{i_k}$. Every element x in the $*$-algebra generated by the S_i (respectively T_i) has a representation as a finite linear combination of the form $x = \sum_{\mu,\nu} \alpha_{\mu,\nu} S_\mu S_\nu^*$ (respectively $x = \sum_{\mu,\nu} \alpha_{\mu,\nu} T_\mu T_\nu^*$). The full Fock space is defined by

$$\Gamma(n) = \bigoplus_{l=0}^{\infty} H^{\otimes l},$$

where H is an n-dimensional Hilbert space and $H^0 := \mathbb{C}\Omega$. Fixing an orthonormal basis $e_1, \ldots, e_n$ of H gives the orthonormal basis $e_\mu = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k}$ of $\Gamma(n)$, where $\mu = i_1 \ldots i_k$ runs through W_n. In fact we may as well define $\Gamma(n) = \ell^2(W_n)$. We denote by M_∞ the $*$-algebra spanned by the matrix units $e_{\mu,\nu}$, where $\mu, \nu \in W_n$.
Clearly, $M_\infty \subseteq \bar{M}_\infty = K(\Gamma(n)) \subseteq B(\Gamma(n))$.

As is well-known [272], T_n acts faithfully on $\Gamma(n)$ with generators $T_i \xi = e_i \otimes \xi$. It contains the matrix units $e_{\mu,\nu} = T_\mu (1 - \sum_{i=1}^n T_i T_i^*) T_\nu^*$ and hence the ideal of compact operators giving the exact sequence

$$0 \to K \to T_n \to \mathcal{O}_n \to 0.$$

As in [290] we can write

$$T_\mu T_\nu^* = \sum_{i=0}^{\infty} e_{\mu,\nu} \otimes 1_{H^{\otimes i}} = \sum_{i=0}^{\infty} e_{\mu,\nu} \otimes 1_i\,,$$

where the sum is to be taken in the strong topology. The map

$$\Lambda(x) = \sum_{i=0}^{\infty} x \otimes 1_{H^{\otimes i}}$$

defined for matrix units x may be regarded as an unbounded completely positive map

$$\Lambda : M_\infty \to T_n.$$

For a fixed integer $k > 0$ define the cut-off Fock space

$$\Gamma(n) := \bigoplus_{l=0}^{k-1} H^{\otimes l}.$$

It gives rise to the factorization

$$\Gamma(n) \cong \Gamma_k(n) \otimes \Gamma(n^k)$$

via $e_\mu \leftrightarrow e_u \otimes e_{\bar{\mu}}$, where $\mu = u\bar{\mu}$ and $|\bar{\mu}|$ is the largest multiple of k below or equal to $|\mu|$. Similarly, if $k_1|k_2$ then $\Gamma_{k_2}(n) \cong \Gamma_{k_1}(n) \otimes \Gamma_{k_2/k_1}(n^{k_1})$.

Corresponding to the first factorization above we consider the C^*-algebra

$$A_k := B(\Gamma_k(n)) \otimes T_{n^k}.$$

Since $\dim \Gamma_k(n) = 1 + n + \ldots + n^{k-1} = \frac{n^k - 1}{n-1} =: d_k$, this algebra is just $M_{d_k}(T_{n^k})$.

As shown in [281] it is also generated by periodic weighted shifts but we don't need this description here.

Important for us is that A_k contains T_n. Indeed, denoting the generators of T_{n^k} by $\hat{T}_v$, where $v \in W_n$ with $|v| = k$, the generators $T_1, \ldots, T_n$ of T_n have the following matrix representation in A_k:

$$T_i = \sum_{j=0}^{\infty} e_{i,0} \otimes 1_j \qquad (in\ \Gamma(n))$$

$$= \left(\sum_{j=0}^{k-2} e_{i,0} \otimes 1_j \right) \otimes 1_{\Gamma(n^k)} + \sum_{|w|=k-1} e_{0,w} \otimes \hat{T}_{iw}\,.$$

Similarly, if $k_1|k_2$ then the generators of $1 \otimes T_{n^{k_1}} \subseteq A_{k_1}$ lie in A_{k_2} and $B(\Gamma_{k_1}(n)) \otimes 1 \subset B(\Gamma_{k_1}(n)) \otimes B(\Gamma_{k_2/k_1}(n^{k_1})) \cong B(\Gamma_{k_2}(n))$ so that $A_{k_1} \subseteq A_{k_2}$. If $k_1 < k_2 < \ldots$ is a sequence of positive integers such that $k_i|k_{i+1}$ then $A((k_i)) = \overline{\bigcup_\iota A_{k_\iota}}$ is a subalgebra of $B(\Gamma(n))$.

Now let $Q(\Gamma(n))$ be the Calkin algebra $B(\Gamma(n))/K$ with quotient homomorphism $q: B(\Gamma(n)) \to Q(\Gamma(n))$ so that $q(T_n) = \mathcal{O}_n$ and $q(A_k) = M_{d_k}(\mathcal{O}_{n^k})$. Notice that the quotient $q(A((k_i))) = A((k_i))/K$ is an inductive limit $B((k_i)) = \lim_i M_{d_{k_i}}(\mathcal{O}_{n^{k_i}})$, which is a simple nuclear purely infinite C^*-algebra.

Moreover, there is a canonical unital inclusion $\mathcal{O}_{n^k} \hookrightarrow \mathcal{O}_n$ given on generators by $s_v \longmapsto s_v$, where $v \in W_n$ with $|v| = k$. (We think of $\mathcal{O}_{n^k}$ as being generated by the isometries $s_v = s_{i_1} \ldots s_{i_k}$.) We obtain a unital embedding $M_{d_k}(\mathcal{O}_{n^k}) \hookrightarrow M_{d_k}(\mathcal{O}_n)$.

It is known from classification theory that a matrix algebra of the form $M_r(\mathcal{O}_s)$ is isomorphic to $\mathcal{O}_s$ if r and $s-1$ are relatively prime ([288] 8.4.11(i)). Since

$$d_k = 1 + n + n^2 + \ldots + n^{k-1} \equiv 1 + 1 + \ldots + 1 = k \bmod (n-1)$$

there are certainly infinitely many k satisfying $M_{d_k}(\mathcal{O}_n) \cong \mathcal{O}_n$.

We will need the following variant of the unbounded completely positive map Λ. Define $\Lambda_k : M_\infty \to B(\Gamma(n))$ by

$$\Lambda_k(x) = \sum_{l=0}^{\infty} x \otimes 1_{H^{\otimes kl}} = \sum_{l=0}^{\infty} x \otimes 1_{kl}.$$

Clearly, $\Lambda = \Lambda_1$.

Lemma (6.1.27)[453]: In the notation above we have:

(i) $\Lambda_k(M_\infty) \subseteq A_k \cong M_{d_k}(T_{n^k})$.

(ii) For a non-negative integer r let

$$\Gamma_{r,r+k} := \bigoplus_{lr}^{r+k-1} H^{\otimes^k} = \Gamma_k(n) \otimes H^{H^{\otimes^r}},$$

so that $\Gamma_{0,k} = \Gamma_k$ and $B(\Gamma_{r,r+k}) \cong M_{n^r d_k}$. Then, $\Lambda_k | B(\Gamma_{r,r+k})$ is a $*$-homomorphism.

Proof. (i) Given $\mu, \nu \in W_n$ there are unique decompositions $\mu = u\bar{\mu}$ and $\nu = v\bar{v}$ such that $|u|, |v| < k$ and $|\bar{\mu}|, |\bar{v}|$ are multiples of k. Then

$$\Lambda_k(e_{\mu,v}) = \sum_{l=0}^{\infty} e_{\mu,v} \otimes 1_{lk}$$

$$= e_{u,v} \otimes \sum_{l=0}^{\infty} e_{\overline{\mu},\overline{v}} \otimes 1_{lk}$$

$$= e_{u,v} \otimes \hat{T}_{\overline{\mu}} \hat{T}_{\overline{v}}^*$$

which proves the claim.

(ii) This follows because for $x \in B(\Gamma_{r,r+k})$ the summands

$$x, x \otimes 1_k, x \otimes 1_{2k}, \ldots$$

of $\Lambda_k(x)$ act $*$-homomorphically on the pairwise orthogonal subspaces

$$\Gamma_{r,r+k}, \Gamma_{r+k,r+2k}, \Gamma_{r+2k,r+3k,\ldots}$$

respectively.

We denote the projection onto $\Gamma_{r,r+k}$by $P_{r,r+k}$. Define $P_k = P_{k,2k}$ and $Q_k = P_{\lceil k/2 \rceil + k, \lceil k/2 \rceil + 2k}$, where, as usual, $\lceil k/2 \rceil = \inf\{n \in \mathbb{Z} \mid n \geq k/2\}$.

We now define the following positive $k \times k$ matrices. For k even let $l = k/2$ and define:

$$\kappa_k = [\kappa_{i,j}] = \frac{1}{l+l} \begin{bmatrix} 1 & 1 & \cdots & & & \cdots & 1 & 1 \\ 1 & 2 & \cdots & & & & 2 & 1 \\ 1 & 2 & & & & & & \vdots \\ \vdots & \vdots & & & & & & \\ & & & l & l & & & \\ & & & l & l & & & \\ \vdots & & & & & & & \vdots \\ 1 & & & & & & & 1 \\ 1 & 1 & \cdots & & & \cdots & 1 & 1 \end{bmatrix}.$$

For k odd let $l = \lceil k/2 \rceil$ and define:

$$\kappa_k = [\kappa_{i,j}] = \frac{1}{l+l}\begin{bmatrix} 1 & 1 & \cdots & & & & \cdots & 1 & 1 \\ 1 & 2 & \cdots & & & & & 2 & 1 \\ 1 & 2 & & & & & & & \vdots \\ \vdots & \vdots & & & & & & & \\ & & & l-1 & l-1 & l-1 & & & \\ & & & l-1 & l & l-1 & & & \\ & & & l-1 & l-1 & l-1 & & & \\ \vdots & & & & & & & & \vdots \\ 1 & 2 & & & & & & & 1 \\ 1 & 1 & \cdots & & & & \cdots & 1 & 1 \end{bmatrix}.$$

Since square matrices with all entries equal to 1 are positive, it is easy to see that the above matrices are positive contractions.

Regarding $x \in B(\Gamma_{r,r+k})$ as a $k \times k$ operator matrix $x = [x_{i,j}]$, where $x_{i,j} \in B(H^{\otimes^{r+j-1}}, H^{\otimes^{r+i-1}})$ we infer that the Schur multiplication $\kappa_k * [x_{i,j}] = [\kappa_{i,j} x_{i,j}]$ defines a completely positive contraction.

With this at hand we define the following completely positive maps

$$\psi_k : T_n \to B(\Gamma_{k,2k}) \oplus B(\Gamma_{\lceil k/2 \rceil + k, \lceil k/2 \rceil + 2k})$$

and

$$\varphi_k : B(\Gamma_{k,2k}) \oplus B(\Gamma_{\lceil k/2 \rceil + k, \lceil k/2 \rceil + 2k}) \to A_k = M_{d_k}(T_{n^k}) \subseteq B(\Gamma(n))$$

by

$$\psi_k(x) = \kappa_k(P_k x P_k) \oplus \kappa_k(Q_k x Q_k)$$

and

$$\varphi_k(x \oplus y) = \Lambda_k(x) + \Lambda_k(y)$$

Clearly $\|\psi_k\| = 1$ and $\|\varphi_k\| = 2$. Finally we consider the composition $q \circ \varphi_k \circ \psi_k$.

Proposition (6.1.28)[453]: For $\mu, \nu \in W_n$ fixed we have

$$q \circ \varphi_k \circ \psi_k(T_\mu T_\nu^*) \to s_\mu s_\nu^*,$$

as $k \to \infty$, where $s_\mu = q(T_\mu)$ are the generators of $\mathcal{O}_n$ in the Calkin algebra $Q(\Gamma(n))$.

Proof. Define the $\mathbb{N}_0 \times \mathbb{N}_0$ matrices

$$A_k = 0_k \oplus \kappa_k \oplus \kappa_k \oplus \dots$$

and

$$B_k = 0_k \oplus 0_l \oplus \kappa_k \oplus \kappa_k \oplus \dots,$$

where $l = \lceil k/2 \rceil$ and 0_k and 0_l denote the $k \times k$ resp. $l \times l$ zero matrices. One checks that the entries $\sigma_{i,j}$ of the matrix $A_k + B_k$ verify $\sigma_{i,i} = 1$ and $|\sigma_{i,i+p} - 1| \leq \frac{2+p}{l+1} \leq \frac{2(2+p)}{k}$, provided $i > k$ and $0 < p < l$. Regard every operator on $\Gamma(n)$ as an operator matrix $[x_{i,j}]$, where $x_{i,j} \in B(H^{\otimes^j}, H^{\otimes^i})$. Then further inspection shows that

$$\varphi_k \circ \psi_k(T_\mu T_\nu^*) = (A_k + B_k) * (T_\mu T_\nu^*),$$

where $*$ denotes again Schur multiplication. Thus provided k is large compared to $|\mu|$ and $|\nu|$ we have

$$\varphi_k \circ \psi_k(T_\mu T_\nu^*) = \sum_{r=0}^{\infty} \sigma_{|\mu|+r, |\nu|+r} e_{\mu,\nu} \otimes I_r \, .$$

By passing to the Calkin algebra (i.e. applying q) we obtain

$$\left\| s_\mu s_\nu^* - q \circ \varphi_k \circ \psi_k(T_\mu T_\nu^*) \right\| \leq 2(2 + |\mu| - |\nu|)k^{-1}.$$

Letting k tend to infinity concludes the proof.

Theorem (6.1.29)[453]: We have $\dim_{nuc} \mathcal{O}_n = 1$ for $n = 2, 3, \ldots$ and $\dim_{nuc} \mathcal{O}_\infty \leq 2$.

Proof. Let $\{x_1, x_2, \ldots, x_N\}$ be a finite subset of $\mathcal{O}_n$ and $\varepsilon > 0$. We need to find a finite dimensional C^*-algebra of the form $F = F^{(0)} \oplus F^{(1)}$, a $c.p.c.$ map $\psi : \mathcal{O}_n \to F$ and $\varphi: F \to \mathcal{O}_n$ $c.p.$ such that $\varphi|_{F^{(0)}}$ and $\varphi|_{F^{(i)}}$ are both order zero contractions and such that $\|x_i - \varphi \circ \psi(x_i)\| < \varepsilon$ for $i = 1, \ldots, N$.

To begin the construction fix $\rho: \mathcal{O}_n \to T_n$, a $u.c.p.$ lift of the quotient map $T_n \to \mathcal{O}_n$, which exists by nuclearity of $\mathcal{O}_n$. For suitable k (to be determined shortly) let $F = B(\Gamma_{k,2k}) \oplus B(\Gamma_{[k/2]+k,[k/2]+2k})$ and define $\psi = \psi_k \circ \rho: \mathcal{O}_n \to F$.

Next observe that $q \circ \psi_k: F \to M_{d_k}(\mathcal{O}_{n^k}) \subseteq q(B(\Gamma(n)))$ which we compose with the inclusion $M_{d_k}(\mathcal{O}_{n^k}) \hookrightarrow M_{d_k}(\mathcal{O}_n) \cong \mathcal{O}_n$ (the latter for suitable k). $M_{d_k}(\mathcal{O}_{n^k})$ contains the copy of $\mathcal{O}_n$ from the inclusion $T_n \subseteq B(\Gamma(n))$; we think of $\{x_1, x_2, \ldots, x_N\}$ as a subset in that copy and then know from (6.1.28) that we may find k such that $\|x_i - q \circ \varphi_k \circ \psi_k \circ \rho(x_i)\| < \varepsilon/2$ for $i = 1, \ldots, N$ and such that d_k and $n - 1$ are relatively prime. (Note that $\varphi_k \circ \psi_k(C) \to 0$ as $k \to \infty$ for any compact $C \in K(\Gamma(n))$ so that the choice of ρ does not really matter.)

Further, we may regard the inclusion given by

$$\mathcal{O}_n \hookrightarrow M_{d_k}(\mathcal{O}_{n^k}) \hookrightarrow M_{d_k}(\mathcal{O}_n) \cong \mathcal{O}_n$$

as a unital $*$-endomorphism σ of $\mathcal{O}_n$. It follows from classification theory that any such endomorphism is approximately unitarily equivalent to the identity map on $\mathcal{O}_n$. Indeed, σ is homotopic to id since it is implemented by a unitary v in $\mathcal{O}_n$ in the sense that $\sigma(s_i) = v s_i$ for all $i = 1, \ldots, n$ and the unitary group of $\mathcal{O}_n$ is connected. By Kirchberg's Classification Theorem ([288] 8.3.3(iii)) σ and id are asymptotically hence approximately unitarily equivalent.

Thus there is a unitary $u \in \mathcal{O}_n$ such that

$$\|u x_i u^* - \sigma(x_i)\| < \varepsilon/2.$$

Define $\varphi(x) = u^*(\beta \circ \varphi_k(x))u$, where β denotes the map from $M_{d_k}(T_{n^k})$ to $\mathcal{O}_n$ discussed above. Then (F, ψ, φ) is as desired.

The estimate for $\mathcal{O}_\infty$ follows since there is an obvious inductive limit representation $\mathcal{O}_\infty = \lim_{n\to\infty} T_n$, and we know that $\dim_{nuc} T_n \leq 2$ because of the exact sequence

$$0 \to K \to T_n \to \mathcal{O}_n \to 0$$

Using Kirchberg–Phillips classification it can be shown that every Kirchberg algebra satisfying the UCT is an inductive limit of C^*-algebras of the form

$$(M_{k_1} \otimes \mathcal{O}_{n_1} \oplus \ldots \oplus M_{k_r} \otimes \mathcal{O}_{n_r}) \otimes C(T),$$

where $n_i \in \{2, 3, \ldots\} \cup \{\infty\}$ and $k_i \in \mathbb{N}$ (cf. [288], 8.4.11). Since the nuclear dimension of any such algebra is at most 5 by Proposition (6.1.6), we obtain the following.

Theorem (6.1.30)[453]: A Kirchberg algebra (i.e., a purely infinite, simple, separable, nuclear C^*-algebra) satisfying the UCT.We explore a connection between the asymptotic dimension of a coarse space and the nuclear dimension of its uniform Roe algebra. Although both concepts may be defined for arbitrary coarse spaces (c.f. [286]) we restrict ourselves to discrete metric spaces of bounded geometry, mostly for simplicity.

Recall that a discrete metric space (X, d) is said to be of bounded geometry if every ball $B_r(x) = \{y \in X \mid d(x, y) \leq r\}$ of finite radius r has finitely many elements, and the number of elements in all balls of a given radius is uniformly bounded, that is, $b_r :=$

$\sup\{|B_r(x)| \mid x \in X\} < \infty$ for all r. This class of coarse spaces includes many interesting examples, e.g. finitely generated discrete groups with a word length metric.

In this setting the uniform Roe algebra $UC_r^*(X)$ associated to (X, d) can be defined as follows:

Consider complex matrices $[\alpha_{x,y}]$ indexed by $x, y \in X$ such that

(i) there is $M \geq 0$ with $|\alpha_{x,y}| \leq M$ for all $x, y \in X$ (i.e. $[\alpha_{x,y}]$ is uniformly bounded);

(ii) there is $r > 0$ such that $\alpha_{x,y} = 0$ whenever $d(x, y) > r$ (i.e. $[\alpha_{x,y}]$ has bounded width).

The smallest r in condition (ii) is called the width of the matrix $a = [\alpha_{x,y}]$, denoted by w(a). Any matrix satisfying (i) and (ii) defines a bounded operator on $\ell^2(X)$, again denoted by a. We have in fact the following elementary estimate.

Lemma (6.1.31)[453]: Let $a = [\alpha_{x,y}]$ be a matrix satisfying (i) and (ii) above and let

$$b(a) := b_{w(a)} = \sup\{|B_{w(a)}(x)| \mid x \in X\}.$$

Then, $\|a\| \leq b(a)M$.

Proof. For $(\beta_x) \in \ell^2(X)$ the sum $\gamma_x = \sum_y \alpha_{x,y}\beta_y$ is well-defined containing at most $b(a)$ many terms for each $x \in X$. Thus

$$|\gamma_x|^2 \leq b(a)M^2 \sum_{y \in B_{w(a)}(x)} |\beta_y|^2 .$$

Since $\sum_{y \in B_{w(a)}(x)} |\beta_y|^2 \leq b(a)\|(\beta_x)\|^2$ we obtain $\|(\gamma_x)\| \leq Mb(a)\|(\beta_x)\|$.

Define the Roe algebra $UC_r^*(X)$ of (X, d) as the concrete C^*-algebra generated by matrices satisfying (i) and (ii) above, that is, the closure of the set of such matrices. Note that if $a \in UC_r^*(X)$ has finite width then the matrix entries are uniformly bounded (by $\|a\|$).

We next recall the definition of the asymptotic dimension of (X, d). Note first that by a uniform cover U of X we mean a family of subsets of X such that $\bigcup_{U \in \mathcal{U}} U = X$ and such that the diameters $d(U)$ of all $U \in \mathcal{U}$ are uniformly bounded. A cover $\mathcal{U}$ has multiplicity or order n if there are $n + 1$ different $U_0, \ldots, U_n \in \mathcal{U}$ such that $U_0 \cap \ldots \cap U_n \neq \emptyset$ but any $n + 2$ different elements in $\mathcal{U}$.

Definition (6.1.32)[453]: Let (X, d) be a metric space. The asymptotic dimension asdim X does not exceed n if for every uniform cover $\mathcal{U}$ there is a uniform cover $\mathcal{V}$ of order n such that $\mathcal{U}$ refines $\mathcal{V}$ (i.e. every $U \in \mathcal{U}$ is contained in a $V \in \mathcal{V}$).

A family $\mathcal{U}$ of subsets of X is said to be r-discrete if the distance $d(U, U') > r$ for any two different $U, U' \in \mathcal{U}$. We need the following characterization of the asymptotic dimension which is part of [265, Theorem 19].

Theorem (6.1.33)[453]: For a metric space (X, d) the following conditions are equivalent.

(i) The asymptotic dimension asdim X does not exceed n.

(ii) For arbitrarily large $r > 0$ there exist r-discrete families

$$\mathcal{U}^{(0)}, \ldots, \mathcal{U}^{(n)}$$

of subsets of X such that $U^{(0)} \cup \ldots \cup \mathcal{U}^{(n)}$ is a uniform cover of X.

It is known that $UC_r^*(X)$ is nuclear if (X, d) is a discrete metric space of bounded geometry and finite asymptotic dimension..

Notice that for $X = \Gamma$ a discrete group, its uniform Roe algebra $UC_r^*(\Gamma)$ is nuclear iff Γ is exact. Also other approximation properties of Γ can be formulated in terms of the uniform Roe algebra ([304]).

Lemma (6.1.34)[453]: Let K be any index set and $(n_k)_{k\in K}$ a bounded family of positive integers. Then $\prod_{k\in K} M_{n_k}$ is an AF algebra.

Proof. Without loss assume (n_k) to be constantly equal to n ($\prod_{k\in K} M_{n_k}$ is a finite direct sum of such). Then any partition $\mathcal{P} = \{P_1,\dots,P_l\}$ of K defines an embedding of

$$\underbrace{M_n \oplus \dots \oplus M_n}_{l} \to \prod_{k\in K} M_{n_k}$$

sending $x_1 \oplus\dots\oplus x_l$ to the family constantly equal to x_i on P_i for $i = 1,\dots,l$. The union of all the ranges of these embeddings for all possible finite partitions is dense in $\prod_{k\in K} M_{n_k}$.

Theorem (6.1.35)[453]: Let (X,d) be a discrete metric space of bounded geometry. Then $\dim_{nuc}(UC_r^*(X)) \le asdim(X)$.

Proof. Let $r \in \mathbb{N}$ and choose, according to Theorem (6.1.33), uniform r-disjoint families $\mathcal{U}^{(0)},\dots,\mathcal{U}^{(n)}$ such that $\bigcup_{i=0}^{n} \mathcal{U}^{(i)}$ covers X. We will define a completely positive contraction

$$\Psi_r: UC_r^*(X) \to A^{(0)} \oplus \dots\oplus A^{(n)},$$

where

$$A^{(i)} = \prod_{U\in\mathcal{U}^{(i)}} M_{|B_{r-1}(U)|}\,.$$

By Lemma (6.1.34) every $A^{(i)}$ is AF and moreover naturally contained in $UC_r^*(X)$. Let

$$\Phi_r: A^{(0)} \oplus \dots\oplus A^{(n)} \to UC_r^*(X)$$

be defined by

$$\Phi_r(a_0 \oplus \dots \oplus a_n) = a_0 + \dots + a_n.$$

Then Φ_r is a completely positive map which is $*$-homomorphic on every $A^{(i)}$. If we can show that $\Phi_r \circ \Psi_r(a) \to a$ for all $a \in UC_r^*(X)$ then we are done since we can combine the Φ_r and Ψ_r with a standard approximating net $(\psi_\lambda, \phi_\lambda)$ of $A^{(0)} \oplus \dots\oplus A^{(n)}$, where the ϕ_λ are order 0, in fact $*$-homomorphic using (6.1.34). In order to define Ψ_r let

$$h^{(i)} = \frac{1}{r}\sum_{U\in\mathcal{U}^{(i)}}\sum_{l=1}^{r} \chi_{B(U,l-1)}\,,$$

where χ_S denotes the characteristic function of S and

$$B(U,s) = \{x \in X \mid d(x,U) \le s\}.$$

Then $h^{(0)},\dots,h^{(n)}$ are commuting positive contractions; moreover

$$1 \le h := \sum_{i=0}^{n} h^{(i)} \le (n+1)1.$$

If $a \in UC_r^*(X)$ is given by the matrix $[\alpha_{x,y}]$ then $[h^{(i)},a]$ is given by the matrix $[(h^{(i)}(x) - h^{(i)}(y))\alpha_{x,y}]$ and if a has finite width $w(a) < r$ then this commutator has still the same width and by Lemma (6.1.31) it follows that

$$\left\|[h^{(i)},a]\right\| \le b(a)\sup\{|h^{(i)}(x) - h^{(i)}(y)| \mid d(x,y) < w(a)\}\|a\| \le \frac{w(a)}{r} b(a)\|a\|$$

and thus

$$\|[h,a]\| \le \frac{n+1}{r} w(a)b(a)\|a\|.$$

Now define

$$h_i = \left(h^{(i)}h^{-1}\right)^{1/2}.$$

Since $[h^{-1}, a] = h^{-1}[a, h]h^{-1}$ we have

$$\|[h^{-1}, a]\| \leq \|h^{-1}\|^2\|[h, a]\| \leq \|[h, a]\|,$$

so that

$$\left\|[h^{(i)}h^{-1}, a]\right\| \to 0$$

as $r \to \infty$.

Approximating the function $t \longmapsto t^{1/2}$ by polynomials and using

$$\|[a, x^n]\| \leq n\|[a, x]\|\|x\|^{n-1}$$

for any x we find that also

$$\left\|\left[\left(h^{(i)}h^{1/2}\right)^{1/2}, a\right]\right\| = \|[h_i, a]\| \to 0$$

as $r \to \infty$, whenever $a \in UC_r^*(X)$ has finite width. But since $\|h_i\| \leq 1$ it follows that this is true for all $a \in UC_r^*(X)$.

Now define the completely positive contraction

$$\Psi_r(a) = h_0ah_0 \oplus h_1ah_1 \oplus \ldots \oplus\ h_nah_n.$$

Then

$$\Phi_r \circ \Psi_r(a) = \sum_{i=0}^{n} h_iah_i\,.$$

Note that $\Phi_r \circ \Psi_r(1) = \sum_{i=0}^{n} h_i^2 = 1$ so that $\Phi_r \circ \Psi_r$ is $u.c.p.$, in particular a contraction. Since for $a \in UC_r^*(X)$ of finite width we have

$$\|\Phi_r \circ \Psi_r(a) - a\| = \left\|\sum_{i=0}^{n} h_iah_i - \sum_{i=0}^{n} h_i^2a\right\|$$

$$= \left\|\sum_{i=0}^{n} h_i[a, h_i]\right\| \leq \sum_{i=0}^{n}\|[a, h_i]\| \to 0,$$

it follows again that $\|\Phi_r \circ \Psi_r(a) - a\| \to 0$ for all $a \in UC_r^*(X)$ because $\|\Phi_r \circ \Psi_r\| \leq 1$ for all r.

We list a number of open problems and possible future developments of the theory.

It follows trivially from the definitions that decomposition rank dominates nuclear dimension, and the (purely) infinite examples show that the two theories do not agree in general. One might ask, however, whether infiniteness is the only obstruction.

A conjecture of Toms relates various regularity properties for separable, simple, finite, unital, and nuclear C^*-algebras. the nuclear dimension enables us to put this conjecture into a broader context.

Conjecture (6.1.36)[453]: For a separable, simple, unital, infinite dimensional and nuclear C^*-algebra A, the following are equivalent:

(i) A has finite nuclear dimension.

(ii) A is Z-stable.

(iii) A has strict comparison of positive elements.

(iv) A has almost unperforated Cuntz semigroup.

As we have mentioned earlier, it will be shown in [293] that crossed products of continuous functions on compact and finite dimensional spaces by the integers via minimal homeomorphisms have finite nuclear dimension. One might ask for similar results when the underlying C^*-algebra is noncommutative, or when the group is more complicated.

Section (6.2): Symmetric Amenability for C^*-Algebras

The general study of tracial states on C^*-algebras has a long history, but recently it gained a renewed interest in connection with the on going classification program for finite nuclear C^*-algebras. We record several facts about tracial C^*-algebras which may be useful in the future study. The results are two-fold. First, we consider Dixmier type approximation property for C^*-algebras and relate it to symmetric amenability. The Dixmier approximation theorem (Theorem III.5.1 in [310]) states a fundamental fact about von Neumann algebras that for any von Neumann algebra N and any element $a \in N$, the norm-closed convex hull of $\{uau^*: u \in \mathcal{U}(N)\}$ meets the center $Z(N)$ of N. Here $\mathcal{U}(N)$ denotes the unitary group of N. If N is moreover a finite von Neumann algebra, then this intersection is a singleton and consists of $ctr(a)$. Here ctr denotes the center-valued trace, which is the unique conditional expectation from N onto $Z(N)$ that satisfies $ctr(xy) = ctr(yx)$. It is proved by Haagerup and Zsido [317] that the Dixmier approximation theorem holds for simple C^*-algebras having at most one tracial states (and obviously does not for simple C^*-algebras having more than one tracial states). Recall that a C^*-algebra A has the quotient tracial state property (QTS property) if every non-zero quotient C^*-algebra of A has a tracial state [327]. We denote by $T(A)$ the space of the tracial states on A, equipped with the weak$*$-topology.

Unlike the case for von Neumann algebras, there is no bound of k in terms of ε and $\|a\|$ that works for an arbitrary element a in a C^*-algebra. Recall that a Banach algebra A is said to be amenable if there is a net $(\varDelta_n)_n$, called an approximate diagonal, in the algebraic tensor product $A \otimes_{\mathbb{C}} A$ (we reserve the symbol $\otimes$ for the minimal tensor product) such that

(a) $sup_n \|\varDelta_n\|_\wedge < +\infty$,

(b) $(m(\varDelta_n))_n$ is an approximate identity,

(c) $\lim_n \|a \cdot \varDelta_n - \varDelta_n \cdot a\|_\wedge = 0$ for every $a \in A$.

Here $\|\cdot\|_\wedge$ is the projective norm on $A \otimes_{\mathbb{C}} A, m: A \otimes_{\mathbb{C}} A \to A$ is the multiplication, and $a \cdot (\sum_i x_i \otimes y_i) = \sum_i ax_i \otimes y_i$ and $(\sum_i x_i \otimes y_i) \cdot a = \sum_i x_i \otimes y_i a$. The celebrated theorem of Connes–Haagerup ([307, 318]) states that a C^*-algebra A is amenable as a Banach algebra if and only if it is nuclear. The Banach algebra A is said to be symmetrically amenable [321] if the approximate diagonal $(\varDelta_n)_n$ can be taken symmetric under the flip $x \otimes y \to y \otimes x$. We characterize symmetric amenability for C^*-algebras.

Recall that a unital C^*-algebra A is strongly amenable if there is an approximate diagonal that consists of convex combinations of $\{u^* \otimes u: u \in \mathcal{U}(A)\}$. This property is formally stronger than symmetric amenability, but it is unclear whether there is really a gap between these properties.

Second, we describe what is the C^*-completion $\bar{A}^U$ of a unital C^*-algebra A under the uniform 2-norm. This work is strongly influenced by the recent works of Kirchberg–Rordam [324], Sato [333], and Toms–White–Winter [335], who studied the central sequence algebra of a C^*-algebra modulo uniformly 2-norm null sequences, in order to extend Matui–Sato's result [326] from C^*-algebras with finitely many extremal tracial states to more general ones. In fact, our result is very similar to theirs (particularly to Kirchberg–Rordam's). Let A be a C^*-algebra and $S \subset T(A)$ be a nonempty metrizable closed face. The

reason we assume S be metrizable is because it makes the description of the boundary measures simpler. We define the uniform 2-norm on A corresponding to S by
$$\|a\|_{2,S} = \sup\{\tau\,(a^*a)^{1/2}: \tau \in S\}.$$
The uniform 2-norm satisfies
$$\|ab\|_{2,S} \leq \min\{\|a\|\|b\|_{2,S}, \|a\|_{2,S}\|b\|\} \text{ and } sup_{\tau\in S}|\tau(a)| \leq \|a\|_{2,S}.$$
The C^*-completion $\bar{A}^U$ is defined to be the C^*-algebra of the norm-bounded uniform 2-norm Cauchy sequences, modulo the ideal of the uniform 2-norm null sequences. For $\tau \in T(A)$, we denote by π_τ the corresponding GNS representation and also $\|a\|_{2,\tau} = \tau\,(a^*a)^{1/2}$. Let $N = (\bigoplus_{\tau\in S} \pi_\tau)(A)''$ be the enveloping von Neumann algebra with respect to S. When $S = T(A)$, it is the finite summand A_f^{**} of the second dual von Neumann algebra A^{**}. The tracial state $\tau \in S$ and the GNS representation π_τ extend normally on N. For the center-valued trace $ctr: N \to Z(N)$, one has $\|a\|_{2,S} = \|ctr(a^*a)\|^{1/2}$ and $\bar{A}^U$ coincides with the closure $\bar{A}^{st}$ of A in N with respect to the strict topology associated with the Hilbert $Z(N)$-module (N, ctr).

Recall that the trace space $T(A)$ of a unital C^*-algebra is a Choquet simplex and so is the closed face S. We denote by $Aff(S)$ the space of the affine continuous functions on S and consider the function system $Aff(S) = \{f|_{\partial S}: f \in Aff(S)\}$ in $B(\partial S)$, where $B(\partial S)$ denotes the C^*-algebra of the bounded Borel functions on ∂S. For every $a \in A$, the formula $\hat{a}(\tau) = \tau\,(a)$ defines a function $\hat{a}$ in $Aff(S)$ (or $Aff(S)$). We note that $\{\hat{a}: a \in A\}$ is dense in $Aff(S)$ (in fact equal, see [308]). Let $M_+^1(\partial S)$ be the space of the probability measures on the extreme boundary ∂S of S. Since S is a metrizable Choquet simplex, every $\tau \in S$ has a unique representing measure $\mu_\tau \in M_+^1(\partial S)$, which satisfies
$$\tau(a) = \int \lambda(a)\, d\mu_\tau(\lambda) = \int \hat{a}(\lambda)\, d\mu_\tau(\lambda)$$
for every $a \in A$ (Theorem II.3.16 in [305]). The center $Z(Aff(S))$ is defined to be
$$Z\big(Aff(S)\big) = \{f \in B(\partial S): f\ \ Aff(S) \subset Aff(S)\} \subset Aff(S).$$
When ∂S is closed (i.e., when S is a Bauer simplex), one has $Aff(S) = C(\partial S)$ and $Z(Aff(S)) = C(\partial S)$. However in general, the center $Z(Aff(S))$ can be trivial (see [305]).

Moreover, if ∂S is closed, then for every $\tau \in \partial S$, one has $\pi_\tau(\bar{A}^{st}) = \pi_\tau(N) = \pi_\tau(A)''$. Takesaki and Tomiyama [334] have studied the structure of a C^*-algebra, for which the set of pure states is closed in the state space, by using a continuous bundle of C^*-algebras (see also [313]). For a C^*-algebra A, for which ∂S is closed, in terms of a continuous W^*-bundle, and present W^*-analogues of a few results for C^*-bundles obtained in [318, 309]. In particular, we give a criterion for a continuous W^*-bundle over a compact space K with all fibers isomorphic to the hyperfinite II_1 factor R to be isomorphic to the trivial bundle $C_\sigma(K, R)$, the C^*-algebra of the norm-bounded and ultrastrongly continuous functions from K into R. We denote the evaluation map at $\lambda \in K$ by $ev_\lambda: C_\sigma(K, R) \to R$. As an application, we show that $\bar{A}^{st} \cong C_\sigma(\partial S, R)$ for certain A.

Let A be a separable C^*-algebra and $S \subset T(A)$ be a closed face. Assume that $\pi_\tau(A)'' \cong R$ for all $\tau \in \partial S$ and that ∂S is a compact space with finite covering dimension. Then, one can coordinatize the isomorphisms $\pi_\tau(A)'' = R$ in such a way that they together give rise to a $*$-homomorphism $\pi: A \to C_\sigma(\partial S, R)$ such that $\pi_\tau = ev_\tau \circ \pi$. The image of π is dense with respect to the uniform 2-norm.

Theorem (6.2.1)[454]: For a unital C^*-algebra A, the following are equivalent.

(i) The C^*-algebra A has the QTS property.
(ii) For every $\varepsilon > 0$ and $a \in A$ that satisfy $sup_{\tau \in T(A)}|\tau(a)| < \varepsilon$, there are k and $u_1, \dots, u_k \in \mathcal{U}(A)$ such that $\left\|\frac{1}{k}\sum_{i=1}^{k} u_i a u_i^*\right\| < \varepsilon$.

Proof . Ad (i) ⇒ (ii). Although the proof becomes a bit shorter if we use Theorem (6.2.4) in [317], we give here a more direct proof of this implication. Let $a \in A$ and $\varepsilon > 0$ be given as in condition (ii). Let $\varepsilon_0 = sup_{\tau \in T(A)}|\tau\,(a)| < \varepsilon$. We decompose the second dual von Neumann algebra A^{**} into the finite summand A_f^{**} and the properly infinite summand A_∞^{**}. We denote the corresponding embedding of A by π_f and π_∞, and the center-valued trace of A_f^{**} by ctr. We note that $\|ctr(\pi_f(a))\| = \varepsilon_0$. By the Dixmier approximation theorem, there are $v_1, \dots, v_k \in \mathcal{U}(A_f^{**})$ such that $\left\|ctr(\pi_f(a)) - \frac{1}{k} v_i \pi_f(a) v_i^*\right\| < \varepsilon - \varepsilon_0$. On the other hand, by Halpern's theorem [318], there are $w_1, \dots, w_l \in U(A_\infty^{**})$ such that $\left\|\frac{1}{l}\sum_{j=1}^{l} w_i \pi_\infty(a) w_i^*\right\| < \varepsilon$. Before giving the detail of the proof of this fact, we finish the proof of (i) ⇒ (ii). By allowing multiplicity, we may assume that $k = l$ and consider $u_i = v_i \oplus w_i \in A^{**}$. Then, $\left\|\frac{1}{k}\sum_{i=1}^{k} u_i a u_i^*\right\| < \varepsilon$ in A^{**}. For each i, take a net $(u_i(\lambda))_\lambda$ of unitary elements in A which converges to $u_i \in A^{**}$ in the ultrastrong $*$-topology. By the Hahn-Banach theorem, $conv\{\frac{1}{k}\sum_{i=1}^{k} u_i(\lambda) a u_i(\lambda)^*\}_\lambda$ contains an element of norm less than ε.

Now, we explain how to apply Halpern's theorem. Let Z (resp. I) be the center (resp. strong radical) of A_∞^{**}. Let Λ be the directed set of all finite partitions of unity by projections in Z, and $\lambda = \{p_{\lambda,i}\}_i \in \Lambda$ be given.

Applying the QTS property to the non-zero $*$-homomorphism $A \ni x \longmapsto p_{\lambda,i}(\pi_\infty(x) + I) \in p_{\lambda,i}((\pi_\infty(A) + I)/I)$, one obtains a (tracial) state $\tau_{\lambda,i}$ on $\pi_\infty(A) + I$ such that $\tau_{\lambda,i}(p_{\lambda,i}) = 1, \tau_{\lambda,i}(I) = 0$, and $|\tau_{\lambda,i}(\pi_\infty(a))| \le \varepsilon_0$. Let $\tilde{\tau}_{\lambda,i}$ be a state extension of it on $p_{\lambda,i}A_\infty^{**}$. We define the linear map $\varphi_\lambda: A_\infty^{**} \to Z$ by $\varphi_\lambda(x) = \sum_i \tilde{\tau}_{\lambda,i}(x) p_{\lambda,i}$, and take a limit point $\varphi: A_\infty^{**} \to Z$. The map φ is a unital positive Z-linear map such that $\varphi(I) = 0$ and $\|\varphi(\pi_\infty(a))\| \le \varepsilon_0$. By Halpern's theorem (Theorem 4.12 in [318]), the norm-closed convex hull of the unitary conjugations of $\pi_\infty(a)$ contains $\varphi(\pi_\infty(a))$.

Ad (ii) ⇒ (i). Suppose that there is a closed two-sided proper ideal I in A such that A/I does not have a tracial state. Let e_n be the approximate unit of I. Then, one has $\tau(1 - e_n) \searrow 0$ for every $\tau \in T(A)$. By Dini's theorem, there is n such that $q = 1 - e_n$ satisfies $\tau(q) < 1/2$ for all $\tau \in T(A)$. By condition (ii), there are $u_1, \dots, u_k \in \mathcal{U}(A)$ such that $\left\|\frac{1}{k}\sum_{i=1}^{k} u_i q u_i^*\right\| < 1/2$, which is in contradiction with the fact that $\frac{1}{k}\sum_{i=1}^{k} u_i q u_i^* \in 1 + I$.

Theorem (6.2.2)[454]: For a unital C^*-algebra A, the following are equivalent.
(i) The C^*-algebra A is nuclear and has the QTS property.
(ii) The C^*-algebra A has an approximate diagonal $\Delta_n = \sum_{i=1}^{k(n)} x_i(n)^* \otimes x_i(n)$ such that $\lim_n \sum_{i=1}^{k(n)} \|x_i(n)\|^2 = 1, m(\Delta_n) = 1$, and $\lim_n \left\|1 - \sum_{i=1}^{k(n)} x_i(n) x_i(n)^*\right\| = 0$.
(iii) The C^*-algebra A is symmetrically amenable.
(iv) The C^*-algebra A has a symmetric approximate diagonal $(\Delta_n)_n$ in

$$\{\sum_i x_i^* \otimes x_i \in A \otimes_{\mathbb{C}} A : \sum_i \|x_i\|^2 \le 1\}.$$

Proof. The implication (iv) ⇒ (iii) is obvious and (iii) ⇒ (i) is standard: Since amenability implies nuclearity by Connes's theorem [307] we only have to prove the QTS property. Let $(\Delta_n)_n$ be a symmetric approximate diagonal and define $m_\Delta(a) = \sum_i x_i a y_i$ for $\Delta = \sum_i x_i \otimes y_i \in A \otimes_{\mathbb{C}} A$ and $a \in A$. Then, for any proper ideal I in A and a state φ on A such that $\varphi(I) = 0$, any limit point τ of $(\varphi \circ m_{\Delta_n})_n$ is a bounded trace such that $\tau(I) = 0$ and $\tau(1) = 1$. By polar decomposition, one obtains a tracial state on A which vanishes on I.

We prove the implication (i) ⇒ (ii) ⇒ (iv). Since A is nuclear, it is amenable thanks to Haagerup's theorem (Theorem 3.1 in [318]), Moreover, there is an approximate diagonal $(\Delta'_n)_n$ in the convex hull of $\{x^* \otimes x: \|x\| \le 1\}$. We note that $\varepsilon_n := \|1 - m(\Delta'_n)\| \to 0$. We fix n for the moment and write $\Delta'_n = \sum_i x_i^* \otimes x_i$. By replacing x_i with $x_i m(\Delta'_n)^{-1/2}$, we may assume $m(\Delta'_n) = 1$ but $\sum_i \|x_i\|^2 \le (1 - \varepsilon_n)^{-1}$. Since $\tau(\sum_i x_i x_i^*) = 1$ for all $\tau \in T(A)$, Theorem (6.2.1) provides $u_1, \ldots, u_l \in \mathcal{U}(A)$ such that $\left\|\frac{1}{l}\sum_{j=1}^{l}\sum_i u_j x_i x_i^* u_j^*\right\| \le 1 + \varepsilon_n$. Thus, $\Delta_n = \frac{1}{l}\sum_{i,j} x_i^* u_j^* \otimes u_j x_i$ satisfies condition (ii). Now, rewrite Δ_n as $\sum_i y_i^* \otimes y_i$. Then, $\Delta_n^{\#} = (\sum_i \|y_i\|^2)^{-2} \sum_{i,j} y_i^* y_j \otimes y_j^* y_i$ is a symmetric approximate diagonal that meets condition (iv).

We consider the trace zero elements in a C^*-algebra. A simple application of the Hahn–Banach theorem implies that $a \in A$ satisfies $\tau(a) = 0$ for all $\tau \in T(A)$ if and only if it belongs to the norm-closure of the subspace $[A, A]$ spanned by commutators $[b, c] = bc - cb, b, c \in A$. Moreover, such a can be written as a convergent sum of commutators ([308 There are many works as to how uniformly this happens ([329,311,312,325,330] just to name a few). The following fact is rather standard.

Unlike the case for von Neumann algebras, there is no bound on k in terms of ε and $\|a\|$ that works for general C^*-algebras. A counterexample is constructed by Pedersen and Petersen (Lemma 3.5 in [329]: the element $x_n - y_n \in [A_n, A_n]$ constructed there has the property that $\|(x_n - y_n) - z\| \ge 1$ for any sum z of n self-commutators). This also means that k in Theorem (6.2.1) depends on the particular element a in A. Nevertheless one can bound k under some regularity condition. Recall that A is said to be Z-stable if $A \cong Z \otimes A$ for the Jiang–Su algebra Z. The Jiang–Su algebra Z is a simple C^*-algebra which is an inductive limit of prime dimension drop algebras and such that $Z \cong Z^{\otimes\infty}$ (Theorem (6.2.8) and Theorem (6.2.4) in [320]).

Theorem (6.2.3)[454]: There is a constant $C > 0$ which satisfies the following. Let A be a C^*-algebra and $a \in A$ and $\varepsilon > 0$ be such that $sup_{\tau\in T(A)}|\tau(a)| < \varepsilon$.

Then, there are $k \in \mathbb{N}$ and b_i and c_i in A such that $\sum_{i=1}^{k}\|b_i\|\|c_i\| \le C\|a\|$ and $\left\|a - \sum_{i=1}^{k}[b_i, c_i]\right\| < \varepsilon$.

Proof . Let $a \in A$. We denote by ctr the centervalued trace from the second dual von Neumann algebra A^{**} onto the center $Z(A_f^{**})$ of the finite summand A_f^{**} of A^{**}. One has $\|ctr(a)\| = sup_{\tau\in T(A)}|\tau(a)| < \varepsilon$ and $a' := a - ctr(a)$ has zero traces. By a theorem of Fack and de la Harpe, for $C = 2 \cdot 12^2$ and $m = 10$, there are $b_i, c_i \in A^{**}$ such that $\sum_{i=1}^{m}\|b_i\|\|c_i\| \le C\|a\|$ and $a' = \sum_{i=1}^{m}[b_i, c_i]$. See [325, 330] for a better estimate of C and m. By Kaplansky's density theorem, there is a net $(b_i(\lambda))_\lambda$ in A such that $\|b_i(\lambda)\| \le \|b_i\|$ and $b_i(\lambda) \to b_i$ ultrastrongly. Likewise for $(c_i(\lambda))_\lambda$. Since

$$\left\| \lim_{\lambda} (a - \sum_{i=1}^{m} [b_i(\lambda), c_i(\lambda)]) \right\| = \|a - a'\| < \varepsilon,$$

there is $a'' \in conv\{\sum_{i=1}^{m}[b_i(\lambda), c_i(\lambda)]\}_\lambda$ which satisfies $\|a - a''\| < \varepsilon$.

Theorem (6.2.4)[454]: There is a constant $C > 0$ which satisfies the following. Let A be an exact Z-stable C^*-algebra, and $\varepsilon > 0$ and $a \in A$ be such that $sup_{\tau\in T(A)}|\tau(a)| < \varepsilon$. Then, for every $R \in \mathbb{N}$, there are $b(r)$ and $c(r)$ in A such that $\sum_{r=1}^{R}\|b(r)\|\|c(r)\| \leq C\|a\|$ and $\|a - \sum_{r=1}^{R}[b(r), c(r)]\| < \varepsilon + C\|a\|R^{-1/2}$.

proof. is inspired by [315] and uses the free semicircular system and random matrix argument of Haagerup–Thorbjornsen ([316]). Let $\mathcal{O}_\infty$ be the Cuntz algebra generated by isometries $l_i(r)$ such that $l_i(r)^* l_j(s) = \delta_{i,j}\delta_{r,s}$, and let $S_i(r) := l_i(r) + l_i(r)^*$ be the corresponding semicircular system. We note that $\mathcal{C} := C^*(\{S_i(r): i, r\})$ is $*$-isomorphic to the reduced free product of the copies of $C([-2, 2])$ with respect to the Lebesgue measure (see [336]), and the corresponding tracial state coincides with the restriction of the vacuum state on $\mathcal{O}_\infty$ to $\mathcal{C}$.

Lemma (6.2.5)[454]: Let $b_i, c_i \in A$ be such that $\|b_i\| = \|c_i\|$. Then, for every $R \in \mathbb{N}$, letting ˜$\tilde{b}(r) = \sum_{i=1}^{n} S_i(r) \otimes b_i$ and $\tilde{c}(r) = \sum_{j=1}^{n} S_j(r) \otimes c_j$, one has

$$\frac{1}{R}\sum_{r=1}^{R}\|\tilde{b}(r)\|\|\tilde{c}(r)\| \leq 4\sum\|b_i\|\|c_i\|$$

and

$$\left\| 1 \otimes \sum_{i=1}^{n}[b_i, c_i] - \frac{1}{R}\sum_{r=1}^{R}[\tilde{b}(r), \tilde{c}(r)] \right\| \leq \frac{6}{\sqrt{R}}\sum_i \|b_i\|\|c_i\|.$$

Proof. For every r, one has

$$\|\tilde{b}(r)\| \leq \left\|\sum l_i(r) \otimes b_i\right\| + \left\|\sum l_i(r)^* \otimes b_i\right\|$$
$$= \left\|\sum b_i^* b_i\right\|^{1/2} + \left\|\sum b_i b_i^*\right\|^{1/2} \leq 2(\sum\|b_i\|^2)^{1/2},$$

and likewise for $\tilde{c}(r)$. It follows that $\|\tilde{b}(r)\|\|\tilde{c}(r)\| \leq 4\sum\|b_i\|\|c_i\|$. Moreover,

$$\tilde{b}(r)\tilde{c}(r) = \sum_{i,j}(\delta_{i,j}1 + l_i(r)l_j(r) + l_i(r)l_j(r)^* + l_i(r)^* l_j(r)^*) \otimes b_i c_j,$$

and

$$\left\|\sum_{r,i,j} l_i(r)l_j(r) \otimes b_i c_j\right\| = \left\|\sum_{r,i,j} c_j^* b_i^* b_i c_j\right\|^{1/2} \leq R^{1/2}\sum_i \|b_i\|\|c_i\|,$$

$$\left\|\sum_{r,i,j} l_i(r)^* l_j(r)^* \otimes b_i c_j\right\| = \left\|\sum_{r,i,j} b_i c_j c_j^* b_i^*\right\|^{1/2} \leq R^{1/2}\sum_i \|b_i\|\|c_i\|,$$

$$\left\|\sum_{r,i,j} l_i(r)l_j(r)^* \otimes b_i c_j\right\| = \max_r \left\|\sum_{i,j} l_i(r)l_j(r)^* \otimes b_i c_j\right\| \leq \sum_i \|b_i\|\|c_i\|.$$

Likewise for $\tilde{c}(r)\tilde{b}(r)$, and one obtains the conclusion.

Proof. Let $a \in A \setminus \{0\}$ be such that $sup_{\tau\in T(A)}|\tau(a)| < \varepsilon$. Since $Z \cong Z^{\otimes\infty}$, we may assume that $A = Z \otimes A_0$ and $a \in A_0$. By Theorem (6.2.4), there are b_i, c_i such that $\|b_i\| = \|c_i\|, \sum_{i=1}^{k}\|b_i\|\|c_i\| \le C\|a\|$, and $\left\|a - \sum_{i=1}^{k}[b_i, c_i]\right\| < \varepsilon$. Recall the theorem of Haagerup and Thorbjornsen ([316]) which states that the C^*-algebra C can be embedded into $\prod \mathbb{M}_n / \oplus \mathbb{M}_n$. By exactness of A_0, there is a canonical $*$-isomorphism

$$(\prod \mathbb{M}_n / \bigoplus \mathbb{M}_n) \otimes A_0 \cong ((\prod \mathbb{M}_n) \otimes A_0)/(\bigoplus \mathbb{M}_n \otimes A_0).$$

Lemma (6.2.5), combined with this fact, implies that there are matrices $s_i^{(n)}(r) \in \mathbb{M}_n$ such that $\tilde{b}^{(n)}(r) = \sum_{i=1}^{k} s_i^{(n)}(r) \otimes b_i$ and $\tilde{c}^{(n)}(r) = \sum_{j=1}^{k} s_j^{(n)}(r) \otimes c_j$ satisfy

$$\limsup_n \frac{1}{R}\sum_{r=1}^{R}\left\|\tilde{b}^{(n)}(r)\right\|\left\|\tilde{c}^{(n)}(r)\right\| \le 4\sum\|b_i\|\|c_i\| \le 4C\|a\|$$

and

$$\limsup_n \left\|1 \otimes a - \frac{1}{R}\sum_{r=1}^{R}[\tilde{b}^{(n)}(r), \tilde{c}^{(n)}(r)]\right\| \le \varepsilon + \frac{6C\|a\|}{\sqrt{R}}.$$

For every relatively prime $p, q \in \mathbb{N}$, the Jiang–Su algebra Z contains the prime dimension drop algebra

$$I(p,q) = \{f \in C([0,1], \mathbb{M}_p \otimes \mathbb{M}_q) : f(0) \in \mathbb{M}_p \otimes \mathbb{C}1, f(1) \in \mathbb{C}1 \otimes \mathbb{M}_q\}$$

and hence $t\mathbb{M}_q$ and $(1-t)\mathbb{M}_p$ also, where $t \in I(p,q)$ is the identity function on $[0,1]$. It follows that there are $b(r), c(r), b'(r), c'(r) \in Z \otimes A_0$ such that

$$\frac{1}{R}\sum_{r=1}^{R}(\|b(r)\|\|c(r)\| + \|b'(r)\|\|c'(r)\|) < 9C\|a\|$$

and

$$\left\|a - \frac{1}{R}\sum_{r=1}^{R}([b(r), c(r)] + [b'(r), c'(r)])\right\| < \varepsilon + \frac{7C\|a\|}{\sqrt{R}}.$$

Here, we note that $\|t \otimes x + (1-t) \otimes y\| = \max\{\|x\|, \|y\|\}$ for any x and y.

Let $(A_n)_n$ be a sequence of C^*-algebras and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We denote by

$$\prod A_n = \{(a_n)_{n=1}^{\infty} : a_n \in A_n, \sup_n\|a_n\| < +\infty\}$$

the ℓ_∞-direct sum of (A_n), and by

$$\prod A_n/\mathcal{U} = (\prod A_n)/\{(a_n)_{n=1}^{\infty} : \lim_{\mathcal{U}}\|a_n\| = 0\}$$

the ultraproduct of A_n. For every m, we view $\tau \in T(A_m)$ as an element of $T(\prod A_n)$ by $\tau((a_n)_n) = \tau(a_m)$. For each $(\tau_n)_n \in \prod T(A_n)$, there is a corresponding tracial state $\tau_{\mathcal{U}} := \lim_{\mathcal{U}} \tau_n$ on $\prod A_n/\mathcal{U}$, defined by

$$\tau_{\mathcal{U}}((a_n)_n) = \lim_{\mathcal{U}} \tau_n(a_n).$$

The set of tracial states that arise in this way is denoted by $\prod T(A_n)/\mathcal{U}$. We note that as soon as $\partial T(\prod A_n/\mathcal{U})$ is infinite, the inclusion $T(\prod A_n)/\mathcal{U} \subset T(\prod A_n/\mathcal{U})$ is proper (see [306]). Moreover, if we take A_n to be the counterexamples of Pedersen and Petersen [329], then $\prod T(A_n)/\mathcal{U}$ (resp. $conv \coprod T(A_n)$) is not weak$*$-dense in $T(\prod A_n/\mathcal{U})$ (resp. $T(\prod A_n)$).

The following theorem is proved by Sato [332] (see also [331]) in the case where A is a simple nuclear C^*-algebra having finitely many extremal tracial states.

Theorem (6.2.6)[454]: Let $(A_n)_n$ be a sequence of exact Z-stable C^*-algebras and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then, $\prod T(A_n)/\mathcal{U}$ (resp. $conv \coprod T(A_n)$) is weak∗-dense in $T(\prod A_n/\mathcal{U})$ (resp. $T(\prod A_n)$). In particular, for every $\tau \in T(\prod A_n/\mathcal{U})$ and every separable C^*-subalgebra $B \subset \prod A_n/\mathcal{U}$, there is $\tau' \in \prod T(A_n)/\mathcal{U}$ such that $\tau|_B = \tau'|_B$.

Proof . Let A be either $\prod A_n$ or $\prod A_n/\mathcal{U}$, and denote by $\Sigma \subset T(A)$ either $conv(\coprod T(A_n))$ or $\prod T(A_n)/\mathcal{U}$ accordingly. Suppose that the conclusion of the theorem is false for $\Sigma \subset T(A)$. Then, by the Hahn–Banach theorem, there are τ in $T(A)$ and a self-adjoint element a_0 in A such that $\gamma := \tau(a_0) - sup_{\sigma\in\Sigma}\, \sigma(a_0) > 0$. Let $\alpha = (|\, inf_{\sigma\in\Sigma}\sigma(a_0)| - \tau(a_0)) \vee 0$, and take $b \in A_+$ such that $\tau(b) = \alpha$ and $\|b\| < \alpha + \gamma$. It follows that $a = a_0 + b$ satisfies $sup_{\sigma\in\Sigma}|\sigma(a)| < \tau(a)$. Now, expand $a \in A$ as $(a_n)_n$. We may assume that $\|a_n\| \le \|a\|$ for all n. Let $I \in \mathcal{U}$ (or $I = \mathbb{N}$ in case $A = \prod A_n$) be such that $\varepsilon_0 := sup_{n\in I}\, sup_{\sigma\in T(A_n)}\, \sigma(a_n) < \tau(a)$. Let $R \in \mathbb{N}$ be such that $\varepsilon_1 := \varepsilon_0 + C\|a\|R^{-1/2} < \tau(a)$. Then, by Theorem (6.2.5), for each $n \in I$ there are $b_n(r), c_n(r) \in A_n$ such that $\sum_{r=1}^{R}\|b_n(r)\|\|c_n(r)\| \le C\|a\|$ and $\|a_n - \sum_{r=1}^{R}[b_n(r), c_n(r)]\| \le \varepsilon_1$. It follows that for $b(r) = (b_n(r))_n$ and $c(r) = (c_n(r))_n \in A$, one has

$$\tau(a) = \tau\left(a - \sum_{r=1}^{R}[b(r), c(r)]\right) \le \left\| a - \sum_{r=1}^{R}[b(r), c(r)] \right\| < \tau(a),$$

which is a contradiction. This proves the first half of the theorem.

For the second half, let τ and B be given. Take a dense sequence $(x(i))_{i=0}^{\infty}$ in B and expand them as $x(i) = (x_n(i))_n$. By the first half, for every m, there is $(\tau_n^{(m)})_n \in \prod T(A_n)$ such that $|\tau(x(i)) - \tau_{\mathcal{U}}^{(m)}(x(i))| < m^{-1}$ for $i = 0, \dots, m$. Let

$$I_m = \{n \in \mathbb{N} : |\tau(x(i)) - \tau_n^{(m)}(x_n(i))| < m^{-1} \text{ for all } i = 0, \dots, m\} \in \mathcal{U}$$

(so $I_0 = \mathbb{N}$), and $J_m = \bigcap_{l=0}^{m} I_l \in \mathcal{U}$. We define τ_n to be $\tau_n^{(m)}$ for $n \in J_m \setminus J_{m+1}$. It is not too hard to check $\tau = \tau_{\mathcal{U}}$ on B.

In passing, we record the following fact.

Lemma (6.2.7)[454]: Let A be a (non-separable) C^*-algebra and $X \subset A$ be a separable subset. Then there is a separable C^*-subalgebra $B \subset A$ that contains X and such that the restriction from $T(A)$ to $T(B)$ is onto.

Proof. We may assume that A is unital. We first claim that for every $x_1, \dots, x_n \in A$ and $\varepsilon > 0$, there is a separable C^*-subalgebra C which satisfies the following property: for every $\tau \in T(C)$ there is $\sigma \in T(A)$ such that $\max_i |\tau(x_i) - \sigma(x_i)| < \varepsilon$. Indeed if this were not true, then for every C there is $\tau_C \in T(C)$ such that $\max_i |\tau_C(x_i) - \sigma(x_i)| \ge \varepsilon$ for all $\sigma \in T(A)$. The set of separable C^*-subalgebras of A is upward directed and one can find a limit point τ of $\{\tau_C\}$. Then, we arrive at a contradiction that $\tau \in T(A)$ satisfies $\max_i |\tau(x_i) - \sigma(x_i)| \ge \varepsilon$ for all $\sigma \in T(A)$. We next claim that for every separable C^*-subalgebra $B_0 \subset A$, there is a separable C^*-subalgebra $B_1 \subset A$ that contains B_0 and such that $Res_{B_0}T(B_1) = Res_{B_0}T(A)$ in $T(B_0)$, where Res is the restriction map. Take a dense sequence $x_1, x_2, \dots$ in B_0, and let $C_0 = B_0$. By the previous discussion, there is an increasing sequence of separable C^*-subalgebras $C_0 \subset C_1 \subset \cdots$ such that for every $\tau \in T(C_n)$ there is $\sigma \in T(A)$ satisfying $|\tau(x_i) - \sigma(x_i)| < n^{-1}$ for $i = 1, \dots, n$. Now, letting $B_1 = \overline{\bigcup_n C_n}$ and we are done. Finally,

we iterate this construction and obtain $X \subset B_0 \subset B_1 \subset \cdots$ such that $Res_{B_n}T(B_{n+1}) = Res_{B_n}T(A)$. The separable C^*-subalgebra $B = \overline{\bigcup B_n}$ satisfies the desired property.
Murphy [327] presents a non-separable example of a unital non-simple C^*-algebra with a unique faithful tracial state and asks whether a separable example of such exists. The above lemma answers it. There is another example, which is moreover nuclear. Kirchberg [323] proves that the Cuntz algebra $\mathcal{O}_\infty$ (or any other unital separable exact C^*-algebra) is a subquotient of the CAR algebra $\mathbb{M}_{2^\infty}$. Namely, there are C^*-subalgebras J and B in $\mathbb{M}_{2^\infty}$ such that J is hereditary in $\mathbb{M}_{2^\infty}$ and is an ideal in B such that $B/J = \mathcal{O}_\infty$. It follows that B is a unital separable nuclear non-simple C^*-algebra with a unique faithful tracial state.

Recall $S \subset T(A), N = (\bigoplus_{\tau\in S} \pi_\tau)(A)''$, and the center-valued trace $ctr: N \to Z(N)$. Since S is a closed face of $T(A)$, any normal tracial state on N restricts to a tracial state on A which belongs to S. Hence, one has

$$\|a\|_{2,S} = \sup\{\|a\|_{2,\tau}: \tau \in S\} = \sup\{\|a\|_{2,\tau}: \tau \in \partial S\} = \|ctr(a^*a)\|^{1/2}.$$

Since S is a metrizable closed face of the Choquet simplex $T(A)$, it is also a Choquet simplex and there is a canonical one-to-one correspondence

$$S \ni \tau \leftrightarrow \mu_\tau \in M_+^1(\partial S), \qquad \tau(a) = \int \lambda(a)\, d\mu_\tau(\lambda) \ for\ a \in A.$$

By uniqueness of the representing measure μ_τ , this correspondence is an affine transformation and extends uniquely to a linear order isomorphism between their linear spans.

Lemma (6.2.8)[454]: For every $\tau \in S$, there is a normal $*$-isomorphism $\theta_\tau: L^\infty(\partial S, \mu_\tau) \to Z(\pi_\tau(A)'')$ such that

$$\tau(\theta_\tau(f)a) = \int f(\lambda)\lambda(a)\, d\mu_\tau\,(\lambda)$$

for $a \in A$.

Proof. Let $f \in L^\infty(\partial S, \mu_\tau)$ be given. The right hand side of the claimed equality defines a tracial linear functional on A whose modulus is dominated by a scalar multiple of τ. Hence, by Sakai's Radon–Nikodym theorem, there is a unique $\theta_\tau(f) \in Z(\pi_\tau(N))$ that satisfies the claimed equality. This defines a unital normal positive map θ_τ from $L^\infty(\partial S, \mu_\tau)$ into $Z(\pi_\tau(A)'')$. Next, let $z \in Z(\pi_\tau(N))_+$ be given. Then, the tracial linear functional $z\tau$ on A defined by $(z\tau)(a) = \tau\,(az)$ is dominated by $\|z\|\tau$. Hence one has $\mu_{z\tau} \le \|z\|\mu_\tau$ and $z = \theta_\tau(d\mu_{z\tau}/d\mu_\tau)$ with $d\mu_{z\tau}/d\mu_\tau \in L^\infty(\partial S, \mu_\tau)$.
This proves θ_τ is a positive linear isomorphism such that $\mu_{\theta_\tau(f)\tau} = f\mu_\tau$.
Therefore, one has $\mu_{\theta\tau\,(fg)\tau} = fg\mu_\tau = f\mu_{\theta_\tau(g)\tau} = \mu_{\theta_\tau(f)\theta_\tau(g)\tau}$, which proves $\theta_\tau(fg) = \theta_\tau(f)\theta_\tau(g)$.

Theorem (6.2.9)[454]: Let A, S, and N be as above. Then, there is a unital $*$-homomorphism $\theta: B(\partial S) \to Z(N)$ with ultraweakly dense range such that $\theta(\hat{a}) = ctr(a)$ and

$$\tau\,(\theta(f)a) = \int f(\lambda)\lambda(a)\, d\mu_\tau(\lambda) = \int f\hat{a} d\mu_\tau$$

for every $a \in A$ and $\tau \in S$. One has

$$\bar{A}^{st} = \{x \in N: ctr(xA) \subset \theta(Aff(S)), ctr(x^*x) \in \theta(Aff(S))\}.$$

In particular,

$$\bar{A}^{st} \cap Z(N) = \{\theta(f): f \in Z(Aff(S))\}.$$

Proof. We first find the $*$-homomorphism $\theta: B(\partial S) \to Z(N)$ that satisfies

$$\tau(\theta(f)a) = \int f(\lambda)\lambda(a)\, d\mu_\tau(\lambda)$$

for every $a \in A$ and $\tau \in S$, or equivalently, $\pi_\tau(\theta(f)) = \theta_\tau(f)$ in $\pi_\tau(A)''$. For this, it suffices to show that the maps $\theta_\tau|_{B(\partial S)}$, given in Lemma (6.2.9), are compatible over $\tau \in S$. We recall that associated with the representation π_τ, there is a unique central projection $p_\tau \in Z(N)$ such that $(1 - p_\tau)N = ker\pi_\tau$. Since $p_\tau \vee p_\sigma = p_{(\tau+\sigma)/2}$, the family $\{p_\tau : \tau \in S\}$ is upward directed and $sup_\tau p_\tau = 1$. We will show that if τ and σ are such that $\tau \le C\sigma$ for some $C > 1$, then $\theta_\tau(f) = p_\tau \theta_\sigma(f)$ in $Z(N)$. We note that p_τ is the support projection of $d\tau/d\sigma \in Z(N)$. For every $f \in B(\partial S)$, one has

$$\begin{aligned}\sigma(\theta_\sigma(\frac{d\mu_\tau}{d\mu_\sigma} f)a) &= \int (\frac{d\mu_\tau}{d\mu_\sigma} f)(\lambda)\lambda(a)\, d\mu_\sigma(\lambda)\\ &= \int f(\lambda)\lambda(a) d\mu_\tau(\lambda)\\ &= \tau(\theta_\tau(f)a)\\ &= \sigma(\frac{d\tau}{d\sigma}\theta_\tau\,(f)a).\end{aligned}$$

This implies $\theta_\sigma(\frac{d\mu_\tau}{d\mu_\sigma} f) = \frac{d\tau}{d\sigma}\theta_\tau(f)$ for every f. In particular, $\theta_\sigma(\frac{d\mu_\tau}{d\mu_\sigma}) = \frac{d\tau}{d\sigma}$ and $p_\tau\theta_\sigma(f) = \theta_\sigma(f)$ in $Z(N)$. Therefore, we may glue $\{\theta_\tau\}_{\tau\in S}$ together and obtain a globally defined $*$-homomorphism $\theta: B(\partial S) \to Z(N)$. Since $\tau(\theta(\hat{a})) = \int \hat{a}(\lambda)\, d\mu_\tau(\lambda) = \tau\,(a)$ for every $\tau \in S$, one has $\theta(\hat{a}) = ctr(a)$ for every $a \in A$. This proves the first part of the theorem.
For the second part, it suffices to prove

$$\bar{A}^{st} \supset \{x \in N: ctr(xA) \subset Aff(\partial S), ctr(x^*x) \in Aff(\partial S)\},$$

as the converse inclusion is trivial. Take x from the set in the right hand side. We will prove a stronger assertion that if a net $(b_j)_j$ in A converges to x ultrastrongly in N, then x is contained in the strict closure of the convex hull of $\{b_j : j\}$. We note that $Aff(S) \ni f \longmapsto f|_{\partial S} \in Aff(\partial S)$ is an affine order isomorphism and that every positive norm-one linear functional μ on $Aff(S)$ is given by the evaluation at a point $\tau_\mu \in S$. (Indeed by the Hahn–Banach theorem, we may regard μ as a state on $C(S)$, which is a probability measure on S by the Riesz–Markov theorem. The point $\tau_\mu = \int \lambda d\mu(\lambda)$ satisfies $f(\tau_\mu) = \mu(f)$ for $f \in Aff(S)$.) Thus, one has $ctr((b_j - x)^*(b_j - x)) \to 0$ weakly in $Aff(\partial S)$. Therefore, by the Hahn–Banach theorem, for every $\varepsilon > 0$ there is a finite sequence $\alpha_j \ge 0, \sum \alpha_j = 1$ such that $\left\|\sum_j \alpha_j\, ctr((b_j - x)^*(b_j - x))\right\| < \varepsilon$. By reindexing, we assume $j = 1, \ldots, k$. Let $b = \sum \alpha_j b_j$. We note that

$$b = \begin{bmatrix}\alpha_1^{1/2} & \cdots & \alpha_m^{1/2}\end{bmatrix}\begin{bmatrix}\alpha_1^{1/2} b_1\\ \vdots\\ \alpha_m^{1/2} b_m\end{bmatrix} =: rc.$$

Hence, $b^*b = c^*r^*rc \le \|r\|^2 c^*c = \sum \alpha_j b_j^* b_j$. It follows that

$$\begin{aligned}ctr((b - x)^*(b - x)) &= ctr(b^*b - b^*x - x^*b + x^*x)\\ &\le ctr(\textstyle\sum \alpha_j b_j^* b_j - \sum \alpha_j b_j^* x - x^* \sum \alpha_j b_j + x^*x)\end{aligned}$$

$$= ctr(\sum \alpha_j (b_j - x)^*(b_j - x))$$
$$< \varepsilon.$$

This proves the claimed inclusion. The last assertion will be proved in more general setting as Theorem (6.2.10).

Let K be a metrizable compact Hausdorff topological space. We call M a (tracial) continuous W^*-bundle over K if the following axiom hold:

(a) There is a unital positive faithful tracial map $E: M \to C(K)$.

(b) The closed unit ball of M is complete with respect to the uniform 2-norm
$$\|x\|_{2,u} = \left\|E(x^*x)^{1/2}\right\|.$$

(c) $C(K)$ is contained in the center of M and E is a conditional expectation.

In case M satisfies only conditions (a) and (b), we say it is a continuous quasi-W^*-bundle. If we denote by π_E the GNS representation of M on the Hilbert $C(K)$-module $L^2(M, E)$, condition (b) is equivalent to that $\pi_E(M)$ is strictly closed in $\mathbb{B}(L^2(M, E))$. For each point $\lambda \in K$, we denote by π_λ the GNS representation for the tracial state $\tau_\lambda := ev_\lambda \circ E$, and also $\|x\|_{2,\lambda} = \tau_\lambda(x^*x)^{1/2}$. We call each $\pi_\lambda(M)$ a fiber of M. A caveat is in order: the system $(M, K, \pi_\lambda(M))$ need not be a continuous C^*-bundle because $ker\, \pi_\lambda$ may not coincide with $C_0(K \setminus \{\lambda\})M$ rather it coincides with the strict closure of that. In particular, for $x \in M$, the map $\lambda \longmapsto \|\pi_\lambda(x)\|$ need not be upper semi-continuous (but it is lower semi-continuous). A continuous quasi-W^*-bundle over S, and by Theorem (6.2.3), it is a continuous W^*-bundle over ∂S if ∂S is closed in S. Conversely, if each fiber $\pi_\lambda(M)$ is a factor, then K can be viewed as a closed subset of the extreme boundary of $T(M)$ and hence the closed convex hull S of K is a metrizable closed face of $T(M)$ such that $\partial S = K$.

Theorem (6.2.10)[454]: Let M be a continuous W^*-bundle over K. Then, $\pi_\lambda(M) = \pi_\lambda(M)''$ for every $\lambda \in K$. Moreover, if a bounded function $f: K \ni \lambda \longmapsto f(\lambda) \in \pi_\lambda(M)$ is continuous in the following sense: for every $\lambda_0 \in K$ and $\varepsilon > 0$, there are a neighborhood O of λ_0 and $c \in M$ such that
$$sup_{\lambda \in O} \|\pi_\lambda(c) - f(\lambda)\|_{2,\lambda} < \varepsilon;$$
then there is $a \in M$ such that $\pi_\lambda(a) = f(\lambda)$.

Proof. Let $\lambda \in K$ be given. By Pedersen's up-down theorem (Theorem (2.4.4) in [328]), it suffices to show that $\pi_\lambda(M)$ is closed in $\pi_\lambda(M)''$ under monotone sequential limits. Let $(x_n)_{n=0}^\infty$ be an increasing sequence in $\pi_\lambda(M)_+$ such that $x_n \nearrow x$ in $\pi_\lambda(M)''$. We may assume that $\|x_n - x\|_{2,\lambda} < 2^{-n}$. We lift $(x_n)_{n=0}^\infty$ to an increasing sequence $(a_n)_{n=0}^\infty$ in M such that $a_n \le \|x\| + 1$. Let $b_n = a_n - a_{n-1}$ for $n \ge 1$. Since $\tau_\lambda(b_n^* b_n) < 4^{-n+2}$, there is $f_n \in C(K)_+$ such that $0 \le f_n \le 1, f_n(\lambda) = 1$, and $E(b_n^* b_n) f_n^2 \le 4^{-n+2}$. It follows that the series $a_0 + \sum_{n=1}^\infty b_n f_n$ is convergent in the uniform 2-norm. Moreover, since $a_0 + \sum_{k=1}^n b_k f_k \le a_0 + \sum_{k=1}^n b_k = a_n \le \|x\| + 1$, the series is norm bounded. Therefore, the series converges in M, by the completeness of the closed unit ball of M. The limit point a satisfies $\pi_\lambda(a) = x$.

We prove the second half. Let us fix n for a while. For each λ, there is $b_\lambda \in M$ such that $\|b_\lambda\| \le \|f(\lambda)\|$ and $\pi_\lambda(b_\lambda) = f(\lambda)$. By continuity, there is a neighborhood O_λ of λ such that $\|\pi_\tau(b_\lambda) - f(\tau)\|_{2,\tau} < n^{-1}$ for $\tau \in O_\lambda$. Since K is compact, it is covered by a finite family $\{O_{\lambda_i}\}$. Let $g_i \in C(K) \subset Z(M)$ be a partition of unity subordinated by it. Then, $a_n := \sum_i g_i b_{\lambda_i} \in M$ satisfies $\|a_n\| \le \|f\|_\infty$ and $sup_\tau \|\pi_\tau(a_n) - f(\tau)\|_{2,\tau} < n^{-1}$.

It follows that (a_n) is a norm bounded and Cauchy in the uniform 2-norm. Hence it converges to $a \in M$ such that $\pi_\lambda(a) = f(\lambda)$ for every $\lambda \in K$.

The following is a W^*-analogue of the result for C^*-algebras in [299], and is essentially the same as Proposition (7.7) in [324].

Corollary (6.2.11)[454]: Let M be a continuous W^*-bundle over K. Assume that each fiber $\pi_\lambda(M)$ has the McDuff property and that K has finite covering dimension. Then, for every k, there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into M, namely a net of unital completely positive maps $\varphi_n: \mathbb{M}_n \to M$ such that $\limsup_n \|\varphi_n(xy) - \varphi_n(x)\varphi_n(y)\|_{2,u} = 0$ and $\limsup_n \|[\varphi_n(x), a]\|_{2,u} = 0$ for every $x, y \in \mathbb{M}_k$ and $a \in M$.

Proof. The proof is particularly easy when K is zero-dimensional: Since $\pi_\lambda(M)$ is McDuff, there is an approximately central embedding of $\mathbb{M}_k$ into $\pi_\lambda(M)$. We lift it to a unital completely positive map $\psi_\lambda: \mathbb{M}_k \to M$. It is almost multiplicative on a neighborhood O_λ of λ. Since K is compact and zero-dimensional, there is a partition of K into finitely many clopen subsets $\{V_i\}$ such that $V_i \subset O_{\lambda_i}$. By Theorem (6.2.10), one can define $\varphi: \mathbb{M}_k \to M$ by the relation $\pi_\lambda \circ \varphi = \pi_\lambda \circ \psi_{\lambda_i}$ for $\lambda \in V_i$. The case $0 < dim K < +\infty$ is more complicated but follows from a standard argument involving orderzero maps. See [324] (or [332, 335]).

Every separable hyperfinite von Neumann algebra with a faithful normal tracial state has a trace preserving embedding into the separable hyperfinite II$_1$ factor R. We consider coordinatization of such embeddings for strictly separable fiberwise hyperfinite continuous quasi-W^*-bundle. We define the C^*-algebra $C_\sigma(K, R)$ to be the subalgebra of $\ell_\infty(K, R)$ which consists of those norm-bounded functions $f: K \to R$ that are continuous from K into $L^2(R, \tau_R)$.

Recall the fact that if (A, τ) is a separable hyperfinite von Neumann algebra with a distinguished tracial state, then a trace-preserving embedding of A into the tracial ultrapower R^ω of the hyperfinite II$_1$ factor is unique up to unitary conjugacy (see [322]). For every n-tuples $x_1, \dots, x_n \in P$ and $y_1, \dots, y_n \in Q$ in hyperfinite II$_1$ factors P and Q, we define

$$d(\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n) = \inf_{\pi,\rho} \max_i \|\pi(x_i) - \rho(y_i)\|_2 ,$$

where the infimum runs over all trace-preserving embeddings of P and Q into R^ω. Then, d is a pseudo-metric and it depends on $(W^*(\{x_1, \dots, x_n\}), \tau)$, i.e., the joint distribution of $\{x_1, \dots, x_n\}$ with respect to τ_P, rather than the specific embedding of $W^*(\{x_1, \dots, x_n\})$ into P. Once $*$-isomorphisms $P \cong Q \cong R$ are fixed, P and Q are embedded into R^ω as constant sequences and

$$d(\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n) = \inf_{U \in \mathcal{U}(R^\omega)} \max_i \|Ad_U(x_i) - y_i\|_2 .$$

It follows that

$$d(\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n) = \inf_{\pi} \max_i \|\pi(x_i) - y_i\|_2 ,$$

where infimum runs over all trace-preserving $*$-homomorphisms π from $W^*(\{x_1, \dots, x_n\})$ into Q, or over all $*$-isomorphisms π from P onto Q. If M is a continuous quasi-W^*-bundle, then for every $a_1, \dots, a_{□} \in M$, the map

$$K \ni \lambda \longmapsto \{\pi_\lambda(a_i)\}_{i=1}^n$$

is continuous with respect to d.

Lemma (6.2.12)[454]: Let $N = C_\sigma(K, R)$ or any other continuous W^*-bundle over K such that $ev_\lambda(N) \cong R$ for every $\lambda \in K$ and such that for every $k \in \mathbb{N}$ there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into N. Let M be a continuous quasi-W^*-bundle over K such that $\pi_\lambda(M)''$ is hyperfinite for every $\lambda \in K$, and let $F_0 \subset F_1$ be finite subsets in the unit ball of M and $\varepsilon > 0$. Assume that there is a map θ_0 from F_0 into the unit ball of N such that

$$\sup_{\lambda\in K} d(\{\pi_\lambda(a)\}_{a\in F_0}, \{ev_\lambda(\theta_0(a))\}_{a\in F_0}) < \varepsilon.$$

Then, for every $\delta > 0$, there is a map θ_1 from F_1 into the unit ball of N such that

$$\sup_{\lambda\in K} d(\{\pi_\lambda(a)\}_{a\in F_1}, \{ev_\lambda(\theta_1(a))\}_{a\in F_1}) < \delta$$

and

$$\max_{a\in F_0}\|\theta_1(a) - \theta_0(a)\|_{2,u} < \varepsilon.$$

Here the symbol ev_λ, instead of π_λ, is used for the N side to make a distinction from the M side.

Proof. For each λ, there is a trace-preserving embedding $\rho_\lambda: \pi_\lambda(M) \to ev_\lambda(N)$. By the remarks preceding this lemma, we may assume that

$$\max_{a\in F_0}\|\rho_\lambda(\pi_\lambda(a)) - ev_\lambda(\theta_0(a))\|_2 < \varepsilon.$$

For each $a \in F_1$, we lift $(\rho_\lambda \circ \pi_\lambda)(a) \in ev_\lambda(N)$ to $a^\lambda \in N$ with $\|a^\lambda\| \le 1$.
There is a neighborhood O_λ of λ such that $\tau \in O_\lambda$ implies

$$d(\{\pi_\tau(a)\}_{a\in F_1}, \{ev_\tau(a^\lambda)\}_{a\in F_1}) < \delta$$

and

$$\max_{a\in F_0}\left\|ev_\tau(a^\lambda) - ev_\tau(\theta_0(a))\right\|_2 < \varepsilon.$$

By compactness, K is covered by a finite family $\{O_{\lambda_j}\}$. Take a partition of unity $g_j \in C(K)$ subordinated by $\{O_{\lambda_j}\}$. Let $h_0 = 0$ and $h_j = \sum_{i=1}^{j} g_i$. For each k, take an approximately central approximately multiplicative embedding $\varphi_{k,n}$ of $\mathbb{M}_k$ into N. Since the closed unit ball of $\mathbb{M}_k$ is norm-compact, one has

$$\forall a \in N \ \limsup_n \ \sup\{\left\|[\varphi_{k,n}(x), a]\right\|_{2,u} : x \in \mathbb{M}_k, \|x\| \le 1\} = 0.$$

For $t \in [0,1]$, we define $p_t \in \mathbb{M}_k$ to be $\mathrm{diag}(1,\dots,1, t - \lfloor t\rfloor, 0,\dots,0)$, with $1s$ in the first $\lfloor t\rfloor$ diagonal entries, $t - \lfloor t\rfloor$ in the $(\lfloor t\rfloor + 1)$-th entry, and $0s$
in the rest. It follows that $t \longmapsto p_t$ is continuous, $0 \le p_t \le 1, tr(p_t) = t$, and $\tau(p_t - p_t^2) \le (4k)^{-1}$. We write $p_{[s,t]} = p_t - p_s$. With the help of Theorem (6.2.10), we define $f_{k,n,j} \in N$ to be the element such that

$$ev_\lambda(f_{k,n,j}) = ev_\lambda(\varphi_{k,n}(p_{[h_{j-1}(\lambda), h_j(\lambda)]})).$$

For $a \in F_1$, we define $\theta_1^{k,n}(a) \in N$ by $\theta_1^{k,n}(a) = \sum_j f_{k,n,j}^{1/2} a^{\lambda_j} f_{k,n,j}^{1/2}$. Since $F_1' := F_1 \cup \{a^{\lambda_j} : a \in F_{1,j}\}$ is finite, it is not too hard to see

$$\limsup_k \limsup_n \max_{a\in F_0}\left\|\theta_1^{k,n}(a) - \theta_0(a)\right\|_{2,u} < \varepsilon.$$

It remains to estimate

$$d(\{\pi_\tau(a)\}_{a\in F_1}, \{ev_\tau(\theta_1^{k,n}(a))\}_{a\in F_1}).$$

Let k be fixed for the moment. Since $(\varphi_{k,n})_n$ is approximately multiplicative, there are unital $*$-homomorphisms $\psi_{k,n}^\tau : \mathbb{M}_k \to ev_\tau(N)$ such that

$$\limsup_n \sup_\tau \sup_{x\in\mathbb{M}_k, \|x\|\le 1}\left\|ev_\tau \circ \varphi_{k,n}(x) - \psi_{k,n}^\tau(x)\right\|_2 = 0.$$

Let $E_{k,n}^{\tau}$ be the trace-preserving conditional expectation from $ev_{\tau}(N)$ onto the relative commutant $\psi_{k,n}^{\tau}(\mathbb{M}_k)' \cap ev_{\tau}(N)$, which is given by $E_{k,n}^{\tau}(b) = |G|^{-1} \sum_{u\in G} \psi_{k,n}^{\tau}(u) b \psi_{k,n}^{\tau}(u)^*$ for the group G of permutation matrices in $\mathcal{U}(\mathbb{M}_k)$. It follows that

$$\lim_n \sup \sup_{\tau} \left\| ev_{\tau}(b) - E_{k,n}^{\tau}(ev_{\tau}(b)) \right\|_2 = 0$$

for every $b \in N$. This implies

$$\lim_n \sup \sup_{j,\tau\in O_{\lambda_j}} d(\{\pi_{\tau}(a)\}_{a\in F_1}, \{E_{k,n}^{\tau}(ev_{\tau}(a^{\lambda_j}))\}_{a\in F_1}) < \delta,$$

$$\lim_n \sup \sup_{j,\tau\in O_{\lambda_j}} d(\{ev_{\tau}(a^{\lambda_j})\}_{a\in F_1}, \{E_{k,n}^{\tau}(ev_{\tau}(a^{\lambda_j}))\}_{a\in F_1}) = 0,$$

and

$$\lim_n \sup \sup_{j,\tau\in O_{\lambda_j}} d(\{ev_{\tau}(\theta_1^{k,n}(a))\}_{a\in F_1},$$
$$\{\sum_j \psi_{k,n}^{\tau}(p_{[h_{j-1}(\lambda),h_j(\lambda)]}) E_{k,n}^{\tau}(ev_{\tau}(a^{\lambda_j}))\}_{a\in F_1}) = 0.$$

If we view $ev_{\tau}(N) = \mathbb{M}_k(\psi_{k,n}^{\tau}(\mathbb{M}_k)' \cap ev_{\tau}(N))$, then $a' = E_{k,n}^{\tau}(ev_{\tau}(a))$ looks like $diag(a', a', \ldots, a')$, and $\psi_{k,n}^{\tau}(p_t)$ looks like $diag(1, \ldots, 1, t - \lfloor t \rfloor, 0 \ldots, 0)$. Hence, one has

$$\sup_{\tau} d(\{\pi_{\tau}(a)\}_{a\in F_1}, \{\sum_j \psi_{k,n}^{\tau}(p_{[h_{j-1}(\lambda),h_j(\lambda)]}) E_{k,n}^{\tau}(ev_{\tau}(a^{\lambda_j}))\}_{a\in F_1})^2$$
$$< \frac{2\left|\left\{O_{\lambda_j}\right\}\right|}{k} + \sum_j g_j(\tau) d(\{\pi_{\tau}(a)\}_{a\in F_1}, \{E_{k,n}^{\tau}(ev_{\tau}(a^{\lambda_j}))\}_{a\in F_1})^2.$$

Altogether, one has

$$\lim_k \sup \lim_n \sup \sup_{\tau} d(\{\pi_{\tau}(a)\}_{a\in F_1}, \{ev_{\tau}(\theta_1^{k,n}(a))\}_{a\in F_1}) < \delta.$$

Therefore, for some k, n, the map $\theta_1 = \theta_1^{k,n}$ satisfies the desired properties.

Theorem (6.2.13)[454]: Let M be a strictly separable continuous quasi-W^*- bundle over K such that $\pi_{\lambda}(M)''$ is hyperfinite for every $\lambda \in K$. Then, there are an embedding $\theta: M \hookrightarrow C_{\sigma}(K, R)$ and embeddings $\iota_{\lambda}: \pi_{\lambda}(M) \hookrightarrow R$ such that $ev_{\lambda} \circ \theta = \iota_{\lambda} \circ \pi_{\lambda}$. If M is moreover a continuous W^*-bundle, then one has

$$\theta(M) = \{f \in C_{\sigma}(K, R): f(\lambda) \in (\iota_{\lambda} \circ \pi_{\lambda})(M)''\}.$$

Proof . Let $(a_n)_{n=1}^{\infty}$ be a strictly dense sequence in the unit ball of M. We use Lemma (6.2.12) recursively and obtain sequences $(\{\theta_n(a_i)\}_{n=1}^{\infty})_{n=1}^{\infty}$ in $C_{\sigma}(K, R)$ such that

$$\sup_{\lambda} d(\{ev_{\lambda}(\theta_n(a_i))\}_{i=1}^{n}, \{\pi_{\lambda}(a_i)\}_{i=1}^{n}) < 2^{-n}$$

and

$$\max_{i=1,\ldots,n-1} \|\theta_n(a_i) - \theta_{n-1}(a_i)\|_{2,u} < 2^{-(n-1)}.$$

Then, each sequence $(\theta_n(a_i))_{n=i}^{\infty}$ converges to an element $\theta(a_i) \in C_{\sigma}(K, R)$. The map θ extends to a $*$-homomorphism from M into $C_{\sigma}(K, R)$, and $ev_{\lambda} \circ \theta$ factors through π_{λ}. This proves the first assertion. The second follows from Theorem (6.2.10).

We give a criterion for a continuous R-bundle to be a trivial bundle.

Theorem (6.2.14)[454]: Let M be a strictly separable continuous W^*-bundle over K such that $\pi_{\lambda}(M) \cong R$ for every $\lambda \in K$. Then, the following are equivalent.

(i) $M \cong C_{\sigma}(K, R)$ as a continuous W^*-bundle.

(ii) There is a sequence $(p_n)_n$ in M such that $0 \le p_n \le 1, \|p_n - p_n^2\|_{2,u} \to 0, \|E(p_n) - 1/2\| \to 0$, and $\|[p_n, a]\|_{2,u} \to 0$ for all $a \in M$.
(iii) For every k, there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into M.
Proof. The implication (i) ⇒ (ii) is obvious. For (ii) ⇒ (iii), we observe that since $\pi_\lambda(M)$'s are all factors, the central sequence $(p_n)_n$ satisfies $\|E(p_n a) - E(p_n)E(a)\| \to 0$ for every $a \in M$. Indeed, let $a \in M$ and $\varepsilon > 0$ be given. By the Dixmier approximation theorem and the proof of Theorem (6.2.9), there are $u_1, \dots, u_k \in \mathcal{U}(M)$ such that $\left\| E(a) - \frac{1}{k}\sum_{i=1}^{k} u_i a u_i^* \right\|_{2,u} < \varepsilon$. It follows that

$$\begin{aligned}\lim_{n\to\infty} \sup \|E(p_n)E(a) - E(p_n a)\| &= \lim_{n\to\infty} \sup \left\| E(p_n E(a)) - \frac{1}{k}\sum_{i=1}^{k} E(u_i p_n a u_i^*) \right\| \\ &= \lim_{n\to\infty} \sup \left\| E(p_n (E(a) - \frac{1}{k}\sum_{i=1}^{k} u_i a u_i^*)) \right\| \\ &< \varepsilon.\end{aligned}$$

Let $m \in \mathbb{N}$ be arbitrary. For a given finite sequence $(p_n)_{n=1}^m, 0 \le p_i \le 1$, and $\nu \in \{0,1\}^m$, we define $q_\nu \in M$ by

$$q_\nu = r_1^{1/2} \cdots r_{m-1}^{1/2} r_m r_{m-1}^{1/2} \cdots r_1^{1/2} \in M,$$

where $r_i = p_i$ or $1 - p_i$ depending on $\nu(i) \in \{0,1\}$. We note that $q_\nu \ge 0$ and $\sum q_\nu = 1$. By choosing $(p_n)_{n=1}^m$ appropriately, we obtain an approximately central approximately multiplicative embedding of $\ell_\infty(\{0,1\}^m)$ into M. Now, condition (iii) follows by choosing at the local level approximately central approximately multiplicative embeddings of $\mathbb{M}_k$ into $\pi_\lambda(M)$ and glue them together, as in the proof of Lemma (6.2.12), by an approximately central approximately projective partition of unity.
The proof of (iii) ⇒ (i) is similar to that of Theorem (6.2.13). Let $(a_n)_{n=1}^\infty$ (resp. $(b_n)_{n=1}^\infty$) be a strictly dense sequence in the unit ball of M (resp. $C_\sigma(K,R)$). We recursively construct finite subsets $F_1 \subset F_2 \subset \cdots$ of M and maps $\theta_n: F_n \to C_\sigma(K,R)$ such that $\{a_1, \dots, a_n\} \subset F_n$,

$$\sup_\lambda d(\{ev_\lambda(\theta_n(a))\}_{a\in F_n}, \{\pi_\lambda(a)\}_{a\in F_n}) < 2^{-n},$$

$$\max_{a\in F_{n-1}} \|\theta_n(a) - \theta_{n-1}(a)\|_{2,u} < 2^{-(n-1)},$$

and $\{b_1, \dots, b_n\} \subset \theta_n(F_n)$. Let $F_0 = \emptyset$ and suppose that we have constructed up to $n-1$.
Let $F_n' = F_n' \cup \{a_n\}$. We use Lemma (6.2.12) and obtain a map $\theta_n': F_n' \to C_\sigma(K,R)$ such that

$$\sup_\lambda d(\{ev_\lambda(\theta_n'(a))\}_{a\in F_n'}, \{\pi_\lambda(a)\}_{a\in F_n'}) < 2^{-(n+1)}$$

and

$$\max_{a\in F_{n-1}} \|\theta_n'(a) - \theta_{n-1}(a)\|_{2,u} < 2^{-(n-1)}.$$

We may assume that θ_n' is injective and $\theta_n'(F_n')$ does not contain any of $b_1, \dots, b_n$. We use Lemma (6.2.12) again but this time to $\theta_n'(F_n') \subset \tilde{F} := \theta_n'(F_n') \cup \{b_1, \dots, b_n\}$ and $(\theta_n')^{-1}$. Then, there is $\psi: \tilde{F} \to M$ such that

$$\sup_\lambda d(\{\pi_\lambda(\psi(b))\}_{b\in\tilde{F}}, \{ev_\lambda(b)\}_{b\in\tilde{F}}) < 2^{-(n+1)}$$

and

$$\max_{a\in F_n'} \|a - \psi(\theta_n'(a))\|_{2,u} < 2^{-(n+1)}.$$

Now, we set $F_n = F'_n \cup \{\psi(b_1),\dots,\psi(b_n)\}$ (which can be assumed to be a disjoint union) and define $\theta_n: F_n \to C_\sigma(K,R)$ by $\theta_n = \theta'_n$ on F'_n and $\theta_n(\psi(b_i)) = b_i$. One has

$$\sup_\lambda d(\{ev_\lambda(\theta_n(a))\}_{a\in F_n}, \{\pi_\lambda(a)\}_{a\in F_n})$$
$$\le \sup_\lambda (d(\{ev_\lambda(b)\}_{b\in\tilde{F}}, \{\pi_\lambda(\psi(b))\}_{b\in\tilde{F}}) + \max_{a\in F'_n}\|\pi_\lambda(\psi(\theta'_n(a))) - \pi_\lambda(a)\|_2)$$
$$< 2^{-n}$$

as desired. By taking the limit of $(\theta_n)_n$, one obtains a $*$-isomorphism θ from M onto $C_\sigma(K,R)$.

By combining Corollary (6.2.11) and Theorem (6.2.13), one obtains the following W^*-analogue of Theorem (6.2.2) in [309]. This also implies Theorem (6.2.3). It is unclear whether the finite-dimensionality assumption is essential.

Corollary (6.2.15)[445]: Let M be a strictly separable continuous W^*-bundle over K. If every fiber $\pi_\lambda(M)$ is isomorphic to R and K has finite covering dimension, then $M \cong C_\sigma(K,R)$ as a continuous W^*-bundle.

Corollary (6.2.16)[462]: For a unital C^*-algebra A, the following are equivalent.

(i) The C^*-algebra A has the QTS property.

(ii) For every $\varepsilon > 0$ and $a^s \in A$ that satisfy $sup_{\tau\in T(A)}|\tau(a^s)| < \varepsilon$, there are k and $u_1^2,\dots,u_k^2 \in \mathcal{U}(A)$ such that $\left\|\frac{1}{k}\sum_{i=1}^k u_i^2 a^s (u_i^*)^2\right\| < \varepsilon$.

Unlike the case for von Neumann algebras, there is no bound of k in terms of ε and $\|a^s\|$ that works for an arbitrary element a^s in a C^*-algebra. Recall that a Banach algebra A is said to be amenable if there is a net $(\Delta_n)_n$, called an approximate diagonal, in the algebraic tensor product $A \otimes_{\mathbb{C}} A$ (we reserve the symbol $\otimes$ for the minimal tensor product) such that

(a) $sup_n\|\Delta_n\|_\wedge < +\infty$,

(b) $(m(\Delta_n))_n$ is an approximate identity,

(c) $\lim_n\|a^s\cdot\Delta_n - \Delta_n\cdot a^s\|_\wedge = 0$ for every $a^s \in A$.

Here $\|\cdot\|_\wedge$ is the projective norm on $A \otimes_{\mathbb{C}} A, m: A \otimes_{\mathbb{C}} A \to A$ is the multiplication, and $a^s \cdot (\sum_i x_i^2 \otimes y_i^2) = \sum_i a^s x_i^2 \otimes y_i^2$ and $(\sum_i x_i^2 \otimes y_i^2)\cdot a^s = \sum_i x_i^2 \otimes y_i^2 a^s$. The celebrated theorem of Connes–Haagerup [307, 314] states that a C^*-algebra A is amenable as a Banach algebra if and only if it is nuclear. The Banach algebra A is said to be symmetrically amenable ([321]) if the approximate diagonal $(\Delta_n)_n$ can be taken symmetric under the flip $x^2 \otimes y^2 \to y^2 \otimes x^2$. We characterize symmetric amenability for C^*-algebras.

Proof. Ad (i) ⇒ (ii). Although the proof becomes a bit shorter if we use Theorem(6.2.4) in [317], we give here a more direct proof of this implication. Let $a^s \in A$ and $\varepsilon > 0$ be given as in condition (ii). Let $\varepsilon_0 = sup_{\tau\in T(A)}|\tau(a^s)| < \varepsilon$. We decompose the second dual von Neumann algebra A^{**} into the finite summand A_f^{**} and the properly infinite summand A_∞^{**}. We denote the corresponding embedding of A by π_f and π_∞, and the center-valued trace of A_f^{**} by ctr. We note that $\|ctr(\pi_f(a^s))\| = \varepsilon_0$. By the Dixmier approximation theorem, there are $v_1^2,\dots,v_k^2 \in \mathcal{U}(A_f^{**})$ such that $\left\|ctr(\pi_f(a^s)) - \frac{1}{k}\pi_f(a^s)(v_i^*)^2\right\| < \varepsilon - \varepsilon_0$. On the other hand, by Halpern's theorem ([314]), there are $w_1^2,\dots,w_k^2 \in U(A_\infty^{**})$ such that $\left\|\frac{1}{l}\sum_{j=1}^l w_i^2\pi_\infty(a^s)(w_i^*)^2\right\| < \varepsilon$. Before giving the detail of the proof of this fact, we finish the proof of (i) ⇒ (ii). By allowing multiplicity, we may assume that $k = l$ and consider $u_i^2 = v_i^2 \oplus w_i^2 \in A^{**}$. Then, $\left\|\frac{1}{k}\sum_{i=1}^k u_i^2 a^s (u_i^*)^2\right\| < \varepsilon$ in A^{**}. For each i, take a net

$(u^2_{\lambda_s})_i\ (\lambda_s)$ of unitary elements in A which converges to $u^2_1 \in A^{**}$ in the ultrastrong$*$-topology. By the Hahn-Banach theorem, $conv\{\frac{1}{k}\sum_{i=1}^{k} u_i^2(\lambda_s)a^s\ ((u_i^*)^2(\lambda_s))\}_{\lambda_s}$ contains an element of norm less than ε.

Now, we explain how to apply Halpern's theorem. Let Z (resp. I) be the center (resp. strong radical) of A^{**}_∞. Let Λ be the directed set of all finite partitions of unity by projections in Z, and $\lambda_s = \{(p_{\lambda_s})_i\}_i \in \Lambda$ be given.

Applying the QTS property to the non-zero $*$-homomorphism $A \ni x^2 \mapsto \left(p_{\lambda_s}\right)_i(\pi_\infty(x^2)+I) \in (p_{\lambda_s})_i((\pi_\infty(A)+I)/I)$, one obtains a (tracial) state $(\tau_{\lambda_s})_i$ on $\pi_\infty(A)+I$ such that $(\tau_{\lambda_s})_i((p_{\lambda_s})_i)=1, (\tau_{\lambda_s})_i(I)=0$, and $|(\tau_{\lambda_s})_i(\pi_\infty(a^s))| \le \varepsilon_0$. Let $(\tilde{\tau}_{\lambda_s})_i$ be a state extension of it on $(p_{\lambda_s})_i\ A^{**}_\infty$. We define the linear map $\varphi_{\lambda_s}: A^{**}_\infty \to Z$ by $\varphi_{\lambda_s}(x^2) = \sum_i (\tilde{\tau}_{\lambda_s})_i(x^2)(p_{\lambda_s})_i$, and take a limit point $\varphi: A^{**}_\infty \to Z$. The map φ is a unital positive Z-linear map such that $\varphi(I)=0$ and $\|\varphi(\pi_\infty(a^s))\| \le \varepsilon_0$. By Halpern's theorem (Theorem 4.12 in [314]), the norm-closed convex hull of the unitary conjugations of $\pi_\infty(a^s)$ contains $\varphi(\pi_\infty(a^s))$.

Ad (ii) ⇒ (i). Suppose that there is a closed two-sided proper ideal I in A such that A/I does not have a tracial state. Let e_n be the approximate unit of I. Then, one has $\tau(1-e_n) \searrow 0$ for every $\tau \in T(A)$. By Dini's theorem, there is n such that $q = 1-e_n$ satisfies $\tau(q) < 1/2$ for all $\tau \in T(A)$. By condition (ii), there are $u_1^2, \dots, u_k^2 \in \mathcal{U}(A)$ such that $\left\|\frac{1}{k}\sum_{i=1}^{k} u_i^2 q(u_i^*)^2\right\| < 1/2$, which is in contradiction with the fact that $\frac{1}{k}\sum_{i=1}^{k} u_i^2 q(u_i^*)^2 \in 1+I$.

Corollary (6.2.17)[462]: For a unital C^*-algebra A, the following are equivalent.

(i) The C^*-algebra A is nuclear and has the QTS property.

(ii) The C^*-algebra A has an approximate diagonal $\Delta_n = \sum_{i=1}^{k(n)} (x_i^*)^2(n) \otimes x_i^2(n)$ such that $\lim_n \sum_{i=1}^{k(n)} \left\|x_i^2(n)\right\|^2 = 1, m(\Delta_n) = 1$, and $\lim_n \left\|1 - \sum_{i=1}^{k(n)} x_i^2(n)(x_i^*)^2(n)\right\| = 0$.

(iii) The C^*-algebra A is symmetrically amenable.

(iv) The C^*-algebra A has a symmetric approximate diagonal $(\Delta_n)_n$ in

$$\{\sum_i (x_i^*)^2 \otimes x_i^2 \in A \otimes_{\mathbb{C}} A: \sum_i \left\|x_i^2\right\|^2 \le 1\}.$$

Proof. The implication (iv) ⇒ (iii) is obvious and (iii) ⇒ (i) is standard: Since amenability implies nuclearity by Connes's theorem [307], we only have to prove the QTS property. Let $(\Delta_n)_n$ be a symmetric approximate diagonal and define $m_\Delta(a^s) = \sum_i x_i^2 a^s y_i^2$ for $\Delta = \sum_i x_i^2 \otimes y_i^2 \in A \otimes_{\mathbb{C}} A$ and $a^s \in A$. Then, for any proper ideal I in A and a state φ on A such that $\varphi(I)=0$, any limit point τ of $(\varphi \circ m_{\Delta_n})_n$ is a bounded trace such that $\tau(I)=0$ and $\tau(1)=1$. By polar decomposition, one obtains a tracial state on A which vanishes on I.

We prove the implication (i) ⇒ (ii) ⇒ (iv). Since A is nuclear, it is amenable thanks to Haagerup's theorem (Theorem 3.1 in [314]). Moreover, there is an approximate diagonal $(\Delta'_n)_n$ in the convex hull of $\{(x^*)^2 \otimes x^2: \|x^2\| \le 1\}$. We note that $\varepsilon_n := \|1 - m(\Delta'_n)\| \to 0$. We fix n for the moment and write $\Delta'_n = \sum_i (x_i^*)^2 \otimes x_i^2$. By replacing x_i^2 with $x_i^2 m(\Delta'_n)^{-1/2}$, we may assume $m(\Delta'_n) = 1$ but $\sum_i \left\|x_i^2\right\|^2 \le (1-\varepsilon_n)^{-1}$. Since $\tau(\sum_i x_i^2\ (x_i^*)^2) = 1$ for all $\tau \in T(A)$, Theorem (6.2.1) provides $u_1^2, \dots, u_i^2 \in \mathcal{U}(A)$ such that

$\left\|\frac{1}{l}\sum_{j=1}^{l}\sum_i u_j^2 x_i^2 (x_i^*)^2 (u_j^*)^2\right\| \leq 1+\varepsilon_n$. Thus, $\Delta_n = \frac{1}{l}\sum_{i,j}(x_i^*)^2(u_j^*)^2 \otimes u_j^2 x_i^2$ satisfies condition(ii).Now, rewrite Δ_n as $\sum_i (y_i^*)^2 \otimes y_i^2$. Then, $\Delta_n^{\#} = (\sum_i \left\|y_i^2\right\|^2)^{-2}\sum_{i,j}(y_i^*)^2\, y_j^2 \otimes (y_j^*)^2\, y_j^2$ is a symmetric approximate diagonal that meets condition (iv).

Corollary (6.2.18)[462]: Let A, S, and N be as above. Then, there is a unital $*$-homomorphism $\theta: B(\partial S) \to Z(N)$ with ultraweakly dense range such that $\theta(\hat{a}^s) = ctr(a^s)$ and

$$\tau\,(\theta(f)a^s) = \int f(\lambda_s)\,\lambda_s(a^s)\,d\mu_\tau(\lambda_s) = \int f(\hat{a})^s d\mu_\tau$$

for every $a^s \in A$ and $\tau \in S$. One has

$$\bar{A}^{st} = \{x^2 \in N: ctr(x^2 A) \subset \theta(Aff(S)), ctr((x^*)^2\, x^2) \in \theta(Aff(S))\}.$$

In particular,

$$\bar{A}^{st} \cap Z(N) = \{\theta(f): f \in Z(Aff(S))\}.$$

If ∂S is closed, then for every $\tau \in \partial S$, one has $\pi_\tau(\bar{A}^{st}) = \pi_\tau(N) = \pi_\tau(A)''$.

Proof. We first find the $*$-homomorphism $\theta: B(\partial S) \to Z(N)$ that satisfies

$$\tau(\theta(f)a^s) = \int f(\lambda_s)\,\lambda_s(a^s)\,d\mu_\tau(\lambda_s)$$

for every $a^s \in A$ and $\tau \in S$, or equivalently, $\pi_\tau(\theta(f)) = \theta_\tau(f)$ in $\pi_\tau(A)''$. For this, it suffices to show that the maps $\theta_\tau|_{B(\partial S)}$, given in Lemma (6.2.9), are compatible over $\tau \in S$. We recall that associated with the representation π_τ, there is a unique central projection $p_\tau \in Z(N)$ such that $(1-p_\tau)N = ker\pi_\tau$. Since $p_\tau \vee p_{\sigma_s} = p_{(\tau+\sigma_s)/2}$, the family $\{p_\tau: \tau \in S\}$ is upward directed and $sup_\tau p_\tau = 1$. We will show that if τ and σ_s are such that $\tau \leq C\sigma_s$ for some $C > 1$, then $\theta_\tau(f) = p_\tau \theta_{\sigma_s}(f)$ in $Z(N)$. We note that p_τ is the support projection of $d\tau/d\sigma_s \in Z(N)$. For every $f \in B(\partial S)$, one has

$$\begin{aligned}\sigma_s(\theta_{\sigma_s}(\frac{d\mu_\tau}{d\mu_{\sigma_s}}f)a^s) &= \int (\frac{d\mu_\tau}{d\mu_{\sigma_s}}f)(\lambda_s)\lambda_s(a^s)\,d\mu_{\sigma_s}(\lambda_s)\\ &= \int f(\lambda_s)\lambda_s(a^s)\,d\mu_\tau(\lambda_s)\\ &= \tau(\theta_\tau(f)a^s)\\ &= \sigma_s(\frac{d\tau}{d\sigma_s}\theta_\tau\,(f)a^s).\end{aligned}$$

This implies $\theta_{\sigma_s}(\frac{d\mu_\tau}{d\mu_{\sigma_s}}f) = \frac{d\tau}{d\sigma_s}\theta_\tau(f)$ for every f. In particular, $\theta_{\sigma_s}(\frac{d\mu_\tau}{d\mu_{\sigma_s}}) = \frac{d\tau}{d\sigma_s}$ and $p_\tau\theta_{\sigma_s}(f) = \theta_{\sigma_s}(f)$ in $Z(N)$. Therefore, we may glue $\{\theta_\tau\}_{\tau\in S}$ together and obtain a globally defined $*$-homomorphism $\theta: B(\partial S) \to Z(N)$. Since $\tau(\theta(\hat{a})^s) = \int(\hat{a})^s(\lambda_s)\,d\mu_\tau(\lambda_s) = \tau\,(a^s)$ for every $\tau \in S$, one has $\theta((\hat{a})^s) = ctr(a^s)$ for every $a^s \in A$. This proves the first part of the theorem.

For the second part, it suffices to prove

$$\bar{A}^{st} \supset \{x^2 \in N: ctr(x^2 A) \subset Aff(\partial S), ctr((x^*)^2\, x^2) \in Aff(\partial S)\},$$

as the converse inclusion is trivial. Take x^2 from the set in the right hand side. We will prove a stronger assertion that if a net $(b_j^s)_j$ in A converges to x^2 ultrastrongly in N, then x^2 is contained in the strict closure of the convex hull of $\{b_j^s: j\}$. We note that $Aff(S) \ni f \mapsto f|_{\partial S} \in Aff(\partial S)$ is an affine order isomorphism and that every positive norm-one linear functional μ on $Aff(S)$ is given by the evaluation at a point $\tau_\mu \in S$. (Indeed by the Hahn–Banach theorem, we may regard μ as a state on $C(S)$, which is a probability measure on S by the Riesz–Markov theorem. The point $\tau_\mu = \int \lambda_s \, d\mu(\lambda_s)$ satisfies $f(\tau_\mu) = \mu(f)$ for $f \in Aff(S)$.) Thus, one has $ctr((b_j^s - x^2)^* (b_j^s - x^2)) \to 0$ weakly in $Aff(\partial S)$. Therefore, by the Hahn–Banach theorem, for every $\varepsilon > 0$ there is a finite sequence $(\alpha_s)_j \geq 0$, $\sum(\alpha_s)_j = 1$ such that $\left\|\sum_j(\alpha_s)_j \, ctr((b_j{}^s - x^2)^* (b_j^s - x^2))\right\| < \varepsilon$. By reindexing, we assume $j = 1, \dots, k$. Let $b^s = \sum(\alpha_s)_j \, b_j^s$. We note that

$$b^s = \left[(\alpha_s)_1^{1/2} \cdots (\alpha_s)_m^{1/2}\right] \begin{bmatrix} (\alpha_s)_1^{1/2} \ b_1^s \\ \vdots \\ (\alpha_s)_m^{1/2} \ b_m^s \end{bmatrix} =: r c^s.$$

Hence, $(b^*)^s \, b^s = (c^*)^s \, r^* r \, c^s \leq \|r\|^2 (c^*)^s \, c^s = \sum(\alpha_s)_j (b_j^*)^s \, b_j^s$. It follows that

$$\begin{aligned} &ctr((b^s - x^2)^*(b^s - x^2)) = ctr((b^*)^s \, b^s - (b^*)^s \, x^2 - (x^*)^2 b^s + (x^*)^2 \, x^2) \\ &\leq ctr\Big(\sum (\alpha_s)_j (b_j^*)^s \, b_j^s - \sum (\alpha_s)_j \, (b_j^*)^s \, x^2 - (x^*)^2 \sum (\alpha_s)_j \, b_j^s + (x^*)^2 \, x^2\Big) \\ &= ctr\Big(\sum (\alpha_s)_j \, (b_j^s - x^2)^*(b_j^s - x^2)\Big) \\ &< \varepsilon. \end{aligned}$$

This proves the claimed inclusion. The last assertion will be proved in more general setting as Theorem (6.2.10).

Corollary (6.2.19)[462]: There is a constant $C > 0$ which satisfies the following. Let A be a C^*-algebra and $a^s \in A$ and $\varepsilon > 0$ be such that $sup_{\tau \in T(A)} |\tau(a^s)| < \varepsilon$. Then, there are $k \in \mathbb{N}$ and b_i^s and c_i^s in A such that $\sum_{i=1}^k \|b_i^s\| \|c_i^s\| \leq C\|a^s\|$ and

$$\left\|a^s - \sum_{i=1}^k [b_i^s, c_i^s]\right\| < \varepsilon.$$

Proof. Let $a^s \in A$. We denote by ctr the centervalued trace from the second dual von Neumann algebra A^{**} onto the center $Z(A_f^{**})$ of the finite summand A_f^{**} of A^{**}. One has $\|ctr(a^s)\| = sup_{\tau \in T(A)} |\tau(a^s)| < \varepsilon$ and $(a')^s := a^s - ctr(a^s)$ has zero traces. By a theorem of Fack and de la Harpe, for $C = 2 \cdot 12^2$ and $m = 10$, there are $b_i^s, c_i^s \in A^{**}$ such that $\sum_{i=1}^m \|b_i^s\| \|c_i^s\| \leq C\|a^s\|$ and $(a')^s = \sum_{i=1}^m [b_i^s, c_i^s]$. See [325, 330] for a better estimate of C and m. By Kaplansky's density theorem, there is a net $(b_i^s(\lambda_s))_{\lambda_s}$ in A such that $\|b_i^s(\lambda_s)\| \leq \|b_i^s\|$ and $b_i^s(\lambda_s) \to b_i^s$ ultrastrongly. Likewise for $(c_i^s(\lambda_s))_{\lambda_s}$. Since

$$\left\|\lim_\lambda \Big(a^s - \sum_{i=1}^m [b_i^s(\lambda_s), c_i^s(\lambda_s)]\Big)\right\| = \|a^s - (a')^s\| < \varepsilon,$$

there is $(a'')^s \in conv\{\sum_{i=1}^m b_i^s(\lambda_s), c_i^s(\lambda_s)]\}_{\lambda_s}$ which satisfies $\|a^s - (a'')^s\| < \varepsilon$.

Corollary (6.2.20)[462]: There is a constant $C > 0$ which satisfies the following. Let A be an exact Z-stable C^*-algebra, and $\varepsilon > 0$ and $a^s \in A$ be such that $sup_{\tau \in T(A)} |\tau(a^s)| < \varepsilon$. Then, for every $R \in \mathbb{N}$, there are $b^s(r)$ and $c^s(r)$ in A such that $\sum_{r=1}^R \|b^s(r)\| \|c^s(r)\| \leq C\|a^s\|$ and $\|a^s - \sum_{r=1}^R [b^s(r), c^s(r)]\| < \varepsilon + C\|a^s\| R^{-1/2}$.

Proof. Let $a^s \in A \setminus \{0\}$ be such that $sup_{\tau \in T(A)} |\tau(a^s)| < \varepsilon$. Since $Z \cong Z^{\otimes \infty}$, we may assume that $A = Z \otimes A_0$ and $a^s \in A_0$. By Theorem (6.2.4), there are b_i^s, c_i^s such that $\|b_i^s\| = \|c_i^s\|, \sum_{i=1}^k \|b_i^s\| \|c_i^s\| \leq C\|a^s\|$, and $\left\|a^s - \sum_{i=1}^k [b_i^s, c_i^s]\right\| < \varepsilon$. Recall the theorem of Haagerup and Thorbjornsen ([316]) which states that the C^*-algebra C can be embedded into $\prod \mathbb{M}_n / \oplus \mathbb{M}_n$. By exactness of A_0, there is a canonical $*$-isomorphism

$$(\prod \mathbb{M}_n / \oplus \mathbb{M}_n) \otimes A_0 \cong ((\prod \mathbb{M}_n) \otimes A_0)/(\oplus \mathbb{M}_n \otimes A_0).$$

Lemma (6.2.5), combined with this fact, implies that there are matrices $s_i^{(n)}(r) \in \mathbb{M}_n$ such that $(\tilde{b}^{(n)})^s(r) = \sum_{i=1}^k s_i^{(n)}(r) \otimes b_i^s$ and $(\tilde{c}^{(n)})^s(r) = \sum_{j=1}^k s_j^{(n)}(r) \otimes c_j^s$ satisfy

$$\lim_n \sup \frac{1}{R} \sum_{r=1}^{R} \left\|(\tilde{b}^{(n)})^s(r)\right\| \left\|(\tilde{b}^{(n)})^s, (\tilde{c}^{(n)})^s(r)\right\| \le 4 \sum \|b_i^s\| \|c_i^s\| \le 4C\|a^s\|$$

and

$$\lim_n \sup \left\| 1 \otimes a^s - \frac{1}{R} \sum_{r=1}^{R} [(\tilde{b}^{(n)})^s(r), (\tilde{c}^{(n)})^s(r)] \right\| \le \varepsilon + \frac{6C\|a^s\|}{\sqrt{R}}.$$

For every relatively prime $p, q \in \mathbb{N}$, the Jiang–Su algebra Z contains the prime dimension drop algebra

$$I(p,q) = \{f \in C([0,1], \mathbb{M}_p \otimes \mathbb{M}_q): f(0) \in \mathbb{M}_p \otimes \mathbb{C}1, f(1) \in \mathbb{C}1 \otimes \mathbb{M}_q\}$$

and hence $t\mathbb{M}_q$ and $(1-t)\mathbb{M}_p$ also, where $t \in I(p,q)$ is the identity function on $[0,1]$. It follows that there are $b^s(r), c^s(r), (b')^s(r), (c')^s(r) \in Z \otimes A_0$ such that

$$\frac{1}{R} \sum_{r=1}^{R} (\|b^s(r)\| \|c^s(r)\| + \|(b')^s(r)\| \|(c')^s(r)\|) < 9C\|a^s\|$$

and

$$\left\| a^s - \frac{1}{R} \sum_{r=1}^{R} ([b^s(r), c^s(r)] + [(b')^s(r), (c')^s(r)]) \right\| < \varepsilon + \frac{7C\|a^s\|}{\sqrt{R}}.$$

Here, we note that $\|t \otimes x^2 + (1-t) \otimes y^2\| = \max\{\|x^2\|, \|y^2\|\}$ for any x^2 and y^2.

Corollary (6.2.21)[462]: Let $b_i^s, c_i^s \in A$ be such that $\|b_i^s\| = \|c_i^s\|$. Then, for every $R \in \mathbb{N}$, letting $(\tilde{b})^s(r) = \sum_{i=1}^n S_i(r) \otimes b_i^s$ and $(\tilde{c})^s(r) = \sum_{j=1}^n S_j(r) \otimes c_j^s$, one has

$$\frac{1}{R} \sum_{r=1}^{R} \left\|(\tilde{b})^s(r)\right\| \|(\tilde{c})^s(r)\| \le 4 \sum \|b_i^s\| \|c_i^s\|$$

and

$$\left\| 1 \otimes \sum_{i=1}^{n} [b_i^s, c_i^s] - \frac{1}{R} \sum_{r=1}^{R} \left[(\tilde{b})^s(r), (\tilde{c})^s(r)\right] \right\| \le \frac{6}{\sqrt{R}} \sum_i \|b_i^s\| \|c_i^s\|.$$

Proof. For every r, one has

$$\left\|(\tilde{b})^s(r)\right\| \le \left\|\sum l_i(r) \otimes b_i^s\right\| + \left\|\sum l_i^*(r) \otimes b_i^s\right\|$$

$$= \left\|\sum (b_i^*)^s \ b_i^s\right\|^{1/2} + \left\|\sum b_i^s (b_i^*)^s \right\|^{1/2} \le 2(\sum \|b_i^s\|^2)^{1/2},$$

and likewise for $(\tilde{c})^s(r)$. It follows that $\left\|(\tilde{b})^s(r)\right\| \|(\tilde{c})^s(r)\| \le 4 \sum \|b_i^s\| \|c_i^s\|$. Moreover,

$$(\tilde{b})^s(r)(\tilde{c})^s(r) = \sum_{i,j} (\delta_{i,j} 1 + l_i(r) l_j(r) + l_i(r) l_j^*(r) + l_i^*(r) l_j^*(r)) \otimes b_i^s c_j^s,$$

and

$$\left\|\sum_{r,i,j} l_i(r)l_j(r)\otimes b_i^s\, c_j^s\right\| = \left\|\sum_{r,i,j}(c_j^*)^s\,(b_i^*)^s\, b_i^s\, c_j{}^s\right\|^{1/2} \leq R^{1/2}\sum_i \|b_i^s\|\|c_i^s\|,$$

$$\left\|\sum_{r,i,j} l_i^*(r)\, l_j^*(r)\otimes b_i^s c_j^s\right\| = \left\|\sum_{r,i,j} b_i^s c_j^s (c_j^*)^s\,(b_i^*)^s\right\|^{1/2} \leq R^{1/2}\sum_i \|b_i^s\|\|c_i^s\|,$$

$$\left\|\sum_{r,i,j} l_i(r)\, l_j^*(r)\otimes b_i^s c_j^s\right\| = \max_r\left\|\sum_{i,j} l_i(r)\, l_j^*(r)\otimes b_i^s c_j^s\right\| \leq \sum_i \|b_i^s\|\|c_i^s\|.$$

Likewise for $(\tilde{c})^s(r)(\tilde{b})^s(r)$, and one obtains the conclusion.

Corollary (6.2.22)[462]: Let $(A_n)_n$ be a sequence of exact Z-stable C^*-algebras and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then, $\prod T(A_n)/\mathcal{U}$ (resp. $conv\coprod T(A_n)$) is weak$*$-dense in $T(\prod A_n/\mathcal{U})$ (resp. $T(\prod A_n)$). In particular, for every $\tau\in T(\prod A_n/\mathcal{U})$ and every separable C^*-subalgebra $B\subset\prod A_n/\mathcal{U}$, there is $\tau'\in\prod T(A_n)/\mathcal{U}$ such that $\tau|_B = \tau'|_B$.

Proof. Let A be either $\prod A_n$ or $\prod A_n/\mathcal{U}$, and denote by $\Sigma\subset T(A)$ either $conv(\coprod T(A_n))$ or $\prod T(A_n)/\mathcal{U}$ accordingly. Suppose that the conclusion of the theorem is false for $\Sigma\subset T(A)$. Then, by the Hahn–Banach theorem, there are τ in $T(A)$ and a self-adjoint element a_0^s in A such that $\gamma_s := \tau\,(a_0^s) - sup_{\sigma_s\in\Sigma}\,\sigma_s(a_n^s) > 0$. Let $\alpha_s = (|\,inf_{\sigma_s\in\Sigma}\sigma_s(a_0^s)| - \tau\,(a_0^s))\vee 0$, and take $b^s\in A_+$ such that $\tau\,(b^s) = \alpha_s$ and $\|b^s\| < \alpha_s + \gamma_s$. It follows that $a^s = a_0^s + b^s$ satisfies $sup_{\sigma_s\in\Sigma}|\sigma_s(a^s)| < \tau(a^s)$. Now, expand $a^s\in A$ as $(a_n^s)_n$. We may assume that $\|a_n^s\|\leq\|a^s\|$ for all n. Let $I\in\mathcal{U}$ (or $I=\mathbb{N}$ in case $A=\prod A_n$) be such that $\varepsilon_0 := sup_{n\in I}\, sup_{\sigma_s\in T(A_n)}\,\sigma_s(a_n^s) < \tau(a^s)$. Let $R\in\mathbb{N}$ be such that $\varepsilon_1 := \varepsilon_0 + C\|a^s\|R^{-1/2} < \tau(a^s)$. Then, by Theorem (6.2.4), for each $n\in I$ there are $b_n^s(r), c_n^s(r)\in A_n$ such that $\sum_{r=1}^R\|b_n^s(r)\|\|c_n^s(r)\|\leq C\|a^s\|$ and $\|a_n^s - \sum_{r=1}^R[b_n^s(r), c_n^s(r)]\|\leq\varepsilon_1$. It follows that for $b^s(r) = (b_n^s(r))_n$ and $c^s(r) = (c_n^s(r))_n\in A$, one has

$$\tau(a^s) = \tau\,(a^s - \sum_{r=1}^R [b^s(r),\, c^s(r)]) \leq \left\|a^s - \sum_{r=1}^R [b^s(r), c^s(r)]\right\| < \tau(a^s),$$

which is a contradiction. This proves the first half of the theorem.

For the second half, let τ and B be given. Take a dense sequence $(x_{i=0}^2(i))^\infty$ in B and expand them as $x^2(i) = (x_n^2(i))_n$. By the first half, for every m, there is $(\tau_n^{(m)})_n\in\prod T(A_n)$ such that $|\tau(x^2(i)) - \tau_{\mathcal{U}}^{(m)}(x^2(i))| < m^{-1}$ for $i=0,\ldots,m$. Let $I_m = \{n\in\mathbb{N}: |\tau\,(x^2(i)) - \tau_n^{(m)}(x_n^2(i))| < m^{-1}$ for all $i=0,\ldots,m\}\in\mathcal{U}$

(so $I_0=\mathbb{N}$), and $J_m = \bigcap_{l=0}^m I_l\in\mathcal{U}$. We define τ_n to be $\tau_n^{(m)}$ for $n\in J_m\setminus J_{m+1}$. It is not too hard to check $\tau = \tau_{\mathcal{U}}$ on B.

Corollary (6.2.23)[462]: Let A be a (non-separable) C^*-algebra and $X\subset A$ be a separable subset. Then there is a separable C^*-subalgebra $B\subset A$ that contains X and such that the restriction from $T(A)$ to $T(B)$ is onto.

Proof. We may assume that A is unital. We first claim that for every $x_1^2,\ldots,x_n^2\in A$ and $\varepsilon > 0$, there is a separable C^*-subalgebra C which satisfies the following property: for every $\tau\in T(C)$ there is $\sigma_s\in T(A)$ such that $\max_i|\tau(x_i^2) - \sigma_s(x_i^2)| < \varepsilon$. Indeed if this were not true, then for every C there is $\tau_{C^s}\in T(C)$ such that $\max_i|\tau_{C^s}(\,x_i^2) - \sigma_s(\,x_i^2)|\geq\varepsilon$ for all $\sigma_s\in$

$T(A)$. The set of separable C^*-subalgebras of A is upward directed and one can find a limit point τ of $\{\tau_{C^s}\}$. Then, we arrive at a contradiction that $\tau \in T(A)$ satisfies $\max_i |\tau(x_i^2) - \sigma_s(x_i^2)| \geq \varepsilon$ for all $\sigma_s \in T(A)$. We next claim that for every separable C^*-subalgebra $B_0 \subset A$, there is a separable C^*-subalgebra $B_1 \subset A$ that contains B_0 and such that $Res_{B_0} T(B_1) = Res_{B_0} T(A)$ in $T(B_0)$, where Res is the restriction map. Take a dense sequence $x_1^2, x_2^2, \dots$ in B_0, and let $C_0 = B_0$. By the previous discussion, there is an increasing sequence of separable C^*-subalgebras $C_0 \subset C_1 \subset \cdots$ such that for every $\tau \in T(C_n)$ there is $\sigma_s \in T(A)$ satisfying $|\tau(x_i^2) - \sigma_s(x_i^2)| < n^{-1}$ for $i = 1, \dots, n$. Now, letting $B_1 = \overline{\bigcup_n C_n}$ and we are done. Finally, we iterate this construction and obtain $X \subset B_0 \subset B_1 \subset \cdots$ such that $Res_{B_n} T(B_{n+1}) = Res_{B_n} T(A)$. The separable C^*-subalgebra $B = \overline{\bigcup B_n}$ satisfies the desired property.

Corollary(6.2.24)[462]: For every $\tau \in S$, there is a normal $*$-isomorphism $\theta_\tau : L^\infty(\partial S, \mu_\tau) \to Z(\pi_\tau(A)'')$ such that

$$\tau(\theta_\tau(f) a^s) = \int f(\lambda_s)\, \lambda_s(a^s)\, d\mu_\tau\,(\lambda_s)$$

For $a^s \in A$.

Proof. Let $f \in L^\infty(\partial S, \mu_\tau)$ be given. The right hand side of the claimed equality defines a tracial linear functional on A whose modulus is dominated by a scalar multiple of τ. Hence, by Sakai's Radon–Nikodym theorem, there is a unique $\theta_\tau(f) \in Z(\pi_\tau(N))$ that satisfies the claimed equality. This defines a unital normal positive map θ_τ from $L^\infty(\partial S, \mu_\tau)$ into $Z(\pi_\tau(A)'')$. Next, let $z^2 \in Z(\pi_\tau(N))_+$ be given. Then, the tracial linear functional $z^2\tau$ on A defined by $(z^2\tau)(a^s) = \tau\,(a^s z^2)$ is dominated by $\|z^2\|\tau$. Hence one has $\mu_{z^2\tau} \leq \|z^2\|\mu_\tau$ and $z^2 = \theta_\tau(d\mu_{z^2\tau}/d\mu_\tau)$ with $d\mu_{z^2\tau}/d\mu_\tau \in L^\infty(\partial S, \mu_\tau)$.
This proves θ_τ is a positive linear isomorphism such that $\mu_{\theta_\tau(f)\tau} = f\mu_\tau$.
Therefore, one has $\mu_{\theta\tau\,(fg)\tau} = fg\mu_\tau = f\mu_{\theta_\tau(g)\tau} = \mu_{\theta_\tau(f)\theta_\tau(g)\tau}$, which proves $\theta_\tau(fg) = \theta_\tau(f)\theta_\tau(g)$.

Corollary (6.2.25)[462]: Let M be a continuous W^*-bundle over K. Then, $(\pi_{\lambda_s})(M) = (\pi_{\lambda_s})(M)''$ for every $\lambda_s \in K$. Moreover, if a bounded function $f : K \ni \lambda_s \longmapsto f(\lambda_s) \in (\pi_{\lambda_s})(M)$ is continuous in the following sense: for every $(\lambda_s)_0 \in K$ and $\varepsilon > 0$, there are a neighborhood O of $(\lambda_s)_0$ and $c^s \in M$ such that

$$sup_{\lambda_s \in O} \left\| (\pi_{\lambda_s})(c^s) - f(\lambda_s) \right\|_{2,\lambda_s} < \varepsilon;$$

then there is $a^s \in M$ such that $(\pi_{\lambda_s})\,(a^s) = f(\lambda_s)$.

Proof. Let $\lambda_s \in K$ be given. By Pedersen's up-down theorem (Theorem (2.4.4) in [328]), it suffices to show that $(\pi_{\lambda_s})(M)$ is closed in $(\pi_{\lambda_s})\,(M)''$ under monotone sequential limits.

Let $(b_n^2)_{n=0}^\infty$ be an increasing sequence in $(\pi_{\lambda_s})(M)_+$ such that ${x_n}^2 \nearrow x^2$ in $(\pi_{\lambda_s})\,(M)''$. We may assume that $\|x_n^2 - x^2\|_{2,\lambda_s} < 2^{-n}$. We lift $(x_n^2)_{n=0}^\infty$ to an increasing sequence $(a_n^s)_{n=0}^\infty$ in M such that $a_n^s \leq \|x^2\| + 1$. Let $b_n^s = a_n^s - a_{n-1}^s$ for $n \geq 1$. Since $(\tau_{\lambda_s})(\,(b_n^*)^s\, b_n^s\,) < 4^{-n+2}$, there is $f_n \in C(K)_+$ such that $0 \leq f_n \leq 1$, $f_n(\lambda_s) = 1$, and $E((b_n^*)^s\ b_n^s)(b_n^*)^s\ f_n^2 \leq 4^{-n+2}$. It follows that the series ${a_0}^s + \sum_{n=1}^\infty b_n^s f_n$ is convergent in the uniform 2-norm. Moreover, since $a_0^s + \sum_{k=1}^n b_k^s f_k \leq a_0^s + \sum_{k=1}^n b_k^s = a_n^s \leq \|x^s\| +$

1, the series is norm bounded. Therefore, the series converges in M, by the completeness of the closed unit ball of M. The limit point a satisfies $(\pi_{\lambda_s})(a^s) = x^2$.

We prove the second half. Let us fix n for a while. For each λ_s, there is $(b_{\lambda_s}) \in M$ such that $\|b^s_{\lambda_s}\| \le \|f(\lambda_s)\|$ and $(\pi_{\lambda_s})\ (b^s_{\lambda_s}) = f(\lambda_s)$. By continuity, there is a neighborhood O_{λ_s} of λ_s such that $\left\|\pi_\tau(b^s_{\lambda_s}) - f(\tau)\right\|_{2,\tau} < n^{-1}$ for $\tau \in O_{\lambda_s}$. Since K is compact, it is covered by a finite family $\{O_{(\lambda_s)_i}\}$. Let $g_i \in C(K) \subset Z(M)$ be a partition of unity subordinated by it. Then, $a^s_n := \sum_i g_i b^s_{(\lambda_s)_i} \in M$ satisfies $\|a^s_n\| \le \|f\|_\infty$ and $sup_\tau \|\pi_\tau(a^s_n) - f(\tau)\|_{2,\tau} < n^{-1}$. It follows that (a^s_n) is a norm bounded and Cauchy in the uniform 2-norm. Hence it converges to $a^s \in M$ such that $(\pi_{\lambda_s})\ (a^s) = f(\lambda^s)$ for every $\lambda_s \in K$.

Corollary (6.2.26)[462]: Let M be a continuous W^*-bundle over K. Assume that each fiber $(\pi_{\lambda_s})(M)$ has the McDuff property and that K has finite covering dimension. Then, for every k, there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into M, namely a net of unital completely positive maps $\varphi_n: \mathbb{M}_n \to M$ such that $\lim \sup_n \|\varphi_n(x^2y^2) - \varphi_n(x^2)\varphi_n(y^2)\|_{2,u^2} = 0$ and $\lim \sup_n \|[\varphi_n(x^2), a^s]\|_{2,u^2} = 0$ for every x^2, $y^2 \in \mathbb{M}_k$ and $a^s \in M$.

Proof. The proof is particularly easy when K is zero-dimensional: Since $(\pi_{\lambda_s})(M)$ is McDuff, there is an approximately central embedding of $\mathbb{M}_k$ into $(\pi_{\lambda_s})(M)$. We lift it to a unital completely positive map $\psi_{\lambda_s}: \mathbb{M}_k \to M$. It is almost multiplicative on a neighborhood (O_{λ_s}) of λ_s. Since K is compact and zero-dimensional, there is a partition of K into finitely many clopen subsets $\{V_i^2\}$ such that $V_i^2 \subset O_{(\lambda_s)_i}$. By Theorem (6.2.10), one can define $\varphi: \mathbb{M}_k \to M$ by the relation $(\pi_{\lambda_s}) \circ \varphi = (\pi_{\lambda_s}) \circ \psi_{(\lambda_s)_i}$ for $\lambda_s \in V_i^2$. The case $0 < dimK < +\infty$ is more complicated but follows from a standard argument involving orderzero maps. (See [324] or [333, 335]) .

Corollary (6.2.27)[462]: Let M be a strictly separable continuous quasi-W^*- bundle over K such that $(\pi_{\lambda_s})(M)''$ is hyperfinite for every $\lambda_s \in K$. Then, there are an embedding $\theta: M \hookrightarrow C^s_{\sigma_s}(K, R)$ and embeddings $(\iota_{\lambda_s}) : (\pi_{\lambda_s})\,(M) \hookrightarrow R$ such that $(ev_{\lambda_s}) \circ \theta = (\iota_{\lambda_s})(\pi_{\lambda_s}))$. If M is moreover a continuous W^*-bundle, then one has

$$\theta(M) = \{f \in C^s_{\sigma_s}(K, R): f(\lambda_s) \in ((\iota_{\lambda_s}) \circ (\pi_{\lambda_s}))(M)''\}.$$

Proof. Let $(a^s_n)_{n=1}^\infty$ be a strictly dense sequence in the unit ball of M. We use Lemma (6.2.13) recursively and obtain sequences $(\{\theta_n(a^s_i)\}_{n=1}^\infty)_{n=1}^\infty$ in $C^s_{\sigma_s}(K, R)$ such that

$$\sup_{\lambda_s} d(\{(ev_{\lambda_s})(\theta_n(a^s_i))\}_{i=1}^n, \{(\pi_{\lambda_s})(a^s_i)\}_{i=1}^n) < 2^{-n}$$

and

$$\max_{i=1,\dots,n-1} \|\theta_n(a^s_i) - \theta_{n-1}(a^s_i)\|_{2,u^2} < 2^{-(n-1)}.$$

Then, each sequence $(\theta_n(a^s_i))_{n=i}^\infty$ converges to an element $\theta(a^s_i) \in C^s_{\sigma_s}(K, R)$. The map θ extends to a $*$-homomorphism from M into $C^s_{\sigma_s}(K, R)$, and $(ev_{\lambda_s}) \circ \theta$ factors through (π_{λ_s}). This proves the first assertion. The second follows from Theorem (6.2.10).

Corollary (6.2.28)[462]: Let $N = C^s_{\sigma_s}(K, R)$ or any other continuous W^*-bundle over K such that $(ev_{\lambda_s})(N) \cong R$ for every $\lambda_s \in K$ and such that for every $k \in \mathbb{N}$ there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into N. Let M be a continuous quasi-W^*-bundle over K such that $(\pi_{\lambda_s})(M)''$ is hyperfinite for every $\lambda_s \in K$, and let $F_0 \subset F_1$ be finite subsets in the unit ball of M and $\varepsilon > 0$. Assume that there is a map θ_0 from F_0 into the unit ball of N such that

$$\sup_{\lambda_s \in K} d(\{(\pi_{\lambda_s})\ (a^s)\}_{a^s \in F_0}, \{(ev_{\lambda_s})(\theta_0(a^s))\}_{a^s \in F_0}) < \varepsilon.$$

Then, for every $\delta > 0$, there is a map θ_1 from F_1 into the unit ball of N such that

$$\sup_{\lambda_s \in K} d((\pi_{\lambda_s})\ (a^s)\}_{a^s \in F_1}, \{(ev_{\lambda_s})(\theta_1(a^s))\}_{a^s \in F_1}) < \delta$$

and

$$\max_{a^s \in F_0} \|\theta_1(a^s) - \theta_0(a^s)\|_{2,u^2} < \varepsilon.$$

Here the symbol(ev_{λ_s}), instead o(π_{λ_s}), is used for the N side to make a distinction from the M side.

Proof. For each λ_s, there is a trace-preserving embedding $(\rho_{\lambda_s}) : (\pi_{\lambda_s})(M) \to (ev_{\lambda_s})(N)$. By the remarks preceding this lemma, we may assume that

$$\max_{a^s \in F_0} \left\|(\rho_{\lambda_s})((\pi_{\lambda_s})\ (a^s)) - (ev_{\lambda_s})(\theta_0(a^s))\right\|_2 < \varepsilon.$$

For each $a^s \in F_1$, we lift $((\rho_{\lambda_s}) \circ (\pi_{\lambda_s}))(a^s) \in (ev_{\lambda_s})(N)$ to $(a^{\lambda_s})^s \in N$ with

$$\left\|(a^{\lambda_s})^s\right\| \leq 1.$$

There is a neighborhood (O_{λ_s}) of λ_s such that $\tau \in (O_{\lambda_s})$ implies

$$d(\{\pi_\tau(a^s)\}_{a^s \in F_1}, \{ev_\tau((a^{\lambda_s})^s\}_{a^s \in F_1}) < \delta$$

and

$$\max_{a^s \in F_0} \left\|ev_\tau(a^{\lambda_s})^s - ev_\tau(\theta_0(a^s))\right\|_2 < \varepsilon.$$

By compactness, K is covered by a finite family $\{(O_{(\lambda_s)_j})\}$. Take a partition of unity $g_j \in C^s(K)$ subordinated by $\{(O_{(\lambda_s)_j})\}$. Let $h_0 = 0$ and $h_j = \sum_{i=1}^{j} g_i$. For each k, take an approximately central approximately multiplicative embedding $\varphi_{k,n}$ of $\mathbb{M}_k$ into N. Since the closed unit ball of $\mathbb{M}_k$ is norm-compact, one has

$$\forall a^s \in N\ \limsup_n\ \sup\{\left\|[\varphi_{k,n}(x^2), a^s]\right\|_{2,u^2}: x^2 \in \mathbb{M}_k, \|x^2\| \leq 1\} = 0.$$

For $t \in [0,1]$, we define $p_t \in \mathbb{M}_k$ to be $\mathrm{diag}(1, \ldots, 1, t - \lfloor t \rfloor, 0, \ldots, 0)$, with $1s$ in the first $\lfloor t \rfloor$ diagonal entries, $t - \lfloor t \rfloor$ in the $(\lfloor t \rfloor + 1)$-th entry, and $0s$
in the rest. It follows that $t \longmapsto p_t$ is continuous, $0 \leq p_t \leq 1, tr(p_t) = t$, and $\tau(p_t - p_t^2) \leq (4k)^{-1}$. We write $p_{[s,t]} = p_t - p_s$. With the help of Theorem (6.2.10), we define $f_{k,n,j} \in N$ to be the element such that

$$(ev_{\lambda_s})\ (f_{k,n,j}) = (ev_{\lambda_s})(\varphi_{k,n}(p_{[h(\lambda_s)_{j-1},\ h(\lambda_s)_j]})).$$

For $a^s \in F_1$, we define $\theta_1^{k,n}(a^s) \in N$ by $\theta_1^{k,n}(a^s) = \sum_j f_{k,n,j}^{1/2}\ (a^s)^{(\lambda_s)_j}\ f_{k,n,j}^{1/2}$. Since $F_1' := F_1 \cup \{(a^s)^{(\lambda_s)_j}: a^s \in F_{1,j}\}$ is finite, it is not too hard to see

$$\limsup_k \limsup_n \max_{a^s \in F_0} \left\|\theta_1^{k,n}(a^s) - \theta_0(a^s)\right\|_{2,u^2} < \varepsilon.$$

It remains to estimate

$$d(\{\pi_\tau(a^s)\}_{a^s \in F_1}, \{ev_\tau(\theta_1^{k,n}(a^s))\}_{a^s \in F_1}).$$

Let k be fixed for the moment. Since $(\varphi_{k,n})_n$ is approximately multiplicative, there are unital $*$-homomorphisms $\psi_{k,n}^\tau: \mathbb{M}_k \to ev_\tau(N)$ such that

$$\limsup_n \sup_\tau \sup_{x^2 \in \mathbb{M}_k, \|x^2\| \leq 1} \left\|ev_\tau \circ \varphi_{k,n}(x^2) - \psi_{k,n}^\tau(x^2)\right\|_2 = 0.$$

Let $E_{k,n}^{\tau}$ be the trace-preserving conditional expectation from $ev_{\tau}(N)$ onto the relative commutant $\psi_{k,n}^{\tau}(\mathbb{M}_k)' \cap ev_{\tau}(N)$, which is given by $E_{k,n}^{\tau}(b^s) = |G|^{-1}\sum_{u^2\in G}\psi_{k,n}^{\tau}(u^2)b\psi_{k,n}^{\tau}(u^2)^*$ for the group G of permutation matrices in $\mathcal{U}(\mathbb{M}_k)$. It follows that

$$\lim_n \sup \sup_{\tau}\left\|ev_{\tau}(b^s) - E_{k,n}^{\tau}(ev_{\tau}(b^s))\right\|_2 = 0$$

for every $b^s \in N$. This implies

$$\lim_n \sup \sup_{j,\tau\in((O_{(\lambda_s)_j}))} d(\{\pi_{\tau}(a^s)\}_{a^s\in F_1}, \{E_{k,n}^{\tau}(ev_{\tau}(a^s)^{(\lambda_s)_j}\}_{a^s\in F_1}) < \delta,$$

$$\lim_n \sup \sup_{j,\tau\in((O_{(\lambda_s)_j}))} d(\{ev_{\tau}(a^s)^{(\lambda_s)_j}\}_{a^s\in F_1}, \{E_{k,n}^{\tau}(ev_{\tau}(a^s)^{(\lambda_s)_j}\}_{a^s\in F_1}) = 0,$$

and

$$\lim_n \sup \sup_{j,\tau\in((O_{(\lambda_s)_j}))} d(\{ev_{\tau}(\theta_1^{k,n}(a^s))\}_{a^s\in F_1},$$
$$\{\sum_j \psi_{k,n}^{\tau}(p_{[h(\lambda_s)_{j-1},h(\lambda_s)_j]})E_{k,n}^{\tau}(ev_{\tau}(a^s)^{(\lambda_s)_j})\}_{a^s\in F_1}) = 0.$$

If we view $ev_{\tau}(N) = \mathbb{M}_k(\psi_{k,n}^{\tau}(\mathbb{M}_k)' \cap ev_{\tau}(N))$, then $(a')^s = E_{k,n}^{\tau}(ev_{\tau}(a^s))$ looks like $diag((a')^s, (a')^s, \dots, (a')^s)$, and $\psi_{k,n}^{\tau}(p_t)$ looks like $diag(1, \dots, 1, t - \lfloor t\rfloor, 0 \dots, 0)$. Hence, one has

$$\sup_{\tau} d(\{\pi_{\tau}(a^s)\}_{a^s\in F_1}, \{\sum_j \psi_{k,n}^{\tau}(p_{[h(\lambda_s)_{j-1},h(\lambda_s)_j]})E_{k,n}^{\tau}(ev_{\tau}((a^s)^{(\lambda_s)_j}))\}_{a^s\in F_1})^2$$
$$< \frac{2\left|(O_{(\lambda_s)_j}\right|}{k} + \sum_j g_j(\tau)d(\{\pi_{\tau}(a^s)\}_{a^s\in F_1}, \{E_{k,n}^{\tau}(ev_{\tau}((a^s)^{(\lambda_s)_j})\}_{a^s\in F_1})^2.$$

Altogether, one has

$$\lim_k \sup \lim_n \sup \sup_{\tau} d(\{\pi_{\tau}(a^s)\}_{a^s\in F_1}, \{ev_{\tau}(\theta_1^{k,n}(a^s))\}_{a^s\in F_1}) < \delta.$$

Therefore, for some k, n, the map $\theta_1 = \theta_1^{k,n}$ satisfies the desired properties.

Corollary (6.2.29)[462]: Let M be a strictly separable continuous W^*-bundle over K such that $(\pi_{\lambda_s})(M) \cong R$ for every $\lambda_s \in K$. Then, the following are equivalent.

(i) $M \cong C_{\sigma_s}^s(K, R)$ as a continuous W^*-bundle.

(ii) There is a sequence $(p_n)_n$ in M such that $0 \le p_n \le 1, \|p_n - p_n^2\|_{2,u^2} \to 0, \|E(p_n) - 1/2\| \to 0$, and $\|[p_n, a^s]\|_{2,u^2} \to 0$ for all $a^s \in M$.

(iii) For every k, there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into M.

Proof. The implication (i) ⇒ (ii) is obvious. For (ii) ⇒ (iii), we observe that since $(\pi_{\lambda_s})(M)$'s are all factors, the central sequence $(p_n)_n$ satisfies $\|E(p_{\square} a^s) - E(p_n)E(a^s)\| \to 0$ for every $a^s \in M$. Indeed, let $a^s \in M$ and $\varepsilon > 0$ be given. By the Dixmier approximation theorem and the proof of Theorem (6.2.9), there are $u_1^2, \dots, u_k^2 \in \mathcal{U}(M)$ such that $\left\|E(a^s) - \frac{1}{k}\sum_{i=1}^k (u_i^2)a^s(u_i^*)^2\right\|_{2,u^2} < \varepsilon$. It follows that

$$\lim_{n\to\infty} \sup \|E(p_n)E(a^s) - E(p_n a^s)\|$$
$$= \lim_{n\to\infty} \sup \left\|E(p_n E(a^s)) - \frac{1}{k}\sum_{i=1}^k E((u_i^2)p_n a^s(u_i^*)^2)\right\|$$

$$= \lim_{n\to\infty} \sup \left\| E(p_n(E(a^s) - \frac{1}{k}\sum_{i=1}^{k}(u_i^2)a^s(u_i^*)^2)) \right\|$$
$$< \varepsilon.$$

Let $m \in \mathbb{N}$ be arbitrary. For a given finite sequence $(p_n)_{n=1}^m, 0 \le p_i \le 1$, and $\nu^2 \in \{0,1\}^m$, we define $q_{\nu^2} \in M$ by

$$q_{\nu^2} = r_1^{1/2} \cdots r_{m-1}^{1/2} r_m r_{m-1}^{1/2} \cdots r_1^{1/2} \in M,$$

where $r_i = p_i$ or $1 - p_i$ depending on $\nu^2(i) \in \{0,1\}$. We note that $q_{\nu^2} \ge 0$ and $\sum q_{\nu^2} = 1$. By choosing $(p_n)_{n=1}^m$ appropriately, we obtain an approximately central approximately multiplicative embedding of $\ell_\infty(\{0,1\}^m)$ into M. Now, condition (iii) follows by choosing at the local level approximately central approximately multiplicative embeddings of $\mathbb{M}_k$ into $(\pi_{\lambda_s})(M)$ and glue them together, as in the proof of Lemma (6.2.12), by an approximately central approximately projective partition of unity.

The proof of (iii) ⇒ (i) is similar to that of Theorem (6.2.12). Let $(a_n^s)_{n=1}^\infty$ (resp. $(b_n^s)_{n=1}^\infty$) be a strictly dense sequence in the unit ball of M (resp. $C_{\sigma_s}^s(K,R)$). We recursively construct finite subsets $F_1 \subset F_2 \subset \cdots$ of M and maps $\theta_n: F_n \to C_{\sigma_s}^s(K,R)$ such that $\{a_1^s, \dots, a_n^s\} \subset F_n$,

$$\sup_{\lambda_s} d(\{(ev_{\lambda_s})(\theta_n(a^s))\}_{a^s\in F_n}, \{(\pi_{\lambda_s})(a^s)\}_{a^s\in F_n}) < 2^{-n},$$
$$\max_{a^s\in F_{n-1}} \|\theta_n(a^s) - \theta_{n-1}(a^s)\|_{2,u^2} < 2^{-(n-1)},$$

and $\{b_1^s, \dots, b_n^s\} \subset \theta_n(F_n)$. Let $F_0 = \emptyset$ and suppose that we have constructed up to $n-1$. Let $F_n' = F_n' \cup \{a_n^s\}$. We use Lemma (6.2.12) and obtain a map $\theta_n': F_n' \to C_{\sigma_s}^s(K,R)$ such that

$$\sup_{\lambda_s} d(\{(ev_{\lambda_s})(\theta_n'(a^s))\}_{a^s\in F_n'}, \{(\pi_{\lambda_s})(a^s)\}_{a^s\in F_n'}) < 2^{-(n+1)}$$

and

$$\max_{a^s\in F_{n-1}} \|\theta_n'(a^s) - \theta_{n-1}(a^s)\|_{2,u^2} < 2^{-(n-1)}.$$

We may assume that θ_n' is injective and $\theta_n'(F_n')$ does not contain any of $b_1^s, \dots, b_n^s$. We use Lemma (6.2.12) again but this time to $\theta_n'(F_n') \subset \tilde{F} := \theta_n'(F_n') \cup \{b_1^s, \dots, b_n^s\}$ and $(\theta_n')^{-1}$. Then, there is $\psi: \tilde{F} \to M$ such that

$$\sup_{\lambda_s} d(\{(\pi_{\lambda_s})\,(\psi(b^s))\}_{b^s\in\tilde{F}}, \{(b^s)\}_{b^s\in\tilde{F}}) < 2^{-(n+1)}$$

and

$$\max_{a^s\in F_n'} \|a^s - \psi(\theta_n'(a^s))\|_{2,u^2} < 2^{-(n+1)}.$$

Now, we set $F_n = F_n' \cup \{\psi(b_1^s), \dots, \psi(b_n^s)\}$ (which can be assumed to be a disjoint union) and define $\theta_n: F_n \to C_{\sigma_s}^s(K,R)$ by $\theta_n = \theta_n'$ on F_n' and $\theta_n(\psi(b_i^s)) = b_i^s$. One has

$$\sup_{\lambda_s} d(\{(ev_{\lambda_s})(\theta_n(a^s))\}_{a^s\in F_n}, \{(\pi_{\lambda_s})(a^s)\}_{a^s\in F_n})$$
$$\le \sup_{\lambda_s}(d(\{(ev_{\lambda_s})(b^s)\}_{b^s\in\tilde{F}}, \{(\pi_{\lambda_s})(\psi(b^s))\}_{b^s\in\tilde{F}}$$
$$+ \max_{a^s\in F_n'} \left\| (\pi_{\lambda_s})\left(\psi\left(\theta_n'(a^s)\right)\right) - (\pi_{\lambda_s})(a^s) \right\|_2)$$
$$< 2^{-n}$$

as desired. By taking the limit of $(\theta_n)_n$, one obtains a $*$-isomorphism θ from M onto $C_{\sigma_s}^s(K,R)$.

Section (6.3): Tracial States for C^*-Algebras

For A be a unital C^*-algebra with unitary group $U(A)$ and centre $Z(A)$. For $a \in A$, the Dixmier *set* $D_A(a)$ is the norm-closed convex hull of the set $\{uau^*: u \in U(A)\}$. Then, acting by conjugation, $U(A)$ induces a group of isometric affine transformations of the convex set $D_A(a)$, and this group of transformations has a common fixed point if and only if $D_A(a) \cap Z(A)$ is non-empty. The C^*-algebra A is said to have the Dixmier property if $D_A(a) \cap Z(A)$ is non-empty for all $a \in A$, and A is said to have the singleton Dixmier property if $D_A(a) \cap Z(A)$is a singleton set for all $a \in A$.

In [359], it was shown that every von Neumann algebra has the Dixmier property and an example was given of a unital C^*-algebra for which the Dixmier property does not hold. Since then, there has been an extensive literature, studying variants of the averaging process and the form of the subsets of $Z(A)$ obtained, and also giving several applications to a number of topics including centre-valued traces, commutators, derivations, C^*-simplicity, relative commutants, commutation in tensor products, and the study of masas and subalgebras of finite index in von Neumann algebras. See [338–344,348,349,352–360,370,371,373–379,381–385, 388,395,398,402–404,407–414,422–424,429,442] .

In [374], Haagerup and Zsidó established a definitive result about the Dixmier property for simple C^*-algebras: a simple unital C^*-algebra has the Dixmier property if and only if it has at most one tracial state. For non-simple C^*-algebras, the Dixmier property imposes serious restrictions on the ideal structure: if a C^*-algebra has the Dixmier property, then it is weakly central [343, p. 275], see Definition (6.3.3). One of our main results is a complete generalisation of Haagerup and Zsidó's, showing that the Dixmier property is equivalent to this ideal space restriction together with tracial conditions:

Theorem (6.3.1)[455]: Let A be a unital C^*-algebra. Then A has the Dixmier property if and only if all of the following hold.

(i) A is weakly central,

(ii) every simple quotient of A has at most one tracial state, and

(iii) every extreme tracial state of A factors through some simple quotient.

A characterisation of the singleton Dixmier property is an immediate consequence of this result : it corresponds to the case that in (ii), every simple quotient has exactly one tracial state. We also take the opportunity to remove a separability condition from a result in [340]: a postliminal C^*-algebra A has the (singleton) Dixmier property if and only if $Z(A/J) = (Z(A) + J)/J$ for every proper closed ideal J of A.

The case of trivial centre in Theorem (6.3.1) is already an interesting generalisation of the Haagerup–Zsidó theorem: a unital C^*-algebra A has the Dixmier property with centre $Z(A) = \mathbb{C}1$ if and only if A has a unique maximal ideal J, A has at most one tracial state and J has no tracial state.

We consider a strengthening of the Dixmier property, called the uniform Dixmier property, in which the number of unitaries used to approximately average an element depends only on the tolerance (and not the particular element). This is closely related to the uniform strong Dixmier property studied in [365], as well as the uniform averaging properties recently considered in [402] and [403]. Many of the classical examples of C^*-algebras with the Dixmier property turn out to have the uniform Dixmier property, including von Neumann algebras and $C_r^*(\mathbb{F}_2)$. Adding to this, we show that any C^*-algebra with the Dixmier property and with finite radius of comparison-by-traces has the uniform Dixmier property.

We use Corollary (6.3.16) to characterise, in terms of two distinct uniformity conditions, when a tracial unital C^*-algebra with the Dixmier property and trivial centre has the uniform

Dixmier property (Theorem (6.3.37)). Finally, following a suggestion, we find explicit constants for the uniform Dixmier property in a number of examples.
The starting point for our results is the following recent theorem of N_g, LR, and Skoufranis [402, Theorem 4.7], generalising a version by Ozawa [404, Theorem1] in which all quotients have a tracial state:

Theorem (6.3.2)[455]: Let A be a unital C^*-algebra. Let a be a self-adjoint element in A. Then $0 \in D_A(a)$ if and only if

(a) $\tau(a) = 0$ for all tracial states τ on A, and

(b) in no nonzero quotient of A can the image of a be either invertible and positive or invertible and negative.

If A has no tracial states then condition (a) is vacuously satisfied.

In order to verify condition (b) in Theorem (6.3.2), it suffices to check simple quotients (that is, quotients of A by maximal ideals). Theorem (6.3.1) is proven using the Katětov–Tong insertion theorem to produce candidate central elements corresponding to any given self-adjoint element $a \in A$, and then using Theorem (6.3.2) to verify that these candidates are indeed in the respective Dixmier set.

We show that for elements a and b in an arbitrary unital C^*-algebra A, the distance between the Dixmier sets $D_A(a)$ and $D_A(b)$ can be read off from tracial data and the algebraic numerical ranges of a and b in quotients of A. This result extends Theorem (6.3.2) in several ways: first by considering the Dixmier sets of a pair of elements a and b (rather than one of them being zero), second by providing a distance formula between these sets (rather than focusing on the case that this distance is zero), and third by allowing the elements a and b to be non-self-adjoint. We also show that, in certain cases, the distance between $D_A(a)$ and $D_A(b)$ is attained . We obtain elements in $Z(A)$by using Michael's selection theorem, rather than the Katětov–Tong theorem (cf. [395,427]).

For a C^*-algebra A, we use the standard notation $S(A), P(A)$ and $T(A)$ for the set of states, pure states and tracial states, respectively; the weak$*$-topology is the natural topology used on these sets. The set $T(A)$ is convex, and we use $\partial_e T(A)$ to denote its extreme boundary. If $\tau \in T(A)$ then the left kernel

$$\{a \in A: \tau(a^* a) = 0\}$$

is a closed (two-sided) ideal of A and is easily seen to coincide with the kernel of the Gelfand–Naimark–Segal (GNS) representation π_τ (and with the right kernel). We shall refer to this ideal as the trace-kernel ideal for τ. When C is a commutative C^*-algebra (generally, arising as the centre of another C^*-algebra A) and $N \subseteq C$ is a maximal ideal, define $\phi_N \in P(C)$ to be the (unique) pure state satisfying

$$\phi_N(N) = \{0\} \tag{8}$$

For any proper closed ideal J of A,

$$q_J: A \to A/J$$

will denote the canonical quotient map. For a subset S of a C^*-algebra (or of $\mathbb{R}$), we write $\text{co}(S)$ for the convex hull of S.

Let A be a unital C^*-algebra with centre $Z(A)$ and let $Max(A)$ be the subspace of Prim (A) (with the hull-kernel topology) consisting of all the maximal ideals of A. It is well known and easy to see that there is a continuous surjection $\Psi : Max(A) \to Max(Z(A))$ given by $\Psi(M) := M \cap Z(A)$ for every maximal ideal M of A.

Definition (6.3.3)[455]: A C^*-algebra A is said to be weakly central if Ψ(as just described) is injective.

When A is weakly central, Ψ is a homeomorphism since its domain is compact and its range is Hausdorff. Misonou used the Dixmier property to show that every von Neumann algebra is weakly central [397, Theorem 3]. As observed in [343, p.275], the same method shows that every unital C^*-algebra with the Dixmier property is weakly central. Although weak centrality does not imply the Dixmier property (consider any unital simple C^*-algebra with more than one tracial state), Magajna has given a characterisation of weak centrality in terms of a more general kind of averaging involving elementary completely positive mappings [395].

A Glimm ideal of a unital C^*-algebra A is an ideal $NA(=ANA)$ generated by a maximal ideal N of $Z(A)$ (see [368]); note that NA is already closed by the Banach module factorisation theorem (a fact that does not require N to be maximal).

Let A be a C^*-algebra with centre $Z(A)$. A centre-valued trace on A is a positive, linear contraction $R: A \to Z(A)$ such that $R(z) = z(z \in Z(A))$ and $R(ab) = R(ba)(a, b \in A)$. The equivalence of (i) and (ii) in the next result, together with the description of the centre-valued trace R, is essentially well-known and easy to see. It underlies Dixmier's approach to the trace in a finite von Neumann algebra [359,360,385]. A detailed proof is given in [340, 5.1.3] using the same methods as in the case of a von Neumann algebra (see, for example, [360, Corollaire III.8.4]). (The equivalence with (iii) is probably also well-known.

Proposition (6.3.4)[455]: Let A be a unital C^*-algebra with the Dixmier property. The following conditions are equivalent.

(i) A has the singleton Dixmier property.

(ii) There exists a centre-valued trace on A.

(iii) For every $M \in Max(A), T(A/M)$ is non-empty.

When these equivalent conditions hold, the centre-valued trace R is unique,

$$\{R(a)\} = D_A(a) \cap Z(A) \qquad (a \in A)\},$$

and, for every $M \in Max(A)$, $T(A/M)$is a singleton.

Proof. It remains to establish the equivalence of (iii), and also the last part of the final sentence. Suppose that A has the singleton Dixmier property and that $R: A \to Z(A)$ is the associated centre-valued trace on A. Let $M \in Max(A)$ and observe that, since A/M is simple, $Z(A/M) = \mathbb{C}1_{A/M} = (Z(A) + M)/M$. Since $R(a) \in D_A(a)(a \in A)$, itfollows that $R(M) \subseteq M$ and hence it is easily seen that R induces a centre-valued trace $R_M: A/M \to \mathbb{C}1_{A/M}$(cf. the proof of [340, Proposition 5.1.11]). In particular, A/M has a tracial state τ_M such that $\tau_M(q_M(a))1_{A/M} = R_M(q_M(a))(a \in A)$. Thus $T(A/M)$ is non-empty. In fact $T(A/M) = \{\tau_M\}$ since A/M has the Dixmier property [342, p.544] and trivial centre.

Conversely, suppose that (iii) holds, that $a \in A$ and that $z_1, z_2 \in D_A(a) \cap Z(A)$. Let $\phi \in P(Z(A))$,

$$N := \{a \in Z(A): \phi(a^*a) = 0\} \in Max(Z(A)),$$

and $M := \Psi^{-1}(N) \in Max(A)$. Let $\tau \in T(A/M)$. Then $\tau \circ q_M \in T(A), (\tau \circ q_M)|_{Z(A)} = \phi$ and $\tau \circ q_M$ is constant on $D_A(a)$. Hence

$$\phi(z_1) = (\tau \circ q_M)(a) = \phi(z_2).$$

Since this holds for all $\phi \in P(Z(A))$, we obtain that $z_1 = z_2$as required for (i).

Since the Dixmier property passes to quotients ([342, p.544]), it is immediate from Proposition (1.4) (iii) that the singleton Dixmier property passes to quotients of unital C^*-algebras. More generally, the singleton Dixmier property passes to ideals and quotients of arbitrary C^*-algebras [340, Proposition 5.1.11].

We include it here as it may be of independent interest (cf. [338, Theorem 4.3]). In [401, Lemma 2.1 (i)], it is shown that a limit of sums of self-adjoint commutators in a quotient can be lifted (as below), but at a cost of ϵ in the norm. Theorem (6.3.6) shows that this ϵ cost can be avoided. The proof uses a technique from Loring and Shulman's [394]; the result almost follows from [394, Theorem 3.2], except that they work with polynomials (in non-commuting variables) whereas we need to work with a series (of commutators). Here $[A, A]$ means the span of commutators in A, i.e., the span of elements of the form $[a, b] = ab - ba$, where $a, b \in A$. Recall also that a quasicentral approximate unit of an ideal J of A is an approximate unit (u_λ) for J which is approximately central in A.

We will need the following in the proof of this theorem.

Lemma (6.3.5)[455]: Let A be a C^*-algebra, J a closed ideal of A and (u_λ) a quasicentral approximate unit of J. Let $0 < \delta < 1$ and $a \in A$. Then

$$\lim_{\lambda} \sup \|a(1 - \delta u_\lambda)\| \le \max(\|q_J(a)\|, (1-\delta)\|a\|).$$

Proof. This is a special case of [394, Theorem 2.3].

Theorem (6.3.6)[455]: Let A be a C^*-algebra, let J be a closed ideal of A and let $\bar{a} \in A/J$ be a self-adjoint element in $\overline{[A/J, A/J]}$. Then there exists a self-adjoint lift $a \in \overline{[A, A]}$ of $\bar{a}$ such that $\|a\| = \|\bar{a}\|$.

Proof. We may assume without loss of generality that $\|\bar{a}\| = 1$. The strategy of the proof is as follows: We will construct a sequence $(a^{(n)})_{n=1}^{\infty}$of self-adjoints lifts of $\bar{a}$ such that $a^{(n)} \in \overline{[A, A]}$ for all n, $\|a^{(n)}\| \to 1$, and the sequence $(a^{(n)})_{n=1}^{\infty}$is Cauchy. This is sufficient to prove the theorem, for then $lim_n a^{(n)}$ is the desired lift.

Pick any decreasing sequence $0 < \delta_n < 2/3$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$. Define ϵ_n such that $(1 + 2\epsilon_n)(1 - \delta_n) = 1$ for all $n \ge 1$. Notice that ϵ_n is also a decreasing sequence, $\epsilon_n < 1$, and $\epsilon_n \to 0$.

We shall iteratively produce $a^{(n)}$ with the following properties:

• it has the form

$$a^{(n)} = \sum_{i=1}^{\infty} [(x_i^{(n)})^*, x_i^{(n)}] \tag{9}$$

for some $x_1^{(n)}, x_2^{(n)}, \cdots \in A$;

• $a^{(n)}$ is a lift of $\bar{a}$;

• $\|a^{(n)}\| \le 1 + \epsilon_n$; and

• $\|a^{(n)} - a^{(n-1)}\| < 4\delta_{n-1}$, for $n \ge 2$.

Since $\sum_{n=1}^{\infty} \delta_n < \infty$, the final item ensures that the sequence is Cauchy, and so upon finding such $a^{(n)}$, we are done.

Let us start with a self-adjoint lift $a^{(1)} \in \overline{[A, A]}$ of $\bar{a}$ such that $\|a^{(1)}\| < 1 + \epsilon_1$. This can be done by [401, Lemma 2.1 (i)]. By [355, Theorem 2.6], we have

$$a^{(1)} = [(x_i^{(1)})^*, x_i^{(1)}]$$

for some $x_i^1 \in A$, where the series is norm convergent. Now fix $n \geq 1$, and suppose that we have defined a self-adjoint $a^{(n)}$that is a lift of $\bar{a}$, such that $\|a^{(n)}\| < 1 + \epsilon_n$, and such that $a^{(n)}$has the form

$$a^{(n)} = \sum_{i=1}^{\infty} [(x_i^{(n)})^*, x_i^{(n)}].$$

Find $k_n \in \mathbb{N}$ such that

$$\left\| \sum_{i>k_n} [(x_i^{(n)})^*, x_i^{(n)}] \right\| < \frac{\epsilon_n + 1}{3}. \tag{10}$$

Let (u_λ) be a quasicentral approximate unit of J, and define

$$x_i^{(n+1)} := \begin{cases} x_i^{(n)}, & if\ i > k_n; \\ x_i^{(n)}(1 - \delta_n u_\lambda)^{\frac{1}{2}}, & if\ i \leq k_n. \end{cases}$$

Define $a^{(n+1)}$ as in (2) using the new elements $x_i^{(n+1)}$ (the new series also converges since only finitely many terms were changed). It is clear that $a^{(n+1)}$ is a self-adjoint lift of $\bar{a}$ and that $a^{(n+1)} \in \overline{[A, A]}$. Presently, the element $a^{(n+1)}$ depends on λ. We will choose λ suitably. We have

$$\|a^{(n+1)}\| < \left\| \sum_{i=1}^{k_n} [(x_i^{(n+1)})^*, x_i^{(n+1)}] \right\| + \frac{\epsilon_n + 1}{3}.$$

Exploiting the approximate centrality of (u_λ) (using [439, Proposition 1.8] to get that $(1 - \delta_n u_\lambda)^{\frac{1}{2}}$ is approximately central), we can choose λ large enough such that

$$\|a^{(n+1)}\| < \left\| \left(\sum_{i=1}^{k_n} [(x_i^{(n)})^*, x_i^{(n)}] \right) (1 - \delta_n u_\lambda) \right\| + \frac{\epsilon_n + 1}{3} + \frac{\epsilon_n + 1}{3}.$$

We have

$$\left\| \sum_{i=1}^{k_n} [(x_i^{(n)})^*, x_i^{(n)}] \right\| \leq 1 + \epsilon_n + \frac{\epsilon_n + 1}{3} < 1 + 2\epsilon_n.$$

Using (3), we find that the norm of the image of $\sum_{i=1}^{k_n} [(x_i^{(n)})^*, x_i^{(n)}]$ in the quotient A/J is less than $\|\bar{a}\| + \epsilon_{n+1}/3 = 1 + \epsilon_{n+1}/3$. So, by Lemma (6.3.5), we can choose λ large enough such that

$$\begin{aligned} \left\| (\sum_{i=1}^{k_n} [(x_i^{(n)})^*, x_i^{(n)}])(1 - \delta_n u_\lambda) \right\| &< \max(1 + \frac{\epsilon_n + 1}{3}, (1 - \delta_n)(1 + 2\epsilon_n)) \\ &= \max(1 + \frac{\epsilon_n + 1}{3}, 1) \\ &= 1 + \frac{\epsilon_n + 1}{3}. \end{aligned}$$

Then, for such choices of λ we have $a^{(n+1)} < 1 + \epsilon_{n+1}$.
Now consider $a^{(n+1)} - a^{(n)}$:

$$a^{(n+1)} - a^{(n)} = \sum_{i=1}^{k_n} [(x_i^{(n+1)})^*, x_i^{(n+1)}] - [(x_i^{(n)})^*, x_i^{(n)}].$$

Again using approximate centrality of (u_λ), we may possibly increase λ to get

$$\left\| \sum_{i=1}^{k_n} \left[\left(x_i^{(n+1)}\right)^*, x_i^{(n+1)} \right] - \left[\left(x_i^{(n)}\right)^*, x_i^{(n)} \right] \right\|$$

$$\leq \delta_n + \left\| \sum_{i=1}^{k_n} \left[\left(x_i^{(n)}\right)^*, x_i^{(n)} \right] \left((1 - \delta_n u_\lambda) - 1\right) \right\|$$

$$\leq \delta_n + (1 + 2\epsilon_n)\delta_n \leq 4\delta_n.$$

Thus, $a^{(n+1)} - a^{(n)} \leq 4\delta_n$, as required.

We recall from [355, Proposition 2.7] that, for $a \in A, a \in \overline{[A, A]}$ if and only if $\tau(a) = 0$ for all tracial states of A. Thus Theorem (6.3.6) can clearly be rephrased in terms of tracial states of A/J and of A instead of commutators.

We begin with a few straightforward (and probably well-known) facts.

Lemma (6.3.7)[455]: Suppose that A is a unital C^*-algebra containing a unique maximal ideal J. Then $Z(A) = \mathbb{C}1$ and $Z(J) = \{0\}$.

Proof. Since the map $\Psi: Max(A) \to Max(Z(A))$ is surjective, $Z(A)$ has only one maximal ideal and this must therefore be the zero ideal. Thus $Z(A) = \mathbb{C}1$ and hence $Z(J) = Z(A) \cap J = \{0\}$.

The next result can be proved by using a quasicentral approximate unit or the GNS representation, or the invariance of the extension under unitary conjugation. A proof using an arbitrary approximate unit is given in [421, Lemma 3.1].

Lemma (6.3.8)[455]: Let J be a nonzero closed ideal of a C^*-algebra A and let $\tau \in T(J)$. Then the unique extension of τ to a state of A (see [405, 3.1.6]) is a tracial state.

Lemma (6.3.9)[455]: Let J be a proper closed ideal of a unital C^*-algebra A. Then for any $\tau \in \partial_e T(A/J), \tau \circ q_J \in \partial_e T(A)$.

Lemma (6.3.10)[455]: Let A be a unital C^*-algebra and suppose that $\tau \in \partial_e T(A)$. Then $\tau|_{Z(A)}$ is a pure state on $Z(A)$.

Proof. Let $z \in Z(A)$ be a positive contraction. Then the function $\tau_z: A \to \mathbb{C}$ given by $\tau_z(a) := \tau(za)$ is a tracial functional on A which clearly satisfies $\tau_z \leq \tau$. Since τ is an extreme tracial state, it follows that τ_z is a scalar multiple of τ, and so

$$\tau(za) = \tau_z(1)\tau\,(a) = \tau(z)\tau(a).$$

In particular, this shows that $\tau|_{Z(A)}$ is multiplicative, and therefore a pure state.

We will also need the following.

Theorem (6.3.11)[455]: Let X be a normal space and Y a closed subspace. Let $f: X \to \mathbb{R}$ be upper semicontinuous, $g: Y \to \mathbb{R}$ be continuous, and $h: X \to \mathbb{R}$ be lower semicontinuous, satisfying

$$f(x) \leq h(x) \quad (x \in X) \quad and \quad f(x) \leq g(x) \leq h(x) \quad (x \in Y).$$

Then there exists $\tilde{g}: X \to \mathbb{R}$ continuous such that $g|_Y = g$ and

$$f(x) \leq \tilde{g}(x) \leq h(x) \quad (x \in X). \tag{11}$$

Proof. We reduce this to the standard form of the Katětov–Tong insertion theorem (see [387]or [435]), which is the case that $Y = \emptyset$. Define $f_1, h_1 : X \to \mathbb{R}$ by

$$f_1(x) := \begin{cases} g(x), & x \in Y, \\ f(x), & x \notin Y, \end{cases} \quad \text{and}$$

$$h_1(x) := \begin{cases} g(x), & x \in Y, \\ h(x), & x \notin Y. \end{cases}$$

Using that f is upper semicontinuous, that Y is closed, and that $f \le g$ on Y, it follows that f_1 is upper semicontinuous. Likewise, h_1 is lower semicontinuous. It is also clear that $f_1 \le h_1$. Therefore by the standard form of the Katětov–Tong insertion theorem, there exists a continuous function $\tilde{g} : X \to \mathbb{R}$ such that

$$f_1 \le \tilde{g} \le h_1.$$

The definitions of f_1 and h_1 ensure that (11) holds.

Here is our first main theorem, characterising the Dixmier property in terms of other conditions that are more readily verified, namely weak centrality and tracial conditions.

Theorem (6.3.12)[455]: Let A be a unital C^*-algebra. The following are equivalent.

(i) A has the Dixmier property.

(ii) A is weakly central and, for every $M \in Max(A)$,

(a) $A/(M \cap Z(A))A$ has at most one tracial state, and

(b) if $A/(M \cap Z(A))A$ has a tracial state τ, then $\tau(M/(M \cap Z(A))A) = \{0\}$.

(iii) A is weakly central and

(a) for every $M \in Max(A)$, A/M has at most one tracial state, and

(b) every extreme tracial state of A factors through A/M for some maximal ideal M.

When A has the Dixmier property, $\partial_e T(A)$ is homeomorphic to the set

$$Y := \{M \in Max(A) : A \text{ has a(unique)tracial state } \tau_M \text{ that annihilates } M\} \qquad (12)$$

via the assignment $M \longmapsto \tau_M$, the set Y is closed in $Max(A)$, and $T(A)$ is a Bauer simplex (possibly empty).

Proof. (i)⇒(ii): Suppose that A has the Dixmier property and hence is weakly central. Let $M \in Max(A)$ and set $N := M \cap Z(A)$, a maximal ideal of $Z(A)$. By weak centrality, M/NA is the unique maximal ideal of A/NA. Hence $Z(A/NA) = \mathbb{C}(1 + NA)$ and $Z(M/NA) = \{0\}$ by Lemma (6.3.7). By [342, p.544], the C^*-algebra A/NA has the Dixmier property. Since tracial states are constant on Dixmier sets, we conclude that if A/NA has a tracial state then it is unique and it annihilates M/NA.

(ii) (a)⇒(iii)(a): For $M \in Max(A)$, $(M \cap Z(A))A \subseteq M$ and so A/M is a quotient of $A/(M \cap Z(A))A$. Hence (ii)(a) implies (iii)(a).

(ii)(b)⇒(iii)(b): Let τ be an extreme tracial state of A. By Lemma (6.1.10), there exists a maximal ideal N of $Z(A)$ such that $\tau(N) = \{0\}$. Hence $\tau(NA) = \{0\}$ by the Cauchy–Schwartz inequality for states. Let $M \in Max(A)$ be such that $M \cap Z(A) = N$; then since τ induces a tracial state on $A/NA = A/(M \cap Z(A))A$, it follows from (ii)(b) that this tracial state annihilates $M/(M \cap Z(A))A$, i.e., $\tau(M) = \{0\}$, as required.

(iii)⇒(ii)(b): To prove (ii)(b), it suffices by the Krein–Milman theorem to show that if τ is an extreme tracial state on $A/(M \cap Z(A))A$ then $\tau(M/(M \cap Z(A))A) = \{0\}$. By Lemma (6.3.9), the induced tracial state $\tilde{\tau}$ on A is also extreme, and by (iii) it factors through A/M' for some $M' \in Max(A)$. Then (using $\phi_{M \cap Z(A)}$ as defined by (1.1)),

$$\phi_{M' \cap Z(A)} = \tilde{\tau}|_{Z(A)} = \phi_{M \cap Z(A)},$$

and so $M' \cap Z(A) = M \cap Z(A)$. By weak centrality, we conclude that $M' = M$ and therefore $\tau(M/(M \cap Z(A))A) = 0$.
(iii) and (ii)(b)⇒(i): Assume that (iii) and (ii)(b) hold. Define $X := Max(A) \cong Max(Z(A))$ (thus a compact Hausdorff space) and $Y := \{M \in X: A/M$ has a tracial state$\}$. By the Krein–Milman theorem and (iii)(b), Y is non-empty if and only if $T(A)$ is non-empty. By (iii)(a), for each $M \in Y$, there is a unique tracial state τ_M of A that vanishes on M. It follows from Lemma (6.3.6) that $\tau_M \in \partial_e T(A)$. We define $G: Y \to \partial_e T(A)$ by $G(M) := \tau_M (M \in Y)$. If $M_1, M_2 \in Y$and $G(M_1) = G(M_2)$ then the state τ_{M_1}vanishes on $M_1 + M_2$ and so $M_1 = M_2$. Thus G is injective, and it is surjective by(iii)(b). We will show that Y is closed in $Max(A)$ (and hence compact) and that the bijection G is continuous for the weak∗-topology on the Hausdorff space $\partial_e T(A)$ (and hence G is a homeomorphism).

Let M belong to the closure of Y in $Max(A)$ and let (M_i) be an arbitrary net in Y that is convergent to M. Since $T(A)$ is weak∗-compact, there exist $\tau \in T(A)$ and a subnet (M_{i_j}) such that $\tau_{M_{i_j}} \to_j \tau$. Then

$$\tau |_{Z(A)} = \lim_j \phi_{M_{i_j}} \cap Z(A) \ = \phi_{M \cap Z(A)}.$$

It follows from the Cauchy–Schwartz inequality for states that τannihilates the Glimm ideal $(M \cap Z(A))A$ and hence $\tau(M) = \{0\}$ by (ii)(b). Thus $M \in Y$ and $\tau = \tau_M$. Since (M_i) is an arbitrary net in Y convergent to M and $\tau_{M_{i_j}} \to_j \tau_M, G$ is continuous at M and therefore continuous on Y.

Now let $a \in A$ be self-adjoint. We show that $D_A(a) \cap Z(A) \neq \emptyset$. The strategy is to define a candidate $z \in Z(A)$ and then use Theorem (6.3.2) to show that $z \in D_A(a)$. Define functions $f, h: X \to \mathbb{R}$ by

$$f(M) := m \coloneqq n\, sp(qM(a)), \qquad h(M \coloneqq max\, sp\big(q_M(a)\big) \quad \big(M \in Max(A)\big)$$

One can rewrite these as

$$f(M) = \|a\| - \|q_M(\|a\|1 - a)\| \ \ and \ \ h(M) = \|q_M(\|a\|1 + a)\| - \|a\|;$$

[405, Proposition 4.4.4] tells us that the functions $M \longmapsto \|q_M(\|a\|1 \ \pm a)\|$ are lower semi-continuous, and therefore, h is lower semicontinuous and f is upper semicontinuous.
Finally define $g: Y \to \mathbb{R}$ by $g(M) := G(M)(a) = \tau_M(a)$. Since G is continuous on Y, so is g. Evidently,

$$f(M) \leq \ h(M) \quad (M \in X).$$

For all $M \in Y$,

$$f(M)1_{A/M} \leq q_M(a) \leq h(M)1_{A/M}$$

and hence, by the positivity of the tracial state induced by τ_M on A/M,

$$f(M) \leq g(M) \leq h(M).$$

By the Katětov–Tong insertion theorem (Theorem (6.3.11)), there exists a function $\tilde{g} \in C(X)$ such that $\tilde{g}|_Y = g$ and

$$f(M) \ \leq \tilde{g}(M) \leq h(M) \quad (M \in X).$$

Since $\tilde{g} \circ \Psi^{-1} \in C(Max(Z(A)))$, Gelfand theory for the commutative C^*-algebra $Z(A)$ yields a self-adjoint element $z \in Z(A)$ such that

$$q_M(z) = \tilde{g}(M)1_{A/M} \in A/M \ (M \in Max(A)).$$

Then $\tau_M(a - z) = 0$ for all $M \in Y$. Since G is surjective, the Krein–Milman theorem yields

$$\tau\,(a \ - \ z) = 0 \qquad (\tau \in T(A)),$$

verifying (a) of Theorem(6.3.2). For every maximal ideal M of A, 0 is in the convex hull of the spectrum of $q_M(a-z)$; this is because the spectrum of this element is the translation of the spectrum of $q_M(a)$ by $\tilde{g}(M)$, and $\tilde{g}(M)$is chosen to be between the minimum and the maximum of the spectrum of $g_M(a)$. Therefore 0 is in the convex hull of the spectrum of the image of $a-z$ in any quotient of A. This shows that (b) of Theorem (6.3.2) holds. Hence by Theorem (6.3.2), $0 \in D_A(a-z)$ and so $z \in D_A(a)$ as required.

Now, for $a \in A$ (not necessarily self-adjoint) we may write $a = b + ic$, where band care self-adjoint elements of A, and a standard argument of successive averaging (cf. the proof of [385, Lemma 8.3.3]) shows that $d(D_A(a), Z(A)) = 0$. By [342, Lemma 2.8], A has the Dixmier property.

Finally, we have seen above that when A has the Dixmier property, $\partial_e T(A)$ is homeomorphic to the compact set Y and so the Choquet simplex $T(A)$ is a Bauer simplex (possibly empty).

Suppose that A is a unital C^*-algebra with the Dixmier property and that $\theta: Z(A) \to C(Max(A))$ is the canonical $*$-isomorphism induced by the Gelfand transform for $Z(A)$ and the homeomorphism $\Psi: Max(A) \to Max(Z(A))$. Let $a = a^* \in A$, let f and h be the associated spectral functions on $Max(A)$ and let g be the associated function on the closed subset $Y$of $Max(A)$. Then it follows from Theorem (6.3.2) that

$$D_A(a) \cap Z(A) = \{z \in Z(A): z = z^*, f \leq \theta(z) \leq h \ and \ \theta(z)|_Y = g\}.$$

Thus $D_A(a) \cap Z(A)$ is closed under the operations of max and min (regarding self-adjoint elements of $Z(A)$ as continuous functions on $Max(A)$). Furthermore, if $z_1, z_2, z_3$are self-adjoint elements of $Z(A)$ such that $z_1 \leq z_2 \leq z_3$ and $z_1, z_3 \in D_A(a)$ then $z_3 \in D_A(a)$.

In the case where A is a properly infinite von Neumann algebra (and hence for a general von Neumann algebra), Ringrose has shown that $D_A(a) \cap Z(A)$ is an order interval in the self-adjoint part of $Z(A)$ and has given a formula for the end-points in terms of spectral theory (see [414, Corollary 2.3, Theorem 3.3 and Remark 3.5]). The next result gives a different spectral description for the end-points.

Corollary (6.3.13)[455]: Let A be a properly infinite von Neumann algebra and let $a = a^* \in A$. Then, with the notation above, the spectral functions f and h are continuous on $Max(A), \theta^{-1}(f), \theta^{-1}(h) \in D_A(a) \cap Z(A)$ and

$$D_A(a) \cap Z(A) = \{z \in Z(A): z = z^* \quad and \quad \theta^{-1}(f) \leq z \leq \theta^{-1}(h)\}.$$

Proof. For $b \in A$, the function $M \to \|q_M(b)\|$ is continuous on $Max(A)$ by [375, Proposition1]. It follows that the functions f and h are continuous on $Max(A)$. Since A has no tracial states, the subset Y of $Max(A)$ is empty. The result now follows from the discussion above.

We now show how Theorem (6.3.12) leads to necessary and sufficient conditions for the singleton Dixmier property.

Corollary (6.3.14)[455]: Let A be a unital C^*-algebra. The following are equivalent.

(i) A has the singleton Dixmier property.

(ii) A is weakly central and, for every $M \in Max(A), A/(M \cap Z(A))A$ has a unique tracial state and this state annihilates $M/(M \cap Z(A))A$.

(iii) A is weakly central and

(a) for every $M \in Max(A), A/M$ has a unique tracial state, and

(b) every extreme tracial state of A factors through A/M for some $M \in Max(A)$.

(iv) (a) for every $M \in Max(A), A/M$ has a unique tracial state, and
(b) the restriction map $r: T(A) \to S(Z(A))$ is a homeomorphism for the weak∗-topologies.
(v) (a) for every $M \in Max(A), T(A/M)$ is non-empty, and
(b) the restriction map $r_e: \partial_e T(A) \to P(Z(A))$ is injective.
Proof. The equivalence of (i), (ii) and (iii) follows from Theorem (6.3.12) and Proposition (6.3.4). It is also clear that (iv) implies (v) (note that r_e maps extreme tracial states into $P(Z(A))$ by Lemma (6.3.10)).
(i)⇒(iv): Suppose that A has the singleton Dixmier property. Then (iv)(a) holds by Proposition (6.3.4). For (iv) (b), we proceed as in the well-known case of a finite von Neumann algebra (cf. [360, Proposition III.5.3]). For the surjectivity of r, we observe that if $\phi \in S(Z(A))$ then $\phi \circ R \in T(A)$, where $R: A \to Z(A)$ is the unique centre-valued trace of A, and $(\phi \circ R)|_{Z(A)} = \phi$. The injectivity of r follows from the facts that A has the Dixmier property and tracial states are constant on Dixmier sets. Since r is a weak∗-continuous bijection from the compact space $T(A)$ to the Hausdorff space $S(Z(A))$, it is a homeomorphism.
(v)⇒(iii): Suppose that A satisfies (v) and let $M \in Max(A)$. By (v) (a) and the KreinMilman theorem, there exists $\tau_M \in \partial_e T(A/M)$. Then $\tau_M \circ q_M \in \partial_e T(A)$ (Lemma (6.3.9)) and $(\tau_M \circ q_M)|_{Z(A)} = \phi_{M\cap Z(A)}$. Since r_e is injective, τ_M is unique and hence $T(A/M) = \{\tau_M\}$ by the Krein–Milman theorem. This establishes (iii) (a).
For weak centrality, suppose that $M_1, M_2 \in Max(A)$ and $M_1 \cap Z(A) = M_2 \cap Z(A)$. Then
$$r_e(\tau_{M_1} \circ q_{M_1}) = \phi_{M_1\cap Z(A)} = \phi_{M_1\cap Z(A)} = r_e(\tau_{M_2} \circ q_{M_2}).$$
Since r_e is injective, $\tau_{M_1} \circ q_{M_1} = \tau_{M_2} \circ q_{M_2}$, which is a state annihilating $M_1 + M_2$. Hence $M_1 = M_2$.
For (iii) (b), let $\tau \in \partial_e T(A)$. By Lemma (6.3.10), there exists $N \in Max(Z(A))$ such that
$$\tau |_{Z(A)} = \phi_N.$$
Let $M \in Max(A)$ satisfy $M \cap Z(A) = N$, so that
$$\tau|_{Z(A)} = \phi_{M\cap Z(A)} = (\tau_M \circ q_M)|_{Z(A)}.$$
Since r_e is injective, $\tau = \tau_M \circ q_M$.
Corollary (6.3.15)[455]: Let A be a unital C^*-algebra with the Dixmier property and suppose that $T(A)$ is non-empty. Then there exists a unique proper closed ideal J of A with the following property: for every proper closed ideal K of $A, A/K$ has the singleton Dixmier property if and only if $K \supseteq J$.
Proof. From Theorem (6.3.12), we have that
$$Y := \{M \in Max(A): T(A/M)\ isnon-empty\}$$
is a non-empty closed subset of $Max(A)$. Let $N := \bigcap_{M\in Y}(M \cap Z(A))$ and $J := NA$. Since Y is non-empty, J is a proper ideal of A.

Let K be a proper closed ideal of A and suppose that A/K has the singleton Dixmier property. Let P be a primitive ideal of A containing K and let M be a maximal ideal of A containing P. Since A/K has the singleton Dixmier property, it follows from Proposition (6.3.4) that $T((A/K)/(M/K))$ is non-empty and hence $M \in Y$. On the other hand, $P \cap Z(A)$ is a prime ideal of $Z(A)$ and hence
$$P \cap Z(A) = M \cap Z(A) \supseteq N.$$
It follows that $P \supseteq NA = J$. Since this holds for all $P \in Prim(A/K)$, we obtain that $K \supseteq J$.

Conversely, suppose that $K \supseteq J$. Since A has the Dixmier property, so does A/K. Let M be a maximal ideal of A that contains K. Since $M \cap Z(A) \supseteq J \cap Z(A) \supseteq N$ and $\Psi(Y)$ is closed in $Max(Z(A))$, we obtain that $M \in Y$. Thus $T((A/K)/(M/K))$ is non-empty and so A/K has the singleton Dixmier property by Proposition (6.3.4). The uniqueness of J is immediate from its stated property.

We highlight the special case of Theorem (6.3.12) in which $Z(A)$ is trivial, which generalises results from [374]. This case plays a crucial role in our investigation of the uniform Dixmier property for C^*-algebras with trivial centre.

Corollary (6.3.16)[455]: Suppose that A is a unital C^*-algebra. The following conditions are equivalent.

(i) $Z(A) = \mathbb{C}1$ and A has the Dixmier property.

(ii) A has a unique maximal ideal J, A has at most one tracial state and J has no tracial states.

(iii) A has a unique maximal ideal $J, A/J$ has at most one tracial state and J has no tracial states.

When these hold, A has the singleton Dixmier property exactly when it has a tracial state τ, and in this case,

$$J = \{x \in A: \tau\,(x^*x) = 0\},$$

the trace-kernel ideal for τ.

If A has the Dixmier property and no tracial states then

$$D_A(a) \cap \mathbb{C}1 = \{t1: t \in co(sp(q_J(a))\}.$$

Proof. Suppose that (i) holds. By Theorem (6.3.12) ((i) ⇒(ii)), A is weakly central and hence, since $Z(A) = \mathbb{C}1, A$ has a unique maximal ideal J. Since $J \cap Z(A) = \{0\}, A$ has at most one tracial state by Theorem (6.3.12) (ii)(a) and if A does have a tracial state then it annihilates J by Theorem (6.3.12) (ii) (b). By Lemma (6.3.8), J has no tracial states. Thus (ii) holds.

Conversely, suppose that (ii) holds. Then $Z(A) = \mathbb{C}1$ (by Lemma (6.3.7)) and A is weakly central. If A has a tracial state then it must annihilate J since J has no tracial states. Thus (i) holds by Theorem (6.3.12) ((ii) ⇒(i)).

(ii)⇔(iii) is immediate.

The statement concerning the singleton Dixmier property follows from Corollary (6.3.14) (i)⇔(ii), and the final statement follows from Theorem(6.3.2).

An example of a non-simple C^*-algebra with a unique maximal ideal, with the Dixmier property but not the singleton Dixmier property is the "Cuntz–Toeplitz algebra" $A := C^*(S_1, \ldots, S_n)$ where $2 \le n < \infty$ and $S_1, \ldots, S_n$ are isometries on an infinite dimensional Hilbert space with mutually orthogonal range projections having sum less than 1(cf. [345, Theorem 11]).

Corollary (6.3.16) above motivates the following question. Is there an example of a unital C^*-algebra A containing a unique maximal ideal J such that A has a unique tracial state and A/J has no tracial states? A non-separable example is the multiplier algebra $M(J)$ where J is a non-unital hereditary subalgebra of a UHF algebra; here, J is simple and has a unique trace, and by [362, Theorem 3.1 and its proof], $M(J)/J$ is simple and infinite. Thus J is the unique maximal ideal of $M(J)$, and the extension of the trace on J is the unique trace on $M(J)$.

For a separable nuclear example, one may utilise a construction of Kirchberg [389] as pointed out by Ozawa at the end of [404]. Thus J and A are C^*-subalgebras of the CAR algebra $\mathbb{M}_{2^\infty}$ such that J is hereditary in $\mathbb{M}_{2^\infty}$ and is an ideal in A such that $A/J \cong \mathcal{O}_\infty$. Since $\mathbb{M}_{2^\infty}$ is simple and has a faithful, unique tracial state, J also has both of these properties (note that any tracial state of J can be extended to a bounded tracial functional on $\mathbb{M}_{2^\infty}$). Suppose that A has a maximal ideal M distinct from J. Then $M \cap J = \{0\}$ and so

$$\mathcal{O}_\infty \cong (M + J)/J \cong M/M \cap J = M,$$

contradicting the fact that A has a faithful tracial state induced from $\mathbb{M}_{2^\infty}$. It follows from a theorem of Cuntz and Pedersen [355, Theorem 2.9], as in the proof of [399, Theorem14], that A has a faithful, unique tracial state. Even though A/J satisfies a strong form of the Dixmier property [355, Theorem 8], A itself does not have the Dixmier property because its tracial state does not vanish on J.

This example also shows that, in Corollary (6.3.14), the condition (v)(a) does not follow from condition (v)(b). On the other hand, to see that condition (v)(a) does not imply condition (v)(b) in Corollary (6.3.14), consider any simple unital C^*-algebra with more than one tracial state.

The following concerns the Dixmier property for non-unital C^*-algebras; a non-unital C^*-algebra A is said to have the (singleton) Dixmier property if the unitization $A + \mathbb{C}1$ has the same property.

Corollary (6.3.17)[455]: Let A be a C^*-algebra with no tracial states. Then the following conditions are equivalent.

(i) A has the Dixmier property and $Z(A) = 0$.

(ii) A has the singleton Dixmier property and $Z(A) = 0$.

(iii) A is the unique maximal ideal of the unitisation $A + \mathbb{C}1$.

Proof.(i)⇔(iii) is Corollary (6.3.16) (i)⇔(iii) applied to $A + \mathbb{C}1$, while the singleton Dixmier property in (ii) is the final sentence of Corollary (6.3.16).

A C^*-algebra A (with or without an identity) is said to have the centre-quotient property if $Z(A/J) = (Z(A) + J)/J$ for every proper closed ideal J of A. Vesterstrøm showed that, for unital A, the centre-quotient property is equivalent to weak centrality [436, Theorems1 and 2]. Dixmier observed that the centre-quotient property is a simple consequence of the Dixmier property in a von Neumann algebra [360, p.259, Ex.7]. Similarly, it is easily seen that if a C^*-algebra has the Dixmier property then it also has the centre-quotient property [340, 2.2.2]. The next result was obtained in [340, 4.3.1, 5.1.9] under the additional assumption that either A is separable or there is a finite bound on the covering dimension of compact Hausdorff subsets of the spectrum $\hat{A}$. The method was very different from that used below.

Theorem (6.3.18)[455]: Let A be a postliminal C^*-algebra. The following conditions are equivalent.

(i) A has the centre-quotient property.

(ii) A has the singleton Dixmier property.

(iii) A has the Dixmier property.

Proof.(i)⇒(ii): Suppose first of all that A is a unital postliminal C^*-algebra with the centre-quotient property. Then A is weakly central ([436]). Furthermore, A automatically satisfies conditions (iii)(a) and (iii)(b) of Corollary (6.3.14). For (iii)(a), recall that a simple, unital

C^*-algebra of type I is $*$-isomorphic to M_n for some $n \in \mathbb{N}$. For (iii)(b), note that if $\tau \in \partial_e T(A)$ then $\pi_\tau(A)$ is a finite factor of type I (see [361, 6.8.7 and 6.8.6]) and so $\ker \pi_\tau$ is maximal. By Corollary (6.3.14), A has the singleton Dixmier property.

Secondly, suppose that A is a non-unital postliminal C^*-algebra with the centre-quotient property. Then it is easily seen that $A + \mathbb{C}1$ has the centre-quotient property (note that if J is a closed ideal of $A + \mathbb{C}1$ then either $J \subseteq A$ or else $(A + \mathbb{C}1)/J$ is canonically $*$-isomorphic to $A/(A \cap J)$). Thus $A + \mathbb{C}1$ is a unital postliminal C^*-algebra with the centre-quotient property and so has the singleton Dixmier property by the first part of the proof.

(iii)⇒(i): For the convenience of the reader, we give the details in the case where A is a non-unital C^*-algebra with the Dixmier property. The unital case is even easier (and could alternatively be obtained via weak centrality and [436]). Let J be a closed ideal of A and let $q: A + \mathbb{C}1 \to (A + \mathbb{C}1)/J$ be the canonical quotient map. Suppose that $a \in A$ and that $q(a) \in Z(A/J) \subseteq Z((A + \mathbb{C}1)/J)$. Since $D_{A+\mathbb{C}1}(a) \subset A$ and $Z(A + \mathbb{C}1) \cap A = Z(A)$, there exists $z \in D_{A+\mathbb{C}1(a)} \cap Z(A)$. Then $q(z) \in D_{q(A+\mathbb{C}1)}(q(a)) = \{q(a)\}$ and so $q(a) \in (Z(A) + J)/J$, as required.

Corollary (6.3.19)[455]: Let A be a postliminal C^*-algebra such that every irreducible representation of A is infinite dimensional. Then A has the singleton Dixmier property.

Proof. As in the proof of [340, 4.3.2], the use of a composition series with liminal quotients shows easily that the centre of A is $\{0\}$. Since the same applies to any nonzero quotient of A, it follows that A has the centre-quotient property and hence the singleton Dixmier property.

we introduce and study the following uniform version of the Dixmier property (cf. [403] and [402]).

Definition (6.3.20)[455]: A unital C^*-algebra A has the uniform Dixmier property if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $a \in A$, there exist unitaries $u_1, \ldots, u_n \in \mathcal{U}(A)$ such that

$$d\left(\sum_{i=1}^{n} \frac{1}{n} u_i a u_i^*, Z(A)\right) \leq \epsilon \|a\|.$$

Theorem (6.3.21)[455]: Let A be a unital C^*-algebra. The following are equivalent:

(i) A has the uniform Dixmier property.

(ii) There exist $m \in \mathbb{N}$ and $0 < \gamma < 1$ such that for every self-adjoint $a \in A$ we have that

$$\left\|\sum_{i=1}^{m} \frac{1}{m} u_i a u_i^* - z\right\| \leq \gamma \|a\|,$$

for some $z \in Z(A)$ and $u_1, \ldots, u_m \in \mathcal{U}(A)$.

(iii) There exist $m \in \mathbb{N}$ and $0 < \gamma < 1$ such that for every self-adjoint $a \in A$ we have that

$$\left\|\sum_{i=1}^{m} t_i u_i a u_i^* - z\right\| \leq \gamma \|a\|, \tag{13}$$

for some $z \in Z(A)$, some $u_1, \ldots, u_m \in U(A)$, and some $t_1, \ldots, t_m \in [0, 1]$ such that $\sum_{i=1}^{m} t_i = 1$.

(iv) There exists a function $\Phi: A \to Z(A)$ such that for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $a \in A$ we have that

$$\left\|\sum_{i=1}^{m^{2k}} \frac{1}{n} u_i a u_i^* - \Phi(a)\right\| \leq \epsilon \|a\|,$$

for some unitaries $u_1, \dots, u_n \in U(A)$.

Proof. This proof uses known ideas from the theory of the Dixmier property and of sequence algebras, and is included for completeness.

The implications (i)⇒(ii)⇒(iii) and (iv)⇒(i) are clear.

Let us prove that (iii)⇒(iv). Given an arbitrary element $a \in A$, we can decompose a as $b + ic$ where b, care self-adjoint and $\|b\|, \|c\| \leq \|a\|$. By a standard argument of successive averaging (cf. the proofs of [385, Lemmas 8.3.2 and 8.3.3]), we deduce from (13) the existence of m^{2k} unitaries $v_1, \dots, v_{m^{2k}}$ such that

$$\left\|\sum_{i=1}^{m^{2k}} t_i v_i a v_i^* - \mathrm{z}\right\| \leq 2\gamma^k \|a\|,$$

for some $z \in Z(A)$ and some scalars $t_i \in [0,1]$ such that $\sum_{i=1}^{m^{2k}} t_i = 1$. In this way, we extend (13) to all $a \in A$ at the expense of changing (m, γ) for $(m^{2k}, 2\gamma^k)$ (where k is chosen so that $2\gamma^k < 1$). Henceforth, let us instead assume, without loss of generality, that the constants (m, γ) are such that (13) is valid for all $a \in A$.

Let $a \in A$. Then there exists $z_1 \in Z(A)$ such that

$$\left\|\sum_{i=1}^{m} t_i u_i a u_i^* - z_1\right\| \leq \gamma \|a\|.$$

for some unitaries $u_1, \dots, u_m \in A$ and scalars $t_1, \dots, t_m \in [0,1]$ such that $\sum_{i=1}^{m} t_i = 1$. Set $a_1 := \sum_{i=1}^{m} t_i u_i a u_i^*$ so that $\|a_1 - z_1\| \leq \gamma \|a\|$. Applying the same argument to $a_1 - z_1$ we find $z_2 \in Z(A)$, and a convex combination of m unitary conjugates of $a_1 - z_1$, call it b_2, such that

$$\|b_2 - z_2\| \leq \gamma \|a_1 - z_1\| \leq \gamma^2 \|a\|.$$

Notice that $b_2 = a_2 - z_1$, where a_2 is a convex combination of munitary conjugates of a_1 (whence, also a convex combination of m^2 unitary conjugates of a). Then

$$\|a_2 - z_1 - z_2\| \leq \gamma^2 \|a\|.$$

Continuing this process ad infinitum we find $a_k \in D(a)$ and $z_k \in Z(A)$ for $k = 1, 2, \ldots$ such that $a_k$is a convex combination of m unitary conjugates of a_{k-1} and

$$\left\|a_k - \sum_{i=1}^{k} z_i\right\| \leq \gamma^k \|a\|$$

for all $k \geq 1$. For each $k \geq 1$ we have that

$$\|z_k\| \leq \gamma^k \|a\| + \left\|a_k - \sum_{i=1}^{k-1} z_i\right\|,$$

and since $a_k - \sum_{i=1}^{k-1} z_i$ is a convex combination of unitary conjugates of $a_{k-1} - \sum_{i=1}^{k-1} z_i$,

$$\|z_k\| \le \gamma^k\|a\| + \left\|a_{k-1} - \sum_{i=1}^{k-1} z_i\right\|$$
$$\le \gamma^k\|a\| + \gamma^{k-1}\|a\|.$$
It follows that $\sum_{i=1}^{\infty} z_i$ is a convergent series. Define $\Phi(a) := \sum_{i=1}^{\infty} z_i$. Let us show that Φ is as desired. We have that

$$\|a_k - \Phi(a)\| \le \left\|a_k - \sum_{i=1}^{k} z_i\right\| + \sum_{i>k}\|z_i\| \le 2\|a\|\frac{\gamma^k}{1-\gamma}.$$

Recall that a_k is a convex combination of m^k unitary conjugates of a. Notice also that the rightmost side tends to 0 as $k \to \infty$. This shows that for each $\epsilon > 0$ there exists n such that $\|a' - \Phi(a)\| \le \epsilon\|a\|$ for all $a \in A$, where a' is a convex combination of nunitary conjugates of a. It remains to show that this convex combination may be chosen to be an average (for a larger n). Let $\epsilon > 0$. Pick $n \in \mathbb{N}$ such that for any $a \in A$ we have

$$\left\|\sum_{i=1}^{n} \lambda_i v_i a v_i^* - \Phi(a)\right\| \le \frac{\epsilon}{2}\|a\|,$$

for some $v_1, \ldots, v_n \in U(A)$ and $\lambda_1, \ldots, \lambda_n \ge 0$ such that $\sum_{i=1}^{n} \mu_i = 1$. Now let $N \ge 2n/\epsilon$. We can find non-negative rational numbers of the form $\mu_i = p_i/N$ for $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \mu_i = 1$ and $|\mu_i - \lambda_i| < \frac{1}{N}$, $\quad i = 1, \ldots, n$.

(To find such μ_i, first set p_1 to be the greatest integer such that $\frac{p_1}{N} \le \lambda_1$; then having picked $p_1, \ldots, p_{i-1}$, pick p_i to be the greatest integer such that $\frac{p_1 + \cdots + p_i}{N} \le \lambda_1 + \cdots + \lambda_i$.) Let $u_1, \ldots, u_N$ be given by listing each unitary v_i a total of p_i times, so that

$$\sum_{i=1}^{N} \frac{1}{N} u_i a u_i^* = \sum_{i=1}^{n} \mu_i v_i a v_i^*.$$

Then

$$\begin{aligned}\left\|\sum_{i=1}^{N} \frac{1}{N} u_i a u_i^* - \Phi(a)\right\| &= \left\|\sum_{i=1}^{n} \mu_i v_i a v_i^* - \Phi(a)\right\| \\ &\le \left\|\sum_{i=1}^{n} (\mu_i - \lambda_i) v_i a v_i^*\right\| + \left\|\sum_{i=1}^{n} \lambda_i v_i a v_i^* - \Phi(a)\right\| \\ &\le \frac{n}{N}\|a\| + \frac{\epsilon}{2}\|a\| \\ &\le \epsilon\|a\|.\end{aligned}$$

Thus, N is as desired.

We will find it useful to keep track of the constants (m, γ) such that Theorem (6.3.21) (ii) is satisfied. If there exist $m \in \mathbb{N}$ and $0 < \gamma < 1$ such that for every self-adjoint $a \in A$ we have that

$$\left\|\sum_{i=1}^{m} \frac{1}{m} u_i a u_i^* - z\right\| \le \gamma\|a\|, \tag{14}$$

for some $z \in Z(A)$ and some $u_1, \ldots, u_m \in U(A)$, then we say that A has the uniform Dixmier property with constants (m, γ).

There have been significant recent advances in the understanding of when $C_r^*(G)$ has the properties of simplicity and of unique trace (for a discrete group G) [352,373,386,388,393]; in particular, if $C_r^*(G)$ is simple, then it also has a unique trace. Therefore, simplicity and the Dixmier property coincide for $C_r^*(G)$; it turns the that, in fact, the Dixmier property is witnessed using only group unitaries to do the averaging ([373, Theorem 4.5]or [388, Theorem 5.3]). However, it is not clear when $C_r^*(G)$ has the uniform Dixmier property.

Question (6.3.22)[455]: Is there a discrete group G for which $C_r^*(G)$has the Dixmier property (i.e., is simple), but not the uniform Dixmier property? Is the uniform Dixmier property for $C_r^*(G)$the same as being able to average uniformly using group unitaries?

In Corollary (6.3.28) below we show that all AF C^*-algebras with the Dixmier property have the uniform Dixmier property. We show that all C^*-algebras with the Dixmier property and finite radius of comparison-by-traces have the uniform Dixmier property. More examples.

Theorem (6.3.23)[455]: Let $m \in \mathbb{N}$ and $0 < \gamma < 1$.

(i) If A is a unital C^*-algebra with the uniform Dixmier property with constants (m, γ), then all of the quotients of A have the uniform Dixmier property, also with constants (m, γ).

(ii) If $A_1, A_2, \ldots$ are unital C^*-algebras with the uniform Dixmier property with constants (m, γ), then $\prod_{n=1}^{\infty} A_n$ has the uniform Dixmier property, also with constants (m, γ).

Proof. This is straightforward.

(i): For every self-adjoint $a \in A/I$ we can find a self-adjoint lift $\tilde{a} \in A$ with the same norm. Then there exist unitaries $u_1, \ldots, u_m \in U(A)$ such that (7) holds for $\tilde{a}$. Passing to the quotient A/I we get the same for a.

(ii): Let $a = (a_n)_n \in \prod_n A_n$be self-adjoint. For each n we may find m unitaries $u_{1,n}, \ldots, u_{m,n} \in U(A_n)$ and $z_n \in Z(A)$ such that

$$\left\| \sum_{i=1}^{m} \frac{1}{m} u_{i,n} a_n u_{i,n}^* - z_n \right\| \le \gamma \|a_n\|.$$

Let $u_i = (u_{i,n})_n$for $i = 1, \ldots, m$ and define $z := (z_n)_n \in \prod_{n=1}^{\infty} Z(A_n)$(note that the sequence $(z_n)_n$is bounded since $\|z_n\| \le (1+\gamma)\|a\|$ for all n). Then

$$\left\| \sum_{i=1}^{m} \frac{1}{m} u_i a u_i^* - z \right\| \le \gamma \|a\|,$$

as desired.

It will be convenient in the proof of Proposition (6.3.24) below to use the following notation from [342]: For a unital C^*-algebra A and a subgroup V of $U(A), A_V(A, V)$ is the set of all mappings (called averaging operators) $\alpha: A \to A$ which can be defined by an equation of the form

$$\alpha(a) = \sum_{j=1}^{n} \lambda_j u_j a u_j^* \qquad (a \in A),$$

where $n \in \mathbb{N}, \lambda_j > 0, u_j \in V (1 \le j \le n)$, and $\sum_{j=1}^{n} \lambda_j = 1$. Elementary properties of such mappings α are described in [342, 2.2].

Proposition (6.3.24)[455]: Let $(A_k)_{k=1}^{\infty}$ be an increasing sequence of C^*-subalgebras of A whose union is dense in A, all containing the unit. Suppose that A_k has the singleton Dixmier property for all k. The following are equivalent:

(i) A has the Dixmier property.

(ii) The limit $lim_{k\to\infty} R_k(a)$ exists for all $a \in \bigcup_{k=1}^{\infty} A_k$, where R_k denotes the centre-valued trace on $A_k$for all k.

(iii) A has the singleton Dixmier property and

$$R(a) = lim_{k\to\infty} R_k(a) \; for \; all \; a \in \bigcup_{k=1}^{\infty} A_k,$$

where R denotes the centre-valued trace on A.

Note that an inductive limit of C^*-algebras with the singleton Dixmier property need not have the Dixmier property (e.g., there exist simple, unital AF algebras with more than one tracial state).

Proof. Glimm's argument for UHF algebras [367, Lemma 3.1] shows that $\bigcup_{k\geq 1} U(A_k)$ is norm-dense in $U(A)$ (in brief, if $a_n \to u$ then $a_n(a_n^* a_n)^{-1/2} \to u$). Since multiplication is jointly continuous for the norm-topology on A, it follows that, for all $a \in A$,

$$D_A(a) = \overline{\bigcup_{k\geq 1}\{\alpha(a) : \alpha \in A_V(A, U(A_k))\}}. \quad (15)$$

We shall use this repeatedly.

(i)⇒(iii): Let us first show that A has the singleton Dixmier property. Suppose that $z_1, z_2 \in D_A(a) \cap Z(A)$ for some $a \in A$. Let $\epsilon > 0$. By (15), there exists $n \in \mathbb{N}$ and $\alpha, \beta \in A_V(A, U(A_n))$ such that

$$\|z_1 - \alpha(a)\| < \frac{\epsilon}{4} \quad \text{and} \quad \|z_2 - \beta(a)\| < \frac{\epsilon}{4}.$$

Enlarging n if necessary, we can find $b \in A_n$such that $\|a - b\| < \epsilon/4$. Notice then that $\|z_1 - \alpha(b)\| < \epsilon/2$. Since z_1 is invariant under conjugation by unitary elements of $A, \|z_1 - R_n(\alpha(b))\| \leq \epsilon/2$. But R_n is constant on Dixmier sets in A_n and so $R_n(\alpha(b)) = R_n(b)$. Thus

$$\|z_1 - R_n(b)\| \leq \frac{\epsilon}{2} \quad \text{and similarly} \quad \|z_2 - R_n(b)\| \leq \frac{\epsilon}{2}.$$

It follows that $\|z_1 - z_2\| \leq \epsilon$ and hence that $z_1 = z_2$, as required.

Let $R: A \to Z(A)$ be the unique centre-valued trace on A. Let $k \geq 1, a \in A_k$ and $\epsilon > 0$. By (15), there exists $M \geq k$ and $\alpha \in A_V(A, U(A_M))$ such that $\|R(a) - \alpha(a)\| < \epsilon/2$. For each $n \geq M$, there exists $\beta_n \in A_V(A_n, U(A_n))$ such that

$$\|R_n(\alpha(a)) - \beta_n(\alpha(a))\| < \frac{\epsilon}{2}.$$

Since R_n is constant on Dixmier sets in $A_n, R_n(\alpha(a)) = R_n(a)$, and since $R(a) \in Z(A), \|R(a) - \beta_n(\alpha(a))\| < \epsilon/2$. Hence

$$\|R(a) - R_n(a)\| \leq \|R(a) - \beta_n(\alpha(a))\| + \|\beta_n(\alpha(a)) - R_n(\alpha(a))\| < \epsilon.$$

Thus $R_n(a) \to R(a)$ as $n \to \infty$.

(iii)⇒(ii) is obvious.

(ii)⇒(i): Let $k \geq 1$ and $a \in A_k$. Then (ii) yields $z \in A$ such that, for $n \geq k, R_n(a) \to z$ as $n \to \infty$. Since $R_n(a) \in Z(A_n), z$ belongs to the relative commutant of $\cup_{j\geq k} A_j$in A and hence $z \in Z(A)$. Since $R_n(a) \in DA_n(a) \subseteq D_A(a)(n \geq k), z \in D_A(a)$. Thus, by [342, Lemma 2.8], A has the Dixmier property.

Suppose that A has the singleton Dixmier property. Let $R: A \to Z(A)$ denote its centre-valued trace. If A also has the uniform Dixmier property then by Theorem (6.3.21) (iv) (applied to $a - R(a)$), there exist $M \in \mathbb{N}$ and $0 < \Upsilon < 1$ such that for every self-adjoint $a \in A$ we have that

$$\left\| \sum_{i=1}^{M} \frac{1}{M} u_i a u_i^* - R(a) \right\| \leq \Upsilon \|a - R(a)\| \qquad (16)$$

for some $u_1, \dots, u_M \in U(A)$. We will find it necessary to keep track of these constants in the theorem below, so we will say in this case that A has the uniform singleton Dixmier property with constants (M, Υ).

Example (6.3.25)[455]: By [357, Lemma1 and Proposition 3], $C_r^*(G)$ has the uniform single Dixmier property with constants $(M, \Upsilon) = (3, 0.991)$ for any Powers group G as defined in [357, p. 244].

Note that if A has the singleton Dixmier property, then A has the uniform singleton Dixmier property if and only if (9) holds for every self-adjoint $a \in A$ such that $R(a) = 0$. But, since tracial states are constant on Dixmier sets, $T(A) = \{\phi \circ R: \phi \in S(Z(A))\}$ and hence $R(a) = 0$ if and only if $\tau(a) = 0$ for all $\tau \in T(A)$. In turn, [355, Proposition 2.7] tells us that $\tau(a) = 0$ for all $\tau \in T(A)$ if and only if $a \in \overline{[A, A]}$. Thus, if A has the singleton Dixmier property, then A has the uniform singleton Dixmier property if and only if (16) holds for every self-adjoint $a \in \overline{[A, A]}$.

As with the uniform Dixmier property constants, if A has the uniform singleton Dixmier property with constants (M, Υ), then it also has the uniform singleton Dixmier property with constants $(M^k, \Upsilon^k)(k = 2, 3, \dots)$. The constants (m, γ) for which we have (7) may not satisfy (9), nor vice versa. However, we do have the following.

Lemma (6.3.26)[455]: Let A be a unital C^*-algebra with the singleton Dixmier property.

(i) If A has the uniform Dixmier property with constants (m, γ) then A has the uniform singleton Dixmier property with constants $M = m^k$ and $\gamma = 2\Upsilon^k$ for all natural numbers k such that $2\gamma^k < 1$.

(ii) If A has the uniform singleton Dixmier property with constants (M, Υ) then A has the uniform Dixmier property with constants $m = M^k$ and $\gamma = 2\Upsilon^k$ for all natural numbers k such that $2\Upsilon^k < 1$.

Proof.(i): Since A has the uniform Dixmier property with constants (m^k, γ^k) for all $k \in \mathbb{N}$, it suffices to show that if $\gamma < 1/2$ then A has the uniform singleton Dixmier property with constants $M = m$ and $\Upsilon = 2\gamma$. Let us prove this. Let $h = h^* \in A$. Then $h - R(h)$ is self-adjoint (where R is the centre-valued trace). Hence there exist $z \in Z(A)$ and $u_1, \dots, u_M \in U(A)$ such that

$$\left\| \sum_{i=1}^{M} \frac{1}{M} u_i h u_i^* - R(h) - z \right\| = \left\| \sum_{i=1}^{M} \frac{1}{M} u_i (h - R(h)) u_i^* - z \right\| \leq \gamma \|h - R(h)\|.$$

Since R is contractive, tracial and fixes elements of $Z(A)$, $\|z\| \leq \gamma \|h - R(h)\|$. Hence

$$\left\| \sum_{i=1}^{M} \frac{1}{M} u_i h u_i^* - R(h) \right\| \leq 2\gamma \|h - R(h)\|.$$

(ii): This is immediate since we always have $\|a - R(a)\| \leq 2\|a\|$.

Theorem (6.3.27)[455]: Let $A_1, A_2, \dots$ be unital C^*-algebras with the uniform singleton Dixmier property, all of them satisfying (9) for some constants (M, Υ). Let $A = \lim A_i$ be a

unital inductive limit C^*-algebra. If A has the Dixmier property, then it has the uniform singleton Dixmier property with constants (M, Υ') for any $\Upsilon < \Upsilon' < 1$.

Proof. The uniform singleton Dixmier property, and indeed the constants (M, Υ), pass to quotients (by the same proof as for Theorem (6.3.23) (i), using Theorem (6.3.6) in place of lifting self-adjoint elements to self-adjoint elements); thus, we may reduce to the case that the connecting maps of the inductive limit are inclusions. So let us assume that the C^*-algebras $(A_k)_{k=1}^{\infty}$ form an increasing sequence of subalgebras of A whose union is dense in A. We denote the centre-valued trace on A_k by R_k. By Proposition (6.3.24), A has the singleton Dixmier property. We denote its centre-valued trace by R.

Let $a \in A$ be a self-adjoint contraction with $R(a) = 0$. Let $\epsilon > 0$. Find a self-adjoint contraction $b \in A_k$, for k large enough, such that $\|a - b\| < \epsilon$. Find $n > k$ such that $\|R_n(b) - R(b)\| < \epsilon$ (its existence is guaranteed by Proposition (6.3.24)). Thus,

$$\|R_n(b)\| \le \|R_n(b) - R(b)\| + \|R(b - a)\| < 2\epsilon.$$

Since A_n has the uniform singleton Dixmier property with constants (M, Υ), we have that

$$\left\| \sum_{i=1}^{M} \frac{1}{M} u_i b u_i^* - R_n(b) \right\| \le \Upsilon \|b - R_n(b)\|$$

for some unitaries $u_1, \dots, u_M \in U(A_n)$. Hence,

$$\begin{aligned} \left\| \sum_{i=1}^{M} \frac{1}{M} u_i a u_i^* \right\| &\le \|a - b\| + \left\| \sum_{i=1}^{M} \frac{1}{M} u_i b u_i^* - R_n(b) \right\| + \|R_n(b)\| \\ &\le \epsilon + \Upsilon \|b - R_n(b)\| + 2\epsilon \\ &\le \Upsilon(1 + 2\epsilon) + 3\epsilon. \end{aligned}$$

Thus, A has the uniform singleton Dixmier property with constants $(M, \Upsilon(1 + 2\epsilon) + 3\epsilon)$ for any sufficiently small $\epsilon > 0$.

Corollary (6.3.28)[455]: All unital AF C^*-algebras with the Dixmier property have the uniform singleton Dixmier property with constants $M = 4$ and $1/2 < \Upsilon < 1$ (i.e., satisfy (9) for $M = 4$ and any $1/2 < \Upsilon < 1$).

Proof. Finite dimensional C^*-algebras have the uniform singleton Dixmier property with constants $M = 4$ and $\Upsilon = 1/2$ by Proposition (6.3.42) below.

Necessary and sufficient conditions for a unital AF C^*-algebra to have the Dixmier property have been given in [343, Theorem 6.6]. The example in [343, Example 6.7] shows how these conditions can be verified by using a Bratteli diagram.

In the following, we let ω be a free ultrafilter on $\mathbb{N}$ and denote by A_ω the ultra-power of A under ω. Generally, many of the arguments used with sequence algebras $\prod_n A_n / \bigoplus_n A_n$ also work with $A\omega$ (and more generally, ultrapowers $\prod_\omega A_n$); for example we could have used ultraproducts in Theorem (6.3.23) instead of sequence algebras. However, A_ω has some advantages in terms of its size. For example, if A is simple and purely infinite then A_ω is simple [418, Proposition 6.2.6], whereas $\prod_n A / \bigoplus_n A$ has a maximal ideal corresponding to each free ultrafilter. Likewise, if A has a unique trace then A_ω has a unique distinguished trace (which is potentially unique-see Theorem (6.3.37)), whereas $\prod_n A / \bigoplus_n A$ has a (distinguished) trace corresponding to each free ultrafilter. For more about ultrapowers, see [391].

Theorem (6.3.29)[455]: Let A be a unital C^*-algebra. The following are equivalent.

(i) A has the uniform Dixmier property.

(ii) A_ω has the Dixmier property and $Z(A_\omega) = Z(A)_\omega$.

Proof.(i)⇒(ii): Let $m \in \mathbb{N}$ and $0 < \gamma < 1$ be such that A has the uniform Dixmier property. By Theorem 6.3.23) (i), $\ell^\infty(A)$ has the uniform Dixmier property (with the same constants), and then by Theorem (6.3.23) (ii), so does the quotient A_ω. Moreover, since $Z(\ell^\infty(A)) = \ell^\infty(Z(A))$, and $\ell^\infty(A)$ has the centre-quotient property (since it has the Dixmier property), $\ell^\infty(Z(A))$ is mapped onto the centre of $A_\omega$by the quotient map. Thus, $Z(A)_\omega = Z(A_\omega)$.

(ii)⇒(i): Suppose that (ii) holds and, for a contradiction, that (i) does not. Using Theorem (6.3.21) (iii)⇒(i), we have that condition (iii) of Theorem (6.3.21) does not hold, and in particular it does not hold for $\gamma = 1/2$. Thus, for each $n \geq 1$ there exists $a_n \in A$ such that $\|a_n\| = 1$ and for all $u_1, \dots, u_n \in U(A)$ and $t_1, \dots, t_n \in [0,1]$ with $\sum_{i=1}^{n} t_i = 1$,

$$d\left(\sum_{i=1}^{n} t_i u_i a_n u_i^*, Z(A)\right) \geq \frac{1}{2}.$$

Let $a \in A_\omega$be the element represented by the sequence $(a_n)_n$. Since $A_\omega$has the Dixmier property, there exist $u_1, \dots, u_k \in U(A_\omega), t_1, \dots, t_k \in [0,1]$ with $\sum_{i=1}^{n} t_i = 1$, and $z \in Z(A_\omega)$ such that

$$\left\|\sum_{i=1}^{k} t_i u_i a u_i^* - z\right\| < \frac{1}{2}.$$

Since $Z(A)_\omega = Z(A_\omega)$, we can lift z to a bounded sequence $(z_n)_n$from $Z(A)$. We may also lift each u_i to a sequence $(u_{i,n})_n$from $U(A)$(either by using [342, Proposition 2.5] in the initial choice of the elements u_i or the fact that unitaries from A_ω always lift to a sequence of unitaries). We have

$$\lim_{n\to\omega} \left\|\sum_{i=1}^{k} t_i u_{i,n} a_n u_{i,n}^* - z_n\right\| < \frac{1}{2}.$$

In particular, for some $n \geq k$ we must have

$$\left\|\sum_{i=1}^{k} t_i u_{i,n} a_n u_{i,n}^* - z_n\right\| < \frac{1}{2},$$

which gives a contradiction.

For a C^*-algebra A, the condition $Z(A_\omega) = Z(A)_\omega$ is related to norms of inner derivations, as follows. Firstly, recall that the triangle inequality shows that $\|ad(a)\| \leq 2d(a, Z(A))$, where $ad(a)$ is the inner derivation of A induced by $a \in A$ (that is, $ad(a)(x) := xa - ax$). In the reverse direction, $K(A)$ is defined to be the smallest number in $[0,\infty]$ such that $d(a, Z(a)) \leq K(A)\|ad(a)\|$ for all $a \in A$ ([343]). It was shown in the proof of [383, Theorem 5.3] that $K(A) < \infty$ if and only if the set of inner derivations of A is norm-closed in the set of all derivations of A. If A is non-commutative (as we shall assume from now on in this summary) then $K(A) \geq \frac{1}{2}$. If A is a von Neumann algebra (or, more generally, an AW^*-algebra) or a unital primitive C^*-algebra (in particular, a unital simple C^*-algebra) then $K(A) = \frac{1}{2}$ ([363,366,381,428,441]). These and other such cases are covered by Somerset's characterisation for unital A: $K(A) = \frac{1}{2}$ if and only if the ideal $P \cap Q \cap R$ is primal whenever P, Q and R are primitive ideals of A such that $P \cap Z(A) = Q \cap$

$Z(A) = R \cap Z(A)$ ([426]). If a unital C^*-algebra A has the Dixmier property then $K(A) \leq 1$ (see [413] and [343, Proposition 2.4]) (this holds more generally if A is weakly central, see [343,89]). On the other hand, in [383, 6.2], an example is given where $K(A) = \infty$. By [425, Corollary 4.6], finiteness of $K(A)$ depends only on the topological space Prim (A). Further information on possible values of $K(A)$ may be found in [346,347].

Proposition (6.3.30)[455]: Let A be a C^*-algebra. The following are equivalent:

(i) $Z(A_\omega) = Z(A)_\omega$.

(ii) $K(A) < \infty$.

Proof.(i)⇒(ii): Suppose that $K(A) = \infty$. For each $n \geq 1$, there exists $b_n \in A$ such that

$$0 < n\|ad(b_n)\| < d(b_n, Z(A)).$$

By scaling, we may assume that $d(b_n, Z(A)) = 1$for all $n \geq 1$. Then, for each $n \geq 1$, there exists $z_n \in Z(A)$ such that $\|b_n - z_n\| < 2$. Let $c_n := b_n - z_n (n \geq 1)$ and let $c \in A_\omega$ correspond to the bounded sequence $(c_n)_n$. Note that

$$d\big(c_n, Z(A)\big) = d\big(b_n, Z(A)\big) = 1 \quad (n \geq 1)$$

and $\|ad(c_n)\| = \|ad(b_n)\| \to 0$ as $n \to \infty$. For any bounded sequence $(a_n)_n$in A, $\lim_{n\to\omega}\|a_n c_n - c_n a_n\| = 0$ and so $c \in Z(A_\omega)$. On the other hand, for any bounded sequence $(y_n)_n$in $Z(A)$, $\lim_{n\to\omega}\|c_n - y_n\| \geq 1$ and so $c \notin Z(A)_\omega$.

(ii)⇒(i): The containment $Z(A)_\omega \subseteq Z(A_\omega)$ is clear. For the other way, let $b \in Z(A_\omega)$ be represented by a bounded sequence $(b_n)_n$in A. For each $n \geq 1$, there exists $z_n \in Z(A)$ such that

$$\|b_n - z_n\| \leq d(b_n, Z(A)) + \frac{1}{2n} \leq K(A)\|ad(b_n)\| + \frac{1}{2n}$$

and there exists $a_n \in A$ such that $\|a_n\| \leq 1$ and

$$K(A)\|a_n(b_n)\| \leq K(A)\|b_n a_n - a_n b_n\| + \frac{1}{2n}.$$

Then, for all $n \geq 1$,

$$\|b_n - z_n\| \leq K(A)\|b_n a_n - a_n b_n\| + \frac{1}{n}.$$

Recalling that $b \in Z(A_\omega)$, we obtain that $\lim_{n\to\omega}\|b_n a_n - a_n b_n\| = 0$ and hence that $\lim_{n\to\omega}\|b_n - z_n\| = 0$. Since $\|z_n\| \leq 2\|b_n\| + \frac{1}{2n}$, $(z_n)_n$is a bounded sequence and so $b \in Z(A)_\omega$.

It is easily seen that the method of proof of Proposition (6.3.30) also shows that $K(A) < \infty$ if and only if the centre of $\ell^\infty(A)/c_0(A)$ is the canonical image of $\ell^\infty(Z(A))/c_0(Z(A))$.

Corollary (6.3.31)[455]: Suppose that A is a unital C^*-algebra with the Dixmier property. The following conditions are equivalent.

(i) A has the uniform Dixmier property.

(ii) A_ω has the Dixmier property.

Proof. Since A has the Dixmier property, $K(A) \leq 1$ (see [413] or [343, Proposition 2.4]) and so $Z(A_\omega) = Z(A)_\omega$ by Proposition (6.3.30). The result now follows from Theorem (6.3.29).

Question (6.3.32)[455]: If A_ω has the Dixmier property, does it follow that A has the Dixmier property? (In other words, by Theorem (6.3.29), if A_ω has the Dixmier property, is $Z(A)_\omega = Z(A_\omega)$?)

Let A be unital with the Dixmier property. If A has strict comparison of positive elements, then if follows from [402, Theorem (6.3.2)] that A has the uniform Dixmier property. We now show that this holds more generally when strict comparison by traces is replaced by finite radius of comparison-by-traces.

Let A be a unital C^*-algebra. For each tracial state τ define $d_\tau: M_n(A)_+ \to [0, \infty)$ by

$$d_\tau(a) := \lim_{n \in \mathbb{N}} \tau\,(a^{1/n}).$$

This is the dimension function associated to τ [350].

Definition (6.3.33)[455]: Let $r \in [0, \infty)$. Let A be a unital C^*-algebra. Let us say that A has radius of comparison-by-traces at most r if for all positive elements $a, b \in \bigcup_{k=1}^{\infty} M_k(A)$, with ba full element, if

$$d_\tau(a) + r' \le d_\tau\,(b) \tag{17}$$

for all $\tau \in T(A)$ and some $r' > r$, then ais Cuntz below b. (Recall that ais said to be Cuntz below b if $d_n b d_m^* \to a$ for some sequence (d_n) in $\bigcup_{k=1}^{\infty} M_k(A)$.) The radius of comparison-by-traces of A is the minimum r such that A has radius of comparison-by-traces at most r. If no such r exists then we say that A has infinite radius of comparison-by-traces. In [351] the radius of comparison of $A$is defined as above, except that in (10) τranges through all 2-quasitraces of A normalised at the unit. We use the name "radius of comparison-by-traces" to emphasise that the comparison of a and b in (10) is done only on tracial states. Clearly, the radius of comparison-by-traces dominates the radius of comparison. If the C^*-algebra A is exact, then by [372] its bounded 2-quasitraces are traces so the two numbers agree.

The seminal examples of simple nuclear C^*-algebras constructed by Villadsen in [437] and [438] have nonzero radius of comparison-by-traces; variations on the first of these examples can be arranged to achieve any possible value of radius of comparison ([434, Theorem 5.11]). Of particular interest here are Villadsen's second examples, which have stable rank in $\{2, 3, \dots\}$; they have nonzero finite radius of comparison while being simple and having unique trace.

Theorem (6.3.34)[455]: Let $A_1, A_2, \dots$ be unital C^*-algebras with radius of comparison-by-traces at most rand let $A := \prod_{i=1}^{\infty} A_i$. The following are true:

(i) A has radius of comparison-by-traces at most r.

(ii) The convex hull of $\bigcup_{i=1}^{\infty} T(A_i)$ is dense in $T(A)$ in the weak∗-topology. (We regard $T(A_i)$ as a subset of $T(A)$ via the embedding induced by the quotient map $A \to A_i$.)

Proof. (i): Let K be the weak∗-closure in $T(A)$ of the convex hull of $\bigcup_{i=1}^{\infty} T(A_i)$. Let $a, b \in M_k(A)$ be positive elements, with b full. Suppose that a and b satisfy (10) for all tracial states $\tau \in K$ and some $r' > r$. We will prove that a is Cuntz below b (which clearly shows that A has radius of comparison-by-traces at most r). Let $\epsilon > 0$ and choose $r < r'' < r'$. We claim that there exists $\delta > 0$ such that

$$d_\tau((a - \epsilon)_+) + r'' \le d_\tau((b - \delta)_+) \text{ for all } \tau \in K. \tag{18}$$

Indeed, let $g_\epsilon \in C_0((0, \|a\|])_+$ be such that $g_\epsilon(t) = 1$ for $t \ge \epsilon$. Then

$$d_\tau((a - \epsilon)_+) \le \tau(g_\epsilon(a)) \le d_\tau(a) \text{ for all } \tau \in T(A).$$

The function $\tau \longmapsto \tau(g_\epsilon(a)) + r''$ is continuous on $T(A)$ while $\tau \longmapsto d_\tau((b - \frac{1}{n})_+)$ is lower semicontinuous for all n. Since

$$\sup_n d_\tau\left(\left(b-\frac{1}{n}\right)_+\right) = d_\tau(b) > \tau(g_\epsilon(a)) + r''$$
for all $\tau \in K$ and K is compact, there exists n such that
$$d_\tau\left(\left(b-\frac{1}{n}\right)_+\right) > \tau(g_\epsilon(a)) + r''$$
for all $\tau \in K$, thus yielding the desired δ. Decreasing δ if necessary, let us also assume that $(b-\delta)_+$ is full. Letting τ range through $T(A_i) \subseteq K$ in (18), and using that A_i has radius of comparison-by-traces at most r, we obtain that $(a_i-\epsilon)_+$is Cuntz below $(b_i-\delta)_+$ for all i. Hence, using [417, Proposition 2.4], we obtain $x_i \in M_k(A_i)$ such that $(a_i - 2\epsilon)_+ = x_i^* x_i$ and $x_i x_i^* \leq M b_i$ for all i, where $M > 0$ is a scalar independent of i. Then $(a-2\epsilon)_+ = x^*x$ and $xx^* \leq Mb$, where $x := (x_i)_i \in \prod_{i=1}^{\infty} M_k(A_i) \cong M_k(A)$. Since ϵ is arbitrary, we get that a is Cuntz below b, as desired.
(ii): Here we follow closely arguments from [400]. We first establish two claims.
Claim 1: If $a, b \in M_k(A)$ are positive elements, with b full, such that $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in K$ then $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(A)$. Let us prove this. Choose a natural number $r' > r$. Then
$$d_\tau(a^{\oplus n}) + r' \leq d_\tau(b^{\oplus n} \oplus 1_{r'})$$
for all $n = 1, 2, \ldots$ and all $\tau \in K$. By the proof of (i), $a^{\oplus n}$ is Cuntz below $b^{\oplus n} \oplus 1_{r'}$for all $n \in \mathbb{N}$. Now let $\tau \in T(A)$. Then $n d_\tau(a) \leq n d_\tau(b) + r'$. Letting $n \to \infty$ we get that $d_\tau(a) \leq d_\tau(b)$, proving our claim.
Claim 2: If $a, b \in A_+$, with b full, are such that $\tau(a) \leq \tau(b)$ for all $\tau \in K$ then $\tau(a) \leq \tau(b)$ for all $\tau \in T(A)$. Let us prove this. Let $\epsilon > 0$. Since
$$\sigma(c) = \int_0^{\|c\|} d_\sigma((c-t)_+)\, dt,$$
for all positive elements $c \in A$ and all $\sigma \in T(A)$ (see for example [364, Proposition 4.2]), one can construct positive elements a_n, b_n (in matrix algebras over A), and find natural numbers r_n, s_n such that
$$\lim_{n\to\infty} \frac{1}{r_n} d_\sigma(a_n) = \sigma((1-\epsilon)a)$$
and
$$\lim_{n\to\infty} \frac{1}{s_n} d_\sigma(b_n) = \sigma(b)$$
for all tracial states σ, with both sequences increasing. Since b is full, we have that
$$\tau\left((1-\epsilon)a\right) \leq \tau\left((1-\epsilon)b\right) < \tau(b)$$
for all $\tau \in K$. Using lower semi-continuity and the compactness of K, as in part (i), we obtain $n \in \mathbb{N}$ such that $\tau((1-\epsilon)a) \leq \frac{1}{s_n} d_\tau(b_n)$ for all $\tau \in K$. Hence,
$$\frac{1}{r_m} d_\tau(a_m) \leq \frac{1}{s_n} d_\tau(b_n), \text{ for all } \tau \in K,$$
for all m and for all sufficiently large n. By the first claim applied to the positive elements $a_m^{\oplus s_n}$and $b_n^{\oplus r_m}$,
$$\frac{1}{r_m} d_\sigma(a_m) \leq \frac{1}{s_n}\, d_\sigma(b_n)$$

for any $\sigma \in T(A)$. Taking the limit as $m \to \infty$, we obtain $\sigma((1-\epsilon)a) \le \sigma(b)$. Letting $\epsilon \to 0$ proves the claim.

Let us now show that $K = T(A)$. By the Hahn–Banach theorem, it suffices to show that for all self-adjoint $a \in A$, if $\tau(a) = 0$ for all $\tau \in K$ then $\tau(a) = 0$ for all $\tau \in T(A)$. If a is a self-adjoint such that $\tau(a) = 0$ for all $\tau \in K$ then $\tau(a + t1) = \tau(t1)$ for all $\tau \in K$. Moreover, for $t > \|a\|$ both $a + t1$ and $t1$ are positive and full. It follows that $\tau(a + t1) = \tau(t1)$ for all $\tau \in T(A)$ and $t > \|a\|$, which yields the desired result.

Theorem (6.3.35)[455]: Let $r \in [0, \infty)$. There exists $M \in \mathbb{N}$ such that if A is a unital C^*-algebra with radius of comparison-by-traces at most rand $a \in A$ is such that $0 \in D_A(a)$, then

$$\left\| \frac{1}{M} \sum_{i=1}^{M} u_i a u_i^* \right\| \le \frac{1}{2} \|a\|$$

for some unitaries $u_1, \ldots, u_M \in A$.

Proof. Suppose, for the sake of contradiction, that there exist unital C^*-algebras $A_1, A_2, \ldots$ with radius of comparison-by-traces at most r, and contractions $a_n \in A_n$ such that $0 \in D_A(a_n)$ for all n, but any average of n unitary conjugates of a_n has norm greater than $1/2$. Let $A := \prod_{n=1}^{\infty} A_n$ and $a := (a_n)_n \in A$. We will show that $0 \in D_A(a)$. To show that $0 \in D_A(a)$, it suffices to check conditions (a) and (b) of Theorem (6.3.50). Notice that $\tau(a) = \tau(a_n) = 0$ for all $\tau \in T(A_n)$ and all n. It follows by Theorem (6.3.34) (ii) that $\tau(a) = 0$ for all $\tau \in T(A)$, i.e., condition (a) holds. In order to show that a satisfies condition (b), we prove that it satisfies the equivalent form (b”), stated right before the proof of Theorem(6.3.50). Let $t', t > 0$ be such $t' > t$ and let $w \in \mathbb{C}$. Since $0 \in D_{A_n}(a_n)$, we have, by condition (b”) applied to an, that $(Re(wa_n) + t)_+$ is a full element of A_n(i.e., it generates A_n as a closed two-sided ideal). For all $\tau \in T(A_n)$ we have

$$d_\tau((Re(wa_n) + t)_+) \ge \frac{1}{|w| + t} \tau((Re(wa_n) + t)_+)$$
$$\ge \frac{1}{|w| + t} \tau(Re(wa_n) + t) = \frac{t}{|w| + t},$$

where we have used that $d_\tau(c) \ge \tau(c)/\|c\|$ for any $c \ge 0$ in the first inequality. Choose $N \ge (2 + r)(|w| + t)/t$. Then

$$d_\tau((Re(wa_n) + t)_+^{\oplus N}) \ge 2 + r.$$

Since A_n has radius of comparison-by-traces at most r, the above (including fullness of $(Re(wa_n) + t)_+$) implies that $1 \in A_n$is Cuntz below $(Re(wa_n) + t)_+^{\oplus N}$. Thus, there exists a partial isometry $v_n \in M_N(A_n)$ such that $1 = v_n^* v_n$ and

$$v_n v_n^* \le C \cdot (Re(wa_n) + t')_+^{\oplus N},$$

where $C > 0$ depends on t and t' but not on n. Then, setting $v := (v_n)_n \in M_N(A)$, we get $1 = v^* v$ and $vv^* \le C \cdot (Re(wa_n) + t')_+^{\oplus N}$. Hence, $(Re(wa) + t')_+$ is full for all $t' > 0$ and $w \in \mathbb{C}$. This proves condition (b”). It follows that $0 \in D_A(a)$. Thus, there is a finite convex combination of unitary conjugates of a whose norm is less than $\frac{1}{2}$. Enlarging the number of terms if necessary, we may assume that this convex combination is an average (see the proof of Theorem (6.3.21) (iii)⇒(iv)). So, there exist $M \in \mathbb{N}$ and unitaries $u_1, \ldots, u_M \in A$ such that

$$\left\|\frac{1}{M}\sum_{i=1}^{M} u_i a u_i^*\right\| \leq \frac{1}{2}.$$

We arrive at a contradiction by projecting onto A_M.

Corollary (6.3.36)[455]: Let $r \in [0, \infty)$. Then there exist constants (m, γ) such that every unital C^*-algebra with the Dixmier property and with radius of comparison-by-traces at most r has the uniform Dixmier property with constants (m, γ). In particular, every simple unital C^*-algebra with at most one tracial state and radius of comparison-by-traces at most r has the uniform Dixmier property with constants (m, γ).

Proof. Let $M \in \mathbb{N}$ be as in Theorem (6.3.35). Suppose that A is a unital C^*-algebra with the Dixmier property and radius of comparison-by-traces at most r. Now let $a \in A$ be a self-adjoint element and choose $z \in D_A(a) \cap Z(A)$. Then $0 \in D_A(a - z)$, and so

$$\left\|\frac{1}{M}\sum_{i=1}^{M} u_i a u_i^* - z\right\| \leq \frac{1}{2}\|a\|,$$

for some unitaries $u_1, \ldots, u_M$. Hence, A has the uniform Dixmier property with constants $(M, 1/2)$.

Let us explain why these examples have finite radius of comparison-by-traces (which is the same as finite radius of comparison, since they are exact). For $n \in \mathbb{N}$, Villadsen's algebra A with stable rank $n + 1$ is constructed, according to [438], as $A = \lim_{\rightarrow} A_i$ where $A_i = p_i(C(X_i) \otimes K)p_i$, with X_i a certain space of dimension $n(1 + 2 \cdot 1! + 4 \cdot 2! + \cdots + 2i \cdot i!)$ and p_i a certain projection of constant rank $(i + 1)!$. We compute

$$\begin{aligned}\frac{dim(X_i) - 1}{2\, rank(p_i)} &\leq \frac{2n(1! + 2! + \cdots + i!)}{2i!} \\ &\leq \frac{n(i-1)(i-1)!}{i!} + \frac{ni!}{i!} \\ &\leq 2n\end{aligned}$$

By [434, Theorem 5.1], it follows that A_i has radius of comparison at most $2n$. Hence by [351, Proposition 3.2.4], the radius of comparison of A is at most $2n$.

If A is a unital C^*-algebra with trivial centre, then by Corollary (6.3.16) A has the Dixmier property if and only if we have one of the following four cases:

(i) A is simple and has no tracial states,

(ii) A is simple and has a unique tracial state,

(iii) A has no tracial states and a unique non-zero maximal ideal,

(iv) A has a unique tracial state and its trace-kernel ideal is the unique nonzero maximal ideal of A.

Cases (ii) and (iv) have the singleton Dixmier property while cases (i) and (iii) do not. Now, since A is unital and has the Dixmier property, $K(A) \leq 1 < \infty$ and so $Z(A_\omega) = Z(A)_\omega = \mathbb{C}1$ by Proposition (6.3.30). (That $\mathbb{C}_\omega = \mathbb{C}$ is because every bounded sequence of complex numbers has a unique limit under ω, i.e., the map taking $(x_n)_{n=1}^{\infty} \in \prod_n \mathbb{C}$ to $\lim_{n\to\omega} x_n$ induces an isomorphism $\mathbb{C}_\omega \to \mathbb{C}$.) Thus, by Corollary (6.3.31), in order for A to have the uniform Dixmier property A_ω must also fall in one of the four cases above. In Theorem (6.3.37) below we take this analysis further to obtain explicit conditions for having the uniform Dixmier property when A falls in cases (ii) and (iv) above.

Suppose that A is in either case (ii) or (iv). Let τ denote the unique tracial state of A. Then τ induces a canonical tracial state τ_ω on A_ω, by

$$\tau_\omega(a) := \lim_{n\to\omega} \tau(a_n),$$

whenever a is represented by the sequence $(a_n)_n$. Let

$$J := \{a \in A_\omega : \tau_\omega(a^*a) = 0\},$$

the trace-kernel ideal for τ_ω. Using the Kaplansky density theorem, one can see that A_ω/J is isomorphic to the tracial von Neumann ultrapower of $\pi_\tau(A)''$, where $\pi_\tau$is the GNS representation associated to τ([391, Theorem 3.3]). In particular, this quotient is a finite factor, and is therefore simple, so that J is a maximal ideal.

In the next result, conditions (i) and (iii) are both expressed purely in terms of the C^*-algebra A. However, in order to show that these conditions are equivalent, we introduce A_ω so that we can apply Corollary (6.3.16) and Corollary (6.3.31).

Theorem (6.3.37)[455]: Let A be a C^*-algebra with the Dixmier property, trivial centre, and unique tracial state τ. The following are equivalent:

(i) A has the uniform Dixmier property.

(ii) τ_W is the unique tracial state on A_ω and the trace-kernel ideal J is the unique maximal ideal of A_ω.

(iii) Both of the following hold:

(a) there exists $m \in \mathbb{N}$ such that if $a \in A$ is a self-adjoint contraction satisfying $\tau(a) = 0$ then there exist contractions $x_1, \ldots, x_m \in A$ such that

$$\|a - \textstyle\sum_{i=1}^m [x_i, x_i^*]\| \le (1 - 1/m)\|a\|, \text{ and}$$

(b) for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that, if $a \in A_+$ is a positive contraction and $\tau(a) > \epsilon$ then there exist contractions $x_1, \ldots, x_n \in A$such that

$$\sum_{i=1}^n x_i a x_i^* = 1.$$

Proof. Recall that since A is unital and has the Dixmier property, $K(A) \le 1 < \infty$ and so $Z(A_\omega) = Z(A)_\omega = \mathbb{C}1$ by Proposition (6.3.30).

(i)⇔(ii): By Corollary (6.3.31), (i) is equivalent to $A_\omega$having the Dixmier property. Thus, (i)⇔(ii) follows from Corollary (6.3.16).

(ii)⇔(iii): We will first show that (a) is equivalent to τ_ω being the unique tracial state on A_ω, then that (b) is equivalent to J being the unique maximal ideal of A_ω.

For a unital C^*-algebra B, set B_0 equal to the norm-closure of the $\mathbb{R}$-span of the set of self-commutators $[x, x^*]$. For a tracial state τ_B on B, by [355, Theorem 2.6 and Proposition 2.7], τ_B is the unique tracial state of B if and only if

$$B_0 = \{b \in B : b \text{ is self-adjoint and } \tau_B(b) = 0\}.$$

Suppose that τ_ω is the unique tracial state on A_ω and, for a contradiction, that (a) doesn't hold. Then for each $n \in N$ there exists a self-adjoint contraction $a_n \in A$ such that $\tau(a_n) = 0$ and

$$\|a_n - \textstyle\sum_{i=1}^n [x_i, x_i^*]\| \ge (1 - 1/n) \tag{19}$$

for all tuples $(x_1, \ldots, x_n)$ of contractions in A.

Since the sequence $(a_n)_n$ is bounded, it defines a self-adjoint element $a \in A_\omega$, and this element clearly satisfies $\tau_\omega(a) = 0$. Since τ_ω is the unique tracial state, it follows (as mentioned above) that there exist $m \in \mathbb{N}$ and $y_1, \ldots, y_m \in A_\omega$ such that

$$\left\| a - \sum_{i=1}^{m} [y_i, y_i^*] \right\| < \frac{1}{2}.$$

By increasing m if necessary, we may assume that all of the elements y_i are contractions. Lifting each y_i to a sequence $(x_{i,n})_n$ of contractions in A, we have for ω-almost all $n \in \mathbb{N}$,

$$\left\| a_n - \sum_{i=1}^{m} [x_{i,n}, x_{i,n}^*] \right\| < \frac{1}{2}.$$

In particular, for some $n \geq m$, we obtain a contradiction to (19). This proves that if A_ω has a unique tracial state then (a) holds.

Now suppose that (a) holds, which provides a number m. If $a \in A_\omega$ is a self-adjoint contraction satisfying $\tau_\omega(a) = 0$, then we may lift a to a sequence $(a_n)_{n=1}^{\infty}$of self-adjoint elements satisfying $\tau(a_n) = 0$ and $\|a_n\| \leq \|a\|$ for all n. (To achieve this, we first lift a to any bounded sequence of self-adjoint elements, then correct the tracial state on each element by adding an appropriate scalar, and finally scale to obtain $\|a_n\| \leq \|a\|$.) By applying (a) to each a_n, we can arrive at elements $x_1, \dots, x_m \in A_\omega$ such that

$$\left\| a - \sum_{i=1}^{m} [x_i, x_i^*] \right\| \leq (1 - 1/m)\|a\|.$$

In other words, this shows that A_ω satisfies (a), with τ_ω in place of τ.

Next, by iterating, we see that if $a \in A_\omega$ is a self-adjoint contraction and satisfies $\tau_\omega(a) = 0$, then for any $k \in \aleph$, there exist mk contractions $x_1, \dots, x_{mk} \in A_\omega$ such that

$$\left\| a - \sum_{i=1}^{mk} [x_i, x_i^*] \right\| \leq (1 - 1/m)^k \|a\|.$$

It follows that $a \in (A_\omega)_0$. By $\mathbb{R}$-linearity,

$$(A_\omega)_0 = \{a \in A_\omega \colon a \text{ is self-adjoint and } \tau_\omega(a) = 0\}$$

and hence $\tau_\omega$is the unique tracial state of A_ω.

Now, suppose that J is the unique maximal ideal of A_ω and let us prove that (b) holds. Suppose for a contradiction that (b) doesn't hold. Then there exists $\epsilon > 0$ and, for each $n \in \mathbb{N}$, a contraction $a_n \in A_+$such that $\tau(a_n) > \epsilon$ yet

$$\sum_{i=1}^{n} x_i a_n x_i^* \neq 1 \tag{20}$$

for all contractions $x_1, \dots, x_n \in A$.

Define $a \in A_\omega$ by the sequence $(a_n)_n$, so that $\tau_\omega(a) \geq \epsilon$. Since J is the unique maximal ideal of A_ω, the ideal generated by a is A_ω. Hence, there exists $y_1, \dots, y_m \in A_\omega$ such that

$$\sum_{i=1}^{m} y_i a y_i^* = 1,$$

and by increasing m if necessary we may assume that all of the elements y_i are contractions. Lift each y_i to a sequence $(y_{i,k})_k$ of contractions. Then, for ω-almost all indices k, we have

$$\left\| \sum_{i=1}^{m} y_{i,k} a_k y_{i,k}^* - 1 \right\| < \frac{1}{2}.$$

Pick $k \geq 2m$ such that this holds. Set

$$b:=\sum_{i=1}^{m} y_{i,k}a_k y_{i,k}^*,$$
so that the spectrum of b is contained in $[1/2,3/2]$. Therefore, $(2b)^{-1/2}y_{i,k}$ is a contraction, and
$$1=2(2b)^{-1/2}y_{i,k}a_k y_{i,k}^*(2b)^{-1/2},$$
in contradiction to (20).

Finally assume that (b) holds, and we'll prove that J is the unique maximal ideal of A_ω. Let I be an ideal of A_ω, such that $I \nsubseteq J$. Therefore, I contains a positive contraction $a \notin J$, so that $r:=\tau_\omega(a)>0$. Using $\epsilon:=r/2$, we get some $n\in\mathbb{N}$ from (b).
We may lift a to a sequence $(a_k)_k$ of positive contractions such that $\tau(a)>r/2$ for each k. Then for each k there exist n contractions $x_{1,k},\ldots,x_{n,k}\in A$ such that
$$1=\sum_{i=1}^{n} x_{i,k}a_k x_{i,k}^*.$$
Letting $x_i\in A_\omega$ be the element represented by the sequence $(x_{i,k})_k$, we have
$$1=\sum_{i=1}^{n} x_i a x_i^* \in I,$$
and therefore $I=A_\omega$. This shows that J is the unique maximal ideal of A_ω.

Under the hypotheses of Theorem (6,3,36), it is unclear whether there is any relation between the conditions that τ_ω is the unique tracial state on A_ω(equivalently, condition(a)) and that J is the unique maximal ideal of A_ω(equivalently, condition (b)).

Question (6.3.38)[455]: Does condition (a) in Theorem (6.3.37) (iii) imply condition (b), or vice versa?

In [416, Theorem 1.4], LR showed that there is a simple unital (and nuclear, in fact AH) C^*-algebra $A$with unique tracial state, which doesn't satisfy (iii)(a) in Theorem (6.3.37) (i.e., A_ω doesn't have a unique tracial state). Since A has the Dixmier property by [374], this shows that the Dixmier property is strictly weaker than the uniform Dixmier property.

Let us briefly discuss the cases when A is unital, has the Dixmier property, trivial centre. If A is simple and purely infinite, then A_ω is also simple and purely infinite [418, Proposition 6.2.6], whence has the Dixmier property, and so A has the uniform Dixmier property. In the cases that A is not simple and purely infinite, we have little to say about whether A has the uniform Dixmier property. In such cases, A_ω has no tracial states either, but it is not simple, for if A_ω is simple and non-elementary then A must be simple and purely infinite [390, Remark 2.4]. Rørdam has constructed examples of simple unital separable (even nuclear) C^*-algebras which are not purely infinite, yet have no tracial states [418].

Question (6.3.39)[455]: Are there simple unital C^*-algebras with the uniform Dixmier property and without tracial states other than the purely infinite ones?

Let A be a simple unital C^*-algebra with no tracial states, which is not purely infinite. Then there is a bounded sequence of self-adjoint elements $(a_n)_{n=1}^{\infty}$with $1\in D_A(a_n)$ for all n, but $1\notin D_{A_\omega}((a_n)_n)$. (However, it is conceivable that $D_{A_\omega}((a_n)_n)$ meets $Z(A_\omega)$in another point, so this does not show that A does not have the uniform Dixmier property.) To see this, first, since $A_\omega$is non-simple, there exists a positive element $a\in A_\omega$of norm 2 that is not full. Lift a to a bounded sequence $(a_n)_n$ of positive elements. Since $\|a\|=2$ and a is not invertible, for ω-almost all n, the convex hull of the spectrum of a_n contains (a). Modifying a_n for n in an ω-null set, we can arrange that the convex hull of the spectrum of a_n contains (a) for all n. Since A is simple, it follows that $1 \in D_A(a_n)$ for all n. However, if $1 \in D_{A_\omega}(A)$ then with $I := Ideal(a), 1 \in D_{A_\omega/I}(q_I(a)) = D_{A_\omega/I}(0)$, which is a contradiction.

Suppose that A is a unital C^*-algebra that has the Dixmier property as well as one of the following properties:

a. finite nuclear dimension, or
b. finite radius of comparison by traces.

Then A has the uniform Dixmier property for suitable constants (m, γ) (i.e., (20) holds). For finite nuclear dimension, this follows from [402, Theorem 5.6]. For finite radius of comparison, this is Corollary (6.3.36) obtained above. These results are proven by contradiction, with repeated use of the Hahn–Banach Theorem, thereby not yielding explicit values for the constants (m, γ). In fact, we do not know explicit values for (m, γ) holding globally in either one of these two cases. (On the other hand, explicit constants may be extracted from the methods used in [424] and [403], for simple C^*-algebras with real rank zero, strict comparison by traces, and a unique tracial state.) Prompted by an interesting question posed by the referee, we find explicit values for the constants (m, γ) for a variety of C^*-algebras with the uniform Dixmier property. When the C^*-algebras have the singleton Dixmier property, we also estimate the constants (M, Υ) (i.e., for which (20) holds).

Let A be a C^*-algebra. Let $h \in A$ be a self-adjoint element and let $[l(h), r(h)]$ be the smallest interval containing the spectrum of h, i.e., the numerical range of h. Set $\omega(h) := r(h) - l(h)$ and note that $\omega(h) \leq 2\|h\|$.

We first consider uniform Dixmier property constants for von Neumann algebras (slightly improving the constants that can be extracted from Dixmier's original argument [360, Lemma1 of §III.5.1]). Let W be a von Neumann algebra and ha self-adjoint element of W. Let $e \in W$ be a central projection. In the next lemma $\omega_e(h)$ denotes $\omega(eh)$ in the von Neumann algebra eW.

Lemma (6.3.40)[455]: Let W be a von Neumann algebra. Let $h \in W$ be a self-adjoint element with finite spectrum. Then there exist central projections $e_1, \ldots, e_n$adding up to 1 and a unitary $u \in W$ such that

$$\omega_{e_k}\left(\frac{h + uhu^*}{2}\right) \leq \frac{1}{2}\omega_{e_k}(h) \; for \; all \; k.$$

Proof. It is shown in [402, Proposition 3.2] that given two self-adjoint elements $h_1, h_2 \in W$ with finite spectrum, it is possible to find projections $P_1, \ldots, P_N$ adding up to 1, a unitary $u \in W$, and self-adjoint central elements $\lambda_1, \mu_1, \ldots, \lambda_N, \mu_N \in Z(W)$ with finite spectrum such that

$$h_1 = \sum_{i=1}^{N} \lambda_i P_i, \qquad uh_2u^* = \sum_{i=1}^{N} \mu_i P_i,$$

and

$$\lambda_1 \geq \ldots \geq \lambda_N, \qquad \mu_1 \geq \ldots \geq \mu_N.$$

(Note: [402, Proposition 3.2] is stated for positive elements but it is easily extended to self-adjoint elements by adding a scalar.) Let us apply this result to the self-adjoint elements h and $-h$. We then get

$$h=\sum_{i=1}^{N}\lambda_i P_i \ and\ uhu^*=\sum_{i=1}^{N}\nu_i P_i\,,$$
where $\lambda_1 \geq \ldots \geq \lambda_N$ and $\nu_1 \leq \ldots \leq \nu_N$. Since all of the λ_i and ν_i have finite spectrum, there exist central projections $e_1,\ldots,e_n$ with sum 1 such that $e_k\lambda_i$ and $e_k\nu_i$ are scalar multiples of e_k for all i and k. Let us show that $e_1,\ldots,e_n$ and u are as desired. Let $\tilde{h}:=(h+uhu^*)/2$. Fix $1\leq k\leq n$. Let $S:=\{i\in\{1,\ldots,N\}: e_kP_i\neq 0\}$. Denote the scalars $e_k\lambda_i$ and $e_k\nu_i$(in e_kW) simply as λ_i and ν_i. Then the spectrum of e_kh in e_kW is $\{\lambda_i: i\in S\}$ and also (since e_kh is unitarily equivalent to e_kuhu^*) $\{\nu_i: i\in S\}$. On the other hand, the spectrum of $e_k\tilde{h}$ in $e_k\tilde{h}$ is the set
$$\left\{\frac{\lambda_i+\nu_i}{2}: i\in S\right\}.$$
Let $i,j\in S$ with $i\leq j$. Then
$$\left|\frac{\lambda_i+\nu_i}{2}-\frac{\lambda_j+\nu_j}{2}\right|=\left|\frac{\lambda_i-\lambda_j}{2}-\frac{\nu_j-\nu_i}{2}\right| \leq \max\left(\frac{\lambda_i-\lambda_j}{2},\frac{\nu_j-\nu_i}{2}\right)\leq\frac{\omega_{e_k}(h)}{2}.$$
Thus, $\omega_{e_k}(\tilde{h})\leq\omega_{e_k}(h)/2$ for all k, as desired.

Theorem (6.3.41)[455]: Let W be a von Neumann algebra. Then W has the uniform Dixmier property with constants (m,γ) for $m=2$ and every $\gamma\in(1/2,1)$. If W is finite, then it has the uniform singleton Dixmier property with constants (M,Υ) for $M=4$ and every $\Upsilon\in(1/2,1)$.

Proof. Let $0<\epsilon<1/2$ and let $0\neq g=g^*\in W$. By the spectral theorem, there is a self-adjoint element $h\in W$ with finite spectrum such that $\|g-h\|<\epsilon\|g\|$ and $\|h\|\leq\|g\|$. Apply Lemma (6.3.40) to h to obtain a unitary $u\in W$ and central projections $e_1,\ldots,e_n$ as in the statement of that lemma, and then define the central element $z:=\sum_{k=1}^{n}\alpha_k e_k$, where α_k is the midpoint of the spectrum of $e_k(h+uhu^*)/2$ in e_kW (that is, the midpoint of the interval $[l(e_k(h+uhu^*)/2), r(e_k(h+uhu^*)/2)]$). Then we see that
$$\begin{aligned}\left\|e_k\left(\frac{h+uhu^*}{2}-z\right)\right\| &= \left\|e_k\,\frac{h+uhu^*}{2}-\alpha_k e_k\right\| \\ &=\frac{1}{2}\omega_{e_k}\left(\frac{h+uhu^*}{2}\right) \\ &\leq\frac{1}{4}\omega_{e_k}(h) \\ &\leq\frac{1}{2}\|e_kh\|\leq\frac{1}{2}\|h\|.\end{aligned}$$
Since the e_k are orthogonal central projections, it follows that $\|(h+uhu^*)/2-z\|\leq\frac{1}{2}\|h\|$. Then
$$\|(g+ugu^*)/2-z\|\leq\|h\|/2+\epsilon\|g\|\leq(1/2+\epsilon)\|g\|.$$
Suppose now that W is finite and hence has the singleton Dixmier property. For all $\epsilon>0$ such that $(1/2+\epsilon)^2<1/2, W$ has the uniform Dixmier property with constants $(2^2,(1/2+\epsilon)^2)$ and hence the uniform singleton Dixmier property with constants $(4,2(1/2+\epsilon)^2)$ (by Lemma (6.3.26)). Since $2(1/2+\epsilon)^2\to 1/2$ as $\epsilon\to 0$, we obtain the required result.

Proposition (6.3.42)[455]: The C^*-algebra M_n has the uniform Dixmier property with constants $m = 2$ and $\gamma = 1/2$ and the uniform singleton Dixmier property with constants $M = 4$ and $\Upsilon = 1/2$.

Proof. That M_n has the uniform Dixmier property with constants $m = 2$ and $\gamma = 1/2$ follows at once from Lemma (6.3.40) above. The constants $M = 4$ and $\Upsilon = 1/2$ are obtained from Lemma (6.3.26).

Theorem (6.3.43)[455]: Let X be a compact Hausdorff space with covering dimension $d < \infty$. Let $n \in \mathbb{N}$. The following are true:

(i) The C^*-algebra $C(X, M_n)$ has the uniform Dixmier property with constants (m, γ) for $m = d + 2$ and every $\gamma \in ((d + 1)/(d + 2), 1)$. It has the uniform singleton Dixmier property with constants (M, Υ) for $M = 3d + 4$ and every $\Upsilon \in ((3d + 2)/(3d + 4), 1)$.

(ii) If $d \leq 2$ and in the Čech cohomology we have $\check{H}^2(X) = 0$ (e.g., $X = [0, 1]$ or $X = [0, 1]^2$), then $C(X, M_n)$ has the uniform Dixmier property with constants (m, γ) for $m = 2$ and every $\gamma \in (1/2, 1)$ and the uniform singleton Dixmier property with constants (M, Υ) for $M = 4$ and every $\Upsilon \in (1/2, 1)$.

Proof. It is well-known that $C(X, M_n)$ has the singleton Dixmier property: for example, the Dixmier property holds by [342, Proposition 2.10] and the singleton Dixmier property is then a consequence of the fact that every simple quotient has a tracial state (see Proposition (6.3.4)). We prove (ii) first, because the argument is more similar to the previous proof.

(ii): Let $h \in C(X, M_n)$ be a self-adjoint element. By [431, Theorem10], h is approximately unitarily equivalent to a diagonal self-adjoint $h' = diag(\lambda_1, \dots, \lambda_n)$, where the eigenvalue functions $\lambda_1, \dots, \lambda_n \in C(X, \mathbb{R})$ are arranged in decreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Note that a self-adjoint element a in a unital C^*-algebra satisfies (14) if and only if every unitary conjugate of a does so (with the same central element z). Hence, by an approximation argument similar to that in the proof of Theorem (6.3.41), it suffices to establish (7) with $m = 2$ and $\gamma = 1/2$ for diagonal self-adjoint elements of the form above. So assume that his diagonal with decreasing eigenvalue functions. Let $u \in M_n$ be the permutation unitary such that $uhu^* = diag(\lambda_n, \dots, \lambda_1)$. Set $\tilde{h} := (h + vhv^*)/2$, where $v \in U(C(X, M_n))$ is given by $v(x) := u(x \in X)$. Then,

$$\tilde{h} = diag\left(\frac{\lambda_1 + \lambda_n}{2}, \frac{\lambda_2 + \lambda_{n-1}}{2}, \dots, \frac{\lambda_n + \lambda_1}{2}\right).$$

The same estimates used in the proof of Lemma (6.3.40) show that $\omega(\tilde{h}) \leq \omega(h)/2$. It follows that

$$\left\|\tilde{h} - \frac{\lambda_1 + \lambda_n}{2} \cdot 1_n\right\| \leq \frac{1}{2}\|h\|.$$

As observed above, this shows that $C(X, M_n)$ has the uniform Dixmier property with $m = 2$ and every $\gamma \in (1/2, 1)$. The constants $M = 4$ and $\Upsilon \in (1/2, 1)$ are then derived from the constants (m, γ) as in the proof of Theorem (6.3.41).

(i): Let $\epsilon > 0$ be given. Let $f \in C(X, M_n)$ be a self-adjoint contraction. For each $x \in X$, by Proposition (6.3.42), we may find $\lambda_x \in \mathbb{R}$ and a unitary $u_x \in M_n$ such that

$$\left\|\frac{1}{2}(f(x) + uxf(x)u_x^*) - \lambda_x 1\right\| \leq \frac{1}{2}.$$

Evidently, we may assume $\lambda_x \in [-1, 1]$. By continuity, we may then find a neighbourhood W_x of x such that

$$\left\|\frac{1}{2}(f(y) + u_x f(y) u_x^*) - \lambda_x 1\right\| < \frac{1}{2} + \epsilon \quad for\ all \quad y \in W_x.$$

From the open cover $\{W_x : x \in X\}$ of X, using compactness and the fact that X has dimension d, we may find a finite refinement of the form $\{W_j^{(i)}\}_{i=0,\dots,d;\, j=1,\dots,r}$ which covers X, and such that

$$\overline{W_j^{(i)}} \cap \overline{W_{j'}^{(i)}} = \emptyset$$

for all $j \neq j'$. Denote $u_j^{(i)}$ the unitary corresponding to the open set $W_j^{(i)}$ and $\lambda_j^{(i)}$ the scalar, i.e., such that

$$\left\|\frac{1}{2}\left(f(y) + u_j^{(i)} f(y) (u_j^{(i)})\right)^* - u_j^{(i)} 1\right\| < \frac{1}{2} + \epsilon$$

for all $y \in W_j^{(i)}$. For $i \in \{0, \dots, d\}$, since all unitaries in M_n are homotopic to the identity, we may produce a unitary $u^{(i)} \in C(X, M_n)$ such that

$$u^{(i)}(y) = u_j^{(i)} \text{ whenever } y \in W_j^{(i)},$$

as follows. We may find disjoint open sets $V_1^{(i)}, \dots, V_r^{(i)}$ containing $\overline{W_1^{(i)}}, \dots, \overline{W_r^{(i)}}$ respectively, and then we may use a homotopy of unitaries to get a unitary in $C(\overline{V_j^{(i)}}, M_n)$ which is identically $u_j^{(i)}$ on $W_j^{(i)}$ and identically 1 on $\partial V_j^{(i)}$. We may then define the continuous unitary $u^{(i)} \in C(X, M_n)$ so that it restricts to the unitary just defined on each $\overline{V_j^{(i)}}$ and is identically 1 outside of $V_1^{(i)} \cup \dots \cup V_r^{(i)}$.

We claim that $\tilde{f} := \frac{1}{d+2}(f + u^{(0)} f (u^{(0)})^* + \dots + u^{(d)} f (u^{(d)})^*)$ has distance at most $(d + 1)/(d + 2) + \epsilon$ to the centre. Note that, by a partition-of-unity argument, the distance from $\tilde{f}$ to the centre is equal to the supremum over all $x \in X$ of the distance from $\tilde{f}(x)$ to $Z(M_n) = \mathbb{C}1_n$ (see [426, Theorem 2.3] for a more general result). For $x \in X$, pick i_0, j such that $x \in W_j^{(i_0)}$. Without loss of generality, $i_0 = 0$. Then

$$\begin{aligned}
\left\|\tilde{f}(x) - \frac{2}{d+2}\lambda_j^{(0)} 1\right\| &\leq \frac{2}{d+2}\left\|\frac{1}{2}(f(x) + u^{(0)}(x) f(x) (u^{(0)}(x))^*) - \lambda_j^{(0)} 1\right\| \\
&\quad + \frac{1}{d+2}\sum_{i=1}^{d} \|u^{(i)}(x) f(x) (u^{(i)}(x))^*\| \\
&= \frac{2}{d+2}\left\|\frac{1}{2}(f(x) + u_j^{(0)} f(x) (u_j^{(0)})^*) - \lambda_j^{(0)} 1\right\| \\
&\quad + \frac{1}{d+2}\sum_{i=1}^{d} \|u^{(i)}(x) f(x) (u^{(i)}(x))^*\| \\
&< \frac{2}{d+2}\left(\frac{1}{2} + \epsilon\right) + \frac{d}{d+2} \\
&\leq \frac{d+1}{d+2} + \epsilon
\end{aligned}$$

as required.

A similar argument is used to get uniform singleton Dixmier property constants. Here we may replace f with $f - R(f)$, so that $f(x)$ has trace 0 for all $x \in X$. Then we use the same argument as above, with the uniform singleton Dixmier property constants $\left(4, \frac{1}{2}\right)$ from Proposition (6.3.42), and with $\lambda_x = 0$ for all x (and thereby $\lambda_j^{(i)} = 0$ for all i, j), to get $M = 3(d+1) + 1 = 3d + 4$ and (for any sufficiently small $\epsilon > 0$)

$$\Upsilon = \frac{1}{2}\frac{4}{3d+4} + \frac{3d}{3d+4} + \epsilon = \frac{3d+2}{3d+4} + \epsilon.$$

Consider the following property of unital C^*-algebras A:

(P): There exist $M \in \mathbb{N}$ and $0 < \Upsilon < 1$ such that if $h \in A$ is a self-adjoint such that $\tau(h) = 0$ for all $\tau \in T(A)$ then

$$\left\| \frac{1}{M} \sum_{i=1}^{M} u_i h u_i^* \right\| \leq \Upsilon \|h\| \tag{20}$$

for some unitaries $u_1, \dots, u_M$.

Note that if A has the property (P) for some (M, Υ) then it also has (P) for $(M^k, \Upsilon^k)(k = 2, 3, \dots)$.

Suppose that A has the Dixmier property and has the property (P) for some (M, Υ). For $h = h^* \in A$ and $z_1, z_2 \in D_A(h) \cap Z(A)$, we have $\tau(z_1 - z_2) = 0$ for all $\tau \in T(A)$ and hence $0 \in D_A(z_1 - z_2) = \{z_1 - z_2\}$. Thus $z_1 = z_2$. An elementary argument with real and imaginary parts shows that A has the singleton Dixmier property. It then follows from (P) that A has the uniform singleton Dixmier property with the same constants (M, Υ) (as introduced in (16)).

Conversely, suppose that A has the uniform singleton Dixmier property with constants (M, Υ). If $h = h^* \in A$ vanishes on all tracial states of A then h also vanishes on the centre-valued trace of A. Thus A has the property (P) with the same (M, Υ).

But (P) may hold much more generally: if every quotient of A has a bounded trace and A has either finite nuclear dimension or finite radius of comparison by traces then A has(P) for some (M, Υ)([402, Theorem 5.6] for the former case, Theorem (6.3.35) in the latter).

In the following results, it will occasionally be convenient to write $a \approx_\epsilon b$ to mean $\|a - b\| < \epsilon$.

Theorem (6.3.44)[455]: Let A be a unital C^*-algebra with decomposition rank one and stable rank one. Then A has(P) with constants (M, Υ) for $M = 15$ and every $\Upsilon \in (11/15, 1)$. Inparticular, if A also has the Dixmier property, then it has the uniform singleton Dixmier property with these constants.

Proof. Let $\epsilon > 0$ be given. Let us factorise the diagonal embedding $\iota: A \to A_\infty$ as $\sum_{i=0}^{1} \phi_i \circ \psi_i$, where $\psi_0, \psi_1$are unital homomorphisms, $\phi_0, \phi_1$are $c.p.c.$ order zero, and where N_0, N_1have the form $\prod_\lambda F_\lambda / \bigoplus_\lambda F_\lambda$, for finite dimensional algebras F_λ. (The existence of such a factorisation follows from the proof of [415, Proposition 2.2], using [392, Proposition 5.1] in place of [440, Proposition 3.2].)

Suppose that, for every $n \in \mathbb{N}$, $M_n$has the uniform Dixmier property with constants (m, γ) (we shall describe suitable values $m > 1$ and γ towards the end of the proof). Then any product of finite-dimensional C^*-algebras has the uniform Dixmier property with constants (m, γ) (Theorem (ii)), hence the Dixmier property and hence the singleton Dixmier property by Proposition (6.3.4) (note that the product of the centre-valued traces is a centre-valued trace on the product). Hence $N_0$and N_1have the uniform Dixmier property with constants (m, γ) (Theorem(6.3.23)) and the singleton Dixmier property (see the remark after Proposition (6.3.4))By Lemma (6.3.26), N_0 and N_1 have the uniform singleton Dixmier property with constants $(m, 2\gamma)$ (and for this we require $2\gamma < 1$). Hence, as noted above, N_0 and N_1have property (P) with constants $M' = m > 1$ and $\Upsilon' = 2\gamma$.

Let $h \in A$ be a self-adjoint contraction that is zero on every tracial state. Then the same is true for $\psi_i(h)$ for $i = 0, 1$. So for $i = 0, 1$ there exist unitaries $u_{i,1}, \ldots, u_{i,M'-1} \in N_i$ such that

$$\left\| \frac{\psi_i(h) + \sum_{j=1}^{M'-1} u_{i,j} \psi_i(h) u_{i,j}^*}{M'} \right\| \leq \Upsilon'. \tag{21}$$

Set $h_i := \phi_i(\psi_i(h))$ for $i = 0, 1$, so that $h = h_0 + h_1$. We set $x_{i,j} = \phi_i^{\frac{1}{n}}(u_{i,j}) \in A_\infty$, so that $x_{i,j}$ depends on n, and

$$|x_{i,j}| h_i \to h_i \text{ and } x_{i,j} h_i x_{i,j}^* \to \phi_i(u_{i,j} \psi_i(h) u_{i,j}^*) \text{ as } n \to \infty.$$

Since A has stable rank one, every element in A_∞ has a unitary polar decomposition, with the unitary element belonging to A_∞. (Indeed, since A has stable rank one every element of A_∞lifts to an element $(a_k) \in \prod_{k=1}^{\infty} A$ such that a_k is invertible for all k; further, such an (a_k) has polar decomposition in $\prod_{k=1}^{\infty} A$.) So $x_{i,j} = U_{i,j}|x_{i,j}|$ for some unitary $U_{i,j} \in A_\infty$. Then, remembering that $x_{i,j}$ depends on n,

$$U_{i,j} h_i U_{i,j}^* \to \phi_i(u_{i,j} \psi_i(h) u_{i,j}^*) \text{ and } U_{i,j} \phi_i(1) U_{i,j}^* \to \phi_i(1) \text{ as } n \to \infty.$$

From (21), for n sufficiently large we get

$$\left\| \frac{h_i + \sum_{j=1}^{M'-1} U_{i,j} h_i U_{i,j}^*}{M'} \right\| \leq \Upsilon' + \epsilon,$$

for $i = 0, 1$. We choose nso that in addition,

$$\left\| U_{i,j} \phi_i(1) U_{i,j}^* - \phi_i(1) \right\| < \epsilon. \tag{22}$$

Consider

$$\tilde{h} := \frac{1}{2M' - 1} \left(h + \sum_{j=1}^{M'-1} U_{0,j} h U_{0,j}^* + \sum_{j=1}^{M'-1} U_{1,j} h U_{1,j}^* \right);$$

we will estimate its norm. We manipulate the sum on the right side:

$$h + \sum_{i=0,1} \sum_{j=1}^{M'-1} U_{i,j} h U_{i,j}^* = (h_0 + \sum_{j=1}^{M'-1} U_{0,j} h U_{0,j}^*) + (h_1 + \sum_{j=1}^{M'-1} U_{1,j} h U_{1,j}^*) + \sum_{j=1}^{M'-1} \left(U_{1,j} h_0 U_{1,j}^* + U_{0,j} h_1 U_{0,j}^* \right). \tag{23}$$

Let $e_0 := \phi_0(1)$ and $e_1 := \phi_1(1)$. Then $h_0 \leq e_0, h_1 \leq e_1$, and $e_0 + e_1 = 1$. Next from (22) and the fact that $e_0 + e_1 = 1$, it follows that $\left\| [U_{i,j}, e_{1-i}] \right\| < \epsilon$ for $i = 0, 1$ and $j = 1, \ldots, M' - 1$. Hence,

$$U_{1,j} h_0 U_{1,j}^* + U_{0,j} h_1 U_{0,j}^* \leq U_{1,j} e_0 U_{1,j}^* + U_{0,j} e_1 U_{0,j}^* \approx_{2\epsilon} e_0 + e_1 = 1,$$

which implies that $U_{1,j} h_0 U_{1,j}^* + U_{0,j} h_1 U_{0,j}^* \leq (1 + 2\epsilon)1.$ A similar argument shows that $U_{1,j} h_0 U_{1,j}^* + U_{0,j} h_1 U_{0,j}^* \geq -(1 + 2\epsilon)1$. The norm of the right side of (23) is at most $2M'(\Upsilon' + \epsilon) + (M' - 1)(1 + 2\epsilon)$ from which we obtain that

$$\|\tilde{h}\| \le \frac{2M'(\Upsilon' + \epsilon) + (M' - 1)(1 + 2\epsilon)}{2M' - 1}.$$

Thus A has property (P) with constants $M = 2M' - 1$ and $\Upsilon = (2M'\Upsilon' + M' - 1)/(2M' - 1) + 2\epsilon$ (provided that this value of Υ is less than 1).

It follows from Proposition (6.3.42) that, for every n, the C^*-algebra M_n has the uniform Dixmier property with constants $m = 2^3 = 8$ and $\gamma = (1/2)^3 = 1/8$. By the discussion above, we may take $M' = 8$ and $\Upsilon' = 1/4$ and hence obtain that A has(P) with constants (M, Υ) for $M = 15$ and every $\Upsilon \in (11/15, 1)$.

Given a self-adjoint element h, let us say that the spectrum of h has gaps of size at most δ if every closed subinterval of $[l(h), r(h)]$ of length δ intersects the spectrum of h.

Lemma (6.3.45)[455]: Let A be simple, unital, and non-elementary. Let $h \in A$ be a self-adjoint element and $\epsilon > 0$. The following are true:

(i) There exists a unitary $u \in A$ such that the spectrum of
$$\frac{1}{3}h + \frac{2}{3}uhu^*$$
has gaps of size at most $\omega(h)/3 + \epsilon$.

(ii) If the spectrum of h has gaps of size at most $\delta > 0$ then there exist a self-adjoint $\tilde{h} \in A$ and $x \in A$ such that $\|x\|^2 = \delta/2$, $x^2 = 0$,
$$\|h - (\tilde{h} + [x^*, x])\| < \epsilon,$$
and the spectrum of $\tilde{h}$ is the interval $[l(h) - \delta/2, r(h) + \delta/2]$.

Proof.(i): The result is trivial if $l(h) = r(h)$. So assume that $l(h) < r(h)$. If the result has been proven for a given h (and arbitrary $\epsilon > 0$) then it at once follows for any $\alpha h + \beta 1$, with $\alpha, \beta \in \mathbb{R}$. Thus, we may assume that $0 \le h \le 1$ and that $[l(h), r(h)] = [0, 1]$. Let us perturb h slightly using functional calculus with a continuous function that is close to the identity function but takes the constant value 0 in a neighbourhood of 0 and the constant value 1in a neighbourhood of 1, so that for the new element k(still a positive contraction) we have that $\|h - k\| < \epsilon/2$ and that $ke = e$ and $kf = 0$ for some non-zero $e, f \in A_+$. Since A is simple (whence prime), there exists a positive element $a \in A$ such that $eaf \ne 0$. Let $x = eaf$. Then $x^*x \in \overline{fAf}$ and $xx^* \in \overline{eAe}$. Since $x^2 = 0, x$ is in the closure of the invertible elements of A. By [406, Theorem 5], for each $t > 0$ there exists a unitary $u \in A$ such that $u(x^*x - t) + u^* = (xx^* - t)_+$. Choose one such u for some $t < \|x\|$. Set $\tilde{e} := (x^*x - t)_+$ and $\tilde{f} := (xx^* - t)_+$. Now consider
$$\tilde{k} := \frac{1}{3}k + \frac{2}{3}(uku^*).$$
Then $\tilde{k}\tilde{e} = (1/3)\tilde{e}$ and $\tilde{k}\tilde{f} = (2/3)\tilde{f}$. Since $\tilde{e}$ and $\tilde{f}$ are nonzero, $1/3$ and $2/3$ are in the spectrum of $\tilde{k}$.

Let $\tilde{h} := \frac{1}{3}h + \frac{2}{3}(uhu^*)$, a positive contraction in A such that $\|\tilde{h} - \tilde{k}\| < \epsilon/2$. Suppose, towards a contradiction, that the spectrum of $\tilde{h}$ does not intersect $(1/3 - \epsilon/2, 1/3 + \epsilon/2)$. Define $b := \tilde{h} - (1/3)1$ and $c := \tilde{k} - (1/3)1$, so that b is self-adjoint, the spectrum of b does not intersect $(-\epsilon/2, \epsilon/2)$ and $\|b - c\| < \epsilon/2$. We have
$$\|1 - b^{-1}c\| \le \|b^{-1}\|\|b - c\| < \frac{\epsilon}{2}\|b^{-1}\| \le 1.$$

Hence $b^{-1}c$ is invertible and so c is invertible, which contradicts that $1/3$ is in the spectrum of $\tilde{k}$. A similar argument shows that the spectrum of $\tilde{h}$ intersects $(2/3 \ - \epsilon/2, 2/3 \ + \epsilon/2)$. It follows that the spectrum of $\tilde{h}$ has gaps of size at most $1/3 + \epsilon$.
(ii): Choose points $l(h) = t_0 < t_1 <\ldots< t_n = r(h)$ in the spectrum of h such that $t_{i+1} - t_i \leq \delta$ for all i. Perturb h by functional calculus using an increasing continuous function close to the identity function that takes the constant value t_i in a small neigh-bourhoodof each t_i, so that the new h' satisfies $\|h' - h\| < \epsilon$, has spectrum contained in $[l(h), r(h)]$ and has the property that there exist pairwise orthogonal non-zero positive elements $e_0, e_1, \ldots, e_n$ such that $h'e_i = t_i e_i$. For each $i = 0, 1, \ldots, n$, choose $x_i \in \overline{e_\iota A e_\iota}$ such that $x_i^2 = 0$ and $x_i^* x_i$ (and hence $x_i x_i^*$) has spectrum equal to $[0, 1]$. This is possible since $\overline{e_\iota A e_\iota}$ is simple and non-elementary. Now let

$$x := \sum_{i=0}^{n} (\delta/2)^{\frac{1}{2}} x_i \ \text{ and } \tilde{h} := h' - [x^*, x].$$

We claim that $\tilde{h}$ and x are as desired. It follows from the pairwise orthogonality of the $e_i$that $x^2 = 0$, that

$$\|x\|^2 = \frac{\delta}{2}\left\|\sum_{i=0}^{n} x_i^* x_i\right\| = \frac{\delta}{2}$$

and that

$$\tilde{h} = h' - \sum_{i=0}^{n} (\delta/2)(x_i^* x_i - x_i x_i^*).$$

Let us show that $\tilde{h}$ has spectrum $[l(h) - \delta/2, \ r(h) + \delta/2]$. Note that all of the elements $h', x_i^* x_i$ and $x_i x_i^* (0 \leq i \leq n)$ lie in a commutative C^*-subalgebra C containing the unit 1 of A. Evaluating the right-hand side of the expression for $\tilde{h}$ on the points of the spectrum of C where $x_i^* x_i$ is supported, all other terms except for h' vanish, while h' takes the constant value t_i. Since the spectrum of $x_i^* x_i$is $[0, 1]$, we obtain the interval $[t_i - \delta/2, t_i]$ in the spectrum of $\tilde{h}$. Evaluating on the points where $x_i x_i^*$ is supported, we obtain the interval $[t_i, t_i + \delta/2]$ in the spectrum of $\tilde{h}$. Doing this for all i, we obtain the interval $[l(h) - \delta/2, r(h) + \delta/2]$ in the spectrum of $\tilde{h}$. Evaluating on any other point in the spectrum of C, we obtain a value in the spectrum of h' which is contained in $[l(h), r(h)]$. Thus the spectrum of $\tilde{h}$ is as required.

Let A be simple, unital, non-elementary, with stable rank one and with strict comparison by traces. Using Cuntz semigroup classification results, one can prove the existence of a nuclear C^*-subalgebra $B \subseteq A$ with rather special properties. [401, Theorem 4.1] spells out the properties of B that we need:

(i) $B \cong C \otimes W$, where C is a simple AF C^*-algebra and W is the Jacelon–Razak algebra.

(ii) Every tracial state τ on B extends uniquely to a tracial state on A.

(iii) Every non-invertible self-adjoint element h in A with connected spectrum is approx-imately unitarily equivalent to a self-adjoint element in B. (Note: In the statement of [401, Theorem 4.1] the hypothesis that the self-adjoint h must be non-invertible is missing, though this is clearly necessary since B is non-unital. Moreover, this hypothesis is tacitly used in the last paragraph of the proof of [401, Theorem 4.1].)

A technique in [380] and [401] involves using these properties to reduce the proof of certain properties of self-adjoint elements in A to the case of self-adjoint elements in B. We will use the same technique to prove the following theorem. In this theorem, the initial hypotheses on A ensure that $T(A)$ is non-empty (surely, if we had $T(A) = \emptyset$ then the strict comparison-by-traces property and the simplicity of A would imply that A is purely infinite, contradicting that A has stable rank one.) Thus the later additional assumption that A has the Dixmier property is equivalent to assuming that A has a unique tracial state [374].

Theorem (6.3.46)[455]: Let A be a simple, unital, non-elementary C^*-algebra with stable rank one and strict comparison by traces. Then A has(P) with constants $M = 3 \cdot 7^3$ and $\Upsilon = 0.86$. Suppose, in addition, that A has the Dixmier property. Then A has the uniform singleton Dixmier property with these constants.

Proof. Suppose that the unitisation $B + \mathbb{C}1$ of B has (P) with constants (M', Υ'). We show how to obtain constants for A. Suppose that $h \in A$ is a self-adjoint element that is zero on every trace and such that $\|h\| = 1$. Let $\epsilon > 0$. From Lemma (6.3.45) (i), we obtain $h_1 = (1/3)h + (2/3)uhu^*$ whose spectrum has gaps at most $2/3 + \epsilon$. Applying Lemma (6.3.45) (ii) to h_1 with $\delta = 2/3 + \epsilon$, we obtain $\tilde{h}_1, x \in A$, as described, such that

$$h_1 \approx_\epsilon \tilde{h}_1 + [x^*, x]. \tag{24}$$

Notice, for later use, that since the positive elements x^*x and xx^* are orthogonal

$$\|[x^*, x]\| = \max\{\|x^*x\|, \|xx^*\|\} = \frac{\delta}{2} = \frac{1}{3} + \frac{\epsilon}{2}.$$

Also, by our choice of δ, the spectrum of $\tilde{h}_1$ is exactly the interval

$$[l(h_1) - \frac{1}{3} - \frac{\epsilon}{2}, r(h_1) + \frac{1}{3} + \frac{\epsilon}{2}].$$

From this and $\|h_1\| \leq 1$, we see that $\|\tilde{h}_1\| \leq 4/3 + \epsilon/2$. Moreover, $\tilde{h}_1$ is non-invertible. Indeed, its spectrum contains the closed interval $[l(h_1), r(h_1)]$ and the latter contains 0 since h_1 is zero on all traces and $T(A) \neq \emptyset$ (as argued before this theorem). By property (iii) of the C^*-subalgebra B above, there is a self-adjoint element $b \in B$ which is approximately unitarily equivalent to $\tilde{h}_1$. Notice, from (24), that $\sup\{|\tau(\tilde{h}_1)| : \tau \in T(A)\} < \epsilon$. Hence $\sup\{|\tau(b)| : \tau \in T(A)\} < \epsilon$. But

$$\sup\{|\tau(b)| : \tau \in T(A)\} = \sup\{|\tau(b)| : \tau \in T(B)\},$$

since tracial states of B extend to tracial states of A (property (ii) of B above). Hence $\sup\{|\tau(b)| : \tau \in T(B)\} < \epsilon$. It follows that there exists a self-adjoint $b' \in B$ such that $\tau(b') = 0$ for all $\tau \in T(B)$ and $\|b - b'\| < \epsilon$ (by [355, Theorem 2.9] and [432, Proof of Lemma 3.1]). Hence there is a unitary conjugate of $\tilde{h}_1$ which has distance from b' less than ϵ. Thus, since B has (P) with constants (M', Υ'), there exists an average of M' unitary conjugates of $\tilde{h}_1$of norm at most

$$\Upsilon'(\|\tilde{h}_1\| + \epsilon) + \epsilon \leq \Upsilon'(\frac{4}{3} + \frac{3\epsilon}{2}) + \epsilon.$$

Applying this average on both sides of (24), we find an average of $3M'$ unitary conjugates of the original element h with norm at most

$$\Upsilon'(\frac{4}{3} + \frac{3\epsilon}{2}) + \epsilon + (\frac{1}{3} + \frac{\epsilon}{2}) + \epsilon.$$

Since ϵ can be chosen arbitrarily small, we find that, provided $\frac{4}{3}\Upsilon' + \frac{1}{3} < 1$, A has (P) with constants

$$M = 3M' \text{ and every } \quad \Upsilon \in \left(\frac{4}{3}\Upsilon' + \frac{1}{3}, 1\right). \qquad (25)$$

Finally, let us find suitable constants for the unitisation $B + \mathbb{C}1$ of B. Since B has decomposition rank 1 and stable rank one, $B + \mathbb{C}1$ has the same properties and so has (P) with constants $M' = 15$ and arbitrary $\Upsilon' \in (11/15, 1)$ by Theorem (6.3.44). Therefore, it also has (P) with constants $M' = 15^3$ and arbitrary $\Upsilon' \in ((11/15)^3, 1)$. Putting the latter constants into the formula (25), we get that A has (P) with constants $M = 3 \cdot 15^3$ and $\Upsilon = 0.86$.

We derive results about the distance between Dixmier sets $D_A(a)$ and $D_A(b)$. Along the way, we obtain a description of $D_A(a) \cap Z(A)$ for C^*-algebras with the Dixmier property and we point out some cases in which the distance between $D_A(a)$ and $D_A(b)$ is attained. Here by the distance between two subsets D_1, D_2 of a C^*-algebra, we mean

$$d(D_1, D_2) := \inf\{\|d_1 - d_2\| : d_1 \in D_1, d_2 \in D_2\}.$$

Lemma (6.3.47)[455]: Let A be a unital C^*-algebra and let $a, b \in A$. The distance between the sets $D_A(a)$ and $D_A(b)$ is equal to the distance between the sets $D_{A^{**}}(a)$ and $D_{A^{**}}(b)$.

Proof. Let $r := d(D_{A^{**}}(a), D_{A^{**}}(b))$. It is clear that $r \le d(D_A(a), D_A(b))$. Let us prove the opposite inequality. Let $\epsilon > 0$ be given. Let $a' \in D_{A^{**}}(a)$ and $b' \in D_{A^{**}}(b)$ be such that $\|a' - b'\| < r + \epsilon$. Approximating a' and b' by averages of unitary conjugates there exists some N and unitaries $u_1, \dots, u_N, v_1, \dots, v_N \in U(A^{**})$ such that

$$\frac{1}{N}\sum_{i=1}^{N} u_i a u_i^* - \frac{1}{N}\sum_{i=1}^{N} v_i b v_i^* < r + \epsilon. \qquad (26)$$

By the version of Kaplansky's density theorem for unitaries [430, Theorem 4.11] (due to Glimm and Kadison, see [369, Theorem 2]), there exist commonly indexed nets of unitaries $(u_i, \lambda)_{\lambda \in \Lambda}, (v_i, \lambda)_{\lambda \in \Lambda} \in U(A)$ such that $u_{i,\lambda} \to u_i$ and $v_{i,\lambda} \to v_i$ in the ultrastrong $*$-topology for $i = 1, \dots, N$. Now consider

$$S := co\left\{\frac{1}{N}\sum_{i=1}^{N} u_{i,\lambda} a u_{i,\lambda}^* - \frac{1}{N}\sum_{i=1}^{N} v_{i,\lambda} b v_{i,\lambda}^* : \lambda \in \Lambda\right\}.$$

The weak$*$-closure of this convex set (in A^{**}) contains an element of norm less than $r + \epsilon$(namely, the element appearing in (26)), so by the Hahn–Banach theorem, S must also contain an element of norm less than $r + \epsilon$. (Otherwise, the Hahn–Banach theorem ensures the existence of a functional $\lambda \in A^*$ such that $Re(\lambda(x)) < r + \epsilon$ for all $\|x\| < r + \epsilon$ and $Re(\lambda(s)) \ge r + \epsilon$ for all $s \in S$; but then $\|\lambda\| \le 1$ and $Re(\lambda(s)) \ge r + \epsilon$ for all s in the weak$*$-closure of S, which is a contradiction.) However, note that $S \subseteq D_A(a) - D_A(b)$, so this shows that $d(D_A(a), D_A(b)) \le r + \epsilon$. Since ϵ is arbitrary, we are done.

Given a unital C^*-algebra A and $a \in A$, let $W_A(a) := \{\rho(a) : \rho \in S(A)\}$, the algebraic numerical range of a. Since the state space $S(A)$ is weak$*$-compact and convex, $W_A(a)$ is a compact convex subset of $\mathbb{C}$.

Lemma (6.3.48)[455]: Let A be a unital C^*-algebra and let $a, b \in A$. The following are true:

(i) $|\tau(a) - \tau(b)| \le d(D_A(a), D_A(b))$ for all $\tau \in T(A)$.

(ii) $d(W_{A/I}(q_I(a)), W_{A/I}(q_I(b))) \le d(D_A(a), D_A(b))$ for all closed ideals $I$of A.

Proof. (i): This is clear from the fact that traces are constant on Dixmier sets.

(ii): Since

$$d(D_{A/I}(q_I(a)), D_{A/I}(q_I(b))) \le d(D_A(a), D_A(a))$$

(because $q_I(D_A(a)) \subseteq D_{A/I}(q_I(a))$ and similarly for b), it suffices to consider the case when $I = 0$. We have

$$\inf\{|\rho_1(a) - \rho_2(b)|: \rho_1, \rho_2 \in S(A)\} \le \sup\{|\rho(a) - \rho(b)|: \rho \in S(A)\}$$
$$\le \|a - b\|.$$

Thus, $d(W_A(a), W_A(b)) \le \|a - b\|$. But if $\alpha, \beta \in A_V(A, U(A))$ are averaging operators then $W_A(\alpha(a)) \subseteq W_A(a)$ and $W_A(\beta(b)) \subseteq W_A(b)$. So

$$d(W_A(a), W_A(b)) \le \|\alpha(a) - \beta(b)\|.$$

Passing to the infimum over all $\alpha, \beta \in A_V(A, U(A))$ we get that

$$d(W_A(a), W_A(b)) \le d(D_A(a), D_A(b)),$$

as desired.

Lemma (6.3.47) allows us to reduce the calculation of the distance between Dixmier sets to the case that the ambient C^*-algebra is a von Neumann algebra. To deal with the von Neumann algebra case we rely on the following theorem of Halpern and Strătilă–Zsidó:

Theorem (6.3.49)[455]: Let M be a properly infinite von Neumann algebra with centre Z and strong radical J (i.e., J is the intersection of all maximal ideals of M). Let $a \in M$. The following are equivalent:

(i) $0 \in D_M(a)$.

(ii) There exists a Z-linear, positive, unital map $\phi: M \to Z$ such that $\phi(J) = 0$ and $\phi(a) = 0$.

(iii) $0 \in W_{M/I}(q_I(a))$ for every maximal ideal I of M.

Proof. (i)⇒(iii): If $0 \in D_M(a)$ then $0 \in D_{M/I}(q_I(a))$ for every maximal ideal I of M. By Lemma (6.3.48), $0 \in W_{M/I}(q_I(a))$, as desired.

(iii)⇒(ii): This is [429, Proposition 7.3].

(ii)⇒(i): This follows from Halpern's [376, Theorem 4.12].

The following theorem extends Theorem (6.3.2) ([402, Theorem 4.7]) to non-self-adjoint elements.

Theorem (6.3.50)[455]: Let A be a unital C^*-algebra and let $a \in A$. Then $0 \in D_A(a)$ if and only if

(a) $\tau(a) = 0$ for all $\tau \in T(A)$, and

(b) in no nonzero quotient of A can the image of $Re(wa)$, with $w \in \mathbb{C}$, be invertible and negative.

Condition (b) need only be checked on all the quotients by maximal ideals of A. A reformulation of (b) is

(b') on every nonzero quotient there exists a state that vanishes on a; i.e., $0 \in W_{A/I}(q_I(a))$ for all closed ideals I of A.

To see this, suppose that $\rho(a) = 0$ for some $\rho \in S(A)$. For all $w \in \mathbb{C}, \rho(wa) = 0$ and so $\rho(Re(wa)) = 0$. Hence $Re(wa)$ is not invertible and negative. Conversely, suppose that $0 \notin W_A(a)$. Then by convexity of $W_A(a)$, for a suitable $w \in \mathbb{C}$ and $\epsilon > 0, Re(\rho(wa)) \le -\epsilon < 0$ for all states ρ, i.e., $\rho(Re(wa)) \le -\epsilon < 0$ for all states ρ. But this implies that $Re(wa)$ is negative and invertible. This equivalence holds similarly in every nonzero quotient. Notice that if every nonzero quotient of A has a tracial state then (b') follows from (a).

*A*nother reformulation of (b) is the following:

$(b")(Re(wa)+t)_+$is a full element (i.e., generates A as a closed two-sided ideal) for all $t>0$ and all $w\in\mathbb{C}$.

To see that this is equivalent to Theorem (6.3.50) (b), notice first that $Re(wa)\le -t1$ in the quotient by the closed two-sided ideal generated by $(Re(wa)+t)_+$, where $a\mapsto\bar{a}$ is the quotient map for this ideal. So, assuming (b), this quotient must be {0}, i.e., $(Re(wa)+t)_+$ is full for all $t>0$. On the other hand, if $Re(w\bar{a})\le -t1$ in the quotient by some ideal I, then clearly $(Re(wa)+t1)_+\in I$. So, if $t>0$, and assuming $(b")$, we get that $I=A$.

Proof. Since traces are constant on Dixmier sets, if $0\in D_A(a)$ then $\tau(a)=0$ for all $\tau\in T(A)$, i.e., (a) holds. Also, if $0\in D_A(a)$ then $0\in D_A(wa)$ for any $w\in\mathbb{C}$ (indeed, any central element) and this prevents $Re(wa)$ from being invertible and negative. The same holds for quotients since $q_I(D_A(a))\subseteq D_{A/I}(q_I(a))$. Thus, (b) holds as well.

Suppose now that $a\in A$ is such that (a) and (b) hold. If $A\subseteq B$(where B is a C^*-algebra with the same unit as A) then (a) and (b) also hold in B. This is clear for condition (a), since traces of B restrict to traces of A. This is also clear for condition (b"), which is equivalent to (b). It follows that a satisfies (a) and (b) in the von Neumann algebra A^{**}. Let $A_f^{**}\ \oplus A_{Pi}^{**}$ be the decomposition of A^{**} into a finite and a properly infinite von Neumann algebra and let $a=a_f+a_{pi}$ be the corresponding decomposition of a. From condition (a) we get that $R(a_f)=0$, where R denotes the centre-valued trace, which in turn implies that $0\in D_{A_f^{**}}(a_f)$. On the other hand, from condition (b) we get that for every maximal ideal I of A_{pi}^{**} there exists a state on A_{pi}^{**}/I that vanishes on $q_I(a_{pi})$. By Theorem (6.3.49), $0\in D_{A_{pi}^{**}}(a_{pi})$. Since we may extend unitary elements in A_f^{**} (respectively A_{pi}^{**}) by adding the unit of A_{pi}^{**} (respectively A_f^{**}), we conclude that $0\in D_{A^{**}}(a)$. By Lemma (6.3.47), $0\in D_A(a)$.

Corollary (6.3.51)[455]: Let A be a unital C^*-algebra with the Dixmier property and let $a\in A$. Let $Y\subseteq Max(A)$ be the closed set of maximal ideals M such that A has a (unique) tracial state $\tau_M$that vanishes on M. Then $D_A(a)\cap Z(A)$is mapped, via the Gelfand transform, onto the set of $f\in C(Max(A))$such that

$$f(M)=\tau_M(a) \text{ if } M\in Y,$$
$$f(M)\in\ W_{A/M}(q_M(a)) \text{ otherwise.}$$

Proof. Let $z\in D_A(a)\cap Z(A)$ and let $f\in C(Max(A))$ be its Gelfand transform (that is, $f=\theta(z)$ where $\theta:Z(A)\to C(Max(A))$ is the canonical $*$-isomorphism discussed prior to Corollary (6.3.13)). Let $M\in Max(A)$. Since $0\in D_A(a-z)$, we have by Lemma (6.3.48) (ii) that

$$0\in W_{A/M}(q_M(a-z))=W_{A/M}(q_M(a))-f(M),$$

i.e., $f(M)\in W_{A/M}(q_M(a))$. Also, $f(M)=\tau_M(z)=\tau_M(a)$ for all $M\in Y$. Thus, f is as required.

Conversely, let $f\in C(Max(A))$ be as in the statement of the corollary. Let $z\in Z(A)$ be the central element whose Gelfand transform is f. Then

$$0\in W_{A/M}(q_M(a))-f(M)=W_{A/M}(q_M(a-z))$$

for all $M\in Max(A)$. Also, $\tau_M(a-z)=0$ for all $M\in Y$, and since $\partial_eT(A)=\{\tau_M: M\in Y\}$ (Theorem (6.3.12)), $\tau(a-z)=0$ for all $\tau\in T(A)$ by the Krein–Milman theorem. By Theorem (6.3.50), $0\in D_A(a-z)$, i.e., $z\in D_A(a)$, as desired.

We extend Theorem (6.3.50) to a distance formula between the sets $D_A(a)$ and $D_A(b)$ (Theorem (6.3.58) below). Note that one cannot reduce the calculation of this distance to the case that one element is 0 by looking at the distance between $D_A(b-a)$ and 0, since $d(D_A(a), D_A(b))$ is in general not the same as $d(D_A(b-a), 0)$. For an example of this, let a be a non-invertible positive element of norm 1 in a simple unital infinite C^*-algebra A and define $b := 1 + a$. Then $D_A(a) \cap Z(A) = [0,1]$ and $D_A(b) \cap Z(A) = [1,2]$ (as sets of scalar elements of A) (see Corollary (6.3.16) or [374]), so that $d(D_A(a), D_A(b)) = 0$. However, $b - a = 1$ so that $d(D_A(b-a), 0) = 1$.

Lemma (6.3.52)[455]: Let $(I_\lambda)_\lambda$ be a collection of closed ideals of a C^*-algebra A and let I be a closed ideal of A such that $\bigcap_\lambda I_\lambda \subseteq I$. Then every state of A which vanishes on I is a weak$*$-limit of convex combinations of states vanishing on the I_λ's.

Recall that for topological spaces X and Y, a set-valued function $\phi: X \to \{subsets\ of\ Y\}$ is defined to be lower semicontinuous if for every open set U of Y, the set

$$\{x \in X: \phi(x) \cap U \neq \emptyset\}$$

is open in X. We will use the Michael selection theorem [396, Theorem 3.1’].

The next lemma is implicit in a strategy mentioned in [395].

Lemma (6.3.53)[455]: Let A be a unital C^*-algebra and let $a \in A$. The set-valued function on $Max(A)$ defined by

$$M \longmapsto W_{A/M}(q_M(a))\ for\ all\ M \in Max(A)$$

is lower semicontinuous.

Proof. Let $\Phi_a(M) := W_{A/M}(q_M(a))$ for all $M \in Max(A)$. Let $M \in Max(A), w \in \Phi_a(M)$ and $\epsilon > 0$. We must show that $\Phi_a(M') \cap B_\epsilon(w)$ is non-empty for all M' in a neighbourhood of M. Suppose, for the sake of contradiction, that there exists a net $M_\lambda \to M$ such that $\Phi_a(M_\lambda) \cap B_\epsilon(w) = \emptyset$ for all λ. For each λ we can separate the sets $\Phi_a(M_\lambda)$ and $B_\epsilon(w)$ by a line $\ell_\lambda$tangent to the circle $\{z: |z - w| = \epsilon\}$. Let $c_\lambda \in \ell_\lambda$ denote the point of tangency. Let us pass to a subnet $M_{\lambda'} \to M$ such that the $c_{\lambda'} \to c$, and let ℓ be the tangent line at c. Since the sets $\Phi_a(M_{\lambda'})$ are uniformly bounded (they are all inside the ball $\overline{B_{\|a\|}(0)}$), there exists λ'_0 such that the sets $\Phi_a(M_{\lambda'})$ for $\lambda' \geq \lambda'_0$ are all separated from the ball $B_{\epsilon/2}(w)$ by a single line ℓ_0 parallel to ℓ (and tangent to the circle $\{z: |z - w| = \epsilon/2\}$). But, since $\bigcup_{\lambda' \geq \lambda'_0} M_{\lambda'} \subseteq M$, we have by the previous lemma that any state of A which vanishes on M is a weak$*$-limit of convex combinations of states vanishing on the $M_{\lambda'}$'s. In particular, $w (= \rho(a)$ for some state ρ of A which vanishes on $M)$ is a limit of convex combinations of elements in $\bigcup_{\lambda' \geq \lambda'_0} \Phi_a(M_{\lambda'})$. This contradicts that we can separate $\bigcup_{\lambda' \geq \lambda'_0} \Phi_a(M_{\lambda'})$ from $B_{\epsilon/2}(w)$ by the line ℓ_0.

Let us describe more specifically how to obtain λ'_0. The line ℓ_0 is parallel to ℓ but closer to w. We may therefore choose λ'_0 such that all the points $\{c_{\lambda'}: \lambda' \geq \lambda'_0\}$ lie on the same side of ℓ_0 and such that the lines $\ell_{\lambda'}$ and ℓ_0 intersect outside of the ball $\overline{B_{\|a\|}(0)}$ for all $\lambda' \geq \lambda'_0$. Then λ'_0 is as desired.

Given a subset S of a metric space and $r > 0$ we denote by S^r the set $\{y: d(y,S) < r\}$.

Lemma (6.3.54)[455]: Let f, g be lower semicontinuous set-valued functions on a topological set X taking values in the subsets of a metric space Y. Let $r > \sup\{d(f(x), g(x)): x \in X\}$. Then the set-valued functions $x \longmapsto f(x) \cap (g(x))^r$ and $x \longmapsto \overline{f(x) \cap (g(x))^r}$ are lower semicontinuous.

Proof. Let us show that $h(x) := f(x) \cap (g(x))^r$ is lower semicontinuous. Let $x \in X, z \in h(x)$ and $\epsilon > 0$. We must show that there exists a neighbourhood U of x such that $h(y) \cap B_\epsilon(z) \neq \emptyset$ for all $y \in U$. Let $w \in g(x)$ be such that $r' := d(z, w) < r$. Let $\delta := \min((r - r')/2, \epsilon)$. By the lower semicontinuity of f and g we can find a neighbourhood U of x such that $f(y) \cap B_\delta(z)$ and $g(y) \cap B_\delta(w)$ are nonempty for all $y \in U$. Let $y \in U$, so that there exist $z' \in f(y) \cap B_\delta(z)$ and $w' \in g(y) \cap B_\delta(z)$. Then, using the triangle inequality, $d(z', w') < r$, so that $z' \in h(y)$. Also by the choice of $\delta, z' \in B_\epsilon(z)$, so $h(y) \cap B_\epsilon(z)$ is nonempty, as required.

Let us show that $x \longmapsto \overline{h(x)}$ is also lower semicontinuous. Let $V \subseteq Y$ be an open set. Suppose that $\overline{h(x)} \cap V \neq \emptyset$ for some $x \in X$. Then $h(x) \cap V \neq \emptyset$, and by the lower semicontinuity of h we find a neighbourhood U of x such that $h(y) \cap V \neq \emptyset$ for all $y \in U$. Then, $h(y) \cap V \neq \emptyset$ for all $y \in U$, as required.

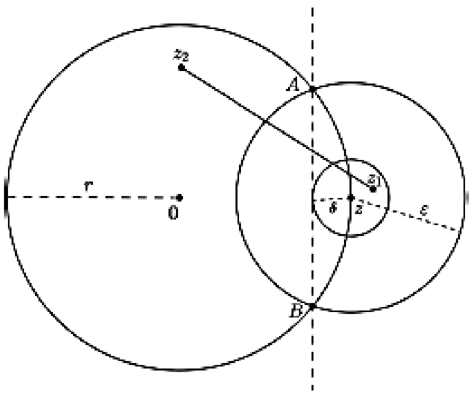
Lemma (6.3.55)[455]: Let $r > 0$. Let f be a lower semicontinuous set-valued function on a topological space X taking values in the convex subsets of C and such that $f(x) \cap \overline{B_r(0)} \neq \emptyset$ for all x. Then

$$h(x) := f(x) \cap \overline{B_r(0)}$$

is lower semicontinuous.

Proof. Let $x \in X$ and $z \in h(x)$. Let $\epsilon > 0$ and, without loss of generality, assume $\epsilon < r$. We must show that $h(y) \cap B_\epsilon(z)$ is nonempty for all y in a neighbourhood of x. Suppose first that $|z| < r$. Let $\delta := \min(\epsilon, r - |z|)$. Then $B_\delta(z) \subseteq B_r(0)$, by the triangle inequality. Since f is lower semicontinuous, $f(y) \cap B_\delta(z) \neq \emptyset$ for all y in a neighbourhood of x, and so $h(y) \cap B_\epsilon(z) \neq \emptyset$ for all such y.

Assume now that $|z| = r$. Let $\delta := \epsilon^2/2r$, as shown in the diagram, so that the circle of centre z and radius δ is tangent to the segment $[A, B]$. Since f is lower semicontinuous, $f(y) \cap B_\delta(z) \neq \emptyset$ for all y in a neighbourhood U of x. Let $y \in U$. Say $z_1 \in f(y) \cap B_\delta(z)$. Recall also that, by assumption, there exists $z_2 \in f(y)$ such that $|z_2| \leq r$. Since the segment $[z_1, z_2]$ is contained in $f(y)$, it suffices to show that $[z_1, z_2]$ intersects $B_\epsilon(z) \cap \overline{B_r(0)}$.



If the points z_1 and z_2 are on the same side of the line AB then $z_2 \in B_\epsilon(z)$. If the points z_1 and z_2 are on different sides of this line AB (as in the figure) then the segment $[z_1, z_2]$ intersects the segment $[A, B]$. (Note for this that the tangents at A and B to the circle centred at 0 are also tangential to the circle centred at z with radius δ.)

Let A be a unital C^*-algebra A with the Dixmier property. Let $Y \subseteq Max(A)$ be the set of maximal ideals M such that A has a (unique) tracial state τ_M that vanishes on M. Recall that

Y is closed and $M \longmapsto \tau_M(a)$ is continuous on Y for all $a \in A$ (Theorem (6.3.12)). Let $a \in A$. Define a set-valued function F_a on $Max(A)$ as follows:

$$F_a(M) := \begin{cases} \{\tau_M(a)\} & if\ M \in Y, \\ W_{A/M}(q_M(a)) & otherwise. \end{cases}$$

The values of F_a are compact convex subsets of $\mathbb{C}$. Since $M \longmapsto W_{A/M}(q_M(a))$ is lower semicontinuous by Lemma (6.3.53), Y is closed, $F_a|_Y$is continuous, and $\tau_M(a) \in W_{A/M}(q_M(a))$ for $M \in Y$, the set-valued function F_a is lower semicontinuous.

The following proposition is trivial in the case of the singleton Dixmier property.

Proposition (6.3.56)[455]: Let A be a unital C^*-algebra with the Dixmier property, and let $a, b \in A$. Set

$$r := sup_{M \in Max(A)}\ d(F_a(M), F_b(M)).$$

Then the distance between $D_A(a)$ and $D_A(b)$ is equal to r. If either a and b are both self-adjoint, or $b = 0$ then this distance is attained by elements in $D_A(a) \cap Z(A)$ and $D_A(b) \cap Z(A)$.

Proof. The inequality $r \leq d(D_A(a), D_A(b))$ follows at once from Lemma (6.3.48).

Let $\epsilon > 0$. By Lemma (6.3.54), the set-valued function

$$F(M) := F_a(M) \cap (F_b(M))^{r+\epsilon}\ for\ M \in Max(A)$$

is lower semicontinuous. Since its values are closed convex sets, by Michael's selection theorem there exists a continuous function $f: Max(A) \to \mathbb{C}$ such that $f(M) \in F(M)$ for all M. Let $z_a \in Z(A)$ be the central element whose Gelfand transform is f. Since $f(M) \in F(M) \subseteq F_a(M)$ for all M we have that $z_a \in D_A(a)$ by Corollary (6.3.51). Let us define

$$G(M) := \overline{\{f(M)\}^{r+2\epsilon} \cap F_b(M)}\ for\ M \in Max(A).$$

Then again this is a lower semicontinuous function taking closed convex set values. So there exists a continuous $g: Max(A) \to \mathbb{C}$ such that $g(M) \in G(M)$ for all M. Let $z_b \in Z(A)$ be the central element whose Gelfand transform is g. As with z_a, we have that $z_b \in D_A(b)$. Also, since $|f(M) - g(M)| \leq r + 2\epsilon$ for all M we have that $\|z_a - z_b\| \leq r + 2\epsilon$. This ends the proof that $r = d(D_A(a), D_A(b))$.

Suppose now that $b = 0$, and let us show that the distance from $D_A(a)$ to 0 is attained. Since $r = \sup\{d(0, F_a(M)): M \in Max(A)\}$, the set $F_a(M) \cap \overline{B_r(0)}$ is nonempty for all M. Thus, by Lemma (6.3.55), the set-valued function $M \longmapsto F_a(M) \cap \overline{B_r(0)}$ is lower semicontinuous. Since it takes values on the closed convex subsets of $\mathbb{C}$, there exists, by Michael's selection theorem, a continuous function $f: Max(A) \to \mathbb{C}$ such that $f(M) \in F_a(M) \cap \overline{B_r(0)}$ for all M. Let zabe the central element whose Gelfand transform is f. Then $z_a \in D_A(a)$ and $\|z_a\| \leq r$, as desired.

Finally, suppose that a and b are self-adjoint. Then

$$W_{A/M}\big(q_M(a)\big) = [f_a(M), h_a(M)],$$
$$W_{A/M}(q_M(b)) = [f_b(M), h_b(M)]$$

for all $M \in Max(A)$. Here $f_a(M) := \min(sp(q_M(a))), h_a(M) := max(sp(q_M(a)))$ and similarly for $f_b(M)$ and $h_b(M)$. As in the proof of Theorem (6.3.12), $f_a, f_b: Max(A) \to \mathbb{R}$ are upper semicontinuous functions and $h_a, h_b: Max(A) \to \mathbb{R}$ are lower semicontinuous. For each $M \in Max(A)$ define

$$G(M) = \begin{cases} \{\tau_M(a)\} & if\ M\ \in\ Y, \\ [f_a(M), h_a(M)] \cap [f_b(M) -\ r, h_b(M) + r] & otherwise. \end{cases}$$

Observe that $G(M)$ is a nonempty closed interval for all M. The assignment $M \longmapsto G(M)$ is a lower semicontinuous set-valued function. Hence, it has a continuous selection $g(M) \in G(M), g \in C(Max(A))$. (Alternatively, we can derive the existence of g from the Katětov–Tong theorem as in the proof of Theorem (6.3.12).) Let z_a denote the central element whose Gelfand transform is g. Then $z_a \in D_A(a)$. Now consider the assignment

$$M \longmapsto [g(M) - r, g(M) + r] \cap F_b(M).$$

It is again lower semicontinuous and takes values in the closed intervals of $\mathbb{R}$. Hence, it has a continuous selection giving rise to a central element $z_b \in D_A(b)$ such that $\|z_a - z_b\| \leq r$.

Example (6.3.57)[455]: For general elements a and b in a C^*-algebra with the Dixmier property, the distance from $D_A(a)$ to $D_A(b)$ need not be attained. Let $A = C([-1,1], \mathcal{O}_2)$. Then $[-1,1]$ is homeomorphic to $Max(A)$ via the assignment

$$s \to M_s := C_0([-1,1]\backslash\{s\}, O_2) \; for \; s \in [-1,1].$$

Since A is weakly central and has no tracial states, it has the Dixmier property by Theorem (6.3.12) (this can also be seen from the fact that A is $*$-isomorphic to the tensor product of $\mathcal{O}_2$ with an abelian C^*-algebra).

Fix a non-invertible positive element $h \in \mathcal{O}_2$ of norm 1and define a continuous function $G: [-1,1] \times [0,1] \to \mathbb{C}$, by

$$G(s,t) := \begin{cases} (1 + si)t & if \; s \in [-1,0], \\ si + (1 - si)t \; if \; s & \in [0,1]. \end{cases}$$

Now define the set-valued function

$$F(s) := \{G(s,t) : t \in [0,1]\}, \text{ for } s \in [-1,1].$$

Observe that the values of $F$are closed intervals in $\mathbb{C}$ (for $s \in [-1,0]$ the set $F(s)$ is an interval swinging like a door with the hinges at 0, while for $s \in [0,1]$ the interval $F(s)$ also swings but with the hinges at 1.)

Now define $a, b \in A$ by $a(s) := G(s,h)$ (functional calculus), and $b(s := h$ for all $s \in [-1,1]$. One can see then that $F_a(M_s) = F(s)$ and $F_b(M_s) = [0,1]$ for all s. It follows by the previous proposition that the distance between $D_A(a)$ and $D_A(b)$ is 0. However, $D_A(a)$ and $D_A(b)$ have no elements it common. For if they did, then $D_A(a) \cap D_A(b) \cap Z(A)$ would be nonempty. By Corollary (6.3.51), elements of $D_A(a) \cap D_A(b) \cap Z(A)$ correspond to continuous selections of $s \longmapsto F_a(M_s) \cap F_b(M_s)$. However, there are no such continuous selections, because

$$F_a(M_s) \cap F_b(M_s) = \begin{cases} \{0\} & for \; s \in [-1,0), \\ [0,1] & for \; s = 0, \\ \{1\} & for \; s \in (0,1]. \end{cases}$$

We now extend the distance formula from Proposition (6.3.56) to arbitrary C^*-algebras. The following result gives a formula for the distance between the Dixmier sets of two elements of an arbitrary unital C^*-algebra. A similar result is [402, Theorem 4.3], which gives a formula for the distance between one self-adjoint element and the Dixmier set of another; these results say the same thing in the case that both elements are self-adjoint and one is central.

Theorem (6.3.58)[455]: Let A be a unital C^*-algebra and let $a, b \in A$. Then the following numbers are equal:

(i) The distance between $D_A(a)$ and $D_A(b)$.

(ii) The minimum number $r \geq 0$ satisfying

(a) $|\tau(a-b)| \leq r$ for all $\tau \in T(A)$, and
(b) $d(W_{A/M}(q_M(a)), W_{A/M}(q_M(b)) \leq r$ for all $M \in Max(A)$.

Proof. The inequality $r \leq d(D_A(a), D_A(b))$ has already been proven in Lemma (6.3.48). We check that (ii)(a) and (ii)(b) with A^{**} in place of A still hold (without changingr). For (ii)(a), this follows since every tracial state on A^{**} restricts to a tracial state on A. Similarly, for any ideal I of A^{**}, since $A/(I \cap A) \subseteq A^{**}/I$,

$$W_{A/I\cap A}(q_{I\cap A}(a)) = W_{A^{**}/I}(q_I(a)).$$

From this we see that (ii)(b) holds for A^{**}. But A^{**}, being a von Neumann algebra, has the Dixmier property. Hence $r \geq d(D_{A^{**}}(a), D_{A^{**}}(b))$ by Proposition (6.3.56). The theorem now follows from Lemma (6.3.47).

List of Symbles

Symble	List of Symbles	page
$\oplus$	Direct Sum	8
Max	Maximum	10
Min	Minimum	14
$\otimes$	Tensor Product	18
ℓ^2	Hilbert Space of Sequences	18
Aut	Automorphism	20
Ker	Kernel	32
H^2	Hardy Space	33
$\ominus$	Direct difference	33
a.e	Al most every where	35
H^∞	Essential Hardy Space	35
L^2	Hilbert Space	35
Sup	Supremum	36
Ran	Range	36
Ind	Index	36
L^∞	Essential Lebesgue Space	42
det	Determinant	51
Deg	Degree	52
Cl	Closure	59
AAK	Adamyan Arov Krein	62
L_a^2	Bergman Space	64
CO_R	Convex	73
SP	Spectrum	87
Red	Reduced	89
UHF	Ultra High Frequency	89
AF	Approximately finite-dimensional	89
Diag	Diagonal	91
Dim	Dimension	93
Im	Imaginary	104
UCT	Coordinated Universal Time B	106
CSO	Complete Symmetric Operators	156

ONB	Orthonormal basis	156
SOT	Strong Operator Topology	157
BDF	Backward Differentiation Formula	176
Nor	Normal	179
Abnor	Abnormal	194
Card	Cardinality	196
UET	Unitary equivalent	196
CS	Complex symmetric	196
Wcs	World Coordinate System	197
CSSO	Completely Symmetric Square Operator	200
m.r.s	Minimal reducing subspaces	202
c.p	Comletely Positive	218
c.p.c	Completely Positive Contraction	219
Nuc	Nuclear	219
Dr	decomposition rank	219
Cov	Covering	229
ℓ_∞	Essential Hilbert Space of Sequences	229
Ann	Annihilator	229
l.s.c	Lower semi continuous	233
Tr	Trace	234
Ctr	Centre-valued trace	244
QTS	Quotient tracial state	244
GNS	Gelfand-Naimark-segal	245
Aff	Affine	252
Prim	Prime	282

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